Information bound for entropy production from detailed fluctuation theorem

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Fluctuation theorems impose fundamental bounds in the statistics of the entropy production, with the second law of thermodynamics being the most notorious. Using information theory, we quantify the information of entropy production and find an upper tight bound as a function of its mean from the strong detailed fluctuation theorem. The bound is given in terms of a maximal distribution, a member of the exponential family with nonlinear argument. We show that the entropy produced by heat transfer using a bosonic mode at weak coupling reproduces the maximal distribution in a limiting case. The upper bound is extended to the continuous domain and verified for the heat transfer using a levitated nanoparticle. Finally, we show that a composition of qubit swap engines satisfies a particular case of the maximal distribution regardless of its size.

Introduction - Fluctuation theorems (FTs) have far reaching consequences in nonequilibrium thermodynamics. As experiments probe smaller setups, entropy production, Σ, is seen as a random variable. In this situation, FTs impose constraints in the distribution \( P(\Sigma) \) by requiring that not only a positive entropy production is more likely to be observed, \( \Sigma \geq 0 \), but also quantifying its chance with respect to the time-reversed event \([1–4]\).

Among some variations of FTs \([5–11]\), we focus our attention in the strong detailed fluctuation theorem (DFT):

\[
P(\Sigma) = \frac{e^\Sigma}{\int e^\Sigma d\Sigma}.
\]

which results from time symmetric protocols in the framework of the exchange fluctuation theorem (EFT) \([12–21]\). In the EFT case, a set of charges \( (Q_1, ..., Q_N) \) are observed in a finite time experiment and, with their respective affinities \( A_i \), they satisfy the EFT, \( P(Q_1, Q_2, ..., Q_N)/P(-Q_1, -Q_2, ..., -Q_N) = \exp(\sum_i A_i Q_i) \). The entropy production random variable is given by \( \Sigma = \sum_i A_i Q_i \). Focusing on the actual distribution of \( \Sigma \) \([12, 13]\), one defines \( P(\Sigma) := \int \delta(\Sigma - \sum A_i Q_i)P(Q_1, ..., Q_N)dQ_1, ..., dQ_N \), which satisfies (1).

Although the DFT (1) still leave plenty of room for a variety of possible distributions \( P(\Sigma) \), some fundamental bounds are imposed in their statistics \([12, 13, 15]\) as well as generic properties \([22, 23]\). For instance, (1) implies the integral fluctuation theorem \( e^{-\Sigma} = 1 \), which in turn results in the second law \( \Sigma \geq 0 \) from Jensen’s inequality. In this case, if the second law is a fundamental bound derived from the FT, perhaps other bounds might also play important roles.

Following this idea, other bounds were obtained recently such as the Thermodynamic Uncertainty Relation (TUR) \([24–28]\), also generalized and obtained directly from the EFT \([12, 13]\). In the tightest form, it reads \( \text{var}(Q_i)/\langle Q_i \rangle^2 \geq f(\Sigma_i) \), for some known function \( f(x) \). From (1), the underlying TUR is also valid for the entropy production itself, \( \text{var}(\Sigma)/\langle \Sigma \rangle^2 \geq f(\Sigma) \). Thus, TUR is seen as another bound concerning the statistics of \( \Sigma \), such as the second law. For the TUR, the uncertainty of \( \Sigma \) (and the currents \( Q_i \)) is quantified in terms of the signal-to-noise ratio.

In this context, it seems opportune to analyse the random variable \( \Sigma \) with other tools that account for uncertainty, and a successful one comes from Information Theory \([29, 30]\). After its debut, the theory was readily recognized as of great importance to statistical mechanics but, in the words of Jaynes, “the exact way in which it should be applied has remained obscure” \([31]\). Notable applications in physics were built in the works that followed \([32–36]\). In particular, a important development was to recognize the Kullback-Leibler divergence \([37]\) (related to Shannon’s entropy) as a Lyapunov function of Markov chains \([38, 39]\), a typical scenario found in the weak coupling approximation of thermodynamics.

