On $\pi$-Product Involution Graphs in Symmetric Groups

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Abstract  Suppose that $G$ is a group, $X$ a subset of $G$ and $\pi$ a set of natural numbers. The $\pi$-product graph $P_\pi(G, X)$ has $X$ as its vertex set and distinct vertices are joined by an edge if the order of their product is in $\pi$. If $X$ is a set of involutions, then $P_\pi(G, X)$ is called a $\pi$-product involution graph. In this paper we study the connectivity and diameters of $P_\pi(G, X)$ when $G$ is a finite symmetric group and $X$ is a $G$-conjugacy class of involutions.

Keywords  Symmetric group · Product · Graph · Diameter · Connectedness

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1 Introduction

There is a cornucopia of combinatorial and geometric structures which are associated with groups. These range from graphs to posets and topological spaces such as simplicial complexes. An example of the latter type arises in a finite group $G$ where for a prime $p$ dividing the order of $G$ we may define the poset of all non-trivial $p$-subgroups of $G$, denoted $S_p(G)$, ordered by inclusion. This poset has a rich structure, as has its

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associated order complex $|S_p(G)|$ known as the Brown complex, after being studied by—among others—Brown in his paper [10]. An analogous order complex, called the Quillen complex, can be defined for the poset $A_p(G)$ of all non-trivial elementary abelian $p$-subgroups of $G$. Indeed, Quillen showed in [16] that the Brown and Quillen complexes are $G$-homotopy equivalent. Thévenez and Webb later showed that the complexes consisting of chains of normal series of $p$-subgroups, and chains of radical $p$-subgroups are also $G$-homotopy equivalent to the Brown and Quillen complexes (see [19,20] for full details). In the case when $G$ is a group of Lie type, the order complex $|S_p(G)|$ is the same as the building of $G$. A good survey of the versatility of such complexes can be found in [18].

We mention a few graphs among the multitude of such structures that we may associate to a given group $G$. Let $X$ be a subset of $G$. The commuting graph $C(G, X)$ has vertex set $X$ and distinct elements $x, y \in X$ are joined by an edge whenever $xy = yx$. The case when $X = G \setminus Z(G)$, first studied in [8], has been the focus of interest recently—see [9,13,14]. When $X$ is taken to be a $G$-conjugacy class of involutions, we get the so-called commuting involution graph, the subject of a number of papers (see [1–4,12,15,17]).

If $\pi$ is a set of natural numbers, then the $\pi$-product graph $P_\pi(G, X)$ again has vertex set $X$, with distinct vertices $x, y \in X$ joined by an edge if the order of $xy$ is in $\pi$. In the case when $X$ is a $G$-conjugacy class of involutions, we note that $P_{[2]}(G, X)$ is just a commuting involution graph. Taking $\pi$ to be the set of all odd natural numbers and $X$ a $G$-conjugacy class, $P_\pi(G, X)$ becomes the local fusion graph $F(G, X)$ which has featured in [5,6].

In the case when $X$ is a set of involutions we refer to $P_\pi(G, X)$ as a $\pi$-product involution graph. It is such graphs when $X$ is a conjugacy class that we consider in this paper for $G = \text{Sym}(n)$, the symmetric group of degree $n$. We use the standard distance metric on $P_\pi(G, X)$, which we denote by $d(\cdot, \cdot)$. For $x \in X$ and $i \in \mathbb{N}$ we denote the set of vertices distance $i$ from $x$ in $P_\pi(G, X)$ by $\Delta_i(x)$. We also denote by $\Omega := \{1, \ldots, n\}$ the underlying set upon which $\text{Sym}(n)$ acts.

We first consider the case when $\pi = \{4\}$. Or, in other words, two distinct involutions $x, y \in X$ are joined by an edge whenever $(x, y) \cong \text{Dih}(8)$, the dihedral group of order 8. In considering this, we are in effect looking at a section of the poset $S_2(\text{Sym}(n))$. Our first result determines when $P_{[4]}(G, X)$ is connected and in such cases, the diameter of $P_{[4]}(G, X)$ is also determined.

**Theorem 1** Suppose $G = \text{Sym}(n)$, $t = (1, 2) \cdots (2m - 1, 2m) \in G$, and let $X$ denote the $G$-conjugacy class of $t$.

(i) The graph $P_{[4]}(G, X)$ is disconnected if and only if one of the following holds:

(a) $n = 2m + 1$;

(b) $m = 1$;

(c) $(n, m) = (4, 2)$ or $(6, 3)$.

(ii) If $P_{[4]}(G, X)$ is connected, then $\text{Diam}(P_{[4]}(G, X)) = 2$.

In (i)(a) of Theorem 1 we observe that $P_{[4]}(G, X)$ consists of $n$ copies of $P_{[4]}(\text{Sym}(2m), Y)$ where $Y$ consists of all involutions of cycle type $2^m$. This corresponds to the $n$ possible fixed points of the involutions of $X$. Cases (i)(b) and (i)(c) result in totally disconnected graphs.
For symmetric groups, the diameters of the connected $\pi$-product involution graphs have been determined when $\pi = \{2\}$—that is the commuting involution graphs—and $\pi = \mathbb{N}_{\text{odd}}$ (= the set of all odd natural numbers)—the local fusion graphs. In the former case the diameter is bounded above by 3 except for three small cases when the diameter is 4. Moreover, the diameter can be 3 infinitely often. In the latter case, the connected local fusion graphs for symmetric groups always have diameter 2. So, from this perspective, $\mathcal{P}_\pi(G, X)$ for $\pi = \{4\}$ and $\pi = \mathbb{N}_{\text{odd}}$ are bedfellows. However, this apparent similarity does not extend to the case that $\pi = \{2^a\}$ for some $a \geq 3$. Indeed, we shall derive the following result.

**Theorem 2** Suppose that $G = \text{Sym}(n)$, $2m = 2^a \leq n$ for some $a \geq 3$, $t = (1, 2)(3, 4) \cdots (2m - 1, 2m)$ and $X$ is the $G$-conjugacy class of $t$. Then

(i) $\mathcal{P}_{\{2m\}}(G, X)$ is connected if and only if $n \geq 2m + 2$; and

(ii) if $\mathcal{P}_{\{2m\}}(G, X)$ is connected, then

$$\min\{m, \lfloor n/2 - m \rfloor\} \leq \text{Diam}(\mathcal{P}_{\{2m\}}(G, X)) \leq 2m - 1$$

(where $\lfloor x \rfloor$ denotes the smallest integer greater than or equal to $x$). Taking $n = 4m$ in this theorem gives an infinite family of $\pi$-product involution graphs whose diameter is unbounded.

Specializing to the case $m = 4$ (so $2m = 8$) we can give precise values for the diameter of $\mathcal{P}_{\{8\}}(G, X)$ in our next theorem.

**Theorem 3** Suppose $G = \text{Sym}(n)$, $t = (1, 2)(3, 4)(5, 6)(7, 8)$ and let $X$ be the $G$-conjugacy class of $t$. Then

(i) for $10 \leq n \leq 14$, $\text{Diam}(\mathcal{P}_{\{8\}}(G, X)) = 3$; and

(ii) for $n \geq 15$, $\text{Diam}(\mathcal{P}_{\{8\}}(G, X)) = 4$.

An analogous version of Theorem 2 also holds for any odd prime power.

**Theorem 4** Suppose that $G = \text{Sym}(n)$, $p$ is an odd prime and $q = p^a$ for some $a \geq 1$. Let $t = (1, 2) \cdots (q - 2, q - 1)$ and $X$ be the $G$-conjugacy class of $t$. Then

(i) $\mathcal{P}_{\{q\}}(G, X)$ is connected if and only if $n \geq q$; and

(ii) if $\mathcal{P}_{\{q\}}(G, X)$ is connected, then

$$\min\{q - 1, n + 1 - q\} \leq \text{Diam}(\mathcal{P}_{\{q\}}(G, X)) \leq q - 1.$$

Our final result combines Theorems 2 and 4.

**Theorem 5** Suppose that $G = \text{Sym}(n)$, and $p_1, \ldots, p_r$ are distinct primes with $p_i < p_{i+1}$ for $i = 1, \ldots, r - 1$. Let $q = p_1^{a_1} \cdots p_r^{a_r}$ for some $a_1, \ldots, a_r \geq 1$ with $a_1 \geq 2$ if $p_1 = 2$ and set

$$q_i = \begin{cases} p_i^{a_i} & \text{if } p_i = 2; \\
 p_i^{a_i} - 1 & \text{otherwise}, \end{cases}$$
and $2m = q_1 + \cdots + q_r$. Assuming $2m \leq n$, let $t = (1, 2) \cdots (2m−1, 2m)$ and $X$ be the $G$-conjugacy class of $t$. Then the graph $\mathcal{P}_{(q)}(G, X)$ is connected if and only if

$$n \geq \begin{cases} q + 2 & \text{if } p_1 = 2; \text{ and } \\ q & \text{otherwise}. \end{cases}$$

This paper is arranged as follows. In Sect. 2 we introduce the notion of the $x$-graph of an element of $X$. These are graphs that encapsulate the $C_G(x)$-orbits of $X$ and were first introduced by Bates et al. [2]. We present a number of their results, and relate the connected components of an $x$-graph to the disc $\Delta_1(t)$ for a fixed involution $t$ of $X$. Sect. 3 begins by considering combinations of connected components of $x$-graphs, and we show that Theorem 1 holds when restricted to the supports of such components. In particular we consider the case when our conjugacy class consists of elements of full support in Lemma 10. We then proceed to give a general proof of Theorem 1 at the end of this section. The paper concludes in Sect. 4 with an analysis of $\pi$-product graphs when $\pi \neq \{4\}$. We begin by considering the case when $\pi = \{2^a\}$ for some $a \geq 3$. Calculations of the sizes of discs $\Delta_i(t)$ for certain $\pi$-product involution graphs are given and these give a direct proof of Theorem 3. This is followed by constructive proofs of Theorems 2 and 4 and a proof of Theorem 5. Finally, we consider some smaller symmetric groups and calculate the sizes of discs of the $\pi$-product graphs $\mathcal{P}_\pi(G, X)$ when $\pi = \{6\}$ or $\{8\}.$

2 Preliminary Results

Throughout this paper, we set $G = \text{Sym}(n)$ and consider $G$ as acting on a set of $n$ letters (or points), $\Omega = \{1, \ldots, n\}$. Let $t \in G$ be a fixed involution and let $X$ be the $G$-conjugacy class of $t$. For an element $g \in G$, we denote the set of fixed points of $g$ on $\Omega$ by $\text{fix}(g)$ and define the support of $g$ to be $\text{supp}(g) := \Omega \setminus \text{fix}(g).$ For the sake of brevity, if $x_1, x_2, \ldots, x_r \in G$ we denote $\text{supp}(x_1) \cup \text{supp}(x_2) \cup \cdots \cup \text{supp}(x_r)$ by $\text{supp}(x_1, x_2, \ldots, x_r).

