Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation

Peter Constantin  
Department of Mathematics  
University of Chicago  
5734 S. University Avenue  
Chicago, IL 60637  
E-mail: const@cs.uchicago.edu

Jiahong Wu  
Department of Mathematics  
Oklahoma State University  
Stillwater, OK 74078  
E-mail: jiahong@math.okstate.edu

Abstract. We present a regularity result for weak solutions of the 2D quasi-geostrophic equation with supercritical ($\alpha < 1/2$) dissipation $(-\Delta)^\alpha$: If a Leray-Hopf weak solution is Hölder continuous $\theta \in C^\delta(\mathbb{R}^2)$ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$.

AMS (MOS) Numbers: 76D03, 35Q35

Keywords: the 2D quasi-geostrophic equation, supercritical dissipation, regularity, weak solutions.
1 Introduction

We discuss the surface 2D quasi-geostrophic (QG) equation
\[ \partial_t \theta + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2, \quad t > 0, \tag{1.1} \]
where \( \alpha > 0 \) and \( \kappa \geq 0 \) are parameters, and the 2D velocity field \( u = (u_1, u_2) \) is determined from \( \theta \) by the stream function \( \psi \) via the auxiliary relations
\[ (u_1, u_2) = (-\partial_{x_2} \psi, \partial_{x_1} \psi), \quad (-\Delta)^{\frac{\alpha}{2}} \psi = -\theta. \tag{1.2} \]
Using the notation \( \Lambda \equiv (-\Delta)^{\frac{\alpha}{2}} \) and \( \nabla^\bot \equiv (\partial_{x_2}, -\partial_{x_1}) \), the relations in (1.2) can be combined into
\[ u = \nabla^\bot \Lambda^{-1} \theta = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \tag{1.3} \]
where \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are the usual Riesz transforms in \( \mathbb{R}^2 \). The 2D QG equation with \( \kappa > 0 \) and \( \alpha = \frac{1}{2} \) arises in geophysical studies of strongly rotating fluids (see [5],[15] and references therein) while the inviscid QG equation ((1.1) with \( \kappa = 0 \)) was derived to model frontogenesis in meteorology, a formation of sharp fronts between masses of hot and cold air (see [7],[10],[15]).

The problem at the center of the mathematical theory concerning the 2-D QG equation is whether or not it has a global in time smooth solution for any prescribed smooth initial data. In the subcritical case \( \alpha > \frac{1}{2} \), the dissipative QG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [8],[16]). In contrast, when \( \alpha \leq \frac{1}{2} \), the issue of global existence and uniqueness is more difficult and has still unanswered aspects. Recently this problem has attracted a significant amount of research ([1],[2],[3],[4],[5],[6],[9],[11],[12],[13],[14],[17],[18],[19],[20],[21],[22],[23]). In Constantin, Córdoba and Wu [6], we proved in the critical case \( \alpha = \frac{1}{2} \) the global existence and uniqueness of classical solutions corresponding to any initial data with \( L^\infty \)-norm comparable to or less than the diffusion coefficient \( \kappa \). In a recently posted preprint in arXiv [13], Kiselev, Nazarov and Volberg proved that smooth global solutions exist for any \( C^\infty \) periodic initial data, by removing the \( L^\infty \)-smallness assumption on the initial data of [6]. Caffarelli and Vasseur (arXiv reference [1]) establish the global regularity of the Leray-Hopf type weak solutions (in \( L^\infty((0, \infty); L^2) \cap L^2((0, \infty); H^{1/2}) \)) of the critical 2D QG equation with \( \alpha = \frac{1}{2} \) in general \( \mathbb{R}^n \).

