Elementary Components of the Quadratic Assignment Problem

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Abstract
The Quadratic Assignment Problem (QAP) is a well-known NP-hard combinatorial optimization problem that is at the core of many real-world optimization problems. We prove that QAP can be written as the sum of three elementary landscapes when the swap neighborhood is used. We present a closed formula for each of the three elementary components and we compute bounds for the autocorrelation coefficient.

1 Introduction
We will define a landscape for a combinatorial problem using a triple \((X, N, f)\), where \(f : X \mapsto \mathbb{R}\) defines the objective function and the neighborhood operator \(N\) assigns a set of neighboring solutions \(N(x) \in X\) to each solution \(x\). If \(y \in N(x)\) then \(y\) is a neighbor of \(x\). The landscape that is induced can be used as a search space for optimization using local search.

There is a special kind of landscape which is of particular interest due to their properties. They are the elementary landscapes, and are characterized by the following equation:

\[
\text{avg}\{f(y)\}_{y \in N(x)} = f(x) + \frac{k}{d}(\bar{f} - f(x)) \tag{1}
\]

where \(d\) is the size of the neighborhood, \(|N(x)|\), which we assume the same for all the solutions in the search space, \(\bar{f}\) is the average solution evaluation over the entire search space, and \(k\) is a characteristic constant. Equation (1) is usually called Grover’s wave equation and makes it possible to compute the average value of the fitness function \(f\) evaluated over all of the neighbors of \(x\); we denote this average using \(\text{avg}\{f(y)\}_{y \in N(x)}\):

\[
\text{avg}\{f(y)\}_{y \in N(x)} = \frac{1}{|N(x)|} \sum_{y \in N(x)} f(y)
\]

Other properties also follow. Assuming \(f(x) \neq \bar{f}\) then

\[
f(x) < \min \left\{ \text{avg}\{f(y)\}_{y \in N(x)}, \bar{f} \right\} \quad \text{and} \quad f(x) > \max \left\{ \text{avg}\{f(y)\}_{y \in N(x)}, \bar{f} \right\}.
\]
This implies that all maxima are greater than $\bar{f}$ and all minima are less than $\bar{f}$ \cite{4}. A landscape $(X, N, f)$ is not always elementary, but even in this case it is possible to characterize the function $f$ as the sum of elementary landscapes \cite{3}, called elementary components of the landscape.

The Quadratic Assignment Problem (QAP) is an NP-hard combinatorial optimization problem \cite{2}. A lot of research has been devoted to analyze and solve the QAP. Some other problems can be formulated as special cases of the QAP. One important example is the Traveling Salesman Problem (TSP). The QAP is not an elementary landscape when the swap neighborhood is considered \cite{1}. But, to the best of our knowledge the exact expressions for the elementary components of the QAP are not known.

Such decomposition could be useful from the theoretical and practical points of view. In theory, the landscape decomposition of QAP can be used to compute the exact expression of the autocorrelation functions and the autocorrelation coefficient \cite{1}. In practice, the landscape decomposition together with the Grover’s wave equation can be used to compute the average value of the objective function in the neighborhood, which can be used as a base for new operators or algorithms. In particular, a new family of selection operators can be designed which select the individuals according to the average fitness value in the neighborhood of a solution $x$ instead of using the fitness value of the solution itself. These selection operators could be especially useful to distinguish solutions that are in plateaus.

We present here the elementary landscape decomposition of QAP. In the next section we present the formal definition of QAP. Section 3 presents the main result and its proof. Finally, we conclude in Section 4 with some conclusions and future work.

\section{Quadratic Assignment Problem}

Let $P$ be a set of $n$ facilities and $L$ a set of $n$ locations. For each pair of locations $i$ and $j$, an arbitrary distance is specified $r_{ij}$ and for each pair of facilities $p$ and $q$, a flow is specified $w_{pq}$. The Quadratic Assignment Problem (QAP) consists in assigning the facilities of $P$ to the locations in $L$ in such a way that the total cost of the assignment is minimized. Each location can only contain one facility. For each pair of facilities the cost is computed as the product of the weight associated to the facilities and the distance between the locations in which the facilities are. The total cost is the sum of all the costs associated to each pair of facilities. One solution to this problem is a bijection between $P$ and $L$, that is, $x : P \rightarrow L$ such that $x$ is bijective. Without loss of generality we can just assume that $P = L = \{1, 2, \ldots, n\}$ and each solution $x$ is a permutation in $S_n$, the set permutations of $\{1, 2, \ldots, n\}$. The cost function to be minimized can be formally defined as:

$$f(x) = \sum_{i,j=1}^{n} r_{ij} w_{x(i)x(j)}$$ \hspace{1cm} (2)
3 Decomposition of QAP

