On the ideals of some sumset semigroups

J. I. García-García*
D. Marín-Aragón†
A. Vigneron-Tenorio‡

Abstract

A sumset semigroup is a non-cancellative commutative monoid obtained from the sumset of finite non-negative integer sets. In this work, an algorithm for computing the ideals associated with some sumset semigroups is provided. Using these ideals, we study some factorization properties of sumset semigroups and some additive properties of sumsets. This approach links computational commutative algebra with additive number theory.

Key words: atomic monoid, elasticity, $h$-fold sumset, non-cancellative semigroup, power monoid, semigroup ideal, sumset.

2020 Mathematics Subject Classification: 11B13, 11P70, 13P25, 20M12, 20M14.

Introduction

Additive number theory is the subfield of number theory concerning the study of subsets of integers and their behaviour under addition. More abstractly, the field of additive number theory includes the study of abelian groups and commutative semigroups with an operation of addition. The principal objects of study are (i) the sumset of two subsets $A$ and $B$ of elements from an abelian group $G$, $A + B = \{a + b \mid a \in A, b \in B\}$, and (ii) to determine the structure and properties of the $h$-fold sumset $hA$ when the set $A$ is known. In an inverse problem, we start with the sumset $hA$ and try to deduce information about the underlying set $A$. An up-to-date reference for inverse problems can be found in [15, Chapter 5]. There is a beautiful and straightforward solution of the direct problem of describing the structure of the $h$-fold sumset $hA$ for any finite set $A$ of integers and for all sufficiently large $h$ (see [8, Theorem 1.1]). This result has implications for the study of Weierstrass semigroups, such as is shown in [3].

Here, we consider the commutative semigroup whose elements are the finite subsets of $\mathbb{N}$, denoted by $(FS(\mathbb{N}), +)$ with $+$ the operation defined as before. This

*Departamento de Matemáticas/INDESS (Instituto Universitario para el Desarrollo Social Sostenible), Universidad de Cádiz, E-11510 Puerto Real (Cádiz, Spain). E-mail: ignacio.garcia@uca.es.
†Departamento de Matemáticas, Universidad de Cádiz, E-11510 Puerto Real (Cádiz, Spain). E-mail: daniel.marin@uca.es.
‡Departamento de Matemáticas/INDESS (Instituto Universitario para el Desarrollo Social Sostenible), Universidad de Cádiz, E-11406 Jerez de la Frontera (Cádiz, Spain). E-mail: alberto.vigneron@uca.es.
semitrivial semigroup is the power monoid of \( \mathbb{N} \) (see [1], [5] and the references therein). A sumset semigroup is a semigroup generated by a finite number of elements of \( \text{FS}(\mathbb{N}) \). We show that the sumset semigroups are atomic reduced semigroups with finite elasticity. It is well known that finitely generated semigroups are finitely presented (see [7]). That is, there exists \( p \in \mathbb{N} \) and a congruence \( \sigma \) in \( \mathbb{N}^p \times \mathbb{N}^p \) such that the semigroup \( S \) is isomorphic to \( \mathbb{N}^p / \sigma \). Equivalently, there exists a binomial ideal \( I_S \) in the polynomial ring \( \mathbb{k}[x_1, \ldots, x_p] \) such that \( S \) is isomorphic to the set of monomials in \( \mathbb{k}[x_1, \ldots, x_p] / I_S \) with the product \( \prec \).

The presentation of \( S \) (or a system of generators of the semigroup ideal \( I_S \)) provides us with a way to obtain the expressions of an element of the semigroup in terms of its generators. This can be done with Gröbner bases and related techniques. For instance, we can check whether the \( h \)-fold of any \( A \in S \) can be expressed as a sum of other elements. By using these techniques, we can build a bridge between computational commutative algebra and additive number theory.

We also present a new Python library [10] that includes an implementation of our algorithms and the examples that illustrate it.

In this work, we show some properties of the semigroup \( \text{FS}(\mathbb{N}) \) and give the ideals of some types of sumset semigroups. The work is organized as follows. In Section 1, we present some definitions and results on Gröbner bases. In Section 2, we introduce the sumset semigroups and study some of their properties. In Sections 3 and 4, by using algebraic commutative algebra tools, we study the ideals of some families of sumset semigroups, thereby allowing us to introduce Algorithm 1 and provide some examples.

## 1 Some results on commutative algebra

For a field \( \mathbb{k} \) and a set of indeterminates \( \{x_1, \ldots, x_l\} \), the polynomial ring \( \mathbb{k}[x_1, \ldots, x_l] \) (also denoted by \( \mathbb{k}[X] \)) is the set of polynomials in \( \{x_1, \ldots, x_l\} \) with coefficients in \( \mathbb{k} \), that is, the set \( \{\sum_{i=1}^m a_i x_1^{a_1} \cdots x_l^{a_l} \mid m \in \mathbb{N}, a_i \in \mathbb{k}, a_1, \ldots, a_l \in \mathbb{N}\} \). We denote by \( X^\alpha \) the monomial \( x_1^{a_1} \cdots x_l^{a_l} \), with \( \alpha = (a_1, \ldots, a_l) \in \mathbb{N}^l \). In this work, some results use Gröbner basis theory and the Elimination Theorem. The necessary background can be found in [2] §2 and §3 but is also provided here so that the present work is self-contained.

It is well known that any ideal in a polynomial ring is finitely generated. In particular, there exists a special generating set associated with the ideals, namely a Gröbner basis. This concept depends on the election of an order on the monomials. A monomial order \( \prec \) on \( \mathbb{k}[X] \) is a multiplicative total order on the set of monomials if for each two monomials \( X^\alpha, X^\beta \) such that \( X^\alpha \prec X^\beta \), then \( X^\alpha X^\gamma \prec X^\beta X^\gamma \) for every monomial \( X^\gamma \).

For a fixed monomial order \( \prec \) on \( \mathbb{k}[X] \), \( \text{In}_\prec(I) \) denotes the set of leading terms of non-zero elements of \( I \), and \( \langle \text{In}_\prec(I) \rangle \) the monomial ideal generated by \( \text{In}_\prec(I) \). A subset \( G \) of \( I \) is a Gröbner basis of \( I \) if \( \langle \text{In}_\prec(g) \mid g \in G \rangle = \langle \text{In}_\prec(I) \rangle \), where \( \text{In}_\prec(g) \) is the leading term of \( g \). An algorithm for computing a Gröbner basis for \( I \) is given in [2], Chapter 2, §7. It is also well known that Gröbner bases of binomial ideals are sets of binomials.

