ASYMPTOTIC AND SPECTRAL PROPERTIES OF EXPONENTIALLY $\phi$-ERGODIC MARKOV PROCESSES

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Abstract. New relations between ergodic rate, $L_p$ convergence rates, and asymptotic behavior of tail probabilities for hitting times of a time homogeneous Markov process are established. For $L_p$ convergence rates and related spectral and functional properties (spectral gap and Poincaré inequality) sufficient conditions are given in the terms of an exponential $\phi$-coupling. This provides sufficient conditions for $L_p$ convergence rates in the terms of appropriate combination of ‘local mixing’ and ‘recurrence’ conditions on the initial process, typical in the ergodic theory of Markov processes. The range of application of the approach includes time-irreversible processes. In particular, sufficient conditions for spectral gap property for Lévy driven Ornstein-Uhlenbeck process are established.

1. Introduction

In this paper, we establish new relations between three topics related to the asymptotic behavior of a time homogeneous Markov process:

- ergodic rate; that is, the rate of convergence of the transition probabilities to the invariant measure of the process;
- $L_p$ convergence rates; that is, rates of convergence for $L_p$-semigroups generated by the process;
- tail probabilities for hitting times of the process.

It is well known that $L_p$ (especially, $L_2$) convergence rates for a Markov process are closely related with a number of intrinsic functional features: the spectral gap property for the generator of the process, the Poincaré inequality for the associated Dirichlet form, Cheeger-type isoperimetric inequality for the invariant measure. On the other hand, the classic methods of the ergodic theory of Markov processes allow one to establish ergodic rates under quite simple and transparent conditions on the process that do not involve any essential limitation on the structure of the state space. Our intent is to extend the range of applications of these methods in order to provide similar conditions for $L_p$ convergence rates.

It looks very unlikely that $L_p$ convergence rates can be deduced from ergodic ones straightforwardly. The ergodic rates are, in fact, norm estimates for a semigroup of operators in $B(X)$. In general, one have no means to expect that such estimates would produce a norm estimate for semigroup of operators in $L_p(X, \pi)$ with some measure $\pi$ when the state space $X$ is of a complicated structure. This guess is supported by concrete examples, see section 4 below.

It is well known that for a Markov process with a finite state space three topics listed above are, in fact, equivalent; see the detailed exposition in [AE], Chapters 2 – 4. For a process with at most countable state space, relations between its ergodic properties and rates of convergence for related $L_2$-semigroup were studied in [Chen00]. However, the methods of [Chen00] exploit the representation of the state space

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as a countable collection of points, and hardly admit a straightforward generalization to a general case. In this paper, we propose a new point of view. Let us explain the main idea of our approach briefly; a more detailed discussion is given in sections 2 and 3 below.

We start our considerations not from the estimate for the ergodic rate of a Markov process itself, but from the auxiliary construction of a coupling, which is a standard tool for proving such an estimate. This construction appears to be an appropriate tool for getting $L_p$ estimates as well, see section 3 below. In such a way, we are able to establish estimates for $L_p$ convergence rates under the typical conditions used in the ergodic theory of Markov processes.

Usually, such conditions include some local mixing conditions, and some recurrence conditions. The former ones are discussed in details in section 2.1; the latter ones can be formulated in the terms of hitting times of some sets by the process $X$. Henceforth, in our framework, estimates for the hitting times are involved, as sufficient conditions, both into ergodic rates and into $L_p$ convergence rates for the process. On the other hand, it is known ([Mat97]) that the functional inequalities like the Poincaré one imply moment estimates for hitting times. Therefore three topics listed at the beginning of the Introduction are closely related indeed. In fact, our approach allows us to give, for some classes of the processes, necessary and sufficient conditions that describe relations between these topics completely.

The range of applications of our approach is not restricted to time-reversible processes. For time-reversible processes respective $L_2$-generators are self-adjoint which makes possible to apply the spectral decomposition theorem in order to get one-to-one correspondence between ergodic rates and $L_2$-convergence rates; see [RR97] and references therein for discrete-time case and [Chen00], Theorem 1.2 for continuous-time case. Our approach does not rely heavily on the spectral decomposition theorem. This makes possible to consider, for instance, solutions to SDE’s with jump noise which typically are irreversible (in time).

The structure of the article is following. In section 2, we give basic notions and constructions required for the main exposition. In particular, we introduce the notion of an exponential $\phi$-coupling, which is the main tool in our approach. Section 3 contains the main part of the paper devoted to the proof of $L_p$ convergence rates and related functional properties in the terms of the exponential $\phi$-coupling property. In section 4 we consider one example of a Markov process and use it to demonstrate main statements, as well as relations between the exponential $\phi$-coupling, growth bound, spectral gap, and Poincaré inequality. Section 5 contains an application of the main results to Lévy driven Ornstein-Uhlenbeck processes. In the recent paper [Kul09], ergodic rates for processes defined by Lévy driven SDE’s are established. Here, we extend these results and describe spectral properties of a generator for some class of such processes. We have already mentioned that solutions to Lévy driven SDE’s, typically, are irreversible (in time). Hence the corresponding theory for $L_p$ semigroups appears to be substantially more complicated than, for instance, respective theory for diffusion processes. For diffusion processes, we establish in section 7 a criterion which gives one-to-one correspondence between three topics mentioned at the beginning of Introduction. This criterion extends, in particular, the sufficient condition from [RW04], Theorem 1.1 for a diffusion process to satisfy the Poincaré inequality. The proof of this criterion is based on the main results from section 3 and exponential integrability of the hitting times under the Poincaré inequality. The latter statement is proved in section 6, and performs an improvement of the integrability result from [Mat97].

2. Notation and basic constructions

2.1. Elements of ergodic theory for Markov processes. We consider a time homogeneous Markov process $X = \{X_t, t \in \mathbb{R}^+\}$ with a locally compact metric space $(X, \rho)$ as the state space. The process $X$
is supposed to be strong Markov and to have cádlág trajectories. The transition function for the process \( X \) is denoted by \( P_t(x, dy), t \in \mathbb{R}^+, x \in \mathbb{X} \). We use standard notation \( P_x \) for the distribution of the process \( X \) conditioned that \( X_0 = x \), and \( E_x \) for the expectation w.r.t. \( P_x \) (\( x \in \mathbb{X} \) is arbitrary).

All the functions on \( \mathbb{X} \) considered in the paper are assumed to be measurable w.r.t. Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{X}) \). The set of probability measures on \( (\mathbb{X}, \mathcal{B}(\mathbb{X})) \) is denoted by \( \mathcal{P}(\mathbb{X}) \). For a given \( \mu \in \mathcal{P}(\mathbb{X}) \) and \( t \in \mathbb{R}^+ \), we denote \( \mu_t(dy) \equiv \int_{\mathbb{X}} P_t(x, dy) \mu(dx) \). Clearly, \( \mu_t \) coincides with the distribution of the value \( X_t \) assuming that the distribution of the initial value \( X_0 \) equals \( \mu \). Probability measure \( \mu \) is called an invariant measure for \( X \) if \( \mu_t = \mu, t \in \mathbb{R}^+ \).

In our considerations, we are mostly interested in the processes on a non-compact state spaces, such as diffusions on non-compact manifolds or solutions to SDE’s with a jump noise. Typically, for such a processes there does not exist a uniform (in \( t \)) estimate for the rate of convergence rate of convergence of \( \mu_t \) to \( \mu \) w.r.t. to the total variation distance. For such a processes, the notion of \((r, \phi)\)-ergodicity appears to be most natural (see [DFG09] and discussion therein). Let us expose this notion and related objects.

Let \( \phi : \mathbb{X} \to [1, +\infty) \) be a Borel measurable function. For a signed measure \( \nu \), its \( \phi \)-variation is defined by \( \|\nu\|_{\phi, \text{var}} = \int_{\mathbb{X}} \phi \, d|\nu| \), where \( |\nu| = \nu_+ - \nu_- \) is the variation of the signed measure \( \nu \). If \( \phi \equiv 1 \), the \( \phi \)-variation is the usual total variation \( \|\cdot\|_{\text{var}} \). Let \( r : \mathbb{R}^+ \to \mathbb{R}^+ \) be some function such that \( r(t) \to 0, t \to \infty \).

**Definition 2.1.** The process \( X \) is called \((r, \phi)\)-ergodic if the class of invariant measures for \( X \) contains exactly one measure \( \pi \), and

\[
\|\mu_t - \pi\|_{\phi, \text{var}} \leq r(t) \int_{\mathbb{X}} \phi \, d\mu, \quad t \in \mathbb{R}^+, \mu \in \mathcal{P}(\mathbb{X}).
\]

We call the process *exponentially \( \phi \)-ergodic* if there exists some positive constants \( C, \beta \) such that \( X \) is \((r, \phi)\)-ergodic with \( r(t) = Ce^{-\beta t} \).

By the common terminology, a *coupling* for a pair of the processes \( U, V \) is any two-component process \( Z = (Z^1, Z^2) \) such that \( Z^1 \) has the same distribution with \( U \) and \( Z^2 \) has the same distribution with \( V \). Following this terminology, for every \( \mu, \nu \in \mathcal{P} \), we consider two versions \( \mu^\nu, \nu^\mu \) of the process \( X \) with the initial distributions equal \( \mu \) and \( \nu \), respectively, and call a \((\mu, \nu)\)-coupling for the process \( X \) any two-component process \( Z = (Z^1, Z^2) \) which is a coupling for \( X^\mu, X^\nu \).

**Definition 2.2.** The process \( X \) admits an exponential \( \phi \)-coupling if there exists an invariant measure \( \pi \) for this process and constants \( C_\phi > 0, \beta > 0 \) such that, for every \( x \in \mathbb{X} \), there exists a \((\delta_x, \pi)\)-coupling \( Z = (Z^1, Z^2) \) with

\[
E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{Z^1_t \neq Z^2_t} \leq C_\phi e^{-\beta t} \phi(x), \quad t \geq 0.
\]

It is a simple observation that a process \( X \) which admits an exponential \( \phi \)-coupling is exponentially \( \phi \)-ergodic. This observation, however, gives an efficient tool for proving exponential \( \phi \)-ergodicity, because explicit sufficient conditions are available that allow one to construct an exponential \( \phi \)-coupling. Let us formulate one statement of such a kind.

**Definition 2.3.** The process \( X \) satisfies the *local Doeblin condition*, if for every compact set \( K \subset \mathbb{X} \) there exists \( T > 0 \) such that

\[
\kappa(T,K) \equiv \sup_{x,y \in K} \frac{1}{2} \|P_T(x, \cdot) - P_T(y, \cdot)\|_{\text{var}} < 1.
\]
Proposition 2.1. Assume process $X$ to satisfy the local Doeblin condition. Let function $\phi : \mathbb{X} \to [1, +\infty)$ be such that $\phi(x) \to +\infty, x \to \infty$ and the process

$$
\phi(X_t) + \int_0^t [\alpha \phi(X_s) - C] \, ds, \quad t \in \mathbb{R}^+
$$

is a supermartingale w.r.t. to every measure $P_x, x \in \mathbb{X}$ for some positive constants $\alpha, C$.

Then the process $X$ admits an exponential $\phi$-coupling.

Clearly, under conditions of Proposition 2.1, the process $X$ is exponentially $\phi$-ergodic. The statements of such a type are well known in the ergodic theory of Markov processes (see e.g. [And91] or [MT93]), but usually the notion of a $\phi$-coupling is not introduced separately. In section 3 below, we show that this notion is of independent interest because it allows one to control $L_p$ convergence rates as well.

Remark 2.1. Frequently, the (exponential) ergodicity results are formulated in the terms of other conditions that guarantee irreducibility of the Markov process $X$ instead of the local Doeblin condition. For instance, in [MT93], [DFG09] such an irreducibility condition is given in the terms of petite sets. However, the local Doeblin condition appears to be more convenient to deal with in the explicit construction of a $\phi$-coupling. In addition, this condition can be verified efficiently for important particular classes of processes, such as diffusions (see [Ver87], [Ver99] and section 7 below) or solutions to SDE’s with jump noise (see [Kul09] and section 5 below).

The more compact, but more restrictive, form of the above condition on the function $\phi$ can be given in the terms of the extended generator $A$ of the process $X$. Recall that a locally bounded function $f : \mathbb{X} \to \mathbb{R}$ belongs to the domain $\text{Dom}(A)$ of the extended generator $A$, if there exists a locally bounded function $g : \mathbb{X} \to \mathbb{R}$ such that the process $X^f_t \overset{df}{=} f(X_t) - \int_0^t g(X_s) \, ds, t \in \mathbb{R}^+$ is a martingale w.r.t. to any measure $P_x, x \in \mathbb{X}$. For such $f, A f \overset{df}{=} g$. Clearly, process (2.1) is a supermartingale w.r.t. to any measure $P_x, x \in \mathbb{X}$ if the function $\phi \in \text{Dom}(A)$ satisfies the following Lyapunov-type condition:

$$
A \phi(x) \leq -\alpha \phi(x) + C, \quad x \in \mathbb{X}.
$$

Conditions of Proposition 2.1 appear to be too restrictive for our further purposes; see more detailed discussion after Proposition 2.2 below. Thereby, we provide milder conditions which still are sufficient for $X$ to admit an exponential $\phi$-coupling.

For a closed set $K$ denote $\tau_K \overset{df}{=} \inf\{t \geq 0 : X_t \in K\}$, the hitting time of the set $K$ by the process $X$.

Theorem 2.1. Assume process $X$ to satisfy the local Doeblin condition. Let there exist function $\phi : \mathbb{X} \to [1, +\infty)$, compact set $K \subset \mathbb{X}$, and $\alpha > 0$ such that

1) $\phi(x) \to +\infty, x \to \infty$;
2) $E_x \phi(X_{\tau_K}) I_{\tau_K > t} \leq e^{-\alpha t} \phi(x), x \in \mathbb{X}$;
3) $\sup_{x \in K, t \in \mathbb{R}^+} E_x \phi(X_t) I_{\phi(X_t) > c} \to 0, \quad c \to +\infty$.

Then the process $X$ admits an exponential $\phi$-coupling.

Condition 1) of Theorem 2.1 is quite natural as long as $\phi$ is considered as a Lyapunov function. However, in some cases this condition may be too restrictive, too. Below, we give a version of Theorem 2.1 that does not require any assumptions on the limit behavior of $\phi$.

We say that process $X$ satisfies the Doeblin condition on a set $A \subset \mathbb{X}$ if there exists $T > 0$ such that $\nu(T, A) < 1$. We also say that process $X$ satisfies the extended Doeblin condition on a set $A \subset \mathbb{X}$ if there
exist $T_1, T_2$ ($0 < T_1 < T_2$) such that

\[(2.3) \quad \kappa(T_1, T_2, K) \overset{df}{=} \sup_{x,y \in K, s,t \in [T_1, T_2]} \frac{1}{2} \|P_s(x, \cdot) - P_t(y, \cdot)\|_{var} < 1.\]

We remark that these definitions are not standard ones, but they look quite natural in the context of Definition 2.3 and the following theorem.

**Theorem 2.2.** Let there exist function $\phi : \mathbb{X} \to [1, +\infty)$, closed set $K \subset \mathbb{X}$, and $\alpha > 0$ such that conditions 2), 3) of Theorem 2.1 hold true. Assume that either $X$ satisfies the Doeblin condition on $K$, or $X$ satisfies the extended Doeblin condition on $K$.

Then the process $X$ admits an exponential $\phi$-coupling.

From Theorems 2.1, 2.2 we deduce the following statement. Denote, for $t > 0$,

$$\tau_K^t \overset{df}{=} \inf\{s \geq 0 : X_{t+s} \in K\}.$$  

**Proposition 2.2.** Assume process $X$ to satisfy the local Doeblin condition. Let there exist compact set $K \subset \mathbb{X}$ and $\alpha > 0$ such that

1) $\lim_{t \to +\infty} \lim_{x \to -\infty} P_x(\tau_K > c) > 0$;
2) $E_x e^{\alpha \tau_K} < +\infty$, $x \in \mathbb{X}$;
3) $\sup_{x \in K, t \in [0, S]} E_x e^{\alpha \tau_K} < +\infty$.

Then, for every $\alpha' \in (0, \alpha)$, the process $X$ admits an exponential $\phi$-coupling with $\phi(x) = E_x e^{\alpha' \tau_K}, x \in \mathbb{X}$.

In addition, if $X$ satisfies the extended Doeblin condition on $K$ then condition 1) is not required.

This proposition demonstrates the difference between Theorems 2.1, 2.2 on one hand, and Proposition 2.1 on another. Typically, a function $\phi$ of the type $\phi(x) = E_x e^{\alpha \tau_K}$ neither belong to the domain of $A$ nor satisfy condition of Proposition 2.1. On the other hand, Theorems 2.1, 2.2 appear to be powerful enough to handle the functions of such a type. This is important for our approach, since we would like to control the construction of a $\phi$-coupling in the terms of the hitting times for the process $X$.

We prove Theorems 2.1, 2.2 and Proposition 2.2 in the Appendix.

### 2.2. Semigroups generated by $X$: growth bounds and spectral properties of generators.

For a function $f : \mathbb{X} \to \mathbb{C}$, we denote

$$T_t f(x) = \int_{\mathbb{X}} f(y) P_t(x, dy), \quad t \in \mathbb{X}, x \in \mathbb{X}$$

assuming the integrals to exist. Typically, the mapping $f \mapsto T_t f$ forms a bounded linear operator in an appropriate functional space. We are mainly interested in the functional spaces $L_p = L_p^c(\mathbb{X}, \pi), p \in (1, +\infty)$, but we also consider some other auxiliary spaces. The Chapman-Kolmogorov equation for the transition function $P_t(x, dy)$ yields the semigroup property for the family $\{T_t\}$: $T_{t+s} = T_t T_s, t, s \in \mathbb{R}^+$.

