POINTWISE SEMICOMMUTATIVE RINGS

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Abstract. We call a ring $R$ pointwise semicommutative if for any element $a \in R$ either $l(a)$ or $r(a)$ is an ideal of $R$. A class of pointwise semicommutative rings is a strict generalization of semicommutative rings. Since reduced rings are pointwise semicommutative, this paper studies sufficient conditions for pointwise semicommutative rings to be reduced.

For a pointwise semicommutative ring $R$, $R$ is strongly regular if and only if $R$ is left SF; $R$ is exchange if and only if $R$ is clean; if $R$ is semiperiodic then $R/J(R)$ is commutative.

1. Introduction

Throughout this paper, unless otherwise mentioned, all rings considered are associative with identity, $R$ represents a ring, and all modules are unital. For any $w \in R$, the notations $r(w)$ ($l(w)$) represents the right (left) annihilator of $w$. We write $C(R)$, $P(R)$, $J(R)$, $N(R)$, $E(R)$, $Z(RR)$ and $U(R)$ respectively, for the set of all central elements, the prime radical, the Jacobson radical, the set of all nilpotent elements, the set of all idempotent elements, the left singular ideal of $R$ and the group of units of $R$. Recall that $R$ is said to be:

1. reduced if $N(R) = 0$.
2. reversible ([4]) if $wh = 0$ implies $hw = 0$ for any $w, h \in R$.
3. semicommutative ([4]) if for each $w \in R$, $r(w)$ is an ideal of $R$.
4. strongly regular ([5]) if for each $w \in R$, $w \in w^2 R$.
5. left (right) weakly regular ([2]) if $w \in RwRw$ ($w \in wRwR$) for any $w \in R$.
6. left (right) quasi duo ([5]) if every maximal left (right) ideal of $R$ is an ideal of $R$.

Let $ME_l(R) = \{e \in E(R) \mid Re$ is a minimal left ideal of $R\}$. $R$ is called left min-abel if for any $e \in ME_l(R)$ re = ere for all $r \in R$. $R$ is called left MC2 if $aRe = 0$ implies $eRa = 0$ for any $a \in R$, $e \in ME_l(R)$. According to [3], $R$ is said to be NCI if $N(R) = 0$ or $N(R)$ contains a non-zero ideal of $R$. $R$ is said to be NI if $N(R)$ is an ideal of $R$, and $R$ is 2-primal if $N(R) = P(R)$. Obviously, NI rings are NCI; nevertheless, the converse is not true (by [3] Example 2.5). $R$ is directly finite if $wh = 1$ implies $hw = 1$ for any $h, w \in R$.

2010 Mathematics Subject Classification. Primary 16U80; Secondary 16S34, 16S36.

Key words and phrases. Pointwise semicommutative rings, semicommutative rings.
Let $\Psi : R \to R$ be an automorphism of $R$. $R[x;\Psi]$ is the ring of polynomials over $R$ with respect to usual polynomial addition and multiplication which is defined by the rule: $xa = \Psi(a)x$. $R[x;\Psi]$ is called skew polynomial ring of $R$.

Over the past several years, semicommutative rings and their generalizations have been studied extensively by many authors. In a semicommutative ring $R$, both the left and the right annihilator of every element of $R$ are ideals of $R$. This motivates us to investigate a ring $R$ in which either the left or the right annihilator (not necessarily both) of any element of $R$ is an ideal of $R$. This paper studies such a class of rings.

2. Main Results

**Definition 2.1.** We call a ring $R$ pointwise semicommutative if for any $w \in R$, either $l(w)$ or $r(w)$ is an ideal of $R$.

It is evident that semicommutative rings are pointwise semicommutative. However, the following example shows that the converse need not be true.

**Example 2.2.** Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ and $a_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $a_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $a_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $a_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $a_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $a_6 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $a_7 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Observe that $l(a_0) = R$, $l(a_1) = l(a_2) = 0$, $l(a_3) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$, $l(a_4) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$, $r(a_5) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$, $l(a_6) = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$, $r(a_7) = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ are ideals. Thus, $R$ is pointwise semicommutative.

