On the Stability of Laminar Flows Between Plates

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Abstract

Consider a two-dimensional laminar flow between two plates, so that \((x_1, x_2) \in \mathbb{R} \times [-1, 1]\), given by \(v(x_1, x_2) = (U(x_2), 0)\), where \(U \in C^4([-1, 1])\) satisfies \(U' \neq 0\) in \([-1, 1]\). We prove that the flow is linearly stable in the large Reynolds number limit, in two different cases:

- \(\sup_{x \in [-1,1]} |U''(x)| + \sup_{x \in [-1,1]} |U'''(x)| \ll \min_{x \in [-1,1]} |U'(x)|\) (nearly Couette flows),
- \(U'' \neq 0\) in \([-1, 1]\).

We assume either no-slip or fixed traction force (Navier-slip) conditions on the plates, and an arbitrary large (but much smaller than the Reynolds number) period in the \(x_1\) direction.

Contents

1. Introduction .................................... 1282
2. Spectral Problem Formulation and Main Results ........................................ 1286
   2.1. Hodge Theory ................................ 1286
   2.2. Zero Flux Solution .......................... 1288
   2.3. Longitudinal Average ........................ 1291
   2.4. The Semigroup \(e^{-t \mathbb{T}_P}\) .................. 1292
   2.5. Main Results .................................. 1293
3. The Orr–Sommerfeld Operator .......................... 1295
   3.1. Stream Function .............................. 1295
   3.2. Inverse Estimates ............................ 1297
4. The Inviscid Operator ............................... 1299
   4.1. The Case \(\Re \lambda = 0\) : Preliminaries .......... 1299
   4.2. Construction of a Right Inverse of \(A_{i,v,\alpha}\) .................. 1301
   4.3. Nearly Couette Velocity Fields ................ 1304
1. Introduction

Consider the incompressible Navier-Stokes equations in the two-dimensional pipe $D = \mathbb{R} \times (-1, 1)$

$$\begin{align*}
\partial_t v + \varepsilon \Delta v + v \cdot \nabla v &= -\nabla p \quad \text{in } \mathbb{R}_+ \times D \\
v &= v_b \hat{i}_1 \quad \text{on } \mathbb{R}_+ \times \partial D,
\end{align*}$$

where $\hat{i}_1 = (1, 0)$ and $v = (v_1, v_2)$.

We require periodicity in the $x_1$ direction, i.e., for all $(t, x_1, x_2) \in \mathbb{R}_+ \times D$, we have

$$v(t, x_1 + L, x_2) = v(t, x_1, x_2)$$  (1.2)
for some $L > 0$, which may be arbitrary large, but must satisfy $L \ll \varepsilon^{-1}$ as $\varepsilon \to 0$. An initial condition on $v$ at $t = 0$ must be placed as well. The vector $v = (v_1, v_2)$ denotes the fluid velocity field which belongs, for all $T > 0$, to

\[ \mathcal{W}_{T,L} = \{ u \in L^2(0, T; H^2_{loc}(D, \mathbb{R}^2)) : \partial_t u \in L^2(0, T; L^2_{loc}(D, \mathbb{R}^2)), \text{div } u = 0, \ u(t, x_1 + L, x_2) = u(t, x_1, x_2) \}, \]

where the divergence is taken in the spatial coordinates.

Here $u \in H^k_{loc}(D)$ for some $k \in \mathbb{N}$ means that for any $\phi \in C_0^\infty(\mathbb{R})$, it holds that $(x_1, x_2) \mapsto \phi(x_1)u(x_1, x_2) \in H^k(D)$. Similarly, $u \in H^k_{0,loc}(D)$ means that for any $\phi \in C_0^\infty(\mathbb{R})$, $(x_1, x_2) \mapsto \phi(x_1)u(x_1, x_2) \in H^k_0(D)$. Recall that

\[ v \cdot \nabla v = v_1 \frac{\partial v}{\partial x_1} + v_2 \frac{\partial v}{\partial x_2}. \]

The pressure $p$ belongs, for all $T > 0$, to

\[ Q_{T,L} = \{ p \in L^2(0, T; H^1_{loc}(D)) : \nabla p(t, x_1 + L, x_2) = \nabla p(t, x_1, x_2) \}. \]

The trace of $v$ on the boundary is constant on each connected component of $\mathbb{R}_+ \times \partial D$:

\[ \{ v_b(-1), v_b(1) \} \in \mathbb{R}^2. \]

The parameter $\varepsilon > 0$ denotes the inverse of the flow’s Reynolds number $Re > 0$. Beyond the above no-slip boundary condition we shall also consider a prescribed constant traction force on the boundary (or Navier-slip conditions [14]), i.e.,

\[ \frac{\partial v_1}{\partial x_2} = s_b, \quad v_2 = 0, \quad (1.3) \]

where $s_b(\pm 1) \in \mathbb{R}$ denote the prescribed traction force on $\partial D = \mathbb{R} \times \{ \pm 1 \}$.

We consider the stability of a stationary pair $(v, p)$ where the flow $v$ is of the form, for $(x_1, x_2) \in D$,

\[ v(x_1, x_2) = U(x_2) \hat{i}_1. \]

For such flows, (1.1) is satisfied for $(v, p)$ if and only if there exist some constants $a$ and $p_0$ such that, for $(x_1, x_2) \in D$,

\[ U''(x_2) \equiv a; \quad p_0(x_1, x_2) = \varepsilon a x_1 + p_0. \quad (1.4) \]

Depending on the boundary condition (no-slip or fixed traction force), $U$ should satisfy an inhomogeneous Dirichlet condition $U(\pm 1) = v_b(\pm 1)$ or an inhomogeneous Neumann condition $U'(\pm 1) = s_b(\pm 1)$.

We shall not confine ourselves, however, in the sequel to such unperturbed velocity fields and discuss a more general class of motions (cf. also [16, p. 154]), which can be obtained if we add a non-uniform body force $b(x_1, x_2) = b(x_2) \hat{i}_2$ to the right-hand-side of (1.1). Such a generalization can be useful if one attempts to examine the stability of a flow in an arbitrary 2D cross-section but uniform in
the longitudinal direction. The linearized operator associated with (1.1), at the flow \((v, p)\), assumes the form

\[(u, q) \mapsto \mathcal{T}_0(u, q) := \mathcal{T}u - \nabla q \tag{1.5} \]

where

\[
\mathcal{T}u = -\varepsilon \Delta u + U \frac{\partial u}{\partial x_1} + u_2 U' \hat{\gamma}_1, \quad \text{in} \ D, \tag{1.6}
\]

\[q \in Q_L \text{ which is given by}
\]

\[Q_L = \{ p \in H^1_{loc}(\overline{D}) \mid \nabla p(\cdot + L, \cdot) = \nabla p(\cdot, \cdot) \text{ and} \int_{(0,L) \times (-1,+1)} p(x_1, x_2)dx_1dx_2 = 0 \}, \]

and \(u = (u_1, u_2)\) belongs to either

\[\mathcal{W} = \{ u \in H^2_{loc}(\overline{D}, \mathbb{R}^2) \mid \text{div} \ u = 0; \ u(\cdot + L, \cdot) = u(\cdot, \cdot) \}. \tag{1.7}\]

\textbf{Remark 1.1.}

\begin{itemize}
  \item Note that when no-slip boundary conditions, introduced in (1.1), are applied, then \(u \in \mathcal{W}_D\). Otherwise, if we select (1.3) instead, then \(u \in \mathcal{W}_S\).
  \item For any \(p \in Q_L\) there exists \(A \in \mathbb{R}\) and \(\tilde{p}\) such that \(p(x_1, x_2) = Ax_1 + \tilde{p}(x_1, x_2)\) with \(\tilde{p}\) satisfying the periodicity condition \(\tilde{p}(\cdot + L, \cdot) = \tilde{p}(\cdot, \cdot)\).
\end{itemize}

We now attempt to define a spectral problem for \(\mathcal{T}_0\). We seek an estimate for the solution \((u, q)\) for some \(\Lambda \in \mathbb{C}\) and \(F\) in a suitable space of the equation

\[\mathcal{T}_0(u, q) - \Lambda u = F. \tag{1.7}\]

To this end we need to define the function space in which the solution should reside, and then to formulate an effective spectral problem involving only \(u\), so that \(q\) is recovered in a final step directly from \(u\).

The local stability of the flow (1.4) has been addressed mostly by physicists and engineers [16,32,40,46]. The Poiseuille flow \((U(x) = 1 - x^2)\), which falls outside the scope of this work, and the Couette flow \((U(x) = x)\), have received some special attention. Thus, for Couette flows it has been established, using a mix of numerical and analytical techniques that Couette flow is always stable [39,43], and [37] establishes numerically that Poiseuille flow looses its stability for \(\varepsilon^{-1} \approx 5772\). In [41] a survey of results published in Russian is presented where the locus of part of the spectrum (but not its left margin) for Couette and Poiseuille flows is approximated in the limit \(\varepsilon \to 0\).
In recent years a number of rigorous mathematical works addressing the spectral problem associated with (1.7) have been published. In particular, in [14], both resolvent and semigroup estimates have been obtained for the case of Couette flow \((U = x_2)\). We shall relate the results in [14] to the present work in more detail in the sequel. More recently in [10] it has been established that the semigroup can be represented as a sum of the one generated by an unbounded Couette flow and an exponentially fast decaying boundary term. For symmetric flows in a channel (including Poiseuille flow) it has been proved in [21] that the laminar flow looses stability for sufficiently small \(\varepsilon\) and sufficiently large \(L\). The stability analysis of a Prandtl boundary layer, leading to a similar problem in \(D = \mathbb{R} \times \mathbb{R}_+\), was studied in [18,22,23]. In three dimensions, the stability of radially symmetric Poiseuille flow for sufficiently small \(\varepsilon\) has been proved in [15]. It is also worthwhile mentioning [29,44] addressing Kolmogorov flows.

The case \(\varepsilon = 0\), or the inviscid case, has been studied much more extensively in the Mathematics literature. Some of the recent works include [30,31,45] which study both the resolvent and the semigroup associated with the linearized operator. Naturally, resolvent estimates in the inviscid case are crucial while attempting to obtain resolvent estimates for \(0 < \varepsilon \ll 1\) (see also [21–23]), but since the results in the above references are specifically designed to obtain stability results for the Euler Equations we derive our own resolvent estimates in Section 4.

It should be emphasized that experimental observations (cf. [13]) conclude that Couette and Poiseuille flows lose their stability for Reynolds numbers that are much lower than \(\varepsilon^{-1} = 5772\). It is commonly believed that these instabilities arise due to finite, though small, initial conditions. Thus, it has been established in [8,9] that unbounded Couette flows (in \(\mathbb{R}\) instead of \([-1, 1]\) as is presently considered), assuming a period \(L = 1\), are finitely stable for sufficiently small initial data. Similar results are obtained in [7] in a three dimensional setting.

To obtain non-linear stability one needs in [7–9] to use semigroup estimates (and not only the locus of the eigenvalues as in [41]), associated with time dependent equation

\[
u_t - T_0(u, q) = 0
\]

in \(\mathcal{W}_\mathcal{D}\) or \(\mathcal{W}_\mathcal{C}\).

Unlike the unbounded Couette flow in [7–9], the semigroup associated with more general laminar flows in \([-1, 1]\) is not explicitly known. In the present contribution we thus consider, in the limit \(\varepsilon \to 0\), velocity fields \(U \in C^4([-1, 1])\) satisfying \(U' \neq 0\) in \([-1, 1]\), of two different types:

- nearly Couette flows, so that

\[
\sup_{x \in [-1, 1]} |U''(x)| + \sup_{x \in [-1, 1]} |U'''(x)| \ll \min_{x \in [-1, 1]} |U'(x)|,
\]

- velocity fields for which \(U'' \neq 0\) in \([-1, 1]\).

We prove that these laminar flows are stable and provide \(C(\mathbb{R}_+; \mathcal{L}(L^2))\) estimates for their associated semigroup norm. We believe that these linear estimates would
be useful when considering the nonlinear stability of these flows in a bounded interval.

The rest of this contribution is arranged as follows:

In the next section we formulate the spectral problem by using Hodge decomposition. Since the standard Hodge decomposition [19] is not a direct sum in this periodic setup, we define a space of zero-flux perturbations, and then formulate our main results in this space.

In Section 3 we present the problem in terms of the stream function and its Fourier coefficients and arrive at the Orr-Sommerfeld operator.

In Section 4 we consider the inviscid problem (where \( \varepsilon = 0 \)) in Fourier space.

Section 5 includes some resolvent estimates obtained for one-dimensional Schrödinger operators on the entire real line and for their Dirichlet realization in \((-1, 1)\).

In Section 6 we consider the same Schrödinger operator on \((-1, 1)\) and \(\mathbb{R}_+\) but this time in a Sobolev space of functions satisfying certain orthogonality conditions (cf. [41]).

Section 7 provides inverse estimates for the Orr-Sommerfeld operator for the fixed-traction problem, whereas Section 8 provides the same estimates for the no-slip realization.

In Section 9 we prove the main results.

Finally in the appendix we bring some auxiliary estimates obtained for Airy functions and generalized Airy functions.

2. Spectral Problem Formulation and Main Results

We now amend the spectral question presented in (1.7) to a more standard spectral problem. To this end we use a variant of Hodge decomposition adapted to our periodic setting (see [19, Theorem 3.4] for the standard case), which allows us to eliminate \( q \). Though expert readers are probably familiar with these ideas, we find it useful to recall them for the general reader’s convenience.

2.1. Hodge Theory

Let

\[
\mathcal{H} = \{ u \in L_{loc}^2 (\overline{D}, \mathbb{R}^2) \mid u(\cdot + L, \cdot) = u(\cdot, \cdot) \},
\]

where the scalar product for the Hilbert space \( \mathcal{H} \) is given by

\[
\mathcal{H} \times \mathcal{H} \ni (u, v) \mapsto \int_{(0,L) \times (-1,1)} \tilde{u} \cdot v \, dx_1 dx_2.
\]

We further define the two closed subspaces of \( \mathcal{H} \)

\[
\mathcal{H}_{\text{curl}} = \{ u \in L_{loc}^2 (\overline{D}, \mathbb{R}^2) \mid \text{curl } u = 0; \ u(\cdot + L, \cdot) = u(\cdot, \cdot) \},
\]

and

\[
\mathcal{H}_{\text{div}} = \{ u \in L_{loc}^2 (\overline{D}, \mathbb{R}^2) \mid \text{div } u = 0; \ u \cdot n|_{\partial D} = 0; \ u(\cdot + L, \cdot) = u(\cdot, \cdot) \}.
\]

We have
Lemma 2.1.
\[ \mathcal{H}_{\text{div}} \cap \mathcal{H}_{\text{curl}} = \text{span}(1, 0). \]

**Proof.** Let \( u \in \mathcal{H}_{\text{div}} \cap \mathcal{H}_{\text{curl}} \), and \( \hat{u} = (\hat{u}_1, \hat{u}_2) \) denote its partial Fourier transform with respect to \( x_1 \), i.e.,
\[
\hat{u}(n, x_2) = \frac{1}{L} \int_0^L u(x_1, x_2) e^{-i2\pi n x_1/L} \, dx_1.
\] (2.2)

It can be easily verified that, for any \( n \in \mathbb{Z} \),
\[
\begin{align*}
  i \frac{2\pi}{L} n \hat{u}_2(n, x_2) - \frac{d}{dx_2} \hat{u}_1(n, x_2) &= 0 \\
  i \frac{2\pi}{L} n \hat{u}_1(n, x_2) + \frac{d}{dx_2} \hat{u}_2(n, x_2) &= 0 \\
  \hat{u}_2(n, -1) &= \hat{u}_2(n, +1) = 0.
\end{align*}
\]
From the above we can conclude that \( \hat{u}(n, x_2) = 0 \) for \( n \neq 0 \), \( \hat{u}_2(0, x_2) = 0 \) and \( \hat{u}_1(0, x_2) = \text{Const.} \). \( \square \)

With Lemma 2.1 in mind, we introduce
\[ \mathcal{H}_{\text{div}}^0 := \{ u \in \mathcal{H}_{\text{div}}, \langle u, (1, 0) \rangle = 0 \}. \] (2.3)

The orthogonality condition reads
\[ \int_{(0,L) \times (-1,+1)} u_1(x_1, x_2) \, dx_1 \, dx_2 = 0. \] (2.4)

We can now prove the following Hodge decomposition:

**Lemma 2.2.** \( \mathcal{H}_{\text{div}}^0 \) and \( \mathcal{H}_{\text{curl}} \) are orthogonal subspaces of \( \mathcal{H} \) and
\[ \mathcal{H} = \mathcal{H}_{\text{curl}} \oplus \mathcal{H}_{\text{div}}^0. \]

**Proof.** Let \( u \in \mathcal{H} \). Let further \( \phi_c \in H^1_{\text{loc}}(\bar{D}) \) and \( \phi_d \in H^1_{\text{loc}}(\bar{D}) \) denote the weak solutions of
\[
\begin{align*}
  -\Delta \phi_c &= \text{div} \, u \quad \text{in } D \\
  \frac{\partial \phi_c}{\partial n} &= 0 \quad \text{on } \partial D \quad (2.5) \\
  \phi_c(\cdot, \cdot) &= \phi_c(\cdot + L, \cdot) + A L \quad \text{in } D,
\end{align*}
\]
and
\[
\begin{align*}
  -\Delta \phi_d &= \text{curl} \, u \quad \text{in } D \\
  \phi_d &= 0 \quad \text{on } \partial D \quad (2.6) \\
  \phi_d(\cdot, \cdot) &= \phi_d(\cdot + L, \cdot) \quad \text{in } D.
\end{align*}
\]
In the above
\[ A = \frac{1}{L} \langle u, (1, 0) \rangle. \] (2.7)
It can be easily verified, by using the Lax-Milgram Lemma, that there exists a unique solution for (2.6). Similarly, by using (see Remark 1.1) the ansatz
\[
\phi_c = A x_1 + \tilde{\phi}_c,
\]
there where \(\tilde{\phi}_c(\cdot, \cdot) = \phi_c(\cdot + L, \cdot)\), it follows that there exists a unique solution of (2.5). (Note that the Neumann condition in (2.5) is satisfied in \(H^{-1/2}_{loc}(\partial D)\) sense.) Equivalently, we can say that \(\tilde{\phi}_c\) is the unique periodic solution, orthogonal to the constant function, of
\[
\int_{(-1,+1) \times (0,L)} (\nabla \tilde{\phi}_c - u) \cdot \nabla v \, dx_1 \, dx_2 = 0,
\]
for every \(L\)-periodic \(v \in H^1((-1,+1) \times (0,L))\).

Clearly, \(\nabla \phi_c \in \mathcal{H}_{curl}, \nabla \phi_d \in \mathcal{H}_{div}^0, \langle u - \nabla \phi_c - \nabla \phi_d, (1,0) \rangle = 0\), and, by the periodicity of \(\phi_d\) and \(\nabla \phi_c\), it holds that
\[
\langle \nabla \phi_c, \nabla \phi_d \rangle = 0.
\]
Finally we set \(u = v + \nabla \phi_c + \nabla \phi_d\) to obtain that \(v \in \mathcal{H}_{curl} \cap \mathcal{H}_{div}^0\) and hence, by Lemma 2.1, \(v \equiv 0\).

2.2. Zero Flux Solution

Since \(\mathcal{W}_# \not\subset \mathcal{H}_{div}^0\) for \(\# \in \{\mathcal{D}, \mathcal{S}\}\), we need to introduce the following spaces, as the domain of the operator in the spectral formulation
\[
\mathcal{W}^0_\mathcal{D} = \left\{ u \in \mathcal{W}^0_L \bigg| u_1 \bigg|_{\partial D} = 0, \ u_2 \bigg|_{\partial D} = 0 \right\},
\]
and
\[
\mathcal{W}^0_\mathcal{S} = \left\{ u \in \mathcal{W}^0_L \bigg| \frac{\partial u_1}{\partial x_2} \bigg|_{\partial D} = 0, \ u_2 \bigg|_{\partial D} = 0 \right\},
\]
where
\[
\mathcal{W}^0_L = \{ u \in H^2_{loc}(\overline{\Omega}, \mathbb{R}^2) \mid \text{div } u = 0; \ u(\cdot + L, \cdot) = u(\cdot, \cdot), \langle u, (1,0) \rangle_H = 0 \}.
\]

**Remark 2.3.** We note that the orthogonality requirement (2.4) is in accordance with the requirement \(u \in L^2(D)\) which should be applied if the periodicity requirement is dropped. Formally, therefore, \(\mathcal{W}^0_L\) should serve as a good approximation for the space
\[
\mathcal{W}^0_\infty = \{ u \in H^2(\overline{\Omega}, \mathbb{R}^2) \mid \text{div } u = 0 \},
\]
in the limit \(L \to \infty\).

We later explain in Remark 2.8 how one can more generally determine all the solutions of (1.7) in \(\mathcal{W}_#\) from its solution in \(\mathcal{W}^0_#\).
Let then \( P : \mathcal{H} \to \mathcal{H}_{\text{div}}^0 \) denote the orthogonal projection on \( \mathcal{H}_{\text{div}}^0 \). We may express \( P \) explicitly for some \( u \in \mathcal{H} \) by
\[
P u = \nabla \phi_d ,
\] (2.10)
where \( \phi_d \) is the solution of (2.6).

Rewriting (1.7) in the form
\[
\mathcal{T} u - \Lambda u - F = \nabla q ,
\]
we observe that \( \nabla q \in \mathcal{H}_{\text{curl}} \). Then, projecting on \( \mathcal{H}_{\text{div}}^0 \), we may write, for \( u \in \mathcal{W}_\sharp^0 \)
\[
P((\mathcal{T} - \Lambda)u - F) = 0 .
\]

With the above in mind, we now define as an unbounded operator on \( \mathcal{H}_{\text{div}}^0 \) whose domain is \( \mathcal{W}_\sharp^0 \) (which is clearly dense in \( \mathcal{H}_{\text{div}}^0 \))
\[
\mathcal{T}_\sharp := P \mathcal{T} .
\] (2.11)
By this definition we have
\[
(\mathcal{T}_\sharp - \Lambda)u = PF ,
\] (2.12)
which appears to be a proper formulation of the resolvent equation.

**Proposition 2.4.** \( \mathcal{T}_\sharp \) is semi-bounded on \( \mathcal{H}_{\text{div}}^0 \) and has compact resolvent. Furthermore,
\[
\| e^{-t \mathcal{T}_\sharp} \| \leq e \frac{1}{2} \| U' \|_{\infty} t .
\] (2.13)

**Proof.** Let \( u \in \mathcal{W}_\sharp^0 \). As \( u \perp \mathcal{H}_{\text{curl}} \) we have
\[
\Re(\langle u, \mathcal{T}_\sharp u \rangle) = \Re(\langle u, \mathcal{T} u \rangle) = \varepsilon \| \nabla u \|_2^2 + \Re(\langle u_2, U' u_1 \rangle) \geq \varepsilon \| \nabla u \|_2^2 - \frac{1}{2} \| U' \|_{\infty} \| u \|_2^2 ,
\] (2.14)
verifying, thereby, semi-boundedness. More precisely, the resolvent set of \( \mathcal{T}_\sharp \) contains \( \{ \lambda \in \mathbb{C} \mid \Re \lambda < -\frac{1}{2} \| U' \|_{\infty} \} \). The semigroup estimate (2.13) is then a consequence of the Hille-Yosida theorem. The compactness of the resolvent is proved by observing that \( \mathcal{W}_\sharp^0 \) is compactly embedded in \( \mathcal{H}_{\text{div}}^0 \). \( \square \)

**Remark 2.5.**
- Suppose that for some \( \Lambda_0 \in \mathbb{R} \) and \( C > 0 \), we have
\[
\sup_{\Re \Lambda \leq \Lambda_0, \Lambda \in \rho(\mathcal{T}_\sharp)} \| (\mathcal{T}_\sharp - \Lambda)^{-1} \| \leq C .
\]
Then, by the compactness of the resolvent the spectrum is discrete, and hence it holds that
\[
\sup_{\Re \Lambda \leq \Lambda_0} \| (\mathcal{T}_\sharp - \Lambda)^{-1} \| \leq C .
\]
It results from (2.14) that there exists \( C > 0 \), such that for any \( \varepsilon \in (0, 1] \), \( \mathbf{F} \in \mathcal{H} \), and \( \# \in \{\mathcal{S}, \mathcal{D}\} \), it holds that

\[
\sup_{\|\Lambda\| \leq -\frac{1}{2} \|U\|_{\infty}^{-1}} \| (\mathfrak{T}^\#_p - \Lambda)^{-1} \mathbf{F} \|_{1,2} \leq \frac{C}{\varepsilon^{1/2}} \| \mathbf{F} \|_2 ,
\]  

(2.15)

where \( \| \cdot \|_{k,p} \) denotes the \( W^{k,p} \((0, L) \times (-1, 1)\) \) norm. In the sequel we use the same notation, depending on context, also for the \( W^{k,p} \((-1, 1)\) \) norm.

If \( \Lambda \) is in the resolvent set of \( \mathfrak{T}^\#_p \), we recover \( u \) by

\[
u = (\mathfrak{T}^\#_p - \Lambda)^{-1} P \mathbf{F} .
\]  

(2.16)

Once we have derived \( u \) we can obtain \( q \) in the following manner:

**Proposition 2.6.** Let \((u, \Lambda, \mathbf{F}) \in \mathcal{W}^0_\# \times \mathbb{C} \times \mathcal{H} \) satisfy (2.16). Then, there exists a unique \( q \in \mathcal{Q}_L \) such that (1.7) holds for \((u, q, \Lambda, \mathbf{F})\).

**Proof.** From (2.12) it follows that

\[(\mathfrak{T} - \Lambda)u - \mathbf{F} = \mathbf{G} \in \mathcal{H}_{curl} .\]

It remains to prove the existence of a unique \( q_G \in \mathcal{Q}_L \) satisfying

\[
\nabla q_G = \mathbf{G} .
\]  

(2.17)

This, however, easily follows from the proof of Lemma 2.2, i.e., one obtain \( q_G \) as the unique solution of (2.5) with \( \mathbf{G} \) in the place of \( u \).

**Corollary 2.7.** If \( \Lambda \) is in the resolvent set of \( \mathfrak{T}^\#_p \), then for any \( \mathbf{F} \in \mathcal{H} \) there exists a unique pair \((u_0, q_0) \in \mathcal{W}^0_\# \times \mathcal{Q}_L \) such that (1.7) holds.

We use the term “the zero flux solution of (1.7)” for this solution.

**Remark 2.8.**

* From the proof of Proposition 2.6 we learn, in addition, that if \( \Lambda \in \sigma(\mathfrak{T}^\#_p ) \) and \( u_\Lambda \) is a corresponding eigenfunction in \( \mathcal{W}^0_\# \) then there exists \( q_\Lambda \) such that \((u_\Lambda, q_\Lambda) \) satisfies (1.7) with \( F = 0 \). We cannot exclude, at the moment, the possibility that \( \Lambda \) is not a simple eigenvalue.

* The proof shows also that, for any \( \Lambda \in \rho(\mathfrak{T}^\#_p ) \), the map \( \mathbf{F} \mapsto (u_0, q_0) \) (as defined in the corollary) is continuous from \( \mathcal{H} \) onto \( \mathcal{W}^0_\# \times \mathcal{Q}_L \).

* Once the zero flux solution of (1.7) has been found in \( \mathcal{W}^0_\# \times \mathcal{Q}_L \), we can also solve the problem more generally in \( \mathcal{W}_\# \times \mathcal{Q}_L \). More precisely, if \( \Lambda \in \rho(\mathfrak{T}^\#_p ) \), then for any \( \mathbf{F} \) and any \( \gamma \in \mathbb{R} \), there exists a unique pair \((u_\gamma, q_\gamma) \in \mathcal{W}_\# \times \mathcal{Q}_L \) satisfying (1.7) and

\[
\langle u_\gamma, (1, 0) \rangle = \gamma .
\]  

(2.18)

Let \((u, q)\) denote the solution of (1.7) in \( \mathcal{W}^0_\# \times \mathcal{Q}_L \). The proof is obtained by adding to \( u \) an appropriate function of \( x_2 \) only. We omit the rather standad details in the interest of brevity.
2.3. Longitudinal Average

We begin by defining the projection
\[ p : L^2((0, L) \times (-1, 1)) \to L^2((0, L) \times (-1, 1)) \]
by
\[ p(u, x_2) = \frac{1}{L} \int_0^L u(s, x_2) \, ds, \tag{2.19} \]
and then extend it to \( \Pi : L^2((0, L) \times (-1, 1); \mathbb{R}^2) \to L^2((0, L) \times (-1, 1); \mathbb{R}^2) \)
by writing
\[ \Pi u = (pu_1, pu_2) \text{ for } u = (u_1, u_2). \tag{2.20} \]

We first show

**Lemma 2.9.** \( \Pi \) is a projection on \( \mathcal{H}^0_{\text{div}} \). Moreover for any \( \# \in \{\mathcal{D}, \mathcal{S}\} \), we have \( \Pi \mathcal{W}^0_{\#} \subset \mathcal{W}^0_{\#} \).

**Proof.** Let \( u \in \mathcal{H}^0_{\text{div}} \). Then \( u = \nabla_{\perp} \phi_d \) where \( \phi_d \) is a solution of (2.6). We may then write, using the periodicity of \( \phi_d \),
\[ \Pi u = \partial x_2 (p\phi_d) \hat{i}_1. \tag{2.21} \]

Obviously, \( \text{div } \Pi u = 0 \), and the orthogonality of \( \Pi u \) to \((1, 0)\) in \( \mathcal{H} \) follows from
\[ \int_{-1}^1 \partial x_2 (p\phi_d) \, dx_2 = 0. \]

Hence \( \Pi u \in \mathcal{H}^0_{\text{div}} \). It can now be easily verified that \( \Pi \mathcal{W}^0_{\#} \subset \mathcal{W}^0_{\#} \). \( \square \)

**Lemma 2.10.** The projector \( P \) commutes with \( \Pi \).

**Proof.** Let \( F \in \mathcal{H} \). Then, by the proof of Lemma 2.2
\[ F = \nabla_{\perp} \phi_d + \nabla \phi_c, \]
where \( \phi_d \) is a solution of (2.6) and \( \phi_c \) is a solution of (2.5). Clearly,
\[ \Pi F = (\partial x_2 (p\phi_d) + A) \hat{i}_1 + \partial x_2 (p\tilde{\phi}_c) \hat{i}_2, \]
where \( A \) is given by (2.7) (with \( F \) instead of \( u \)).

Hence
\[ \Pi F = \nabla_{\perp} (p\phi_d) + A \hat{i}_1 + \nabla (p\tilde{\phi}_c), \]
and by uniqueness of the Hodge decomposition we obtain
\[ P \Pi F = \partial x_2 (p\phi_d) \hat{i}_1. \]

Next, we compute \( P F \). Observing (see (2.10)) that
\[ P F = \nabla_{\perp} \phi_d, \]
we get
\[ \Pi P F = \partial x_2 (p\phi_d) \hat{i}_1 = P \Pi F. \tag{2.22} \]
\( \square \)
We can now prove the following commutation result

**Lemma 2.11.** For any \( \# \in \{ \mathcal{S}, \mathcal{D} \} \), \( \Xi^\#_p \) commutes with \( \Pi \).

**Proof.** We simply observe that for all \( u \in \mathcal{W}^0_\# \)

\[
\Xi \Pi u = \Pi \Xi u ,
\]

and use the commutation of \( P \) and \( \Pi \).

\( \square \)

An immediate consequence follows

**Proposition 2.12.** For any \( \# \in \{ \mathcal{D}, \mathcal{S} \} \) it holds that

\[
\Pi e^{-t \Xi^\#_p} = e^{-t \Xi^\#_p} \Pi .
\]

2.4. The Semigroup \( \Pi e^{-t \Xi^\#_p} \)

Since our main results are stated for \( (I - \Pi)e^{-t \Xi^\#_p} \) we bring, for the convenience of the reader, the following, rather straightforward, estimate for \( \| \Pi e^{-t \Xi^\#_p} \| \).

**Proposition 2.13.** Let \( U \in C^1([-1,1]) \). Then,

\[
\| e^{-t \Xi^\#_p(U,s,L)} \Pi \| \leq e^{-\varepsilon \pi^2/4} .
\]

**Proof.** Let \( u \in L^2(0, T; \mathcal{W}^0_\#) \), s.t \( \partial_t u \in L^2(0, T; L^2_{\text{loc}}(\mathcal{D})) \), where \( \# \in \{ \mathcal{S}, \mathcal{D} \} \), and \( q(\cdot, \cdot) \in Q_{T,L} \) satisfy

\[
u_t - \Xi u = \nabla q .
\]

Since \( \text{div} \ u = 0 \), we can conclude, as in (2.21) that

\[
\Pi u = (pu_1, 0) . \tag{2.26}
\]

Hence, using (1.6) and (2.23), we conclude that

\[
\Pi \Xi u = e^\varepsilon \frac{\partial^2 (pu_1)}{\partial x_2^2} ,
\]

Thus, since \( q(\cdot, \cdot) \in Q_{T,L} \), there exists, by (2.8), a function \( A \in L^2(0, T) \) such that

\[
(pu_1)_t - \varepsilon \frac{\partial^2 (pu_1)}{\partial x_2^2} = A(t) , \tag{2.27}
\]

in \( L^2((0, T) \times (-1, 1+1)) \).

Taking the inner product with \( pu_1 \) in \( L^2(-1, 1) \) then yields, for any \( \# \in \{ \mathcal{S}, \mathcal{D} \} \), in view of (2.4)

\[
\frac{1}{2} \frac{\partial \| pu_1 \|_2^2}{\partial t} + \varepsilon \| \frac{\partial (pu_1)}{\partial x_2} \|_2^2 = 0 . \tag{2.28}
\]
In the case \( \# = D \) we use Poincaré inequality to obtain
\[
\frac{1}{2} \frac{\partial \| p u_1 \|^2_2}{\partial t} + \varepsilon \lambda_D^1 \| p u_1 \|^2_2 \leq 0 ,
\]
where \( \lambda_D^1 \) is the first eigenvalue of the Dirichlet problem in \((-1, 1)\), or,
\[\lambda_D^1 = \frac{\pi^2}{4} .\]

In the case \( \# = S \) we have \((p u_1)'(\pm 1)\), and hence we can write
\[
\frac{1}{2} \frac{\partial \| p u_1 \|^2_2}{\partial t} + \varepsilon \lambda_N^2 \| p u_1 \|^2_2 \leq 0 ,
\]
relying on the fact that \( x_2 \mapsto (p u_1)(x_2) \) is orthogonal, by (2.4), to the first eigenfunction of the Neumann problem in \((-1, 1)\). Note that
\[\lambda_N^2 = \frac{\pi^2}{4} .\]

From the above, together with (2.26), we conclude (2.25). \(\square\)

### 2.5. Main Results

Throughout this work we make the following assumption:

**Assumption 2.14.** \( U' \) does not vanish in \([-1, 1] \), or
\[
m := \inf_{x \in [-1,1]} |U'(x)| > 0 . \tag{2.29}
\]

The statement of the main results below involves the spectral properties of the complex Airy operator on \( \mathbb{R}_+ \)
\[
\mathcal{L}_+ = -\frac{d^2}{dx^2} + ix , \tag{2.30a}
\]
defined on
\[
D(\mathcal{L}_+) = \{ u \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \mid xu \in L^2(\mathbb{R}_+) \} . \tag{2.30b}
\]
We denote its leftmost eigenvalue [3] by \( \nu_1 \). We further set
\[
\mathfrak{I}_m(U) = \min(|U'(-1)|, |U'(1)|) \tag{2.30c}
\]
We also need below
\[
\delta_2(U) := \| U'' \|_{1,\infty} , \tag{2.31}
\]
where \( \| u \|_{1,\infty} = \| u \|_\infty + \| u' \|_\infty \).
Finally we define, for any \( r > 1 \) and \( k \geq 2 \),
\[
S_r^k = \{ v \in C^k([-1, 1]), \inf_{x \in [-1,1]} |v'(x)| \geq r^{-1} \text{ and } \| v \|_{k,\infty} \leq r \} , \tag{2.32}
\]
and then set for convenience of notation

\[ S_r = S_r^4. \]  

(2.33)

For \( U \in S_r, \varepsilon > 0 \) and \( L > 0 \), we recall that

\[ \Xi^\varepsilon_p := \Xi^\varepsilon_p(U, \varepsilon, L) \]

is defined in (2.9b) and (2.11) (where \( \varepsilon \) appears in the definition of \( \Xi \) and \( L \) is the \( x_1 \) periodicity).

For \( \beta > 0 \), we introduce

\[ \Omega(\beta) := \{ (\varepsilon, L) \in (0, 1] \times \mathbb{R}_+^+, (L\varepsilon)^{-1} \geq \beta/(2\pi) \}. \]

**Theorem 2.15.** The following statements hold for any \( r > 1 \):

1. Let \( U \in S_r \) satisfy

\[ \inf_{x \in [-1,1]} |U''(x)| \geq 1/r. \]  

(2.34)

Then, for any \( \hat{\delta} > 0 \), there exist \( \Upsilon > 0, \beta_0 > 0 \) and \( C > 0 \) such that for all \( (\varepsilon, L) \in \Omega(\beta_0) \) and \( t > 0 \) we have

\[ \|e^{-t \Xi^\varepsilon_p(U,\varepsilon,L)(I - \Pi)}\| \leq CL^{\frac{1}{3} - \hat{\delta} - \frac{7}{6} - \hat{\delta} - \varepsilon \Upsilon[L\varepsilon]^{-2/3} t}, \]  

(2.35)

where \( \Pi \) is given by (2.19).

2. For all \( \Upsilon < \frac{2^{1/3}}{3m} \Re \nu_1 \), there exist \( \delta > 0, \beta_0 > 0 \) and \( C > 0 \) such that, for any \( U \in S_r \) satisfying \( \delta_2(U) < \delta, (\varepsilon, L) \in \Omega(\beta_0) \) and \( t > 0 \), it holds that

\[ \|e^{-t \Xi^\varepsilon_p(U,\varepsilon,L)(I - \Pi)}\| \leq CL^{2/3} \varepsilon^{-\frac{2}{6}} e^{-\varepsilon \Upsilon[L\varepsilon]^{-2/3} t}. \]  

(2.36)

For the case \# = \( \mathcal{D} \), we first define, for some \( \theta > 0 \), the operator \( \mathcal{L}^\theta \) whose differential operator is given by (2.30a) and its domain by

\[ D(\mathcal{L}^\theta) = \{ u \in H^2(\mathbb{R}_+) \mid \langle e^{-\theta \cdot}, u \rangle = 0, \quad xu \in L^2(\mathbb{R}_+) \}. \]

We show later (see Proposition 6.4 and Corollary 6.7) that \( \mathcal{L}^\theta \) is a closed operator and that

\[ \mu_m := \inf_{\theta \geq 0} (\inf \Re \sigma(\mathcal{L}^\theta) + \frac{\theta^2}{2}) \]  

(2.37)

is finite and positive. For \( U \in S_r, \varepsilon > 0 \) and \( L > 0 \), we recall that

\[ \Xi^\varepsilon_p := \Xi^\varepsilon_p(U, \varepsilon, L) \]

is defined in (2.9a) and (2.11).

**Theorem 2.16.** For all \( r > 1 \), the following properties hold:

1. Let \( U \in S_r \) satisfy (2.34). Then, for any \( \hat{\delta} > 0 \), there exist \( \Upsilon > 0, \beta_0 > 0 \) and \( C > 0 \) such that for all \( (\varepsilon, L) \in \Omega(\beta_0) \) and \( t > 0 \) we have

\[ \|e^{-t \Xi^\varepsilon_p(U,\varepsilon,L)(I - \Pi)}\| \leq CL^{\frac{1}{3} - \hat{\delta} - \frac{7}{6} - \hat{\delta} - \varepsilon \Upsilon[L\varepsilon]^{-2/3} t}. \]  

(2.38)
2. For all $\Upsilon < \frac{2}{3} \hat{\mu}_m$, there exist $\delta > 0$, $\beta_0 > 0$ and $C > 0$ such that, for any $U \in \mathcal{S}_r$ satisfying $\delta_2(U) < \delta$, $(\varepsilon, L) \in \Omega(\beta_0)$ and $t > 0$, it holds that

$$
\|e^{-t \mathcal{D}_P(U, \varepsilon, L)}(I - \Pi)\| \leq C \frac{L^{2/3}}{\varepsilon} e^{-\frac{5}{6} \varepsilon \Upsilon [\lim_{\varepsilon \to 0} \varepsilon]^{-2/3} t}.
$$

Note that together with Proposition 2.13, Theorems 2.15 and 2.16 provide stability of the semigroup $e^{-t \mathcal{D}_P}$ for any $\# \in \{\mathcal{S}, \mathcal{D}\}$.

**Remark 2.17.** For Couette flow, where $\delta_2(U) = 0$ and $\hat{\mu}_m = 1$, one obtains that (2.39) is true for any $\Upsilon < \hat{\mu}_m$, and that (2.36) is true for all $\Upsilon < \Re \nu_1$. This provides better estimate for the exponential rate of decay than in [14, Proposition 6.1] which proves semigroup decay only for sufficiently small $\Upsilon$.

## 3. The Orr–Sommerfeld Operator

We focus attention in the sequel on $\mathcal{D}_P$ and its resolvent.

### 3.1. Stream Function

When considering a two-dimensional incompressible fluid flow, it is customary to introduce a stream function, i.e., to let $u = \nabla \perp \psi$; its introduction is again related to Hodge decomposition theory.

**Lemma 3.1.** Let $u \in \mathcal{H}^0_{\text{div}}$. Then, there exists a unique $\psi \in H^1_{\text{loc}}(D, \mathbb{R}^2)$ such that $\psi(x_1 + L, x_2) = \psi(x_1, x_2)$ and $u = \nabla \perp \psi$. If in addition, $u \in \mathcal{W}^0_{\#}$, then $\psi \in H^3_{\text{loc}}(D, \mathbb{R}^2)$ and $\psi$ satisfies $\partial_{x_2} \psi = 0$ on $\partial D$ if $\# = \mathcal{S}$ and $\partial^2_{x_2} \psi = 0$ on $\partial D$ if $\# = \mathcal{D}$.

**Proof.** Existence and uniqueness of $\psi$ follow from the proof of Lemma 2.2. In particular, for any $u \in \mathcal{H}^0_{\text{div}}$, we have $\psi = \phi_d$ where $\phi_d$ is a solution of (2.6). The second part of the lemma is immediate.

We next substitute $u = \nabla \perp \psi$ into (1.6) and take the curl of the ensuing equation, which leads to the following equation, in a distributional sense,

$$
\mathcal{P}^\#_{\Lambda, \varepsilon} \psi = \text{curl } F \text{ in } D, \quad (3.1)
$$

with $\# \in \{\mathcal{D}, \mathcal{S}\}$,

$$
\mathcal{P}^\#_{\Lambda, \varepsilon} := -\varepsilon \Delta^2 + U \frac{\partial}{\partial x_1} \Delta - U'' \frac{\partial}{\partial x_1} - \Lambda \Delta. \quad (3.2)
$$

We treat $\mathcal{P}^\#_{\Lambda, \varepsilon}$ as an unbounded operator on $L^2_{\text{per}}(D)$, where

$$
L^2_{\text{per}}(D) := \{u \in L^2_{\text{loc}}(D), u(x_1 + L, x_2) = u(x_1, x_2)\}.
$$
is equipped with the $L^2([-1, 1] \times [0, L])$ norm. Note that additional regularity is needed while attempting to use results obtained on $L^2_{\text{per}}(D)$ for spectral problem associated with $\Sigma^\#_p$. We shall obtain the necessary regularity at a later stage. Similarly, we introduce for $k = 1, 2$ and $s \geq 0$

$$H^s_{\text{per}}(D, \mathbb{C}^k) := \{ u \in H^s_{\text{loc}}(\bar{D}, \mathbb{C}^k) \mid u(x_1 + L, x_2) = u(x_1, x_2) \},$$

and assign to it the $H^s([-1, 1] \times [0, L])$ norm. In the interest of brevity we write in the sequel $H^s_{\text{per}}(D, \mathbb{C}) = H^s_{\text{per}}(D)$. For no-slip boundary conditions we take

$$D(P^\#_{\Lambda, \varepsilon}) = \{ \psi \in H^4_{\text{per}}(D), \psi = 0 \text{ and } \partial_{x_2} \psi = 0 \text{ on } \partial D \}.$$ 

For fixed traction, the domain of $P^\#_{\Lambda, \varepsilon}$ is given by

$$D(P^\#_{\Lambda, \varepsilon}) = \{ \psi \in H^4_{\text{per}}(D), \psi = 0 \text{ and } \partial_{x_2}^2 \psi = 0 \text{ on } \partial D \}. $$

We can now make the following statement:

**Lemma 3.2.** The operator $P^\#_{\Lambda, \varepsilon}$ is invertible for each $\# \in \{ \mathcal{S}, \mathcal{D} \}$ and $\Lambda \in \rho(\Sigma^\#_p)$.

**Proof.** Let $\psi \in D(P^\#_{\Lambda, \varepsilon})$ and $f \in L^2_{\text{per}}(D)$ satisfy $P^\#_{\Lambda, \varepsilon} \psi = f$ for some $\# \in \{ \mathcal{S}, \mathcal{D} \}$ and $\Lambda \in \rho(\Sigma^\#_p)$. Let $F$ denote the unique vector field in $\mathcal{H}^0_{\text{div}}$ satisfying $\text{curl } F = f$. As

$$(\Sigma - \Lambda) \nabla \psi - F \in \mathcal{H}_{\text{curl}},$$

it follows that

$$\| \nabla \psi \|_2 \leq \| (\Sigma - \Lambda)^{-1} \| \| F \|_2 \leq \| (\Sigma^\#_p - \Lambda)^{-1} \| \| f \|_2.$$ 

$\square$

Due to the periodicity of our function spaces and the fact that the coefficients of the differential operator $P^\#_{\Lambda, \varepsilon}$ and the associated boundary conditions do not depend on $x_1$, it is natural to consider the operator in a Fourier space. Hence, we introduce $L^2_{\text{per}}(D) \ni \psi \mapsto (F\psi) \in \ell^2(\frac{2\pi}{L}\mathbb{Z}) \otimes L^2(-1, 1)$ satisfying

$$(F\psi)(\alpha_n, x_2) = \frac{1}{L} \int_0^L e^{-i\alpha_n x_1} \psi(x_1, x_2) \, dx_1,$$ 

for $\alpha_n = 2\pi n / L$ ($n \in \mathbb{Z}$).

We then obtain the Hilbertian sum

$$\mathcal{F}(I - \Pi) P^\#_{\Lambda, \varepsilon} \mathcal{F}^{-1} = \varepsilon \left( \oplus_{n \in \mathbb{Z} \setminus \{0\}} \mathcal{B}^\#_{\lambda, \alpha_n, \beta_n} \right).$$

The unbounded operator $\mathcal{B}^\#_{\lambda, \alpha, \beta}$ on $L^2(-1, 1)$, which is commonly referred to as the Orr-Sommerfeld operator is given by

$$\phi \mapsto \mathcal{B}^\#_{\lambda, \alpha, \beta} \phi = (L_\beta - \beta \lambda) \left( \frac{d^2}{dx^2} - \alpha^2 \right) \phi - i\beta U'' \phi.$$ 

(3.4)
in which
\[ \beta = \alpha \varepsilon^{-1}, \]  
(3.5)

\[ \mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U, \]  
(3.6)

and, for \( \beta \neq 0 \),
\[ \lambda = \beta^{-1}(\hat{\Lambda} - \alpha^2). \]  
(3.7)
in which
\[ \hat{\Lambda} = \frac{\Lambda}{\varepsilon}. \]  
(3.8)

In the sequel, unless stated otherwise, we consider \( \beta \) and \( \alpha \) as independent parameters.

