A truncation method for trace distance between fermionic Gaussian states

Jiaju Zhang\textsuperscript{1*} and M. A. Rajabpour\textsuperscript{2†}

\textsuperscript{1} Center for Joint Quantum Studies and Department of Physics, School of Science, Tianjin University, 135 Yaguan Road, Tianjin 300350, China
\textsuperscript{2} Instituto de Fisica, Universidade Federal Fluminense, Av. Gal. Milton Tavares de Souza s/n, Gragoatá, 24210-346, Niterói, RJ, Brazil
\textsuperscript{*}jiajuzhang@tju.edu.cn, \textsuperscript{†}mohammadali.rajabpour@gmail.com

October 24, 2022

Abstract

We develop a truncation method to calculate the trace distance between two Gaussian states in fermionic systems. The Gaussian states are fully determined by their corresponding correlation matrices. To calculate the trace distance between two Gaussian states, we truncate the corresponding correlation matrices according to the von Neumann entropies and the difference of the two correlation matrices. We find two classes of cases for which the method works. The first class are the cases that the von Neumann entropies of the two states are not too large and the two corresponding correlation matrices nearly commute. The other class are the cases that the two states are nearly orthogonal in the way that are characterized by the canonical values of the correlation matrix difference. We apply the method to examples of subsystems in eigenstates of Ising and XX spin chains, and when the method works we obtain the subsystem trace distances with considerably large subsystem sizes.

Contents

1 Introduction 3
2 Truncated canonicalized correlation matrix method 4
  2.1 Canonicalized correlation matrix method 4
  2.2 Truncated canonicalized correlation matrix method 5
    2.2.1 Maximal entropy strategy 5
    2.2.2 Maximal difference strategy 6
    2.2.3 Mixed strategy 7
3 Examples of RDMs in Ising chain 8
  3.1 Ground and excited states 8
  3.2 Two few-quasiparticle states 11
  3.3 Two logarithmic-law states 11
  3.4 Two volume-law states 12
  3.5 One few-quasiparticle state and one logarithmic-law state 14
  3.6 One few-quasiparticle state and one volume-law state 14
1 Introduction

In quantum information theory, it is necessary to distinguish quantitatively two quantum states \([1, 2]\). In quantum many-body systems, quantum field theories and gravity, the quantitative distinguishability of states also plays important roles \([3-33]\). There are many such quantities such as relative entropy, Schatten distances, trace distance and fidelity. In this paper we focus on the trace distance. For two density matrices \(\rho\) and \(\sigma\), the trace distance is defined as

\[
D(\rho, \sigma) = \frac{1}{2} \text{tr}|\rho - \sigma|.
\] (1.1)

In quantum many-body systems, trace distance has advantages over the other distinguishability measures not only because trace distance is a mathematically well-defined distance \([1, 2]\) but also because in the scaling limit it can distinguish states that the other measures can not distinguish \([23]\), despite the fact that it is notoriously difficult to calculate for large systems due to the fact that Hilbert space dimension grows exponentially with the number of qubits. We will also consider the fidelity

\[
F(\rho, \sigma) = \text{tr}\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}},
\] (1.2)

which serves as a benchmark to be compared with the trace distance. Especially, fidelity gives both an upper bound and a lower bound of trace distance \([1, 2]\)

\[
1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}.
\] (1.3)

For recent developments on the calculation of the fidelity and the trace distance using variational techniques and quantum computers see \([34-38]\).

In this paper we develop a truncation method to calculate the trace distance between two Gaussian states in fermionic systems. It is trivial for two pure states. The Gaussian states can also be mixed states of the total system, or the reduced density matrices (RDMs) of some subsystems. A Gaussian state is fully determined by all the two-point correlation functions of the fermions in the system or subsystem, which can be arranged properly forming a correlation matrix. To calculate the trace distance, we truncate two Gaussian states by truncating their correlation matrices according to the von Neumann entropies of the two states, which we call the maximal entropy strategy, or the difference of the correlation matrices, which we call the maximal difference strategy. We also truncate the correlation matrices by a mix of the maximal entropy strategy and the maximal difference strategy. From the contractive property of the trace distance, it is easy to see that the mixed strategy of the truncation method always gives the best approximation of the trace distance.

We test the truncation method in several examples, including the eigenstate RDMs in the Ising chain and XX chain and the ground state RDMs in Ising chains with different transverse fields. Based on the examples, we summarize two classes of cases that the truncation method works. The method works when both the condition that the von Neumann entropies of the two states are small and the condition that the two relevant correlation functions nearly commute are satisfied. The method also works for two nearly orthogonal states which are characterized by the largest canonical value of the correlation matrix difference of the states approaching 2.

The remaining part of the paper is organized as follows. In section 2, we elaborate the truncated canonicalized correlation matrix method for two general fermionic Gaussian states. In sections 3, 4 and 5, we check the truncation method using respectively examples of eigenstate RDMs in the Ising chain, examples of low-lying state RDMs in critical Ising chain, and examples of ground
state RDMs in the Ising chains with different transverse fields. We summarize the conditions that truncation method works in section 6. In section 7, we present the truncated diagonalized correlation matrix method, a special version of the truncation method, for two Gaussian states in the free fermionic theory. In sections 8 and 9, we check the truncated diagonalized correlation matrix method using examples of eigenstate RDMs in XX chain and examples of low-lying state RDMs in half-filled XX chain. We conclude with discussions in section 10. In appendix A, we give a simple example of two orthogonal Gaussian states in the scaling limit for which no finite truncation method for trace distance is possible.

2 Truncated canonicalized correlation matrix method

In this section we first review the canonicalized correlation matrix method in [33], which is an exact method but is efficient only for cases of low-rank states. Then we generalize the canonicalized correlation matrix method to the so-called truncated canonicalized correlation matrix method, which also applies to the cases of high-rank states under some specific conditions.

2.1 Canonicalized correlation matrix method

We consider a system of \( \ell \) spinless fermions \( a_j, a_j^\dagger, j = 1, 2, \cdots, \ell \), which can be the whole system or a subsystem. It is convenient to define the Majorana modes

\[
d_{2j-1} = a_j + a_j^\dagger, \quad d_{2j} = i(a_j + a_j^\dagger), \quad j = 1, 2, \cdots, \ell.
\]

The system is in a general Gaussian state with density matrix \( \rho \) characterized by a correlation function matrix \( \Gamma \) with entries

\[
\Gamma_{m_1 m_2} = \langle d_{m_1} d_{m_2} \rangle - \delta_{m_1 m_2}, \quad m_1, m_2 = 1, 2, \cdots, 2\ell.
\]

All other correlation functions in the Gaussian state could be obtained from the correlation matrix \( \Gamma \) by Wick contractions.

The correlation matrix \( \Gamma \) is purely imaginary and skew-symmetric and can be transformed in the canonical form following [39, 40]. There are equations

\[
\Gamma u_j = -i\gamma_j v_j, \quad \Gamma v_j = i\gamma_j u_j, \quad j = 1, 2, \cdots, \ell,
\]

with the real numbers \( \gamma_j \in [0, 1] \) and the real \( 2\ell \)-component vectors \( u_j, v_j \). For each \( j = 1, 2, \cdots, \ell \), one can call \( \gamma_j \) a canonical value of \( \Gamma \) and call \( u_j \) and \( v_j \) the corresponding canonical vectors. The canonical vectors satisfy

\[
u_j^T u_{j_2} = \delta_{j_1 j_2}, \quad u_j^T v_{j_2} = 0, \quad j_1, j_2 = 1, 2, \cdots, \ell.
\]

We define the \( 2\ell \times 2\ell \) orthogonal matrix

\[
Q = (u_1, v_1, u_2, v_2, \cdots, u_\ell, v_\ell),
\]

with each vector written as a column vector. The matrix \( Q \) transforms \( \Gamma \) into the canonical form

\[
Q^T \Gamma Q = \bigoplus_{j=1}^\ell \begin{pmatrix} 0 & i\gamma_j \\ -i\gamma_j & 0 \end{pmatrix}.
\]
The density matrix of the Gaussian state can be written as [41–43]

\[ \rho = \sqrt{\det \frac{1-\Gamma}{2}} \exp\left( -\frac{1}{2} \sum_{m_1, m_2=1}^{2\ell} W_{m_1, m_2} d_{m_1} d_{m_2} \right), \tag{2.7} \]

with the matrix

\[ W = \arctanh \Gamma. \tag{2.8} \]

The matrix \( W \) is also purely imaginary and skew-symmetric and can be transformed as

\[ Q^T W Q = \bigoplus_{j=1}^{\ell} \begin{pmatrix} 0 & i\delta_j \\ -i\delta_j & 0 \end{pmatrix}, \quad \delta_j = \arctanh \gamma_j, \quad j = 1, 2, \cdots, \ell. \tag{2.9} \]

We define the new Majorana modes

\[ \tilde{d}_{m_1} \equiv \sum_{m_2=1}^{2\ell} Q_{m_2 m_1} d_{m_2}, \quad m_1 = 1, 2, \cdots, 2\ell, \tag{2.10} \]

and write the density matrix as

\[ \rho = \prod_{j=1}^{\ell} \frac{1 - i\gamma_j \tilde{d}_{2j-1} \tilde{d}_{2j}}{2}. \tag{2.11} \]

Then we calculate the trace distance between two Gaussian states using their explicit density matrices. To calculate the fidelity between two Gaussian states, it is convenient to use the formula

\[ \sqrt{\rho} = \prod_{j=1}^{\ell} \left( \sqrt{1 + \gamma_j} + \sqrt{1 - \gamma_j} \right) - i(\sqrt{1 + \gamma_j} - \sqrt{1 - \gamma_j}) \tilde{d}_{2j-1} \tilde{d}_{2j} \frac{2\sqrt{2}}{2\sqrt{2}}. \tag{2.12} \]

### 2.2 Truncated canonicalized correlation matrix method

To calculate the approximate trace distance between two fermionic Gaussian states, we use three strategies to implement the truncated canonicalized correlation matrix method.

#### 2.2.1 Maximal entropy strategy

For the correlation matrix \( \Gamma \) with (2.6), the corresponding density matrix \( \rho \) is similar with [44,45]

\[ \rho \approx \bigotimes_{j=1}^{\ell} \begin{pmatrix} \frac{1-\gamma_j}{2} & \frac{1+\gamma_j}{2} \\ \frac{1+\gamma_j}{2} & \frac{1-\gamma_j}{2} \end{pmatrix}. \tag{2.13} \]

The von Neumann entropy of \( \rho \) is just sum of the Shannon entropy of each effective probability distribution \( \left( \frac{1-\gamma_j}{2}, \frac{1+\gamma_j}{2} \right), \, j = 1, 2, \cdots, \ell \)

\[ S = \sum_{j=1}^{\ell} \left( -\frac{1-\gamma_j}{2} \log \frac{1-\gamma_j}{2} - \frac{1+\gamma_j}{2} \log \frac{1+\gamma_j}{2} \right). \tag{2.14} \]
For a low-rank state, the entropy $S$ is not large and one may truncate the above effective probability distribution to a few values of smallest $\gamma_j$. For two Gaussian states, we follow the same idea and truncate the two density matrices and calculate their trace distance.

We consider two general Gaussian states $\rho_1$ and $\rho_2$ with correlation matrices $\Gamma_1$ and $\Gamma_2$. The matrix $\Gamma_1$ has canonical values and vectors $(\gamma_j, u_j, v_j)$ with $j = 1, 2, \cdots, \ell$, and the matrix $\Gamma_2$ has canonical values and vectors $(\gamma_j, u_j, v_j)$ with $j = \ell+1, \ell+2, \cdots, 2\ell$. We sort the $2\ell$ sets of canonical values and vectors $(\gamma_j, u_j, v_j)$ with $j = 1, 2, \cdots, 2\ell$ according to the values of $\gamma_j$ from the smallest to the largest one, and choose the first $2\ell$ vectors and rename them as $w_i$ with $i = 1, 2, \cdots, 2\ell$.

We define the $2\ell \times 2\ell$ matrix $V$ with entries

$$V_{i_1,i_2} = w_{i_1}^T w_{i_2}, \quad i_1, i_2 = 1, 2, \cdots, 2\ell. \quad (2.15)$$

We sort the eigenvalues and eigenvectors $(\alpha_i, \alpha_i)$ of $V$ according the values of $\alpha_i$ from the largest to the smallest and discard the ones smaller than a cutoff, say $10^{-9}$. When the number of remaining vectors is odd, we just add one extra vector with the largest eigenvalue smaller than the cutoff. After the discard, we have $2\ell$ sets of eigenvalues and eigenvectors $(\alpha_i, \alpha_i)$ with $i = 1, 2, \cdots, 2\ell$, from which we define the $2\ell$-component orthonormal vectors

$$\tilde{w}_i \equiv \frac{1}{\sqrt{\alpha_i}} \sum_{j=1}^{2\ell} [\alpha_i]_j w_{j,i}, \quad i = 1, 2, \cdots, 2\ell. \quad (2.16)$$

Then we define the $2\ell \times 2\ell$ matrix

$$\tilde{Q} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{2\ell}), \quad (2.17)$$

and the new truncated correlation matrices of size $2\ell \times 2\ell$

$$\tilde{\Gamma}_1 = \tilde{Q}^T \Gamma_1 \tilde{Q}, \quad \tilde{\Gamma}_2 = \tilde{Q}^T \Gamma_2 \tilde{Q}. \quad (2.18)$$

We construct the $2\ell \times 2\ell$ truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$ from the $2\ell \times 2\ell$ truncated correlation matrices $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ using the canonicalized correlation matrix method in the previous subsection. With the truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$, we calculate the approximate trace distance and fidelity

$$D_{t,0}(\rho_1, \rho_2) \equiv D(\tilde{\rho}_1, \tilde{\rho}_2), \quad (2.19)$$

$$F_{t,0}(\rho_1, \rho_2) \equiv F(\tilde{\rho}_1, \tilde{\rho}_2). \quad (2.20)$$

Here we use the subscript $t, 0$ for later convenience.

