ARTIN APPROXIMATION OVER BANACH SPACES

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Abstract. We give examples showing that the usual Artin Approximation theorems valid for convergent series over a field are no longer true for convergent series over a commutative Banach algebra. In particular, we construct an example of a commutative integral Banach algebra \( R \) such that the ring of formal power series over \( R \) is not flat over the ring of convergent power series over \( R \).

1. Introduction

The classical Artin Approximation Theorem is the following:

**Theorem 1.1.** [1] Let \( F(x, y) \) be a vector of convergent power series over \( \mathbb{C} \) in two sets of variables \( x \) and \( y \). Assume given a formal power series solution \( \hat{y}(x) \) vanishing at 0,

\[
F(x, \hat{y}(x)) = 0.
\]

Then, for any \( c \in \mathbb{N} \), there exists a convergent power series solution \( y(x) \) vanishing at 0,

\[
F(x, y(x)) = 0
\]

which coincides with \( \hat{y}(x) \) up to degree \( c \),

\[
y(x) \equiv \hat{y}(x) \mod (x)^c.
\]

The main tools for proving this theorem are the implicit function theorem and the Weierstrass division theorem. But in the case the equations \( F(x, y) \) are linear in \( y \), this theorem is equivalent to the faithful flatness of the morphism \( \mathbb{C}\{x\} \to \mathbb{C}[x] \) (see [13, Example 1.4] for instance or [1] I. 3 Proposition 13). In fact the faithful flatness of this morphism comes from the fact that \( \mathbb{C}\{x\} \) is a Noetherian local ring. And the Noetherianity of \( \mathbb{C}\{x\} \) is usually proved by using the Weierstrass division theorem.

Another version of this theorem is the following one:

**Theorem 1.2.** [2] [10] Let \( F(x, y) \) be a vector of convergent power series over \( \mathbb{C} \) in two sets of variables \( x \) and \( y \). Then for any integer \( c \) there exists...
an integer $\beta$ such that for any given approximate solution $\overline{y}(x)$ at order $\beta$, $\overline{y}(0) = 0$,

$$F(x, \overline{y}(x)) \equiv 0 \text{ modulo } (x)^\beta,$$

there exists a formal power series solution $y(x)$ vanishing at 0,

$$F(x, y(x)) = 0$$

which coincides with $\overline{y}(x)$ up to degree $c$,

$$y(x) \equiv \overline{y}(x) \text{ modulo } (x)^c.$$

In particular this result implies that, if $F(x, y) = 0$ has approximate solutions at any order, then it has a formal (even convergent by Theorem 1.1) power series solution.

Let us mention that these results remain valid when we replace $\mathbb{C}$ by a complete valued field, or when we replace the ring of convergent power series over $\mathbb{C}$ by the ring of algebraic power series over a field. In fact these results remain true in the more general setting of excellent Henselian local rings by [12] (see [13] for a review of all these different results).

The aim of this note is to show that these results are no longer true when we replace $\mathbb{C}$ by a commutative Banach algebra over $\mathbb{R}$ or $\mathbb{C}$. In the first part we construct a commutative Banach algebra $R$ such that $R[t] \rightarrow R[[t]]$ is not flat, showing that Artin approximation theorem is not true for linear equations with coefficients in $R[t]$.

Let us mention here that $R[[t]]$ is flat over $R$, for a commutative ring $R$, if and only if $R$ is coherent (indeed $R[[t]]$ is a direct product of copies of $R$ - see [5, Theorem 2.1]). And there are several known examples of Banach algebras which are not coherent (in fact most of the known Banach algebras are not coherent; see for instance [9] or [8] and the references herein). But the flatness of $R[t] \rightarrow R[[t]]$ is a different property that is not related to the coherence of $R$.

In the second part we provide an example of one polynomial $F(y)$ with coefficients in $R[t]$, where $R$ is the Banach algebra of holomorphic functions over a disc, with the following property: $F(y)$ has approximate solutions up to any order but has no solution in $R[[t]]$. This shows that Theorem 1.2 does not hold for convergent power series over a Banach algebra. Let us mention that this example is a slight modification of an example of Spivakovsky related to a similar problem [15].

Nevertheless we mention that in the case where $R$ is a complete valuation ring of rank one (in particular a non-archimedean Banach algebra), Schoutens and Moret-Bailly proved several extensions of Theorems 1.1 and 1.2 (see [14] and [11]).

