Interactions of Massless Higher Spin Fields
from String Theory

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Abstract

We construct vertex operators for massless higher spin fields in RNS superstring theory
and compute some of their three-point correlators, describing gauge-invariant cubic inter-
actions of the massless higher spins. The Fierz-Pauli on-shell conditions for the higher
spins (including tracelessness and vanishing divergence) follow from the BRST-invariance
conditions for the vertex operators constructed in this paper. The gauge symmetries of
the massless higher spins emerge as a result of the BRST nontriviality conditions for these
operators, being equivalent to transformations with the traceless gauge parameter in the
Fronsdal’s approach. The gauge invariance of the interaction terms of the higher spins
is therefore ensured automatically by that of the vertex operators in string theory. We
develop general algorithm to compute the cubic interactions of the massless higher spins
and use it to explicitly describe the gauge-invariant interaction of two $s = 3$ and one $s = 4$
massless particles.

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1. Introduction

Constructing the gauge field theories describing interacting particles of higher spins (with $s > 2$) is a fascinating and complicated problem that has attracted a profound interest over many years since the 30s. Despite strong efforts by some leading experts in recent years [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], there are still key issues about these theories that remain unresolved (even for the non-interacting particles; much more so in the interacting case). There are several reasons why the higher spin theories are so complicated. First of all, in order to be physically meaningful, these theories need to possess sufficiently strong gauge symmetries, powerful enough to ensure the absence of unphysical (negative norm) states. For example, in the Fronsdal’s description [32] the theories describing symmetric tensor fields of spin $s$ are invariant under gauge transformations with the spin $s - 1$ traceless parameter. Theories with the vast gauge symmetries like this are not trivial to construct even in the non-interacting case, when one needs to introduce a number of auxiliary fields and objects like non-local compensators [2], [3], [33], [34], [35]. Moreover, as the gauge symmetries in higher spin theories are necessary to eliminate the unphysical degrees of freedom, they must be preserved in the interacting case as well, i.e. one faces a problem (even more difficult) of introducing the interactions in a gauge-invariant way. In the flat space things are further complicated because of the no-go theorems (such as Coleman-Mandula theorem [36], [37]) imposing strong restrictions on conserved charges in interacting theories with a mass gap, limiting them to the scalars and those related to the standard Poincare generators. Thus Coleman-Mandula theorem in $d = 4$ makes it hard to construct consistent interacting theories of higher spin, at least as long as the locality is preserved, despite several examples of higher spin interaction vertices constructed over the recent years [8], [13], [38]. In certain cases, such as in AdS backgrounds, the Coleman-Mandula theorem can be bypassed (since there is no well-defined S-matrix in the AdS geometry) and gauge-invariant interactions can be introduced consistently - as it has been done in the Fradkin-Vasiliev construction [4], [5], [6], [39], [40], [41]. The AdS case is particularly interesting since, in the context of the AdS/CFT correspondence, the higher spin currents in $AdS_4$ have been found to be dual to the operators in $d = 3$ CFT described by the O(N) model [12]; also the higher spin dynamics in $AdS_5$ is presumed to be relevant to the weakly coupled limit of $N = 4$ super Yang-Mills theory in $d = 4$. In non-AdS geometries, however (such as in the flat case),
the no-go theorems do lead to complications, implying, in particular, that the interacting
gauge-invariant theories of higher spins have to be essentially non-local.

In this paper we approach this problem from the string theory side by constructing
vertex operators for massless higher spin fields. It has already been observed some time ago
that string theory is a particularly effective and natural framework to approach the problem
of higher spins \[43\], \[44\], \[45\] at least in the massive case, since the higher spin modes
naturally appear in the massive sector of the theory. Thus one can hope to obtain the higher
field spin theories in the low energy limit of string theory, by analyzing the worldsheet
correlators of the appropriate vertex operators. In the massless case, discussed in this work,
things, however, are more subtle. While it is well-known that the massive string modes
include higher spin fields, that can be emitted by the standard vertex operators (with the
standard stringy mass to spin relation), describing massless higher spin modes in terms of
strings is a challenge since the only vertex operator in open string theory, decoupled from
superconformal ghost degrees of freedom (and therefore existing at zero ghost picture)
has spin 1. Therefore the massless operators for the higher spins are inevitably those that
couple to the worldsheet ghost degrees of freedom and violate the picture equivalence. The
geometrical reasons for the existence of such picture-dependent operators and questions of
their BRST invariance and non-triviality have been discussed in a number of our previous
works (particularly in \[46\], \[47\]). In this paper we apply the formalism developed in
\[47\], \[48\] to construct physical vertex operators describing emissions of massless higher
spin fields by an open string. We mostly restrict ourselves to totally symmetric higher
spin fields, although it seems to be relatively straightforward to extend the construction,
performed in this work, to the higher spins corresponding to more general Young tableau,
as well as to the case of the multiple families of indices (e.g. considered in \[44\]). The
BRST-invariance constraints for the vertex operators, considered in this paper, lead to the
on-shell Pauli-Fierz conditions for the higher spin fields in space-time, coupled to these
operators. The gauge symmetries of the higher spin fields, on the other hand, follow from
the BRST nontriviality constraints on the appropriate vertex operators. In particular,
the BRST nontriviality conditions for massless symmetric operators of integer spins from
3 to 9, considered in this work, entail the gauge symmetries equivalent to those in the
Fronsdal’s approach (with the tracelessness condition imposed on the gauge parameter).
Thus the correlation functions of these operators, computed in this paper, lead to the
interaction terms for the higher spin fields, that are gauge-invariant by construction. The
paper is organized as follows. In the Sections 2-5 we present the expressions for the
vertex operators describing emissions of massless symmetric higher spin fields in RNS string theory and analyze their BRST invariance and nontriviality conditions, leading to the gauge symmetries and the on-shell conditions for the higher spins fields. In Section 6 we develop a technique to calculate the 3-point correlation functions of these operators, particularly using it to derive the gauge-invariant cubic interaction terms for $s = 3$ and a $s = 4$ higher spins. In the concluding section we comment on higher order interaction terms and on the directions for the future work.

2. Vertex Operators for Massless Higher Spins and BRST Conditions

We start with presenting the expressions for the vertex operators of massless higher spins in RNS superstring formalism. As was noted above, these operators are essentially coupled to the worldsheet ghost fields (in order to ensure the appropriate conformal dimension) and violate the equivalence of pictures (being the elements of nontrivial superconformal ghost cohomologies, particularly described in [46], [47]). To compute the their matrix elements, we shall need both negative and positive ghost picture representations of these operators (to ensure the ghost anomaly cancellation). The expressions for the symmetric massless higher spin operators for the spin values $3 \leq s \leq 9$ at their minimal negative $\phi$-pictures (i.e. with no local versions at pictures above the minimal one) are given by:

$$
V_{s=3}(p) = H_{a_1a_2a_3}(p)ce^{-3\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\vec{p}\vec{X}}
$$

$$
V_{s=4}(p) = H_{a_1...a_4}(p)\eta e^{-4\phi}\partial X^{a_1}\partial X^{a_2}\psi^{a_3}\psi^{a_4}e^{i\vec{p}\vec{X}}
$$

$$
V_{s=5}(p) = H_{a_1...a_5}(p)e^{-4\phi}\partial X^{a_1}...\partial X^{a_3}\psi^{a_4}\psi^{a_5}e^{i\vec{p}\vec{X}}
$$

$$
V_{s=6}(p) = H_{a_1...a_6}(p)c\eta e^{-5\phi}\partial X^{a_1}...\partial X^{a_3}\partial^2\psi^{a_4}\psi^{a_5}\psi^{a_6}e^{i\vec{p}\vec{X}}
$$

$$
V_{s=7}(p) = H_{a_1...a_7}(p)e^{-5\phi}\partial X^{a_1}...\partial X^{a_4}\partial^2\psi^{a_5}\psi^{a_6}\psi^{a_7}e^{i\vec{p}\vec{X}}
$$

$$
V_{s=8}(p) = H_{a_1...a_8}(p)c\eta e^{-5\phi}\partial X^{a_1}...\partial X^{a_7}\psi^{a_8}e^{i\vec{p}\vec{X}}
$$

$$
V_{s=9}(p) = H_{a_1...a_9}(p)e^{-5\phi}\partial X^{a_1}...\partial X^{a_8}\psi^{a_9}e^{i\vec{p}\vec{X}}
$$

where $X^a$ and $\psi^a$ are the RNS worldsheet bosons and fermions ($a = 0, ..., d - 1$), the ghost fields are bosonized as usual, according to

$$
\begin{align*}
    b &= e^{-\sigma}, c = e^{\sigma} \\
    \gamma &= e^{\phi-\chi} \equiv e^{\phi}\eta \\
    \beta &= e^{\chi-\phi}\partial\chi \equiv \partial\xi e^{-\phi}
\end{align*}
$$

The vertices for the massless spin fields with $s > 9$ can be constructed similarly, by using the combinations of $\partial X$’s and symmetrized products of $\psi$’s and their derivatives.
Obviously (from simple conformal dimension arguments) they would have to carry bigger values of minimal negative ghost numbers, which would make them technically cumbersome objects to work with.

For simplicity, in this work we shall concentrate on the totally symmetric polarization tensors $H_{a_1...a_s}(p)$, although it should be relatively straightforward to generalize the vertices (1) to less symmetric cases. For example, the operators with 2 families of indices can be obtained by separating the indices carried by the derivatives of $X$’s and $\psi$’s into 2 independent groups. Let us now turn to the question of the BRST-invariance and the non-triviality of the vertex operators (1). We start from the BRST-invariance condition. For simplicity, consider the $s = 3$ vertex operator first, all other operators can be analyzed similarly. For our purposes it is convenient to cast the BRST operator as

$$Q_{brst} = Q_1 + Q_2 + Q_3$$

where

$$Q_1 = \oint \frac{dz}{2i\pi} \{ cT - bc\partial c \}$$

$$Q_2 = -\frac{1}{2} \oint \frac{dz}{2i\pi} \gamma_\psi \partial X^a$$

$$Q_3 = -\frac{1}{4} \oint \frac{dz}{2i\pi} b\gamma^2$$

where $T$ is the full stress-energy tensor. It is easy to demonstrate that all the vertex operators (1) commute with $Q_2$ and $Q_3$ of $Q_{brst}$. The commutation with $Q_1$, however, requires the constraints on the on-shell fields. Since all the operators (1) are the worldsheet integrals of operators of conformal dimension 1, they commute with $Q_1$ if the integrands are the primary fields, i.e. their OPEs with $T$ don’t contain singularities stronger than double poles (along with the on-shell $(\vec{p})^2 = 0$ condition). Since $H_{a_1a_2a_3}$ is fully symmetric, the OPE is given by

$$T(z)\partial X^{(a_1} \partial X^{a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p) \sim -\frac{\eta^{(a_1a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p)}{(z-w)^4}$$

$$+ i \frac{p^{(a_1} \partial X^{a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p)}{(z-w)^3} + O((z-w)^{-2})$$

$$Q_{brst} = Q_1 + Q_2 + Q_3$$

$$Q_1 = \oint \frac{dz}{2i\pi} \{ cT - bc\partial c \}$$

$$Q_2 = -\frac{1}{2} \oint \frac{dz}{2i\pi} \gamma_\psi \partial X^a$$

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$$T(z)\partial X^{(a_1} \partial X^{a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p) \sim -\frac{\eta^{(a_1a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p)}{(z-w)^4}$$

$$+ i \frac{p^{(a_1} \partial X^{a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p)}{(z-w)^3} + O((z-w)^{-2})$$

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$$T(z)\partial X^{(a_1} \partial X^{a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p) \sim -\frac{\eta^{(a_1a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p)}{(z-w)^4}$$

$$+ i \frac{p^{(a_1} \partial X^{a_2} \psi^{a_3)} e^{i\vec{p}\vec{X}}(w)H_{a_1a_2a_3}(p)}{(z-w)^3} + O((z-w)^{-2})$$
Therefore the BRST-invariance conditions for the $s = 3$ vertex:

\[
H_{a_1a_3}^1(p) = 0
\]

\[
p^2 H_{a_1a_2a_3}(p) = 0
\]

\[
p^2 H_{a_1a_2a_3}(p) = 0
\]

are precisely the Pauli-Fierz conditions for the symmetric massless higher spins.

