Brane-cosmology dynamics with induced gravity

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The Friedmann equations for a brane with induced gravity are analyzed and compared with the standard general relativity and Randall-Sundrum cases. Randall-Sundrum gravity modifies the early universe dynamics, whereas induced gravity changes the late universe evolution. The early and late time limits are investigated. Induced gravity effects can contribute to late-universe acceleration. The conditions for this are found. Qualitative analysis is given for a range of scalar field potentials.

I. INTRODUCTION

Recently, the idea of the universe as 3+1-dimensional braneworld embedded in extra dimensions (called the bulk) has been proposed in order to solve the hierarchy problem. Matter particles live on this brane but only gravitons can travel out to the bulk. Due to the leaking of gravity into the extra dimensions, one observes higher dimensional gravity on small scales. The scale could be as large as sub-millimeter and the fundamental scale of gravity could be as low as 1 TeV \[1\]. In the Randall-Sundrum (RS) II scenario \[2\], our universe is a positive tension brane. There is one non-compact extra dimension warped by a negative bulk cosmological constant, \(\Lambda_5\). Hence the bulk metric is \(\text{AdS}_5\). From the effect of warped geometry, the fundamental gravity scale, \(M_5\), is below the four-dimensional Planck scale, \(M_4\). A modified braneworld scenario was proposed by Dvali, Gabadadze and Porrati (DGP) \[3\] (see also \[4\]). The key idea of the DGP model is the inclusion of a four-dimensional Ricci scalar into the action. The conventional four-dimensional gravity is recovered on scales smaller than a crossover scale, \(r_c\). Beyond this scale gravity becomes five dimensional and weaker due to the suppression of this term. The scale \(r_c\) could be as large as the Hubble radius. The merit of the DGP model is to explain late-time acceleration \[5\].

We investigate the modification of solutions to the Friedmann equations in braneworld scenarios in comparison to the standard cosmology, especially in the DGP case, and including the case where the universe is dominated by a scalar field. The scalar field phase space with potentials that are of interest in quintessential models, inflation and cold dark matter will be illustrated in the contexts of standard cosmology, RSII and DGP braneworlds.

II. BASIC EQUATIONS

In braneworld scenarios, the gravity model is modified. However, on our brane universe, cosmology should be consistent with the standard FRW model. The standard Friedmann equation is

\[ H^2 + \frac{k}{a^2} = \kappa^2 \rho + \frac{\Lambda}{3} \tag{1} \]

where \(\kappa^2 = M_4^{-2} = 8\pi G\). Nucleosynthesis already imposes the strong constraint that the standard Friedmann equation should govern the expansion after \(\rho \sim 1\) MeV. Therefore at this energy scale, the brane-modified Friedmann equations should effectively approach the standard Friedmann equation. Table-top gravity experiments tell us that the 4D Newton law is still valid at least down to 0.1 mm, hence 4D gravity can only be modified for distances \(l < 0.1\) mm.

On the brane in both RSII and DGP models the conservation of energy and momentum holds:

\[ \dot{\rho} = -3H(\rho + p) \tag{2} \]

A. Randall-Sundrum II braneworld

In the RSII braneworld, the 5D Einstein equation in the bulk is \(G_{AB}^{(5)} = -\Lambda_5^{(5)} g_{AB}\). Projecting the 5D curvature and imposing Darmois-Israel junction conditions and \(Z_2\) symmetry, Shiromizu et al. \[6\] have found the effective Einstein equation on the brane

\[ G_{ab} = -\Lambda g_{ab} + \kappa^2 T_{ab} + 6\kappa^2 \frac{k^2}{A} S_{ab} - \mathcal{E}_{ab} \tag{3} \]
where $S_{ab}$ is the high-energy correction term which is quadratic in the brane energy momentum tensor, $E_{ab}$ is the projected bulk Weyl tensor on the brane and $\lambda$ is the brane tension. Assuming that the brane is FRW, the RSII Friedmann equation on the brane is

$$H^2 + \frac{k}{a^2} = \kappa^2 \rho \left( 1 + \frac{\rho}{2\lambda} \right) + \frac{m}{a^4} + \frac{\Lambda}{3}$$

