Tangent structures: sector-forms, jets and connections

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Abstract. To define a higher order connection on a fibered manifold one can use the sections of nonholonomic jet prolongations. However, a more natural approach seems to be the one assuming the structure of a higher-order tangent bundle and using White’s sector-forms on these bundles.

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1. Iterations of a tangent functor

The tangent functor $T$ assigns to a smooth manifold $M$ a tangent bundle $TM \to M$ (first order tangent bundle), and to a mapping $\varphi : M_1 \to M_2$ between two smooth manifolds $M_1$, $M_2$ its differential (the tangent mapping $T\varphi : TM_1 \to TM_2$ as a fibered morphism). If the functor $T$ is applied $k$ times, then the $k$-th iteration assigns to a manifold $M$ its $k$-th order tangent bundle $T^kM \to M$ and to a mapping $\varphi$ its $k$-th differential $T^k\varphi : T^kM_1 \to T^kM_2$. In short we have $T^k : \begin{cases} M \leadsto T^kM, \\ \varphi \leadsto T^k\varphi. \end{cases}$

The total space $T^kM$ of a fibre bundle $T^kM \to M$, as a manifold, has a dimension of $2^kn$ where $n = \dim M$. Each time the tangent functor $T$ is applied, the dimension is doubled.

Proposition 1. The bundle $T^kM \to T^{k-1}M$ is equipped with the structure of a $k$-fold vector bundle. Particularly, $T^kM$ admits $k$ different projections to $T^{k-1}M$, 

$$\rho_s := T^{k-s}\pi_s : T^kM \to T^{k-1}M,$$

where $\pi_s$ is the natural projection $T^sM \to T^{s-1}M$, $s = 1, 2, \ldots, k$. Each projection defines a vector bundle with basis $T^{k-1}M$ and the total space composed of $2^{k-1}n$-dimensional vector spaces as fibers.

For every order $l$, $l = 2, \ldots, k$, the fibers of $T^lM \to T^{l-1}M$ have common $n$-dimensional intersections as equalizers of all possible projections. Using this fact, on an order $k$ tangent bundle $T^kM \to M$ one can define a $(k-1)n$-dimensional subbundle, the osculating bundle of the manifold $M$, denoted by $\text{Osc}^{k-1}M$, $k > 1$. 

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Definition 1.1. The subbundle \(\text{Osc}^{k-1}M \subset T^kM\) is defined by the equality of projections
\[
\rho_1 = \rho_2 = \ldots = \rho_k.
\]
This means that the osculating bundle \(\text{Osc}^{k-1}M\) consists of exactly those elements of \(T^kM\) for which all \(k\) projections on \(T^{k-1}M\) coincide.

2. Local coordinates
The local coordinates on the neighborhoods
\[
T^sU \subset T^sM, \quad \text{where} \quad T^{s-1}U = \pi_s(T^sU), \quad s = 1, 2, \ldots, k,
\]
are derived from coordinates, or coordinate mappings, \((u^i)\), which are given on the neighborhood \(U \subset M:\)
\[
U: \quad (u^i), \quad i = 1, 2, \ldots, n,
\]
\[
TU: \quad (u^i, u^i_1), \quad \text{where} \quad u^i := u^i \circ \pi_1, \quad u^i_1 := \text{d}u^i,
\]
\[
T^2U: \quad (u^i, u^i_1, u^i_2, u^i_{12}),
\]
where \(u^i := u^i \circ \pi_1 \pi_2, \quad u^i_1 := \text{d}u^i \circ \pi_2, \quad u^i_2 := \text{d}(u^i \circ \pi_1), \quad u^i_{12} := \text{d}^2u^i,
\]
etc.

Proposition 2. Coordinate mappings given on the neighborhood \(T^{s-1}U\) can be transformed into coordinate mappings defined on the neighborhood \(T^sU\) with respect to the projection \(\pi_s\) by adding the differentials of these mappings.

Local coordinates are obtained by the following principle:
to the coordinates of a point of a manifold we attach the coordinates of the vector tangent to the manifold at that point. We use the following notation: the coordinates of a neighborhood \(T^kU\) consist of two copies of local coordinates on \(T^{k-1}U\) where the second copy is equipped with an additional subscript \(k\). This principle is suitable in the sense that the coordinates with index \(s\) are recognized as the coordinates of the kernels of projections \(\rho_s, \quad s = 1, 2, \ldots, k\), i.e. the coordinates with index \(s\) disappear after the application of projection \(\rho_s\).