Given two polynomials \( f \) and \( g \), their \( S \)-polynomial is defined by \( S(f, g) = \frac{a X^\alpha}{\text{In}_\prec(f)} f - \frac{a X^\alpha}{\text{In}_\prec(g)} g \), where \( a X^\alpha \) is the least common multiple of \( \text{In}_\prec(f) \) and \( \text{In}_\prec(g) \).

Also, for \( k \) non-zero polynomials \( f_1, \ldots, f_k \), we say that \( S(f_1, f_2) = \sum_{i=1}^k g_i f_i \)
This ideal is usually called the semigroup ideal of \( k \) field \( 0 \) has an lcm representation. A semigroup \( A \) is finitely generated if there exists a finite set of generators \( \{a_1, \ldots, a_t\} \subseteq S \) such that \( A = \langle a_1, \ldots, a_t \rangle = \{a_1a_1 + \cdots + a_t a_t \mid a_1, \ldots, a_t \in \mathbb{N} \} \) (\( \sum_{i=1}^{t} a_i \)). For a field \( k \), \( S \) has associated the binomial ideal in \( k[x_1, \ldots, x_t] \),

\[ I_S = \left\{ X^\alpha - X^\beta \mid \sum_{i=1}^{t} \alpha_i a_i = \sum_{i=1}^{t} \beta_i a_i \right\}. \]

This ideal is usually called the semigroup ideal of \( S \), and it has an important role in studying some properties of the semigroup. Note that \( I_S \) codifies the relationships among the elements of \( S \). Associated to these ideals we have the lattice \( M \) of \( \mathbb{Z}^t \) generated by the elements \( \{\alpha - \beta \mid X^\alpha - X^\beta \in I_S\} \). We say that
Let us begin this section by recalling some standard definitions in semigroup theory. Assume $S$ is a commutative semigroup, $S$ is cancellative if $x + z = y + z$ for some $x, y, z \in S$, implies $x = y$. An element $x \in S$ is a unit if $x + y = 0$ for some $y \in S$. The set of units of $S$ is denoted by $\mathcal{U}(S)$. When $S \cap (-S) = \{0\}$, $S$ is named reduced. An atom in $S$ is any non-unit $x \in S$ such that there do not exist two non-units $y, z \in S$ with $x = y + z$. The semigroup is atomic if $S \setminus \mathcal{U}(S)$ is generated by its atoms. The set of atoms of $S$ is denoted by $\mathcal{A}(S)$.

Let $\text{FS}(\mathbb{N})$ be the set whose elements are the finite non-empty subsets of $\mathbb{N}$. Recall that on $\text{FS}(\mathbb{N})$, the binary operation $+$ is defined as $A + B = \{a + b \mid a \in A, b \in B\}$ for all $A, B \in \text{FS}(\mathbb{N})$. The pair $(\text{FS}(\mathbb{N}), +)$ is a commutative monoid with identity element equal to $\{0\}$. Every finitely generated submonoid of $(\text{FS}(\mathbb{N}), +)$ is called a sumset semigroup. If $A \in \text{FS}(\mathbb{N})$ and $\alpha, \beta \in \mathbb{N}$, denote by $\alpha \otimes A$ the subset $\sum_{i=1}^{\alpha} A$.

The monoid $\text{FS}(\mathbb{N})$ satisfies the following interesting properties:

- Since $\{1, 3\} + \{1, 2, 3\} = \{1, 2, 3\} + \{1, 2, 3\}$, it is non-cancellative;
- it is a reduced monoid;
- since $2 \otimes \{1, 2, 4, 5\} = 2 \otimes \{1, 2, 3, 4, 5\}$, it is not torsion free;
- by Proposition 3.2 of [2], this monoid is atomic.

The operation $\otimes$ has good properties, as shown in the following lemma.

**Lemma 4.** Let $A, B$ be in $\text{FS}(\mathbb{N})$ and $\alpha, \beta \in \mathbb{N}$, then:

1. $\alpha \otimes (\beta \otimes A) = (\alpha \beta) \otimes A$;
2. $\alpha \otimes (A + B) = \alpha \otimes A + \alpha \otimes B$.

Let $S$ be a sumset semigroup minimally generated by $\{A_1, \ldots, A_t\}$. By definition, the elasticity of a non-unit $A \in S$ is $\rho(A) = \sup \{m/n \mid \exists a_1, \ldots, a_m, b_1, \ldots, b_n \in A(S) \text{ with } A = \sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i\}$, and the elasticity of $S$ is $\rho(S) = \sup\{\rho(A) \mid A \in S \setminus \{0\}\}$. If there is an element in the monoid whose elasticity “reaches” that of the whole monoid, we say that the monoid has acceptable elasticity.
Note that the ideal associated to $S$ is

$$I_S = \left\{ X^\alpha - X^\beta \mid \sum_{i=1}^t \alpha_i \otimes A_i = \sum_{i=1}^t \beta_i \otimes A_i \right\} \subset k[x_1, \ldots, x_t].$$

Let $\alpha \in M \cap \mathbb{N}_t$ be a non-zero element, thus there exists $\beta \in \mathbb{N}_t$ such that $X^\alpha + \beta = X^\beta (X^\alpha - 1) \in I_S$, and then $\sum_{i=1}^t (\alpha_i + \beta_i) A_i = \sum_{i=1}^t \beta_i A_i$. Therefore, $\max_{i=1}^t (\alpha_i + \beta_i) A_i = \sum_{i=1}^t (\alpha_i + \beta_i) \max A_i > \sum_{i=1}^t \beta_i \max A_i = \max \sum_{i=1}^t \beta_i A_i$, which it is not possible. Hence, $M \cap \mathbb{N}_t$ is $\{0\}$, and the ideal $I_S$ is strongly reduced.

Since $I_S$ is strongly reduced, we have that $S$ is an atomic reduced semigroup with finite elasticity (see Theorem 15 in [13]). Moreover,

$$\rho(S) = \max \left\{ \frac{\sum_{i=1}^t \alpha_i}{\sum_{i=1}^t \beta_i} \mid (\alpha, \beta) \in A(I_M) \right\}.$$  \hspace{1cm} (1)

If $A = \{a_1 < \cdots < a_n\} \in FS(\mathbb{N})$, then $A = \{a_1\} + \{0, a_2 - a_1, \ldots, a_n - a_1\}$ (denote $\{0, a_2 - a_1, \ldots, a_n - a_1\}$ by $A$). The semigroup $S = \langle A_1, \ldots, A_t \rangle$, with $A_i = \{a_i < \cdots < a_{i+1}\}$, is a submonoid of $\langle \{a_1\}, \ldots, \{a_i\}, A_i, \ldots, A_t \rangle$. Trivially, the sumset semigroup $\langle \{a_1\}, \ldots, \{a_t\} \rangle$ is isomorphic to the semigroup $\langle a_1, \ldots, a_t \rangle$, thus $I_{\langle a_1, \ldots, a_t \rangle} = I_{\langle \{a_1\}, \ldots, \{a_t\} \rangle}$.