We assume process $X$ to be stochastically continuous, which yields that $\{T_t\}$, considered as a semigroup in $L_p$ with any $p \in (1, +\infty)$, is strongly continuous. We denote by $A$ the generator of the semigroup $\{T_t\}$. By the definition,

$$Af \overset{df}{=} \lim_{t \to 0+} \frac{1}{t} [T_t f - f],$$

where the convergence holds in the sense of respective functional space, and the domain of $A$ consists of all functions $f$ such that the limit exists.
Definition 2.4. Let \( \{ T_t \} \) be a strongly continuous semigroup of bounded linear operators on some complex Banach space \( \mathcal{X} \). A number \( \gamma \in \mathbb{R} \) is called

a) a \textit{spectral bound} for the generator \( A \) of \( \{ T_t \} \), if every point \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > -\gamma \) belongs to the resolvent set of \( A \);

b) an (exponential) \textit{growth bound} for \( \{ T_t \} \), if there exists \( C \in \mathbb{R}^+ \) such that

\[
\| T_t f \| \leq C e^{-\gamma t} \| f \|, \quad t \in \mathbb{R}^+, f \in \mathcal{X}.
\]

The terminology introduced in Definition 2.4 differs slightly from the standard one in the general spectral theory of semigroups. Namely, the constant in the standard definition of a growth bound may depend on \( f \) ([Nag86], Chapter A-III). However, this modified terminology appears to be more convenient in our framework. We remark that the following condition is equivalent to (2.4) and, in some cases, can be verified more easily:

\[
\| (T_t f, g) \| \leq C e^{-\gamma t} \| f \| \| g \|_{\mathcal{X}^*}, \quad t \in \mathbb{R}^+, f \in \mathcal{X}, g \in \mathcal{X}^*,
\]

where \( \mathcal{X}^* \) is the dual space for \( \mathcal{X} \).

The following statement is quite standard, but, for the sake of completeness, we give the sketch of the proof here.

Proposition 2.3. If \( \gamma \in \mathbb{R} \) is a growth bound for \( \{ T_t \} \), then \( \gamma \) is a spectral bound for its generator.

\[\text{Proof.}\] Take any \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda > -\gamma \) and consider the mapping

\[ R_{\lambda} : f \mapsto \int_0^\infty e^{-\lambda s} T_s f \, ds = \lim_{s \to +\infty} \int_0^s e^{-\lambda s} T_s f \, ds. \]

The integrals under the limit are defined in the Riemannian sense, and the limit exists in the sense of the norm convergence. The operator \( R_{\lambda} \) is bounded with its norm being dominated by \( C(\text{Re} \lambda + \gamma)^{-1} \), where the constant \( C \) comes from the definition of a growth bound. On the other hand, by standard arguments, \( T_t R_{\lambda} = R_{\lambda} T_t = e^{\lambda t} \left( R_{\lambda} - f_t^* e^{-\lambda s} T_s \, ds \right) \) and \( R_{\lambda} \) is the inverse operator for \( \lambda - A \), i.e., \( \lambda \) belongs to the resolvent set of \( A \). \( \square \)

For semigroups defined by a (conservative) Markov process \( X \) in \( L_p, p \in (1, +\infty) \), the point \( \lambda = 0 \) is a trivial eigenvalue with the corresponding eigenfunction \( f_\lambda = 1 \) (i.e., the function that equals 1 in every point). If this eigenvalue is simple and the rest of the spectrum of the generator \( A \) is separated from zero, then it is said that this generator (resp., semigroup or process) possesses a \textit{spectral gap}. This motivates the following terminology. Denote, for \( p \in (1, +\infty) \),

\[ L_p^0 = \{ f \in L_p^0(\mathcal{X}, \pi) : \int_{\mathcal{X}} f \, d\pi = 0 \} = \langle 1 \rangle^\perp, \]

where in the last expression \( \mathbf{1} \) is interpreted as an element of \( L_p^* = L_q, q^{-1} + p^{-1} = 1 \). Since \( \pi \) is an invariant measure for \( X \), one has

\[ \int_{\mathcal{X}} T_t f(x) \pi(dx) = \int_{\mathcal{X}} \int_{\mathcal{X}} f(y) P_t(x, dy) \pi(dx) = \int_{\mathcal{X}} f(y) \pi(dy), \]

which means, in particular, that \( L_p^0 \) is invariant under \( \{ T_t \} \).

For a given \( \gamma > 0, p \in (1, +\infty) \), we say that process \( X \) possesses either property \( SG_p(\gamma) \) or property \( GB_p(\gamma) \), if, for the restriction of its semigroup \( \{ T_t \} \) to the space \( \mathcal{X} = L_p^0 \) the number \( \gamma \) is a spectral bound or a growth bound, respectively. Also, we say that the process possesses an exponential \( L_p \) rate if

\[
\| T_t f \|_p \leq C e^{-\gamma t} \| f \|_p, \quad t \in \mathbb{R}^+, f \in L_p^0
\]
with some $C > 0, \gamma > 0$ (here and below, we denote $\| \cdot \|_p \overset{df}{=} \| \cdot \|_{L_p}$).

Some authors (e.g. [Chen00]) say that the process $X$ possesses an exponential $L_2$ rate if
\begin{equation}
\| T_t f \|_2 \leq e^{-\gamma t} \| f \|_2, \quad t \in \mathbb{R}^+, f \in L_2^0
\end{equation}
with some $\gamma > 0$. This terminology does not look to be perfectly adjusted with the matter of the problem discussed above, because the constant $C$ in (2.6) with $p = 2$ does not play an essential role in the asymptotic behavior of the semigroup; in particular, the estimate (2.6) is already strong enough to provide existence of a spectral gap for $X$.

On the other hand, it makes sense to consider estimate of the type (2.7) separately. Let us express (2.7) in the terms of the Dirichlet form $\mathcal{E}$ associated with the process $X$. Recall that the Dirichlet form $\mathcal{E}$ corresponding to the $L_2$-semigroup $\{ T_t \}$ generated by $X$ is defined as the completion of the bilinear form $\text{Dom}(A) \times \text{Dom}(A) \ni (f, g) \mapsto -(Af, g)_{L_2}$ with respect to the norm $\| \cdot \|_{\mathcal{E}, 1}$ (e.g. [MR92], Chapter 2). It can be verified easily that, for $c = \gamma^{-1}$, (2.7) is equivalent to the functional inequality
\begin{equation}
\int_X |f|^2 d\pi - \left| \int_X f d\pi \right|^2 \leq c \mathcal{E}(f, f), \quad f \in \text{Dom}(\mathcal{E}),
\end{equation}
called the Poincaré inequality. The Poincaré inequality is one of the most important in the field, and this motivates the interest to the inequality (2.7). One can say that (2.7) is a kind of a differential estimate, while (2.6) with $p = 2$ is an integral one. In section 4 below we give an example which demonstrates that these estimates are non-equivalent.

For a given $\gamma > 0$, we say that process $X$ possesses the property $PI(\gamma)$ if (2.7) holds true. We have the following implications:

$$PI(\gamma) \Rightarrow GB_2(\gamma), \quad GB_p(\gamma) \Rightarrow SG_p(\gamma).$$

Examples are available, where a number being a spectral bound is not a growth bound ([Nag86], Example 1.4, [Chen00], Example 2.3), and thus $SG_p(\gamma) \not\Rightarrow GB_p(\gamma)$. As we have already mentioned, $GB_2(\gamma) \not\Rightarrow PI(\gamma)$ (see section 4). Therefore, in general, each of three properties formulated above requires a separate investigation.

3. $L_p$ convergence rates and Poincaré inequality for a process that admits an exponential $\phi$-coupling

In this section, we assume that, for a given function $\phi$, the process $X$ admits an exponential $\phi$-coupling. We denote by $C_\phi, \beta$ the constants from the definition of a $\phi$-exponential coupling, and write $C$ for any constant which can be, but is not, expressed explicitly. The value of the constant $C$ can vary from line to line. We denote by $\pi$ the unique invariant measure for $X$ and assume $\phi \in L_1(X, \pi)$. This assumption is not restrictive; it holds true under conditions of either Theorem 2.1, Theorem 2.2 or Proposition 2.2 (see Remark A.1 in the Appendix).

We separate our investigation into several parts. First, we establish rates of convergence of the semigroups generated by $X$ in auxiliary spaces $L_{p, \phi}, L_{p, \phi}^*$. Then we consider $L_p$-semigroups with arbitrary $p \in (1 + \infty)$. Finally, we investigate the $L_2$-semigroup, considering separately the cases of a reversible and an irreversible (in time) process $X$ separately.
3.1. Spaces $L_{p,\phi}, L_{p,\phi}^{*}, p \in (1, +\infty)$. For $p \in (1, +\infty)$, denote by $L_{p,\phi}, p \in (1, +\infty)$ the set of functions $f$ such that

$$\|f\|_{p,\phi} \stackrel{df}{=} \left[ \int_{\mathcal{X}} \left| \frac{f(x)}{\phi(x)} \right|^p d\pi \right]^{\frac{1}{p}} < +\infty,$$

where $q$ is adjoint to $p$, i.e. $p^{-1} + q^{-1} = 1$. The set $L_{p,\phi}$ is a Banach space with the norm $\| \cdot \|_{p,\phi}$. The dual space $L_{p,\phi}^{*}$ to $L_{p,\phi}$ with respect to the natural duality $(f, g) \mapsto \langle f, g \rangle \stackrel{df}{=} \int_{\mathcal{X}} f g d\pi$ coincides with the space of functions $f$ such that

$$\|f\|_{p,\phi}^{*} \stackrel{df}{=} \left[ \int_{\mathcal{X}} |f|^q \phi d\pi \right]^{\frac{1}{q}} < +\infty.$$

The space $L_{p,\phi}^{*}$ is a subset of $L_q$ since $\phi \geq 1$. On the other hand, $\phi$ may be unbounded, and in this case $L_{p,\phi}$ is strictly larger than $L_p$. Nevertheless, in any case $L_{p,\phi} \subset L_1$ because

$$\int_{\mathcal{X}} |f|^p d\pi \leq \left[ \int_{\mathcal{X}} \left| \frac{f}{\phi(x)} \right|^p d\pi \right]^{\frac{1}{p}} \left[ \int_{\mathcal{X}} \phi q d\pi \right]^{\frac{1}{q}} = \|f\|_{p,\phi} \|\phi\|_{1,\mathcal{L}}^{\frac{1}{q}}.$$

Define $L_{p,\phi}^0$ and $L_{p,\phi}^{*,0}$ as the subspaces of the elements $f$ of $L_{p,\phi}$ and $L_{p,\phi}^{*}$, respectively, such that $\int_{\mathcal{X}} f d\pi = 0$. Clearly, $L_{p,\phi}^{*,0}$ is the dual to $L_{p,\phi}^0$ w.r.t. duality $(\cdot, \cdot)$.

**Theorem 3.1.** For every $p \in (1, +\infty)$, $\{T_t\}$ is a semigroup of bounded operators in $L_{p,\phi}$. The subspace $L_{p,\phi}^0$ is invariant w.r.t. to $\{T_t\}$, and $\frac{\beta}{q} = \beta - \frac{\beta}{p}$ is a growth bound for the restriction of $\{T_t\}$ on $L_{p,\phi}^0$.

**Proof.** In the representation $L_{p,\phi} = \langle 1 \rangle \bigoplus L_{p,\phi}^0$, both summands are invariant subspaces for the semigroup $\{T_t\}$. Clearly, every $T_t$ is an identity operator on the one-dimensional subspace $\langle 1 \rangle$. Let us investigate the restriction of $\{T_t\}$ on $L_{p,\phi}^0$.

Let us prove that, for every $f \in L_{p,\phi}^0$ and $x \in \mathcal{X}$,

$$|T_t f(x)|^p \leq 2^{p-1} C_{\phi}^{\frac{p}{q}} e^{-\frac{\beta t}{q}} \phi(x) \left( T_t \left( \frac{|f|^p}{\phi(x)} \right) (x) + \|f\|^p_{p,\phi} \right).$$

Consider an exponential $\phi$-coupling $Z = (Z^1, Z^2)$ that exists by assumption. We have

$$T_t f(x) = T_t f(x) - \int_{\mathcal{X}} f(y) \pi(dy) = E \left[ f(Z_t^1) - f(Z_t^2) \right],$$

here in the last equality we have used that $\pi$ is an invariant measure and thus $Z_t^2$ has the distribution $\pi$ for every $t$. Then

$$|T_t f(x)|^p = \left| E \left[ f(Z_t^1) - f(Z_t^2) \right] I_{Z_t^1 \neq Z_t^2} \right|^p \leq E \left| f(Z_t^1) - f(Z_t^2) \right|^p \frac{\phi(Z_t^1) + \phi(Z_t^2)}{[\phi(Z_t^1) + \phi(Z_t^2)]^{\frac{1}{q}}} \times \left( E \left[ \phi(Z_t^1) + \phi(Z_t^2) \right] I_{Z_t^1 \neq Z_t^2} \right)^{\frac{1}{q}} \leq C_{\phi}^{\frac{p}{q}} e^{-\frac{\beta t}{q}} \phi(x) E \left| f(Z_t^1) - f(Z_t^2) \right|^p \frac{\phi(Z_t^1) + \phi(Z_t^2)}{[\phi(Z_t^1) + \phi(Z_t^2)]^{\frac{1}{q}}}.$$

We have $\phi(Z_t^1) + \phi(Z_t^2) \geq \max \left( \phi(Z_t^1), \phi(Z_t^2) \right)$. Hence,

$$E \left| f(Z_t^1) - f(Z_t^2) \right|^p \frac{1}{\phi(Z_t^1) + \phi(Z_t^2)} \leq 2^{p-1} E \left[ \frac{|f(Z_t^1)|^p}{\phi(Z_t^1)} I_{Z_t^1 \neq Z_t^2} \right] + 2^{p-1} \left| f \right|^p_{p,\phi} = 2^{p-1} T_t \left( \frac{|f|^p}{\phi(x)} \right)(x) + 2^{p-1} \|f\|^p_{p,\phi},$$

which proves (3.1).
As a corollary, we get the following estimate valid for every \( f \in L^0_{p, \phi} \):

\[
\|f\|_{p, \phi}^p = \int_X |T_t f(x)|^p \phi(x)^{-\frac{p}{q}}(dx) \leq 2^{p-1}C_\phi^p e^{-\frac{\beta p t}{q}} \int_X \left( T_t \left( \frac{|f|^p}{\phi^q} \right)(x) + \|f\|_{p, \phi}^p \right) \phi(x)^{-\frac{p}{q}}(dx)
\]

(3.2)

\[
= 2^{p-1}C_\phi^p e^{-\frac{\beta p t}{q}} \left( \int_X \frac{|f(x)|^p}{\phi^q(x)} \phi(x)^{-\frac{p}{q}}(dx) + \|f\|_{p, \phi}^p \right) = 2^{p}C_\phi^p e^{-\frac{\beta p t}{q}} \|f\|_{p, \phi}^p
\]

(here, the invariance property for \( \pi \) is used).

By (3.2), every \( T_t \) is bounded on \( L^0_{p, \phi} \), and thus it is bounded on whole \( L_{p, \phi} \). Moreover, (3.2) immediately implies inequality (2.4) from the definition of a growth bound.

By standard duality arguments, Theorem 3.1 yields the following corollary for the adjoint semigroup \( \{T_t^*\} \).

**Corollary 3.1.** For every \( p \in (1, +\infty) \), \( \{T_t^*\} \) is a semigroup of bounded operators in \( L^*_{p, \phi} \). The subspace \( L^*_{p, \phi} \) is invariant w.r.t to \( \{T_t\} \), and \( \frac{\beta}{q} = \beta - \frac{\beta}{p} \) is a growth bound for the restriction of \( \{T_t\} \) on \( L^*_{p, \phi} \).

3.2. Spaces \( L_{p, p} \in (1, +\infty) \). Theorem 3.1 in fact, provides that the generator of \( \{T_t\} \), considered as a semigroup in \( L_{p, \phi} \), possesses a spectral gap. The following simple corollary shows that, in a particular case, this yields existence of a spectral gap for the generator of the respective \( L_{p, p} \)-semigroup.

**Corollary 3.2.** If the function \( \phi \) is bounded, then the process \( X \) satisfies \( GB_p \left( \beta - \frac{\beta}{p} \right), p \in (1, +\infty) \).

**Proof.** Since \( 1 \leq \phi \leq C \), the norms \( \| \cdot \|_{p, \phi} \) and \( \| \cdot \|_p \) are equivalent. □

However, the general situation is more complicated, and under conditions of Theorem 3.1 respective \( L_{p, p} \) generators may fail to possess a spectral gap (see section 4). Here we provide existence of a spectral gap under additional assumptions formulated in the terms of the dual process \( X^* \) to the Markov process \( X \).

Recall that if \( \pi \) is an invariant measure for the Markov process \( X \), then, on appropriate probability space, a stationary process \( \tilde{X}_t, t \in \mathbb{R} \) can be constructed in such a way that \( \tilde{X}_0 \sim \pi \) and \( \tilde{X} \) is a Markov process with the transition function \( P_t(x, dy) \). The process \( X^*_t \) generated by \( X \) in \( L_p \) coincides with the Markov process generated by \( X^* \) in \( L_q \).

