Twisting the $q$-deformations of compact semisimple Lie groups

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Abstract. Given a compact semisimple Lie group $G$ of rank $r$, and a parameter $q > 0$, we can define new associativity morphisms in $\text{Rep}(G_q)$ using a 3-cocycle $\Phi$ on the dual of the center of $G$, thus getting a new tensor category $\text{Rep}(G_q)^\Phi$. For a class of cocycles $\Phi$ we construct compact quantum groups $G_{\tau q}$ with representation categories $\text{Rep}(G_q)^\Phi$. The construction depends on the choice of an $r$-tuple $\tau$ of elements in the center of $G$. In the simplest case of $G = SU(2)$ and $\tau = -1$, our construction produces Woronowicz’s quantum group $SU_{-q}(2)$ out of $SU_q(2)$. More generally, for $G = SU(n)$, we get quantum group realizations of the Kazhdan–Wenzl categories.

Introduction.

A known problem in the theory of quantum groups is classification of quantum groups with fusion rules of a given Lie group $G$, see e.g. [Wor88], [WZ94], [Ban96], [Ohn99], [Bic03], [Ohn05], [Mro15]. Although this problem has been completely solved in a few cases, most notably for $G = SL(2, \mathbb{C})$ [Ban96], [Bic03], as the rank of $G$ grows the situation quickly becomes complicated. Already for $G = SL(3, \mathbb{C})$, even when requiring the dimensions of the representations to remain classical, one gets a large list of quantum groups that is not easy to grasp [Ohn99], [Ohn05]. A categorical version of the same problem turns out to be more manageable. Namely, the problem is to classify semisimple rigid monoidal $\mathbb{C}$-linear categories with fusions rules of $G$. As was shown by Kazhdan and Wenzl [KW93], for $G = SL(n, \mathbb{C})$ such categories $\mathcal{C}$ are parametrized by pairs $(q, \tau)$ of nonzero complex numbers, defined up to replacing $(q, \tau)$ by $(q^{-1}, q^{-1})$, such that $q^{n(n-1)/2} = \tau^n$ and $q$ is not a nontrivial root of unity. Concretely, these are twisted representation categories $\mathcal{C} = \text{Rep}(SL_q(n))^\zeta$, where $q$ is a nontrivial root of unity and $\zeta$ is a root of unity of order $n$; the corresponding parameters are $q = q^2$ and $\tau = \zeta^{-1}q^{n-1}$. The twists are defined by choosing a $T$-valued 3-cocycle on the dual of the center of $SL(n, \mathbb{C})$ and by using this cocycle to define new associativity morphisms in $\text{Rep}(SL_q(n))$. The third cohomology group of the dual of the center is cyclic of order $n$, and this explains the parametrization of twists of $\text{Rep}(SL_q(n))$ by roots of unity. A partial extension of the result of Kazhdan and Wenzl to types BCD was obtained by Tuba and Wenzl [TW05].

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This is not how the result is formulated in [KW93]. There is a known mistake in [KW93, Proposition 5.1], see [PR11, Section 7] for a discussion.
Although two problems are clearly related, a solution of the latter does not immediately say much about the former. The present work is motivated by the natural question whether there exist quantum groups with representation categories $\text{Rep}(SL_q(n))^\zeta$ for all $\zeta$ such that $\zeta^n = 1$. Equivalently, do the categories $\text{Rep}(SL_q(n))^\zeta$ always admit fiber functors? For $n = 2$ there is essentially nothing to solve, since for $q \neq 1$ the category $\text{Rep}(SL_q(2))^{-1}$ is equivalent to $\text{Rep}(SL_{-q}(2))$. For $q = 1$ the answer is also known: the quantum group $SU_{-1}(2)$ defined by Woronowicz (which has nothing to do with the quantized universal enveloping algebra $U_q(\mathfrak{sl}_2)$ at $q = -1$) has representation category $\text{Rep}(SL(2, \mathbb{C}))^{-1}$. For $n \geq 2$, quantum groups with fusion rules of $SL(n, \mathbb{C})$ have been studied by many authors, see e.g. [Hai00] and the references therein. Usually, one starts by finding a solution of the quantum Yang–Baxter equation satisfying certain conditions, and from this derives a presentation of the algebra of functions on the quantum group [RTF89]. This approach cannot work in our case, since the category $\text{Rep}(SL_q(n))^\zeta$ does not have a braiding unless $\zeta^2 = 1$.

The approach we take works, to some extent, for any compact semisimple simply connected Lie group $G$. Assume that $\Phi$ is a $\mathbb{T}$-valued 3-cocycle on the dual of the center of $G$. To construct a fiber functor $\varphi$ from the category $\text{Rep}(G_q)^\Phi$ with associativity morphisms defined by $\Phi$, such that $\dim \varphi(U) = \dim U$, is the same as to find an invertible element $F$ in a completion $U(G_q \times G_q)$ of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying

$$\Phi = (t \otimes \hat{\Delta}_q)(F^{-1})(1 \otimes F^{-1})(F \otimes 1)(\hat{\Delta}_q \otimes t)(F).$$

Then, using the twist (or a pseudo-2-cocycle in the terminology of [EV96]) $F$, we can define a new comultiplication on $U(G_q)$, thus getting a new quantum group with representation category $\text{Rep}(G_q)^\Phi$.

Our starting point is the simple remark that to solve the above cohomological equation we do not have to go all the way to $G_q$, it might suffice to pass from the center $Z(G)$ to a (quantum) subgroup of $G_q$, for example, to the maximal torus $T$. For simple $G$ this is indeed enough: any 3-cocycle on $\widehat{Z(G)}$ becomes a coboundary when lifted to the dual $P = \hat{T}$ of $T$. The reason is that, for simple $G$, the center is contained in a torus of dimension at most 2. However, a 2-cochain $f$ on $P$ such that $\partial f = \Phi$ is unique only up to a 2-cocycle on $P$. Already for trivial $\Phi$ this leads to deformations of $G_q$ by 2-cocycles on $P$ that are not very well studied [AST91], [LS91], with associated $C^*$-algebras of functions (for $q > 0$) that are typically not of type I.

Our next observation is that, for arbitrary $G$, if $\Phi$ lifts to a coboundary on $P$, then the cochain $f$ can be chosen to be of a particular form. This leads to a very special class of quantum groups $G_q^+$, whose construction depends on the choice of elements $\tau_1, \ldots, \tau_r \in Z(G)$, where $r$ is the rank of $G$. We show that the quantum groups $G_q^+$ are as close to $G_q$ as one could hope. For example, they can be defined in terms of finite central extensions of $U_q(\mathfrak{g})$.

Since we are, first of all, interested in compact quantum groups in the sense of Woronowicz, we will concentrate on the case $q > 0$, when the categories $\text{Rep}(G_q)^\Phi$ have a $C^*$-structure and, correspondingly, $G_q^+$ become compact quantum groups. We then show that the $C^*$-algebras $C(G_q^+)$ are KK-isomorphic to $C(G)$, they are of type I,
and their primitive spectra are only slightly more complicated than that of \( C(G_q) \). For \( G = SU(n) \) we also find explicit generators and relations of the algebras \( \mathbb{C}[SU_q^n(n)] \) of regular functions on \( SU_q^n(n) \).