We now define two different realizations associated with the differential operator appearing in (3.4). The domain of \( \mathcal{B}^{\Sigma}_{\lambda,\alpha,\beta} \), corresponding to the prescribed traction force boundary condition, is given by
\[ D(\mathcal{B}^{\Sigma}_{\lambda,\alpha,\beta}) = \{ u \in H^4(-1, 1), u(\pm 1) = 0 \text{ and } u''(\pm 1) = 0 \}, \]  
(3.9)

while the operator \( \mathcal{B}^{\Omega}_{\lambda,\alpha,\beta} \), corresponding to the no-slip condition, is defined on
\[ D(\mathcal{B}^{\Omega}_{\lambda,\alpha,\beta}) = \{ u \in H^4(-1, 1), u(\pm 1) = 0 \text{ and } u'(\pm 1) = 0 \}. \]  
(3.10)

### 3.2. Inverse Estimates

The Orr-Sommerfeld operator given by (3.4) has extensively been studied in the Physics literature \([16,40,46]\). Very few rigorous studies, however, address its spectrum (cf. \([39]\) in the Couette case \( U'' = 0 \)) and none, to the best of our knowledge, provide estimates for its inverse norm \( \| (\mathcal{B}^{\#}_{\lambda,\alpha,\beta})^{-1} \| \) in \( L(L^2(-1, 1)) \). Assuming that \( \Lambda \in \rho(T^\#) \), the inverse of \( \mathcal{P}_{\Lambda,M}(I - \Pi) \) is bounded. To estimate its norm, one needs a proper uniform bound of \( \| (\mathcal{B}^{\#}_{\lambda,\alpha,\beta})^{-1} \| \) for all \( \alpha > 0 \). Hence we write,
\[ \| (\mathcal{P}_{\Lambda,\varepsilon})^{-1}(I - \Pi) \| \leq \varepsilon^{-1} \sup_{n \in \mathbb{Z} \setminus \{0\}} \| (\mathcal{B}^{\#}_{\lambda_n,\alpha_n,\beta_n})^{-1} \|, \]
where, for \( n \in \mathbb{Z} \setminus \{0\} \),
\[ \lambda_n = \beta_n^{-1}(\hat{\Lambda} - \alpha_n^2) = \varepsilon\alpha_n^{-1}(\hat{\Lambda} - \alpha_n^2) = \alpha_n^{-1}\Lambda - \varepsilon\alpha_n. \]

Clearly, for any \( \varepsilon > 0 \) and \( \hat{\Lambda} \in \mathbb{R} \),
\[
\sup_{n \in \mathbb{Z} \setminus \{0\}} \| (\mathcal{B}^{\#}_{\lambda_n,\alpha_n,\beta_n})^{-1} \| \leq \sup_{\alpha \geq \alpha_1} \| (\mathcal{B}^{\#}_{\lambda,\alpha,\alpha^{-1}})^{-1} \| \leq \sup_{\beta \geq \beta_1} \sup_{\alpha \geq \alpha_1} \| (\mathcal{B}^{\#}_{\beta,\alpha,\beta})^{-1} \|.
\]
with
\[ \beta_1(\varepsilon, L) = \varepsilon^{-1}\alpha_1 = 2\pi/(L\varepsilon). \]
Consequently, for any $\lambda \in \mathbb{R}_+$ and $\varepsilon > 0$, we have, the following inequality

$$\sup_{\Re{\lambda} \leq \varepsilon / \beta_1(\varepsilon, L)^{2/3}}\| (T_{\lambda, \varepsilon}^\#)^{-1}(I - \Pi)\| \leq \varepsilon^{-1} \sup_{\beta \geq \beta_1(\varepsilon, L)} \sup_{\Re{\lambda} \leq \beta^{-1}(\varepsilon / \beta_1(\varepsilon, L)^{2/3} - o(1))} \| (B_{\lambda, \varepsilon}^\#)^{-1}\|. \quad (3.11)$$

Note that $B_{\lambda, \varepsilon}^\# = B_{\lambda, -\beta, \varepsilon}^\#$ and hence, it is sufficient to consider $\alpha \geq 0$ in the above. We emphasize that the supremum with respect to $\lambda$, $\beta$ and $\alpha$ of $\| (B_{\lambda, \alpha, \beta}^\#)^{-1}\|$ is obtained while ignoring the dependence of $\beta$ on $\alpha$. Note also that $\beta_1 = \beta_1(\varepsilon)$ tends to $+\infty$ as $L \varepsilon \to 0$. Hence we shall attempt to obtain a bound on $\| (B_{\lambda, \alpha, \beta}^\#)^{-1}\|$ for large values of $\beta$, since we are interested in the small $\varepsilon$ limit.

Throughout this work we always presume Assumption 2.14. Without any loss of generality we can assume that

$$U' > 0 \text{ on } [-1, +1]. \quad (3.12)$$

Indeed, the case $U' < 0$ can similarly be treated after applying the transformation $x \to -x$.

In view of (3.11), we attempt to obtain, in the large $\beta$ limit, a bound on $\| B_{\lambda, \alpha, \beta}^{-1}\|$. To this end, we introduce

$$A_{\lambda, \alpha} \overset{\text{def}}{=} (U + i\lambda)(-\frac{d^2}{dx^2} + \alpha^2) + U''. \quad (3.13)$$

We define $A_{\lambda, \alpha}$, for $\Re{\lambda} \neq 0$ or when $\Re{\lambda} = 0$ and $\Im{\lambda} \notin [U(-1), U(1)]$, on

$$D(A_{\lambda, \alpha}) = H^2(-1, 1) \cap H_0^1(-1, 1). \quad (3.14)$$

In the case $\Re{\lambda} = 0$ and $\Im{\lambda} \in [U(-1), U(1)]$ we set

$$D(A_{i\nu, \alpha}) = H^2((-1, 1); |U - \nu|^2dx) \cap H_0^1(-1, 1), \quad (3.15)$$

where $\nu \in [U(-1), U(1)]$.

Note that $A_{\lambda, \alpha} = A_{\lambda, -\alpha}$ and hence we consider it only in the case $\alpha \geq 0$.

One can formally obtain $A_{\lambda, \alpha}$ from $B_{\lambda, \alpha, \beta}$ by dividing it by $\beta$ and taking the limit $\beta \to \infty$ which corresponds to the limit $\alpha^{-1}\varepsilon \to 0$. This is why it has been commonly referred to as the “inviscid operator” in [16].

The formal limit of the Orr-Sommerfeld operator as $\beta \to \infty$ is very different from that of the Schrödinger operator $-\frac{d^2}{dx^2} + i\beta(U - \nu)$ (where $\nu = \Im{\lambda}$). In the latter case, we expect the resolvent to be small away from the set were $U = \nu$. This fact was used in [3,4], for instance, to obtain resolvent estimates via localization techniques. For $B_{\lambda, \alpha, \beta}$, the best one can expect is that $\nu = A_{\lambda, \alpha}\phi$ would be small outside a close neighborhood of the set where $U = \Im{\lambda}$. We note that $A_{\lambda, \alpha}$ raises considerable interest independently of the viscous operator $B_{\lambda}^\#$ (cf. [16, Sections 21-24] and [35]).
4. The Inviscid Operator

We consider here the inviscid operator $A_{\lambda,\alpha}$ (often called the Rayleigh operator) associated with the differential operator (3.13), whose domain of definition is given either by (3.14) where $\Re \lambda \neq 0$ or $\Im \lambda \notin [U(-1), U(1)]$, or by (3.15) in the case $\lambda = i\nu$ for $\nu \in [U(-1), U(1)]$. The spectrum and the inverse of $A_{\lambda,\alpha}$ have been studied in the context of both inviscid (Euler) and viscous flows and their stability [16, 21, 22, 30, 31, 35, 45]. In particular, the inverse norm has been estimated in [21] for symmetric shear flows in a channel in the limit $\alpha \ll 1$. Similar estimates are obtained in [22] for boundary-layer type flows in a semi-infinite domain. In [30, 31, 45] weighted norm estimates are obtained for $A_{\lambda,\alpha}^{-1}$. These norms are hard to implement when seeking inverse estimates for the Orr-Sommerfeld operator (3.4). The purpose of this section is therefore to offer a systematic study of $\|A_{\lambda,\alpha}^{-1}\|$ using standard Sobolev norms, with emphasis on the limit $\Re \lambda \to 0$ for $\nu \in [U(-1), U(1)]$.

4.1. The Case $\Re \lambda = 0$: Preliminaries

We begin by showing that $A_{\lambda,\alpha}$ is a closed operator on $L^2(-1, 1)$. In the cases where $D(A_{\lambda,\alpha})$ is given by (3.14) the proof is standard and will therefore be omitted.

**Proposition 4.1.** Let $U$ satisfy Assumption 2.14 and $\nu \in [U(-1), U(1)]$. Then $A_{i\nu,\alpha}$, whose domain is given by (3.15), is closed as an unbounded operator on $L^2(-1, +1)$ and the space $V = \{u \in C^\infty([-1, +1]), \text{s.t. } u(-1) = u(1) = 0\}$

is dense in $D(A_{i\nu,\alpha})$ under the graph norm.

**Proof.** Let $\{u_n\}_{n=1}^\infty$ denote a sequence in $V$ such that

$$u_n \to \hat{u} \text{ in } L^2(-1, +1) \text{ and } A_{i\nu,\alpha}u_n \to \hat{v} \text{ in } L^2(-1, +1).$$

(4.1)

To prove the proposition we need to show that $\hat{u} \in D(A_{i\nu,\alpha})$ and that $A_{i\nu,\alpha}\hat{u} = \hat{v}$.

Let $U(x, \nu) = \nu$. We use Hardy’s inequality for weighted Sobolev spaces associated with the intervals $(-1, x_v)$ and $(x_v, 1)$ separately.

Set then, for $k \in \{1, 2\}$

$$W^k_{1,v}(x_v, 1) := \{u \in H^k-1(x_v, 1), (U - \nu)u^{(k)} \in L^2(x_v, 1), u(1) = 0\}.$$  

Recall the one-dimensional Hardy inequality (see for example [33, Eq. 0.6] or [11, Lemma 2.1]) which holds for any $v \in W^1_{1,v}(x_v, 1)$,

$$\|v\|_{L^2(x_v, 1)} \leq 2 \|(x - x_v)v'\|_{L^2(x_v, 1)}. \quad (4.2)$$

Note that (4.2) follows by extension from Hardy’s inequality in $(0, +\infty)$, given for any $v \in L^2(\mathbb{R}+) \text{s.t. } xv' \in L^2(\mathbb{R}+)$, we have

$$\|v\|_{L^2(0, +\infty)} \leq 2 \|xv'\|_{L^2(0, +\infty)}. \quad (4.3)$$
Hence, for any $u \in W_{1,v}^2(x_v, 1)$ (and hence for the restriction to $[x_v, 1]$ of any $u \in \mathcal{V}$)

$$
\|u'\|_{L^2(x_v, 1)} \leq 2\|(x - x_v)u''\|_{L^2(x_v, 1)} \leq \frac{C}{m} \|(U - v)u''\|_{L^2(x_v, 1)}.
$$

(4.4)

From (4.1) and (4.4) we deduce that $u_\nu$ is a Cauchy sequence in $H^1(x_v, 1)$ and in $H^1(-1, x_v)$. Hence there are two corresponding limits $u_+ \in H^1(x_v, 1)$ and $u_- \in H^1(-1, x_v)$, with $\hat{u}/(-1, x_v) = u_- \text{ and } \hat{u}/(x_v, 1) = u_+$. By continuity we have $u_-(1) = 0, u_+(1) = 0$ and $u_-(x_v) = u_+(x_v)$. This shows that $\hat{u} \in H^1(-1, +1)$. Finally it is clear that $\hat{v} = A_{i_v, \alpha} \hat{u}$ in $\mathcal{D}'(-1, +1)$ and hence $\hat{u} \in D(A_{i_v, \alpha})$.

The density argument is a consequence, after localization, of Proposition 2.1 in [12].

The cases $x_v = \pm 1$ are easier, since one can apply Hardy’s inequality only once, in $(-1, 1)$.

Consider the case of Couette flow $U = x$ in $(-1, 1)$, which is one of the most popular examples of uniaxial flows ([6,39,43]. In this case

$$
A_{i_v, \alpha}^c \overset{def}{=} (x - v)(-\frac{d^2}{dx^2} + \alpha^2).
$$

We can construct in some cases the explicit solution of the inhomogeneous problem

$$
A_{i_v, \alpha}^c u = f.
$$

(4.5)

For example, when $\alpha = 0, v \in (-1, +1)$ and $f \equiv 1$, let

$$
u_{i_v}(x) = A_1(x - v) + A_2(x - v) \log |x - v|,
$$

(4.6a)

where

$$
A_1 = -\frac{1}{2}\left[(1 - v) \log |1 - v| + (1 + v) \log |1 + v|\right],
$$

$$
A_2 = \frac{1}{2}\left[(1 - v) \log |1 - v| - (1 + v) \log |1 + v|\right].
$$

(4.6b)

It can be easily verified that $\nu_{i_v}$ satisfies (4.5).

While it seems at first glance that $A_{i_v, 0}^c : D(A_{i_v, 0}^c) \to L^2(-1, 1)$ is injective for all $v \in \mathbb{R}$, it is wrong for $v \in (-1, +1)$. It has indeed a non-trivial solution of the form

$$
\varphi_v(x) = \begin{cases} 
\frac{x - v}{1 - v} - 1 & \text{if } v \leq x, \\
\frac{x - v}{1 + v} - 1 & \text{if } x < v.
\end{cases}
$$

While $\varphi_v \notin H^2(-1, 1)$ for all $v \in (-1, 1)$, it does satisfy (4.8) in the sense of distributions.

We now look at the injectivity of $A_{i_v, \alpha}^c$ when $v \in (-1, +1)$. We have already explicitly obtained a non trivial element in $\ker A_{i_v, 0}^c$. More generally, we prove

**Lemma 4.2.** For all $v \in (-1, 1), \alpha \geq 0$, it holds that

$$
\dim \ker A_{i_v, \alpha}^c = 1.
$$

(4.7)
**Proof.** We observe that if \( u \) is in the kernel, we have \( u \in H^1_0(-1, +1) \) and
\[
\left( -\frac{d^2}{dx^2} + \alpha^2 \right) u = c \delta(x - \nu),
\]
where \( c \) is the jump \( u' \) undergoes through \( x = \nu \). Since \( \delta(x - \nu) \) belongs to \( H^{-1}(-1, +1) \), we may use the Lax-Milgram Lemma for the Dirichlet problem to obtain a unique solution \( u_1 \in H^1_0(-1, +1) \) for
\[
\left( -\frac{d^2}{dx^2} + \alpha^2 \right) u_1 = \delta(x - \nu).
\]
It follows that \( u = cu_1. \)
\[ \square \]

Using [12, Proposition 3.1] and the same argument as in the proof of [12, Theorem 3.1] one can show that \( A_{i\nu,\alpha}^c \), which is a bounded operator from the Hilbert space \( D(A_{i\nu,\alpha}^c) \) into \( L^2(-1, +1) \), is a Fredholm operator of index 1 for \( \nu \in (-1, +1) \). With the aid of Lemma 4.2, we can then conclude the surjectivity of \( A_{i\nu,\alpha}^c \) for any \( \nu \in (-1, +1) \) or more precisely the existence of a right inverse. We note that surjectivity of \( A_{i\nu,\alpha}^c \) follows from the surjectivity \( A_{i\nu,\alpha} \) we prove in the sequel.

**Remark 4.3.** When \( f \in C^1([-1, 1]) \), we can prove (4.7) in the following alternative manner. Recall the solution of \( A_{i\nu,0}^c u_\nu = 1 \) we have obtained in (4.6). If \( f(\nu) = 0 \), we can solve (4.5) by dividing it by \( (x - \nu) \). When \( f(\nu) \neq 0 \) we write \( u = f(\nu)u_\nu + \tilde{u} \), to obtain
\[
A_{i\nu,\alpha}^c \tilde{u} = \tilde{f}, \text{ with } \tilde{f}(x) := f(x) - f(\nu) - \alpha^2(x - \nu)u_\nu,
\]
which can be easily solved as \( \tilde{f}(\nu) = 0 \). It is not clear, however, how to extend by density the above solution to any \( f \in L^2(-1, +1) \).

In the next subsection, we obtain the surjectivity of \( A_{i\nu,\alpha} \) (for any \( U \) satisfying (3.12)) via a non-explicit compactness argument.

### 4.2. Construction of a Right Inverse of \( A_{i\nu,\alpha} \)

We begin by establishing the surjectivity of \( A_{i\nu,\alpha} \).

**Lemma 4.4.** Suppose that (3.12) holds, and that \( \lambda = i\nu \) for some \( \nu \in [U(-1), U(1)] \). Then, for any \( \nu \in L^2(-1, 1) \) and \( \alpha \geq 0 \) there exists some \( \phi \in D(A_{\lambda,\alpha}) \) satisfying \( A_{\lambda,\alpha} \phi = \nu \). Furthermore, there exists \( C > 0 \), such that for all \( \nu, \alpha \) and \( \nu \in L^2, \phi \) satisfies
\[
\|\phi\|_{1,2} + [1 + \alpha^2]^{-1} \|(U - \nu)\phi''\|_2 \leq C \|\nu\|_2, \quad (4.8)
\]
where \( \|\phi\|_{1,2} \) denotes the norm of \( \phi \) in the Sobolev space \( H^1(-1, 1) \).
Proof. Let $\kappa > 0$. With (3.13) in mind, we can use the following alternative form of $A_{\lambda, \alpha} \phi = v$, which is valid wherever $U \neq v$,

$$-\left( (U - v)^2 \left( \frac{\phi}{U - v} \right) \right)' + \alpha^2 (U - v) \phi = v. \quad (4.9)$$

We look first for some $\phi_\kappa \in H^2(-1, +1) \cap H^1_0(-1, +1) \subset D(A_{i v, \alpha})$ satisfying the regularized equation

$$-\left( [(U - v)^2 + \kappa^2] \left( \frac{\phi_\kappa}{U - v + i \kappa} \right) \right)' + \alpha^2 (U - v - i \kappa) \phi_\kappa = v. \quad (4.10)$$

We may set $w_\kappa = (U - v + i \kappa)^{-1} \phi_\kappa$ to obtain

$$-\left( [(U - v)^2 + \kappa^2] w'_\kappa \right)' + \alpha^2 [(U - v)^2 + \kappa^2] w_\kappa = v.$$

Taking the inner product with $w_\kappa$ then yields

$$\|[(U - v)^2 + \kappa^2]^{1/2} w'_\kappa\|^2 + \alpha^2 \|[(U - v)^2 + \kappa^2]^{1/2} w_\kappa\|^2 = \langle w_\kappa, v \rangle. \quad (4.11)$$

This immediately implies, by the Lax-Milgram Lemma, that $w_\kappa$ (and hence also $\phi_\kappa$) uniquely exists in $H^2(-1, 1) \cap H^1_0(-1, 1)$. Since $v \in [U(-1), U(1)]$, there exists $x_v \in [-1, 1]$ such that $U(x_v) = v$. Let $m > 0$ be defined by (2.29). Clearly, we have

$$(U - v)^2 + \kappa^2 \geq m^2 (x - x_v)^2.$$

Hence

$$\|[(U - v)^2 + \kappa^2]^{1/2} w'_\kappa\|^2 \geq m^2 \|(x - x_v) w'_\kappa\|^2,$$

$$= m^2 \left[ \|(x - x_v) w'_\kappa\|^2_{L^2(-1, x_v))} + \|(x - x_v) w'_\kappa\|^2_{L^2(x_v, 1)} \right].$$

Upon translation we apply Hardy’s inequality (4.3), and the fact that $w_\kappa(1) = 0$ (permitting the extension $w_k = 0$ in $(1, +\infty)$ so that $w_k \in H^1(x_v, \infty)$), to obtain that

$$\|(x - x_v) w'_\kappa\|^2_{L^2(x_v, 1)} \geq \frac{1}{4} \|w_k\|^2_{L^2(x_v, 1)}.$$

A similar bound can be established on $(-1, x_v)$. Hence, by (4.11),

$$\|w_k\|_2 \leq \frac{4}{m^2} \|v\|_2. \quad (4.12)$$

Substituting once again into (4.11) yields, in addition,

$$\|[(U - v)^2 + \kappa^2]^{1/2} w'_\kappa\|_2 \leq \frac{2}{m} \|v\|_2. \quad (4.13)$$

We note that (4.12) and (4.13) do not imply convergence of $w_k$ in $L^2(-1, 1)$ to $(U - v)^{-1} \phi$ where $\phi$ is a solution of (3.13) (with $\lambda = iv$). As a matter of fact it
is expected that, in most cases, $(U - v)^{-1} \phi$ will be unbounded in $L^2$. We thus use (4.13) and (4.12) to obtain that

$$\|\phi'_k\|_2 = \|((U - v + i\kappa)w_k)'\|_2 \leq \|(U - v)^2 + \kappa^2\|^{1/2}w_k'\|_2 + \|U'w_k\|_2 \leq C\|v\|_2.$$  (4.14)

It follows, either by Poincaré’s inequality or by (4.12), that $\{\phi_k\}_{k=1}^{\infty}$ is bounded in a ball of size $\hat{C}\|v\|_2$ in $H^1(\mathcal{V})$. By weak compactness, there exists a sequence $\kappa_n > 0$ tending to 0 such that $\phi_{\kappa_n}$ converges weakly in $H^1(\mathcal{V})$ to a limit $\phi_0$ in the same ball, and hence

$$\|\phi_0\|_{H^1_0(-1,1)} \leq \hat{C}\|v\|.$$  (4.15)

It remains to establish that $\phi_0$ is a weak solution of (3.13) for $\lambda = i\nu$. We thus write (4.10) in its weak form for some $\psi \in C_0^\infty(-1,1)

$$\langle \psi', (U - v - i\kappa)\phi_k' \rangle - \alpha^2 \langle \psi, (U - v - i\kappa)\phi_k \rangle = \langle \psi, v \rangle.$$  (4.16)

Letting $\kappa = \kappa_n$ and then taking the limit $n \to +\infty$ yields by the above established weak convergence in $H^1$ that

$$\langle \psi', (U - v)\phi_0' \rangle - \alpha^2 \langle \psi, (U - v)\phi_0 \rangle = \langle \psi, v \rangle.$$  

Consequently, $\phi_0$ is a weak solution of (3.13) satisfying (3.12). Finally, since for all $x \neq x_\nu$ we have

$$(U - v)[\phi''_0 - \alpha^2\phi_0] = U''\phi_0,$$

we easily obtain from (4.15) that

$$\|(U - v)\phi''_0\|_2 \leq C(1 + \alpha^2)\|v\|_2.$$  

The lemma is proved.

Remark 4.5. As in the case of a Couette flow, one can show that the index of $A_{i\nu,\alpha}$ as a Fredholm operator from the Hilbert space $D(A_{i\nu,\alpha})$ (equipped with the graph norm) into $L^2(-1,1)$ is one for $v \in [U(-1), U(1)]$. Since the multiplication by $U''$ is a compact operator from $D(A_{i\nu,\alpha})$ into $L^2(-1,1)$ we may conclude from Fredholm theory that

$$\tilde{A}_{\nu,\alpha} = (U - v)\left(-\frac{d^2}{dx^2} + \alpha^2\right)$$

and $A_{i\nu,\alpha}$ have the same index. Then, we observe that the multiplication operator by $(U(x) - v)/(x - x_\nu)$ has index 0, and hence $A_{\nu,\alpha}$ and $A_{i\nu,\alpha}$ again have the same index. Consequently, the index of $A_{i\nu,\alpha}$ equals to one, and by Lemma 4.4 it holds that $\dim \ker A_{i\nu,\alpha} = 1$. We note that one can construct a direct proof of injectivity with much greater difficulty.
We have proved above, the surjectivity of \( A = A_{i\nu,\alpha} \). By Fredholm theory, there is a natural right inverse \( E_{i\nu,\alpha} \) which associates with \( v \) the solution \( \phi \) of \( A\phi = v \) which is orthogonal to \( \text{Ker} A_{i\nu,\alpha} \) in \( D(A_{i\nu,\alpha}) \) for the scalar product

\[
(\phi, \psi) \mapsto \langle \phi, \psi \rangle_{\nu,\alpha} := \langle A\phi, A\psi \rangle_{L^2} + \langle \phi, \psi \rangle_{L^2}.
\]

Note that \( \langle \phi, \psi \rangle_{\nu,\alpha} \) coincides with the ordinary \( L^2 \) product whenever \( \psi \in \text{Ker} A_{i\nu,\alpha} \).

Employing the estimates of Lemma 4.4 we now prove:

**Proposition 4.6.** Suppose that (3.12) holds, and that \( \lambda = i\nu \) for some \( \nu \in [U(-1), U(1)] \). Then, there exists \( C > 0 \), such that for all \( \nu, \alpha \)

\[
\|E_{i\nu,\alpha}v\|_{L(L^2, H^1_0)} + \|E_{i\nu,\alpha}v\|_{L(L^2, D(A_{i\nu,\alpha}))} \leq C. \tag{4.17}
\]

**Proof.** We first observe that the solution constructed in Lemma 4.4 satisfies

\[
\|\phi\|_{\alpha,\nu} \leq C \|v\|_2.
\]

To obtain \( E_{i\nu,\alpha}v \) we now need to subtract \( \pi_{\alpha,\nu}\phi \) from \( \phi \), where \( \pi_{\alpha,\nu} \) is the orthogonal projector from \( D(A_{i\nu,\alpha}) \) onto \( \text{Ker} A_{i\nu,\alpha} \). Obviously,

\[
\|E_{i\nu,\alpha}v\|_{\alpha,\nu}^2 = \|\phi - \pi_{\alpha,\nu}\phi\|_{\alpha,\nu}^2 \\
\leq \|\phi\|_{\alpha,\nu}^2 \\
\leq C \|v\|_2^2,
\]

establishing, thereby, a bound on the right inverse in \( L(L^2, D(A_{i\nu,\alpha})) \). To estimate \( E_{i\nu,\alpha} \) in \( L(L^2, H^1_0) \) we observe that by (4.4) there exists \( C_0 > 0 \) such that for any \( \alpha \in \mathbb{R}, \nu \in [U(-1), U(1)] \), and \( \phi \in D(A_{i\nu,\alpha}) \)

\[
\|\phi\|_{1,2} \leq C_0 \|\phi\|_{\alpha,\nu}.
\]

\( \square \)

### 4.3. Nearly Couette Velocity Fields

We now attempt to estimate \( \|A_{\lambda,\alpha}^{-1}\| \) in the case where \( \Re \lambda \neq 0 \). We shall begin with the case where \( \delta_2(U) \), defined in (2.31), is small. To this end, we introduce

\[
I(\phi, \lambda) = \frac{1}{2} \|\phi'\|_{2}^2 + \left( \frac{U''(U - v)\phi}{(U - v)^2 + \mu^2} \right)^2 \tag{4.18a}
\]

and

\[
\gamma_m(\lambda, U) = \inf_{\phi \in H^1_0(-1,1) \setminus \{0\}} \frac{I(\phi, \lambda)}{\|\phi\|_{2}^2}. \tag{4.18b}
\]

The following result proves that under suitable assumptions on \( U \), the infimum of \( \gamma_m(\lambda) \) over \( \mathbb{C} \setminus \overline{\mathcal{J}} \), where \( \mathcal{J} = \{\lambda \in \mathbb{C} | \Re \lambda = 0, \nu \in [U(-1), U(1)]\} \) is positive:
Lemma 4.7. For any $r > 1$, there exists $\delta_0 > 0$ and $\gamma_0 > 0$ such that for any $U \in S^3_r$ satisfying $\delta_2(U) \leq \delta_0$, we have
\[
\inf_{\lambda \in \mathbb{C}\setminus\mathcal{J}} \gamma_m(\lambda, U) \geq \gamma_0 > 0.
\]

Proof. Writing $I(\phi, \lambda)$ in the form
\[
I(\phi, \lambda) = \frac{1}{2} \|\phi'\|_2^2 + \Re\left\langle \phi, \frac{U''\phi}{U' + i\lambda}\right\rangle.
\]
we attempt to estimate the second term on the right hand side. For some $\nu_0 > 0$ to be chosen later, we consider two different cases depending on the size of $|\lambda|$.

**Case I:** $\sqrt{|\mu|^2 + |\nu|^2} < \nu_0$.
Integration by parts yields, accounting for the Dirichlet condition $\phi$ satisfies at $x = \pm 1$,
\[
\left\langle \phi, \frac{U''\phi}{U' + i\lambda}\right\rangle = \left(\left(\frac{U''|\phi|^2}{U'}\right)', \log (U + i\lambda)\right).
\]
We estimate the right-hand-side as follows:
\[
\left|\left(\left(\frac{U''|\phi|^2}{U'}\right)', \log (U + i\lambda)\right)\right| \leq C(\nu_0) \delta_2(U) [\|\phi'\|_2 \|\phi\|_\infty + \|\phi\|^2_\infty]. \tag{4.20}
\]
Sobolev’s embedding and Poincaré’s inequality then yield that for some positive $\hat{C}(\nu_0, r) > 0$,
\[
\left|\Re\left\langle \phi, \frac{U''\phi}{U' + i\lambda}\right\rangle\right| \leq \hat{C}(\nu_0, r) \delta_2(U) \|\phi'\|^2_2. \tag{4.21}
\]

**Case II:** $\sqrt{|\mu|^2 + |\nu|^2} \geq \nu_0$.
As $U \in S_r$, we conclude that there exists $C(r) > 0$ such that
\[
\left|\Re\left\langle \phi, \frac{U''\phi}{U' + i\lambda}\right\rangle\right| \leq \left|\left\langle \phi, \frac{U''\phi}{U'}\right\rangle\right| \leq \frac{C}{|\nu_0|} \|\phi\|^2_2 \leq \frac{\hat{C}(r)}{\nu_0} \|\phi'\|^2_2. \tag{4.22}
\]
Poincaré’s inequality was applied to obtain the last estimate. We can now set $\nu_0 = 8 \hat{C}$ in (4.22) and $\delta_2(U) \leq \delta_0 = \frac{1}{8C(\nu_0, r)}$ in (4.22), to obtain
\[
I(\phi, \lambda) \geq \frac{1}{4} \|\phi'\|^2_2.
\]
The lemma is proved by using Poincaré’s inequality once again. \qed

Remark 4.8. Let $\alpha \geq 0$. Suppose that there exist $\lambda_0 \in \mathfrak{J}$ and $\phi \in D(A_{\lambda_0, \alpha})$ satisfying $A_{\lambda_0, \alpha}\phi = 0$ or, in other words, that there exists an embedded eigenvalue $\lambda_0 \in \mathfrak{J}$. Taking the scalar product of $A_{\lambda_0, \alpha}\phi$ with $(U - i\lambda_0)^{-1}\phi$, we now observe that
\[
I(\phi, \lambda_0) \leq 0. \tag{4.23}
\]
On the other hand, since $I(\phi, \lambda)$ depends continuously on $\lambda$, we obtain a contradiction, for a sequence $\lambda_n$ in $\mathbb{C}\setminus\mathfrak{J}$ tending to $\lambda_0$, between (4.23) and (4.19) for sufficiently small $\delta_2(U)$. Hence $A_{\lambda, \alpha}$ does not possess any eigenvalue in $\mathfrak{J}$ [45, Section 3].
Without the assumption that \( \delta_2(U) \) is small we may still show

**Lemma 4.9.** For any \( r > 1 \), there exists \( \gamma_0 \in \mathbb{R} \) such that for any \( U \in S^3_r \) we have

\[
\inf_{\lambda \in \mathbb{C} \setminus \mathfrak{J}} \gamma_m(\lambda, U) \geq \gamma_0.
\]

**Proof.** Indeed, we obtain by (4.20), Poincaré’s inequality and Sobolev’s embeddings, that for any \( \nu > 0 \) and any \( \nu_0 > 0 \) there exists \( C_{\nu, \nu_0} \) such that, for any \( \lambda \in \mathbb{C} \setminus \mathfrak{J} \) and \( \phi \in H^1(\mathbb{R}, 0, \mathbb{R}) \),

\[
I(\phi, \lambda) \geq \frac{1}{2} \| \phi' \|_2^2 - C \| \phi' \|_2 \| \phi \|_2^{3/2} \geq \left( \frac{1}{2} - \nu \right) \| \phi' \|_2^2 - C_{\nu, \nu_0} \| \phi \|_2^2.
\]

For \( |\lambda| > \nu_0 \), semiboundedness of \( I \) follows immediately from (4.22).

\( \Box \)

We now obtain an estimate for \( \| A^{-1}_{\lambda, \alpha} \| \) which is neither singular as \( |\mu| \to 0 \) (unlike (4.50)), nor does it necessitate any assumption on the injectivity of \( A_{i, \nu, \alpha} \). Instead, we simply assume (4.19). Observe that \( m \geq 1/r \) for any \( U \in S_r^2 \) where \( m \) is defined in (2.29).

**Proposition 4.10.** For any \( r > 1 \) and \( p > 1 \), there exists a constant \( C \) such that for any \( U \in S_r^2 \) satisfying (4.19) we have:

1. For all \( v \in H^1(-1, 1) \),

\[
\sup_{|\Re \lambda| \leq (2r)^{-1} \setminus \{ |\Re \lambda| = 0 \}} \frac{1}{0 \leq \alpha} \| A^{-1}_{\lambda, \alpha} v \|_1,2 \leq C \| v \|_\infty .
\] (4.24)

2. For all \( v \in W^{1,p}(-1, 1) \),

\[
\sup_{|\Re \lambda| \leq (2r)^{-1} \setminus \{ |\Re \lambda| = 0 \}} \| A^{-1}_{\lambda, \alpha} v \|_1,2 \leq C (\| v' \|_p + \| v \|_\infty).\] (4.25)

3. For all \( v \in L^p(-1, +1) \),

\[
\sup_{|\Re \lambda| \leq (2r)^{-1} \setminus \{ |\Re \lambda| = 0 \}} |\Re \lambda|^{1/p} \| A^{-1}_{\lambda, \alpha} v \|_1,2 \leq C \| v \|_p .
\] (4.26)

**Proof of (4.24).** Let \( v \in H^1(-1, 1) \) and \( \phi = A^{-1}_{\lambda, \alpha} v \) for some \( \lambda = \mu + i \nu \) with \( 0 < |\mu| \leq m/2 \). We begin by observing that

\[
\Re \left( \phi, \frac{v}{U - v + i \mu} \right) = \| \phi' \|_2^2 + \alpha^2 \| \phi \|_2^2 + \Re \left( \phi, \frac{U'' \phi}{U + i \lambda} \right),
\] (4.27)

and note that by (4.19), which is assumed to hold here,

\[
\Re \left( \phi, \frac{v}{U - v + i \mu} \right) = \frac{1}{2} \| \phi' \|_2^2 + \alpha^2 \| \phi \|_2^2 + I(\phi, \lambda) \geq \frac{1}{2} \| \phi' \|_2^2.
\] (4.28)
The left hand side of (4.28) can be estimated as follows:

\[
\|N(\phi, \frac{v}{U + i\lambda})\|_1 \leq \|\phi\|_{\infty}\|v\|_{\infty}\|\frac{1}{U + i\lambda}\|_1 \leq C \log (m|\mu|^{-1})\|\phi\|_{\infty}\|v\|_{\infty}.
\]

(4.29)

To obtain (4.29), we consider two different cases.

We first apply the following computation, valid whenever \(v \in [U(-1), U(1)]\), and \(0 < |\mu| \leq m/2\),

\[
\left\|\frac{1}{U+i\lambda}\right\|_1 = \int_{-1}^{1} \frac{1}{\sqrt{(U(x)-v)^2+\mu^2}} \, dx
\leq m^{-1} \int_{-1}^{1} \frac{1}{\sqrt{|x-x_v|^2+|\mu/m|^2}} \, dx
\leq C m^{-1} \log (m|\mu|^{-1}).
\]

(4.30)

In the case \(v < U(-1)\) we may write

\[
\left\|\frac{1}{U+i\lambda}\right\|_1 = \int_{-1}^{1} \frac{1}{\sqrt{(U(x)-v)^2+\mu^2}} \, dx
\leq m^{-1} \int_{-1}^{1} \frac{1}{\sqrt{(U(x)-U(-1))^2+\mu^2}} \, dx
\leq C m^{-1} \log (m|\mu|^{-1}).
\]

and proceed as before. The case \(U(1) > v\) is similar. Hence (4.29) is proved in both cases.

Sobolev’s embedding and Poincaré’s inequality then yield

\[
\left\|\phi, \frac{v}{U + i\lambda}\right\|_1 \leq C \log (m|\mu|^{-1})\|\phi\|_2\|v\|_\infty.
\]

(4.31)

Together with (4.28), it implies for \(0 < |\mu| \leq m/2\),

\[
\|\phi\|_{1,2} \leq C \log (m|\mu|^{-1})\|v\|_\infty,
\]

(4.32)

and hence also (4.24).

\[ \square \]

**Proof of (4.25).** Let \(|v| > 2 \max(|U(1)|, |U(-1)|)\). Then we have

\[
\left|N(\phi, \frac{v}{U + i\lambda})\right| \leq C \|\phi\|_2\|v\|_2.
\]

(4.33)

If \(|v| \leq 2 \max(|U(1)|, |U(-1)|)\) we integrate by parts to obtain

\[
\left\langle \phi, \frac{v}{U + i\lambda} \right\rangle = -\left\langle \log (U + i\lambda), \left(\frac{\bar{\phi} v}{U'}\right)' \right\rangle.
\]

Use of Hölder’s inequality and Sobolev embedding lead, for every \(p > 1\), to the conclusion that there exists \(C_p > 0\) such that

\[
\left|\left\langle \left(\frac{\bar{\phi} v}{U'}\right)', \log (U + i\lambda) \right\rangle \right| \leq C_p (\|\phi\|_{\infty}\|v\|_p + \|\phi\|_p\|v\|_\infty).
\]
where we have used the fact that the $L^q$-norm ($q$ being the Hölder conjugate) of \( \log(U+i\lambda) \) can be uniformly bounded for
\[
(\|\Re\lambda\|, |\Im\lambda|) \in [0, m/2] \times [0, 2\max(|U(1)|, |U(-1)|)].
\]

Combining the above with (4.33), and making use of Sobolev embedding once again yield
\[
\left| \left\langle \phi, \frac{v}{U+i\lambda} \right\rangle \right| \leq C_p(\|v\|_p \|\phi\|_\infty + \|v\|_\infty \|\phi\|_p). \quad (4.34)
\]

By (4.28) we obtain that
\[
\frac{1}{2} \|\phi\|_2^2 \leq C \|\phi\|_{1,p} \min(\|v\|_p, \|v\|_\infty), \quad (4.35)
\]
from which (4.25) easily follows for $1 < p \leq 2$, and then for $p > 2$ by the inequality $\|v\|_2^2 \leq 2^{1-2/p} \|v\|_p$. \(\square\)

Proof of (4.26). We first observe that
\[
\left| \left\langle \phi, \frac{v}{U+i\lambda} \right\rangle \right| \leq \|1\|_q \left\| \frac{1}{U+i\lambda} \right\|_q \|\phi\|_\infty \|v\|_p, \quad (4.36)
\]
where $q = p/(p-1)$.

If $v \in [U(-1), U(1)]$, we estimate the right-hand-side as follows
\[
\left\| \frac{1}{U+i\lambda} \right\|_q \leq m^{-1} \left[ \int_{-1}^{1} \frac{dx}{[(x-x_0)^2+|\mu/m|^2]^{q/2}} \right]^{1/q}
\leq m^{-1} \left[ \int_{-\infty}^{\infty} \frac{dx}{[(x+1)^2+|\mu/m|^2]^{q/2}} \right]^{1/q}
= m^{-1} \left[ \frac{|\mu|}{m} \right]^{1/q-1} \left[ \int_{-\infty}^{\infty} \frac{dx}{[(x+1)^2+1]^{q/2}} \right]^{1/q}
= C_q m^{-\frac{1}{q}} |\mu|^{-1/p}. \quad (4.37)
\]

If $v < U(-1)$, we write
\[
\left\| \frac{1}{U+i\lambda} \right\|_q \leq m^{-1} \left[ \int_{-1}^{1} \frac{dx}{[(x+1)^2+|\mu/m|^2]^{q/2}} \right]^{1/q}
\]
and proceed as before. The case $v > U(-1)$ is similar. Thus, without any restriction on $v$, we get
\[
\left\| \frac{1}{U+i\lambda} \right\|_q \leq \hat{C}_q |\mu|^{-1/p}. \quad (4.38)
\]

Substituting the above into (4.36) yields
\[
\left| \left\langle \phi, \frac{v}{U+i\lambda} \right\rangle \right| \leq C |\mu|^{-1/p} \|\phi\|_\infty \|v\|_p. \quad (4.39)
\]

Combining the above with (4.28), we obtain
\[
\frac{1}{2} \|\phi\|_2^2 \leq C |\mu|^{-1/p} \|\phi\|_2 \|v\|_p, \quad (4.40)
\]
from which (4.26) easily follows \(\square\)
We shall need in Section 8 the following immediate consequence of Proposition 4.10.

**Corollary 4.11.** For any \( r > 1 \) there exists \( C > 0 \) such that for all \( v \in L^2(-1, 1) \), \( U \in S_r \) satisfying (4.19), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), and \( \alpha \geq 0 \) it holds that

\[
\|A_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C \|v\|_{1/2} \frac{v}{U + i\lambda}.
\] (4.41)

**Proof.** Let \( v \in L^2(-1, 1) \) and \( \phi = A_{\lambda,\alpha}^{-1} v \). By (4.28) we have that

\[
\frac{1}{2} \|\phi\|_2^2 \leq \Re \left( \phi, \frac{v}{U + i\lambda} \right).
\] (4.42)

As \( \phi \in H^1_0((-1, 1)) \), we may write

\[
\Re \left( \phi, \frac{v}{U + i\lambda} \right) = \Re \left( \phi - \phi(\pm 1), \frac{v}{U + i\lambda} \right) \leq \|\phi\|_2 \left( 1 \pm x \right)^{1/2} \frac{v}{U + i\lambda}.
\]

Combining the above with (4.42) yields (4.41) with the aid of Poincaré’s inequality. \( \square \)

In the next lemma we address the optimality of (4.26).

**Lemma 4.12.** Let \( U \in C^3([-1, 1]) \) satisfy (3.12). Then, (4.26) is optimal, i.e., there exists a sequence \( (\lambda_k, v_k)_{k=1}^\infty \in \mathbb{C}^N \times \{L^p(-1, 1)\}^N \) and \( \alpha \in \mathbb{R}_+ \) such that

\[
\|v_k\|_p = 1, \mu_k = \Re \lambda_k \to 0, \text{ and } \liminf_{k \to \infty} |\mu_k|^{1/p} \|A_{\lambda_k,\alpha}^{-1} v_k\|_{1,2} > 0.
\] (4.43)

**Proof.** We prove (4.43) for \( \alpha = 0 \). Consider then \( A_{\lambda,0} \) for some \( \lambda = \mu + i\nu \) with \( \mu \neq 0 \) and \( v \in (U(-1), U(1)) \). Let \( (\phi, v) \) satisfy

\[
A_{\lambda,0} \phi = v, \text{ with } v \in L^p(-1, 1) \text{ and } \|v\|_p = 1.
\]

We may rewrite this equation in the form

\[
\left( (U + i\lambda)^2 \left( \frac{\phi}{U + i\lambda} \right) \right)' = v.
\]

Integrating once yields

\[
(U(t) + i\lambda)^2 \left( \frac{\phi}{U + i\lambda} \right)'(t) = A_1 + \int_{-1}^{t} v(\tau)d\tau.
\]

Integrating again leads to

\[
\frac{\phi(x)}{U(x) + i\lambda} = A_1 \int_{-1}^{x} \frac{dt}{(U(t) + i\lambda)^2} + \int_{-1}^{x} \frac{1}{(U(t) + i\lambda)^2} \left( \int_{-1}^{t} v(\tau)d\tau \right)dt.
\]

The Dirichlet boundary condition at \( x = 1 \) is then satisfied through the requirement that \( A_1 \) satisfies

\[
A_1 \int_{-1}^{1} \frac{dt}{(U(t) + i\lambda)^2} + \int_{-1}^{1} \frac{1}{(U(t) + i\lambda)^2} \left( \int_{-1}^{t} v(\tau)d\tau \right)dt = 0.
\]
Making use of Fubini’s Theorem, we finally obtain

\[ \phi(x) = (U(x) + i\lambda) \left[ \int_{-1}^{x} \frac{v(t) \int_{t}^{x} \frac{d\tau}{(U(\tau) + i\lambda)^2} d\tau}{(U(\tau) + i\lambda)^2} - A_1 \int_{-1}^{x} \frac{d\tau}{(U(\tau) + i\lambda)^2} \right], \]

where

\[ A_1 = \frac{\int_{-1}^{1} v(t) \int_{t}^{1} \frac{d\tau}{(U(\tau) + i\lambda)^2} d\tau}{\int_{-1}^{1} (U(\tau) + i\lambda)^2}. \]

(4.44a)

We now write

\[ \int_{t}^{x} \frac{d\tau}{(U(\tau) + i\lambda)^2} = -\frac{1}{U'(U + i\lambda)} \bigg|_{t}^{x} - \int_{t}^{x} \frac{U''(\tau) d\tau}{|U'(\tau)|^2(U(\tau) + i\lambda)}. \]

(4.45)

It can be verified (as in the proof of (4.30)) that for some positive, independent of \( \mu \), constants \( C_1 \) and \( C \),

\[ \left| \int_{t}^{x} \frac{U''(\tau) d\tau}{|U'(\tau)|^2(U(\tau) + i\lambda)} \right| \leq C_1 \int_{t}^{1} |U(\tau) + i\lambda|^{-1} d\tau \leq C \left( 1 + |\log |\mu|| \right). \]

Consequently, as \( \mu \to 0 \) we have

\[ \left| \int_{-1}^{x} v(t) \int_{t}^{x} \frac{d\tau}{(U(\tau) + i\lambda)^2} d\tau - \int_{-1}^{x} v(t) \frac{1}{U'(U + i\lambda)} \bigg|_{t}^{x} \right| d\tau \leq C \log |\mu| \| v \|_1. \]

(4.46)

We seek a sequence \( \{\lambda_k, v_k\} \) with \( v_k \in L^p(-1, 1) \) and \( \mu_k > 0 \) a decreasing sequence tending to 0 for which (4.43) holds. Let then

\[ \lambda_k = \mu_k + i\nu. \]

For sufficiently large \( k \) we have that \( x_\nu + 2\mu_k < 1 \). We then define \( v_k \) by

\[ v_k(x) = \begin{cases} \mu_k^{-1/p} & x \in [x_\nu, x_\nu + \mu_k] \\ 0 & \text{otherwise}. \end{cases} \]

Clearly \( \| v_k \|_p = 1 \), and \( \| v_k \|_1 = \mu_k^{1-1/p} \).

By (4.46) we then have as \( \mu_k \) tends to 0,

\[ \int_{-1}^{x} v_k(t) \int_{t}^{x} \frac{d\tau}{(U(\tau) + i\lambda_k)^2} d\tau = \int_{-1}^{x} v_k(t) \left( \frac{1}{U'(U + i\lambda_k)} \bigg|_{t}^{x} \right) d\tau + O(|\mu_k|^{1-1/p} \log |\mu_k|). \]

(4.47)

We now write, for \( (x_\nu + 1)/2 < x \leq 1 \),

\[ \int_{-1}^{x} v_k(t) \left( \frac{1}{U'(U + i\lambda_k)} \bigg|_{t}^{x} \right) d\tau = \frac{1}{U'(x)(U(x) + i\lambda_k)} \mu_k^{-1/p} - \mu_k^{-1/p} \int_{x_\nu}^{x_\nu + \mu_k} \frac{1}{U'(U + i\lambda_k)} d\tau. \]
As
\[ \left| \int_{x_v}^{x_v+\mu_k} \frac{d\tau}{U'(U+i\lambda_k)} - \int_{x_v}^{x_v+\mu_k} \frac{d\tau}{J_v[J_v(\tau-x_v) + i\mu_k]} \right| \leq C \mu_k (1 + \log \mu_k^{-1}) , \]
where \( J_v = U'(x_v) \), we obtain, for \((x_v + 1)/2 < x \leq 1, \)
\[ \int_{-1}^{x} v_k(t) \left( \frac{1}{U'(U+i\lambda_k)} \right)^x \, dt \]
\[ = -\frac{\mu_k^{-1/p}}{J_v} \log [J_v(\tau-x_v) + i\mu_k] \bigg|_{x_v}^{x_v+\mu_k} + O(\mu_k^{1-1/p}) . \]
Consequently,
\[ \left( 4.49 \right) \text{ from the above.} \]

Substituting the above into (4.44a) and (4.44b) yields, for all \((x_v + 1)/2 < x \leq 1, \)
\[ \phi_k(x) = -(U(x) + i\lambda_k) \frac{\mu_k^{-1/p}}{J_v} \log (1 - iJ_v) + O(\mu_k^{1-1/p} |\log \mu_k|) , \]
where \( \phi_k := \mathcal{A}_{\lambda_k,0}^{-1} v_k . \)
Clearly, for all \((x_v + 1)/2 < x \leq 1, \)
\[ \lim_{k \to \infty} \int_{x_v}^{x} (U(\tau) + i\lambda_k)^{-2} \, d\tau = \int_{x_v}^{x} (U(\tau) - \nu)^{-2} \, d\tau > 0 , \]
the convergence being uniform in \( x . \)

We now prove (4.43) by establishing that
\[ \lim \inf_{k \to \infty} |\mu_k|^{1/p} \|\phi_k\|_{L^2((x_v+1)/2,1)} > 0 , \]
which will immediately imply \( \lim \inf_{k \to \infty} |\mu_k|^{1/p} \|\phi_k\|_{L^2(-1,1)} > 0 \) and consequently (4.43).
To this end we need to prove that
\[ \left| \int_{-1}^{1} (U(\tau) + i\lambda_k)^{-2} \, d\tau \right| \leq C . \] (4.49)

We now use (4.45) together with the fact that \( U \in C^3([-1, 1]) \) to obtain that
\[ \int_{-1}^{1} \frac{d\tau}{(U(\tau) + i\lambda_k)^2} = \left[ -\frac{1}{U'(U+i\lambda_k)} - \frac{U'' \log (U+i\lambda_k)}{|U'|^3} \right]_{-1}^{1} \]
\[ + \int_{-1}^{1} \left( \frac{U''}{|U'|^3} \right)^' \log (U+i\lambda_k) \, d\tau . \]
Since \( U' > 0 \) and \( |U(\pm 1) + i\lambda_k)| \geq C(1 \pm x_v) , \) for some \( C > 0 , \) we can conclude (4.49) from the above. \( \square \)
4.4. Non-vanishing $U''$

We dedicate this subsection to the case when $U \in C^2([-1, +1]$ and $U'' \neq 0$ which may result from a combination of non-vanishing pressure gradient and relative velocity between the plates at $x_2 = \pm 1$. Note that in this case there are no eigenvalues of $A_{\lambda, \alpha}$ embedded in $\mathcal{J}$ (see [45, Section 3]). We begin by establishing the following, rather straightforward, result.