The coordinate form of three projections \(\rho_s : T^3U \rightarrow T^2U, \quad s = 1, 2, 3,\) is given by the following diagram:
\[
\begin{array}{cccc}
(u^i, u^i_1, u^i_2, u^i_{12}, u^i_{13}, u^i_{23}) & (u^i, u^i_1, u^i_2, u^i_{12}) & (u^i, u^i_1, u^i_2) & (u^i, u^i_1, u^i_2, u^i_{12}) \\
\rho_1 & \rho_2 & \rho_3 \\
(u^i, u^i_2, u^i_{23}) & (u^i, u^i_1, u^i_3) & (u^i, u^i_1, u^i_2, u^i_{12})
\end{array}
\]

Proposition 3. The osculating bundle \(\text{Osc}^{k-1}M\) is defined on the neighborhood \(T^kU \subset T^kM\) by equality of the coordinates with the same number of subscripts.

For the osculating bundle \(\text{Osc}^{k-1}M\) we use the coordinate notation
\[
(u^i, \text{d}u^i, \text{d}^2u^i, \ldots, \text{d}^k u^i).
\]
Thus for \(k = 3\) the osculating bundle \(\text{Osc}^2M\) in third order tangent bundle \(T^3M \rightarrow M\) is defined by the equality of the projections
\[
\rho_1 = \rho_2 = \rho_3,
\]
and on the neighborhood $T^3U \subset T^3M$ this leads to the equality of osculating coordinates and to the notation: $du_i := u_1^i = u_2^i = u_3^i$, $d^2u_i := u_4^i = u_5^i = u_6^i$.

We describe the embedding $\text{Osc}^{k-1}M \hookrightarrow T^kM$. In case $k = 3$ this embedding is constructed by means of the immersion $\zeta : \text{Osc}M \to T^2M$ and its tangent mapping $T\zeta : T\text{Osc}M \to T^3M$ described as follows:

$$
\begin{pmatrix}
 u^i \\
 u_1^i \\
 u_2^i \\
 u_3^i \\
 u_4^i \\
 u_5^i \\
 u_6^i 
\end{pmatrix}
\circ \zeta =
\begin{pmatrix}
 u^i \\
 \frac{\partial u^i}{\partial u^i} \\
 \frac{\partial u^i}{\partial (d^2u^i)} \\
 \frac{\partial u^i}{\partial (d^2u^i)} \\
 \frac{\partial u^i}{\partial (d^3u^i)} \\
 \frac{\partial u^i}{\partial (d^3u^i)} \\
 \frac{\partial u^i}{\partial (d^3u^i)} 
\end{pmatrix}
\circ T\zeta =
\begin{pmatrix}
 du^i \\
 d^2u^i \\
 d^3u^i \\
 d^2u^i \\
 d^3u^i \\
 d^3u^i \\
 d^3u^i 
\end{pmatrix}.
$$

The fibers of $\text{Osc}M$ are the integral surfaces of a distribution

$$
\langle \partial_1^i + \partial_2^i, \partial_3^i \rangle, \text{ where } \partial_1^i + \partial_2^i := \frac{\partial}{\partial u_1^i} + \frac{\partial}{\partial u_2^i}, \partial_1^{12} := \frac{\partial}{\partial u_1^{12}}.
$$

The functions $(u_1^i - u_2^i)$ vanish on $\text{Osc}M$,

$$
(\partial_1^i + \partial_2^i)(u_1^i - u_2^i) = \partial_1^{12}(u_1^i - u_2^i) = 0.
$$

In case $k = 3$ the appropriate distribution is $\langle \partial_1^i + \partial_2^i + \partial_3^i, \partial_1^{12} + \partial_2^{13} + \partial_3^{123} \rangle$, etc.

