On complementary channels and the additivity problem

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Abstract

We explore complementarity between output and environment of a quantum channel (or, more generally, CP map), making an observation that the output purity characteristics for complementary CP maps coincide. Hence, validity of the multiplicativity/additivity conjecture for a class of CP maps implies its validity for complementary maps. The class of CP maps complementary to entanglement-breaking ones is described and is shown to contain diagonal CP maps as a proper subclass, resulting in new class of CP maps (channels) for which the multiplicativity/additivity holds. Covariant and Gaussian channels are discussed briefly in this context.

In what follows $\mathcal{H}_A, \mathcal{H}_B, \ldots$ will denote (finite dimensional) Hilbert spaces of quantum systems $A, B, \ldots$. $\mathcal{M}(\mathcal{H})$ denotes the algebra of all operators, $\mathcal{S}(\mathcal{H})$ — the convex set of density operators (states) and $\mathcal{P}(\mathcal{H}) = \text{ext} \mathcal{S}(\mathcal{H})$ — the set of pure states (one-dimensional projections) in $\mathcal{H}$. For a natural $d$, $\mathcal{H}_d$ denotes the Hilbert space of $d$—dimensional complex vectors, and $\mathcal{M}_d$ — the algebra of all complex $d \times d$— matrices.

Given three finite spaces $\mathcal{H}_A, \mathcal{H}_B, \mathcal{H}_C$ and a linear operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$, the relation

$$\Phi_B(\rho) = \text{Tr}_{\mathcal{H}_C} V \rho V^*, \quad \Phi_C(\rho) = \text{Tr}_{\mathcal{H}_B} V \rho V^*; \quad \rho \in \mathcal{M}(\mathcal{H}_A)$$

(1)

defines two CP maps $\Phi_B : \mathcal{M}(\mathcal{H}_A) \to \mathcal{M}(\mathcal{H}_B)$, $\Phi_C : \mathcal{M}(\mathcal{H}_A) \to \mathcal{M}(\mathcal{H}_C)$, which will be called mutually complementary. If $V$ is an isometry, both maps are trace preserving (TP) i.e. channels. The name “complementary
channels” is taken from the paper [4], where they were used to define quantum version of degradable channels.

The Stinespring dilation theorem implies that for given a CP map (channel) a complementary always exists. In the Appendix we give a proof which also clarifies in what sense the complementary map is unique. It follows that for a given CP map $\Phi_B$, any two channels $\Phi_C, \Phi_C'$ complementary to $\Phi_B$ are equivalent in the sense that there is a partial isometry $W : \mathcal{H}_C \to \mathcal{H}_C'$ such that

$$
\Phi_{C'}(\rho) = W \Phi_C(\rho) W^*, \quad \Phi_C(\rho) = W^* \Phi_{C'}(\rho) W, \quad (2)
$$

for all $\rho$. Dilations with the minimal dimensionality $d_C$ are called minimal. Any two minimal dilations are isometric (i.e. $W$ is an isometry from $\mathcal{H}_C$ onto $\mathcal{H}_{C'}$). By performing a Stinespring dilation for a complementary CP map one obtains a map equivalent to the initial one in the sense (2). Thus the complementarity is a relation between the equivalence classes of CP maps.

To simplify formulas we shall also use the notation $\tilde{\Phi}$ for the map which is complementary to $\Phi$.

Consider the following “measures of output purity” of a CP map $\Phi$

$$
\nu_p(\Phi) = \max_{\rho \in \mathcal{S}(|\Phi|)} \left[ \text{Tr} \Phi(\rho)^p \right]^{1/p}, \quad 1 \leq p, \quad (3)
$$

introduced in [4]. For $p = \infty$ one puts $\nu_\infty(\tilde{\Phi}) = \max_{\rho \in \mathcal{S}(\mathcal{H})} \| \Phi(\rho) \|$. In the case of channel $\Phi$, further useful characteristics are the minimal output entropy

$$
\hat{H}(\Phi) = \min_{\rho \in \mathcal{S}(\mathcal{H})} H(\Phi(\rho)),
$$

where $H(\sigma) = -\text{Tr} \sigma \ln \sigma$ is the von Neumann entropy of a density operator $\sigma$, and its convex closure

$$
\hat{H}_\Phi(\rho) = \min_{\rho = \sum_x \pi(x) \rho(x)} \sum_x \pi(x) H(\Phi(\rho(x))),
$$

where the minimum is taken over all possible convex decompositions of the density operator $\rho$ into pure states $\rho(x) \in \mathcal{S}(\mathcal{H})$ [5]. By convexity argument, all these quantities remain unchanged if we replace $\mathcal{S}(\mathcal{H})$ by $\mathcal{P}(\mathcal{H})$ in their definitions.

