The ideal intersection property for essential groupoid C*-algebras

Matthew Kennedy
joint work with Se-Jin Kim, Xin Li, Sven Raum and Dan Ursu

University of Waterloo, Waterloo, Canada

March 17, 2022
Let $A$ be a $C^*$-algebra constructed in terms of some “combinatorial” data. For example, a group or group action.
Motivation

Let $A$ be a C*-algebra constructed in terms of some “combinatorial” data. For example, a group or group action.

**Basic question:** When is $A$ simple? Prefer an answer in terms of the data.
Motivation

Let $A$ be a C*-algebra constructed in terms of some “combinatorial” data. For example, a group or group action.

**Basic question:** When is $A$ simple? Prefer an answer in terms of the data.

**Theorem (Murray-von Neumann 1936)**

*For a discrete group $G$, the group von Neumann algebra $L(G)$ is simple (i.e. factorial) iff $G$ is ICC.*
Theorem (Powers 1975, ..., Olshanski-Osin 2014)

The $C^*$-algebra $C^*_\lambda(G)$ is simple if $G$ satisfies a “Powers-type” condition.
Theorem (Powers 1975, ..., Olshanski-Osin 2014)

The C*-algebra $\mathcal{C}_\lambda^*(G)$ is simple if $G$ satisfies a “Powers-type” condition.

Note: First established by Powers for $\mathbb{F}_2$. Culminating in work of Olshanskii-Osin for certain free Burnside groups.
Reduced discrete group C*-algebras

Theorem (Powers 1975, ..., Olshanski-Osin 2014)

The C*-algebra $C^*_{\lambda}(G)$ is simple if $G$ satisfies a “Powers-type” condition.

Note: First established by Powers for $\mathbb{F}_2$. Culminating in work of Olshanski-Osin for certain free Burnside groups.

Theorem (Kalanatar-K 2017)

The C*-algebra $C^*_{\lambda}(G)$ is simple if and only if $G$ has a topologically free boundary action.
Theorem (Powers 1975, ..., Olshanski-Osin 2014)

The $C^*$-algebra $C^*_\lambda(G)$ is simple if $G$ satisfies a “Powers-type” condition.

Note: First established by Powers for $\mathbb{F}_2$. Culminating in work of Olshanski-Osin for certain free Burnside groups.

Theorem (Kalanatar-K 2017)

The $C^*$-algebra $C^*_\lambda(G)$ is simple if and only if $G$ has a topologically free boundary action.

Note: Equivalent to freeness of the Furstenberg boundary $\partial_F G$. 
Theorem (Powers 1975, ..., Olshanski-Osin 2014)

The C*-algebra $\mathcal{C}^*_\lambda(G)$ is simple if $G$ satisfies a “Powers-type” condition.

Note: First established by Powers for $\mathbb{F}_2$. Culminating in work of Olshanskii-Osin for certain free Burnside groups.

Theorem (Kalanatar-K 2017)

The C*-algebra $\mathcal{C}^*_\lambda(G)$ is simple if and only if $G$ has a topologically free boundary action.

Note: Equivalent to freeness of the Furstenberg boundary $\partial_F G$.

Theorem (K 2020)

The C*-algebra $\mathcal{C}^*_\lambda(G)$ is simple if and only if $G$ has no amenable confined subgroups.
Reduced discrete group C*-algebras

**Theorem (Powers 1975, ..., Olshanski-Osin 2014)**

The $C^\ast$-algebra $C^\ast_\lambda(G)$ is simple if $G$ satisfies a “Powers-type” condition.

Note: First established by Powers for $F_2$. Culminating in work of Olshanskii-Osin for certain free Burnside groups.

**Theorem (Kalanatar-K 2017)**

The $C^\ast$-algebra $C^\ast_\lambda(G)$ is simple if and only if $G$ has a topologically free boundary action.

Note: Equivalent to freeness of the Furstenberg boundary $\partial_F G$.

**Theorem (K 2020)**

The $C^\ast$-algebra $C^\ast_\lambda(G)$ is simple if and only if $G$ has no amenable confined subgroups.

Note: $H \leq G$ is confined if it is non-trivial and “almost normal” in the sense that

$$1 \notin \{gHg^{-1}\}.$$
Theorem (Elliott 1980, ..., Archbold-Spielberg 1993)

If $G$ is amenable then $C(X) \rtimes G$ is simple if and only if $G \curvearrowright X$ is minimal and topologically free.

