FOCK SPACE ASSOCIATED WITH QUADRABASIC HERMITE ORTHOGONAL POLYNOMIALS

WIKTOR EJSMONT

Abstract. This paper introduces a new idea for constructing operators associated with a certain class of probability measures. Special cases include several known classical and noncommutative probability. The main example is derived from Feller [31, Page 503, Example 10], i.e. the hyperbolic secant distribution. In probability theory and statistics, the hyperbolic secant distribution is a continuous probability distribution whose probability density function and characteristic function are proportional to the hyperbolic secant function.

1. Introduction

The study of \(q\)-Gaussian distributions [18] has been an active field of research during the last decade. A noncommutative analog of a Brownian motion (or Gaussian process, more generally) is the family of operators \((a^*_q(x) + a_q(x))_{x \in H}\). When equipped with the vacuum expectation state \(\langle \Omega, \cdot \rangle_q\), the \(q\)-Gaussian algebra yields a rich non-commutative probability space. For \(q = 1\) (corresponding to the Bose statistics) the operator \(a^*_1(x) + a_1(x)\) is the standard Gaussian random variable, i.e. its spectral measure relative to the vacuum state satisfies

\[
\langle (a^*_1(x) + a_1(x))^n \Omega, \Omega \rangle_1 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} t^n e^{-\frac{t^2}{2}} dt
\]

when \(\|x\| = 1\). Moreover, \(\{a^*_1(x) + a_1(x)\}_{x \in H}\) are commutative in the classical sense. The case \(q = -1\) corresponds to the Fermi statistics. It should be stressed that, for \(q \neq \pm 1\), the \(q\)-modification of the (anti)symmetrization operator is a strictly positive operator. Therefore, unlike the classical Bose and Fermi cases, there are no commutation relations between the creation operators. For \(q = 0\), the \(q\)-Fock space recovers the full Fock space of Voiculescu’s free probability [48]. For \(q = 0\), the \(q\)-Gaussian random variables are distributed according to the semi-circle law

\[
\langle (a^*_0(x) + a_0(x))^n \Omega, \Omega \rangle_0 = \frac{1}{2\pi} \int_{-2}^{2} t^n \sqrt{4 - t^2} dt
\]

when \(\|x\| = 1\).

The study of the noncommutative Brownian motion \(\{a^*_q(x) + a_q(x)\}_{x \in H}\) was initiated in [18] [19] [20]. For further generalizations of a \(q\)-Brownian motion, see [32] [10] [14] [15]. In particular, this setting gives rise to \(q\)-deformed versions of the stochastic calculus [3] [22] [4] [29] [28].

One of the most beautiful and important results in this area was initiated by Blitvič [10] where a second-parameter refinement of the \(q\)-Fock space, formulated as a \((q, t)\)-Fock space \(\mathcal{F}_{q,t}(H)\) was introduced. It is constructed via a direct generalization of Bożejko and Speicher’s framework [18], yielding the \(q\)-Fock space when \(t = 1\). These are the defining relations of the Chakrabarti-Jagannathan deformed quantum oscillator algebra; see [10] and references therein for more details. The moments of the deformed Gaussian process \(\{a_{q,t}(x) + a^*_{q,t}(x)\}_{x \in H}\) are encoded by the joint statistics of crossings and nestings in pair partitions. In particular, it is shown that the distribution of a single Gaussian operator orthogonalizes the \((q, t)\)-Hermite polynomials.

The goal of this paper is to introduce quadrabasic Fock space. Our approach is to replace the permutation group by the tensor product of two permutation groups. The orthogonal
polynomials of a Gaussian type arising in the present framework satisfying the recurrence relation

\[(1.1) \quad xQ_n^{(q,t,v,w)}(x) = Q_{n+1}^{(q,t,v,w)}(x) + \sum [n]_{q,t} [v]_{v,w} Q_n^{(q,t,v,w)}(x), \quad n = 0, 1, 2, \ldots \]

where \(Q_{-1}^{(q,t,v,w)}(x) = 0, Q_0^{(q,t,v,w)}(x) = 1, |q| \leq t \leq 1, |v| \leq w \leq 1 \) and \([n]_{q,t} \) is the \(q,t\)-number

\[ [n]_{q,t} := t^{n+1} + qt^{n-2} + \cdots + q^{n-2}t + q^{n-1}, \quad [n]_q := [n]_{q,1} \quad n \geq 1. \]

We call these polynomials quadrabasic Hermite orthogonal polynomials, because they depend on four parameters and it is a natural extension of \(q\) or \((q,t)\)-Hermite orthogonal polynomials. Considering families built around more general hypergeometric functions, the quadrabasic Hermite sequence belongs to the octabasic Laguerre family (or its symmetric version) introduced by Simion and Stanton \cite{SimionStanton} and recently extended by Blitvić and Steingrimsson \cite{BlitvicSteingrimsson} or Sokal and Zeng \cite{SokalZeng} (see also the earlier work \cite{Blitvic}). This formula recovers the hyperbolic secant case when \(q = t = v = w = 1\). The hyperbolic secant function is equivalent to the reciprocal hyperbolic cosine, and thus this distribution is also called the hyperbolic cosine distribution. This measure orthogonalizes a special class of Meixner-Pollaczek polynomials which satisfy the recurrence relation

\[(1.2) \quad xQ_n(x) = Q_{n+1}(x) + n^2Q_{n-1}(x), \quad n = 0, 1, 2, \ldots \]

with initial conditions \(Q_{-1}(x) = 0\) and \(Q_0(x) = 1\). To the best of our knowledge, the relation \((1.2)\) appears for the first time in the literature in \cite{Sokal} eq. (5.4)]. In literature it is also possible to find a relatively large number of works addressing the relation \((1.2)\) for example \cite{GasperRahman} eq. (4.7), with rescaling \(Q_n(x) = n!A_n((x-1)/2)\); it was shown that these polynomials satisfy a symbolic orthogonality relation with respect to the Euler numbers. Moreover, we investigate this construction in the context of a Poisson-type operator and apply this idea to introduce a new class of noncommutative Lévy processes.

The plan of the paper is following: first we present definitions of the \((q,t)\)-Fock space and corresponding creation, annihilation and gauge operators. By using this we present the definition of quadrabasic Fock space and the creation and annihilation operators acting on it. In Section 3 we present a new type of partitions and the relevant statistics. Next, in Sections 3 and 4 we introduce the Generalized Gaussian process and gauge operators and some of their natural properties, including norm estimates and the self-adjointness. In these two sections we mainly study an explicit Wick formula for the mixed moments. Finally, in Section 5 we apply this technique for constructing new Lévy processes, which allows us to define a \((q,t,v,w)\)-convolution for a large class of probability measures.

2. Preliminaries and Quadrabasic Fock space

2.1. The Blitvić \((q,t)\)-Fock space \cite{BlitvicSteingrimsson} \cite{Blitvics}. Let \(H_\mathbb{R}\) be a separable real Hilbert space and let \(H\) be its complexification with inner product \((\cdot,\cdot)\) linear on the right component and anti-linear on the left. When considering elements in \(H_\mathbb{R}\), it holds true that \((x,y) = \langle y,x \rangle\). Let \(\mathcal{F}_{alg}(H)\) be its algebraic full Fock space, \(\mathcal{F}_{alg}(H) = \bigoplus_{n=0}^{\infty} H^{\otimes n}\), where \(H^0 = \mathbb{C}\Omega\) and \(\Omega\) is the vacuum vector. For each \(n \geq 0\), define the operator \(P_n\) on \(H^\otimes n\) by

\[ P^{(0)}_{q,t}(\Omega) = \Omega, \]

\[ P^{(n)}_{q,t}(\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n) = \sum_{\sigma \in \mathcal{S}_n} q^{l_1(\sigma)} t^{l_2(\sigma)} \eta_{\sigma(1)} \otimes \eta_{\sigma(2)} \otimes \ldots \otimes \eta_{\sigma(n)}, \quad \text{for } q,t \in [-1,1], \quad \eta \leq t, \]

where \(\mathcal{S}_n\) is the group of permutations of \(1,2,\ldots,n\) elements, and \(l_1(\sigma)\) (inversions) is the minimal number of \(\sigma_i, 1 \leq i \leq n-1\), appearing in \(\sigma\) and \(l_2(\sigma)\) is \({n \choose 2} - l_1(\sigma)\) (co-inversions). Remember that the symmetric group is generated by \(\sigma_i = (i,i+1), i = 1,\ldots,n-1\), which satisfy the generalized braid relations \(\sigma_i^2 = e, 1 \leq i < n-1\) and \((\sigma_i\sigma_j)^2 = e\) if \(|i-j| \geq 2, 0 \leq i,j \leq n-1\).

For \(q = t = 0\) each \(P^{(n)}_{q,t} = I\). For \(q = t = 1\), \(P^{(n)}_{q,t} = n! \times \) the projection onto the subspace...
of symmetric tensors. For \( q = -1, t = 1 \), \( P_{q,t}^{(n)} = n! \times \) the projection onto the subspace of anti-symmetric tensors.

Define the \((q, t)\)-deformed inner product on \( \mathcal{F}_{\text{alg}}(H) \) by the rule that for \( \zeta \in H^{\otimes k}, \eta \in H^{\otimes n}, \)
\[
\langle \zeta, \eta \rangle_{q,t} := \delta_{nk} \langle \zeta, P_{q,t}^{(n)} \eta \rangle,
\]
where the inner product on the right-hand-side is the usual inner product induced on \( H^{\otimes n} \). All inner products are linear in the second variable. It is a result of [10] that the inner product \( \langle \cdot, \cdot \rangle_{q,t} \) is positive definite for \( q, t \in (-1, 1) \) and \( |q| < t \), while for \( |q| = t \) it is positive semi-definite. Let \( \mathcal{F}_{q,t}(H) \) be the completion of \( \mathcal{F}_{\text{alg}}(H) \) with respect to the norm corresponding to \( \langle \cdot, \cdot \rangle_{q,t} \). For \( |q| = t \) one first needs to quotient out by the vectors of norm 0 and then complete; the result is the anti-symmetric, respectively, symmetric Fock space, with the inner product multiplied by \( n! \) on the \( n \)-particle space. For \( \xi \in H_{\mathbb{R}} \), define the (left) creation and annihilation operators on \( \mathcal{F}_{\text{alg}}(H) \) by, respectively,
\[
a_{q,t}^*(\xi)\Omega = \xi,
\]
\[
a_{q,t}(\xi)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \xi \otimes \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n,
\]
and
\[
a_{q,t}(\xi)\Omega = 0,
\]
\[
a_{q,t}(\xi)\eta = \langle \xi, \eta \rangle \Omega,
\]
\[
a_{q,t}(\xi)\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_{i=1}^{n} q^{i-1} t^{n-i} \langle \xi, \eta_i \rangle \eta_1 \otimes \ldots \otimes \hat{\eta}_i \otimes \ldots \otimes \eta_n,
\]
where as usually \( \hat{\eta}_i \) means omitting the \( i \)-th term. They satisfy the commutation relations
\[
a_{q,t}(\xi)a_{q,t}^*(\eta) - qa_{q,t}^*(\eta)a_{q,t}(\xi) = \langle \xi, \eta \rangle t^N,
\]
where \( t^N \) is the operator on \( \mathcal{F}_{q,t}(H) \) defined by the linear extension of \( t^N\Omega = 0 \) and \( t^N\eta_1 \otimes \ldots \otimes \eta_n = t^n\eta_1 \otimes \ldots \otimes \eta_n \).

In the following range of parameters [10] Lemma 5] the operators \( a_{q,t} \) and \( a_{q,t}^* \) extend to bounded linear operators (adjoints of each other) on \( \mathcal{F}_{q,t}(H) \), with the norm
\[
\|a_{q,t}^*(\xi)\|_{q,t} = \begin{cases}
\|\xi\| & -t \leq q \leq 0 < t \leq 1 \\
\frac{\sqrt{1-q}}{\sqrt{t-q}}\|\xi\| & 0 < q < t = 1 \\
\frac{\sqrt{n(t+1-q^{n(t+1)}}}{t-q}\|\xi\| & 0 < q = t < 1
\end{cases}
\]
where \( \xi \neq 0, n(t) := [t/(1-t)] \) and \( \hat{n}(q, t) := \frac{\log(1-t)-\log(1-q)}{\log t-\log q} \).

For \( q = \pm 1 \) and \( t = 1 \) we first need to compress the operators by the projection onto the symmetric/anti-symmetric Fock space, respectively, and the resulting operators differ from the usual ones by \( \sqrt{n} \), but satisfy the usual commutation relations (thanks to a different inner product). For \( q = t = 1 \) the resulting operators are unbounded, but still adjoints of each other.

2.2. \((q, t)\)-gauge operators. In this subsection we recall a differential second quantization operator which is partially investigated in [30]. In order to simplify the computation we recall the properties the symmetric group. There is a natural embedding \( \mathcal{S}(n-1) = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle \subset \mathcal{S}(n) = \langle \sigma_1, \ldots, \sigma_n \rangle \), which allows us to decompose the operator
\[
P_{q,t}^{(n)} = (I \otimes P_{q,t}^{(n-1)})R_{q,t}^{(n)} = R_{q,t}^{(n)}(I \otimes P_{q,t}^{(n-1)}) \mbox{ on } H^{\otimes n},
\]
where \( \overline{R}_{q,t}^{(n)} = 1 + \sum_{k=1}^{n-1} q^k \sigma_1 \cdots \sigma_k \) (see [19]). From this we have
\[
P_{q,t}^{(n)} = t(q)P_{q,t}^{(n)} = t(q)(I \otimes P_{q,t}^{(n-1)})R_{q,t}^{(n)} = (I \otimes t(q^{-1})P_{q,t}^{(n-1)})t^{-1}R_{q,t}^{(n)}
\]
\[
= (I \otimes P_{q,t}^{(n-1)})R_{q,t}^{(n)} = R_{q,t}^{(n)}(I \otimes P_{q,t}^{(n-1)})
\]
where
where $P_{q,t}^{(n)} = t^{n-1} + \sum_{k=1}^{n-1} q^k t^{n-k-1} \sigma_1 \cdots \sigma_k$ and $n \geq 1$. Let us further observe that $\|P_{q,t}^{(n)}\|_{0,0} \leq \|\cdot\|_{[n]}$ and so

\begin{equation}
\|P_{q,t}^{(n)}\|_{0,0} \leq [n]_{q,t}\|P_{q,t}^{(n-1)}\| \otimes I \|_{0,0} \leq \prod_{i=1}^{n} [i]_{q,t} \leq n!.
\end{equation}

First, we introduce an operator which acts on $(q,t)$-Fock space as

$p_0(T)\Omega = 0,$

$p_0(T)(\xi_1 \otimes \ldots \otimes \xi_n) = T(\xi_1) \otimes \ldots \otimes \xi_n,$

where $T$ is an operator on Hilbert space $H$ with dense domain $D$. The adjoint of this operator satisfies $\langle p_0(T)f|\xi\rangle_{0,0} = (\langle f|p_0(T^*)\rangle_0,0$, and allows us to define a gauge operator (preservation or differential second quantization). Let $p_T^{(q,t)} := p_0(T)R_{q,t}^{(n)}$. Let us observe that directly from the generalized braid relations for $k \in [n-1]$, we have $\sigma_1 \cdots \sigma_k = (1 \cdots k \cdots n)$, which allows us to rewrite action of $p_T$ on $H^{\otimes n}$ as

$p_T^{(q,t)}(\xi_1 \otimes \ldots \otimes \xi_n) = \sum_{i=1}^{n} q^{-i}t^{n-i}T(\xi_i) \otimes \xi_1 \otimes \ldots \otimes \xi_{i-1} \otimes \ldots \otimes \xi_n.$

2.2.1. $p_T^{(q,t)}$ is symmetric operator on $\mathcal{F}_{q,t}(D)$ and bounded for $q < 1$. First, we explain the symmetry properties of $p_T^{(q,t)}$. Observe that $p_0(T^*)(I \otimes P_{q,t}^{(n-1)}) = (I \otimes P_{q,t}^{(n-1)})p_0(T^*)$; indeed for $\xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n \in D^{\otimes n}$, we have

$p_0(T^*)(I \otimes P_{q,t}^{(n-1)})(\xi_1 \otimes \ldots \otimes \xi_n) = T^*(\xi_1) \otimes P_{q,t}^{(n-1)}(\xi_2 \otimes \ldots \otimes \xi_n) = (I \otimes P_{q,t}^{(n-1)})(T^*(\xi_1) \otimes \ldots \otimes \xi_n) = (I \otimes P_{q,t}^{(n-1)})p_0(T^*)(\xi_1 \otimes \ldots \otimes \xi_n).