**Theorem 1** If one of the relations

$$
\nu_p(\Phi_1 \otimes \Phi_2) = \nu_p(\Phi_1) \nu_p(\Phi_2), \quad (4)
$$


\[ \hat{H}(\Phi_1 \otimes \Phi_2) = \hat{H}(\Phi_1) + \hat{H}(\Phi_2), \]  

holds for the CP maps (channels) \( \Phi_1, \Phi_2 \), then similar relation holds for the pair of their complementary maps \( \tilde{\Phi}_1, \tilde{\Phi}_2 \). If one of these relations holds for given \( \Phi_1 \) and arbitrary \( \Phi_2 \), then similar relation holds for complementary \( \tilde{\Phi}_1 \) and arbitrary \( \Phi_2 \).

**Remark.** Let us recall that for two given channels \( \Phi_1, \Phi_2 \), the property (4) with \( p \in [1, 1 + \varepsilon] \) implies (5) by differentiation [1]. The property (5), which is equivalent to the additivity of the \( \chi \)-capacity (the Holevo capacity) with arbitrary input constraints [5], implies both additivity of the \( \chi \)-capacity and (5) by the arguments similar to that for the superadditivity of entanglement of formation, see e.g. [17]. On the other hand, assuming that (4) with \( p \in [1, 1 + \varepsilon] \) holds for all CP maps \( \Phi_1, \Phi_2 \) implies (5), (6) for all channels, and these two properties, as well as additivity of the \( \chi \)-capacity, are globally equivalent, i.e. if one holds for all channels, another holds for all channels as well [17].

**Proof.** If \( \rho = |\psi\rangle\langle\psi| \) for some \( |\psi\rangle \in \mathcal{H}_A \), then Hermitian operators \( \Phi(\rho), \tilde{\Phi}(\rho) \) have the same nonzero eigenvalues. Indeed, \( \Phi(\rho), \tilde{\Phi}(\rho) \) are partial traces of the operator \( |\psi_{BC}\rangle\langle\psi_{BC}| \), where \( |\psi_{BC}\rangle = V|\psi\rangle \in \mathcal{H}_B \otimes \mathcal{H}_C \), then the proof goes in the same way as in the case of normalized vectors (see, e.g. [15], Theorem 2.7).

Both \( \text{Tr} \sigma^p \) and \( H(\sigma) \) are universal functions of nonzero eigenvalues of a Hermitian operator \( \sigma \). From the definitions of \( \nu^p, \hat{H} \) and \( \tilde{\hat{H}} \) it follows that for arbitrary CP map \( \Phi \)

\[ \nu^p(\tilde{\Phi}) = \nu^p(\Phi). \]  

Moreover, if \( \Phi \) is a channel, then

\[ \hat{H}(\tilde{\Phi}) = \hat{H}(\Phi), \]

\[ \tilde{\hat{H}}(\tilde{\Phi}) = \tilde{\hat{H}}(\Phi). \]

