Noncooperative Games for Autonomous Consumer Load Balancing over Smart Grid

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Abstract

Traditionally, most consumers of electricity pay for their consumptions according to a fixed rate. With the advancement of Smart Grid technologies, large-scale implementation of variable-rate metering becomes more practical. As a result, consumers will be able to control their electricity consumption in an automated fashion, where one possible scheme is to have each individual maximize its own utility as a noncooperative game. In this paper, noncooperative games are formulated among the electricity consumers in Smart Grid with two real-time pricing schemes, where the Nash equilibrium operation points are investigated for their uniqueness and load balancing properties. The first pricing scheme charges a price according to the average cost of electricity borne by the retailer and the second one charges according to a time-variant increasing-block price, where for each scheme, a zero-revenue model and a constant-rate revenue model are considered. In addition, the relationship between the studied games and certain competitive routing games from the computer networking community, known as atomic flow games, is established, for which it is shown that the proposed noncooperative game formulation falls under the class of atomic splittable flow games. The Nash equilibrium is shown to exist for four different combined cases corresponding to the two pricing schemes and the two revenue models, and is unique for three of the cases under certain conditions. It is further shown that both pricing schemes lead to similar electricity loading patterns when consumers are only interested in minimizing the electricity costs without any other profit considerations. Finally, the conditions under which the increasing-block pricing scheme is preferred over the average-cost based pricing scheme are discussed.

Index Terms

Game Theory, Noncooperative Game, Atomic Splittable Flow Game, Nash Equilibrium, Smart Grid, Real Time Pricing, Increasing-Block Pricing.

I. INTRODUCTION

In the traditional power market, electricity consumers usually pay a fixed retail price for their electricity usage. This price only changes on a seasonal or yearly basis. However, it has been long recognized in the economics community that charging consumers a flat rate for electricity creates allocative inefficiencies, i.e., consumers do not pay equilibrium prices according to their consumption levels [1]. This was shown through an example in [2], which illustrates how flat pricing causes deadweight loss at off-peak times and excessive demand at the peak times. The latter may lead to small-scale blackouts in a short run and excessive capacity buildup over a long run. As a solution, variable-rate metering that reflects the real-time cost of power generation can be used to influence consumers to defer their power consumption away from the peak times. The reduced peak-load can significantly reduce the need for expensive backup generation during peak times and excessive generation capacity.

The main technical hurdle in implementing real-time pricing has been the lack of cost-effective two-way smart metering, which can communicate real-time prices to consumers and their consumption levels back to the energy provider. In addition, the claim of social benefits from real-time pricing also assumes that the consumer demand is elastic and responds to price changes while traditional consumers do not possess the equipments that enable them to quickly alter their demands according to the changing power prices. Significant research efforts on real-time pricing have involved estimating the consumer demand elasticity and the level of benefits that real time pricing can achieve [1], [3], [4]. Fortunately, the above requirements on smart metering and consumer adaptability are being fulfilled [5] as technology advances in cyber-enabled metering, power generation, power storage, and manufacturing automation, which is driven by the need for a Smart Grid.

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Such real-time pricing dynamics have been studied in the literature mainly with game theory [6]–[8]. In particular, the authors in [6] provided a design mechanism with *revelation principle* to determine the optimal amount of incentive that is needed for the customers to be willing to enter a contract with the utility and accept power curtailment during peak periods. However, they only considered a fixed pricing scheme. In [7], the authors studied games among consumers under a certain class of demand profiles at a price that is a function of day long aggregate cost of global electricity load of all consumers. However, the case with real-time prices was not investigated in [7]. In [8], a noncooperative game was studied to tackle the real-time pricing problem, where the solution was obtained by exploring the relationship with the congestion games and potential games. However, the pricing schemes that we study are not amenable to transformations described in [8].

In this paper we formulate noncooperative games [9], [10] among the consumers with two real-time pricing schemes under more general load profiles and revenue models. The first pricing scheme charges a price according to the instantaneous average cost of electricity production and the second one charges according to a time-varying version of increasing-block price [11]. We investigate consumer demands at the Nash equilibrium operation points for their uniqueness and load balancing properties. Furthermore, two revenue models are considered for each of the schemes, and we show that both pricing schemes lead to similar electricity loading patterns when consumers are interested only in the minimization of electricity costs. We also demonstrate the relationship between these games and certain competitive routing games [12], known as atomic flow games [13] from the computer networking community. We show that the proposed noncooperative game formulation falls under the class of atomic splittable flow games [14]. Specifically, we show that the noncooperative game amongst the consumers has the same structure as that in the atomic splittable flow game over a two-node network with multiple parallel links between them. Finally we discuss the conditions under which the increasing-block pricing scheme is preferred over the average-cost based pricing scheme.

The rest of the paper is organized as follows. The system model, formulation of the noncooperative game, and its relationship with the atomic flow games are presented in Section II. The game is analyzed with different real-time pricing schemes under different revenue models in Sections III and IV, where the Nash equilibrium properties are investigated. We conclude the paper in Section V.

II. SYSTEM MODEL

We study the transaction of energy between a single electricity retailer and multiple consumers. In each given time slot, each consumer has a demand for electric energy (measured in Watt-hour, Wh). The job of the retailer is to satisfy demands from all the consumers. The electricity supply of the retailer is purchased from a variety of sources over a wholesale electricity market and the retailer may possess some generation capacity as well. These sources may use different technologies and fuels to generate electricity, which leads to different marginal costs of electricity at the retailer, where the marginal cost is the incremental cost incurred to produce an additional unit of output [15]. Mathematically, the marginal cost function is expressed as the first derivative of the total cost function. Examples of the marginal cost function and the corresponding total cost are presented in Fig. 1(a) and Fig. 1(b), respectively, which are based on real world data from the wholesale electricity market [3]. Naturally, the retailer attempts to satisfy demands by procuring the cheapest source first. This results in a non-decreasing marginal cost of the supply curve, as illustrated through the example in Fig. 1(a). The retailer charges each consumer a certain price for its consumption in order to cover the cost, where the sum payments by all the consumers should be enough to cover the total cost and certain profit margin set by the retailer or regulatory body. In our model we assume that all these are incorporated within the marginal cost of electricity.