Let us fix $n$, and $f,g \in D^{\otimes n}$, then

$\langle p_T^{(q,t)}f, g \rangle_{q,t} = \langle p_T^{(q,t)}f, P_{q,t}^{(n)}g \rangle_{0,0} = \langle p_0(T)P_{q,t}^{(n)}f, (I \otimes P_{q,t}^{(n-1)})P_{q,t}^{(n)}g \rangle_{0,0} = \langle P_{q,t}^{(n)}f, (I \otimes P_{q,t}^{(n-1)})P_{q,t}^{(n)}g \rangle_{0,0}$

by Equation (2.3), we have

$\langle f, P_{q,t}^{(n)}(I \otimes P_{q,t}^{(n-1)})p_0(T^*)R_{q,t}^{(n)}g \rangle_{0,0} = \langle f, P_{q,t}^{(q,t)}g \rangle_{q,t}.$

The following proposition is inspired by [30, Proposition 1.9]. The proof is almost identical to that of [30, Proposition 1.9], and it can be omitted (the only difference is estimate of norm from the end).

**Proposition 2.1.** If $T$ is a bounded operator on $H$ and $|q| < 1$, then $p_T^{(q,t)}$ is bounded in $\mathcal{F}_{q,t}(H)$.

2.3. **Diagonal full Fock space.** Let $[n]$ be the set $\{1, \ldots, n\}$. We also consider additional numbers $\bar{1}, \bar{2}, \ldots, \bar{n}$ and define $\bar{[n]}$ as the set $\{\bar{1}, \ldots, \bar{n}\}$. We associate these indices with natural ordering $\bar{1} < \cdots < \bar{n}$. Our strategy is to define the action on the Weyl group $\mathfrak{S}_{\pm n} = \mathfrak{S}_n \times \mathfrak{S}_\bar{n}$ which permutes separately indices of positive and negative part, as in the example: 2 1 4 5 3 × 5 2 1 3 4.

Let $H$ and $\bar{H}$ be two separable Hilbert spaces with inner product $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_{\bar{H}}$, respectively. We further assume that these Hilbert spaces have real Hilbert subspaces, so that $H$ ($\bar{H}$) is the complexification of $H_R$ ($\bar{H}_R$).

Then the Hilbert space $\mathcal{H} := H \otimes \bar{H}$ is the complexification of its real subspace $\mathcal{H}_R := H_R \otimes \bar{H}_R$, with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle_H \langle \cdot, \cdot \rangle_{\bar{H}}$ (and so $\mathcal{H}$ has a natural conjugation defined on it). We define $\mathcal{H}_n := H^{\otimes n} \otimes \bar{H}^{\otimes n}$ and an action of $\mathfrak{S}_{\pm n}$ on it by $U_n(\sigma) \otimes U_n(\gamma)$, where

$U_n(\sigma)x_1 \otimes \cdots \otimes x_n = x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$

is a unitary representation of the symmetric group. Moreover, we
Moreover, let \( \xi_1 \otimes \cdots \otimes \xi_n \in H^\otimes n \) and \( \xi_1 \otimes \cdots \otimes \xi_n \in \tilde{H}^\otimes n \), by \( \xi_n \) and \( \tilde{\xi}_n \), respectively.

We introduce the algebraic diagonal full Fock space over \( H \)

\[
F_{\text{dig}}(H) := \bigoplus_{n=0}^{\infty} H_n = \bigoplus_{n=0}^{\infty} H^\otimes n \otimes \tilde{H}^\otimes n.
\]

with convention that \( H^\otimes 0 = \mathbb{C} \Omega \otimes \tilde{\Omega} \) is a one-dimensional normed space along a unit vector \( \Omega \otimes \tilde{\Omega} \). Note that elements of \( F_{\text{dig}}(H) \) are finite linear combinations of the elements from \( H^\otimes n, n \in \mathbb{N} \cup \{0\} \) and we do not take the completion. We define the following inner product \( \langle \cdot \rangle \) with respect to the norm corresponding to \( \Omega \).

\[
\langle \xi_n \otimes \tilde{\xi}_n, \eta_m \otimes \tilde{\eta}_m \rangle_{0,0,0,0} := \delta_{n,m} \prod_{i=1}^{n} \langle \xi_i, \eta_i \rangle_H \langle \xi_i, \eta_i \rangle_{\tilde{H}}.
\]

**Remark 2.2.** We may think that the elements of \( F_{\text{dig}}(H) \) arise from the diagonal elements of full Fock space \( (\bigoplus_{n=0}^{\infty} H^\otimes n) \otimes (\bigoplus_{n=0}^{\infty} \tilde{H}^\otimes n) \), that is, it is the sum of the boxed entries in the table

\[
\begin{array}{cccc}
H^\otimes 0 & \bigotimes & \tilde{H}^\otimes 0 & \\
H^\otimes 1 & \bigotimes & \tilde{H}^\otimes 1 & H^\otimes 0 \otimes \tilde{H}^\otimes 1 \\
H^\otimes 2 & \bigotimes & \tilde{H}^\otimes 2 & H^\otimes 1 \otimes \tilde{H}^\otimes 2 \\
H^\otimes 3 & \bigotimes & \tilde{H}^\otimes 3 & H^\otimes 2 \otimes \tilde{H}^\otimes 3 \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

It is why we called them the diagonal Fock space.

### 2.4. Quadrabasic Fock space creation and annihilation operators.

Now we deform the inner product on \( F_{\text{dig}}(H) \). For \( q, t, v, w \in [-1, 1], |q| \leq t \) and \( |v| \leq w \) we define the \( (q, t, v, w) \)-symmetrization operator on \( H_n \)

\[
P^{(n)}_{q,t,v,w} := P^{(n)}_{q,t} \otimes P^{(n)}_{v,w}.
\]

Moreover, let \( P_{q,t,v,w} := \bigoplus_{n=0}^{\infty} P^{(n)}_{q,t,v,w} \). We equip \( F_{\text{dig}}(H) \) with the inner product by using the deformed operator:

\[
\langle \xi_n \otimes \tilde{\xi}_n, \eta_m \otimes \tilde{\eta}_m \rangle_{q,t,v,w} := \langle \xi_n \otimes \tilde{\xi}_n, P^{(n)}_{q,t,v,w} \eta_m \otimes \tilde{\eta}_m \rangle_{0,0,0,0}
\]

\[
= \delta_{n,m} \langle \xi_n, \eta_m \rangle_{q,t} \langle \tilde{\xi}_n, \tilde{\eta}_m \rangle_{v,w}.
\]

Remember that a strictly positive operator means that it is positive and \( \text{Ker}(P^{(n)}_{q,t,v,w}) = \{0\} \), and directly from above definition and information from Section 2.1 we can state the following

**Proposition 2.3.** The operator \( P_{q,t,v,w} \)

(a) is positive for \( |q| \leq t \) and \( |v| \leq w \);

(b) is strictly positive for \( |q| < t \) and \( |v| < w \).

If \( \xi \otimes \eta \in H_\mathbb{R} \), then the adjoint of \( a_{q,t}(\xi) \otimes a_{v,w}(\eta) \) is \( a_{q,t}^*(\xi) \otimes a_{v,w}^*(\eta) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{q,t,v,w} \). This allows us to define the generalized creator and annihilator.

**Definition 2.4.** We define \( \mathcal{F}^\otimes_{\text{dig}}(H) \) the quadrabasic Fock space which is completion of \( F_{\text{dig}}(H) \) with respect to the norm corresponding to \( \langle \cdot, \cdot \rangle_{q,t,v,w} \). Note that for \( |q| = t \) or \( |v| = w \) we first have to divide by the kernel of \( P_{q,t,v,w} \) before taking the completion. For \( \xi \otimes \eta \in H_\mathbb{R} \) we define \( A^{\otimes \otimes n}_{\xi \otimes \eta} := a_{q,t}^*(\xi) \otimes a_{v,w}^*(\eta) \). Let \( A^{\otimes \otimes n} \) be its adjoint with respect to the inner product \( \langle \cdot, \cdot \rangle_{q,t,v,w} \). Denote by \( \varphi \) the vacuum vector state \( \varphi(\cdot) := \langle \Omega \otimes \tilde{\Omega}, \cdot \rangle_{q,t,v,w} \).
Example 2.5. For $\xi \otimes \eta \in \mathcal{H}_{\mathbb{R}}$ and $\bar{\xi_n} \otimes \bar{\eta_n} = (\xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_n) \in \mathcal{H}_{\mathbb{R}}$ this works as follows:

\[
A_{\xi \otimes \eta}^* \bar{\xi_n} \otimes \bar{\eta_n} = (\xi \otimes \xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta \otimes \eta_1 \otimes \cdots \otimes \eta_n),
\]

\[
A_{\xi \otimes \eta} \bar{\xi_n} \otimes \bar{\eta_n} = \sum_{i,j \in \mathbb{N}} q^{i-1} t^{n-i} v^{j-1} w^{n-j} \langle \xi \otimes \xi_i, \eta \otimes \eta_j \rangle_{\mathcal{H}}
\times (\xi_1 \otimes \cdots \otimes \xi_i \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_j \otimes \cdots \otimes \eta_n),
\]

\[
A_{\xi \otimes \eta} \Omega \otimes \bar{\Omega} = 0.
\]

Remark 2.6. (1). Quadrabasic Fock space creator and annihilator operators depend on four parameters $q, t, v, w$. If it is necessary to emphasize this dependence then we often write $A_{(q, t, v, w)}$ or $A_{(q, t, v, w)}^*$. We will omit the dependence on $q, t, v, w$ in the notation if it is clear from the context. The same remark applies to other objects, which appear in further parts of the work.

(2). It is easy to see that $A^* : \mathcal{H} \to \mathbb{B}(\mathcal{F}_{\text{dig}}(\mathcal{H}))$ is linear and $A : \mathcal{H} \to \mathbb{B}(\mathcal{F}_{\text{dig}}(\mathcal{H}))$ is anti-linear.

2.5. Properties of creation and annihilation operators. If $t = w = 1$, then we get a nice commutation relation.

Proposition 2.7. For $\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \in \mathcal{H}_{\mathbb{R}}$, we have the commutation relation

\[
A_{(q, t, v, w)}(\xi_1 \otimes \eta_1) - q v A_{(q, t, v, w)}(\xi_2 \otimes \eta_2) = A_1 \otimes B_2 + B_1 \otimes A_2 + \langle \xi_1, \eta_1, \xi_2, \eta_2 \rangle_{\mathcal{H}} I \otimes I
\]

where $A_1 = q a_{q,1}^*(\xi_1) a_{q,1}(\xi_1)$, $A_2 = v a_{v,1}(\eta_2) a_{v,1}(\eta_1)$, $B_1 = \langle \xi_1, \xi_2 \rangle_{\mathcal{H}} I$ and $B_2 = \langle \eta_1, \eta_2 \rangle_{\mathcal{H}} I$.

Remark 2.8. Operators $A_1 \otimes B_2, B_1 \otimes A_2$ and $I \otimes I$ commute.

Proof. The proof follows directly from relation (3.1) between $a_{q,1}^*$ and $a_{v,1}$, and we see that

\[
A_{(q, t, v, w)}(\xi_1 \otimes \eta_1) - q v A_{(q, t, v, w)}(\xi_2 \otimes \eta_2) = a_{q,1}(\xi_1) a_{q,1}(\xi_2) \otimes v a_{v,1}(\eta_1) a_{v,1}(\eta_2) = q a_{q,1}^*(\xi_1) a_{q,1}(\xi_1) \otimes v a_{v,1}(\eta_2) a_{v,1}(\eta_1)
\]

and the conclusion follows.

The norm of the creation operators follows directly from equation (3.2). For $\xi \otimes \eta \in \mathcal{H}$, $\xi \otimes \eta \neq 0$, we have

\[
\|A_{\xi \otimes \eta}^*\|_{q, t, v, w} = \|a_{q,t}^*(\xi)\|_{q,t} \|a_{v,w}^*(\eta)\|_{v,w}
\]

because $\|A_{\xi \otimes \eta}^* \bar{\xi_n} \otimes \bar{\eta_n}\|^2_{q,t,v,w} = \langle a_{q,t}^*(\xi) \bar{\xi_n}, a_{q,t}^*(\xi) \bar{\xi_n} \rangle_{q,t,v,w} \langle a_{v,w}^*(\eta) \bar{\eta_n}, a_{v,w}^*(\eta) \bar{\eta_n} \rangle_{v,w}$.

3. Combinatorial and partitions

3.1. Diagonal partitions. For an ordered set $S$, let $\mathcal{P}(S)$ denote the lattice of set partitions of that set. We write $B \in \pi$ if $B$ is a class of $\pi$ and we say that $B$ is a block of $\pi$. A block of $\pi$ is called a singleton if it consists of one element. Similarly, a block of $\pi$ is called a pair if it consists of two elements. Let Sing($\pi$) and Pair($\pi$) denote the set of all singletons and pairs of $\pi$, respectively. The maximal element of $\mathcal{P}(\pi)$ under this order is the partition consisting of only one block and it is denoted by $\bar{1}_n$. On the other hand, the minimal element $\bar{0}_n$ is the unique partition where every block is a singleton. Given a partition $\pi$ of the set $[n]$, we write $\text{Arc}(\pi)$ for the set of pairs of integers $(i, j)$ which occur in the same block of $\pi$ such that $j$ is the smallest element of the block greater than $i$. The same notation $\text{Arc}(B)$ is applied to a block $B \in \pi$. Thus, when we draw the points of block then we think that consecutive elements in every block (bigger than one) are connected by arcs above the x axis. For a given block of partition $\pi \in \mathcal{P}(\pi)$ every element is an opener, a closer, a middle point or a singleton for this block – see Figure 1.
The set of such diagonal blocks is denoted by $\pi$ and the collections of closers coincide as well.

Remark 3.3. One should note that the Definition 3.1 means that left legs of arcs are the same between different parts of $[n]$ and $[\bar{n}]$.

When $n$ is even, we call $\pi \in \mathcal{P}^\odot(n)$ a pair partition of $[n] \sqcup [\bar{n}]$. If $\pi$ is a pair partition, then each block consists of one arc. The set of diagonal pair partitions of $[n] \sqcup [\bar{n}]$ is denoted by $\mathcal{P}^\odot_2(n)$ and the set of pairs or singletons of $[n] \sqcup [\bar{n}]$ is denoted by $\mathcal{P}^\odot_{1,2}(n)$.

From Definition 3.1 it follows that for every block in $B \in \pi|_{[n]}$ there exists a unique conjugate block $\bar{B} \in \pi|_{[\bar{n}]}$, which starts from the same point. This leads to one more definition. We call block $B$ a diagonal block if $B = B \otimes \bar{B} := (B, \bar{B})$, where $B \in \pi|_{[n]}$ and $\bar{B} \in \pi|_{[\bar{n}]}$ are conjugate. The set of such diagonal blocks is denoted by $\mathcal{D}_{\text{block}}(\pi)$. A block $B$ of $\mathcal{D}_{\text{block}}(\pi)$ is called a singleton if $B = (a) \otimes (\bar{a})$. We note that the diagonal block is not a block of $\pi$, but a pair of conjugate blocks.

Example 3.2. In Figure 3 (a), (b), (c) and (e) we have:

\[
\{(1, 4) \otimes (\bar{1}, \bar{6}), (2, 6) \otimes (\bar{2}, \bar{5}), (3, 5) \otimes (\bar{3}, \bar{4})\} \in \mathcal{D}_{\text{block}}(\pi),
\{(1, 6) \otimes (\bar{1}, \bar{6}), (2, 5) \otimes (\bar{2}, \bar{5}), (3, 4) \otimes (\bar{3}, \bar{4})\} \in \mathcal{D}_{\text{block}}(\pi),
\{(1, 3, 6) \otimes (\bar{1}, \bar{4}, \bar{6}), (2, 4, 5) \otimes (\bar{2}, \bar{3}, \bar{5})\} \in \mathcal{D}_{\text{block}}(\pi),
\{(1, 3) \otimes (\bar{1}, \bar{4}, \bar{5}, \bar{6}), (2, 4, 5, 6) \otimes (\bar{2}, \bar{3})\} \in \mathcal{D}_{\text{block}}(\pi).
\]

Remark 3.3. (1) One should note that the Definition 3.1 means that left legs of arcs are the same between different parts of $[n]$ and $[\bar{n}]$. This means that partitions on $[n]$ and $[\bar{n}]$ are not independent, see Figure 3 (a), (b), (c), (e).
(2). A set partition $\pi$ is noncrossing if $rc(\pi) = 0$ (for definition of $rc$ see Subsection 3.2.2). From Definition 3.1 it follows that a set of partitions $\pi$ with respect to $[n] \sqcup [\overline{n}]$ is noncrossing if and only $\pi|_{[n]}$ is noncrossing and $B = (b_1, \ldots, b_n) \in \pi|_{[n]} \iff \overline{B} = (\overline{b}_1, \ldots, \overline{b}_n) \in \pi|_{[\overline{n}]}$; see Figure (2) (b). Indeed, if $\pi|_{[n]}$ is noncrossing partition on $[n]$ then there exists an arc of consecutive integers $(a, a + 1) \in \text{Arc}(\pi|_{[n]})$. By definition there exists an arc $(\overline{a}, \overline{c}) \in \text{Arc}(\pi|_{[\overline{n}]}).$ If $c \neq \overline{a} + 1$, then $\overline{a} + \overline{1}$ can not be a left leg or singleton of some block in $\pi|_{[\overline{n}]}$ and thus arc $(\overline{a}, \overline{c})$ must cross $(\overline{a}, \overline{c})$; hence we get contradiction and thus $(\overline{a}, \overline{c}) = (\overline{a}, \overline{a} + 1).$ Further, we proceed with the same algorithm. The set of noncrossing blocks is denoted by $\mathcal{NC}^\circ(n).