For a functions \( \phi, \psi : \mathbb{X} \to [1, +\infty) \), we write \( \phi \asymp \psi \) if

\[
\inf_x \frac{\phi(x)}{\psi(x)} > 0, \quad \sup_x \frac{\phi(x)}{\psi(x)} < +\infty.
\]

**Theorem 3.2.** Assume that there exist functions \( \phi \) and \( \phi^* \) such that the process \( X \) admits an exponential \( \phi \)-coupling, the dual process \( X^* \) admits an exponential \( \phi^* \)-coupling, and \( \phi \asymp \phi^* \).

Then, for every \( p \in [2, +\infty) \) and \( \gamma < \frac{\beta}{2p-1} \), process \( X \) satisfies \( GB_p (\gamma) \).
Proof. We will show that
\begin{equation}
\langle T_t f, g \rangle \leq C e^{-\gamma t} \| f \|_p \| g \|_q, \quad f \in L^0_p, g \in L^0_q, t \in [1, +\infty).
\end{equation}
Since \( T_t \) is a contraction semigroup in \( L_p \), this will provide that (3.5) holds true for the restriction of \( \{ T_t \} \) to \( L^0_p \) (with some other constant \( C \)), and thus will prove the required statement.

Let us verify first that
\begin{equation}
\langle T_t f, g \rangle - \int_X f d\pi \int_X g d\pi \leq C e^{-\gamma t} \| f \|_{p,\phi} \| g \|_{p,\phi}, \quad f \in L^*_{p,\phi}, g \in L^*_{p,\phi}.
\end{equation}
For \( f \in L^0_{p,\phi}, g \in L^*_{p,\phi} \), inequality (3.4) with \( C = 2C^\frac{1}{q} \) follows from (3.2). For arbitrary \( f \in L^*_{p,\phi}, g \in L^*_{p,\phi} \), one has
\[ \langle T_t f, g \rangle - \int_X f d\pi \int_X g d\pi = \langle \Pi f, \Pi g \rangle, \]
where \( \Pi f \overset{df}{=} f - \int_X f d\pi \). Since \( \Pi \) is bounded both as an operator \( L^0_{p,\phi} \rightarrow L^0_{p,\phi} \) and as an operator \( L^*_{p,\phi} \rightarrow L^*_{p,\phi} \), this yields (3.4) in the general case.

Let us proceed with the proof of (3.3). Take \( \gamma \in \left( 0, \frac{\beta}{2p-1} \right) \). Denote \( I_k = \{ x : e^{\gamma k t} \leq \phi(x) < e^{\gamma(k+1)t} \} \), \( f_k = f I_k, g_k = g I_k, k \geq 0 \). We have
\begin{equation}
\langle T_t f, g \rangle = \sum_{k,j=0}^{\infty} \langle T_t f_k, g_j \rangle = \sum_{k=0}^{\infty} \langle T_t f_k, g_k \rangle
+ \sum_{r=1}^{\infty} \sum_{k=0}^{\infty} \langle T_t f_{k+r}, g_k \rangle + \sum_{r=1}^{\infty} \sum_{k=0}^{\infty} \langle T_t f_k, g_{k+r} \rangle.
\end{equation}
Let us estimate the summands in the right hand side of (3.5) separately.

We have from (3.4)
\[ |\langle T_t f_k, g_k \rangle| \leq \left| \int_X f_k d\pi \right| \left| \int_X g_k d\pi \right| + C e^{-\frac{\gamma k}{q}} \| f_k \|_{p,\phi} \| g_k \|_{p,\phi}.\]
By the construction, \( g_k = 0 \) on the set \( \{ \phi > e^{\gamma(k+1)t} \} \), thus
\[ \| g_k \|_{p,\phi} = \left[ \int_X |g_k|^q \phi d\pi \right]^{\frac{1}{q}} \leq \| g_k \|_q \cdot e^{\frac{\gamma(k+1)}{q}}. \]
Analogously,
\[ \| f_k \|_{p,\phi} = \left[ \int_X |f_k|^p \phi^{-\frac{p}{q}} d\pi \right]^{\frac{1}{p}} \leq \| f_k \|_p \cdot e^{-\frac{\gamma k}{q}}. \]
For \( k \geq 1 \), we have
\[ \left| \int_{I_k} f_k d\pi \right| = \left| \int_{I_k} f d\pi \right| \leq \| f \|_p \phi^{-\frac{1}{q}}(I_k) \leq \| f \|_p \phi^{-\frac{1}{q}} \| \phi \|_1 e^{-\frac{\gamma k}{q}}. \]
For \( k = 0 \), since \( \int_X f d\pi = 0 \),
\[ \left| \int_{X \setminus I_0} f_k d\pi \right| \leq \| f \|_p \phi^{-\frac{1}{q}}(X \setminus I_0) \leq \| f \|_p \phi^{-\frac{1}{q}} \| \phi \|_1 e^{-\frac{\gamma k}{q}}. \]

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Analogously,

\[ \left| \int_X g_k \, d\pi \right| \leq \|g_k\|_q \|\phi\|_p^\frac{1}{q} \min\{e^{-\frac{k}{p}}, e^{-\frac{k}{q}}\}. \]

Therefore,

\[
\sum_{k=0}^{\infty} |\langle T_k f, g_k \rangle| \leq C(e^{-\gamma t} + e^{-\frac{2}{p} t + \frac{\gamma}{q} t}) \sum_{k=0}^{\infty} \|f_k\|_p \|g_k\|_q \leq \\
\leq C(e^{-\gamma t} + e^{-\frac{2}{p} t + \frac{\gamma}{q} t}) \left[ \sum_{k=0}^{\infty} \|f_k\|_p^p \right]^\frac{1}{p} \left[ \sum_{k=0}^{\infty} \|g_k\|_q^q \right]^\frac{1}{q} = C(e^{-\gamma t} + e^{-\frac{2}{p} t + \frac{\gamma}{q} t}) \|f\|_p \|g\|_q.
\]

In the last equality, we have used that the family \( \{I_k\} \) is disjoint, and hence

\[
\sum_{k} \|f_k\|_p^p = \sum_{k} \int_{I_k} |f|^p \, d\pi = \|f\|_p^p, \quad \sum_{k} \|g_k\|_q^q = \sum_{k} \int_{I_k} |g|^q \, d\pi = \|g\|_q^q.
\]

By the choice of \( \gamma \), we have \( \frac{\beta}{p} - \frac{\gamma}{q} > \gamma \left( \frac{2p-1}{p} - \frac{1}{q} \right) = \gamma \left( 2 - p^{-1} - q^{-1} \right) = \gamma \). Therefore, finally, (3.6)

\[
\sum_{k=0}^{\infty} |\langle T_k f_k, g_k \rangle| \leq C e^{-\gamma t} \|f\|_p \|g\|_q, \quad f \in L_{p}, g \in L_{q}, t \in \mathbb{R}^+.
\]

Analogously, for every \( k \geq 0, r \geq 1 \), we have

\[
|\langle T_k f_{k+r}, g_k \rangle| \leq C \|f_{k+r}\|_p \|g_k\|_q \left( e^{-\frac{(k+r)}{q} t} \min\{e^{-\frac{2}{p} t}, e^{-\frac{k}{p}}\} + e^{-\frac{2}{p} t + \frac{\gamma(k+1)}{q} t - \frac{\gamma(k+1)}{q} t} \right)
\]

\[
\leq C \|f_{k+r}\|_p \|g_k\|_q e^{-\gamma t} e^{-\frac{\gamma(r-1)}{q} t}.
\]

For every given \( r \geq 1 \),

\[
\sum_{k=0}^{\infty} \|f_{k+r}\|_p \|g_k\|_q \leq \left[ \sum_{k=0}^{\infty} \|f_{k+r}\|_p^p \right]^\frac{1}{p} \left[ \sum_{k=0}^{\infty} \|g_k\|_q^q \right]^\frac{1}{q} \leq \|f\|_p \|g\|_q.
\]

In addition,

\[
\sum_{r=1}^{\infty} e^{-\frac{\gamma(r-1)}{q} t} \leq [1 - e^{-\frac{\gamma}{q}}], \quad t \in [1, +\infty).
\]

Hence, finally,

(3.7) \[
\sum_{r=1}^{\infty} \sum_{k=0}^{\infty} |\langle T_k f_{k+r}, g_k \rangle| \leq C e^{-\gamma t} \|f\|_p \|g\|_q, \quad f \in L_{p}, g \in L_{q}, t \in [1, +\infty).
\]

Up to this moment, we have not used the assumption that \( X^* \) admits an exponential \( \phi^* \)-coupling. Now, we use this assumption in order to estimate the last summand in the right hand side of (3.5). We replace \( X, \phi, p \) by \( X^*, \phi^*, q \) and write, under this assumption, the following estimate analogous to (3.4):

\[
|\langle T_k g, f \rangle| - \int_X g \, d\pi \int_X \bar{f} \, d\pi \leq C e^{-\frac{\gamma}{2} t} \|g\|_{q, \phi^*} \|f\|_{q, \phi^*}, \quad g \in L_{q, \phi^*}, f \in L_{q, \phi^*}.
\]

From the condition \( \phi \prec \phi^* \) we conclude that

\[
\|g_k\|_{q, \phi^*} = \left[ \int_{I_k} |g_k|^q (\phi^*)^{-\frac{q}{p}} \, d\pi \right]^\frac{1}{q} \leq C \|g_k\|_q e^{-\frac{\gamma k}{p} t},
\]
Corollary 3.3. Under conditions of Theorem 3.2, for or, equivalently,

\[ T \text{ represented as} \]

Theorem 3.3. Consider the semigroup \( \{T_t\} \) generated by \( X \) in \( L_2 \). Assume that \( X \) is time-reversible, or, equivalently, \( T_t = T^*_t, t \in \mathbb{R}^+ \).

Then \( X \) satisfies \( PI \left( \frac{\beta}{q} \right) \).

Proof. Since \( T_t, t \in \mathbb{R}^+ \) is a contraction semigroup of self-adjoint non-negative operators, it can be represented as \( T_t = e^{-At}, t \in \mathbb{R}^+ \), where \( A \) is the \( L_2 \)-generator of the process \( X \), and \( A \) is self-adjoint and non-negative. Let \( P(d\lambda) \) be the projector-valued measure from the spectral decomposition for the operator \( A: \)

\[ A = \int_0^\infty \lambda P(d\lambda). \]

Then

\[ T_t = \int_0^\infty e^{-\lambda t} P(d\lambda), \]

and, for every \( f \in L_2 \),

\[ \|T_t f\|_2^2 = \int_0^\infty e^{-\lambda t} (P(d\lambda)f, f), \quad t \in \mathbb{R}^+, \]

(3.9) where \((\cdot, \cdot)\) denotes scalar product in \( L_2 \).

We have \( \| \cdot \|_2 \leq \| \cdot \|^*_{2, \phi} \), and hence Corollary 3.1 provides that, for every \( f \in L^*_{2, \phi} \), there exists a constant \( C(f) \in \mathbb{R}^+ \) such that

\[ \|T_t f\|_L^2 \leq C(f)e^{-\beta t}, \quad t \in \mathbb{R}^+. \]
The latter inequality and (3.9) implies that, for such \( f \), the measure \((P(d\lambda)f, f)\) is supported by \([\beta, +\infty)\). One has \((P(\Delta)f_n, f_n) \to (P(\Delta)f, f)\) for every Borel set \(\Delta\) and every sequence \(f_n \to f\) in \(L_2\). In addition, the set \(L_{2,\phi}\) is dense in \(L_2^0 \setminus \{1\}\). Therefore, the measure \((P(d\lambda)f, f)\) is supported by \([\beta, +\infty)\) for every \(f \in L^0_2\), and (3.9) yields (2.7) with \(\gamma = \frac{\beta}{2}\).

Note that all the properties \(SG_2(\gamma), GB_2(\gamma),\) and \(PI(\gamma)\) coincide for a time-reversible process \(X\). This can be verified easily, and the argument here is similar to the previous proof. The spectral decomposition theorem is the key tool here, and the claim for the generator \(A\) of \(\{T_t\}\) to be self-adjoint (or, at least, normal) is crucial. This claim is closely related with the structure of the process. For instance, it is satisfied when \(X\) is a diffusion process. On the other hand, for \(X\) being a solution to SDE with a jump noise, this claim is highly restrictive. This motivates the following modification of Theorem 3.3, that extends the domain of its applications. The construction exposed below is an appropriate modification of the one introduced in [Chen00].

The rough idea is to replace the \(L_2\)-generator \(A\) by the operator \(A^0 \overset{df}{=} \frac{1}{2}(A + A^*)\). Since the symmetric part of the Dirichlet form generated by \(A\) coincides with the Dirichlet form generated by \(A^0\), the \(PI(\gamma)\) property for the process \(X\) would be equivalent to the \(PI(\gamma)\) property for the process \(X^0\) corresponding to \(A^0\). In the formal realisation of this idea, one needs, at least, to take care of the domains of various generators.

**Theorem 3.4.** Assume there exists a time-reversable Markov process \(X^0\) that admits a \(\phi\)-coupling for some \(\phi\). Assume also that there exists a set \(D \subset L_2\) such that

(i) \(D \cap L^0_2\) is dense in \(L^0_2\);

(ii) \(D\) is invariant w.r.t. \(L_2\)-semigroup corresponding to \(X\);

(iii) \(D\) belongs to the domains to the \(L_2\)-generators \(A, A^*\), and \(A^0\) corresponding to \(X, X^*, \) and \(X^0\), respectively, and

\[A^0f = \frac{1}{2}(Af + A^*f), \quad f \in D.\]

Then \(X\) satisfies \(PI_2\left(\frac{\beta}{2}\right)\), where \(\beta\) is the constant from the definition of the \(\phi\)-coupling for \(X^0\).

**Proof.** Denote by \(\{T^0_t\}\) the \(L_2\)-semigroup generated by \(X^0\). It follows from the previous theorem that, for every \(f \in D \cap L^0_2\),

\[(A^0f, f) \leq -\frac{\beta}{2}\|f\|_2^2.\]

Since \((A^0f, f) = \frac{1}{2}[(Af, f) + (A^*f, f)] = \text{Re}\ (Af, f)\), this yields

\[\text{Re}\ (Af, f) \leq -\frac{\beta}{2}\|f\|_2^2, \quad f \in D \cap L^0_2.\]

Then, for \(f \in D \cap L^0_2\), we have

\[
\frac{d}{dt}\|T_tf\|^2 = 2\text{Re}\ (AT_tf, T_tf) \leq -\beta\|T_tf\|^2, \quad t \in \mathbb{R}^+,
\]

here we have used that, by the condition (ii), \(T_tf \in D \cap L^0_2\). Hence,

\[(3.10)\quad \|T_tf\|^2 \leq e^{-\beta t}\|f\|^2_2\]

for every \(f \in D \cap L^0_2\). Since \(D \cap L^0_2\) is dense in \(L^0_2\), (3.11) holds true for every \(f \in L^0_2\). \(\square\)
4. One example

In this section, we give an example of a Markov process which demonstrates relations between the objects considered in our main exposition. We will see that process that admits an exponential \( \phi \)-coupling may fail to possess a spectral gap property. This would make more clear the statements of section 3.2: in general, in order to control growth bounds and spectral properties of \( L_p \) semigroups, one should control ergodic properties both for the process \( X \) and for the dual process \( X^* \). Also, we will see that the exponential \( L_2 \) growth bound (2.6) is not equivalent to the Poincaré inequality (2.7). Consequently, for time-irreversible processes these two inequalities should be studied separately.

Let \( X = [0, +\infty) \) and the extended generator of the process \( X \) be defined on the functions \( f \in C^1 \) by the formula

\[
\mathcal{A} f(x) = -a(x)f'(x) + \theta(x) \sum_{k=1}^{\infty} (1-p)p^{k-1}[f(x_k) - f(x)], \quad x \in X,
\]

where \( \{x_k, k \geq 1\} \subset [1, +\infty), p \in (0, 1), \) and \( a, \theta \in C^1 \) are functions taking values in \([0, 1]\). We assume that

\[
a(0) = 0, \quad a(x) > 0, \quad x > 0, \quad a(x) = 1, \quad x \geq 1 \quad \text{and} \quad \theta(x) = 0, \quad x \geq 1, \quad \theta(x) = 1, \quad x < \frac{1}{2}.
\]

It is also assumed that \( x_k < x_{k+1}, k \geq 1; \) that is, the points \( x_k, k \geq 1 \) are naturally ordered.

The dynamics of the process \( X \) contains two parts. The first (deterministic) component is given by the ordinary differential equation (ODE) \( dx = -a(x)dt \). The second (jump) part corresponds to possibility for the process to jump at one of the positions \( x_k, k \geq 1 \). The intensity for such a jump depends on \( k \) and the current position \( x \), and is equal \( (1-p)p^{k-1}\gamma \).

For this model, ergodic and spectral properties can be expressed completely in the terms of \( p \) and \( \{x_k, k \geq 1\} \); let us formulate corresponding statements.

1. If there exists \( \alpha > 0 \) such that \( \sum_{k \geq 1} p^k e^{\alpha x_k} < +\infty \), then \( \phi(x) = e^{\alpha x} \).

