TOROIDAL AND BOUNDARY-REDUCING DEHN FILLINGS

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Abstract. Let $M$ be a simple 3-manifold with a toral boundary component $\partial_0 M$. If Dehn filling $M$ along $\partial_0 M$ one way produces a toroidal manifold and Dehn filling $M$ along $\partial_0 M$ another way produces a boundary-reducible manifold, then we show that the absolute value of the intersection number on $\partial_0 M$ of the two filling slopes is at most two. In the special case that the boundary-reducing filling is actually a solid torus and the intersection number between the filling slopes is two, more is said to describe the toroidal filling.

1. Introduction

Following [W2], let us call a compact, orientable 3-manifold simple if it contains no essential sphere, disk, torus or annulus. Let $M$ be such a manifold, and let $\partial_0 M$ be a torus component of $\partial M$. If $\gamma$ is a slope on $\partial_0 M$ (the isotopy class of an essential unoriented circle), then as usual $M(\gamma)$ will denote the manifold obtained by $\gamma$-Dehn filling on $M$. Thus $M(\gamma) = M \cup V_\gamma$, where $V_\gamma$ is a solid torus, glued to $M$ by a homeomorphism from $\partial_0 M$ to $\partial V_\gamma$ taking $\gamma$ to the boundary of a meridian disk of $V_\gamma$.

If $\gamma, \delta$ are two slopes on $\partial_0 M$ such that $M(\gamma)$ and $M(\delta)$ are not simple, then there are several results giving upper bounds for $\Delta(\gamma, \delta)$ (the minimal geometric intersection number of $\gamma$ and $\delta$) for the various possible pairs of essential surfaces that arise, which in many cases are best possible (see [W2] for more details). In the present paper we dispose of one of the remaining cases, namely, that in which the surfaces in question are a torus and a disk.

Theorem 1.1. Let $M$ be a simple 3-manifold such that $M(\gamma)$ is toroidal and $M(\delta)$ is boundary-reducible. Then $\Delta(\gamma, \delta) \leq 2$.

Examples showing that this bound is best possible are given in [HM1] and [MM].

It was previously known that $\Delta(\gamma, \delta) \leq 3$, by [W1]. We shall therefore assume from now on that $\Delta(\gamma, \delta) = 3$, and eventually obtain a contradiction. We shall also assume that $\partial M$ has exactly two components $\partial_0 M$ and $\partial_1 M$, each a torus, and that $M$ is a $\mathbb{Q}$-homology cobordism between them, since otherwise $\Delta(\gamma, \delta) \leq 1$ by [W2, Theorem 4.1].

In [GLu3] it was shown that if $M(\gamma)$ is toroidal and $M(\delta) \cong S^3$ then $\Delta(\gamma, \delta) \leq 2$, and much of the proof of Theorem 1.1 consists of carrying over the arguments there to the

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present context. (In [GLu3] we analyzed the intersection of the punctured surfaces in \( M \) coming from an essential torus in \( M(\gamma) \) and a Heegaard sphere in \( M(\delta) \); here the Heegaard sphere is replaced by an essential disk.) This quickly leads to a proof of Theorem 1.1 when \( t \), the number of points of intersection of the essential torus in \( M(\gamma) \) with the core of \( V_\gamma \), is at least 4. This is completed in Section 4. Sections 2, 3 and 4 parallel the corresponding Sections of [GLu3]; (in Section 2 it is also pointed out that Sections 6 and 7 of [GLu3] carry over without change). The main divergence in the proofs is in the case \( t = 2 \). In [GLu3] this case was handled by showing that the associated knot \( K \) in \( S^3 \) was strongly invertible, and then appealing to a result of Eudave-Muñoz [E-M]. Since no analog of this result is available in our present setting, we have to argue this case directly. This is done in Section 5.

In Section 6 we specialize to the case that the boundary-reducible filling of \( M \) is a solid torus. Infinitely many examples of simple 3-manifolds \( M \) and slopes \( \gamma, \delta \) with \( \Delta(\gamma, \delta) = 2 \), \( M(\gamma) \) toroidal, and \( M(\delta) \) a solid torus, are given in [MM]. In these examples, there is a twice-punctured essential torus in \( M \) whose boundary slope is \( \gamma \). Section 6 is devoted to showing this always happens:

**Theorem 6.1.** Let \( M \) be a simple 3-manifold such that \( M(\gamma) \) is toroidal, \( M(\delta) \) is a solid torus, and \( \Delta(\gamma, \delta) = 2 \). Let \( K_\gamma \) be the core of the attached solid torus in \( M(\gamma) \) and \( \hat{T} \) be an essential torus in \( M(\gamma) \) that intersects \( K_\gamma \) minimally. Then \( |K_\gamma \cap \hat{T}| = 2 \).

We thank the referee for his helpful comments, and in particular for suggesting a simpler proof of Lemma 5.9.

### 2. The graphs of intersection

Let \( K_\gamma(K_\delta) \) be the core of \( V_\gamma \) (resp. \( V_\delta \)). Let \( \hat{T} \) be an essential torus in \( M(\gamma) \). We assume that \( \hat{T} \) meets \( K_\gamma \) transversely, and that \( t = |\hat{T} \cap K_\gamma| \) is minimal (over all essential tori \( \hat{T} \) in \( M(\gamma) \)). We may also assume that \( \hat{T} \cap V_\gamma \) consists of meridian disks, so \( T = \hat{T} \cap M \) is a punctured torus, with \( t \) boundary components, each having slope \( \gamma \) on \( \partial_0 M \).