To summarize, our construction produces quantum groups with nice properties and with representation category \( \text{Rep}(G_q)^\Phi \) for any 3-cocycle \( \Phi \) on \( \widehat{Z(G)} \) that lifts to a coboundary on \( \widehat{T} \). This covers the cases when \( G \) is simple, but in the general semisimple case there exist cocycles that do not have this property. For such cocycles the existence of fiber functors for \( \text{Rep}(G_q)^\Phi \) remains an open problem.

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1. **Preliminaries.**

1.1. **Compact quantum groups.**

A compact quantum group \( G \) is given by a unital \( C^* \)-algebra \( \mathbb{C}(G) \) together with a coassociative unital \( * \)-homomorphism \( \Delta: \mathbb{C}(G) \to \mathbb{C}(G) \otimes \mathbb{C}(G) \) satisfying the cancellation condition

\[
[\Delta(\mathbb{C}(G))(\mathbb{C}(G) \otimes 1)] = \mathbb{C}(G) \otimes \mathbb{C}(G) = [\Delta(\mathbb{C}(G))(1 \otimes \mathbb{C}(G))],
\]

where brackets denote the closed linear span. Here we only introduce the relevant terminology and summarize the essential results, see e.g. [NT13] for details.

A theorem of Woronowicz gives a distinguished state \( h \), the Haar state, which is an analogue of the normalized Haar measure over compact groups. Denote by \( C_r(G) \) the quotient of \( \mathbb{C}(G) \) by the kernel of the GNS-representation defined by \( h \). We will be interested in the case where \( h \) is faithful, so that \( C_r(G) = \mathbb{C}(G) \). This condition is automatically satisfied for coamenable compact quantum groups. The quantum groups studied in this paper will be coamenable thanks to Banica’s theorem [Ban99, Proposition 6.1] and [NT13, Theorem 2.7.14].

A finite dimensional unitary representation of \( G \) is given by a unitary element \( U \in B(\mathcal{H}_U) \otimes \mathbb{C}(G) \) satisfying the condition \( U_{13}U_{23} = (\iota \otimes \Delta)(U) \). The tensor product of two representations is defined by \( U \otimes V = U_{13}V_{23} \). The category \( \text{Rep}(G) \) of finite dimensional unitary representations of \( G \) has the structure of a rigid \( C^* \)-tensor category with a unitary fiber functor (‘forgetful functor’) \( U \mapsto \mathcal{H}_U \) to the category Hilb of finite dimensional Hilbert spaces. Woronowicz’s Tannaka–Krein duality theorem states that the reduced quantum group \( (C_r(G), \Delta) \) can be axiomatized in terms of \( \text{Rep}(G) \) and the fiber functor.

We denote by \( \mathbb{C}[G] \subset \mathbb{C}(G) \) the Hopf \( * \)-algebra of matrix coefficients of finite dimensional representations of \( G \). Denote by \( U(G) \) the dual \( * \)-algebra of \( \mathbb{C}[G] \), so \( U(G) = \prod_{\pi \in \text{irrep}(G)} B(\mathcal{H}_\pi) \). It can be considered from many different angles: as the algebra of functions on the dual discrete quantum group \( \widehat{G} \), as the algebra of endomorphisms of the forgetful functor, as the multiplier algebra of the convolution algebra \( \widehat{\mathbb{C}[G]} \) of \( G \). We also write \( U(G^n) \) for \( n \geq 2 \) to denote the ‘tensor product’ multipliers, such as
\[ \mathcal{U}(G^2) = \prod_{U,V \in \text{Irrep}(G)} B(\mathcal{H}_U) \otimes B(\mathcal{H}_V). \]

By duality, the multiplication map \( m : \mathbb{C}[G] \otimes \mathbb{C}[G] \to \mathbb{C}[G] \) defines a ‘coproduct’ \( \hat{\Delta} : \mathcal{U}(G) \to \mathcal{U}(G^2) \).

### 1.2. Twisting of quantum groups.

Let \( G \) be a compact quantum group, and \( \Phi \) be an invariant unitary 3-cocycle over the discrete dual of \( G \) \([\text{NT} 13, \text{Chapter 3}]\). Thus, \( \Phi \) is a unitary element in \( \mathcal{U}(G^3) \) satisfying the cocycle condition

\[
(1 \otimes \Phi)(\iota \otimes \hat{\Delta} \otimes \iota)(\Phi)(\Phi \otimes 1) = (\iota \otimes \iota \otimes \hat{\Delta})(\Phi)(\hat{\Delta} \otimes \iota \otimes \iota)(\Phi) \tag{1.1}
\]

and the invariance condition \([\Phi, (\hat{\Delta} \otimes \iota)\hat{\Delta}(x)] = 0 \) for \( x \in \mathcal{U}(G) \).

Then, the representation category \( \text{Rep}(G) \) can be twisted into a new \( C^* \)-tensor category \( \text{Rep}(G)^\Phi \), by using the action by \( \Phi \) on \( \mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W \) as the new associativity morphism \( (U \oplus V) \otimes W \to U \oplus (V \otimes W) \) for \( U, V, W \in \text{Rep}(G) \). The category \( \text{Rep}(G)^\Phi \) can be considered as the module category of the discrete quasi-bialgebra \( (\mathbb{C}[G], \hat{\Delta}, \Phi) \) \([\text{Dri} 89]\).

Suppose the category \( \text{Rep}(G)^\Phi \) is rigid. This is equivalent to the condition that the central element

\[ \Phi_1 \hat{S}(\Phi_2)\Phi_3 = m(m \otimes \iota)(\iota \otimes \hat{S} \otimes \iota)(\Phi) \]

in \( \mathcal{U}(G) \) is invertible. Suppose also that there exists a unitary \( F \in \mathcal{U}(G^2) \) such that

\[ \Phi = (\iota \otimes \hat{\Delta})(F^*)(1 \otimes F^*)(F \otimes 1)(\hat{\Delta} \otimes \iota)(F). \tag{1.2} \]

Then the discrete quantum group \( \mathcal{U}(G) \) can be deformed into another one, with the new coproduct \( \hat{\Delta}_F(x) = F\hat{\Delta}(x)F^* \). By duality, the function algebra \( \mathbb{C}[G] \) can be endowed with the new product

\[ x \cdot_F y = m(F^* \triangleright(x \otimes y) \triangleleft F). \]

Here, \( \triangleright \) and \( \triangleleft \) are the natural actions of \( \mathcal{U}(G) \) on \( \mathbb{C}[G] \) given by \( X \triangleright a = \langle X, a_{[2]} \rangle a_{[1]} \) and \( a \triangleleft X = \langle X, a_{[1]} \rangle a_{[2]} \). We denote the corresponding compact quantum group by \( G_F \). Note that in general the involution on \( \mathbb{C}[G_F] \) differs from the original one, see \([\text{NT} 13, \text{Example 2.3.9}]\).

