On reversal of centrifugal acceleration in special relativity

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Abstract

The basic principles of General Theory of Relativity historically have been tested in gedanken experiments in rotating frame of references. One of the key issues, which still evokes a lot of controversy, is the centrifugal acceleration. Machabeli & Rogava (1994) argued that centrifugal acceleration reverse direction for particles moving radially with relativistic velocities within a ”bead on a wire” approximation. We show that this result is frame-dependent and reflects a special relativistic dilution of time (as correctly argued by de Felice (1995)) and is analogous to freezing of motion on the black hole horizon as seen by a remote observer. It is a reversal of coordinate acceleration; there is no such effect as measured by a defined set of observers, e.g., proper and/or comoving. Frame-independent velocity of a ”bead” with respect to stationary rotating observers increases and formally reaches the speed of light on the light cylinder. In general relativity, centrifugal force does reverse its direction at photon circular orbit, $r = 3M$ in Schwarzschild metric, as argued by Abramowicz (1990).
I. INTRODUCTION

Since the conception of the Special and General Theories of Relativity, rotating frames served as conceptual testbed of our understanding of effects of motion and gravitation on measured quantities. For over a century this has lead to a number of paradoxes, most notably the Ehrenfest paradox \cite{1} of the circumference length of a rotating disk. The Ehrenfest paradox involved a discussion between such prominent physicists as Born, Plank, Kaluza, Einstein, Becquerel, and Langevin among others \cite{2}. Kinematics and especially dynamics in rotating frame continues to be a source of confusion. In this article we aim to elucidate one of the “paradoxes”, the reversal of centrifugal acceleration.

Following the work \cite{3} on reversal of centrifugal force in general relativity, Machabeli & Rogava \cite{4} suggested that a somewhat similar effect, reversal of centrifugal acceleration, occurs in special relativity. This suggestion we taken up in a number of astrophysically-related works on particle acceleration around rotating black hole and neutron star magnetospheres \cite{5, 6, 7, 8} and others. In this Letter we show that the effect discussed by Machabeli & Rogava \cite{4} is not frame-invariant and disappears if one uses frame-invariant quantities. Thus, in special relativity, there is no reversal of centrifugal acceleration. The effect seen by Machabeli & Rogava is a time dilation, as correctly argued by \cite{9}. It describes an unphysical coordinate acceleration.

The motivation for this work comes from numerous astrophysical cites (e.g., magnetospheres of various black holes and neutron stars), where both strong gravity, magnetic field, and rotation are all important ingredients. The effects of magnetic field on a single particle motion are often approximated as a solid guiding wire, which restricts particle motion across the field. This simple approximation neglecting various cross-field drifts. The key question that we will address is “what is the behavior of the parallel momentum of the particle?".

II. ROTATING WIRE

A. Motion in coordinate time

To elucidate the key problems, consider a bead on a radial wire inclined at angle $\pi/2$ to the rotation axis. Let us first neglect gravitation. Using standard methods of general relativity, we transform to rotating coordinates by changing the azimuthal variable $\phi \rightarrow \phi' - \omega t$ and
assume that in rotating coordinates the motion is strictly radial, \( d\phi' = d\theta = 0 \). The non-trivial element of the metric tensor is then

\[ g_{00} = - \left( 1 - \omega^2 r^2 \right) \tag{1} \]

The Hamilton-Jacobi equation \( \partial_t S + H = 0 \), where \( H \) is Hamiltonian and \( S \) is generating function, then becomes

\[ \frac{1}{1 - \omega^2 r^2} (\partial_t S)^2 - (\partial_r S)^2 = 1 \tag{2} \]

(we set \( G = c = 1 \), use \((-1, 1, 1, 1)\) sign convention and assume that the mass of a test particle is unity.) Since the two-dimensional motion in \( r - t \) plane has a conserved quantity – the product of the particle momentum and the time-like Killing vector (time is a cyclic variable), we can look for separable solutions in a form \( S = -E_0 t + S(r) \). After differentiating with respect to \( E_0 \), Eq. (2) gives

\[ (\partial_t r)^2 = \left( 1 - \omega^2 r^2 \right) \left( 1 - \frac{1 - \omega^2 r^2}{E_0^2} \right) \tag{3} \]

By differentiating with respect to time, and eliminating constant \( E_0 \), we find expression for coordinate acceleration in terms of coordinate velocity

\[ \ddot{r} = r\omega^2 \left( 1 - \frac{2v_r^2}{1 - r^2\omega^2} \right) \tag{4} \]

where \( v_r = \partial r / \partial t \). (This result can also be heuristically obtained from Newtonian centrifugal acceleration formula \( \partial_t (m_{\text{eff}} \partial_r r) = r\omega^2 m_{\text{eff}} \) with \( m_{\text{eff}} = 1/\sqrt{1 - r^2\omega^2 - \dot{r}^2} \).) This is the result of Machabeli & Rogava [4], who argued that at \( r = 0 \), for \( v_r > 1/\sqrt{2} \) centrifugal acceleration reverses its sign and becomes centrifugal deceleration. Indeed, for \( v_r > 1/\sqrt{2} \) we have \( \ddot{r} < 0 \). In addition, Eq. (1) does have a solution in terms of elliptic sinus function with formal reversal of velocity occurring at the light cylinder.

