Correlance and Discordance: Computable Measures of Nonlocal Correlation

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We present six new measures of nonlocal correlation for discrete multipartite quantum systems; correlance, statance, probablance, strong discordance, discordance, and diagonal discordance. The correlance measures all nonlocal correlation (even bound entanglement), and is exactly computable for all pure and mixed states. Statance and probablance are not yet computable, but motivate the strong discordance (for nonlocal correlation beyond that achievable by a strictly classical state), discordance (a measure of all nonlocal correlation in distinguishably quantum states), and diagonal discordance (for nonlocal correlation in diagonal states), all of which are exactly computable for all states. We discuss types of correlation and notions of classicality, and compare correlance, strong discordance, and discordance to quantum discord. We also define diagonal correlance to handle strictly classical probability distributions, providing a powerful tool with wide-ranging applications.

I. INTRODUCTION

The strong nonlocal correlations achievable in quantum systems can cause novel physical effects that are impossible in classical systems, and have led to an intense worldwide search for applications of these properties, particularly in quantum computing [1, 2], quantum cryptography [3–5], and quantum communications [6–9].

While entanglement has been the star of this show, other types of quantum nonlocal correlation exist that may also be useful. Thus, there is a need to quantify such nonlocal correlation to assess which states are most useful for a given application. One such measure is quantum discord, which we now briefly summarize.

In the case of bipartite mixed states $\rho \equiv \rho^{(1,2)}$, where parenthetical superscripts denote labels of subsystems (modes) of a Hilbert space $\mathcal{H} \equiv \mathcal{H}^{(1,2)} \equiv \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ of dimension $n \equiv \text{dim}(\rho) = n_1n_2$, where $n_m$ is the dimension of mode $m$, the quantum mutual information $I$ of $\rho$ is

$$I(\rho) \equiv S(\bar{\rho}^{(1)}) + S(\bar{\rho}^{(2)}) - S(\rho),$$

where $\bar{\rho}^{(m)}$ is the reduced state for mode $m$ (see App. A), and $S(\rho) \equiv -\text{tr}[\rho \log_2(\rho)]$ is the von Neumann entropy [10]. It has been suggested that $I(\rho)$ can be partitioned as

$$I(\rho) = C(\rho) + Q(\rho),$$

where $C \equiv C(\rho)$ is a measure of distinctly classical nonlocal correlation, and $Q \equiv Q(\rho)$ is a measure of all quantum (nonclassical) correlations called quantum discord (see App. B for a full definition of quantum discord) [11–13].

It has been shown that $C$ is not necessarily only due to entanglement, nor is it only due to nonentanglement quantum correlations (so-called quantum nonlocality without entanglement) [14–16].

However, quantum discord is not the only way to quantify nonclassical correlations. In this paper, we identify several mechanisms of nonlocal correlations (not necessarily all distinct), and propose six measures to quantify them. The most general of these, correlance, is computable for all possible input states and provides a single measure that can detect all forms of nonlocal correlation, while discordance offers an alternative to quantum discord, yet is also computable for all states.

A. General Mechanisms of Nonlocal Correlation

Here we define the condition of no nonlocal correlation, and three general mechanisms of nonlocal correlation.

1. Absence of Nonlocal Correlation: An $N$-mode state $\rho$ (mixed or pure) has no nonlocal correlation if and only if (iff) it can be decomposed in product form,

$$\rho = \bigotimes_{m=1}^{N} \rho^{(m)} = \rho^{(1)} \otimes \cdots \otimes \rho^{(N)},$$

where each $\rho^{(m)}$ is a possibly mixed state in mode $m$. See App. C for details about purity.

2. Entanglement Correlation: An $N$-mode state $\rho \equiv \rho^{(1,\ldots,N)}$ is separable (fully $N$-partite separable) iff it has a set of decomposition probabilities and pure decomposition states $\{p_j, \rho_j\}$ such that

$$\rho = \sum_j p_j \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)},$$

otherwise it is entangled (fully $N$-partite entangled). See App. D for details and basic examples.

3. Decomposition-State Correlation: A necessary condition for $\rho$ to have product form is that it has a decomposition with pure decomposition states $\{\rho_j\}$ such that, for each $j$, the $\rho_j$ are pure product states,

$$\rho_j = \bigotimes_{m=1}^{N} \rho_j^{(m)} = \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)} \quad \forall j,$$

where each $\rho_j^{(m)}$ also has mode independence,

$$\rho_j^{(m)} = \rho_{j_1,\ldots,j_{m-1},j_{m+1},\ldots,j_N}^{(m)} \quad \forall m \in 1,\ldots,N.$$  (6)

If (5–6) are not satisfied, then $\rho$ must have some nonlocal correlation, even if it is separable, and we say there is decomposition-state correlation. See App. E for an example illustrating mode independence.

4. Probability Correlation: A necessary condition for $\rho$ to have product form is that it has a decomposition with probabilities $\{p_j\}$ with $N$ factors,

$$p_j = \prod_{m=1}^{N} p_j^{(m)} = p_j^{(1)} \cdots p_j^{(N)} \quad \forall j.$$  (7)
and also with mode independence, \( \forall j \),
\[ p^{(m)}_j = p^{(m)}_{(j_1, \ldots, j_N)} = p^{(m)}_{j_m}; \quad \sum_m p^{(m)}_j = 1; \quad \forall m \in 1, \ldots, N. \]  

(8)

Any state with no decomposition-state-correlation-minimizing decomposition (see Sec. IV) able to fulfill (7–8) must have some nonlocal correlation (even if separable), a feature we call probability correlation.

**Theorem 1:** Together, (5–8) form a set of necessary and sufficient conditions for \( \rho \) to have product form.

*Proof:* Convert between general \( \rho \) and (3), as in App. F.

The combinations of (5–8) yield the six families of nonlocal correlation in Table I, visualized in Fig. 1. These families are not mutually exclusive and represent only the most general mechanisms of nonlocal correlation; more specific mechanisms are merely subsets of these, such as entanglement which is a subset of Families 1 and 2. Note that since \( p_j = p_j^{(1)} \cdots p_j^{(N)} \) \( \forall j \) is always possible, that does not generate more families, as proved in App. G.

**TABLE I:** Six families of nonlocal correlation, meaning states that can be decomposed satisfying (5–8) in these various ways. In all cases, \( j \equiv j_1, \ldots, j_N \). We abbreviate “family” as F, “product-form” as PF, and “mode-independent” as MI.

| F | Description and State Form |
|---|---------------------------|
| 1 | Generally Nonlocally Correlated: \( \rho = \sum_j p_j \rho_j \) |
| 2 | MI Probabilities: \( \rho = \sum_j (p_j^{(1)} \cdots p_j^{(N)}) \rho_j \) |
| 3 | PF Decomposition States: \( \rho = \sum_j p_j \rho_j^{(1)} \cdots \rho_j^{(N)} \) |
| 4 | MI Probabilities, PF Decomposition States: \( \rho = \sum_j (p_j^{(1)} \cdots p_j^{(N)}) \rho_j^{(1)} \cdots \rho_j^{(N)} \) |
| 5 | MI Decomposition States: \( \rho = \sum_j p_j \rho_j^{(1)} \cdots \rho_j^{(N)} \) |
| 6 | MI Probabilities, MI Decomposition States: \( \rho = \sum_j (p_j^{(1)} \cdots p_j^{(N)}) \rho_j^{(1)} \cdots \rho_j^{(N)} = \rho^{(1)} \cdots \rho^{(N)} \) |

**FIG. 1:** (color online) Depiction of the six families of nonlocal correlation from Table I. Not all set relations are written here; for example 5 \( \subset \) 1 and 6 \( \subset \) 4 are implied, etc. Not shown are Quasi-Families 2\[3, 2\[5, \text{ and 4\[5], which consist of states with different decompositions in different distinct families, but with no decomposition in their intersection; see App. G 3 for details. For multipartite generalizations, see Sec. VII.

**B. Correlation: A Measure of All Nonlocal Correlation**

Given any \( N \)-mode state \( \rho \in \mathcal{H} = \mathcal{H}(1) \otimes \cdots \otimes \mathcal{H}(N) \), a measure of all possible nonlocal \( N \)-mode correlation (full \( N \)-partite correlation) is the correlation, given by

\[ \mathcal{X}(\rho) \equiv \frac{\tilde{X}(\rho)}{\mathcal{N}_X}, \]

(9)

where the raw correlation (unnormalized correlation) is

\[ \tilde{X}(\rho) \equiv \text{tr}[(\rho - \varsigma)^2], \]

(10)

where we define the reduction product,

\[ \varsigma \equiv \varsigma(\rho) \equiv \bigotimes_{m=1}^N \tilde{\rho}^{(m)} = \tilde{\rho}^{(1)} \otimes \cdots \otimes \tilde{\rho}^{(N)}, \]

(11)

where \( \tilde{\rho}^{(m)} \) is the mode-\( m \) reduction of \( \rho \) (see App. A), and the normalization factor is

\[ \mathcal{N}_X \equiv \max_{\forall \rho' \in \mathcal{H}} [\tilde{X}(\rho')] = \tilde{X}(\rho_{\text{ME}}) = \tilde{X}(\rho_{\text{ME-TGX}}), \]

(12)

where \( \rho_{\text{ME}} \) is any maximally entangled (ME) state in \( \mathcal{H} \) and \( \rho_{\text{ME-TGX}} \) is any maximally entangled true-generalized X (TGX) state where the TGX states, first defined in [17] are generalizations of the Bell states (see App. H). The 13-step algorithm of [18] makes ME TGX states methodically to compute \( \mathcal{N}_X \), while Sec. II B gives exact forms.

**FIG. 2:** (color online) Correlance of arbitrary two-qubit mixed states for each of the six families of nonlocal correlation from Table I, \( 10^5 \) states each (colors not related to Fig. 1). This gives a sense of the upper limits of each family, and shows that merely having mode independence in either probabilities or decomposition states alone is not enough to get independent modes; both must have mode independence to get product form and \( \mathcal{X} = 0 \). Families 1 and 2 are generally entangled, Families 3–5 are generally nonlocally correlated without entanglement, and Family 6 has no nonlocal correlation at all.

The correlance \( \mathcal{X}(\rho) \) is 0 iff \( \rho \) is expressible in product form as \( \rho = \varsigma = \tilde{\rho}^{(1)} \otimes \cdots \otimes \tilde{\rho}^{(N)} \), meaning that all modes are uncorrelated. Furthermore, \( \mathcal{X}(\rho) = 1 \) iff \( \rho \) is maximally entangled (meaning fully \( N \)-partite entangled here; see Sec. VII for generalizations), meaning that the \( N \) modes share the most nonlocal correlation possible. Figure 2 explores the correlance for the families of Table I.

Correlance \( \mathcal{X} \) measures all forms of nonlocal correlation; entanglement, decomposition-state correlation (includes entanglement), and probability correlation. See Sec. II for details, including proofs and numerical tests.
II. TESTS AND PROOFS OF CORRELANCE

Here, we test the correlance to check its performance and give details on its normalization. For proof that correlance is necessary and sufficient for measuring all nonlocal correlation, see App. I.

A. Normalization Tests of Correlance

Figure 3 shows the results of a necessary test of the normalization of the correlance $\mathcal{X}$, which is that if it is properly normalized, then we should not be able to find any states with a value of $\mathcal{X}$ that exceeds 1. The proof that $\mathcal{X}$ is properly normalized is given in App. J.

B. Exact Calculation of Correlance Normalization

The correlance normalization factor for all systems can be computed without using any ME states as

$$\mathcal{N}_X = 1 - \prod_{m=1}^{N} P_{\text{MP}}^{(m)}(L_*), \quad (13)$$

as proved in App. K, where $P_{\text{MP}}^{(m)}(L_*)$ is the minimum physical reduction purity of mode $m$ given a pure maximally entangled parent state of $L_*$ levels of equal nonzero probability such that the combination of all $P_{\text{MP}}^{(m)}(L_*)$ is minimized, the calculation of which is given in App. K.

For systems where more than one mode has the largest mode size $n_{\text{max}} = \max(n) = \max\{n_1, \ldots, n_N\}$,$$
\mathcal{N}_X = 1 - \frac{1}{n}, \quad \text{(multiple modes with } n_{\text{max}}),$$

for systems such as $2 \times 3 \times 3$ or $N$-qudit systems like $2 \times 2$ or $3 \times 3 \times 3$. However for systems with exactly one largest mode, like $2 \times 3$ or $2 \times 2 \times 3$, we must use (13) or (12).

Table II: Examples of normalization factors $\mathcal{N}_X$ for the correlance of (9) as computed by (13), all of which were checked and found correct by using the 13-step algorithm $A_{13}$ of [18] to generate ME TGX states $\rho_{\text{ME TGX}}$ for use in (12).

| $n$  | $n_1 \times \cdots \times n_N$ | $\mathcal{N}_X$ | $n$  | $n_1 \times \cdots \times n_N$ | $\mathcal{N}_X$ |
|------|-------------------------------|----------------|------|-------------------------------|----------------|
| 4    | $2 \times 2$                  | $3/4$          | 20   | $2 \times 10$                | $3/4$          |
| 6    | $2 \times 3$                  | $3/4$          | 20   | $4 \times 5$                 | $15/16$        |
| 8    | $2 \times 4$                  | $3/4$          | 20   | $2 \times 2 \times 5$        | $15/16$        |
| 8    | $2 \times 2 \times 2$         | $7/8$          | 21   | $3 \times 7$                 | $8/9$          |
| 9    | $3 \times 3$                  | $8/9$          | 22   | $2 \times 11$                | $3/4$          |
| 10   | $2 \times 5$                  | $3/4$          | 24   | $2 \times 12$                | $3/4$          |
| 12   | $2 \times 6$                  | $3/4$          | 24   | $3 \times 8$                 | $8/9$          |
| 12   | $3 \times 4$                  | $8/9$          | 24   | $4 \times 6$                 | $15/16$        |
| 12   | $2 \times 2 \times 3$         | $29/32$        | 24   | $2 \times 2 \times 6$        | $15/16$        |
| 14   | $2 \times 7$                  | $3/4$          | 24   | $2 \times 3 \times 4$        | $103/108$      |
| 15   | $3 \times 5$                  | $8/9$          | 24   | $2 \times 2 \times 2 \times 3$| $23/24$        |
| 16   | $2 \times 8$                  | $3/4$          | 25   | $5 \times 5$                 | $24/25$        |
| 16   | $4 \times 4$                  | $15/16$        | 26   | $2 \times 13$                | $3/4$          |
| 16   | $2 \times 2 \times 4$         | $15/16$        | 27   | $3 \times 9$                 | $8/9$          |
| 16   | $2 \times 2 \times 2 \times 2$| $15/16$        | 27   | $3 \times 3 \times 3$        | $26/27$        |
| 18   | $2 \times 9$                  | $3/4$          | 28   | $2 \times 14$                | $3/4$          |
| 18   | $3 \times 6$                  | $8/9$          | 28   | $4 \times 7$                 | $15/16$        |
| 18   | $2 \times 3 \times 3$         | $17/18$        | 28   | $2 \times 2 \times 7$        | $15/16$        |

C. Correlance Adapted for Diagonal-Only Input

If the input states $\rho$ are definitely always diagonal, we can define the diagonal correlance as

$$\mathcal{X}_D(\rho) \equiv \frac{\tilde{\mathcal{X}}(\rho)}{\mathcal{N}_{X_D}}, \quad (15)$$

where $\tilde{\mathcal{X}}(\rho)$ is from (10), but the normalization is

$$\mathcal{N}_{X_D} \equiv \max_{\{\rho_D\}}[\tilde{\mathcal{X}}(\rho_D)] = \tilde{\mathcal{X}}(\rho_{D_{\text{max}}}), \quad (16)$$

where $\{\rho_D\}$ is the set of all diagonal states, and $\rho_{D_{\text{max}}}$ is any diagonal state that maximizes $\tilde{\mathcal{X}}(\rho)$.

Surprisingly, the states $\rho_{D_{\text{max}}}$ that maximize $\mathcal{X}_D$ are not merely maximally dephased maximally entangled states in general. Instead, one simple example that maximizes $\mathcal{X}_D$ in all systems is

$$\rho_{D_{\text{max}}} = \frac{1}{2}(|1\rangle\langle 1| + |n\rangle\langle n|), \quad (17)$$

yielding the normalization factor, valid for all systems,

$$\mathcal{N}_{X_D} = \tilde{\mathcal{X}}(\rho_{D_{\text{max}}}) = \frac{1}{2} - \frac{1}{2^{N}}. \quad (18)$$

Figure 4 gives a necessary test showing strong evidence that (18) is not wrong, while App. L gives a sketched proof that (17) is a valid $\rho_{D_{\text{max}}}$, and uses it to derive (18).
$X_D$ can be used for strictly classical probability distributions, for quantum states with no coherence, or for diagonal states exhibiting combinations of both classical and quantum features (see Sec. III). $X_D$ is useful because in strictly classical situations, entanglement is not possible, so ME states would not be a reasonable standard to use for normalization.

![Diagram](image)

**FIG. 4:** (color online) Normalization test of (15), the diagonal correlation $X_D$ of arbitrary general diagonal mixed states $\rho_D$ for all discrete multipartite systems up to $n = 18$ levels, 30,000 states for each system. This shows 540,000 consecutive examples that do not produce $X_D$ higher than 1, providing strong evidence that $X_D$ is not improperly normalized, where $N_{X_D}$ of (18) was used. For a *sketched proof* that (17) is a valid $\rho_{D_{\max}}$, and a derivation of $N_{X_D}$ in (18), see App. L.

## D. Diagonal Correlation as a Measure of General Nonlocal Correlation of Strictly Classical Data

For $N$ classical random variables (RVs), meaning $N$-tuples of variables $x \equiv (x^{(1)}, \ldots, x^{(N)})$, where the data consists of $n_N$ samples of $N$-tuple data points $X \equiv \{x_j\} \equiv \{(x_j^{(1)}, \ldots, x_j^{(N)})\}$ for $j = 1$, we can model the variables as a discrete system of $N$ modes, where mode $m$ represents RV $x^{(m)}$. The size of mode $m$ is $n_m$ (the number of discrete values $x^{(m)}$ can have or its number of histogram bins when viewed as a quantized continuous variable), so the total $N$-mode system has size $n = n_1 \cdots n_N$.

For simplicity, we will refer to the algorithm for constructing a density matrix $\rho$ from strictly classical data $X$ as $A_{rho} \equiv A_{rho}(n, X)$, and its output is

$$\rho = A_{rho}(n, X);$$  \hspace{1cm} (19)

App. M shows how to implement $A_{rho}$. Figure 5 shows four examples of bivariate data treated by this method, and compares the traditional Pearson correlation coefficient $r_p$ (see App. N) [19–21] to the diagonal correlation $X_D$.

As seen in Fig. 5, Fig. 5a is where RVs $x$ and $y$ are completely independent, and both $|r_p| \approx 0.01$ and $X_D \approx 0.01$, which is appropriately near zero for a finite sample of uncorrelated variables. In Fig. 5b, the data has some nonlinear correlation, and $|r_p| \approx 0.03$, while $X_D \approx 0.24$, showing that $X_D$ correctly detects the presence of nonlocal correlation, while $|r_p|$ has trouble detecting it because $|r_p|$ is merely a measure of *linear* correlation. Figure 5c shows the linear focus of $|r_p|$ since the linear data causes $|r_p| \approx 0.99$, close to its maximum value of 1, while $X_D \approx 0.57$, showing that $X_D$ acknowledges this correlation but does not consider it to be maximal.

![Diagram](image)

**FIG. 5:** (color online) The left column of plots show four bivariate dimensionless data sets of 200 points for random variables (RVs) $x$ and $y$, each quantized to four outcomes (not shown) before further calculation. Each RV has a uniformly distributed random deviation of 0.05 about ideal functions: (a) $(x, y) = (\text{rand, rand})$, (b) $(x, y) = (t, \frac{1}{2}(1 + \cos(2\pi t)))$, (c) $(x, y) = (t, t)$, and (d) $(x, y) = (\text{round}[t], \text{round}[t])$, where rand is a random number on $[0, 1]$, and $t \in [0, 1]$. The center column of plots are the density matrices $\rho$ for the quantized 2-mode data in single-index form, and the right column of plots shows the magnitude of the Pearson correlation coefficient $|r_p|$ of App. N and the diagonal correlance $X_D$ from (15).

Finally, Fig. 5d shows data that is indeed more correlated than linear since both $x$ and $y$ tend to have matching values while also producing a $\rho$ that does not factor into a product state, giving $|r_p| \approx 1.00$ and $X_D \approx 1.00$, showing that $|r_p|$ cannot distinguish the data of Fig. 5d from that of Fig. 5c, while $X_D$ correctly detects the difference and values the data of Fig. 5d as being maximally correlated for a diagonal state.

*Important:* States such as $\rho$ in Fig. 5d or (17) in general only have maximal diagonal correlance for the *set of all diagonal states*, and are not generally maximally correlated in the context of all quantum states, as maximally entangled states are. Thus, if the data comes from a system with novel quantum correlations, application of $X_D$ could yield values greater than unity, even though this is guaranteed not to happen for strictly classical data.

Thus, the diagonal correlance $X_D$ provides a more sensitive measure of correlations in data with strictly diagonal density matrices than the Pearson correlation coefficient, and is not limited to two RVs.
III. CLASSICAL VS. QUANTUM

Before we can develop other measures capable of distinguishing the types of correlation from Sec. I A, we must first confront the issue of classicality that is raised by the diagonal correlance. Our main questions here are What does it mean for a state to be diagonal in the context of classicality? and Is there a precise and unambiguous definition for classicality?

These questions and their answers are extremely relevant in any discussion on quantum correlations that includes quantum discord, since it defines requirements that we identify the part of a state that represents “distinctly classical nonlocal correlation.” Therefore here, we will make a series of observations, and from these deduce some concrete answers to these questions that will then allow us to construct the desired measures.

A. Diagonal States Are Not Necessarily Classical

Quantum superposition is an inherently nonclassical, distinctly quantum phenomenon. The idea that an object’s state could simultaneously have multiple exclusive outcomes at once is nonsense in any classical model.

For pure quantum states, superposition appears in density matrices as nonzero off-diagonal elements. For mixed quantum states, having nonzero off-diagonals is called coherence. However, as shown in App. H of [22], we can also have superposition without coherence.

For example, given a general two-qubit pure parent state expanded as $|\psi\rangle = a_1|1\rangle + a_2|2\rangle + a_3|3\rangle + a_4|4\rangle$, where $|1\rangle \equiv |1,1\rangle$, $|2\rangle \equiv |1,2\rangle$, $|3\rangle \equiv |2,1\rangle$, $|4\rangle \equiv |2,2\rangle$, if that state is the Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |4\rangle)$, where $a_1 = a_4 = \frac{1}{\sqrt{2}}$ and $a_2 = a_3 = 0$, then its density matrix is

$$
\rho = |\psi\rangle \langle \psi| = \begin{pmatrix}
a_1a_1^* & \cdots & a_1a_3^* \\
\cdots & \cdots & \cdots \\
a_3a_1^* & \cdots & a_4a_4^*
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & \cdots & \frac{1}{2} \\
\cdots & \cdots & \cdots \\
\frac{1}{2} & \cdots & \frac{1}{2}
\end{pmatrix},
$$

and therefore its mode-1 reduction is diagonal,

$$
\rho^{(1)} = \begin{pmatrix}
\rho_{1,1} & \rho_{1,2} & \rho_{1,3} & \rho_{1,4} \\
\rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \rho_{2,4}
\end{pmatrix} = \begin{pmatrix}
a_1a_1^* & 0 & 0 & a_4a_4^* \\
0 & 0 & a_3a_3^* & 0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2} & \cdots & \frac{1}{2} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{2} & \cdots & \frac{1}{2}
\end{pmatrix},
$$

but its probabilities are directly inherited from the wavefunction overlaps of the superposition amplitudes of its Bell parent state as $p_1^{(1)} \equiv \rho_{1,1} = a_1a_1^*$ and $p_2^{(1)} = \rho_{2,2} = a_4a_4^*$, where $a_k \equiv \langle k|\psi\rangle$ are wavefunction overlaps of basis states $|k\rangle$ with the pure parent state $|\psi\rangle$, which has matrix elements $p_{j,k} = a_ja_k^*$. Thus, (21) is an example of superposition without coherence.

In contrast, the probabilities of classical discrete states are estimators of which outcome to expect from a step function of pure computational basis states on average. Therefore, in principle, there is a sample-time window which could reveal the true step-function behavior of any truly classical mixed state.

For diagonal quantum states such as (21), no such time window exists, because the probabilities inherit the instantaneous nature of superposition from the wavefunction overlaps of the pure parent state, as in (21).

A popular misconception is that Fock states [23] (photon number states $|n\rangle$, not to be confused with our generic basis states used in the rest of this paper) are nonclassical. This comes from the preconception that coherent states [24, 25] are somehow somehow classical simply because they fulfill some necessary conditions for what we think the word “classical” should mean, such as having all quantum coherence functions equal 1 which only measures similarity to coherent states [24–27] (see App. O).

The problem of misdiagnosing coherent states as classical and then mistakenly using them as standards of classicality is really a language problem, as we explain next.

B. The Need for a New Term: Strictly Classical

The main reason for confusion about classicality is that “classical” is already a colloquial word in everyday speech. Similarly, “work” has an everyday meaning that has nothing to do with physics, but it also has a well-defined physics-meaning (though it is aptly named).

In physics, all models predating quantum theory were unofficially lumped into a category colloquially referred to as “classical,” where its use is just an adjective describing those models as part of the pre-quantum mindset.

However, the colloquial use of the word “classical” is often incorrectly interpreted as a formal physics definition, and this has led to some ridiculous conclusions, such as considering coherent states to be classical even though they have quantum superposition in the Fock basis.

Therefore, since improper use of the word “classical” is already ubiquitous, we need a new term with a clear, formal definition that lists the rules that a physical model must obey in a hypothetical world that is truly without quantum-mechanical properties.

Thus, we propose the term strictly classical as being the simplest, most transparent label for this idea that also lets existing literature have the word “classical” as either a colloquial adjective or a colloquial term improp-erly used as a formal physics term. Figure 6 summarizes these terms, while Sec. III C presents a formal definition of strict classicality.

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C. Definition of Strictly Classical States

Almost all (if not all) of the strange behavior of quantum mechanics comes from its distinctive feature of quantum superposition, the primary idea that caused so much reluctance to accept quantum mechanics, because it is at such odds with classical physics. Therefore we use the absence of superposition as the core criterion for what constitutes strict classicality;

**Strict Classicality:** Let a *strictly classical system* be a hypothetical physical system for which a necessary condition is that quantum superposition plays no role, is impossible, and cannot be created. A system that is strictly classical can be said to have *strict classicality.*

While there may be other conditions we need to identify for strict classicality, the above definition immediately leads to the following rules;

1. **Strictly classical states must have no coherence (off-diagonals of the density matrix must be zero, so the density matrix is diagonal).**
2. **Strictly classical states must have no diagonal superposition in any complete orthonormal pure product-state basis (no cases of superposition without coherence as described in Sec. IIIA and seen in (21)).**
3. **The actual instantaneous state of any strictly classical system must be a pure computational basis state.**
4. **Mixed** strictly classical states (having purity less than 1) must be averages of step functions of computational basis states, and the lack of purity comes entirely from lack of human knowledge of the instantaneous state (which is always a pure computational basis state).
5. **(Corollary to Rule 3)** The only transformations possible for a discrete strictly classical system are permutation unitaries; no transformation can convert a strictly classical state to a state with quantum superposition. Thus, strictly classical states are strictly classical in all reference frames, and reference frames themselves are defined only in terms of strictly classical states.

Thus, by Rule 2 we must be careful when using the word “classical.” A list of probabilities is not enough information to determine whether the system is strictly classical or quantum; we must also be told or find more information about the origins of the physical system itself.

A strictly classical situation often arises in the macroscopic world, where we happen to know that outcomes are already in step functions of particular basis states, and we simply sample them over certain longer time windows and build estimators of the probabilities yielding a diagonal statistical mixture. In such cases, we can use the diagonal correlation $X_D$ exclusively, but our physical interpretation of “nonlocal correlations” must be due to distinctly nonquantum features, meaning they have nothing to do with superposition and its related effects such as entanglement or relative quantum phase.

If both superposition and classical probability exist, such as a step function of quantum states sampled with a long time window giving a mixture containing both wave-function overlap probabilities and ignorance-based probabilities, our interpretation of $X$ or $X_D$ must acknowledge that both probability mechanisms are part of the model. Figure 7 depicts some types of hybrid quantum/classical states.

