On the \(N\)-Solitons Solutions in the Novikov–Veselov Equation

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Abstract. We construct the \(N\)-solitons solution in the Novikov–Veselov equation from the extended Moutard transformation and the Pfaffian structure. Also, the corresponding wave functions are obtained explicitly. As a result, the property characterizing the \(N\)-solitons wave function is proved using the Pfaffian expansion. This property corresponding to the discrete scattering data for \(N\)-solitons solution is obtained in [arXiv:0912.2155] from the \(∂\)-dressing method.

Key words: Novikov–Veselov equation; \(N\)-solitons solutions; Pfaffian expansion; wave functions

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1 Introduction

The Novikov–Veselov equation [3, 9, 34, 41] is defined by

\[
U_t = \partial_z^3 U + \partial_{\bar{z}} U + 3\partial_z (VU) + 3\partial_{\bar{z}} (V^* U),
\]

(1)

\[
\bar{\partial}_z V = \partial_z U, \quad \partial_z V^* = \bar{\partial}_z U.
\]

When \(z = \bar{z} = x\), we get the famous KdV equation \((U = \bar{U} = V = \bar{V})\)

\[
U_t = 2U_{xxx} + 12UU_x.
\]

The equation (1) can be represented as the form of Manakov’s triad [24]

\[
H_t = [A, H] + BH,
\]

where \(H\) is the two-dimension Schrödinger operator

\[
H = \partial_z \bar{\partial}_z + U
\]

and

\[
A = \partial_z^3 + V \partial_z + \bar{V} \bar{\partial}_z, \quad B = V_z + \bar{V}_z.
\]

It is equivalent to the linear representation

\[
H \psi = 0, \quad \partial_t \psi = A \psi.
\]

We see that the Novikov–Veselov equation (1) preserves a class of the purely potential self-adjoint operators \(H\). Here the pure potential means \(H\) has no external electric and magnetic fields. The periodic inverse spectral problem for the two-dimensional Schrödinger operator \(H\)
was investigated in terms of the Riemann surfaces with some group of involutions and the corresponding Prym $\Theta$-functions [5, 10, 22, 27, 28, 33, 37]. On the other hand, it is known that the Novikov–Veselov hierarchy is a special reduction of the two-component BKP hierarchy [23, 36, 40] (and references therein). In [23], the authors showed that the Drinfeld–Sokolov hierarchy of D-type is a reduction of the two-component BKP hierarchy using two different types of pseudo-differential operators, which is different from Shiota’s point of view [37]. Also, in [26], it is shown that the Tzitzeica equation is a stationary symmetry of the Novikov–Veselov equation. Finally, it is worthwhile to notice that the Novikov–Veselov equation \((1)\) is a special reduction of the Davey–Stewartson equation \([20, 21]\).

Let $H\psi = H\omega = 0$. Then via the Moutard transformation \([1, 29, 30, 31]\)

$$U(z, \bar{z}) \rightarrow \hat{U}(z, \bar{z}) = U(z, \bar{z}) + 2\partial \bar{\partial} \ln \omega,$$

$$\psi \rightarrow \theta = \frac{1}{\omega} \int (\psi \partial \omega - \omega \partial \psi) dz - (\psi \bar{\partial} \omega - \omega \bar{\partial} \psi) d\bar{z}, \tag{3}$$

one can construct a new Schrödinger operator $\hat{H} = \partial_z \bar{\partial}_{\bar{z}} + \hat{U}$ and $\hat{H} \theta = 0$. We remark that the Moutard transformation \((3)\) is utilized to construct the $N$-solitons solutions of the Tzitzeica equation \([15]\).

The extended Moutard transformation was established such that $\hat{U}(t, z, \bar{z})$ and $\hat{V}(t, z, \bar{z})$ defined by \([13, 25]\)

$$\hat{U}(t, z, \bar{z}) = U(t, z, \bar{z}) + 2\partial \bar{\partial} \ln W(\psi, \omega), \quad \hat{V}(t, z, \bar{z}) = V(t, z, \bar{z}) + 2\partial \bar{\partial} \ln W(\psi, \omega),$$

where the skew product (alternating bilinear form) $W$ is defined by

$$W(\psi, \omega) = \int (\psi \partial \omega - \omega \partial \psi) dz - (\psi \bar{\partial} \omega - \omega \bar{\partial} \psi) d\bar{z} + \left[\psi \partial^3 \omega - \omega \partial^3 \psi + \omega \bar{\partial}^3 - \psi \bar{\partial}^3 \omega\right] + 2(\partial^2 \psi \partial \omega - \partial \psi \partial^2 \omega) - 2(\bar{\partial}^2 \psi \bar{\partial} \omega - \bar{\partial} \psi \bar{\partial}^2 \omega) + 3V(\psi \partial \omega - \omega \partial \psi) \tag{4}$$

will also satisfy the Novikov–Veselov equation.

In \([2, 6, 7, 8]\), the rational solutions and line solitons of the Novikov–Veselov equation \((1)\) are constructed by the $\bar{\partial}$-dressing method. To get these kinds of solutions, the scattering datum have to be delta-type and the reality of $U$ also puts some extra constraints on them. In \([39]\), the singular rational solutions are obtained using the extended Moutard transformation \((4)\); however, the non-singular rational solutions are constructed in \([4]\).