Theorem (Kawabe 2017)

The $C^*$-algebra $C(X) \times \lambda G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.

Note: Here $G_x$ is the stabilizer subgroup $G_x = \{g \in G : gx = x\}$.

Theorem (Kalantar-Scarparo 2021)

The $C^*$-algebra $C^0(X) \times \lambda G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.

Note: Key idea is that the inclusion $C^0(X) \times \lambda G \subseteq C(X)^* \times \lambda G$ is essential, where $X^* = X \sqcup \{\infty\}$.
| Theorem (Elliott 1980, ..., Archbold-Spielberg 1993) |
|--------------------------------------------------|
| If $G$ is amenable then $\mathbb{C}(X) \rtimes \lambda G$ is simple if and only if $G \curvearrowright X$ is minimal and topologically free. |

| Theorem (Kawabe 2017) |
|------------------------|
| The $C^*$-algebra $\mathbb{C}(X) \rtimes \lambda G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups. |
Reduced discrete crossed product $\text{C}^*$-algebras

**Theorem (Elliott 1980, ..., Archbold-Spielberg 1993)**

If $G$ is amenable then $\mathcal{C}(X) \rtimes \lambda G$ is simple if and only if $G \curvearrowright X$ is minimal and topologically free.

**Theorem (Kawabe 2017)**

The $\text{C}^*$-algebra $\mathcal{C}(X) \rtimes \lambda G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.

Note: Here $G_x$ is the stabilizer subgroup

$$G_x = \{g \in G : gx = x\}.$$
Reduced discrete crossed product C*-algebras

**Theorem (Elliott 1980, ..., Archbold-Spielberg 1993)**

If $G$ is amenable then $\mathbb{C}(X) \rtimes G$ is simple if and only if $G \curvearrowright X$ is minimal and topologically free.

**Theorem (Kawabe 2017)**

The C*-algebra $\mathbb{C}(X) \rtimes G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.

Note: Here $G_x$ is the stabilizer subgroup

$$G_x = \{ g \in G : gx = x \}.$$

**Theorem (Kalantar-Scarparo 2021)**

The C*-algebra $\mathbb{C}_0(X) \rtimes G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.
Reduced discrete crossed product $C^*$-algebras

**Theorem (Elliott 1980, ..., Archbold-Spielberg 1993)**

If $G$ is amenable then $C(X) \rtimes G$ is simple if and only if $G \curvearrowright X$ is minimal and topologically free.

**Theorem (Kawabe 2017)**

The $C^*$-algebra $C(X) \rtimes G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.

Note: Here $G_x$ is the stabilizer subgroup

$$G_x = \{g \in G : gx = x\}.$$

**Theorem (Kalantar-Scarparo 2021)**

The $C^*$-algebra $C_0(X) \rtimes G$ is simple if and only if $X$ is minimal and for every $x \in X$, $G_x$ has no amenable confined subgroups.

Note: Key idea is that the inclusion $C_0(X) \rtimes G \subseteq C(X^*) \rtimes G$ is essential, where $X^* = X \sqcup \{\infty\}$. 
Next step: reduced étale groupoid $\text{C}^*$-algebras
Étale groupoids

A **groupoid** is an algebraic structure \((\mathcal{G},^{-1},\ast)\) consisting of a set of objects \(\mathcal{G}\), an inverse map \(-^1 : G \rightarrow G\) and a (potentially only partially defined) multiplication \(\ast : G \times G \rightarrow G\).
Étale groupoids

A groupoid is an algebraic structure $(G,^{-1},*)$ consisting of a set of objects $G$, an inverse map $^{-1} : G \rightarrow G$ and a (potentially only partially defined) multiplication $* : G \times G \rightarrow G$.

The range and source maps $r : G \rightarrow G$ and $s : G \rightarrow G$ are defined by

$$r(g) = gg^{-1}, \quad s(g) = g^{-1}g.$$ 

The unit space is $G^{(0)} = r(G)$. 

