1. The function \( c : \text{Irr} W \rightarrow \mathbb{N} \)

1.1. Let \( G \) be a simple algebraic group over an algebraically closed field whose characteristic is 0 or a good prime for \( G \). Let \( W \) be the Weyl group of \( G \). For any unipotent class \( C \) of \( G \) let \( E_C \) be the representation of \( W \) defined by Springer [S76] on the top \( l \)-adic cohomology of the Springer fibre \( B_u \) at an element \( u \in C \). (Here \( l \) is a fixed prime number \( \neq p \).) For any Weyl group \( W' \) let \( \text{Irr} W' \) be a set of representatives for the irreducible representations of \( W' \) over the \( l \)-adic numbers and let \( R_W \) be the free abelian group with basis \( \text{Irr} W' \). We can view \( E_C \) as an element of \( R_W \). According to Springer, there is a unique partition \( \text{Irr} W = \bigsqcup C \text{Irr}_C W \) (the “Springer partition”) where \( C \) runs over the unipotent classes of \( G \) such that \( \text{Irr}_C W \) consists of all \( E \in \text{Irr} W \) which appear in \( E_C \) with coefficient \( >0 \). We define \( \gamma : \text{Irr} W \rightarrow \mathbb{N} \) by \( \gamma(E) = \dim B_u \) where \( E \in \text{Irr}_C \) and \( u \in C \).

The purpose of this paper is to propose a definition of the Springer partition, of the collection of representations \( E_C \) and of the function \( \gamma : \text{Irr} W \rightarrow \mathbb{N} \) which is purely algebraic (without use of geometry) and which is suitable for computer calculations. The same objects can be obtained (without use of geometry) from the approach of [LY]; but that approach is more complicated than the present one and seems to be unsuitable for computer calculations.

1.2. Let \( W' \) be a Weyl group. For \( E' \in \text{Irr} W' \) the generic degree of the irreducible Hecke algebra representation \( E'_q \) corresponding to \( E' \) is of the form \( (1/m_{E'}) q^{a_{E'}} + \) terms of strictly higher degree in \( q \). (Here \( m_{E'} \) is a constant integer \( \geq 1 \).) In [L79] we have defined a partition of \( \text{Irr} W' \) into subsets called families. The definition is purely algebraic; it involves induction from parabolic subgroups, tensoring by the sign representation and the knowledge of the function \( \text{Irr} W' \rightarrow \mathbb{N}, E' \mapsto a_{E'} \).

It is known that \( E' \mapsto a_{E'} \) is constant on each family. Let \( \text{Irr}_{sp} W' \) be the subset of \( \text{Irr} W' \) consisting of the special representations. If \( E' \in \text{Irr} W' \) then there is a unique \( E'_0 \in \text{Irr}_{sp} W' \) in the same family as \( E' \) and \( n_{E'} = m_{E'_0}/m_{E'} \) is an integer \( \geq 1 \).

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1.3. Let $W_{af}$ be the affine Weyl group associated to the reductive group over $\mathbb{C}$ of type dual to that of $G$. Let $S_{af}$ be the set of simple reflections of $W_{af}$. We can identify $W$ with the quotient of $W_{af}$ by the group of translations. For any $K \not\subset S_{af}$, the subgroup of $W_{af}$ generated by $K$ is a finite Weyl group; we will identify it with its image $W_K$ under the canonical surjection $W_{af} \to W$.

For $E \in \text{Irr}W$ let $\Gamma(E)$ be the set of all pairs $(K, E')$ where $K \not\subset S_{af}$ and $E' \in \text{Irr}W_K$ is such that the multiplicity $(E : \text{Ind}_{W_K}^W(E'))$ is nonzero. We have $\Gamma(E) \not\subset \emptyset$. Let $c_E = \max\{a_{E'}; (K, E') \in \Gamma(E)\}$; here $a_{E'}$ is computed in terms of $W_K$. Let $\Gamma_M(E) = \{(K, E') \in \Gamma(E); a_{E'} = c_E\}$. We have $\Gamma_M(E) \not\subset \emptyset$. Let $N_E = \max\{n_{E'}; (K, E') \in \Gamma_M(E)\}$. Let $\sim$ be the equivalence relation on $\text{Irr}W$ generated by the relation $E_1 \sim E_2$ if there exist $(K_1, E'_1) \in \Gamma_M(E_1), (K_2, E'_2) \in \Gamma_M(E_2)$ such that $K_1 = K_2$ and $E'_1, E'_2$ are in the same family of $W_{K_1} = W_{K_2}$. The equivalence classes on $\text{Irr}W$ for $\sim$ are called $c$-families. Note that the function $E \mapsto c_E$ is constant on $c$-families. For any $c$-family $f \subset \text{Irr}W$ we set $e_f = \sum_{E \in f} N_E E \in R_W$.