Similarly, let \( \hat{Q} \) be an essential disk in \( M(\delta) \), meeting \( K_\delta \) transversely, with \( q = |\hat{Q} \cap K_\delta| \) minimal over all essential disks in \( M(\delta) \). Then \( Q = \hat{Q} \cap M \) is a punctured disk, with one boundary component, \( \partial \hat{Q} \), on \( \partial_1 M \), and \( q \) boundary components on \( \partial_0 M \), each with slope \( \delta \).

By standard arguments we may assume that:

(i) \( Q \) meets \( T \) transversely, in properly embedded arcs and circles;

(ii) each component of \( \partial Q \cap \partial_0 M \) meets each component of \( \partial T \) in \( \Delta = \Delta(\gamma, \delta) \) points;

(iii) no arc component of \( Q \cap T \) is boundary parallel in either \( Q \) or \( T \);

(iv) no circle component of \( Q \cap T \) bounds a disk in either \( Q \) or \( T \).

Then, as described in [GLu3, Section 2], the arc components of \( Q \cap T \) define graphs \( G_Q \) in \( \hat{Q} \) and \( G_T \) in \( \hat{T} \).

The definitions and terminology of [GLu1, Section 2] and [GLu3, Section 2], in particular the notion of a \( q \)-type, carry over to our present context. We assume familiarity with this terminology.
Thus we may designate the corners of faces of $G_T$ as either $++$, $--$, or $+-$, according to the signs of the corresponding pair of labels. However, we have the following lemma.

**Lemma 2.1.** Let $D$ be a set of disk faces of $G_T$ representing all $q$-types. Then there exists $D' \subset D$ such that $D'$ represents all $q$-types and all the corners appearing in faces in $D'$ are $+-$.

**Proof.** Let $C_0$ be the set of $++$ and $--$ corners, and $C_1$ the set of $+-$ corners, that appear in faces belonging to $D$. Let $\tau_0$ be the $C_0$-type defined by $\tau_0|(+ + \text{ corner}) = +$, $\tau_0|(-- \text{ corner}) = -$.

Now let $\tau_1$ be any $C_1$-type, and let $\tau$ be the $(C_0 \cup C_1)$-type $(\tau_0, \tau_1)$. By hypothesis, there exists a face $D \in D$ which represents $\tau$. We may assume, by definition of $\tau_0$, that the character (see [GLu1, p.386]) of each edge endpoint at a corner of $D$ belonging to $C_0$ is $+$. Since edges join points of opposite character, and since the edge endpoints at a corner in $C_1$ have distinct characters, it follows that no corner of $D$ can belong to $C_0$. We thus obtain our desired subcollection $D' \subset D$ such that $D'$ represents all $C_1$-types, and hence all $q$-types. \(\square\)

**Theorem 2.2.** $G_T$ does not represent all $q$-types.

**Proof.** Let $D$ be a set of disk faces of $G_T$ representing all $q$-types. By Lemma 2.1, we can assume that all corners appearing in faces in $D$ are $+-$, i.e., correspond to points of intersection of $K_\delta$ with $\hat{Q}$ of opposite sign. By [GLu1, Lemma 3.1], there exists $D_0 \subset D$ representing all $q$-types such that each face in $D_0$ is locally on the same side of $\hat{Q}$. By [GLu2, Lemma 4.4], there exists $D_1 \subset D_0$ such that $\{[\partial D]: D \in D_1\}$ is a basis for $\mathbb{R}^{c(D_1)}$. Here $c(D_1)$ is the number of corners appearing in faces belonging to $D_1$, and $[\partial D]$ is the element of $\mathbb{Z}^{c(D_1)} \subset \mathbb{R}^{c(D_1)}$ (defined up to sign) obtained in the obvious way, by taking the algebraic sum of the corners in $D$. Tubing $\hat{Q}$ along the annuli in $\partial V_\delta$ corresponding to the corners appearing in faces belonging to $D_1$ and surgering by the disks in $D_1$ then gives a disk $\hat{Q}' \subset M(\delta)$ with $\partial \hat{Q}' = \partial \hat{Q}$, and $|\hat{Q}' \cap K_\delta| < |\hat{Q} \cap K_\delta| = q$, contradicting the minimality of $q$. \(\square\)

The definition of a web $\Lambda$ in $G_Q$ is exactly as in [GLu3, p.601]. If $U$ is the component of $\hat{Q} - \text{nhd}(\Lambda)$ containing $\partial \hat{Q}$ then we say $D_\Lambda = \hat{Q} - U$ is the disk bounded by $\Lambda$. A great web is a web $\Lambda$ such that $\Lambda$ contains all the edges of $G_Q$ that lie in $D_\Lambda$.