Next, we construct Pfaffian-type solutions. Given any $N$ wave functions $\psi_1, \psi_2, \psi_3, \ldots, \psi_N$ (or their linear combinations) of \((2)\) for fixed potential $U(z, \bar{z}, t)$, the $N$-step extended Moutard transformation can be obtained in the Pfaffian \([1, 31]\) (also see \([12, 35]\))

$$P(\psi_1, \psi_2, \psi_3, \ldots, \psi_N) = \begin{cases} \text{Pf}(\psi_1, \psi_2, \psi_3, \ldots, \psi_N) & \text{if } N \text{ even}, \\ \text{Pf}(\psi_2, \psi_3, \ldots, \psi_N) & \text{if } N \text{ odd}, \end{cases} \tag{5}$$

$$\text{Pf}(\psi_1, \psi_2, \psi_3, \ldots, \psi_N) = \sum_{\sigma} \epsilon(\sigma) W_{\sigma_1 \sigma_2} W_{\sigma_3 \sigma_4} \cdots W_{\sigma_{N-1} \sigma_N}, \tag{5}$$

$$\text{Pf}(\psi_1, \psi_2, \psi_3, \ldots, \psi_N) = \sum_{\sigma} \epsilon(\sigma) W_{\sigma_1 \sigma_2} W_{\sigma_3 \sigma_4} \cdots W_{\sigma_{N-2} \sigma_{N-1} \sigma_N}, \tag{6}$$

where $W_{\sigma_i \sigma_j} = W(\psi_{\sigma(i)}, \psi_{\sigma(j)})$ is defined by the skew product \((4)\). The summations $\sigma$ in \((5)\) and \((6)\) run from over the permutations of \(\{1, 2, 3, \ldots, N\}\) such that $\sigma_1 < \sigma_2 < \sigma_3 < \sigma_4, \sigma_5 < \sigma_6, \ldots$ and $\sigma_1 < \sigma_3 < \sigma_5 < \sigma_7 < \ldots$, with $\epsilon(\sigma) = 1$ for the even permutations and $\epsilon(\sigma) = -1$ for the odd permutations. Then the solution $U$ and $V$ can be expressed as \([1]\)

$$U = U_0 + 2\partial \bar{\partial} \ln \text{Pf}(\psi_1, \psi_2, \psi_3, \ldots, \psi_N), \quad V = V_0 + 2\partial \bar{\partial} \ln \text{Pf}(\psi_1, \psi_2, \psi_3, \ldots, \psi_N).$$
and the corresponding wave function is
\[ \varphi = \frac{P(\psi_1, \psi_2, \psi_3, \ldots, \psi_N, \vartheta)}{P(\psi_1, \psi_2, \psi_3, \ldots, \psi_N)}, \tag{7} \]
where \( \vartheta \) is an arbitrary wave function different from \( \psi_1, \psi_2, \psi_3, \ldots, \psi_N \).

The paper is organized as follows. In Section 2, we obtain the \( N \)-solitons solutions using the extended Moutard transformation and the Pfaffian expansion. Several examples are given. In Section 3, the \( N \)-solitonic wave function is derived using (7) and the Pfaffian expansion. Section 4 is used to prove a special property to characterize the \( N \)-solitons wave function. Section 5 is devoted to the concluding remarks.

## 2 \( N \)-solitons solutions

In this section, one uses successive iterations of the extended Moutard transformation (4) to construct \( N \)-solitons solutions.

To obtain the \( N \)-solitons solutions, we assume that \( V = 0 \) in (1) and recall that \( \partial \overline{\partial} = \frac{i}{4} \Delta \). One considers \( U = -\epsilon \neq 0 \), i.e.,
\[ \partial \overline{\partial} \varphi = \epsilon \varphi, \quad \varphi_t = \varphi_{zzz} + \varphi_{\overline{\partial} \overline{\partial}}, \]  
where \( \epsilon \) is non-zero real constant. The general solution of (8) can be expressed as
\[ \varphi(z, \overline{z}, t) = \int_{\Gamma} e^{i\lambda z + \frac{i}{4} \overline{\lambda} \overline{z} + \frac{\epsilon^3}{(i\lambda)^3} t} \nu(\lambda) d\lambda, \tag{9} \]
where \( \nu(\lambda) \) is an arbitrary distribution and \( \Gamma \) is an arbitrary path of integration such that the r.h.s. of (9) is well defined.

Next, using (5) and (9), one can construct the \( N \)-solitons solutions. Let’s take \( \nu_m(\lambda) = \delta(\lambda - p_m) \) and \( \nu_n(\lambda) = a_n \delta(\lambda - q_n) \), where \( p_m, a_n, q_n \) are complex numbers. Then one defines
\[ \phi_m = \frac{\varphi(p_m)}{\sqrt{3}} = \frac{1}{\sqrt{3}} e^{F(p_m)}, \quad \psi_n = \frac{a_n \varphi(q_n)}{\sqrt{3}} = \frac{a_n}{\sqrt{3}} e^{F(q_n)}, \]
where
\[ F(\lambda) = (i\lambda)z + (i\lambda)^3 t + \frac{\epsilon}{i\lambda} \overline{z} + \frac{\epsilon^3}{(i\lambda)^3} t. \]

Then a direct calculation of the extended Moutard transformation (4) can yield
\[ W(\phi_m, \psi_n) = ia_n \frac{q_n - p_m}{q_n + p_m} e^{F(p_m) + F(q_n)}, \quad W(\phi_m, \phi_n) = i \frac{p_n - p_m}{p_n + p_m} e^{F(p_m) + F(p_n)}, \quad W(\psi_m, \psi_n) = ia_m a_n \frac{q_n - q_m}{q_n + q_m} e^{F(q_m) + F(q_n)}. \tag{10} \]