**Proposition 4.13.** Suppose that $U'' \neq 0$ on $[-1, +1]$, then, for any $\lambda \in \mathbb{C}$ for which $\Re \lambda \neq 0$ and $\alpha \geq 0$, $A_{\lambda, \alpha}$ is invertible. Moreover, for any $r > 1$ there exists $C > 0$, such that, for any $\lambda$ with $\Re \lambda \neq 0$, $\alpha \geq 0$, and $U \in S^2_r$ satisfying (2.34), it holds that

$$\|A_{\lambda, \alpha}^{-1}\| + \left\| \frac{d}{dx} A_{\lambda, \alpha}^{-1} \right\| \leq C \frac{1 + |\Re \lambda|^{1/2}}{|\Re \lambda|}. \quad (4.50)$$

**Proof.** For a pair $(\phi, v)$ such that $v = A_{\lambda, \alpha} \phi$, with $\lambda = \mu + i \nu$, we write,

$$\Im \left\{ \phi, \left. v \right| U - \nu + i \mu \right\} = -\mu \left( \frac{U'' \phi}{(U - \nu)^2 + \mu^2}, \phi \right). \quad (4.51)$$

Consequently, since $U'' \neq 0$ (and hence has a constant sign) we obtain

$$\left\| \phi \right\|_2 \leq C \left( \frac{\nu_0}{|\mu|} \right) \|v\|_2. \quad (4.52)$$

Let $\nu_0 > 0$ be chosen at a later stage and consider the following two cases.

In the case $\sqrt{|\mu|^2 + |\nu|^2} < \nu_0$, we immediately deduce from (4.52) that there exists $C(\nu_0)$ such that

$$\|\phi\|_2 \leq C(\nu_0) \|v\|_2. \quad (4.53)$$

In the case $\sqrt{|\mu|^2 + |\nu|^2} \geq \nu_0$, as

$$\Re \left\{ \phi, \left. v \right| U - \nu + i \mu \right\} = \|\phi\|^2_2 + \alpha^2 \|\phi\|^2_2 + \left( \frac{U''(U - \nu)\phi}{(U - \nu)^2 + \mu^2}, \phi \right). \quad (4.54)$$

we can use (4.52) once again to obtain that, for $\nu_0 \geq 2\|U\|_\infty$, there exists $C > 0$ such that

$$\|\phi\|^2_2 + \left( \alpha^2 - \frac{C}{\nu_0} \right) \|\phi\|^2_2 \leq C \left( \frac{1}{|\mu|} \right) \|v\|^2_2.$$

Since $\phi \in H^1_0(-1, 1)$ we can use Poincaré’s inequality to obtain for sufficiently large $\nu_0$ that there exists $C > 0$ such that, for any $\alpha \geq 0$ (and $|\lambda| \geq \nu_0$),

$$\|\phi\|_{1,2} \leq C \left( \frac{1}{|\mu|^{1/2}} \right) \|v\|_2, \quad (4.55)$$

which, combined with (4.53) yields (4.50).

Once injectivity of $A_{\lambda, \alpha}$ is established, we may apply Fredholm theory to prove its surjectivity. By the compactness of the multiplication with $U''$ from $D(A_{\lambda, \alpha})$ into $L^2(-1, +1)$, we can conclude, as in Remark 4.5, that the index of $A_{\lambda, \alpha}$ is the same
as the index of \((U + i\lambda)(-\frac{d^2}{dx^2} + \alpha^2)\). Since for \(\mu \neq 0\), \(U + i\lambda \neq 0\) on \([-1, +1]\), it follows that the indices of \(A_{\lambda,\alpha}\) and \(-\frac{d^2}{dx^2} + \alpha^2\) from \(H^2(-1, 1) \cap H^0(-1, 1)\) onto \(L^2(-1, 1)\) are the same. Consequently, the index of \(A_{\lambda,\alpha}\) is 0 and surjectivity follows from injectivity.

It should be noted that (4.9) is unsatisfactory. Clearly, it is significantly inferior to (4.24)-(4.26), where a bound of \(O(|\Re \lambda|^{-1/2})\) for \(\|A_{\lambda,\alpha}^{-1}\|\) is obtained. We seek, therefore, a better estimate for \(\|A_{\lambda,\alpha}^{-1}\|\) that will be applicable in Sections 7 and 8.

**Proposition 4.14.** Let \(r > 1\) and \(p > 1\). There exist \(\mu_0 > 0\) and \(C > 0\) such that for all \(v \in W^{1,p}(-1, 1)\) and \(U \in S^3_r\) satisfying (2.34) we have

\[
\sup_{|\Re \lambda| \leq \mu_0, |\Im \lambda| = 0} \frac{1}{|\Re \lambda|} \|A_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C \|v\|_{\infty},
\]

(4.56a)

\[
\sup_{|\Re \lambda| \leq \mu_0, |\Im \lambda| = 0} \|A_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C (\|v\|_{p} + \|v\|_{\infty}),
\]

(4.56b)

and

\[
\sup_{|\Re \lambda| \leq \mu_0, |\Im \lambda| = 0} |\Re \lambda|^{1/p} \|A_{\lambda,\alpha}^{-1} v\|_{1,2} \leq C \|v\|_{p}.
\]

(4.56c)

**Proof.** In the case where \(v \in [U(-1), U(1)]\) we (uniquely) select \(x_v \in [-1, 1]\) where \(U(x_v) = v\). Otherwise if \(v > U(1)\) \((v < U(-1))\) we set \(x_v = 1\) \((-1)\).

**Step 1:** For \(p > 1\) and \(\Re \lambda \neq 0\) define \(N^{\pm}_{m,p}\) by

\[
v \mapsto N^{\pm}_{m,p}(v, \lambda) := \min \left(\left\| (1 \pm \cdot)^{1/2} \frac{v}{U + i\lambda} \right\|_1, \|v\|_{1,p} \right).
\]

We prove that there exists \(C > 0\) such that, for all \(\varepsilon > 0\) and \(0 < |\mu| \leq 1\) it holds that

\[
|\phi(x_v)| \leq C \left(\varepsilon^{-1/2} N^{\pm}_{m,p}(v, \lambda) + (|\mu|^{1/2} + \varepsilon^{1/2})|\phi'|_{2}\right),
\]

(4.57)

for all pairs \((\phi, v) \in D(A_{\lambda,\alpha}) \times W^{1,r}(-1, 1)\) satisfying \(A_{\lambda,\alpha} \phi = v\).

As

\[
|\phi(x)|^2 \geq \frac{1}{2} |\phi(x_v)|^2 - |\phi(x) - \phi(x_v)|^2,
\]

we may use (4.51) to obtain

\[
|\Im \left(\phi, \frac{v}{U - v + i\mu}\right)| \geq |\mu| \left(\left|\frac{|U''|}{(U - v)^2 + \mu^2}\right| \frac{1}{2} |\phi(x_v)|^2 - |\phi(x) - \phi(x_v)|^2\right).
\]

(4.58)

We note that, for any \(1 < p < 2\), there exists \(C > 0\) such that

\[
\left|\phi, \frac{v}{U + i\lambda}\right| = \left|\left(\frac{\phi v}{U'}\right)', \log (U + i\lambda)\right| \leq C \left(\|\phi'\|_2 \|v\|_{\infty} + \|\phi\|_{\infty} \|v\|_{p}\right) \leq C \|\phi'\|_2 \|v\|_{1,p}.
\]
Note that
\[ | \log(U + i\lambda) | \leq | \log m | + | \log | x - x_v | | + | \log \| U + i\lambda \|_\infty | , \]
and consequently, the constant \( C \), which depends on \( \log \| U + i\lambda \|^{p/(p-1)} \) is uniformly bounded for all \( U \in \mathcal{S}_r^3 \) satisfying (2.34) whenever \( |\lambda| < 2 \| U \|_\infty \). In the case where \( |\lambda| \geq 2 \| U \|_\infty \) we may use (4.33) and Sobolev embeddings.

On the other hand,
\[ | \phi(x) | = | \phi(x) - \phi(\pm 1) | \leq \| \phi' \|_2 (1 \pm x)^{1/2} , \]
we may conclude that
\[ \left\| \phi, \frac{v}{U + i\lambda} \right\| \leq \| \phi' \|_2 (1 \pm \cdot)^{1/2} \frac{v}{U + i\lambda} \]
and hence, there exists \( C > 0 \), such that
\[ \left\| \phi, \frac{v}{U + i\lambda} \right\| \leq C \| \phi' \|_2 N_{m,p}^\pm (v, \lambda) . \] (4.59)

Substituting the above into (4.58) yields
\[ \frac{1}{2} \left| \phi(x_v) \right|^2 \left\| \frac{U''}{U + i\lambda} \right\|_2^{1/2} \leq |\mu| \sup |U''| \left\| \frac{\phi - \phi(x_v)}{U + i\lambda} \right\|_2^2 + C \| \phi' \|_2 N\pm_{m,p} (v, \lambda) . \]

We now observe, as in (4.37) (but with a lower bound in mind), that, for some positive constants \( C \) and \( \hat{C} \) (note that \( |\mu| \leq 1 \)), it holds
\[ \left\| \frac{U''}{U + i\lambda} \right\|_2^{1/2} \geq \frac{1}{C} \left\| \frac{1}{(x - x_v)^2 + \mu^2} \right\|_1 \geq \frac{1}{\hat{C}|\mu|} . \]

Hence, for another constant \( C > 0 \), we get
\[ |\phi(x_v)|^2 \leq C \left[ |\mu| \left\| \frac{\phi - \phi(x_v)}{U + i\lambda} \right\|_2^2 + \| \phi' \|_2 N\pm_{m,p} (v, \lambda) \right] . \] (4.60)

To estimate the first term on the right-hand-side of (4.60) we first observe that for some \( C > 0 \) we have
\[ \left| \frac{1}{U(x) + i\lambda} \right| \leq \frac{C}{|x - x_v|} , \forall x \in (-1, +1) . \]

Then we notice that for any \( w \in L^2(\mathbb{R}_+) \) such that \( xw' \in L^2(\mathbb{R}_+) \) we have by (4.3) and some integration by parts
\[ \| (xw)' \|_2^2 = \| xw' \|_2 \geq \frac{1}{4} \| w \|_2^2 . \]

Recalling that \( \phi(-1) = \phi(1) = 0 \), we thus apply the above inequality to
\[ w(x) = \begin{cases} \frac{\phi(x) - \phi(x_v)}{x - x_v} & -1 < x < 1 \\ \frac{\phi(x_v)}{x - x_v} & 1 \leq |x| \end{cases} . \]
in \((x_v, +\infty)\) and \((-\infty, x_v)\) to obtain
\[
\left\| \frac{\phi - \phi(x_v)}{U + i\lambda} \right\|_2^2 \leq C\|\phi'\|_2^2,
\]
which when substituted into \((4.60)\) readily yields \((4.57)\) via Cauchy’s inequality. Note that, for \(v \not\in [U(-1), U(1)]\), \((4.57)\) is trivial as \(\phi(x_v) = 0\).

**Step 2:** Let \(d = \min(1 - x_v, 1 + x_v)\). We prove that for any \(A > 0\), and \(d_1 > 0\), there exists \(C_{A,d_1}\) and \(\mu_{A,d_1}\) such that, for \(\alpha^2 \leq A\), \(x_v \in (-1, 1)\), \(|\mu| \leq \mu_{A,d_1}\), and \(d \geq d_1\),
\[
\|\phi\|_{1,2} \leq C_{A,d_1} N_{m,p}^m(v, \lambda) \quad (4.61)
\]
holds for any pair \((\phi, v)\) satisfying \(A_{\lambda,\alpha}\phi = v\).

Let \(\chi \in C_0^\infty(\mathbb{R}, [0, 1])\) satisfy
\[
\chi(x) = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1. \end{cases}
\]

Let \(\chi_d(x) = \chi((x - x_v)/d)\) and set
\[
\phi = \phi + \phi(x_v)\chi_d. \quad (4.62)
\]

Note that by the choice of \(d\), \(\phi\) satisfies also the boundary condition at \(\pm 1\). It can be easily verified that
\[
A_{\lambda,\alpha}\phi = v + \phi(x_v)((U + i\lambda)(\chi''_d - \alpha^2 \chi_d) - U''\chi_d).
\]

By construction we have that \(w = (U - v)^{-1}\phi \in H^2(-1, 1)\), and hence we can rewrite the above equality (using \((3.13)\) twice) in the form
\[
-(U - v)^2 \left( \frac{\phi}{U - v} \right)' + \alpha^2(U - v)\phi = v + \phi(x_v)((U - v)(\chi''_d - \alpha^2 \chi_d) - U''\chi_d) + i\mu(\phi'' - \alpha^2\phi) = \frac{(U - v)v}{U + i\lambda} + \phi(x_v)((U - v)(\chi''_d - \alpha^2 \chi_d) - U''\chi_d) + i\mu \frac{U''\phi}{U + i\lambda}.
\]
Taking the scalar product with \(w\) and integrating by parts then yield
\[
\|(U - v)w'\|_2^2 + \alpha^2\|\phi\|_2^2 = \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle - \langle w, \phi(x_v)U''\chi_d \rangle + \phi(x_v)\langle \phi, \chi''_d - \alpha^2 \chi_d \rangle + i\mu \left\langle w, \frac{U''\phi}{U + i\lambda} \right\rangle. \quad (4.63)
\]

As in the proof of \((4.59)\), the first term on the right-hand side is estimated as follows
\[
\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq \|\phi'\|_2 N_{m,p}^m(v, \lambda) \leq C(\|\phi'\|_2 + d^{-1/2}|\phi(x_v)|) N_{m,p}^m(v, \lambda). \quad (4.64)
\]
To estimate the second term on the right-hand-side, we note that by Hardy’s inequality, we have
\[ \|w\|_2 \leq C \|\varphi\|_2 \leq \hat{C} \left( \|\varphi\|_2 + \frac{1}{d^{1/2}} |\phi(x_v)| \right). \] (4.65)

From (4.65) we get
\[ |\langle w, \phi(x_v) U'' \chi_d \rangle| \leq \hat{C} d^{1/2} |\phi(x_v)| \left( \|\varphi\|_2 + \frac{1}{d^{1/2}} |\phi(x_v)| \right). \] (4.66)

Then, we write for the third term on the right-hand-side of (4.63), using integration by parts and the upper bound on \( \alpha^2 \)
\[ |\langle \varphi, \chi_d'' - \alpha^2 \chi_d \rangle| \leq \|\varphi\|_2 \|\chi_d\|_2 + C \|\varphi\|_2 \leq \hat{C} \left( \frac{1}{d^{1/2}} \|\varphi\|_2 + \|\varphi\|_2 \right). \]

Consequently, by (4.62),
\[ |\phi(x_v)| \|\varphi, \chi_d'' - \alpha^2 \chi_d \| \leq C |\phi(x_v)| \left( \frac{1}{d^{1/2}} \|\varphi\|_2 + \frac{1}{d^{1/2}} |\phi(x_v)| \right) + \left( \|\varphi\|_2 + d^{1/2} |\phi(x_v)| \right). \]

Hence, using Poincaré’s inequality,
\[ |\phi(x_v)| \|\varphi, \chi_d'' - \alpha^2 \chi_d \| \leq C \frac{1}{d^{1/2}} |\phi(x_v)| \left( \|\varphi\|_2 + \frac{1}{d^{1/2}} |\phi(x_v)| \right), \]
from which we conclude the existence of \( C \) such that for any \( \varepsilon_1 \in (0, 1) \), we have
\[ |\phi(x_v)| \|\varphi, \chi_d'' - \alpha^2 \chi_d \| \leq C \left( \varepsilon_1 |\varphi\|_2^2 + \frac{1}{\varepsilon_1 d} |\phi(x_v)|^2 \right). \] (4.67)

To estimate the last term on the right-hand-side of (4.63), we use (4.38) and (4.65) to obtain
\[ \left\| \left( \frac{w}{U + i\lambda} \phi \right) \right\| \leq C \|w\|_2 \|\phi\|_2 \left\| \frac{1}{U + i\lambda} \right\|_2 \leq C \|\mu\|^{-1/2} \|\phi\|_2 \left( \|\varphi\|_2 + \frac{1}{d^{1/2}} |\phi(x_v)| \right). \] (4.68)

Substituting (4.68) together with (4.64), (4.66), and (4.67) into (4.63) yields that there exists \( C > 0 \) such that for every \( 0 < \varepsilon_1 < 1 \) it holds that
\[ \| (U - v) w' \|_2^2 + \alpha^2 \|\varphi\|_2^2 \leq C \left( \left( \varepsilon_1 + |\mu|^{1/2} \right) \|\varphi\|_2^2 + \frac{1}{\varepsilon_1 d} |\phi(x_v)|^2 + \varepsilon_1^{-1} N_{m,p}^\pm (v, \lambda)^2 \right). \] (4.69)

By Hardy’s inequality (4.2), Poincaré’s inequality, and (4.57) we obtain, for \( 0 < |\mu| \leq 1, \varepsilon \in (0, 1) \), and \( \varepsilon_1 \in (0, 1) \), that
\[ \|w\|_2 \leq C \left( |\mu|^{1/4} + \varepsilon^{1/2} + \frac{\varepsilon^{1/2} + |\mu|^{1/2}}{[\varepsilon_1 d]^{1/2}} \right) \|\varphi\|_2 + [\varepsilon_1 d \varepsilon]^{-1/2} N_{m,p}^\pm (v, \lambda). \]

Selecting \( \varepsilon = \varepsilon_1^2 \) then yields, for any \( |\mu| \leq \mu_{A,d_1} \), and \( \varepsilon_1 \in (0, 1) \),
\[ \|w\|_2 \leq C(|\mu|^{1/4}(1 + |\mu|^{1/4}|\varepsilon_1 d|^{-1/2}) + d^{-1/2}\varepsilon_1^{1/2})\|\phi'\|_2 + C[\varepsilon_1 d]^{-1/2}N_{m,p}(v, \lambda). \]

As \( d \geq d_1 \) we then write

\[ \|w\|_2 \leq C(|\mu|^{1/4}(1 + |\mu|^{1/4}|\varepsilon_1 d|^{-1/2}) + d_1^{-1/2}\varepsilon_1^{1/2})\|\phi'\|_2 + C[\varepsilon_1 d_1]^{-1/2}N_{m,p}(v, \lambda). \]

Recalling the definitions of \( \phi, \varphi \) and \( w \), we immediately conclude that

\[ \|\phi'\|_2 \leq \|(U - v)w'\|_2 + \|U'w\|_2 + C d^{-1/2}d(x_v)\|, \]

which together with (4.57) gives

\[ \|\phi'\|_2 \leq \|(U - v)w'\|_2 + \|U'w\|_2 + C(d_1^{1/2}\varepsilon_1^{-1}N_{m,p}(v, \lambda) + d_1^{-1/2}(|\mu|^{1/2} + \varepsilon_1))\|\phi'\|_2. \]

Substituting (4.70) and (4.69) into (4.71) yields

\[ \|\phi'\|_2 \leq C(d_1)(\varepsilon_1^{-3/2}N_{m,p}(v, \lambda) + (|\mu|^{1/2} + |\mu|^{1/2}\varepsilon_1^{1/2} + \varepsilon_1^{1/2})\|\phi'\|_2). \]

Hence, we can choose first \( \varepsilon_1 \) and then \( \mu_{A,d_1} > 0 \) such that (4.61) follows for \( |\mu| \leq \mu_{A,d_1} \) and \( d \geq d_1 \).

**Step 3:** We prove (4.61) under the assumption that

\[ \alpha^2 \geq C_U := \frac{1}{4}C_0^4\delta_2(U)^4, \]

with

\[ C_0 = 2 \sup_{|\mu| \leq 1} \log(U + i\lambda)_2. \]

We recall (4.27) which reads as

\[ \Re\left\{ \phi, \frac{v}{U + i\lambda} \right\} = \|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 - \Re\left\{ U''\phi, \frac{\phi}{U + i\lambda} \right\}. \]

For the last term we have, using Poincaré’s inequality and Sobolev’s embeddings

\[ \left| \left\langle U''\phi, \frac{\phi}{U + i\lambda} \right\rangle \right| \leq \left| \left\langle \left( \frac{U''}{U} ||\phi||^2 \right)' \log(U + i\lambda) \right\rangle \right| \leq \| \log(U + i\lambda)_2 \| \| \phi \|_\infty \left( 2 \left\| \frac{U''}{U} \right\|_\infty \| \phi' \|_2 + \left\| \left( \frac{U''}{U} \right)' \right\|_\infty \| \phi \|_2 \right) \leq C_0 \delta_2(U) \| \phi' \|_2^{3/2} \| \phi \|_2^{1/2}, \]

where \( C_0 \) is given by (4.73).

Consequently, by (4.59)

\[ \|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C_0\delta_2(U)\|\phi'\|_2^{3/2}\|\phi\|_2^{1/2} + C\|\phi'\|_2N_{m,p}(v, \lambda). \]
Using Young’s inequality we obtain
\[
\frac{1}{8} \|\phi'\|_2^2 \leq \left( \frac{1}{4} C_0^4 \delta_2 (U)^4 - \alpha^2 \right) \|\phi\|_2^2 + \hat{C} N_{m,p}^\pm (v, \lambda)^2 \tag{4.74}
\]
Hence, for \( \alpha^2 \geq C_U \) (4.61) follows immediately from the above inequality in conjunction with Poincaré’s inequality.

**Step 4:** We prove that there exist \( d_0 > 0 \), \( \mu_0 > 0 \) and \( C > 0 \) such that, for all \( d \leq d_0 \), \( v \in [U(-1), U(1)] \), \( \alpha \geq 0 \) and \( |\mu| \leq \mu_0 \),
\[
\|\phi\|_{1,2} \leq C N_{m,p}^\pm (v, \lambda), \tag{4.75}
\]
holds for any pair \((\phi, v)\) such that \( A_{\lambda, \alpha} \phi = v \).

Without any loss of generality we can assume that \( d = 1 - x_v \). As
\[
(U - U(1))(-\phi'' + \alpha^2 \phi) - U'' \phi = v - (U(1) - v + i\mu)(-\phi'' + \alpha^2 \phi)
\]
or equivalently, by (3.13),
\[
-(U - U(1))^2 \left( \frac{\phi}{U - U(1)} \right)' + \alpha^2 (U - U(1)) \phi = \frac{(U - U(1))v}{U + i\lambda} + (U(1) - v + i\mu) \frac{U'' \phi}{U + i\lambda}.
\]
Taking the scalar product with \((U - U(1))^{-1} \phi\) and integrating by parts then yield
\[
\| (U - U(1)) \left( \frac{\phi}{U - U(1)} \right)' \|_2^2 + \alpha^2 \|\phi\|_2^2 = \left( \phi, \frac{v}{U + i\lambda} \right) + (U(1) + i\lambda) \left( \int \frac{\phi}{U - U(1)} \right) \frac{U'' \phi}{U + i\lambda}. \tag{4.76}
\]
To estimate the first term on the right-hand-side of (4.76) we use (4.59).

Next, we estimate the second inner product on the right-hand-side of (4.76) by splitting the domain of integration in two sub-intervals: \((1 - 2d, 1)\) and \((-1, 1 - 2d)\).

**The integral over** \((1 - 2d, 1)\). To estimate the integral over \((1 - 2d, 1)\) we use the identity
\[
\frac{1}{[U(1) - U](U + i\lambda)} = \frac{1}{U(1) + i\lambda} \left[ \frac{1}{U + i\lambda} + \frac{1}{U(1) - U} \right],
\]
to obtain that
\[
(U(1) + i\lambda) \int_{1-2d}^1 \frac{U'' |\phi|^2}{[U(1) - U](U + i\lambda)} \, dx = \int_{1-2d}^1 \frac{U'' |\phi|^2}{U + i\lambda} \, dx + \int_{1-2d}^1 \frac{U'' |\phi|^2}{U(1) - U} \, dx. \tag{4.77}
\]
As
\[
\int_{1-2d}^1 \frac{U'' |\phi|^2}{U + i\lambda} \, dx = -\frac{U''}{U'} |\log(U + i\lambda)| \mid_{1-2d}^1 - \int_{1-2d}^1 \left( \frac{U''}{U'} |\phi|^2 \right)' \mid \log(U + i\lambda) \mid \, dx,
\]
we may conclude, having in mind that $\mu$ and $d$ are bounded, that

$$
\left| \int_{1-2d}^{1} \frac{U''|\phi|^2}{U + i\lambda} \, dx \right| 
\leq C \left\| \frac{U''}{U'} \right\|_{1, \infty} \left( d \log d \|\phi'\|_2^2 + \int_{1-2d}^{1} \left( |\phi| |\phi'| + |\phi|^2 \right) |\log (U + i\lambda)| \, dx \right) 
\leq \tilde{C} d^{1/2} \|\phi'\|_2^2.
$$

(4.78)

Furthermore, employing Hardy’s inequality and Cauchy-Schwarz inequality yields

$$
\left| \int_{1-2d}^{1} \frac{U''|\phi|^2}{U - U(1)} \, dx \right| \leq C d^{1/2} \|\phi'\|_2 \|\phi\|_\infty.
$$

Substituting the above inequalities into (4.77) yields

$$
\left| (U(1) + i\lambda) \int_{1-2d}^{1} \frac{U''|\phi|^2}{[U(1) - U(U + i\lambda)] dx} \right| \leq Cd^{1/2} \|\phi'\|_2^2.
$$

(4.79)

**The integrals over** $(-1, 1 - 2d)$. We now estimate the integrals over $[-1, 1 - 2d]$ for the inner products on the right-hand-side of (4.76). To this end we write, using Hardy’s inequality (4.2), the lower bound $|U + i\lambda| \geq CD$, and (4.38)

$$
\left| \int_{-1}^{-1-2d} \frac{U''|\phi|^2}{[U(1) - U(U + i\lambda)]} \, dx \right| 
\leq C \|\phi'\|_2 \|\phi\|_\infty \left[ \int_{-1}^{-1-2d} \frac{1}{|U + i\lambda|^2} \right]^{1/2} 
\leq \tilde{C} (\max(d, |\mu|))^{-1/2} \|\phi'\|_2^2.
$$

(4.80)

Returning to the estimate of the right hand side of (4.76), we use (4.80), and the fact that $|U(1) - v| \leq \tilde{C} d$ (following from by definition of $d$) together with Poincaré’s inequality, to obtain

$$
|U(1) - v| \left| \int_{-1}^{-1-2d} \frac{U''|\phi|^2}{[U(1) - U(U + i\lambda)]} \, dx \right| \leq Cd^{1/2} \|\phi'\|_2^2.
$$

and

$$
|\mu| \left| \int_{-1}^{-1-2d} \frac{\phi U'' \phi}{[U(1) - U(U + i\lambda)]} \, dx \right| \leq C |\mu|^{1/2} \|\phi'\|_2^2.
$$

Combining the above and (4.79) yields

$$
\left| (U(1) - v + i\mu)\left( \frac{\phi}{U - U(1)}, \frac{U'' \phi}{U + i\lambda} \right) \right| \leq C [d^{1/2} + |\mu|^{1/2}] \|\phi'\|_2^2.
$$

(4.81)

Substituting (4.81) together with (4.59) into (4.76), yields

$$
\| (U - U(1)) \left( \frac{\phi}{U - U(1)} \right) \|_2^2 \leq C \left[ [d^{1/2} + |\mu|^{1/2}] \|\phi'\|_2^2 + \|\phi'\|_2 \mathcal{N}_{m,p}(v, \lambda) \right].
$$
From Hardy’s inequality (4.2) we then conclude
\[
\left\| \frac{\phi}{U - U(1)} \right\|_2^2 \leq C([d^{1/2} + |\mu|^{1/2}]\|\phi'\|_2^2 + \|\phi'\|_{2N_m,p}(v, \lambda)).
\]
Combined with the following straightforward observation:
\[
\|\phi'\|_2 \leq \left\| (U - U(1))\left(\frac{\phi}{U - U(1)}\right)' \right\|_2 + \left\| U' \frac{\phi}{U - U(1)} \right\|_2,
\]
this yields
\[
\|\phi'\|_2^2 \leq C([d^{1/2} + |\mu|^{1/2}]\|\phi'\|_2^2 + \|\phi'\|_{2N_m,p}(v, \lambda)).
\]
Hence, there exists \(d_0 > 0\) and \(\mu_0 > 0\) such that (4.75) holds for all \(d \leq d_0\) and \(|\mu| \leq \mu_0\).

**Step 5:** We prove that there exist \(C > 0\) and \(\mu_0 > 0\) such that (4.75) holds for all \(\nu \in \mathbb{R} \setminus [U(-1), U(1)]\) and \(|\mu| \leq \mu_0\).

Without any loss of generality we assume \(\nu > U(1)\). We begin by rewriting \(A_{\lambda, \alpha} \phi = v\) in the form
\[
-((U - v)^2 \left(\frac{\phi}{U - v}\right)' + \alpha^2(U - v)\phi = v - i\mu \frac{v + U'' \phi}{U + i\lambda} = (U - v)v - i\mu U'' \phi.
\]
Taking the inner product with \(\phi/(U - v)\) on the left yields
\[
\left\| (U - v)\left(\frac{\phi}{U - v}\right)' \right\|_2^2 + \alpha^2\|\phi\|_2^2 = \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle - i\mu \left(\frac{\phi}{U - v}, \frac{U'' \phi}{U + i\lambda}\right). 
\]
Let
\[
\tilde{\phi}(x) = \begin{cases} 
\phi(x) & x \in [-1, 1] \\
0 & |x| > 1.
\end{cases}
\]
Let
\[
\hat{U} = \begin{cases} 
U(x) & x \in [-1, 1] \\
U(1) + U'(1)(x - 1) & x > 1.
\end{cases}
\]
Let \(\hat{x}_\nu > 1\) denote the unique zero of \(\hat{U} - \nu\). We may now use Hardy’s inequality to obtain the existence of \(C > 0\) such that for all \(\nu \in \mathbb{R} \setminus [U(-1), U(1)]\), all \(\phi \in H_0^1([-1, 1])\),
\[
\left\| (U - v)\left(\frac{\phi}{U - v}\right)' \right\|_2^2 = \left\| (\hat{U} - v)\left(\frac{\tilde{\phi}}{\hat{U} - v}\right)' \right\|_{L^2(-1, \hat{x}_\nu)}^2 
\geq \frac{1}{C} \left\| \frac{\phi}{U - v} \right\|_{L^2(-1, \hat{x}_\nu)}^2 
= \frac{1}{C} \left\| \frac{\phi}{U - v} \right\|_2^2.
\]
(4.84)
Next we use the analog of (4.82)
\[ \| \phi' \|^2_2 \leq 2 \left\| (U - v) \left( \frac{\phi}{U - v} \right)' \right\|^2_2 + 2 \left\| U' \frac{\phi}{U - v} \right\|^2_2, \]
which leads, together with (4.84), to
\[ \| \phi' \|_2 \leq \hat{C} \| (U - v) \left( \frac{\phi}{U - v} \right)' \|_2, \] (4.85)
On the other hand we have from (4.83) and (4.38)
\[ \left\| (U - v) \left( \frac{\phi}{U - v} \right)' \right\|^2_2 \leq \| \phi' \|_2 N_{m,p}^\pm (v, \lambda) + |\mu| \| \phi \|_\infty \frac{1}{\| U + i\lambda \|_2} \]
\[ \leq C \| \phi' \|_2 \left( N_{m,p}^\pm (v, \lambda) + |\mu|^{1/2} \left\| \frac{\phi}{U - v} \right\|_2 \right). \]
Combining the above with (4.84) yields first that
\[ \left\| (U - v) \left( \frac{\phi}{U - v} \right)' \right\|_2 \leq \tilde{C} \left( N_{m,p}^\pm (v, \lambda) + |\mu|^{1/2} \left\| (U - v) \left( \frac{\phi}{U - v} \right)' \right\|_2 \right), \]

hence, for sufficiently small \(|\mu|\),
\[ \left\| (U - v) \left( \frac{\phi}{U - v} \right)' \right\|_2 \leq 2 \tilde{C} N_{m,p}^\pm (v, \lambda). \] (4.86)
Finally, using (4.85) once again leads to the existence of \( C > 0 \) and \( \mu_0 > 0 \) such that if \(|\mu| \leq \mu_0 \)
\[ \| \phi' \|_2 \leq C N_{m,p}^\pm (v, \lambda), \] (4.87)
and we obtain (4.75) in this case as well.
Step 6: Prove (4.56).
By (4.75) (established in steps 4 and 5 for \( d \leq d_0 \)) and (4.61) (proved in steps 2 and 3 for \( d \geq d_1 = d_0 \)), there exist \( C > 0 \) and \( \mu_0 > 0 \) such that for \(|\mu| \leq \mu_0 \) we have
\[ \| \phi \|_{1,2} \leq C N_{m,p}^\pm (v, \lambda), \] (4.88)
As \( N_{m,p}^\pm (v, \lambda) \leq \| v \|_{1,p} \), we can immediately conclude (4.56b). To conclude (4.56a,c), we first observe that
\[ N_{m,p}^\pm (v, \lambda) \leq 2 \left\| \frac{v}{U + i\lambda} \right\|_1, \]
and then use Hölder’s inequality to get that
\[ \left\| \frac{v}{U + i\lambda} \right\|_1 \leq \| v \|_p \left\| \frac{1}{U + i\lambda} \right\|_q, \]
which is valid for any \( 1 < p \leq \infty \), together with (4.38), and (4.30).
For later reference we also need the following estimate which can be deduced from the proofs of Propositions 4.13 and 4.14.
Proposition 4.15. Let $r > 1$. Then, there exists $C > 0$ such that, for all $v \in L^2(-1, 1)$, $\lambda$ such that $\Re \lambda \neq 0$, and $U \in S^3_r$ satisfying (2.34) it holds that
\[
\|A_{\lambda, \alpha}^{-1}v\|_{1, 2} \leq C \left\| (1 \pm x)^{1/2} \frac{v}{U + i\lambda} \right\|_1.
\] (4.89)

Proof. Let $\mu_0$ be as in the statement of Proposition 4.14. Let $\lambda = \mu + i\nu$, $v \in L^2(-1, 1)$, and $\phi = A_{\lambda, \alpha}^{-1}v$. For $|\mu| < \mu_0$, (4.89) is an immediate result of (4.88).

Consider then the case $|\mu| \geq \mu_0$. By (4.51) and (4.59), we obtain that
\[
|\mu| \min_{x \in [-1, 1]} |U''| \left(\frac{\phi}{(U - v)^2 + \mu^2}, \phi\right) \leq \left| \left( \frac{\phi}{U - v + i\mu} \right) \right| \leq C \|\phi\|_{2N^\pm_{m, p}(v, \lambda)}.
\]

Consequently, as $|\mu| > \mu_0$, there exists $C(\mu_0) > 0$ such that
\[
\left\| \frac{\phi}{U + i\lambda} \right\|_2 \leq C \|\phi\|_{2N^\pm_{m, p}(v, \lambda)}.
\] (4.90)

Then, we use (4.54) to establish that
\[
\|\phi'\|_{2}^2 + \alpha^2 \|\phi\|_{2}^2 \leq \left( \left( \frac{\phi}{U - v + i\mu} \right) \right) + \left\| \frac{\phi}{U + i\lambda} \right\|_2 \|\phi\|_2,
\]
from which we conclude, with the aid of (4.90), that
\[
\|\phi'\|_{2}^2 + \alpha^2 \|\phi\|_{2}^2 \leq C \left( \left\| \phi'\right\|_{2N^\pm_{m, p}(v, \lambda)} \right)^{1/2} \|\phi\|_2 + \|\phi'\|_{2N^\pm_{m, p}(v, \lambda)}.
\] (4.91)

Using Poincaré’s inequality we can now establish (4.89). \qed

Corollary 4.16. Under the assumptions of Proposition 4.15 there exists $C > 0$ such that
\[
\sup_{\Re \lambda \neq 0} \frac{1}{|\Re \lambda| - 1} \|A_{\lambda, \alpha}^{-1}v\|_{1, 2} \leq C \|v\|_\infty,
\] (4.92a)
\[
\sup_{\Re \lambda \neq 0} \|A_{\lambda, \alpha}^{-1}v\|_{1, 2} \leq C (\|v'\|_p + \|v\|_\infty),
\] (4.92b)
and
\[
\sup_{\Re \lambda \neq 0} |\Re \lambda|^{1/p} \|A_{\lambda, \alpha}^{-1}v\|_{1, 2} \leq C \|v\|_p.
\] (4.92c)

The proof follows immediately from (4.91) and step 6 of the proof of Proposition 4.14.

5. Some Schrödinger Operators and Their Resolvents

In this section we derive several refinements of estimates obtained in [4,28] for the resolvent of $L_\beta = -\frac{d^2}{dx^2} + i\beta U$, (as in (3.6)) defined over different domains.
As in the rest of this contribution, we are assuming (2.29). These estimates will be used in Sections 7 and 8.

5.1. The Entire Real Line

We begin by stating the following result on \( \mathbb{R} \).

**Proposition 5.1.** For \( \tilde{U} \in C^1(\mathbb{R}) \), let \( \tilde{\mathcal{L}}_{\beta, \mathbb{R}} \) be given by

\[
\tilde{\mathcal{L}}_{\beta, \mathbb{R}} = -\frac{d^2}{dx^2} + i\beta \tilde{U},
\]

with domain

\[
D(\tilde{\mathcal{L}}_{\beta, \mathbb{R}}) = \{ u \in H^2(\mathbb{R}) \mid xu \in L^2(\mathbb{R}) \}.
\]

Then, for all positive \( \Upsilon, m, M, C, \) and \( \epsilon \), there exist \( \beta_0 > 0 \) and \( \hat{C} \), such that for all \( \beta \geq \beta_0 \) and \( \tilde{U} \in C^1(\mathbb{R}) \) satisfying

\[
\begin{align*}
0 < m &\leq \tilde{U}''(x) \leq M \text{ for all } x \in \mathbb{R} \\
|\tilde{U}'(x) - \tilde{U}'(y)| &\leq C|x - y|^\epsilon, \text{ for all } x, y \in \mathbb{R} \text{ s.t. } |x - y| \leq 1,
\end{align*}
\]

it holds that

\[
\sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} (1 + \beta^{1/3}|\Re \lambda|) \| (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)^{-1} \| + \beta^{-1/3} \sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \| \frac{d}{dx}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)^{-1} \| \leq \frac{\hat{C}}{\beta^{2/3}},
\]

and

\[
\sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \| (\tilde{U} - \Re \lambda)(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)^{-1} \| \leq \frac{\hat{C}}{\beta}.
\]

**Proof.** The estimation of the first term on the left-hand-side of (5.4), can be obtained, for \(-1 < \beta^{1/3} \mu < \Upsilon \) (with \( \mu = \Re \lambda \)) as in the proof of [28, Theorem 1.1 (ii)]. The difference is that the interval is infinite and that we impose here less regularity on the potential. To accommodate \( C^{1,\epsilon} \) potential in the proof in [28], we construct a partition of unity composed of intervals of size \( \beta^{-2/3} \), and select \( \rho \in \left( \frac{2}{3(1+\epsilon)}, \frac{2}{3} \right) \) instead of \( \rho \in \left( \frac{1}{3}, \frac{2}{3} \right) \) (as in p. 16, line 6 in [28]). The remaining details are skipped.

We now observe that

\[
\Re \langle u, (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)u \rangle = \| u' \|_2^2 - \beta \mu \| u \|_2^2.
\]

For \( \mu \leq -\beta^{-1/3} \), we deduce

\[
\| u \|_2^2 \leq \frac{C}{\beta|\mu|} \| (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)u \|_2 \leq \frac{2C}{\beta^{2/3}(1 + \beta^{1/3}|\mu|)} \| (\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)u \|_2,
\]

which gives the estimate of the first term on the left-hand-side of (5.4) for \( \beta^{1/3} \mu \leq -1 \).
To estimate the second term on the right-hand-side of (5.4), we return to (5.6) to conclude from the bound of 
\[ \| (\tilde{L}_\beta, R - \beta \lambda)^{-1} \|, \]
we have just obtained, that
\[ \| u' \|_2 \leq \frac{C}{\beta^{1/3}} \| (\tilde{L}_\beta, R - \beta \lambda)u \|_2. \] (5.7)

Finally, to prove (5.5) we use the identity
\[ \Im \langle (\tilde{U} - \nu)u, (\tilde{L}_\beta, R - \beta \lambda)u \rangle = \beta \| (\tilde{U} - \nu)u \|_2^2 + \Im \langle \tilde{U}'u, u \rangle, \]
to obtain with the aid of (5.4) that
\[ \beta \| (\tilde{U} - \nu)u \|_2^2 \leq \| (\tilde{L}_\beta, R - \beta \lambda)u \|_2 \| (\tilde{U} - \nu)u \|_2 + \frac{C}{\beta} \| (\tilde{L}_\beta, R - \beta \lambda)u \|_2^2, \]
from which (5.5) easily follows.

5.2. A Dirichlet Problem

We now obtain some resolvent estimate for the Dirichlet realization \( L_D \beta \) of \( L_\beta \) in \((-1, 1)\).

**Proposition 5.2.** For any \( r > 1 \) and \( \Upsilon < \frac{2}{3} m \Im v_1 \), there exist \( C > 0 \) and \( \beta_0 > 0 \) such that for all \( U \in S_r^2 \) and \( \beta \geq \beta_0 \)
\[ \sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \left[ \| (L_D \beta - \beta \lambda)^{-1} \| + \beta^{-1/3} \frac{d}{dx} (L_D \beta - \beta \lambda)^{-1} \right] \leq \frac{C}{\beta^{2/3}}, \] (5.8)
and
\[ \sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \| (U - \Im \lambda)(L_D \beta - \beta \lambda)^{-1} \| \leq \frac{C}{\beta}. \] (5.9)

Furthermore, for every \( 1 < p < 2 \) there exists \( C_p > 0 \) such that for all \( f \in L^2(-1, 1) \)
\[ \sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \left\| \frac{d}{dx} (L_D \beta - \beta \lambda)^{-1} f \right\|_p \leq \frac{C_p}{\beta^{2+p/3}} \| f \|_2. \] (5.10)

**Proof.** By [28, Theorem 1.1] we have, under the assumptions of the proposition,
\[ \| (L_D \beta - \beta \lambda)^{-1} \| \leq \frac{C}{\beta^{2/3}}. \] (5.11)

We next observe that for any \( u \in D(L_D \beta) \)
\[ \Re \langle u, (L_D \beta - \beta \lambda)u \rangle = \| u' \|_2^2 - \beta \mu \| u \|_2^2, \]
which together with \((5.11)\) yields
\[
\|u\|_2^2 \leq \frac{C}{\beta^{1/3}} \| (L^D_\beta - \beta \lambda) u \|_2^2 .
\] (5.12)

To complete the proof of \((5.8)\) we write, with \(\mu = \Re \lambda\),
\[
- \Im \langle u'' , (L^D_\beta - \beta \lambda) u \rangle = \|u''\|_2^2 - \beta \mu \|u'\|_2^2 - \beta \Im \langle U' u , u' \rangle .
\]

From \((5.11)\) and \((5.12)\) we then obtain that
\[
\|u''\|_2 \leq C \| (L^D_\beta - \beta \lambda) u \|_2 ,
\]
completing, thereby, the proof of \((5.8)\). We establish \((5.9)\) in the same way we have established \((5.5)\).

It remains to prove \((5.10)\). Let \(f \in L^2 (-1, 1)\) and \(u \in D(L^D_\beta)\) satisfy
\[
(L^D_\beta - \beta \lambda) u = f .
\] (5.13)

Let \(\eta\) and \(\tilde{\eta}\) denote \(C^\infty\) functions such that
\[
\tilde{\eta}(t) = \begin{cases} 0 & |t| < 1/2 \text{ and } \eta(t) = \sqrt{1 - \tilde{\eta}(t)^2}. \\ 1 & |t| > 1, \end{cases}
\] (5.14)

Let \(x_v \in [-1, 1]\) be such that \(U(x_v) = v\) (otherwise, if \(U(x) \neq v\) for all \(x \in [-1, 1]\), we arbitrarily set \(x_v = -2\)).

Let
\[
\eta_v(x) = \tilde{\eta}(\beta^{1/3} (x - x_v)) \mathbf{1}_{\mathbb{R}_+} (x - x_v) .
\] (5.15)

As
\[
\Re \langle \eta_v^2 (U - v) u , (L^D_\beta - \beta \lambda) u \rangle = \| \eta_v |U - v|^{1/2} u'\|_2^2 + \Re \langle \eta_v (U' \eta_v + 2(U - v) \eta_v') u , u' \rangle - \mu \beta \| \eta_v |U - v|^{1/2} u\|_2^2 ,
\]
we obtain, observing that \(|(U - v)\eta_v'|\) is uniformly bounded,
\[
\| \eta_v |U - v|^{1/2} u'\|_2^2 \leq \| (U - v) u \|_2 \| f \|_2 + C \left( \beta^{2/3} \| \eta_v |U - v|^{1/2} u\|_2^2 + \| u \|_2 \| u' \|_2 \right) .
\] (5.16)

Furthermore, since
\[
\Im \langle \eta_v^2 u , (L^D_\beta - \beta \lambda) u \rangle = \beta \| \eta_v |U - v|^{1/2} u\|_2^2 + 2 \Im \langle \eta_v' u , \eta_v u' \rangle ,
\] (5.17)
we obtain
\[
\| \eta_v |U - v|^{1/2} u\|_2^2 \leq \frac{1}{\beta} (\| u \|_2 \| f \|_2 + \beta^{1/3} \| u \|_2 \| u' \|_2 )
\]
and hence, by \((5.8)\),
\[
\| \eta_v |U - v|^{1/2} u\|_2 \leq \frac{C}{\beta^{5/6}} \| f \|_2 .
\]
Substituting the above into (5.16) yields, with the aid of (5.8) and (5.9),
\[ \| \eta_v | U - v |^{1/2} u' \|_2 \leq \frac{C}{\beta^{1/2}} \| f \|_2 . \]

Setting \( \eta_v^-(x) = \eta_v(-x) \), we obtain in a similar manner
\[ \| \eta_v^- | U - v |^{1/2} u' \|_2 \leq \frac{C}{\beta^{1/2}} \| f \|_2 , \]
and hence, with the aid of (5.8), we can conclude that
\[ \| U + i \lambda |^{1/2} u' \|_2 \leq \| \eta_v | U - v |^{1/2} u' \|_2 + \| \eta_v^- | U - v |^{1/2} u' \|_2 + C \beta^{-1/6} \| u' \|_2 \leq \hat{C} \beta^{-1/2} \| f \|_2 . \]

We can now conclude, assuming first \( |\mu| \geq \Upsilon \beta^{-1/3}/2 \), with the aid (4.37),
\[ \| u' \|_p \leq \| U + i \lambda |^{1/2} u' \|_2 \| U + i \lambda |^{-1/2} \|_p \leq C \beta^{-2 + \beta/3} \| f \|_2 . \] (5.18)

Otherwise, if \( |\mu| < \Upsilon \beta^{-1/3}/2 \), we rewrite (5.13) in the form
\[ (\mathcal{L}_{\beta}^D - \beta \Upsilon \beta^{-1/3} + i \mu) u = f - (\Upsilon \beta^{-1/3} - \mu) \beta u \]
to obtain from (5.18) that
\[ \| u' \|_p \leq C \beta^{-2 + \beta/3} \| f \|_2 + \beta^{2/3} \| u \|_2 . \]

The proof of (5.10) can now be completed using (5.8). \( \square \)

We can now deduce the following corollary

**Corollary 5.3.** For any \( r > 1 \) and \( \Upsilon < \mathfrak{S}_0^{2/3} \Re \nu_1 \), there exist \( C > 0 \) and \( \beta_0 > 0 \) such that for all \( U \in \mathcal{S}_r^2 \), \( \beta \geq \beta_0 \), \( 2 < p \leq \infty \), and \( f \in L^2(-1, 1) \) it holds that
\[ \sup_{\beta \lambda \leq \Upsilon \beta^{-1/3}} \| (\mathcal{L}_{\beta}^D - \beta \lambda)^{-1} f \|_p \leq C \beta^{-3p/2 + 2 + \beta/3} \| f \|_2 . \] (5.19)

**Proof.** Let \( f \in L^2(-1, 1) \) and set \( v = (\mathcal{L}_{\beta}^D - \beta \lambda)^{-1} f \). By (5.8), there exist \( C > 0 \) and \( \beta_0 \) such that, for \( \beta \geq \beta_0 \)
\[ \| v \|_2 \leq \frac{C}{\beta^{2/3}} \| f \|_2 , \quad \text{and} \quad \| v' \|_2 \leq C \beta^{-1/3} \| f \|_2 . \] (5.20)

Consequently, using the interpolation inequality
\[ \| v \|_p \leq \frac{2^{1/2}}{p} \| v \|_2 \| v \|_\infty^{1/2} , \]
and the Sobolev embedding
\[ \| v \|_\infty \leq \sqrt{2} \| v \|_2^{1/2} \| v' \|_2^{1/2} , \]
we can conclude that
\[ \| v \|_p \leq 2^{p-2} \frac{p-2}{2p} \| v \|_2 \| v' \|_2 \| v \|_\infty \leq C \beta^{-3p/2 + 2} \| f \|_2 . \]

\( \square \)
For later reference we also need the following refined estimate.

