Fairly Constricted Particle Swarm Optimization

1st Anwesh Bhattacharya  
Department of Physics, Department of CS&IS  
BITS-Pilani, Pilani  
Rajasthan, India  
f2016590@pilani.bits-pilani.ac.in

2nd Snehanshu Saha  
Department of CS&IS, APPCAIR  
BITS-Pilani, Goa  
Goa, India  
snehanshus@goa.bits-pilani.ac.in

Abstract—We have adapted the use of exponentially averaged momentum in PSO to multi-objective optimization problems. The algorithm was built on top of SMPSO, a state-of-the-art MOO solver, and we present a novel mathematical analysis of constriction fairness. We extend this analysis to the use of momentum and propose rich alternatives of parameter sets which are theoretically sound. We call our proposed algorithm “Fairly Constricted PSO with Exponentially-Averaged Momentum” — FCPSO-em.

I. PARTICLE SWARM OPTIMIZATION

A. Vanilla PSO

Particle Swarm Optimization (PSO) was first proposed by Kennedy and Eberhart [1], [2] in 1995 as an attempt to model the behaviour of bird flocks. N particles are initialised at random positions/velocities in the search space, i\(^\text{th}\) particle updates its trajectory according to —

\[
v_i^{(t+1)} = w v_i^{(t)} + c_1 r_1 (pbest_i^{(t)} - x_i^{(t)}) + c_2 r_2 (gbest^{(t)} - x_i^{(t)})
\]

(1)

\[
x_i^{(t+1)} = x_i^{(t)} + v_i^{(t+1)}
\]

(2)

\(r_1\) and \(r_2\) are D-dimensional random vectors with each component in \(U(0,1)\). \(pbest_i^{(t)}\) is the best position that particle \(i\) has visited up to time \(t\) i.e. achieved the lowest value in the objective function \(f(x)\). \(gbest^{(t)}\) is the best position among all particles that has been achieved.

B. Exponentially-Averaged Momentum

One technique to avoid premature convergence in PSO is to introduce an exponentially-averaged momentum (EM) term that tracks the histories of a particle’s velocities. Hence, if a particle is near a local optima, the influence of momentum will push it out and it will subsequently contribute to the global search. With the momentum term as \(M_i^{(t)}\) —

\[
M_i^{(t+1)} = \beta M_i^{(t)} + (1 - \beta) v_i^{(t)}
\]

(3)

\[
v_i^{(t+1)} = M_i^{(t+1)} + c_1 r_1 (pbest_i^{(t)} - x_i^{(t)}) + c_2 r_2 (gbest^{(t)} - x_i^{(t)})
\]

(4)

The position update equation remains the same as eq (2). The factor \(\beta\) controls the preference of momentum over velocity. \(\beta = 0\) degenerates to vanilla PSO. By recursively expanding eq (3), a particle’s momentum is an exponentially weighted sum of all its previous velocities —

\[
M_i^{(t+1)} = (1 - \beta) v_i^{(t)} + \beta (1 - \beta) v_i^{(t-1)} + \ldots + \beta^{t-2} (1 - \beta) v_i^{(2)} + \beta^{t-1} (1 - \beta) v_i^{(1)}
\]

(5)

Such a PSO algorithm has been devised in [3] and they have reported significant reduction in the number of iterations for single-objective problems (upto 50%), and avoidance of premature convergence.

Definition 1 - A PSO system that uses EM is called an EMPSO algorithm.

C. Stability

Since PSO is a randomized search algorithm, it is important to analyse whether its convergence is reliable if the swarm is subject to small perturbations. In this light, PSO can be looked at as a finite difference scheme. There is a theoretical framework for analysing the stability of such schemes, which come from the field of numerical partial differential equations. In particular, we are interested in Von-Neumann stability [4]. Vanilla PSO is known to be stable [5] when its parameters obey the following condition —

\[
0 \leq c_1 + c_2 \leq 2(1 + w)
\]

(6)

Analogously, the stability criteria for EMPSO is [3] —

\[
0 < \beta < 1
\]

\[
0 \leq c_1 + c_2 \leq 2
\]

(7)

D. Multi-Objective Optimization

Multi-Objective Optimization (MOO) is a frontier research topic in evolutionary computation. NSGA-II [6] was the most popular MOO solver and the state of the art, based on the genetic algorithm.

An efficient adaptation of the PSO scheme for multi-objective problems is SMPSO [7]. Its advantage over other PSO-based solvers such as OMOPSO [8] is its velocity constriction of the swarm. It enjoys smooth pareto front convergence and has beaten other SOTA optimizers in terms of the widely used quality indicators [9].
II. Motivation

A. Derivation of Constriction Co-efficient for EMPSO

The modified velocity update equation with constriction —
\[ v_i^{(t+1)} = \chi (wv_i^{(t)} + c_1r_1(pbest_i^{(t)} - x_i^{(t)}) + c_2r_2(gbest^{(t)} - x_i^{(t)})) \] (8)

\( \chi \) is the constriction co-efficient and \( \chi = 1 \) retrieves the original PSO equation, and thus this is a trivial constriction factor. A non-trivial constriction factor, in this paper, is defined to be \( 0 < \chi < 1 \). Setting \( \phi = c_1 + c_2 \) and drawing \( c_1, c_2 \) from appropriate ranges, the landmark paper [10] has shown that vanilla PSO has a non-trivial constriction-coefficient—
\[ \chi^{(v)} = \begin{cases} \frac{2}{2 - \phi - \sqrt{\phi^2 - 4\phi}} & \phi > 4 \\ 1 & \phi \leq 4 \end{cases} \] (9)

The starting point of this paper is the derivation of a constriction co-efficient for EMPSO in section (III) that would be subsequently used to introduce EM in SMPSO.

B. Sign of the Constriction Co-efficient

\( \chi^{(v)} \) is strictly negative whenever \( \phi > 4 \). A negative constriction would imply that a particle reverses its velocity whenever it’s velocity has crossed a threshold. Negating its velocity would cause the particle to backtrack its shoot-off from its ideal direction due to its previously high velocities. One of the reasons that SMPSO performs well is it constricts the particles fairly — an exact mathematical notion that is subsequently developed in this paper.

A lack of fairness in constriction would be fatal to the collective behaviour of the swarm. Over-constriction would cause the search to be chaotic as particles would be switching velocities too frequently. Under-constriction would cause the converged pareto-front to lack smoothness, or not to converge at the optimal pareto front at all. The mathematical setup for fairness analysis is described in section (IV).

C. Probabilistic Analysis

For each iteration of the swarm in SMPSO, the parameters \( c_1, c_2 \) are drawn from the distribution \( U(1.5, 2.5) \) and the constriction is applied to the swarm based on eq (9). This introduces a layer of randomization on top of the PSO process that endows variability/richness in the swarm dynamics which is also fair.

Fairness analysis for Vanilla PSO — on which SMPSO is based — is simple as it only depends on \( \phi \). The influence of momentum adds complexity to the form of the constriction factor. In section (V), we have rigorously studied the fairness of certain PSO systems. In section (VI), we propose parameters for \( c_1, c_2, \beta \) with theoretical justifications and section (VII) contains experimental results on various problems.

