ELECTROMAGNETIC WAVE PROPAGATION IN GLHUA INVISIBLE SPHERE BY GL NO SCATTERING FULL WAVE MODELING AND INVERSION

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Using GL no scattering full wave modeling and inversion, we create a GLHUA pre cloak electromagnetic (EM) material in the virtual sphere that makes the sphere is invisible. The invisible sphere is called GLHUA sphere. In GLHUA sphere, the Pre cloak relative parameter is not less than 1; the parameters and their derivative are continuous across the boundary r=R2 and the parameters are going to infinity at origin r=0. The phase velocity of EM wave in the sphere is less than light speed and going to zero at origin. The EM wave field excited in the outside of the sphere can not be disturbed by GLHUA sphere. By GL full wave method, we rigorously proved the incident EM wave field excited in outside of GLHUA sphere and propagation through the sphere without any scattering by the sphere, the total EM field in outside of the sphere equal to the incident wave field. Moreover, we prove that in GLHUA sphere with the pre cloak material, when r is going to origin, EM wave field propagation in GLHUA sphere is going to zero. All copyright and patent of the GLHUA EM cloaks,GLHUA sphere and GL modeling and inversion methods are reserved by authors in GL Geophysical Laboratory.

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I. INTRODUCTION

Using GL no scattering modeling and inversion and many GL method simulation [1] [2] [3], we find a class nonzero relative anisotropic parameter solution of EM zero scattering inversion in the sphere r ≤ R2. We create a novel material in GLHUA sphere with relative EM parameter not less than 1 that makes the sphere is invisible. The parameters and their derivative are continuous across the boundary r = R2 and the parameters are going to infinity at origin r = 0. The phase velocity of EM wave in the sphere is less than light speed and going to zero at origin. We discovered and proved an essential property that in the local sphere r ≤ RO without EM source, including R2 < r < RO annular layer in free space and r ≤ R2 in GLHUA sphere, on any spherical surface with radius r < RO, the spherical surface integral of Er sin θ and Hr sin θ is zero, (Er is radial electric wave field, Hr is radial magnetic wave field). The essential property of EM wave is a key difference from the acoustic wave and seismic wave. Based on the essential property, by GL full wave method, we rigorously proved the incident EM wave field excited in the outside of GLHUA sphere and propagation through the sphere without any scattering from GLHUA sphere, the total EM field in the outside of GLHUA sphere equal to the incident wave field. EM wave field excited in the outside of the sphere can not be disturbed by GLHUA sphere. GLHUA sphere is complete invisible. Moreover, based on the above essential property of EM wave, we prove that in GLHUA sphere with the pre cloak material, when r is going to origin, EM wave field propagation in GLHUA sphere is going to zero. We propose a special GL transform to map GLHUA outer annular layer cloak to GLHUA sphere. The anisotropic relative parameters of EM material in physical GLHUA outer annular layer cloak, εp,r = μp,r = 1, εp,θ = 1/2 (r-R1/R2-R1 + R2-R1/r-R1), εp,φ = εp,θ = μp,θ = μp,φ are mapping to anisotropic relative parameter material in GLHUA invisible sphere that satisfy GLHUA pre invisible cloak material conditions. From the theoretical proof in GLHUA invisible sphere, we can rigorously prove GLHUA outer annular layer cloak is invisible cloak with concealment. The major new ingredients in this paper are proposed in 7 sections. The introduction is presented in the section 1. Second order Maxwell electromagnetic equation in anisotropic material in spherical coordinate, and the basic fundamental electromagnetic wave in free space is presented in the section 2. The Global and Local (GL) method for radial electromagnetic equation and the essential property of radial EM wave are presented in the section 3. In the section 4, we propose GL EM wave Greens equation and GL EM Greens function. GL EM Integral equation and its theoretical proof are proposed in the section 5. In the section 6, we propose GLHUA EM invisible sphere, in this section, we rigorously proved that EM wave field propagation is going to zero at origin in GLHUA sphere. In section 7, the discussion and conclusion is presented.
II. SECOND ORDER MAXWELL ELECTROMAGNETIC EQUATION IN SPHERICAL COORDINATE

A. Electromagnetic Maxwell Equation In Sphere Coordinate System

\[
\nabla \times \vec{E} = \frac{1}{r^2 \sin \theta} \left[ \begin{array}{c}
\vec{r} \cdot \frac{\partial}{\partial r} \sin \theta \vec{\phi} \\
\frac{\partial}{\partial \theta} - \frac{1}{r} \vec{\phi} \\
\frac{\partial}{\partial \phi} 
\end{array} \right] \begin{bmatrix}
\varepsilon_r & \mu_r & \mu \phi \\
\mu_r & \varepsilon_\theta & \mu_\phi \\
\mu_\phi & \mu_\phi & \mu_0
\end{bmatrix} \begin{bmatrix}
E_r \\
E_\theta \\
E_\phi
\end{bmatrix}
\]

\[
\nabla \times \vec{H} = \frac{1}{r^2 \sin \theta} \left[ \begin{array}{c}
\vec{r} \cdot \frac{\partial}{\partial r} \sin \theta \vec{\phi} \\
\frac{\partial}{\partial \theta} - \frac{1}{r} \vec{\phi} \\
\frac{\partial}{\partial \phi} 
\end{array} \right] \begin{bmatrix}
\varepsilon_r & \mu_r & \mu \phi \\
\mu_r & \varepsilon_\theta & \mu_\phi \\
\mu_\phi & \mu_\phi & \mu_0
\end{bmatrix} \begin{bmatrix}
H_r \\
H_\theta \\
H_\phi
\end{bmatrix}
\]

\[
\nabla \cdot \vec{B} = \frac{1}{r^2 \sin \theta} \left( \varepsilon_r \mu_r \frac{\partial^2 E_r}{\partial r^2} + \varepsilon_\theta \mu_\theta \frac{\partial^2 E_\theta}{\partial \theta^2} + \varepsilon_\phi \mu_\phi \frac{\partial^2 E_\phi}{\partial \phi^2} \right)
\]

where \( \rho \) is the electric charge, \( \varepsilon_0 \) is the basic constant electric permittivity, \( \mu_0 \) is the basic constant magnetic permeability, \( \varepsilon_r \), relative radial electric permittivity, \( \varepsilon_\theta \) and \( \varepsilon_\phi \), relative angular electric permittivity, \( \mu_r \), relative radial magnetic permeability, \( \mu_\theta \) and \( \mu_\phi \), relative angular magnetic permeability.