Exactly as in [GLu3, proof of Theorem 2.5], Theorem 2.2 implies

**Theorem 2.3.** $G_Q$ contains a great web.

Since $\Delta = 3$, the arguments in [GLu3, Sections 6 and 7] apply here without change to show

**Theorem 2.4.** $M(\gamma)$ does not contain a Klein bottle.

Since $M(\delta)$ is boundary-reducible and $\gamma \neq \delta$, $M(\gamma)$ is irreducible by [S].
Finally we note that since $M$ is a $\mathbb{Q}$-homology cobordism between $\partial_0 M$ and $\partial_1 M$, $M(\gamma)$ is a $\mathbb{Q}$-homology $S^1 \times D^2$. Hence $\hat{T}$ separates $M(\gamma)$, into $X$ and $X'$, say. It follows that the faces of $G_Q$ may be shaded alternately black and white, with the black faces lying in $X$ and the white faces lying in $X'$.

3. Scharlemann cycles and extended Scharlemann cycles

In this section we follow [GLu3, Section 3] and show that the relevant statements there hold in our present setting. For ease of reference, here and in Section 4 we will give lemmas etc. the same numbers as the corresponding statements in [GLu3].

**Lemma 3.1.** The edges of a Scharlemann cycle in $G_Q$ cannot lie in a disk in $\hat{T}$.

**Proof.** Otherwise, as in [GLu3, proof of Lemma 3.1], $M(\gamma)$ would have a lens space summand, and hence be reducible. □

**Theorem 3.2.** If $t \geq 4$ then $G_Q$ does not contain an extended Scharlemann cycle.

**Proof.** The proof of [GLu3, Claim 3.3] goes through verbatim in our present context.

The proof of [GLu3, Claim 3.4] remains valid here once we note in addition that $\partial M_4$ cannot be parallel to $\partial M(\gamma)$ ($= \partial_1 M$). For then $\hat{T}$ would lie in a collar of $\partial M(\gamma)$ in $M(\gamma)$, contradicting the essentiality of $\hat{T}$.

We now follow the remainder of the proof of [GLu3, Theorem 3.2] as closely as possible.

Let $M_2 = \overline{X - M_1}$. Then $X = M_1 \cup_B M_2$. Similarly, since $M(\gamma)$ is a $\mathbb{Q}$-homology $S^1 \times D^2$, $A$ separates $X'$, into $M_3$ and $M_4$, say.

Let $T_2, T_3, T_4$ be the tori defined as $T_2, \hat{T}_3, \hat{T}_4$ in [GLu3, pp.609–610]. These satisfy $|T_i \cap K_\gamma| < |\hat{T} \cap K_\gamma|$, $i = 2, 3, 4$, and hence, by the minimality of $t$, $T_i$ is either compressible, and therefore (since $M(\gamma)$ is irreducible) bounds a solid torus in $M(\gamma)$, or is peripheral, i.e., parallel to $\partial M(\gamma)$, $i = 2, 3, 4$. It follows that exactly one of $M_2, M_3, M_4$ is a collar of $\partial M(\gamma)$ in $M(\gamma)$, and the other two are solid tori.

Now let $T_0$ be the torus $A \cup (C_1 - A_1) \cup B$. Then $T_0$ separates $M(\gamma)$ into $Y = M_1 \cup_A M_3$ and $Y' = M_2 \cup_{C_2} M_4$. Also $|T_0 \cap K_\gamma| < t$. Hence again $T_0$ either bounds a solid torus or is peripheral.

If $T_0$ is peripheral, then $M(\gamma) \cong Y$ or $Y'$, the union of two solid tori along an annulus essential in the boundary of each. Since $M(\gamma)$ is irreducible, this implies that $M(\gamma)$ is a Seifert fiber space over the disk with at most two singular fibers, contradicting the fact that it contains an essential torus.

If $Y$, say, is a solid torus, then $M_2$ or $M_4$ (say $M_2$) is a collar of $\partial M(\gamma)$, and hence $M(\gamma) \cong Y \cup M_4$ is a union of two solid tori, which is a contradiction as before. We get a similar contradiction if $Y'$ is a solid torus. □

We remark that we will only use Theorem 3.2 for extended Scharlemann cycles of length 2; see Section 4. However, we have given the proof of the general statement, for possible future reference.

Since $M(\gamma)$ does not contain a Klein bottle by Theorem 2.4, the proof of [GLu3, Lemma 3.10] applies verbatim to give the same statement here:
Lemma 3.10. Let $\sigma_1$ and $\sigma_2$ be Scharlemann cycles of length 2 in $G_Q$ on distinct label-pairs. Then the loops on $\hat{T}$ formed by the edges of $\sigma_1$ and $\sigma_2$ respectively are not isotopic on $\hat{T}$.

Similarly, the proof of [GLu3, Theorem 3.11] gives

Theorem 3.11. At most three labels can be labels of Scharlemann cycles of length 2 in $G_Q$.

4. The case $t \geq 4$

We obtain a contradiction in this case exactly as in [GLu3, Section 4].

Let $\Lambda$ be a great web in $G_Q$, as guaranteed by Theorem 2.3, and $D_\Lambda$ be the disk that it bounds. For every label $x$ of $G_Q$ let $\Lambda_x$ be the graph in $D_\Lambda$ consisting of the vertices of $\Lambda$ along with all $x$-edges in $\Lambda$.

Let $V$ be the number of vertices of $\Lambda$.

