LATTICES OF FLATS FOR SYMPLECTIC MATROIDS

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Abstract. We are interested in expanding our understanding of symplectic matroids by exploring the properties of a class of symplectic matroids with a "lattice of flats". Taking a well-behaved family of subdivisions of the cross polytope we obtain a construction of lattices, resembling a known definition for the geometric lattice corresponding to ordinary matroid. We construct a correspondence to a set of enveloped symplectic matroids, we denote ranked symplectic matroids. As a byproduct of our construction, we also obtain a new way of finding symplectic matroids from ordinary ones and an embedding theorem into geometric lattices.

The second part of this paper is dedicated to the properties of ranked symplectic matroids and their enveloping ordinary matroids. We focus on establishing a geometric approach to the study of ranked symplectic matroids, demonstrating the ability to take minors, and proving shellability. We finish with a characterization of ranked symplectic matroids using recursive atom orderings.

1. Introduction

Many books have been written on the subject of matroid theory. Developed in order to generalize the concept of independent sets of vector spaces, matroids now have numerous applications in many fields. A very useful property of matroids is their large amount of cryptographic characterizations, which contributes a great deal to their many uses. Below is one such characterization using flats:

1.1. Definition: A collection \( L \) of subsets of \( [n] \) is a lattice of flats for a matroid, also called a geometric lattice if

1. \( \emptyset, [n] \in L \)
2. \( \forall A, B \in L \) we have \( A \cap B \in L \),
3. For every \( A \in L \) let \( \{B_1, \ldots, B_m\} \) be the set of elements in \( L \) covering \( A \) then \( B_i \cap B_j = A \) for all \( i \neq j \), and \( \cup_{i=1}^{m} B_i = [n] \).

Some of our readers may be more familiar with the following definition: a poset is a geometric lattice iff it is a finite atomic graded and submodular lattice, we encourage the readers to use [1] for more background on matroid theory.

Gelfand and Serganova [2] introduced the concept of coxeter matroids, where matroids are a special case in which the coxeter group is the symmetric group. Symplectic matroids are another important example, namely the case where the coxeter group is the hyperoctahedral group, the group of symmetries of the \( n \)-cube.

In contrast to the case of matroids, which will be referred to from now on as ordinary matroids, we know very little about the symplectic case. In this work, we attempt to give a "lattice of flats" characterization for a class of symplectic matroids, which

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we call ranked. The "lattices of symplectic flats" we construct, denoted $C_n$ lattices, take a similar form to Definition 1.1 and have many similar properties. Our hope is that we can lay the groundwork for geometric constructions and properties in the spirit of the many simplicial complexes associated with ordinary matroids.

Instead of the algebraic property used to define coxeter matroids, we use a more combinatorial approach introduced in [3]. We also make use of a characterization of symplectic matroids using independent sets, introduced by T. Chow in [4]. Another characterization that exists but is not used in this work is a circuits definition introduced in [5].

The structure of this paper is as follows. In Section 1 we define a $C_n$ lattice in a way similar to geometric lattices and introduce some of its basic properties. In Section 2 we present a general discussion of $NBB$ sets which can be of interest and will be the main tool used in Section 3. Section 3 is the heart of this paper that interprets $C_n$ lattices as lattices of flats for ranked symplectic matroids. Another byproduct of this construction is a new way of obtaining symplectic matroids. In section 4 we prove $C_n$ lattices are lexicographically shellable. We end the paper with a characterization of $C_n$ lattices by atom orderings, resembling the way it was done for geometric lattices in [8].

Throughout the paper, we use lowercase letters when referring to elements, uppercase letters when referring to sets, and uppercase letters in calligraphic font when referring to collections of sets. For example, we may have $x \in X \in \mathcal{X}$.

2. Lattice properties

Geometric lattices were defined on the ground set $[n]$, in the case of $C_n$ lattices, we use a different ground set, the disjoint union of two copies of $[n]$: $J = [n] \cup [n]^* = \{1, 2, ..., n\} \cup \{1^*, 2^*, ..., n^*\}$

We also introduce the map $*: [n] \rightarrow [n]^*$ defined by $i \mapsto i^*$ and $*: [n]^* \rightarrow [n]$ defined by $i^* \mapsto i$. Throughout this paper, we apply $*$ to sets and collections of sets. A set $A \subseteq P(J)$ is called admissible if $A \cap A^* = \emptyset$ and a maximal admissible set will be called a transversal. We denote the collection of all admissible subsets of $J$ by $P^{ad}(J)$. The following is the main subject of this work.

2.1. Definition: A collection $\mathcal{L}$ of subsets of $J$ is a $C_n$ lattice if:

1. $\emptyset, J \in \mathcal{L}$
2. $\forall A \in \mathcal{L} \setminus \{J\}$ we have $A \in P^{ad}(J)$.
3. $\forall A, B \in \mathcal{L}$ we have $A \cap B \in \mathcal{L}$.
4. For every $A \in \mathcal{L}$ let $\{B_1, ..., B_m\}$ be the set of elements in $\mathcal{L}$ covering $A$ then $B_i \cap B_j = A$ for all $i \neq j$, and $\cup_{i=1}^m B_i \supseteq J \setminus A^*$.

$C_n$ lattices are sublattices of the face lattice of the cross-polytope, resembling the ordinary case in which geometric lattices are sublattices of the face lattice of the simplex.

Before continuing with our main results, we establish some basic properties of these lattices. A partially ordered set $S$ in which every pair of elements has a unique supremum (also called join and denoted $\lor$) and a unique infimum (also called meet...
and denoted $\wedge$ is called a lattice. We say that a lattice is bounded if it has a greatest element (denoted $\hat{1}$) and a least element (denoted $\hat{0}$).

All $C_n$ lattices are bounded by $\emptyset$ and $J$. For the meet and join, we have:

1. $A \wedge B = A \cap B$
2. $A \lor B = \min \{C \in L \mid A \cup B \subseteq C\}$

We say $B$ covers $A$ and denote $A \preceq B$ if $A < B$ and for all $C$ such that $A \leq C \leq B$ we have $C = B$ or $C = A$. $A$ is an atom if it covers the least element $\hat{0}$. A bounded lattice is called atomic if every element is a join of atoms. To prove $C_n$ lattices are atomic we show every element $A \in L$ is the join of all the atoms contained in it.

We have the following:

$$\bigvee_{i=1}^m A_i \subseteq A$$

As each $A_i$ is contained in $A$. We now use property (4) to get that every $x \in J$ is in some atom (as they partition $J \setminus \emptyset^* = J$). So if $x \in A$ we have some atom $B$ s.t. $x \in B$. If $B \setminus A \neq \emptyset$ we will have $\emptyset \subseteq B \cap A \subseteq B$ which is a contradiction to $B$ being an atom and so $B \subseteq A$ and we have:

$$A \subseteq \bigvee_{i=1}^m A_i \Rightarrow A = \bigvee_{i=1}^m A_i$$

we finish this section by showing that $C_n$ lattices are graded. In Section 3 we will prove that there is an underlying family of independent sets which determines this rank function. This should remind our readers of the ordinary case.

existence of rank function: Let $L$ be a $C_n$ lattice, the existence of a rank function is equivalent to having all maximal chains between every two elements $A, B \in L$ such that $A < B$ be of the same length. We prove this by induction on the maximal length of a maximal chain.

For length 1 this is obvious as there is no chain of length 0. For some arbitrary element $A \in L$ assume we have two maximal chains $\mathcal{X}$ and $\mathcal{Y}$. We can assume $\mathcal{X} \cap \mathcal{Y} = \{A, B\}$ as otherwise we are reduced to looking at maximal chains from $C \in \mathcal{X} \cap \mathcal{Y}$ to $B$ by the induction hypothesis.