**B. Coordinate and physical acceleration**

In the previous section we derived equations of motion of a bead on a wire and obtained fully analytical and mathematically correct solutions. Does it mean that a particle experiences a reversal of centrifugal accelerations and can never leave the light cylinder of a rigidly rotating wire? The answer, which is physically obvious, but given the above derivation is a bit surprising, is no. The key moment missed by Machabeli & Rogava is that observed
quantities must be formulated in a frame-invariant, but observer-dependent form. Thus, quantities measured in terms of, e.g., coordinate time are, in some sense, the least physical. On the other hand, quantities measured by a defined set of observers can be cast in a frame-independent form using the four-velocities of those observers (e.g., notion of ZAMOs in [10]). Expression (4) is coordinate-dependent, and thus is not physically useful. Physically important are velocities and acceleration measured by a defined set of observers. For example, we can define a set of local stationary observers rotating with the wire. For such observers $dr_s = dr$, $dt_s = \sqrt{1 - \omega^2 r^2} dt$,

$$
(\partial_t r_s)^2 = 1 - \frac{1 - \omega^2 r^2}{E_0^2}
$$

$$
\frac{\partial^2 r_s}{\partial t_s^2} = \frac{r \omega^2}{E_0^2} = \frac{r \omega^2 (1 - (\partial_t r_s)^2)}{1 - \omega^2 r^2}
$$

Eqns (5) clearly shows that centrifugal acceleration of the bead, as measured by a set of observers stationary with respect to the rotating wire, is always directed away from the axis of rotation.

We can also find equations of motions and acceleration in terms of proper time $\tau$ of the bead:

$$
(\partial_\tau r)^2 = \frac{E_0^2}{1 - r^2 \Omega^2} - 1
$$

$$
\partial_\tau^2 r = r \Omega^2 \frac{E_0^2}{(1 - r^2 \Omega^2)^2} > 0
$$

So that proper velocity and proper acceleration are always positive.

As the question under consideration is controversial, we next show that that velocity (5), i.e. the velocity of a bead with respect to rotating observer stationary with respect to the wire, is frame-invariant. Recalling that a frame-independent value of relative velocity of two observers $V_{rel}$ moving with four-velocities $U$ and $V$ can be calculated according to

$$
V_{rel}^2 = 1 - \frac{1}{(U \cdot V)^2}.
$$

Using (1), the velocity of a stationary observer at $r$ in coordinates $\{t, r\}$ is

$$
U^\mu = \{-\frac{1}{\sqrt{1 - r^2 \omega^2}}, 0\}
$$

Radial velocity in coordinate time is (3), so that

$$
V^\mu = \{-\frac{E_0}{1 - r^2 \omega^2}, \sqrt{\frac{E_0^2}{1 - r^2 \omega^2} - 1}\}
$$
(recall, that it is the use of this expression that leads to ”reversal” of centrifugal acceleration.)

The relative velocity between the bead and local stationary observer is

$$V_{rel}^2 = 1 - \frac{1 - r^2 \omega^2}{E_0^2},$$

consistent with (5).

Eqns. (5) and (10) shows that velocity of the bead as measured by a defined set of observers, e.g. stationary with respect to the wire, increases toward the light cylinder and becomes $c$. This is a frame-independent statement: velocity of a bead measured by any observer reaches $c$ on the light cylinder.

Eq. (5) can be integrated, assuming that a particle starts with velocity $v_0$ on the axis:

$$r_s = r = \sinh(\omega t_s/\gamma_0) \frac{v_0 \gamma_0}{\omega},$$

where $\gamma_0 \equiv E_0 = 1/\sqrt{1 - v_0^2}$. Thus, in terms of observer time $t_s$, motion of a particle is nearly exactly the same as if we were to solve the non-relativistic equation of motion $\ddot{r} = r \omega^2$. Qualitatively, the reason is that centrifugal force increases with $\gamma$, so that even though a particle becomes heavier, the centrifugal force increases proportionally. In terms of local observers time, a particle starting from the axis with velocity $v_0$ reaches light cylinder in finite time $\Delta t_s = \frac{2\omega}{\omega} \arcsin(1/(v_0 \gamma_0))$, beyond which (11) is inapplicable.