B. Second Order Maxwell Electromagnetic Equation On Angular Electric Wave In Spherical Coordinate

We use Electric-Magnetic-Electric EME translate process to translate Maxwell Equation in Spherical Coordinate (S1) and (S2) with anisotropic material into the Second order Maxwell Electromagnetic Equation on Radial Electric Wave Field In Spherical Coordinate

\[
\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( r^2 \varepsilon_r E_r \right) + \frac{\sin \theta}{r^2 \sin \theta} \frac{\partial^2 E_r}{\partial \theta^2} + \frac{\varepsilon_r \mu_r \mu_\theta}{r^2 \sin \theta} \frac{\partial^2 E_r}{\partial \phi^2} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( k^2 \mu_\phi \varepsilon_r E_r \right)
\]

C. Second order Maxwell Electromagnetic Equation On Radial Magnetic Wave Field In Spherical Coordinate

We use MEM Magnetic-Electric-Magnetic translate process to translate Maxwell equation in spherical Coordinate (S1) and (S2) into the Second order Maxwell magnetic equation on radial magnetic wave field in spherical coordinate with anisotropic material,

\[
\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( \frac{1}{\mu_\phi} \frac{\partial}{\partial r} H_r \right) + \frac{\varepsilon_\theta}{r^2 \sin \theta} \frac{\partial^2 H_r}{\partial \theta^2} + \frac{k^2 \varepsilon_\phi \mu_r}{r^2 \sin \theta} \frac{\partial^2 H_r}{\partial \phi^2} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( \mu_\phi \mu_\theta \frac{\partial H_r}{\partial r} \right)
\]

D. Second Order Maxwell Electromagnetic Equation On Angular Magnetic Wave Field In Spherical Coordinate

After solving equation (5) and (6) and obtaining the radial electric wave field \( E_r(r, \theta, \phi) \) and radial magnetic wave field \( H_r(r, \theta, \phi) \), the following second order Maxwell electromagnetic equation govern angular electric wave field \( E_\theta(r, \theta, \phi) \) and \( E_\phi(r, \theta, \phi) \)

\[
\frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( \frac{1}{\mu_\phi} \frac{\partial}{\partial r} E_\theta \right) + \frac{k^2 \varepsilon_\phi \mu_r}{r^2 \sin \theta} \frac{\partial^2 E_\theta}{\partial \phi^2} = \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial r} \left( \mu_\phi \mu_\theta \frac{\partial E_\theta}{\partial r} \right)
\]

E. Second Order Maxwell Electromagnetic Equation On Angular Magnetic Wave Field In Spherical Coordinate

After solving equation (5) and (6) and obtaining the radial electric wave field \( E_r(r, \theta, \phi) \) and radial magnetic wave field \( H_r(r, \theta, \phi) \), the following second order Maxwell electromagnetic equation govern angular magnetic wave field \( H_\theta(r, \theta, \phi) \) and \( H_\phi(r, \theta, \phi) \)
\[ \frac{\partial}{\partial t} \left( \frac{1}{\varepsilon_0} \frac{\partial E_j}{\partial r} r H_\theta \right) + k^2 \mu_0 r H_\theta = i \omega \varepsilon_0 \frac{1}{\sin \theta} \frac{\partial E_j}{\partial \theta} + \frac{\partial}{\partial t} \left( \frac{1}{\varepsilon_0} \frac{\partial H_j}{\partial \varphi} \right) \]
\[ + \frac{\partial}{\partial r} \left( \frac{1}{\varepsilon_0} \frac{\partial r}{\partial r} J_\theta \right), \]
\[ (9) \]

F. Fundamental acoustic wave field

Let \( g(\vec{r}, \vec{r}_s) \) is the fundamental acoustic wave field, which is solution of following acoustic equation
\[ \frac{\partial^2 g}{\partial t^2} - \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial g}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 g}{\partial \varphi^2} + k^2 r^2 g = \delta(\vec{r} - \vec{r}_s) \delta(\theta - \theta_s) \delta(\phi - \phi_s), \]
\[ (11) \]
\[ g(\vec{r}, \vec{r}_s) = \frac{1}{4\pi \sqrt{|\vec{r} - \vec{r}_s|^2}} \]
\[ (12) \]
\[ |\vec{r} - \vec{r}_s| = \sqrt{|\vec{r} - \vec{r}_s|^2}, \]
\[ |\vec{r} - \vec{r}_s|^2 = r^2 + r_s^2 - 2rr_s \sin \theta \sin \theta_s \cos(\phi - \phi_s) - 2rr_s \cos \theta \cos \theta_s, \]
\[ (13) \]

where \( \vec{r}_s = (r_s, \theta_s, \phi_s) \) is point source location, \( \vec{r} = (r, \theta, \phi) \) is variable spherical coordinate, i.e. observation point.

\section{G. Basic fundamental electromagnetic wave field in free space}

The basic fundamental electromagnetic wave field in free space can be excited by current source \( \vec{J} = \delta(\vec{r} - \vec{r}_s) \vec{e}_j \), or magnetic moment source \( \vec{M} = \delta(\vec{r} - \vec{r}_s) \vec{e}_j \). We consider the current source in this paper, similarly theorem and proof are suitable for the magnetic moment source. Let \( \varepsilon_\perp = \varepsilon_\theta = \varepsilon_\phi = 1 \) and \( \mu_\perp = \mu_\theta = \mu_\phi = 1 \), electric current source \( \vec{J} = \delta(\vec{r} - \vec{r}_s) \vec{e}_j \). The basic fundamental electromagnetic wave field in free space are solutions of equations (5)-(6) in free space,
\[ E^b_j = \frac{1}{i \omega \varepsilon_0} \left( \nabla \nabla \cdot (g \vec{e}_j) + k^2 g \vec{e}_j \right), \]
\[ (14) \]
\[ E^b_{j,r} = \frac{1}{i \omega \varepsilon_0} \left( \varepsilon_\perp \left( \frac{\partial^2 g}{\partial r^2} + k^2 g \right) + \varepsilon_\perp \frac{\partial}{\partial t} \frac{\partial g}{\partial \varphi} \right) + \varepsilon_\perp \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \frac{\partial g}{\partial \theta} \]
\[ + \varepsilon_\perp \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \frac{\partial g}{\partial \varphi}, \]
\[ (15) \]
\[ H^b = \nabla \times (g \vec{e}_j), \]
\[ H^b_{r,r} = \left( \varepsilon_\perp \frac{1}{\sin \theta} \frac{\partial g}{\partial \theta} - \varepsilon_\perp \frac{1}{\sin \theta} \frac{\partial g}{\partial \varphi} \right), \]
\[ (16) \]
where the above equation are in the spherical coordinate system, \( g(\vec{r}, \vec{r}_s) \) is denoted in (11)-(13), \( \nabla \cdot \) is diverge operator \( E^b_j \) is basic fundamental electric wave, , \( H^b \) is basic fundamental magnetic wave, which is excited by source \( \vec{J} = \delta(\vec{r} - \vec{r}_s) \vec{e}_j \), the up script \( b \) means basic fundamental electric wave, the lower script \( j \) means source with excited vector \( \vec{e}_j \). \( E^b_j \) is radial basic fundamental electric wave, i.e. \( r \) component of \( E^b_j \) is radial basic fundamental magnetic wave, i.e. \( r \) component of \( H^b_j \), \( \vec{e}_j \), \( j=1,2,3 \) is a unite vector,
\[ e_1 = e_x = (\sin \theta \cos \phi \cos \theta \cos \phi - \sin \phi), \]
\[ e_2 = e_y = (\sin \theta \sin \phi \cos \theta \sin \phi \cos \phi), \]
\[ e_3 = e_z = (\cos \theta - \sin \theta \sin \phi), \]
\[ (17) \]

\section{III. GLOBAL AND LOCAL (GL) METHOD FOR RADIAL ELECTROMAGNETIC EQUATION AND THE ESSENTIAL PROPERTY OF THE RADIAL ELECTROMAGNETIC WAVE}

In the section 1, we proposed second order Maxwell electromagnetic equation in anisotropic material in spherical coordinate. The fundamental electromagnetic wave field in free space in sphere coordinate is presented in section 2. These basic equations are used in this and next sections.