(3). Let $S = (s_{a_i})_{i \in I}$ and $T = (t_{b_i})_{i \in I}$ be two indexing sets $a_i, b_i \in \mathbb{N}$ with the same cardinality $|S| = |T|$. The diagonal Cartesian product of two sets $S$ and $T$, denoted $S \times_\Delta T$ is the set of all ordered pairs $(s_{a_i}, t_{b_i})$ where $s_{a_i}$ is in $S$ and $t_{b_i}$ is in $T$. In terms of set-builder notation, that is

$$S \times_\Delta T = \{(s_{a_i}, t_{b_i}) \mid s_{a_i} \in S, t_{b_i} \in T\}.$$ 

Clearly $S \times_\Delta T \subset S \times T$, where $S \times T$ is the Cartesian product of $S$ with $T$. For a class $B \in \pi$, denote by $f(B)$ its first element. Let us number the block of $\pi$ according to the order of their first elements, i.e.

$$\pi = \{B_{f(B)} \mid B \in \pi\}, \quad \pi \in \mathcal{P}([n]).$$

Then for $\pi \in \mathcal{P}^\circ(n)$ we have

$$\mathcal{D}_{\text{block}}(\pi) = \pi|_{[n]} \times_\Delta \pi|_{[\overline{n}]}$$

That is why we called it the diagonal partitions. For example if

$$\pi = \{(1, 4) \ (2, 6), (3, 5), (\overline{1}, \overline{6}), (\overline{2}, \overline{5}), (\overline{3}, \overline{4})\}$$

then $\pi|_{[n]} = \{(1, 4) \ (2, 6), (3, 5)\}$, $\pi|_{[\overline{n}]} = \{(\overline{1}, \overline{6}), (\overline{3}, \overline{4})\}$ and

$$\mathcal{D}_{\text{block}}(\pi) = \pi|_{[n]} \times_\Delta \pi|_{[\overline{n}]} = \{((1, 4), (\overline{1}, \overline{6})), ((2, 6), (\overline{2}, \overline{5})), ((3, 5), (\overline{3}, \overline{4}))\}$$

$$= \{(1, 4) \otimes (\overline{1}, \overline{6}), (2, 6) \otimes (\overline{2}, \overline{5}), (3, 5) \otimes (\overline{3}, \overline{4})\}.$$ 

(4). From Definition 3.1 it follows that whenever

$$\{\hat{1}_n \otimes \overline{B}\} \in \mathcal{D}_{\text{block}}(\pi) \text{ or } \{B \otimes \hat{1}_n\} \in \mathcal{D}_{\text{block}}(\pi) \text{ for } \pi \in \mathcal{P}^\circ(n)$$

then $\hat{1}_n \otimes \overline{B} = B \otimes \hat{1}_n = \hat{1}_n \otimes \overline{1}_n$. 

(5). The Definition 3.1 say that

I. Each block of $\pi$ is associated with $[n]$ or $[\overline{n}]$ (no connection between them);

II. $B = (a, \ldots, b)$ is a block of at least two elements in $\pi|_{[n]} \iff \overline{B} = (\overline{a}, \ldots, \overline{b})$ is a block of at least two elements in $\pi|_{[\overline{n}]}$;

III. $(a, b)$ is an arc in $\pi|_{[n]}$ if and only if $(\overline{a}, \overline{c})$ is an arc in $\pi|_{[\overline{n}]}$;

IV. $a$ is singleton in $\pi|_{[n]}$ if and only if $\overline{a}$ is singleton in $\pi|_{[\overline{n}]}$.

(6). It is worth to emphasize that full Fock space $(\bigoplus_{n=0}^{\infty} H^{\otimes n}) \otimes (\bigoplus_{n=0}^{\infty} \overline{H}^{\otimes n})$ is studied in the context of semi-meander polynomials which are used in the enumeration of semi-meandric systems; see [38]. In the context of enumeration of such objects we can say that the diagonal pair partition corresponds to a special subset of the self-intersecting meandric system $(t = w = 1)$, such that closed pairs which intersect the x-axis have the same parity. We skip formal description of it and just heuristically explain that we can combine negative and positive parts by $i \leftrightarrow \overline{i}$ as in the figure:

It is worth to mention that the number of such pair partitions is the Euler number; see Corollary 4.9.
3.2. Statistics. We introduce some partition statistics for \( \pi \in \mathcal{P}(n) \) which are necessary to obtain the moment-cumulant formula. These statistics are naturally extended to the set \( \mathcal{P}^\otimes(n) \), i.e. separately for the parts \([n]\) and \([\bar{n}]\). We say that an arc (or pair) \((i, j)\) is crossing the arc \((i', j')\) if \(i < i' < j < j'\) or \(i' < i < j' < j\) and similarly, we say that an arc (or pair) \((i, j)\) nests \((i', j')\) if \(i < k < j\) for any \(k \in \{i', j'\}\). Similarly we say that an arc (or pair) \((i, j)\) covers singleton \(k\) if \(i < k < j\).

3.2.1. Statistics related to Gaussian operator – pairs and singletons. For a set partition \( \pi \in \mathcal{P}_{1,2}(n) \) let \( \text{cr}(\pi) \) be the number of crossings of \( \pi \), i.e.

\[
\text{cr}(\pi) = \#\{(V, W) \in \text{Pair}(\pi) \times \text{Pair}(\pi) \mid V \text{ is crossing } W\}.
\]

Let \( \text{CS}(\pi) \) be the number of pairs of a singleton and a covering block:

\[
\text{CS}(\pi) = \#\{(V, W) \in \text{Sing}(\pi) \times \text{Pair}(\pi) \mid W \text{ covers } V\}.
\]

Let \( \text{nest}(\pi) \) be the number of pairs of a nesting block:

\[
\text{nest}(\pi) = \#\{(V, W) \in \text{Pair}(\pi) \times \text{Pair}(\pi) \mid W \text{ nests } V\}.
\]

Let \( \text{SR}(\pi) \) be the number of singleton to the right of pairs:

\[
\text{SR}(\pi) = \#\{(V, W) \in \text{Sing}(\pi) \times \text{Pair}(\pi) \mid V = (i) \text{ and } i > j \text{ for all } j \in W\}.
\]

3.2.2. Statistics related to gauge operator – block of size at least three. For \( \pi \in \mathcal{P}(n) \) we define two statistics, which are related to the gauge operator.

**Restricted crossings.** We use the same definition of restricted crossings as given in Biane \([9]\), namely

\[
\text{rc}(B, \widetilde{B}) := \#\{(V, W) \in \text{Arc}(B) \times \text{Arc}(\widetilde{B}) \mid \text{such that } V \text{ is crossing } W\}.
\]

For a set partition \( \pi \in \mathcal{P}(n) \) let \( \text{rc}(\pi) \) be the number of restricted crossings of \( \pi \):

\[
\text{rc}(\pi) := \sum_{i<j} \text{rc}(B_i, B_j),
\]

where \( \pi \setminus \text{Sing}(\pi) = \{B_1, \ldots, B_l\} \).

**Restricted nestings.** Now we define the number of restricted nestings of the partition \( \pi \in \mathcal{P}(n) \). The set of restricted nestings of \( B, \widetilde{B} \) is

\[
\text{rnest}(B, \widetilde{B}) := \#\{(V, W) \in \text{Arc}(B) \times \text{Arc}(\widetilde{B}) \mid V \text{ nests } W \text{ or } W \text{ nests } V\},
\]

and the set of restricted nestings of \( \pi \) is

\[
\text{rnest}(\pi) := \sum_{i<j} \text{rnest}(B_i, B_j),
\]

where \( \pi \setminus \text{Sing}(\pi) = \{B_1, \ldots, B_l\} \).

4. Moments of Gaussian operator

We present an example of a generalized Gaussian operator. It is given by creation and annihilation operators on a quadrabasic Fock space. We show that the distribution of these operators with respect to the vacuum expectation is a generalized Gaussian distribution, in the sense that all moments can be calculated from the second moments with the help of a combinatorial formula.
4.1. Gaussian operator.

Definition 4.1. The operator

\[ G_{\xi \otimes \eta} = A_{\xi \otimes \eta} + A_{\xi \otimes \eta}^*, \quad \xi \otimes \eta \in \mathcal{H}_R \]

on \( \mathcal{F}_{\text{dig}}(\mathcal{H}) \) is called the quadrabasic Gaussian operator.

Remark 4.2. (1). From estimating the norm of the creation (annihilator) operators we conclude that \( G_{\xi \otimes \eta} \) is bounded, whenever \(|q|, |v| < 1\).

(2). If \( q = t = v = w = 1 \), then the kernel of the symmetrization is nontrivial. In this case the square of the operator \( P_{(1,1,1,1)}^{(n)} \) given by the expression \( \left[ P_{(1,1,1,1)}^{(n)} \right]^2 = \frac{1}{n!} P_{(1,1,1,1)}^{(n)} \) projects \( \mathcal{H}_n \) to the space of special symmetric functions \( \mathcal{H}_n \)

\[ \tilde{\xi}_n \tilde{\eta}_n := (\xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_n) \]

\[ = P_{(1,1,1,1)}^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n) \otimes (\eta_1 \otimes \cdots \otimes \eta_n) \]

\[ = (\sum_{\sigma \in \mathfrak{S}_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}) \otimes (\sum_{\sigma \in \mathfrak{S}_n} \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(n)}) \]

i.e. we have

\[ \sigma(\tilde{\xi}_n \tilde{\eta}_n) = \tilde{\xi}_n \tilde{\eta}_n \quad \sigma \in \mathfrak{S}_n. \]

More precisely, the operator \( A_{\xi_1 \otimes \eta_1}^* \) for \( \tilde{\xi}_n \otimes \tilde{\eta}_m \) is \((\xi_2 \otimes \cdots \otimes \xi_{n-1}) \otimes (\eta_2 \otimes \cdots \otimes \eta_{m-1}) \) in \( \mathcal{H}_n \) and \( \xi_1 \otimes \eta_1 \in \mathcal{H}_R \) works in the following way

\[ A_{\xi_1 \otimes \eta_1}^* \tilde{\xi}_n \otimes \tilde{\eta}_m = (\xi_1 \otimes \cdots \otimes \xi_{n-1}) \otimes (\eta_1 \otimes \cdots \otimes \eta_{m-1}) \]

\[ = (\sum_{\sigma \in \mathfrak{S}_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}) \otimes (\sum_{\sigma \in \mathfrak{S}_n} \eta_{\sigma(1)} \otimes \cdots \otimes \eta_{\sigma(n)}). \]

Thus the scalar product automatically implements the additional relations \( A_{\xi_1 \otimes \xi_2}^* A_{\xi_2 \otimes \xi_1}^* = A_{\xi_2 \otimes \xi_2} \) on \( \mathcal{H}_n \). From this we conclude that when \( q = t = v = w = 1 \), the operators \( G_{\xi_1 \otimes \xi_2} \) and \( G_{\xi_2 \otimes \xi_1} \) are commutative on \( \mathcal{H}_n \). Notice that in this case if \( \xi_1 \perp \xi_2 \) and \( \xi_1 \perp \xi_2 \), then \( \varphi(G_{\xi_1 \otimes \xi_1}^n G_{\xi_2 \otimes \xi_2}^m) = \varphi(G_{\xi_1 \otimes \xi_1}^n \varphi(G_{\xi_2 \otimes \xi_2}^m) \) which coincides with classical independence.

4.2. Orthogonal polynomials. For a probability measure \( \mu \) with finite moments of all orders, let us orthogonalize the sequence \((1, x, x^2, x^3, \ldots)\) in the Hilbert space \( L^2(\mathbb{R}, \mu) \), following the Gram-Schmidt method. This procedure yields orthogonal polynomials \((P_0(x), P_1(x), P_2(x), \ldots)\)

with the convention that \( P_{-1}(x) = 0 \). The coefficients \( \beta_n \) and \( \gamma_n \) are called Jacobi parameters and they satisfy \( \beta_n \in \mathbb{R} \) and \( \gamma_n \geq 0 \). It is known that

\[ x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x), \quad n = 0, 1, 2, \ldots \]

with the convention that \( P_{-1}(x) = 0 \). The coefficients \( \beta_n \) and \( \gamma_n \) are called Jacobi parameters and they satisfy \( \beta_n \in \mathbb{R} \) and \( \gamma_n \geq 0 \). It is known that

\[ \gamma_0 \cdots \gamma_n = \int_{\mathbb{R}} |P_{n+1}(x)|^2 \mu(dx), \quad n \geq 0. \]

Moreover, the measure \( \mu \) has a finite support of cardinality \( N \) if and only if \( \gamma_{N-1} = 0 \) and \( \gamma_n > 0 \) for \( n = 0, \ldots, N-2 \).

The continued fraction representation of the Cauchy transform can be expressed in terms of the Jacobi parameters:

\[ \int_{\mathbb{R}} \frac{\mu(dt)}{z - t} = \frac{1}{z - \beta_0 - \frac{\gamma_0}{z - \beta_1 - \frac{\gamma_1}{z - \beta_2 - \cdots}}}. \]
This representation is useful to calculate the Cauchy transform when Jacobi parameters are given. More details can be found in [33]. Notice that if \( \lim_{n \to \infty} \beta_n = \beta > 0 \) and \( \lim_{n \to \infty} \gamma_n = \gamma \) then absolutely continuous part of measure \( \mu \) is supported on \( \left(-2\sqrt{\beta + \gamma}, 2\sqrt{\beta + \gamma}\right) \), see [25].

Let \( (Q_{n,q,t,v,w}^{(q,t,v,w)}(x))_{n=0}^{\infty} \) be the quadrabasic Hermite orthogonal polynomials (1.4). The orthogonalizing probability measure \( \mu_{q,t,v,w} \) is unknown in general case but from Section 4.2 we see that if \( t = w = 1 \), then absolutely continuous part of \( \mu_{q,1,v,1} \) is supported on \( \left(\frac{2}{\sqrt{1 - q}}, \frac{2}{\sqrt{1 - q}}\right) \).

**Theorem 4.3.** Suppose that \( q, v \in (-1, 1) \), \( \xi \otimes \eta \in H_\mathbb{R} \) and \( \|\xi \otimes \eta\| = 1 \). Let \( \kappa_{q,t,v,w} \) be the probability distribution of \( G_{\xi \otimes \eta} \) with respect to the vacuum state. Then \( \kappa_{q,t,v,w} \) is equal to \( \mu_{q,t,v,w} \).

