Solvability of a class of mean-field BSDEs with quadratic growth

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Abstract: In this paper, we study the multi-dimensional mean-field backward stochastic differential equations (BSDEs, for short) with quadratic growth. Under small terminal value, the existence and uniqueness are proved for the multi-dimensional situation when the generator \( f(t, Y, \mathbb{E}[Y], Z, \mathbb{E}[Z]) \) is of quadratic growth with respect to the last four items, using some new methods. Besides, a kind of comparison theorem is obtained.

Key words: mean-field backward stochastic differential equation, backward stochastic differential equation, quadratic growth.

AMS subject classifications. 60H10, 60H30

1 Introduction

Mean-field stochastic differential equations (SDEs, for short), also called McKean-Vlasov equations, can be traced back to the works of Kac [19] in the 1950s. Recently, mean-field backward stochastic differential equations (BSDEs, for short) were introduced by Buckdahn, Djehiche, Li and Peng [6] and Buckdahn, Li and Peng [7], owing to that mathematical mean-field approaches play an important role in many fields, such as economics, physics, and game theory (see Lasry and Lions [21]). Since then, mean-field BSDEs have received intensive attention. In order to present the work more clearly, we describe the problem in detail.

Assume that \( \{W_t; 0 \leq t < \infty\} \) is a \( d \)-dimensional standard Brownian motion defined on a complete filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), where \( \mathbb{F} = \{\mathcal{F}_t; 0 \leq t < \infty\} \) is the filtration...
generated by $W$ and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. Consider the following mean-field backward stochastic differential equations over a finite horizon $[0, T]$:

$$
\begin{align*}
& -dY_t = f(t, Y_t, E[Y_t], Z_t, E[Z_t])dt - Z_tdW_t, \quad t \in [0, T]; \\
& Y_T = \xi,
\end{align*}
$$

(1.1)

where the random variable $\xi$ is called the terminal value and the mapping $f(\cdot)$ is called the generator. The pair of unknown processes $(Y, Z)$, called an adapted solution of (1.1), are $\mathbb{F}$-adapted with values in $\mathbb{R}^m \times \mathbb{R}^{m \times d}$. For convenience, hereafter, by quadratic mean-field BSDEs, or mean-field BSDEs with quadratic growth, we mean that in (1.1), the generator $f(\cdot)$ grows in $Z$ quadratically. Meanwhile, we call $\xi$ the bounded terminal value if it is bounded. In addition, we call $\xi$ the small terminal value, if there exists a small positive constant $\varepsilon$ such that $\|\xi\|_{\infty} \leq \varepsilon$.

As a natural extension of BSDEs (see below for precise description), the mean-field BSDEs (1.1) were introduced by Buckdahn et al. [6, 7], where they established the existence and uniqueness of adapted solutions under the condition that $f(\cdot)$ is uniformly Lipschitz in the last four arguments. From then on, the theory and applications of mean-field BSDEs have been developed significantly. For example, Carmona and Delarue provided a detailed probabilistic analysis of the optimal control of nonlinear stochastic dynamical systems of the McKean-Vlasov type in [10], and studied some special class of quadratic forward-backward stochastic differential equations (FBSDEs, for short) of mean-field type in [11]. Buckdahn et al. [8] studied the general mean-field stochastic differential equations and their relation with the associated PDEs. Zhang, Sun and Xiong [30] obtained a general stochastic maximum principle for a Markov regime switching jump-diffusion model of mean-field type. Briand, Elie and Hu [3] introduced and studied the BSDEs with mean reflection. Li, Sun and Xiong [22] established the results for linear quadratic optimal control problems for mean-field backward stochastic differential equations. Douissi, Wen and Shi [13] and Shi, Wen and Xiong [26] studied the optimal control problem of mean-field forward-backward stochastic systems driven by fractional Brownian motion.

Now, we recall the following nonlinear BSDEs:

$$
\begin{align*}
& -dY_t = f(t, Y_t, Z_t)dt - Z_tdW_t, \quad t \in [0, T]; \\
& Y_T = \xi,
\end{align*}
$$

(1.2)

When the generator $f(\cdot)$ is of linear growth with respect to $(Y, Z)$, the existence and uniqueness of (1.2) were firstly proved by Pardoux and Peng [24]. Since then, a lot of researchers have found that BSDEs have important applications in mathematical finance, stochastic optimal control and partial differential equation (see [14, 23, 29], to mention a few). Meanwhile, owing to many important applications, a lot of efforts have been made to relax the conditions on the generator $f(\cdot)$ of (1.2) with respect to $Z$. For example, Kobylianski [20] proved the existence and uniqueness of one-dimensional BSDE with bounded terminal condition and with $f(\cdot)$ growing quadratically in $Z$. The well-posedness of one-dimensional quadratic BSDE with unbounded terminal value was obtained by Briand and Hu [4, 5]. Hibon et al. [15] studied a class of quadratic BSDEs with mean reflection. Hu, Li and Wen [18] obtained the existence and uniqueness of anticipated BSDEs with quadratic
growth. Some other recent developments of quadratic BSDEs can be found in Bahlali, Eddahbi and Ouknine [1], Barrieu and El Karoui [2], Cheridito and Nam [9], Hibon, Hu and Tang [16], Hu and Tang [17], Tevzadze [27], Xing and Zitkovic [28], and references cited therein.

