Threshold Between Short and Long-range Potentials for Non-local Schrödinger Operators

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Abstract
We develop scattering theory for non-local Schrödinger operators defined by functions of the Laplacian that include its fractional power \((-\Delta)^\rho\) with \(0 < \rho \leq 1\). In particular, our function belongs to a wider class than the set of Bernstein functions. By showing the existence and non-existence of the wave operators, we clarify the threshold between the short and long-range decay conditions for perturbational potentials.

Keywords Bernstein function · Scattering theory · Wave operators

Mathematics Subject Classification (2010) 47B25 · 81Q10 · 81U05

1 Introduction
We consider the set of functions
\[
\tilde{\mathcal{B}} = \left\{ \Psi \in C^1(0, \infty) \mid \Psi(\sigma) \geq 0, \frac{d\Psi}{d\sigma}(\sigma) \geq 0, \sigma \in (0, \infty) \right\}.
\]
(1.1)

For \(\Psi \in \tilde{\mathcal{B}}\), the free Hamiltonian we consider in this paper is given by
\[
H_0^\Psi = \Psi(-\Delta) = \Psi\left(|D|^2\right),
\]
(1.2)
a self-adjoint operator acting on $L^2(\mathbb{R}^n)$, where $D = -i (\partial_{x_1}, \ldots, \partial_{x_n})$. Let $V$ be a real-valued function satisfying suitable conditions. We call operator $H_0^\Psi + V$ a non-local Schrödinger operator.

An element of the following central set of functions

$$B = \left\{ \Phi \in C^\infty (0, \infty) \mid \Phi (\sigma) \geq 0, (-1)^k \frac{d^k \Phi}{d\sigma^k} (\sigma) \leq 0, \sigma \in (0, \infty), k \in \mathbb{N} \right\}$$

(1.3)

is called a Bernstein function. Clearly, the relation

$$B \subsetneq \tilde{B}$$

(1.4)

holds. Therefore, we treat much more general operators than those defined as Bernstein functions.

Consider $\Phi \in B$ that satisfy $\lim_{\sigma \to +0} \Phi (\sigma) = 0$. One important property of $\Phi$ is that the semigroup generated by $\Phi \left( (|D|^2) + V \right)$ is expressible via a stochastic process

$$\left( e^{-t\Phi(|D|^2)+V} f \right) (x) = \mathbb{E}_x \left[ e^{-\int_0^t V(BT(s))ds} f \left( B_T(t) \right) \right]$$

(1.5)

where $\{B_t \}_{t \geq 0}$ is the $n$-dimensional Brownian motion starting at $x \in \mathbb{R}^n$ and $T$ is the subordinator associated with $\Phi$. The above expression is often called Feynman-Kac formula or path integral representation.

The Feynman–Kac formula enables us to analyze $\Phi \left( (|D|^2) + V \right)$ in terms of stochastic calculus. In particular, it is useful to investigate some properties of the eigenfunctions of $\Phi \left( (|D|^2) + V \right)$. In this direction, there already exist several results [3, 4, 11]. Comprehensive results related to the Feynman–Kac formula are summarized in [15].

With regard to quantum scattering theory for non-local Schrödinger operators, there are only a few results. For instance, [2] considered a function $\Phi \in \hat{B}$, where

$$\hat{B} = \left\{ \Phi \in C^\infty (0, \infty) \mid \frac{d\Phi}{d\sigma} (\sigma) \geq 0, \sigma \in (0, \infty), \lim_{\sigma \to \infty} \Phi (\sigma) = \infty \right\}.$$  

(1.6)

Giere [2] discusses the asymptotic completeness of the wave operators under a short-range perturbation $V$ by investigating the semigroup differences and proves the absence of the singular continuous spectrum of $\Phi \left( (|D|^2) + V \right)$. Of note, instances of fractional powers are specific examples of functions of the Laplacian. Kitada [13, 14] first constructed long-range scattering theory for $\left( -\Delta \right)^{\rho}$ with $1/2 \leq \rho \leq 1$. Recently, inverse scattering problems involving the short-range potentials were investigated in [7] for exponents $\rho$ satisfying $1/2 < \rho \leq 1$.