Let us now turn to the question of the BRST nontriviality of the $V_s$ operators (1). We look for the conditions to ensure that $V_s$ cannot be represented as a BRST commutators with operators in small Hilbert space, i.e. for a given $V_s$ there is no operator $W_s$ such that $V_s = \{Q_{\text{brst}}, W_s\}$. We start with the operators for massless fields with odd spin values ($s = 3, 5, 7, 9$) that have the following structure if taken at minimal negative ghost pictures $-n$ ($n = 3$ for $s = 3$, $n = 4$ for $s = 5$ and $n = 5$ for $s = 7, 9$):

\[
V_s = c e^{-n \phi} F_{\frac{n^2}{2} - n + 1}(X, \psi)
\]

where $F_{\frac{n^2}{2} - n + 1}(X, \psi)$ is the primary matter field of conformal dimension $\frac{n^2}{2} - n + 1$ (suppressing all the indices). Then there are only two possible sources of $W_s$. The first possibility is that $W_s$ is proportional to the ghost factor $\partial^2 \xi \partial^2 \xi e^{-(n+2)\phi}$. Then there is a possibility that $V_s$ could be obtained as a BRST commutator with

\[
W_s = \partial^2 \xi \partial^2 \xi e^{-(n+2)\phi} G^{(2n-3)}(\phi, \chi, \sigma) F_{\frac{n^2}{2} - n + 1}(X, \psi)
\]

where $G^{(2n-3)}(\phi, \chi, \sigma)$ is the conformal dimension $2n - 3$ polynomial in the derivatives of the bosonized ghost fields $\phi, \chi$ and $\sigma$ that must be chosen so that

\[
\{Q_1, W_s\} = 0
\]

Provided that $G^{(2n-3)}(\phi, \chi, \sigma)$ are chosen to satisfy (9), it is easy to check that the $W_s$-operators also satisfy

\[
\{Q_2, W_s\} = 0
\]

\[
\{Q_3, W_s\} = \alpha_n V_s
\]

and therefore

\[
\{Q_{\text{brst}}, W_s\} = \alpha_n V_s
\]

where $\alpha_n$ are the numerical coefficients that depend on the structure of $G^{(2n-3)}(\phi, \chi, \sigma)$. A lengthy but straightforward computation shows, however, that for all the choices of
$G^{(2n-3)}(\phi, \chi, \sigma)$, consistent with the condition (9) for $n = 3, 4, 5$ (that are relevant for the higher spin operators (1) with $3 \leq s \leq 9$) one has

$$\alpha_n = 0$$

$$n = 3, 4, 5$$  \hspace{1cm} (12)

and therefore the higher spin operators cannot be written as commutators of $Q_{brst}$ with the $W_s$ operators with the structure (8). The details of the calculation for $n = 3$ case are given in [48]; the $n = 4$ and $n = 5$ cases are treated totally similarly, producing $\alpha_n = 0$. At present, we do not know if the $\alpha_n$ constants also vanish for $n > 5$. This question is important in relation with the massless spin operators with $s > 9$. Thus there are no BRST nontriviality conditions on the higher spin fields of the $V_s$-operators of the type (7) due to the $W_s$-operators with the structure (8). The second, and the only remaining possibility for $V_s$ to be written as BRST commutators stems from the $W_s$-operators with the ghost structure $\sim c\partial\xi e^{-(n+1)\phi}$, satisfying

$$[Q_1, W_s] = 0$$

$$[Q_2, W_s] \sim V_s$$

$$[Q_3, W_s] = 0$$  \hspace{1cm} (13)

The only possible construction for $W_s$ with such a structure is given by

$$W_s = c\partial\xi e^{-(n+1)\phi} F_{\frac{2}{n+1}}(X, \psi)(\psi_\alpha \partial X^\alpha)$$  \hspace{1cm} (14)

The operators of this type always commute with $Q_3$ and produce $V_s$ when commuted with $Q_2$. Therefore $V_s$ are trivial as long as $W_s$ commute with $Q_1$. So $V_s$ are physical operators only if the commutator $[Q_1, W_s] \neq 0$, which, in turn, imposes constraints on the space-time fields and entails the gauge symmetries for the higher spins. Let us consider the particular case of $s = 3$, other operators are analyzed similarly. The $W_s$-operator of the type (14) for $V_{s=3}$ (1) is

$$W_{s=3}(p) = c\partial\xi e^{-4\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} (\bar{\psi} \partial X) e^{i\bar{\psi}X} H_{a_1a_2a_3}(p)$$  \hspace{1cm} (15)

where, as previously, the $H$ three-tensor is symmetric and satisfies the on-shell conditions (6) Using the Pauli-Fierz constraints (6) on $H$, one easily finds that $W_{s=3}$ satisfies:

$$[Q_1, W_{s=3}(p)] = -\frac{i}{2} \partial^2 c \partial\xi e^{-4\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} (\bar{\psi} \partial X) e^{i\bar{\psi}X} H_{a_1a_2a_3}(p)$$

$$[Q_2, W_{s=3}(p)] = \frac{d}{2} V_{s=3}(p)$$

$$[Q_2, W_{s=3}(p)] = 0$$  \hspace{1cm} (16)
So the nontriviality of $V_{s=3}$ requires that the right hand side of the commutator $[Q_1, W_{s=3}(p)]$ is nonzero. This leads to the following nontriviality conditions on the $H$-tensor:

$$p_{[a_4}H_{a_3]a_1a_2} \neq 0 \quad (17)$$

The analysis of the nontriviality constraints for all other odd spin operators ($s = 5, 7, 9$) with the structure (7) is totally similar and leads to the same conditions on $H_{a_1...a_s}(p)$:

$$p_{[a_{s+1}}H_{a_s]a_1...a_{s-1}} \neq 0. \quad (18)$$

Next, consider the even spin operators ($s = 4, 6, 8$) that, if taken at their minimal superconformal ghost pictures $-n$ ($n = 4$ for $s = 4$ and $n = 5$ for $s = 6, 8$), have the structure

$$V_s = ce^{-n}F_{\frac{n^2}{2} - n}(X, \psi) \quad (19)$$

where $F_{\frac{n^2}{2} - n}(X, \psi)$ is again the primary matter field of conformal dimension $\frac{n^2}{2} - n$. The nontriviality analysis for these operators doesn’t differ from the odd spin case that we have just described. As before, there are two potential sources $W_s$ that could imply the triviality of $V_s$, the first is

$$W_s(p) = \partial cc\partial \xi e^{-(n+2)\phi}F_{\frac{n^2}{2} - n}(X, \psi)G^{(2n)}(\phi, \chi, \sigma)H(p) \quad (20)$$

satisfying (9) - (11) and the second is

$$W_s(p) = ce^{-n}(\bar{\psi}\partial \bar{X})F_{\frac{n^2}{2} - n}(X, \psi)G^{(n-1)}(\phi, \chi, \sigma)e^{i\vec{p}\vec{X}}H(p) \quad (21)$$

satisfying (13) where, as before, $G^{(h)}(\phi, \chi, \sigma)$ are the conformal dimension $h$ polynomials in derivatives of the bosonized ghost fields, chosen so that $W_s$ and $Q_1$ commute. As previously, lengthy but straightforward analysis (with some help of Mathematica) shows that all the ghost operators $G^{(2n)}(\phi, \chi, \sigma)$ of (20), leading to $[Q_1, W_s] = 0$ for $W_s$ of the type (20), imply $\alpha_n = 0$ ($n = 4, 5$), implying the nontriviality of $V_s$ without any conditions on $H(p)$. At the same time, the nontriviality of $V_s$ due to $W_s$ of the type (21) imply the constraints on $H(p)$ identical to (18). Thus the BRST nontriviality constraints for the massless higher spin operators are summarized by the condition (18) on $H_{a_1...a_s}(p)$ for both even and odd values of $s$. The constraints (18) entail, in turn, the gauge symmetry transformations for $H_{a_1...a_s}(p)$ that will be analyzed in the next section.
3. BRST Nontriviality Conditions and Gauge Symmetries for Higher Spins

The gauge symmetry for the higher spin fields is the consequence of the nontriviality condition (18) for their vertex operators. It is not difficult to show that the condition (18) entails the gauge symmetry transformations

\[ H_{a_1...a_n}(p) \rightarrow H_{a_1...a_n}(p) + p(a_1\Lambda_{a_2...a_n}) \]  

(22)
i.e. the gauge symmetry transformations for a spin \( n \) massless field in the Fronsdal’s formalism.

To show this, consider, for simplicity, the \( s = 3 \) operator, other cases can be analyzed similarly. Consider first the case of an arbitrary (not necessarily symmetric) polarization tensor \( H_{a|bc} \) which symmetry in \( b \) and \( c \) is the consequence of the multiplication by \( \partial X^b \partial X^c \) in the vertex operator for \( s = 3 \).

Then the constraint (18) implies that \( H_{a|bc}(p) \) can be shifted by the gauge transformation

\[ H_{a|bc}(p) \rightarrow H_{a|bc}(p) + p_a\Lambda_{bc}(p) \]  

(23)
provided that the symmetric rank 2 gauge parameter \( \Lambda^{bc} \) is traceless:

\[ \eta_{bc}\Lambda^{bc} = 0 \]  

(24)
due to the BRST-invariance conditions (6). Renaming the indices \( a \leftrightarrow b, a \leftrightarrow c \) we get:

\[ H_{b|ac}(p) \rightarrow H_{b|ac}(p) + p_b\Lambda_{ac}(p) \]  

(25)
and

\[ H_{c|ab}(p) \rightarrow H_{c|ab}(p) + p_c\Lambda_{ab}(p) \]  

(26)

Summing together (23), (25), (26) we obtain the transformations

\[ H_{(a|bc)}(p) \rightarrow H_{(a|bc)}(p) + p(a\Lambda_{bc})(p) \]  

(27)
leading to (22). Alternatively, one could start with (23), decomposing the left and the right hand side into two Young diagrams, one fully symmetric (single row) and another \( \Gamma \)-like with two rows. Interestingly, straightforward calculation of the S-matrix elements involving the vertex operators with the double-row polarizations shows them to vanish, so the tensors of \( \Gamma \)-like diagrams do not contribute to correlation functions of the \( s = 3 \) vertex operators (1), with only the symmetric part of (23) left.
This concludes the proof that the vertex operators (1) are the sources of the massless higher spin fields of spin values $3 \leq s \leq 9$ with Pauli-Fierz on-shell conditions and with suitable gauge symmetries equivalent to those of the Fronsdal’s description. All these properties are consequences of the BRST invariance and nontriviality conditions for the appropriate vertex operators. Therefore the correlation functions of these operators, describing the interactions of the massless higher spins, will by construction lead to the interaction terms, consistent with the basic properties of the massless higher spins, including the gauge invariance. In the following sections we shall particularly concentrate on the three-point correlation functions of the operators (1) leading to the consistent gauge-invariant cubic terms for interacting massless higher spins.

4. Vertex Operators for Higher spins: representations at positive ghost pictures

Before we start the computation of the correlators of the higher spin operators, it is necessary to obtain their representations in positive ghost pictures, in order to ensure the appropriate ghost number balance in the correlation functions.