To study the graph $\mathcal{P}_{\{4\}}(G, X)$, we first introduce another type of graph known as an $x$-graph. Indeed, let $x \in X$. The $x$-graph corresponding to $x$, denoted $G_x$, has vertex set given by the orbits of $\Omega$ under $\langle t \rangle$. Two vertices $\sigma, \gamma$ are joined in $G_x$ if there exists $\sigma_0 \in \sigma$ and $\gamma_0 \in \gamma$ such that $\{\sigma_0, \gamma_0\}$ is an orbit of $\Omega$ under $\langle x \rangle$. We call the vertices corresponding to transpositions of $t$ black vertices, denoted $\bullet$, and those corresponding to fixed points of $t$ white vertices, denoted $\circ$. As an example, let $n = 15$, $t = (1, 2)(3, 4)(5, 6)(7, 8)(9, 10)(11, 12)$ and $x = (1, 7)(2, 3)(4, 10)(8, 9)(11, 13)(12, 14)$. The $x$-graph $G_x$ is given by

We note that we could swap the roles of $t$ and $x$ to produce another $x$-graph which we...
denote by $G^x_t$. In general the $x$-graph $G^x_t$ has vertices the orbits of $\Omega$ under $(y)$, and edges defined by the orbits of $\Omega$ under $(x)$.

The concepts of $x$-graphs were first introduced in [2] as a tool for studying the commuting involution graphs of the symmetric groups. More recently they have also been used in the study of local fusion graphs for the symmetric groups (see [6] for further details). The versatility of $x$-graphs in calculations arises from the simple observation that each black vertex has valency at most two and each white vertex has valency at most one. Consequently, we may fully determine the possible connected components of a given $x$-graph.

**Lemma 1** Let $x \in X$. The possible connected components of $G^x_t$ are

In the subsequent discussion, we will consider $x$-graphs up to isomorphism. It is implicit that such an isomorphism will preserve vertex colours. We also fix $t = (1, 2) \cdots (2m - 1, 2m) \in G$.

Bates, Bundy, Perkins and Rowley’s interest in $x$-graphs stemmed from the following elementary result.

**Lemma 2** (i) Every graph with $b$ black vertices of valency at most two, $w$ white vertices of valency at most one and exactly $b$ edges is the $x$-graph for some $x \in X$ (with $m = b$ and $n = 2b + w$).

(ii) Let $x, y \in X$. Then $x$ and $y$ are in the same $C_G(t)$-orbit if and only if $G^x_t$ and $G^y_t$ are isomorphic graphs.

**Proof** See Lemma 2.1 of [2].

Part (i) of Lemma 2 is of particular interest, as it confirms that when employing a combinatorial approach using the connected components of $x$-graphs, we must consider all possible connected components given in Lemma 1. This approach will be used repeatedly in the proof of Theorem 1.

An immediate consequence of the definition of $G^x_t$ is that the number of black vertices is equal to the number of edges. Consequently the number of connected components of the form $\bullet-\bullet-\bullet-\bullet$ containing at least one black vertex must be equal to the number of connected components of the form $\circ-\bullet-\bullet-\bullet-\circ$ and $\circ-\bullet-\bullet-\bullet-\circ$.

Lemma 1 allows a combinatorial approach to be used when considering conjugate involutions. Indeed, given a connected component $C_i$ of $G^x_t$, we may define $\Omega_i$ to be the union of all vertices of $C_i$. We may then define the $i$-part of $t$, denoted $t_i$, to be the product of those transpositions of $t$ that occur in $\text{Sym}(\Omega_i)$. We define $x_i$ similarly. By analysing the structure of the connected components given in Lemma 1 it is possible to relate the order of $tx$ to the $x$-graph $G^x_t$. 

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Lemma 3 Suppose that $x \in X$ and that $C_1, \ldots, C_k$ are the connected components of $G_x$. Denote the number of black vertices, white vertices and cycles in $C_i$ by $b_i$, $w_i$ and $c_i$ respectively. Then

(i) the order of $tx$ is the least common multiple of the orders of $t_i x_i$ (for $i = 1, \ldots, k$); and
(ii) the order of $t_i x_i$ is $(2b_i + w_i)/(1 + c_i)$ for each $i = 1, \ldots, k$.

Proof See Proposition 2.2 of [2].

We have the following immediate corollary to Lemmas 1 and 3.

Corollary 1 For $P_{\{4\}}(G, X)$ the disc $\Delta_1(t)$ consists of all $x \in X$ whose $x$-graphs have at least one connected component of the form

and all other components have the form

Proof The element $x$ lies in $\Delta_1(t)$ precisely when $tx$ has order 4. The result then follows from Lemmas 1 and 3.

We conclude this section by noting that we can define an $x$-graph for any two (not-necessarily conjugate) involutions. This we will do frequently in Sect. 3. However, in such a situation it is no longer the case that the number of edges of $G_x$ is equal to the number of black vertices.

3 Proof of Theorem 1

In this section, we prove Theorem 1. Note that for $m \geq 2$ and $t = (1, 2) \cdots (2m - 1, 2m)$, the involution $x = (1, 3)(2, 4)(5, 6) \cdots (2m - 1, 2m) \in X$ satisfies $d(t, x) \geq 2$. Thus it suffices to prove when $P_{\{4\}}(G, X)$ is connected, that for all $x \in X$ we have $d(t, x) \leq 2$. To do this we consider pairs or triples of connected components $C_i, C_j$ and $C_k$ of $G_x$ and the corresponding parts $t_i, t_j, t_k, x_i, x_j, x_k$ of $t$ and $x$. We then construct an element $y_{ijk} \in H$, where $H := \text{Sym}(\supp(t_i, t_j, t_k, x_i, x_j, x_k))$, which is $H$-conjugate to $t_i t_j t_k$ and such that the $x$-graphs $G_{y_{ijk}}$ and $G_{x_i x_j x_k}$ have connected components featuring in Corollary 1.

We begin by proving a few preliminary results, dealing with the case $n = 2m$.

Lemma 4 Let $m \geq 5$, $n = 2m$ and suppose that $x \in X$ is such that $G_x$ is connected. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.

Proof Without loss of generality we may assume that $x = (1, 2m)(2, 3) \cdots (2m - 2, 2m - 1)$. If $m = 5$, then taking $y = (1, 10)(2, 6)(3, 4)(5, 8)(7, 9)$ we see that $G_y$ and $G_y^y$ are given respectively by

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If $m = 6$, we take $y = (1, 3)(2, 4)(5, 7)(6, 12)(8, 9)(10, 11)$. Then $G_y$ and $G_x^y$ are, respectively

In the general case when $m \geq 7$, we take $y = (1, 3)(2, 4)\cdots m+5\cdots (m+4, m+5)$. The exact nature of the associated $x$-graphs is dependent on the parity of $m$. If $m$ is even, then $G_y$ is given by

and $G_x^y$ is given by
If $m$ is odd, the graphs $G_y$ and $G_x^y$ are, respectively

\[
\begin{align*}
\{1,2\} & \quad \{3,4\} & \quad \{5,6\} & \quad \{7,8\} & \quad \{9,10\} \\
\{11,12\}\{2m-3,2m-2\} & \quad \{m+2,m+3\}\{m+6,m+7\}\{m+4,m+5\}
\end{align*}
\]

and

\[
\begin{align*}
\{6,2m\} & \quad \{1,3\} & \quad \{2,4\} & \quad \{5,7\} & \quad \{8,9\}\{10,2m-1\}\{11,2m-2\} & \quad \{m+3,m+6\} & \quad \{m+4,m+5\}
\end{align*}
\]

In all cases, the given graphs satisfy the conditions of Corollary 1, whence $d(t, y) = d(y, x) = 1$.

The proof of Lemma 4 illustrates a general feature that the actual $x$-graphs constructed may vary depending on the parity and values of the given parameters (such as the parameter $m$ above). However, in using Corollary 1 we are only interested in the connected components of the $x$-graph. Thus for the sake of brevity, in all future proofs we will only describe the connected components of each $x$-graph.

**Lemma 5** Suppose that $m = 3$, $n = 6$ and $x \in X$. If $G_x$ is connected, then there exists $y \in X$ such that the $x$-graphs $G_y$ and $G_x^y$ are isomorphic to

\[
\begin{align*}
\text{Graph 1}
\end{align*}
\]

*Proof* Without loss of generality, we may assume that $x = (1, 6)(2, 3)(4, 5)$. Then $y = (1, 2)(3, 6)(4, 5)$ is the required element. □

**Lemma 6** Let $m = 4$ and $n = 8$. Suppose that $x \in X \setminus \{t\}$ has a disconnected $x$-graph, $G_x$. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.