While the retailer aims to procure sufficient supply to meet the sum demand of its consumers in each time slot, in reality, the supply is limited by the generation capacity available in the wholesale electricity market. Thus, the maximum sum load that the retailer can service bears an upper limit and we model this capacity limit by setting the marginal cost of electricity to infinity when the sum load exceeds a predetermined threshold. Each consumer has an energy demand in each time slot and it pays the retailer at a price that is set by the retailer such that, in each time slot, the sum of payments made by all consumers meets the total cost in that slot. As such, a particular

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1In real life the base load, i.e., the regular power that is demanded by the consumers, is satisfied from sources such as hydro, coal or nuclear, as they are cheap. The fluctuating components of the demand are satisfied from sources such as oil, as the power-plants based on oil are more flexible to control.
consumer’s share of this bill depends on the retailer’s pricing scheme, which is a function of the demands from all the consumers. Accordingly, as the total load varies over time, each consumer operates over a time-variant price with time-slotted granularity. We assume that each consumer has a total demand for electricity over each day, which can be distributed throughout the day in a time-slotted manner, to maximize certain utility function. Next, we model such individual load balancing behaviors as a noncooperative game.

A. Noncooperative Load Balancing Game

The noncooperative game between these consumers is formulated as follows. Consider a group of $N$ consumers, who submit their daily demands to a retailer in a time-slotted pattern at the beginning of the day (which contains $T$ time slots). These consumers are selfish and aim to maximize their individual utility/payoff functions; hence they do not cooperate with each other to manage their demands. Each consumer $i$ has a minimum total daily requirement of energy, $\beta_i \geq 0$, which is split over the $T$ time slots. Let $x^i_t$ denote the $i$th consumer’s demand in the $t$th time slot. A consumer can demand any value $x^i_t \geq 0$ (negativity constraint) with $\sum_t x^i_t \geq \beta_i$ (demand constraint). Let

$$x^i = \{x^i_1, x^i_2, \ldots, x^i_T\},$$

represent the $i$th consumer’s demand vector, which is called the strategy for the $i$th consumer. Let

$$x_t = \{x^1_t, \ldots, x^N_t\},$$

represent the demand vector from all consumers in time slot $t$ with $x_t = \sum_i x^i_t$. Let $x$ represent the set $\{x^1, \ldots, x^N\}$.

The payoff or utility for consumer $i$ is denoted by $\pi^i$ which is the difference between the total revenue it generates from the purchased electricity and its cost. In particular, let $E^i_t$, a function of $x^i_t$, represent the revenue generated by the $i$th consumer in the $t$th time slot and $M^i_t$, a function of $x_t$, represent its payment to the retailer for purchasing $x^i_t$. Then the payoff $\pi^i$, to be maximized by consumer $i$, is given by

$$\pi^i = \sum_{t \in \{1, \ldots, T\}} (E^i_t - M^i_t).$$

Since $M^i_t$ is a function of $x_t$, we see that the consumer payoff is influenced by its load balancing strategy and those of other consumers.

\footnote{Here we adopt one day as an operation period that contains a certain number of time slots. Obviously, such a choice has no impact on the analytical results in this paper.}
We consider the problem of maximizing the payoff at each consumer by designing the distributed load balancing strategy $x^t$'s, under two real-time pricing schemes set by the retailer. The first one is the average-cost based pricing scheme and the second one is the increasing-block pricing scheme. Specifically, for the first scheme the retailer charges the consumers the average cost of electricity procurement that is only dependent on the sum demands, $x_t$, from all the consumers. For the second scheme, the retailer charges according to a marginal cost function that depends on the vector of demands from all consumers, $x_t$.

Let $C(x)$ represent the cost of $x$ units of electricity, to the retailer, from the wholesale market (an example function is plotted in Fig. 1(b)). Then under the average-cost based pricing, the price per unit charged to the consumers is given by
\[
A(x_t) = \frac{C(x_t)}{x_t},
\]
and at time $t$ consumer $i$ pays
\[
M^i_t = x^i_t A(x_t)
\]
for consuming $x^i_t$ units of electricity. It is easy to see that $\sum_i M^i_t = C(x_t)$, i.e., with average-cost based pricing the total payment made by the consumers covers the total cost to the retailer. Note that $C'(x_t)$ gives the marginal cost function in the wholesale market, henceforth denoted by $C'(x_t) = C(x_t)$ in the context of increasing-block pricing (an example marginal cost curve is plotted in Fig. 1(a)). For reasons we discussed earlier, in the context of electricity market, the marginal cost $C(x_t)$ is always non-negative and non-decreasing such that $C(x_t)$ is always positive, non-decreasing, and convex. Briefly, we note that as the retailer capacity is constrained by a predetermined upper limit $U$, we model this constraint as $C(x_t) = \infty, \forall x_t > U$; obviously $x^i_t \leq U$ is an implicit constraint on the demand $x^i_t$ for any rational consumer.

The second scheme is a time-variant version of the increasing-block pricing scheme [11]. With a typical increasing-block pricing scheme, consumer $i$ is charged a certain rate $b_1$ for its first $z_1$ units consumed, then charged rate $b_2$ ($> b_1$) for additional $z_2$ units, and charged rate $b_3$ ($> b_2$) for additional $z_3$ units, and so on. The $b$'s and $z$'s describe the marginal cost price for the commodity. In our scheme we design a marginal cost function, which retains the increasing nature of increasing-block pricing, such that it depends on $x_t$ and the function $C(\cdot)$. Consumer $i$ pays an amount determined by the marginal cost function $M(x, x_t)$, applicable to all consumers at time slot $t$. In particular consumer $i$ pays
\[
M^i_t = \int_0^{x^i_t} M(x, x_t) dx
\]
for consuming $x^i_t$ units of electricity where $M(\cdot)$ is chosen as
\[
M(x, x_t) = C \left( \sum_j \min (x, x^j_t) \right),
\]
such that $\sum_i M^i_t = C(x_t)$ is satisfied. An intuition behind this pricing scheme is to penalize consumers with relatively larger demands. Note that in this case, $x^i_t \leq U$ is implicitly assumed by letting $C(\cdot) = \infty \forall x^i_t > U$ and hence $M^i_t = \infty \forall x^i_t > U$.