We have a unitary monoidal equivalence of the \( C^* \)-tensor categories \( \text{Rep}(G)^\Phi \) and \( \text{Rep}(G_F) \). The tensor functor \( \varphi : \text{Rep}(G)^\Phi \to \text{Rep}(G_F) \) is given by the identity map on objects and morphisms, but with the nontrivial tensor transformation \( \varphi(U \oplus V) \to \varphi(U \oplus V) \) defined by

\[ \mathcal{H}_U \otimes \mathcal{H}_V \to \mathcal{H}_U \otimes \mathcal{H}_V, \quad \xi \otimes \eta \mapsto F^*(\xi \otimes \eta). \]
In terms of fiber functors, \( F \) gives a tensor functor \( \text{Rep}(\mathbb{G})^\Phi \to \text{Hilb}_t \) which is the same as that of \( \text{Rep}(\mathbb{G}) \) on objects and morphisms, but with the modified tensor transformation \( \mathcal{H}_U \otimes \mathcal{H}_V \to \mathcal{H}_{U \oplus V} \) given by \( \xi \otimes \eta \mapsto F^*(\xi \otimes \eta) \).

Examples of invariant 3-cocycles can be obtained as follows. Assume \( \mathbb{H} \) is a closed central subgroup of \( \mathbb{G} \), so \( \mathbb{H} \) is a compact abelian group and we are given a surjective homomorphism \( \pi: \mathbb{C}[\mathbb{G}] \to \mathbb{C}[\mathbb{H}] \) of Hopf \( * \)-algebras such that the image of \( \mathcal{U}(\mathbb{H}) \) under the dual homomorphism \( \mathcal{U}(\mathbb{H}) \to \mathcal{U}(\mathbb{G}) \) is a central subalgebra of \( \mathcal{U}(\mathbb{G}) \), or equivalently, for any irreducible unitary representation \( U \) of \( \mathbb{G} \) the element \( (\pi \otimes \pi)\xi(U) \) has the form \( 1 \otimes \chi_U \) for a character \( \chi_U \) of \( \mathbb{H} \). Unitary 3-cocycles in \( \mathcal{U}(\mathbb{H}^3) \) are nothing else than \( \mathbb{T} \)-valued 3-cocycles on the Pontryagin dual \( \hat{\mathbb{H}} \). Any such cocycle defines an invariant cocycle \( \Phi \) in \( \mathcal{U}(\mathbb{G}^3) \); when \( \mathbb{G} \) is itself compact abelian, this is just the usual pullback homomorphism \( Z^3(\hat{\mathbb{H}}; \mathbb{T}) \to Z^3(\hat{\mathbb{G}}; \mathbb{T}) \). Explicitly, the action of \( \Phi \) on \( \mathcal{H}_U \otimes \mathcal{H}_V \otimes \mathcal{H}_W \) is by multiplication by \( \Phi(\chi_U, \chi_V, \chi_W) \). For such cocycles \( \Phi \) the \( C^* \)-tensor category \( \text{Rep}(\mathbb{G})^\Phi \) is always rigid.

### 1.3. Quantized universal enveloping algebra.

Throughout the whole paper \( G \) denotes a semisimple simply connected compact Lie group, and \( \mathfrak{g} \) denotes its complexified Lie algebra. We fix a maximal torus \( T \) in \( G \), and denote the corresponding Cartan subalgebra by \( \mathfrak{h} \). The root lattice is denoted by \( Q \), and the weight lattice by \( P \). We fix a choice of positive roots, and denote the corresponding positive simple roots by \( \{ \alpha_1, \ldots, \alpha_r \} \). We also fix an \( \text{ad} \)-invariant symmetric form on \( \mathfrak{g} \) such that it is negative definite on the real Lie algebra of \( G \). If \( G \) is simple, we assume that this form is standardly normalized, meaning that \( (\alpha, \alpha) = 2 \) for every short root \( \alpha \). The Cartan matrix is denoted by \( (a_{ij})_{1 \leq i, j \leq r} \), and the Weyl group is denoted by \( W \).

The center \( Z(G) \) of \( G \) is contained in \( T \) and can be identified with the dual of \( P/Q \).

In what follows the variable \( q \) ranges over the strictly positive real numbers, although many results remain true for all \( q \neq 0 \) such that the numbers \( q_i = q^{(\alpha_i, \alpha_i)/2} \) are not nontrivial roots of unity. For \( q \neq 1 \), the quantized universal enveloping algebra \( \mathcal{U}_q(\mathfrak{g}) \) is the universal algebra over \( \mathbb{C} \) generated by the elements \( E_i, F_i \) and \( K_i^{\pm 1} \) for \( 1 \leq i \leq r \) satisfying the relations

\[
[K_i, K_j] = 0, \quad K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j,
\]

\[
[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \end{bmatrix}_k q_i E_i^k E_j F_i^{1-a_{ij}-k} = 0,
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \end{bmatrix}_k q_i F_i^k F_j F_i^{1-a_{ij}-k} = 0.
\]

It has the structure of a Hopf \( * \)-algebra defined by the operations

\[
\hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \hat{\Delta}_q(K_i) = K_i \otimes K_i,
\]

\[
\hat{\check{S}}_q(E_i) = -K_i^{-1} E_i, \quad \hat{\check{S}}_q(F_i) = -F_i K_i^{-1}, \quad \hat{\check{S}}_q(K_i) = K_i^{-1},
\]
\[ \hat{\epsilon}_q(E_i) = \hat{\epsilon}_q(F_i) = 0, \quad \hat{\epsilon}_q(K_i) = 1, \]
\[ E_i^* = F_i K_i, \quad F_i^* = K_i^{-1} E_i, \quad K_i^* = K_i. \]

A representation \((\pi, V)\) of \(U_q(g)\) is said to be \textit{admissible} when \(V\) admits a decomposition \(\bigoplus_{x \in \rho} V_x\) such that \(\pi(K_i)|_{V_x}\) is equal to the scalar \(q^{(\alpha, \chi)}\). The category of finite dimensional admissible \(*\)-representations of \(U_q(g)\) is a \(C^*\)-tensor category with the forgetful functor. We denote the associated compact quantum group by \(G_q\). There is a natural inclusion of \(T\) into \(G(G_q)\). Then the set \(Z(G_q)\) of group-like central elements in \(G(G_q)\) coincides with \(Z(G)\). The class of representations of \(G_q\) on which \(Z(G)\) acts trivially corresponds to a quotient quantum group denoted by \(G_q/Z(G)\).

2. Twisted \(q\)-deformations.

2.1. Extension of the QUE-algebra.

For \(q > 0\), we let \(\tilde{U}_q(g)\) denote the universal \(*\)-algebra generated by \(U_q(g)\) and unitary central elements \(C_1, \ldots, C_r\). It is not difficult to check that for \(q \neq 1\) the following formulas define a Hopf \(*\)-algebra structure on \(\tilde{U}_q(g)\):

\[ \hat{\Delta}(E_i) = E_i \otimes C_i + K_i \otimes E_i, \quad \hat{\Delta}(K_i) = K_i \otimes K_i, \quad \hat{\Delta}(C_i) = C_i \otimes C_i. \]

Similarly, for \(q = 1\), we define

\[ \hat{\Delta}(E_i) = E_i \otimes C_i + 1 \otimes E_i, \quad \hat{\Delta}(K_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \hat{\Delta}(C_i) = C_i \otimes C_i. \]

There is a Hopf \(*\)-algebra homomorphism from \(\tilde{U}_q(g)\) onto \(U_q(g)\), defined by \(C_i \mapsto 1\) and by the identity map on the copy of \(U_q(g)\). There is also a Hopf \(*\)-algebra homomorphism onto \(\mathbb{C}[[C_i^r]]_{i=1}^r\), given by \(E_i \mapsto 0, F_i \mapsto 0, K_i \mapsto 1\), and by the identity map on the \(C_i\)'s. We regard representations of \(U_q(g)\) and of \(\mathbb{C}[[C_i^r]]_{i=1}^r\) as the ones of \(\tilde{U}_q(g)\) via these homomorphisms.