Now notice that if \( \Phi_j, \tilde{\Phi}_j, j = 1, 2 \), are two pairs of complementary CP maps, then \( \Phi_1 \otimes \Phi_2 \) and \( \tilde{\Phi}_1 \otimes \tilde{\Phi}_2 \) are complementary. For this take \( \mathcal{H}_B = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}, \mathcal{H}_C = \mathcal{H}_{C_1} \otimes \mathcal{H}_{C_2} \) and \( V = V_1 \otimes V_2 \). Summarizing all these facts, we get the statement. \( \square \)
Assume that a CP map \( \Phi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}') \) is given by Kraus representation
\[
\Phi(\rho) = \sum_{\alpha=1}^{\hat{d}} V_\alpha \rho V_\alpha^*,
\]
then a complementary map \( \tilde{\Phi} : \mathcal{M}(\mathcal{H}) \to \mathcal{M}_{\hat{d}} \) is given by
\[
\tilde{\Phi}(\rho) = [\text{Tr} V_\alpha \rho V_\beta^*]_{\alpha,\beta=1,d} = [\text{Tr} \rho V_\beta^* V_\alpha]_{\alpha,\beta=1,d},
\]
since \( V = \sum_{\alpha=1}^{\hat{d}} \oplus V_\alpha \) is a map from \( \mathcal{H} \) to \( \sum_{\alpha=1}^{\hat{d}} \oplus \mathcal{H}' \simeq \mathcal{H}' \otimes \mathcal{H}_{\hat{d}} \) for which \( \Phi, \tilde{\Phi} \) are given by the partial traces \([11]\), see \([7]\). By writing the trace in \( \mathcal{H}' \) with respect to an orthonormal basis \( \{e'_j\} \), we have the Kraus representation
\[
\tilde{\Phi}(\rho) = \sum_{j=1}^{d'} \tilde{V}_j \rho \tilde{V}_j^*,
\]
where \( (\tilde{V}_j)_\alpha = \langle e'_j | V_\alpha \rangle \). One can check by direct computation that applying the same procedure to \( \tilde{\Phi} \), one obtains the map \( \tilde{\tilde{\Phi}} \) which is isometric to \( \Phi \).

A CP map \( \Phi : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}') \) is entanglement-breaking if it has a Kraus representation with rank one operators \( V_\alpha \) \([10]\):
\[
\Phi(\rho) = \sum_{\alpha=1}^{\hat{d}} |\varphi_\alpha \rangle \langle \psi_\alpha | \rho |\psi_\alpha \rangle \langle \varphi_\alpha |.
\]
Such a CP map is channel if and only if the (over)completeness relation
\[
\sum_{\alpha=1}^{\hat{d}} |\psi_\alpha \rangle \langle \varphi_\alpha | \psi_\alpha \rangle = I
\]
is fulfilled. The complementary map \( \tilde{\Phi} : \mathcal{M}(\mathcal{H}) \to \mathcal{M}_{\hat{d}} \) is
\[
\tilde{\Phi}(\rho) = [c_{\alpha\beta} |\psi_\beta \rangle \langle \psi_\beta | \rho |\psi_\beta \rangle \langle \psi_\beta |]_{\alpha,\beta=1,d},
\]
where \( c_{\alpha\beta} = \langle \varphi_\beta | \varphi_\alpha \rangle \). Notice that by the Kolmogorov decomposition, arbitrary nonnegative definite matrix can be represented in such form. In the special case where \( \{\psi_\alpha\}_{\alpha=1,d} \) is an orthonormal base in \( \mathcal{H} \), \([11]\) is diagonal CP map \([11]\). Diagonal channels, which are characterized by additional
property \( c_{\alpha\alpha} \equiv 1 \), were also earlier considered in \([4]\) under the name of dephasing channels. From \([13]\) we see that the diagonal maps are complementary to a particular class of entanglement-breaking maps, namely to c-q maps. For another special subclass of entanglement-breaking maps, the q-c maps, \( \{ \varphi_\alpha \}_{\alpha=1,d} \) is an orthonormal base in \( \mathcal{H} \), so that \( c_{\alpha\beta} = \delta_{\alpha\beta} \), and the complementary map is easily seen to be of the same subclass.

Let us rewrite (14) in the form

\[
\tilde{\Phi}(\rho) = \sum_{\alpha,\beta = 1}^{d} c_{\alpha\beta}|e_\alpha\rangle\langle\rho|e_\beta\rangle |\psi_\alpha\rangle |\psi_\beta\rangle
\]

where \( \{ e_\alpha \} \) is the canonical base for \( \mathcal{H}_{\tilde{d}} \). Representing \( c_{\alpha\beta} = \sum_{j=1}^{d'} \bar{v}_{\beta j} v_{\alpha j} \) by Kolmogorov decomposition and denoting

\[
\tilde{V}_j = \sum_{\alpha = 1}^{d} v_{\alpha j}|e_\alpha\rangle |\psi_\alpha\rangle
\]

we have the Kraus representation \([12]\) for the complementary map. For the diagonal maps \( |\psi_\alpha\rangle = |e_\alpha\rangle \), hence from \([15]\) one sees that the diagonal maps are characterized by the property of having a Kraus representation with simultaneously diagonalizable (i.e. commuting normal) operators \( \tilde{V}_j \). Somewhat more generally, \( \{|\psi_\alpha\rangle\} \) can be an orthonormal base different from \( \{|e_\alpha\rangle\} \), in which case both \( \tilde{V}_j^* \tilde{V}_j \) and \( \tilde{V}_j^* \tilde{V}_j \) are families of commuting normal operators.