In the previous section we defined the search space and the objective function. In order to completely define a landscape we need to define the neighborhood. The neighborhood $\mathcal{N}$ considered here is the swap or 2-exchange neighborhood, in which two solutions are neighboring if one can be obtained from the other one by a swap (exchange of two elements) in the permutation. In this section we prove that the QAP is composed at most by three elementary components and we give an expression for them.

Let us start rewriting (2). In order to analyze the elementary components of the fitness functions associated to a problem class, like QAP, it is useful to separate in the formal definition of the objective function the information that is particular of a given instance (the data of the instance) from the general issues that characterize the class of the problem. In the case of QAP, the information related to the particular instance is included in the distance matrix ($r_{ij}$) and the weight matrix ($w_{pq}$). The question now is: how to separate the data of the instance in the fitness formulation. There are different ways to do it, but we are interested in linear combinations of functions where the coefficients of the functions are associated to the particular instances. The reason for this is that any linear combination of elementary functions (with the same characteristic constant $k$) is also an elementary function. With this idea in mind, it is not difficult to see that Equation (2) can be written using the following linear combination:

$$f(x) = \sum_{i,j=1}^{n} \sum_{p,q=1}^{n} r_{ij} w_{pq} \delta^p_{x(i)} \delta^q_{x(j)}$$

where we used the Kronecker’s delta. The problem-related part of the fitness function is, thus, the product $\delta^p_{x(i)} \delta^q_{x(j)}$. At this point we can go further and deal with a more general objective function. In (3) the value of the product $r_{ij} w_{pq}$ depends on $i$, $j$, $p$, and $q$ in a particular way, but is not the most general one. Using multilinear algebra concepts, the previous product is a four-rank tensor that has been computed as a tensor product of two two-rank tensors (matrices), which is a special case of four-rank tensor. In the most general case we can define a four-rank tensor to replace the product. Let us call $\psi_{ijpq}$ the new general four-rank tensor and let us define the parameterized function $\varphi_{(i,j),(p,q)}(x) = \delta^p_{x(i)} \delta^q_{x(j)}$. Then we can rewrite the fitness function as:

$$f = \sum_{i,j,p,q=1}^{n} \psi_{ijpq} \varphi_{(i,j),(p,q)}$$

and we can focus our analysis on $\varphi_{(i,j),(p,q)}$, since any result on it can be extended to any linear combination of $\varphi$ functions, and, thus, to $f$. Now, the objective function of the QAP is just a particular case of our new objective function $f$, in which $\psi_{ijpq} = r_{ij} w_{pq}$.
If \( i = j \) and \( p = q \) in the functions \( \varphi_{(i,j),(p,q)} \), then we have \( \varphi_{(i,i),(p,p)}(x) = \delta_{x(i)}^p \). For this particular case we have the following

**Lemma 1.** Considering the swap neighborhood the function \( \varphi_{(i,i),(p,p)} \) is an elementary landscape with \( k = n \).

**Proof.** In the following, for the sake of clarity we will remove all the parameters from the name of the function when there is no confusion. The function \( \varphi \) is elementary if and only if there exist two constants \( a \) and \( b \) such that the following expression holds for all the solutions:

\[
\text{avg}\{\varphi(y)\} = a\varphi(x) + b \\
y \in N(x)
\]

In order to reduce the expressions we multiply the previous expression by the size of the neighborhood, which is \( d = \frac{n(n-1)}{2} \). We then obtain:

\[
\sum_{y \in N(x)} \varphi(y) = c\varphi(x) + e \tag{5}
\]

where \( c = ad \) and \( e = bd \). Now, we compute the exact expression of \( \sum_{y \in N(x)} \varphi(y) \) for the two different values that \( \varphi \) can take:

- **Case** \( \varphi(x) = 1 \) (in this case \( x(i) = p \)). From the neighboring solutions there are \( n-1 \) with \( \varphi(y) = 0 \) and the remaining neighbors have a value \( \varphi(y) = 1 \). Then we can write:

\[
\sum_{y \in N(x)} \varphi(y) = (d - n + 1)
\]