Remark 2.1. The Hopf algebra \(\tilde{U}_q(g)\) is closely related to the Drinfeld double \(D(U_q(b_+))\) of \(U_q(b_+)\) \(\langle E_i, K_i \mid 1 \leq i \leq r \rangle\). Namely, put

\[ X_i^+ = E_i C_i^{-1}, \quad K_i^+ = K_i C_i^{-1}, \quad X_i^- = F_i, \quad K_i^- = K_i C_i. \]

Then we see that the elements \(X_i^+\) and \(K_i^+\) generate a copy of \(U_q(b_+)\), while the \(X_i^-\) and \(K_i^-\) generate a copy of \(U_q(b_-)\), and taking together these subalgebras give a copy of \(D(U_q(b_+))\) in \(\tilde{U}_q(g)\). The homomorphism \(\tilde{U}_q(g) \rightarrow U_q(g)\) is an extension of the standard projection \(D(U_q(b_+)) \rightarrow U_q(g)\). If we add square roots of \(K_i^\pm\) to \(D(U_q(b_+))\), thus getting a Hopf algebra \(D(U_q(b_+))\), we can recover \(U_q(g)\) by letting \(C_i = (K_i^-)^{1/2}(K_i^+)^{-1/2}\). Therefore we have inclusions of Hopf algebras \(D(U_q(b_+)) \subset \tilde{U}_q(g) \subset D(U_q(b_+))\).

Let \(\tau = (\tau_1, \ldots, \tau_r)\) be an \(r\)-tuple of elements in \(Z(G)\). We say that a representation \((\pi, V)\) of \(U_q(g)\) is \(\tau\)-\textit{admissible} if its restriction to \(U_q(g)\) is admissible and the elements \(C_i\) act on the weight spaces \(V_\chi\) as scalars \(\langle \tau_i, \chi \rangle\). The category of \(\tau\)-admissible repre-
sentations is a rigid $C^*$-tensor category with forgetful functor. Moreover, the $G_q/Z(G)$-representations are naturally included in the \( \tau \)-admissible representations as a $C^*$-tensor subcategory.

**Definition 2.2.** We let $G_q^\tau$ denote the compact quantum group realizing the category of finite dimensional \( \tau \)-admissible \( * \)-representations of $\tilde{U}_q(\mathfrak{g})$ together with its canonical fiber functor.

In other words, $\mathbb{C}[G_q^\tau] \subset \tilde{U}_q(\mathfrak{g})^*$ is spanned by matrix coefficients of finite dimensional $\tau$-admissible representations, and the Hopf \( * \)-algebra structure on $\mathbb{C}[G_q^\tau]$ is defined by duality using that of $\tilde{U}_q(\mathfrak{g})$.

Since every admissible representation of $U_q(\mathfrak{g})$ extends uniquely to a $\tau$-admissible representation of $\tilde{U}_q(\mathfrak{g})$, and every $\tau$-admissible representation is obtained this way, we can identify the \( * \)-algebra $U(G_q^\tau)$ with $U(G_q)$. The image $\tilde{U}_q^\tau(\mathfrak{g})$ of $\tilde{U}_q(\mathfrak{g})$ in $U(G_q^\tau) = U(G_q)$ plays the role of a quantized universal enveloping algebra for $G_q^\tau$. As an algebra it is generated by $E_i, F_i, K_i^{\pm 1}$ and $\tau_i$ (which is the image of $C_i$), but is endowed with a modified coproduct

\[
\hat{\Delta}(E_i) = E_i \otimes \tau_i + K_i \otimes E_i, \quad \hat{\Delta}(K_i) = K_i \otimes K_i, \quad \hat{\Delta}(\tau_i) = \tau_i \otimes \tau_i. \tag{2.1}
\]

To put it differently, as a \( * \)-algebra, $\tilde{U}_q^\tau(\mathfrak{g})$ is the tensor product of $\tilde{U}_q(\mathfrak{g})$ and the group algebra of the group $T_\tau \subset Z(G)$ generated by $\tau_1, \ldots, \tau_r$, while the coproduct is defined by (2.1). As a quotient of $\tilde{U}_q(\mathfrak{g})$, the Hopf \( * \)-algebra $\tilde{U}_q^\tau(\mathfrak{g})$ is obtained by requiring that the unitaries $C_1, \ldots, C_r$ satisfy the same relations as $\tau_1, \ldots, \tau_r \in Z(G)$.

**2.2. Twisting and associator.**

Given $\tau = (\tau_1, \ldots, \tau_r) \in Z(G)^r$, we obtain a 3-cocycle on $\widehat{Z(G)} = P/Q$ as follows.

First, let $f(\lambda, \mu)$ be a $\mathbb{T}$-valued function on $P \times P$ satisfying

\[
f(\lambda, \mu + Q) = f(\lambda, \mu), \quad f(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle f(\lambda, \mu). \tag{2.2}
\]

These conditions imply that $f$ can be determined by its restriction to the image of a set-theoretic section $(P/Q)^2 \rightarrow P^2$. For example, if $\lambda_1, \ldots, \lambda_n$ is a system of representatives of $P/Q$, then we can put

\[
f\left(\lambda_i + \sum_{j=1}^r m_j \alpha_j, \mu\right) = \prod_{j=1}^r \langle \tau_j, \mu \rangle^{m_j}
\]

for all $1 \leq i \leq n$ and $(m_1, \ldots, m_r) \in \mathbb{Z}^r$.

Using (2.2), the coboundary of $f$,

\[
(\partial f)(\lambda, \mu, \nu) = f(\mu, \nu) f(\lambda + \mu, \nu)^{-1} f(\lambda, \mu + \nu) f(\lambda, \mu)^{-1},
\]

is seen to be invariant under the translation by $Q$ in each variable. Thus, $\partial f$ can be considered as a 3-cocycle on $P/Q$ with values in $\mathbb{T}$. By construction, it is a cocycle. If
f′ satisfies the same condition as f above, the difference f′f−1 is \(Q^2\)-invariant, that is, it defines a function on \((P/Q)^2\). Thus, the cohomology class of \(\partial f\) in \(H^3(P/Q; \mathbb{T})\) depends only on \(\tau\). It also follows that the twisted coproduct \(\hat{\Delta}_f(x) = f\Delta_q(x)f^*\) does not depend on the choice of \(f\).

