On locally finite-dimensional traces

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Abstract
We partially resolve two open questions on approximation properties of traces on simple $C^*$-algebras. We answer a question raised by Nate Brown by showing that locally finite-dimensional (LFD) traces form a convex set for simple $C^*$-algebras. We prove that all the traces on the reduced $C^*$-algebra $C^*_r(\Gamma)$ of a discrete amenable ICC group $\Gamma$ are LFD, and conclude that $C^*_r(\Gamma)$ is strong-NF in the sense of Blackadar–Kirchberg in this case. This partially answers another open question raised by Brown.

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1 | INTRODUCTION

In his monumental paper [10], Alain Connes introduced the notion of amenable trace (cf., Definition 1 below) under the title of “hyper_TRACE” and proved that a type $\text{II}_1$-factor is hyperfinite if and only if its unique tracial state is amenable. Exploiting this fact, Connes proves in [11] the nonexistence of finite summable Fredholm modules on reduced group $C^*$-algebra of a free group. In the same vein, Eberhard Kirchberg defined liftable traces on $C^*$-algebras and proved that a discrete group with Kazhdan property $(T)$ is residually finite if and only if the canonical trace of full group $C^*$-algebra is liftable [15]. Also, he proves that a general discrete group $\Gamma$ has factorization property if and only if its canonical trace of full group algebra, $C^*(\Gamma)$, is liftable. Eventually, it was Nate Brown who recognized the central role of amenable tracial states and elaborated on how they could be employed in studying finite-dimensional approximation properties of $C^*$-algebras, including a characterization of the central notion of liftability [5]. Based on the pioneering work of Sorin Popa [21], Brown also introduced and studied the notions of quasi-diagonal and locally finite-dimensional (LFD) traces and their uniform counterparts, but unlike his thorough analysis of amenable traces, he gave no characterization of quasi-diagonality or local finite dimensionality.
In their celebrated paper [2], Blackadar and Kirchberg examined the class of $C^*$-algebras that can be written as a generalized inductive limit of finite-dimensional $C^*$-algebras. In particular, they introduced the class of strong-NF algebras, where the connecting maps are complete order embeddings, and characterized them as exactly the class of nuclear and inner-QD $C^*$-algebras, for example, every RFD $C^*$-algebra is inner-QD; hence, any nuclear and RFD algebra is strong-NF. They further showed that for separable $C^*$-algebras, inner quasi-diagonality is equivalent to the existence of a separating sequence of irreducible quasi-diagonal representations [4]. For this reason, inner quasi-diagonality is not homotopy invariant and does not pass to subalgebras or quotients in general.†

One of the central problems in approximation theory of traces on $C^*$-algebras is the question that in what circumstances all amenable traces are quasi-diagonal [29, 7.3.2], [6]. The first major breakthrough was a result of Tikuisis–White–Winter, who showed that faithful traces on separable nuclear $C^*$-algebras in the UCT class are quasi-diagonal [26]. In particular, as examples in [8] and [24] suggest, it is plausible that for a discrete group $G$, the full $C^*$-algebra $C^*(G)$ is inner quasi-diagonal (inner-QD), provided that $G$ is amenable. A partial answer to this question is provided by Proposition D below.

The main objective of the current paper is to give affirmative answers to some natural questions along the above lines. First, we obtain a new characterization of LFD traces, as follows.

**Theorem A.** A tracial state $\tau$ on a simple $C^*$-algebra $A$ is LFD if and only if for every $\epsilon > 0$, every positive integer $n \geq 1$, and every $a_1, \ldots, a_n \in A$, there are disjoint irreducible representations $\pi_j : A \to B(H_j)$ and finite rank projections $p_j \in B(H_j)$, for $1 \leq j \leq n$, such that

$$\left| \tau(a_i) - \frac{\sum_{j=1}^n Tr(p_j \pi_j(a_i))}{\sum_{j=1}^n Tr(p_j)} \right| < \epsilon,$$

for each $1 \leq i \leq n$, and

$$\max_{1 \leq j \leq n} \|[(p_j, \pi_j(a_i))\| < \epsilon,$$

where $Tr$ is the canonical trace on the matrix algebra $p_j B(H_j) p_j$, that is, the sum of eigenvalues counted with multiplicities.

This is then used to prove that for simple $C^*$-algebras, these tracial states form a convex set, partially answering an open question raised by Brown [5, page 29].