- **Case** \( \varphi(x) = 0 \) (in this case \( x(i) \neq p \)). From the neighboring solutions there is only one with \( \varphi(y) = 1 \). The remaining neighbors have a value \( \varphi(y) = 0 \). Then we can write:

\[
\sum_{y \in N(x)} \varphi(y) = 1
\]

Now we use Equation (5) to obtain the following linear equation system:

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
c \\
e
\end{pmatrix} =
\begin{pmatrix}
d - n + 1 \\
1
\end{pmatrix}
\]

The solution of the previous system is \( c = d - n \) and \( e = 1 \); so we have \( a = 1 - n/d \) and \( b = 1/d \). Then, we can write

\[
\text{avg}\{\varphi(y)\} = \left(1 - \frac{n}{d}\right)\varphi(x) + \frac{1}{d} = \varphi(x) + \frac{n}{d} \left(\frac{1}{n} - \varphi(x)\right) \tag{6}
\]

and we conclude that \( \varphi_{(i,i),(p,p)} \) is an elementary landscape with \( k = n \) and average \( \bar{\varphi}_{(i,i),(p,p)} = 1/n \). \( \square \)
Before proving the main result of this section we need to introduce a family of auxiliary functions that map permutations to \( \mathbb{R} \):

\[
\phi_{(i, j), (p, q)}^{\alpha, \beta, \gamma, \epsilon, \zeta}(x) = \begin{cases} 
\alpha & \text{if } x(i) = p \land x(j) = q \\
\beta & \text{if } x(i) = q \land x(j) = p \\
\gamma & \text{if } x(i) = p \oplus x(j) = q \\
\epsilon & \text{if } x(i) = q \oplus x(j) = p \\
\zeta & \text{if } x(i) \neq p, q \land x(j) \neq p, q
\end{cases}
\] (7)

where \( 1 \leq i, j, p, q \leq n \) are integer values with \( i \neq j \) and \( p \neq q \) and \( \alpha, \beta, \gamma, \epsilon, \zeta \in \mathbb{R} \). All the previous values are parameters of the family of functions. We denote with \( \oplus \) the exclusive-or operator. The previous functions are valuable thanks to the following

**Lemma 2.** Considering the swap neighborhood the function \( \phi_{(i, j), (p, q)}^{\alpha, \beta, \gamma, \epsilon, \zeta} \) is an elementary landscape in the following cases:

- \( \alpha = n - 3, \beta = 1 - n, \gamma = -2, \epsilon = 0, \zeta = -1 \) with \( k = 2n \)
- \( \alpha = n - 3, \beta = n - 3, \gamma = 0, \epsilon = 0, \zeta = 1 \) with \( k = 2(n - 1) \)
- \( \alpha = 2n - 3, \beta = 1, \gamma = n - 2, \epsilon = 0, \zeta = -1 \) with \( k = n \)

**Proof.** In the following, for the sake of clarity we will remove all the parameters from the name of the function when there is no confusion. The function \( \phi \) is elementary if and only if there exist two constants \( a \) and \( b \) such that the following expression holds for all the solutions:

\[
\text{avg}_{y \in N(x)} \{ \phi(y) \} = a\phi(x) + b
\]

In order to reduce the expressions we multiply the previous expression by the size of the neighborhood, which is \( d = \frac{n(n - 1)}{2} \). We then obtain:

\[
\sum_{y \in N(x)} \phi(y) = c\phi(x) + e
\] (8)

where \( c = ad \) and \( e = bd \).

We distinguish five different cases which are symbolically represented in Figure 1. In the figure, each node represents the set of solutions for which one of the five branches in (7) is true. We label the nodes with the value that \( \phi \) takes for all the solutions in that node. There exists an arc \((i, j)\) if all the solutions in node \( i \) have at least one neighboring solution in node \( j \). The label of arc \((i, j)\) is the number of neighbors that any solution in \( i \) has in \( j \). Now, we compute the exact expression of \( \sum_{y \in N(x)} \phi(y) \) for the five different values that \( \phi \) can take:

- **Case \( \phi(x) = \alpha \).** In this case \( x(i) = p \) and \( x(j) = q \). From the neighboring solutions there is one with \( \phi(y) = \beta \) and \( 2(n - 2) \) solutions with \( \phi(y) = \gamma \). The remaining neighbors have a value \( \phi(y) = \alpha \). Then we can write:

\[
\sum_{y \in N(x)} \phi(y) = \beta + 2(n - 2)\gamma + (d - 2n + 3)\alpha
\]
Figure 1: Transition graph for functions $\phi_{(i,j),(p,q)}^{\alpha,\beta,\gamma,\epsilon,\zeta}$.