The \( N \)-solitons solutions are defined by
\[ U(z, \overline{z}, t) = -\epsilon + 2 \partial \overline{\partial} \ln \tau_N(z, \overline{z}, t), \quad V(z, \overline{z}, t) = 2 \partial \overline{\partial} \ln \tau_N(z, \overline{z}, t), \]
and then
\[ U(z, \overline{z}, t) \rightarrow -\epsilon \quad \text{as} \quad \overline{z} \overline{z} \rightarrow \infty, \]
where \( t \) is fixed. The \( \tau \)-functions are defined as follows. For simplicity, let’s denote

\[
W(p_m, q_n) = W(\phi_m, \psi_n), \quad W(p_m, p_n) = W(\phi_m, \phi_n), \quad W(q_m, q_n) = W(\psi_m, \psi_n),
\]

and notice that \( F(-\lambda) = -F(\lambda) \). The \( \tau_N \) is defined as

\[
\tau_N(z, \bar{z}, t) = \text{Pf}(-p_1, q_1, -p_2, q_2, -p_3, q_3, \ldots, -p_N, q_N),
\]

where

\[
\begin{align*}
(-p_m, -p_n) &= W(-p_m, -p_n), \quad (-p_m, q_n) = W(-p_m, q_n) + \delta_{mn}, \\
(q_m, q_n) &= W(q_m, q_n).
\end{align*}
\]

To get the expansion of (12), we use the following useful formula [14, 38]

\[
\text{Pf}(A + B) = \sum_{r=0}^{s} \sum_{\alpha \in \mathbb{F}_2} (-1)^{|\alpha|-r} \text{Pf}(A_{\alpha}) \text{Pf}(B_{\alpha}),
\]

where \( A \) and \( B \) are \( m \times m \) matrices and \( s = \lfloor m/2 \rfloor \) is the integer part of \( m/2 \); moreover, we denote by \( \alpha^c \) the complementary set of \( \alpha \) in the subset \( \{1, 2, 3, \ldots, m\} \) which is arranged in increasing order, and \( |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_{2r} \) for \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2r}) \). For the case (12), one has

\[
A_N(z, \bar{z}, t) = \begin{bmatrix}
0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) & \cdots & (-p_1, q_N) \\
(q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) & \cdots & (q_1, q_N) \\
(-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) & \cdots & (-p_2, q_N) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
(q_N, -p_1) & (q_N, q_1) & (q_N, -p_2) & (q_N, q_2) & \cdots & 0
\end{bmatrix},
\]

and

\[
B_N = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & -1 & 0
\end{bmatrix},
\]

where \( A_N \) and \( B_N \) are \( 2N \times 2N \) matrices. Hence by (14) one can have the expansion of (12), i.e.,

\[
\tau_N = 1 + \sum_{\ell=1}^{N} f_{\ell} + \sum_{m=2} \left( \sum_{1 \leq \ell_1 < \ell_2 < \cdots < \ell_m \leq N} f_{\ell_1} f_{\ell_2} \cdots f_{\ell_m} \prod_{1 \leq j < k \leq m} \mathbb{P}_{\ell_j \ell_k} \right),
\]

where

\[
f_{\ell} = i a_{\ell} p_{\ell} + \frac{q_{\ell}}{q_{\ell} - p_{\ell}} F(q_{\ell} - F(p_{\ell})), \quad \mathbb{P}_{\ell_j \ell_k} = \frac{(p_{\ell_j} - p_{\ell_k})(q_{\ell_j} - q_{\ell_k})(p_{\ell_j} + q_{\ell_k})(q_{\ell_j} + p_{\ell_k})}{(p_{\ell_j} + p_{\ell_k})(q_{\ell_j} + q_{\ell_k})(p_{\ell_j} - q_{\ell_k})(q_{\ell_j} - p_{\ell_k})}.
\]

Here we have utilized the formula that if \( C \) is a \( 2N \times 2N \) matrix with \((i, j)\)-th entry \( \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \), then one has the Schur identity [32, 36]

\[
\text{Pf}(C) = \prod_{1 \leq i < j \leq 2N} \left( \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right).
\]
Next, we illustrate the formula (15) (or 12) with several examples.

(1) One-soliton solution:

\[
A_1(z, \bar{z}, t) = \begin{bmatrix}
0 & (-p_1, q_1) \\
(q_1, -p_1) & 0
\end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

Then

\[
\tau_1 = Pf(A_1 + B_1) = 1 + ia_1 \frac{q_1 + p_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}.
\]

(2) Two-solitons solution:

\[
A_2(z, \bar{z}, t) = \begin{bmatrix}
0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) \\
(q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) \\
(-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) \\
(q_2, -p_1) & (q_2, q_1) & (q_2, -p_2) & 0
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix}.
\]

Then

\[
\tau_2 = Pf(A_2 + B_2) = 1 + ia_1 \frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)} + ia_2 \frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)}
+ ia_1 a_2 \frac{p_1 + q_1 p_2 + q_2 - p_1 q_2 - q_1 p_1 + q_2 p_2 + q_1 e^{F(q_1) - F(p_1)} + F(q_2) - F(p_2)}{q_1 - p_1 q_2 - p_2 + p_1 q_2 + q_1 - p_2 - q_1 p_1 - q_2}
\]

or

\[
\tau_2 = 1 + f_1 + f_2 + P_{12} f_1 f_2,
\]

where

\[
f_1 = ia_1 \frac{p_1 + q_1}{q_1 - p_1} e^{F(q_1) - F(p_1)}, \quad f_2 = ia_2 \frac{p_2 + q_2}{q_2 - p_2} e^{F(q_2) - F(p_2)},
\]

\[
P_{12} = \frac{p_1 - p_2 - q_1 - q_2 + q_1 + p_2 + q_1 p_1 - q_2 q_1 - p_2}{p_1 + p_2 q_1 + q_2 p_1 - q_2 q_1 - p_2}.
\]

The \( \tau_2 \) soliton (17) is also found in [2] using the \( \bar{\partial} \)-dressing method.