In this paper, we use concepts of information theory to tackle the following problem: for a given mean \( \Sigma \), how much information, or surprise, should one expect in the distribution \( P(\Sigma) \) that satisfies the DFT (1)? As it turns out, the information of \( P(\Sigma) \) is upper bounded in terms of the mean, \( \Sigma \). More precisely, for a given discrete support \( s = \{\Sigma_i\} \), we quantify the information of the entropy production in terms of its Shannon’s entropy:

\[
H[\Sigma] := -\sum_i P(\Sigma_i) \log P(\Sigma_i),
\]

here simply called information, where the sum is over \( \Sigma_i \in s \). Then, we find a tight upper bound for (2) from the DFT (1), namely

\[
H[\Sigma] \leq M(\Sigma).
\]

The bound is given in terms of the information (2) of the following maximal distribution:

\[
P_M(\Sigma) = \frac{1}{Z(\lambda)} \exp\left(\frac{\Sigma}{2} - \lambda \frac{\Sigma}{2} \tanh\left(\frac{\Sigma}{2}\right)\right),
\]

defined over the discrete support \( s \), which also can be written in terms of its mean, \( M(\langle \Sigma \rangle) = \log Z(\lambda) + (\lambda - 1)/2 \langle \Sigma \rangle \), using the constraints \( Z(\lambda) = \frac{1}{2} \sum_i \exp(\Sigma_i/2 - \lambda \Sigma_i/2 \tanh(\Sigma_i/2)) \). For continuous distributions \( P(\Sigma) \), the upper bound also holds for differential entropy, \( h[\Sigma] = -\int P(\Sigma) \log P(\Sigma) d\Sigma \), with full support in the real line. Notice that the maximal distribution (4) is a member of the exponential family \([40]\), but it has a nonlinear argument that seems unusual at first glance. As a matter of fact, we argue that this nonlinear structure is rather intuitive when combining the information maximization with the DFT (1), as discussed below.

Formalism - In this section, we find the upper bound for the information (2) of the entropy production. We consider
a general point mass function $P(\Sigma)$ over a discrete support, $s = \{\Sigma_i\} \subset \mathbb{R}$, i.e., $P(\Sigma) > 0$ for all $\Sigma_i \in s$ and $P(\Sigma) = 0$ otherwise. Without loss of generality, we consider $0 \in s$. Additionally, $P(\Sigma)$ satisfies normalization, $\sum_i P(\Sigma_i) = 1$, and known mean $\sum_i \Sigma_i P(\Sigma_i) = \langle \Sigma \rangle$, where summation is assumed over $s$. Finally, $P(\Sigma)$ also satisfies a detailed fluctuation theorem (1) in $s$, which means the support is symmetric (for all $\Sigma_i \in s$, we have $-\Sigma_i \in s$). In the text, we use the terms distribution and point mass function (pmf) interchangeably.

First, define new variables $\sigma = \text{sign}(\Sigma)$, for $\Sigma \neq 0$, and $\varepsilon = |\Sigma|$ with support $\{-1,+1\}$ and $\geq 0$ respectively, and distributions $p(\sigma)$ and $q(\varepsilon)$. Using Bayes theorem, one has $P(\Sigma) = p(\sigma|\varepsilon)q(\varepsilon)$, where the fluctuation theorem (1) defines $p(\sigma|\varepsilon)$ uniquely

$$p(\sigma|\varepsilon) = \frac{e^{\sigma \varepsilon / 2}}{e^{\varepsilon / 2} + e^{-\varepsilon / 2}},$$

for $\varepsilon > 0$ and $p(\sigma|0) = 1$. Notice that $\langle \Sigma \rangle = \langle \sigma \varepsilon \rangle = \langle \varepsilon \tanh(\varepsilon / 2) \rangle$, using (5). Also notice that $\sigma \tanh(\Sigma / 2) = e\tanh(\varepsilon / 2)$ by definition of $\varepsilon$, which leads to the following identity

