The dynamics of the Schrödinger flow from the point of view of semiclassical measures

Nalini Anantharaman and Fabricio Macià

Abstract. On a compact Riemannian manifold, we study the various dynamical properties of the Schrödinger flow \( e^{it\Delta/2} \), through the notion of semiclassical measures and the quantum-classical correspondence between the Schrödinger equation and the geodesic flow. More precisely, we are interested in its high-frequency behavior, as well as its regularizing and unique continuation-type properties. We survey a variety of results illustrating the difference between positive, negative and vanishing curvature.

1. Introduction

Let \( (M,g) \) be a smooth, \( d \)-dimensional, complete manifold. Denote by \( \Delta = \text{div} (\nabla g) \) the Laplace-Beltrami operator and consider the following linear Schrödinger equation on \( M \):

\[
\begin{aligned}
&i\partial_t u(t, x) + \frac{1}{2} \Delta u(t, x) = 0, \\
&u|_{t=0} = u^0 \in L^2(M).
\end{aligned}
\]  

Since \( (M,g) \) is complete, \( \Delta \) is an essentially self-adjoint operator on \( L^2(M) \) and the initial value problem (1.1) has a unique solution \( u \in C(\mathbb{R}; L^2(M)) \). The corresponding flow, the Schrödinger flow, is denoted by \( e^{it\Delta/2} \); recall that each operator \( e^{it\Delta/2} \) is unitary on \( L^2(M) \), and in particular, for every \( t \in \mathbb{R} \) and \( u^0 \in L^2(M) \),

\[
\left\| e^{it\Delta/2} u^0 \right\|_{L^2(M)} = \| u^0 \|_{L^2(M)}. 
\]

When \( M \) is compact a little more can be said: the solutions to (1.1) can be expressed in terms of eigenvalues and eigenfunctions of \( \Delta \) and the dynamics of \( e^{it\Delta/2} \) turns out to be almost periodic: if \( (\varphi_j)_{j \in \mathbb{N}} \) is an orthonormal basis formed of eigenfunctions on \( \Delta \) (with \( \Delta \varphi_j = -\lambda_j \varphi_j \)), we can write

\[
e^{it\Delta/2} u^0 = \sum_{j \in \mathbb{N}} e^{-it\lambda_j/2} (u^0 | \varphi_j) \varphi_j, 
\]

Key words and phrases. Semiclassical (Wigner) measures; linear Schrödinger equation on a manifold; semiclassical limit; dispersive estimates; observability.

N. Anantharaman wishes to acknowledge the support of Agence Nationale de la Recherche, under the grant ANR-09-JCJC-0099-01.

F. Macià was supported by grants MTM2007-61755, MTM2010-16467 (MEC).
where \((\cdot, \cdot)\) denotes the scalar product in \(L^2(M)\). However, this expression brings little geometric information about the propagation properties of \(e^{it\Delta/2}\) : usually, the eigenfunctions \(\varphi_j\) are not explicit, and even when they are (for instance, in the case of a flat torus), oscillatory sums such as \(\sum_j \varphi_j(x)\) are complicated objects. We shall return to the problem of characterising the structure of eigenfunctions at the end of this introduction and throughout the rest of this article; however, let us state, from the very beginning, that this is not the point of view that we are adopting (although, obviously, the dynamics of the propagator \(e^{it\Delta/2}\) and the properties of eigenfunctions are closely related).

The issues we shall address here are aimed to obtain a better understanding of the dynamics of \(e^{it\Delta/2}\) and its relation to the geometry of \((M, g)\). In particular, we shall be dealing with those aspects related to the high-frequency behavior of \(e^{it\Delta/2}\).

Let us describe precisely the main object of our study, before we discuss in more detail the motivations that have guided us. Consider a sequence \((u_n^0)\) of initial data in \(L^2(M)\) with \(\|u_n^0\|_{L^2(M)} = 1\); we shall focus on the asymptotic behavior, as \(n \to \infty\), of the densities

\[
(1.4) \quad \left| e^{it\Delta/2} u_n^0 \right|^2.
\]

As we shall see, it is difficult to understand the behavior of this quantity for individual \(t\), but much more can be said if we average w.r.t. \(t\). Note that \(\left| e^{it\Delta/2} u_n^0 \right|^2 \in L^1(M)\) for every \(t \in \mathbb{R}\), and, because of \((1.2)\) it can be identified to an element of \(P(M)\), the set of probability measures in \(M\). Therefore, \(\left(\left| e^{it\Delta/2} u_n^0 \right|^2\right)\) is a sequence in \(C(\mathbb{R}; P(M))\), and the Banach-Alaoglu theorem ensures that it is compact in \(L^\infty(\mathbb{R}; M(M))\) for the weak*-topology.

In particular, there always exist a subsequence \((u_{n'}^0)\) and a measure \(\nu \in L^\infty(\mathbb{R}; M_+(M))\) such that

\[
(1.5) \quad \int_a^b \int_M \chi(x) \left| e^{it\Delta/2} u_{n'}^0 \right|^2 dt \to \int_a^b \int_M \chi(x) \nu(t, dz) dt, \quad \text{as } n' \to \infty,
\]

for every \(\chi \in C_c(M)\) and \(a, b \in \mathbb{R}\). If \(M\) is compact then \(\nu(t, \cdot)\) is in fact a probability measure for a.e. \(t \in \mathbb{R}\). In general, the sequence \((u_{n'}^0)\) does not converge strongly in \(L^2(M)\) and in consequence, the measure \(\nu\) may be singular with respect to the Riemannian measure. The singular part of \(\nu\) describes the regions in \(M\) on which the sequences of densities \((u_n^0)\) concentrates.

Here we shall describe some results related to the question of understanding the structure of the measures \(\nu\) that arise in this way. More precisely, we shall focus in aspects such as:

- The dependence of \(\nu\) on the initial data \((u_n^0)\). Is there a propagation law relating \(\nu\) to some limiting object obtained from the sequence \((u_n^0)\)?
- The regularity of \(\nu\). Under which conditions on the geometry of \((M, g)\) or on the structure of \((u_n^0)\) is it possible to ensure that the measure \(\nu\) is more regular than \textit{a priori} expected? For instance, \(\nu \in L^p([a, b] \times M)\) for some \(p > 2\) and \(a, b \in \mathbb{R}\).

1In fact, identity \((1.3)\) is often used to obtain information about the eigenfunctions from the geometric description of the propagator \(e^{it\Delta/2}\) [Ana08, AN07, Mac08].

2Given a metric space \(X\), we shall respectively denote by \(M(X), M_+(X)\) and \(P(X)\) the set of Radon measures, positive Radon measures and probability Radon measures on \(X\).
The structure of the support of $\nu$. Which closed sets $U \subset M$ can be the support of a measure $\nu$ obtained through (1.5) for some sequence $(u^n_0)$?

One expects that the answer to these questions will strongly depend on the geometry of $(M,g)$ and, in particular, on the dynamics of the geodesic flow on the cotangent bundle $T^*M$. Here we shall review some results obtained by the authors in two different, and somewhat extremal, situations: the cases of completely integrable (Sections 3, 4) and Anosov geodesic flows (Section 5). From the point of view of manifolds of constant sectional curvature, this corresponds to the cases of nonnegative and negative sectional curvature, respectively. These results are expressed in terms of semiclassical (or Wigner) measures, whose main properties are recalled in Section 2.

As we already mentioned, our motivation for addressing these issues comes from the study of the dynamics of the linear Schrödinger equation, and more precisely on the following three aspects.

1. The high-frequency dynamics of $e^{it\Delta/2}$ and its relation to the quantum-classical correspondence principle and the semiclassical limit of quantum mechanics.
2. The analysis of the dispersive properties of $e^{it\Delta/2}$, and in particular the validity of Strichartz estimates on a general Riemannian manifold.
3. The validity of observability or quantitative unique continuation estimates for $e^{it\Delta/2}$.

Before proceeding to describe the results, let us give a more detailed description of each of these questions and, to conclude this introduction, clarify how the problem addressed here is related to other questions in Spectral Geometry that have been widely studied in the literature (random initial data, eigenfunction limits, and pair-correlation eigenvalue statistics).

1.1. The quantum-classical correspondence principle and the semiclassical limit. The Schrödinger equation (1.1) is a mathematical model for the propagation of a free quantum particle whose motion is constrained to $M$. If $u$ is a solution to (1.1) then for every measurable set $U \subset M$ and every $t \in \mathbb{R}$, the quantity

\[
\int_U |u(t,x)|^2 \, dx
\]

is the probability for the particle that was at $t = 0$ at the state $u^0$, to be in the region $U$ at time $t$. The quantum-classical correspondence principle asserts that if the characteristic length of the oscillations of $u$ is very small, then the dynamics of $|u(t,\cdot)|^2$ can be deduced from that of the corresponding classical system, that is, the geodesic flow $g^t$ on the cotangent bundle $T^*M$ of $(M,g)$.