### 2.2.2 Maximal difference strategy

We sort the canonical values and canonical vectors $(\gamma_j, u_j, v_j)$ with $j = 1, 2, \cdots, \ell$ of the correlation matrix difference $\Gamma_1 - \Gamma_2$ according to canonical values from the largest to the smallest and rename the canonical vectors corresponding to the first $t = \ell$ canonical values as $w_j$ with $j = 1, 2, \cdots, 2\ell$. Notice that for one canonical value there are two corresponding canonical vectors. We construct the $2\ell \times 2\ell$ matrix

$$\tilde{Q} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{2\ell}), \quad (2.21)$$

and the new truncated correlation matrices of size $2\hat{\ell} \times 2\hat{\ell}$

$$\tilde{\Gamma}_1 = \tilde{Q}^T \Gamma_1 \tilde{Q}, \quad \tilde{\Gamma}_2 = \tilde{Q}^T \Gamma_2 \tilde{Q}. \quad (2.22)$$
We construct the $2^\ell \times 2^\ell$ truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$ from the $2\tilde{\ell} \times 2\tilde{\ell}$ truncated correlation matrices $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ using the canonicalized correlation matrix method. With the truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$, we calculate the approximate trace distance and fidelity

$$D_{0,t}(\rho_1, \rho_2) \equiv D(\tilde{\rho}_1, \tilde{\rho}_2), \quad (2.23)$$
$$F_{0,t}(\rho_1, \rho_2) \equiv F(\tilde{\rho}_1, \tilde{\rho}_2). \quad (2.24)$$

Here we use the subscript $0, t$ for later convenience.

### 2.2.3 Mixed strategy

We combine the maximal entropy strategy and maximal difference strategy as a mixed strategy. The matrix $\Gamma_1$ has canonical values and vectors $(\mu_j, u_j, v_j)$ with $j = 1, 2, \cdots, \ell$, and the matrix $\Gamma_2$ has canonical values and vectors $(\mu_j, u_j, v_j)$ with $j = \ell + 1, \ell + 2, \cdots, 2\ell$. We sort the $2\ell$ sets of canonical values and vectors $(\mu_j, u_j, v_j)$ with $j = 1, 2, \cdots, 2\ell$ according to the canonical values $\mu_j$ from the smallest to the largest, and choose the $2t_1$ canonical vectors corresponding to the first $t_1$ canonical values and rename them as $w_i$ with $i = 1, 2, \cdots, 2t_1$. We choose $2t_2$ canonical vectors of $\Gamma_1 - \Gamma_2$ corresponding to the $t_2$ largest canonical values and rename them as $w_i$, $i = 2t_1 + 1, 2t_1 + 2, \cdots, 2(t_1 + t_2)$. In total, we have the $2t = 2(t_1 + t_2)$ vectors $w_i$, $i = 1, 2, \cdots, 2t$.

We define the $2t \times 2t$ matrix $V$ with entries

$$V_{i_1 i_2} = w_{i_1}^T w_{i_2}, \quad i_1, i_2 = 1, 2, \cdots, 2t. \quad (2.25)$$

We sort the eigenvalues and eigenvectors $(\alpha_i, \alpha_i)$ of $V$ according the values of $\alpha_i$ from the largest to the smallest and discard the ones smaller than a cutoff, say $10^{-9}$. When the number of remaining vectors is odd, we just add one extra vector with the largest eigenvalue that is smaller than the cutoff. Then we have $2\tilde{\ell}$ sets of eigenvalues and eigenvectors $(\alpha_i, \alpha_i)$ with $i = 1, 2, \cdots, 2\tilde{\ell}$, from which we define the $2\tilde{\ell}$-component orthonormal vectors

$$\tilde{w}_i \equiv \frac{1}{\sqrt{\alpha_i}} \sum_{i' = 1}^{2t} [\alpha_i]_{i'} w_{i'}, \quad i = 1, 2, \cdots, 2\tilde{\ell}. \quad (2.26)$$

Then we define the $2\tilde{\ell} \times 2\tilde{\ell}$ matrix

$$\tilde{Q} = (\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_{2\tilde{\ell}}), \quad (2.27)$$

and the new truncated correlation matrices of size $2\tilde{\ell} \times 2\tilde{\ell}$

$$\tilde{\Gamma}_1 = \tilde{Q}^T \Gamma_1 \tilde{Q}, \quad \tilde{\Gamma}_2 = \tilde{Q}^T \Gamma_2 \tilde{Q}. \quad (2.28)$$

We construct the $2^\ell \times 2^\ell$ truncated density matrices $\bar{\rho}_1$ and $\bar{\rho}_2$ from the $2\tilde{\ell} \times 2\tilde{\ell}$ truncated correlation matrices $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ using the canonicalized correlation matrix method. With the truncated density matrices $\bar{\rho}_1$ and $\bar{\rho}_2$, we calculate the approximate trace distance and fidelity

$$\bar{D}_{t_1, t_2}(\rho_1, \rho_2) \equiv D(\bar{\rho}_1, \bar{\rho}_2), \quad (2.29)$$
$$\bar{F}_{t_1, t_2}(\rho_1, \rho_2) \equiv F(\bar{\rho}_1, \bar{\rho}_2). \quad (2.30)$$

When $t_2 = 0$, the mixed strategy reduces to the maximal entanglement strategy. When $t_1 = 0$, the mixed strategy reduces to the maximal difference strategy.
As the trace distance is contractive under partial trace and the fidelity is expansive under partial trace \[1\], the exact trace distance of the two states is always larger than the approximate trace distance from the truncation method and the exact fidelity is always smaller than the approximate fidelity from the truncation method. For fixed \(t < \ell\) we get the best approximate trace distance and fidelity from the mixed truncation strategy

\[
\tilde{D}_t(\rho_1, \rho_2) \equiv \max_{0 \leq t_1 \leq t} \tilde{D}_{t_1, t-t_1}(\rho_1, \rho_2), \\
\tilde{F}_t(\rho_1, \rho_2) \equiv \min_{0 \leq t_1 \leq t} \tilde{F}_{t_1, t-t_1}(\rho_1, \rho_2). \tag{2.31}
\]

For \(\ell \leq t\), we do not need to impose any truncation and we just calculate the exact trace distance and fidelity from the canonicalized correlation matrix method.

The mixed strategy of the truncation matrix method always has better precision than the other two strategies. When we later say the truncated canonicalized correlation matrix method without specifying the strategy, we just mean the mixed strategy of the truncation method.

### 3 Examples of RDMs in Ising chain

We use the truncation method to calculate the subsystem trace distance between the ground and excited states in Ising chain with transverse field.

#### 3.1 Ground and excited states

The Ising chain with a transverse field has the Hamiltonian

\[
H = -\frac{1}{2} \sum_{j=1}^{L} (\sigma_j^x \sigma_{j+1}^x + \lambda \sigma_j^z), \tag{3.1}
\]

with periodic boundary conditions of the Pauli matrices \(\sigma_{j+L}^x = \sigma_j\). From Jordan-Wigner transformation, it becomes a fermionic chain \([46-48]\)

\[
H = \sum_{j=1}^{L} \left[ \lambda \left( a_j^\dagger a_j - \frac{1}{2} \right) - \frac{1}{2} (a_j^\dagger a_{j+1} + a_j^\dagger a_{j+1}^\dagger + a_j a_{j+1}) \right], \tag{3.2}
\]

with addition of proper combination of boundary conditions and projections. From further Fourier transformation and Bogoliubov transformation, it becomes

\[
H = \sum_k \varepsilon_k (c_k^\dagger c_k - \frac{1}{2}), \quad \varepsilon_k = \sqrt{\lambda^2 - 2\lambda \cos \frac{2\pi k}{L} + 1}. \tag{3.3}
\]

We focus on the case that \(L\) is an even integer. For anti-periodic boundary condition of the fermions, i.e. the Neveu-Schwarz (NS) sector, we have

\[
k = \frac{L-1}{2}, \ldots, -\frac{1}{2}, \frac{1}{2}, \ldots, -\frac{L-1}{2}. \tag{3.4}
\]

For periodic boundary condition of the fermions, i.e. the Ramond (R) sector, we have

\[
k = \frac{-L}{2}, \frac{-L}{2} + 1, \ldots, -1, 0, 1, \ldots, \frac{L}{2} - 1. \tag{3.5}
\]
The energy eigenstates are characterized by the set of the momenta of the excited quasiparticles \( K = \{ k_1, k_2, \ldots, k_r \} \)

\[
|K\rangle = c_{k_1}^\dagger c_{k_2}^\dagger \cdots c_{k_r}^\dagger |G\rangle.
\] (3.6)

Note that in the NS and R sectors there are different ground states \(|G\rangle\)’s. The RDM of a quasiparticle excited state is also a Gaussian state. For the subsystem \( A = [1, \ell] \) in state \(|K\rangle\), one can define the \( 2\ell \times 2\ell \) correlation matrix \( \Gamma_{A,K} \) following (2.2) with entries

\[
[\Gamma_{A,K}]_{j_1,j_2-1} = f^K_{j_2-j_1}, \quad [\Gamma_{A,K}]_{j_2,j_1-1} = -f^K_{j_2,j_1-1} = g^K_{j_2-j_1}, \quad j_1, j_2 = 1, 2, \ldots, \ell,
\] (3.7)

where we have defined \([44, 45, 49–51]\)

\[
f^K_j \equiv \frac{2i}{L} \sum_{k \in K} \sin \left( \frac{2\pi j k}{L} \right), \quad g^K_j \equiv -\frac{i}{L} \sum_{k \not\in K} \cos \left( \frac{2\pi j k}{L} - \theta_k \right) + \frac{i}{L} \sum_{k \in K} \cos \left( \frac{2\pi j k}{L} - \theta_k \right).
\] (3.8)

The angle \( \theta_k \) is determined by

\[
e^{i\theta_k} = \frac{\lambda - \cos \frac{2\pi k}{L} + i \sin \frac{2\pi k}{L}}{\sqrt{\lambda^2 - 2\lambda \cos \frac{2\pi k}{L} + 1}}.
\] (3.9)

To check the efficiency of the truncated canonicalized correlation matrix method with different strategies, we calculate the approximate subsystem trace distance and fidelity between various typical states

\[
K_0 = \emptyset, \quad K_1 = \left\{ \frac{1}{2}, \frac{3}{2} \right\}, \quad K_2 = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2} \right\},
\]

\[
K_3 = \left[-\frac{L-1}{2}, -\frac{1}{2}\right], \quad K_4 = \left[\frac{1}{2}, \frac{L-1}{2}\right],
\]

\[
K_5 = \left[-\frac{L}{4} + \frac{1}{2}, \frac{L}{4} - \frac{1}{2}\right], \quad K_6 = \left[-\frac{L}{6} + \frac{1}{2}, \frac{L}{6} - \frac{1}{2}\right],
\]

\[
K_7 = K_3 \cup K_1, \quad K_8 = K_3 \cup K_2,
\]

\[
K_9 = \left[-\frac{L-1}{2}, -\frac{3}{2}\right], \quad K_{10} = \left[\frac{3}{2}, \frac{L-1}{2}\right],
\]

\[
K_{11} = \left[-\frac{L-1}{2}, -\frac{L-1}{2}\right], \quad K_{12} = \left[-\frac{L-1}{2}, \frac{L-1}{2}, 3\right],
\]

\[
K_{13} = K_9 \cup K_1, \quad K_{14} = K_9 \cup K_2.
\] (3.10)

Here \( K_0 \) is the state with no modes excited. By \([k_1, k_2]\) with \( k_1 \leq k_2 \) we mean the set \( \{ k_1, k_2+1, k_2+2, \ldots, k_2 \} \). By \([k_1, k_2, \delta k]\) with \( k_1 \leq k_2 \) we mean the set \( \{ k_1, k_2 + \delta k, k_2 + 2\delta k, \ldots, k_2' \} \) with \( k_2' \) satisfying \( k_2' \leq k_2 \) and \( k_2' + \delta k > k_2 \). According to the rule, there is \([k_1, k_2] = [k_1, k_2, 1]\). For general consistency, we only consider the cases with \( L \) being multiples of 12. We show the scheme of these states in figure 1.

