JACOB’S LADDERS, CROSSBREEDING AND NEW SYNERGETIC FORMULAS FOR THE CLASS OF MORE COMPLICATED EXTERNAL PARTS OF $\zeta$-FACTORIZATION FORMULAS

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Abstract. In this paper we obtain new canonical synergetic formula, namely an $\zeta$-analogue of next elementary trigonometric formula. This one describes cooperative interactions between corresponding class of elementary functions and the Riemann’s zeta-function on a class of disconnected sets on the critical line.

1. Introduction

1.1. In this paper we obtain new results of the following type: the set of elementary functions

\[
\{t \sin^2 t, t \cos^2 t, t \cos(2t)\},
\]

\[t \in [\pi L, \pi L + U],\ U \in (0, \pi/4),\ L \in \mathbb{N}\]

generates the following synergetic (cooperative) formula

\[
\{\alpha_0^{2,1} \tilde{Z}^2(\alpha_1^{2,1})\} \cos^2(\alpha_0^{2,1}) - \{\alpha_0^{1,1} \tilde{Z}^2(\alpha_1^{1,1})\} \sin^2(\alpha_0^{1,1}) =
\]

\[
= \{\alpha_0^{3,1} \tilde{Z}^2(\alpha_1^{3,1})\} \cos(2\alpha_0^{3,1}),\ L \geq L_0 > 0,
\]

with $L_0 \in \mathbb{N}$ being sufficiently big, where

\[
\tilde{Z}^2(t) = \frac{\zeta \left(\frac{1}{2} + it\right)^2}{\omega(t)},\ \omega(t) = \left\{1 + O \left(\frac{\ln \ln t}{\ln t}\right)\right\} \ln t,
\]

(see [2], (6.1), (6.7), (7.7), (7.8), (9.1)), next

\[
\alpha_0^{1,1}, \alpha_0^{2,1}, \alpha_0^{3,1} \in (\pi L, \pi L + U),
\]

\[
\alpha_1^{1,1}, \alpha_1^{2,1}, \alpha_1^{3,1} \in (\pi L, \pi L + U),
\]

and the segment

\[
\frac{1}{[\pi L, \pi L + U]}
\]

is the first reverse iteration (by means of Jacob’s ladder $\varphi_1(t)$, see [3]) of the basic segment

\[
[\pi L, \pi L + U] = [0, 0].
\]

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1.2. Let us notice that in our theory the following is true: the components of the main \(\zeta\)-disconnected set (that is (1.2) in our case)

\[
\Delta(\pi L, U, 1) = [\pi L, \pi L + U] \cup \frac{1}{\pi L, \pi L + U} \cup \frac{1}{\pi L, \pi L + U}
\]

are separated each from other by the gigantic distance \(\rho\):

\[
\rho([\pi L, \pi L + U]; \frac{1}{\pi L, \pi L + U}) \sim (1 - c)\pi(\pi L) \sim \\
\sim \pi(1 - c)\frac{L}{\ln L} \to \infty, \ L \to \infty,
\]

(\(c\) stands for Euler’s constant and \(\pi(x)\) for the prime-counting function).

1.3. Since (see (1.4))

\[
\frac{\pi L}{\pi L + U} < \frac{\alpha_0^{1.1}}{\alpha_0^{2.1}}, \frac{\alpha_0^{3.1}}{\alpha_0^{2.1}} < \frac{\pi L + U}{\pi L} \Rightarrow \frac{\alpha_0^{1.1}}{\alpha_0^{2.1}}, \frac{\alpha_0^{3.1}}{\alpha_0^{2.1}} \to L \to \infty
\]

then we have (see (1.2), (1.3), (1.7)) the following canonical synergetic formula

\[
\left| \zeta \left( \frac{1}{2} + i\alpha_1^{2.1} \right) \right|^2 \cos^2 \alpha_0^{1.1} - \left| \zeta \left( \frac{1}{2} + i\alpha_1^{1.1} \right) \right|^2 \sin^2 \alpha_0^{1.1} \sim \\
\sim \left| \zeta \left( \frac{1}{2} + i\alpha_1^{3.1} \right) \right|^2 \cos 2\alpha_0^{3.1}, \ L \to \infty.
\]

Remark 1. Our formula (1.8) is:

(a) simple case of formula that is generated by immediate metamorphosis of the main formula,

(b) \(\zeta\)-analogue of the elementary trigonometric formula

\[
\cos^2 x - \sin^2 x = \cos 2x,
\]

(c) synergetic one (as well as (1.2)) since it is generated by interactions between the continuum sets (see our interpretation in [8])

\[
\left\{ \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 \right\}, \left\{ t \sin^2 t \right\}, \left\{ t \cos^2 t \right\}, \left\{ t \cos 2t \right\}, \ t \geq L_0
\]

(some analogue of the classical Belousov-Zhabotiski chemical oscillations),

(d) new type of result simultaneously in the theory of Riemann’s zeta-function

and in the theory of real continuous functions.