Since \((\partial f)^*\) belongs to \(U(Z(G)^3)\), as we discussed in Section 1.2, it can be regarded as an invariant 3-cocycle in \(U(G_q^3)\) which is denoted by \(\Phi^\tau\). Similarly, \(f\) can be considered as a unitary in \(U(G_q^2)\), and we have

\[
\Phi^\tau = (\iota \otimes \hat{\Delta}_q)(f^*)(1 \otimes f^*)(f \otimes 1)(\hat{\Delta}_q \otimes \iota)(f).
\]

**Proposition 2.3.** The coproduct \(\hat{\Delta}_f\) on \(U(G_q)\) coincides with the coproduct \(\hat{\Delta}\) defined by (2.1).

**Proof.** Since \(f\) is contained in \(U(T^2) \subset U(G_q^2)\), \(\hat{\Delta}_f = \hat{\Delta}_q\) on the elements \(K_i\). For \(E_i\), since the action of \(E_i\) on an admissible module increases the weight of a vector by \(\alpha_i\), identities (2.2) imply that \(f(K_i \otimes E_i)f^* = K_i \otimes E_i\) and \(f(E_i \otimes 1)f^* = E_i \otimes \tau_i\). Comparing these identities with (2.1), we obtain the assertion. \(\square\)

**Corollary 2.4.** The representation category of \(G_q^\tau\) is unitarily monoidally equivalent to \(\text{Rep}(G_q)^{\Phi^\tau}\), the representation category of \(G_q\) with associativity morphisms defined by \(\Phi^\tau\).

This result can also be interpreted as follows. Let \(\Phi_{KZ,q} \in U(G^3)\) be the Drinfeld associator coming from the Knizhnik–Zamolodchikov equations associated with the parameter \(h = \log(q)/\pi i\). The representation category of \(G_q\) is equivalent to that of \(G\) with associativity morphisms defined by \(\Phi_{KZ,q}\). The equivalence is given by a unitary Drinfeld twist \(F_D \in U(G^2)\) satisfying (1.2) for \(\Phi_{KZ,q}\) [NT13, Chapter 4]. It follows that \(\text{Rep}(G_q^\tau)\) is unitarily monoidally equivalent to the category \(\text{Rep}(G)\) with associativity morphisms defined by

\[
\Phi_{KZ,q}^\tau = (\iota \otimes \hat{\Delta})(F_D^\tau)(1 \otimes F_D^\tau)\Phi^\tau(F_D \otimes 1)(\hat{\Delta} \otimes \iota)(F_D) = \Phi^\tau \Phi_{KZ,q},
\]

where we now consider \(\Phi^\tau\) as an element of \(U(G^3)\). Correspondingly, the unitary \(F_D^\tau = fF_D \in U(G^2)\) plays the role of a Drinfeld twist for \(G_q^\tau\).

**Remark 2.5.** The construction of [NT10] can be carried out for \(G_q^\tau\) to obtain a spectral triple over \(C[G_q^\tau]\) as an isospectral deformation of the spin Dirac operator on \(G\). Indeed, it is enough to verify the boundedness of \([1 \otimes (\iota \otimes \gamma)(t), (\pi \otimes \iota \otimes \text{ad})(\Phi_{KZ,q})]\) for any irreducible representation \(\pi\), where \(t\) is the standard symmetric tensor \(\sum_i x_i \otimes x_i\) [NT10, Corollary 3.2]. Since \((\pi \otimes \iota \otimes \text{ad})(\Phi^\tau) \in C \otimes U(Z(G)) \otimes C\) commutes with \(1 \otimes (\iota \otimes \gamma)(t)\), we can reduce the proof to the case of trivial \(\tau\).

A natural question is how large the class of cocycles of the form \(\Phi^\tau\) is. These cocycles are analyzed in detail in Appendix. Using that analysis we point out the following.

**Proposition 2.6.** A \(\mathbb{T}\)-valued 3-cocycle \(\Phi\) on \(P/Q\) is cohomologous to \(\Phi^\tau\) for some...
$\tau_1, \ldots, \tau_r \in Z(G)$ if and only if $\Phi$ lifts to a coboundary on $P$. This is always the case if $P/Q$ can be generated by not more than two elements. For example, this is the case if $G$ is simple.

**Proof.** The first statement is proved in Corollary A.4. It is also shown there that another equivalent condition on $\Phi$ is that it vanishes on $\bigwedge^3(P/Q) \subset H_3(P/Q; \mathbb{Z})$. This condition is obviously satisfied if $P/Q$ can be generated by two elements. Finally, if $G$ is simple, then it is known that $P/Q$ is cyclic in all cases except for $G = \text{Spin}(4n)$, in which case $P/Q \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. □

Therefore for simple $G$ the quantum groups $G^\tau_q$ realize all possible associativity morphisms on $\text{Rep}(G_q)$ defined by 3-cocycles on the dual of the center. In the semisimple case this is not true as soon as the center becomes slightly more complicated, namely, as soon as $\bigwedge^3(P/Q) \neq 0$. We conjecture that in this case, if we take a cocycle $\Phi$ on $P/Q$ that does not lift to a coboundary on $P$, then there are no unitary fiber functors on $\text{Rep}(G)^\Phi$, that is, there are no compact quantum groups with this representation category. Note that by Corollary A.5 any such cocycle $\Phi$ is cohomologous to product of a cocycle $\Phi^\tau$ and a 3-character on $P/Q$ that is nontrivial on $\bigwedge^3(P/Q) \subset (P/Q)^{\otimes 3}$.

**2.3. Isomorphisms of twisted quantum groups.**

Denote the cohomology class of the cocycle $\Phi^\tau$ in $H^3(P/Q; \mathbb{T})$ by $\Theta(\tau)$. This way we obtain a homomorphism

$$\Theta: Z(G)^r \to H^3(P/Q; \mathbb{T}).$$

Assume $\tau \in \ker \Theta$. Let $f$ be a function satisfying (2.2). Then there exists a 2-cochain $g: (P/Q)^2 \to \mathbb{T}$ such that $\partial f = \partial g$, so that $fg^{-1}$ is a 2-cocycle on $P$. Another choice of $f$ and $g$ would give us a cocycle that differs from $fg^{-1}$ by a 2-cocycle on $P/Q$. Therefore taking the cohomology class of $fg^{-1}$ we get a well-defined homomorphism

$$\Upsilon: \ker \Theta \to H^2(P; \mathbb{T})/H^2(P/Q; \mathbb{T}).$$

**Proposition 2.7.** Assume $\tau', \tau \in Z(G)^r$ are such that

$$\tau' \tau^{-1} \in \ker \Theta \quad \text{and} \quad \tau' \tau^{-1} \in \ker \Upsilon.$$

Then the quantum groups $G^{\tau'}_q$ and $G^\tau_q$ are isomorphic.

**Proof.** Denote by $\hat{\Delta}'$ and $\hat{\Delta}$ the coproducts on $\mathcal{U}(G_q)$ defined by $\tau'$ and $\tau$, see (2.1). Let $f'$ and $f$ be functions satisfying (2.2) for $\tau'$ and $\tau$, respectively, so that $\hat{\Delta}' = \hat{\Delta}_{f'}$ and $\hat{\Delta} = \hat{\Delta}_f$. The assumptions of the proposition mean that there exists a 2-cochain $g$ on $P/Q$ such that $f'f^{-1}g$ is a coboundary on $P$. In other words, there exists a unitary $u \in \mathcal{U}(T^2) \subset \mathcal{U}(G^2_q)$ such that

$$f'g = (u \otimes u)f\hat{\Delta}_q(u)^*.$$
Then $\text{Ad}_u$ is an isomorphism of $(\mathcal{U}(G_q), \hat{\Delta})$ onto $(\mathcal{U}(G_q), \hat{\Delta}')$, hence $G_q^r \cong G_q^{r'}$. \hfill $\square$

Apart from the isomorphisms given by this proposition, we have $G_q^r \cong G_q^{r-1}$. There also are isomorphisms induced by symmetries of the based root datum of $G$. Finally, for $q = 1$ there can be additional isomorphisms defined by conjugation by elements in $\mathcal{U}(G)$ that lie in the normalizer of the maximal torus.

