STABLE RANK FOR CROSSED PRODUCTS BY ACTIONS OF
FINITE GROUPS ON C*-ALGEBRAS

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Abstract. Let $G$ be a finite group, $A$ a unital separable finite simple nuclear
C*-algebra, and $\alpha$ an action of $G$ on $A$. Assume that $A$ absorbs the Jiang-Su
algebra $Z$, the extremal boundary of the trace space of $A$ is compact and finite
dimensional and that $\alpha$ fixes any tracial state of $A$. Then $\text{tsr}(A \rtimes_\alpha G) = 1$. In
particular, when $A$ has a unique tracial state, we conclude it without above
conditions on a tracial state space of $A$.

1. Introduction

Rieffel [16] defined the (topological) stable rank, $\text{tsr}(A)$, of a C*-algebra, which is
the noncommutative analogue of the complex dimension of topological spaces. That
is, for the continuous functions on a compact Hausdorff $X$ one has $\text{tsr}(C(X)) = \left\lfloor \frac{1}{2} \dim X \right\rfloor + 1$, where $\dim X$ is the covering dimension of $X$. For a unital C*-algebra
$A$ the stable rank $\text{tsr}(A)$ is either $\infty$ or the smallest possible integer $n$ such that
each $n$-tuple in $A^n$ can be approximated in norm by $n$-tuples $(b_1, \ldots, b_n)$ such that
$\sum_{i=1}^n b_i^* b_i$ is invertible. For a nonunital C*-algebra $A$ we define $\text{tsr}(A) = \text{tsr}(\tilde{A})$, $\tilde{A}$ means the unitaization of $A$.

Rieffel [16] showed that $\text{tsr}(A) = 1$ if and only if $\text{tsr}(M_n(A)) = 1$ for $n \in \mathbb{N}$
and $\text{tsr}(A) = 1$ if and only if $\text{tsr}(A \otimes \mathbb{K}) = 1$ for the C*-algebra $\mathbb{K}$ of compact
operators on a separable infinite dimensional Hilbert space. Related to crossed
products we have in general that $\text{tsr}(A \rtimes_\alpha Z) \leq \text{tsr}(A) + 1$ by [16] Theorem 7.1
and $\text{tsr}(A \rtimes_\alpha G) \leq \text{tsr}(A) + \text{card}(G) - 1$ for any action $\alpha$ from a finite group $G$ on
$A$ by [8] Theorem 2.4. However, it is not easy to determine when those crossed
products have stable rank one, except crossed products by "strongly" outer actions
as (tracial) Rokhlin property [18], [14], [15]. See [11], [1], [3] and their bibliography
for basic properties about stable ranks.

In this note we determine stable rank one for crossed products $A \rtimes_\alpha G$ of any
action $\alpha$ from a finite group $G$ on a separable finite simple nuclear unital C*-algebra
$A$, when $A$ is the Jiang-Su absorbing with some conditions on a tracial state space
of $A$, using observations by Sato [19] and by Rørdam [17]. In particular, when $A$ has
a unique tracial state, we conclude it without above conditions on a tracial state space
of $A$. Here, the Jiang-Su algebra $Z$ is a unital separable simple infinite-dimensional
nuclear C*-algebra with a unique tracial state whose K-theoretic invariants are
same as that of the complex numbers $\mathbb{C}$. In the current classification theorem of

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C*-algebras, the absorption of $\mathbb{Z}$ is regarded as one of the regular properties of classifiable C*-algebras. See [5], [6], [20].

2. Stable rank for inclusions of unital C*-algebras

Let $A \subset B$ be an inclusion of unital C*-algebras and $E : G \to A$ be a conditional expectation of index-finite type as defined in Definition 1.2.2. of [21].

The following is a general formula for stable rank for an inclusion of unital C*-algebras of index-finite type.

**Theorem 2.1.** [8, Theorem 2.1] 1 \in A \subset B of unital C*-algebras, let $E : B \to A$ be a conditional expectation with index-finite type, and let $((v_k, v_k^\ast))_{1 \leq k \leq n}$ be a quasi-basis for $E$. Then $\text{tsr}(B) \leq \text{tsr}(A) + n - 1$.

The inclusion 1 \in A \subset B of unital C*-algebras of index-finite type is said to have finite depth $k$ if the derived tower obtained by iterating the basic construction $A' \cap A \subset A' \cap B_2 \subset A' \cap B_3 \subset \cdots$ satisfies $(A' \cap B_k)e_k(A' \cap B_k) = A' \cap B_{k+1}$, where $\{e_k\}_{k \in \mathbb{N}}$ are projections obtained by iterating the basic construction, so that $B_1 = B$, $e_1 = e_A$, and $B_{k+1} = C^*(B_k, e_k)$. When $G$ is a finite group and $\alpha$ an action of $G$ on $A$, it is well known that an inclusion 1 \in A \subset A \rtimes_\alpha G$ is of depth 2. (See [12, Lemma 3.1].)

In the case of an infinite dimensional simple unital C*-algebra $A$ with Property (SP), that is, any nonzero hereditary C*-subalgebra of $A$ has nonzero projection, we have the following estimate.