**Proposition 1.4.** (a) For any $E \in \text{Irr}W$ we have $c_E = \gamma_E$.

(b) The Springer partition of $\text{Irr}W$ coincides with the partition of $\text{Irr}W$ into $c$-families.

(c) The collection of Springer representations $E_C$ of $W$ for various unipotent classes $C$ coincides with the collection of elements $e_f \in R_W$ for various $c$-families $f$ in $\text{Irr}W$.

1.5. For any $K \not\subset S_{af}$ we define a homomorphism $\mathcal{J}_{W_K}^W : R_{W_K} \to R_W$ by setting for any $E' \in \text{Irr}W_K$

$$\mathcal{J}_{W_K}^W(E') = \sum_{E \in \text{Irr}W : c_E = a_{E'}} (E : \text{Ind}_{W_K}^W(E')) E \in R_W.$$ 

Let $\phi$ be a family of $W_K$ and let $f$ be a $c$-family. We say that $(K, \phi)$ is adapted to $f$ if there exists a subset $\phi_*$ of $\phi$ such that

(a) $\mathcal{J}_{W_K}^W(E') \in f$ (and in particular is in $\text{Irr}W$) for any $E' \in \phi_*$;

(b) $E' \mapsto E := \mathcal{J}_{W_K}^W(E')$ is a bijection $\phi_* \sim \to f$;

(c) if $E', E$ are as in (b) then $n_{E'} = N_E$;

(d) if $E' \in \phi - \phi_*$ and $E \in \text{Irr}W$ appears in $\text{Ind}_{W_K}^W(E')$, then $c_E > a_{E'}$.

Note that $\phi_*$ above is unique (if it exists). Moreover the value of the $a$-function on $\phi$ must be equal to the value of the $c$-function on $f$. For $(K, \phi)$ adapted to $f$ we set $E'_{\phi_*} = \sum_{E' \in \phi_*, n_{E'} \in R_{W_K}} E' \in R_{W_K}$. We then have $\mathcal{J}_{W_K}^W(E'_{\phi_*}) = e_f$. Note that $\phi_*$ is not determined by the Coxeter group $W_K$ (it depends on $f$).

**Proposition 1.6.** Let $f \subset \text{Irr}W$ be a $c$-family. There exist $K \not\subset S_{af}$ and a family $\phi$ of $W_K$ such that $(K, \phi)$ is adapted to $f$.

1.7. Let $W'$ be a Weyl group; let $S'$ be the set of simple reflections in $W'$. For any $H \subset S'$ let $W_H'$ be the subgroup of $W'$ generated by $H$. Let $E' \in \text{Irr}_{sp}W'$. We say that $E'$ is non-rigid if there exists $H \not\subset S'$ and $E'' \in \text{Irr}_{sp}W_H'$ such that $E'$
appears in \(\text{Ind}_{W_H}^W(E')\) and \(a_{E''} = a_{E'}, m_{E''} = m_{E'}\). (Here \(a_{E''}, m_{E''}\) are defined relative to \(W'_H\).) We say that \(E' \in \text{Irr}_{sp}W'\) is rigid if it is not non-rigid. For any \(E' \in \text{Irr}_{sp}W'\) one can find \(H \subset S'\) and \(E'' \in \text{Irr}_{sp}W'_H\) such that \(E''\) is rigid, \(E'\) appears in \(\text{Ind}_{W_H}^W(E'')\) and \(a_{E''} = a_{E'}, m_{E''} = m_{E'}\). Moreover, \((H, E'')\) is uniquely determined by \(E'\) up to conjugation by an element of \(W'\). (A statement close to this appears in [L84, 13.1].) A family of \(W'\) is said to be rigid if the unique special representation in it is rigid. If \(W'\) is a product of two Weyl groups \(W'_1, W'_2\), the rigid special representations of \(W'\) are precisely the external tensor products of a rigid special representation of \(W'_1\) with one of \(W'_2\).