Because the operators (1) violate picture equivalence, higher picture versions cannot be obtained by straightforward picture-changing transformation (which simply annihilates these operators). Moreover, there are no local (unintegrated) analogues of the operators (1) at higher ghost pictures, so all of their higher picture versions always appear in the integrated form. In particular, in this paper we shall need to use, in addition to unintegrated higher spin vertex operators (1) at negative ghost pictures $-n - 2$ with $n = 1, 2, 3$, their integrated counterparts at positive ghost pictures $n$. These counterparts can be constructed by using the $K$-transformation procedure \cite{46,47} which we shall briefly review below. Consider one of unintegrated vertex operators (1) for odd spins at minimal negative picture $-n - 2$ (the even spin case is considered analogously). Such an operator has a structure

$$V_{-n-2} = ce^{-(n+2)\phi} F_{\frac{n^2}{2}+n+1}(X, \psi)$$ (28)

where, as previously, $F_{\frac{n^2}{2}+n+1}(X, \psi)$ the is matter primary field of conformal dimension $\frac{n^2}{2} + n + 1$. Using the fact that the operators $e^{-(n+2)\phi}$ and $e^{n\phi}$ have the same conformal dimension $-\frac{n^2}{2} - n$, one starts with constructing the charge

$$\oint V_n \equiv \oint dze^{n\phi} F_{\frac{n^2}{2}+n+1}(X, \psi)$$ (29)
This charge commutes with $Q_1$ since it is a worldsheet integral of dimension 1 and $b - c$ ghost number zero but doesn’t commute with $Q_2$ and $Q_3$. To make it BRST-invariant, one has to add the correction terms by using the following procedure \cite{46, 47}. We write

$$[Q_{brst}, V_n(z)] = \partial U(z) + W_1(z) + W_2(z) \tag{30}$$

and therefore

$$[Q_{brst}, \oint dz V_n] = \oint dz(W_1(z) + W_2(z)) \tag{31}$$

where

$$U(z) \equiv cV_n(z)$$

$$[Q_1, V_n] = \partial U$$

$$W_1 = [Q_2, V_n]$$

$$W_2 = [Q_3, V_n] \tag{32}$$

Introduce the dimension 0 $K$-operator:

$$K(z) = -4ce^{2x-2\phi}(z) \equiv \xi\Gamma^{-1}(z) \tag{33}$$

satisfying

$$\{Q_{brst}, K\} = 1 \tag{34}$$

It is easy to check that this operator has a non-singular operator product with $W_1$:

$$K(z_1)W_1(z_2) \sim (z_1 - z_2)^{2n}Y(z_2) + O((z_1 - z_2)^{2n+1}) \tag{35}$$

where $Y$ is some operator of dimension $2n+1$. Then the complete BRST-invariant operator can be obtained from $\oint dz V_n(z)$ by the following transformation:

$$\oint dz V_n(z) \rightarrow A_n(w) = \oint dz V_n(z) + \frac{1}{(2n)!} \oint dz(z - w)^{2n} : K\partial^{2n}(W_1 + W_2) : (z)$$

$$+ \frac{1}{(2n)!} \oint dz\partial_z^{2n+1}[(z - w)^{2n}K(z)]K\{Q_{brst}, U\} \tag{36}$$

where $w$ is some arbitrary point on the worldsheet. It is then straightforward to check the invariance of $A_n$ by using some partial integration along with the relation (34) as well as the obvious identity

$$\{Q_{brst}, W_1(z) + W_2(z)\} = -\partial(\{Q_{brst}, U(z)\}) \tag{37}$$
Although the invariant operators $A_n(w)$ depend on an arbitrary point $w$ on the worldsheet, this dependence is irrelevant in the correlators since all the $w$ derivatives of $A_n$ are BRST exact - the triviality of the derivatives ensures that there will be no $w$-dependence in any correlation functions involving $A_n$. Equivalently, the positive picture representations $A_n$ (36) for higher spin operators can also be obtained from minimal negative picture representations $V_{-n-2}$ by straightforward, but technically more cumbersome procedure by using the combination of the picture-changing and the $Z$-transformation (the analogue of the picture-changing for the $b - c$-ghosts).

Namely, the $Z$-operator, transforming the $b - c$ pictures (in particular, mapping integrated vertices to unintegrated) given by [19]

$$Z(w) = b\delta(T)(w) = \oint dz(z - w)^3(bT + 4c\partial\xi\xi e^{-2\phi T^2})(z)$$

where $T$ is the full stress-energy tensor in RNS theory. The usual picture-changing operator, transforming the $\beta - \gamma$ ghost pictures, is given by $\Gamma(w) =: \delta(\beta)G : (w) =: e^\phi G : (w)$. Introduce the integrated picture-changing operators $R_n(w)$ according to

$$R_n(w) = Z(w) : \Gamma^n : (w)$$

where $: \Gamma^n :$ is the $n$th power of the standard picture-changing operator:

$$: \Gamma^n : (w) =: e^{n\phi}\partial^{n-1}G...\partial GG : (w)$$

$$\equiv: \partial^{n-1}\delta(\beta)...\partial\delta(\beta) :$$

Then the positive picture representations for the higher spin operators $A_n$ can be obtained from the negative ones $V_{-n-2}$ (1) by the transformation:

$$A_n(w) = (R_2)^{n+1}(w)V_{-n-2}(w)$$

Since both $Z$ and $\Gamma$ are BRST-invariant and nontrivial, the $A_n$-operators by construction satisfy the BRST-invariance and non-triviality conditions identical to those satisfied by their negative picture counterparts $V_{-2n-2}$ and therefore lead to the same Pauli-Fierz on-shell conditions (6) and the gauge symmetries (22), (23) for the higher spin fields.
Below we shall list some concrete examples of the $K$-transformation (36) applied to
the spin $s = 3$ and $s = 4$ operators that will be used in our calculations. For the $s = 3$
operator the above procedure gives

\[
V_{s=3} = c e^{-3\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}} H_{a_1 a_2 a_3} (p) \to \int dz V_1
\]

\[
= H_{a_1 a_2 a_3} (p) \int e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}
\]

\[
[Q_1, V_1] = \partial U = H_{a_1 a_2 a_3} (p) \partial (c e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}})
\]

\[
[Q_2, V_1] = W_1 = \frac{1}{2} H_{a_1 a_2 a_3} (p) e^{2\phi - x} \{(-\vec{\psi} \partial \vec{X}) + i(\vec{p} \vec{\psi}) P_{\phi-\chi} \} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}
\]

\[
\frac{1}{2} \partial X^{a_1}(\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} - \partial X^{a_1} \partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)}) \} e^{i\vec{p}\vec{X}}
\]

\[
[Q_3, V_1] = W_2 = -\frac{1}{4} H_{a_1 a_2 a_3} (p) e^{3\phi - 2x} P_{2\phi-2\chi-\sigma}^{(1)} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}
\]

(42) where the conformal weight $n$ polynomials in the derivatives of the ghost fields $\phi, \chi, \sigma$ are
defined according to $^{16, 17}$:

\[
P_{f(\phi, \chi, \sigma)}^{(n)} (z) = e^{-f(\phi(z), \chi(z), \sigma(z))} \frac{\partial^n}{\partial z^n} e^{f(\phi(z), \chi(z), \sigma(z))}
\]

(43) where $f$ is some linear function in $\phi, \chi, \sigma$. For example, $P_{\phi-\chi}^{(1)} = \partial \phi - \partial \chi$, etc. Note that
the product (43) is defined in the algebraic sense (not as an operator product).

Accordingly,

\[
K \partial^2 W_1 := 4 H_{a_1 a_2 a_3} (p) \partial c \xi \{(-\vec{\psi} \partial \vec{X}) + i(\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} \} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}
\]

\[
+ \partial X^{a_1}(\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} - \partial X^{a_1} \partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)}) \} e^{i\vec{p}\vec{X}}
\]

\[
: K \partial^2 W_2 := H_{a_1 a_2 a_3} (p) \{ -\partial^2 (e^\phi \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}) + P_{2\phi-2\chi-\sigma}^{(2)} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}} \}
\]

(44) and

\[
: \partial^{2n+1} KK\{Q_{brst}, U\} := -24 H_{a_1 a_2 a_3} (p) \partial c c \partial \xi \xi e^{-\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}
\]

\[
: \partial^m KK\{Q_{brst}, U\} := 0 (m < 2n + 1)
\]

(45) and therefore, upon integrating out total derivatives, the complete BRST-invariant expression for the $s = 3$
operator at picture 1 is

\[
A_{s=3}(w) = H_{a_1 a_2 a_3} (p) \int dz (z - w)^2 \{ \frac{1}{2} P_{2\phi-2\chi-\sigma}^{(2)} P_{\phi-\chi}^{(1)} \partial X^{a_1} \partial X^{a_2} \psi^{a_3}
\]

\[
+ 2c \xi \{(-\vec{\psi} \partial \vec{X}) + i(\vec{p} \vec{\psi}) P_{\phi-\chi}^{(1)} \} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}
\]

\[
+ \partial X^{a_1}(\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P_{\phi-\chi}^{(1)}) \psi^{a_3} - \partial X^{a_1} \partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3} P_{\phi-\chi}^{(1)}) \} e^{i\vec{p}\vec{X}}
\]

(46)
To abbreviate notations for our calculations of the correlation functions in the following sections, it is convenient to write the vertex operator \( A_{s=3} \) (46) as a sum

\[
A_{s=3} = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6
\]  

(47)

where

\[
A_0(w) = \frac{1}{2} H_{a_1a_2a_3}(p) \oint dz(z - w)^2 P^{(2)}_{2\phi - \chi - \sigma} e^{\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}(z)
\]  

(48)

and

\[
A_6(w) = -12 H_{a_1a_2a_3}(p) \oint dz(z - w)^2 \partial c c \partial \xi \xi e^{-\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}(z)
\]  

(49)

have ghost factors proportional to \( e^\phi \) and \( \partial c c \partial \xi \xi e^{-\phi} \) respectively and the rest of the terms carry ghost factor proportional to \( c \xi \):

\[
A_1(w) = -2 H_{a_1a_2a_3}(p) \oint dz(z - w)^2 c \xi (\vec{p} \vec{\psi}) P^{(1)}_{\phi - \chi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}(z)
\]

\[
A_2(w) = 2i H_{a_1a_2a_3}(p) \oint dz(z - w)^2 c \xi (\vec{p} \vec{\psi}) P^{(1)}_{\phi - \chi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}(z)
\]

\[
A_3(w) = 2i H_{a_1a_2a_3}(p) \oint dz(z - w)^2 c \xi (\vec{p} \vec{\psi}) P^{(1)}_{\phi - \chi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{i\vec{p}\vec{X}}(z)
\]

(50)

\[
A_4(w) = 2 H_{a_1a_2a_3}(p) \oint dz(z - w)^2 c \xi (\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P^{(1)}_{\phi - \chi}) \psi^{a_3} e^{i\vec{p}\vec{X}}(z)
\]

\[
A_5(w) = -2 H_{a_1a_2a_3}(p) \oint dz(z - w)^2 c \xi \partial X^{a_1} \partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3} P^{(1)}_{\phi - \chi}) e^{i\vec{p}\vec{X}}(z)
\]

Analogously, the \( K \)-operator procedure applied to the \( s = 4 \) vertex operator in (1) leads to the positive picture representation of the \( s = 4 \) operator given by

\[
B_{s=4} = B_0 + B_1 + B_2 + B_3 + B_4 + B_5 + B_6
\]  

(51)

where

\[
B_0(w) = \frac{1}{2} H_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 P^{(2)}_{2\phi - \chi - \sigma} \eta e^{2\phi} \partial X^{a_1} \partial X^{a_2} \partial \psi^{a_3} \psi^{a_4} e^{i\vec{p}\vec{X}}(z)
\]  

(52)

and

\[
B_7(w) = -12 H_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 \partial c c \xi \partial X^{a_1} \partial X^{a_2} \partial \psi^{a_3} \psi^{a_4} e^{i\vec{p}\vec{X}}(z)
\]  

(53)
carry the ghost factors $\sim \eta e^{2\phi}$ and $\sim \partial \xi \partial \xi$ respectively, while the rest of the terms carry the ghost factor $\sim ce^{\phi}$:

$$B_1(w) = -2H_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 ce^{\phi}(\bar{\psi}\partial \bar{X})\partial X^{a_1}\partial X^{a_2}\partial \psi^{a_3}\psi^{a_4}e^{i\bar{p}\bar{X}}(z)$$

$$B_2(w) = 2iH_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 ce^{\phi}(\bar{p}\partial \bar{\psi})P^{(1)}_{\phi - \chi}\partial X^{a_1}\partial X^{a_2}\partial \psi^{a_3}\psi^{a_4}e^{i\bar{p}\bar{X}}(z)$$

$$B_3(w) = 2iH_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 ce^{\phi}(\bar{p}\partial \bar{\psi})\partial X^{a_1}\partial X^{a_2}\partial \psi^{a_3}\psi^{a_4}e^{i\bar{p}\bar{X}}(z)$$

$$B_4(w) = 2H_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 P^{(2)}_{\phi - \chi}ce^{\phi} \partial X^{a_1}\partial^2 \psi^{a_2}\partial \psi^{a_3}\psi^{a_4}e^{i\bar{p}\bar{X}}(z)$$

$$B_5(w) = 2H_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 ce^{\phi}\partial X^{a_1}\partial X^{a_2} (\frac{1}{2}\partial^3 X^{a_3}$$
$$+ \partial^2 X^{a_3}P^{(1)}_{\phi - \chi} + \frac{1}{2} \partial X^{a_3}P^{(2)}_{\phi - \chi})\psi^{a_4}e^{i\bar{p}\bar{X}}(z)$$

$$B_6(w) = -2H_{a_1a_2a_3a_4}(p) \oint dz(z - w)^2 ce^{\phi}\partial X^{a_1}\partial X^{a_2} (\partial^2 X^{a_3} + \partial X^{a_3}P^{(1)}_{\phi - \chi})\partial \psi^{a_4}e^{i\bar{p}\bar{X}}(z)$$

(54)

The procedure is totally similar for the operators in (1) with $s \geq 5$ which positive picture representations can be constructed analogously; however, higher ghost number operators generally consist of bigger number of terms, so the manifest expressions for operators with higher $n$ become quite cumbersome.