(a) **Strictly Classical and its Time Average:**

\[ |k_1⟩⟨k_1| | k_2⟩⟨k_2| | k_3⟩⟨k_3| \quad \Rightarrow \quad \rho \]

(b) **Pure Quantum and its Time Average:**

\[ |ψ_1⟩⟨ψ_1| | ψ_2⟩⟨ψ_2| | ψ_3⟩⟨ψ_3| \quad \Rightarrow \quad ρ \]

(c) **Mixed Quantum and its Time Average:**

\[ ρ \quad \Rightarrow \quad \rho \]

(d) **Constant Mixed Quantum and its Time Average:**

\[ t_1 \quad t_2 \quad t_3 \quad \Rightarrow \quad \rho \]

**FIG. 7:** (color online) Depiction of density matrices of some types of physical systems (omitting quantum phase for simplicity). The first three columns of plots show the instantaneous state as a step function in time, and the fourth column shows the density matrix as it would be constructed from ideal tomography after using measurement time windows that were much longer than the duration of each step (to show the effect of human-induced mixing). The state in (21) is a special case of scenario (d) where the constant mixed quantum state is diagonal. In contrast, a strictly classical state can never be instantaneously mixed, as seen in (a). In these examples, the instantaneous states are ontic, while the time-averaged states are epistemic [28–30], as defined and discussed in App. O.

At this point, a keen reader might object to nonlocal correlations in strictly classical systems. However, the concept of *state,* which belongs to both classical and quantum physics, is inherently *global* and therefore nonlocal, so nonlocality is an inherent part of classical physics. For example, even in a purely classical world, if you sneeze here on Earth, the state of the entire universe changes instantaneously. Similarly, solutions to the classical heat equation propagate with *infinite* velocity because the *state* is global and therefore changes everywhere at once. Thus, nonlocal correlations are perfectly permissible in mixed strictly classical states, because nonlocality is built-in to the concept of state.

Next, we define a few more specific measures which are not practical to calculate, but help to justify several fully computable measures similar to quantum discord.
IV. STATANCE AND PROBANCE

Here we propose two measures of nonlocal correlation, **statance** and **probance**, that are unlikely to be practical to calculate in general, but which nevertheless give us sufficient insight into the mechanisms of nonlocal correlation to define three more measures, **strong discordance**, **discordance**, and **diagonal discordance**, that are exactly computable for all mixed and pure states and may be preferable to quantum discord.

A. Statance

The statance \( \hat{S}(\rho) \) for any \( N \)-mode quantum state \( \rho \) is

\[
\hat{S}(\rho) = \frac{1}{N_S} \min_{\{U\}} \{S(\rho)\},
\]

over all unitary matrices \( U \) of \( D \in r, \ldots, r^2 \) levels where \( r \equiv \text{rank}(\rho), N_S \) normalizes \( \min_{\{U\}} \{S(\rho)\} \) over all \( \rho \), and \( S \) is the unoptimized statance,

\[
S(\rho) \equiv \sum_{j_1,\ldots,j_N=1}^{D_1,\ldots,D_N} \text{tr}[\{\rho(j_1,\ldots,j_N) - \mu(j_1,\ldots,j_N)^2\}],
\]

where \( D_m \in 1, \ldots, n^2 \) s.t. \( \prod_{m=1}^{N} D_m = D \) (see App. P), and \( \mu_j \equiv \mu(j_1,\ldots,j_N) \) are

\[
\mu_j \equiv \frac{N}{\otimes_{m=1}^{N} \text{tr}(\rho_{j_1,\ldots,j_1})} \left( \frac{1}{D_m} \sum_{j_1^{(m)}, \ldots, j_N^{(m)}} \delta_{j_1^{(m)},j_1^{(m)}} \rho_{j_1^{(m)},\ldots,j_N^{(m)}} \right),
\]

where \( D_m = \frac{D}{\text{tr}(A)} \) and \( \text{tr}(A) \) is the partial trace of \( A \) over all modes that are not mode \( m \) (see App. A). Superscripts \( \{m\} \) in \( j_1^{(m)}, \ldots, j_N^{(m)} \) distinguish indices inside each factor of the tensor product, and the pure decomposition states \( \rho_j \equiv \rho(j_1,\ldots,j_N) \) are

\[
\rho_j = \frac{1}{p_j} \sum_{k,l=1}^{r} U_{j,k} U_{k,j}^* \sqrt{\lambda_k \rho_{e_k}} |e_k\rangle \langle e_k|,
\]

where \( \{\lambda_k,|e_k\rangle\} \) are the eigenvalues and eigensetates of \( \rho \) s.t. \( \lambda_1 \geq \cdots \geq \lambda_r > 0 \), and \( \sum_{k=1}^{r} |U_{j,k}|^2 = 0 \) \( \forall j \) as explained in App. P, and \( p_j = \sum_{k=1}^{r} \lambda_k |U_{j,k}|^2 \). See [31] for a useful parameterization of unitary matrices.

Basically, \( \mu_j \) uses the \( p_j \) to form states [inside the partial trace in (24)] constructed to have mode-independent (MI) **probabilities** (due here to the fact that they are all equal) so that only the way the tensor product of the mode-\( m \) reductions of those states could also be MI would be if all the \( p_j \) were MI (see App. Q).

The statance \( \hat{S}(\rho) = 0 \) iff \( \rho \) has a decomposition where all of the decomposition states \( \rho_j \) have MI product form as \( \rho_j = \otimes_{m=1}^{N} \rho_{j_1^{(m)}} = \rho_{j_1^{(1)}} \otimes \cdots \otimes \rho_{j_N^{(N)}} \) \( \forall j \), so \( \hat{S}(\rho) = 0 \) is guaranteed in Families 5 and 6 of Table I. If \( \hat{S}(\rho) > 0 \), then \( \rho \) has decomposition-state correlation, meaning that it has no decomposition for which all \( \rho_j \) have both product form and mode independece, so its optimal decomposition has at least one \( \rho_j \) that violates either (5) and (6) or violates just (6). Thus, statance \( \hat{S}(\rho) \) is a necessary and sufficient measure of decomposition-state correlation (see App. Q for proof, and App. R for a special example).

B. Probance

Our goal here is to define a measure of probability correlation, but there are several fine points to consider.

1. Recalling that Family 5 of Table I, with defining form \( \rho = \sum_{j} p_j \rho_{j_1^{(1)}} \otimes \cdots \otimes \rho_{j_N^{(N)}} \), has the ability to yield \( X(\rho) > 0 \) in Fig. 2, this shows that there are indeed some states whose only type of correlation is probability correlation (since those states were **constructed** with MI decomposition states so they have no decomposition-state correlation). Moreover, that means that for Family 5 only, \( X(\rho) > 0 \) exclusively indicates probability correlation.

2. Probability correlation cannot be judged by itself over all decompositions, because it is always possible to make a decomposition with equal probabilities which are therefore MI and have no probability correlation, as we will show in the next few paragraphs.

3. The fact that Family 5 has MI decomposition states and yet can have \( X(\rho) > 0 \) means that to get the defining optimal form of that family with respect to probability correlation, we **must first find decompositions that minimize the decomposition-state correlation**, and then minimize over the probability sets of those decompositions to judge probability correlation. Thus, for Family 5, since this restriction causes us to only consider decompositions with MI decomposition states when calculating the probabilities, then by Fine Point 1, if \( X(\rho) > 0 \), we are **guaranteed to be unable to find MI probabilities from that set of decompositions**.

Keeping the above in mind, we define the **probance** for any \( N \)-mode quantum state \( \rho \) as

\[
\hat{P}(\rho) \equiv \frac{1}{N_P} \min_{\{U\}} \{P(\rho)\}; \quad \{U'\} \equiv \arg \left( \min_{\{U\}} \{S(\rho)\} \right),
\]

where \( \{U'\} \) is the set of all unitaries that minimize \( S(\rho) \) from (23), \( N_P \) normalizes over all \( \rho \), and \( P(\rho) \) is the unoptimized probance,

\[
P(\rho) \equiv \sum_{j=1}^{D} \left| p'_j - q'_j \right|^2,
\]

where \( D \) and \( j \) are as in Sec. IV A, and \( q'_j \equiv q'_{(j_1,\ldots,j_N)} \) is

\[
q'_{(j_1,\ldots,j_N)} \equiv \prod_{m=1}^{N} g'_{j_1^{(m)}}, \quad g' \equiv \sum_{l=1}^{D} p'_j |l\rangle \langle l|
\]

\[
= \prod_{m=1}^{N} \left( \sum_{j_1^{(m)},\ldots,j_N^{(m)}} \delta_{j_1^{(m)},j_1^{(m)}} p'_{(j_1^{(m)},\ldots,j_N^{(m)})} \right),
\]

where \( |l\rangle = |l_1\rangle \otimes \cdots \otimes |l_N\rangle \) are computational basis states in a Hilbert space of mode sizes \( n' \equiv (D_1, \ldots, D_N) \), and

\[
p'_j \equiv p'_{(j_1,\ldots,j_N)} \equiv \sum_{k=1}^{r} \lambda_k |U'_{(j_1,\ldots,j_N)}| k^2,
\]

where \( U' \) is defined in (26), based on \( U \) from Sec. IV A.
To see how $\hat{P}(\rho)$ works, $\rho_j'$ in (28) creates a $D$-level diagonal state that automatically has $D$ MI decomposition states so that the only way $\rho_j'$ can have product form is if its diagonal elements are MI, and those are the probabilities of a given decomposition specified by $U'$. The quantity $q_j'$ then exploits this fact by making a product that equals $p_j'$ if the set of $p_j'$ is MI (see App. S for proof).

The reason we cannot simply ignore statance in this definition is that if we did ignore it, we could just choose $U$ to be an $r$-dimensional Fourier matrix, and then (29) would yield $p_j' = \frac{1}{r}$ for $j = 1, \ldots, r$ (so $D = r$), and since any set of equal probabilities is automatically MI, that would yield $P(\rho) = 0$ for all $\rho$, which cannot be true for any valid measure of probability correlation, because the nonzero correlance of some states in Family 5 demonstrates that probability correlation exists.

Therefore, we must first find the particular $\{U'\}$ that minimize $S(\rho)$ from (23) and then find a particular $U'$ that minimizes $P(\rho)$ to define the $p_j'$ for $\hat{P}(\rho)$.

Probablance $P(\rho)$ measures how far the decomposition probabilities are from having mode-independent factorizability for a decomposition that minimizes the decomposition-state correlation. For a given $\rho$ with decompositions $\rho = \sum j p_j \rho_j$, we get $\hat{P}(\rho) = 0$ if there exists a decomposition-state-correlation-minimizing decomposition for which $p_j = \prod_{m=1}^{n_j} [p_{j,m}]$, which $\hat{P}(\rho) = 0$ is guaranteed for Families 2, 4, and 6 of Table I. If $\hat{P}(\rho) > 0$, then there is no decomposition-state-correlation-minimizing decomposition of $\rho$ for which all $p_j$ have mode-independent product form. Thus, $\hat{P}(\rho)$ is a valid measure of probability correlation as defined in (7–8) (see App. S for proof, and App. T for an example).

Caution: While it is tempting to call $\hat{P}(\rho)$ the “classical correlation,” that would be inaccurate because although strictly classical states must have no superposition and are therefore diagonal in the computational basis, there are quantum states of the same form with inherently different meaning, as explained in Sec. III. Therefore, the correlation measured by $\hat{P}(\rho)$ can either be classical, quantum, or a composite of both depending on the physical system described by $\rho$, so we just call this correlation the “probability correlation,” since that is the most transparent term, and makes no extra assumptions.

For strictly classical states, since they are step functions of computational basis states according to Rule 3 in Sec. III C, only decompositions for which all decomposition states have no superposition in the computational basis are allowed, so the only permissible decompositions have $D = r$, $U$ is an $r$-level permutation unitary, and $\rho$'s eigenstates are required to be the computational basis states. The reason $D = r$ is because $U$ is limited to a permutation unitary (because superposition is not allowed), which would lead to some $p_j = 0$ if $D > r$, and that would cause our measures to include $p_j$ that do not contribute to $\rho$. Thus, for strictly classical states, statance $S(\rho)$ only considers the finite set of $r!$ permutation unitaries of dimension $r$ as $\{U\}$ and probablance $P(\rho)$ only considers the $\{U'\}$ from that set that minimize statance.

C. Observations about Statance and Probablance

Here, we use statance and probablance to make simple observations and do some special-case examples to lay the groundwork for a new measure that is comparable to quantum discord, but with several advantages, thereby showing statance and probablance to be worthwhile measures despite the difficulties in their calculation.

Theorem 2: Entanglement is merely sufficient for causing $S(\rho) > 0$, but is not necessary for that [i.e., states with $S(\rho) > 0$ do not all have entanglement, but all states with entanglement have $S(\rho) > 0$]. Proof: By definition, decomposition states of an entangled state cannot achieve product form, so they also cannot achieve mode independence as in (6), and therefore entangled states can never have $S(\rho) = 0$, which proves that entanglement is sufficient to cause $S(\rho) > 0$. The fact that there exist separable states $\rho = \sum_j (p_j^{(1)} \cdots p_j^{(N)}) p_j^{(1)} \otimes \cdots \otimes p_j^{(N)}$ which are not entangled by definition, but lack any decomposition states with mode independence (so they have $S(\rho) > 0$ as in Fig. 2 for Family 4 [for which $X(\rho) > 0$ implies $S(\rho) > 0$]), proves that entanglement is not necessary to achieve $S(\rho) > 0$. (Thus, entanglement is merely a special type of decomposition-state correlation.)

Theorem 3: Entanglement is the only kind of nonlocal correlation that pure states can have. Proof: If $\rho$ is pure, then $S(\rho)$ would only measure entanglement because all pure states have no probability correlation (since its only decomposition is it itself with probability 1), and any decomposition-state correlation (due to mode-independence violation) in a pure state is the same as violation of product form since its optimal decomposition is itself up to global phase, and violation of product form is the definition of entanglement in a pure state.

All of this prompts the question: Can a strictly classical state violate mode independence with its decomposition states alone? In other words, are there (diagonal) strictly classical states with $\hat{P} = 0$ but $S > 0$?

The answer is yes, but it depends on which basis states correspond to nonzero eigenvalues, and is limited by $r$.

For a strictly classical two-qubit state $\rho$, the case of $r = 1$ always involves a single pure computational basis state, which always has product form, so $\hat{S} = 0$ and $\hat{P} = 0$. For $r = 4$, the complete set of computational basis states will always form a mode-independent (MI) set for some of the permutation unitaries $U$, so $\hat{S} = 0$ while $\hat{P} \geq 0$.

For $r = 2$ and $r = 3$, the strictly-classical limitation of $D = r$ means that the probabilities will always appear MI, since for example a general set of $\{p_1, p_2, p_3\}$ can be interpreted as $\{p_1^{(1)} p_1^{(2)} p_2^{(1)} p_2^{(2)} p_3^{(1)} p_3^{(2)}\}$, where $p_1^{(1)} = 1$ and $\sum_{j=2}^{3} p_j^{(2)} = 1$ where $D = (1, 3)$ since $D_1 D_2 = D$. Therefore for $r = 2$ and $r = 3$ we get $\hat{P} = 0$ always.

But for statance, in $r = 3$ no group of three computational basis states can form an MI set and all 3-level permutation unitaries yield the same result. For example, suppose the decomposition states involved are

Therefore for $r = 2$ and $r = 3$ we get $\hat{P} = 0$ always.
\[ \rho_1 \equiv \rho_{(1,1)} = |1\rangle\langle 1| \otimes |1\rangle\langle 1| \\
\rho_2 \equiv \rho_{(1,2)} = |1\rangle\langle 1| \otimes |2\rangle\langle 2| \\
\rho_3 \equiv \rho_{(1,3)} = |2\rangle\langle 2| \otimes |1\rangle\langle 1|. \tag{30} \]

Then, putting (30) into (24) gives (for the 3-level \( U = I \)),
\[ \mu_1 = \frac{5}{2} |1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{1}{2} |2\rangle\langle 2| \otimes |1\rangle\langle 1| \\
\mu_2 = \frac{5}{2} |1\rangle\langle 1| \otimes |2\rangle\langle 2| + \frac{1}{2} |2\rangle\langle 2| \otimes |2\rangle\langle 2|, \tag{31} \]

which, when put into (23) and (22) yields
\[ \hat{S}(\rho) = \frac{1}{N_S} \min_{\{U\}} |S(\rho)| = \frac{4}{3N_S} > 0, \tag{32} \]

since \( S(\rho) = \frac{4}{3} \) here for all 3-level permutation unitaries, and thus we get another powerful result:

**Theorem 4:** Strictly classical states can have nonzero statance even though all of the decomposition states are computational basis states.

For \( r = 2 \), the factorizability of the set of basis states belonging to nonzero eigenvalues determines the statance. For example if \( \rho = \rho_1 |1\rangle\langle 1| \otimes |1\rangle\langle 1| + \rho_2 |1\rangle\langle 1| \otimes |2\rangle\langle 2|, \) then \( \hat{S} = 0 \) since the mode-1 state is a common factor, but if \( \rho = \rho_1 |1\rangle\langle 1| \otimes |1\rangle\langle 1| + \rho_2 |2\rangle\langle 2| \otimes |2\rangle\langle 2|, \) then \( \hat{S} > 0 \).

However, it is important to acknowledge that mixed strictly classical states are by necessity epistemic, so that the probabilities strictly represent observer ignorance of the actual ontic state. Since the ontic state of a strictly classical system is always pure, it can have no probability correlation, and since it is also a computational basis state, then it has no entanglement, and by Theorem 3 it also has no decomposition-state correlation. Therefore we can summarize all of this as:

**Theorem 5:** All ontic strictly classical states have no nonlocal correlation of any kind. (Where again, ontic means the actual state, which is instantaneously pure as defined in Sec. III C.)  
*Proof:* See the above text.

**Theorem 6:** Mixed strictly classical states (which are epistemic by necessity) can have nonlocal correlation as any combination of decomposition-state correlation and probability correlation, but can never have entanglement correlation, and all of this nonlocal correlation is strictly observer-induced. (Again, epistemic means that this is a state of observer knowledge, so any mixture of a strictly classical state is induced by the observer’s ignorance of the actual state.)  
*Proof:* See the text before Theorem 5.

Note that since strict classicality forbids superposition, reduction cannot induce mixture from pure strictly classical states; it can only remix the pre-existing observer-induced mixture of a mixed strictly classical parent state.

Theorem 5 is exactly what we should expect of the ontic state of a strictly classical system; nonlocality is impossible. Theorem 6 is significant because it shows that nonlocality can arise in mixed strictly classical states.

However, while these theorems are true for strictly classical states (which are all diagonal) they do not generally also apply to diagonal quantum states (for which general decomposition unitaries are allowed in statance and probability calculations, not merely permutation unitaries).

So far, we have developed one general measure of nonlocal correlation (the correlance \( \mathcal{X} \)), and two measures of specific kinds of nonlocal correlation (statance \( \hat{S} \) for decomposition-state correlation, and probablance \( \hat{P} \) for probability correlation). For entanglement correlation, we can use any valid entanglement measure, such as the ent \( \Upsilon \) or its generalization for distinguishing distinctly different types of multipartite entanglement as the ent-concurrence \( \Upsilon_C \).

Thus, we have measures for each of the main mechanisms of nonlocal correlation from Sec. I.A, although among these measures, only the correlance \( \mathcal{X} \) is computable for mixed states at this time.

Now we return to the essential question that prompted the idea of quantum discord: is it possible to construct a measure of nonlocal correlation that can distinguish between quantum and classical nonlocal correlation?

**V. STRONG DISCORDANCE, DISCORDANCE, AND DIAGONAL DISCORDANCE**

Before we can answer the question at the end of Sec. IV C, we need to define what we mean by classical correlations. Generalizing from Werner’s work, any fully N-partite separable state in the form of (4) is “classically correlated,” however we cannot accept this as accurate terminology here because separable quantum states in general fail the requirements for being strictly classical states in Sec. III C, even when such states are diagonal, as explained in Sec. III. Instead, we must recognize that strictly classical states are statistical mixtures built up from estimators of probabilities generated from using measurement time windows that are too long to show the true nature of the classical system as a step function of pure computational basis states, and contain no quantum superposition in any reference frame. In other words, the most general truly classical state is represented by a strictly classical epistemic mixed diagonal density matrix.

However, since ontic quantum states can have the same form as epistemic strictly classical mixed states (with the key difference that no choice of measurement time window could reveal a quantum state to be a step function of pure computational basis states unless that were the reality of the system) then we get another theorem:

**Theorem 7:** Decomposition-state correlation and probability correlation are not exclusively strictly-classical phenomena.  
*Proof:* There exist ontic quantum states of the same form as epistemic strictly classical mixed states but with inherently different meaning that nevertheless have nonentanglement-decomposition-state correlation and/or probability correlation as their only source of nonlocal correlation, such as the diagonal two-qubit reductions of some fully N-partite entangled N-qubit state [similar to (21) but where the reduction is a two-qubit system]. In other words; reductions of ontic quantum states can lead to ontic quantum mixed states of the same form as epistemic strictly classical mixed states.
A. Strong Discordance: A Measure of Nonlocal Correlation Beyond that Achievable by a Strictly Classical State

Despite the fact that Theorem 4, Theorem 6, and Theorem 7 seem to indicate that quantum states cannot be so easily distinguished from strictly classical states, we can still use these theorems as motivation to create a measure that reveals whether a quantum state has correlations beyond those of a strictly classical strength, which we will refer to as distinctly non-strictly-classical nonlocal correlation, or more compactly, distinctly nonclassical nonlocal correlation, or just distinctly quantum nonlocal correlation. (Essentially, this means that we acknowledge Theorem 7 but will still consider any nonlocal correlation in quantum states that is not greater than the correlation achievable by a strictly classical state to have classical strength regardless of the inherently quantum physical origins that cause it.)

Therefore we define the strong discordance of \( \rho \) as a measure of nonlocal correlation beyond that achievable by a strictly classical state, given by

\[
D_S(\rho) = \frac{1}{\mathcal{N}_{DS}} \max \{0, \chi(\rho) - \chi(\rho_{D_{max}})\}, \tag{33}
\]

where \( \chi(\rho) \) is the correlance from (9), with normalization

\[
\mathcal{N}_{DS} = \max_{\{\rho'\}} \{\chi(\rho') - \chi(\rho_{D_{max}})\} = 1 - \frac{2^{N-1} - 1}{2^N \chi_N}, \tag{34}
\]

where \( \rho_{D_{max}} \) are any diagonal states that maximize \( \chi_D(\rho) \) such as (17), and \( \chi(\rho_{D_{max}}) = \chi(\rho_{D_{max}})/\chi_N \), where \( \chi(\rho_{D_{max}}) = \chi_D(\rho_{D_{max}}) = \frac{1}{2} - \frac{1}{2^N} \) from (18), and \( \chi_N \) is from (12), (13), or (K57), so then (33) becomes

\[
D_S(\rho) = \frac{2^N \chi_N}{2^N \chi_N - 2^{N-1} + 1} \max \{0, \chi(\rho) - \frac{2^{N-1} - 1}{2^N \chi_N}\}. \tag{35}
\]

Thus, (35) gives strong discordance in exactly computable form for all pure and mixed \( N \)-partite states. The definition of \( D_S(\rho) \) is such that the only states that can attain \( D_S(\rho) > 0 \) are those with correlance \( \chi(\rho) \) that exceeds that of the correlance-maximizing diagonal states \( \rho_{D_{max}} \) (since those have the form of the most correlated strictly classical states). Thus, strong discordance \( D_S(\rho) \) is a measure of how much nonlocal correlation exists in \( \rho \) beyond that achievable by a mixed strictly classical state. \( D_S(\rho) = 0 \) iff there exists a mixed strictly classical state that can achieve the same amount of nonlocal correlation as \( \rho \). Similarly, if \( D_S(\rho) > 0 \), then there is definitely some nonlocal correlation that is distinctly quantum and not achievable by a strictly classical state, and \( D_S(\rho) = 1 \) iff \( \rho \) is maximally entangled. However, having \( D_S(\rho) = 0 \) does not guarantee that there is no nonlocal correlation, and it does not necessarily mean that the nonlocal correlation has nonquantum origins; as stated earlier, states with \( D_S(\rho) = 0 \) can still have nonlocal correlation arising from distinctly quantum mechanisms such as entanglement, but the key point is that for these states, the correlation is weak enough that the same amount could be generated by a mixed strictly classical state, and that is why we consider it to be “not distinctly quantum.” Also as mentioned earlier, states with \( D_S(\rho) > 0 \) do not necessarily have exclusively quantum origins of nonlocal correlation; there may be multiple mechanisms, some of which can be nonquantum in nature (such as probability correlation arising from an epistemic mixture of nondiagonal quantum states.) Figure 8 tests strong discordance \( D_S(\rho) \) for the same families as Fig. 2.

In summary, a few caveats for strong discordance are:

1. The definition of \( D_S(\rho) \) in (33–35) implies that \( D_S(\rho) > 0 \) guarantees some distinctly nonclassical nonlocal correlation, but it does not necessarily mean there are no sources of strictly classical nonlocal correlation contributing to the total correlation.

2. A state for which \( D_S(\rho) = 0 \) is not necessarily a strictly classical state or even a diagonal state, but rather it is a state with no nonlocal correlations stronger than those of a strictly classical state, meaning that it may have any combination of probability correlation and decomposition-state correlation (including entanglement), but that there exists a strictly classical state that could achieve the same amount of nonlocal correlation using a combination of only probability correlation and nonentanglement-decomposition-state correlation.

3. A value of \( D_S(\rho) > 0 \) guarantees some nondiagonality of \( \rho \), but some nondiagonal states can have \( D_S(\rho) = 0 \) (so \( D_S(\rho) = 0 \) does not imply diagonality). Proof: The maximum correlance of all diagonal states is the threshold for \( D_S(\rho) = 0 \), so any states with \( D_S(\rho) > 0 \) must have more correlance than the most correlated diagonal states, and therefore must not be diagonal. For the second claim, a proof by example is that \( D_S(\rho) = 0 \) for a product state of nondiagonal mode states such as \( \rho = \frac{1}{2} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \otimes \frac{1}{2} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \).
4. The evidence that $D_S(\rho) = 0$ for all separable states in Fig. 8 strongly supports the requirement that strictly classical states must be diagonal, because their value of $\mathcal{X}(\rho_{\text{max}})$ defines the threshold of distinctly quantum correlation strength in (33).

Therefore, strong discordance is a more selective measure than quantum discord, but because it rejects some weakly entangled states, it is too strong to be a workable alternative to quantum discord. Therefore, next we develop a more inclusive measure that is closer in spirit to quantum discord, but more conceptually consistent.

### B. Discordance: A Measure of Nonlocal Correlation in a Distinguishably Quantum State

Here, we define a more inclusive measure of nonlocal correlation that can never report a zero for distinctly quantum states such as entangled states, even if strictly classical states exist that have the same correlance.

To achieve this, we need a way to distinguish strictly classical states from nonlocally correlated quantum states with novel quantum features. A practical feature for this purpose is coherence (here, coherence means that $\rho$ has some nonzero off-diagonal elements in the computational basis). Even though superposition and coherence do not imply nonlocal correlation, they do imply the presence of novel quantum effects, and are necessary for some nonlocal correlations such as entanglement. One caveat here is that, as we showed in (21), diagonal quantum states can have “superposition without coherence” (such as reductions of maximally entangled states), so if we use coherence as a criterion for being a distinguishably quantum state, we would still be rejecting diagonal quantum states with nonzero correlance.

However, as discussed in Sec. III C, it is not generally possible to determine whether a diagonal quantum state is quantum or strictly classical without doing a potentially impossible set of experiments or having more information beyond the state, such as its parent state before a reduction. Therefore, for the purpose of measuring nonlocal correlation in a state that is definitely quantum using the state alone, the best we can do is use coherence as a criterion to reject strictly classical states at the expense of also rejecting diagonal quantum states that may have some nonlocal correlation.

As a preliminary step, we define raw nondiagonality as

$$\bar{\nu}(\rho) \equiv \text{tr}[\{(\rho - \Delta)^2\}],$$

(36)

where $\Delta$ is the maximally dephased input state,

$$\Delta \equiv \Delta(\rho) \equiv \sum_{a=1}^{n} \rho_{a,a} |a\rangle \langle a|,$$

(37)

so that $\bar{\nu}(\rho) = 0$ iff $\rho$ is diagonal, and $\bar{\nu}(\rho) > 0$ iff $\rho$ is not diagonal. We could normalize $\bar{\nu}(\rho)$ either over all states [by dividing $\bar{\nu}(\rho)$ by $1 - \frac{1}{n}$] or to only the subspace for which the diagonal elements are nonzero, but for our purposes, we are only interested in the sign of $\bar{\nu}(\rho)$, so we do not need to normalize it.