Clearly, $R$ is not semicommutative.

**Proposition 2.3.** Let $R$ be a pointwise semicommutative ring. Then:

1. $R$ is directly finite.
2. $R$ is left min-abel.

**Proof.**

(1) Suppose $w, h \in R$ with $wh = 1$. Take $k = h - h^2w$. Since $R$ is pointwise semicommutative and $k^2 = 0$, $0 = k(w - h^2w)k = (h - h^2w)(w - h^2w)(h - h^2w)$. This implies that $hw = 1$.

(2) Let $e \in ME_l(R)$ and $w \in R$. Take $h = we - ewe$. If possible, assume that $h \neq 0$. Clearly, $he = h$ and $h^2 = 0$. Observe that $0 \neq Rh \subseteq Re$. Since $e \in ME_l(R)$, $Rh = Re$. As $R$ is pointwise semicommutative and $h^2 = 0$, $hRh = 0$. Thus, $0 = RhRh = ReRe = Re$, a contradiction. Thus, $h = 0$.

**Proposition 2.4.** Let $R$ be a ring. Then $R$ is a domain if and only if $R$ is prime and pointwise semicommutative.
Proof. The necessary part is obvious. Conversely, assume that \( R \) is a prime and pointwise semicommutative ring and \( w, h \in R \) be such that \( wh = 0 \). Since \( R \) is pointwise semicommutative and \((hw)^2 = 0, hwRhw = 0\). By hypothesis, \( hw = 0 \). So for any \( r \in R \), \((wrh)^2 = 0\) which further implies that \( wrhRwrh = 0 \), that is, \( wrh = 0 \). As \( R \) is prime, \( w = 0 \) or \( h = 0 \). \( \square \)

It is well known that the ring

\[
R_n = \left\{ \begin{bmatrix} a & a_{12} & \cdots & a_{1(n-1)} & a_{1n} \\ 0 & a & \cdots & a_{2(n-1)} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & a_{(n-1)n} \\ 0 & 0 & \cdots & 0 & a \end{bmatrix} \mid a, a_{ij} \in R, \ i < j \right\}
\]

is semicommutative whenever \( R \) is reduced and \( n = 3 \). However, \( R_n \) is not semicommutative for \( n \geq 4 \) even if \( R \) is reduced (see \[4\] Example 1.3). So one might suspect whether \( R_n \) is pointwise semicommutative for \( n \geq 4 \) whenever \( R \) is reduced. Nevertheless, the following example obliterates the possibility.

**Example 2.5.** Let \( R = \mathbb{Z}_6 \), \( R_4 = \left\{ \begin{bmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & g \\ 0 & 0 & 0 & a \end{bmatrix} \mid a, b, c, d, e, f, g \in R \right\}, \)

where \( \mathbb{Z}_6 \) is the ring of integers modulo 6. Take \( A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \in R_4 \).

Note that \( B = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in r(A) \). Let \( C = \begin{bmatrix} 1 & 5 & 5 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in R_4 \).

Then \( ACB = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq 0 \). Now, observe that \( E = \begin{bmatrix} 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in R_4 \).

Thus, neither \( r(A) \) nor \( l(A) \) is an ideal of \( R_4 \), that is, \( R_4 \) is not a pointwise semicommutative ring.

**Proposition 2.6.** Let \( R \) be a pointwise semicommutative ring. Then \( R \) is NCI.

Proof. Let \( R \) be a pointwise semicommutative ring. Suppose \( N(R) \neq 0 \). Then there exists \( w \neq 0 \in N(R) \) such that \( w^n = 0 \) for some integer \( n \geq 2 \).
and \( w^{n-1} \neq 0 \). Since \( R \) is pointwise semicommutative, either \( w^{n-1}Rw = 0 \) or \( wRw^{n-1} = 0 \). So \( w^{n-1}Rw^{n-1} = 0 \). Hence \( Rw^{n-1}R \) is a non-zero nilpotent ideal of \( R \). Thus, \( R \) is NCI.

\[ \square \]

Observe that \( R_4 \) (in Example 2.5) is NCI. Hence the converse is not true.

