On shear and torsion factors in the theory of linearly elastic rods

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Abstract
Lower bounds for the factors entering the standard notions of shear and torsion stiffness for a linearly elastic rod are established in a new and simple way. The proofs are based on the following criterion to identify the stiffness parameters entering rod theory: the rod’s stored-energy density per unit length expressed in terms of force and moment resultants should equal the stored-energy density per unit length expressed in terms of stress components of a Saint–Venant cylinder subject to either flexure or torsion, according to the case. It is shown that the shear factor is always greater than one, whatever the cross section, a fact that is customarily stated without proof in textbooks of structure mechanics; and that the torsion factor is also greater than one, except when the cross section is a circle or a circular annulus, a fact that is usually proved making use of Saint–Venant’s solution in terms of displacement components.

Keywords: Rod theory, shear stiffness, torsion stiffness, shear factor, torsion factor

1 Introduction

When a direct approach is chosen to expound the standard theory of linearly elastic rods with a straight axis $L$, the Principle of Virtual Working is laid down:

$$
\int_{L} (T\gamma + N\varepsilon + M\psi + M_{t}\psi_{t})dx =: W^{i} = W^{e} := \int_{L} (pv + qw + c\varphi + c_{t}\vartheta)dx.
$$

The internal working $W^{i}$ collects four pairs $(T, \gamma), \ldots, (M_{t}, \psi_{t})$ of kinematic and dynamic fields in duality, with $T, N, M,$ and $M_{t}$ the shear force, normal

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force, bending moment and twisting moment and with $\gamma, \varepsilon, \psi$ and $\psi_t$ the shear, extension, flexion and torsion measures. The geometric compatibility conditions are:

$$\gamma = v' + \varphi, \quad \varepsilon = w', \quad \psi = \varphi' \text{ and } \psi_t = \vartheta',$$

where $v$ and $w$ are the transverse and axial displacements, $\varphi$ and $\vartheta$ are the rotations about, respectively, the $x_2$-axis and the $x_3$-axis (a superscript prime denotes differentiation with respect to $x_3$; see Fig. 1); the external

\begin{figure}[h!]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Figure 1:}
\end{figure}

working $W^e$ sets these kinematic variables in duality with the applied loads per unit length $p, q, c$ and $c_t$. The constitutive equations are:

$$T = s_s \gamma, \ldots, M_t = s_t \psi_t,$$

all the stiffness moduli $s_s, \ldots, s_t$ being assumed positive; the stored-energy density per unit length is:

$$\frac{1}{2} \frac{T^2}{s_s} + \ldots + \frac{1}{2} \frac{M_t^2}{s_t}$$

(see, e.g., [1]).

\footnote{To keep things simple, in laying down (1) we have ignored the contributions to external and internal workings of concentrated loads, axial pin-junctions et similia.}
To make use of such a theory — in fact, to make use of any structure theory formulated by way of a direct approach — the issue of parameter identification must be dealt with. In particular, one would like to gather from some suitable equilibrium theory for a rod-like three-dimensional body subject to flexure and torsion enough information to choose the ‘right’ shear and torsion stiffness parameters $s_s, s_t$, that is to say, to choose those parameters so as to guarantee that the qualitative and quantitative predictions of the ensuing rod theory allow for a quick, but technically sufficient, evaluation of the behavior of the corresponding three-dimensional body.

Customarily, the three-dimensional theory one picks is the treatment of the cases of flexure and torsion of the Saint–Venant Problem in classic linearly isotropic elasticity. With this choice, one is led to set:

$$s_s = \frac{GA}{\chi_s}, \quad s_t = \frac{GJ_o}{\chi_t},$$

where $G$ is the shear modulus of the material Saint–Venant’s prismatic cylinder is comprised of, $A$ and $J_o$ are the area and the polar moment of inertia of the cylinder’s cross section, and the dimensionless coefficients $\chi_s$ and $\chi_t$ are the shear and torsion factors. Generally, these identifications are justified by inspection of the Saint–Venant solutions for flexure and torsion in terms of displacement components. Each of these solutions is constructed by means of Saint–Venant’s semi-inverse method, starting from an educated guess about a representation for the displacement field depending on an as-short-as-possible list of parameter functions. Since the analytic forms of these functions depend solely on the shape of the cross section, the same is automatically true for the shear and torsion factors.