1.4. Finally, we notice the following.

Remark 2. The formulations of all results and proofs in this paper are based on new notions and methods in the theory of Riemann’s zeta-function we have introduced in our series of 47 papers concerning Jacob’s ladders. These can be found in arXiv [math.CA] starting with the paper [1].

Here we use especially the following notions: Jacob’s ladder, \(\zeta\)-disconnected set that generates the Jacob’s ladder, (see [3]), algorithm for generating the \(\zeta\)-factorization formulas (see [4]), crossbreeding, secondary crossbreeding, exact and asymptotic complete hybrid formula (see [6] – [9]). Short survey of these notions are listed in papers [5], [8].
2. Lemmas

By making use of our algorithm for generating \( \zeta \)-factorization formulas (see [5], (3.1) – (3.11), comp. [4]) we obtain the following set of results.

2.1. Since

\[
\frac{1}{U} \int_{\pi L}^{\pi L + U} t \sin^2 t \, dt = \frac{1}{4} (2\pi L + U) - \frac{1}{2} (\pi L + U) \frac{\sin 2U}{2U} + \frac{1}{4} \frac{\sin^2 U}{U},
\]

then we obtain the following statement.

Lemma 1. For the function

\[
f_1(t) = t \sin^2 t \in \hat{C}_0[\pi L, \pi L + U],
\]

\( U \in (0, \pi/4), \ L \in \mathbb{N} \)

there are vector-valued functions

\[
(\alpha_0^{1,k_1}, \alpha_1^{1,k_1}, \ldots, \alpha_r^{1,k_1}, \beta_1^{k_1}, \ldots, \beta_{k_1}^{k_1}),
\]

\( 1 \leq k_1 \leq k_0, \ k_1, k_0 \in \mathbb{N} \)

(here, we fix arbitrary integer \( k_0 \)) such that the following exact \( \zeta \)-factorization formula

\[
\prod_{r=1}^{k_1} \hat{Z}^2(\alpha_r^{1,k_1}) = \frac{1}{\alpha_0^{1,k_1} \sin^2 \alpha_0^{1,k_1}} \times \\
\times \left\{ \frac{1}{4} (2\pi L + U) - \frac{1}{2} (\pi L + U) \frac{\sin 2U}{2U} + \frac{1}{4} \frac{\sin^2 U}{U} \right\}, \ \forall L \geq L_0 > 0
\]

(\( L_0 \) is a sufficiently big one) holds true, where

\[
\alpha_r^{1,k_1} = \alpha_r(U, \pi L, k_1; f_1), \ r = 0, 1, \ldots, k_1,
\]

\[
\beta_r^{k_1} = \beta_r(U, \pi L, k_1), \ r = 1, \ldots, k_1,
\]

\[
\alpha_r^{1,k_1} \in (\pi L, \pi L + U), \alpha_r^{1,k_1}, \beta_r^{k_1} \in (\pi L, \pi L + U), \ r = 1, \ldots, k_1,
\]

and the segment

\[
\frac{r}{[\pi L, \pi L + U]}
\]

is the \( r \)-th reverse iteration by means of the Jacob’s ladder, see [3], of the basic segment

\[
[\pi L, \pi L + U] = \frac{0}{[\pi L, \pi L + U]}.
\]

2.2. Since

\[
\frac{1}{U} \int_{\pi L}^{\pi L + U} t \cos^2 t \, dt = \frac{1}{4} (2\pi L + U) + \frac{1}{2} (\pi L + U) \frac{\sin 2U}{2U} - \frac{1}{4} \frac{\sin^2 U}{U},
\]

then we obtain the following statement.
Lemma 2. For the function

\[ f_2(t) = t \cos^2 t \in \tilde{C}_0[\pi L, \pi L + U], \]
\[ U \in (0, \pi/4) \]