The note has been motivated by questions from Nefton Pali and Wei Xia.
2. A Banach algebra $R$ such that $R\{t\} \rightarrow R\|t\|$ is not flat

Let $K = \mathbb{R}$ or $\mathbb{C}$. We begin by the following definition of power series in countable many indeterminates:

**Definition 2.1.** Let $\mathbb{N}^{(N)}$ be the submonoid of $\mathbb{N}^N$ formed by the sequences whose all but finitely terms are 0. Let $(x_i)_{i\in\mathbb{N}}$ be a countable family of indeterminates. Then $K[[x_i]]_{i\in\mathbb{N}}$ is the set of series $\sum_{\alpha\in\mathbb{N}^{(N)}} a_\alpha x^\alpha$ where $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \cdots$. This former product is finite since $\alpha_i = 0$ for $i$ large enough.

This set is a commutative ring since the sum of sequences $N(\mathbb{N}) \times N(\mathbb{N}) \rightarrow N(\mathbb{N})$ has finite fibers (see [3, Chapter III, § 2, 11]). Let us mention that this ring is not the $(x)-adic completion of K[x]$, the ring of polynomials in the $x_i$ (see [17] for instance).

Let $x, y, z$ and $w_k$ for $k \in \mathbb{N}$ be indeterminates. For simplicity we denote by $w$ the vector of indeterminates $(w_0, w_1, \ldots)$. We denote by $K[x, y, z, w]$ the ring of polynomials in the indeterminates $x, y, z, w$.

For a polynomial $p = \sum_{k,l,m,\alpha} a_{k,l,m,\alpha} x^k y^l z^m w^\alpha \in K[x, y, z, w]$ we set

$$\|p\| := \sum_{k,l,m,\alpha} |a_{k,l,m,\alpha}|.$$

This is well defined because the sum is finite. This defines a norm on $K[x, y, z, w]$.

We denote by $K\{x, y, z, w\}$ the completion of $K[x, y, z, w]$ for this norm. This is the following commutative Banach algebra:

$$\left\{ \sum_{k,l,m,\alpha} a_{k,l,m,\alpha} x^k y^l z^m w^\alpha \mid \sum_{k,l,m,\alpha} |a_{k,l,m,\alpha}| < \infty \right\}$$

and the norm of an element $f := \sum_{k,l,m,\alpha} a_{k,l,m,\alpha} x^k y^l z^m w^\alpha$ is

$$\|f\| := \sum_{k,l,m,\alpha} |a_{k,l,m,\alpha}|.$$

In particular $K\{x, y, z, w\}$ is a subring of $K[x, y, z, w]$ for this norm.

We denote by $I$ the ideal of $K[x, y, z, w]$ generated by the polynomials $xw_0 - z^2$ and $yw_k - (k+1)xw_{k+1}$ for all $k \geq 0$.

The ideal $I K\{x, y, z, w\}$ is not closed since it is not finitely generated. Thus, we denote by $\overline{I}$ its closure. This is the set of sums

$$\sum_{k \in \mathbb{N}} f_k(x, y, z, w)$$

such that $f_k(x, y, z, w) \in I K\{x, y, z, w\}$ and $\sum_k \|f_k(x, y, z, w)\| < \infty$. 

Definition 2.2. We denote by $R$ the Banach $K$-algebra $K\{x, y, z, w\}/I$.

In order to denote that two series $f$ and $g \in K\{x, y, z, w\}$ have the same image in $R$, we write $f \equiv_R g$. The norm of the image $\overline{f}$ of an element $f \in K\{x, y, z, w\}$ is

$$\|\overline{f}\| = \inf_{g \in I} \|f + g\| = \inf_{g \in I} \|f + g\|.$$ 

Now we denote by $R\{t\}$ the ring of convergent series in the indeterminate $t$ with coefficients in $R$. We have the following result:

Proposition 2.3. The linear equation

$$(2.1) \quad (x - yt)f(t) = z^2$$

has a unique solution $f(t)$ in $R[[t]]$ and this solution is not convergent.

From this we will deduce the following result:

Theorem 2.4. The Banach $K$-algebra $R$ is an integral domain and the morphism $R\{t\} \rightarrow R[[t]]$ is not flat.

2.1. Proofs of Proposition 2.3 and Theorem 2.4. We begin by giving the following key result:

Lemma 2.5. $x$ is not a zero divisor in $R$.