$$\langle \Sigma \rangle = \langle \sigma \tanh(\Sigma / 2) \rangle,$$

which will be useful later, in analogy to similar treatments [12, 15]. Now we find the upper bound of the information (2) using calculus of variations. Usually, in the MaxEnt recipe, we have constraints (for instance, $\langle \Sigma \rangle$), but (1) is not integral: it is a detailed relation that couples the negative and positive parts of the support. This symmetry allows the information (2) to be written as a functional of $P(\Sigma)$ for $s > 0 := \{\Sigma_i \in s | \Sigma_i > 0\}$:

$$H[\Sigma] = -\sum_{\Sigma} P(\Sigma) \left[ \log P(\Sigma)(1 + e^{-\Sigma}) - \Sigma e^{-\Sigma} \right] - P_0 \log P_0,$$

for $P_0 = P(0)$. The same idea is applied to the constraints, resulting in the following functionals,

$$\sum_{\Sigma} \Lambda(1 - e^{-\Sigma})P(\Sigma) = \langle \Sigma \rangle,$$

and

$$\sum_{\Sigma} (1 + e^{-\Sigma})P(\Sigma) + P_0 = 1.$$
1/(k_b T_1). The system is prepared in equilibrium with temperature T_1 and at t = 0 it is placed in thermal contact with the second reservoir (at temperature T_2). Using a two point measurement scheme (at t = 0 and t > 0), in the absence of work, the entropy production is given in terms of the energy variation \[ \Sigma = \Delta \beta \Delta E \] and the distribution \[ \Sigma \] slightly above it. The system is prepared in thermal equilibrium \[ \langle \Sigma \rangle = \Sigma \] for several values of \[ \Sigma \] and \[ \Sigma \] is uniquely defined from the normalization and mean constraints. Notice that (16) satisfies (1). The information (2) of the distribution (16) is given by

\[ H[\Sigma] = \log A(\alpha) - \frac{\langle \Sigma \rangle}{2} + \alpha \frac{\langle \Sigma \rangle^2}{2}, \]

where \[ \langle \Sigma \rangle \] may be written in closed form in terms of \( \alpha \) using the geometric series. In Fig. 1, we compute the information (17) and the mean, \( \langle \Sigma \rangle \), for several values of \( \alpha \) for the distribution (16), then we plot the curve \( H[\Sigma] \) vs. \( \langle \Sigma \rangle \) (blue curve). We repeat the same process with the maximal distribution (4), computing its information and mean \( \langle \Sigma \rangle \) for several values of \( \lambda \), then we also plot \( M(\Sigma) \) vs. \( \langle \Sigma \rangle \) (dashed curve). In Fig.1, notice that the upper bound is always above the information of the system's entropy production by a small amount. Actually, the entropy production in this case has the general form \[ \Sigma = -\Delta \beta \Delta E = AQ \] where \( A = \Delta \beta \) and \( Q = -x \text{hom} \). For the limiting case, \( \Delta \beta \Delta E \gg 1 \), one has \( |\pm \Sigma| = m(\Delta \beta \Delta E) \gg 1 \) and the following approximation holds

\[ \frac{\Sigma_m}{2} \tanh \frac{\Sigma_m}{2} \approx \frac{|\Sigma_m|}{2}, \]

valid for \( |\Sigma_m| \), which makes the exponent of the maximal distribution (4) similar to the observed (16), \( P_m(\Sigma) \approx 1/2 \exp(\Sigma/2 - \lambda |\Sigma|/2). \)

We conclude that, depending on the support, ie, the interplay between quantum energy levels and affinities, the maximal distribution is approximately attained for entropy production in the heat transfer using a bosonic mode at weak coupling.