2. Condition \( \sup_{k}(x_{k+1} - x_k) < +\infty \) is necessary for \( X \) to satisfy \( SG_p(\gamma) \) with some \( \gamma > 0 \), and sufficient for \( X \) to satisfy \( GB_p(\gamma') \) with some \( \gamma' > 0 \).

3. For any sequence \( \{x_k, k \geq 1\} \), the process \( X \) does not satisfy the Poincaré inequality.

**Proof of statement (1).** It is clear that the Lyapunov-type condition (2.2) holds true with \( \phi(x) = e^{\alpha x} \). Hence, it is enough to prove that the local Doeblin condition holds and then use Proposition 2.3. Denote by \( \psi_t(x), t \in \mathbb{R}^+, x \in \mathbb{X} \) the flow generated by ODE \( dx = -a(x)dt \). For a given compact \( K \subset \mathbb{X} \), there exists \( T_K > 0 \) such that \( \psi_t(x) \leq \frac{1}{2}, x \in K, t \geq T_K \). This together with the Chapman-Kolmogorov equation yields that we need to prove Doeblin condition for the compact \( K = [0, 2^{-1}] \), only.

On the segment \([0, 2^{-1}]\), the intensity of a jump to a point \( x_k, k \geq 1 \) is constant and equals \( (1-p)p^{k-1} \). If the starting point \( x \) belongs to this segment, then the process spends inside this segment a random time that has exponential distribution with intensity \( 1 \). As soon as the process jumps to \( x_2 \), it moves with the constant speed \( a = -1 \) and does not have any jumps up to any time moment \( t \leq x_2 - 1 \). This means that, for \( t \leq x_2 - 1, x \in [0, 2^{-1}] \),

\[
P_t(x, dy) \geq (1-p)pP(\eta \leq t, \eta \in dy + t - x_2) = (1-p)e^{-y-t+x_2}I_{y \leq dy},
\]

where \( \eta \) denotes an exponential random variable with intensity \( 1 \). Therefore, for \( T = x_2 - 1 \)

\[
\sup_{x,x' \in [0,2^{-1}]} \|P_T(x, \cdot) - P_T(x_2, \cdot)\|_{\text{var}} \leq 2 - (1-p)p \int_0^T e^{-y+1}dy < 2,
\]
which gives the Doeblin condition on $[0, 2^{-1}]$.

**Proof of statement (2): sufficiency.** Let us determine the dual process $X^*$. Note that under assumption $\sup_k (x_{k+1} - x_k) < +\infty$ one has $\sum_{k \geq 1} p^k e^{\alpha x_k} < +\infty$ for sufficiently small $\alpha > 0$. Hence, by statement (1), the invariant measure $\pi$ is unique.

Denote $I_0 = [0, x_1), I_k = (x_k, x_{k+1}), k \geq 1$. The invariant measure $\pi$ is determined by the relations

$$\int_X Af \, d\pi = 0, \quad f \in \text{Dom}(A) = \{ f : f \text{ and } Af \text{ are bounded} \}.$$ 

Taking in this relation $f \in C^1$ with $\text{supp} \, f \subset I_k$, we get that $\pi|_{I_k}$ has a density $\rho_k$. In addition, this density is constant for $k \geq 1$ and has the form

$$C[a(x)]^{-1} \exp \left[ \int_1^x \frac{\theta(y)}{a(y)} \, dy \right]$$

for $k = 0$. On the other hand, taking $f \in C^1$ with $\text{supp} \, f \subset I_{k-1} \cup I_k$, we obtain that $\pi\{\{x_k\}\} = 0, k \geq 1$, and

$$\rho_k(x_k) = p \rho_{k-1}(x_k), \quad k \geq 1.$$ 

These relations and normalizing condition $\pi(X) = 1$ determine the invariant measure $\pi$ uniquely.

As soon as $\pi$ is determined, one can find the transition probability for the dual process using the relations

$$\int_A P^*_k(x, B)\pi(dx) = \int_B P_k(x, A)\pi(dx), \quad A, B \in \mathcal{B}(X).$$ 

Without a detailed exposition of this standard step, we just give the description of the dual process. Its dynamics also contains two components. The deterministic component is given by the ODE $dx = a(x)\, dt$. When the process comes to one of the points $x_k, k \geq 1$, it can either continue its move or make a jump into the segment $[0, 1]$. The probability of a jump is equal $(1 - p)$, and the distribution of the position of the process after a jump has the density

$$\frac{\theta(x)}{a(x)} \exp \left[ \int_1^x \frac{\theta(y)}{a(y)} \, dy \right].$$

Consider the function $\phi^*(x) = E_x e^{\alpha \tau^*}$, where $\tau^*$ is the hitting time of the segment $[0, 1]$ by the dual process $X^*$. For $x \leq 1$, $\phi^*(x) = 1$. If the starting point is $X^*_0 = x > 1$, the process $X^*_k$ moves with the constant speed 1 and, at every point $x_k, k \geq 1$, gets a chance to jump into the segment $[0, 1]$ with probability $(1 - p)$. Hence for $x > 1$ one has

$$\phi^*(x) = \sum_{k=K(x)}^{\infty} (1 - p)p^{k-K(x)} e^{\alpha(x_k - x)},$$

where $K(x) = \inf\{k : x_k > x\}$. Therefore, for $\alpha > 0$ small enough, the function $\phi^*$ is bounded:

$$\phi^*(x) \leq \sum_{k=K(x)}^{\infty} (1 - p)p^{k-K(x)} e^{\alpha(k-K(x)+1) \sup_k (x_{k+1} - x_k)}$$

$$\leq \sum_{k=1}^{\infty} (1 - p)p^{k-1} e^{\alpha \sup_k (x_{k+1} - x_k)} < +\infty.$$ 

Like it was done in the proof of statement (1), one can verify that $X^*$ satisfies the local Doeblin condition. Then by Proposition 2.2 we get that $X^*$ admits an exponential $\phi^*$-coupling. By Corollary 3.2 we get the required statement.
Remark 4.1. It can be verified that the initial process $X$ does not admit an exponential $\phi$-coupling for any bounded $\phi$. One can say that, in the example in the discussion, the ergodic properties of the dual process $X^*$ are better than those of the process $X$ itself. On the other hand, $L_p$ estimates for the process $X$ are equivalent to $L_q$ estimates for the process $X^*$ ($p^{-1} + q^{-1} = 1$). Hence, the one interested in $L_p$ rates can choose to start the investigation either from $X$ or from $X^*$ depending on their ergodic properties. This is exactly what we have done in our proof. Another possibility is provided by Theorem 4.2, where ergodic properties of $X$ and $X^*$ are exploited jointly. We will use this possibility in section 5 below.

Proof of statement (2): necessity. Let $X$ satisfy $SG_p(\gamma)$ with some $p \in (1, +\infty)$, $\gamma > 0$. Then $0$ is a resolvent point for the restriction of $\{T_t\}$ to $L^0_p$, and therefore there exists $C_1 \in \mathbb{R}^+$ such that

$$
\limsup_{\lambda \to 0^+} \int_0^\infty \int_X e^{-\lambda T_t f(x)g(x)}\pi(dx)dt \leq C_1 \|f\|_p \|g\|_q, \quad f \in L^0_p, \ g \in L^0_q.
$$

(4.1)

For every given $Q > 0$, (4.1) also holds true with $f$ replaced by $T_Q f$. Then easy transformation gives

$$
\limsup_{\lambda \to 0^+} \int_0^Q \int_X e^{-\lambda T_t f(x)g(x)}\pi(dx)dt \leq 2C_1 \|f\|_p \|g\|_q, \quad Q \in \mathbb{R}^+, \ f \in L^0_p, \ g \in L^0_q.
$$

(4.2)

Denote $d_k = x_{k+1} - x_k$, $y_k = x_k + \frac{1}{4}d_k$, $z_k = x_k + \frac{3}{4}d_k$ and put

$$
f_k = g_k = I_{(x_k,y_k)} - I_{(y_k,z_k)}, \quad k \geq 1.
$$

For $t \leq \frac{1}{2}d_k$, we have $T_t f_k(x) = f_k(x) - t = I_{(x_k+t,y_k+t)}(x) - I_{(y_k+t,z_k+t)}(x)$. Recall that the invariant measure $\pi$ has a positive constant density $\rho_k$ on every segment $I_k = (x_k, x_{k+1})$. Then straightforward calculations show that

$$
\int_X T_t f(x)g(x)\pi(dx) \geq 4^{-1}d_k \rho_k, \quad t \leq 4^{-1}d_k.
$$

On the other hand, $\|f_k\|^p = \|g_k\|^q = \frac{1}{2}d_k \rho_k$. Therefore, inequality (4.2) with $Q = \frac{1}{2}d_k$ gives the estimate

$$
8^{-1}d_k^2 \rho_k \leq 2C_1 d_k \rho_k, \quad k \geq 1,
$$

which implies that the sequence \{d_k = x_{k+1} - x_k\} is bounded.

Proof of statement (3). For a fixed $k \geq 1$, consider the function $f_k$ introduced in the previous proof. We have

$$
\|T_t f_k\|^2 = \|f_k(\cdot - t)\|^2 = \|f_k\|^2, \quad t \leq 2^{-1}d_k.
$$

But under the Poincaré inequality (2.7) one should have

$$
\|T_t f\|^2 < \|f\|^2, \quad t > 0, \quad f \in L^2_2, \ f \neq 0.
$$

Therefore, for the process $X$ the Poincaré inequality fails.

5. Solutions to SDE’s with jump noise

In this section we apply the general results of Section 3 to solution to SDE of the type

$$
dx(t) = a(X(t))dt + \int_{\|u\| \leq 1} c(X(t-), u)\nu(dt, du) + \int_{\|u\| > 1} c(X(t-), u)\nu(dt, du).
$$

(5.1)

Here $\nu$ is a Poisson point measure on $\mathbb{R}^+ \times \mathbb{R}^d$ with the intensity measure $dt\mu(du)$, $\mu$ is the corresponding Lévy measure, $\nu(dt, du) = \nu(dt, du) - dt\mu(dt)$ is the compensated point measure, and coefficients $a, c$ satisfy standard conditions sufficient for existence and uniqueness of a strong condition (e.g. local Lipschitz and linear growth conditions).
We start the discussion mentioning that, for the process $X$ defined by \((5.1)\), efficient tools to provide existence of an exponential $\phi$-coupling are available. In [Kul09], it was demonstrated that, for such processes, the local Doeblin condition can be verified efficiently. This condition follows from appropriate support condition ([Kul09], condition $S$) and a (partial) continuity in variation of the law of the solution to SDE w.r.t. initial value. The latter property means that there exists a subset $\Omega'$ of the initial probability space $\Omega$ such that $P(\Omega') > 0$ and the law of the solution conditioned by $\Omega'$ is continuous in total variation norm. This property holds under a non-degeneracy condition formulated in terms of the random point measure ([Kul09], condition $N$), and the main tool in its proof is a certain version of a stochastic calculus of variations for SDE's with jumps. We do not give a detailed overview here, referring interested reader to [Kul09].

At the same time, the Lyapunov-type condition for solutions to SDE's of the type \((5.1)\) is quite transparent (see [Mas07], [Kul09] and discussion therein). Therefore, for solutions to SDE's with jump noise, one can prove existence of an exponential $\phi$-coupling using Proposition 2.1.

However, solutions to SDE's with jump noise, typically, are not time-reversible. The example given in section 4 indicates that, to investigate $L_p$ convergence rates and spectral properties for a time-irreversible Markov process, it may be insufficient to have an exponential $\phi$-coupling for the process itself. In general, an analysis of the ergodic properties of the the dual processes is also required. In this section, we provide such an analysis and give sufficient condition for the process $X$ defined by \((5.1)\) to possess a spectral gap property.

In order to keep exposition reasonably short, we restrict our considerations by a particular, but important class of one-dimensional Lévy driven Ornstein-Uhlenbeck processes; that is, solutions to \((5.1)\) with lineal drift and additive jump noise. Henceforth, in the rest of this section, $X$ is a real-valued process solution to SDE

\[(5.2)\]

$$dX_t = -aX_t \, dt + dZ_t,$$

where $a > 0$ and $Z_t = \int_0^t \int_{|u| \geq 1} u\nu(ds, du) + \int_0^t \int_{|u| < 1} u\tilde{\nu}(ds, du)$ is a Lévy process.

Ergodic properties for Lévy driven Ornstein-Uhlenbeck processes are well studied. It is known that a Lévy driven Ornstein-Uhlenbeck process $X$ is ergodic if and only if $\int_{|u| \geq 1} \ln |u|\mu(du) < +\infty$ (see [SY84]). Sufficient conditions for exponential ergodicity for $X$ are also available (see [Mas07] and references therein). Our intent is to establish a spectral gap property for the (unique) stationary version of $X$. We give one sufficient condition of that type. Remark that this condition is not strongest possible and allows various generalizations; see Remarks 5.1 and 5.2 after the proof of Theorem 5.1.

**Theorem 5.1.** Assume that

1) $\mu(\mathbb{R}^-) = \mu(\mathbb{R}^+) = \infty$;
2) $\mu$ is supported by a bounded set;
3) $\int_{|u| \leq 1} |u|\mu(du) < +\infty$.

Then, for every $p > 1$, process $X$ satisfies $GB_p(\gamma)$ for sufficiently small $\gamma > 0$.

**Proof.** We will prove that $X$ and $X^*$ admit an exponential $\phi$-coupling with the same $\phi(x)$; then Theorem 3.2 would yield the required statement.
It is known (see [KK09], Proposition 2.1) that under condition 1) the invariant distribution $\pi$ admits a $C^\infty$ density; denote this density by $\rho$. In the sequel, we need the following asymptotic result. Denote $M_1(\xi) = \int_{\mathbb{R}} u(e^{\xi u} - 1)\mu(du)$, $M_2(\xi) = \int_{\mathbb{R}} u^2e^{\xi u}\mu(du)$, $\xi \in \mathbb{R}$, 

$$M_k(\xi) = \int_{\mathbb{R}} M_k(e^{-\frac{\sigma_*}{\xi}}) ds \quad \xi \in \mathbb{R}, k = 1, 2.$$ 

Clearly, 

$$\frac{d}{d\xi} M_1(\xi) = M_2(\xi) > 0$$

and condition 1) yields $M_1(\xi) \to \pm\infty, \xi \to \pm\infty$. Then for every $x \in \mathbb{R}$ there exists unique solution $\xi = \xi(x)$ to the equation 

$$M_1(\xi) = x.$$ 

Proposition 5.1. ([KK09], Theorem 7.1) Under conditions 1) and 3) of Theorem 5.1, 

$$\begin{bmatrix} \rho(x+y) \\ \rho(y) \end{bmatrix} e^{y \xi(x)} \to 1, \quad x \to \infty$$

uniformly by $y \in Y$ for every bounded set $Y \subset \mathbb{R}$.

We proceed with the proof of the theorem. Our first step is to specify the dual process $X^\ast$. Since this step is quite standard, we sketch the argument and omit technical details.

Every $f \in C^1$ with at most polynomial growth of its derivative belongs to the domain of the extended generator $A$ and 

$$Af(x) = -axf'(x) + \int_{\mathbb{R}} [f(x+u) - f(x)]\mu(du).$$

We have 

$$\int_{\mathbb{R}} Af d\pi = 0$$

for every such $f$. This yields that the invariant density $\rho$ satisfies the relation 

(5.3) 

$$ax\rho'(x) + a\rho(x) + \int_{\mathbb{R}} [\rho(x-u) - \rho(x)]\mu(du) = 0$$

The formally adjoint operator to $A$ is given by the formula 

$$A^\ast f(x) = \rho^{-1}(x) \left( \frac{d}{dx} [ax\rho(x)f(x)] + \int_{\mathbb{R}} [f(x-u)\rho(x-u) - f(x)\rho(x)]\mu(du) \right).$$

This relation combined with (5.3) provides that the extended generator of the dual process $X^\ast$ is given by 

$$A^\ast f(x) = axf'(x) + \int_{\mathbb{R}} \left( (f(x-u) - f(x))\frac{\rho(x-u)}{\rho(x)} \right) \mu(du).$$

Note that the domain of $A^\ast$ is yet to be determined. It can be verified by additional investigation that $\rho(x) > 0, x \in \mathbb{R}$, but this would lead to unnecessary complication of the proof. For our needs, it is sufficient to refer to Proposition 5.1 which implies that $\rho(x) > 0$ outside some segment $[-I, I]$. Furthermore it is easy to verify that, for every $\varepsilon > 0$,

$$\mathcal{M}_1(\xi)e^{-(\sigma_*+\varepsilon)|\xi|} \to 0, \quad \xi \to \infty$$
with $\sigma_\ast \overset{df}{=} \inf\{ \sigma : \mu(|u| > \sigma) = 0 \}$ (see [KK09], Example 5.2), and consequently
\[
\sup_{|u| \leq \sigma_\ast} \frac{\rho(x - u)}{\rho(x)} \leq C(1 + |x|^p),
\]
where $C, p > 0$ are some constants. Hence every $f \in C^1$ with at most polynomial growth of its derivative, being constant on $[-I - \sigma_\ast, I + \sigma_\ast]$, belongs to the domain of $A^\ast$.