For each of the two pricing schemes, we study two different revenue models. For the first one we set $E^i_t$ as zero for all consumers over all time slots, which leads to payoff maximization being the same as cost minimization from the point of view of the consumers. For the second one we assign consumer $i$ a constant revenue rate $\phi^i_t$ at each time slot $t$, which gives $E^i_t = \phi^i_t x^i_t$ and leads to payoff maximization being the same as profit maximization.

B. Atomic Flow Games with Splittable Flows

The noncooperative game that we have formulated in the previous section is related to the following problem in the network routing literature [12], [13]. Consider several agents each trying to establish paths from a specific source node to some destination node in order to transport a fixed amount of traffic. In the context of Internet, each agent can be viewed as a manager of packet routing. In the context of transportation, each agent can be a company routing its fleet vehicles across the network of roads. The problem here is of competitive routing between agents,
where each agent needs to deliver a given amount of flow over the network from its designated origin node to the corresponding destination node. An agent can choose how to divide its flow amongst the available routes. On each link the agents experience a certain delay. In the case of computer networks, if many agents collectively route a large number of packets through a particular link, the packets will experience larger delays; and beyond a certain level, the link may even start dropping packets, resulting in infinite delay. Such a delay can be referred to as cost, which is a function of the link congestion or the total flow through the link. The cost of a path is the sum of the link costs along the route.

To show the relationship between our noncooperative consumer load balancing problem and the above routing problem, we can reformulate the load balancing problem into the following routing game over a network with two nodes and multiple links \[12\]. We use notations similar to there in \[12\] in the interest of readability. Let there be \(N\) agents who share a common source node and a common destination over a two-node network connected by \(T\) parallel links (see Fig. 2). It is assumed that the agents do not cooperate. Each agent \(i \in \{1, \ldots, N\}\) has a minimum throughput demand \(\beta_i\), which can be split among the \(T\) links as chosen by the agent. Let \(x^i_t \geq 0\) denote the flow that agent \(i\) sends through link \(t \in \{1, \ldots, T\}\). The sum of \(x^i_t\) should add up to \(\beta_i\), i.e., \(\beta_i = \sum_t x^i_t\). Let \(x_t = \sum_i x^i_t\), and \(x = \{x^1_t, \ldots, x^i_t, \ldots, x^T_t\}\). The flow vector for agent \(i\) is denoted by the vector \(x^i = \{x^i_1, \ldots, x^i_t, \ldots, x^i_T\}\). The system flow vector is the collection of all agent flow vectors, denoted by \(x = \{x^1, \ldots, x^i, \ldots, x^N\}\). A given \(x^i\) is feasible if its components obey the non-negativity constraint and the demand constraints. Let \(X^i\) be the set of all feasible choices of \(x^i\) for agent \(i\), and \(\mathcal{X}\) be the set of all feasible choices of \(x\).

Let \(J^i(x)\) denote the cost for each agent \(i\), which it wishes to minimize. Since \(J^i(x)\) is a function of the flow vector of all the agents, the best response of a given agent is a function of the responses from all the other agents; and hence we can have a noncooperative game formulation. The Nash solution of the game is defined as the system flow vector such that none of the agents can unilaterally improve their performance. Formally, \(\hat{x} \in \mathcal{X}\) is a Nash Equilibrium Point (NEP) if the following condition holds for all agents

\[
J^i(\hat{x}) = \min_{x^i \in X^i} J^i(\hat{x}^1, \ldots, \hat{x}^{i-1}, x^i, \hat{x}^{i+1}, \ldots, \hat{x}^N),
\]

where \(\hat{x}^j\) is the demand vector for the \(j\)th agent. The above noncooperative game is known as an atomic splittable flow game \[14\], \[16\]. In \[12\], the existence of NEP is proved for the atomic splittable flow game over the two-node network with parallel links when the following five assumptions (G1-G5) are satisfied for the cost function.

G1: \(J^i\) is the sum of link cost functions, i.e., \(J^i(x) = \sum_{t=1}^{T} J^i_t(x_t)\).
G2: \(J^i_t : [0, \infty)^N \rightarrow [0, \infty]\) is a continuous function.
G3: \(J^i_t\) is convex over \(x^i_t\).
G4: Wherever finite, \(J^i_t\) is continuously differentiable over \(x^i_t\).
G5: Sum capacities of all links is greater than the sum demands from all the agents.

The last assumption (G5) mentioned here is a simplification of the original assumption mentioned in \[12\], applicable to two-node networks, while the original form of the assumption applies to more general networks. The consequence of this assumption is that, at Nash equilibrium, all users incur finite link costs, i.e., \(J^i_t < \infty, \forall i, t\).
As a side-note, in [12], the uniqueness of NEP is further imposed for the two-node network if the cost function \( J_t^i \) additionally complies with the following assumptions:

**A1:** \( J_t^i \) is a function of two arguments, namely agent \( i \)'s flow on link \( t \) and the total flow on that link, i.e., \( J_t^i(x_t) = J_t^i(x_t^i, x_t) \).

**A2:** \( J_t^i \) is increasing over each of its two arguments.

**A3:** Let \( K_t^i = \frac{\partial J_t^i}{\partial x_t^i} \). Wherever \( J_t^i \) is finite, \( J_t^i \) is finite, and \( K_t^i = K_t^i(x_t^i, x_t) \) is strictly increasing in each of its two arguments.