We denote by \(\epsilon\) the sign representation of \(W'\). It is rigid special. If \(W'\) is of type \(A_n, \epsilon\) is the only rigid special representation of \(W'\).

In the next subsection we list the rigid special representation for \(W'\) irreducible of low rank. We use the following notation. If \(W'\) is of type \(E_6, E_7\) or \(E_8\), an element \(E' \in \text{Irr}W'\) can be represented uniquely in the form \(\delta_x\) where \(d = \dim(E')\) and \(x\) is the smallest integer \(\geq 0\) such that \(E'\) appears in the \(x\)-th symmetric power of the reflection representation of \(W'\). The same notation can be used in type \(F_4, G_2\) but in these cases there may be two elements \(E' \in \text{Irr}W'\) with the same \(\delta_x\). (This ambiguity does not appear for rigid special representations.) If \(W'\) is of type \(D_n, n \geq 4\) we represent an \(E' \in \text{Irr}W'\) as a sequence \(a_1a_2...a_{2k}\) where \([a_2k < a_{2k-2} < \cdots < a_2, a_{2k-1} < a_{2k-3} < \cdots < a_1]\) is the symbol representing \(E'\) in [L84]. If \(W'\) is of type \(B_n, n \geq 2\) we represent an \(E' \in \text{Irr}W'\) as a sequence \(a_0a_1a_2...a_{2k}\) where \([a_2k < a_{2k-2} < \cdots < a_0, a_{2k-1} < a_{2k-3} < \cdots < a_1]\) is the symbol representing \(E'\) in [L84]. When \(n = 2\) this notation depends on the order of \(S'\); the representations 201, 120 are interchanged when the order of \(S'\) is reversed; again this ambiguity does not appear for rigid special representations.

1.8. Here are the rigid special representations for \(W'\) assumed to be irreducible of low rank.

Type \(B_2\): 210, 22110 = \(\epsilon\) (with \(a = 1, 4\) respectively);
Type \(B_3\): 32110, 332110 = \(\epsilon\) (with \(a = 4, 9\) respectively);
Type \(B_4\): 32210, 4322110, 443322110 = \(\epsilon\) (with \(a = 6, 9, 16\) respectively);
Type \(D_5\): 3210, 332110, 43322110 = \(\epsilon\) (with \(a = 3, 7, 12\) respectively);
Type \(D_5\): 432110, 44322110, 5443322110 = \(\epsilon\) (with \(a = 7, 13, 20\) respectively);
Type \(D_6\): 432210, 54322110, 443322110, 5543322110, 655443322110 = \(\epsilon\) (with \(a = 10, 13, 16, 21, 30\) respectively);
Type \(E_7\): 433210, 54332110, 6543322110, 55443322110, 665443322110 = \(\epsilon\) (with \(a = 12, 16, 21, 24, 29, 42\) respectively);
Type \(E_7\): 433210, 54332110, 6543322110, 55443322110, 665443322110, 77655443322110, 87665443322110 = \(\epsilon\) (with \(a = 13, 18, 21, 24, 29, 31, 34, 42, 43, 56\) respectively);
Type \(E_8\): 807, 3015, 625, 136 = \(\epsilon\);
Type \(E_7\): 51211, 31516, 12025, 5630, 2737, 746, 163 = \(\epsilon\);
Type \(E_8\): 448016, 420034, 409626, 224028, 140032, 140037, 70042, 56047, 21052, 11263, 3574, 891, 1120 = \(\epsilon\);
Type $F_4$: $12_4, 9_{10}, 4_{13}, 1_{24} = \epsilon$

Type $G_2$: $2_1, 1_6 = \epsilon$

We now state a refinement of Proposition 1.6.