In this paper we present a regularity result of weak solutions of the dissipative QG equation with \( \alpha < \frac{1}{2} \) (the supercritical case). The result asserts that if a Leray-Hopf weak solution \( \theta \) of (1.1) is in the Hólder class \( C^\delta \) with \( \delta > 1 - 2\alpha \) on the time interval \([t_0, t]\), then it is actually a classical solution on \([t_0, t]\). The proof involves representing the functions in Hólder space in terms of the Littlewood-Paley decomposition and using Besov space techniques. When \( \theta \) is in \( C^\delta \), it also belongs to the Besov space \( B^{(1-2/\alpha)}_{p,\infty} \) for any \( p \geq 2 \). By taking \( p \) sufficiently large, we have \( \theta \in C^{\delta_1} \cap B^{\delta_1}_{p,\infty} \) for \( \delta_1 > 1 - 2\alpha \).
The idea is to show that $\theta \in C_{\delta_2}^\delta \cap \mathring{B}_{p,\infty}^\delta$ with $\delta_2 > \delta_1$. Through iteration, we establish that $\theta \in C^\gamma$ with $\gamma > 1$. Then $\theta$ becomes a classical solution.

The results of this paper can be easily extended to a more general form of the quasi-geostrophic equation in which $x \in \mathbb{R}^n$ and $u$ is a divergence-free vector field determined by $\theta$ through a singular integral operator.

The rest of this paper is divided into two sections. Section 2 provides the definition of Besov spaces and necessary tools. Section 3 states and proves the main result.

2 Besov spaces and related tools

This section provides the definition of Besov spaces and several related tools. We start with some notation. Denote by $\mathcal{S}(\mathbb{R}^n)$ the usual Schwarz class and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. $\hat{f}$ denotes the Fourier transform of $f$, namely

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx.$$  

The fractional Laplacian $(-\Delta)^\alpha$ can be defined through the Fourier transform

$$\overset{\longrightarrow}{(\Delta)}^\alpha f = |\xi|^{2\alpha} \hat{f}(\xi).$$

Let

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^n} \phi(x) x^\gamma \, dx = 0, \, |\gamma| = 0, 1, 2, \ldots \right\}.$$  

Its dual $\mathcal{S}'_0$ is given by

$$\mathcal{S}'_0 = \mathcal{S}'/\mathcal{S}^\perp_0 = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ is the space of polynomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}'_0$ if their difference is a polynomial.

It is a classical result that there exists a dyadic decomposition of $\mathbb{R}^n$, namely a sequence $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\text{supp} \, \hat{\Phi}_j \subset A_j, \quad \hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j} \xi) \quad \text{or} \quad \Phi_j(x) = 2^{jn} \Phi_0(2^j x)$$

and

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_k(\xi) = \left\{ \begin{array}{ll} 1 & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\
0 & \text{if } \xi = 0, \end{array} \right.$$  

where

$$A_j = \{ \xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1} \}.$$
As a consequence, for any \( f \in \mathcal{S}'_0 \),
\[
\sum_{k=\infty}^{\infty} \Phi_k \ast f = f. \tag{2.1}
\]

For notational convenience, set
\[
\Delta_j f = \Phi_j \ast f, \quad j = 0, \pm 1, \pm 2, \ldots. \tag{2.2}
\]

**Definition 2.1** For \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), the homogeneous Besov space \( \dot{B}^s_{p,q} \) is defined by
\[
\dot{B}^s_{p,q} = \{ f \in \mathcal{S}'_0 : \| f \|_{\dot{B}^s_{p,q}} < \infty \},
\]
where
\[
\| f \|_{\dot{B}^s_{p,q}} = \begin{cases} 
\left( \sum_j \left( 2^{js} \| \Delta_j f \|_{L^p} \right)^q \right)^{1/q} & \text{for } q < \infty, \\
\sup_j 2^{js} \| \Delta_j f \|_{L^p} & \text{for } q = \infty.
\end{cases}
\]

For \( \Delta_j \) defined in (2.2) and \( S_j \equiv \sum_{k<j} \Delta_k \),
\[
\Delta_j \Delta_k = 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j (S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j - k| \geq 3.
\]

The following proposition lists a few simple facts that we will use in the subsequent section.

**Proposition 2.2** Assume that \( s \in \mathbb{R} \) and \( p, q \in [1, \infty] \).

1) If \( 1 \leq q_1 \leq q_2 \leq \infty \), then \( \dot{B}^s_{p,q_1} \subset \dot{B}^s_{p,q_2} \).