Then, we can define a measure of nonlocal correlation in a distinguishably quantum state as the discordance,

$$D(\rho) \equiv \text{sgn}[\bar{\nu}(\rho)]\mathcal{X}(\rho),$$

(38)

where $\bar{\nu}(\rho)$ is from (36) and $\mathcal{X}(\rho)$ is the correlance from (9). Thus, only states with some nondiagonality can have nonzero discordance, which ensures that all entangled states are recognized having discordance, and only diagonal quantum states with nonzero correlance have their nonlocal correlation unrecognized (with the justifiability for that being that such states have density matrices with identical form to a strictly classical state, meaning that the state alone is not sufficient to distinguish them from strictly classical systems).

$D(\rho) = 0$ if $\rho$ has no nonlocal correlation or if the state is diagonal (whether it has nonlocal correlation or not), and $D(\rho) = 1$ iff $\rho$ is maximally entangled. Figure 9 plots the discordance for the families of Table I.

![FIG. 9: (color online) Discordance $D$ of arbitrary two-qubit mixed states for each of the six families of nonlocal correlation of Table I, $10^5$ states each (colors not related to Fig. 1). This shows that only Family 6 (product states, pure and mixed) has $D = 0$ always, while Families 3–5 (all separable) can have $D > 0$ (indicating some instances of quantum nonlocality without entanglement [14–16]). Essentially, discordance is correlation recognized for nondiagonal states only. Again, only if entanglement is allowed, as in Families 1 and 2, can states achieve the full range of discordance up to $D = 1$.](image)

Thus, we now have three main computable measures on all mixed and pure states to measure different degrees of general nonlocal correlation:

1. Strong discordance $D_S(\rho)$ (nonzero for states with more nonlocal correlation [as measured by correlance] than what is achievable by any strictly classical state).

2. Discordance $D(\rho)$ (nonzero for quantum states with nonlocal correlation [as measured by correlance] that have the distinguishably quantum feature of coherence and are thus nondiagonal).

3. Correlance $\mathcal{X}(\rho)$ (nonzero for any nonlocally correlated state, whether quantum or strictly classical).

Ironically, as far as we know, all evidence suggests that there is really no such thing as a strictly classical state.
in the real world, and that all states are truly quantum. Therefore, correlance is really all we ever need to measure the presence of any nonlocal correlation in a state.

However, for the purpose of distinguishing quantum from strictly classical scenarios, the above measures give us powerful tools with different degrees of specificity. In fact, we will show that they obey a similar relationship to (2), but first we will do some simple examples to compare these measures to quantum discord.

C. Comparisons of Discordance Measures with Quantum Discord

Here, we briefly compare correlation, discordance, and strong discordance with quantum discord, and also concurrence for reference. Keep in mind that none of these new measures is meant to calculate quantum discord, but rather they are intended as different measures of the same intended qualitative features, with various restrictions that might be useful in particular applications.

First, we consider two-qubit Werner states [32],

\[ \rho = a |\Psi^- \rangle \langle \Psi^-| + \frac{1-a}{4} I, \]  

where \( a \in [0,1] \), \( |\Psi^- \rangle \equiv \frac{1}{\sqrt{2}}(|1,2\rangle - |2,1\rangle) \) is the (maximally entangled) singlet state (where the generic basis for each qubit is \{1,2\}), and \( \frac{1}{2} I \) is the maximally mixed state. Using the quantum discord from [13], Fig. 10 compares the various measures, showing that discordance and correlance behave similarly to quantum discord.

Next, consider the mixture (from [13]),

\[ \rho = \frac{1}{3} \left( (1-a) |1,1\rangle \langle 1,1| + 2 |\Psi^+ \rangle \langle \Psi^+| + a |2,2\rangle \langle 2,2| \right), \]

where \( a \in [0,1] \) and \( |\Psi^+ \rangle \equiv \frac{1}{\sqrt{2}}(|1,2\rangle + |2,1\rangle) \). Using its \( Q \) from [13], Fig. 11 compares it to our various measures, again showing that \( D \) and \( X \) have similar behavior to \( Q \).

Thus, these (limited) tests in Fig. 10 and Fig. 11 agree with our earlier reasoning that for the purpose of finding a substitute for quantum discord \( Q \), strong discordance \( D_S \) is too exclusive while correlance \( X \) is too inclusive (since it recognizes nonlocal correlation in diagonal states, which is not encountered in these examples). However, discordance \( D \) seems to be the most inclusive it is possible to be without having more information beyond the state itself, making it the most appropriate substitute for quantum discord. In fact, all of these measures are related in an elegant way that formally parallels the theory of quantum discord, as we will show next.

D. Analogous Quantities to Quantum Mutual Information, Classical Correlation, and Quantum Discord

Since discordance \( D \) only recognizes the nonlocal correlation of nondiagonal states, here we define a measure for only diagonal states as the diagonal discordance,

\[ D_D(\rho) \equiv (1 - \text{sgn}[\nu(\rho)])X(\rho), \]

which is just the correlance \( X(\rho) \) of (9) with a sifting factor to map all nondiagonal states to 0, where \( \nu \) is the raw nondiagonality from (36). [We could define (41) in terms of \( X_D \) from (15), but as we show next, it is more useful to leave it in this “unnormalized” form. Also, \( X_D \) does not have the sifting factor \((1 - \text{sgn}[\nu(\rho)])\) because \( X_D \) is a stand-alone measure on diagonal states. The sifting factors in \( D_B \) and \( D \) are justified because they are meant to be used together, as shown next.] Thus, diagonal discordance \( D_D \) measures the nonlocal correlation
in diagonal states, be they strictly classical or diagonal quantum, and is loosely the analog of “classical correlation" \( \mathcal{C} \) from the quantum discord theory. 

Interestingly, comparing (38) and (41), we see that 

\[
\mathcal{X}(\rho) = D_D(\rho) + D(\rho),
\]

in formal analogy to (2) which was \( \mathcal{I}(\rho) = \mathcal{C}(\rho) + Q(\rho) \) where \( \mathcal{I} \) is the quantum mutual information, \( \mathcal{C} \) is the classical correlation, and \( Q \) is the quantum discord.

However, the similarity of (42) to (2) is a deceptive parallel for several reasons:

1. Classical correlation \( \mathcal{C} \) inherently contains a conceptual flaw: the rank-1 projection operators (of the von Neumann measurements involved in its definition) are generally allowed to have superposition, which sabotages the goal of measuring distinctly classical correlation by allowing it to depend on novel quantum properties. In contrast, classical discordance \( D_D \) only registers as nonzero for diagonal states, which have no coherence, making them indistinguishable from strictly classical states, so \( D_D \) is a much more appropriate measure of classical correlation (with the caveat that diagonal quantum states can have “superposition without coherence” as discussed in Sec. III).

2. Since quantum discord \( Q \) is defined as \( Q \equiv \mathcal{I} - \mathcal{C} \), it inherits the conceptual flaw of \( \mathcal{C} \) from Reason 1. In contrast, \( D \) of (38) is designed to be the most general and inclusive measure of nonlocal correlation in distinguishably quantum states through its rejection of diagonal states, since, barring further information beyond the state, nondiagonality is the defining feature separating quantum from classical. Again, we acknowledge the existence of diagonal quantum states, but they are indistinguishable from strictly classical states unless further information is known.

3. Classical correlation \( \mathcal{C} \) and quantum discord \( Q \) are not mutually exclusive since they can be simultaneously nonzero. While this is not necessarily a problem since one can imagine simultaneous influences of classical and quantum mechanisms of correlation, the conceptual flaw mentioned in Reason 1 raises the question of whether there could be some overlap of what these two measures detect. Meanwhile, \( D_D \) and \( D \) are mutually exclusive, since they depend on whether or not the state has coherence (with the value also depending on \( \mathcal{X} \)). However, as mentioned earlier, since all states really are fundamentally quantum, it makes more sense to divide based on the novel quantum feature of coherence (nondiagonality) than it does to define classicality based on measurements allowing projectors with superposition as in the definition of \( \mathcal{C} \) and thus \( Q \).

4. Quantum mutual information \( \mathcal{I} \) is only defined for bipartite systems, whereas correlation \( \mathcal{X} \) can handle multipartite systems (and is extended to further multipartite generalizations in Sec. VII).

Therefore, while (42) does not exactly parallel the quantum-discord relation of (2), it does have a solid grounding in the well-defined notion of nonlocal correlation in terms of actual quantum states and their distance from achieving mode-independence, which is the basis of correlation \( \mathcal{X} \). It makes sense to use diagonality as the criterion for classicality since that is the form of strictly classical states of Sec. III which are carefully defined to avoid superposition both internally and through transformation.

Furthermore, since we already showed in Theorem 4, Theorem 6, and Theorem 7 that stateance \( S \) and probability \( P \) can arise in both diagonal and nondiagonal states, it makes sense that the general measure of correlation \( \mathcal{X} \) is used to calculate the actual value of correlation for both diagonal states in \( D_D \) and nondiagonal states in \( D \), since \( \mathcal{X} \) measures all forms of nonlocal correlation whether or not the state is quantum or classical.

Thus, our definitions divide correlations in a much more conceptually consistent way than quantum discord simply by focusing on the state itself and carefully acknowledging the distinctly quantum feature of coherence.

VI. HIDDEN VARIABLES ARE NOT ANOTHER KIND OF NONLOCAL CORRELATION

Throughout this paper, we use the term nonlocal correlation to mean a state’s inability to be factored into a mode-independent product form as defined in Sec. I.A. However, there is a more subtle way to achieve nonlocal correlation that we must consider.

It is often said that “correlation does not imply causation” (meaning that correlation does not imply variable-dependence), but that does not mean that causation cannot give rise to correlation (meaning that variable-dependence can lead to correlation, just not always).

This is essentially the idea of hidden-variable (HV) theories [30, 33–35], where all states of quantum theory can be recast for each observable as diagonal states with variable-dependent probabilities that yield the same mean values of that observable as quantum theory, implying all possible correlations, including those perceived as entanglement. Furthermore, HV models produce deterministic theories, in which the future is exactly predictable if one knows the initial conditions of some set of “hidden variables,” whose trajectories, when sampled, yield the variable-dependent probabilities. Hidden-variable theories are generally regarded with skepticism for many reasons [36–38], and a full treatment of this subject is beyond the scope of this paper.

However, since correlation measures nonlocal correlation as a state’s distance from mode-independent product form, then the variable-dependence of an implied quantum state \( \rho \) constructed from a complete HV theory only affects nonlocal correlation insofar as it affects the state’s closeness to mode-independent product form. In other words, HV theories do not constitute a new mechanism of nonlocal correlation, so all measures in this paper also apply to states of HV theories.
VII. MULTIPARTITE GENERALIZATIONS BEYOND N-MODE CORRELATIONS

So far, correlation, diagonal correlation, statance, probability, strong discordance, discordance, and diagonal discordance only measure nonlocal correlations over all $N$ modes of an $N$-mode coincidence system (see [18], App. A). For instance, the correlation $\mathcal{X}(\rho)$ of (9) measures how far $\rho$ is from being a product state of $N$ modes as its distance from its own reduction product (since only product states are their own reduction products).

But in general, nonlocal correlation can arise in multipartite systems in many different ways, and they are not all equivalent. For example, from [22], a 4-qubit GHZ state [39–41] and a Bell-product state can have the same full $N$-partite entanglement by some necessary and sufficient measure of full $N$-partite entanglement, but the separability of the two Bell states in the Bell-product state shows that it has a distinctly different kind of multipartite entanglement than the GHZ state for which no partitions of the $N$ modes are separable.

Furthermore, we may only be interested in determining a correlation property for a subset of the total modes, or we may want to view groups of modes as if they were single modes and determine that property between those groups, or between several groups and a single mode.

In [22], a preliminary study was done on how best to adapt the ent from an $N$-mode entanglement measure to a more general measure called the ent-concurrence to measure distinctly multipartite entanglement. The ent-concurrence (equal to the concurrence [42, 43] for the case of two qubits) was shown to have the ability to distinguish between distinctly different types of multipartite entanglement for states that may have the same amount of full $N$-partite entanglement via the ent, such as a 4-qubit GHZ state and a Bell-product state.

This section explains how to generalize the correlation to measure the more general distinctly multipartite features described above, based on the formalism of [22]. Generalizations of diagonal correlation, statance, probability, strong discordance, discordance, and diagonal discordance follow that of correlation by direct analogy, and are all computable on all states, except for generalizations of statance and probability, for which computable expressions are not yet known.

A. Multicorrelation

The multicorrelation for any pure or mixed $\rho$ is

$$M_X(\rho) \equiv \frac{1}{N_M} \sum_{k=2}^N M^{(N^{(k)})}_X(\rho),$$

where $N_M$ is a normalization factor from maximizing the unnormalized $M_X(\rho)$ over all possible input states, and we define the $N$-mode $k$-partitional multicorrelation,

$$M^{(N^{(k)})}_X(\rho) \equiv \frac{1}{N_M} \sum_{h=1}^{N^{(k)}} \sqrt{\mathcal{X}^{(N^{(k)})}_h(\rho)},$$

where $\mathcal{X}^{N_M}_k$ is a normalization factor again over all possible input states, $\{k\} \equiv \{1\} \cup \sum_{j=2}^N (-1)^{k-j} \binom{k}{j} N^{(k)}_j$ are Stirling numbers of the second kind where $\binom{k}{j} \equiv \frac{k^j}{j!}$, and $\{\mathcal{X}^{(N^{(k)})}_h(\rho)\}$ is the set of all $N$-mode $k$-partitional correlations (defined below), where $h$ is an index arbitrarily assigning a label to a given unique partition of modes.

To define the $N$-mode $k$-partitional correlations, first we need a more explicit definition of correlation as

$$\mathcal{X}(\rho) \equiv \mathcal{X}(\rho, n) \equiv \frac{1}{N_{X}(n)} \text{tr} ((\rho - \zeta(\rho, n))^2),$$

where $N_X(n) = \max_{\rho' \in \mathcal{H}} \{\text{tr}[\rho' - \zeta(\rho', n)]^2\}$, where the maximizing states $\rho'_{\text{max}} \equiv \{\rho_{\text{ME}}, \{ho_{\text{ME-TOKX}}\}\}$ are maximally full-$N$-partite-entangled states of an $N$-mode $n$-level system where $n \equiv (n_1, \ldots, n_N)$, mode $m$ has $n_m$ levels, $n \equiv n_1 \cdots n_N$, and the reduction product is

$$\zeta(\rho, n) \equiv \sum_{m=1}^N \rho^{(m)}.$$ \hspace{1cm} (46)

Then, recalling that $\rho^{(m)}$ is the $n_m$-level single-mode reduction of $\rho$ for mode $m$, and that $\rho^{(m)}$ is the multimode reduction where $m \equiv (m_1, \ldots, m_S)$ for $S \leq 1, \ldots, N$, we see that the indices of argument $n$ are the mode labels $m$ (with $S = N$) that label the reductions in $\zeta(\rho, n)$, so that $n$ implicitly governs the tensor product in (46).

Given the above, let the partitional correlation be

$$\mathcal{X}(\rho^{(m)}) \equiv \mathcal{X}(\rho^{(m)}, n^{(m)}) \equiv \frac{1}{N_{X}(n^{(m)})} \text{tr} \left[(\rho^{(m)} - \zeta(\rho^{(m)}, n^{(m)}))^2\right],$$

which lets us specify as input any reduction to mode group $m$ of the input state with any repartitioning of its mode structure into new mode groups as $n^{(m)} \equiv (n^{(1)}_m, \ldots, n^{(T)}_m)$ where $T \in \{1, \ldots, S\}$, and $N_{X}(n^{(m)}) \equiv \max_{\rho' \in \mathcal{H}(m)} \{\text{tr}[\rho' - \zeta(\rho', n^{(m)})]^2\}$ where the maximizing states $\rho_{\text{max}}^{(m)} \equiv \{\rho_{\text{ME}}, \rho_{\text{ME-TOKX}}\}$ are maximally $T$-partite entangled states of a $T$-mode $n^{(m)}$-level system of mode-structure $n^{(m)} \equiv (n^{(1)}_m, \ldots, n^{(T)}_m)$, where the input states can be specified in terms of the general reduction structure and underlying original modes as $\rho^{(m)}$ where $m \equiv (m_1, \ldots, m_S)$ since it is true by definition that $n^{(m)} = n^{(m)}_T$ where $n^{(m)} \equiv (n^{(1)}_m, \ldots, n^{(T)}_m)$ and $n^{(m)}_T \equiv n^{(1)}_m \cdots n^{(T)}_m$. See ([22], App. C) for more details about the notation. We do not use the reduction symbol over density matrices in the normalization because this maximization is over all states in the Hilbert space $\mathcal{H}(m)$ of the reduced system, not merely reductions from the parent state. The reduction product in (47) is then

$$\zeta(\rho^{(m)}; n^{(m)}) \equiv \sum_{q=1}^T \rho^{(m^{(q)})}.$$ \hspace{1cm} (48)

where each multimode reduction $\rho^{(m^{(q)})}$ has internal mode structure $m^{(q)} \equiv (m_1^{(q)}, \ldots, m_S^{(q)})$ where $Q^{(q)} \in \{1, \ldots, S\}$, in terms of the original indivisible modes $m_j$.
such that all of them appear exactly once among all new mode groups $m^{(q)}_s$ for $q = 1,...,T$. Thus, in the case of the nonreduction ($S = N$), the input state to the partitional correlance is the full state $\tilde{\rho}^{(N)} = \tilde{\rho}^{(1,...,N)} = \rho$, and the new mode-group vector is (in this particular case) any element of the set $m^{(T)} \in \{N^{(h)}_k\} \equiv \{(N^{(h)}_1^{(1)},...,N^{(h)}_n^{(k)})\}_{h=1}^k$, meaning it is the set of all $\{N^{(k)}_h\}$ unique partitions of $N$ objects partitioned into $k$ groups.

As an example of how the notation works, for a tripartite system, $N = 3$ and $\{1,...,N\} = \{1,2,3\}$, so the possible $N$-mode partitions into $k = 2$ groups are

$$m^{(T)} \in \{(1\mid 2\mid 3),(2\mid 1\mid 3),(3\mid 1\mid 2)\} \equiv \{\{N^{(1)}_1\},\{N^{(2)}_2\}\}, \{\{N^{(1)}_1\},\{N^{(2)}_2\}\}, \{\{N^{(1)}_1\},\{N^{(2)}_2\}\},$$

(49)

where notice that reordering is not considered unique here, so for instance we do not list $(3\mid 2\mid 1)$ as a unique option since we have already listed $(3\mid 1\mid 2)$. Therefore, specifying $m^{(T)} \in \{N^{(h)}_k\}$ in the partitional correlance gives the $N$-mode $k$-partitional correlances as

$$\lambda^{(N^{(h)}_k)}(\rho) = \lambda^{(N^{(h)}_k)}(\tilde{\rho}^{(N)}),$$

$$\equiv \frac{1}{\lambda^{(N^{(h)}_k)}} \text{Tr} \left( |\rho - \lambda^{(N^{(h)}_k)}(\tilde{\rho}^{(N)})|^{2} \right),$$

(50)

with $N$-mode $k$-partitional reduction products,

$$\gamma^{(N^{(h)}_k)}(\rho) \equiv \prod_{q=1}^{k} \lambda^{(N^{(h)}_k)}(\rho).$$

(51)

Therefore, (50) appears in the terms of (44). The multicorrelance $\lambda^{(N^{(h)}_k)}(\rho)$ measures the simultaneous amount of all $N$-mode $k$-partitional correlances $\lambda^{(N^{(h)}_k)}(\rho)$, while each individual $\lambda^{(N^{(h)}_k)}(\rho)$ measures not only the combination of all possible $N$-mode $k$-partitional correlances, but it also measures how equally distributed they are by use of the square root. Alternatively, we may also define the $h$th $N$-mode $k$-partitional root-correlance as

$$\lambda^{(N^{(h)}_k)}(\rho) \equiv \sqrt{\lambda^{(N^{(h)}_k)}(\rho)},$$

(52)

which simplifies the notation for later concepts. Note that in this nomenclature, the prefix multi implies a sum over all varieties of something, whereas the absence of multi means no sum over varieties. For example, the $N$-mode $k$-partitional multicorrelance implies a sum over all $N$-mode $k$-partitional root-correlances which are each single-term quantities for a specific $N$-mode $k$-partition.

The multicorrelance $\lambda^{(N^{(h)}_k)}(\rho)$ is built to reveal whether there is any nonlocal correlation at all in a given state at its $N$-mode scale, even including different perspectives created by grouping those modes together in a way that the total size of the input state remains the same.

As an example, for a 4-partite system, the (unnormalized) multicorrelance is, from (43),

$$\overline{\lambda}^{(N^{(h)}_k)}(\rho) = \lambda^{(N^{(2)}_1)}(\rho) + \lambda^{(N^{(3)}_1)}(\rho) + \lambda^{(N^{(4)}_1)}(\rho),$$

(53)

where $\lambda^{(N^{(h)}_k)}(\rho) \equiv \overline{\lambda}^{(N^{(h)}_k)}(\rho)/N^{(M)}$ are the $N$-mode $k$-partitional multicorrelances from (44), given in unnormalized form (with correlance inputs suppressed) by

$$\overline{\lambda}^{(N^{(h)}_k)}(\rho) = \sqrt{\lambda^{(N^{(h)}_k)}(1)} + \sqrt{\lambda^{(N^{(h)}_k)}(2)} + \sqrt{\lambda^{(N^{(h)}_k)}(3)} + \sqrt{\lambda^{(N^{(h)}_k)}(4)}$$

$$+ \sqrt{\lambda^{(N^{(h)}_k)}(5)} + \sqrt{\lambda^{(N^{(h)}_k)}(6)} + \sqrt{\lambda^{(N^{(h)}_k)}(7)} + \sqrt{\lambda^{(N^{(h)}_k)}(8)}$$

(54)

where the radicands above are obtained from (50).

The specification of the structure vectors for each term then determines how each one is specifically calculated by controlling which effective modes are recognized for each term. The result is that if any nonlocal correlation exists between any partitioning of the $N$ modes, the multicorrelance will report the presence of that correlation as a value relative to some maximum over all states.

The square root of each partitional correlance is used here in analogy with the definition of ent-concurrency from [22], where it was shown to be able to distinguish different types of distinctly multipartite entanglement, whereas without the square root, states like the 4-qubit GHZ state and Bell-product states did not appear to have different multipartite entanglement. In this context, tests of the 4-qubit tier-1, tier-2, and tier-3 maximally entangled states of [22] have shown that the square root is necessary to distinguish the $N$-mode 3-partitional correlances of the tier-2 and tier-3 states (GHZ state and Bell-product states) which are identical if the square root is omitted. See [22] for mathematical details about why the square root is appropriate for the ent-concurrency; the same argument applies here as well.

The purpose of the detailed notation is just to keep track of partitions for the purpose of telling the correlation function over which groups of modes we want to measure nonlocal correlations. As we will soon see, this degree of specificity will allow us to extract information about all possible nonlocal correlations (with the limitation that the correlation cannot distinguish between particular types of nonlocal correlation such as free entanglement and bound entanglement [44–46]). For a more in-depth exposition of partitions in this same notation, see ([22], App. C), and for further details on multipartite reductions see ([18], App. B).

B. $N$-mode Partitional Multicorrelance Vector

For a finer-grained picture of nonlocal correlation that tells us between which mode groups correlation exists, we define the $N$-mode partitional multicorrelance vector,

$$\lambda^{(N^{(h)}_k)}(\rho) \equiv \begin{pmatrix} \lambda^{(N^{(h)}_k)}(\rho) \\ \vdots \\ \lambda^{(N^{(h)}_k)}(\rho) \end{pmatrix},$$

(55)
where \( \mathcal{M}_N^{(N_k)}(\rho) \) is the set of all \( N \)-mode \( k \)-partitional root-correlances, and is valid for both pure and mixed states \( \rho \). For example, in a 4-mode system (suppressing input arguments),

\[
\Xi_{\mathcal{M}_X}^{(4)} \equiv \Xi_{\mathcal{M}_X}^{(1,2,3,4)} = \left( \begin{array}{c}
\mathcal{M}_X^{(1|2,3,4)} \\
\mathcal{M}_X^{(2|1,3,4)} \\
\mathcal{M}_X^{(3|1,2,4)} \\
\mathcal{M}_X^{(4|1,2,3)}
\end{array} \right),
\]

where each particular \( \mathcal{M}_N^{(N_k)}(\rho) \) is given by (52). The top row gives each \( N \)-mode 2-partitional root-correlance, and each row farther down treats increasing partitions until the bottom row gives the \( N \)-mode \( N \)-partitional root-correlance which is just the square root of the actual correlance from (9). Thus, \( \Xi_{\mathcal{M}_X}^{N} \) gives a fine-grained, location-specific view of nonlocal correlation.

For a more intermediate picture of nonlocal correlation, the \( N \)-mode \( k \)-partitional multicorrelances from (44), expressible as [with the help of (52)],

\[
\mathcal{M}_X^{(N_k)}(\rho) \equiv \frac{1}{N_{\mathcal{M}_X}} \sum_{h=1}^{N_h} \mathcal{M}_X^{(N_k)}(\rho),
\]

give a measure of the total possible nonlocal correlance over all partitions of a certain \( k \)-value for an \( N \)-mode system, such as all possible bipartitions.

Note that the more aggregated measures such as multicorrelance and \( N \)-mode \( k \)-partitional multicorrelance do not necessarily imply that all of these nonlocal correlation resources are available for use simultaneously; rather they indicate that such correlations are present, and typically only some of them may be used simultaneously.

C. Absolute Multicorrelance

Again following [22], while the multicorrelance \( \mathcal{M}_X(\rho) \) and its more specific \( N \)-mode partiational multicorrelance vector \( \Xi_{\mathcal{M}_X}^{N}(\rho) \) and the even more specific \( N \)-mode \( k \)-partitional root-correlances \( \mathcal{M}_X^{(N_k)}(\rho) \) give us a sense of the nonlocal correlance present in the full input state within its full Hilbert space, we can get an even more in-depth picture of the resources available in a state by evaluating the nonlocal correlations within reductions of the input state.

To this end, rather than look at all \( N \) modes, we focus on a subset of modes denoted by mode group \( \mathbf{m} \equiv (m_1, \ldots, m_S) \) for \( S \in 2, \ldots, N \) (we exclude \( S = 1 \) here since nonlocal correlance is correlation between at least two modes), and thus now want to look for multitarget nonlocal correlances of some \( S \)-mode reduction of \( N \)-mode state \( \rho \). Thus, we define the \( S \)-mode particional multicorrelance vector as

\[
\Xi_{\mathcal{M}_X}^{(\mathbf{m})}(\rho) \equiv \left( \begin{array}{c}
\mathcal{M}_X^{(m_1)}(\rho) \\
\cdots \\
\mathcal{M}_X^{(m_S)}(\rho)
\end{array} \right),
\]

where \( \mathcal{M}_X^{(m_1)}(\rho) \) is the set of all \( S \)-mode \( T \)-partitional root-correlances of a given reduction \( \tilde{\rho}^{(m)} \), where \( T \in 2, \ldots, S \) here and each particular partitioning is labeled by \( h \), and each \( S \)-mode \( T \)-partitional root-correlance is

\[
\mathcal{M}_X^{(m_T)}(\rho) \equiv \sqrt{\chi_{\mathcal{M}_X}(m_T)(\rho)},
\]

where the \( h \)th \( S \)-mode \( T \)-partitional correlance is

\[
\chi_{\mathcal{M}_X}(m_T)(\rho) \equiv \chi_{\tilde{\rho}^{(m)}}(\rho, n^{(m_T)}),
\]

\[
= \frac{1}{N_{\mathcal{M}_X}} \text{tr} \left( (\tilde{\rho}^{(m)} - \zeta(\tilde{\rho}^{(m)}, n^{(m_T)}))^2 \right),
\]

with \( S \)-mode \( T \)-partitional reduction products,

\[
\zeta(\tilde{\rho}^{(m)}, n^{(m_T)}) \equiv \frac{T}{q=1} \tilde{\rho}^{(m)(q)},
\]

where (61) is written in terms of a general input state \( \rho^{(m)} \) in the Hilbert space of the reduced system to help show that the reductions of this state must be taken to calculate this quantity. Therefore, in (60) we are taking reductions of reductions to calculate \( \zeta(\tilde{\rho}^{(m)}, n^{(m_T)}). \)

Again, since correlance can handle mixed states, all of these definitions work for general states both pure and mixed. Thus, in (58), row \( T - 1 \) is a list of all \( S \)-mode \( T \)-partitional root-correlances of \( \tilde{\rho}^{(m)} \) which is the mode-\( \mathbf{m} \) reduction of \( \rho \).

