Existence and optimality conditions for relaxed mean-field stochastic control problems

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Abstract

We consider optimal control problems for systems governed by mean-field stochastic differential equations, where the control enters both the drift and the diffusion coefficient. We study the relaxed model, in which admissible controls are measure-valued processes and the relaxed state process is driven by an orthogonal martingale measure, whose covariance measure is the relaxed control. This is a natural extension of the original strict control problem, for which we prove the existence of an optimal control. Then, we derive optimality necessary conditions for this problem, in terms of two adjoint processes extending the known results to the case of relaxed controls.

Key words: Mean-field stochastic differential equation; relaxed control; martingale measure; adjoint process; stochastic maximum principle; variational principle.

MSC 2010 subject classifications, 93E20, 60H30.

1 Introduction

In this paper, we deal with optimal control of systems driven by mean-field stochastic differential equations (MFSDE), where the coefficients depend not only on the state but also on its distribution. This mean-field equation, represents in some sense the average behavior of an infinite number of particles, see 13, 20 for details. Since the earlier papers 12, 14, mean-field control theory has raised a lot of interest, motivated by applications to various fields such as game theory, mathematical finance, communications networks, management of oil resources. Mean-field control problems occur in many applications, such as in a continuous-time Markowitz’s mean–variance portfolio selection model where the variance term involves a quadratic function of the expectation. The inclusion of this mean-field terms in the coefficients introduces time inconsistency, leading to the failure of Bellmann principle. For this kind of problems, the stochastic maximum principle, provides a powerful tool to solve them, see 4, 3 and the references therein. The first objective of the present paper is to investigate the problem of existence of an optimal control. It is well known that in the absence of convexity assumptions, this problem has no optimal solution. Therefore it is natural to embedd the set of strict controls into a wider class of measure-valued controls, enjoying good compactness properties, called relaxed controls. We show that the right state process associated with a relaxed control, satisfies a MFSDE driven by an orthogonal martingale measure rather than a Brownian motion. For this model, we prove that the strict and relaxed control problems have the same value function and that an optimal relaxed control exists. Our result extends in particular 2, 8, 10, 16 to mean field controls and 11 to the case of a MFSDE with a controlled diffusion coefficient. The proof is based on tightness properties of the underlying processes and the Skorokhod selection theorem. In a second step, we establish necessary conditions for optimality in the form of a relaxed stochastic maximum principle, obtained via the first and second order adjoint processes. This result generalizes Peng’s stochastic maximum principle 18, to mean field control problems and 4 to relaxed controls. The other advantage is that our maximum principle applies to a natural class of controls, which is the closure of the class of strict controls, for which we have existence of

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an optimal control. The proof of the main result is based on the approximation of the relaxed control problem by a sequence of strict control problems. Then Ekeland’s variational principle is applied to get necessary conditions of near-optimality, for the sequence of near optimal strict controls. The result is obtained by a passage to the limit in the state equation as well as in the adjoint processes. The resulting first and second order adjoint processes are solutions of linear BSDEs driven by a Brownian motion and an orthogonal square integrable martingale. Moreover, our result is given via an approximation procedure, so that it could be convenient for numerical computation.

2 Assumptions and preliminaries

Let \((\Omega, \mathcal{F}, P)\) be a probability space, equipped with a filtration \((\mathcal{F}_t)\), satisfying the usual conditions and \((W_t)\) a \((\mathcal{F}_t, P)\)-Brownian motion. Let \(A\) be some compact metric space called the action space. A strict control \((u_t)\) is a measurable, \(\mathcal{F}_t\)-adapted process with values in the action space \(A\). We denote \(\mathcal{U}_{ad}\) the space of strict controls.

The state process corresponding to a strict control is the unique solution, of the mean-field stochastic differential equations (MFSDE)

\[
dx_t = b(t, X_t, E(X_t), u_t)dt + \sigma(t, X_t, E(X_t), u_t)dW_t; \quad X_0 = x
\]

and the corresponding cost functional is given by

\[
J(u) = E \left( \int_0^T b(t, X_t, E(X_t), u_t) \, dt + g(X_T, E(X_T)) \right).
\]

The coefficients of the state equation as well as of the cost functional are of mean-field type, in the sense that they depend not only on the state process, but also on its marginal law, through its expectation. The objective is to minimize \(J(u)\) over the space \(\mathcal{U}_{ad}\), that is to find \(u^* \in \mathcal{U}_{ad}\) such that \(J(u^*) = \inf \{J(u), u \in \mathcal{U}_{ad}\}\).

Let us consider the following assumptions which will be used in different combinations throughout the paper.

- \((H_1)\) \(b : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \to \mathbb{R}, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \to \mathbb{R} \) are bounded continuous functions such that \(b(t, \ldots, a)\) and \(\sigma(t, \ldots, a)\) are Lipschitz continuous in \((x, y)\).
- \((H_2)\) \(h : [0, T] \times \mathbb{R} \times \mathbb{R} \times A \to \mathbb{R}\) and \(g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\), are bounded continuous functions such that \(h(t, \ldots, a)\) and \(g(\ldots)\) are Lipschitz continuous in \((x, y)\).
- \((H_3)\) \(b(t, \ldots, a), \sigma(t, \ldots, a), h(t, \ldots, a)\) and \(g(\ldots)\) are twice continuously differentiable with respect to \((x, y)\), and their derivatives are bounded and continuous in \((x, y, a)\).

Without loss of generality, the coefficients are assumed to be one dimensional as in [4], to avoid heavy notations in the definition of adjoint processes. Under assumption \((H_1)\), according to [13] Prop.1.2, for each \(u \in \mathcal{U}_{ad}\) the MFSDE has a unique strong solution, such that for every \(p > 0\) we have \(E(|X_t|^p) < +\infty\). Moreover the cost functional is well defined.

