The effective spacetime geometry of graviton condensates in $f(\mathcal{R})$ gravity

Andy Octavian Latief, Fiki Taufik Akbar, and Bobby Eka Gunara

Theoretical Physics Laboratory, Theoretical High Energy Physics and Instrumentation Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung, Jl. Ganesha no. 10 Bandung, Indonesia, 40132

Email: latief@fi.itb.ac.id, ftakbar@fi.itb.ac.id, bobby@fi.itb.ac.id

ABSTRACT

Working in static spherically symmetric setup, recent study has demonstrated that the effective spacetime geometry of a Bose-Einstein condensate of weakly interacting gravitons is analogous to a gravastar, hence providing a bridge between these two attempts in describing the black hole interior. In this paper we make three generalizations: introducing a composite of two graviton condensates so that the exterior spacetime is not necessarily asymptotically flat, working in $f(\mathcal{R})$ gravity, and extending the calculations to higher dimensions. We find that the effective spacetime geometry is again analogous to a gravastar, but the interior can be de Sitter or Anti-de Sitter and the exterior can be Schwarzschild, Schwarzschild-de Sitter, or Schwarzschild-Anti de Sitter, where the cosmological constant for the exterior must be smaller than the one for the interior. These geometries are determined by the modified gravity function $f(\mathcal{R})$, in contrast to previous works where they were selected by hand. We also presented a new possible value for the size of the interior condensate provided a certain restriction is satisfied, which would not be met if we are still working in ordinary gravity with four spacetime dimensions. This restriction can be seen manifested in the behavior of the interior graviton wavelength as a function of spacetime dimension.
1 Introduction

According to classical general relativity, black hole is a very dense object with a curvature singularity at the origin and a coordinate singularity known as the event horizon at some radius. Any particle, and even light, cannot escape from the black hole once it enters the event horizon, making the black hole interior inaccessible to outside observers. Although the existence of curvature singularity is already a controversial issue, the picture of black hole becomes more problematic when quantum effects are included (see Ref. [1] and [2] for reviews). For example, Hawking discovered that black holes can evaporate due to a mechanism later known as the Hawking radiation [3]. This implies that an information in the form of a pure quantum state can transform into a mixed state, which contradicts the unitarity of quantum mechanics.

Several ideas have been proposed to remove these problems or even to change drastically the physical picture inside the black hole interior [2]. One of them is the gravastar, which stands for the gravitational vacuum star, proposed by Mazur and Mottola [4, 5, 6]. The idea is that when an astronomical object undergoes a gravitational collapse, a phase transition occurs at the expected position of the event horizon to form a spherical thin shell of stiff fluid with equation of state $p = \rho$. The interior is a de Sitter (dS) condensate phase obeying $p = -\rho$, while the exterior is a Schwarzschild vacuum obeying $p = \rho = 0$. There are also other proposals to generalize the interior and exterior regions of a gravastar, ranging from changing only the interior to an Anti-de Sitter (AdS) spacetime [7] and a Born-Infeld phantom [8] to even changing both the interior and exterior regions by including also the case of AdS for the interior as before and generalizing the Schwarzschild exterior to Schwarzschild-de Sitter (Sch-dS), Schwarzschild-Anti de Sitter (Sch-AdS), and Reissner-Nordström spacetimes [9]. These varieties of gravastars have been proved to be stable under radial perturbations by the corresponding authors. Especially for the case where the interior is dS and the exterior is Sch-dS, there is a relation between the cosmological constants of the interior and the exterior regions: the latter must be smaller than the former [10].

Another idea to solve black hole paradoxes is a proposal by Dvali and Gómez that black holes are Bose-Einstein condensates (BEC) of weakly interacting gravitons at the critical point of a quantum phase transition [11, 12, 13, 14, 15]. They explained that even a macroscopic black hole is a quantum object and therefore treatments using semiclassical reasoning (with $\hbar \neq 0$ and $1/N = 0$, where $N$ is the number of quantum constituents) is not adequate; one has to use more quantum treatments ($\hbar \neq 0$ and $1/N \neq 0$) to avoid paradoxes. The Bekenstein entropy and Hawking radiation now have natural explanations; the former is quantum degeneracy of the condensate at the quantum critical point [14] and the latter is quantum depletion and leakage of the condensate [11, 15].

Since this language of microscopic graviton condensate should translate into the geometrical language of general relativity in the classical regime, one may ask about the effective spacetime geometry that this condensate generates. Working in the static spherically symmetric setup, Cunillera and Germani in Ref. [16] adapted the derivation of the Gross-Pitaevskii (GP) equation for ordinary BEC to this graviton condensate, namely by varying the condensate energy obtained from the Arnowitt-Deser-Misner (ADM) formalism while the number of gravitons is kept fixed. They found that the interior of the condensate is described by the dS spacetime while the exterior is described by the Schwarzschild spacetime, which is analogous to the picture of gravastar explained earlier.
Therefore, this method provides a bridge between the theory of graviton condensates and the model of gravastars [2].