Under these situations, our aim in this paper is to develop scattering theory for non-local Schrödinger operators defined by functions belonging to the wider set $\tilde{B}$ with some additional assumptions. We first prove the existence of the wave operators in which the potential function $V$ has a short-range decay. Giere [2] also proved these existence for $\Phi \in \hat{B}$ by way of semigroups $e^{-it\Phi(|D|^2)}$ and $e^{-it\Phi(|D|^2)+V}$. However, our proof of the existence of the wave operators is simple and intuitive, and is obtained directly from the propagation estimate for the time-evolution of $e^{-itH_0^\Psi}$. We next clarify the threshold between the short and long-range decay exponents by
providing a concrete example of the potential functions for which the wave operators do not exist.

For quantum scattering theory, it is important to distinguish the threshold between the short and long-range conditions. Historically, [1] considered the non-existence of the standard wave operators for the Coulomb interaction by showing that the weak limits of the pair of propagators were equal to zero (see also [17]). Ishida [6, 8] applied this method to the fractional Laplacian and massive relativistic operator. From a different perspective, [16] invented an original approach to prove the non-existence of the wave operators for the Stark Hamiltonian. This approach was applied to various quantum systems (repulsive Hamiltonian [5], time-dependent harmonic oscillator [9, 10] and 1D quantum walk [18]). Our approach in this paper follows [16].

First of all, we provide the basic properties associated with the spectrum of \( H_0^\Psi \). The absolute continuity of \( \Phi (|D|^2) \) for \( \Phi \in \mathcal{H} \) was mentioned in [2] (Remark 2.2). We also state several other properties of \( H_0^\Psi = \Psi (|D|^2) \) for \( \Psi \in \mathcal{H} \) in the following proposition.

**Proposition 1.1** Assume that \( \Psi \in \mathcal{H} \).

1. The spectrum of \( H_0^\Psi \) coincides with

   \[
   \left[ \lim_{\sigma \to +0} \Psi(\sigma), \lim_{\sigma \to \infty} \Psi(\sigma) \right]
   \]

   if \( \lim_{\sigma \to \infty} \Psi(\sigma) < \infty \),

   \[
   \left[ \lim_{\sigma \to +0} \Psi(\sigma), \infty \right)
   \]

   if \( \lim_{\sigma \to \infty} \Psi(\sigma) = \infty \).

2. If the set

   \[
   A = \left\{ \sigma \in (0, \infty) \mid \frac{d \Psi}{d \sigma}(\sigma) = 0 \right\}
   \]

   is at most discrete, the pure point spectrum of \( H_0^\Psi \) is empty. If there exists a proper interval \( I \subset A \), for \( \sigma \in I \) fixed, \( \Psi(\sigma) \) is an eigenvalue of \( H_0^\Psi \) with infinite multiplicity.

3. If the set \( A \) is at most discrete, \( H_0^\Psi \) is absolutely continuous.

**Proof** 1. By the usual Fourier transform on \( L^2(\mathbb{R}^n) \), we represent

\[
H_0^\Psi = \mathcal{F}^* \Psi (|\xi|^2) \mathcal{F}.
\]

Therefore, its spectrum is given by the closure of the range of \( \Psi (|\xi|^2) \). Because \( \Psi \) is continuous and monotonically increasing, the spectrum of \( H_0^\Psi \) coincides with (1.7) or (1.8).