Then, to account for the fact that we can make \( \Xi_{\mathcal{M}_X}^{(\mathbf{m})}(\rho) \) for many different reduction mode-groups \( \mathbf{m} \) for a given parent state \( \rho \), we can collect them all in a larger object called the multicorrelance array \( \overline{\Xi}_{\mathcal{M}_X} \) (not a gradient), the elements of which are all possible \( S \)-mode particional multicorrelance vectors,

\[
(\overline{\Xi}_{\mathcal{M}_X})_{k,l}(\rho) \equiv \Xi_{\mathcal{M}_X}^{(\mathbb{N}_k \mathbb{C}_k \mathbb{N}_k \mathbb{C}_k \cdots)}(\rho),
\]

where again \( \mathbb{N} \equiv (1, \ldots, N) \), and \( k \) \( \in \{1, \ldots, N\} \), and \( (\mathbb{N}_k \mathbb{C}_k \mathbb{N}_k \mathbb{C}_k \cdots) \) is the vectorized “\( n \)-choose-\( k \)” function yielding the matrix whose rows are each unique combinations of the elements of \( \mathbb{N} \) chosen \( k \) at a time, and we use the notation \( A_{l, \ldots} \) to mean the \( l \)th row of matrix \( A \). Note that the tilde in (62), in keeping with earlier notation, implies that this quantity has not been normalized over all states. As we will soon see, this is actually preferable in this case, since each smallest element of the resulting object is already normalized, and therefore it is most useful to just see the actual root-correlance values as they are.

Continuing our example from earlier, a 4-partite multicorrelance array has the form (suppressing inputs)

\[
(\overline{\Xi}_{\mathcal{M}_X})_4(\rho) = \left( \begin{array}{c}
\Xi_{\mathcal{M}_X}^{(1,2)} \\
\Xi_{\mathcal{M}_X}^{(1,3)} \\
\Xi_{\mathcal{M}_X}^{(1,4)} \\
\cdots
\end{array} \right)
\]

where the 4-mode particional multicorrelance vector \( \Xi_{\mathcal{M}_X}^{(1,2,3,4)} \) was given in (56), while the 3-mode particional
multicorrelance vectors \( \Xi^{(a,b,c)}_{\mathcal{M}_X} \) have the form
\[
\Xi^{(a,b,c)}_{\mathcal{M}_X} = \left( \mathcal{M}_X^{(a,b,c)} \mathcal{M}_X^{(b,a,c)} \mathcal{M}_X^{(c,a,b)} \right),
\]
and the 2-mode partitional multicorrelance vectors all have just one element of the form
\[
\Xi^{(a,b)}_{\mathcal{M}_X} = \mathcal{M}_X^{(a,b)} = \mathcal{M}_X^{(b,a)}.
\]

The multicorrelance array \( \nabla_{\mathcal{M}_X}(\rho) \) gives the most detailed picture of the nonlocal correlations available in \( \rho \), showing in which reductions such correlations exist, as well as where (between which mode groups) in those reductions they exist. Again, it is unlikely that all correlations within a state are available to be exploited simultaneously, but this treatment may give us a powerful way to categorize states for their potential resources of nonlocal correlation.

Lastly, for the most aggregated measure of the total potential nonlocal correlation resources within a state, we can define the absolute multicorrelance as
\[
\mathcal{M}_{X_{abs}}(\rho) \equiv \frac{||\nabla_{\mathcal{M}_X}(\rho)||_1}{\text{max}_{\rho' \in \mathcal{N}} \{||\nabla_{\mathcal{M}_X}(\rho')||_1\}},
\]
which is the normalized 1-norm of the multicorrelance array, where the 1-norm of vector \( \mathbf{v} \) is \( ||\mathbf{v}||_1 \equiv \sum_{k=1}^{\dim(\mathbf{v})} |v_k| \), and the 1-norm is taken over the smallest scalar elements of \( \nabla_{\mathcal{M}_X}(\rho) \), meaning for example that even though \( \nabla_{\mathcal{M}_X}(\rho) \) in (63) has 11 “elements,” the fact that each of them is an object containing several scalar elements means that the total number of scalar-element terms in the 1-norm in \( \mathcal{M}_{X_{abs}}(\rho) \) for a 4-partite system is 36.

Note that all of this multipartite generalization follows the formalism of [22] and [18] closely, with the exception that here there is no difficulty in handling mixed states and our measure considers all nonlocal correlation, not just entanglement.

Keep in mind that all state measures rate states based on different criteria, and in general, the decision for which measure to use depends on the application. For example, if we only intend on using a particular state \( \rho \) in its full N-mode form for the correlations that exist between partitions of its N modes, then none of the measures in Sec. VII C are directly relevant for that application, and the measures of Sec. VII A and Sec. VII B are more appropriate. However, even then, if we do not care about the total available correlations among the N modes and instead want it only between a specific partition of mode groups, then we would only need a particular N-mode k-partitional root-correlance as our measure of choice to compare candidate states for our application. Therefore, care should be taken to precisely define the needs of a given application before jumping in and applying one of these measures; otherwise the value of “goodness” they give on a scale of 0 to 1 may mean something unsuitable to a given application.

Finally, all of the definitions in Sec. VII can be applied to diagonal correlation, strong discordance, discordance, and diagonal discordance as well (and also statance and probabilance although they are not yet generally computable); to do so in these equations simply replace all occurrences of \( \mathcal{X} \) with \( \mathcal{L} \) where \( \mathcal{L} \) is the label for that particular measure such as \( \mathcal{D} \) for discordance or \( \mathcal{D}_D \) for diagonal discordance etc. For the nomenclature, just replace all occurrences of “correlation” (including within compound words) with the name of new measure, such as in “multidiscordance” or “multi-diagonal-discordance,” etc.

VIII. CONCLUSIONS

In this paper, we have introduced several measures of nonlocal correlation for discrete N-partite systems, with explicit generalizations of them for multipartite correlations beyond merely N-partite correlations in an N-mode system. Most of these measures are computable for all states both mixed and pure, giving them distinct advantages over other measures such as quantum discord.

To facilitate this goal, in Sec. I A we first defined several mechanisms of nonlocal correlation, not all of which are mutually exclusive, but all of which are important to acknowledge. The two main kinds identified are called decomposition-state correlation and probability correlation. We also identified entanglement correlation (which we later showed to be a special kind of decomposition-state correlation), and we defined general mixed product states as the absence of nonlocal correlation.

From the above general mechanisms of nonlocal correlation, we were able to identify the Six Families of Nonlocal Correlation in Table I as a set of forms that density matrices can take that yield nonlocal correlation. While these families are not all mutually exclusive, identifying them was an extremely helpful conceptual aid to investigating nonlocal correlations in multipartite systems.

From these initial observations, we immediately constructed the main measure of interest as the correlance \( \mathcal{X} \) in Sec. I B, which measures all possible N-mode nonlocal correlation by gauging a state’s distance from having mode-independent product form, meaning its distance from its own reduction product (since only product states are their own reduction products), the conditions for which are established in Sec. I A. The only drawback of \( \mathcal{X} \) is that it cannot distinguish between different kinds of correlation. Its advantages are that it is computable for all states, and it is capable of detecting all nonlocal correlation (including, for instance, the notoriously difficult-to-detect bound entanglement [44-46]).

In Sec. I I, we proved the validity of and derived the normalization factor of \( \mathcal{X} \) (contained in various appendices), and did extensive numerical tests in a wide variety of multipartite systems as a set of necessary checks against the proofs. Furthermore, we constructed a related measure as the diagonal correlance \( \mathcal{X}_D \) intended for only diagonal states and we sketched a similar proof of
its normalization from which we derived its normalization factor, and numerically tested its normalization as well. We also compared $X_D$ to the well-known Pearson correlation and gave extensive instructions for how to handle classical data for use with $X_D$ in App. M. The diagonal correlance $X_D$ is intended for classical probability distributions only, since those must always be diagonal since superposition and thus coherence (nonzero off-diagonal elements of a density matrix) are not possible in classical physics. (This also provided a nice segue into a discussion of what we really mean by classical states; a crucial topic for any paper on nonlocal correlations including quantum discord and notions of classicality.)

Section III delves deeply into the distinction between classical and quantum, and ultimately proposes the new term strictly classical to provide a well-defined and conceptually consistent meaning for true classicality beyond the often careless and ambiguously used term “classical.” Ultimately, we conclude that the main defining feature for strict classicality must be the absence of quantum superposition, both intrinsically and through transformation. This notion is a departure from the popular belief that coherent states are the most appropriate standards of classicality despite having quantum superposition. This topic is discussed in depth in App. O which describes the consistency of strict classicality with established physics and also explains why using coherent states as standards of classicality is not conceptually consistent.

Section III also makes the important observation that there exist diagonal states that are quantum, and identifies the property of a state having “superposition without coherence,” meaning that in cases such as when a diagonal state is a reduction of a larger maximally entangled parent state, the probabilities of the reduction are directly inherited from the superposition coefficients of the parent state, which are themselves overlaps of the parent state with the basis states. Thus we can have diagonal states that still have superposition in an inherited sense. All of these ideas relating classicality to the edge of distinctly quantum properties form powerful conceptual tools for the remaining discussions.

In Sec. IV, we present the hard-to-compute measures statance $S(\rho)$ and probablance $P(\rho)$, which measure the decomposition-state correlation and probability correlation respectively. From these definitions, we learn several very useful new facts, one of the most interesting of which is that strictly classical states (which are diagonal) can have nonzero $S(\rho)$ and $P(\rho)$, just as general quantum states can. Furthermore, we showed that these types of correlation can exist independently of each other in strictly classical states as well. This taught us that correlations of a classical kind cannot be purely ascribed to one mechanism such as probability correlation or decomposition-state correlation, because in the cases of diagonal quantum states where we have “superposition without coherence,” both of these types of nonlocal correlation are entirely quantum in origin, having come from the superposition of the parent state before the reduction. Therefore, since there is no fundamental way to distinguish strictly classical states from diagonal quantum states without having more information beyond the state, this established that diagonal quantum states are not distinguishable from strictly classical states.

Section V then began by defining the strong discordance $D_S$ as a measure of nonlocal correlation beyond that achievable by a strictly classical state, as a reasonable first step towards constructing a measure sensitive to uniquely quantum correlations. However, despite being exactly computable, we soon found that $D_S$ is too exclusive, since it rejects weak quantum correlation such as entanglement that produces a value of correlance $X$ that is also achievable by some strictly classical states.

Therefore, since we desired a measure that can detect all novel quantum correlation such as entanglement, and since we established that diagonality is the fundamental defining feature of strictly classical states but also the point at which diagonal quantum states cannot be distinguished from strictly classical states, then this prompted the definition of discordance $D$ in Sec. V B as a measure of nonlocal correlation in a distinguishably quantum state. The fully computable measure of discordance $D$ uses non-diagonality as the threshold for recognizing nonlocal correlation, so that it is guaranteed to report “zero discordance” for strictly classical states (which are all diagonal), while it is also guaranteed to correctly report the net correlation for all entangled states as well as for all quantum states that are distinguishably quantum by being nondiagonal. Its only weakness is that $D$ rejects the nonlocal correlation of diagonal quantum states as being “not quantum enough” to qualify as discordance.

However, that prompted the definition in Sec. V D of another computable measure, the diagonal discordance $D_D$ which only recognizes the nonlocal correlation in diagonal states, and rejects all distinguishably quantum states (nondiagonal states) as being “too quantum.”

Together, the three computable measures of correlance $X$, diagonal discordance $D_D$, and discordance $D$ were shown to obey the relation $X = D_D + D$ in (42), which formally parallels the quantum-discord relation from (2), $I = C + Q$, where $I$ is the quantum mutual information, $C$ is the classical correlation, and $Q$ is the quantum discord.

We then showed that this similarity is only one of form, but not meaning, since we identified the main conceptual flaw of quantum discord $Q$ is that it is defined in terms of classical correlation $C$ which is based on von Neumann projectors that are allowed to have quantum superposition, something that is completely at odds with classical physics. Therefore, it may be that $C$ and $Q$ have some conceptually inconsistent overlaps in what they measure, making them conceptually less preferable, beyond the fact that they are difficult to calculate.

Alternatively, we could treat the raw strong discordance $D_S(\rho) = \max\{0, X(\rho) - X(\rho_{\text{max}})\}$ as an analog of $Q$, and then define an analog to $C$ as the raw diagonal strong discordance $D_{DS}(\rho) = X(\rho) - D_S(\rho) = \min\{X(\rho), X(\rho_{\text{max}})\}$. Then $X = D_{DS} + D_S$ would par-
correlated diagonal-state threshold, as for ing zero for weakly entangled states until the maximally-
the \tilde{\mathcal{D}}
since
\mathcal{X}
relance
related feature of coherence. The main measures of cor-
the distinctly quantum feature of superposition and its
the possible mechanisms of nonlocal correlation and also
normalized computation sequence as the input state.

Since then the standard state must undergo the same pre-
ition standard, which may help compensate for the errors
in those cases it may be beneficial to find a maximally
to increasingly large errors in normalization. Therefore,
mulated numerical errors in typical computers may lead
X
excellent precision. However, no such difficulty arose for
X
correlance
a more explicit proof of the normalization of diagonal

Appendix B: Quantum Discord Definition

A suggested definition for quantum discord \cite{11, 13} is
\begin{equation}
\mathcal{Q}(\rho) \equiv \mathcal{I}(\rho) - \mathcal{C}(\rho),
\end{equation}
with \mathcal{I}(\rho) from (1), where the classical correlation is
\begin{equation}
\mathcal{C}(\rho) \equiv \max_{\{P_k^{(2)}\}} \left[ \mathcal{I}(\rho|\{P_k^{(2)}\}) \right],
\end{equation}
where we define
\begin{equation}
\mathcal{I}(\rho|\{P_k^{(2)}\}) \equiv S(\hat{\rho}^{(1)}) - S(\rho|\{P_k^{(2)}\})
\end{equation}
as the quantum mutual information of the von Neumann measurement with projection operators \{P_k^{(2)}\} for mode
2, where
\begin{equation}
S(\rho|\{P_k^{(2)}\}) \equiv \sum_k p_k S(\rho_k)
\end{equation}
is the conditional von Neumann entropy given this measurement, where \(p_k \triangleq \text{tr}[(I^{(1)} \otimes P_k^{(2)})\rho(I^{(1)} \otimes P_k^{(2)})^\dagger]\) are the probabilities of conditional measurement-outcome states \(\rho_k \equiv \frac{1}{p_k}(I^{(1)} \otimes P_k^{(2)})\rho(I^{(1)} \otimes P_k^{(2)})^\dagger\).

Appendix C: Purity

Recall that the purity of \(\rho\) is the trace of its square,
\begin{equation}
P \equiv P(\rho) \equiv \text{tr}(\rho^2),
\end{equation}
with range \(P \in [\frac{1}{n}, 1]\) for an isolated system, and \(\rho\) is pure iff \(P = 1\). Otherwise, if \(P < 1\) then \(\rho\) is mixed. Thus, for mode \(m\), \(P(\rho^{(m)}) \in [\frac{1}{n^m}, 1]\) for an isolated mode \(m\).

Beware: if the parent state \(\rho\) is not a product state, then its reductions \(\tilde{\rho}^{(m)}\) can have constraints on the lim-
its of their purity, as proved in \cite{18}. The reason (3) has
no nonlocal correlation is that its tensor product ensures that each reduction (marginal state) is \(\tilde{\rho}^{(m)} = \rho^{(m)}\), and therefore has no dependence on any other modes.

Appendix D: Some Basic Details about Entanglement

In separable states, as in (4), notice that the proba-
bilities \(p_j\) do not need any special structure for (4) to
be satisfied, and that the decomposition states \(\rho_j\) merely need product form, but not mode independence. There-
fore, product form of the full state is not necessary for

Appendix A: Brief Review of Reduced States

Let the multimode reduction from \(N\)-mode parent state
\(\rho\) to a composite subsystem of \(S \in 1, \ldots, N\) possibly non-
contiguous and reordered modes \(m \equiv (m_1, \ldots, m_S)\) be
\begin{equation}
\rho^{(m)} \equiv \text{tr}_m(\rho),
\end{equation}
where the \(\sim\) symbol in \(\tilde{\rho}^{(m)}\) indicates that it is a reduction of \(\rho\) (and not merely an isolated system of the same size as mode group \(m\)), and the bar in \(\bar{m}\) means “not \(m\,” meaning that we trace over all modes whose labels are not in \(m\). See App. B of \cite{18} for details.
separability in general. However, for pure states, product form is both necessary and sufficient for separability.

For example, in a two-qubit system where each qubit has a generic basis with labels starting on 1 (our convention in this paper as \(\{|1\}, |2\}\)), the pure state \(|\psi\rangle = ac|1\rangle \otimes |1\rangle + ad|1\rangle \otimes |2\rangle + bc|2\rangle \otimes |1\rangle + bd|2\rangle \otimes |2\rangle\) is separable because it can be factored as \(|\psi\rangle = (a|1\rangle + b|2\rangle) \otimes (c|1\rangle + d|2\rangle)\), so that \(\rho = |\psi\rangle \langle \psi| = \rho^{(1)} \otimes \rho^{(2)}\) where \(\rho^{(m)} = \langle \psi^{(m)}| \langle \psi^{(m)}|\) and \(|\psi^{(m)}\rangle = a|1\rangle + b|2\rangle\) and \(|\psi^{(2)}\rangle = c|1\rangle + d|2\rangle\). In contrast, the pure state \(|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}(|1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle\) is not separable so it is entangled (and happens to be maximally entangled) because it cannot be factored into a product form (since this example is for pure states).

**Appendix E: Example of Product Form**

To get product form for a two-qubit state \(\rho = \sum_j p_j \rho_j\) (and to demonstrate mode independence), Theorem 1 requires (5–8) by setting \(p_{1j} \equiv p_j^{(1)} \otimes \rho_j^{(2)}\) and

\[
p_1 = p_{11} \equiv p_1^{(1)} (1^{(2)}), \quad p_1^{(1)} (1^{(2)}) = \rho_{11}^{(1)} \equiv \rho_{11}^{(1)} \otimes \rho_{11}^{(2)}, \quad p_2 = p_{12} \equiv p_2^{(1)} (1^{(2)}), \quad p_2^{(1)} (1^{(2)}) = \rho_{12}^{(1)} \equiv \rho_{12}^{(1)} \otimes \rho_{12}^{(2)}.
\]

where all states in (E1) are pure. Then \(\rho\) becomes

\[
\rho = (p_1^{(1)} (1^{(2)}) \otimes \rho_{11}^{(2)}) + p_2^{(1)} (1^{(2)}) \otimes \rho_{11}^{(2)}\]

\[
= (p_1^{(1)} (1^{(2)}) \otimes \rho_{11}^{(2)}) + (p_2^{(1)} (1^{(2)}) \otimes \rho_{11}^{(2)}).
\]

with \(\rho^{(1)} \equiv p_1^{(1)} (1^{(2)}) + p_2^{(1)} (1^{(2)})\) and \(\rho^{(2)} \equiv p_1^{(1)} (1^{(2)}) + p_2^{(1)} (1^{(2)})\).

Notice the redundant probabilities and decomposition states in (E1); this is a consequence of applying Theorem 1. Also, if either or both mode reductions have lower rank, some of the probabilities in (E1) will be zero.

Thus, the smallest maximum number of decomposition states required for any mixed product state is

\[
r_1 \cdots r_N = r,
\]

(wh)ich is also their minimum number of decomposition states) where the ranks of each single-mode reduction of the product state are \(r_m \in \{1, \ldots, r_m\}\), and \(r \equiv \text{rank}(\rho)\), which is just an application the well-known property for tensor products that \(\text{rank}(A \otimes B) = \text{rank}(A) \text{rank}(B)\). In fact, by making mode-independent (MI) tensor products of the eigenstates of the reductions, and forming the corresponding probabilities from MI products of the eigenvalues of the reductions, we can always construct an \(r\)-member MI decomposition of any product state.

**Appendix F: Proof of Theorem 1**

Starting with a general mixed state,

\[
\rho = \sum_j p_j \rho_j, \quad (F1)
\]

applying (5) gives a separable state,

\[
\rho = \sum_j p_j \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)}, \quad (F2)
\]

which is not enough to definitely get product form. If we then also use (7), we get

\[
\rho = \sum_j (p_j^{(1)} \cdots p_j^{(N)}) \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)} = \sum_j p_j^{(1)} \rho_j^{(1)} \otimes \cdots \otimes p_j^{(N)} \rho_j^{(N)}, \quad (F3)
\]

which is still not enough for product form because the indices are “locked” across the modes, indicating nonlocal correlation. Therefore, vectorizing the indices as

\[
\rho = \sum_{j_1, \ldots, j_N} p_{j_1, \ldots, j_N} \rho_{j_1, \ldots, j_N}, \quad (F4)
\]

and then applying both (8) and (6), we get

\[
\rho = \sum_{j_1, \ldots, j_N} \sum_{i_1, \ldots, i_N} p_{j_1, \ldots, j_N} \rho_{j_1, \ldots, j_N} i_1 \otimes \cdots \otimes i_N = \sum_{j_1, \ldots, j_N} p_{j_1} \rho_{j_1} \otimes \cdots \otimes p_{j_N} \rho_{j_N}, \quad (F5)
\]

where \(\rho^{(m)} \equiv \sum_{j_m} p_{j_m} \rho_{j_m}^{(m)}\), which proves the sufficiency of (5–8) for product form, and thus the absence of all nonlocal correlation.

To prove the necessity of (5–8) for product form, we simply start from the product-form definition in (3),

\[
\rho = \rho^{(1)} \otimes \cdots \otimes \rho^{(N)}. \quad (F6)
\]

Then, since each mode’s state can be generally mixed as

\[
\rho^{(m)} = \sum_{j_m} p_{j_m} \rho_{j_m}^{(m)}, \quad (F7)
\]

we can put (F7) into (F6) to get

\[
\rho = \sum_{j_1} \rho_{j_1} \otimes \cdots \otimes \sum_{j_N} \rho_{j_N} \otimes \rho_{j_N}. \quad (F8)
\]

Rearranging shows that product states must have form

\[
\rho = \sum_{j_1, \ldots, j_N} \sum_{i_1, \ldots, i_N} p_{j_1, \ldots, j_N} \rho_{j_1, \ldots, j_N} i_1 \otimes \cdots \otimes p_{j_N} \rho_{j_N} = \sum_{j_1, \ldots, j_N} (p_{j_1} \cdots p_{j_N} \rho_{j_1}) \otimes \cdots \otimes \rho_{j_N}. \quad (F9)
\]

Thus, line 2 of (F9) verifies that this form is equivalent to product-form. Then, to show how (F9) relates to the notation of a general mixed state, we can unify these all with a vector index only if each mode has redundantly defined quantities of the forms \(p_{j_m}^{(m)} = \rho_{j_1, \ldots, j_N}^{(m)}\).
and \( \rho_{(m)}^{(N)} = \rho_{(\theta_{1}, \ldots, \theta_{N})}^{(N)} \), regardless of the values of the indices of labels other than \( m \), so then (F9) becomes

\[
\rho = \sum_{j_{1}, \ldots, j_{N}}(p_{(1)}^{j_{1}} \cdot \ldots \cdot p_{(N)}^{j_{N}})\rho_{(j_{1}, \ldots, j_{N})}^{(1)} \otimes \ldots \otimes \rho_{(j_{1}, \ldots, j_{N})}^{(N)},
\]

and then if we abbreviate \( p_{j} = p_{j}^{(1)} \cdot \ldots \cdot p_{j}^{(N)} \) and \( \rho_{j} = \rho_{j}^{(1)} \otimes \ldots \otimes \rho_{j}^{(N)} \), then (F10) takes the general form

\[
\rho = \sum_{j} p_{j}\rho_{j},
\]

which shows that since we started from product states in (F6) and used (5–8) to recover the form of a general state in (F11), then (5–8) are necessary for product form. Thus, we have now proven that (5–8) are necessary and sufficient conditions for a state to have product form.

**Appendix G: Proofs and Caveats for the Six Families of Nonlocal Correlation**

Here we develop some important facts about the six families of nonlocal correlation from Table I.

1. **Proof that All Decomposition Probabilities are Expressible in Product Form Without Mode Normalization**

Given a set of normalized decomposition probabilities \( \{p_{j}\} \), we can always express them in product form as

\[
p_{j} = \prod_{m=1}^{N} p_{j}^{(m)} = p_{j}^{(1)} \cdots p_{j}^{(N)} \quad \forall j,
\]

by using the parameterization

\[
p_{j}^{(m)} = x_{m}^{2}(\theta^{(j)}) \quad \forall j; \quad m = 1, \ldots, N,
\]

where for each \( j \), the \( x_{m}(\theta^{(j)}) \) for \( m = 1, \ldots, N \) are \( N \)-dimensional unit-hyperspherical coordinates \([31]\) with \( N - 1 \) angles \( \theta^{(j)} = (\theta_{1}^{(j)}, \ldots, \theta_{N-1}^{(j)}) \), which can be restricted to \( \theta_{k}^{(j)} \in [0, \frac{\pi}{2}] \) for this application, with a different set for each \( j \). These coordinates have the property that \( \sum_{m=1}^{N} x_{m}^{2}(\theta^{(j)}) = 1 \) \( \forall j \). Notice that in (G2), we are setting each mode-specific factor \( p_{j}^{(m)} \) equal to the full probability \( p_{j} \) raised to the power \( x_{m}^{2}(\theta^{(j)}) \). Thus,

\[
p_{j} = p_{j}^{1} = p_{j}^{2} \cdots x_{N}^{2}(\theta^{(j)}) = p_{j}^{2}(\theta^{(j)}) \cdots x_{N}^{2}(\theta^{(j)}) = p_{j}^{(1)} \cdots p_{j}^{(N)} \quad \forall j,
\]

which also works right to left, given (G2). Therefore, we have proven that all decomposition probabilities are always expressible in product form as in (G1), where we do not require mode normalization over \( j \) as \( \sum_{j} p_{j}^{(m)} = 1 \) \( \forall m \), which is actually impossible as we prove in App. G2.

The consequence of (G3) is that we do not need to define separate sets of families based on whether they have decompositions with product-form probabilities as \( p_{j} = p_{j}^{(1)} \cdots p_{j}^{(N)} \), since such families would be redundant to other families with general probabilities \( p_{j} \) and the same decomposition-state forms. For example, defining a family as \( \rho = \sum_{j} p_{j}^{(1)} \cdots p_{j}^{(N)} \rho_{j} \) would be exactly the same as Family 1, which is \( \rho = \sum_{j} p_{j}\rho_{j} \) of Table I.

2. **Proof that Product-Form Probabilities with Mode Normalization on the Full Decomposition Index Are Impossible Except Trivially**

Suppose we require that in addition to having product form (PF), the decomposition probabilities \( p_{j} \) must also have mode normalization over the full decomposition index \( j \), so that

\[
p_{j} = \prod_{m=1}^{N} p_{j}^{(m)} = p_{j}^{1} \cdots p_{j}^{(N)} \quad \forall j; \quad \sum_{j} p_{j}^{(m)} = 1 \quad \forall m,
\]

which is not the same thing as mode independence (MI) from (8) due to the sum over the full \( j \) rather than mode-specific index \( j_{m} \) where \( j = (j_{1}, \ldots, j_{N}) \).

First, for the PF part, we can use the parameterization of App. G1, which, together with the mode normalization over \( j \), requires that

\[
\sum_{j} p_{j}^{(m)} = \sum_{j} x_{m}^{2}(\theta^{(j)}) = 1 \quad \forall m.
\]

For decompositions with \( D \) states, this requires that

\[
p_{j}^{k} = x_{m}^{2}(\theta^{(j)}) + \ldots + x_{D}^{2}(\theta^{(j)}) = 1 \quad \forall m.
\]

Now, given that any collection of \( D \) real numbers on \((0, 1]\) such as \( \{p_{j}\} \) can be viewed as eigenvalues of some physical \( D \)-level state, then they are always exactly determined by a set of \( D \) equations for their power sums as

\[
p_{1}^{k} + \ldots + p_{D}^{k} = \sum_{j=1}^{D} p_{j}^{k} \equiv P_{k},
\]

for integers \( k = 1, \ldots, D \). Thus, (G6) constitutes a set of generally additional constraints beyond the main set in (G7), so together (G6) and (G7) form a generally overdetermined set of nonlinear equations.

It turns out that this overdetermined set does have solutions, but only for cases that are irrelevant for the purpose of defining nontrivial families of decompositions for probability correlation, as we now briefly explain.