In this paper we will perform three generalizations to the work outlined in Ref. [16]:

(i) introducing a composite of two condensates instead of one to construct the effective non-trivial geometry for the interior and exterior spacetimes;

(ii) working in $f(\mathcal{R})$ gravity;

(iii) extending the calculations to higher dimensions.

In Sec. 2 we will derive the Gross-Pitaevskii equations governing the effective metrics of the graviton condensates. The combination of points (i) and (ii) above will enable us to have dS and AdS spacetimes for the interior region and Schwarzschild, Sch-dS, and Sch-AdS spacetimes for the exterior region. However, unlike previous works, these spacetimes are not selected by hand; here their cosmological constants are determined by the modified gravity function $f(\mathcal{R})$. In Sec. 3 following Ref. [16], we will discuss a method to determine the size of the interior condensate and demonstrate that the generalization to the $f(\mathcal{R})$ gravity again gives us richer results compared to the case of ordinary gravity. Then in Sec. 4 we will study some special cases of interior and exterior geometries, starting from the conventional case of dS interior and Schwarzschild exterior, followed by the case of dS interior and Sch-(A)dS exterior, and completed by the case of AdS interior. The behavior of the interior graviton wavelength as a function of spacetime dimension is studied for each of these cases. The paper is then concluded in Sec. 5.

2 The effective metrics for the interior and exterior condensates

Consider a composite of two condensates of weakly interacting gravitons in a static spherically symmetric setup of $d$-dimensional spacetime described by the ansatz metric,

$$ds_a^2 = -L_a(r)^2 dt^2 + \frac{dr^2}{\xi_a(r)} + r^2 d\Omega_{d-2}^2,$$

(2.1)

where the index $a = 1, 2$ identifies the two condensates, $d\Omega_{d-2}^2$ is the metric of $(d-2)$-dimensional compact smooth manifold $\mathcal{M}$, and the functions $L_a(r)$ and $\xi_a(r)$ are arbitrary. The first condensate, denoted by $a = 1$, with size $R_1$ and average wavelength of its gravitons $\lambda_1$, is localized around the origin $r = 0$ and will become an object that looks like a black hole as seen by outside observers in the semiclassical limit [17], while the second condensate, with characteristic length scale $R_2 > R_1$ and average wavelength of its gravitons $\lambda_2$, will become a curved exterior background.

Following Ref. [16], we want to find the GP equations describing these two condensates by varying the gravitational Hamiltonian $H_a$ while fixing the number of gravitons $N_a$ in each condensate, so that we need to vary the equation $H_a - \mu_a N_a$, with $\mu_a$ the chemical potential of the two condensates. In this article we work in the $f(\mathcal{R})$ gravity such that the gravitational Hamiltonian of the condensate, obtained using the ADM formalism [18], has the form [19]

$$H_a = -\frac{\Omega_{d-2}}{16\pi G_d} \int dr^d r^{-2} \frac{L_a(r)}{\sqrt{\xi_a(r)}} f^{(d-1)}(\mathcal{R}_a).$$

(2.2)
Here $\Omega_{d-2}$ is the volume of $\mathcal{M}$ and $G_d$ is the $d$-dimensional gravitational constant. If $\mathcal{M}$ is a $(d - 2)$-dimensional sphere $S^{d-2}$, then $\Omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma\left(\frac{d-1}{2}\right)$, where $\Gamma$ is the gamma function. The $(d - 1)$-dimensional Ricci scalar curvature $(d-1)\mathcal{R}_a$, again for $a = 1, 2$, has the form

$$(d-1)\mathcal{R}_a = -\frac{(d-2)}{r^2} \left[ \xi'_a(r) r + (d-3)\xi_a(r) \right] + \frac{(d-2)\mathcal{R}_a}{r^2},$$

(2.3)

where $(d-2)\mathcal{R}_a$ is the Ricci scalar curvature of $\mathcal{M}$, which is taken from now on to be constant, $(d-2)\mathcal{R}_a \equiv A_0$ for all $a$. In the ordinary case where $\mathcal{M} = S^{d-2}$, then $A_0 = (d-2)(d-3)$.

The number of gravitons $N_a$ can be found from the relation $N_a = \langle E_a \rangle \lambda_a$ where $\langle E_a \rangle$ is the spatial average of the energy of the condensate and $1/\lambda_a$ is the energy of each graviton [16]. Due to the gravitational redshift, the energy $E_a$ is related to the energy measured at infinity $E_{a,\infty}$ through the relation

$$E_a(r) = L_a(r)E_{a,\infty}.$$  

(2.4)

As in Ref. [16], the chemical potential can be written as $\mu_a = -\gamma_a/\lambda_a$ for a constant $\gamma_a$. Therefore, the term $\mu_a N_a$ takes the form

$$\mu_a N_a = -\beta_a^2 \int d^d r \frac{L_a(r)}{\sqrt{\xi_a(r)}},$$

(2.5)

where all constants are absorbed to $\beta_a^2$.