2. If \( A \) is a discrete set, then because \( \Psi \) is continuous and monotonic, for \( \lambda \geq \lim_{\sigma \to +0} \Psi(\sigma) \), there is only one \( \sigma_\lambda \in (0, \infty) \) such that \( \Psi(\sigma_\lambda) = \lambda \) and the
$n$-dimensional Lebesgue measure of
\[
\left\{ \xi \in \mathbb{R}^n \mid |\xi|^2 = \sigma \right\}
\]  
(1.11)
is zero. Therefore, if we assume that there exists $u \in L^2(\mathbb{R}^n)$ such that
\[
\left( \Psi \left( |\xi|^2 \right) - \lambda \right) u(\xi) = 0,
\]  
(1.12)
then $u = 0$ holds. This implies that the pure point spectrum of $H_0^\Psi$ is empty. Alternatively, if $I \subset A$ is a proper interval, for $\sigma \in I$, the Lebesgue measure of
\[
\left\{ \xi \in \mathbb{R}^n \mid \Psi \left( |\xi|^2 \right) = \Psi(\sigma) \right\}
\]  
(1.13)
is positive. In this case, $0 \neq u \in L^2(\mathbb{R}^n)$, which has support in (1.13), satisfies
\[
\left( \Psi \left( |\xi|^2 \right) - \Psi(\sigma) \right) u(\xi) = 0.
\]  
(1.14)
This implies that $\Psi(\sigma)$ is an eigenvalue of $H_0^\Psi$ and $u$ is the corresponding eigenfunction. In particular, (1.13) has infinite disjoint subsets for which the Lebesgue measures are positive. This also implies that $\Psi(\sigma)$ has infinite multiplicity.

3. If the one-dimensional Lebesgue measure of the Borel set $B$ is equal to zero, the $n$-dimensional Lebesgue measure of
\[
\left\{ \xi \in \mathbb{R}^n \mid \Psi \left( |\xi|^2 \right) \in B \right\}
\]  
(1.15)
is also zero because $A$ is discrete. Therefore,
\[
\int_{\Psi \left( |\xi|^2 \right) \in B} |(\mathcal{F} u)(\xi)|^2 d\xi = 0
\]  
(1.16)
holds for $u \in L^2(\mathbb{R}^n)$. This shows the absolute continuity of $H_0^\Psi$.

Remark 1.2 Statements 1 and 2 in Proposition 1.1 can be replaced by the following. If the Lebesgue measure of $A$ is zero, the pure point spectrum of $H_0^\Psi$ is empty and $H_0^\Psi$ is absolutely continuous. If the Lebesgue measure of $A$ is positive, $H_0^\Psi$ has an eigenvalue with infinite multiplicity. These proofs are demonstrated in a similar manner to that above.

Assumption 1.3 Let $\Psi_\pm \in \mathring{\mathcal{B}}$ be fixed and suppose $\Psi'_\pm = d\Psi_\pm/d\sigma > 0$. In addition, we assume that $\Psi'_\pm(\sigma^2)\sigma$ increases monotonically and $\Psi'_\pm(\sigma^2)\sigma$ decreases monotonically, for $\sigma \in (0, \infty)$.

Remark 1.4 Under this assumption, $H_0^{\Psi_\pm}$ do not have any eigenvalues and are absolutely continuous because $\Psi'_\pm > 0$.

The monotonicity in Assumption 1.3 is not extraordinary and it is not difficult to remove this assumption (see Remark 2.2 immediately following Proposition 2.1). For
example, $\Psi_+(\sigma) = \sqrt{\sigma}$ is allowed although $\Psi'_+(\sigma^2)\sigma$ is always equal to 1/2. In this case, as is well known,

$$\Psi_+\left(|D|^2\right) = \sqrt{-\Delta}$$  \hspace{1cm} (1.17)

is the massless relativistic Schrödinger operator. More generally, Assumption 1.3 admits fractional Schrödinger operators $\Psi_+\left(|D|^2\right) = (-\Delta)^\rho$ with $1/2 \leq \rho \leq 1$ and $\Psi_-\left(|D|^2\right) = (-\Delta)^\rho$ with $0 < \rho < 1/2$. We assume that $\Psi'_+(\sigma^2)\sigma$ is monotonic when keeping in mind fractional exponents.