From this point we will assume $A = \emptyset$, as we will prove in Section 4 independently that the set of elements containing $A$ is order isomorphic to a $C_n$ lattice. We now show that every element of $\mathcal{X}$ is disjoint from every element in $\mathcal{Y}$. Assuming that this is not the case, and so they both contain an element covering some atom $C$ (since every element is a join of atoms) we can then produce chains of the same length as $\mathcal{X}, \mathcal{Y}$ from $\emptyset$ to $B$ containing $C$. This will obviously be a contradiction to the induction hypothesis as we now get two chains of different length from $C$ to $B$.

We are now left with the case where every element in $\mathcal{X}$ is disjoint from every element in $\mathcal{Y}$. Each chain contains an atom, assume $C \in \mathcal{X}$ and $C' \in \mathcal{Y}$. We can assume $C' \neq C^*$, otherwise we can choose another atom in the third element of $\mathcal{X}$ to replace $C$. As we know that the elements covering $C$ without $C^*$ partition $J \setminus C \cup C^*$ and every element is a join of atoms we must have some element $D \in L$ such that $C < D$ and $C' \subseteq D$. Therefore there is a maximal chain $Z_i$ between $C'$ and $D$. We do the same thing with $D'$ the element in $\mathcal{X}$ covering $C$ (which is not $D$ as it does not contain $C'$), we again have $C'$ contained in one of the elements...
covering $D'$, denote this element $F$. We now show that $D' \subseteq F$, if not, we can observe $D' \cap F$ which is strictly contained in $D'$ and contains $C$ (this is also not $C$ as it contains $C'$) which is a contradiction as $C \ll D'$.

Hence, we can build a maximal chain from $\emptyset$ to $F$ containing $C'$ denoted $Z_2$. After finitely many steps of the same process, we get a maximal chain from $\emptyset$ to $B$ going through $C'$ with length larger than or equal to the length of $X$, and since the length of $X$ is strictly larger than the length of $Y$ we get a contradiction.

3. **Independent sets of a lattice**

In [9] Andreas Blass and Bruce E. Sagan introduced the concept of $NBB$ sets as a way to calculate the Möbius function of a finite bounded lattice. These sets can be thought of as a generalization of $NBC$ sets for ordinary matroids. In this section, we show that if these $NBB$ sets form a matroid we can order embed the lattice in a geometric lattice, respecting the rank function of the original lattice.

3.1. **Definition:** Let $(\mathcal{L}, \omega)$ be a pair of a finite bounded lattice $\mathcal{L}$ and a partial order $\omega$ on its set of atoms $A(\mathcal{L})$. A nonempty set $D \subseteq A(\mathcal{L})$ of atoms is bounded below or $BB$ if there exist $a \in A(\mathcal{L})$ such that $a$ is a strict lower bound for all $d \in D$ in the order $\omega$ and $a \leq \lor D$ in $\mathcal{L}$.

A set $B \subseteq A(\mathcal{L})$ is called $NBB$ if $B$ does not contain any bounded below subset. Given a bounded lattice $\mathcal{L}$ and $P$ the set of all linear orders on $A(\mathcal{L})$ we define the family of independent sets $I(\mathcal{L}, P)$ of $\mathcal{L}$ as follows:

$$I(\mathcal{L}, P) = \{A \in P(A(\mathcal{L})) \mid \exists \omega \in P \text{ s.t. } A \text{ is an } NBB \text{ set in } (\mathcal{L}, \omega)\}$$

We refer to elements in $I(\mathcal{L}, P)$ as independent sets of $\mathcal{L}$, note that sets containing at most two atoms are always independent. The constructions in the section can be thought of as a generalization of the following observation:

3.2. **Theorem:** If $\mathcal{L}$ is a geometric lattice, then $I(\mathcal{L}, P)$ is the family of independent sets of the ordinary matroid corresponding to $\mathcal{L}$.

For the remainder of this section, we fix a finite, bounded, atomic, and graded lattice $\mathcal{L}$. If $\lor A = X \in \mathcal{L}$ for a set of atoms $A \subseteq A(\mathcal{L})$ we say that $A$ spans $X$ and set $\text{rank}(A) := \text{rank}(X)$.

3.3. **Lemma:** If $I$ is an independent set then there exist an atom $a \in I$ such that $\text{rank}(I \setminus \{a\}) \leq \text{rank}(I)$.

*Proof:* If it were true that $\text{rank}(I \setminus \{a\}) = \text{rank}(I)$ for every atom $a \in I$ then by our previous discussion:

$$a \leq \lor (I \setminus \{a\}), \forall a \in I$$

Let $\omega \in P$ be a linear order for which $I$ is $NBB$ and let $a$ be the first element of $I$ with respect to $\omega$. we have $a \in \lor (I \setminus \{a\})$ and so $I \setminus \{a\}$ is $BB$ by $a$ with respect to $\omega$, which is a contradiction to $I$ being an independent set.

3.4. **Lemma:** $\text{rank}(I) \geq |I|$ for every independent set $I$. 
Proof: The theorem follows from Lemma [3.3] and an induction argument on the rank of $I$.

In the case of geometric lattices, the rank of a flat is the size of a maximal independent set contained in it. The next two lemmas establish the same property for our lattice $\mathcal{L}$. For this statement to be true we first observe that if $I$ is an independent set and $a \in A(\mathcal{L})$ such that $\text{rank}(I) \preceq \text{rank}(I \cup \{a\})$ then $I \cup \{a\}$ is also independent, which follows again by taking the linear order for which $I$ is NBB and placing $a$ first. This observation implies that every element of $\mathcal{L}$ is spanned by some independent set.

3.5. Figure: The dual lattice of the partition lattice on 4 elements (a) and its induced geometric lattice (b).

3.6. Lemma: The rank of an element $X \in \mathcal{L}$ is the maximum size of a spanning independent set $I$.

Proof: As $\forall I \leq X$, we must have $|I| \leq \text{rank}(I) \leq \text{rank}(X)$. Next we find a spanning independent set of $X$ of size $\text{rank}(X)$. Let $Y \ll X$, as $\mathcal{L}$ is atomic, there exists an atom $a \leq X$ such that $Y \lor a = X$. By an induction argument, we have an independent set $J$ spanning $Y$ with $|J| = \text{rank}(Y) = n$ and as $a \not\preceq \lor J$ we must have $I = J \cup \{a\}$ an independent set of size $n + 1$ spanning $X$.

As we mentioned before, in the case of geometric lattices the rank of an element is the size of a maximal independent set contained in it. In the other direction, starting with the family of independent sets, the elements of the lattice (called flats and ordered by inclusion) are the closed subsets. Formally, $F$ is a flat of the matroid $\mathcal{I}$ on the ground set $[n]$ if $\text{rank}(F) \preceq \text{rank}(F \cup \{i\})$ for all $i \notin F$. 
3.7. **Theorem:** If \( I(\mathcal{L}, P) \) is the family of independent sets of an ordinary matroid then \( \mathcal{L} \) can be uniquely extended to a geometric lattice \( \mathcal{P} \), denoted the induced geometric lattice of \( \mathcal{L} \), with \( I(\mathcal{L}, P) = I(\mathcal{P}, P) \). In this case the restriction \( f|_{\mathcal{L}} \) of the rank function \( f \) of \( \mathcal{P} \) is the original rank function on \( \mathcal{L} \).