2) \textbf{(Besov embedding)} If \( 1 \leq p_1 \leq p_2 \leq \infty \) and \( s_1 = s_2 + n \left( \frac{1}{p_1} - \frac{1}{p_2} \right) \), then
\[
\dot{B}^s_{p_1,q_1}(\mathbb{R}^n) \subset \dot{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n).
\]

3) If \( 1 < p < \infty \), then
\[
\dot{B}^s_{p,\min(p,2)} \subset \dot{W}^{s,p} \subset \dot{B}^s_{p,\max(p,2)},
\]
where \( \dot{W}^{s,p} \) denotes a standard homogeneous Sobolev space.

We will need a Bernstein type inequality for fractional derivatives.

**Proposition 2.3** Let \( \alpha \geq 0 \). Let \( 1 \leq p \leq q \leq \infty \).

1) If \( f \) satisfies
\[
\text{supp} \, \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq K 2^j \},
\]
for some integer \( j \) and a constant \( K > 0 \), then
\[
\| (-\Delta)^\alpha f \|_{L^q(\mathbb{R}^n)} \leq C_1 2^{2\alpha j + jn \left( \frac{1}{p} - \frac{1}{q} \right)} \| f \|_{L^p(\mathbb{R}^n)}.
\]
2) If $f$ satisfies
\[ \text{supp} \hat{f} \subset \{ \xi \in \mathbb{R}^n : K_1 2^j \leq |\xi| \leq K_2 2^j \} \tag{2.3} \]
for some integer $j$ and constants $0 < K_1 \leq K_2$, then
\[
C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^n)} \leq \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \leq C_2 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)},
\]
where $C_1$ and $C_2$ are constants depending on $\alpha, p$ and $q$ only.

The following proposition provides a lower bound for an integral that originates from the dissipative term in the process of $L^p$ estimates (see [20],[4]).

**Proposition 2.4** Assume either $\alpha \geq 0$ and $p = 2$ or $0 \leq \alpha \leq 1$ and $2 < p < \infty$. Let $j$ be an integer and $f \in S'$. Then
\[
\int_{\mathbb{R}^n} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f \, dx \geq C 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p
\]
for some constant $C$ depending on $n, \alpha$ and $p$.

## 3 The main theorem and its proof

**Theorem 3.1** Let $\theta$ be a Leray-Hopf weak solution of (1.1), namely
\[
\theta \in \mathcal{L}^\infty([0, \infty); L^2(\mathbb{R}^2)) \cap L^2([0, \infty); \dot{H}^\alpha(\mathbb{R}^2)). \tag{3.1}
\]
Let $\delta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If
\[
\theta \in \mathcal{L}^\infty([t_0, t]; C^{\delta}(\mathbb{R}^2)), \tag{3.2}
\]
then
\[
\theta \in C^\infty((t_0, t] \times \mathbb{R}^2).
\]

**Proof.** First, we notice that (3.1) and (3.2) imply that
\[
\theta \in \mathcal{L}^\infty([t_0, t]; B_p^{\delta_1}(\mathbb{R}^2)),
\]
for any $p \geq 2$ and $\delta_1 = \delta(1 - \frac{2}{p})$. In fact, for any $\tau \in [t_0, t]$,
\[
\|\theta(\cdot, \tau)\|_{B_p^{\delta_1}} = \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p}
\leq \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^\infty}^{\frac{1}{p}} \|\Delta_j \theta\|_{L^2}^{\frac{2}{p}}
\leq \|\theta(\cdot, \tau)\|_{C^{\delta_1}}^{\frac{1}{p}} \|\theta(\cdot, \tau)\|_{L^2}^{\frac{2}{p}}.
\]
Since $\delta > 1 - 2\alpha$, we have $\delta_1 > 1 - 2\alpha$ when

$$p > p_0 \equiv \frac{2\delta}{\delta - (1 - 2\alpha)}.$$  

Next, we show that

$$\theta \in L^\infty([t_0, t]; \dot{B}_{p,\infty}^{\delta_1} \cap C^{\delta_1})$$  

implies

$$\theta(\cdot, t) \in \dot{B}_{p,\infty}^{\delta_2} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$ to be specified. Let $j$ be an integer. Applying $\Delta_j$ to (1.1), we get

$$\partial_t \Delta_j \theta + \kappa \Lambda^{2\alpha} \Delta_j \theta = -\Delta_j (u \cdot \nabla \theta).$$  

