INTERIOR REGULARITY FOR TWO-DIMENSIONAL STATIONARY 
Q-VALUED MAPS

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Abstract. We prove that 2-dimensional Q-valued maps that are stationary with respect to outer and inner variations of the Dirichlet energy are Hölder continuous and that the dimension of their singular set is at most one. In the course of the proof we establish a strong concentration-compactness theorem for equicontinuous maps that are stationary with respect to outer variations only, and which holds in every dimensions.

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1. Introduction

Recall that a Q-valued map \( f \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n)) \), \( \Omega \subset \mathbb{R}^m \) open, is stationary (with respect to outer and inner variations of the Dirichlet energy) if it is a critical point of the Dirichlet energy, that is it satisfies an outer variation formula

\[
\mathcal{O}(f, \psi) := \int \sum_i \langle Df_i(x) : D_x \psi(x, f_i(x)) \rangle \, dx + \int \sum_i \langle Df_i(x) : D_u \psi(x, f_i(x)) \cdot Df_i(x) \rangle \, dx = 0,
\]

(1.1)
where \( \psi \in C^\infty(\Omega \times \mathbb{R}^n; \mathbb{R}^n) \) with \( \Omega \) compact and
\[
|D_u \psi| \leq C < \infty \quad \text{and} \quad |\psi| + |D_x \psi| \leq C (1 + |u|),
\]
and an inner variation formula
\[
\mathcal{I}(f, \phi) := 2 \sum_{i=1}^{Q} \langle D f_i : D f_i \cdot D \phi \rangle - \hat{\int} |Df|^2 \ \text{div} \ \phi = 0, \quad \forall \phi \in C_\infty^c(\Omega, \mathbb{R}^m).
\] (1.2)

For the derivation of such formulas as the Euler-Lagrange equations of the appropriate Dirichlet energy and the related Sobolev theory of \( Q \)-valued maps, we refer the reader to [7, Sections 2 and 3], whose notations we will follow.

Multivalued maps that minimize an appropriate Dirichlet energy were introduced by Almgren in [1] in his celebrated proof of the optimal bound on the dimension of the singular set of area minimizing currents in high codimension, as the appropriate linearized problem. More recently, De Lellis and Spadaro revisited the theory with modern techniques in [3, 4, 5, 6, 7]. In order to extend Almgren’s proof and techniques to the study of the singular set of stationary varifolds and currents, it is therefore a natural first step to investigate the regularity of stationary multivalued maps: this is the goal of this note in the 2-dimensional case. The only known result in this setting is due to Lin who proved in [11] continuity of such maps. In this paper we give a shorter and intrinsic proof of Lin’s result (that is without the use of Almgren’s embedding) and we improve it to Hölder regularity. We then prove a concentration compactness result for maps that are stationary with respect to outer variations only, and equicontinuous, and which holds in every dimension. Finally we combine these results to prove the optimal bound on the dimension of the singular set of 2-dimensional stationary \( Q \)-valued maps.

1.1. Main results. Our first result is a continuity result for 2-dimensional stationary maps. As mentioned above this has already been proven in [11], following ideas from [9]. Our proof also follows the blueprint of Grueter, but with respect to Lin’s proof it has the advantage of avoiding Almgren’s projection, thus being completely intrinsic, and moreover we obtain an explicit modulus of continuity that will be important in the proofs of the results below.

**Theorem 1.1 (Continuity).** Let \( f \in W^{1,2}(B_1, A_Q(\mathbb{R}^n)) \) be stationary in \( B_1 \subset \mathbb{R}^2 \). Then \( f \) is continuous in \( B_{1/2} \), more precisely
\[
G(f(x), f(y))^2 \leq C \left( \int_{B_{3|x-y|}} |Df|^2 \right) + C \left( \int_{B_1} |Df|^2 \right) |x - y|^2, \quad \forall x, y \in B_{1/2}, |x - y| < \frac{1}{4},
\] (1.3)
where \( C > 0 \) depends only on \( Q \).

Next we prove higher integrability for 2d stationary maps. A similar result for Dir-minimizing multivalued maps in any dimension has been proven with different techniques in [3, Theorem 5.1].

**Theorem 1.2 (Higher integrability).** There exist \( p > 2 \) and a constant \( C = C(Q) > 0 \) such that if \( f \in W^{1,2}(B_1, A_Q(\mathbb{R}^n)) \) is stationary in \( B_1 \subset \mathbb{R}^2 \) then
\[
\left( \int_{B_{1/2}} |Df|^p \right)^{\frac{1}{p}} \leq C \int_{B_1} |Df|^2.
\] (1.4)
As a corollary of Theorems 1.1 and 1.2, we can immediately deduce Hölder regularity of 2-dimensional stationary maps. It is interesting to notice that our proof makes no use of Almgren’s embedding, at difference from the proof in the minimizing case.

Corollary 1.3 (Hölder continuity). There exist positive constant $C$ and $\alpha$, depending only on $Q$, such that if $f \in W^{1,2}(B_1, A_Q(\mathbb{R}^n))$ is a stationary map in $B_1 \subset \mathbb{R}^2$, then

$$G(f(x), f(y)) \leq C \left( \int_{B_1} |Df|^2 \, dx \right)^{\frac{1}{2}} |x - y|^\alpha, \quad \forall x, y \in B_{1/2}. \quad (1.5)$$

Next we prove a concentration compactness result for equicontinuous maps stationary with respect to outer variations only. Given the decomposition in Subsection 6.1, we will not need the full power of the following statement in the rest of the paper (i.e., we will only need the compactness part). However we state it in the most general form with possible future applications in mind. It is worth noticing that it holds in every dimension.

Theorem 1.4 (Concentration compactness for equicontinuous and outer stationary maps). Let $(f_j)_j \subset W^{1,2}(B_1, A_Q(\mathbb{R}^n))$, $B_1 \subset \mathbb{R}^n$, be a sequence of maps such that for every $j \in \mathbb{N}$ the following properties hold:

1. there exists a modulus continuity $\omega$, independent of $j$, such that

$$G(f_j(x), f_j(y)) \leq \omega(|x - y|), \quad \forall x, y \in B_1,$$

2. $\int_{B_1} |Df_j|^2 \, dx \leq C < \infty$;

3. $\mathcal{O}(f_j, B_1) = 0$.

Then there exist $q \in \mathbb{N}$, vectors $a_1(k), \ldots, a_q(k) \in \mathbb{R}^n$ and multivalued functions $h_1, \ldots, h_q \in W^{1,2}(B_1, A_{Q_i}(\mathbb{R}^n))$, continuous with modulus of continuity $\omega$ and such that

$$\sum_{i=1}^q Q_i = Q \quad \text{and} \quad \mathcal{O}(h_i, B_1) = 0, \quad \text{for all } i = 1, \ldots, q,$$

and satisfying

$$\lim_{k \to \infty} \sup_{B_r} G(g_k(x), f_k(x)) = 0 \quad \text{(1.6)}$$

$$\lim_{k \to \infty} \int_{B_r} |Df_k|^2 - |Dg_k|^2 = 0, \quad \text{(1.7)}$$

where $g_k(x) := \sum_{i=1}^q a_i(k) \oplus h_i$.

Finally we will use the previous theorems to prove the optimal bound on the dimension of the singular set of 2-dimensional stationary $Q$-valued maps. We recall that a point $x \in \Omega$ is regular if there exists a neighborhood $B \subset \Omega$ of $x$ and $Q$ analytic functions $f_i : B \to \mathbb{R}^n$ such that

$$f(y) = \sum_{i=1}^Q \|f_i(y)\| \quad \text{for almost every } y \in B,$$

and either $f_i(x) = f_j(x)$ for every $x \in B$ or $f_i \equiv f_j$. The singular set $\Sigma_f$ of $f$ is the complement in $\Omega$ of the set of regular points.

Theorem 1.5 (Dimension of the singular set). Let $f \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n))$, with $\Omega \subset \mathbb{R}^2$, be a stationary map. Then $\dim_H(\Sigma_f) \leq 1$. 

It should be possible to improve the above result to 1-rectifiability of the singular set and locally finite length, by using the same techniques as in [2] (see Remark 6.8).

1.2. Differences with Dir-minimizing case. The proofs of the above theorems in the Dir-minimizing case are achieved via comparison with suitable competitors and using induction on the multiplicity $Q$. Of course, no competitor argument is available in our setting. Moreover, while if a Dir-minimizing map splits as the sum of two maps, then such maps are Dir-minimizing separately, this is not the case for stationary maps, since the inner variation doesn’t split, as the following example illustrates. In fact Corollary 1.3 together with Remark 2.1 shows that $Q \geq 4$ is indeed necessary.