**Proof:** We know that there exist a unique ordinary matroid \( M_L \) on the ground set \( A(\mathcal{L}) \) corresponding to the family of independent sets \( I(\mathcal{L}, P) \) of \( \mathcal{L} \). We identify each element \( X \in \mathcal{L} \) with the set \( D = \{ a \in A(\mathcal{L}) \mid a \leq X \} \). To see \( D \) is a flat of \( M_L \) observe that for every atom \( a /\in D \) we have \( X \not\succ \lor(D \cup \{a\}) \) and so \( \text{rank}(X) \not\preceq \text{rank}(D \cup \{a\}) \).

As the rank function of the matroid agrees with the rank function of \( \mathcal{L} \) we must then have the lattice of flats of \( M_L \) be the desired geometric lattice.

3.8. **Remark:** It is not always true that \( I(\mathcal{L}, P) \) is a family of independent sets of an ordinary matroid. A wide class of examples can be found in the study of the adjoint of a matroid. The first example I could find is by A. C. Cheung and can be observed in [10]. Another known example is the dual lattice of the Vamos matroid.

In addition to the previous embedding theorem, we give a characterization of geometric lattices using independent sets. We call an independent set \( I \) geometric if \(|I| = \text{rank}(I)\).

3.9. **Theorem:** \( \mathcal{L} \) is a geometric lattice if and only if every independent set \( I \) of \( \mathcal{L} \) is geometric.

**Proof:** This property is known for the geometric lattice corresponding to the ordinary matroid \( I(\mathcal{L}, P) \). Let \( X, Y \in \mathcal{L} \) and \( I \) be an independent set that spans \( X \land Y \). We can extend \( I \) to independent sets \( I \cup J \) that span \( X \) and \( I \cup J' \) that span \( Y \). Note also that \( I \cup J \cup J' \) spans \( X \lor Y \) and therefore contains an independent set \( K \) that spans \( X \lor Y \). This proves that \( \mathcal{L} \) is sub-modular as:

\[
\text{rank}(X) + \text{rank}(Y) = 2|I| + |J| + |J'| \geq \text{rank}(X \land Y) + \text{rank}(X \lor Y)
\]

In figure [3.4] we can see an example of the induced geometric lattice of the dual lattice of the partition lattice on the set of 4 elements. The element \( X \) added to the lattice reduces the rank of the size two subsets of \{124/3, 13/24, 123/4\}, which are the only independent sets with different size and rank.

4. **Correspondence to Symplectic matroids**

4.1. **Remark:** For the purpose of this section we do not consider the full \( C_n \) lattice \( P^{ad}(J) \cup \{J\} \) to be a \( C_n \) lattice. The reason is that \( P^{ad}(J) \cup \{J\} \) is the only \( C_n \) lattice in which there are no maximal admissible independent sets. As a result, the symplectic matroid corresponding to \( P^{ad}(J) \cup \{J\} \) is the same as the one corresponding to the lattice of all admissible subsets of \( J \) of size \( \leq n - 1 \). To avoid our correspondence not being injective, we choose to discard \( P^{ad}(J) \cup \{J\} \).

We move on to showing the correspondence between \( C_n \) lattices and ranked symplectic matroids, starting with some definitions.
An ordering $\leq$ on $J = [n] \sqcup [n]^*$ is called admissible if and only if $\leq$ is a linear ordering and from $i \leq j$ it follows that $j^* \leq i^*$. A linear order on $J$ induces an order on size $k$ subsets of $J$ in the following way. Let $A = \{a_1 < a_2 < ... < a_k\}$ and $B = \{b_1 < b_2 < ... < b_k\}$, we set $A \leq B$ if

$$a_1 \leq b_1, a_2 \leq b_2, ..., a_k \leq b_k$$

4.2. **Definition:** A symplectic matroid $B$ on $J$ is a family of equi-numerous admissible subsets of $J$ called bases, which contains a unique maximal element with respect to every admissible ordering. A symplectic matroid will be called loop-free if every element of $J$ is contained in some basis.

An equivalent definition in terms of independent sets was introduced by Timothy Y. Chow in [4]:

4.3. **Definition:** A subset-closed family $I$ of admissible subsets of $J$ is the family of independent sets of a symplectic matroid if and only if it adheres the following:

1. For every transversal $T$, $\{I \cap T \mid I \in I\}$ is the family of independent sets of an ordinary matroid with ground set $T$, and
2. If $I_1$ and $I_2$ are members of $I$ such that $|I_1| \leq |I_2|$, then either there exists $a \in I_2 \setminus I_1$ such that $\{a\} \cup I_1 \in I$ or there exists $a \notin I_1 \cup I_2$ such that both $\{a\} \cup I_1 \in I$ and $\{a^*\} \cup I_2 \setminus I_2^* \in I$.

In this section we use a slightly different notion of $\text{NBB}$ and independent sets defined in section 2. The reason is that we want independent sets to be subsets of the ground set $J$ and not of the set of atoms $A(\mathcal{L})$. For a subset $X \subseteq J$ we define $A(X) \subseteq A(\mathcal{L})$ to be the set of atoms that contain the elements of $X$, we say $X$ is disjoint in $\mathcal{L}$ if $A(X) \neq A(X \setminus \{a\})$ for all $a \in X$. Notice that a linear order on $J$ induces a linear orders on the set of atoms of a $C_n$ lattice, giving the desired connection between the definitions. All of the theorems in the previous section hold with a small change in formulation using the new definitions.

4.4. **Definition:** Let $(\mathcal{L}, \omega)$ be a bounded lattice of subsets on a finite ground set $E$ with a partial order $\omega$ on $E$. A nonempty disjoint in $\mathcal{L}$ set $D \subseteq E$ is bounded below or $BB$ if there exist $a \in E$ such that $a$ is a strict lower bound for all $d \in D$ in the order $\omega$ and $a \in \forall D$.

A set $B \subseteq E$ is called $\text{NBB}$ if $B$ does not contain any bounded below subset.

We define the family of independent sets $I(\mathcal{L}, P)$ of a bounded lattice $\mathcal{L}$ on the ground set $E$ with $P$ the set of all linear orders on $E$.

$$I(\mathcal{L}, P) = \{A \in P(\mathcal{L}) \mid \exists \omega \in P \text{ s.t. } A \text{ is an NBB set in } (\mathcal{L}, \omega)\}$$

For a $C_n$ lattice $\mathcal{L}$, for which the ground set is $J$, we denote by $I^{ad}(\mathcal{L}, P)$ the family of admissible independent sets of $\mathcal{L}$ and by $\overline{I^{ad}}(\mathcal{L}, P)$ the family of admissible sets of $\mathcal{L}$ with admissible join. We are ready to state the main theorem of this section:

4.5. **Theorem:** If $\mathcal{L}$ is a $C_n$ lattice then $I^{ad}(\mathcal{L}, P)$ is a family of independent sets of a symplectic matroid $B$. 
In order to prove this theorem we need to work with two types of ordinary matroids induced by \( \mathcal{L} \). We start with the first type of ordinary matroid, corresponding to the family of independent sets of the lattice \( \mathcal{L} \cap T = \{ X \cap T \mid X \in \mathcal{L} \} \) for a transversal \( T \). We introduce the following lemma:

4.6. **Lemma:** For any \( A \in P^{ad}(J) \), we have \( I(\mathcal{L}, P) \cap A = I(\mathcal{L} \cap A, P |_{A}) \).