Étale groupoids

A **groupoid** is an algebraic structure \((\mathcal{G}, -1, \ast)\) consisting of a set of objects \(\mathcal{G}\), an inverse map \(-1 : G \rightarrow G\) and a (potentially only partially defined) multiplication \(\ast : G \times G \rightarrow G\).

The range and source maps \(r : \mathcal{G} \rightarrow \mathcal{G}\) and \(s : \mathcal{G} \rightarrow \mathcal{G}\) are defined by

\[
r(g) = gg^{-1}, \quad s(g) = g^{-1}g.
\]

The unit space is \(\mathcal{G}^{(0)} = r(\mathcal{G})\).

The groupoid \(\mathcal{G}\) is **topological** if it is equipped with a locally compact topology for which the above maps are continuous and \(\mathcal{G}^{(0)}\) is Hausdorff in the relative topology. It is **étale** if, in addition, \(r\) is a local homeomorphism.
Étale groupoids

A **groupoid** is an algebraic structure \((\mathcal{G},^{-1}, \ast)\) consisting of a set of objects \(\mathcal{G}\), an inverse map \(-1 : G \rightarrow G\) and a (potentially only partially defined) multiplication \(\ast : G \times G \rightarrow G\).

The range and source maps \(r : \mathcal{G} \rightarrow \mathcal{G}\) and \(s : \mathcal{G} \rightarrow \mathcal{G}\) are defined by

\[
 r(g) = gg^{-1}, \quad s(g) = g^{-1}g.
\]

The unit space is \(\mathcal{G}^{(0)} = r(\mathcal{G})\).

The groupoid \(\mathcal{G}\) is **topological** if it is equipped with a locally compact topology for which the above maps are continuous and \(\mathcal{G}^{(0)}\) is Hausdorff in the relative topology. It is **étale** if, in addition, \(r\) is a local homeomorphism.

**Intuition:** Think about homeomorphisms between open subsets of a topological space with composition as multiplication, defined whenever it makes sense.
Étale groupoids

A **groupoid** is an algebraic structure \((\mathcal{G}, {^{-1}}, \ast)\) consisting of a set of objects \(\mathcal{G}\), an inverse map \(\ast^{-1} : G \rightarrow G\) and a (potentially only partially defined) multiplication \(\ast : G \times G \rightarrow G\).

The range and source maps \(r : \mathcal{G} \rightarrow \mathcal{G}\) and \(s : \mathcal{G} \rightarrow \mathcal{G}\) are defined by

\[
r(g) = gg^{-1}, \quad s(g) = g^{-1}g.
\]

The unit space is \(\mathcal{G}^{(0)} = r(\mathcal{G})\).

The groupoid \(\mathcal{G}\) is **topological** if it is equipped with a locally compact topology for which the above maps are continuous and \(\mathcal{G}^{(0)}\) is Hausdorff in the relative topology. It is **étale** if, in addition, \(r\) is a local homeomorphism.

**Intuition:** Think about homeomorphisms between open subsets of a topological space with composition as multiplication, defined whenever it makes sense.

Connes and Renault showed that **étale** groupoids give rise to an extremely rich class of C*-algebras: the reduced C*-algebra \(C^{*}\lambda(\mathcal{G})\) is of \(\mathcal{G}\) is the C*-completion of a convolution algebra of functions on \(\mathcal{G}\), or equivalently the C*-algebra generated by the left regular representation of \(\mathcal{G}\).
Reduced (Hausdorff) étale groupoid C*-algebras

Hausdorff case: $G$ is a Hausdorff étale groupoid

Theorem (Brown-Clark-Farthing-Sims 2014)
If $G$ is amenable, then $C^*_\lambda(G)$ is simple if and only if $G$ is minimal and topologically principal.

Theorem (Borys 2019)
The C*-algebra $C^*_\lambda(G)$ is simple if $G$ has compact unit space, is minimal and has no amenable confined subgroups of isotropy groups.

Note: One direction only. We will return to this point.
Reduced (Hausdorff) étale groupoid C*-algebras

**Hausdorff case:** $\mathcal{G}$ is a Hausdorff étale groupoid

**Theorem (Brown-Clark-Farthing-Sims 2014)**

If $\mathcal{G}$ is amenable, then $C^*_\lambda(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and topologically principal.

