On the local metric dimension of corona product graphs

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Abstract

A vertex \( v \in V(G) \) is said to distinguish two vertices \( x, y \in V(G) \) of a nontrivial connected graph \( G \) if the distance from \( v \) to \( x \) is different from the distance from \( v \) to \( y \). A set \( S \subset V(G) \) is a local metric generator for \( G \) if every two adjacent vertices of \( G \) are distinguished by some vertex of \( S \). A minimum local metric generator is called a local metric basis for \( G \) and its cardinality, the local metric dimension of \( G \). In this paper we study the problem of finding exact values for the local metric dimension of corona product of graphs.

Keywords: Metric generator; metric dimension; local metric set; local metric dimension, corona product graph.

1 Introduction

A generator of a metric space is a set \( S \) of points in the space with the property that every point of the space is uniquely determined by the distances from the elements of \( S \). Given a simple and connected graph \( G = (V, E) \), we consider the function \( d_G : V \times V \to \mathbb{R}^+ \), where \( d_G(u, v) \) is the length of a shortest path between \( u \) and \( v \). Clearly, \( (V, d_G) \) is a metric space, i.e., \( d_G \) satisfies \( d_G(x, x) = 0 \) for all \( x \in V \), \( d_G(x, y) = d_G(y, x) \) for all \( x, y \in V \) and \( d_G(x, y) \leq d_G(x, z) + d_G(z, y) \) for all \( x, y, z \in V \). A vertex \( v \in V \) is said to distinguish two vertices \( x \) and \( y \) if \( d_G(v, x) \neq d_G(v, y) \). A set \( S \subset V \) is said to be a metric generator for \( G \) if any pair of vertices of \( G \) is distinguished by some element of \( S \). A minimum generator is called a metric basis, and its cardinality the metric dimension of \( G \), denoted by \( \dim(G) \).

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [18], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced by Harary and Melter in [9], where metric generators were called resolving sets. Applications of this invariant to the navigation of robots in networks are discussed in [13] and applications to chemistry in [11, 12]. This invariant was studied further in a number of other papers including, for instance [11, 14, 15, 16, 19]. Several variations of metric generators including resolving dominating sets [13], independent resolving sets [6], local metric sets [15], strong resolving sets [17], etc. have since been introduced and studied.
In this article we are interested in the study of local metric generators, also called local metric sets [15]. A set \( S \) of vertices in a connected graph \( G \) is a local metric generator for \( G \) if every two adjacent vertices of \( G \) are distinguished by some vertex of \( S \). A minimum local metric generator is called a local metric basis for \( G \) and its cardinality, the local metric dimension of \( G \), is denoted by \( \dim_l(G) \). The following main results were obtained in [15].

**Theorem 1.** [15] Let \( G \) be a nontrivial connected graph of order \( n \). Then \( \dim_l(G) = n - 1 \) if and only if \( G \) is the complete graph of order \( n \) and \( \dim_l(G) = 1 \) if and only if \( G \) is bipartite.

The clique number \( \omega(G) \) of a graph \( G \) is the order of a largest complete subgraph in \( G \).

**Theorem 2.** [15] Let \( G \) be connected graph of order \( n \). Then \( \dim_l(G) = n - 2 \) if and only if \( \omega(G) = n - 1 \).

In this paper we study the local metric dimension of corona product graphs. We begin by giving some basic concepts and notations. For two adjacent vertices \( u, v \) of \( G = (V, E) \) we use the notation \( u \sim v \). For a vertex \( v \) of \( G \), \( N_G(v) \) denotes the set of neighbors that \( v \) has in \( G \), i.e., \( N_G(v) = \{ u \in V : u \sim v \} \). The set \( N_G(v) \) is called the open neighborhood of \( v \) in \( G \) and \( N_G[v] = N_G(v) \cup \{ v \} \) is called the closed neighborhood of \( v \) in \( G \). The degree of a vertex \( v \) of \( G \) will be denoted by \( \delta_G(v) \), i.e., \( \delta_G(v) = |N_G(v)| \). Given a set \( S \subset V \), we denote by \( \langle S \rangle_G \) the subgraph of \( G \) induced by \( S \) and by \( N_G(S) = \cup_{v \in S} N_G(v) \) the open neighborhood of \( S \). In particular, if \( S = \{ x \} \) we will use the notation \( \langle x \rangle \) instead of \( \{ \{ x \} \} \).

Let \( G \) and \( H \) be two graphs of order \( n \) and \( n_1 \), respectively. Recall that the corona product \( G \odot H \) is defined as the graph obtained from \( G \) and \( H \) by taking one copy of \( G \) and \( n \) copies of \( H \) and joining by an edge each vertex from the \( i \)-th copy of \( H \) with the \( i \)-th vertex of \( G \). We will denote by \( V = \{ v_1, v_2, \ldots, v_n \} \) the set of vertices of \( G \) and by \( H_i = (V_i, E_i) \) the copy of \( H \) such that \( v_i \sim x \) for every \( x \in V_i \). The join \( G + H \) is defined as the graph obtained from disjoint graphs \( G \) and \( H \) by taking one copy of \( G \) and one copy of \( H \) and joining by an edge each vertex of \( G \) with each vertex of \( H \). Notice that the corona graph \( K_1 \odot H \) is isomorphic to the join graph \( K_1 + H \). The vertex of \( K_1 \) will be denoted by \( v \).

## 2 General results

To begin with, we consider some straightforward cases. If \( H \) is an empty graph, then \( K_1 \odot H \) is a star graph and \( \dim_l(K_1 \odot H) = 1 \). Moreover, if \( H \) is a complete graph of order \( n \), then \( K_1 \odot H \) is a complete graph of order \( n + 1 \) and \( \dim_l(K_1 \odot H) = n \).

**Theorem 3.** Let \( G \) be a connected nontrivial graph. For any empty graph \( H \),

\[
\dim_l(G \odot H) = \dim_l(G).
\]

**Proof.** Let \( B \) be a local metric basis for \( G \). Since in \( G \odot H \) every pair of adjacent vertices of \( G \) is distinguished by some vertex of \( B \) and every vertex of \( B \) distinguishes every pair of adjacent vertices composed by one vertex of \( G \) and one vertex of \( H \), we conclude that \( B \) is a local metric generator for \( G \odot H \).