Proof. First of all, we will determine a subset of $K\{x, y, z, w\}$ such that every element of $K\{x, y, z, w\}$ is equal modulo $I$ to a unique series of this subset. First we remark that

$$(2.2) \quad M_1 := yw_iw_j \equiv_R (i + 1)zw_{i+1}w_j \equiv_R \frac{i + 1}{j}yw_{i+1}w_{j-1} =: M_2$$

for all integers $i$ and $j$ with $i < j$. If $j = i + 1$ these two monomials are equal, otherwise the largest index of a monomial $w_j$ appearing in the expression of $M_2$ is strictly less than for $M_1$.

Now we have, for $i > 0$:

$$(2.3) \quad z^2w_i \equiv_R xw_0w_i \equiv_R \frac{1}{i}yw_0w_{i-1}.$$ 

A well chosen composition of these operations transforms any monomial of the form $Cx^ay^bz^cy^lw_0^{n_0} \ldots w_i^{n_i}$ into a monomial of the form $rCx^a'y^b'z^c'y^l'w_0'^{n_0'} \ldots w_j'^{n_j'}$ where $j$ is minimal and $r \in (0, 1]$.

By repeating these two operations we may reduce every monomial to a constant times one of the following monic monomials:

$$(2.4) \quad \begin{cases} 
  z^a x^ay^lw_0^{n_0} & \text{with } a > 1, l, n_0 \geq 0 \text{ and } \varepsilon \in \{0, 1\}, \\
  z^a y^lw_i^{n_i} & \text{with } l > 0, i > 0, n_i > 0 \text{ and } \varepsilon \in \{0, 1\}, \\
  z^a y^lw_i^{n_i} w_{i+1}^{n_{i+1}} & \text{with } l > 0, i \geq 0, n_i, n_{i+1} > 0 \text{ and } \varepsilon \in \{0, 1\}, \\
  z^a w_0^0 \ldots w_i^i & \text{with } n_i > 0 \text{ with } \varepsilon \in \{0, 1\},
\end{cases}$$

We denote by $E$ the subset of $K[x, y, z, w]$ of polynomials that are sums of monomials of (2.4) (up to multiplicative constants), and by $\overline{E}$ the closure
of $E$ in $\mathbb{K}\{x, y, z, w\}$, that is the set of convergent power series whose non zero monomials are those of $2^4$ (up to multiplicative constants). We have shown that every polynomial is equivalent to a polynomial of $E$ modulo $I$. To prove the unicity we proceed as follows.

We set

$$F_0 := xw_0 - z^2, F_{k+1} := yw_k - (k+1)xw_{k+1} \text{ for } k \geq 0$$

$$G_{k,l} := (l+1)yw_kw_{l+1} - (k+1)yw_lw_{k+1} \text{ for all } k < l.$$  

Then we consider the following monomial order: We define

$$x^a y^b z^c w^d, \quad x^a y^b z^c w^d > x'^a y'^b z'^c w'^d,$$

if

$$a + k + l + \sum_{i} \alpha_i > a' + k' + l' + \sum_{i} \alpha'_i, \text{ or } a + k + l + \sum_{i} \alpha_i = a' + k' + l' + \sum_{i} \alpha'_i$$

and $(l, a, k, \alpha_n, \ldots, \alpha_0) >_{\text{lex}} (l', a', k', \alpha'_n, \ldots, \alpha'_0)$

where $>_{\text{lex}}$ denotes the lexicographic order. That is, we first compare the total degree of two monomials, then we order the indeterminates as

$$z > x > y > w_l > w_k \text{ for all } l < k.$$ 

We claim that $\{F_j, G_{k,l}\}_{j,k \in \mathbb{N}, l > k}$ is a Gröbner basis of $I$ for this order. In order to prove this, we only need to compute the $S$-polynomials of the elements of this set of polynomials, and then their reduction (see [6] for the terminology). This is Buchberger’s Algorithm which is very classical in the Noetherian case. The case of polynomial rings in countably many indeterminates works identically, cf. [7 Proposition 1.13] for instance. The only S-polynomials we have to consider are those of polynomials whose leading terms are not coprime, that is, for $l > k$,

$$S(F_{k+1}, F_{l+1}); S(G_{k,l}, F_{l+1}); S(G_{k,l}, F_k).$$

We have $S(F_{k+1}, F_{l+1}) = G_{k,l}$. Moreover

$$S(G_{k,l}, F_{l+1}) = y(yw_kw_l - (k+1)xw_{l+1}).$$

This leading term of $S(G_{k,l}, F_{l+1})$ is $-(k+1)y^2w_kw_{l+1}$, and it is equal to $y(F_{k+1}w_l - yw_kw_l)$. Therefore $S(G_{k,l}, F_{l+1}) = F_{k+1}yw_l$.