**Proposition 5.** For every $\bar{S} = \{\{a_1\}, \ldots, \{a_s\}, \bar{A}_1, \ldots, \bar{A}_t\}$, with $a_i \neq 0$ and $\min \bar{A}_i = 0$, we have $I_{\bar{S}} = I_{\langle a_1, \ldots, a_s \rangle} + I_{\langle \bar{A}_1, \ldots, \bar{A}_t \rangle}$, where $I_{\langle a_1, \ldots, a_s \rangle} \subset k[x_1, \ldots, x_s]$ and $I_{\langle \bar{A}_1, \ldots, \bar{A}_t \rangle} \subset k[y_1, \ldots, y_t]$.

**Proof.** Trivially, $I_{\langle a_1, \ldots, a_s \rangle}, I_{\langle \bar{A}_1, \ldots, \bar{A}_t \rangle} \subset I_{\bar{S}}$.

Let $X^\alpha Y^\beta - X^\gamma Y^\delta \in I_{\bar{S}}$, then $\sum_{i=1}^s \alpha_i \otimes \{a_i\} + \sum_{i=1}^t \beta_i \otimes \bar{A}_i = \sum_{i=1}^s \gamma_i \otimes \{a_i\} + \sum_{i=1}^t \delta_i \otimes \bar{A}_i$. Since $\min \bar{A}_i = 0$, we then have $\sum_{i=1}^s \alpha_i \otimes \{a_i\} = \sum_{i=1}^s \gamma_i \otimes \{a_i\}$, and $\sum_{i=1}^t \beta_i \otimes \bar{A}_i = \sum_{i=1}^t \delta_i \otimes \bar{A}_i$. That is to say, $X^\alpha - X^\gamma \in I_{\langle a_1, \ldots, a_s \rangle}$, and $Y^\beta - Y^\delta \in I_{\langle \bar{A}_1, \ldots, \bar{A}_t \rangle}$. Note that $X^\alpha Y^\beta - X^\gamma Y^\delta = Y^\beta (X^\alpha - X^\gamma) + X^\gamma (Y^\beta - Y^\delta) \in I_{\langle a_1, \ldots, a_s \rangle} + I_{\langle \bar{A}_1, \ldots, \bar{A}_t \rangle} \subset k[x_1, \ldots, x_t, y_1, \ldots, y_t].$ \hfill $\square$

Since the semigroup $\langle a_1, \ldots, a_s \rangle$ is isomorphic to a numerical semigroup, there exist algorithms for computing $I_{\langle a_1, \ldots, a_s \rangle}$. Thus, to compute a presentation of $\bar{S}$ we need an algorithm to calculate $I_{\langle \bar{A}_1, \ldots, \bar{A}_t \rangle}$. In the next sections, we provide some algorithms for computing the ideals of some families of sumset semigroups.

### 3  Ideals of a fundamental family of sumset semigroups

In this section, we give explicitly the ideals associated with the sumset semigroups generated by the elements $\{0, ka\}$ and $\{0, kb\}$, where $a < b$ are two positive co-prime integers, and $k \in \mathbb{N} \setminus \{0\}$. These semigroups are key to providing an algorithm to compute the semigroup ideals of more types of sumset semigroups.
Fix $a < b$ as two positive co-prime integers and $k \in \mathbb{N} \setminus \{0\}$, and consider the semigroup $\mathcal{S} = (ka, kb)$ and the sunset semigroup $\mathcal{S}$ minimally generated by $\{0, ka\}$ and $\{0, kb\}$. We prove that $I_{\mathcal{S}} \subset \mathbb{k}[x, y]$ is a principal ideal providing its generator. Note that $I_{\mathcal{S}} = \langle x^{b} - y^{a} \rangle$.

**Lemma 6.** Set $x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} \in I_{\mathcal{S}} \setminus \{0\}$. Then, $\alpha \neq \gamma, \beta \neq \delta$ and $\alpha \cdot \beta \cdot \gamma \cdot \delta \geq 1$.

**Proof.** Note that, since $f = x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} \in I_{\mathcal{S}}$, $\alpha \otimes \{0, ka\} + \beta \otimes \{0, kb\} = \gamma \otimes \{0, ka\} + \delta \otimes \{0, kb\}$.

Suppose $\alpha = \gamma$, then we have $\alpha ka + \beta kb = \max \{\alpha \otimes \{0, ka\} + \beta \otimes \{0, kb\}\} = \max \{\alpha \otimes \{0, ka\} + \delta \otimes \{0, kb\}\} = \alpha ka + \delta kb$, and $\beta = \delta$. Since $f \neq 0$, this is not possible, and therefore $\alpha \neq \gamma$. Analogously, it can be proved that $\beta \neq \delta$.

Suppose $\alpha = 0$, then we have $kb = \min \{\beta \otimes \{0, kb\}\} = \{\gamma \otimes \{0, ka\} + \delta \otimes \{0, kb\}\} \setminus \{0\}$. If $\gamma$ is non-zero, then $kb = \min \{\{\gamma \otimes \{0, ka\} + \delta \otimes \{0, kb\}\}\} = ka$. Therefore, the integers $\gamma, \beta, \delta$ are zero and $f = 0$. Similarly, $\beta, \gamma, \delta \geq 1$ can be proved.

In the sequel, we assume $\alpha > \gamma \geq 1$, and $x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} \in I_{\mathcal{S}} \setminus \{0\}$. Since $I_{\mathcal{S}} \subset \langle x^{b} - y^{a} \rangle$, $\alpha \geq b$ and $\delta \geq a$.

**Lemma 7.** If $\alpha > \gamma$, then $\delta > \beta$. Additionally, there exists a positive integer $n$ such that $\alpha = nb + \gamma$ and $\beta = na + \beta$.

**Proof.** Since $x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} \in I_{\mathcal{S}}$, $\alpha \otimes \{0, ka\} + \beta \otimes \{0, kb\} = \gamma \otimes \{0, ka\} + \delta \otimes \{0, kb\}$, and $\alpha ka + \beta kb = \max \{\alpha \otimes \{0, ka\} + \beta \otimes \{0, kb\}\} = \max \{\gamma \otimes \{0, ka\} + \delta \otimes \{0, kb\}\} = \gamma ka + \delta kb$. So, $(\beta - \gamma)kb = (\alpha - \gamma)ka > 0$.

Furthermore, $(\alpha - \gamma)/\delta - \beta = \gamma/a$. Since $\gcd(a, b) = 1$, there exist two positive integers $n$ and $m$ such that $\alpha = nb + \gamma$ and $\delta = ma + \beta$. From the equality $\alpha ka + \beta kb = \gamma ka + \delta kb$, we deduce that $n = m$.