The states \( K_0, K_1 \) and \( K_2 \) are few-quasiparticle states in which no quasiparticle or only a few quasiparticles are excited. The states from \( K_3 \) to \( K_{14} \) are many-quasiparticle states in which the number of excited modes is proportional to \( L \) in the scaling limit. When the Ising chain is gapped, the von Neumann entropies of the RDMs, i.e. the entanglement entropies, in few-quasiparticle states are finite in the scaling limit \([26, 27, 52–59]\). When the Ising chain is critical, the entanglement entropies in few-quasiparticle states follow the logarithmic law. For many-quasiparticle states, there are two typical scaling laws for the entanglement entropy. In states
Figure 1: Scheme of the typical states (3.10) we consider in the paper. The $L$ circles denote the quasiparticle modes from $-\frac{L-1}{2}$ to $\frac{L-1}{2}$. The empty circles denote the modes that are not excited and the red filled circles denote the excited modes. Here $K_0$, $K_1$ and $K_2$ are few-quasiparticle states, $K_3$, $K_4$, $K_5$, $K_6$, $K_7$ and $K_8$ are logarithmic-law states, and $K_9$, $K_{10}$, $K_{11}$, $K_{12}$, $K_{13}$ and $K_{14}$ are volume-law states. We have set $L = 60$ as an example.

from $K_3$ to $K_8$ almost all the excited modes are successive and the entropies follow the logarithmic law [44, 45, 60, 61]. We call them logarithmic-law many-quasiparticle states or in short logarithmic-law states. In states from $K_9$ to $K_{14}$, some fixed patterns are excited successively in an extended region of the momentum space. In such patterns there are both modes excited and modes not excited. For example, in state $K_9$ every other mode is excited in the range $[-\frac{L-1}{2}, -\frac{3}{2}]$, and in state $K_{12}$ every third mode is excited in the range $[-\frac{L-1}{2}, \frac{L-1}{2}]$. In such states, the entropy follows the volume law in the regime $1 \ll \ell \ll L$ [49], and we call them volume-law many-quasiparticle state or in short volume-law states. Note that when finite number of modes are changed in one state, the scaling behavior of the entanglement entropy does not change. In summary, we have the typical states (3.10) classified as follows.

\[
\begin{align*}
\text{Few-quasiparticle states} & \quad \text{States } K_0 \text{ to } K_2, \\
\text{Many-quasiparticle states} & \quad \text{Logarithmic-law states States } K_3 \text{ to } K_8. \\
& \quad \text{Volume-law states States } K_9 \text{ to } K_{14}.
\end{align*}
\]

Later we will also call few-quasiparticle states and the logarithmic-law states as small-entropy states and call the volume-law states as large-entropy states.

Whether the model is gapped or gapless depends on the value of the transverse field $\lambda$. The values of the transverse field does not affect the efficiency of the truncation method. We will just show examples with $\lambda = 1$ in this paper. We plot the canonical values of the correlation matrices of these states in figure 2. For the correlation matrix of a few-quasiparticle state or a logarithmic-law state, most of the canonical values are 1, and only finite number of the canonical values are in the range $[0, 1)$. For a volume-law state, the number of the canonical values in the range $[0, 1)$ is proportional to $L$ in the scaling limit, and in some cases all the canonical values are in the range $[0, 1)$. 

\[\text{Figure 1: Scheme of the typical states (3.10) we consider in the paper. The } L \text{ circles denote the quasiparticle modes from } -\frac{L-1}{2} \text{ to } \frac{L-1}{2}. \text{ The empty circles denote the modes that are not excited and the red filled circles denote the excited modes. Here } K_0, K_1 \text{ and } K_2 \text{ are few-quasiparticle states, } K_3, K_4, K_5, K_6, K_7 \text{ and } K_8 \text{ are logarithmic-law states, and } K_9, K_{10}, K_{11}, K_{12}, K_{13} \text{ and } K_{14} \text{ are volume-law states. We have set } L = 60 \text{ as an example.} \]
The canonical values of the correlation matrices of the few-quasiparticle states (left), logarithmic-law states (middle) and volume-law states (right) in (3.10) and figure 1. The canonical values of $\Gamma_{A,\mathcal{K}_4}$ and $\Gamma_{A,\mathcal{K}_5}$ are the same as those of $\Gamma_{A,\mathcal{K}_7}$. The canonical values of $\Gamma_{A,\mathcal{K}_{10}}$ are the same as those of $\Gamma_{A,\mathcal{K}_9}$. We have set $\lambda = 1$, $L = 360$ and $\ell = 120$.

[0, 1). These results indicate that the maximal entropy strategy of the truncation method may give good approximate RDMs for the small-entropy states and may not work for the large-entropy states.

We consider six classes of trace distances and fidelities among the typical states (3.10), as indicated by the titles of the following six subsections. We show the canonical values of the correlation matrix difference for the six classes in figure 3. We show the trace distance and fidelity from the truncated correlation matrix method with different strategies in figure 4.

### 3.2 Two few-quasiparticle states

The canonical values of the correlation matrix difference for the classes between two few-quasiparticle states are shown in panel (a) of figure 3. The number of nonvanishing canonical values is just the number of different modes of the two few-quasiparticle states. The approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the first row of figure 4. We compare the approximate fidelity from the truncated correlation matrix method with the exact results from the correlation matrix [62]

$$F_{\text{fer}}(\rho_\Gamma, \rho_\Gamma') = \left( \det \frac{1 - \Gamma_1}{2} \right)^{1/4} \left( \det \frac{1 - \Gamma_2}{2} \right)^{1/4} \left[ \det \left( 1 + \sqrt{\frac{1 + \Gamma_1}{1 - \Gamma_1} \frac{1 + \Gamma_2}{1 - \Gamma_2} \frac{1 + \Gamma_1}{1 - \Gamma_1}} \right) \right]^{1/2}, \quad (3.12)$$

where we have used $\rho_\Gamma$ to represent the corresponding RDM of the correlation matrix $\Gamma$. The exact fidelity gives lower and upper bounds of the trace distance according to (1.3).

In the first row of figure 4, the maximal entropy strategy works well for most of the subsystem sizes except when subsystem is almost the whole system. The maximal difference strategy does not work well due to missing of the modes that are excited in both states. The mixed strategy overcomes the drawbacks of both the maximal entropy strategy and the maximal difference strategy and works well for all the subsystem sizes.

### 3.3 Two logarithmic-law states

The canonical values of the correlation matrix difference for the classes between two logarithmic-law states are shown in panel (b) of figure 3. We see that almost all the canonical values of the
correlation matrix difference are 0 or 2. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the second row and the third row of figure 4.

For the cases $K_3$ vs $K_4$, the number of difference modes is proportional to $L$ in the scaling limit, and for large enough $\ell$ there exist canonical values equal to 2. The maximal entropy strategy fails well for the chosen truncation number $t = 10$, while the maximal difference strategy works well.

For the case $K_7$ vs $K_8$, the number of different modes is finite in the scaling limit. The maximal entropy strategy works well for most of the subsystem sizes except when subsystem is almost the whole system. The maximal difference strategy does not work well due to possible missing of the modes that are excited in both states. The mixed strategy overcomes the drawbacks of both the maximal entropy strategy and the maximal difference strategy and works well for all the subsystem sizes.

### 3.4 Two volume-law states

The canonical values of the correlation matrix difference for the classes between two volume-law states are shown in panel (c) of figure 3. All the canonical values are in the range $[0, 2)$, and there
Figure 4: The symbols are examples of the approximate trace distance ($D$) and fidelity ($F$) from the truncated canonicalized correlation matrix method with the maximal entropy strategy (MES, left), maximal difference strategy (MDS, middle) and the mixed strategy (MS, right) in Ising chain with transverse field. The solid lines are exact results of fidelity calculated from (3.12). The red dotted lines are lower and upper bounds of trace distance from fidelity. We have set $\lambda \equiv 1$, $L = 120$ and $t = 10$. 
are no canonical values being equal to 2 at all. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the fourth row of figure 4. Generally all the three strategies fail.

3.5 One few-quasiparticle state and one logarithmic-law state

The canonical values of the correlation matrix difference for the classes between one few-quasiparticle state and one logarithmic-law state are shown in panel (d) of figure 3. For all the cases there exist large number of canonical values being equal to 2. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the fifth row of figure 4. With the increase of the subsystem size, the trace distance increases quickly to 1 and the fidelity decreases quickly to 0. The maximal entropy strategy generally does not always works well, while the maximal difference strategy always works well.

3.6 One few-quasiparticle state and one volume-law state

The canonical values of the correlation matrix difference for the classes between one few-quasiparticle state and one volume-law state are shown in panel (e) of figure 3. All the canonical values are in the range $[0, 2)$. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the sixth row of figure 4. Generally all the three strategies fail.

3.7 One logarithmic-law state and one volume-law state

The canonical values of the correlation matrix difference for the classes between one logarithmic-law state and one volume-law state are shown in panel (f) of figure 3. For the cases $K_3$ vs $K_9$ and $K_3$ vs $K_{11}$, there are no canonical values being equal to 2, while for the cases $K_3$ vs $K_{10}$ and $K_5$ vs $K_9$ there exist large number of canonical values being equal to 2. The approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the last two rows of figure 4. The maximal entropy strategy fails for all the cases. The maximal difference strategy works well for the case $K_3$ vs $K_{10}$, while it fails for the case $K_3$ vs $K_{11}$. The mixed strategy does not improve the results for the case $K_3$ vs $K_{11}$.

3.8 Summary

From above examples, we see that the mixed strategy of the truncation method works well for the cases with only few-quasiparticle states and logarithmic-law states are involved. As the ranks of the small-entropy states are relatively small, it is understandable that there exist approximate truncated density matrices that produce the trace distance with a high precision.

It is tricky when at least one large-entropy state is involved. For all the cases that involve at least one large-entropy state and the maximal difference strategy of the truncation method works, there exist large numbers of the canonical values of the correlation matrix difference being equal to 2, for example, the cases $K_3$ vs $K_{10}$ and $K_5$ vs $K_9$ as can be seen in figure 3. For the cases with only small-entropy states and the maximal difference strategy of the truncation method works, there exist large numbers of the canonical values of the correlation matrix difference being equal to 2, for example, the cases $K_3$ vs $K_4$, $K_5$ vs $K_6$, $K_1$ vs $K_3$, $K_1$ vs $K_5$, $K_1$ vs $K_6$. From the contractive property of the trace distance and expansive property of the fidelity under partial trace [1], it is easy to see that the trace distance is just 1 and the fidelity is just 0 if there exists at least one
canonical value of the correlation matrix difference being equal to 2. The maximal difference strategy of the truncation method works well for all these cases, and with the increase of the subsystem size the trace distance increases quickly to 1 and the fidelity decreases quickly to 0. For all the cases there exists large consecutive segment in the momentum space that the modes are all excited in one state and not excited in the other state. We expect that this is the sufficient condition that there exist large numbers of the canonical values of the correlation matrix difference being equal to 2 in the scaling limit.

We formulate the conjecture in the previous paragraph precisely. We consider two energy eigenstates $|K\rangle$ and $|K'\rangle$ which are represented as excited momentum modes $K$ and $K'$ in a general fermionic theory. The total system has size $L$, and the subsystem $\Lambda$ has $\ell$ consecutive sites. For the Gaussian states $|K\rangle$ and $|K'\rangle$ of the whole system, there are respectively Gaussian RDMs $\rho_{AK}$ and $\rho_{AK'}$. The correlation matrices for the states $|K\rangle$ and $|K'\rangle$ are $\Gamma_K$ and $\Gamma_{K'}$, and the correlation matrices for the RDMs $\rho_{AK}$ and $\rho_{AK'}$ are $\Gamma_{AK}$ and $\Gamma_{AK'}$. For the two states $|K\rangle$ and $|K'\rangle$, if there exists a large consecutive segment in the momentum space in which the modes are all excited in one state and all not excited in the other state, then there exist large number of the canonical values of the RDM correlation matrix difference $\Gamma_{AK} - \Gamma_{AK'}$ being equal to 2. The maximal difference strategy of the truncation method works well for all these cases, and with the increase of the canonical value of the correlation matrix difference being equal to 2.

We may see more clearly the statement in the previous paragraph with several examples. For the case $K_3$ vs $K_4$ we have two segments $S_1 = [-\frac{L-1}{2}, -\frac{1}{2}]$ and $S_2 = [\frac{1}{2}, \frac{L-1}{2}]$ with $y_1 = y_2 = \frac{1}{2}$ and $\ell = 1$, nearly all the canonical values of the correlation matrix difference $\Gamma_{AK_3} - \Gamma_{AK_4}$ are 2 as shown in panel (b) of figure 3, and the maximal difference strategy of the truncation method works as shown in panel (e) of figure 4. For the case $K_1$ vs $K_5$ we have one segment $S = [-\frac{L}{4} + \frac{L}{2}, \frac{L}{4} - \frac{1}{2}]$ with $y = \frac{1}{2}$, nearly half of the canonical values of the correlation matrix difference $\Gamma_{AK_1} - \Gamma_{AK_5}$ are 2 as shown in panel (d) of figure 3, and the maximal difference strategy of the truncation method works as shown in panel (n) of figure 4. For the case $K_3$ vs $K_{10}$ we have one segment $S = [-\frac{L-1}{2}, -\frac{1}{2}]$ with $y = \frac{1}{2}$, nearly half of the canonical values of the correlation matrix difference $\Gamma_{AK_3} - \Gamma_{AK_{10}}$ are 2 as shown in panel (f) of figure 3, and the maximal difference strategy of the truncation method works as shown in panel (t) of figure 4.

