Forbidding Couples of Tournaments and the Erdős–Hajnal Conjecture

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Abstract

A celebrated unresolved conjecture of Erdős and Hajnal states that for every undirected graph $H$ there exists $\epsilon(H) > 0$ such that every undirected graph on $n$ vertices that does not contain $H$ as an induced subgraph contains a clique or a stable set of size at least $n^{\epsilon(H)}$. This conjecture has a directed equivalent version stating that for every tournament $H$ there exists $\epsilon(H) > 0$ such that every $H$-free $n$-vertex tournament $T$ contains a transitive subtournament of size at least $n^{\epsilon(H)}$. Recently the conjecture was proved for all six-vertex tournaments, except $K_6$. In this paper we construct two infinite families of tournaments for which the conjecture is still open for infinitely many tournaments in these two families—the family of so-called super nebulas and the family of so-called super triangular galaxies. We prove that for every super nebula $H_1$ and every $\Delta$-galaxy $H_2$ there exist $\epsilon(H_1, H_2)$ such that every $\{H_1, H_2\}$-free tournament $T$ contains a transitive subtournament of size at least $|T|^{\epsilon(H_1, H_2)}$. We also prove that for every central triangular galaxy $H$ there exist $\epsilon(K_6, H)$ such that every $\{K_6, H\}$-free tournament $T$ contains a transitive subtournament of size at least $|T|^{\epsilon(K_6, H)}$. And we give an extension of our results.

Keywords  Erdős–Hajnal conjecture · Ordering · Tournament · Transitive set

Mathematics Subject Classification  05C20 · 05C69
1 Introduction

Let $G$ be an undirected graph. We denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. We call $|G| = |V(G)|$ the size of $G$. A clique in $G$ is a set of pairwise adjacent vertices and a stable set in $G$ is a set of pairwise nonadjacent vertices.

A digraph is a pair $D = (V, E)$ of sets such that $E \subseteq V \times V$, and such that for every $(x, y) \in E$ we must have $(y, x) \notin E$, in particular if $(x, y) \in E$ then $x \neq y$. $E$ is the arc set and $V$ is the vertex set and they are denoted by $E(D)$ and $V(D)$ respectively.

Let $D'$ be a subdigraph of a digraph $D$ if $V(D') \subseteq V(D)$ and $E(D') \subseteq E(D)$.

Let $X \subseteq V(D)$, the subdigraph of $D$ induced by $X$ is denoted by $D \mid X$, that is the digraph with vertex set $X$, such that for $x, y \in X$, $(x, y) \in E(D \mid X)$ if and only if $(x, y) \in E(D)$. Denote by $D \setminus X$ the subdigraph of $D$ induced by $V(D) \setminus X$.

We say that $D$ contains $D'$ if $D'$ is isomorphic to a subdigraph of $D$. A tournament is a directed graph (digraph) such that for every pair $u$ and $v$ of vertices, exactly one of the arcs $(u, v)$ or $(v, u)$ exists. A tournament is transitive if it contains no directed cycle. Let $T$ be a tournament. We write $|T|$ for $|V(T)|$ and we say that $|T|$ is the size of $T$. If $(u, v) \in E(T)$ then we say that $u$ is adjacent to $v$ (alternatively: $v$ is an outneighbor of $u$) and we write $u \to v$, also we say that $v$ is adjacent from $u$ (alternatively: $u$ is an inneighbor of $v$) and we write $v \leftarrow u$.

For two sets of vertices $V_1, V_2$ of $T$ we say that $V_1$ is complete to (resp. from) $V_2$ if every vertex of $V_1$ is adjacent to (resp. from) every vertex of $V_2$. We write $V_1 \to V_2$ (resp. $V_1 \leftarrow V_2$).

We say that a vertex $v$ is complete to (resp. from) a set $V$ if $|v|$ is complete to (resp. from) $V$ and we write $v \to V$ (resp. $v \leftarrow V$). Given a tournament $H$, we say that $T$ contains $H$ if $H$ is isomorphic to $T \mid X$ for some $X \subseteq V(T)$.

Erdős and Hajnal proposed the following conjecture (EHC) [6]:

Conjecture 1 For any undirected graph $H$ there exists $\epsilon(H) > 0$ such that every $n$-vertex undirected graph that does not contain $H$ as an induced subgraph contains a clique or a stable set of size at least $n^{\epsilon(H)}$.

In 2001 Alon et al. proved [1] that Conjecture 1 has an equivalent directed version, as follows:

Conjecture 2 For any tournament $H$ there exists $\epsilon(H) > 0$ such that every $H$-free tournament with $n$ vertices contains a transitive subtournament of size at least $n^{\epsilon(H)}$.

A class of tournaments $\mathcal{F}$ satisfy the Erdős–Hajnal conjecture (EHC) (equivalently: $\mathcal{F}$ has the Erdős–Hajnal property) if there exists $\epsilon(\mathcal{F}) > 0$ such that every $\mathcal{F}$-free tournament $T$ with $n$ vertices contains a transitive subtournament of size at least $n^{\epsilon(\mathcal{F})}$.

If $\{H\}$ satisfy EHC we simply say that $H$ satisfies EHC.

Instead of forbidding just one tournament, one can state the analogous conjecture where we forbid two tournaments. The only results in this setting are in [5, 8].

Conjecture 2 has not yet been proved for super nebulas, $K_6$, and super triangular galaxies. That motivates the work of this paper. In this paper we prove that $\{H_1, H_2\}$-free tournaments $T$ contain transitive subtournaments of size at least $|T|^{\epsilon(H_1,H_2)}$ for some $\epsilon(H_1, H_2) > 0$ and infinite number of couples of tournaments $\{H_1, H_2\}$. Before stating formally our results, we need to introduce some definitions and notations.

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Let \( \theta = (v_1, \ldots, v_n) \) be an ordering of the vertex set \( V(D) \) of an \( n \)-vertex digraph \( D \). An arc \((v_i, v_j) \in E(D)\) is a backward arc of \( D \) under \( \theta \) if \( i > j \). We say that a vertex \( v_j \) is between two vertices \( v_i, v_k \) under \( \theta = (v_1, \ldots, v_n) \) if \( i < j < k \) or \( k < j < i \). The graph of backward arcs under \( \theta \), denoted by \( B(D, \theta) \), is the undirected graph that has vertex set \( V(D) \), and \( v_i v_j \in E(B(D, \theta)) \) if and only if \((v_i, v_j) \) is a backward arc of \( D \) under \( \theta \). A tournament \( S \) on \( p \) vertices with \( V(S) = \{u_1, u_2, \ldots, u_p\} \) is a right star (resp. left star) if there exist an ordering \( \theta^* = (u_1, u_2, \ldots, u_p) \) of its vertices such that the backward arcs of \( S \) under \( \theta^* \) are \((u_p, u_i)\) for \( i = 1, \ldots, p-1 \) (resp. \((u_i, u_1)\) for \( i = 2, \ldots, p \)) (resp. \((u_i, u_r)\) for \( i = r+1, \ldots, p \) and \((u_r, u_i)\) for \( i = 1, \ldots, r-1 \), where \( 2 \leq r \leq p-1 \)). In this case we write \( S = \{u_1, u_2, \ldots, u_p\} \) and we call \( \theta^* = (u_1, u_2, \ldots, u_p) \) a right star ordering (resp. left star ordering) of \( S, \) \( u_p \) (resp. \( u_1 \)) (resp. \( u_r \) or \( u_p \)) the center of \( S \), and \( u_1, \ldots, u_{p-1} \) (resp. \( u_2, \ldots, u_p \)) the leaves of \( S \). A star is a left star or a right star or a middle star. A star ordering is a left star ordering or a right star ordering or a middle star ordering. Note that in the case \( p = 2 \) we may choose arbitrarily any one of the two vertices to be the center of the star, and the other vertex is then considered to be the leaf. A frontier star is a left star or a right star (note that a frontier star is not a middle star, a frontier star is either left or right). A star \( S = \{v_{i_1}, \ldots, v_{i_t}\} \) of \( D \) under \( \theta \) (where \( i_1 < \cdots < i_t \)) is the subdigraph of \( D \) induced by \( \{v_{i_1}, \ldots, v_{i_t}\} \) such that \( S \) is a star and \( S \) has the star ordering \((v_{i_1}, \ldots, v_{i_t})\) under \( \theta \) (i.e. \((v_{i_1}, \ldots, v_{i_t})\) is the restriction of \( \theta \) to \( V(S) \) and \((v_{i_1}, \ldots, v_{i_t})\) is a star ordering of \( S \)).

A tournament \( T \) is a galaxy if there exists an ordering \( \theta \) of its vertices such that \( V(T) \) is the disjoint union of \( V(Q_1), \ldots, V(Q_t) \), where \( Q_1, \ldots, Q_t \) are the frontier stars of \( T \) under \( \theta \), and for every \( x \in X \), \( \{x\} \) is a singleton component of \( B(T, \theta) \), and no center of a star is between leaves of another star under \( \theta \). In this case we also say that \( T \) is a galaxy under \( \theta \). If \( X = \emptyset \), we say that \( T \) is a regular galaxy under \( \theta \).

The condition concerning the type of the stars in galaxies (that is middle stars are not allowed) and that no center of a star appears in the ordering between leaves of another star is necessary to make the proof of the following theorem work, and it remains an open problem if these conditions are abandoned:

**Theorem 1** [2] Every galaxy satisfies the Erdős–Hajnal conjecture.

A tournament \( T \) is a nebula if there exists an ordering \( \theta \) of its vertices such that \( V(T) \) is the disjoint union of \( V(Q_1), \ldots, V(Q_t) \), where \( Q_1, \ldots, Q_t \) are the stars of \( T \) under \( \theta \), and for every \( x \in X \), \( \{x\} \) is a singleton component of \( B(T, \theta) \) (the star does not have to be necessarily left or right, and there is no condition concerning the location of the centers of the stars). We also say that \( T \) is a nebula under \( \theta \). If \( X = \emptyset \), we say that \( T \) is a regular nebula. Notice that every galaxy is a nebula. The following is still a conjecture:

**Conjecture 3** Every nebula satisfies the Erdős–Hajnal conjecture.

In 2015 Choromanski confirmed Conjecture 3 for every constellation [4] (every constellation is a nebula). Indeed, in constellation Choromanski relaxed the condition concerning the location of the centers of the stars to a much weaker one, but middle stars are still not allowed. Constellations are fully characterized in [4]. Recently we proved
Conjecture 3 for every galaxy with spiders [9]. In galaxies with spiders we relax both conditions, we allow the existence of middle stars and having center of a star between leaves of another star under some conditions. Note that the relaxed condition for the centers in constellation in different than that in galaxy with spiders. Galaxies with spiders are fully characterized in [9] (every galaxy with spiders is a nebula).

In what follows we define formally the family of super nebulas and the family of super triangular galaxies.

To make it easier define the family of super nebulas, we would like first to define special nebulas called 2-nebulas (2 stands for the number of stars in the 2-nebulas), and tournaments called super 2-nebulas obtained from 2-nebulas by reversing the orientation of one arc. Let $s$ be a $\{0,1\}$-vector. Denote $s_c$ the vector obtained from $s$ by replacing every subsequence of consecutive 1’s by single 1. Let $T$ be a regular nebula under $\theta = (v_1, \ldots, v_n)$ and let $Q_1, \ldots, Q_l$ be the stars of $T$ under $\theta$. Let $s^{T, \theta}$ be the $\{0,1\}$-vector such that $s^{T, \theta}_c = 1$ if and only if $v_i$ is a leaf of one of the stars of $T$ under $\theta$. A tournament $T$ is a 2-nebula if it is a regular nebula under $\theta$ and besides $V(T)$ is the disjoint union of $V(Q_1)$ and $V(Q_2)$, where $Q_1$ and $Q_2$ are the frontier stars of $T$ under $\theta$, and $s^{T, \theta}_c$ is one of the following vectors: $(0,0,1)$, $(0,1,0)$, or $(1,0,0)$. A 2-nebula $T$ under $\theta$ is a left (resp. middle) (resp. right) 2-nebula under $\theta$ if $s^{T, \theta}_c = (1,0,0)$ (resp. $s^{T, \theta}_c = (0,1,0)$) (resp. $s^{T, \theta}_c = (0,0,1)$). A tournament $H$ is a super left 2-nebula (resp. super middle 2-nebula) (resp. super right 2-nebula) under $\theta = (v_1, \ldots, v_n)$ if it is obtained from a left 2-nebula (resp. middle 2-nebula) (resp. right 2-nebula) $T$ under $\theta$ by reversing the orientation of the arc joining the centers of the two frontier stars of $T$ under $\theta$. In this case we write $H = \{v_1, \ldots, v_n\}$, we call $\theta$ a super left 2-nebula ordering (resp. super middle 2-nebula ordering) (resp. super right 2-nebula ordering) of $H$, the leaves of the two frontier stars of $T$ under $\theta$ are called the leaves of $H$, and the centers of the two frontier stars of $T$ under $\theta$ are called the centers of $H$. A super 2-nebula is a super left 2-nebula or a super middle 2-nebula or a super right 2-nebula. A super 2-nebula ordering is a super left 2-nebula ordering or a super middle 2-nebula ordering or a super right 2-nebula ordering. Let $v_{q_1}$ and $v_{q_2}$ be the centers of $Q_1$ and $Q_2$ respectively. The leaves of $Q_1$ (resp. $Q_2$) are called the leaves of $H$ incident to the center $v_{q_1}$ (resp. $v_{q_2}$) of $H$.