In particular, functions that comply with the assumptions G1-G5 and A1-A3 are referred to as type-A functions in [12]. In the following sections we will apply some of the results in [12] to facilitate our analysis over the noncooperative consumer load balancing game. The cost functions in our formulation do not satisfy all of the assumptions A1-A3, and hence we use other means to prove uniqueness of NEP.

With our load balancing problem for each of the two pricing schemes in our game two different revenue models are studied to provide more design insights, which leads to two different payoff structures. In the first model the revenue is set to zero, such that payoff maximization is cost minimization. In the second model, the rate of revenue generation at each consumer is set as a non-zero constant, such that payoff maximization is profit maximization.

### III. Nash Equilibrium with Average-Cost Pricing

For the average-cost pricing, the payment to the retailer in slot \( t \) by consumer \( i \) is given by (2).

#### A. Zero-Revenue Model

In this case the revenue is set to zero as \( E_t^i = 0 \), which results in payoff maximization being the same as cost minimization for each consumer. Specifically, the payoff for consumer \( i \) is given by

\[
\pi_t^i = -\sum_t M_t^i.
\]

The consumer load balancing problem for consumer \( i \), for \( i = 1, \ldots, N \), is given by the following optimization problem:

\[
\begin{align*}
\text{maximize} & \quad \pi_t^i(x_t^i) = -\sum_t M_t^i \\
\text{subject to} & \quad M_t^i = x_t^i A(x_t) , \quad \forall t, \\
& \quad \sum_t x_t^i \geq \beta_t, \\
& \quad x_t = \sum_j x_t^j , \quad \forall t, \\
& \quad 0 \leq x_t^i , \quad \forall t.
\end{align*}
\]

As cost to the retailer becomes infinity whenever the total demand goes beyond the capacity threshold for the wholesale market, i.e., when

\[
C(x_t) = \infty \quad \forall x_t > U,
\]

the price to consumers will become infinite and their payoff will go to negative infinity. Thus any consumer facing an infinite cost at a particular time slot can manipulate the demand vector such that the cost becomes finite, which is always feasible as long as assumption G5 holds. This implies that, at Nash equilibrium, sum demand \( x_t \) will be less than the capacity threshold \( U \), \( \forall t \), which allows for a redundant constraint \( x_t^i \leq U, \forall i, t, \) as \( x_t^i \leq \sum_i x_t^i = x_t \leq U \). Such a redundant but explicit constraint in turn makes the feasible region for \( x \), denoted by \( \mathcal{X} \), finite and hence compact. The compactness property will be later utilized to prove the Kakutani’s theorem [17].
We have already shown that this game is similar to the routing game described in [12]. With the average-cost based pricing and the zero-revenue model, the effective cost function for agent $i$ to minimize in the routing game is

$$J^i_t = M^i_t = x^i_t A(x^i_t) = \frac{x^i_t}{x_t} C(x_t).$$

This cost function satisfies the assumptions G1-G5 given earlier. In particular, G1 holds as the total payment made by the consumers satisfies

$$M^i = \sum_t M^i_t,$$

which is the cost to the agents in the routing formulation. In addition, G2 trivially holds by the definition of $M^i_t$.

In order to satisfy G3, i.e., to show that $J^i_t$ is convex over $x^i_t$, we show that $\frac{\partial^2 J^i_t}{\partial x^i_t^2} \geq 0$. First we evaluate

$$\frac{\partial J^i_t}{\partial x^i_t} = \frac{\partial}{\partial x^i_t} \left( \frac{x^i_t}{x_t} C(x^i_t) \right)$$

$$= \ldots$$

$$= \left( x_t - x^i_t \right) A(x^i_t) + x^i_t C'(x^i_t).$$

(4)

Then we evaluate

$$\frac{\partial^2 J^i_t}{\partial x^i_t^2} = \frac{1}{x_t^2} \left[ 2 \left( x_t - x^i_t \right) \left( C'(x^i_t) - \frac{C(x^i_t)}{x_t} \right) + x_t x^i_t C''(x^i_t) \right].$$

(5)

Given $C(x)$ is convex, both $\left( C'(x^i_t) - \frac{C(x^i_t)}{x_t} \right) \geq 0$ and $C''(x^i_t) \geq 0$ hold; and therefore $\frac{\partial^2 J^i_t}{\partial x^i_t^2} \geq 0$. Thus $J^i_t$ is convex over $x^i_t$ and G3 holds. The above also shows that $J^i_t$ is continuously differentiable over $x^i_t$ and hence G4 holds. Finally, we assume that G5 holds by construction. By the results in [12] we know that if the cost function satisfies assumption G2 and G3, and $X$, is compact, there exists an NEP strategy for all agents. Therefore, the NEP solution exists for the proposed noncooperative consumer load balancing game.

On the other hand, the cost function $J^i_t$ does not satisfy the assumption A3; so it is disqualified as a type-A function defined in [12]. Therefore, the corresponding uniqueness result in [12] cannot be extended to our formulation. Next, we prove the uniqueness of the NEP solution by extending the result in [14]. The number of player types in a particular game refers to the number of different values for $\beta_i$’s. Thus, a single type of players implies that all players have the same value for $\beta_i$’s. Next, we first introduce some definitions and show that our load balancing problem satisfies the conditions for NEP uniqueness as described in [14]. We now begin with some definitions from [14].

**Definition 1.** The component-join operation for two given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ consists of merging any two vertices $v_1 \in V_1$ and $v_2 \in V_2$ into a single vertex $v$.

**Definition 2.** Consider two nodes, named hubs, then a hub-component is a connected graph formed by connecting edges and nodes to the hubs such that all paths between the hubs remain vertex-disjoint.

![Hub-components](image)

Fig. 3. Hub-components. These five basic units are used to construct nearly-parallel graphs.
Definition 3. A generalized nearly-parallel graph is any graph that can be constructed from hub-components applying component-join operations.