**Proposition 5.4.** For any $r > 1$ and $\Upsilon < \Re\nu_1$, there exist $C > 0$ and $\beta_0 > 0$ such that, for all $U \in S_r$, $\beta \geq \beta_0$, and $f \in L^\infty(-1, 1)$,

$$\sup_{\Re \lambda \leq \Upsilon \frac{2}{3} \beta^{-1/3}} \|(\mathcal{D}_{\beta}^\circ - \beta \lambda)^{-1} f\|_2 \leq \frac{C}{\beta^{5/6}} \|f\|_\infty. \quad (5.21)$$

**Proof.** Let $\eta$ be given by (5.14) and for any $x \in [-1, 1]$,

$$\eta_x(t) = \eta(\beta^{1/3}(t - x))1_{[-1,1]}(t) .$$

Let further

$$\eta_{x,2}(t) = \eta(\beta^{1/3}(t - x)/2)1_{[-1,1]}(t) ,$$

implying that

$$\eta_x \eta_{x,2} = \eta_x .$$

**Step 1:** We prove that, for any $\nu \in [U(-1), U(1)]$, there exist $C > 0$ and $\beta_0$ such that, for all $\beta \geq \beta_0$, $x \in [-1, 1]$, $f \in L^\infty(-1, 1)$, $\Re \lambda \leq \Upsilon \beta^{-\frac{1}{3}}$,

$$\|(U - \nu)\eta_x v\|_2^2 \leq \frac{C}{\beta^2} \|\eta_x f\|_2^2 + C \frac{\beta^{1/3}|x - x_v| + 1}{\beta^{2/3}} \|\eta_{x,2} v\|_2^2 , \quad (5.22)$$

where $x_v \in [-1, +1]$ is chosen so that $U(x_v) = \nu$ and

$$v = (\mathcal{D}_{\beta}^\circ - \beta \lambda)^{-1} f . \quad (5.23)$$

Clearly,

$$\Im(\eta_x^2(U - \nu)v, (\mathcal{D}_{\beta}^\circ - \beta \lambda)v) = \beta\|(U - \nu)\eta_x v\|_2^2 + \Im([\eta_x^2(U - \nu)]'v, v'),$$

which implies

$$\|(U - \nu)\eta_x v\|_2^2 \leq \frac{C}{\beta^2} \|\eta_x f\|_2^2 + C \beta \left(\sup_t |\eta_x'(t)(U(t) - \nu)|\right) \|\eta_x v\|_2^2 \|\eta_{x,2} v\|_2^2 . \quad (5.24)$$

To estimate $\|\eta_x v\|_2^2$ we use the identity

$$\Re(\eta_x^2 v, (\mathcal{D}_{\beta}^\circ - \beta \lambda)v) = \|(\eta_x v)'\|_2^2 - \|\eta_x v\|_2^2 - \mu \beta \|\eta_x v\|_2^2 , \quad (5.25)$$

from which we easily conclude, using the fact that $\beta^{\frac{1}{3}} \mu$ is bounded by assumption, that

$$\|\eta_x v\|_2^2 \leq C(\beta^{-2/3}\|\eta_x f\|_2^2 + \beta^{2/3}\|\eta_{x,2} v\|_2^2) . \quad (5.26)$$

Finally we note that

$$C^{-1}|x - x_v| \leq |U(x) - \nu| \leq C|x - x_v| . \quad (5.27)$$
Suppose first that \(|x - x_v| < 3\beta^{-1/3}\).
Since by (5.27) it holds that \((\sup_{t} |\eta'_x(t)(U(t) - v)|) \leq C\) we may use (5.24) to obtain
\[
\|(U - v)\eta_x v\|^2 \leq \frac{C}{\beta^2} \|\eta_x f\|^2 + \frac{C}{\beta} \|\eta_x v'\|_2 \|\eta_x v\|_2 ,
\]
which together with (5.26) yields (5.22).

Suppose now that \(|x - x_v| \geq 3\beta^{-1/3}\).
This time we have by (5.27) that \((\sup_{t} |\eta'_x(t)(U(t) - v)|) \leq C\beta^{1/3}|x - x_v|\), and hence we get from (5.24)
\[
\|(U - v)\eta_x v\|^2 \leq \frac{C}{\beta^2} \|\eta_x f\|^2 + \frac{C|x - x_v|}{\beta^{2/3}} \|\eta_x v'\|_2 \|\eta_x v\|_2 .
\]
(5.28)
To estimate the last term on the right-hand-side of (5.28), we use (5.25) once again and get instead of (5.26)
\[
\|\eta_x v'\|^2 \leq C\left(\beta^{2/3} \|\eta_x v\|^2 + \|(U - v)\eta_x v\|_2 \|\eta_x f\|_2 \right),
\]
or, alternatively, for any \(\delta > 0\),
\[
\|\eta_x v'\|_2 \leq C\left(\beta^{1/3} \|\eta_x v\|_2 + \frac{\delta \beta^{1/2}}{|x - x_v|^{1/2}} \|(U - v)\eta_x v\|_2 + \frac{|x - x_v|^{1/2}}{\delta \beta^{1/2}} \|\eta_x f\|_2 \right).  
\]
Substituting the above into (5.28) then yields
\[
\|(U - v)\eta_x v\|^2 \leq C\left(\frac{1}{\beta^2} \|\eta_x f\|^2 + \frac{|x - x_v|}{\beta^{1/3}} \|\eta_x v\|^2 + \right.
\]
\[
+ \frac{|x - x_v|^{3/2}}{\delta \beta^{1/6}} \|\eta_x v\|_2 \|\eta_x f\|_2 \|\eta_x v\|_2 + \frac{\delta |x - x_v|^{1/2}}{\beta^{1/6}} \|\eta_x v\|_2 \|(U - v)\eta_x v\|_2 \right). 
\]
Observe that
\[
\|\eta_x f\|_2 \leq C|x - x_v|^{-1} \|\eta_x f\|_2 ,
\]
(5.29)
yields (5.22) by choosing a sufficiently small value of \(\delta\).

To proceed to the next step, we need to define, yet, two additional \(\beta\) dependent cutoff functions. Let then for \(s \geq 2\), \(\eta_s\) and \(\tilde{\eta}_s\) in \(C^\infty(\mathbb{R}, [0, 1])\) satisfy
\[
\eta_s(t) = \begin{cases} 
1 & |t - x_v| \leq s\beta^{-1/3} \\
0 & |t - x_v| \geq (s + 1)\beta^{-1/3} \end{cases} \text{ and } \tilde{\eta}_s = \sqrt{1 - \eta_s^2}.
\]
We further require that there exists \(C\) and \(\beta_0\) such that for any \(s \geq 2\) and \(\beta \geq \beta_0\)
\[
\|\eta'_s\|_\infty \leq C\beta^{1/3} ; \quad \|\eta''_s\|_\infty \leq C\beta^{2/3}.
\]

Step 2: We prove that there exist \(s_0 > 0\), and \(C > 0\) such that, for all \(s \geq s_0\), there exists \(\beta_s\) such that \(\beta \geq \beta_s\)
\[
\|\tilde{\eta}_s v\|^2_2 \leq C(\beta^{-5/3} \|f\|_\infty^2 + s^{-1} \|v\|_2^2) ,
\]
(5.30)
for any pair \((f, v)\) satisfying \((5.23)\).

By \((5.22)\) we have

\[
\|\eta_x v\|_2^2 \leq \frac{C}{|x - x_v|^2 \beta^2} \|\eta_x f\|_2^2 + \frac{C}{|x - x_v| \beta^{1/3}} \|\eta_x v\|_2^2.
\]

We now integrate the above inequality with respect to \(x\) over \((-1, x_v - s\beta^{-1/3}/2) \cup (x_v + s\beta^{-1/3}/2, 1)\). By changing the order of integration we obtain that for all \(s > 4\)

\[
\int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \|\eta_x v\|_2^2 \, dx \\
= \int_{-1,1} |v|^2(t) \, dt \int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \eta^2(\beta^{1/3}(x - t)) \, dx \\
\geq \int_{-1,1} |\tilde{\eta}_x v|^2(t) \, dt \int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \eta^2(\beta^{1/3}(x - t)) \, dx \\
\geq \beta^{-1/3} \|\tilde{\eta}_x v\|_2^2.
\]

Note that

\[
\int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \eta^2(\beta^{1/3}(x - t)) \, dx \\
= \beta^{-1/3} \left( \int_{s/2 < |\tau + \beta^{1/3}(t - x_v)|} \eta^2(\tau) \, d\tau \right) \geq \beta^{-1/3}
\]

for any \(t\) in the support of \(\tilde{\eta}_x\). We rewrite the above in the form

\[
\|\tilde{\eta}_x v\|_2^2 \leq \beta^{1/3} \int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \|\eta_x v\|_2^2 \, dx.
\]

As \(\|\eta_x f\|_2 \leq 2\beta^{-1/3} \|f\|_\infty\) we have

\[
\int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \frac{1}{|x - x_v|^2 \beta^2} \|\eta_x f\|_2^2 \, dx \\
\leq C \beta^{-7/3} \|f\|_\infty^2 \int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \frac{1}{|x - x_v|^2} \, dx \\
\leq \hat{C} \beta^{-2} \|f\|_\infty^2.
\]

Finally, we have for all \(s \geq 16\),

\[
\int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \frac{1}{|x - x_v| \beta^{1/3}} \|\eta_x, v\|_2^2 \, dx \\
\leq \int_{-1,1} |v|^2(t) \, dt \int_{s\beta^{-1/3}/2 < |x - x_v|} \mathbf{1}_{[-1, 1]}(x) \eta^2(\beta^{1/3}(x - t)/2) \frac{1}{|x - x_v| \beta^{1/3}} \, dx \\
\leq \beta^{-1/3} \int_{s\beta^{-1/3}/4 < |t - x_v|} \mathbf{1}_{[-1, 1]}(t) \log \left( \frac{|t - x_v| + 2\beta^{-1/3}}{|t - x_v| - 2\beta^{-1/3}} \right) |v(t)|^2 \, dt \\
\leq C \beta^{-1/3} \frac{1}{s} \|v\|_2^2.
\]
Note that \( \eta(\beta^{1/3}(x-t)/2) \) vanishes for \( s\beta^{-1/3}/2 < |x-x_v| \) (for \( s \geq 16 \)) and \( |t-x_v| < s\beta^{-1/3}/4 \). Note further that
\[
0 < \log \frac{|t-x_v|+2\beta^{-1/3}}{|t-x_v|-2\beta^{-1/3}} \leq \log \frac{1+8/s}{1-8/s} \leq \frac{C}{s}.
\]
Combining the above with (5.33), (5.32), and (5.31) easily yields (5.30).

**Step 3:** We now prove (5.21).

Writing
\[
(L_D - \beta \lambda)(\eta_s v) = \eta_s f + 2\eta_s' v' + \eta_s'' v,
\]
we obtain from (5.8) and the definitions and properties of the cut-off functions \( \eta_s \) and \( \eta_x \),
\[
\| \eta_s v \|_2^2 \leq C(\beta^{-4/3}\|\eta_s f\|_2^2 + \beta^{-2/3}\|\eta_x v'\|_2^2 + \|\tilde{\eta}_s/2v\|_2^2),
\]
where \( x_s = x_v + \frac{s}{2} / 2 \beta^{-1/3} \).

We now use (5.29) (with \( x = x_s \)) together with (5.22) to obtain that
\[
\| \eta_s v' \|_2^2 \leq C(\beta^{2/3}\|\eta_{x_s}v'\|_2^2 + (U-v)\eta_{x_s}v_\|_2\|\eta_{x_s}v\|_2)^2 \leq \hat{C}(\beta^{2/3}\|\eta_{x_s}v'\|_2^2 + s^{-1}\beta^{1/3}[\beta^{-1}\|\eta_{x_s}f\|_2 + s\beta^{-1/3}\|\eta_{x_s}v'\|_2\|\eta_{x_s}f\|_2],
\]
from which we can conclude, using the inequality \( \tilde{\eta}_s/2 \geq \eta_{x_s} \) for \( s > 16 \), that
\[
\| \eta_s v' \|_2^2 \leq C(\beta^{2/3}\|\tilde{\eta}_s/2v\|_2^2 + \beta^{-2/3}\|\eta_{x_s}f\|_2^2).
\]
Substituting the above into (5.34), yields that, for some \( C > 0 \), it holds:
\[
\| \eta_s v \|_2^2 \leq Cs(\beta^{-5/3}\|f\|_\infty^2 + \|\tilde{\eta}_s/2v\|_2^2).
\]
Using (5.30) (used with \( s \) replaced by \( s/2 \) and for \( s \) large enough) then yields
\[
\| \eta_s v \|_2^2 \leq C(\beta^{-5/3}\|f\|_\infty^2 + s^{-1}\|v\|_2^2).
\]
Combining the above with again (5.30) yields (5.21), by choosing a sufficiently large value of \( s \). \( \square \)

### 5.3. \( L^1 \) Estimates

In this subsection we establish new \( L^1 \) estimates for the resolvent of \( \tilde{L}_{\beta,R} \), defined by (5.1) and (5.2). We first observe that the proof of Proposition 5.4 can be applied to the entire real line case, replacing the estimates of the resolvent for the Dirichlet problem \( (L_D - \beta \lambda)^{-1} \) by the corresponding estimates of the resolvent \( (\tilde{L}_{\beta,R} - \beta \lambda)^{-1} \) for \( \Re \lambda \leq \beta^{-1/3} \). Hence, we may state
Lemma 5.5. For any $r > 1$, any $\Upsilon > 0$ and $a > 0$ there exists $C > 0$ and $\beta_0 > 0$ such that, for all $U \in S_r$, $g \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and $\beta \geq \beta_0$,

$$\sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \| (\widetilde{L}_\beta, \mathbb{R} - \beta \lambda)^{-1} g \|_{L^2(-a,a)} \leq \frac{C}{\beta^{5/6}} \| g \|_\infty.$$  (5.35)

We continue this subsection with the following $L^1$ estimate:

Lemma 5.6. For any $r > 1$ and $\Upsilon > 0$, there exist $C > 0$ and $\beta_0 > 0$ such that, for all $\beta \geq \beta_0$, $U \in S_r^2$, $\Re \lambda \leq \Upsilon \beta^2/3 \beta^{-1/3}$, and any $g \in L^\infty(\mathbb{R})$ supported in $[-1, 1]$, we have

$$\| (\widetilde{L}_\beta, \mathbb{R} - \beta \lambda)^{-1} g \|_{L^1(-1,1)} \leq C \min(\beta^{-5/6} \| g \|_2, \beta^{-1} \log \| g \|_\infty).$$  (5.36)

Proof. Let $\Upsilon$ denote a fixed positive value and $\Re \lambda \leq \Upsilon \beta^{-1/3}$. Let further $g \in L^\infty(\mathbb{R})$, supported on $[-1, 1]$, and $v \in D(\widetilde{L}_\beta, \mathbb{R})$ satisfy

$$(\widetilde{L}_\beta, \mathbb{R} - \beta \lambda)v = g.$$  (5.37)

By (5.4)-(5.5) and (4.38), we have

$$\| v \|_{L^1(-1,1)} \leq \| (\widetilde{U} - v + i\beta^{-1/3})^{-1} \|_{L^2(-1,1)} \| (\widetilde{U} - v + i\beta^{-1/3})v \|_2 \leq C \beta^{-5/6} \| g \|_2,$$

which proves the first inequality of (5.36).

Let $x_v \in \mathbb{R}$ satisfy $\widetilde{U}(x_v) = v$. Let $\eta$ and $\eta_v$ be given by (5.14) and (5.15) and $\zeta_v(x) = \eta_v(x) \eta(|x - x_v|/6)$.

Taking the inner product of (5.37) with $\zeta_v^2 v$ yields for the imaginary part (see (5.17) with $\eta_v$ instead of $\zeta_v$)

$$\beta \| \zeta_v \| \widetilde{U} - v \|^{1/2} \| v \|_2^2 + 2 \Im \langle \zeta_v', v, (\zeta_v v)' \rangle = \langle \zeta_v v, \zeta_v g \rangle.$$  (5.38)

By (5.35) we have that

$$\| \zeta_v v \|_2 \leq \frac{C}{\beta^{1/2}} \| g \|_\infty.$$  (5.39)

To estimate $\| (\zeta_v v)' \|_2$ we use the identity

$$\| (\zeta_v v)' \|_2^2 - \Re \lambda \beta \| \zeta_v v \|_2^2 - \| \zeta_v' v \|_2^2 = \Re \langle \zeta_v v, \zeta_v g \rangle,$$

to obtain from (5.4), (5.35), and (5.39)

$$\| (\zeta_v v)' \|_2^2 \leq \frac{C}{\beta} \| g \|_\infty^2 + \| \zeta_v v \|_1 \| g \|_\infty.$$

Substituting the above into (5.38) yields

$$\| \zeta_v \| \widetilde{U} - v \|^{1/2} \| v \|_2 \leq C \left( \frac{1}{\beta} \| g \|_\infty + \beta^{-1/2} \| \zeta_v v \|_{1/2} \| g \|_{1/2} \right).$$  (5.40)
As
\[
\|\xi v\|_1 \leq \|\xi v\|_2 \|\tilde{U} - v\|_2^{1/2} \|1_{(x_v + \frac{1}{2} \beta^{-1/3}, x_v + 6)}\|_2 \|\tilde{U} - v\|_2^{-1/2}
\]
\[
\leq C (\log \beta)^{1/2} \|\xi v\|_2 \|\tilde{U} - v\|_2^{1/2},
\]
(5.41)
we obtain from (5.40) that
\[
\|\xi v\|_2 \|\tilde{U} - v\|_2^{1/2} \leq \tilde{C} (\log \beta)^{1/2} \|g\|_\infty.
\]
Then, using (5.41) once again yields
\[
\|\xi v\|_1 \leq \hat{C} \frac{\log \beta}{\beta} \|g\|_\infty,
\]
from which we easily conclude that
\[
\|1_{(x_v, x_v + 3)} v\|_1 \leq \|\xi v\|_1 + C \beta^{-1/6} \|1_{(x_v, x_v + \beta^{-1/3})} v\|_2 \leq \tilde{C} \frac{\log \beta}{\beta} \|g\|_\infty.
\]
In a similar manner, we obtain that
\[
\|1_{(x_v - 3, x_v)} v\|_1 \leq \tilde{C} \frac{\log \beta}{\beta} \|g\|_\infty.
\]
This proves (5.36) in the case where \((-1, +1) \subset (x_v - 3, x_v + 3)\).

It remains to prove (5.36) in the case \(|x_v| \geq 2\). By (5.5) and the fact that \(g\) is supported on \([-1, 1]\), we have that
\[
\|v\|_{L^1(-1,1)} \leq \sqrt{2} \|v\|_{L^2(-1,1)} \leq C \|\tilde{U} - v\|_{L^2(-1,1)} \leq \frac{C}{\beta} \|g\|_2 \leq \frac{C}{\beta} \|g\|_\infty.
\]
The lemma is proved \(\square\)

A similar statement can be proved in the Dirichlet case.

**Lemma 5.7.** For any \(r > 1\) and \(\Upsilon \in (0, \Re v_1)\), there exist \(C > 0\) and \(\beta_0 > 0\) such that, for all \(\beta \geq \beta_0\), \(U \in S^2_r\) and \(\Re \lambda \leq \frac{2}{3} \Upsilon \beta^{-1/3}\), and for any \(g \in L^\infty(-1, 1)\) we have
\[
\|(L^D_\beta - \beta \lambda)^{-1} g\|_1 \leq C \min(\beta^{-5/6} \|g\|_2, \beta^{-1} \log \beta \|g\|_\infty).
\]
(5.42)
The proof is similar to the proof of Lemma 5.6 and is therefore skipped.
6. No-Slip Resolvent Estimates

6.1. A No-Slip Schrödinger Operator

We begin by providing a short explanation of the difficulties arising when the no-slip boundary condition (3.10) is prescribed. Comprehensive details will be given in Section 8.

In the zero-traction case, estimating \( \phi \in D(B_{\lambda,\alpha,\beta}) \) satisfying \( B_{\lambda,\alpha,\beta} \phi = f \) for some \( f \in L^2(-1, 1) \), we may write, by (3.4),

\[
\left( -\frac{d^2}{dx^2} + i\beta U - \beta\lambda \right)(-\phi'' + \alpha^2 \phi) = i\beta U''\phi + f.
\]

Since by (3.9), it holds that \(-\phi'' + \alpha^2 \phi\) satisfies a Dirichlet condition at \( \pm 1 \), one can now use, for instance, (5.8) and (5.21) to obtain

\[
\| -\phi'' + \alpha^2 \phi \|_2 \leq C\beta^{1/6}\|\phi\|_\infty + \beta^{-2/3}\|f\|_2.
\]

Such an estimate is particularly useful in the case \( \alpha \gg \beta^{1/6} \), but also in other cases (detailed in Section 7).

Similar estimates can be obtained for \( v = A_{\lambda,\alpha} \phi \) and \( \tilde{v} = (U + i\lambda)^{-1} v \).

If we now consider the same problem in the no-slip case the above approach is inapplicable. Thus, for \( \phi \in D(B_{\lambda,\alpha,\beta}^D) \) satisfying \( B_{\lambda,\alpha,\beta}^D \phi = f \), we can no longer use neither (5.8) nor (5.21), as \(-\phi'' + \alpha^2 \phi\) does not satisfy a Dirichlet condition at \( x = \pm 1 \). However, integration by parts easily yields that for all \( \phi \in D(B_{\lambda,\alpha,\beta}^D) \) we have

\[
\langle e^{\pm\alpha x}, -\phi'' + \alpha^2 \phi \rangle = 0.
\]

If we consider \( v \) or \( \tilde{v} \) instead of \(-\phi'' + \alpha^2 \phi\) we can still obtain similar orthogonality conditions (see (8.22) and (8.32c)). These conditions read

\[
\langle v, \xi_{\pm} \rangle_{L^2(-1, +1)} = 0,
\]

where \( \xi_{-} \) and \( \xi_{+} \) are linearly independent, \( \beta \) dependent, and belong to \( H^1(-1, +1) \).

With (3.4) in mind, we let \( L_{\beta}^x \) be the differential operator \(-d^2/dx^2 + i\beta U\) with domain

\[
D(L_{\beta}^x) = \{ u \in H^2(-1, 1) \mid \langle \xi_{\pm}, u \rangle = 0 \}.
\]

For convenience we require that \( \xi_{\pm} \) satisfy

\[
\begin{bmatrix}
\xi_+(1) \\
\xi_+(-1)
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Note that \( \xi_{\pm} = e^{\pm \alpha x} \) do not satisfy the above requirement, and we shall therefore need to replace them by a pair of proper linear combinations of them [41] (a more detailed explanation is brought in Section 8). We seek resolvent estimates for \( L_{\beta}^x \) in
the following. In the absence of a Dirichlet boundary condition, it seems reasonable to approximate the solution of

\[ (\mathcal{L}_\beta^\varepsilon - \beta \lambda)v = g, \tag{6.4} \]

by a sum of a solution in \( \mathbb{R} \) of the inhomogeneous equation and a linear combination of two independent approximate solutions of the homogeneous equation whose coefficients will be determined by the above integral conditions. Using affine approximations of \( U \) in \((-1, +1)\) or extensions outside of \((-1, +1)\), the approximate solutions can be described by a pair of dilated and translated Airy functions in \((-1, +\infty)\) and \((-\infty, +1)\).

**The solution in \( \mathbb{R} \).**

We now explain our construction of an approximate inverse in \( \mathbb{R} \) by defining first a natural \( C^1 \)-extension \( \tilde{U} \) of \( U \) outside of \([-1, +1]\), satisfying (5.3), by

\[ \tilde{U}(x) = \begin{cases} U(x) & x \in [-1, 1] \\ U(1) + U'(1)(x - 1) & x > 1 \\ U(-1) + U'(-1)(x + 1) & x < -1. \end{cases} \]

We note that \( \tilde{U} \) satisfies the conditions of Proposition 5.1. We also extend \( g \) by

\[ \tilde{g}(x) = \begin{cases} g(x) & x \in [-1, 1] \\ 0 & \text{otherwise}, \end{cases} \]

and set

\[ u = \Gamma_{(-1, 1)}(\tilde{\mathcal{L}}_{\beta, \mathbb{R}} - \beta \lambda)^{-1}\tilde{g}, \tag{6.5} \]

where \( \tilde{\mathcal{L}}_{\beta, \mathbb{R}} \) is defined by (5.1) and (5.2) and \( \Gamma_{(-1,1)} \) denotes the restriction to \((-1, 1)\).

**Boundary terms.**

To obtain the boundaries effect, we replace \( U(x) \) by its affine approximation at \( \pm 1 \) and consider the \( L^2 \cap L^1 \) solutions \( \psi_\pm \) of the approximate problems

\[ \begin{cases} \left(-d^2/dx^2 + i \beta[U(-1) + J_-(x + 1)] - \beta \lambda \right)\psi_- = 0 \text{ in } (-1, +\infty) \\ \int_{-1}^{+\infty} \psi_-(x) \, dx = (J_-\beta)^{-1/3}, \tag{6.6} \end{cases} \]

and

\[ \begin{cases} \left(-d^2/dx^2 + i \beta[U(1) + J_+(x - 1)] - \beta \lambda \right)\psi_+ = 0 \text{ in } (-\infty, 1) \\ \int_{-\infty}^{1} \psi_+(x) \, dx = (J_+\beta)^{-1/3}, \tag{6.7} \end{cases} \]

with

\[ J_\pm = U'(\pm 1). \]

Except, perhaps for some particular values of \( \lambda \), the above solutions are unique, and \( \psi_\pm \) rapidly decays as \( x \to \mp\infty \), but their existence (due to the additional integral condition) could depend on \((\beta, \lambda, J_\pm)\) as is clarified below. We express \( \psi_\pm \) using Airy functions. Having in mind the definition of the generalized Airy
functions [43, eq. (39)] or [39, Lemma 2] (for more details see [16, Appendix] or our short review in Appendix A). These solutions are given, assuming that the denominator does not vanish, by

$$\psi_-(x) = e^{i\pi/6} \frac{\text{Ai}((J_\beta)^{1/3}e^{i\pi/6}[(1 + x) + i J^{-1}_-(\lambda - i U(-1))])}{A_0(i\beta^{1/3} J^{-2/3}_-(\lambda - i U(-1)))}, \quad (6.8a)$$

and

$$\psi_+(x) = -e^{i\pi/6} \frac{\text{Ai}((J_+\beta)^{1/3}e^{i\pi/6}[(1 - x) + i J^{-1}_+(\lambda + i U(1))])}{A_0(i\beta^{1/3} J^{2/3}_+(\lambda + i U(1)))}, \quad (6.8b)$$

where $A_0$ is the holomorphic extension to $\mathbb{C}$ of $x \mapsto A_0(x) = e^{i\pi/6} \int^{+\infty}_x \text{Ai}(e^{i\pi/6}t) \, dt$. (6.9)

Much of the properties of $A_0$ are recalled (mainly from Wasow’s paper [43]) in Appendix A. It has been established in [43] (see also Appendix A) that the zeroes of $z \mapsto A_0(iz)$ are located in the sector $\arg z \in (\pi/6, \pi/2)$. Let

$$S_\lambda = \{ z \mid A_0(iz) = 0 \} .$$

and further define

$$\varrho^r_1 := \inf_{z \in S_\lambda} \Re z. \quad (6.10)$$

In addition, we prove in Appendix A.2 that $S_\lambda \neq \emptyset$ and (relying on [43]) that $\varrho^r_1 > 0$. It follows that the denominators in (6.8b) and (6.8a) do not vanish, if

$$\Re \lambda < \varrho^r_1 \beta^{-1/3} \Im^{2/3}_m,$$

where $\Im_m$ is given by (2.30c).

The functions $\psi_\pm$ are not exact solutions of $(-d^2/dx^2 + i\beta(U - \beta\lambda))\psi = 0$ and hence we must introduce a correction term. We thus consider

$$g_\pm = \begin{cases} (-d^2/dx^2 + i\beta(U + i\lambda))\psi_\pm & \text{for } x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}, \quad (6.11)$$

and then introduce

$$\tilde{v}_\pm = \Gamma(\beta, 1)(\tilde{L}_{\beta, \mathbb{R}} - \beta\lambda)^{-1}g_\pm. \quad (6.12)$$

This correction term can be estimated as follows

**Lemma 6.1.** For any $r > 1$ and $\Upsilon < \varrho^r_1$, there exist $C$ and $\beta_0$ such that, for all $U \in S^2_r$, $\lambda \in \mathbb{C}$ satisfying

$$\beta^{1/3}\Re \lambda \leq \Upsilon \Im^{2/3}_m,$$

and $\beta \geq \beta_0$, we have

$$\|(U + i\lambda)\tilde{v}_\pm\|_2 + \beta^{-1/3}\|\tilde{v}_\pm\|_2 \leq C\beta^{-5/6}. \quad (6.13)$$
Proof. A simple computation shows that:

\[ g_{\pm} = i\beta [U - U(\pm 1) - J_{\pm}(x \mp 1))] \psi_{\pm} \] in \((-1, 1)\).

Let

\[ \lambda_{\pm} = \mu - i(U(\pm 1) - v). \] (6.15)

We note that

\[ \psi_{\pm}(x) = e^{i\frac{\pi}{2} J_{\pm}}(J_{\pm})^{\frac{1}{2}} \lambda_{\pm}^{-1} (J_{\pm})^{\frac{1}{2}}(1 \mp x)), \] (6.16)

where \( \Psi_{\lambda} \) is defined in the appendix (see (A.42)).

Using translation, dilation, and (A.43a), we can conclude that, under the assumptions of Lemma 6.1, it holds, for \( k \in [0, 4] \), that

\[ \| (1 \mp x)^k \psi_{\pm} \|_2 \leq C [1 + |\lambda_{\pm}|\beta^{1/3}]^{1-2k} \beta^{-(1+2k)/6}. \] (6.17)

Hence, as

\[ |g_{\pm}(x)| \leq C\beta (1 \mp x)^2 \| \psi_{\pm}(x) \|, \]

we have

\[ \| g_{\pm} \|_2 \leq C \beta^{1/6} [1 + |\lambda_{\pm}|\beta^{1/3}]^{-3/4}. \] (6.18)

By (5.4) and (5.5) we have

\[ \| (U + i\lambda) \tilde{v}_{\pm} \|_2 + \beta^{-1/3} \| \tilde{v}_{\pm} \|_2 \leq C\beta^{-5/6} [1 + |\lambda_{\pm}|\beta^{1/3}]^{-3/4}, \] (6.19)

establishing thereby (6.14).

We are now ready for introducing a solution of (6.4) in the form

\[ v = A_+(\psi_+ - \tilde{v}_+) + A_-(\psi_- - \tilde{v}_-) + u. \] (6.20)

We observe that \((L_\beta - \beta \lambda)v = g\) for any pair \((A_- , A_+)\). Therefore, one can attempt to find two linear forms \(g \mapsto A_-(g)\) and \(g \mapsto A_+(g)\) such that \(v\) belongs to the domain of \(L_\beta^\xi\), and hence is the solution of (6.4). This is the object of the next lemma.

Lemma 6.2. Let \( \theta > 0 \) and \( C_\xi > 0 \), and suppose that \( \zeta_- , \zeta_+ \in H^1(-1, 1) \) satisfy the conditions (6.3),

\[ \| \xi_{\pm} \|_\infty \leq C_\xi, \] (6.21)

and

\[ \| \xi_{\pm} \|_2 \leq \theta \beta^{1/6}. \] (6.22)

Let further \( r > 1 \) and \( \Upsilon < \theta_r^\xi \). Then, there exist \( \beta_0 > 0 \) and \( \theta_0 > 0 \) such that for all \( \beta \geq \beta_0 \), \( 0 < \theta \leq \theta_0 \), \( U \in S_r^2 \) and \( \lambda \in \mathbb{C} \) satisfying

\[ \beta^{1/3} \Re \lambda \leq \frac{2}{3} \Upsilon, \] (6.23)

(6.4) and (6.20) hold true with \( v \in D(L_\beta^\xi) \) and \( A_\pm = A_\pm(g) \) denoting a pair of linear forms \( A_\pm(g) : L^\infty(-1, 1) \to \mathbb{C} \). Furthermore, there exists \( C > 0 \) such that, for all \( \beta \geq \beta_0 \), \( U \in S_r^2 \) and \( g \in L^\infty(-1, +1) \), we have

\[ |A_\pm(g)| \leq C \min(\beta^{-1/2} \| g \|_2, \beta^{-2/3} \log \beta \| g \|_\infty). \] (6.24)
Proof. In view of the discussion preceding the statement of the lemma, it remains to show the existence of $A_{\pm}(g)$ satisfying (6.24).

Taking the inner product of (6.20) in $L^2(-1, +1)$, first by $\zeta_+$ and then by $\zeta_-$ while having (6.1) in mind yields the following system

$$
\begin{bmatrix}
\langle \zeta_+, (\psi_+ - \tilde{v}_+) \rangle & \langle \zeta_+, (\psi_- - \tilde{v}_-) \rangle \\
\langle \zeta_-, (\psi_+ - \tilde{v}_+) \rangle & \langle \zeta_-, (\psi_- - \tilde{v}_-) \rangle
\end{bmatrix}
\begin{bmatrix}
A_+(g) \\
A_-(g)
\end{bmatrix}
= \begin{bmatrix}
\langle \zeta_+, u \rangle \\
\langle \zeta_-, u \rangle
\end{bmatrix}.
$$

We now write

$$
\langle \zeta_{\pm}, \psi_{\pm} \rangle = \langle 1, \psi_{\pm} \rangle + \langle \zeta_{\pm} - 1, \psi_{\pm} \rangle.
$$

For the first term on the right-hand-side we have

$$
\langle 1, \psi_- \rangle = (J_- \beta)^{-1/3} - \int_1^{\infty} \psi_-(x) \, dx.
$$

The integral on the other side can be estimated as follows: we first write

$$
\left| \int_1^{\infty} \psi_-(x) \, dx \right| \leq \|(1 + x)^3 \psi_-\|_1.
$$

Then, using (A.43b), (6.16) and dilation, we obtain for all $s \in [0, 3]$,

$$
\|(1 \mp x)^s \psi_\pm\|_1 \leq C \left[ 1 + |\lambda_{\pm}| \beta^{1/3} \right]^{-s/2} \beta^{-s(s+1)/3}.
$$

(6.27)

The above estimate for $s = 3$ yields,

$$
\left| \int_1^{\infty} \psi_-(x) \, dx \right| \leq C \beta^{-4/3}.
$$

A similar estimate can be obtained for $\langle 1, \psi_+ \rangle$. Consequently we have

$$
\langle 1, \psi_{\pm} \rangle = (J_{\pm} \beta)^{-1/3} [1 + O(\beta^{-1})].
$$

(6.28)

For the second term on the right-hand-side of (6.26) we use the fact that, for all $x \in [-1, 1]$, we have by (6.3)

$$
|\xi_{\pm}(x) - 1| \leq [1 \mp x]^{1/2} \|\xi_\pm'(\cdot)\|_2.
$$

We obtain, using (6.27) with $s = \frac{1}{2}$ and (6.22),

$$
|\langle \zeta_\pm - 1, \psi_\pm \rangle| \leq \|\xi_\pm'(\cdot)\|_2 \|[1 \mp x]^{1/2} \psi_\pm\|_1 \leq C(\Upsilon) \theta_0 \beta^{-1/3}.
$$

(6.29)

Furthermore, by (4.37) and (6.19), we have

$$
|\langle \xi_\pm, \tilde{v}_\pm \rangle| \leq 2 \|[U - \nu + i \beta^{-1/3}]^{-1}\|_2 \|\tilde{v}_\pm\|_2 \leq C \beta^{-2/3} \left[ 1 + |\lambda_{\pm}| \beta^{1/3} \right]^{-3/4}.
$$

(6.30)

By the above, (6.28), and (6.29), there exists $C > 0$ and $\beta_0$, such that, for any $\beta \geq \beta_0$ and any $\lambda$ satisfying $\Re \lambda \leq \Upsilon \beta^{-1/3}$, we have

$$
|\langle \xi_\pm, (\psi_\pm - \tilde{v}_\pm) \rangle - (J_{\pm} \beta)^{-1/3}| \leq C(\theta_0 \beta^{-1/3} + \beta^{-2/3}).
$$

(6.31)
As $\zeta_{\pm}(\mp 1) = 0$, we obtain as in (6.29)

$$|\langle \zeta_{\pm}, \psi_{\mp} \rangle| \leq C \theta_0 \beta^{-1/3}. \quad (6.32)$$

Furthermore, as in (6.30) we obtain that

$$|\langle \zeta_{\pm}, \tilde{\psi}_{\mp} \rangle| \leq C \beta^{-2/3} [1 + |\lambda_{\pm}| \beta^{1/3}]^{-3/4}. \quad (6.33)$$

Substituting the above, together with (6.31) and (6.32) into (6.25) then yields, for $\theta_0$ small enough, and $\beta$ large enough, the invertibility of (6.25) together with the estimate

$$|A_{\pm}(g)| \leq |\langle \zeta_{\pm}, u \rangle|(J_{\pm} \beta)^{1/3} [1 + C \theta_0] + C \theta_0 \beta^{-1/3} |\langle \zeta_{\mp}, u \rangle|. \quad (6.34)$$

By (5.36) we obtain that

$$|\langle \zeta_{\pm}, u \rangle| \leq C \min(\beta^{-1/2} \|g\|_2, \beta^{-2/3} \log \beta \|g\|_\infty). \quad (6.35)$$

6.2. A No-Slip Schrödinger in $\mathbb{R}_+$

In the previous subsection we have considered a space of functions satisfying the orthogonality condition (6.1). We have assumed that the functions spanning the orthogonal space $\zeta_{+}$ and $\zeta_{-}$ satisfy the bound

$$\|\zeta_{\pm}'\|_2 \leq \theta \beta^{1/6},$$

where $\theta \in (0, \theta_0]$ for some sufficiently small $\theta_0 > 0$.

Of particular interest is the example $\zeta_{\pm} = e^{-\alpha(1 \mp x)}$ (or a proper linear combination of them satisfying (6.3)). In this case, we have

$$\|(e^{-\alpha(1 \mp x)})'\|_2 \simeq \sqrt{\alpha/2},$$
for sufficiently large $\alpha$.
Consequently, as long as $\alpha \ll \beta^{1/3}$, Lemma 6.2 is applicable in this case. We, however, need to consider also the case where $\alpha \sim \beta^{1/3}$, or even $\alpha \gg \beta^{1/3}$. These cases can, nevertheless, be treated using localization techniques as in [4, 27]. To this end we have to consider a localized version of $L_\xi$ near $x = \pm 1$. This subsection is devoted therefore to the study of the ensuing linearized operator.

We begin by establishing a proper spectral formulation for the problem.

**Proposition 6.4.** Let, for some $\theta > 0$,

$$L^\theta = -\frac{d^2}{dx^2} + ix ,$$

be defined on

$$D(L^\theta) = \{ u \in H^2(\mathbb{R}_+) | \langle e_\theta, u \rangle = 0 , \ xu \in L^2(\mathbb{R}_+) \} ,$$

where

$$e_\theta(x) := e^{-\theta x} .$$

Then, $L^\theta$ is a closed operator with non empty resolvent set and compact resolvent. Moreover $L^\theta$ has index 0.

Before proceeding to the proof of the proposition we establish the following $H^1$ estimate of any $v \in D(L^\theta)$ in term of the $L^2$-norm of $L^\theta v$ and $v$.

**Lemma 6.5.** There exists some constant $C(\theta)$ such that, for any $\lambda \in \mathbb{C}$ and any $v \in D(L^\theta)$, we have

$$\| v' \|_2 + |v(0)| \leq C(\theta) [(1 + |\Im\lambda|^{1/2} \text{sign } \Im\lambda)\| v \|_2 + \|(L^\theta - \lambda)v\|_2] .$$

**Proof.** Let $v \in D(L^\theta)$ and $g \in L^2(\mathbb{R}_+)$ satisfy

$$(L^\theta - \lambda)v = g .$$

Taking the inner product with $v$ yields

$$\| v' \|_2^2 - \lambda \| v \|_2^2 + i \langle xv, v \rangle + v'(0)\tilde{v}(0) = \langle v, g \rangle .$$

To obtain an estimate for the fourth term on the left-hand-side of (6.39) we need an effective bound on $v'(0)$.

Integration by parts yields, with the aid of the fact that $\langle e_\theta, v \rangle = 0$,

$$-\langle e_\theta, v'' \rangle = v'(0) + \theta \ v(0) .$$

Taking the inner product of (6.38) with $e_\theta$, we then obtain

$$v'(0) + \theta v(0) + i \langle xe_\theta, v \rangle = \langle e_\theta, g \rangle ,$$

from which we conclude that

$$|\tilde{v}(0)v'(0)| \leq \theta |v(0)|^2 + \frac{1}{2\theta^{3/2}}|v(0)| (\| v \|_2 + \theta^{1/2} \| g \|_2 ) .$$
As

\[ |v(0)|^2 \leq \|v'\|_2 \|v\|_2, \]  

(6.42)

we obtain, using (6.41) that

\[ |\tilde{v}(0)v'(0)| \leq \frac{\theta}{2} \|v'\|_2 \|v\|_2 + \frac{1}{2\theta^{3/2}} |v(0)| (\|v\|_2 + \theta \sqrt{2} \|g\|_2). \]  

(6.43)

Combining the real part of (6.39) with (6.43) yields

\[ \|v'\|_2^2 - \mu \|v\|_2^2 \leq \frac{\theta}{2} \|v'\|_2 \|v\|_2 + \|v\|_2 \|g\|_2 + \frac{1}{2\theta^{3/2}} |v(0)| (\|v\|_2 + \theta \sqrt{2} \|g\|_2). \]  

(6.44)

Consequently, we obtain (6.37). \( \square \)

Proof of Proposition 6.4. Step 1: We prove that \( L^\theta \) is a closed operator.

Let \( \{ (v_n, L^\theta v_n) \}_{n=1}^\infty \in [D(L^\theta)]^N \times [L^2(\mathbb{R}_+)]^N \) converge, as \( n \to \infty \), in \( L^2(0, +\infty) \times L^2(0, +\infty) \) to \( (v_\infty, g_\infty) \). We need to establish that \( v_\infty \in D(L^\theta) \).

The orthogonality of \( v_\infty \) with \( e_\theta \) immediately follows from the \( L^2 \) convergence. From (6.37) (with \( \lambda = 0 \)) we conclude that \( v_n \) is a Cauchy sequence in \( H^1(\mathbb{R}_+) \) and hence must converge to \( v_\infty \) in the \( H^1(\mathbb{R}_+) \) norm.

Let \( \chi \in C^\infty(\mathbb{R}) \) be supported on \( [\frac{1}{2}, +\infty) \) and satisfy \( \chi = 1 \) for \( x > 1 \).

Clearly,

\[ \left( -\frac{d^2}{dx^2} + ix \right) (\chi v_n) = \chi g_n + 2\chi' v'_n + \chi'' v_n. \]

Since we can smoothly extend \( \chi v_n \) to \( H^2(\mathbb{R}) \), it follows from (5.4) and (5.5) (with \( \beta = 1 \) and \( U = x \)) that \( \chi v_n \) and \( \chi x v_n \) are Cauchy sequences in \( H^2(\mathbb{R}) \) and in \( L^2(\mathbb{R}) \) respectively. Hence, its limit \( \chi v_\infty \) satisfies \( \chi v_\infty \in H^2(\mathbb{R}) \) and \( \chi x v_\infty \in L^2(\mathbb{R}) \).

By the \( H^1(\mathbb{R}_+) \) convergence of \( \{ v_n \}_{n=1}^\infty \) it follows that \( v_\infty \) is a weak solution of

\[ v''_\infty = -g_\infty + ix v_\infty. \]

Since the right-hand-side is in \( L^2(\mathbb{R}_+) \), it follows that \( v_\infty \in H^2(\mathbb{R}_+) \) and hence \( L^\theta \) is closed.

Step 2: We prove that \( L^\theta - \lambda \) has index 0.

Let \( \tilde{L} : H^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^2 dx) \to L^2(\mathbb{R}_+) \) be associated with the same differential operator as \( L^\theta \). Clearly, \( \tilde{L} \) is a Fredholm operator of index 1. Indeed, it is clearly surjective (we can find a unique solution satisfying a Dirichlet condition at \( x = 0 \)) and it is easy to see that the kernel has dimension 1 (\( \text{span}\{ Ai(e^{i\pi/6}) \} \)). Consequently, for any \( \lambda \in \mathbb{C} \), \( \tilde{L} - \lambda \) has index 1. We now observe that \( L^\theta \) is obtained by imposing a single orthogonality condition in the domain. Hence the index of \( L^\theta - \lambda \) equals 0.

Step 3: We show that \( \rho(L^\theta) \neq \emptyset \).

We prove that there exists \( \mu_0 < 0 \) such that for all \( \Re \lambda < \mu_0 \) the operator \( L^\theta - \lambda \) is injective. Combined with the above zero index property, it would yield that the resolvent set contains the half plane \( \Re \lambda < \mu_0 \). The injectivity follows from (6.42) and (6.44), by which there exist \( \mu_0 \) and \( C > 0 \) such that for all \( \Re \lambda < \mu_0 \), we have

\[ \|v\|_2 + \|v'\|_2 + |v(0)| \leq C \|g\|_2. \]  

(6.45)
Finally, the compactness of the resolvent follows from the fact that $D(L^\theta)$ is compactly embedded in $L^2(\mathbb{R}_+)$. \hfill \Box

The previous proof has also shown to us that $L^\theta$ is semi-bounded. The next proposition provides a more explicit lower bound for the spectrum as a function of $\theta$.