Let $\theta = (v_1, \ldots, v_n)$ be an ordering of the vertex set $V(G)$ of an $n$-vertex tournament $G$. A super 2-nebula $T = \{v_{i_1}, \ldots, v_{i_l}\}$ of $G$ under $\theta$ is the subtournament of $G$ induced by $\{v_{i_1}, \ldots, v_{i_l}\}$ such that $T$ has the super 2-nebula ordering $(v_{i_1}, \ldots, v_{i_l})$ under $\theta$ (note that $i_1 < \cdots < i_l$). Now we are ready to define super nebulas:

A tournament $T$ is a super nebula if there exists an ordering $\theta$ of its vertices such that $V(T)$ is the disjoint union of $V(Q_1), \ldots, V(Q_m), V(\Sigma_1), \ldots, V(\Sigma_l), X$ where $Q_1, \ldots, Q_m$ are the stars of $T$ under $\theta$, $\Sigma_1, \ldots, \Sigma_l$ are the super 2-nebulas of $T$ under $\theta$, no center of a star is between leaves of a super 2-nebula under $\theta$, no center of a super 2-nebula is between leaves of another super 2-nebula under $\theta$, and for every $x \in X$, $\{x\}$ is a singleton component of $B(T, \theta)$. In this case we say that $T$ is a super nebula under $\theta$ and $\theta$ is called a super nebula ordering of $T$. If $X = \emptyset$ then $T$ is called regular super nebula (see Fig. 1). In case $l = 0$ (that is $\bigcup_{i=1}^{l} V(\Sigma_i) = \emptyset$), $T$ is a nebula.
Denote by $K_6$ the six-vertex tournament with $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ such that under ordering $(v_1, v_2, v_3, v_4, v_5, v_6)$ of its vertices the set of backward arcs is: $\{(v_4, v_1), (v_6, v_3), (v_6, v_1), (v_5, v_2)\}$. We call this ordering of vertices of $K_6$ the canonical ordering of $K_6$. $K_6$ is the only tournament on at most six vertices for which the conjecture is still open [3]. Notice that $K_6$ is obviously a super nebula and its canonical ordering is its super nebula ordering (see Fig. 2).

Let $\theta = (v_1, \ldots, v_n)$ be an ordering of the vertex set $V(T)$ of an $n$-vertex tournament $T$. A triangle $C = \{v_{i_1}, v_{i_2}, v_{i_3}\}$ of $T$ under $\theta$ is a transitive subtournament of $T$ induced by $\{v_{i_1}, v_{i_2}, v_{i_3}\}$ such that $(v_{i_3}, v_{i_2}, v_{i_1})$ is its transitive ordering (i.e. $v_{i_1} \prec v_{i_2}$ and $v_{i_1}, v_{i_2} \prec v_{i_3}$), and $i_1 < i_2 < i_3$. We call $v_{i_2}$ the center of the $C$, $v_{i_1}$ the left exterior of $C$, and $v_{i_3}$ the right exterior of $C$.

A tournament $T$ is a triangular galaxy if there exists an ordering $\theta$ of its vertices such that $V(T)$ is the disjoint union of $V(\Delta_1), \ldots, V(\Delta_l), X$ where $\Delta_1, \ldots, \Delta_l$ are the triangles of $T$ under $\theta$, and $T \mid X$ is a galaxy under $\theta_X$ where $\theta_X$ is the restriction of $\theta$ to $X$, and no vertex of a triangle is between leaves of a star of $T$ under $\theta$. In this case we say that $T$ is triangular galaxy under $\theta$ and $\theta$ is a triangular galaxy ordering of $T$. If $T \mid X$ is a regular galaxy under $\theta_X$ then we say that $T$ is a regular triangular galaxy under $\theta$. If $l = 1$ then $T$ is called $\Delta$galaxy under $\theta$ and $\theta$ is a $\Delta$galaxy ordering of $T$. If $l = 1$ and $T \mid X$ is a regular galaxy under $\theta_X$ then $T$ is called regular $\Delta$galaxy under $\theta$. If for every $x \in X, \{x\}$ is a singleton component of $B(T, \theta)$ we say that $T$ is a triangular tournament. If $X = \phi$, we say that $T$ is a regular triangular tournament. If the condition concerning the location of the vertices of the triangles is weakened such that the centers (resp. right exteriors) (resp. left exteriors) of the triangles are allowed to be between leaves of a star of $T$ under $\theta$ then $T$ is called central triangular galaxy under $\theta$ (resp. right triangular galaxy under $\theta$) (resp. left triangular galaxy under $\theta$) and $\theta$ is called a central triangular galaxy ordering of $T$ (resp. right triangular galaxy ordering of $T$) (resp. left triangular galaxy ordering of $T$) (see Fig. 3). If the condition concerning the vertices of the triangles is abandoned or weakened then $T$ is called a super triangular galaxy under $\theta$.

Unfortunately the following are still open:

**Problem 1** Does the tournament $K_6$ satisfy EHC?
Problem 2 Does every nebula (resp. super nebula) satisfy EHC?

Problem 3 Does every triangular tournament (resp. triangular galaxy) (resp. $\Delta$ galaxy) (resp. super triangular galaxy) satisfy EHC?

However if we exclude both:

– An arbitrary super nebula and an arbitrary $\Delta$ galaxy, or
– An arbitrary central triangular galaxy and $K_6$, or
– An arbitrary middle $\Sigma$-galaxy and an arbitrary central triangular galaxy, or
– An arbitrary left $\Sigma$-galaxy and an arbitrary left triangular galaxy, or
– An arbitrary right $\Sigma$-galaxy and an arbitrary right triangular galaxy,

then we proved that the conjecture is satisfied. More results in this setting concerning nebulas, super nebulas, and super triangular galaxies are formally stated in Sect. 5.

Middle (resp. left) (resp. right) $\Sigma$-galaxies are defined in Sect. 5. Every middle (resp. left) (resp. right) $\Sigma$-galaxy is a super nebula.

In [8] we proved that some 7-vertex super triangular galaxy tournaments satisfy EHC and in [9] we proved that if the super nebula tournament with some additional conditions, then it satisfies EHC.

The main results of this paper are the following, and it turns out that the technique of key tournaments remains useful for the extensions of the following two theorems:

Theorem 2 If $\mathcal{N}$ is a super nebula and $\mathcal{G}$ is a $\Delta$ galaxy, then $\{\mathcal{N}, \mathcal{G}\}$ satisfy the Erdős–Hajnal Conjecture.

Theorem 3 If $H$ is a central triangular galaxy, then $\{K_6, H\}$ satisfy the Erdős–Hajnal Conjecture.

The results concerning the extensions of Theorems 2 and 3 are formally stated in Sect. 5.

Let $T$ be a nebula under $\theta$ and let $Q_1, \ldots, Q_l$ be the stars of $T$ under $\theta$. $T$ is a left nebula (resp. right nebula) (resp. central nebula) under $\theta$ if for all $i \in \{1, \ldots, l\}$, $Q_i$ is a 3-vertex left star (resp. 3-vertex right star) (resp. 3-vertex middle star). In the setting of forbidding pairs of tournaments, Choromanski proved [5] the following:

Theorem 4 [5] If $H_1$ and $H_2$ are: a left nebula and a right nebula, or: a left nebula and a central nebula, or: a right nebula and a central nebula, then $\{H_1, H_2\}$ satisfies the Erdős–Hajnal Conjecture.

This paper is organized as follows:
In Sect. 2 we give some definitions and preliminary lemmas, and we prove some lemmas needed in the proof of the main results in this paper.
In Sect. 3 we introduce some tools useful in the proof of Theorem 2 and we prove Theorem 2.
In Sect. 4 we introduce some definitions and we prove Theorem 3.
In Sect. 5 we give extensions of our results.

2 Definitions and Preliminary Lemmas

Denote by $tr(T)$ the largest size of a transitive subtournament of a tournament $T$. For $X \subseteq V(T)$, write $tr(X)$ for $tr(T \mid X)$. Let $X, Y \subseteq V(T)$ be disjoint. Denote by $e_{X,Y}$ the number of directed arcs $(x, y)$, where $x \in X$ and $y \in Y$. The directed density from $X$ to $Y$ is defined as $d(X, Y) = \frac{e_{X,Y}}{|X||Y|}$. We call $T$ $\epsilon$-critical for $\epsilon > 0$ if $tr(T) < |T|^\epsilon$ but for every proper subtournament $S$ of $T$ we have: $tr(S) \geq |S|^\epsilon$.

Lemma 1 [7] Every tournament on $2^{k-1}$ vertices contains a transitive subtournament of size at least $k$.

Lemma 2 [2] For every $N > 0$, there exists $\epsilon(N) > 0$ such that for every $0 < \epsilon < \epsilon(N)$ every $\epsilon$-critical tournament $T$ satisfies $|T| \geq N$.

Lemma 3 [2] Let $T$ be an $\epsilon$-critical tournament with $|T| = n$ and $\epsilon, c > 0$ be constants such that $\epsilon < \log_2(\frac{1}{2})$. Then for every two disjoint subsets $X, Y \subseteq V(T)$ with $|X| \geq cn$, $|Y| \geq cn$ there exist an integer $k \geq \frac{cn}{2}$ and vertices $x_1, \ldots, x_k \in X$ and $y_1, \ldots, y_k \in Y$ such that $y_i$ is adjacent to $x_i$ for $i = 1, \ldots, k$.

Lemma 4 [2] Let $T$ be an $\epsilon$-critical tournament with $|T| = n$ and $\epsilon, c, f > 0$ be constants such that $\epsilon < \log_2(1 - f)$. Then for every $A \subseteq V(T)$ with $|A| \geq cn$ and every transitive subtournament $G$ of $T$ with $|G| \geq f \cdot tr(T)$ and $V(G) \cap A = \emptyset$, we have: $A$ is not complete from $V(G)$ and $A$ is not complete to $V(G)$.

Lemma 5 Let $f_1, \ldots, f_m, l_1, \ldots, l_t, c, \epsilon > 0$ be constants, where $0 < f_1, \ldots, f_m, l_1, \ldots, l_t, c < 1$ and $0 < \epsilon < \min\{\log_2(\frac{1}{2}), (1 - f_1), \ldots, (1 - f_m), \log_2(\frac{1}{2}) (1 - l_1), \ldots, \log_2(\frac{1}{2}) (1 - l_t)\}$. Let $T$ be an $\epsilon$-critical tournament with $|T| = n$, and let $S_1, \ldots, S_m, P_1, \ldots, P_t$ be $m + t$ disjoint transitive subtournaments of $T$ with $m, t \in \mathbb{N}$, $|S_i| \geq f_i \cdot tr(T)$ for $i = 1, \ldots, m$, and $|P_i| \geq l_i \cdot tr(T)$ for $i = 1, \ldots, t$. Let $A \subseteq V(T) \setminus ((\bigcup_{i=1}^m V(S_i)) \cup (\bigcup_{i=1}^t V(P_i)))$ with $|A| \geq cn$. Then there exist vertices $s_1, \ldots, s_m, p_1, \ldots, p_t, g$ such that $g \in A$, $s_i \in S_i$ for $i = 1, \ldots, m$, $p_i \in P_i$ for $i = 1, \ldots, t$, and $\{s_1, \ldots, s_m\} \leftarrow g \leftarrow \{p_1, \ldots, p_t\}$.