**Proposition 2.7.** Let \( \{ R_i \}_{i \in \Delta} \) be a class of rings and \( \Delta \) an index set. If \( R = \Pi_{i \in \Delta} R_i \) is pointwise semicommutative, then \( R_i \) is pointwise semicommutative for each \( i \in \Delta \).

**Proof.** Let \( a_j \in R_j, \ j \in \Delta \). Suppose \( l((0, \ldots, 0, a_j, 0, \ldots)) \) is an ideal of \( R = \Pi_{i \in \Delta} R_i \) and \( x_j \in l(a_j) \). Note that \( (0, 0, \ldots, x_j, 0, \ldots) \in l((0, \ldots, 0, a_j, 0, \ldots)) \). So \( (0, 0, \ldots, x_j, 0, \ldots)(r_i)_{i \in \Delta}(0, \ldots, a_j, 0, \ldots) = 0 \) for any \((r_i)_{i \in \Delta} \in \Pi_{i \in \Delta} R_i \). So \( x_jr_j \in l(a_j) \) for all \( r_j \in R_j \). Thus, \( l(a_j) \) is an ideal of \( R_j \). Similarly, \( r(a_j) \) is an ideal of \( R_j \) whenever \( r((0, \ldots, 0, a_j, 0, \ldots)) \) is an ideal of \( R \). Therefore, for each \( j \in \Delta \), \( R_j \) is pointwise semicommutative.

\[ \square \]

However, the converse is not true (see the following example).

**Example 2.8.** Let \( R_i = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}, i \in \{1, 2\} \). Then \( R_i \) is pointwise semicommutative (see Example 2.2). Take \( A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \in R_1 \times R_2 \). Then \( l(A) = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \times \begin{bmatrix} 0 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix} \). Note that \( X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \in l(A) \). \( l(A) \) and take \( Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \in R_1 \times R_2 \). Then \( XY = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \notin l(A) \). Thus, \( l(A) \) is not an ideal of \( R_1 \times R_2 \).

Observe that \( r(A) = \begin{bmatrix} x & y \\ 0 & z \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & z \\ z & 0 \\ 0 & 0 \end{bmatrix} \). So, \( P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \in r(A) \). \( YP = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \notin r(A) \). So \( r(A) \) is not an ideal of \( R_1 \times R_2 \). Therefore, \( R_1 \times R_2 \) is not pointwise semicommutative.

**Proposition 2.9.** Let \( R \) be a pointwise semicommutative, and every nilpotent element is of index \( \leq 2 \). Then \( R \) is 2-primal.

**Proof.** It is obvious that \( P(R) \subseteq N(R) \). Let \( w \in N(R) \). Then \( w^2 = 0 \). Since \( R \) is pointwise semicommutative, \( wRw = 0 \subseteq P(R) \) and so \( w \in P(R) \). Therefore \( P(R) = N(R) \).

Following [6], an element \( w \) of a ring \( R \) is said to be clean if \( w \) is a sum of a unit, and an idempotent of \( R \), \( w \) is said to be exchange if there exist \( e \in E(R) \) such that \( e \in wR \) and \( 1 - e \in (1 - w)R \). \( R \) is said to be clean if every element of \( R \) is clean, and \( R \) is said to be exchange if every element of \( R \) is exchange. It is well known that clean rings are exchange.
Proposition 2.10. Let $R$ be a pointwise semicommutative exchange ring, then $R$ is clean.