Consider $\phi \in C^1$ such that $\phi \geq 1$, $\phi$ is constant on $[-I - \sigma_\ast, I + \sigma_\ast]$, and $\phi(x) = |x|$ for $|x|$ large enough. Then for $|x|$ large enough
\[
A\phi(x) = -ax \sign x + \int_{-\sigma_\ast}^{\sigma_\ast} u\mu(du) \leq -\frac{a}{2}|x| = -\frac{a}{2}\phi(x);
\]
that is, $\phi$ satisfies the Lyapunov-type condition \(\text{(2.2)}\) w.r.t. the process $X$. On the other hand, from Proposition 5.1 we get that, for every $\delta > 0$, there exist $C > 0$ such that
\[
A^\ast \phi(x) \leq ax \sign x - (1 - \delta) \int_{\mathbb{R}} (|x - u| - |x|)e^{-u\xi(x)}\mu(du)
\]
\[
= \sign x [ax - (1 - \delta)M_1(\xi(x))] = |x|[a(1 - \delta)x^{-1}M_1(\xi(x))].
\]
It can be verified easily (e.g. see the proof of Theorem 7.1) that for every $\sigma \in (0, 1)$
\[
M_1(\sigma\xi)|M_1(\xi)|^{-1} \to 0, \quad \xi \to \infty
\]
and consequently
\[
M_1(\xi)|M_1(\xi)|^{-1} \to 0, \quad \xi \to \infty.
\]
Therefore $x^{-1}M_1(\xi(x)) \to +\infty$, $x \to \infty$ because $M_1(\xi(x)) = x$. This and \(\text{(5.4)}\) yield that $\phi$ satisfies the Lyapunov-type condition \(\text{(2.2)}\) w.r.t. the process $X$.

For the process $X$, the local Doeblin condition holds; we have already mentioned that one can derive this condition using results of [Kul09]. In particular, in the case under consideration one can deduce the local Doeblin condition from Theorem 1.3 [Kul09] using literally the same arguments with those given in the proof of Proposition 0.1 [Kul09].

On the other hand, $X^\ast$ is not a solution to SDE of the type \(\text{(5.1)}\). It is a process with non-constant rate of jumps, and such processes were not considered in [Kul09]. Henceforth, one can not deduce the local Doeblin condition for $X^\ast$ from the results of [Kul09]. However, the stochastic calculus of variations that provides (partial) continuity in variation is available for the processes with non-constant rate of jumps as well, see [Kul08], and the main results from [Kul09] can be extended for such processes without principal changes. In particular, one can prove the local Doeblin condition for $X^\ast$ following the proof of Proposition 0.1 [Kul09] and using within this proof Theorem 4.2 [Kul08] instead of Theorem 1.3 [Kul09].

Now we apply Proposition \(\text{(2.1)}\) twice, and get that both $X$ and $X^\ast$ admit an exponential $\phi$-coupling. Applying Theorem \(\text{(3.2)}\) completes the proof.

\textbf{Remark 5.1.} (On the class of equations). In Theorem 5.1 we restrict our consideration by the linear SDE’s with jump noise. The only point in the proof where this structural assumption was used substantially – the Lyapunov-type condition for $\phi$ w.r.t. the dual process – is based on the estimates for the ratio $\frac{\rho(x+y)}{\rho(x)}$. In [KK09], these estimates are obtained via harmonic analysis arguments, and here is the point where the linear structure of SDE’s under investigation is substantial. However, this is not the only possible technique. Supposedly, using ‘stochastic calculus of variations’ tools similar to those given in [Kul06], Section 6, one can extend such estimates to non-linear SDE’s with jump noise as well, and then give an extension of Theorem 5.1 to this more general class of equations. We postpone such a generalization to a further publication.
Remark 5.2. (On conditions). Conditions 1) and 2) of Theorem 5.1 come from [KK09] Theorem 7.1. The first condition is rather mild, and the second one allows a wide field of modifications. For instance, it can be replaced by an appropriate condition on the ‘exponential tails’ of the Lévy measure \( \mu \) (see [KK09], Proposition 6.1 and discussion before Theorem 7.1). On the other hand, condition 3), though not used explicitly, is crucial in our framework. Without this condition one can not apply Theorem 4.2 [Kul08] which is required to get the local Doeblin condition for the dual process.

6. Exponential moments for hitting times under Poincaré inequality

The results of section 3 allows one to establish spectral gap property for a given process \( X \) in the following way: first, prove the local Doeblin condition to hold true; second, find some \( \phi \) such that the recurrence conditions 1) – 3) from Theorem 2.1 holds true; then, if \( X \) is time-irreversible, repeat this procedure for the dual process \( X^* \); and, finally, deduce the required property using Theorems 3.2 – 3.4.

We have already mentioned that the local Doeblin condition is straightforward, and can be verified efficiently for important classes of processes like diffusions or solutions to SDE’s with jump noise. The second part in the framework outlined above – the recurrence conditions – looks less transparent since there is a lot of freedom in the choice of \( \phi \). Proposition 2.2, in fact, reduces such a choice to the class of functions of the form \( \phi(x) = E_x e^{\alpha \tau_K} \). In this section we demonstrate that this reduction well corresponds to the matter of the problem.

Considerations of this section are mainly motivated by the paper [Mat97], where the relation between the family of certain weak versions of the Poincaré inequality, on one hand, and the moments of the hitting times

\[
\tau_K = \inf \{ t : X_t \in K \},
\]
on the other hand, is investigated.

In what follows, we suppose an invariant measure \( \pi \) for the process \( X \) to be fixed, and consider the Dirichlet form \( \mathcal{E} \) on \( L^2_\pi \) corresponding to the process \( X \) (see section 2.2). The form \( \mathcal{E} \) is supposed to be regular; that is, the set \( \mathcal{E}(\mathcal{D}) \cap C_0(\mathbb{X}) \) is claimed to be dense both in \( \mathcal{D}(\mathcal{E}) \) w.r.t. the norm \( \| \cdot \|_{\mathcal{E},1} \) and in \( C_0(\mathbb{X}) \) w.r.t. uniform convergence on a compacts (\( C_0(\mathbb{X}) \) is the set of continuous functions with compact supports). We also assume that the sector condition holds true:

\[
\exists D \in \mathbb{R}^+ : |\mathcal{E}(f,g)| \leq D \|f\|_{\mathcal{E},1} \|g\|_{\mathcal{E},1}, \quad f, g \in \mathcal{D}(\mathcal{E}).
\]

It is well-known (see the discussion in Introduction to [Mat97] and references therein) that the hitting times \( \tau_K \) have natural application in the probabilistic representation for the family of \( \alpha \)-potentials for the Dirichlet form \( \mathcal{E} \). The \( \alpha \)-potential, for given \( \alpha > 0 \) and closed \( K \subset \mathbb{X} \), is defined as the function \( h^K_\alpha \in \mathcal{D}(\mathcal{E}) \) such that \( h^K_\alpha = 1 \) quasi-everywhere on \( K \), and \( \mathcal{E}(h^K_\alpha, u) = -\alpha (h^K_\alpha, u) \) for every quasi-continuous function \( u \in \mathcal{D}(\mathcal{E}) \) such that \( u = 0 \) quasi-everywhere on \( K \). On the other hand,

\[
h^K_\alpha(x) = E_x e^{-\alpha \tau_K}, \quad x \in \mathbb{X}.
\]

It is a straightforward corollary of the part (i) of the main theorem from [Mat97] that, if \( X \) possesses \( PI(\gamma) \) with some \( \gamma > 0 \), then \( E_\pi \tau_K < +\infty \) for every \( K \) with \( \pi(K) > 0 \) (here and below, \( E_\pi = \int_\mathbb{X} E_x \pi(dx) \)). We will prove the following stronger version of this statement.

Theorem 6.1. Assume \( X \) possess \( PI(\gamma) \) with some \( \gamma > 0 \). Then for every closed set \( K \subset \mathbb{X} \) with \( \pi(K) > 0 \)

\[
E_\pi e^{\alpha \tau_K} < +\infty, \quad \alpha < \frac{\gamma \pi(K)}{2}.
\]
Moreover, the function \( h^K_{-\alpha}(x) \overset{df}{=} E_x e^{\alpha x} \), \( x \in \mathbb{X} \) possesses the following properties:

a) \( h^K_{-\alpha} \in \text{Dom}(\mathcal{E}) \) and \( h^K_{-\alpha} = 1 \) on \( K \);

b) \( \mathcal{E}(h^K_{-\alpha}, u) = \alpha(h^K_{-\alpha}, u) \) for every quasi-continuous function \( u \in \text{Dom}(\mathcal{E}) \) such that \( u = 0 \) quasi-everywhere on \( K \).

**Proof.** We assume \( K \) to be fixed and omit the respective index in the notation, e.g. write \( \tau \) for \( \tau^K \) and \( h_\alpha \) for \( h^K_{-\alpha} \). For \( z \in \mathbb{C} \) with \( \Re z > 0 \), define respective \( z \)-potential:

\[
h_z(x) = E_x e^{-z x}, \quad x \in \mathbb{X}.
\]

Denote by \( H_\mathcal{E} \) the \( \text{Dom}(\mathcal{E}) \) considered as a Hilbert space with the scalar product \( (f, g)_{\mathcal{E}, 1} \overset{df}{=} (f, g)_{L^2} + \mathcal{E}(f, g) \). The following lemma shows that \( \{h_z, \Re z > 0\} \) can be considered as an analytical extension of the family of \( \alpha \)-potentials \( \{h_\alpha, \alpha > 0\} \subset H_\mathcal{E} \) that, in addition, keeps the properties of this family.

**Lemma 6.1.** 1) The function \( z \mapsto h_z \) is analytic as a function taking values in the Hilbert space \( H_\mathcal{E} \).

2) For every \( z \) with \( \Re z > 0 \), the following properties hold:
   
   (i) \( h_z = 1 \) quasi-everywhere on \( K \);
   
   (ii) \( \mathcal{E}(h_z, u) = -z(h_z, u) \) for every quasi-continuous function \( u \in \text{Dom}(\mathcal{E}) \) such that \( u = 0 \) quasi-everywhere on \( K \).

**Proof.** Denote \( h^m_z(x) = (-1)^m E_x^{m} e^{-z x}, x \in \mathbb{X}, m \geq 1 \). One can verify easily that, for every \( m \in \mathbb{N} \),

\[
\frac{d^m}{dz^m} h_z = h^m_z
\]

on the set \( \{z : \Re z > 0\} \), with the function \( z \mapsto h_z \) is considered as a function taking values in \( L_2 \). In addition,

\[
\|h^m_z\|_2 \leq E_x \left| \tau^m e^{z x} \right|^2 = E_x \tau^m e^{-2z \Re z} \leq \frac{(2m)!}{(2m)!	au^{2m} e^{2z \Re z}},
\]

since \( \frac{(2z \Re z)^2m}{(2m)!} \leq e^{2z \Re z} \). Therefore,

\[
\frac{\|h^m_z\|_2^2}{m!} \leq \sqrt{\frac{C_{2m}^m}{2^{2m}}} (\Re z)^{-m} \leq (\Re z)^{-m}, \quad m \in \mathbb{N},
\]

and hence the function

\[
\{z : \Re z > 0\} \ni z \mapsto h_z \in L_2
\]

is analytic.

For every \( \alpha, \alpha' > 0 \) we have \( h_\alpha - h_{\alpha'} = 0 \) quasi-everywhere on \( K \). Hence

\[
\mathcal{E}(h_\alpha - h_{\alpha'}, h_\alpha - h_{\alpha'}) = \mathcal{E}(h_\alpha - h_{\alpha'}, h_\alpha) - \mathcal{E}(h_{\alpha'}, h_\alpha - h_{\alpha'})
\]

\[
= -\alpha (h_\alpha, h_\alpha - h_{\alpha'}) + \alpha' (h_{\alpha'}, h_\alpha - h_{\alpha'})
\]

\[
= (\alpha' - \alpha) (h_{\alpha'}, h_\alpha - h_{\alpha'}) + \alpha (h_{\alpha'} - h_\alpha, h_\alpha - h_{\alpha'}).
\]

Therefore, for a given \( \alpha > 0 \) and \( \alpha' \to \alpha \), the family \( \left\{h_{\alpha'}, h_\alpha - h_{\alpha'}\right\} \) is bounded in \( H_\mathcal{E} \), and thus is weakly compact in \( H_\mathcal{E} \). On the other hand, this family converges to \( h^1_\alpha \) in \( L_2 \). This yields that the function \( (0, +\infty) \ni \alpha \mapsto h_\alpha \in H_\mathcal{E} \) is differentiable in a weak sense, and \( h^1_\alpha \) equals its (weak) derivative at the point \( \alpha \).
We have \(h^1_\alpha = 0\) quasi-everywhere on \(K\), since
\[h_\alpha(x) = 1 \iff e^{-\alpha x} = 1 \quad P_x - \text{a.s.} \iff \tau = 0 \quad P_x - \text{a.s.} \iff h^1_\alpha(x) = 0.
\]
In addition, since \(h^1_\alpha\) is a weak derivative of \(h_\alpha\), we have \(E(h^1_\alpha, u) = -\langle h_\alpha, u \rangle - \alpha(h^1_\alpha, u)\) for every quasi-continuous function \(u \in \text{Dom}(E)\) such that \(u = 0\) quasi-everywhere on \(K\). Now, repeating the same arguments, we get by induction that, for every \(m \geq 1\), the function \((0, +\infty) \ni \alpha \mapsto h^m_\alpha \in H_\alpha\) is \(m\) times weakly differentiable, \(h^m_\alpha\) is the corresponding weak derivative of the \(m\)-th order, and the following properties hold:

\((i^m)\) \(h^m_\alpha = 0\) quasi-everywhere on \(K\);

\((ii^m)\) \(E(h^m_\alpha, u) = -(h^{m-1}_\alpha, u) - \alpha(h^m_\alpha, u)\) for every quasi-continuous function \(u \in \text{Dom}(E)\) such that \(u = 0\) quasi-everywhere on \(K\).

Property (ii) with \(u = h^m_\alpha\) and estimate (6.1) yield that, for a given \(\alpha\), series
\[
H_\alpha = h_\alpha + \sum_{m=1}^{\infty} \frac{z^m}{m!} h^m_\alpha \in H_\alpha
\]
converge in the circle \(\{|z - \alpha| < \alpha\}\). The sum is a weakly analytic \(H_\alpha\)-valued function, and hence is analytic ([Rud73], Theorem 3.31). On the other hand, the same series converge in \(L_2\) to \(h_\alpha\). This yields that \(h_\alpha = H_\alpha\) in the circle \(\{|z - \alpha| < \alpha\}\). By taking various \(\alpha \in (0, +\infty)\), we get that the function \(z \mapsto h_\alpha\) is an \(H_\alpha\)-valued analytic function inside the angle \(D_{1,\alpha} = \{z : \text{Re } z > |\text{Im } z|\}\). In addition, properties \((i^m)\), \((ii^m)\) of the \(m\)-th coefficients of the series \((m \geq 1)\) provide that \(h_\alpha\) satisfy (i),(ii) inside the angle.

Now, we complete the proof using the following iterative procedure. Assume that the function \(z \mapsto h_\alpha\) is analytic in some domain \(D \subset \{z : \text{Re } z > 0\}\) and satisfy (i),(ii) in this domain. Then the same arguments with those used above show that, for every \(z_0 \in D\), the domain \(D\) can be extended to \(D' = D \cup \{z : |z - z_0| < \text{Re } z_0\}\) with the function \(z \mapsto h_\alpha\) still being analytic in \(D'\) and satisfying (i),(ii) in the extended domain. Therefore, we prove iteratively that the required statement holds true in every angle \(D_{k,\alpha} = \{z : \text{Re } z > \frac{1}{k} |\text{Im } z|\}\). Since \(\bigcup_k D_{k,\alpha} = \{z : \text{Re } z > 0\}\), this completes the proof. \(\square\)

Next, we consider "\(\psi\)-potentials" that correspond to functions \(\psi : \mathbb{R}^+ \to \mathbb{R}\). Denote
\[h_\psi(x) = E_x \psi(t), \quad x \in \mathbb{X}.
\]
The following statement is an appropriate modification of the inversion formula for the Laplace transform.

**Lemma 6.2.** Let \(\psi \in C^2(\mathbb{R})\) have a compact support and \(\text{supp } \psi \subset [0, +\infty)\). Denote \(\Psi(z) = \int_{\mathbb{R}} e^{zt} \psi(t) \, dt, z \in \mathbb{C}.
\]

The function \(h_\psi\) belongs to \(H_\alpha\) and admits integral representation
\[
(6.3) \quad h_\psi = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z) h_\alpha \, dz,
\]
where \(\sigma > 0\) is arbitrary, and the integral is well defined as an improper Bochner integral of an \(H_\alpha\)-valued function.