Let us introduce a motivation to study (1.1) at the particles level. Consider the following particle system with $N$ particles:

$$Y_i^t = \xi_i + \int_t^T f^i(s, Y_s^i, \frac{1}{N} \sum_{i=1}^N Y_s^i, \frac{1}{N} \sum_{i=1}^N Z_s^{ii}) ds - \sum_{j=1}^N \int_t^T Z_s^{ij} dW_s^j, \quad t \in [0, T],$$

where for each $i, j = 1, \ldots, N$, $Z_s^{ij}$ is an $m \times d$ matrix; $\{W_s^j; 1 \leq j \leq N\}$ are $N$ independent $d$-dimensional Brownian motions; $\xi_i$ and $f^i$ are $N$ independent copies of $\xi$ and $f$, respectively. Following Lions’s idea and the law of large numbers, we conjecture that when $N$ tends to $\infty$, the mean-field limit of solutions of the above particle system corresponds to that of the mean-field BSDE (1.1).

On the other hand, in the last two decades, stimulated by the broad applications and the open problem proposed by Peng [25], a lot of efforts have been made to relax the conditions on the generator $f$ of the mean-field BSDEs. Hibon, Hu and Tang [16] considered quadratic mean-field BSDEs in one-dimensional situation when the generator $f$ depends on the expectation of $(Y, Z)$, and studied the existence and uniqueness of related equations. However, the mean-field BSDEs (1.1) with quadratic growth in the multi-dimensional situation is still open. In this paper, along with the work of [16], we study the solvability of mean-field BSDEs (1.1) with quadratic growth in the multi-dimensional situation. First, borrowing some ideas from Tevzadze [27], we construct an artful method to prove that, under small terminal value, the mean-field BSDE (1.1) admits a unique adapted solution in the multi-dimensional situation. It should be pointed out that, besides the multidimensional situation, our contribution also includes the generator $f(\cdot)$ is of quadratic growth with respect to all items of $(Y, \mathbb{E}[Y], Z, \mathbb{E}[Z])$. Then, a comparison theorem for such equations is obtained for the one-dimensional situation.

This article is organized as follows. Some preliminaries are presented in Section 2. The existence and uniqueness of multi-dimensional quadratic mean-field BSDEs (1.1) with small terminal value are proved by the fixed point argument in Section 3. A comparison theorem for such BSDEs is given in Section 4.

2 Preliminaries

Recall that $\{W_t; 0 \leq t < \infty\}$ is a $d$-dimensional standard Brownian motion defined on the complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is the filtration generate generated by $W$ and augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. We denote by $\mathbb{R}^{m \times d}$ the space of the $m \times d$-matrix $C$ with Euclidean norm $|C| = \sqrt{tr(CC^*)}$. In the following, for Euclidean space $\mathbb{H}$ and $t \in [0, T]$, we denote

$$L_{\mathcal{F}_t}^\infty(\Omega; \mathbb{H}) = \left\{ \theta: \Omega \to \mathbb{H} \mid \theta \text{ is } \mathcal{F}_t\text{-measurable, } \|\theta\|_\infty \triangleq \operatorname{esssup}_{\omega \in \Omega} |\theta(\omega)| < \infty \right\},$$

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\[ L^2_F(t, T; \mathbb{H}) = \left\{ \varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi \text{ is } \mathbb{F}-\text{progressively measurable,} \right\}, \]

\[ \|\varphi\|_{L^2_F(t, T)} \triangleq \left( \mathbb{E} \int_t^T |\varphi_s|^2 ds \right)^{\frac{1}{2}} < \infty, \]

\[ L^\infty_F(t, T; \mathbb{H}) = \left\{ \varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi \text{ is } \mathbb{F}-\text{progressively measurable,} \right\}, \]

\[ \|\varphi\|_{L^\infty_F(t, T)} \triangleq \sup_{(s, \omega) \in [t, T] \times \Omega} |\varphi_s(\omega)| < \infty, \]

\[ S^2_F(t, T; \mathbb{H}) = \left\{ \varphi : [t, T] \times \Omega \to \mathbb{H} \mid \varphi \text{ is } \mathbb{F}-\text{adapted, continuous,} \right\}, \]

\[ \|\varphi\|_{S^2_F(t, T)} \triangleq \left\{ \mathbb{E} \left( \sup_{s \in [t, T]} |\varphi_s|^2 \right)^{\frac{1}{2}} < \infty \right\}. \]

Let \( M = (M_t, \mathcal{F}_t) \) be a uniformly integrable martingale with \( M_0 = 0 \), and for \( p \in [1, \infty) \), we set

\[ \|M\|_{BMO_p} \triangleq \sup_{\tau} \left\| \mathbb{E}_\tau \left[ \left( \langle M \rangle_\tau \right)^{\frac{p}{2}} \right] \right\|_{\infty}, \]

where the supremum is taken over all \( \mathbb{F} \)-stopping times \( \tau \), and \( \mathbb{E}_\tau \) is the conditional expectation given \( \mathcal{F}_\tau \). The class \( \{ M : \| M \|_{BMO_p} < \infty \} \) is denoted by \( BMO_p(\mathbb{P}) \). Observe that \( \| \cdot \|_{BMO_p} \) is a norm on this space and \( BMO_p(\mathbb{P}) \) is a Banach space. In the sequel, we denote \( BMO(\mathbb{P}) \) the space of \( BMO_2(\mathbb{P}) \) for simplicity. Next, for any \( Z \in L^2_F(0, T; \mathbb{H}) \), by Burkholder-Davis-Gundy’s inequalities, one has

\[ c_2 \mathbb{E}_\tau \left[ \left( \int_\tau^T |Z_s|^2 ds \right) \right] \leq \mathbb{E}_\tau \left[ \sup_{t \in [\tau, T]} \int_\tau^t Z_s dW_s \right]^2 \leq C_2 \mathbb{E}_\tau \left[ \left( \int_\tau^T |Z_s|^2 ds \right) \right], \]