Performing the variation of $H_a - \mu_a N_a$ with respect to $L_a(r)$ gives us the first GP equation,

$$f'((d-1)\mathcal{R}_a) = l_a,$$

(2.6)

with $l_a \equiv 16\pi G_d \beta_a^2 / \Omega_{d-2}$. It means that $(d-1)\mathcal{R}$ is constant,

$$(d-1)\mathcal{R}_a = B_a,$$

(2.7)

where $B_a$ is the root of the equation $f(\mathcal{R}_a) = l_a = 0$ (see Fig. [1]). Therefore, we are always dealing with a space of constant $(d-1)$-dimensional Ricci scalar curvature $(d-1)\mathcal{R}_a$ no matter which model of $f(\mathcal{R})$ gravity that we choose. By substituting Eq. (2.3) to Eq. (2.7), we find that the function $\xi_a(r)$ takes the form

$$\xi_a(r) = \frac{A_0}{(d-2)(d-3)} - \frac{B_a}{(d-1)(d-2)} r^2 - \frac{C_a}{r^{d-3}},$$

(2.8)

where $C_a$ is an integration constant. For the interior condensate, we need to set $C_1 = 0$ to ensure a regular solution at the origin $r = 0$.

Performing the variation of $H_a - \mu_a N_a$ with respect to $\xi_a(r)$ gives us the second GP equation,

$$-\Omega_{d-2} \frac{16\pi G_d}{2(d-2)\xi_a(r) \frac{d}{dr} \left[ f'((d-1)\mathcal{R}_a) L_a(r) \right] - (d-2)f'((d-1)\mathcal{R}_a) L_a(r) \xi(r)}{f((d-1)\mathcal{R}_a) L_a(r) r} - \beta_a^2 L_a(r) r = 0,$$

(2.9)
Figure 1: An illustration of finding the constant $B_a$ for the hyperbolic model of the $f(R)$ gravity, $f(R) = R - bR_0 \tanh(R/R_0)$, with constants $b, R_0 > 0$ [20]. Since $B_a$ is the root of the equation $f(R_a) - l_a = 0$, we have three possible solutions for $B_a$ in the figure above: two solutions correspond to AdS ($B_{a,1}, B_{a,2} < 0$) and one solution corresponds to dS ($B_{a,3} > 0$). Note that $f'(B_{a,1}) > 0$, $f'(B_{a,2}) < 0$, and $f'(B_{a,3}) > 0$, which will be useful later to determine which solution that should be ruled out. If we only have one condensate, work in ordinary gravity where $f(R_1) = R_1$, and expect $l_1 > 0$, the only possible solution is dS for the interior spacetime ($B_1 > 0$), as in Ref. [16].
where the primed function always denotes its derivative with respect to its argument. Substituting Eq. (2.6) to Eq. (2.9) and solving the resulting equation, we find

\[ L_a(r) = \sqrt{\xi_a(r)}, \]  

(2.10)

up to a proportionality constant, which can been absorbed to the time parameter in the metric. Therefore, the interior can be dS \((B_1 > 0)\) or AdS \((B_1 < 0)\), while the exterior can be Schwarzschild \((B_2 = 0)\), Sch-dS \((B_2 > 0)\), or Sch-AdS \((B_2 < 0)\), provided \(C_2 > 0\), which will be assumed throughout the remainder of this paper. This is in contrast to the system discussed in Ref. [16], where there is only one condensate and the ordinary gravity \(f(R_1) = R_1\) is used. Expecting \(l_1 > 0\), the solution is therefore always dS for the interior spacetime \((B_1 > 0)\).

3 The size of the interior condensate

The size \(R_1\) of the interior condensate can be determined using two considerations. First is by matching the interior and exterior metrics at \(r = R_1\) to ensure the continuity at the boundary, namely \(\xi_1(R_1) = \xi_2(R_1)\). It yields

\[ C_2 = \frac{B_1 - B_2}{(d - 1)(d - 2)} R_1^{d-1}. \]  

(3.1)

From the equation above it is clear that the condition \(C_2 > 0\) puts a restriction \(B_1 > B_2\). If we interpret \(B_a/2\) as the cosmological constant, it means that the cosmological constant for the exterior must be smaller than the one for the interior, as in Ref. [10]. Hence, we obtain:

(i) If the interior is dS \((B_1 > 0)\), then the exterior can be Sch-dS \((B_2 > 0)\), Schwarzschild \((B_2 = 0)\), or Sch-AdS \((B_2 < 0)\).

(ii) If the interior is AdS \((B_1 < 0)\), then the exterior must be also AdS \((B_2 < 0)\).