**Corollary B.** The set of all LFD traces on a simple separable $C^*$-algebra form a convex set.

To highlight the importance of LFD traces, we observe that if for each separable $C^*$-algebra $A$, uniformly amenable QD traces are LFD, it follows that hyperfinite type $\text{II}_1$-factor $R$ is QD, something which seems to be wide-open at present.

†For the lack of homotopy invariance and passing to subalgebras, consider the cone $C_0((0,1], \varnothing_n)$ of the Cuntz algebra, and note that its irreducible representations factor through point evaluations on $(0,1]$ and so cannot be QD, since $\varnothing_n$ is not QD, also the cone embeds into an AF algebra that is inner-QD (actually strong QD) [19]. For the lack of passing to quotients, consider $C^*(F_2)$, the maximal group $C^*$-algebra of free group on two generators, which is RFD [9] and $\varnothing_2$ arises as its quotient.
Proposition C. The following statements are equivalent:

(i) For every separable C*-algebra C, \( \text{UAT}(C) \cap \text{AT}_{\text{QD}}(C) \subset \text{AT}_{\text{LFD}}(C) \).

(ii) The hyperfinite-type II_1-factor \( \mathcal{R} \) is quasi-diagonal, and for every unital inclusion of separable C*-algebras \( B \subset A \) if \( \tau \in \text{UAT}(A) \cap \text{AT}_{\text{LFD}}(A) \), then \( \tau|_B \in \text{AT}_{\text{LFD}}(B) \).

Our last result gives a characterization of inner quasi-diagonality of the reduced group C*-algebra of a discrete group, giving an inner-QD analog of the celebrated Rosenberg conjecture, already settled by Tikuisis–White–Winter in [26]. Indeed, we prove a little more.

Proposition D. Let \( \Gamma \) be a discrete amenable ICC group, then all traces on the reduced C*-algebra \( C^*_r(\Gamma) \) are automatically LFD, and, in particular, \( C^*_r(\Gamma) \) is strong-NF.

2 | PRELIMINARIES AND CONVENTIONS

In this section, we recall a number of preliminary facts on approximation properties of traces and inner quasi-diagonality, needed in the rest of this paper. In this paper, traces are always meant to be tracial states and \( T(A) \) denotes the collection of all traces. We write \( \mathbb{M}_n \) for the C*-algebra of all \( n \times n \) complex matrices with canonical non-normalized tracial functional \( T \). We denote the normalized trace by \( tr \) or more specifically by \( tr_n \). For \( x \in \mathbb{M}_n \), we write \( \|x\|_2 := tr_n(x^*x)^{1/2} \). Also, for a C*-algebra \( A \), we denote the positive cone of \( A \) by \( A^+ \).

When two representations \( \pi: A \to B(H) \) and \( \sigma: A \to B(K) \) are (unitarily) equivalent via a unitary \( v: H \to K \), we write \( \pi \sim_v \sigma \). For a representation \( \pi: A \to B(H) \) and projection \( p \in B(H) \), \( A_p \) is the set of all \( a \in A \) with \( p\pi(a) = \pi(a)p \).

For a nondegenerate representation \( \rho \), we denote the central cover of \( \rho \) by \( c(\rho) \). This is defined to be the central projection \( c(\rho) := 1_{A^{**}} - e_{\rho} \), where \( e_{\rho} \) is the unit of the W*-algebra \( \ker(\rho)^{**} \). It is known that \( c(\rho)A^{**} \) is isomorphic to \( \rho(A)^{''} \) and that \( c(\rho) \) determines \( \rho \) up to quasi-equivalence [7, Page 6]. We say that an irreducible representation \( \rho \) is GCR if its range contains \( \mathbb{K}(H_{\rho}) \). Finally, we say that a projection \( p \in A^{**} \) is in the socle of \( A^{**} \) if the cut-down \( pA^{**}p \) is finite-dimensional. Note that every finite rank projection in \( B(H_{\rho}) \) actually lives in the socle of \( A^{**} \).

In the rest of this section, we recall the definition of certain classes of traces and use the same notations as of the ones used by Brown in [5].

