Renormalization aspects of chaotic strings

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Abstract. Chaotic strings are a class of non-hyperbolic coupled map lattices, exhibiting a rich structure of complex dynamical phenomena with a surprising correspondence to physical contents. In this paper we introduce different types and models for chaotic strings, where $2B$-strings with finite length are considered in detail. We demonstrate possibilities to extract renormalized quantities, which are expected to describe essential properties of the string.

1. Introduction
Coupled map lattices (CMLs) were introduced in seminal papers by Kaneko and Kapral [1, 2] as a relatively simple model of spatiotemporal chaos. They exhibit a rich structure of complex dynamical phenomena [3, 4, 5, 6, 7] and their applications range from hydrodynamical flows, turbulence, chemical reactions, biological systems to quantum field theories, see e.g. reviews in [3, 4]. Of particular interest are CMLs where the local map exhibits chaotic behaviour. For weakly coupled hyperbolic local maps it can be proved [8, 9, 10, 11] that the dynamics is ergodic and there exists a smooth invariant density of the entire CML. However, the case of nonhyperbolic local maps, e.g. one-dimensional maps for which the slope is equal to zero at some point, is more complicated from a mathematical point of view and fewer analytical results are known [12, 13, 14, 15, 16, 17, 18].

Chaotic strings are one-dimensional CMLs of diffusively coupled Chebychev maps introduced by Christian Beck [18]. These CMLs are non-hyperbolic, since the Chebychev maps have an extremum with vanishing slope. They can be used to model vacuum fluctuations [4, 19, 20, 21] by replacing the noise in stochastically quantized field theories by a deterministic but chaotic dynamics [22, 23, 24]. A possible physical embedding is to associate the vacuum energy of chaotic strings with dark energy [25]. Surprisingly, these models could also improve our understanding about the observed mass spectrum of the standard model (see Ref. [4] for more details).

This paper, however, will not deal with the physical aspects but will merely concentrate on the mathematical properties of chaotic strings. The main object of study is an interesting selfsimilar dependence of the invariant density on the coupling constant $a$ of the CML. For weakly coupled $N$-th order Chebyshev maps [18, 24, 25, 26, 27] it was shown that certain scaling properties with respect to the coupling $a$ can be calculated perturbatively [19, 20]. However, the fine structure of the coupling dependence for important characteristics of the chaotic string as for instance the self energy could not be explained by perturbative methods. It has been illustrated numerically that the fine structure does occur only for CMLs of dimension 1, and not on lattices of higher dimension [28]. In Ref. [29] chaotic strings with different boundary conditions were studied.
Figure 1. Invariant density for the closed 2B-string with $L = 10\,000$ at $a = 0.03$ numerically. It was shown that some short strings (with lengths between 3 and 5 lattice sites) show the same fine structure as long strings with lengths of the order of 10\,000 sites, while for other short strings the fine structure had a totally different shape.

2. Overview
The dynamics of chaotic strings is given by the recurrence relation

$$\phi_{n+1} = (1 - a)T_N(\phi_n^i) + s \frac{a}{2}(T_M(\phi_{n+1}^i) + T_M(\phi_n^{i-1}))$$

where $T_N(x) = \cos(N \arccos(x))$ is the $N$-th order Chebychev polynomial of the first kind, $a \in [0, 1]$ is a coupling constant, $s = \pm 1$ distinguishes between diffuse and anti-diffuse coupling, and $n$ resp. $i$ are the temporal and spatial indices. The strings are assumed to have infinite spatial extent. The field $\phi_n^i$ takes values in the interval $[-1, 1]$. Chaotic strings are divided up into type A strings with $M = N$ and type B strings with $M = 1$, denoted by $NA$ and $NB$, respectively. The most important cases are $N = 2, 3$ corresponding to

$$T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x$$

Due to the ergodicity of the map (2.1), local averages can be given by

$$\langle f(\phi) \rangle_a = \lim_{k \to \infty} \frac{1}{k} \sum_{n=1}^{k} f(\phi_n^i) = \int_{-1}^{1} d\phi \rho_a(\phi) f(\phi)$$

with $\rho_a(\phi)$ being the invariant density of the field variable $\phi$. As an example, in Fig. 1 we show the invariant density for the 2B-string for a closed chaotic string of length $L = 10\,000$. The density is independent of $i$, since the map (2.1) is invariant under shifts in $i$. The interesting objects to study are the coupling dependences of averages $\langle V \rangle_a$ of formal potentials defined by $T_N(x) = -\frac{d}{dx}V_N(x)$. For $N = 2, 3$ one has

$$V_2(\phi) = -\frac{2}{3} \phi^3 + \phi, \quad V_3(\phi) = -\phi^4 + \frac{3}{2} \phi^2$$
The quantity $\langle V_N \rangle_a$ can be interpreted as (formal) self energy of the field $\phi$. The dependence of the expectation values $\langle V_2 \rangle_a$ and $\langle V_3 \rangle_a$ for the $2B$- and $3B$-strings on $a$ are shown in Fig. 2 in a double logarithmic plot.