**Proposition 6.6.** For all $\theta \in \mathbb{R}_+$, we have

$$\mu_0(\theta) = \inf \Re \sigma (L^\theta) \geq -\frac{1}{2} \min(\theta^2, \theta^{-2}).$$

(6.46)

**Proof.** Suppose that for some positive $\hat{\theta}_0$ there exist $\lambda_0 \in \sigma (L^{\hat{\theta}_0})$ and $v_0 \in D(L^{\hat{\theta}_0})$ such that

$$(L^{\hat{\theta}_0} - \lambda_0)v_0 = 0.$$  
(6.47)

Since $v_0$ is an $L^2$ solution of the complex Airy equation in $\mathbb{R}_+$ it is expressible, up to a multiplicative constant, in the form

$$v_0(x) = Ai (e^{i\pi/6} (x + i\lambda_0)).$$  
(6.48a)

The orthogonality condition for $v_0$ reads

$$F(\lambda, \theta) = 0,$$  
(6.48b)

where

$$F(\lambda, \theta) = \int_{\mathbb{R}_+} e^{\theta x} Ai (e^{i\pi/6} (x + i\lambda)) \, dx.$$  
(6.48c)

**Step 1:** We prove that the set \{ $\theta \in (0, +\infty), \exists \lambda \in \sigma (L_\theta)$ with $\Re \lambda < \Re \nu_1$ \} is open.

We use the implicit function theorem. If indeed $F(\lambda_0, \hat{\theta}_0) = 0$, we get after integration by parts that

$$\frac{\partial F}{\partial \lambda}(\lambda_0, \hat{\theta}_0) = -i Ai (e^{i2\pi/3} \lambda_0) \neq 0.$$  

Hence there exists a neighborhood of $\hat{\theta}_0$ and a $C^1$-solution $\lambda(\theta)$ in this neighborhood such that $\lambda(\hat{\theta}_0) = \lambda_0$.

**Step 2:** Let $\varepsilon < \Re \nu_1$. Consider the set

$$\Sigma(\varepsilon) := \{ (\lambda, \theta) \in \mathbb{C} \times (0, +\infty), \Re \lambda < \varepsilon \text{ and } F(\lambda, \theta) = 0 \},$$

which can be described as a countable (or finite) union of simple analytic curves denoted by \{ $\lambda_k(\theta)$ $\}_{k \in \mathcal{K}}$, each with an interval of definition $(\theta_k, \theta_k^*)$. Let further $\mu_k(\theta) = \Re \lambda_k(\theta)$. We prove that for all $k \in \mathcal{K}$ and $\theta \in (\theta_k, \theta_k^*)$

$$\mu_k(\theta) + \theta^2/2 \geq (\mu_k(\theta_k) + \theta_k^2/2) \exp \left\{ -\int_{\theta_k}^{\theta} \frac{\|v_k(\cdot, \tau)\|^2}{\|v_k(0, \tau)\|^2} \, d\tau \right\},$$

(6.49)

where

$$v_k(x, \theta) := Ai (e^{i\pi/6} (x + i\lambda_k(\theta))).$$
Let $k \in \mathcal{K}$. For convenience of notation we set
\[ \lambda(\theta) = \lambda_k(\theta), \quad \mu(\theta) = \Re \lambda(\theta) \text{ and } v(\theta) = v_k(\theta). \]

By (6.48b,c) we have that
\[ F(\lambda(\theta), \theta) = 0, \quad \forall \theta \in \Sigma(0). \quad (6.50) \]

Differentiating this identity with respect to $\theta$ yields
\[ e^{2\pi i/3} \frac{d\lambda}{d\theta}(\theta) I_1(\theta) - I_2(\theta) = 0, \quad (6.51a) \]
where
\[ I_1(\theta) = \int_{\mathbb{R}^+} e_\theta(x) \operatorname{Ai}'(e^{i\pi/6}(x + i\lambda(\theta))) \, dx, \quad (6.51b) \]
and
\[ I_2(\theta) = \int_{\mathbb{R}^+} x e_\theta(x) \operatorname{Ai}(e^{i\pi/6}(x + i\lambda(\theta))) \, dx. \quad (6.51c) \]

Integration by parts yields, in conjunction with (6.50),
\[ I_1(\theta) = -e^{-i\pi/6} \operatorname{Ai}(e^{2\pi i/3}\lambda(\theta)). \quad (6.52) \]

We now write, with the aid of (6.50) and Airy’s equation
\[ I_2(\theta) = \int_{\mathbb{R}^+} (x + i\lambda(\theta)) e_\theta(x) \operatorname{Ai}(e^{i\pi/6}(x + i\lambda(\theta))) \, dx \]
\[ = e^{-i\pi/6} \int_{\mathbb{R}^+} e_\theta(x) \operatorname{Ai}''(e^{i\pi/6}(x + i\lambda(\theta))) \, dx. \]

Integration by parts and (6.52) then yield
\[ I_2(\theta) = -e^{-i\pi/3} \operatorname{Ai}'(e^{2\pi i/3}\lambda(\theta)) + i\theta \operatorname{Ai}(e^{2\pi i/3}\lambda(\theta)). \quad (6.53) \]

Taking the inner product of (6.47) with $v(\cdot, \theta)$ we obtain for the real part
\[ \| \partial_x v(\cdot, \theta) \|^2_2 - \mu(\theta) \| v(\cdot, \theta) \|^2_2 + \Re \{ \bar{v}(0, \theta) \partial_x v(0, \theta) \} = 0, \]
where $\mu(\theta) = \Re \lambda(\theta)$.

Combining the above with (6.53) then yields
\[ \frac{d\mu}{d\theta}(\theta) + \theta = \frac{\| \partial_x v(\cdot, \theta) \|^2_2}{\| v(0, \theta) \|^2_2} - \frac{\mu(\theta) \| v(\cdot, \theta) \|^2_2}{\| v(0, \theta) \|^2_2}. \]

We then have on the branch
\[ \frac{d(\mu + \theta^2/2)}{d\theta} + \frac{\| v(\cdot, \theta) \|^2_2}{\| v(0, \theta) \|^2_2} (\mu + \theta^2/2) > 0. \]

Solving in $[\hat{\theta}_0, \hat{\theta}_0^*]$ yields (6.49).
Step 3: We prove that along every curve in $\Sigma(0)$

$$\Re \lambda(\theta) = \mu(\theta) \geq -\theta^{-2}/2.$$  \hspace{1cm} (6.54)

From (6.44) with $g = 0$, and (6.42) we obtain that

$$\frac{1}{2} \|v'\|_2^2 - \mu(\theta) \|v\|_2^2 \leq \frac{1}{2\theta^{3/2}} \|v'\|_2^{1/2} \|v\|_2^{3/2}.$$  \hspace{1cm} (6.54)

The above, in conjunction with Young’s inequality yields $\mu(\theta) \geq -\theta^{-2}/2$ which is precisely (6.54). In particular it implies that

$$\liminf_{\theta \to +\infty} \inf_{F(\lambda, \theta) = 0} \Re \lambda \geq 0.$$  \hspace{1cm} (6.55)

Step 4: We prove (6.46).

If $\inf_{\theta} \inf_{F(\lambda, \theta) = 0} \Re \lambda \geq 0$ then (6.46) readily follows. Hence, we can assume that there exists $(\lambda_0, \hat{\theta}_0)$ such that $\Re \lambda_0 < 0$ and $F(\lambda_0, \hat{\theta}_0) = 0$.

We then look at $\{\lambda_k(\theta)\}_{k \in K}$ inside $\Sigma(-\varepsilon)$ where $\Re \lambda_0 < -\varepsilon < 0$. By (6.55), all branches exit $\Sigma(-\varepsilon)$ for sufficiently large $\theta$.

We now observe that

$$F(\lambda(0), 0) = 0 \Rightarrow \mu(0) > 0.$$  \hspace{1cm} (6.56)

Indeed, as

$$F(\lambda, 0) = e^{-i\pi} A_0(i\lambda)$$

we can apply Corollary A.4. Hence, these branches must lie outside $\Sigma(0)$ for $\theta \in [0, \theta_{\inf})$ for sufficiently small $\theta_{\inf} > 0$. Assume that $\varepsilon$ is chosen small enough so that

$$-\varepsilon + \frac{1}{2} \theta_{\inf}^2 > 0.$$  \hspace{1cm} (6.56)

Consider any branch $\lambda(\theta)$ in $\Sigma(-\varepsilon)$ with $\theta$ in some interval $[\theta(\varepsilon), \theta^*(\varepsilon)] \subset (0, +\infty)$ such that $\Re \lambda(\theta(\varepsilon)) = \Re \lambda(\theta^*(\varepsilon)) = -\varepsilon$.

We can then use (6.49) to obtain

$$\mu(\theta) + \theta^2/2 \geq (\mu(\theta(\varepsilon)) + \theta(\varepsilon)^2/2) \exp \left\{-\int_{\theta(\varepsilon)}^{\theta} \frac{\|v\|_2^2}{|v(0)|^2} d\theta \right\}$$

$$\geq (-\varepsilon + \theta(\varepsilon)^2/2) \exp \left\{-\int_{\theta(\varepsilon)}^{\theta} \frac{\|v\|_2^2}{|v(0)|^2} d\theta \right\} \hspace{1cm} (6.56)$$

$$> 0.$$  \hspace{1cm} (6.56)

This completes the proof of (6.46).  \hspace{1cm} $\Box$

Corollary 6.7. It holds that

$$\hat{\mu}_m := \inf_{\theta \in \mathbb{R}^+} \inf_{\lambda \in \sigma(L^\theta)} \left(\Re \lambda + \frac{\theta^2}{2}\right) > 0.$$  \hspace{1cm} (6.57)
The proof of (6.57) with the non strict inequality follows immediately from (6.46). Note now that by (6.56), taking into account that \(\mu(\theta) + \theta^2 \xrightarrow[\theta \to +\infty]{} +\infty\) and that \(\mu(0) > 0\) we can conclude the strict inequality in (6.57).

**The adjoint operator**

We note that \(D(L^\theta)\) is not dense in \(L^2(\mathbb{R}_+)\). Let \(A : D(A) \to \mathcal{H}\). We introduce 

\[ H_0 = D(A), \]

and then define

\[ D(A^*) = \{ v \in \mathcal{H} \text{ s.t. } D(A) \ni u \mapsto \langle v, Au \rangle \text{ extends as a continuous linear map on } H_0 \}, \]

(6.58)

and set \(A^*v\) to be the unique (by Riesz theorem) \(y \in H_0\) for which

\[ \langle v, Au \rangle = \langle y, u \rangle. \]

The standard definition is recovered when \(H_0 = H\). Note that

\[ (A - \lambda I)^* = A^* - \overline{\lambda} \Pi_{H_0}, \]

where \(\Pi_{H_0}\) is the projector on \(H_0\). We further note that \(A^*A\) is an unbounded operator on \(H_0\).

In the particular case \(A = L^\theta\), \(H_0\) is the orthogonal complement in \(L^2(\mathbb{R}_+)\) of \(e_\theta\). Hence, \(\Pi_{H_0} = I - P_\theta\), where, for any \(u \in L^2(\mathbb{R}_+)\),

\[ P_\theta u(x) = \frac{1}{2\theta} \langle e_\theta, u \rangle e_\theta. \]

We next provide a more explicit representation of \((L^\theta)^*\).

**Lemma 6.8.** We have

\[ D((L^\theta)^*) = H_0^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^2 dx), \]

(6.59a)

and for any \(v \in D((L^\theta)^*)\),

\[ (L^\theta)^* v = (I - P_\theta) \left( -\frac{d^2}{dx^2} - ix \right) v. \]

(6.59b)

**Proof.** The proof is reminiscent of the analysis of selfadjointness for (1D)-problems in [38]. Let \(\phi \in C_0^\infty(\mathbb{R}_+)\) and then set \(u = (I - P_\theta)\phi \in D(L_\theta)\). Let \(v \in D(L_\theta^*)\), where \(D(L_\theta^*)\) is defined by (6.58). From the definition we deduce that the distribution

\[ C_0^\infty(\mathbb{R}_+) \ni \phi \mapsto \langle v, \mathcal{L}_\theta(I - P_\theta)\phi \rangle_{L^2(\mathbb{R}_+)} \]

should extend as a continuous linear map on \(L^2(\mathbb{R}_+)\). We then observe that

\[ \langle v, \mathcal{L}_\theta(I - P_\theta)\phi \rangle = \left( \left( -\frac{d^2}{dx^2} - ix \right) v, \phi \right) - \left( v, \left( -\frac{d^2}{dx^2} + ix \right) e_\theta \right) \langle e_\theta, \phi \rangle. \]

The second term on the right hand side defines a linear form on \(L^2(\mathbb{R}_+)\). Hence, from (6.58) we get that \(\phi \mapsto \langle (-d^2/dx^2 - ix)v, \phi \rangle\) is a distribution in \(L^2(\mathbb{R}_+)\).
Hence, it holds that $(-d^2/dx^2 - ix)v \in L^2(\mathbb{R}_+)$. We can thus conclude that $v \in H^2(\mathbb{R}_+), xv \in L^2(\mathbb{R}_+)$. We now compute $\langle v, \mathcal{L}_0 u \rangle$ using integration by parts to obtain

$$\langle v, \left( -\frac{d^2}{dx^2} + ix \right) u \rangle = -u'(0) \bar{v}(0) + u(0) \bar{v}'(0) + \left\langle \left( -\frac{d^2}{dx^2} - ix \right) v, u \right\rangle.$$ 

To conform with (6.58) $u \mapsto -u'(0) \bar{v}(0) + u(0) \bar{v}'(0)$ must be a continuous map on $(I - P_0)L^2$. This, however, is possible only if $v(0) = v'(0) = 0$ (consider the sequence $u_n = \chi_n - P_0 \chi_n$ with $\chi_n(x) = \sqrt{n} \chi(nx)$), leading thereby to (6.59a). Consequently for any $v \in D(L^*_0)$, we have

$$\langle v, \left( -\frac{d^2}{dx^2} + ix \right) u \rangle = \left\langle \left( -\frac{d^2}{dx^2} - ix \right) v, u \right\rangle.$$ 

Having in mind that

$$\langle v, \mathcal{L}_0 u \rangle = \langle L^*_0 v, u \rangle,$$ 

leads to

$$L^*_0 v = (I - P_0) \left( -\frac{d^2}{dx^2} - ix \right) v.$$ 

We can then extend (6.60) by density to any $u \in \mathcal{H}_0$. □

**Proposition 6.9.** The eigenfunctions of $\mathcal{L}_0$ are complete in $(I - P_0)L^2(\mathbb{R}_+)$. 

**Proof.** We take a similar approach to the one in [5]. 

**Step 1:** By the semi-boundedness of $\mathcal{L}_0$ and (6.59a) there exists $c_0 > 0$ and $\hat{\mu}_0 \in \mathbb{R}_-$ such that for all $u \in D(\mathcal{L}_0)$

$$c_0 \| -u'' + xu \|_2^2 \leq 2c_0 \left( \| xu \|_2^2 + \| u \|_{L^2(\mathbb{R}_+)}^2 \right) \leq \| -u'' + (ix - \hat{\mu}_0)u \|_2^2. \quad (6.61)$$

**Step 2:** We now show that the resolvent of $\mathcal{L}_0$ is in $S_p$ for every $p > 3/2$, where $S_p$ denotes the Schatten of order $p$.

By the Max-Min principle the singular values $(\mu_n)_{n \in \mathbb{N}^*}$ of the operator $(\mathcal{L}_0 - \hat{\mu}_0)^{-1}$ satisfy for $k \in \mathbb{N}$

$$\mu_{k+1}^{-2} = \max_{U_k \in \mathcal{H}_0^k} \min_{u \in U_k \cap D(\mathcal{L}_0)} \frac{\| -u'' + (ix - \hat{\mu}_0)u \|_2^2}{\| u \|_2^2}.$$ 

Let further

$$\kappa_{k+1}^{-2} = \max_{U_k \in (L^2(\mathbb{R}_+))^k} \min_{u \in U_k \cap D(\mathcal{L}_0)} \frac{\| -u'' + xu \|_2^2}{\| u \|_2^2}.$$ 

By (6.61) we have, for $n \in \mathbb{N}^*$,

$$\kappa_n^{-2} \leq c_0 \mu_n^{-2}. $$
Finally, let
\[ \tilde{\kappa}_{k+1}^{-2} = \max_{U_k \in L^2(\mathbb{R}_+)^k, u \in U_k^+ \cap H^2(\mathbb{R}_+) \cap \mathcal{L}^2(\mathbb{R}_+; x^2 \cdot dx)} \min \left\{ \frac{\| -u'' + xu\|^2}{\|u\|^2} : \right\} \]

In view of the additional constraint embedded in \( D(\mathcal{L}^\theta) \) we have
\[ \kappa_n \leq \tilde{\kappa}_n. \]

By the Max-Min principle the \( \tilde{\kappa}_n^{-2} \) are eigenvalues of
\[ \mathcal{A}_N := (-d^2/dx^2 + x)^2 \]
defined on
\[ D(\mathcal{A}_N) = \{ u \in H^4(\mathbb{R}_+) \cap L^2(\mathbb{R}_+; x^4 \cdot dx) : u''(0) = -(u'' + xu)'(0) = 0 \}. \]

Let \( \lambda = \alpha^2 \), where \( \alpha > 0 \), denote an eigenvalue of \( \mathcal{A}_N \). (Note that \( \lambda = 0 \) is an eigenvalue.) Let \( u_\alpha \) denote the corresponding eigenfunction. As
\[ \left( -\frac{d^2}{dx^2} + x \right)^2 - \alpha^2 = \left( -\frac{d^2}{dx^2} + x + \alpha \right) \left( -\frac{d^2}{dx^2} + x - \alpha \right), \]
we easily obtain that, up to a product by an arbitrary constant,
\[ u_\alpha = -\frac{1}{2\alpha} \text{Ai} \left( x + \alpha \right) + A_1 \text{Ai} \left( x - \alpha \right), \]
where \( A_1 \) has to be determined from the requirement \( u_\alpha \in D(\mathcal{A}_N) \). It can now be easily verified that \( \alpha \in \sigma(\mathcal{A}_N) \) if and only if
\[ \delta(\alpha) := -\frac{\text{Ai}'(\alpha)}{\text{Ai}(\alpha)} + \frac{\text{Ai}'(-\alpha)}{\text{Ai}(-\alpha)} = 0. \]

Let \( \{\omega_n\}_{n=1}^{\infty} \subset \mathbb{R}_- \) denote the zeroes of Airy’s function \( \text{Ai} \left( x \right) \). By computation of its derivative \( \delta(\alpha) \) is a monotone increasing for \( \alpha \in (-\omega_n, -\omega_{n+1}) \) and tends to \( \pm \infty \) at the edges. Consequently, there is precisely one solution of (6.62) in \(( -\omega_n, -\omega_{n+1} )\).

As \( -\omega_n \sim n^{2/3} \) we may conclude from the foregoing discussion that \( \tilde{\kappa}_n^{-1} \sim n^{2/3} \) as well. Consequently, there exists \( C > 0 \) such that, for sufficiently large \( n \),
\[ \mu_n \leq \frac{C}{n^{2/3}}. \]

As a result, for all \( p > 3/2 \) it holds that
\[ \sum_{n=1}^{\infty} \mu_n^p < \infty. \]

**Step 3:** We complete the proof of the proposition.
We take a similar approach to the one in [5]. By (6.42) and (6.44) we have, for sufficiently large $-\Re \lambda$,
\[
\left\| (L^\theta - \lambda)^{-1} \right\| \leq \frac{C(\theta)}{|\lambda|}.
\]
Let $v = (L^\theta - \lambda)^{-1} g$ for some $g \in L^2(\mathbb{R}_+)$. From the imaginary part of (6.39) we learn that
\[
-\Im \|v\|^2_2 + \|x^{1/2}v\|^2_2 = \Im \langle v, g \rangle - \Im \{v'(0)\overline{v}(0)\}.
\]
With the aid of (6.37) and (6.43) we then obtain that
\[
-\Im \|v\|^2_2 \leq C(\theta)[(1 + |\mu|^{1/2})\|v\|^2_2 + \|g\|^2_2].
\]

Hence, there exists $\hat{C}$ such that if $-\Im \lambda \geq \hat{C} (1 + |\mu|^{1/2})$, then
\[
\left\| (L^\theta - \lambda)^{-1} \right\|_2 \leq \frac{C(\theta)}{|\lambda|}.
\]

From the foregoing discussion we may conclude that every direction where $\pi/2 < \arg \lambda < 2\pi$ is a direction of minimal growth for $(L^\theta - \lambda)^{-1}$. Following the arguments of the proof of [2, Theorem 16.4] (cf. also [20, Theorem X.3.1 ] or [17, Corollary XI.9.31]) we can conclude that the eigenspace of $L^\theta$ is given by
\[
\mathcal{D}(L^\theta) = (I - P_\theta)L^2(\mathbb{R}_+).
\]

\section*{Proposition 6.10.}
Let $\mu_0(\theta) = \inf_{\lambda \in \sigma(L^\theta)} \Re \lambda$. Then,
\[
\lim_{\theta \to +\infty} \mu_0(\theta) = \Re \nu_1,
\]
where $\nu_1$ the left most eigenvalue of $L_+ \equiv \text{Dirichlet realization of } L \text{ in } \mathbb{R}_+$.

\textbf{Proof.} We begin the proof by applying Rouché’s Theorem, in the large $\theta$ limit, to the holomorphic functions $\theta F(\lambda, \theta)$ and $\text{Ai}(e^{2i\pi/3} \lambda)$ inside a disk of radius $r > 0$ centered at $\nu_1$ and containing no other eigenvalue of $L^\theta$. As $\lambda \mapsto \text{Ai}(e^{2i\pi/3} \lambda)$ has a unique zero in this disk, Rouché’s Theorem would show the same for the zeros of $F(\cdot, \theta)$. It is therefore necessary to compare the two functions for $\lambda \in \partial B(\nu_1, r)$. We thus write
\[
\theta F(\lambda, \theta) = \text{Ai}(e^{i2\pi/3} \lambda) - \theta \int_{R_+} e^{-\theta x} \left[ \text{Ai}(e^{i2\pi/3} \lambda + e^{i\pi/6} x) - \text{Ai}(e^{i2\pi/3} \lambda) \right] dx.
\]

We bound the right-hand-side in the following manner
\[
\theta \int_{R_+} e^{-\theta x} \left| \text{Ai}(e^{i2\pi/3} \lambda + e^{i\pi/6} x) - \text{Ai}(e^{i2\pi/3} \lambda) \right| dx
\]
\[
\leq \theta \int_{R_+} e^{-\theta x} x^{1/2} \| \text{Ai}'(e^{i\pi/6} \cdot + i\lambda) \|_{L^2(0, x)} dx
\]
\[
\leq \frac{C}{\theta^{1/2}} \| \text{Ai}'(e^{i\pi/6} \cdot + i\lambda) \|_2.
\]
From (6.65), we obtain the existence of \( r_0 > 0 \) and \( C > 0 \) such that, for any \( r \in (0, r_0] \) and any \( \lambda \in \partial B(v_1, r) \), we have

\[
|\theta F(\lambda, \theta) - \text{Ai}(e^{2i\pi/3\lambda})| \leq \frac{C}{r\theta^{1/2}}|\text{Ai}(e^{2i\pi/3\lambda})|. 
\]

It follows from Rouché’s Theorem that for sufficiently large \( \theta \), \( F(\lambda, \theta) \) has a unique zero in \( B(v_1, r) \).

At this stage we have obtained

\[
\lim \sup_{\theta \to +\infty} \mu_0(\theta) \leq \Re v_1.
\]

Using the arguments as above and supposing now that \( r < |\lambda - v_1| < R \) and \( \Re \lambda \leq \Re v_1 \), we can establish that

\[
\left| \frac{\theta F(\lambda, \theta)}{\text{Ai}(e^{2i\pi/3\lambda})} - 1 \right| \leq \frac{C(R, r)}{\theta^{1/2}}. \tag{6.66}
\]

Consequently, we obtain that, there exists \( \theta_1(R, r) \) such that, for all \( \theta > \theta_1(R, r) \) \( F(\lambda, \theta) \) does not vanish in \( (B(v_1, R) \setminus B(v_1, r)) \cap \{\Re \lambda \leq \Re v_1\} \).

To complete the proof we need yet to establish that there exists \( R_0 > 0 \), and \( \theta_2(R_0) > 0 \) such that for all \( \theta > \theta_2(R_0) \) we have that

\[
\inf_{\Re \lambda \leq \Re v_1, |\lambda - v_1| > R_0} \left| \frac{\theta + (-\lambda)^{1/2}}{\text{Ai}(e^{2i\pi/3\lambda})} F(\lambda, \theta) - 1 \right| > 0. \tag{6.67}
\]

To this end we set as in (A.6)

\[
\frac{F(\lambda, \theta)}{\text{Ai}(e^{i2\pi/3\lambda})} = \int_{\mathbb{R}^+} e^{-\theta x} e^{-(\lambda)_{1/2} x} \, dx + \int_{\mathbb{R}^+} e^{-\theta x} w(x) \, dx.
\]

To bound the second term we use (A.14) and (A.17) (with \( \mu_0 = \Re v_1 \)) together with Sobolev embeddings to obtain

\[
\|w\|_{L^\infty}^2 \leq \|w\|_2 \|w'\|_2 \leq C|\lambda|^{-3/2}.
\]

Hence,

\[
\left| \frac{\theta + (-\lambda)^{1/2}}{\text{Ai}(e^{i2\pi/3\lambda})} F(\lambda, \theta) - 1 \right| \leq C|\lambda|^{-1/4}, \tag{6.68}
\]

from which (6.67) easily follows. \( \square \)
6.3. No-Slip Operator on $(-1, 1)$ for Large $\alpha$

Consider $\mathcal{L}_\beta^\xi$, defined in (6.2), with $\xi = \bar{z}_\pm$, where $\bar{z}_\pm \in C^2(-1, 1)$ is the solution of
\[
\begin{cases}
-\bar{z}'_\pm'' + \alpha^2 \bar{z}_\pm = 0 & \text{for } x \in (-1, 1) \\
\bar{z}_\pm(\pm 1) = 1 \text{ and } \bar{z}_\pm(\mp 1) = 0.
\end{cases}
\]
An immediate computation gives
\[
\bar{z}_+(x) = \frac{\sinh \alpha(1 + x)}{\sinh 2\alpha},
\]
and a similar formula for $\bar{z}_-$. We attempt to obtain a resolvent estimate for $\mathcal{L}_\beta^\xi$ in the case $\alpha \geq \theta_1 \beta^{1/3}$ where $\theta_1 > 0$. If we try to use the arguments of Subsection 6.1 we would encounter a problem while attempting to use (6.29). It can be verified from (6.70) that
\[
\|\bar{z}_\pm - e^{-\alpha(1 \mp x)}\|_\infty \leq Ce^{-2\alpha} \leq Ce^{-2\theta_1 \beta^{1/3}}.
\]
Then, one can deduce in the same manner that for some $C > 0$, $C_1 \in (0, 1)$, and sufficiently large $\beta$,
\[
\|\bar{z}_\pm'\|_2 \geq \alpha^{1/2}(1 - Ce^{-2\alpha}) \geq C_1 \theta_1^{1/2} \beta^{1/6}.
\]
Thus, the error introduced by (6.29) is not necessarily small and one needs an alternate route for the estimation of $\| (\mathcal{L}_\beta^\xi - \lambda)^{-1} \|$.

Since for $\alpha \geq \theta_1 \beta^{1/3}$ we need to consider, in the next section, only the case $\xi_\pm \approx \bar{z}_\pm$, we focus attention here on the resolvent of $\mathcal{L}_\beta^\xi$ in that case. Thus, we no longer approximate $\xi_\pm$ near $x = \pm 1$ by 1, as in Subsection 6.1, and use instead the approximation $\xi_\pm \approx e^{-\alpha(1 \mp x)}$ as observed in (6.71). Note that $\mathcal{L}_\beta^\xi$ depends on $\alpha$ through the orthogonality conditions appearing in the definition of its domain. Consequently, we need to renormalize $\psi_\pm$ from (6.8b) and (6.8a) in a manner that would suit the approximation used for $\xi_\pm$.

For some $\theta > 0$, the renormalization factor will be defined by
\[
\omega_\pm(\beta, \lambda, \theta) := \frac{F(\tilde{\lambda}_\pm, 0)}{F(\tilde{\lambda}_\pm, \theta J_\pm^{-1/3})},
\]
where (see (6.15) for the definition of $\lambda_\pm$)
\[
\tilde{\lambda}_\pm = \beta^{1/3} J_\pm^{-2/3} [\lambda - iU(\pm 1)] = \beta^{1/3} J_\pm^{-2/3} \lambda_\pm.
\]
We now define
\[
\psi_{\pm, \theta} = \omega_\pm(\beta, \lambda, \theta) \psi_\pm,
\]
where $\psi_\pm$ was introduced in (6.8b)-(6.8a).

The above normalization provides the approximation $(\xi_\pm, \psi_{\pm, \theta}) \sim (J_\pm \beta)^{-1/3}$, in the limit $\beta \to \infty$, as in Subsection 6.1 (see below (6.85)). We similarly introduce with the notation of (6.11) and (6.12)
\[
g_{\pm, \theta} = \omega_\pm(\beta, \lambda, \theta) g_\pm \text{ and } \tilde{v}_{\pm, \theta} = \omega_\pm(\beta, \lambda, \theta) \tilde{v}_\pm.
\]
We can now state
Proposition 6.11. Let $r > 1$, $θ_1 > 0$ and $x > 0$. Then, there exist $β_0 > 0$ and $C(x) > 0$ such that, for all $U ∈ S_r^2$, $β ≥ β_0$ and $θ = αβ^{-1/3} ≥ θ_1$,

$$
\sup_{g λ ≤ (μ_0(θ)−x)β^{-1/3}} \|(L_β^x − βλ)^{-1}\| ≤ C(x)β^{-2/3},
$$

(6.76)

where

$$
μ_0(θ) = \min(J_2^{2/3} − μ_0(J_0^{−1/3} θ), J_1^{2/3} + μ_0(J_0^{−1/3} θ)).
$$

(6.77)

Remark 6.12. In the sequel we apply Proposition 6.11 with $θ_1 = θ_0$ where $θ_0$ is defined in the statement of Lemma 6.2.

Proof. The proof goes along similar lines to the proof of Lemma 6.2. Let $λ ∈ C$ satisfy

$$
β^{1/3} θ ≤ γ(θ, x) := μ_0(θ) − x.
$$

Furthermore, let the pair $(g, v)$ in $L^2(−1, 1) × D(L_β^x)$ satisfy the relation

$$(L_β^x − βλ)v = g.
$$

(6.78)

We assume, as in (6.20), that

$$
v = A_+(g)(ψ_{+,θ} − ˜v_{+,θ}) + A_−(g)(ψ_{−,θ} − ˜v_{−,θ}) + u,
$$

(6.79)

where $u$ is given by (6.5), and then estimate $A_±(g)$ in the relevant regime of $α$ values.

We first estimate the renormalization factor. We note that

$$
ω_±(β, λ, θ) = e^{-iπ/6} \frac{A_0(i ˜λ_±)}{F(˜λ_±, θ J_±^{−1/3})},
$$

and from (A.44), which reads

$$
\sup_{γ(θ)} \left| \frac{A_0(i ˜λ)}{F(˜λ, θ)} \right| ≤ C(γ)(1 + ˜θ),
$$

we obtain that there exist $C(γ) > 0$ and $β_0(γ)$, such that for all $θ ≥ θ_1$, $γ(θ) ≤ γ(θ, x)β^{-1/3}$, and $β ≥ β_0(γ)$ we have

$$
|ω_±(β, λ, θ)| ≤ C γ.
$$

(6.80)

We can now use (6.17), (6.18) and (6.19) to obtain that

$$
\|ψ_{±,θ}\|_2 ≤ C γ [1 + \|λ_{±}\|^{1/2} β^{1/6}]^{1/2} β^{-1/6},
$$

(6.81)

and

$$
\|g_{±,θ}\|_2 ≤ C γ \frac{β^{1/6}}{[1 + \|λ_{±}\|^{1/2} β^{1/6}]^{3/2}},
$$

(6.82)

and

$$
\|(U − ν + i β^{-1/3}) ˜v_{±,θ}\|_2 + β^{-1/3} \|v_{±,θ}\|_2 ≤ C γ \frac{β^{-5/6}}{[1 + \|λ_{±}\|^{1/2} β^{1/6}]^{3/2}}.
$$
We note that (6.25) remain valid in the case \( \theta > 0 \), i.e.,
\[
\begin{bmatrix}
\langle 3_+, (\psi_+, \theta - \tilde{\psi}_+, \theta) \rangle \\
\langle 3_-, (\psi_-, \theta - \tilde{\psi}_-, \theta) \rangle 
\end{bmatrix}
\begin{bmatrix}
A_+ \\
A_- 
\end{bmatrix}
= \begin{bmatrix}
\langle 3_+, u \rangle \\
\langle 3_-, u \rangle 
\end{bmatrix},
\]
(6.83)

We now write
\[
\langle 3_\pm, \psi_{\pm, \theta} \rangle = (e^{-\alpha(1+\varepsilon)}, \psi_{\pm, \theta}) + \langle 3_\pm - e^{-\alpha(1+\varepsilon)}, \psi_{\pm, \theta} \rangle.
\]

Since by (6.79) and (A.43b) we have that
\[
\| \psi_{\pm, \theta} \|_1 \leq C(z) \| \psi_{\pm, \theta} \|_1 \leq C \theta \beta^{-1/3},
\]
(6.84)
we can easily deduce using (6.71) that for sufficiently large \( \beta \)
\[
|3_\pm - e^{-\alpha(1+\varepsilon)}, \psi_{\pm, \theta}| \leq \hat{C} e^{-2\theta \beta^{1/3}} \| \psi_{\pm, \theta} \|_1 \leq \hat{C} \beta^{-\frac{3}{2}} e^{-\theta_0 \beta^{1/3}}.
\]
Furthermore, as
\[
\int_{-1}^{+\infty} e^{-\alpha(1+\varepsilon)} \text{Ai}(J(-\beta)^{1/3} e^{i\pi/6} [(1 + x) + i J^{-1}(\lambda - i U(-1))] \rangle dx
\]
\[
= (J(-\beta)^{-1/3} \int_{\mathbb{R}_+} e^{-\theta J^{1/3} x} \text{Ai}(e^{i\pi/6} [\xi + i J^{-2/3} \beta^{1/3} (\lambda - i U(-1))] \rangle d\xi
\]
\[
= (J(-\beta)^{-1/3} F(\tilde{\lambda}_\pm, \theta J^{-1/3}),
\]
we have that
\[
\langle e^{-\alpha(1+\varepsilon)}, \psi_{-, \theta} \rangle = (J(-\beta)^{-1/3} - \langle 1_{1, \infty} e^{-\alpha(1+\varepsilon)}, \psi_{-, \theta} \rangle.
\]

Since by (6.84) we have
\[
|\langle 1_{1, \infty} e^{-\alpha(1+\varepsilon)}, \psi_{-, \theta} \rangle| \leq C e^{-2\theta \beta^{1/3}} \| \psi_{-, \theta} \|_1 \leq \hat{C} \beta^{-\frac{1}{2}} e^{-\theta_0 \beta^{3/3}}.
\]
we obtain
\[
\langle e^{-\alpha(1+\varepsilon)}, \psi_{-, \theta} \rangle = (J(-\beta)^{-1/3} [1 + O(e^{-\theta_0 \beta^{1/3}})],
\]
(6.85)
and a similar estimate can be obtained for \langle e^{-\alpha(1-\varepsilon)}, \psi_{+, \theta} \rangle.

We now write
\[
|\langle 3_+, \psi_{-, \theta} \rangle| \leq \| \psi_{-, \theta} \|_{L^1(-1, 0)} e^{-\theta \beta^{1/3}} + \| \psi_{-, \theta} \|_{L^1(0, 1)} \| 3_i \|_{L^1(0, 1)} ,
\]
and then use (6.84), (6.79), and (A.43c) to obtain that
\[
|\langle 3_+, \psi_{-, \theta} \rangle| \leq C \beta^{-5/3}.
\]

**Bounds for \langle 3_\pm, u \rangle.**

Let,
\[
\hat{\lambda} = \begin{cases}
\lambda & \text{if } |\Re \lambda| > \max \left( 1, \frac{\| U^\prime \|_{\infty}}{\theta} \right) \beta^{-1/3} \\
\max \left( 1, \frac{\| U^\prime \|_{\infty}}{\theta} \right) \beta^{-1/3} + i \Im \lambda & \text{if } |\Re \lambda| \leq \max \left( 1, \frac{\| U^\prime \|_{\infty}}{\theta} \right) \beta^{-1/3}.
\end{cases}
\]
(6.87)
As $\beta^{\frac{1}{3}} |\Re \hat{\lambda}| \geq 1$ and $|\Im \hat{\lambda}| \geq |\Re \lambda|$, it can be verified that
\[ 1 + |\Re \lambda|^2 \beta^{2/3} \leq 2 |\Re \hat{\lambda}|^2 \beta^{2/3}, \]
and hence
\[ 1 + \beta^{2/3} |\lambda|_\pm^2 \leq 2 \beta^{2/3} |U(\pm 1) + i \hat{\lambda}|^2. \] (6.88)
We now verify that
\[ \|\beta_+^{-1}(U + i \hat{\lambda})^{-1/2}\|_1 = |U(1) + i \hat{\lambda}|^{-1/2}. \] (6.89)
Indeed, we have
\[ [\beta_+^{-1}(U + i \hat{\lambda})^{-1/2}]^\prime = \beta_+^{-1}(U + i \hat{\lambda})^{-4} \left( -2 \beta^{1/3} (U - \Re \lambda)^2 + |\Re \hat{\lambda}|^2 + 2U'(U - \Re \lambda) \right) \]
\[ = \beta_+^{-2}(U + i \hat{\lambda})^{-4} \left( -2 \beta^{1/3} \left( (U - \Re \lambda) + \frac{U'}{2 \beta^{1/3}} \right)^2 + \frac{|U'|^2}{2 \beta^{1/3} - 2 \beta^{1/3} |\Re \hat{\lambda}|^2} \right), \]
which is non positive by (6.87).
Hence the maximum of $\beta_+^{-1/2}|U + i \hat{\lambda}|^{-2}$ is obtained at $x = -1$. A similar inequality can be established for $\beta_+^{-1/2}|U + i \hat{\lambda}|^{-2}$. Combining (6.88) and (6.89) yields
\[ \|\beta_\pm(U + i \hat{\lambda})^{-1/2}\|_1 \leq \|\beta_\pm(U + i \hat{\lambda})^{-1/2}\|_\infty \]
\[ \leq C \frac{\beta^{-1/6}}{\theta} |\beta^{1/3}(U(\pm 1) + i \hat{\lambda})|^{-1/2} \]
\[ \leq \hat{C} \frac{\beta^{-1/6}}{\theta(1 + \beta^{1/3} |\lambda|_\pm)|^{1/2}}. \] (6.90)
Note that the replacement of $\lambda$ by $\hat{\lambda}$ avoids the burden of a vanishing denominator.

We now write
\[ |\langle \beta_\pm, u \rangle| \leq \|\beta_\pm(U + i \hat{\lambda})^{-1/2}\|_1 \|U + i \hat{\lambda})^{-1/2}u\|_\infty. \] (6.91)
To estimate $\|U + i \hat{\lambda})^{-1/2}u\|_\infty$ in the right hand side of (6.91), we first obtain a bound for $\|U - v|^{1/2}u\|_\infty$. Thus, integration by parts for all $(x, x_0) \in [-1, 1]^2$
\[ (U - v)|u|^2|_{x_0}^x = 2\Re \int_{x_0}^x (U - v)\bar{u}u' \, dt - \int_{x_0}^x U'|u|^2 \, dt. \]
From this we conclude, by integrating the above over $(-1, +1)$ and Cauchy-Schwarz inequality, that
\[ \|U - v|^{1/2}u\|_\infty \leq C(\|U - v_2\|_2 ||u'||_2 + \|u\|_2^2 + \|U - v|^{1/2}u\|_2^2). \]
By (5.4) and (5.5) we then have
\[ \|\|U - v|^{1/2}u\|_\infty^2 \leq \frac{C}{\beta^{4/3}} \|g\|_2^2. \]
We now write
\[
\| (U + i \hat{\lambda})^{1/2} u \|_\infty^2 \leq 2 \left( \| U - \nu \|^{1/2} u \|_\infty^2 + (|\Re \nu \lambda| + 2 \beta^{-1/3}) \| u \|_\infty^2 \right) \\
\leq C (\beta^{-4/3} \| g \|_2^2 + (|\Re \nu \lambda| + 2 \beta^{-1/3}) \| u \|_2 \| u' \|_2).
\]
By (5.4) we then have
\[
(|\Re \nu \lambda| + 2 \beta^{-1/3}) \| u \|_2 \leq \frac{C}{\beta} \| g \|_2,
\]
and hence, using (5.4) once again to estimate \( \| u' \|_2 \), we may conclude that
\[
\| (U + i \hat{\lambda})^{1/2} u \|_\infty \leq \frac{C}{\beta^{2/3}} \| g \|_2.
\] (6.92)
Combining the above with (6.90) and (6.91) yields
\[
| \langle \tilde{3}, u \rangle | \leq \frac{C}{\theta [1 + |\lambda| |\beta^{1/3}|]^{1/2}} \beta^{-5/6} \| g \|_2.
\] (6.93)

**Bounds for** \( \langle \tilde{3}_{\pm}, \tilde{v}_{\pm, \theta} \rangle \).

The estimation of \( \langle \tilde{3}_{\pm}, \tilde{v}_{\pm, \theta} \rangle \) follows a similar path to that of \( \langle \tilde{3}, u \rangle \). We begin by writing
\[
| \langle \tilde{3}_{\pm}, \tilde{v}_{\pm, \theta} \rangle | \leq \| \tilde{3}_{\pm} (U + i \hat{\lambda})^{-1/2} \|_1 \| (U + i \hat{\lambda})^{1/2} \tilde{v}_{\pm, \theta} \|_\infty.
\]
Since \( \tilde{v}_{\pm, \theta} \), given by (6.75) satisfies the same problem as \( u \) with \( g \) replaced by \( g_{\pm, \theta} \) we may conclude as in (6.92) that
\[
\| (U + i \hat{\lambda})^{1/2} \tilde{v}_{\pm, \theta} \|_\infty \leq \frac{C}{\beta^{2/3}} \| g_{\pm, \theta} \|_2.
\]
Consequently, by (6.81) and (6.90)
\[
| \langle \tilde{3}_{\pm}, \tilde{v}_{\pm, \theta} \rangle | \leq \frac{C \beta^{-2/3}}{[1 + |\lambda| |\beta^{1/3}|]^{5/4}}.
\] (6.94)
In a similar manner we can obtain that
\[
| \langle \tilde{3}_{\mp}, \tilde{v}_{\mp, \theta} \rangle | \leq \frac{C \beta^{-2/3}}{[1 + |\lambda| |\beta^{1/3}|]^{3/4} [1 + |\lambda_{\mp}| |\beta^{1/3}|]^{1/2}} \leq C \beta^{-2/3}.
\] (6.95)
As in (6.31) we can now write, in view of (6.94) and
\[
| \langle \tilde{3}_{\pm}, (\psi_{\pm, \theta} - \tilde{v}_{\pm, \theta}) \rangle - (J_{\pm})^{-1/3} \| \leq C \beta^{-2/3}.
\]
Combining the above with (6.86), (6.94), and (6.95) yields
\[
| A_{\pm} (g) | \leq C \beta^{1/3} | \langle \tilde{3}_{\pm}, u \rangle | + C \beta^{-2/3} | \langle \tilde{3}_{\mp}, u \rangle |.
\]
The above, together with (6.93) yields
\[
| A_{\pm} (g) | \leq \frac{C \beta^{-1/2}}{\theta} \left( \frac{1}{[1 + |\lambda| |\beta^{1/3}|]^{1/2}} + \beta^{-1} \frac{1}{[1 + |\lambda_{\mp}| |\beta^{1/3}|]^{1/2}} \right) \| g \|_2.
\]
As \( |\lambda_{\pm} - \lambda_{\mp}| = \| U(1) - U(-1) \| \), we obtain for sufficiently large \( \beta \)
\[
| A_{\pm} (g) | \leq \frac{C \beta^{-1/2}}{\theta} \frac{1}{[1 + |\lambda| |\beta^{1/3}|]^{1/2}} \| g \|_2.
\] (6.96)
Combining the above with (6.78), (6.80), and (6.82) yields (6.76). \( \Box \)
7. Zero Traction Orr–Sommerfeld Operator

7.1. A Short Reminder

We recall for the commodity of the reader that $B_{\lambda,\alpha,\beta}^\Phi$ is defined by (3.4) and (3.10) i.e.

$$B_{\lambda,\alpha,\beta} := (\mathcal{L}_\beta - \beta \lambda) \left( \frac{d^2}{dx^2} - \alpha^2 \right) - i \beta U'' ,$$

with domain

$$D(B_{\lambda,\alpha,\beta}^\Phi) = \{ u \in H^4(-1, 1) \cap H_0^1(-1, 1) | u'' \in H_0^1(-1, 1) \} .$$

Here

$$\mathcal{L}_\beta = - \frac{d^2}{dx^2} + i \beta U ,$$

and $\mathcal{L}_\beta^D$ is the Dirichlet realization of $\mathcal{L}_\beta$ in $(-1, +1)$.

Finally we recall that the inviscid operator $A_{\lambda,\alpha}$ associated with $U$ is defined by

$$A_{\lambda,\alpha} = (U + i \lambda) \left( - \frac{d^2}{dx^2} + \alpha^2 \right) + U''$$

with domain $D(A_{\lambda,\alpha}) = H^2(-1, +1) \cap H_0^1(-1, +1)$.

7.2. The Case $U'' \neq 0$

We now prove

**Proposition 7.1.** For all $r > 1$ and $\hat{\delta} > 0$ there exist positive $\beta_0$, $\Upsilon$, and $C$ such that, for any $\beta \geq \beta_0$ and $U \in S_r$ satisfying (2.34), it holds that

$$\sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \| (B_{\lambda,\alpha,\beta}^\Phi)^{-1} \| + \left\| \frac{d}{dx} \circ (B_{\lambda,\alpha,\beta}^\Phi)^{-1} \right\| \leq C \beta^{-\left(\frac{1}{2} - \hat{\delta}\right)} . \quad (7.1)$$

**Proof.** Let $\lambda = \mu + i \nu$. Let further $0 < \Upsilon < \Re \nu_1$ and suppose that $\mu \leq \Upsilon \beta^{-1/3}$. Let $\phi \in D(B_{\lambda,\alpha,\beta}^\Phi)$ and $f \in L^2(-1, +1)$ satisfy

$$B_{\lambda,\alpha,\beta}^\Phi \phi = f \quad (7.2)$$

and

$$v := A_{\lambda,\alpha} \phi . \quad (7.3)$$

We note that $v \in D(\mathcal{L}_\beta^D)$ and, defining $g$ by

$$g := (\mathcal{L}_\beta^D - \lambda \beta) v , \quad (7.4)$$
we have
\[ g = -(U + i\lambda)f + 2U'(\phi^{(3)} - \alpha^2\phi') + U''(\phi'' - \alpha^2\phi) - (U''\phi)'. \] (7.5)

By (5.19), there exist \( C > 0 \) and \( \beta_0 > 0 \) such that, for \( \beta \geq \beta_0 \) and \( p \in (2, +\infty] \) we have
\[ \|v\|_p \leq \hat{C}\beta^{-\frac{3p^2}{6p}}\|g\|_2. \] (7.6)

As
\[ \Re\langle (U'')^{-1}(\phi'' - \alpha^2\phi), B_{\lambda}^S\phi \rangle = \|(U'')^{-1/2}(\phi^{(3)} - \alpha^2\phi')\|_2^2 + \Re\|(U'')^{-1/2}(\phi'' - \alpha^2\phi, \phi^{(3)} - \alpha^2\phi') - \beta\mu\phi'' - \alpha^2\phi\|_2^2, \] (7.7)
we easily obtain that
\[ \|\phi^{(3)} - \alpha^2\phi'\|_2 \leq C(\|f\|_2 + |\mu|^{1/2}\beta^{1/2}\|\phi'' - \alpha^2\phi\|_2). \] (7.8)

We now write,
\[ (L_D^\beta - \beta\lambda)(\phi'' - \alpha^2\phi) = i\beta U''\phi + f. \] (7.9)

With the aid of (5.8) and (5.21) we then obtain
\[ \|\phi'' - \alpha^2\phi\|_2 \leq C(\beta^{1/6}\|\phi\|_\infty + \beta^{-2/3}\|f\|_2), \] (7.10)

Substituting the above into (7.8) then yields
\[ \|\phi^{(3)} - \alpha^2\phi'\|_2 \leq C(\|f\|_2 + |\mu|^{1/2}\beta^{2/3}\|\phi\|_\infty). \] (7.11)

To bound \( \|(U''\phi)''\|_2 \) we first use the fact that
\[ \langle \phi'', \phi'' - \alpha^2\phi \rangle = \|\phi''\|_2^2 + \alpha^2\|\phi'\|_2^2. \] (7.12)

Then, by (7.10) we obtain that
\[ \|\phi''\|_2 \leq C(\beta^{1/6}\|\phi\|_\infty + \beta^{-2/3}\|f\|_2), \] (7.13)
and
\[ \|(U''\phi)''\|_2 \leq C(\beta^{1/6}\|\phi\|_\infty + \beta^{-2/3}\|f\|_2 + \|\phi_{1,2}\|). \] (7.14)

We continue the proof by considering \( B_{\lambda,\alpha,\beta}^S \) in a few different regimes of \( \lambda \) values.

