Independent nonclassical tests for states and measurements in the same experiment

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Abstract

We show that a single experiment can test simultaneously and independently both the nonclassicality of states and of measurements by the violation or fulfillment of classical bounds on the statistics. Nonideal measurements affected by imperfections can be characterized by two bounds depending on whether we test the ideal measurement or the real one.

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1. Introduction

Within standard quantum theory, quantum states play two dissimilar but complementary roles: (i) they express the state of the system, represented by a density matrix $\rho$, and (ii) they determine the statistics of measurements, typically by projection of the system state on the eigenstates of the measured observable. More precisely, any observable event is represented by a non-negative Hermitian operator $\Delta$ (maybe part of a larger positive operator-valued measure) that determines the event probability as $p = \text{tr}(\rho \Delta)$, where $\rho$ is the state of the system. In many relevant situations, $\Delta$ is proportional to a suitable state, such as photon-number and quadrature measurements in quantum optics. Positive operators playing the role of $\Delta$ can be turned into the role $\rho$ as shown in [1].

While making reference to nonclassical states is quite common, not so much effort has been devoted to nonclassical measurements [2–5]. A customary criterion of nonclassicality for states $\rho$ is the failure of the Glauber–Sudarshan $P$ function to exhibit all the properties of a classical probability density [6]. This occurs when $P$ takes negative values or when it fails to be a proper function becoming more singular than the delta function. Accordingly, we may say that the event represented by $\Delta$ in Hilbert space is nonclassical when its $P$ phase-space representative takes negative values or is more singular than the delta function.

Although the nonclassicality of states and of measurements are different things, both may be tested simultaneously within a single experiment in terms of its statistics $p$. This possibility is addressed in this paper by means of a simple example: nonefficient single-photon detection in photon-added thermal states [7]. A key point of this example is feasibility, since these states have already been generated in experiment [8], and it explicitly includes typical imperfections such as losses and thermalization.

As nonclassicality criteria we will consider the simple and robust practical tests recently introduced where nonclassicality is revealed by breaking classical bounds on probabilities satisfied by all classical states and measurements [4, 9] (see [10] for other nonclassicality criteria). The main features of these tests are summarized in section 2.

2. Classical bounds on probabilities

For definiteness let us focus on a single mode of the electromagnetic field with complex-amplitude operator $a$.

To derive the nonclassical tests we will use the $P$ and $Q$ phase-space representatives associated with any operator $A$

$$A = \int d^2 \alpha P_A(\alpha) |\alpha\rangle \langle \alpha|, \quad Q_A(\alpha) = \frac{1}{\pi} \langle \alpha| A |\alpha\rangle,$$

(1)
where $|\alpha\rangle$ are coherent states, $a|\alpha\rangle = \alpha|\alpha\rangle$. They are suitably normalized

$$
\int d^2\alpha P_\alpha(\alpha) = \int d^2\alpha Q_\alpha(\alpha) = \text{tr} A,
$$

with $d^2\alpha = dx\,dy$, where $x$ and $y$ are the real and imaginary parts of $\alpha = x + iy$.

Exploiting the $\rho \leftrightarrow \Delta$ symmetry, the same probability $p = \text{tr} (\rho \Delta)$ can be expressed by two equivalent formulæ

$$
p = \pi \int d^2\alpha P_\rho(\alpha)Q_\Delta(\alpha) = \pi \int d^2\alpha P_\Delta(\alpha)Q_\rho(\alpha).
$$

By using the first equality we are able to derive bounds sensitive to the nonclassicality of the state $\rho$, while the second equality leads to bounds sensitive to the nonclassicality of the measurement $\Delta$. Note that the $Q$ function is always positive and well behaved.

### 2.1. Nonclassical test for states

For classical states, i.e. for ordinary non-negative functions $P_\rho(\alpha) \geq 0$, we obtain

$$
P_\rho(\alpha)Q_\Delta(\alpha) \leq P_\rho(\alpha)Q_{\Delta,\max},
$$

where $Q_{\Delta,\max}$ is the maximum of $Q_\Delta(\alpha)$ when $\alpha$ is varied. Applying this to the first equality in (3) and taking into account (2), we obtain the following upper bound $S$ for $p$,

$$
p \leq S = \pi Q_{\Delta,\max}
$$

which holds for every $P_\rho(\alpha)$ compatible with classical physics. If this condition is violated it means that (4) is false and the state is not classical.