In the case of non-coupled maps ($a = 0$) the recurrence relation (2.1) reduces effectively to the Chebyshev map $\phi^{n+1} = T_N(\phi^n)$ for which the dynamics is well understood. In this case the process is ergodic and $\phi$ behaves as a pseudorandom variable [3, 4] with the invariant density

$$\rho_0(\phi) = \frac{1}{\pi \sqrt{1 - \phi^2}} \quad (2.2)$$

which is independent of $N$.

2.1. Scaling behaviour

In general, the invariant density depends on the coupling in a highly non-trivial way. However, it was shown analytically using perturbative methods, that self energies exhibit a scaling property for weak couplings [19, 20]. To be more precise, the self energies behave in the scaling region $0 \leq a \leq 0.039$ as

$$\langle V \rangle_a - \langle V \rangle_0 = f_N(\ln(a)) \sqrt{a}$$

where $f_N$ is periodic function with the period $2 \ln(N)$. The function $f_N$ describes the fine structure of the self energies, as shown in Fig. 3 for the $2B$-string in the scaling interval $a \in [1.5, 6.0] \times 10^{-4}$.

2.2. Boundary conditions

The chaotic strings are taken to have infinite spatial extent, but this case is usually complicated to study in practice. The number of possible sites $i$ is called the length $L$ of the string. It is expected that for smaller couplings the spatial correlations should be weaker and, therefore, the
Figure 3. Self energy of the 2B-string in the scaling interval \(a \in [1.5, 6.0] \times 10^{-4}\)

Invariant density should be less sensitive to the length of the string. The effect of imposing different boundary conditions was studied in Ref. [29] where it was proved numerically that invariant density is relatively insensitive to the length of the string if \(L \geq 3\). There are many different possible boundary conditions:

- **Cyclic boundary condition** (closed strings) for which \(\phi^n_1 = \phi^n_{L+1}\). This boundary condition preserves translational invariance. However, low-\(L\) models can have stable fixed points which strongly change the shape of the invariant density. The fixed points disappear when \(L \to \infty\). The fine structure of the self energies can be reproduced for \(L \geq 3\).

- **Chaotic boundary condition** (open strings) for which the endpoints of the string are uncoupled from their neighbours and evolve as \(\phi^{n+1} = T_N(\phi^n)\). This boundary condition mimics the behaviour of the infinite string with negligible long range spatial correlations. The case \(L \geq 3\) works well in the scaling region as it reproduces the fine structure.\(^1\) Since the endpoints evolve independently, this boundary condition also avoids the appearance of fixed points.

- **Fixed boundary conditions** (fixed strings) for which the endpoints of the string are given fixed values. These boundary conditions are probably the simplest, but numerical studies show they are able to reproduce the fine structure only for \(L \geq 5\).

- **Random boundary conditions** (random strings) for which the endpoints of the string are independent random variables distributed by Eq. (2.2). These boundary conditions approximate the behaviour of \(\phi^{n+1} = T_N(\phi^n)\), i.e. the chaotic boundary conditions, with the important difference that they also neglect temporal correlations of the endpoints. This model does not reproduce the fine structure for \(L = 3\).

These boundary conditions could also be used to study the effects of different correlations of the invariant density. In Fig. 4 we show the self energy of the 2A-string for two different boundary conditions and two different lengths \(L\).

\(^1\) Note that due to a numerical error, in Ref. [29] is was erroneously stated that the open string of length 3 does not reproduce the fine structure. This will be corrected in an erratum to Ref. [29].
2.3. Perturbative method

The perturbative method developed in Refs. [19, 20] is based on open strings of length \( L = 3 \) given by

\[
\phi^{n+1} = (1 - a)T_2(\phi^n) + \frac{a}{2}(\phi^n_+ + \phi^n_-), \quad \phi^{n+1}_\pm = T_2(\phi^n_\pm) \tag{2.3}
\]

Note that this map does not distinguish between strings of type \( A \) and \( B \). The idea is to analytically understand the origin of the scaling property. By choosing the initial distribution to be the uncoupled density (2.2), after a few \( p \) iterations the scaling of the invariant density \( \rho_a^{(p)}(\phi) \) (like the one in Fig. 5) close to the boundaries \( \phi = \pm 1 \) can be obtained. As a consequence this method explains the scaling behaviour of the self energy. However, the perturbative method is unable to reproduce the fine structure.