(3) Three-solitons solution:

\[
A_3(z, \bar{z}, t) = \begin{bmatrix}
0 & (-p_1, q_1) & (-p_1, -p_2) & (-p_1, q_2) & (-p_1, -p_3) & (-p_1, q_3) \\
(q_1, -p_1) & 0 & (q_1, -p_2) & (q_1, q_2) & (q_1, -p_3) & (q_1, q_3) \\
(-p_2, -p_1) & (-p_2, q_1) & 0 & (-p_2, q_2) & (-p_2, -p_3) & (-p_2, q_3) \\
(q_2, -p_1) & (q_2, q_1) & (q_2, -p_2) & 0 & (q_2, -p_3) & (q_2, q_3) \\
(-p_3, -p_1) & (-p_3, q_1) & (-p_3, -p_2) & (-p_3, q_2) & 0 & (-p_3, q_3) \\
(q_3, -p_1) & (q_3, q_1) & (q_3, -p_2) & (q_3, q_2) & (q_3, q_3) & 0
\end{bmatrix},
\]

\[
B_3 = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}.
\]
Then
\[ \tau_3 = \text{Pf}(A_3 + B_3) = 1 + f_1 + f_2 + f_3 + P_{12}f_1f_2 + P_{13}f_1f_3 + P_{23}f_2f_3 + P_{12}P_{13}P_{23}f_1f_2f_3, \] (18)
where
\[ f_1 = ia_1 \frac{p_1 + q_1 e^{F(q_1) - F(p_1)}}{q_1 - p_1}, \quad f_2 = ia_2 \frac{p_2 + q_2 e^{F(q_2) - F(p_2)}}{q_2 - p_2}, \]
\[ f_3 = ia_3 \frac{p_3 + q_3 e^{F(q_3) - F(p_3)}}{q_3 - p_3}, \quad P_{12} = \frac{p_1 - p_2 q_1 - q_2 p_1 + q_2 q_1 + p_2}{p_1 + p_2 q_1 + q_2 p_1 - q_2 q_1 - p_2}, \]
\[ P_{13} = \frac{p_1 + q_3 q_1 + p_3 q_1 - q_3 p_1 + p_3 q_1 + q_3}{p_1 - q_3 q_1 - p_3 q_1 - q_3}, \quad P_{23} = \frac{p_2 + q_3 q_2 + p_3 q_2 - q_3 p_2 + q_3 q_2 - q_3}{p_2 - q_3 q_2 - p_3 q_2 + p_3 q_2 + q_3}. \]

### 3 The wave functions

In this section, one uses (7) to construct the corresponding wave function of the τ function (15).

From (7), one knows that the corresponding wave function of the \( N \)-solitons (15) can be written as
\[ \varphi_N = \frac{P(\varphi(-p_1), \varphi(q_1), \varphi(-p_2), \varphi(q_2), \ldots, \varphi(-p_N), \varphi(q_N), \varphi(\lambda))}{\tau_N}. \]

Using the notations in (11), (12) and (13), we can express \( \varphi_N \) as
\[ \varphi_N = \frac{P(-p_1, q_1, -p_2, q_2, \ldots, -p_N, q_N, \lambda)}{\tau_N}. \] (19)

But we notice that
\[ (-p_m, \lambda) = W(-p_m, \lambda), \quad (q_m, \lambda) = W(q_m, \lambda), \] (20)
where \((\cdot)^2\) means there is no \( \delta_{mn} \) here when compared with (13). Now, let’s compute \( P(-p_1, q_1, -p_2, q_2, \ldots, -p_N, q_N, \lambda) \) using (14). In this case,
\[ P(-p_1, q_1, -p_2, q_2, \ldots, -p_N, q_N, \lambda) = \text{Pf}(M_N + Q_N), \]
where
\[
M_N = \begin{bmatrix}
(-p_1, \lambda) & \varphi(-p_1)/\sqrt{3} \\
(q_1, \lambda) & \varphi(q_1)/\sqrt{3} \\
\vdots & \vdots \\
(q_N, \lambda) & \varphi(q_N)/\sqrt{3} \\
(\lambda, -p_1) & \varphi(\lambda)/\sqrt{3} \\
-\varphi(-p_1)/\sqrt{3} & -\varphi(q_1)/\sqrt{3} & \cdots & -\varphi(q_N)/\sqrt{3} & 0 \end{bmatrix}
\]
and
\[
Q_N = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & \vdots \\
0 & \vdots & \vdots \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}
\]
Using (16), a simple calculation yields

\[
Pf(M_N + Q_N) = \phi \left[ 1 + \sum_{\ell=1}^{N} h_\ell(\lambda) + \sum_{m=2}^{N} \left( \sum_{1 \leq \ell_1 < \ell_2 < \ldots < \ell_m \leq N} h_{\ell_1} h_{\ell_2} \cdots h_{\ell_m} \prod_{1 \leq j < k \leq m} P_{\ell_j \ell_k} \right) \right] = \phi \hat{X}_N(\lambda),
\]

where

\[
\phi = \frac{\varphi(\lambda)}{\sqrt{3}}, \quad h_\ell(\lambda) = i \alpha \left( \frac{p_\ell}{q_\ell} + p_\ell p_\ell + \lambda q_\ell - \lambda \frac{F(q_\ell) - F(p_\ell)}{q_\ell - p_\ell} \right) = \frac{f_\ell}{q_\ell - p_\ell} \frac{p_\ell + \lambda q_\ell - \lambda}{p_\ell - \lambda q_\ell + \lambda},
\]

\[
\hat{X}_N(\lambda) = 1 + \sum_{\ell=1}^{N} h_\ell(\lambda) + \sum_{m=2}^{N} \left( \sum_{1 \leq \ell_1 < \ell_2 < \ldots < \ell_m \leq N} h_{\ell_1} h_{\ell_2} \cdots h_{\ell_m} \prod_{1 \leq j < k \leq m} P_{\ell_j \ell_k} \right),
\]

and \( P_{\ell_j \ell_k} \) is defined in (15).