**Application to a Gaussian distribution** - The Gaussian distribution has a broad range of applications also in the context of entropy production \[ [23, 45] \]. The DFT (1) allows one to write its standard deviation as a function of the mean and the resulting pdf is

\[ P(\Sigma) = \frac{1}{2 \sqrt{\pi} \lambda} \exp\left( -\frac{(\Sigma - \langle \Sigma \rangle)^2}{4 \langle \Sigma \rangle} \right). \]

where it clearly satisfies (1), \[ \int P(\Sigma) d\Sigma = 1 \] and \[ \int \Sigma P(\Sigma) d\Sigma = \langle \Sigma \rangle \). Therefore, the differential entropy (12) for the Gaussian case (19) must satisfy the upper bound

\[ h[\Sigma] = \frac{1}{2} \log(4 \pi e \langle \Sigma \rangle) \leq m(\langle \Sigma \rangle), \]

which is a general inequality for the function \( m(\langle \Sigma \rangle) \) defined in (13). This inequality is depicted in Fig. 3.

**Application to a levitated nanoparticle** - The highly under-damped limit of the Langevin equation represents the dynamics of a levitated nanoparticle \[ [46-48] \]. Consider the Langevin dynamics with potential \( U(x) = mkx^2/2 \), in one dimensional for simplicity. The particle’s dynamics is given by

\[ \ddot{x} + \Gamma \dot{x} + \Omega_0^2 x = \frac{1}{m} F_{\text{ituc}}(t), \]

for position \( x(t) \), with Gaussian noise \( \langle F_{\text{ituc}}(t)F_{\text{ituc}}(t') \rangle = 2mT \delta(t - t') \), where \( \Gamma \) is a friction coefficient, \( m = 1 \) is the particle mass, \( T \) is the reservoir temperature and \( k = \Omega_0^2 \). Define a \( d \) dimensional system energy, \( E = \sum_{i=1}^{d} (p_i^2/2 + U(x_i)) \), with momentum \( p_i = \dot{x}_i \), the following SDE was obtained
for the total energy in the highly underdamped limit [46], \( \Omega_0 \gg \Gamma \):

\[
dE = -\Gamma (E - \frac{f}{2} T) \, dt + \sqrt{2 \Gamma T E} \, dW_t,
\]

with degrees of freedom \( f = 2d \) and \( dW_t \) is a Wiener increment. Using the same setup of Fig. 1, but now with the levitated nanoparticle as working medium for the heat transfer, one defines the entropy production as \( \Sigma = \Delta \beta \Delta E \), where \( P_\epsilon(\Delta E) = \int P(E_1) \Pi_\epsilon(E_1 \rightarrow E_2) \delta(\Delta E - (E_2 - E_1)) \). The propagator \( \Pi_\epsilon \) is known [47, 49] for the SDE (22), which yields the following distribution

\[
P(\Sigma) = \frac{1}{B(\alpha)} \exp\left(\frac{\Sigma}{2}\right) \exp(d+1/2) K_{d+1/2}(\alpha | \Sigma),
\]

defined for the real line, for constant \( \alpha \) defined in terms of parameters \( (T_1, T_2, \Gamma t) \), \( B(\alpha) \) is a normalization constant and \( K \) is the modified Bessel function of the second kind. In Fig. 2, the information \( H(\Sigma) \) and the mean \( \langle \Sigma \rangle \) of the distribution (23) are numerically computed for several values of \( \alpha \) and different sizes \( d = 1, 2, 3 \). For each value of \( \alpha \), we plot the blue curves \( H(\Sigma) \) vs. \( \langle \Sigma \rangle \), one for each \( d = 1, 2, 3 \). The same process is repeated for the upper bound (13), resulting in the dashed curve. For comparison the Gaussian distribution was included (red curve), using (20). Inspecting Fig. 2, one sees that the observed entropy production is close to the upper bound, especially for the case \( d = 1 \), also with good agreement in the tails (inset). Larger systems \( d = 2, 3 \) and the Gaussian case have lower information and misses the bound by a larger amount for \( \Sigma \gg 1 \).