It is somewhat surprising that an observer located infinitely close to the axis of rotation, and thus moving with infinitely small velocity with respect to the stationary observer on the axis, measures a qualitatively different acceleration (positive for rotating, negative for observer on the axis). This is due to the fact that the unit frame vectors describing the physical experience of rotating observers are not Fermi-Walker transported along the world line; these observers are spinning as well as non-inertial.

C. Radially falling particle in Schwarzschild metric

As yet another way to look at this controversial issue, let us discuss briefly a very similar problem with a known answer: radial falling of a particle into Schwarzschild black hole. In coordinate time (11)

$$\begin{align*}
(\partial_t r)^2 &= (1 - 2M/r)^2(1 - (1 - 2M/r)E_0^2) \\
\partial_t^2 r &= (1 - 2M/r) \frac{M}{E_0^2 r^2} \left( \frac{6M}{r} - 3 + 2E_0^2 \right)
\end{align*}$$

(12)
Thus, coordinate acceleration reverses at \( r = 6M/(3 - 2E_0^2) \). For a particle starting at rest at infinity this reversal occurs at \( r = 6M \). But for \( E_0 > \sqrt{3/2} \), acceleration is always positive, directed away from the black hole. Thus, if at infinity a particle is shot towards black hole with \( \beta > 1/\sqrt{3} \) the coordinate acceleration is always directed away from black hole. If we were to take this mathematically correct result literally, it would mean that gravitational force becomes repulsive. Of course, the resolution of the "paradox" in this case is obvious, and is similar to the "reversal" of acceleration in rotating frame: one cannot use coordinate acceleration to infer physically relevant quantities; one needs to use a defined set of observers. For any observer at fixed radius,

\[
(\partial_t r)^2 = (1 - 2M/r)(1 - (1 - 2M/r)E_0^2)
\]

\[
\partial^2_t r = -\frac{M}{E_0^2 r^2}
\]  

acceleration is always negative, towards a black hole (proper acceleration is also negative \( \partial^2 r = -M/r^2 \)).

D. Photon motion

Finally, let us show that radial motion of a photon in rotating frame (e.g. along an optical fiber attached to the wire) experiences the same "deceleration" when measured in terms of coordinate time, as that of a relativistic particle. A condition \( ds = 0 \) gives in rotating frame

\[
\frac{dr_{ph}}{dt} = \sqrt{1 - \omega^2 r^2}
\]  

This has formal solution \( r_{ph} = (1/\omega) \sin \omega t \) for a photon emitted from \( r = 0 \) at \( t = 0 \). Surely, it does not mean that a photon bounces back from the light cylinder! Eq. (14) measures coordinate velocity of a photon, which is not surprisingly differs from \( c \).

III. CENTRIFUGAL EFFECT IN GENERAL RELATIVITY

It is straightforward to repeat the previous derivations in a coordinate frame rotating in Schwarzschild metric. Making a coordinate transformation \( \phi \rightarrow \phi' - \omega t \) and assuming that in rotating coordinates motion is strictly radial, \( d\phi' = d\theta = 0 \), the non-vanishing components
of the metric tensor are

\[ g_{00} = -\left(1 - \frac{2M}{r} - \omega^2 r^2 \right) \]
\[ g_{rr} = \left(1 - \frac{2M}{r}\right)^{-1} \] (15)

There are two light cylinders, inner and outer, solutions of 
\[ 1 - \frac{2M}{r} - \omega^2 r^2 = 0. \]
Since determinant of the metric tensor is \(< 0\) beyond the light cylinders, approximation of a rigidly rotating wire is inapplicable in those regions (formally it becomes applicable again inside the horizon). Inner and outer light cylinders coincide when \( \omega = \omega_{\text{ph}} \), angular velocity of a photon orbit \( r = 3M \). In case of Schwarzschild black hole it is required that \( \omega < 1/(3\sqrt{3}M) \).