For entanglement-breaking channels the additivity property \([5]\) (and in fact, \([3]\), although not explicitly stated) with arbitrary second channel was established by Shor \([16]\) and the multiplicativity property \([4]\) for all \( p > 1 \) by King \([12]\), using the Lieb-Thirring inequality. This proof of multiplicativity can be generalized with almost no changes to the case of entanglement-breaking CP maps. Note that for diagonal channels (expression \([14]\) with \( \{|\psi_\alpha\rangle\} = \{|e_\alpha\rangle\} \) and \( c_{\alpha\alpha} \equiv 1 \) the properties \([4]\), \([6]\) can be established easily because these channels leave invariant the canonical base in \( \mathcal{H}_{\tilde{d}} \), hence \( \nu_p(\Phi) = 1, \tilde{H}(\Phi) = 0 \) for such channels. Let us prove for example \([4]\). (Results for a more general class involving channels of such kind are given in \([5]\)).
Let $\Phi_2$ be an arbitrary CP map, and $\Phi_1$ a channel such that $\nu_p(\Phi_1) = 1$. We have

$$\nu_p(\Phi_1 \otimes \Phi_2) = \nu_p((\text{Id}_1 \otimes \Phi_2) \circ (\Phi_1 \otimes \text{Id}_2)) \leq \nu_p(\text{Id}_1 \otimes \Phi_2),$$

where $\text{Id}$ denotes the identity channel. Applying the equality $\nu_p(\text{Id} \otimes \Phi) = \nu_p(\Phi)$ established in [1], we get

$$\nu_p(\Phi_1 \otimes \Phi_2) \leq \nu_p(\Phi_2) = \nu_p(\Phi_1)\nu_p(\Phi_2),$$

whence the multiplicativity follows.

However the proof of multiplicativity for diagonal CP maps, that are not necessarily channels, given in [11], is substantially more complicated (it uses the same method as for the entanglement-breaking maps). Moreover, this proof seems not to be extendable to the more general class of CP maps [14] where $\{\psi_\alpha\}$ is not an orthonormal base, but an arbitrary system of vectors. On the other hand, theorem 1 implies all the multiplicativity/additivity properties for this more general class simply by their complementarity to entanglement-breaking maps and a reference to results in [16, 12]. Specifically, it implies the superadditivity property (6), which so far was known only for direct convex sums of the identity and entanglement-breaking channels (e.g. erasure channel), see [8]. More precisely, theorem 1 combined with proposition 3 from [8] implies property (6) for convex mixtures of either identity or its complementary – completely depolarizing channel – with either entanglement-breaking channel or its complementary. Therefore additivity of (constrained) $\chi$–capacity holds as well for such convex mixtures.

4. Let $G$ be a group and $g \rightarrow U^A_g, U^B_g; g \in G; j = 1, 2$, be two (projective) unitary representations of $G$ in $\mathcal{H}_A, \mathcal{H}_B$. The CP map $\Phi : \mathcal{M}(\mathcal{H}_A) \rightarrow \mathcal{M}(\mathcal{H}_B)$ is covariant if

$$\Phi[U^A_g \rho U^A_{g^*}] = U^B_g \Phi[\rho] U^B_{g^*}$$

for all $g \in G$ and all $\rho$. The structure of covariant CP maps was studied in the context of covariant dynamical semigroups, see e. g. [6]. In particular, for arbitrary covariant CP map there is the Kraus representation (10), where $V_j$ are the components of a tensor operator for the group $G$, i. e. satisfy the equations

$$U^B_g V_j U^A_{g^*} = \sum_k d_{jk}(g)V_k,$$
where \( g \rightarrow D(g) = [d_{jk}(g)] \) is a matrix unitary representation of \( G \). It follows that the map complementary to covariant CP map is again covariant, with \( D(g) \) playing the role of the second unitary representation.