We give several examples here.

1. The one-soliton wave function:

\[
M_1 = \begin{bmatrix}
A_1(z, \bar{z}, t) & (-p_1, \lambda) \\
(\lambda, -p_1) & (q_1, \lambda)
\end{bmatrix}
\]

\[
Q_1 = \begin{bmatrix}
B_1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Then

\[
\varphi_1 = P(-p_1, q_1, \lambda) = \frac{M_1 + Q_1}{\tau_1} = \frac{\phi}{\tau_1} \left( 1 + \frac{ia_1}{q_1 - p_1} \frac{p_1 + \lambda q_1 - \lambda}{q_1 - p_1} e^{F(q_1) - F(p_1)} \right).
\]

We remark that

\[
\varphi_1 = \frac{\phi}{1 + f_1} \left[ 1 + \frac{ia_1}{q_1 - p_1} \frac{p_1 + \lambda q_1 - \lambda f_1}{p_1 - \lambda q_1 + \lambda} \right] = \frac{\phi}{1 + f_1} \left[ 1 + \frac{ia_1}{q_1 - p_1} \frac{2p_1 - \lambda - 1}{q_1 + \lambda - 1} \right]
\]

\[
= \phi \left[ 1 + \frac{2ia_1}{\lambda - p_1} \frac{p_1 + q_1}{\lambda + q_1} e^{F(q_1) - F(p_1)} \right].
\]

This is the one-soliton wave function in [2, p. 9].

2. The two-soliton wave function:

\[
M_2 = \begin{bmatrix}
A_2(z, \bar{z}, t) & (-p_1, \lambda) \\
(\lambda, -p_1) & (q_1, \lambda)
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
Then using (21), one has
\[
\varphi_2 = \frac{P(-p_1, q_1, -p_2, q_2, \lambda)}{\tau_2} = \frac{\text{Pf}(M_2 + Q_2)}{\tau_2} = \frac{\phi}{\tau_2} \left[ 1 + h_1(\lambda) + h_2(\lambda) + \mathbb{P}_{12} h_1(\lambda) h_2(\lambda) \right] \\
= \frac{\phi}{\tau_2} \left( 1 + i a_1 q_1 p_1 + \lambda q_1 - \lambda q_2 - \lambda q_1 - \lambda q_2 - \lambda q_2 - \lambda q_2 + \lambda e^{F(q_1) - F(p_1)} + i a_2 p_2 + q_2 p_2 + \lambda q_2 - \lambda q_2 + \lambda e^{F(q_2) - F(p_2)} \right) \\
+ i a_1 i a_2 p_1 + q_1 p_1 + \lambda q_1 - \lambda p_2 + q_2 p_2 + \lambda q_2 - \lambda q_2 + \lambda e^{F(q_1) + F(q_2) - F(p_1) - F(p_2)} \right). \\
\]

(3) Three-solitons wave function:
\[
M_3 = 
\begin{bmatrix}
(\lambda, -p_1) & (\lambda, q_1) & (\lambda, -p_2) & (\lambda, q_2) & (\lambda, -p_3) & (\lambda, q_3) & 0 \\
-\varphi(-p_1) & \varphi(q_1) & -\varphi(-p_2) & \varphi(q_2) & -\varphi(-p_3) & \varphi(q_3) & -\varphi(\lambda) \\
\varphi(-p_1) & \varphi(q_1) & \varphi(-p_2) & \varphi(q_2) & \varphi(-p_3) & \varphi(q_3) & \varphi(\lambda) \\
\end{bmatrix},
\]
\[
Q_3 = 
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.
\]

From (21), we get
\[
\varphi_3 = \frac{P(-p_1, q_1, -p_2, q_2, -p_3, q_3, \lambda)}{\tau_3} = \frac{\text{Pf}(M_3 + Q_3)}{\tau_3} = \frac{\phi}{\tau_3} \left[ 1 + h_1(\lambda) + h_2(\lambda) + h_3(\lambda) + \mathbb{P}_{12} h_1(\lambda) h_2(\lambda) + \mathbb{P}_{13} h_1(\lambda) h_3(\lambda) + \mathbb{P}_{23} h_2(\lambda) h_3(\lambda) \right] \\
+ \mathbb{P}_{12} \mathbb{P}_{23} h_1(\lambda) h_2(\lambda) h_3(\lambda)],
\]
where
\[
h_1 = i a_1 p_1 + \lambda q_1 - \lambda p_1 + q_1 e^{F(q_1) - F(p_1)}, \quad h_2 = i a_2 p_2 + \lambda q_2 - \lambda p_2 + q_2 e^{F(q_2) - F(p_2)},
\]
\[
h_3 = i a_3 p_3 + \lambda q_3 - \lambda p_3 + q_3 e^{F(q_3) - F(p_3)},
\]
and \(\mathbb{P}_{12}, \mathbb{P}_{13}\) and \(\mathbb{P}_{23}\) are defined in (18).
4 A property of N-solitons wave function

In this section, we will express the wave function (19) as another form to generalize the equation (22) to N-solitons case.

Firstly, according to the Pfaffian expansion in [11], it is not difficult to see that

\[ \widetilde{\text{Pf}}(b_1, b_2, b_3, b_4, \ldots, b_{2n-1}, b_{2n}, b_{2n+1}) = \sum_{m=1}^{2n+1} (-1)^{j+m} (b_j, b_m) \text{Pf}(b_1, b_2, \ldots, \hat{b}_j, \ldots, \hat{b}_m, \ldots, b_{2n}, b_{2n+1}), \]  
for \( j = 1, 2, \ldots, 2n + 1, \)

where \( \hat{b}_j \) and \( \hat{b}_m \) mean these two terms are omitted.