for some constants \( c_2, C_2 > 0 \). Thus,

\[ c_2 \sup_{\tau \in \mathcal{F}[t, T]} \left\| \mathbb{E}_\tau \left[ \left( \int_\tau^T |Z_s|^2 ds \right) \right] \right\|_{\infty} \leq \sup_{\tau \in \mathcal{F}[t, T]} \left\| \mathbb{E}_\tau \left[ \sup_{t \in [\tau, T]} \int_\tau^t Z_s dW_s \right]^2 \right\|_{\infty} \]

\[ \leq C_2 \sup_{\tau \in \mathcal{F}[t, T]} \left\| \mathbb{E}_\tau \left[ \left( \int_\tau^T |Z_s|^2 ds \right) \right] \right\|_{\infty}, \]

where \( \mathcal{F}[t, T] \) denotes the set of all \( \mathbb{F} \)-stopping times \( \tau \) valued in \( [t, T] \). Note that the above quantities could be infinite. Therefore, we introduce the following:

\[ Z^2[t, T] = \left\{ Z \in L^2_F(t, T; \mathbb{H}) \mid \|Z\|_{Z^2[t, T]} \equiv \sup_{\tau \in \mathcal{F}[t, T]} \left\| \mathbb{E}_\tau \left[ \int_\tau^T |Z_s|^2 ds \right] \right\|^{\frac{1}{2}} < \infty \right\}. \]

Recall that for \( Z \in Z^2[0, T] \), the process \( s \mapsto \int_0^s Z_r dW_r \) (denoted by \( Z \cdot W \), \( s \in [0, T] \), is a \( BMO \)-martingale. Moreover, note that on \( [0, T] \), \( Z \cdot W \) belongs to \( BMO(\mathbb{P}) \) if and only if \( Z \in Z^2[0, T] \), that is,

\[ \|Z \cdot W\|_{BMO(\mathbb{P})}^2 = \|Z\|_{Z^2[0, T]}^2. \]

**Definition 2.1.** A pair of processes \((Y, Z) \in S^2_F(0, T; \mathbb{R}^m) \times L^2_F(0, T; \mathbb{R}^{m \times d})\) is called an adapted solution of BSDE (1.1), if \( \mathbb{P} \)-almost surely, it satisfies (1.1). Moreover, if \((Y, Z) \in L^2_F(0, T; \mathbb{R}^m) \times Z^2[0, T]\), it is called a bounded adapted solution.
3 Existence and Uniqueness

In this section, we study multi-dimensional mean-field BSDEs with quadratic growth and small terminal value. By the theory of ordinary differential equations, we see that if the generator \( f(\cdot) \) is super-linear with respect to \( Y \), then the equations may not have global solutions. However, we pointed out that, under small bounded value, the generator could be of quadratic growth with respect to \( Y \) and \( Z \) (see Hu, Li and Wen [18]). Now let us introduce the following hypothesis.

**Assumption 3.1.** Let \( C \) be a positive constant. For all \( s \in [0, T] \), \( y_i, \bar{y}_i \in \mathbb{R}^m \), \( z_i, \bar{z}_i \in \mathbb{R}^{m \times d} \) with \( i = 1, 2 \), \( f(s, 0, 0, 0, 0) \) is bounded and

\[
|f(s, y_1, \bar{y}_1, z_1, \bar{z}_1) - f(s, y_2, \bar{y}_2, z_2, \bar{z}_2)| \leq C \left( |y_1| + |\bar{y}_1| + |y_2| + |\bar{y}_2| + |z_1| + |\bar{z}_1| + |z_2| + |\bar{z}_2| \right).
\]

**Example 3.2.** Assumption 3.1 implies that the generator \( f(\cdot) \) could be of quadratic growth with respect to the last four arguments. The following generator

\[
f(s, y_1, y_2, z_1, z_2) = y^2 + \bar{y}^2 + z^2 + \bar{z}^2, \quad s \in [0, T], y, \bar{y} \in \mathbb{R}^m, z, \bar{z} \in \mathbb{R}^{m \times d}
\]
satisfies such an assumption.

In the following, we state and prove the main result of this section, which establishes the existence and uniqueness of multi-dimensional BSDE (1.1) with quadratic growth and small terminal value.

**Theorem 3.3.** Under the Assumption 3.1, for any bounded \( \xi \in L^\infty_{\mathcal{F}_T}(\Omega; \mathbb{R}^m) \) satisfying

\[
\|\xi\|_{\infty} + \left\| \int_0^T |f(t, 0, 0, 0, 0)| dt \right\|_{\infty} \leq \rho, \tag{3.1}
\]

where

\[
\rho^2 = \frac{1}{4097C^2(T^2 + 1)},
\]

we have that mean-field BSDE (1.1) admits a unique adapted solution \((Y, Z)\) in \( \mathcal{B}_\rho \), where

\[
\mathcal{B}_\rho \triangleq \left\{ (Y, Z) \in L^\infty_{\mathcal{F}_T}(0, T; \mathbb{R}^m) \times \mathcal{Z}\mathcal{Z}^2[0, T] \left| \|Y\|_{L^\infty_{\mathcal{F}_T}(0, T)} + \|Z\|_{\mathcal{Z}\mathcal{Z}^2[0, T]} \leq \rho^2 \right. \right\}.
\]

**Proof.** The proof is divided into two steps.

**Step 1.** We firstly consider the existence and uniqueness of the following BSDE

\[
Y_t = \xi + \int_t^T \left( f(s, Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s]) - f(s, 0, 0, 0, 0) \right) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{3.2}
\]