Lemma 4.2. Let $x$ be a label of $G_Q$. If $\Lambda_x$ has at least $3V - 4$ edges, then $\Lambda_x$ contains a bigon.

Proof. The proof of [GLu3, Lemma 4.2] applies here without change.

We take this opportunity to thank Chuichiro Hayashi and Kimihiko Motegi for pointing out that the case where the face $f$ is a monogon was not explicitly addressed in [GLu3]. However, in that case $\Lambda_x$ could have at most one ghost $x$-label, since otherwise $\Lambda$ would have more than $t$ ghost labels. Hence $E \geq 3V - 1$. Also, if $\Lambda_x$ does not contain a bigon then $1 + 3(F - 1) \leq 2E$. Therefore $2 = V - E + F \leq (E + 1)/3 - E + 2(E + 1)/3 = 1$, a contradiction. □

Lemma 4.2 and Theorems 3.2 and 3.11 then lead to a contradiction exactly as in [GLu3, p.615].

5. The case $t = 2$

As in the previous section, let $\Lambda$ be a great web in $G_Q$.

Lemma 5.1. $\Lambda$ contains a bigon.

Proof. Regard $\Lambda$ as a graph in the disk, with $V$ vertices, $E$ edges, and $F$ faces.

Since $\Lambda$ has at most two ghost labels, $2E \geq 6V - 2$, so $E \geq 3V - 1$. Suppose $\Lambda$ contains no bigon; then $3F < 2E$. Therefore $1 = V - E + F \leq (E + 1)/3 - E + 2E/3 = \frac{1}{3}$, a contradiction. □

By the parity rule, each edge of $\Lambda$ must connect different vertices of $G_T$ when considered as edges lying in $G_T$. Recall that there are four edge classes in $G_T$, i.e., isotopy classes of non-loop edges of $G_T$ in $\hat{T}$ rel{vertices of $G_T$}, which we call $1, \alpha, \beta, \alpha\beta$, as illustrated in [GLu3, Figure 7.1]. The ordering of these classes around vertices 1 and 2 of $G_T$ is indicated in Figure 5.1. We may label an edge $e$ of $G_Q$ by the class of the corresponding edge of $G_T$; we refer to this label as the edge class label of $e$. 
Lemma 5.2. Any two black (white) bigons in $\Lambda$ have the same pair of edge class labels.

Proof. Let $f, f'$ be two (say) black bigons in $\Lambda$, with vertices $x, x'$ and $y, y'$, and edge class labels $\lambda, \mu$ and $\lambda', \mu'$, respectively. See Figure 5.2. Note that $\lambda \neq \mu$ and $\lambda' \neq \mu'$ by Lemma 3.1.

Figure 5.2

Let $H_{12}$ be the 1-handle $V_\gamma \cap X$. The anticlockwise ordering of the labels $x, x', y, y'$ around vertex 1 of $G_T$ must agree with their clockwise ordering around vertex 2, since the corners of $f$ and $f'$ give four disjoint arcs on the annulus $\partial H_{12} - \tilde{T}$ joining corresponding labels. However, if $\lambda = \lambda'$ but $\mu \neq \mu'$ then it is easy to see that these orderings do not agree. For example, the case $\lambda = \lambda' = 1$, $\mu = \alpha$, $\mu' = \beta$ is illustrated in Figure 5.3. Similarly, the pairs $\{1, \alpha \beta\}, \{\alpha, \beta\}$ give incompatible orderings of the labels.

Figure 5.3

Thus the only possible distinct pairs of edge class labels for $f$ and $f'$ are $\{1, \alpha\}$ and $\{\alpha \beta, \beta\}$ (or $\{1, \beta\}$ and $\{\alpha \beta, \alpha\}$). Now by shrinking $H_{12}$ to its core, $H_{12} \cup f$ becomes a M"obius band $B$ in $X$ such that $\partial B$ is the loop on $\tilde{T}$ formed by the edges of $f$. Similarly $f'$ gives rise to a M"obius band $B'$. If the edge class labels for $f$ and $f'$ are as above, then $\partial B$ and $\partial B'$ may be isotoped on $\tilde{T}$ to be disjoint. This produces a Klein bottle in $M(\gamma)$, contradicting Theorem 2.4. $\square$

Lemma 5.3. Let $e, e'$ be edges of $\Lambda$ incident to a vertex $v$ of $\Lambda$, with the same label at $v$. Then $e$ and $e'$ have distinct edge class labels.

Proof. If not, then $e$ and $e'$ would be parallel in $G_T$, and hence would cobound a family of
\( q + 1 \) parallel edges of \( G_T \). The argument of [GLi, p.130, Case (2)] now constructs a cable space in \( M \), contradicting the assumption that \( M \) is simple. □

If \( f \) is a white (black) face of \( \Lambda \) such that each edge of \( f \) belongs to a black (resp. white) bigon, then we say that \( f \) is surrounded by bigons.

**Lemma 5.4.** \( \Lambda \) does not contain a face surrounded by bigons.

**Proof.** Suppose that \( \Lambda \) contains, say, a white face \( f \) surrounded by black bigons.

By Lemma 5.2 (and Lemma 3.1) all the black bigons have the same pair of (distinct) edge class labels \( \lambda, \mu \).