III. Constriction Factor for EMPSO

**Theorem 1**: EMPSO has a non-trivial constriction co-efficient

**Proof**: Consider a deterministic version of unconstricted PSO equations with \([x, v]\) as a 2-D discrete-time map —

\[ v(t + 1) = v(t) + \phi(g - x(t)) \] (10)
\[ x(t + 1) = x(t) + v(t + 1) \] (11)

where \( y(t) = g - x(t) \) and \( p_{best} = g_{best} = g \), \( w = 1 \) for simplicity. If we let \( y(t) = g - x(t) \), and introduce a momentum series \( m(t) \), we get a 3-D discrete-time map in \([v, y, m]\) —

\[ v(t + 1) = (1 - \beta)v(t) + \phi y(t) + \beta m(t) \] (12)
\[ y(t + 1) = (\beta - 1)v(t) + (1 - \phi)y(t) - \beta m(t) \] (13)
\[ m(t + 1) = (1 - \beta)v(t) + \beta m(t) \] (14)

The evolution matrix of this system is —

\[ U = \begin{bmatrix} 1 - \beta & \phi & \beta \\ \beta - 1 & 1 - \phi & -\beta \\ 1 - \beta & 0 & \beta \end{bmatrix} \] (15)

On solving for the eigenvalues of \( U \) with \(|\lambda| > 0\), we obtain the following quadratic equation —

\[ \lambda^2 + (\phi - 2)\lambda + (1 - \beta\phi) = 0 \] (16)

where \( \Delta = \phi^2 - 4\phi \) is the discriminant. We also define the term \( k = 4(1 - \beta) \). The solutions to eq (16) are —

\[ \lambda = \frac{(2 - \phi) \pm \sqrt{\phi^2 - k\phi}}{2} \] (17)

According to [10], constriction entails finding the scaling factor \( \chi \) for the eigenvalues \( \lambda \) such that setting \( \lambda' = \chi \lambda \) gives \(|\lambda'| \leq 1 \). Hence it is sufficient to set —

\[ \chi = \min \left( \frac{1}{|\lambda_+|}, \frac{1}{|\lambda_-|} \right) = \frac{1}{\max(|\lambda|)} \] (18)

where \( \lambda_{\pm} \) is either of the roots defined in eq (17). Based on the discriminant \( \Delta \), we obtain two cases that give us real/complex roots for \( \lambda \).

**Case 1** \((\Delta \leq 0)\) —

\[ |\lambda| = \sqrt{\frac{(2 - \phi)^2 - \Delta}{4}} = \sqrt{1 - \beta^2} \] (19)

Since \( \sqrt{1 - \beta^2} \) is an absolute modulus, we must have —

\[ \phi \leq \frac{1}{\beta} \] (20)

Moreover, \( \Delta \leq 0 \implies \phi \leq k = 4(1 - \beta). \) Also, eq (20) must be satisfied simultaneously. To check whether this is true, construct a function in the range \( \beta \in (0, 1) \) —

\[ f(\beta) = \frac{1}{\beta} - 4(1 - \beta) \] (21)
and its derivative —

\[ f'(\beta) = 4 - \frac{1}{\beta^2} \]  

(22)

The critical point is \( \beta = \frac{1}{2} \) which is also the global minimum in \( \mathbb{R}^+ \) due to \( f''(\beta) = \frac{2}{\beta^3} > 0 \). Moreover, from \( f\left(\frac{1}{2}\right) = 0 \), we have \( f(\beta) \geq 0 \) in its domain and —

\[ \frac{1}{\beta} \geq 4(1 - \beta) \]

\[ \implies \phi \leq k \leq \frac{1}{\beta} \]

Eq (20) is thus satisfied and \( |\lambda| = \sqrt{1 - \beta \phi} \leq 1 \), hence we can set \( \chi = 1 \).

**Case 2 \( (\Delta > 0) \) —

Define \( \lambda_m = \max(|\lambda_\pm|) \). Depending on whether \( (2 - \phi) \) and \( \sqrt{\Delta} \) have opposing signs in eq (17), we obtain —

\[ \lambda_m = \begin{cases} 
|\lambda_+| & \phi < 2 \\
|\lambda_-| & \phi \geq 2
\end{cases} \]  

(23)

It is instructive to find an explicit formula for \( \lambda_m \) according to the above piece-wise definition —

\[ \lambda_m = \frac{|\phi - 2| + \sqrt{\phi^2 - k\phi}}{2} \]  

(24)

Note that \( \beta = 0 \implies k = 4 \), which signifies zero contribution of the momentum, and it is easily verified that this corresponds to \( \chi^{(m)} = \chi^{(c)} \). Thus, our derivation is consistent with that of vanilla PSO. The existence of a non-trivial constriction factor demands \( \lambda_m > 1 \) —

\[ \frac{|\phi - 2| + \sqrt{\phi^2 - k\phi}}{2} > 1 \]

\[ \implies (8 - k)\phi + 4|\phi - 2| > 8 \quad \text{(squaring)} \]

With \( \phi > 2 \) in the equation above, it simplifies to —

\[ (8 - k)\phi > 16 \]

\[ \implies \phi > \omega \]  

(25)

where \( \omega = 16(8 - k)^{-1} = 4(1 + \beta)^{-1} \). We have defined \( \omega \) explicitly as it plays an important role in the probabilistic analysis. Since \( 0 < \beta < 1 \), \( 2 < \omega < 4 \). Thus, there exists an interval of \( \beta, \phi \) where eq (25) is satisfied for a suitable distribution of \( \phi \). For the sake of example, \( \beta \in (0, 1) \) and \( \phi \in [2, 4] \). This completes the proof.

For the sake of implementation, we adopt a negative constriction co-efficient as has been used in SMPSO for our formulation —

\[ \chi^{(m)} = \begin{cases} 
-\lambda_m^{-1} & \Delta > 0 \text{ and } \lambda_m > 1 \\
1 & \text{otherwise}
\end{cases} \]  

(26)

From a theoretical standpoint, adopting a positive/negative constriction co-efficient are equivalent — only the modulus \( |\lambda| \) is significant. However, we have briefly explained the intuition for favoring a negative constriction in section (II-B).

### A. Stability and Constriction is Impossible

**Corollary 1**: EMPSO cannot simultaneously obey Von-Neumann stability and have a non-trivial constriction coefficient.

**Proof**: The stability condition for EMPSO is \( 0 \leq \phi \leq 2 \). Enforcing this on \( \phi > \omega \), we get —

\[ (4 - k)\phi - 4(\phi - 2) > 8 \]

\[ \implies k\phi < 0 \]

which is a contradiction as \( k > 0, \phi > 0 \).