More generally, for two Gaussian states $\rho$ and $\rho'$, when at least one of the canonical values of the correlation matrix difference $\Gamma - \Gamma'$ approaches 2, the two density matrices $\rho$ and $\rho'$ are nearly orthogonal, and the maximal difference strategy of the truncation method works.
Notice that for a density matrix with finite entropy or logarithmic-law entropy, we call it small-entropy state, and for a density matrix with volume-law entropy, we call it large-entropy state. The lesson in the previous section and this section is that the truncation method works between two small-entropy states. The truncation method tends to break down when at least one large-entropy state is involved, except when the two states are nearly orthogonal in the way that the maximal canonical value of the correlation matrix difference is nearly 2.

4 Examples of low-lying state RDMs in critical Ising chain

As an application of the truncation method, we consider the trace distance and fidelity among several low-lying states in the critical Ising chain. The continuum limit of the critical Ising chain is just the two-dimensional free fermion theory. We consider the states in the critical Ising chain corresponding to the ground and excited states \(|\psi\rangle, |\sigma\rangle, |\mu\rangle, |\psi\rangle, |\tilde{\psi}\rangle, |\epsilon\rangle\) in the two-dimensional free fermion theory.

In [20, 22], there are results of the trace distances

\[
D(\rho_{A,G}, \rho_{A,\sigma}) = D(\rho_{A,G}, \rho_{A,\mu}) = \frac{x}{2} + o(x),
\]

\[
D(\rho_{A,\sigma}, \rho_{A,\mu}) = x,
\]

\[
D(\rho_{A,\sigma}, \rho_{A,\psi}) = D(\rho_{A,\sigma}, \rho_{A,\tilde{\psi}}) = D(\rho_{A,\mu}, \rho_{A,\psi}) = D(\rho_{A,\tilde{\psi}}, \rho_{A,\psi}) = \frac{x}{2} + o(x),
\]

\[
D(\rho_{A,\sigma}, \rho_{A,\epsilon}) = D(\rho_{A,\mu}, \rho_{A,\epsilon}) = \frac{x}{2} + o(x),
\]

\[
D(\rho_{A,G}, \rho_{A,\psi}) = D(\rho_{A,G}, \rho_{A,\tilde{\psi}}) = D(\rho_{A,\psi}, \rho_{A,\epsilon}) = D(\rho_{A,\tilde{\psi}}, \rho_{A,\epsilon}) \approx 0.0916 \times 2\pi^2 x^2 + o(x^2),
\]

\[
D(\rho_{A,\psi}, \rho_{A,\epsilon}) \approx 0.115 \times 2\pi^2 x^2 + o(x^2),
\]

\[
D(\rho_{A,G}, \rho_{A,\epsilon}) \approx 0.153 \times 2\pi^2 x^2 + o(x^2),
\]

and fidelities

\[
F(\rho_{A,G}, \rho_{A,\sigma}) = F(\rho_{A,G}, \rho_{A,\mu}) = \left( \cos \frac{\pi x}{2} \right)^{\frac{1}{3}},
\]

\[
F(\rho_{A,\sigma}, \rho_{A,\mu}) = \left( \cos \frac{\pi x}{2} \right)^{\frac{1}{2}},
\]

\[
F(\rho_{A,\sigma}, \rho_{A,\psi}) = F(\rho_{A,\sigma}, \rho_{A,\tilde{\psi}}) = F(\rho_{A,\mu}, \rho_{A,\psi}) = F(\rho_{A,\mu}, \rho_{A,\tilde{\psi}}) = 1 - \frac{\pi^2 x^2}{64} + o(x^2),
\]

\[
F(\rho_{A,\sigma}, \rho_{A,\epsilon}) = F(\rho_{A,\mu}, \rho_{A,\epsilon}) = 1 - \frac{\pi^2 x^2}{64} + o(x^2),
\]

\[
F(\rho_{A,G}, \rho_{A,\psi}) = F(\rho_{A,G}, \rho_{A,\tilde{\psi}}) = F(\rho_{A,\psi}, \rho_{A,\epsilon}) = F(\rho_{A,\tilde{\psi}}, \rho_{A,\epsilon}) = \frac{\Gamma\left(\frac{3+\csc \frac{\pi x}{4}}{4}\right)}{\Gamma\left(\frac{1+\csc \frac{\pi x}{4}}{4}\right)} \sqrt{2 \sin(\pi x)},
\]

\[
F(\rho_{A,\psi}, \rho_{A,\epsilon}) = \frac{\Gamma^2\left(\frac{3+\csc \frac{\pi x}{4}}{4}\right)}{\Gamma^2\left(\frac{1+\csc \frac{\pi x}{4}}{4}\right)} 2 \sin(\pi x).
\]

Note that the fidelity between the ground state and a general primary excited state in two-dimensional conformal field theory was firstly obtained in [5]. The calculations of the trace distance and fidelity in the critical Ising chain in [20, 22] are exact but only for the cases with very small subsystem sizes. We recalculate the trace distance and fidelity for much larger subsystem sizes using
the mixed strategy of the truncation method and show the results in figure 5. There are perfect matches with the analytical results in the free fermion theory.

5 Examples of ground state RDMs in different Ising chains

In the previous two sections, we calculate the subsystem trace distance between eigenstates of the same Hamiltonians. In this section, we calculate trace distance between eigenstate RDMs of different Hamiltonians. Explicitly, we consider the subsystem trace distance among the NS sector ground states in Ising chains with different transverse fields. The NS sector ground state is defined by

$$c_k |G\rangle = 0, \quad k = \frac{L-1}{2}, \ldots, \frac{L-1}{2},$$

and it can be written as

$$|G\rangle = \sum_{k=1/2}^{(L-1)/2} \left( \frac{\cos \theta_k}{2} + i \frac{\sin \theta_k}{2} b_k b_k^\dagger \right) |\uparrow \cdots \uparrow\rangle.$$

Figure 5: The approximate trace distance and fidelity between low-lying states in the critical Ising chain from the mixed strategy of the truncation method (symbols) and the corresponding exact results in the 2D free fermion theory (solid lines). Note that in panel (e) the fidelity $F(\rho_{A,\psi}, \rho_{A,\tilde{\psi}})$ overlaps with $F(\rho_{A,G}, \rho_{A,\psi})$. The critical Ising chain has the transverse field $\lambda = 1$, we have also set $L = 360$ and $t = 10$. 

17
The state $|\Gamma\rangle = |G(\lambda)\rangle$ depend on the transverse field. In the following, we will use $|\lambda\rangle \equiv |G(\lambda)\rangle$ to denote the NS sector ground state in the Ising chain with transverse field $\lambda$. We will also use the notation $\Gamma_{A,\lambda} \equiv \Gamma_{A,G(\lambda)}$.

The overlap between the ground states $|\lambda_1\rangle$ and $|\lambda_2\rangle$ is

$$
|\langle \lambda_1 | \lambda_2 \rangle |^2 = \frac{1}{2^{l/2}} \prod_{k=1/2}^{(l-1)/2} \left( 1 + \frac{\lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) \cos \frac{2\pi k}{L} + 1}{\sqrt{(\lambda_1^2 - 2\lambda_1 \cos \frac{2\pi k}{L} + 1)(\lambda_2^2 - 2\lambda_2 \cos \frac{2\pi k}{L} + 1)}} \right).
$$

(5.3)

The trace distance and fidelity for the whole system are

$$
D(\rho_{\lambda_1}, \rho_{\lambda_2}) = \sqrt{1 - |\langle \lambda_1 | \lambda_2 \rangle |^2}, \quad F(\rho_{\lambda_1}, \rho_{\lambda_2}) = |\langle \lambda_1 | \lambda_2 \rangle |.
$$

(5.4)

In the scaling limit, there should be

$$
\lim_{x \to 1} D(\rho_{A,\lambda_1}, \rho_{A,\lambda_2}) = D(\rho_{\lambda_1}, \rho_{\lambda_2}), \quad \lim_{x \to 1} F(\rho_{A,\lambda_1}, \rho_{A,\lambda_2}) = F(\rho_{\lambda_1}, \rho_{\lambda_2}).
$$

(5.5)

We choose the examples of NS sector ground states

$|\lambda_1\rangle$ with $\lambda_1 = -5$, $|\lambda_2\rangle$ with $\lambda_2 = -2$, $|\lambda_3\rangle$ with $\lambda_3 = -1$, $|\lambda_4\rangle$ with $\lambda_4 = -0.5$, $|\lambda_5\rangle$ with $\lambda_5 = 0$, $|\lambda_6\rangle$ with $\lambda_6 = 0.5$, $|\lambda_7\rangle$ with $\lambda_7 = 1$, $|\lambda_8\rangle$ with $\lambda_8 = 2$, $|\lambda_9\rangle$ with $\lambda_9 = 5$.

(5.6)

For the Ising chain in transverse field, there are three phases separated by two critical points. For $\lambda < -1$ it is the downward paramagnetic phase, for $-1 < \lambda < 1$ it is the ferromagnetic phase, and for $\lambda > 1$ it is the upward paramagnetic phase. At $\lambda = -1$ and $\lambda = 1$, they are respectively the downward and upward critical points. Here by downward we mean $\langle \sigma_z^i \rangle < 0$ and by upward we mean $\langle \sigma_z^i \rangle > 0$. We will call these ground states according to the phases they are in. For example, there are noncritical states and critical states. A noncritical state can be a downward paramagnetic state, or a ferromagnetic state, or an upward paramagnetic state. A critical state can be a downward critical state or an upward critical state.

We consider the RDM of the subsystem $A = [1, \ell]$. We show the canonical values of the subsystem correlation matrices in the ground states (5.6) in figure 6. For all the cases, almost all canonical values are 1. This is because the ground state entanglement entropy is small, i.e. that there are finite entropy in gapped phase and logarithmic-law entropy at critical point.

We classify the trace distance and fidelity among the ground states (5.6) according to the positions of the two relevant states in the phase diagram, as shown in table 1. From class I case to class VI case, the two relevant states are kind of more and more far away from each other in the phase diagram. We show the canonical values of the correlation matrix difference for the six classes in figure 7. Note that besides 0 and 2, there exists another special point $\sqrt{2}$ for the canonical values of the correlation matrix difference. We show examples of the trace distance and fidelity from the truncated correlation matrix method with different strategies of the truncation method in figure 8.

### 5.1 Class I case

For the class I case, the canonical values of the correlation matrix difference are shown in panel (a) of figure 7, and the canonical values go from 0 to some value in the range $(0, \sqrt{2})$. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the first two rows of figure 8. All strategies of the truncation method fail.
Figure 6: The canonical values of the correlation matrices of the ground states in Ising chains with different transverse fields in (5.6). In the left panel there are downward paramagnetic state, in the middle panel it is the downward critical point, and in the right panel they are ferromagnetic states. Note that the canonical values of $\Gamma_{A,\lambda}$ are the same as those of $\Gamma_{A,-\lambda}$. We have set $L = 360$ and $\ell = 120$.

5.2 Class II case

For the class II case, the canonical values of the correlation matrix difference are shown in panel (b) of figure 7, and the canonical values go from 0 to $\sqrt{2}$. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the third row and the fourth row of figure 8. The truncation method fails.

5.3 Class III case

For the class III case, the canonical values of the correlation matrix difference are shown in panel (c) of figure 7, and the canonical values go from 0 to 2. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the fifth row of figure 8. The maximal entropy strategy fails while the maximal difference strategy works well.

5.4 Class IV case

For the class IV case, the canonical values of the correlation matrix difference are shown in panel (d) of figure 7, and almost all the canonical values are $\sqrt{2}$ except very few canonical values in the range $(0, \sqrt{2})$. The approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the sixth row of figure 8. The truncation method fails.

5.5 Class V case

For the class V case, the canonical values of the correlation matrix difference are shown in panel (e) of figure 7, and the canonical values go from $\sqrt{2}$ to 2 except very few canonical values in the range $(0, \sqrt{2})$. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the seventh row of figure 8. The maximal difference strategy of the truncation method works well.
| Class | Description                                                                 | Scheme |
|-------|------------------------------------------------------------------------------|--------|
| I     | Two noncritical states in the same phase, including the cases of two downward paramagnetic states, two ferromagnetic states, and two upward paramagnetic states. | ![Diagram](image) |
| II    | One noncritical state and its nearby critical state, including the cases of one downward paramagnetic state and one downward critical state, one ferromagnetic state and one downward critical state, one ferromagnetic state and one upward critical state, and one upward paramagnetic state and one upward critical state. | ![Diagram](image) |
| III   | Two noncritical states in different phases separated by one critical point, including the cases of one downward paramagnetic state and one ferromagnetic state, and one upward paramagnetic state and one ferromagnetic state. | ![Diagram](image) |
| IV    | Two critical states separated by one noncritical phase, i.e. the downward critical state and the upward critical state. | ![Diagram](image) |
| V     | One noncritical state and one critical state separated by another noncritical phase and another critical state, including the cases of one downward paramagnetic state and the upward critical state, and one upward paramagnetic state and the downward critical state. | ![Diagram](image) |
| VI    | Two noncritical states in different phases separated by another noncritical phase and two critical points, i.e. the case of one downward paramagnetic state and one upward paramagnetic state. | ![Diagram](image) |

Table 1: The classification of the trace distance and fidelity among the ground states (5.6) according to the positions of the relevant two states in the phase diagram of Ising chain. In the third column, in each scheme, a line connects the two states in the phase diagram, where the black solid dots denote the two relevant states and the red empty circles denote the critical points.