**Proof:** For the first inclusion it is enough to prove that if \( D \) is a minimal \( BB \) set in \( \mathcal{L} \cap A \) with respect to a partial order \( \omega \) on \( A \) then \( D \) is a \( BB \) in \( \mathcal{L} \) with respect to any extension of \( \omega \) to \( J \). This is the case as the lower bound for \( D \) in \( \mathcal{L} \cap A \) will also be a lower bound in \( \mathcal{L} \). For the second inclusion let \( I \) be \( NBB \) in \( \mathcal{L} \cap A \) with respect to an order \( \omega \), we have that with respect to a partial order starting with \( \omega \), \( I \) is \( NBB \) in \( \mathcal{L} \).

We first describe the inadmissible independent sets in \( \mathcal{L} \). For that purpose, we introduce the following lemma:

4.7. **Lemma:** Let \( I \in I^{ad}(\mathcal{L}, P) \) and \( a \in I \) then exactly one of \( I \cup \{ a^* \} \in I(\mathcal{L}, P) \) or \( rank(I) = rank(\mathcal{L}) \) holds.

**Proof:** Assume \( I \cup \{ a^* \} \notin I(\mathcal{L}, P) \) then by definition [4.4] we have \( a^* \in \forall I \). Therefore, \( \forall I \) is inadmissible and we get from definition [2.1] that \( rank(I) = rank(\mathcal{L}) \).

We can conclude that \( I \cup \{ a, a^* \} \in I(\mathcal{L}, P) \) for \( I \in I^{ad}(\mathcal{L}, P) \) if \( I \cup \{ a \}, I \cup \{ a^* \} \in I^{ad}(\mathcal{L}, P) \). On the other hand, if \( a, a^*, b, b^* \in B \subseteq A(\mathcal{L}) \) then \( a \in \forall B \setminus \{ a \} \) and \( b \in \forall B \setminus \{ b \} \), therefore \( B \) is not an independent set. We have thus proven the following theorem:

4.8. **Theorem:** Let \( \mathcal{L} \) be a \( C_n \) lattice, then:

\[
I(\mathcal{L}, P) = I^{ad}(\mathcal{L}, P) \cup \left\{ I \cup \{ a, a^* \} \mid I \cup \{ a \}, I \cup \{ a^* \} \in I^{ad}(\mathcal{L}, P) \right\}
\]

The second type of ordinary matroid that we introduce is the one induced by the original family of independent sets of \( \mathcal{L} \), including the inadmissible independent sets. We use Theorem [3.7] to induce a unique geometric lattice \( \mathcal{P} \) on the same ground set with the same family of independent sets such that \( \mathcal{L} \subseteq \mathcal{P} \) denotes the geometric lattice induced by \( \mathcal{L} \).

4.9. **Theorem:** If \( \mathcal{L} \) is a \( C_n \) lattice, then \( I(\mathcal{L}, P) \) is a family of independent sets of an ordinary matroid.

**Proof:** We prove that \( I(\mathcal{L}, P) \) has the augmentation property for independent sets, stating that for a smaller independent set \( I_1 \) there exists \( a \in I_2 \setminus I_1 \) such that \( I_1 \cup \{ a \} \) is independent. Let \( I_1, I_2 \in I(\mathcal{L}, P) \) be such that \( |I_1| \leq |I_2| \). Notice that if \( rank(I_1) \neq rank(\mathcal{L}) \), then \( rank(I_1) = |I_1| \leq rank(I_2) \), and we can therefore assume \( rank(I_1) = rank(\mathcal{L}) \).

Let \( a \in I_1 \) be an atom such that \( rank(I_1 \setminus \{ a \}) \leq rank(I_1) \) is guaranteed by lemma [3.3]. It is enough to find an atom \( b \in I_2 \) for which the following strict inequality holds:

\[
rank(I_1 \setminus \{ a \}) \leq rank((I_1 \setminus \{ a \}) \cup \{ b \}) \leq rank(\mathcal{L})
\]

Following lemma [4.7], the inequality above holds for \( b \in I_2 \) iff \( b, b^* \notin \forall(I_1 \setminus \{ a \}) \).

As we have subtracted an element from \( I_1 \) we can do the same for \( I_2 \) while keeping it larger, so assume \( rank(I_2) \leq rank(\mathcal{L}) \). Therefore, applying lemma [4.7] we
have \((I_2 \setminus \{b\}) \cup \{b^*\} \in \mathcal{I}^{ad}(\mathcal{L}, P)\) and \(\text{rank}(I_2 \setminus \{b\} \cup \{b^*\}) \leq \text{rank}(\mathcal{L})\) for every \(b \in I_2\). If there exist no \(b \in I_2\) such that \(b, b^* \notin \vee(I_1 \setminus \{a\})\) then by replacing all \(b \in I_2\) with \(b^* \in \vee(I_1 \setminus \{a\})\) we obtain an admissible independent set \(I_3\) such that \(\vee I_3 \subseteq \vee I_1 \subseteq J\) but \(|I_1| \leq |I_2| = |I_3|\), contradicting Lemma [3.4].

4.10. **Theorem:** Let \(\mathcal{L}\) be a \(C_n\) lattice and \(\mathcal{P}\) the induced geometric lattice then:

\[
\mathcal{L} = \mathcal{P} \cap \mathcal{P}^{ad}(J)
\]

**Proof:** Assume \(X\) is an admissible element of \(\mathcal{P}\) that covers (in \(\mathcal{P}\)) an element \(Y\) in \(\mathcal{L}\). As the union of the elements of \(\mathcal{L}\) covering \(Y\) in \(\mathcal{L}\) is \(J \setminus (Y \cup Y^*)\) we must have some element \(Z\) in \(\mathcal{L}\) covering \(Y\) and \(X \cap Z \in \mathcal{L}\) covering \(Y\). Therefore, \(Y \subseteq X \subseteq Z\) and we have changed the rank of \(X\) which contradicts theorem [3.8]. As \(A(\mathcal{L}) = A(\mathcal{P})\), if we had \(\mathcal{L} \neq \mathcal{P} \cap \mathcal{P}^{ad}(J)\), there would have been such \(X\).

We can now use theorems [4.8], [4.9], and [4.10] to formulate the characterization of symplectic matroids corresponding to \(C_n\) lattices using rank functions.

4.11. **Definition:** Let \(\mathcal{I}\) be the family of independent sets of a symplectic matroid. We call \(I \in \mathcal{I}\) a delta-independent set if \(I \cup \{a\} \in \mathcal{I}\) or \(I \cup \{a^*\} \in \mathcal{I}\) for every \(a \in J\). We denote the family of delta-independent sets by \(\mathcal{I}^\Delta\).