In order to solve the above equation, for every \((y, z) \in L^\infty_{\mathcal{F}}(0, T; \mathbb{R}^m) \times \mathcal{Z}\mathcal{Z}^2[0, T]\), we define the mapping \((Y, Z) = \Gamma(y, z)\) by

\[
Y_t = \xi + \int_t^T \left( f(s, y_s, \mathbb{E}[y_s], z_s, \mathbb{E}[z_s]) - f(s, 0, 0, 0, 0) \right) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{3.3}
\]
For (3.3), using Itô’s formula to $|Y|^2$ on $[t, T]$, we obtain

$$|Y_t|^2 + \int_t^T |Z_r|^2 \, dr = |\xi|^2 + \int_t^T 2Y_r \cdot \left( f(r, y_r, E[y_r], z_r, E[z_r]) - f(r, 0, 0, 0, 0) \right) \, dr - 2 \int_t^T Y_r \cdot Z_r \, dW_r.$$ 

Taking the conditional expectation and using the inequality $2ab \leq \frac{1}{4}a^2 + 2b^2$, we get

$$|Y_t|^2 + \mathbb{E}_t \int_t^T |Z_r|^2 \, dr \leq \|\xi\|_\infty^2 + 2\|Y\|_{L^\infty(0,T)} \left( \mathbb{E}_t \int_t^T |f(r, y_r, E[y_r], z_r, E[z_r]) - f(r, 0, 0, 0, 0)| \, dr \right)$$

$$\leq \|\xi\|_\infty^2 + 2\|Y\|_{L^\infty(0,T)}^2 \left( \mathbb{E}_t \int_t^T |f(r, y_r, E[y_r], z_r, E[z_r]) - f(r, 0, 0, 0, 0)| \, dr \right)^2 \tag{3.4}.$$ 

It follows from Jensen’s inequality that the last term of (3.4) naturally reduces to

$$\mathbb{E}_t \int_t^T |f(r, y_r, E[y_r], z_r, E[z_r]) - f(r, 0, 0, 0, 0)| \, dr \leq C \mathbb{E}_t \int_t^T \left( |y_r| + |E[y_r]| + |z_r| + |E[z_r]| \right)^2 \, dr$$

$$\leq 4C \mathbb{E}_t \int_t^T \left( |y_r|^2 + |E[y_r]|^2 + |z_r|^2 + |E[z_r]|^2 \right) \, dr$$

$$\leq 4C \left( \mathbb{E}_t \int_t^T (|y_r|^2 + |z_r|^2) \, dr + \mathbb{E} \int_t^T (|y_r|^2 + |z_r|^2) \, dr \right). \tag{3.5}$$ 

Hence, combining (3.4) and (3.5), we can obtain

$$\frac{1}{2} \|Y\|_{L^\infty(0,T)}^2 + \|Z\|_{L^2_{[0,T]}}^2 \leq \|\xi\|_\infty^2 + 32C^2 \limsup_{(t,\omega)\in[0,T] \times \Omega} \left[ \mathbb{E}_t \int_t^T (|y_r|^2 + |z_r|^2) \, dr + \mathbb{E} \int_t^T (|y_r|^2 + |z_r|^2) \, dr \right]^2$$

$$\leq \|\xi\|_\infty^2 + 64C^2 \left( T^2 \|y\|_{L^\infty(0,T)}^4 + \|z\|_{L^2(0,T)}^4 \right).$$ 

Then, it follows from the elementary inequality $a^2 + b^2 \leq (|a| + |b|)^2$ that

$$\|Y\|_{L^\infty(0,T)}^2 + \|Z\|_{L^2_{[0,T]}}^2 \leq 4\|\xi\|_\infty^2 + \beta^2 \left( \|y\|_{L^\infty(0,T)}^2 + \|z\|_{L^2(0,T)}^2 \right)^2,$$

where $\beta \triangleq 16C \sqrt{T^2 + 1}$. Now, we would like to pick $R$ such that

$$4\|\xi\|_\infty^2 + \beta^2 R^4 \leq R^2.$$
In fact, the above inequality is solvable if and only if
\[
\|\xi\|_\infty \leq \frac{1}{4\beta}.
\] (3.6)

For example, we can take
\[
R = 2\sqrt{2}\|\xi\|_\infty
\]
in order to satisfy this quadratic inequality. Then the ball
\[
\mathcal{B}_R \triangleq \left\{ (Y, Z) \in L^\infty_T (0, T; \mathbb{R}^m) \times Z^2_T | \| Y \|^2_{L^\infty_T (0, T)} + \| Z \|^2_{Z^2_T (0, T)} \leq R^2 \right\}
\]
is such that \( \Gamma (\mathcal{B}_R) \subset \mathcal{B}_R \).

**Step 2.** We prove that the mapping \( \Gamma \) is a contraction on \( \mathcal{B}_R \).

For every \((y, z)\), \((\bar{y}, \bar{z})\) \(\in\mathcal{B}_R\), let \((Y, Z) = \Gamma (y, z)\) and \((\bar{Y}, \bar{Z}) = \Gamma (\bar{y}, \bar{z})\). For simplicity of presentation, we denote
\[
\dot{y} = y - \bar{y}, \quad \dot{z} = z - \bar{z}, \quad \dot{Y} = Y - \bar{Y}, \quad \dot{Y} = Y - \bar{Y}.
\]