Remark 1.6. Consider the the 2-valued map

$$g(x) = \begin{cases} [x] + [-x] & \text{for } x > 0 \\ 2[0] & \text{for } x < 0. \end{cases}$$

One can directly check that $g$ satisfies the outer variation since $g$ is the composition of linear hence harmonic functions on $x > 0$ and $x < 0$ furthermore it satisfies $\lim_{x \uparrow 0} g(x) = 2[0] = \lim_{x \downarrow 0} g(x)$. Nonetheless it does not satisfy the inner variation since

$$|g'(x)|^2 = \begin{cases} 2 & \text{for } x > 0 \\ 0 & \text{for } x < 0. \end{cases}$$

If instead we consider for any value $p \in \mathbb{R}$ the 4-valued map

$$f(x) = (g(x) \oplus a) + (g(-x) \oplus (-a))$$

we obtain a stationary map, since $g(x)$ and $g(-x)$ satisfy the outer variation for themselves and $f$ satisfies trivially the inner variations since $|f'(x)|^2 = 2$ for all $x$. Choosing $a$ sufficient large we obtain therefore a stationary map that splits but the components themselves are not stationary.

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2. Preliminaries and splitting $Q$-points

In this section we recall some basic notations and definitions about stationary $Q$-valued maps, and we introduce a useful combinatoric lemma.

2.1. Preliminaries and notations. For the sake of completeness we recall some of the basic notations and definitions concerning $Q$-valued maps. For a detailed treatment we refer the reader to [7], whose notation and conventions we follow.

We denote by $[p_i]$ the Dirac mass in $p_i \in \mathbb{R}^n$ and we define the space of $Q$-points

$$A_Q(\mathbb{R}^n) := \left\{ \sum_{i=1}^Q [p_i] : p_i \in \mathbb{R}^n \text{ for every } i = 1 \ldots, Q \right\}.$$
We define a distance function on \( A_Q(\mathbb{R}^n) \) by setting for every \( T_1, T_2 \in A_Q(\mathbb{R}^n) \), with \( T_1 = \sum_{i=1}^{Q} [p_i] \) and \( T_2 = \sum_{i=1}^{Q} [s_i] \),
\[
G(T_1, T_2) := \min_{\sigma \in \mathcal{P}_Q} \sqrt{\sum |p_i - s_{\sigma(i)}|},
\]
where \( \mathcal{P}_Q \) denotes the group of permutations of \( \{1, \ldots, Q\} \). If \( T = \sum_{i=1}^{Q} [p_i] \), then we define the support, the diameter and the separation of \( T \) respectively by
\[
spt(T) = \{p_1, \ldots, p_Q\}, \quad \text{diam}(T) := \max_{i,j} |p_i - p_j| \quad \text{and} \quad \text{sep}(T) := \min\{|p_i - p_j| : p_i \neq p_j\}.
\]
Moreover given \( T = \sum_{i=1}^{Q} [p_i] \) and a vector \( v \in \mathbb{R}^n \), we define
\[
T \oplus v := \sum_{i=1}^{Q} [p_i + v] \quad \text{and} \quad T \ominus v := \sum_{i=1}^{Q} [p_i - v].
\]
We define the space of average free \( Q \)-points by
\[
A^0_Q(\mathbb{R}^n) := \{T \in A_Q(\mathbb{R}^n) : \eta \circ T = 0\},
\]
where \( \eta \circ T := \frac{1}{Q} \sum_{i=1}^{Q} p_i \) is the average of \( T = \sum_{i=1}^{Q} [p_i] \).

Any map \( f : \Omega \to A_Q(\mathbb{R}^n) \) will be called a \( Q \)-valued map. We refer the reader to [7] for the Sobolev theory of \( Q \)-valued maps. In particular, we observe that [7, Chapter 4] provides such a theory independently of Almgren’s embedding. Finally the derivation of (1.1) and (1.2) are contained in [7, Section 3.1].

In the sequel we will often use the following useful observation:

**Remark 2.1.** Let \( h : \Omega \to \mathbb{R}^n \) be a smooth single-valued map. Then a straightforward calculation shows that for any \( Q \)-valued map \( f \in W^{1,2}(\Omega, A_Q) \) and any admissible test-vectorfields \( \psi \) and \( \phi \) we have
\[
\mathcal{O}(f \oplus h, \psi_h) = \mathcal{O}(f, \psi) + Q \int_{\Omega} (Dh : \eta \circ \psi(x, f)) \, dx \tag{2.1}
\]
\[
\mathcal{I}(f \oplus h, \phi) = \mathcal{I}(f, \phi) + Q \mathcal{I}(h, \phi). \tag{2.2}
\]
where \( \psi_h(x,z) = \psi(x,z - h(x)) \).

In particular we deduce that adding an harmonic function \( h \) to a stationary map \( f \) will preserve stationarity.

### 2.2. Splitting \( Q \)-points

Given a point \( S \in A_Q(\mathbb{R}^n) \), with \( S = \sum_{i=1}^{q} Q_i [s_i] \), we define
\[
r_i(S) = r_i := \min\{\text{dist}(s_i, s_j) : j \neq i\}, \quad i = 1, \ldots, q,
\]
and we define the projection neighborhoods of \( S \) by
\[
\mathcal{P}_\varepsilon(S) := \bigcup_{i=1}^{q} B_{r_i}(s_i), \quad 0 < \varepsilon < \frac{1}{4}.
\]

**Remark 2.2** (Balanced splitting). Notice that the map \( P_\varepsilon \) that projects each ball \( B_{\varepsilon r_i}(s_i) \) on the center \( S_i \) is a well defined Lipschitz retraction, and moreover if \( T \in A_Q(\mathbb{R}^n) \) is such that
\[
spt(T) \subset \mathcal{P}_\varepsilon(S)
\]
then we have a natural splitting of $T = \sum_{i=1}^{q} T_i$, where
\[ \text{spt}(T_i) \subset B_{\epsilon r_i}(s_i), \quad \forall i = 1 \ldots q. \] (2.3)

Using the notation of currents we have $T_i = T \cdot B_{\epsilon r_i}(s_i)$. We will call such a $T$ balanced if $M(T_i) = Q_i = M(T \cdot B_{\epsilon r_i}(s_i))$ for all $i$.

Furthermore note that if $T \in P_\varepsilon(S)$ is balanced and $\varepsilon < \frac{1}{4}$ then
\[ G^2(S, T) = \sum_{i=1}^{q} G^2(T_i, Q_i \llbracket s_i \rrbracket). \] (2.4)

This follows from the pigeonhole property. Indeed let $T = \sum_{j=1}^{Q} \left\llbracket t_j \right\rrbracket$ and let $(I_i)_{i=1}^{q}$ be the partition of $\{1, \ldots, Q\}$ such that $Q_i \llbracket s_i \rrbracket = \sum_{j \in I_i} \left\llbracket s_j \right\rrbracket$ and select a permutation $\sigma \in P_Q$ that realises the distance, i.e.
\[ G^2(S, T) = \sum_{i=1}^{q} \sum_{j \in I_i} |s_i - t_{\sigma(j)}|^2. \]

Assume by contradiction it is not true that $\sigma(I_i) = I_i$, then by the pigeonhole property there must be a “k-chain” (after relabeling the $I_i$’s) of indices $j_1, \ldots, j_k$ with the property that
\begin{enumerate}
  \item $j_i \in I_i$;
  \item $\sigma(j_i) \in I_{i+1}$ for $i = 1, \ldots, k-1$ and $\sigma(j_k) \in I_1$.
\end{enumerate}

Then we have on the one hand
\[ \sum_{i=1}^{k} |s_{i+1} - t_{\sigma(j_i)}|^2 \geq \frac{(1 - \varepsilon)^2}{2} \sum_{i=1}^{k} r_i^2, \]
but on the other hand we have as a competitor
\[ \sum_{i=1}^{k} |s_i - t_{j_i}|^2 \leq \varepsilon^2 \sum_{i=1}^{k} r_i^2. \]

This contradicts the optimality of $\sigma$.

With a similar proof one can show that if $S, S' \in P_\varepsilon(T)$ are balanced and $\varepsilon < \frac{1}{8}$ then
\[ G^2(S, S') = \sum_{i=1}^{q} G^2(S_i, S'_i), \] (2.5)
but we will not use this in the sequel.

Next we recall the following splitting lemma from [3].

**Lemma 2.3** ([3, Lemma 3.8]). For every $0 < \varepsilon < 1$, we set $\beta(\varepsilon, Q) = (\frac{\varepsilon}{Q})^{3Q}$. Then, for every $T \in A_Q$ there exists a point $S \in A_Q$ such that
\[ \beta(\varepsilon, Q) \cdot \text{diam}(T) \leq \text{sep}(S) < \infty, \] (2.6)
\[ G(S, T) \leq \varepsilon \cdot \text{sep}(S), \] (2.7)
\[ \text{spt}(S) \subset \text{spt}(T). \] (2.8)

Next we state and prove a combinatorial lemma that will be used later in the proof of the continuity of stationary maps. The proof is based on inductive applications of Lemma 2.3.
Lemma 2.4 (Key combinatoric lemma). Let $0 < \varepsilon < 1/8$ and set $\tilde{\beta}(\varepsilon, Q) = \left( \varepsilon + \frac{Q-1}{G(\varepsilon, Q)} \right)^{-1}$, with $\beta(\varepsilon, Q)$ as in Lemma 2.3. Then for every $T \in A_Q$ there exist $q \leq Q$ and a family of points $\{S_i \in A_Q : i = 1, \ldots, q\}$ such that $S_0 = Q \llbracket t \rrbracket$, $S_q = T$, and, for every $k = 0, \ldots, q$, the following properties hold:

1. $\text{spt}(S_k) \subseteq \text{spt}(T)$;
2. $T \in \mathcal{P}_\varepsilon(S_k)$ and it is balanced, as defined in Remark 2.2;
3. $\tilde{\beta}(\varepsilon, Q)^k G(S_{k-1}, S_k) \leq \text{sep}(S_k)$.