**Proof.** Let \( \gamma_{n-1} = [n]_q, [n]_{v,w} \), then

\[
\|\xi^{\otimes n} \otimes \eta^{\otimes n}\|^2_{q,t,v,w} = \langle \xi^{\otimes n}, P_{q,t}^{(n)} \xi^{\otimes n}\rangle_0,0(\eta^{\otimes n}, P_{v,w}^{(n)} \eta^{\otimes n})_0,0 = \langle \xi^{\otimes n}, (I \otimes P_{q,t}^{(n-1)})R_{q,t}^{(n)} \xi^{\otimes n}\rangle_0,0(\eta^{\otimes n}, (I \otimes P_{v,w}^{(n-1)})R_{v,w}^{(n)} \eta^{\otimes n})_0,0
\]

We remind that the operator \( R_{q,t}^{(n)} \) given by \( R_{q,t}^{(n)} = t^{n-1} + \sum_{k=1}^{n-1} q^{k} t^{n-k-1} \sigma_1 \cdots \sigma_k \) acts as follows

\[
R_{q,t}^{(n)} \xi^{\otimes n} = [n]_{q,t} \xi^{\otimes n}, \text{because } \sigma_1 \cdots \sigma_k \xi^{\otimes n} = \xi^{\otimes n} \text{ and } [n]_{q,t} = t^{n-1} + \sum_{k=1}^{n-1} q^{k} t^{n-k-1}. \text{ We now expand further } R_{q,t}^{(n)} \xi^{\otimes n} = [n]_{q,t} \xi^{\otimes n}, R_{v,w}^{(n)} \eta^{\otimes n} = [n]_{v,w} \eta^{\otimes n} \text{ and obtain}
\]

\[
\|\xi^{\otimes n} \otimes \eta^{\otimes n}\|^2_{q,t,v,w} = \sqrt{\bar{\Omega}} \gamma_1 \cdots \gamma_{n-1}.
\]

Hence from (4.3) it follows that

\[
\|\xi^{\otimes n} \otimes \eta^{\otimes n}\|^2_{q,t,v,w} = \|Q_{n}^{(q,t,v,w)}(x)\|_{L^2}, \quad n \in \mathbb{N} \cup \{0\}.
\]

Therefore, the map \( \Phi: (\text{span}\{\xi^{\otimes n} \otimes \eta^{\otimes n} \mid n \geq 0, \|\cdot\|_{q,t,v,w}) \to L^2(\mathbb{R}, \mu_{q,t,v,w}) \) defined by

\[
\Phi(\xi^{\otimes n} \otimes \eta^{\otimes n}) = Q_{n}^{(q,t,v,w)}(x)
\]

is an isometry. Note that

\[
G_{\xi \otimes \eta} \xi^{\otimes n} \otimes \eta^{\otimes n} = A_{\xi \otimes \eta} \xi^{\otimes n} \otimes \eta^{\otimes n} + A_{\xi \otimes \eta} \xi^{\otimes n} \otimes \eta^{\otimes n} = \xi^{\otimes(n+1)} \otimes \eta^{\otimes(n+1)} + \delta_{q,t}(\xi) \xi^{\otimes(n+1)} \otimes (\delta_{v,w}(\eta) \eta^{\otimes(n+1)}) = \xi^{\otimes(n+1)} \otimes \eta^{\otimes(n+1)} + [n]_{q,t} \xi^{\otimes(n+1)} \otimes [n]_{v,w} \eta^{\otimes(n+1)} = \xi^{\otimes(n+1)} \otimes \eta^{\otimes(n+1)} + [n]_{q,t} [n]_{v,w} \xi^{\otimes(n+1)} \otimes \eta^{\otimes(n+1)}.
\]

Hence, by induction we can compute \( G_{\xi \otimes \eta}^{n} \otimes \Omega \otimes \Omega \) and show that \( \Phi(G_{\xi \otimes \eta}^{n} \otimes \Omega \otimes \Omega) = x^n \). Since \( \Phi \) is an isometry we get \( \langle \Omega \otimes \Omega, G_{\xi \otimes \eta}^{n} \otimes \Omega \otimes \Omega \rangle_{q,t,v,w} = m_n(\mu_{q,t,v,w}) \) for \( n \in \mathbb{N} \). Since \( \mu_{q,t,v,w} \) is compactly supported, probability measures giving the moment sequence \( m_n(\mu_{q,t,v,w}) \) are unique and hence \( \mu_{q,t,v,w} = \kappa_{q,t,v,w} \).

**4.2.1. Interesting cases of orthogonal polynomials.** We present a class of orthogonal polynomials corresponding to known measures.

**q-Meixner-Pollaczek polynomials.** Let \( t = w = 1 \). For \(-1 < \alpha < 1 \) let \( (\tilde{P}_n^{(\alpha)}(x))_{n=0}^{\infty} \) be the orthogonal polynomials with the recursion relation

\[
(\tilde{P}_n^{(\alpha)}(x))_{n=0}^{\infty}
\]

where \( \tilde{P}_0^{(\alpha)}(x) = 0, \tilde{P}_1^{(\alpha)}(x) = 1 \). These polynomials are called \( q \)-\textit{Meixner-Pollaczek polynomials}. The orthogonalizing probability measure MP_\alpha is known in [33] (14.9.4), supported on \((-2/\sqrt{1-q}, 2/\sqrt{1-q})\) and absolutely continuous with respect to the Lebesgue measure with density

\[
\frac{d\text{MP}_{\alpha,q}}{dt}(x) = \frac{(q; q)_\infty (\beta^2; q)_\infty}{2\pi \sqrt{4/(1-q) - x^2}} g(x, 1; q) g(x, -1; q) g(x, \sqrt{q}; q) g(x, -\sqrt{q}; q) g(x, i\beta; q) g(x, -i\beta; q)
\]

\[
(4.6)
\]
Carleman’s theorem states that the moment problem is determined if the known criterion (in this case) of Hamburger moment problem is due to Carleman [23, 44, 26]. Now we will explain that the moment problem is determined in this situation. Probably the best

\[ g(x, b; q) = \prod_{k=0}^{\infty} (1 - 4bx(1 - q)^{-1/2}q^k + b^2q^{2k}), \]

\[ (s; q)_\infty = \lim_{n \to \infty} (s; q)_n = \prod_{k=0}^{\infty} (1 - sq^k), \quad s \in \mathbb{R}, \]

\[ \beta = \begin{cases} \sqrt{-\alpha}, & \alpha \leq 0, \\ i\sqrt{\alpha}, & \alpha \geq 0. \end{cases} \]

**Proposition 4.4.** The measure \( \mu_{(q,1,q,1)} \) belongs to the family \( \text{MP}_{-q,q} \) for \( |q| < 1 \).

**Proof.** If we put \( \alpha = -q \) in the recurrence (4.5), then

\[ yU_n(y) = U_{n+1}(y) + [n]_q (1 - q^n) U_{n-1}(y), \quad n \geq 1. \]

Now, let us substitute \( L_n(y) = U_n(y\sqrt{1-q})/(1-q)^{\frac{n}{2}} \), multiply (4.10) by \((1-q)^{-\frac{n}{2}}\) and replace \( y \) by \( \sqrt{1-q}y \), then we get the recursion

\[ yL_n(y) = L_{n+1}(y) + [n]^2_q L_{n-1}(y), \quad n \geq 1, \]

with \( L_0(y) = 1 \) and \( L_1(y) = y \), so we see that \( L_n(y) = Q_n^{(q,1,q,1)}(y) \). This observation means that the recurrence (4.10) corresponds to monic orthogonal polynomials which orthogonalize the distribution of \( \sqrt{1-q}G^{(q,1,q,1)}_{\xi\infty} \).

**Hyperbolic secant.** Professor M. Ismail let us know that the case \( v = w = 1 \) is interesting because in this situation we get a measure interpolated between classical normal \((q = t = 0)\) and hyperbolic secant distribution \((q = t = 1)\) i.e. we have the relation \((Q_n^{(q,t)} := Q_n^{(q,t,1,1)})\)

\[ xQ_n^{(q,t)}(x) = Q_{n+1}^{(q,t)}(x) + [n]_{q,t} nQ_{n-t}^{(q,t)}(x), \quad n = 0, 1, 2, \ldots. \]

Now we will explain that the moment problem is determined in this situation. Probably the best known criterion (in this case) of Hamburger moment problem is due to Carleman [23, 11, 26]. Carleman’s theorem states that the moment problem is determined if

\[ \sum_{n \geq 1} \gamma_n^{-\frac{1}{2}} = \infty, \]

where \( \gamma_n \) are Jacobi parameters from (4.2). In this case we have

\[ \infty = \sum_{n \geq 1} \frac{1}{n} = \sum_{n \geq 1} (n^2)^{-\frac{1}{2}} \leq \sum_{n \geq 1} ([n]_{q,t,n})^{-\frac{1}{2}}. \]

Hence, the moments uniquely determine the measure and we can use the argument from the proof of Theorem 4.3 to conclude that \( G^{(q,t,1,1)}_{\xi\infty} \) has the distribution \( \mu_{q,t,1,1}. \)

In particular we show that \( G^{(1,1,1,1)}_{\xi\infty} \) has the hyperbolic secant distribution; see (1.2). A random variable follows a hyperbolic secant distribution if its probability density function can be related to the following standard form \( \rho(dx) = \frac{1}{2\cosh(\pi x/2)}dx. \) We note that the characteristic function of this distribution is \( \frac{1}{\cosh(x)} \) i.e. the density and its characteristic function differ only by scale parameters (the normal distribution is the prime example for this phenomenon) – see [7, 17]. The secant distribution is infinitely divisible distribution generated by some particular processes with stationary independent increments (Lévy processes – see [39]). The moments \( E_n = m_{2n}(\rho) \) are Euler numbers with positive signs (see [39, eq. (8)] or [1, Chapter 23])

\[ (m_0(\rho), m_2(\rho), m_4(\rho), m_6(\rho), m_8(\rho), \ldots) = (1, 1, 5, 61, 1385, 50521, \ldots). \]

**Discrete \( q \)-Hermite I polynomials.** Important classes of orthogonal polynomials studied here are the continuous and the discrete \( q \)-Hermite I polynomials, which are both special cases of the Al-Salam–Chihara polynomials. We recover them for \( q = w, \ t = 1 \) and \( v = 0; \) see
4.3. Gaussian moment. In the proof below, we also use additional notation and definitions.

**Definition 4.5.** Let $\mathcal{PS}_{1,2}(n)$ be a set of diagonal partitions of $[n] \sqcup \bar{[n]}$ such that every block is a pair or a singleton, each block of $\pi$ is associated with $[n]$ or $\bar{[n]}$ and the collections of openers of pair coincide.

We emphasize that from the definition above it follows that for $\pi \in \mathcal{PS}_{1,2}(n)$ the number of singletons in $\pi_{|[n]}$ is the same as in $\pi_{|\bar{[n]}}$, which we use in the proof, and if $\alpha$ is a singleton in $\pi_{|[n]}$, then it does not mean that $\bar{\alpha}$ is a singleton in $\pi_{|\bar{[n]}}$; For these partitions we define

$$D_{\text{block}}(\pi) = \{ B \otimes \bar{B} \mid B \text{ is a pair and } \bar{B} \text{ is a conjugate pair} \}.$$ 

**Example 4.6.** For example if $\pi = \{(1,3),(2)\} \sqcup \{(1,\bar{2}), (\bar{3})\}$, then $\pi \in \mathcal{PS}_{1,2}(3)$ and $\pi \notin \mathcal{P}^\circ_{1,2}(3)$. In this case we also have $D_{\text{block}}(\pi) = \{(1,3) \otimes (\bar{1},\bar{2})\}$ and on this set $D_{\text{block}}(\pi)$ is undefined because singletons are not coincide.

For a given diagonal pair $B = (a, b) \otimes (\bar{a}, \bar{c})$, we denote the left and right legs of $B$ by $l_B = a$, $\bar{l}_B = \bar{a}$, $r_B = b$ and $\bar{r}_B = \bar{c}$.

**Theorem 4.7.** Suppose that $\xi_i \otimes \xi_i \in \mathcal{H}_R$, $i \in \{1, \ldots, n\}$, then

$$\varphi(G_{\xi_1 \otimes \xi_1} \cdots G_{\xi_n \otimes \xi_n}) = \sum_{\pi \in \mathcal{PS}_{1,2}(n)} q^{r(\pi_{|[n]})} l^{\text{nest}(\pi_{|[n]})} u^{r(\pi_{|\bar{[n]}})} u^{\text{nest}(\pi_{|\bar{[n]}})} \prod_{B \in D_{\text{block}}(\pi)} \langle \xi_B \otimes \xi_B, \xi_{\bar{B}} \otimes \xi_{\bar{B}} \rangle _{\mathcal{H}}.$$ 

**Remark 4.8.** Let $S = \{i_1, \ldots, i_k\}$ such that $i_1 < \cdots < i_k$ be a finite subset of natural numbers and $\{T_i\}_{i \in \mathbb{N}}$ be the sequence of operators then when we write $\prod_{i \in S} T_i$ we mean that $T_{i_1} \cdots T_{i_k}$.

**Proof.** Given $\varepsilon = (\varepsilon(1), \ldots, \varepsilon(n)) \in \{1,*\}^n$, let $\mathcal{PS}_{1,2;\varepsilon}$ be the set of partitions $\pi \in \mathcal{PS}_{1,2}$ such that

- if $(a, b)$ is a pair in $\pi_{|[n]}$ then $\varepsilon(b) = *$, $\varepsilon(a) = 1$,
- if $(\bar{a}, \bar{c})$ is a pair in $\pi_{|\bar{[n]}}$ then $\varepsilon(\bar{c}) = *$, $\varepsilon(\bar{a}) = 1$,
- if $c$ is a singleton in $\pi$ then $\varepsilon(\{c\}) = *$.

Let $\varepsilon(i) \in \{1,*\}$. We will prove the result for $Z = \{1, \ldots, n\}$ by induction

$$\prod_{i \in Z} A^{\varepsilon(i)}_{\xi_i \otimes \xi_i} \Omega \otimes \bar{\Omega} = \sum_{\pi \in \mathcal{PS}_{1,2;\varepsilon}(Z)} q^{r(\pi_{|Z})+CS(\pi_{|Z})} l^{\text{nest}(\pi_{|Z})+SR(\pi_{|Z})} u^{r(\pi_{|Z})+CS(\pi_{|Z})} u^{\text{nest}(\pi_{|Z})+SR(\pi_{|Z})} \prod_{B \in D_{\text{block}}(\pi)} \langle \xi_B \otimes \xi_B, \xi_{\bar{B}} \otimes \xi_{\bar{B}} \rangle _{\mathcal{H}} \times \langle \otimes_{i \in \text{Sing}(\pi_{|Z})} \xi_i \rangle \otimes \langle \otimes_{i \in \text{Sing}(\pi_{|Z})} \xi_i \rangle,$$

it is assumed to be true for $Z = \{2, \ldots, n\}$. When $n = 1$, $A_{\xi_1 \otimes \xi_1} \Omega \otimes \bar{\Omega} = 0$ and $A^{\varepsilon(1)}_{\xi_1 \otimes \xi_1} \Omega \otimes \bar{\Omega} = \xi_1 \otimes \xi_1$ and hence the formula is true. Suppose that the formula (4.12) is true for $Z = \{2, \ldots, n\}$.

We will show that the action of $A^{\varepsilon(1)}_{\xi_1 \otimes \xi_1}$ corresponds to the inductive pictorial description of set partitions. We fix $\pi \in \mathcal{PS}_{1,2;\varepsilon}(\{2, \ldots, n\})$ and run the argument below over all partitions of this type. Suppose that

- $\pi_{|\{2, \ldots, n\}}$ has singletons $s_1 < \cdots < s_{p_1} < \cdots < s_{r}$ and pair blocks $W_1, \ldots, W_{u_1}$ which cover $s_{p_1}$ and pairs $U_1, \ldots, U_{t_1}$ to the left of $s_{p_1}$,
- $\pi_{|\{2, \ldots, n\}}$ has singletons $k_1 < \cdots < k_{p_2} < \cdots < k_{r}$ and pair blocks $\bar{W}_1, \ldots, \bar{W}_{u_2}$ which cover $k_{p_2}$ and pairs $\bar{U}_1, \ldots, \bar{U}_{t_2}$ to the left of $k_{p_2}$.
Note that when there is no singleton, the arguments below can be modified easily.

Case 1. If $\varepsilon(1) = 1$, then the operator $A^\varepsilon_{\xi_1 \otimes \xi_1}$ acts on the tensor product, putting $\xi_1 \otimes \xi_1$ on the left. This operation pictorially corresponds to adding the singletons as follows

$$\{s_1, \ldots, s_r\} \cup \{k_1, \ldots, k_r\} \mapsto \{1, s_1, \ldots, s_r\} \cup \{\bar{1}, k_1, \ldots, k_r\}.$$ 

This yields a new partition $\tilde{\pi} \in \mathcal{P}\mathcal{S}_{1,2}^{\otimes \varepsilon}(n)$. This map $\pi \mapsto \tilde{\pi}$ does not change the numbers related to our statistic like $\text{cr}(\cdot)$, which is compatible with the fact that the action of $A^\varepsilon_{\xi_1 \otimes \xi_1}$ does not change the coefficient, hence the formula (4.12) holds when we moved form $n - 1$ to $n$ and $\varepsilon(1) = 0$.