*Proof* If $G_x$ has a connected component of the form $\bullet \longrightarrow \bullet$, then we may assume that $x = (1, 6)(2, 3)(4, 5)(7, 8)$. The element $y = (1, 8)(2, 4)(3, 6)(5, 7)$ is then the desired $y$. The other possibilities occur when $G_x$ has one or two connected components of the form $\bullet \longrightarrow \bullet$, corresponding respectively to $x = (1, 3)(2, 4)(5, 6)(7, 8)$ and $x = (1, 3)(2, 4)(5, 7)(6, 8)$. The $y$ satisfying the lemma for both such $x$ is $y = (1, 8)(2, 3)(4, 5)(6, 7)$. □

**Lemma 7** Suppose that $m \geq 5$, $n = 2m$ and that $G_x$ consists entirely of components of the form $\bullet \longrightarrow \bullet$ and $\bullet \longrightarrow \bullet$. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.

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Proof We consider three separate cases. First assume that $G_x$ contains at least two components, $C_i$ and $C_j$, of the form $\bullet-\bullet$. Without loss, we may take $t_i = (1, 2)(3, 4)(5, 6)$, $t_j = (7, 8)(9, 10)(11, 12)$, $x_i = (1, 6)(2, 3)(4, 5)$ and $x_j = (7, 12)(8, 9)(10, 11)$. Defining $y_{ij} \in \text{Sym}(\text{supp}(t_i, t_j, x_i, x_j))$ to be $y_{ij} = (1, 2)(3, 4)(5, 7)(6, 12)(8, 9)(10, 11)$, we see that both $G_{y_{ij} t_i}$ and $G_{y_{ij} x_i x_j}$ are isomorphic to $\bullet-\bullet$.

Denote the remaining parts of $t$ and $x$ by $t_k$ and $x_k$. Applying Lemmas 5 and 6 to $t_k$ and $x_k$ produces an element $y_k \in \text{Sym}(\text{supp}(t_k, x_k))$ such that $y := y_{ij} y_k$ is the desired element of $X$.

In the case that $G_x$ contains a unique component, $C_i$, of the form $\bullet-\bullet$, then there exists at least one component, say $C_j$, of the form $\bullet-\bullet$. Taking $t_i = (1, 2)(3, 4)(5, 6)$, $t_j = (7, 8)(9, 10)$, $x_i = (1, 6)(2, 3)(4, 5)$ and $x_j = (7, 9)(8, 10)$, then the element $y_{ij} \in \text{Sym}(\text{supp}(t_i, t_j, x_i, x_j))$ given by $y_{ij} = (1, 2)(3, 10)(4, 6)(5, 8)(7, 9)$ results in $\mathcal{X}$-graphs $G_{y_{ij} t_i}$ and $G_{y_{ij} x_i x_j}$ which are isomorphic to $\bullet-\bullet$.

Denoting the remaining part of $t$ by $t_k$ and setting $y := y_{ij} t_k \in X$ we have that $d(t, y) = d(y, x) = 1$ as required.

Finally, assume that all connected components of $G_x$ are of the form $\bullet-\bullet$ and let $C_i$ and $C_j$ be two such components. Without loss of generality we have that $t_i = (1, 2)(3, 4)$, $t_j = (5, 6)(7, 8)$, $x_i = (1, 3)(2, 4)$ and $x_j = (5, 7)(6, 8)$, and as above denote the remaining part of $t$ by $t_k$. Setting $y_{ij} = (1, 5)(2, 7)(3, 8)(4, 6)$, we see that both $G_{y_{ij} t_i}$ and $G_{y_{ij} x_i x_j}$ consist of a single connected component of isomorphism type $\bullet-\bullet$. Hence, $y := y_{ij} t_k$ is our desired element of $X$. \hfill \Box

Corollary 2 If $n = 2m$, then Theorem 1 holds.

Proof Let $x \in X$. If $G_x$ has connected components containing precisely 4 black vertices then we leave the parts of $t$ and $x$ corresponding to such components alone. We then apply Lemma 4 to any connected component containing at least 5 black vertices, and Lemma 5 to any connected component containing 3 black vertices to obtain the desired result. Otherwise all connected components have at most 3 black
vertices. Applying Lemmas 6 and 7 to a pair of components containing a total of 4, 5 or 6 black vertices, Lemma 5 to any remaining connected components containing 3 black vertices, and leaving all other connected components invariant gives the result.

\[ \square \]

Before presenting the proof of Theorem 1 we give a further three intermediate results.

**Lemma 8** Let \( x \in X \). Suppose that \( G_x \) has connected components \( C_i \) and \( C_j \) of the given forms. Then there exists \( y_{ij} \in H \), where \( H := \text{Sym}(\text{supp}(t_i, t_j, x_i, x_j)) \), which is \( H \)-conjugate to \( t_i t_j \) and such that the connected components of the \( x \)-graphs \( G_{ti t_j}^{y_{ij}} \) and \( G_{x_i x_j}^{y_{ij}} \) satisfy the conditions of Corollary 1.

**(i)** \( C_i \) : \( \bullet \ldots \bullet \) (with \( q \geq 3 \) black vertices),

\[ C_j : \circ \ldots \circ \] (with \( r \geq 0 \) black vertices);

**(ii)** \( C_i \) : \( \bullet \ldots \bullet \) (with \( q \geq 2 \) black vertices), \( C_j : \emptyset \);

**(iii)** \( C_i \) : \( \bullet \ldots \bullet \) (with \( q \geq 2 \) black vertices), \( C_j : \emptyset \);

**(iv)** \( C_i : \bullet \ldots \bullet \) (with \( q \geq 1 \) black vertices),

\[ C_j : \circ \ldots \circ \] (with \( r \geq 1 \) black vertices); and

**(v)** \( C_i \) and \( C_j \) are both of the form \( \bullet \ldots \bullet \circ \) (with \( q, r \geq 1 \) black vertices respectively).

**Proof** For each case, without loss of generality we give explicit formulations of the \( t_i \) and \( x_j \). For ease of notation, where parameters \( q \) and \( r \) have been defined we set \( v = 2(q + r) \).

In case (i) assume that \( t_i = (1, 2) \cdots (2q - 1, 2q), t_j = (2q + 1, 2q + 2) \cdots (v - 1, v)(v + 1)(v + 2), \) and \( x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q) \). We now consider three possibilities. If \( r = 0 \), then we may assume that \( x_j = (2q + 1, 2q + 2) \) and we take

\[ y_{ij} := (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q + 1, 2q + 2). \]

If \( r = 1 \), then taking \( x_j = (2q + 1, v + 1)(2q + 2, v + 2) \) we define

\[ y_{ij} := (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(v + 1, v + 2)(2q + 1, 2q + 2). \]

Finally, if \( r > 1 \), then we assume that \( x_j = (2q + 1, v + 1)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, v + 2) \) and define

\[ y_{ij} := (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(v + 1, v + 2) \]

\[ (2q + 1, v)(v + 2)(2q + r, 2q + r + 1). \]

We see that the \( x \)-graph \( G_{y_{ij}}^{ti t_j} \) has connected components of the form \( \bullet \ldots \bullet \circ \) and—depending on the values of \( q \) and \( r \) —also \( \bullet \ldots \bullet \) and \( \circ \). Similarly \( G_{y_{ij}}^{x_i x_j} \) has connected components of the form \( \circ \ldots \circ \bullet \) and in some cases also \( \bullet \) and \( \circ \) as required.

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For (ii) we may set $t_i = (1, 2) \cdots (2q - 1, 2q)(2q + 1)(2q + 2)$ and $x_i = (1, 2q + 2)(2, 3) \cdots (2q, 2q + 1)$. Then the element

$$y_i = (1)(2, 2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q)(2q + 1, 2q + 2)$$

results in the $x$-graph $G_{y_i}^{t_i}$ having connected components $\circ \circ$, $\bullet \bullet \bullet$ and in some cases $\bullet \bullet \circ$ and $\bullet - \bullet$—depending on the value and parity of $q$. The graph $G_{x_i}^{y_i}$ has connected components $\circ \bullet \bullet \circ$ and possibly $\bullet \bullet \bullet \bullet$ and $\bullet \bullet$.

Considering case (iii), we take $t_i = (1, 2) \cdots (2q - 1, 2q)(2q + 1)$, $x_i = (1)(2, 3) \cdots (2q, 2q + 1)$ and $t_j = x_j = (2q + 2)$. If $q = 2$, define

$$y_{ij} = (1, 4)(2)(3)(5, 6),$$

whilst if $q \geq 3$ define

$$y_{ij} = (1, 2q)(2)(2q - 1)(3, 2q - 2) \cdots (q, q + 1)(2q + 1, 2q + 2).$$

Then the permutation $y_{ij}$ gives the desired $x$-graphs. Indeed, $G_{y_{ij}}^{t_i t_j}$ has connected components of the form $\circ \circ \circ$ and $\bullet \bullet \bullet \bullet$, $G_{x_i x_j}^{y_{ij}}$ has components of the form $\bullet \bullet \bullet$ and $\circ \bullet \bullet \circ$ (with the black vertex omitted if $q = 2$) and both $x$-graphs may also have connected components of the form $\bullet \bullet \bullet$ and $\bullet$ depending on the value and parity of $q$.