Similar to the above discussion, we deduce that
\[
\frac{1}{2} \| \dot{Y} \|^2_{L^\infty_T (0, T)} + \| \dot{Z} \|^2_{Z^2_T (0, T)}
\]
\[
\leq 2 \text{ esssup}_{(t, \omega) \in [0, T] \times \Omega} \left[ \mathbb{E}_t \int_t^T \left( f(r, y, \mathbb{E}[y], z, \mathbb{E}[z]) - f(r, \bar{y}, \mathbb{E}[\bar{y}], \bar{z}, \mathbb{E}[\bar{z}]) \right) dr \right]^2
\]
\[
\leq 2C^2 \text{ esssup}_{(t, \omega) \in [0, T] \times \Omega} \left[ \mathbb{E}_t \int_t^T \left( \| y \| + \| \bar{y} \| + \mathbb{E}[\| y \|] + \mathbb{E}[\| \bar{y} \|] + \| z \| + \| \bar{z} \| + \mathbb{E}[\| z \|] + \mathbb{E}[\| \bar{z} \|] \right) \right]^2 dr
\]
\[
\leq 2C^2 \text{ esssup}_{(t, \omega) \in [0, T] \times \Omega} \left[ \mathbb{E}_t \int_t^T \left( \| y \| + \| \bar{y} \| + \mathbb{E}[\| y \|] + \mathbb{E}[\| \bar{y} \|] + \| z \| + \| \bar{z} \| + \mathbb{E}[\| z \|] + \mathbb{E}[\| \bar{z} \|] \right)^2 dr
\]
\[
\leq 64C^2 \text{ esssup}_{(t, \omega) \in [0, T] \times \Omega} \left[ \mathbb{E}_t \int_t^T \left( \| y \|^2 + \| \bar{y} \|^2 + \mathbb{E}[\| y \|^2] + \mathbb{E}[\| \bar{y} \|^2] + \| z \|^2 + \| \bar{z} \|^2 + \mathbb{E}[\| z \|^2] + \mathbb{E}[\| \bar{z} \|^2] \right) dr
\]
\[
\times \left( \| y \|^2 + \| \bar{y} \|^2 + \mathbb{E}[\| y \|^2] + \mathbb{E}[\| \bar{y} \|^2] + \| z \|^2 + \| \bar{z} \|^2 + \mathbb{E}[\| z \|^2] + \mathbb{E}[\| \bar{z} \|^2] \right)
\]
\[
\leq 256C^2(T^2 + 1) \left[ \| y \|^2_{L^\infty_T (0, T)} + \| z \|^2_{Z^2_T (0, T)} + \| y \|^2_{L^\infty_T (0, T)} + \| z \|^2_{Z^2_T (0, T)} \right]
\]
\[
\times \left( \| y \|^2_{L^\infty_T (0, T)} + \| z \|^2_{Z^2_T (0, T)} \right).
\]

Noting that
\[
\| y \|^2_{L^\infty_T (0, T)} + \| z \|^2_{Z^2_T (0, T)} \leq R^2, \quad \| y \|^2_{L^\infty_T (0, T)} + \| z \|^2_{Z^2_T (0, T)} \leq R^2,
\]
we obtain
\[ \|\dot{Y}\|_{L^\infty_T(0,T)}^2 + \|\dot{Z}\|_{L^2_T(0,T)}^2 \leq MR^2 (\|\dot{Y}\|_{L^\infty_T(0,T)}^2 + \|\dot{Z}\|_{L^2_T(0,T)}^2), \]
where
\[ M \triangleq 512C^2(T^2 + 1). \]
Now, note that \( R = 2\sqrt{3}\|\xi\|_{\infty} \), we have that if
\[ \|\xi\|_{\infty} < \frac{1}{8M}, \quad (3.7) \]
then
\[ MR^2 < 1, \]
which implies that \( \Gamma \) is a contraction on \( B_R \). By the contraction principle, under the condition (3.7), the mapping \( \Gamma \) admits a unique fixed point, which is the solution of (3.2).

Finally, we come back to BSDE (1.1), which can be rewritten into the form of (3.2) as follows:
\[ Y_t = \tilde{\xi} + \int_t^T \left( f(s, Y_s, E[Y_s], Z_s, E[Z_s]) - f(s, 0, 0, 0, 0) \right) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad (3.8) \]
where
\[ \tilde{\xi} \triangleq \xi + \int_t^T f(s, 0, 0, 0, 0) ds. \]
Note that (3.7) is stronger than (3.6). So if we define \( \rho > 0 \) by letting
\[ \rho^2 \triangleq \frac{1}{4097C^2(T^2 + 1)}, \]
then BSDE (1.1) admits a unique adapted solution \((Y, Z) \in B_\rho\) under the following condition
\[ \|\xi\|_{\infty} + \left\| \int_0^T |f(t, 0, 0, 0, 0)| dt \right\|_{\infty} \leq \rho. \quad (3.9) \]
This completes the proof. \( \square \)

**Remark 3.4.** Comparing with the results of Carmona and Delarue [11], where they studied some kind of quadratic FBSDE of mean-field type, we would like to show two differences.

(i) The equations of mean-field type are different between this paper and [11]. In [11], motivated by the problem of the mean-field game, Carmona and Delarue proved the solvability of the following FBSDE with quadratic growth:
\[
\begin{cases}
    dX_t = b(t, X_t, \mathcal{L}(X_t), \alpha(t, X_t, \mathcal{L}(X_t), (\sigma(t, X_t, \mathcal{L}(X_t))^{-1})^T Z_t)) dt \\
        + \sigma(t, X_t, \mathcal{L}(X_t)) dW_t, \\
    dY_t = -f(t, X_t, \mathcal{L}(X_t), \alpha(t, X_t, \mathcal{L}(X_t), (\sigma(t, X_t, \mathcal{L}(X_t))^{-1})^T Z_t)) dt + Z_t dW_t, \\
    X_0 = \xi \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d), \quad Y_T = g(X_T, \mathcal{L}(X_T)),
\end{cases}
\]
where \( \mathcal{L}(X_t) \) denotes the law of the process \( X_t \). Comparing the backward equation of (3.10) with (1.1), it is easy to see that the generator \( f \) of (3.10) depends on the law of the process \( X \), however, the generator \( f(\cdot) \) of (1.1) depends on the expectations of \( Y \) and \( Z \). So the backward equation of (3.10) and (1.1) are two different equations of mean-field type.
The circumstances are different. The circumstance of Carmona and Delarue [11] is Markovian, however, our model is non-Markovian. Besides, the backward equation of (3.10) studied in [11] is a one-dimensional BSDE, however, (1.1) is a multi-dimensional BSDE.