**Application to swap engines** - We consider a pair of qubits with energy gaps \( \epsilon_A \) and \( \epsilon_B \) initially prepared in thermal equilibrium, \( p(\sigma) = \exp(\sigma \beta \epsilon) / (\exp(\beta \epsilon) + \exp(-\beta \epsilon)) \), for \( \sigma = \pm 1 \), \( \beta \in \{\beta_1, \beta_2\} \) and \( \epsilon \in \{\epsilon_A, \epsilon_B\} \), with reservoirs at temperature \( T_1 \) and \( T_2 \), respectively. A two point energy measurement is performed before and after a swap operation [21] takes place, defined as \( |xy\rangle \rightarrow |yx\rangle \), for \( x,y \in \{-,+\} \) and the entropy production in the process is given [13, 21] by

\[
\Sigma = \beta_1 \Delta E_A + \beta_2 \Delta E_B,
\]

where \( \Delta E_A = E_A^f - E_A^i \), \( \Delta E_B = E_B^f - E_B^i \) are the variations of energy measurements before and after the swap. Therefore, in this measurement scheme, the three possible outcomes are \( \Sigma \in s = \{0, \pm 2 \alpha \} \), for \( 2\alpha = 2(\beta_2 \epsilon_B - \beta_1 \epsilon_A) \). The distribution \( P(\Sigma) \) for the swap operation follows from initial state distributions directly:

\[
P(\Sigma) = \frac{1}{Z_0} \exp\left(\frac{\Sigma}{2}\right),
\]

for \( \Sigma \in s \), which satisfies the DFT (1) and it is a particular case of the maximal distribution (4) for \( \lambda = 0 \). In this case, the information reads \( H(\Sigma) = M(\Sigma) = \log Z_0 - \langle \Sigma \rangle / 2 \), for \( Z_0 = \Sigma \exp(\Sigma_0 / 2) \). This is a trivial example of the maximal distribution, because any distribution satisfying the DFT in a support \( s \) with only three values always has the form (25).

However, we show that larger swap engines preserve the form (25) with nontrivial supports. For instance, consider the double swap engine formed by four qubits with energy gaps \( \epsilon_A, \epsilon_B, \epsilon_C, \epsilon_D \) as arranged in Fig. 3. Qubits \( A \) and \( B \) (\( C \) and \( D \)) are in thermal equilibrium with a reservoir at temperature \( T_1 \) (\( T_2 \)). A swap operation takes places between qubits \( A \) and \( C \) (pair 1). Simultaneously, another swap is performed with qubits \( B \) and \( C \) (pair 2). We choose the energy gaps such that \( r := \epsilon_A / \epsilon_C = \epsilon_B / \epsilon_D \), i.e., the independent engines \( (A,C) \) and \( (B,D) \) operate similarly. Additionally, the independent engines are related by \( \epsilon_A / \epsilon_B = 2 / 3 \). For simplicity, let \( \beta_1 / \beta_2 = 1 / 2 \). The entropy production now is given by

\[
\Sigma = \beta_1 (\Delta E_A + \Delta E_B) + \beta_2 (\Delta E_C + \Delta E_D) = \Sigma_1 + \Sigma_2,
\]