**Proof.** First, let us show that the integral in the right hand side of (6.3) is well defined. We have by condition (ii) of Lemma 6.1 that
\[
E(h_\alpha, h_\alpha) = E(h_\alpha, h_\alpha - 1) = -z(h_\alpha, h_\alpha - 1).
\]
For any $z$ with $\text{Re } z > 0$, we have $|h_z(x)| \leq E_x e^{-\text{Re } z}$, and thus $|h_z(x) - 1| \leq 2$. Hence,

$$\|h_z\|_{H^\sigma} = \sqrt{\|h_z\|^2 + \mathcal{E}(h_z, h_z)} \leq \sqrt{1 + 2|z|}.$$ 

On the other hand, for $\psi$ satisfying conditions of the lemma,

$$z^2 \Psi(z) = \int_{0}^{\infty} e^{zt} \psi''(t) \, dt, \quad |z^2 \Psi(z)| \leq \int_{0}^{\infty} e^{Re z} |\psi''(t)| \, dt.$$ 

Thus, on the line $\sigma + i \mathbb{R} = \{ z : \text{Re } z = \sigma \}$, the function $z \mapsto \Psi(z) h_z \in H^\sigma$ admits the following estimate:

$$\|\Psi(z) h_z\|_{H^\sigma} \leq C|z|^{-\frac{3}{2}},$$

and therefore it is integrable on $\sigma + i \mathbb{R}$. Denote by $g_\psi \in H^\sigma$ corresponding integral. In order to prove that $h_\psi = g_\psi$, it is sufficient to prove that $h_\psi$ and $g_\psi$ coincide as elements of $L^2$. Hence, we have reduced the proof of the lemma to verification of the following "weak $L^2$-version" of (6.3):

$$(6.4) \quad \int_{\mathbb{X}} h_\psi v \, d\pi = \frac{1}{2\pi i} \int_{\mathbb{X}} \int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z) h_z(x)v(x) \, dz \, \pi(dx), \quad v \in L^2.$$ 

Recall that $h_z(x) = E_x e^{-z \tau}$, and hence the right hand side of (6.4) can be rewritten to the form

$$\frac{1}{2\pi i} \int_{\mathbb{X}} \int_{\sigma - i\infty}^{\sigma + i\infty} E_x \Psi(z)e^{-z \tau}v(x) \, dz \, \pi(dx) = \frac{1}{2\pi i} \int_{\mathbb{X}} E_x \int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z)e^{-z \tau}v(x) \, dz \, \pi(dx).$$

Here, we have changed the order of integration using Fubini’s theorem. This can be done, because $|\Psi(z)| \leq C|z|^{-2}$, and therefore

$$E_x \int_{\sigma - i\infty}^{\sigma + i\infty} |\Psi(z)e^{-z \tau}| \, dz = h_\sigma(x) \int_{\sigma - i\infty}^{\sigma + i\infty} |\Psi(z)| \, dz \leq Ch_\sigma(x).$$

The function $\Psi$ is the (two-sided) Laplace transform for $\psi$, up to the change of variables $p \mapsto -z$. We write the inversion formula for the Laplace transform in the terms of $\Psi$ and, after the change of variables, get

$$\psi(t) = \frac{1}{2\pi i} \int_{-\sigma - i\infty}^{\sigma + i\infty} e^{pt} \Psi(-p) \, dp = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{-z t} \Psi(z) \, dz, \quad t \in \mathbb{R}^+.$$ 

Hence, the right hand side of (6.4) is equal

$$\int_{\mathbb{X}} E_x \psi(x) \pi(dx) = \int_{\mathbb{X}} h_\psi v \, d\pi,$$

that proves (6.4).

\textbf{Corollary 6.1.} Let $\psi \in C^2(\mathbb{R})$ and supp $\psi' \subset [0, +\infty)$. Then $h_\psi \in \text{Dom}(\mathcal{E})$ and

$$(6.5) \quad \mathcal{E}(h_\psi, u) = (h_\psi, u)$$

for every $u \in \text{Dom}(\mathcal{E})$ such that $u = 0$ quasi-everywhere on $\mathcal{K}$.

\textbf{Proof.} Assume first that $\int_{\mathbb{R}^+} \psi'(x) \, dx = 0$. Then both $\psi$ and $\psi'$ satisfy conditions of Lemma 6.2. We have $\tilde{\Psi}(z) = \int_{\mathbb{R}} e^{zt} \psi'(t) \, dt = -z \Psi(z)$. Hence, from the representation (6.3) for $h_\psi$ and $h_\psi'$ and relation $\mathcal{E}(h_z, u) = -z(h_z, u)$, Re $z > 0$, we get

$$\mathcal{E}(h_\psi, u) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \Psi(z) \mathcal{E}(h_z, u) \, dz = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{\Psi}(z)(h_z, u) \, dz = (h_\psi', u).$$
The general case can be reduced to the one considered above by the following limit procedure. Since sup \( \psi' \subset [0, +\infty) \), there exist \( C \in \mathbb{R} \) and \( x_* \in \mathbb{R}^+ \) such that \( \psi(x) = C, x \geq x_* \). Take a function \( \chi \in C^3(\mathbb{R}) \) with sup \( \chi \subset [0,1] \), and put
\[
\psi_t(x) = \psi(x) - C\chi(x - t), \quad x \in \mathbb{R}, t > x_*.
\]
Then every \( \psi_t \) satisfies the additional assumption \( \int_\mathbb{R}^+ \psi_t'(x) \, dx = 0 \), and thus \( h_{\psi_t} \) belongs to \( Dom(\mathcal{E}) \) and satisfies (6.5). It can be verified easily that \( u \), (here, we have used (6.5) with \( \psi \)), \( \psi \in \mathbb{R} \) and satisfies (6.5). Hence, \( \psi \rightarrow \psi_{\infty} \), \( t \rightarrow \infty \) weakly in \( L_2 \) sense. In addition,
\[
\mathcal{E}(h_{\psi_t}, h_{\psi_t}) = (h_{[\psi_t]'}, h_{\psi_t}) + (h_{[\psi_t]'}, h_{\psi}) < +\infty, \quad t \rightarrow +\infty
\]
(here, we have used \( (6.5) \) with \( u = h_{\psi_t} \)). This means that the family \( \{h_{\psi_t}\} \) is bounded in \( H_\mathcal{E} \), and hence is weakly compact in \( H_{\mathcal{E}} \). Therefore, \( h_{\psi_t} \rightarrow h_{\psi_{\infty}} \), \( t \rightarrow \infty \), weakly in \( H_{\mathcal{E}} \). Since \( h_{[\psi_t]' \rightarrow h_{\psi_{\infty}}}, t \rightarrow \infty \) in \( L_2 \) sense, \( (6.5) \) for \( \psi \) follows from \( (6.5) \) for \( \psi_t \).

Now, we are ready to complete the proof of the theorem. Let us fix \( \alpha < \frac{\gamma_{K\mathcal{E}}}{2} \), and construct the family of the functions \( \rho_t, t \geq 1 \) that approximate the function \( \varphi : x \leftrightarrow e^{\alpha x} - 1 \) appropriately. First, we take function \( \chi \in C^3(\mathbb{R}) \) such that \( \chi \geq 0, \chi' \leq 0, \chi(x) = 1, x \leq 0, \) and \( \chi(x) = 0, x \geq 1 \). We put
\[
\rho_t(x) = \int_{0}^{x} e^{\alpha y} \chi(y - t) \, dy, \quad x \geq 0, t \geq 1.
\]
By the construction, the derivatives of the functions \( \rho_t, t \geq 1 \) have the following properties:
\begin{enumerate}
  \item \( [\rho_t]' \geq 0 \) and \( [\rho_t]'(x) = 0, x \geq t + 1; \)
  \item \( [\rho_s]' \leq [\rho_t]' \), \( s \leq t. \)
\end{enumerate}
Since \( \rho_t(0) = 0, t \geq 1 \), the latter property yields that \( \rho_s \leq \rho_t, s \leq t. \) In addition,
\[
[\rho_t]'(x) = \alpha e^{\alpha x} \chi'(x) + \alpha^2 e^{\alpha x} \chi(x) \leq \alpha^2 e^{\alpha x} \chi(x) = \alpha [\rho_t]'(x),
\]
since \( \chi' \leq 0 \). This and relation \( [\rho_t]'(0) = \alpha (\rho_t(0) + 1) \) provide
\begin{equation}
[\rho_t]' \leq \alpha (\rho_t + 1).
\end{equation}
At last, we take function \( \theta \in C^3(\mathbb{R}) \) such that \( \theta' \geq 0, \theta(x) = 0, x \leq 0, \) and \( \theta(x) = 1, x \geq 1 \). We put
\[
\varrho_t(x) = \begin{cases} 
\theta (xt) \rho_t(x), & x \geq 0 \\
0, & x < 0,
\end{cases} \quad t \geq 1.
\]
We have \( \varrho_t \uparrow \varrho, t \uparrow \infty \). In addition, by \( (6.6) \),
\begin{equation}
[\varrho_t]'(x) = t \theta'(tx) \rho_t(x) + \theta(tx)[\rho_t]'(x) \leq t \sup_y \theta'(y) \rho_t(t^{-1}) + \alpha (\rho_t(x) + 1) \leq \alpha \varrho_t(x) + C
\end{equation}
with an appropriate constant \( C \) (recall that \( t \rho_t(t^{-1}) = t (e^{\alpha t^{-1}} - 1) \rightarrow \alpha^2, t \rightarrow \infty \)).

Every \( \varrho_t \) satisfies conditions of Corollary 6.1 and hence
\[
\int_{\mathbb{R}} h_{\varrho_t}^2 \, d\pi - \left( \int_{\mathbb{R}} h_{\varrho_t} \, d\pi \right)^2 \leq \frac{2}{\gamma} \mathcal{E}(h_{\varrho_t}, h_{\varrho_t}) = \frac{2}{\gamma} [h_{[\varrho_t]'}, h_{\varrho_t}] \leq \frac{2\alpha}{\gamma} (h_{\varrho_t}, h_{\varrho_t}) + C \int_{\mathbb{R}} h_{\varrho_t} \, d\pi.
\]
Here, we have used subsequently property \( PI_2(\varrho) \), equality \( (6.6) \) with \( u = h_{\varrho_t} \), and \( (6.7) \).
We have $h_{gt} = 0$ on $K$ because $g_t(0) = 0$. Then, by the Cauchy inequality,
\[
\int_{X} h_{gt}^2 \, d\pi - \left( \int_{X} h_{gt} \, d\pi \right)^2 = \int_{X} h_{gt}^2 \, d\pi - \left( \int_{X \setminus K} h_{gt} \, d\pi \right)^2 \geq (1 - \pi(X \setminus K)) \int_{X} h_{gt}^2 \, d\pi = \pi(K)(h_{gt}, h_{gt}).
\]
Therefore,
\[
(h_{gt}, h_{gt}) \leq \frac{2\alpha}{\gamma \pi(K)}(h_{gt}, h_{gt}) + C \int_{X} h_{gt} \, d\pi,
\]
which implies that
\[
(6.8) \quad (h_{gt}, h_{gt}) \leq C \int_{X} h_{gt} \, d\pi
\]
(recall that $\alpha < \frac{\gamma \pi(K)}{2}$). One can verify easily that (6.8) yields that the $L_2$-norms of the functions $h_{gt}$ are uniformly bounded. Since $g_t \uparrow \varrho$, this implies that the function
\[
h_{\varrho}(x) \overset{df}{=} E_x e^{\alpha t} - 1, \quad x \in X
\]
belongs to $L_2$, and $h_{gt} \rightarrow h_{\varrho}, t \rightarrow \infty$ in $L_2$. Similarly to the proof of Corollary 6.1, one can verify that \{h_{gt}\} is a bounded subset in $H_\varrho$, and hence $h_{gt} \rightarrow h_{\varrho}, t \rightarrow \infty$ weakly in $H_\varrho$. This proves statement a) of the theorem. In order to prove statement b), we apply (6.5) to $\psi = g_t$, and pass to the limit as $t \rightarrow +\infty$. The theorem is proved. □

7. Poincaré inequality for diffusions: criterion in the terms of hitting times

In this section, we apply our general results to diffusion processes on non-compact manifolds. The Poincaré inequality for diffusions was studied extensively by numerous authors. We refer to \cite{RW04}, \cite{Wang00} for various sufficient conditions for this inequality and further references. The main result of this section – Theorem 7.1 – is a refinement of Theorem 3.3, Theorem 6.1, and Proposition 2.2. It gives necessary and sufficient condition for the Poincaré inequality in the terms of hitting times of the diffusion process.

Let $X$ be a connected locally compact Riemannian manifold of dimension $d$, and $X$ be a diffusion process on $X$. On a given local chart of the manifold $X$, the generator of the process $X$ has the form
\[
A = \sum_{j=1}^{d} a_j \partial_j + \frac{1}{2} \sum_{j,k=1}^{d} b_{jk} \partial_j \partial_k,
\]
where $a = \{a_j\}_{j=1}^{d}$ and $b = \{b_{jk}\}_{j,k=1}^{d}$ are the drift and diffusion coefficients of the process $X$ on this chart, respectively. We assume the coefficients $a, b$ to be Hölder continuous on every local chart, and the drift $b$ coefficient to satisfy ellipticity condition
\[
\sum_{j,k=1}^{d} b_{jk} v_j v_k \geq c \sum_{j=1}^{d} v_j^2
\]
uniformly on every compact. Under these conditions, the transition function of the process $X$ has a positive density w.r.t. Riemannian volume, and this density is a continuous function on $(0, +\infty) \times X \times X$. One can easily deduce this from the same statement for diffusions in $\mathbb{R}^d$ (e.g. \cite{IKO62}) and strong Markov property of $X$. This implies that $X$ satisfies the extended Doeblin condition on every compact subset of $X$. 

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Let $\pi \in \mathcal{P}(\mathbb{X})$ be an invariant measure for the process $X$ (we assume invariant measure to exist). Denote by $\mathcal{E}$ the Dirichlet form on $L_2(\mathbb{X}, \pi)$ corresponding to $X$.

**Theorem 7.1.** The following statements are equivalent:

1) the Poincaré inequality holds true with some constant $c$:

$$\int_{\mathbb{X}} |f|^2 d\pi - \left| \int_{\mathbb{X}} f d\pi \right|^2 \leq c \mathcal{E}(f, f), \quad f \in \text{Dom}(\mathcal{E}).$$

2) the process $X$ admits an exponential $\phi$-coupling for some function $\phi$;

3) for every closed subset $K \subset \mathbb{X}$ with $\pi(K) > 0$, there exists $\alpha > 0$ such that

$$E_{\pi} e^{\alpha \tau_K} < +\infty.$$ 

In addition, 1) – 3) hold true assuming that

3') there exists a compact subset $K \subset \mathbb{X}$ and $\alpha > 0$ such that

$$E_{\pi} e^{\alpha \tau_K} < +\infty \quad \text{for } \pi\text{-almost all } x \in \mathbb{X}.$$ 

**Proof.** Implications 2) $\Rightarrow$ 1) and 1) $\Rightarrow$ 3) are proved in Theorems 3.3 and 6.1, respectively. Hence, we need to prove implication 3' $\Rightarrow$ 2), only. We will prove it using Proposition 2.2. In order to simplify exposition, we consider the case $\mathbb{X} = \mathbb{R}^d$, only. One can easily extend the proof to the general case by a standard localization procedure.

We take $\tilde{\alpha} \in (0, \alpha)$ and put $\phi(x) = E_x e^{\tilde{\alpha} \tau_K}, x \in \mathbb{X}$. Let us show that $\phi$ is locally bounded; that is, condition 2) of Proposition 2.2 holds true with $\alpha$ replaced by $\tilde{\alpha}$.

Let $x_0 \in \mathbb{R}^d$ and $0 < r_0 < r_1$ be such that $K \subset \{ x : \| x - x_0 \| < r_0 \}$. Denote $D = \{ x : \| x - x_0 \| < r_1 \}\setminus K$, $\theta = \inf \{ t : X_t \in \partial D \}$, and $\mu_x(dy) \overset{df}{=} P_x(X_{\theta} \in dy), x \in D$.

Consider auxiliary function

$$h(x) = \int_{\partial D} E_y e^{\alpha \tau_K} \mu_x(dy), \quad x \in D.$$ 

This function is $A$-harmonic in $D$, hence it satisfies the Harnack inequality (see [KS81]). Namely, there exists $C \in \mathbb{R}^+$ such that

$$h(x_1) \leq C \, h(x_2)$$ 

for every $y \in D$, and $x_1, x_2 \in \{ x : \| x - y \| < \frac{1}{2} \text{dist}(y, \partial D) \}$. On the other hand, by the strong Markov property of $X$, we have

$$E_x e^{\alpha \tau_K} = E_x \left( e^{\alpha \theta \phi(X_{\theta})} \right) \geq E_x \phi(X_{\theta}) = h(x), \quad x \in D.$$ 

Hence, under condition 3'), $h(x) < +\infty$ for $\pi$-a.a. $x \in D$. In addition, supp $\pi = \mathbb{X}$; one can easily verify this fact using positivity of the transition probability density. Therefore, the function $h$ is bounded on every compact $S \subset D$.

The function $h$ can be written in the form

$$h(x) = E_x e^{\alpha \tau_K^\theta}, \quad \tau_K^\theta = \inf \{ t \geq 0 : X_{t+\theta} \in K \}.$$ 

For $x \in D$, we have $\tau_K = \theta + \tau_K^\theta$, $P_x$-a.s., and therefore

$$E_x e^{\alpha \tau_K} \leq \left[ E_x \left( e^{\frac{\alpha \theta}{\alpha - \theta}} \right) \right]^{\frac{\alpha - \theta}{\alpha}} [h(x)]^{\frac{\alpha}{\alpha - \theta}}.$$
Using the Kac formula one can show that, for every $a > 0$, the function $x \mapsto E_x e^{a\phi}$ is bounded on $D$ (this fact is quite standard and hence we do not go into details here). Therefore, the function $\phi$ is bounded on every compact $S \subset D$.

Next, consider closed ball $E = \{ x : \| x - x_0 \| \leq r_0 \}$ with the boundary $S = \{ x : \| x - x_0 \| = r_0 \} \subset D$, and put $\sigma = \inf \{ t : X_t \in S \}$.