In textbooks of structural mechanics, the shear factor is interpreted as a measure of the non-uniformity in the distribution over the cross section of Saint–Venant’s cylinder of the tangential stresses induced by the shear force; it is stated, but as a rule not proved, that

$$\chi_s > 1.$$  \hfill (3)

As to the torsion factor, it is generally stated, and proved, that

$$\chi_t \geq 1,$$  \hfill (4)

with equality holding if and only if the cross section is a circle or a circular annulus. While this facts suggests that ($\chi_t - 1$) measures the deviation from central symmetry of the cross section, a precise interpretation is generally non attempted; at best, Saint-Venant’s conjecture that a circular section
minimizes the twist angle among all simply-connected sections with the same area is mentioned.

In this note we approach the issue of parameter identification as is done in [2]: we require that a linearly elastic rod stores the same energy per unit length as a Saint-Venant cylinder subject to end loads having the same force and moment resultant. The expression we use for the rod energy is in terms of either the shear resultant force or the twisting moment, that for Saint-Venant’s cylinder is in terms of stress components. Both to prove (1) and to characterize the torsion optimality of center-symmetric cross sections, we make no use of Saint-Venant’s solutions in terms of displacement; the only assumptions we borrow from Saint-Venant’s problem are those restricting the admissible external forces and the class of stress fields.

2 The shear and torsion factors

Given a cartesian reference frame \( \{O ; x_1, x_2, x_3\} \) and the associated orthonormal basis \( \{e_1, e_2, e_3\} \), we consider a Saint–Venant’s cylinder whose axis is parallel to \( e_3 \) and whose cross section has its barycenter \( O \) on the \( x_3 \)-axis. The vector \( x = x_1 e_1 + x_2 e_2 \) yields the position of a point of a typical cross section \( A \) with respect to \( O \). On denoting by \( S_{ij} \) \( (i, j = 1, 2) \) the cartesian components of the stress tensor, with Saint–Venant we restrict attention to cases when both external distance forces and external contact forces over the cylinder’s mantle are null and, moreover,

\[
S_{\alpha\beta} \equiv 0 \quad (\alpha, \beta = 1, 2). \tag{5}
\]

Consequently, in order to satisfy the equilibrium equations, not only the fields \( S_{3\alpha} \) \( (\alpha = 1, 2) \) must be independent of \( x_3 \) but also

\[
s \cdot n = S_{3\alpha} n_\alpha = 0, \quad x \in \partial A, \tag{6}
\]

where \( s = S_{3\alpha} e_\alpha \) is the cross-sectional traction, \( n = n_\alpha e_\alpha \) is the normal to the mantle, and \( \partial A \) is the boundary curve of \( A \).

When expressed in terms of stress components, the stored-energy density for unit volume of an isotropic material is:

\[
w(S) = \frac{1}{4\mu} \left( |S|^2 - \frac{\lambda}{3\lambda + 2\mu} (\text{tr} S)^2 \right), \tag{7}
\]

where \( \lambda \) and \( \mu \equiv G \) are the Lamé coefficients, \( |S| = (S_{ij} S_{ij})^{1/2} \), and \( \text{tr} S = S_{11} + S_{22} + S_{33} \); under Saint-Venant’s assumption [2], [2] becomes:

\[
w_{SV} = \frac{1}{2G} (S_{31}^2 + S_{32}^2) + \frac{1}{2E} S_{33}^2, \quad E := \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu},
\]

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where $E$ is the Young modulus. Thus, in Saint-Venant’s cylinder, the stored-energy density splits additively into two terms: the first one is non-null if and only if the applied loads induce cross-sectional stress, while the second one is non-null if and only if the applied loads induce axial stress. On the other hand, the stored-energy density per unit length of a linearly elastic rod is:

$$w_s = \frac{1}{2} \frac{T^2}{s_s}$$

when the rod is subjected to a shear force of magnitude $T$, and is:

$$w_t = \frac{1}{2} \frac{M_t^2}{s_t}$$

when the rod is subjected to a torsion moment of magnitude $M_t$.

Following [2], we identify the shear-stiffness parameter by imposing that

$$\frac{1}{2} \frac{T^2}{s_s} = \left( w_s = \int_A w_{SV} \, dA \right) = \frac{1}{2G} \int_A (S_{31}^2 + S_{32}^2) \, dA,$$

with the integrand on the right side proportional to $T^2$; hence,

$$s_s := G \frac{T^2}{\int_A (S_{31}^2 + S_{32}^2) \, dA}. \quad (8)$$

Next, we compare this relation with the first of (1): since the factor multiplying the shear modulus $G$ has the dimension of an area, we find it natural to give $s_s$ the form (1) by defining

$$\chi_s := A \frac{\int_A (S_{31}^2 + S_{32}^2) \, dA}{T^2}, \quad A = \int_A dA. \quad (9)$$