Proof. We will prove the lemma replacing (3) with the stronger conclusion

(3*) $\tilde{\beta}_k \max (G(T, S_k), G(T, S_{k-1})) \leq \text{sep}(S_k)$,

where $\tilde{\beta}_k = \tilde{\beta}(\varepsilon, Q)^k$. Notice that indeed (3) follows from (3)* and the triangular inequality.

We prove the lemma by induction. At several steps in the proof we will use the following inequality for $T \in A_Q$ and $p \in \text{spt}(T)$, which follows directly from the definition of $G$:

$$\frac{1}{\sqrt{2}} \text{diam}(T) \leq G(T, Q \llbracket p \rrbracket) \leq \sqrt{Q-1} \text{diam}(T).$$

For the base step we let $t \in \text{spt}(T)$ and set $S_0 := Q \llbracket t \rrbracket$. Next we apply Lemma 2.3 to $T$ and find a point $S_1 = S \in A_Q$ satisfying (2.6), (2.7) and (2.8). In particular (1) follows from (2.8), (2) follows from (2.7), since $\text{sep}(S) \leq r_i(S)$ for every $i$, giving that $T \in \mathcal{P}_\varepsilon(S_1)$. Furthermore $T$ needs to be balanced, otherwise $G(T, S_1) \geq (1-\varepsilon)\text{sep}(S_1)$, contradicting (2.7). Finally (3)* follows since from (2.7) we have $G(S_1, T) \leq \varepsilon \text{sep}(S_1)$ and, from (2.7) and (2.6) we have

$$\frac{\beta(\varepsilon, Q)}{\sqrt{Q-1}} G(T, S_0) \leq \beta(\varepsilon, Q) \text{diam}(T) \leq \text{sep}(S_1).$$

Next assume we have $S_{k-1} = \sum_{i=1}^l Q_i \llbracket s_i \rrbracket$, $l \leq Q$, satisfying (1), (2) and (3)*. By Remark 2.2 with $T$ and $S = S_{k-1}$, we can write

$$T = \sum_{i=1}^l T_i$$

with $T$ balanced. Without loss of generality we can assume that

$$G(T_1, Q_1 \llbracket s_1 \rrbracket) \geq G(T_2, Q_2 \llbracket s_2 \rrbracket) \geq \ldots \geq G(T_l, Q_l \llbracket s_l \rrbracket).$$

Next we apply Lemma 2.3 with $T = T_1$ to get $S' = \sum_{j=1}^m Q_j' \llbracket s_j' \rrbracket \in A_{Q_1}$ with $\sum_{j=1}^m Q_j' = Q_1$ and we set

$$S_k := S' + \sum_{i=2}^l Q_i \llbracket s_i \rrbracket \in A_Q.$$

Then (1) and (2) follow exactly as in the base step. To prove (3)*, notice that

$$\tilde{\beta}_{k-1} \text{sep}(S') \leq \tilde{\beta}_{k-1} G(T_1, Q_1 \llbracket s_1 \rrbracket) \leq \tilde{\beta}_{k-1} G(T, S_{k-1}) \leq \text{sep}(S_{k-1}),$$

where the next to last inequality follows from the inductive assumption (3)* for $S_{k-1}$. Moreover we have

$$(1-\varepsilon)\text{sep}(S_{k-1}) \leq (1-\varepsilon)|s_1 - \hat{s}| \leq |s' - \hat{s}|, \quad \forall s' \in \text{spt}(S'), \hat{s} \in \text{spt}(S_{k-1}) \setminus \{s_1\}.$$ 

Therefore we conclude that

$$\tilde{\beta}_{k-1} \text{sep}(S') \leq \text{sep}(S_k) \leq \text{sep}(S'),$$

(2.11)
where the first inequality follows from (2.9), (2.10) and the fact that $\tilde{\beta}_{k-1}\text{sep}(S') < \text{sep}(S')$, while the second inequality follows by the definition of $S_k$. Next notice that

$$\frac{\beta(\varepsilon,Q_1)}{\sqrt{1+|Q_1|^2}} G(T,S_{k-1}) \leq \frac{\beta(\varepsilon,Q_1)}{\sqrt{|Q_1|^2}} G(T,Q_1[s_1]) \leq \beta(\varepsilon,Q_1) \text{diam}(T_1) \leq \text{sep}(S'), \quad (2.12)$$

where the last inequality follows from (2.6). Moreover

$$G(T,S_k) \leq G(T_1,S_1) + G(T,S_{k-1}) \leq \left(\varepsilon + \frac{\sqrt{1+|Q_1|^2}}{\beta(\varepsilon,Q_1)}\right) \text{sep}(S'). \quad (2.13)$$

The conclusion now follows combining (2.11), (2.12) and (2.13), with

$$\tilde{\beta}_k^{-1} \geq \tilde{\beta}_{k-1}^{-1} \left(\varepsilon + \frac{Q-1}{\beta(\varepsilon,Q)}\right) \geq \left(\varepsilon + \frac{Q-1}{\beta(\varepsilon,Q)}\right)^k. \quad \Box$$

3. Average conformal maps and monotonicity of the energy

In this section we introduce average conformal maps, and then we show that every stationary map in $\mathbb{R}^2$ is average conformal. This will follow from the inner variation formula.

3.1. Average conformal maps. We introduce the following class of multivalued maps.

**Definition 3.1.** We say that a map $f \in W^{1,2}(B_1 \subset \mathbb{R}^2; A_Q(\mathbb{R}^n))$ is average conformal if

$$\sum_{i=1}^Q |D_1 f_i|^2(x) = \sum_{i=1}^Q |D_2 f_i|^2(x) \quad \text{and} \quad \sum_{i=1}^Q D_1 f_i(x) \cdot D_2 f_i(x) = 0 \quad (3.1)$$

for almost every $x$ in the domain of $f$.

Next we consider the following modified energy density on a set $\Omega$

$$\Theta_f(\Omega,S,s) := \frac{1}{s^2} \int_\Omega \phi \left(\frac{G(f,S)}{s}\right) |Df|^2 \, dx,$$

where $\phi(t)$ is smooth non-negative, non-increasing, with $\phi = 1$ on $t < \frac{1}{2}$ and $\phi = 0$ for $t \geq 1$. A nice property of average conformal maps is the following monotonicity of the energy density (notice that the domains are not balls in the domain but in the target).

**Proposition 3.2** (Energy monotonicity). Let $f \in W^{1,2}(\mathbb{R}^2; A_Q(\mathbb{R}^n))$ be an average conformal map, then $r \mapsto \theta_f(\Omega,S,r)$ is monotone non-decreasing on $[0,r_S]$, where $r_S > 0$ satisfies

1. $r_S \leq \frac{1}{4} \text{sep}(S)$;
2. $\{G(f,S) < r_S\} \subset \Omega$.

**Proof.** Let $P_S$ be the Lipschitz retraction of Remark 2.2. We test the outer variation (1.1) with the vector field

$$\psi(x,z) = \phi \left(\frac{G(f(x),S)}{s}\right) (z - P_S(z)),$$

which is an admissible vector field by approximation and since it is 0 on the boundary of $\Omega$ due to the assumptions on $r_S$. Hence we have

$$\int_\Omega \phi \left(\frac{G(f,S)}{s}\right) |Df|^2 \, dx + \int_\Omega \phi' \left(\frac{G(f,S)}{s}\right) \frac{G(f,S)}{s} |DG(f,S)|^2 = 0 \quad (3.2)$$
where we used that for a.e. \( x \in \{ G(f, S) < \frac{1}{4} \text{sep}(S) \} \) the following holds
\[
\sum_{i=1}^{Q} Df_i(x): (f_i(x) - P_S(f_i(x))) = G(f(x), S)DG(f(x), S).
\]

To justify the above calculations one can use a sequence of Lipschitz function \( \hat{f}^k \) that approximate \( f \) strongly in \( W^{1,2} \). Passing to \( f^k = P_S \circ \hat{f}^k \) we may even assume that \( G(f^k(x), S) \leq \frac{1}{4} \text{sep}(S) \) for all \( S \), and the Lipschitz functions \( f^k \) still approximate \( f \) strongly in \( W^{1,2} \) on the set \( \{ G(f, S) < r_S \} \).