Definition 1. A trace \( \tau \) on a C*-algebra \( A \) is amenable if there exists a net of contractive completely positive (c.c.p.) maps \( \varphi_n: A \to \mathbb{M}_{k(n)} \) such that

\[
\lim_{n \to \infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\|_2 = 0, \quad \tau(a) = \lim_{n \to \infty} tr_{k(n)} \circ \varphi_n(a),
\]

for \( a, b \in A \). An amenable trace \( \tau \) is called QD (quasi-diagonal) if, moreover,

\[
\lim_{n \to \infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0,
\]

for \( a, b \in A \), in the operator norm. Finally, an amenable trace \( \tau \) on a C*-algebra \( A \) is called LFD if there are c.c.p. maps \( \varphi_n: A \to \mathbb{M}_{k(n)} \) with

\[
d(a, A_{\varphi_n}) \to 0, \quad \tau(a) = \lim_{n \to \infty} tr_{k(n)} \circ \varphi_n(a),
\]

for \( a \in A \), where \( A_{\varphi_n} \) is the multiplicative domain of \( \varphi_n \).
When \( A \) is unital, in the above definition, one could always arrange the required maps to be unital, and when \( A \) is separable, there is a sequence (instead of a net) witnessing the above approximations. The sets of amenable, QD, and LFD traces are denoted by \( AT(A) \), \( AT(A)_{QD} \), and \( AT(A)_{LFD} \), respectively.

Next, we recall the notion of QD \( C^* \)-algebras, introduced and studied first by Javier Thayer [25] and later by Norberto Salinas [23] and Dan Voiculescu [27, 28]. It includes the natural subclass of inner-QD \( C^* \)-algebras of Blackadar–Kirchberg [3].

**Definition 2.** A separable \( C^* \)-algebra \( A \) is QD (quasi-diagonal) if there exists a sequence of c.c.p. maps \( \varphi_n : A \to M_k(n) \) such that

\[
\lim_{n \to \infty} \| \varphi_n(ab) - \varphi_n(a)\varphi_n(b) \| = 0, \quad \lim_{n \to \infty} \| \varphi_n(a) \| = \| a \|,
\]

for \( a, b \in A \), and it is called inner quasi-diagonal if for each \( \epsilon > 0 \), each \( n \geq 1 \), and every \( x_1, \ldots, x_n \in A \), there is a projection \( p \) in the socle of \( A^* \) with \( \| px_j p \| > \| x_j \| - \epsilon \) and \( \| [p, x_j] \| < \epsilon \), for each \( 1 \leq j \leq n \).

First, let us recall a characterization of inner quasi-diagonality, due to Blackadar and Kirchberg (cf., [7, Corollary 11.3.7]).

**Proposition 2.1.** A separable \( C^* \)-algebra \( A \) is inner-QD if and only if there is a sequence of c.c.p. maps \( \varphi_n : A \to M_k(n) \) such that \( \| a \| = \lim \| \varphi_n(a) \| \) and \( d(a, A_{\varphi_n}) \to 0 \), for \( a \in A \).

For a free ultrafilter \( \omega \) on the set of natural numbers, let \( Q_\omega \) be the ultrapower of the universal UHF algebra \( Q \). Let \( q_\omega : \ell^\infty(Q) \to Q_\omega \) be the canonical quotient map and define the canonical trace \( \tau_{Q_\omega} \) on \( Q_\omega \) by

\[
\tau_{Q_\omega}(q_\omega(a_1, a_2, \ldots)) = \lim_{n \to \omega} \tau_Q(a_n).
\]

This is the unique tracial state on \( Q_\omega \) ([20, Theorem 8]). A c.c.p. map \( \Phi : A \to Q_\omega \) is said to be liftable if there is a c.c.p. map \( \hat{\Phi} : A \to \ell^\infty(Q) \) such that \( q_\omega \circ \Phi = \hat{\Phi} \).

Next, we recall further characterizations of QD \( C^* \)-algebras and traces in terms of liftability (see [26, Proposition 1.4], [16, Theorem 4], and [14, Proposition 3.4]).

**Proposition 2.2.** For a separable \( C^* \)-algebra \( A \),

(i) \( A \) is quasi-diagonal if and only if there is a liftable *-monomorphism \( \Phi : A \to Q_\omega \).

(ii) If \( A \) is unital, then \( A \) is inner QD if and only if there is a sequence of u.c.c.p maps \( \varphi_n : A \to M_k(n) \) with irreducible minimal Stinespring’s dilation such that the induced map \( \Phi : A \to \prod M_k(n) / \sum M_k(n) \) is a faithful *-homomorphism.