3 The relaxed control problem

3.1 The space of relaxed controls

As it is well known in control theory, in the absence of convexity conditions, an optimal control may fail to exist in the set \(\mathcal{U}_{ad}\) of strict controls (see e.g. [10]). This suggests that the set of strict controls is too narrow and should be embedded into a wider class of relaxed controls, with nice compactness properties. For the relaxed model, to be a true extension of the original control problem, the following both conditions must be satisfied:

i) The value functions of the original and the relaxed control problems must be equal.
ii) The relaxed control problem must have an optimal solution.

The idea of relaxed control is to replace the \(A\)-valued process \((u_t)\) with a \(\mathbb{P}(A)\)-valued process \((\mu_t)\), where \(\mathbb{P}(A)\) is the space of probability measures equipped with the topology of weak convergence. Then \((\mu_t)\) may be identified as a random product measure on \([0, T] \times A\), whose projection on \([0, T]\) coincides with Lebesgue measure. Let \(\mathcal{V}\) be the set of product measures \(\mu\) on \([0, T] \times A\) whose projection on \([0, T]\) coincides with the Lebesgue measure \(dt\). It is clear that every \(\mu\) in \(\mathcal{V}\) may be disintegrated as \(\mu = dt \cdot \mu_t(da)\), where \(\mu_t(da)\) is a transition probability. The elements of \(\mathcal{V}\) are called Young measures in deterministic theory. \(\mathcal{V}\) as a closed subspace of the space of positive Radon measures \(\mathcal{M}_+([0, T] \times A)\) is compact for the topology of weak
convergence. In fact it can be proved that it is compact also for the topology of stable convergence, where test functions are measurable, bounded functions \( f(t, a) \) continuous in \( a \), see \[8\] for further details.

**Definition 3.1.** A relaxed control on the filtered probability space \((Ω, F, F_t, P)\) is a random variable \( \mu = dt.\mu_t(da) \) with values in \( \mathbb{V} \), such that \( \mu_t(da) \) is progressively measurable with respect to \((F_t)\) and such that for each \( t, 1_{(0,t]}\mu \) is \( F_t \)-measurable.

**Remark 3.2.** The set \( U_{ad} \) of strict controls is embedded into the set of relaxed controls by identifying \( u_t \) with \( dt\delta_{u_t}(da) \).

### 3.2 The relaxed state equation

The question now is to define the natural state process associated to a relaxed control. In deterministic control or in the stochastic theory where only the drift is controlled, one has just to replace in equation (2.1) the drift by the same drift integrated against the relaxed control. Now we are in a situation where both the drift and the diffusion coefficient are controlled. Following \[13\] Prop. 1.10, the existence of a weak solution \( f \) for every \( \mu \) is not difficult to prove that Equation (3.1) has a unique solution such that for every \( \nu \) such that \( \mu_t(da) \) is progressively measurable with respect to \((F_t)\) and such that for each \( t, 1_{(0,t]}\mu \) is \( F_t \)-measurable.

#### 3.2.1 The relaxed state equation

The following theorem gives a pathwise representation of the solution of the relaxed martingale problem, in terms of a mean-field stochastic differential equation driven by an orthogonal martingale measure.

**Theorem 3.3.** 1) Let \( P \) be a solution of the relaxed martingale problem. Then \( P \) is the law of an adapted, continuous process \( X \) defined on an extension of the space \((Ω, F, F_t, P)\), which is a solution of the following MFSDE:

\[
dx_t = \int_{\mathbb{A}} b(t, x, E(X_t), a) \mu_t(da) dt + \int_{\mathbb{A}} \sigma(t, x, E(X_t), a) M_t(da, dt) \quad \text{for } 0 \leq t < \infty
\]

where \( M \) is an orthogonal continuous martingale measure, with intensity \( dt\mu_t(da) \).

2) If the coefficients \( b \) and \( \sigma \) are Lipschitz in \( x, y \), uniformly in \( t \) and \( a \), equation (3.1) has a unique pathwise solution.

**Proof.** 1) The proof is based essentially on the same arguments as in \[7\], Theorem IV-2 and \[13\] Prop. 1.10.

2) Since the coefficients are Lipschitz continuous, then following the same steps as in \[13\] and \[7\], it is not difficult to prove that Equation (3.1) has a unique solution such that for every \( p > 0 \) we have \( E(\sup_{t\in[0,T]} |X_t|^p) < +\infty \).

**Remark 3.4.** i) Note that the orthogonal martingale measure corresponding to the relaxed control \( dt\mu_t(da) \) is not unique.

ii) From now on, the probability space is an extension of the initial probability space. The Brownian motion \((W_t)\) remains a Brownian motion on this new probability space, but the filtration is no longer the natural filtration of \((W_t)\).

**Definition 3.5.** Let \((Ω, F, F_t, P)\) be a filtered probability space and \( M(t, B) \) a random process, where \( B \in \mathcal{B}(\mathbb{A}) \). \( M \) is a \((F_t, P)\)-martingale measure if:

1) For every \( B \in \mathcal{B}(\mathbb{A}) \), \( (M(t, B))_{t \geq 0} \) is a square integrable martingale, \( M(0, B) = 0 \).

2) For every \( t > 0 \), \( M(t, \cdot) \) is a \( \sigma \)-finite \( L^2 \)-valued measure.