Next notice that
\[
\frac{d}{ds} \Theta_f(\Omega, S, s) = -\frac{1}{s^3} \left( \int_{\Omega} \phi' \left( \frac{G(f, S)}{s} \right) \frac{G(f, S)}{s} |Df|^2 + 2 \int_{\Omega} \phi \left( \frac{G(f, S)}{s} \right) |Df|^2 \right)
\]
\[
= -\frac{1}{s^3} \int_{\Omega} \phi' \left( \frac{G(f, S)}{s} \right) \frac{G(f, S)}{s} (|Df|^2 - 2|DG(f, S)|^2) \geq 0,
\]
where the last inequality follows from the fact that \( f \) is averaged conformal in the following way. Let \( x \) be a Lebesgue point of \( f \), and let \( f_i \) be a selection such that \( Df = \sum_{i=1}^{Q} ||Df_i|| \) and \( G(f, S)^2 = \sum_{i=1}^{Q} |f_i(x) - s_i|^2 \), where \( s_i \in \text{spt}(S) \). Define the \( R^{Qn} \) vectors
\[
e_i = \frac{(\partial_i f_1(x), \partial_i f_2(x), \ldots, \partial_i f_Q(x))}{\sqrt{|Df(x)|^2}}, \quad i = 1, 2
\]
\[
b = (f_1(x) - s_1, f_2(x) - s_2, \ldots, f_Q(x) - s_Q).
\]

Average conformality implies that \( |e_1|^2 = |e_2|^2 = 1, \langle e_1, e_2 \rangle = 0 \), hence \( (e_i)_{i=1}^2 \) are orthonormal. Therefore from (3.3) we have
\[
G(f(x), S)^2|DG(f(x), S)|^2 = \frac{1}{2} |Df(x)|^2 \sum_{i=1}^{2} \langle e_i, b \rangle^2 \leq \frac{1}{2} |Df(x)|^2 |b|^2 = \frac{1}{2} G(f(x), S)^2 |Df(x)|^2,
\]
which concludes the proof. \( \square \)

We will also need the following

**Lemma 3.3** (Energy lower bound). Let \( f \in W^{1,2}(B_2, A_Q) \) and \( \Omega \subset B_2 \), then
\[
\lim_{s \to 0} \Theta_f(\Omega, f(x_0), s) \geq \frac{1}{2} \int_{\Omega} \phi(x) \, dx > 0 \quad \text{for a.e.} \quad x_0 \in \Omega \setminus \{|Df| = 0\}.
\]

**Proof.** Let us fix a point \( x_0 \in \Omega \) with the properties that

1. \( x_0 \) is a full density point of \( \Omega \),
2. \( f \) is approximate differentiable in \( x_0 \),
3. \( x_0 \) is a Lebesgue point of \( |Df(x)|^2 \).

These three properties hold for a.e. point in \( \Omega \) by the Lebesgue differentiation theorem and [?], Corollary 2.7.

After translating we may assume \( x_0 = 0 \) and \( d^2 = |Df(0)|^2 \). Let \( Tf(x) = \sum_{i=1}^{Q} [f_i(0) + Df_i(0) \cdot x] \) the first order approximation of \( f \) in 0. In particular (1), (2) and (3) read

1. \( \lim_{r \to 0} r^{-2} |\Omega \cap B_r(x_0)| = \omega_2 \)
(2) for all $\lambda > 0$ one has $\lim_{r \to 0} r^{-2} \mathbb{1}_{\{x: \mathcal{G}(f(x), T_1 f(x)) \geq \lambda |x|\}} \cap B_r = 0$;

(3) $\lim_{r \to 0} r^{-2} \int_{B_r} |Df(x)|^2 - d^2 |dx = 0$.

Let $0 < \varepsilon < \frac{1}{4}$ be given and set $U = \{x: \mathcal{G}(f(x), T_1 f(x)) \geq \varepsilon d |x|\}$, so that by (1)

$$
\lim_{r \to 0} r^{-2} |U \cap B_r| = 0. \tag{3.5}
$$

Then for each $x \not\in U$ we have

$$
\mathcal{G}(f(x), f(0)) \leq \mathcal{G}(f(x), T_1 f(x)) + \mathcal{G}(T f(x), f(0)) \leq (1 + \varepsilon) d |x|,
$$

where we used that $\mathcal{G}(T f(x), f(0))^2 \leq \sum_{i=1}^{Q} |D f_i(0) x|^2 = d^2 |x|^2$. Hence we have for $s = (1 + \varepsilon) dr$

$$
s^{-2} \int_{\Omega} \phi \left( \frac{\mathcal{G}(f, f(0))}{s} \right) |Df|^2 \, dx \geq s^{-2} \int_{\Omega \setminus B_r} \phi \left( \frac{\mathcal{G}(f, f(0))}{s} \right) |Df|^2 \, dx
$$

$$
\geq s^{-2} \int_{\Omega \setminus B_r} \phi \left( \frac{\mathcal{G}(f, f(0))}{s} \right) d^2 \, dx - \frac{2d^2}{r^2} \int_{B_r} |Df|^2 - d^2 \, dx
$$

$$
\geq ((1 + \varepsilon)r)^{-2} \int_{\Omega \setminus B_r} \phi \left( \frac{|x|}{r} \right) - o(1) \geq ((1 + \varepsilon)r)^{-2} \int_{B_r} \phi \left( \frac{|x|}{r} \right) - o(1)
$$

$$
= (1 + \varepsilon)^{-2} \int \phi(x) \, dx - o(1),
$$

where in the next to last inequality we used (3), in the last inequality we used (3.6) and (1), and in the last equality we used (3.5). \qed

3.2. 2-d stationary maps are average conformal. Using an analogue of the “Hopf differential” we show that 2-dimensional stationary maps can be made average conformal. Similar ideas have been used in [12, Theorem 3.2], [10] and [11, Proposition 1], in chronological order.

We define the average Hopf differential of a map $f: \mathbb{C} \to \mathcal{A}_Q$ by

$$
\sum_{i=1}^{Q} \partial_\nu f_i \partial_\nu f = \sum_{i=1}^{Q} |D_1 f_i(z)|^2 - |D_2 f_i(z)|^2 - 2iD_1 f_i(z) \cdot D_2 f_i(z),
$$

and we observe that a map is average conformal if its average Hopf differential is zero.

Proposition 3.4 (Stationarity implies conformality). Let $f \in W^{1,2}(B_1, \mathcal{A}_Q)$ be stationary with respect to the Dirichlet energy, then there exists an harmonic function $v \in C^\infty(B_1, \mathbb{R}^2)$ such that

$$
F = f \oplus (v_1 e_{n+1} + v_2 e_{n+2}) = \sum_{i=1}^{Q} [(f_i, v)] \in W^{1,2}(B_1, \mathcal{A}_Q(\mathbb{R}^{n+2})). \tag{3.7}
$$

is average conformal. In particular $F$ is stationary with respect to the Dirichlet energy and moreover

$$
\int_{B_r} |DF|^2 \, dx \leq \int_{B_r} |Df|^2 \, dx + C_Q r^2 \int_{B_1} |Df|^2 \, dx \quad \forall B_r \subset B_\frac{1}{2} . \tag{3.8}
$$
Proof. We can just repeat the proof suggested by Gruter in [10] with a modification to control the energy. It is well-known that the inner variation can be written as
\[ \int \Phi(z) \partial_z \eta = 0, \quad \forall \eta \in C_0^\infty(\Omega, \mathbb{C}), \]
where \( \Phi(z) = \sum_{i=1}^Q |D_1 f_i(z)|^2 - |D_2 f_i(z)|^2 - 2iD_1 f_i(z) \cdot D_2 f_i(z) \) is the average Hopf differential of \( f \). In particular this implies that \( \Phi(z) \) is holomorphic. Then let \( \Psi \) be a primitive to \( \Phi \) and set \( E^2 = \int_{\Omega} |Df|^2 \) to be the Dirichlet energy of \( f \), and define the function
\[ v = v_1 + iv_2 = \frac{1}{2\sqrt{Q}} E \bar{z} - \frac{1}{2\sqrt{Q}} E \Psi. \]
Notice that \( v \) is harmonic, since \( \Phi \) is holomorphic. Reasoning as in [10], the Hopf differential of \( v \) is given by
\[ \partial_z v \partial_{\bar{z}} v = \partial_{\bar{z}} v \partial_z v = -\frac{1}{4Q} \Phi. \]
Furthermore its Dirichlet energy is calculated to be for any \( \Omega' \subset \Omega \)
\[ \int_{\Omega'} |\partial_z v|^2 + |\partial_{\bar{z}} v|^2 = \frac{E^2}{2Q} |\Omega| + \frac{1}{2Q E} \int_{\Omega'} |\Phi|^2 \leq \frac{E^2}{2Q} |\Omega| + \frac{C}{4Q E} \left( \int_{\Omega} |\Phi| \right)^2 \leq CE^2, \]
where we used regularity for holomorphic functions and the constant \( C \) depends only on \( \Omega \) and \( \Omega' \). Finally we define \( F \) as in (3.7), and we notice that its averaged Hopf differential is vanishing pointwise since
\[ \Phi_F(z) = \Phi(z) + 4Q \partial_z v \partial_{\bar{z}} \bar{v} = 0, \]
so that \( F \) is averaged conformal.

\[ \square \]

4. Continuity, Higher integrability and Hölder regularity

In the first subsection we prove continuity of 2-d stationary maps. We then use it to prove higher integrability, while Hölder regularity is a straightforward corollary of these two results.