**Lemma 8.** Let $x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} \in I_{\mathcal{S}} \setminus \{0\}$. Then, $\gamma \geq b - 1, \beta \geq a - 1$, and there is a positive integer $n \in \mathbb{N}$ such that $x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} = x^{\gamma}y^{\delta}(x^{nb} - y^{na})$.

**Proof.** Assume $\gamma < b - 1$. Take $(\gamma + 1)ka + \beta kb$ in $\alpha \otimes \{0, ka\} + \beta \otimes \{0, kb\}$ (recall $\alpha > \gamma$). For that element, there exist two integers $i \in [0, \gamma]$ and $j \in [0, \delta]$ such that $(\gamma + 1)ka + \beta kb = (\gamma - i)ka + (\delta - j)kb \in \gamma \{0, ka\} + \delta \{0, kb\}$, hence $(1 + i)ka = (\delta - \beta - j)kb$. Since $\gcd(a, b) = 1, i + 1 \geq b$ and $i \geq b - 1 > \gamma$, but $i \leq \gamma$, which is a contradiction. Analogously, the fact $\beta \geq a - 1$ can be proved.

By Lemma 7, there exists $n \in \mathbb{N} \setminus \{0\}$ such that

$$x^{\alpha}y^{\beta} - x^{\gamma}y^{\delta} = x^{\alpha + \gamma}y^{\beta} - x^{\gamma}y^{\delta} = x^{\alpha}y^{\beta}(x^{nb} - y^{na}).$$

**Theorem 9.** Let $1 \leq a < b$ be two co-prime integers, $k \in \mathbb{N} \setminus \{0\}$, and $\mathcal{S}$ be the sunset semigroup $\{(0, ka), \{0, kb\}\}$. The ideal $I_{\mathcal{S}} \subset \mathbb{k}[x, y]$ is principal and is generated by $x^{b - 1}y^{a - 1}(x^{b} - y^{a})$.

**Proof.** Observe that $x^{b - 1}y^{a - 1}(x^{b} - y^{a}) = x^{2b - 1}y^{a - 1} - x^{b - 1}y^{2a - 1}$. To prove this theorem, we describe explicitly the sets $A = (2b - 1) \otimes \{0, ka\} + (a - 1) \otimes \{0, kb\}$ and $B = (b - 1) \otimes \{0, ka\} + (2a - 1) \otimes \{0, kb\}$ associated with the monomials $x^{2b - 1}y^{a - 1}$ and $x^{b - 1}y^{2a - 1}$ (respectively), to achieve $A = B$. 

6
Note that the first set \( A = (2b - 1) \otimes \{0, ka\} + (a - 1) \otimes \{0, kb\} \) is equal to
\[
\{0, ka, \ldots, (2b - 1)ka\} + \{0, kb, \ldots, (a - 1)kb\} = \\
k\langle\{0, a, \ldots, (2b - 1)a\} \cup \{0, b, \ldots, (a - 1)b\}\cup \\
\{a + b, \ldots, a + (a - 1)b\} \cup \{2a + b, \ldots, 2a + (a - 1)b\} \cup \ldots \\
\cup((2b - 1)a + b, \ldots, (2b - 1)a + (a - 1)b) \rangle \\
= k\langle\{0, a, \ldots, (b - 1)a, ba, (b + 1)a, \ldots, (2b - 1)a\}\cup \\
\{0, b, \ldots, (a - 1)b\} \cup \{a + b, \ldots, a + (a - 1)b\} \cup \ldots \\
\cup((b - 1)a + b, \ldots, (b - 1)a + (a - 1)b) \cup \\
\cup\{ba + b, \ldots, ba + (a - 1)b\} \cup \ldots \\
\cup((2b - 1)a + b, \ldots, (2b - 1)a + (a - 1)b) \rangle \\
= \cup_{i=1}^{b}C_i.
\]
We denote \( C_1 = \{0, a, \ldots, (b - 1)a\}, C_2 = \{0, b, \ldots, (a - 1)b\}, C_3 = \cup_{i=1}^{b-1}\{ia + b, \ldots, ia + (a - 1)b\}, C_4 = \{ab\} \cup \{ba + b, \ldots, ba + (a - 1)b\}, C_5 = \{(b + 1)a, \ldots, (2b - 1)a\}, \) and \( C_6 = \cup_{i=b+1}^{2b-1}\{ia + b, \ldots, ia + (a - 1)b\}. \) The set \( A \) is the union \( \cup_{i=1}^{b}kC_i. \)

The set \( B = (b - 1) \otimes \{0, ka\} + (2a - 1) \otimes \{0, kb\} \) is
\[
\{0, ka, \ldots, (b - 1)ka\} + \{0, kb, \ldots, (2a - 1)kb\} = \\
k\langle\{0, a, \ldots, (b - 1)a\} \cup \{0, b, \ldots, (a - 1)b\}, ab, \ldots, (2a - 1)b\} \cup \\
\{a + b, \ldots, a + (a - 1)b, a + ab, \ldots, a + (2a - 1)b\} \cup \ldots \\
\cup\{(b - 1)a + b, \ldots, (b - 1)a + (a - 1)b, (b - 1)a + ab, \ldots, (b - 1)a + (2a - 1)b\} \rangle \\
= \cup_{i=1}^{b}kC_i.
\]
Thus, \( A = B, \) and \( x^{b-1}y^{a-1}(x^{b} - y^{a}) \in I_{\mathbb{S}}. \)

To finish the proof, we use Lemma \( \mathbb{S} \). If \( n = 1, \) then \( x^{a}y^{b} - x^{a}y^{b} = x^{b}y^{a+1}y^{a} - x^{b}y^{a+1}y^{a}(x^{b} - y^{a}). \) In case \( n > 1, \) by factorizing the binomial \( x^{n}y^{b} - y^{n}a, \) we obtain
\[
x^{a}y^{b} - x^{a}y^{b} = \\
x^{b}y^{a+1}y^{a} - x^{b}y^{a+1}y^{a}(x^{b} - y^{a}).
\]
In any case, \( I_{\mathbb{S}} = \langle x^{b-1}y^{a-1}(x^{b} - y^{a}) \rangle. \)

\begin{corollary}
Let \( a_1, \ldots, a_s \) be positive integers, and \( S \) be the sumset semigroup generated by \( \{\{a_1\}, \ldots, \{a_s\}, \{0, ka\}, \{0, kb\}\}. \) Then,
\[
I_{\mathbb{S}} = I_{\langle a_1, \ldots, a_s \rangle} + \langle x^{b-1}y^{a-1}(x^{b} - y^{a}) \rangle \subset k[x_1, \ldots, x_s, x, y].
\]
\end{corollary}

\begin{proof}
From Proposition \( \mathbb{S} \) and Theorem \( \mathbb{S} \) we obtain the result.
\end{proof}
4 Computing the ideals of sumset semigroups

The aim of this section is to determine an algorithm for computing the ideals associated with some families of sumset semigroups. As in the previous section, we consider two positive co-prime integers $a < b$, $k \in \mathbb{N} \setminus \{0\}$, the semigroup $\mathcal{S} = \langle ka, kb \rangle$, and the sumset semigroup $\tilde{S} = \langle \{0, ka\}, \{0, kb\} \rangle$.