The second consideration to determine the size \(R_1\) is by matching the energy due to the Gibbons-Hawking-York boundary term \(E_{\text{GHY}}\) [21] with the energy in Eq. (2.4) evaluated at \(r = R_1\), namely \(E_1(R_1) = E_{1,\infty}\sqrt{\xi_1(R_1)}\). We identify the energy \(E_{1,\infty}\) as the Komar mass, which is the same for the case of Schwarzschild, Sch-dS, and Sch-AdS [22]. Therefore, \(E_{1,\infty} = (d - 2)\Omega_{d-2}C_2/(16\pi G_d)\). The energy \(E_{\text{GHY}}\) in \(f(R)\) gravity takes the form [23]

\[ E_{\text{GHY}} = \frac{1}{8\pi G_d} \int_M d^{d-2}x \sqrt{h} f'((d-1)R_1) K_1, \]

(3.2)

where \(h\) is the determinant of the induced metric on \(M\) and \(K_1\) is the trace of the extrinsic curvature on \(M\). For our case, \(E_{\text{GHY}}\) is given by

\[ E_{\text{GHY}} = \frac{(d - 2)\Omega_{d-2}}{8\pi G_d} f'(B_1) R_1 \sqrt{\xi_1(R_1)}, \]

(3.3)
where \( f'(B_1) \) is the first derivative \( f'((d-1)R_1) \) evaluated at \((d-1)R_1 = B_1\). Requiring \( E_{GHY} = E_1(R_1) \), we find

\[
\left[ \frac{(d-2)\Omega_d-2}{8\pi G_d} f'(B_1) R_1 - E_{1,\infty} \right] \sqrt{\xi_1(R_1)} = 0.
\]

(3.4)

This equation gives us two possible values for \( R_1 \),

\[
R_1 = r_{H1},
\]

(3.5)

\[
R_1 = \frac{C_2}{2f'(B_1)},
\]

(3.6)

where \( r_{H1} \) is the horizon of the interior metric, \( \xi_1(r_{H1}) = 0 \). Since we assume \( C_2 > 0 \) and expect \( R_1 > 0 \), we find that \( f'(B_1) > 0 \). Hence, the solution \( B_1 \) with \( f'(B_1) < 0 \) should be ruled out, such as the root \( B_{a,2} \) in Fig. [1].

4 Special cases for the interior and exterior space-times

4.1 dS interior and Schwarzschild exterior spacetimes

Let us first consider the case where the interior is dS \((B_1 > 0)\) and the exterior is Schwarzschild \((B_2 = 0)\). The continuity condition of the metrics at the boundary, Eq. (3.1), now reads

\[
C_2 = \frac{B_1}{(d-1)(d-2)} R_1^{d-1}.
\]

(4.1)

If \( r_{H1} \) and \( r_{H2} \) are the horizons of the interior and exterior metrics, respectively, which satisfy \( \xi_1(r_{H1}) = 0 \) and \( \xi_2(r_{H2}) = 0 \), then

\[
r_{H1} = \frac{(d-1)A_0}{(d-3)B_1},
\]

(4.2)

\[
r_{H2} = \left[ \frac{(d-2)(d-3)}{A_0} C_2 \right]^{1/(d-3)}.
\]

(4.3)

From these three equations, we get

\[
r_{H1}^2 r_{H2}^{d-3} = R_1^{d-1}.
\]

(4.4)

Choosing the first possible value for \( R_1 \) from the \( E_{GHY} = E_1(R_1) \) requirement, namely Eq. (3.5), we obtain \( R_1 = r_{H1} = r_{H2} \), which tells us that there is no horizon formation. Therefore, the effective geometry of this composite of graviton condensates is analogous to the gravastar picture, as has been demonstrated previously in Ref. [16].

We also need to require that the volume of the interior condensate is equal to \( \frac{\Omega_d-2}{d-1} \lambda_1^{d-1} \) [16]. Mathematically,

\[
\Omega_{d-2} \int_0^{R_1} \frac{r^{d-2}}{\sqrt{\xi_1(r)}} \, dr = \frac{\Omega_d-2}{d-1} \lambda_1^{d-1},
\]

(4.5)
which then yields

\[
\left(\frac{\lambda_1}{R_1}\right)^{d-1} = \sqrt{\frac{(d-2)(d-3)}{A_0}} \cdot 2F_1\left(\frac{1}{2}, \frac{d-1}{2}; \frac{d+1}{2}; \frac{R_1^2}{r_{H1}^2}\right),
\]

(4.6)

where \(2F_1\) is the hypergeometric function. As discussed above, here we want to set \(R_1 = r_{H1}\). Note that \(\lambda_1 \to R_1\) for very large spacetime dimension \(d\). Throughout the remainder of this paper, we will assume that \(\mathcal{M}\) is a maximally symmetric space, such that \(A_0 = k(d-2)(d-3)\), for a constant \(k\). The case \(k = 1\) is where \(\mathcal{M} = S^{d-2}\), in which we get the value \(\lambda_1 = 1.33 R_1\) for \(d = 4\). We plot the ratio \(\lambda_1/R_1\) versus the spacetime dimension \(d\) for various values of \(k\) in Fig. 2.