From the classical mechanics aspect, $\nabla_\xi \Psi_+\left(|\xi|^2\right)$ means the velocity of the free particle. Therefore, in the monotonically increasing case, high-energy is expressed by $|\nabla_\xi \Psi_+\left(|\xi|^2\right)| = 2\Psi_+'\left(|\xi|^2\right)|\xi| \to \infty$ as $|\xi| \to \infty$, whereas in monotonically decreasing case, high-energy becomes $2\Psi_-\left(|\xi|^2\right)|\xi| \to \infty$ as $|\xi| \to 0$. This reverse trend in the momentum space is interesting and is described by the difference of two inequalities (2.1) and (2.2) in Proposition 2.1.

**Assumption 1.5** Let $V^S \in L^\infty(\mathbb{R}^n)$. There exist positive constant $C$ and exponents $\gamma_S > 1$ such that

$$|V^S(x)| \leq C \langle x \rangle^{-\gamma_S},$$  \hspace{1cm} (1.18)

where $\langle x \rangle = \sqrt{1 + |x|^2}$. For $0 \neq \kappa \in \mathbb{R}$ and $0 < \gamma_L \leq 1$, we also define $V^L \in L^\infty(\mathbb{R}^n)$ by

$$V^L(x) = \kappa \langle x \rangle^{-\gamma_L}.$$  \hspace{1cm} (1.19)

Throughout this paper, $\phi \in L^2(\mathbb{R}^n)$ satisfies $\mathcal{F}\phi \in C^\infty_0(\mathbb{R}^n \setminus \{0\})$. In particular, for $R > \epsilon > 0$ fixed, we assume that

$$\text{supp } \mathcal{F}\phi \subset \{\xi \in \mathbb{R}^n \mid \epsilon \leq |\xi| \leq R\}.$$  \hspace{1cm} (1.20)

$F(\cdots)$ denotes the characteristic function of the set $\{\cdots\}$. Moreover, we write the full Hamiltonians in the form

$$H^{\Psi}_S = H^0_{\Psi^\pm} + V^S, \quad H^{\Psi}_L = H^0_{\Psi^\pm} + V^L.$$  \hspace{1cm} (1.21)

**Theorem 1.6** The wave operators

$$\text{s-lim}_{t \to \infty} e^{itH^{\Psi}_S} e^{-itH^0_{\Psi^\pm}}, \quad \text{s-lim}_{t \to -\infty} e^{itH^{\Psi}_S} e^{-itH^0_{\Psi^\pm}}$$  \hspace{1cm} (1.22)

exist. However,

$$\text{s-lim}_{t \to \infty} e^{itH^{\Psi}_L} e^{-itH^0_{\Psi^\pm}}, \quad \text{s-lim}_{t \to -\infty} e^{itH^{\Psi}_L} e^{-itH^0_{\Psi^\pm}}$$  \hspace{1cm} (1.23)

do not exist. This means that the threshold between short and long-range depends on whether the decay exponent of the potential function is less than $-1$, or greater than or equal to $-1$

To prove Theorem 1.6, several Propositions and Lemmas are needed. In the following, we only consider the limit $t \to \infty$ because the other case is proved similarly.
2 Existence of Wave Operators

In this section, we prove the existence of the wave operators. Although the following propagation estimates for free evolution $\mathrm{e}^{-it\mathcal{H}_0^\Psi}$ are simple, these estimates also work well in the next section.

**Proposition 2.1** Let $t > 0$ and $N \in \mathbb{N}$. There exist positive constants $C_{\pm,N,\varepsilon,R}$ such that

$$\left\| F \left( \left| x \right| \leqslant \Psi'_+(\varepsilon^2)\varepsilon \right) \mathrm{e}^{-it\mathcal{H}_0^\Psi} \phi \right\|_{L^2(\mathbb{R}^n)} \leqslant C_{+,N,\varepsilon,R} t^{-N} \left\| \langle x \rangle^N \phi \right\|_{L^2(\mathbb{R}^n)} \quad (2.1)$$

and

$$\left\| F \left( \left| x \right| \leqslant \Psi'_-(R^2)R \right) \mathrm{e}^{-it\mathcal{H}_0^\Psi} \phi \right\|_{L^2(\mathbb{R}^n)} \leqslant C_{-,N,\varepsilon,R} t^{-N} \left\| \langle x \rangle^N \phi \right\|_{L^2(\mathbb{R}^n)} \quad (2.2)$$

hold.