One way to get a solution to an overdetermined set is to find conditions for which the additional constraints simplify to the main constraints (G7). Since the powers of \( p_{j} \) in (G6) are squared unit-hyperspherical coordinates and therefore are each no greater than 1, then for a given \( m \) in (G6), we can never achieve the condition of having both all powers being equal and all being integers, without preventing that for all other \( m \). Specifically, for a particular \( m \), the only way to involve integer powers is to set \( x_{m}^{2}(\theta^{(j)}) = 1 \) \( \forall j \) and thus satisfy (G6) for that \( m \), but then for all other \( m \), we would have \( x_{m}^{2}(\theta^{(j)}) = 0 \) \( \forall j \),
so that we would have $N - 1$ unsatisfied equations of the form of (G6).

There are only two ways that could work. It could work for a unipartite system (one with $N = 1$ mode, meaning no physical coincidence behavior, which is a system that can never have nonlocal correlation of any kind). Alternatively, it could work with multiple modes if $D = 1$, but that means the state is pure, in which case the question of achieving a special new kind of decomposition probability correlation through (G4) is irrelevant since there is only one unique decomposition state with probability 1.

Therefore, we have outlined the proof that expressing decomposition probabilities $p_j$ as products of $N$ factors where each is separately normalized over full decomposition index $j$ is impossible in all cases except pure states or unipartite states, neither of which can have probability correlation. (An alternative proof would be to use $p$-norms to achieve $DN$ separate inequalities, which lead to the same trivial exceptions.)

3. Quasi-Families

Since state decompositions are generally not unique, it may happen that for some states, they belong to multiple partially-intersecting families from Table I, but only through different decompositions, so that they do not belong to the intersection of those families since they do not have a single decomposition in that intersection.

For example, since Family 2 is $\rho = \sum_j (p^{(1)}_j \cdots p^{(N)}_j) \rho_j$, while Family 3 is $\rho = \sum_j p_j \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)}$, we see that the probabilities of Family 2 are a strict subset of Family 3, but the decomposition states of Family 2 are a strict superset of Family 3, meaning that neither Family 2 nor Family 3 can be strict subsets of each other, although they can have an intersection, which is Family 4 $\rho = \sum_j (p^{(1)}_j \cdots p^{(N)}_j) \rho_j^{(1)} \otimes \cdots \otimes \rho_j^{(N)}$. While any state with a decomposition in Family 4 is in both Family 2 and Family 3, there may exist some states that have different decompositions in Family 2 and Family 3 separately, but no decomposition in Family 4.

Thus we define a quasi-family as being a set of states with the property of belonging to multiple partially intersecting families without belonging to their intersection.

To identify all quasi-families, we have to check every possible set of every possible number of families, keeping in mind that if any family is a proper subset of another family, then that pair does not constitute a quasi-family, since then membership in the subset guarantees membership in the superset for the same decomposition.

Checking all family combinations in Table I shows that there are only three quasi-families in this group,

Quasi-Family 2|3,  
Quasi-Family 2|5,  
Quasi-Family 4|5,  

where Quasi-Family $x_1 \cdots x_M$ is a quasi-family involving $M$ families; Family $x_1$ through Family $x_M$. Note that due the various subset memberships in this small set of families, there are no quasi-families between more than two families in Table I.

The quasi-families may not be as important as the main families. In particular, being able to satisfy multiple family definitions with a single decomposition is what leads to such extreme behavior as achieving product form, whereas a state that only achieved mode-independent (MI) probabilities and MI decomposition states with separate decompositions alone would not achieve product form. Therefore, the fact that a state can have membership to multiple families but not their intersection does not diminish the significance of the families regarding their roles in producing nonlocal correlation.

Ultimately, since a state needs to have at least a single decomposition that satisfies a given family definition for it to have the nonlocal correlation achievable by that family, then quasi-families are not as important as families regarding nonlocal correlation.

Furthermore, due to the difficulty of constructing states that have different decompositions of particular forms, we do not have examples of quasi-family states at this time (keeping in mind that states within the intersection of the two families involved in a quasi-family are not part of the quasi-family, so we cannot use such intersections to generate examples of quasi-family members.) It may be that no states exist in any quasi-families, in which case they can be ignored as physically irrelevant.

Nevertheless, we mention quasi-families here in case they turn out to be physically meaningful in some way, and thus this is a possible area for further research.

Appendix H: True Generalized X (TGX) States

True-generalized X (TGX) states are defined as a special family of states that are conjectured to be related to all general states (both pure and mixed) by an entanglement-preserving unitary (EPU) transformation, so that the TGX state and the general state connected by such an EPU have the same entanglement, a property called EPU equivalence.

The name TGX means "the true generalization of X states with respect to entanglement for all systems as big as or larger than two qubits," meaning that, just as the X states are EPU equivalent to general states for $2 \times 2$ systems (which is now proven by two independent methods as detailed below), TGX states (if they exist) are generally the larger-system analog of that two-qubit family, having the defining property of EPU equivalence with the set of general states.

The leading candidates for TGX states in all systems are called simple states, defined as those states for which their $N$ single-mode reductions are all diagonal in the computational basis, such that all of the off-diagonal parent-state matrix elements appearing in the formal off-diagonals of those reductions are identically zero (meaning that those parent elements do not merely add to zero, but are each themselves zero).
As a simple nontrivial example of TGX states, in $2 \times 3$, since the single-mode reductions are diagonal, but defines a $\rho$ of these reductions are $\rho$ then the parent elements contributing to the off-diagonals of these reductions are $\rho_{41}, \rho_{52}, \rho_{63}, \rho_{44}, \rho_{55}, \rho_{66}, \rho_{64}, \rho_{31}, \rho_{65}$, and their index-swapped counterparts, so setting all of these to zero not only makes the reductions diagonal, but defines a simple parent state as

$$\rho = \begin{pmatrix}
\rho_{11} & \cdots & \rho_{15} & \rho_{16} \\
\cdots & \cdots & \cdots & \cdots \\
\rho_{51} & \cdots & \rho_{55} & \cdots \\
\rho_{61} & \rho_{62} & \cdots & \rho_{66}
\end{pmatrix},$$

which we take as a working hypothesis to be the family of $2 \times 3$ TGX states (where dots represent zeros to help show its form). Note that in all of the work on TGX states so far, all evidence strongly supports the hypothesis that simple states are TGX states, so the two terms are often used interchangeably. However, if simple states are ever proved not to have EPU equivalence, the idea of TGX states can then be reserved for EPU equivalent states if they exist. See [17] for many other examples.

A Brief History of TGX States:

1. (2013) [17] gave the first definition of TGX states, and the general form was conjectured to be that of simple states. The idea of EPU equivalence was also introduced, and strong numerical evidence was shown that simple TGX states are EPU equivalent to all states for $2 \times 2$ and $2 \times 3$ systems, and this property was conjectured to hold for TGX states in all quantum systems. Numerical evidence was also given showing that literal X states cannot in general be EPU equivalent to general states. The maximally entangled-basis (MEB) theorem was conjectured and shown to be fulfilled by TGX states for several example systems.

2. (2014) [47] presented the Bloch-vector form of simple candidates for TGX states.

3. (2014) [48] proved the conjecture of [17] for the $2 \times 2$ case by showing the implicit existence of an EPU connecting all general states to X states (which are TGX states in $2 \times 2$ systems).

4. (2016) [18] presented the multiparticle entanglement measure the ent. TGX states (simple candidates) were used to prove the MEB theorem for all discrete quantum systems. It was proved that ME TGX states have the special property of having balanced superposition. Furthermore, ME TGX states were shown to yield indexing patterns that can function as a multipartite Schmidt decomposition state for full N-partite entanglement. This also presented the 13-step algorithm as a method for deterministically constructing all possible ME TGX states in all discrete quantum systems.

5. (2017) [49] proved that in $2 \times 3$ systems, literal X states definitely cannot achieve EPU equivalence to general states, which also proves that literal X states cannot have EPU equivalence in general. This study also added further numerical evidence agreeing with that of [17] suggesting that the TGX states may indeed achieve EPU equivalence in $2 \times 3$ systems.

6. (2018) [50] presented an explicit family of $2 \times 2$ X states parameterized by concurrence and spectrum and proved it to be EPU-equivalent to the set of all states, providing an explicit proof of the original conjecture of [17], and proving the existence of an explicit formula for the EPU of the transformation, as well as yielding an explicit ready-to-use EPU-equivalent state family.

Appendix I: Proof that Correlance is a Necessary and Sufficient Measure of All Nonlocal Correlation

First, from the definition of correlance $\mathcal{X}$ in (9–12),

$$\mathcal{X}(\rho) = 0 \text{ iff } \rho = \varsigma(\rho),$$

since $\mathcal{X}(\rho) = \text{tr}[(\rho - \varsigma)^2]/N_\mathcal{X}$ is proportional to the sum of the square magnitudes of the difference of the Bloch vectors of $\rho$ and $\varsigma(\rho)$, which is zero iff $\rho = \varsigma(\rho)$. Next, by Theorem 1, proven in App. F,

$$\rho = \varsigma(\rho) \text{ iff } \rho \text{ has product form.}$$

Furthermore, from (3),

$$\rho \text{ has no nonlocal correlation iff } \rho \text{ has product form,}$$

which is true since discarding any modes by partial tracing leaves the states of the remaining modes unchanged. Then, putting (I3) into (I2) and the result into (I1) gives

$$\mathcal{X}(\rho) = 0 \text{ iff } \rho \text{ has no nonlocal correlation,}$$

which is what we set out to prove. This means we can use $\mathcal{X}$ to detect any and all kinds of nonlocal correlation. The only drawback is that it cannot tell us which kind(s) of correlation is(are) present. Nevertheless, if $\mathcal{X}(\rho) > 0$, we are guaranteed that $\rho$ has some nonlocal correlation.

Appendix J: Proof that Correlance is Properly Normalized

Here we prove that it is valid to normalize correlance $\mathcal{X}$ with maximally entangled (ME) states (see App. K for a derivation of the normalization factor). For ease of display, the proof’s steps are given as numbered facts.

1. $\mathcal{X}$ measures nonlocal correlation, as proved in App. I. (“Nonlocal correlation” abbreviates “full N-partite nonlocal correlation” here; see Sec. VII for generalizations to distinctly multipartite nonlocal correlation).
2. From (10), the raw (unnormalized) correlance \( \tilde{X}(\rho) \equiv tr[(\rho - \varsigma)^2] = tr(\rho^2) - 2tr(\rho\varsigma) + tr(\varsigma^2) \) is a function of input state \( \rho \) and its reduction product \( \varsigma \equiv \rho(1) \otimes \cdots \otimes \rho(N) \). As we will see later, \( \tilde{X}(\rho) \) is simply a function of the squared Euclidean distance between the Bloch vectors of \( \rho \) and \( \varsigma(\rho) \). Thus correlance \( X(\rho) \equiv X(\rho)/N_X \) is a measure of distance between \( \rho \) and its reduction product \( \varsigma \).

3. Pure states can be used to maximize \( X(\rho) \). Proof: \( X(\rho) \) is proportional to a squared distance between Bloch vectors (BVs) as \( X(\rho) = \frac{1}{N_X} \sum_{j=1}^{N} |\mathbf{G} - \mathbf{G}_j|_2 \) (see App. K for details about our BV notation) where \( \mathbf{G} \) is the BV of \( \rho \) and \( \mathbf{G}_j \) is the BV of \( \varsigma \equiv \rho(1) \otimes \cdots \otimes \rho(N) \), both of which have real components in a Hermitian operator basis. \( X(\rho) \) can be adapted for BV input as \( X'(\mathbf{G}) \equiv X(\rho) \), and since both terms in the reduction difference vector \( \Delta \equiv \Delta(\mathbf{G}) \equiv \mathbf{G} - \mathbf{G}_j \) depend on the input state, we can rewrite it as \( X''(\Delta) \equiv X'(\Delta(\mathbf{G})) = \frac{1}{N_X} \sum_{j=1}^{N} |\Delta|_2 \). Then, recall that \( f(x) \) is strongly convex iff for all \( x \) and \( y \) in its domain and \( p \geq 0 \), there exists some scalar \( m \geq 0 \) such that \( f(px + (1-p)y) \leq pf(x) + (1-p)f(y) - \frac{1}{2}mp(1-p)||x-y||_2^2 \), and that any strongly convex function is also convex. Thus, given reduction difference vectors \( \mathbf{X} \) and \( \mathbf{Y} \), since the quantities \( X''(p\mathbf{X} + (1-p)\mathbf{Y}) \) and \( pX''(\mathbf{X}) + (1-p)X''(\mathbf{Y}) - \frac{1}{2}mp(1-p)||\mathbf{X} - \mathbf{Y}||_2^2 \) are equal if \( m = \frac{2}{N_X} \sum_{j=1}^{N} p_j \), then \( X''(\Delta) \) and thus \( X(\rho) \) are both strongly convex and convex. Then, recalling Jensen’s inequality for convex \( f(x) \), that \( f(\sum_{j=1}^{D} p_j x_j) \leq \sum_{j=1}^{D} p_j f(x_j) \), to which a corollary is \( f(\sum_{j=1}^{D} p_j x_j) \leq \max\{f(x_1), \ldots, f(x_D)\} \), then for any mixed input \( \rho \), \( X(\rho) \leq \max\{X(\rho_1), \ldots, X(\rho_D)\} \) where \( \rho_j \) are pure decomposition states of \( \rho \). Thus maximizers of \( X(\rho) \) over all \( \rho \) are pure.

4. Since pure states have the trivial decomposition probability of \( p_1 = 1 \), with only one pure decomposition state up to global phase, then the only mechanism for nonlocal correlation in pure states is the nonfactorizability of the state itself, meaning its entanglement.

5. Since entanglement is the only nonlocal correlation possible for pure states by Fact 4, the pure states of highest \( X \) are those of highest entanglement. (Since App. I proved that nonlocal correlation is what \( X \) measures, this also means that for pure states, the states of highest nonlocal correlation are the states of highest entanglement.)

6. The pure states with the highest entanglement are any \( \rho \) for which all of its single-mode reductions \( \tilde{\rho}^{(m)} \) have the lowest simultaneous purities possible for them to have, given their pure parent state \( \rho \). We define these states as maximally full-\( N \)-partite-entangled states, or just “maximally entangled” (ME) states here, and represent them as \( \rho_{ME} \). (This definition led to the derivation of the automatically normalized entanglement measure the ent \( \Upsilon(\rho) \) in [18].)

7. Therefore, by Fact 5 and Fact 6, the highest \( X \) for pure states is achieved by pure ME states \( \rho_{ME} \), so then by Fact 3, the pure ME states maximize \( X \) over all states (mixed and pure), which proves the part of (12) that says \( N_X = \tilde{X}(\rho_{ME}) \).

8. In [18] it was shown that the simplest maximally full-\( N \)-partite-entangled states are ME TGX states \( \rho_{ME TGX} \), since they achieve the minimum simultaneous single-mode purities while also having equal superposition coefficients, with not all levels being nonzero. Therefore, since ME TGX states have the same entanglement as general ME states, then by Fact 7, this proves the part of (12) that says \( N_X = \tilde{X}(\rho_{ME TGX}) \).

Thus, we have proven that any pure ME state can be used to normalize \( X \) over all states, both mixed and pure. The reason for using ME TGX states is that they are generally simpler than general ME states, since it was proved in [18] that ME TGX states always have balanced superposition and not all levels are nonzero. Furthermore, the ME TGX states can be methodically generated, using the 13-step algorithm of [18]. Regarding the distance interpretation of \( X \) from Fact 2, see [51] for the intimate connection between distance measures of entanglement and convex-roof extensions of entanglement monotones.

Appendix K: Proof and Calculation of Explicit Correlance Normalization Factors

This proof makes extensive use of multipartite Bloch vectors [47, 52], and therefore this appendix has two parts: App. K1 reviews multipartite-Bloch-vector formalism, and App. K2 derives the normalization factors.

1. Review of Bloch-Vector Quantities

The concept of a Bloch vector is simply to use a set of operators that is somehow complete in that it allows us to expand any operator as a linear combination of that set of operators. The Bloch vector is then the list of scalar coefficients of that expansion.

The idea of Bloch vectors actually originated as Stokes parameters in 1852 [53], which were used as an operational method of describing classical light. However, the quantum-mechanical density matrix was not invented until 1927, by von Neumann [54], and the modern idea of Bloch vectors for quantum states came from Felix Bloch’s 1946 treatment of mixed-state qubits [52], which were soon-after connected with the density matrix. The 1961 development of the Gell-Mann (GM) matrices by Ne’eman and Gell-Mann [55, 56] paved the way for describing states larger than a qubit, but it was not until 1981 that a unipartite \( n \)-level Bloch vector was devised, by Hioe and Eberly [57]. Soon after, many multipartite descriptions were attempted and many works treat simple cases of these. Therefore, here we present a brief general treatment of multipartite Bloch vectors, from the more complete work in [47] from 2014.
Consider a multipartite system of $N$ subsystems (modes), with Hilbert space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \cdots \otimes \mathcal{H}^{(N)}$, where $\mathcal{H}^{(m)}$ is the Hilbert space of mode $m$, which $\mathcal{H}$ has $n$ total levels such that $n = n_1 \cdots n_N$, where $n_m$ is the number of levels of mode $m$.

Now let $\{\nu_k^{(m)}\}$ be a complete basis of $n_m^2$ operators for mode $m$, such that all operators in $\mathcal{H}^{(m)}$ can be expanded as linear combinations of $\{\nu_k^{(m)}\}$, and where $\nu_0^{(m)} = I^{(m)}$ is the identity for $\mathcal{H}^{(m)}$. Furthermore, under the Hilbert-Schmidt (HS) inner product $A \cdot B \equiv \text{tr}(A^\dagger B)$, suppose that $\{\nu_k^{(m)}\}$ has uniform orthogonality,

$$\text{tr}(\nu_j^{(m)} \nu_k^{(m)}) = n_m \delta_{j,m,k,m}, \quad (K1)$$

for all $j, k, m \in 0, \ldots, n_m^2 - 1$, where “uniform” means that only one case is needed to cover all indices, which allows the simplest transition to a multipartite basis. Thus, we can define a multipartite basis as

$$\nu_k \equiv \nu_k^{(1)} \otimes \cdots \otimes \nu_k^{(N)} \quad (K2)$$

for $k \in 0, \ldots, n_N^2 - 1 \forall m \in \{1, \ldots, N\}$, where the vector-index subscripts indicates the multipartite nature of the basis, where the number of elements in the vector is the number of modes $N$. The set $\{\nu_k\}$ inherits the uniform orthogonality of its modes as

$$\text{tr}(\nu_j^{\dagger} \nu_k) = n \delta_{j,k}, \quad (K3)$$

valid for all $k \in 0, \ldots, n_m^2 - 1 \forall m \in \{1, \ldots, N\}$, and where $\delta_{j,k} \equiv \delta_{j_1,k_1} \cdots \delta_{j_N,k_N}$.

The HS completeness of $\{\nu_k\}$ lets us express all density operators as $\rho = \sum_k \alpha_k \nu_k = \frac{1}{n} (\nu_0 + A \sum_{k \neq 0} \Gamma_k \nu)$, where $\alpha_k$ are generally complex scalars, and $\Gamma_k \equiv \Gamma_{k \neq 0} \equiv \Gamma_{k_1,\ldots,k_N} \equiv \frac{1}{n} \nu^{\dagger} \nu$ are the $n^2 - 1$ scalars known as Bloch components that constitute a Bloch vector $\Gamma$, and we choose $\Gamma_0 \equiv \sum_0 \equiv \frac{1}{n}$ by convention. Then, computing the purity $P \equiv \text{tr}(\rho^2)$ and applying the unitization condition that $|\Gamma| = 1$ for pure states $\rho$, then $A = \sqrt{n} - I$, and we obtain the multipartite Bloch-vector expansion of $\rho$ as

$$\rho = \frac{1}{n} \nu_0 + \sqrt{n - 1} \Gamma \cdot \nu, \quad (K4)$$

where $\Gamma$ is the list of scalars $\{\Gamma_{k \neq 0}\}$, and $\nu$ is the list of operators $\{\nu_{k \neq 0}\}$ in the same order as $\{\Gamma_{k \neq 0}\}$, and the dot product in the context of (K4) is just an abbreviation for $\sum_{k \neq 0} \Gamma_k \nu_k$. Then, applying (K3) to (K4) gives the multipartite Bloch components as

$$\Gamma_{k \neq 0} = \frac{1}{\sqrt{n - 1}} \text{tr}(\rho \nu_k^{\dagger}). \quad (K5)$$

Note that if $\{\nu_k\}$ consists entirely of Hermitian operators, then the $\Gamma_{k \neq 0}$ will all be real. The Bloch vector $\Gamma$ contains all of the same information as $\rho$ and can be used as an alternate method of representing any physical state.

The purity of $\rho$ is $P \equiv \text{tr}(\rho^2) = \frac{1}{n} (1 + (n - 1)|\Gamma|^2)$, so we can define the Bloch purity as

$$P_B \equiv |\Gamma|^2 = \sum_{k \neq 0} |\Gamma_k|^2 = \frac{n \lambda(n^2 - 1)}{n^2}, \quad (K6)$$

which obeys $0 \leq P_B \leq 1$, such that $P_B = 1$ for pure states, $0 \leq P_B < 1$ for general strictly mixed states, and $P_B = 0$ for the maximally mixed state.

So far, we have merely specified properties of $\{\nu_k\}$ without explaining how to make it. One simple way to construct it is, for each $m \in \{1, \ldots, N\}$, let

$$\nu_0^{(m)} \equiv \lambda_0^{(m)} \equiv I^{(m)} \quad \text{and} \quad \nu_k^{(m)} \equiv \sqrt{n - 2} \nu_k^{(m)}, \quad (K7)$$

where $k_m \in 1, \ldots, n_m^2 - 1$, and the $\lambda_0^{(m)}$ are generalized Gell-Mann (GM) matrices in mode $m$ of $n_m$ levels, given by the implicit equations

$$\lambda^{(m)}_{\alpha^2 - 2(\alpha - \beta) - 1} \equiv \lambda^{(m)}_{\alpha^2 - 2(\alpha - \beta)} \equiv \frac{E_{\alpha^2}(\nu_k^{(m)}) + E_{\alpha^2}(\nu_k^{(m)})}{E_{\alpha^2}(\nu_k^{(m)})} ; \alpha > \beta,$$

where $2 \leq \alpha \leq n_m$ and $1 \leq \beta \leq \alpha - 1$, and $E_{\alpha^2}(\nu_k^{(m)}) \equiv |\alpha\rangle \langle \beta|$ is the $n_m \times n_m$ matrix with a 1 in the row-$\alpha$, column-$\beta$ entry and 0 elsewhere, given that the top row is row 1, and the left column is column 1, and $\lambda_0^{(m)} = I^{(m)}$ is the $n_m \times n_m$ identity matrix for mode $m$. A given pair of integers $\alpha, \beta$ determines the conventional label for each GM matrix.

Note that the GM matrices are not preferable as a basis for general multipartite Bloch vectors (though they are often used for that), because their orthogonality relations require two cases to include the identity, resulting in $2^n$ cases for the orthogonality of a general $N$-partite system. Thus, putting (K8) into (K7) and using that in (K2), we obtain a realization for the multipartite basis $\{\nu_k\}$, which has only one orthogonality case, which is given by (K3).

We now develop some formalism that will be useful in establishing the results we will use to prove the normalization factors of the correlance.

For multipartite systems, the implicit-basis representation of $\Gamma$ is not intuitive, since any vector or matrix representation uses relative positions on a page to encode the basis, so listing out components with vector indices is not helpful. Therefore, we will simply keep basis-explicit notation using a vector-indexed basis, and to that end we introduce a uniform standard basis (USB) as

$$b_k \equiv b_k^{(1, \ldots, N)} \equiv \frac{1}{\sqrt{n}} \nu_k = \frac{1}{\sqrt{n}} \nu_k^{(1)} \otimes \cdots \otimes \nu_k^{(N)}. \quad (K9)$$

If we then define the Hilbert-Schmidt inner product between Bloch vectors $A$ and $B$ in the USB as

$$A \cdot B \equiv \text{tr}(A^\dagger B), \quad (K10)$$

then any USB as defined above has orthonormality

$$b_j \cdot b_k = \delta_{j,k}. \quad (K11)$$

Then, in the USB, Bloch vectors have explicit-basis form,

$$\Gamma = \sum_{k \neq 0} \Gamma_k b_k, \quad (K12)$$
where $K_m \equiv n_m^2 - 1$ and $k_m \in 0, \ldots, K_m$, so only the case of all $k_m = 0$ is excluded from the sum. Thus, in the USB, $\Gamma$ is a matrix, and $\Gamma_k$ have the same values as in (K5) but are now given by
\[
\Gamma_{k \neq 0} = \Gamma \cdot b_k = \text{tr}(\Gamma b_k^\dagger) = \sqrt{\frac{n}{n-1}} \text{tr}(\rho b_k^\dagger),
\tag{K13}
\]
and the density matrix can be expanded more simply as
\[
\rho = \frac{1}{n} (I + \sqrt{n(n-1)} \Gamma),
\tag{K14}
\]
where note that we use boldness to distinguish Bloch-vector objects from density matrices, despite both being represented as matrices here.

The (informal) overlap of any two states $\rho_A$ and $\rho_B$ is then (using their Hermiticity in the HS inner product),
\[
\text{tr}(\rho_A \rho_B) = \frac{1}{n} (1 + (n-1) \Gamma_A \cdot \Gamma_B),
\tag{K15}
\]
where, since all $\Gamma$ are also Hermitian, using (K12) and supposing that the $b_k$ are Hermitian as well so that all Bloch components are real, then
\[
\Gamma_A \cdot \Gamma_B = \text{tr}(\Gamma_A \Gamma_B) = \sum_{k \neq 0} (\Gamma_A)_k (\Gamma_B)_k,
\tag{K16}
\]
and so the Bloch purity of a Bloch vector in the USB is
\[
P_B \equiv |\Gamma|^2 = \Gamma \cdot \Gamma = \text{tr}(\Gamma^2) = \sum_{k \neq 0} |\Gamma_k|^2.
\tag{K17}
\]

For reference, a useful way to obtain $\Gamma$ in the USB is
\[
\Gamma = \frac{1}{\sqrt{n(n-1)}} (n \rho - I).
\tag{K18}
\]

The multiparte reduction to $S \in 1, \ldots, N$ modes notated by mode-label vector $m \equiv (m_1, \ldots, m_S)$ is given in density-matrix form expanded by its USB reduced Bloch vector $\vec{\Gamma}(m)$ as
\[
\vec{\rho}(m) = \frac{1}{n_m} (I^n + \sqrt{n_m(n_m-1)} \vec{\Gamma}(m)),
\tag{K19}
\]
where $n_m \equiv n_{m_1} \cdots n_{m_S}$, and $\vec{\Gamma}(m)$ has components
\[
\vec{\Gamma}_k(m)_{k \neq m} = \vec{\Gamma}(m) \cdot b_k(m),
\tag{K20}
\]
where $k_m \equiv (k_{m_1}, \ldots, k_{m_S})$, $0_m \equiv (0_{m_1}, \ldots, 0_{m_S})$, and the USB of the reduced system is
\[
b_k(m) = \frac{1}{\sqrt{n_m}} \nu^{(m_1)}_{k_{m_1}} \cdots \nu^{(m_S)}_{k_{m_S}},
\tag{K21}
\]
where the mode-specific basis operators $\nu^{(m)}_{k_m}$ are defined in (K7). Note that modes in $m$ are not necessarily contiguous or ordered, but if order is changed, permutation unitaries are needed; see ([18], App. B).

Here is where our formalism will start to show benefits (with more to follow below). First, note that these definitions cause $\vec{\Gamma}(m)$ to be automatically unititized, meaning that $|\vec{\Gamma}(m)| = 1$ iff $\vec{\Gamma}(m)$ is a pure state and $|\vec{\Gamma}(m)| = 0$ iff $\vec{\Gamma}(m)$ is ideally maximally mixed for a general system of mode structure $m$.