**Theorem (Borys 2019)**

The C*-algebra $C^*_\lambda(\mathcal{G})$ is simple if $\mathcal{G}$ has compact unit space, is minimal and has no amenable confined subgroups of isotropy groups.

Note: One direction only. We will return to this point.
Reduced (Hausdorff) étale groupoid
C*-algebras

Hausdorff case: $\mathcal{G}$ is a Hausdorff étale groupoid

**Theorem (Brown-Clark-Farthing-Sims 2014)**

If $\mathcal{G}$ is amenable, then $\mathcal{C}^*_{\lambda}(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and topologically principal.

**Theorem (Borys 2019)**

The $C^*$-algebra $\mathcal{C}^*_{\lambda}(\mathcal{G})$ is simple if $\mathcal{G}$ has compact unit space, is minimal and has no amenable confined subgroups of isotropy groups.
Hausdorff case: \( \mathcal{G} \) is a Hausdorff étale groupoid

**Theorem (Brown-Clark-Farthing-Sims 2014)**

If \( \mathcal{G} \) is amenable, then \( C^*_\lambda(\mathcal{G}) \) is simple if and only if \( \mathcal{G} \) is minimal and topologically principal.

**Theorem (Borys 2019)**

The \( C^* \)-algebra \( C^*_\lambda(\mathcal{G}) \) is simple if \( \mathcal{G} \) has compact unit space, is minimal and has no amenable confined subgroups of isotropy groups.

Note: One direction only. We will return to this point.
Reduced (potentially non-Hausdorff) étale groupoid C*-algebras

**General case:** \( \mathcal{G} \) is a (potentially non-Hausdorff) étale groupoid
Reduced (potentially non-Hausdorff) étale groupoid C*-algebras

**General case:** $\mathcal{G}$ is a (potentially non-Hausdorff) étale groupoid

More appropriate to work with the essential groupoid C*-algebra $C_{\text{ess}}^*(\mathcal{G})$ introduced by Exel-Pitts. This is the quotient of $C^*(\mathcal{G})$ by ideal of singular elements, so coincides with $C^*_\lambda(\mathcal{G})$ in the Hausdorff case.
Reduced (potentially non-Hausdorff) étale groupoid C*-algebras

**General case:** $\mathcal{G}$ is a (potentially non-Hausdorff) étale groupoid

More appropriate to work with the essential groupoid C*-algebra $C^*_{\text{ess}}(\mathcal{G})$ introduced by Exel-Pitts. This is the quotient of $C^*_\lambda(\mathcal{G})$ by ideal of singular elements, so coincides with $C^*_\lambda(\mathcal{G})$ in the Hausdorff case.

In the amenable case, characterizations of simplicity of $C^*_{\text{ess}}(\mathcal{G})$ can be deduced from work of Clark-Exel-Pardo-Sims-Starling (2019) Kwaśniewski-Meyer (2021).
(Speculative) strategy for characterizing simplicity

Let $G$ be an étale groupoid with compact Hausdorff unit space $G^{(0)}$. 

Preliminary strategy: Motivated by the strategy for groups.

1. Construct Furstenberg boundary $\partial F_G$ of $G$, i.e. the minimal injective object $C(\partial F_G)$ in the category of $G$-C*-algebras.

2. Replace $G$ by the "Furstenberg groupoid" $\partial F_G \rtimes G$, which is more tractable. Essentiality of the inclusion $C^\text{ess}(G) \subseteq C^\text{ess}(\partial F_G \rtimes G)$ implies simplicity of smaller C*-algebra equivalent to simplicity of larger C*-algebra.

3. Classify $G$-maps from $C^\text{ess}(\partial F_G \rtimes G)$ to $C(\partial F_G)$. Minimality of $C(\partial F_G)$ implies correspondence between maps and quotients of $C^\text{ess}(\partial F_G \rtimes G)$.

Major obstructions: What is the Furstenberg boundary? More generally, what is the right category to work in? $C^\text{ess}(G)$ is typically not a $G$-C*-algebra in the traditional sense, even when $G$ is Hausdorff. Note: Traditionally, a $G$-C*-algebra is fibered over $G^{(0)}$ and so contains a central copy of $C(G^{(0)})$. 
(Speculative) strategy for characterizing simplicity

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}^{(0)}$.