where $m$ is a constant of integration obtained from the bulk Weyl tensor. The $m/a^4$ term is called dark radiation. We can fine-tune the cosmological constant on the brane by the relation

$$\Lambda = \frac{1}{2} (\Lambda_5 + \kappa^2 \lambda)$$

The negative bulk cosmological constant defines the $AdS_5$ curvature scale $l$ via $\Lambda_5 = -\frac{6}{l^2}$. The curvature scale and $\kappa_5$ relate to each other via $\frac{1}{l} = \kappa_5^2 \lambda / 6$. The relation between the 5D Planck mass, brane tension and 4D Planck mass is

$$M_3^5 = M_4^5 \sqrt{\lambda / 6}.$$ 

In this paper, we consider only scalar fields that live on the brane; the RS braneworld with bulk scalar field has been discussed [8].

B. Dvali-Gabadadze-Porrati braneworld

In the DGP braneworld, unlike the RSII model, the infinite extra dimension is flat ($\Lambda_5 = 0$). Brane tension is assumed to be zero or cancelled out with a brane cosmological constant. (The model can be generalized to include nonzero $\Lambda_5$ and $\lambda$ (see Ref. [4, 14, 11, 12]) but we do not consider this case here.) The model considers effects of induced gravity on the brane through the one-loop process of graviton and brane matter interaction. Induced gravity yields a Ricci scalar term in the Einstein action [3]. The 5D Einstein equation has the form

$$(5)G_{AB} \equiv (5)R_{AB} - \frac{1}{2} (5)R g_{AB} = \kappa_5^2 \left[ (5)T_{AB} + (5)U_{AB} \right]$$

where $(5)U_{AB}$ is a contribution of the induced brane scalar curvature. In the DGP model we have the usual potential $V(r) \sim 1/r$ at small scales while gravity becomes 5D at scales larger than a crossover scale

$$r_c = \frac{M_4^2}{\sqrt{\lambda} M_5^2}$$

where the potential transforms to $V(r) \sim 1/r^2$. The DGP Friedmann equation is

$$H^2 + \frac{k}{a^2} = \left( \sqrt{\kappa^2 \rho \left( 1 + \frac{\rho}{4r_c^2} \right) + \epsilon} \right)^2$$

where $\epsilon = \pm 1$ gives two branches of solutions [3]. The signs correspond to how the brane is embedded into the bulk [3]. Cosmological implications of the DGP braneworld have been analyzed [13, 14, 15].

III. SOLUTION OF FRIEDMANN EQUATION

From now on we assume that the universe is flat with negligible cosmological constant. For the RSII Friedmann equation with $k = \Lambda = m = 0$ and constant $w$, the exact solution is

$$a = \text{const} \left[ t(t + t_\lambda) \right]^{1/(3(1+w))}, \quad t_\lambda = \frac{\sqrt{6}}{2\sqrt[3]{\lambda}} M_4 < 10^{-9} \text{sec}$$

where $w \neq -1$. If we include the dark radiation, the solution for the radiation era is [16]

$$a = \text{const} \left[ t(t + t_\lambda) \right]^{1/4}, \quad t_\lambda = \frac{\sqrt{6} M_4}{2\sqrt{\lambda} (1 + \rho^*/\rho)}$$

These solutions recover the standard solution in the radiation era when $t \gg t_\lambda$, as $a \propto t^{1/2}$. The solution is different from standard cosmology when $t \ll t_\lambda$ and it is $a \propto t^{1/4}$. Here we will solve the DGP Friedmann equation in the high-energy and low-energy regimes of the universe.
A. Solution in the late universe

The DGP Friedmann equation \[ \Box \] with \( k = 0 \) can be rewritten as

\[
H^2 = \frac{1}{4r_c^2} \left[ \sqrt{1 + \frac{4\rho_r^2}{3M_4^2} + \epsilon} \right]^2
\]

\[
= \frac{1}{4r_c^2} \left[ \left( 1 + \frac{4\rho_r^2}{3M_4^2} \right) + 2\epsilon \sqrt{1 + \frac{4\rho_r^2}{3M_4^2} + 1} \right]
\]

We solve the above equation by expanding it in terms of \( \rho_r^2/M_4^2 \ll 1 \). At zeroth order, the equation becomes

\[
H^2 \rightarrow \frac{1}{2r_c^2} (1 + \epsilon)
\]

If \( \epsilon = +1 \), \( H^2 \rightarrow 1/r_c^2 \) in agreement with \[ \Box \]. This yields \( a \sim \exp(r_c^{-1}t) \), i.e. accelerating late-universe expansion. In the case \( \epsilon = -1 \), \( H^2 \rightarrow 0 \), implying that \( a \sim \text{constant} \), i.e. the universe is asymptotically static. At the next order, we consider the two branches of solution separately.