A. Global and Local GL radial electric second order differential equation

For \( R_2 > 0 \), we consider electromagnetic equation (5)-(10) in the sphere \( r \leq R_2 \) with anisotropic media, in the outside of the sphere, \( r \geq R_2 \) with basic isotropic electric permittivity \( \varepsilon_0 \) and magnetic permeability \( \mu_0 \) in free space. In our paper, we suppose that the electromagnetic source set is bounded, the bounded source set \( \Omega_s \) is in outside of the large sphere with radius \( R_O \), \( R_O \geq R_2 \), for example, the point source located in outside of the sphere, \( r_s \geq R_O \). Using Global and Local (GL) field method, we propose radial GL electromagnetic field and GL electromagnetic equation. and study the following GL radial electromagnetic equation in the sphere \( r \leq R_2 \).

Definition of GL radial electromagnetic wave field
\[ E(\vec{r}) = \varepsilon_\perp r^2 E_s(\vec{r}), \]
\[ H(\vec{r}) = \mu_\perp r^2 H_s(\vec{r}), \]
\[ (18) \]
From the radial electric equation (5) , we propose GL radial electric second order differential equation
\[ \frac{\partial}{\partial r} \frac{1}{\varepsilon_\perp} \frac{\partial E}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \frac{1}{\sin \theta} \frac{\partial E}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \frac{\partial E}{\partial \varphi} + k^2 \mu_\perp E = J_s, \]
\[ (19) \]
where \( k = 2\pi f \sqrt{\varepsilon_0 \mu_0} \).

\[
J_s = \delta(r - r_s) \overline{\delta(\theta - \theta_s)} \delta(\phi - \phi_s) e_r
\]
electric point source \( r_s > R_O \). The incident GL electric wave field in free space,

\[
E^b_j(\vec{r}) = r^2 E^b_j, \quad E^b = \frac{1}{r^2} r e_j \left( \frac{\partial^2}{\partial r^2} + k^2 g \right)
\]

\[
+ \frac{1}{r^2} \left( e_j \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + e_j \frac{\partial}{\partial r} \frac{\partial}{\partial \phi \sin \theta} \frac{\partial}{\partial \phi} \right),
\]

where \( g = g(r, r_s) \) is the fundamental solution of the acoustic wave equation, (11-13). Let \( E^b(\vec{r}) \) to denote one of the \( E^b_j(\vec{r}) = r^2 E^b_j, \quad j = 1, 2, 3 \), it obvious that

\[
\lim_{r \to 0} E^b(\vec{r}) = 0,
\]

\[
\lim_{r \to 0} \frac{\partial}{\partial r} E^b(\vec{r}) = 0,
\]

B. Global and Local GL radial magnetic field second order differential equation

By definition of GL magnetic wave \( H(\vec{r}) = \mu_s r^2 H_s(\vec{r}) \) in (18), from the radial magnetic equation (6), we propose GL radial magnetic field second order differential equation

\[
\frac{\partial}{\partial r} \left( \frac{1}{\mu_s} \frac{\partial}{\partial r} H \right) + \frac{1}{\mu_s r^2} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} H = M_s,
\]

\( M_s \) is magnetic point source, incident GL magnetic wave in free space is,

\[
H^b_j = r^2 H^b_j, \quad r \left( e_j \frac{\partial}{\partial \phi} g - e_j \frac{\partial}{\partial \phi} \sin \theta \frac{\partial}{\partial \phi} g \right).
\]

Let \( H^b(\vec{r}) \) to denote one of the \( H^b_j(\vec{r}) = r^2 H^b_j, \quad j = 1, 2, 3 \), it obvious

\[
\lim_{r \to 0} H^b(\vec{r}) = 0,
\]

\[
\lim_{r \to 0} \frac{\partial}{\partial r} H^b(\vec{r}) = 0,
\]

C. Spherical surface integral of incident GL electromagnetic wave is vanished

Define spherical surface integral of incident GL electric wave in free space as

\[
E^b_0(\vec{r}) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} E^b(\vec{r}) \sin \theta d\theta d\phi,
\]

D. Essential property of GL radial electromagnetic wave

Theorem 3.1: Suppose that the electromagnetic source set is bounded, the bounded source set is in outside of the sphere with large radius \( R_O, r_s > R_O \). In the no source domain, with weight \( \sin \theta \) spherical surface integral of incident radial GL electromagnetic wave is zero.

\[
E_0^b(r) = 0, \quad H^b_0(r) = 0,
\]

The spherical surface integral of radial GL electromagnetic wave is zero,

\[
E_0(r) = 0, \quad H_0(r) = 0,
\]

Proof: By equation (19), the GL radial electric second order differential equation in free space is

\[
\frac{\partial^2}{\partial r^2} E^b + \frac{1}{r^2} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} E^b
\]

\[
+ \frac{1}{\sin^2 \theta} \frac{\partial^2 E^b}{\partial \phi^2} + k^2 E^b = S_E
\]

The electromagnetic source \( S_E \) is denoted by (5), (6), because the bounded source set is in outside of the sphere, with large radius \( R_O, r_s > R_O \), there exist the no source domain \( r < R_O \), in the no source domain or for plane electromagnetic wave without source, \( S_E = 0 \). Use \( \sin \theta \) times both sides of (31) and take spherical surface integral and by integral by parts, we get Linville ordinary equation

\[
\frac{\partial^2}{\partial r^2} E^b + k^2 E^b = 0,
\]

From (21) and (22), the initial condition is

\[
\lim_{r \to 0} E^b_0(r) = 0
\]

\[
\lim_{r \to 0} \frac{\partial}{\partial r} E^b_0(r) = 0,
\]

The equation system (32)-(34) has only zero solution. We have proved that \( E_0^b(r) \) is complete vanished

\[
E_0(r) = 0,
\]
Similarly, we have proved that \( H_0^b(r) \) is complete vanished,
\[
H_0^b(r) = 0.
\]
The first part (29) of the theorem 3.1 is proved. Next, we prove the second part of the theorem. Suppose that the electromagnetic source set is bounded, the bounded source set is in outside of the sphere with radius \( R_O \), in the GLHUA sphere, \( r \leq R_2 < R_O \), the source term is zero in the right hand of GL electric equation (19). Use \( \sin \theta \) times both sides of (19) and take sphere surface integral and by integral by parts, we have ordinary equation
\[
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} E_0(r) \right) + k^2 \mu_0 E_0(r) = 0, \quad r \leq R_2. \tag{35}
\]
By above proof, we have initial condition \( E_0^b(R_2) = 0 \) and \( \frac{\partial}{\partial r} E_0^b(R_2) = 0 \). Because in the GLHUA sphere, electromagnetic material parameters and their derivative are continuous across the outer boundary \( r = R_2 \), the radial GL electromagnetic wave and their derivative are continuous across the boundary \( r = R_2 \).
\[
E_0(R_2) = E_0^b(R_2) = 0, \tag{36}
\]
\[
\frac{\partial}{\partial r} E_0(R_2) = \frac{\partial}{\partial r} E_0^b(R_2) = 0, \tag{37}
\]
The solution of ordinary differential equation (35) with zero initial boundary condition (36) and (37) must be zero. Therefore, \( E_0(r) = 0 \) similar \( H_0(r) = 0 \), we proved theorem 3.1 that spherical surface integral of GL electromagnetic wave is vanished.