4.1. Continuity: proof of Theorem 1.1. By Proposition 3.4 we can assume that \( f = F \) is average conformal and still stationary.

Let \( y_0 \in B_{1/2} \) be fixed and let \( 0 < r < 1/2 \). Replacing \( f(\cdot) \) with \( f(y_0 + r \cdot) \) we can assume that \( r = 1 \) and \( y_0 = 0 \). By Fubini, there exists \( \rho \in (1, \frac{3}{2}) \) such that
\[ \int_{B_{\rho}} |Df|^2 \leq \int_{B_{\frac{3}{2}}} |Df|^2 dx =: E_0. \]
In particular a straightforward computation using Cauchy-Schwarz and Taylor expansion gives
\[ G(f(x), f(y))^2 \leq C_1 \left( \int_{B_{\frac{3}{2}}} |Df|^2 dx \right) |x - y|, \quad \forall x, y \in \partial B_{\rho}. \tag{4.1} \]

1Direct computation shows that
\[ \sum_{i,j} (2D_i f \cdot D_j f - \delta_{ij}|Df|^2) D_i \phi^j = \text{Re} \left( \Phi_f(z) \partial_z \phi \right) \]
where we have set \( \Phi_f(z) = (\partial f(z))^2 = |D_1 f(z)|^2 - |D_2 f(z)|^2 - 2iD_1 f(z) \cdot D_2 f(z) \) and used the classical identification of \( \mathbb{R}^2 \) with \( \mathbb{C} \) such that \( \phi \equiv \phi_1 + i\phi_2 \). Now choosing \( \phi \) and \( i\phi \) one obtains the claimed equation.
Claim 1: Let \( x_0 \in B_p \) be as in Lemma 3.3, with \( \Omega = B_p \), then we claim that

\[
\inf_{y \in \partial B_p} G(f(x_0), f(y))^2 \leq C Q \int_{B_{\frac{1}{2}}} |Df|^2 \, dx. \tag{4.2}
\]

Indeed, let us denote with \( r_0^2 \) the value on the left hand side. We may assume that \( r_0^2 > 0 \) otherwise there is nothing to show. Notice that \( G(f(x), f(x_0)) \geq r_0 \) for all \( x \in \partial B_p \), so that

\[
\{ x \in B_p : G(f(x), f(x_0)) < r_0 \} \subset B_p.
\]

Hence we have that

\[
c \leq \Theta_f(B_p, f(x_0), s_0) \leq \frac{1}{s_0} \int_{B_p} |Df|^2, \quad s_0 = \min\{r_0, \text{sep}(f(x_0))\}.
\]

If \( s_0 = r_0 \) the claim follows.

If not we make use of the combinatoric Lemma 2.4 as follows. Let \( S_0 = Q \llbracket \varepsilon \rrbracket \), \( S_1, \ldots, S_{q-1} \), \( S_q = \Theta \) be the sequence of points constructed there with parameter \( \varepsilon = \frac{1}{4} \). Note that we have by property (3) of Lemma 2.4

\[
\{ P \mid G(P, S_1) \leq s \} \subset \left\{ P \mid G(P, S_{k-1}) \leq \frac{1}{2} M s \right\}, \quad \text{for any } s > \frac{1}{8} \text{sep}(S_k),
\]

where \( M = 1 + 8 \beta(\frac{1}{4}, Q)^{-1} \), and therefore

\[
\Theta_f(B_p, S_k, s) \leq M^2 \Theta_f(B_p, S_{k-1}, M s), \quad \text{for any } s > \frac{1}{8} \text{sep}(S_k). \tag{4.3}
\]

Let us define the sequence of “stopping times” \( b_k \) by setting \( b_q = \frac{1}{4} \text{sep}(f(x_0)) \), and then inductively backwards for \( k < q \) by

\[
b_k = \max \left\{ M b_{k+1}, \min \left\{ \frac{1}{4} \text{sep}(S_k), s_k \right\} \right\}, \quad s_k := \min_{y \in \partial B_p} G(f(y), S_k).
\]

Furthermore let \( k^* - 1 \) the last \( k \) such that \( b_k < s_k \), so that \( b_{k^*} \geq s_{k^*} = \text{sep}(f(y_k), S_k) \), for some \( y_k \in \partial B_p \).

Now we will show by backwards induction that for each \( k_s \leq k \leq q \) we have

\[
M^{2(k-q)} c^2 \leq \Theta_f(B_p, S_k, s), \quad \text{for } M b_{k+1} \leq s \leq b_k. \tag{4.4}
\]

This is obtained combining (4.3) with the the energy monotonicity, Proposition 3.2. Note that (4.4) implies that

\[
b_k^2 M^{2(k-q)} c^2 \leq \int_{B_p} |Df|^2 \, dx, \quad \forall k \geq k^*,
\]

where we used that

\[
\{ x \in B_p : G(f, S_k) < s_k \} \subset B_p.
\]

Using (3) of Lemma 2.4 and the fact that \( \text{sep}(S_k) \leq 4 b_k \), for \( k^* - 1 \leq k \leq q \), by the choice of \( k^* \), we then conclude

\[
G(f(x_0), f(y_k)) \leq \sum_{l \geq k^* + 1} G(S_l, S_{l-1}) + G(f(y_k), S_{k^*}) \leq \frac{4}{\beta(\varepsilon, Q) Q} \sum_{l \geq k^*} b_k \leq C \left( \int_{B_p} |Df|^2 \, dx \right)^{\frac{1}{2}}.
\]
Claim 2: For a.e. \( x \in B_\rho \) we have (4.2).

Fix \( y_1 \in \partial B_\rho \) and note that for each \( x_0 \in B_\rho \) as in Claim 1 or any other \( x_0 \in \partial B_\rho \) we have due to Claim 1 and (4.1) that

\[
\mathcal{G}(f(x_0), f(y_1))^2 \leq C \int_{B^{1/2}_\rho} |Df|^2 \, dx = CE_0.
\]

Consider the \( W^{1,1} \) function defined on \( \overline{B}_\rho \) by

\[
g(x) = ((\mathcal{G}(f(x), f(y_1)) - 2CE_0)^+)^2.
\]

We clearly have that \( g = 0 \) on \( \partial B_\rho \) and due to Claim 1 we have for each \( x_0 \) as in Lemma 3.3

\[
Dg(x_0) = 0.
\]

On the other hand for a.e. other point \( x \in B_\rho \) we have \( Df(x) = 0 \) hence \( Dg(x) = 0 \), which implies \( g \equiv 0 \) proving the claim.

We can now conclude the proof of the theorem. Indeed notice that after translating and scaling back, we have proved that

\[
\mathcal{G}(f(x), f(y))^2 \leq \int_{B^{1/2}_\rho, (y_0)} |Df|^2 \, dx \text{ for all } x, y \in B_r(y_0).
\]

Combining this with (3.8), yields the conclusion. \( \square \)

4.2. Higher integrability: proof of Theorem 1.2. Theorem 1.2 will be a consequence of a Gehring’s type lemma and the following inequality. Notice the difference with [3, Proposition 5.2], which is obtained with a smoothed competitor constructed via the Almgren’s embedding, while our proof is independent of this embedding.

Lemma 4.1. There is a constant \( C = C(Q) > 0 \) such that whenever \( f \in W^{1,2}(B_2, A_Q) \) is stationary with respect to outer variations and continuous in \( B_2/4 \) then

\[
\int_{B_1 + \frac{1}{2}Q} |Df|^2 \, dx \leq C \left( \left( \int_{B_2} |Df|^2 \, dx \right)^{1/2} + \omega_f(1) \right) \left( \int_{B_2} |Df| \, dx \right)
\]

where \( \omega_f \) is a modulus of continuity for \( f \) in \( B_2/4 \), i.e. \( \mathcal{G}(f(x), f(y)) \leq \omega_f(|x - y|) \) for all \( x, y \in B_{1 - \frac{1}{4}} \).