Now, suppose that \( A \) is a local metric basis for \( G \odot H \) such that \( |A| < |B| \). Since \( H \) is an empty graph, if there exists \( x \in A \cap V_i \), for some \( i \), then the pairs of vertices of \( G \odot H \) which are distinguished by \( x \) can be distinguished also by \( v_i \). So, we consider the set \( A' \) obtained from \( A \) by replacing by \( v_i \) each vertex \( x \in A \cap V_i \), where \( i \in \{ 1, \ldots, n \} \). Thus, \( A' \) is a local metric generator for \( G \) and \( |A'| \leq |A| < |B| = \dim_l(G) \), which is a contradiction. Therefore, \( B \) is a local metric basis for \( G \odot H \). \( \square \)
We present now the main result on the local metric dimension of corona graphs \( G \odot H \) for the case where \( H \) is a non-empty graph.

**Theorem 4.** Let \( H \) be a non-empty graph. The following assertions hold.

(i) If the vertex of \( K_1 \) does not belong to any local metric basis for \( K_1 + H \), then for any connected graph \( G \) of order \( n \),

\[
\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).
\]

(ii) If the vertex of \( K_1 \) belongs to a local metric basis for \( K_1 + H \), then for any connected graph \( G \) of order \( n \geq 2 \),

\[
\dim_l(G \odot H) = n(\dim_l(K_1 + H) - 1).
\]

**Proof.** If \( n = 1 \), then \( G \odot H \cong K_1 + H \) and we are done. We consider \( n \geq 2 \). Let \( S_i \) be a local metric basis for \( \langle v_i \rangle + H_i \) and let \( S'_i = S_i - \{v_i\} \). Note that \( S'_i \neq \emptyset \) because \( H_i \) is a non-empty graph and \( v_i \) does not distinguish any pair of adjacent vertices belonging to \( V_i \).

In order to show that \( X = \bigcup_{i=1}^{n} S'_i \) is a local metric basis for \( G \odot H \) we differentiate the following cases for two adjacent vertices \( x, y \).

Case 1. \( x, y \in V_i \). Since \( v_i \) does not distinguish \( x, y \), there exists \( u \in S'_i \) such that \( d_{G \odot H}(x, u) = d_{\langle v_i \rangle + H_i}(x, u) \neq d_{\langle v_i \rangle + H_i}(y, u) = d_{G \odot H}(y, u) \).

Case 2. \( x \in V_i \) and \( y = v_i \). For \( u \in S'_j, j \neq i \), we have \( d_{G \odot H}(x, u) = 1 + d_{G \odot H}(y, u) > d_{G \odot H}(y, u) \).

Case 3. \( x = v_i \) and \( y = v_j \). For \( u \in S'_j \), we have \( d_{G \odot H}(x, u) = 2 = d_{G \odot H}(x, y) + 1 > 1 = d_{G \odot H}(y, u) \).

Hence, \( X \) is a local metric basis for \( G \odot H \).

Now we shall prove (i). If the vertex of \( K_1 \) does not belong to any local metric basis for \( K_1 + H \), then \( v_i \not\in S_i \) for every \( i \in \{1, \ldots, n\} \) and, as a consequence,

\[
\dim_l(G \odot H) \leq |X| = \sum_{i=1}^{n} |S'_i| = \sum_{i=1}^{n} \dim_l(\langle v_i \rangle + H_i) = n \cdot \dim_l(K_1 + H).
\]

Now we need to prove that \( \dim_l(G \odot H) \geq n \cdot \dim_l(K_1 + H) \). In order to do this, let \( W \) be a local metric basis for \( G \odot H \) and let \( W_i = V_i \cap W \). Consider two adjacent vertices \( x, y \in V_i - W_i \). Since no vertex \( a \in W - W_i \) distinguishes the pair \( x, y \), there exists \( u \in W_i \) such that \( d_{\langle v_i \rangle + H_i}(x, u) = d_{G \odot H}(x, u) \neq d_{G \odot H}(y, u) = d_{\langle v_i \rangle + H_i}(y, u) \). So we conclude that \( W_i \cup \{v_i\} \) is a local metric generator for \( \langle v_i \rangle + H_i \). Now, since \( v_i \) does not belong to any local metric basis for \( \langle v_i \rangle + H_i \), we have that \( |W_i| + 1 = |W_i \cup \{v_i\}| > \dim_l(\langle v_i \rangle + H_i) \) and, as a consequence, \( |W_i| \geq \dim_l(\langle v_i \rangle + H_i) \). Therefore,

\[
\dim_l(G \odot H) = |W| \geq \sum_{i=1}^{n} |W_i| \geq \sum_{i=1}^{n} \dim_l(\langle v_i \rangle + H_i) = n \cdot \dim_l(K_1 + H),
\]

and the proof of (i) is complete.

Finally, we shall prove (ii). If the vertex of \( K_1 \) belongs to a local metric basis for \( K_1 + H \), then we assume that \( v_i \in S_i \) for every \( i \in \{1, \ldots, n\} \). Suppose that there exists \( B \) such that \( B \) is a local metric basis for \( G \odot H \) and \( |B| < |X| \). In such a case, there exists \( i \in \{1, \ldots, n\} \) such that the set \( B_i = B \cap V_i \) satisfies \( |B_i| < |S'_i| \). Now, since no vertex of \( B - B_i \) distinguishes the
pairs of adjacent vertices belonging to \( V \), the set \( B_i \cup \{ v_i \} \) must be a local metric generator for \( \langle v_i \rangle + H_i \). So, \( \text{dim}_l(\langle v_i \rangle + H_i) \leq |B_i| + 1 < |S'_i| + 1 = |S_i| = \text{dim}_l(\langle v_i \rangle + H_i) \), which is a contradiction. Hence, \( X \) is a local metric basis for \( G \) and, as a consequence, 

\[
\text{dim}_l(G \circ H) = |X| = \sum_{i=1}^{n} |S'_i| = \sum_{i=1}^{n} (\text{dim}_l(\langle v_i \rangle + H_i) - 1) = n(\text{dim}_l(K_1 + H) - 1).
\]

The proof of (ii) is now complete. \( \square \)

As a direct consequence of Theorem 4 we obtain the following results.

**Corollary 5.** The following assertions hold for any connected graph \( G \) of order \( n \geq 2 \).

(i) For any integer \( t \geq 2 \), \( \text{dim}_l(G \circ K_t) = n(t - 1) \).

(ii) For any positive integers \( r \) and \( s \), \( \text{dim}_l(G \circ K_{r,s}) = n \).

(iii) Let \( t \geq 4 \) be an integer. If \( t \equiv 1(4) \), then \( \text{dim}_l(G \circ P_t) = n \left\lceil \frac{t}{4} \right\rceil \) and if \( t \not\equiv 1(4) \), then \( \text{dim}_l(G \circ P_t) = n \left\lceil \frac{t}{4} \right\rceil \).