Proof Let $A_i \subseteq A$ such that $A_i$ is complete from $S_i$ for $i = 1, \ldots, m$ and let $A^i \subseteq A$ such that $A^i$ is complete to $P_i$ for $i = 1, \ldots, t$. Let $1 \leq j \leq m$. If $|A_j| \geq \frac{|A|}{2^{(m+t)}} \geq \frac{c}{2^{(m+t)}} n$, then this will contradicts Lemma 4 since $|S_j| \geq f_j \cdot tr(T)$ and $\epsilon < \log_2(\frac{1}{2}) (1 - f_j)$. Then $\forall i \in \{1, \ldots, m\}, |A_i| < \frac{|A|}{2^{(m+t)}}$. Similarly we prove that $\forall i \in \{1, \ldots, t\}, |A^i| < \frac{|A|}{2^{(m+t)}}$. Let $A^* = A \setminus ((\bigcup_{i=1}^m A_i) \cup (\bigcup_{i=1}^t A^i))$, then $|A^*| > |A| - (m+t). \frac{|A|}{2^{(m+t)}} \geq \frac{|A|}{2}$. Then $A^* \neq \emptyset$. Fix $g \in A^*$. So there exist vertices
Lemma 6 Let $f, c, t, \epsilon > 0$ be constants, where $0 < f, c < 1$, $t$ a positive integer, and $0 < \epsilon < \min\{\log \frac{1}{2}\sqrt[4]{t}, \log \frac{1}{2}\sqrt[4]{t}\}$. Let $T$ be an $\epsilon$-critical tournament with $|T| = n$, and let $S_1, \ldots, S_m, S_{m+1}, \ldots, S_t$ be $t$ disjoint transitive subtournaments of $T$ with $|S_i| \geq f.\text{tr}(T)$ for $i = 1, \ldots, t$. Let $A_1, A_2$ be two disjoint subsets of $V(T)$ with $|A_1| \geq cn$, $|A_2| \geq cn$, and $A_1, A_2 \subseteq V(T) \setminus \bigcup_{i=1}^t S_i$. Then there exist vertices $x, y, s_1, \ldots, s_t$ such that $x \in A_1, y \in A_2, s_i \in S_i$ for $i = 1, \ldots, t, \{u_1, \ldots, u_m, p\} \leftarrow q,$ and $\{u_{m+1}, \ldots, u_t\} \leftarrow p$. Similarly there exist vertices $v, z_1, \ldots, z_t$ such that $g \in A_1, v \in A_2, z_i \in S_i$ for $i = 1, \ldots, t, g \leftarrow \{v, z_1, \ldots, z_m\}$, and $v \leftarrow \{z_{m+1}, \ldots, z_t\}$.

Proof We will prove only the first statement because the rest can be proved analogously. Let $A^*_1 = \{x \in A_1; \exists s_i \in S_i \text{ for } i = 1, \ldots, m \text{ and } x \leftarrow \{s_1, \ldots, s_m\}\}$ and let $A^*_2 = \{y \in A_2; \exists s_i \in S_i \text{ for } i = m+1, \ldots, t \text{ and } \{s_{m+1}, \ldots, s_t\} \leftarrow y\}$. Since $\epsilon < \log \frac{1}{2}\sqrt[4]{t} - f$, then $|A^*_1| > \frac{|A_1|}{\log c} \geq \frac{c}{2}n$ and $|A^*_2| > \frac{|A_2|}{\log c} \geq \frac{c}{2}n$. Now since $\epsilon < \log \frac{1}{2}\sqrt[4]{t}$, then Lemma 3 implies that $\exists k \geq \frac{c}{2}n, \exists x_1, \ldots, x_k \in A^*_1, \exists y_1, \ldots, y_k \in A^*_2$, such that $x_i \leftarrow y_i$ for $i = 1, \ldots, k$. Fix $i_1 \in \{1, \ldots, k\}$. So, there exist vertices $s_1, \ldots, s_t$ such that $s_i \in S_i$ for $i = 1, \ldots, t, x_{i_1} \leftarrow \{y_{i_1}, s_1, \ldots, s_m\}$, and $\{s_{m+1}, \ldots, s_t\} \leftarrow y_{i_1}$.

Lemma 7 [2] Let $A_1, A_2$ be two disjoint sets such that $d(A_1, A_2) \geq 1 - \lambda$ and let $0 < \eta_1, \eta_2 \leq 1$. Let $\hat{\lambda} = \frac{\lambda}{\eta_1 \eta_2}$. Let $X \subseteq A_1, Y \subseteq A_2$ be such that $|X| \geq \eta_1$ $|A_1|$ and $|Y| \geq \eta_2$ $|A_2|$. Then $d(X, Y) \geq 1 - \hat{\lambda}$.

The following is introduced in [3].

Let $c > 0, 0 < \lambda < 1$ be constants, and let $w$ be a $\{0, 1\}$-vector of length $|w|$. Let $T$ be a tournament with $|T| = n$. A sequence of disjoint subsets $\chi = (S_1, S_2, \ldots, S_{|w|})$ of $V(T)$ is a smooth $(c, \lambda, w)$-structure if:

- whenever $w_i \gamma 0$ we have $|S_i| \geq cn$ (we say that $S_i$ is a linear set).
- whenever $w_i = 1$ the tournament $T | S_i$ is transitive and $|S_i| \geq c.\text{tr}(T)$ (we say that $S_i$ is a transitive set).
- $d(\{v\}, S_j) \geq 1 - \lambda$ for $v \in S_i$ and $d(S_i, \{v\}) \geq 1 - \lambda$ for $v \in S_j, i < j$ (we say that $\chi$ is smooth).

Theorem 5 [3] Let $S$ be a tournament, let $w$ be a $\{0, 1\}$-vector, and let $0 < \lambda_0 < \frac{1}{2}$ be a constant. Then there exist $\epsilon_0, c_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, every $S$-free $\epsilon$-critical tournament contains a smooth $(c_0, \lambda_0, w)$-structure.

Let $\{S_1, \ldots, S_{|w|}\}$ be a smooth $(c, \lambda, w)$-structure of a tournament $T$, let $i \in \{1, \ldots, |w|\}$, and let $v \in S_i$. For $j \in \{1, 2, \ldots, |w|\} \setminus \{i\}$, denote by $S_{j,v}$ the set of the vertices of $S_j$ adjacent from $v$ for $j > i$ and adjacent to $v$ for $j < i$.

Lemma 8 Let $0 < \lambda < 1, 0 < \gamma \leq 1$ be constants and let $w$ be a $\{0, 1\}$-vector. Let $\{S_1, \ldots, S_{|w|}\}$ be a smooth $(c, \lambda, w)$-structure of a tournament $T$ for some $c > 0$. Let $j \in \{1, \ldots, |w|\}$. Let $S_j^* \subseteq S_j$ such that $|S_j^*| \geq \gamma |S_j|$ and let $A = \{x_1, \ldots, x_k\} \subseteq
∪_{i \neq j} S_i \) for some positive integer \( k \). Then \( \left| \cap_{x \in A} S^*_j \right| \geq (1 - \frac{k}{\gamma}) \left| S^*_j \right| \). In particular \( \left| \cap_{x \in A} S^*_j \right| \geq (1 - k\lambda) \left| S_j \right| \).

**Proof** The proof is by induction on \( k \). Without loss of generality assume that \( x_1 \in S_i \) and \( j < i \).

Since \( \left| S^*_j \right| \geq \gamma \left| S_j \right| \) then by Lemma 7, \( d(S^*_j, \{x_1\}) \geq 1 - \frac{k}{\gamma} \). So \( 1 - \frac{k}{\gamma} \leq d(S^*_j, \{x_1\}) = \frac{|S^*_j \setminus \{x_1\}|}{|S^*_j|} \).

Then \( \left| S^*_j \setminus \{x_1\} \right| \geq (1 - \frac{k}{\gamma}) \left| S^*_j \right| \) and so true for \( k = 1 \). Suppose the statement is true for \( k - 1 \).

\[
\left| \cap_{x \in A} S^*_j \right| = \left| \left( \cap_{x \in A \setminus \{x_1\}} S^*_j \right) \setminus \{x_1\} \right| + \left| S^*_j \setminus \{x_1\} \right| \geq (1 - (k - 1)\frac{k}{\gamma}) \left| S^*_j \right| + (1 - \frac{k}{\gamma}) \left| S^*_j \right| - \left| S^*_j \right| = (1 - k\frac{k}{\gamma}) \left| S^*_j \right|.
\]

\( \square \)

### 3 Super Nebulas and Δ-galaxies

#### 3.1 Definitions and Tools

In this section we introduce the notion of **bad triplets**, the notion of **key tournaments**, and the notion of **corresponding structures under a tournament and an ordering of its vertices** that will be very crucial in our later analysis.

Let \( s \) be a \([0, 1]\)-vector. Denote \( s_c \) the vector obtained from \( s \) by replacing every subsequence of consecutive 1’s by single 1. Let \( N \) be a regular super nebula under \( \theta = (v_1, \ldots, v_n) \). Let \( Q_1, \ldots, Q_m \) be the stars of \( N \) under \( \theta \) and let \( \Sigma_1, \ldots, \Sigma_\ell \) be the super 2-nebula of \( N \) under \( \theta \). Let \( s^N,\theta \) be the \([0, 1]\)-vector such that \( s_i \leq 1 \) if and only if \( v_i \) is a leaf of one of the stars of \( N \) under \( \theta \) or a leaf of one of the super 2-nebula of \( N \) under \( \theta \). Let \( \omega = s^N,\theta \) and let \( i_t \) be such that \( \omega_{i_t} = 1 \). Let \( j \) be such that \( s_j \leq 1 \). We say that \( s_j \) corresponds to \( \omega_{i_t} \) if \( s_j \leq 1 \) belongs to the subsequence of consecutive 1’s that is replaced by the entry \( \omega_{i_t} \). Define \( R_{i_t} = \{ v_i \in V(N) ; s^N,\theta \} \) and \( s_{i_t} \leq 1 \) and \( s_{i_t} \) corresponds to \( \omega_{i_t} \). Let \( i_1, \ldots, i_t \) be the non zero entries of \( \omega \). Define \( R = \bigcup_{i=1}^{t} R_{i_t} \). Notice that \( R \) is the set of the leaves of the stars and super 2-nebula of \( N \) under \( \theta \). Let \( 1 \leq i \leq m \). Let \( L_i \) be the set of leaves of the star \( Q_i \) and let \( v_{q_i} \) be the center of \( Q_i \). For all \( v_j, v_k \in L_i \), if \( v_j \in R_{i_t} \) and \( v_k \notin R_{i_t} \), for some \( 1 \leq r \leq t \) then the triplet \( \{v_q_i, v_j, v_k\} \) is called a **bad triplet** of \( Q_i \). Denote by \( B_i \) the set of all bad triplets of \( Q_i \). Define \( T^N,\theta = \{ T_j = \{ v_{j_1}, v_{j_2}, v_{j_3} \} \subseteq \bigcup_{i=1}^{t} V(\Sigma_i) ; E(\Sigma_i) \} \) and \( E(B(\Sigma) \cap \{ v_{j_1}, v_{j_2}, v_{j_3} \}) \) is the restriction of \( \theta \) to \( T_j \). Denote by \( B^N,\theta = \bigcup_{i=1}^{m} B_i \) and \( T^N,\theta \) the set of all **bad triplets** of \( N \) under \( \theta \).

Let \( G \) be a regular \( \Delta \)-galaxy under \( \alpha = (z_1, \ldots, z_{n_2}) \) be the triangle of \( G \) under \( \alpha \). Define \( G_{\alpha} = \{ G ; G \text{ is obtained from } G \text{ by reversing the orientation of exactly one arc of } \Delta \} \). Notice that \( |G_{\alpha}| = 3 \). Let \( G_{\alpha} = G \setminus \{ z_{j_1}, z_{j_2}, z_{j_3} \} \) and let \( \alpha_{\Delta} \) be the restriction of \( \alpha \) to \( V(G_{\Delta}) \).

Let \( N \) be a regular super nebula under \( \theta = (v_1, \ldots, v_n) \) and let \( B^N,\theta = \{ t^i_n \} = (v_{i_1}, v_{i_2}, v_{i_3}) ; i = 1, \ldots, s \} \)(note that for all \( i \in \{ 1, \ldots, s \}, i_1 < i_2 < i_3 \)).
∀1 ≤ i ≤ s, |E(B(\mathcal{N} | \{v_{i1}, v_{i2}, v_{i3}\}, (v_{i1}, v_{i2}, v_{i3}))| = 2. ∀1 ≤ i ≤ s, let e_i \in E(\mathcal{N} | \{v_{i1}, v_{i2}, v_{i3}\}) such that e_i is forward under \theta. e_i is called the forward arc of (v_{i1}, v_{i2}, v_{i3}). Define \mathcal{E}_{\mathcal{N},\theta} = \{e_1, \ldots, e_s\}. The mutant super nebula \tilde{\mathcal{N}} under \theta is the digraph obtained from \mathcal{N} by deleting all the arcs in \mathcal{E}_{\mathcal{N},\theta}.