In order to develop a rigorous mathematical theory, we must precise what we mean by a characteristic length of oscillations. To do so, it is convenient to replace the initial datum $u^0$ by a sequence $(u^n_0)$ of initial data with $\|u^n_0\|_{L^2(M)} = 1$. Let $1_{[0,1]}$ denote the characteristic function of the interval $[0,1]$. Chose a sequence $(h_n)$ of positive reals such that $h_n \to 0$ as $n \to \infty$ and:

\[
\lim_{n \to \infty} \|1_{[0,1]}(h_n \sqrt{-\Delta}) u^n_0\|_{L^2(M)} = 1;
\]
note that such a sequence always exists, by the spectral theorem for self-adjoint operators on Hilbert space. If (1.7) holds we say that \((h_n)\) is a characteristic length-scale for the oscillations of \((u_n^0)\), or, following the terminology in [Gér91, GL93], that \((u_n^0)\) is \((h_n)\)-oscillating.

As an example, let \((x_0, \xi_0) \in T^*M\) and \(u_n^0 \in L^2(M)\) be supported on a coordinate patch around \(x_0\) such that in coordinates:

\[
u_n^0(x) = \frac{1}{h_n^{d/4}} \rho \left( \frac{x - x_0}{\sqrt{h_n}} \right) e^{i \frac{\xi_0}{h_n} \cdot x},\tag{1.8}
\]

where \((h_n)\) is a sequence of positive reals tending to zero and \(\rho\) is taken to have \(\|u_n^0\|_{L^2(M)} = 1\). The function \(u_n^0\) is usually called a wave-packet or coherent state centered at \((x_0, \xi_0)\). If \(\|\xi_0\|_{x_0} = 1\) then \((u_n^0)\) is \((h_n)\)-oscillating in the sense introduced above. A manifestation of the correspondence principle is the following classical result: for any fixed \(t \in \mathbb{R}\):

\[
\left| e^{ih_n t \frac{\Delta}{2}} u_n^0 \right|^2 \to \delta_{x(t)}, \quad \text{as} \ n \to \infty,
\]

where \(x(t)\) is the projection on \(M\) of the orbit \(g^t(x_0, \xi_0)\) of the geodesic flow. Therefore, in the limit \(n \to \infty\) the probability densities \(\left| e^{ih_n t \frac{\Delta}{2}} u_n^0 \right|^2\) become concentrated on the classical trajectory \(x(t)\). Note that the time scale considered in the limit (1.9) is \(h_n t\), which is proportional to the characteristics length of oscillations of \((u_n^0)\) and therefore tends to zero. An analogous result holds for more general, \(h_n\)-oscillating sequences of initial data, see Section 2 below; we shall refer to this as the semiclassical limit.

The convergence in (1.9) is locally uniform in \(t \in \mathbb{R}\). Due to the dispersive nature of \(e^{it \Delta/2}\) one cannot expect that (1.9) holds uniformly in time: for fixed \(n\) and as \(t\) increases, the wave-packet \(e^{ih_n t \frac{\Delta}{2}} u_n^0\) will become less and less concentrated around \(x(t)\). The study of the simultaneous limits \(h_n \to 0\) and \(t \to \infty\) is a notoriously difficult problem. In the most general framework, it is known [CR97, BGP99, HI00, BR02] that (1.9) holds uniformly for

\[
|t| \leq T_{E}^{h_n} := (1 - \delta) \lambda_{\max}^{-1} \log (1/h_n),
\]

where \(\delta \in (0, 1)\) and \(\lambda_{\max}\) stands for the maximal expansion rate of the geodesic flow on the spheres \(\{\|\xi\|_x = \|\xi_0\|_{x_0}\}\). This upper bound \(T_{E}^{h_n}\), known as the Ehrenfest time, has been shown to be optimal for some one-dimensional systems, see [BR03, Lab11].

For the Euclidean space \(\mathbb{R}^d\) or the torus \(\mathbb{T}^d\) equipped with the flat metric (or, more generally, when the geodesic flow of \((M, g)\) is completely integrable), it is possible to show that the convergence in (1.9) is uniform up to times \(|t| \leq C h^{-1/2+\delta}\) for any \(\delta > 0\), see [BR02]. We stress the fact that having an explicit expression of \(e^{it \Delta/2}\) does not necessarily make the study of the probability measures (1.6) easier. To illustrate this phenomenon, let us consider a gaussian coherent state on \(\mathbb{R}^d\):

\[
u_n^0(x) = \frac{1}{h_n^{d/4}} e^{-\frac{|x - x_0|^2}{2h_n}} e^{i \frac{\xi_0}{h_n} \cdot x}.\]

\(^{3}\)In what follows, the expression \(\|\xi\|_x\) will denote the norm of \((x, \xi) \in T^*M\) induced by the Riemannian metric of \((M, g)\).
One finds by an explicit calculation
\begin{equation}
\label{eq:1.11}
e^{i t h_n \Delta/2} u_n^0 (x) = \frac{1}{h_n^{d/4} (1 + it)^{d/2}} e^{-|x - x_0 - t\xi_0|^2 / 2h_n (1 + t^2)} e^{i \phi(t, x, x_0, \xi_0) / 2h_n},
\end{equation}
with
\[\phi(t, x, x_0, \xi_0) := t \frac{|x - x_0 - t\xi_0|^2}{(1 + t^2)} - t |\xi_0|^2 + 2\xi_0 \cdot x.\]

On the torus \( T^d = \mathbb{R}^d / 2\pi \mathbb{Z}^d \), this means that the evolution of the periodic wave packet
\[u_n^0 (x) = \frac{1}{h_n^{d/4}} \sum_{k \in 2\pi \mathbb{Z}^d} e^{-|x - x_0 - k|^2 / 2h_n} e^{i \phi_n (x-k) / h_n}\]
is given by the explicit expression
\begin{equation}
\label{eq:1.12}
e^{i t h_n \Delta/2} u_n^0 (x) = \frac{1}{h_n^{d/4} (1 + it)^{d/2}} \sum_{k \in 2\pi \mathbb{Z}^d} e^{-|x - x_0 - k - t\xi_0|^2 / 2h_n (1 + t^2)} e^{i \phi(t, x, x_0, \xi_0) / 2h_n}.
\end{equation}

For \(|t| \leq C h_n^{-1/2 + \delta}\) it is clear that the associated probability measure \(\rho_n\) concentrates on the trajectory \(x_0 + t\xi_0\) (or on its image on the torus), but for \(|t| \geq h_n^{-1/2}\) these probability measures become complicated objects due to the interferences between the different terms in the sum \(\rho_n\). On compact negatively curved manifolds, there are also examples of initial coherent or lagrangian states whose time evolution is explicit up to times \(t \sim h_n^{-2}\) \cite{Paul11, Sch07}, but for which the associated probabilities \(\rho_n\) are extremely complicated oscillatory sums.

One of our motivations is to study the probability measures \(\rho_n\) at times \(t\) for which the convergence \(\rho_n\) fails. Although this a very difficult question for fixed \(t\), it becomes more tractable if one performs a time average. The problem we study consists in averaging the probability measure \(\rho_n\) over a fixed time interval, which means, with the semiclassical normalisation of time, to average \(|e^{i t h_n \Delta/2} u_n^0|^2\) over time intervals of size \(t \sim h_n^{-1}\) or larger. In particular, this averaging procedure allows us to go much beyond the times where the individual \(|e^{i t h_n \Delta/2} u_n^0|^2\) have been previously studied.

We shall discuss and compare various geometries: Zoll manifolds (Section 3), flat tori (Section 4) and negatively curved manifolds (Section 5). Even when the geodesic flow is completely integrable, important differences may occur, as the analysis for the sphere \(S^d\) (or more generally, of manifolds with periodic geodesic flow) and the torus \(T^d\) shows, see Sections 3 and 4.

1.2. Dispersive properties of the Schrödinger flow on a Riemannian manifold. By the word “dispersion”, we mean that any solution to the Schrödinger equation \((1.1)\) can be expressed as a superposition of waves propagating at different speeds, depending on the characteristic frequencies of the initial datum. For instance, when \(M = \mathbb{R}^d\) any solution to \((1.1)\) can be written as:
\begin{equation}
\label{eq:1.13}
e^{it \Delta/2} u_0 (x) = \int_{\mathbb{R}^d} \hat{u}_0 (\xi) e^{i \xi \cdot (x - t \delta / 2)} \frac{d\xi}{(2\pi)^d},
\end{equation}
where \(\hat{u}_0\) stands for the Fourier transform of \(u_0\). This formula shows indeed that \(e^{it \Delta/2} u_0\) is built as a superposition of plane waves \(e^{i \xi \cdot (x - t \delta / 2)}\), travelling at velocity \(\xi / 2\). The dispersion property is also seen very clearly in the expression \((1.11)\), where
we see that a coherent state initially microlocalised around $(x_0, \xi_0)$ is less and less localized as time evolves, while its $L^\infty$-norm decreases accordingly.