4.12. **Theorem:** Let \(\mathcal{I}\) be a family of independent sets of a symplectic matroid of rank \(d\) with no loops, then \(\mathcal{I}\) corresponds to a \(C_n\) lattice and denoted a ranked symplectic matroid iff the function \(r : \mathcal{P}(J) \to \mathbb{Z}_+\), sending a set \(A\) to \(\min(d, r(A))\), is the rank function of a loop-free ordinary matroid on the ground set \(J\) for the following function \(r\):

\[
r(A) = \min\left(d, \max_{I \subseteq A} \left\{\frac{|I| + 2}{|I|} : \exists \{a, a^*\} \subseteq A \setminus I, I \cup \{a\}, I \cup \{a^*\} \in \mathcal{I} \right\}\right)
\]

**Proof:** The "only if" direction is Theorem [4.8]. For the "if" direction it is enough to show, again using theorem [4.8], that \(\mathcal{P} \cap \mathcal{P}^{ad}(J)\) is a \(C_n\) lattice for the induced geometric lattice \(\mathcal{P}\) of \(r\). We note that every atom of \(\mathcal{P}\) is admissible by the definition of \(r\) and \(\mathcal{I}\) being without loops. Therefore, we are reduced to showing that any non-maximal admissible flat \(F\) of \(\mathcal{P}\) is covered by exactly one inadmissible flat \(F \cup F^*\). To see that any inadmissible flat covering \(F\) is contained in \(F \cup F^*\), let \(I \cup \{a\}\) be an independent set spanning \(F\) such that \(I \cup \{a^*\}\) is also independent. If \(b, b^* \notin F\), then we must have \(b, b^* \notin cl_r(I \cup \{a, a^*\})\). As both \(I \cup \{a, b\}\) and \(I \cup \{a, b^*\}\) are independent and \(I \cup \{a\} \in \mathcal{I}^\Delta\), contradicting \(cl_r(I \cup \{a, a^*\}), cl_r(I \cup \{b, b^*\})\) being of the same rank. As the rank function \(r\) does not allow independent sets with more than one pair of inadmissible elements, we must also have \(F \cup F^*\) contained in the inadmissible flat covering \(F\).

**Proof of theorem 4.5:** By definition, \(NBB\) is a hereditary property, so \(I(\mathcal{L}, P)\) is a subset closed family.

1. Given a transversal \(T \subseteq J\) we have by lemma [4.5] that \(I(\mathcal{L}, P) \cap T = I(\mathcal{L} \cap T, P|_T)\) and the restriction of \(P\) to \(T\) the set of all linear orders on \(T\). By Theorem [3.6] we then have \(I(\mathcal{L}, P) \cap T\) the family of independent sets of an ordinary matroid with a ground set \(T\).
(2) Let \( I_1, I_2 \in I^{ad}(\mathcal{L}, P) \) be such that \( |I_1| \leq |I_2| \). As \( I_1, I_2 \in I(\mathcal{L}, P) \), which is a family of independent sets of a matroid, there exist \( a \in I_2 \) such that \( I_1 \cup \{a\} \in I(\mathcal{L}, P) \). Assume all such \( a \) has \( a^* \in I_1 \) and observe that \( |I_1 \setminus I_2^*| \leq |I_2 \setminus I_1^*| \) for the smaller independent sets \( I_1 \setminus I_2^*, I_2 \setminus I_1^* \). Let \( b \in I_2 \setminus I_1^* \) be such that \( (I_1 \setminus I_2) \cup \{b\} \in I^{ad}(\mathcal{L}, P) \). We actually have \( I_1 \setminus I_2^* \in I(\mathcal{L}, P) \) because we have removed all inadmissible pairs in \( I_1 \cup I_2 \).

Using Lemma [4.7], we see that, as \( I_2 \cup \{a\} \in I(\mathcal{L}, P) \) is inadmissible, \( rank(I_2) \leq rank(\mathcal{L}) \). Using Lemma [4.7] once more, we get \( I_1 \cup \{b^*\} \in I(\mathcal{L}, P) \) satisfying the second part of condition (2) of definition [4.3].

Using ordinary matroid terminology, we have defined a "cryptomorphism" \( f \) sending a \( C_n \) lattice to the symplectic matroid defined by its family of admissible independent sets. By Theorems [3.7] and [4.10] the family of admissible independent sets is uniquely determined by the induced geometric lattice, and so we have \( f \) injective, see remark [4.1].

The inverse of \( f \) is obtained by taking the sublattice of admissible flats of the geometric lattice corresponding to the rank function of the ranked symplectic matroid.

The next two remarks now easily follow:

4.13. **Remark 1:** An ordinary matroid on \( J \) of rank \( \geq 3 \) with admissible bases, and so also a symplectic matroid, is never a ranked symplectic matroid. The rank function from Theorem [4.11] will never be submodular.

4.14. **Remark 2:** If \( \mathcal{B} \) is an ordinary matroid with a partition of its atoms into pairs \( 1,1^*,...,k,k^* \) such that its rank function identifies with the induced rank function of its admissible independent sets (in the sense of Theorem [4.11]) then \( \mathcal{B} \cap P^{ad}(J) \) is a ranked symplectic matroid.

The second remark introduces a new way of constructing families of symplectic matroids using ordinary matroids. This can be helpful because the theory of ordinary matroids is much more developed. We show an example of how any spike with no tip gives rise to a ranked symplectic matroid.

In [6] T. Zaslavsky introduced lift matroids on bias graphs for which spikes are a special case.

4.15. **Definition:** A biased graph is a pair \((G, \mathcal{C})\) where \( G \) is a finite undirected multigraph and \( \mathcal{C} \) is a set of cycles of \( G \) satisfying the following theta property.

**Theta property:** For every two cycles \( C_1 \) and \( C_2 \) in \( \mathcal{C} \) that intersect in a nonempty path, the third cycle in \( C_1 \cup C_2 \) is also in \( \mathcal{C} \).

The cycles of \( \mathcal{C} \) are called balanced and the other cycles are unbalanced. The lift matroid represented by \( \mathcal{M}(G, \mathcal{C}) \) has for its ground set the set of edges of \( G \) and, as independent sets, the sets of edges containing at most one cycle, which is unbalanced. A spike with no tip is a lift matroid with \( G \) the graph obtained from the cycle on \( n \) vertices by doubling every edge and \( C \) some set of cycles that do not contain any pair of double edges satisfying the theta property.

To see that any spike with no tip \( \mathcal{M}(G, \mathcal{C}) \) is a simple matroid of the form defined in Theorem [4.8], notice that \( J = [n] \cup [n^*] \) with \( n \) being the size of the original cycle in \( G \). Every simple cycle in \( G \) is a set of the form \( \{i, i^*\} \) for some edge \( i \)
or a transversal $T$. Therefore, every non maximal admissible independent set $I$ of $\mathcal{M}(G, \mathcal{C})$ does not contain a cycle and $I \cup \{i^*\}$ is an independent set of $\mathcal{M}(G, \mathcal{C})$ for every $i \in I$. Moreover, as $\{i, i^*, j, j^*\}$ contains two cycles it cannot be extended to an independent set.

4.16. **Figure**: The graph defining a spike with no tip, with the blue and red cycles being balanced (a), the $C_n$ lattice induced by it (b), and the convex polytopes associated with its ranked symplectic matroid (c).

In [4] T. Chow introduced graphic symplectic matroids. We note that our construction of spikes with no tip is not necessarily graphic symplectic matroids, as the transversal being even does not correspond to the number of starred elements.

5. **Shellability**

The purpose of this section is to prove that the $C_n$ lattices are shellable. We start with a few definitions.

5.1. **Definition**: A simplicial complex $\Delta$ is a set of simplices satisfying the following conditions:

1. Every face of a simplex from $\Delta$ is also in $\Delta$.
2. The nonempty intersection of any two simplices $\sigma, \tau \in \Delta$ is a face of both $\sigma$ and $\tau$.

5.2. **Definition**: Let $\Delta$ be a finite simplicial complex. We say that $\Delta$ is pure $d$ dimensional if all its facets are of dimension $d$. A pure $d$ dimensional simplicial complex $\Delta$ is said to be shellable if its facets can be ordered $F_1, ..., F_t$ in such a way that:

$$F_j \cap \bigcup_{i=1}^{j-1} F_i$$

is a pure $(d-1)$ dimensional complex for $j = 2, 3, ..., t$, with:

$$\mathcal{T}_j = \{G \text{ a simplicial complex} \mid G \subseteq F_j\}$$

To a finite poset $\mathcal{L}$ one can associate a simplicial complex $\Delta(\mathcal{L})$ (the order complex) of all unrefinable chains of $\mathcal{L}$. Also, if $\mathcal{L}$ is a graded poset of rank $d$ then $\Delta(\mathcal{L})$ is pure $d$ dimensional. We say that a finite and graded poset $\mathcal{L}$ is shellable if its order complex $\Delta(\mathcal{L})$ is shellable.