where \( \Sigma_i \) is the entropy production of the independent pair \( i = 1, 2 \) (24). One can easily check that the supports of \( \Sigma_1 \) and \( \Sigma_2 \) are \( s_1 = \{0, \pm 2b\} \) and \( s_2 = \{0, \pm 3b\} \), and their composition results in nine different outcomes for (26), \( s = \{0, \pm b, \pm 2b, \pm 3b, \pm 5b\} \), all multiples of \( b = \beta_2 \epsilon_C (1 - r / 2) \). In this specific case, the distribution is also given by (25), which follows from \( P(\Sigma) = P_1(\Sigma_1)P_2(\Sigma_2) = \exp(\Sigma_1 / 2) C_1 \cdot \exp(\Sigma_2 / 2) C_2 = C \cdot \exp(\Sigma / 2) \), which is a maximal distribution. In summary, a composite microscopic swap engine, now with nine possible outcomes in the support of the entropy production, still behaves as a particular case of the maximal distribution. This is particularly interesting, since the swap operation is the optimal unitary operation that outputs the most work per cycle [21]. The argument is easily generalized for larger compositions of swap engines, for suitable choices of energy gaps.

**Other applications** - It is worth noticing that the strong
DFT (1) also holds for deterministic dynamical ensembles [9, 12]. In this case, one has $N$ particles described by a deterministic trajectory in the phase space, where randomness is encoded in the initial distribution. It was proved that, for some assumptions in the distribution and dynamics, the system satisfies (1). Therefore, the upper bound (13) is expected to hold for such systems.

**Conclusions** - In this paper, we used information to quantify the uncertainty in the entropy production. We obtained an upper tight bound for a given mean in terms of a proposed maximal distribution, $P_M(\Sigma)$. We argued that the non-linearity observed in $P_M(\Sigma)$ is a result of a symmetry derived from the DFT. Then, we verified the behavior of some relevant distributions in comparison to the maximal. Namely, transferring heat between two reservoirs using a bosonic mode results in a distribution close to $P_M(\Sigma)$, specially in a limiting case. In the same setup, a levitated nanoparticle yielded a distribution close to the maximal, but now in the continuous domain. For the composite swap qubit engine, we found that a case of the maximal distribution is always observed.

In this context, analyzing the role of mutual information to quantify dependencies between thermodynamic variables is left for future research.