The representation formula (1.13) leads to the estimate:

\begin{equation}
\left\| e^{it\Delta} u_0 \right\|_{L^\infty(\mathbb{R}^d)} \leq C \frac{\|u_0\|_{L^1(\mathbb{R}^d)}}{|t|^{d/2}},
\end{equation}

which quantifies the decay in time of solutions to (1.13) due to dispersion. That estimate is in turn used to derive, by interpolation with the conservation property (1.2), the commonly known as Strichartz estimate:

\begin{equation}
\left\| e^{it\Delta/2} u_0 \right\|_{L^p([0,1] \times \mathbb{R}^d)} \leq C \|u_0\|_{L^2(\mathbb{R}^d)},
\end{equation}

where

\begin{equation}
p = 2 \left(1 + \frac{2}{d}\right).
\end{equation}

Estimate (1.15) expresses that the singularities (quantified by a Lebesgue norm) developed by a solution to the Schrödinger equation are better than what one would initially expect based on the fact that $u_0 \in L^2(\mathbb{R}^d)$. These estimates play a key role in the well-posedness theory of semi-linear Schrödinger equations, see for instance [Gin96, Bou99, Caz03, Tao06, Gér06] for an introduction to this wide area of active research.

It is natural to wonder under which circumstances an estimate such as (1.15) holds if $\mathbb{R}^d$ is replaced by a more general Riemannian manifold $(M, g)$. Or more generally, how the geometry of $M$ affects the dispersive character of the Schrödinger flow. A first difficulty arises in generalizing (1.15) to a compact manifold: as mentioned above, if $M$ is compact the dynamics of $e^{it\Delta/2}$ turns out to be almost-periodic; therefore, there is no hope for a global-in-time estimate to hold in that case (clearly, no decay in time estimate as (1.14) holds). But even if the time integral is replaced by a local one, an estimate such as in (1.15) may still fail for any choice of $p > 2$, as the example of the sphere $\mathbb{S}^2$ shows, see [BGT02].

The validity of a Strichartz estimate:

\begin{equation}
\left\| e^{it\Delta/2} u_0 \right\|_{L^p([0,1] \times M)} \leq C \|u_0\|_{L^2(M)},
\end{equation}

on a compact manifold $M$ is related (in a somewhat loose manner) to the regularity properties of the limit measures we introduced in (1.5). Suppose that the Strichartz estimate (1.17) holds for some $p > 2$, and let $\nu$ be a measure obtained as in (1.5), for some sequence $(u_0^n)$ of initial data in $L^2(M)$. Then the Strichartz inequality (1.17) automatically implies that $\nu \in L^{p/2}(\mathbb{R}^d \times M)$, and in particular, that $\nu$ is absolutely continuous with respect to the Riemannian volume measure. In other words, if one is able to construct a sequence $(u_0^n)$ that admits a measure $\nu$ as its limit (1.5), such that $\nu$ has a non-trivial singular part, then this immediately shows that no estimate such as (1.17) holds for any $p > 2$.

This is in fact the case when $(M, g)$ has periodic geodesic flow (such a $(M, g)$ is called a Zoll manifold), which proves that Strichartz estimates are false in that case [Mac11]. Note however that frequency-dependent estimates (that is, with the $L^2(M)$-norm in the right-hand side of (1.17) replaced by a Sobolev norm $H^s(M)$) still hold in that case for exponents $s$ smaller than the one given by the Sobolev
embedding (see the works of Burq, Gérard and Tzvetkov [BGT04, BGT05a, BGT05b]).

The situation is a bit different in the case of the flat torus $\mathbb{T}^d$. For $d = 1$, a simple and elegant argument due to Zygmund [Zyg74] shows that (1.17) holds for $p = 4$. However, this is no longer the case for $d = 1$, $p = 6$ and $d = 2$, $p = 4$ which are the exponents corresponding to the Euclidean space (1.16). Estimate (1.17) fails in those cases as shown by Bourgain [Bou93b, Bou07]. Our results in that case [Mac10, AM10a], developed in Section 4, imply that the measures obtained through (1.5) are absolutely continuous with respect to the Lebesgue measure, for any $d \geq 1$; a proof of this fact based on results on the distribution of lattice points on paraboloids is indicated in the final remark of the article by Bourgain [Bou97]. Moreover, in [AM10a] it is shown that this absolute continuity result holds even for a more general class of Hamiltonians defined on the flat torus.

1.3. Observability and unique continuation for the Schrödinger flow.

A third aspect of the dynamics of the Schrödinger flow, also related to the properties of the limits (1.5), is the validity of the observability property, a quantitative version of the unique continuation property that is relevant, for instance, in Control Theory [Lio88], or Inverse Problems [Isa06].

Let $T > 0$ and $U \subset M$ be an open set; we say that the Schrödinger flow on $(M, g)$ satisfies the observability property for $T$ and $U$ if a constant $C = C(T, U) > 0$ exists, such that the inequality

$$
\|u_0\|_{L^2(M)}^2 \leq C \int_0^T \int_U |e^{it\Delta/2}u_0(x)|^2 \, dx \, dt
$$

holds for every initial datum $u_0 \in L^2(M)$. Clearly, the unique continuation property $\left. e^{it\Delta/2}u_0 \right|_{(0, T) \times U} \equiv 0 \implies u_0 = 0$ is a consequence of (1.18). However, (1.18) also implies a stronger stability property for the Schrödinger flow: two solutions to (1.1) that are close to each other in $(0, T) \times U$ (with respect to the $L^2((0, T) \times U)$-norm) must necessarily be issued from initial data that are also close in $L^2(\mathbb{R}^d)$.

The following condition on $U$, sometimes referred to as the Geometric Control Condition for observability, is sufficient for observability to hold for every $T > 0$, as shown by Lebeau [Leb92] (see also [RT74, DGL06]).

There exists $L_U > 0$ such that

$$
\text{every geodesic of } (M, g) \text{ of length larger than } L_U \text{ intersects } \mathbb{T}.
$$

In the particular case in which the geodesic flow of $(M, g)$ is periodic, it has been shown in [Mac10] that (1.19) turns out to be also necessary for observability.

However, this is not the case in general. For instance, when $(M, g)$ is the torus $\mathbb{T}^d$ equipped with the flat canonical metric, a result of Jaffard [Jal90] (see also [BZ04]) shows that the observability property holds for every $T > 0$ and every open set $U \subset \mathbb{T}^d$, even if the Geometric Control Condition fails. In the same direction, as proved in [AR10], observability holds under conditions weaker than (1.19) for the case of manifolds of constant negative sectional curvature, see Theorem 5.1.

The observability property and the analysis of the limits (1.5) are related as follows. Suppose that (1.18) holds for some $T$ and $U$. Any measure $\nu$ obtained as a limit (1.5) would then satisfy $\nu((0, T) \times U) \geq 1/C$. In particular, the open set $(0, T) \times U$ must intersect the support of every measure $\nu$ obtained by (1.5) for any sequence of initial data $(u^n_0)$. Therefore, it is relevant in this context to have
detailed information on localization properties of the measures obtained through (1.5).

1.4. Relations to other problems studied in the literature.

1.4.1. Deterministic vs. random sequences of initial data. In this article we have addressed the problem of understanding how the properties of the limits (1.5) depend on the geometry of the ambient manifold. From this point of view, what we seek is to prove properties of the limits \( \nu \) that hold for any arbitrary sequence \((u^0_n)\), thus reflecting the geometric features of the propagator \( e^{it\Delta/2} \).

In the present context arbitrary is not synonymous to random. By the term random one can mean random sequences of initial data, as in [Zel92] where it is shown that for almost all orthonormal base of eigenfunctions of the laplacian on the sphere, the limit (1.5) coincides with the standard Riemannian volume. Random can also refer to the fact that the coefficients \((u^0|\varphi_j)\) in (1.3) are random variables: for instance, independent centered gaussians. If one is interested in the high-frequency régime, one should restrict to \( \lambda_j \) in some interval \([E - \delta E, E + \delta E]\) and take the limit \( E \to +\infty \). If \( \delta E \gg E^{1/2} \), it is easy to show that, for any given \( t \) and almost surely, \(|u(t,x)|^2dx\) converges to the uniform measure on \( M \) as \( E \to +\infty \). Note that this type of result is independent of the geometry of \( M \), and its scope is different from the type of problem previously described here, namely that of characterising the limits (1.5) for every possible bounded sequence. As particular cases, we deal with coherent states, Dirac states, and the eigenfunctions of the laplacian \( \varphi_j \) themselves.

1.4.2. The case of eigenfunctions. When the initial condition \( u^0 \) is an eigenfunction of the laplacian \( \varphi_j \), the probability measure (1.4) does not depend on \( t \) and simply reads \(|\varphi_j(x)|^2dx\). The behavior of these measures as \( j \to +\infty \) has been the center of much attention recently, in particular in the context of the Quantum Unique Ergodicity conjecture (see Section 5 and the references therein, as well as Sections 3 and 4 for results and references on the completely integrable case). The study of the limit (1.4) is more general, and thus all the results we mention below also apply to eigenfunctions.

We stress again the fact that having an explicit expression of the eigenfunctions does not necessarily make the problem easy: consider for instance the case of the flat torus \( T^d = \mathbb{R}^d/2\pi \mathbb{Z}^d \). An eigenfunction that satisfies \( \Delta \varphi = -\lambda \varphi \) can be decomposed as

\[
\varphi(x) = \sum_{k \in \mathbb{Z}^d, |k|^2 = \lambda} c_k e^{ik \cdot x}.
\]

The spectral degeneracy of \( \lambda \), that is, the number of integral solutions of \(|k|^2 = \lambda\), gets unbounded as \( \lambda \) grows. Sums of the form (1.20) and the corresponding squares \(|\varphi(x)|^2\) have been studied in [Jak97, Mar05].