Finally we have

$$S(G_{k,l}, F_k) = kx((l+1)yw_kw_{l+1} - (k+1)yw_kw_{k+1}) + (l+1)yw_{l+1}(yw_{k+1} - kxw_k)$$

$$= (l+1)y^2w_{k+1}w_{l+1} - k(k+1)yw_kw_{k+1}.$$ 

Its leading term is $-k(k+1)yw_kw_{k+1}$ and it is divisible by the leading term of $F_{k+1}$. The remainder of the division of $S(G_{k,l}, F_k)$ by $F_{k+1}$ is

$$(l+1)y^2w_{k-1}w_{l+1} - ky^2w_kw_l = yG_{k-1,l}.$$
Therefore the reductions of these S-polynomials is always zero, hence the family \( \{F_j, G_k\}_{j,k \in \mathbb{N}, j > k} \) is a Gröbner basis of \( I \). Thus, the initial ideal of \( I \) is generated by the monomials
\[ z^2, xw_{k+1}, yw_{k+1} \text{ for } 0 \leq k < l. \]
Therefore every polynomial of \( \mathbb{K}[x, y, z, w] \) is equivalent modulo \( I \) to a unique polynomial of \( E \).

Now let \( f \in \mathbb{K}\{x, y, z, w\} \). We can write \( f = \sum_{n \in \mathbb{N}} C_n x^{a_n} y^{b_n} z^{c_n} w^{\alpha_n} \) where the \( C_n \) are in \( \mathbb{K}^* \). In particular \( \sum_n |C_n| < \infty \). For every \( n \in \mathbb{N} \), there is a unique \((a'_n, b'_n, c'_n, \alpha'_n)\) and a unique \( r_n \in (0, 1] \) such that
\[ C_n x^{a_n} y^{b_n} z^{c_n} w^{\alpha_n} - r_n C_n x^{a'_n} y^{b'_n} z^{c'_n} w^{\alpha'_n} \in I \]
and \( x^{a_n} y^{b_n} z^{c_n} w^{\alpha_n} \) has one the forms given in (2.4). Now, for every \( n \in \mathbb{N} \), we set
\[ g_n := \sum_{k=0}^{n-1} r_k C_k x^{a'_k} y^{b'_k} z^{c'_k} w^{\alpha'_k} + \sum_{k \geq n} C_k x^{a_k} y^{b_k} z^{c_k} w^{\alpha_k}. \]
In particular we have that \( P_n := f - g_n \in I \) and the sequence \((g_n)_n\) converges in \( \mathbb{K}\{x, y, z, w\} \) to the series \( g = \sum_{k \in \mathbb{N}} r_k C_k x^{a'_k} y^{b'_k} z^{c'_k} w^{\alpha'_k} \in \mathbb{K}\{x, y, z, w\} \).

Therefore the sequence \((P_n)_n\) converges in \( \mathbb{K}\{x, y, z, w\} \), and its limits is in \( \mathcal{T} \). Therefore, every power series of \( \mathbb{K}\{x, y, z, w\} \) can be written as a sum of a power series in \( \mathcal{T} \) and a convergent power series whose monomials are as in (2.4) (up to multiplicative constants).

We remark that, by repeating (2.2) \( \lfloor \frac{j-i-1}{2} \rfloor \) times, we have
\[ yw_i w_j \equiv_R r yw_{i+\lfloor \frac{j-i}{2} \rfloor} w_{j-\lfloor \frac{j-i}{2} \rfloor} \]
for some constant \( r \in (0, 1] \). Moreover applying (2.3) reduces by 2 the degree in \( z \) of a monomial. Therefore, a monomial of the form
\[ C x^a y^b z^c w_1^{\alpha_1} \cdots w_j^{\alpha_j} \]
of total degree \( d = a + b + c + \sum_k \alpha_k \), is not equal to a monomial involving only the indeterminates
\[ x, y, z, \text{ and } w_i \text{ for } i < \frac{j - \frac{d}{2}}{2}. \]
Moreover (2.2) and (2.3) transforms monomials into monomials of the same degree since \( I \) is generated by homogeneous binomials. Therefore, given a monomial \( M \) among those of (2.4) (up to some multiplicative constant), there is finitely many monomials that are equal to \( M \) modulo \( I \).