**Proposition 1.9.** Let $\mathfrak{f} \subset \text{Irr} W$ be a $c$-family. There exist $K \subseteq S_{af}$ and a rigid family $\phi$ of $W_K$ such that $(K, \phi)$ is adapted to $\mathfrak{f}$.

## 2. Tables

**2.1.** In the tables below, for $W$ of type $E_8, E_7, E_6, F_4, G_2$ we make a list of of the elements $e_\mathfrak{f}$ (see 1.3) for each $c$-family $\mathfrak{f}$ of $W$. (Note that $\mathfrak{f}$ is determined by $e_\mathfrak{f}$.) In each case we list the value of the $c$-function on $\mathfrak{f}$ and we specify the pairs $(K, \phi)$ with $\phi$ a rigid family of $W_K$ such that $(K, \phi)$ is adapted to $\mathfrak{f}$. When there are several such pairs $(K, \phi)$ which are obtained one from another by conjugating by an element of $W$ or by applying an automorphism of the affine diagram of $W_{af}$, we list only one pair out of these. In each case there remain one or two pairs to be listed. In each case we specify $K$ by the type of the corresponding subdiagram, as a name or as a picture (with the elements of $K$ marked by $\bullet$). The irreducible representations which enter in $\phi$ are denoted using the conventions in 1.7. In type $F_4$ the notation $\delta_x$ in 1.7 is ambiguous; when it represents two objects, we distinguish them by denoting them by $\delta'_x, \delta''_x$ (in a way compatible with the conventions in [Ch]); the table in 2.5 could be taken as definition of $\delta'_x, \delta''_x$ as well as a definition of $201, 120$ for type $B_2$.

In type $G_2$ there are two objects represented by $1_3$; we write them as $1_3, \tilde{1}_3$ (they are defined by the table in 2.6).

**2.2. Table for type $E_8$.**

$c = 0; 1_0 = J_{W_K}(\epsilon), K = \emptyset$

$c = 1; 8_1 = J_{W_K}(\epsilon), K = (\bullet \cdots \bullet);$

$c = 2; 35_2 = J_{W_K}(\epsilon), K = (\bullet \bullet \bullet \cdots);$

$c = 3; 112_3 + 28_8 = J_{W_K} (3210 + 3201), K = (\bullet \bullet \bullet \bullet \cdot \cdots);$

$c = 4; 84_4 = J_{W_K}(\epsilon), K = (\bullet \bullet \bullet \bullet \cdots);$ 

$c = 4; 210_4 + 160_7 = J_{W_K} (3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon), K = (\bullet \bullet \bullet \bullet \cdot \cdots);$ 

$c = 5; 560_5 + 50_8 = J_{W_K} (3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon), K = D_4 \times (A_1 \times A_1);$ 

$c = 6; 567_6 = J_{W_K}(\epsilon), K = (\bullet \bullet \bullet \bullet \bullet \bullet);$ 

$c = 6; 700_6 + 300_8 = J_{W_K} (3210 \boxtimes \epsilon + 3201 \boxtimes \epsilon), K = D_4 \times A_2;$ 

$c = 7; 400_7 = J_{W_K}(\epsilon), K = (\bullet \bullet \bullet \bullet \bullet);$ 

$c = 7; 1400_7 + 2 \times 1008_9 + 561_9 = J_{W_K} (80_7 + 2 \times 90_8 + 201_10), K = E_6;$ 

$c = 8; 1344_8 = J_{W_K}(\epsilon), K = (\bullet \bullet \bullet \bullet \bullet \bullet);$ 

$c = 8; 1400_8 + 2 \times 1575_{10} + 350_{14} = J_{W_K} (80_7 \boxtimes \epsilon + 2 \times 90_8 \boxtimes \epsilon + 201_10 \boxtimes \epsilon), K = E_6 \times A_1;$ 