Turning to (iv), if $q = 1$, then without loss of generality we have that $t_i = (1, 2)(v + 1)$, $t_j = (3, 4) \cdots (v - 1, v)(v + 2)(v + 3)$, $x_i = (1)(2, v + 1)$ and $x_j = (3, v + 2)(4, 5) \cdots (v - 2, v - 1)(v, v + 3)$ (take $x_j = (3, 6)(4, 7)$ if $r = 1$). When $r = 1$, define

$$y_{ij} = (1)(2)(3, 5)(4, 7)(6).$$

The $x$-graphs $G_{y_{ij}}^{t_i t_j}$ and $G_{x_i x_j}^{y_{ij}}$ are isomorphic to

$$\circ \bullet \bullet \circ \bullet \circ \circ \circ \circ \bullet \bullet \circ \circ$$

respectively as required. If $r > 1$, then

$$y_{ij} = (1)(2)(3, v - 2) \cdots (r + 1, r + 2)(v - 1, v + 2)(v, v + 1)(v + 3)$$

is our desired element. Indeed in this case $G_{y_{ij}}^{t_i t_j}$ has connected components of the form $\circ$, $\bullet$ and $\circ \bullet \bullet \circ$ in addition to components of the form $\bullet$ and/or
\begin{itemize}
  \item \textbullet \textbullet (depending on the value of \(r\)), whilst \(G_{y_{ij}}^{x_{i},x_{j}}\) has components of the forms \(\textbullet \textbullet \textbullet \) and/or \(\textbullet \textbullet \textbullet \textbullet \) and/or \(\textbullet \) and/or \(\textbullet \\
  \textbullet \textcircled{\textbullet} \textbullet \textbullet \textbullet \) in addition to components of the form \(\textbullet \) and/or \(\textbullet \textbullet \textbullet \)
  (depending on the value of \(r\)).
  
  If \(q > 1\), then we define \(t_{i} = (1, 2) \cdots (2q - 1, 2q)(v + 1)\), \(t_{j} = (2q + 1, 2q + 2) \cdots (v - 1, v)(v + 2)(v + 3)\), \(x_{i} = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q, v + 1)\) and \(x_{j} = (2q + 1, v + 2)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, v + 3)\). Our desired element is then

\[ y_{ij} = (1)(2, 2q - 1) \cdots (q + 1)(2q)(2q + 1, v - 2) \cdots (2q + r - 1, 2q + r)(v - 1, v + 2)(v, v + 1)(v + 3). \]

It follows that the connected components of \(G_{y_{ij}}^{x_{i},x_{j}}\) and \(G_{x_{i},x_{j}}^{y_{ij}}\) have the form \(\textbullet \textbullet \textcircled{\textbullet} \textbullet \textbullet \) and \(\textbullet \textbullet \) and possibly \(\textbullet \textbullet \) and \(\textbullet \textbullet \textbullet \), whilst \(G_{y_{ij}}^{x_{i},x_{j}}\) has an additional connected component of the form \(\textbullet \textbullet \textbullet \textbullet \).

For case (v) we assume without loss of generality that \(q \geq r\). We consider the subcases \(q = r = 1\), \(q > r = 1\), \(r = r > 1\) and \(q > r > 1\) in turn. If \(q = r = 1\), then we take \(t_{i} = (1, 2)(5)\), \(t_{j} = (3, 4)(6)\), \(x_{i} = (1)(2, 5)\) and \(x_{j} = (3)(4, 6)\). Defining

\[ y_{i,j} = (1, 3)(5, 6)(2)(4), \]

we see that \(G_{y_{ij}}^{x_{i},x_{j}}\) has isomorphism type \(\textbullet \textbullet \textbullet \textbullet \textbullet \textbullet \) and \(G_{x_{i},x_{j}}^{y_{ij}}\) has isomorphism type \(\textbullet \textbullet \textbullet \textbullet \textbullet \textbullet \) as required.

If \(q > r = 1\), then setting \(t_{i} = (1, 2) \cdots (2q - 1, 2q)(v + 1)\), \(t_{j} = (2q + 1, 2q + 2)(v + 2)\), \(x_{i} = (1)(2, 3) \cdots (2q, v + 1)\) and \(x_{j} = (2q + 1)(2q + 2, v + 2)\) we define

\[ y_{ij} = (1, 2(q - 1))(2, 2(q - 1) - 1) \cdots (q - 1, q)(2q - 1, 2q + 1) \]
\[ (2q)(2q + 2)(v + 1, v + 2). \]

Consequently \(G_{y_{ij}}^{x_{i},x_{j}}\) has connected components of the form \(\textbullet \textbullet \textbullet \), \(\textbullet \textbullet \textbullet \textbullet \textbullet \) and \(\textbullet \textbullet \textbullet \) and/or \(\textbullet \textbullet \textbullet \textbullet \textbullet \) (depending on the value of \(q\)), whilst \(G_{x_{i},x_{j}}^{y_{ij}}\) has components of the form \(\textbullet \textbullet \textbullet \textbullet \textbullet \textbullet \), \(\textbullet \textbullet \textbullet \) and \(\textbullet \textbullet \textbullet \textbullet \textbullet \) and/or \(\textbullet \textbullet \textbullet \textbullet \textbullet \).

When \(r > 1\) we may assume that \(t_{i} = (1, 2) \cdots (2q - 1, 2q)(v + 1)\), \(t_{j} = (2q + 1, 2q + 2)(v - 1, v)(v + 2)\), \(x_{i} = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q, v + 1)\) and \(x_{j} = (2q + 1)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, v + 2)\). Define

\[ (\textcircled{\textbullet} \textbullet \) Springer
\[ y_{ij} = (1, 2q + 1)(2, 2q + 2) \cdots (2q - 1, v - 1)(2q)(v)(v + 1, v + 2) \]

if \( q = r \) and

\[
y_{ij} = (1, 2(q - r))(2, 2(q - r) - 1) \cdots (q - r, q - r + 1) \]
\[
(2(q - r) + 1, 2q + 1) \cdots (2q - 1, v - 1)(2q)(v)(v + 1, v + 2)
\]

if \( q \neq r \). We see that \( G_{y_{ij}}^{t_it_j} \) has connected components \( \circ-\circ, \bullet-\bullet-\bullet \) and \( \bullet-\circ-\bullet-\bullet-\circ \) in addition to \( \bullet-\bullet \) for some values of \( q \) and \( r \). The \( x \)-graph \( G_{y_{ij}}^{x_i;x_j} \) also has the desired properties having a component of the form \( \circ-\bullet-\circ-\bullet-\circ \), some components of the form \( \bullet-\bullet-\bullet \) and possibly also \( \bullet-\bullet \) depending on the values of \( q \) and \( r \) and the parity of \( q - r \).

\[ \square \]

We note that in cases (ii) and (iv) above, \( t_it_j \) and \( x_i;x_j \) have different cycle types. This is a fact which we will utilise in the proof of Theorem 1.

In a similar vein to Lemma 8 we next consider collections of three connected components simultaneously.

**Lemma 9** Let \( x \in X \). Suppose that \( G_x \) has connected components \( C_i, C_j \) and \( C_k \) of the given forms and define \( H := \text{Sym}(\text{supp}(t_i, t_j, t_k, x_i, x_j, x_k)) \). Then there exists \( y_{ijk} \in H \) which is \( H \)-conjugate to \( t_it_j \) and \( x_i;x_j \) and \( x_i;x_j;x_k \) satisfy the conditions of Corollary 1:

(i) \( C_i, C_j \) and \( C_k \) are each of the form \( \circ-\bullet-\bullet-\bullet-\bullet-\bullet-\circ \) (having \( q, r, s \geq 1 \) black vertices respectively); and

(ii) \( C_i: \bullet-\bullet-\bullet-\bullet-\bullet-\bullet-\circ \) (with \( q \geq 1 \) black vertices), \( C_j: \bullet-\bullet-\bullet-\bullet-\bullet-\circ \) (with \( r \geq 1 \) black vertices), \( C_k: \circ-\circ-\circ \).

**Proof** We follow the approach of the proof of Lemma 8 and construct the appropriate \( t_i \) and \( x_i \). We also set \( v = 2(q + r) \) and \( w = 2(q + r + s) \).

For case (i) we may assume without loss of generality that \( q \geq r \geq s \geq 1 \) and set

\[
t_i = (1, 2) \cdots (2q - 1, 2q)(w + 1), \quad t_j = (2q + 1, 2q + 2) \cdots (v - 1, v)(w + 2)
\]
\[
\text{and} \quad t_k = (v + 1, v + 2) \cdots (w - 1, w)(w + 3).
\]

We also set

\[
x_i = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q, w + 1),
\]
\[
x_j = (2q + 1)(2q + 2, 2q + 3) \cdots (v - 2, v - 1)(v, w + 2), \text{ and}
\]
\[
x_k = (v + 1)(v + 2, v + 3) \cdots (w - 2, w - 1)(w, w + 3),
\]

taking \( x_i = (1)(2, w + 1) \) in the case when \( q = 1, x_j = (2q + 1)(2q + 2, w + 2) \) when \( r = 1 \) and \( x_k = (v + 1)(v + 2, w + 3) \) when \( s = 1 \). There are three subcases to
consider. If \( q = r = s = 1 \), then taking \( y_{ijk} = (1, 4)(5, 8)(6, 7)(2, 3)(9) \) we see that the \( x \)-graphs \( G_{y_{ijk}}^{t_{ij}l_{jk}} \) and \( G_{y_{ijk}}^{s_{x_{i}x_{j}x_{k}}} \) are both isomorphic to

\[ 
\begin{array}{c}
\circ \quad \bullet \quad \circ \\
\end{array}
\]

which has the desired form. If \( s = 1 \), but \( q \neq 1 \), we set

\[
y_{ijk} = (1, w + 2)(2, w + 1)(3, 2q)(4, 2q - 1) \cdots (q + 1, q + 2)
\]

\[
(2q + 1, v) \cdots (2q + r, 2q + r + 1)(v + 1)(w)(w + 3),
\]

whilst if \( s > 1 \), we define

\[
y_{ijk} = (1, w + 2)(2, w + 1)(3, 2q)(4, 2q - 1) \cdots (q + 1, q + 2)
\]

\[
(2q + 1, v) \cdots (2q + r, 2q + r + 1)
\]

\[
(v + 2, w - 1) \cdots (v + s, v + s + 1)(v + 1)(w)(w + 3).
\]

It follows that the \( x \)-graph \( G_{y_{ijk}}^{t_{ij}l_{jk}} \) has connected components of the form

\[
\begin{array}{c}
\circ \quad \bullet \quad \circ \\
\end{array}
\]

and \( \circ \) with further components of the form \( \bullet \) (if \( s = 1 \)), \( \bullet \) and \( \circ \) depending on the values of \( q, r \) and \( s \). Meanwhile, \( G_{y_{ijk}}^{s_{x_{i}x_{j}x_{k}}} \) has connected components of the form \( \bullet \) and \( \circ \) and \( \circ \), in addition to \( \bullet \) and \( \circ \) depending on the parameters \( q, r, s \).