For any two non-negative integers $n$ and $m$, $A_{nm}$ denotes $\{\alpha ka + \beta kb \mid \alpha \in \{0, \ldots, n\}, \beta \in \{0, \ldots, m\}\}$.

**Theorem 11.** Let $\{(n_i, m_i) \mid n_i, m_i \in \mathbb{N}, n_i + m_i > 0, i = 1, \ldots, t\}$ be a non-empty subset of $\mathbb{N}^2$, $b_1, \ldots, b_p, a_1, \ldots, a_s \in \mathbb{N} \setminus \{0\}$ with $s \leq t$, and consider $S$ the sumset semigroup generated by

$$\{\{b_1\}, \ldots, \{b_p\}, \{a_1\}, \ldots, \{a_s\}, \{0, ka\}, \{0, kb\}\},$$

then, $I_S \subset k[x_1, \ldots, x_p, z_1, \ldots, z_t]$ is

$$\left(I_{S'} + (z_1 - w_1 x^n y^m, \ldots, z_s - w_s x^n y^m, z_{s+1} - x_{s+1} y^{m+1}, \ldots, z_t - x^n y^m)\right) \cap k[x_1, \ldots, x_p, z_1, \ldots, z_t],$$

with $I_{S'} \subset k[x_1, \ldots, x_p, w_1, \ldots, w_s, x, y]$.

**Proof.** Denote $J$ to the ideal

$$(z_1 - w_1 x^n y^m, \ldots, z_s - w_s x^n y^m, z_{s+1} - x_{s+1} y^{m+1}, \ldots, z_t - x^n y^m).$$

Any monomials $Z^3$ and $Z^5$ can be rewritten as follows. Denote $f_i = \prod_{j=1}^{k} z_j^{b_j}$, $\hat{f}_i = \prod_{j=1}^{k} z_j^{\beta_j}$, $g_i = \prod_{k=1}^{j} (w_k x^n y^m)^{\beta_k}$, $\hat{g}_i = \prod_{k=1}^{j} (w_k x^n y^m)^{\beta_k}$, $h_i = \prod_{i=1}^{j} (x^n y^m)^{\beta_k}$, $\hat{h}_i = \prod_{i=1}^{j} (x^n y^m)^{\beta_k}$, and suppose $\beta_0 = \delta_0 = 0$, then we have that

$$Z^3 = z_1^{\beta_1} \cdots z_t^{\beta_t} = \sum_{i=0}^{s-1} f_i + 2g_i (z_1^{\beta_1} - g_i^{t+1}) + \sum_{i=s+1}^{t} \hat{f}_i + g_i h_i (z_i^{\beta_i} - h_i) + g_i h_i^{t+1} + t$$

and

$$Z^5 = z_1^{\beta_1} \cdots z_t^{\beta_t} = \sum_{i=0}^{s-1} f_i + 2g_i h_i (z_1^{\beta_1} - g_i^{t+1}) + \sum_{i=s+1}^{t} \hat{f}_i + g_i h_i (z_i^{\beta_i} - h_i) + g_i h_i^{t+1} + t.$$
Therefore,

\[ \sum_{i=1}^{p} \alpha_i \otimes \{b_i\} + \sum_{i=1}^{s} \beta_i \otimes (\{a_i\} + A_{n,m_i}) + \sum_{i=s+1}^{t} \beta_i A_{n,m_i} = \sum_{i=1}^{p} \gamma_i \otimes \{b_i\} + \sum_{i=1}^{s} \delta_i \otimes (\{a_i\} + A_{n,m_i}) + \sum_{i=s+1}^{t} \delta_i A_{n,m_i}, \]

and \( X^\alpha Z^\beta - X^\gamma Z^\delta \in I_S \).

Analogously, if \( X^\alpha Z^\beta - X^\gamma Z^\delta \in I_S \), then \( F \in I_{S'}, \) and \( X^\alpha Z^\beta - X^\gamma Z^\delta \in I_{S'} + J. \) This completes the proof.

The above proof can also be done by using \([12, \text{Proposition 4}]\). In our proof, we employ the language of polynomials, ideals and Gröbner bases, avoiding congruences.

From Theorem \([11]\) we obtain an algorithm (Algorithm \([11]\)) for computing the ideal of the subset semigroup generated by \( \{b_1, \ldots, b_p\}, \{a_1\} + A_{n,m_1}, \ldots, \{a_s\} + A_{n,m_s}, A_{n+1,m+1}, \ldots, A_{n,m} \}.

**Algorithm 1:** Computation of \( I_S \).

**Data:** \( \{b_1, \ldots, b_p\}, \{a_1\} + A_{n,m_1}, \ldots, \{a_s\} + A_{n,m_s}, A_{n+1,m+1}, \ldots, A_{n,m} \}, \) the generating set of \( S \).

**Result:** \( \mathcal{G} \), a generating set of the semigroup ideal of \( S \).

**begin**

\( S' \leftarrow \{b_1, \ldots, b_p\}, \{a_1\}, \ldots, \{a_s\}, \{0, kA\}, \{0, k\mathbf{b}\}; \)

\( S_1 \leftarrow \{b_1, \ldots, b_p\}, \{a_1\}, \ldots, \{a_s\}; \)

\( G_1 \leftarrow \) a generating set of \( I_{S_1} \subset k[X,W]; \)

\( G' \leftarrow G_1 \cup \{x^{p+1}y^{n+1}(x^b - y^b)\} \subset k[X,W,x,y] \), it is a generating set of \( I_{S'} \) (Corollary \([11]\));

\( G_2 \leftarrow G' \cup \{z_1 - w_1 x^{m_1} y^{n_1}, \ldots, z_s - w_s x^{m_s} y^{n_s}, z_{s+1} - x^{m_{s+1}} y^{n_{s+1}}, \ldots, z_t - x^{m_t} y^{n_t}\} \subset k[X,W,x,y,Z]; \)

\( G_3 \leftarrow \) a Gröbner basis of \( G_2 \) respect to a monomial order with \( x, y, w \) for every \( i = 1, \ldots, p, j = 1, \ldots, t, \) and \( q = 1, \ldots, s; \)

\( \mathcal{G} \leftarrow \{X^\alpha Z^\beta - X^\gamma Z^\delta | X^\alpha Z^\beta - X^\gamma Z^\delta \in G_3\}, \) it is a generating set (Gröbner basis) of \( I_S; \)

**return** \( \mathcal{G} \);

**end**

We show how this algorithm works with an example.