A regular super nebula \mathcal{K} = \mathcal{N} \otimes \mathcal{G} under \tilde{\theta} is a key tournament corresponding to \mathcal{N} and \mathcal{G} under \theta and \alpha respectively if \mathcal{K} under \tilde{\theta} satisfies all the following:

- V(\mathcal{K}) = V(\mathcal{N}) \cup \bigcup_{i=1}^{s} U_i where U_i = \{u_{i1}^1, \ldots, u_{n_i}^i\} and s = |B^{\mathcal{N},\theta}|.
- \mathcal{K} | V(\mathcal{N}) is isomorphic to \tilde{\mathcal{N}} and the restriction of \tilde{\theta} to V(\mathcal{N}) is the super nebula ordering \theta of \mathcal{N}.
- \mathcal{K} | U_i is the galaxy tournament \mathcal{G}_{\tilde{\theta}} and the restriction of \tilde{\theta} to U_i (say \tilde{\theta}_{U_i}) is the galaxy ordering \alpha_{\tilde{\theta}} of \mathcal{G}_{\tilde{\theta}} for i = 1, \ldots, s.
- B^{\mathcal{K},\tilde{\theta}} = B^{\mathcal{N},\theta} = \{(v_{i1}, v_{i2}, v_{i3}); i = 1, \ldots, s\}.
- \mathcal{K} | X_i is isomorphic to \hat{\mathcal{G}} for some \hat{\mathcal{G}} \in \mathcal{G}_{\tilde{\theta},\alpha}, where X_i = \{v_{i1}, v_{i2}, v_{i3}\} \cup U_i for i = 1, \ldots, s. Moreover, \tilde{\theta}_{X_i}, the restriction of \tilde{\theta} to X_i for i = 1, \ldots, s verifies the following: let e_i \in \mathcal{K} | \{v_{i1}, v_{i2}, v_{i3}\} such that e_i is forward under \tilde{\theta}. The tournament obtained from \mathcal{K} | X_i by reversing the orientation of e_i is the tournament \mathcal{G} and the ordering \tilde{\theta}_{X_i} is then the triangular ordering \alpha of \mathcal{G}. Here we say that U_i corresponds to t_i \in B^{\mathcal{K},\tilde{\theta}} for i = 1, \ldots, s.

In Fig. 4 we draw a super nebula \mathcal{H}_1 under \theta_1 = (1, \ldots, 10), a \Delta galaxy \mathcal{H}_2 under \theta_2 = (1, \ldots, 7), and a key tournament \mathcal{H}_1 \otimes \mathcal{H}_2 under \theta = (1, \ldots, 26) corresponding to \mathcal{H}_1 and \mathcal{H}_2 under \theta_1 and \theta_2 respectively (note that we have more than one key tournament corresponding to \mathcal{H}_1 and \mathcal{H}_2 under \theta_1 and \theta_2 respectively).

Let \mathcal{N} be a regular super nebula under \theta = (v_1, \ldots, v_n). Let Q_1, \ldots, Q_m be the stars of \mathcal{N} under \theta and let \Sigma_1, \ldots, \Sigma_l be the super 2-nebulas of \mathcal{N} under \theta. Let \Sigma Springer
Let $N$ be the number of consecutive 1's corresponding to $N$ under $\theta$, and $N_2$ under $\theta_2$. All the arcs that are not drawn are forward.

In the example in Fig. 5 we have: $s = 1, V_s = \{v_1, v_2, \ldots, v_6\}$, $\theta = 1, \theta_1 = 0, \theta_2 = 1$, $N_k = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $N_k^1 = \{v_1, v_2, v_3, v_4, v_5\}$, $N_k^2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, and $N_k^0$ is the empty tournament. For $k \in \{0, \ldots, m\}$ define $N_k = \bigcup_{i=1}^k V(Q_i)$ and let $\theta_k$ be the restriction of $\theta$ to $V(N_k)$, where $N_k^1 = N_1$, and $N_k^0$ is the empty tournament. For $k_1, k_2, k_3, \ldots, k_{t+1}$ be the super $t$-nebulas of $N_k$, and let $s_{N_k}^{N_1, \theta}$ be the vector obtained from $s_{N_k}^{N_1, \theta}$ by replacing every subsequence of consecutive 1's corresponding to the same entry of $s_{N_k}^{N_1, \theta}$ by single 1.

In the example in Fig. 5 we have: $s_{N_k}^{N_1, \theta} = (0, 0, 0, 1, 1, 1, 0, 0, 1, 1, 1, 1, 0, 1)$, $s_{N_k}^{N_1, \theta} = (0, 0, 0, 0, 0, 1, 0, 0, 1, 1, 1, 0, 0)$, $s_{N_k}^{N_1, \theta} = (0, 1, 1, 0, 0, 1, 1, 1, 0)$, $s_{N_k}^{N_1, \theta} = (0, 0, 0, 1, 1, 0, 0)$, $s_{N_k}^{N_1, \theta} = (0, 1, 0, 0, 0, 1, 1)$.

We say that a smooth $(c, \lambda, w)$-structure of a tournament $T$ corresponds to $N$ under $\theta$ if $w = s_{N_k}^{N_1, \theta}$. For $i \in \{1, 2\}$, we say that a smooth $(c, \lambda, w)$-structure of a tournament $T$ corresponds to $N_k^i$ under $(N, \theta)$ if $w = c_{N_k^i}^{N, \theta}$.

Let $\delta_{N_k}^{N, \theta} : \{j : s_{N_k}^{N, \theta} = 1\} \to \mathbb{N}$ be a function that assigns to every nonzero entry of $s_{N_k}^{N, \theta}$ the number of consecutive 1's of $s_{N_k}^{N, \theta}$ replaced by that entry of $s_{N_k}^{N, \theta}$. Similarly let $\delta_{v}^{v} : \{j : v^j = 1\} \to \mathbb{N}$ be a function that assigns to every nonzero entry of $v^j$ the number of consecutive 1's of $s_{N_k}^{N, \theta}$ replaced by that entry of $v^j$ for $i = 1, 2$.

Let $N$ be a regular super nebula under $\theta = (v_1, \ldots, v_n)$. Let $Q_1, \ldots, Q_m$ be the stars of $N$ under $\theta$ and let $S_1, \ldots, S_{\ell}$ be the super 2-nebulas of $N$ under $\theta$. Let $N_1 = N \setminus \bigcup_{i=1}^\ell V(S_i)$ and let $N_2 = N \setminus \bigcup_{i=1}^m V(Q_i)$. Fix $k \in \{0, \ldots, l\}$ and $z \in \{0, \ldots, m\}$. Let $N_k^1 = N \setminus V(N_k^1)$ and let $N_k^2 = N \setminus V(N_k^2)$, where $N_k$ is the mutant super nebula obtained from $N$ under $\theta$. Let $\theta_k = (v_{k_1}, \ldots, v_{k_{\ell}})$ and $\theta_2 = (v_z, \ldots, v_{z_{\ell}})$. Let $(S_1, \ldots, S_{\ell})$ be a smooth $(c, \lambda, w)$-structure corresponding to $N$ under $\theta$ (resp. corresponding to $N_k^{\ell}$ under $(N, \theta)$) (resp. corresponding to $N_2^{\ell}$).
under \((\mathcal{N}, \theta)\). Let \(i_r\) be such that \(w(i_r) = 1\). Assume that \(S_{i_r} = \{s_{i_r}^1, \ldots, s_{i_r}^{|S_{i_r}|}\}\) and \((s_1^1, \ldots, s_{|S_{i_r}|}^1)\) is a transitive ordering. Write \(m(i_r) = \lfloor \frac{|S_{i_r}|}{\delta w(i_r)} \rfloor\).

Denote \(S_{i_r}^j = \{s_{i_r}^{(j-1)m(i_r)+j}, \ldots, s_{i_r}^{jm(i_r)}\}\) for \(j \in \{1, \ldots, \delta w(i_r)\}\). For every \(v \in S_{i_r}\) denote \(\xi(v) = (| \{ k < i_r : w(k) = 0 \} | + \sum_{k < i_r : w(k) = 1} \delta w(k)) + j\). For every \(v \in S_{i_r}\) such that \(w(i_r) = 0\) denote \(\xi(v) = (| \{ k < i_r : w(k) = 0 \} | + \sum_{k < i_r : w(k) = 1} \delta w(k)) + 1\). We say that \(\tilde{\mathcal{N}}\) (resp. \(\tilde{\mathcal{N}}_1^k\) (resp. \(\tilde{\mathcal{N}}_2^k\)) is well-contained in \((S_1, \ldots, S_{|w|})\) that corresponds to \(\mathcal{N}\) under \(\theta\) (resp. corresponds to \(\mathcal{N}_1^k\) under \((\mathcal{N}, \theta)\)) (resp. corresponds to \(\mathcal{N}_2^k\) under \((\mathcal{N}, \theta)\)) if there is an injective homomorphism \(f\) of \(\tilde{\mathcal{N}}\) (resp. \(\tilde{\mathcal{N}}_1^k\) (resp. \(\tilde{\mathcal{N}}_2^k\)) into \(T\) \(\cup_{i=1}^{w}| S_i\) such that \(\xi(f(v_j)) = j\) for every \(j \in \{1, \ldots, n\}\) (resp. \(\xi(f(v_j)) = j\) for every \(j \in \{1, \ldots, q_k\}\) (resp. \(\xi(f(v_j)) = j\) for every \(j \in \{1, \ldots, p_z\}\)).

### 3.2.2 Proof of Theorem 2

We start by the following technical lemma:

**Lemma 9** Let \(\mathcal{N}\) be a regular super nebula under \(\theta_1\) with \(|\mathcal{N}| = \mu_1\) and let \(G\) be a regular \(\Delta\) galaxy under \(\theta_2\). Let \(K = \mathcal{N} \otimes G\) under \(\theta\) be a key tournament corresponding to \(\mathcal{N}\) and \(G\) under \(\theta_1\) and \(\theta_2\) respectively. Let \(Q_1, \ldots, Q_m\) be the stars of \(K\) under \(\theta\) and let \(\Sigma_1, \ldots, \Sigma_l\) be the super-2 nebulas of \(K\) under \(\theta\). Let \(K_1 = K | \cup_{i=1}^{m} V(\Sigma_i)\) and let \(K_2 = K | \cup_{i=1}^{m} V(Q_i)\). Let \(0 < \lambda < \frac{1}{(2\mu_1)^{n+1}}, c > 0\) be constants, and \(w\) be a \(\{0, 1\}\)-vector. Fix \(k \in \{0, \ldots, l\}\) and let \(\gamma = (2\mu_1)^{-k}\lambda\) and \(\tilde{c} = \frac{c}{(2\mu_1)^{n-1}}\). There exist \(\epsilon_k > 0\) such that \(\forall 0 < \epsilon < \epsilon_k\), for every \(\epsilon\)-critical tournament \(T\) with \(|T| = n\) containing \(\chi = (S_1, \ldots, S_{|w|})\) a smooth \((\tilde{c}, \lambda, w)\)-structure corresponding to \(K_1^k\) under \((K, \theta)\), we have \(K_1^k\) is well-contained in \(\chi\).