4 Comparison Theorem

In this section, we study the comparison theorem of mean-field BSDEs with quadratic growth of the following form:

\[ Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \]  

(4.1)

We consider BSDE (4.1) in one-dimensional case only, i.e., \( m = 1 \). For simplicity of presentation, we let \( d = 1 \) too. For this situation, we have the following lemma.

**Lemma 4.1.** Let \( C \) be a positive constant, and suppose that there are two increasing functions \( \lambda : \mathbb{R}^+ \to \mathbb{R}^+ \) and \( \bar{\lambda} : \mathbb{R}^+ \to \mathbb{R}^+ \), bounded on all bounded subsets, and a predictable process \( k \in \mathbb{Z}^2[0, T] \) such that for all \( s \in [0, T] \), \( y, \bar{y}, z \in \mathbb{R} \),

\[ |f(s, y, \bar{y}, z)| \leq k_s^2 [\lambda(|y|) + \bar{\lambda}(|\bar{y}|)] + Cz^2. \]  

(4.2)

Then, for bounded terminal value \( \xi \in L^\infty(\Omega; \mathbb{R}) \), the martingale part of any bounded solution of BSDE (4.1) belongs to the space \( \text{BMO} \) (\( P \)), i.e., \( Z \in \mathbb{Z}^2[0, T] \).

**Proof.** Let \( Y \) be a solution of BSDE (4.1) and there be a positive constant \( M \) such that \( Y_t \leq M \), a.s. for all \( t \in [0, T] \).

So we have that \( \|\xi\|_\infty \leq M \). Applying Itô formula to \( \exp\{\beta Y_s\} \) on \( [\tau, T] \), we have

\[ \frac{\beta^2}{2} \int_\tau^T e^{\beta Y_s} Z_s^2 ds - \beta \int_\tau^T e^{\beta Y_s} f(s, Y_s, \mathbb{E}[Y_s], Z_s) ds + \beta \int_\tau^T e^{\beta Y_s} Z_s dW_s = e^{\beta \xi} - e^{\beta T \tau} \leq e^{\beta M}, \]

or

\[ \frac{\beta^2}{2} \int_\tau^T e^{\beta Y_s} Z_s^2 ds + \beta \int_\tau^T e^{\beta Y_s} Z_s dW_s \leq e^{\beta M} + \beta \int_\tau^T e^{\beta Y_s} f(s, Y_s, \mathbb{E}[Y_s], Z_s) ds, \]

where \( \beta \) is a constant which will be determined later. Now, if \( Z \cdot W \) is square integrable martingale, then taking conditional expectations on the above inequality, we obtain that

\[ \frac{\beta^2}{2} \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} Z_s^2 ds \leq e^{\beta M} + \beta \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} f(s, Y_s, \mathbb{E}[Y_s], Z_s) ds. \]

Using the estimate (4.2) we obtain that

\[ \frac{\beta^2}{2} \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} Z_s^2 ds \leq e^{\beta M} + \beta (\lambda(M) + \bar{\lambda}(M)) \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} k_s^2 ds + \beta C \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} |Z_s|^2 ds. \]
Thus we have
\[
\| \hat{\beta}^2 - \beta C \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} Z_s^2 ds \| \leq e^{\beta M} + \beta [\lambda(M) + \bar{\lambda}(M)] \mathbb{E}_\tau \int_\tau^T e^{\beta Y_s} k_s^2 ds.
\]
Taking \(\beta = 4C\), we deduce that
\[
4C^2 \mathbb{E}_\tau \int_\tau^T e^{4CY_s} Z_s^2 ds \leq e^{4CM} + 4C[\lambda(M) + \bar{\lambda}(M)] \mathbb{E}_\tau \int_\tau^T e^{4CY_s} k_s^2 ds \leq e^{4CM} \left(1 + 4C[\lambda(M) + \bar{\lambda}(M)] \|k\|_{Z^2[0,T]} \right).
\]
Note that \(-M \leq Y \leq M\), from the latter inequality we deduce that for any stopping time \(\tau\),
\[
\mathbb{E}_\tau \int_\tau^T Z_s^2 ds \leq \frac{e^{8CM} \left(1 + 4C[\lambda(M) + \bar{\lambda}(M)] \|k\|_{Z^2[0,T]} \right)}{4C^2}.
\]
Hence \(Z \cdot W \in \text{BMO}\), i.e., \(Z \in \mathcal{Z}^2[0,T]\). This completes the proof.