### IV. DEFINING THE SWARM DYNAMICS

For a PSO system with evolution matrix\( U \) and its eigenvalues \( \lambda_U \), define the following events —

\[ E_g : |\lambda_U| > 1 \text{ (Non-trivial constriction)} \]

\[ E_1 : |\lambda_U| < 1 \text{ (Trivial constriction)} \]

We define two additional events —

\[ E_+ : \Delta_U > 0 \text{ (Positive discriminant)} \]

\[ E_- : \Delta_U < 0 \text{ (Negative discriminant)} \]

Note that \( E_+ \cap E_- = E_g \cap E_1 = \emptyset \) as they are mutually exclusive. It is also generally true that \( E_g \neq E_+ \& E_1 \neq E_- \).

**Definition 2** - A PSO system with its parameters \( \{p_1, p_2, \ldots\} \) being drawn from distributions with probability density functions \( \{p_{r1}(\omega_1), p_{r2}(\omega_2), \ldots\} \) is said to be fairly constrained if \( P(E_g) = \frac{1}{2} \) and is called an FCPSO algorithm.

**Example** - The parameter sets are \( \{e_1, e_2, w\} \) (or equivalently \( \{\phi, w\} \) with \( \phi = e_1 + e_2 \) and \( \{\phi, \beta\} \) for Vanilla PSO and EMPSO respectively. In the specific case of SMPSO (which is Vanilla PSO adapted to multi-objective optimization), \( \phi \sim U(3, 5) \) and \( p_w(w) = \delta(w - 0.1) \) where \( \delta \) is the Dirac-delta function.

#### A. Vanilla PSO

Although not derived here explicitly, the eigenvalue of evolution matrix of vanilla PSO is [10] —

\[ |\lambda| = \frac{|\phi - 2| + \sqrt{\phi^2 - 4\phi}}{2} \]  

(27)

For \( \phi > 4 \), it is easy to see that \( |\lambda| > 1 \). Otherwise, for \( \phi \leq 4 \), we have \( |\lambda| = 1 \) — it precisely corresponds to the case of \( \beta = 0 \) in eq (19). Thus the discriminant would take either signs with equal probability if \( \phi \) is sampled from a uniform distribution centred at \( \phi = 4 \), as in the case of SMPSO where \( \phi \sim U(3, 5) \). In other words, \( P(E_+) = P(E_-) = \frac{1}{2} \).

#### B. EMPSO

**Theorem 2**: If \( \phi \sim U(2 - \delta, 2 + \delta) \) and \( \beta \sim U\left(\frac{1}{2}, \frac{1}{2} + \epsilon\right) \) for some \( 0 < \delta < 2 \) \& \( 0 < \epsilon < \frac{1}{2} \), \( P(E_+) = P(E_-) = \frac{1}{2} \).

**Proof**: The proof is trivial, but it has been exposited as a natural segue to the analyses in the later sections. Due to \( k = 4(1 - \beta) \), it is easily seen that \( k \sim U(2 - 4\epsilon, 2 + 4\epsilon) \). A positive discriminant corresponds to \( \phi > k \)

\[ P(E_+) = \int_{\phi > k} \int_{\phi > k} p_\phi(\phi)p_k(k) \, d\phi \, dk \]

\[ = \int_{\phi > k} \int_{\phi > k} p_\phi(\phi)p_k(k) \, d\phi \, dk \]

\[ = \int_{\phi > k} \int_{\phi > k} p_\phi(\phi)p_k(k) \, d\phi \, dk \]
where \( \phi_k(k) = \max(k, 2 - \delta) \). Without loss of generality, assume \( 4\epsilon > \delta \). If not, the argument presented below can be applied with order of integration reversed. The integral can be broken up as (in the region \( \phi > k \))

\[
\int_{2-4\epsilon}^{2+\delta} \int_{2-\delta}^{2+\delta} d\phi \int_{k}^{2+\delta} d\phi_2 \int_{k}^{2+\delta} d\phi_1
\]

With \( p_\phi(\phi) = \frac{1}{2\phi} \) and \( p_\phi(k) = \frac{1}{8\phi} \), we conclude the proof —

\[
P(E_+) = \frac{1}{16\epsilon \delta} \left[ 2\delta(4\epsilon - \delta) + 2\delta^2 \right] = \frac{1}{2} = P(E_-)
\]

\( E_+ \cap E_- = \emptyset \)

**C. Fairness of SMPSO**

**Theorem 3** - SMPSO is fairly constricting.

**Proof** - From eq (26), \( E_g = E_+ \cap (\lambda_m > 1) \). For SMPSO, the events \( E_+ \) and \( (\phi > 4) \) are equivalent as shown in section (IV-A) and \( \phi > 4 \implies \lambda_m > 1 \). Thus \( E_g = E_+ \& E_l = E_- \) and \( P(E_g) = \frac{1}{2} \).

**D. A General Recipe For Assessing Constriction Fairness**

The specific analyses for Vanilla PSO and EMPSO serve as a precursor to the more general problem — Given an EMPSO system with parameters \( \phi \sim U(\phi_1, \phi_2) \) and \( \beta \sim U(\beta_1, \beta_2) \), quantify the unfairness parameter \( \mu = P(E_g) - \frac{1}{2} \). This parameter lies in the range \(-\frac{1}{2} \leq \mu \leq \frac{1}{2} \). A value of \( \mu = 0 \) means that a PSO system is fairly constricted. On the other hand, \( \mu > 0 \) indicates that a PSO system is over-constricted and \( \mu < 0 \) under-constricted. Define —

\[
\omega(\beta) = \frac{4}{1 + \beta}
\]

From eq (25), \( P(E_g) = P(\phi > \omega) \) and we have to evaluate the integral —

\[
\int \int_{\phi > \omega(\beta)} p_\phi(\phi)p_\beta(\beta) \, d\phi \, d\beta
\]

It is useful to define the inverse function \( \omega^{-1}(\phi) = 4\phi^{-1} - 1 \), and reverse the order of integration —

\[
\int \int_{\beta > \omega^{-1}(\phi)} p_\beta(\beta)p_\phi(\phi) \, d\beta \, d\phi
\]

Since \( \omega^{-1} \) is a monotonically decreasing function of \( \phi \), we can break up the integral in a similar fashion as in section (IV-B) —

\[
\int \int_{\beta > \omega^{-1}(\phi)} = \int_{\phi_1}^{\phi_2} \int_{\omega^{-1}(\phi)}^{\beta_2} d\beta \, d\phi + \int_{\phi_1}^{\phi_2} d\phi
\]

where we define the following —

\[
\phi_1 = \max(\phi_1, \omega(\beta_2))
\]

\[
\phi_2 = \min(\omega(\beta_1), \phi_2)
\]

**V. FAIRNESS ANALYSIS OF CONSTRICTION**

In this section, we consider 4 EMPSO systems, describe their motivations for their range of parameters and carry out the probabilistic analysis based on the aforementioned recipe.

