Approximate Solutions to a Class of Reachability Games

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Abstract—In this paper, we present a method for finding approximate Nash equilibria in a broad class of reachability games. These games are often used to formulate both collision avoidance and goal satisfaction. Our method is computationally efficient, running in real-time for scenarios involving multiple players and more than ten state dimensions. The proposed approach forms a family of increasingly exact approximations to the original game. Our results characterize the quality of these approximations and show operation in a receding horizon, minimally-invasive control context. Additionally, as a special case, our method reduces to local optimization in the single-player (optimal control) setting, for which a wide variety of efficient algorithms exist.

I. INTRODUCTION

Optimal control problems are often written with running, or time-additive, cost functions. That is, the objective of interest is typically a sum of time-varying functions over a fixed time horizon. Although this structure is reasonably general and easily amenable to both locally optimal and globally contractive optimal control methods, not all scenarios of interest can be expressed with a running cost. For example, in problems which encode properties like goal-reaching and collision-avoidance (Fig. 1), a time-additive objective can indicate safety even for an unsafe trajectory. Encoding these types of requirements with time-additive costs requires the introduction of constraints, which complicate solution methods. On the other hand, by considering a maximum-over-time objective structure we can accurately assess the safety of trajectories without introducing explicit constraints. Aside from reducing problem (and solution) complexity, this extremum-over-time formulation is easily amenable to minimally-invasive control applications.

Fig. 1 illustrates the importance of extremum-over-time objectives for encoding “safety” (understood as constraint satisfaction for all time); one natural application is collision-avoidance. It is also common to use a minimum-over-time to express whether a constraint is ever satisfied. Encoding constraint satisfaction in this manner as an extremum-over-time is inherently a reachability formulation. More generally, reachability problems are concerned with whether a system always remains within (or ever enters) a “target” set. By contrast, other reachability problems are concerned with entering the target set at the final time and are expressed as terminal costs rather than extrema over time.

In this paper, we consider the $N$-player general-sum dynamic game variant of these problems, in which each player has an extremum-over-time cost. Our method finds approximate Nash equilibria of the game efficiently and in real-time. It relies upon making a family of approximations to the original game, which become increasingly precise and in the limit recover the original game. Further, as a special case, the dynamic game formulation reduces to an optimal control problem in the single-player setting. Our method also applies here; however, we note that it is substantially similar to existing, well-developed optimization algorithms that can be readily applied in this setting.

The paper proceeds as follows. Sec. II provides a more formal description of the problem and a brief summary of the most related literature. Sec. III presents our approach in the multi-player, general-sum game context, and Sec. IV discusses its reduction to standard optimization techniques in the single-player optimal control setting. We conclude the paper with Sec. V by noting several shortcomings of our method and interesting directions for future research.

II. BACKGROUND

In this section, we shall present the core mathematical foundation of our approach and the corresponding related
work. We shall treat the cases of dynamic games and optimal control separately. Further, we note that although some of the prior work we reference deals in continuous-time, our methods operate in discrete-time.

A. Multi-Player Reachability Games

We address both the multi-player and the single-player settings in this paper. A multi-player dynamic game with $N$ players evolving over discrete-time $t \in \{1, \ldots, T\} \equiv [T]$ is primarily defined by its dynamics, information pattern, and cost structure. Similar to Sec. II-B, the dynamics are specified as a difference equation, $x_{t+1} = f_i(x_t, u_i^{1:N})$, which describes the evolution of the state $x \in X_t \subseteq \mathbb{R}^n$ with each player $i$’s control input $u_i \in U_i \subseteq \mathbb{R}^{m_i}$. The information pattern or strategy space specifies what each player knows at each timestep; for our purposes, we shall presume a feedback structure in which each player knows all the state of the game $x_t$ and chooses a corresponding control input according to their strategy, i.e., $u_i(t) \in \Gamma_{i}$ and $\Gamma_{i} \subseteq \mathbb{R}^{m_i}$. A measurable map from state to input for each player. Finally, the cost structure of the game may be defined arbitrarily for each player, i.e.,