### 5.6 Class VI case

For the class VI case, the canonical values of the correlation matrix difference are shown in panel (f) of figure 7, and the canonical values go from some value in the range $(\sqrt{2}, 2)$ to the maximal value 2. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the last row of figure 8. All the strategies of the truncation method work well.

### 5.7 Summary

As we mentioned before, both density matrices with finite entropies and density matrices with logarithmic entropies are called small-entropy states, according to which all the ground state RDMs we consider in this section are all small-entropy states. However, the truncation method does not necessarily work, except the cases that the two density matrices are nearly orthogonal in the way that at least one of the canonical values of the correlation matrix difference approaches 2.
Figure 7: The canonical values of the correlation matrix difference for the six classes of two ground states in Ising chain with different transverse fields in table 1. We have set \( L = 360 \) and \( \ell = 120 \).

6 Conditions that the truncation method works

From the examples in the previous sections, we summarize the conditions when the truncation method works.

It is easily understandable when the truncation method fails for large-entropy states. Generally, the truncation method in this paper fails but it works for the special case that the two Gaussian states are nearly orthogonal in the way that the largest canonical value of the correlation matrix difference approaches 2.

It is not apparent why the truncation method in this paper fails in section 5 for two ground state RDMs even when both of the RDMs are small-entropy states. For two small-entropy states \( \rho \) and \( \rho' \), if they commute, i.e. that they are diagonal under the same basis, the corresponding correlation matrices \( \Gamma \) and \( \Gamma' \) also commute, i.e. that they are in canonical forms under the same basis. In this case the number of effective modes that contribute to the entropy and trace distance is small and the truncation method in this paper must work. Furthermore, the truncation method should also work for two nearly commuting small-entropy states \( \rho \) and \( \rho' \). We use the commutativity and noncommutativity of the correlation matrices to represent the commutativity and noncommutativity of the corresponding density matrices. To quantify the noncommutativity of two correlation matrices \( \Gamma \) and \( \Gamma' \), we calculate the noncommutativity trace norm

\[
N(\Gamma, \Gamma') = ||\Gamma\Gamma' - \Gamma'\Gamma||_1. \quad (6.1)
\]

We show examples of the noncommutativity for the small-entropy states in the same Hamil-
Figure 8: The trace distances and fidelities between two ground states of Ising chain with different transverse fields from the truncated correlation matrix method (symbols), the exact results of fidelity (solid lines), and lower and upper bounds for trace distance (red dotted lines). The red dashed lines are the trace distance and fidelity for the whole system (5.4). We have set $L = 120$ and $t = 10$. 

22
tonian, which is the critical Ising chain with $\lambda = 1$ in figure 9. For some cases with the increase of the total system size $L$ the noncommutativity trace norms of the correlation matrices approach some constants, while for the other cases the trace norms grow very slowly with $L$, not faster than $\log \log \log L$ as we have checked numerically. The truncation method works for all these cases.

Interestingly, it seems that the two different patterns of the trace norms depend on some kind of edge structure of the excited states in the momentum space. For a small-entropy state, we define the edge of a state as a point in the momentum space where there exist successively excited modes on one side and successively modes not excited on the other side. We call all the edges of a state the edge structure. We require that exciting or annihilating finite number of modes does not change the edge structure. States $K_0$ to $K_3$ in (3.10) and figure 1 have the same edge structure $\emptyset$. States $K_3$, $K_4$, $K_7$ and $K_8$ have the same edge structure $\{-\frac{L}{5}, 0\}$. State $K_5$ has the edge structure $\{-\frac{L}{5}, \frac{L}{8}\}$. State $K_6$ has the edge structure $\{L, \frac{L}{5}\}$. From the examples in figure 9 and some other examples we do not show here, it seems that for two small-entropy states with the same edge structure the noncommutativity trace norms approach some constants with the increase of $L$, while for two small-entropy states with different edge structures the noncommutativity trace norms grow slowly with $L$.

We show examples of the noncommutativity trace norms for the NS sector ground state RDMs in different Hamiltonians, which are the Ising chains with different transverse fields, in figure 10. In the scaling limit all the noncommutativity trace norms grow linearly with the size of the total system $L$. The truncation method fails for all these cases unless the two RDMs are nearly orthogonal in the way that the maximal canonical value of the correlation matrix difference is nearly 2.

Based on all the above results, we propose the protocol of using the truncation method in this paper as follows.

Two Gaussian states

\[
\begin{align*}
\text{Small entropy condition & nearly commutativity condition} & \quad \text{It works.} \\
\text{Nearly orthogonal condition} & \quad \text{It works.} \\
\text{Otherwise} & \quad \text{Otherwise} \quad \text{It possibly fails.}
\end{align*}
\]

(6.2)
By small entropy condition, we mean finite entropy or logarithmic-law entropy for both states in the scaling limit. By nearly commutativity condition, we mean the noncommutativity trace norm of the two correlation matrices approaches a constant or grows very slowly with the size of the system. By nearly orthogonal condition, we mean the special case that the maximal canonical value of the correlation matrix difference is nearly 2.

Note that for some orthogonal Gaussian states, there not necessary exist canonical values of the correlation matrix difference being nearly 2. A simple example is given in appendix A. There are two Gaussian states that are orthogonal in the scaling limit, while all the canonical values of the correlation matrix difference may be smaller than 2. For this example there is no method with finite truncation that could produce the trace distance with a good approximation.

7 Truncated diagonalized correlation matrix method

For Gaussian states in a free fermionic theory, there is a special variation of the truncation method that we call truncated diagonalized correlation matrix method. In this section we first review the diagonalized correlation matrix method in [33], which only applies to the cases of low-rank states. Then we generalize the diagonalized correlation matrix method to the truncated diagonalized correlation matrix method, which also applies to the cases of high-rank states under some specific conditions.

7.1 Diagonalized correlation matrix method

We consider a system of $\ell$ spinless fermions $a_j, a_j^\dagger, j = 1, 2, \cdots, \ell$. A Gaussian state is characterized by a correlation function matrix $C$ with entries

$$C_{j_1,j_2} = \langle a_{j_1}^\dagger a_{j_2} \rangle, \quad j_1, j_2 = 1, 2, \cdots, \ell. \quad (7.1)$$

The trace distance and fidelity between two such Gaussian states can be calculated using the diagonalized correlation matrix method in [33]. One correlation matrix $C$ has the eigenvalue
equation
\[ Cu_j = \mu_j u_j, \quad j = 1, 2, \cdots, \ell, \tag{7.2} \]
with the real eigenvalues \( \mu_j \in [0, 1] \) and eigenvectors \( u_j \). There is the unitary matrix defined as
\[ U \equiv (u_1, u_2, \cdots, u_\ell), \tag{7.3} \]
with each of the eigenvector written as a column vector. The correlation matrix \( C \) can be diagonalized as
\[ U^\dagger C U = \text{diag}(\mu_1, \mu_2, \cdots, \mu_\ell). \tag{7.4} \]
The density matrix can be written in terms of the modular Hamiltonian [41, 63]
\[ \rho = \det(1 - C) \exp\left(- \sum_{j_1, j_2 = 1}^\ell M_{j_1 j_2} \hat{a}^\dagger_{j_1} \hat{a}_{j_2} \right), \tag{7.5} \]
with the matrix
\[ M = \log(C^{-1} - 1). \tag{7.6} \]
The matrix \( M \) is also diagonal under the same basis
\[ U^\dagger M U = \text{diag}(\nu_1, \nu_2, \cdots, \nu_\ell), \quad \nu_j = \log(\mu_j^{-1} - 1), \quad j = 1, 2, \cdots, \ell. \tag{7.7} \]
We define the new modes
\[ \tilde{a}_j = \sum_{j' = 1}^\ell U_{j j'} a_{j'}, \quad \tilde{a}^\dagger_j = \sum_{j' = 1}^\ell U_{j' j} a^\dagger_{j'}, \tag{7.8} \]
in terms of which the density matrix takes the form
\[ \rho = \prod_{j = 1}^\ell [(1 - \mu_j) + (2\mu_j - 1)\tilde{a}^\dagger_j \tilde{a}_j]. \tag{7.9} \]

With the explicit density matrices, we calculate the trace distance and fidelity. To calculate the fidelity, it is convenient to use the square root of the density matrix
\[ \sqrt{\rho} = \prod_{j = 1}^\ell [(\sqrt{1 - \mu_j} + (\sqrt{\mu_j} - \sqrt{1 - \mu_j})\tilde{a}^\dagger_j \tilde{a}_j]. \tag{7.10} \]

### 7.2 Truncated diagonalized correlation matrix method

We refine the truncated correlation matrix method in [33] to many-quasiparticle excited states. The truncation strategy in [33] is an exact method but it only applies to low-rank excited states. The truncation strategies introduced below are approximate methods but apply also to many-quasiparticle excited states.
7.2.1 Maximal entropy strategy

For the correlation matrix $C$ with (7.4), the corresponding density matrix $\rho$ is similar with

$$
\rho \equiv \bigotimes_{j=1}^{\ell} \begin{pmatrix}
1 - \mu_j \\
\mu_j
\end{pmatrix}.
$$

(7.11)

The von Neumann entropy for $\rho$ is just the Shannon entropy of each effective probability distribution $\{1 - \mu_j, \mu_j\}, j = 1, 2, \cdots, \ell$

$$
S = \sum_{j=1}^{\ell} \left[-(1 - \mu_j) \log(1 - \mu_j) - \mu_j \log \mu_j \right].
$$

(7.12)

When the entropy $S$ is not so large, one may truncate the above effective probability distribution to a few cases with $\mu_j$ that are the most close to $\frac{1}{2}$. We follow the same idea and truncate two density matrices and calculate their distance.

We consider two states $\rho_1$ and $\rho_2$ with correlation matrices $C_1$ and $C_2$. The matrix $C_1$ has $\ell$ pairs of eigenvalues and eigenvectors $(\mu_j, u_j)$ with $j = 1, 2, \cdots, \ell$, and the matrix $C_2$ has $\ell$ pairs of eigenvalues and eigenvectors $(\mu_j, u_j)$ with $j = \ell + 1, \ell + 2, \cdots, 2\ell$. We sort the $2\ell$ pairs of eigenvalues and eigenvectors $(\mu_j, u_j)$ with $j = 1, 2, \cdots, 2\ell$ according to the values of $|\mu_j - \frac{1}{2}|$ from the smallest one to the largest one, and choose the first $t$ vectors and rename them as $v_i$ with $i = 1, 2, \cdots, t$.

The vectors $v_i$ with $i = 1, 2, \cdots, t$ generally are not orthogonal. We define the $t \times t$ matrix $V$ with entries

$$
V_{i_1i_2} \equiv v_{i_1}^\dagger v_{i_2}, \quad i_1, i_2 = 1, 2, \cdots, t.
$$

(7.13)

Note that the rank of the matrix $V$ is possibly smaller than $t$. We sort the $t$ pairs of eigenvalues and eigenvectors $(\alpha_i, a_i)$ with $i = 1, 2, \cdots, t$ of $V$ according to the values of $\alpha_i$ from the largest to the smallest and discard the ones smaller than a cutoff, say $10^{-9}$. After the discard, we have $\tilde{\ell}$ sets of eigenvalues and eigenvectors $(\alpha_i, a_i)$ with $i = 1, 2, \cdots, \tilde{\ell}$, from which we define $\tilde{\ell}$ orthonormal vectors

$$
\tilde{u}_i \equiv \frac{1}{\sqrt{\alpha_i}} \sum_{i'=1}^{t} [a_{i'}]_i v_{i'}, \quad i = 1, 2, \cdots, \tilde{\ell}.
$$

(7.14)

Note that each of the vector $\tilde{u}_i$ has $\tilde{\ell}$ components.