(3.3)

By Bony’s notion of paraproduct,

$$\Delta_j (u \cdot \nabla \theta) = \sum_{|j-k| \leq 2} \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) + \sum_{|j-k| \leq 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta)$$

$$+ \sum_{k \geq j-1} \sum_{|k-l| \leq 1} \Delta_j (\Delta_k u \cdot \nabla \Delta_l \theta).$$  

(3.4)

Multiplying (3.3) by $p|\Delta_j \theta|^{p-2} \Delta_j \theta$, integrating with respect to $x$, and applying the lower bound

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f \, dx \geq C 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p$$

of Proposition 2.4, we obtain

$$\frac{d}{dt} \|\Delta_j \theta\|_{L^p}^p + C \kappa 2^{2\alpha j} \|\Delta_j \theta\|_{L^p}^p \leq I_1 + I_2 + I_3,$$  

(3.5)

where $I_1$, $I_2$ and $I_3$ are given by

$$I_1 = -p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) \, dx,$$

$$I_2 = -p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta) \, dx,$$

$$I_3 = -p \sum_{k \geq j-1} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot \sum_{|k-l| \leq 1} \Delta_j (\Delta_k u \cdot \nabla \Delta_l \theta) \, dx.$$  

We first bound $I_2$. By Hölder’s inequality

$$I_2 \leq C \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} \|\nabla S_{k-1} \theta\|_{L^\infty}.$$  

6
Applying Bernstein’s inequality, we obtain
\[
I_2 \leq C \| \Delta_j \|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^p} \sum_{m \leq k-1} 2^m \| \Delta_m \theta \|_{L^\infty} \\
\leq C \| \Delta_j \|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^p} 2^{(1-\delta_1)k} \sum_{m \leq k-1} 2^{(m-k)(1-\delta_1)} 2^{m\delta_1} \| \Delta_m \theta \|_{L^\infty}.
\]

Thus, for \(1 - \delta_1 > 0\), we have
\[
I_2 \leq C \| \Delta_j \|_{L^p}^{p-1} \| \theta \|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \| \Delta_k u \|_{L^p} 2^{(1-\delta_1)k}.
\]

We now estimate \(I_1\). The standard idea is to decompose it into three terms: one with commutator, one that becomes zero due to the divergence-free condition and the rest.
That is, we rewrite \(I_1\) as
\[
I_1 = -p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot [\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta \, dx \\
- p \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot (S_j u \cdot \nabla \Delta_j \theta) \, dx \\
- p \sum_{|j-k| \leq 2} \int |\Delta_j \theta|^{p-2} \Delta_j \theta \cdot (S_{k-1} u - S_j u) \cdot \nabla \Delta_j \Delta_k \theta \, dx
\]
= \(I_{11} + I_{12} + I_{13}\),
where we have used the simple fact that \(\sum_{|k-j| \leq 2} \Delta_k \Delta_j \theta = \Delta_j \theta\), and the brackets \([\cdot]\) represent the commutator, namely
\[
[\Delta_j, S_{k-1} u \cdot \nabla] \Delta_k \theta = \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) - S_{k-1} u \cdot \nabla \Delta_j \Delta_k \theta.
\]
Since \(u\) is divergence free, \(I_{12}\) becomes zero. \(I_{12}\) can also be handled without resort to the divergence-free condition. In fact, integrating by parts in \(I_{12}\) yields
\[
I_{12} = \int |\Delta_j \theta|^p \nabla \cdot S_j u \, dx \leq \| \Delta_j \theta \|_{L^p}^p \| \nabla \cdot S_j u \|_{L^\infty}.
\]

By Bernstein’s inequality,
\[
|I_{12}| \leq \| \Delta_j \theta \|_{L^p}^p \sum_{m \leq j-1} 2^m \| \Delta_m u \|_{L^\infty} \\
= \| \Delta_j \theta \|_{L^p}^p 2^{(1-\delta_1)j} \sum_{m \leq j-1} 2^{(1-\delta_1)(m-j)} 2^{m\delta_1} \| \Delta_m u \|_{L^\infty}.
\]