there are vector-valued functions

\[ (\alpha_{0, k_2}^{2,k_2}, \alpha_{1, k_2}^{2,k_2}, \ldots, \alpha_{k_2, k_2}^{2,k_2}, \beta_{1, k_2}^{k_2}, \ldots, \beta_{k_2, k_2}^{k_2}), \]
\[ 1 \leq k_2 \leq k_0, \ k_2 \in \mathbb{N} \]

such that the following exact \( \zeta \)-factorization formula

\[
\prod_{r=1}^{k_2} \frac{Z^2(\alpha_{r, k_2}^{2,k_2})}{Z(\beta_{r, k_2}^{k_2})} = \frac{1}{\alpha_{0, k_2}^{2,k_2} \cos^2 \alpha_{0, k_2}^{2,k_2}} \times \\
\left\{ \frac{1}{4} (2\pi L + U) + \frac{1}{2} (\pi L + U) \frac{\sin 2U}{2U} - \frac{1}{4} \frac{\sin^2 U}{U} \right\}, \forall L \geq L_0 > 0
\]

holds true, where

\[ \alpha_{r, k_2}^{2,k_2} = \alpha_r(U, \pi L, k_2; f_2), \ r = 0, 1, \ldots, k_2, \]
\[ \beta_{r, k_2}^{k_2} = \beta_r(U, \pi L, k_2), \ r = 1, \ldots, k_2, \]
\[ \alpha_{0, k_2}^{2,k_2} \in (\pi L, \pi L + U), \alpha_{r, k_2}^{2,k_2}, \beta_{r, k_2}^{k_2} \in (\pi L, \pi L + U), \ r = 1, \ldots, k_2. \]

2.3. Since

\[
\frac{1}{U} \int_{\pi L}^{\pi L + U} t \cos 2t \, dt = (\pi L + U) \frac{\sin 2U}{2U} - \frac{1}{2} \frac{\sin^2 U}{U},
\]

then we obtain the following statement.

Lemma 3. For the function

\[ f_3(t) = t \cos 2t \in \tilde{C}_0[\pi L, \pi L + U], \]
\[ U \in (0, \pi/4) \]

there are vector-valued functions

\[ (\alpha_{0, k_3}^{3,k_3}, \alpha_{1, k_3}^{3,k_3}, \ldots, \alpha_{k_3, k_3}^{3,k_3}, \beta_{1, k_3}^{k_3}, \ldots, \beta_{k_3, k_3}^{k_3}), \]
\[ 1 \leq k_3 \leq k_0, \ k_3 \in \mathbb{N} \]

such that the following exact \( \zeta \)-factorization formula

\[
\prod_{r=1}^{k_3} \frac{Z^2(\alpha_{r, k_3}^{3,k_3})}{Z(\beta_{r, k_3}^{k_3})} = \frac{1}{\alpha_{0, k_3}^{3,k_3} \cos(2\alpha_{0, k_3}^{3,k_3})} \times \\
\left\{ (\pi L + U) \frac{\sin 2U}{2U} - \frac{1}{2} \frac{\sin^2 U}{U} \right\}, \forall L \geq L_0 > 0
\]

holds true, where

\[ \alpha_{r, k_3}^{3,k_3} = \alpha_r(U, \pi L, k_3; f_3), \ r = 0, 1, \ldots, k_3, \]
\[ \beta_{r, k_3}^{k_3} = \beta_r(U, \pi L, k_3), \ r = 1, \ldots, k_3, \]
\[ \alpha_{0, k_3}^{3,k_3} \in (\pi L, \pi L + U), \alpha_{r, k_3}^{3,k_3}, \beta_{r, k_3}^{k_3} \in (\pi L, \pi L + U), \ r = 1, \ldots, k_3. \]
3. Exact complete hybrid formula

3.1. We start with the following.

Remark 3. Our description of the operation of crossbreeding (see \[6\] and \[8\], subsection 3.3) contains the following expression:

\[
\ldots \text{that is: after finite number of eliminations of the external functions}
\]

\[
E_m(U, T), \quad m = 1, \ldots, M, \ldots
\]

However, this is not exact. The exact phrase is as follows:

\[
\ldots \text{that is: after finite number of eliminations of the variables } U, T \text{ from the set of external functions . . .}
\]

We shall call these variables as external ones.

3.2. Now, we make the crossbreeding on the set

(3.1)

\[
\{(2.3), (2.7), (2.11)\}, \quad U \in (0, \pi/4)
\]

of exact \(\zeta\)-factorization formulas.