1.4.3. Level spacings and pair-correlation statistics. Let \( \text{sp}(-\Delta) \) denote the spectrum of the Laplace-Beltrami operator and for \( \lambda \in \text{sp}(-\Delta) \), write \( P_\lambda \) to denote the orthogonal projection from \( L^2(M) \) onto the eigenspace associated to \( \lambda \). As before, denote by \((\lambda_j)_{j \in \mathbb{N}}\) the eigenvalues of \(-\Delta\) counted with their multiplicities
and by \((\varphi_j)_{j \in \mathbb{N}}\) an orthonormal basis consisting of eigenfunctions indexed accordingly. Using (1.3), we see that for any \(\theta \in L^1(\mathbb{R})\) and \(\chi \in C(M)\), the expression

\[
\int_{\mathbb{R}} \theta(t) \int_M \chi(x) |e^{it\Delta/2}u|^2(x) dx
\]

can be expanded into:

\[
\sum_{\lambda, \lambda' \in \text{sp}(-\Delta)} \hat{\theta} \left( \frac{\lambda - \lambda'}{2} \right) \int_M \chi(x) P_{\lambda} u(x) \overline{P_{\lambda'} u(x)} dx,
\]

or, equivalently,

\[
\sum_{j, j' \in \mathbb{N}} \hat{\theta} \left( \frac{\lambda_j - \lambda_{j'}}{2} \right) (u|\varphi_j)(u|\varphi_{j'}) \int_M \chi(x) \varphi_j(x) \overline{\varphi_{j'}}(x) dx,
\]

where \(\hat{\theta}\) denotes the Fourier transform of \(\theta\). In particular, if \(\hat{\theta}\) is compactly supported, this restricts our sum to bounded \(\lambda - \lambda'\) (resp. \(\lambda_j - \lambda_{j'}\)), and two natural questions arise:

1. Can the study of (1.21), (1.22) be reduced to that of the matrix elements \(\int_M \chi(x) P_{\lambda} u(x) \overline{P_{\lambda'} u(x)} dx\) (for bounded \(\lambda - \lambda'\))?
2. Does the knowledge of the distribution of the pair correlations \(\lambda_j - \lambda_{j'}\) help to gain some insight in (1.21), (1.23)?

To answer (1), it is quite clear from (1.22) that the study of (1.21) amounts in some sense to a study of matrix elements; however, it is not necessarily easier to study the matrix elements than to study the time-dependent equation. We can note that, in the (very special) cases where the minimal spacing \(\inf \{\lambda - \mu : \lambda, \mu \in \text{sp}(-\Delta), \lambda \neq \mu\}\) is strictly positive (as is the case of the sphere or the flat torus, for instance), (1.22) takes a particularly simple form:

\[
\int_{\mathbb{R}} \theta(t) dt \sum_{\lambda \in \text{sp}(-\Delta)} \int_M \chi(x) |P_{\lambda} u(x)|^2 dx.
\]

This property has been exploited in [Mac08], on certain classes of Zoll manifolds, to characterise the accumulation points of sequences of the form \(|P_{\lambda} u(x)|^2 dx\) from the knowledge of the structure of the limits of (1.21) (this can also be deduced from the fine study of the structure of \(P_{\lambda}\) performed in [Zel97]). Also in [Mac08] it is shown how the study of (1.21) can be used to obtain information on the off-diagonal matrix elements (1.22) in the case of Zoll manifolds. The relations between time averaging and eigenvalue level spacing are further explored in the forthcoming article [AFKM11] in the context of completely integrable systems.

In answer to (2), we first recall that the pair correlation distribution is conjectured to be Poissonian in the completely integrable case [BT77]; this has been proved in a certain number of cases [Sar97, Van99b, Van99a, Van00, Mar98, Mar03, Mar02, EMM05]. At the opposite end of “chaotic systems”, e.g. the case of the laplacian on negatively curved surfaces, the pair correlation distribution is conjectured to be given by Random Matrix Theory [BGS84, HODA84]; there is no mathematical proof of this fact, but this is a field of active current research [BK96, Sie02, SR01]. In any case, we do not think that the pair correlation distribution bears any obvious relevance to the understanding of (1.21) or (1.23). As we said, the study of (1.21) is already of interest when \(u\) is itself an eigenfunction \(\varphi_j\), in which case (1.23) is just \(\int_M \chi(x)|\varphi_j(x)|^2 dx\) and the pair correlations play
acts on functions by multiplication by \(a\). Note that when \(a\) have results about (1.21) \((\S 5)\) of (1.21). We can also add, in the case of negatively curved manifolds, that our knowledge of the pair correlation distribution is purely conjectural, whereas we do have results about \((1.21)\) \((\S 5)\). Thus, the link between the two problems that seems to arise when rewriting (1.21) in the form (1.23) is only apparent. We believe that the study of the pair correlation problem is even more difficult.

2. Semiclassical measures

Consider again the example of a wave-packet sequence of initial data \((u_n^0)\) centered at point \((x_0,\xi_0)\) in the cotangent bundle \(T^*M\) as defined in (1.8). In this case,

\[
|u_n^0|^2 \to \delta_{x_0}, \quad \text{as } n \to \infty;
\]

note that this holds independently of the direction of oscillation \(\xi_0 \in T^*_{x_0}M\). However, as we saw (1.9), the densities \(|e^{i\hbar \Delta/2}u_n^0|^2\) corresponding to the evolution concentrate on the point \(x(t)\) of the geodesic of \(M\) issued from \((x_0,\xi_0)\). Therefore, their limit does effectively depend on \(\xi_0\). This shows that there is no propagation law relating the limit of the densities \(|e^{it\Delta/2}u_n^0|^2\) to that of the initial densities \(|u_n^0|^2\).

This difficulty is overcome by lifting the measure \(|u_n^0|^2\,dx\) to phase space \(T^*M\), which allows to keep track of the characteristic oscillation frequencies of \(u_n^0\). There are different procedures to accomplish this, but all of them are equivalent for our purposes. Here we shall focus on the one based on the Weyl quantization (see [Fol89] for a comprehensive introduction). Let us first discuss the definition in the case \(M = \mathbb{R}^d\) and then give the general case.

Starting from a function (a classical observable) \(a \in C_c^\infty(T^*\mathbb{R}^d)\), the Weyl quantization associates to \(a\) the operators \(\text{Op}_h(a)\) (with the “semiclassical” parameter \(h > 0\)), that act on tempered distributions \(u \in \mathcal{S}'(T^*\mathbb{R}^d)\) as follows:

\[
\text{Op}_h(a) u(x) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a\left(\frac{x+y}{2}, h\xi\right) u(y) e^{i\xi \cdot (x-y)} dy \frac{d\xi}{(2\pi)^d}.
\]

Those operators are uniformly bounded from \(L^2(\mathbb{R}^d)\) into itself, in fact (see [GL93]):

\[
\|\text{Op}_h(a)\|_{\mathcal{L}(L^2)} \leq C_d \|a\|_{C^{d+1}(T^*M)}.
\]

Note that when \(a\) only depends on the variable \(x\), the corresponding Weyl operator acts on functions by multiplication by \(a\). On the other hand, if \(a\) only depends on \(\xi\), then \(\text{Op}_h(a)\) is simply the Fourier multiplier:

\[
a \,(hD_x) u(x) = \int_{\mathbb{R}^d} a(h\xi) \hat{u}(\xi) e^{i\xi \cdot x} \frac{d\xi}{(2\pi)^d}.
\]

One can extend this definition to functions \(a \in C_c^\infty(T^*M)\) for a general manifold \(M\) by means of local coordinates and partitions of unity, see for instance [GL93, EZ10].

In what follows, \((u_n^0)\) will be a bounded sequence in \(L^2(M)\) and \((h_n)\) a sequence of positive reals tending do zero such that the \(h_n\)-oscillation condition (1.7).
is fulfilled. We shall define a distribution \( w_{h_n} \) on \( T^*M \), which is a lift of the measure \( |u_n^0|^2 \, dx \), in the sense that it projects down to \( |u_n^0|^2 \, dx \) under the canonical projection \( T^*M \to M \). The action of the distribution \( w_{h_n} \in \mathcal{D}'(T^*M) \) on a test function \( a \in C_c^\infty(T^*M) \) is given by:

\[
\langle w_{h_n}, a \rangle := \left( \text{Op}_{h_n}(a) u_n^0 | u_n^0 \right).
\]

Usually, because of E.P. Wigner’s seminal work \cite{Wig32}, \( w_{h_n} \) is called the Wigner distribution of the function \( u_n^0 \). The sequence of distributions \( (w_{h_n}) \) is uniformly bounded, as a consequence of \eqref{eq:Wigner}. It turns out that any accumulation point of \( (w_{h_n}) \) (in the weak topology of distributions) is a positive measure \( \mu_0 \in \mathcal{M}_+(T^*M) \) despite the fact that the \( w_{h_n} \) are not positive. See \cite{CdV85,Zel87,Ger91,LP93,EZ10} for different proofs of this non-trivial result. Moreover, if some subsequence of \( (|u_n^0|^2) \) and \( (w_{h_n}) \) converges respectively to some measures \( \nu_0 \in \mathcal{M}_+(M) \) and \( \mu_0 \in \mathcal{M}_+(T^*M) \) then:

\[
\nu_0(x) = \int_{T^*M} \mu_0(x, d\xi).
\]

This means that \( \mu_0 \) is also a lift of \( \nu_0 \). Usually, \( \mu_0 \) is called a semiclassical measure of the sequence \( (u_n^0) \). \textbf{A priori}, there can be several semiclassical measures, as different subsequences may have different limits. It is easy to show that a wave-packet \eqref{eq:wave-packet} has a unique semiclassical measure which is \( \delta_{(x_0,\xi_0)} \).