(iv) For any integer \( t \geq 4 \), \( \text{dim}_l(G \circ C_t) = n \left\lceil \frac{t}{4} \right\rceil \).

**Proof.**

(i) If \( H \cong K_t \), then \( K_1 + K_t \cong K_{t+1} \) and the vertex of \( K_1 \) can belong to a local metric basis for \( K_1 + K_t \). Thus,

\[
\text{dim}_l(G \circ K_t) = n \cdot (\text{dim}_l(K_{t+1}) - 1) = n \cdot (t - 1).
\]

(ii) If \( H = (U_1 \cup U_2, E) \cong K_{r,s} \) then for every \( a \in U_1 \) (or \( a \in U_2 \)) the set \( \{a, v\} \) is a local metric basis for \( \langle v \rangle + H \). Therefore,

\[
\text{dim}_l(G \circ K_{r,s}) = n \cdot (\text{dim}_l(K_1 + K_{r,s}) - 1) = n.
\]

(iii) Notice that a set \( B \) is a local metric basis for \( K_1 + P_t \) if and only if for every pair of adjacent vertices \( x, y \in V(P_t) \), vertex \( x \) is adjacent to an element of \( B \) or vertex \( y \) is adjacent to an element of \( B \). Thus, for any subgraph \( H' \) of \( P_t \) isomorphic to \( P_t \), we have \( B \cap V(H') \neq \emptyset \). With this observation in mind, we consider the following two cases.

Case 1. \( 4 \leq t \leq 5 \). In this case we have that \( \text{dim}_l(\langle v \rangle + P_t) = 2 \) and \( v \) belongs to any local metric basis. Thus, \( \text{dim}_l(G \circ P_t) = n = n \left\lceil \frac{t}{4} \right\rceil \).

Case 2. \( t \geq 6 \). For \( t = 4k + r \), where \( 0 \leq r \leq 3 \), we obtain

\[
\text{dim}_l(K_1 + P_t) = \begin{cases} 
  k, & \text{if } r = 0 \text{ or } r = 1 \\
  k + 1, & \text{if } r = 2 \text{ or } r = 3
\end{cases}
\]

(1)

Therefore, since in this case vertex \( v \) does not belong to any local metric basis for \( \langle v \rangle + P_t \), we obtain

\[
\text{dim}_l(G \circ P_t) = n \cdot \text{dim}_l(K_1 + P_t) = \begin{cases} 
  n \cdot \left\lceil \frac{t}{4} \right\rceil, & \text{if } t \equiv 1(4) \\
  n \cdot \left\lceil \frac{t}{4} \right\rceil, & \text{if } t \not\equiv 1(4).
\end{cases}
\]
Lemma 8. For any graph $H$ it follows $\dim_l(\langle v \rangle + C_t) = 2$. Since $v$ belongs to any local metric metric basis for $\langle v \rangle + C_4$ and $v$ does not belong to any local metric basis for $\langle v \rangle + C_5$, we have

$$\dim_l(G \circ C_4) = n$$

and

$$\dim_l(G \circ C_5) = 2n = n \left\lceil \frac{5}{4} \right\rceil.$$ 

Now we consider the case where $t \geq 6$. As in the proof of (iii), for any local metric basis $B$ of $\langle v \rangle + C_t$ and any subgraph $H'$ of $C_t$, isomorphic to $P_4$, we have $B \cap V(H') \neq \emptyset$. Hence, for $t = 4k + r$, where $0 \leq r \leq 3$, we deduce

$$\dim_l(K_1 + C_t) = \begin{cases} k, & \text{if } r = 0 \\ k + 1, & \text{otherwise.} \end{cases}$$  

(2)

Then, since for $t \geq 6$ vertex $v$ does not belong to any local metric basis for $\langle v \rangle + C_t$,

$$\dim_l(G \circ C_t) = n \cdot \dim_l(K_1 + C_t) = n \cdot \left\lceil \frac{t}{4} \right\rceil.$$ 

Corollary 6. For any connected graph $H$ and any connected graph $G$ of order $n \geq 2$, $\dim_l(G \circ H) \geq n \cdot \dim_l(H)$.

Proof. Let $B$ be a local metric basis for $K_1 + H$. Since the vertex $v$ of $K_1$ does not distinguish any pair of adjacent vertices $x, y \in V(H)$, $B - \{v\}$ is a local metric generator for $H$. Thus, if $v \in B$, then $\dim_l(K_1 + H) - 1 \geq \dim_l(H)$ and, if $v \notin B$, then $\dim_l(K_1 + H) \geq \dim_l(H)$. Therefore, Theorem 4 leads to $\dim_l(G \circ H) \geq n \cdot \dim_l(H)$.

Corollary 7. For any graph $H$ of diameter two and any connected graph $G$ of order $n \geq 2$, $\dim_l(G \circ H) = n \cdot \dim_l(H)$.

Proof. Since $H$ has diameter two, for every $x, y \in V(H)$ it follows $d_H(x, y) = d_{K_1 + H}(x, y)$. So, if the vertex of $K_1$ does not belong to any local metric basis for $K_1 + H$, then every local metric basis for $H$ is a local metric basis for $K_1 + H$ and vice versa. Hence, in such a case, Theorem 4(i) leads to $\dim_l(G \circ H) = n \cdot \dim_l(H)$.

Now we suppose that there exists a local metric basis $B$ of $K_1 + H$ such that the vertex $v$ of $K_1$ belongs to $B$. Since $v$ does not distinguish any pair of vertices of $H$, $B' = B - \{v\}$ is a local metric generator for $H$. Moreover, if there exists $A \subset V(H)$ such that $|A| < |B'|$ and $A$ is a local metric basis for $H$, then $A \cup \{v\}$ is a local metric generator for $K_1 + H$, which is a contradiction because $|A| + 1 < |B'| + 1 = |B| = \dim_l(K_1 + H)$. Therefore, $B'$ is a local metric basis for $H$ and, as a result, $\dim_l(K_1 + H) = 1 + \dim_l(H)$. So, by Theorem 4(ii) we obtain $\dim_l(G \circ H) = n \cdot \dim_l(H)$.

Lemma 8. Let $H$ be a graph of radius $r(H)$. If $r(H) \geq 4$ then the vertex of $K_1$ does not belong to any local metric basis for $K_1 + H$.

Proof. Let $B$ be a local metric basis for $K_1 + H$. We suppose that the vertex $v$ of $K_1$ belongs to $B$. Note that $v \in B$ if and only if there exists $u \in V(H) - B$ such that $B \subset N_{K_1 + H}(u)$.