Remark 4. In the case (3.1) we see that the corresponding external functions contain the pair \(\pi L, U\) of external variables, comp. Remark 3.

First, elimination of the block \(\ldots\) (see (2.11)) from (2.3), (2.11) gives

\[
\alpha_0^{1,k_1} \sin^2 \alpha_0^{1,k_1} \prod_{r=1}^{k_1} \frac{\tilde{Z}^2(\alpha_r^{1,k_1})}{Z^2(\beta_r^{k_1})} +
\]

(3.2)

\[
+ \frac{1}{2} \alpha_0^{3,k_1} \cos(2\alpha_0^{3,k_1}) \prod_{r=1}^{k_3} \frac{\tilde{Z}^2(\alpha_r^{3,k_3})}{Z^2(\beta_r^{k_3})} = \frac{1}{2} \left(\pi L + \frac{U}{2}\right),
\]

and secondly, (2.3) and (2.7) imply

\[
\alpha_0^{2,k_2} \cos^2 \alpha_0^{2,k_2} \prod_{r=1}^{k_2} \frac{\tilde{Z}^2(\alpha_r^{2,k_2})}{Z^2(\beta_r^{k_2})} +
\]

(3.3)

\[
+ \alpha_0^{1,k_1} \sin^2 \alpha_0^{1,k_1} \prod_{r=1}^{k_1} \frac{\tilde{Z}^2(\alpha_r^{1,k_1})}{Z^2(\beta_r^{k_1})} = \pi L + \frac{U}{2}.
\]

Finally, we obtain from (3.2) and (3.3) the following

Theorem 1. The set

(3.4)

\[
\{t \sin^2 t, t \cos^2 t, t \cos(2t)\},
\]

\[
t \in [\pi L, \pi L + U], \quad U \in (0, \pi/4), \quad L \in \mathbb{N}
\]

of elementary functions generates the following exact complete hybrid formula

\[
\left\{ \frac{\alpha_0^{2,k_2} \prod_{r=1}^{k_2} \frac{\tilde{Z}^2(\alpha_r^{2,k_2})}{Z^2(\beta_r^{k_2})}}{Z^2(\beta_r^{k_2})} \right\} \cos^2 \alpha_0^{2,k_2} - \left\{ \frac{\alpha_0^{1,k_1} \prod_{r=1}^{k_1} \frac{\tilde{Z}^2(\alpha_r^{1,k_1})}{Z^2(\beta_r^{k_1})}}{Z^2(\beta_r^{k_1})} \right\} \sin^2 \alpha_0^{1,k_1} =
\]

(3.5)

\[
= \left\{ \frac{\alpha_0^{3,k_3} \prod_{r=1}^{k_3} \frac{\tilde{Z}^2(\alpha_r^{3,k_3})}{Z^2(\beta_r^{k_3})}}{Z^2(\beta_r^{k_3})} \right\} \cos(2\alpha_0^{3,k_3}),
\]

\[\forall L \geq L_0 > 0, \quad 1 \leq k_1, k_2, k_3 \leq k_0, \]
(we fix arbitrary $k_0 \in \mathbb{N}$ and $L_0$ is a sufficiently big one), where
\[
\alpha_0^{1,k_1}, \alpha_0^{2,k_2}, \alpha_0^{3,k_3} \in (\pi L, \pi L + U),
\]
\[
\alpha_r^{1,k_1}, \beta_r^{k_1} \in (\pi L, \pi L + U), \quad r = 1, \ldots, k_1,
\]
\[
\alpha_r^{2,k_2}, \beta_r^{k_2} \in (\pi L, \pi L + U), \quad r = 1, \ldots, k_2,
\]
\[
\alpha_r^{3,k_3}, \beta_r^{k_3} \in (\pi L, \pi L + U), \quad r = 1, \ldots, k_3,
\]
(i.e.)
\[
r = 0, 1, \ldots, \tilde{k} : \quad \alpha_r^{1,k_1}, \alpha_r^{2,k_2}, \alpha_r^{3,k_3}, \beta_r^{k_1}, \beta_r^{k_2}, \beta_r^{k_3} \in \in \Delta(U, \pi L, \tilde{k}) = \bigcup_{r=0}^{\tilde{k}} [\pi L, \pi L + U], \quad \tilde{k} = \max\{k_1, k_2, k_3\}, \quad \tilde{k} \leq k_0,
\]
and
\[
\Delta(U, \pi L, \tilde{k}) \subset \Delta(U, \pi L, k_0) = \bigcup_{r=0}^{k_0} [\pi L, \pi L + U],
\]
where the last $\zeta$-disconnected set is basic one (for every fixed $k_0 \in \mathbb{N}$).