**Case 1: Bounded \(|\lambda|\)**

Suppose first that
\[ |v| \leq C_0; \quad 0 < |\mu| \leq \gamma\beta^{-1/3}, \] (7.15a,b)
where \( C_0 \in \mathbb{R}_+ \). The value of \( C_0 \) above will be determined at a later stage.

Using (7.10), (7.11), and (7.14), we obtain from (7.5) that
\[ \|g\|_2 \leq C(\|f\|_2 + (|\mu|^{1/2}\beta^{2/3} + \beta^{1/6})[\|\phi\|_\infty + \|\phi_{1,2}\|]), \]
or by Sobolev’s embedding
\[ \|g\|_2 \leq C(\|f\|_2 + (|\mu|^{1/2}\beta^{2/3} + \beta^{1/6})\|\phi_{1,2}\|). \] (7.16)
In view of (7.3), we may apply Proposition 4.14 to the pair \((\phi, v)\) to conclude, by (4.25), that for any \(q > 1\) there exists \(C > 0\) such that
\[
\|\phi\|_{1,2} \leq C (\|v\|_q + \|v\|_\infty). \tag{7.17}
\]
Similarly, by (4.26), for any \(p > 2\) there exists \(C > 0\) such that
\[
|\mu|^{1/2} \beta^{2/3} \|\phi\|_{1,2} \leq |\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p. \tag{7.18}
\]
Hence,
\[
\|g\|_2 \leq C \left( \|f\|_2 + |\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p + \beta^{1/6} (\|v\|_q + \|v\|_\infty) \right).
\]
We may now use (7.6) to obtain, for \(p > 2\),
\[
|\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p \leq C |\mu|^{1/2-1/p} \beta^{2/3} \|v\|_p \leq \hat{C} \gamma^{1/2-1/p} \beta^{2/3 - \frac{p-2}{6p}} \|g\|_2 \leq \hat{C} \gamma^{1/2-1/p} \|g\|_2
\]
Applying (7.6) (which is valid for \(p = +\infty\) as well) once again yields
\[
\beta^{1/6} \|v\|_\infty \leq C \beta^{-1/2} \|g\|_2 = C \beta^{-1/2} \|g\|_2.
\]
Finally, we apply (5.10) (with \(q = p\)) to the pair \((v, g)\) satisfying (7.4)) to conclude, for \(1 < q < 2\), that
\[
\beta^{1/6} \|v\|_q \leq \frac{C_q}{\beta^{7/6}} \|g\|_2.
\]
Combining the above we then obtain, for sufficiently small \(\gamma\) and \(\beta^{-1}\),
\[
\|g\|_2 \leq C \|f\|_2. \tag{7.19}
\]
From (5.10) and (7.19) we now get, for any \(p \in (1, 2)\),
\[
\|v'\|_p \leq C \beta^{-\frac{2+p}{6p}} \|g\|_2 \leq \hat{C} \beta^{-\frac{2+p}{6p}} \|f\|_2.
\]
From this we deduce, for \(p\) sufficiently close to 1 and sufficiently small \(\delta > 0\), the existence of \(C_\delta > 0\) such that
\[
\|v'\|_p + \|v\|_\infty \leq C_\delta \beta^{-\left(\frac{1}{2}-\delta\right)} \|f\|_2.
\]
We now return to (7.17) to conclude that
\[
\|\phi\|_{1,2} \leq \hat{C}_\delta \beta^{-\left(\frac{1}{2}-\delta\right)} \|f\|_2. \tag{7.20}
\]
Hence (7.1) is proven in Case 1 for sufficiently small \(\gamma > 0\).
Case 2: $\Re \lambda$ unbounded negative
Next, consider the case where (7.15a) is kept in place but instead of (7.15b),
$\mu < -\Upsilon \beta^{-1/3}$ is assumed. In this case we return to (7.7) and use the positivity of $-\beta \mu$ for the last term on its right hand side. In a similar manner to the one used to derive (7.8) we establish the existence of $\beta(\Upsilon) > 0$ such that for all $\beta \geq \beta(Y)$,

$$\|\phi'' - \alpha^2 \phi\|_2 \leq \frac{C}{\Upsilon \beta^{2/3}} \|f\|_2.$$  

Since $\phi \in H^1_0(-1, 1)$ it can be easily verified that

$$\|\phi\|_{1,2} \leq \frac{4}{\pi} \|\phi'' - \alpha^2 \phi\|_2,$$  

and hence for any $\Upsilon > 0$, there exist $C$ and $\beta_0$ such that, for all $\alpha \geq 0$ and all $\beta \geq \beta_0$

$$\|\phi\|_{1,2} \leq C \beta^{-2/3} \|f\|_2,$$

establishing (7.1) in Case 2.

Remark 7.2. Case 3: $|\Im \lambda|$ unbounded
Consider next the case, where instead of (7.15) we have

$$|v| \geq C_0.$$  

(7.22)

In this case we write

$$-\Im \langle \phi'' - \alpha^2 \phi, B_\lambda^\phi \phi \rangle = \beta [\langle \phi'' - \alpha^2 \phi, (\nu - U)(\phi'' - \alpha^2 \phi) \rangle + \langle \phi'' - \alpha^2 \phi, U'' \phi \rangle].$$

As $\phi \in H^1_0(-1, 1)$ we have that

$$\|\phi\|_2 \leq \frac{4}{\pi^2} \|\phi'' - \alpha^2 \phi\|_2,$$

and hence, under (7.22),

$$|\langle \phi'' - \alpha^2 \phi, (\nu - U)(\phi'' - \alpha^2 \phi) \rangle + \langle \phi'' - \alpha^2 \phi, U'' \phi \rangle| \geq (C_0 - \|U\|_{\infty} - 4\pi^{-2} \|U''\|_{\infty}) \|\phi'' - \alpha^2 \phi\|_2^2.$$  

Selecting $C_0 > \|U\|_{\infty} + 4\pi^{-2} \|U''\|_{\infty}$ we then obtain

$$\|\phi'' - \alpha^2 \phi\|_2 \leq \frac{C}{\beta} \|f\|_2.$$  

We can now conclude from (7.21) that

$$\|\phi\|_{1,2} \leq \frac{C}{\beta} \|f\|_2.$$
As (4.24)-(4.26) are not valid for $\mu = 0$, we need to establish (7.20) for the case $\mu = 0$. Let

$$B_{i\nu, \alpha}^{S} \phi = f.$$ 

Then, we may write by (7.7) that

$$\|\phi'' + \alpha^2 \phi\|_2 \leq C \|f\|_2.$$ 

Then, as

$$B_{i\nu + \beta - 1, \alpha}^{S} \phi = \phi'' + \alpha^2 \phi + f,$$

we may conclude (7.1) from (7.20).

\[\square\]

7.3. The Nearly Couette Case

We now proceed to consider the nearly Couette case addressed by both Theorems 2.15 and 2.16. Let $U \in C^4([-1, 1])$ satisfy (2.29) and recall the definition of $\delta_2(U)$ from (2.31),

$$\delta_2(U) := \|U''\|_{1, \infty}.$$ 

We next recall from (2.33), for some $r > 1$,

$$S_r = \{v \in C^4([-1, 1]) \mid \inf_{x \in [-1, 1]} v' \geq 1/r \text{ and } \|v\|_{4, \infty} \leq r \}.$$ 

We shall consider the case where $\delta_2(U)$ is small. Unlike the Couette case, where $U(x) = x$, $B_{\lambda, \alpha, \beta}^{S}$ is no longer accretive when $\delta_2(U) > 0$ [14, Subsection 6.1]. Note that in contrast with the assumptions of Proposition 7.1, $U''$ may change its sign.

**Proposition 7.3.** For any $r > 1$ and $\Upsilon < \Re \nu_1$, there exist $\delta \in (0, \frac{1}{4})$, and positive $\beta_0$ and $C$, such that, for all $U \in S_r$ satisfying $0 < \delta_2(U) \leq \delta$ and all $\beta \geq \beta_0$,

$$\sup_{\beta^{1/3} \Re \lambda \leq \delta_m^{2/3} \Upsilon} \left\| (B_{\lambda, \alpha, \beta}^{S})^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^{S})^{-1} \right\| \leq \frac{C}{\beta^{5/6}}.$$ 

(7.23)

**Proof.** Note that by (2.33)

$$r \geq m \geq \frac{1}{r}. \quad (7.24)$$

As in the proof of Proposition 7.1 we separately consider different regimes of $\lambda \in \mathbb{C}$.

**Case 1:** $\Re \nu_1 / 2 \leq \delta_m^{-2/3} \beta^{1/3} \mu \leq \Upsilon$ or $\beta^{1/3} \mu \leq -\delta_m^{2/3} \Re \nu_1 / 2$

Let $f \in L^2(-1, 1)$. Using the definitions (7.2)-(7.3) we further set

$$\tilde{v} = \frac{v}{U + i \lambda}.$$
Clearly
\[(\mathcal{L}_\beta^2 - \beta \lambda) \tilde{v} = f + \left( \frac{U'' \phi}{U + i \lambda} \right)'' = h. \tag{7.25}\]

We now write
\[\|v\|_2 \leq \|(U + i \lambda)(\mathcal{L}_\beta^2 - \beta \lambda)^{-1} h\|_2 \]

By (5.8) and (5.9) we then have
\[\|v\|_2 \leq \frac{C}{\beta} \|h\|_2. \tag{7.26}\]

Next, we turn to estimate \(\|h\|_2\). Clearly, by (7.25),
\[\|h\|_2 \leq \|f\|_2 + C(r) \left( \left\| \frac{\phi''}{|U + i \lambda|} \right\|_2 + \left\| \phi \left( 1 + \frac{1}{|U + i \lambda|^2} \right) \right\|_2 + \left\| \frac{U'' \phi}{|U + i \lambda|^3} \right\|_2 + \left\| \phi \left( 1 + \frac{1}{|U + i \lambda|^2} \right) \right\|_2 \right), \tag{7.27}\]

where we have used that fact that
\[\frac{1}{|U + i \lambda|} \leq \frac{1}{2} \left( 1 + \frac{1}{|U + i \lambda|^2} \right). \]

For the third term in the right-hand-side of (7.27) we have
\[\left\| \phi' \left( 1 + \frac{1}{|U + i \lambda|^2} \right) \right\|_2 \leq C \beta^{2/3} \|\phi'\|_2. \tag{7.28}\]

We now turn to estimate the second term on the right-hand-side of (7.27). Here we write
\[\left\| \frac{\phi''}{U + i \lambda} \right\|_2 \leq C \beta^{1/3} \|\phi''\|_2. \]

Then, we use (7.13) (which remains valid in the nearly-Couette case) to obtain
\[\left\| \frac{\phi''}{U + i \lambda} \right\|_2 \leq C(r) (\beta^{1/2} \|\phi\|_\infty + \beta^{-1/3} \|f\|_2). \]

Sobolev embedding then yields
\[\left\| \frac{\phi''}{U + i \lambda} \right\|_2 \leq C(r) (\beta^{1/2} \|\phi\|_{1,2} + \beta^{-1/3} \|f\|_2). \tag{7.29}\]

Finally, we turn to estimate the last term on the right-hand-side of (7.27). Suppose first that for some \(x_v \in [-1, 1]\) we have \(U(x_v) = v\). Clearly, as
\[|U(x) + i \lambda| \geq \left| m(x - x_v) + \frac{1}{2} i \Im m_3 \beta^{-1/3} \right|, \tag{7.30}\]

it holds by (7.24) that
\[\left\| \frac{U'' \phi}{|U + i \lambda|^3} \right\|_2 \leq C \delta \left\| \frac{\phi}{(x - x_v) + i \Im m_3 \beta^{-1/3}} \right\|_2. \]
Hence, we obtain
\[ \left\| \frac{U'' \phi}{|U + i \lambda|^3} \right\|_2 \leq \hat{C} \delta \beta^{5/6} \| \phi \|_\infty . \] (7.31)
Substituting the above together with (7.29) and (7.28) into (7.27) yields with the aid of Sobolev embedding and Poincare inequality
\[ \| h \|_2 \leq \| f \|_2 + C(r)(\delta \beta^{5/6} + \beta^{2/3})\| \phi \|_{1,2} , \]
which for \( \beta \geq \beta_0(\delta) \) gives
\[ \| h \|_2 \leq \| f \|_2 + \hat{C} \delta \beta^{5/6} \| \phi \|_{1,2} . \] (7.32)
By (7.26) we then have
\[ \| v \|_2 \leq C \beta \| f \|_2 + C(r) \delta \beta^{-1/6} \| \phi \|_{1,2} . \] (7.33)
The proof can now be completed, for \( \Re \nu_{1/2} \leq 3_m^{-2/3} \beta^{1/3} \mu \leq \gamma \) or \( \beta^{1/3} \mu \leq -3_m^{2/3} \Re v_1/2 \), by using (4.26) with \( p = 2 \). In the case where \( U(x) - v \neq 0 \) on \([-1, 1]\), we may still apply the previous arguments by replacing (7.30) with
\[ |U(x) + i \lambda| \geq \left| m(x \pm 1) + \frac{1}{2} i 3_m^{2/3} \Re v_1 \beta^{-1/3} \right| . \] (7.34)
where \( \pm \) denotes + for \( \nu < \inf U(x) \) and − for \( \nu > \sup U(x) \).

**Case 2:** \( \beta^{1/3} |\mu| < 3_m^{2/3} \Re v_1/2 \)
Let
\[ s = \beta^{1/3} \left( \frac{3_m^{2/3} \Re v_1}{2} - \mu \right) , \]
and then write
\[ B_{\lambda + s \beta^{-1/3}, \nu} \Phi = f - s \beta^{2/3} (\phi'' - \alpha^2 \phi) . \]
Since \( \Re (\lambda + s \beta^{-1/3}) = \beta^{-1/3} 3_m^{2/3} \Re v_1/2 \), we can use (7.23) to obtain that
\[ \| \phi \|_{1,2} \leq C (\beta^{-5/6} \| f \|_2 + \beta^{-1/6} \| \phi'' - \alpha^2 \phi \|_2 ) . \] (7.35)
We now recall (7.9)
\[ (\mathcal{L}_{\beta}^D - \beta \lambda)(\phi'' - \alpha^2 \phi) = i \beta U'' \phi + f , \]
and then use (5.8) and (5.21) to establish that
\[ \| \phi'' - \alpha^2 \phi \|_2 \leq C (\beta^{-2/3} \| f \|_2 + \delta \beta^{1/6} \| \phi \|_{\infty}) . \] (7.36)
Substituting the above into (7.35) yields
\[ \| \phi \|_{1,2} \leq C (\beta^{-5/6} \| f \|_2 + \delta \| \phi \|_{\infty}) . \]
Using Sobolev embedding yields (7.23) , for sufficiently small \( \delta \) and \( \beta^{1/3} \mu \leq \gamma \).
\[ \Box \]
8. No Slip Orr–Sommerfeld Operator

In contrast with the prescribed traction condition, the auxiliary function \( v = A_{\lambda, \alpha} \phi \), does not satisfy, if \( \phi \in D(B^2_{\lambda, \alpha, \beta}) \), a Dirichlet boundary condition. Consequently, special attention must be given to the behaviour of \( v \) near the boundary through (6.20) for instance. Since \( \phi \) satisfies a Dirichlet boundary condition, we expect that the rapidly decaying boundary terms \( \psi_\pm - \tilde{v}_\pm \) in (6.20) should have a negligible contribution to \( \phi \) compared with that of \( u \) (i.e., \( \|A_{\lambda, \alpha}^{-1}(\psi_\pm - \tilde{v}_\pm)\|_{1,2} \ll \|A_{\lambda, \alpha}^{-1}u\|_{1,2} \)). The next subsection is dedicated to the establishment of such estimates.

8.1. Preliminaries

We recall that \( J_\pm = U'(\pm 1) \) and that \( \partial_x^r \) and \( \psi_\pm \) are respectively defined by (6.10), (6.6) and (6.7). The next lemma holds true under the assumptions of either Proposition 4.10 or Proposition 4.14.

**Lemma 8.1.** For any \( r > 1 \) and \( \Upsilon < \partial_x^r \), there exist positive constants \( C \) and \( \beta_0 \) such that, for all \( \beta \geq \beta_0, \lambda \in \mathbb{C} \) for which \( \Re \lambda \leq \beta^{-1/3} J_{1/3}^2 \Upsilon, \) and \( U \in \mathcal{S}_r \) satisfying either (2.34) or (4.19), it holds that

\[
\|A_{\lambda, \alpha}^{-1}(U + i\lambda)\psi_\pm\|_{1,2} \leq C \left[ 1 + \beta^{1/3} |\lambda_\pm| \right]^{-1/4} \beta^{-1/2},
\]

(8.1)

where

\[
\lambda_\pm = \mu + i(\nu - U(\pm 1)).
\]

(8.2)

**Proof.** Let \( \phi = A_{\lambda, \alpha}^{-1}(U + i\lambda)\psi_\pm \). We write

\[
\left| \left( \frac{\phi}{U + i\lambda}, (U + i\lambda)\psi_\pm \right) \right| \leq \|\phi - \phi(\pm 1)\|_{1,2} \psi_\pm \|_{1,2} .
\]

(8.3)

As

\[
|\phi(x) - \phi(\pm 1)| \leq \|\phi\|_{1,2}(1 + x)^{1/2}, \quad \forall x \in (-1, 1),
\]

we may write

\[
\|\phi - \phi(\pm 1)\|_{1,2} \|\psi_\pm\|_{1,2} \leq \|\phi\|_{1,2} \|\psi_\pm\|_{1,2} \|1 + x\|_{1/2} .
\]

It follows from (6.27) (with \( s = \frac{1}{2} \)) that for some positive \( C \)

\[
\|1 + x\|_{1/2} \|\psi_\pm\|_{1} \leq C \left[ 1 + |\lambda_\pm| \beta^{1/3} \right]^{-1/4} \beta^{-1/2} .
\]

(8.4)

In the case where (4.19) is satisfied, we may use (4.41) so that

\[
\|\phi\|_{1,2} \leq C \|1 + x\|_{1/2} \|\psi_\pm\|_{1} ,
\]

which, combined with (8.4) yields (8.1).

In the case where (2.34) is satisfied, we may use (4.89) for \( v = (U + i\lambda)\psi_\pm \) to obtain that

\[
\|A_{\lambda, \alpha}^{-1}(U + i\lambda)\psi_\pm\|_{1,2} \leq C \|1 + x\|_{1/2} \|\psi_\pm\|_{1} .
\]

Then, we apply (8.4) to obtain (8.1). □
We shall also need the following:

**Lemma 8.2.** Let \( r > 1 \) and \( \Upsilon < \vartheta_r^1 \). Let further \((\lambda, \tilde{\lambda}) \in \mathbb{C}^2\) satisfy \( \lambda - \tilde{\lambda} \in \mathbb{R}\), \(\Re \lambda\) and \(\Re \tilde{\lambda}\) are in \((\infty, \beta^{-1/3} J^{2/3}_\pm \Upsilon]\) and \(|\Re \lambda - \Re \tilde{\lambda}| \leq 2 \vartheta_r^1 \beta^{-1/3}\). Then, there exist positive \( C \) and \( \beta_0 \) such that, for all \( \beta \geq \beta_0 \), and all \( U \in \mathcal{S}_r \) satisfying either (2.34) or (4.19), it holds that

\[
\| A_{r,\alpha}^{-1} (U + i\tilde{\lambda}) \Gamma_{[-1,1]} (\tilde{L}_{\beta,\mathbb{R}} - \beta \tilde{\lambda})^{-1} \tilde{\psi}_\pm (\lambda) \|_{1,2} \leq C \left[ 1 + |\lambda \pm |\beta^{1/3}| \right]^{-1} \beta^{-7/6},
\]

(8.5)

where

\[
\tilde{\psi}_\pm (x, \lambda) = \begin{cases} 
\psi_\pm (x, \lambda) & x \in [-1, 1] \\
0 & |x| > 1.
\end{cases}
\]

**Proof.** For later reference we note that the requirement set above that \( |\lambda - \tilde{\lambda}| \leq 2 \vartheta_r^1 \beta^{-1/3} \) implies the existence of \( C > 0 \) such that

\[
\frac{1}{C} \left[ 1 + |\lambda \pm |\beta^{1/3}| \right] \leq \left[ 1 + |\tilde{\lambda} \pm |\beta^{1/3}| \right] \leq C \left[ 1 + |\lambda \pm |\beta^{1/3}| \right]
\]

and

\[
\frac{1}{C} (1 + |\beta^{1/3}| \mu) \leq (1 + |\beta^{1/3}| \tilde{\mu}) \leq C(1 + |\beta^{1/3}| \mu),
\]

where \( \tilde{\mu} = \Re \tilde{\lambda} \).

**Step 1:** We prove that there exist positive \( C, \beta_0 \) such that for all \( \beta > \beta_0 \) and \( k \in \{0, 1, 2\} \)

\[
\|(1 \mp x)^k (\tilde{L}_{\beta,\mathbb{R}} - \beta \tilde{\lambda})^{-1} \tilde{\psi}_\pm (\lambda)\|_2 \leq C \beta^{-(5/6+k)/2} \left[ 1 + |\beta^{1/3}| \lambda_\pm \right]^{-3/4-k/2}. \tag{8.6}
\]

For convenience of notation we prove (8.6) only for \((\tilde{L}_{\beta,\mathbb{R}} - \beta \tilde{\lambda})^{-1} \tilde{\psi}_+ (\lambda)\). The proof for \((\tilde{L}_{\beta,\mathbb{R}} - \beta \tilde{\lambda})^{-1} \tilde{\psi}_- (\lambda)\) can be obtained in a similar manner.

Let \( u_k = (1 - x)^k (\tilde{L}_{\beta,\mathbb{R}} - \beta \tilde{\lambda})^{-1} \tilde{\psi}_+ (\lambda), \) for \( k \in \{0, 1, 2, 3\} \). For convenience we also set \( u_k \equiv 0 \) for all \( k \leq -1 \).

**The case** \( k = 0 \).

For \( k = 0 \), we observe from (5.4) and (6.17), that we have

\[
\| u_0 \|_2 \leq \frac{C}{\beta^{5/6}(1 + |\beta^{1/3}| \mu)} \left[ 1 + |\beta^{1/3}| \lambda_+ \right]^{1/4}. \tag{8.7}
\]

We note that (8.6) for \( k = 0 \) does not follow from (8.7). Some additional estimates for large values \( \beta^{1/3} |v| \) should be obtained to this end. For \( k \in \{1, 2, 3\} \), we now write:

\[
(\tilde{L}_{\beta,\mathbb{R}} - \beta \tilde{\lambda}) u_k = (1 - x)^k \tilde{\psi}_\pm (\lambda) - 2 k u_{k-1} + k (k - 1) u_{k-2}. \tag{8.8}
\]

By (5.4) it holds that

\[
\| u_k \|_2 \leq \frac{C}{\beta^{2/3}(1 + |\beta^{1/3}| \mu)} \left( \|(1 - x)^k \tilde{\psi}_\pm (\lambda)\|_2 + \| u'_{k-1} \|_2 + (k - 1) \| u_{k-2} \|_2 \right). \tag{8.9}
\]
As
\[ \Re(a_k, (\tilde{\alpha} - \beta \tilde{\psi}) u_k) = \| u_k' \|_2^2 - \beta \tilde{\mu} \| u_k - \tilde{\psi} \|_2^2, \tag{8.10} \]
we obtain, as \( \tilde{\mu} \leq \beta^{-1/3} J_{\pm}^{2/3} \gamma \), that
\[
\| u_k' \|_2 \leq C \left( \beta^{1/3} \| u_k \|_2 + \beta^{-1/3} ((1 - x)^{k - 1} \tilde{\psi}_\pm(\lambda))_2 + (k - 1) \| u_{k-2} \|_2 + (k - 1)(k - 2) \| u_{k-3} \| \right). \tag{8.11}
\]
Substituting the above into (8.9) yields, for \( k = 1 \),
\[
\| u_1 \|_2 \leq C \beta^{-7/3} (1 + \beta^{1/3} |\mu|)^{-1} \left( \| (1 - x) \tilde{\psi}_\pm(\lambda) \|_2 + \beta^{1/3} \| u_0 \|_2 + \beta^{-1/3} \| \tilde{\psi}_\pm(\lambda) \|_2 \right).
\]
Using (6.17) for \( k = 0 \) and \( k = 1 \), which holds as \( \beta^{1/3} \tilde{\psi}_\pm \lambda \leq \gamma < J_{+}^{2/3} \varphi_{1}^r \), yields
\[
\| u_1 \|_2 \leq \frac{C}{\beta^{2/3} (1 + \beta^{1/3} |\mu|)} \left( [1 + \beta^{1/3} |\lambda_+|]^{1/4} \beta^{-1/2} + \beta^{1/3} \| u_0 \|_2 \right). \tag{8.12}
\]
From (8.7) and (8.12) we then get
\[
\| u_1 \|_2 \leq C \beta^{-7/6} (1 + \beta^{1/3} |\mu|)^{-1} [1 + \beta^{1/3} |\lambda_+|]^{1/4}. \tag{8.13}
\]
Using (8.13) we can now complete the proof of (8.6) for \( k = 0 \). To this end we write
\[
[\beta^{-1/3} + |\lambda_+|] \| u_0 \|_2 \leq (\beta^{-1/3} + |\mu|) \| u_0 \|_2 + \| U - U(1) u_0 \|_2 + \| (U - v) u_0 \|_2.
\]
From (5.5) and (6.17), we deduce that
\[
\| (U - v) u_0 \|_2 \leq C \beta^{-1} \| \tilde{\psi}_\pm(\lambda) \|_2 \leq \tilde{C} \beta^{-7/6} [1 + \beta^{1/3} |\lambda_+|]^{-3/4}.
\]
On the other hand, we have, by (8.13)
\[
\| U - U(1) u_0 \|_2 \leq C \| u_1 \|_2 \leq \tilde{C} \beta^{-7/6} (1 + \beta^{1/3} |\mu|)^{-1} [1 + \beta^{1/3} |\lambda_+|]^{1/4}
\leq \tilde{C} \beta^{-7/6} [1 + \beta^{1/3} |\lambda_+|]^{1/4}.
\]
Together with (8.7), we obtain
\[
\| u_0 \|_2 \leq \frac{C}{\beta^{5/6} [1 + \beta^{1/3} |\lambda_+|]^{-3/4}}, \tag{8.14}
\]
which proves (8.6) for \( k = 0 \).

**The case** \( k = 1 \).

By (8.10) we may write, instead of (8.11),
\[
\| u_0' \|_2 \leq C \left( \beta^{1/3} [1 + \beta^{1/3} |\lambda_+|]^{1/2} \| u_0 \|_2 + \beta^{-1/3} [1 + \beta^{1/3} |\lambda_+|]^{-1/2} \| \tilde{\psi}_\pm(\lambda) \|_2 \right). \]
Hence, by (6.17), it holds that
\[
\| u_0' \|_2 \leq \frac{C}{\beta^{1/2} [1 + \beta^{1/3} |\lambda_+|]^{-1/4}}, \tag{8.15}
\]

which, when substituted into (8.9) for \( k = 1 \), yields
\[
\|u_1\|_2 \leq \frac{C}{\beta^{7/6}(1 + |\mu|)} [1 + \beta^{1/3}|\lambda_+|]^{-1/4}.
\]
Substituting into (8.11) yields, for \( k = 2 \),
\[
\|u_1'\|_2 \leq \frac{C}{\beta^{7/6}} [1 + \beta^{1/3}|\lambda_+|]^{-1/4},
\] (8.16)
and hence, by (8.9) for \( k = 2 \),
\[
\|u_2\|_2 \leq \frac{C}{\beta^{3/2}(1 + |\mu|)} [1 + \beta^{1/3}|\lambda_+|]^{-3/4}.
\] (8.17)

As above we now write
\[
[\beta^{-1/3} + |\lambda_+|]\|u_1\|_2 \leq (\beta^{-1/3} + |\mu|)\|u_1\|_2 + C\|u_2\|_2 + \|U - v\|u_1\|_2,
\] (8.18)
to obtain from (8.17), (8.16), (5.5), and (6.17) that
\[
\|u_1\|_2 \leq \frac{C}{\beta^{7/6}} [1 + \beta^{1/3}|\lambda_+|]^{-5/4}.
\] (8.19)

**The case \( k = 2 \)**
We briefly repeat the same argument as in the case \( k = 1 \). By (8.10) with \( k = 2 \) we may write (instead of using (8.11))
\[
\|u_1'\|_2 \leq C (\beta^{1/3}[1 + \beta^{1/3}|\lambda_+|]^{1/2}\|u_1\|_2
\]
\[
+ \beta^{-1/3} [1 + \beta^{1/3}|\lambda_+|]^{-1/2} [\|U - v\|u_1\|_2 + \|u_0\|_2]).
\]

From this we obtain, with the aid of (8.19) and (8.15)
\[
\|u_1'\|_2 \leq \frac{C}{\beta^{3/6}} [1 + \beta^{1/3}|\lambda_+|]^{-3/4}.
\] (8.20)
Substituting (8.20) into (8.9) with \( k = 2 \) then yields with the aid of (8.14)
\[
\|u_2\|_2 \leq \frac{C}{\beta^{3/2}(1 + \beta^{1/3}|\mu|)} [1 + \beta^{1/3}|\lambda_+|]^{-3/4}.
\]

Substituting the above, (8.20), and (8.14) into (8.11), with \( k = 3 \) yields
\[
\|u_2'\|_2 \leq \frac{C}{\beta^{7/6}(1 + \beta^{1/3}|\mu|)} [1 + \beta^{1/3}|\lambda_+|]^{-3/4}.
\]

From (8.9) with \( k = 3 \) we then obtain
\[
\|u_3\|_2 \leq \frac{C}{\beta^{11/6}(1 + \beta^{1/3}|\mu|)^2} [1 + \beta^{1/3}|\lambda_+|]^{-3/4}.
\]

As in (8.18) and (8.19) we can now obtain that
\[
\|u_2\|_2 \leq \frac{C}{\beta^{3/2}} [1 + \beta^{1/3}|\lambda_+|]^{-7/4}.
\]
Combining the above with (8.19) and (8.14) yields (8.6).

Step 2: We prove (8.5).

We first observe that by interpolation (8.6) holds for any \( k \in [0, 2] \). If the conditions of (4.10) are met, we may now obtain (8.5) from (4.19) as in the proof of (8.1). Otherwise if the assumptions of Proposition 4.14 are met, we may conclude (8.1) from (4.89).

8.2. Nearly Couette Flows

We begin by considering the case where the flow \( U \) is nearly linear, as in Subsection 7.3.

8.2.1. The Case \( 0 \leq \alpha \leq \tilde{\alpha} \beta^{1/3} \)

Let \( B_{\lambda,\alpha,\beta}^D \) be defined by (3.4). We can now state and prove

**Proposition 8.3.** For every \( r > 1 \) and \( \Upsilon < \theta^r_1 \), there exist positive \( \delta, \beta_0, \alpha_0 \), and \( C \) such that, for all \( U \in \mathcal{S}_r \) satisfying \( 0 < \delta_2(U) \leq \delta \) (where \( \delta_2 \) is given by (2.31)) and \( \beta \geq \beta_0 \), it holds that

\[
\sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \| (B_{\lambda,\alpha,\beta}^D)^{-1} \| + \left\| \frac{d}{dx} (B_{\lambda,\alpha,\beta}^D)^{-1} \right\| \leq C \beta^{-5/6}.
\]

**Proof.** Let \( \phi \in D(B_{\lambda,\alpha,\beta}^D) \) and \( f \in L^2(-1, 1) \) satisfy \( B_{\lambda,\alpha,\beta}^D \phi = f \). Let \( \lambda = \mu + i \nu \). Let further \( v \) be given by (7.3), and set \( \tilde{v} = (U + i \lambda)^{-1} v \). Without any loss of generality we select \( \Upsilon > \theta^r_1/2 \).

**Step 1:** We prove (8.21) for \( \lambda \in \mathbb{C} \) satisfying either \( \theta^r_1/2 \leq \theta^{-2/3} \beta^{-1/3} \Re \lambda \leq \Upsilon \) or \( -3r \leq \Re \lambda \leq -3 \theta^{-2/3} \beta^{-1/3} \theta^r_1/2 \), and \( |\Im \lambda| \leq 3r \).

Let \( \xi_\pm \in C^2([-1, 1]) \) satisfy the problem

\[
\begin{cases}
-(U + i \lambda) \xi''_\pm + (\alpha^2[U + i \lambda] + U'') \xi_\pm = 0 & \text{in } (-1, 1) \\
\xi_\pm(\pm 1) = 1 \text{ and } \xi_\pm(\mp 1) = 0.
\end{cases}
\]

Note that the differential operator on the left-hand-side is identical with that of \( A_{\lambda,\alpha} \) given by (3.13). We now write

\[
\xi_\pm(x) = \xi^0_\pm(x) + \exp[-\alpha(1 \mp x)] \tilde{\eta}_\pm(x),
\]

in which \( \tilde{\eta}_\pm(x) = \eta(1 \mp x) \), where \( \eta \) is given by (5.14), and \( \xi^0_\pm \in D(A_{\lambda,\alpha}) \).

\( \xi^0_\pm \) and \( \xi_\pm \) estimates

We begin by establishing \( L^\infty \) and \( W^{1,q} \) estimates for \( \xi^0_\pm \). To this end we first write

\[
A_{\lambda,\alpha} \xi^0_\pm = -(U'' \tilde{\eta}_\pm + 2 \alpha (U + i \lambda) \tilde{\eta}_\pm' + \tilde{\eta}_\pm'' (U + i \lambda)) \exp[-\alpha(1 \mp x)].
\]
Note that since the derivatives of $\tilde{\eta}$ are supported on $[-1/2, 1/2]$ we have, since $|\mu|$ is bounded,
\[
\left\| (2\alpha(U + i\lambda)\tilde{\eta}_+'' + \tilde{\eta}_-''(U + i\lambda)) \exp\{-\alpha(1 \mp \cdot)\} \right\|_{1,\infty} \leq C(1 + |v|)(1 + \alpha^2)e^{-\alpha/2} \leq C(1 + |v|) .
\]
Note further that, for all $1 \leq \alpha \leq 2$, we have
\[
\left\| U''\tilde{\eta}_\pm \exp\{-\alpha(1 \mp \cdot)\} \right\|_q \leq C(1 + \alpha)^{-1/q} ,
\]
and
\[
\left\| U''\tilde{\eta}_\pm \exp\{-\alpha(1 \mp \cdot)\} \right\|_{1,q} \leq C(1 + \alpha)^{-1-1/q} .
\]
Consequently, for all $1 \leq q \leq 2$, we have
\[
\left\| A_{\lambda,\alpha}\xi_\pm^0 \right\|_q \leq C(1 + |v| + (1 + \alpha)^{-1/q}) \leq \hat{C}(1 + |v|) . \tag{8.25a}
\]
and
\[
\left\| A_{\lambda,\alpha}\xi_\pm^0 \right\|_{1,q} \leq C(1 + |v| + \alpha^{1-1/q}) . \tag{8.25b}
\]

Observing that
\[
\Re\langle \xi_\pm^0, (U + i\lambda)^{-1}A_{\lambda,\alpha}\xi_\pm^0 \rangle = \frac{1}{2} |(\xi_\pm^0)'|^2 + \alpha^2 |\xi_\pm^0|^2 + I(\xi_\pm^0, \lambda) ,
\]
we deduce from Lemma 4.7 that, for sufficiently small $\delta$ there exist $\gamma_0 > 0$ and $C > 0$ such that, for any $\alpha \geq 0$ and any $\lambda \in \mathbb{C} \setminus J$, (recall that $|\Re\lambda| > \delta^{2/3} \beta^{-1/3} \vartheta^2/2$ in this step)
\[
\Re\langle \xi_\pm^0, (U + i\lambda)^{-1}A_{\lambda,\alpha}\xi_\pm^0 \rangle \geq \frac{1}{2} \left\| (\xi_\pm^0)' \right\|^2_2 + (\gamma_0 + \alpha^2) \left\| \xi_\pm^0 \right\|^2_2 \geq C(1 + \alpha)^{1/2} \left\| (\xi_\pm^0)' \right\|^3/2_2 \left\| \xi_\pm^0 \right\|^{1/2}_2 . \tag{8.26}
\]

Consequently, by (4.34) (which is applied with $q = p, \phi = \xi_\pm^0$ and $v = A_{\lambda,\alpha}\xi_\pm^0$) and (8.25), for any $1 < q < 2$ there exists $C_q > 0$ such that
\[
\left\| (\xi_\pm^0)' \right\|^3_2 \left\| \xi_\pm^0 \right\|^1_2 \leq C_q (1 + \alpha)^{-1/2} \left( \left\| (A_{\lambda,\alpha}\xi_\pm^0)' \right\|_q \left\| \xi_\pm^0 \right\|_\infty + \left\| A_{\lambda,\alpha}\xi_\pm^0 \right\|_\infty \left\| (\xi_\pm^0)' \right\|_q \right) \leq C_q (1 + |v|)(1 + \alpha)^{-1/q+1/2} \left\| \xi_\pm^0 \right\|_2 .
\]
(We use the fact that $q \leq 2$ and Poincaré’s inequality to obtain the last inequality.) We may thus conclude that
\[
\left\| (\xi_\pm^0)' \right\|^3/2_2 \left\| \xi_\pm^0 \right\|^1/2_2 \leq C_q (1 + |v|)(1 + \alpha)^{-1/q+1/2} .
\]
Hence, for $q = \frac{3}{2}$, we get
\[
\left\| \xi_\pm^0 \right\|_\infty \leq \left\| (\xi_\pm^0)' \right\|^1/2_2 \left\| \xi_\pm^0 \right\|^1/2_2 \leq C_q (1 + |v|)(1 + \alpha)^{-1/6} ,
\]
which finally implies
\[
\left\| \xi_\pm^0 \right\|_\infty \leq C (1 + |v|) . \tag{8.27}
\]
By (8.26) we also have that
\[ \Re\langle \xi_0^0, (U + i\lambda)^{-1}A_{\lambda, \alpha} \xi_0^0 \rangle \geq \frac{1}{2} \| (\xi_0^0)' \|^2, \]
and hence
\[ \| (\xi_0^0)' \|_2 \leq C_0 (1 + |\nu|) (1 + \alpha)^{-1/q + 1}. \]

For \( q = \frac{3}{2} \) we may thus conclude
\[ \| (\xi_0^0)' \|_2 \leq C (1 + |\nu|) (1 + \alpha)^{1/2}. \] (8.28)

As (8.27) and (8.28) are unsatisfactory for large values of \( \nu \), we use the fact that
\[ \Re\langle (U - \nu)^2 \xi_0^0, (U + i\lambda)^{-1}A_{\lambda, \alpha} \xi_0^0 \rangle = \| (U - \nu)\xi_0^0 \|_2^2 - \| (U - \nu)' \xi_0^0 \|_2^2 + \alpha \| (U - \nu)\xi_0^0 \|_2^2 - \Re\langle (U - \nu)^2 \xi_0^0, (U + i\lambda)^{-1}U'' \xi_0^0 \rangle. \]

Assuming \( |\nu| \geq 2 |U|_\infty \), we first obtain, for any \( \varepsilon > 0 \)
\[ \varepsilon \| (U - \nu)\xi_0^0 \|_2^2 + \frac{4}{\varepsilon} \| A_{\lambda, \alpha} \xi_0^0 \|_2^2 \geq \| (U - \nu)\xi_0^0 \|_2^2 - C \| (U - \nu)\xi_0^0 \|_2 \| \xi_0^0 \|_2. \]
Poincaré’s inequality then yields, for sufficiently small \( \varepsilon > 0 \)
\[ \nu \| \xi_0^0 \|_{1,2} \leq C (\| A_{\lambda, \alpha} \xi_0^0 \|_2 + \| \xi_0^0 \|_2), \]
which implies, for sufficiently large \( |\nu| \)
\[ \| \xi_0^0 \|_{1,2} \leq C. \] (8.29)

We can now use (8.25a), (8.27) and (8.29) to obtain, for any \( \nu \),
\[ \| \xi_0^0 \|_{1,2} \leq C (1 + \alpha)^{1/2}. \] (8.30)

Combining (8.29) with (8.27) implies, in addition, the following refinement of (8.27) to any \( \nu \)
\[ \| \xi_0^0 \|_\infty \leq C. \] (8.31)

Substituting (8.30) and (8.31) into (8.23) respectively yields
\[ \| \xi_0 \|_{1,2} \leq C (1 + \alpha_0^{1/2} \beta^{1/6}), \] (8.32a)
and
\[ \| \xi_0 \|_\infty \leq C. \] (8.32b)

**Application of Lemma 6.2**

It can be easily verified that
\[ \langle \xi_0, \bar{v} \rangle = 0. \] (8.32c)

We can thus consider the problem
\[ \left( - \frac{d^2}{dx^2} + i\beta (U + i\lambda) \right) \bar{v} = h, \]
where \( \tilde{v} \) satisfies (8.32c) and \( h \) is given by (see (7.25))

\[
h = f + \left( \frac{U''\phi}{U + i\lambda} \right)''.
\]

By (8.32) we may now use Lemma 6.2 to conclude from (6.20) (applied for \((g, u, v) = (h, \tilde{u}, \tilde{v})\)) that \( \tilde{v} \) admits the decomposition

\[
\tilde{v} = A_+(h)(\psi_+ - \tilde{v}_+) + A_-(h)(\psi_- - \tilde{v}_-) + \tilde{u},
\]

where \( A_\pm(h) \) and \( \tilde{v}_\pm \) respectively satisfy (6.24) and (6.14), and \( \tilde{u} \) is given by

\[
\tilde{u} = \Gamma_{[-1, 1]}(\tilde{L}_\beta, \beta - \tilde{v})^{-1} \tilde{h},
\]

where

\[
\tilde{h}(x) = \begin{cases} 
  h(x) & x \in [-1, 1] \\
  0 & \text{otherwise}
\end{cases}.
\]

By (5.4) we then have

\[
\|\tilde{u}\|_2 \leq C \beta^{-2/3} \|h\|_2.
\]

**Estimate of \( h \).**

Since (7.27), (7.28), and (7.31) are still valid, we obtain

\[
\|h\|_2 \leq \|f\|_2 + C \left( \delta \beta^{5/6} + \beta^\frac{2}{3} \right) \|\phi\|_{1, 2} + \left\| \frac{\phi''}{U + i\lambda} \right\|_2,
\]

which leads for \( \beta \geq \beta_0(\delta) \) to

\[
\|h\|_2 \leq \|f\|_2 + \hat{C} \delta \beta^{5/6} \|\phi\|_{1, 2} + \left\| \frac{\phi''}{U + i\lambda} \right\|_2.
\]

To bound the last term on the right-hand-side we write, recalling that \( |\mu| \geq \Im \beta \beta^{-1/3} \phi_1^2/2 \) in this step,

\[
\left\| \frac{\phi''}{U + i\lambda} \right\|_2 \leq C \beta^{1/3} \|\phi''\|_2.
\]

Then, we use the fact that by (7.3) and (3.13)

\[
\|\phi'' - \alpha^2\phi\|_2 \leq \|\tilde{v}\|_2 + \|U''(U + i\lambda)^{-1}\phi\|_2.
\]

As in the proof of (7.31), we get

\[
\|U''(U + i\lambda)^{-1}\phi\|_2 \leq C \delta \beta^{1/6} \|\phi\|_\infty.
\]

We now obtain a bound for the norm of \( \tilde{v} \) by estimating each term appearing in the right hand side of (8.33). By (6.14) we get for \( \tilde{v}_\pm \),

\[
\|\tilde{v}_\pm\| \leq C \beta^{-\frac{1}{2}}.
\]
By (6.17) (for \( k = 0 \)) we get for \( \psi_{\pm} \),
\[
\| \psi_{\pm} \| \leq C \beta^{-\frac{1}{6}} [1 + |\lambda_{\pm}| \beta^{1/3}] \frac{1}{3},
\]
(8.38)
and by (6.24) we get for \( A_{\pm}(h) \)
\[
|A_{\pm}(h)| \leq C \beta^{-\frac{1}{2}} \| h \|_2.
\]
(8.39)
Together with (8.35) for \( \tilde{u} \), we then have
\[
\| \tilde{v} \|_2 \leq C \beta^{-2/3} [1 + |\lambda_{\pm}| \beta^{1/3}] \frac{1}{3} \| h \|_2,
\]
where \( \lambda_{\pm} \) is given by (6.15).
Since (7.12) remains valid for no-slip conditions we may conclude that
\[
\| \phi'' \|_2 \leq C (\beta^{-7/12} \| f \|_2 + \delta \beta^{5/6} \| \phi \|_{1,2}).
\]
(8.40)
Combining the above with (8.36) and (8.37) yields
\[
\| h \|_2 \leq (1 + C \beta^{-1/4}) \| f \|_2 + C \delta \beta^{5/6} \| \phi \|_{1,2}.
\]
(8.41)
For later reference we note that by (7.31), (8.40) and (8.41) we have
\[
\left\| \left( \frac{U'' \phi}{U + i \lambda} \right)'' \right\|_2 \leq C (\beta^{-1/4} \| f \|_2 + \delta \beta^{5/6} \| \phi \|_{1,2}).
\]
(8.42)

**Estimates of \( \phi \)**

Next we set, with (8.33) in mind,
\[
v = v_o + v_+ + v_-,
\]
where
\[
v_{\pm} = A_{\pm}(h)(U + i \lambda)(\psi_{\pm} - \tilde{v}_{\pm}),
\]
(8.43)
and
\[
v_o = (U + i \lambda) \Gamma_{[1-1,1]} \tilde{u} = (U + i \lambda) \Gamma_{[1-1,1]}(\tilde{L}_{\beta, R} - \beta \lambda)^{-1} \tilde{h}.
\]
(8.44)
Set further
\[
\phi_{\pm} = A_{\lambda, \alpha}^{-1} v_{\pm}; \quad \phi_o = A_{\lambda, \alpha}^{-1} v_o.
\]
Clearly,
\[
\phi = \phi_o + \phi_+ + \phi_-.
\]
(8.45)
To estimate \( \phi_{\pm} \) we write
\[
\phi_{\pm} = A_{\pm}(h) A_{\lambda, \alpha}^{-1} (U + i \lambda) \psi_{\pm} - A_{\pm}(h) A_{\lambda, \alpha}^{-1} (U + i \lambda) \tilde{v}_{\pm}.
\]
(8.46)
For the first term in the right hand side, we use (8.1) to obtain
\[ \|A_\pm(h)A_{\lambda,\alpha}^{-1}(U + i\lambda)\psi_\pm\|_{1,2} \leq C\beta^{-\frac{1}{2}}|A_\pm(h)|. \]
Using (4.26) with \( p = 2 \) and (6.14), yields for the second term
\[ \|A_\pm(h)A_{\lambda,\alpha}^{-1}(U + i\lambda)\tilde{v}_\pm\|_{1,2} \leq C\beta^{-1}|A_\pm(h)|. \]
Hence using (8.39), we obtain
\[ \|\phi_\pm\|_{1,2} \leq C(\beta^{-1}\|f\|_2 + \delta\beta^{-1/6}\|\phi\|_{1,2}). \]  
(8.47)
To estimate \( \phi_0 \) we first recall that by (8.34) and (8.44)
\[ v_o = (U + i\lambda)\Gamma_{[-1,1]}\tilde{u} = (U + i\lambda)\Gamma_{[-1,1]}(\tilde{L}_{\beta,R} - \beta\lambda)^{-1}\tilde{h}. \]
By (5.4) and (5.5) we then have
\[ \|v_o\|_2 \leq C\beta\|h\|_2, \]
and hence by (8.41)
\[ \|v_o\|_2 \leq C\beta^{1/2}\|f\|_2 + C\delta\beta^{-1/6}\|\phi\|_{1,2}. \]  
(8.48)
By (4.26) (with \( p = 2 \)), and the definition of \( \phi_o \) we obtain
\[ \|\phi_o\|_2 \leq C(\beta^{-5/6}\|f\|_2 + \delta\|\phi\|_{1,2}). \]  
(8.49)
Substituting (8.49) and (8.47) into (8.45) yields, for sufficiently small \( \delta \)
\[ \|\phi\|_{1,2} \leq C\beta^{-5/6}\|f\|_2, \]
which is precisely (8.21) established in this step for all \( \lambda \in \mathbb{C} \) such that \(|\Im\lambda| \leq 3r\), and either \( \vartheta'_1/2 \leq \beta^{1/3}\gamma_m^{-2/3}m\lambda \leq \vartheta'_2 \) or \(-3r \leq \beta^{1/3}\gamma_m^{-2/3}m\lambda \leq -\vartheta'_1/2 \).