Let us consider in some detail the extreme transpose-depolarizing channel

\[
\Phi(\rho) = \frac{1}{d-1} \left[ \text{Tr} \rho - \rho^T \right]
\]

where \( \rho^T \) is transpose of \( \rho \) in an orthonormal basis \( \{e_j\} \) in \( \mathcal{H} = \mathcal{H}_A = \mathcal{H}_B, \dim \mathcal{H} = d \). This channel breaks the multiplicativity (4) with \( \Phi_1 = \Phi_2 = \Phi \) for \( d > 3 \) and large enough \( p \) [18]. At the same time it fulfills the multiplicativity for \( 1 \leq p \leq 2 \) [2] and the additivity (5), see [14], [3]. It has the covariance property

\[
\Phi(U \rho U^*) = \bar{U} \Phi(\rho) \bar{U}^*
\]

for arbitrary unitary \( U \). Since

\[
\Phi(\rho) = \frac{1}{2(d-1)} \sum_{j,k=1}^d (|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|) \rho (|e_k\rangle\langle e_j| - |e_j\rangle\langle e_k|),
\]

introducing the index \( \alpha = (j,k) \), we have the Kraus representation (10) with operators

\[
V_\alpha = \frac{1}{\sqrt{2(d-1)}} (|e_j\rangle\langle e_k| - |e_k\rangle\langle e_j|).
\]

Hence

\[
\tilde{\Phi}(\rho) = \left[ \text{Tr} V_\alpha \rho V_\beta^* \right]_{\alpha,\beta=1,d} = \frac{1}{2(d-1)} \left[ \delta_{jj'}\langle e_k|\rho|e_{k'}\rangle - \delta_{jj'}\langle e_k|\rho|e_{j'}\rangle - \delta_{kk'}\langle e_j|\rho|e_k\rangle + \delta_{kk'}\langle e_j|\rho|e_{j'}\rangle \right].
\]

The space \( \mathcal{H}_{12} \) in which this matrix acts is tensor product of two \( d \)-dimensional coordinate spaces with vectors indexed by \( k(k') \) and \( j(j') \). Let \( F \) be the operator in \( \mathcal{H}_{12} \) which flips the indices \( j \) and \( k \). The expression above takes the form

\[
\tilde{\Phi}(\rho) = \frac{1}{2(d-1)} (I_{12} - F)(\rho \otimes I_2)(I_{12} - F).
\]

This is the complementary channel which shares the multiplicativity/additivity properties with the channel (17).
By using the decomposition $I_2 = \sum_{j=1}^{d} |e_j\rangle \langle e_j|$, we have the Kraus representation \[ \sum \] for the complementary channel, where
\[
\tilde{V}_j |\psi\rangle = \frac{1}{\sqrt{2(d-1)}} (I_{12} - F) (|\psi\rangle \otimes |e_j\rangle)
\]
\[
= \frac{1}{\sqrt{2(d-1)}} (|\psi\rangle \otimes |e_j\rangle - |e_j\rangle \otimes |\psi\rangle).
\]

The covariance property of the channel is
\[
\tilde{\Phi}(U\rho U^*) = (U \otimes \tilde{U})\tilde{\Phi}(\rho)(U^* \otimes \tilde{U})^*,
\]
as follows from the fact that $F(U \otimes U) = (U \otimes U)F$.

The case of depolarizing channel
\[
\Phi(\rho) = (1-p)\rho + \frac{p}{d} I \text{Tr} \rho, \quad 0 \leq p \leq \frac{d^2}{d^2 - 1},
\]
can be considered along similar lines\(^1\). We give only the final result
\[
\tilde{\Phi}(\rho) = S(\rho \otimes I_2)S,
\]
where
\[
S = \sqrt{\frac{p}{d}}I_{12} + \sqrt{d} \left[ -\frac{\sqrt{p}}{d} + \sqrt{1 - p \left( \frac{d^2 - 1}{d^2} \right)} |\Omega_{12}\rangle \langle \Omega_{12}| \right],
\]
with $|\Omega_{12}\rangle$ the maximally entangled vector in $\mathcal{H} \otimes \mathcal{H}$.