Proof. We can assume \( \omega_f(1) > 0 \), otherwise there is nothing to prove since \( f \) would be constant. We will prove the lemma by induction on \( Q \). Let us fix a radius \( \frac{2Q - 2}{4} < r < \frac{2Q - 1}{4} \) such that

\[
\int_{\partial B_r} |Df| \leq 4Q \int_{B_1} |Df| \, dx.
\]

Base step: This follows from classical regularity theory. Subtracting a constant we may assume that \( f_{|\partial B_r} \equiv 0 \), then we have from the outer variation

\[
\int_{B_r} |Df|^2 = \int_{\partial B_r} \partial_r f \left( f - \int_{\partial B_r} f \right) \leq C \left( \sup_{x \in \partial B_r} |Df| \right) \int_{B_1} |Df| \leq C \left( \int_{B_1} |Df|^2 \right)^{1/2} \int_{B_1} |Df|.
\]

Inductive step: Notice that we can assume \( \eta \circ f = 0 \). Indeed the average \( h = \eta \circ f \) is harmonic, and so as pointed out in Remark 2.1 the map \( f \circ h \) is still stationary with respect
to the outer variations. Moreover it is still continuous, with modulus of continuity estimated by
\[ \omega_{f\circ h}(r) \leq \omega_f(r) + r \left( \int_{B_1} |Df|^2 \right)^{\frac{1}{2}} \]
since the Lipschitz constant of \( h \) in \( B_r \) is controlled by \( \left( \int_{B_1} |Dh|^2 \right)^{\frac{1}{2}} \leq \left( \int_{B_1} |Df|^2 \right)^{\frac{1}{2}} \). Since we have the pointwise identities
\[ G^2(f(x), f(y)) = G^2((f \circ h)(x), (f \circ h)(y)) + Q|h(x) - h(y)|^2 \]
\[ |Df(x)|^2 = |D(f \circ h)(x)|^2 + Q|Dh(x)|^2, \]
we see that the lemma is proved if it holds for \( f \circ h \), since \( f \circ h \) has the pointwise estimates
\[ \omega_{f \circ h}(r) \leq \omega_f(r) + r \left( \int_{B_1} |Df|^2 \right)^{\frac{1}{2}}. \]

Let \( M > 0 \) be a large number chosen later and let
\[ D = \sup_{x \in B_r} |f(x)| = |f(x_0)|^2. \]
for some \( x_0 \in \overline{B}_r \).

If \( D < M \omega_f(1) \), we can just argue as above using the outer variations:
\[ \int_{B_r} |Df|^2 = \int_{\partial B_r} \partial_r f \cdot f \leq M \omega_f(1) \int_{B_1} |Df|. \]

If \( D > M \omega_f(1) \), we apply Lemma 2.3 with \( T = f(x_0) \) and \( \varepsilon = \frac{1}{16} \) and obtain a point \( S \)
such that
\[ \beta(\varepsilon, Q)D \leq \mathrm{sep}(S) \quad \text{and} \quad G(f(x_0), S) \leq \frac{1}{16} \mathrm{sep}(S). \]
Using this and triangular inequality, we have for \( M \) sufficiently large
\[ G(f(x), S) \leq 2 \omega_f(1) + \frac{1}{16} \mathrm{sep}(S) \leq \left( \frac{2}{M\beta} + \frac{1}{16} \right) \mathrm{sep}(S) \leq \frac{1}{8} \mathrm{sep}(S). \]
Hence, since \( \mathrm{sep}(S) > 0 \), as pointed out in Remark 2.2 \( f \) splits in \( B_r \), i.e. \( f = [f_1] + [f_2] \).
Furthermore the remark implies that \( \omega_{f_i} \leq \omega_f \) for \( i = 1, 2 \). Since the outer variation splits, \( f_i \) are both stationary with respect to the outer variation. Hence we can apply the lemma to \( f_i \) in \( B_r \), separately and obtain
\[ \int_{B_r} |Df_i|^2 dx \leq \frac{C}{r} \left( \frac{r^{2-n}}{\int_{B_r} |Df_i|^2 dx} \right)^{\frac{1}{2}} + \omega_{f_i}(1) \left( \int_{B_r} |Df_i|^2 dx \right)^{\frac{1}{2}}. \]
Adding on \( i = 1, 2 \) gives the result. \( \square \)

**Proof of Theorem 1.2.** First we prove that for any \( B_{2r} \subset B_1 \) we have
\[ \int_{B_r} |Df|^2 \leq C \left( \int_{B_{2r}} |Df|^2 \right)^{\frac{1}{2}} \int_{B_{2r}} |Df|. \quad (4.6) \]
Since this is scale invariant it is sufficient to prove it for \( r = 1 \). Theorem 1.1 implies that \( f \) is continuous on \( B_{2Q+1} \) with modulus of continuity controlled by
\[ \omega_f(r) \leq C \left( \int_{B_2} |Df|^2 \right)^{\frac{1}{2}} (1 + r). \]
Hence we can apply the lemma above, Lemma 4.1 and obtain the result.
Now Theorem 1.2 follows from a standard modification of [8, Theorem 6.38], with equation (6.48) there replaced with (4.6) i.e.

\[
\int_{B_R(x_0)} f^q \leq b \left( \int_{B_{2R}(x_0)} f^q \right)^{\frac{q-1}{q}} \int_{B_{2R}(x_0)} f
\]

Indeed the only place where (6.48) is used is equation (6.53), which in our case should be replaced with

\[
t^q \leq \left( \int_{Q^i_{k,j}} (\phi f)^q \right) \left( \int_{Q^i_{k,j}} (\phi f)^q \right)^{q-1} \leq \sigma^2 \left( \int_{Q^i_{k,j}} (\phi f)^q \right) t^{q-1},
\]

where the last inequality follows from equation (6.51) in [8]. The rest of the proof is exactly the same. □

5. Concentration compactness: proof of Theorem 1.4

In this section we prove Theorem 1.4. We proceed by induction on \(Q\).

Base step: Up to subtracting a constant, this follows from the regularity of harmonic functions.

Inductive step: Notice that the averages \(\eta \circ f_k\) are harmonic, by outer variation equal to 0, and moreover

\[
|Df_k|^2 = |Df_k \circ (\eta \circ f_k)|^2 + Q|\eta \circ f_k|^2,
\]

so we can assume without loss of generality that \(\eta \circ f_k = 0\) for every \(k\). We split the rest of the proof in two steps.

Step 1: We first prove the theorem under the additional assumption that

\[
|f_k(0)| \leq C < \infty. \tag{5.1}
\]

By the equicontinuity, assumption (1), we know that there exists \(f \in A_Q\), such that, up to a not relabelled subsequence, \(f_k\) converges to \(f\) in \(C^0\) and weakly in \(W^{1,2}\). Next let

\[
\mu = \lim_{k \to \infty} |Df_k|^2 - |Df|^2 \geq 0,
\]

the limit intended in the sense of measures, that is \(\mu\) is the defect measure and it is non-negative by lower semicontinuity of the energy. We are going to prove that \(\mu = 0\), which gives the desired result.

Indeed, if \(x \in \{x \in B_1 : |f| > 0\}\), then by continuity of \(f\) there exists a radius \(r > 0\), depending on \(x\), such that

\[
f(y) = \left[ f^1(y) \right] + \left[ f^2(y) \right] \quad \forall y \in B_r(x),
\]

with \(f^i \in A_{Q_i}, Q_i \geq 1\). By the uniform convergence, for \(k\) sufficiently large, we have

\[
f_k(y) = \left[ f^1_k(y) \right] + \left[ f^2_k(y) \right] \quad \forall y \in B_r(x).
\]

By Remark 2.2, we also have

\[
G^2(f, f) = G^2(f^1_k, f^1) + G^2(f^2_k, f^2),
\]
so that $|f_k^i(x)| \leq C < \infty$. In particular $(f_k^i)_k$ satisfies the inductive assumptions, including (5.1), and we have that

$$
\lim_{k \to \infty} \int_{B_r(x)} |Df_k|^2 = \int_{B_r(x)} |Df|^2 \quad i = 1, 2,
$$

so that $\mu(B_r(x)) = 0$. This implies that

$$
\mu(\{x \in B_1 : |f| > 0\}) = 0. \quad (5.2)
$$

Next suppose that $x \in \{x \in B_1 : |f| = 0\}$. Let $\theta : [0, \infty) \to [0, \infty)$ be a nonnegative function equal to 1 on $[0, 1]$ and 0 on $[2, \infty]$. Moreover let $\eta$ be a smooth function which is 1 in the ball of radius $r$, less than or equal to 1, and supported in $B_1$. Then we test the outer variation (1.1) with the test function

$$
\psi(x, u) := \eta(x) \theta \left( \frac{|u|}{\delta} \right) u,
$$
to obtain

$$
\int_{\{|f| < \delta\}} \eta |Df_k|^2 \leq \int \eta \theta \left( \frac{|f_k|}{\delta} \right) |Df_k|^2 = - \int \theta \left( \frac{|f_k|}{\delta} \right) f_k Df_k \nabla \eta - \int \eta \theta' \left( \frac{|f_k|}{\delta} \right) \frac{|f_k Df_k f_k Df_k|}{|f_k|} \leq \delta \left( \|Df_k\|_{L^2(B_1)} \|\nabla \eta\|_{L^2(B_1)} + \|Df_k\|_{L^2(B_1)}^2 \right),
$$

where the equality follows from the outer variation, the first inequality from the choice of $\eta$, and the last inequality from Hölder. Passing to the limit we obtain

$$
\mu(B_r(x) \cap \{|f| < \delta\}) \leq \mu(\{|f| > 0\}) \leq C \delta
$$

so that we can conclude

$$
\mu(\{|f| = 0\}) = \lim_{\delta \to 0} \mu(\{|f| < \delta\}) = 0,
$$

which combined with (5.2) concludes the proof of Step 1.