Historically, osculating bundles were investigated much earlier than higher order tangent bundles. The work begun by V.V. Vagner, [9], 60 years ago and has recently been finalized in the theory of Miron-Atanasiu, [2]. Recently, more attention has been given to iterations of the functor $T$, i.e. to higher order tangent bundles and sector forms. It is fully justified, since the above concepts form the base of differential and global analysis. The structural approaches given by Ehresmann and Pradini, [5, 7], although important, are not efficient. The geometric structure of higher order tangent bundles $T^kM$ suggests applications in mechanics, such as higher order movements. From this point of view, sector calculus and the generalization of Cartan differential forms seem to be suitable tools. By White [10], a sector-form is a scalar function on $T^kM$, linear on the fibers of all $k$ projections $T^kM \to T^{k-1}M$. Sector-forms include not only Cartan forms but, as coefficients, they also use generally non-holonomic jets. Bertran in [3] presented an original approach to these structures. In our recently published book [1], we tried to unify these approaches.

3. Sector forms

Definition 3.1. A general $k$-sector form is defined as a scalar function on $T^kM$ that is linear and homogenous on the fibers of all projections $\rho_1, \ldots, \rho_k : T^kM \to T^{k-1}M$, with arbitrary coefficients defined by the coordinate maps of manifold $M$. The coefficients are formed by a non-holonomic jet.

Proposition 4. $k$-sector forms, same as covectors and 1-forms, form a vector space, i.e. an arbitrary linear combination of $k$-sector forms is again a $k$-sector form.

Examples of sector forms.

1. Any 1-form on a manifold $M$ is a 1-sector form on the tangent bundle $TM$.

2. A sector form $\Phi$ on a neighborhood $T^2U$ and a sector form $\Psi$ on a neighborhood $T^3U$ can generally be written as:

$$
\Phi = \varphi_{ij} u_1^i u_2^j + \varphi_i u_1^i 12, \\
\Psi = \psi_{ijk} u_1^i u_2^j u_3^k + \psi_{ij} u_1^i u_2^j u_3^k + \psi_{ij} u_1^i u_2^j u_3^k + \psi_{ij} u_1^i u_2^j u_3^k + \psi_{ij} u_1^i u_2^j u_3^k + \psi_{ij} u_1^i u_2^j u_3^k.
$$
with arbitrary coefficients $\varphi_{ij} \cdot \varphi_i$ and $\psi_{ijk} \cdot \psi_{ij} \cdot \psi_{i}$. Note that, in each expression, the lower subscript 1 and 2, or 1, 2 and 3, respectively, occur exactly once, which means that these functions are really linear on the fibers of projections $\rho_1$ and $\rho_2$ or $\rho_1, \rho_2$ and $\rho_3$, respectively.

3. Locally, on the neighborhoods $U, TU, T^2U, \ldots$, the differentials of a function $f$, defined on manifold $M$, can be written as a sector form:

$$f_1 := df = f_1 u_1^i,$$

$$f_{12} := d^2 f = f_{ij} u_1^i u_2^j + f_{1i} u_{12}^i,$$

$$f_{123} := d^3 f = f_{ijk} u_1^i u_2^j u_3^k + f_{ij} (u_1^i u_{23} + u_{12}^i + u_3 u_{12}) + f_i u_{123},$$

$$\ldots \ldots \ldots$$

The coefficients are formed by the holonomic jet of the function $f$,

$$f_i := \frac{\partial f}{\partial u^i}, \quad f_{ij} := \frac{\partial^2 f}{\partial u^i \partial u^j}, \quad f_{ijk} := \frac{\partial^3 f}{\partial u^i \partial u^j \partial u^k}, \ldots$$

4. The differentials of a 1-form $\Phi_1 = \varphi_i u_1^i$ defined on $M$, as scalar functions on $TM$, can be written as sector forms:

$$\Phi_2 := d\Phi = \varphi_i, j u_1^i u_2^j + \varphi_i u_{12}^i,$$

$$\Phi_{23} := d^2 \Phi = \varphi_{ijk} u_1^i u_2^j u_3^k + \varphi_i (u_1^i u_{23} + u_{12}^i + u_3 u_{12}) + \varphi_i u_{123},$$

$$\ldots \ldots \ldots$$

having as coefficients partial derivatives of functions $\varphi_i$,

$$\varphi_{ij} := \frac{\partial \varphi_i}{\partial u^j}, \quad \varphi_{ijk} := \frac{\partial^2 \varphi_i}{\partial u^j \partial u^k}, \ldots$$

The coefficients form a semi-holonomic (pseudo-, quasi-) jet.