$c = 9; 3240_9 + 1050_{10} = J_{W_K} (3210 \boxtimes \epsilon + 3120 \boxtimes \epsilon) = J_{W_{K'}} (432110 \boxtimes \epsilon + 423110 \boxtimes \epsilon), K = D_4 \times A_3, K' = D_5 \times (A_1 \times A_1);$
\[ c = 9; \quad 448_9 = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 10; \quad 2240_{10} + 2 \times 17512 + 840_{13} = \mathcal{J}^W_{W_K}(80_7 \boxplus \epsilon + 2 \times 10_9 \boxplus \epsilon + 20_{10} \boxplus \epsilon), \quad K = E_6.A_2; \]

\[ c = 10; \quad 2268_{10} + 1296_{13} = \mathcal{J}^W_{W_K}(432210 + 432201), \quad K = D_6; \]

\[ c = 11; \quad 4096_{11} + 4096_{12} = \mathcal{J}^W_{W_K}(512_{11} + 512_{12}) = \mathcal{J}^W_{W_K'}(432210 \boxplus \epsilon + 432201 \boxplus \epsilon), \quad K = E_7, \quad K' = D_6.A_1; \]

\[ c = 11; \quad 1400_{11} = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 12; \quad 525_{12} = \mathcal{J}^K_{W_K}(e), \quad K = D_4; \]

\[ c = 12; \quad 972_{12} = \mathcal{J}^K_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 12; \quad 4200_{12} + 3360_{13} = \mathcal{J}^W_{W_K}(4323210 + 433201) = \mathcal{J}^W_{W_K'}(512_{11} \boxplus \epsilon + 512_{12} \boxplus \epsilon), \quad K = D_7, \quad K' = E_7 \times A_1; \]

\[ c = 13; \quad 4536_{13} + 840_{14} = \mathcal{J}^W_{W_K}(432110 \boxplus \epsilon + 431201 \boxplus \epsilon) = \mathcal{J}^W_{W_K'}(443210 + 443201), \quad K = D_5 \times A_3, \quad K' = D_8; \]

\[ c = 13; \quad 2800_{13} + 2100_{16} = \mathcal{J}^W_{W_K}(54322110 + 54321201), \quad K = D_6; \]

\[ c = 14; \quad 6075_{14} + 700_{16} = \mathcal{J}^W_{W_K}(54322110 \boxplus \epsilon + 53422110 \boxplus \epsilon), \quad K = D_6 \times A_1; \]

\[ c = 14; \quad 2835_{14} = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 15; \quad 4200_{15} = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 15; \quad 5600_{15} + 2400_{17} = \mathcal{J}^W_{W_K}(30_{15} + 15_{17}), \quad K = E_6; \]

\[ c = 16; \quad 4480_{16} + 5 \times 4536_{18} + 4 \times 5670_{18} + 5 \times 1400_{20} + 6 \times 1680_{22} + 4 \times 70_{32} = \mathcal{J}^W_{W_K}(4480_{16} + 5 \times 4536_{18} + 4 \times 5670_{18} + 5 \times 1400_{20} + 6 \times 1680_{22} + 4 \times 70_{32}, \quad K = E_8; \]

\[ c = 16; \quad 3200_{16} = c^W_{W_K}(e), \quad K = A_5 \times A_1; \]

\[ c = 17; \quad 7168_{17} + 5600_{19} + 448_{25} = \mathcal{J}^W_{W_K}(315_{16} \boxplus \epsilon + 2 \times 280_{18} \boxplus \epsilon + 35_{22} \boxplus \epsilon), \quad K = E_7 \times A_1; \]

\[ c = 18; \quad 3150_{18} + 1134_{20} = \mathcal{J}^W_{W_K}(30_{15} \boxplus \epsilon + 15_{17} \boxplus \epsilon), \quad K = E_6 \times A_2; \]

\[ c = 18; \quad 4200_{18} + 2688_{20} = \mathcal{J}^W_{W_K}(54432110 + 54431201), \quad K = D_8; \]

\[ c = 19; \quad 1344_{19} = \mathcal{J}^W_{W_K}(44322110 \boxplus \epsilon), \quad K = D_5 \times A_3; \]

\[ c = 19; \quad 2016_{19} = \mathcal{J}^W_{W_K}(e), \quad K = A_5 \times A_2 \times A_1; \]

\[ c = 20; \quad 4200_{20} = \mathcal{J}^W_{W_K}(e), \quad K = A_4 \times A_4; \]