**Proof** The proof is by induction on \(k\). For \(k = 0\) the statement is obvious since \(K_1^0\) is the empty digraph. Suppose that \(\chi = (S_1, \ldots, S_{|w|})\) is a smooth \((\tilde{c}, \lambda, w)\)-structure in \(T\) corresponding to \(K_1^k\) under \((K, \theta)\) with \(\theta = (h_1, \ldots, h_{|K|})\) and \(|K| = h\). Let \(\theta_1^{k_1} = (h_{p_{b_1}}, \ldots, h_{p_{b_2}})\) be the restriction of \(\theta\) to \(V(K_1^k)\) (notice that \(q \leq \mu_1\)). Let \(h_{b_1}, \ldots, h_{b_2}\) be the center of \(\Sigma_k\) and let \(h_{p_{b_2}}, \ldots, h_{p_{b_2}}\) be its leaves for some integer \(z > 0\) such that \(h_{p_{b_2}}, \ldots, h_{p_{b_2}}\) are the leaves incident to \(h_{p_{b_1}}\) and \(h_{p_{b_1}}, \ldots, h_{p_{b_2}}\) are the leaves incident to \(h_{p_{b_1}}\) (note that \(\forall 2 \leq i < j \leq z, \text{we don't have necessarily}\). \(b_1 < b_2). \forall 0 \leq i \leq z, \text{let} D_i = \{v \in \cup_{j=1}^{w} S_j; \xi(v) = b_i\}. \text{Then} \exists x \in \{1, \ldots, |w|\}, \exists y \in \{1, \ldots, |w|\}, \exists f \in \{1, \ldots, |w|\} \text{with} x < y, w(x) = w(y) = 0, \text{and} w(f) = 1, \text{such that} D_0 = S_x, D_1 = S_y, \text{and} \forall 2 \leq i \leq z, D_i \subseteq S_f. \text{Since we can assume that} \epsilon < \min\{\log \frac{\tilde{c}}{2} (1 - \frac{\epsilon}{\mu_1}), \log \frac{\tilde{c}}{2} (\frac{1}{4})\}, \text{then by Lemma 6 there exists}\}

vertices \(d_0, d_1, \ldots, d_z\) such that \(d_i \in D_i\) for \(i = 0, 1, \ldots, z\) and 
* \(d_0 \leftarrow \{d_1, d_2, \ldots, d_r\}\) and \(d_{r+1}, \ldots, d_z\) \(\leftarrow d_1\) if \(x < f < y\).
* \{d_0, d_{r+1}, \ldots, d_z\} \(\leftarrow d_1\) and \{d_2, \ldots, d_r\} \(\leftarrow d_0\) if \(f < x < y\).
* \(d_0 \leftarrow \{d_1, d_2, \ldots, d_r\}\) and \(d_{r+1}, \ldots, d_z\) \(\leftarrow d_1\) if \(x < y < f\).

So \(T | \{d_0, d_1, d_2, \ldots, d_z\}\) contains a copy of \(K_1^k | V(\Sigma_k)\). Denote this copy by \(Y\).
\[
\forall i \in \{1, \ldots, |w|\} \setminus \{x, y, f\}, \text{let } S_i^* = \bigcap_{p \in V(Y)} S_{i,p} \text{. Then by Lemma 8, } |S_i^*| \\
\geq (1 - \frac{1}{2\mu_1}) |S_i| \geq (1 - \mu_1^\lambda) |S_i| \geq \frac{1}{2\mu_1} |S_i| \text{ since } \lambda \leq \frac{2\mu_1 - 1}{2\mu_1^2} \text{.}
\]
Write \(\mathcal{H} = \{1, \ldots, q\} \setminus \{b_0, \ldots, b_s\}\). If \(|v| \in S_f : \xi(v) \in \mathcal{H}\) \(\neq \emptyset\), then define \(J_f = \{\eta \in \mathcal{H} : \exists v \in S_f \text{ and } \xi(v) = \eta\}\). Now \(\forall \eta \in J_f, \text{ let } S_\eta^f = \{v \in S_f : \xi(v) = \eta \text{ and } v \in \bigcap_{i \in [0,1]} S_f, d_i\}\). Then by Lemma 8, \(\forall \eta \in J_f, \text{ we have } |S_\eta^f| \geq \frac{1 - 2\mu_1^\lambda}{\mu_1^2} |S_f| \geq \frac{|S_f|}{2\mu_1^2}\) since \(\lambda \leq \frac{1}{2\mu_1^2}\). Now \(\forall \eta \in J_f, \text{ select arbitrary } \frac{|S_f|}{2\mu_1^2}\) vertices of \(S_\eta^f\) and denote the union of these \(|J_f| \text{ sets by } S^*_f\). So we have defined \(t\) sets \(S_1^*, \ldots, S_t^*\), where \(t = |w| - 2\) if \(S^*_f\) is defined and \(t = |w| - 3\) if \(S^*_f\) is not defined. We have \(|S_i^*| \geq \frac{c}{2\mu_1} tr(T)\) for every defined \(S^*_i\) with \(w(i) = 1\), and \(|S_i^*| \geq \frac{c}{2\mu_1} n\) for every defined \(S^*_i\) with \(w(i) = 0\). Now Lemma 7 implies that \(\chi^* = (S_1^*, \ldots, S_t^*)\) form a smooth \((\frac{c}{2\mu_1}, 2\mu_1^\lambda, w^*)\)-structure of \(T\) corresponding to \(K_1^{k-1}\) under \((K, \theta)\), where \(\frac{c}{2\mu_1} = \frac{c}{(2\mu_1)^{n-k}\lambda^\theta}, 2\mu_1^\lambda = (2\mu_1)^{1-(k-1)\lambda}, \text{ and } w^*\) is an appropriate \(\{0, 1\}\)-vector. Now take \(\epsilon_k < \min(\epsilon_{k-1}, \log \frac{c}{2\mu_1}(1 - \frac{c}{2\mu_1}), \log \frac{1}{\epsilon^{1/2}})\). So by induction hypothesis \(K_1^{k-1}\) is well-contained in \(\chi^*\). Now by merging the well-contained copy of \(K_1^{k-1}\) and \(Y\) we get a well-contained copy of \(K_1^k\). \(\square\)

We also need the following technical lemma:

**Lemma 10** Let \(N\) be a regular super nebula under \(\theta_1\) with \(|N| = \mu_1\) and let \(G\) be a regular \(\Delta\)-galaxy under \(\theta_2\) with \(|G| = \mu_2\). Let \(\delta = \mu_1\mu_2\). Let \(K = N \otimes G\) under \(\theta\) be a key tournament corresponding to \(N\) and \(G\) under \(\theta_1\) and \(\theta_2\) respectively. Let \(Q_1, \ldots, Q_m\) be the stars of \(K\) under \(\theta\) and let \(\Sigma_1, \ldots, \Sigma_l\) be the super-2-nebula of \(K\) under \(\theta\). Let \(K_1 = K \cup \bigcup_{i=1}^m V(\Sigma_i)\) and let \(K_2 = K \cup \bigcup_{i=1}^n V(Q_i)\). Let \(0 < \lambda < \frac{1}{(2\delta)^{n-k}\lambda^\theta}, c > 0\) be constants, and \(\theta\) be a \(\{0, 1\}\)-vector. Fix \(k \in \{0, \ldots, m\}\) and let \(\lambda = (2\delta)^{m-k}\lambda\) and \(\frac{c}{(2\delta)^{m-k}}\). There exist \(\epsilon_k > 0\) such that \(\forall 0 < \epsilon < \epsilon_k\), for every \(\epsilon\)-critical tournament \(T\) with \(|T| = n\) containing \(\chi = (S_1, \ldots, S_{|w|})\) as a smooth \((\frac{c}{2\mu_1}, \lambda, w)\)-structure corresponding to \(K_2^k\) under \((K, \theta)\), we have \(K_1^k\) is well-contained in \(\chi\).

**Proof** The proof is by induction on \(k\). For \(k = 0\) the statement is obvious since \(K_0^0\) is the empty digraph. Suppose that \(\chi = (S_1, \ldots, S_{|w|})\) is a smooth \((\frac{c}{2\mu_1}, \lambda, w)\)-structure in \(T\) corresponding to \(K_2^k\) under \((K, \theta)\) with \(\theta = (h_1, \ldots, h_{|K|})\) and \(|K| = h\). Let \(\theta_2 = (h_{q_1}, \ldots, h_{q_p})\) be the restriction of \(\theta\) to \(V(K_2^k)\) (notice that \(p \leq \delta\)). Let \(h_{q_0}\) be the center of \(Q_k\), and let \(h_{q_{a1}}, \ldots, h_{q_{ad}}\) be its leaves for some integer \(d > 0\). \(\forall 0 \leq i \leq d, \text{ let } R_i = \{v \in \bigcup_{j=1}^{|w|} S_j : \xi(v) = a_i\}\). We have \(|R_0| \geq \chi n\) and \(R_i \geq \frac{c}{2\delta} tr(T)\) for \(i = 1, \ldots, d\). Since we can assume that \(\epsilon < \log \frac{c}{2\delta}(1 - \frac{c}{2\delta})\), then by Lemma 5 there exists vertices \(r_0, r_1, \ldots, r_d\) such that \(r_i \in R_i\) for \(i = 0, 1, \ldots, d\) and

- \(r_0 \leftarrow \{r_1, \ldots, r_d\}\) if \(Q_k\) is a left star of \(K\) under \(\theta\).
- \(\{r_1, \ldots, r_d\} \leftarrow r_0\) if \(Q_k\) is a right star of \(K\) under \(\theta\).
- \(\{r_1, \ldots, r_d\} \leftarrow \{r_1, \ldots, r_d\} \) if \(Q_k\) is a middle star of \(K\) under \(\theta\).

So \(T |\{r_0, r_1, \ldots, r_d\}\) contains a copy of \(K_2^k \cup V(Q_k)\). Denote this copy by \(Y\). Let \(x \in \{1, \ldots, |w|\} \setminus \{x\}\) such that \(R_0 = S_x\) and let \(y_1, \ldots, y_d \in \{1, \ldots, |w| \}\)\(\setminus\{x\}\) such
that $R_i \subseteq S_{yi}$ for $i = 1, \ldots, d$. Notice that we have: $w(x) = 0$ and $w(y_i) = 1$ for $i = 1, \ldots, d$. Also notice that we don’t have necessarily that $y_1, \ldots, y_d$ are distinct.

\[ \forall i \in \{1, \ldots, d\}, \text{let } S_i^* = \bigcap_{j=0}^{d} S_{rij}. \]

Then by Lemma 8, $|S_i^*| \geq |1 - \frac{1}{\lambda}| |S_i| \geq (1 - \frac{1}{\lambda}) |S_i| \geq \frac{\Delta}{2 \delta} |S_i| \geq |1 - \frac{2\delta}{8\lambda}| S_i | \geq \frac{\Delta}{2 \delta} S_i $. Write $\mathcal{H} = \{1, \ldots, p\} \{a_0, \ldots, a_q\}$. Let $Y_i = \{v \in V(Y) : v \in S_{yi}\}$ for $i = 1, \ldots, d$.

\[ \forall 1 \leq l \leq d, \text{if } \{v \in S_{yi} : \xi(v) \in \mathcal{H}\} \neq \emptyset, \text{then define } J_{yi} = \{\eta \in \mathcal{H} : \exists v \in S_{yi}, \text{and } \xi(v) = \eta\}. \]

Now $\forall \eta \in J_{yi},$ let $S_{yi}^\eta = \{v \in S_{yi} : \xi(v) = \eta \}$ and $v \in \bigcap_{q \in V(Y)} J_{yi}$. Then by Lemma 8, we have $|S_{yi}^\eta| \geq (1 - (d - 1)\delta \lambda) |S_{yi}| - \frac{1 - \delta \lambda^2}{\delta} S_i |$. Since $\lambda \geq \frac{1}{2\delta}$, we get the union of these $J_1$ sets by $S_{yi}^\eta$.

\[ \text{From Lemmas 9 and 10 we get the following lemma:} \]

**Lemma 11** Let $N$ be a regular super nebula under $\theta_1$ with $|N| = \mu_1$ and let $G$ be a regular $\Delta$ galaxy under $\theta_2$ with $|G| = \mu_2$. Let $\delta = \mu_1 \mu_2$. Let $K = N \otimes G$ under $\theta = (h_1, \ldots, h_h)$ be a key tournament corresponding to $N$ and $G$ under $\theta_1$ and $\theta_2$ respectively. Let $0 < \lambda_0 < \frac{1}{(2\delta)^{h+1}}, \c > 0$ be constants, and $w$ be a $(0, 1)$-vector. There exist $\epsilon_0 > 0$ such that for every $\epsilon$-critical tournament $T$ with $|T| = n$ containing $\chi = (A_1, \ldots, A_{|w|})$ as a smooth $(c_0, \lambda_0, w)$-structure corresponding to $K$ under the ordering $\theta$, we have $\hat{K}$ is well-contained in $\chi$.