It is called continuous if for each \( B \in \mathcal{B}(\mathbb{A}) \), \( M(t, B) \) is continuous and orthogonal if \( M(t, B).M(t, C) \) is a martingale whenever \( B \cap C = \phi \).
Lemma 3.7. We show in this section that the strict and the relaxed control problems have the same value function. This

\[ \mu \] and accordingly the relaxed cost functional is given by

\[ \text{The relaxed control problem is now driven by equation} \]

\[ dX_t = \sum_{i=1}^{n} b(t, X_t, E(X_t), a_i) \alpha_i^t dt + \sum_{i=1}^{n} \sigma(t, X_t, E(X_t), a_i) (\alpha_i^t)^{1/2} dW_i^t, \quad X_0 = x, \]

where the \( W^i \) are independent Brownian motions, defined on an extension of the initial probability space.

The process \( M \) given by \( M([0, t] \times A) = \sum_{i=1}^{n} \int_{0}^{t} (\alpha_i^t)^{1/2} 1_{\{a_i \in A\}} dW_i^t \) is in fact an orthogonal continuous martingale measure (cf. [7]) with intensity \( \mu_t(\mathrm{d}a)dt = \sum \alpha_i^t \delta_{a_i}(\mathrm{d}a)dt \). Thus, the last SDE can be expressed in terms of \( M \) and \( \mu \) as follows:

\[ dX_t = \int_{A} b(t, X_t, E(X_t), a) \mu_t(\mathrm{d}a)dt + \int_{A} \sigma(t, X_t, E(X_t), a) M(\mathrm{d}a, dt) \]

3.3 Approximation of the relaxed model

The relaxed control problem is now driven by equation

\[ dX_t = \int_{A} b(t, X_t, E(X_t), a) \mu_t(\mathrm{d}a)dt + \int_{A} \sigma(t, X_t, E(X_t), a) M(\mathrm{d}a, dt), \quad X_0 = x, \tag{3.2} \]

and accordingly the relaxed cost functional is given by

\[ J(\mu) = E \left( \int_{0}^{T} \int_{A} h(t, X_t, E(X_t), a) \mu_t(\mathrm{d}a)dt + g(X_T, E(X_T)) \right). \tag{3.3} \]

We show in this section that the strict and the relaxed control problems have the same value function. This is based on the chattering lemma and the stability of the state process with respect to the control variable.

Lemma 3.7. (Chattering lemma) i) Let \( (\mu_t) \) be a relaxed control. Then there exists a sequence of adapted processes \( (\alpha_i^t) \) with values in \( \mathbb{A} \), such that the sequence of random measures \( (\delta_{\alpha_i^t}(\mathrm{d}a)dt) \) converges in \( \mathbb{V} \) to \( \mu_t(\mathrm{d}a)dt \), \( P - a.s. \).

ii) For any \( g \) continuous in \([0, T] \times \mathbb{M}_1(\mathbb{A}) \) such that \( g(t, \cdot) \) is linear, we have

\[ \lim_{n \to \infty} \int_{0}^{t} g(s, \delta_{\alpha_i^s})ds = \int_{0}^{t} g(s, \mu_s)ds \]

uniformly in \( t \in [0, T] \), \( P - a.s. \).

Proof. See [8] and [10] Lemma 1 page 152.

Proposition 3.8. 1) Let \( \mu = \mu_t(\mathrm{d}a)dt \) a relaxed control. Then there exist a continuous orthogonal martingale measure \( M(\mathrm{d}t, \mathrm{d}a) \) whose covariance measure is given by \( \mu_t(\mathrm{d}a)dt \).

2) If we denote \( M^n(t, B) = \int_{0}^{t} \int_{B} \delta_{\alpha_i^s}(\mathrm{d}a)dW_s \), where \( (u^n) \) is defined as in the last Lemma, then for every bounded predictable process \( \varphi : \Omega \times [0, T] \times \mathbb{A} \to \mathbb{R} \), such that \( \varphi(\omega, t, \cdot) \) is continuous, we have

\[ E \left[ \left( \int_{0}^{t} \int_{A} \varphi(\omega, t, a)M^n(\mathrm{d}t, \mathrm{d}a) - \int_{0}^{t} \int_{A} \varphi(\omega, t, a)M(\mathrm{d}t, \mathrm{d}a) \right)^2 \right] \to 0 \text{ as } n \to \infty, \]

for a suitable Brownian motion \( W \) defined on an eventual extension of the probability space.

Proof. See [15] pages 196-197.

Proposition 3.9. 1) Let \( X_n, X^n_n \) be the solutions of state equation (3.3) corresponding to \( \mu \) and \( u^n \), where \( \mu \) and \( u^n \) are defined as in the chattering lemma. Then \( \lim_{n \to \infty} E(\sup_{0 \leq t \leq T} |X^n_n - X_t|^2) = 0 \).

2) Let \( J(u^n) \) and \( J(\mu) \) be the expected costs corresponding respectively to \( u^n \) and \( \mu \), then \( J(u^n) \) converges to \( J(\mu) \).

Proof. Similar to [1], Proposition 2.
Remark 3.10. As a consequence of the last proposition, it holds that the infimum among relaxed controls is equal to the infimum among strict controls, that is the value functions for the relaxed and strict models are the same.

3.4 Existence of an optimal relaxed control

The following theorem, which is the main result of this section, extends \cite{2,8,10} to systems driven by mean field SDEs and \cite{1} to mean field SDEs with controlled diffusion coefficient.

Theorem 3.11. Under assumptions (H₁), (H₂), there exist an optimal relaxed control.

Proof. Let \((\mu^n)_{n \geq 0}\) be a minimizing sequence, that is \(\lim_{n \to \infty} J(\mu^n) = \inf_{\mu \in \mathcal{R}} J(\mu)\) and let \(X^n\) be the unique solution of (\ref{eq:3.2}), associated with \(\mu^n\) and \(M^n\) where \(M^n\) is a continuous orthogonal martingale measure with intensity \(\mu^n\). We will prove that the sequence \((\mu^n, M^n, X^n)\) is tight and then show that we can extract a subsequence, which converges in law to a process \((\hat{\mu}, \hat{M}, \hat{X})\), satisfying the same MFSDE. To finish the proof we show that the sequence of cost functionals \((J(\mu^n))_{n}\) converges to \(J(\hat{\mu})\) which is equal to \(\inf_{\mu \in \mathcal{R}} J(\mu)\) and then we conclude that \((\hat{\mu}, \hat{M}, \hat{X})\) is optimal.