**Example 12.** Let \( S \) be the subset semigroup generated by

\( \{\{3\}, \{4\}, \{6,12\}, \{7,10,13\}, \{0,3,6,9\}\}. \)

Then, from the first steps of Algorithm \([11]\)

- \( S' = \{\{3\}, \{4\}, \{6,7\}, \{0,3\}, \{0,6\}\}, \)
- \( S_1 = \{\{3\}, \{4\}, \{6,7\}\}. \)
If we compute a generating set of the ideal of \( S_1 \), then we get the following one, \( G_1 = \{ w_1^7 - w_6^9, w_2^2 x_2 - w_1^7, w_1^7 x_2 - w_2^7, w_3 x_2^2, x_2^3 - w_1^7, w_2 x_1, w_7^2 x_1 - w_2 x_1^2, x_1 x_2 - w_2, x_2^4 - w_1 \} \). Therefore, \( G' = G_1 \cup \{ x(x^2 - y) \} \) and \( G_2 = G' \cup \{ z_1 - w_1 y, z_2 - w_2 x^2, z_3 - x^3 \} \).

Now, we compute a Gröbner basis of \( G_2 \) respect to the lexicographical order where \( x > y > w_i > x_j > z_k \) for all \( i, j, \) and \( k \), and we obtain

\[
G_3 = \left\{ z_1^{21} z_3 - z_2^{18} z_3, z_1^{21} z_2 - z_2^{19} z_3, x_2 z_2^{14} z_3 - z_1^{17} z_3, x_2 z_2^{15} z_2 - z_1^{17} z_2, \\
x_2 z_2^{11} z_3 - z_1^{13} z_3, x_2 z_2^{10} z_2 - z_1^{13} z_2, x_2 z_2^{11} z_3 - z_1^{13} z_2, \\
x_2 z_2^{6} z_3 - z_1^{9} z_3, x_2 z_2^{7} z_3 - z_1^{9} z_2, x_2 z_2^{2} z_3 - z_1^{5} z_3, x_2 z_2^{3} z_3 - z_1^{5} z_2, \\
x_2 z_2^{2} - z_1^{3} z_2, x_1 z_2 z_3 - x_2 z_1 z_3, x_1 z_2 z_2 - x_2 z_1 z_2, x_1 z_2 z_3 - z_2^3 z_3, \\
x_1 z_2 z_2 - z_2^4, x_1 x_2 z_3 - z_2^3 z_2, x_1 x_2 z_2 - x_2 z_2 z_2, x_1 x_2 z_2 - z_2^2 z_3, \\
x_1^2 x_2 z_3 - z_1^3 z_3, x_1^2 z_1 z_3 - z_2^2 z_3, x_1^2 z_1 z_2 - z_2^2 z_2, x_1^2 x_2 z_3 - z_2^2 z_3, \\
x_1^2 - x_1^2, x_2 w_2 - x_1 x_2, w_1 - x_1^2, y z_2^3 - x_1^2 z_2, y z_2 - x_1^2 z_1, \\
y z_2^3 - x_2^3 z_3, y x_2 z_2 z_3 - z_1^2 z_3, y x_2 z_2 z_3 - z_2^2 z_3, y x_2 z_2 - x_1^2 z_1, \\
x_1 x_2 x_2 z_2 - z_2 z_2, y x_1 x_2 x_2 z_2 - z_2^3, y x_1 - z_1, y^2 z_2 - x_1 x_2 x_2 z_2, y^2 z_1 z_2 - x_1^2 z_3, \\
y z_1^3 - z_3 z_3, x z_2^3 - y^2 z_3, x z_2 - x_1 x_2 x_2 z_2, x z_2 - z_2^3 z_3, x z_2^2 z_3 - x_1^2 z_1, \\
x z_2 x_2 z_3 - y z_2^2, x y - z_3, x^2 z_2 - y z_3, x^2 z_1 - x_1^2 z_2, x^2 x_1 x_2 z_2 - z_2^3, \right\}
\]

Finally, the output of the algorithm is

\[
\left\{ z_1^{21} z_3 - z_2^{18} z_3, z_1^{21} z_2 - z_2^{19} z_3, x_2 z_2^{14} z_3 - z_1^{17} z_3, x_2 z_2^{15} z_2 - z_1^{17} z_2, \\
x_2 z_2^{11} z_3 - z_1^{13} z_3, x_2 z_2^{10} z_2 - z_1^{13} z_2, x_2 z_2^{11} z_3 - z_1^{13} z_2, \\
x_2 z_2^{6} z_3 - z_1^{9} z_3, x_2 z_2^{7} z_3 - z_1^{9} z_2, x_2 z_2^{2} z_3 - z_1^{5} z_3, x_2 z_2^{3} z_3 - z_1^{5} z_2, \\
x_2 z_2^{2} - z_1^{3} z_2, x_1 z_2 z_3 - x_2 z_1 z_3, x_1 z_2 z_2 - x_2 z_1 z_2, x_1 z_2 z_3 - z_2^3 z_3, \\
x_1 z_2 z_2 - z_2^4, x_1 x_2 z_3 - z_2^3 z_2, x_1 x_2 z_2 - x_2 z_2 z_2, x_1 x_2 z_2 - z_2^2 z_3, \\
x_1^2 x_2 z_3 - z_1^3 z_3, x_1^2 z_1 z_3 - z_2^2 z_3, x_1^2 z_1 z_2 - z_2^2 z_2, x_1^2 x_2 z_3 - z_2^2 z_3, \\
x_1^2 - x_1^2, x_2 w_2 - x_1 x_2, w_1 - x_1^2, y z_2^3 - x_1^2 z_2, y z_2 - x_1^2 z_1, \\
y z_2^3 - x_2^3 z_3, y x_2 z_2 z_3 - z_1^2 z_3, y x_2 z_2 z_3 - z_2^2 z_3, y x_2 z_2 - x_1^2 z_1, \\
x_1 x_2 x_2 z_2 - z_2 z_2, y x_1 x_2 x_2 z_2 - z_2^3, y x_1 - z_1, y^2 z_2 - x_1 x_2 x_2 z_2, y^2 z_1 z_2 - x_1^2 z_3, \\
y z_1^3 - z_3 z_3, x z_2^3 - y^2 z_3, x z_2 - x_1 x_2 x_2 z_2, x z_2 - z_2^3 z_3, x z_2^2 z_3 - x_1^2 z_1, \\
x z_2 x_2 z_3 - y z_2^2, x y - z_3, x^2 z_2 - y z_3, x^2 z_1 - x_1^2 z_2, x^2 x_1 x_2 z_2 - z_2^3, \right\}
\]

Since the binomial \( x_1 z_2 z_3 - x_2 z_1 z_3 \in I_S, \{ 3 \} + \{ 7, 10, 13 \} + \{ 0, 3, 6, 9 \} = \{ 4 \} + \{ 6, 12 \} + \{ 0, 3, 6, 9 \}, \) but \( \{ 3 \} + \{ 7, 10, 13 \} \neq \{ 4 \} + \{ 6, 12 \}, \) the semigroup \( S \) is non-cancellative.