Now, if $r(H) \geq 4$, then we take $u' \in V(H)$ such that $d_H(u, u') = 4$ and a shortest path $uu_1u_2u_3u'$. In such a case for every $b \in B - \{v\}$ we will have that $d_{K_1 + H}(b, u_3) = d_{K_1 + H}(b, u') = 2$, which is a contradiction. Hence, $v$ does not belong to any local metric basis for $K_1 + H$. 

□
The converse of Lemma 8 is not true. In Figure 1 we show a graph $H$ of radius three where the vertex of $K_1$ does not belong to any metric basis for $K_1 + H$.

The following result is a direct consequence of Theorem 4 (i) and Lemma 8.

**Theorem 9.** For any connected graph $G$ of order $n$ and any graph $H$ of radius $r(H) \geq 4$,

$$\dim_l(G \odot H) = n \cdot \dim_l(K_1 + H).$$

Another consequence of Theorem 4 is the following result.

**Corollary 10.** For any non-empty graph $H$ of order $n'$ and any connected graph $G$ of order $n \geq 2$,

$$n \leq \dim_l(G \odot H) \leq n(n' - 1).$$

The aim of the next section is the study of the limit cases of Corollary 10.

### 2.1 Extremal values for $\dim_l(G \odot H)$

**Theorem 11.** Let $H$ be a graph of order $n'$ and let $G$ be a connected graph of order $n \geq 2$. Then $\dim_l(G \odot H) = n(n' - 1)$ if and only if $H \cong K_{n'}$ or $H \cong K_1 \cup K_{n' - 1}$.

**Proof.** By Theorem 4 we conclude that $\dim_l(G \odot H) = n(n' - 1)$ if and only if exactly one of the following cases hold:

Case a: the vertex $v$ of $K_1$ does not belong to any local metric basis for $K_1 + H$ and $\dim_l(K_1 + H) = n' - 1$.

Case b: the vertex $v$ of $K_1$ belongs to a local metric basis for $K_1 + H$ and $\dim_l(K_1 + H) = n'$.

We first consider Case a. By Theorem 2 $\dim_l(K_1 + H) = n' - 1$ if and only if $\omega(H) = n' - 1$. Let $V(H) = \{u_1, u_2, \ldots, u_{n'}\}$. If $\langle V(H) - \{u_1\} \rangle$ is a clique and $u_i \sim u_1$, then $\{v\} \cup V(H) - \{u_1, u_i\}$ is a local metric basis for $K_1 + H$, which is a contradiction. Hence $u_1$ is an isolated vertex of $H$. So, Case a holds if and only if $H \cong K_1 \cup K_{n' - 1}$.

Finally, by Theorem 4 we deduce that Case b holds if and only if $H \cong K_{n'}$. \hfill\Box

The radius $r(G)$ of a graph $G$ is the minimum eccentricity of any vertex of $G$. The center of $G$, denoted by $C(G)$, is the set of vertices of $G$ with eccentricity equal to $r(G)$.

**Theorem 12.** Let $H$ be a non-empty graph and let $G$ be a connected graph of order $n \geq 2$. Then $\dim_l(G \odot H) = n$ if and only if $H$ is a bipartite graph having only one non-trivial connected component $H^*$ and $r(H^*) \leq 2$.

**Proof.** Since $\langle v \rangle + H$ is not bipartite, by Theorem 4 we deduce $\dim_l(\langle v \rangle + H) \geq 2$. So, if $\dim_l(G \odot H) = n$, then by Theorem 4 we have that $\dim_l(\langle v \rangle + H) = 2$ and $v$ belongs to a local metric basis for $\langle v \rangle + H$, say $B = \{u, v\}$. So, $B \cap V(H) = \{u\}$ must be a local
metric generator for $H$ and, by Theorem 1, we conclude that $H$ is a bipartite graph having only one non-trivial connected component. Moreover, if the non-trivial component of $H$ has radius $r > 2$, then there exists $u_3 \in V(H)$ such that $d_H(u, u_3) = 3$ and, as a consequence, for any shortest path $uu_1u_2u_3$ we have $d_{(v)+H}(u, u_2) = d_{(v)+H}((u, u_3)$, i.e., the pair of adjacent vertices $u_2, u_3$ is not distinguished by the elements of $B$, which is a contradiction. Therefore, $r \leq 2$.

Conversely, let $H$ be a bipartite graph where having only one non-trivial component $H^*$. Let $r(H^*) \leq 2$, let $a$ be a vertex belonging to the center of $H^*$ and let $v$ be the vertex of $K_1$. Since $H$ is a triangle free graph, $a$ distinguishes every pair of adjacent vertices $x, y \in V(H^*)$. So, $\{v, a\}$ is a local metric generator for $K_1 + H$, which is a local metric basis because $\dim_l(K_1 + H) \geq 2$. We conclude the proof by Theorem 4 (ii).

\section{The value of $\dim_l(G \odot H)$ when $H$ is a bipartite graph of radius three}

Theorems 12 and 9 suggest to consider the case where $H$ is a bipartite graphs of radius three. To do that, we need the following additional notation. For any $a \in V(H)$, we denote

$$N^{(i)}_H(a) = \{w \in V(H) : d_H(w, a) = i\}.$$ 

We also define $N^{(i)}_H[a] = N^{(i)}_H(a) \cup \{a\}$. Note that $N^{(1)}_H(a) = N_H(a)$ and $N^{(1)}_H[a] = N_H[a]$. Given two sets $A, B \subset V(H)$ we say that $A$ dominates $B$ if every vertex in $B - A$ is adjacent to some vertex belonging to $A$. From now on we will use the notation $A \succ B$ to indicate that $A$ dominates $B$. For every $x \in C(H)$, let $\beta(x) = \min\{\vert A \vert : A \subseteq N_H(x) \text{ and } A \succ N^{(2)}_H(x)\}$ and let

$$\delta'(H) = \min_{x \in C(H)} \{\beta(x)\}.$$ 

\textbf{Lemma 13.} For any bipartite graph $H$ of radius three,

$$\dim_l(K_1 + H) \leq \delta'(H) + 1.$$ 

Moreover, $\dim_l(K_1 + H) = \delta'(H) + 1$ if and only if the vertex of $K_1$ belongs to a local metric basis for $K_1 + H$.