Let \( v \) be a vertex of \( f \). By Lemma 5.3 the edges of the two black bigons incident to \( v \) having label 1 (say) at \( v \) have distinct edge class labels. Therefore the two edges of \( f \) incident to \( v \) have the same edge class label (see Figure 5.4). Hence all the edges of \( f \) have the same edge class label, contradicting Lemma 3.1. □

![Figure 5.4](image)

**Lemma 5.5.** Not all black faces of \( \Lambda \) are bigons. Similarly, not all white faces of \( \Lambda \) are bigons.

**Proof.** Suppose, for example, that all black faces of \( \Lambda \) are bigons.

Define a graph \( \Sigma \) in the disk \( \hat{Q} \) as follows. The vertices of \( \Sigma \) are the vertices of \( \Lambda \), and the edges of \( \Sigma \) are in one-one correspondence with the black bigons of \( \Lambda \), each edge joining the vertices at the corners of the corresponding bigon and lying in its interior. Then each vertex of \( \Sigma \) has valency at least 2, except that if \( \Lambda \) has a vertex at which there are two ghost labels, then that vertex may have valency 1 in \( \Sigma \). Hence if \( V \) and \( E \) are the number of vertices and edges of \( \Sigma \) respectively, then \( 2E \geq 2V - 1 \), and so \( 2E \geq 2V \). Therefore

\[
1 \leq V - E + \sum \chi(f) \text{ (the sum being over all faces } f \text{ of } \Sigma) \\
\leq \sum \chi(f)
\]

implying that \( \Sigma \) has a disk face. But such a face corresponds to a white face of \( \Lambda \) surrounded by black bigons, contradicting Lemma 5.4. □

We shall say that two faces \( f_1, f_2 \) of \( \Lambda \) of the same color are isomorphic if the cyclic sequences of edge class labels obtained by reading around the boundaries of \( f_1 \) and \( f_2 \) in the same direction, are equal.

**Lemma 5.6.** Either all black faces of \( \Lambda \) are isomorphic or all white faces of \( \Lambda \) are isomorphic.

**Proof.** Recall that \( \hat{T} \) separates \( M(\gamma) \), into \( X \) and \( X' \). We suppose without loss of generality that \( \partial M(\gamma) \subset X \).
Recall also that $H_{12}$ is the 1-handle $V_i \cap X$. Let $f$ be any black face of $\Lambda$, and let $N = \text{nhd}(\hat{T} \cup H_{12} \cup f) \subset X$. Then (since all the vertices of $\Lambda$ have the same sign) $\partial N = \hat{T} \cup T_1$, where $T_1$ is a torus. Since $T_1 \cap K_\gamma = \emptyset$, and since $M$ is irreducible and atoroidal, $T_1$ is parallel to $\partial_1 M = \partial M(\gamma)$. Hence $W = X - H_{12}$ is a compression body, with $\partial W$ the genus 2 surface obtained by adding the tube $\partial H_{12} - \hat{T}$ to $T$, and $\partial - W = \partial M(\gamma)$. It follows that $f$ is the unique non-separating disk in $W$, up to isotopy.

Let $f'$ be any other black face of $\Lambda$. Then $\partial f$ and $\partial f'$ are isotopic in $\partial W$, and hence (freely) homotopic in $\hat{T} \cup H_{12}$. Now $\pi_1(\hat{T} \cup H_{12}) \cong \pi_1(\hat{T}) \ast \mathbb{Z}$, where, taking as base-“point” a disk neighborhood in $\hat{T}$ of an edge in $G_T$ in edge class 1, $\pi_1(\hat{T}) \cong \mathbb{Z} \times \mathbb{Z}$ has basis $\{\alpha, \beta\}$, represented by edges in the correspondingly named edge classes, oriented from vertex 2 to vertex 1, and $\mathbb{Z}$ is generated by $x$, say, represented by an arc in $H_{12}$ going from vertex 1 to vertex 2. Then, if the sequence of edge class labels around $\partial f$ (in the appropriate direction) is $(\gamma_1, \ldots, \gamma_n)$, $\partial f$ represents $\gamma_1 x \gamma_2 x \ldots \gamma_n x \in \pi_1(\hat{T}) \ast \mathbb{Z}$, and similarly for $\partial f'$. Since $\partial f$ and $\partial f'$ are homotopic in $\hat{T} \cup H_{12}$, we conclude that the corresponding cyclic sequences $(\gamma_1, \ldots, \gamma_n)$ and $(\gamma'_1, \ldots, \gamma'_n)$ are equal, i.e., $f$ and $f'$ are isomorphic. Thus all black faces of $\Lambda$ are isomorphic, and the lemma is proved. □

By Lemma 5.1, $\Lambda$ contains a bigon. We may therefore assume from now on that $\Lambda$ contains a black bigon.