Secondly, noticing (10) and letting \( \lambda = -p_\alpha \) or \( \lambda = q_\alpha, \alpha = 1, 2, \ldots, n, \) we have

\[ \widetilde{\text{Pf}}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, -p_\alpha) = \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, -p_\alpha) \]

\[ = \phi(-p_\alpha) \chi_N(-p_\alpha), \]

\[ \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, q_\alpha) = \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N) = a_\alpha \phi(q_\alpha) \chi_N(q_\alpha). \]  

They can be seen as follows. By (23), one has

\[ \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, -p_\alpha) = -(q_\alpha, -p_1) \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, -p_\alpha) \]

\[ + (q_\alpha, q_1) \text{Pf}(p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, -p_\alpha) - \cdots \]

\[ - (q_\alpha, -p_\alpha) \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N, -p_\alpha) + \cdots \]

\[ + (q_\alpha, -p_\alpha)^2 \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N) \]

\[ = -(q_\alpha, -p_\alpha) + (q_\alpha, -p_\alpha)^2 \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N) \]

\[ = \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_\alpha, q_\alpha, -p_{\alpha+1}, q_{\alpha+1}, \ldots, -p_N, q_N), \]

where \( (q_\alpha, -p_\alpha)^2 \) is defined in (20) and we know that

\[ \text{Pf}(\ldots, -p_\alpha, \ldots, -p_\alpha) = 0. \]

The second equation of (24) can be proved similarly.

Finally, from (23), one yields

\[ \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_N, q_N, \lambda) = (\lambda, -p_1) \text{Pf}(q_1, -p_2, q_2, \ldots, -p_N, q_N) \]

\[ - (\lambda, q_1) \text{Pf}(-p_1, q_1, -p_2, q_2, \ldots, -p_N, q_N) + (\lambda, -p_2) \text{Pf}(q_1, -p_1, q_2, \ldots, -p_N, q_N) \]

\[ - (\lambda, q_2) \text{Pf}(q_1, -p_1, -p_2, \ldots, -p_N, q_N) + \cdots \]

\[ + (\lambda, -p_N) \text{Pf}(-p_1, -p_2, q_2, \ldots, -p_N-1, q_N-1, q_N) \]

\[ - (\lambda, q_N) \text{Pf}(-p_1, -p_2, q_2, \ldots, -p_N-1, q_N-1, -p_N) + \phi(\lambda) \tau_N. \]

Also, the wave function (21) can be written as

\[ \varphi_N(\lambda) = \phi(\lambda) \chi_N(\lambda), \]
where $\chi_N(\lambda) = \frac{\hat{\chi}_N(\lambda)}{\tau_N}$. Therefore, using (24) and letting $\Delta F_n = F(q_n) - F(p_n)$, we get

$$
\chi_N(\lambda) = 1 - \frac{p_1 + \lambda}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_N(q_1) - \frac{q_1 - \lambda}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_N(-p_1)
- \frac{p_2 + \lambda}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_N(q_2) - \frac{q_2 - \lambda}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots
- \frac{p_N + \lambda}{\lambda - p_N} ia_N e^{\Delta F_N} \chi_N(q_N) - \frac{q_N - \lambda}{q_N + \lambda} ia_N e^{\Delta F_N} \chi_N(-p_N)
= 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_N(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_N(-p_1)
- \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_N(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots
- \frac{2p_N}{\lambda - p_N} ia_N e^{\Delta F_N} \chi_N(q_N) - \frac{2q_N}{q_N + \lambda} ia_N e^{\Delta F_N} \chi_N(-p_N)
+ \left[ -ia_1 e^{\Delta F_1} \chi_N(q_1) + ia_1 e^{\Delta F_1} \chi_N(-p_1) - ia_2 e^{\Delta F_2} \chi_N(q_2) + ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots - ia_N e^{\Delta F_N} \chi_N(q_N) + ia_N e^{\Delta F_N} \chi_N(-p_N) \right].
$$

(25)

Since $\tilde{\text{P}}f(-p_1, q_1, -p_2, q_2, \ldots, -p_N, q_N, 0) = \frac{\tau_N}{\tau_1}$, we have $\hat{\chi}_N(0) = \tau_N$. Then the last term in $[\cdots]$ of (25) is zero (or $\lim_{\lambda \to \infty} \chi_N(\lambda) = 1$). Hence one has

$$
\chi_N(\lambda) = 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_N(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_N(-p_1)
- \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_N(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_N(-p_2) - \cdots
- \frac{2p_N}{\lambda - p_N} ia_N e^{\Delta F_N} \chi_N(q_N) - \frac{2q_N}{q_N + \lambda} ia_N e^{\Delta F_N} \chi_N(-p_N).
$$

This formula is also obtained by the d-bar dressing method when the d-bar data is the degenerate delta kernel [2, p. 6].