5. $\xi$-dependence of Higher Spin Vertices: a comment

One property of the higher spin vertices which may seem unusual is their manifest dependence on the zero mode of $\xi$ in positive picture representations which poses a question whether the states created by these operators are outside the small Hilbert space. It is not difficult to see, however, that the manifest $\xi$-dependence of the operators (46)-(54) is just the matter of the gauge and can be removed by suitable picture-changing transformation. Indeed, it is straightforward to show that all the operators with the structure (36) can be represented as BRST commutators in the large Hilbert space:

$$A_n = const \times \{Q_{brst}, \oint dz(z - w)^{2n} \xi \partial \xi ce^{(n-2)\phi}F_{a_1^{a_2}a_3a_4\cdots a_n}(X, \psi)\}$$

(55)

with the matter operators $F_{a_1^{a_2}a_3a_4\cdots a_n}(X, \psi)$ taken the same as in (1). But since $\{Q_{brst}, \xi\} = \Gamma$ is a picture-changing operator, the expression (55) is actually given by picture-changing transformation of an operator inside the small Hilbert space. In fact, the $\xi$-dependence of the $A_n$-vertices is inherited from the structure of the $Z$-operator in the map (41). The
Z-operator (38) relating the $b - c$ ghost pictures also manifestly depends on $\xi$ but, as one can cast it as a BRST commutator in the large Hilbert space [17]:

$$Z(w) = 16\{Q_{brst}, \int dz(z - w)^3bc\partial\xi\xi e^{-2\phi T}\}$$

such a dependence is also the matter of the gauge, in analogy to the case explained above.

6. Three-Point Correlation Functions and Cubic Interactions of Higher Spin Fields

In this section we derive the gauge-invariant cubic interaction terms for the higher spinors by computing the three-point functions of the vertex operators (1), (36), (46), (51). The gauge invariance is the consequence of the BRST non-triviality conditions for the vertex operators and is thus ensured by construction. For simplicity, we shall particularly concentrate on derivation of the cubic interaction of $s = 3$ and $s = 4$ spin fields - mainly because the vertex operators for these fields have relatively simple structure; however the calculation performed in this section is straightforward to generalize to cases of other massless integer higher spins. Before we start, it is useful to introduce an object that we shall refer to as “interaction block” $T_{p,q,r}^{a_1...a_p|b_1...b_q|c_1...c_r}(p_1,p_2,p_3)$ and that will play an important role in our calculations. Consider a three-point correlation function

$$A^{a_1...a_p|b_1...b_q|c_1...c_r}(z,w,u;p_1,p_2,p_3) = <V_1(z,p_1)V_2(w,p_2)V_3(u,p_3)>$$

$$=<\partial X^{a_1}...\partial X^{a_p}e^{i\vec{p}_1\cdot\vec{X}}(z)\partial X^{b_1}...\partial X^{b_q}e^{i\vec{p}_2\cdot\vec{X}}(w)\partial X^{c_1}...\partial X^{c_r}e^{i\vec{p}_3\cdot\vec{X}}(u)>$$

(56)

with the momenta $\vec{p}_1, \vec{p}_2, \vec{p}_3$ satisfying $p_1^2 = p_2^2 = p_3^2 = 0$, so the operators have conformal dimensions $p,q$ and $r$ respectively. Take the limit $u \to \infty$ in which $A^{a_1...a_p|b_1...b_q|c_1...c_r}$ becomes function of $u$ and $z - w$ It is not difficult to see that it will consist of terms which asymptotic behaviour $u^{-s}$ will range from $s = r$ to $s = p + q + r$, depending on pairing arrangements of $\partial X$’s. Then the interaction blocks $T_{p,q,r}^{a_1...a_p|b_1...b_q|c_1...c_r}(p_1,p_2,p_3)$ are defined as the coefficients in the expansion:

$$\lim_{u \to \infty} A^{a_1...a_p|b_1...b_q|c_1...c_r}(z,w,u;p_1,p_2,p_3) = (z - w)\vec{p}_1\cdot\vec{p}_2\sum_{s=r}^{p+q+r} \frac{T_{p,q,r}^{a_1...a_p|b_1...b_q|c_1...c_r}(p_1,p_2,p_3)}{u^s(z-w)^{p+q+r-s}}$$

(57)

It is not difficult to obtain the manifest expressions. for $T_{p,q,r}^{a_1...a_p|b_1...b_q|c_1...c_r}(p_1,p_2,p_3)$ Consider the contribution defined by $n_1$ pairings between $\partial X$’s of $V_1(z)$ and those of $V_3(u)$; $n_2$ pairings between $\partial X$’s of $V_2(w)$ and those of $V_3(u)$ and $n_3$ pairings between $\partial X$’s of $V_1(z)$ and those of $V_2(w)$. In addition, let this contribution be characterized.
by the numbers $m_1, ..., m_6$ where $m_1$ and $m_2$ are the numbers of pairings of $\partial X$’s in $V_3(u)$ with the exponents $e^{i\vec{p}_1 \vec{X}}(z)$ and $e^{i\vec{p}_2 \vec{X}}(w)$ respectively; $m_3$ and $m_4$ are the numbers of pairings of $\partial X$’s in $V_1(z)$ with the exponents $e^{i\vec{p}_2 \vec{X}}(w)$ and $e^{i\vec{p}_3 \vec{X}}(u)$ respectively and, finally, $m_5$ and $m_6$ are the numbers of pairings of $\partial X$’s in $V_2(w)$ with the exponents $e^{i\vec{p}_1 \vec{X}}(z)$ and $e^{i\vec{p}_3 \vec{X}}(u)$. It is not difficult to see that $T_{a_1...a_p|b_1...b_q|c_1...c_r}^{p,q,r|s}(p_1,p_2,p_3)$ is given by the sum of the diagrams with each diagram completely characterized by the set $\{n_i\}, \{m_j, j = 1, ..., 6\}$ with the following constraints on $n_i$ and $m_j$, defined by the number of $\partial X$’s each of the vertices ($p, q$ and $r$), as well as by the $u$-asymptotics, given by $s$

\begin{align*}
n_1 + n_2 + m_1 + m_2 &= r \\
n_1 + n_3 + m_3 + m_4 &= p \\
n_2 + n_3 + m_5 + m_6 &= q \\
2n_1 + 2n_2 + m_1 + m_2 + m_4 + m_6 &= s
\end{align*}

\begin{align*}
0 &\leq m_1, m_2 \leq r \\
0 &\leq m_3, m_4 \leq p \\
0 &\leq m_5, m_6 \leq q \\
0 &\leq n_1 \leq \text{min}(p, r) \\
0 &\leq n_2 \leq \text{min}(q, r) \\
0 &\leq n_3 \leq \text{min}(p, q)
\end{align*}

(58)

The symmetry factor for each diagram is easily calculated to give

$$N_{symm} = \frac{p!q!r!}{\prod_{i=1}^{3} n_i! \prod_{j=1}^{6} m_j!}$$

(59)

Using the operator products:

\begin{align*}
\partial X^a(z)\partial X^b(w) &\sim -\frac{\eta^{ab}}{(z-w)^2} \\
\partial X^a(z)e^{i\vec{p}\vec{X}}(w) &\sim \frac{-ip^ae^{i\vec{p}\vec{X}}(w)}{z-w}
\end{align*}

(60)
it is straightforward to find

\[
\sum_{\{m\},\{n\}} (-1)^{n_1+n_2+n_3+m_1+m_2+m_4} \prod_{i=1}^3 n_i! \prod_{j=1}^6 m_j! \prod_{k=1}^{n_1} \eta^{a_{k+c}} \prod_{k=1}^{n_3} \eta^{a_{n_1+k+c}} \prod_{k=1}^{n_2} \eta^{b_{n_3+k+c_{n_2+k}}} \
\prod_{k=1}^{m_1} (ip_1)^{c_{n_1+n_2+k}} \prod_{k=1}^{m_2} (ip_2)^{c_{n_1+n_2+m_1+k}} \prod_{k=1}^{m_3} (ip_2)^{a_{n_3+n_1+k}} \
\prod_{k=1}^{m_4} (ip_2)^{a_{n_3+n_3+k}} \prod_{k=1}^{m_5} (ip_1)^{b_{n_2+n_3+k}} \prod_{k=1}^{m_6} (ip_3)^{b_{n_2+n_3+m_5+k}}
\]

\[\text{Symm}\{(a_1, ..., a_p); (b_1, ..., b_q); (c_1, ..., c_r)\}\]

where the sum over \(n_i\) and \(m_j\) is taken over all the values satisfying (58) and the symmetrization is performed within each family of indices \((a_1, ..., a_p), (b_1, ..., b_q)\) and \((c_1, ..., c_r)\) (note that this symmetrization absorbs the factor of \(p!q!r!\) in the numerator of (59)) Note that, in the particular case of \(s = 2\) the blocks (57), (61) simply define the 3-point correlators of massive fully symmetric higher spin particles in bosonic string theory, with the spins \(p, q\) and \(r\) respectively and with the square masses \(p - 1, q - 1\) and \(r - 1\) respectively. Unlike the massless case, these massive interactions are not in conflict with the no-go theorems and are described by the standard massive vertex operators with the elementary ghost structure. Having obtained the expressions for the interaction blocks (57), (61), we are now prepared to proceed with the computation of the three-point correlators of the vertex operators (1), (36) that determine the gauge-invariant cubic interactions of massless higher spins. As was noted above, we shall concentrate on the three-point correlation function of two \(s = 3\) and one \(s = 4\) operators; other three-point functions of higher spin operators (1), (36) can be obtained in a similar way. In order to ensure the cancellation of all the ghost number anomalies, the correct ghost number balance requires that each correlation function has total \(b - c\) ghost number equal to 3, superconformal \(\phi\)-ghost number equal to \(-2\) and superconformal \(\chi\)-ghost number equal to \(1\). This means that in the three-point correlation function \(\langle V_{s=3} V_{s=3} V_{s=4} \rangle\) two operators must be taken in the positive picture representation (46), (51) and one at the negative picture (1). It is particularly convenient to choose \(V_{s=4}\) and one of \(V_{s=3}\) at positive pictures and the remaining \(V_{s=3}\) at the negative. So we need to consider the correlator \(\langle A_{s=3}(p_1)B_{s=4}(p_2)C(p_3) \rangle\) where, for simplicity of notations, \(C \equiv V_{s=3}\) is the \(s = 3\) unintegrated operator at picture \(-3\) while \(A_{s=3}\) and \(B_{s=4}\)
are given by (46) and (51). Using the decompositions (47) and (51) - (54), simple analysis of ghost number balance shows that $<A_{s=3}B_{s=4}C>$ is contributed by the correlators

$$<A_{s=3}B_{s=4}C> = <A_0B_7C> + <A_6B_0C> + \sum_{i=1}^{5} \sum_{j=1}^{6} <A_iB_jC>$$

(62)

while all other correlators (e.g. such as $<A_0B_jC>$, $j = 1, \ldots, 6$) vanish due to the total ghost number constraints. Below we shall perform the computation of the correlators (62), one by one. We start with $<A_0B_7C>$. Using the expressions (1), (46) - (54) for the operators and performing conformal mapping of the worldsheet to the upper half plane (so the operators are located at the worldsheet boundary which is the real axis) this correlation function is given by

$$<A_0(p_1;z_1)B_7(p_2;w_1)C(p_3,u)>
= \frac{-1}{2} \times 12H_{a_1a_2a_3}(p_1)H_{b_1\ldots b_4}(p_2)H_{c_1c_2c_3}(p_3) \int_0^1 dw \int dz \int_{0 \leq z < w} (z - z_1)^2(w - w_1)^2 \{ P^{(2)}_{2\phi-2\chi-\sigma} e^{\phi} \partial X^{a_1} \partial X^{a_2} \psi^{a_3} e^{ip_1\bar{X}}(z) \partial c c c \partial X^{b_1} \partial X^{b_2} \partial \psi^{b_3} \psi^{b_4} e^{ip_2\bar{X}}(w) \}
= \frac{1}{2} \eta^{a_3(b_3)\eta^{b_4)c_3}(z - w)^2(w - u)^2 + \frac{1}{2} \eta^{a_3(b_3)\eta^{b_4)c_3}(z - w)(w - u)^2
= \frac{1}{2} \eta^{a_3(b_3)\eta^{b_4)c_3}(z - u)^2(w - u)^2$$