![Figure 2: Plot of the ratio between the graviton wavelength \(\lambda_1\) and the size \(R_1\) of the interior condensate versus the spacetime dimension \(d\) for the case of dS interior and Schwarzschild exterior when the first possible value for \(R_1\) as in Eq. (3.5) is chosen, namely where the horizons of the interior and exterior spacetimes coincide with the interior condensate radius, \(R_1 = r_{H1} = r_{H2}\). Here we set \(\mathcal{M}\) to be a maximally symmetric space such that \(A_0 = k(d-2)(d-3)\), for a constant \(k\). For each value of \(k\) above, \(\lambda_1 \to R_1\) for very large \(d\). Since the wavelength \(\lambda_1\) is always finite in the plot above, we conclude that the interior condensate radius \(R_1\) is always possible to have a value \(R_1 = r_{H1} = r_{H2}\) for spacetime dimension \(d \geq 4\).](image)

If we choose the second possible value for \(R_1\) as in Eq. (3.6), then from the continuity condition of the metrics at the boundary we find an equation that can be used to determine the value of \(C_2\) given the values of \(B_1\) and \(f'(B_1)\),

\[
1 = \frac{B_1}{(d-1)(d-2)} \frac{C_2^{d-2}}{[2f'(B_1)]^{d-1}}.
\]

(4.7)
However, if we want this solution to be distinct from the previous case while still preventing the horizon formation, then we require the horizon of the exterior to be smaller than \( R_1 \) so that, using Eq. (4.4), the horizon of the interior is automatically larger than \( R_1 \), \( \rho_{H2} < R_1 < \rho_{H1} \). It yields
\[
[2f'(B_1)]^{d-3} < \frac{A_0}{(d-2)(d-3)} C_2^{d-4}.
\] (4.8)

Combined with the previous equation, we arrive at the inequality
\[
f'(B_1) < f'(B_1)_{\text{max}},
\] (4.9)
where
\[
f'(B_1)_{\text{max}} = \frac{1}{2} \frac{A_0}{(d-2)(d-3)} \left[ \frac{(d-1)A_0}{(d-3)B_1} \right]^{\frac{d-2}{d}}.
\] (4.10)

When \( \mathcal{M} \) is a maximally symmetric space characterized by the constant \( k \), and if we set \( d = 4 \), we find that \( f'(B_1)_{\text{max}} \) is constant with value \( k/2 \), independent of the value of \( B_1 \). For \( k = 1 \), which is the case when \( \mathcal{M} = S_{d-2} \), we have \( f'(B_1)_{\text{max}} = 1/2 \). This bound cannot be satisfied for ordinary gravity where \( f'(B_1) = 1 \), as has been pointed out in Ref. [16]. For \( d > 4 \) and arbitrary value of \( k \), \( f'(B_1)_{\text{max}} \) is monotonically decreasing as \( B_1 \) increases and approaching zero as \( B_1 \to \infty \) (see Fig. 3 where we focus on the case \( k = 1 \)).

The graviton wavelength in this case is again given by Eq. (4.6), with the value of \( R_1^2/\rho_{H1}^2 \) now becomes
\[
\frac{R_1^2}{\rho_{H1}^2} = \left[ \frac{f'(B_1)}{f'(B_1)_{\text{max}}} \right]^{\frac{2}{d-2}}.
\] (4.11)

Therefore, we find
\[
\left( \frac{\lambda_1}{R_1} \right)^{d-1} = \sqrt{\frac{(d-2)(d-3)}{A_0}} \cdot _2F_1 \left( \frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}; \frac{f'(B_1)}{f'(B_1)_{\text{max}}} \right)^{\frac{2}{d-2}}.
\] (4.12)

For very large spacetime dimension \( d \gg 1 \), we again have \( \lambda_1 \to R_1 \). We plot the ratio \( \lambda_1/R_1 \) versus the spacetime dimension \( d \) for various values of \( k \) in Fig. 5. We choose to set \( f'(B_1) = 1 \) to produce that plot, since \( f'(R) \to \infty \) for \( R \to \infty \) in many models of \( f(R) \) gravity (for example, in the hyperbolic model [20], the Hu-Sawicki model [21], the exponential gravity model [25, 26], and the Appleby-Battye model [27]). Notice that the wavelength \( \lambda_1 \) may blow up at smaller values of \( d \), which indicates that it is not possible for the interior condensate radius \( R_1 \) to have a value as in Eq. (3.6) at those values of \( d \).