**Preliminary strategy**: Motivated by the strategy for groups.
(Speculative) strategy for characterizing simplicity

Let $G$ be an étale groupoid with compact Hausdorff unit space $G^{(0)}$.

**Preliminary strategy:** Motivated by the strategy for groups.

1. Construct Furstenberg boundary $\partial_F G$ of $G$, i.e. the minimal injective object $C(\partial_F G)$ in the category of $G$-$C^*$-algebras.
(Speculative) strategy for characterizing simplicity

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}^{(0)}$.

**Preliminary strategy:** Motivated by the strategy for groups.

1. Construct Furstenberg boundary $\partial F \mathcal{G}$ of $\mathcal{G}$, i.e. the minimal injective object $C(\partial F \mathcal{G})$ in the category of $\mathcal{G}$-$C^*$-algebras.

2. Replace $\mathcal{G}$ by the “Furstenberg groupoid” $\partial F \mathcal{G} \rtimes \mathcal{G}$, which is more tractable. Essentiality of the inclusion $C^{\text{ess}}(\mathcal{G}) \subseteq C^{\text{ess}}(\partial F \mathcal{G} \rtimes \mathcal{G})$ implies simplicity of smaller $C^*$-algebra equivalent to simplicity of larger $C^*$-algebra.

Major obstructions:
What is the Furstenberg boundary? More generally, what is the right category to work in?

$C^{\text{ess}}(\mathcal{G})$ is typically not a $\mathcal{G}$-$C^*$-algebra in the traditional sense, even when $\mathcal{G}$ is Hausdorff.

Note: Traditionally, a a $\mathcal{G}$-$C^*$-algebra is fibered over $\mathcal{G}^{(0)}$ and so contains a central copy of $C(G^{(0)})$. 
(Speculative) strategy for characterizing simplicity

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}(0)$.

**Preliminary strategy:** Motivated by the strategy for groups.

1. Construct Furstenberg boundary $\partial_F \mathcal{G}$ of $\mathcal{G}$, i.e. the minimal injective object $C(\partial_F \mathcal{G})$ in the category of $\mathcal{G}$-C*-algebras.

2. Replace $\mathcal{G}$ by the “Furstenberg groupoid” $\partial_F \mathcal{G} \rtimes \mathcal{G}$, which is more tractable. Essentiality of the inclusion $C_{\text{ess}}^*(\mathcal{G}) \subseteq C_{\text{ess}}^*(\partial_F \mathcal{G} \rtimes \mathcal{G})$ implies simplicity of smaller C*-algebra equivalent to simplicity of larger C*-algebra.

3. Classify $\mathcal{G}$-maps from $C_{\text{ess}}^*(\partial_F \mathcal{G} \rtimes \mathcal{G})$ to $C(\partial_F \mathcal{G})$. Minimality of $C(\partial_F \mathcal{G})$ implies correspondence between maps and quotients of $C_{\text{ess}}^*(\partial_F \mathcal{G} \rtimes \mathcal{G})$. 

Major obstructions: What is the Furstenberg boundary? More generally, what is the right category to work in? $C_{\text{ess}}^*(\mathcal{G})$ is typically not a $\mathcal{G}$-C*-algebra in the traditional sense, even when $\mathcal{G}$ is Hausdorff.

Note: Traditionally, a a $\mathcal{G}$-C*-algebra is fibered over $\mathcal{G}(0)$ and so contains a central copy of $C(\mathcal{G}(0))$. 
(Speculative) strategy for characterizing simplicity

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}(0)$.

Preliminary strategy: Motivated by the strategy for groups.

1. Construct Furstenberg boundary $\partial_F \mathcal{G}$ of $\mathcal{G}$, i.e. the minimal injective object $C(\partial_F \mathcal{G})$ in the category of $\mathcal{G}$-C*-algebras.

2. Replace $\mathcal{G}$ by the “Furstenberg groupoid” $\partial_F \mathcal{G} \rtimes \mathcal{G}$, which is more tractable. Essentiality of the inclusion $C^*_{\text{ess}}(\mathcal{G}) \subseteq C^*_{\text{ess}}(\partial_F \mathcal{G} \rtimes \mathcal{G})$ implies simplicity of smaller C*-algebra equivalent to simplicity of larger C*-algebra.