In particular, we get a simple relationship that connects the Bloch components of the reduction to the full Bloch-vector components by a common factor as
\[
\vec{\Gamma}(m)_{k \neq m} = \frac{\sqrt{n-1}}{n_m-1} \Gamma_{k \neq m} \cdot \vec{b}(m),
\tag{K22}
\]
where $\vec{b}(m)$ is the ordered set of all full-system mode labels not in $m$, and where $k \equiv (k_1, \ldots, k_N)$, $k_{\vec{m}} \equiv (k_{m_1}, \ldots, k_{m_{N-S}})$, and $\delta_{k_{\vec{m}},0_m} = \delta_{k_{\vec{m}},0_1} \cdots \delta_{k_{\vec{m}},0_{N-S}}$, where $\vec{m}_j$ is the $j$th mode label in $\vec{m}$. Thus, the explicit-basis form of a multipartite reduced Bloch vector is
\[
\vec{\Gamma}(m) = \sum_{m_{\neq 0}} \vec{\Gamma}_k(m) b_k(m) = \sqrt{\frac{n-1}{n_m-1}} \sum_{m_{\neq 0}} \Gamma_{k \neq m} \cdot b_k(m),
\tag{K23}
\]
where $K_m \equiv (K_{m_1}, \ldots, K_{m_S})$, where $K_{m_j} \equiv n_{m_j} - 1$. The Bloch purity of a reduction in a Hermitian USB is
\[
P_B \equiv |\vec{\Gamma}(m)|^2 = \vec{\Gamma}(m) \cdot \vec{\Gamma}(m) = \text{tr}(\vec{\Gamma}(m)^2) = \sum_{m \neq 0} |\vec{\Gamma}(m)|^2.
\tag{K24}
\]

It is often tidier to simply organize the full Bloch-vector components corresponding to certain reductions into groups by defining correlation vectors (in the USB of the full Bloch vector) as
\[
X^{(m)} = \sum_{k_m = 1_m}^{K_m} \Gamma_{k \neq m} b_k(m) \cdot b_k(m),
\tag{K25}
\]
where $1_m \equiv (1_{m_1}, \ldots, 1_{m_S})$, with the property that
\[
\Gamma = \sum_{k=1}^N \sum_{l=1}^C \chi^{((nC_k)[c,k])},
\tag{K26}
\]
where $c \equiv (1, \ldots, N)$, and $nC_k[v,k]$ is the vectorized $n$-choose-$k$ function that gives a matrix whose rows are the unique combinations of the elements of $v$ chosen $k$ at a time, and $A_{l,\ldots}$ is the $l$th row of a matrix $A$. So, for example, for a tripartite system (meaning $N = 3$),
\[
\Gamma = X^{(1)} + X^{(2)} + X^{(3)} + X^{(1,2)} + X^{(2,3)} + X^{(1,3)} + X^{(1,2,3)},
\tag{K27}
\]
where, just expanding a few terms as examples,
\[
X^{(2)} = \sum_{k_2=1}^{K_2} \Gamma_{0,k_2,0} b_{0,k_2,0},
\quad
X^{(1,3)} = \sum_{k_1,k_3=1}^{K_1,K_3} \Gamma_{k_1,0,k_3} b_{k_1,0,k_3},
\quad
X^{(1,2,3)} = \sum_{k_1,k_2,k_3=1}^{K_1,K_2,K_3} \Gamma_{k_1,k_2,k_3} b_{k_1,k_2,k_3}.
\tag{K28}
\]
We can even define \textit{unitized} correlation vectors as
\[
\tilde{X}^{(m)} = \sqrt{\frac{n - 1}{n_m - 1}} X^{(m)} = \sum_{k_m=1}^{K_m} \tilde{\Gamma}^{(m)}_{k_m} b_{k_m} o_m,
\] (K29)
with the property that the Bloch purity of \(\tilde{X}^{(m)}\) matches that of \(\tilde{\Gamma}^{(m)}\) so that \(|\tilde{X}^{(m)}|^2 = |\tilde{\Gamma}^{(m)}|^2\), and the only difference between them is that the matrix basis of \(\tilde{X}^{(m)}\) lives in the full space of parent state \(\Gamma\), whereas the matrix basis of \(\tilde{\Gamma}^{(m)}\) lives in the space of mode group \(m\).

As proved in [47], we only need the single-mode reductions to quantify full \(N\)-partite entanglement, and as such, it is useful to define the \textit{liaison vector},
\[
\Lambda \equiv \Gamma - \sum_{m=1}^{N} X^{(m)} = \tilde{\Gamma} - \sum_{m=1}^{N} \sqrt{\frac{n_m - 1}{n - 1}} \tilde{X}^{(m)},
\] (K30)
which is the sum of all strictly multipartite reductions, so for example, in a tripartite system,
\[
\Lambda = X^{(1,2)} + X^{(2,3)} + X^{(1,3)} + X^{(1,2,3)}.
\] (K31)
Since \(\Lambda\) is the group of all \textit{strictly multimode} Bloch components, the vector indices \(k\) of its components \(\Lambda_k\) always have at least two nonzero indices \(k_m\). For example, \(X^{(1,3)}\) and \(X^{(1,2,3)}\) of (K28) both have two or more nonzero indices, as do all correlation vectors that make up \(\Lambda\).

Note that the Bloch purity is expressible as
\[
|\Gamma|^2 = |\Lambda|^2 + \sum_{m=1}^{N} |X^{(m)}|^2,
\] (K32)
which we use in our correlation-normalization proof. Another useful fact is that all \textit{separable} states obey
\[
\Gamma_{k \neq 0} = \frac{A_k}{\sqrt{n - 1}} \sum_{j_k} \bar{p}_{j_k} \tilde{\Gamma}^{(1)}_{j_k} \cdots \tilde{\Gamma}^{(N)}_{j_k},
\] (K33)
where \(\bar{\Gamma}^{(m)}_{j_k} \equiv \tilde{\Gamma}^{(m)}_{j_k}\), and
\[
A_k \equiv \Lambda_{k_1} \cdots \Lambda_{k_N}; \quad A_{k_m} \equiv (\sqrt{n_m - 1}) \text{sgn}(k_m).
\] (K34)
For \textit{product-form} states, using (5–8) in (K33) leads to
\[
\Gamma_{k \neq 0} = \frac{A_k}{\sqrt{n - 1}} \tilde{\Gamma}^{(1)}_{k_1} \cdots \tilde{\Gamma}^{(N)}_{k_N},
\] (K35)
where we used the fact that \(\bar{\tilde{\Gamma}}^{(m)}_{k_m} = \sum_{j_m} p_{j_m} \tilde{\Gamma}^{(m)}_{j_m k_m}\).

There are many useful applications of this formalism (and much more to say about it), which was introduced in the present form in [47], but this will suffice as a good working reference both here and for future research.

\section{Proof of Explicit Correlation Normalization Factors}

Expanding (12) in its simplest form in \(\text{ME TGX}\) states (since they have nice properties such as balanced superposition, multiple levels of zero probability, and diagonal reductions), we get
\[
\mathcal{N}_X = 1 - 2 \text{tr}(\rho_{\text{ME TGX}} \times \text{ME TGX}) + \text{tr}(\varsigma^2_{\text{ME TGX}}),
\] (K36)
where \(\varsigma_{\text{ME TGX}} \equiv \varsigma(\rho_{\text{ME TGX}})\) where \(\varsigma(\rho)\) is the reduction product from (11). Since the reduction product has product form, then its purity also has product form as
\[
\text{tr}(\varsigma^2_{\text{ME TGX}}) = P(\tilde{\rho}_{\text{ME TGX}}^{(1)}) \cdots P(\tilde{\rho}_{\text{ME TGX}}^{(N)}) = \prod_{m=1}^{N} P_{\text{MP}}^{(m)}(L_s),
\] (K37)
where \(P(\rho) \equiv \text{tr}(\rho^2)\), and \(P_{\text{MP}}^{(m)}(L_s)\) is the minimum physical reduction purity of mode \(m\) given a pure maximally full-\(N\)-partite-entangled parent state with \(L_s\) levels of equal nonzero probabilities.

To calculate each \(P_{\text{MP}}^{(m)}(L_s)\), first make the more general function,
\[
P_{\text{MP}}^{(m)}(L) \equiv \text{mod}(L, n_m) \left( \frac{1 + \text{floor}(L/n_m)}{L} \right)^2 + \left( n_m - \text{mod}(L, n_m) \right) \left( \frac{\text{floor}(L/n_m)}{L} \right)^2,
\] (K38)
where \(\text{mod}(a, b) \equiv a - \text{floor}(a/b)b\), and use that to define
\[
M(L) \equiv 1 - \frac{1}{N} \sum_{m=1}^{N} \frac{n_m P_{\text{MP}}^{(m)}(L) - 1}{n_m - 1}.
\] (K39)
Then find the set \(L_s \equiv \{ L_s \} \) which are the values of \(L\) that satisfy
\[
\min_{L \in 2, 3, \ldots, n_{\max}} [1 - M(L)],
\] (K40)
where \(n_{\max} \equiv n/n_{\max}\) and \(n_{\max} \equiv \max(n) = \{n_1, \ldots, n_N\}\). Then by convention let \(L_s \equiv \min(L_s)\) and use that to compute each \(P_{\text{MP}}^{(m)}(L_s)\). Note that we could simplify things slightly, but these quantities have physical significance in the context of the entanglement measure the \textit{ent} [18], so we use these forms for conceptual consistency. Thus, putting (K37) into (K36) gives
\[
\mathcal{N}_X = 1 - 2 \text{tr}(\rho_{\text{ME TGX}} \times \text{ME TGX}) + \prod_{m=1}^{N} P_{\text{MP}}^{(m)}(L_s).
\] (K41)

To get the overlap term in (K41), it is helpful to use a Bloch-vector formalism, such as that given in App. K1. First, we note that Bloch components of \(\varsigma\) have the product form of (K35) as
\[
\Gamma_{k \neq 0}(\varsigma) = \frac{A_k}{\sqrt{n - 1}} \tilde{\Gamma}^{(1)}_{k_1}(\varsigma) \cdots \tilde{\Gamma}^{(N)}_{k_N}(\varsigma),
\] (K42)
where \(k = (k_1, \ldots, k_N)\) is a vector index for multipartite Bloch vectors where \(k_m \in 0, \ldots, n_m^2 - 1\) for \(m = 1, \ldots, N\), (so \(k \neq 0\) means only the case of all \(k_m = 0\) is excluded), \(A_k\) is given in (K34), and \(\tilde{\Gamma}^{(m)}_{k_m}(\varsigma)\) are Bloch components of the mode-\(m\) reduction of \(\tilde{\Gamma}(\varsigma)\) which is the Bloch vector of \(\varsigma\) (see App. K1 for more details).

Now, consider the following facts:
1. If the parent state is an ME TGX state and multiple modes have size \( n_{\text{max}} \), then, as proved in ([18], App. D.4.c), all mode-\( m \) reductions are ideally maximally mixed, having purities \( P(\tilde{\rho}^{(m)}) = \frac{1}{n_m} \) for \( m \in 1, \ldots, N \).

2. If the parent state is an ME TGX state and exactly one mode (call it mode \( N \)) has size \( n_{\text{max}} \), then, as proved in ([18], App. D.4.c), all nonlargest mode-\( m \) reductions are ideally maximally mixed, with purities \( P(\tilde{\rho}^{(m)}) = \frac{1}{n_m} \) for \( m \in 1, \ldots, N - 1 \), while \( \tilde{\rho}^{(N)} \) has purity \( P^{(m)}(L_*) \) as given in (K38–K40), which is larger than that of the ideal maximally mixed state for an isolated system of size \( n_m \) levels, which is due to the purity and maximal entanglement of its parent state (see [18], App. D for full explanations). Note that \( P^{(m)}(L_*) \) simplifies to the correct value for all modes in this case, not just the largest mode.

3. From Step 1 and Step 2, all maximally full-\( N \)-partite entangled states have at least one single-mode reduction that is ideally maximally mixed.

4. As given in App. K.1, any mode-\( m \) ideally maximally-mixed Bloch vector has magnitude \( |\Gamma^{(m)}| = 0 \), and thus all its components are \( \Gamma_{k_n>0}^{(m)} = 0 \).

5. From Steps 1, 2, 3, and 4, for ME states, there always exist at least \( N - 1 \) reduced Bloch vectors with \( \{\Gamma^{(m)}(\rho_{\text{ME}})\} = 0 \) and thus \( \Gamma_{k_n>0}^{(m)}(\rho_{\text{ME}}) = 0 \) for \( m \in 1, \ldots, N - 1 \). (Note, here and throughout we use the convention that the system is organized in increasing mode size, so that mode \( N \) is always the largest even if there are multiple largest modes.)

6. The liaison vector \( \Lambda \) from (K30) is composed entirely of components of the full Bloch vector \( \Gamma \) that have two or more nonzero Bloch indices \( k_n \).

7. From (K42) and Step 6, the components of \( \Lambda(\varsigma_{\text{ME}}) \) take the form

\[
\Lambda_{k \neq 0}(\varsigma_{\text{ME}}) = \frac{\Lambda_k}{\sqrt{n} - 1} \tilde{\Gamma}^{(1)}_{k_1}(\varsigma_{\text{ME}}) \cdots \tilde{\Gamma}^{(N)}_{k_N}(\varsigma_{\text{ME}}) = 0, \quad (K43)
\]

where \( \varsigma_{\text{ME}} \equiv \varsigma(\rho_{\text{ME}}) \), because even though the largest mode can have nonzero components \( \tilde{\Gamma}^{(N)}_{k_N>0}(\varsigma_{\text{ME}}) \) (if it is the only largest mode), all other modes have \( \tilde{\Gamma}^{(m<\bar{N})}_{k_n>0}(\varsigma_{\text{ME}}) = 0 \), so since \( \mathbf{k} \) of \( \Lambda_k \) must always have at least two nonzero indices, at least one of the factors in \( \Lambda_k(\varsigma_{\text{ME}}) \) will always be zero. In the case of multiple largest modes, at least two factors in \( \Lambda_k(\varsigma_{\text{ME}}) \) will always be zero, so (K43) holds true for all systems. Therefore (K43) means that

\[
\Lambda(\varsigma_{\text{ME}}) = 0. \quad (K44)
\]

Now, putting (K44) into (K30), we see that the full Bloch vector of \( \varsigma_{\text{ME}} \) is a function of only single-mode reduced Bloch components as

\[
\Gamma(\varsigma_{\text{ME}}) = X^{(1)}(\varsigma_{\text{ME}}) + \cdots + X^{(N)}(\varsigma_{\text{ME}}), \quad (K45)
\]

where from putting (K22) into (K25), we see that, in general, \( X^{(m)} \) are Bloch vectors whose only nonzero components are proportional to single-mode reduction Bloch components, but whose basis matrices are in the space of the full Bloch vector, given by

\[
X^{(m)} = \sqrt{\frac{n_m - 1}{n - 1}} \sum_{k_m=1}^{K_m} \tilde{\Gamma}_{k_m}(m) b_{k_m} \cdot a_m, \quad (K46)
\]

where \( b_{k_m} \cdot a_m \equiv b_{k_1 \ldots k_n=0} \) is part of a matrix basis for the full Bloch vector, as defined in App. K.1. Thus, (K45) becomes

\[
\Gamma(\varsigma_{\text{ME}}) = \sum_{m=1}^{N} \sqrt{\frac{n_m - 1}{n - 1}} \sum_{k_m=1}^{K_m} \tilde{\Gamma}_{k_m}(m) b_{k_m} \cdot a_m. \quad (K47)
\]

However, this is the same as the Bloch purity of \( \varsigma_{\text{ME}} \),

\[
|\Gamma(\varsigma_{\text{ME}})|^2 = \Gamma(\varsigma_{\text{ME}}) \cdot \Gamma(\varsigma_{\text{ME}}) = \sum_{m=1}^{N} \frac{n_m - 1}{n - 1} |\tilde{\Gamma}^{(m)}(\rho_{\text{ME}})|^2, \quad (K50)
\]

so we have

\[
\Gamma(\rho_{\text{ME}}) \cdot \Gamma(\varsigma_{\text{ME}}) = \Gamma(\varsigma_{\text{ME}}) \cdot \Gamma(\varsigma_{\text{ME}}). \quad (K51)
\]

Then, putting (K51) into the fact from (K15) that \( \text{tr}(\rho_A \rho_B) = \frac{1}{n} (1 + (n - 1) \Gamma_A \cdot \Gamma_B) \), we obtain the important result that

\[
\text{tr}(\rho_{\text{ME}} \varsigma(\rho_{\text{ME}})) = P(\varsigma(\rho_{\text{ME}})). \quad (K52)
\]

Then, since this holds true for pure ME TGX states as well, then using (K37) in (K52) gives

\[
\text{tr}(\rho_{\text{ME TGX}} \varsigma(\rho_{\text{ME TGX}})) = \prod_{m=1}^{N} \frac{p^{(m)}_{\text{ME}}(L_*)}, \quad (K53)
\]
which, when put into (K41) yields
\[ N_X = 1 - \prod_{m=1}^{N} P_{MP}^{(m)}(L_m), \] \hspace{1cm} (K54)
which proves the result from (13), valid for all systems, where \( P_{MP}^{(m)}(L_m) \) is given by (K38) with \( L = L_m \) where \( L_m \) is determined from (K40).

For the special-case systems where more than one mode has size \( n_{\text{max}} \), we can use the fact, proved in ([18], App. D.4c), that for maximally entangled states, all reductions are ideally maximally mixed, so then (K53) is
\[ \prod_{m=1}^{N} P_{MP}^{(m)}(L_m) = \frac{1}{n^N} \prod_{m=1}^{N} I^{(N)} = \frac{1}{n}, \] \hspace{1cm} (K55)
which, when put into (K54) gives
\[ N_X = 1 - \frac{1}{n}, \] \hspace{1cm} (K56)
in agreement with (14), but again, is only valid for systems with multiple modes of size \( n_{\text{max}} \). For systems with exactly one largest mode, we must use (K54).

Finally, since all cases of mode structures have the property that all modes except for the nominally largest mode (which we will call \( n_{\text{max}} \) here for generality, meaning the single mode with size \( n_{\text{max}} \), designated as the nominally largest mode regardless of whether other modes also have size \( n_{\text{max}} \)) have minimal reductions that are ideally maximally mixed when the parent is maximally full-N-partite entangled, then (K54) can be simplified to
\[ N_X = 1 - \frac{P_{MP}(n_{\text{max}})}{n_{\text{max}}}, \] \hspace{1cm} (K57)
which may be used as an alternative to (13) or (K54), where \( \frac{n_{\text{max}}}{n} \equiv n/n_{\text{max}}. \)

**Appendix I: Proof-Sketch and Derivation of the Normalization of Diagonal Correlance**

Here we sketch a proof of why (17) is valid for finding the normalization of diagonal correlance \( \chi_D \), and derive the explicit result in (18).

1. Since all diagonal pure states are pure product states with no nonlocal correlation, that disqualifies pure states as maximizers of \( \chi_D \).
2. Therefore, by Step 1, maximizers of \( \chi_D \) (called \( \rho_{D_{\text{max}}} \)) must have rank 2 or higher.
3. By definition, \( \rho_{D_{\text{max}}} \) maximize \( \chi_D \) as in (16). Therefore, while \( \rho_{D_{\text{max}}} \) must have rank \( r \geq 2 \) by Step 2, they must also \textit{not} have product form by definition, and their Bloch vectors must \textit{also} have the largest distance from their own reduction products \( \varsigma_D \equiv \varsigma(\rho_{D_{\text{max}}}) \) under the Hilbert-Schmidt inner product, out of all states compared to their own reduction products. (Note: \( \chi_D \) and \( \chi \) are \textit{not} measures of distance to the set of all product states from the given input \( \rho \); instead they are measures of the distance between \( \rho \) and its \textit{own} reduction product. Therefore, all \( \rho_{D_{\text{max}}} \) have the biggest distance between themselves and their own \( \varsigma_D \); out of all diagonal states \( \rho_D \), but their \( \varsigma_D \) are generally neither the farthest nor the closest product states from themselves.)
4. Since all \( \rho_D \) are diagonal, then by the definition of partial trace, their reductions \( \rho_{D_{\text{red}}} \) are also diagonal.
5. From (16), the function to be maximized by \( \rho_{D_{\text{max}}} \) is \( \chi(\rho_D) = P(\rho_D) + P(\rho_{D_{\text{red}}}) - 2\text{tr}[\rho_{D_{\text{red}}}^2] \) over all \( \rho_D \), where \( P(\sigma) \equiv \text{tr}(\sigma^2) \) is the purity of \( \sigma \), and the overlap is \( \text{tr}[\rho_{D_{\text{red}}}^2] = \text{tr}[\rho_{D_{\text{red}}}^2] \).
6. By Step 5, maximizers of \( \chi(\rho_D) \) need to simultaneously fulfill the conditions of having highest purity, highest single-mode reduction purities, and lowest overlap of parent state and reduction product.
7. By Step 6, we need to minimize \( \text{tr}[\rho_{D_{\text{red}}}^2] \) (while also keeping the purity terms as high as possible). So first, expanding it as \( \text{tr}[\rho_{D_{\text{red}}}^2] = \sum_{a=1}^{n} \rho_D(a,a)(\varsigma_D)_{a,a} = \sum_{a_1,\ldots,a_N=1}^{n} \rho_D(a_1,a_1)\ldots\rho_D(a_N,a_N) \) where \( a \equiv (a_1,\ldots,a_N) \), shows that each term is a product of the probabilities of \( \rho_D \) with the probabilities of each of the \( N \) single-mode reductions of \( \rho_D \) (which are also functions of probabilities of \( \rho_D \)), by Step 4. This suggests that the more diagonal elements of \( \rho_D \) are zero, the lower this overlap function will be, which would also increase the purity of \( \rho_D \), but not necessarily the reduction purity product. Therefore this suggests that we reduce rank as much as possible while also looking at the particular ways to choose nonzero elements to maximize \( \chi(\rho_D) \). For example, in 2 \times 3, abbreviating with \( \rho \equiv \rho_{D_{\text{max}}} \) and \( \varsigma_D \equiv \varsigma(\rho_D) \),
\[
(\varsigma_D)_{11} = P_{11}^{(1)}P_{11}^{(2)}, \quad (\varsigma_D)_{12} = P_{12}^{(1)}P_{12}^{(2)} = (P_{11} + P_{21} + P_{31})(P_{12} + P_{22} + P_{32}),
\]
\[
(\varsigma_D)_{21} = P_{21}^{(1)}P_{21}^{(2)} = (P_{11} + P_{21} + P_{31})(P_{22} + P_{23} + P_{32}),
\]
\[
(\varsigma_D)_{33} = P_{33}^{(1)}P_{33}^{(2)} = (P_{11} + P_{21} + P_{31})(P_{33} + P_{33} + P_{36}),
\]
\[
(\varsigma_D)_{14} = P_{11}^{(1)}P_{11}^{(2)} = (P_{44} + P_{54} + P_{64})(P_{11} + P_{44}),
\]
\[
(\varsigma_D)_{55} = P_{55}^{(1)}P_{55}^{(2)} = (P_{44} + P_{54} + P_{64})(P_{55} + P_{56} + P_{56}),
\]
\[
(\varsigma_D)_{66} = P_{66}^{(1)}P_{66}^{(2)} = (P_{44} + P_{54} + P_{64})(P_{66} + P_{66} + P_{66}),
\]
which shows that some choices of nonzero parent elements lead to more nonzero elements in \( \varsigma_D \) than others. Since the rank of \( \rho \) determines the number of terms in \( \text{tr}[\rho_{D_{\text{red}}}^2] \) anyway, then to minimize it for a given rank, we need to choose the nonzero parent elements in a way that maximizes the number of nonzero terms in \( \varsigma_D \) so that their values are lower due to normalization, which allows a lower value of \( \text{tr}[\rho_{D_{\text{red}}}^2] \).
8. Considering rank and choice of nonzero elements, notice that for all states, \( \rho_{11} \) always appears in \( \rho_{11}^{(1)} \) and
\( \rho_{n,n} \) always appears in \( \tilde{\rho}_{n,n,n} \) \( \forall m \in 1, \ldots, N \). Thus for any number of \( N \) modes, if a rank-2 parent state only has \( \rho_{1,1} \neq 0 \) and \( \rho_{n,n} \neq 0 \), then \( s_{1,1} \) and \( s_{n,n} \) will always be nonzero and several other elements of \( \varsigma \) may still be nonzero as well, as in (L1) which has four nonzero elements in this case. This is important because since the parent state has all other \( n-2 \) diagonal elements as 0, then \( n-2 \) terms of the negative overlap term in Step 5 vanish as seen by Step 7, reducing its ability to lessen the objective function, while the factors from \( \varsigma \) are lower than they would be for other choices of nonzero parent elements, since there are more than two nonzero elements of \( \varsigma \). Also, the parent purity has its largest minimum since \( P(\rho) \in (\frac{1}{2},1) \), while \( P(\varsigma) \in (\frac{1}{2},1) \) with possibly a larger minimum due to the parent state’s correlation (as shown for reduction products of maximally entangled parent states in [18]), where we used \( \rho \equiv \rho_D \) and \( \varsigma \equiv \varsigma_D \).

9. By Step 7, since every term of the objective function \( \tilde{X}_D \) (for any rank) is a function of parent elements, their normalization and nonnegativity makes \( \tilde{X}_D \) a linear combination of products of squares of \( r \) unit hyperspherical coordinates \( \{x_k \} \). It is well-known that the sum of even powers of \( \{x_k \} \) is minimized when \( x_k = \frac{1}{\sqrt{r}} \forall k \in 1, \ldots, r \) (see [18], App. D.2). Here, this same solution maximizes the rank-2 case of \( \tilde{X}_D \), and for ranks \( r \geq 3 \), neither balanced probabilities nor any other combination of values of hyperspherical coordinates can cause \( \tilde{X}_D \) to get as large as the rank-2 case, because they generally cause a smaller parent-state purity and a smaller reduction-purity product than the rank-2 case. The overlap term behaves less consistently, however its combination with the other terms in \( \tilde{X}_D \) is such that the rank-2 value of \( \tilde{X}_D \) is always the largest when maximized over all combinations of nonzero-levels for the equal-probabilities case. This was confirmed by brute-force combinatorial comparison of all ranks and nonzero-element combinations for several multipartite systems in equal-probability states, while Fig. 4 provides strong numerical evidence that it is true over all \( \rho_D \). However, a rigorous proof of this is still lacking.

10. Therefore, Step 9 gives evidence that \( \rho_{D_{\text{max}}} \) must have rank 2 and equal probabilities (note that if either probability is larger, the state would be closer to being a pure product state, so it would have nonmaximal correlation). Furthermore, Step 8 suggests one way to choose particular nonzero elements that will minimize the overlap the most out of all possible pairs, since it causes the most terms in the overlap to be zero, while causing the most terms of \( \varsigma \) to be nonzero which lowers the values of the surviving overlap terms. Therefore, our candidate maximizer states are \( \rho_{D_{\text{max}}} = \text{diag} \{ \frac{1}{2^1}, 0, \ldots, 0_{n_m-1}, \frac{1}{2^N} \} \).

11. To see that the candidate \( \rho_{D_{\text{max}}} \) from Step 10 also satisfies the requirement from Step 3 that it not have product form, note that in any diagonal state of equal nonzero probabilities, the factorizability of basis elements of the nonzero terms completely determines whether it has product form. A rank-2 diagonal state of equal nonzero elements that does not have product form in all multipartite systems is \( \rho_{D_{\text{max}}} = \frac{1}{2}(|1\rangle\langle 1| + |n\rangle\langle n|) \) of (17), since its basis states of nonzero probabilities yield the coincidence form \( \rho_{D_{\text{max}}} = \frac{1}{2}(|1\rangle\langle 1| \otimes \cdots \otimes |n_N\rangle\langle n_N|) + |n_1\rangle\langle n_1| \otimes \cdots \otimes |n_N\rangle\langle n_N|) \), which is unfactorizable because the projectors in each mode are different in each term, preventing any single-mode projectors from being factored out. Using the first and last basis element guarantees that the projectors will always have different labels in each mode between the two terms, which is the simplest way to guarantee this property for all systems. Thus, since this is exactly the candidate state of Step 10, we have justified why it is a prototypical \( \rho_{D_{\text{max}}} \).