(iii) A trace \( \tau \) on \( A \) is QD if and only if there is a liftable *-homomorphism \( \Phi : A \to Q_\omega \) with \( \tau = \tau_{Q_\omega} \circ \Phi \).

We have the following de facto result which we omit its straightforward proof (using Propositions 2.1 and 2.2).
Proposition 2.3. Let $A$ be a separable $C^*$-algebra admitting a faithful LFD trace, then $A$ is inner-QD. Moreover, any unital, inner-QD $C^*$-algebra has at least one LFD trace.

By a well-known result of Voiculescu, the cone of every $C^*$-algebra is quasi-diagonal (more is true, as quasi-diagonality is homotopy invariant). An analogous result in the context of QD traces is known as “Gabe’s order-zero quasi-diagonality” (cf., [6, Proposition 3.2]).

Proposition 2.4. Let $A$ be a separable $C^*$-algebra and $\tau \in \text{AT}(A)$, then there is a c.c.p. order-zero map $\Phi : A \to Q_\omega$, which is liftable and $\tau(a) = \tau_{Q_\omega}(\Phi(a)\Phi(1)^{n-1})$ for $n = 1, 2, \ldots$ and $a \in A$. In particular, every amenable trace on the cone of $A$ is quasi-diagonal.

3 | PROOF OF THE MAIN RESULTS

In this section, we show that the set of LFD traces is convex. For this, we need a characterization of LFD traces in terms of irreducible representations (to control the multiplicative domain of the underlying c.c.p. maps). First we recall a result of Blackadar–Kirchberg [3], which gives a way to calculate the distance to the multiplicative domain (cf., [7, Proposition 11.3.6]).

Lemma 3.1. For a separable $C^*$-algebra $A$, let $p \in A^{**}$ be in the socle and $A_p$ be the multiplicative domain of the map $a \mapsto p a p$. Then $d(a, A_p) = \| [a, p] \|$, for each $a \in A$.

Next, let us prove the first main result of this paper.

Proof of Theorem A. Let $\tau$ be a LFD trace, and for $\epsilon > 0$, and contractions $a_1, \ldots, a_n \in A$, choose positive integer $N \geq 1$ and a c.c.p. map $\varphi : A \to M_N$ with

$$d(a_i, A_\varphi) < \epsilon/4, \quad \left| \frac{\tau(a_i) - \text{Tr}(\varphi(a_i))}{N} \right| < \epsilon,$$

for $1 \leq i \leq n$. Choose contractions $b_1, \ldots, b_n \in A_\varphi$ such that $\|a_i - b_i\| < \epsilon/2$. Since the restriction $\varphi|_{A_\varphi}$ of $\varphi$ is a finite-dimensional representation, it decomposes into a direct sum of finitely many irreducible representations of finite dimension say, $\varphi|_{A_\varphi} = \sigma_1 \oplus \cdots \oplus \sigma_r$, by [17, Theorem 5.5.1] there are irreducible representations $\tilde{\sigma}_i : A \to B(H_i)$ and closed vector subspaces $H'_i \subset H_i$ which are invariant for $\sigma_i(A_\varphi)$ such that the restriction of $\tilde{\sigma}_i$ to $A_\varphi$ and Hilbert space $H'_i$ is unitarily equivalent to $\sigma_i$. After obvious identifications, let $p_i$ be the corresponding orthogonal projection of $H_i$ onto $H'_i$ which has a finite rank as $\text{dim}H_i$ is finite. Since $A$ is simple, each $\tilde{\sigma}_i$ is non-GCR, hence by [3, Theorem A.2], there are uncountably many disjoint irreducible representations say $\pi_j$ with the same kernel as $\tilde{\sigma}_i$ (hence approximate unitarily equivalent by Voiculescu’s theorem [12, Theorem II.5.8.]). Let $u_i$ be a unitary operator implementing $\| \tilde{\sigma}_i(b_j) - u_i^* \pi_j(b_j) u_i \| < \epsilon/3$ for $j = 1, 2, \ldots, n$. Put $q_i := u_i p_i u_i^*$, then

$$\left| \frac{\tau(b_j) - \sum_{i=1}^r \text{Tr}(q_i \pi_i(b_j))}{\sum_{i=1}^r \text{Tr}(q_i)} \right| < 3\epsilon,$$

$$\max_i \| [q_i, \pi_i(b_j)] \| < \epsilon,$$

for $j = 1, 2, \ldots, n$. 