Quite similarly, we identify the torsion-stiffness parameter by imposing that

$$\frac{1}{2} \frac{M_t^2}{s_t} = \frac{1}{2G} \int_A (S_{31}^2 + S_{32}^2) \, dA,$$

with the integrand on the right side proportional to $M_t^2$; hence,

$$s_t := G \frac{M_t^2}{\int_A (S_{31}^2 + S_{32}^2) \, dA}. \quad (10)$$

This time, the factor multiplying $G$ has the dimension of a moment of inertia; to recover the second of (1), we set:

$$\chi_t := J_o \frac{\int_A (S_{31}^2 + S_{32}^2) \, dA}{M_t^2}, \quad J_o = \int_A \|x\|^2 \, dA. \quad (11)$$
3 Lower bounds

In this section we prove the lower bounds (11) and (11) for, respectively, $\chi_s$ and $\chi_t$.

3.1 $\chi_s > 1$.

Consider the following chain of inequalities:

$$A \int_A (S_{31}^2 + S_{32}^2) \, dA \geq A \int_A S_{32}^2 \, dA \geq \left( \int_A S_{32} \, dA \right)^2.$$  \hspace{1cm} (12)

Note that the first inequality in (3.1) holds true with the equality sign if and only if $S_{31}$ is identically null on $A$; and that the second, which is established by making use of Jensen’s inequality for convex functions, reduces to an equality if and only if $S_{32}$ is identically constant on $A$. Next, without any loss of generality, take the shear force parallel to $e_2$, so that

$$T = \int_A S_{32} \, dA,$$

(3.1) yields:

$$A \int_A (S_{31}^2 + S_{32}^2) \, dA \geq T^2,$$

and (2) reduces to

$$\chi_s \geq 1,$$

with

$$\chi_s = 1 \iff S_{31} \equiv 0 \text{ and } S_{32} \equiv \text{const}.$$  

This set of conditions on the stress field is incompatible with the boundary condition (2) no matter the shape of the cross section, because the stress field must be continuous up to the boundary of the cross section itself. We conclude that the strict inequality (11) must hold.

3.2 $\chi_t \geq 1$.

By definition,

$$M_t = \int_A (x_1 S_{32} - x_2 S_{31}) \, dA.$$  

\footnote{A use of this second inequality in similar circumstances is found on p. 475 of [3].}
With an application of Fubini’s theorem, the denominator in (2) can be written as
\[
M_i^2 = \int_{A \times A} \left( x_1 S_{32}(x) - x_2 S_{31}(x) \right) \left( y_1 S_{32}(y) - y_2 S_{31}(y) \right) \, dA \, dA,
\]
or rather, equivalently, given that
\[
\int_{A \times A} x_1 y_2 S_{32}(x) S_{31}(y) \, dA \, dA = \int_{A \times A} y_1 x_2 S_{32}(y) S_{31}(x) \, dA \, dA,
\]
as
\[
M_i^2 = \int_{A \times A} \left( x_1 y_1 S_{32}(x) S_{32}(y) + x_2 y_2 S_{31}(x) S_{31}(y) - 2 x_1 y_2 S_{32}(x) S_{31}(y) \right) \, dA \, dA.
\]
(13)

An iterated use of the inequality:
\[
\pm 2ab \leq a^2 + b^2
\]
yields:
\[
\left\{ \begin{array}{l}
\int_{A \times A} x_1 y_1 S_{32}(x) S_{32}(y) \, dA \, dA \leq \frac{1}{2} \int_{A \times A} \left( x_1^2 S_{32}^2(y) + y_1^2 S_{32}^2(x) \right) \, dA \, dA = \\
\int_{A \times A} y_1 x_2 S_{32}(y) S_{31}(x) \, dA \, dA = \int_{A \times A} x_1^2 S_{32}^2(y) \, dA \, dA, \\
\int_{A \times A} x_2 y_2 S_{31}(x) S_{31}(y) \, dA \, dA \leq \frac{1}{2} \int_{A \times A} \left( x_2^2 S_{31}^2(y) + y_2^2 S_{31}^2(x) \right) \, dA \, dA = \\
\int_{A \times A} x_2^2 S_{31}^2(y) \, dA \, dA, \\
\int_{A \times A} -2 x_1 S_{32}(x) y_2 S_{31}(y) \, dA \, dA \leq \int_{A \times A} \left( x_1^2 S_{31}^2(y) + y_2^2 S_{31}^2(x) \right) \, dA \, dA = \\
\int_{A \times A} x_1^2 S_{31}^2(y) + x_2^2 S_{31}^2(y) \, dA \, dA.
\end{array} \right.
\]