The two-node network with parallel links in Fig. [2] is the first of the five basic units (as drawn in Fig. [3]) of nearly-parallel graphs [14], [18]; and by definitions a hub-component is also a generalized nearly-parallel graph.

Definition 4. A cost function \( f(x) \) is (strictly) semi-convex if \( xf(x) \) is (strictly) convex.

For the load balancing game with average-cost based pricing and zero revenue, the cost of consumer \( i \) is given by (2) where the price \( A(x_i) \) is given by (1), with \( A(x_i) \) a non-negative and non-decreasing function. For the function \( A(x_i) \) to be strictly semi-convex, \( x_i A(x_i) \) needs to be strictly convex. Since \( C(x_i) = x_i A(x_i) \) is the total cost of electricity to the retailer, and we assume that the marginal cost price \( C'(x) \) is a monotonically increasing function, \( C(x) \) is strictly convex. Thus our problem is equivalent to an atomic flow game with splittable flows and different player types (i.e., each player controls a different amount of total flow) over a generalized nearly-parallel graph, which has strictly semi-convex, non-negative, and non-decreasing functions for cost per unit flow. By the results of [14], the NEP solution for the load balancing game is unique.

In the following, we discuss the properties for the unique NEP solution for the proposed load balancing game.

Lemma 1. With the average-cost based pricing and zero revenue, at the Nash equilibrium the price of electricity faced by all consumers is the same over all time slots.

Proof: Consider two arbitrary time slots \( t_1 \) and \( t_2 \). At the Nash equilibrium the sum demands in the system over the two time slots are either the same or different. If the sum demands are equal over the two time slots, by (1), we know that the price of electricity will be same over the two slots. If the sum demands are not equal, without losing generality, let us assume \( x_{t_1} < x_{t_2} \) such that

\[
A(x_{t_1}) < A(x_{t_2}) \tag{6}
\]

holds. Then any consumer \( j \) with \( x^j_{t_2} > 0 \) can reduce cost by reducing \( x^j_{t_2} \) and increasing \( x^j_{t_1} \) by the same small quantity. This contradicts our assumption that the system is in equilibrium. Hence \( A(x_{t_1}) = A(x_{t_2}) \).

Lemma 2. If \( C(\cdot) \) is strictly convex, at the Nash equilibrium, the sum of demands on the system, \( x_t \), keeps the same across different time slots.

Proof: In order to prove this we will show that \( x_{t_1} \neq x_{t_2} \) leads to contradiction. At Nash equilibrium we have \( A(x_{t_1}) = A(x_{t_2}) \) from Lemma [1] for all possible \( t_1 \) and \( t_2 \). Thus, we obtain

\[
A(x_{t_1}) = A(x_{t_2}) \iff C(x_{t_1})/x_{t_1} = C(x_{t_2})/x_{t_2} \iff C(x_{t_1}) = C(x_{t_2}) \frac{x_{t_1}}{x_{t_2}}.
\]

If \( C(\cdot) \) is strictly convex and \( x_{t_1} \neq x_{t_2} \) (without loss of generality say \( x_{t_1} < x_{t_2} \)), by definition of strict convexity, we have

\[
C(x_{t_1}) < \frac{x_{t_2} - x_{t_1}}{x_{t_2}} C(0) + \frac{x_{t_1}}{x_{t_2}} C(x_{t_2}),
\]

\[
\Rightarrow C(x_{t_1}) \frac{x_{t_2}}{x_{t_2}} < \frac{x_{t_2} - x_{t_1}}{x_{t_2}} \times 0 + \frac{x_{t_1}}{x_{t_2}} C(x_{t_2})
\]

\[
\Rightarrow C(x_{t_2}) \frac{x_{t_1}}{x_{t_2}} < \frac{x_{t_1}}{x_{t_2}} C(x_{t_2})
\]

which is a contradiction and hence \( x_{t_1} = x_{t_2} \).

Lemma 3. If \( C(\cdot) \) is strictly convex, at Nash equilibrium, each consumer will distribute its demands equally over the \( T \) time slots.

Proof: As the Nash equilibrium is unique, by symmetry over all the time slots, for consumer \( i \) we shall have

\[
x^i_{t_1} = x^i_{t_2}, \quad \forall t_1, t_2,
\]
as otherwise we could swap the demand vectors \( x_{t_1} \) and \( x_{t_2} \) in time slots \( t_1 \) and \( t_2 \) without altering the Nash equilibrium conditions and get another distinct NEP, thus contradicting the uniqueness. Thus, with \( \sum_t x_t^i \geq \beta_i \) and the fact that the consumer is trying to minimize their cost such that \( \sum_t x_t^i = \beta_i \) holds at equilibrium, we have the solution \( x_t^i = \beta_i / T \) for all consumers \( i \) and time slots \( t \).

**Remark:** Under the average-cost based pricing scheme with zero revenue, if one particular consumer increases its total demand of electricity, the price \( A(\cdot) \) increases, which in turn increases the payments for all other consumers as well. Theoretically one consumer may cause indefinite increases in the payments of all others; and in this sense this scheme does not protect the group from reckless action of some consumer(s). This issue will be addressed by our second pricing scheme as we will show in Section IV.

### B. Constant-Rate Revenue Model

In this case, the rate of revenue generation for each consumer at each time slot is taken as a non-negative constant \( \phi_i^t \). Thus,

\[
E_t^i = \phi_i^t \times x_t^i.
\]

The consumer load balancing problem for each consumer \( i \) is given by the following optimization problem:

\[
\begin{align*}
\text{maximize} & \quad \pi^i(x^i) = \sum_t (E_t^i - M_t^i) \\
\text{subject to} & \quad E_t^i = \phi_i^t x_t^i, \ \forall t, \\
& \quad M_t^i = x_t^i A(x_t^i), \ \forall t, \\
& \quad \sum_t x_t^i \geq \beta_i, \\
& \quad x_t = \sum_j x_t^j, \ \forall t, \\
& \quad 0 \leq x_t^i, \ \forall t.
\end{align*}
\]

We assume that \( \beta_i = 0, \ \forall i \), and the rate of revenue is larger than the price of electricity such that we do not end up with any negative payoff or the trivial solution \( x_t^i = 0, \ \forall i, t \).