$$J^i(x_1, \gamma_{1:T}^{i}) \equiv J^i(x) := \max_{t \in [T]} g_i^t(x_t),$$

where we have defined $x := x_{1:T+1}$, and $g_i^t(x_t)$ encodes an arbitrary state cost at each time (such as distance-to-collision). For clarity, we shall use maxima throughout the paper, although minima may also be used, and for convenience. We shall use the shorthand $\gamma^{i} \equiv \gamma_{1:T}^{i}$ to represent each player’s strategy over time. Additionally, note that player $i$’s cost function $J^i$ explicitly depends upon the initial condition $x_1$ which determines the entire game trajectory $x$. Further, note that we have presumed control-independence to restrict our attention to state reachability problems. For a much more detailed introduction to dynamic game theory, please refer to [1] and for their initial conception [3]. As in general games, dynamic games admit a variety of solution concepts. We shall consider the well-known Nash equilibrium concept, in which we seek a set of strategies $\gamma^{1:N,*}$ where no player has an incentive to deviate from its strategy, i.e.

Definition 1: (Nash Equilibrium, see e.g. [1], Chapter 6)
A set of strategies $\gamma^{1:N,*}$ is a Nash equilibrium if

$$J^i(x_1, \gamma_{1:T}^{1:N,*}) \leq J^i(x_1, \gamma_i, \gamma_{1:T}^{1:N,-i}), \forall i \in [N], \forall \gamma_{1:T}^{1:N,-i} \in \Gamma_{1:T}^{1:N,-i},$$

where by $\gamma_{1:T}^{1:N,-i}$ we denote the Nash strategies of all players other than $i$.

Assumption 1: We shall presume the existence of a Nash equilibrium. Conditions under which this is guaranteed are detailed in [1, Chapter 6], but as a practical matter it is often possible to construct continuous dynamic games in which equilibria must exist.

We note that, in a single-player optimal control setting where $N = 1$, a Nash equilibrium is precisely a globally optimal strategy. In this sense then, the method we develop for dynamic games also applies to optimal control problems. Since the single-player case is extremely well studied in its own right, we present it separately.

B. Single-Player Setting

In the single-player case, our work can be understood as an approximate method for finding optimal trajectories for a class of Hamilton-Jacobi reachability problems [4][5]. More precisely, we are concerned with choosing a discrete-time control signal which minimizes the extremum of a target function over time, and as before we restrict our attention to maxima although our contributions also apply for minima. As described above and illustrated in Fig. 1 this problem structure can be used to encode safety constraints.

These problems are characterized by dynamics and cost structure. The dynamics describe the evolution of the state variable $x_t \in X_t \subseteq \mathbb{R}^n$ with control variable $u_t \in U_t \subseteq \mathbb{R}^{m}$ as $x_{t+1} = f_i(x_t, u_t), \forall t \in [T]$. The cost structure and resulting optimal control problem are as follows:

$$x^*, u^* = \arg \min_{x, u} \left[ J^i(x_1) = \max_{t \in [T]} g_i(x_t) \right]$$

such that $u_t \in U_t, \forall t \in [T]$, $x_t \in X_t, \forall t \in [T+1]$, $x_{t+1} = f_i(x_t, u_t), \forall t \in [T]$, $x_1 = x_1$ (given initial state).

with optimal trajectory given by $(x^*, u^*)$. Note that we use shorthand $u := u_{1:T}$ (i.e., in the multi-player context, $u^{1:N} \equiv u_{1:T}^{1:N}$).

Assumption 2: We shall presume that feasible optima always exist in (3). This assumption is often automatically satisfied, e.g., if $X_t$ and $U_t$ are compact.

This type of problem has been studied extensively in the literature. Broadly, approaches fall into three categories.