Case 2. If $\varepsilon(1) = 1$, then $A^\varepsilon_{\xi_1 \otimes \xi_1}$ acts on the tensor product, contributing to new $r^2$ terms. In the $(p_1, p_2)\text{th}$ term in the action we obtain the term $q^{p_1 - 1}q^{p_2 - 1}q^{r - p_1}w^{r - p_2} \langle \xi_1, s_{p_1} \rangle H(\xi_1, \xi_{p_2}) H$, with tensor product

$$(\xi_1 \otimes \cdots \otimes \hat{\xi}_{s_{p_1}} \otimes \cdots \otimes \xi_{s_r}) \otimes (\xi_1 \otimes \cdots \otimes \hat{\xi}_{k_{p_2}} \otimes \cdots \otimes \xi_{k_r}).$$

Pictorially this corresponds to getting a set partition $\tilde{\pi} \in \mathcal{P}\mathcal{S}_{1,2}^{\otimes \varepsilon}(n)$ by adding the most left singletons 1 to set $\{s_1, \ldots, s_r\}$ and $\bar{1}$ to the set $\{k_1, \ldots, k_r\}$ and creating the diagonal pair

$$B = (1, s_{p_1}) \otimes (\bar{1}, k_{p_2}) \in \mathcal{D}^{\text{block}}_{\text{PS}}(\tilde{\pi}),$$

with the same left legs 1 and $\bar{1}$. Note also that

$$\langle \xi_{\text{in}} \otimes \xi_{\text{in}}, \xi_{\text{in}} \otimes \xi_{\text{in}} \rangle H = \langle \xi_1, s_{p_1} \rangle H(\xi_1, \xi_{p_2}) H.$$ 

The diagonal pair $B$ crosses the blocks $W_1, \ldots, W_{u_1}, \bar{W}_1, \ldots, \bar{W}_{u_2}$ and so increases the number of crossings by $u_1$ in $\pi|_{[n]}$ and by $u_2$ in $\pi|_{[\bar{n}]}$, but decreases the number of inner singletons by $u_1$ and $u_2$. The diagonal pair $B$ covers the pairs $U_1, \ldots, U_{l_1}, \bar{U}_1, \ldots, \bar{U}_{l_2}$ so increases the nesting by $l_1$ in $\pi|_{[n]}$ and by $l_2$ in $\pi|_{[\bar{n}]}$ and create a new $r - p_1$ right singletons in $\pi|_{[n]}$ and $r - p_2$ in $\pi|_{[\bar{n}]}$. In the new situation $s_{p_1}$ and $k_{p_2}$ are not singletons, so the number of right singletons of $U_1, \ldots, U_{l_1}$ and $\bar{U}_1, \ldots, \bar{U}_{l_2}$ decreases by $l_1$ and $l_2$, respectively. Now some new inner singletons $s_1, \ldots, s_{p_1 - 1}, k_1, \ldots, k_{p_2 - 1}$ appear. Altogether we have:

$$\text{cr}(\tilde{\pi}|_{[n]}) = \text{cr}(\pi|_{(2, \ldots, n)}) + u_1,$$

$$\text{cr}(\tilde{\pi}|_{[\bar{n}]}) = \text{cr}(\pi|_{(2, \ldots, \bar{n})}) + u_2,$$

$$\text{CS}(\tilde{\pi}|_{[n]}) = \text{CS}(\pi|_{(2, \ldots, n)}) - u_1 + p_1 - 1,$$

$$\text{CS}(\tilde{\pi}|_{[\bar{n}]}) = \text{CS}(\pi|_{(2, \ldots, \bar{n})}) - u_2 + p_2 - 1,$$

$$\text{nest}(\tilde{\pi}|_{[n]}) = \text{nest}(\pi|_{(2, \ldots, n)}) + l_1,$$

$$\text{nest}(\tilde{\pi}|_{[\bar{n}]}) = \text{nest}(\pi|_{(2, \ldots, \bar{n})}) + l_2,$$

$$\text{SR}(\tilde{\pi}|_{[n]}) = \text{SR}(\pi|_{(2, \ldots, n)}) + r - p_1 - l_1,$$

$$\text{SR}(\tilde{\pi}|_{[\bar{n}]}) = \text{SR}(\pi|_{(2, \ldots, \bar{n})}) + r - p_2 - l_2.$$ 

So we see that the exponent of $q'$s, $\nu$'s, $t$'s and $w$'s increases by $q^{p_1 - 1}w^{p_2 - 1}q^{r - p_1}w^{r - p_2}$ respectively; see Figure 3. Note that as $\pi$ runs over $\mathcal{P}\mathcal{S}_{1,2}(\varepsilon(2), \ldots, \varepsilon(n))(n - 1)$, every set partition $\tilde{\pi} \in \mathcal{P}\mathcal{S}_{1,2}^{\otimes \varepsilon}(n)$ appears exactly once either in Case 1 or in Case 2 as shown by induction that the formula (4.12) holds for all $n \in \mathbb{N}$. Finally, formula (4.11) follows from (4.12) by taking the sum over all $\varepsilon$ such that $\text{Sing}(\pi) = \emptyset$ (in this case we understand that $(\otimes_{i \in \text{Sing}(\pi|_z)} \xi_i) \otimes (\otimes_{i \in \text{Sing}(\pi|_{\bar{z}})} \xi_i) = \Omega \otimes \bar{\Omega}$) and applying the state action, because then $\mathcal{P}\mathcal{S}_{1,2}^{\otimes \varepsilon}(n) = \mathcal{P}\mathcal{S}_{2}^{\otimes \varepsilon}(n)$ and $\mathcal{D}^{\text{block}}_{\text{PS}}(\pi) = \mathcal{D}^{\text{block}}_{\text{PS}}(\pi)$.\hfill \qed

---

**Figure 3.** The main structure of partition $\tilde{\pi} \in \mathcal{P}\mathcal{S}_{1,2}^{\otimes \varepsilon}(n)$ in the induction step.
Corollary 4.9. It is interesting to compare the moment of $G^{(1,1,1,1)}_{\xi \otimes \eta}$, where $\|\xi \otimes \eta\| = 1$, with Euler numbers $E_n$ with positive signs (see moment of hyperbolic secant distribution in Subsection 4.2.1) because it provides a new combinatorial interpretation of these numbers:

$$E_n = \#P_2^\circ(2n).$$

These numbers also occur in combinatorics, specifically when counting the number of alternating permutations of a set with an even number of elements and it is not clear why they are the same.

4.4. Trace. Let $\nu N(G(H_\mathbb{R}))$ be the von Neumann algebra generated by $\{G_{\xi \otimes \eta} \mid \xi \otimes \eta \in H_\mathbb{R}\}$ acting on the completion of $\mathcal{F}_{\text{alg}}(H)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{q,t,v,w}$.

Proposition 4.10. Suppose that $\dim(H_\mathbb{R}) \geq 2$, $\dim(H_\mathbb{R}) \geq 2$ and $q,v \in (-1,1)$. Then the vacuum state is a trace on $\nu N(G(H_\mathbb{R}))$ if and only if $q = v = 0$ and $t = w = 1$.

Proof. By using Theorem 4.7, we obtain

$$\varphi(G_{\xi_1 \otimes \xi_1}G_{\xi_2 \otimes \xi_2}G_{\xi_3 \otimes \xi_3}G_{\xi_4 \otimes \xi_4}) = \langle \xi_1, \xi_2 \rangle_H \langle \xi_3, \xi_4 \rangle_H \langle \xi_1, \xi_3 \rangle_H \langle \xi_2, \xi_4 \rangle_H + qv\langle \xi_1, \xi_3 \rangle_H \langle \xi_2, \xi_4 \rangle_H \langle \xi_1, \xi_4 \rangle_H \langle \xi_2, \xi_3 \rangle_H + tw\langle \xi_1, \xi_4 \rangle_H \langle \xi_2, \xi_3 \rangle_H \langle \xi_1, \xi_2 \rangle_H \langle \xi_3, \xi_4 \rangle_H$$

and by permuting

$$\varphi(G_{\xi_1 \otimes \xi_1}G_{\xi_2 \otimes \xi_2}G_{\xi_3 \otimes \xi_3}G_{\xi_4 \otimes \xi_4}) = \langle \xi_2, \xi_3 \rangle_H \langle \xi_4, \xi_1 \rangle_H \langle \xi_2, \xi_1 \rangle_H \langle \xi_3, \xi_4 \rangle_H + qv\langle \xi_2, \xi_4 \rangle_H \langle \xi_3, \xi_1 \rangle_H \langle \xi_2, \xi_3 \rangle_H \langle \xi_4, \xi_1 \rangle_H + tw\langle \xi_2, \xi_4 \rangle_H \langle \xi_3, \xi_1 \rangle_H \langle \xi_2, \xi_3 \rangle_H \langle \xi_4, \xi_1 \rangle_H.$$

Since $\dim(H_\mathbb{R}) \geq 2$, there are two orthogonal unit eigenvectors $e_1, e_2$ and we put $\xi_1 = \xi_2 = e_1$ and $\xi_3 = \xi_4 = e_2$. We also take $\xi_1 = \xi_2 = \xi_3 = \xi_4 = \eta \neq 0$ and so the difference is

$$\varphi(G_{\xi_1 \otimes \eta}G_{\xi_2 \otimes \eta}G_{\xi_3 \otimes \eta}G_{\xi_4 \otimes \eta}) - \varphi(G_{\xi_1 \otimes \eta}G_{\xi_2 \otimes \eta}G_{\xi_3 \otimes \eta}G_{\xi_4 \otimes \eta}) = (1 - t - tw)||\eta||^4.$$

Therefore, the traciality of the vacuum state implies $1 - t - tw = 0$. Now we do the same analysis with orthogonal vectors $\xi_1 = \xi_2 = e_1$ and $\xi_3 = \xi_4 = e_2$, and obtain $1 - t - tw = 0$, which by the restriction $|t|,|w| \leq 1$ implies that $t = w = 1$. If we substitute this into the first equation, we see that $v = 0$. It is clear by symmetry that using the same argument as above, we will show that $\varphi$ is not a trace when $q \neq 0$. When $q = v = 0$ and $t = w = 1$, we get

$$\varphi(G_{\xi_1 \otimes \xi_1} \cdots G_{\xi_n \otimes \xi_n}) = \sum_{\pi \in NC_2(n)} \prod_{B \in \mathcal{P}_{\text{block}}(\pi)} \langle \xi_B \otimes \xi_B, \xi_B \otimes \xi_B \rangle_H$$

by Remark 3.3 (2), we have

$$\sum_{\pi \in NC_2(n)} \prod_{(i,j) \in \pi} \langle \xi_i \otimes \xi_j, \xi_j \otimes \xi_i \rangle_H$$

where the traciality is known because $\langle \xi_i \otimes \xi_j, \xi_j \otimes \xi_i \rangle_H = \langle \xi_j \otimes \xi_j, \xi_i \otimes \xi_i \rangle_H$. \(\square\)

Remark 4.11. (1). The vacuum expectation $\varphi$ is a trace on $\nu N(G(H_\mathbb{R}))$ if

$$(q,t,v,w) = (q,1,0,1) \text{ and } \dim(H_\mathbb{R}) = 1 \text{ or } \dim(H_\mathbb{R}) = 1.$$

Indeed, if $\xi_i = \xi$ and $||\xi||_{\mathbb{R}} = 1$ then by Theorem 4.7 we have

$$\varphi(G_{\xi_1 \otimes \xi_1} \cdots G_{\xi_n \otimes \xi_n}) = \sum_{\pi \in P_2^n} q^{cr(\pi|_\eta)} \prod_{B \in \mathcal{P}_{\text{block}}(\pi)} \langle \xi_B \otimes \xi_B, \xi_B \otimes \xi_B \rangle_H$$
Let $\pi \in \mathcal{P}_2(n)$. There exists a unique non-crossing partition $\hat{\pi} \in \mathcal{P}_2(n)$, such that the set of right and left legs of the pairs of $\pi$ and $\hat{\pi}$ coincide – see [16, Page 215]. In our situation this means that the set of all partitions $\pi \in \mathcal{P}_2(\bar{n})$ such that $\pi|_{[\bar{n}]}$ is noncrosing is isomorphic to $\mathcal{P}_2(n)$. From this and $\|\xi\|_{\bar{H}} = 1$ we can deduce

$$
= \sum_{\pi \in \mathcal{P}_2(n)} q^{cr(\pi)} \prod_{(i,j) \in \pi} \langle \xi_i, \xi_j \rangle_H,
$$

and in this case the traciality is known – see [19]. Unfortunately, this argument can not be applied under the assumption that $\dim(H_R) \geq 2$ and $\dim(\bar{H}_R) \geq 2$ (the reason is given in the next point).

(2). At first glance the impression may be that there should a trace, when $t = 1$ and $w = 1$, because just then the crossing partition plays a role. The essence of the problem is that under the cyclic permutation action of a partition of the form $\mathcal{P}^\otimes(n)$ is not a map to itself; see Figure 4 for specific example.

![Figure 4. A cyclic permutation action of $[n]$ and $[\bar{n}]$](image)

5. Poisson-type operators

5.1. Gauge operator. Now we define differential second quantization operator on $F_{\text{dig}}^S(\mathcal{H})$. In order to this, we introduce some special operators. Let $T$ and $\bar{T}$ be the operators on Hilbert spaces $H$ and $\bar{H}$ with dense domains $D$ and $\bar{D}$, respectively. We also assume that $T(D) \subset D$ and $\bar{T}($\bar{D}$) \subset \bar{D}$ and $D := D \otimes \bar{D}$. The following gauge operator is motivated by the papers [3, 30].

**Definition 5.1.** The gauge operator $p_{T \otimes \bar{T}}$ is an operator on $F_{\text{dig}}^S(\mathcal{H})$ defined by

$$p_{T \otimes \bar{T}} := p_{\alpha_{T}} \otimes p_{\bar{\alpha}_{T}}$$

with a dense domain $F_{\text{dig}}^S(\mathcal{H})$.

In this part let us recall the properties of the gauge operator from [30, Proposition 1.8] (the proof is almost identical and it can be omitted).

**Proposition 5.2.** If $T \otimes \bar{T}$ is essentially self-adjoint on a dense domain $D$, then $p_{T \otimes \bar{T}}$ is essentially self-adjoint on a dense domain $F_{\text{dig}}^S(\mathcal{H})$.

Directly from Proposition 2.1 we can state the following proposition.

**Proposition 5.3.** If $T$ and $\bar{T}$ are bounded operators on $\mathcal{H}$, then $p_{T \otimes \bar{T}}$ is a bounded operator on the $F_{\text{dig}}^S(\mathcal{H})$.

5.2. Quadrabasic operators and cumulants. In non-commutative setting, random variables are understood to be the elements of the *-algebra generated by creator, annihilator or gauge operators. Particularly interesting are their joint mixed moments. In order to work effectively on this object we need to combine joint moments with corresponding cumulants. This topic in the case of $q$-deformed Fock space was deeply analyzed in the literature; see [3, 4, 9, 37]. Our approach is close to [3, 30]. We define $\lambda_1 \otimes \lambda_i$ to be $\lambda_i \lambda_i I \otimes I$. We also use a special convention that $\bar{T}_i := T_i$ where $i \in [\bar{n}]$. 
Definition 5.5. The operator
\[ X_{\xi_i, \lambda_i, T_i}^\xi := A_{\xi_i} \otimes \xi_i + A_{\lambda_i}^* \otimes \lambda_i + \rho_{T_i} \otimes T_i + \lambda_i \otimes \lambda_i, \quad \xi_i \otimes \xi_i \in \mathcal{H}_R, \quad \lambda_i, \lambda_i \in \mathbb{R}, \]
onumber on \( \mathcal{F}_{\text{dis}}(\mathcal{H}) \) is called a quadrabasic operator.

Definition 5.6. Let \( \pi \in \mathcal{P}^{\circ}(n) \), \( B = B \otimes \bar{B} = \{i_1, \ldots, i_m\} \otimes \{\bar{i}_1, \ldots, \bar{i}_k\} \in \mathcal{D}_{\text{block}}(\pi) \), \( \lambda_i, \lambda_i \in \mathbb{R} \) and \( \xi_i \otimes \xi_i \in \mathcal{H}_R \) and the diagonal cumulant is defined by
\[ R_\pi^{\xi, T}(B) := \begin{cases} \lambda_i \lambda_i, & \text{if } B \text{ is a singleton,} \\ \langle \xi_{i_1}, T_{i_2} \cdots T_{i_{m-1}} \xi_{i_m}, \lambda_i \rangle_R, & \text{otherwise.} \end{cases} \]
\[ R_\pi^{\xi, T} := \prod_{B \in \mathcal{D}_{\text{block}}(\pi)} R_\pi^{\xi, T}(B). \]

The following theorem is the main result of this section. Its proof is given in Subsection 5.4.

Theorem 5.6. Suppose that \( \xi_i \otimes \xi_i \in \mathcal{H}_R^n \), then
\[ \varphi(X_{\xi_i, \lambda_i, T_i}^\xi \cdots X_{\xi_n, \lambda_n, T_n}^\xi) = \sum_{\pi \in \mathcal{P}^{\circ}(n)} q^{r_c(\pi|_n)} t^{r_{\text{nest}}(\pi|_n)} w^{r_{\text{nest}}(\pi|_n)} R_\pi^{\xi, T}. \]

Corollary 5.7. (1) For \( t = w = 1, v = 0, \xi = \xi, \|\xi\| = 1 \) and \( \lambda_i = \lambda_i = 0 \), we obtain the \( q \)-deformed formula for moments of random variable on \( q \)-Fock space (see 3 or 1 Proposition 6). (2) For \( T_i \otimes T_i = 0 \) (which is equivalent to \( T_i = 0 \) or \( T_i = 0 \)) and \( \lambda_i \otimes \lambda_i = 0 \), we get the formula (4.11).

5.3. The orthogonal polynomial. \((q, t, v, w)\)-Poisson polynomials are defined by the recursion relations
\[ x \hat{Q}_n^{q,t,v,w}(x) = \hat{Q}_n^{q,t,v,w}(x) + [n]_{q,t} [n]_{v,w} \hat{Q}_n^{q,t,v,w}(x) + [n]_{q,t} [n]_{v,w} \hat{Q}_{n-1}^{q,t,v,w}(x), \quad n \geq 1 \]
onumber with initial conditions \( \hat{Q}_0^{q,t,v,w}(x) = 0 \), \( \hat{Q}_1^{q,t,v,w}(x) = 1 \) and \( \hat{Q}_1^{q,t,v,w}(x) = x \). There exists a probability measure \( \tilde{\mu}_{q,t,v,w} \) which is associated to the orthogonal polynomials \( \hat{Q}_n^{q,t,v,w} \).