Let us now list several important properties of sector forms.

**Proposition 5.** For any $k, l \in \mathbb{N}$, each $(k + l)$-sector form on $T^{k+l}M$ is an $l$-sector form on $T^k M$. Particularly, a sector form on $T^{k+1}M$ can be the first differential of a sector form on $T^k M$.

**Proposition 6.** For $k > 1$ each exterior Cartan $k$-form, as a $k$-sector form on $T^k M$, vanishes on the osculating bundle $Osc^{k-1} M$. Cartan forms define an exterior geometry of the osculating bundle $Osc^{k-1} M$ in $T^k M$.

See also Proposition 3.

**Proposition 7.** The embedding $\zeta : Osc^{k-1} \hookrightarrow T^k M$ turns each $k$-sector form into a differential form of order $k$ with symmetric coefficients.

If $k = 2$, for example, we have:

$$\Phi = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i \hookrightarrow \Phi \circ \zeta = \varphi_{ij} du^i du^j + \varphi_i d^2 u^i.$$

**Proposition 8.** Each $(k-1)$-sector form can be lifted from $T^{k-1}M$ in $k$ different ways onto $T^k M$ where these images may be seen as $k$-sector forms.

If $k = 3$, for instance, there are three different lifts onto $T^3 M$:

$$\Phi \circ \rho_1 = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i \quad \Phi \circ \rho_2 = \varphi_{ij} u_1^i u_3^j + \varphi_i u_{13}^i \quad \Phi \circ \rho_3 = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i$$

$$\rho_1 \quad \rho_2 \quad \rho_3$$

$$\Phi = \varphi_{ij} u_1^i u_2^j + \varphi_i u_{12}^i$$
4. Bundle connections

**Definition 4.1.** A connection on the bundle $\pi : M_1 \to M$ is defined by the structure $\Delta_h \oplus \Delta_v$ on a manifold $M_1$ where $\Delta_v = \ker T\pi$ is vertical distribution tangent to the fibers and $\Delta_h$ is horizontal distribution complementary to the distribution $\Delta_v$. The transport of the fibers along the path $\gamma \subset M$ is realized by the horizontal lifts given by the distribution $\Delta_h$ on the surface $\pi^{-1}(\gamma)$. If the bundle is a vector one and the transport of fibers along an arbitrary path is linear, then the connection is called linear.

We will assume that the base manifold $M$ is of dimension $n$ and the fibers are of dimension $r$. Then

$$\dim \Delta_h = n, \quad \dim \Delta_v = r.$$ 

On the neighborhood $U \subset M_1$, let us consider local base and fiber coordinates:

$$(u^i, u^\alpha), \ i = 1, 2, \ldots, n; \ \alpha = n + 1, \ldots, n + r.$$ 

Base coordinates $(u^i)$ are determined by the projection $\pi$ and the coordinates $(\bar{u}^i)$ on a neighborhood $\bar{U} = \pi(U), \ u^i = \bar{u}^i \circ \pi$.

**Proposition 9.** On a neighborhood $U \subset M_1$ we define a local (adapted) basis of the structure $\Delta_h \oplus \Delta_v$,

$$(X_i, X_\alpha) = \left( \frac{\partial}{\partial u^j} \frac{\partial}{\partial u^\beta} \right) \cdot \left( \delta_i^j \begin{pmatrix} \Gamma_1^\beta & 0 \\ 0 & \delta_\alpha^\beta \end{pmatrix} \right), \ \left( \omega^i \omega^\alpha \right) = \left( \delta_i^j - \Gamma_1^\delta \delta_\alpha^\beta \right) \cdot \left( du^j \right).$$

The horizontal distribution $\Delta_h$ is the linear span of the vector fields $(X_i)$ and annihilator of the forms $(\omega^\alpha)$,

$$X_i = \partial_i + \Gamma_1^\delta \partial_\delta, \ \omega^\alpha = du^\alpha - \Gamma_i^\alpha du^i.$$ 

**Proposition 10.** Any tensor field of type $(p, q)$ on a manifold $M_1$ is decomposed in $\Delta_h \oplus \Delta_v$ into $2^{p+q}$ invariant components.