(63)

Using the $SL(2, R)$ symmetry, it is convenient to set $z_1 = 0$, $w_1 = 1$, $u \to \infty$. Note that $SL(2, R)$ symmetry is equivalent to the fact that $z_1$, $w_1$ and $u$ derivatives of the vertex operators (36), (46)-(54) are BRST-trivial (so that the correlation function is invariant under the change of the operator’s locations). On the other hand, due to the ghost structure of the higher spin vertices, the 3-point function already contains 2 out of 3 integrated operators despite gauge fixing $SL(2, R)$. This is in contrast with the standard case when the $SL(2, R)$ symmetry ensures that all the operators in 3-point function are unintegrated, leading to the usual Koba-Nielsen’s determinant. The correlator in the integrand of (63) is the direct product of the $\psi$, $X$ and ghost correlators. Using the symmetries in $b_3$ and $b_4$ indices (since all the H-tensors, including the one of $s = 4$, are fully symmetric) the $\psi$ part can be written as

$$<\psi^{a_3}(z)\partial \psi^{b_3}\psi^{b_1}(w)\psi^{c_3}(u)>
= \frac{1}{2} \eta^{a_3(b_3)\eta^{b_4)c_3}(z - w)^2(w - u) + \frac{1}{2} \eta^{a_3(b_3)\eta^{b_4)c_3}(z - w)(w - u)^2$$

(64)

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In the limit \( u \to \infty \) we have

\[
\lim_{u \to \infty} \frac{1}{2} \langle \psi^{a3}(z) \partial \psi^{(b_3 \psi^{b_4})}(w) \psi^{c_3}(u) \rangle = -\frac{1}{2} \frac{\eta^{a_3(b_3 \eta^{b_4}) c_3}}{(z-w)^2 u}
\]  

(65)

Next, the ghost part of the correlator (63) is given by

\[
\langle P^{(2)} \partial \phi - 2 \chi - \sigma \rangle = (z-u)^3(w-u)^2\left(\frac{4}{(z-w)^2} - \frac{5}{(z-w)^2} - \frac{40}{(z-w)(z-u)}\right)
\]  

(66)

where we used

\[
P^{(2)}_{2\phi-2\chi-\sigma} = \partial P^{(1)}_{2\phi-2\chi-\sigma} + (P^{(1)}_{2\phi-2\chi-\sigma})^2
\]

(67)

along with the OPEs:

\[
P^{(1)}_{2\phi-2\chi-\sigma}(z) \psi^{c_3}(u) \sim -\frac{4 \partial \psi^{c_3}}{z-w}
\]

(68)

In the limit \( u \to \infty \) the ghost correlator becomes

\[
\lim_{u \to \infty} \langle P^{(2)} \partial \phi - 2 \chi - \sigma \rangle = -u^5\left(\frac{4}{(z-w)^2} - \frac{5}{u^2} + \frac{40}{(z-w)u}\right)
\]

(69)

Therefore in the limit \( u \to \infty \) the ghost-\( \psi \)-part of the correlator (63) is expressed as

\[
\langle \psi^{a_3}(z) \partial \psi^{b_3} \psi^{b_4}(w) \psi^{c_3}(u) \rangle < P^{(2)}_{2\phi-2\chi-\sigma} \partial cc \xi(w) ce^{-3\phi}(u) >
\]

\[
\left(\frac{4}{(z-w)^2} - \frac{5}{u^2} + \frac{40}{(z-w)u}\right)
\]

(70)

The only remaining part to compute is the \( X \)-correlator given by

\[
\langle \partial X^{a_1} \partial X^{a_2} e^{ip_1 X}(z) \partial X^{b_1} \partial X^{b_2} e^{ip_2 X}(w) \partial X^{c_1} \partial X^{c_2} e^{ip_3 X}(z) \rangle
\]

This correlator has the structure (56) and is thus given by the combination of

\[
T_{p,q,r|s}^{a_1...a_p|b_1...b_q|c_1...c_r}(p_1, p_2, p_3)
\]

with \( p = q = r = 2 \) but with the different values of \( s \). There is, however, a considerable simplification due to the conformal invariance of the theory. Namely, consider the \( \psi \)-ghost part (70) of the correlator (63) which, in the limit \( u \to \infty \), is given by the order 4 polynomial containing positive powers of \( u \). On the other hand, the conformal invariance only allows
the terms behaving as $u^0$ when $u \to \infty$; terms with the asymptotics $\sim u^n, n > 0$ cannot appear on-shell, as they are prohibited by the conformal invariance; terms of the type $\frac{1}{u^n}, n > 0$ vanish as $u \to \infty$. This condition very much limits the on-shell contributions from the $X$-correlator received by the non-vanishing terms in the overall correlator. That is, note that, given the $\psi$-ghost factor (70) the overall correlator has the following structure at $u \to \infty$:

$$< A_0(p_1; z_1)B_7(p_2; w_1)C(p_3, u) > \sim \frac{\eta^{a_3(b_3, b_4)c_3}}{2} \frac{4u^4}{(z-w)^4} - \frac{5u^2}{(z-w)^2} + \frac{40u^3}{(z-w)^3} \sum_s T^{a_1\ldots a_p|b_1\ldots b_q|c_1\ldots c_r}_{p,q,r|s}(p_1, p_2, p_3)$$

(71)

Since by definition $\lim_{u \to \infty} T^{a_1\ldots a_p|b_1\ldots b_q|c_1\ldots c_r}_{p,q,r|s}(p_1, p_2, p_3) \sim u^{-s}$, it is clear that the only non-vanishing contribution picked from the $X$-correlator is the one with $s = 4$, i.e. with the value of $s$ equal to the leading order of the $u$-asymptotics of the $\psi$-ghost factor (here and in a number of places below, $s$ refers to the order of the asymptotics and not the spin value, we hope that the difference shall be clear to the reader from the context). Those with $s < 4$ are prohibited by the conformal invariance (since they lead to positive powers of $u$ in the asymptotics) and thus we know in advance that they must vanish on-shell; those with $s > 4$ are gauged away in the limit $u \to \infty$. Therefore substituting (64), (66), (71) in the integral (63) we obtain the following expression for the overall correlator:

$$< A_0(p_1)B_7(p_2)C(p_3) >$$

$$= -24I(\vec{p}_1 \vec{p}_2)\eta^{a_3b_3}\eta^{b_4c_4}T_{2,2,2|4}^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3)$$

$$\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3}(p_2)H_{c_1c_2c_3}(p_3)\delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$

(72)

where

$$I(\vec{p}_1 \vec{p}_2) = \int dw \int dz_{0 \leq z < w \leq 1} z^2(w-1)^2(z-w)(\vec{p}_1 \vec{p}_2)^{-4}$$

(73)

The integral (73) is easy to evaluate. We have:

$$I(\vec{p}_1 \vec{p}_2) = \int_0^1 dw (w-1)^2 \int_0^w dz z^2(z-w)(\vec{p}_1 \vec{p}_2)^{-4}$$

$$= \int_0^1 dw (w-1)^2 w^2(\vec{p}_1 \vec{p}_2)^{-4} \int_0^w dz z^2(\frac{z}{w} - 1)(\vec{p}_1 \vec{p}_2)^{-4}$$

$$= \int_0^1 dw (w-1)^2 w(\vec{p}_1 \vec{p}_2)^{-4} \int_0^1 dx x^2(x-1)(\vec{p}_1 \vec{p}_2)^{-4}$$

(74)

$$= \frac{\Gamma(3)\Gamma((\vec{p}_1 \vec{p}_2) - 3)}{\Gamma((\vec{p}_1 \vec{p}_2))} \times \frac{\Gamma(3)\Gamma((\vec{p}_1 \vec{p}_2))}{\Gamma((\vec{p}_1 \vec{p}_2) + 3)} = 4 \prod_{n=-3}^{2} \frac{1}{(\vec{p}_1 \vec{p}_2) + n}$$

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where in the process we changed the integration variable \( x = \frac{\vec{w}}{w} \). In the on-shell limit one has \((\vec{p}_1 \vec{p}_2) \to 0\) and the integral becomes

\[
I(\vec{p}_1 \vec{p}_2) \approx -\frac{1}{3(\vec{p}_1 \vec{p}_2)} - \frac{1}{9} + O(\vec{p}_1 \vec{p}_2)
\]  

(75)

with the first term reflecting the non-localities well-known in the theories of higher spins (e.g. \([3], [4], [1], [7], [2], [44]\)) and the second term corresponding to the local part of the cubic interactions. Note that the non-localities appear as a result of the ghost structure of the vertex operators (36), (46) - (54) leading to appearance of the integrated terms in the three-point function and thus the deformation of the standard Koba-Nielsen’s measure. The interaction terms in the position space are straightforward to obtain by the Fourier transform. For example, using (72), (74) and the expression (61) for \( T_{\alpha \beta \gamma | s}^{a_1...a_p | b_1...b_q | c_1...c_r} (p_1, p_2, p_3) \) the cubic interaction term in the higher spin Lagrangian due to the correlator (63) is given by

\[
\sim -24\eta^{a_3 b_3} \eta^{b_4 c_3} I(\vec{\partial}_1 \vec{\partial}_2) \sum_{\{m\}, \{n\}} (-1)^{n_1 + n_2 + n_3 + m_1 + m_2 + m_4} \prod_{k=1}^{n_1} \eta^{a_k c_k} \prod_{k=1}^{n_3} \eta^{a_{n_1+k} b_k} \\
\times \prod_{k=1}^{m_2} \eta^{b_2 n_1 + k c_2 n_1 + k} \left( \partial^{a_1 + n_2 + 2k} \partial^{b_2 n_2 + 3l} H_{a_1 a_2 a_3} \right) \prod_{l=1}^{m_4} \partial^{a_3 + n_3 + 3k} \partial^{b_2 n_3 + 5l} H_{c_1 c_2 c_3} + \text{Symm}\{(a_1, ..., a_p); (b_1, ..., b_q); (c_1, ..., c_r)\}
\]

(76)

where, according to the notation of (76), the space-time derivatives \( \partial_1 \) and \( \partial_2 \) of \( I(\vec{\partial}_1 \vec{\partial}_2) \) act on \( H_{a_1 a_2 a_3} \) and \( H_{b_1...b_4} \) respectively. The cubic interaction terms corresponding to other correlators in (62), that we shall consider below, can be obtained in a totally similar way.

We now turn to the next correlator contributing to the cubic interaction, \( < A_6 B_0 C > \) of (62). The calculation using the vertex operators (1), (46)-(54) is totally similar to the one described above. The result is given by

\[
< A_6(p_1) B_0(p_2) C(p_3) > = 24I(\vec{p}_1 \vec{p}_2) \eta^{a_3 b_3} \eta^{b_4 c_3} T_{2, 2, 2|4}^{a_1 a_2 b_1 b_2 | c_1 c_2} (p_1, p_2) H_{a_1 a_2 a_3} (p_1) H_{b_1...b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \\
\times \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)
\]

(77)

so the sum of the first two contributions to the cubic interaction vertex is

\[
< A_0(p_1) B_7(p_2) C(p_3) > + < A_6(p_1) B_0(p_2) C(p_3) > = 48I(\vec{p}_1 \vec{p}_2) \eta^{a_3 b_3} \eta^{b_4 c_3} T_{2, 2, 2|4}^{a_1 a_2 b_1 b_2 | c_1 c_2} (p_1, p_2) H_{a_1 a_2 a_3} (p_1) H_{b_1...b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \\
\times \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)
\]

(78)
Finally, we need to analyze the correlators of the type $< A_j B_j C >$ in (62) $(i = 1, ..., 5; j = 1, ..., 6)$ which have different ghost structure and a bit more cumbersome structure of the matter part. The calculation of these correlators is also performed according to the same procedure as above, in the gauge $z_1 = 0, w_1 = 1, u = \infty$. Below we shall present the results for these correlators, one by one. The correlator $< A_1 B_1 C >$ is given by

$$< A_1(p_1)B_1(p_2)C(p_3) > = 4H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3}(p_2)H_{c_1c_2c_3}(p_3) \times \int_0^1 dw \int_{0 \leq z < w} dz z^2 (w - 1)^2 \{ < c\xi(z)ce^\phi(w)ce^{-3\phi}(u) > \times (\bar{\psi}\partial X)\partial X^{a_1}\partial X^{a_2}\psi^{a_3}e^{i\bar{p}_1 X}(z)(\bar{\psi}\partial X)\partial X^{b_1}\partial X^{b_2}\psi^{b_3}\psi^{b_4}e^{i\bar{p}_2 X}(w) \times \partial X^{c_1}\partial X^{c_2}\psi^{c_3}e^{i\bar{p}_3 X}(u) > \}$$

(79)

where, in order to maintain the order of indices in $T_{3,3,2|4}^{a_1a_2a_3|b_1b_2b_3|c_1c_2}$, consistent with the expression (61), we have renamed the indices in some of the contractions (e.g. $b_3 \rightarrow b_4$, $b_4 \rightarrow b_5$ while reserving $b_3$ for the contraction of $\psi$ and $\partial X$ in the second vertex operator).