3. Function algebras of twisted quantum groups.

3.1. Crossed product description.

As before, assume $\tau = (\tau_1, \ldots, \tau_r) \in Z(G)^r$. Recall that we denote by $T_\tau$ the subgroup of $Z(G)$ generated by the elements $\tau_1, \ldots, \tau_r$. There is a homomorphism

$$\psi: \hat{T}_\tau \to T/Z(G)$$

defined as follows. Given $\chi \in \hat{T}_\tau$, we define a character on the root lattice $Q$ by $\alpha_i \mapsto \chi(\tau_i)$. It can be extended to $P$, and we obtain an element $\tilde{\psi}(\chi) \in \hat{P} = T$. The ambiguity of this extension is in $Q^\perp \cap T = Z(G)$. Thus, the image $\psi(\chi)$ of $\tilde{\psi}(\chi)$ in $T/Z(G)$ is well-defined.

The homomorphism $\psi$ allows us to define an action of $\hat{T}_\tau$ by conjugation on $G_q$, that is, we have an action $\text{Ad}_\psi$ of $\hat{T}_\tau$ on $C(G_q)$ defined by

$$(\text{Ad}_\psi(\chi))(a) = \langle \tilde{\psi}(\chi^{-1}), a_{[1]} \rangle \langle \tilde{\psi}(\chi), a_{[2]} \rangle;$$

recall that the elements of $T$ define characters of $C(G_q)$, that is, they are group-like unitary elements in $\mathcal{U}(G_q)$.

**Theorem 3.1.** There is a canonical isomorphism

$$C(G_q^r) \cong (C(G_q) \rtimes_{\text{Ad}_\psi} \hat{T}_\tau)^{T_\tau},$$

where the group $T_\tau$ acts on $C(G_q) \rtimes_{\text{Ad}_\psi} \hat{T}_\tau$ by right translations $\rho$ on $C(G_q)$ and by the dual action on $C^*(\hat{T}_\tau)$.

**Proof.** Let us first identify the compact quantum group $\tilde{G}_q^r$ defined by the category of finite dimensional representations of $\mathcal{U}_q^r(\mathfrak{g})$ such that their restrictions to $\mathcal{U}_q(\mathfrak{g})$ are admissible. Any such irreducible representation is tensor product of an irreducible admissible representation of $\mathcal{U}_q(\mathfrak{g})$ and a character of $T_\tau^r$; recall that these can be regarded as representations of $\mathcal{U}_q^r(\mathfrak{g})$. It follows that the Hopf $*$-algebra $\mathbb{C}[\tilde{G}_q^r]$ contains copies of $\mathbb{C}[G_q]$ and $C^*(\hat{T}_\tau)$, and as a space it is tensor product of these Hopf $*$-subalgebras. It remains to find relations between elements of $\mathbb{C}[G_q]$ and $C^*(\hat{T}_\tau)$ inside $\mathbb{C}[\tilde{G}_q^r]$.

Let $(\pi, V)$ be a finite dimensional admissible representation of $\mathcal{U}_q(\mathfrak{g})$, and $\chi$ be a character of $T_\tau$. Then, on the one hand, $\pi \otimes \chi$ is a representation on $V$ with $E_i$ acting by $\chi(\tau_i)\pi(E_i)$. On the other hand, $\chi \otimes \pi$ is also a representation on the same space $V$ with $E_i$ acting by $\pi(E_i)$. From this we see that the operator $\pi(\tilde{\psi}(\chi))$, where we consider
the standard extension of $\pi$ to $\mathcal{U}(G_q)$, intertwines $\chi \otimes \pi$ with $\pi \otimes \chi$. In other words, if $U_\pi \in B(V) \otimes \mathbb{C}[G_q]$ is the representation of $G_q$ defined by $\pi$, then in $B(V) \otimes \mathbb{C}[\tilde{G}_q^\tau]$ we have

$$(\pi(\tilde{\psi}(\chi)) \otimes u_\chi) U_\pi = U_\pi (\pi(\tilde{\psi}(\chi)) \otimes u_\chi).$$

Since

$$(\pi(\tilde{\psi}(\chi)^{-1}) \otimes 1) U_\pi (\pi(\tilde{\psi}(\chi)) \otimes 1) = (\iota \otimes \text{Ad}_\psi(\chi))(U_\pi),$$

this exactly means that if $a \in \mathbb{C}[G_q]$ is a matrix coefficient of $\pi$, then $u_\chi a = (\text{Ad}_\psi(\chi))(a) u_\chi$. Therefore $\mathbb{C}[\tilde{G}_q^\tau] = \mathbb{C}[G_q] \rtimes_\text{Ad}_\psi \tilde{T}_\tau$.

Now, the quantum group $G_q^\tau$ is the quotient of $\tilde{G}_q^\tau$ defined by the category of $\tau$-admissible representations. By definition, a representation $\pi \otimes \chi$ of $U_q^\tau(g)$ is $\tau$-admissible if $\pi(\tau_i) = \chi(\tau_i)$. Therefore $\mathbb{C}[G_q^\tau] \subset \mathbb{C}[\tilde{G}_q^\tau] = \mathbb{C}[G_q] \rtimes_\text{Ad}_\psi \tilde{T}_\tau$ is spanned by elements of the form $au_\chi$, where $a$ is a matrix coefficient of an admissible representation $\pi$ such that $\pi(\tau_i) = \chi(\tau_i)$. If $\pi$ is irreducible, then $\pi(\tau_i)$ is scalar, and we have $\rho(\tau_i)(a) = \pi(\tau_i) a$. Hence $\mathbb{C}[G_q^\tau] = \langle \mathbb{C}[G_q] \rtimes_\text{Ad}_\psi \tilde{T}_\tau \rangle^\tau$. □

**Corollary 3.2.** The $C^*$-algebra $C(G_q^\tau)$ is of type I.

**Proof.** Since $C(G_q^\tau) \subset C(G_q) \rtimes_\text{Ad}_\psi \tilde{T}_\tau$, this follows from the known fact that the $C^*$-algebra $C(G_q)$ is of type I. □

Recall that the family $(C(G_q))_{0 < q < \infty}$ has canonical structure of a continuous field of $C^*$-algebras [NT11].