For example, a vector field $X$ and a 1-form $\Phi$ on $M_1$ can be decomposed both in natural and adapted bases:

$$X = \bar{x}^i \partial_i + \bar{x}^\alpha \partial_\alpha = x^i X_i + x^\alpha \partial_\alpha, \quad \bar{x}^i = \bar{x}^i, \ x^\alpha = \bar{x}^\alpha - \Gamma^\alpha_i \bar{x}^i,$$

$$\Phi = \bar{\varphi}_i du^i + \bar{\varphi}_\alpha du^\alpha = \varphi_i du^i + \varphi_\alpha \omega^\alpha, \quad \bar{\varphi}_i = \bar{\varphi}_i + \bar{\varphi}_\alpha \Gamma_1^\alpha,$$ 

While the adapted components are invariant, the natural components are not. The components $x^\alpha$ and $\varphi_i$ can be seen as the prototypes of covariant derivatives.

The appropriate affinor decomposes into four blocks with two of them being equal to zero:

$$d := \partial_i \otimes du^i + \partial_\alpha \otimes du^\alpha = X_i \otimes du^i + \partial_\alpha \otimes \omega^\alpha.$$ 

The invariant form is suitable for the description of differentials of functions in an adapted basis:

$$df = X_i f du^i + \partial_\alpha f \omega^\alpha.$$ 

**Definition 4.2.** Generally, $\Gamma_1^\alpha$ are functions of base and fiber coordinates $(u^i, u^\alpha)$. The curvature of a connection can be defined by $K_{ij}^\alpha = X_i \Gamma_{ij}^\alpha$ appearing in the formulae

$$[X_i, X_j] = K_{ij}^\alpha \partial_\alpha,$$

$$d\omega^\alpha = -K_{ij}^\alpha du^i \wedge du^j + \partial_\beta \Gamma_i^\alpha du^i \wedge \omega^\beta.$$ 


In the case of a linear connection the functions \( \Gamma^i_\alpha \) and \( K^i_{ij} \) are linear on the fibers:

\[
\Gamma^i_\alpha = \Gamma^i_{\alpha \beta} u^\beta, \\
K^i_{ij} = K^i_{i\beta j} u^\beta, \quad K^i_{ij\beta} := \partial_{[i} \Gamma^i_{j\beta]} - \Gamma^i_{[i} \Gamma^i_{j\beta]}.
\]

**Proposition 11.** The Pfaff system \( \omega^\alpha = 0 \) is equivalent to a system of differential equations with unknown functions \( \omega^\alpha \) and independent variables \( u^i \). In the case of a linear connection we have the following system of linear differential equations

\[
\omega^\alpha = 0 \iff \frac{\partial u^\alpha}{\partial u^i} = \Gamma^\alpha_i \implies \left( \frac{\partial u^\alpha}{\partial u^i} = \Gamma^\alpha_i u^\beta \right).
\]

In the case that the curvature of a connection equals to zero, \( K^i_{ij} = 0 \), the Pfaff system \( \omega^\alpha = 0 \) is fully integrable.

**Remark.** If the system of first order differential equations is not of the form \( \omega^\alpha = 0 \), the following substitution can be made \( \frac{\partial u^\alpha}{\partial u^i} \sim \Gamma^\alpha_i \) which translates the problem into the structure \( \Delta_h \oplus \Delta_v \), see, e.g., [1], p. 85.

**Proposition 12.** A classical affine connection on manifold \( M \) is seen as a linear connection on the bundle \( \pi_1 : TM \to M \). On the tangent bundle \( TM \to M \) one can define the structure \( \Delta_h \oplus \Delta_v \). The indices in the formulae are denoted by Latin letters all of them ranging from 1 to \( n \). The functions \( \Gamma^i_\alpha, X_i, \omega^\alpha \) are of the form (in \( \Gamma^i_\alpha \) the sign is changed to comply with the classical theory):

\[
\Gamma^i_\alpha \sim -\Gamma^i_{jk} u^k_1, \\
X_i = \partial_i + \Gamma^i_\alpha \partial_\alpha \sim X_i = \partial_i - \Gamma^i_{jk} u^j_1 \partial_k, \\
\omega^\alpha = du^\alpha - \Gamma^\alpha_i du^i \sim U^i_{12} = u^i_{12} + \Gamma^i_{jk} u^j_1 u^k_2.
\]

The forms \( \omega^\alpha \) become 2-sector forms on a neighborhood \( T^2 U \).