In addition, here and elsewhere below we shall suppress the common factor of $\delta(\bar{p}_1 + \bar{p}_2 + \bar{p}_3)$ in all the amplitudes. The next correlator to consider is $< A_1(p_1)(B_2 + B_3)(p_2)C(p_3) >$.

Using the expressions (1), (46) - (54) for the corresponding operators, we have:

$$< A_1(p_1)(B_2 + B_3)(p_2)C(p_3) > = -4ip_2^{b_5}H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3}(p_2)H_{c_1c_2c_3}(p_3) \times \int_0^1 dw \int_{0 \leq z < w} dz z^2 (w - 1)^2 \{ < c\xi\psi^{a_3}\psi^{a_4}(z)ce^\phi\psi^{b_3}\psi^{b_4}(w)\psi^{c_3}(u) > \times (\bar{\psi}\partial X^{a_1}\partial X^{a_2}\partial X^{a_3}e^{i\bar{p}_1 X}(z)\partial X^{b_1}\partial X^{b_2}e^{i\bar{p}_2 X}(w)\partial X^{c_1}\partial X^{c_2}e^{i\bar{p}_3 X}(u) > \}$$

(80)
The next correlator, \(< A_1(p_1)B_4(p_2)C(p_3) >\), vanishes:

\[
< A_1(p_1)B_5(p_2)C(p_3) > = -4H_{a_1a_2a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \int_0^1 dw \int_{0 < z < w} dzz^2(1-w)^2 \times \left\{ \right. \\
\times c\xi(z)|P^{(2)}_{\phi-\chi}|c\phi(w)c^{-3\phi}(u) > \left. \psi^{a_3}\psi^{a_4}(z)\partial^2\psi^{b_2}\partial\psi^{b_3}\psi^{b_4}(w)\psi^{c_3}(u) > \times < \partial X^{a_1}\partial X^{a_2}\partial X^{a_3}e^{i\vec{p}_1\vec{X}}(z)\partial X^{b_1}e^{i\vec{p}_2\vec{X}}(w)\partial X^{c_1}\partial X^{c_2}e^{i\vec{p}_3\vec{X}}(u) > \right\} = 0
\]

due to the vanishing \(\psi\)-correlator (which is equal to zero as the fermions \(\psi^{a_3}\psi^{a_4}\) at \(z\), antisymmetric in the \(a_3, a_4\) indices always contract with 2 out of 3 fermions \(\partial^2\psi^{b_2}\partial\psi^{b_3}\psi^{b_4}\) at \(w\) which are symmetric in \(b_2, b_3, b_4\) (since \(H_{b_1...b_4}\) is totally symmetric). The next correlator,

\[
< A_1(p_1)B_5(p_2)C(p_3) > = -4H_{a_2a_3a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \int_0^1 dw \int_{0 < z < w} dzz^2(1-w)^2 \times \left\{ \right. \\
\times c\xi(\partial X^{a_1}\partial X^{a_2}\partial X^{a_3}e^{i\vec{p}_1\vec{X}}(z)ce^{\phi}(X^{b_1}\partial X^{b_2}(\frac{1}{2}\partial^3X^{b_3} + \partial^2X^{b_3}P^{(1)}_{\phi-\chi}) + \frac{1}{2}\partial X^{b_3}P^{(2)}_{\phi-\chi})e^{i\vec{p}_2\vec{X}}(w)\partial X^{c_1}\partial X^{c_2}e^{i\vec{p}_3\vec{X}}(u) > \right\} = I(\vec{p}_1\vec{p}_2)\eta^{b_4[a_4\eta^{a_1}c_3}\{ 48\eta^{\delta_{a_3b_3}}T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3) + 8\eta^{\delta_{b_1c_2}}T^{a_1a_2a_3|b_1b_2|c_1}(p_1, p_2, p_3) - 16\eta^{\delta_{b_1\eta_1}}T^{a_1a_2a_3|b_1b_2|c_1}(p_1, p_2, p_3) - 4\eta^{\delta_{b_2\eta_1}}T^{a_1a_2a_3|b_1b_2|c_1}(p_1, p_2, p_3) \times H_{a_2a_3a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]

The next correlator,

\[
< A_1(p_1)B_6(p_2)C(p_3) > = 4H_{a_2a_3a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \int_0^1 dw \int_{0 < z < w} dzz^2(1-w)^2 \times \left\{ \right. \\
\times c\xi(\partial X^{a_1}\partial X^{a_2}\partial X^{a_3}e^{i\vec{p}_1\vec{X}}(z)e^{\phi}(X^{b_1}\partial X^{b_2}(\partial^2X^{b_3}) + \partial X^{b_3}P^{(1)}_{\phi-\chi})e^{i\vec{p}_2\vec{X}}(w)e^{-3\phi}(X^{c_1}\partial X^{c_2}e^{i\vec{p}_3\vec{X}}(u) > \right\} = I(\vec{p}_1\vec{p}_2)\eta^{b_4[a_4\eta^{a_1}c_3}\{ -24\eta^{\delta_{a_3b_3}}T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3) + 8\eta^{\delta_{b_1c_2}}T^{a_1a_2a_3|b_1b_2|c_1c_2}(p_1, p_2, p_3) + 8\eta^{\delta_{b_1\eta_1}c_3}\{ -24\eta^{\delta_{a_3b_3}}T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3) + 8\eta^{\delta_{b_1c_2}}T^{a_1a_2a_3|b_1b_2|c_1c_2}(p_1, p_2, p_3) + 4\eta^{\delta_{b_2\eta_1}c_3}\{ -24\eta^{\delta_{a_3b_3}}T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3) + 8\eta^{\delta_{b_1c_2}}T^{a_1a_2a_3|b_1b_2|c_1c_2}(p_1, p_2, p_3)\} H_{a_2a_3a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]

23
and therefore

\[
< A_1(p_1)(B_5 + B_6)(p_2)C(p_3) > = 
I(p_1 \bar{p}_2)\eta^{b_4[a_1\eta^{a_1}c_3}(24\eta^{a_3b_3}T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3) \\
+ 16\eta^{b_3c_2}T^{a_1a_2a_3|b_1b_2|c_1}(p_1, p_2, p_3) - 8iP^{b_3}_1T^{a_1a_2a_3|b_1b_2|c_1c_2}(p_1, p_2, p_3)) \\
\times H_{a_2a_3a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]  

(84)

This concludes the list of all the correlators involving \( A_1(p_1) \). Next,

\[
< (A_2 + A_3)(p_1)(B_2 + B_3)(p_2)C(p_3) > = -4iP^{a_4}_1H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \\
\times \int_0^1 dw \int_{0 \leq z < w} dzz^2(w - 1)^2 \{ c\xi(\partial \psi^{a_4} + \psi^{a_4}P^{(1)}_{\phi - \chi})\psi^{a_3}(z)ce^{\phi}\partial \psi^{b_3}\psi^{b_4}(\psi^{b_5}P^{(1)}_{\phi + \chi} + \partial \psi^{b_5})(w)ce^{-3\phi}\psi^{c_3}(u) > \\
\times \partial X^{a_1} \partial X^{a_2}e^{i\bar{p}_3X}(z)\partial X^{b_1} \partial X^{b_2}e^{i\bar{p}_2X}(w)\partial X^{c_1} \partial X^{c_2}e^{i\bar{p}_1X}(u) > \\
= -4I(p_1 \bar{p}_2)P^{b_4}_1P^{b_5}_2(5\eta^{a_3b_4}\eta^{a_4b_5}\eta^{b_5c_3} - 6\eta^{a_3b_4}\eta^{a_4b_5}\eta^{b_4c_3} + \eta^{a_3b_4}\eta^{a_4b_5}\eta^{b_3c_3}) \\
\times T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3)H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]  

(86)

The next correlator is

\[
< (A_2 + A_3)(p_1)(B_2 + B_3)(p_2)C(p_3) >
\]

\[
= -4P^{a_4}_1P^{b_4}_2H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \int_0^1 dw \int_{0 \leq z < w} dzz^2(w - 1)^2 \{ c\xi(\partial \psi^{a_4} + \psi^{a_4}P^{(1)}_{\phi - \chi})\psi^{a_3}(z)ce^{\phi}\partial \psi^{b_3}\psi^{b_4}(\psi^{b_5}P^{(1)}_{\phi + \chi} + \partial \psi^{b_5})(w)ce^{-3\phi}\psi^{c_3}(u) > \\
\times \partial X^{a_1} \partial X^{a_2}e^{i\bar{p}_3X}(z)\partial X^{b_1} \partial X^{b_2}e^{i\bar{p}_2X}(w)\partial X^{c_1} \partial X^{c_2}e^{i\bar{p}_1X}(u) > \\
= -4I(p_1 \bar{p}_2)P^{b_4}_1P^{b_5}_2(5\eta^{a_3b_4}\eta^{a_4b_5}\eta^{b_5c_3} - 6\eta^{a_3b_4}\eta^{a_4b_5}\eta^{b_4c_3} + \eta^{a_3b_4}\eta^{a_4b_5}\eta^{b_3c_3}) \\
\times T^{a_1a_2|b_1b_2|c_1c_2}(p_1, p_2, p_3)H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]  

(87)

Next,

\[
< (A_2 + A_3)(p_1)B_4(p_2)C(p_3) >
\]

\[
= 4iP^{a_4}_1H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \int_0^1 dw \int_{0 \leq z < w} dzz^2(w - 1)^2 \{ c\xi(\partial \psi^{a_4} + \psi^{a_4}P^{(1)}_{\phi - \chi})\psi^{a_3}(z)ce^{\phi}\partial \psi^{b_3}\psi^{b_4}(\psi^{b_5}P^{(1)}_{\phi + \chi} + \partial \psi^{b_5})(w)ce^{-3\phi}\psi^{c_3}(u) > \\
\times \partial X^{a_1} \partial X^{a_2}e^{i\bar{p}_3X}(z)\partial X^{b_1} \partial X^{b_2}e^{i\bar{p}_2X}(w)\partial X^{c_1} \partial X^{c_2}e^{i\bar{p}_1X}(u) > \\
= -16iI(p_1 \bar{p}_2)P^{a_4}_1\eta^{a_3b_3}\eta^{a_4b_2}\eta^{b_4c_3}T^{a_1a_2|b_1|c_1c_2}(p_1, p_2, p_3) \\
\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]  

(87)
Next,

\[< (A_2 + A_3)(p_1)B_5(p_2)C(p_3) > = 4ip^{a_1}_{1}H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \]
\[\times \int_0^1 dw \int_{0 \leq z < w} dwz^2(w - 1)^2 \{ < c\xi(\partial\psi^{a_1} + \psi^{a_1}P^{(1)}_{\phi - \chi})\psi^{a_3}(z)\partial X^{a_1}\partial X^{a_2}e^{i\vec{p}_1\vec{X}}(z) \]
\[ce^{\phi}\psi^{b_1}\partial X^{b_1}\partial X^{b_2}(\frac{1}{2}\partial^3 X^{b_3} + \partial^2 X^{b_3}P^{(1)}_{\phi - \chi} + \frac{1}{2}\partial X^{b_3}P^{(2)}_{\phi - \chi})e^{i\vec{p}_2\vec{X}}(w) \]
\[ce^{-3\phi}\psi^{c_1}\partial X^{c_1}\partial X^{c_2}e^{i\vec{p}_3\vec{X}}(u) > \] (88)

\[= I(\vec{p}_1\vec{p}_2)\{48i(p_1^c,\eta^{a_3b_4}\eta^{b_1a_2} - 2p_1^b,\eta^{a_2b_3}\eta^{a_3c_3})T_{a_1b_1c_1c_2}(p_1, p_2, p_3) + 8i(p_1^c,\eta^{a_3b_4}\eta^{b_1c_2} - 2p_1^b,\eta^{a_2b_3}\eta^{b_1c_3})T_{a_1b_1c_1c_2}(p_1, p_2, p_3) + 4p_3^c, p_1^b,\eta^{a_3b_4}T_{a_1b_1c_1c_2}(p_1, p_2, p_3) - 16p_3^c, p_1^b,\eta^{a_3b_4}T_{a_1b_1c_1c_2}(p_1, p_2, p_3) - 32p_3^c, p_1^b,\eta^{a_3c_4}T_{a_1b_1c_1c_2}(p_1, p_2, p_3) - 8p_3^c, p_1^b,\eta^{a_3c_4}T_{a_1b_1c_1c_2}(p_1, p_2, p_3) \]
\[\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \]