**A. Discriminant-Centred**

With reference to the PSO system in (IV-B), in the interval of parameters, \( \omega(\frac{1}{2} + \epsilon) > 2 > 2 - \delta \). On the other hand, \( \frac{8}{3} < \omega(\frac{1}{2} - \epsilon) < 4 \) is independent of \( 2 + \delta \) with respect to comparison. According to the framework for computing the probability integral —

\[
\phi_I = \max(2 - \delta, \omega(1/2 + \epsilon)) = \frac{8}{3 + 2\epsilon}
\]

\[
\phi_g = \min(\omega(1/2 - \epsilon), 2 + \delta)
\]

On exercising the choice \( \phi_g = 2 + \delta \), we need to obey the condition —

\[
2 + \delta < \omega(1/2 - \epsilon)
\]

\[
\implies \delta < 2 + 4\epsilon
\]

The probability integral computes as —

\[
\begin{align*}
&\int_{8/(3+2\epsilon)}^{2+\delta} \int_{4/\phi-1}^{1/2+\epsilon} \frac{1}{2\epsilon} \, d\nu \, d\phi = \frac{1}{4\epsilon \delta} \left\{ \frac{2\epsilon \delta + 4\epsilon + 3\delta - 2}{2} - 4 \ln \left[ \frac{(2 + \delta)(3 + 2\epsilon)}{8} \right] \right\} \\
&= \frac{1}{4\epsilon \delta} \left\{ -4\epsilon \delta + 12\epsilon + 6\delta - 4 - 4 \ln \left[ \frac{3 + 2\epsilon}{3 - 2\epsilon} \right] \right\}
\end{align*}
\]

Using the alternative choice for \( \phi_g = \omega(1/2 - \epsilon) = \frac{8}{3 - 2\epsilon} \), we need to obey \( \delta \geq \frac{2 + 4\epsilon}{3 - 2\epsilon} \), and the probability integral —

\[
\begin{align*}
&\int_{8/(3+2\epsilon)}^{8/(3-2\epsilon)} \int_{4/\phi-1}^{1/2+\epsilon} \frac{1}{2\epsilon} \, d\nu \, d\phi + \int_{8/(3-2\epsilon)}^{2+\delta} \frac{1}{2\epsilon} \, d\phi \\
&= \frac{1}{4\epsilon \delta} \left\{ -2\epsilon \delta + 4\epsilon + 3\delta - 2 - 4 \ln \left[ \frac{(2 + \delta)(3 + 2\epsilon)}{8} \right] \right\}
\end{align*}
\]

Overall, the unfairness as a function \( \mu(\epsilon, \delta) \) —

\[
\begin{cases}
\frac{1}{4\epsilon \delta} \left\{ -2\epsilon \delta + 4\epsilon + 3\delta - 2 - 4 \ln \left[ \frac{(2 + \delta)(3 + 2\epsilon)}{8} \right] \right\}, & \delta < \frac{2 + 4\epsilon}{3 - 2\epsilon} \\
\frac{1}{4\epsilon \delta} \left\{ -4\epsilon \delta + 12\epsilon + 6\delta - 4 - 4 \ln \left[ \frac{3 + 2\epsilon}{3 - 2\epsilon} \right] \right\}, & \delta \geq \frac{2 + 4\epsilon}{3 - 2\epsilon}
\end{cases}
\]

Clearly, this 2-variable function is not conducive to further analysis. As an example, we use a discriminant-centred parameter set that has been proposed in section (VI). It corresponds to \( \epsilon = \frac{1}{2} \), \( \delta = 1 \) and \( \mu = \frac{1}{2} - 2\ln(\frac{3}{2}) \approx -0.3109 \). The system is under-constricted.

**B. Restricted Momentum Injection**

We wish to carry forward \( \phi \sim U(3, 5) \) from SMPSO and restrict \( \beta \sim U(0, \epsilon) \) for \( 0 < \epsilon < 1 \). It is hoped that for \( \epsilon << 1 \), the system would be fairly constricted. In other words, we are injecting a restricted amount of momentum.

**Theorem 4** - SMPSO with momentum injection cannot be fairly constricted.
Proof: We need to find the unfairness as a function of the injected momentum, \( \mu(\epsilon) \). Following the recipe in the previous section, we obtain

\[
\phi_L = \max(3, \omega(\epsilon)) = \frac{4}{1+\epsilon},
\]
\[
\phi_R = \min(\omega(0), 5) = 4
\]
where have exercised the choice \( \epsilon < \frac{1}{3} \). With \( p_3(\beta) = \frac{1}{\epsilon} \) and \( p_\phi(\phi) = \frac{1}{2} \), the probability integral is

\[
P(E_g) = \int_3^4 \int_{4/\phi-1}^\epsilon \frac{d\beta}{2} \frac{d\phi}{2} + \int_4^5 \frac{d\phi}{2}
\]
\[
= \frac{5}{2} - \frac{2\ln(1+\epsilon)}{\epsilon}
\]

On the other hand, exercising the choice of \( \epsilon \geq \frac{1}{3} \), we need to modify \( \phi_L = 3 \), and the probability integral

\[
P(E_g) = \int_3^4 \int_{4/\phi-1}^\epsilon \frac{d\beta}{2} \frac{d\phi}{2} + \int_4^5 \frac{d\phi}{2}
\]
\[
= 1 - \frac{4\ln(4/3) - 1}{2\epsilon}
\]

Finally, the unfairness function

\[
\mu(\epsilon) = \begin{cases} 
2 \left[ 1 - \frac{\ln(1+\epsilon)}{\epsilon} \right] & \epsilon < \frac{1}{3} \\
\frac{1}{2} \left[ 1 - \frac{4\ln(4/3) - 1}{\epsilon} \right] & \epsilon \geq \frac{1}{3}
\end{cases}
\]

This is piece-wise continuous at \( \epsilon = \frac{1}{3} \). From elementary calculus, \( \lim_{\epsilon \to 0} \frac{\ln(1+\epsilon)}{\epsilon} = 1 \) and hence

\[
\lim_{\epsilon \to 0} \mu(\epsilon) = 0
\]

Is there an \( \epsilon > 0 \) such that \( \mu = 0 \)? Consider the derivative

\[
\frac{d\mu}{d\epsilon} = \begin{cases} 
\frac{2}{\epsilon} \left[ \frac{\ln(1+\epsilon)}{\epsilon} - \frac{1}{1+\epsilon} \right] & 0 < \epsilon < \frac{1}{3} \\
\frac{2}{\epsilon} \left[ \frac{\ln(1+\epsilon)}{\epsilon} - \frac{1}{1+\epsilon} \right] & \frac{1}{3} \leq \epsilon < 1
\end{cases}
\]

Clearly, \( \frac{d\mu}{d\epsilon} > 0 \) for \( \frac{1}{3} \leq \epsilon < 1 \). On the other interval, we first compute the limit (applying L'Hôpital rule twice)

\[
\lim_{\epsilon \to 0} \frac{d\mu}{d\epsilon} = 1
\]

Secondly, we construct the function for \( 0 < \epsilon < \frac{1}{3} \)

\[
g(\epsilon) = \ln(1+\epsilon) - \frac{\epsilon}{1+\epsilon}
\]

With \( g'(\epsilon) = \frac{\epsilon}{(1+\epsilon)^2} > 0 \), and \( \lim_{\epsilon \to 0} g(\epsilon) = 0 \)

\[
\ln(1+\epsilon) - \frac{\epsilon}{1+\epsilon} > 0
\]

\[
\Rightarrow \frac{d\mu}{d\epsilon} > 0 \quad (\epsilon \neq 0)
\]

Based on eqs (39, 41), we can conclude that \( \mu(\epsilon) \neq 0 \) for \( 0 < \epsilon < 1 \). This completes our proof that momentum injection cannot guarantee fairness.