We now define a recursive atom ordering.
5.3. **Definition:** A graded poset $L$ is said to admit a recursive atom ordering if the rank of $L$ is 1 or if the rank of $L$ is greater than 1 and there is an ordering $A_1, ..., A_t$ of the atoms of $L$ that satisfies:

1. For all $j = 2, ..., t$ we have $[A_j, \hat{1}]$ admit a recursive atom ordering in which the atoms of $[A_j, \hat{1}]$ that come first in the ordering are those that cover some $A_i$ where $i < j$.
2. For all $i < j$, if $A_i, A_j < B$ for some $B \in L$ then there is some $k < j$ and an element $B \geq C \in L$ such that $C$ covers $A_k$ and $A_j$.

5.4. **Theorem:** In [7] A. Bjorner and M. Wachs proved that if a graded poset $L$ admits a recursive atom ordering, then the order complex $\Delta(L)$ is shellable.

We are now ready to start proving that $C_n$ lattices are shellable, using induction on the number of atoms. We start by introducing some technical lemmas.

5.5. **Remark:** It is possible to prove a stronger type of lexicographic shellability called EL-shellability by taking the known EL-labeling for geometric lattices (can be found in [10]) of its enveloping ordinary matroid, starting with the atoms of an admissible coatom. We decided to prove CL-shellability, equivalent to admitting a recursive atom ordering, because we obtain more shelling orders and the characterization in part 6.

5.6. **Lemma:** If $A$ is an atom of a $C_n$ lattice $L$ of rank $\geq 3$ then $A^*$ is also an atom of $L$.

**Proof:** Assume that $A, B$ are atoms of $L$ with $B^* \cap A \neq \emptyset$. If $B^* \setminus A \neq \emptyset$ we must have for every $c \in A \setminus B^*$ some element $c \in C \in L$ with $C$ covering $B$, but as we proved that $L$ is atomic and so we must have $A \subseteq C$ which is a contradiction as:

$$B^* \cap A \neq \emptyset \Rightarrow C^* \cap A \neq \emptyset$$

$C$ will not be admissible and not of full rank as $L$ is of rank $\geq 3$. So we must have:

$$B^* \cap A \neq \emptyset \Rightarrow B^* = A$$

And so if $a \in A$ there exist some atom $B \in L$ such that $a^* \in B$ (as the set of atoms partition $J$), but then $B = A^*$.

5.7. **Lemma:** Let $L$ be a $C_n$ lattice and $A \in L$ then the restriction $[A, \hat{1}] \subseteq L$ is order isomorphic to a $C_n$ lattice on the ground set $J' = J \setminus (A \cup A^*)$.

**Proof:** We define

$$[A, \hat{1}] \setminus A = (\{X \setminus A \mid X \in [A, \hat{1}]\} \setminus \{J \setminus A\}) \cup \{J'\}$$

This is obviously isomorphic to $[A, \hat{1}]$ by the lattice morphism:

$$\psi : [A, \hat{1}] \rightarrow [A, \hat{1}] \setminus A$$

$$X \mapsto X \setminus A$$

(At first glance, it is not clear that $[A, \hat{1}] \setminus A$ and $[A, \hat{1}]$ are lattices, but this becomes obvious by proving $[A, \hat{1}] \setminus A$ is a $C_n$ lattice).

We continue with proving the four conditions:
(1) $\emptyset, J' \in L$ as we have $A \mapsto \emptyset$ and $J \mapsto J'$.
(2) Every element $B \in [A, \hat{A}] \setminus A$ is admissible as it is an element of $L$ with some elements removed.
(3) $\forall B, C \in [A, \hat{A}] \setminus A$ we have:
$$A \subseteq (A \cup B) \cap (A \cup C) \in L \Rightarrow (A \cup B) \cap (A \cup C) \in [A, \hat{A}] \Rightarrow A \cap B \in [A, \hat{A}] \setminus A$$
(4) For $B \in [A, \hat{A}] \setminus A$ let $\{B_1, ..., B_m\}$ the set of elements in $[A, \hat{A}] \setminus A$ covering $B$. Therefore, we have $B \cup A \in L$ and a set of elements $\{C_1, ..., C_k\}$ covering $B \cup A$ in $L$. To see $C_i \setminus A$ covers $B$ in $[A, \hat{A}] \setminus A$, assume that there is some $C \in [A, \hat{A}] \setminus A$ such that $B \leq C \leq C_i \setminus A$. Hence $C \cup A \in L$ and so $B \cup A \leq C \cup A \leq C_i$ in $L$. Resulting in $C \cup A = B \cup A \Rightarrow C = B$ or $C \cup A = C_i \Rightarrow C = C_i \setminus A$. In a similar way, we show that $B_i \cup A \in L$ covers $B \cup A$ in $L$ and so we get:
$$\{B_1, ..., B_m\} = \{C_1 \setminus A, ..., C_m \setminus A\}$$
Therefore, we have:
$$B_i \cap B_j = (C_i \setminus A) \cap (C_j \setminus A) = (C_i \cap C_j) \setminus A = (B \cup A) \setminus A = B$$
And also:
$$\bigcup_{i=1}^{m} B_i = \bigcup_{i=1}^{m} (C_i \setminus A) = (\bigcup_{i=1}^{m} C_i) \setminus A = (J \setminus (B \cup A)^*) \setminus A = J' \setminus B^*$$

5.8. Lemma: If the number of atoms in a $C_n$ lattice $L$ is $t$, then the number of atoms of $[A, \hat{A}] \subseteq L$ for $A$ an atom of $L$ is $\leq t - 2$.

Proof: As $L$ is atomic, we have for each of its elements a join of atoms. By property (4) of $C_n$ lattices the elements that cover $A$ pairwise disjoint (excluding the elements of $A$) and so every element covering $A$ is a choice of some atoms from a set of size $t - 2$, namely the set $A(L) \setminus \{A, A^*\}$. There can be at most $t - 2$ choices.

To finish our technical preparations we recall the definition of an independent set following Section 4. Using Lemma [5.5], we have for a $C_n$ lattice $L$ of rank $\geq 3$, $A(L) = A' \cup A^*$. Therefore, since $L$ is also atomic it is order isomorphic to a $C_n$ lattice on the ground set $A(L)$ sending the atom $A_i$ in each element to $i$ and the atom $A_i^*$ to $i^*$. In the following theorems, we take independent sets as sets of atoms using the correspondence above. In addition, using lemmas [5.5] and [5.6] we can give a $C_n$ lattice structure to the restriction $[B, \hat{B}]$ for every $B \in L$. As the $*$ function on the restriction is induced by the original function on $L$, we get the following relation between admissible independent sets:

5.9. Lemma: Let $\omega$ be a linear order on the atoms of a $C_n$ lattice $L$ of rank $d \geq 4$ with the first $d - 1$ atoms forming a geometric independent set $I$. Then for every $A_i \in A(L)$ there exists a linear order $\omega_i$ on the atoms of $L_i = [A_i, \hat{A}]$ with the atoms that come first in the ordering being those that cover some $A_j$ where $j < i$ and the first $d - 2$ atoms form a geometric independent set of $L_i$. Moreover, if $\omega$ has that no atom is the star of the previous atom we can construct $\omega_i$ also having this property. We denote such an order $\omega$ as an admissible ordering.
Proof: Let \( \omega \) be such an order and define the induced order \( \omega_i \) on \( A(\mathcal{L}_i) \) taking \( B_j < B_k \) if \( B_j \) covers an atom smaller than any atom covered by \( B_k \) (excluding \( A_i \)) with respect to \( \omega \). As the atoms of \( \mathcal{L}_i \) partition \( A(\mathcal{L}_i) \setminus \{ A_i, A_i^* \} \), \( \omega_i \) is a well-defined linear order. We now observe two cases:

1. If \( i \leq d - 1 \), then using Theorem [4.9], the first \( d - 2 \) atoms of \( \omega_i \) will be \( A_1 \lor A_i, ..., A_{d-1} \lor A_i \) and so a geometric independent set of \( \mathcal{L}_i \) and \( \omega_i \) the required order.