For \(1 - \delta_1 > 0\),
\[
|I_{12}| \leq C \| \Delta_j \theta \|_{L^p}^p 2^{(1-\delta_1)j} \| u \|_{C^{\delta_1}} \leq C \| \Delta_j \theta \|_{L^p}^{p-1} 2^{(1-2\delta_1)j} \| \theta \|_{B^\delta_{p,\infty}} \| u \|_{C^{\delta_1}}.
\]
We now bound $I_{11}$ and $I_{13}$. By Hölder’s inequality,

$$|I_{11}| \leq p\|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|\Delta_j, S_{k-1}u \cdot \nabla\|_{L^p} \|\Delta_k \theta\|_{L^p}. $$

To bound the the commutator, we have by the definition of $\Delta_j$

$$[\Delta_j, S_{k-1}u \cdot \nabla] \Delta_k \theta = \int \Phi_j(x-y) (S_{k-1}(u)(x) - S_{k-1}(u)(y)) \cdot \nabla \Delta_k \theta(y) dy. $$

Using the fact that $\theta \in C^{\delta_1}_t$, and thus

$$\|S_{k-1}(u)(x) - S_{k-1}(u)(y)\|_{L^\infty} \leq \|u\|_{C^{\delta_1}} |x-y|^{\delta_1},$$

we obtain

$$\|\Delta_j, S_{k-1}u \cdot \nabla\|_{L^p} \|\Delta_k \theta\|_{L^p} \leq 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} 2^k \|\Delta_k \theta\|_{L^p}. $$

Therefore,

$$|I_{11}| \leq C_p \|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} 2^k \|\Delta_k \theta\|_{L^p}. $$

The estimate for $I_{13}$ is straightforward. By Hölder’s inequality,

$$|I_{13}| \leq p\|\Delta_j \theta\|_{L^p}^{p-1} \sum_{|j-k| \leq 2} \|S_{k-1}u - S_j u\|_{L^p} \|\nabla \Delta_j \theta\|_{L^\infty} \leq C_p \|\Delta_j \theta\|_{L^p}^{p-1} 2^{(1-\delta_1) j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p}. $$

We now bound $I_3$. By Hölder’s inequality and Bernstein’s inequality,

$$|I_3| \leq p\|\Delta_j \theta\|_{L^p}^{p-1} \|\Delta_j \nabla \cdot (\sum_{k \geq j-1} \sum_{|j-k| \leq 1} \Delta_k u \Delta_k \theta)\|_{L^p} \leq p\|\Delta_j \theta\|_{L^p}^{p-1} 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p}. (3.6)$$

Inserting the estimates for $I_1$, $I_2$ and $I_3$ in (3.5) and eliminating $p\|\Delta_j \theta\|_{L^p}^{p-1}$ from both sides, we get

$$\frac{d}{dt}\|\Delta_j \theta\|_{L^p} + C \kappa 2^{2\alpha j} \|\Delta_j \theta\|_{L^p} \leq C 2^{(1-2\delta_1) j} \|\theta\|_{\dot{B}^{\delta_1}_{p,\infty}} \|u\|_{C^{\delta_1}}$$

$$+ C 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p}$$

$$+ C \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k}$$

$$+ C 2^{(1-\delta_1) j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p}$$

$$+ C 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p}. (3.7)$$

8
The terms on the right can be further bounded as follows.

\[
C 2^{-\delta_1 j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^k \|\Delta_k \theta\|_{L^p} = C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k \theta\|_{L^p} 2^{(k-j)(1-\delta_1)} \leq C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \|\theta\|_{\hat{B}_{p,\infty}^{\delta_1}}
\]

\[
C \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} = C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k u\|_{L^p} 2^{(k-j)(1-2\delta_1)} \leq C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \|u\|_{\hat{B}_{p,\infty}^{\delta_1}}
\]