\textbf{Proof.} Let $u$ be a vertex belonging to the center of $H$ and $A \subseteq N_H(u)$ such that $A \succ N^{(2)}_H(u)$ and $\vert A \vert = \delta'(H)$. Let us show that $B = A \cup \{v\}$ is a local metric generator for $\langle v \rangle + H$. We first note that since $H$ is bipartite, for two adjacent vertices $x, y \notin B$ it follows $d_H(u, x) \neq d_H(u, y)$. Hence, without loss of generality, we may consider the following three cases for two adjacent vertices $x, y \notin B$.

Case 1: $x = u$ and $y \sim u$. In this case for every $z \in A$ it follows $d_{K_1+H}(x, z) = 1$ and $d_{K_1+H}(y, z) = 2$.

Case 2: $d_H(u, x) = 1$ and $d_H(u, y) = 2$. In this case $y \in N^{(2)}_H(u)$ and there exists $x' \in A$ such that $x' \sim y$ and, since $H$ is a bipartite graph, $x' \not\sim x$. So, $d_{K_1+H}(x, x') = 2$ and $d_{K_1+H}(y, x') = 1$.

Case 3: $d_H(u, x) = 2$ and $d_H(u, y) = 3$. In this case $x \in N^{(2)}_H(u)$ and there exists $x' \in A$ such that $ux'y$ is a shortest path in $H$. So, $d_{K_1+H}(x, x') = 1$ and $d_{K_1+H}(y, x') = 2$. 

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Thus, $B$ is a local metric generator for $K_1 + H$ and, as a consequence, $\dim_l(K_1 + H) \leq \delta'(H) + 1$.

Moreover, if $\dim_l(K_1 + H) = \delta'(H) + 1$, then $B$ is a local metric basis for $K_1 + H$ which contains the vertex of $K_1$.

Conversely, let $S$ be a local metric basis for $K_1 + H$ which contains the vertex $v$ of $K_1$. In this case there exists $w \in V(H)$ such that $N_H(w) \supset S - \{v\}$. If $w \not\in C(H)$, then there exists $w' \in V(H)$ such that $d_H(w, w') \geq 4$ and for every shortest path $ww_1w_2w_3w'$ from $w$ to $w'$ the pair of vertices $w_3, w'$ is not resolved in $K_1 + H$ by any $s \in S$, which is a contradiction. Hence, $w \in C(H)$ and $S - \{v\} \supset N_H^{(2)}(w)$. The minimality of the cardinality of $S$ leads to $|S - \{v\}| = \delta'(H)$. Therefore, $\delta'(H) + 1 = |S| = \dim_l(K_1 + H)$. \hfill \Box

As a direct consequence of Theorem 4 and Lemma 13 we obtain the following result.

**Theorem 14.** Let $H$ be a bipartite graph of radius three and let $G$ be a connected graph of order $n \geq 2$. Then

\[ \dim_l(G \odot H) \leq n \cdot \delta'(H). \]

### 2.2.1 The maximum value of $\dim_l(G \odot H)$ when $H$ is a bipartite graph of radius three

In this section we show that the above bound is attained for a subfamily of bipartite graphs of diameter three that does not contain a square (a subgraph isomorphic to $K_{2,2}$). In such a case, the girth of $H$ must be six and $H = (U_1 \cup U_2, E)$ satisfies the following property:

✧ For any $i \in \{1, 2\}$ and any two distinct vertices $a, b \in U_i$, $|N_H(a) \cap N_H(b)| = 1$.

Therefore, $H$ is the incidence graph of a finite projective plane. So, we have two possibilities (see, for instance, [2]):

(P1) $H = (U_1 \cup U_2, E)$ is the incidence graph of a degenerate projective plane. In this case $|U_1| = |U_2| = t$, $t \geq 3$, and $H$ is a pseudo sphere graph $S_t$ (also called near pencil) defined as follows: we consider $t - 1$ path graphs of order 4 and we identify one extreme of each one of the $t - 1$ path graphs in one pole $a$ and all the other extreme vertices of the paths in a pole $b$. In particular, $S_3$ is the cycle graph $C_6$.

(P2) $H = (U_1 \cup U_2, E)$ is the incidence graph of a non-degenerate projective plane of order $q$. In this case $H$ is a regular graph of degree $\delta_H = q + 1$ and $|U_1| = |U_2| = q^2 + q + 1$. Note that $|U_1| = |U_2| = \delta_H^2 - \delta_H + 1$.

In the case (P1) the set $B = \{a, b\}$ composed by both poles of the pseudo sphere is a dominating set of $S_t$. Thus, $B$ is a local metric basis for $\langle v \rangle + S_t$ and $N_{S_t}(a) \cap N_{S_t}(b) = \emptyset$. Also, there are no local metric generators composed by two vertices at distance two, so the vertex $v$ does not belong to any local metric basis for $\langle v \rangle + S_t$ and, by Theorem 4 (i), we obtain that for any connected graph $G$ of order $n \geq 2$, $\dim_l(G \odot S_t) = 2n$.

The rest of this section covers the study of case (P2), i.e., the case where $H$ is the incidence graph of a non-degenerate projective plane.

**Lemma 15.** For any bipartite graph $H \not\cong S_t$ of diameter three and girth six,

\[ \delta'(H) = \delta_H. \]
Proof. Let $x \in U_i$, $i \in \{1, 2\}$. Since for any $y, z \in N_H(x)$ we have $N_H(y) \cap N_H(z) = \{x\}$, we deduce that for any $A \subseteq N_H(x)$,

$$
|N_H^{(2)}(x)| = |U_i - \{x\}| \geq \left| \bigcup_{y \in A} (N_H(y) - \{x\}) \right| = \sum_{y \in A}(|N_H(y)| - 1) = (\delta_H - 1)|A|.
$$

Therefore, since $|U_i| = \delta_H^2 - \delta_H + 1$, we have that $A \succeq N_H^{(2)}(x)$ if and only if $A = N_H(x)$. □

**Lemma 16.** Let $H = (U_1 \cup U_2, E) \not\cong S_t$ be a bipartite graph of diameter three and girth six. For any local metric basis $B$ of $K_1 + H$, either $B \cap U_1 = \emptyset$ or $B \cap U_2 = \emptyset$.

**Proof.** We proceed by contradiction. Suppose that $B_1 = B \cap U_1 \neq \emptyset$ and $B_2 = B \cap U_2 \neq \emptyset$. We differentiate two cases.