**Step 1:** \((\mu^n)_{n}\) is relatively compact in \(V\).

The relaxed controls \(\mu^n\) are random variables with values in the space \(V\) which is compact. Then Prohorov’s theorem yields that the family of distributions associated to \((\mu^n)_{n \geq 0}\) is tight, then \((\mu^n)_{n}\) is relatively compact in \(V\).

**Step 2:** \((M^n)\) is tight in \(C([0,1], \mathcal{S}')\) \(\mathcal{S}'\) where \(\mathcal{S}'\) is the Schwartz space of rapidly decreasing functions. By \cite{17}, Theorem 5.1, it is sufficient to show that for every \(\varphi \in \mathcal{S}\) the family \((M^n(\varphi), n \geq 0)\) is tight in \(C([0,T], \mathcal{S}')\) where \(M^n(\omega, t, \varphi) = \int_0^\omega \varphi(a)M^n(\omega, t, da)\) and \(p > 1\) and \(s < t\), let the Burkeholder-Davis-Gundy inequality we have

\[
E \left( |M^n_0(\varphi) - M^n_t(\varphi)|^{2p} \right) \leq C_p \sup_{a \in A} |\varphi(a)|^{2p} |t - s|^p = K_p |t - s|^p,
\]

where \(K_p\) is a constant depending on \(p\) and \(\varphi\). Then, the Kolmogorov tightness criteria in \(C([0,T], \mathcal{S}')\) is fulfilled and the sequence \((M^n(\varphi))\) is tight. Therefore \((M^n)\) is tight in \(C([0,1], \mathcal{S}')\).

**Step 3:** \((X^n)_{n \geq 0}\) is tight in \(C([0,1], \mathcal{S})\).

Let \(p > 1\) and \(s < t\). Using argument iterations from stochastic calculus and the boundness of the coefficients \(b\) and \(\sigma\), it is easy to show that

\[
E \left( |X^n_0 - X^n_t|^{2p} \right) \leq C_p |t - s|^p
\]

which yields the tightness of \((X^n_0, n \geq 0)\) in \(C([0,T], \mathcal{S})\).

**Step 4:** The sequence of processes \((\mu^n, M^n, X^n)\) is tight on the space \(\Gamma = \mathcal{V} \times C([0,1], \mathcal{S}') \times C([0,T], \mathcal{S})\), then by the Skorokhod representation theorem, there exists a probability space \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), a sequence \(\hat{\gamma}^n = (\hat{\mu}^n, \hat{M}^n, \hat{X}^n)\) and \(\hat{\gamma} = (\hat{\mu}, \hat{M}, \hat{X})\) defined on this space such that:

(i) for each \(n \in \mathbb{N}\), \(\text{law}(\gamma^n) = \text{law}(\hat{\gamma}^n)\),

(ii) there exists a subsequence \((\hat{\gamma}^{n_k})\) of \((\hat{\gamma}^n)\), still denoted by \((\hat{\gamma}^n)\), which converges to \(\hat{\gamma}, \mathbb{P}\)-a.s. on the space \(\Gamma\).

This means in particular that the sequence of relaxed controls \((\hat{\mu}^n)\) converges in the weak topology to \(\hat{\mu}\), \(\mathbb{P}\)-a.s. and \((\hat{M}^n, \hat{X}^n)\) converges to \((\hat{M}, \hat{X})\), \(\mathbb{P}\)-a.s. in \(C([0,1], \mathcal{S}') \times C([0,T], \mathcal{S})\).

According to property (i), we get

\[
\hat{X}^n = x + \int_0^t \int_0^s b \left( s, \hat{X}^n_t, E(\hat{X}^n_t), a \right) \hat{\mu}_s(da)ds + \int_0^t \int_0^s \sigma \left( s, \hat{X}^n_t, E(\hat{X}^n_t), a \right) \hat{M}_s(ds, da), \hat{X}_0^n = x.
\]

Since the coefficients \(b, \sigma\) are Lipschitz continuous in \((x, y)\), then according to property (ii) and using similar arguments as in \cite{19} page 32, it holds that the first and second terms converge in probability to the corresponding terms without the superscript \(n\). Now, since \(b\) and \(\sigma\) are Lipschitz continuous, then \(\hat{X}\) is the unique solution of the MFSDE

\[
\hat{X} = x + \int_0^t \int_0^s b \left( s, \hat{X}_s, E(\hat{X}_s), a \right) \hat{\mu}_s(da)ds + \int_0^t \int_0^s \sigma \left( s, \hat{X}_s, E(\hat{X}_s), a \right) \hat{M}_s(ds, da), \hat{X}_0 = x.
\]
To finish the proof of Theorem 3.11, it remains to check that \( \hat{\mu} \) is an optimal control. The functions \( b \) and \( \sigma \) are Lipschitz continuous, then according to the above properties (i)-(ii) we get

\[
\inf_{\mu \in \mathcal{R}} J(\mu) = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \int_A h(t, X^n_t, E(X^n_t), a) \mu^n_t(da) dt + g(X^n_T, E(X^n_T)) \right] \\
= \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \int_A h\left(t, X^n_t, E\left(X^n_t\right), a\right) \bar{\mu}^n_t(da) dt + g(X^n_T, E(X^n_T)) \right] \\
= \mathbb{E} \left[ \int_0^T \int_A h\left(t, X_t, E(X_t), a\right) \bar{\mu}_t(da) dt + g(X_T, E(X_T)) \right].
\]

Hence \( \hat{\mu} \) is an optimal control. \( \blacksquare \)

**Remark 3.12.** The proof of the last Theorem is based on tightness and weak convergence techniques. Then it is possible to prove it by using the non linear martingale problem, following the same steps as in [8], without using the pathwise representation of the solution.