**Proof** There exists a function $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ with supp $f \subset \{\varepsilon \leqslant |\xi| \leqslant R\}$ such that $\phi = f(D)\phi$. We then find

$$F \left( \left| x \right| \leqslant \Psi'_+(\varepsilon^2)\varepsilon \right) \mathrm{e}^{-it\mathcal{H}_0^\Psi} \phi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \mathrm{e}^{i(x \cdot \xi - t\Psi'_+(|\xi|^2))} F \left( \left| x \right| \leqslant \Psi'_+(\varepsilon^2)\varepsilon \right) f(\xi) \hat{\phi}(\xi) d\xi. \quad (2.3)$$

When $|\xi| \geqslant \varepsilon$ and $|x|/t \leqslant \Psi'_+(\varepsilon^2)\varepsilon$ hold, we see that

$$\left| \nabla_\xi \left( x \cdot \xi - t\Psi'_+(|\xi|^2) \right) \right| \geqslant 2t \Psi'_+(|\xi|^2) |\xi| - |x| \geqslant t\Psi'_+(\varepsilon^2)\varepsilon \quad (2.4)$$

because $\Psi'_+(\sigma^2)\sigma$ is monotonically increasing for $\sigma > 0$. Using this inequality and the relation

$$-i \frac{\nabla_\xi \left( x \cdot \xi - t\Psi'_+(|\xi|^2) \right) \cdot \nabla_\xi e^{i(x \cdot \xi - t\Psi'_+(|\xi|^2))}}{\left| \nabla_\xi \left( x \cdot \xi - t\Psi'_+(|\xi|^2) \right) \right|^2} = e^{i(x \cdot \xi - t\Psi'_+(|\xi|^2))}, \quad (2.5)$$

equation (2.1) follows from the standard integration by parts method (see Kitada [12] for example). As for (2.2), when $|\xi| \leqslant R$ and $|x|/t \leqslant \Psi'_-(R^2)R$ hold, we see that

$$\left| \nabla_\xi \left( x \cdot \xi - t\Psi'_-(|\xi|^2) \right) \right| \geqslant t\Psi'_-(R^2)R. \quad (2.6)$$

We therefore also obtain (2.2). \qed

**Remark 2.2** If we do not assume the monotonicity of $\Psi'_+(\sigma^2)\sigma$, the estimates (2.1) and (2.2) are replaced by

$$\left\| F \left( \left| x \right| \leqslant \inf_{\varepsilon \leqslant \sigma \leqslant R} \Psi'(\sigma^2)\sigma \right) \mathrm{e}^{-it\mathcal{H}_0^\Psi} \phi \right\|_{L^2(\mathbb{R}^n)} \leqslant C_{N,\varepsilon,R} t^{-N} \left\| \langle x \rangle^N \phi \right\|_{L^2(\mathbb{R}^n)}, \quad (2.7)$$

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for \( \Psi \in \mathcal{B} \) which satisfies \( \Psi' > 0 \). The following proofs also proceed without monotonicity.

Proposition 2.1 yields the existence of the wave operators immediately.