**Case 1:** $B_1 \cup N_H(B_2) \neq U_1$ or $B_2 \cup N_H(B_1) \neq U_2$. We take, without loss of generality, $x \in U_1$ such that $x \notin B_1 \cup N_H(B_2)$. Since $B$ is a local metric basis for $K_1 + H$ and $N_H(x) \cap B_2 = \emptyset$, the set $N_H(x)$ must be dominated by $B_1$. Moreover, since $H$ is a square free graph, for any $b \in B_1$ there exists only one vertex $y_b \in N_H(x) \cap N_H(b)$. Thus, $\delta_H = |N_H(x)| \leq |B_1|$. On the other hand, by Lemmas 13 and 15 we have $|B \cap (U_1 \cup U_2)| \leq \delta_H$. Hence, the assumption $B_2 = B \cap U_2 \neq \emptyset$ leads to $|B_1| \leq \delta_H - 1$, which is a contradiction with the fact that $|B_1| \geq \delta_H$.

**Case 2:** $B_1 \cup N_H(B_2) = U_1$ and $B_2 \cup N_H(B_1) = U_2$. If $|B_1| = |B_2| = 1$, then $\delta_H^2 - \delta_H + 1 = |U_1| = |B_1 \cup N_H(B_2)| \leq 1 + \delta_H$, which is a contradiction for $\delta_H > 2$. Thus, without loss of generality, we assume that $|B_2| \geq 2$. Let $a, b \in B_2$ and let $c \in U_1$ such that $\{c\} = N_H(a) \cap N_H(b)$. We define $B'_1 = B_1 \cup \{c\}$, $B'_2 = B_2 - \{a, b\}$ and $B' = B'_1 \cup B'_2$. Note that $|B'| < |B|$. We take two adjacent vertices $x, y$ such that $x \in U_1 - B'_1$ and $y \in U_2 - B'_2$. Now, if $y \in \{a, b\}$, then $c \in B'$ distinguishes the pair $x, y$ and if $y \notin \{a, b\}$, then there exists $y' \in B_1 \subseteq B'$ such that $y'$ is adjacent to $y$. Thus, $B'$ is a local metric basis for $K_1 + H$, which is a contradiction.

Since both cases lead to a contradiction, the proof is complete. □

**Lemma 17.** Let $H \not\cong S_t$ be a bipartite graph of diameter three and girth six. Then the vertex of $K_1$ belongs to any local metric basis for $K_1 + H$.

**Proof.** Let $B$ be a local metric basis for $\langle v \rangle + H$. We proceed by contradiction. Suppose that $v \notin B$. By Lemmas 13 and 15 we have $|B| \leq \delta_H$. By Lemma 16 we can assume that $B \subseteq B_1$. Now, if $|B| \leq \delta_H - 1$, then

$$
|N_H(B)| = \left| \bigcup_{b \in B} N_H(b) \right| \leq \sum_{b \in B} |N_H(b)| = (\delta_H - 1)\delta_H < |U_2|,
$$

which is a contradiction because if there exist two adjacent vertices $x, y$ such that $x \in U_1 - B$ and $y \in U_2 - N_H(B)$, then the pair $x, y$ is not distinguished by the elements of $B$. Hence, we conclude $|B| = \delta_H$.

Now, if there exists $a \in U_2$ such that $N_H(a) = B$, then the pair of adjacent vertices $a, v$ is not distinguished by the elements of $B$, which is a contradiction. Thus, let $b, b' \in B$, $a \in N_H(b) \cap N_H(b')$, and $x_a \in N_H(a) - B$. Since $B$ is a local metric basis and $H$ is a square free graph, for every $y, z \in N_H(x_a)$, there exist two vertices $b_y \in (B - \{b, b'\}) \cap N_H(y)$ and $b_z \in (B - \{b, b'\}) \cap N_H(z)$ such that $b_y \neq b_z$. Hence,

$$
\delta_H - 1 = |N_H(x_a) - a| \leq |B - \{b, b'\}| = \delta_H - 2,
$$

which is a contradiction. Therefore, $v$ must belong to $B$. □
\textbf{Theorem 18.} Let $H \not\cong S_i$ be a bipartite graph of diameter three and girth six. Then for any connected graph $G$ of order $n \geq 2$,

$$\dim_l(G \circ H) = n \cdot \delta_H.$$ 

\textit{Proof.} By Lemma 17 we know that the vertex of $K_1$ belongs to every local metric basis for $K_1 + H$, by Lemmas 13 and 15 we have $\dim_l(K_1 + H) = \delta_H + 1$ and by Theorem 4 (ii) we conclude $\dim_l(G \circ H) = n \cdot \delta_H$. \hfill \qed

Let $\pi = (P, L)$ be a finite non-degenerate projective plane of order $q$, where $P$ is the set of points and $L$ is the set of lines. Given two sets $P' \subset P$ and $L' \subset L$, we say that $P' \cup L'$ satisfies the property $G$, if for any point $p_0$ and any line $l_0$ such that $p_0 \in l_0$ we have

- there exists $p \in P'$ such that $p \in l_0$, or
- there exists $l \in L'$ such that $p_o \in l$.

We define $\Upsilon(\pi) = \min\{|P' \cup L'| \text{ such that } P' \cup L' \text{ satisfies the property } G\}$.

We have that if $H$ is the incidence graph of $\pi$, then a set $P' \cup L'$ satisfies the property $G$ if and only if $P' \cup L' \cup \{v\}$ is a local metric generator for $\langle v \rangle + H$. Therefore, according to Lemmas 13, 15 and 17 we conclude

$$\Upsilon(\pi) = \delta_H = q.$$

Note that if $P' \cup L'$ satisfies the property $G$ and its cardinality is the minimum among all the sets satisfying this property, then either $P' = \emptyset$ and $L'$ is the set of lines incident to one point or $L' = \emptyset$ and $P'$ is the set composed by all the points laying on one line.

\subsection*{2.2.2 The minimum value of $\dim_l(G \circ H)$ when $H$ is a bipartite graph of radius three}

As a direct consequence of Theorems 4 and 12 we derive the following result.

\textbf{Remark 19.} For any connected graph $H$ of radius $r(H) \geq 3$ and any connected graph $G$ of order $n \geq 2$,

$$\dim_l(G \circ H) \geq 2n.$$

In this section we study the limit case of the above bound for the case where $H$ is bipartite.

\textbf{Lemma 20.} If $H$ is a graph of radius three and $\dim_l(K_1 + H) = 2$, then the vertex of $K_1$ does not belong to any local metric basis for $K_1 + H$.