The last correlator involving \((A_2 + A_3)(p_1)\) is

\[< (A_2 + A_3)(p_1)B_6(p_2)C(p_3) > = -4ip^{a_1}_{1}H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \]
\[\times \int_0^1 dw \int_{0 \leq z < w} dwz^2(w - 1)^2 \{ < c\xi(\partial\psi^{a_1} + \psi^{a_1}P^{(1)}_{\phi - \chi})\psi^{a_3}(z)\partial X^{a_1}\partial X^{a_2}e^{i\vec{p}_1\vec{X}}(z) \]
\[ce^{\phi}\partial X^{b_1}\partial X^{b_2}\partial X^{b_3}P^{(1)}_{\phi - \chi}ce^{-3\phi}\psi^{c_1}\partial X^{c_1}\partial X^{c_2}e^{i\vec{p}_3\vec{X}}(u) > \] (89)

\[= I(\vec{p}_1\vec{p}_2)\{(72i(p_1^c,\eta^{a_3b_4}\eta^{b_1a_2} - 24ip_1^c,\eta^{a_2b_3}\eta^{a_3b_4})T_{a_1b_1c_1c_2}(p_1, p_2, p_3) + (24p_3^c, p_1^b,\eta^{a_3b_4} - 8p_3^c, p_1^b,\eta^{a_3b_4})T_{a_1b_1c_1c_2}(p_1, p_2, p_3) + (12p_3^c, p_1^b,\eta^{a_3c_4} - 4ip_3^c, p_1^b,\eta^{a_3c_4})T_{a_1b_1c_1c_2}(p_1, p_2, p_3) + (24ip_1^c,\eta^{a_3c_4}\eta^{c_1b_2} - 8ip_1^c,\eta^{a_3c_4}\eta^{c_1b_2})T_{a_1b_1c_1c_2}(p_1, p_2, p_3) \]
\[\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \]

Next, we shall consider the correlators with \(A_4(p_1)\). We have:

\[< A_4(p_1)B_1(p_2)C(p_3) > = -4H_{a_1a_2a_3}(p_1)H_{b_1b_2b_4b_5}(p_2)H_{c_1c_2c_3}(p_3) \]
\[\times \int_0^1 dw \int_{0 \leq z < w} dwz^2(w - 1)^2 \{ < c\xi\partial X^{a_1}(\partial^2\psi^{a_2} + 2\partial\psi^{a_2}P^{(1)}_{\phi - \chi})\psi^{a_3}e^{i\vec{p}_1\vec{X}}(z) \]
\[ce^{\phi}(\bar{\psi}\partial \bar{X})\partial X^{b_1}\partial X^{b_2}\partial X^{b_3}\psi^{b_4}e^{i\vec{p}_2\vec{X}}(w)ce^{-3\phi}\psi^{c_4}\partial X^{c_1}\partial X^{c_2}e^{i\vec{p}_3\vec{X}}(u) > \] \[= 24I(\vec{p}_1\vec{p}_2)\eta^{a_2b_4}\eta^{a_3b_5}\eta^{b_1c_3}T_{a_1b_1b_3c_1c_2}(p_1, p_2, p_3)H_{a_1a_2a_3}(p_1)H_{b_1b_2b_4b_5}(p_2)H_{c_1c_2c_3}(p_3) \] (90)
The next correlator is
\[ < A_4(p_1)(B_2 + B_3)(p_2) C(p_3) > = 4i p_2^{b_5} a_{23}^a a_{1}^a a_{23}^a (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]
\[ \times \int_0^1 dw \int_{0 \leq z < w} dwz^2 (w - 1)^2 \{ < c \xi \partial X^{a_1} (\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P^{(1)}_{\phi - \chi}) \psi^{a_3} e^{i \vec{p} \cdot \vec{X}} (z) \}
ce^{\phi} \partial X^{b_1} \partial X^{b_2} \partial \psi^{b_3} \psi^{b_4} (\partial \psi^{b_5} + \psi^{b_5} P^{(1)}_{\phi - \chi}) e^{i \vec{p} \cdot \vec{X}} (w) ce^{-3 \phi^{c_3} \partial X^{c_1} \partial X^{c_2} e^{i \vec{p} \cdot \vec{X}} (u) > \}
\]
\[ = -24i p_2^{b_5} I (\vec{p} \vec{p}_2) \{ (\eta^{a_3 b_3} \eta^{a_2 b_4} \eta^{b_5 c_3} + \eta^{a_3 b_3} \eta^{a_2 b_4} \eta^{b_5 c_3}) T_{1,2,2|4} (p_1, p_2, p_3)
+ \eta^{a_3 b_4} \eta^{a_2 b_5} \eta^{b_3 c_3} T_{1,1,1|4} (p_1, p_2, p_3) \} H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]

Next,
\[ < A_4(p_1) B_4 (p_2) C(p_3) > = 4H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]
\[ \times \int_0^1 dw \int_{0 \leq z < w} dwz^2 (w - 1)^2 \{ < c \xi \partial X^{a_1} (\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P^{(1)}_{\phi - \chi}) \psi^{a_3} e^{i \vec{p} \cdot \vec{X}} (z) \}
\]
\[ ce^{\phi} \partial X^{b_1} \partial X^{b_2} (\partial \psi^{b_3} + \psi^{b_3} P^{(1)}_{\phi - \chi}) e^{i \vec{p} \cdot \vec{X}} (w) ce^{-3 \phi^{c_3} \partial X^{c_1} \partial X^{c_2} e^{i \vec{p} \cdot \vec{X}} (u) > \}
\]
\[ = 128 I (\vec{p} \vec{p}_2) \eta^{a_2 b_4} \eta^{a_3 b_3} \eta^{b_3 c_3} T_{1,1,1|4} (p_1, p_2, p_3) H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]

Next,
\[ < A_4(p_1) B_5 (p_2) C(p_3) > = 4H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]
\[ \times \int_0^1 dw \int_{0 \leq z < w} dwz^2 (w - 1)^2 \{ < c \xi \partial X^{a_1} (\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P^{(1)}_{\phi - \chi}) \psi^{a_3} e^{i \vec{p} \cdot \vec{X}} (z) \}
\]
\[ ce^{\phi} \partial X^{b_1} \partial X^{b_2} \partial \psi^{b_3} \psi^{b_4} (\partial \psi^{b_5} + \psi^{b_5} P^{(1)}_{\phi - \chi}) e^{i \vec{p} \cdot \vec{X}} (w) ce^{-3 \phi^{c_3} \partial X^{c_1} \partial X^{c_2} e^{i \vec{p} \cdot \vec{X}} (u) > \}
\]
\[ = 16 I (\vec{p} \vec{p}_2) \eta^{a_2 b_4} \eta^{a_3 b_3} \{ 6 \eta^{a_1 b_3} T_{0,2,2|4} (p_1, p_2, p_3) + 2 \eta^{a_3 b_3} T_{1,1,1|2} (p_1, p_2, p_3)
+ 4i p_1^{b_3} T_{1,1,1|2} (p_1, p_2, p_3) + i p_3 T_{1,2,2|3} (p_1, p_2, p_3) \}\]
\[ \times H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]

The last correlator involving \( A_4(p_1) \) is
\[ < A_4(p_1) B_6 (p_2) C(p_3) > = -4H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]
\[ \times \int_0^1 dw \int_{0 \leq z < w} dwz^2 (w - 1)^2 \{ < c \xi \partial X^{a_1} (\partial^2 \psi^{a_2} + 2 \partial \psi^{a_2} P^{(1)}_{\phi - \chi}) \psi^{a_3} e^{i \vec{p} \cdot \vec{X}} (z) \}
\]
\[ ce^{\phi} \partial X^{b_1} \partial X^{b_2} (\partial^2 X^{b_3} + \partial X^{b_3} P^{(1)}_{\phi - \chi}) e^{i \vec{p} \cdot \vec{X}} (w) ce^{-3 \phi^{c_3} \partial X^{c_1} \partial X^{c_2} e^{i \vec{p} \cdot \vec{X}} (u) > \}
\]
\[ = 16 I (\vec{p} \vec{p}_2) \eta^{a_2 b_4} \eta^{a_3 b_3} \{ -6 \eta^{a_1 b_3} T_{0,0,2|4} (p_1, p_2, p_3) - 4 \eta^{a_3 b_3} T_{1,1,1|2} (p_1, p_2, p_3)
+ 2i p_1^{b_3} T_{1,1,1|2} (p_1, p_2, p_3) + i p_3 T_{1,2,2|3} (p_1, p_2, p_3) \}\]
\[ \times H_{a_1 a_2 a_3} (p_1) H_{b_1 b_2 b_3 b_4} (p_2) H_{c_1 c_2 c_3} (p_3) \]
and therefore

\[
< A_4(p_1)(B_5 + B_6)(p_2)C(p_3) > = 16I(\bar{p}_1\bar{p}_2)\eta^{a_2b_4}\eta^{a_3c_3}\{ -2\eta^{b_4c_2}T_{1,2,1|2}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3) \\
+ 6\eta^{b_3}T_{1,2,2|4}^{a_1|b_1b_2|c_2}(p_1,p_2,p_3) + 2\eta^{b_3}T_{1,2,2|3}^{a_1|b_1b_2|c_1c_2}(p_1,p_2,p_3) \} (95)
\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]

This concludes the list of the correlators involving \(A_4(p_1)\). Finally, we are left to consider the set of the correlators involving \(A_5(p_1)\). Note that the expression for \(A_5(p_1)\) contains no \(\psi\)'s at all while the one for \(C(p_3)\) has only one \(\psi\). At the same time, the operators \(B_1, B_2, B_3\) and \(B_4\) are all cubic in \(\psi\). For this reason,

\[
< A_5(p_1)B_1(p_2)C(p_3) > = < A_5(p_1)B_2(p_2)C(p_3) > \\
= < A_5(p_1)B_3(p_2)C(p_3) > = < A_5(p_1)B_4(p_2)C(p_3) > = 0
\] (96)

and the only non-vanishing correlators with \(A_5\) are \(< A_5(p_1)B_5(p_2)C(p_3) > \) and \(< A_5(p_1)B_6(p_2)C(p_3) > \). For these remaining correlators we obtain

\[
< A_5(p_1)B_5(p_2)C(p_3) > = -4H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \\
\times \int_0^1 dw \int_{0\leq z<w} dwz^2(w - 1)^2 \{ < c\xi\partial X^{a_1}\partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3}P^{(1)}_{\phi-\chi})e^{i\bar{p}_1\bar{X}}(z) \\
ce^{\phi}\partial X^{b_1}\partial X^{b_2}(\frac{1}{2}\partial^3 X^{b_3} + \partial X^{b_3}P^{(1)}_{\phi-\chi}) + \frac{1}{2}\partial X^{b_3}P^{(2)}_{\phi-\chi})\psi^{b_4}e^{i\bar{p}_2\bar{X}}(w) \\
ce^{-3\phi}\psi^{c_3}\partial X^{c_1}\partial X^{c_2}e^{i\bar{p}_3\bar{X}}(u) > \}
\]