**Theorem 2.2.** [13, Theorem 3.2] 1 \in A \subset B of unital C*-algebras of index-finite type and depth 2. Suppose that $A$ is an infinite dimensional simple C*-algebra with $\text{tsr}(A) = 1$ and Property (SP). Then $\text{tsr}(B) \leq 2$.

**Remark 2.3.** When $A$ is not simple, the estimate in Theorem 2.1 is the best possible. Indeed, in [2, Example 8.2.1] Blackadar constructed a symmetry action $\alpha$ on the CAR $\mathcal{U}$ algebra such that $(C[0, 1] \otimes \mathcal{U}) \rtimes_{id \otimes \alpha} \mathbb{Z}/2\mathbb{Z} \cong C[0, 1] \otimes B$, where $\text{tsr}(B) = 1$ and $K_1(B) \neq 0$. Hence we know that $\text{tsr}(C[0, 1] \otimes B) = 2$ by [10, Corollary 7.2] and [10, Proposition 5.2].

Let $\mathbb{Z}$ be the Jiang-Su algebra. When a C*-algebra $B$ in Theorem 2.2 is Jiang-Su absorption, that is, $A \otimes \mathbb{Z} \cong A$, we can conclude that $\text{tsr}(B) = 1$ as follows.

**Theorem 2.4.** 1 \in A \subset B of unital C*-algebras of index-finite type. Suppose that $A$ is an infinite dimensional simple C*-algebra with $\text{tsr}(A) < \infty$ and $B$ is Jiang-Su absorption. Then $\text{tsr}(B) = 1$.

We use the following simple observation to prove Theorem 2.4.

**Lemma 2.5.** Let $A$ be a simple unital C*-algebra. Then $A$ is finite if $\text{tsr}(A) < \infty$.

**Proof.** Suppose that $A$ is infinite. Then from [3] there are orthogonal projections $p, q$ such that $1 \sim p \sim q$, where $\sim$ means the Murray-von Neumann equivalence. Hence $\text{tsr}(A) = \infty$ by [16, Proposition 6.5], and a contradiction. \[\square\]
Proof of Theorem 2.4

Since $A$ is simple, $B$ can be decomposed into finite direct sums $\oplus B_i$ of simple (unital) closed ideals by [7, Theorem 3.3]. By Theorem 2.1 we know $\text{tsr}(B) < \infty$, that is, $\text{tsr}(B_i) < \infty$ for each $i$. Hence each $B_i$ is a finite simple C*-algebra by Lemma 2.5.

Let $Z$ be the Jiang-Su algebra. Then, since each $B_i$ is a finite simple C*-algebra, by [17, Theorem 6.7] each $B_i \otimes Z$ has stable rank one. From the assumption since $B \cong B \otimes Z = \oplus B_i \otimes Z$, we conclude that $\text{tsr}(B) = \max\{\text{tsr}(B_i \otimes Z)\} = 1$.

3. Stable rank for C*-crossed products

Very recently, Sato [19] gives the sufficient condition for the Jiang-Su absorption of crossed products by actions of amenable groups on $\mathcal{Z}$-absorbing C*-algebras. Using this observation we can prove the stable rank one property for the crossed product $A \rtimes \alpha G$ by an action $\alpha$ of a finite group on a $\mathcal{Z}$-absorbing C*-algebra $A$ under some condition.

Theorem 3.1. Let $G$ be a finite group, $A$ a unital separable finite simple nuclear C*-algebra, and $\alpha$ an action of $G$ on $A$. Assume that $A$ absorbs the Jiang-Su algebra $\mathcal{Z}$, the extremal boundary of the trace space of $A$ is compact and finite dimensional and that $\alpha$ fixes any tracial state of $A$. Then $\text{tsr}(A \rtimes \alpha G) = 1$.

Proof. Note that an inclusion $A \subset A \rtimes \alpha G$ is of a finite-index type.

By [19, Theorem 1.1] $A \rtimes \alpha G$ is the Jiang-Su absorbing. Since $A$ is a finite simple unital C*-algebra with $A \otimes \mathcal{Z} \cong A$, $\text{tsr}(A) = 1$ by [17, Theorem 6.7]. Hence, by Theorem 2.4 $\text{tsr}(A \rtimes \alpha G) = 1$.

Remark 3.2. When $A$ is a unital simple C*-algebra with $\text{tsr}(A) = 1$ and Property (SP), then we know that $\text{tsr}(A \rtimes \alpha G) \leq 2$ for any action $\alpha$ of a finite group on $A$ by Theorem 2.2. Moreover, if $\alpha$ is ”strongly” outer like the tracial Rokhlin property in the sense of N.C. Phillips [15], then $\text{tsr}(A \rtimes \alpha G) = 1$. (For example see [14, Proposition 4.13].)

Corollary 3.3. Let $G$ be a finite group, $A$ a unital separable finite simple nuclear $\mathcal{Z}$-absorbing C*-algebra with a unique tracial state, and $\alpha$ an action of $G$ on $A$. Then $\text{tsr}(A \rtimes \alpha G) = 1$.

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