We expected to regain fairness if there is no contribution of momentum i.e \( \epsilon = 0 \). Indeed, the closest one could get to it being fairly constricted is to choose as small an \( \epsilon \) as possible (eq 37). One could call this system almost constricted and this is demonstrated by the constriction curve in figure (1).

C. Fully Ranged Momentum

Although one could create a EMPSO system that is almost fair by drawing \( \beta \sim U(0, \epsilon) \) for \( \epsilon << 1 \), it does not take any advantage the exponentially-averaged momentum has to offer. We would like to draw \( \beta \) from a wider range, and also ensure that fair constriction is met. Being optimistic, we set \( \beta \sim U(0, 1) \) to utilize the full range of momentum. However, we also wish to centre \( \phi \) at \( 3 \), as in SMPSO, and hence choose \( \phi \sim U(3-\delta, 3+\delta) \) for \( 0 \leq \delta \leq 3 \). Our recipe suggests

\[
\phi_L = \max(3-\delta, 2) = 3-\delta
\]
\[
\phi_R = \min(4, 3+\delta) = 3+\delta
\]

We have exercised the choice \( \delta \leq 1 \). The probability integral evaluates as follows with \( p_3(\beta) = 1 \), \( p_\phi(\phi) = \frac{1}{2\delta} \)

\[
P(E_g) = \frac{1}{2\delta} \int_{3-\delta}^{3+\delta} \int_3^{4/\phi-1} \frac{d\beta}{2\delta} \frac{d\phi}{2\delta}
\]
\[
= 2 \left[ 1 + \frac{1}{\delta} \ln \left( \frac{3-\delta}{3+\delta} \right) \right]
\]
With the other choice of $\delta \geq 1$, $\phi_1 = 2$, $\phi_g = 4$ and —

$$ P(E_g) = \int_{x_1}^{x_2} \int_{0}^{1} \frac{d\phi}{2\delta} + \int_{4}^{\infty} \frac{d\phi}{2\delta} $$

$$ = \frac{1}{\delta} \left[ \frac{3}{2} - 2\ln 2 \right] + \frac{1}{2} \tag{43} $$

Thus, the unfairness as a function $\mu(\delta)$ is —

$$ \mu(\delta) = \begin{cases} \frac{3}{2} + \frac{2}{\delta} \ln \left( \frac{3 - \delta}{\delta + 3} \right) & 0 \leq \delta < 1 \\ \frac{1}{3} \left[ \frac{3}{2} - 2\ln 2 \right] & 1 \leq \delta \leq 3 \end{cases} \tag{44} $$

It is piece-wise continuous at $\delta = 1$ and clearly, $\mu(\delta)$ is a monotonically decreasing in $1 \leq \delta \leq 3$. In $0 \leq \delta < 1$, it is sufficient for $\ln \left( \frac{3 - \delta}{\delta + 3} \right)$ to be monotonically decreasing for $\mu(\delta)$ to be as well —

$$ \frac{d}{d\delta} \left[ \ln \left( \frac{3 - \delta}{\delta + 3} \right) \right] = - \left[ \frac{1}{3 - \delta} + \frac{1}{3 + \delta} \right] < 0 $$

$$ \lim_{\delta \to 0^+} \frac{1}{3} \ln \left( \frac{3 - \delta}{\delta + 3} \right) = -\frac{2}{3}, \text{ which gives } \mu(0) = \frac{1}{5}. \text{ Moreover, } \mu(3) = \frac{1}{3} - \frac{2}{3} \ln 2 \approx 0.037 > 0. \text{ This system is close to being fairly constricted. Figure (2) is the constriction curve.} $$

D. Finding an FCPSO-em Algorithm

**Definition 3** - Any EMPSO system that is fairly constricting is an FCPSO-em algorithm.

Until now, we have been unable to conform to the parameters of SMPSO, while including momentum and being fairly constricting. However, we shall prove that an FCPSO-em algorithm exists.

**Theorem 5** - There exists a set of probability distributions $\{p_\phi(\phi), p_\beta(\beta)\}$ for which EMPSO is fairly constricted.

**Proof** - The proof is by construction. Let $\beta_1 = 0$, $\beta_2 = 1$.

In computing the probability integral, we posit $\phi_1 = \phi_2$ and $\phi_g = \phi_2$ which amounts to exercising the choices of $\phi_1 \geq 2$ and $\phi_2 \leq 4$ respectively. Hence —

$$ P(E_g) = \int_{\phi_1}^{\phi_2} \int_{0}^{1} \frac{d\beta}{2\delta} \cdot \frac{d\phi}{\phi_2 - \phi_1} $$

$$ = 2 - 4 \ln(\phi_2/\phi_1) \quad \phi_2 - \phi_1 \tag{45} $$

Let us arbitrarily assign $\phi_1 = 2$. This is reasonable as per the previous parameter sets studied —

$$ \mu(\phi_2) = \frac{3}{2} - 4 \ln(\phi_2/2) \quad \phi_2 \geq 2 \tag{46} $$

We transform $\frac{\phi_2}{2} \rightarrow x$ and set $\mu(x) = 0$ to obtain the following transcendental equation —

$$ \psi(x) = x - \frac{1}{\ln x} - \frac{4}{3} = 0 \quad \mu(\phi_2) = 0 \quad \phi_2 \geq 2 \tag{47} $$

A solution $x = \bar{x}$ to this equation must lie in $1 < \bar{x} \leq 2$. Note that $\lim_{x \to 1^+} \psi(x) = -\frac{1}{\ln 2} < 0$ and $\psi(2) = \frac{1}{\ln 2} - \frac{4}{3} > 0$. It is well known that $\psi(x)$ is monotonically increasing — it is of the form of the asymptotic prime counting function [11]— and thus a unique solution exists in our domain of interest. Wolfram Alpha [12] outputs the solution as $\bar{x} \approx 1.7336$ and we obtain $\phi_2 = 2\bar{x} \approx 3.4672$. Hence $c_1, c_2 \sim U(1, \bar{x})$ and $\beta \sim U(0, 1)$ fairly constricts the system and our proof is complete.

Eq (46) is plotted in figure (3), and eq (45) in the $(\phi_1, \phi_2)$ plane is plotted in figure (4). Note that there exist other parameter sets that are also fairly constricting. For instance, a more detailed analysis of the discriminant-centred PSO system may reveal a fair parameter set. In this work, we have derived only one such set and subsequently used it for benchmarking.

VI. PROPOSAL OF PARAMETER SETS

We describe 7 parameter sets for use in boosting SMPSO with exponentially-averaged momentum. They are summarized in table (I) —

1) **Naïve Injection** (NI) — Momentum injection with the full range of $\beta$ i.e $c_1, c_2 \left[ 1.5, 2.5 \right]$, $\beta \left[ 0, 1 \right]$. Unfairness $\mu = 1 - 2\ln \left( \frac{4}{7} \right) \approx 0.425$ from the formula derived in (V-B).