(a) Conservative, often geometric, methods approximate the reachable set which (depending upon the type of extremum) contains states from which either every or at least one trajectory ends in a given target set; examples include [6][7]. (b) Approximate dynamic programming methods (e.g., [8][9]) attempt to find a parameterized “value” function which summarizes the cost-to-go from any state. (c) Finally, grid-based Hamilton-Jacobi methods are one such method (summarized in [10]), in which the value function is represented on a grid; while resolution-complete, these methods suffer from Bellman’s “curse of dimensionality” [11] and do not generally scale to high-dimensional problems. Reducing our game-theoretic approach to the single-player setting, we recover yet a fourth category of local optimization methods, which are commonly used to solve other optimal control and model predictive control problems, but to our knowledge not widely used in reachability. That is, our approach reduces to a straightforward nonlinear program which may be solved to local optimum using a wide variety of efficient algorithms [12]. We discuss specifics in Sec. IV.
III. APPROXIMATE OPTIMAL TRAJECTORIES FOR MULTI-AGENT REACHABILITY

In this section, we consider an $N$-player dynamic game with maximum-over-time cost structure as in (2), in which each player wishes to minimize this same form of objective, though now with different instantaneous costs $q^i$ for each player $i \in [N]$. Note that our approach also applies when some players instead have minimum-over-time or even sum-over-time objectives.

A. Implementation

In practice, it is generally intractable to compute Nash equilibria \[\text{(13)}\]. Instead, we settle for “approximate local” feedback Nash equilibria. These solutions are “local” in the sense that they only attempt to satisfy Def. \[\text{1}\] within a small neighborhood in the strategy space, and “approximate” in the sense that they may still be a small distance from a local Nash equilibrium. However, making these relaxations facilitates a much more computationally tractable algorithm for finding solutions—the iterative linear-quadratic (ILQ) method of \[\text{(14)}\].

This approach is similar to the ILQ regulator \[\text{(15–18)}\] and differential dynamic programming \[\text{(19, 20)}\], and generalizes the two-player zero-sum approach of \[\text{(21)}\]. Furthermore, it is worthwhile to note that the approach of \[\text{(14)}\] finds open-loop solutions found in, e.g., \[\text{(22, 23)}\].

B. Quadratic Approximation

This ILQ approach relies upon an efficient computation of the global feedback Nash equilibrium of a linear-quadratic game. Such an efficient solution is only known for Lagrange problems \[\text{(1)}\] with sum-over-time structure, not maximum-over-time problems as in \[\text{(2)}\]. To express the cost structure in a Lagrange form, we recognize that, at each iteration of the overall ILQ algorithm, a quadratic approximation to the maximum-over-time objective is only known for finding solutions—the iterative linear-quadratic (ILQ) method of \[\text{(14)}\].

Equipped with the quadratic approximation of player $i$’s cost, which is zero at all time except for at $t^i$, at which it is specified by $q^i_{t^i}$ and $Q^i_{t^i}$ from \[\text{(14)}\], we have formed a local approximation of the original game which is compatible with the ILQ game algorithm from \[\text{(14)}\]. Taken together, this algorithm applied to such approximated games identifies approximate, local feedback Nash equilibria.

C. Relaxation

To apply the ILQ method from \[\text{(14)}\] and avoid matrix singularities, we must also incorporate nontrivial control dependence in each player’s objective. That is, we must further approximate the instantaneous cost $g^i_t(x_t)$ by adding a small but nonzero dependence upon control $u^i_{t^i}$. Concretely, we approximate each player’s objective at level $\epsilon > 0$ as follows:

$$J^i(x_1:T, u^i_1:T) = \bar{g}^i_t, \epsilon \approx \max_{\epsilon \in [\mathcal{T}]} \left\{ g^i_t(x_t) + \frac{1}{2} \epsilon \|u^i_t\|^2 \right\}. \quad (5)$$

As we take $\epsilon \to 0$, we recover the original problem and, assuming Assumptions \[\text{3} \text{ and } \text{4}\] and taking $\epsilon \to 0$, we show the following result.

**Assumption 3:** Nash strategies for games with costs as in both \[\text{(2)} \text{ and } \text{(5)}\] exist and are feasible, for all positive $\epsilon$.