Remark 5.8. The measure of orthogonality of the above polynomial sequence is not known. In special cases, we can identify this measure:
(1) the measure \( \tilde{\mu}_{1,1,0,0} \) is the classical Poisson law;
(2) the measure \( \tilde{\mu}_{0,1,0,1} \) is the Marchenko-Pastur distribution;
(3) the measure \( \tilde{\mu}_{q,1,0,1} \) is the \( q \)-Poisson law and the orthogonal polynomials \( \hat{Q}_n^{q,1,0,1}(x) \) are called \( q \)-Poisson-Charlier polynomials (see 3).

Using the same argument as in Theorem 4.3, we can prove the following.

Proposition 5.9. Let \( \xi \otimes \eta \in \mathcal{H}_R \) and \( \|\xi \otimes \eta\| = 1 \) and \( T = \overline{T} = I \). Then the probability distribution of \( X_{\xi,0}^{q,0,1} \) with respect to the vacuum state is given by \( \tilde{\mu}_{q,t,v,w} \).

5.4. Proof of Theorem 5.6. We begin with some special notations.

In order to prove Theorem 5.6 we need to set \( \mathcal{P}_{\bar{E}}^\circ(n) \) of so-called extended partitions. Here some blocks can be additionally marked by \( E \) and so we consider additional blocks denoted by \( \{i_1, \ldots, i_m\} \cup \{\bar{i}_1, \ldots, \bar{i}_k\} \).

Definition 5.10. We denote by \( \mathcal{P}_{\bar{E}}(n) \) the set partition of \( [n] \cup \bar{[n]} \) such that each block of \( \pi \) is associated with \( [n] \) or \( \bar{[n]} \) and the collections of openers points of arcs of \( \pi|[n] \) and \( \pi|\bar{[n]} \) coincide. Additionally, every block of partitions of \( \pi \) is regular or extended. All singletons are denoted as extended. If for the block \( B \in \pi|[n] \) exist conjugate block \( \bar{B} \in \pi|\bar{[n]} \), then both of them must be denoted as regular or extended as \( (a, \ldots, b), (\bar{a}, \ldots, \bar{c}) \) or \( (a, \ldots, b), (\bar{a}, \ldots, \bar{c}) \).

All other blocks are denoted as extended.
Example 5.11. For example
\[ \pi = \{(1, 4)_E, (2)_E, (3)_E, (5)_E \} \cup \{(1, 3), (2)_E, (5)_E \} \in \mathcal{P}_E^E(5) \]
\[ \pi = \{(1, 4), (2)_E, (3)_E, (5)_E \} \cup \{(1, 3), (2)_E, (5)_E \} \in \mathcal{P}_E^E(5) \]
\[ \pi = \{(1, 4, 5)_E, (2)_E, (3)_E \} \cup \{(1, 3), (2)_E, (4, 5)_E \} \in \mathcal{P}_E^E(5) \]
\[ \pi = \{(1, 4, 5), (2)_E, (3)_E \} \cup \{(1, 3), (2)_E, (4, 5)_E \} \notin \mathcal{P}_E^E(5) \]

For \( \pi \in \mathcal{P}_E^E(n) \) we denote by \( \text{Block}_E(\pi) \) the tensor blocks of \( \pi \) which are marked by \( E \) and \( \text{Block}(\pi) = \pi \setminus \text{Block}_E(\pi) \). Thus we can decompose an extended partition as a disjoint subset
\[ \pi = \text{Block}_E(\pi) \cup \text{Block}(\pi). \]

Remark 5.12. Note that if \( \mathcal{P}_E^E(n) \) consists of pairs and singletons, then it is not the same as \( \mathcal{P}S_{1,2}^E(n) \). The objects \( \mathcal{P}S_{1,2}^E(n) \) are the partitions from \( \mathcal{P}_E^E(n) \) consisting of pairs and singletons, such that all pairs are regular, and all singletons are expanded.

Cover and left of max. We also need to extend the definition of \( \text{CS}(\pi|Z) \) and \( \text{SR}(\pi|Z) \) for \( \pi \in \mathcal{P}_E^E(n) \) where \( Z = [n] \) or \([\bar{n}]\), i.e. we define
\[ \text{CS}(\pi|Z) := \#\{(V, W) \in \text{Block}_E(\pi|Z) \times \text{Arc}(\pi|Z) \mid i < \min V < j \text{ for } i, j \in W\}, \]
\[ \text{SR}(\pi|Z) := \#\{(V, W) \in \text{Block}_E(\pi|Z) \times \text{Arc}(\pi|Z) \mid \max V > j \text{ for } j \in W\}. \]

Remark 5.13. Note that \( \text{CS}(\pi) \) represents the number of covered singletons and \( \text{SR}(\pi) \) the number of singletons to the right of arcs, whenever all extended block are singletons (which is the reason we use the same notation).

In order to simplify notation, we define the following operators, which map \( H \) (\( \bar{H} \)) into \( H \) (\( \bar{H} \)) and which are indexed by the block \( B_E = \{i_1, \ldots, i_m\}_E \in \text{Block}_E(\pi|_{[n]} \mid i. e. \) \( \bar{T}^E_{B_E} = T_{i_1} \cdots T_{i_{m-1}} \) and for \( B = \{i_1, \ldots, i_m\} \in \text{Block}(\pi|_{[n]} \mid i. e. \) \( T^E_B = T_{i_1} \cdots T_{i_{m-1}} \).

We use the same notation for \([\bar{n}]\), i.e. \( \bar{T}^E_B \) or \( \bar{T}^E_{B_E} \). With the notation above we also introduce:
\[ K^\xi_T^E = \prod_{B \in \pi|_{[n]}} \langle x_{\min B}, T^E_B \xi_{\max B} B \rangle_H \prod_{B \in \pi|_{[n]}} \langle x_{\min B}, T^E_B \xi_{\max B} B \rangle_{\bar{H}}, \]
\[ \hat{K}^\xi_T^E = \left( \bigotimes_{B \in \pi|_{[n]}} \left( T^E_{B_E} \xi_{\max B_E} \right)_{\min B_E} \right) \otimes \left( \bigotimes_{B \in \pi|_{[n]}} \left( T^E_{B_E} \xi_{\max B_E} \right)_{\min B_E} \right). \]

Notice that in the above formula we use the following bracket notation \( \{\cdot\}_{\min B_E} \), which should be understood that the position of \( \cdot \) (in the tensor product) is ordered with respect to the min \( B_E \).

Example 5.14. For the partition
\[ \pi = \{(1, 4, 6, 7)_E, (2)_E, (3, 5)_E, (9)_E, (8, 10)\} \cup \{(1, 3, 4, 6, 10)_E, (2)_E, (5)_E, (7)_E, (8, 9)\}, \]
we have
\[ K^\xi_T^E = \langle x_8, \xi_{10} \rangle_H \langle x_8, \xi_9 \rangle_{\bar{H}}, \]
\[ \hat{K}^\xi_T^E = [T_1 T_4 T_6 \xi_7 \otimes \xi_2 \otimes T_5 \xi_5 \otimes \xi_9] \otimes [T_1 T_3 T_5 \xi_{10} \otimes \xi_2 \otimes \xi_5 \otimes \xi_7]. \]

We also use the following convention for \( \varepsilon \in \{1, *, E\} \)
\[ A^\varepsilon_{E \otimes E} = \begin{cases} A^*_{E \otimes E} & \text{if } \varepsilon = *, \\ A^1_{E \otimes E} & \text{if } \varepsilon = 1, \\ pT^E \otimes T^E & \text{if } \varepsilon = E. \end{cases} \]

The main idea of the proof is similar to that of Theorem 4.7 so, for brevity, we will leave out some of the combinatorial details.
Proof of Theorem 5.7. Observe that when \( n = 1 \), then \( A_{\xi_1 \otimes \xi_1} \Omega \otimes \bar{\Omega} = pt_1 \otimes T \Omega \otimes \bar{\Omega} = 0 \) and \( A_{\xi_1 \otimes \xi_1}^* \Omega \otimes \bar{\Omega} = \xi_1 \otimes \xi_1 \). Suppose that \( \xi_1 \otimes \xi_1 \in \mathcal{H}_R, i \in \{2, \ldots, n\} \) and any \( \varepsilon = (\varepsilon(2), \ldots, \varepsilon(n)) \in \{1, *\}^n \), we have

\[
A_{\xi_2 \otimes \xi_2}^{(2)} \cdots A_{\xi_n \otimes \xi_n}^{(n)} \Omega \otimes \bar{\Omega} = \sum_{\pi \in \mathcal{P}_{E,\varepsilon}^\otimes \{2, \ldots, n\}} q^{r(\pi) + CS((\pi)_{\varepsilon})} t^{\text{nest}(\pi)_{\varepsilon}} + SR(\pi)_{\varepsilon} \cdot \hat{K}_E T \tilde{K}_E T.
\]

We will show that the action of \( A_{\xi_1 \otimes \xi_1} \) corresponds to the inductive graphic description of set tensor partitions. We fix \( \pi \in \mathcal{P}_{E,\varepsilon}^\otimes \{2, \ldots, n\} \) and run the argument below over all partitions of this type. Suppose that

- \( \pi|_{\{2, \ldots, n\}} \) has blocks in \( \text{Block}_E(\pi|_{\{2, \ldots, n\}}) \) on the positions \( s_1 < \cdots < s_{p_1} < \cdots < s_r \) and arcs \( W_1, \ldots, W_{s_1} \) which cover \( s_{p_1} \), arcs \( U_1, \ldots, U_{s_1} \) to the left of \( s_{p_1} \),
- \( \pi|_{\{2, \ldots, n\}} \) has blocks in \( \text{Block}_E(\pi|_{\{2, \ldots, n\}}) \) on the positions \( k_1 < \cdots < k_{p_2} < \cdots < k_r \), arcs \( W_1, \ldots, W_{s_2} \) which cover \( k_{p_2} \) and arcs \( U_1, \ldots, U_{s_2} \) to the left of \( k_{p_2} \).

Suppose that a partition \( \pi \) has blocks \( S_E, K_E \in \text{Block}_E(\pi) \) on the \( (s_{p_1}, k_{p_2}) \)th position, i.e. \( (s_{p_1}, k_{p_2}) = (\min S_E, \min K_E) \). In this case blocks \( S_E, K_E \) have the following contribution to \( \hat{K}_E T \):

\[
\{1\} s_1 \cdots \{1\} s_2 \cdots \{1\} s_r \otimes \{1\} k_1 \cdots \{1\} k_2 \cdots \{1\} k_r
\]

Case 1. If \( \varepsilon(1) = \ast \), then the operator \( A_{\xi_1 \otimes \xi_1} \) acts on the tensor product, putting \( \xi_1 \otimes \xi_1 \) by adding (expanded) singleton on the left as in Case 1 of the proof of Theorem 1.7.

Case 2. If \( \varepsilon(1) = 1 \), then \( A_{\xi_1 \otimes \xi_1} \) acts on the tensor product, then new \( r^2 \) terms appear. In terms \( s_{p_1} \) and \( s_{p_2} \) the inner product

\[
\langle \xi_1, \hat{T}_{S_E}^{\xi_{\text{max} S_E}} \hat{T}_{K_E}^{\xi_{\text{max} K_E}} \rangle H \langle \xi_1, \hat{T}_{S_E}^{\xi_{\text{max} S_E}} \hat{T}_{K_E}^{\xi_{\text{max} K_E}} \rangle H
\]

appears with coefficient \( q^{p_1 - 1 + p_2 - 1 + r - p_1 + r - p_2} \). Graphically this corresponds to getting a set partition \( \bar{\pi} \in \mathcal{P}_{E,\varepsilon}^\otimes(n) \) by adding 1 and \( \bar{\Pi} \) to \( \pi \) and creating a new regular block \( (1, S) \) and \( (\bar{1}, \bar{K}) \) by adding first arcs \( (1, s_{p_1}) \) to \( S_E \) and the second \( (\bar{1}, k_{p_2}) \) to \( K_E \). We see that Equation (5.6) can be written in the form:

\[
\langle \bar{x}_{\text{min}(1, S)}, \hat{T}_{(1, S)}^{\xi_{\text{max}(1, S)}} \rangle H \langle \bar{x}_{\text{min}(1, K)}, \hat{T}_{(1, \bar{K})}^{\xi_{\text{max}(1, K)}} \rangle H.
\]

We can calculate the change in the statistic generated by the new arcs in the same way as in Case 2 of the proof of Theorem 1.7. Indeed it suffices to repeat all steps of counting changes with arcs instead of pairs, and \( rc, \text{nest} \) in place of \( cr, \text{nest} \). In this procedure we can think that extended blocks are singletons.

Case 3. If \( \varepsilon(1) = E \), then we use the equation (5.1), delete the element

\[
\hat{T}_{S_E}^{\xi_{\text{max} S_E}} \hat{T}_{K_E}^{\xi_{\text{max} K_E}}
\]

from \( \hat{K}_E T \), and then a new component \( \hat{K}_E T \) appears in the tensor product with coefficient \( q^{p_1 - 1 + p_2 - 1 + r - p_1 + r - p_2} \) in the first position as shown in Figure 5.

![Figure 5](image-url)

The visualization of the action of \( pt_1 \otimes T_1 \) on the tensor product \( \hat{K}_E T \).

Then we get a new partition \( \tilde{\pi} \in \mathcal{P}_{E,\varepsilon}^\otimes(n) \) by adding 1 to \( S_E \), \( \bar{1} \) to \( K_E \) (with the first arc \( (1, s_{p_1}) \) and \( (1, k_{p_2}) \)) and creating two blocks in \( \text{Block}_E(\tilde{\pi}) \). Now the minimum of newly created blocks are 1 and \( \bar{1} \) and so we can calculate the change in the statistic generated by the new arc.
in Case 2, because new arcs cannot be covered or be to the right of some arc. This situation is also compatible with changes inside the tensor product, i.e.

\[ \hat{T}^{ε}_{(1,S)E} ε_{\text{max}} S_E = T_I (\hat{T}^{ε}_{S_E} ε_{\text{max}} S_E) \text{ and } \hat{T}^{ε}_{(I,K)E} ε_{\text{max}} K_E = T_I (\hat{T}^{ε}_{K_E} ε_{\text{max}} K_E) \]

We now present the final step. First, let us notice that for \( ε \in \{1, *, E\} \) we have

\[ (5.7) \quad φ(A^{(1)}_{ε_{I} I_{1} \otimes I_{1}} \cdots A^{(n)}_{ε_{n} I_{n} \otimes I_{n}}) = \sum_{π \in P_{n}} q^{ν(π)} u^{ν_{ε} (π)} u^{ν_{ε} (\pi_{|n|})} K^{π}_{ν} T^{π} \]

Indeed, from equation (1.12) we see that the following condition must hold: \( \hat{K}^{π}_{ν} T^{π} = Ω \otimes \hat{Ω} \).
This will happen if and only if \( Block_E(π) = \emptyset \), which implies (5.7). If \( Block_E(π) = \emptyset \), then \( R^{π}_{ν} T^{π} = K^{π}_{ν} T^{π} \) so by taking the sum over all \( ε \) from equation (5.7), we see that

\[ \varphi\left(\left(X^{ε_{I_{1}, I_{1}}, T_{I_{1}}} - λ_{1} \otimes λ_{1}\right) \cdots \left(X^{ε_{n}, λ_{n}, T_{n}} - λ_{n} \otimes λ_{n}\right)\right) = \sum_{π \in P_{n}} q^{ν(π)} u^{ν_{ε} (π)} u^{ν_{ε} (\pi_{|n|})} R^{π}_{ν} T^{π}. \]

We also see that

\[ \varphi\left(\left(X^{ε_{I_{1}, I_{1}}, T_{I_{1}}} - λ_{1} \otimes λ_{1} + λ_{1} \otimes λ_{1}\right) \cdots \left(X^{ε_{n}, λ_{n}, T_{n}} - λ_{n} \otimes λ_{n} + λ_{n} \otimes λ_{n}\right)\right) \]

by equation (5.7), we get

\[ \sum_{ν \in \{1, \ldots, n\}} \prod_{ι \in ν} λ_{ι} λ_{ι} \sum_{π \in P_{n}} q^{ν(π)} u^{ν_{ε} (π)} u^{ν_{ε} (\pi_{|n|})} R^{π}_{ν} T^{π}. \]

\[ □ \]

6. Application to the Lévy Process

The main goal of this section is to investigate a new class of noncommutative Lévy processes. To make it clear, we use the following Anshelevich notation.