\[ c = 20; \quad 2100_{20} = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 21; \quad 4200_{21} + 168_{24} = \mathcal{J}^W_{W_K}(54322110 + 54342120), \quad K = D_8; \]

\[ c = 21; \quad 5600_{21} + 2400_{23} = \mathcal{J}^W_{W_K}(6543322110 + 654321201), \quad K = D_7; \]

\[ c = 22; \quad 2835_{22} = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 22; \quad 3200_{22} = \mathcal{J}^W_{W_K}(e), \quad K = D_6 \times (A_1 \times A_1); \]

\[ c = 22; \quad 6075_{22} = \mathcal{J}^W_{W_K}(5543322110 \boxplus \epsilon), \quad K = D_6 \times A_1; \]

\[ c = 23; \quad 4536_{23} = \mathcal{J}^W_{W_K}(e), \quad K = (\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet); \]

\[ c = 24; \quad 4200_{24} + 3360_{25} = \mathcal{J}^W_{W_K}(6543322110 + 6544231201) = \mathcal{J}^W_{W_K'}(4200_{24} + 3360_{25}), \quad K = D_8, \quad K' = E_8; \]

\[ c = 25; \quad 2800_{25} + 2100_{28} = \mathcal{J}^W_{W_K}(120_{25} + 105_{26}), \quad K = E_7; \]

\[ c = 26; \quad 840_{26} = \mathcal{J}^W_{W_K}(e), \quad K = D_5.A_3; \]

\[ c = 26; \quad 4096_{26} + 4096_{27} = \mathcal{J}^W_{W_K}(4096_{26} + 4096_{27}) = \mathcal{J}^W_{W_K'}(120_{25} \boxplus \epsilon + 105_{26} \boxplus \epsilon), \]
$K = E_8$, $K' = E_7 \times A_1$;
$c = 28$; $700_{28} = J_{W_K}^W (\epsilon) = J_{W_{K'}}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet) =$ $A_8$;
$c = 30$; $2240_{28} + 840_{31} = J_{W_K}^W (2240_{28} + 840_{31}), K = E_8$;
$c = 29$; $1400_{29} = J_{W_K}^W (\epsilon) = J_{W_{K'}}^W (5544332110), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet) =$ $K' = D_8$;
$c = 30$; $2268_{30} + 1296_{33} = J_{W_K}^W (56_{30} + 21_{33}), K = E_7$;
$c = 31$; $3240_{31} + 972_{32} = J_{W_K}^W (765443322110 + 756443322110) =$ $J_{W_{K'}}^W (56_{30} \otimes \epsilon + 35_{31} \otimes \epsilon), K = D_8, K' = E_7 \times A_1$;
$c = 32$; $1400_{32} + 2 \times 1575_{34} + 350_{38} = J_{W_K}^W (1400_{32} + 2 \times 1575_{34} + 350_{38}), K = E_8$;
$c = 34$; $1050_{34} = J_{W_K}^W (665543322110), K = D_8$;
$c = 36$; $175_{36} = J_{W_K}^W (\epsilon), K = A_8$;
$c = 36$; $525_{36} = J_{W_K}^W (\epsilon), K = E_6$;
$c = 37$; $1400_{37} + 2 \times 1008_{39} + 564_{49} = J_{W_K}^W (1400_{37} + 2 \times 1008_{39} + 564_{49}), K = E_8$;
$c = 38$; $1344_{38} = J_{W_K}^W (2737 \otimes \epsilon), K = E_7 \times A_1$;
$c = 39$; $448_{39} = J_{W_K}^W (\epsilon), K = E_6 \times A_2$;
$c = 42$; $700_{42} + 300_{44} = J_{W_K}^W (700_{42} + 300_{44}), K = E_8$;
$c = 43$; $400_{43} = J_{W_K}^W (77655443322110), K = D_8$;
$c = 46$; $567_{46} = J_{W_K}^W (746), K = E_7$;
$c = 47$; $560_{47} = J_{W_K}^W (746 \otimes \epsilon) = J_{W_{K'}}^W (560_{47}), K = E_7 \times A_1, K' = E_8$;
$c = 52$; $210_{52} + 160_{55} = J_{W_K}^W (210_{52} + 160_{55}), K = E_8$;
$c = 56$; $50_{56} = J_{W_K}^W (877655443322110), K = D_8$;
$c = 63$; $112_{63} + 28_{68} = J_{W_K}^W (112_{63} + 28_{68}), K = E_8$;
$c = 64$; $84_{64} = J_{W_K}^W (\epsilon), K = E_7 \times A_1$;
$c = 74$; $35_{74} = J_{W_K}^W (35_{74}), K = E_8$;
$c = 91$; $89_{91} = J_{W_K}^W (89_{91}), K = E_8$;
$c = 120$; $1120 = J_{W_K}^W (\epsilon), K = E_8$.