**Theorem 3.2:** Spherical surface integral of incident radial electromagnetic wave is zero. Spherical surface integral of radial electromagnetic wave is zero.

**Proof:**
\[
E_0^b(r) = \frac{1}{4 \pi \varepsilon_0} \int_0^\pi \int_0^{2\pi} E^b(\vec{r}) \sin \theta d\theta d\phi = 0, \tag{38}
\]
\[
H_0^b(r) = \frac{1}{4 \pi \mu_0} \int_0^\pi \int_0^{2\pi} H^b(\vec{r}) \sin \theta d\theta d\phi = 0, \tag{39}
\]
\[
E_0(\vec{r}) = \frac{1}{4 \pi \varepsilon_0} \int_0^\pi \int_0^{2\pi} E(\vec{r}) \sin \theta d\theta d\phi = 0, \tag{40}
\]
\[
H_0(\vec{r}) = \frac{1}{4 \pi \mu_0} \int_0^\pi \int_0^{2\pi} H(\vec{r}) \sin \theta d\theta d\phi = 0 \tag{41}
\]
For EM point source and \( r_s > R_O \), and \( r < R_O \), also from (15) and (16), by direct integral, we can calculate
\[
E_0^b(r) = \frac{1}{4 \pi} \int_0^\pi \int_0^{2\pi} E^b(\vec{r}) \sin \theta d\theta d\phi = 0, \tag{42}
\]
and
\[
H_0^b(r) = \frac{1}{4 \pi} \int_0^\pi \int_0^{2\pi} H^b(\vec{r}) \sin \theta d\theta d\phi = 0, \tag{43}
\]
Here, we direct integral of (43), because (16)
\[
H_j^b(\vec{r}) = \frac{1}{4 \pi} \int_0^\pi \int_0^{2\pi} H_j^b(\vec{r}) \sin \theta d\theta d\phi, j = 1, 2, 3,
\]
\( \vec{e}_j \) is an unit vector of the source, \( j = 1, 2, 3 \), in (17),
\[
H_0^b(\vec{r}) = \frac{1}{4 \pi} \int_0^\pi \int_0^{2\pi} H_0^b(\vec{r}) \sin \theta d\theta d\phi
\]
\[
E(\vec{r}) = \varepsilon_0 r^2 E_r(\vec{r}),
\]
\[
H(\vec{r}) = \mu_0 r^2 H_r(\vec{r}),
\]
we have
\[
E_0(\vec{r}) = \frac{1}{\varepsilon_0 r^2} E_0(\vec{r}) = 0
\]
and
\[
H_0(\vec{r}) = \frac{1}{\mu_0 r^2} H_0(\vec{r}) = 0.
\]
. The theorem 3.2 is proved.

The spherical surface integral of incident radial electromagnetic wave is zero that is essential property. The key property is essential different between electromagnetic wave and acoustic wave. Note that GL electromagnetic wave (18) is not Maxwell electromagnetic field wave, also is not flux nor displace current. In the GL method, Global and Local virtual wave \( E \) and \( H \) in (18) is convenient under any coordinate transform. Global and Local virtual wave (18) and GL second order differential equation (19) (23) are important for study GLHUA sphere and GLHUA cloak. It is shown that the GL electromagnetic differential equation and their incident wave (18)-(21) and (22)-(25) have same equation form and theoretical properties. For simply, we study GL electric differential equation and its incident wave (18)-(21) in detail.
IV. GL ELECTROMAGNETIC GREENS EQUATION

A. We propose GL electromagnetic Greens equation for GL electric equation (18) and GL magnetic equation (23)

\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \theta} G(\vec{r}, \vec{r}') + \frac{1}{r} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} G(\vec{r}, \vec{r}') + k^2 G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}'), \tag{44}
\]

We find spherical surface integral of GL Greens function (44),

\[
G(\vec{r}, \vec{r}') = rr'g(\vec{r}, \vec{r}'), \tag{45}
\]

\[
g(\vec{r}, \vec{r}') = -\frac{1}{4\pi} e^{-ik|\vec{r} - \vec{r}'|}, \tag{46}
\]

Where

\[
|\vec{r} - \vec{r}'| = \sqrt{\vec{r} - \vec{r}'^2},
\]

\[
|\vec{r} - \vec{r}'|^2 = r^2 - 2rr' \sin \theta \sin \theta' \cos(\phi - \phi') - 2rr' \cos \theta \cos \theta' + r'^2,
\]

The GL Greens equation (44) and GL Greens function (45) are suitable for all global free space which is background of local cloak.

B. GL electromagnetic Greens function

We find and propose GL Greens function \(G(\vec{r}, \vec{r}')\) which is the solution of above GL electromagnetic Greens equation (44),

\[
G(\vec{r}, \vec{r}') = \chi r^{2\chi} - \frac{1}{4\pi} e^{-ik|\vec{r} - \vec{r}'|}, \tag{47}
\]

Take sphere surface integral of GL electromagnetic Greens equation (44), then sphere surface integral of Greens function, \(G_0(r, r')\), in (44) satisfy

\[
\frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \theta} G_0(r, r') + k^2 G_0(r, r') = \delta(r - r'), \tag{48}
\]

We find spherical surface integral of GL Greens function \(G_0(r, r')\) which is the solution of above equation (48),

\[
G_0(r, r') = \chi r^{2\chi} J_{2\chi}(kr) J_{2\chi}(kr') - \frac{1}{4\pi} e^{-ik|\vec{r} - \vec{r}'|},
\]

\[
J_0(kr) = \frac{\sin kr}{kr}, \quad y_0(kr) = -\frac{\cos kr}{kr}, \tag{50}
\]

V. GL ELECTROMAGNETIC INTEGRAL EQUATION

We propose the Global and Local GL integral equation on GL electric wave field \(E(\vec{r})\) in (18) in the sphere body \(r \leq R_2\)

\[
E(\vec{r}) = E_b(\vec{r}) - \int \frac{1}{S(r \leq R_2)} \left(1 - \frac{1}{\mu_H}\right) \frac{\partial}{\partial r} G(\vec{r}, \vec{r}') \frac{\partial}{\partial r} E dV
\]

\[
+ \int \frac{1}{S(r \leq R_2)} \left(1 - \frac{1}{\mu_E}\right) \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \theta^2} \right) E dV
\]

\[
+ \int k^2 (1 - \epsilon) \mu_0 G E dV, \tag{51}
\]

The equivalent between GL integral equation (51) and GL electric wave differential equation (19) is proved next theorem 5.1. In integral equation (51), we change \(E\) to \(H, \epsilon\) to \(\mu_r, \epsilon\) to \(\mu_\theta\), we obtain GL magnetic integral equation.