Proof. Let $w \in R$. Then there exists $e \in E(R)$ satisfying $e \in wR$ and $1-e \in (1-w)R$. So $e = wh$ and $1-e = (1-w)k$ for some $h$, $k \in R$ such that $h = he$, $k = k(1-e)$. Then $(w-(1-e))(h-k) = wh-wk-(1-e)h+(1-e)k = wh + (1-w)k - (1-e)h - ek = 1 - (1-e)h - ek$. Since $R$ is pointwise semicommutative, either $r(e)$ or $l(e)$ is an ideal of $R$. If $r(e)$ is an ideal of $R$, then $0 = hes(1-e) = hs(1-e)$, that is, $(1-e)hs \in N(R)$ for all $s \in R$ and $0 = ek(1-e) = ek$. So $(1-e)h$, $ek \in J(R)$. If $l(e)$ is an ideal of $R$, then $(1-e)h = 0$ and $ek \in J(R)$. Therefore, $1 - (1-e)h - ek$ is unit and by Proposition 2.3(i), $w - (1-e)$ is unit. Thus, $w$ is clean. □

A ring $R$ is said to be semiperiodic ([1]) if for each $w \in R \setminus (J(R) \cup C(R))$, $w^p - w^q \in N(R)$ for some positive integers $p$ and $q$ of opposite parity.

Lemma 2.11. Let $R$ be a pointwise semicommutative ring. If $R$ is semiperiodic then $N(R) \subseteq J(R)$.

Proof. Suppose $w \neq 0$, $x \in R$ and $w^k = 0$ for some positive integer $k$. If $wx \in J(R)$, then $wx$ is right quasi-regular. If $wx \in C(R)$, then $wx$ is nilpotent, and so $wx$ is right quasi-regular. Assume that $wx \notin J(R) \cup C(R)$. Then by [1, Lemma 2.3(iii)], there exist a positive integer $p$ and $e \in E(R)$ such that $(wx)^p = (wx)^pe$ and $e = wy$ for some $y \in R$. Observe that $e = wy = ewy = ew(1-e)y + ewy = ew(1-e)y + ew^2y^2 = ... = \sum_{i=1}^{k-1} ew^i(1-e)y^i$. Since $R$ is pointwise semicommutative, $r(e)$ or $l(e)$ is an ideal of $R$. If $r(e)$ is an ideal of the $R$, then $ew^i(1-e)y^i = 0$ for all $i$ and hence $e = 0$. If $l(e)$ is an ideal of $R$ then $(1-e)re = 0$ for any $r \in R$ and hence $ew^i(1-e)s \in N(R)$ for all $i$ and $s \in R$, that is, $ew^i(1-e) \in J(R)$. Hence $e = \sum_{i=1}^{k-1} ew^i(1-e)y^i \in J(R)$, that is, $e = 0$. Consequently, we obtain $(wx)^p = 0$ and so $wx$ is right quasi-regular. Thus, $w \in J(R)$. □

Proposition 2.12. Let $R$ be a pointwise semicommutative ring. If $R$ is semiperiodic, then:

(1) $R/J(R)$ is commutative.
(2) $R$ is NI.
(3) $R$ is commutative whenever $J(R) \neq N(R)$.

Proof. (1) By Lemma 2.11, $N(R) \subseteq J(R)$. Write $\bar{R} = R/J(R)$ and let $\bar{w} \in \bar{R}$ with $\bar{w}^2 = 0$. Then by [1 Lemma 2.6], $w^2 \in J(R) \subseteq N(R) \cup C(R)$. If $w^2 \in N(R)$, then $w \in N(R) \subseteq J(R)$ (see Lemma 2.11), that is, $\bar{w} = 0$. Suppose $w^2 \notin N(R)$, then $w^2 \in C(R)$. If $w \in Z(R)$, then $\bar{w}R\bar{w} = 0$. Since $\bar{R}$ is semiprime, $\bar{w} = 0$. Assume, if possible, that $\bar{w} \notin C(\bar{R})$ then $w \notin J(R) \cup C(R)$. By [1 Lemma 2.3(iii)], there
exist a positive integer \( p \) and \( e \in E(R) \) such that \( w^p = w^p e \) and \( e = wy \) for some \( y \in R \). Hence \( e = ewy = ew(1 - e)y + ewy = ew(1 - e)y + ewy \). Since \( R \) is pointwise semicommutative, \( e \in J(R) \), that is, \( e = 0 \). This yields that \( w^p = 0 \) and so \( w \in N(R) \subseteq J(R) \), a contradiction. Therefore \( w \in C(R) \) and so \( w = 0 \). Thus, \( R \) is reduced. Since \( R \) is semiperiodic, \( R \) is commutative (by [1, Theorem 4.4]).