For $x \in E$, we have by the strong Markov property of $X$ that

$$
\phi(x) \leq E_x (e^{\tilde{\alpha} \sigma} \phi(X_\sigma)) \leq (E_x e^{\tilde{\alpha} \sigma}) \sup_{y \in S} \phi(y).
$$

The function $x \mapsto E_x e^{\tilde{\alpha} \sigma}$ is bounded on $E$ (again, we do not give a detailed discussion here). Hence $\phi$ is bounded on $E$. Since $r_0$ can be taken arbitrarily large, this means that that $\phi$ is locally bounded.

Now, let us verify that condition 3) of Proposition 2.2 holds true with $\alpha$ replaced by $\tilde{\alpha}$. We put $\sigma^0 = 0, \sigma^{2n-1} = \inf \{ t \geq \sigma^{2n-2} : X_t \in S \}, \sigma^{2n} = \inf \{ t \geq \sigma^{2n-1} : X_t \in K \}, n \geq 1$.

For any $a > 0$, one has

$$
q \overset{\text{def}}{=} \max \left[ \sup_{x \in K} E_x e^{-a\tau_S} < 1, \sup_{x \in S} E_x e^{-a\tau_K} < 1 \right] < 1
$$

because $\text{dist} (K, S) > 0$ and $X$ is a Feller process with continuous trajectories. Therefore,

$$
(7.1) \quad E \left[ e^{-a(\sigma^{k+1}-\sigma^k)} \mid \mathcal{F}_{\sigma^k} \right] \leq q \quad \text{a.s.,} \quad k \geq 0.
$$

We have

$$
E_x e^{\tilde{\alpha} \tau_K} = \sum_{k=0}^{\infty} E_x e^{\tilde{\alpha} \tau_K} e_{\sigma^k \leq t < \sigma^{k+1}}, \quad x \in K.
$$

For $k$ even, $X_t \in E$ a.s. on the set $C_{k,t} = \{ \sigma^k \leq t < \sigma^{k+1} \}$. In addition, $C_{k,t} \in \mathcal{F}_t$. Hence

$$
E_x e^{\tilde{\alpha} \tau_K} e_{\sigma^k \leq t < \sigma^{k+1}} = E_x \left( e^{\tilde{\alpha} \tau_K} e_{\sigma^k \leq t < \sigma^{k+1}} \phi(X_t) \right) = E_x \left( e^{\tilde{\alpha} \tau_K} \left[ X_{\sigma^k} \right] \right) \leq \sup_{y \in E} \phi(y) P_x (\sigma^k \leq t < \sigma^{k+1}), \quad k = 2n.
$$

For $k$ odd, $\tau_K = \sigma^{k+1} - t \leq \sigma^{k+1} - \sigma^k$ a.s. on the set $C_{k,t}$. Hence

$$
E_x e^{\tilde{\alpha} \tau_K} e_{\sigma^k \leq t < \sigma^{k+1}} \leq E_x e^{\tilde{\alpha} \tau_K} \phi(X_{\sigma^k}) \leq \sup_{y \in E} \phi(y) P_x (\sigma^k \leq t).
$$

Therefore,

$$
E_x e^{\tilde{\alpha} \tau_K} \leq \sup_{y \in E} \phi(y) \sum_{k=0}^{\infty} P_x (\sigma^k \leq t), \quad x \in K.
$$

It follows from (7.1) that $E_x e^{-a\sigma^k} \leq q^k, x \in K$. Then

$$
P_x (\sigma^k \leq t) = P_x (-\sigma^k \geq -t) \leq e^{at} q^k, \quad k \geq 0, x \in K,
$$

and consequently

$$
\sup_{x \in K, t \in [0,S]} E_x e^{\tilde{\alpha} \tau_K} \leq e^{\tilde{\alpha} S} (1 - q)^{-1} \sup_{y \in E} \phi(y) < +\infty.
$$

We have verified that conditions 2), 3) of Proposition 2.2 hold true with $\alpha$ replaced by $\tilde{\alpha}$. Also, we have already seen that $X$ satisfies the extended Doeblin condition on $K$. We complete the proof of the theorem applying Proposition 2.2.
Remark 7.1. The criterion given in Theorem 7.1 extends, in particular, the sufficient condition from [RW04, Theorem 1.1]. Indeed, under condition (1.1) of the latter theorem one can verify that there exists a function \( \Phi : \mathbb{R} \to \mathbb{R} \) such that \( \Phi(x) \to +\infty, |x| \to \infty \) and the function \( \phi = \Phi(\rho) \) satisfies the Lyapunov-type condition (2.2). This yields existence of an exponential \( \phi \)-coupling and hence the spectral gap property.

On the other hand, it is worth to compare Theorem 7.1 with the necessary and sufficient condition given in [Mat97]. The principal difference is that Theorem 7.1 deals with the Poincaré inequality itself while in the part (ii) of the main theorem in [Mat97] some weak version of this inequality is established. In addition, sufficient condition of [Mat97] involves the whole collection of hitting times \( \{\tau_K : K \text{ is closed and } \pi(K) \geq \frac{1}{2}\} \), while in Theorem 7.1 condition 3') is imposed on one hitting time \( \tau_K \), which makes this theorem much easier in application.

Appendix A. Proofs of Theorems 2.1, 2.2 and Proposition 2.2

A.1. Proof of Theorem 2.1. Under conditions of Theorem 2.1 consider two independent copies \( Y^1, Y^2 \) of the process \( X \) with \( Y^1_0 = y^1, Y^2_0 = y^2 \) (\( y^1, y^2 \in X \) are arbitrary). It follows from condition 3) that \( \sup_{x \in K, t \in \mathbb{R}^+} E_x \phi(X_t) < +\infty \). In particular, \( \phi \) is bounded on \( K \). Denote
\[
D_1 = \sup_{x \in K} \phi(x), \quad D_2 = \sup_{x \in K, t \in \mathbb{R}^+} E_x \phi(X_t).
\]
Take arbitrary \( \gamma \in (0, \alpha) \) and choose \( c > D_1 \) such that
\[
\delta \overset{df}{=} \sup_{x \in K, t \in \mathbb{R}^+} E_x \phi(X_t) 1_{\phi(X_t) > c} < 1 - \frac{\gamma}{\alpha}.
\]
Define
\[
K' \overset{df}{=} \{ \phi \leq c \}, \quad \theta \overset{df}{=} \inf\{ t : Y^1_t \in K', Y^2_t \in K' \}.
\]

Lemma A.1.
\[
E[\phi(Y^1_t) + \phi(Y^2_t)] 1_{\theta > t} \leq D_3 e^{-\gamma t[\phi(y^1) + \phi(y^2)]}, \quad y^1, y^2 \in X,
\]
\[
D_3 = 2D_2 + 3 + 2(D_2 + 1)^2 \left( 1 - \frac{\gamma}{\alpha} - \delta \right)^{-1}.
\]

Proof. We consider stopping time
\[
\tau^1 = \inf\{ t : Y^1_t \in K \text{ or } Y^2_t \in K \},
\]
and define the sequence of random variables \( \iota_n, n \geq 1 \) taking values in \( \{1, 2\} \) by
\[
\begin{cases}
\iota_n = n \pmod{2}, & \text{if } Y^1_{\tau^n} \in K \\
\iota_n = n + 1 \pmod{2}, & \text{otherwise}, \quad n \geq 1.
\end{cases}
\]
Then, we define iteratively the sequence of stopping times
\[
\tau^{n+1} = \inf\{ t > \tau^n : Y^1_{\tau^{n+1}} \in K \}, \quad n \geq 1.
\]
We put \( \tau^0 = 0, \tau^\infty = \lim_n \tau^n \). Obviously, \( \tau^\infty = \inf\{ t : Y^1_t \in K, Y^2_t \in K \} \geq \theta \). Hence,
\[
E[\phi(Y^1_t) + \phi(Y^2_t)] 1_{\theta > t} = \sum_{n=0}^{\infty} E[\phi(Y^1_t) + \phi(Y^2_t)] 1_{\tau^n \leq t < \tau^{n+1}, \theta > t}.
\]

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Let us estimate separately summands in the right-hand side of (A.2). Note that every process \( Y^1, Y^2 \) is strongly Markov, and every stopping time \( \tau^n \), given the values \( Y_{\tau^n}^1 \) and \( \tau^{n-1} \), is completely defined by the trajectory of one component of the process \( Y = (Y^1, Y^2) \). Because these components are independent, this yields that \( Y \) has strong Markov property at every stopping time \( \tau^n \).

We have \( \tau^1 = \tau^1_K \land \tau^2_K \), where \( \tau^i_K \) denotes the hitting time for the process \( Y^i, i = 1, 2 \). Since \( Y^1, Y^2 \) are independent, we get from condition 2):

\[
E[\phi(Y^1_t) + \phi(Y^2_t)]_{\tau^1 > t} \leq E[\phi(Y^1_t) + \phi(Y^2_t)]_{\tau^1 > t} = E_y \phi(X_t)_{\tau^1 > t} + E_y \phi(X_t)_{\tau^2 > t} \leq e^{-\alpha t} [\phi(y^1) + \phi(y^2)].
\]

Next, consider the summand

\[
E[\phi(Y^1_t) + \phi(Y^2_t)]_{\tau^1 \leq t < \tau^2, \theta > t} \leq E[\phi(Y^1_t) + \phi(Y^2_t)]_{\tau^1 \leq t, \tau^2 > t + \tau^2 \leq t, \tau^1 > t}
\]

\[
= \left(E_y \phi(X_t)_{\tau^1 \leq t}\right)_{\tau^2 > t} P_{y^2}(\tau^2 > t) + \left(E_y \phi(X_t)_{\tau^1 > t}\right)_{\tau^2 > t} P_{y^2}(\tau^1 \leq t) + E_y \phi(X_t)_{\tau^1 \leq t} P_{y^2}(\tau^1 > t) + \left(E_y \phi(X_t)_{\tau^1 > t}\right)_{\tau^1 \leq t} P_{y^1}(\tau^2 > t).
\]

Recall that \( \phi \geq 1 \). Then condition 2) yields

\[
P_y(\tau_K > t) \leq e^{-\alpha t} \phi(y).
\]

By the strong Markov property of \( X \), we have

\[
E_x \phi(X_t)_{\tau^1 \leq t} = E_x \left[ E_y \phi(X_{t-s}) \bigg| s = \tau_K, y = X_{\tau_K} \right] \leq D_2.
\]

Therefore,

\[
E[\phi(Y^1_t) + \phi(Y^2_t)]_{\tau^1 \leq t \leq t < \tau^2, \theta > t} \leq 2(D_2 + 1)[\phi(y^1) + \phi(y^2)]e^{-\alpha t}.
\]

Remark that, in fact, we have proved inequality

\[
E[\phi(Y^1_t) + \phi(Y^2_t)]_{\tau^1 \leq t \leq t < \tau^2} \leq 2(D_2 + 1)[\phi(y^1) + \phi(y^2)]e^{-\alpha t}
\]

which yields

\[
(\text{A.3})
\]

\[
E e^{\gamma \tau^2} \leq 2(D_2 + 1) \left( \frac{\alpha}{\alpha - \gamma} \right) [\phi(y^1) + \phi(y^2)].
\]

We have estimated two first summands in (A.2). The other summands can be estimated iteratively in the following way. We have

\[
E \phi(Y^{t^n+1}_t)_{\tau^n \leq t < \tau^{n+1}, \theta > t} \leq E \phi(Y^{t^n+1}_t)_{\tau^n \leq t < \tau^{n+1}, \theta > \tau^n}
\]

\[
= e^{-\gamma t} E e^{\gamma \tau^n} \phi(Y^{t^n+1}_t)_{\tau^n \leq \theta > \tau^n} E \left[ \phi(Y^{t^n+1}_t)_{\tau^n \leq \tau^{n+1}} \bigg| \mathcal{F}_{\tau^n} \right]
\]

\[
\leq e^{-\gamma t} E e^{\gamma \tau^n} \phi(Y^{t^n+1}_t)_{\tau^n \leq \theta > \tau^n} \phi(Y^{t^n+1}_t).
\]

Here, \( \{\mathcal{F}_t\} \) denotes the natural filtration for \( Y \). We have used strong Markov property at the point \( \tau^n \) and inequality

\[
E_x e^{\gamma t} \phi(X_t)_{\tau_K > t} \leq \phi(x)
\]

that follows from 2).

Next, the processes \( U^n_t \overset{df}{=} Y^n_{t_{\tau^n}}, V^n_t \overset{df}{=} Y^n_{t_{\tau^n+1}} \) are conditionally independent w.r.t. \( \mathcal{F}_{\tau^n} \). Denote \( \varsigma^n \) the first time for \( V^n \) to hit \( K \). Then \( \tau^{n+1} = \varsigma^n + \tau^n \).
We have

\[
E(\phi(Y_{t}^{1,n})I_{t}^{n}\leq t<\tau^{n+1},\theta>\tau^{n}) \leq E(\phi(Y_{t}^{1,n})I_{t}^{n}\leq t<\tau^{n+1},\theta>\tau^{n})
\]

\[
= E\left(E[\phi(U_{t}^{n})|F_{\tau^{n}}]E[I_{t}^{n}\leq t-\tau_{n}^{n}|F_{\tau^{n}}]\right)I_{t}^{n}\leq t,\theta>\tau^{n}
\]

\[
\leq D_{2}E\left(E[I_{t}^{n}\leq t-\tau_{n}^{n}|F_{\tau^{n}}]\right)I_{t}^{n}\leq t,\theta>\tau^{n}.
\]

In the last inequality we have used that, by the construction, \(U_{t}^{0} = Y_{\tau^{n}}^{n} \in K\), and hence

\[
E\left[\phi(U_{t}^{n})\right|F_{\tau^{n}}]I_{t}^{n}\leq t, \theta \geq \sup_{x\in K,t\in R^{+}}E\phi(X_{t}) = D_{2}.
\]

Then, since \(\phi \geq 1\),

\[
E(\phi(Y_{t}^{1,n})I_{t}^{n}\leq t<\tau^{n+1},\theta>\tau^{n}) \leq D_{2}e^{-\gamma t}E(e^{\gamma t}I_{t}^{n}\leq t,\theta>\tau^{n}E(\phi(Y_{t}^{1,n+1})e^{\gamma(t-\tau^{n})}I_{t<\tau^{n+1}}|F_{\tau^{n}}))
\]

\[
\leq D_{2}e^{-\gamma t}e^{\gamma \tau^{n}}I_{t}^{n}\leq t,\theta>\tau^{n}\phi(Y_{t}^{1,n+1}) \leq D_{2}e^{\gamma \tau^{n}}I_{t}^{n}\leq t,\theta>\tau^{n} \phi(Y_{t}^{1,n+1}).
\]

Let us estimate

\[
Ee^{\gamma \tau^{n}}I_{\theta>\tau^{n}}\phi(Y_{\tau^{n+1},n}).
\]

We have \(Y_{\tau^{n+1},n} \in K\), and hence inequality \(\theta > \tau^{n}\) implies that \(Y_{\tau^{n+1},n} \notin K'\). Recall that \(\phi > c\) outside \(K'\), and \(c\) is chosen in such a way that \(E_{x}\phi(X_{t})I_{\phi(X_{t})>c} < \delta\) for any \(x \in K, t \in R^{+}\). Therefore, the same arguments with those that lead to (A.5) provide

\[
Ee^{\gamma \tau^{n}}I_{\theta>\tau^{n}}\phi(Y_{\tau^{n+1},n}) \leq \delta Ee^{\gamma \tau^{n-1}}I_{\theta>\tau^{n-1}}E\left[e^{\gamma(t-\tau^{n})}I_{t<\tau^{n+1}}|F_{\tau^{n-1}}\right].
\]

It can be verified easily that, under condition 2),

\[
E_{x}e^{\gamma \tau^{n}} \leq \frac{\alpha}{\alpha - \gamma} \phi(x), \quad x \in X.
\]

Hence, for \(n \geq 2\),

\[
E^{\gamma \tau^{n}}I_{\theta>\tau^{n}}\phi(Y_{\tau^{n+1},n}) \leq \frac{\delta \alpha}{\alpha - \gamma} E^{\gamma \tau^{n-1}}I_{\theta>\tau^{n-1}}\phi(Y_{\tau^{n},n-1}) \leq \cdots
\]

\[
\leq \left(\frac{\delta \alpha}{\alpha - \gamma}\right)^{n-2} E^{\gamma \tau^{2}}\phi(Y_{\tau^{2}}^{1,n}) \leq D_{2} \left(\frac{\delta \alpha}{\alpha - \gamma}\right)^{n-2} E^{\gamma \tau^{2}}.
\]

The latter estimate and (A.3) provide

\[
E[\phi(Y_{t}^{1}) + \phi(Y_{t}^{2})]I_{t}^{n}\leq t<\tau^{n+1},\theta>\tau^{n} \leq 2e^{-\gamma t}(D_{2} + 1)^{2} \left(\frac{\alpha}{\alpha - \gamma}\right) \left(\frac{\delta \alpha}{\alpha - \gamma}\right)^{n} [\phi(y^{1}) + \phi(y^{2})], \quad n \geq 2.
\]

This inequality, together with (A.4) and (A.5), gives (A.1) after summation by \(n\). \(\square\)

The rest of the proof of Theorem 2.1 is based on the construction described in [Kul09], Section 3.2. Here, we give the sketch of the construction, referring interested reader to [Kul09] for details, discussion and references.