2. If \( i \geq d \), let \( \mathcal{T}' \) be an extension of \( A_i \) to a geometric independent set of size \( d-1 \) contained in \( \mathcal{T} \cup \{ A_i \} \). We remark that if \( A_i^* \in \mathcal{T} \) then \( (\mathcal{T} \setminus \{ A_i^* \}) \cup \{ A_i \} \) is a geometric independent set according to the lemma [4.7]. Otherwise, we have one of \( A_i, A_i^* \) not in \( \mathcal{T} \) and by Theorem [4.9] there exist some \( j \leq d-2 \) such that \( A_i, A_i^* \notin \mathcal{T} \setminus \{ A_j \} \). Therefore, \( (\mathcal{T} \setminus \{ A_j \}) \cup \{ A_i \} \) is a geometric independent set. Let \( \omega' \) be the order on \( A(\mathcal{L}) \) obtained from \( \omega \) by moving \( A_i, A_j \) between \( A_{d-1} \) and \( A_d \). We then have \( \omega'_i \), the induced order by \( \omega' \) on \( A(\mathcal{L}_i) \) being the required order.

As no non-maximal geometric independent set contains an atom and its star, we can extend \( \mathcal{T} \) to a linear order \( \omega \) with \( A_i \neq A_i^* + 1 \) for every \( i \leq |A(\mathcal{L})| \). Notice that we cannot always extend the \( \mathcal{T} \) admissible independent set constructed above in \( \mathcal{L}_i \) in the way specified. However, since we can always rearrange the atoms of the independent set (it always contains more than two elements), a problem occurs only if the atoms of \( \mathcal{L}_{d+1} \) covering \( A_{d+1} \) are exactly our independent set + an extra atom and its star. In this case, we have \( \{ A_{d+1}, A_j \} \) (as above) an independent set and we can change \( \mathcal{T}' \) so that it contains \( \{ A_i, A_j \} \). As there is only one star of \( A_d \lor A_{d+1} \) in \( \mathcal{L}_{d+1} \), this solves our problem.

5.10. Corollary: If \( \{ a_1, ..., a_k \} \) is a geometric independent set of a graded, bounded, and atomic lattice \( \mathcal{S} \), then \( \{ a_1 \lor a_i, ..., a_k \lor a_i \} \setminus \{ a_i \} \) is a geometric independent set of \( [a_i, 1] \).

5.11. Theorem: Every \( C_n \) lattice \( \mathcal{L} \) admits a recursive atom ordering.

Proof: First notice that the conditions for a recursive atom ordering are met if the rank of \( \mathcal{L} \) is 2, since then all atoms are covered by \( 1 \). Another separated case is rank(\( \mathcal{L} \)) = 3, in this case taking any linear order \( \omega \) on \( A(\mathcal{L}) \) with \( A_i \neq A_i^* + 1 \) for every \( i \leq |A(\mathcal{L})| \) is a recursive atom ordering. The first condition is obvious, as all the elements of rank 2 are covered by \( 1 \). The second condition is true, as every two consecutive atoms are covered by a rank 2 element and the only two elements not covered by a rank 2 element are stars of each other for which their join is \( 1 \).

We now prove the theorem by induction on \( t \), the number of atoms of \( \mathcal{L} \). For the base case, we take \( t = 2 \). This is a degenerate case in which the atoms are also the maximal admissible elements. Assume \( A, B \) are atoms of \( \mathcal{L} \), if there is some element \( C \in \mathcal{L} \) that is not an atom we would have \( C = A \lor B = A \lor B \) since \( \mathcal{L} \) is atomic and since the set of atoms must partition \( J \), we have \( C = J \).

We now prove that any linear order on the atoms of a \( C_n \) lattice \( \mathcal{L} \) of rank \( d \) with its first \( d-1 \) atoms forming a geometric independent set and no atom being the star of the previous atom is a recursive atom ordering. We assume that the
statement is true for every \(C_n\) lattice \(\mathcal{L}'\) with \(\leq t - 2\) atoms and prove it for a \(C_n\) lattice \(\mathcal{L}\) with \(t\) atoms.

Let \(A_1, ..., A_t\) be such an atom ordering. For the first condition, we just apply lemma [5.8] and lemma [5.7] to ensure that \([A_1, 1] \subseteq \mathcal{L}\) has \(\leq t - 2\) atoms. Therefore, if \(\text{rank}(\mathcal{L}) \geq 5\) we get a recursive atom ordering using the induction hypothesis, and otherwise we get a recursive atom ordering by our separate cases.

The proof of the second condition is the same as in the separate case \(\text{rank}(\mathcal{L}) = 3\) as by Lemma [5.8] no atom is the star of the previous atom in linear order \(\omega_1\).

6. CHARACTERIZATION BY RECURSIVE ATOM ORDERING:

In this section we prove a characterization of \(C_n\) lattices by recursive atom ordering inspired by the work in [8], our goal is to give a minimal list of orders that are always and never recursive atom orderings equivalent to being a \(C_n\) lattice. We start with a definition of strongly admissible sets characterizing non maximal geometric independent sets in \(C_n\) lattices.

6.1. Definition: A set \(A\) of atoms will be called strongly admissible if it is obtained from a size \(|A| - 1\) strongly admissible set \(B\) with \(a^* \not< \vee B\) for \(a \in A \setminus B\).

We observe that every strongly admissible set is indeed admissible and that for any admissible element \(X\) of a bounded, atomic, and graded lattice \(\mathcal{L}\) we have a spanning strongly admissible independent set. Next, we present a characterization of \(C_n\) lattices similar to the characterization of geometric lattices given in Theorem [3.9].

6.2. Theorem: A bounded, atomic, and graded lattice \(\mathcal{L}\) is order isomorphic to a \(C_n\) lattice iff there is a partition of the atoms of \(\mathcal{L}\) to pairs \(1, 1^*, 2, 2^*, ..., k, k^*\) such that the non-maximal geometric independent sets are exactly the strongly admissible non maximal independent sets.