**Lemma 5.7.** All white faces of $\Lambda$ are isomorphic.

**Proof.** If not, then by Lemma 5.6 all black faces of $\Lambda$ are isomorphic, and therefore bigons. But this contradicts Lemma 5.5. □

**Lemma 5.8.** $\Lambda$ does not contain a white bigon.

**Proof.** This follows from Lemmas 5.7 and 5.5. □

Recall that $D_\Lambda$ is the disk bounded by the great web $\Lambda$. Travelling around $\partial D_\Lambda$ in some direction defines in the obvious way a cyclic sequence of edges of $\Lambda$, $(e_1, \ldots, e_n)$. (Note that the same edge may appear twice, with opposite orientations.) If the subgraph of $\Lambda$ consisting of these edges has no cut vertex, define $\Lambda_0 = \Lambda$. (Recall that a cut vertex of a connected graph is one whose removal disconnects the graph.) Otherwise, let $v_0$ be an outermost cut vertex of this subgraph. This vertex cuts off a subdisk of $D_\Lambda$ which is the disk $D_{\Lambda_0}$ bounded by a subgraph $\Lambda_0$ of $\Lambda$ (see Figure 5.5). We shall call $\Lambda_0$ an extremal subgraph of $\Lambda$. If $\Lambda_0 \neq \Lambda$ we call $v_0$ the attaching vertex of $\Lambda_0$. 
An interior vertex of $\Lambda$ is a vertex of $\Lambda$ that is not an endpoint of any of the edges $e_1, \ldots, e_n$. All six corners at such a vertex belong to faces of $\Lambda$.

**Lemma 5.9.** $\Lambda$ contains an interior vertex.

*Proof.* The following proof was suggested by the referee.

Since $\Lambda$ has at most two ghost labels, $\Lambda$ has an extremal subgraph $\Lambda_0$ such that either $\Lambda_0 = \Lambda$ or $\Lambda_0$ contains at most one ghost label of $\Lambda$ at a vertex other than the attaching vertex $v_0$ of $\Lambda_0$. If $\Lambda = \Lambda_0$, we can also choose a vertex $v_0$ of $\Lambda$ such that $\Lambda$ has at most one ghost label at a vertex other than $v_0$. We shall show that $\Lambda_0$ contains an interior vertex.

Suppose for contradiction that $\Lambda_0$ has no interior vertices. Let $m$ be the number of vertices of $\Lambda_0$. Note that $m > 1$, since $G_\hat{Q}$ has no trivial loops. Let $\overline{\Lambda}_0$ be the reduced graph of $\Lambda_0$, obtained by amalgamating each family of parallel edges to a single edge. If $m = 2$ or 3, then each vertex of $\overline{\Lambda}_0$ has valency 1 or 2 respectively. If $m \geq 4$, the interior edges of $\overline{\Lambda}_0$ decompose $D_{\Lambda_0}$ into subdisks, and by considering outermost such subdisks we see that $\overline{\Lambda}_0$ has at least two vertices of valency 2. Hence in all cases $\overline{\Lambda}_0$ has a vertex $v \neq v_0$ of valency 1 or 2. Since there is at most one ghost label at $v$, $\Lambda_0$ has a family of at least three parallel edges at $v$, and hence contains a white bigon, contradicting Lemma 5.8. \hfill $\Box$

**Lemma 5.10.** Each white face of $\Lambda$ has at least three distinct edge class labels.

*Proof.* By Lemma 5.9, $\Lambda$ contains an interior vertex $v$. Then all three white corners at $v$ belong to faces of $\Lambda$. By Lemma 5.3, the three edges with label (say) 1 at $v$ have distinct edge class labels. The result now follows from Lemma 5.7. \hfill $\Box$

**Lemma 5.11.** Each white face of $\Lambda$ has length at least 4.

*Proof.* The argument in the first paragraph of [GLu3, proof of Lemma 3.7] shows that a face of $\Lambda$ of length 3 has only two distinct edge class labels, contradicting Lemma 5.10. \hfill $\Box$

Recall that $\Lambda$ has either 0 or 2 ghost labels. In the former case, let $\Lambda^+ = \Lambda$. In the latter case, let $\Lambda^+ \subset \hat{Q}$ be obtained from $\Lambda$ by adjoining two disjoint arcs running from the ghost labels to $\partial \hat{Q}$. The complementary regions of $\Lambda^+$ in $\hat{Q}$ are the faces of $\Lambda$ together with either one or two outside regions, i.e., regions that meet $\partial \hat{Q}$. The black/white shading of the faces of $\Lambda$ extends to a shading of the complementary regions of $\Lambda^+$.

Define a graph $\Gamma$ in the disk $\hat{Q}$ as follows. The vertices of $\Gamma$ are the (fat) vertices of $\Lambda$, together with dual vertices, one in the interior of each black complementary region of $\Lambda^+$.
that is either a face of \( \Lambda \) of length at least 3, or an outside region. The edges of \( \Gamma \) consist of edges joining each dual vertex to the fat vertices in the boundary of the corresponding region, together with an edge in the interior of each black bigon in \( \Lambda \), joining the (fat) vertices at the ends of the bigon. (If \( \Lambda^+ = \Lambda \) and the outside region is black, then there is a choice involved in defining the edges incident to the dual vertex corresponding to that region.)