Conversely, let such representations $\pi_j$ and projections $p_j$ exist and put

$$N := \sum_{j=1}^{n} Tr(p_j), \quad \varphi(x) := \bigoplus_{j=1}^{n} p_j \pi_j(x) p_j, \quad p := p_1 + \cdots + p_n.$$  

Let us identify each $B(H_j) = \pi_j(A)''$ with a unique summand of $A''$, then each $p_j$ lives in the socle of $A''$. Since the representations $\pi_j$ are disjoint, these summands are different; hence, the projections $p_j$ are orthogonal and sum to a projection $p$ in the socle of $A''$. Now $A_\varphi$ consists of those elements commuting with projections $p_j$ and hence with $p$, therefore, $A_\varphi \subset A_p$. By Lemma 3.1,

$$d(a, A_p) = ||[a, p]|| = \max_{1 \leq j \leq n} ||[p_j, \pi_j(a)]||,$$

and we are done. \qed

Blackadar and Kirchberg introduced the notion of pure matricial states in [4], here we need an analogous notion.

**Definition 3.** Let $A$ be a $C^*$-algebra, $\mathcal{G} \subset A$ be a finite set and $\varepsilon > 0$. We call $f \in A^*$ an $(\mathcal{G}, \varepsilon)$-ATS (approximate tracial state) if there are finitely many disjoint irreducible representations $\sigma_i : A \to B(H_i)$, $i = 1, 2, \ldots, n$ and finite rank projections $p_i \in B(H_i)$, such that $f(a) = \sum_i Tr(p_i \sigma_i(a)) / \sum_i Tr(p_i)$, and $\max_i ||[p_i, \sigma_i(x)]|| < \varepsilon$, for every $a \in A$ and $x \in \mathcal{G}$.

The following lemma is the key technical tool to prove convexity of $AT_{LFD}$.

**Lemma 3.2.**

(i) For disjoint irreducible representations $\sigma_i : A \to B(H_i)$, $i = 1, 2, \ldots, n$, and finite rank projections $p_i \in B(H_i)$ as above, the map

$$\varphi : A \to B(\bigoplus_i p_iH_i); \quad \varphi(a) := \bigoplus_{i=1}^{n} p_i \sigma_i(a) p_i \quad (a \in A),$$

is c.c.p. with $d(x, A_\varphi) < \varepsilon$, for all $x \in \mathcal{G}$.

(ii) If $A$ is simple and $f$ and $g$ are $(\mathcal{G}, \varepsilon)$-ATS, for $\delta > 0$ there is a $(\mathcal{G}, \varepsilon + \delta)$-ATS, $h$, with $|h(x) - \frac{1}{2}(f(x) + g(x))| < \delta$, for all $x \in \mathcal{G}$.

**Proof.** The first assertion of part (i) directly follows from the definition, and the second follows from Lemma 3.1. For part (ii), let $f(x) = \frac{\sum_i Tr(p_i \sigma_i(x))}{\sum_i Tr(p_i)}$ and $g(x) = \frac{\sum_r Tr(p'_{r} \sigma'_{r}(x))}{\sum_r Tr(p'_{r})}$, with $\max_i ||[p_i, \sigma_i(x)]|| < \varepsilon$, $\max_r ||[p'_{r}, \sigma'_{r}(x)]|| < \varepsilon$, for $x \in \mathcal{G}$ and $i = 1, 2, \ldots, n$, $r = 1, 2, \ldots, n'$. Set $d := \sum_i Tr(p_i)$ and $d' := \sum_r Tr(p'_{r})$. Then,

$$\frac{1}{2}(f(x) + g(x)) = \frac{d' \sum_i Tr(p_i \sigma_i(x)) + d \sum_r Tr(p'_{r} \sigma'_{r}(x))}{2dd'},$$