**Definition 4.3.** Higher order connections are defined as follows:

on the tangent bundle \( TM \) the structure \( \Delta \oplus \Delta_1 \) is defined where \( \ker T \rho_1 = \Delta_1 \),

on \( T^2M \) the structure \( \Delta \oplus \Delta_1 \oplus \Delta_2 \oplus \Delta_{12} \) is defined where \( \ker T \rho_s = \Delta_s \oplus \Delta_{12} \), \( s = 1, 2, \) etc.

See [1, 8].

Even if the theory of invariants can only be applied to tensors, algebraic methods can also be used for smooth mappings provided that the components of the jets that are not tensors can be replaced by covariant derivatives.

Let \( M_1, M_2 \) be two smooth manifolds. Let us consider the jet of the mapping \( f : M_1 \to M_2 \),

\[
f^\alpha, f^i_\alpha, f^i_{ij}, \ldots
\]

We assume that the neighborhoods \( U \subset M_1 \) and \( V = f(U) \subset M_2 \) are equipped with coordinates

\[
(u^i), \ i = 1, \ldots, \dim M_1, \ (v^\alpha), \ \alpha = 1, \ldots, \dim M_2.
\]

The tangent mapping \( Tf \) is defined by the vector form

\[
F_1 = \partial_\alpha \otimes u^\alpha_1, \text{ where } v^\alpha_1 := df^\alpha = f^\alpha_i u^i_1,
\]
the second order tangent mapping $T^2f$ is defined by the vector form
\[
\mathcal{F}_2 = (\partial_i \partial^j) \otimes \left( \frac{\partial^j}{v^j} \right), \quad \text{where} \quad \left( \begin{array}{c} v^j \\ v^j \end{array} \right) = \left( \begin{array}{cc} f_i^a & 0 \\ \frac{df_i^a}{df^a} & f_i^a \end{array} \right) \cdot \left( \begin{array}{c} u^j \\ u^j \end{array} \right),
\]
noting that \( df_i^a = f_i^a u_1^j \), \( v_1^a = f_i^a u_1^j u_2^j + f_i^a u_1^j \). If we define on the manifolds \( M_1 \) and \( M_2 \) affine connections with coefficients \( \Gamma^k_{ij} \) and \( \Lambda^\alpha_{\beta\gamma} \) on the neighborhoods \( U \) and \( V \), then the vector form \( \mathcal{F}_2 \) can be written, rather than in the natural, in the adapted bases as follows:
\[
\mathcal{F}_2 = (X_\alpha \partial^\alpha) \otimes \left( \frac{\partial^\alpha}{v^\alpha} \right), \quad \text{where} \quad \left( \begin{array}{c} v^\alpha \\ v^\alpha \end{array} \right) = \left( \begin{array}{cc} f_i^\alpha & 0 \\ \frac{df_i^\alpha}{df^\alpha} & f_i^\alpha \end{array} \right) \cdot \left( \begin{array}{c} u^\alpha \\ u^\alpha \end{array} \right).
\]
The following invariant functions are considered:
\[
\begin{align*}
  u_1^i & \leadsto U_1^i = \Gamma^i_{jk} u_1^j u_1^k + u_1^i, \\
  v_1^\alpha & \leadsto V_1^\alpha = \Lambda^\alpha_{\beta\gamma} v_1^\beta v_1^\gamma + v_1^\alpha, \\
  v_1^\alpha & = f_i^\alpha u_1^j u_2^j + f_i^\alpha u_1^j \leadsto V_1^\alpha = f_i^\alpha u_1^j u_2^j + f_i^\alpha U_1^j, \\
  \text{where} \quad F_i^\alpha & = f_i^\alpha - f_i^\alpha \Gamma^k_{ij} + (\Lambda^\alpha_{\beta\gamma} \circ f) f_i^\beta f_j^\gamma.
\end{align*}
\]