\[
C 2^{(1-\delta_1 j)} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} \|\Delta_k u\|_{L^p} = C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \sum_{|j-k| \leq 2} 2^{\delta_1 k} \|\Delta_k u\|_{L^p} 2^{(j-k)\delta_1} \leq C 2^{(1-2\delta_1)j} \|\theta\|_{C^{\delta_1}} \|u\|_{\hat{B}_{p,\infty}^{\delta_1}}
\]

and

\[
C 2^j \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-\delta_1 k} \|\Delta_k \theta\|_{L^p} = C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \sum_{k \geq j-1} 2^{-2\delta_1 (k-j)} 2^{\delta_1 k} \|\Delta_k \theta\|_{L^p} \leq C 2^{(1-2\delta_1)j} \|u\|_{C^{\delta_1}} \|\theta\|_{\hat{B}_{p,\infty}^{\delta_1}}.
\]

We can write (3.7) in the following integral form

\[
\|\Delta_j \theta(t)\|_{L^p} \leq e^{-C \kappa 2^{2\alpha j}(t-t_0)} \|\Delta_j \theta(t_0)\|_{L^p} + C \int_{t_0}^t e^{-C \kappa 2^{2\alpha j}(t-s)2^{(1-2\delta_1)j}} (\|\theta\|_{C^{\delta_1}} \|u\|_{\hat{B}_{p,\infty}^{\delta_1}} + \|u\|_{C^{\delta_1}} \|\theta\|_{\hat{B}_{p,\infty}^{\delta_1}}) ds.
\]

Multiplying both sides by \(2^{(2\alpha+2\delta_1-1)j}\) and taking the supremum with respect to \(j\), we get

\[
\|\theta(t)\|_{\hat{B}_{p,\infty}^{2\delta_1+2\alpha-1}} \leq \sup_j \left\{ e^{-C \kappa 2^{2\alpha j}(t-t_0)2^{(2\delta_1+2\alpha-1)j}} \|\theta(t_0)\|_{\hat{B}_{p,\infty}^{\delta_1}} + C \kappa^{-1} \sup_j \left\{ (1 - e^{-C \kappa 2^{2\alpha j}(t-t_0)}) \right\} \max_{s \in [t_0, t]} \|\theta(s)\|_{\hat{B}_{p,\infty}^{\delta_1}} \right\} \|\theta\|_{C^{\delta_1}}
\]

Here we have used the fact that

\[
\|u\|_{C^{\delta_1}} \leq \|\theta\|_{C^{\delta_1}} \quad \text{and} \quad \|u\|_{\hat{B}_{p,\infty}^{\delta_1}} \leq \|\theta\|_{\hat{B}_{p,\infty}^{\delta_1}}
\]

Therefore, we conclude that if

\[
\theta \in L^\infty([t_0, t]; \hat{B}_{p,\infty}^{\delta_1} \cap C^{\delta_1}),
\]
then
\[ \theta(\cdot, t) \in \dot{B}_{p,\infty}^{2\delta_1 + 2\alpha - 1}. \] (3.8)

Since \( \delta_1 > 1 - 2\alpha \), we have \( 2\delta_1 + 2\alpha - 1 > \delta_1 \) and thus gain regularity. In addition, according to the Besov embedding of Proposition 2.2,
\[ \dot{B}_{p,\infty}^{2\delta_1 + 2\alpha - 1} \subset \dot{B}_{\infty,\infty}^{\delta_2}, \]
where
\[ \delta_2 = 2\delta_1 + 2\alpha - 1 - \frac{2}{p} = \delta_1 + \left( \delta_1 - \left(1 - 2\alpha + \frac{2}{p}\right) \right). \]

We have \( \delta_2 > \delta_1 \) when
\[ p > p_1 \equiv \frac{2}{\delta_1 - (1 - 2\alpha)}. \]

Noting that
\[ \dot{B}_{\infty,\infty}^{\delta_2} \cap L^\infty = C^{\delta_2}, \]
we conclude that, for \( p > \max\{p_0, p_1\} \),
\[ \theta(\cdot, t) \in \dot{B}_{p,\infty}^{\delta_2} \cap C^{\delta_2} \]
for some \( \delta_2 > \delta_1 \). The above process can then be iterated with \( \delta_1 \) replaced by \( \delta_2 \). A finite number of iterations allow us to obtain that
\[ \theta(\cdot, t) \in C^\gamma \]
for some \( \gamma > 1 \). The regularity in the spatial variable can then be converted into regularity in time. We have thus established that \( \theta \) is a classical solution to the supercritical QG equation. Higher regularity can be proved by well-known methods.