- In the case of \( \epsilon = +1 \) we obtain

\[
H^2 \simeq \frac{1}{r_c^2} + \frac{2\rho}{3M_4^2}
\]

Assuming matter domination \( \rho = \rho_0(a_0/a)^3 \), the solution is

\[
a \simeq a_0 \left( \frac{2\rho r_c^2}{3M_4^2} \right)^{1/3} \sinh^{2/3} \left[ \frac{3}{2r_c}(t - \tau) \right]
\]

(14)

where \( \tau \) is an arbitrary constant of integration. At very late times, extra-dimension effects dominate the expansion completely, and the expansion accelerates as \( a \simeq a_0 \left[ \rho_0 r_c^2/(6M_4^2) \right]^{1/3} \exp [(t - \tau)/r_c] \).

- When \( \epsilon = -1 \), we obtain at lowest order

\[
H \simeq \frac{\rho r_c}{3M_4^2}
\]

with solution

\[
a \simeq a_0 \left[ 3(1 + w) \frac{\rho_0 r_c}{3M_4^2} \right]^{1/3(1+w)} (t - \tau)^{1/3(1+w)}
\]

(16)

For matter domination at late times, we obtain \( a \propto t^{1/3} \). This solution implies slower expansion than the standard cosmology for which \( a \propto t^{2/3} \). In this case, the late time DGP brane universe does not give acceleration and one needs dark energy to dominate in order to obtain acceleration at late time. Note that equation (15) looks similar to the RSII Friedmann equation when the quadratic term dominates at high energy. Therefore it is not surprising that the expansion of this case is similar to equation \[ \Box \] of the RSII model.

B. Solution in the early universe

We rearrange the Friedmann equation \[ \Box \] with \( k = 0 \) as

\[
H^2 = \frac{\rho}{3M_4^2} \left[ \sqrt{1 + \frac{3M_4^2}{4\rho r_c^2} + \frac{\epsilon}{2r_c} \sqrt{\frac{3M_4^2}{\rho}}} \right]^2
\]

(17)
At high energy $\rho r_c^2/M_4^2 \gg 1$, we can expand this as

$$H^2 = \frac{\rho}{3M_4^2} \left[ 1 + \frac{1}{2} \left( \frac{3M_4^2}{4\rho r_c^2} \right) + \cdots + \frac{\epsilon}{2r_c} \sqrt{\frac{3M_4^2}{\rho}} \right]^2$$  \hspace{1cm} (18)$$

To zeroth order, we obtain the standard cosmology Friedmann equation $H^2 \simeq \rho/3M_4^2$. However if we include the $\epsilon/r_c$ term,

$$H^2 \simeq \frac{\rho}{3M_4^2} \left( 1 + \frac{\epsilon}{r_c} \sqrt{\frac{3M_4^2}{\rho}} \right)$$  \hspace{1cm} (19)$$

Assuming radiation domination, this can be solved to obtain

$$a \simeq a_0 \left( \frac{4\rho_0}{3M_4^2} \right)^{1/4} (t - \tau)^{1/2} \left[ 1 + \frac{\epsilon}{4r_c} (t - \tau) \right]$$  \hspace{1cm} (20)$$

where the second term in square brackets is the correction from the extra dimension to the standard evolution.

**IV. CONDITION FOR ACCELERATION**

The condition for accelerating expansion is

$$\dot{H} + H^2 = \frac{\ddot{a}}{a} > 0$$  \hspace{1cm} (21)$$

After differentiating $H^2$ with respect to time and using equation (2), the acceleration condition for standard cosmology takes the form:

$$\dot{H} + H^2 = -\frac{\kappa^2}{6} (\rho + 3p) > 0$$  \hspace{1cm} (22)$$

and this implies $p < -\rho/3$. In the braneworld cosmologies, the conditions are modified.