The following example introduces an algorithm to obtain an expression for an integer set as a sum of other given integer sets, if possible. In particular, the \( i \)-fold sumset of a set is studied.

**Example 13.** We now use the above presentation of \( S \) to check whether the element \( i \in \{ 7, 10, 13 \} \) can be expressed in terms of the other generators of the semigroup \( S \). We compute the Gröbner basis with respect to the order given by the matrix

\[
A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

...
and we obtain the set
\[
G_A = \{ x_1^4 - x_2^3, x_1^2 x_2^3 - z_1 z_3, x_2^6 - z_1^4 z_1, z_1^2 z_2 - x_1 x_2^4 z_3, \\
x_1 z_2 z_3 - x_2 z_1 z_3, x_2^2 z_2 z_3 - x_1^4 z_1 z_3, z_1^2 z_2 - x_2^5 z_3, x_1^2 z_2 - x_2^5 z_3, \\
x_2^2 z_2 z_3 - x_1^4 z_1 z_3, x_2^2 z_2 z_3 - x_1^4 z_1 z_3, z_2^3 - x_1^3 x_2^4 z_3 \}. \]

In Table 1 we show some elements \( z_i^2 \) that when reduced with respect to the basis \( G_A \) are expressed by using only the variables \( x_1, x_2, z_1, \) and \( z_3, \) and the expression of \( i \otimes \{7, 10, 13\} \) in terms of the elements of the set \( \{\{3\}, \{4\}, \{6, 12\}, \{0, 3, 6, 9\}\}. \)

| \( i \) | Reduction of \( z_i^2 \) | \( i \otimes \{7, 10, 13\} = \) |
|---|---|---|
| 3 | \( x_1^2 x_2^3 z_3^2 \) | \( 3 \otimes \{3\} + 3 \otimes \{4\} + 2 \otimes \{0, 3, 6, 9\} \) |
| 4 | \( x_1^2 x_2^3 z_3^2 \) | \( 2 \otimes \{3\} + 4 \otimes \{4\} + 1 \otimes \{0, 6, 12\} + 2 \otimes \{0, 3, 6, 9\} \) |
| 5 | \( x_1^2 x_2^3 z_3^2 \) | \( 1 \otimes \{3\} + 5 \otimes \{4\} + 2 \otimes \{0, 6, 12\} + 2 \otimes \{0, 3, 6, 9\} \) |
| 6 | \( z_1^2 z_2 \) | \( 6 \otimes \{4\} + 3 \otimes \{6, 12\} + 2 \otimes \{0, 3, 6, 9\} \) |
| 7 | \( x_1^2 x_2^3 z_3^2 \) | \( 3 \otimes \{3\} + 4 \otimes \{4\} + 4 \otimes \{0, 6, 12\} + 2 \otimes \{0, 3, 6, 9\} \) |
| 8 | \( x_1^2 x_2^3 z_3^2 \) | \( 2 \otimes \{3\} + 5 \otimes \{4\} + 5 \otimes \{0, 6, 12\} + 2 \otimes \{0, 3, 6, 9\} \) |

Table 1: Expressions of some elements \( i \otimes \{7, 10, 13\} \) in terms of the subset of generators \( \{\{3\}, \{4\}, \{6, 12\}, \{0, 3, 6, 9\}\}. \)

In general, the reduction of \( z_i^2 \) with respect to \( G_A \) is
\[
x_1^{(2-i) \mod 4} \cdot x_2^{\left\lfloor \frac{i+1}{4} \right\rfloor + 2 + (i-3 \mod 4)} \cdot z_1^{(i-3) \mod 3} z_3^{2-i}.
\]

Therefore, for every \( i \geq 3, \)
\[
i \otimes \{7, 10, 13\} = ((2 - i) \mod 4) \otimes \{3\} + \\
\left(\left\lfloor \frac{i+1}{4} \right\rfloor + 2 + (i-3 \mod 4)\right) \otimes \{4\} + (i - 3) \otimes \{6, 12\} + 2 \otimes \{0, 3, 6, 9\}.
\]

The last examples are dedicated to study the elasticity of a sumset semigroup.

**Example 14.** Again, consider the semigroup \( S \) given in example [12]. From its ideal, we compute a generating set of its associated lattice \( M, \)
\[
\{0, 0, 21, -18, -2\}, \{0, 1, -17, 14, 2\}, \{0, 1, 4, -4, 0\}, \{0, 2, -13, 10, 2\}, \\
\{0, 3, -9, 6, 2\}, \{0, 4, -5, 2, 2\}, \{0, 5, -1, -2, 2\}, \{1, -1, -1, 1, 0\}, \\
\{1, 0, 3, -3, 0\}, \{1, 4, -2, -1, 2\}, \{2, -1, 2, -2, 0\}, \{2, 3, -3, 0, 2\}, \\
\{3, -2, 1, -1, 0\}, \{3, 3, 0, -3, 2\}, \{4, -3, 0, 0, 0\}\},
\]
and its system of linear homogeneous equations,
\[
Ax = \begin{pmatrix} -3 & -4 & 2 & 1 & 12 \\ -6 & -8 & 2 & 0 & 21 \end{pmatrix} x = 0.
\]

We already know that \( S \) is strongly reduced, but this fact is far for being clear from the above equations. We can check it by computing with Normaliz [1] its Hilbert basis:
>>> c1=Cone(equations=[[[-3,-4,2,1,12],[-6,-8,2,0,21]])

>>> c1.HilbertBasis()
[]

Since the above output is the empty list, we have \( M \cap \mathbb{N}^5 = \{0\} \). The Hilbert basis of \((A | -A)(x, y) = 0\) has 109 elements:

\[
HB = \{(0, 1, 4, 0, 0, 0, 0, 0, 0, 0), (1, 0, 3, 0, 0, 0, 0, 0, 0, 3), (0, 0, 1, 0, 0, 0, 0, 0, 1, 0), (0, 0, 1, 0, 0, 0, 0, 0, 0, 18, 2),
(0, 21, 0, 8, 0, 0, 0, 0, 12, 0), (0, 0, 0, 12, 0, 0, 21, 0, 0, 8), (0, 0, 1, 0, 0, 0, 0, 1, 0, 0), (0, 0, 1, 0, 0, 0, 0, 0, 1, 0)
\}