We now turn to the problem of computing semiclassical measures of sequences of trajectories to the Schrödinger flow. We shall denote by \( w_{h_n}(t) \) the Wigner distribution of \( e^{it\Delta/2}u_n^0 \). The main tool in this context is Egorov’s theorem, which relates the Schrödinger group \( e^{it\Delta/2} \) to the geodesic flow \( g^t \) (see \cite{EZ10} for a proof).

**Theorem 2.1.** For every \( a \in C_c^\infty(T^*M) \) there exists a family \( R_h(t) \) of bounded operators on \( L^2(M) \) such that

\[
e^{-ith\Delta/2} \text{Op}_h(a) e^{ith\Delta/2} = \text{Op}_h(a \circ g^t) + R_h(t),
\]

and \( \|R_h(t)\|_{L(L^2(M))} \leq \rho(|t|) h \) for some non-negative continuous function \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \).

With this result at our disposal, it is not hard to derive a propagation law for the time-scaled Wigner distributions \( w_{h_n}(h_nt) \).

**Theorem 2.2.** Let \( (u_n^0) \) and \( (h_n) \) be as above. It is possible to extract a subsequence such that, for every \( t \in \mathbb{R} \),

\[
w_{h_{n'}}(h_{n'}t) \to \mu_t^{sc}, \quad \text{as } n' \to \infty,
\]

where \( \mu_t^{sc} \) is a continuous family of positive measures in \( \mathcal{M}_+(T^*M) \). Moreover, \( \mu_t^{sc} \) is transported along the geodesic flow of \( (M, g) \):

\[
\mu_t^{sc} = (g^t)_* \mu_0, \quad \text{for every } t \in \mathbb{R}.
\]

Identity \eqref{eq:transport} means that \( \mu_t^{sc} \) is obtained as the push-forward of \( \mu_0 \) by the geodesic flow:

\[
\int_{T^*M} a(x, \xi) \mu_t^{sc}(dx, d\xi) = \int_{T^*M} a \circ g^t(x, \xi) \mu_0(dx, d\xi),
\]

for every \( a \in C_c(T^*M) \).
As a consequence of this result, the semiclassical measure of the evolution of a
wave-packet initial datum (1.8) is $\delta_{g^t(x_0,\xi_0)}$. Using the projection identity (2.2) we
deduce the propagation law (1.9) stated in the introduction.

More generally, Theorem 2.2 can be used to compute the limit of $|e^{it\Delta/2}u_n^0|^2$
but does not apply to obtain that of $|e^{it\Delta/2}u_n^0|^2$. This is related to the sensitivity to
time dependence of Egorov’s theorem. The identity that is relevant to our analysis
is obtained by rescaling time of a factor $1/h_n$ in (2.3). The remainder $R_h(t)$ is only
known to go to zero as $h \to 0^+$ uniformly for $|t| \leq T_{E_n}^h$, where $T_{E_n}^h$ is the Ehrenfest
time defined in [10], see [BR02]. Therefore, it is not possible to ensure that
$R_h(t/h_n)$ will tend to zero as $h \to 0^+$. But even if (2.3) is exact (i.e. $R_h(t) \equiv 0$, as
is the case when $M = T^d$), it is not easy to deal with the operators $O_{h_n}(a \circ g^{t/h_n})$,
due to the fact that the functions $a \circ g^{t/h_n}$ depend on $h_n$ and vary very rapidly as
$h_n$ goes to zero.

This problem has been widely studied when $M$ is compact and the initial data
are normalized eigenfunctions of the Laplacian:
$$-\Delta u_n^0 = \lambda_n u_n^0, \quad \|u_n^0\|_{L^2(M)}^2 = 1,$$
corresponding to a sequence of eigenvalues $(\lambda_n)$ that tends to infinity as $n \to \infty$.
In this case, because of (1.7), it is natural to set $h_n := \lambda_n^{-1/2}$ and it turns out that,
for every $t \in \mathbb{R}$:
$$|e^{it\Delta/2}u_n^0|^2 = |u_n^0|^2, \quad w_{h_n}(t) = w_{h_n}(0).$$
Since these quantities do not depend on $t$, Theorem 2.2 shows that any semiclassical
measure $\mu_0$ of $(u_n^0)$ is invariant by the geodesic flow: $(g^t)_* \mu_0 = \mu_0$ for every $s \in \mathbb{R}$.
Moreover, it can easily be proved that $\mu_0$ is supported on the cosphere bundle
$S^*M := \{(x, \xi) \in T^*M : \|\xi\|_x^2 = 1\}$; therefore we can view $\mu_0$ as an element of
$\mathcal{P}(S^*M, g^t)$, the set of $g^t$-invariant probability measures on $S^*M$.

The problem of identifying those measures in $\mathcal{P}(S^*M, g^t)$ that arise as semiclassical
measures of some sequence of eigenfunctions of the Laplacian has proven to
be very hard in general. In contrast with the propagation law (2.3), global aspects
of the dynamics of $g^t$ play a role; in particular, the global geometry of $(M, g)$ is
relevant for this problem. Some results on this issue will be reviewed later on in
this article.

Let us now turn to the case of arbitrary initial data in (1.4). A first difficulty
one encounters when dealing with the Wigner distributions $w_{h_n}(t)$ of $e^{it\Delta/2}u_n^0$ is
that, due to the highly oscillating nature of the propagator $e^{it\Delta/2}$, it is in general
not possible to extract a subsequence such that $w_{h_n}(t)$ converges for every $t \in \mathbb{R}$.
This difficulty can be overcome by viewing $(w_{h_n})$ as a sequence in $L^\infty(\mathbb{R}; \mathcal{D}'(T^*M))$
and considering its accumulation points with respect to the weak-* topology in that
space. This is nothing else but considering time averages of $w_{h_n}$. The following
result holds (see [Mac09]).

**Theorem 2.3.** Let $(u_n^0)$ and $(h_n)$ be as above. Then there exist a subsequence
$(u_n^0)$ and a measure $\mu \in L^\infty(\mathbb{R}; \mathcal{M}_+(T^*M))$ such that, for every $\varphi \in L^1(\mathbb{R})$,
$$\int_{\mathbb{R}} \varphi(t) w_{h_n}(t) \, dt \to \int_{\mathbb{R}} \varphi(t) \mu(t) \, dt, \quad \text{as } n \to \infty.$$ 
Moreover, for a.e. $s \in \mathbb{R}$ the measure $\mu(s)$ is invariant by the geodesic flow.
The measure \( \mu \in L^\infty (\mathbb{R}; M (T^*M)) \) will be called a \textit{semiclassical measure} associated to \((e^{it\Delta/2}u_n^0)\) (or \textit{time-dependent semiclassical measure} when we want to stress the difference with the previous definition). In full generality, there is no propagation law relating the semiclassical measures of the initial data \((u_n^0)\) and the semiclassical measures of \((e^{it\Delta/2}u_n^0)\), as we shall see in §4.

It should be noted that Theorem 2.3 still holds for the time rescaled distributions \(w_n(h_n, \alpha_n t)\) where \((\alpha_n)\) is any sequence that tends to infinity as \(n \to \infty\) (see [Mac09]). Here we restrict ourselves to the case \(\alpha_n = 1/h_n\). The rest of the article is devoted to understanding how the structure of the measures \(\mu(t)\) depends on the geometry of \((M, g)\).

### 3. Zoll manifolds

In this section we shall deal with manifolds whose geodesic flow has the simplest possible dynamics. We shall assume that \((M, g)\) is a compact manifold all of whose geodesics are closed. These are called Zoll manifolds, and the book [Bes78] provides a comprehensive treatment of these geometries. It is known that the geodesic flow of a Zoll manifold is periodic. Every manifold of positive constant sectional curvature (that is, the sphere \(S^d\) and its quotients [Wol67]) is a Zoll manifold; the same holds for compact symmetric spaces of rank one, as the complex projective spaces. O. Zoll constructed a real analytic Riemannian metric on the sphere \(S^2\), which is not isometric to the canonical one, but still has the property that every geodesic is closed. It should be noted that the geodesic flow on the cotangent bundle of a Zoll manifold is a completely integrable Hamiltonian system [Dur97].