**A. Condition for acceleration in RSII brane cosmology**

During an inflationary phase, the dark radiation term of the RSII model is diluted away very quickly and can be considered negligible. By the same procedure as in standard cosmology, one can find the RSII acceleration condition as

$$\dot{H} + H^2 = -\frac{\kappa^2}{6} \left[ \rho \left( 1 + \frac{2\rho}{\lambda} \right) + 3p \left( 1 + \frac{\rho}{\lambda} \right) \right] > 0$$  \hspace{1cm} (23)$$

which implies \[17\]

$$p < -\frac{\rho}{3} \left[ \frac{1 + 2\rho/\lambda}{1 + \rho/\lambda} \right]$$  \hspace{1cm} (24)$$

At low energies, $\rho/\lambda \ll 1$, hence

$$p \lesssim -\frac{\rho}{3} \left( 1 + \frac{2\rho}{\lambda} \right) \left( 1 - \frac{\rho}{\lambda} \right) \simeq -\frac{\rho}{3} \left( 1 + \frac{\rho}{\lambda} \right)$$  \hspace{1cm} (25)$$

showing the correction to standard cosmology. On the other hand at high energies, $\rho/\lambda \gg 1$, the condition becomes

$$p \lesssim -\frac{\rho}{3} \left( \frac{\lambda}{\rho} + 2 \right) \left( 1 - \frac{\lambda}{\rho} \right) \simeq -\frac{\rho}{3} \left( 2 - \frac{\lambda}{\rho} \right)$$  \hspace{1cm} (26)$$

or $p \lesssim -2\rho/3$. Hence it modifies the standard cosmology condition significantly at high energies.
B. Condition for acceleration in DGP brane cosmology

By a similar method, the acceleration condition for DGP brane cosmology is

$$\dot{H} + H^2 = -\frac{\kappa^2}{6}(\rho + p) \left[ 1 + \left( \frac{\kappa^2 \rho}{3} + \frac{1}{4r_c^2} \right)^{-1/2} \frac{\epsilon}{2r_c} \right] + \left[ \sqrt{\kappa^2 \rho + \frac{1}{4r_c^2} + \frac{\epsilon}{2r_c}} \right]^2 > 0 \quad (27)$$

which implies

$$p < -\rho + 2\frac{\kappa^2}{\kappa^2} \left[ \left( \sqrt{\kappa^2 \rho + \frac{1}{4r_c^2} + \frac{\epsilon}{2r_c}} \right)^2 \left( 1 + \left[ \frac{\kappa^2 \rho}{3} + \frac{1}{4r_c^2} \right]^{-1/2} \frac{\epsilon}{2r_c} \right) \right]^{-1} \quad (28)$$

At high energy, the $1/r_c$ term is small compared to the density term and one can obtain the standard cosmology condition $p < -\rho/3$. In the late universe, the extra-dimension effect cannot be neglected. We will use the approximation of the DGP Friedmann equation previously derived instead of using the full form. For the case $\epsilon = +1$, using the first-order late-time DGP Friedmann equation (13), the acceleration condition becomes

$$\dot{H} + H^2 \simeq -\kappa^2(\rho + p) + \frac{1}{r_c^2} + \frac{2n^2 \rho}{3} > 0 \quad (29)$$

or

$$p < -\frac{\rho}{3} + \frac{1}{r_c^2 \kappa^2} \quad (30)$$

In the case $\epsilon = -1$, using equation (15), the condition is simply

$$\dot{H} + H^2 \simeq -\frac{\kappa^4 r_c^2}{3} \rho(\rho + p) + \frac{\kappa^4 \rho^2 r_c^2}{9} > 0 \quad (31)$$

or

$$p < -\frac{2\rho}{3} \quad (32)$$

These acceleration conditions will be used for analyzing the DGP brane universe in next section when a scalar field is the dominant component in the universe.