2) **Restricted Injection** (RI) — As derived in section (V-B), $c_1, c_2 \left[ 1.5, 2.5 \right]$, $\beta \left[ 0, \epsilon \right]$. We have tested for $\epsilon = \frac{1}{5}, \frac{1}{3}, \frac{1}{2}$. This parameter set is always overly constricted in general but almost constricted for small $\epsilon$
3) Discriminant-Centred (DC) — A system with \( P(E_+) = \frac{1}{2} \). We draw \( c_1, c_2 \leftarrow [0.5, 1.5] \) and \( \beta \leftarrow (0, 1) \). It is under-constricted.

4) Eigen-Skewed (ES) — We choose \( \delta = 2 \) in section (V-C) to obtain \( \phi \sim U(1, 5) \). Hence we can choose \( c_1, c_2 \leftarrow [0.5, 2.5] \) and \( \beta \leftarrow (0, 1) \). It is over-constricted.

5) Eigen-Centred-Beta (ECB) — From section (V-D), \( \phi \sim U(2, 2\bar{x}) \) is a fairly constricted system. This corresponds to \( c_1, c_2 \leftarrow [1, \bar{x}] \) and \( \beta \leftarrow (0, 1) \) and is an FCPSO-em algorithm.

6) Eigen-Centred-Omega (ECO) — If we sample \( \omega \) (instead of \( \beta \)) and \( \phi \) from a uniform distribution centred around the same point, then by analogy to Theorem 2, \( P(E_\phi) = P(E_\omega) = \frac{1}{2} \) and the system would be fairly constricted. However, we would need to transform \( \omega \rightarrow \beta \) via \( \omega^{-1} \) and the distribution of \( \beta \) would be no more uniform. We use \( c_1, c_2 \leftarrow [1, 2], \omega \leftarrow (2, 4) \) and it is an FCPSO-em algorithm.

7) Stability-Respecting (SR) — Obey Von-Neumann stability by drawing \( c_1, c_2 \leftarrow [0, 1] \) and \( \beta \leftarrow (0, 1) \). Constriction never occurs here (Corollary 1) and \( \mu = -\frac{1}{2} \) making this a maximally under-constricted system.

| Param   | \( c_1, c_2 \)     | \( \beta \) | Unfairness |
|---------|---------------------|-------------|------------|
| NI      | [1.5, 2.5]          | (0, 1)      | Over       |
| RI      | [1.5, 2.5]          | (0, \frac{1}{2}) | Over       |
| DC      | [0.5, 1.5]          | (0, 1)      | Under      |
| ECB     | [1.1, 1.3, 0.6]     | (0, 1)      | Fair       |
| ECO     | [1, 2]              | \omega \leftarrow [2, 4] | Fair       |
| SR      | [0, 1]              | (0, 1)      | Under      |

TABLE I: Parameter Sets

We have built momentum-boosted SMPSO and FCPSO-em algorithms jmetalpy framework [13]. To be consistent with the definitions in this paper, both are EMPSO systems adapted for MOO problems. The former uses over/under constricting parameter sets whereas the latter is fairly constricting — ECB/ECO parameter sets.

VII. RESULTS

We provide results for the ECB/ECO parameter sets only as they constitute a fairly constricting algorithm. We respectively name them as FCPSO-em-Beta/FCPSO-em-Omega.

A. Monte-Carlo Verification of Constriction Fairness

To verify our theoretical development, we have written two procedures to check whether constriction fairness is experimentally met.

Algorithm (1) simulates iterations of the PSO Algorithm FCPSO-em-Beta without evolving any underlying swarm. Lines 5, 6 draw \( \phi, \beta \) respectively from a uniform distribution and line 9 is the condition for the event \( E_c \) to occur as \( \lambda > 1 \implies \chi < 1 \). Sampling occurs for a total of 10,000 times. On the exit of the while loop, the variable \( pr \) holds the value of \( P(E_c) \) for the simulated evolution of the swarm.

Algorithm 1 Verify-Fairness-Beta

Input: \( \phi_1, \phi_2, \beta_1, \beta_2 \)

Output: \( pr \)

1: total \( \leftarrow \) 10,000
2: \( pr \leftarrow 0 \)
3: \( i \leftarrow 1 \)
4: while \( i \leq \) total do
5: \( \phi \leftarrow \text{random}(\phi_1, \phi_2) \)
6: \( \beta \leftarrow \text{random}(\beta_1, \beta_2) \)
7: \( k \leftarrow 4^*(1-\beta) \)
8: if \( \phi > k \) then
9: \( \lambda \leftarrow (\text{abs}(\phi - 2) + \text{sqrt}(\phi**2 - k*\phi))/2 \)
10: if \( \lambda > 1 \) then
11: \( pr \leftarrow pr + 1 \)
12: end if
13: end if
14: \( i \leftarrow i + 1 \)
15: end while
16: \( pr \leftarrow pr/\text{total} \)
17: \( \text{return} \ pr \)

Algorithm (2) is the analogue of algorithm (1) to FCPSO-em-Beta. On running both these procedures repeatedly with the appropriate arguments, we find that the output is invariably centred around 0.5 with a negligible deviation from it.

Algorithm 2 Verify-Fairness-Omega

Input: \( \phi_1, \phi_2, \omega_1, \omega_2 \)

Output: \( pr \)

1: total \( \leftarrow \) 10,000
2: \( pr \leftarrow 0 \)
3: \( i \leftarrow 1 \)
4: while \( i \leq \) total do
5: \( \phi \leftarrow \text{random}(\phi_1, \phi_2) \)
6: \( \omega \leftarrow \text{random}(\omega_1, \omega_2) \)
7: \( \beta \leftarrow 4/\omega - 1 \)
8: \( k \leftarrow 4^*(1-\beta) \)
9: if \( \phi > k \) then
10: \( \lambda \leftarrow (\text{abs}(\phi - 2) + \text{sqrt}(\phi**2 - k*\phi))/2 \)
11: if \( \lambda > 1 \) then
12: \( pr \leftarrow pr + 1 \)
13: end if
14: end if
15: \( i \leftarrow i + 1 \)
16: end while
17: \( pr \leftarrow pr/\text{total} \)
18: \( \text{return} \ pr \)

B. Single-Objective

We have compared the following algorithms —

1) Vanilla PSO [1]
2) EMPSO [3]
3) FCPSO
4) FCPSO-em-Beta
5) FCPSO-em-Omega

For algorithms 1 and 2, we have implemented velocity clipping, in each dimension of a particle, up to 10% of the allowed search space in that dimension. For the case of algorithms 3, 4 and 5, the value is set to 50% as it conforms to [7]. More details on velocity clipping can be found in [14].

Boundary handling is a key determinant for the performance of a PSO algorithm as the global optima of any objective could lie near the search boundary. There might be a considerable propensity for a swarm to migrate towards the boundary if the objective function landscape is such, or a particular instantiation of the swarm (since it is randomized) causes such a migration. This is a reasonable possibility as the heart of PSO lies in its frequent exchange of local/global information amongst the particles. We have used the Inverse Parabolic Confined Distribution technique [15] for boundary handling on all the aforementioned algorithms.