[1] U. Seifert, Reports on progress in physics. Physical Society (Great Britain) 75, 126001 (2012), arXiv:1205.4176v1.
[2] M. Campisi, P. Hänggi, and P. Talkner, Reviews of Modern Physics 83, 771 (2011).
[3] C. Bustamante, J. Liphardt, and F. Ritort, Physics Today 58, 43 (2005).
[4] C. Jarzynski, The European Physical Journal B 64, 331 (2008).
[5] C. Jarzynski, Physical Review Letters 78, 2690 (1997).
[6] C. Jarzynski, Journal of Statistical Physics 98, 77 (2000).
[7] G. E. Crooks, Journal of Statistical Physics 90, 1481 (1998).
[8] G. Gallavotti and E. G. D. Cohen, Journal of Statistical Physics 80, 931 (1995).
[9] D. J. Evans, E. G. D. Cohen, and G. P. Morriss, Physical Review Letters 71, 2401 (1993).
[10] P. Hänggi and P. Talkner, Nature Physics 11, 108 (2015), arXiv:1311.0275.
[11] K. Saito and Y. Utsumi, Physical Review B - Condensed Matter and Materials Physics 78, 115429 (2008).
[12] Y. Hasegawa and T. Van Vu, Phys. Rev. Lett. 123, 110602 (2019).
[13] A. M. Timpanaro, G. Guarnieri, J. Goold, and G. T. Landi, Physical Review Letters 123, 90604 (2019), arXiv:1904.07374.
[14] D. J. Evans and D. J. Searles, Advances in Physics 51, 1529 (2002).
[15] N. Merhav and Y. Kafri, Journal of Statistical Mechanics: Theory and Experiment 2010, 10.1088/1742-5468/2010/12/P12022.
[16] R. García-García, D. Domínguez, V. Lecomte, and A. B. Kolton, Physical Review E - Statistical, Nonlinear, and Soft Matter Physics 82, 30104 (2010), arXiv:1007.1435.
[17] B. Cleuren, C. Van den Broeck, and R. Kawai, Physical Review E 74, 21117 (2006).
[18] U. Seifert, Physical Review Letters 95, 40602 (2005), arXiv:0505368 [cond-mat].
[19] C. Jarzynski and D. K. Wójcik, Physical Review Letters 92, 230602 (2004).
[20] D. Andrieux, P. Gaspard, T. Monnai, and S. Tasaki, New Journal of Physics 11, 43014 (2009).
[21] M. Campisi, J. Pekola, and R. Fazio, New Journal of Physics 17, 35012 (2015).
[22] I. Neri, É. Roldán, and F. Jülicher, Phys. Rev. X 7, 11019 (2017).
[23] S. Pigolotti, I. Neri, É. Roldán, and F. Jülicher, Phys. Rev. Lett. 119, 140604 (2017).
[24] A. C. Barato and U. Seifert, Physical Review Letters 114, 158101 (2015).
[25] T. R. Gingrich, J. M. Horowitz, N. Perunov, and J. L. England, Physical Review Letters 116, 120601 (2016), arXiv:1512.02212.
[26] K. Macieszczak, K. Brandner, and J. P. Garrahan, Physical Review Letters 121, 130601 (2018), arXiv:1803.01904.
[27] M. Polettini, A. Lazarescu, and M. Esposito, Phys. Rev. E 94, 52104 (2016).
[28] P. Pietzonka and U. Seifert, Physical Review Letters 120, 190602 (2017), arXiv:1705.05817.
[29] C. E. Shannon and W. W. The Mathematical Theory of Communication (University of Illinois Press, Urbana, IL, 1949).
[30] T. M. Cover and T. A. J. Elements of Information Theory (Wiley, New York, 1991).
[31] E. T. Jaynes, Physical Review 106, 620 (1957), arXiv:arXiv:1011.1669v3.
[32] V. Vedral, Rev. Mod. Phys. 74, 197 (2002).
[33] A. Peres and D. R. Terno, Rev. Mod. Phys. 76, 93 (2004).
[34] G. Adesso, N. Datta, M. J. W. Hall, and T. Sagawa, Journal of Physics A: Mathematical and Theoretical 52, 320201 (2019).
[35] K. Maruyama, F. Nori, and V. Vedral, Rev. Mod. Phys. 81, 1 (2009).
[36] J. M. R. Parrondo, J. M. Horowitz, and T. Sagawa, Nature Physics 11, 131 (2015), arXiv:0903.2792.
[37] R. A. Kullback, S. and Leibler, Annals of Mathematical Statistics 22, 79 (1951).
[38] F. Schlögl, Zeitschrift für Physik A Hadrons and nuclei 243, 303 (1971).
[39] J. Schnakenberg, Reviews of modern physics 48, 571 (1976).
[40] Robert W. Keener, Theoretical Statistics, 1st ed. (Springer-Verlag New York, New York, 2010) p. 538.
[41] J. P. Santos, G. T. Landi, and M. Paternostro, Physical Review Letters 118 (2017).
[42] D. Salazar, A. Macêdo, and G. Vasconcelos, Physical Review E 99 (2019), 10.1103/PhysRevE.99.022133.
[43] T. Denzler and E. Lutz, , 1arXiv:arXiv:1807.03572v1.
[44] N. A. Sinitsyn, Journal of Physics A: Mathematical and Theoretical 44, 405001 (2011).
[45] H.-M. Chun and J. D. Noh, Phys. Rev. E 99, 12136 (2019).
[46] J. Gieseler, B. Deutsch, R. Quidant, and L. Novotny, Phys. Rev. Lett. 109, 130630 (2012).
[47] J. Gieseler and J. Millen, Entropy 20, 326 (2018), arXiv:1805.02927.
[48] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Reviews of Modern Physics 86, 1391 (2014).
[49] D. Salazar and S. Lira, Journal of Physics A: Mathematical and Theoretical 49 (2016), 10.1088/1751-8113/49/46/465001.