\[
H(\vec{r}) = H_b(\vec{r}) - \int \frac{1}{S(r \leq R_2)} \left(1 - \frac{1}{\mu_H}\right) \frac{\partial}{\partial r} G(\vec{r}, \vec{r}') \frac{\partial}{\partial r} H dV
\]

\[
+ \int \frac{1}{S(r \leq R_2)} \left(1 - \frac{1}{\mu_E}\right) \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \theta^2} \right) H dV
\]

\[
+ \int k^2 (1 - \epsilon) \mu_0 G H dV, \tag{52}
\]

A. The equivalent between GL integral equation (51) and GL electric wave differential equation (19)

**Theorem 5.1 :** Suppose that the radial electric wave \(E(\vec{r})\) is solution of the GL radial electric wave differential equation (19) with incident wave (20)-(22), Greens function \(G(\vec{r}, \vec{r}')\) in (45) satisfy the GL electromagnetic Greens differential equation (44), then \(E(\vec{r})\) satisfy the GL integral equation (51)

**Proof :** By using Greens function \(G(\vec{r}, \vec{r}')\) in (45) to time the GL electric differential equation (19) and after
doing some calculation and integral by part, we have

\[ \int_{S(r \leq R_2)} \frac{\partial}{\partial r} \left( \frac{1}{\varepsilon_0} \left( \frac{\partial}{\partial r} E \right) G \right) dV \]
\[- \int_{S(r \leq R_2)} \frac{1}{\varepsilon_0} \frac{\partial}{\partial r} E \frac{\partial}{\partial r} G dV \]
\[+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon, r} \right) \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) dV \]
\[+ \int_{S(r \leq R_2)} k^2 \mu G E dV = E^b(\vec{r}) \quad (53) \]

By using unknown wave function \( E(\vec{r}, \vec{r}_s) \) to time the GL Greens equation (44) and after do some calculation and integral by part, we have

\[ \int_{S(r \leq R_2)} \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r} G \right) E dV \]
\[- \int_{S(r \leq R_2)} \frac{\partial}{\partial r} G \frac{\partial}{\partial r} E dV \]
\[+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \right) \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) E dV \]
\[+ \int_{R^3} k^2 G E dV = E(\vec{r}) \quad (54) \]

To subtract (53) from (54)

\[ E(\vec{r}') = E^b(\vec{r}') \]
\[+ \frac{e^{-ikr'}}{4\pi} \int_0^{2\pi} \int_0^\pi E(0, \theta, \phi) \sin \theta d\theta d\phi \]
\[- \int_{S(r \leq R_2)} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \frac{\partial}{\partial r} G dV \]
\[+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon} \right) \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) E dV \]
\[+ \int_{S(r \leq R_2)} k^2 \mu_0 G E dV \quad (55) \]

Substitute (27) for sphere surface integral of electric wave into the (55), because the anisotropic inhomogeneous electromagnetic relative material parameter are variable in the sphere, \( r \leq R_2 \) The outside of sphere, \( r > R_2 \) is free space with relative electric permittivity \( \text{diag}(1, 1, 1) \) and magnetic permeability \( \text{diag}(1, 1, 1) \), because (30)

\[ \frac{e^{-ikr'}}{4\pi} \int_0^{2\pi} \int_0^\pi E(0, \theta, \phi) \sin \theta d\theta d\phi = 0 \]

the integral equation (55) will become

\[ E(\vec{r}') = E^h(\vec{r}') \]
\[- \int_{S(r \leq R_2)} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \frac{\partial}{\partial r} G dV \]
\[+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon} \right) \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) E dV \]
\[+ \int_{S(r \leq R_2)} k^2 \mu(1 - \mu_0) G E dV \quad (51) \]

The theorem 5.1 is proved.

VI. THEORY OF GLHUA ELECTROMAGNETIC INVISIBLE SPHERE AND BEHAVIOR OF THE ELECTROMAGNETIC WAVE FIELD PROPAGATION

A. GLHUA pre cloak material conditions on relative electric permittivity and magnetic permeability for invisible sphere

In this section, we propose GLHUA pre cloak material conditions of anisotropic relative electric permittivity and magnetic permeability for invisible virtual sphere \( r \leq R_2 \), which are of the following properties for invisible sphere \( r \leq R_2 \). The electric permittivity is the product of relative electric permittivity and \( \varepsilon_0 \), the magnetic permeability is the product of relative magnetic permeability and \( \mu_0, \varepsilon_0 \) is the basic constant electric permittivity, \( \mu_0 \) is basic constant magnetic permeability. GLHUA pre cloak material conditions in invisible virtual sphere \( r \leq R_2 \) are as follows:

\[ (6.1) \mu_r(r) = \varepsilon_r(r), \]
\[ \mu_\theta(r) = \mu_\phi(r) = \varepsilon_\phi(r) = \varepsilon_\phi(r) \]

in the sphere \( r \leq R_2 \) are continuous differentiable function of \( r \),

\[ (6.2) \] these parameter functions and their derivative functions are continuous across boundary \( r = R_2 \),

\[ (6.3) \lim_{r \to 0} r^2 \mu_r(r) = \infty, \]

\[ (6.4) \mu_\theta(r) = \mu_\phi(r) = \varepsilon_\phi(r) = f(r) \frac{1}{r}, \]

and their derivative are continuous across boundary \( r = R_2, \lim_{r \to 0} f(r) = \frac{R^2}{2} \),

\[ \lim_{r \to 0} r^2 f'(r) = 0, \]
\textbf{B. GL electromagnetic field and are approaching to zero at } r=0 \text{ in invisible sphere}

\textit{Theorem 6.1:} Suppose that the anisotropic relative electric permittivity \( \varepsilon_r(r), \varepsilon_{\theta}(r), \varepsilon_{\phi}(r) \) and magnetic permeability, \( \mu_r(r), \mu_{\theta}(r), \mu_{\phi}(r) \), satisfy the above GL-HUA pre cloak material conditions in invisible sphere (6.1) to (6.4), also we suppose that

\[
\int_{S(r \leq R_2)} \left( |E(\vec{r})|^2 + |H(\vec{r})|^2 \right) dV \text{ is finite}, \quad (60)
\]

then GL radial electromagnetic wave field is vanished at origin, \( r = 0 \),

\[
\lim_{r \to 0} E(\vec{r}) = 0, \quad (61)
\]

\[
\lim_{r \to 0} H(\vec{r}) = 0, \quad (62)
\]

\textbf{Proof}

\[
\lim_{r \to 0} r^2 E(\vec{r}) = 0, \quad \lim_{r \to 0} r^2 H(\vec{r}) = 0, \quad (63)
\]

is derived from (60). the additional condition (60) is reasonable finite energy condition. From GL radial electric integral equation (51)), we have

\[
E(\vec{r}) = E_b(\vec{r}) - \int_{S(r \leq R_2)} \left( \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G_E \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) \right) EdV
+ \int_{S(r \leq R_2)} k^2 \left( 1 - \mu_\theta \right) G E dV, \quad (64)
\]

The integral equation (51) is translated to

\[
\frac{1}{\varepsilon_0} E(\vec{r}') = E_b(\vec{r}') - \int_{S(r \leq R_2)} \left( \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G_E \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) \right) EdV
+ \int_{S(r \leq R_2)} k^2 \left( 1 - \mu_\theta \right) G E dV, \quad (65)
\]