(2) Let \( w, h \in N(R) \) and \( x \in R \). By Lemma 2.11, \( w - h, wx \in J(R) \). By [1, Lemma 2.6], \( w - h, wx \in N(R) \cup C(R) \). If \( w - h, wx \in N(R) \), then nothing to prove. Suppose \( w - h, wx \in C(R) \). Now, observe that \( (w - h)w = w(w - h) \) and \( (wx)^m = w^m x^m \) for all integer \( m \geq 1 \). This implies that \( wh = hw \) and hence \( w - h, wx \in N(R) \). Hence \( N(R) \) is an ideal.

(3) By [1, Lemma 2.6], \( J(R) = (J(R) \cap N(R)) \cup (J(R) \cap C(R)) \). Note that \( R \) is NI (by [2], and \( J(R) \cap N(R) \) and \( J(R) \cap C(R) \) are additive subgroups of \( R \), so \( J(R) = J(R) \cap N(R) \) or \( J(R) \cap C(R) \). This yields that \( J(R) \subseteq N(R) \) or \( J(R) \subseteq C(R) \). By hypothesis and Lemma 2.11, \( J(R) \subseteq C(R) \). Let \( w \in R \). Suppose \( w \notin C(R) \). Then \( w \notin J(R) \cup C(R) \). Then there exist positive integers \( p, q \) \((p \geq q)\) of opposite parity such that \( w^p = w^q \in N(R) \). So \((w^p - w^q)^k = 0 \) for some \( k \geq 1 \). Then \((w^p - w^q - 1)^k = 0\), this gives \( w - w^p - q + 1 \in N(R) \subseteq J(R) \subseteq C(R) \). By Herstein’s Theorem [1], \( R \) is commutative.

The following examples show that the skew polynomial ring and the polynomial ring over a pointwise semicommutative ring need not be pointwise semicommutative.

**Example 2.13.**

(1) Let \( D \) be a division ring and \( R = D \oplus D \) with componentwise multiplication. Clearly, \( R \) is reduced, so \( R \) is pointwise semicommutative. Define \( \sigma(h, w) = (w, h) \). Then \( \sigma \) is an automorphism of \( R \). Let \( f(x) = (1, 0)x \in R[x; \sigma] \). Observe that \( f(x)^2 = 0 \) but \( f(x) f(x) \neq 0 \). Hence \( R[x; \sigma] \) is not pointwise commutative.

(2) Take \( \mathbb{Z}_2 \) as the field of integers modulo 2 and let \( A = \mathbb{Z}_2[a_0, a_1, a_2, b_0, b_1, b_2, c] \) be the free algebra of polynomials with zero constant terms in noncommuting indeterminates \( a_0, a_1, a_2, b_0, b_1, b_2 \) and \( c \) over \( \mathbb{Z}_2 \). Take an ideal \( I \) of the ring \( \mathbb{Z}_2 + A \) generated by \( a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, a_1 b_2 + a_2 b_1, a_1 a_0 b_0, a_2 r b_2, b_0 a_0, b_0 a_1 + b_1 a_0, b_0 a_2 + b_1 a_1 + b_2 a_0, b_1 a_2 + b_2 a_1, b_2 b_0 a_0, b_2 r a_0, b_2 r a_2, (a_0 + a_1 + a_2) r (b_0 + b_1 + b_2), (b_0 + b_1 + b_2) r (a_0 + a_1 + a_2) \) and \( r_1 r_2 r_3 r_4 \) where \( r, r_1, r_2, r_3, r_4 \in A \). Take \( R = (\mathbb{Z}_2 + A) / I \). Then we have \( R[x] \cong (\mathbb{Z}_2 + A)[x] / I[x] \). By [1, Example 2.1], \( R \) is reversible and hence pointwise semicommutative. Observe that \( (b_0 + b_1 x + b_2 x^2)(a_0 + a_1 x + a_2 x^2) \in I[x] \). But \( (b_0 + b_1 x + b_2 x^2) \notin I[x] \), since \( b_0 c a_1 + b_1 c a_2 \notin I \). Hence \( I ((a_0 + a_1 x + a_2 x^2)) \) is not an ideal of \( R[x] \). Again, \( (a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2) \notin I[x] \). But \( (a_0 + a_1 x + a_2 x^2)(b_0 + b_1 x + b_2 x^2) \notin
Proof. (1) Assume that \( R \) is pointwise semicommutative. Let \( \Delta \) be a multiplicatively closed subset of \( \Delta^{-1}R \) consisting of central non-zero divisors. For any \( u^{-1}a \in \Delta^{-1}R \), \( l(u^{-1}a) \) is an ideal of \( \Delta^{-1}R \) if and only if \( l(a) \) is an ideal of \( R \) and \( r(u^{-1}a) \) is an ideal of \( \Delta^{-1}R \) if and only if \( r(a) \) is an ideal of \( R \).