While the depolarizing channel is globally unitarily covariant, the complementary channel has the covariance property
\[
\tilde{\Phi}[U\rho U^*] = (U \otimes \tilde{U})\tilde{\Phi}[\rho](U \otimes \tilde{U})^*
\]
for arbitrary unitary operator $U$ in $\mathcal{H}$.

Notice that in both cases the complementary channels have the form
\[
\Phi_C(\rho) = S(\rho \otimes I_B)S^*,
\]
where $S : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_C$ is such that $\text{Tr}_{\mathcal{H}_B}S^*S = I_A$. There is a simple general relation between this representation and the second formula in [1] for
\[^1\text{This case was elaborated jointly with N. Datta.}\]
arbitrary CP map. Namely, given $V: \mathcal{H}_A \rightarrow \mathcal{H}_B \otimes \mathcal{H}_C$ choose an orthonormal basis $\{e'_j\}$ in $\mathcal{H}_B$ and define $S: \mathcal{H}_A \otimes \mathcal{H}_B \rightarrow \mathcal{H}_C$ by the relation $\langle e'_j | V = S | e'_j \rangle$, or, more precisely,

$$\langle \bar{\psi}_B \otimes \psi_C | V | \psi_A \rangle = \langle \psi_C | S | \psi_A \otimes \psi_B \rangle,$$

where $\bar{\psi}_B$ is complex conjugate in the basis $\{e_j\}$. By interchanging the roles of $\mathcal{H}_B$, $\mathcal{H}_C$ we of course obtain a similar representation for the initial map $\Phi_B$. This is in fact nothing but the dual form (21) of the Stinespring representation, if $\Phi_B$, $\Phi_C$ are considered as maps in Heisenberg rather than in Schrödinger picture.

The next important class is Bosonic Gaussian channels [9]. Any such channel can be described as resulting from a quadratic interaction with Gaussian environment. It follows that complementary channel is again Gaussian (see [9], Sec. IVB, for an explicit description). As an example consider attenuation channel with coefficient $k < 1$ described by the transformation

$$a' = ka + \sqrt{1 - k^2} a_0$$

in the Heisenberg picture (to simplify notations we write $a$ instead of $a \otimes I_0$ and $a_0$ instead of $I \otimes a_0$), where the mode $a_0$ is in a Gaussian state. Complementing this transformation with

$$a'_0 = \sqrt{1 - k^2} a - ka_0,$$

we get a canonical (Bogoljubov) transformation implementable by a Hamiltonian quadratic in $a, a_0, a', a_0^\dagger$. It follows that the complementary channel is again attenuation channel with the coefficient $\sqrt{1 - k^2}$. In the same way, the linear amplifier with coefficient $k > 1$ described by the transformation

$$a' = ka + \sqrt{k^2 - 1} a_0^\dagger,$$

complements to

$$a'_0 = \sqrt{k^2 - 1} a^\dagger + ka_0.$$

More detail on complementary covariant and Gaussian channels will be given in a subsequent work.

Note added in replacement: Similar ideas, in the context of channels, are independently developed in the work of C. King, K. Matsumoto, M. Natanson and M. B. Ruskai [13].

Appendix
Theorem 2 For a CP map $\Phi_B : \mathcal{M}(\mathcal{H}_A) \to \mathcal{M}(\mathcal{H}_B)$, there exist a Hilbert space $\mathcal{H}_C$ of dimensionality $d_C \leq d_A d_B$ and an operator $V : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$, such that the first relation in (1) holds. For any other such operator $V' : \mathcal{H}_A \to \mathcal{H}_B \otimes \mathcal{H}_C$, there is a partial isometry $W : \mathcal{H}_C \to \mathcal{H}_C'$ such that

$$V' = (I_B \otimes W)V, \quad V = (I_B \otimes W^*)V'. \quad (19)$$

Proof. Consider the algebraic tensor product $L = \mathcal{H}_A \otimes \mathcal{M}(\mathcal{H}_B)$ generated by the elements $\psi \otimes X, \psi \in \mathcal{H}_A, X \in \mathcal{M}(\mathcal{H}_B)$. Let us introduce pre-inner product in $L$ with the corresponding square of norm

$$\| \sum_j \psi_j \otimes X_j \|^2 = \sum_{j,k} \langle \psi_j | \Phi^*(X_j^* X_k) | \psi_k \rangle = \text{Tr} \sum_{j,k} X_k \Phi(|\psi_k \rangle \langle \psi_j |) X_j^*,$$

where $\Phi^*$ is the dual map. This quantity is nonnegative for CP map $\Phi$. After factorizing with respect to the subspace $L_0$ of zero norm, we obtain the Hilbert space $\mathcal{K} = L/L_0$. By construction, $\dim \mathcal{K} \leq d_A d_B$.