Finally, we consider case (ii) and note that in this case \( w = v + 2 \). Assume that \( t_{1} = (1, 2) \cdots (2q - 1, 2q) \), \( t_{j} = (2q + 1, 2q + 2) \cdots (v + 1)(v + 1) \), \( t_{k} = (v + 2)(v + 3) \), \( x_{i} = (1)(2, 3) \cdots (2q - 2, 2q - 1)(2q) \), \( x_{j} = (2q + 1)(2q + 2, 2q + 3) \cdots (v + 1) \) and \( x_{k} = (v + 2, v + 3) \). If \( q = r = 1 \), then \( x = (1)(2)(3)(4, 5)(6, 7) \), and defining

\[
y_{ijk} = (1, 3)(2)(4)(5, 6)(7)
\]

we see that the \( x \)-graphs \( G_{y_{ijk}}^{t_{ij}l_{jk}} \) and \( G_{y_{ijk}}^{s_{x_{i}x_{j}x_{k}}} \) are of isomorphism type

\[
\begin{array}{c}
\bullet \quad \circ \quad \circ \\
\end{array}
\]

and

\[
\begin{array}{c}
\bullet \quad \circ \quad \circ \quad \bullet \quad \circ \\
\end{array}
\]

respectively. Meanwhile, if \( q = 1 \) and \( r > 1 \), then \( x = (1)(2)(3)(4, 5) \cdots (v, v + 1)(v + 2, v + 3) \), and so setting

\[
y_{ijk} = (1)(2, v - 3) \cdots (r, r + 1)(v - 2)(v - 1, v + 2)(v, v + 1)(v + 3)
\]

results in the \( x \)-graph \( G_{y_{ijk}}^{t_{ij}l_{jk}} \) having connected components \( \circ \) and possibly \( \bullet \) and \( \circ \). Moreover, \( G_{y_{ijk}}^{s_{x_{i}x_{j}x_{k}}} \) has connected components \( \circ \), \( \circ \) and for some values of \( r \) also \( \bullet \) and/or \( \bullet \).
If $q = 2$, then we take
\[ y_{ijk} = (1, v + 1)(2, v + 2)(3)(4)(2q + 1, v) \cdots (2q + r, 2q + r + 1)(v + 3), \]
whilst if $q > 2$ we take
\[ y_{ijk} = (1, v + 1)(2, v + 2)(3)(4, 2q - 1) \cdots (q + 1, q + 2)(2q) \\
(2q + 1, v) \cdots (2q + r, 2q + r + 1)(v + 3). \]

Consequently, the $x$-graph $G_{y_{ijk}}^{x_{ij}x_{jk}}$ has connected components $\circ\bullet\circ$ and $\circ$
and possibly also $\bullet, \bullet\bullet, \bullet\bullet\bullet$ and $\bullet$. Meanwhile, $G_{x_{ij}x_{jk}}^{y_{ijk}}$ has connected components $\circ\bullet\circ, \bullet, \bullet$ and $\circ$ and in some cases also $\bullet\bullet\bullet$ and/or $\bullet$.

Lemma 10 Suppose that $m \geq 2, n \geq 7$ with $n \neq 2m + 1$ and let $x \in X \setminus \{t\}$ be such that $\text{fix}(t) = \text{fix}(x)$. Then there exists $y \in X$ such that $d(t, y) = d(y, x) = 1$.

Proof By considering $t, x \in \text{Sym}(\text{supp}(t))$ and appealing to Corollary 2 we may assume that $m = 2$ or $3$ and hence that $|\text{fix}(t)| \geq 2$. If $G_x$ contains a connected component of the form $\bullet\bullet\bullet$, then without loss of generality we have
\[ t = (1, 2)(3, 4)(5, 6)(7) \cdots (n) \quad \text{and} \quad x = (1, 3)(2, 4)(5, 6)(7) \cdots (n) \]
(where the transposition $(5, 6)$ is replaced by $(5)(6)$ if $m = 2$). We take $y = (1, 4)(2)(3)(5, 6)(7) \cdots (n - 2)(n - 1, n)$ (again replacing $(5, 6)$ by $(5)(6)$ if $m = 2$). If $m = 3$ and $G_x$ contains a cycle of three black vertices, then we may assume that
\[ t = (1, 2)(3, 4)(5, 6)(7) \cdots (n) \quad \text{and} \quad x = (1, 6)(2, 3)(4, 5)(7) \cdots (n). \]

In this case, we set $y = (1)(2, 3)(4)(5, 6)(7, 8)(9) \cdots (n)$. In all cases we have that $G_y$ has one connected component of the form $\bullet\bullet\bullet$, $G_x^{y}$ has one connected component of the form $\circ\bullet\circ\circ$ and all other connected components of these $x$-graphs are of the form $\circ, \bullet, \circ\circ\circ$ and $\bullet$. Thus $d(t, y) = d(y, x) = 1$ by Corollary 1.

We are now in a position to prove Theorem 1 in the general case. For $x \in X$ we proceed by considering collections of connected components $\{C_i\}_{i \in I}$ of $G_x$ for some set $I$, and then finding an element $y_I \in \text{Sym}(\bigcup_{i \in I} \text{supp}(C_i))$ that is conjugate to $t_I := \prod_{i \in I} t_i$ such that the connected components of $G_y^{t_I}$ and $G_x^{y_I}$ satisfy the conditions of Corollary 1 (where $x_I := \prod_{i \in I} x_i$). The product of all such $y_I$ will then be our desired element of $X$.

Proof of Theorem 1 Let $x \in X$. 

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First assume that $n = 2m + 1$. We observe that the product of two elements of $X$ that fix distinct elements of $\Omega$ cannot have order 4. Thus $P_{[4]}(G, X)$ consists of $n$ copies of the $[4]$-product involution graph $P_{[4]}(\text{Sym}(2m), Y)$, where $Y$ is the conjugacy class of $\text{Sym}(2m)$ consisting of elements of cycle type $2^m$.

If $m = 1$, then $P_{[4]}(G, X)$ is clearly totally disconnected.

In the case that $(n, m) = (4, 2)$ [respectively $(n, m) = (6, 3)$], then any $x$-graph will contain 2 (respectively 3) black vertices and 2 (respectively 3) edges. It follows from Corollary 1 that $P_{[4]}(G, X)$ is totally disconnected.

(ii) Assume that $m \neq 1$ and $(n, m) \neq (4, 2), (6, 3)$ or $(2m+1, m)$. We first consider the case that $(n, m) = (6, 2)$. If distinct involutions $t = (1, 2)(3, 4)$ and $x$ of cycle type $2^2$ do not have product of order 4, then the reader may check that the $x$-graph $G_x$ will be isomorphic to one given in Table 1 and that the given element $y$ satisfies $d(t, y) = d(y, x) = 1$. Hence $P_{[4]}(G, X)$ is connected and $\text{Diam}(P_{[4]}(G, X)) = 2$.

By Corollary 2, as $n \neq 2m + 1$, we may assume that $|\text{fix}(t)| \geq 2$, and so $G_x$ contains at least 2 white vertices. Moreover, by Lemma 10 we only need to consider the case when $\text{fix}(t) \neq \text{fix}(x)$. Let $\alpha, \beta, \gamma$ and $\delta$ denote the number of connected components (containing at least 1 black vertex and 1 edge) of $G_x$ of the form

![Diagram](https://via.placeholder.com/150)

and let $\epsilon$ denote the number of connected components of the form $\bigcirc \bigcirc$. For ease of reading, we shall refer to components of type $\alpha$ instead of components of the form $\bigcirc \bigcirc \bigcirc \bigcirc$. Similarly for $\beta, \gamma, \delta$ and $\epsilon$. Note that $\alpha \leq \beta + \epsilon$, and as $\text{fix}(t) \neq \text{fix}(x)$ it follows that $\beta, \gamma$ and $\epsilon$ are not all zero.

If $\gamma \geq 2$, then partitioning the components of type $\gamma$ into pairs or triples we obtain a suitable $y_J$ from Lemmas 8(v) and 9(i). Indexing the remaining connected components by $J$, a suitable $y_J$ such that $t_J y_J$ and $x_J y_J$ have orders 1, 2 or 4 may be constructed using Lemmas 4, 5, and 8(i),(ii). In the forthcoming cases, when referring to the construction of $y_J$, it will be implicit that $t_J y_J$ and $x_J y_J$ have orders 1, 2 or 4.
If \( \gamma = 1 \) and \( \beta \neq 0 \), then we pair the unique component of type \( \gamma \) with one of type \( \beta \) to obtain an element \( y_I \) via Lemma 8(iv). An element \( y_J \) for the remaining components follows from Lemmas 4, 5 and 8(i),(ii).

If \( \gamma = 1, \beta = 0 \) and \( \alpha \geq 1 \), then \( \epsilon \geq 1 \). Hence we may use Lemma 9(ii) to construct the element \( y_I \) and Lemmas 4, 5 and 8(i) to obtain a suitable \( y_J \).

If \( \gamma = 1 \) and \( \alpha = \beta = 0 \), then there must be a connected component of \( G_x \) consisting of a single vertex. Assume first that the connected component of type \( \gamma \) contains at least two black vertices. If there is an isolated white vertex in \( G_x \), then the existence of \( y_I \) follows from Lemma 8(iii). Conversely, if there is an isolated black vertex, then - as the number of black vertices equals the number of edges - there must be a connected component of type \( \epsilon \). Applying Lemma 9(ii) to this connected component, the connected component of type \( \gamma \) and an isolated black vertex results in our element \( y_I \). Applying Lemmas 4 and 5 to our remaining components as appropriate gives our desired element \( y_J \).