\[= 4I(\bar{p}_1\bar{p}_2)\{6\eta^{a_2b_3}\eta^{b_4c_3}(6\eta^{a_2b_3}T_{1,1,2|4}^{a_1|b_1|c_1}(p_1,p_2,p_3) + 6\eta^{b_3c_2}T_{2,1,2|1}^{a_1|b_1|c_2}(p_1,p_2,p_3) - 4i\eta^{b_3}T_{2,1,2|4}^{a_1|b_1|c_2}(p_1,p_2,p_3) + ip_1^{b_3}T_{2,1,2|3}^{a_1|b_1|c_2}(p_1,p_2,p_3) + \eta^{a_3c_2}\eta^{b_4c_3}( -24\eta^{a_2b_3}T_{1,2,1|2}^{a_1|b_1|c_1}(p_1,p_2,p_3) - 8i\eta^{b_3}T_{2,2,1|2}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3) + 2ip_1^{b_3}T_{2,2,1|1}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3)) \\
+ 26\eta^{a_3b_3}\eta^{b_4c_3}( -26\eta^{a_2b_3}T_{2,2,2|4}^{a_1|b_1|b_2|c_2}(p_1,p_2,p_3) + 12\eta^{a_2b_3}\eta^{b_4c_3}( -2ip_1^{a_3}T_{2,1,2|4}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3) + ip_1^{a_3}T_{1,2,2|3}^{a_1|b_1b_2|c_2}(p_1,p_2,p_3)) \\
+ 2\eta^{b_4c_3}\eta^{b_3c_2}( -2ip_1^{a_3}T_{2,2,1|2}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3) + ip_1^{a_3}T_{1,2,2|3}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3)) + \eta^{b_4c_3}( -2p_1^{a_3} + p_3^{a_3})(4p_1^{b_3}T_{2,2,2|4}^{a_1|b_1|b_2|c_2}(p_1,p_2,p_3) + p_3^{b_3}T_{2,2,2|3}^{a_1|b_1b_2|c_1}(p_1,p_2,p_3)) \}
\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)
\]
and finally

\[
\begin{align*}
&< A_5(p_1)B_6(p_2)C(p_3) > = 4H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \\
&\int_0^1 dw \int_{0 \leq z < w} dw z^2(w-1)^2 \{ -c\xi \partial X^{a_1} \partial X^{a_2}(\partial^2 X^{a_3} + \partial X^{a_3} P^{(1)}_{\phi-\chi})e^{i\vec{p}_1 \cdot \vec{X}(z)} \\
&ce^{\phi} \partial X^{b_1} \partial X^{b_2}(\partial^2 X^{b_3} + \partial X^{b_3} P^{(1)}_{\phi-\chi})\partial\psi^{b_4}e^{i\vec{p}_2 \cdot \vec{X}(w)}ce^{-3\phi} \psi^{c_3} \partial X^{c_1} \partial X^{c_2}e^{i\vec{p}_3 \cdot \vec{X}(u)} > \\
&= 4I(\vec{p}_1\vec{p}_2) \{ -11\eta^{a_3b_3}\eta^{b_4c_3}T_{2,1,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&+\eta^{b_4c_3}\eta^{a_3b_2}(36\eta^{b_3a_2}\eta^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3)) \\
&-12ip_3^{b_3}T_{2,1,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) - 6ip_3^{b_3}T_{2,1,2;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&-12ip_1^{b_1c_1}\eta^{b_3a_2}a_3^{a_1}T_{2,1,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) + 4i\eta^{b_1c_3}\eta^{b_3a_2}T_{2,2,1;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&+4ip_1^{b_2a_2}\eta^{b_1c_3}T_{2,2,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) + 2ip_3^{b_3a_2}\eta^{b_1c_3}T_{2,2,2;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&-6ip_3^{a_3}\eta^{b_1c_3}\eta^{a_2b_2}T_{1,2,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) - 2\eta^{b_4c_3}\eta^{b_3a_2}p_1^{a_1}T_{2,2,2;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \} \\
&= 4I(\vec{p}_1\vec{p}_2) \{ -11\eta^{a_3b_3}\eta^{b_4c_3}T_{2,1,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&+\eta^{b_4c_3}\eta^{a_3b_2}(36\eta^{b_3a_2}\eta^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3)) \\
&-12ip_3^{b_3}T_{2,1,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) - 6ip_3^{b_3}T_{2,1,2;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&-12ip_1^{b_1c_1}\eta^{b_3a_2}a_3^{a_1}T_{2,1,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) + 4i\eta^{b_1c_3}\eta^{b_3a_2}T_{2,2,1;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&+4ip_1^{b_2a_2}\eta^{b_1c_3}T_{2,2,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) + 2ip_3^{b_3a_2}\eta^{b_1c_3}T_{2,2,2;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \\
&-6ip_3^{a_3}\eta^{b_1c_3}\eta^{a_2b_2}T_{1,2,2;3}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) - 2\eta^{b_4c_3}\eta^{b_3a_2}p_1^{a_1}T_{2,2,2;2}^{a_1a_2b_1b_2c_1c_2}(p_1,p_2,p_3) \} \\
&\times H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3) \tag{98}
\end{align*}
\]

It is straightforward to check, by using the momentum conservation and by substituting the appropriate expressions for \( T_{p,q,r}^{a_1...a_p,b_1...b_q,c_1...c_r}(p_1,p_2,p_3) \) entering (98) that the correlator (98) vanishes identically on-shell. In fact, this vanishing is a direct consequence of the conformal invariance: comparing the correlators (97), (98) and, when necessary, using the momentum conservation \( \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0 \), it is easy to see that each term in (98) has a counterpart in (97) with precisely the same index structure but with higher value of \( s \) in the appropriate \( T_{p,q,r}^{a_1...a_p,b_1...b_q,c_1...c_r}(p_1,p_2,p_3) \). For this reason, any term appearing in (98) is forbidden by the conformal invariance and has to vanish on-shell (see the discussion above in this Section).

This concludes the computation of all the correlators contributing to the gauge-invariant cubic interaction of one \( s = 4 \) and two \( s = 3 \) fields.

Summing over all of the contributions from the three-point correlators (63)-(98), using the momentum conservation along with the symmetry of the polarization tensors and eliminating terms that vanish on-shell, we obtain the final answer for the gauge-invariant
3-point amplitude:

\[
< V_{s=3}(p_1)V_{s=4}(p_2)V_{s=3}(p_3) >
\]

\[
= \{ 272q^{a_3b_2}q^{a_2b_3}q^{b_4c_3}T_{1,1,2}\mid_4^{a_1}\mid_{b_1c_2}(p_1,p_2,p_3)
\]

\[
+ 144q^{a_3b_2}q^{b_3c_2}q^{b_4c_3}T_{2,1,1}\mid_2^{a_1a_2}\mid_{b_1c_1}(p_1,p_2,p_3)
\]

\[
- 128q^{a_2b_2}q^{a_3c_2}q^{b_4c_3}T_{1,2,1}\mid_2^{a_1}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
- (16i\rho_{a_3}q^{b_3c_2}q^{b_4c_3} + 24i\rho_{a_3}q^{b_3c_2}q^{b_4c_3})T_{2,2,1}\mid_2^{a_1}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
- 32i\rho_{a_3}q^{b_3c_2}q^{b_4c_3}T_{2,1,2}\mid_4^{a_1a_2}\mid_{b_1c_1}(p_1,p_2,p_3) + (48i\rho_{a_3}q^{a_3b_4}q^{a_2b_3}
\]

\[
+ 72i\rho_{a_3}q^{a_2b_4}q^{a_1c_3} - 144i\rho_{a_3}q^{a_2b_4}q^{b_4c_3}T_{1,2,2}\mid_4^{a_1}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
= ((56 - 20(\rho_{a_3}\rho_{a_4}))q^{a_3b_3}q^{b_4c_3} - 24\rho_{a_3}\rho_{a_4})
\]

\[
\times I(p_{a_3}\rho_{a_4})H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)\delta(p_1 + p_2 + p_3)
\]

\[
+ \{ 24i\rho_{a_3}q^{b_3c_2}q^{b_4c_3}T_{1,3,2}\mid_4^{a_1a_2}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
+ 8q^{b_4c_3}(i\rho_{a_3}q^{b_3c_5} - i\rho_{a_4}q^{a_3b_5})T_{2,3,2}\mid_4^{a_1a_2}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
\times I(p_{a_3}\rho_{a_4})H_{a_1a_2a_3}(p_1)H_{b_1b_2b_3b_5}(p_2)H_{c_1c_2c_3}(p_3)\delta(p_1 + p_2 + p_3)
\]

\[
+ \{ 24i\rho_{a_3}q^{b_3c_2}q^{b_4c_3}T_{2,2,2}\mid_4^{a_1a_2}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
+ 16i\rho_{a_3}q^{a_4}q^{a_1}T_{3,2,1}\mid_4^{a_1a_2a_3}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
\times I(p_{a_3}\rho_{a_4})H_{a_2a_3a_4}(p_1)H_{b_1b_2b_3b_4}(p_2)H_{c_1c_2c_3}(p_3)\delta(p_1 + p_2 + p_3)
\]

\[
+ \{ 24i\rho_{a_3}q^{b_3c_2}q^{b_4c_3}T_{3,3,2}\mid_4^{a_1a_2a_3}\mid_{b_1b_2c_1}(p_1,p_2,p_3)
\]

\[
\times I(p_{a_3}\rho_{a_4})H_{a_1a_2a_4}(p_1)H_{b_1b_2b_3b_5}(p_2)H_{c_1c_2c_3}(p_3)\delta(p_1 + p_2 + p_3)
\]

The gauge-invariant cubic interaction term in the position space is then easy to obtain from (99) by usual Fourier transform.

### 7. Conclusion and Discussion

In this paper we have constructed vertex operators in open string theory, describing massless higher spin fields and computed the three-point correlation function describing the gauge-invariant cubic interaction of \( s = 4 \) with two \( s = 3 \) particles. The computation performed in this paper is straightforward to generalize to obtain the gauge-invariant cubic interactions of other massless higher spins, although technically in certain cases practical computations could be quite complicated due to the picture changing issue. The BRST-invariance conditions for the vertex operators, constructed in this paper, lead to Fierz-Pauli
on-shell conditions for the space-time higher spin fields. The BRST-nontriviality constraints on these operators lead to the gauge transformations for the space-time fields: as the gauge transformations imply shifting higher operators by BRST-trivial terms, the correlators are automatically invariant under these transformations and so are the interaction terms induced by these correlators. As we have pointed out, the gauge transformations for spin $s$ fields, implied by the BRST non-triviality of their vertex operators, are equivalent to the transformations given by the symmetrized derivatives of spin $s-1$ gauge parameter, restricted by the tracelessness constraints. So the vertex operators, considered in this paper, give a description of interacting higher spins, isomorphic to Fronsdal’s framework [32] rather than a well-known alternative approach involving non-local compensators, traded for the tracelessness constraints on the gauge symmetries [3], [43], [44], [45]. It would be interesting to try to interpret the compensator approach in the language of string theory. Interestingly, the nonlocality of the massless higher spin interactions, observed in this paper (already in the cubic case), is the direct consequence of the non-standard ghost structure of the vertex operators, leading to the deformation of the usual Koba-Nielsen’s measure and the appearance of the integrated vertices in three-point amplitudes - while typically, as far as the standard lower spin cases are concerned (such as photon, graviton, etc.) three-point amplitudes only involve unintegrated vertices and thus no nonlocalities. It should be noted, however, that in the massive case one shouldn’t expect nonlocalities for higher spins on the cubic level either, as the massive higher spin vertex operators appear naturally in the massive sector of string theory and have standard ghost structures, not different from the lower spin case. Thus the nonlocalities in cubic interactions, observed in this work, appear to be specific to the massless case only.

In this work we have considered, for simplicity, the higher spin vertex operators for the values of $s$ from 3 to 9 in the totally symmetric case. It should be, however, quite a direct excercise to extend our calculation to less symmetric cases, including those involving several families of indices, although matching the gauge symmetries of vertex operators on the string theory side (as a consequence of the BRST conditions on the operators) to the standard gauge symmetries observed in higher spin field theories in space-time, is an open question. As we have pointed out, this matching does work out in totally symmetric case, considered in this paper; in less symmetric cases this will require additional careful analysis of the BRST constraints on the appropriate vertex operators. We hope to perform this analysis in our future work. Another important direction for the future work is to extend the formalism developed in this work to compute the higher order gauge-invariant
interaction terms of the higher spin fields, such as the quartic interactions. As in the cubic case, one would expect the non-localities, stemming from the ghost structures of the vertex operators; in addition the calculation of the four-point functions will require careful analysis of ghost number balance and (whenever necessary) insertions of appropriate picture changing operators. It is also possible that the ghost number balance conditions shall impose certain selection rules for the higher order interactions. In general though, it appears that string theory provides us with a set of powerful tools to investigate the interacting theories of the higher spin fields. What seems especially attractive about the string-theoretic approach, is that the traditionally difficult issues about the higher spin field theories (such as the gauge invariance of the interaction terms) appear to be under control in string theory - e.g. with the BRST conditions automatically ensuring the gauge invariance in the space-time effective action. In this work we have restricted ourselves to the totally symmetric higher spin fields, appearing in open string theory framework. Considering higher spins related to less trivial Young tableau and with several families of indices will particularly require to extend the analysis and the formalism, developed in this paper, to the closed string case. This is another direction for the future research and the subject for the work currently in progress.

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