### 4 The relaxed maximum principle

In this section, we shall derive necessary conditions for optimality, satisfied by an optimal relaxed control. To achieve this goal, we begin by deriving necessary conditions for near optimality, satisfied by a minimizing sequence of strict controls, which converges to the relaxed control. Then we pass to the limit in the state equation as well as in the adjoint processes to obtain the relaxed maximum principle.

Throughout this section assumptions \((H_1), (H_2)\) and \((H_3)\) will be in force.

#### 4.1 Necessary conditions for near optimality

Let \( \mu = dt \mu_t(da) \) be an optimal relaxed control and \( X \) be the corresponding state process, solution of [32].

According to the optimality of \( \mu \) and the chattering lemma, there exists a sequence \( (u_n) \subset U_{ad} \) such that:

\[
J(u^n) = J(\mu^n) \leq \inf \{ J(\mu); \mu \in \mathcal{R} \} + \varepsilon_n,
\]

where \( \mu^n = dt \delta_{u^n} (da) \) and \( \lim_{n \to +\infty} \varepsilon_n = 0 \).

In this section, we apply Ekeland’s variational principle [4], to establish necessary conditions for near-optimality satisfied by the minimizing sequence \( (u^n) \).

**Lemma 4.1.** (Ekeland) Let \((V,d)\) be a complete metric space and \( F : V \rightarrow \mathbb{R} \cup \{+\infty\} \) be lower-semicontinuous and bounded from below. Given \( \epsilon > 0 \), suppose \( u^* \in V \) satisfies \( F(u^*) \leq \inf \{ F(v); v \in V \} + \epsilon \). Then for any \( \lambda > 0 \), there exists \( v \in V \) such that \( F(v) \leq F(u^*) \), \( d(u^*, v) \leq \lambda \) and \( \forall w \neq v ; F(w) < F(u^*) + \varepsilon / \lambda d(w, v) \).

For \( u, v \in U_{ad} \), define \( d(u, v) = P \otimes dt \{ (\omega, t) \in \Omega \times [0, T]; u(\omega, t) \neq v(\omega, t) \} \), where \( P \otimes dt \) is the product of \( P \) with the Lebesgue measure. It is clear that \( d \) defines a metric in \( U_{ad} \).

**Lemma 4.2.** i) \( U_{ad}, d \) is a complete metric space.

ii) For any \( p \geq 1 \), there exists \( M > 0 \) such that for any \( u, v \in U_{ad} \),

\[
E \left[ \sup_{0 \leq t \leq T} |X^u_t - X^v_t|^{2p} \right] \leq M. (d(u, v))^{1/2},
\]

where \( X^u_t, X^v_t \) are the solutions of [21] corresponding to \( u \) and \( v \).

iii) For any \( u, v \in U_{ad} \), \( |J(u) - J(v)| \leq C. (d(u, v))^{1/2} \).

**Proof.** The proof goes as in [21] Lemma 3.1 and uses classical arguments from stochastic calculus, such as Burkholder-Davis-Gundy, Hölder’s inequality and Gronwall’s lemma. The fact that the coefficients are of mean-field type depending on the expectation of the state, does not add new difficulties. \( \blacksquare \)
Let us define the Hamiltonian of the system associated to a random variable \( X \):

\[
H(t, X, E(X), u, p, q) = b(t, X, E(X), u).p + \sigma(t, X, E(X), u).q - h(t, X, E(X), u)
\]

For any strict control \( u \in \mathcal{U} \), we denote \((p, q)\) and \((P, Q)\) the first and second order adjoint processes satisfying the following backward SDEs

\[
\begin{align*}
    dp(t) &= -\left[b_x(t)p(t) + E(b_y(t)p(t)) + \sigma_x(t)q(t) + E(\sigma_y(t)q(t))\right] dt + q(t)dW_t + dM_t \\
    p(T) &= g_x(T) - E(g_y(T)) \\
    -dP(t) &= -\left[2b_x(t)P(t) + \sigma_x^2(t)P(t) + 2\sigma_x(t)Q(t) + H_{xx}(t)\right]dt + Q(t)dW_t + dN_t \\
    P(T) &= -g_{xx}(x(T))
\end{align*}
\]

where \( X(t) \) is the state process associated with \( u \), \( f_x(t) = f_x(t, X_t, E(X_t), u_t) \) for \( f = b, \sigma, h \) and \( M \) and \( N \) are square integrable martingales which are orthogonal to the Brownian motion and are parts of the solutions. The appearance of such martingales is due to the fact that \((\mathcal{F}_t)\) is not necessarily the Brownian filtration. Note that BSDEs (4.1) and (4.2) have been introduced for the first time in [4], without the orthogonal martingales \( M \) and \( N \).

Equation (4.1) is a mean field backward stochastic differential equation (MFBSDE), whose driver is Lipschitz continuous, then by [7] Theorem 3.1, it has a unique \( \mathcal{F}_t \)-adapted solution \((p, q, M)\) such that:

\[
E \left[ \sup_{0 \leq t \leq T} |p(t)|^2 + \int_0^T |q(t)|^2 dt + [M, M]_T \right] < +\infty
\]

Note that in [5] Theorem 3.1, \((\mathcal{F}_t)\) is the Brownian filtration. Considering general filtrations on which a Brownian motion is defined does not bring additional difficulties in the proof of existence and uniqueness (see e.g [9] Theorem 5.1, page 54).