Proof: If \(\mathcal{L}\) is a \(C_n\) lattice, then equality is a consequence of Lemma [4.7]. In the other direction, every atomic lattice \((\mathcal{L}, \leq)\) is order isomorphic to a set of subsets, ordered by inclusion by defining \(x = \{a \in A(\mathcal{L}) \mid a \leq x\}\) for every \(x \in \mathcal{L}\). We prove the four conditions in Definition [2.1]:

1. As \(\mathcal{L}\) is bounded, we just have to denote the minimal set \(\phi\) and the maximal set \(J\). Notice that since \(\mathcal{L}\) is atomic, we must have \(J = [k] \cup [k]^*\).
2. Let \(A \in \mathcal{L}\) be an inadmissible set, \(\{i, i^*\} \subset A\). As \(\{i, i^*\}\) is an independent set we can extend it to a size \(\text{rank}(A)\) independent set \(I\) that spans \(A\) (this extension only exists for sizes \(\leq 2\) sets of atoms). As \(I\) is inadmissible and geometric, we must have \(\text{rank}(I) = \text{rank}(A) = \text{rank}(\mathcal{L})\) and so \(A = J\).
3. This property is always true for an atomic and bounded lattice of subsets. \(A \land B \subseteq A \cap B\) and if there is an atom \(i \in A \cap B\) not in the meet then \(A \land B \not\subseteq (A \land B) \lor \{i\} \subset A, B\) which is a contradiction to the definition of \(A \land B\), therefore \(A \land B = A \cap B\).
4. Let \(B, C \in \mathcal{L}\) be two elements that cover some \(A \in \mathcal{L}\), since \(B \cap C \in \mathcal{L}\) we must have \(B \cap C = A\). Furthermore, if \(A\) is not covered by \(J\), then for \(i \in A\) we have \(I \lor i^*\) not geometric for every spanning geometric set of \(A\) containing \(i\). Therefore, there is no element covering \(A\) and containing \(i\). It remains to prove that if \(i \in J \setminus (A \cup A^*)\) then there exists \(B \in \mathcal{L}\) covering
A with \( i \in B \). Take a strongly admissible independent set \( I \) that spans \( A \), making \( I \cup \{ i \} \) also a strongly admissible independent set. Therefore, \( I \cup \{ i \} \) is geometric and we have \( \text{rank}(I \cup \{ i \}) = \text{rank}(A) + 1 \) and the element of \( \mathcal{L} \) spanned by \( I \cup \{ i \} \) is the desired \( B \).

6.3. **Corollary:** A bounded, atomic, and graded lattice \( \mathcal{L} \) is order isomorphic to a \( C_n \) lattice iff there is a partition of the atoms of \( \mathcal{L} \) into pairs \( 1, 1^*, 2, 2^*, ..., k, k^* \) such that the non-maximal geometric independent sets are admissible and strongly admissible independent sets are geometric.

We are now ready to prove the main theorem of this section.

6.4. **Theorem:** A bounded, atomic and graded lattice \( \mathcal{L} \) of rank \( d \leq k = |A(\mathcal{L})| \) is order isomorphic to a \( C_n \) lattice iff there is a partition of the atoms of \( \mathcal{L} \) to pairs \( 1, 1^*, 2, 2^*, ..., k, k^* \) such that the following conditions hold:

1. Every strongly admissible independent set of size \( d - 1 \) (in any order) can be extended to a recursive ordering.
2. A linear order \( a_1, ..., a_{2k} \) starting with a geometric independent set of size \( d - 1 \) is a recursive atom ordering if \( a_i \neq a_{i+1}^* \) for \( i \in [2k-1] \).

**Proof:** If \( \mathcal{L} \) is a \( C_n \) lattice, every strongly admissible independent set of size \( d - 1 \) is geometric, and we have proven in Theorem [5.10] that such recursive atom orderings exist.

For the other direction, we use corollary [6.3]. Throughout the proof, we assume that \( d \geq 4 \) as the cases \( d = 2, 3 \) are obvious. Every such lattice of rank 2 is a \( C_n \) lattice, and for \( d = 3 \) we only need to check that admissible pairs are of rank 2 and inadmissible pairs are of rank 3.

We first observe that every independent set \( I \) for which every linear order of its elements can be extended to a recursive atom ordering is geometric. Therefore, every strongly admissible independent set of size \( d - 1 \) is geometric.

We proceed to prove that no element of the form \( i \vee i^* \) is of rank 2. Assume there is such an \( i \) and notice that if \( \text{rank}(i \vee i^*) \leq d \) we can always extend \( \{ i, i^* \} \) to a size \( d - 1 \) geometric independent set \( I \). Therefore, we can construct two recursive atom ordering of the form \( i^*, j, i, \omega \) and \( i, j, i^*, \omega \), with \( j \in I \) and \( \omega \) being a linear ordering of \( A(\mathcal{L}) \setminus \{ i, i^*, j \} \). We will now prove that if \( \text{rank}(i \vee i^*) = 2 \) then \( j, i, i^*, \omega \) is a recursive atom ordering. Using the two previously constructed orderings, it remains to show that \( \{ i, \bar{i} \} \) has a recursive atom ordering starting with \( i \vee \bar{j} \). If \( d \geq 5 \) we can take a recursive atom ordering on \( \mathcal{L} \) starting with \( j, i, t, i^* \) for some \( t \in I \) to obtain a recursive atom ordering of \( \{ i, \bar{i} \} \) starting with \( i \vee \bar{j} \). Otherwise we need to consider the case \( d = 4 \), it is enough to find some atom \( t \) with \( \{ i, j, t \} \) a strongly admissible independent set. This set exists unless \( i \vee j = \vee T \) with \( T \) a transversal. Using a symmetry argument, we must also have \( i^* \vee j = \vee T \), which results in \( A(\mathcal{L}) = \{ i, i^*, j, j^* \} \) contradicting \( \text{rank}(\mathcal{L}) \leq |A(\mathcal{L})| \).

We now know that no element of the form \( i \vee i^* \) is of rank 2 and using the recursive atom ordering \( i, j, i^*, \omega \) we get \( j \leq i \vee i^* \). As this is true for every \( j \in I \), we must have \( \text{rank}(i \vee i^*) \geq d - 1 \). Let \( j \notin i \vee i^* \), we have \( \{ i, j \} \) a strongly geometric independent set and so of rank 2. Therefore, we can extend \( \{ i, j \} \) to a geometric independent set of rank \( d - 1 \), either containing \( j \) or not, in both cases we can construct a recursive atom ordering containing the sequence \( i, j, i^* \). Consequently \( \text{rank}(i \vee i^*) = d \) and there does not exist a non-maximal inadmissible independent
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References

[1] White, Neil, G-C. Rota, and Neil M. White, eds. Theory of matroids. No. 26. Cambridge University Press, 1986.
[2] I.M. Gelfand, V.V. Serganova, Combinatorial geometries and torus strata on homogeneous compact manifolds, Russian Math. Surveys 42 (1987) 133–168.
[3] A.V. Borovik, I.M. Gelfand, N.L. White, Symplectic matroids, J. Algebraic Combin. 8 (1998) 235–252.
[4] Timothy Y. Chow "Symplectic matroids, independent sets, and signed graphs", Discrete Mathematics 263 (2003) 35 – 45.
[5] Tu, Zhexiu. “Symplectic Matroids, Circuits, and Signed Graphs.” arXiv: Combinatorics (2018): n. pag.
[6] T. Zaslavsky, Biased graphs. II. The three matroids, J. Combin. Theory Ser. B 51 (1991), 46-72.
[7] A. Björner, M. Wachs, On lexicographically shellable posets, Trans. Amer. Math. Soc. 277 (1983), no. 1, 323–341.
[8] Davidson, Ruth, and Patricia Hersh. "A lexicographic shellability characterization of geometric lattices". Journal of Combinatorial Theory, Series A 123.1 (2014): 8-13.
[9] Andreas Blass, Bruce E. Sagan. "Mobius functions of lattices". Adv. in Math. 127 (1997), 94-123.
[10] Björner, A., Garsia, A., Stanley, R.: An introduction to the theory of Cohen-Macaulay partially ordered sets. In: I. Rival: Ordered sets (pp. 583–615) Dordrecht Boston London: Reidel 1982.