Lastly, the stopping criteria for the swarm is not evidently obvious from the PSO update equations. The stopping criteria for a swarm of $n$ particles is as follows —

$$\forall i, j \in \{1, 2 \ldots n\}, i \neq j \ni x_i \in H^d_i(x_j)$$

where $H_i(x_i)$ is an $e$-hypercube about particle $x_i$ in $d$-dimensions. We have chosen $e = 10^{-2}$. If this criterion isn’t satisfied for 10,000 iterations, the swarm is terminated.

The algorithms were benchmarked on a suite of 2-D and 5-D SO functions [16]. A swarm of 25 particles was run for 100 instances on each algorithm and the following indicators were noted —

- **Iters** - Iterations to convergence
- **Min** - Best objective value achieved by algorithm
- **Conv** - Number of particles converged at the $g_{best}$ of the swarm on termination
- **Succ** - Success ratio $\text{viz}$. of the total instances, how many found the theoretical global optimum at $10^{-2}$ tolerance

Overall, FCAs are more precise and reliable, especially in higher dimensions. However, this comes at a cost of increased iterations $\text{viz}$. at least factor $> 10$ in all cases.

### C. Multi-Objective

We have tested the following algorithms —

1) NSGA-II [6]
2) SMPSO [7] (or FCPSO)
3) FCPSO-em-Beta
4) FCPSO-em-Omega

The quality indicator (QIs) used are Inverted Generational Distance (IGD), Spacing (SP), Hypervolume (HV) and the $\epsilon$-indicator (EPS). A thorough description of these indicators can be found in [9]. The QIs were computed after evolving the swarm for 25,000 function evaluations (FE). The theoretically optimal Pareto fronts, needed as a reference for the QI computations, were obtained from [17], [18] and [19].

For NSGA-II, we have used polynomial mutation with a probability $p_m = \frac{1}{n}$ where $n$ is the dimension of the input space. The cross-over probability was set to 1. The offspring population size was set equal to the original swarm size. In the case of SMPSO and FCAs, the leaders were stored in a CrowdingDistanceArchive [13]. Mutation probability was set to $p_m = \frac{1}{n}$ with a distribution_index= 20.

A comprehensive coverage of synthesized MOO problems can be found in [20]. In this work, we focus on the 2-objective

In the case of number of particles, FCAs are comparable to Vanilla PSO or EMPSO. Note that FCPSO-em-Beta has significantly less iterations than FCPSO, which is attributed to the velocity accumulation of EM. The Omega variant is worse than the Beta variant, and it can be clearly attributed to the non-uniform distribution of the momentum factor $\beta$. FCAs are a clear success for higher dimensional objectives and thus have reliable/reproducible performance. Note that the disparity between the success ratio of Vanilla-PSO/EMPSO and FCAs increases with the rise in the dimensionality of the problem (2 to 5).

### TABLE II: Rastrigin-2D ($min = 0$)

|                | Iters | Min    | Conv | Succ |
|----------------|-------|--------|------|------|
| Vanilla PSO    | 140.57| 1.09 x 10^{-1} | 25   | 0.89 |
| EMPSO          | 909.13| 4.08 x 10^{-1}  | 24.99| 0.66 |
| FCPSO          | 9702.35| 2.44 x 10^{-1}  | 4.35 | 1    |
| FCPSO-em-Beta  | 2443.96| 1.62 x 10^{-1}  | 25   | 1    |
| FCPSO-em-Omega | 9887.74| 3.21 x 10^{-4}  | 4.76 | 1    |

### TABLE III: Goldstein ($min = 3$)

|                | Iters | Min    | Conv | Succ |
|----------------|-------|--------|------|------|
| Vanilla PSO    | 82.06 | 2.81   | 25   | 0.97 |
| EMPSO          | 46.28 | 5.97   | 0.91 |      |
| FCPSO          | 9134.26| 8.12   | 1    |      |
| FCPSO-em-Beta  | 1394.4 | 3      | 25   | 1    |
| FCPSO-em-Omega | 9543.01| 6.62   | 1    |      |

Goldstein (table III) is a 2-D function like Rastrigin-2D (table II), however the latter has numerous and closely-spaced turning points due to its highly sinusoidal nature. Although our fairly constricted algorithms (FCAs $\text{viz}$. FCPSO, FCPSO-em-Beta and FCPSO-em-Omega) are worse off in iterations, they are incomparably more precise in the objective minima. This is also the case for the 5-D functions, and more pronounced (tables IV, V).

### TABLE IV: Rastrigin-5D ($min = 0$)

|                | Iters | Min    | Conv | Succ |
|----------------|-------|--------|------|------|
| Vanilla PSO    | 192.35| 2.85   | 25   | 0.07 |
| EMPSO          | 1449.17| 5.28   | 24.95| 0    |
| FCPSO          | 10000 | 2.01 x 10^{-7} | 1.3  | 1    |
| FCPSO-em-Beta  | 6246.16| 1.51 x 10^{-9} | 20.87| 1    |
| FCPSO-em-Omega | 10000 | 2.47 x 10^{-7} | 1.26 | 1    |

### TABLE V: Alpine-5D ($min = -174.617$)

|                | Iters | Min    | Conv | Succ |
|----------------|-------|--------|------|------|
| Vanilla PSO    | 260.05| -96.11 | 25   | 0.07 |
| EMPSO          | 899.54| -95.16 | 24.99| 0    |
| FCPSO          | 9936.97| -174.61| 1.87 | 1    |
| FCPSO-em-Beta  | 5072.76| -172.56| 23.52| 0.97 |
| FCPSO-em-Omega | 10000 | -174.61| 1.22 | 1    |
ZDT [21] and DTLZ (3, 5 and 10 objective) [22] problem suites. Note that these two problem sets are outdated and have been replaced by a harder set of problems called the WFG-suite, also proposed in [20]. However, it is important to evaluate our proposed FCAs on the easier problems to show that it is at least of comparable performance to that of the SOTA.

Benchmarking for FE is a separate process as compared to benchmarking for QIs. This is due to the very nature of the stopping criteria. In the case of QIs, we terminate the swarm on reaching 25,000 FEs — done by the StoppingByEvaluation() call in jmetalpy. Whereas to evaluate for FEs, we need to monitor some QI to reach a threshold value, at which point we could safely conclude that the swarm has reached the optimal Pareto front — for which there exists an analogous StoppingByQualityIndicator(). Hence, in our scheme, QIs and FEs are mutually incompatible for benchmarking. A commonly adopted resolution to this incompatibility in the literature is to monitor the HV achieved by the swarm and terminate it when it has reached 95% of that of the optimal Pareto front.