3. Classify $\mathcal{G}$-maps from $C^*_{\text{ess}}(\partial_F \mathcal{G} \rtimes \mathcal{G})$ to $C(\partial_F \mathcal{G})$. Minimality of $C(\partial_F \mathcal{G})$ implies correspondence between maps and quotients of $C^*_{\text{ess}}(\partial_F \mathcal{G} \rtimes \mathcal{G})$.

Major obstructions: What is the Furstenberg boundary? More generally, what is the right category to work in? $C^*_{\text{ess}}(\mathcal{G})$ is typically not a $\mathcal{G}$-C*-algebra in the traditional sense, even when $\mathcal{G}$ is Hausdorff.
(Speculative) strategy for characterizing simplicity

Let $G$ be an étale groupoid with compact Hausdorff unit space $G^{(0)}$.

Preliminary strategy: Motivated by the strategy for groups.

1. Construct Furstenberg boundary $\partial_F G$ of $G$, i.e. the minimal injective object $C(\partial_F G)$ in the category of $G$-C*-algebras.

2. Replace $G$ by the “Furstenberg groupoid” $\partial_F G \rtimes G$, which is more tractable. Essentiality of the inclusion $C^{\text{ess}}(G) \subseteq C^{\text{ess}}(\partial_F G \rtimes G)$ implies simplicity of smaller C*-algebra equivalent to simplicity of larger C*-algebra.

3. Classify $G$-maps from $C^{\text{ess}}(\partial_F G \rtimes G)$ to $C(\partial_F G)$. Minimality of $C(\partial_F G)$ implies correspondence between maps and quotients of $C^{\text{ess}}(\partial_F G \rtimes G)$.

Major obstructions: What is the Furstenberg boundary? More generally, what is the right category to work in? $C^{\text{ess}}(G)$ is typically not a $G$-C*-algebra in the traditional sense, even when $G$ is Hausdorff.

Note: Traditionally, a $G$-C*-algebra is fibered over $G^{(0)}$ and so contains a central copy of $C(G^{(0)})$. 
New category of groupoid C*-algebras

Solution: Instead of acting by elements of $\mathcal{G}$, act by elements of the pseudogroup $\Gamma(\mathcal{G})$ consisting of open bisections of $\mathcal{G}$, i.e. open subsets of $\mathcal{G}$ on which the range and source maps restrict to homeomorphisms. This permits a natural definition of composition.
New category of groupoid C*-algebras

Solution: Instead of acting by elements of \( \mathcal{G} \), act by elements of the pseudogroup \( \Gamma(\mathcal{G}) \) consisting of open bisections of \( \mathcal{G} \), i.e. open subsets of \( \mathcal{G} \) on which the range and source maps restrict to homeomorphisms. This permits a natural definition of composition.

Definition (sketch)

A \( \mathcal{G} \)-C*-algebra \( A \) is a C*-algebra containing a (not necessarily central) copy of \( C(\mathcal{G}^{(0)}) \) along with compatible families of hereditary subalgebras \( (A_U) \) and *-isomorphisms \( (\alpha_\gamma)_{\gamma \in \Gamma(\mathcal{G})} \) satisfying

\[
\alpha_\gamma : A_{\text{supp}(\gamma)} \to A_{\text{im}(\gamma)}.
\]
New category of groupoid C*-algebras

Solution: Instead of acting by elements of $\mathcal{G}$, act by elements of the pseudogroup $\Gamma(\mathcal{G})$ consisting of open bisections of $\mathcal{G}$, i.e. open subsets of $\mathcal{G}$ on which the range and source maps restrict to homeomorphisms. This permits a natural definition of composition.

Definition (sketch)

A $\mathcal{G}$-C*-algebra $A$ is a C*-algebra containing a (not necessarily central) copy of $C(\mathcal{G}^{(0)})$ along with compatible families of hereditary subalgebras $(A_U)$ and *-isomorphisms $(\alpha_\gamma)_{\gamma \in \Gamma(\mathcal{G})}$ satisfying

$$\alpha_\gamma : A_{\text{supp}(\gamma)} \to A_{\text{im}(\gamma)}.$$  

Theorem (KKLRU 2021)

There is a minimal injective $\mathcal{G}$-C*-algebra $C(\partial_F \mathcal{G})$ in the category of $\mathcal{G}$-C*-algebras.
Ideal intersection property

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$. 