The above equation (65) becomes

\[
\frac{1}{\varepsilon_0} E(\vec{r}') = E_b(\vec{r}') - \int_{S(r \leq R_2)} \left( \frac{\partial}{\partial r} G_E \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) \right) EdV
+ \int_{S(r \leq R_2)} k^2 \left( 1 - \mu_\theta \right) G E dV, \quad (66)
\]

The GL electric integral equation (51) becomes to

\[
\frac{1}{\varepsilon_0} E(\vec{r}') = E_b(\vec{r}') - \int_{S(r \leq R_2)} \left( \frac{\partial}{\partial r} G_E \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) \right) EdV
+ \int_{S(r \leq R_2)} k^2 \left( 1 - \mu_\theta \right) G E dV, \quad (67)
\]

Let

\[
G_L(r, r') = G(\vec{r}, \vec{r}') - G_0(r, r') \quad (68)
\]

to substitute (68) for \( G(\vec{r}, \vec{r}') \) into (67) , and by using \( E_0(r) = 0 \) in (30), the integral equation (67) becomes

\[
\frac{1}{\varepsilon_0} E(\vec{r}') = E_b(\vec{r}') - \int_{S(r \leq R_2)} \left( \frac{\partial}{\partial r} G_E \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} G \right) E dV
+ \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_0} \left( 1 - \frac{1}{\varepsilon_0} \right) \frac{\partial}{\partial r} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial G}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2 G}{\partial \phi^2} \right) \right) EdV
+ \int_{S(r \leq R_2)} k^2 \left( 1 - \mu_\theta \right) G L E dV, \quad (69)
\]

where

\[
G_L(r, r') = G(\vec{r}, \vec{r}') - G_0(r, r') \quad (70)
\]

\[
g_l(r, r') = r r' \sum_{l=1}^{\infty} g_l(r, r') \sum_{m=-l}^{l} Y_{l}^{m*}(\theta, \phi) Y_{l}^{m}(\theta', \phi'), \quad (70)
\]

\[
g_l(r, r') = i k j_l(k r) j_l(k r') - i y_l(k r'), r \leq r' \quad (71)
\]

\[
g_l(r, r') = i k (j_l(k r) - i y_l(k r)) j_l(k r'), r \geq r' \quad (72)
\]
Substitute the \( G_L(r, r') \) in (70-72) into the integral equation (69), the equation (69) becomes to the following integral equation,

\[
\frac{1}{\varepsilon^2} E(r') = E_0(r')
\]

\[
r' \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} \left( \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \varepsilon_r} \right) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi')
\]

\[
+ r' \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} \frac{1}{r'} \left( \frac{1}{\varepsilon_r^2} - \frac{1}{r_r^2} \right) l(l+1) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi')
\]

\[
+ r' \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} k^2 \left( \frac{1}{\varepsilon_r^2} - \mu \right) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi')
\]

\[
- \lim_{r' \to 0} \frac{1}{\varepsilon_r} E(r') = \lim_{r' \to 0} E_0(r')
\]

\[
r' \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} \left( \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \varepsilon_r} \right) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi')
\]

\[
- \lim_{r' \to 0} \frac{1}{\varepsilon_r} E(r') = 0
\]

From the equation (60)

\[
\lim_{r' \to 0} \frac{1}{\varepsilon_r} E(r') = 0
\]

From the equation (21)

\[
\lim I = \lim_{r' \to 0} E_0(r')
\]

In next, we will prove

\[
\lim_{r' \to 0} II =
\]

\[
- \lim_{r' \to 0} \frac{1}{\varepsilon_r} \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} \left( \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial \varepsilon_r} \right) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi') = 0,
\]

\[
\lim III =
\]

\[
= \lim_{r' \to 0} \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} \left( \frac{1}{\varepsilon_r^2} - \frac{1}{r_r^2} \right) (l+1) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi') = 0,
\]

\[
\lim IV =
\]

\[
= \lim_{r' \to 0} \sum_{l=1}^{\infty} \int_{S(r \leq R_2)} k^2 \frac{1}{r_r^2} r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{lm}^m(\theta, \phi) E_{lm}^m(\theta', \phi') = 0,
\]

Substitute (70) and (71) into the (77) and using L'HOSPITAL ROLE, we prove the limitation equation (77) in detail,
By LHOPITAL ROLE

\[
\lim_{r' \to 0} II = \\
\lim_{r' \to 0} \sum_{l=1}^{\infty} \frac{(kr')^2}{2(l+1)(2l+3)} \left( \frac{1}{\sigma} \right) \left( \frac{1}{\tau} \right)
\]

\[
\sum_{m=-l}^{l} \int_0^{\pi} 2\pi Y_{m}^{l}(\theta, \phi) E(r, \theta, \phi) \sin \theta d \theta d \phi Y_{m}^{l}(\theta', \phi')
\]

\[
- \lim_{r' \to 0} \sum_{l=1}^{\infty} \frac{(kr')^2}{2(l+1)(2l+3)} \left( \frac{1}{\sigma} \right) \left( \frac{1}{\tau} \right)
\]

\[
\sum_{m=-l}^{l} \int_0^{\pi} 2\pi Y_{m}^{l}(\theta, \phi) E(r, \theta, \phi) \sin \theta d \theta d \phi Y_{m}^{l}(\theta', \phi')
\]

\[
- \lim_{r' \to 0} \sum_{l=1}^{\infty} \frac{(kr')^2}{2(l+1)(2l+3)} \left( \frac{1}{\sigma} \right) \left( \frac{1}{\tau} \right)
\]

\[
\sum_{m=-l}^{l} \int_0^{\pi} 2\pi Y_{m}^{l}(\theta, \phi) E(r, \theta, \phi) \sin \theta d \theta d \phi Y_{m}^{l}(\theta', \phi')
\]

\[
= 0,
\]

Therefore,

\[
\lim_{r' \to 0} II = \lim_{r' \to 0} \sum_{l=1}^{\infty} \int_{r \leq R_2} \frac{1}{\sigma} \left( \frac{1}{\tau} \right) l(l+1) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{m}^{l}(\theta, \phi) EdV Y_{m}^{l}(\theta', \phi') = 0,
\]

limitation equation (77) is proved. Similarly, we can prove limitation equation (78)

\[
\lim_{r' \to 0} III = \lim_{r' \to 0} \sum_{l=1}^{\infty} \int_{r \leq R_2} k^2 \left( \frac{1}{\sigma} \right) \left( \frac{1}{\tau} \right) l(l+1) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{m}^{l}(\theta, \phi) EdV Y_{m}^{l}(\theta', \phi') = 0,
\]

(86)

and (79),

\[
\lim_{r' \to 0} IV = \lim_{r' \to 0} \sum_{l=1}^{\infty} \int_{r \leq R_2} k^2 \left( \frac{1}{\sigma} \right) \left( \frac{1}{\tau} \right) l(l+1) r g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{m}^{l}(\theta, \phi) EdV Y_{m}^{l}(\theta', \phi') = 0,
\]

(87)

Substitute (75)-(79) into the limitation equation (74), the limitation equation (74) induces

\[
\lim_{r' \to 0} V = - \lim_{r' \to 0} \sum_{l=1}^{\infty} \int_{r \leq R_2} k^2 \mu g_l(r, r')
\]

\[
\sum_{m=-l}^{l} Y_{m}^{l}(\theta, \phi) EdV Y_{m}^{l}(\theta', \phi') = 0,
\]