**Proof** Let $Q_1, \ldots, Q_m$ be the stars of $K$ under $\theta$ and let $\Sigma_1, \ldots, \Sigma_l$ be the super 2-nebula of $K$ under $\theta$. Let $K_1 = K \cup \bigcup_{i=1}^{l} V(S_i)$ and let $K_2 = K \cup \bigcup_{m=1}^{m} V(Q_i)$. Let $\theta_1 = (h_1, \ldots, h_{p_q})$ be the restriction of $\theta$ to $V(K_1)$ and let $\theta_2 = (h_{q_1}, \ldots, h_{q_p})$ be the restriction of $\theta$ to $V(K_2)$. $\forall 1 \leq l \leq q$, let $S_i = \{v \in \bigcup_{j=1}^{w_i} A_j : \xi(v) = p_i\}$. Then $1 \leq j \leq |w_i|$, let $S_i^* = \bigcup_{S_i \subseteq A_j} S_i$. (notice that we may have: $S_i^* = \emptyset$). Let $S_i^*, \ldots, S_{w_i}^*$ denote the non empty subsets $S_i^*$. Then $\chi^* = (S_1^*, \ldots, S_{w_i}^*)$ is a smooth $(c_0, \delta \lambda_0, w^*)$-structure of $T$ corresponding to $K_1$ under $(K, \theta)$. Let $\epsilon_1$ be the $\epsilon$ from Lemma 9 taken for $c = \frac{c_0}{\delta}$. Taking $\epsilon < \epsilon_1$ and since $\lambda_0 < \frac{1}{(2\delta)^{h+1}}$, then we can use Lemma 9 and conclude taking $k = l$ that $\hat{K}_1$ is well-contained in $\chi^*$. Denote this well-contained copy of $\hat{K}_1$ by $G$. $\forall 1 \leq l \leq p$, let $R_i^* = \bigcap_{x \in V(G)} R_{i,x}$, where $R_i = \{v \in \bigcup_{j=1}^{w_i} A_j : \xi(v) = q_i\}$. Let $1 \leq i \leq p$. If $R_i = A_{j_1}$ for some $1 \leq j_1 \leq |w_i|$ with $w(j_1) = 0$, then by Lemma 8, $|R_i^*| \geq |(1 - p \lambda_0) |A_{j_1}| - (1 - \delta \lambda_0) |A_{j_1}| \geq \frac{|A_{j_1}|}{2} \geq \frac{|A_{j_1}|}{28}$ since $\lambda_0 \leq \frac{1}{2\delta}$. In this case we only rename the set $R_i^*$ by $R_i^*$. Let $1 \leq i \leq p$. If $R_i \subseteq A_{j_2}$ for some $1 \leq j_2 \leq |w_i|$ with $w(j_2) = 1$, then by Lemma 8, $|R_i^*| \geq \frac{|A_{j_2}|}{2} \geq \frac{|A_{j_2}|}{28}$ since $\lambda_0 \leq \frac{1}{2\delta}$.
$\geq (1 - \delta^2 \lambda_0)\frac{|A_{ij}|}{\delta} \geq \frac{|A_{ij}|}{2\delta}$ since $\lambda_0 \leq \frac{1}{2\delta^2}$. In this case we select arbitrary $\left\lceil \frac{|A_{ij}|}{2\delta} \right\rceil$ vertices from $R^*_i$ and we denote by $R^*_{ij}$ the set of the selected $\left\lceil \frac{|A_{ij}|}{2\delta} \right\rceil$ vertices. Now $\forall 1 \leq j \leq |w|$, let $M_j = \bigcup R^*_{\leq A_j} R^*_{ij}$ (notice that we may have: $M_j = \emptyset$). Let $M_1, \ldots, M_{|\pi|}$ denote the non empty sets $M_j$. Also notice that for all $1 \leq j \leq |w|$, $|M_j| \geq \frac{|A_{ij}|}{2\delta}$ for some $1 \leq s \leq |w|$. Then $\chi' = (M_1, \ldots, M_{|\pi|})$ form a smooth $(\frac{c_0}{2\delta}, 2\delta \lambda_0, w)$-structure of $T$ corresponding to $K_2$ under $(\mathcal{K}, \theta)$ for an appropriate $\{0, 1\}$-vector $\bar{w}$. Let $\epsilon_2$ be the $\epsilon$ from Lemma 10 taken for $c \geq \frac{c_0}{2\delta}$. Taking $\epsilon < \epsilon_2$ and since $\lambda_0 < \frac{1}{(2\delta)^{p+1}}$, then we can use Lemma 10 and conclude taking $k = m$ that $\tilde{K}_2$ is well contained in $\chi'$. Denote this copy well-contained of $\tilde{K}_2$ by $G_2$. Now by merging $G_1$ and $G_2$ we get a well-contained copy of $\tilde{K}$ in $\chi$. This completes the proof.  

From the previous lemma we get the following lemma:

**Lemma 12** Let $\mathcal{N}$ be a regular super nebula under $\theta_1$ with $|\mathcal{N}| = \mu_1$ and let $\mathcal{G}$ be a regular $\Delta$galaxy under $\theta_2$ with $|\mathcal{G}| = \mu_2$. Let $\delta = \mu_1/\mu_2$. Let $K = \mathcal{N} \otimes \mathcal{G}$ under $\theta = (h_1, \ldots, h_h)$ be a key tournament corresponding to $\mathcal{K}$ and $\mathcal{G}$ under $\theta_1$ and $\theta_2$ respectively. Let $0 < \lambda_0 < \frac{1}{(2\delta)^{p+1}}$, $c_0 > 0$ be constants, and let $w$ be a $\{0, 1\}$-vector. Suppose that $\chi = (S_1, \ldots, S_{|w|})$ is a smooth $(c_0, \lambda_0, w)$-structure of an $\epsilon$-critical tournament $T$ ($\epsilon$ is small enough) corresponding to $K$ under the ordering $\theta$. Then

- $T$ contains $\mathcal{N}$ or
- $T$ contains $\mathcal{G}$.

**Proof** Taking $\epsilon > 0$ small enough, we conclude using the previous lemma that $\tilde{K}$ is well-contained in $\chi$. Denote by $G$ the well-contained copy of $\tilde{K}$ in $\chi$. Let $\tilde{\theta} = (x_1, \ldots, x_h)$ be the ordering of the vertices of $G$ according to their appearance in $\chi$ (i.e $\forall 1 \leq i \leq h, \xi(x_i) = i$). Let $T = T | V(G)$ and let $E = E(T) \setminus E(G)$. We have two cases:

Case 1: For all $e \in E$, $e$ is a forward arc of $T$ under $\tilde{\theta}$. Then $T$ under $\tilde{\theta}$ is the key tournament $\mathcal{K} = \mathcal{N} \otimes \mathcal{G}$ under $\theta = (h_1, \ldots, h_h)$ corresponding to $\mathcal{N}$ and $\mathcal{G}$ under $\theta_1$ and $\theta_2$ respectively (i.e $\tilde{\theta}$ is the super nebula ordering $\theta$ of $\mathcal{K}$). Then $T$ under $\tilde{\theta}$ satisfies the following: $V(G) = V(T) = \{x_1, \ldots, x_h\}$ is partitioned as follows: $V(T) = P \cup U_1 \cup \ldots \cup U_s$, where $s = |B^{\mathcal{K}, \theta}| = |B^{T, \tilde{\theta}}|$ and $U_i$ corresponds to the bad triplet $t_i = (x_{i_1}, x_{i_2}, x_{i_3}) \in B^{T, \tilde{\theta}}$ for $i = 1, \ldots, s$, such that: $T | P$ is isomorphic to $\mathcal{N}$ and the restriction of $\tilde{\theta}$ to $P$ is the super nebula ordering $\tilde{\theta}$ of $\mathcal{N}$. So $T$ contains $\mathcal{N}$ and we are done.

The partition of $V(G)$ that we get in case 1 will be very useful and helpful in our analysis in case 2 (i.e useful in identifying precisely the vertices of $G$, such that the subtournament of $T$ induced by these chosen vertices form a copy of $G$).

Case 2: There exist $e_r = (x_{r_{j_1}}, x_{r_{j_2}}) \in E$ with $1 \leq r \leq s$ and $j_1, j_2 \in \{1, 2, 3\}$, such that $e_r$ is a backward arc of $T$ under $\tilde{\theta}$. In this case $T | (U_r \cup \{x_{r_1}, x_{r_2}, x_{r_3}\})$ is isomorphic to $\mathcal{G}$ and the restriction of $\tilde{\theta}$ to $U_r \cup \{x_{r_1}, x_{r_2}, x_{r_3}\}$ is the $\Delta$galaxy ordering $\theta_2$ of $\mathcal{G}$ (see the 5th property of key tournaments in page 11). So $T$ contains $\mathcal{G}$ and we are done. This completes the proof.  

Now we are ready to prove Theorem 2
Proof of Theorem 2. Let \( \mathcal{N} \) be a super nebula under \( \theta_1 \) with \( |\mathcal{N}| = \mu_1 \) and let \( G \) be a \( \Delta \)galaxy under \( \theta_2 \) with \( |G| = \mu_2 \). Let \( \delta = \mu_1 \mu_2 \) and let \( 0 < \lambda_0 < \frac{1}{(2\delta)^{\frac{1}{3}}} \). We may assume that \( \mathcal{N} \) is a regular super nebula and \( G \) is a regular \( \Delta \)galaxy since every super nebula is a subtournament of a regular super nebula and every \( \Delta \)galaxy is a subtournament of a regular \( \Delta \)galaxy. Let \( K = \mathcal{N} \otimes G \) under \( \theta \) be a key tournament corresponding to \( \mathcal{N} \) and \( G \) under \( \theta_1 \) and \( \theta_2 \) respectively. Let \( \epsilon > 0 \) be small enough. Assume that \( \{\mathcal{N}, G\} \) does not satisfy \( EHC \), then there exists an \( \{\mathcal{N}, G\}\)-free \( \epsilon \)-critical tournament \( T \). By Lemma 2, \( |T| \) is large enough. By Theorem 5, \( T \) contains a smooth \((c_0, \lambda_0, w)\)-structure \((S_1, \ldots, S_{|w|})\) corresponding to \( K \) under \( \theta \) for some \( c_0 > 0 \) and appropriate \( \{0, 1\}\)-vector \( w \). Then by the previous lemma, \( T \) contains \( \mathcal{N} \) or \( T \) contains \( G \), a contradiction. \( \square \)

4 Central Triangular Galaxies and \( K_6 \)

4.1 Definitions

Let \( \beta = (v_1, \ldots, v_f) \) be an ordering of the vertex set \( V(T) \) of an \( f \)-vertex tournament \( T \). A \( K_6 \) = \( \{v_{i_1}, \ldots, v_{i_6}\} \) of \( T \) under \( \beta \) (where \( i_1 < \cdots < i_6 \)) is the subtournament of \( T \) induced by \( \{v_{i_1}, \ldots, v_{i_6}\} \) such that \( T \mid \{v_{i_1}, \ldots, v_{i_6}\} \) is the tournament \( K_6 \), \( (v_{i_1}, \ldots, v_{i_6}) \) is the canonical ordering of \( K_6 \), \( i_2 = i_1 + 1, i_4 = i_3 + 1, \) and \( i_6 = i_5 + 1 \). We call \( v_{i_1} \) and \( v_{i_6} \) the centers of \( K_6 \).

Let \( K_6^j = \{v_{i_1}, \ldots, v_{i_6}\} \) be a \( K_6 \) of \( T \) under \( \beta \). Define operation \( K_6, \theta \) by deleting the vertices \( v_{i_2}, v_{i_3}, v_{i_5} \) and reversing the orientation of the arc \( (v_{i_4}, v_{i_6}) \). This \( K_6, \beta \) in operation \( K_6, \beta \) is because this operation is applied for the tournaments \( K_6 \) of \( T \) under \( \beta \).

A tournament \( T \) is a \( GK_6 \) if there exist an ordering \( \beta = (v_1, \ldots, v_f) \) of its vertices such that \( V(T) \) of \( V(K_6^1), \ldots, V(K_6^l), X \), and such that \( K_6^1, \ldots, K_6^l \) are the \( K_6 \) tournaments of \( T \) under \( \beta \), \( T \mid X \) is a regular galaxy under the restriction of \( \beta \) to \( X \), and no center of a \( K_6 \) of \( T \) under \( \beta \) is between leaves of a star of \( T \) under \( \beta \). In this case we also say that \( T \) is a \( GK_6 \) under \( \beta \). Obviously notice that every \( GK_6 \) under \( \beta \) is a super nebula under \( \beta \).