Now let \( f \in \mathcal{E} \cap \mathcal{T}, f = \sum_{(a,b,c,\alpha)} f_{(a,b,c,\alpha)} x^a y^b z^c w^\alpha \). Let us fix such \((a, b, c, \alpha)\) such that \( x^a y^b z^c w^\alpha \) is one of the monic monomials of (2.4). There is only a finite number of distinct monomials that are equal to \( f_{(a,b,c,\alpha)} x^a y^b z^c w^\alpha \) modulo \( I \). Let us denote them by
\[ C_1 x^{a_1} y^{b_1} z^{c_1} w^{\alpha_1}, \ldots, C_N x^{a_N} y^{b_N} z^{c_N} w^{\alpha_N}. \]
We can remark that there is only a finite number of \( F_i \) that have a monomial that divides at least one of the following monic monomials
\[
(2.5) \quad x^a y^b z^c w^\alpha, x^a y^b z^c w^\alpha n_1, \ldots, x^a y^b z^c w^\alpha n_N.
\]

We denote them by \( F_{i_1}, \ldots, F_{i_p} \). Because \( f \in \overline{I} \), we can write \( f = \sum_{i \in \mathbb{N}} f_i F_i \) where the \( f_i \) are in \( \mathbb{K}[x, y, z, w] \). For every \( i \in \{1, \ldots, p\} \) we remove from \( f_i \) all the monomials that do not divide one of the monomials \( (2.5) \), and we denote by \( f_i' \) the resulting polynomial. Then we have that
\[
P := \sum_{i=1}^p f_i F_i \in I.
\]

By construction the coefficients of the monomials \( (2.5) \) in the expansion of \( P \) are the corresponding coefficients in the expansion of \( f \), that is
\[
f_{(a, b, c, \alpha), 0, \ldots, 0}
\]
respectively. Therefore, the coefficient of \( x^a y^b z^c w^\alpha \) in the expansion of the unique \( Q \in E \) such that \( Q \equiv_R P \), is equal to \( f_{(a, b, c, \alpha)} \) because no other monomial than those listed in \( (2.5) \) (up to some multiplicative constants) is equivalent to a monomial of the form \( C x^a y^b z^c w^\alpha \) where \( C \in \mathbb{K}^* \). But \( Q = 0 \) since \( P \in I \), thus \( f_{(a, b, c, \alpha)} = 0 \). Hence \( f = 0 \) and \( E \cap \overline{I} = 0 \).

Therefore every series of \( \mathbb{K}\{x, y, z, w\} \) is equivalent modulo \( \overline{I} \) to a unique series of \( E \).

Now take \( f \in \mathbb{K}\{x, y, z, w\} \) such that \( x \equiv_R 0 \). We can write \( f = xp(x, y, z, w_0) + q(y, z, w) \) and assume that the monomials in the expansion of \( xp(x, y, z, w_0) + q(y, z, w) \) are only those of \( (2.4) \). Then
\[
x^2 p(x, y, z, w_0) + xq(y, z, w) \equiv_R 0.
\]

The representation of \( x^2 p(x, y, z, w_0) + xq(y, z, w) \) as a sum of monomials as in \( (2.6) \) has the form
\[
x^2 p(x, y, z, w_0) + xq(y, z, w, 0) + \overline{q}(y, z, w) = 0
\]
where \( \overline{q}(y, z, w) \) is the series obtained from \( xq(y, z, w) - xq(y, z, w, 0) \) by replacing the monomials as follows (using the two previous operations \( (2.2) \) and \( (2.3) \)):
\[
(2.7) \begin{cases}
x z^e y^i w_{n_i}^{m_i} & \rightarrow \frac{1}{i+1} z^e y^{l+1} w_{i+1}^{m+1} w_{i+1}^{m+1-1}, \text{ if } i > 0 \\
x z^e y^i w_{n_i}^{m_i+1} w_{i+1} & \rightarrow \frac{1}{i+1} z^e y^{l+1} w_{i+1}^{m+1} w_{i+1}^{m+1-1}, \text{ if } i > 0 \\
x z^e w_{n_0}^{m_0} \ldots w_{n_i}^{m_i} & \rightarrow C z^e y_{m_j}^{w_{j+1}} w_{j+1}^{m_{j+1}} \text{ or } C z^e y_{m_j}^{m_{j+1}} \\
\text{for } i > 0 \text{ and } n_i > 0, & \text{for some } C \in \mathbb{K}, |C| \leq 1, j \geq 0
\end{cases}
\]