Here \( I_f \) is the indicator function of the set \( I \), considered both as a vector in \( L^2(\mathbb{R}^{+}) \) and a multiplication operator on it. Let \( K \) be a Hilbert space, and let \( H \) be the Hilbert space \( L^2(\mathbb{R}^{+}, dx) \otimes K \). Let \( η \) be in \( K \), and let \( T \) be an essentially self-adjoint operator on a dense domain \( D \subset K \) so that \( D \) is equal to the linear span of \( \{ T^n η \}_{n=0}^{∞} \). Moreover \( η \) is an analytic vector for \( T \). Let \( H = H \otimes H \), where we assume that \( H \) is a one-dimensional Hilbert space spanned by such \( ν \) that \( ∥ν∥ = 1 \), \( T = I \) and \( D = D \otimes D \). Given a half-open interval \( I \subset \mathbb{R}^{+} \) denote \( p_I(T) = p_{(I, \otimes I) T} \). For \( λ \in \mathbb{R} \) and \( (I_f \otimes ξ) \otimes η \in H_{\mathbb{R}} \) we define

\[ p_I(η \otimes η, T, λ) := A^{(1)}_{I_f \otimes ξ} + A^{∗}_{I_f \otimes ξ} + p_I(T) + |I| λ \otimes 1. \]

We will call a process of the form \( I \mapsto p_I(η \otimes η, T, λ) \) a quadrabasic Lévy process or \( (q, t, v, w) \)-Lévy process.

Remark 6.1. (1). For \( t = w = 1, v = 0 \) this is indeed a \( q \)-Lévy process in a sense of Anshelevich [3, 4].

(2). We assume \( T = I \) for several reasons. Firstly, all theorems below are not true for general \( T \). Secondly, cumulants in a general sense are not conditionally positive in a sense of Hamburger moment problem for the one-parameter moment-problems but maybe this analysis can be considered in the two-parameter case in the context of papers [27, 40, 47].

Definition 6.2. Denote by \( \mathbb{C}(x) = \mathbb{C}(x_1, x_2, \ldots, x_k) \) the algebra of polynomials in \( k \) formal noncommuting variables with complex coefficients. We denote by \( \theta_0(f) \) the constant term of \( f \in \mathbb{C}(x) \). While we take \( V \) to be \( k \)-dimensional, the same arguments will work for an arbitrary \( V \), as long as we use a more functional definition of a process, namely for \( f = \sum a_i x_i \in V \), we would define \( T(f) = \sum a_i T_i, \xi(f) = \sum a_i \xi_i, \lambda(f) = \sum a_i λ_i. \) We define a process

\[ X^{κ(i)}(I^{κ(i)}) := p_{I^{κ(i)}}(ξ^{κ(i)} \otimes η, T^{κ(i)}; λ^{κ(i)}) \text{ for a multi- indices } v \text{ and } u. \]
Denote by $X(s)$ the appropriate objects corresponding to the interval $[0, s)$. We define the functional $M$ on $C(x)$ by the following action on monomials: $M(1, s; X) = 1$ and

$$M(x_1, s; X) = \varphi(X_1(s) \ldots X_n(s)),$$

and extend linearly. We will call $M(\cdot, s; X)$ the moment functional of the process $X$ at time $s$. If we equip $C(x)$ with a conjugation * extending the conjugation on $C$ so that each $x_i^* = x_i$, it is clear that $M$ is a positive functional, i.e. $M(f^*, s; X) \geq 0$ for all $f \in C(x)$.

In order to keep the essentially self-adjoint operators, we should make an additional assumption on the family of operators $\{T_j\}_{j=1}^k$ (see \[3\] Subsection 2.5). Moreover, we emphasize that, since the second operator $T$ is bounded and self-adjoint no additional restriction on it are necessary.

**Assumption 6.3.** Now fix a $k$-tuple $\{T_j\}_{j=1}^k$ of essentially self-adjoint operators on a common dense domain $D \subset V$, $T_j(D) \subset D$, a $k$-tuple $\{\xi_j\}_{j=1}^k$ of vectors, and $\{\lambda_j\}_{j=1}^k \subset \mathbb{R}$. We will make an extra assumption that

$$\forall i, j \in [1 \ldots k], l \in \mathbb{N}, u \in [1 \ldots k]^l,$$

$$T_u \xi_i = T_u(1)T_u(2) \ldots T_u(l) \xi_i \text{ is an analytic vector for } T_j,$$

and $D = \text{span} \left\{ T_u \xi_i : i \in [1 \ldots k], l \in \mathbb{N}, u \in [1 \ldots k]^l \right\}.$

Now we define the joint cumulants of $X$.

**Definition 6.4.** The cumulant corresponding to the partition $\pi \in \mathcal{P}^\infty(n)$, the block $B = (i_1, \ldots, i_k) \in \pi|_n$ and the sub-monomial $\lambda_{(B,u)} := x_u(i_1) \ldots x_u(i_k)$ is

$$R(x_{(B,u)}, s) := \begin{cases} s\lambda_{(i_1)} & \text{if } k = 1, \\ s(\xi_{u(i_1)}, T_u(i_2) \ldots T_u(i_k-1) \xi_{u(i_k)})_H & \text{if } k \geq 2. \end{cases}$$

$$R_{\pi|_n}(x_u, s; X) := \prod_{B \in \pi|_n} R(x_{(B,u)}, s)$$

Sometimes for a one-dimensional process we will write

$$R_n(X(s)) = R_n((X(s), \ldots, X(s)).$$

In particular, we have

$$R_{1,n}(x_{(B,u)}; s; X) = \langle \xi_{(1)}, T_u(2) \ldots T_u(n-1) \xi_{u(n)} \rangle_H$$

i.e. the $n$-th joint cumulant of $X$ at time $s$. Note that the functional $R(\cdot, s; X)$ can be linearly extended to all of $C(x)$. We call this functional the cumulant functional of the process $X$ at time $s$.

An explicit formula for moments in terms of cumulants, involving the number of restricted crossings and nestings of a partition follows from Theorem \[5.6\] and we have

$$M(x_u, s; X) = \sum_{\pi \in \mathcal{P}^\infty(n)} q^{nc(\pi|_n)} t^{nest(\pi|_n)} q^{rc(\pi|_n)} u^{nest(\pi|_n)} R_{\pi|_n}(x_u, s; X).$$

This is because all diagonal cumulants of order at least two from Definition \[5.5\] involved with $H$ are equal to one.

**Remark 6.5.** We emphasize that the general algebraic notation of independence introduced by Kämmerer \[35\] (pyramidally independent increments) is not true for quadrabasic Lévy process, therefore we do not use this argument in our proof. Note that in the special case $(q, 1, 0, 1)$-Lévy process has this property which follows directly from equation \[6.2\].
6.1. Multiple stochastic measures. Rota and Wallstrom [12] introduced the notion of partition-dependent stochastic measures. Their approach unifies a number of combinatorial results in probability theory, for example the Itô multi-dimensional stochastic integrals through the usual product measures, by employing the Möbius inversion on the lattice of all partitions. We shall show that, in contrast, such an approach has also some potential.

Fix $s > 0$. For $N$ and a subdivision of $[0, s)$ into disjoint ordered half-open intervals $\mathcal{I} = \{I_1, I_2, \ldots, I_N\}$, let $\delta(I) = \max_{1 \leq i \leq N} |I_i|$. Fix a monomial $x_u \in \mathbb{C}\langle x_1, x_2, \ldots, x_k \rangle$ of degree $n$.

**Definition 6.6.** The $n$-dimensional diagonal measure corresponding to the monomial $x_u$ and the subdivision $\mathcal{I}$ is

$$\Delta_n(x_u, s; \mathcal{I}) = \sum_{i=1}^{N} \prod_{j=1}^{n} X^{u(j)}(I_{\mathcal{I}(i)}) = \sum_{i=1}^{N} X^{u(1)}(I_{\mathcal{I}(i)}) \cdots X^{u(n)}(I_{\mathcal{I}(i)})$$

where $n \in \mathbb{N}$. The $n$-dimensional diagonal measure corresponding the monomial $x_u$ is

$$\Delta_n(x_u, s; X) = \lim_{\delta(I) \to 0} \Delta_n(x_u, s; \mathcal{I})$$

if the limit, along the net of subdivisions of the interval $[0, s)$, exists.

**Remark 6.7.** If an element of $X$ does not depend on $u$, i.e. this is a one-dimensional process, then we write $\Delta_n(s; X)$.

**Proposition 6.8.** For the monomial $x_u$ of degree $n$, the cumulant functional of the quadrabasic Lévy process $X$ is given by

$$R_n(x_u, s; X) = \lim_{\delta(I) \to 0} \varphi(\Delta_n(x_u, s; X)).$$

**Remark 6.9.** We emphasize that the existence of the limit $\lim_{\delta(I) \to 0} \Delta_n(x_u, s; \mathcal{I})$ is not essential in Proposition [6.8].

**Proof.** By definition, we have to calculate

$$\lim_{\delta(I) \to 0} \sum_{i=1}^{N} \varphi(X^{u(1)}(I_{\mathcal{I}(i)}) \cdots X^{u(n)}(I_{\mathcal{I}(i)})).$$

Let us denote $R_{\pi|\pi}(x_u, \mathcal{I}; X) := R_{\sigma|\sigma}(x_u, 1; X) \prod_{B \in \sigma|\sigma} |I_{\mathcal{I}(B)}|$, where we write $\nu(D)$ for any $\nu(i), i \in B$. Using this and Theorem [5.6] with $\pi \in P(n)$ we see that

$$\sum_{i=1}^{N} \varphi(X^{u(1)}(I_{\mathcal{I}(i)}) \cdots X^{u(n)}(I_{\mathcal{I}(i)}))$$

$$= \sum_{i=1}^{N} \sum_{\sigma \in \mathcal{D}(n)} q^{rc(\sigma|\pi)} t^{nest(\sigma|\pi)} u^{rc(\sigma|\pi)} w^{nest(\sigma|\pi)} R_{\sigma|\sigma}(x_u, \mathcal{I}; X)$$

$$= \sum_{i=1}^{N} \sum_{\sigma \in \mathcal{D}(n)} q^{rc(\sigma|\pi)} t^{nest(\sigma|\pi)} u^{rc(\sigma|\pi)} w^{nest(\sigma|\pi)} R_{\sigma|\sigma}(x_u, \mathcal{I}; X)$$

$$+ \sum_{i=1}^{N} \sum_{\sigma \in \mathcal{D}(n)} q^{rc(\sigma|\pi)} t^{nest(\sigma|\pi)} u^{rc(\sigma|\pi)} w^{nest(\sigma|\pi)} R_{\sigma|\sigma}(x_u, \mathcal{I}; X)$$
By Remark 3.3 (4) if \( \sigma \in \mathcal{P}(n) \) and \( \sigma|_{[1]} = \hat{1}_n \) then \( \sigma = \hat{1}_n \otimes \hat{1}_T \). We now expand further and obtain

\[
R_n(x_n, 1; X) = \sum_{i=1}^{N} \sum_{\sigma \in \mathcal{P}(n) \backslash B \in \sigma} \prod_{\sigma|_{[1]} < \hat{1}_n} |I_{\mathfrak{g}(B)}| q^{\mathfrak{r}(\sigma|_{[1]})} t^{\mathfrak{r}(\sigma|_{[1]})} u^{\mathfrak{r}(\sigma|_{[1]})} R_{\sigma|_{[1]}}(x_n, 1; X).
\]

Now we show that the limit of each of the remaining terms (*) is 0. Indeed, if \( \sigma|_{[1]} < \hat{1}_n \), then \( \#\sigma|_{[1]} > 1 \) and assume that the number of them is \( d \), where \( d \geq 2 \). We may assume \( \delta(I) < 1 \), and thus for each fixed \( \pi \) the term (*) is bounded by

\[
C \sum_{i=1}^{N} |I_i|^d \leq C\delta(I)s^{d-1},
\]

where \( C \) is a constant independent of the subdivision \( I \). Therefore such a term converges to 0 as \( \delta(I) \to 0 \).

6.1.1. The higher diagonal measures. Now we calculate all the higher diagonal measures (this object appears in the functional Itô formula for Lévy processes).

**Proposition 6.10.** For a one-dimensional self-adjoint process \( X(s) = p_s(\xi \otimes \eta, T, \lambda) \) the \( n \)-dimensional diagonal measure with \( n \geq 2 \) exists in the \( L^2 \)-norm with respect to \( \varphi(\cdot) \), and equals

\[
\Delta_n(s; X) = p_s(T^{n-1}\xi \otimes \eta, T^n, \langle \xi, T^{n-2}\xi \rangle).
\]

**Proof.** Let \( Y_n(I) = p_I(T^{n-1}\xi \otimes \eta, T^n, \langle \xi, T^{n-2}\xi \rangle) \) be the process from the right-hand side of above theorem. We will show that

\[
\lim_{\delta(I) \to 0} \|\Delta_n(s; X, I) - Y_n(s)\|_2 = 0.
\]

where \( \Delta_n(s; X, I) = \sum_{1 \leq i \leq N} X^n(I_{\mathfrak{g}(i)}) \). First expand

\[
[\Delta_n(s; X, I) - Y_n(s)]^2 = \Delta_n^2(s; X, I) - \Delta_n(s; X, I)Y_n(s) - Y_n(s)\Delta_n(s; X, I) + Y_n^2(s).
\]

We will show that in the limit (as \( \delta(I) \to 0 \)) the first two factors of above expansion disappear. We start with the first factor

\[
\varphi(\Delta_n^2(s; X, I)) = \varphi(\sum_{1 \leq i \leq N} X^n(I_{\mathfrak{g}(i)})X^n(I_{\mathfrak{g}(i)}))
\]

\[
= \varphi(\sum_{i=1}^{N} X^{2n}(I_{\mathfrak{g}(i)})) + \varphi(\sum_{i \neq j} X^n(I_{\mathfrak{g}(i)})X^n(I_{\mathfrak{g}(j)})) \xrightarrow{\delta(I) \to 0} R_{2n}(X(s)) + R_n^2(X(s)).
\]
Indeed, by Proposition 6.8 it follows that the first factor converges to \( R_{2n}(X(s)) \). Now using cumulant expansions of the second factor we get

\[
\sum_{i \neq j}^N \varphi(X^n(I_{i,j}(s))) \varphi(X^n(I_{i,j}(s))) = \sum_{i \neq j}^N R_n(X(I_{i,j}(s))) R_n(X(I_{i,j}(s))) + \sum_{i \neq j}^N \sum_{\sigma \in \mathcal{P} \cap \mathcal{P}(2n)} q^{rc(\sigma)}(\sigma) R^{n \text{rest}}(\sigma) R^{n \text{rest}}(\sigma) R^n(X(I_{i,j}(s))) = R^n(X(s))
\]

by the same argument as in Proposition 6.8 we have that each term in the expression \((*)\) converges to zero. Now we focus on the second factor, i.e.

\[
\varphi(\Delta_n(s; X, I) Y_n(s)) = \varphi \left( \sum_{1 \leq i \leq N} X^n(I_{i,j}(s)) Y_n(I_{i,j}(s)) \right).
\]

By similar argument as we used in the first part, we conclude that

\[
\delta(I) \rightarrow 0 \quad R\left( X(s), \ldots, X(s), Y_n(I) \right) + R_n(X(s)) R_1(Y_n(s))
\]

By Definition 6.1 we observe that mixed cumulants of \( X(I) \) and \( Y_n(I) \) are coincident in a sense that for \( n \geq 2 \) and \( k \geq 0 \) we have

\[
R\left( X(I), \ldots, X(I), Y_n(I) \right) = R_{k+n}(X(I)).
\]

So, the last limit expression reduces to \( R_{2n}(X(s)) + R^n(X(s)) \). Hence, the first two factors disappear and similarly we conclude for the two remaining elements. \( \square \)

6.2. Generators. The analysis in this subsection is partially motivated by papers [3] and [13].

Definition 6.11. I. A functional \( \psi \) on \( \mathbb{C}(x) \) is conditionally positive if its restriction to the subspace of polynomials with zero constant term is positive semi-definite.

II. We say that a functional \( \psi \) on \( \mathbb{C}(x) \) is a generators of \( (q, t, v, w) \)-Lévy process if it is a derivative of the moment functional at zero.

III. We say that the functional \( \psi \) is analytic if for any \( i \) and any multi-index \( u \),

\[
\lim_{n \rightarrow \infty} \frac{1}{n!} \psi([x_u]^s d_x^{2n} x_u]^{1/2n} < \infty.
\]

Remark 6.12. The family of the moment functionals of a \( (q, t, v, w) \)-Lévy process is determined by its cumulant functional. Indeed, by equation (6.2), we have

\[
M(x_u, s; X) = \sum_{\pi \in \mathcal{P} \cap \mathcal{P}(n)} q^{rc(\pi)}(\pi) R^{\text{rest}}(\pi) R^n(\pi)(x_u, 1; X),
\]

which implies that this is a polynomial in \( s \) for \( \pi = \hat{1}_n \otimes \hat{1}_n \), and so by differentiating this equality, we obtain

\[
\frac{d}{ds} M(x_u, s; X) \bigg|_{s=0} = R_{1_n}(x_u, 1; X) = \langle \xi_{\pi(1)}, T_{\pi(2)} \cdots T_{\pi(n-1)} \xi_{\pi(n)} \rangle_H.\]
The following proposition is an analog of the Schoenberg correspondence for our context. The proof is almost identical to one of [3, 43] and will be omitted. We reproduce the proof on arXiv version for the reader’s convenience.