V. BRANEWORLD SCALAR FIELD PHASE SPACE

The energy density and pressure of the scalar field are $
\rho = \frac{1}{2} \dot{\phi}^2 + V(\phi)$ and $p = \frac{1}{2} \dot{\phi}^2 - V(\phi)$. When the scalar field dominates the universe, the conservation equation takes the form of the Klein-Gordon equation

$$\ddot{\phi} + 3H \dot{\phi} = -\frac{dV}{d\phi} \quad (33)$$

Inflation requires a scalar field to drive the acceleration. The standard acceleration condition for a scalar field dominated universe is $\dot{\phi}^2 < V(\phi)$. In the RSII model when a scalar field dominates, the condition can be derived from equation (24) to obtain (see Ref. 17)

$$\dot{\phi}^2 - V + \frac{\dot{\phi}^2 + 2V}{8\lambda} (5\dot{\phi}^2 - 2V) < 0 \quad (34)$$

The third term on the left is the modified effect from the quadratic density term in the Einstein equation 17, 18. There are some classes of potentials that are too steep to give inflation in standard cosmology. However in the RSII scenario with the presence of the additional quadratic term in the Friedmann equation, inflation can occur and ends with a kinetic regime when the field has stiff equation of state 19.

In the case of the DGP braneworld, since equation (28) reduces to standard cosmology at high energy, it gives the usual acceleration condition of the standard cosmology in the early universe. Considering a scalar field dominated late universe and using the approximate condition (30), when $\epsilon = +1$ we obtain

$$\dot{\phi}^2 - V - \frac{3}{2r_c^2 \kappa^2} < 0 \quad (35)$$
The scalar field needs less potential energy than in the standard case in order to accelerate the expansion. Acceleration can happen even if $\dot{\phi}^2$ is equal to or larger than $V$, as long as it is smaller than $V + 3/2 r_c^2 \kappa^2$. However at very late times, when the field is moving very slowly, we can use the slow-roll approximation $\ddot{\phi} \sim 0$ and we can approximate that $\dot{\phi}^2 \ll V + 3/2 r_c^2 \kappa^2$. Applying these conditions to the DGP Friedmann equation (13) and the Klein-Gordon equation, we obtain

$$\dot{\phi}^2 \simeq \frac{V'^2}{6\kappa^2 V + 9/r_c^2}$$

(36)

This equation is the late time attractor trajectory in phase space. In the case $\epsilon = -1$, using the condition (32) we obtain

$$5\dot{\phi}^2 - 2V < 0$$

(37)

Applying the DGP Friedmann equation (15) and the Klein-Gordon equation at very late time, with similar assumptions as above, a late time phase space attractor trajectory can be found from

$$\dot{\phi} \simeq -\frac{1}{r_c \kappa^2} \frac{V'}{V}$$

(38)

Now we consider the phase space for some particular types of potentials $V(\phi)$. Here we will consider mainly the DGP braneworld model but illustration of the phase-space in standard, RSII and DGP models will be presented for comparison. In our numerical plots, we include the curve showing the acceleration condition. The curve divides acceleration from non-acceleration regions.
FIG. 2: The phase space of a scalar field with \( V(\phi) = \mu^{n+4}/\phi^n \), \( n = 2 \), for the standard cosmology, RSII and DGP brane cosmologies. The curves marking the acceleration condition are the solid lines. (The acceleration curve for the DGP case is beyond the scales on the axes; see Fig. 4.)

### A. Inverse-power potential

The potential

\[
V(\phi) = \frac{\mu^{n+4}}{\phi^n}
\]  

is one candidate for a quintessential potential. This potential was considered in some supersymmetry breaking models \[20\]. The potential can give tracking behavior in which the scalar field energy density evolves along with the density of matter and radiation \[22\]. The ratio of scalar field density and total density of matter and radiation increases slowly as \( \rho_\phi/(\rho_m + \rho_r) \propto t^{4/(n+2)} \) \[23\]. For early universe inflation, this potential is too steep for standard inflation. However the RSII braneworld scenario can give inflation at high energy but with large magnitude relic gravitational waves \[24\]. Solutions of the scalar field dominated era and radiation dominated era of RSII brane cosmology have been found \[21\]. In this subsection we consider phase spaces of scalar field dominated universe and their acceleration conditions in three cases: early universe RSII brane cosmology, low energy standard cosmology limit of RSII brane cosmology, and DGP brane cosmology. Since observations constrain \( n \lesssim 2 \) \[22\], we choose \( n = 1 \) and 2 in our numerical plots.