3. Renormalization

In looking at the gradual changes of the invariant density while decreasing the coupling \( a \), we became aware of a possible reason for the fine structure of the self energy of the \( 2B \)-string. Decreasing the coupling, more and more peaks appear from the left hand border of the interval (i.e. at \( \phi = -1 \)) and move to the right, while the distance between existing peaks decreases. As a result a selfsimilar structure develops [19, 20]. However, a closer look reveals an opposite movement. Before the peaks are generated at the left hand side, they arrive as wide peaks from the right. When such a newly-formed peak (in Fig. 5 found close to \( \phi = -0.15 \)) passes existing peaks, it modulates the distribution and could thereby cause a minimum in the self energy.

3.1. Partitioning the interval \([-1,1]\)

In extending the perturbative method based on Eq. (2.3) onto the whole interval \([-1,1]\) we could construct the self energy as a three-fold integral of a polynomial of very high degree. The integration is taken over the central field \( \phi \) and the fields \( \phi_\pm \) of the next neighbours, while the
Figure 5. Comparison of the rescaled $\rho^p_a(\phi)$ ($p = 5, 6, 7$) with the numerically obtained distribution. In this plot the numerical distribution is shifted by $+0.2$ for better visibility.

The integrand is generated by the $(p + 1)$-fold iteration of Eq. (2.3). For $a = 0$, the integrand is just the $(p + 1)$-fold iteration of the Chebyshev map $T_{2}^{p+1}(\phi) = T_{2p+1}(\phi)$, displayed on the left hand side of Fig. 6 for $p = 5$. For $a \neq 0$, there are slight deviations from this shape. As shown on the right hand side of Fig. 6 for $a = 0.00122$, some of the maxima (if not all) are damped, i.e. reach lower values than for the uncoupled integrand. It turns out that the heights of the maxima are directly related to the positions of the peaks of the invariant density. If the coupling increases, the maximum for instance at $\phi = 0$ will disappear, and the two neighbouring minima will merge and cause the created minima to raise. This corresponds to a peak moving to the left. The same will happen later on to the maxima at $\pm \sin(\pi/4) = \pm 1/\sqrt{2}$ if the coupling is increased further. But what is the role of disappearing maxima? The positions of the maxima constitute a partition of the interval $[-1, 1]$. If one of the maxima disappears, this partition will be changed.

3.2. The model function

In the following we were able to construct two characterizing functions, the draft function $\tilde{d}^{(k)}(t_0; \phi_+, \phi_-)$ and the blunt function $b_{p,k}(\phi_+, \phi_-)$, that measure the gradual displacement resp. damping of the maxima. For this, the maxima are divided into classes with common behaviour. For $a = 0$ the maxima in class $k$ are located at $\sin(\pi t_0/2)$, where $t_0 = \pm(2n + 1)/2^k$, $n = 0, 1, \ldots, 2^k - 1$) [30]. Using the draft and blunt functions it’s possible to construct a model function

$$f(\Delta t; t_0) = \int_{-1}^{1} \rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_- \times$$
$$\times \cos\left(2^p \sqrt{\pi^2 \left(\Delta t - a\tilde{d}^{(k)}(t_0; \phi_+, \phi_-)\right)^2 + 4^{1-k}ar_2(-T_{2k}(\phi_+), -T_{2k}(\phi_-))}\right)$$
that depends on the coupling $a$. This is a quite precise approximation for the integrand $f(t_0 + \Delta t)$ in the neighbourhood of $t_0$. The draft is given by

$$d^{(k)}(t_0; \phi_+, \phi_-) = \frac{1}{2} \left( d^{(k)}(t_0; \phi_+) + d^{(k)}(t_0; \phi_-) \right)$$

with

$$d^{(k)}(t_0; \phi_\pm) = d^{(k)}(t_0) + \sum_{l=0}^{k-1} T_{2l}(-\phi_\pm) d^{(k)}_l(t_0)$$

and

$$d^{(k)}(t_0) = 2^{-k} d^{(k)}_{-1}(t_0) - \sum_{l=0}^{k-1} \left( 2^{1-k+l} + \cos(2l_\pi t_0) \right) d^{(k)}_l(t_0), \quad d^{(k)}_l(t_0) = \frac{1}{2^{l} \pi \sin(2^l \pi t_0)}$$

and the blunt is characterized by

$$r_2(\phi_+, \phi_-) := \frac{1}{2} \left( r_2(\phi_+) + r_2(\phi_-) \right), \quad r_2(\phi) := \frac{1}{2} \sum_{p=0}^{\infty} \frac{1 - T_{2p}(\phi)}{4^p}$$