**Remark 5.** It is true in our theory (see [3]): consecutive components of the basic disconnected set are separated each from other by gigantic distances $\rho$:
\[
\rho \left(\frac{r}{\pi L, \pi L + U}, \frac{r+1}{\pi L, \pi L + U}\right) \sim (1 - c)\pi(\pi L) \sim \pi \cdot (1 - c) \frac{L}{\ln L} \xrightarrow{L \to \infty} \infty, \quad r = 0, 1, \ldots, k_0 - 1,
\]
($c$ is the Euler’s constant and $\pi(x)$ stands for the prime-counting function).

**Remark 6.** By our interpretation given in the paper [8] the formula (3.5) is the synergetic (cooperative) one in the following sense: it is the result of interactions between the following continuum sets
\[
\left\{ \left( \frac{1}{2} + it \right)^2 \right\}, \{t \sin^2 t\}, \{t \cos^2 t\}, \{t \cos 2t\}, \quad t \geq L_0
\]
and these interactions are excited by the Jacob’s ladder $\varphi_1(t)$. We call these interactions (see [9]) as the $\zeta$-chemical reaction between sets [3.3].

**Remark 7.** The result of above mentioned $\zeta$-chemical reactions (the $\zeta$-chemical compound) is our synergetic formula (3.5). This interpretation represents a $\zeta$-analogue of the classical Belousov-Zhabotiski chemical oscillations (see our paper [8] as the starting point in this direction).

4. **Immediate metamorphosis of the formula (3.5) into asymptotic secondary complete hybrid formula**

4.1. If we rewrite the formula (3.5) in the form
\[
\{\ldots\} \cos^2 \alpha_0^{2,k_2} = \{\ldots\} \sin^2 \alpha_0^{1,k_1} + \{\ldots\} \cos(2\alpha_0^{3,k_3})
\]
and use (1.3), (1.7) and some small algebra (comp. [8], Section 8.2), then we obtain the following
Corollary 1.

\[
\left\{ \prod_{r=1}^{k_2} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{2,k_2} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{2,k_2} \right)} \right|^2 \right\} \cos^2 \alpha_0^{2,k_2} - \left\{ \prod_{r=1}^{k_1} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{1,k_1} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{1,k_1} \right)} \right|^2 \right\} \sin^2 \alpha_0^{1,k_1} \sim \\
\sim \left\{ \prod_{r=1}^{k_3} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{3,k_3} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{3,k_3} \right)} \right|^2 \right\} \cos(2\alpha_0^{3,k_3}), \; L \to \infty.
\]

(4.1)

4.2. Now, in the case

\[ k_1 = k_2 = k_3 = k; \; 1 \leq k \leq k_0 \]

we obtain the following

Corollary 2.

\[
\left\{ \prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{2,k} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{2,k} \right)} \right|^2 \right\} \cos^2 \alpha_0^{2,k} - \left\{ \prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{1,k} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{1,k} \right)} \right|^2 \right\} \sin^2 \alpha_0^{1,k} \sim \\
\sim \left\{ \prod_{r=1}^{k} \left| \frac{\zeta \left( \frac{1}{2} + i\alpha_r^{3,k} \right)}{\zeta \left( \frac{1}{2} + i\beta_r^{3,k} \right)} \right|^2 \right\} \cos(2\alpha_0^{3,k}), \; L \to \infty.
\]

(4.3)

Remark 8. Formula (4.3) expresses the result of immediate metamorphosis of the asymptotic complete hybrid formula (4.1) into asymptotic secondary complete hybrid formula in the case (4.2).

Remark 9. The case \( k = 1 \) in (4.3) gives the formula (1.8) that has been used in Introduction to inform about the content of this paper.