**Corollary 3.3.** The $C^*$-algebras $(C(G_q^\tau))_{0 < q < \infty}$ form a continuous field of $C^*$-algebras.

### 3.2. Primitive spectrum.

Let us turn to a description of the primitive spectrum of $C(G_q^\tau)$. We will concentrate on the case $q \neq 1$, the case $q = 1$ can be treated similarly. First of all observe that the action of $T_\tau$ on $C(G_q) \rtimes_\text{Ad}_\psi \tilde{T}_\tau$ is saturated, since every spectral subspace contains a unitary. We thus obtain a strong Morita equivalence

$$C(G_q^\tau) \sim_M C(G_q) \rtimes_\text{Ad}_\psi \tilde{T}_\tau \rtimes_{\rho, \text{Ad}_\psi} T_\tau \cong C(G_q) \rtimes_{\rho} T_\tau \rtimes_{\text{Ad}_\psi, \rho} \tilde{T}_\tau.$$ (3.1)

Recall how to describe primitive spectra of crossed products, see e.g. [Wil07]. Let $\Gamma$ be a finite group acting on a separable $C^*$-algebra $A$. Then any primitive ideal $J$ of $A \rtimes \Gamma$ is determined by the $\Gamma$-orbit of an ideal $I \in \text{Prim}(A)$ and an ideal $J_0 \in \text{Prim}(A \rtimes \text{Stab}_\Gamma(I))$ by the condition $J_0 \cap A = I$ and $J = \text{Ind} J_0$.

If $A$ is of type I, the ideals $J_0$ can, in turn, be described as follows. Put $\Gamma_0 = \text{Stab}_\Gamma(I)$. We want to describe irreducible representations of $A \rtimes \Gamma_0$ whose restrictions to $A$ have kernel $I$. Let $H$ be the space of an irreducible representation $\pi$ of $A$ with kernel $I$. Then the action of $\Gamma_0$ on $A/I$ is implemented by a projective unitary representation
\[ \gamma \mapsto u_{\gamma} \] of \( \Gamma_0 \) on \( H \). Let \( \omega \) be the corresponding 2-cocycle. Consider the regular \( \bar{\omega} \)-representation \( \gamma \mapsto \lambda_{\gamma}^{\bar{\omega}} \) of \( \Gamma_0 \) on \( \ell^2(\Gamma_0) \). Then \( A \rtimes \Gamma_0 \) has a representation on \( H \otimes \ell^2(\Gamma_0) \) defined by \( a \mapsto \pi(a) \otimes 1 \), \( \gamma \mapsto u_{\gamma} \otimes \lambda_{\gamma}^{\bar{\omega}} \). Any irreducible representation of \( A \rtimes \Gamma_0 \) whose restriction to \( A \) corresponds with irreducible representations of \( B \) defined by \( \gamma \) on \( A \). \( \gamma \mapsto \lambda_{\gamma}^{\bar{\omega}} \) of \( \Gamma_0 \) on \( \ell^2(\Gamma_0) \). Then \( A \times \Gamma_0 \) has a representation on \( H \otimes \ell^2(\Gamma_0) \) into irreducible subrepresentations. The von Neumann algebra generated by the image of \( A \times \Gamma_0 \) is \( B(H) \otimes C^*(\Gamma_0; \bar{\gamma}) \). Therefore the representations we are interested in are in a one-to-one correspondence with irreducible representations of \( C^*(\Gamma_0; \bar{\gamma}) \).

To summarize, if \( A \) is a separable \( C^* \)-algebra of type I and \( \Gamma \) is a finite group acting on \( A \), then the primitive spectrum \( \text{Prim}(A \rtimes \Gamma) \) can be identified with the set of pairs \( ([I], J) \), where \( [I] \) is the \( \Gamma \)-orbit of an ideal \( I \in \text{Prim}(A) \), \( J \in \text{Prim}(C^*(\Gamma_I; \bar{\omega}_I)) \), and \( \omega_I \) is the 2-cocycle on \( \Gamma_I = \text{Stab}_\Gamma(I) \) defined by a projective representation of \( \Gamma_I \) implementing the action of \( \Gamma_I \) on the image of \( A \) under an irreducible representation with kernel \( I \).

Returning to \( C(G_q^\tau) \), for an element \( w \in W \) of the Weyl group and a character \( \chi \in \hat{T}_\tau \), put \( \theta_w(\chi) = w^{-1}(\bar{\psi}(\chi))\bar{\psi}(\chi)^{-1} \). This defines a homomorphism from \( \hat{T}_\tau \) to \( T \).

**Proposition 3.4.** For \( q > 0 \), \( q \neq 1 \), the primitive spectrum of \( C(G_q^\tau) \) can be identified with

\[
\prod_{w \in W} (\theta_w(\hat{T}_\tau) \backslash T/T_\tau) \times \theta_{\bar{w}^{-1}}(T_\tau).
\]

**Proof.** In view of the strong Morita equivalence (3.1) it suffices to describe the primitive spectrum of

\[ C(G_q) \rtimes_{\rho} T_\tau \rtimes_{\text{Ad}\psi, \hat{\rho}} \hat{T}_\tau \]

Recall that the spectrum of \( C(G_q) \) is \( W \times T \). The right translation action of \( T_\tau \) on \( C(G_q) \) defines an action on \( W \times T \) that is simply the action by translations on \( T \). Therefore \( \text{Prim}(C(G_q) \rtimes_{\rho} T_\tau) \) can be identified with \( W \times T/T_\tau \), and every irreducible representation of \( C(G_q) \rtimes_{\rho} T_\tau \) is induced from an irreducible representation of \( C(G_q) \).

Next, we have to understand the action of \( \hat{T}_\tau \) on \( \text{Prim}(C(G_q) \rtimes_{\rho} T_\tau) \). Since the dual action preserves the equivalence class of any induced representation, we just have to look at the action \( \text{Ad}\psi \). Given a representation \( \pi_w \otimes \pi_t \) of \( C(G_q) \) corresponding to \( (w, t) \in W \times T \), we have

\[ (\pi_w \otimes \pi_t)(\text{Ad}\psi(\chi^{-1})) \sim \pi_w \otimes \pi_{\theta_w(\chi)t} \]

by [NT12, Lemma 3.4] and [Yam13, Lemma 8]. It follows that the action of \( \hat{T}_\tau \) on \( \text{Prim}(C(G_q) \rtimes_{\rho} T_\tau) = W \times T/T_\tau \) is by translations on \( T/T_\tau \) via the homomorphisms \( \theta_w: \hat{T}_\tau \to T \). Hence the space of \( T_\tau \)-orbits is \( \prod_{w \in W} \theta_w(\hat{T}_\tau) \backslash T/T_\tau \), and the stabilizer of a point \( (w, tT_\tau) \) is \( \theta_{\bar{w}^{-1}}(T_\tau) \subset \hat{T}_\tau \).