Now assume that our connected component of type \( \gamma \) contains precisely one black vertex. If all other white vertices are isolated, then \( G_x \) contains a connected component consisting of a cycle of \( u \geq 1 \) black vertices. We may consider one such cycle, an isolated white vertex and the connected component of type \( \gamma \) to correspond to those components indexed by \( I \). Thus without loss of generality

\[
\begin{align*}
t_I &= (1, 2) \cdots (2u - 1, 2u) (2u + 1, 2u + 2) (2u + 3) (2u + 4); \\
x_I &= (1, 2u) (2, 3) \cdots (2u - 2, 2u - 1) (2u + 1) (2u + 2, 2u + 3) (2u + 4),
\end{align*}
\]

unless \( u = 1 \) when we let \( t_I = (1, 2) (3, 4) (5, 6) \) and \( x_I = (1, 2) (3) (4, 5) (6) \). Taking \( y_I = (1)(2u)(2u - 1) \cdots (u, u + 1)(2u + 1, 2u + 3)(2u + 2, 2u + 4) \) (or \( y_I = (1)(2)(3, 5)(4, 6) \) if \( u = 1 \)) it follows that \( G_{x_I}^{t_I} \) has one connected component of the form \( \bullet \cdots \bullet \circ \), \( G_{x_I}^{x_I} \) has one connected component of the form \( \bullet \cdots \bullet \) and

\[
\begin{align*}
\text{the remaining components of these } x \text{-graphs are of the form } \bullet, \bullet, \bullet, \circ, \circ \text{ and } \bullet \cdots \bullet,
\end{align*}
\]

thus satisfying the conditions of Corollary 1.

Conversely, if there exists a white vertex that is not isolated, then it will be in a component of type \( \epsilon \). Again, as the number of edges and black vertices must be equal, there exists an isolated black vertex. We take the connected component of type \( \gamma \) along with one of type \( \epsilon \) and an isolated black vertex to be those indexed by \( I \). The existence of \( y_I \) then follows from Lemma 9(ii). Finally applying Lemmas 4 and 5 as appropriate to the remaining connected components, we obtain an element \( y_J \) as required.

If \( \gamma = 0 \), but \( \beta \neq 0 \), then consider the connected components of type \( \alpha \). If there exist connected components of type \( \alpha \) containing at least 3 black vertices, then we may pair these up with connected components of type \( \beta \) and \( \epsilon \) and apply Lemma 8(i) to obtain our element \( y_I \). If all connected components of type \( \alpha \) contain at most 2 black vertices, then we simply apply Lemma 8(ii) (if required) to the connected components of type \( \beta \) to obtain \( y_I \). Finally, applying Lemmas 4 and 5 to the remaining connected components ensures the existence of \( y_J \).
If $\beta = \gamma = 0$ and $\alpha \neq 0$, then we apply Lemma 8(i) to the connected components of type $\alpha$ and $\epsilon$ (if required) to obtain $y_I$ and Lemmas 4 and 5 to the remaining connected components to find a suitable $y_J$.

If $\alpha = \beta = \gamma = 0$, then $\epsilon \geq 1$ as by assumption $\text{fix}(t) \neq \text{fix}(x)$. As the number of edges of $G_x$ equals the number of black vertices, there exists an isolated black vertex. Moreover, as $m \geq 2$, there are two possible cases. If every black vertex is isolated, then there exists $m$ connected components of type $\epsilon$. Take two such components and two isolated black vertices as the components corresponding to our indexing set $I$, and leave all other components of type $\epsilon$ alone. Without loss of generality, we may assume that the parts of $t$ and $x$ corresponding to $I$ are

\[ t_I = (1, 2)(3, 4)(5)(6)(7)(8) \quad \text{and} \quad x_I = (1)(2)(3)(4)(5, 6)(7, 8) . \]

Let $y_I = (1, 5)(2, 6)(3)(4)(7)(8)$. Then the connected components of $G_{y_I}^U$ and $G_{x_I}^V$ satisfy the conditions of Corollary 1.

Conversely, if there is only one isolated black vertex, then there exists a connected component which is a cycle of $u \geq 1$ black vertices. Thus taking the components indexed by $I$ to be an isolated black vertex, a cycle of $u \geq 1$ black vertices and a component of type $\epsilon$, and leaving all other components of type $\epsilon$ alone, we may take $t_I$ and $x_I$ to be

\[ t_I = (1, 2)\cdots(2u - 1, 2u)(2u + 1, 2u + 2)(2u + 3)(2u + 4); \quad \text{and} \quad x_I = (1, 2u)(2, 3)\cdots(2u - 2, 2u - 1)(2u + 1)(2u + 2)(2u + 3, 2u + 4), \]

or $t_I = (1, 2)(3, 4)(5)(6)$ and $x_I = (1, 2)(3)(4)(5, 6)$ if $u = 1$. Taking $y_I = (1)(2u)(2, 2u - 1)\cdots(u, u + 1)(2u + 1, 2u + 3)(2u + 2, 2u + 4)$ (or if $u = 1$, taking $y_I = (1)(2)(3, 5)(4, 6)$), we see that the connected components of $G_{y_I}^U$ and $G_{x_I}^V$ satisfy the conditions of Corollary 1. Finally, applying Lemmas 4 and 5 to the remaining components of type $\delta$ (that is cycles of black vertices) gives the desired $y_J$.

Since all possible $x$-graphs have been analysed, this completes the proof of Theorem 1. \hfill \Box

4 The Cases $\pi \neq \{4\}$

We illustrate the exceptional nature of $P_{[4]}(G, X)$ with a brief exploration of other $\pi$-product involution graphs. We begin by considering the case that $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geq 3$. The simplest such case arises when $a = 3$. Thus for $G = \text{Sym}(n)$, we consider the $G$-conjugacy class of $t = (1, 2)(3, 4)(5, 6)(7, 8)$, which we denote by $X$. As supp$(t)$ has size 8, it suffices to consider $8 \leq n \leq 16$. We calculate the sizes of the discs $\Delta_4(t)$ of $P_{[8]}(G, X)$ using the computer algebra package MAGMA (see [7,11]). Theorem 3 is an immediate consequence of our calculations, which are summarised in Table 2.

The situation observed is indicative of the situation which arises when $\pi = \{2m\}$ and $2m = 2^a$ for some $a \geq 3$. This leads to the formulation of Theorem 2, a proof of which we now give.
Table 2  The sizes of the discs $\Delta_i(t)$ for $P_{[8]}(G, X)$, where $G := \text{Sym}(n)$ and $X$ is the $G$-conjugacy class of $t := (1, 2)(3, 4)(5, 6)(7, 8)$

| $n$ | $|\Delta_1(t)|$ | $|\Delta_2(t)|$ | $|\Delta_3(t)|$ | $|\Delta_4(t)|$ | $|X|$ | Diam($P_{[8]}(G, X)$) |
|-----|-----------------|-----------------|-----------------|-----------------|-----|-----------------|
| 8   |                 |                 |                 |                 | 105 | Totally disconnected |
| 9   |                 |                 |                 |                 | 945 | Totally disconnected |
| 10  | 384             | 4308            | 32              |                 | 4725| 3                |
| 11  | 1152            | 16,076          | 96              |                 | 17,325| 3               |
| 12  | 2304            | 49,382          | 288             |                 | 51,975| 3               |
| 13  | 3840            | 123,974         | 7320            |                 | 135,135| 3              |
| 14  | 5760            | 267,014         | 42,540          |                 | 315,315| 3              |
| 15  | 8064            | 512,630         | 154,140         | 840             | 675,675| 4               |
| 16  | 10,752          | 902,012         | 431,760         | 6825            | 1,351,350| 4            |

Proof of Theorem 2 (i) We note that as the support of $t$ has size $2m$, any $x$ in $X$—the $G$-conjugacy class of $t$—will have an $x$-graph, $G_x$, containing $m$ black vertices. If the order of $tx$ is $2m$, then $G_x$ must be of the form

\begin{equation}
(1) \quad \includegraphics{fig1}
\end{equation}

(2) \quad \includegraphics{fig2}

(where additional isolated white vertices may be present). In both cases, we see that $|\text{supp}(t) \cap \text{supp}(x)| = 2m - 2$. It follows that if $n = 2m$ or $2m + 1$, then $P_{[2m]}(G, X)$ is totally disconnected. Thus assume that $n \geq 2m + 2$. Denote the $m$ transpositions that $x$ is comprised of by $x_1, \ldots, x_m$, where $\min \text{supp}(x_i) \leq \min \text{supp}(x_{i+1})$ for all $i = 1, \ldots, m - 1$. We will construct elements $y_i \in X$ for $i = 1, \ldots, m$ such that

\[ y_i = x_1 \cdots x_i w_{i+1} \cdots w_m \quad (1) \]

for some transpositions $w_{i+1}, \ldots, w_m$ and such that $y_i$ is connected to $t$ in $P_{[2m]}(G, X)$. At each stage, if $y_i$ involves the transposition $x_{i+1}$ we are done, so we will assume that this is not the case.

Assume first that $|\text{supp}(x_1) \cap \text{supp}(t)| = 2$. If every transposition of $x$ satisfying $|\text{supp}(x_i) \cap \text{supp}(t)| = 2$ also feature in $t$, then by relabelling the elements of $\text{supp}(t)$ we may assume that $x_1$ is a transposition of $t$. Hence without loss we may assume that $x_1 = (1, 3)$. Define

\[ y_1 = (1, 3)(4, 5)(6, 7) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2). \]

Thus $G_{y_1}$ has the form (2). If $|\text{supp}(x_1) \cap \text{supp}(t)| = 1$, then we may assume that $x_1 = (1, 2m + 1)$. Taking

\[ y_1 = (1, 2m + 1)(2, 3)(4, 5) \cdots (2m - 4, 2q - 3)(2m - 2, 2m + 2), \]
it follows that $\mathcal{G}_{y_1}$ is isomorphic to (1). Finally, if $|\text{supp}(x_1) \cap \text{supp}(t)| = 0$, then we assume that $x_1 = (2m + 1, 2m + 2)$. The element

$$y_1 = (2, 3)(4, 5) \cdots (2m - 2, 2m - 1)(2m + 1, 2m + 2)$$

results in an x-graph $\mathcal{G}_{y_1}$ of type (2). In all cases we see that $t$ and $y_1$ are adjacent and hence connected vertices of $\mathcal{P}_{[2m]}(G, X)$.