**Proof of the existence of the wave operators** Let us first consider the existence of

\[
\text{s-lim}_{t \to \infty} e^{itH_+} e^{-itH_0^+}.
\]

The derivative at \( t \) of \( e^{itH_+} e^{-itH_0^+} \phi \) is

\[
\frac{d}{dt} e^{itH_+} e^{-itH_0^+} \phi = i e^{itH_+} V e^{-itH_0^+} \phi
\]

\[
= i e^{itH_+} F \left( \left| \frac{x}{t} \right| > \Psi' + (\epsilon^2) \epsilon \right) V e^{-itH_0^+} \phi
\]

\[
+ i e^{itH_+} F \left( \left| \frac{x}{t} \right| \leq \Psi' + (\epsilon^2) \epsilon \right) V e^{-itH_0^+} \phi.
\]

We abbreviate the \( L^2 \)-norm \( \| \cdot \|_{L^2(\mathbb{R}^n)} \) to \( \| \cdot \| \) for simplicity and estimate

\[
\left\| \frac{d}{dt} e^{itH_+} e^{-itH_0^+} \phi \right\| \leq C \left( t \Psi' + (\epsilon^2) \epsilon \right) \| \phi \| + \| V \|_{L^\infty(\mathbb{R}^n)} C_N, \epsilon, R t^{-N} \left\| \langle x \rangle \right\|_N \phi,
\]

where we have used the decay assumption (1.18) and Proposition 2.1. With \( \gamma_S > 1 \), we can choose \( N \in \mathbb{N} \) as \( N \geq 2 \). Then (2.10) implies the existence of (2.8) by the Cook–Kuroda method and a density argument. The existence of

\[
\text{s-lim}_{t \to \infty} e^{itH_+} e^{-itH_0^+}.
\]

is proved by simply replacing \( \Psi' + (\epsilon^2) \epsilon \) with \( \Psi' + (R^2) R \) inside the characteristic functions of (2.9).

\[\square\]

### 3 Non-existence of Wave Operators

This section is devoted to proving the non-existence of the wave operators when the potential function \( V^L \) satisfies (1.19).

**Lemma 3.1** For \( t \geq 1 \), there exist positive constants \( c_{\pm,1} \) and \( c_{\pm,2} \) such that

\[
\frac{1}{\kappa} \left( V^L e^{-itH_0^+} \phi, e^{-itH_0^+} \phi \right)_{L^2(\mathbb{R}^n)} \geq c_{\pm,1} t^{-\gamma} \| \phi \|_{L^2(\mathbb{R}^n)}^2 - c_{\pm,2} t^{-2-\gamma} \| x \phi \|_{L^2(\mathbb{R}^n)}^2,
\]