Now that we have a motivation for why \( \rho_{D_{\text{max}}} \) from Step 11 is a nonfactorizable diagonal state furthest from its own reduction product, we need to calculate its raw diagonal correlation to get the normalization factor \( N_{X_D} \) of (18). Therefore, we start with the raw correlate of \( \rho_{D_{\text{max}}} = \frac{1}{2}(|1\rangle\langle 1| + |n\rangle\langle n|) \) from Step 11 above, as

\[
N_{X_D} = \tilde{\chi}(\rho_{D_{\text{max}}}) = \text{tr}[\rho_{D_{\text{max}}}/\varsigma_{D_{\text{max}}})^2] = P(\rho_{D_{\text{max}}}) - 2\text{tr}[\rho_{D_{\text{max}}}\varsigma_{D_{\text{max}}}] + P(\varsigma_{D_{\text{max}}}),
\]

where \( \varsigma_{D_{\text{max}}} \equiv \varsigma(\rho_{D_{\text{max}}}) \), and

\[
\rho_{D_{\text{max}}} = \text{diag} \{ \frac{1}{2^1}, 0_2, \ldots, 0_{n_m-1}, \frac{1}{2_n} \},
\]

(where we use subscripts to help indicate which basis element corresponds to each matrix element), and thus

\[
P(\rho_{D_{\text{max}}}) = \frac{1}{2}.
\]

Then, using the fact from Step 8 that writing any single-mode reduction in terms of the matrix elements of the parent state always features the first and last diagonal parent matrix elements in separate diagonal matrix elements of the reduced states (specifically the first and last element of each, where the multipartite basis is ordered by the standard register-counting convention), then each mode-\( m \) reduction mirrors the parent state as

\[
\tilde{\rho}^{(m)}_{D_{\text{max}}} = \text{diag} \{ \frac{1}{2^1}, 0_2, \ldots, 0_{n_m-1}, \frac{1}{2} \},
\]

which means that the reduction product is

\[
\varsigma_{D_{\text{max}}} = \otimes_{m=1}^{N} \tilde{\rho}^{(m)}_{D_{\text{max}}} = \otimes_{m=1}^{N} \text{diag} \{ \frac{1}{2^1}, 0_2, \ldots, 0_{n_m-1}, \frac{1}{2_n} \},
\]

and therefore has purity

\[
P(\varsigma_{D_{\text{max}}}) = \prod_{m=1}^{N} \frac{1}{2} = \frac{1}{2^N}.
\]
For the overlap in (L2), since all but the first and last elements of \( \rho_{D_{\max}} \) are zero, only the first and last elements of \( \Delta_{D_{\max}} \) affect the overlap, and since those elements are both \( \frac{1}{2N} \) by expanding (L6), then

\[
\text{tr}(\rho_{D_{\max}} \Delta_{D_{\max}}) = \text{tr}(\text{diag}(\frac{1}{2}, \frac{1}{2N}, 0, \ldots, \frac{1}{2N})) = \frac{1}{2N}.
\]

Thus, putting (L4), (L7), and (L8) into (L2) gives

\[
\mathcal{N}_{D} = \frac{1}{2} - 2 \frac{1}{2N} + \frac{1}{2N} = \frac{1}{2} - \frac{1}{2N}, \tag{L9}
\]

which is the result in (18).

Note that any given diagonal state \( \rho_{D} \) can be converted to another diagonal state of equal \( \mathcal{X}_{D} \) by a local-permutation unitary (LPU) (which can generally be complex), meaning a tensor-product of permutation unitaries of each mode, since these operators cause neither superpositions between modes nor superpositions within the modes, preserving both locality and diagonality. Local unitaries (LU) also preserve \( \mathcal{X}_{D} \), but not diagonality, so they cannot be used to reach all states of a given \( \mathcal{X}_{D} \) from a single \( \rho_{D} \). Thus, since \( \rho_{D_{\max}} \) only has rank 2 and balanced probabilities, the set of all \( \rho_{D_{\max}} \) is found by applying any LPU to the prototypical \( \rho_{D_{\max}} \) of (17).

**Appendix M: Algorithm \( \mathcal{A}_{\rho} \) to Form a Multipartite Density Matrix from Strictly Classical Data**

The purpose of this algorithm is to generate a density-matrix estimator \( \rho \) from the data set of measurements of a strictly classical multivariable system, so that we can then measure the general nonlocal correlation in the data using the diagonal correlation \( \mathcal{X}_{D}(\rho) \) of (15).

Continuing from the description leading up to (19), the main idea of this algorithm is that the data for each of the \( N \) random variables (RVs) \( \{x^{(m)}\} \) is first quantized separately, and then combined as a set of \( n_{S} \) quantized \( N \)-tuple data points \( \{x^{(m)}\} \). During the quantization, we make a list of the center values of every bin of each mode as the representative bin values \( \bar{x}^{(m)} \equiv (\bar{x}^{(m)}_{1}, \ldots, \bar{x}^{(m)}_{n_{m}})^{T} \). Then, we create the density matrix \( \rho \) for the quantized data by cycling through \( N \)-tuples of representative bin values in “register format” (for example, \( \{(\bar{x}^{(1)}, \bar{x}^{(2)}), (\bar{x}^{(1)}, \bar{x}^{(2)}), (\bar{x}^{(1)}, \bar{x}^{(2)}), (\bar{x}^{(1)}, \bar{x}^{(2)})\} \) in the simplest nontrivial case), and counting the number of quantized data points that match each of them as an entire \( N \)-tuple, forming a multipartite histogram, which is then normalized and placed in the main-diagonal elements of an \( n \times n \) matrix, giving us \( \rho \).

The following steps constitute algorithm \( \mathcal{A}_{\rho} \):

1. Identify the \( N \) RVs of a given data set as

\[
x \equiv (x^{(1)}, \ldots, x^{(N)}).
\]

2. Organize the data as the \( n_{S} \times N \) matrix,

\[
X \equiv \begin{pmatrix}
x_{1}^{(1)} & \cdots & x_{1}^{(N)} \\
\vdots & \ddots & \vdots \\
x_{n_{S}}^{(1)} & \cdots & x_{n_{S}}^{(N)}
\end{pmatrix}
\]

where the \( j \)-th \( N \)-dimensional data point \( x_{j} \) is an \( N \)-tuple of measured values, for a total of \( n_{S} \) measurements of \( N \) values each. Thus, elements of \( X \) are \( X_{j,m} \equiv x_{j}^{(m)} \) for \( j \in \{1, \ldots, n_{S}\} \) and \( m \in \{1, \ldots, N\} \).

3. Choose quantization (bin) numbers \( n_{m} \) for each RV as

\[
n_{m} \equiv (n_{1}, \ldots, n_{N}). \tag{M3}
\]

For discrete RVs, \( n_{m} \) is the number of possible outcomes for RV \( x^{(m)} \). For continuous RVs, \( n_{m} \) is the finite number of recognized possible values for RV \( x^{(m)} \), and there are many techniques for choosing \( n_{m} \) appropriately based on the data. Choice of \( n_{m} \) sets the bin number in a histogram of the data in column \( m \) of \( X \). Note: in most cases, the precision of the measuring devices used to obtain the data already imposes some quantization on the RVs.

4. Define the domain of each RV as a bounded a pair of extreme values \( [x^{(m)}_{\min}, x^{(m)}_{\max}] \). In some cases it might be preferable to choose bounds that hug the extremes of the data itself, such as by using

\[
x_{\min}^{(m)} \equiv \min\{X_{:,m}\}, \quad x_{\max}^{(m)} \equiv \max\{X_{:,m}\}, \quad X_{:,m} \equiv \begin{pmatrix} X_{1,m} \\ \vdots \end{pmatrix} \quad \text{for all } m \quad \text{(M4)}.
\]

However, note that defining \( [x^{(m)}_{\min}, x^{(m)}_{\max}] \) as the extreme values of what could happen for each RV (regardless of whether data reaches those values) is technically the more correct method here, so these should generally be specified inputs instead of using (M4).

5. Get bin-edge lists \( e^{(m)} \) for each RV as

\[
e^{(m)} \equiv (e^{(m)}_{1}, \ldots, e^{(m)}_{n_{m}+1})^{T}, \tag{M5}
\]

where the \( n_{m} + 1 \) bin edges for RV \( x^{(m)} \) are

\[
e^{(m)}_{k_{m}} \equiv x^{(m)}_{\min} + (k_{m} - 1) \frac{x^{(m)}_{\max} - x^{(m)}_{\min}}{n_{m}}, \tag{M6}
\]

for \( k_{m} \in \{1, \ldots, n_{m} + 1\} \).

6. Get representative bin-value lists for each RV as

\[
\bar{x}^{(m)} \equiv \begin{pmatrix} \bar{x}^{(m)}_{1} \\ \vdots \end{pmatrix}, \tag{M7}
\]

with elements

\[
\bar{x}^{(m)}_{q} \equiv \frac{e^{(m)}_{q} + e^{(m)}_{q+1}}{2}; \quad q \in \{1, \ldots, n_{m}\}. \tag{M8}
\]
7. Quantize the data as an \( n_S \times N \) matrix \( X' \) using
\[
X' = \{0^{[n_S \times N]} \}
\]
for \( m = 1, \ldots, N \)
for \( j = 1, \ldots, n_S \)
for \( q = 1, \ldots, n_m \)
if \( (q == 1) \& (X_{j,m} < e_2(m)) \)
\[
X'_{j,m} = \tilde{x}^{(m)}_1
\]
else if \( (q > 1) \& (q < n_m) \)
\[
X'_{j,m} = \tilde{x}^{(m)}_q
\]
else if \( (q == n_m) \& (X_{j,m} > e_{n_q+1}(m)) \)
\[
X'_{j,m} = \tilde{x}^{(m)}_{n_m}
\]
where the notation \( M^{[a \times b]} \) means an \( a \times b \) matrix \( M \).
Notice that for bin 1 and bin \( n_m \), the conditions assign data outside the RV extremes to those end bins;
that is to allow variations in measured values to go under or over the extremes, but again, this may not be applicable
in all situations, so modify as appropriate.

8. Generate the multipartite histogramic \( n \times n \) density matrix \( \rho \) (where \( n = n_1 \cdots n_N \)) using
\[
\rho = \{0^{[n \times n]} \}
\]
for \( a \in 1, \ldots, n \)
\[
a = a^{[N \times n]}_a
\]
for \( j = 1, \ldots, n_S \)
\[
x_j = X'_j \cdots v = \{0^{[1 \times N]} \}
\]
for \( m = 1, \ldots, N \)
\[
v_m = \tilde{x}^{(m)}_{a m}
\]
if \( \text{sum}(x_j == v) == N \)
\[
\rho_{a,a} = \rho_{a,a} + 1
\]
\[
\rho = \rho / \text{tr}(\rho)
\]
where \( a^{[N \times n]}_a \) is the inverse register function given in App. U that maps scalar index \( a \) to vector index
\( \mathbf{a} \equiv (a_1, \ldots, a_N) \), and \( \mathbf{v} \equiv (v_1, \ldots, v_N) \) is a temporary vector to hold the multipartite representative bin
corresponding to scalar index \( a \).

This step counts all the occurrences of each multipartite representative bin value that arises in the quantized
data \( X' \) and tabulates it as a relative frequency in the main diagonal of a density matrix indexed with the corresponding
classical scalar row-column indices for that value. Note that, being diagonal by design, this \( \rho \) can be
stored as a vector of length \( n \), since it is just a classical discrete probability density function, but we keep it in matrix form here for conceptual continuity
with the full quantum problem.

The result of this algorithm is \( \rho \), an estimator of the
strictly classical density matrix, which can then be used to calculate the diagonal correlation \( \mathcal{X}_D(\rho) \) in (15).

The above steps may not necessarily be the most efficient implementation of \( \mathcal{A}_D \); however in this form, the
steps are at least conceptually clear.

### Appendix N: Pearson Correlation Coefficient

Consider a system of \( N \) classical random variables (RVs) \( \mathbf{x} \equiv (x^{(1)}, \ldots, x^{(N)}) \), for a dataset \( X \) defined as a matrix of \( N \) columns representing the RVs, and \( n_S \) rows each consisting of an \( N \)-tuple sample of particular measured (observed) values \( x_j \equiv (x^{j(1)}, \ldots, x^{j(N)}) \) for the \( N \) RVs, for a total sample number of \( n_S \) measured \( N \)-tuples [see (M2)]. Thus, \( X_{j,m} \) is the \( j \)th observed value for the \( m \)th RV \( x^{(m)}(\cdot) \).

The sample Pearson correlation coefficient [19–21], which is defined only for the case of two variables, is
given for any two RVs \( x^{(a)} \) and \( x^{(b)} \) by
\[
r_p \equiv r_p^{(a,b)} \equiv \frac{V_S^{(a,b)}}{s^{[a]s^{[b]}}},
\]
where \( V_S^{(a,b)} \equiv \text{cov}_S(x^{(a)}, x^{(b)}) \) is the (unbiased) sample
covariance between \( x^{(a)} \) and \( x^{(b)} \),
\[
V_S^{(a,b)} \equiv \frac{1}{n_S-1} \sum_{j=1}^{n_S} (x^{j(a)} - \mu_S^{(a)})(x^{j(b)} - \mu_S^{(b)})
\]
\[
= \frac{1}{n_S-1} \sum_{j=1}^{n_S} (X_{j,a} - \mu_S^{(a)})(X_{j,b} - \mu_S^{(b)}),
\]
where the sample mean of RV \( x^{(m)} \) is
\[
\mu_S^{(m)} \equiv \bar{x}^{(m)} \equiv \frac{1}{n_S} \sum_{j=1}^{n_S} x^{j(m)} = \frac{1}{n_S} \sum_{j=1}^{n_S} X_{j,m},
\]
and the sample standard deviation is
\[
s^{(m)} \equiv \sqrt{V_S^{(m,m)}} \equiv \sqrt{\frac{1}{n_S-1} \sum_{j=1}^{n_S} (x^{j(m)} - \mu_S^{(m)})^2}
\]
\[
= \sqrt{\frac{1}{n_S-1} \sum_{j=1}^{n_S} (X_{j,m} - \mu_S^{(m)})^2}.
\]

We can also form an \( N \)-by-\( N \) sample covariance matrix \( V_S \) with elements \( (V_S)^{a,b}_j \equiv V_S^{(a,b)} \) and an \( N \)-by-\( N \) sample Pearson correlation matrix \( r_p \) with elements \( (r_p)^{a,b}_j \equiv r_p^{(a,b)} \), each containing all pairwise covariances and Pearson correlations between the \( N \) RVs. However, it is common practice to simply write \( r_p \equiv r_p^{(a,b)} \), as in (N1), to denote the scalar value of the sample Pearson correlation coefficient between two particular variables when those are the only two variables in the problem and no Pearson correlation matrix is used elsewhere in the analysis, which is the convention we use. Also, it is not standard to use the “\( P \)” subscript, but we use it to distinguish it from our use of \( r \) to mean rank.

The Pearson correlation has range \([-1, 1]\), and is merely a measure of linear correlation between two RVs only. Thus, two variables can have strong nonlinear correlations and yet produce \( r_p = 0 \), so it is not a measure of general correlation. Furthermore, the old adage that “correlation does not imply causation” means that having a nonzero \( r_p \) does not mean that the RVs share
some functional dependence on a common parameter, but rather it means that the data of the two RVs have a linear correspondence whether they are functions of the same underlying variables or not.

Appendix O: Consistency of Strict Classicality with Established Physics

Here, we list a few additional reasons why the strictly classical states defined in Sec. III C are compatible with existing ideas in physics, and are a much more appropriate standard for classicality than coherent states.

1. For multiple modes of the same size, the set of all distinct pure strictly classical states has the same form as the set of possible wave functions that yield Maxwell-Boltzmann statistics [58, 59], which arise from a collection of identical but distinguishable particles, and is a regime of statistics that would apply if the universe were not quantum-mechanical, i.e. in a strictly classical universe. For example, for two identical, distinguishable two-level particles, the possible wave functions are \( \{ \psi_1(x_1) \otimes \psi_1(x_2), \psi_1(x_1) \otimes \psi_2(x_2), \psi_2(x_1) \otimes \psi_1(x_2), \psi_2(x_1) \otimes \psi_2(x_2) \} \), where \( \psi_n(x_m) \equiv \langle x^{(m)} | \psi_n^{(m)} \rangle \) is the nth possible state of particle \( m \), and \( |x^{(m)} \rangle \) is the position eigenstate of particle \( m \). These states form a set of pure computational basis states, which qualify (in form) as strictly classical states as defined in Sec. III C.

2. In regards to probability representing the lack of knowledge of the observer, there are really two general kinds of states with different meanings [28–30]:

a. Ontic states: the actual state of a system, part of fundamental reality.

b. Epistemic states: the state in the context of an observer, given limitations and lack of full knowledge; is usually a statistical mixture of ontic states each assigned probabilities based on observer ignorance.

Pure strictly classical states are ontic, since the state is definite. Mixed strictly classical states are epistemic, since by the definition of strict classicality, any probability distributions arise from lack of specific knowledge (i.e., taking sample measurements with time windows too long to notice the fact that the system is instantaneously pure). This dichotomy fits nicely with the classical notion of probability; the only kind of probability in a strictly classical system is the ignorance-induced kind, and there is no fundamental quantum probability such as that which arises in pure quantum states with superposition (we exclude the possibility of hidden variable theories until Sec. VI). Furthermore, since pure strictly classical states are pure product states of computational basis states in all reference frames, nonlocal correlation is impossible in an ontic strictly classical state. In Sec. III–Sec. V we discuss the more interesting case of mixed strictly classical nonlocal correlation in detail.

3. Quantum coherence functions arise from finding mean values of quantities composed of electric field operators in analogy to classical coherence functions. What this leads to is a certification of whether or not the state of the electric field is a coherent state (which is only a necessary and sufficient certification of being in a coherent state if the field has \( m \)-th order quantum coherence values of \( \langle |g^{(m)}(x_1, \ldots, x_m)\rangle = 1 \) \( \forall m \in \{1, \ldots, \infty\} \), meaning that it is “infinite-order coherent” [27], where here \( x_m \equiv (\eta_m, t_m) \) where \( \eta_m \equiv r - r_m \) is a separation vector from source point \( r_m \) to field point \( r \) and \( t_m \equiv t - |\eta_m| \) is the retarded time for light to reach \( r \) from \( r_m \). However, measuring how close a state is to infinite-order coherence does not prove anything about classicality; rather it is a test of how close the state is to being a coherent state (with lower-order failures such as \( \langle g^{(2)}(x_1, x_2)\rangle \neq 1 \) being sufficient to conclude that a state is not a coherent state, thereby making such a test practical to use).

The fallacy of assuming that a state’s classicality is determined by its similarity to a coherent state as quantified by quantum coherence functions is evident for several reasons. The main red flag indicating this fallacy is that a coherent state requires quantum superposition in the Fock basis, an impossibility in truly classical physics. Another red flag is that quantum-coherence-function results are often interpreted in conjunction with quantum-phase space probability distributions where nonclassicality is said to be evident by the appearance of negative probabilities, which are not truly allowed in quantum mechanics; a density matrix with any negative diagonal elements is nonphysical. In contrast, the definition of strict classicality forbids quantum superposition, a restriction that is a necessary requirement of classicality not satisfied by coherent states, and strict classicality requires no negative probabilities to certify it.

Regarding Reason 3 above, the classical coherence functions arise by defining abbreviations for quantities not naturally expressible in terms of electric field intensities at single spacetime points alone. The quantum coherence functions simply promote the classical field quantities to operators and promote the time average to a quantum ensemble average; thus its main claim to classicality is through a generalization of Ehrenfest’s theorem [60, 61], which is merely an approximate condition for correspondence of quantum operators to classical quantities, but not sufficient (nor exactly necessary) to determine classicality since it is not a fundamental law of quantum mechanics. In other words; even though construction of quantum coherence functions is permissible, the ability of certain quantum states (such as coherent states) to produce mean values that we expect in a classical theory is not sufficient to conclude that such states
are truly classical; at best we could say that they exhibit classical behavior, and as such, coherent states might be considered the most classical of all quantum states that possess superposition in a computational basis.

Appendix P: Limits of Decomposition Indices

In Sec. IV A, the decomposition unitary $U$ is limited to dimension $D$ of at least $r \equiv \text{rank}(\rho)$ (guaranteed by the existence of the spectral decomposition), and a minimax of at most $r^2$ (proved by P. Horodecki using the Caratheodory theorem justified by a Bloch-vector expansion [44, 62], which sets the maximum number of decomposition states needed to describe a separable state with mode-independent (MI) decomposition states [Family 5 from Table I]), so that the number of pure decomposition states can always be limited to $D \in r, \ldots, r^2$, even though we are free to use $D \in r, \ldots, \infty$.

Note that in [44], the term “dimension” is used to mean “rank” which can be seen in a later work by the same author [63] in which the upper decomposition limit is clearly identified as $r^2$. The fact that this limit should be $r^2$ rather than $n^2$ is clear when we consider the case of pure states, which have $r = 1$ and only need one decomposition state (the pure state itself up to global phase), regardless of dimension. However, if for some reason this interpretation is incorrect and the use of $\max(D) = r^2$ turns out to be too restrictive, then we would simply use $\max(D) = n^2$ instead. Since none of the main results in this paper rely on actually computing statance or probablance, this issue has no bearing on the main results.

Then, since $j \equiv (j_1, \ldots, j_N)$ is our index for decomposition states of an $N$-mode state, we need to determine the bounds of each $j_m$ such that we achieve a given total of $D$ decomposition states. Therefore, if we let the range of each mode label be $j_m \in 1, \ldots, D_m$, we can define a vector $\mathbf{D} \equiv (D_1, \ldots, D_N)$ such that $D = \prod_{m=1}^N D_m = D_1 \cdots D_N$ [due to the nested sums in quantities like (24)] so that as $j \equiv (j_1, \ldots, j_N)$ counts over all allowed values, it has exactly $D$ of them.

Now we just need to determine the value of a given $D_m$. By the Caratheodory theorem [44, 62], the upper necessary limit $D_m$ for each mode label for a Family-5 state is $n^2_m$, where $n_m$ is the number of levels of mode $m$. Thus, by this line of reasoning, we need to use

$$D_m \in 1, \ldots, n^2_m \quad \text{s.t.} \quad \prod_{m=1}^N D_m = D \quad \text{for} \quad D \in r, \ldots, r^2.$$  \hspace{1cm} (P1)

However, note that in some cases, the Caratheodory theorem’s application by P. Horodecki may be too restrictive for the following reason. Recall that in Wootters’s full concurrence paper [43], he included an explicit method showing that it is always possible to decompose any separable two-qubit state with exactly $r$ separable decomposition states (he used $n$ to mean rank there) of the form of Family 3 from Table I. Yet in P. Horodecki’s paper [44], he placed an upper bound on the number of necessary separable decomposition states of $r^2$, but he used states of the form of Family 5 from Table I, which are only a small subset of all separable states, the full set being Family 3.

The interesting thing about Wootters’s decomposition is that it also applies to the Family-5 states, meaning that even states for which statance is zero (since they have MI decomposition states) have a separable decomposition of the Family-5 form with only $r$ members! But there is no contradiction here; the reason the Wootters decomposition works for all separable states is that it is based merely on minimizing average entanglement, a special kind of decomposition-state correlation, which means that at least for two qubits, all states with an MI set of decomposition states (Families 5 and 6 from Table I) are guaranteed to have a separable decomposition of only $r$ members. But because statance measures all decomposition-state correlation, of which entanglement is only a part, the decomposition that minimizes statance is generally more restrictive and therefore requires more members, so that is why we must use the Caratheodory theorem to set the limits in the statance calculation. Basically, P. Horodecki used a set of states that is only a subset of separable states; his proof applies only to Family 5 and its subset, Family 6. However, his result is still correct; it is just too loose of an upper bound for more general separable states such as those of Families 3 and 4 that are not also in Families 5 and 6.

To see how (P1) fits into the decomposition of zero-statance states (since they are the standard by which zero statance is achieved), Table III shows all the possible decomposition vectors $\mathbf{D}$ allowable for two qubits for the purpose of keeping $D \in r, \ldots, r^2$.

| $r$  | $\{\mathbf{D}\}$                          | $\{D\}$                     |
|-----|----------------------------------------|-----------------------------|
| 1   | $\{1\}$                                | $\{(1,1)\}$                |
| 2   | $\{2,3,4\}$                            | $\{(1,2),(2,1),(1,3),(3,1),(1,4),\}$ |
|     |                                         | $(4,1),(2,2)\}$             |
| 3   | $\{3,4,6,8,9\}$                        | $\{(1,3),(3,1),(1,4),(4,1),(2,2),\}$ |
|     |                                         | $(2,3),(3,2),(2,4),(4,2),(3,3)\}$ |
| 4   | $\{4,6,8,9,12,16\}$                    | $\{(1,4),(4,1),(2,2),(2,3),(3,2),(2,4),(4,2),(3,3)\}$ |

Notice that $D$ is never a prime number for $D > n$, and for the same reason, any numbers whose only factors include values that exceed any $n^2_m$ are also excluded, such as $D = 10$ whose only factors are 1-10 and 2-5, since both 10 and 5 exceed $n^2_2 = 4$. We can use decompositions such as that, but if P. Horodecki’s proof is correct, then we are guaranteed to always be able to find a decomposition of member numbers from the set in Table III instead.
For example, if a decomposition for a rank-3 state is known to have 10 MI states, we should always be able to find a different MI decomposition with only $D \in \{3, 4, 6, 8, 9\}$ MI states. However, in a rank-4 state, starting with a known decomposition of 10 MI states may require that we increase to $D = 12$ or $D = 16$ states, depending on whether a different MI decomposition with fewer MI states than 10 exists for the input state. But if we allow things like $D = 10$ here, then our single-mode limits would need to exceed the single-mode Caratheodory bounds. Therefore, the single-mode Caratheodory bounds of $n_m^r$ may actually lead to larger decompositions (or they may instead always allow a smaller decomposition), but we do not have a proof for the convertibility of such decompositions at this time.

To see how $D$ relates to a state with zero statance, consider a two-qubit state such as

$$
\rho = p_1\rho_1^{(1)} \otimes \rho_2^{(1)} + p_6\rho_6^{(1)} \otimes \rho_1^{(2)}
+ p_2\rho_2^{(1)} \otimes \rho_2^{(2)} + p_7\rho_7^{(1)} \otimes \rho_2^{(2)}
+ p_3\rho_3^{(1)} \otimes \rho_3^{(2)} + p_8\rho_8^{(1)} \otimes \rho_3^{(2)}
+ p_4\rho_4^{(1)} \otimes \rho_4^{(2)} + p_9\rho_9^{(1)} \otimes \rho_4^{(2)}
+ p_5\rho_5^{(1)} \otimes \rho_5^{(2)} + p_{10}\rho_{10}^{(1)} \otimes \rho_5^{(2)}.
\tag{P2}
$$

Since statance ignores the decomposition probabilities, then regardless of what they are, we can merely look at the decomposition states themselves, and here we notice that we can generate the same set from

$$
\{\rho_1^{(1)}, \rho_2^{(1)}\} \otimes \{\rho_1^{(2)}, \rho_2^{(2)}, \rho_3^{(2)}, \rho_4^{(2)}, \rho_5^{(2)}\} \equiv \{\rho_{(j_1,j_2)}\},\tag{P3}
$$

where the tensor product distributes over commas (if you prefer, the commas can be replaced by plus signs, and then the terms of the resulting sum comprise $\{\rho_{(j_1,j_2)}\}$.

We can simplify (P3) further using the notation

$$
\{1, 2\} \otimes \{1, 2, 3, 4, 5\} \equiv \{j_1, j_2\}
= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5)\}.
\tag{P4}
$$

The decomposition vector $D \equiv (D_1, \ldots, D_N)$ is a list of the maximal index labels in each mode that are needed to decompose a zero-statance state whose optimal decomposition has $D = D_1 \cdots D_N$ MI decomposition states. Thus, in this example, the decomposition in (P2) has

$$
D \equiv (2, 5),
\tag{P5}
$$

[which are also the sizes of the mode sets in (P4)]. According to P. Horodecki’s decomposition limit, which applies to states such as this, $D_2 = 5$ is larger than is needed for the mode-2 limit, so we should be able to find a different decomposition with mode-2 limit $D_2 \leq 4$. If $r = 3$, then Table III shows that we should be guaranteed to find such an MI decomposition of fewer members where $D \in \{3, 4, 6, 8, 9\}$, but if $r = 4$, then we may or may not be able to find a different MI decomposition with fewer members, as mentioned earlier, though we are guaranteed to find one that has at most $D = r^2 = 16$ MI members.

Regarding the discrepancy between Wootters’s decompositions and P. Horodecki’s limits, we may hypothesize that Wootters’s result is true for all systems (not just two qubits); that all full $N$-partite separable states have a decomposition of exactly $r$ separable pure decomposition states in the form of Family 3, but that for states with zero statance, they require at least between $r$ and $r^2$ pure decomposition states to achieve a set of MI decomposition states (and furthermore, states belonging to Families 5 and 6 all have both types of decompositions; an $r$-member separable decomposition, and an $(r, \ldots, r^2)$-member zero-statance decomposition).

Note that in cases where the spectral decomposition is also a statance-minimizing decomposition, it is not necessarily $U = I^r$ that achieves this (where square bracketed superscripts indicate matrix dimension), but rather it is the $r$-level permutation unitary that minimizes statance. This is important when constructing examples using orthogonal decomposition states, since the descending-order convention (DOC) of eigenvalues used to define their labels does not always produce the statance-minimizing decomposition when $U = I^r$; therefore even when constructing such states, we still need to search the order of their labels to truly minimize the statance for that decomposition. Furthermore, since the eigenvalues in the probability formula $p_j = \sum_{k=1}^{r} \lambda_k |U_{j,k}|^2$ in the DOC prevent any columns of $U$ above column $r$ from affecting the probability, the factors $|U_{j,k}|^2$ must not all be zero for a given $j$ so that we consider only sets where $p_j > 0 \ \forall j$. A necessary and sufficient way to ensure this is to specify that $\sum_{k=1}^{r} |U_{j,k}|^2 \neq 0 \ \forall j$. Then, since there is always at least one nonzero eigenvalue, all probabilities are nonzero.