### 4.2 dS interior and Sch-(A)dS exterior spacetimes

Now we will discuss the case of dS interior where the exterior spacetime can be Sch-dS or Sch-AdS. From the expression for the horizon of the interior metric \( \rho_{H1} \) given in Eq. (4.2) and choosing the first solution \( R_1 = \rho_{H1} \) as in Eq. (3.5), we obtain
\[
R_1^2 = \frac{(d-1)A_0}{(d-3)B_1}.
\] (4.13)
Figure 3: Plot of $f'(B_1)_{\text{max}}$ as a function of $B_1$ for various spacetime dimensions $d$, in the case of dS interior and Schwarzschild exterior when $R_1$ is chosen to have a value as in Eq. (3.6). Here we set $\mathcal{M} = S^{d-2}$, such that $k = 1$. For $d = 4$, $f'(B_1)_{\text{max}}$ is constant with value $1/2$, independent of the value of $B_1$. However, for $d > 4$, $f'(B_1)_{\text{max}}$ is monotonically decreasing to zero as $B_1 \rightarrow \infty$. 
Inserting this expression to the continuity condition of the metrics at the boundary given in Eq. (3.1) yields
\[
\frac{A_0}{(d-2)(d-3)} - \frac{B_2}{(d-1)(d-2)} R_1^2 - \frac{C_2}{R_1^{d-3}} = 0, \tag{4.14}
\]
which is essentially a statement that \(\xi_2(R_1) = 0\). Therefore, we again find that \(R_1 = r_{H1} = r_{H2}\), which, in the case of Sch-dS, \(r_{H2}\) means the smaller positive horizon. As before, this indicates that there is no horizon formation. The graviton wavelength \(\lambda_1\) is given by Eq. (4.6) with \(R_1^2/r_{H1}^2 = 1\), hence in this case the plot of \(\lambda_1/R_1\) as a function of \(d\) will be identical to Fig. 2.

If we choose the second possible value for \(R_1\) as in Eq. (3.6), then we first need to require \(r_{H2} < R_1\) to ensure distinct solution while still preventing the horizon formation.

Using the equation \(\xi_2(r_{H2}) = 0\) and the continuity condition of the metrics, we obtain
\[
\left( \frac{R_1}{r_{H2}} \right)^{d-1} = \frac{s_1^2}{s_2^2 \pm s_1^2} \left( \frac{s_2}{r_{H2}} \pm 1 \right), \tag{4.15}
\]
with the plus (minus) sign is for the case of Sch-AdS (Sch-dS) exterior, and we have defined
\[
s_a \equiv \sqrt{\frac{(d-1)A_0}{(d-3)|B_1|}}. \tag{4.16}
\]
Note that \(s_1\) is actually \(r_{H1}\) and \(s_2\) is the cosmological horizon, in the case of Sch-dS, of the exterior spacetime. Since \(R_1\) must be positive, for the case of Sch-dS exterior we have \(s_2 > s_1\) or \(B_1 > B_2\), which is automatically satisfied. Using Eq. (4.15), the requirement \(r_{H2} < R_1\) gives us \(r_{H2} < r_{H1}\).

There are now two possible choices of inequalities, \(r_{H2} < R_1 < r_{H1}\) and \(r_{H2} < r_{H1} < R_1\), where only the former which will give us physical results. The inequality \(R_1 < r_{H1}\) implies
\[
f'(B_1) < f'(B_1)_{\text{max}}, \tag{4.17}
\]
where
\[
f'(B_1)_{\text{max}} = \frac{1}{2} \frac{A_0}{(d-2)(d-3)} \left( \frac{(d-1)A_0}{(d-3)B_1} \right) \left( \frac{1 - B_2}{B_1} \right) \left[ \frac{(d-1)A_0}{(d-3)B_1} \right]^{\frac{d-2}{2}}. \tag{4.18}
\]
When \(\mathcal{M}\) is a maximally symmetric space characterized by the constant \(k\) and if we set \(d = 4\), \(f'(B_1)_{\text{max}}\) is monotonically increasing as \(B_1\) increases and approaching \(k/2\) as \(B_1 \to \infty\), in contrast to the previous result for the case of dS interior and Schwarzschild exterior. For \(d > 4\) and arbitrary value of \(k\), \(f'(B_1)_{\text{max}}\) displays a non-monotonic behavior as \(B_1\) increases and approaches zero as \(B_1 \to \infty\) (see Fig. 4 where we focus on the case \(k = 1\)).

The graviton wavelength in this case is given by Eq. (4.12) but Eq. (4.18) is now used for \(f'(B_1)_{\text{max}}\). We plot the ratio \(\lambda_1/R_1\) versus the spacetime dimension \(d\) for various values of \(k\) in Fig. 5. Again, as in the case of dS interior and Schwarzschild exterior, we notice that the wavelength \(\lambda_1\) may blow up at smaller values of \(d\). This indicates that it is not possible for the interior condensate radius \(R_1\) to have a value as in Eq. (3.6) at those values of \(d\).
Figure 4: Plot of $f'(B_1)_{\text{max}}$ as a function of $B_1$ for various spacetime dimensions $d$, in the case of dS interior and Sch-dS exterior (with $B_2 = 2$ in this plot) when $R_1$ is chosen to have a value as in Eq. (3.6). Here we set $\mathcal{M} = S^{1-\varepsilon}$, such that $k = 1$. For $d = 4$, $f'(B_1)_{\text{max}}$ is monotonically increasing as $B_1$ increases and approaching $1/2$ as $B_1 \to \infty$. However, for $d > 4$, $f'(B_1)_{\text{max}}$ displays non-monotonic behavior as $B_1$ increases and approaches zero as $B_1 \to \infty$. Notice that the wavelength $\lambda_1$ has finite values only when $B_1 > B_2$, since the cosmological constant for the exterior must be smaller than the one for the interior.
Figure 5: Plot of the ratio between the graviton wavelength $\lambda_1$ and the size $R_1$ of the interior condensate versus the spacetime dimension $d$ for the case of dS interior with Schwarzschild and Sch-dS exterior when the second possible value for $R_1$ as in Eq. (3.6) is chosen. In this plot, $B_1 = 30$, $f'(B_1) = 1$, and $B_2 = 25$. Here we again set $\mathcal{M}$ to be a maximally symmetric space such that $A_0 = k(d - 2)(d - 3)$, for a constant $k$. For each value of $k$ above, $\lambda_1 \to R_1$ for $d \gg$. Notice that the wavelength $\lambda_1$ may blow up at smaller values of $d$, hence it is not possible for the interior condensate radius $R_1$ to have a value as in Eq. (3.6) at those values of $d$. This is the reason why this value of $R_1$ was ruled out in Ref. [16].
4.3 AdS interior and Sch-AdS exterior spacetimes