### 2.2. Nonclassical test for measurements

For classical measurements, i.e. for ordinary non-negative functions $P_\Delta(\alpha) \geq 0$, it holds that

$$
P_\Delta(\alpha)Q_\rho(\alpha) \leq P_\Delta(\alpha)Q_{\rho,\max},
$$

where $Q_{\rho,\max}$ is the maximum of $Q_\rho(\alpha)$ when $\alpha$ is varied. Applying this to the second equality in (3) we obtain the following upper bound $M$ for $p$, provided that tr$\Delta$ is finite,

$$
p \leq M = \pi Q_{\rho,\max}\text{tr} \Delta.
$$

Equation (7) can be violated if $P_\Delta(\alpha)$ fails to be positive or when it becomes a generalized function (this is a nonclassical measurement), since in both cases (6) fails to be true.

### 3. Inefficient photon detection on photon-added thermal states

The same probability $p$ may serve to test both the nonclassicality of $\rho$ and $\Delta$. Let us demonstrate this by applying the above formalism to nonefficient single-photon detection in photon-added thermal states.

#### 3.1. Single-photon-added thermal states

The single-photon-added thermal states read, in the photon-number basis [7, 8],

$$
\rho = (1 - \xi)|\alpha\rangle\langle\alpha| + |\xi\rangle\langle\xi| + \sum_{n=1}^{\infty} |n\rangle\langle n|,
$$

where $\rho_{\xi}$ is a thermal chaotic state

$$
\rho_{\xi} = (1 - \xi)|\xi\rangle\langle\xi| + \sum_{n=0}^{\infty} |n\rangle\langle n|,
$$

with mean number of photons

$$
\bar{n} = \frac{\xi}{1 - \xi}.
$$

The $P$ representative of $\rho$ is well behaved but nonpositive

$$
P_{\rho}(\alpha) = \frac{1}{\pi \bar{n}^\frac{3}{2}} \left[ (\bar{n} + 1)|\alpha|^2 - \bar{n} \right] \exp\left( - \frac{|\alpha|^2}{\bar{n}} \right),
$$

while the $Q$ function is

$$
Q_{\rho}(\alpha) = \frac{|\alpha|^2}{\pi (\bar{n} + 1)^{\frac{3}{2}}} \exp\left( - \frac{|\alpha|^2}{\bar{n} + 1} \right),
$$

so that its maximum occurs for $|\alpha|^2 = \bar{n} + 1$,

$$
Q_{\rho,\max} = \frac{1}{\pi e^{\bar{n} + 1}}.
$$

These states present three relevant features for our purposes:

(i) their nonclassical behavior is independent of other typical nonclassical features, because there is no quadrature squeezing, they present super-Poissonian photon-number statistics for all $\bar{n} > 1/\sqrt{2}$ and they have no oscillatory statistics [4];

(ii) they can be generated experimentally [8];

(iii) their definition embodies a typical source of practical imperfection such as thermalization.

#### 3.2. Ideal single-photon detection

For ideal single-photon detection, we have, in the photon-number basis, $\Delta = |1\rangle\langle 1|$ with tr$\Delta = 1$. The $P$ representative is nonclassical being more singular than the delta function

$$
P_{\Delta}(\alpha) = \left( 1 + \frac{\partial^2}{\partial \alpha^2 \partial \alpha^*} \right) \delta^{(2)}(\alpha).
$$

The $Q$ function is

$$
Q_{\Delta}(\alpha) = \frac{|\alpha|^2}{\pi} \exp\left( - |\alpha|^2 \right)
$$

and the maximum occurs at $|\alpha| = 1$,

$$
Q_{\Delta,\max} = \frac{1}{e\pi}.
$$

If the measured state is classical, the single-photon probability $p$ is bounded by [4, 9]

$$
p \leq S = \frac{1}{e}.
$$
3.3. Inefficient single-photon detection

In figure 1, we illustrate the case of inefficient single-photon detection. A detector with quantum efficiency \( \eta \) can be modeled by a beam splitter of amplitude-transmission coefficient \( t = \sqrt{\eta} \), mixing the input state \( \rho \) with vacuum, placed before an ideal detector \( \Delta \) with \( \eta = 1 \) [11]. Following this model two different routes can be taken:

(i) we can test the underlying ideal detection \( \Delta \) regarding inefficiency as a handicap of practical origin. This is to say, we have ideal detection on the state \( \tilde{\rho} \) after the beam splitter that carries the effect of inefficiency, so that the probability is \( p = \text{tr}(\tilde{\rho} \Delta) \). Since the transformation of coherent states through lossless beam splitters is \( |\alpha\rangle \rightarrow |\sqrt{\eta}\alpha\rangle \) the state \( \tilde{\rho} \) is

\[
\tilde{\rho} = \int d^2\alpha P_\rho(\alpha)|\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha|,
\]

where \( P_\rho(\alpha) \) is in (11). From this expression, we obtain by direct computation the \( Q \) function of \( \tilde{\rho} \)

\[
Q_\rho(\alpha) = \frac{1}{\pi} \left[ \frac{(\bar{n} + 1)|\alpha|^2}{(n\bar{n} + 1)^3} + \frac{1 - \eta - |\alpha|^2}{(n\bar{n} + 1)^2} \right] \exp\left( \frac{-|\alpha|^2}{n\bar{n} + 1} \right).
\]