The first result on the structure of the set of semiclassical measures for solutions to the Schrödinger equation is due to Jakobson and Zelditch [JZ99]. These authors consider the case \(M = S^d\), equipped with its canonical metric, and study semiclassical measures arising from sequences of eigenfunctions of the Laplacian: they show that any invariant probability measure in \(P(S^*S^d, g^t)\) can be obtained as the semiclassical measure of some sequence of eigenfunctions. At the origin of the proof of this result, there is the easy remark that the restriction to the sphere of the following harmonic polynomials:

\[
\psi_n(x) = C_n (x_1 + ix_2)^n, \quad \|\psi_n\|_{L^2(S^d)} = 1,
\]

\((n \in \mathbb{Z}, |n| \to \infty)\) concentrates on the maximal circle \(x_1^2 + x_2^2 = 1\) on \(S^d\). The semiclassical measure of \((\psi_n)\) is concentrated on one of the two orbits of the geodesic flow, that lie above the aforementioned geodesic in the unit cotangent bundle (the orientation depends on the sign of \(n\)). Thus, any invariant measure carried by a closed geodesic is a semiclassical measure arising from a sequence of eigenfunctions. Jakobson and Zelditch then use the fact that the closed convex hull of such measures is the set of all invariant measures. Following these ideas, the result of Jakobson and Zelditch was extended in [AM10b] to manifolds of constant positive sectional curvature: any invariant measure can be obtained as the semiclassical measure arising from a sequence of eigenfunctions.

The spectrum of the Laplacian consists of clusters of bounded width, centered at the points \((k + \beta)^2, k \in \mathbb{Z}\), where \(\beta > 0\) is a constant depending on the geometry of \(M\) — note that the spectrum of the sphere \(S^d\) is exactly of this form. This was proved in [DG75, Wei77, CdV79], see also [UZ93, Zel96, Zel97] for more precise results on the structure of the spectrum. Using this fact, it is possible to
show that Jakobson and Zelditch’s result also holds for compact rank one symmetric spaces, see [Mac08]. However, the statement for eigenfunctions of the Laplacian on general Zoll manifolds does not seem to be known.

The situation for the time-dependent equation (1.4) is clearer. Consider a general sequence of initial data \((u_0^n)\) normalized in \(L^2(M)\) and chose \((h_n)\) as in Section 2. Suppose moreover that \(\mu_0\) is the unique semiclassical measure of this sequence and that \(w_n\) converges to \(\mu(t)\) as given by Theorem 2.3. Macià has proved in [Mac09] the following result relating \(\mu_0\) to \(\mu(t)\).

**Theorem 3.1.** Let \((M, g)\) be a Zoll manifold and \(\mu_0\) and \(\mu(t)\) be as above. Suppose \(\mu_0 \{\xi = 0\}\) = 0. Then, for every \(a \in C_c(T^*M)\) and a.e. \(t \in \mathbb{R}\) the following holds:

\[
\int_{T^*M} a(x, \xi) \mu(t) (dx, d\xi) = \int_{T^*M} \langle a \rangle (x, \xi) \mu_0 (dx, d\xi),
\]

where \(\langle a \rangle\) is the average of \(a\) along the geodesic flow\(^4\).

Note, in particular, that \(\mu(t)\) does not depend on \(t\); if in addition \(\mu_0\) is an invariant measure then \(\mu(t) = \mu_0\) for almost every \(t \in \mathbb{R}\). When \((u_0^n)\) is a wave-packet \((1.8)\) whose semiclassical measure is \(\delta_{(x_0, \xi_0)}\), Theorem 3.1 implies that

\[
\mu(t) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \delta_{g^s(x_0, \xi_0)} ds,
\]

in other words, \(\mu(t)\) is the orbit measure on the geodesic issued from \((x_0, \xi_0)\). From the fact that the closed convex hull of such measures is the whole set of invariant measures, the following consequence is obtained.

**Corollary 3.2.** Suppose \((M, g)\) is a Zoll manifold. Then every invariant measure in \(P(T^*M, g^t)\) can be obtained as the semiclassical measure (in the sense of Theorem 2.3) of some sequence of initial data in \(L^2(M)\).

As mentioned in the introduction, this shows that Strichartz estimates fail in Zoll manifolds. Combining Corollary 3.2 with Lebeau’s result [Leb92] gives the following.

**Corollary 3.3.** [Mac11] Let \((M, g)\) be a Zoll manifold, \(T > 0\) and \(U \subset M\) an open set. Then condition \((1.19)\) holds for \(U\) if and only if the observability property for the Schrödinger flow holds for \(U\) and \(T\).

Note that in this result, \(T > 0\) can be chosen arbitrarily small, since it does not play a role in condition \((1.19)\).

### 4. The flat torus

The geodesic flow on the cotangent bundle of the flat torus \(\mathbb{T}^d := \mathbb{R}^d / 2\pi \mathbb{Z}^d\) is the prototype of a non-degenerate completely integrable Hamiltonian system. It has a simple explicit expression:

\[
g^s(x, \xi) = (x + s\xi, \xi).
\]

\(^4\)That is:

\[
\langle a \rangle (x, \xi) := \lim_{T \to \infty} \frac{1}{T} \int_0^T a \circ g^s(x, \xi) ds.
\]
For each $\xi \in \mathbb{R}^d$ the torus $\mathbb{T}^d \times \{\xi\}$ is an invariant Lagrangian submanifold of $T^*\mathbb{T}^d \cong \mathbb{T}^d \times \mathbb{R}^d$. In each of these tori, the dynamics of the geodesic flow can be described in terms of the order of resonance of $\xi$. More precisely, consider the following primitive submodule of $\mathbb{Z}^d$:

$$\Lambda_\xi := \{k \in \mathbb{Z}^d : k \cdot \xi = 0\}.$$ 

Then the orbit issued from $(x, \xi)$, for any $x \in \mathbb{T}^d$, is dense in a torus of dimension $d - \text{rk} \Lambda_\xi$; this quantity is sometimes called the order of resonance of $\xi$. In particular, such a trajectory is periodic (and non-constant) if $\text{rk} \Lambda_\xi = d - 1$, and dense on $\mathbb{T}^d$ when $\Lambda_\xi = \{0\}$.

If $\text{rk} \Lambda_\xi > 0$ then $\xi$ is said to be resonant; this means that there exists $k \in \mathbb{Z}^d \setminus \{0\}$ such that $k \cdot \xi = 0$. We shall denote by $\Omega$ the set of all $\xi \in \mathbb{R}^d$ that are resonant; they play an important role in the results we present below.

Let us first recall some existing results for the case in which the sequence of initial data consists of eigenfunctions of the Laplacian. Let $(u_n)$ be such that $-\Delta u_n = \lambda_n u_n$, with $\|u_n\|_{L^2(\mathbb{T}^d)} = 1$ and $\lambda_n \to \infty$. Clearly, $\lambda_n = |k_n|^2$ for some $k_n \in \mathbb{Z}^d$ and the corresponding eigenfunction $u_n$ is a linear combination of exponentials $e^{ik \cdot x}$ with $|k| = |k_n|$. When $d = 1$, the multiplicity of $\lambda_n > 0$ is equal to two and it follows that the weak limits of the densities $|u_n|^2$ are constant. As soon as $d \geq 2$ the multiplicity of $\lambda_n$ tends to infinity as $n \to \infty$, and the structure of the limits becomes less evident. The following inequality is due to Cooke [Coo71] and Zygmund [Zyg74]: there exists $C > 0$ such that if $u$ is an eigenfunction of the Laplacian on $\mathbb{R}^2$ then

$$\|u\|_{L^4(\mathbb{T}^2)} \leq C \|u\|_{L^3(\mathbb{T}^2)}.$$ 

In particular, this implies that any accumulation point $\nu$ of $|u_n|^2$, in the weak* topology of $\mathcal{M}_+(\mathbb{T}^2)$, is in $L^2(\mathbb{T}^2)$. This result was greatly improved by Jakobson [Jak97], who showed that $\nu$ is in fact a trigonometric polynomial whose frequencies lie in at most two circles centered at the origin. It is not known whether an estimate such as (4.1) holds when $d \geq 3$ (for frequency dependent estimates see [Bou93a]). However, Bourgain has proved that any limit measure $\nu$ is absolutely continuous with respect to the Lebesgue measure; in fact, $\nu$ has additional regularity: if $\nu(x) = \sum_{k \in \mathbb{Z}^d} c_k e^{ik \cdot x}$ then $\sum_{k \in \mathbb{Z}^d} |c_k|^{d-2} < \infty$. These results are proved in [Jak97, NTJ01, A’E99], and rely on a deep understanding of the geometry of lattice points in $\mathbb{R}^d$. The proof does not use semiclassical analysis nor the relation between the Schrödinger equation and the geodesic flow.

When the sequence of initial data $(u_0^n)$ is not formed by eigenfunctions much less is known. The analogue of (4.1) is this setting is the estimate for $p = 4$ and $d = 1$:

$$\left( \int_0^1 \|e^{i\Delta/2_t} u\|_{L^4(\mathbb{T}^d)}^p \, dt \right)^{1/p} \leq C \|u\|_{L^2(\mathbb{T}^d)}.$$ 

However, no such inequality is known to hold when $d \geq 2$ (Bourgain has made some conjectures in that direction [Bou93b]).

The following holds.