### TABLE VI: 2-DTL & 3-DTL : FEs (in 100s)

|        | SMPSO | NSGA-II | Fem-Beta | Fem-Omega |
|--------|-------|---------|----------|-----------|
| zdt1   | 5.36  | 3.66    | 3.66     | 3.66      |
| zdt2   | 5.33  | 3.33    | 3.33     | 3.33      |
| zdt3   | 4.40  | 4.40    | 4.38     | 4.39      |
| zdt4   | 3.85  | 3.85    | 3.85     | 3.85      |
| zdt5   | 3.17  | 3.15    | 3.15     | 3.17      |
| zdt6   | 3.28  | 3.24    | 3.27     | 3.27      |
| zdt7   | 3.74  | 3.74    | 3.73     | 3.74      |
| zdt8   | 3.69  | 3.69    | 3.69     | 3.69      |
| zdt9   | 3.33  | 3.25    | 3.25     | 3.25      |
| zdt10  | 4.26  | 4.26    | 4.26     | 4.26      |
| zdt11  | 4.26  | 4.26    | 4.26     | 4.26      |

We have additionally tested with $c_1, c_2 \leftarrow \{1.5, 2.5\}$ and $\beta \leftarrow (0, 1)$ viz. SMPSO with EM enforced without any prior consideration. Although not benchmarked, we noted that the Pareto front for the ZDT/DTLZ problems evaluated by it included just one point as mentioned in (section II). Hence, we did not expect it to perform any better on problems with more objectives. Moreover, this phenomenon was one of the primary reasons that led to the consideration of constriction fairness of SMPSO. It was suspected that the parameter choices for EM on SMPSO needed to satisfy a similar property to perform up to the mark.

We have evaluated HV for problems upto 5 objectives, however we have monitored FEs for only upto 3 objectives. It was found that the HV stopping criterion led to $> 2 \times 10^{6}$ FEs for a swarm of size 100 in the former case, and deemed unreasonably high. For 10 objectives, the HV computation in jmetalpy would not terminate and hence have not been included. Respective p-values (compared to SMPSO) have been included as a subscript for statistical significance.

1) 2-DTL and 3-DTLZ: In table (VII), it is clear that FCAs are comparable with SMPSO in terms of hypervolume — and hence no cell has been emboldened. With respect to FEs, FCAs are again in the same ballpark. In fact, the FCAs perform significantly better than SMPSO in ZDT1, ZDT6, DTLZ3 and DTLZ7 problems. However, in DTLZ6 they are worse off. Otherwise, they are comparable and we conclude that FCAs are at least as good as SMPSO.

A peculiarity to be noted for the DTLZ-2, 4 and 5 problems is that the number of FEs is not characteristic of the other problems and hence we have not included their statistical p-values. This is probably due to the particular nature of the problem, or the swarm initialisation implemented in jmetalpy.
**TABLE XII: 10-DTLZ : IGD**

| dtlz | SMPSO | NSGA-II | Fem-Beta | Fem-Omega |
|------|-------|---------|----------|-----------|
| dtlz1 | 5.61  | 16.11  | 1.57  | 2.98e-10  |
| dtlz2 | 0.65  | 0.68   | 0.56   | 0.50e-06  |
| dtlz3 | 33.16 | 30.37  | 21.56  | 1.45e-04  |
| dtlz4 | 0.65  | 0.72   | 0.52   | 0.53e-00  |
| dtlz5 | 0.08  | 0.17   | 0.10   | 6.33e-05  |
| dtlz6 | 0.62  | 1.72   | 0.15   | 2.56e-12  |
| dtlz7 | 1.54  | 0.92   | 0.95   | 1.64e-02  |

**TABLE XIII: 10-DTLZ : EPS**

| dtlz | SMPSO | NSGA-II | Fem-Beta | Fem-Omega |
|------|-------|---------|----------|-----------|
| dtlz1 | 13.92 | 54.48  | 10.29  | 1.55e-10  |
| dtlz2 | 0.44  | 0.67   | 0.02   | 0.02e-00  |
| dtlz3 | 90.24 | 202.02 | 93.83  | 7.96e-02  |
| dtlz4 | 0.43  | 0.64   | 0.44   | 1.16e-03  |
| dtlz5 | 0.22  | 0.32   | 0.30   | 0.00e-00  |
| dtlz6 | 0.80  | 1.24   | 0.95   | 1.33e-15  |
| dtlz7 | 0.92  | 1.19   | 0.74   | 6.02e-13  |

**TABLE XIV: 10-DTLZ : SP**

| 2) 5/10-DTLZ: Tables (VIII-XI) contains QIs for the 5-DTLZ problems, and tables (XII-XIV) for that of 10-DTLZ. FCAs have substantial improvement in IGD and EPS indicators, and this relatively less pronounced in SP. In 5-DTLZ, the HV is improved in half the test problems.

3) 5/10-WFG: The WFG test suite was proposed in [20] to overcome the limitations of ZDT/DTLZ test suites. For one, ZDT is limited to 2 objectives only. Secondly, the DTLZ problems are not deceptive — a notion developed by the authors — and none of them feature a large flat landscape. Moreover, they state that the nature of the Pareto-front for DTLZ-5,6 is unclear beyond 3 objectives. Lastly, the complexity of each of the previous mentioned problem is fixed for a particular problem. Instead of defining a problem by a myriad of function transformations that lead an input vector to a point in the function space. The WFG test suite is harder and a rigorous attempt at an infallible, robust assessment of MOO solvers.

In our results, a peculiarity is that in 5-WFG, NSGA-II appears to be better than SMPSO/FCAs in terms of HV/IGD, but this improvement is not carried forward in EPS/SP. An absolute improvement in NSGA-II in all indicators would have been an anomaly as it is clearly worse off in 2-3 objectives. Otherwise, FCAs are better in half the test problems in IGD/EPS, except SP for 5/10-WFG.

**VIII. DISCUSSION AND IMPACT OF OUR WORK**

With the motivation of introducing EM in SMPSO, we have built FCAs that lie in the same ballpark as SMPSO in the 2/3-objective realm, if not better. We have found that the advantages of FCAs are emphasized in problems of greater objectives (5 and more) and are significantly better than SMPSO in terms of IGD, EPS and SP. This trend is akin to the increase in precision/reliability of FCAs in SO problems with increasing dimensionality of the objective.

Thus, EM plays a significant role in searching a large input space and also pushing the Pareto-envelope in a large function space. We have also developed the novel notion of constriction fairness and shown experimentally that this property is key in the performance of swarm MOO solvers. Without it, the performance falls drastically. This opens up a groundwork for proposing more complex swarm solvers whose parameter sets can be theoretically backed. Complete results can be found at the footnote on the first page of this manuscript.

**IX. CONCLUSION AND FUTURE WORKS**

In this paper, we have discussed the motivations for introducing exponentially-averaged momentum in the SMPSO
The framework. Having defined specific notions for constriction fairness, which is an integral part of SMPSO, we have successfully incorporated exponentially-averaged momentum to SMPSO and demonstrated its equivalence/superiority in problems with 2, 3, 5 and 10 objectives. Moreover, we also experimentally demonstrate EM’s improvement in single-objective problems, which was our primary motivation for its extension to MOO.

It would be beneficial to develop a large number of parameter schemes that are also fairly constricting and compare their performance. The sign of the constriction coefficient affects the swarm behaviour, and provide a theoretical backing to our observed results.

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