Step 2: It remains to consider the case $\limsup_{k \to \infty} |f_k(0)| = \infty$. We apply Lemma 2.3 to $f_k(0)$ with parameter $\varepsilon = \frac{1}{2}$ and obtain points $S_k = \sum_{k=1}^{\infty} Q_k \left[ s_k \right]$. Passing to a subsequence, we may assume that $q_k, Q_k$ do not depend on $k$. By assumptions

$$
\beta \left( \frac{1}{\sqrt{n}} Q \right) \left| f_k(0) \right| \leq \beta \left( \frac{1}{\sqrt{n}} Q \right) \text{diam}(f_k(0)) \leq \text{sep}(S_k)
$$

hence $\lim_{k \to \infty} \text{sep}(S_k) = \infty$. By the equicontinuity and Remark 2.2, for sufficiently large $k$ the functions $f_k$ split, i.e. $f_k = \sum_{k=1}^{\infty} \left[ f_k^i \right]$, with $f_k^i$ themselves satisfying the assumptions. Therefore we can apply the inductive step once again and conclude.

□

Corollary 5.1 (Compactness for 2-d stationary maps). Let $f_k \in W^{1,2}(B_R, A_Q(\mathbb{R}^n))$, $B_R \subset \mathbb{R}^2$, be a sequence of stationary maps satisfying

$$
f_k(0) = Q[0] \quad \text{and} \quad \sup_k \int_{B_R} |Df_k|^2 < \infty,
$$

...
then there exists a stationary map \( f \in W^{1,2}(B_R, \mathcal{A}_Q(\mathbb{R}^n)) \) such that for every \( 0 < r < R \)

\[
\lim_{k \to \infty} \sup_{B_r} \mathcal{G}(f_k(x), f(x)) = 0 \quad \text{and} \quad \lim_{k \to \infty} \int_{B_r} |Df_k| - |Df|^2 = 0.
\]

Proof. Since all the maps \( f_k \) are stationary with uniformly bounded Dirichlet energy they are uniformly Hölder continuous, compare 1.5. Furthermore the assumption \( f_k(0) = Q[0] \) ensures together with Hölder regularity ensures that the functions don’t split. Hence we are in the setting of concentration compactness without splitting and the claimed convergence follows. But since the inner and outer variation are continuous with respect to strong \( W^{1,2} \) convergence it follows that \( f \) is stationary.

\[\square\]

6. Analysis of the singular set

In the first subsection we show how given a stationary map we can construct a new one whose points of cardinality \( k \) are collapsed. In the second section we recall the frequency function and its consequence in our setting. Finally in the last subsection we conclude the proof of Theorem 1.5.

6.1. Splitting of a stationary map. Given a stationary map we construct a new one whose points of fixed cardinality are collapsed as follows:

**Proposition 6.1.** Let \( f \) be a stationary map with \( \text{card}(f(0)) = K \), that is

\[
f(0) = \sum_{k=1}^{K} Q_t \llbracket p_t \rrbracket, \quad \text{with} \quad p_k \neq p_l, \forall l \neq k.
\]

Then there exist \( r_0 > 0 \) and

1. harmonic functions \( h_k : B_{r_0} \to \mathbb{R}^n \), with \( h_k(0) = p_k \),
2. average free \( Q_k \)-valued maps \( g_k \in W^{1,2}(B_{r_0}, \mathcal{A}_{Q_k}(\mathbb{R}^n)) \) that are stationary with respect to the outer variations

such that

\[
f = \sum_{k=1}^{K} h_k \oplus g_k \quad \text{in} \quad B_{r_0}.
\]

Furthermore, the new map

\[
\tilde{f} = \sum_{k=1}^{K} g_k \in W^{1,2}(B_{r_0}, \mathcal{A}_Q(\mathbb{R}^n))
\]

has the properties that

1. \( \tilde{f} \) is stationary with respect to inner and outer variations;
2. \( f(0) = Q[0] \);
3. for all \( z \in B_{r_0} \) with \( \text{card}(\text{supp} f(z)) = k \) we have \( \tilde{f}(z) = Q[0] \);
4. \( \text{Sing}(f) \cap B_{r_0} \subset \text{Sing}(\tilde{f}) \cap B_r \);

Proof. Due to the continuity of \( f \) we may choose \( r_0 > 0 \) such that

\[
\mathcal{G}(f(x), f(0)) < \frac{1}{16} \text{sep}(f(0)) \quad \forall |z| < r_0.
\]
This implies that $f$ splits, see for instance Remark 2.2, i.e. there are $Q_k$-valued continuous functions $f_k$ on $B_{r_0}$ such that

$$f(z) = \sum_{k=1}^{K} f_k(z) \quad z \in B_{r_0}.$$ 

Since $f$ is stationary with respect to outer variations and $\text{dist}(\text{spt}(f_k(z)), \text{spt}(f_j(z))) > \frac{1}{16} \text{sep}(f(0))$ on $B_{r_0}$, we deduce that each $f_k$ is itself stationary w.r.t to the outer variations, i.e.

$$\mathcal{O}(f_k, \psi) = 0 \quad \forall k \in \mathbb{N}.$$ 

In particular this implies that the averages $h_k = \eta \circ f_k$ are all harmonic in $B_{r_0}$, i.e. $\Delta h_k = 0$.

Now we set $g_k = f_k \circ h_k$ and we observe that, by Remark 2.1, each $g_k$ satisfies

$$\mathcal{O}(g_k, \psi) = \mathcal{O}(f_k, \psi_{h_k}) = 0 \quad \text{for all outer variations } \psi$$

$$\mathcal{I}(g_k, \phi) = \mathcal{I}(f_k, \phi) \quad \text{for all domain variations } \phi,$n

where we have used that the remainders in Remark 2.1 vanish since $h_k$ is harmonic. In particular we have established (1), (2) and (6.1).

Next we notice that $\tilde{f}$ is stationary since

$$0 = \mathcal{I}(f, \phi) = \sum_{k=1}^{K} \mathcal{I}(f_k, \phi) = \sum_{k=1}^{K} \mathcal{I}(g_k, \phi) = \mathcal{I}(\tilde{f}, \phi).$$

and analogously, given any admissible outer variation $\psi \in C^\infty(\Omega_x \times \mathbb{R}^n; \mathbb{R}^n)$, we have

$$\mathcal{O}(\tilde{f}, \psi) = \sum_{k=1}^{K} \mathcal{O}(g_k, \psi) = 0.$$ 

Hence (4) is proven. (5) follows by construction. To deduce (6) we only observe that due to (6.1) we have $\text{card}(\text{spt}(f(z))) \geq k$ for all $z \in B_{r_0}$ and equality implies that $g_k(z) = Q_k[0]$ for all $k$, since the $g_k$ are average free. Finally (7) is an immediate consequence of the combination of Lemma 6.2 and Corollary 6.3 below.

\begin{lemma}
Let $f : \Omega \to \mathcal{A}_Q(\mathbb{R}^n)$ be continuous on a simply connected domain $\Omega \subset \mathbb{R}^m$ then the following are equivalent:

(1) $x \mapsto \text{card}(\text{spt}(f(x)))$ is constant.

(2) there are maps $g_j \in C^0(\Omega, \mathbb{R}^n)$, $j = 1, \ldots, Q$, such that $f = \sum_{j=1}^{Q} [g_j]$ and either $g_j \equiv g_i$ or $g_j(x) \neq g_i(x)$, for every $x \in \Omega$.

\end{lemma} 

\begin{proof}
Clearly (2) implies (1). To show that (1) implies (2), we assume that card(spt($f(x)$)) $\equiv K$ on $\Omega$.

First we claim that each $x \in \Omega$ has a neighbourhood $U_x \subset \Omega$ where $f$ admits a decomposition as described in (2).

This is true, since $f(x) = \sum_{j=1}^{K} Q_j [t_j]$, with $t_i \neq t_j$ for $i \neq j$, and so by the continuity of $f$ we can apply Remark 2.2 to find in a neighbourhood $U_x$ a decomposes into $K$ maps, i.e. there are continuous maps $\tilde{g}_j : U_x \mapsto \mathcal{A}_{Q_j}(\mathbb{R}^n)$ such that $f = \sum_{j=1}^{K} \tilde{g}_j$ and $\text{spt}(\tilde{g}_i) \cap \text{spt}(\tilde{g}_j) = \emptyset$ for $i \neq j$. This implies that card(spt($f(y)$)) $\geq K$ on $U_x$. Since by assumption card(spt($f(y)$)) $= K$, we deduce that $\tilde{g}_j = Q_j [g_j]$ on $U_x$, thus proving the claim.