#### 1. High energy regime of RSII brane cosmology

During the high energy regime, \( \rho \gg \lambda \), the quadratic term in the RSII Friedmann equation is dominant. From the acceleration condition in equation (34), dropping the first two terms at high energy and using a slow-roll approximation
FIG. 3: The phase space of a scalar field with exponential potential, equation (45), for the standard cosmology, RSII and DGP brane cosmologies. The solid lines are the acceleration condition curves. (The acceleration curve for the DGP case is beyond the scales on the axes; see Fig. 4.) We use \( \kappa = 1, \ p = 1/8, \ \lambda = 10^{-4}, \ r_c = 2, \) and \( V_0 = 1. \)

(\( \ddot{\phi} \sim 0 \)), one can get the attractor trajectory \(^{21}\)

\[
\dot{\phi} = \frac{n}{3\kappa} \sqrt{6\lambda} \tag{40}
\]

For \( n \leq 2 \) inflation can happen. For \( n = 2 \) the kinetic term and potential terms are balanced and this yields power law expansion. For \( n > 2 \) the kinetic term is dominant (see \(^{21, 26}\)). The attractor trajectory equation (40) matches the RSII phase plot of Figs. 1 and 2 at late times.

2. Low energy standard cosmology limit of RSII brane cosmology

In the late universe, the RSII Friedmann equation approaches the standard cosmology limit. When a quintessential scalar field dominates the universe at late times, the Friedmann equation (with small value of \( \dot{\phi} \)) is

\[
H = \frac{\kappa}{\sqrt{3}} \left( \frac{\mu^{n+4}}{\phi^n} \right)^{1/2} \tag{41}
\]

Together with the Klein-Gordon equation in the slow-roll regime, we find that the attractor curve is described by

\[
\dot{\phi} \phi^{n+1} = \frac{n}{\sqrt{3} \kappa} \mu^{n+2} \tag{42}
\]

The attractor trajectory agrees well with the numerical plots at late times as seen in the standard cosmology case in Figs. 1 and 2.
FIG. 4: The phase space of a scalar field with exponential potential, equation (45), in a DGP brane universe with $r_c = 20$. The crossover scale term dominates the potential term at $\dot{\phi} \simeq 0.061$. The solid line is the acceleration condition curve.

3. DGP brane cosmology

DGP cosmology recovers standard cosmology in the early universe when $\rho \gg 3/4r_c^2\kappa^2$. However when the extra-dimension effects become significant in the late universe, i.e. $\rho \leq 3/4r_c^2\kappa^2$, the modification of its dynamics is crucial. We here consider the low-energy late universe with scalar field domination. Consider the case $\epsilon = +1$. Using equation (36), the slow-roll attractor trajectory at late times is

$$\dot{\phi}^2 = \frac{n^2\mu^{2n+8}\phi^{-2n-2}}{6\kappa^2[\mu^{n+4}\phi^{-n} + 3/2r_c^2\kappa^2]}$$

(43)

At small $\dot{\phi}$, equation (43) fits the numerical plots in both $n = 1$ and 2 cases, as seen in Figs. 1 and 2.

In the case $\epsilon = -1$, the attractor trajectory at late time is

$$\phi \frac{\dot{\phi}}{r_c\kappa^2} = \frac{n}{r_c\kappa^2}$$

(44)

and the solution is $\phi = \left[2n(t-\tau)/\kappa^2 r_c\right]^{1/2}$. Indeed in the case $\epsilon = -1$, the equation (15) has a similar form to that of RSII at high energies. Therefore, all subsequent results looks similar to those of the RSII case which has been found before (see [21]). For example in our $\epsilon = -1$ DGP case, the slow-roll condition can be applied only when $n < 2$ since $\ddot{\phi} \sim t^{-3/2} \ll V' \sim t^{-(n+1)/2}$ and $\dot{\phi}^2 \sim t^{-1} \ll V \sim t^{-n/2}$.