If the interior metric is AdS, then the exterior metric must be Sch-AdS due to the inequality $B_1 > B_2$. The interior metric now does not have horizon, so we have to choose the second possible value for $R_1$ as in Eq. (3.6). Using the continuity condition of the metrics, Eq. (3.1), and the equation defining $r_{H2}$, namely $\xi_2(r_{H2}) = 0$, the relation between $R_1$ and $r_{H2}$ can be obtained as

$$
\left( \frac{R_1}{r_{H2}} \right)^{d-1} = \frac{s_1^2}{s_2^2 - s_1^2} \left( \frac{s_2^2}{r_{H2}^2} + 1 \right),
$$

(4.19)

Since $B_1 > B_2$, or $|B_2| > |B_1|$, we find $s_1 > s_2$. Therefore, $R_1 > r_{H2}$, which means that in this case there is also no horizon formation. The value of $R_1$ can be obtained using

$$
R_1^{d-2} = \frac{(d-1)(d-2)}{|B_2| - |B_1|} \left[ 2f'(B_1) \right].
$$

(4.20)

This expression then can be used to calculate the graviton wavelength, which in this case has the form

$$
\left( \frac{\lambda_1}{R_1} \right)^{d-1} = \sqrt{\frac{(d-2)(d-3)}{A_0}} \cdot {}_2F_1 \left( \frac{1}{2}, \frac{d-1}{2}; \frac{d+1}{2}; -\frac{R_1^2}{s_1^2} \right).
$$

(4.21)

We note again that for very large spacetime dimension $d \gg 1$, $\lambda_1 \to R_1$. We plot the ratio $\lambda_1/R_1$ versus the spacetime dimension $d$ for various values of $k$ in Fig. 6. Unlike the case of dS interior, here the wavelength $\lambda_1$ never blows up, which indicates that the interior condensate radius $R_1$ is always possible to have a value as in Eq. (3.6).
Figure 6: Plot of the ratio between the graviton wavelength $\lambda_1$ and the size $R_1$ of the interior condensate versus the spacetime dimension $d$ for the case of AdS interior with Sch-AdS exterior when the second possible value for $R_1$ as in Eq. (3.6) is chosen. In this plot, $B_1 = -25$, $f'(B_1) = 1$, and $B_2 = -30$. Here we again set $\mathcal{M}$ to be a maximally symmetric space such that $A_0 = k(d - 2)(d - 3)$, for a constant $k$. For each value of $k$ above, $\lambda_1 \to R_1$ for $d \gg 1$. Notice that the wavelength $\lambda_1$ is always finite for all values of $d \geq 4$, in contrast to the case of dS interior and Sch-(A)dS exterior. Therefore, we conclude that the interior condensate radius $R_1$ is always possible to have a value as in Eq. (3.6) for spacetime dimension $d \geq 4$. 
5 Conclusions

In this paper we make three generalizations to the work outlined in Ref. [16]: (i) introducing a composite of two condensates to construct the effective interior and exterior spacetime geometry, so that the exterior background is not necessarily asymptotically flat; (ii) working in $f(R)$ gravity; and (iii) extending the calculations to higher dimensions. Remaining in the static spherically symmetric setup, we found that the effective spacetime geometry is again analogous to a gravastar. However, these generalizations give more possibilities to the effective spacetime geometry both in the interior and the exterior. The interior spacetime now can be dS or AdS, while the exterior can be Schwarzschild, Sch-dS, or Sch-AdS. These geometries are determined by the modified gravity function $f(R)$, unlike in previous works where they were selected by hand. A continuity condition of the metrics at the boundary of the interior condensate provides a relation between the interior and exterior spacetimes: the cosmological constant for the exterior must be smaller than the one for the interior.

We also found a new value for the radius of the interior condensate, which is only possible if we make the generalizations above. Otherwise, if one is still working in ordinary gravity with four spacetime dimensions, this new value would be unphysical. To keep it distinct from the previous value in which the interior condensate radius coincides with the interior and exterior horizons, while still preventing the horizon formation, certain inequality has to be satisfied. This restriction manifests clearly when the behavior of the interior graviton wavelength versus the spacetime dimension is considered. If the wavelength blows up, it means that the inequality is not satisfied, which then implies that the interior condensate radius cannot have this new value. We found that the wavelength may blow up at some values of spacetime dimension in the case of dS interior, but never blows up in the case of AdS interior.