- Case $\phi(x) = \beta$. In this case $x(i) = q$ and $x(j) = p$. From the neighboring solutions there is one with $\phi(y) = \alpha$ and $2(n-2)$ solutions with $\phi(y) = \epsilon$. The remaining neighbors have a value $\phi(y) = \beta$. Then we can write:
  \[
  \sum_{y \in N(x)} \phi(y) = \alpha + 2(n-2)\epsilon + (d-2n+3)\beta
  \]

- Case $\phi(x) = \gamma$. In this case $x(i) = p$ or $x(j) = q$, but not both. From the neighboring solutions there is one with $\phi(y) = \alpha$, two neighbors with $\phi(y) = \epsilon$, and $n-3$ neighbors with $\phi(y) = \zeta$. The remaining neighbors have a value $\phi(y) = \gamma$. Then we can write:
  \[
  \sum_{y \in N(x)} \phi(y) = \alpha + 2\epsilon + (n-3)\zeta + (d-n)\gamma
  \]

- Case $\phi(x) = \epsilon$. In this case $x(i) = q$ or $x(j) = p$, but not both. From the neighboring solutions there is one with $\phi(y) = \beta$, two neighbors with $\phi(y) = \gamma$, and $n-3$ neighbors with $\phi(y) = \zeta$. The remaining neighbors have a value $\phi(y) = \epsilon$. Then we can write:
  \[
  \sum_{y \in N(x)} \phi(y) = \beta + 2\gamma + (n-3)\zeta + (d-n)\epsilon
  \]

- Case $\phi(x) = \zeta$. In this case $x(i) \neq p,q$ and $x(j) \neq p,q$. From the neighboring solutions there are two with $\phi(y) = \gamma$ and two neighbors with $\phi(y) = \epsilon$. The remaining neighbors have a value $\phi(y) = \zeta$. Then we can write:
  \[
  \sum_{y \in N(x)} \phi(y) = 2\gamma + 2\epsilon + (d-4)\zeta
  \]

Now we use Equation (8) to obtain the following linear equation system:
The previous system has five equations and two variables, $c$ and $e$, so it could be unsolvable. However, the system can be solved for some value combinations of $\alpha$, $\beta$, $\gamma$, $\varepsilon$, $\zeta$. In particular, the system can be solved for the value combinations mentioned in the statement, that is:

1. $\alpha = n - 3$, $\beta = 1 - n$, $\gamma = -2$, $\varepsilon = 0$, $\zeta = -1$
2. $\alpha = n - 3$, $\beta = n - 3$, $\gamma = 0$, $\varepsilon = 0$, $\zeta = 1$
3. $\alpha = 2n - 3$, $\beta = 1$, $\gamma = n - 2$, $\varepsilon = 0$, $\zeta = -1$

This does not mean that these are the only combinations of parameter values for which the system can be solved. They are just three combinations of special interest for the goal of this section. It should be noticed here that the linear system does not depend on the values of $i$, $j$, $p$, and $q$. Thus, the solutions to the system are also independent of the values of the mentioned parameters.

Let us study the values of $a, b, c, e$ for the first parameter combination, that is, $\alpha = n - 3$, $\beta = 1 - n$, $\gamma = -2$, $\varepsilon = 0$, and $\zeta = -1$. The solution of the linear system is $c = \frac{n(n-3)}{2}$ and $e = -2n$, and, thus: $a = (1 - 2n/d)$ and $b = -2n/d$.

In order to simplify the notation, let us define $\Omega^1_{(i,j),(p,q)} = \phi^{n-3,1-n,-2,0,-1}_{(i,j),(p,q)}$. Then, we can write

$$\begin{pmatrix} \alpha & 1 \\ \beta & 1 \\ \gamma & 1 \\ \varepsilon & 1 \\ \zeta & 1 \end{pmatrix} \begin{pmatrix} c \\ e \end{pmatrix} = \begin{pmatrix} \beta + 2(n-2)\gamma + (d-2n+3)\alpha \\ \alpha + 2(n-2)\varepsilon + (d-2n+3)\beta \\ \alpha + 2\varepsilon + (n-3)\zeta + (d-n)\beta \\ \beta + 2\gamma + (n-3)\zeta + (d-n)\varepsilon \\ 2\gamma + 2\varepsilon + (d-4)\zeta \end{pmatrix}$$

The solutions to this system are independent of the values of the mentioned parameters.