(88)
Substitute (71) and (72) for \( g_l(r, r') \) into the (88),

\[
\lim_{r' \to 0} V = \\
\lim_{r' \to 0} k^3 \sum_{l=1}^{\infty} \int_0^{r'} \mu r j_l(kr) dr \\
\sum_{m=-l}^{l} \int_0^{\pi} \int_0^{2\pi} Y_{lm}^*(\theta, \phi) E(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \, Y_l^m(\theta', \phi') \\
- \lim_{r' \to 0} k^3 \sum_{l=1}^{\infty} \int_0^{r'} \mu r j_l(kr) dr \\
\sum_{m=-l}^{l} \int_0^{\pi} \int_0^{2\pi} Y_{lm}^*(\theta, \phi) E(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \, Y_l^m(\theta', \phi') \\
- \lim_{r' \to 0} i k^3 \sum_{l=1}^{\infty} \int_0^{r'} \mu r j_l(kr) dr \\
\sum_{m=-l}^{l} \int_0^{\pi} \int_0^{2\pi} Y_{lm}^*(\theta, \phi) E(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \, Y_l^m(\theta', \phi') \\
= 0. \\
(89)
\]

By LHOPITAL ROLE

\[
\lim_{r' \to 0} V = \\
\lim_{r' \to 0} k^3 \sum_{l=1}^{\infty} \int_0^{r'} \mu r j_l(kr) dr \\
\sum_{m=-l}^{l} \int_0^{\pi} \int_0^{2\pi} Y_{lm}^*(\theta, \phi) E(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \, Y_l^m(\theta', \phi') \\
- \lim_{r' \to 0} k^3 \sum_{l=1}^{\infty} \int_0^{r'} \mu r j_l(kr) dr \\
\sum_{m=-l}^{l} \int_0^{\pi} \int_0^{2\pi} Y_{lm}^*(\theta, \phi) E(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \, Y_l^m(\theta', \phi') \\
- \lim_{r' \to 0} i k^3 \sum_{l=1}^{\infty} \int_0^{r'} \mu r j_l(kr) dr \\
\sum_{m=-l}^{l} \int_0^{\pi} \int_0^{2\pi} Y_{lm}^*(\theta, \phi) E(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \, Y_l^m(\theta', \phi') \\
= 0. \\
(90)
\]

Because for incident wave with electric current point source \( \delta(\vec{r} - \vec{r}_s) \hat{e}_s \) at \( (r_s, \theta_s, \phi_s) \) \( r_s > R_1 > R_2, E_0(r) = 0 \) in (30),

\[
E(r', \theta', \phi') = \sum_{l=1}^{\infty} E_l(r') \\
\sum_{m=-l}^{l} Y_{lm}^*(\theta', \phi') Y_l^m(\theta_s, \phi_s) = 0. \\
(95)
\]

Substitute (95) into the (94), we have

\[
\lim_{r' \to 0} k^2 R_2^2 \sum_{l=1}^{\infty} \frac{1}{l(2l+1)} E_l(r') \\
\sum_{m=-l}^{l} Y_{lm}^m(\theta', \phi') Y_l^m(\theta_s, \phi_s) = 0. \\
(96)
\]
Let
\[ \Re(\theta, \phi) = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}, \]  
(97)

To use \( \Re(\theta, \phi) \) in (97) to make action to both side of the limitation equation (96), we have

\[ \lim_{r' \to 0} k^2 R^2_2 \Re(\theta', \phi') \sum_{l=1}^{\infty} \frac{1}{l(l+1)} E_l(r') \sum_{m=-l}^{l} Y_l^m(\theta', \phi') Y_l^m(\theta, \phi) = 0 \]  
(98)

\[ \lim_{r' \to 0} k^2 R^2_2 E(r', \theta', \phi') = 0, \]  
(99)

For every \((\theta', \phi') \) and \(k \geq k_0 > 0 \)

\[ \lim_{r' \to 0} E(r', \theta', \phi') = 0, \]  
(100)

Other proof approach is as follows: By using above similar with proof process of (98), for every term \(l \) and \(k \), we can prove that

\[ \lim_{r' \to 0} k^2 R^2_2 \frac{1}{l(l+1)} E_l(r') = 0, \]  
(101)

Because \( E_0(r) = 0 \) in (30), \(l \geq 1 \) and for \( k \geq k_0 > 0 \)

\[ \lim_{r' \to 0} E_l(r') = 0, \]  
(102)

\[ \lim_{r' \to 0} E(r', \theta', \phi') = \sum_{l=1}^{\infty} \lim_{r' \to 0} E_l(r') \sum_{m=-l}^{l} Y_l^m(\theta', \phi') Y_l^m(\theta, \phi) = 0, \]  
(103)

The limitation equation (61) is proved. Similarly, we can prove limitation equation (62)

\[ \lim_{r' \to 0} H(r', \theta', \phi') = \sum_{l=1}^{\infty} \lim_{r' \to 0} H_l(r') \sum_{m=-l}^{l} Y_l^m(\theta', \phi') Y_l^m(\theta, \phi) = 0, \]  
(104)

Similarly, for incident plane electromagnetic wave , we also can prove limitation equation (61) and (62). The theorem 6.1 is proved.

**Theorem 6.2.** Suppose that the anisotropic relative electric permittivity \( \varepsilon_r(r), \varepsilon_\theta(r), \varepsilon_\phi(r) \) and magnetic permeability, \( \mu_r(r), \mu_\theta(r), \mu_\phi(r) \), satisfy the above GL-HUA pre cloak material conditions in invisible sphere (6.1) to (6.4), and finite energy condition (60), then

\[ \lim_{r' \to 0} \frac{1}{r} \frac{\partial}{\partial r} E(r') = 0, \]  
(105)

\[ \lim_{r' \to 0} \frac{1}{r} \frac{\partial}{\partial r} H(r') = 0, \]  
(106)

**Proof :** Using similar proof process, the theorem 6.2 can be proved.

**Theorem 6.3.** Suppose that the anisotropic relative electric permittivity \( \varepsilon_r(r), \varepsilon_\theta(r), \varepsilon_\phi(r) \) and magnetic permeability, \( \mu_r(r), \mu_\theta(r), \mu_\phi(r) \), satisfy the above GL-HUA pre cloak material conditions in invisible sphere (6.1) to (6.4), also angular electromagnetic wave satisfy following finite energy condition (63)

\[ \int_{S(r \leq R_2)} \left( |rE_\theta(\vec{r})|^2 + |rH_\phi(\vec{r})|^2 \right) dV, \]  
(107)

then

\[ \lim_{r' \to 0} rE_\theta(\vec{r}) = 0, \]  
(108)

\[ \lim_{r' \to 0} rH_\phi(\vec{r}) = 0, \]  
(109)

**Proof :** Because the source is outside sphere, \( r_s > R_O > R_2 \), In the Sphere \( 0 < r \leq R_2 \), from (4), we have

\[ \nabla \cdot \vec{D} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \varepsilon_r E_r \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \varepsilon_\theta E_\theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \varepsilon_\phi E_\phi = 0 \]  
(110)