Equation (4.2) is a classical backward stochastic differential equation, whose driver is Lipschitz continuous, then by [9] Theorem 5.1, it has a unique \( \mathcal{F}_t \)-adapted solution \((P, Q, N)\) such that:

\[
E \left[ \sup_{0 \leq t \leq T} |P(t)|^2 + \int_0^T |Q(t)|^2 dt + [N, N]_T \right] < +\infty.
\]

The following lemma is a stability result of the adjoints processes with respect to the control variable.

**Lemma 4.3.** For any \( 0 < \alpha < 1 \) and \( 1 < p < 2 \) satisfying \((1 + \alpha) < 2\), there exists a constant \( C_1 = C_1(\alpha, p) > 0 \) such that for any strict controls \((u, u')\) along with the corresponding trajectories \((X, X')\) and the solutions \((p, q, P, Q, M, N), (p', q', P', Q', M', N')\) of the backward SDEs (4.1) and (4.2), the following estimates hold

\[
E \left[ \int_0^T \left( |p(t) - p'(t)|^p + |q(t) - q'(t)|^p \right) dt + [M - M', M - M']_T^{p/2} \right] \leq C_1 d(u, u')^{\frac{p}{4}}
\]

\[
E \left[ \int_0^T \left( |P(t) - P'(t)|^p + |Q(t) - Q'(t)|^p \right) dt + [N - N', N - N']_T^{p/2} \right] \leq C_2 d(u, u')^{\frac{p}{4}}
\]

**Proof.**
The proof goes as in [21] Lemma 3.2. The only difference is that the driver is of mean-field type. But this does not add new difficulties, as the driver is linear and then Lipschitz in the state variable as well as in its expectation. 

\[\square\]
Let us define the $H$–function or generalized Hamiltonian, associated with a strict control $u$ and its state process $X$, is defined as follows:

\[
H(X(t),u(t),r(t),a) = H(t,Y,E(Y),a,p(t),q(t) - P(t)\sigma(t,X_t,E(X_t),u(t)))
\]

where $(p(t), q(t)), (P(t), Q(t))$ are solutions of the adjoint equations \[4.1\] and \[4.2\). According to the Chattering lemma and Proposition 3.9, for every relaxed optimal control $\mu$, and for every $\varepsilon_n > 0$, there exist a strict control $u^n$ such that:

\[
J(u^n) = J(\mu^n) \leq \inf \{J(\mu) : \mu \in \mathcal{R}\} + \varepsilon_n.
\]

$u^n$ is called an $\varepsilon_n$–optimal control.

The next Proposition gives necessary conditions for near optimality satisfied by an $\varepsilon_n$–optimal control.

**Proposition 4.4.** Let $u^n$ be an $\varepsilon_n$–optimal strict control. Then there exist adapted processes $(p^n, q^n, M^n)$ and $(P^n, Q^n, N^n)$, solutions of the adjoint equations \[4.1\] and \[4.2\], corresponding to the admissible pair $(u^n, X^n)$ such that:

\[
E \left( \int_0^T H(X(t),u^n(t),t,X^n(t),E(X^n(t)),u^n(t))dt \right) \geq \sup_{u \in \mathcal{A}} E \left( \int_0^T H(X(t),u(t),t,X^n(t),E(X^n(t)),u(t))dt \right) - T\varepsilon_n^{1/3}
\]

**Proof.** According to Lemma 4.2, the cost functional $J(u)$ is continuous with respect to the topology induced by the metric $d$. Then by applying Ekeland’s variational principle for $u^n$ with $\lambda_n = \varepsilon_n^{2/3}$, there exists an admissible control $v^n$ such that $d(u^n, v^n) \leq \varepsilon_n^{1/3}$ and $J(v^n) \leq J(u)$ for all $u \in \mathcal{U}$, where $\tilde{J}(u) = J(u) + \varepsilon_n^{1/3}d(u, v^n)$.

The control $v_n$, which is $\varepsilon_n$–optimal is in fact optimal for the new cost functional $\tilde{J}(u)$. We proceed as in the classical mean-field maximum principle \[4\] to derive a maximum principle for $v^n$. Let $t_0 \in (0, 1)$, $a \in \mathcal{A}$ and define the spike variation of $v^n = a$ on $(t_0, t_0 + \delta)$ and $v^n = v_n$ otherwise.

The fact that $\tilde{J}(v^n) \leq \tilde{J}(u)$ for all $u \in \mathcal{U}$ and $d(u^n, v^n) \leq \delta$ imply that $J(v^n) - J(u^n) \geq -\varepsilon_n^{1/3}\delta$.

Proceeding as in \[4\], we can expand $Y^n(t)$ (the solution of \[2.1\] corresponding to $v^n$) to the second order, to get the following inequality

\[
E \left[ \int_{t_0}^{t_0+\delta} \left( \sigma(t,Y^n(t),E(Y^n(t)),a) - \sigma(t,Y^n(t),E(Y^n(t)),v^n) \right)^2 P^n \right.
\]

\[
+ P^n \left( b(t,Y^n(t),E(Y^n(t)),a) - b(t,Y^n(t),E(Y^n(t)),v^n) \right)
\]

\[
+ 2P^n \left( \sigma(t,Y^n(t),E(Y^n(t)),a) - \sigma(t,Y^n(t),E(Y^n(t)),v^n) \right)
\]

\[- (h(t,Y^n(t),E(Y^n(t)),a) - h(t,Y^n(t),E(Y^n(t)),v^n)) \right) dt + o(\delta) \leq \varepsilon_n^{1/3}\delta
\]

where $Y^n(t)$ is the state process (solution of \[3.3\]) corresponding to the control $v^n$ and $(p^n, q^n)$ and $(P^n, Q^n)$ are the first and second order adjoint processes, solutions of \[4.1\] and \[4.2\] corresponding to $(v^n, Y^n)$.