\end{proof}
Hence we can consider the set \( \text{spt}(G_f(\Omega)) \), that is the support of the graph of \( f \), as a covering space of the domain \( \Omega \). Since \( \Omega \) is simply connected, we obtain the desired decomposition.

**Corollary 6.3.** Let \( f : \Omega \to A_Q(\mathbb{R}^n) \) be continuous and stationary with respect to the outer variation. Then the following are equivalent

1. \( y \mapsto \text{card}(\text{spt}(f(y))) \) is constant in a neighbourhood of \( x \),
2. \( x \in \text{Reg}(f) \).

**Proof.** That (2) implies (1) follows directly from the implication (2) implies (1) in Lemma 6.2.

To prove the converse we may assume that \( U_x \subset \Omega \) is a simply connected domain where \( y \mapsto \text{card}(\text{spt}(f(y))) \) is constant. Hence we find a decomposition as described in (2) of Lemma 6.2. But then we can use the outer variation to deduce that each of these maps must be harmonic.

**6.2. The frequency function and blow-ups.** Let \( f \) be a \( Q \)-valued function, and assume that \( f(0) = Q[0] \) and \( \int_{B_\rho} |Df|^2 > 0 \). Following [7, section 3.4], we define the quantities

\[
D_{x,f}(r) = \int_{B_r(x)} |Df|^2, \quad H_{x,f}(r) = \int_{\partial B_r(x)} |u|^2 \quad \text{and} \quad I_{x,f}(r) = \frac{r D_{x,f}(r)}{H_{x,f}(r)}.
\]

Since the proof concerning the monotonicity of the frequency function only relies on the outer and inner variation formulas, (1.1)-(1.2), we have

**Theorem 6.4.** [7, Theorem 3.15] Let \( f \) be stationary and \( x \in \Omega \). Either there exists \( \rho \) such that \( f|_{B_\rho(x)} = Q[0] \) or \( I_{x,f}(r) \) is an absolutely continuous non-decreasing function on \([0, \text{dist}(x,\partial \Omega)]\).

Next we define the blow-ups of \( f \) at \( y \) by \( f_{y,\rho}(x) := \frac{\rho^{\frac{m-2}{2}} f(\rho x + y)}{\sqrt{D_{y,f}(\rho)}} \). Then we have the following

**Theorem 6.5** ([7, Theorem 3.19]). Let \( f \in W^{1,2}(B_1, A_Q) \) be a stationary map, with \( B_1 \subset \mathbb{R}^2 \). Assume \( f(0) = Q[0] \) and \( D_f(\rho) > 0 \) for every \( \rho \leq 1 \). Then, for any sequence \( \{f_{\rho_k}\} \) with \( \rho_k \downarrow 0 \), a subsequence, not relabeled, converges locally uniformly to a function \( g : \mathbb{R}^2 \to A_Q \) with the following properties:

1. \( D_g(1) = 1 \) and \( g \) is stationary;
2. \( g(x) = |x|^{\alpha} g \left( \frac{x}{|x|} \right) \), where \( \alpha = I_{0,f}(0) > 0 \) is the frequency of \( f \) at 0.

**Proof.** The proof follows from the same arguments as in the proof of [7, Theorem 3.19], replacing Theorem 3.9 about Hölder continuity of minimizers with Corollary 1.3 on the Hölder continuity of stationary maps, and Proposition 3.20 on the compactness of minimizers, with our Corollary 5.1 for stationary maps.

Finally we will need the following elementary lemma on the classification of homogeneous stationary map on \( \mathbb{R} \).

**Lemma 6.6.** Let \( h \in W^{1,2}(\mathbb{R}, A_Q(\mathbb{R}^n)) \) be an \( \alpha \)-homogeneous map that is stationary with respect to the outer variation then
(1) $\alpha = 1$
(2) there are two points $T_{\pm} \in \mathcal{A}_Q(\mathbb{R}^n)$ such that

$$h(t) = \begin{cases} tT_+ & \text{if } t > 0 \\ tT_- & \text{if } t \leq 0 \end{cases}.$$ 

Furthermore $h$ is stationary w.r.t the inner variations if and only if $|T_+| = |T_-|$.

In general the problem of classifying homogeneous stationary maps seems rather difficult, as illustrated by the following example.

**Remark 6.7.** Let us fix two points $T_{\pm} \in \mathcal{A}_Q(\mathbb{R}^n)$ such that $|T_+| \neq |T_-|$. Consider the 1-homogeneous stationary map

$$h(x) = \begin{cases} xT_+ & \text{if } x > 0 \\ xT_- & \text{if } x \leq 0 \end{cases}.$$ 

and its extension to $\mathbb{C}$ by $h(x,y) = h(x)$. Finally we may consider for every $Q_1 > 1$ the map

$$f(z) = \sum_{w:Q_1 = z} h(z).$$

It is an exercise to check that $f$ is stationary itself.

**6.3. Estimate on the size of the singular set: proof of Theorem 1.5.** Having established all the tools we can combine them to show the estimate on the size of the singular set of a stationary map $f$, i.e. $\dim(\text{Sing}(f)) \leq 1$. In fact with the established tools the argument is nowadays almost “classical” and is very close to the original argument presented in [7, section 3.6]. Hence we only outline the argument and highlight the needed adaptations.

Having Corollary 6.3 in mind, it is natural to decompose the singular set into the subsets

$$S_k = \text{Sing}(f) \cap \{\text{card}(f) = k\}, \quad \text{for } k = 1, \ldots, Q.$$

Note that since $x \mapsto \text{card}(f(x))$ is lower semicontinuous, all the sets $S_k$ are relatively open in $\text{Sing}(f)$. Furthermore $S_Q = \emptyset$ and $S_1$ corresponds to the set $\Sigma_Q$ studied in [7, section 3.6].

Assume now by contradiction that $\mathcal{H}^t(\text{Sing}(f)) > 0$ for some $t > 1$. This implies that there is at least one $k < Q$ with $\mathcal{H}^t(S_k) > 0$. Hence there is a point $x_0 \in \Omega$ with positive density, i.e.

$$\limsup_{r \to 0} \frac{\mathcal{H}^t(S_k \cap B_r(x_0))}{r^t} > 0.$$ 

After translation we may assume that $x_0 = 0$ and note that, due to Corollary 6.3, $x_0$ cannot be an interior point of $S_k$. We apply Proposition 6.1 to $f$ around 0 and obtain the map $\tilde{f}$, such that $\tilde{f} \neq Q [0]$, since 0 is not an interior point of $S_k$, and $\eta \circ \tilde{f} = 0$ by (2). Furthermore due to (6) and (7) we have

$$\limsup_{r \to 0} \frac{\mathcal{H}^t(\text{Sing}(\tilde{f}) \cap \{\tilde{f} = Q [0]\}) \cap B_r}{r^t} > 0.$$ 

(6.4)

Now the conclusion follows by the same arguments presented in [7, Subsection 3.6.2]: let $r_k \downarrow 0$ be a subsequence realising the lim sup and consider the corresponding blow-up sequence.
\[ \hat{f}_{\rho_0}. \] By Theorem 6.5, we find a nontrivial \( \alpha \)-homogeneous stationary tangent map \( g: \mathbb{R}^2 \to A_Q(\mathbb{R}^n) \). Moreover by (6.4), \( g \) satisfies
\[ \mathcal{H}^t_{\infty}(B_1 \cap \{g = Q[0]\}) > 0. \]
Therefore we may pick \( y \in \partial B_1 \cap \partial \{g = Q[0]\} \) once again with positive \( \mathcal{H}^t_{\infty} \)-density. Since this must be a singular point we can perform a second blow-up as described in [7, Lemma 3.24]. However the corresponding tangent function \( k \in W^{1,2}(\mathbb{R}, A_Q(\mathbb{R}^n)) \) is homogeneous and such that \( \mathcal{H}^t_{\infty}(B_1 \cap \{k = Q[0]\}) > 0 \), contradicting Lemma 6.6. This proves the estimate on the singular set.

**Remark 6.8.** It is very likely that applying the arguments presented in [2] to any stationary map \( f \) would actually lead to Minkowski-type bounds
\[ L^2(\overline{B}_{\rho_0} \cap B_r(\{f = Q[0]\})) \leq C(\hat{f}, \rho_0) r. \]
Additionally we expect that \( \{f = Q[0]\} \) is countable 1-rectifiable in \( B_{\rho_0} \). In fact, applying the reasoning to \( \hat{f} \) should lead to the rectifiability of \( S_k \cap B_{\rho_0} \) and therefore to the rectifiability of the singular set of any 2-d stationary map \( f \).

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