Proof. Easy to prove. \( \square \)

**Proposition 2.15.** Let \( R \) be a ring and \( \Delta \) be a multiplicatively closed subset of \( R \) consisting of central non-zero divisors. Then \( R \) is pointwise semicommutative if and only if \( \Delta^{-1}R \) is pointwise semicommutative.

Proof. Suppose \( R \) is pointwise semicommutative. Let \( u^{-1}a \in \Delta^{-1}R \). Then, either \( l(a) \) or \( r(a) \) is an ideal of \( R \). By the Lemma 2.14, either \( l(u^{-1}a) \) or \( r(u^{-1}a) \) is an ideal of \( \Delta^{-1}R \). Thus, \( \Delta^{-1}R \) is pointwise semicommutative. Observe that the converse is trivial. \( \square \)

**Corollary 2.16.** \( R[x] \) is pointwise semicommutative if and only if \( R[x,x^{-1}] \) is so.

Proof. Note that \( R[x,x^{-1}] = \Delta^{-1}R[x] \), where \( \Delta = \{1,x,x^2,...\} \). Hence the result follows. \( \square \)

A left \( R \)-module \( M \) is said to be \( Wnil-injective \) if for any \( w (\neq 0) \in N(R) \), there exists a positive integer \( m \) such that \( w^m \neq 0 \) and any \( R \)-homomorphism \( \Psi : Rw^m \rightarrow M \) extends to one from \( R \) to \( M \). In order to probe some properties of pointwise semicommutative rings, we investigate \( Wnil \)-injective modules over a pointwise semicommutative ring in the following.

**Proposition 2.17.** Let \( R \) be a pointwise semicommutative ring, and every simple singular left \( R \)-module is \( Wnil \)-injective then:

1. \( R \) is left non-singular.
2. \( R \) is left weakly regular whenever \( R \) is left MC2.
3. \( R \) is reduced if any \( e \in E(R) \), \( er = ere \) for all \( r \in R \).

Proof. (1) Assume that \( Z(RR) \neq 0 \). Then there exists \( w (\neq 0) \in Z(RR) \) such that \( w^2 = 0 \). So, \( l(w) \subseteq M \) for some maximal left ideal \( M \) of \( R \). Since \( w \in Z(RR) \), \( M \) is essential. Now, define an \( R \)-homomorphism \( \Psi : Rw \rightarrow R/M \) via. \( \Psi(rw) = r + M \). By hypothesis, \( R/M \) is \( Wnil \)-injective and so there exists \( h \in R \) with \( 1 - wh \in M \). Since \( R \) is pointwise semicommutative and \( w^2 = 0 \), \( whw = 0 \), that is, \( wh \in l(w) \), which further implies that \( 1 \in M \), a contradiction. Therefore \( Z(RR) = 0 \).