where \( \cdot, \cdot \)_{L^2(\mathbb{R}^n)} \) denotes the scalar product on \( L^2(\mathbb{R}^n) \).
Proof By a straightforward computation, we see that the Heisenberg representation of the position $x$ is
\[
e^{itH_0^\pm_0} x e^{-itH_0^\pm_0} = x + 2t \Psi'(|D|^2) D. \tag{3.2}
\]
Therefore, its time evolution for the monotonically increasing case is estimated to be
\[
\|x e^{-itH_0^\pm_0} \phi\|^2 = \left\| \left( x + 2t \Psi' \left( |D|^2 \right) D \right) f(D) \phi \right\|^2 
\leq 2 \|x \phi\|^2 + 8nt^2 \Psi'(R^2)^2 R^2 \|\phi\|^2. \tag{3.3}
\]
Take $\Gamma_+ \in \mathbb{R}$ such that
\[
\Gamma_+ \geq \max \left\{ 4\sqrt{n} \Psi'(R^2) R, 1 \right\}, \tag{3.4}
\]
then the estimate of $e^{-itH_0^\pm_0} \phi$ outside a sphere of radius $\Gamma_+ t$ is
\[
\int_{|x|>\Gamma_+ t} \left| \left( e^{-itH_0^\pm_0} \phi \right)(x) \right|^2 dx \leq \int_{|x|>\Gamma_+ t} \frac{|x|^2}{\Gamma_+^2 t^2} \left| \left( e^{-itH_0^\pm_0} \phi \right)(x) \right|^2 dx 
\leq \frac{1}{\Gamma_+^2 t^2} \left\| x e^{-itH_0^\pm_0} \phi \right\|^2 \leq \frac{1}{\Gamma_+^2 t^2} \left\{ 2 \|x \phi\|^2 + 8nt^2 \Psi'(R^2)^2 R^2 \|\phi\|^2 \right\}
\leq \frac{2}{\Gamma_+^2 t^2} \|x \phi\|^2 + \frac{1}{2} \|\phi\|^2. \tag{3.5}
\]
We write $(\cdot, \cdot)_{L^2(\mathbb{R}^n)} = (\cdot, \cdot)$ and compute
\[
\frac{1}{\kappa} \left( V^L e^{-itH_0^\pm_0} \phi, e^{-itH_0^\pm_0} \phi \right) \geq \frac{1}{\kappa} \int_{|x|>\Gamma_+ t} (x)^{-\gamma_1} \left| \left( e^{-itH_0^\pm_0} \phi \right)(x) \right|^2 dx 
\geq \int_{|x|>\Gamma_+ t} (x)^{-\gamma_1} \left| \left( e^{-itH_0^\pm_0} \phi \right)(x) \right|^2 dx 
\geq (\Gamma_+ t)^{-\gamma_1} \int_{|x|>\Gamma_+ t} \left| \left( e^{-itH_0^\pm_0} \phi \right)(x) \right|^2 dx 
= (\Gamma_+ t)^{-\gamma_1} \|\phi\|^2 - (\Gamma_+ t)^{-\gamma_1} \int_{|x|>\Gamma_+ t} \left| \left( e^{-itH_0^\pm_0} \phi \right)(x) \right|^2 dx. \tag{3.6}
\]
Using the inequality (3.5), we have
\[
\frac{1}{\kappa} \left( V^L e^{-itH_0^\pm_0} \phi, e^{-itH_0^\pm_0} \phi \right) \geq \frac{1}{2} (\Gamma_+ t)^{-\gamma_1} \|\phi\|^2 - (\Gamma_+ t)^{-\gamma_1} \times \frac{2}{\Gamma_+^2 t^2} \|x \phi\|^2 
\geq c_+ t^{-\gamma_1} \|\phi\|^2 - c_+ t^{-2-\gamma_1} \|x \phi\|^2. \tag{3.7}
\]
For the last inequality in (3.7), we set $c_{+,1}$ and $c_{+,2}$ using
\[
\frac{1}{2} (\Gamma_+ t)^{-\gamma_n} \geq \frac{1}{2} \left( 1 + \Gamma_+ t \right)^{-\gamma_n} \geq \frac{1}{2} \left( 2 \Gamma_+ \right)^{-\gamma_n} t^{-\gamma_n} = c_{+,1} t^{-\gamma_n}.
\]
\[
2 \frac{(\Gamma_+ t)^{-\gamma_n}}{\Gamma_+^2 t^2} \leq 2 \frac{(\Gamma_+ t)^{-\gamma_n}}{\Gamma_+^2 t^2} = 2 \Gamma_+^{-2} t^{-2} - \gamma_n = c_{+,2} t^{-2} - \gamma_n.
\] (3.8)

In contrast, for the monotonically decreasing case, we have
\[
\left\langle \frac{1}{\Gamma_1} + t \right\rangle - \gamma_n \leq \left\langle \frac{1}{\Gamma_1} + t \right\rangle - \gamma_n = c_{+,2} t^{-2} - \gamma_n.
\] (3.9)

Therefore,
\[
\| x e^{-itH_0^{\Psi_1} - \frac{t}{\Gamma_1}} \phi \|^{2} \leq 2 \| x \phi \|^{2} + 8nt^2 \Psi'_1(\epsilon^2)^2 \epsilon^2 \| \phi \|^2.
\] (3.10)

\[
\frac{1}{\kappa} \left( V^L e^{-itH_0^{\Psi_1}} \phi, e^{-itH_0^{\Psi_1}} \phi \right) \geq c_{-,1} t^{-\gamma_n} \| \phi \|^2 - c_{-,2} t^{-2} - \gamma_n \| x \phi \|^2.
\] (3.11)

holds for $\Gamma_1 \in \mathbb{R}$ which satisfies
\[
\Gamma_1 \geq \max \left\{ 4\sqrt{n} \Psi'_1(\epsilon^2) \epsilon, 1 \right\}.
\] (3.12)