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$.

More generally, we can ask when the only ideals in $C^*_\chi(\mathcal{G})$ are the "obvious" ones. This property was first considered by Sierakowski (2010) for crossed products.
Ideal intersection property

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$.

More generally, we can ask when the only ideals in $C^*_\lambda(\mathcal{G})$ are the “obvious” ones. This property was first considered by Sierakowski (2010) for crossed products.

**Definition**

The $C^*$-algebra $C^*_{\text{ess}}(\mathcal{G})$ has the ideal intersection property if every nonzero ideal of $C^*_{\text{ess}}(\mathcal{G})$ has nonzero intersection with $C_0(\mathcal{G}^{(0)})$.
Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}(0)$.

More generally, we can ask when the only ideals in $C^*_\lambda(\mathcal{G})$ are the “obvious” ones. This property was first considered by Sierakowski (2010) for crossed products.

**Definition**

The $C^*$-algebra $C^*_{\text{ess}}(\mathcal{G})$ has the ideal intersection property if every nonzero ideal of $C^*_{\text{ess}}(\mathcal{G})$ has nonzero intersection with $C_0(\mathcal{G}(0))$.

**Intuition:** The only ideals in $C^*_{\text{ess}}(\mathcal{G})$ are the “obvious” ones corresponding to invariant subspaces of $\mathcal{G}(0)$. 
Dynamical characterization of simplicity

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}^{(0)}$. Theorem (KKLRU 2021)

The following are equivalent:

1. The C*-algebra $C^*_{ess}(\mathcal{G})$ has the ideal intersection property.
2. The C*-algebra $C^*_{ess}(\partial F \mathcal{G} \rtimes \mathcal{G})$ has the ideal intersection property.
3. The Furstenberg boundary $\partial F \mathcal{G}$ is free.
4. There is a unique $\mathcal{G}$-pseudoexpectation from $C^*_{ess}(\mathcal{G})$ to $C(\partial F \mathcal{G})$.

Corollary

The C*-algebra $C^*_{ess}(\mathcal{G})$ is simple if and only if $\partial F \mathcal{G}$ is minimal and free.

Note: We can apply this result when $\mathcal{G}^{(0)}$ is not compact by replacing $\mathcal{G}$ with a suitable one-point compactification.
Dynamical characterization of simplicity

Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}^{(0)}$.

**Theorem (KKLRU 2021)**

The following are equivalent:
1. The C*-algebra $C^*_\text{ess}(\mathcal{G})$ has the ideal intersection property.
2. The C*-algebra $C^*_\text{ess}(\partial_F \mathcal{G} \rtimes \mathcal{G})$ has the ideal intersection property.
3. The Furstenberg boundary $\partial_F \mathcal{G}$ is free.
4. There is a unique $\mathcal{G}$-pseudoeexpectation from $C^*_\text{ess}(\mathcal{G})$ to $C(\partial_F \mathcal{G})$.

**Corollary**

The C*-algebra $C^*_\text{ess}(\mathcal{G})$ is simple if and only if $\partial_F \mathcal{G}$ is minimal and free.

Note: We can apply this result when $\mathcal{G}^{(0)}$ is not compact by replacing $\mathcal{G}$ with a suitable one-point compactification.
Dynamical characterization of simplicity

Let \( \mathcal{G} \) be an étale groupoid with compact Hausdorff unit space \( \mathcal{G}^{(0)} \).

**Theorem (KKLRU 2021)**

The following are equivalent:

1. The C*-algebra \( C^*_\text{ess}(\mathcal{G}) \) has the ideal intersection property.
2. The C*-algebra \( C^*_\text{ess}(\partial_{F}\mathcal{G} \rtimes \mathcal{G}) \) has the ideal intersection property.
3. The Furstenberg boundary \( \partial_{F}\mathcal{G} \) is free.
4. There is a unique \( \mathcal{G} \)-pseudoexpectation from \( C^*_\text{ess}(\mathcal{G}) \) to \( C(\partial_{F}\mathcal{G}) \).

**Corollary**

The C*-algebra \( C^*_\text{ess}(\mathcal{G}) \) is simple if and only if \( \partial_{F}\mathcal{G} \) is minimal and free.