**Assumption 4:** Limiting Nash strategies exist for the game with cost \[\text{(5)}\] as $\epsilon \to 0$ and are feasible.

**Theorem 1:** If $\gamma^i_{1:N, \epsilon}$ are Nash strategies in the limit $\epsilon \to 0$ for the game with relaxed objective \[\text{(5)}\], then they are also Nash strategies in the original game with each players’ cost as in \[\text{(2)}\].

**Proof:** We rewrite player $i$’s problem in the original game with costs \[\text{(2)}\] as

$$\gamma^i_{\epsilon} = \arg \min_{\epsilon \in [\mathcal{T}]} \max_{\gamma^i} \gamma^i \to 0 g^i_t(x_t) = \arg \min_{\epsilon \to 0} \max_{\gamma^i} \gamma^i \to 0 \bar{g}^i_t, \epsilon = \lim_{\epsilon \to 0} \arg \min_{\epsilon \to 0} \max_{\gamma^i} \gamma^i \to 0 \bar{g}^i_t, \epsilon = \gamma^i_{\epsilon}, \quad (6)$$

where we can interchange the limit and minimum safely by Assumptions \[\text{3} \text{ and } \text{4}\].

Thus, if these assumptions hold, the limiting Nash strategies in the relaxed game \[\text{(5)}\] are Nash in the original game with player costs from \[\text{(2)}\].

**Remark 1:** We note that this approach does not generally work for zero sum games, in which a Nash equilibrium of the LQ approximation to the unconstrained, relaxed game may not exist for small $\epsilon$. A sufficient condition for the existence of an equilibrium may be found in \[\text{(1)}\], Remark 6.4.

Incorporation of constraints in general-sum feedback games is an important direction of future research (Sec. V-B).

D. Example: Three-Player Avoidance Game

Consider a three-player dynamic game where each player has bicycle dynamics, i.e.

$$\dot{x}^i = \begin{bmatrix} p^1_x \ p^2_x \ p^3_x \\ p^1_\theta \ p^2_\theta \ p^3_\theta \\ \phi^1 \ \phi^2 \ \phi^3 \\ \omega^i \ \alpha^i \end{bmatrix} = \begin{bmatrix} v^i \cos(\theta^i) \\ v^i \sin(\theta^i) \\ v^i \tan(\phi^i) / L^i \\ \omega^i \ \alpha^i \\ \end{bmatrix}. \quad (6)$$

Here, player $i$’s input $u^i = (\omega^i, \alpha^i)$ controls the front wheel angular rate ($\phi^i$) and linear acceleration ($v^i$), respectively. Each bicycle has inter-axle distance $L^i > 0$, which enforces a minimum nonzero turning radius.

We construct a dynamic game in which each player has Euler-discretized dynamics according to \[\text{(6)}\] with a sampling time of 0.1 s and time horizon 2 s (used throughout), and the following cost structure (written for player 1 without loss of generality), which penalizes player 1 for the smallest relative distance between it and players 2 and 3:

$$g^i_t = d - \min\{d^{12}_t, d^{13}_t\}, \quad (7)$$

where $d^{12}_t := \| (p^1_{x,t}, p^1_{\theta,t}) - (p^2_{x,t}, p^2_{\theta,t}) \|_2$, $d^{13}_t := \| (p^1_{x,t}, p^1_{\theta,t}) - (p^3_{x,t}, p^3_{\theta,t}) \|_2$, and $d^{11}_t := \| (p^1_{x,t}, p^1_{\theta,t}) - (p^1_{x,t}, p^1_{\theta,t}) \|_2$. 

and define $\tilde{g}_{t,\epsilon}^i$ as in (5). Here, $\tilde{d}$ is a desired separation between the vehicles.

We plot the approximately optimal strategies for the relaxed game constructed according to (9) in Fig. 2 showing the affect of decreasing $\epsilon \rightarrow 0$. As we approach this limit, the players take more extreme avoidance maneuvers. Additionally, for each $\epsilon$ our method finds a solution reliably in real-time under 1 s.