Suppose that for some $1 \leq i < m$ an element $y_i$ of the form (1) exists with $y_i$ connected to $t$ in $\mathcal{P}_{[2m]}(G, X)$. Note that $y_i$ fixes at least two elements of $\Omega$ as $n \geq 2m + 2$. We denote these elements by $f_1, f_2$. Define

$$x_{j,1} := \min \text{supp}(x_j) \quad \text{and} \quad x_{j,2} := \max \text{supp}(x_j). \quad (2)$$

for $1 \leq j \leq i$ and

$$w_{j,1} := \min \text{supp}(w_j) \quad \text{and} \quad w_{j,2} := \max \text{supp}(w_j). \quad (3)$$

for $i + 1 \leq j \leq m$. Set $\alpha = x_{i+1,1}$, $\beta = x_{i+1,2}$ and $w = w_{i+1} \cdots w_m$. Thus $x_{i+1} = (\alpha, \beta)$.

We follow an analogous approach to that used to define $y_1$. If $|\text{supp}(x_{i+1}) \cap \text{supp}(w)| = 2$, then without loss we have that $w_{i+1,1} = \alpha$ and $w_{i+2,1} = \beta$. First we construct an element $z_{i+1} \in X$ given by

$$z_{i+1} = (x_{i,1}, w_{m,2})(x_{i,1}, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1})$$

$$\quad (x_{i,2}, \alpha)(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-1,2}, w_{m,1})(f_1, f_2)$$

for $i > 1$ and

$$z_2 = (x_{1,1}, w_{m,2})(x_{1,2}, \alpha)(w_{3,2}, w_{4,1}) \cdots (w_{m-1,2}, w_{m,1})(f_1, f_2).$$

Consequently $\mathcal{G}_{z_{i+1}}^{y_i}$ has the form given in (2). The element

$$y_{i+1} = x_1x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2})w_{i+3} \cdots w_m$$

results in an x-graph $\mathcal{G}_{y_{i+1}}^{z_{i+1}}$ of isomorphism type (1). We deduce that $y_{i+1}$ is connected to $y_i$ and hence to $t$, and that $d(y_i, y_{i+1}) \leq 2$.

In the case that $|\text{supp}(x_{i+1}) \cap \text{supp}(w)| = 1$, we may assume that $w_{i+1,1} = \alpha$ and that $\beta = f_1 \in \text{fix}(y_i)$. Taking $z_2 = (\beta, x_{2,1})(x_{1,2}, w_{m,2})(w_{2,2}, w_{3,1}) \cdots (w_{m-1,2}, w_{m,1})(\alpha, f_2)$,

$$z_{i+1} = (\beta, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{m,2})$$

$$\quad (w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-1,2}, w_{m,1})(\alpha, f_2)$$

for $i > 1$ and

$$y_{i+1} = x_1 \cdots x_{i+1}w_{i+2} \cdots w_m$$
we see that $G_{z_{\ell_1+1}}^{z_{\ell_1}}$ is of isomorphism type (1), whilst $G_{x_{i+1}}^{x_i}$ is of isomorphism type (2). Thus $d(y_i, y_{i+1}) \leq 2$ and $y_{i+1}$ is connected to $t$.

The final possibility is that $|\text{supp}(x_{i+1}) \cap \text{supp}(w)| = 0$. Consequently, defining $z_2 = (x_{1,1}, x_{1,2})(x_{1,2}, w_{m,2})(x_{2,2}, w_{3,1})(w_{3,2}, w_{4,1}) \cdots (w_{m-1,2}, w_{m,1}),$

$$z_{i+1} = (x_{1,1}, x_{i+1,1})(x_{1,2}, x_{2,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{m,2})$$

$$(x_{i+1,2}, w_{i+2,2})(w_{i+2,2}, w_{i+3,1}) \cdots (w_{m-1,2}, w_{m,1})$$

for $i > 1$ and

$$y_{i+1} = x_1 \cdots x_{i+1} w_{i+1} w_{i+2} \cdots w_{m-1}$$

we obtain $x$-graphs $G_{z_{\ell_1+1}}^{z_{\ell_1}}$ and $G_{x_{i+1}}^{x_i}$ of types (1) and (2) respectively. We conclude that $d(y_i, y_{i+1}) \leq 2$, and hence $y_{i+1}$ is connected to $t$ in $P_{\{2m\}}(G, X)$ as required.

(ii) If $P_{\{2m\}}(G, X)$ is connected, then $n \geq 2m + 2$ by (i). Moreover, the above argument shows that $d(t, y_1) \leq 1$ and for $1 \leq i \leq m - 1$ we have $d(y_i, y_{i+1}) \leq 2$. Thus as $x = y_m$, we conclude that $\text{Diam}(P_{\{2m\}}(G, X)) \leq 2m - 1$. For the lower bound, we note that $x \in \Delta_1(t)$ precisely when $G_x$ is of type (1) or (2). In particular $|\text{supp}(t) \cap \text{supp}(x)| = 2m - 2$. Arguing iteratively we deduce that if $d(t, x) \leq s$, then $|\text{supp}(t) \cap \text{supp}(x)| \geq 2m - 2s$. Since $X$ contains an involution $x$ satisfying $|\text{supp}(t) \cap \text{supp}(x)| = \max \{0, 4m - n\}$ we deduce that

$$\text{Diam}(P_{\{2m\}}(G, X)) \geq \min \{m, [n/2 - m]\}$$

as required. □

We now consider Theorem 4. A non-constructive proof of the connectivity of $P_{\{q\}}(G, X)$ using Jordan’s theorem is contained in the proof of [6, Theorem 4.1]. Here we give a constructive proof in a similar vein to the proof of Theorem 2 above.

**Proof of Theorem 4** (i) We first note that the elements of the disc $\Delta_1(t)$ are precisely those elements $x \in X$ whose $x$-graph, $G_x$, is of isomorphism type $\bullet \rightarrow \rightarrow \cdots \rightarrow \circ$. Consequently, $P_{\{q\}}(G, X)$ is totally disconnected if $n = q - 1$. Thus assume that $n \geq q$ and hence that $|\text{fix}(t)| \geq 1$.

We proceed as in the proof of Theorem 2 and set $2m = q - 1$. Let $x \in X$ be given and denote the transpositions of $x$ as $x = x_1 x_2 \cdots x_m$. As in the proof of Theorem 2, we will construct elements $y_i \in X$ for $i = 1, \ldots m$ such that

$$y_i = x_1 \cdots x_i w_{i+1} \cdots w_m$$

for some transpositions $w_j$ and such that $y_i$ is connected to $t$ in $P_{\{q\}}(G, X)$. Mirroring the situation of the proof of Theorem 2, we may assume that $x_{i+1}$ is not a transposition of $y_i$. We continue to use the notation $x_{j,1}, x_{j,2}$ and $w_{j,1}, w_{j,2}$ previously introduced in (2) and (3) respectively. For convenience we define $y_0 := t$ and for each $y_i$ we consider the cases $\text{supp}(y_i) = \text{supp}(x)$ and $\text{supp}(y_i) \neq \text{supp}(x)$ separately.
Assume that $\text{supp}(t) = \text{supp}(x)$. Thus $|\text{supp}(t) \cap \text{supp}(x)| = 2$. If every transposition of $x$ satisfying $|\text{supp}(x_i) \cap \text{supp}(t)| = 2$ also feature in $t$, then by relabelling the elements of $\text{supp}(t)$ we may assume that $x_i$ is a transposition of $t$. Hence without loss we may take $x_1 = (1, 3)$. Define $z_1$ and $y_1$ by

\[ z_1 = (2, 3)(4, 5) \cdots (2m, 2m + 1); \quad \text{and} \quad y_1 = x_1(2, 4)(5, 6) \cdots (2m - 1, 2m). \]

The $x$-graphs $G_{z_1}$ and $G_{y_1}^{z_1}$ are both of isomorphism type $\begin{array}{cccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}$ and hence $t$ and $y_1$ are connected in $\mathcal{P}_{\{q\}}(G, X)$ with $d(t, y_1) \leq 2$.

When $\text{supp}(t) \neq \text{supp}(x)$ we consider three subcases. First, suppose that $|\text{supp}(t) \cap \text{supp}(x_1)| = 2$, then without loss we have $x_1 = (1, 3)$ and we take

\[ y_1 = x_1(4, 5)(6, 7) \cdots (2m, 2m + 1). \]

If $|\text{supp}(t) \cap \text{supp}(x_1)| = 1$, then we may assume that $x_1 = (1, 2m + 1)$ and thus take

\[ y_1 = x_1(2, 3)(4, 5) \cdots (2m - 2, 2m - 1). \]

In both cases, $G_{y_1}$ is of the required form. Consequently $t$ and $y_1$ are adjacent in $\mathcal{P}_{\{q\}}(G, X)$. Finally, suppose that $|\text{supp}(t) \cap \text{supp}(x_1)| = 0$ and hence that $x_1 = (2m + 1, 2m + 2)$. Defining

\[ z_1 = (1, 2m + 1)(2, 3) \cdots (2m - 2, 2m - 1); \quad \text{and} \quad y_1 = x_1(1, 2)(3, 4) \cdots (2m - 1, 2m - 2), \]

we have that $G_{z_1}$ and $G_{y_1}^{z_1}$ are of the aforementioned isomorphism type, and so $t$ and $y_1$ are connected in $\mathcal{P}_{\{q\}}(G, X)$. Moreover, $d(t, y_1) \leq 2$.