Acknowledgments

The work in this paper is supported by Riset ITB 2019. BEG acknowledges the Abdus Salam ICTP for Associateships 2019 and for the warmest hospitality where the final part of this paper has been done.

References

[1] R. M. Wald, The thermodynamics of black holes, Living Reviews in Relativity 4 (07, 2001) 6.

[2] R. Brustein and A. J. M. Medved, Non–singular black holes interiors need physics beyond the standard model, arXiv:1902.07990 (02, 2019).

[3] S. W. Hawking, Particle creation by black holes, Communications in Mathematical Physics 43 (08, 1975) 199–220.

[4] P. O. Mazur and E. Mottola, Gravitational condensate stars: An alternative to black holes, arXiv:gr-qc/0109035 (02, 2002).
[5] P. O. Mazur and E. Mottola, Dark energy and condensate stars: Casimir energy in the large, arXiv:gr-qc/0405111 (05, 2004).

[6] P. O. Mazur and E. Mottola, Gravitational vacuum condensate stars, Proceedings of the National Academy of Sciences 101 (2004), no. 26 9545–9550 [https://www.pnas.org/content/101/26/9545.full.pdf].

[7] M. Visser and D. L. Wiltshire, Stable gravastars—an alternative to black holes?, Classical and Quantum Gravity 21 (01, 2004) 1135–1151.

[8] N. Bilić, G. B. Tupper and R. D. Viollier, Born–Infeld phantom gravastars, Journal of Cosmology and Astroparticle Physics 2006 (02, 2006) 013.

[9] B. M. N. Carter, Stable gravastars with generalized exteriors, Classical and Quantum Gravity 22 (10, 2005) 4551–4562.

[10] R. Chan, M. da Silva and P. Rocha, How the cosmological constant affects gravastar formation, Journal of Cosmology and Astroparticle Physics 2009 (12, 2009) 017.

[11] G. Dvali and C. Gomez, Landau–Ginzburg limit of black hole’s quantum portrait: Self–similarity and critical exponent, Physics Letters B 716 (2012), no. 1 240–242.

[12] G. Dvali and C. Gomez, Black hole macro–quantumness, arXiv:1212.0765 (12, 2012).

[13] G. Dvali and C. Gomez, Black hole’s 1/N hair, Physics Letters B 719 (2013), no. 4 419–423.

[14] G. Dvali and C. Gomez, Black hole’s quantum N–portrait, Fortschritte der Physik 61 (2013), no. 78 742–767 [https://onlinelibrary.wiley.com/doi/pdf/10.1002/prop.201300001].

[15] G. Dvali and C. Gomez, Black holes as critical point of quantum phase transition, The European Physical Journal C 74 (02, 2014) 2752.

[16] F. Cunillera and C. Germani, The Gross–Pitaevskii equations of a static and spherically symmetric condensate of gravitons, Classical and Quantum Gravity 35 (04, 2018) 105006.

[17] G. Dvali and C. Gomez, Quantum compositeness of gravity: Black holes, AdS and inflation, Journal of Cosmology and Astroparticle Physics 2014 (01, 2014) 023.

[18] R. Arnowitt, S. Deser and C. W. Misner, The dynamics of general relativity, in Gravitation: An Introduction to Current Research (L. Witten, ed.), ch. 7, pp. 227–265. John Wiley & Sons Inc., New York, London, 1962.

[19] C. Gao, Modified gravity in Arnowitt–Deser–Misner formalism, Physics Letters B 684 (2010), no. 2 85–91.

[20] S. Tsujikawa, Observational signatures of f(R) dark energy models that satisfy cosmological and local gravity constraints, Phys. Rev. D 77 (01, 2008) 023507.
[21] S. W. Hawking and G. T. Horowitz, *The gravitational hamiltonian, action, entropy and surface terms*, Classical and Quantum Gravity **13** (06, 1996) 1487–1498.

[22] B. E. Gunara, D. M. Akbar, F. T. Akbar and H. Susanto, *Higher dimensional SU(2) static skyrme black holes*, arXiv:1810.06123 (10, 2018).

[23] A. Guarnizo, L. Castañeda and J. M. Tejeiro, *Boundary term in metric f(R) gravity: Field equations in the metric formalism*, General Relativity and Gravitation **42** (11, 2010) 2713–2728.

[24] W. Hu and I. Sawicki, *Models of f(R) cosmic acceleration that evade solar system tests*, Phys. Rev. D **76** (Sep, 2007) 064004.

[25] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, *Class of viable modified f(R) gravities describing inflation and the onset of accelerated expansion*, Phys. Rev. D **77** (02, 2008) 046009.

[26] E. V. Linder, *Exponential gravity*, Phys. Rev. D **80** (12, 2009) 123528.

[27] S. A. Appleby and R. A. Battye, *Do consistent F(R) models mimic general relativity plus Λ?*, Physics Letters B **654** (2007), no. 1 7–12.