A. Motion of a particle along the radial wire

The Hamilton - Jacobi equation,

\[ \frac{1}{1 - \frac{2M}{r} - \omega^2 r^2} (\partial_t S)^2 - \left(1 - \frac{2M}{r}\right) (\partial_r S)^2 = 1 \] (16)

gives

\[ (\partial_r r)^2 = \left(1 - \frac{2M}{r}\right) \left(1 - \frac{2M}{r} - \omega^2 r^2\right) \times \left(1 - \frac{1 - \frac{2M}{r} - \omega^2 r^2}{E^2_0}\right) \] (17)

For completeness we also give the relevant Christoffels

\[ \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2g_{tt}} \partial_r g_{tt} = \frac{r \omega^2 - M/r^2}{1 - 2M/r - r^2 \omega^2} \]
\[ \Gamma^r_{tt} = -\frac{1}{2g_{rr}} \partial_r g_{tt} = (1 - 2M/r)(r \omega^2 - M/r^2) \]
\[ \Gamma^r_{rr} = \frac{1}{2g_{rr}} \partial_r g_{rr} = -M/(r(r - 2M)) \] (18)

Transforming to a local stationary observer rotating with the wire

\[ dr_s = \frac{dr}{\sqrt{1 - \frac{2M}{r}}} \]
\[ dt_s = \sqrt{1 - \frac{2M}{r} - \omega^2 r^2} dt_s, \] (19)
we find

\[(\partial_t r_s)^2 = 1 - \frac{1 - \frac{2M}{r} - \omega^2 r^2}{E_0^2}\]

\[\frac{\partial^2 r_s}{\partial t_s^2} = -\sqrt{\frac{1 - \frac{2M}{r}}{E_0}} \left( \frac{E_0}{r^2} - \omega^2 \right) =
\]

\[-\sqrt{1 - \frac{2M}{r}} \frac{(1 - (\partial_t r_s)^2)^2}{1 - \frac{2M}{r} - \omega^2 r^2} \left[ \frac{M}{r^2} - r\omega^2 \right] \]

(20)

Again, velocity (20) is an invariant, as can be verified directly using the four-velocity of a bead (which follows from (17)) and \(U_2 = \{ -1/\sqrt{1 - \frac{2M}{r} - \omega^2 r^2}, 0 \} \), the four-velocity of a stationary observer. The first term in square brackets can be identified with gravitational acceleration, the second term - with centrifugal acceleration. Inside the light cylinders they do not change signs.

Finally, the equations of motion in terms of a proper time read

\[\left( \frac{\partial r}{\partial \tau} \right)^2 = \left( 1 - \frac{2M}{r} \right) \left( \frac{E_0^2}{1 - \frac{2M}{r} - \omega^2 r^2} - 1 \right)\]

\[\frac{\partial^2 r}{\partial \tau^2} = -\frac{M}{r^2} + \frac{(r - 3M)\omega^2 E_0^2}{\left( 1 - \frac{2M}{r} - \omega^2 r^2 \right)^2} = -\frac{M}{r^2} + (r - 3M)\omega^2 \]

(21)

where the last equality uses the fact that for circular orbit \(E_0 = -g_{00} = 1 - \frac{2M}{r} - \omega^2 r^2\). Eq. (21) shows the reversal of centrifugal force at the photon circular orbit \(r = 3M\). Thus, the proper observer sees a reversal at \(r = 3M\).

For Kerr black hole, no clear separation can be made between the effects of the wire rotation and rotation of space-time, so that a notion of a centrifugal force becomes not well defined, see [3, 12, 13, 14] for discussion.

IV. DISCUSSION

We have discussed a subtle special relativistic effect, the seeming reversal of centrifugal acceleration for relativistically moving particle. Straightforward analysis seems to indicate that centrifugal acceleration reverses its direction for fast moving particles, and becomes centrifugal deceleration, which seems to prevent a particle escaping from the system. This conclusion was drawn by Machabeli & Rogava [4] and applied to a number of astrophysical cases. It is mathematically correct, but physical interpretation that centrifugal acceleration
reverses in rotating frame is wrong, since the motion was defined in a frame-dependent way. It is the coordinate acceleration which reverses, while any physically relevant acceleration, e.g. measured by a set of stationary observers and/or proper acceleration remain directed away from the axis of rotation. As a result, a change in the velocity Machabeli & Rogava \cite{4} found reflects mostly the changing rate of time measured by locally stationary observers and not the motion of a bead. This is similar to freezing of motion on the horizon of a black hole for a free-falling particle, when considered in Schwarzschild coordinates.

The centrifugal acceleration controversy provides an excellent illustration of one of the principal issues in GR, that physical effects should be formulated in a frame-independent, but observer-dependent form (e.g. a set of ZAMO observers). For a defined set of observers, e.g. stationary with respect to the wire, a particle always accelerates and reaches the speed of light when crossing the light cylinder. The analogy between bead on a wire and free fall motion in Schwarzschild geometry is nearly exact: in both cases a particle reaches a speed of light while approaching the point where $g_{00} = 0$, light cylinder or horizon. The only difference is that in case of a rotating wire beyond the light cylinder the determinant of the metric tensor becomes negative, so that the system becomes unphysical, while the determinant of the metric tensor remains positive when crossing the horizon.

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