B. Exponential potential

A scalar field with exponential potential

$$V(\phi) = V_0 \exp \left( -\sqrt{\frac{\phi}{p M_4}} \right)$$

(45)

is known to yield power-law inflation in standard cosmology, i.e. $a = a_0 (t/t_0)^p$ [27]. The slow-roll parameters in this case are $\epsilon = \eta/2 = 1/p$ and the inflation condition requires that $p > 1$ [28]. The exponential potential that drives inflation could be the same potential that is responsible for quintessence and cold dark matter unless the quintessential and cold dark matter potentials are too steep ($p < 1$) for inflation to happen [29]. As quintessence, an exponential potential can yield tracking behavior. However it evolves with constancy of the ratio between background matter density and scalar field density, so it is not able to yield a quintessence dominated universe at late time. Some
FIG. 5: The phase space of a scalar field with $V(\phi) = V_0 [\cosh(\alpha \phi/M_4) - 1]$ in standard cosmology. The numerical values are $\kappa = 1$, $\alpha = 3$ and $V_0 = 5 \times 10^{-3}$.

FIG. 6: The phase space of a scalar field with $V(\phi) = V_0 [\cosh(\alpha \phi/M_4) - 1]$ in RSII brane cosmology, with $\lambda = 10^{-6}, 10^{-5}, 10^{-4}$ and $10^{-3}$. The solid lines are the acceleration condition.

modification of this model for standard cosmology has been proposed, e.g. the potential $V(\phi) = V_0 [\cosh(\alpha \phi/M_4) - 1]^q$.

In the RSII braneworld, a steep potential, i.e. $p < 1$, is able to drive inflation. The quadratic term in the RSII Friedmann equation is dominant at high energy, contributing to more friction in the Klein-Gordon equation. As a result, slow-roll inflation can be obtained. After the quadratic term stops dominating, the field moves faster and enters a kinetic regime where the inflaton field can redshift faster than the produced particles and radiation, i.e. $\rho_\phi \propto a^{-6}$. This results in a natural exit and reheating by gravitational particle production.

In the DGP brane universe, acceleration of the universe is an effect from both quintessence and existence of the
FIG. 7: The phase space of a scalar field with \( V(\phi) = V_0 \cosh(\alpha \phi/M_4) - 1 \) in DGP brane cosmology, with \( r_c = 2, 6, 10 \) and 20.

extra-dimension. Using equation (46), the slow-roll attractor trajectory in phase space is

\[
\dot{\phi}^2 = \frac{\alpha^2 V_0 e^{-\alpha \phi}}{6\kappa^2 + 9e^{\alpha \phi}/V_0 r_c^2}
\]

(46)

where \( \alpha = \sqrt{2/p M_4^{-1}} \). The standard general relativity limit of equation (46) is recovered when \( r_c \to \infty \), and yields the standard cosmology solution \( \phi(t) \sim \ln(t) \). The full solution of equation (46) is

\[
t - \tau = \frac{\sqrt{A e^{\alpha \phi} + B e^{2\alpha \phi}}}{\alpha} + \frac{A}{2\alpha \sqrt{B}} \ln \left\{ \frac{8}{\alpha^2} \left[ \frac{A^2}{2} + B e^{\alpha \phi} + \sqrt{B} \sqrt{A e^{\alpha \phi} + B e^{2\alpha \phi}} \right] \right\}
\]

(47)

where \( A = 6\kappa^2/\alpha^2 V_0 \) and \( B = (3/\alpha V_0 r_c)^2 \). The solution is found when \( B \) is nonzero.

Numerical plots of the standard, RSII and DGP (\( \epsilon = +1 \)) braneworld phase spaces are illustrated in Fig. 3. In this figure, it can be seen that the RSII phase space is similar to the standard cosmology phase space at late time. In RSII case of Fig. 3 if we start from some suitable initial conditions, the field moves to the attractor trajectory in the acceleration phase and gives slow-roll inflation. After that, it enters a non-acceleration phase where the attractor trajectory becomes too steep for inflation and the field is in a kinetic regime. All regions of the DGP case of Fig. 3 are in the acceleration phase (the non-accelerating region is off the scale of the plot).