Then we define the $\ell \times \tilde{\ell}$ matrix

$$
\tilde{U} \equiv (\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_\ell).
$$

(7.15)

We define the new truncated correlation matrices of size $\ell \times \tilde{\ell}$

$$
\tilde{C}_1 \equiv \tilde{U}^\dagger C_1 \tilde{U}, \quad \tilde{C}_2 \equiv \tilde{U}^\dagger C_2 \tilde{U}.
$$

(7.16)

We construct the $2^\tilde{\ell} \times 2^\tilde{\ell}$ truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$ from the $\ell \times \tilde{\ell}$ truncated correlation matrices $\tilde{C}_1$ and $\tilde{C}_2$ using the diagonalized correlation matrix method in the above subsection. With the truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$, we calculate the approximate trace distance and fidelity

$$
\tilde{D}_{t,0}(\rho_1, \rho_2) \equiv D(\tilde{\rho}_1, \tilde{\rho}_2),
$$

(7.17)

$$
\tilde{F}_{t,0}(\rho_1, \rho_2) \equiv F(\tilde{\rho}_1, \tilde{\rho}_2).
$$

(7.18)
7.2.2 Maximal difference strategy

For two states $\rho_1$ and $\rho_2$, we choose the $\ell = t < \ell$ vectors $\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_\ell$ of the correlation matrices $C_1 - C_2$ corresponding to the $\ell$ largest absolute values of the eigenvalues and define the $\ell \times \ell$ matrix

$$\bar{U} \equiv (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_\ell).$$

The truncated correlation matrices are defined as

$$\bar{C}_1 \equiv \bar{U}^\dagger C_1 \bar{U}, \quad \bar{C}_2 \equiv \bar{U}^\dagger C_2 \bar{U}. \quad (7.20)$$

We construct the $2\ell \times 2\ell$ truncated density matrices $\bar{\rho}_1$ and $\bar{\rho}_2$ from the $\ell \times \ell$ truncated correlation matrices $\bar{C}_1$ and $\bar{C}_2$ using the diagonalized correlation matrix method introduced above. With the truncated density matrices $\bar{\rho}_1$ and $\bar{\rho}_2$, we calculate the approximate trace distance and fidelity

$$\bar{D}_{0,t}(\rho_1, \rho_2) \equiv D(\bar{\rho}_1, \bar{\rho}_2),$$

$$\bar{F}_{0,t}(\rho_1, \rho_2) \equiv F(\bar{\rho}_1, \bar{\rho}_2). \quad (7.22)$$

7.2.3 Mixed strategy

We also use a strategy that is a mix of the maximal entropy strategy and maximal difference strategy. The matrix $C_1$ has eigenvalues and eigenvectors $(\mu_j, \nu_j)$ with $j = 1, 2, \ldots, \ell$, and the matrix $C_2$ has eigenvalues and eigenvectors $(\mu_j, \eta_j)$ with $j = 1, 2, \ldots, \ell$. We sort the $2\ell$ sets of eigenvalues and eigenvectors $(\mu_j, \nu_j)$ with $j = 1, 2, \ldots, 2\ell$ according to the values of $|\mu_j - \frac{1}{2}|$ from the smallest to the largest one, and choose the first $t_1$ vectors and rename them as $v_i$ with $i = 1, 2, \ldots, t_1$. We choose $t_2$ eigenvectors of $C_1 - C_2$ corresponding to the $t_2$ largest absolute values of the eigenvalues and rename them as $v_i$, $i = t_1 + 1, t_1 + 2, \ldots, t_1 + t_2$. In total, we get $t = t_1 + t_2$ vectors $v_i$, $i = 1, 2, \ldots, t$.

We define the $t \times t$ matrix $V$ with entries

$$V_{i_1 i_2} \equiv v_{i_1}^\dagger v_{i_2}, \quad i_1, i_2 = 1, 2, \ldots, t. \quad (7.23)$$

We sort the eigenvalues and eigenvectors $(\alpha_i, \nu_i)$ of $V$ according to the values of $\alpha_i$ from the largest to the smallest and discard the ones smaller than a cutoff, say $10^{-9}$. After the discard, we have $\ell$ sets of eigenvalues and eigenvectors $(\alpha_i, \nu_i)$ with $i = 1, 2, \ldots, \ell$, from which we define the $\ell$-component orthonormal vectors

$$\bar{u}_i \equiv \frac{1}{\sqrt{|\alpha_i|}} \sum_{j=1}^{\ell} [a_i]_j \nu_j, \quad i = 1, 2, \ldots, \ell. \quad (7.24)$$

Then we define the $\ell \times \ell$ matrix

$$\bar{U} \equiv (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_\ell), \quad (7.25)$$

and the new truncated correlation matrices of size $\ell \times \ell$

$$\bar{C}_1 \equiv \bar{U}^\dagger C_1 \bar{U}, \quad \bar{C}_2 \equiv \bar{U}^\dagger C_2 \bar{U}. \quad (7.26)$$
We construct the $2^{\tilde{\ell}} \times 2^{\tilde{\ell}}$ truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$ from the $\tilde{\ell} \times \tilde{\ell}$ truncated correlation matrices $\tilde{C}_1$ and $\tilde{C}_2$ using the diagonalized correlation matrix method in the above section. With the truncated density matrices $\tilde{\rho}_1$ and $\tilde{\rho}_2$, we calculate the approximate trace distance and fidelity

$$\tilde{D}_{t_1,t_2}(\rho_1, \rho_2) \equiv D(\tilde{\rho}_1, \tilde{\rho}_2),$$

$$\tilde{F}_{t_1,t_2}(\rho_1, \rho_2) \equiv F(\tilde{\rho}_1, \tilde{\rho}_2).$$ (7.27)

For fixed $t < \ell$, we get the best approximate trace distance and fidelity from the mixed truncation strategy

$$\tilde{D}_t(\rho_1, \rho_2) \equiv \max_{0 \leq t_1 \leq t} \tilde{D}_{t_1,t-t_1}(\rho_1, \rho_2),$$ (7.29)

$$\tilde{F}_t(\rho_1, \rho_2) \equiv \min_{0 \leq t_1 \leq t} \tilde{F}_{t_1,t-t_1}(\rho_1, \rho_2).$$ (7.30)

For $\ell \leq t$, we just calculate the exact trace distance and fidelity from the diagonalized correlation matrix method.

### 8 Examples of RDMs in XX chain

We use the truncation method to calculate the subsystem trace distance between eigenstates in the XX chain.

#### 8.1 Ground and excited states

The XX chain with a transverse field has the Hamiltonian

$$H = -\sum_{j=1}^{L} \left[ \frac{1}{4}(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \frac{\lambda}{2} \sigma_j^z \right].$$ (8.1)

From Jordan-Wigner transformation, it becomes a chain of free spinless fermions

$$H = \sum_{j=1}^{L} \left[ \lambda \left( a_j^\dagger a_{j+1} - \frac{1}{2} \right) - \frac{1}{2}(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) \right],$$ (8.2)

with combination of proper boundary conditions and projections. From Fourier transformation, it becomes

$$H = \sum_k \left( \lambda - \cos \frac{2\pi k}{L} \right) (b_k^\dagger b_k - \frac{1}{2}).$$ (8.3)

We focus on anti-periodic boundary condition for the fermions $a_{j+L} = -a_j$, $a_{j+L}^\dagger = -a_j^\dagger$, i.e. the NS sector, and the case that the total number of sites $L$ is an even integer, and so there are half-integer momenta

$$k = -\frac{L-1}{2}, \cdots, -\frac{1}{2}, \frac{1}{2}, \cdots, \frac{L-1}{2}. \quad (8.4)$$

The energy eigenstates are characterized by the set of the momenta of excited quasiparticles $K = \{k_1, k_2, \cdots, k_r\}$

$$|K\rangle = b_{k_1}^\dagger b_{k_2}^\dagger \cdots b_{k_r}^\dagger |G\rangle. \quad (8.5)$$

For many-quasiparticle states, we consider two typical scaling laws of the entropy. In states from few-quasiparticle states and logarithmic-law states and may not work for the volume-law states. These results indicate that the maximal entropy strategy may give good approximate RDMs for the few-quasiparticle states and logarithmic-law states and may not work for the volume-law states.

For the subsystem $A = [1, \ell]$ in the state $K$, there is the $\ell \times \ell$ Hermitian correlation matrix $C_{A,K}$ with the entries \([41, 44, 45, 63, 64]\)

\[
[C_{A,K}]_{j_1,j_2} = \langle a_{j_1}^\dagger a_{j_2} \rangle_K = h^K_{j_1-j_2}, \quad j_1, j_2 = 1, 2, \ldots, \ell, \tag{8.6}
\]

and the definition

\[
h^K_j \equiv \frac{1}{L} \sum_{k=K} e^{-\frac{2\pi i jk}{L}}. \tag{8.7}
\]

To check the efficiency of the truncated correlation matrix method with different strategies we calculate the approximate trace distance and fidelity between several typical states (3.10) as shown in figure 1 in XX chain.

In state $K_0$, the von Neumann entropy is trivially vanishing. The states $K_1$ and $K_2$ are few-quasiparticle states in which only a few quasiparticle modes are excited. The states from $K_3$ to $K_{14}$ are many-quasiparticle states in which the number of excited modes are proportional to $L$ in the scaling limit. In few-quasiparticle states, the entropy is finite in the scaling limit \([26, 27, 52–59]\). For many-quasiparticle states, we consider two typical scaling laws of the entropy. In states from $K_3$ to $K_8$ almost all the excited modes are successive and the entropy follows the logarithmic law \([44, 45, 60, 61]\). We still call them logarithmic-law many-quasiparticle states or logarithmic-law states. In states from $K_9$ to $K_{14}$, some fixed patterns are excited successively in an extended region of the momentum space. In such states, the entropy follows the volume law in the regime $1 \ll \ell \ll L$ \([49]\), and we still call them volume-law states. In a word, the classification in (3.11) still applies to the states in the XX chain.

We plot the eigenvalues of the correlation matrices of these states in figure 11. For the correlation matrix of the few-quasiparticle state, most of the eigenvalues are 0, and only finite number of the eigenvalues are in the range (0, 1). For the correlation matrix of the logarithmic-law state, most of the eigenvalues are either 0 or 1, and only a few of eigenvalues are in the range (0, 1). For the correlation matrix of the volume-law state, the number of the eigenvalues in the range (0, 1) is proportional to $L$ in the scaling limit, and in some cases all the eigenvalues are in the range (0, 1). These results indicate that the maximal entropy strategy may give good approximate RDMs for the few-quasiparticle states and logarithmic-law states and may not work for the volume-law states.
Figure 12: The eigenvalues of the correlation matrix difference for the classes between two few-quasiparticle states (top left), between two logarithmic-law states (top middle), between two volume-law states (top right), between one few-quasiparticle state and one logarithmic-law state (bottom left), between one few-quasiparticle state and one volume-law state (bottom middle), and between one logarithmic-law state and one volume-law state (bottom right) in the XX chain. We have set $L = 360$ and $\ell = 120$.

We consider six classes of trace distance and fidelity among the states (3.10), as indicated by the titles of the following six subsections. We show the eigenvalues of the correlation matrix difference for the six classes in figure 12. We show examples of the trace distance and fidelity from the truncated diagonalized correlation matrix method with different strategies in figure 13.

### 8.2 Two few-quasiparticle states

The eigenvalues of the correlation matrix difference for two few-quasiparticle states are shown in panel (a) of figure 12. The number of nonvanishing eigenvalues of the correlation matrix difference is just the number of different modes of the two few-quasiparticle states. Examples of approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the first row of figure 13. We compare the approximate trace distance and fidelity with the exact results from the subsystem mode method in [57–59]. The maximal entropy strategy works well, as already checked in [33]. The maximal difference strategy can find the different modes but may miss the modes that are excited in both states, and so it may fail giving the precise trace distance and fidelity.
Figure 13: The symbols are the trace distance ($D$) and fidelity ($F$) in the XX chain from the truncated diagonalized correlation matrix method with the maximal entropy strategy (MES, left), maximal difference strategy (MDS, middle) and the mixed strategy (MS, right). The solid lines are the exact results from the subsystem mode method or the formula (8.8). The dotted lines are lower and upper bounds for trace distance coming from exact results of fidelity. We have set $\mathcal{B}_T = 120$ and $t = 10$. 
8.3 Two logarithmic-law states

The eigenvalues of the correlation matrix difference for the classes between two logarithmic-law states are shown in panel (b) of figure 12. We see that nearly all the eigenvalues of the correlation matrix difference are 0, ±1. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the second and third rows of figure 13. We compare the approximate fidelity from the truncated correlation matrix method with the exact results from correlation matrices [65]

\[
F(\rho_{C_1}, \rho_{C_2}) = \sqrt{\det(1 - C_1)} \sqrt{\det(1 - C_2)} \det \left( 1 + \sqrt{\frac{C_1}{1 - C_1} \frac{C_2}{1 - C_2} \left( \frac{C_1}{1 - C_1} \right)} \right),
\]  

(8.8)

where we have used \(\rho_C\) to represent the corresponding RDM of the correlation matrix \(C\). The exact fidelity gives both lower and upper bounds for the trace distance (1.3).

For the case \(K_3 vs K_4\), the number of different excited modes is proportional to \(L\) in the scaling limit, and for large enough \(\ell\) there exist eigenvalues equal to ±1. The maximal entropy strategy does not work well for the chosen truncation number \(t = 10\), while the maximal difference strategy works well.

For the case \(K_7 vs K_8\), the number of different excited modes is small. The maximal entropy strategy works well for most of the subsystem sizes except when subsystem is almost the whole system. The maximal difference strategy does not work well due to the missing of the modes that are excited in both states. The mixed strategy overcomes the drawbacks of both the maximal entropy strategy and it works well for all the subsystem sizes.

8.4 Two volume-law states

The eigenvalues of the correlation matrix difference for the classes between two volume-law states are shown in panel (c) of figure 12. All the eigenvalues are in the range \((-1, 1)\), and there are no eigenvalues being equal to ±1. Examples of the approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the fourth row of figure 13. Generally all the three strategies fail.

8.5 One few-quasiparticle state and one logarithmic-law state

The eigenvalues of the correlation matrix difference for the classes between one few-quasiparticle state and one logarithmic-law state are shown in panel (d) of figure 12. For all the cases there exist large number of eigenvalues being equal to ±1. Examples of approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the fifth row of figure 13. With the increase of the subsystem size, the trace distance increases quickly to 1 and the fidelity decreases quickly to 0. The maximal entropy strategy does not always work well, while the maximal difference strategy always works.