E. Example: Receding Horizon, Minimally-Invasive Control

Perhaps the most practical usage of reachability-based controllers is in receding horizon and minimally-invasive settings, where a single “ego” agent overrides its nominal controller whenever safety is nearly violated. In receding horizon problems, players’ strategies at time $t$ only match those for the most recently solved game, not earlier ones; hence, collision-avoidance is not generally guaranteed. For a recent analysis of this information structure, please refer to [25]. Fig. 6 demonstrates our method’s operation in a receding horizon, minimally-invasive setting for a three-player intersection game resembling that of [14].

Here, the ego vehicle (at bottom, red trajectory) and another car (both system dynamics are as above) navigate an intersection while a pedestrian crosses a crosswalk (its dynamics are those of a standard planar unicycle model). We set up costs for a nominal game as in [14], and in order to emphasize the role of a safety controller, we reduce the cost weight for maintaining sufficient proximity between agents. We also construct an identical safety game, except in this game the ego vehicle’s objective is of the form (7), and its equilibrium trajectory is shown in dotted red. Note that it avoids proximity to a much greater degree (i.e., behaves much more conservatively) than the nominal strategy; this is typical of safety strategies. As shown in Fig. 3 over time the ego vehicle switches to the safety strategy as its planned (nominal) trajectory nears other agents. $t_0$ refers to the initial time in the each planning invocation’s time horizon, which is of length 2 s. As above, our method operates in real-time, and since each receding-horizon invocation can be warm-started with the previous solution the amortized speed per invocation is typically on the order of 0.1 s or less.

IV. APPROXIMATE OPTIMAL TRAJECTORIES FOR SINGLE-AGENT REACHABILITY

The discrete-time optimal control problem (3) is already in the form of a nonconvex, nonlinear program, and our approach from Sec. III reduces to the well-known ILQ regulator of [15–18] in this case. However, a wider variety of methods for approximately solving such nonlinear problems can be found, e.g., in [12, 26]. Here, solution methods are approximate in the sense that they find local optima. However, they are significantly more efficient than global solution techniques such as [4, 7] and, to our knowledge, not widely used in the reachability literature. Due to this approximation, solutions are conservative as discussed below.

Theorem 2: Define optimal cost or value $V(\hat{x}_1)$ as the globally minimum cost $J^*(\hat{x}_1)$ attained by (3) at initial state $\hat{x}_1$, and $\hat{V}(\hat{x}_1)$ a local minimum identified by a particular nonlinear programming algorithm. For any scalar threshold $\alpha \in \mathbb{R}$, the following inequalities hold:

\[ J^*(\hat{x}_1) \equiv V(\hat{x}_1) \leq \hat{V}(\hat{x}_1), \]
\[ \{ \hat{x}_1 : V(\hat{x}_1) \leq \alpha \} \subseteq \{ \hat{x}_1 : \hat{V}(\hat{x}_1) \leq \alpha \}, \quad \forall \hat{x}_1 \in \mathcal{X}_1. \] (9)

Sublevel sets of this form are called reachable sets and these inequalities imply that the solution is conservative, i.e., any state it concludes is safe (with sufficiently small cost-to-go) would also be safe for the globally optimal controller.

Proof: The first inequality is a consequence of the local optimality of solutions to nonconvex programming, and the second inequality follows from the first.

Further, in the special case in which (5) is convex, we have the following result.

Theorem 3: Define $V$ and $\hat{V}$ as before, and suppose that problem (3) is convex in the decision variables, i.e., that sets $\mathcal{X}_t, \mathcal{U}_t$ for all $t$, dynamics $f$ are affine in $x_t, u_t$ for all $t$, and instantaneous cost $g_t$ is a convex function of its argument (and that the cost has a maximum-over-time structure). Then, the local minimum cost $\hat{V}$ is equivalent to the globally minimal $V$.

Proof: Follows from the global optimality of solutions to convex programs.