\textbf{Step 2:} We prove (8.21) for sufficiently small \( \alpha_0 \) and \( \lambda \) satisfying
\[ \beta^{1/3}\gamma_m^{-2/3}|m\lambda| < \vartheta'_1/2 \] and \(|\Im\lambda| \leq 3r\).
Let \( \tilde{z}_\pm \in C^2([-1, 1]) \) be given by (6.69). Note that
\[ \|\tilde{z}_\pm\|_\infty = 1 ; \quad \|\tilde{z}_\pm\|_{1,2} \leq C(1 + \alpha^{1/2}) \leq C(1 + \alpha_0^{1/2} \beta^{1/6}). \]  
(8.50)
By (3.10), (6.69), and two integrations by parts, it holds that
\[ \langle \tilde{z}_\pm, -\phi'' + \alpha^2\phi \rangle = 0 . \]  
(8.51)
As (7.9) is still valid, and in view of (8.50) and (8.51), we can apply Lemma 6.2, assuming that \( \alpha_0 \) is small enough, with \( \xi_\pm \) replaced by \( \tilde{z}_\pm \), to obtain for
\[ g = f + i\beta U''\phi \] and \( v = -\phi'' + \alpha^2\phi \),
that
\[ -\phi'' + \alpha^2\phi = B_+ (\psi_+ - \tilde{v}_+) + B_- (\psi_- - \tilde{v}_-) + \Gamma_{(-1,1)}(\tilde{L}_{\beta,R} - \beta\lambda)^{-1}\tilde{g}, \]  
(8.52)
where
\[ B_\pm = A_\pm (f + i\beta U'' \phi), \]
and
\[ \tilde{g} := \gamma R(f + i\beta U'' \phi)(x) = \begin{cases} (f + i\beta U'' \phi)(x) & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}. \]

By (6.24) and the fact that \( \|U''\|_\infty \leq \delta \) in this nearly Couette case, it holds that
\[ |B_\pm| \leq C(\beta^{-1/2}\|f\|_2 + \delta\beta^{1/3} \log \beta \|\phi\|_\infty). \]  

(8.53)

By (6.14) together with (8.53), we have
\[ \|B_\pm \tilde{v}_\pm\|_2 \leq C(\beta^{-1}\|f\|_2 + \delta\beta^{-1/6} \log \beta \|\phi\|_\infty). \]  

(8.54)

By (5.4) and (5.35) it holds that
\[ \| L_\beta R - \beta \lambda \|_2 \leq C(\beta^{-2/3}\|f\|_2 + \delta\beta^{1/6} \|\phi\|_\infty). \]

Combining the above with (8.52) and (8.54) then yields
\[ \|w_0\|_2 \leq C(\beta^{-2/3}\|f\|_2 + \delta\beta^{1/6} \|\phi\|_\infty), \]  

(8.55)

where
\[ w_0 = -\phi'' + \alpha^2 \phi - B_\pm \psi_\pm - B_\mp \psi_\mp. \]

As in the proof of Proposition 7.3, we use the result of the previous step by considering \( \lambda + \gamma_0 \beta^{-1/3} \) for a suitable value of \( \gamma_0 \). We choose \( \gamma_0 \) such that
\[ \frac{3\bar{m}^{2/3} \theta \Gamma}{2} \leq \beta^{1/3} \Re \lambda + \gamma_0 \leq \frac{3\bar{m}^{2/3} \gamma}{2} \]  

(8.56)

We now write
\[ B_{\lambda + \gamma_0 \beta^{-1/3}, \alpha}^2 \phi = f - \gamma_0 \beta^{2/3}(w_0 + B_+ \psi_+ + B_- \psi_-), \]
and introduce the following decomposition of \( \phi \)
\[ \phi = \chi_0 - \gamma_0 \beta^{2/3}(B_+ \chi_+ + B_- \chi_-), \]  

(8.57)

where
\[ \chi_0 = (B_{\lambda + \gamma_0 \beta^{-1/3}, \alpha}^2)^{-1}(f - \gamma_0 \beta^{2/3}w_0) \quad \text{and} \quad \chi_\pm = (B_{\lambda + \gamma_0 \beta^{-1/3}, \alpha}^2)^{-1}\psi_\pm. \]

For convenience we set
\[ \tilde{\lambda} = \lambda + \gamma_0 \beta^{-1/3}. \]

We may now apply (8.21) (with \( \lambda \) replaced by \( \tilde{\lambda} \)) to obtain, with the aid of (8.55), that
\[ \|\chi_0\|_2 \leq C(\beta^{-5/6}\|f\|_2 + \delta\|\phi\|_\infty). \]  

(8.58)
Estimate of $\chi_{\pm}$.

We seek an estimate of $\|\chi_{\pm}\|_{1,2}$. To this end we repeat the same procedure applied in step 1. For convenience of notation we consider only $\chi_{\pm}$ in the following. The same estimates for $\chi_{-}$ can be obtained in a similar manner. Let then

$$\hat{w}_{\pm} = -\chi_{\pm}'' + \alpha^2 \chi_{\pm} + \frac{U''}{U + i\tilde{\lambda}} \chi_{\pm},$$

and

$$H_+ := \left(-\frac{d^2}{dx^2} + i\beta(U + i\tilde{\lambda})\right)\hat{w}_+.$$

(8.59)

It can be easily verified that

$$H_+ = \psi_+ + \left(\frac{U'' \chi_{\pm}}{U + i\tilde{\lambda}}\right)''.$$

and that $\langle \zeta_{\pm}, \hat{w}_+ \rangle = 0$. Consequently, we can use Lemma 6.2 with $\lambda$ replaced by $\tilde{\lambda}$. With the notation $\psi_\pm = \psi_\pm(\lambda)$, $\hat{w}_\pm = \hat{w}_\pm(\tilde{\lambda})$ (as defined in (6.12)) Lemma 6.2 yields

$$\hat{w}_+ = \hat{B}_+(\tilde{\psi}_+ - \tilde{v}_+) + \hat{B}_-(\tilde{\psi}_- - \tilde{v}_-) + (\tilde{\mathcal{L}}_{\beta,\tilde{\lambda}} - \beta\tilde{\lambda})^{-1}\tilde{H}_+,$$

where $\hat{B}_\pm = A_{\pm}(H_+)$ and

$$\tilde{H}_+(x) = \begin{cases} H_+(x) & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}.$$

Moreover, we have that

$$|\hat{B}_\pm| \leq C\beta^{-\frac{1}{2}} \|H_+\|_2.$$

(8.60)

We now follow the arguments of the previous step with $(\chi_{\pm}, \tilde{\psi}_+, H_+, \hat{w}_+)$ respectively replacing $(\phi, f, h, \tilde{v})$. We then reach by the equivalent of (8.42)

$$\|H_+\|_2 \leq (1 + C\beta^{-1/4})\|\psi_+\|_2 + C\delta\beta^{5/6}\|\chi_+\|_{1,2},$$

from which, using (6.17), we get

$$\|H_+\|_2 \leq C\beta^{-1/6}[1 + |\lambda_\pm|^{1/4} \beta^{1/2}] + C\delta\beta^{5/6}\|\chi_+\|_{1,2}.$$

Hence, by (8.60) for $k = 0$, we obtain

$$|\hat{B}_\pm| \leq C(\beta^{-2/3}[1 + |\lambda_\pm|^{1/4} \beta^{1/2}] + \delta\beta^{1/3}\|\chi_+\|_{1,2}).$$

(8.61)

As above we set

$$\chi_+ = \hat{\phi}_+ + \hat{\phi}_- + \hat{\phi}_0,$$

(8.62a)

where

$$\hat{\phi}_\pm = \hat{B}_\pm A_{\lambda,\alpha}^{-1}(U + i\tilde{\lambda})(\tilde{\psi}_\pm - \tilde{v}_\pm); \quad \hat{\phi}_0 = A_{\lambda,\alpha}^{-1}(U + i\tilde{\lambda})\Gamma_{[-1,1]}(\tilde{\mathcal{L}}_{\beta,\tilde{\lambda}} - \beta\tilde{\lambda})^{-1}\tilde{H}_+.$$

(8.62b)
By (8.1) and (8.61) we have
\[ |\hat{B}_{\pm}||A_{\hat{\lambda},\alpha}^{-1}(U + i\hat{\lambda})\hat{\psi}_{\pm}||_{1,2} \leq C\left(\beta^{-7/6} + \delta \frac{\beta^{-1/6}}{[1 + |\hat{\lambda}_{\pm}|^{1/4}\beta^{1/2}]}\right)\|\chi_{+}\|_{1,2}.\]

Note here that as \(|\lambda - \tilde{\lambda}| = \gamma_0\beta^{-1/3}\) we have used that, for some \(C > 1\),
\[ C^{-1}[1 + |\lambda_{\pm}|^{1/4}\beta^{1/2}] \leq [1 + |\tilde{\lambda}_{\pm}|^{1/4}\beta^{1/2}] \leq C[1 + |\lambda_{\pm}|^{1/4}\beta^{1/2}].\]

Moreover, by (8.61), (6.14), and (4.26) (with \(p = 2\)) it holds that
\[ |\hat{B}_{\pm}||A_{\hat{\lambda},\alpha}^{-1}(U + i\hat{\lambda})\hat{\nu}_{\pm}||_{1,2} \leq C(\beta^{-4/3}[1 + |\tilde{\lambda}_{\pm}|^{1/4}\beta^{1/2}] + \delta\beta^{-1/3}\|\chi_{+}\|_{1,2}).\]

Combining the above yields
\[ \|\hat{\phi}_{\pm}\|_{1,2} \leq C\left(\beta^{-7/6} + \delta \frac{\beta^{-1/6}}{[1 + |\lambda_{\pm}|^{1/4}\beta^{1/2}]}\right)\|\chi_{+}\|_{1,2}. \tag{8.63} \]

Next, we estimate \(\|\hat{\phi}_o\|_{1,2}\). By (8.5) we have
\[ \|A_{\hat{\lambda},\alpha}^{-1}(U + i\hat{\lambda})\Gamma[-1,1](\tilde{\mathcal{L}}_{\beta,R} - \beta\hat{\lambda})^{-1}\hat{\psi}_+\|_{1,2} \leq C\frac{\beta^{-7/6}}{1 + |\nu - U(+1)|^{1/3}}. \tag{8.64} \]

Furthermore, by (5.5), (5.4), and (4.26), we have
\[ \left\|A_{\hat{\lambda},\alpha}^{-1}(U + i\hat{\lambda})\Gamma[-1,1](\tilde{\mathcal{L}}_{\beta,R} - \beta\hat{\lambda})^{-1}\left(\frac{U''\chi_+}{U + i\hat{\lambda}}\right)''\right\|_{1,2} \leq C\frac{\beta^{5/6}}{\|\chi_+\|_{1,2}}. \tag{8.65} \]

By (8.42) with \(\chi_+\) instead of \(\phi\) and \(\psi_+\) replacing \(f\), it holds that
\[ \left\|\left(\frac{U''\chi_+}{U + i\hat{\lambda}}\right)''\right\|_2 \leq C\left(\beta^{-5/12}[1 + |\lambda_{\pm}|^{1/4}\beta^{1/2}] + \delta\beta^{5/6}\|\chi_{+}\|_{1,2}\right). \]

Combining the above with (8.65) leads to
\[ \left\|A_{\hat{\lambda},\alpha}^{-1}(U + i\hat{\lambda})\Gamma[-1,1](\tilde{\mathcal{L}}_{\beta,R} - \beta\hat{\lambda})^{-1}\left(\frac{U''\chi_+}{U + i\hat{\lambda}}\right)''\right\|_{1,2} \leq C\left(\beta^{-5/4}[1 + |\lambda_{\pm}|^{1/4}\beta^{1/2}] + \delta\|\chi_{+}\|_{1,2}\right), \]
which, together with (8.64), yields
\[ \|\hat{\phi}_o\|_{1,2} \leq C\left(\beta^{-7/6} + \delta\|\chi_{+}\|_{1,2}\right). \]

Combining the above with (8.62) and (8.63) yields, for \(\delta > 0\) small enough,
\[ \|\chi_{+}\|_2 \leq C\beta^{-7/6}. \]

In a similar manner we obtain
\[ \|\chi_-\|_2 \leq C\beta^{-7/6}. \]
Substituting the above, (8.58), and (8.53) into (8.57) yields that (8.21) holds for 
\[ |\Re \lambda| \leq \beta^{-1/3} \frac{\alpha^2 \rho_1}{3} r / 2. \]

**Step 3:** We prove (8.21) for \( \lambda \in \mathbb{C} \) satisfying \( \Re \lambda \leq -3r \).
As \( f = B_{\lambda, \alpha, \beta} \phi \) and
\[ -\Re \langle \phi, B_{\lambda, \alpha, \beta} \phi \rangle = \| \phi'' \|_2^2 - (\beta \mu - \alpha^2) \| \phi' \|_2^2 - \beta \mu \alpha^2 \| \phi \|_2^2 - \beta \Re \langle U' \phi, \phi' \rangle, \]
we can conclude that
\[ \beta (3r \| \phi' \|_2^2 - \| U' \|_\infty \| \phi' \|_2 \| \phi \|_2) \leq \| \phi \|_2 \| f \|_2. \]

Since \( \phi \in H_0^1(-1, 1) \) it holds that
\[ \| \phi' \|_2 \geq \frac{\pi}{2} \| \phi \|_2, \]
and hence (recall that \( \| U' \|_\infty < r \) and \( 3r > 3 > 2/\pi \)) we can conclude that there exists \( C > 0 \) such that
\[ \beta (3r \| \phi' \|_2^2 - \| U' \|_\infty \| \phi' \|_2 \| \phi \|_2) \leq \| \phi \|_2 \| f \|_2. \]

**Step 4:** We prove (8.21) for \( \lambda \in \mathbb{C} \) satisfying \( |\Im \lambda| \geq 3r \).
An integration by parts yields
\[ \Im \langle \phi, B_{\lambda, \alpha, \beta} \phi \rangle = \beta \nu \left( \| \phi' \|_2^2 + \alpha^2 \| \phi \|_2^2 \right) - \beta \left( \langle U \phi', \phi' \rangle + \alpha^2 \langle U \phi, \phi \rangle + \Re \langle U' \phi, \phi' \rangle + \langle U'' \phi, \phi \rangle \right), \]
and as
\[ \Re \langle U' \phi, \phi' \rangle = -\frac{1}{2} \langle U'' \phi, \phi \rangle, \]
we obtain that
\[ \beta \left( |\nu| - \| U \|_\infty - \frac{1}{2} \| U'' \|_\infty \right) \| \phi' \|_2^2 + \alpha^2 \beta (|\nu| - \| U \|_\infty) \| \phi \|_2^2 \leq \| \phi \|_2 \| f \|_2. \]

As
\[ |\nu| - \| U \|_\infty - \frac{1}{2} \| U'' \|_\infty \geq |\nu| - \frac{3}{2} r, \]
we establish (8.66) whenever \( |\nu| \geq 3r \).
The proposition is proved. \( \square \)
8.2.2. The Case $\alpha \geq \alpha^# \beta^{1/6}$ We separately treat the case $\#=D$ and the case $\# = \mathcal{G}$.

Proposition 8.4. For any $r > 1$ and any $\kappa > 0$ there exist $\beta_0 > 0$, $\alpha^D > 0$ and $C$ such that for all $\beta \geq \beta_0$ and $U \in \mathcal{S}_r$

$$\sup_{\alpha D^\beta_1/6 \leq \alpha} \left( (1 + \alpha) \| (B^{\alpha D}_\alpha, \beta)^{-1} \| + \left\| \frac{d}{dx} (B^{\alpha D}_\alpha, \beta)^{-1} \right\| \right) \leq C \beta^{-5/6}.$$  

(8.67)

Proof. Let $\phi \in D(B^{\alpha D}_\alpha, \beta)$ and $f \in L^2(-1, 1)$ satisfy $B^{\alpha D}_\alpha, \beta \phi = f$. Let $\mu_\pm \in C^2([-1, 1])$ be given by (6.69). By (6.76), (7.9), and (8.51), we have

$$\| - \phi'' + \theta^2 \beta^{2/3} \phi \|_2 \leq C (\beta^{1/3} \| \phi \|_2 + \beta^{-2/3} \| f \|_2),$$

(8.68)

where $\theta = \alpha \beta^{-1/3}$. Hence,

$$\| \phi' \|_2^2 + \theta^2 \beta^{2/3} \| \phi \|_2^2 = \langle -\phi'' + \theta^2 \beta^{2/3} \phi, \phi \rangle \leq C (\beta^{1/3} \| \phi \|_2^2 + \beta^{-2/3} \| f \|_2 \| \phi \|_2).$$

(8.69)

As $\theta \geq \alpha^D \beta^{-1/6}$, we obtain that for sufficiently large $\alpha^D$ and $\beta$,

$$\| \phi \|_2 \leq \frac{C}{\theta^2 \beta^{4/3} \| f \|_2},$$

which implies

$$\alpha \| \phi \|_2 \leq \frac{C}{\theta \| f \|_2} \leq \frac{C}{\alpha^D \beta^{5/6} \| f \|_2}.$$  

Returning to (8.69), we get

$$\| \phi' \|_2^2 \leq \beta^{-\frac{3}{2}} \| f \|_2 \| \phi \|_2 \leq \frac{C}{\theta^2 \beta^2} \| f \|_2^2,$$

hence

$$\| \phi' \|_2 \leq \frac{C}{\theta \| f \|_2} \leq \frac{C}{\alpha^D \beta^{5/6} \| f \|_2}.$$  

\[ \square \]

Remark 8.5. Using the definition of $\hat{\mu}_0$ from (6.77) yields that

$$\hat{\mu}_0(\beta^{-1/3} \alpha) \beta^{2/3} + \alpha^2 = \beta^{2/3} \min \left( J_{\alpha}^{2/3} \mu_0(J_{\alpha}^{-1/3} \theta) + \theta^2, J_{\alpha}^{2/3} \mu_0(J_{\alpha}^{-1/3} \theta) + \theta^2 \right),$$

where $\theta = \beta^{-1/3} \alpha$. Let $\theta_{\pm} = J_{\alpha}^{-1/3} \theta$.

Using the definition of $\hat{\mu}_m$ from (6.57) we then conclude

$$J_{\alpha}^{2/3} \mu_0(J_{\alpha}^{-1/3} \theta) + \theta^2 = J_{\alpha}^{2/3} (\mu_0(\theta_{\pm}) + \theta_0^2) \geq J_{\alpha}^{2/3} \hat{\mu}_m.$$
Consequently we obtain that
\[ \hat{\mu}_0 (\beta^{-1/3} \alpha) \beta^{2/3} + \alpha^2 \geq \beta^{2/3} \Delta m \hat{\mu}_m. \]

By the foregoing discussion, we may conclude from (8.67) that
\[ \sup_{\Re \lambda \leq \beta^{-1/3} (\Delta m \mu_m - \alpha - \beta^{-2/3} \alpha^2)} (1 + \alpha) \| (B_{\lambda, \alpha}^{\mathcal{S}})^{-1} \| + \left\| \frac{d}{dx} (B_{\lambda, \alpha}^{\mathcal{S}})^{-1} \right\| \leq \frac{C}{\beta^{5/6}}. \]

(8.70)

A similar estimate holds true also for \( B_{\lambda, \alpha, \beta}^{\mathcal{S}} \).

**Proposition 8.6.** For any \( r > 1 \) and \( \gamma < \Re \nu_1 \) there exist positive \( \beta_0, \alpha^{\mathcal{S}} \) and \( C \) such that for all \( \beta \geq \beta_0 \) and \( U \in \mathcal{S}_r \) such that
\[ \sup_{\Re \lambda \leq \gamma^{2/3} \beta^{-1/3}} (1 + \alpha) \| (B_{\lambda, \alpha, \beta}^{\mathcal{S}})^{-1} \| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^{\mathcal{S}})^{-1} \right\| \leq C \beta^{-5/6}. \]

\[ (8.71) \]

**Proof.** Let \( \phi \in D(B_{\lambda, \alpha, \beta}^{\mathcal{S}}) \) and \( f \in L^2(-1, 1) \) satisfy \( B_{\lambda, \alpha, \beta}^{\mathcal{S}} \phi = f \). To prove (8.71) we note that by (7.9) we have
\[ \| \phi'' - \alpha^2 \phi \|_2 \leq C (\beta^{1/3} \| \phi \|_2 + \beta^{-2/3} \| f \|_2). \]

Consequently,
\[ \| \phi' \|_2^2 + \alpha^2 \| \phi \|_2^2 \leq C (\beta^{1/3} \| \phi \|_2^2 + \beta^{-2/3} \| f \|_2 \| \phi \|_2), \]
from which (8.71) follows as in the \( \mathcal{D} \)- case. \( \square \)

### 8.3. Strictly Convex/Concave Flows

#### 8.3.1. Large \( \alpha \) or \( |\Im \lambda| \)

If we assume large \( \alpha \) or \( |\nu| \) we may obtain resolvent estimates for all \( U \in C^4([-1, 1]) \), without the necessity to assume any further restrictions on \( U \) as in the nearly Couette case or in the case \( U'' \neq 0 \).

**Proposition 8.7.** Let \( r > 1 \), \( \gamma < \min(\Re \nu_1, \Re \nu_1^0) \), and \( \# \in \{\mathcal{S}, \mathcal{D}\} \). Then, there exist \( \alpha_0 > 0, \alpha_1 > 0, \nu > 0, \beta_0 > 0, \) and \( C \) such that, for all \( \beta \geq \beta_0 \) and \( U \in \mathcal{S}_r \) satisfying (4.24)-(4.26), it holds that
\[ \sup_{\Re \lambda \leq \gamma^{2/3} \beta^{-1/3}} \left\| (B_{\lambda, \alpha, \beta}^{\#})^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^{\#})^{-1} \right\| \leq C \beta^{-5/6}, \]

\[ (8.72) \]

and
\[ \sup_{d(\Im \lambda, [U(-1), U(1)]) \geq \nu \beta^{-1/3}} \left\| (B_{\lambda, \alpha, \beta}^{\#})^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^{\#})^{-1} \right\| \leq C \beta^{-5/6}. \]

\[ (8.73) \]
Proof. Let \( f \in L^2(-1, 1) \), \( \phi \in D(B^\#, \alpha, \beta) \) and \( h \) satisfy
\[
B^\#, \alpha, \beta \phi = f \quad \text{and} \quad h = f + \left( \frac{U'' \phi}{U + i \lambda} \right)''.
\]
\[\Box\]

**Proof of (8.73).** Consider first the case \( d(\Im \lambda, [U(-1), U(1)]) \geq \tilde{\nu} \beta^{-1/3} \). Here, following the same derivation as in (7.27)-(7.31) (for \( \# = \mathcal{S} \)) and (8.37)-(8.41) (for \( \# = \mathcal{D} \)) (the role of \( \delta \) being replaced by \( \frac{1}{\tilde{\nu}} \), which is small too) we obtain
\[
\|h\|_2 \leq \|f\|_2 + C \tilde{\nu} \beta^{5/6} \|\phi\|_{1, 2}.
\]
(8.74)

For sufficiently large \( \tilde{\nu} \) we can thus follow the same steps of either the proof of Proposition 7.3 starting from (7.32) now replaced by (8.74), or the proof of Proposition 8.3 starting from (8.41), now replaced by (8.74), to obtain (8.73).
\[\Box\]

**Proof of (8.72).** Case 1: \(|\Re \lambda| > \frac{2^2}{3} \beta^{-1/3} \min(\vartheta_f, \vartheta v_1)/2 \). To prove (8.72) for \( \# = \mathcal{D} \) we repeat the same procedure used in the proof of Proposition 8.3 to establish (8.48) with \( \delta = r \) (note that we always have \( \delta_2(U) \leq r \) and recall that \( v_0 \) is defined by (8.44))
\[
\|v_o\|_2 \leq C \frac{\beta}{\tilde{\nu}} \|f\|_2 + C \beta^{-1/6} \|\phi\|_{1, 2}.
\]
(8.75)

In the case \( \# = \mathcal{S} \) we repeat the same steps as in the proof of Proposition 7.3 to establish (7.33) with \( \delta = r \). For convenience we use the notation \( v_o \) instead of \( v \) (which is defined by (7.3)).

We now estimate \( \phi_o = A_{\lambda, \alpha}^{-1} v_o \). To this end we rewrite (4.28) in the form
\[
\Im \left( \frac{\phi_o}{U - v + i \mu}, v_o \right) \geq \frac{1}{2} \|\phi_o'\|_2^2 + (\alpha^2 + \gamma_m(\lambda, U)) \|\phi_o\|_2^2.
\]
The left-hand-side can be bounded by (4.39) (for \( p = 2 \)) to obtain
\[
\frac{1}{2} \|\phi_o'\|_2^2 + (\alpha^2 + \gamma_m(\lambda, U)) \|\phi_o\|_2^2 \leq C \beta^{1/6} \|\phi_o\|_\infty \|v_o\|_2.
\]
Since, by Lemma 4.9, \( \gamma_m(\lambda, U) \geq \vartheta_0 > -\infty \), we pick \( \alpha \in \mathbb{R} \) such that
\[
\alpha^2 \geq \sup(1, -2 \vartheta_0).
\]
Then we write
\[
\alpha \|\phi_o\|_2^2 \leq \alpha \|\phi_o'\|_2 \|\phi_o\|_2 \leq \frac{1}{2} (\|\phi_o'\|_2^2 + \alpha^2 \|\phi_o\|_2^2) \leq C \beta^{1/6} \|\phi_o\|_\infty \|v_o\|_2.
\]
(8.76)

Consequently,
\[
\|\phi_o\|_\infty \leq \frac{C}{\alpha} \beta^{\frac{1}{6}} \|v_o\|_2.
\]
and

\[ \| \phi_0 \|_{1,2}^2 \leq \frac{2C}{\alpha} \beta^{1/3} \| v_0 \|_2^2. \tag{8.77} \]

By (8.75) and (8.77), we deduce

\[ \| \phi_0 \|_{1,2} \leq \frac{\hat{C}}{\sqrt{\alpha}} \beta^{-5/6} \| f \|_2 + \frac{\hat{C}}{\sqrt{\alpha}} \| \phi \|_{1,2}. \tag{8.78} \]

In the case \( \# = S \) we have \( \phi_0 = \phi \) and hence (8.72) immediately follows. In the case \( \# = D \) we continue as in the proof of Proposition 8.3 to establish (8.47) with \( \delta = r \), or explicitly,

\[ \| \phi \|_{1,2} \leq C(\beta^{-1} \| f \|_2 + \beta^{-1/6} \| \phi \|_{1,2}). \]

The above, combined with (8.78) and (8.45) yields for \( \alpha \geq \bar{\alpha}_1 \) with \( \bar{\alpha}_1 \) large enough

\[ \| \phi \|_{1,2} \leq C \left( \frac{1}{\sqrt{\alpha}} + \beta^{-1/6} \right) \beta^{-5/6} \| f \|_2, \tag{8.79} \]

from which (8.72) readily follows.

**Case 2:** \( |\Re \lambda| \leq \min(\Re \nu_1, \vartheta_1 r) 2^{-1/3} \beta^{-1/3} \).

For \( \# = S \) we use (7.36) for \( \delta = r \), i.e.,

\[ \| \phi'' - \alpha^2 \phi \|_2 \leq C(\beta^{-2/3} \| f \|_2 + r \beta^{1/6} \| \phi \|_\infty). \tag{8.80} \]

Then, as

\[ B_{\lambda+s\beta^{-1/3},\alpha,\beta}^S \phi = f - s \beta^{2/3} (\phi'' - \alpha^2 \phi), \]we may use (8.78), having in mind that \( \phi_0 = \phi \), to obtain that

\[ \| \phi \|_{1,2} \leq \frac{\hat{C}}{\sqrt{\alpha}} \beta^{-5/6} (\| f \|_2 + \beta^{2/3} \| \phi'' - \alpha^2 \phi \|_2). \]

Substituting (8.80) into the above yields (8.72).

For \( \# = D \) we first obtain (8.55) with \( \delta = r \), which implies

\[ \| w_0 \|_2 \leq C(\beta^{-2/3} \| f \|_2 + r \beta^{1/6} \| \phi \|_\infty), \tag{8.81} \]

where

\[ w_0 = -\phi'' + \alpha^2 \phi - B_+ \psi_+ - B_- \psi_-, \]

and

\[ B_{\pm} = A_{\pm}(f + i \beta U'' \phi). \]

As

\[ B_{\lambda+\gamma_0 \beta^{-1/3},\alpha,\beta}^D \phi = f - \gamma_0 \beta^{2/3} (w_0 + B_+ \psi_+ + B_- \psi_+), \]
where $\Upsilon_0$ satisfies (8.56), we can write as in (8.57) that

$$\phi = \chi_0 - \Upsilon_0 \beta^{2/3} (B_+ \chi_+ + B_- \chi_-),$$  \tag{8.82}

where

$$\chi_0 = (B^D_{\lambda_0 \gamma_0 \beta^{-1/3}, \alpha, \beta})^{-1} (f - \Upsilon_0 \beta^{2/3} w_0) \quad \text{and} \quad \chi_\pm = (B^D_{\lambda_0 \gamma_0 \beta^{-1/3}, \alpha})^{-1} \psi_\pm.$$

The estimation of $\chi_0$ can be done with the aid of (8.79) and (8.81), yielding

$$\| \chi_0 \|_{1,2} \leq C \left( \frac{1}{\sqrt{\alpha}} + \beta^{-1/6} \right) \beta^{-5/6} (\| f \|_2 + \| \phi \|_\infty).$$  \tag{8.83}

To estimate $\chi_+$ (or $\chi_-$) we write

$$\chi_+ = \chi_1^1 + \chi_1^2,$$

where

$$\chi_1^1 = \mathcal{A}_{\lambda_0, \alpha}^{-1} \left( U + i \tilde{\lambda} \right) \Gamma_{[-1,1]}(\tilde{L}_{\beta, \Re} - \beta \tilde{\lambda})^{-1} \left( \frac{U'' \chi_+}{U + i \tilde{\lambda}} \right).$$

As

$$\left\| \left( \frac{U'' \chi_+}{U + i \tilde{\lambda}} \right)^{\prime} \right\|_2 \leq C \left( \beta^{-1/3} + \beta^{5/6} \| \chi_+ \|_{1,2} \right),$$

we can conclude from (5.4), (5.5), and (8.77) that

$$\| \chi_1^1 \|_{1,2} \leq \frac{C}{\sqrt{\alpha}} (\beta^{-7/6} + \| \chi_+ \|_{1,2}).$$

The estimation of $\chi_1^2$ in the proof of Proposition 8.3 does not involve $\delta$ at all, but only $r$ and hence we can conclude by (8.63) and (8.64) that

$$\| \chi_1^2 \|_{1,2} \leq C (\beta^{-7/6} + \beta^{-1/6} \| \chi_+ \|_{1,2}).$$

Consequently we obtain that for sufficiently large $\alpha$

$$\| \chi_+ \|_{1,2} \leq C \beta^{-7/6}.$$

A similar estimate holds for $\chi_-$, and hence we can conclude (8.72) from (8.83), (8.82), and (8.53).
8.3.2. The Case $U'' \neq 0$

Lemma 8.8. Let $r > 1$ and $\hat{\delta} \in (0, \frac{1}{\gamma}]$. Then, there exist $\beta_0 > 0$, $\Upsilon > 0$, and $C > 0$ such that for all $\beta \geq \beta_0$ and $U \in S_r$ satisfying (2.34), it holds that

$$\sup_{\gamma \lambda \leq \gamma \delta_m^{2/3} \beta^{-1/3}} \|B_{\lambda, \alpha, \beta}^D \|^{-1} + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^D)^{-1} \right\| \leq \frac{C}{\beta^{1/2 - \delta}}, \quad (8.84)$$

where $\tilde{\alpha}_1$ is the same as in Proposition 8.7.

Proof. Let $\phi \in D(B_{\lambda, \alpha, \beta}^D), \alpha \leq \tilde{\alpha}_1, f = B_{\lambda, \alpha, \beta}^D \phi$ and $v_\Delta \in H^2(-1, +1)$ defined by

$$v_\Delta = A_{\lambda, \alpha} \phi + (U + i \lambda)(\phi''(1)^+ \psi_+ + \phi''(1)^- \psi_-), \quad (8.85)$$

where

$$\hat{\psi}_\pm = \psi_\pm / \Theta_\pm(\pm 1) \quad (8.86)$$

in which $\psi_\pm$ is defined in (6.8) and $\Theta_\pm(x) = 1 - \tilde{\eta}(1 \mp x)$, with $\tilde{\eta}$ given by (5.14). We note that by (A.43d) we have that for some $C > 0$

$$\frac{1}{C} [1 + |\lambda| \beta^{1/3}]^{1/2} \leq |\psi(\pm 1)| \leq C [1 + |\lambda| \beta^{1/3}]^{1/2}. \quad (8.87)$$

Note that $v_\Delta \in H^1_0(-1, 1)$ and hence we may introduce

$$g_\Delta := (L_\beta^D - \beta \lambda)v_\Delta.$$

We have

$$g_\Delta = (U + i \lambda)(-f + \phi''(1)^+ \hat{g}_+ + \phi''(1)^- \hat{g}_-) - (U'' \phi)^+ - 2U' \hat{v}_\Delta + U'' \tilde{v}_\Delta. \quad (8.88)$$

wherein

$$\hat{g}_\pm = \left(-\frac{d^2}{dx^2} + i \beta U - \beta \lambda\right)\hat{\psi}_\pm,$$

and

$$\tilde{v}_\Delta = \frac{v_\Delta - U'' \phi}{U + i \lambda} = -\phi'' + \alpha^2 \phi + \phi''(1)^+ \psi_+ + \phi''(1)^- \psi_-.$$

We note that

$$(L_\beta^D - \beta \lambda)\tilde{v}_\Delta - i \beta U'' \phi = -f + \phi''(1)^+ \hat{g}_+ + \phi''(1)^- \hat{g}_-. \quad (8.89)$$
As in the proof of Proposition 7.1 (see in particular (7.7)) we can integrate by parts to obtain

\[ \mathfrak{R}\langle (U'')^{-1}\bar{v}_D, (L^D_{\beta} - \beta \lambda)\bar{v}_D - i\beta U''\phi \rangle = \| (U'')^{-1/2}\bar{v}'_D \|^2 + \\
+ \mathfrak{R}\langle (U'')^{-1}[\bar{v}_D, \bar{v}'_D] - \beta \mu \|\bar{v}_D\|^2 + \beta \mathfrak{R}\langle \phi''(1)\hat{\psi}_+ + \phi''(-1)\hat{\psi}_-, i\phi \rangle. \]

(8.90)

We begin the estimation by obtaining a bound for the last term on the right-hand-side.

**Estimate of** \( \beta \mathfrak{R}\langle \phi''(1)\hat{\psi}_+ + \phi''(-1)\hat{\psi}_-, i\phi \rangle. \)

We first write

\[
\phi(x) = \int_x^1 (\xi - x)\phi''(\xi) \, d\xi = \phi''(1) \int_x^1 (\xi - x)\hat{\psi}_+(\xi) \, d\xi \\
+ \int_x^1 (\xi - x)[\phi''(\xi) - \phi''(1)\hat{\psi}_+(\xi)] \, d\xi.
\]

Let

\[ w_+(x) = \int_x^1 (\xi - x)\hat{\psi}_+(\xi) \, d\xi. \]

By (A.43c) and (8.87) there exists \( C > 0 \) such that \( \|\hat{\psi}_+\|_\infty \leq C \), and hence

\[ |w_+(x)| \leq C |1 - x|^2. \]

Thus,

\[ |\mathfrak{R}\langle \phi''(1)\hat{\psi}_+, i\phi''(1)w_+ \rangle| \leq C|\phi''(1)|^2\|1 - x\|^2\|\hat{\psi}_+\|_1. \]

Using (A.43b), (8.87), translation, and dilation (see also (6.17)), we may conclude that

\[ \| (1 - x)^s\hat{\psi}_+ \|_1 \leq CB^{-(s+1)/3}[1 + |\lambda_+|^{1/2}\beta^{1/6}]^{-(s+1)} \quad \forall s \leq 3, \]

(8.91)

and hence

\[ |\mathfrak{R}\langle \phi''(1)\hat{\psi}_+, i\phi''(1)w_+ \rangle| \leq \frac{C}{\beta}[1 + |\lambda_+|^{1/2}\beta^{1/6}]^{-3}|\phi''(1)|^2. \]

(8.92)

We then obtain for \( x \in (-1, +1) \), using the fact that \( \hat{\psi}_+\hat{\psi}_- \equiv 0 \),

\[
\left| \overline{\hat{\psi}_+(x)} \int_x^1 (\xi - x)[\phi''(\xi) - \phi''(1)\hat{\psi}_+(\xi)] \, d\xi \right| \\
= \left| \overline{\hat{\psi}_+(x)} \int_x^1 (\xi - x)[\bar{v}_D(\xi) + \alpha^2\phi(\xi)] \, d\xi \right| \\
\leq C (1 - x)^{5/2} |\hat{\psi}_+(x)| \left( \|\bar{v}_D'\|_2 + \alpha^2\|\phi\|_2 \right).
\]
Consequently, from (8.91) and (8.92), we thus get, as \( \alpha \leq \tilde{\alpha} \),
\[
\beta |\mathfrak{R}(\phi''(1)v_+ + i\phi)| \leq C \left[ 1 + |\lambda_+|\beta^{1/3}\right]^{-3/2}[|\phi''(1)|^2 \\
+ [1 + |\lambda_+|\beta^{1/3}]^{-1/4} \beta^{-1/6} \Vert \phi''(1)| \left( \Vert \tilde{v}_D' \Vert_2 + \Vert \phi' \Vert_2 \right) .
\]  
(8.93)

**Estimate of** \( \Vert \tilde{v}_D' \Vert \).

Next we obtain from (8.90) and (8.93) that
\[
\Vert \tilde{v}_D' \Vert_2^2 \leq C \Vert \tilde{v}_D \Vert_2 (\Vert f \Vert_2 + |\phi''(1)| \Vert \hat{g}_+ \Vert_2 + |\phi''(-1)| \Vert \hat{g}_- \Vert_2 \\
+ C \max(0, \mu \beta) \Vert \tilde{v}_D \Vert_2^2 \\
+ C \left[ 1 + |\lambda_+|\beta^{1/3}\right]^{-3/2}[|\phi''(1)|^2 \\
+ [1 + |\lambda_+|\beta^{1/3}]^{-1/4} \beta^{-1/6} \Vert \phi''(1)| \left( \Vert \tilde{v}_D' \Vert_2 + \Vert \phi' \Vert_2 \right) .
\]  
(8.94)

To estimate \( |\phi''(\pm 1)| \) we first write, as in (7.9)
\[
(L^\xi_{\beta} - \beta \lambda)(\phi'' - \alpha^2 \phi) = i \beta U'' \phi + f,
\]
where \( \xi_\pm = \xi_\pm \) is given by (6.69). Then, in view of Remark 6.3, we may use (6.35) to obtain with \( i \beta U'' \phi + f \), \( g = i \beta U'' \phi + f \),
\[
|\phi''(\pm 1)| \leq C \left[ 1 + |\lambda_+|^{1/2} \beta^{1/6}\right] (\beta^{-1/6} \Vert f \Vert_2 + \beta^{1/3} \log \beta \Vert \phi \Vert_\infty) .
\]  
(8.95)

Furthermore, using the fact that \( \hat{g}_\pm(x) = g_\pm(x) / \psi_\pm(\pm 1) \), we may use (6.18) and (8.87) to obtain that
\[
\Vert \hat{g}_\pm \Vert_2 \leq C \beta^{1/6} \left[ 1 + |\lambda_+|^{1/2} \beta^{1/6}\right]^{-5/2} .
\]  
(8.96)

which, when substituted into (8.94), yields, with the aid of Sobolev's embeddings,
\[
\Vert \tilde{v}_D' \Vert_2^2 \leq C (\Vert f \Vert_2^2 + \beta^{1/2} \log \beta \Vert \phi \Vert_\infty \Vert \tilde{v}_D \Vert_2 + \beta^{1/3} \log^2 \beta \Vert \phi \Vert_1^2 + \max(0, \mu \beta) \Vert \tilde{v}_D \Vert_2^2) .
\]  
(8.97)

**Estimate of** \( \Vert \tilde{v}_D \Vert \).

By (5.21) and (8.89) we have
\[
\Vert \tilde{v}_D \Vert_2 \leq C \left( \beta^{1/6} \Vert \phi \Vert_\infty + \beta^{-2/3} \left[ \Vert f \Vert_2 + |\phi''(1)| \Vert \hat{g}_+ \Vert_2 + |\phi''(-1)| \Vert \hat{g}_- \Vert_2 \right] \right) .
\]

Hence, by (8.96) and (8.95), it holds that
\[
\Vert \tilde{v}_D \Vert_2 \leq C (\beta^{1/6} \Vert \phi \Vert_\infty \beta^{-2/3} \Vert f \Vert_2) .
\]

Substituting the above into (8.97) yields
\[
\Vert \tilde{v}_D' \Vert_2 \leq C \left( \Vert f \Vert_2 + \left( \max(\mu, 0)^{1/2} \beta^{2/3} \beta^{1/3} \log \beta \right) \Vert \phi \Vert_1 \right) .
\]  
(8.98)

We now combine (8.88), (8.98), (8.97), (8.95), and (8.96) to obtain that
\[
\Vert g_\beta \Vert_2 \leq C (\Vert f \Vert_2 + \left( \max(\mu, 0)^{1/2} \beta^{2/3} \beta^{1/3} \log \beta \right) \Vert \phi \Vert_1) .
\]  
(8.99)
Proof of (8.84). We continue as in the proof of Proposition 8.3. We first write, in view of (8.85), that
\[
\phi = \phi_\mathcal{D} + \tilde{\phi}_+ + \tilde{\phi}_- .
\] (8.100)

where
\[
\phi_\mathcal{D} = A^{-1}_{\lambda, \alpha} v_\mathcal{D} ; \quad \tilde{\phi}_\pm = -A^{-1}_{\lambda, \alpha} ([U + i\lambda] \phi''(\pm 1) \hat{\psi}_\pm).
\]

By (8.1) and (8.95) we have
\[
\|\tilde{\phi}_\pm\|_{1,2} \leq C [1 + |\lambda_\pm|^{1/2} \beta^{1/6}]^{-1/2} (\beta^{-2/3} \|f\|_2 + \beta^{-1/6} \log \beta \|\phi\|_{1,2}),
\]
and hence, by (8.100),
\[
\|\tilde{\phi}_\pm\|_{1,2} \leq C [1 + |\lambda_\pm|^{1/2} \beta^{1/6}]^{-1/2} (\beta^{-2/3} \|f\|_2 + \beta^{-1/6} \log \beta \|\phi_\mathcal{D}\|_{1,2}) .
\] (8.101)

Substituting the above into (8.99) yields, with the aid of (8.100)
\[
\|g_\mathcal{D}\|_2 \leq C (\|f\|_2 + [(\max(\mu, 0)^{1/2}) \beta^{2/3} + \beta^{1/3} \log \beta] \|\phi_\mathcal{D}\|_{1,2}).
\]

By (4.92b,c) we then have for any \(q > 1\) and \(p > 2\) (recall that \(\mu \leq 3^{2/3} \gamma \beta^{-1/3}\))
\[
\|g_\mathcal{D}\|_2 \leq C (\|f\|_2 + \gamma^{1/2 - 1/p} \beta^{-\frac{3p+2}{6p}} \|v_\mathcal{D}\|_p + \beta^{1/3} \log \beta (\|v'_\mathcal{D}\|_q + \|v_\mathcal{D}\|_\infty)) .
\] (8.102)

By (5.19) there exists \(C > 0\) such that for all \(p > 2\) (including \(p = \infty\)) it holds that
\[
\|v_\mathcal{D}\|_p \leq C \beta^{-\frac{3p+2}{6p}} \|g_\mathcal{D}\|_2 .
\] (8.103)

Similarly, by (5.10), for all \(1 < q < 2\), there exists \(C_q > 0\) such that
\[
\|v'_\mathcal{D}\|_q \leq C_q \beta^{-\frac{2+q}{6q}} \|g_\mathcal{D}\|_2 .
\] (8.104)

Substituting (8.103) and (8.104) into (8.102) yields, choosing \(\gamma > 0\) small enough and \(\beta_0\) large enough, the existence of \(C > 0\) such that for \(\beta \geq \beta_0\)
\[
\|g_\mathcal{D}\|_2 \leq C \|f\|_2 .
\]

Using (4.92b,c) once again upon (8.103) and (8.104) and the above inequality readily verifies (8.84).

We can now conclude.

Proposition 8.9. Let \(r > 1\) and \(\hat{\delta} > 0\). Then, there exist \(\beta_0 > 0\), \(\gamma > 0\), and \(C > 0\), such that for all \(\beta \geq \beta_0\) and \(U \in S_r\) satisfying (2.34), it holds that
\[
\sup_{\|\lambda + \beta^{-1} \alpha^2 \| \leq \gamma J^{3/2} \beta^{-1/3}} \| (B_{\lambda, \alpha, \beta})^{-1} \| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta})^{-1} \right\| \leq \frac{C}{\beta^{1/2 - \hat{\delta}}}. \] (8.105)

The proof follows immediately by combining (8.70), (8.72), and (8.84) for a sufficiently small value of \(\gamma > 0\).
9. Semigroup Estimates

In this section we prove Theorems 2.15 and 2.16.

9.1. Preliminaries

For $\sharp \in \{S, D\}$, let $\mathbf{F} = (F_1, F_2) \in H^1_{loc}(\tilde{D}, \mathbb{R}^2) \cap \mathcal{H}, \Lambda \in \mathbb{C}$, and $u \in \mathcal{W}^0_\sharp$ satisfy

$$(\mathcal{T}_\sharp(U, \varepsilon, L) - \Lambda)u = P\mathbf{F}. \quad (9.1)$$

Recall that by writing $u = \nabla \psi$ for some $\psi \in D(P_{\Lambda, \varepsilon})$ we have established in (3.1) that

$$P_{\Lambda, \varepsilon}^\sharp \psi = \text{curl} \mathbf{F} \quad \text{in } D. \quad (9.2)$$

This gives, for $\Lambda \in \rho(\mathcal{T}_\sharp)$,

$$u = \nabla (P_{\Lambda, \varepsilon}^\sharp)^{-1} \text{curl} \mathbf{F} = (\mathcal{T}_\sharp(U, \varepsilon, L) - \Lambda)^{-1} P\mathbf{F}. \quad (9.3)$$

To estimate $(\mathcal{T}_\sharp(U, \varepsilon, L) - \Lambda)^{-1}$ we seek therefore a bound for $(P_{\Lambda, \varepsilon}^\sharp)^{-1}$ for $\Lambda$ in a suitable region of $\mathbb{C}$. We later derive the properties of the semi-group $e^{-t\mathcal{T}_\sharp}$ from these resolvent estimates.

We seek a bound for the $L^2_{\text{per}}, H^1_{\text{per}}$ norm of $(P_{\Lambda, \varepsilon}^\sharp)^{-1}$ for $\Lambda$ in a suitable region of $\mathbb{C}$. We later derive the properties of the semi-group $e^{-t\mathcal{T}_\sharp}$ from these resolvent estimates.

We seek a bound for the $\mathcal{L}(L^2_{\text{per}}, H^1_{\text{per}})$ norm of $(P_{\Lambda, \varepsilon}^\sharp)^{-1}$ in the domain in $\Re \Lambda \leq \varepsilon \gamma \beta_1^{2/3}$ for some $\gamma > 0$. Recall that $P_{\Lambda, \varepsilon}$ depends on $L > 0$ through the periodicity condition appearing in the definition of its domain. As in Section 3 we can rewrite (3.11), which provides an $L^2_{\text{per}}$ bound for $(P_{\Lambda, \varepsilon}^\sharp)^{-1}$, as the following $L^2_{\text{per}}, H^1_{\text{per}}$ estimate

$$\sup_{\Re \lambda \leq \varepsilon \gamma \beta_1^{2/3}} \| (P_{\Lambda, \varepsilon}^\sharp)^{-1} (I - \Pi) \|_{\mathcal{L}(L^2_{\text{per}}, H^1_{\text{per}})} \leq \varepsilon^{-1} B_*^\sharp(\gamma, \varepsilon, L), \quad (9.4)$$

where,

$$B_*^\sharp(\gamma, \varepsilon, L) = \sup_{\beta \geq \beta_1(\varepsilon, L)} \sup_{\alpha \geq 0} \left[ \|(1 + \alpha)(B_{\lambda, \alpha, \beta}^\sharp)^{-1}\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\sharp)^{-1} \right\| \right], \quad (9.5)$$

in which $\beta_1(\varepsilon, L) = (2\pi)/(L \varepsilon)$.