8.6 One few-quasiparticle state and one volume-law state

The eigenvalues of the correlation matrix difference for the classes between one few-quasiparticle state and one volume-law state are shown in panel (e) of figure 12. All the eigenvalues are in the range \((-1, 1)\). Examples of approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the sixth row of figure 13. All the three strategies fail.
8.7 One logarithmic-law state and one volume-law state

The eigenvalues of the correlation matrix difference for the classes between one logarithmic-law state and one volume-law state are shown in panel (f) of figure 12. For the cases $K_3$ vs $K_9$ and $K_3$ vs $K_{11}$, there are no eigenvalues being equal to $\pm 1$, while for the cases $K_3$ vs $K_{10}$ and $K_5$ vs $K_9$ there exist large number of eigenvalues being equal to $\pm 1$. Examples of approximate trace distance and fidelity from the truncated correlation matrix method with different strategies are shown in the last two rows of figure 13. The maximal entropy strategy fails for all the cases. The maximal difference strategy works well for the case $K_3$ vs $K_{10}$, while it fails for the case $K_3$ vs $K_{11}$. The mixed strategy does not improve the results for the case $K_3$ vs $K_{11}$.

8.8 Summary

The results and summary for the XX chain in this section are similar to those for the Ising chain in section 3. The mixed strategy of the truncation method works when both Gaussian RDMs are small-entropy states. The maximal difference strategy of the truncation method works for two special nearly orthogonal Gaussian RDMs. For the special nearly orthogonal Gaussian RDMs, the pattern that there exist large number of consecutive excited modes in the momentum space is the same as that in the Ising chain. The different point is that the orthogonal Gaussian RDMs are characterized by the existence of eigenvalues of the correlation matrix difference being very close to $-1$ or $1$, as shown for the cases $K_3$ vs $K_4$, $K_5$ vs $K_6$, $K_1$ vs $K_2$, $K_1$ vs $K_5$, $K_1$ vs $K_6$, $K_3$ vs $K_{10}$ and $K_5$ vs $K_9$ in figure 12. For these cases the two states are nearly orthogonal and the maximal difference strategy of the truncation method works, with the increase of the subsystem size the trace distance increases quickly to 1 and the fidelity quickly decreases to 0 as shown for the cases $K_3$ vs $K_4$, $K_1$ vs $K_5$ and $K_3$ vs $K_{10}$ in panels (e), (n) and (t) in figure 13.

9 Examples of low-lying state RDMs in half-filled XX chain

We consider the trace distance and fidelity among several low-lying states in the XX chain without the transverse field, whose ground state is half-filled in the momentum space. The continuum limit of the critical XX chain is just the 2D free compact boson theory. The states we consider in the XX chain are the ones corresponding to the ground states $|G\rangle$, current states $|J\rangle$ and $|\bar{J}\rangle$, state $|\bar{J}\bar{J}\rangle$, and vertex operator state $|\alpha,\bar{\alpha}\rangle \equiv V_{\alpha,\bar{\alpha}}|G\rangle$. Note that $|0,0\rangle = |G\rangle$.

In [20, 22], there are results for the trace distance

\[ D(\rho_{A,\alpha,\bar{\alpha}}, \rho_{A,\alpha',\bar{\alpha}'}) = x, \quad \text{if } (\alpha - \alpha')^2 + (\bar{\alpha} - \bar{\alpha}')^2 = 1, \]
\[ D(\rho_{A,\alpha,\bar{\alpha}}, \rho_{A,\alpha',\bar{\alpha}'}) = \sqrt{(\alpha - \alpha')^2 + (\bar{\alpha} - \bar{\alpha}')^2} x + o(x), \]
\[ D(\rho_{A,J}, \rho_{A,\alpha,\bar{\alpha}}) = D(\rho_{A,J}, \rho_{A,\bar{\alpha},\alpha}) = \sqrt{(\alpha - \alpha')^2 + (\bar{\alpha} - \bar{\alpha}')^2} x + o(x), \]
\[ D(\rho_{A,J}, \rho_{A,\alpha,\bar{\alpha}}) = \sqrt{\alpha^2 + \bar{\alpha}^2} x + o(x), \]
\[ D(\rho_{A,G}, \rho_{A,J}) = D(\rho_{A,G}, \rho_{A,J}) = D(\rho_{A,J}, \rho_{A,J,J}) = D(\rho_{A,J}, \rho_{A,J,J}) \approx 0.107 \times 2\sqrt{2}\pi^2 x^2 + o(x^2), \]
\[ D(\rho_{A,J}, \rho_{A,J}) \approx 0.141 \times 2\sqrt{2}\pi^2 x^2 + o(x^2), \]
\[ D(\rho_{A,G}, \rho_{A,J,J}) \approx 0.166 \times 2\sqrt{2}\pi^2 x^2 + o(x^2), \]

(9.1)
and fidelity

\[
F(\rho_{A,a,\tilde{a}}, \rho_{A,a',\tilde{a}'}) = \left( \cos \frac{\pi x}{2} \right)^{\frac{(a-a')^2+(\tilde{a}-\tilde{a}')^2}{4}},
\]

\[
F(\rho_{A,\tilde{a}}, \rho_{A,\tilde{a}}) = F(\rho_{A,a}, \rho_{A,a}) = 1 - (\alpha^2 + \bar{\alpha}^2) \frac{\pi^2 x^2}{16} + o(x^2),
\]

\[
F(\rho_{A,\bar{a}}, \rho_{A,a}) = 1 - (\alpha^2 + \bar{\alpha}^2) \frac{\pi^2 x^2}{16} + o(x^2),
\]

\[
F(\rho_{A,G}, \rho_{A,J}) = F(\rho_{A,G}, \rho_{A,J}) = F(\rho_{A,J}, \rho_{A,J}) = \frac{\Gamma^2(\frac{3+\csc \frac{x}{2}}{4})}{\Gamma^4(\frac{1+\csc \frac{x}{2}}{4})} 2 \sin(\pi x),
\]

\[
F(\rho_{A,J}, \rho_{A,J}) = F(\rho_{A,G}, \rho_{A,J}) = \frac{\Gamma^4(\frac{3+\csc \frac{x}{2}}{4})}{\Gamma^4(\frac{1+\csc \frac{x}{2}}{4})} 4 \sin^2(\pi x).
\]

(9.2)

We repeat that the fidelity between the ground state and a general primary excited state in two-dimensional conformal field theory was firstly obtained in [5]. We show the results in the critical XX chain and the free compact boson theory in figure 14 and find perfect matches.

10 Conclusion and discussions

In this paper we have developed a truncation method to calculate the trace distance between two fermionic Gaussian states, and checked the method with various examples of RDMs in Ising and XX chains. Based on the examples, we found that the method works at least for two conditions as summarized in (6.2). The first condition is that the von Neumann entropies of the two states are small and the correlation matrices of the two states nearly commute. The second condition is that the two states are nearly orthogonal in the sense that the maximal canonical value of the correlation difference is nearly 2.

As an application of the truncation method, we calculated the subsystem trace distances among the ground state and the low-lying excited states in critical Ising chain and half-filled XX chain in respectively section 4 and section 9 and found perfect matches with the predictions in the corresponding two-dimensional conformal field theories. This improves the calculations of subsystem trace distances in spin chains in [20,22,33], from subsystems with very small sizes to much larger subsystems.

It should be noted that a finite truncation method is not always possible for two Gaussian states with volume law entropies, as shown with an example in appendix A. Still there exist fermionic Gaussian states for which finite truncation method is possible but the truncation method presented in this paper fails. For examples see the section 5 in which we studied the trace distance between RDMs coming from the ground states of the Ising chains with different transverse fields. It would be interesting to generalize the truncation method presented in this paper and apply it to a wider range of fermionic Gaussian states and even non-Gaussian states. We hope to be able to come back to these problem in the future.
Figure 14: The trace distance and fidelity between low-lying states in the 2D free compact boson theory (solid lines) and the corresponding approximate trace distance and fidelity from the mixed truncation strategy in the half-filled XX chain (symbols). Note that in panel (d) the fidelity $F(\rho_{A,J}, \rho_{A,J})$ overlap with the fidelity $F(\rho_{A,G}, \rho_{A,J})$. In the XX chain we have set $L = 360$ and $t = 10$.

Acknowledgements

We thank Markus Heyl and Reyhaneh Khaseh for reading a previous version of the draft and helpful comments. MAR thanks CNPq and FAPERJ (grant number 210.354/2018) for partial support. JZ thanks support from NSFC grant number 12205217.

A An example of orthogonal states for which no truncation is possible

In this appendix we give an example of two orthogonal Gaussian states in the scaling limit for which no finite truncation method of trace distance is possible.

We consider the Gaussian state density matrix in simple form

$$\rho(p) = \bigotimes_{j=1}^{\ell} \left( \begin{array}{cc} p & \alpha_j \sqrt{1-p} \\ \alpha_j \sqrt{1-p} & 1-p \end{array} \right)$$

(A.1)

with $p \in [0, 1]$. For $p = 0$ or $p = 1$, it is trivially a pure state, and we focus on the case $p \in (0, 1)$. 

35
Figure 15: The trace distance and fidelity (A.4) between two toy model states (A.1) with $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{2}$.

The corresponding correlation matrix is simply

$$\Gamma(p) = \bigoplus_{j=1}^\ell \left( \begin{array}{cc} -i(1-2p) & i(1-2p) \\ -i(1-2p) & -i(1-2p) \end{array} \right).$$

(A.2)

The von Neumann entropy of the density matrix follows the volume law

$$S(\rho(p)) = \ell \left[ -p \log p - (1-p) \log(1-p) \right].$$

(A.3)

It is easy to get the trace distance and fidelity

$$D(\rho(p_1), \rho(p_2)) = \frac{1}{2} \sum_{i=0}^\ell C_i \left| p_1^i (1-p_1)^{\ell-i} - p_2^i (1-p_2)^{\ell-i} \right|,$$

$$F(\rho(p_1), \rho(p_2)) = \left[ \sqrt{p_1 p_2} + \sqrt{(1-p_1)(1-p_2)} \right]^\ell.$$  

(A.4)

We show the example with $p_1 = \frac{1}{3}$ and $p_2 = \frac{1}{2}$ in figure 15. As long as $p_1 \neq p_2$, even when $p_1$ and $p_2$ are very close, there are

$$\lim_{\ell \to +\infty} D(\rho(p_1), \rho(p_2)) = 1,$$

$$\lim_{\ell \to +\infty} F(\rho(p_1), \rho(p_2)) = 0.$$  

(A.5)

With a finite truncation there is no way to reach this limit. This means that the two states $\rho(p_1)$ and $\rho(p_2)$ with $p_1 \neq p_2$ are orthogonal in the scaling limit $\ell \to +\infty$. The correlation matrix difference is

$$\Gamma(p_1) - \Gamma(p_2) = \bigoplus_{j=1}^\ell \left( \begin{array}{cc} -2i(p_1-p_2) & 2i(p_1-p_2) \\ 2i(p_1-p_2) & -2i(p_1-p_2) \end{array} \right).$$

(A.6)

All the canonical values of the correlation matrix difference is $2|p_1 - p_2|$, which is 2 only for the special trivial cases $p_1 = 1$, $p_0 = 0$ and $p_1 = 0$, $p_0 = 1$. 

36
References

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK, 10th anniversary edn., doi:10.1017/CBO9780511976667 (2010).

[2] J. Watrous, *The Theory of Quantum Information*, Cambridge University Press, Cambridge, UK, doi:10.1017/9781316848142 (2018).

[3] M. Fagotti and F. H. Essler, *Reduced density matrix after a quantum quench*, Phys. Rev. B 87(24), 245107 (2013), doi:10.1103/PhysRevB.87.245107, 1302.6944.

[4] J. Cardy, *Thermalization and Revivals after a Quantum Quench in Conformal Field Theory*, Phys. Rev. Lett. 112, 220401 (2014), doi:10.1103/PhysRevLett.112.220401, 1403.3040.

[5] N. Lashkari, *Relative Entropies in Conformal Field Theory*, Phys. Rev. Lett. 113, 051602 (2014), doi:10.1103/PhysRevLett.113.051602, 1402.3216.

[6] N. Lashkari, *Modular Hamiltonian for Excited States in Conformal Field Theory*, Phys. Rev. Lett. 117(4), 041601 (2016), doi:10.1103/PhysRevLett.117.041601, 1601.03506.

[7] D. L. Jafferis, A. Lewkowycz, J. Maldacena and S. J. Suh, *Relative entropy equals bulk relative entropy*, JHEP 06, 004 (2016), doi:10.1007/JHEP06(2016)004, 1512.06431.

[8] X. Dong, D. Harlow and A. C. Wall, *Reconstruction of Bulk Operators within the Entanglement Wedge in Gauge-Gravity Duality*, Phys. Rev. Lett. 117, 021601 (2016), doi:10.1103/PhysRevLett.117.021601, 1601.05416.

[9] G. Sárosi and T. Ugajin, *Relative entropy of excited states in two dimensional conformal field theories*, JHEP 07, 114 (2016), doi:10.1007/JHEP07(2016)114, 1603.03057.