---

A submodule $\Lambda \subset \mathbb{Z}^d$ is primitive if it equals the intersection of $\mathbb{Z}^d$ with its linear span $\langle \Lambda \rangle$ over $\mathbb{R}$.
Theorem 4.1. Let \((u_0^n)\) be a bounded sequence in \(L^2(T^d)\). If \(\nu \in L^\infty(\mathbb{R}; M_+(T^d))\) is obtained as the weak limit of \(|e^{it\Delta/2}u_0^n|^2\) (in the sense of (1.5)) then \(\nu\) is absolutely continuous with respect to Lebesgue measure.

This result was proved by Bourgain [Bou97] using fine results on the distribution of lattice points on paraboloids. It can be also deduced as a consequence of the results of Macià [Mac10] and Anantharaman and Macià [AM10a] which we describe below. In [AM10a] it is shown that Theorem 4.1 also holds for more general Hamiltonians of the form \(\frac{\Delta}{2} + V(t, x)\). In the case of a coherent state (1.12) or of a Lagrangian state on a torus, the explicit computations of the densities (1.4) and of their limits (1.5) are presented in Propositions 13 and 14 of [Mac10].

The proof of Theorem 4.1 given in [AM10a, Mac10] relies on the structure of the geodesic flow on the torus and is better understood in terms of semiclassical measures.

Suppose \(\mu_0\) is a semiclassical measure of \((u_0^n)\) and that \(\mu \in L^\infty(\mathbb{R}; M_+(T^\ast T^d))\) is a time-dependent semiclassical measure for \((e^{it\Delta/2}u_0^n)\), obtained as a weak-\(\ast\) limit as in Theorem 2.3. When \(\mu_0(T^d \times \Omega) = 0\) (where \(\Omega\) is the set of resonant vectors defined above), it has been shown in [Mac09] that for almost every \(t \in \mathbb{R}\):

\[
\mu(t) = \frac{1}{(2\pi)^d} dx \otimes \int_{T^d} \mu_0(dy, \cdot).
\]

This can be seen as an analogue of Theorem 3.1 in this context, since for \(\xi \in \mathbb{R}^d \setminus \Omega\) and \(a \in C_c(T^\ast T^d)\) one has for every \(x \in T^d\):

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T a \circ g^t(x, \xi) \, ds = \frac{1}{(2\pi)^d} \int_{T^d} a(y, \xi) \, dy.
\]

Therefore, the non-trivial part of the structure of \(\mu(t)\) is that corresponding to its restriction to \(T^d \times \Omega\). A first insight on the complexity of this restriction is provided by a construction in [Mac09]. Two sequences of initial data exist such that both are \(h_n\)-oscillating for a common scale \((h_n)\) and have as semiclassical measure:

\[
\mu_0(x, \xi) = |\rho(x)|^2 \, dx \delta_{\xi_0}(\xi)
\]

with \(\xi_0 \in \Omega\) and \(\rho \in L^2(T^d)\) with \(\|\rho\|_{L^2(T^d)} = 1\). However, the time-dependent measures corresponding to their orbits by the Schrödinger flow differ, and are respectively equal to:

\[
\left|e^{it\Delta/2} \rho\right|^2 \, dx \otimes \delta_{\xi_0}, \quad \text{and} \quad \frac{1}{(2\pi)^d} dx \otimes \delta_{\xi_0}.
\]

Therefore, in contrast with the situation in Theorem 3.1, the semiclassical measure of the initial data no longer determines that of the evolutions. Moreover, those limiting measures \(\mu(t)\) may have a non-trivial dependence on \(t\).

A precise formula relating \(\mu(t)\) to the sequence of initial data is presented in [Mac10] for the two-dimensional case and, more generally, in [AM10a] for any dimension. For the sake of simplicity, let us give here the result corresponding to \(d = 2\).

Start noticing that the set of resonant directions can be written as a disjoint union

\[
\Omega = \bigsqcup_{\omega \in \mathcal{R}} R_\omega \cup \{0\},
\]
where $\mathcal{R}$ is formed by those vectors in $\mathbb{Z}^2$ whose components are relatively prime integers and $R_ω := \omega^⊥ \setminus \{0\}$. If a measure $μ \in \mathcal{P}(T\omega × T\mathbb{R}^2, g')$ is invariant, its restriction to $T\omega × (\mathbb{R}^2 \setminus \Omega)$ and $T\omega × R_ω$ enjoy more regularity on the $x$-variable than it is expected \textit{a priori}. In fact, the restriction of $μ$ to these sets is constant with respect to $x$ along certain directions [AM10a]: given $v ∈ \mathbb{R}^2$ write $τ_v(x, \xi) := (x + v, \xi)$; then $(τ_v)_∗ μ |_{T\omega × (\mathbb{R}^2 \setminus \Omega)} = μ |_{T\omega × (\mathbb{R}^2 \setminus \Omega)}$ for all $v ∈ \mathbb{R}^2$ (and therefore, $μ |_{T\omega × (\mathbb{R}^2 \setminus \Omega)}$ is constant in $x$), whereas $(τ_v)_∗ μ |_{T\omega × R_ω} = μ |_{T\omega × R_ω}$ holds for every $v ∈ ω^⊥$.

If $μ(t)$ is a time-dependent semiclassical measure (as given by Theorem 2.3) of a sequence $(e^{it\Delta/2}u_n^0)$ then it turns out that, besides from the fact that $μ(t)$ is invariant, $μ(t) |_{T^2 × R_ω}$ enjoys additional regularity in the directions in $\mathbb{R}ω$. The reason for this is that time-averaging produces a second microlocalization around the lines $ω^⊥$ which neglects the contribution of the fraction of the energy of $(e^{it\Delta/2}u_n^0)$ that goes to infinity in the direction $ω$. In other words, if $a ∈ C^∞(\mathbb{T}^2)$ is a function whose non vanishing Fourier modes correspond to frequencies in $ω$ and $φ ∈ C^∞_c(\mathbb{R}^2)$ vanishes in a neighborhood of $ω^⊥$ then one has (see [Mac10]):

$$\int_{a}^{b} e^{-it\Delta/2} Op_{h} (a ∗ φ) e^{it\Delta/2} dt = \frac{b - a}{(2π)^2} \int_{\mathbb{T}^2} adx \phi (hD_x) + O(h).$$

Denote by $L^p_{ω}(\mathbb{T}^2)$ the space of functions $a ∈ L^p(\mathbb{T}^2)$ such that $a ∗ τ_v = a$ for $v ∈ ω^⊥$. The following result holds (see [Mac10, AM10a]).

**Theorem 4.2.** Let $(u_n^0)$ be a sequence normalized in $L^2(\mathbb{T}^2)$. Suppose that $μ(t)$ is a semiclassical measure of $(e^{it\Delta/2}u_n^0)$, in the sense of Theorem 2.3. Then for every $ω ∈ \mathcal{R}$ there exists a measure $ρ_ω$, defined on $R_ω$ and taking values in the space of trace-class operators on $L^2_ω(\mathbb{T}^2)$ such that for a.e. $t ∈ \mathbb{R}$, every $a ∈ L^∞(\mathbb{T}^2)$ and every $φ ∈ C^∞_c(\mathbb{R}^2)$ we have:

$$\int_{T^2×R_ω} a(x) φ(\xi) μ(t) (dx, dξ) = \int_{R_ω} φ(\xi) tr \left( m_a e^{-it\Delta/2} ρ_ω (dξ) e^{it\Delta/2} \right),$$

where $m_a$ denotes the operator acting by multiplication by $a$ in $L^2_ω(\mathbb{T}^2)$.

From this, it follows that $μ(t) |_{T^2 × R_ω}$ is absolutely continuous with respect to the $x$-variable, and because of (4.3), that $μ(t) |_{T^2 × (\mathbb{R}^2 \setminus \{0\})}$ is absolutely continuous. It is also possible to prove that $μ(t) |_{T^2 × \{0\}}$ is also given by a formula similar to (4.3) from which Theorem 4.1 follows. The “measures” $ρ_ω$ depend only on the (sub)sequence $(u_n^0)$, however they are not determined by the semiclassical measure of the initial data alone, and are responsible for the phenomena presented in example 4.2.

The generalization of Theorem 4.2 to dimensions higher than two is non-trivial. The main reason for that is that in the general case there is not a decomposition of the set of resonant frequencies as simple as (4.3). Therefore, the analogues of identities (4.3) in this case are obtained by an iterative procedure that requires to perform successive two-microlocalizations along nested sequences of linear subspaces contained in the resonant set $\Omega$, see [AM10a].

As pointed out in the introduction, Theorem 4.1 cannot be used to obtain counterexamples to the validity of Strichartz estimates for the Schrödinger flow. On the other hand, it was used in [Mac11] to obtain an alternative proof of Jaffard’s
result [Ja90] on the observability of the Schrödinger flow on the bidimensional flat torus described in Section 1.3.

5. Negatively curved manifolds

In the case of negatively curved compact Riemannian manifolds, we can make two contradictory remarks. The fact that the geodesic flow has well-understood chaotic properties (to be precise, has the Anosov property) makes one very optimistic about the good dispersive properties of the Schrödinger flow. This motivates some very strong conjectures, such as the quantum unique ergodicity conjecture (QUE) described below. On the other hand, these same chaotic properties make it difficult to approximate the Schrödinger dynamics by the geodesic dynamics: the quantum-classical correspondence is only valid for a relatively short range of time (the Ehrenfest time), and this leaves little hope to use it to prove those conjectures.