Put $V \psi = \psi \otimes I$, and $\pi(Y) \psi = \pi(Y) (\psi \otimes X) = \psi \otimes YX$. Then $\pi$ is a $*$-homomorphism $\mathcal{M}(\mathcal{H}_B) \to \mathcal{M}(\mathcal{K})$, i.e. a linear map preserving the algebraic operations and the involution: $\pi(XY) = \pi(X) \pi(Y), \pi(X^*) = \pi(X)$. Moreover,

$$\langle \varphi | \Phi^*(X) | \psi \rangle = \langle \varphi \otimes I | \psi \otimes X \rangle = \langle \varphi | V^* \pi(X) V | \psi \rangle, \quad X \in \mathcal{M}(\mathcal{H}_B). \quad (20)$$

However any $*$-homomorphism of the algebra $\mathcal{M}(\mathcal{H})$ is unitary equivalent to the ampliation $\pi(X) = X \otimes I_C$, where $I_C$ is the unit operator in a Hilbert space $\mathcal{H}_C$, i.e. we can take $\mathcal{K} = \mathcal{H}_B \otimes \mathcal{H}_C$, and (20) takes the form

$$\langle \varphi | \Phi^*(X) | \psi \rangle = \langle \varphi | V^* (X \otimes I_C) V | \psi \rangle, \quad X \in \mathcal{M}(\mathcal{H}_B),$$

or

$$\Phi^*(X) = V^* (X \otimes I_C) V, \quad (21)$$

which is equivalent to the first equation in (1) with $\Phi_B = \Phi$. It also follows that $\dim \mathcal{H}_C \leq d_A d_B$.

To prove the second statement, consider the subspace

$$\mathcal{M} = \{(X \otimes I_C) V \psi : \psi \in \mathcal{H}_A, X \in \mathcal{M}(\mathcal{H}_B)\} \subset \mathcal{K} = \mathcal{H}_B \otimes \mathcal{H}_C. \quad (22)$$

It is invariant under multiplication by operators of the form $Y \otimes I_C$, hence it has the form $\mathcal{M} = \mathcal{H}_B \otimes \mathcal{M}_C, \mathcal{M}_C \subset \mathcal{H}_C$. For a minimal representation we should have $\mathcal{M}_C = \mathcal{H}_C$, because otherwise there would be a proper subrepresentation.
Consider a similar subspace \( \mathcal{M}' = \mathcal{H}_B \otimes \mathcal{M}_{C'} \) of the space \( \mathcal{K}' = \mathcal{H}_B \otimes \mathcal{H}_{C'} \) for the second dilation. Define the operator \( R \) from \( \mathcal{M} \) to \( \mathcal{M}' \) by

\[
R(X \otimes I_C)V\psi = (X \otimes I_{C'})V'\psi.
\]

(23)

Then \( R \) is isometric, since the norms of the vector and of its image under \( R \) are both equal to \( \langle \psi | \Phi^*(X^*X) | \psi \rangle \) by (20). From (23) we obtain for all \( Y \in \mathfrak{M}(\mathcal{H}_B) \)

\[
R(YX \otimes I_C)V\psi = (Y \otimes I_{C'})R(X \otimes I_C)V'\psi
\]

and hence

\[
R(Y \otimes I_C) = (Y \otimes I_{C'})R
\]

(24)

on \( \mathcal{M} \). Extend \( R \) to the whole of \( \mathcal{K} \) by letting it equal to zero on the orthogonal complement to \( \mathcal{M} \), then (24) holds on \( \mathcal{K} \). Therefore \( R = I_C \otimes W \), where \( W \) isometrically maps \( \mathcal{M}_{C} \) onto \( \mathcal{M}_{C'} \). Relation (23) implies (19). □

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