Let us now focus on the second parameter combination, that is, $\alpha = \beta = n - 3$, $\gamma = \varepsilon = 0$, and $\zeta = 1$. The solution of the linear system is $c = \frac{(n-1)(n-4)}{2}$ and $e = 2(n-3)$, and, thus: $a = 1 - 2n/2$ and $b = 2(n-3)/d$. Introducing the notation $\Omega^2_{(i,j),(p,q)} = \phi^{n-3,n-3,0,0,1}_{(i,j),(p,q)}$ we can write

$$\begin{align*}
\text{avg}\left\{\Omega^1_{(i,j),(p,q)}(y)\right\} & = \left(1 - \frac{2n}{d}\right) \Omega^1_{(i,j),(p,q)}(x) - \frac{2n}{d} \\
& = \Omega^1_{(i,j),(p,q)}(x) + \frac{2n}{d} \left(-1 - \Omega^1_{(i,j),(p,q)}(x)\right) \\
\text{avg}\left\{\Omega^2_{(i,j),(p,q)}(y)\right\} & = \left(1 - \frac{2(n-1)}{d}\right) \Omega^2_{(i,j),(p,q)}(x) + \frac{2(n-3)}{d} \\
& = \Omega^2_{(i,j),(p,q)}(x) + \frac{2(n-1)}{d} \left(\frac{n-3}{n-1} - \Omega^2_{(i,j),(p,q)}(x)\right)
\end{align*}$$
and we conclude that $\Omega^3_{(i,j),(p,q)}$ is an elementary landscape with $k = 2(n-1)$ and $\bar{\Omega}^3_{(i,j),(p,q)} = (n-3)/(n-1)$.

Finally, let us analyze the third parameter combination, that is, $\alpha = 2n-3$, $\beta = 1$, $\gamma = n-2$, $\varepsilon = 0$ and $\zeta = -1$. The solution of the linear system is $c = \frac{n(n-3)}{2}$ and $e = n$, and, thus: $a = 1 - n/d$ and $b = n/d$. Introducing the notation $\Omega^3_{(i,j),(p,q)} = \phi^2_{(i,j),(p,q)}\Omega^{2n-3,1,n-2,0,-1}$ we can write

$$\text{avg}\{\Omega^3_{(i,j),(p,q)}(y)\} = \left(1 - \frac{n}{d}\right)\frac{\Omega^3_{(i,j),(p,q)}(x)}{\bar{\Omega}^{2n}N(x)} + \frac{n}{d} =$$

$$= \frac{\Omega^3_{(i,j),(p,q)}(x)}{\bar{\Omega}^{2n}N(x)} + \frac{\Omega^{3,1,n-2,0,-1}(x)}{\bar{\Omega}^{2n}N(x)}$$

(11)

and we conclude that $\Omega^3_{(i,j),(p,q)}$ is an elementary landscape with $k = n$ and $\bar{\Omega}^3_{(i,j),(p,q)} = 1$.

Now, we are in conditions of presenting the main result of the paper, which is the following

**Theorem 1.** For the swap neighborhood defined above, the function $f$ defined in (4) is the sum of at most three elementary landscapes with constants $k_1 = 2n$, $k_2 = 2(n-1)$, and $k_3 = n$.

**Proof.** The functions $\varphi_{(i,j),(p,q)}$ defined above can be written using the auxiliary functions $\Omega^1_{(i,j),(p,q)}$, $\Omega^2_{(i,j),(p,q)}$, and $\Omega^3_{(i,j),(p,q)}$ when $i \neq j$ and $p \neq q$ as (we omit the subindices for clarity)

$$\varphi = \frac{1}{2n}\Omega^1 + \frac{1}{2(n-2)}\Omega^2 + \frac{1}{n(n-2)}\Omega^3$$

(12)

This can be easily appreciated with the help of Table 1.