The \( \vec{D} \) is displacement electric in spherical coordinate, by definition of GL electromagnetic wave (18), equation (110) becomes

\[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta r E_\theta + \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} r E_\phi = -\varepsilon_0 \frac{\partial}{\partial r} E_r \]  
(111)

By Maxwell equation (1) and (18), we have

\[ -\varepsilon_0 \frac{\partial}{\partial \phi} r E_\phi + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} r E_\theta = -i \omega \mu_0 H, \]  
(112)
Rewrite (111) and (112) as matrix equation

\[
\begin{bmatrix}
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta & \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \\
-\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta & -\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \sin \theta
\end{bmatrix}
\begin{bmatrix}
rE_\theta \\
rE_\phi
\end{bmatrix} = \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \right) \left( -\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \sin \theta \right) \left( \frac{r{E}_\theta}{r{E}_\phi} \right) \sin \theta \theta \theta \phi \phi \phi
\]

(113)

The adjoint Greens equation of equation (113) on \([0, \pi; 0, 2\pi]\) is

\[
\begin{bmatrix}
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta & \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \\
-\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta & -\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \sin \theta
\end{bmatrix}
\begin{bmatrix}
G_{11}(\theta, \theta', \phi, \phi') & G_{21}(\theta, \theta', \phi, \phi') \\
G_{12}(\theta, \theta', \phi, \phi') & G_{22}(\theta, \theta', \phi, \phi')
\end{bmatrix}
\begin{bmatrix}
\delta(\theta, \theta', \phi, \phi') \\
\delta(\theta, \theta', \phi, \phi')
\end{bmatrix} = \frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} \sin \theta \right)
\]

(114)

It is different from GL Green equation in (44), the adjoint Greens equation is novel GLHUA angular Green equation. It is different from GL Green function in (45), the GLHUA angular Green function matrix is in the following

\[
G_{11}(\theta, \phi, \theta', \phi') = \sum_{l=1}^{\infty} \frac{1}{(l+1)} \sum_{m=-l}^{l} \frac{\partial}{\partial \theta} Y_{l}^{m*}(\theta, \phi) Y_{l}^{m*}(\theta', \phi'),
\]

\[
G_{12}(\theta, \phi, \theta', \phi') = -\sum_{l=1}^{\infty} \frac{1}{(l+1)} \sum_{m=-l}^{l} \frac{\partial}{\partial \theta} Y_{l}^{m*}(\theta, \phi) Y_{l}^{m*}(\theta', \phi'),
\]

\[
G_{21}(\theta, \phi, \theta', \phi') = \sum_{l=1}^{\infty} \frac{1}{(l+1)} \sum_{m=-l}^{l} \frac{\partial}{\partial \phi} Y_{l}^{m*}(\theta, \phi) Y_{l}^{m*}(\theta', \phi'),
\]

\[
G_{22}(\theta, \phi, \theta', \phi') = \sum_{l=1}^{\infty} \frac{1}{(l+1)} \sum_{m=-l}^{l} \frac{\partial}{\partial \phi} Y_{l}^{m*}(\theta, \phi) Y_{l}^{m*}(\theta', \phi').
\]

(115)

To use product of GLHUA Greens function matrix by \(\sin \theta, G(\theta, \theta', \phi, \phi') \sin \theta\), to multiply the matrix equation (113) and take sphere surface integral of resulted equation on \([0, \pi; 0, 2\pi]\)

\[
\int_{0}^{\pi} \int_{0}^{2\pi} \left( \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix} \right) \left( \begin{bmatrix}
rE_\theta \\
rE_\phi
\end{bmatrix} \right) \sin \theta \theta \theta \phi \phi \phi
\]

(116)

To use vector \([rE_\theta, rE_\phi] \sin \theta\) to multiply the adjoint Greens equation of equation (114) and take sphere surface integral of resulted adjoint equation on \([0, \pi; 0, 2\pi]\),

\[
\int_{0}^{\pi} \int_{0}^{2\pi} \left( \begin{bmatrix}
\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta & \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \\
-\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta & -\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \sin \theta
\end{bmatrix} \right) \left( \begin{bmatrix}
rE_\theta \\
rE_\phi
\end{bmatrix} \right) \sin \theta \theta \theta \phi \phi \phi
\]

(117)

To subtract equation (116) from (117), we have

\[
\int_{0}^{\pi} \int_{0}^{2\pi} \left( \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix} \right) \left( \begin{bmatrix}
rE_\theta \\
rE_\phi
\end{bmatrix} \right) \sin \theta \theta \theta \phi \phi \phi
\]

(118)

Based on theorem 6.1 and theorem 6.2, we have

\[
\lim_{r \to 0} \left( \begin{bmatrix}
rE_\theta(r, \theta', \phi') \\
rE_\phi(r, \theta', \phi')
\end{bmatrix} \right) = \int_{0}^{\pi} \int_{0}^{2\pi} \left( \begin{bmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{bmatrix} \right) \left( \begin{bmatrix}
rE_\theta(r) \\
rE_\phi(r)
\end{bmatrix} \right) \sin \theta \theta \theta \phi \phi \phi
\]

(119)

We have proved the first part of (108) of theorem 6.3, similarly, we can prove second part of (109). The theorem 6.3 is proved.

**Theorem 6.4.** Suppose that the anisotropic relative electric permittivity \(\varepsilon_r(r), \varepsilon_\phi(r)\) and magnetic permeability, \(\mu_r(r), \mu_\phi(r)\), satisfy the above GLHUA pre cloak material conditions in invisible sphere (6.1) to (6.4), also angular electromagnetic wave satisfy following finite energy condition (63), then

\[
\lim_{r \to 0} \frac{1}{\varepsilon_\phi(r)} E_\theta(r) = 0,
\]

\[
\lim_{r \to 0} \frac{1}{\varepsilon_\phi(r)} E_\phi(r) = 0,
\]

\[
\lim_{r \to 0} \frac{1}{\mu_\phi(r)} H_0(r) = 0,
\]

\[
\lim_{r \to 0} \frac{1}{\mu_\phi(r)} H_\phi(r) = 0,
\]

(120)

(121)

**Proof:** By using the similar proof process on the 8.3, we can prove the theorem 6.4.

**VII. DISCUSSION AND CONCLUSION**

The pre cloak material conditions (6.1) to (6.4) in GLHUA sphere is from GL zero scattering inversion and GL no scattering modeling. Many GL no scattering modeling simulations show that under the conditions
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[1] Xie, G., J.H. Li, E. Majer, D. Zuo, M. Oristaglio. "3-D electromagnetic modeling and nonlinear inversion," Geophysics, Vol. 65, No. 3, 804-822, 2000.

[2] Xie, G., F. Xie, L. Xie, and J. Li. "New GL method and its advantages for resolving historical difficulties," Progress In Electromagnetics Research, PIER 63, 141–152, 2006.

[3] Xie, G., J. Li, L. Xie, and F. Xie. "GL metro carlo EM Inversion," Journal of Electromagnetic Waves and Applications, Vol. 20, No. 14, 1991–2000, 2006.

[6.1] to (6.4), \( \lim E(r) = 0 \), and \( \lim H(r) = 0 \), are verified. These conditions is not unique and can be relaxed.