Indeed for the third monomial we have
\[
x z^e w_{n_0}^{m_0} \ldots w_{n_i}^{m_i} \equiv_R \frac{1}{i+1} z^e y_{n_0}^{m_0} \ldots w_{n_i-1}^{m_i+1} w_{n_i}^{m_i-1}
\]
and this monomial on the right side can be transformed into a monomial of the form \( C z^e y_{m_j}^{w_{j+1}} w_{j+1}^{m_{j+1}} \) or \( C z^e y_{m_j}^{m_{j+1}} \) for some \( C \in \mathbb{K} \), \( |C| \leq 1 \), and \( j \geq 0 \).
by using the two operations \((2.2)\) and \((2.3)\) on monomials.
This shows that the three types of monomials that we obtain after multipli-
cation by \(x\) are all distinct, that is the map defined by \((2.7)\) is injective. By
\((2.6)\) we have \(q(y, z, w) = 0\), therefore \(q(y, z, w) - q(y, z, w_0) = 0\).
Moreover, again by \((2.6)\), we have
\[x^2 p(x, y, z, w_0) + xq(y, z, w_0) = 0.\]
This shows that \(x^2 p(x, y, z, w_0) + xq(y, z, w_0) = 0\). Therefore \(x\) is not a zero
divisor in \(R\). □

**Proof of Proposition 2.3.** Let \(f(t) \in R[[t]]\) such that
\((x - yt)f(t) = z^2.\)
By writing \(f = \sum_{k=0}^{\infty} f_k t^k\) with \(f_k \in R\) for every \(k\), we have
\[xf_0 = z^2\]
\[xf_k - yf_{k-1} = 0 \quad \forall k \geq 1.\]
Thus
\[xf_0 = z^2 = xw_0\]
so \(x(f_0 - w_0) = 0\) and \(f_0 = w_0\) by Lemma 2.5. Then we will prove by
induction on \(k\) that \(f_k = k! w_k\) for every \(k\). Assume that this is true for an
integer \(k \geq 0\). Then we have
\[xf_{k+1} = yf_k = k! yw_k = (k + 1)! xw_{k+1}.\]
Hence \(x(f_{k+1} - (k + 1)! w_{k+1}) = 0\) and \(f_{k+1} = (k + 1)! w_{k+1}\) by Lemma 2.5.
Therefore the only solution of
\[(x - yt)f(t) = z^2\]
is the series \(\sum_{k=0}^{\infty} k! w_k t^k\), and this one is divergent because \(\|w_k\| = 1.\)
This holds because in every element of \(I\), the monomial \(w_k\) has coefficient 0.
□

Now we can give the proof of Theorem 2.4:

**Proof of Theorem 2.4.** Since \(x\) is not a zero divisor in \(R\) by Lemma 2.5, the
localization morphism
\[R \rightarrow R_1/x\]
is injective. But \(R_1/x\) is isomorphic to \(K\{x, y, z\}_{1/x}\) since in \(R_1/x\) we have
\[w_0 = z^2/x \quad \text{and} \quad \forall k \geq 0, w_k = \frac{1}{k!} y^2 x^{k+1}.\]
But \(K\{x, y, z\}_{1/x}\) is an integral domain (this is a localization of the integral
domain \(K\{x, y, z\}\)), therefore so is \(R\).

Now assume that the morphism \(R\{t\} \rightarrow R[[t]]\) is flat. By [10, Theorem
applied to the linear equation \((x - yt)F - z^2 G = 0\), there exist an integer \(s \geq 1\), and convergent series
\[a_1(t), \ldots, a_s(t), b_1(t), \ldots, b_s(t) \in R\{t\}\]
such that
\[(x - yt)a_i(t) - z^2 b_i(t) = 0 \text{ for every } i,\]
and formal power series
\[h_1(t), \ldots, h_s(t) \in R[t]\]
such that
\[f(t) = \sum_{i=1}^{s} a_i(t)h_i(t), \quad 1 = \sum_{i=1}^{s} b_i(t)h_i(t).\]
Indeed the vector \((f(t), 1)\) is a solution of the linear equation
\[(x - yt)f(t) - z^2 g(t) = 0\]
with \(f(t) := \sum_{k=0}^{\infty} k! w_k t^k\).
Then
\[\tilde{g}(t) := \sum_{i=1}^{s} b_i(t)h_i(0) = 1 + t\varepsilon(t)\]
for some \(\varepsilon(t) \in R\{t\}\). Since 1 is a unit of \(R\), \(1 + t\varepsilon(t)\) is a unit in \(R\{t\}\).
Set \(\tilde{f}(t) := \sum_i a_i(t)h_i(0)\). By (2.8), \((\tilde{f}(t), \tilde{g}(t))\) is a solution of the equation
\[(x - yt)\tilde{f}(t) - z^2 \tilde{g}(t) = 0.\]
Since \(\tilde{g}(t)\) is a unit in \(R\{t\}\) we have
\[(x - yt)\tilde{f}(t)\tilde{g}(t)^{-1} = z^2.\]
This contradicts Theorem 2.3. Therefore \(R\{t\} \rightarrow R[[t]]\) is not flat.