For \( \eta \neq 0 \), its maximum holds for

\[
|\alpha|^2 = 1 + \eta \bar{n} = \frac{(1 - \eta)(1 + \eta\bar{n})}{\eta(n + 1)},
\]

yielding

\[
Q_{\tilde{\rho},\text{max}} = \frac{\eta(n + 1)}{\pi(n\bar{n} + 1)^2} \exp\left[ -\frac{\eta\bar{n} + 2\eta - 1}{\eta(n + 1)} \right];
\]

(ii) alternatively, we can test the real measurement embodying the inefficiency as part of the measuring apparatus. The real measurement is represented by a Hermitian non-negative operator \( \Delta \) to be determined such that the probability can be expressed as \( p = \text{tr}(\rho \Delta) \). From (3), (18) and the equality \( p = \text{tr}(\tilde{\rho} \Delta) = \text{tr}(\rho \Delta) \), we obtain

\[
Q_\Delta(\alpha) = Q_\Delta(\sqrt{\eta}\alpha) = \frac{\eta|\alpha|^2}{\pi} \exp(-\eta|\alpha|^2),
\]

so that

\[
Q_{\Delta,\text{max}} = Q_{\Delta,\max} = \frac{1}{\pi e}.
\]

Figure 1. Illustration of inefficient single-photon detection.

Figure 2. Plots of \( \rho \) (solid), \( M_{\Delta} \) (dashed), \( M_{\tilde{\Delta}} \) (dotted) and \( S \) (dash-dotted) as a function of \( \eta \) for fixed \( \eta = 0.4 \) (a) and \( \eta = 0.9 \) (b).

From (2), (22) and \( \text{tr}\Delta = 1 \), we readily obtain

\[
\text{tr} \Delta = \frac{1}{\eta}.
\]

Moreover, by expressing the exponential in (22) as \( \exp(-\eta|\alpha|^2) = \exp[(1 - \eta)|\alpha|^2] \exp(-|\alpha|^2) \) and expanding the first exponential in power series, we obtain

\[
Q_{\Delta}(\alpha) = \frac{\eta}{\pi} \sum_{n=0}^{\infty} (n + 1)(1 - \eta)\frac{|\alpha|^{2(n+1)}}{(n + 1)!} \exp(-|\alpha|^2),
\]

which readily provides the expression of \( \tilde{\Delta} \) in the photon-number basis

\[
\tilde{\Delta} = \eta \sum_{n=0}^{\infty} (n + 1)(1 - \eta)^n|n + 1\rangle\langle n + 1| = \eta(1 - \eta)a^\dagger a^\dagger a.
\]

The probability \( p \) is independent of the interpretations (i) and (ii), being (from (8) and (26) for example)

\[
p = \text{tr}(\tilde{\rho} \Delta) = \text{tr}(\rho \Delta) = \frac{1 + 2\eta - \eta\bar{n}}{(1 + \eta\bar{n})^3}.
\]

Neither do the interpretations affect the classical upper bound for states, from (5) and (23),

\[
p \leq S = \pi Q_{\Delta,\max} = \frac{1}{e}.
\]
An exception is when \( \eta \) is rather low, since for small \( \eta \), thermal photons may increase the probability of photon detection, as illustrated in figure 2(a);

(d) for low values of \( \bar{\eta} \), increasing \( \eta \) favors the violation of the classical bounds by increasing \( p \) and decreasing \( M_\Delta \) and \( M_\bar{\Delta} \), as illustrated in figure 3(a). On the other hand, for larger \( \bar{\eta} \) we find that increasing \( \eta \) increases the probability of detecting more than one photon, decreasing \( p \) as illustrated in figure 3(b);

(e) for large \( \eta \) and small \( \bar{\eta} \) it is possible to have \( p > M_\Delta, M_\bar{\Delta}, S \) simultaneously, so that one and the same measurement can reveal at the same time the nonclassical character of the \( \rho, \Delta \) and \( \bar{\Delta} \) as illustrated in figures 2(b) and 3(a).

4. Conclusions

We have shown that the same experiment can test simultaneously and independently both the nonclassicality of states and of measurements. This is because the nonclassicality of states and measurements manifests itself via the violation of different and independent bounds to the same statistics. We have shown that practical imperfections lead to two bounds testing the ideal and real measurements.

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