Let \( H \) be a regular central triangular galaxy with \( |H| = h \) and let \( (u_1, \ldots, u_h) \) be a central triangular galaxy ordering of \( H \). Denote this ordering by \( \theta \). Let \( \Delta_1, \ldots, \Delta_l \) be the triangles of \( H \) under \( \theta \) and let \( Q_1, \ldots, Q_m \) be the frontier stars of \( H \) under \( \theta \). A \( GK_6 \) tournament \( K \) under \( \beta \) is a key tournament corresponding to \( H \) under \( \theta \) if the tournament obtained from \( K \) under \( \beta \) after performing operation \( K_6, \beta \) to every \( K_6 \) of \( K \) under \( \beta \) is the tournament \( H \) and the obtained ordering is the central triangular galaxy ordering \( \theta \) of \( H \) (see Fig. 6).
Let $H$ be a regular central triangular galaxy under $\theta$ with $|H| = h$. Let $\Delta_1, \ldots, \Delta_l$ be the triangles of $H$ under $\theta$ and let $Q_1, \ldots, Q_m$ be the frontier stars of $H$ under $\theta$. Assume without loss of generality that $m = l$. Let $K$ under $\beta = (v_1, \ldots, v_f)$ be the key tournament corresponding to $H$ under $\theta$. Let $K_1^1, \ldots, K_1^h$ be the $K_6$ tournaments of $K$ under $\beta$. Let $X = V(K) \setminus \bigcup_{j=1}^l V(K_j^l)$ and let $Q_1, \ldots, Q_l$ be the stars of $K \setminus X$ under $\beta$ ($\beta_X$ is the restriction of $\beta$ to $X$). Notice that $f = h + 3l$. For $k \in \{0, \ldots, l\}$ define $K^k = K \setminus \bigcup_{j=1}^k (V(K_j^l) \cup V(Q_j))$ where $K_1^l = K$, and $K_0$ is the empty tournament. For $k \in \{1, \ldots, l\}$ let $\beta_k = (v_{k_1}, \ldots, v_{k_6})$ be the restriction of $\beta$ to $V(K^k)$. Recall that $s_{K^k}^{k}$ is a $\{0, 1\}$-vector such that $s_{K^k}^{k}(i) = 0$ if and only if $v_i \in C$, where $C$ is the set of all chosen centers of the stars of $K$ under $\beta$ and the centers of the super middle 2-nebulas of $K$ under $\beta$. Let $s_{K^k}^{1 \#}$ be the restriction of $s_{K^k}^{k}$ to the $0$'s and $1$'s corresponding to $V(K^k)$ (notice that $s_{K^k}^{1 \#} = s_{K^k}^{k}$) and let $s_{K^k}^{1 \#}$ be the vector obtained from $s_{K^k}^{1 \#}$ by replacing every subsequent of consecutive $1$'s corresponding to the same entry of $s_{K^k}^{1 \#}$ by single $1$ (see page 10 for the definition of $s_{K^k}^{1 \#}$). We say that a smooth $(c, \lambda, w)$-structure of a tournament $T$ corresponds to $K^k$ under $(K, \beta)$ if $w = c s_{K^k}^{1 \#}$. Notice that $s_{K^k}^{1 \#} = s_{K^k}^{1 \#}$ and $s_{K^k}^{1 \#} = s_{K^k}^{1 \#}$.

Let $\nu = c s_{K^k}^{1 \#}$. Let $\nu : \{j : v_j = 1\} \to \mathbb{N}$ be a function that assigns to every nonzero entry of $\nu$ the number of consecutive $1$'s of $s_{K^k}^{1 \#}$ replaced by that entry of $\nu$.

Fix $k \in \{0, \ldots, l\}$. Let $\tilde{K}^k = \tilde{K} \setminus V(K^k)$, where $\tilde{K}$ is the mutant super nubula obtained from $K$ under $\theta$ (see page 10). Let $(S_1, \ldots, S_{|w|})$ be a smooth $(c, \lambda, w)$-structure of a tournament $T$ corresponding to $K^k$ under $(K, \beta)$. Let $i_r$ be such that $w(i_r) = 1$. Assume that $S_{i_r} = \{s_{i_r}^1, \ldots, s_{i_r}^{|S_{i_r}|}\}$ and $(s_{i_r}^1, \ldots, s_{i_r}^{|S_{i_r}|})$ is a transitive ordering. Write $m(i_r) = |\sum_{j=1}^{|S_{i_r}|} s_{i_r}^j|$. Denote $S_{i_r}^j = \{s_{i_r}^j(m(i_r)+1), \ldots, s_{i_r}^j(m(i_r)+1)\}$ for $j \in \{1, \ldots, \delta^w(i_r)\}$. For every $v \in S_{i_r}^j$ denote $\delta(v) = (|\{k < i_r : w(k) = 0\} + \sum_{k < i_r, w(k) = 0} \delta^w(k)) + j$. For every $v \in S_{i_r}$ such that $w(i_r) = 0$ denote $\delta^w(v) = (|\{k < i_r : w(k) = 0\} + \sum_{k < i_r, w(k) = 0} \delta^w(k)) + 1$. We say that $\tilde{K}^k$ is well-contained in $(S_1, \ldots, S_{|w|})$ that corresponds to $K^k$ under $(K, \beta)$ if there is an injective homomorphism $f$ of $\tilde{K}^k$ into $T \setminus \bigcup_{i=1}^{|w|} S_i$ such that $\delta^w(f(v_{k_j})) = j$ for every $j \in \{1, \ldots, q_k\}$.

4.2 Proof of Theorem 3

We start by the following technical lemma:

**Lemma 4** Let $H$ be a regular central triangular galaxy under $\theta$ with $|H| = h$. Let $\Delta_1, \ldots, \Delta_l$ be the triangles of $H$ under $\theta$ and let $Q_1, \ldots, Q_l$ be the stars of $H$ under $\theta$. Let $K$ under $\beta$ be the key tournament corresponding to $H$ under $\theta$ (\{K \setminus \theta = h + 3l\}). Let $K_1^1, \ldots, K_1^h$ be the $K_6$ tournaments of $K$ under $\beta$ and let $Q_1, \ldots, Q_l$ be the stars of $K \setminus X$ under $\beta_X$, where $X = V(K) \setminus \bigcup_{j=1}^l V(K_j^l)$ and $\beta_X$ is the restriction of $\beta$ to $X$. Let $0 < \lambda < \frac{1}{(4h)^{l+1}}$, $c > 0$ be constants, and $w$ be a $\{0, 1\}$-vector. Fix $k \in \{0, \ldots, l\}$ and let $\hat{\lambda} = (4h)^{l-k}\lambda$ and $\hat{c} = \frac{c}{(4h)^{l-k}}$. There exist $\epsilon_k > 0$ such that $\forall 0 < \epsilon < \epsilon_k$.
for every $\epsilon$-critical tournament $T$ with $|T| = n$ containing $\chi = (S_1, \ldots, S_{|w|})$ as a smooth $(\hat{C}, \hat{\lambda}, w)$-structure corresponding to $K^k$ under $(K, \beta)$, we have $\tilde{K}^k$ is well-contained in $\chi$.

**Proof** The proof is by induction on $k$. For $k = 0$ the statement is obvious since $\tilde{K}^0$ is the empty digraph. Suppose that $\chi = (S_1, \ldots, S_{|w|})$ is a smooth $(\hat{C}, \hat{\lambda}, w)$-structure in $T$ corresponding to $K^k$ under $(K, \beta)$ with $\beta = (h_1, \ldots, h_{|K|})$ and $|K| = h + 3l$. Let $\beta_k = (h_{k1}, \ldots, h_{kp})$ be the restriction of $\beta$ to $V(K^k)$. Let $K^k_{6} = \{h_{k1}, \ldots, h_{k16}\}$. Assume without loss of generality that the star $\{h_{k2}, \ldots, h_{k15}\}$ is considered as a left star of $K$ under $\beta$. Let $h_{k0}$ be the center of $Q_k$ and $h_{kp}, \ldots, h_{kp}$ be its leaves for some integer $q > 0$. Let $D_i = \{v \in \bigcup_{j=1}^{|w|} S_j; \xi(v) = s_j\}$ for $i = 1, \ldots, 6$. Then $\exists x_1, x_2, x_3, z_1, z_2 \in \{1, \ldots, |w|\}$ with $x_1 < x_2 < z_1 < z_2 < x_3, w(x_1) = w(x_2) = w(x_3) = 0$, and $w(z_1) = w(z_2) = 1$, such that $D_1 = S_{x_1}, D_2 = S_{x_2}, D_i \subseteq S_{z_1}$ for $i = 3, 4, 5, D_5 \subseteq S_{z_2}$, and $D_6 = S_{z_3}$. Let $R_i = \{v \in \bigcup_{j=1}^{|w|} S_j; \xi(v) = p_i\}$. Then $\exists x_4 \in \{1, \ldots, |w|\}\{x_1, x_2, x_3, z_1, z_2\}$ with $w(x_4) = 0$, and $\exists z_3 \in \{1, \ldots, |w|\}\{x_1, x_2, x_3, x_4, z_3\}$ with $w(z_3) = 1$, such that $R_0 = S_{x_4}$ and $\forall 1 \leq i \leq q, R_i \subseteq S_{x_3}$. Since we can assume that $\epsilon < \min\{\log_{\frac{3}{2}}(1 - \frac{c}{2n}), \log_{\frac{3}{4}}(\frac{1}{2})\}$, then by Lemma 6 there exists vertices $d_1, d_3, d_4, d_6$ such that $d_i \in D_i$ for $i = 1, 3, 4, 6$ and $d_1 \leftrightarrow d_4$ and $\{d_1, d_3\} \leftrightarrow d_6$. Also notice that $d_3 \rightarrow d_4$. One of the following holds:

- $z_1 < z_2$ and $z_3 \notin \{z_1, z_2\}$, or
- $z_1 < z_2$ and $z_3 = z_1$, or
- $z_1 < z_2$ and $z_3 = z_2$, or
- $z_1 = z_2$ and $z_3 \neq z_1$, or
- $z_1 = z_2 = z_3$.

Assume that $z_1 < z_2$ and $z_3 \notin \{z_1, z_2\}$. Else, the argument is similar and we omit it.

Let $D_2^* = \{d_2 \in D_2; d_1 \rightarrow d_2 \rightarrow \{d_3, d_4, d_6\}\}$ and $D_5^* = \{d_5 \in D_5; \{d_1, d_3, d_4\} \rightarrow d_5 \rightarrow d_6\}$. Then by Lemma 8, $|D_2^*| \geq (1 - 4\hat{\lambda})\hat{n} \geq \frac{c}{2} n$ since $\hat{\lambda} \leq \frac{1}{8}$, and $|D_5^*| \geq 1 - \frac{8h\hat{\lambda}}{2} tr(T) \geq \frac{c}{4} tr(T)$. Since $\hat{\lambda} \leq \frac{1}{12}$, we can assume that $\epsilon < \log_{\frac{3}{4}}(1 - \frac{c}{4n})$, then Lemma 4 implies that there exist vertices $d_2 \in D_2^*$ and $d_5 \in D_5^*$ such that $d_2 \leftrightarrow d_5$. Then $T \rightarrow \{d_1, \ldots, d_6\}$ contains a copy of $\tilde{K}^k \rightarrow V(K^k_{6})$. Denote this copy by $W$. $\forall 0 \leq i \leq q$, let $R_i^* = \bigcap_{x \in V(W)} R_i, x$. Then by Lemma 8, $|R_i^*| \geq (1 - 6\hat{\lambda}) |R_0| \geq \frac{|R_0|}{2} \geq \frac{c}{2} n$ since $\hat{\lambda} \leq \frac{1}{12}$, and $\forall 1 \leq i \leq q, |R_i^*| \geq \frac{1 - 12h\hat{\lambda}}{2} |S_{z_3}| \geq \frac{c}{4} tr(T)$. Since we can assume that $\epsilon < \log_{\frac{3}{4}}(1 - \frac{c}{4n})$, then by Lemma 5 there exists vertices $r_0, r_1, \ldots, r_q$ such that $r_i \in R_i^*$ for $i = 0, 1, \ldots, q$ and

* $r_1, \ldots, r_q$ are all adjacent from $r_0$ if $x_4 > z_3$.
* $r_1, \ldots, r_q$ are all adjacent to $r_0$ if $x_4 < z_3$.

So $T \rightarrow \{d_1, \ldots, d_6, r_0, r_1, \ldots, r_q\}$ contains a copy of $\tilde{K}^k \rightarrow (V(K^k_{6}) \cup V(Q_k))$. Denote this copy by $Y$.