Here again, if the sum demand in a given time slot \( t \) exceeds the retailer’s capacity threshold \( U \), the consumers will face an infinite price for their consumption. This implies that, at Nash equilibrium the sum demand \( x_t \) will never exceed the capacity threshold \( U \), as G5 holds. This again allows for the redundant constraint \( x_t^i \leq U, \ \forall i, t \), as \( x_t^i \leq \sum_j x_t^j = x_t \leq U \), which in turn makes the feasible region for \( x, \mathcal{X} \), finite and hence compact.

We briefly show that under these assumptions there exists an NEP for this game. In particular, the effective cost function for the corresponding routing game is given as

\[
J_t^i = M_t^i - \phi_i^t \times x_t^i.
\]

As \( M_t^i \) is continuous in \( x_t \), \( J_t^i \) is continuous in \( x_t \) as well and satisfies assumption G2. We have already shown that \( M_t^i \) under the average-cost based pricing scheme is convex in \( x_t^i \) through (5). The function \( -\phi_i^t x_t^i \) is linear and hence convex in \( x_t^i \). Thus, by the property that the summation of two convex functions is convex, \( J_t^i \) from (7) is convex in \( x_t^i \) and hence satisfies assumption G3. Following the proof in [12], we consider the point-to-set mapping \( x \in \mathcal{X} \rightarrow \Gamma(x) \subset \mathcal{X} \) defined as

\[
\Gamma(x) = \{ \hat{x} \in \mathcal{X} : \hat{x}^i \in \arg\min_{z^i \in \mathcal{X}} J^i(x^1, \ldots, z^i, \ldots, x^N) \},
\]

where \( \Gamma \) is an upper semicontinuous mapping (by the continuity assumption G2) that maps each point of the convex compact set \( \mathcal{X} \) into a closed (by G2) convex (by G3) subset of \( \mathcal{X} \). By the Kakutani Fixed Point Theorem [17], there exists a fixed point \( x \in \Gamma(x) \) and such a point is an NEP [19].

**Lemma 4.** At the Nash equilibrium, the consumer(s) with the highest revenue rate \( (\phi_i^t) \) within the time slot, may be the only one(s) buying the power in that time slot.
**Proof:** For a given time slot $t$ consumer $i$ has an incentive to increase its demand $x^i_t$ as long as the payoff can increase, i.e., as long as 

$$\frac{\partial \pi_i}{\partial x^i_t} > 0.$$ 

Therefore at the equilibrium the following holds for all consumers.

$$\frac{\partial \pi_i}{\partial x^i_t} \leq 0 \Rightarrow \frac{\partial (E^i_t - M^i_t)}{\partial x^i_t} \leq 0$$

$$\Rightarrow \frac{\partial E^i_t}{\partial x^i_t} - \frac{\partial M^i_t}{\partial x^i_t} \leq 0$$

$$\Rightarrow \phi^i_t \leq \frac{\partial M^i_t}{\partial x^i_t} = \frac{C(x^i_t)}{x^i_t} = A(x^i_t)$$

For the consumers with a strict inequality $\phi^i_t < A(x^i_t)$, the rate of revenue is less than the price per unit of electricity at time $t$; hence the revenue is less than the cost, $E^i_t < M^i_t$, such that buying electricity will incur them a negative payoff and hence all such consumers, with $\phi^i_t < A(x^i_t)$, will not buy any power in that time slot, i.e., $x^i_t = 0$. Therefore only the set of consumers $\{\arg\max_k \phi^k_t\}$, i.e., the consumers who enjoy the maximum rate of revenue may be able to purchase electricity.

Thus if consumer $i$ has the maximum rate of revenue, either it is the only consumer buying non-zero power $x^i_t$ such that $\phi^i_t = A(x^i_t)$ or $\phi^i_t < C'(0)$ and hence $x^i_t = 0$ in that time slot, which leads to a unique Nash equilibrium for the sub-game. If in a given time slot multiple consumers experience the same maximum rate of revenue, the sub-game will turn into a Nash Demand Game [20] between the set of consumers given by $\{\arg\max_k \phi^k_t\}$, which is well known to admit multiple Nash equilibriums. Thus the overall noncooperative game has a unique Nash equilibrium if and only if, in each time slot, at most one consumer experiences the maximum rate of revenue.

**IV. Nash Equilibrium with Increasing-Block Pricing**

In this section we study the load balancing game with the time-variant increasing-block pricing scheme. Under this scheme consumer $i$ pays $M^i_t$ for $x^i_t$ units of electricity, which is given by (3) with $\mathcal{M}(x, x^i_t)$ the marginal cost function posed to the consumer. Thus, as defined before, we have

$$\mathcal{M}(x, x^i_t) = C \left( \sum_j \min(x, x^j_t) \right).$$

As an example, if the demands from different consumers at time slot $t$ are identical, i.e., if $x^i_t = x^j_t$, $\forall i, j$, we have,

$$\mathcal{M}(x, x^i_t) = C(Nx).$$

**A. Zero-Revenue Model**

In this case the payment by consumer $i$ is given by (3)

$$M^i_t = \int_0^{x^i_t} \mathcal{M}(x, x^i_t)dx.$$ 

The consumer load balancing problem for each consumer $i$ is given by the following optimization problem:

maximize \quad $\pi^i(x^i) = - \sum_t M^i_t$

subject to \quad $M^i_t = \int_0^{x^i_t} \mathcal{M}(x, x^i_t)dx$, $\forall t$,

$\sum_t x^i_t \geq \beta_i$, $\forall t$,

$0 \leq x^i_t$, $\forall t$. 