For simplicity of presentation, in the following we introduce some more notations. Let \((Y, Z)\) and \((\tilde{Y}, \tilde{Z})\) be two pairs of processes, and the coefficients \((f, \xi)\) and \((\tilde{f}, \tilde{\xi})\) be two pairs of generators. We define
\[
\delta f = f - \tilde{f}, \quad \delta \xi = \xi - \tilde{\xi},
\]
\[
\delta_y f(t) \equiv \delta_y f(t, Y_t, \tilde{Y}_t, \mathbb{E}[Y_t], \tilde{Z}_t) = \frac{f(t, Y_t, \mathbb{E}[Y_t], Z_t) - f(t, \tilde{Y}_t, \mathbb{E}[Y_t], Z_t)}{Y_t - \tilde{Y}_t},
\]
\[
\delta_y f(t) \equiv \delta_y f(t, \tilde{Y}_t, \mathbb{E}[Y_t], \mathbb{E}[\tilde{Y}_t], Z_t) = \frac{f(t, \tilde{Y}_t, \mathbb{E}[Y_t], Z_t) - f(t, \tilde{Y}_t, \mathbb{E}[\tilde{Y}_t], Z_t)}{\mathbb{E}[Y_t] - \mathbb{E}[\tilde{Y}_t]},
\]
\[
\delta_z f(t) \equiv \delta_z f(t, \tilde{Y}_t, \mathbb{E}[\tilde{Y}_t], Z_t, \tilde{Z}_t) = \frac{f(t, \tilde{Y}_t, \mathbb{E}[\tilde{Y}_t], Z_t) - f(t, \tilde{Y}_t, \mathbb{E}[\tilde{Y}_t], \tilde{Z}_t)}{Z_t - \tilde{Z}_t},
\]
and \(\delta Y\) and \(\delta Z\) could be defined similarly. Then we have
\[
f(t, Y_t, \mathbb{E}[Y_t], Z_t) - f(t, \tilde{Y}_t, \mathbb{E}[\tilde{Y}_t], \tilde{Z}_t) = \delta_y f(t) \delta Y_t + \delta_y f(t) \mathbb{E}[\delta Y_t] + \delta_z f(t) \delta Z_t. \tag{4.3}
\]

For one dimensional BSDE (4.1), we have the following comparison theorem.

**Theorem 4.2.** Let \((Y, Z)\) and \((\tilde{Y}, \tilde{Z})\) be the bounded solutions of BSDE (4.1) with generators \((f, \xi)\) and \((\tilde{f}, \tilde{\xi})\) respectively, which satisfy the condition of Lemma 4.1. Moreover, if \(\xi \leq \tilde{\xi}\) a.e., \(f(t, y, \bar{y}, z) \leq \tilde{f}(t, y, \bar{y}, z)\) a.e., and the following conditions hold:

(A1) for every \(Y, \tilde{Y}, Z, \delta_y f(t), \delta_y f(t) \in L^\infty_T(0, T; \mathbb{R})\),

(A2) for every \(Z, \tilde{Z} \in \mathcal{Z}^2[0,T]\), and any bounded process \(Y, \delta_z f(t) \in \mathcal{Z}^2[0,T]\),

(A3) one of the two coefficients \(f\) and \(\tilde{f}\) is nondecreasing in \(\bar{y}\),

then we have \(Y_t \leq \tilde{Y}_t\) a.s. for every \(t \in [0,T]\).
Remark 4.3. The conditions (A1) and (A2) hold if there exists a positive constant $C$ such that

$$|f(t, y_1, \bar{y}_1, z_1) - f(t, y_2, \bar{y}_2, z_2)| \leq C(|y_1 - y_2| + |\bar{y}_1 - \bar{y}_2|) + C(1 + |z_1| + |z_2|)|z_1 - z_2|$$

for all $t \in [0, T]$ and $y_1, \bar{y}_1, y_2, \bar{y}_2, z_1, z_2 \in \mathbb{R}$. Moreover, (A1) and (A2) hold too if $f(t, y, \bar{y}, z)$ satisfies the global Lipschitz condition.

Proof of Theorem 4.2. Without lose of generality, we would like to let that $f$ is nondecreasing in $\bar{y}$. Taking the difference of BSDE (4.1) with coefficients $(f, \xi)$ and $(\tilde{f}, \tilde{\xi})$ respectively, we obtain that

$$Y_t - \tilde{Y}_t = Y_0 - \tilde{Y}_0 - \int_0^t [f(s, Y_s, \mathbb{E}[Y_s], Z_s) - f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)]ds$$

$$- \int_0^t [f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s) - \tilde{f}(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)]ds + \int_0^t (Z_s - \tilde{Z}_s)dW_s. \quad (4.4)$$

We define the measure $Q$ by $dQ = \mathcal{E}_T(\Lambda)d\mathbb{P}$, where

$$\Lambda_t = \int_0^t \delta z(t)dW_s.$$ 

By Lemma 4.1, we have $Z, \tilde{Z} \in \mathbb{Z}^2[0, T]$. So the conditions (A1) and (A2) imply that $\Lambda \in \text{BMO}$ and hence $Q$ is a probability measure equivalent to $\mathbb{P}$.

We denote by $\tilde{\Lambda}$ the martingale part of $\delta Y = Y - \tilde{Y}$, in other words,

$$\tilde{\Lambda}_t = \int_0^t (Z_s - \tilde{Z}_s)dW_s, \quad t \in [0, T].$$

Therefore, on the one hand, from (4.3), we have that the process

$$\delta Y_t + \int_0^t [f(s, Y_s, \mathbb{E}[Y_s], Z_s) - f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)]ds$$

$$+ \int_0^t [f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s) - \tilde{f}(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)]ds$$

$$= \delta Y_t + \int_0^t \left( \delta_y f(s)\delta Y_s + \delta_y f(s)\mathbb{E}[\delta Y_s] + \delta_z f(s)\delta Z_s \right)ds + \int_0^t \delta f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)ds$$

$$= \delta Y_t + \int_0^t \left( \delta_y f(s)\delta Y_s + \delta_y f(s)\mathbb{E}[\delta Y_s] + \delta f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s) \right)ds + \int_0^t \delta_z f(s)\delta Z_sds.$$ 