**Acknowledgment**: PC was partially supported by NSF-DMS 0504213. JW thanks the Department of Mathematics at the University of Chicago for its support and hospitality.

**References**

[1] L. Caffarelli and A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, ArXiv: Math.AP/0608447 (2006).

[2] D. Chae, On the regularity conditions for the dissipative quasi-geostrophic equations, *SIAM J. Math. Anal.* 37 (2006), 1649-1656.
[3] D. Chae and J. Lee, Global well-posedness in the super-critical dissipative quasi-geostrophic equations, *Commun. Math. Phys.* **233** (2003), 297-311.

[4] Q. Chen, C. Miao and Z. Zhang, A new Bernstein inequality and the 2D dissipative quasi-geostrophic equation, to appear in *Commun. Math. Phys.*.

[5] P. Constantin, Euler equations, Navier-Stokes equations and turbulence. Mathematical foundation of turbulent viscous flows, 1–43, Lecture Notes in Math., 1871, Springer, Berlin, 2006.

[6] P. Constantin, D. Córdoba and J. Wu, On the critical dissipative quasi-geostrophic equation, *Indiana Univ. Math. J.* **50** (2001), 97-107.

[7] P. Constantin, A. Majda, and E. Tabak, Formation of strong fronts in the 2-D quasi-geostrophic thermal active scalar, *Nonlinearity* **7**(1994), 1495-1533.

[8] P. Constantin and J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, *SIAM J. Math. Anal.* **30** (1999), 937-948.

[9] A. Córdoba and D. Córdoba, A maximum principle applied to quasi-geostrophic equations, *Commun. Math. Phys.* **249** (2004), 511-528.

[10] I. Held, R. Pierrehumbert, S. Garner, and K. Swanson, Surface quasi-geostrophic dynamics, *J. Fluid Mech.* **282** (1995), 1-20.

[11] N. Ju, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, *Commun. Math. Phys.* **255** (2005), 161-181.

[12] N. Ju, Global solutions to the two dimensional quasi-geostrophic equation with critical or super-critical dissipation, *Math. Ann.* **334** (2006), 627–642.

[13] A. Kiselev, F. Nazarov and A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, ArXiv: Math.AP/0604185 (2006).

[14] F. Marchand and P.G. Lemarié-Rieusset, Solutions auto-similaires non radiales pour l’équation quasi-géostrophique dissipative critique, *C. R. Math. Acad. Sci. Paris* **341** (2005), 535–538.

[15] J. Pedlosky, “Geophysical fluid dynamics”, Springer, New York, 1987.

[16] S. Resnick, Dynamical problems in nonlinear advective partial differential equations, Ph.D. thesis, University of Chicago, 1995.

[17] M. Schonbek and T. Schonbek, Asymptotic behavior to dissipative quasi-geostrophic flows, *SIAM J. Math. Anal.* **35** (2003), 357-375.
[18] M. Schonbek and T. Schonbek, Moments and lower bounds in the far-field of solutions to quasi-geostrophic flows, *Discrete Contin. Dyn. Syst.* 13 (2005), 1277-1304.

[19] J. Wu, The quasi-geostrophic equation and its two regularizations, *Commun. Partial Differential Equations* 27 (2002), 1161-1181.

[20] J. Wu, Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces, *SIAM J. Math. Anal.* 36 (2004/2005), 1014-1030.

[21] J. Wu, The quasi-geostrophic equation with critical or supercritical dissipation, *Nonlinearity* 18 (2005), 139-154.

[22] J. Wu, Solutions of the 2-D quasi-geostrophic equation in Hölder spaces, *Nonlinear Analysis* 62 (2005), 579-594.

[23] J. Wu, Existence and uniqueness results for the 2-D dissipative quasi-geostrophic equation, *Nonlinear Analysis*, in press.