3.3. The renormalization procedure

The invariant density of the chaotic string is obtained in the limit $p \to \infty$. In order to perform this limit in a scale invariant way, certain parameters have to be replaced by reduced parameters which stay unaltered under this limit. These parameters are the reduced order $\hat{p} = p - 2k$, the reduced coupling $\hat{a} = 4^{p-k-a}$, and the reduced deviation $\hat{\Delta}$. In terms of these parameters the model function can be rewritten as

$$f^{(\hat{p})}(\hat{\Delta} t; t_0) := f(2^{-p} \Delta t; t_0) = \int \rho_0(\phi_+) d\rho_+ \rho_0(\phi_-) d\phi_- \times$$

$$\times \cos \left( \sqrt{\pi^2 \left( \Delta \hat{\phi}_0 d^{(k)}(t_0; \phi_+, \phi_-) \right)^2 + 2\hat{a} \hat{r}_2(\hat{\Delta} t_0; \phi_+, \phi_-)} \right)$$

Figure 6. Integrand with $p = 5$ for $a = 0$ (left hand side) and $a = 0.00122$ (right hand side) at $\phi_+ = \phi_- = 1$ as a function of $\phi \in [-1, 1]$. 
Figure 7. A fish-type diagram containing bosons (twiggles) and fermions (straight lines)

Finally, in performing the limit $p \to \infty$ while keeping $\hat{p}$ constant, one can separate the high-frequency blunt from the low-frequency draft, leading to a (yet unexplained) effective blunt of $8/3$ and a model function

$$\hat{f}(\hat{p})(\Delta \hat{t}; t_0) = \int \rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_- \cos \left( \pi^2 \left( \Delta \hat{t} - 2 - \hat{p} \hat{a} \tilde{d}(k)(t_0; \phi_+, \phi_-) \right)^2 + \frac{8}{3} \hat{a} \right)$$

which in turn can be expressed via Fourier transformation as

$$\hat{f}(\hat{p})(\Delta \hat{t}; t_0) = \int \rho_0(\phi_+)d\phi_+ \rho_0(\phi_-)d\phi_- \left[ \cos \left( \pi \left( \Delta \hat{t} - 2 - \hat{p} \hat{a} \tilde{d}(k)(t_0; \phi_+, \phi_-) \right) \right) ight]$$

$$- \int_{-1}^{+1} \frac{4 \hat{a} d\omega}{3\sqrt{8\hat{a}(1-\omega^2)/3}} J_1 \left( \sqrt{8\hat{a}(1-\omega^2)/3} \right) \cos \left( \pi \omega \left( \Delta \hat{t} - 2 - \hat{p} \hat{a} \tilde{d}(k)(t_0; \phi_+, \phi_-) \right) \right)$$

An interesting observation is the appearance of the Bessel function $J_\lambda(x)$ of degree $\lambda = 1$. A Bessel function of the same degree appears as integration measure for $D = 4$ space-time dimensions (see e.g. Ref. [31]). This observation can provide us with some clues about the physical embedding of the chaotic string dynamics into generalized versions of quantum field theory [4]. Could it be possible to consider $\lambda = 1$ as a reference to four-dimensional space-time without the explicit emergence of space-time itself? If we could interpret the integrand of the Bessel function as a correlator in (four-dimensional) configuration space, the integration over $\phi_+$ and $\phi_-$ should mean the insertion of two vertices into the diagram which are connected by a propagator, i.e. a virtual particle transition. Still, one has to be careful, since the analogy is far from being exact. However, the fish-type diagram in Fig. 7 is a compactified version of the Feynman web that was used in Ref. [4] to give a physical meaning to the chaotic string dynamics.

4. Conclusions and Outlook

We gave guidelines for a detailed analysis of the dynamics of chaotic strings, taking as an example the $2B$-string. It turned out that an open string of length $L = 3$ proves useful in simplifying numerical and analytical calculations for small couplings, i.e in the scaling region. We have introduced a model function which allows to analyse the scaling behaviour in the stochastic limit $q \to \infty$ using a renormalization procedure. Renormalized quantities are the reduced degree $\hat{p}$, the reduced coupling $\hat{a}$ and the reduced deviation $\Delta \hat{t}$. Results obtained by us earlier via a
perturbative approach can be reproduced for larger values of $\hat{p}$ while nonperturbative effects are seen for smaller (even negative) values of $\hat{p}$. These nonperturbative effects are expected to explain the fine structure of the self energy of the $2B$-string.

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