| Condition | $\Omega^1_{2n}$ | $\Omega^2_{2(n-2)}$ | $\Omega^3_{n(n-2)}$ | $\sum$ |
|-----------|----------------|---------------------|---------------------|-------|
| $x(i) = p \wedge x(j) = q$ | $\frac{1}{2n}$ | $\frac{1}{2(n-2)}$ | $\frac{2n-3}{n(n-2)}$ | 1     |
| $x(i) = q \wedge x(j) = p$ | $\frac{1}{2n}$ | $\frac{1}{2(n-2)}$ | $\frac{2n-3}{n(n-2)}$ | 0     |
| $x(i) = p \oplus x(j) = q$ | $\frac{1}{2n}$ | $\frac{1}{2(n-2)}$ | $0$ | 0     |
| $x(i) = q \oplus x(j) = p$ | $\frac{1}{2n}$ | $\frac{1}{2(n-2)}$ | $0$ | 0     |
| $x(i) \neq p, q \wedge x(j) \neq p, q$ | $\frac{1}{2n}$ | $\frac{1}{2(n-2)}$ | $-\frac{1}{n(n-2)}$ | 0     |

Table 1: Elementary components and their sum.

Since the $\Omega$ family of functions are elementary, the $\varphi$ family of functions are a sum of three elementary components, namely: $\frac{1}{2n}\Omega^1$, $\frac{1}{2(n-2)}\Omega^2$, and $\frac{1}{n(n-2)}\Omega^3$. This decomposition of $\varphi$ allows us to write the fitness function $f$ as a decomposition of elementary landscapes in the following way:
The elementary components of \( f \) are:

\[
\begin{align*}
f_{c1} &= \sum_{i,j,p,q}^{n} \psi_{ijpq} \phi_{i,j}^{(p,q)} \frac{\Omega_{ij}^{1}(p,q)}{2n} \\
f_{c2} &= \sum_{i,j,p,q}^{n} \psi_{ijpq} \phi_{i,j}^{(p,q)} \frac{\Omega_{ij}^{2}(p,q)}{2(n-2)} \\
f_{c3} &= \sum_{i,j,p,q}^{n} \psi_{ijpq} \phi_{i,j}^{(p,q)} \frac{\Omega_{ij}^{3}(p,q)}{n(n-2)} + \sum_{i,p}^{n} \psi_{iipp} \phi_{i,i}^{(p,p)} \phi_{i,i}^{(p,p)} 
\end{align*}
\]

where the functions \( f_{c1}, f_{c2}, \) and \( f_{c3} \) are elementary with constants \( k_1 = 2n, \ k_2 = 2(n-1) \) and \( k_3 = n \), respectively, because they are a linear combination of elementary functions. Thus, \( f \) can be written in a compact form as

\[
f = f_{c1} + f_{c2} + f_{c3}.\]

In the statement of the theorem we say that the number of elementary components is three at most. That is, this number cannot be larger than three, but it could be lower. It is possible that for some particular instances the number of elementary landscapes could be reduced (we will see later that this happens for the TSP).

Unlike elementary landscapes, when the fitness function is the sum of several elementary components with different characteristic constants, the average value in the neighborhood does not linearly depend on the fitness function. Since, \( f_{c1}, f_{c2}, \) and \( f_{c3} \) are elementary components, the Grover’s wave equation (1) can be applied to them. And we can compute the average value in the neighborhood in the following way:
\[
\text{avg}\{f(y)\} = \text{avg}\{f_{c_1}(y)\} + \text{avg}\{f_{c_2}(y)\} + \text{avg}\{f_{c_3}(y)\}
\]
\[
= f_{c_1}(x) + f_{c_2}(x) + f_{c_3}(x) + \frac{k_1}{d} (\bar{f}_{c_1} - f_{c_1}(x)) + \frac{k_2}{d} (\bar{f}_{c_2} - f_{c_2}(x)) + \frac{k_3}{d} (\bar{f}_{c_3} - f_{c_3}(x)) =
\]
\[
f(x) + \frac{4}{n-1} (\bar{f}_{c_1} - f_{c_1}(x)) + \frac{4}{n} (\bar{f}_{c_2} - f_{c_2}(x)) + \frac{2}{n-1} (\bar{f}_{c_3} - f_{c_3}(x))
\]

4 Conclusions and Future Work

We have proven that QAP can be decomposed as a sum of three elementary landscapes when the swap neighborhood is used (Theorem 1). We have presented the exact expressions for the three components in Equations (13), (14), and (15). The elementary components of QAP allow one to compute the average value of the objective function in the neighborhood of a given solution \(x\) using the evaluation of the three components in \(x\).

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