**Appendix Q: Proof that Statance is a Necessary and Sufficient Measure of Decomposition-State Correlation**

Here we prove that achieving zero statance $\hat{S}(\rho) = 0$ is necessary and sufficient for any mixed or pure state $\rho$ to have no decomposition-state correlation as defined in Sec. 1A, and that having $\hat{S}(\rho) > 0$ is necessary and sufficient for $\rho$ to have some decomposition-state correlation, proving that statance is a necessary and sufficient measure of decomposition-state correlation.

First, the unoptimized statance $S(\rho)$ of (23) has form

$$
S(\rho) \equiv ||w||_1 = \sum_{j=1}^{D} |w_j|,\tag{Q1}
$$

which is just a 1-norm of the vector $w$ with components

$$
w_j \equiv \text{tr}[\rho_j - \mu_j^2],\tag{Q2}
$$

where the absolute value $|w_j|$ in (Q1) is because the operator $\Delta_j \equiv \rho_j - \mu_j$ is Hermitian and therefore has only real eigenvalues, so the trace of its square as $w_j \equiv \text{tr}[\Delta_j^2]$ in (Q2) is just the sum of the squares of its real eigenvalues, which is always nonnegative. Thus, it is always true that
$w_j = |w_j|$. Therefore, since the necessary and sufficient condition for any 1-norm of a vector $w$ to be zero is that $w = 0$ which means that all of its components are zero $w_j = 0 \forall j$, then the necessary and sufficient condition for $\mathcal{S}(\rho)$ to be zero is that $w_j = 0 \forall j$, or equivalently, that $\rho_j = \mu_j \forall j$ (since that is the unique condition that causes $\rho_j - \mu_j = 0$ which means that $w_j = \text{tr}[\rho_j - \mu_j] = 0 \forall j$).

Then, due to the definition in (22) of statance $\mathcal{S}(\rho)$ as being the minimum value of $\mathcal{S}(\rho)$ over all decompositions of $\rho$, then the necessary and sufficient condition for $\mathcal{S}(\rho) = 0$ can be stated as

$$\mathcal{S}(\rho) = 0 \iff \exists \{p_j, \rho_j\} \text{ s.t. } \rho_j = \mu_j \forall j,$$  

(Q8)

where $\{p_j, \rho_j\}$ is some particular decomposition set of $\rho$.

To show that statance $\mathcal{S}(\rho)$ is a necessary and sufficient measure of decomposition-state correlation, we will first show that a sufficient condition for $\mathcal{S}(\rho) = 0$ is that $\rho$ has a decomposition with mode-independent decomposition states $\{\text{MI}\{\rho_j\}\}$, which means that $\mathcal{S}(\rho) = 0$ is necessary for $\text{MI}\{\rho_j\}$. Then, we will show that $\mathcal{S}(\rho) = 0$ is also sufficient for $\text{MI}\{\rho_j\}$, which means that $\text{MI}\{\rho_j\}$ is necessary for $\mathcal{S}(\rho) = 0$.

To prove that having $\text{MI}\{\rho_j\}$ is sufficient for $\mathcal{S}(\rho) = 0$, we need to show that a $\rho$ with $\text{MI}\{\rho_j\}$ causes $\rho_j = \mu_j \forall j$. Therefore, suppose that $\rho$ has an optimal decomposition with $\text{MI}\{\rho_j\}$. Then, by definition of $\text{MI}$, combining (5) and (6) (using a different dummy index to keep things distinct), the decomposition states take the special form

$$\rho_j \equiv \rho_{(j_1, \ldots, j_N)} = \bigotimes_{q=1}^N \rho_{j_q} \forall j,$$  

(Q4)

For $\mu_j \equiv \mu_{(j_1, \ldots, j_N)}$ in (24), we need the quantity

$$\rho_{(j_1^{(m)}, \ldots, j_m^{(m)}, j_{m+1}^{(m)}, \ldots, j_N^{(m)})} = \bigotimes_{q=1}^n \rho_{j_q^{(m)}},$$  

(Q5)

so then, the quantity

$$R_{jm}^{(m)} = \sum_{j_1^{(m)}, \ldots, j_m^{(m)}, j_{m+1}^{(m)}, \ldots, j_N^{(m)}} \delta_{jm} \rho_{(j_1^{(m)}, \ldots, j_m^{(m)})}$$  

(Q6)

becomes

$$R_{jm}^{(m)} = \sum_{j_1^{(m)}, \ldots, j_m^{(m)}, j_{m+1}^{(m)}, \ldots, j_N^{(m)}} \delta_{jm} \rho_{(j_1^{(m)}, \ldots, j_m^{(m)}, j_{m+1}^{(m)}, \ldots, j_N^{(m)})}$$  

and since (24) has the form

$$\mu_j \equiv \sum_{m=1}^N \text{tr} \left( \frac{D_1 \cdots D_{m-1} D_{m+1} \cdots D_N}{D_1 \cdots D_{m-1}} R_{jm}^{(m)} \right),$$  

(Q8)

then putting (Q7) into (Q8) gives

$$\mu_j = \sum_{m=1}^N \left[ \text{tr} \left( \frac{1}{D_1} \sum_{j_1^{(m)}=1}^{D_1} \rho_{j_1^{(m)}} \right) \cdots \text{tr} \left( \frac{1}{D_{m-1}} \sum_{j_{m-1}^{(m)}=1}^{D_{m-1}} \rho_{j_{m-1}^{(m)}} \right) \right] \cdot \text{tr} \left( \frac{1}{D_1 \cdots D_{m-1}} R_{jm}^{(m)} \right)$$

and since (Q4) holds for each $j$, then (Q9) does as well, so then putting (Q9) into (Q4) we see that

$$\rho_j = \mu_j \forall j,$$  

(Q10)

which, by (Q3) proves that having $\text{MI}\{\rho_j\}$ is sufficient to cause $\mathcal{S}(\rho) = 0$.

To show that having $\text{MI}\{\rho_j\}$ is also necessary to cause $\mathcal{S}(\rho) = 0$, we will simply show that $\mathcal{S}(\rho) = 0$ is sufficient for $\rho$ to have $\text{MI}\{\rho_j\}$. Therefore, suppose that $\mathcal{S}(\rho) = 0$, then by (Q3), $\rho_j = \mu_j \forall j$, which by (24) is

$$\rho_{(j_1, \ldots, j_N)} = \mu_{(j_1, \ldots, j_N)} = \sum_{m=1}^N \beta_{jm}^{(m)},$$  

(Q11)

where each $\beta_{jm}^{(m)}$ is a physical state in mode $m$ given by

$$\beta_{jm}^{(m)} = \text{tr} \left( \frac{1}{D_m} \sum_{j_1^{(m)}, \ldots, j_{m-1}^{(m)}, j_{m+1}^{(m)}, \ldots, j_N^{(m)}} \rho_{j_1^{(m)}, \ldots, j_{m-1}^{(m)}, j_{m+1}^{(m)}, \ldots, j_N^{(m)}} \right),$$  

(Q12)

where note that in (Q12) we did not expand the decomposition states further because we are making no assumption about their form in this part of the proof.

Thus, (Q11) shows that the condition of $\mathcal{S}(\rho) = 0$ automatically leads to a situation where the $\rho_j = \beta_{jm}^{(m)}$ have a (tensor) product form of $N$ factors guaranteed for all $j$, and each factor depends on a different mode-specific index $j_m$, which is the definition of the form of mode independence for decomposition states. Furthermore, since each $\beta_{jm}^{(m)}$ is a convex sum of physical states (which are just the pure decomposition states with special mode-specific index arguments) with equal probabilities such that $\text{tr}(\beta_{jm}^{(m)}) = 1$, then each $\beta_{jm}^{(m)}$ qualifies as a physical state, and we can simply rename them as states

$$\beta_{jm}^{(m)} \equiv \rho_{jm}^{(m)},$$  

(Q13)

which, put into (Q11), is the definition of $\text{MI}\{\rho_j\}$, and so we have proven that $\mathcal{S}(\rho) = 0$ is a sufficient condition for $\text{MI}\{\rho_j\}$, and thus $\text{MI}\{\rho_j\}$ is necessary for $\mathcal{S}(\rho) = 0$.

We have now shown that the condition of a state having a decomposition with mode-independent decomposition states $\{\text{MI}\{\rho_j\}\}$ is both necessary and sufficient to
cause $\hat{S}(\rho) = 0$. Then, since a violation of $\text{MI}\{\rho_j\}$ implies $\hat{S}(\rho) \neq 0$, and by its definition this means that $\hat{S}(\rho) > 0$, we have proven that statance $\hat{S}(\rho)$ is a valid measure of decomposition-state correlation, since the violation of $\text{MI}\{\rho_j\}$ is the definition of decomposition-state correlation, and the achievement of $\text{MI}\{\rho_j\}$ is the necessary and sufficient condition for $\hat{S}(\rho) = 0$.

Appendix R: Statance Example

Suppose we have a two-qubit state of the type in Family 5 of Table I, with the statance-minimizing decomposition $\rho = \sum_j p_j \rho_{j,1}^{(1)} \otimes \rho_{j,2}^{(2)}$, where the probabilities $p_j \equiv p(j_{1,2})$ in this example happen to not have mode independence, while by definition the decomposition states do have mode independence. Therefore, in this particular example, the decomposition states are

$$\begin{align*}
\rho_1 &= \rho_{1,1}^{(1)} = \rho_{1,1}^{(1)} \otimes \rho_{2,1}^{(2)} \\
\rho_2 &= \rho_{1,2}^{(1)} = \rho_{1,2}^{(1)} \otimes \rho_{2,2}^{(2)} \\
\rho_3 &= \rho_{2,1}^{(1)} = \rho_{2,1}^{(1)} \otimes \rho_{2,2}^{(2)} \\
\rho_4 &= \rho_{2,2}^{(1)} = \rho_{2,1}^{(1)} \otimes \rho_{2,2}^{(2)}.
\end{align*}$$

(R1)

To find the statance of $\rho$, first write the general form of $\mu_j \equiv \mu(j_{1,2})$ from (24) with $D = (2,2)$ as

$$\mu_j = \text{tr}(\rho_j \rho_{1,1}^{(1)} + \rho_{j,1,2}^{(2)})) \equiv \text{tr}(\rho_{1,1}^{(1)} + \rho_{j,2,1}^{(2)} \rho_{j,2,2}^{(2)})).$$

(R2)

Writing these out for each index and then plugging-in the particular states in (R1) gives

$$\begin{align*}
\mu_1 &= \text{tr}(\rho_1^{(1)} \otimes \rho_2^{(2)}) = \rho_{1,1}^{(1)} \otimes \rho_{2,2}^{(2)} \\
\mu_2 &= \text{tr}(\rho_1^{(1)} \otimes \rho_2^{(2)}) = \rho_{1,1}^{(1)} \otimes \rho_{2,2}^{(2)} \\
\mu_3 &= \text{tr}(\rho_2^{(1)} \otimes \rho_1^{(2)}) = \rho_{2,1}^{(1)} \otimes \rho_{1,2}^{(2)} \\
\mu_4 &= \text{tr}(\rho_2^{(1)} \otimes \rho_1^{(2)}) = \rho_{2,1}^{(1)} \otimes \rho_{1,2}^{(2)}.
\end{align*}$$

(R3)

Then, putting (R1) and (R3) into (23), we get

$$\begin{align*}
S(\rho) &= \sum_{j_1,j_2=1}^{(1)} \sum_{j_2=1}^{(2)} \text{tr}[(\rho_{j_1,j_2}^{(1,2)} - \mu_{j_1,j_2}^{(1,2)})^2] \\
&= \sum_{j_1,j_2=1}^{(1)} \sum_{j_2=1}^{(2)} \text{tr}[(\rho_{j_1}^{(1)} \otimes \rho_{j_2}^{(2)} - \rho_{j_1}^{(1)} \otimes \rho_{j_2}^{(2)})^2] = 0,
\end{align*}$$

(R4)

since for this example, each $\mu_j$ is exactly equal to $\rho_j$ regardless of what the probabilities were. Then, since (R4) was calculated with an optimal decomposition [because it minimizes the unoptimized statance $\hat{S}(\rho)$], $S(\rho)$ yields the statance from (22) as

$$\hat{S}(\rho) \equiv \frac{1}{N_S} \min_{\{U\}}[S(\rho)] = 0.$$ 

(R5)

Thus, this example shows that any states belonging to Families 5 or 6 from Table I have zero statance regardless of their decomposition probabilities, because those states all have mode-independent sets of decomposition states for their statance-minimizing decompositions.

Appendix S: Proof that Probablance is a Necessary and Sufficient Measure of Probability Correlation

Here we prove that achieving zero probablance $\mathcal{P}(\rho) = 0$ is necessary and sufficient for any mixed or pure state $\rho$ to have no probability correlation as defined in Sec. IA, and that having $\mathcal{P}(\rho) > 0$ is necessary and sufficient for the state to have some probability correlation, proving that probablance is a necessary and sufficient measure of probability correlation.

First, note that the unoptimized probablance $\mathcal{P}(\rho)$ from (27) can be written as

$$\mathcal{P}(\rho) \equiv |v|^2 = \sum_j |v_j|^2,$$

(S1)

which is just a nonzero scalar times a 2-norm of the vector

$$v \equiv p' - q',$$

(S2)

where $p'$ has components $p'_j = p'(j_{1,2})$ which is the set of decomposition probabilities of an arbitrary statance-minimizing (SM) decomposition of $\rho$ as given by (29), and $q'$ has components $q'_j = q'(j_{1,2})$ defined in (28), so $v$ has components $v_j \equiv p'_j - q'_j$. Therefore, since the necessary and sufficient condition for any 2-norm of a vector $v$ to be zero is that $v = 0$, which means that all of its components are zero $v_j \equiv 0 \forall j$, then the necessary and sufficient condition for $\mathcal{P}(\rho)$ to be zero is that $v_j \equiv 0 \forall j$, or equivalently, that $p'_j = q'_j \forall j$. Then, due to the definition in (26) of $\mathcal{P}(\rho)$ as being the minimum value of $\mathcal{P}(\rho)$ over all SM decompositions of $\rho$, the necessary and sufficient condition for $\mathcal{P}(\rho) = 0$ can be stated as

$$\mathcal{P}(\rho) = 0 \iff \exists \{p'_j, q'_j\} \text{ s.t. } p'_j = q'_j \forall j \text{ and } \mathcal{S}(\rho) = \mathcal{S}_\text{min}.$$

(S3)

where $\{p'_j, q'_j\}$ is some particular SM decomposition of $\rho$, $\mathcal{S}(\rho)$ is the statance from Sec. IV A, and $\mathcal{S}_\text{min}$ is the minimal $\mathcal{S}(\rho)$ over all decompositions of $\rho$. (The requirement that $\{p'_j, q'_j\}$ causes $\mathcal{S}_\text{min}$ is actually already part of the definition of $\{p'_j, q'_j\}$, so it is redundant in (S3), but stated anyway to emphasize its importance).

To show that $\mathcal{P}(\rho)$ is a necessary and sufficient measure of probability correlation, we will first show that a sufficient condition for $\mathcal{P}(\rho) = 0$ is that $\rho$ has an SM decomposition with mode-independent probabilities (MI$\{p_j\}_{SM} \equiv$ MI$\{p'_j\}$), meaning that $\mathcal{P}(\rho) = 0$ is necessary for MI$\{p_j\}_{SM}$. Then, we will show that $\mathcal{P}(\rho) = 0$ is also sufficient for MI$\{p_j\}_{SM}$, meaning that MI$\{p_j\}_{SM}$ is also necessary for $\mathcal{P}(\rho) = 0$.

To prove that having MI$\{p_j\}_{SM}$ is sufficient for $\mathcal{P}(\rho) = 0$, we need to show that a state $\rho$ with MI$\{p_j\}_{SM}$
causes $p_j' = q_j' \forall j$. Therefore, suppose that $\rho$ has an SM decomposition with $\text{MI}(p_j')_{SM}$. Then, by definition, combining (7) and (8) (using a different dummy index to keep things distinct),
\begin{equation}
  p_j' \equiv p_j'(j_1, \ldots, j_N) = \prod_{q=1}^{N} p_j^{(q)} \quad \forall j,
\end{equation}
where $\sum_j p_j^{(q)} = 1; \forall q \in 1, \ldots, N$. For $p_j' \equiv q_j'(j_1, \ldots, j_N)$ in (28), we need the quantity
\begin{equation}
  p_j'(j_1, \ldots, j_N) = \prod_{q=1}^{N} p_j^{(q)} \quad \forall j,
\end{equation}
so then, the quantity
\begin{equation}
  Q_j^{(m)} \equiv \sum_{j_1^{(m)}, \ldots, j_N^{(m)}} \delta_{j_1^{(m)}j_1, \ldots, j_N^{(m)}j_N} p_j'(j_1^{(m)}, \ldots, j_N^{(m)})
\end{equation}
takes on the value
\begin{equation}
  Q_j^{(m)} = \frac{D_1, \ldots, D_N}{d_1, \ldots, d_N, \ldots, D_{m-1}, d_{m-1}, D_m, \ldots, D_N} = \sum_{j_1^{(m)}, \ldots, j_N^{(m)}} \delta_{j_1^{(m)}j_1, \ldots, j_N^{(m)}j_N} p_j'(j_1^{(m)}, \ldots, j_N^{(m)}) = \left[ \prod_{j_1^{(m)}=1}^{D_1} p_j^{(1)} \right] \cdots \left[ \prod_{j_N^{(m)}=1}^{D_N} p_j^{(N)} \right] = p_j^{(m)},
\end{equation}
which, when put into (28), gives
\begin{equation}
  q_j' = q_j'(j_1, \ldots, j_N) = \prod_{m=1}^{N} p_j^{(m)},
\end{equation}
Since (S4) holds for each $j$, then (S8) does as well, so then putting (S8) into (S4), we see that
\begin{equation}
  p_j' = q_j' \forall j,
\end{equation}
which, by (S3) proves that having $\text{MI}(p_j')_{SM}$ is sufficient to cause $\bar{P}(\rho) = 0$.

To show that having $\text{MI}(p_j')_{SM}$ is also necessary to cause $\bar{P}(\rho) = 0$, we will simply show that $\bar{P}(\rho) = 0$ is sufficient for $\rho$ to have $\text{MI}(p_j')_{SM}$. Therefore, supposing that $\bar{P}(\rho) = 0$, then by (S1) and (S2), $p_j' = q_j' \forall j$, which expands using (28) as
\begin{equation}
  p_j'(j_1, \ldots, j_N) = \prod_{m=1}^{N} b_j^{(m)}
\end{equation}
where we define the quantity
\begin{equation}
  b_j^{(m)} = \sum_{j_1^{(m)}, \ldots, j_N^{(m)}} \prod_{j=m+1}^{N} p_j^{(j_1^{(m)}, \ldots, j_{m-1}^{(m)}j_{m+1}, \ldots, j_N^{(m)})},
\end{equation}
where we cannot simplify the probabilities further because we are making no assumptions about their form here. Also, since the sum of all the $b_j^{(m)} = p_j'(j_1, \ldots, j_N)$ is 1, and the sum in (S11) is missing one of the mode sums that would complete that sum to 1, and since the terms in (S11) are all probabilities existing on $[0, 1]$, then these quantities $b_j^{(m)}$ generally satisfy
\begin{equation}
  0 \leq b_j^{(m)} \leq 1.
\end{equation}

Thus, (S10) shows that the condition of $\bar{P}(\rho) = 0$ automatically leads to a situation where each $p_j' = p_j'(j_1, \ldots, j_N)$ has a product form of $N$ factors guaranteed for all $j$, and each factor depends on a different mode-specific index $j_m$, which is the definition of the form of mode independence. Furthermore, since $b_j^{(m)} \in [0, 1]$ by (S12), then by the normalization property of the $p_j' = p_j'(j_1, \ldots, j_N)$, the mode-independence leads to all the sums producing positive factors, seen by summing (S10) as
\begin{equation}
  \sum_{j_1, \ldots, j_N=1, \ldots, 1}^{D_1 \ldots D_N} p_j(j_1, \ldots, j_N) = \left[ \prod_{j_1=1}^{D_1} b_j^{(1)} \right] \cdots \left[ \prod_{j_N=1}^{D_N} b_j^{(N)} \right] = B^{(1)} \cdots B^{(N)},
\end{equation}
where $B^{(m)} \equiv \sum_j b_j^{(m)}$ such that $B^{(m)} \in [0, 1]$. Then, the only way that the right-side product, with each factor on $[0, 1]$, could equal the left is if all of the factors are 1, meaning $B^{(m)} = 1 \forall m$, which means that each $b_j^{(m)}$ then satisfies all of the conditions for being a probability since $b_j^{(m)} \in [0, 1]$ and $\sum_j b_j^{(m)} = 1$. [Also, $\sum_j b_j^{(m)} = 1$ due to the definition in (S11) because it supplies the missing mode sum for the full probabilities to sum to 1.] Therefore, we can rename them as mode-specific probabilities
\begin{equation}
  b_j^{(m)} = p_j^{(m)},
\end{equation}
obeying $\sum_j p_j^{(m)} = 1$ so that (S10) becomes
\begin{equation}
  p_j'(j_1, \ldots, j_N) = \prod_{m=1}^{N} p_j^{(m)} : \sum_j p_j^{(m)} = 1 \forall n \forall j,
\end{equation}
which is the definition of $\text{MI}(p_j')_{SM}$, and therefore we have proven that $\bar{P}(\rho) = 0$ is a sufficient condition for $\text{MI}(p_j')_{SM}$, and thus $\text{MI}(p_j')_{SM}$ is necessary for $\bar{P}(\rho) = 0$.

We have now shown that the condition of a state having an SM decomposition with mode-independent decomposition probabilities (MI $p_j'$) is both necessary and sufficient to cause $\bar{P}(\rho) = 0$. Then, since a violation of $\text{MI}(p_j')_{SM}$ implies $\bar{P}(\rho) \neq 0$, and by its definition this means that $\bar{P}(\rho) > 0$, we have proven that proballance $\bar{P}(\rho)$ is a valid measure of probability correlation, since violation of $\text{MI}(p_j')_{SM}$ is the definition of probability correlation, and achievement of $\text{MI}(p_j')_{SM}$ is the necessary and sufficient condition for $\bar{P}(\rho) = 0$. See Sec. IVB for the explanation of why we must use a statement-minimizing decomposition to compute proballance.
Appendix T: Probablance Example

To see how probablance works for a state with mode-independent (MI) probabilities of a statance-minimizing (SM) decomposition, consider the special case of some product-form two-qubit input mixed state \( \rho \), which, due to its product form, has an optimal decomposition [one that minimizes the unoptimized probablance of (27)] with probabilities of the form

\[
\begin{align*}
 p'_{(1,1)} &= p_1^{(1)} p_2^{(2)} \\
 p'_{(1,2)} &= p_1^{(1)} p_2^{(2)} \\
 p'_{(2,1)} &= p_2^{(1)} p_1^{(2)} \\
 p'_{(2,2)} &= p_2^{(1)} p_1^{(2)},
\end{align*}
\]  

(T1)

where \( p_1^{(m)} + p_2^{(m)} = 1 \), and these \( p'_{(j_1,\ldots,j_N)} \) are found from solving the optimization problem over all the unitaries \( U' \) of an SM decomposition that produce \( p'_{(j_1,\ldots,j_N)} \) from (29) to minimize \( \mathcal{P}(\rho) \) in (27) to give the probablance \( \mathcal{P}(\rho) \) in (26). (In practice, we would just get a list of numbers that have this structure, but we would not know it at this point in the calculation; it would only be clear when we finish computing the probablance.)

In App. E we proved that any mixed product-form state has an optimal decomposition of \( r = \text{rank}(\rho) \) decomposition states, where the rank of each single-mode reduction is \( r_m \) and \( r_1 \cdots r_N = r \). Using this fact, then \( D = (2, 2) \), and the overall form of \( q'_j \) from (28) is

\[
q'_{(j_1, j_2)} = (p'_{(j_1,1)} + p'_{(j_1,2)})(p'_{(j_2,1)} + p'_{(j_2,2)}).
\]  

(T2)

Then, using the particular values of (T1) in (T2) gives

\[
q'_{(1,1)} = (p_1^{(1)} p_1^{(2)} + p_1^{(1)} p_2^{(2)})(p_2^{(1)} p_1^{(2)} + p_2^{(1)} p_2^{(2)}) = p_1^{(1)} p_2^{(2)},
\]  

(T3)

and similarly,

\[
q'_{(1,2)} = (p_1^{(1)} p_1^{(2)} + p_2^{(1)} p_2^{(2)})(p_1^{(1)} p_2^{(2)} + p_2^{(1)} p_2^{(2)}) = p_1^{(1)} p_2^{(2)},
\]  

(T4)

which are exactly the same as the optimal decomposition probabilities themselves [which only happens when the optimal decomposition probabilities have mode independence as in (T1)]. Thus, putting (T3–T4) into (27),

\[
\mathcal{P}(\rho) = \sum_{j_1, j_2=1,1} |p'_{(j_1, j_2)} - q'_{(j_1, j_2)}|^2 = 0,
\]  

(T5)

which also yields the fully optimized probablance as \( \mathcal{P}(\rho) = 0 \) here, since we specified that the \( U' \) we used to get the \( p'_{(j_1, j_2)} \) produced an SM decomposition, and since \( 0 \) is the lowest value that \( \mathcal{P}(\rho) \) can have, then this decomposition minimizes the unoptimized probablance.

Thus, this example shows the mechanism by which states with SM decompositions of mode-independent product-form probabilities produce a probablance of 0.

Appendix U: Register and Inverse Register Functions

Register counting is any ordered set of sets of characters for which the rightmost character increases in its ordered set until it reaches the end, and then the character to its immediate left can increment by one member in its ordered set, but the character to its right simultaneously resets and must go through another cycle before the character to its left can increment again.

For example, in a two-character word where each character has an alphabet of two letters (a system with structure 2 \( \times \) 2 such as two qubits), register counting goes as \( \{1,1,2,2\} \). We can relabel this vector-index list with the scalar-index labels \( \{1, 2, 3, 4\} \).

For an \( N \)-mode discrete quantum system with structure \( n = (n_1, \ldots, n_N) \), where mode \( m \) has \( n_m \) outcomes in increasing order from 1 to \( n_m \), the map from a particular vector-index outcome labeled as \( a = (a_1, \ldots, a_N) \), where \( a_m \in 1, \ldots, n_m \) \( \forall m \in 1, \ldots, N \) to the scalar-index outcome in the set 1, \ldots, \( n \), where \( n \equiv n_1 \cdots n_N \), is given by the indicial register function [18, 47],

\[
R_{a}^{(N,n)} \equiv 1 + \sum_{m=1}^{N} (a_m - 1) \prod_{j=m+1}^{N} n_j
\]

(U1)

which maps the vector index \( a \) to scalar index \( n \) as \( n = R_{a}^{(N,n)} \).

Thus (U2) maps scalar index \( a \) to vector index \( a \) as \( a = a_{a}^{(N,n)} \). For example, in a two-qubit system, for which \( n = (2, 2) \), (U1) maps the vector index (2, 1) to the scalar index 3.

In other situations, we know a scalar-index value \( a \) and want to know to which vector-index \( a \) it corresponds. For that, we use the inverse indicial register function [18],

\[
a_{a}^{(N,n)} = (a_1, \ldots, a_N); \text{ where } a_m = \text{floor}(\frac{a_m - 1}{d_m}) + 1, \quad v_m = v_{m-1} - (a_m - 1)m, \quad (U2)
\]

for \( m \in 1, \ldots, N \), where \( v_0 \equiv a \), and \( d_m \equiv \Pi_{n=1}^{N} n_N \).

Thus (U2) maps scalar index \( a \) to vector index \( a \) as \( a = a_{a}^{(N,n)} \). For example, in a two-qubit system, (U2) maps scalar index 2 to vector index (1, 2).

Note that throughout this paper, although \( n \) is the mode-size vector to use for register counting of the basis functions with (U1) and (U2), in the case of register counting the decomposition indices \( j \equiv (j_1, \ldots, j_N) \), we need to use the decomposition-limit vector \( D \equiv (D_1, \ldots, D_N) \) [see App. F] instead of \( n \) in (U1) and (U2) to convert between scalar decomposition indices \( j \) and vector decomposition indices \( (j_1, \ldots, j_N) \) [where again all indices start on 1]. Thus we have \( j = j_{j}^{(N,D)} \) and \( j = j_{j}^{(N,D)} \), where \( n_m \to D_m, a_m \to j_m, v_0 \equiv j, \) and we use \( j \) to specifically mean the vector index as a collection of many indices \( j \equiv (j_1, \ldots, j_N) \) rather than as a single equivalent number \( j \).