$\forall i \in \{0, 1, \ldots, p\}\{p_0, \ldots, p_q, s_1, \ldots, s_6\}$. Let $Z_i = \{v \in V(Y); v \in S_{z_i}\}$
for $i = 1, 2, 3$. \forall 1 \leq i \leq 3$, if $\{v \in S_{zi} : \xi(v) \in \mathcal{H}\} \neq \emptyset$, then define $J_{zi} = \{\eta \in \mathcal{H} : \exists v \in S_{zi}$ and $\xi(v) = \eta\}$. Now $\forall \eta \in J_{zi}$, let $S^\eta_{zi} = \{v \in S_{zi} : \xi(v) = \eta$ and $v \in \bigcap_{x \in V(Y) \setminus z_i} S_{x,i,x}\}$. Then by Lemma 8, $\forall \eta \in J_{zi}$, we have $|S^\eta_{zi}| \geq \frac{1-2h^2\sqrt{v}}{2h} |S_{zi}| \geq \frac{|S_{zi}|}{4h}$ since $\frac{\lambda}{4h} \leq \frac{1}{4h^2}$. Now $\forall \eta \in J_{zi}$, select arbitrary $\big[\frac{|S^\eta_{zi}|}{4h}\big]$ vertices of $S^\eta_{zi}$ and denote the union of these $|J_{zi}|$ sets by $S^\eta_{zi}$. So we have defined some number of sets. Denote by $t$ the number of these defined sets and by $S^1_i, \ldots, S^t_i$ these sets. We have $|S^1_i| \geq \frac{S_i}{4h}$ for every defined set. Now Lemma 7 implies that $\chi^* = (S^1_i, \ldots, S^t_i)$ form a smooth $(\frac{c_i}{4h}, 4h\hat{\lambda}, w^*)$-structure of $T$ corresponding to $\hat{K}^{k-1}$ under $(K, \beta)$, where $\frac{c_i}{4h} = \frac{c}{(4h)^{k-(k-1)}}, 4h\hat{\lambda} = (4h)^{l-\lambda},$ and $w^*$ is an appropriate $\{0, 1\}$-vector. Now take $\epsilon_k < \min\{\epsilon_{k-1}, \log_\frac{4}{h} (1 - \frac{c_i}{4h}), \log_\frac{4}{h} (\frac{1}{2}), \log_\frac{4}{h} (1 - \frac{c_i}{4h})\}$. So by induction hypothesis $\hat{K}^{k-1}$ is well-contained in $\chi^*$. Now by merging the well-contained copy of $\hat{K}^{k-1}$ and $Y$ we get a well-contained copy of $\hat{K}^k$.

From the above lemma we get the following lemma:

**Lemma 14** Let $H$ be a regular central triangular galaxy under $\theta$ with $|H| = h$. Let $K$ under $\beta$ be the key tournament corresponding to $H$ under $\theta$. Let $0 < \lambda < \frac{1}{(4h)^{k+1}}$, $c > 0$ be constants, and let $w$ be a $\{0, 1\}$-vector. Suppose that $\chi = (S_1, \ldots, S_{|w|})$ is a smooth $(c, \lambda, w)$-structure of an $\epsilon$-critical tournament $T$ ($\epsilon$ is small enough) corresponding to $K$ under the ordering $\beta$. Then $T$ contains $H$ or $T$ contains $K_6$.

**Proof** Let $K^1_6, \ldots, K^6_6$ be the $K_6$ tournaments of $K$ under $\beta$ and let $Q_1, \ldots, Q_l$ be the stars of $K \mid X$ under $\beta_X$. Taking $\epsilon > 0$ small enough and $k = l$, we conclude using the previous lemma that $\hat{K}$ is well-contained in $\chi$. Denote by $G$ the well-contained copy of $\hat{K}$ in $\chi$. $\forall 1 \leq i \leq l$, let $D_i = \{d^1_i, \ldots, d^6_i\}$ be the copy of $\hat{K} \mid V(K^i_6)$ in $\chi$ and let $Q_i$ be the copy of $\hat{K} \mid V(Q_i)$ in $\chi$. Let $\hat{\beta}$ be the ordering of the vertices of $G$ according to their appearance in $\chi$. Notice that $\forall 1 \leq i \leq l$, we don’t know the orientation of the edges $d^4_i d^6_i$ and $d^4_i d^3_i$.

Assume first that $\forall 1 \leq i \leq l$, at least one of the following holds:

- $d^1_i \leftarrow d^3_i$
- $d^4_i \leftarrow d^6_i$

Let $j_i \in \{3, 4\}$, such that $d^1_i \leftarrow d^j_i \leftarrow d^6_i$ for $i = 1, \ldots, l$. But then the restriction of $\hat{\beta}$ to $\bigcup_{i=1}^l (V(Q_i) \cup \{d^1_i, d^j_i, d^6_i\})$ is the central triangular galaxy ordering $\theta$ of $H$.

So $T$ contains $H$, and we are done.

Otherwise there exist $i \in \{1, \ldots, l\}$ such that $d^1_i \rightarrow d^3_i$ and $d^4_i \rightarrow d^6_i$. But then $(d^1_i, d^2_i, d^3_i, d^4_i, d^5_i, d^6_i)$ is the canonical ordering of $K_6$. So $T$ contained $K_6$. This completes the proof.

Now we are ready to prove Theorem 3:

**Proof of Theorem 3.** Let $H$ be a regular central triangular galaxy under $\theta$ with $|H| = h$. We may assume that $H$ is a regular central triangular galaxy since every central triangular galaxy is a subtournament of a regular central triangular galaxy. Let $K$ under $\beta$ be the key tournament corresponding to $H$ under $\theta$. Let $\epsilon > 0$ be small enough and let $0 < \lambda < \frac{1}{(4h)^{k+1}}$ be constants. Assume that $\{H, K_6\}$ does not satisfy $EHC$, then there exists an $\{H, K_6\}$-free $\epsilon$-critical tournament $T$. By Lemma 2, $|T| <
is large enough. By Theorem 5, \( T \) contains a smooth \((c, \lambda, w)\)-structure \((S_1, \ldots, S_{|w|})\) corresponding to \( K \) under under \((\mathcal{K}, \beta)\), for some \( c > 0 \) and appropriate \((0, 1)\)-vector \( w \). Then by the previous lemma, \( T \) contains \( H \) or \( T \) contains \( K_6 \), a contradiction. □

5 Extension of the Results

5.1 Extension of Theorem 2

Let \( S = \{u_1, u_2, \ldots, u_p\} \) be a middle star and let \( u_r \) be the center of \( S \) with \( 2 \leq r \leq p - 1 \) (note that \((u_1, u_2, \ldots, u_p)\) is its star ordering). If \( r = 2 \) then \( S \) is called 1-left middle star and if \( r = p - 1 \) then \( S \) is called 1-right middle star.

**Theorem 6** Let \( \mathcal{N} \) be a super nebula under \( \theta \) and besides for every star \( Q_i \) of \( \mathcal{N} \) under \( \theta \), \( Q_1 \) is a 1-right middle star or a right star, and all the super 2-nebulas of \( \mathcal{N} \) under \( \theta \) are left super 2-nebulas. Let \( G \) be a left triangular galaxy under \( \alpha \) and besides \( G \) has only one triangle under \( \alpha \). Then \( \{\mathcal{N}, G\} \) satisfy EHC.

**Theorem 7** Let \( \mathcal{N} \) be a super nebula under \( \theta \) and besides for every star \( Q_i \) of \( \mathcal{N} \) under \( \theta \), \( Q_1 \) is a 1-left middle star star or a left star, and all the super 2-nebulas of \( \mathcal{N} \) under \( \theta \) are right super 2-nebulas. Let \( G \) be a right triangular galaxy under \( \alpha \) and besides \( G \) has only one triangle under \( \alpha \). Then \( \{\mathcal{N}, G\} \) satisfy EHC.

**Theorem 8** Let \( \mathcal{N} \) be a super nebula under \( \theta \) and besides all the stars of \( \mathcal{N} \) under \( \theta \) are frontier stars, and for every super 2-nebula \( \Sigma_i \) of \( \mathcal{N} \) under \( \theta \), \( \Sigma_i \) is a middle super 2-nebula. Let \( G \) be a central triangular galaxy under \( \alpha \) and besides \( G \) has only one triangle under \( \alpha \). Then \( \{\mathcal{N}, G\} \) satisfy EHC.

We say that a tournament \( H \) is a super \( \Delta \)galaxy under \( \theta \) if it is a super triangular galaxy under \( \theta \) and besides \( H \) has only one triangle under \( \theta \). We say that a tournament \( H \) is an \( LR-\Delta \)galaxy under \( \theta \) if it is a super \( \Delta \)galaxy under \( \theta \) and besides the vertices of the triangle \( \Delta \) of \( H \) under \( \theta \) that are allowed to be in the ordering \( \theta \) between leaves of a star of \( H \) under \( \theta \) are only the exteriors of \( \Delta \). We say that a tournament \( H \) is a \( CR-\Delta \)galaxy (resp. \( CL-\Delta \)galaxy) under \( \theta \) if it is a super \( \Delta \)galaxy under \( \theta \) and besides the vertices of the triangle \( \Delta \) of \( H \) under \( \theta \) that are allowed to be in the ordering \( \theta \) between leaves of a star of \( H \) under \( \theta \) are the right (resp. left) exterior and the center of \( \Delta \), such that: if \( Q_i \) and \( Q_j \) are frontier stars of \( H \) under \( \theta \), such that the center of \( \Delta \) is in the ordering between the leaves of \( Q_i \) and the right (resp. left) exterior of \( \Delta \) is in the ordering between the leaves of \( Q_j \), then no leaf of \( Q_i \) is between the leaves of \( Q_j \) under \( \theta \) and no leaf of \( Q_j \) is between the leaves of \( Q_i \) under \( \theta \). And if the center of \( \Delta \) is between the leaves of \( Q_i \) for some star \( Q_i \) of \( H \) under \( \theta \), then the right (resp. left) exterior of \( \Delta \) is not between the leaves of \( Q_i \) under \( \theta \).

We say that a tournament \( H \) is a super left nebula (resp. super right nebula) under \( \theta \) if it is a nebula under \( \theta \) and besides all the stars of \( H \) under \( \theta \) are left stars (resp. right stars).

**Theorem 9** If \( H_1 \) and \( H_2 \) are: a central nebula and a \( LR-\Delta \)galaxy, or: a super left nebula and a \( CR-\Delta \)galaxy, or: a super right nebula and a \( CL-\Delta \)galaxy, then \( \{H_1, H_2\} \) satisfies the Erdős–Hajnal Conjecture.
We omit the proof of Theorems 6, 7, 8, 9 because they have completely the same proof of Theorem 2.

5.2 Generalization of Theorem 3

Let $H$ be a tournament such that there exists an ordering $\theta$ of its vertices such that $V(H)$ is the disjoint union of $V(\Sigma)$, $V(Q_1)$, ..., $V(Q_m)$ where $Q_1, \ldots, Q_m$ are the frontier stars of $H$ under $\theta$, $\Sigma$ is the super 2-nebula of $H$ under $\theta$, no center of a star is between leaves of $\Sigma$ under $\theta$, no center of $\Sigma$ is between leaves of a star of $H$ under $\theta$, and no center of a star appears in the ordering $\theta$ between leaves of another star. In this case $H$ is called a $\Sigma$-galaxy under $\theta$ and $\theta$ is called a $\Sigma$-galaxy ordering of $H$. If $\Sigma$ is a super middle 2-nebula (resp. super left 2-nebula) (resp. super right 2-nebula) then $H$ is called a middle $\Sigma$-galaxy (resp. left $\Sigma$-galaxy) (resp. right $\Sigma$-galaxy). Obviously one can notice that $K_6$ is a middle $\Sigma$-galaxy and its canonical ordering is its $\Sigma$-galaxy ordering. Also notice that every $\Sigma$-galaxy is a super nebula. The following theorem is a generalization of Theorem 3:

**Theorem 10** If $H_1$ and $H_2$ are: a middle $\Sigma$-galaxy and a central triangular galaxy, or: a left $\Sigma$-galaxy and a left triangular galaxy, or: a right $\Sigma$-galaxy and a right triangular galaxy, then $\{H_1, H_2\}$ satisfies the Erdős–Hajnal Conjecture.

We omit the proof of Theorem 10, since it is completely analogous to the proof of Theorem 3. The proof uses the notion of key tournaments $K$ under $\theta$ corresponding to $H_1$ under its $\Sigma$-galaxy ordering and $H_2$ under its super triangular galaxy ordering. The problem we face is that when looking for a well-contained copy of $H_1$ (resp. $H_2$) in an appropriate smooth $(c, \lambda, w)$-structure, there are a group of arcs that we know nothing about their orientation. This is the place where we need to use key tournaments constructed depending on both $H_1$ and $H_2$ (we construct it following the same principle in Sect. 4.1). We first find this mutant key tournament $\tilde{K}$ as a well-contained copy in a smooth $(c, \lambda, w)$-structure corresponding to $K$ under $\theta$. Then we extract $H_1$ or $H_2$ depending on the orientation of the arcs where the problem is faced.

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