3. An Example concerning the strong Artin approximation theorem

Let \(n\) be a positive integer, \(x = (x_1, \ldots, x_n)\) and \(\rho > 0\). We set \(K = \mathbb{R}\) or \(\mathbb{C}\). Then
\[B^\alpha_\rho := \left\{ f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \mid \|f\|_\rho := \sum_{\alpha \in \mathbb{N}^n} |a_\alpha|\rho^{\alpha} < \infty \right\}\]
is a Banach space equipped with the norm \(\| \cdot \|_\rho\). Of course \(K[x] \subset B^\alpha_\rho\).

Remark 3.1. We do not have
\[B^\alpha_\rho[t] \cap K\{x, t\} = B^\alpha_\rho\{t\}.\]
For instance, the power series

\[ f = \sum_{k \in \mathbb{N}} x_1^k t^k \]

is a convergent power series in \((x, t)\), belongs to \(B_{2}^n[t]\), but

\[ \sum_{k} \|x_1^k\|_2 \tau^k = \sum_{k} 2^k \tau^k = \infty \]

for every \(\tau > 0\). Therefore \(f \notin B_{n}^0\{t\}\).

We provide two examples based on an example of Spivakovsky concerning the extension of Theorem 1.2 to the nested case (see [15]).

**Example 3.2.** Let \(n = 1\) and set

\[ F(x, t, y_1, y_2) := xy_1^2 - (x + t)y_2^2 \in B_{\rho}\{t\}[y_1, y_2]. \]

Let

\[ \sqrt{1 + t} = 1 + \sum_{n \geq 1} a_n t^n \in \mathbb{Q}\{t\} \]

be the unique power series such that \((\sqrt{1 + t})^2 = 1 + t\) and whose value at the origin is 1. For every \(c \in \mathbb{N}\) we set \(y_2^{(c)}(t) := x^c\) and \(y_1^{(c)}(t) := x^c + \sum_{n=1}^{c} a_n x^{c-n} t^n \in B_{\rho}\{t\}\). Then

\[ F(x, t, y_1^{(c)}(t), y_2^{(c)}(t)) \in (t)^{c+1}. \]

On the other hand the equation \(f(x, t, y_1(t), y_2(t)) = 0\) has no solution \((y_1(t), y_2(t)) \in B_{\rho}\{t\}^2\) but \((0, 0)\). Indeed let us denote by \(T_0\) the Taylor map at 0:

\[ T_0 : B_{\rho}\{t\} \rightarrow \mathbb{K}[x, t]. \]

If \(f(x, t, y_1(t), y_2(t)) = 0\) then

\[ xT_0(y_1(t))^2 - (x + t)T_0(y_2(t))^2 = 0. \]

But since \(\mathbb{K}[x, t]\) is a unique factorization domain, this equality implies that \(T_0(y_1(t)) = T_0(y_2(t)) = 0\), hence \(y_1(t) = y_2(t) = 0\).