The variational inequality is obtained for $v^n$ by dividing by $\delta$ and tending $\delta$ to 0.

The same claim can be proved for $u^n$ by using the stability of the state equations and the adjoint processes with respect to the control variable (Lemma 4.2 and Lemma 4.3). \[\blacksquare\]

**Remark 4.5.** The variational inequality \[2.1\] can be proved with the supremum over $a \in \mathcal{A}$ replaced by the supremum over $u \in \mathcal{U}$, by simply putting $u(t)$ in place of $a$ in the definition of the strong perturbation.

### 4.2 The relaxed maximum principle

We know that the relaxed control problem has an optimal solution $\mu$. Let $X$ be the corresponding optimal state process. Let $(p,q,M)$ and $(P,Q,N)$ the solutions of the first and second order adjoint equations, associated with the optimal relaxed pair $(\mu, X)$.

\[
\begin{align*}
dp(t) &= -\bar{p}_y(t)p(t) + E(\bar{p}_y(t)p(t)) + \sigma_x(t)q(t) + E(\bar{\sigma}_x(t)q(t)) \\
- \bar{p}_x(t) &= - E(\bar{p}_x(t)) \ dt + q(t)dW_t + dM_t \\
p(T) &= -\bar{p}_x(T) - E(\bar{p}_x(T))
\end{align*}
\]

\[
\begin{align*}
-dP(t) &= - \bar{p}_y(t)P(t) + \sigma_y(t)P(t) + 2\bar{\sigma}_y(t)Q(t) + \bar{H}_{yx}(t)dt \\
+ Q(t)dW_t + dN_t \\
P(T) &= -\bar{H}_{yx}(x(T))
\end{align*}
\]
where we denote $T(t) = f(t, x(t), E(x(t)), \mu(t)) = \int_A f(t, x(t), E(x(t)), a)\mu(t, da)$ and $f$ stands for $b_x, \sigma_x, h_x, b_y, \sigma_y, h_y, H_{xx}$.

$(M_t)$ and $(N_t)$ are square integrable martingales which are orthogonal to the Brownian motion $(W_t)$.

The drivers of the BSDEs (4.4) and (4.5) being Lipschitz continuous, then by Theorem 3.1 and Theorem 5.1, they admit unique solutions $(p, q, M)$ and $(P, Q, N)$ satisfying:

$$
E \left[ \sup_{0 \leq t \leq T} |p(t)|^2 + \int_0^T |q(t)|^2 \, dt + [M, M]_T \right] < +\infty,
$$

$$
E \left[ \sup_{0 \leq t \leq T} |P(t)|^2 + \int_0^T |Q(t)|^2 \, dt + [N, N]_T \right] < +\infty.
$$

Define the generalized Hamiltonian function associated with the optimal pair $(\mu, X(\cdot))$ and the corresponding adjoint processes,

$$
H^{(X(\cdot)), (\mu(\cdot))}(t, Y, E(Y), a) = H(t, Y, E(Y), a, p(t), q(t) - P(t).\sigma(t, X(t), E(X(t)), \mu(t)))
$$

$$
-\frac{1}{2}\sigma^2(t, Y, E(Y), a)P(t)
$$

The main result of this paper is the following.

**Theorem 4.6.** (Relaxed maximum principle)

Let $(\mu, X)$ be an optimal relaxed pair, then there exist adapted processes $(p, q, M)$ and $(P, Q, N)$, solutions of the adjoint equations (4.6) and (4.7) respectively, such that

$$
E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \mu(t)) \, dt \right) = \sup_{a \in A} E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), a) \, dt \right) \tag{4.6}
$$

The proof of Theorem 4.6 will be given later.

**Corollary 4.7.** Under the same conditions as in Theorem 4.6 it holds that

$$
E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \mu(t)) \, dt \right) = \sup_{\mu \in \mathcal{P}(A)} E \left[ \int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \mu(t)) \, dt \right]
$$

where $H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \mu(t)) = \int_A H^{(X(t), \mu(t))}(t, X(t), a)\nu(da)$ and $\mathcal{P}(A)$ is the space of probability measures on $A$.

**Proof.** The inequality

$$
\sup_{\mu \in \mathcal{P}(A)} E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \nu) \, dt \right) \geq \sup_{a \in A} E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), a) \, dt \right)
$$

is obvious. Indeed it suffices to take $\mu = \delta_a$ where $a$ is any element of $A$. Now if $\nu \in \mathcal{P}(A)$ is a probability measure on $A$, then

$$
\int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \nu) \, dt \in \text{conv} \left\{ E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), a) \, dt \right), a \in A \right\}
$$

Hence, by using Fubini’s theorem, it holds that

$$
\int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \nu) \, dt \leq \sup_{a \in A} E \left( \int_0^T H^{(X(t), \mu(t))}(t, X(t), E(X(t)), a) \, dt \right)
$$

**Remark.** Since $\mathcal{P}(A)$ is a subspace of $\mathbb{V}$ whose elements are constant (in $(\omega, t)$) relaxed controls, then one can replace in Corollary 4.8, the supremum over $\mathcal{P}(A)$ by the supremum over $\mathbb{V}$.