As above, our method is functionally equivalent to a host of nonlinear programming algorithms for problems in the form of (3). Hence, we use well-known existing tools such as IPOPT [27] and SNOPT [28, 29], accessed via YALMIP [30] in MATLAB®, for the following high-dimensional example.

A. Example: High-Dimensional Quadrotor

To demonstrate the computational advantages of this local method, we showcase it in a high-dimensional quadrotor example with 14D state space. The continuous-time dynamics
are may be found in [31], and as before, we use an Euler discretization:

\[ x = (p_x, p_y, p_z, \psi, \phi, v_x, v_y, v_z, \zeta, \xi, p, q, r) \]

\[
\begin{align*}
\frac{d}{dt} \begin{bmatrix} p_x \\ v_x \\ p_y \\ v_y \\ p_z \\ v_z \\ \psi \\ \phi \\ \theta \\ \theta' \\ \theta'' \\ \gamma \end{bmatrix} &= \begin{bmatrix} v_x \\ \zeta \\ v_y \\ \xi \\ v_z \\ \zeta - g \\ g_y \\ g_z - g \\ \xi \\ \xi \\ \zeta \\ \gamma \\ \alpha_x \\ \alpha_y \\ \alpha_z \\ I_x \\ I_y \\ I_z \end{bmatrix},
\end{align*}
\]  

(10)

with \( g_x = \frac{(\sin(\phi) \sin(\psi) + \cos(\phi) \cos(\psi) \sin(\theta))}{m}, \)
\( g_y = \frac{(\cos(\phi) \sin(\psi) \sin(\theta) - \cos(\psi) \sin(\phi))}{m}, \)
\( g_z = \frac{\cos(\phi) \cos(\theta)}{m}, \)

where \( g = 9.81 \text{ m/s}^2 \) is the acceleration due to gravity. Here, \((m, I_x, I_y, I_z)\) are mass and inertia parameters (in our example, set to unity), and the states include positions, angles, their derivatives, and a double integrator on thrust. The controls are the second-derivative of thrust \((\tau)\), and angular accelerations \((\alpha_x, \alpha_y, \alpha_z)\).

The instantaneous cost records the signed distance from the boundary of a cube, i.e. \( g(x) = \| (p_x, p_y, p_z) \|_\infty - B \). Marking the boundary of the unsafe region where \( g > 0 \) is shown in red. The solid black dot denotes the quadrotor’s initial position.

**V. DISCUSSION**

This paper presents a novel approach to solving extremum-over-time reachability games. Our method is computationally tractable and yields real-time approximate solutions to high-dimensional problems, both in single- and multi-player settings. In the optimal control setting, we provide a conservativeness guarantee, and in the game setting, we introduce an arbitrarily accurate relaxation of the original objective structure and provide a guarantee of limiting convergence. We have demonstrated our approach in several examples,
including a high-dimensional optimal control example, a three-player dynamic game, and a receding horizon game. In the remainder of this section, we shall outline promising extensions and directions for future work.

A. Numerical Stability

The choice of $\epsilon$ may have significant impact upon the numerical stability of the ILQ algorithm, since the underlying LQ feedback Nash solution relies upon solving linear systems of equations which become singular when $\epsilon \approx 0$. Further work is needed to improve numerical stability in these cases.

B. Constraints

A related, promising direction is the incorporation of constraints on both states and inputs. Constrained games require a distinct notion of equilibrium; the most straightforward of which is the generalized Nash equilibrium. Our current work investigates efficient methods for solving these types of feedback games, although it is worth noting that solutions already exist for the simpler open-loop information pattern $\epsilon$.

C. Annealing

Although we have not investigated it in this work, we believe that annealing $\epsilon \to 0$ may improve asymptotic approximation quality. Such annealing may encourage the iterative ILQ method of Sec. [III-A] to find globally optimal solutions of the original reachability problem. We note that, because each subproblem with fixed $\epsilon$ may be solved so rapidly, the computational burden of annealing is minimal.

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