When $n=1$, we have (22). For $n=2$, from (21), one knows that

$$
\chi_2(-p_1) = \left[ 1 + ia_2 \frac{p_2 + q_2 p_2 - p_1 q_2 + p_1}{q_2 - p_2 + q_1 - q_2 + p_2} e^{\Delta F_2} \right] / \tau_2,
$$

$$
\chi_2(q_1) = \left[ 1 + ia_2 \frac{p_2 + q_2 q_2 - p_1 q_2 + q_1}{q_2 - p_2 + q_1 - q_2 + p_2} e^{\Delta F_2} \right] / \tau_2,
$$

$$
\chi_2(-p_2) = \left[ 1 + ia_1 \frac{p_1 + q_1 p_1 - p_2 q_1 + p_2}{q_1 - p_1 + q_2 - q_1 + p_2} e^{\Delta F_1} \right] / \tau_2,
$$

$$
\chi_2(q_2) = \left[ 1 + ia_1 \frac{p_1 + q_1 q_1 + p_1 - q_2}{q_1 - p_1 + q_2 - q_1 + q_2} e^{\Delta F_1} \right] / \tau_2.
$$

Then

$$
\chi_2(\lambda) = 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_2(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_2(-p_1)
- \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_2(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_2(-p_2).
$$

We remark that this formula also appears in [2, p. 10], the parameters being different. For $n=3$, by (21), we obtain

$$
\chi_3(-p_1) = \left[ 1 + ia_2 \frac{p_2 + q_2 p_2 - p_1 q_2 + p_1}{q_2 - p_2 + q_1 - q_2 + p_2} e^{\Delta F_2} + ia_3 \frac{p_3 + q_3 p_3 - p_1 q_3 + p_1}{q_3 - p_3 + q_2 - q_3 + p_3} e^{\Delta F_3} \right] / \tau_3.
$$

Then

$$
\chi_3(\lambda) = 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_3(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_3(-p_1)
- \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_3(q_2) - \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_3(-p_2)
- \frac{2p_3}{\lambda - p_3} ia_3 e^{\Delta F_3} \chi_3(q_3) - \frac{2q_3}{q_3 + \lambda} ia_3 e^{\Delta F_3} \chi_3(-p_3).
$$

We remark that this formula also appears in [2, p. 10], the parameters being different.
where $\mathbb{P}_{12}$, $\mathbb{P}_{13}$ and $\mathbb{P}_{23}$ are defined in (18). Then

$$
\chi_3(\lambda) = 1 - \frac{2p_1}{\lambda - p_1} ia_1 e^{\Delta F_1} \chi_3(q_1) - \frac{2q_1}{q_1 + \lambda} ia_1 e^{\Delta F_1} \chi_3(-p_1) - \frac{2p_2}{\lambda - p_2} ia_2 e^{\Delta F_2} \chi_3(q_2)
$$

$$
- \frac{2q_2}{q_2 + \lambda} ia_2 e^{\Delta F_2} \chi_3(-p_2) - \frac{2p_3}{\lambda - p_3} ia_3 e^{\Delta F_3} \chi_3(q_3) - \frac{2q_3}{q_3 + \lambda} ia_3 e^{\Delta F_3} \chi_3(-p_3).
$$

5 Concluding remarks

In this paper, we have used the extended Moutard transformation to construct the $N$-solitons solutions. The basic idea comes from the successive iterations of solitons solutions, as remains to be the simple method to obtain the $N$-solitons solutions. Also, the corresponding wave functions are constructed by the Pfaffian expansion of the sum of two anti-symmetric matrices [14] when compared with the $\tilde{\mathcal{D}}$-dressing method [2].

To obtain real $N$-solitons solutions of the Novikov–Veselov equation (1), one has to put extra relations between $-p_i$ and $q_i$ [2]. It could be interesting to investigate these real solutions for the Schrödinger operator (self-adjoint). On the other hand, the resonance of $N$-solitons solutions of DKP or KP theory has been studied in [16, 17, 18, 19]. And then the resonance of $N$-solitons solutions of Pfaffian type (15) deserves to be investigated. These issues will be published elsewhere.

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References

[1] Athorne C., Nimmo J.J.C., On the Moutard transformation for integrable partial differential equations, Inverse Problems 7 (1991), 809–826.

[2] Basalaev M.Yu., Dubrovin V.G., Topovsky A.V., New exact solutions with constant asymptotic values at infinity of the NVN integrable nonlinear evolution equation via $\partial$-dressing method, arXiv:0912.2155.

[3] Bogdanov L.V., Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg–de Vries equation, Theoret. Math. Phys. 70 (1987), 219–223.

[4] Chang J.H., The Gould–Hopper polynomials in the Novikov–Veselov equation, J. Math. Phys. 52 (2011), 092703, 15 pages, arXiv:1011.1614.

[5] Dubrovin B.A., Krichever I.M., Novikov S.P., The Schrödinger equation in a periodic field and Riemann surfaces, Sov. Math. Dokl. 17 (1976), 947–952.

[6] Dubrovin V.G., Formusatik I.B., New lumps of Veselov–Novikov integrable nonlinear equation and new exact rational solutions of two-dimensional stationary Schrödinger equation via $\partial$-dressing method, Phys. Lett. A 313 (2003), 68–76.

[7] Dubrovin V.G., Formusatik I.B., The construction of exact rational solutions with constant asymptotic values at infinity of two-dimensional NVN integrable nonlinear evolution equations via the $\partial$-dressing method, J. Phys. A: Math. Gen. 34 (2001), 1837–1851.

[8] Grinevich P.G., Rational solitons of the Veselov–Novikov equations are reflectionless two-dimensional potentials at fixed energy, Theoret. Math. Phys. 69 (1986), 1170–1172.

[9] Grinevich P.G., Manakov S.V., Inverse scattering problem for the two-dimensional Schrödinger operator, the $\partial$-method and nonlinear equations, Funct. Anal. Appl. 20 (1986), 94–103.

[10] Grinevich P.G., Mironov A.E., Novikov S.P., New reductions and nonlinear systems for 2D Schrödinger operators, arXiv:1001.4300.

[11] Hirota R., The direct method in soliton theory, Cambridge Tracts in Mathematics, Vol. 155, Cambridge University Press, Cambridge, 2004.

[12] Hu H.C., Lou S.Y., Construction of the Darboux transformaiton and solutions to the modified Nizhnik–Novikov–Veselov equation, Chinese Phys. Lett. 21 (2004), 2073–2076.