**Corollary 4.8.** (The Pontriagin relaxed maximum principle). If $(\mu, X)$ denotes an optimal relaxed pair, then there exists a Lebesgue negligible subset $N$ such that for any $t$ not in $N$

$$
H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \mu(t)) = \sup_{\mu \in \mathcal{P}(A)} H^{(X(t), \mu(t))}(t, X(t), E(X(t)), \nu), \quad P-a.s.
$$

**Proof.** Let $\theta \in [0, T]$ and $B \in \mathcal{F}_\theta$, for small $h > 0$ define the relaxed control $\mu^h_t = \nu 1_B$ for $\theta < t < \theta + h$ and $\mu^h_\theta = \mu_\theta$ otherwise, where $\nu$ is a probability measure on $A$. It follows from Theorem 4.6, that
Proof. i) Let us write down the drivers of the first order adjoint equations (4.4) and (4.5) associated to $(\mu, X)$ for any $\theta$ (adjoint equations (4.1) and (4.2) associated with the pair $(\mu, X)$). Let us treat the first term in the right hand side of (4.7).

\[ f_0^T \left( |p^n(t) - p(t)|^2 + |q^n(t) - q(t)|^2 \right) dt + |M^n - M|_T = 0 \]

where

\[ G^n(t, p^n, q^n) = -b^n_x(t)p^n(t) + E(b^n_x(t)p^n(t)) + \sigma^n_x(t)q^n(t) + E(\sigma^n_x(t)q^n(t)) - h^n_y(t) - E(h^n_y(t)) \]

\[ G(t, p, q) = -b_x(t)p(t) + E(\bar{b}_y(t))p(t) + \bar{\sigma}_x(t)q(t) + E(\bar{\sigma}_y(t)q(t)) - \bar{b}_x(t) - E(\bar{b}_y(t)) \]

where

\[ f^n(t) = \int f(t, X^n_t, E(X^n_t), u^n) \, da \]

and

\[ \bar{f}(t) = \int f(t, X^n_t, E(X^n_t), \mu(t)) \, da \]

for $f$ stands for $b_x, \sigma_x, b_y, \sigma_y, h_y$.

By using a sublarity stability result of Hu and Peng [11], Theorem 2.1, it is sufficient to show that:

\[ \lim_{n \to \infty} E \left[ \int_0^T \left( G^n(t, p^n, q^n) - G(t, p, q) \right) dt \right]^2 = 0 \]

We have

\[ \left| \int_0^T \left( G^n(t, p^n, q^n) - G(t, p, q) \right) dt \right| \leq \int_0^T \left( |b^n_x(t) - b_x(t)| \right) \left| p(t) \right| dt \]

\[ + \int_0^T \left( |\sigma^n_x(t) - \sigma_x(t)| \right) \left| q(t) \right| dt \]

\[ + \int_0^T \left( |h^n_y(t) - \bar{b}_y(t)| \right) dt \]

Let us treat the first term in the right hand side of (4.7).

\[ \int_0^T \left( b^n_x(t) - b_x(t) \right) \right| \left| p(t) \right| dt \]

\[ = \int_0^T \left( \int f(t, X^n_t, E(X^n_t), a) \delta_\omega^n (da) \right) - \int f(t, X^n_t, E(X^n_t), a) \mu (da) \right) \left| p(t) \right| dt \]

\[ = \int_0^T \left( \int f(t, X^n_t, E(X^n_t), a) \delta_\omega^n (da) \right) \left| p(t) \right| dt \]

\[ + \int_0^T \left( \int f(t, X^n_t, E(X^n_t), a) \mu (da) \right) \left| p(t) \right| dt \]
The facts that \( b_x \) is Lipschitz continuous in \((x, y)\) and \((X^n_t)\) converges to \(X_t\) uniformly in \(t\) in probability, imply that the first term in the right hand side of the last inequality converges to 0 in probability.

Furthermore \( E \left( \sup_{0 \leq t \leq T} |p(t)|^2 \right) < +\infty \), then \( \sup_{0 \leq t \leq T} |p(t)| < +\infty, \) \( P - a.s., \) which implies the existence of a \( P \)-negligible set \( N, \) such that for each \( \omega \notin N, \) there exist \( M(\omega) < +\infty \) \( s.t \) \( \sup_{0 \leq t \leq T} |p(t)| \leq M(\omega). \) In particular for each \( \omega \notin N, \) the function \( b_x(t, X_t, E(X_t), a)p(t)1_{[0,1]} \) is a measurable bounded function in \((t, a)\) and continuous in \(a, \) therefore it is a test function for the stable convergence. Hence by using the fact that \( (\delta_\omega(da)) dt \) converges in \( \mathbb{V} \) to \( \mu_\omega(da) dt, \) \( P - a.s., \) it follows that the second term in the right hand side tends to 0, \( P - a.s. \)

We conclude by using the Lebesgue dominated convergence theorem.

The other terms containing \( p(t) \) can be handled by using the same techniques.

The terms in \((4.7)\) containing \( q(t) \) can be treated similarly. However one should pay a little more attention as \( q(t) \) is just square integrable \( (in (t, \omega)) \). More precisely

\[
\left| \int_0^T (\sigma_n^a(t) - \sigma_x(t)) q(t) dt \right| \leq \left| \int_0^T (\sigma_n^a(t) - \sigma_x(t)) q(t)1_{\{|q(t)| \leq N\}} dt \right| + \left| \int_0^T (\sigma_n^a(t) - \sigma_x(t)) q(t)1_{\{|q(t)| \geq N\}} dt \right|
\]

The first integral in the right hand side may be handled by using similar argument as precedently as the function \( (\sigma_n^a(t) - \sigma_x(t)) q(t)1_{\{|q(t)| \leq N\}} \) is measurable bounded and continuous in \(a.\) The second term tends to 0 by Tchebychev’s inequality, using the square integrability of \( q(t). \)

ii) and iii) are proved by using similar arguments. \( \blacksquare \)

**Proof of Theorem 4.9.** The result is proved by passing to the limit in inequality \((4.3)\) and using Lemma 4.9.

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