Note: We can apply this result when \( \mathcal{G}^{(0)} \) is not compact by replacing \( \mathcal{G} \) with a suitable one-point compactification.
Let $\mathcal{G}$ be an étale groupoid with compact Hausdorff unit space $\mathcal{G}^{(0)}$.

**Theorem (KKLRU 2021)**

The following are equivalent:

1. The C*-algebra $C^*_{\text{ess}}(\mathcal{G})$ has the ideal intersection property.
2. The C*-algebra $C^*_{\text{ess}}(\partial_F \mathcal{G} \rtimes \mathcal{G})$ has the ideal intersection property.
3. The Furstenberg boundary $\partial_F \mathcal{G}$ is free.
4. There is a unique $\mathcal{G}$-pseudoexpectation from $C^*_{\text{ess}}(\mathcal{G})$ to $C(\partial_F \mathcal{G})$.

**Corollary**

The C*-algebra $C^*_{\text{ess}}(\mathcal{G})$ is simple if and only if $\partial_F \mathcal{G}$ is minimal and free.

Note: We can apply this result when $\mathcal{G}^{(0)}$ is not compact by replacing $\mathcal{G}$ with a suitable one-point compactification.
Intrinsic characterization of simplicity

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$. 
Intrinsic characterization of simplicity

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$.

For $x \in \mathcal{G}^{(0)}$, the isotropy group $\mathcal{G}_x^x$ is

$$\mathcal{G}_x^x = \{ g \in \mathcal{G} : s(x) = r(x) \}.$$

An isotropy subgroup is a subgroup of some $\mathcal{G}_x^x$. 
Intrinsic characterization of simplicity

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$.

For $x \in \mathcal{G}^{(0)}$, the isotropy group $\mathcal{G}_x$ is

$$\mathcal{G}_x = \{ g \in \mathcal{G} : s(x) = r(x) \}.$$  

An isotropy subgroup is a subgroup of some $\mathcal{G}_x$.

**Theorem (KKLRU 2021)**

The $C^*$-algebra $C^*_{\text{ess}}(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is minimal and has no amenable confined isotropy subgroups.
Intrinsic characterization of the ideal intersection property

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$. 
Intrinsic characterization of the ideal intersection property

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$.

A section of isotropy subgroups is a collection

$$\Lambda = \{(x, H_x) : x \in \mathcal{G}^{(0)}, H_x \leq \mathcal{G}_x^x\}.$$  

The section $\Lambda$ is essentially confined if $\mathcal{G}^{(0)} \not\subseteq \overline{\mathcal{G}\Lambda}$. 
Intrinsic characterization of the ideal intersection property

Let $\mathcal{G}$ be an étale groupoid with locally compact Hausdorff unit space $\mathcal{G}^{(0)}$.

A section of isotropy subgroups a collection

$$\Lambda = \{(x, H_x) : x \in \mathcal{G}^{(0)}, H_x \leq \mathcal{G}^x_x\}.$$

The section $\Lambda$ is essentially confined if $\mathcal{G}^{(0)} \not\subseteq \overline{\mathcal{G} \Lambda}$.

**Theorem (KKLRU 2021)**

The C*-algebra $C^*_\text{ess}(\mathcal{G})$ has the ideal intersection property if and only if $\mathcal{G}$ has no amenable essentially confined sections of isotropy subgroups.
1. In the non-Hausdorff case, what is the precise relationship between $\mathbb{C}_\lambda^*(G)$ and $\mathbb{C}_{\text{ess}}^*(G)$?
1. In the non-Hausdorff case, what is the precise relationship between $C^*_\lambda(G)$ and $C^*_{\text{ess}}(G)$?

2. How does the triviality of $\partial_F G$ relate to the amenability of $G$?
1. In the non-Hausdorff case, what is the precise relationship between $C^*_\lambda(G)$ and $C^*_{\text{ess}}(G)$?

2. How does the triviality of $\partial_F G$ relate to the amenability of $G$?

3. When is a twisted reduced crossed product of a group simple?
1. In the non-Hausdorff case, what is the precise relationship between $C^*_\lambda(G)$ and $C^*_{\text{ess}}(G)$?

2. How does the triviality of $\partial_F G$ relate to the amenability of $G$?

3. When is a twisted reduced crossed product of a group simple?
Thanks!