[10] G. Sárosi and T. Ugajin, *Relative entropy of excited states in conformal field theories of arbitrary dimensions*, JHEP 02, 060 (2017), doi:10.1007/JHEP02(2017)060, 1611.02959.

[11] R. Arias, D. Blanco, H. Casini and M. Huerta, *Local temperatures and local terms in modular Hamiltonians*, Phys. Rev. D 95(6), 065005 (2017), doi:10.1103/PhysRevD.95.065005, 1611.08517.

[12] A. Dymarsky, N. Lashkari and H. Liu, *Subsystem eigenstate thermalization hypothesis*, Phys. Rev. E 97, 012140 (2018), doi:10.1103/PhysRevE.97.012140, 1611.08764.

[13] S. He, F.-L. Lin and J.-j. Zhang, *Subsystem eigenstate thermalization hypothesis for entanglement entropy in CFT*, JHEP 08, 126 (2017), doi:10.1007/JHEP08(2017)126, 1703.08724.

[14] P. Basu, D. Das, S. Datta and S. Pal, *Thermality of eigenstates in conformal field theories*, Phys. Rev. E 96, 022149 (2017), doi:10.1103/PhysRevE.96.022149, 1705.03001.

[15] R. Arias, H. Casini, M. Huerta and D. Pontello, *Anisotropic Unruh temperatures*, Phys. Rev. D 96(10), 105019 (2017), doi:10.1103/PhysRevD.96.105019, 1707.05375.

[16] S. He, F.-L. Lin and J.-j. Zhang, *Dissimilarities of reduced density matrices and eigenstate thermalization hypothesis*, JHEP 12, 073 (2017), doi:10.1007/JHEP12(2017)073, 1708.05090.
[17] Y. Suzuki, T. Takayanagi and K. Umemoto, **Entanglement Wedges from Information Metric in Conformal Field Theories**, Phys. Rev. Lett. **123**(22), 221601 (2019), doi:10.1103/PhysRevLett.123.221601, 1908.09939.

[18] J. Zhang and P. Calabrese, **Subsystem distance after a local operator quench**, JHEP **02**, 056 (2020), doi:10.1007/JHEP02(2020)056, 1911.04797.

[19] Y. Kusuki, Y. Suzuki, T. Takayanagi and K. Umemoto, **Looking at Shadows of Entanglement Wedges**, PTEP **2020**(11), 11B105 (2020), doi:10.1093/ptep/ptaa152, 1912.08423.

[20] J. Zhang, P. Ruggiero and P. Calabrese, **Subsystem Trace Distance in Quantum Field Theory**, Phys. Rev. Lett. **122**(14), 141602 (2019), doi:10.1103/PhysRevLett.122.141602, 1901.10993.

[21] T. Mendes-Santos, G. Giudici, M. Dalmonte and M. A. Rajabpour, **Entanglement Hamiltonian of quantum critical chains and conformal field theories**, Phys. Rev. B **100**(15), 155122 (2019), doi:10.1103/PhysRevB.100.155122, 1906.00471.

[22] J. Zhang, P. Ruggiero and P. Calabrese, **Subsystem trace distance in low-lying states of (1 + 1)-dimensional conformal field theories**, JHEP **10**, 181 (2019), doi:10.1007/JHEP10(2019)181, 1907.04332.

[23] J. Zhang, P. Calabrese, M. Dalmonte and M. A. Rajabpour, **Lattice Bisognano-Wichmann modular Hamiltonian in critical quantum spin chains**, SciPost Phys. Core **2**, 007 (2020), doi:10.21468/SciPostPhysCore.2.2.007, 2003.00315.

[24] R. Arias and J. Zhang, **Rényi entropy and subsystem distances in finite size and thermal states in critical XY chains**, J. Stat. Mech. **2008**, 083112 (2020), doi:10.1088/1742-5468/ababfd, 2004.13096.

[25] J. de Boer, V. Godet, J. Kastikainen and E. Keski-Vakkuri, **Quantum hypothesis testing in many-body systems**, SciPost Phys. Core **4**, 019 (2021), doi:10.21468/SciPostPhysCore.4.2.019, 2007.11711.

[26] J. Zhang and M. A. Rajabpour, **Excited state Rényi entropy and subsystem distance in two-dimensional non-compact bosonic theory. Part I. Single-particle states**, JHEP **12**, 160 (2020), doi:10.1007/JHEP12(2020)160, 2009.00719.

[27] J. Zhang and M. A. Rajabpour, **Excited state Rényi entropy and subsystem distance in two-dimensional non-compact bosonic theory. Part II. Multi-particle states**, JHEP **08**, 106 (2021), doi:10.1007/JHEP08(2021)106, 2011.11006.

[28] R.-Q. Yang, **Gravity duals of quantum distances**, JHEP **08**, 156 (2021), doi:10.1007/JHEP08(2021)156, 2102.01898.

[29] J. Kudler-Flam, **Relative Entropy of Random States and Black Holes**, Phys. Rev. Lett. **126**(17), 171603 (2021), doi:10.1103/PhysRevLett.126.171603, 2102.05053.

[30] H.-H. Chen, **Symmetry decomposition of relative entropies in conformal field theory**, JHEP **07**, 084 (2021), doi:10.1007/JHEP07(2021)084, 2104.03102.
[31] L. Capizzi and P. Calabrese, *Symmetry resolved relative entropies and distances in conformal field theory*, JHEP 10, 195 (2021), doi:10.1007/JHEP10(2021)195, 2105.08596.

[32] J. Kudler-Flam, V. Narovlansky and S. Ryu, *Distinguishing Random and Black Hole Microstates*, PRX Quantum 2(4), 040340 (2021), doi:10.1103/PRXQuantum.2.040340, 2108.00011.

[33] J. Zhang and M. A. Rajabpour, *Subsystem distances between quasiparticle excited states*, JHEP 07, 119 (2022), doi:10.1007/JHEP07(2022)119, 2202.11448.

[34] P. J. Coles, M. Cerezo and L. Cincio, *Strong bound between trace distance and Hilbert-Schmidt distance for low-rank states*, Phys. Rev. A 100(2), 022103 (2019), doi:10.1103/PhysRevA.100.022103, 1903.11738.

[35] M. Cerezo, A. Poremba, L. Cincio and P. J. Coles, *Variational Quantum Fidelity Estimation*, Quantum 4, 248 (2020), doi:10.22331/q-2020-03-26-248, 1906.09253.

[36] R. Chen, Z. Song, X. Zhao and X. Wang, *Variational quantum algorithms for trace distance and fidelity estimation*, Quantum Sci. Technol. 7, 015019 (2021), doi:10.1088/2058-9565/ac38ba, 2012.05768.

[37] S.-J. Li, J.-M. Liang, S.-Q. Shen and M. Li, *Variational quantum algorithms for trace norms and their applications*, Commun. Theor. Phys. 73, 105102 (2021), doi:10.1088/1572-9494/ac1938.

[38] R. Agarwal, S. Rethinasamy, K. Sharma and M. M. Wilde, *Estimating distinguishability measures on quantum computers* (2021), 2108.08406.

[39] L.-K. Hua, *On the theory of automorphic functions of a matrix variable i-geometrical basis*, Amer. J. Math. 66(3), 470 (1944), doi:10.2307/2371910.

[40] H. G. Becker, *On the transformation of a complex skew-symmetric matrix into a real normal form and its application to a direct proof of the Bloch-Messiah theorem*, Lett. Nuovo Cimento 8, 185 (1973), doi:10.1007/BF02906230.

[41] I. Peschel, *Calculation of reduced density matrices from correlation functions*, J. Phys. A: Math. Gen. 36(14), L205 (2003), doi:10.1088/0305-4470/36/14/101, cond-mat/0212631.

[42] T. Barthel, M. C. Chung and U. Schollwöck, *Entanglement scaling in critical two-dimensional fermionic and bosonic systems*, Phys. Rev. A 74(2), 022329 (2006), doi:10.1103/PhysRevA.74.022329, cond-mat/0602077.

[43] M. Fagotti and P. Calabrese, *Entanglement entropy of two disjoint blocks in XY chains*, J. Stat. Mech. 1004, P04016 (2010), doi:10.1088/1742-5468/2010/04/P04016, 1003.1110.

[44] G. Vidal, J. I. Latorre, E. Rico and A. Kitaev, *Entanglement in Quantum Critical Phenomena*, Phys. Rev. Lett. 90, 227902 (2003), doi:10.1103/PhysRevLett.90.227902, quant-ph/0211074.

[45] J. I. Latorre, E. Rico and G. Vidal, *Ground state entanglement in quantum spin chains*, Quant. Inf. Comput. 4, 48 (2004), doi:10.26421/QIC4.1, quant-ph/0304098.

[46] E. H. Lieb, T. Schultz and D. Mattis, *Two soluble models of an antiferromagnetic chain*, Annals Phys. 16, 407 (1961), doi:10.1016/0003-4916(61)90115-4.
[47] S. Katsura, *Statistical mechanics of the anisotropic linear Heisenberg model*, Phys. Rev. 127(5), 1508 (1962), doi:10.1103/PhysRev.127.1508.

[48] P. Pfeuty, *The one-dimensional Ising model with a transverse field*, Annals Phys. 57(1), 79 (1970), doi:10.1016/0003-4916(70)90270-8.

[49] V. Alba, M. Fagotti and P. Calabrese, *Entanglement entropy of excited states*, J. Stat. Mech. 10, P10020 (2009), doi:10.1088/1742-5468/10/10/P10020, 0909.1999.

[50] F. C. Alcaraz, M. I. Berganza and G. Sierra, *Entanglement of low-energy excitations in Conformal Field Theory*, Phys. Rev. Lett. 106, 201601 (2011), doi:10.1103/PhysRevLett.106.201601, 1101.2881.

[51] M. I. Berganza, F. C. Alcaraz and G. Sierra, *Entanglement of excited states in critical spin chains*, J. Stat. Mech. 01, P01016 (2012), doi:10.1088/1742-5468/2012/01/P01016, 1109.5673.

[52] I. Pizorn, *Universality in entanglement of quasiparticle excitations* (2012), 1202.3336.

[53] R. Berkovits, *Two-particle excited states entanglement entropy in a one-dimensional ring*, Phys. Rev. B 87(7), 075141 (2013), doi:10.1103/PhysRevB.87.075141, 1302.4031.

[54] J. Mölter, T. Barthel, U. Schollwöck and V. Alba, *Bound states and entanglement in the excited states of quantum spin chains*, J. Stat. Mech. 2014(10), 10029 (2014), doi:10.1088/1742-5468/2014/10/P10029, 1407.0066.

[55] O. A. Castro-Alvaredo, C. De Fazio, B. Doyon and I. M. Szécsényi, *Entanglement Content of Quasiparticle Excitations*, Phys. Rev. Lett. 121(17), 170602 (2018), doi:10.1103/PhysRevLett.121.170602, 1805.04948.

[56] O. A. Castro-Alvaredo, C. De Fazio, B. Doyon and I. M. Szécsényi, *Entanglement content of quantum particle excitations. Part I. Free field theory*, JHEP 10, 039 (2018), doi:10.1007/JHEP10(2018)039, 1806.03247.

[57] J. Zhang and M. A. Rajabpour, *Universal Rényi entanglement entropy of quasiparticle excitations*, EPL 135(6), 60001 (2021), doi:10.1209/0295-5075/ac130e, 2010.13973.

[58] J. Zhang and M. A. Rajabpour, *Corrections to universal Rényi entropy in quasiparticle excited states of quantum chains*, J. Stat. Mech. 2109, 093101 (2021), doi:10.1088/1742-5468/ac1f28, 2010.16348.

[59] J. Zhang and M. A. Rajabpour, *Entanglement of magnon excitations in spin chains*, JHEP 02, 072 (2022), doi:10.1007/JHEP02(2022)072, 2109.12826.

[60] C. Holzhey, F. Larsen and F. Wilczek, *Geometric and renormalized entropy in conformal field theory*, Nucl. Phys. B 424, 443 (1994), doi:10.1016/0550-3213(94)90402-2, hep-th/9403108.

[61] P. Calabrese and J. L. Cardy, *Entanglement entropy and quantum field theory*, J. Stat. Mech. 06, P06002 (2004), doi:10.1088/1742-5468/2004/06/P06002, hep-th/0405152.

[62] L. Banchi, P. Giorda and P. Zanardi, *Quantum information-geometry of dissipative quantum phase transitions*, Phys. Rev. E 89(2), 022102 (2014), doi:10.1103/PhysRevE.89.022102, 1305.4527.
[63] S.-A. Cheong and C. L. Henley, *Many-body density matrices for free fermions*, Phys. Rev. B 69(7), 075111 (2004), doi:10.1103/PhysRevB.69.075111, cond-mat/0206196.

[64] M.-C. Chung and I. Peschel, *Density-matrix spectra of solvable fermionic systems*, Phys. Rev. B 64, 064412 (2001), doi:10.1103/PhysRevB.64.064412, cond-mat/0103301.

[65] H. Casini, R. Medina, I. Salazar Landea and G. Torroba, *Rényi relative entropies and renormalization group flows*, JHEP 09, 166 (2018), doi:10.1007/JHEP09(2018)166, 1807.03305.