Consider two types of "elementary couplings": a "simple coupling" and a "gluing coupling". The simple coupling is just a two-component Markov process \(Z = (Z^{1}, Z^{2})\) such that either \(Z^{1}, Z^{2}\) are independent if \(Z_{0}^{1} = z^{1}, Z_{0}^{2} = z^{2}, z^{1,2} \in X\), and \(z^{1} \neq z^{2}\), or \(Z^{1} = Z^{2}\) if \(Z_{0}^{1} = Z_{0}^{2} = z \in X\). The gluing coupling is constructed on a given time interval \([0, T]\) for fixed \(z^{1}, z^{2} \in X\) in such a way that \(Z_{0}^{1} = z^{1}, Z_{0}^{2} = z^{2}\), and

\[
P(Z_{T}^{1} = Z_{T}^{2}) = 1 - \frac{1}{2}\|P_{T}(z^{1}, \cdot) - P_{T}(z^{2}, \cdot)\|_{var}.
\]
Next, we construct the ”switching coupling” $Z$ as an appropriate mixture of these elementary ones. Namely, for a given $z^1, z^2 \in \mathbb{X}$ we consider a simple coupling $Z^\ast = (Z^\ast_1, Z^\ast_2)$ with $Z^\ast_0^1 = z^1, Z^\ast_0^2 = z^2$ and define $\theta^1 = \min\{t : Z^\ast_t \in K' \times K'\}$ (the set $K'$ is defined above). Then the value of $Z^\ast$ at the random time moment $\theta^1$ is substituted, as the starting position, into an independent copy of the gluing coupling $Z^g$. The switching coupling $Z$ is defined, up to the random moment of time $\theta^2 = \theta^1 + T$, as

$$Z_t = \begin{cases} Z^s_t, & t \leq \theta_1, \\ Z^g_{t-\theta_1}, & t \in (\theta_1, \theta^2]. \end{cases}$$

Then this construction is iterated: the value $Z^g_{\theta^2}$ is substituted, as the starting position, into an independent copy of the simple coupling, etc. This construction gives a coupling $Z$ and a sequence of stopping times $\theta^k, k \geq 1$ such that

(a) if $Z^1_{\theta^k} = Z^2_{\theta^k}$ for some $k$, then $Z^1_t = Z^2_t$ for $t > \theta^k$;

(b) for every $k$,

$$P(Z^1_{\theta^k} \neq Z^2_{\theta^k} | F_{\theta^k-1}) \leq \varkappa(T, K') \quad \text{a.s.,}$$

where $\{F_t\}$ denotes the natural filtration for $Z$.

Recall that $\phi(x) \rightarrow \infty, x \rightarrow \infty$, hence $K' = \{\phi \leq c\}$ has a compact closure. Therefore, by the local Doeblin condition, $T$ can be chosen in such a way that $\varkappa(T, K') \leq \varkappa(T, \text{closure}(K')) < 1$.

Let us estimate the value

$$E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{Z^1_t \neq Z^2_t}. $$

Property (a) allows one to write

$$E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{Z^1_t \neq Z^2_t} \leq E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{\theta^2 > t}$$

$$+ \sum_{k=1}^{\infty} E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{\theta^2 \leq t < \theta^{k+2}} I_{\theta^1 = Z^1_{\theta^k} \neq Z^2_{\theta^k}}. \quad (A.6)$$

Take arbitrary $\beta \in (0, \alpha)$. It follows immediately from (A.1) with $\gamma = \beta$ that the first summand in the right hand side of (A.6) is estimated by $Ce^{-\beta T} [\phi(z^1) + \phi(z^2)]$. The same inequality yields that

$$E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{t < \theta^{k+2}} | F_{\theta^{k+2}} = E \left[ \phi(Z^1_t) + \phi(Z^2_t) \right] I_{t < \theta^{k+2}} | F_{\theta^{k+2}}$$

$$\leq C \left[ \phi(Z^1_{\theta^{k+2}}) + \phi(Z^2_{\theta^{k+2}}) \right] e^{\beta (\theta^{k+2} - T)}. $$

Hence, we can estimate the $k$-th summand in the sum in the right hand side of (A.6) by

$$Ce^{-\beta T} e^{\beta T} \left[ \phi(Z^1_{\theta^{k+2}}) + \phi(Z^2_{\theta^{k+2}}) \right] e^{\beta \theta^{k+2}} I_{\theta^1 = Z^1_{\theta^{k+2}} \neq Z^2_{\theta^{k+2}}}. $$

Next, we remove the function $\phi$ from this estimate:

$$E \left[ \phi(Z^1_{\theta^{k+2}}) + \phi(Z^2_{\theta^{k+2}}) \right] e^{\beta \theta^{k+2}} I_{\theta^1 = Z^1_{\theta^{k+2}} \neq Z^2_{\theta^{k+2}}} \leq E \left[ \phi(Z^1_{\theta^{k+2}}) + \phi(Z^2_{\theta^{k+2}}) \right] e^{\beta \theta^{k+1} - \beta T} I_{\theta^1 = Z^1_{\theta^{k+2}} \neq Z^2_{\theta^{k+2}}}$$

$$= E e^{\beta \theta^{k+1} - \beta T} I_{\theta^1 = Z^1_{\theta^{k+2}} \neq Z^2_{\theta^{k+2}}} E \left[ \phi(Z^1_{\theta^{k+1}+T}) + \phi(Z^2_{\theta^{k+1}+T}) | F_{\theta^{k+1}} \right]$$

$$\leq C e^{\beta T} E e^{\beta \theta^{k+1} - \beta T} I_{\theta^1 = Z^1_{\theta^{k+2}} \neq Z^2_{\theta^{k+2}}}. $$
Here, we have used condition 3) and notation \( \theta^0 = 0 \) (recall that \( \phi(Z_{g2k-1}^1) \leq c, \phi(Z_{g2k-1}^1) \leq c \) by the construction of the coupling \( Z \)). Hence, \((A.6)\) can be rewritten as

\[
E\left[ \phi(Z_1^1) + \phi(Z_1^2) \right] I_{Z_1^1 \neq Z_1^2} \leq Ce^{-\beta t} \left[ 1 + \sum_{k=1}^{\infty} Ee^{\beta g_{2k-1}} I_{Z_{g2k-2}^1 \neq Z_{g2k-2}^2} \right].
\]

Next, from the property (b) of the coupling \( Z \), we have

\[
Ee^{\beta g_{2k-1}} I_{Z_{g2k-2}^1 \neq Z_{g2k-2}^2} \leq \left[ Ee^{2\beta g_{2k-1}} \right]^{\frac{1}{2}} P_{T,K} (Z_{g2k-2}^1 \neq Z_{g2k-2}^2) \leq \left[ Ee^{2\beta g_{2k-1}} \right]^{\frac{1}{2}} \times \frac{k+1}{k} (T, K').
\]

Up to this moment, \( \beta \in (0, \alpha) \) was taken in an arbitrary way. On the other hand, \((A.1)\) yields that, for fixed \( \gamma < \alpha \) and \( k > 1 \),

\[
E[I_{g2k-2} - g2k-3} > t | F_{g2k-3}] = E[I_{g2k-2} - g2k-3} > t - T | F_{g2k-3}] \leq Ce^{-\gamma t}.
\]

Hence, for every \( q > 1 \), one can take \( \beta > 0 \) small enough for \( E[e^{2\beta t} | F_{g2k-3}] \leq q \) a.s. At last, by \((A.1)\),

\[
Ee^{2\beta g_{1}} \leq C[\phi(z^1) + \phi(z^2)]
\]

for \( \beta < \frac{\alpha}{2} \). This, finally, provides the estimate

\[
(A.7) \quad E\left[ \phi(Z_1^1) + \phi(Z_1^2) \right] I_{Z_1^1 \neq Z_1^2} \leq Ce^{-\beta t} \left[ \phi(z^1) + \phi(z^2) \right] \left[ 1 + \sum_{k=1}^{\infty} \left( q \kappa(T, K') \right)^{\frac{k+1}{k}} \right] = C'e^{-\beta t} \left[ \phi(z^1) + \phi(z^2) \right],
\]

where \( C' = C \left[ 1 + \sum_{k=1}^{\infty} \left( q \kappa(T, K') \right)^{\frac{k+1}{k}} \right] \). Note that \( C' < +\infty \) if, in the construction described before, \( q > 1 \) is taken in such a way that \( \kappa(T, K') < 1 \).

Now, we put \( z^1 = x \) and assume \( z^2 \) to be random and have its distribution equal to \( \pi \). Then \( Z \) is a \((\delta, \pi)\)-coupling, and, by \((A.7)\),

\[
E\left[ \phi(Z_1^1) + \phi(Z_1^2) \right] I_{Z_1^1 \neq Z_1^2} \leq C'e^{-\beta t} \left[ \phi(x) + \int_{\mathbb{X}} \phi \, d\pi \right] \leq Ce^{-\beta t} \phi(x),
\]

here we took into account that \( \phi \geq 1 \) and \( \int_{\mathbb{X}} \phi \, d\pi < +\infty \). The proof of Theorem 2.1 is complete.

**Remark A.1.** Condition 3) of Theorem 2.1 yields \( \sup_{x \in K, t \in \mathbb{R}} E_x \phi(X_t) < +\infty \). On the other hand, existence of exponential \( \phi \)-coupling provides that \( P_t(x, dy) \to \pi(dy), t \to \infty \) in variation for every \( x \in \mathbb{X} \). Consequently, under conditions of Theorem 2.1, \( \int_{\mathbb{X}} \phi \, d\pi < +\infty \). One can easily deduce similar statement under conditions of Theorem 2.2 and Proposition 2.2.

A.2. **Proof of Theorem 2.2** One can see that, in the previous arguments, the only place where condition 1) of Theorem 2.1 was used is that the set \( K' = \{ \phi \leq c \} \) has a compact closure. This property was not required straightforwardly: we use it only to verify that \( \kappa(T, K') < 1 \), i.e. that \( X \) satisfies the Doeblin condition on \( K' \). Hence, literally the same arguments ensure that the process \( X \) admits an exponential \( \phi \)-coupling assuming that \( X \) satisfies the Doeblin condition on every set of the type \( \{ \phi \leq c \} \). Therefore, the following statement yields Theorem 2.2.

**Lemma A.2.** Assume that conditions 2), 3) of Theorem 2.1 hold true and \( X \) satisfies the extended Doeblin condition on \( K' \).

Then \( X \) satisfies the Doeblin condition on every set of the type \( \{ \phi \leq c \} \).
Proof. We use an auxiliary construction of the extended gluing coupling. This coupling is defined, for fixed $z^1, z^2 \in \mathbb{X}, t_1, t_2 \in \mathbb{R}$, in such a way that $Z^1_0 = z^1, Z^2_0 = z^2$, and

$$P(Z^1_{t_1} \neq Z^2_{t_2}) = 1 - \frac{1}{2}\|P_{t_1}(z^1, \cdot) - P_{t_2}(z^2, \cdot)\|_{\text{var}}.$$

One can construct this coupling using literally the same arguments with those used in the construction of the (usual) gluing coupling (see [Kul09], Section 3.2), with the terminal time moment $T$ replaced by $t_1$ for the component $Z^1$ and $t_2$ for the component $Z^2$. It can be verified that such a construction can be made in a jointly measurable way w.r.t. probability variable and $z^{1,2}, t^{1,2}$ (we refer for a more detailed discussion of the measurability problems to [Kul09], Section 3.2).

Under condition 2) of Theorem 2.1,

$$P_x(\tau_K > t) \leq e^{-\alpha t}\phi(x).$$

Therefore, for $Q \in \mathbb{R}^+ \text{ large enough}$,

$$P_x(\tau_K \leq Q) \geq \frac{1}{2}, \quad x \in K' = \{\phi \leq c\}.$$

Consider two independent copies $Y^1, Y^2$ of the process $X$ starting from the points $x^1, x^2 \in K'$. Denote

$$\tau^{1,2} = \inf\{t \geq 0 : Y_t^{1,2} \in K\}.$$

Since $P(\tau^1 \leq Q, \tau^2 \leq Q) \geq \frac{1}{4}$, one of the following inequalities hold:

$$P(\tau^1 \leq \tau^2 \leq Q) \geq \frac{1}{8}, \quad P(\tau^2 \leq \tau^1 \leq Q) \geq \frac{1}{8}.$$

Assume that the first inequality holds (this does not restrict generality). Then we put $T = Q + T_1$ (here $T_1$ comes from (2.3)) and construct the coupling $Z_t, t \in [0, T]$ in the following way. If inequality $\tau^1 \leq \tau^2 \leq Q$ does not hold, then $Z_t^{1,2} = Y_t^{1,2}$. Otherwise we consider an independent copy of the extended gluing coupling, and substitute in it $Z_t^{x_1}, Z_t^{x_2}$ instead of the initial values $z^1, z^2$, and $T - \tau^1, T - \tau^2$ instead of the terminal time moments $t^1, t^2$. Under such a construction,

$$P(Z^1_t = Z^2_t) \geq \frac{1}{8}\sup_{z^1, z^2 \in K, t^1, t^2 \in [T_1, T_1 + Q]} \|P_{t_1}(z^1, \cdot) - P_{t_2}(z^2, \cdot)\|_{\text{var}}.$$

Therefore,

$$(A.8) \quad 1 - \kappa(T, K') \leq \frac{1}{8}\left(1 - \kappa(T_1, T_1 + Q, K)\right).$$

Clearly, $\kappa(T_1, T_2, K) \leq \kappa(T_1, T_2, K)$ for every $T_2' \in [T_1, T_2]$. On the other hand, using Chapman-Kolmogorov equation, one can verify easily that inequality (2.3) implies the same inequality with $T_2$ replaced by arbitrary $T_2' > T_2$. Hence, under condition (2.3), we can put $T_2' = T_1 + Q$ and get $\kappa(T_1, T_1 + Q, K) < 1$. This, together with (A.8), provides that $\kappa(T, K') < 1$. \hfill \square

A.3. Proof of Proposition 2.2 It can be verified easily that condition 1) of Proposition 2.2 implies that $\phi(x) = E_x e^{\alpha^\prime \tau_K}$ satisfies condition 1) of Theorem 2.1 By the Markov property of $X$,

$$\phi(X_t) = \left[ E_y e^{\alpha^\prime \tau_K} \right]_{y = X_t} = E \left[ e^{\alpha^\prime \tau_K} \mid \mathcal{F}_t \right], \quad t \geq 0$$

(see Section 2.1 for the notation $\tau_K^x$). We have $\tau_K^x = \tau_K - t$ on the set $\{\tau_K > t\}$. Therefore,

$$E_x \phi(X_t) \mathbf{1}_{\tau_K > t} = E_x e^{\alpha^\prime \tau_K} \mathbf{1}_{\tau_K > t} = E_x e^{\alpha^\prime (\tau_K - t)} \mathbf{1}_{\tau_K > t} \leq e^{-\alpha t} \phi(x).$$

Hence, condition 2) of Theorem 2.1 holds true with $\alpha$ replaced by $\alpha'$.  

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Condition 3) of Theorem 2.1 in fact, is the claim for the function \( \phi \) to be uniformly integrable w.r.t. the family of distributions \( \{ \mathcal{P}_t(x, \cdot), x \in K, t \in \mathbb{R}^+ \} \). Clearly, it is satisfied if
\[
\sup_{x \in K, t \in \mathbb{R}^+} E_x \phi^r(X_t) < +\infty.
\]
for some \( r > 1 \). Therefore, for the function \( \phi(x) = E_x e^{\tau_K}, \) condition 3) of Theorem 2.1 holds true provided that
\[
\sup_{x \in K, t \in \mathbb{R}^+} E_x \phi^{\alpha/\alpha'}(X_t) < +\infty.
\]
(recall that \( \alpha' \in (0, \alpha) \)). By the Hölder inequality,
\[
\phi^{\alpha/\alpha'}(y) \leq E_y e^{\tau_K}.
\]
Therefore, Proposition 2.2 is provided by Theorems 2.1, 2.2 and the following statement.

**Lemma A.3.** Let function \( \psi : \mathbb{X} \to [1, +\infty) \) be such that
\[
E_x \psi(X_t) \mathbf{1}_{\tau_K > t} \leq e^{-\alpha t} \psi(x), \quad x \in \mathbb{X};
\]
\[
\exists S > 0 : \sup_{x \in K, t \leq S} E_x \psi(X_t) < +\infty.
\]
Then
\[
\sup_{x \in K, t \in \mathbb{R}^+} E \psi(X_t) < +\infty.
\]

**Proof.** For \( t > S \), one has
\[
E_x \psi(X_t) = E_x \psi(X_t) \mathbf{1}_{\tau_K > t} + E_x \psi(X_t) \mathbf{1}_{\tau_K \leq t} \leq \int_{\mathbb{X}} \left[ E_y \psi(X_t) \mathbf{1}_{\tau_K > t - S} \right] P_S(x, dy) + E_x \psi(X_t) \mathbf{1}_{\tau_K \leq t}.
\]
Denote \( \mathcal{T}_k = [kS, (k+1)S] \). It follows from (A.9) that
\[
\sup_{t \in \mathcal{T}_k} E_x \psi(X_t) \leq e^{-\alpha (k-1)S} E_x \psi(X_S) + \sup_{t \in \mathcal{T}_{k-1}} E_x \psi(X_t), \quad k \geq 1,
\]
and, consequently,
\[
\sup_{t \in \mathcal{T}_k} E_x \psi(X_t) \leq (e^{-\alpha (k-1)S} + \cdots + 1) E_x \psi(X_S) + \sup_{t \leq S} E_x \psi(X_t) \leq [1 + (1 - e^{-\alpha S})^{-1}] \sup_{t \leq S} E_x \psi(X_t).
\]

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