5. Secondary exact complete hybrid formula

We choose the following exact complete hybrid formula (see [9], (3.7))

\[
(1 + \Delta_4)^{1/\Delta_4} (a_0^{4,k_4} - \pi L) \left\{ \prod_{r=1}^{k_4} \frac{2(a_0^{4,k_4})}{Z^2(\beta_r^{4,k_4})} \right\}^{1/\Delta_4} = \\
= (1 + \Delta_5)^{1/\Delta_5} (a_0^{5,k_5} - \pi L) \left\{ \prod_{r=1}^{k_5} \frac{2(a_0^{5,k_5})}{Z^2(\beta_r^{5,k_5})} \right\}^{1/\Delta_5}
\]

\( \forall L \geq L_0 > 0, \; \Delta_4, \Delta_5 > 0, \; \Delta_5 + \Delta_4, \; 1 \leq k_4, k_5 \leq k_0. \)

(5.1)

Now, we make the use of operation of secondary crossbreeding (see [8]) on the set

\{(3.5), (5.1)\}

as follows. First of all, we put

\[ k_4 = k_5 = k; \; 1 \leq k \leq k_0 \]
in the formula (6.1) that gives the result
\[
\prod_{r=1}^{k} \tilde{Z}^2(\beta_r^k) = 
(5.2)
\]
\[
\left[ (1 + \Delta_4)^{1/\Delta_4} (1 + \Delta_5)^{1/\Delta_5} \right] \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \left\{ \prod_{r=1}^{k} \tilde{Z}^2(\alpha_r^{4,k}) \right\}^{-\frac{\Delta_\alpha}{\Delta_{\alpha_0} \Delta_{\alpha_k}}} \times 
\]
\[
\left\{ \prod_{r=1}^{k} \tilde{Z}^2(\alpha_r^{5,k}) \right\}^{-\frac{\Delta_\alpha}{\Delta_{\alpha_0} \Delta_{\alpha_k}}} .
\]
For the second, we put consecutively
\[
k = k_1, k_2, k_3
\]
in (5.2), and the corresponding results we substitute into the formula (3.5). The final result is expressed by the following

**Theorem 2.** The two sets of elementary functions
\[
\{ t \sin^2 t, t \cos^2 t, t \cos 2t \}, \{ (t - \pi L) \Delta_4, (t - \pi L) \Delta_5 \},
\]
\[
t \in [\pi L, \pi L + U], \ U \in (0, \pi/4), \Delta_4, \Delta_5 > 0, \Delta_4 \neq \Delta_5
\]
generate the following secondary exact complete hybrid formula
\[
\alpha_0^{2,k} \prod_{r=1}^{k_2} \tilde{Z}^2(\alpha_r^{2,k}) \tilde{Z}^2(\alpha_r^{2,k}) \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \tilde{Z}^2(\alpha_r^{4,k}) \left\{ \prod_{r=1}^{k} \tilde{Z}^2(\alpha_r^{4,k}) \right\}^{-\frac{\Delta_\alpha}{\Delta_{\alpha_0} \Delta_{\alpha_k}}} \cos^2 \alpha_0^{4,k_2} - 
\]
\[
\left( \frac{\alpha_0^{2,k_2} - \pi L}{\alpha_0^{2,k_2} - \pi L} \right) \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \sin^2 \alpha_0^{2,k_1} = 
\]
\[
\alpha_0^{1,k_1} \prod_{r=1}^{k_1} \tilde{Z}^2(\alpha_r^{1,k_1}) \tilde{Z}^2(\alpha_r^{1,k_1}) \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \tilde{Z}^2(\alpha_r^{4,k_1}) \left\{ \prod_{r=1}^{k} \tilde{Z}^2(\alpha_r^{4,k}) \right\}^{-\frac{\Delta_\alpha}{\Delta_{\alpha_0} \Delta_{\alpha_k}}} \sin^2 \alpha_0^{1,k_1} = 
\]
\[
\alpha_0^{4,k_1} \prod_{r=1}^{k_1} \tilde{Z}^2(\alpha_r^{3,k_1}) \tilde{Z}^2(\alpha_r^{3,k_1}) \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \frac{\Delta_{\alpha_0} \Delta_{\alpha_k}}{\Delta_{\alpha_0} \Delta_{\alpha_k}} \tilde{Z}^2(\alpha_r^{4,k_1}) \left\{ \prod_{r=1}^{k} \tilde{Z}^2(\alpha_r^{4,k}) \right\}^{-\frac{\Delta_\alpha}{\Delta_{\alpha_0} \Delta_{\alpha_k}}} \cos(2\alpha_0^{1,3,k_1}),
\]
\[
\forall L \geq L_0, 1 \leq k_1, k_2, k_3 \leq k_0 .
\]

**Remark 10.** Let us notice explicitly that two complicated types of \( \zeta \)-modulation (of amplitude and also phase) of the elementary trigonometric formula
\[
\cos^2 x - \sin^2 x = \cos 2x
\]
are expressed by the synergetic formulae (3.5), (5.3).

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