2.3. Table for type $E_7$.
$c = 0$; $1_0 = J_{W_K}^W (\epsilon), K = \emptyset$;
$c = 1$; $7_1 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 2$; $27_2 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 3$; $56_3 + 21_6 = J_{W_K}^W (3210 + 3201), K = D_4$;
$c = 3$; $21_3 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 4$; $35_4 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 4$; $120_4 + 105_5 = J_{W_K}^W (3210 \otimes \epsilon + 3201 \otimes \epsilon), K = D_4 \times A_1$;
$c = 5$; $189_5 + 15_7 = J_{W_K}^W (3210 \otimes \epsilon + 3120 \otimes \epsilon), K = D_4 \times (A_1 \times A_1)$;
$c = 6$; $105_6 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 6$; $168_6 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 6$; $210_6 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$;
$c = 7$; $315_7 + 2 \times 280_9 + 351_3 = J_{W_K}^W (807 + 2 \times 909 + 2010), K = E_6$;
$c = 7$; $189_7 = J_{W_K}^W (\epsilon), K = (\ldots \bullet \bullet \bullet \bullet \bullet \bullet \bullet)$.
2.4. Table for type $E_6$.

$c = 0$; $1_0 = \mathcal{J}_{W_K}^W(\epsilon), K = \emptyset$;

$c = 1$; $6_1 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \end{pmatrix}$;

$c = 2$; $20_2 = \mathcal{J}_{W_K}^W(\epsilon), K = \begin{pmatrix} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \end{pmatrix}$;
$c = 3; \ 30_3 + 15_5 = J^{W}_{W, K}(3210 + 3201), K = D_4;$
$c = 4; \ 15_4 = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 4; \ 64_4 = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 5; \ 60_5 = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 6; \ 24_6 = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 6; \ 81_6 = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 7; \ 80_7 + 2 \times 90_8 + 20_{10} = J^{W}_{W, K}(807 + 2 \times 908 + 2010), K = E_6;$
$c = 8; \ 60_8 = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 9; \ 10_9 = J^{W}_{W, K}(e), K = A_2 \times A_2 \times A_2;$
$c = 10; \ 81_{10} = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 11; \ 60_{11} = J^{W}_{W, K}(e), K = \bullet \bullet \bullet \bullet$;
$c = 12; \ 24_{12} = J^{W}_{W, K}(e), K = D_4;$
$c = 13; \ 64_{13} = J^{W}_{W, K}(44322110), K = D_5;$
$c = 15; \ 30_{15} + 15_{17} = J^{W}_{W, K}(30_{15} + 15_{17}), K = E_6;$
$c = 16; \ 15_{16} = J^{W}_{W, K}(e), K = A_5 \times A_1;$
$c = 20; \ 20_{20} = J^{W}_{W, K}(e), K = D_5;$
$c = 25; \ 6_{25} = J^{W}_{W, K}(6_{25}), K = E_6;$
$c = 36; \ 13_{6} = J^{W}_{W, K}(13_{6}), K = E_6;$

2.5. Table for type $F_4$.

In this table we denote by $A_2$ (resp. $A'_2$) a subset $K \subseteq S_{af}$ of type $A_2$ which is contained (resp. not contained) in a subset of $S_{af}$ of type $A_3$.