**Proposition 6.13.** A functional $\psi$ is analytic and conditionally positive if and only if it is the generator of the family of the moment functionals for some $(q, t, v, w)$-Lévy process.

**Proof.** Suppose $\psi$ is the generator of the family of moment functionals $M(\cdot, s; X)$ for a $(q, t, v, w)$-Lévy process $X(s)$. By Definition [6.11] we have $\psi(x) = R_{\nu}(x, 1; X)$, which is means that the cumulant functional is conditionally positive indeed:

$$R_{\nu}(\langle x \rangle^* x, 1; X) = \|T^{(1)} \ldots T^{(n)} \xi| H \geq 0.$$ 

For $x$ of degree $m$ we have

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \psi[(\langle x \rangle^* x)^{2n}]^{1/2n} = \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \langle \xi^{(m)} \parallel \prod_{j=m-1}^{m} T^{(j)} T^{(m)} \parallel \xi^{(m)} \rangle^{1/2n}$$

$$= \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \|T^n \xi|^{1/n} < \infty$$

by Assumption [6.13]

Now suppose $\psi$ is conditionally positive and analytic.

The first step in the proof is that of Anshelevich [3, Proposition 4.3] to show the existence of $T_i$ and space $K$ (the proof is practically identical to that of this result and we provide an outline of the details for the reader’s convenience). Since $\psi$ is positive then we can define semi-definite inner product on the space $\mathbb{C} \langle x \rangle$ by

$$\langle f, g \rangle_\psi = \psi[(f - \delta_0(f))^*(g - \delta_0(g))].$$

Let $N_\psi = \{a \in \mathbb{C} \mid \langle a, a \rangle_\psi = 0\}$ and let $K$ be the Hilbert space obtained by completing the quotient $\mathbb{C} \langle x \rangle / N_\psi$ with respect to this inner product. Denote by $\rho$ the canonical mapping $\mathbb{C} \langle x \rangle \to K$, let $D$ be its image, and for $f, g \in \mathbb{C} \langle x \rangle$ define the operator $\Gamma(a) : D \to D$ by

$$\Gamma(f) \rho(g) = \rho(fg) - \rho(f) \delta_0(g).$$

The operator $\Gamma$ is well defined since, by the Cauchy-Schwartz inequality,

$$\|\Gamma(f) \rho(g)\|_\psi = \psi[(g - \delta_0(g))^* f^* \langle (f - \delta_0(g)) \rangle] \leq \|\rho(g)\|_\psi \|f^* \rho(f - \delta_0(g))\|_\psi.$$ 

Clearly $D$ is dense in $K$, invariant under $\Gamma(a)$, and $\Gamma(a)$ is symmetric on it if $a$ is symmetric. We define $\lambda_i = \psi[x_i], \xi_i = \rho(x_i), T_i = \Gamma(x_i)$. Each $T_i$ takes $D$ to itself. By construction, $\Gamma(x_i) \rho(x) = \rho(x_i x)$, and so

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n!}} \|T^n \rho(x)\|_\psi^{1/n} = \limsup_{n \to \infty} \frac{1}{\sqrt{n!}} \|x^n x\|_\psi^{1/n}$$

$$= \limsup_{n \to \infty} \frac{1}{\sqrt{n!}} \psi[(\langle x \rangle^* x)^{2n}]^{1/2n} < \infty$$

since the functional $\psi$ is analytic. Therefore each of these vectors is analytic for $T_i$, and the linear span of these vectors is $D$. In particular, $T_i$ is essentially self-adjoint on $D$. Let $H = H \otimes \hat{H}$ where $H = L^2(\mathbb{R}, dx) \otimes K$, $\hat{H}$ is a one-dimensional Hilbert space spanned by the $\eta$ with norm one and $D = D \otimes \hat{H}$. Finally, we can define the $(q, t, v, w)$-Lévy process by $X_{\nu(i)}(s) = \rho_1(\xi_{\nu(i)} \otimes \eta, T_{\nu(i)}(x), \lambda_{\nu(i)})$ and by Remark [6.12] we obtain $R(x, 1; X) = \psi[x]$. 

□
6.3. Convolution. First, we introduce a product state which reduces to a usual (tensor) product state, for \( q = t = w = 1 \) and \( v = 0 \) while for \( q = v = 0 \) and \( t = w = 1 \) it is the (reduced) free product state.

**Definition 6.14.** I. For functional \( \Phi \) on \( \mathbb{C}(x) \) we define the functional \( \Psi = \Psi(\Phi) \) on \( \mathbb{C}(x) \) by induction

\[
\Psi(x_\mu) = \Phi(x_\mu) - \sum_{\pi \in \mathcal{P}^{\ell}(\{n\})} q^\langle \pi\rangle l^{\text{rest}(\pi)} u^{w^\text{rest}(\pi)} \prod_{B \in \pi_{\{n\}}} \Psi(x_{(\beta;2)})
\]

and extend linearly. Let \( \Phi_1, \Phi_2 \) be functionals on \( \mathbb{C}(x_1, \ldots, x_k) \) and \( \mathbb{C}(y_1, \ldots, y_l) \), respectively. On \( \mathbb{C}(xy) := \mathbb{C}(x_1, \ldots, x_k, y_1, \ldots, y_l) \) we define their product functional by rule that mixed cumulants of independent quantities equal zero

\[
\Phi_1 \times_{q,t,v,w} \Phi_2 : \mathbb{C}(xy) \to \mathbb{C}
\]

\[
\Psi(\Phi_1 \times_{q,t,v,w} \Phi_2)(xy) \to \begin{cases} 
\Psi(\Phi_1)(xy) & \text{if } xy_\mu \in \mathbb{C}(x_1, \ldots, x_k) \\
\Psi(\Phi_2)(xy) & \text{if } xy_\nu \in \mathbb{C}(y_1, \ldots, y_l) \\
0 & \text{otherwise.}
\end{cases}
\]

II. We define \( \mathcal{ID}_{q,t,v,w} \), i.e. the set of all infinitely divisible functionals on \( \mathbb{C}(x) \) by

\[
\mathcal{ID}_{q,t,v,w}(k) := \{ \Phi : \Phi(\cdot) = M(\cdot, 1; X) \} = \{ \Phi : \Psi(\Phi) \text{ is conditionally positive and analytic} \}.
\]

From the definition above it is clear that \( R(\cdot, s; X) = \Psi(M(\cdot, s; X)) \).

**Proposition 6.15.** For \( \Phi_1 \in \mathcal{ID}_{q,t,v,w}(k) \) and \( \Phi_2 \in \mathcal{ID}_{q,t,v,w}(l) \), their product functional is a state.

**Proof.** The proof follows by direct construction. From Proposition 6.13 we know that there exist processes \( X^{(i,1)}(s) \) and \( Y^{(i,2)}(s) \) which may be identified with the \( (q, t, v, w) \)-Lévy processes whose distributions at time 1 are \( \Phi_1 \in \mathcal{ID}_{q,t,v,w}(k) \) on \( \mathbb{C}(x_1, \ldots, x_k) \) and \( \Phi_2 \in \mathcal{ID}_{q,t,v,w}(l) \) on \( \mathbb{C}(y_1, \ldots, y_l) \), respectively. We will explain that we can choose these processes in such a way that the product functional conditions are met. Let \( \xi_{i,1} \otimes \eta \in V_1 \otimes H \) and \( T_{i,1} \otimes I \) is an operator on \( V_1 \otimes H \) with domain \( D_1 = D_1 \otimes H \). Similarly \( \xi_{i,2} \otimes \eta \in V_2 \otimes H \) and \( T_{i,2} \otimes I \) is an operator on \( V_2 \otimes H \) with domain \( D_2 = D_2 \otimes H \). We identify

\[
\begin{align*}
\xi_{i,1} \otimes \eta & \text{ with } (\xi_{i,1} \otimes 0) \otimes \eta, \\
(\xi_{i,2} \otimes 0) & \text{ with } (0 \otimes \xi_{i,2}) \otimes \eta, \\
T_{i,1} \otimes I & \text{ with } (T_{i,1} 0) \otimes I, \\
T_{i,2} \otimes I & \text{ with } (0 T_{i,2}) \otimes I.
\end{align*}
\]

Let \( V = (V_1 \oplus V_2) \otimes H \) and \( X^{(i,1)}(s) = p_s(\xi_{i,1} \otimes \eta; T_{i,1}, \lambda_{i,1}), \quad Y^{(i,2)}(s) = p_s(\xi_{i,2} \otimes \eta; T_{i,2}, \lambda_{i,2}) \). By Definition 6.4 we know that this identification does not change the mixed cumulants of \( X^{(i,1)}(s) \) and \( Y^{(i,2)}(s) \). From this identification it follows that

\[
\varphi(A_1 \ldots A_{k+l}) = \Phi_1 \times_{q,t,v,w} \Phi_2(A_1 \ldots A_{k+l}),
\]

for all \( A_i \in \{X^{(1,1)}, \ldots, X^{(k,1)}, Y^{(1,2)}, \ldots, Y^{(l,2)}\} \) and

\[
a_i = \begin{cases} 
x_i & \text{if } A_i \in \{X^{(1,1)}, \ldots, X^{(k,1)}\} \\
y_i & \text{if } A_i \in \{Y^{(1,2)}, \ldots, Y^{(l,2)}\}.
\end{cases}
\]

Thus we prove that

\[
\Phi_1 \times_{q,t,v,w} \Phi_2 \in \mathcal{ID}_{q,t,v,w}(k+l).
\]

\[ \square \]
6.3.1. \((q, t, v, w)\)-convolution and the Lévy-Hinchin representation. Now we consider the \((q, t, v, w)\)-Lévy processes in the simplest case of one-dimensional \(K\). We use the following notations in this subsection:

1. \(\mathcal{M}\) denotes the space of finite positive Borel measures on \(\mathbb{R}\);
2. \(\mathcal{M}_P \subset \mathcal{M}\) denotes the subset of probability measures;
3. For \(\mu \in \mathcal{M}_P\) and \(n \geq 1\), we define a cumulant \(r_n(\mu)\) by
   \[
   r_n(\mu) = m_n(\mu) - \sum_{\substack{\pi \in \mathcal{P}^{(n)} \ni \pi \neq \mathbf{1}_n \otimes \mathbf{1}_n}} q^{\text{rec}(\pi)} l^{\text{nest}(\pi)} w^{\text{rec}(\pi)} \prod_{\beta \in \pi[r]} r|_\beta.
   \]
4. Let \(\tau \in \mathcal{M}_u \subset \mathcal{M}\) such that
   \[
   \limsup_{n \to \infty} \frac{1}{\sqrt{n!}} m_{2n}^{1/2n}(\tau) < \infty.
   \]
   This condition means that a measure in \(\mathcal{M}_u\) is uniquely determined by its moments. Indeed Carleman’s theorem for moments (see \([2]\)) states that the moment problem is determined if the following condition holds:
   \[
   \sum_{n \geq 0} m_{2n}^{1/2n}(\tau) = \infty.
   \]
   In our case \(m_{2n}^{-1/2n}(\tau) \geq \frac{C}{\sqrt{n!}}\) for some \(C\), so the conclusion follows. Equivalently, under our assumption \(\tau \in \mathcal{M}_u \iff\) there exists the operator \(T\) which has the distribution \(\tau\) with respect to the vector functional \(\langle \xi, \cdot \rangle^*_H\) and moreover, \(\xi\) is an analytic vector for \(T\).

Definition 6.16. For \(\tau \in \mathcal{M}_u\) such that the operator \(T\) has the distribution \(\tau\) with respect to the vector functional \(\langle \xi, \cdot \rangle^*_H\) and \(\lambda \in \mathbb{R}\) we define an injective map
   \[
   \mathcal{IF} : \mathbb{R} \times \mathcal{M}_u \to \mathcal{M}_P;
   \]
   \[
   (\lambda, \tau) \mapsto \mu \text{ such that } \mu \text{ is the distribution of } p_1(\xi \otimes \eta, T, \lambda);
   \]
   Let \(\mathcal{IF}_{q, t, v, w}(\cdot) := \{ \mathcal{IF}(\lambda, \tau) \mid (\lambda, \tau) \in \mathbb{R} \times \mathcal{M}_u \}\). We define the analog of the Lévy-Hinchin representation \(\mathcal{LH}_{q, t, v, w} : \mathcal{IF}_{q, t, v, w}(\cdot) \to \mathbb{R} \times \mathcal{M}_u\) to be the inverse of \(\mathcal{IF}(\lambda, \tau)\).

II. For \(\mu, \nu \in \mathcal{IF}_{q, t, v, w}(\cdot)\), define their \((q, t, v, w)\)-convolution \(\mu *_{q, t, v, w} \nu\) by the rule that
   \[
   \mathcal{LH}_{q, t, v, w}(\mu *_{q, t, v, w} \nu) = \mathcal{LH}_{q, t, v, w}(\mu) + \mathcal{LH}_{q, t, v, w}(\nu).
   \]

Corollary 6.17. The \((q, t, v, w)\)-convolution of two positive measures is positive, i.e.
   \[
   \mathcal{IF}(\lambda_1, \tau_1) *_{q, t, v, w} \mathcal{IF}(\lambda_2, \tau_2) = \mathcal{IF}(\lambda_1 + \lambda_2, \tau_1 + \tau_2).
   \]

Finally, we present the relation between the convolution of measures and product states.

Proposition 6.18. For \(\mu_1, \mu_2 \in \mathcal{IF}_{q, t, v, w}(\cdot)\), and \(\Phi_1, \Phi_2 \in \mathcal{ID}_{q, t, v, w}(1)\) we have
   \[
   (\mu_1 *_{q, t, v, w} \mu_2)(x^n) = (\Phi_1 \times_{q, t, v, w} \Phi_2)((x_1 + x_2)^n).
   \]

Proof. We use the representation described in the proof of Proposition 6.15 and obtain
   \[
   M(x^n, 1; Z) = M((x_1 + y_1)^n, 1; (X^{(1)}, Y^{(2)})).
   \]
where \(Z = X^{(1)} + Y^{(2)}\), because \(r_n(\mu_Z) = r_n(\mu_X^{(1)}) + r_n(\mu_Y^{(2)})\). \(\Box\)

Corollary 6.19. (1). Let \(V = \mathbb{C}, \xi = 1 \in V, T = 0\) and \(\lambda = 0\). Then the \((q, t, v, w)\)-Brownian motion is the process \(X(s) = p_s(\xi \otimes \eta, 0, 0)\). The distribution \(\mu\) of \(X(s)\) is the \((q, t, v, w)\)-Gaussian distribution with parameter \(s\), given by \(\mathcal{LH}_{q, t, v, w}(\mu) = (0, s \delta_0)\).

(2). Let \(V = \mathbb{C}, \xi = 1 \in V, T = 1\) and \(\lambda = 1\). The \((q, t, v, w)\)-Poisson process is the process \(X(s) = p_s(\xi \otimes \eta, I, s)\). The distribution \(\mu\) of \(X(s)\) is the \((q, t, v, w)\)-Poisson distribution with parameter \(s\), given by \(\mathcal{LH}_{q, t, v, w}(\mu) = (s, s \delta_1)\).
7. Concluding Remark

Finally, we summarize our conclusions and contributions and give some perspectives for future research directions.

(1) The construction presented in this article can be extended for $\mathfrak{S}_n^k$ with multipolar Hermite orthogonal polynomial of the type

$$xP_n(x) = P_{n+1}(x) + \underbrace{[n] \cdots [n]}_{k \text{ times}} P_{n-1}(x).$$

In these cases combinatorics and partitions are of the same type as those described in Section 3 except in the limit case because then the measure is not necessarily uniquely determined, for example when $P_n(x) = P_{n+1}(x) + n^2 P_{n-1}(x)$.

(2) Let $P_n$ be a family of orthogonal polynomials. One standard combinatorial task is to calculate the linearization coefficients, when we are interested in the expectations $\varphi(P_{n_1} P_{n_2} \ldots P_{n_k})$. The name stems from the fact that these are the coefficients in the expansion of products of this type in the basis $P_n$, that is, expansions as the sums of orthogonal polynomials. Many of these coefficients are positive integers, and so they count something; see [5, 6]. We expected that for $t = w = 1$, we can obtain a nice result for the polynomials by using diagonal pair partition because just one crossing plays a role.

(3) It is worth to find the central limit theorem for the quadrabasic Gaussian operator as in [11]. Our initial investigation shows that this problem is nontrivial.

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**Department of Telecommunications and Teleinformatics, Wrocław University of Science and Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland**

*Email address: wiktorejsmont@gmail.com*