In Fig. 4 where \( r_c = 20 \), the DGP acceleration curve is shown. Due to the high \( r_c \) value in this figure, the phase space looks similar to the standard case at early time, but it approaches the DGP attractor trajectory at late time. Most of the area of the figure is in acceleration phase. Moving from any initial conditions, the field will eventually be in acceleration phase.
FIG. 8: The phase space of a scalar field with $V(\phi) = \frac{1}{2}m^2\phi^2$ for the standard cosmology, RSII and DGP brane cosmologies. The solid lines are the acceleration condition. We set $\epsilon = +1$, $\kappa = m = \sigma = 1$, $\lambda = 10^{-4}$, $r_c = 2$.

C. Modified exponential potential

The potential

$$V(\phi) = V_0 \left[ \cosh\left( \frac{\alpha}{M_4} \phi \right) - 1 \right]^q$$

(48)

is too steep to yield inflation in standard cosmology but in the RSII braneworld with $q = 1$, it can give steep inflation (with large amplitude gravitational waves). For $q = 1$ the potential has asymptotic forms $V(\phi) \approx V_0 e^{\alpha \phi / M_4} / 2$ when $\alpha / M_4 \gg 1$, and $V(\phi) \approx V_0 (\alpha \phi / M_4)^2 / 2$ when $\alpha / M_4 \ll 1$. The exponential potential can not give oscillating inflaton reheating, but the cosh potential gives chaotic inflation behavior [24].

The potential has also been used for quintessence and cold dark matter models in standard cosmology [30]. During the oscillation phase at late times, the mean equation of state is $\langle w_\phi \rangle = (q - 1) / (q + 1)$. The equation of state dictates that dark energy can be achieved if $q \leq 1/2$ and that to obtain dust (cold dark matter) we need $q = 1$ [30, 51].

In Figs. 5, 6 and 7, we illustrate phase diagrams of a scalar field with potential $V(\phi) = V_0 \left[ \cosh(\alpha \phi / M_4) - 1 \right]$, for standard cosmology (where it plays the role of cold dark matter), RSII (where it could be regarded as the inflaton field) and DGP ($\epsilon = +1$). We vary the brane tension for the RSII case and vary the crossover scale for the DGP case. It can be seen that at larger brane tension or larger crossover scale, the RSII and DGP phase spaces become similar to the phase space of standard cosmology.

D. Power law potentials

The power law potentials [32]

$$V(\phi) = \frac{1}{2}m^2\phi^2 \quad \text{and} \quad V(\phi) = \frac{1}{4}\sigma\phi^4$$

(49)
FIG. 9: The phase space of a scalar field with $V(\phi) = \frac{1}{4}\sigma\phi^4$ for the standard cosmology, RSII and DGP brane cosmologies. The solid lines are the acceleration condition.

are the simplest models for inflation that can generate gravitational waves with large enough magnitude to be observed in the cosmic microwave background. The plots in Figs. 8 and 9 show the phase spaces in standard cosmology and the two types of brane cosmologies. At late times, the RSII trajectories are similar to the standard case, but they differ markedly from the standard case at early times. The reverse holds for the DGP trajectories.

VI. CONCLUSION

We have investigated the dynamics of a braneworld with induced gravity, i.e. the DGP model. Unlike the RSII braneworld, which modifies general relativity at early times and then recovers it at late times, the DGP braneworld is like general relativity in the early universe but modifies the dynamics at late times. The extra-dimensional gravity effect in the DGP ($\epsilon = 1$) model contributes to late-time acceleration, which provides a possible way to avoid dark energy. We used approximations to solve the DGP Friedmann equation in both the early and late universe (equations (13), (15) and (19)). The conditions for acceleration in the DGP model were derived, equations (28), (35) and (37). At early times, the DGP Friedmann equation approaches the standard general relativity limit. Modifications appear at late times. When $\epsilon = +1$, the solutions give accelerating expansion at late time. On the other hand, when $\epsilon = -1$, the universe expands slower than standard cosmology. We analyzed the phase planes for a range of potentials used for inflation, quintessence and cold dark matter, and compared the dynamics in the DGP case to the RSII and standard cases. These phase planes show clearly the qualitatively different behavior of the DGP model. The phase space in the case of the potential $V(\phi) = V_0[cosh(\alpha\phi/M_4) - 1]$, with various values of brane tension (RSII braneworld) and crossover scale (DGP braneworld), showed how the standard cosmology scenario is approached in the limit.
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