**Lemma 3.2** For $t > 0$ and $N \in \mathbb{N}$, there exist positive constants $c_{\pm,3}$ and $c_{\pm,4}$ such that
\[
\left\| V^L e^{-itH_0^{\Psi_1}} \phi \right\|_{L^2(\mathbb{R}^n)} \leq c_{\pm,3} t^{-\gamma_n} \| \phi \|_{L^2(\mathbb{R}^n)} + c_{\pm,4} t^{-N} \left\| \langle x \rangle^N \phi \right\|_{L^2(\mathbb{R}^n)}.
\] (3.13)

**Proof** This proof follows in almost the same way as the proof for the existence of the wave operators (see (2.10)). For the monotonically increasing case, (3.13) follows by setting $c_{+,3} = |\kappa| (\Psi'_1(\epsilon^2)^2 \epsilon)^{-\gamma_n}$ and $c_{+,4} = |\kappa| C_{+,N,\epsilon,R}$. For the monotonically decreasing case, (3.13) follows by $c_{-,3} = |\kappa| (\Psi'_1(R^2)^{R})^{-\gamma_n}$ and $c_{-,4} = |\kappa| C_{-,N,\epsilon,R}$. \hfill \qed

We have now gathered everything required to prove the non-existence of the wave operators.

**Proof of the nonexistence of the wave operators** We assume that
\[
\lim_{t \to \infty} e^{itH_L^{\Psi_1}} e^{-itH_0^{\Psi_1}}
\] (3.14)
exists and put
\[
\phi_\pm = \lim_{t \to \infty} e^{itH_L^{\Psi_1}} e^{-itH_0^{\Psi_1}} \phi \in L^2(\mathbb{R}^n).
\] (3.15)

There exist $T_\pm > 0$ such that
\[
\left\| e^{itH_L^{\Psi_1}} e^{-itH_0^{\Psi_1}} \phi - \phi_\pm \right\| \leq \frac{|\kappa| c_{\pm,1} \| \phi \|}{2c_{\pm,3}}.
\] (3.16)
for all $t \geq T$. We take $t_1$ and $t_2$ such that $t_2 \geq t_1 \geq \max\{T, 1\}$ and compute

$$\left| \left\{ e^{it_2 H_L} \phi, e^{-it_2 H_L} \phi \right\} \right| = \left| \int_{t_1}^{t_2} \frac{d}{dt} \left( e^{-it H_L} \phi, e^{-it H_L} \phi \right) dt \right| = \left| \int_{t_1}^{t_2} \left( V_L e^{-it H_L} \phi, e^{-it H_L} \phi \right) dt \right| + \left| \int_{t_1}^{t_2} \left( V_L e^{-it H_L} \phi, e^{-it H_L} \phi \right) dt \right|$$

$$\geq |\kappa| \int_{t_1}^{t_2} \left( V_L e^{-it H_L} \phi, e^{-it H_L} \phi \right) dt.$$

By virtue of Lemmas 3.1 and 3.2, we conclude that

$$2\|\phi\|\|\phi\| \geq \frac{|\kappa| c_{1,1}}{2} \|\phi\|^2 \int_{t_1}^{t_2} t^{-\gamma_L} dt$$

$$-|\kappa| c_{2,1} \|x\| \|\phi\|^2 \int_{t_1}^{t_2} t^{-2-\gamma_L} dt - \frac{|\kappa| c_{4,1} c_{4,2}}{2c_{3,1}} \|\phi\| \|x\|^N \int_{t_1}^{t_2} t^{-N} dt$$

$$\rightarrow \infty$$

as $t_2$ goes to infinity because $\gamma_L \leq 1$, and we can choose $N \in \mathbb{N}$ as $N \geq 2$. This leads to a contradiction.

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