**Remark.** The move to adapted bases transforms the Jacobi matrix (on the neighborhood \( TU \)):
\[
\left( \begin{array}{cc} \delta^\alpha_{\beta\gamma} & 0 \\ \Lambda^\alpha_{\beta\gamma} v_1^\gamma & \delta^\alpha_{\beta} \end{array} \right) \cdot \left( \begin{array}{cc} f_i^\beta & 0 \\ \frac{df_i^\beta}{df^\beta} & f_i^\beta \end{array} \right) \cdot \left( \begin{array}{cc} \delta^k_i & 0 \\ -\Gamma^k_{ij} u_1^j & \delta^k_i \end{array} \right) = \left( \begin{array}{cc} f_i^\alpha & 0 \\ \frac{df_i^\alpha}{df^\alpha} & f_i^\alpha \end{array} \right).
\]
The functions \( F_i^\alpha \), as the covariant derivatives of the Jacobi matrix \( (f_i^\alpha) \), replace the partial derivatives \( f_i^\alpha \). The tensors \( F_i^\alpha, \ldots \) are constructed in the same way, but this time using higher order connections and thus higher order tangent bundles, see [8], p. 155-168.

Let us list some cases in which \( (f_i^\alpha) \) and \( (F_i^\alpha) \) play a major role, see [1], p. 101-110:
- Riemannian and co-Riemannian geometry,
- geodesics and co-geodesics fields,
- symmetries and motions in spaces with connection,
- Cartan-Laptev method.

5. **Analytical mechanics**
The special role played by the tangent bundles \( TM \) and \( T^2M \) in analytical mechanics was pointed out by Godbillon in [4].

To the function \( H = H(u, u_1) \) on \( TM \), the following vector field is associated:
\[
X = \sum_i H_{u_i} \partial_i - \sum_i H_{u^i} \partial^i, \quad H_i := \frac{\partial H}{\partial u_i}, \quad H_{u_1} := \frac{\partial H}{\partial u_1},
\]

**Proposition 13.** A flow \( a_t \) of a field \( X \) in \( TM \) is defined by the following differential equations
\[
\left\{ \begin{array}{l}
\dot{u}^i = H_{u_1}, \\
\dot{u}_1 = -H_{u^i}, \\
u_t^i := \frac{du^i}{dt}, \quad u_t^i := \frac{du_1}{dt}.
\end{array} \right.
\]

This system is called a Hamiltonian system and the function $H$ a Hamiltonian. In view of
\[
(u^i, u_1^i, u_2^i, u_{12}^i) \mapsto (u^i, u_1^i, \dot{u}^i_1),
\]
the Hamiltonian system defines a section of $\pi_2 : T^2M \to TM$ of dimension $2n$.

**Proposition 14.** In the flow of the vector field $X$, the Hamiltonian $H$ and the symplectic form $\Omega = du^i \wedge du_1^i$, see [6], are invariant:
\[
XH = 0, \quad L_X \Omega = 0.
\]

**Proposition 15.** The Poisson bracket of two Hamiltonians $H$ and $G$ coincides with the derivative of the function $G$ with respect to the field $X$,
\[
XG = \Sigma_i H u^i_1 G u_i - \Sigma_i H u^i G u^i_1 = \{H, G\}.
\]

**Proposition 16.** If a Hamiltonian system defined on $T^2M$ is restricted to the osculating bundle $\text{Osc}M$, this leads to the following Lagrange equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{u}^i} \right) - \frac{\partial L}{\partial u^i} = 0.
\]

**Proof.** By definition, a Hamiltonian $H = H(u, u_1)$ is connected to a Lagrangian $L = L(u, u_2)$ by the (Legendre transform):
\[
H(u, u_1) - \Sigma_i u^i_1 u^i_2 + L(u, u_2) = 0.
\]

However, on $T^2M$, this equation generally makes no sense:
\[
d(H - \Sigma_i u^i_1 u^i_2 + L) = 0 \implies H_{u^i} + L_{u^i} = 0, \quad H_{u^i_1} = u^i_2, \quad L_{u^i_2} = u^i_1.
\]

and only on the osculating bundle $\text{Osc}M$ where $u^i_1 = u^i_2 = \dot{u}$, the transformation $H \mapsto L$ is possible. On $\text{Osc}M$ the Hamiltonian system yields a Lagrange system.

A Lagrange system defines in $\text{Osc}M \to TM$ a section of the same dimension $2n$ as a Hamiltonian system in $T^2M \to TM$.

Hamiltonian geometry on $T^kM$ and Lagrange geometry on osculating bundles $\text{Osc}^{k-1}M$ with $k > 2$ are related to each other in a similar way.

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