(2) Suppose there is an element \( w \in R \) such that \( RwR + l(w) \neq R \). So, \( Rrw + l(w) \subseteq M \) for some maximal left ideal \( M \) of \( R \). If \( M \) is not essential in \( RR \), then \( M = Re = l(1-e) \) for some \( e \in E(R) \). As
\( R(1-e) \cong R/l(1-e) = R/M \) is a simple left \( R \)-module, \( R(1-e) \) is a minimal left ideal of \( R \). By Proposition 2.3 (2), \( R \) is a left min-abel ring. Since \( R \) is a left MC2 ring, \( 1-e \in Z(R) \) by [8] Theorem 1.8.

As \( w \in RwR + l(w) \subseteq M = l(1-e) \), \( w(1-e) = 0 = (1-e)w \). So \( 1-e \in l(w) \subseteq M = l(1-e) \), a contradiction. Therefore, \( M \) is essential left ideal of \( R \). Thus, \( R/M \) is W-nil-injective. As in the proof of (1), \( 1-wh \in M \) for some \( h \in R \). Since \( wh \in RwR \subseteq M \), \( 1 \in M \), a contradiction. Therefore, \( RwR + l(w) = R \) for any \( w \in R \), that is, \( RwRw = R \). Hence \( R \) is a left weakly regular ring.

(3) Suppose there exists \( w (\neq 0) \in R \) satisfying \( w^2 = 0 \). Then \( l(w) \subseteq M \) for some maximal left ideal \( M \) of \( R \). If \( M \) is not essential, then \( M = l(e) \) for some \( 0 \neq e \in E(R) \). So \( we = 0 \) and by hypothesis, \( ew = ewe = 0 \). This implies that \( e \in l(w) \subseteq M = l(e) \), a contradiction. Hence \( M \) is essential, and \( R/M \) is simple singular left \( R \)-module. As in the proof of (1), \( 1-wh \in M \) for some \( h \in R \). Since \( R \) is pointwise semicommutative and \( w^2 = 0 \), \( wh \in l(w) \subseteq M \). This implies that \( 1 \in M \), a contradiction. Therefore, \( w = 0 \).

\[ \square \]

\( R \) is called a left (right) SF if all simple left (right) \( R \)-modules are flat. [7] Remark 3.13 shows that \( R \) is strongly regular whenever \( R \) is a reduced SF ring. We extend this result as follows.

**Proposition 2.18.** Let \( R \) be a pointwise semicommutative ring. If \( R \) is left SF, then \( R \) is strongly regular.

**Proof.** By [7] Proposition 3.2, \( R/J(R) \) is left SF. Let \( w^2 \in J(R) \) such that \( w \notin J(R) \). Assume, if possible, \( Rr(w) + J(R) = R \), then

\[ 1 = x + \sum_{\text{finite}} r_is_i, x \in J(R), r_i \in R, s_i \in r(w). \]

Then \( w = xw + \sum_{\text{finite}} r_is_iw \). Take \( t_i = s_iw \). So \( t_i^2 = 0 \). Since \( R \) is pointwise semicommutative, \( t_iRt_i = 0 \). Suppose \( t_i \notin J(R) \). Then \( M + Rt_i = R \) for some maximal left ideal \( M \) of \( R \) with \( t_i \notin M \). So \( m + pt_i = 1, m \in M, p \in R \). This yields that \( (1-m)^2 = 0 \), that is, \( 1 \in R \). This is a contradiction. Therefore, \( t_i \in J(R) \). This further yields that \( w \in J(R) \), a contradiction. Hence \( Rr(w) + J(R) \neq R \). There exist some maximal left ideal \( H \) satisfying \( Rr(w) + J(R) \subseteq H \). Note that \( w^2 \in H \). By [7] Lemma 3.14, \( w^2 = w^2x \) for some \( x \in H \), that is, \( w - wx \in r(w) \subseteq H \). So \( w \in H \). Hence there exists \( y \in H \) satisfying \( w = wy \), that is, \( 1 - y \in r(w) \subseteq H \). This implies that \( 1 \in H \), a contradiction. Therefore \( R/J(R) \) is reduced. Therefore by [7] Remark 3.13, \( R/J(R) \) is strongly regular. This implies that \( R \) is left quasi-duo, and hence by [7] Theorem 4.10, \( R \) is strongly regular. \[ \square \]

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