This shows that there is no \(\beta : \mathbb{N} \rightarrow \mathbb{N}\) such that for every \(y(t) \in B_{\rho}\{t\}^2\) and every \(k \in \mathbb{N}\) with

\[ F(x, t, y(t)) \in (t)^{\beta(k)} \]

there exists \(\tilde{y}(t) \in B_{\rho}\{t\}^2\) such that

\[ F(x, t, \tilde{y}(t)) = 0 \]

and \(\tilde{y}(t) - y(t) \in (t)^k\).
Example 3.3. We can modify a little bit the previous example to construct a $F$ as before that does not depend on $t$. We set

$$G(x, y_1, y_2, y_3):= xy_1^2 - (x + y_3)y_2^2 \in B_\rho[y_1, y_2, y_3].$$

For every $c \in \mathbb{N}$ we set

$$y_2^{(c)}(t) := x^c, \quad y_1^{(c)}(t) := x^c + \sum_{n=1}^c a_n x^{c-n} t^n \quad \text{and} \quad y_3^{(c)}(t) := t \in B_\rho \{t\}.$$ 

Then

$$G(x, y_1^{(c)}(t), y_2^{(c)}(t), y_3^{(c)}(t)) \in (t)^c.$$ 

Now if $\tilde{y}(t) \in B_\rho \{t\}^3$ satisfies $G(x, \tilde{y}(t)) = 0$ and

$$\tilde{y}(t) - y(t) \in (t)^2$$

then $\bar{y}_3(t) = x + t + \epsilon(t)$ with $\epsilon(t) \in (t^2)$. Thus $x + \bar{y}_3(t)$ is an irreducible power series in $x$ and $t$, and it is coprime with $x$. By the same argument based on the Taylor map as in Example 3.2 the relation

$$x \bar{y}_1(t)^2 - (x + t + \epsilon(t)) \bar{y}_2(t)^2 = 0$$

implies that $\bar{y}_1(t) = \bar{y}_2(t) = 0$.

This shows that there is no $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $y(t) \in B_\rho \{t\}^3$ and every $k \in \mathbb{N}$ with

$$G(x, y(t)) \in (t)^{\beta(k)}$$

there exists $\tilde{y}(t) \in B_\rho \{t\}^3$ such that

$$G(x, \tilde{y}(t)) = 0$$

and $\tilde{y}(t) - y(t) \in (t)^k$.

References

[1] M. Artin, On the solutions of analytic equations, *Invent. Math.*, 5, (1968), 277-291.
[2] M. Artin, Algebraic approximation of structures over complete local rings, *Publ. Math. IHES*, 36, (1969), 23-58.
[3] N. Bourbaki, Algèbre, Chapitres 1 à 3, Hermann, Paris, 1970.
[4] N. Bourbaki, Algèbre commutative, Chapitres 1 à 4, Masson, Paris, 1985.
[5] S. U. Chase, Direct products of modules, *Trans. Amer. Math. Soc.*, 97, (1960), 457-473.
[6] D. Cox, J. Little, and D. OShea, Ideals, varieties, and algorithms. Undergraduate-Texts in Mathematics. Springer-Verlag, New York, second edition, 1997.
[7] K. Iima, Y, Yoshino, Gröbner bases for the polynomial ring with infinite variables and their applications, *Comm. Algebra*, 37, (2009), no. 10, 3424-3437.
[8] M. Hickel, Noncohérence de certains anneaux de fonctions holomorphes, *Illinois J. Math.*, 34, (1990), no. 3, 515-525.
[9] W. S. McVoy, L. A. Rubel, Coherence of some rings of functions, *J. Func. Anal.*, 21, (1976), 76-87.
[10] H. Matsumura, Commutative Ring Theory, *Cambridge studies in advanced mathematics*, 1989.
[11] L. Moret-Bailly, An extension of Greenberg’s theorem to general valuation rings, *Manuscripta math.*, 139, n 1 (2012), 153-166.
[12] D. Popescu, General Néron desingularization and approximation, *Nagoya Math. J.*, 104, (1986), 85-115.
[12] P. Ribenboim, Fields: algebraically closed and others, *Manuscripta Math.*, **75**, (1992), 115-150.
[13] G. Rond, Artin Approximation, *J. Singul.*, **17**, (2018), 108-192.
[14] H. Schoutens, Approximation properties for some non-noetherian local rings, *Pacific J. Math.*, **131**, (1988), 331-359.
[15] M. Spivakovsky, Non-existence of the Artin function for Henselian pairs, *Math. Ann.*, **299**, (1994), 727-729.
[16] J. J. Wavrik, A theorem on solutions of analytic equations with applications to deformations of complex structures, *Math. Ann.*, **216**, (1975), 127-142.
[17] A. Yekutieli, On flatness and completion for infinitely generated modules over Noetherian rings, *Comm. Algebra*, **39**, (2011), 4221-4245.
[18] A. Yekutieli, Flatness and Completion Revisited, *Algebras and Representation Theory*, **21**, Issue 4, (2018), 717-736.

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