$c = 0; \ 1_0 = J^W_{W, K}(e), K = \emptyset;$
$c = 1; \ 4_1 + 2''_4 = J^W_{W, K}(210 + 201), K = B_2;$
$c = 2; \ 9_2 + 2'_4 = J^W_{W, K}(210 \boxtimes e + 120 \boxtimes e), K = B_2 \times A_1;$
$c = 3; \ 8''_3 = J^W_{W, K}(e), K = A'_2;$
$c = 3; \ 8'_3 = J^W_{W, K}(e) = J^W_{W, K'}(e), K = A_2, K' = A_1 \times A_1 \times A_1;$
$c = 4; \ 12_4 + 3 \times 9'_6 + 2 \times 6''_6 + 3 \times 1''_1 = J^W_{W, K}(12_4 + 3 \times 9'_6 + 2 \times 6''_6 + 3 \times 1''_1), K = F_4;$
$c = 5; \ 16_5 + 4''_7 = J^W_{W, K}(32110 \boxtimes e + 31201 \boxtimes e), K = C_3 \times A_1;$
$c = 6; \ 9'_6 + 4_8 = J^W_{W, K}(32210 + 23120), K = B_4;$
$c = 6; \ 6'_6 = J^W_{W, K}(e), K = A_2 \times A'_2;$
$c = 7; \ 4'_4 = J^W_{W, K}(e), K = A_3 \times A_1;$
$c = 9; \ 8'_9 + 1_12 = J^W_{W, K}(4322110 + 4231201), K = B_4;
2.6. Table for type $G_2$.

$c = 9; \ 8''_9 = J_{W_K}(\epsilon), K = C_3$;
$c = 10; \ 9_{10} = J_{W_K}(\epsilon) = J_{W_K}(9_{10}), K = C_3 \times A_1, K' = F_4$;
$c = 13; \ 4_{13} + 2''_{16} = J_{W_K}(4_{13} + 2''_{16}), K = F_4$;
$c = 16; \ 2_{16} = J_{W_K}(\epsilon), K = B_4$;
$c = 24; \ 1_{24} = J_{W_K}(1_{24}), K = F_4$.

3. Comments

3.1. The tables in §2 can be obtained by combining the induction/restriction tables in [A05] with the explicit knowledge of the $a$-function in [L84]. This is how I first tried to obtain them. (While doing that I found that, in [L84], the entry $[189c]$ on p.364, line -5, should be replaced by $[189'c]$.) But eventually I used instead the induction/restriction tables in the CHEVIE package [Ch] combined with a package to compute the $a$-function kindly supplied to me by Meinolf Geck. I am very grateful to Gongqin Li for doing the necessary programming.

3.2. Now Propositions 1.4, 1.9 (and hence 1.6) follow (in the case of exceptional groups) from the tables in §2 and the explicit determination of the Springer representations (see [S85] and the references there). The case of classical groups requires additional arguments; it will be considered elsewhere. One ingredient in the proof is the interpretation [L89] of the function $\gamma : \text{Irr} \to \mathbb{N}$ in 1.1 in terms of the $a$-function on $W_{af}$; this can be used to prove the inequality $a_{E'} \leq \gamma(E)$ for any $E \in \text{Irr}G$ and any $(K, E') \in \Gamma(E)$. In addition to this, one has to use combinatorial arguments similar to those used in [L09].

3.3. Recall that in [L86] an algorithm is given which determines the (ordinary) Green functions in terms of the function $\gamma : \text{Irr} \to \mathbb{N}$. Since by Proposition 1.4, this function is equal to the explicitly computable function $c : \text{Irr} \to \mathbb{N}$, we see that the Green functions can now be calculated by a computer without the input of the Springer correspondence.

3.4. Now Proposition 1.4 can be viewed as a refinement of a statement in [L79a] according to which the unipotent classes of $G$ can be indexed in terms of pairs $(K, \phi)$ with $K \subseteq S_{af}$ and $\phi$ is a family of $W_K$. (That statement has been verified in [AL] and [L09].) Proposition 1.6 can be viewed as a refinement of a statement in [L84, p.346] according to which the size of the group of components of the centralizer of an element in a unipotent class $C$ of $G$ can be described in terms of pairs $(K, \phi)$ as above. (That statement has been verified in [L09].)
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