\textit{Proof.} Let $\{a, b\}$ be a local metric basis for $\langle v \rangle + H$. Since $r(H) = 3$, no vertex of $H$ distinguishes every pair of adjacent vertices of $H$. Thus, $a \neq v$ and $b \neq v$. \hfill \qed

\textbf{Theorem 21.} Let $H = (U_1, U_2, E)$ be a bipartite graph of radius three and let $G$ be a connected graph of order $n$. Then $\dim_l(G \circ H) = 2n$ if and only if $\dim_l(K_1 + H) = 2$ or for some $i \in \{1, 2\}$, there exist $a, b \in U_i$ such that $N_H(a) \cup N_H(b) = U_j$, where $j \in \{1, 2\} - \{i\}$. 

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Proof. By Theorem 4 we know that \( \dim_l(G \odot H) = 2n \) if and only if either \( \dim_l(\langle v \rangle + H) = 2 \) and \( v \) does not belong to any local metric basis for \( \langle v \rangle + H \) or \( \dim_l(\langle v \rangle + H) = 3 \) and there exists a local metric basis \( B \) of \( \langle v \rangle + H \) such that \( v \in B \).

If \( \dim_l(\langle v \rangle + H) = 2 \), then we are done (note that by Lemma 20 we have that \( v \) does not belong to any local metric basis for \( \langle v \rangle + H \)).

Let \( B = \{a, b, v\} \) be a local metric basis of \( \langle v \rangle + H \). Since \( v \in B \), we have \( N_H(a) \cap N_H(b) \neq \emptyset \). So, \( a \) and \( b \) must belong to the same color class, set \( a, b \in U_1 \). Hence, if there exists \( y \in U_2 \setminus (N_H(a) \cup N_H(b)) \), then for every \( x \in N_H(y) \), the pair \( x, y \) is distinguished by \( a \) or by \( b \). So, \( \{a, b, v\} \) is a local metric basis for \( \langle v \rangle + H \) and, as a consequence, \( \dim_l(\langle v \rangle + H) = 2 \) or \( \{a, b, v\} \) is a local metric basis of \( \langle v \rangle + H \).

Conversely, if there exists \( a, b \in U_i \) such that \( N_H(a) \cup N_H(b) = U_j \), where \( j \in \{1, 2\} \setminus \{i\} \), then for every \( y \in U_j \) and \( x \in N_H(y) \), the pair \( x, y \) is distinguished by \( a \) or by \( b \). Hence, \( \{a, b, v\} \) is a local metric basis of \( \langle v \rangle + H \). Therefore, either \( \dim_l(\langle v \rangle + H) = 2 \) or \( \{a, b, v\} \) is a local metric basis of \( \langle v \rangle + H \).

Consider the following decision problem. The input is an arbitrary bipartite graph \( H = (U_1 \cup U_2, E) \) of radius three. The problem consists in deciding whether \( H \) satisfies \( \dim_l(K_1 + H) = 2 \), or not. According to the next remark we deduce that the time complexity of this decision problem is at most \( O(|U_1|^2|U_2|^2) \). Although this remark is straightforward, we include the proof for completeness.

**Remark 22.** Let \( H = (U_1, U_2, E) \) be a bipartite graph of radius three. Consider the following statements:

(i) For some \( i \in \{1, 2\} \), there exist \( a, b \in U_i \) such that \( \{N_H(a), N_H(b)\} \) is a partition of \( U_j \), where \( j \in \{1, 2\} \setminus \{i\} \).

(ii) There exist two vertices \( a \in U_1 \) and \( b \in U_2 \) such that for every edge \( xy \in E \), where \( x \in U_1 \) and \( y \in U_2 \), it follows \( y \in N_H(a) \) or \( x \in N_H(b) \).

Then \( \dim_l(K_1 + H) = 2 \) if and only if (i) or (ii) holds.

**Proof.** We first note that since \( K_1 + H \) is not bipartite, Theorem 1 leads to \( \dim_l(K_1 + H) \geq 2 \).

(Sufficiency) If (i) holds, then \( \{a, b\} \supset U_j \) and \( N_H(a) \cap N_H(b) = \emptyset \). Hence, \( \{a, b\} \) is a local metric basis of \( K_1 + H \) and, as a consequence, \( \dim_l(K_1 + H) = 2 \).

Now, if (ii) holds, it is straightforward to see that \( \{a, b\} \) is a local metric basis of \( K_1 + H \) and, as a consequence, \( \dim_l(K_1 + H) = 2 \).

(Necessity) Let \( \{a, b\} \) be a local metric basis for \( \langle v \rangle + H \). By Lemma 20 we know that \( v \notin \{a, b\} \). Then we have two possibilities.

1. \( a \) and \( b \) belong to the same color class of \( H \), say \( a, b \in U_1 \). Since for every \( x \in V(H) \), the pair \( x, v \) must be distinguished by \( a \) or by \( b \), we conclude that \( N_H(a) \cap N_H(b) = \emptyset \).

   Also, since every pair of adjacent vertices \( x \in U_1 \) and \( y \in U_2 \) must be distinguished by \( a \) or by \( b \), we conclude that \( y \sim a \) or \( y \sim b \) and, as a result, \( \{a, b\} \supset U_2 \). Hence, we conclude that \( \{N_H(a), N_H(b)\} \) is a partition of \( U_2 \).

2. \( a \) and \( b \) belong to different color classes of \( H \), say \( a \in U_1 \) and \( b \in U_2 \). Since \( \{a, b\} \) is a local metric basis for \( \langle v \rangle + H \), for every edge \( xy \in E \), where \( x \in U_1 \) and \( y \in U_2 \), it follows \( y \in N_H(a) \) or \( x \in N_H(b) \).
Note that if \( H = (U_1 \cup U_2, E) \) is a bipartite graph of diameter \( D(H) = 3 \), then for any \( i \in \{1, 2\} \) and \( x, y \in U_i \) we have \( N_H(x) \cap N_H(y) \neq \emptyset \). Hence, we deduce the following consequence of Remark 22.

**Corollary 23.** Let \( H \) be a bipartite graph where \( D(H) = r(H) = 3 \). If \( B = \{a, b\} \) is a local metric basis for \( K_1 + H \), the \( a \) and \( b \) belong to different color classes.

Other direct consequence of Remark 22 is the following.

**Corollary 24.** Let \( H = (U_1, U_2, E) \) be a bipartite graph of radius three. If for some \( i \in \{1, 2\} \), there exist \( a \in U_i \) such that \( \delta_H(a) = |U_j| - 1 \), where \( j \in \{1, 2\} - \{i\} \), then \( \dim_l(K_1 + H) = 2 \).