Now suppose that a $y_i$ of the form (4) has been defined for some $i < m$ with $y_i$ connected to $t$ in $\mathcal{P}_{\{q\}}(G, X)$. First assume that $\text{supp}(y_i) = \text{supp}(x)$ and that $\alpha \in \text{fix}(y_i) \cap \text{fix}(x)$. Without loss we may assume that $x_{i+1,1} = w_{i+1,1}$ and $x_{i+1,2} = w_{i+2,1}$. Define $z_2 = (x_1, 2, w_2, 1)(w_2, 2, w_3, 1) \cdots (w_{m-1,2}, w_{m,1})(w_{m,2}, \alpha)$,

\[ z_{i+1} = (x_{i,2}, x_{2,1})(x_{2,2}, x_{3,1}) \cdots (x_{i-1,2}, x_{i,1})(x_{i,2}, w_{i+1,1}) \]

\[ (w_{i+1,2}, w_{i+2,1}) \cdots (w_{m-1,2}, w_{m,1})(w_{m,2}, \alpha) \]

for $i > 1$, and

\[ y_{i+1} = x_1 x_2 \cdots x_{i+1}(w_{i+1,2}, w_{i+2,2})w_{i+3} \cdots w_m. \]

We have that the $x$-graphs $G_{z_{i+1}}^{y_i}$ and $G_{y_{i+1}}^{z_{i+1}}$ have the required form and hence $d(y_i, y_{i+1}) \leq 2$.

It remains to consider the case that $\text{supp}(y_i) \neq \text{supp}(x)$. Let $f_i \in \text{fix}(y_i) \setminus \text{fix}(x)$. In the case when $|\text{supp}(y_i) \cap \text{supp}(x_{i+1})| = 2$, assume that $x_{i+1} = (w_{i+1,1}, w_{i+2,1})$ and set $z_2 = (x_{1,1}, w_{m,2})(x_{1,2}, w_{2,1})(w_{3,1}, f_i)(w_{3,2}, w_{4,1}) \cdots (w_{m-1,2}, w_{m,1})$.
\[
z_{i+1} = (x_{1,1}, w_{m, 2}) (x_{1,2}, x_{2,1}) \cdots (x_{i-1, 2}, x_{i, 1}) (x_{i, 2}, w_{i+1, 1}) (w_{i+2, 1}, f_i) \\
(w_{i+2, 2}, w_{i+3, 1}) \cdots (w_{m-1, 2}, w_{m, 1})
\]
for \(i > 1\), and
\[
y_{i+1} = x_1 x_2 \cdots x_{i+1} (w_{i+1, 2}, w_{i+2, 2}) w_{i+3} \cdots w_m.
\]
If \(|\supp(y_i) \cap \supp(x_{i+1})| = 1\), then without loss we have \(x_{i+1} = (w_{i+1, 1}, f_i)\). Hence we define \(z_2 = (x_{1,1}, w_{m, 2}) (x_{1,2}, x_{2,1}) (w_{3,1}, f_i) (w_{3,2}, w_{4,1}) \cdots (w_{m-1, 2}, w_{m, 1})\),
\[
z_{i+1} = (x_{1,1}, w_{m, 2}) (x_{1,2}, x_{2,1}) \cdots (x_{i-1, 2}, x_{i, 1}) (x_{i, 2}, x_{i+1, 1}) \\
(w_{i+2, 1}, f_i) (w_{i+2, 2}, w_{i+3, 1}) \cdots (w_{m-1, 2}, w_{m, 1})
\]
for \(i > 1\), and
\[
y_{i+1} = x_1 x_2 \cdots x_{i+1} w_{i+2} \cdots w_{m-1} (w_{m, 1}, w_{i+1, 2}).
\]
Finally, if \(x_{i+1}\) and \(y_i\) are disjoint we set \(z_2 = (x_{1,1}, x_{2,1}) (w_{2,1}, w_{2,2}) (w_{3,1}, w_{3,2}) \cdots (w_{m-1, 2}, w_{m, 1})\),
\[
z_{i+1} = (x_{1,1}, x_{i+1, 1}) (x_{1,2}, x_{2,1}) \cdots (x_{i-1, 2}, x_{i, 1}) (x_{i, 2}, w_{i+1, 1}) \\
(w_{i+2, 1}, w_{i+2, 1}) \cdots (w_{m-1, 2}, w_{m, 1})
\]
for \(i > 1\), and
\[
y_{i+1} = x_1 x_2 \cdots x_{i+1} w_{i+1} \cdots w_{m-1}.
\]
For each pair \((z_{i+1}, y_{i+1})\) the \(x\)-graphs \(G_{z_{i+1}}^y\) and \(G_{y_{i+1}}^{z_{i+1}}\) have isomorphism type \(\bullet \cdots \bullet \circ\). Consequently \(d(y_i, y_{i+1}) \leq 2\) and the elements \(t\) and \(y_{i+1}\) are connected in \(P_{[q]}(G, X)\).

(ii) By part (i) we have that \(\text{Diam}P_{[q]}(G, x) \leq q - 1\). For the lower bound, we note that for \(x \in X\) to be adjacent to \(t\) we have \(|\supp(t) \cap \supp(x)| = |\supp(t)| - 1\). Taking \(x \in X\) such that \(|\supp(t) \cap \supp(x)|\) is minimal we have that \(|\supp(t) \cap \supp(x)| = \max\{0, 2q - 2 - n\}\) and hence \(d(t, x) \geq \min\{q - 1, n + 1 - q\}\) as required. \(\square\)

The proofs of Theorems 2 and 4 are utilised in the proof of Theorem 5.

Proof of Theorem 5 The \(x\)-graph \(G_x\) of any \(x \in \Delta_1(t)\) must consist of a connected component of isomorphism type \(\bullet \cdots \bullet \circ\) containing \(q_i/2\) black vertices for each \(i = 2, \ldots, r\). In addition, there will be connected components of types (1) or (2) from the proof of Theorem 2 if \(p_1 = 2\), or a component of type \(\bullet \cdots \bullet \circ\) containing \(q_1/2\) black vertices if \(p_1 \neq 2\). It follows that
\[
|\text{fix}(t)| \geq \begin{cases} r + 1 & \text{if } p_1 = 2; \\ r & \text{otherwise.} \end{cases}
\]
We conclude that if

\[ n \geq \begin{cases} 
q + 2 & \text{if } p_1 = 2; \\
q & \text{otherwise} 
\end{cases} \]

(6)
does not hold, then \( P_{\{q\}}(G, X) \) is totally disconnected.

Conversely, assume that (6) holds, and denote the connected component of \( P_{\{q\}}(G, X) \) containing \( t \) by \( P_t \). By arranging the fixed points of \( t \) appropriately we may consider \( t = t_1 t_2 \cdots t_r \in \text{Sym}(p_1^{a_1}) \times \text{Sym}(p_2^{a_2}) \times \cdots \times \text{Sym}(p_r^{a_r}) \), where \( t_i \in \text{Sym}(p_i^{a_i}) \). Applying Theorems 2 and 4, we see that \( \text{Sym}(p_1^{a_1}) \times \text{Sym}(p_2^{a_2}) \times \cdots \times \text{Sym}(p_r^{a_r}) \) is a subgroup of \( \text{Stab}_G(P_t) \). However, we may also interchange the fixed points of the \( t_i \) to obtain different copies of \( \text{Sym}(p_1^{a_1}) \times \text{Sym}(p_2^{a_2}) \times \cdots \times \text{Sym}(p_r^{a_r}) \) contained in \( \text{Stab}_G(P_t) \). Combining all such subgroups, it follows that \( \text{Stab}_G(P_t) = G \) and hence \( P_{\{q\}}(G, X) \) is connected. \( \square \)

We briefly consider an example of a case when \( \pi \) consists of a composite number. The smallest such situation arises when \( \pi = \{6\} \). If \( t \in G \) is an involution and \( X \) is the \( G \)-conjugacy class of \( t \), then any \( x \in X \) will be adjacent to \( t \) in \( P_{\{6\}}(G, X) \) if the connected components of \( G_x \) consist of components of the form

Table 3 The sizes of the discs \( \Delta_i(t) \) for \( P_{\{6\}}(G, X) \), where \( X \) is the \( G \)-conjugacy class of \( t = (1, 2) \cdots (2m - 1, 2m) \in G := \text{Sym}(n) \)

| \( n \) | \( m \) | \( |\Delta_1(t)| \) | \( |\Delta_2(t)| \) | \( |\Delta_3(t)| \) | \( |X| \) | Diam(\( P_{\{6\}}(G, X) \)) |
|---|---|---|---|---|---|---|
| 6 | 2 | 45 | |
| 3 | 15 | |
| 7 | 2 | 12 | 38 | 54 | 105 | 3 |
| 3 | 12 | 60 | 32 | 105 | 3 |
| 8 | 2 | 48 | 158 | 3 | 210 | 3 |
| 3 | 72 | 347 | 420 | 2 |
| 4 | 105 | |
| 9 | 2 | 120 | 242 | 15 | 378 | 3 |
| 3 | 216 | 1043 | 1260 | 2 |
| 4 | 48 | 836 | 60 | 945 | 3 |
| 10 | 2 | 240 | 389 | 630 | 2 |
| 3 | 624 | 2525 | 3150 | 2 |
| 4 | 416 | 4308 | 4725 | 2 |
| 5 | 160 | 784 | 945 | 2 |

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Moreover, either one component is of isomorphism type (iv), or there exists at least one component of type (ii) and one component of type (iii).

Finally, Table 3 gives the sizes of the discs $\Delta_i(t)$ of $P_{[6]}(G, X)$ for the symmetric groups $G := \text{Sym}(n)$ ($6 \leq n \leq 10$), when $X$ is the $G$-conjugacy class of an involution $t \in G$.

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