[13] Hu H.C., Lou S.Y., Liu Q.P., Darboux transformation and variable separation approach: the Nizhnik–Novikov–Veselov equation, Chinese Phys. Lett. 20 (2003), 1413–1415, nlin.SI/0210012.

[14] Ishikawa M., Wakayama M., Applications of minor-summation formula. II. Pfaffians and Schur polynomials, J. Combin. Theory Ser. A 88 (1999), 136–157.

[15] Kaptsov O.V., Shan’ko Yu.V., Trilinear representation and the Moutard transformation for the Tzitzéica equation, solv-int/9704014.

[16] Kodama Y., KP solitons in shallow water, J. Phys. A: Math. Gen. 43 (2010), 434004, 54 pages, arXiv:1004.4607.

[17] Kodama Y., Maruno K., N-soliton solutions to the DKP equation and Weyl group actions, J. Phys. A: Math. Gen. 39 (2006), 4063–4086, nlin.SI/0602031.

[18] Kodama Y., Williams L.K., KP solitons and total positivity for the Grassmannian, arXiv:1106.0023.

[19] Kodama Y., Williams L.K., KP solitons, total positivity, and cluster algebras, Proc. Natl. Acad. Sci. USA 108 (2011), 8984–8989, arXiv:1105.4170.

[20] Konopelchenko B.G., Introduction to multidimensional integrable equations. The inverse spectral transform in 2 + 1 dimensions, Plenum Press, New York, 1992.

[21] Konopelchenko B.G., Landolfi G., Induced surfaces and their integrable dynamics. II. Generalized Weierstrass representations in 4D spaces and deformations via DS hierarchy, Stud. Appl. Math. 104 (2000), 129–169.

[22] Krichever I.M., A characterization of Prym varieties, Int. Math. Res. Not. 2006 (2006), Art. ID 81476, 36 pages, math.AG/0506238.

[23] Liu S.Q., Wu C.Z., Zhang Y., On the Drinfeld–Sokolov hierarchies of D type, Int. Math. Res. Not. 2011 (2011), 1952–1996, arXiv:0912.5273.

[24] Manakov S.V., The method of the inverse scattering problem, and two-dimensional evolution equations, Russian Math. Surveys 31 (1976), no. 5, 245–246.
On the $N$-Solitons Solutions in the Novikov–Veselov Equation

[25] Matveev V.B., Salle M.A., Darboux transformations and solitons, *Springer Series in Nonlinear Dynamics*, Springer-Verlag, Berlin, 1991.

[26] Mironov A.E., A relationship between symmetries of the Tzitzéica equation and the Veselov–Novikov hierarchy, *Math. Notes* **82** (2007), 569–572.

[27] Mironov A.E., Finite-gap minimal Lagrangian surfaces in $\mathbb{CP}^2$, in Riemann Surfaces, Harmonic Maps and Visualization, *OCAMI Stud.*, Vol. 3, Osaka Munic. Univ. Press, Osaka, 2010, 185–196, arXiv:1005.3402.

[28] Mironov A.E., The Veselov–Novikov hierarchy of equations, and integrable deformations of minimal Lagrangian tori in $\mathbb{CP}^2$, *Sib. Electron. Math. Rep.* **1** (2004), 38–46, math.DG/0607700.

[29] Moutard M., Note sur les équations différentielles linéaires du second ordre, *C.R. Acad. Sci. Paris* **80** (1875), 729–733.

[30] Moutard M., Sur la construction des équations de la forme \( \frac{1}{z} \frac{\partial^2 z}{\partial x \partial y} = \lambda(xy) \), qui admettent une intégrale générale explicite, *J. de l’Éc. Polyt.* **28** (1878), 1–12.

[31] Nimmo J.J.C., Darboux transformations in $(2 + 1)$-dimensions, in Applications of Analytic and Geometric Methods to Nonlinear Differential Equations (Exeter, 1992), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, Vol. 413, Kluwer Acad. Publ., Dordrecht, 1993, 183–192.

[32] Nimmo J.J.C., Hall–Littlewood symmetric functions and the BKP equation, *J. Phys. A: Math. Gen.* **23** (1990), 751–760.

[33] Novikov S.P., Two-dimensional Schrödinger operators in periodic fields, *J. Sov. Math.* **28** (1985), 1–20.

[34] Novikov S.P., Veselov A.P., Two-dimensional Schrödinger operator: inverse scattering transform and evolutionary equations, *Phys. D* **18** (1986), 267–273.

[35] Ohta Y., Pfaffian solutions for the Veselov–Novikov equation, *J. Phys. Soc. Japan* **61** (1992), 3928–3933.

[36] Orlov A.Yu., Shiota T., Takasaki K., Pfaffian structures and certain solutions to BKP hierarchies. I. Sums over partitions, arXiv:1201.4518.

[37] Shiota T., Prym varieties and soliton equations, in Infinite-Dimensional Lie Algebras and Groups (Luminy-Marseille, 1988), *Adv. Ser. Math. Phys.*, Vol. 7, World Sci. Publ., Teaneck, NJ, 1989, 407–448.

[38] Stembridge J.R., Nonintersecting paths, Pfaffians, and plane partitions, *Adv. Math.* **83** (1990), 96–131.

[39] Taimanov I.A., Tsarev S.P., Two-dimensional rational solitons and their blowup via the Moutard transformation, *Theoret. Math. Phys.* **157** (2008), 1525–1541, arXiv:0801.3225.

[40] Takasaki K., Dispersionless Hirota equations of two-component BKP hierarchy, *SIGMA* **2** (2006), 057, 22 pages, nlin.SI/0604003.

[41] Veselov A.P., Novikov S.P., Finite-zone, two-dimensional, potential Schrödinger operators. Explicit formulas and evolution equations, *Sov. Math. Dokl.* **30** (1984), 588–591.