### 2.2.3 Closed formulae for \( \dim_l(G \odot H) \) when \( H \) is a tree of radius three

In order to study the particular case when \( H \) is a tree of radius three, we introduce the following additional notation. Let \( T \) be a tree of radius three. For the particular case when \( C(T) = \{u\} \) we consider the forest \( F_u = \cup_{w \in N_T(u)} T_w \) composed of all the rooted trees \( T_w = (V_w, E_w) \), of root \( w \in N_T(u) \), obtained by removing the central vertex \( u \) from \( T \). The height of \( T_w \) is \( h_w = \max_{x \in V(T_w)} \{d(w, x)\} \). We denote by \( \varsigma(T) \) the number of trees in \( F_u \) with \( h_w \) equal to two, i.e., \( \varsigma(T) = |S(T)| \), where

\[
S(T) = \{w \in N_T(u) : h_w = 2\}.
\]

Note that if \( h_w \neq 1 \), for every \( w \in N_T(u) \), then \( \varsigma(T) = \delta'(T) \). So, as the following result shows, the bound \( \dim_l(G \odot T) \leq n \cdot \delta'(T) \) is tight.

**Theorem 25.** Let \( T \) be a tree of radius three and center \( C(T) \). The following assertion hold for any connected graph \( G \) of order \( n \geq 2 \).

(i) If \( |C(T)| = 2 \), then \( \dim_l(G \odot T) = 2n \)

(ii) If \( C(T) = \{u\} \), then

\[
\dim_l(G \odot T) = \begin{cases} 
n \cdot (\varsigma(T) + 1), & \text{if there exists } w \in N_T(u) \text{ such that } h_w = 1, \\
n \cdot \varsigma(T), & \text{otherwise.}
\end{cases}
\]

**Proof.** It is well-known that the center of a tree consists of either a single vertex or two adjacent vertices.

We first consider the case where \( C(T) \) consists of two adjacent vertices, say \( C(T) = \{u', u''\} \). Note that in this case, if we remove the edge \( \{u', u''\} \) from \( T \), we obtain two rooted trees \( T' = (V', E') \) and \( T'' = (V'', E'') \), with roots \( u' \) and \( u'' \), respectively, where the distance from the root to the leaves is at most two. Hence, in \( K_1 + T \) every pair of adjacent vertices \( x, y \in V' \) is distinguished by \( u' \) and every pair of adjacent vertices \( x, y \in V'' \) is distinguished by \( u'' \). Also, for every \( x \in V' - \{u'\} \) the pair \( v, x \) is distinguished by \( u'' \) and for every \( x \in V'' - \{u''\} \), the pair \( v, x \) is distinguished by \( u' \), where \( v \) is the vertex of \( K_1 \). So, \( C(T) \) is a local metric generator for \( K_1 + T \). Hence, \( \dim_l(K_1 + T) \leq 2 \) and, since \( K_1 + T \) is not bipartite, by Theorem 1 we conclude that \( \dim_l(K_1 + T) = 2 \). Now, in this case, if the vertex of \( K_1 \) belongs to a local metric basis for \( K_1 + T \), then there exists \( z \in V(T) \) such that \( z \) distinguishes any pair of adjacent vertices \( x, y \in V(T) \), and as a consequence \( r(T) \leq 2 \), which is a contradiction. Thus, we conclude that the vertex of \( K_1 \) does not belong to any
local metric basis for $K_1 + T$. Therefore, as a consequence of Theorem 4 (i) we obtain $\dim_d(G \odot T) = 2n$.

Now let us consider the case where the center of $T$ consists of a single vertex, say $C(T) = \{u\}$. Let $B$ be a local metric basis for $K_1 + T$. We first note that for every rooted tree $T_w = (V_w, E_w)$ of height two we have $|B \cap V_w| = 1$, due to the fact that in $K_1 + T$ the vertex $w \in N_T(u)$ distinguishes every pair of adjacent vertices $x, y \in V_w$ and no vertex of $V(K_1 + T) - V_w$ distinguishes a pair of adjacent vertices where one vertex is a leaf. Hence, $\dim_d(K_1 + T) \geq \varsigma(T)$. Now we differentiate the following cases.

Case 1. There exists $w \in N_T(u)$ such that $h_w = 1$. In this case, the subgraph of $T$ induced by the set $X = \cup_{h_w \leq 1} V_w \cup \{u\}$ is a tree of root $u$ and height two. Hence, as above we conclude that $|B \cap X| = 1$. So, $\dim_d(K_1 + T) \geq \varsigma(T) + 1$. In order to show that the set $A = \{u\} \cup S(T)$ is a local metric basis for $K_1 + T$ we only need to observe that $N_T(w) \cap N_T(u) = \emptyset$ and, as a consequence, for every $x \in V(T)$ the pair $x, v$ is distinguished by some $z \in A$. Thus, $\dim_d(K_1 \odot T) = \varsigma(T) + 1$.

Moreover, since for every metric basis $A$ of $K_1 + T$ we have $|A \cap X| = 1$ and for every rooted tree $T_w = (V_w, E_w)$ of height two, $|A \cap V_w| = 1$, we conclude that the vertex of $K_1$ does not belong to any local metric basis for $K_1 + T$. Therefore, as a consequence of Theorem 4 (i) we obtain $\dim_d(G \odot T) = n(\varsigma(T) + 1)$.

Case 2. For every $w \in N_T(u)$, $h_w \neq 1$. In this case we define

$$\varphi(T_w) = |\{z \in N_{T_w}(w) : \delta_T(z) \geq 2\}|.$$

Suppose there exists $w_i \in N_T(u)$ such that $\varphi(T_{w_i}) = 1$. With this assumption we define

$$A' = \{z\} \cup S(T) - \{w_i\},$$

where $z \in V_{w_i}$ and $\delta_T(z) \geq 2$. Note that every pair of adjacent vertices $x, y \in \{u\} \cup V_{w_i}$ is distinguished by $z$. So, by analogy to Case 1 we show that $A'$ is a local metric basis for $K_1 + T$ and the vertex of $K_1$ does not belong to any local metric basis for $K_1 + T$. Therefore, as a consequence of Theorem 4 (i) we obtain $\dim_d(G \odot T) = n \cdot \varsigma(T)$.

On the other hand, if for every $w \in S(T)$ it follows $\varphi(T_w) \geq 2$, then $w$ is the only vertex of $V_w$ which distinguishes every pair of adjacent vertices $x, y \in V_w$. Thus, in such a case $S(T)$ is a subset of any local metric basis for $K_1 + T$ and, as a consequence, the only two local metric basis for $K_1 + T$ are $\{u\} \cup S(T)$ and $\{v\} \cup S(T)$. Therefore, as a consequence of Theorem 4 (ii) we obtain $\dim_d(G \odot T) = n \cdot \varsigma(T)$.

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