If the sum demand $x_t$ in a time slot $t$ exceeds $U$, the price of electricity for the consumer with the highest demand (indexed by $j$) becomes infinite. As we retain the assumption G5, consumer $j$ can rearrange its demand vector such that either the sum demand becomes within the capacity threshold or consumer $j$ is no longer the highest demand consumer (then the new customer with the highest demand performs the same routine until the sum demand is under the threshold). This implies that, at the Nash equilibrium point we have $x_t \leq U$. Similarly, we now have the redundant constraint $x_t^i \leq U$, $\forall i, t$, which in turn makes the feasible region $\mathcal{X}$ finite and hence compact.

As $M_t^i$ is continuous in $x_t$, in the corresponding routing game we have that

$$J_t^i = M_t^i$$

is continuous in $x_t$ and satisfies assumption G2. In addition, $M_t^i$ is convex in $x_t^i$ as its derivative, the marginal cost function $\mathcal{M}(x, x_t)$, is non-decreasing. Thus, $J_t^i$ is convex in $x_t^i$ and hence satisfies assumption G3. Following the proof in [12], we consider the point-to-set mapping $x \in \mathcal{X} \rightarrow \Gamma(x) \subset \mathcal{X}$ defined the same as in (8). By the Kakutani Fixed Point Theorem [17], there exists a fixed point $x \in \Gamma(x)$ and such a point is an NEP.

When each consumer tries to minimize its total cost while satisfying its minimum daily energy requirement $\beta_i$, we have the following result.

**Lemma 5.** If $C(\cdot)$ is strictly convex, the Nash equilibrium is unique and each consumer distributes its demand uniformly over all time slots.

*Proof:* For the equilibrium conditions to be satisfied,

$$\mathcal{M}(x_t^i, x_{t_1}, x_{t_2}) = \mathcal{M}(x_{t_1}^i, x_{t_2}), \quad \forall i, t_1, t_2,$$

should hold; otherwise consumer $i$ can increase payoff by varying $x_t^i$, and $x_{t_2}$, in a similar argument to that for Lemma [1] This condition can be rewritten after expanding $\mathcal{M}(\cdot)$ as

$$C \left( \sum_j \min(x_t^i, x_{t_1}^j) \right) = C \left( \sum_j \min(x_{t_2}^i, x_{t_2}^j) \right), \quad \forall i, t_1, t_2.$$  

(10)

Given that $C(\cdot)$ is strictly convex, we have $C(\cdot) = C'(\cdot)$ monotonically increasing, which gives

$$C(z_1) = C(z_2) \Leftrightarrow z_1 = z_2.$$  

(11)

Therefore, (10) implies

$$\sum_j \min(x_t^i, x_{t_1}^j) = \sum_j \min(x_{t_2}^i, x_{t_2}^j), \quad \forall i, t_1, t_2.$$  

(12)

Now assume that there exists an NEP $x$ with demand vectors $x_{t_1} \neq x_{t_2}$. Let $\mathcal{P}$ represent the subset of consumers with unequal demands in time slots $t_1$ and $t_2$, i.e.,

$$\mathcal{P} = \{k | x_{t_1}^k \neq x_{t_2}^k, k \in \{1, 2, \ldots, N\}\}.$$  

Then let $a$ represent the consumer from subset $\mathcal{P}$ with the highest value of demand in time slot $t_1$, i.e.,

$$a = \arg \max_{k \in \mathcal{P}} x_{t_1}^k,$$  

(13)

and let $b$ represent the consumer from subset $\mathcal{P}$ with the highest value of demand in time slots $t_2$, i.e.,

$$b = \arg \max_{k \in \mathcal{P}} x_{t_2}^k.$$  

(14)

From (12) we have

$$\sum_j \min(x_{t_1}^a, x_{t_1}^j) = \sum_j \min(x_{t_2}^a, x_{t_2}^j), \quad \forall t_1, t_2,$$  

(15)

and

$$\sum_j \min(x_{t_1}^b, x_{t_1}^j) = \sum_j \min(x_{t_2}^b, x_{t_2}^j), \quad \forall t_1, t_2.$$  

(16)
Combining (12) and (13) leads to
\[
\sum_j \min (x_{t_1}^a, x_{t_1}^j) \geq \sum_j \min (x_{t_2}^b, x_{t_2}^j); \quad (17)
\]

combining (12) and (14) leads to
\[
\sum_j \min (x_{t_2}^a, x_{t_2}^j) \leq \sum_j \min (x_{t_2}^b, x_{t_2}^j). \quad (18)
\]

If \( x_{t_1}^a \neq x_{t_1}^b \) or \( x_{t_2}^a \neq x_{t_2}^b \), (17) holds with strict inequality. With (15), (16), and (17), we have
\[
\sum_j \min (x_{t_1}^a, x_{t_1}^j) > \sum_j \min (x_{t_2}^b, x_{t_2}^j),
\]
which contradicts (18). If \( x_{t_1}^a = x_{t_1}^b \) and \( x_{t_2}^a = x_{t_2}^b \), (15) and (16) imply \( x_{t_1}^a = x_{t_2}^a \) and \( x_{t_1}^b = x_{t_2}^b \), respectively, which contradicts that \( a, b \in \mathcal{P} \). This implies that the set \( \mathcal{P} \) is empty, which contradicts that \( x_{t_1} \neq x_{t_2} \).

Hence we have
\[
x_{t_1}^a = x_{t_2}^a \quad \forall i, t_1, t_2,
\]
and the solution is given by \( x_t^{i} = \beta_i / T \), \( \forall i, t \). Under the necessary conditions for NEP (10), this is the only solution for the set \( x \), hence NEP is unique.