9.2. Proof of Theorem 2.15

9.2.1. $(P_{\Lambda, \varepsilon}^{\Sigma})^{-1}$ Estimates

Estimation of $B_*^{\Sigma}(\gamma, \varepsilon, L)$.

We begin by showing that for any $\hat{\delta} \in (0, 1/3)$, and $\alpha_M > 0$, there exist $\beta_0 > 0$, $C > 0$ and $\gamma_0 > 0$, such that, for all $\beta \geq \beta_0$ and $\gamma \leq \gamma_0$

$$\sup_{0 \leq \alpha \leq \alpha_M \beta^{1/6}} \sup_{\Re \lambda \leq \gamma \beta^{-1/3}} \left[ \|(1 + \alpha)(B_{\lambda, \alpha, \beta}^{\Sigma})^{-1}\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^{\Sigma})^{-1} \right\| \right] \leq C \beta^{-1/3 + \hat{\delta}}. \quad (9.6)$$
Let
\[ S(\beta, \Upsilon) = \{(\Re \lambda, \alpha) \in \mathbb{R}^2 \mid \Re \lambda \leq \beta^{-1/3} \Upsilon \ ; \ 0 \leq \alpha \leq \alpha_M \beta^{1/6}\}. \]

By (7.1) we have that for sufficiently small \( \Upsilon > 0 \) there exists \( \beta'_0(\Upsilon) > 0 \) and \( C > 0 \) such that
\[
\sup_{(\Re \lambda, \alpha) \in S(\beta, \Upsilon)} \left\| (1 + \alpha)(B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| \leq C \beta^{-1/3 + \delta}, \quad (9.7)
\]
for all \( \beta \geq \beta'_0(\Upsilon) \).

On the other hand, it follows from (8.71) in a similar manner that for any \( 0 < \Upsilon < \Re \nu_1 \) there exist \( \alpha^\Upsilon > 0 \), \( \beta''_0 > 0 \), and \( C > 0 \), such that for all \( \beta \geq \beta''_0 \) we have
\[
\sup_{\alpha^\Upsilon \beta^{1/6} \leq \alpha \ \Re \lambda \leq \beta^{-1/3} \Upsilon} \left\| (1 + \alpha)(B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| \leq C \beta^{-5/6}. \quad (9.8)
\]

As a result, for all \( L > 0 \) and \( \varepsilon > 0 \) such that \( L \varepsilon \leq \varepsilon_0 := 2\pi/\max(\beta''_0, \beta'_0) \) (implying that \( \beta_1(\varepsilon, L) \geq \max(\beta''_0, \beta'_0) \)) it holds that
\[
\sup_{\beta \geq \beta_1(\varepsilon, L)} \sup_{\Re \lambda \leq \beta^{-1/3} \Upsilon} \left\| (1 + \alpha)(B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| \leq C \beta_1(\varepsilon, L)^{-5/6}. 
\]

Combining the above with (9.6) for \( \alpha_M = \alpha^\Upsilon \) yields that for any \( \hat{\delta} \in (0, \frac{1}{3}) \) there exists \( C > 0 \), \( \Upsilon_0 > 0 \), and \( \varepsilon_0 > 0 \) such that for all \( L > 0 \) and \( \varepsilon > 0 \) satisfying \( L \varepsilon \leq \varepsilon_0 \) and \( \Upsilon \leq \Upsilon_0 \), we have
\[
\sup_{\beta \geq \beta_1(\varepsilon, L)} \sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \left\| (1 + \alpha)(B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| \leq C \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}. 
\]

We now observe that
\[
B_{*}^{\Upsilon}(\Upsilon, \varepsilon, L) \leq \sup_{\beta \geq \beta_1(\varepsilon, L)} \sup_{\Re \lambda \leq \Upsilon \beta^{-1/3}} \left\| (1 + \alpha)(B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\Upsilon)^{-1} \right\|.
\]

Hence we obtain that for any \( \hat{\delta} \in (0, \frac{1}{3}) \) there exist \( \Upsilon_0 > 0 \), \( \varepsilon_0 > 0 \), and \( C > 0 \) such that for all \( L > 0 \), \( \varepsilon > 0 \) satisfying \( 0 < L \varepsilon \leq \varepsilon_0 \) it holds that
\[
B_{*}^{\Upsilon}(\Upsilon, \varepsilon, L) \leq C \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}. \quad (9.9)
\]

In conclusion, we have established the following

**Proposition 9.1.** For any \( \hat{\delta} \in (0, \frac{1}{3}) \) there exist \( \varepsilon_0 > 0 \) and \( \Upsilon_0 > 0 \), such that for all positive \( L \) and \( \varepsilon \) for which \( 0 < L \varepsilon \leq \varepsilon_0 \), \( \Upsilon \leq \Upsilon_0 \), and \( U \in S_r \) satisfying (2.34), it holds that
\[
\sup_{\Re \lambda \leq \Upsilon \beta_1^{-2/3}} \| (P_{\lambda, \varepsilon}^{\Upsilon})^{-1}(I - \Pi) \|_{L_2(L_{\text{per}}^2, H_{\text{per}}^1)} \leq \frac{C}{\varepsilon} \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}. \quad (9.10)
\]
9.2.2. Estimation of $(I - \Pi)e^{-t\Sigma_P^\natural}(U, \varepsilon, L)$

**Proof of (2.35).** Let $\delta \in (0, 1/3)$ and $\Upsilon > 0$ be as in the statement of Proposition 9.1. We can now combine (9.2), (9.3), and (9.10), to obtain for all $\Upsilon \leq \Upsilon_0$

$$
sup_{\Re \Lambda \leq \varepsilon \Upsilon_0^{2/3}} \| (I - \Pi)u \|_2 \leq \| (I - \Pi)F \|_2^\natural \leq \frac{C}{\varepsilon} \beta_1(\varepsilon, L)^{-1/3+\delta} \| \text{curl} (I - \Pi)F \|_2
$$

$$
\leq \frac{C}{\varepsilon} \beta_1(\varepsilon, L)^{-1/3+\delta} \| (I - \Pi)F \|_{1,2}.
$$

(9.11)

\[\square\]

We now establish a bound on $\| \nabla (I - \Pi)u \|_2$ for $\Re \Lambda < -\frac{1}{2} \| U' \|_\infty - 1$. To this end, we first recall Lemma 2.11 to obtain

$$
\Sigma_P^\natural (U, \varepsilon, L) - \Lambda) (I - \Pi)u = (I - \Pi)F
$$

Using (2.14) with $u$ replaced by $(I - \Pi)u$ (note that, by Lemma 2.9 $(I - \Pi)u \in W_0^\natural$) we obtain

$$
\varepsilon \| \nabla (I - \Pi)u \|_2^2 + \langle (I - p)u_1, U'(I - p)u_2 \rangle - \Lambda \| (I - \Pi)u \|_2^2
$$

$$
= \langle (I - \Pi)u, (I - \Pi)F \rangle,
$$

(9.12)

which implies

$$
\varepsilon \| \nabla (I - \Pi)u \|_2^2 \leq (\Re \Lambda + \frac{1}{2} \| U' \|_\infty) \| (I - \Pi)u \|_2^2 + \| (I - \Pi)u \| \| (I - \Pi)F \|,
$$

(9.13)

Since, by (2.13), we have

$$
\| e^{-t\Sigma_P^\natural} (I - \Pi) \| \leq e^{\frac{1}{2} \| U' \|_\infty^t},
$$

we may use the resolvent-semigroup relation to obtain that

$$
sup_{\Re \Lambda \leq -\frac{1}{2} \| U' \|_\infty - 1} \| (I - \Pi)u \| \leq \| (I - \Pi)F \|.
$$

We may therefore conclude that

$$
sup_{\Re \Lambda \leq -\frac{1}{2} \| U' \|_\infty - 1} \| (I - \Pi)u \|_{1,2} \leq \frac{C}{\varepsilon^{1/2}} \| (I - \Pi)F \|_2.
$$

(9.14)

Consequently,

$$
sup_{\Re \Lambda \leq -\frac{1}{2} \| U' \|_\infty - 1} \| (I - \Pi)(\Sigma_P^\natural - \Lambda)^{-1} \| \mathcal{L}(H, H^1_{per}(D, \mathbb{C}^2)) \leq \frac{C}{\varepsilon^{1/2}}.
$$

(9.15)

To estimate the resolvent for $\Lambda \in \mathbb{C}$ satisfying $-\frac{1}{2} \| U' \|_\infty - 1 \leq \Re \Lambda \leq \varepsilon \Upsilon_0^{2/3}$, we introduce $\omega = -\frac{1}{2} \| U' \|_\infty - 1$ and use the resolvent identity, composed on the left by $(I - \Pi)$,

$$
(I - \Pi)(\Sigma_P^\natural - \Lambda)^{-1} = (I - \Pi)(\Sigma_P^\natural - \omega - i\Im \Lambda)^{-1}
$$

$$
+ (\Re \Lambda - \omega)(I - \Pi)(\Sigma_P^\natural - \Lambda)^{-1}(\Sigma_P^\natural - \omega - i\Im \Lambda)^{-1},
$$

(9.16)
to obtain, with the aid of Lemma 2.11, that
\[
\|(I - \Pi)(\overline{\Sigma}_p - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \\
\leq \|(I - \Pi)(\overline{\Sigma}_p - \omega - i\gamma\Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \\
+ |\Re\Lambda - \omega| \|(I - \Pi)(\overline{\Sigma}_p - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}^{per}(D, \mathbb{C}^2), \mathcal{H})} \\
\|(I - \Pi)(\overline{\Sigma}_p - \omega - i\gamma\Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{H}^{per}(D, \mathbb{C}^2))}.
\]

Using (9.11) and (9.15) we then obtain that for each \( \hat{\delta} \in (0, 1/3) \) there exist \( \Upsilon_0 > 0, C > 0, \) and \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, 1], L > 0 \) satisfying \( 0 < \varepsilon L \leq \varepsilon_0 \)
\[
\sup_{-\|U\|_{\infty}/2 - 1 \leq |\Re\Lambda| \leq \varepsilon \Upsilon \beta_1^{2/3} \to} \|(I - \Pi)(\overline{\Sigma}_p(U, \varepsilon, L) - \Lambda)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{\varepsilon^{3/2} \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}}.
\]

(9.17)

We can now use [26, Proposition 2.1] (cf. also [25, Proposition 13.31] or [42, Theorem 11.3.5]), which we repeat here for the benefit of the reader.

**Proposition 9.2.** Let \( S(t) \) be a strongly continuous semigroup, defined on a Hilbert space \( H \), which satisfies for some \( M \geq 1 \) and \( \hat{\omega} \in \mathbb{R} \),
\[
\|S(t)\| \leq \hat{M} e^{-\hat{\omega}t}.
\]

(9.18)

Let \(-A : D(A) \to H \) denote the generator of \( S(t) \). Suppose further that for some \( \omega > \hat{\omega} \)
\[
\sup_{|z| \leq \omega} \| (A - z)^{-1} \| := \frac{1}{r(\omega)} < +\infty.
\]

(9.19)

Then,
\[
\|S(t)\| \leq \hat{M} \left(1 + \frac{2\hat{M}(\omega - \hat{\omega})}{r(\omega)}\right) e^{-\omega t}.
\]

(9.20)

By Remark 1.4 in [26] (or Remark 11.3.4 in [42]), we may conclude that
\[
\sup_{|z| = \omega} \| (A - z)^{-1} \| = \sup_{|z| \leq \omega} \| (A - z)^{-1} \|.
\]

(9.21)

Note that since \((A - z)^{-1}\) is bounded by (9.19), we can obtain the above from its analyticity and the Phragmén-Lindelöf Theorem. We now apply the proposition with \( A = (I - \Pi)\overline{\Sigma}_p(U, \varepsilon, L) \). By (2.13) it follows that (9.18) holds for \( \hat{M} = 1 \) and \( \hat{\omega} = -\|U\|_{\infty}/2 \). By (9.17) and (9.21), (9.19) holds for \( \omega = \varepsilon \Upsilon \beta_1^{2/3} \) with \( r(\omega) \geq \varepsilon^{3/2} \beta_1^{1/3 - \hat{\delta}} \). Consequently by (9.20), we obtain that for any \( \hat{\delta} \in (0, 1/3) \), there exist \( \Upsilon > 0, C > 0, \) and \( \varepsilon_0 > 0 \) such that for all \( \varepsilon \in (0, 1] \) and \( L > 0 \) satisfying \( 0 < L \varepsilon \leq \varepsilon_0 \) we have
\[
\|e^{-t} \overline{\Sigma}_p(U, \varepsilon, L)(I - \Pi)\| \leq \frac{C}{\varepsilon^{3/2} \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}} e^{-\varepsilon \Upsilon \beta_1^{2/3}}.
\]

This completes the proof of (2.35).
Remark 9.3. Note that while (9.16) allows for an estimate in $L^1(H^1)$ of $e^{-t \mathcal{P}^\#(U, \varepsilon, L)}(I - \Pi)$, it contributes an additional $\varepsilon^{-1/2}$ factor to the coefficient of the exponent in (2.35). If we manage to obtain estimates for $e^{-t \mathcal{P}^\#(U, \varepsilon, L)}(I - \Pi)$ in $\mathcal{L}(H^1_{per}, D, \mathbb{C}^2, \mathcal{H})$ the additional factor could be avoided. In fact, in [14], the initial conditions are assumed to be in $H^2_{per}$, resulting in improved estimates for the semigroup.

Proof of Part 2.
The proof of part 2 is obtained in the same manner. We begin by replacing Proposition 9.1 by the following result

Proposition 9.4. Let $r > 1$. For any $\Upsilon < \Re \nu_1$ there exist $\delta > 0$, $\varepsilon_0 > 0$, and $C > 0$ such that for all positive $L$ and $\varepsilon$ for which $0 < L \varepsilon \leq \varepsilon_0$, and $U \in S_r$ satisfying $\delta_2(U) < \delta$, it holds that

$$\sup_{0 \leq \alpha \leq \alpha_M \beta_1^{1/6}} \| \mathcal{P}^\#_{\lambda, \varepsilon}^{-1} \|_{\mathcal{L}(L^2_{per}, H^1_{per})} \leq \frac{C}{\varepsilon} \beta_1^{-2/3}.$$  \hfill (9.22)

Proof. We use (7.23) which gives

$$\sup_{\beta \geq \beta_1(\varepsilon, L)} \sup_{0 \leq \alpha \leq \alpha_M \beta_1^{1/6}} \left\| (1 + \alpha)(B_{\lambda, \alpha, \beta}^\#)^{-1} \right\| + \left\| \frac{d}{dx} (B_{\lambda, \alpha, \beta}^\#)^{-1} \right\| \leq C \beta_1(\varepsilon, L)^{-2/3}.$$  \hfill (9.23)

Combining the above with (9.8) yields

$$B_{\alpha}^\#(\Upsilon, \varepsilon, L) \leq C \beta_1(\varepsilon, L)^{-2/3},$$

which together with (9.4) yields (9.22). \qed

The proof of (2.36) proceeds from here in the same manner as in Part 1.

9.3. Proof of Theorem 2.16

Since the proof is similar to the proof of Theorem 2.15 we address here only the main ingredients.

Proof of Part 1 For the first part of the theorem we use (8.70) and (8.105) to establish that for any $\hat{\delta} \in (0, 1/3)$ there exist $\Upsilon_0 > 0$, $\varepsilon_0 > 0$ and $C > 0$ such that for all positive $L$ and $\varepsilon$ satisfying $0 < L \varepsilon \leq \varepsilon_0$ and $\Upsilon \leq \Upsilon_0$ we have

$$B_{\alpha}^\#(\Upsilon, \varepsilon, L) \leq C \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}.$$  \hfill (9.24)

This is similar to (9.9) in the case $\# = \mathcal{G}$.

Combining (9.24) with (9.4), we have

Proposition 9.5. For any $\hat{\delta} \in (0, 1/3)$ there exist $\varepsilon_0 > 0$ and $\Upsilon_0 > 0$, such that for all positive $L$ and $\varepsilon > 0$ for which $0 < L \varepsilon \leq \varepsilon_0$, $\Upsilon \leq \Upsilon_0$, and $U \in S_r$ satisfying (2.34), it holds that

$$\sup_{0 \leq \alpha \leq \alpha_M \beta_1^{1/6}} \| (I - \Pi)(\mathcal{P}^\#_{\lambda, \varepsilon})^{-1} \|_{\mathcal{L}(L^2_{per}, H^1_{per})} \leq \frac{C}{\varepsilon} \beta_1(\varepsilon, L)^{-1/3 + \hat{\delta}}.$$
We may now proceed as in the proof of (2.35) to establish (2.38).

**Proof of Part 2** To establish (2.39) in Part 2 we use (8.21) and (8.70) to show that

$$B_{\alpha}^{D}(\gamma, \varepsilon, L) \leq C \beta_1(\varepsilon, L)^{-2/3}.$$  

Then we can continue in the same manner as in the proof of (2.36) in the first part of the theorem.

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### A. Basic Properties of the Airy Function and Wasow’s Results on $A_0$

#### A.1. Airy Function Properties

In this subsection, we summarize some of the basic properties of Airy function $\text{Ai}(z)$, and the generalized Airy function $A_0(z)$, that are being used throughout this work (see [1] for details) and establish some new inequalities satisfied by these functions. We recall that Airy function is the unique solution of

$$(D_x^2 + x)u = 0,$$

on the line such that $u(x)$ tends to 0 as $x \to +\infty$ and $\text{Ai}(0) = 1/\left(3^{\frac{2}{3}} \Gamma\left(\frac{2}{3}\right)\right)$.

Standard ODE theory shows that Airy function is entire and strictly decreasing on $\mathbb{R}_+$, but has an infinite number of zeros in $\mathbb{R}_-$. Airy function satisfies different asymptotic expansions as $|z| \to \infty$ depending on $\arg z$. We bring two of them here

1. \( \text{Ai}(z) = \frac{1}{2} \pi^{-\frac{1}{2}} z^{-1/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \left(1 + \mathcal{O}(|z|^{-3/2})\right), \quad \text{for } |\arg z| < \pi, \quad (A.1a) \)

2. \( \text{Ai}(-z) = \pi^{-\frac{1}{2}} z^{-1/4} \sin\left(\frac{2}{3} z^{3/2} + \frac{\pi}{4}\right) \left(1 + \mathcal{O}(|z|^{-3/2})\right), \quad \text{for } |\arg z| < \frac{2\pi}{3}, \quad (A.1b) \)

Moreover the $\mathcal{O}(|z|^{-3/2})$ estimate is, for any $\hat{\delta} > 0$, uniform when $|\arg z| \leq \pi - \hat{\delta}$ in (A.1a) or $|\arg z| \leq 2\pi/3 - \hat{\delta}$ in (A.1b). In particular, $\text{Ai}(z)$ is rapidly decreasing at $\infty$ if $z$ belongs to a sector $|\arg z| \leq \frac{\pi}{3} - \hat{\delta}$, with $\hat{\delta} > 0$.

The following moment estimates are needed in Subsection 6.1:
**Proposition A.1.** Let $\langle \lambda \rangle := \sqrt{1 + |\lambda|^2}$. For every $\mu_0 > 0$ there exists $C > 0$ such that, for any $\lambda$ with $\Re \lambda \leq \mu_0$, we have

$$
\int_0^\infty x^{2k} |\mathrm{Ai}(e^{i\pi/6}[x+i\lambda])|^2 \, dx \leq C \langle \lambda \rangle^{-k-1} e^{-\frac{4}{3} \Re((e^{i2\pi/3}\lambda)^{3/2})}, \quad \text{for } 0 \leq k \leq 4, \quad (A.2)
$$

and

$$
\int_0^\infty x^s |\mathrm{Ai}(e^{i\pi/6}[x+i\lambda])| \, dx \leq C \langle \lambda \rangle^{-3+2s} e^{-\frac{2}{3} \Re((e^{i2\pi/3}\lambda)^{3/2})} \quad \text{for } 0 \leq s \leq 3. \quad (A.3)
$$

**Proof.** If $|\lambda| \leq 3\mu_0$ then, by (A.1a), all the estimates of the proposition are satisfied for some $C = C(\mu_0)$. Hence, we can consider from now on the case where $|\lambda| > 3\mu_0$. Note that by interpolation, it is sufficient to consider $k \in \mathbb{Z}$.

**Proof of (A.2) for $k = 0$.**

Let

$$
u(x) = \frac{\mathrm{Ai}(e^{i\pi/6}[x+i\lambda])}{\mathrm{Ai}(e^{i2\pi/3}\lambda)} \quad (A.4)
$$

which is well-defined for

$$
\mathcal{V}(\mu_0) := \{\Re \lambda \leq \mu_0\} \cap \{|\lambda| > 3\mu_0\},
$$

since the denominator can vanish only when $\arg \lambda = \pi/3$.

It can be easily verified that

$$
\begin{cases}
-u'' + (ix - \lambda)u = 0 & \text{for } x > 0 \\
u(0) = 1.
\end{cases} \quad (A.5)
$$

Further we let

$$
w = u - e_\lambda(x) \quad \text{with } e_\lambda(x) := e^{-(\lambda)^{1/2}x}, \quad (A.6)
$$

where $(-\lambda)^{1/2}$ is well-defined for $\lambda \in \mathcal{V}(\mu_0)$ as

$$
-\pi + \arccos\left(\frac{1}{3}\right) < \arg(-\lambda) < \pi - \arccos\left(\frac{1}{3}\right).
$$

Note that for $\lambda \in \mathcal{V}(\mu_0)$

$$
\Re(-\lambda)^{1/2} \geq |\lambda|^{1/2} \sin\left(\frac{1}{2} \arccos\left(\frac{1}{3}\right)\right) \geq |3\mu_0|^{1/2} \sin\left(\frac{1}{2} \arccos\left(\frac{1}{3}\right)\right) > 0, \quad (A.7)
$$

implying, for any $\mu_0 > 0$, the existence of $C(\mu_0) > 0$, such that, for all $\lambda \in \mathcal{V}(\mu_0)$, it holds that

$$
\|e_\lambda\|_2 \leq C(\mu_0) |\lambda|^{-\frac{1}{4}}. \quad (A.8)
$$

Substituting into (A.5) yields

$$
(\mathcal{L}_+ - \lambda)w = -ixe_\lambda(x), \quad (A.9)
$$
where $\mathcal{L}_+$ is associated with the differential operator $-d^2/dx^2 + ix$ and is defined on the domain

$$D(\mathcal{L}_+) = \{ u \in H^2(\mathbb{R}_+) \cap H^1_0(\mathbb{R}_+) \mid xu \in L^2(\mathbb{R}_+) \}.$$  

It has been established in [24, §5] that, for any $\hat{\mu} \in \mathbb{R}_+$, there exist $\nu_0(\hat{\mu}) > 0$ and $C(\hat{\mu})$ such that

$$\| (\mathcal{L}_+ - \lambda)^{-1} \| \leq C(\hat{\mu}) \text{, for } |\Im \lambda| \geq \nu_0(\hat{\mu}) \text{ and } |\Re \lambda| \leq \hat{\mu}. \quad (A.10)$$

In addition, we have

$$\| (\mathcal{L}_+ - \lambda)^{-1} \| \leq C, \text{ for } \Re \lambda \leq 0. \quad (A.11)$$

Denote by $\{ \nu_n \}_{n=1}^{\infty}$ the eigenvalues of $\mathcal{L}_+$, and recall that they are located on the ray $\arg \lambda = \pi/3$ (see [3, §2.2]). Observing that $\mathcal{V}(\mu_0)$ does not contain any eigenvalue (a consequence of the fact that $\arccos 1/3 > \pi/3$) and combining the above with

$$\text{(A.10)-(A.11)} \text{ yield the existence of } C(\mu_0) > 0 \text{ such that }$$

$$\sup_{\lambda \in \mathcal{V}(\mu_0)} \| (\mathcal{L}_+ - \lambda)^{-1} \| \leq C(\mu_0). \quad (A.12)$$

Hence, for $\lambda \in \mathcal{V}(\mu_0)$,

$$\| w \|_2 \leq C(\mu_0) \| xe_\lambda \|_2.$$  

Using (A.7), we obtain that

$$\| xe_\lambda \|_2 \leq \hat{C}(\mu_0)|\lambda|^{-3/4}, \quad (A.13)$$

and hence

$$\| w \|_2 \leq \hat{C}(\mu_0)|\lambda|^{-3/4}. \quad (A.14)$$

By the above, (A.6), and (A.8), we thus have

$$\| u \|_2 \leq C(\mu_0)|\lambda|^{-1/4}. \quad (A.15)$$

To prove (A.2) for $k = 0$, we need to establish yet an upper bound for $\text{Ai} (e^{2i\pi/3} \lambda)$, for $\lambda \in \mathcal{V}(\mu_0)$. This is an immediate consequence of (A.1a). We observe indeed that for any $\varepsilon > 0$, arg $\lambda \not\in (-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon)$ as $|\lambda| \to +\infty$ with $\Im \lambda \leq \mu_0$. This implies that, for any $\varepsilon > 0$, there exists $\lambda_0(\varepsilon, \mu_0) > 0$ such that for $|\lambda| \geq \lambda_0(\varepsilon)$ and $\Im \lambda \leq \mu_0$, arg $\left( \lambda \exp \frac{2i\pi}{3} \right) \in (-\frac{5\pi}{6} - \varepsilon, \frac{5\pi}{6} + \varepsilon)$.

Consequently, for any $\mu_0 > 0$, there exists a constant $C(\mu_0)$, such that for all $\lambda \in \mathcal{V}(\mu_0)$

$$|\text{Ai} (e^{2i\pi/3} \lambda)| \leq C(\mu_0) \text{, for } \lambda > -\frac{1}{2} e^{-\frac{3}{2} \Im (e^{i2\pi/3})^{3/2}}. \quad (A.16)$$

Together with (A.15), (A.16) yields (A.2) for $k = 0$ and $\lambda \in \mathcal{V}(\mu_0)$.

Proof of (A.2) for $k = 1$.

We begin by deriving an estimate of $xw$. To this end we observe that

$$(D^2_\lambda + ix - \lambda)(xw) = -ix^2 e_\lambda - 2i w'. $$
Following the same procedure applied in the previous proof, we need an estimate of \( w' \). To achieve this end, we first observe that
\[
\Re\langle (D_x^2 + ix - \lambda)w, w \rangle = \|w'\|^2 - \Re \lambda \|w\|^2/2,
\]
which leads to the estimate
\[
\|w'\|^2 \leq \mu_0 \|w\|^2 + \|xe\| \|w\|_2.
\]
Implementing (A.13) and (A.14) leads to
\[
\|w'\|^2 \leq C(\mu_0) |\lambda|^{-3/4}. \tag{A.17}
\]
We observe in addition that
\[
\|x^2e\|_2 \leq C(\mu_0) |\lambda|^{-3/4}.
\]
Proceeding as in the proof of (A.2) for \( k = 0 \), we obtain (A.2) for \( k = 1 \).

**Proof of (A.2) for \( k = 2 \).**

In this case the approximation of \( u \) by \( e_\lambda \) is unsatisfactory. To improve it, in light of (A.9), we solve in \( H^2(\mathbb{R}^+) \) the problem
\[
\begin{cases}
\left( -\frac{d^2}{dx^2} - \lambda \right) f_\lambda = -ixe_\lambda & \text{for } x > 0, \\
f_\lambda(0) = 0.
\end{cases} \tag{A.18}
\]
We look for \( f_\lambda \) in the form \( f_\lambda = p_\lambda e_\lambda \) which means that \( p_\lambda \) must satisfy
\[
p_\lambda''(x) - 2(-\lambda)^{3/2} p_\lambda'(x) = ix, \ p_\lambda(0) = 0.
\]
We search for a polynomial solution. A simple computation leads to
\[
f_\lambda = p_\lambda e_\lambda \text{ with } p_\lambda(x) = \frac{i}{4} \left( (-\lambda)^{-1/2} x^2 - \lambda^{-1} x \right). \tag{A.19}
\]
We now write
\[
u = e_\lambda + w = e_\lambda + f_\lambda + w_1, \tag{A.20}
\]
to obtain
\[
(L_+ - \lambda)w_1 = -ixe_\lambda = -ixe_\lambda. \tag{A.21}
\]
In order to prove (A.2) \(( k = 2) \) we observe first that
\[
\sum_{\ell=0}^3 <\lambda >^{\ell/2} \|x^\ell f_\lambda\| \leq C <\lambda >^{-7/4}. \tag{A.22}
\]
Hence, it remains necessary to obtain an estimate of \( x^2w_1 \) in \( L^2 \). Here we use the fact that (A.9) is similar to (A.21), the only difference being that the right-hand-side is given by \(-ixe_\lambda \) instead of \(-ixe_\lambda \). Note that \( \|xf_\lambda\|_2 \) is much smaller than \( \|xe_\lambda\|_2 \) as \( |\lambda| \to \infty \).
By (A.12), (A.21) and (A.22) (for $\ell = 1$), we then have

$$\|w_1\|_2 \leq C |\lambda|^{9/4},$$

(A.23)

which is significantly smaller than the bound provided by (A.14) for $w$. We continue as in the case $k = 1$. We first use the identity

$$\Re \langle w_1, (L_+ - \lambda)w_1 \rangle = \|w_1'\|^2_2 - \mu \|w_1\|^2_2,$$

to conclude with the aid of (A.21), (A.22), (A.23), and recalling that $\mu \leq \mu_0$,

$$\|w_1'\|_2 \leq C |\lambda|^{9/4}.$$

(A.24)

Then we write

$$(L_+ - \lambda)(xw_1) = x(L_+ - \lambda)w_1 - 2w_1' = -ix^2 f_\lambda - 2w_1'.$$

(A.25)

Combining (A.24), (A.21), and (A.12) yields

$$\|xw_1\|_2 \leq C |\lambda|^{9/4}.$$

(A.26)

From (A.26) and (A.23) we get, in addition, that

$$\|x^{1/2}w_1\|_2 \leq \|xw_1\|_2^{1/2} \|w_1\|_2^{1/2} \leq C |\lambda|^{9/4}.$$  

(A.27)

Next, we write

$$\Re \langle xw_1, (L_+ - \lambda)(xw_1) \rangle = \|(xw_1)'\|^2_2 - \mu \|xw_1\|^2_2,$$

to conclude from (A.25) and (A.26) that

$$\|(xw_1)'\|_2 \leq C |\lambda|^{9/4}.$$  

(A.28)

Upon writing

$$(L_+ - \lambda)(x^2w_1) = x^2(L_+ - \lambda)w_1 - 2(xw_1)' = -ix^3 f_\lambda - 2(xw_1)'$$

we use (A.28), (A.21), and (A.12) to obtain

$$\|x^2w_1\|_2 \leq C |\lambda|^{9/4},$$

(A.29)

which can be used, together with (A.20) and (A.6) to obtain (A.2) for $k = 2$.

In a similar manner we obtain the estimates for $k \in \{3, 4\}$. We note in particular that

$$\|x^4w_1\|_2 + \|(x^4w_1)'\|_2 \leq C |\lambda|^{9/4},$$
and hence, by Sobolev embeddings,
\[ \| x^4 w_1 \|_\infty \leq \frac{C}{|\lambda|^{9/4}}. \]

Combining the above with (A.6) and (A.20) yields, for \( k \in [0, 4] \),
\[ \| x^k u \|_\infty \leq \frac{C}{|\lambda|^{k/4}}. \quad (A.30) \]

**Weighted \( L^1 \) estimates.**
Recall that in deriving (A.2) we needed to establish that, for \( k = 0, \ldots, 4 \), it holds that
\[ \| x^k u \|_{L^2} \leq C < \lambda >^{2k+1/2}. \quad (A.31) \]

By interpolation it is enough to treat the case when \( k \) is an integer. Then we use Hölder inequality and (A.31) for \( k = s \) and \( k = s + 1 \) to obtain, for \( s \leq 3 \),
\[ \| x^s u \|_{L^1} \leq \left( \| x^{s+1} u \|_{L^2} < \lambda >^{-1/2} \right) \| x^s u \|_{L^2} \quad (A.32) \]

We then recover (A.3) by using (A.16).

\[ \square \]

### A.2. Definition of \( A_0 \) and the Locus of its Zeroes

Let \( A_0 : \mathbb{C} \to \mathbb{C} \) be given (see (6.9)) by the holomorphic extension to \( \mathbb{C} \) of
\[ \mathbb{R} \ni z \mapsto A_0(z) = e^{i\pi/6} \int_{z}^{+\infty} \text{Ai} (e^{i\pi/6} t) \, dt. \]

To use the results of Wasow [43] (see [43, Eq. (39)]) and justify this holomorphic extension we observe the following relation:

**Lemma A.2.**
\[ A_0(z) = -\psi(-e^{i\pi/6} z), \quad (A.33) \]

where \( \psi \) is the holomorphic extension of the real function
\[ \mathbb{R} \ni x \mapsto \psi(x) := \int_{-\infty}^{x} \text{Ai} (-t) \, dt = \frac{1}{3} + \int_{0}^{x} \text{Ai} (-t) \, dt. \quad (A.34) \]

It has been proved by Wasow in [43, Section 3] that the zeroes of \( \psi \) are all located in the sector \( |\arg z| < \frac{\pi}{6} \).

**Proposition A.3.** The zeroes of \( \psi \) belongs to \( -\pi/6 < \arg z < \pi/6 \). Moreover \( \psi \) has no real zeroes.
Sketch of the proof. To establish that result is a combination of the argument principle and Rouché’s. The change of arg $\psi$ is estimated along the path $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ where, for some $R > 0$

$$\Gamma_1 = \{ z = re^{i\pi/6} \mid 0 < r \leq R \} \quad \Gamma_2 = \{ z = Re^{i\theta} \mid \pi/6 < \theta < \pi \} \quad \Gamma_3 = [-R, 0].$$

Since $\text{Ai}(z)$ is real and positive for $z \in \mathbb{R}_+$ it follows that arg $\psi$ does not change along $\Gamma_3$. Along $\Gamma_2$ one uses (A.1), with a bound on the remainder. Finally, to estimate $\Delta \arg \psi$ along $\Gamma_1$ one uses the power series of $\text{Ai}(t)$, for $0 < |t| < 9$, and (A.1) for $|t| > 9$. The tails of the ensuing power series of $\Re \psi$ and $\Im \psi$ are Leibniz series with terms of alternating sign and decreasing moduli. Thus, one may truncate the series into finite sums, and the remainders can be easily estimated. Once the above procedure is applied, one can establish that $|\Delta \arg \psi| < 2\pi$ and hence that $\Delta \arg \psi = 0$ along $\Gamma$. Since $\psi(z) = 0 \Rightarrow \psi(\bar{z}) = 0$, the first statement of the proposition follows.

The second statement is proved in [43, p. 199].

Corollary A.4. Let $A_0(iz) = 0$. Then, $\pi/6 < \arg z < \pi/2$.

We continue with the following result stated in [43] (Eq. (35)) which allows us to obtain additional information on the location of the zeroes of $A_0$; it is also serves as a useful tool in some of the proofs in Subsection 6.1.

Lemma A.5. Let $\tilde{\psi}(z) = \psi(-z)$. For any $0 < \hat{\delta} < \pi$ there exists $C_{\hat{\delta}} > 0$ and $r_0(\hat{\delta}) > 0$ such that for all $|z| > r_0(\hat{\delta})$ in the sector $|\arg z| < \pi - \hat{\delta}$ it holds that

$$\left| \tilde{\psi}(z) - \frac{1}{2} \pi^{-1} z^{-3/4} \exp\left(-\frac{2}{3} z^{3/2}\right) \right| \leq \frac{C_{\hat{\delta}}}{|z|^{7/4}} \exp\left(-\frac{2}{3} z^{3/2}\right).$$

(A.35)

The proof is a rather standard application of the method of steepest descent method [36, Chapter 4] and is therefore being skipped.

A.3. Asymptotics of the Zeroes

In Subsection 6.1 we also need to establish the following lemma about the asymptotic behavior of the zeroes of $A_0$. A similar statement for $\psi$ is made in [43, § 3] without a clear proof.

Proposition A.6. Let $S_0$ denote the set of points $\lambda \in \mathbb{C}$ satisfying $A_0(i\lambda) = 0$. Then, for any $R > 0$, $S_0 \cap B(0, R)$ is finite and its cardinality tends to $+\infty$ as $R$ tends to $+\infty$. In particular $S_0$ is non empty. Moreover, for any $\varepsilon > 0$, there exists $R$ such that, for $\lambda \in S_0 \cap B(0, R)^\varepsilon$,

$$|\arg \lambda - \pi/3| \leq \varepsilon.$$  

(A.36)
Proof. We apply Jensen’s formula [34, Theorem 1.7] to \( \tilde{\psi}(z) = \psi(-z) \).

\[
\log \tilde{\psi}(0) + N_{\tilde{\psi}}(R, 0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta .
\] (A.37)

In the above \( N_{\tilde{\psi}}(R, 0) \) is the counting function,

\[
N_{\tilde{\psi}}(R, 0) = \sum_{|a| < R, \psi(a) = 0} \log \left| \frac{R}{a} \right| ,
\]

where we have used the fact that all zeroes of \( \tilde{\psi} \) are simple. We recall from [43] that none of them is real, and all the zeroes of \( \tilde{\psi}' = -\text{Ai} \) lie on the negative real axis. Note that if \( S_0 = \emptyset \) then \( N_{\tilde{\psi}}(R, 0) \equiv 0 \) for all \( R > 0 \). Recall from the definition that \( \tilde{\psi}(0) = 1/3 \) and hence if we show that the right-hand-side of (A.37) is unbounded as \( R \to \infty \), we may conclude the first statement of the proposition.

It follows from (A.35) that for any \( C_0 \in (0, \frac{8}{3}) \) there exist \( \hat{\delta} > 0 \) and \( R_0 > 0 \) such that for all \( R > R_0 \) we have

\[
\int_{\pi-\hat{\delta}}^{\pi+\hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta \geq C_0 R^{3/2} .
\] (A.38)

Indeed, using Lemma A.5, we establish the existence for any \( \hat{\delta} > 0 \) of \( R_0(\hat{\delta}) \) and \( C \) such that, for all \( R \geq R_0(\hat{\delta}) \),

\[
\int_{-\pi+\hat{\delta}}^{\pi-\hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta \geq \frac{2}{3} R^{3/2}(1 - CR^{-1}) \int_{-\pi+\hat{\delta}}^{\pi-\hat{\delta}} -\cos \left( \frac{3\theta}{2} \right) \, d\theta \\
\geq \frac{8}{9} R^{3/2}(1 - CR^{-1}) \cos(3\hat{\delta}/2) .
\]

To estimate the integral for \( |\arg z| \in (\pi - \hat{\delta}, \pi) \) we write, owing to the concavity of \( R \mapsto x \mapsto \log x \),

\[
\int_{|\theta| > \pi-\hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta = 2 \int_{\pi-\hat{\delta}}^{\pi} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta \leq 2\hat{\delta} \log \left( \frac{1}{\hat{\delta}} \int_{\pi-\hat{\delta}}^{\pi} |\tilde{\psi}(Re^{i\theta})| \, d\theta \right) .
\] (A.39)

Since by (A.34) we have, for any \( 0 < \hat{\delta} < \pi/3 \) and \( R > R_0 \),

\[
\int_{\pi-\hat{\delta}}^{\pi} |\tilde{\psi}(Re^{i\theta})| \, d\theta \leq \int_{0}^{R_0} \int_{\pi-\hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| \, d\theta \, ds + \int_{R_0}^{R} \int_{\pi-\hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| \, d\theta \, ds ,
\]

where \( R_0 \) is fixed, but sufficiently large so that \( \text{Ai}(se^{i\theta}) \) obeys (A.1b) for all \( s > R_0 \). For the first term on the right-hand side there exists \( C(R_0) > 0 \) such that

\[
\int_{0}^{R_0} \int_{\pi-\hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| \, d\theta \, ds \leq C(R_0) \hat{\delta} .
\]

For the second term we use (A.1b) to obtain the rather crude estimate for \( R > R_0 \)

\[
\int_{R_0}^{R} \int_{\pi-\hat{\delta}}^{\pi} |\text{Ai}(se^{i\theta})| \, d\theta \, ds \leq C\hat{\delta}(R - R_0)e^{-\frac{3}{2} R^{3/2} \cos(3(\pi-\hat{\delta})/2)} .
\]
Consequently, for every $C_1 > 1$ there exists $R_1$, such that, for all $R > R_1$,
\[
\log \left( \frac{1}{\delta} \int_{\pi - \delta}^{\pi} |\tilde{\psi}(Re^{i\theta})| \, d\theta \right) \leq \frac{2}{3} C_1 R^{3/2} |\cos(\pi - \delta)/2| \leq C_1 R^{3/2} \delta. \tag{A.40}
\]

This implies, by (A.39), the existence, for any $0 < \hat{\delta} < \pi/3$, of $R_2 > 0$, such that for $R \geq R_2$,
\[
\int_{|\theta| > \pi - \hat{\delta}} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta \leq 2\hat{\delta}^2 C_1 R^{3/2}.
\]

Combining the above with (A.38) yields, by fixing $\hat{\delta}$ small enough, the existence of $R_3$ and $C_0 > 0$ such that for all $R > R_3$
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |\tilde{\psi}(Re^{i\theta})| \, d\theta \geq C_0 R^{3/2}.
\]

The proof shows that the above bound holds true for any $\hat{C}_0 \in (0, \frac{8}{9})$ for $R_3(\hat{C}_0)$ large enough. Hence, we get a lower bound for $N_\tilde{\psi}(R, 0)$ which implies the first statement of the proposition.

To prove (A.36) we notice that for any $\hat{\delta} > 0$, it holds by (A.35) that there exists $r_0 > 0$ such that $\tilde{\psi}$ cannot have any zeroes, for $|z| > r_0$ in the sector $|\arg z| < \pi - \hat{\delta}$.

We then observe that $A_0(i\lambda) = -\tilde{\psi}(e^{2i\pi/3}z)$.

We can now immediately draw the following conclusion:

\textbf{Corollary A.7.}
\[
\inf \Re S_0 = \vartheta_1^r > 0. \tag{A.41}
\]

\textit{A.4. Normalized Airy Functions}

We complete the appendix with some corollaries of (A.2)-(A.3) and (A.35) needed in Subsection 6.1 and with other estimates needed in Subsection 6.3.

\textbf{Proposition A.8.} Let $\Psi_\lambda \in L^2(\mathbb{R}_+)$ be defined by
\[
\Psi_\lambda(x) = \frac{\text{Ai}(e^{i\pi/6}(x + i\lambda))}{A_0(i\lambda)}. \tag{A.42}
\]

Then, for any $\hat{\delta}_1 > 0$ there exists $C > 0$ such that for all $\Re \lambda \leq \vartheta_1^r - \hat{\delta}_1$
\[
\|x^k \Psi_\lambda\|_2 \leq C < \lambda > \frac{1-2k}{4} \quad \text{for} \ k \in [0, 4], \tag{A.43a}
\]
\[
\|x^s \Psi_\lambda\|_1 \leq C < \lambda > \frac{s}{2} \quad \text{for} \ s \in [0, 3], \tag{A.43b}
\]
\[
\|x^s \Psi_\lambda\|_\infty \leq C < \lambda > \frac{s-1}{2} \quad \text{for} \ s \in [0, 4], \tag{A.43c}
\]
\[
\frac{1}{C} < \lambda > 1/2 \leq \Psi_\lambda(0) \leq C < \lambda > 1/2. \tag{A.43d}
\]
Proof. Let $|\lambda| > 3\theta_1^2$ (the proof for $|\lambda| \leq 3\theta_1^2$ follows by continuity as the denominator is bounded away from zero). As $A_0(i\lambda) = -\tilde{\psi}(e^{i2\pi/3\lambda})$ we obtain from (A.35) that there exists $C > 0$ such that for any $\delta_1 > 0$ and $|\arg\lambda + 2\pi/3| < \pi$

$$
\frac{1}{C < \lambda > 3/4} \exp \left\{ - \frac{2}{3} \Re\{e^{i2\pi/3\lambda}\}^{3/2} \right\} \leq |A_0(i\lambda)|
$$

$$
\leq \frac{C}{C < \lambda > 3/4} \exp \left\{ - \frac{2}{3} \Re\{e^{i2\pi/3\lambda}\}^{3/2} \right\},
$$

which combined with (A.2) and (A.3) yields (A.43a,b), and with the aid of (A.1) gives (A.43d). The proof of (A.43c) follows from (A.30). \qed

We also need, in Subsection 6.3 the following estimate:

**Lemma A.9.** Let $\varkappa > 0$ and, for $\theta > 0$, $\Upsilon(\theta) = \mu_0(\theta) - \varkappa$ where

$$
\mu_0(\theta) := \inf \Re(C^0).
$$

Let $F(\lambda, \theta)$ be defined by (6.48c). There exists $C(\varkappa) > 0$ such that, for all $\theta > 0$,

$$
\begin{align*}
\sup_{\Re\lambda \leq \Upsilon(\theta)} \left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| &\leq C(\varkappa)(1 + \theta). 
\end{align*}
$$

(A.44)

**Proof.** Let $R > 0$ be determined later.

**The case** $|\lambda| < R$.

Let $\theta_1(\varkappa)$ satisfy (see (6.63))

$$
\sup_{\theta > \theta_1(\varkappa)} \mu_0(\theta) \leq \Re v_1 + \frac{\varkappa}{2}.
$$

By the continuity of $(\lambda, \theta) \mapsto A_0(i\lambda)/F(\lambda, \theta)$ on $B(0, R) \times [0, \theta_1(\varkappa)]$, we have

$$
\sup_{\{\Re\lambda \leq \Upsilon(\theta)\} \cap B(0, R), 0 \leq \theta \leq \theta_1} \left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq C(\varkappa, R).
$$

(A.45)

In the case $\theta > \theta_1(\varkappa)$ we write, as in Subsection 6.3 (see (6.64))

$$
\theta F(\lambda, \theta) - \text{Ai} (e^{i2\pi/3\lambda}) = \theta \int_{\mathbb{R}_+} e^{-\theta x} \left[ \text{Ai} (e^{i2\pi/3\lambda} + e^{i\pi/6} x) - \text{Ai} (e^{i2\pi/3\lambda}) \right] dx,
$$

and deduce (see (6.65)) that, for $|\lambda| \leq R$, we have

$$
|\theta F(\lambda, \theta) - \text{Ai} (e^{i2\pi/3\lambda})| \leq C(R)\theta^{-1}.
$$

We may now use the fact that $\Upsilon(\theta) < \Re v_1 - \varkappa/2$ to get the existence of $\theta(\varkappa, R) \geq \theta_1(\varkappa)$ and $C(\varkappa, R) > 0$ such that, for $\theta \geq \theta(\varkappa, R)$, $|\lambda| \leq R$ and $\Re\lambda \leq \Upsilon(\theta)$ we have

$$
|F(\lambda, \theta)| \geq \frac{C(\varkappa, R)}{1 + \theta}.
$$
For \( \theta \in [\theta_1(\varpi), \theta(\varpi, R)] \), we use the continuity of \(|F(\lambda, \theta)|\) to get a uniform lower bound for it. Consequently, we obtain that
\[
\sup_{|R\lambda \leq \gamma(\theta)| \cap |\lambda| \leq R} \left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq C(\varpi, R)(1 + \theta). \tag{A.46}
\]

**The case** \(|\lambda| \geq R\).

If \(|\lambda| \geq R\) we may use (6.68) which reads as
\[
\left| \frac{[\theta + (-\lambda)^{1/2}]F(\lambda, \theta)}{\text{Ai}(e^{12\pi/3\lambda})} - 1 \right| \leq C |\lambda|^{-1/4}.
\]
to obtain for \( R \) large enough with the aid of (A.1), (A.33), and (A.35) that,
\[
\left| \frac{A_0(i\lambda)}{F(\lambda, \theta)} \right| \leq \left| \frac{A_0(i\lambda)}{\text{Ai}(e^{12\pi/3\lambda})} \right| \left| \text{Ai}(e^{12\pi/3\lambda}) \right| F(\lambda, \theta) \leq C \frac{\theta + |\lambda|^{1/2}}{|\lambda|^{1/2}}.
\]
Combining the above with (A.45) and (A.46) yields (A.44). \( \square \)

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