We first state the Snirelman theorem, whose proof can be found in [Sni74, Zel87, CdV85]. On a smooth compact Riemannian manifold $(M,g)$, take an orthonormal basis $(u_n)$ of $L^2(M)$, formed of eigenfunctions of the Laplacian $(-\Delta u_n = \lambda_n u_n)$, and $\lambda_n \to \infty$. Assume that the geodesic flow $g^t$ is ergodic with respect to the Liouville measure. Write \( h_n := \lambda_n^{-1/2} \) and let $w_{h_n}$ denote the Wigner distribution of $u_n$. Then, there exists a subset $S \subset \mathbb{N}$, of density 1, such that the sequence $(w_{h_n})_{n \in S}$ converges weakly to the Liouville measure. Thus, the result says that a typical sequence of eigenfunctions becomes equidistributed, both in the “$x$-variable” and in the “$\xi$-variable”. At this level of generality, it is not well understood if the whole sequence converges, or if there can be exceptional subsequences with a different limiting behavior. There are manifolds (or Euclidean domains) with ergodic geodesic flows, but with exceptional subsequences of eigenfunctions [Has10]. But these examples have only been found very recently, and we stress the fact that the proof is not constructive; a fortiori, the exceptional subsequences, whose existence is proved, are not exhibited explicitly. Thus, one cannot say that the phenomenon is fully understood.

The statement of the Snirelman theorem can be adapted to solutions of the time-dependent Schrödinger equation [AR10]: take a sequence of initial conditions $(u_0^n(t))$ chosen randomly from a “generalized orthonormal family”, with characteristic lengths of oscillations $h_n$ going to 0. Denote by $w_{h_n}(t)$ the Wigner distribution associated with $(e^{it\Delta/2}u_0^n(t))$. Then the sequence $\int_0^1 w_{h_n}(t)dt$ converges to the Liouville measure, in the probabilistic sense (the paper [AR10] provides a detailed statement, and a rate of convergence for negatively curved manifolds).

Negatively curved manifolds have ergodic geodesic flows, but actually the understanding of the chaotic properties of the flow is so good that one could hope to go beyond the Snirelman theorem. It may seem surprising that the question is still widely open, even in the case of manifolds of constant negative curvature (where the local geometry is completely explicit). The QUE conjecture was stated by Rudnick and Sarnak [Sar95, RS94] for eigenfunctions of the Laplacian on a negatively curved compact manifold. If $(u_n)$ is a sequence of eigenfunctions of the Laplacian $(-h_n^2\Delta u_n = u_n$ with $h_n \to 0$) and $(w_{h_n})$ the associated Wigner distributions, the conjecture says that $(w_{h_n})$ converges to the Liouville measure. In other words, there are no exceptional subsequences of eigenfunctions for which $(w_{h_n})$ converges to an invariant measure other than Liouville. So far, the only complete result is due to E. Lindenstrauss [BL03, Lin06], who proved the conjecture in the case where
$M$ is an arithmetic congruence surface, and the eigenfunctions $(u_n)$ are common eigenfunctions of $\Delta$ and of the Hecke operators. Unfortunately, his proof relies a lot on the use of the Hecke operators, and cannot be adapted to more general situations.

There is a partial result, due to N. Anantharaman and S. Nonnenmacher, which holds in great generality, on any compact negatively curved manifold [Ana08, AN07, Riv10]. Let again $(u_n)$ be a sequence of eigenfunctions, and $\mu$ be a limit point of the corresponding Wigner distributions $(w_n)$. The following result deals with the Kolmogorov-Sinai entropy of the invariant measure $\mu$.

The Kolmogorov-Sinai entropy is a functional $h_{KS} : \mathcal{P}(S^* M, g^*) \to \mathbb{R}_+$, from the set of $g^*$-invariant probability measures to $\mathbb{R}_+$. The shortest (though not always the most convenient) definition of the entropy results from a theorem due to Brin and Katok [BK83]. For any time $T > 0$, introduce a distance on $S^* M$,

$$d_T(\rho, \rho') = \max_{t \in [-T/2, T/2]} d(g^t \rho, g^t \rho'),$$

where $d$ is the distance built from the Riemannian metric. For $\epsilon > 0$, denote by $B_T(\rho, \epsilon)$ the ball of center $\rho$ and radius $\epsilon$ for the distance $d_T$. When $\epsilon$ is fixed and $T$ goes to infinity, it looks like a thinner and thinner tubular neighborhood of the geodesic segment $[g^{-T} \rho, g^T \rho]$.

Let $\mu$ be a $g^*$-invariant probability measure on $T^* M$. Then, for $\mu$-almost every $\rho \in T^* M$, the limit

$$\lim_{\epsilon \to 0} \liminf_{T \to +\infty} -\frac{1}{T} \log \mu(B_T(\rho, \epsilon)) = \lim_{\epsilon \to 0} \limsup_{T \to +\infty} -\frac{1}{T} \log \mu(B_T(\rho, \epsilon)) =: h_{KS}(\mu, \rho)$$

exists and it is called the local entropy of the measure $\mu$ at the point $\rho$ (it is independent of $\rho$ if $\mu$ is ergodic). The Kolmogorov-Sinai entropy is the average of the local entropies:

$$h_{KS}(\mu) = \int h_{KS}(\mu, \rho) d\mu(\rho).$$

In the case when $\mu$ is obtained from a limit of Laplace eigenfunctions, the result of Anantharaman-Nonnenmacher says that $h_{KS}(\mu) > 0$. This is a strong restriction, for instance, $\mu$ cannot be entirely concentrated on a countable union of closed geodesics. The result has been extended to the time-dependent context of Theorem 2.3 by Anantharaman–Rivièere, who showed that, for $a.e \, t \in \mathbb{R}$, $\mu(t)$ has positive entropy. In the case of manifolds of constant curvature $-1$, and dimension $d$, there is actually an explicit lower bound, for the eigenfunction case [AN07] and more generally for the time-dependent case [AR10]. In the latter case, one can disintegrate $\mu(t)(x, \xi) = \int \mu_E(t)(x, \xi) \nu(dE)$ where $\nu$ is a positive measure, and $\mu_E(t)$ is a $g^*$-invariant probability measure supported on the energy layer $\{\|\xi\|^2 = E\}$. Then, one has, $dt \otimes \nu$ almost everywhere,

$$h_{KS}(\mu_E(t)) \geq \frac{d-1}{2} \sqrt{E}.$$

We note that $\sqrt{E}$ is the speed of the geodesics on $\{\|\xi\|^2 = E\}$, and that $(d-1) \sqrt{E}$ is the maximal entropy for invariant measures carried by this set (the maximum is achieved only for the Liouville measure). The statement for eigenfunctions is similar, with $\mu$ concentrated on one single energy layer.
One can use this result to improve the Geometric Control condition \([1.19]\) of \(\S 1.3\):

**Theorem 5.1.** \([\text{AR10}]\) Let \(M\) be a compact Riemannian manifold of dimension \(d\) and constant curvature \(\equiv -1\). Let \(a\) be a smooth function on \(M\), and define the closed \(g^s\)-invariant subset of \(S^*M\),

\[
K_a = \{ \rho \in S^*M, a(g^s(\rho)) = 0 \ \forall s \in \mathbb{R} \}.
\]

Assume that the topological entropy of \(K_a\) is \(< \frac{d-1}{2}\). Then, for all \(T > 0\), there exists \(C_{T,a} > 0\) such that, for all \(u\):

\[
\|u\|_{L^2(M)}^2 \leq C_{T,a} \int_0^T \|ae^{it\Delta/2}u\|_{L^2(M)}^2 dt.
\]

For the purposes of this survey, we simply define the topological entropy of \(K_a\) as the supremum of Kolmogorov-Sinai entropies, for all invariant measures supported on \(K_a\). For manifolds of constant curvature \(\equiv -1\), the topological entropy is closely related to the more familiar notion of Hausdorff dimension, at least if \(K_a\) is locally maximal: saying that the topological entropy of \(K_a\) is \(< \frac{d-1}{2}\) is equivalent to \(K_a\) having Hausdorff dimension \(< d\) \([\text{PS01}]\).

We see that the Geometric Control condition \((K_a \text{ empty})\) is weakened by only assuming that \(K_a\) have small Hausdorff dimension. Examples of such functions \(a\) on a negatively curved surface of genus \(g\) are given in \([\text{AR10}]\): one takes a decomposition of the surface into \(2g - 2\) “hyperbolic pairs of pants” with very long boundary components, and takes \(a\) to be non-zero in a neighbourhood of these \(3g - 3\) curves. It would be interesting if one could enrich this list of examples.

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Université Paris-Sud 11, Mathématiques, Bât. 425, 91405 ORSAY CEDEX, FRANCE

E-mail address: Nalini.Anantharaman@math.u-psud.fr

Universidad Politécnica de Madrid. DCAIN, ETSI Navales. Avda. arco de la Victoria s/n. 28040 MADRID, SPAIN

E-mail address: Fabricio.Macia@upm.es