for $x \in \mathcal{G}$. Since $A$ is simple, there are mutually disjoint irreducible representations $\{\pi_{i,j}\}$, $i = 1, \ldots, n$, $j = 1, \ldots, d'$, and $\{\pi'_{r,s}\}$, $r = 1, \ldots, n'$, $s = 1, \ldots, d$ such that $\pi_{i,j}$ are approximate unitarily equivalent to $\sigma_i$ and the same happens for $\pi'_{r,s}$ and $\sigma'_{r}$, that is, there are families $\{u_{i,j}\}$ and $\{v_{r,s}\}$ of
unitary operators such that
\[ \| \sigma_i(x) - u_{i,j}^* \pi_{i,j}(x) u_{i,j} \| < \delta/2, \quad \| \sigma'_{r,s}(x) - v_{r,s}^* \pi'_{r,s}(x) v_{r,s} \| < \delta/2, \quad (x \in \mathcal{F}). \]

Then, for \( q_{i,j} := u_{i,j} p_i u_{i,j}^* \) and \( q'_{r,s} := v_{r,s} p'_r v_{r,s}^* \),
\[ |Tr(p_i \sigma_i(x)) - Tr(q_{i,j} \pi_{i,j}(x))| < \delta Tr(q_{i,j}) = \delta Tr(p_i), \]
\[ |Tr(p'_r \sigma'_r(x)) - Tr(q'_{r,s} \pi'_{r,s}(x))| < \delta Tr(p'_r), \]
and
\[ \| [q_{i,j}, \pi_{i,j}(x)] \| \leq \| [p_i, \sigma_i(x)] \| + 2 \| \sigma_i(x) - u_{i,j}^* \pi_{i,j}(x) u_{i,j} \| < \epsilon + \delta, \]
for \( x \in \mathcal{F} \). Similarly,
\[ \| [q'_{r,s}, \pi'_{r,s}(x)] \| < \epsilon + \delta, \quad (x \in \mathcal{F}). \]

To finish the proof let us put,
\[ h(a) := \frac{\sum_{i,j} Tr(q_{i,j} \pi_{i,j}(a)) + \sum_{r,s} Tr(q'_{r,s} \pi'_{r,s}(a))}{\sum_{i,j} Tr(q_{i,j}) + \sum_{r,s} Tr(q'_{r,s})}, \]
for \( a \in A \). Then,
\[ |h(x) - \frac{1}{2} (f(x) + g(x))| \leq \frac{1}{2dd'} \left( \sum_{i,j} |Tr(q_{i,j} \pi_{i,j}(x)) - Tr(p_i \sigma_i(x))| \right. \]
\[ \left. + \sum_{r,s} |Tr(q'_{r,s} \pi'_{r,s}(x)) - Tr(p'_r \sigma'_r(x))| \right) \]
\[ \leq \frac{1}{2dd'} (\sum_{i,j} \delta Tr(p_i) + \sum_{r,s} \delta Tr(p'_r)) < \delta, \]
for all \( x \in \mathcal{F} \), as required.

The convexity now follows.

**Proof of Corollary B.** Given LFD traces \( \tau, \tau', \epsilon > 0 \), and a finite set \( \mathcal{F} \) of contractions, \( \tau \) and \( \tau' \) are \((\mathcal{F}, \epsilon/3)\)-ATS by Theorem A. Hence, \( \frac{1}{2} (\tau + \tau') \) is \((\mathcal{F}, \epsilon)\)-ATS, by Lemma 3.2(ii) for \( \delta = \epsilon/3 \), and so is LFD by Theorem A. Now the set of LFD traces is closed under taking dyadic convex combinations, and so is convex by weak*-closedness.

Next, we prove Proposition C. This follows from Gabe’s order-zero quasi-diagonality for separable subalgebras \( A \) of \( R \) and equivalence of inner quasi-diagonality for \( A \) and its cone.
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Proof of Proposition C.

(i) \(\Rightarrow (ii)\). Given an inclusion \(B \subset A\) and \(\tau \in UAT(A) \cap AT_{LFD}(A)\), since \(\tau|_B\) is quasi-diagonal and uniformly amenable, \(\tau|_B \in AT_{LFD}(B)\). Let \(C \subset R\) be a separable simple subalgebra, then by Gabe’s order-zero quasi-diagonality, \(C_0(0,1] \otimes C\) has a faithful, uniformly amenable, quasi-diagonal trace. Indeed, \(C\) has a faithful, uniformly amenable tracial state coming from the normalized tracial state of \(C\). This trace is then LFD by (i), which implies that \(C_0(0,1] \otimes C\) is inner quasi-diagonal by Corollary B. Since inner quasi-diagonality of \(C\) and \(C_0(0,1] \otimes C\) are equivalent [3, Corollary 3.11], \(C\) is quasi-diagonal. Now since all separable subalgebras of \(R\) are quasi-diagonal and every separable subalgebra embeds into a simple subalgebra (by simplicity of \(R\)), it follows that \(R\) is quasi-diagonal.