On the other hand, from (4.4), the process

$$\delta Y_t + \int_0^t [f(s, Y_s, \mathbb{E}[Y_s], Z_s) - f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)]ds$$

$$+ \int_0^t [f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s) - \tilde{f}(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s)]ds$$

$$= \delta Y_0 + \int_0^t (Z_s - \tilde{Z}_s)dW_s.$$
So, by Girsanov’s theorem, noting \( \delta Y_0 \in \mathbb{R} \), we obtain that the process
\[
\delta Y_t + \int_0^t \left( \delta_y f(s) \delta Y_s + \delta_{\tilde{y}} f(s) \mathbb{E}[\delta Y_s] + \delta f(s, \tilde{Y}_s, \mathbb{E}[\tilde{Y}_s], \tilde{Z}_s) \right) ds
\]
\[
= \delta Y_0 + \int_0^t (Z_s - \tilde{Z}_s) dW_s - \int_0^t \delta z f(s) \delta Z_s ds
\]
\[
= \delta Y_0 + \tilde{A}_t - \langle \Lambda, \tilde{A} \rangle_t
\]
is a local martingale under the measure \( \mathbb{Q} \). Moreover, Proposition 11 of [12] implies that
\[
\tilde{A}_t - \langle \Lambda, \tilde{A} \rangle_t \in \text{BMO}(\mathbb{Q}).
\]
Finally, using the martingale property, the duality principle between SDEs and BSDEs, and the boundary conditions \( Y_T = \xi, \tilde{Y}_t = \tilde{\xi} \), we have that
\[
Y_t - \tilde{Y}_t = \mathbb{E}_t^\mathbb{Q} \left( e^{\int_t^T \delta_y f(r) dr} (\xi - \tilde{\xi}) + \int_t^T e^{\int_r^T \delta_{\tilde{y}} f(s) ds} \delta y f(s) \mathbb{E}[Y_s - \tilde{Y}_s] ds \right)
\]
\[
+ \int_t^T e^{\int_r^T \delta_{\tilde{y}} f(r) dr} \delta y f(s) \mathbb{E}[Y_s - \tilde{Y}_s] ds
\]
which, noting \( \xi \leq \tilde{\xi} \) a.s. and \( f(t, y, z) \leq \tilde{f}(t, y, z) \) a.e., implies that
\[
Y_t - \tilde{Y}_t \leq \mathbb{E}_t^\mathbb{Q} \int_t^T e^{\int_r^T \delta_{\tilde{y}} f(r) dr} \delta y f(s) \mathbb{E}[Y_s - \tilde{Y}_s] ds
\]
\[
= \int_t^T \mathbb{E}_t^\mathbb{Q} \left( e^{\int_r^T \delta_{\tilde{y}} f(r) dr} \delta y f(s) \right) \mathbb{E}[Y_s - \tilde{Y}_s] ds.
\]
Since \( \delta y f(t) \) and \( \delta_{\tilde{y}} f(t) \) are bounded processes, there exists a positive constant \( K \) such that
\[
\left\| e^{\int_r^T \delta_{\tilde{y}} f(r) dr} \delta y f(s) \right\|_{L^\infty(0, T)} \leq K.
\]
Hence,
\[
Y_t - \tilde{Y}_t \leq K \int_t^T \mathbb{E}[Y_s - \tilde{Y}_s] ds.
\]
Notice that \( \mathbb{E}[Y_s - \tilde{Y}_s] \leq \mathbb{E}[(Y_s - \tilde{Y}_s)^+] \) and then
\[
\int_t^T \mathbb{E}[Y_s - \tilde{Y}_s] ds \leq \int_t^T \mathbb{E}[(Y_s - \tilde{Y}_s)^+] ds,
\]
which implies that
\[
\left( \int_t^T \mathbb{E}[Y_s - \tilde{Y}_s] ds \right)^+ \leq \left( \int_t^T \mathbb{E}[(Y_s - \tilde{Y}_s)^+] ds \right)^+ = \int_t^T \mathbb{E}[(Y_s - \tilde{Y}_s)^+] ds.
\]
Note the inequality \( (ab)^+ \leq a \cdot b^+ \) when \( a \geq 0 \), it follows
\[
(Y_t - \tilde{Y}_t)^+ \leq K \left( \int_t^T \mathbb{E}[Y_s - \tilde{Y}_s] ds \right)^+ \leq K \int_t^T \mathbb{E}[(Y_s - \tilde{Y}_s)^+] ds, \quad t \in [0, T],
\]
from which we can conclude with the help of Gronwall’s lemma that \( Y_t \leq \tilde{Y}_t, \ t \in [0, T], \mathbb{P}\text{-a.s.} \) □
Remark 4.4. We point out that the generator $f$ here is of quadratic growth with respect to $z$, which is weaker than that of [7] which is of linear growth with respect to $z$. On the other hand, the terminal value $\xi$ is bounded in our paper, which is stronger than that of the terminal value $\xi$ is in $L^2_F(t, T; \mathbb{R})$ used in Buckdahn, Li and Peng [7].

5 Conclusion

The multi-dimensional BSDEs with quadratic growth is a difficult yet important topic in the field of BSDEs. In this work, using an artful method to construct the contracting mapping principle, we proved the existence and uniqueness of multi-dimensional mean-field BSDEs with quadratic growth under a small terminal value. Moreover, a comparison theorem is obtained. For the general mean-field BSDEs (see Carmona and Delarue [10, 11]) with quadratic growth, the relevant results are under our investigation.

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