**Remark:** Notice that under the zero-revenue model, the NEP point is the same with both increasing-block pricing and average-cost based pricing. For both the cases, at NEP, we have \( x_t^{i} = \beta_i / T \), \( \forall i, t \). However, even though the loading pattern is similar, the payments \( M_t^i \) made by the consumers will differ and, with increasing-block pricing, will likely be lesser for consumers with relatively lower consumption. In addition, with increasing-block pricing, the maximum payment \( M_t^i \) made by the \( i \)th consumer given \( x_t^{i} \) demand will be \( C(Nx_t^{i}) / N \), irrespective of what other consumers demand and consume. Thus this addresses the issue faced under the average-cost based pricing and zero-revenue model, in which one consumer can increase their demand indefinitely and cause indefinite increase in the payments of all other consumers.

**B. Constant-Rate Revenue Model**

The consumer load balancing problem for consumer \( i \) is given by the following optimization problem:

\[
\text{maximize} \quad \pi^i(x^i) = \sum_t (E_t^i - M_t^i)
\]

subject to
\[
E_t^i = \phi_t x_t^i, \quad \forall t,
\]
\[
M_t^i = \int_0^{x_t^i} \mathcal{M}(x, x_t) dx, \quad \forall t,
\]
\[
\sum_t x_t^i \geq \beta_i,
\]
\[
0 \leq x_t^i, \quad \forall t.
\]

Here again, we assume \( \beta_i = 0 \), \( \forall i \), to avoid any negative payoffs and we could agree for the redundant constraint \( x_t^i \leq U, \forall i, t \), which in turn makes the feasible region for \( \mathcal{X} \) finite and hence compact.

In this case, we briefly show that with increasing-block pricing and a constant-rate for revenue, there exists an NEP solution for this game. First, the cost function in the corresponding routing game is given by
\[
J_t^i = M_t^i - \phi_t x_t^i, \quad (19)
\]
where \( M_t^i \) is continuous in \( x_t \), and therefore \( J_t^i \) is continuous in \( x_t \) and satisfies assumption G2. We have already shown that \( M_t^i \) under the increasing-block pricing scheme is convex in \( x_t^i \) in the previous subsection. The function \( -\phi_t x_t^i \) is linear and hence convex in \( x_t^i \) as well. Thus, \( J_t^i \) from (19) is convex in \( x_t^i \) and hence satisfies assumption G3. By the same point-to-set mapping argument as that for Lemma 1, we can have that there exists a fixed point \( x \in \Gamma(x) \) and such a point is NEP.
With the average-cost based pricing scheme under the constant-rate revenue model, we see that in a given time slot, if a single consumer enjoys the maximum rate of revenue, it will be the only consumer who is able to purchase power. We show here that with the increasing-block pricing scheme under constant-rate revenue model, the result is different.

For a given time slot $t$, consumer $i$ has an incentive to increase their demand $x_i^t$ as long as the payoff increases, i.e.,

$$\frac{\partial \pi_i}{\partial x_i^t} > 0.$$  

Therefore at the equilibrium the following holds for all consumers:

$$\frac{\partial \pi_i}{\partial x_i^t} \leq 0 \Rightarrow \phi_i^t \leq \frac{\partial M_i}{\partial x_i^t} = M(x_i^t, x_t).$$  \hspace{1cm} (20)

Additionally, if $\phi_i^t < M(x_i^t, x_t)$, $J_i$ can be reduced by reducing $x_i^t$. This implies that if $x_i^t > 0$, at the equilibrium we have

$$\phi_i^t \geq M(x_i^t, x_t).$$  \hspace{1cm} (21)

Thus (20) and (21) together imply that, if $x_i^t > 0$, we have

$$\phi_i^t = M(x_i^t, x_t).$$

Together we can write the following set of necessary conditions for equilibrium,

$$\phi_i^t = M(x_i^t, x_t) \quad \text{if} \quad \phi_i^t \geq M(0, x_t),$$

$$x_i^t = 0 \quad \text{if} \quad \phi_i^t < M(0, x_t).$$  \hspace{1cm} (22)

For illustration, we simulate a scenario consisting of 100 consumers, who have their rate of revenue $\phi_i^t$ generated from a uniform distribution ranging over $\$0 - \$100/MWh, where the marginal cost to the retailer $C(\cdot)$ is given by Fig. 1(a). In Fig. 4 we plot the demand $x_i^t$ versus the rate of revenue ($\phi_i^t$) at a given time slot $t$, where $x_i^t$ is evaluated over $i = \{1, \ldots, 100\}$. The equilibrium is obtained by iterative updates of $M(\cdot)$ and $x_t$ until convergence within an error tolerance as in (22).

![Fig. 4](image_url)

Fig. 4. Demand $x_i^t$ versus the rate of revenue ($\phi_i^t$) at equilibrium. Each dot represents a particular consumer $i = \{1, \ldots, 100\}$.

Thus, unlike with the average-cost pricing, where only the consumer with the maximum rate of revenue could purchase electricity at the equilibrium, any consumer may procure a non-zero amount of energy as long as its own rate of revenue is larger than $M(0, x_t)$.
V. CONCLUSION

In this paper we formulated noncooperative games among the consumers of Smart Grid with two real-time pricing schemes to derive autonomous load balancing solutions. The first pricing scheme charges consumers a price that is equal to the average cost of electricity borne by the retailer and the second scheme charges consumers an amount that is dependent on the incremental marginal cost which is shown to protect consumers from irrational behaviors. Two revenue models were considered for each of the pricing schemes, for which we investigated the Nash equilibrium operation points for their uniqueness and load balancing properties. For the zero-revenue model, we showed that when consumers are interested only in the minimization of electricity costs, the Nash equilibrium point is unique with both the pricing schemes and leads to similar electricity loading patterns in both cases. For the constant-rate revenue model, we showed the existence of Nash equilibrium with both the pricing schemes and the uniqueness results with the average-cost based pricing scheme. Throughout the paper, we utilized the relationship between the load balancing games and the atomic splittable flow games from the computer networking community to prove the properties at the Nash Equilibrium solutions.

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