**Lemma 5.12.** \( \Gamma \) has a face of length at most 5.

**Proof.** The valency of each fat vertex of \( \Gamma \) is 3. The valency of each dual vertex corresponding to a face of \( \Lambda \) is at least 3. There is at most one dual vertex corresponding to an outside region, and that vertex has valency at least 1. Hence, if \( \Gamma \) has \( V \) vertices and \( E \) edges then \( 2E \geq 3V - 2 \). If \( \Gamma \) has \( F \) faces, and each face has length at least 6, then \( 6F < 2E \). Therefore \( 1 = V - E + F < 2(\frac{E}{2} + 1)/3 - E + E/3 = \frac{2}{3} \), a contradiction. \( \square \)

Let \( g \) be a face of \( \Gamma \) as in Lemma 5.12. Then \( g \) contains a unique white face \( f \) of \( \Lambda \). If \( f \) has \( b \) edges which are edges belonging to black bigons of \( \Lambda \), and \( c \) other edges, then the length of \( g \) is \( b + 2c \leq 5 \). On the other hand \( b + c \geq 4 \) by Lemma 5.11. Also, \( c > 0 \) by Lemma 5.4; hence \( c = 1 \). Thus every edge of \( f \) except one belongs to a black bigon, and the argument in the proof of Lemma 5.4 shows that every such edge has the same edge class label. Therefore \( f \) has at most two edge class labels, contradicting Lemma 5.10.

We have thus shown that the case \( t = 2 \) is impossible.

This completes the proof of Theorem 1.1.

### 6. Knots in solid tori

In [GLu3] and [GLu4] it is shown that if \( M \) is simple, \( M(\gamma) \) is toroidal, \( M(\delta) \) is \( S^3 \), and \( \Delta(\gamma, \delta) = 2 \), then \( t = 2 \). [MM] gives examples of simple manifolds, \( M \), for which \( M(\gamma) \) is toroidal, \( M(\delta) = S^1 \times D^2 \), and \( \Delta(\gamma, \delta) = 2 \). In these examples \( t = 2 \). It is natural then to ask if this is always the case.

**Theorem 6.1.** Let \( M \) be simple 3-manifold such that \( M(\gamma) \) is toroidal, \( M(\delta) \) is \( S^1 \times D^2 \), and \( \Delta(\gamma, \delta) = 2 \). Let \( K_\gamma \) be the core of the attached solid torus in \( M(\gamma) \) and \( \hat{T} \) be an essential torus in \( M(\gamma) \) that intersects \( K_\gamma \) minimally. Then \( |K_\gamma \cap \hat{T}| = 2 \).

The remainder of this section is devoted to the proof of Theorem 6.1. Let \( \lambda = S^1 \times * \), \( \mu = * \times \partial D^2 \) in \( \partial_1 M = \partial M(\delta) \) be a longitude and meridian respectively of \( S^1 \times D^2 \).

For any integer \( n \), let \( M_n \) be the \((\lambda + n\mu)\)-filling of \( M \).

**Lemma 6.2.** \( M_n \) is simple for \( n \) sufficiently large.

**Proof.** By [Th], \( M \) is hyperbolic and all but finitely many fillings on a single component of \( \partial M \) are hyperbolic. See also Theorem 1.3 of [G]. \( \square \)

Let \( N = M(\gamma) \), and parametrize slopes (as rational numbers) on \( \partial N = \partial_1 M \) using \( \lambda, \mu \). Thus \( M_n(\gamma) = N(n) \).

Let \( \hat{T} \) be an essential torus in \( N \) that minimizes \( |K_\gamma \cap \hat{T}| \).
Lemma 6.3. If \( \hat{T} \) compresses in \( N(n_1) \) and \( N(n_2) \) then either \( |K_\gamma \cap \hat{T}| = 2 \) or \( |n_1 - n_2| \leq 2 \).

Proof. We assume for contradiction that every essential torus in \( N \) (which must separate since \( \Delta(\gamma, \delta) = 2 \) intersects \( K_\gamma \) at least four times and that \( \hat{T} \) compresses in \( N(n_1), N(n_2) \) where \( |n_1 - n_2| \geq 3 \). Applying [CGLS, Theorem 2.4.2] to the side of \( \hat{T} \) containing \( \partial_1 M \), this last condition guarantees an essential annulus \( A' \subset N \) such that one boundary component of \( A' \) is \( A' \cap \hat{T} \), the other is \( A' \cap \partial_1 M \), and \( A' \cap \partial_1 M \) is distance 1 from each of the integral slopes \( n_1 \) and \( n_2 \). Since \( |n_1 - n_2| \geq 3 \), \( A' \cap \partial_1 M \) must be a copy of \( \mu \). Surgering \( \hat{T} \) along \( A' \) gives a properly embedded annulus \( A'' \subset N \) whose boundary consists of two copies of \( \mu \). Since \( \hat{T} \) is essential in \( N \), so is \( A'' \).

Let \( A \) be an essential annulus in \( N \), with boundary consisting of two copies of \( \mu \), such that \( |K_\gamma \cap A| \) is minimal over all such annuli. Since \( M \) contains no essential annulus, \( K_\gamma \cap A \neq \emptyset \). Let \( D \) be a meridian disk of \( M(\delta) = S^1 \times D^2 \), chosen so as to minimize \( |K_\delta \cap D| \). We may isotope \( D \) so that \( \partial D \cap \partial A = \emptyset \). Then \( T' = A \cap M \) and \( Q = D \cap M \) are essential planar surfaces in \( M \), whose “outer” boundaries, \( \partial A \) and \( \partial D \) lie on \( \partial_1 M \) and are disjoint, and whose “inner” boundaries \( \partial T' \cap \partial_0 M, \partial Q \cap \partial_0 M \) consist of, say, \( t' \) copies of \( \gamma \) and \( q \) copies of \( \delta \) respectively.