(ii) \(\Rightarrow (i)\). Given a separable \(C^*\)-algebra \(C\) and \(\tau \in UAT(C) \cap AT_{QD}(C)\), by a result of Brown [5, Proposition 3.2.2], \(\pi_{\tau}(C)''\) is hyperfinite. Since every separable, finite and hyperfinite von Neumann algebra embeds into \(R\) (see the proof of [18, Theorem A.1]), we may assume without loss of generality that \(\pi_{\tau}(C)'' \subset R\). Now let \(\pi_{\tau}(C) \subset D \subset R\) be a separable, simple, monotracial \(C^*\)-algebra, then by quasi-diagonality of \(R\), \(D\) is strongly quasi-diagonal with a unique LFD trace [5, 6.1.14]. In particular, \(\tau \in AT_{LFD}(C)\).

The next proposition illustrates that simplicity is rather technical obstruction. Indeed, the lamp-lighter group \(\Gamma := \mathbb{Z}_2 \ast \mathbb{Z}\) is an amenable, ICC group such that \(C^*(\Gamma)\) is RFD hence far from being simple but every trace on \(C^*(\Gamma)\) is LFD.

Proof of Proposition D. First note that the reduced \(C^*\)-algebra \(C^*(\Gamma)\) has a faithful irreducible representation \(\pi\) such that \(\pi(C^*(\Gamma)) \cap \mathcal{K}(H_\pi) = 0\). Indeed, \(C^*(\Gamma)'' = R\) since \(\Gamma\) is ICC, and by amenability of \(\Gamma\), \(C^*(\Gamma) = C^*(\Gamma)\), hence \(C^*(\Gamma)\) admits a faithful, \(II_1\)-factor representation. Therefore, by a result of Glimm [3, Theorem A.2], there are uncountably many non-GCR irreducible representations that are faithful (note that by separability, prime ideals and primitive ideals are the same [1, II.6.5.15]). Invoking the main result of Tikuisis–White–Winter [26, Corollary 6.1], we conclude that every trace on \(C^*(\Gamma)\) is QD. Combining this with a result of Nate Brown [5, Proposition 3.3.2], we obtain an increasing sequence \((p_n)\) of finite rank projections in \(B(H_\pi)\) such that for each \(a \in C^*(\Gamma)\), \(\|[p_n, \pi(a)]\| \to 0\), and for every trace \(\tau \in T(C^*(\Gamma))\), there is a subsequence \(p_n(k)\) with \(\tau(a) = \lim_{k \to \infty} \frac{Tr(p_n(k)\pi(a))}{Tr(p_n(k))}\). Therefore, \(\tau\) satisfies the conditions of (the proof of) Theorem A, and so, it is LFD. In particular, the (faithful) canonical trace of \(C^*(\Gamma)\) is LFD. Hence, \(C^*(\Gamma)\) is inner-QD (by Proposition 2.3) and nuclear, which is equivalent to being strong-NF.

Remark.

(i) The main result of [13] and similar argument as above reveals that for separable, unital, residually finite dimensional \(C^*\)-algebras \(A_1\) and \(A_2\) with \((\dim(A_1) - 1)(\dim(A_2) - 1) \geq 2\), the full free product, \(A = A_1 \ast A_2\) satisfies \(AT_{QD}(A) = AT_{LFD}(A)\) and every LFD trace has the form described in Theorem A.

(ii) The simplicity assumption on \(A\) is used in the proof of Theorem A to make sure that \(A\) has no GCR irreducible representation. In particular, this result is also valid in the nonsimple case, as long as this extra condition holds. The same holds for Corollary B and Proposition C. A practical instance is a nonsimple \(\mathcal{Z}\)-stable \(C^*\)-algebra, where \(\mathcal{Z}\) is the Jiang–Su algebra (as easily seen by the Kirchberg slice lemma [22, Lemma 4.1.9]).
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