With this and another use of Fubini’s Theorem, it follows from (3.2) that
\[
M_i^2 \leq \int_{A \times A} \left( x_1^2 S_{32}^2(y) + x_2^2 S_{31}^2(y) + x_1^2 S_{31}^2(y) + x_2^2 S_{32}^2(y) \right) \, dA \, dA = \\
\int_{A \times A} \|x\|^2 (S_{32}^2(y) + S_{31}^2(y)) \, dA \, dA = \left( \int_{A} \|x\|^2 \, dA \right) \left( \int_{A} (S_{32}^2(x) + S_{31}^2(x)) \, dA \right) = \\
J_0 \int_{A} (S_{32}^2 + S_{31}^2) \, dA,
\]

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which is tantamount to having from (2) that $\chi_t \geq 1$ whatever the shape of the cross section.

Now, given that equality holds in (3.2) if and only if $a = \mp b$, $\chi_t = 1$ if and only if all of the following conditions are identically satisfied in $A \times A$:

\[ x_1 S_{32}(y) = y_1 S_{32}(x), \quad x_2 S_{31}(y) = y_2 S_{31}(x), \quad x_1 S_{31}(y) = -y_2 S_{32}(x); \]

the first two are implied by the last, which can be written as:

\[ \frac{S_{32}(x)}{x_1} = \frac{-S_{31}(y)}{y_2} = c, \]

or rather, equivalently,

\[ s = c (-x_2 e_1 + x_1 e_2) = c e_3 \times x, \quad (15) \]

with $c$ a constant. Hence, the cross-sectional traction $s$ must be orthogonal to the position vector $x$ all over $A$ up to the boundary, where on the other hand it has to be orthogonal to the normal $n$ to satisfy (2). Because of the assumed continuity of the stress field up to the boundary, $x$ must then be parallel to $n$ all over $\partial A$, which is possible if and only if $A$ is a circle or a circular annulus. Thus, as anticipated in the introduction, $(\chi_t - 1)$ quantifies the reduction in torsional stiffness due to the defect in polar symmetry of the cross section and the accompanying deviation from the form (3.2) of the cross-sectional traction.

4 Final remarks

Remark 1 One can define the extension and bending stiffnesses $s_e$ and $s_b$ and the relative factors $\chi_e$ and $\chi_b$ of a rod by posing, respectively,

\[ \frac{1}{2} s_e := \frac{1}{2E} \int_A S_{33}^2 \, dA, \quad \chi_e := \frac{A \int_A S_{33}^2 \, dA}{N^2}; \quad (16) \]

for

\[ N = \int_A S_{33} \, dA, \]

and

\[ \frac{1}{2} s_b := \frac{1}{2E} \int_A S_{33}^2 \, dA, \quad \chi_b := \frac{J \int_A S_{43}^2 \, dA}{M^2}; \quad (17) \]

\[ ^4 \text{A reviewer of our paper remarked that the inequality } M_t^2 \leq J_0 \int_A (S_{31}^2 + S_{32}^2) \, dA \text{ can be proved by a straightforward application of the Cauchy-Bunyakovsky-Schwarz inequality.} \]
for
\[ M = \int_A x_2 S_{33} dA, \quad J = \int_A x_2^2 dA. \]

To prove that
\[ \chi_c \geq 1 \quad \text{and} \quad \chi_b \geq 1 \quad \tag{18} \]
is left as an exercise for the interested reader. We note that equality is realized in relations (1) whatever the shape of the cross section \( A \) if and only if, respectively, the field \( S_{33} \) is constant-valued or linear in \( x_2 \) over \( A \), just as it happens to be in Saint-Venant’s cases of normal force and pure bending.

Remark 2 The recipe for parameter identification embodied in our definitions (2), (2), (1), and (1) for rod stiffnesses is applicable as such to any one-dimensional counterpart of a three-dimensional rod-like material body, no matter its axis were straight, its cross section constant or its mechanical response spatially uniform, as is the case for Saint-Venant’s prismatic cylinder; while linearity in the elastic response is crucial, isotropy is not: all those definitions make sense also for transverse isotropy with respect to the axial direction.

References

[1] J.M. Gere and S.P. Timoshenko, Mechanics of Materials, 3d Edition. Chapman & Hall (1991).

[2] P. Podio–Guidugli, Lezioni di Scienza delle Costruzioni, Seconda Edizione. Aracne (2009).

[3] G. Romano, Scienza delle Costruzioni, Tomo II: Elasticità e resistenza dei materiali. Hevelius Edizioni (2002).