Recall that \( \Delta(\gamma, \delta) = 2 \). As in Section 2 the arc components of \( T' \cap Q \) give rise to graphs \( G_{T'}, G_Q \) in \( A, D \) (resp.). By abstractly identifying the two boundary components of \( A \), we may regard \( G_{T'} \) as a graph in a torus, and we are now exactly in the combinatorial set-up of Section 2. In particular we may apply Theorem 2.3 to conclude that \( G_Q \) contains a great web, \( \Lambda \). (Note that the one face of \( G_{T'} \) in the torus which is not a face of \( G_{T'} \) in \( A \) is not a disk face, hence would not be involved in any collection of faces representing all types). Because \( \Lambda \) has at most \( t' \) ghost labels and \( \Delta(\gamma, \delta) = 2 \), there is a label \( x \) of \( G_Q \) and a vertex \( y_0 \) of \( \Lambda \) such that the following conditions are satisfied:

(i) for any vertex \( y \) of \( \Lambda \) other than \( y_0 \), there is an edge of \( \Lambda \) incident to \( y \) at each occurrence of the label \( x \) on \( y \), and

(ii) there is an edge of \( \Lambda \) incident to \( y_0 \) at some occurrence of the label \( x \) at \( y_0 \).

If we let \( \Sigma \) be the set of all \( x \)-edges of \( \Lambda \), then \( \Sigma \) is a great \( x \)-web in the sense of [GLu2, p.390]. The argument of Theorem 2.3 of [GLu2] now shows that \( \Lambda \) is separating and either (a) \( G_Q \) must contain Scharlemann cycles on distinct label pairs or (b) \( t' = 2 \). (This argument uses Lemma 2.2 of the same paper, which requires the fact that consecutive labels on \( G_Q \) represent vertices of \( G_{T'} \) of opposite sign. This assumption was shown not to be necessary in [HM2, Proposition 5.1]. However, one may guarantee this condition in the present context by showing that \( \Lambda \) is separating. This is done in the same way as in Theorem 2.3: show that \( \Lambda \) contains a Scharlemann cycle, then use the face of \( \Lambda \) bounded by this Scharlemann cycle to tube and compress \( A \) to get a new annulus \( A' \) such that \( |A' \cap K_\delta| < |A \cap K_\delta| \). If \( A \) were non-separating, \( A' \) would be also, contradicting the minimality of \( A \).)

Lemma 3.1 of [HM1] shows that conclusion (a) contradicts the minimality of \( |A \cap K_\gamma| \). Thus we must have that \( t' = 2 \). Let \( X_1 \) and \( X_2 \) be the components into which \( A \) separates \( N \). Then each \( \partial X_i \) is a torus meeting \( K_\gamma \) twice, and hence is compressible. Since \( N \) is
irreducible by [S], $X_i$ must be a solid torus. As $N = X_1 \cup A \cup X_2$, $N$ is Seifert-fibered over the disk with at most two exceptional fibers. But this contradicts the fact that $N$ contains an essential torus. This contradiction finishes the proof of Lemma 6.3.

**Remark.** In fact, the possibility that $N$ contains an essential torus intersecting $K_\gamma$ twice can be excluded from the conclusion of Lemma 6.3. For example one can argue as above to show the existence of $A$, then argue as in [HS] to arrive at a contradiction. In particular, Sections 4 and 5 of that paper along with Lemma 3.1 of [HM1] show that the great $x$-web $\Lambda$ that we see in the above argument cannot occur.

**Proof of Theorem 6.1.** Lemmas 6.2 and 6.3 show that if $|K_\gamma \cap \hat{T}| \neq 2$, then there is an integer $J$ such that for all $n > J$, $M_n$ is a simple manifold and $M_n(\gamma)$ contains an essential torus. On the other hand $M_n(\delta) = S^3$, so $M_n$ is the exterior of a simple knot in $S^3$. By Theorem 1.2 of [GLu3], (which is proved in [GLu3] and [GLu4]), each $M_n$ contains an essential, twice-punctured torus, $T_n$, where $\partial T_n$ is two copies of $\gamma$. If, for some $n$, $T_n$ can be isotoped to lie inside $M$, then we are done. So assume not. Then for each $n > J$, there is an incompressible, $\partial$-incompressible surface $T'_n$ in $M$ whose boundary is a non-empty collection of copies of $\lambda + n\mu$ on $\partial_1 M$ and two copies of $\gamma$ on $\partial_0 M$. But this contradicts [H].

In the above proof of Theorem 6.1, the direct appeal to Theorem 1.2 of [GLu3] was possible because of the assumption that $M(\delta)$ is a solid torus. Conceivably Theorem 6.1 is still true if it is only assumed that $M(\delta)$ is boundary-reducible. If so, a proof in this generality might be obtainable by suitably modifying the proof, in [GLu3] and [GLu4], of Theorem 1.2 of [GLu3].

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