Now, using the formula (1), we conclude that the elasticity of \( S \) is 3. To know if \( S \) has acceptable elasticity, we use Algorithm 28 of [13] which is implemented in [https://github.com/D-marina/CommutativeMonoids/blob/master/Sumsetssemigroups/](https://github.com/D-marina/CommutativeMonoids/blob/master/Sumsetssemigroups/). Running the commands,

```python
>>> gb14=computationIS([3],[4],[6,12],[7,10,13],[0,3,6,9])
>>> hasAcceptableElasticity(gb14, debug=True)
... We compute a Groebner basis with the lex ordering of the variables [x2, z1, x1, z2, z3]
GroebnerBasis([-x1**4 + x2**3, x1**4*x2**2*z3**2 - 1*z1**2*x2**2, 
-x1**3*z11*z2 + x2**2*z2**2, -x1**3*z1*z3 + x2**2*x2*z3, 
-x1**2*z2**2 + x2*z21*z2, -x1**2*z2 + x21*z11*, -x1**2*z2 + x21*z11, 
-x1**2*z21 + x21*z11, -x1**2*z2 + x21*z11, 
-x1**2*z21 + x21*z11, -x1**2*z2 + x21*z11, 
-x1**2*z21 + x21*z11, -x1**2*z2 + x21*z11],
,x2, z1, x1, z2, z3, domain='ZZ', order='lex')
Once removed the variables [x2, z1], we obtain
GroebnerBasis([x1**7*z3**2 - z2**3, x1, z2, z3, domain='ZZ', order='lex'])
... True
```
we obtain that the monoid has acceptable elasticity. Moreover, from the above output, we see that the binomial $x_1^7 + x_2^3 - x_3^2$ belongs to $I_S$. Since the quotient of the addition of the exponents of these two monomials is $(7 + 2)/3 = 3 = \rho(S)$, the element $7 \otimes \{3\} + 2 \otimes \{0, 3, 6, 9\} = 3 \otimes \{7, 10, 13\}$ reaches the elasticity.

We see now an example of sumset semigroup without acceptable elasticity.

**Example 15.** Let $S$ be the semigroup

\[ \langle \{0, 3\}, \{0, 4\}, \{7, 10, 11, 13, 14, 15, 17, 18, 19, 1, 22, 25\} \rangle. \]

Analogously as in the preceding example, we use function `hasAcceptableElasticity` to check if $S$ has acceptable elasticity. The display output of this function shows some steps of Algorithm 28 of [13] and returns `False`, that is, the semigroup has not acceptable elasticity.

```python
>>> gb15=computationIS([0,3],[0,4],
                      [7,10,11,13,14,15,17,18,19,1,22,25])
>>> hasAcceptableElasticity(gb15, debug=True)
Positive cone of M (the semigroup is strongly reduced): []
Equations of M
[[3, 4, 0], [0, 0, 1]]
Matrix (A|-A) (equations of M \cap N^n, n=5):
[[3, 4, 0, -3, -4, 0], [0, 0, 1, 0, 0, -1]]
Generator system of M \cap N^n (number of generators 5):
[[0, 0, 1, 0, 0, 1], [0, 1, 0, 0, 1, 0], [0, 3, 0, 4, 0, 0],
 [1, 0, 0, 1, 0, 0], [4, 0, 0, 0, 3, 0]]
Elasticity of S: 4/3
Atoms of A(I_M) that reach the elasticity:
[[4, 0, 0, 0, 3, 0]]
S has not acceptable elasticity (the monoid C is empty, see step 6 of algorithm in Algorithm 28 in http://doi.org/10.1007/s00233-002-0022-4)
False
```

**Acknowledgements.** The authors were supported partially by Junta de Andalucía research groups FQM-343 and FQM-366, and by the project MTM2017-84890-P (MINECO/FEDER, UE).

**References**

[1] W. Bruns, B. Ichim, C. Söger and U. von der Ohe. *Normaliz. Algorithms for rational cones and affine monoids.* Available at https://www.normaliz.uni-osnabrueck.de

[2] D. A. Cox, J. Little, and D. O’Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Undergraduate Texts in Mathematics. Springer, Cham, 2015.

[3] S. Eliahou, J. I. García-García, D. Marín-Aragón, and A. Vigneron-Tenorio. The Buchweitz set of a numerical semigroup, 2020; arXiv:2011.09187
[4] Y. Fan, A. Geroldinger, F. Kainrath, and S. Tringali. Arithmetic of commutative semigroups with a focus on semigroups of ideals and modules. J. Algebra Appl. 16 (2017), no. 12, 1750234, 42 pp.

[5] Y. Fan and S. Tringali. Power monoids: A bridge between factorization theory and arithmetic combinatorics. J. Algebra 512 (2018), 252–294.

[6] J.I. García-García, D. Marín-Aragón, A. Sánchez-R.-Navarro, and A. Vigneron-Tenorio. CommutativeMonoids, a Python library for computations in finitely generated commutative monoids. Available at https://github.com/D-marina/CommutativeMonoids

[7] R. Gilmer. Commutative Semigroup Rings. Chicago Lectures in Mathematics. 1984.

[8] M. B. Nathanson. Additive Number Theory: Inverse Problems and the Geometry of Sumsets. Graduate Texts in Mathematics. Springer, Vol. 165, 1996.

[9] P. Pisón-Casares, and A. Vigneron-Tenorio. N-solutions to linear systems over Z. Linear Algebra Appl. 384 (2004), 135–154.

[10] J. C. Rosales y P. A. García-Sánchez, Finitely generated commutative monoids. Nova Science Publishers, Inc., New York, 1999.

[11] J. C. Rosales and P. A. García-Sánchez. Numerical Semigroups. Developments in Mathematics, 20. Springer, New York, 2009.

[12] J. C. Rosales, P. A. García-Sánchez, and J. I. García-García. Presentations of finitely generated submonoids of finitely generated commutative monoids. Internat. J. Algebra Comput. 12 (2002), no. 5, 659–670.

[13] J. C. Rosales, P. A. García-Sánchez, and J. I. García-García. Atomic commutative monoids and their elasticity. Semigroup Forum 68 (2004), no. 1, 64–86.

[14] J. C. Rosales, P. A. García-Sánchez, and J. M. Urbano-Blanco. On presentations of commutative monoids. Internat. J. Algebra Comput 9 (1999), 539–553.

[15] T. Tao and V. Vu. Additive Combinatorics: 105. Cambridge; New York, 2006.