To finish the proof of the proposition it remains to show that the action \( (\text{Ad}\psi, \hat{\rho}) \) of \( \theta_{\bar{w}^{-1}}(T_\tau) \) on \( C(G_q) \rtimes_{\rho} T_\tau \) can be implemented in the space of the induced representation.
Ind(\(\pi_w \otimes \pi_t\)) by a unitary representation of \(\theta_w^{-1}(T_\tau)\). For this, in turn, it suffices to show that the equivalences

\[(\pi_w \otimes \pi_{t'}) (\text{Ad} t^{-1}) \sim \pi_w \otimes \pi_{w^{-1}(t) t^{-1} t'}\]

from [NT12, Lemma 3.4] and [Yam13, Lemma 8] can be implemented by a unitary representation \(t \mapsto v_t\) of \(T/\mathbb{Z}(G)\) on the space of representation \(\pi_w\). But this is easy to see. Specifically, using the notation of [NT12] and [Yam13], if \(w = s_i\) is the reflection corresponding to a simple root \(\alpha_i\), then the required representation \(t \mapsto v_t\) on \(\ell^2(\mathbb{Z}_+)\) can be defined by \(v_t e_n = \langle t, \alpha_i \rangle^n e_n\). For arbitrary \(w\) we just have to take tensor products of such representations. \(\square\)

**Remark 3.5.** A description of the topology on \(\text{Prim}(C(G_q))\) is given in [NT12]. The above argument is, however, not quite enough to understand the topology on \(\text{Prim}(C(G_q^*)\).

### 3.3. K-theory.

The maximal torus \(T\) is embedded in \(U(G_q^*)\), so it can be considered as a subgroup of \(G_q^*\). Let us consider the right translation action \(\rho\) of \(T\) on \(C(G_q^*)\). The crossed product \(C(G_q^*) \rtimes_\rho T\) is a \(T\)-C*-algebra with respect to the dual action.

**Proposition 3.6.** The dual action of \(\hat{T}\) on \(C(G_q^*) \rtimes_\rho T\) is equivariantly strongly Morita equivalent to an action on \(C(G_q) \rtimes_\rho T\) that is homotopic to the dual action.

**Proof.** If we identify \(C(G_q^*)\) with \((C(G_q) \rtimes_{\text{Ad} \psi} \hat{T}_\tau)^T\), then the action of \(T\) by right translations on \(C(G_q^*)\) extends to an action on \(C(G_q) \rtimes_{\text{Ad} \psi} \hat{T}_\tau\) that is trivial on \(C^*(T_\tau)\) and coincides with the action by right translations on \(C(G_q)\). This action of \(T\) on \(C(G_q) \rtimes_{\text{Ad} \psi} \hat{T}_\tau\) commutes with the action of \(T_\tau\). Hence the strong Morita equivalence (3.1) is \(T\)-equivariant, and taking crossed products we get a \(\hat{T}\)-equivariant strong Morita equivalence

\[C(G_q^*) \rtimes_\rho T \sim_M C(G_q) \rtimes_{\text{Ad} \psi} \hat{T}_\tau \rtimes_{\rho, \text{Ad} \psi} T \rtimes_\rho T.\] (3.2)

Denote the C*-algebra on the right hand side by \(A\). We claim that \(A\) is isomorphic to

\[B = C(G_q) \rtimes_{\text{Ad} \psi} \hat{T}_\tau \rtimes_{\text{Ad} \psi} T \rtimes_\rho T.\]

Indeed, the map \(u_{t}(\chi)u_{t'}(\chi) \mapsto au_{t}(\chi)u_{t'}(\chi)\) for \(a \in C(G_q)\), \(\chi \in \hat{T}_\tau\), \(t \in T_\tau\) and \(t' \in T\) is the required isomorphism. The dual action of \(\hat{T}\) on \(A\) corresponds to an action \(\beta\) on \(B\) which is given by the dual action on the copy of \(C^*(T_\tau)\) and by the dual action on the copy of \(C^*(T_\tau)\) via the canonical homomorphism \(r: \hat{T} \to \hat{T}_\tau\).

The map \(\hat{T} \ni \chi \mapsto u_{r(\chi)}(\chi) \in C^*(\hat{T}_\tau) \subset M(B)\) is a 1-cocycle for the action \(\beta\). Therefore \(\beta\) is strongly Morita equivalent to the action \(\gamma\) defined by \(\gamma_{\chi} = (\text{Ad} u_{r(\chi)}(\chi))\beta_{\chi}\). This action is already trivial on \(C^*(T_\tau)\), while on \(C(G_q)\) it is given by \(\text{Ad} \psi(r(\chi))\), and on \(C^*(T)\) it coincides with the dual action.
Denote by $\delta$ the restriction of $\gamma$ to $C(G_q) \rtimes_T \mathbb{T} \subset M(B)$. Then, similarly to (3.2), the actions $\delta$ and $\gamma$ are strongly Morita equivalent.

Combining the Morita equivalences that we have obtained, we conclude that the dual action of $\hat{T}$ on $C(G_q^\tau) \rtimes_T \mathbb{T}$ is strongly Morita equivalent to the action $\hat{\delta} = (\text{Ad}\hat{\psi}(r(\cdot)), \hat{\rho})$ on $C(G_q) \rtimes_T \mathbb{T}$. Choosing a basis in $\hat{T} = P$ and paths from $\hat{\psi}(r(\chi))$ to the neutral element in $\hat{T}$ for every basis element $\chi$, we see that $\hat{\delta}$ is homotopic to the dual action on $C(G_q) \rtimes_T \mathbb{T}$.

**Theorem 3.7.** The C*-algebra $C(G_q^\tau)$ is $KK$-isomorphic to $C(G_q)$, hence to $C(G)$.

**Proof.** Since the torsion-free commutative group $\hat{T}$ satisfies the strong Baum–Connes conjecture, the functor $A \mapsto A \rtimes \hat{T}$ maps homotopic actions into $KK$-isomorphisms of the corresponding crossed products. By the previous proposition, this, together with the Takesaki–Takai duality, implies that $C(G_q^\tau)$ and $C(G_q)$ are $KK$-isomorphic. By [NT12] we also know that $C(G_q)$ is $KK$-isomorphic to $C(G)$. $\square$

**Remark 3.8.**

(i) The above proof shows that the continuous field of Corollary 3.3 is a $KK$-fibration in the sense of [ENO00]. The argument of [NT11] applies to the Dirac operator $D$ given by Remark 2.5, and we obtain that the $K$-homology class of $D$ is independent of $q$. The bi-equivariance of $D$ and the construction in the proof of Proposition 3.6 imply that the $K$-homology class of $D$ is also independent of $\tau$ up to the isomorphism of Theorem 3.7.

(ii) For the group $\hat{T}$ the strong Baum–Connes conjecture is a consequence of the Pimsner–Voiculescu sequence in $KK$-theory. Therefore the proof of Theorem 3.7 can be written such that it relies only on this sequence, see e.g. [San11, Section 5.1] for a related argument.

4. **Twisted $SU_q(n)$.

4.1. **Special unitary group.**

Let us review the structure of $SU(n)$, see e.g. [FH91, Chapter 15]. For the sake of presentation, it is convenient to consider also the unitary group $U(n)$. We take the subgroup of the diagonal matrices $T$ as a maximal torus of $U(n)$, and take $T = \bar{T} \cap SU(n)$ as a maximal torus of $SU(n)$. We will often identify $\bar{T}$ with $\mathbb{T}^n$. We write the corresponding Cartan subalgebras as $\mathfrak{h} \subset \mathfrak{gl}_n$ and $\mathfrak{h} \subset \mathfrak{sl}_n$.

Let $\{e_{ij}\}_{i,j=1}^n$ be the matrix units in $M_n(\mathbb{C}) = \mathfrak{gl}_n$, and $\{\tilde{L}_i\}_{i=1}^n$ be the basis in $\mathfrak{h}^*$ dual to the basis $\{e_{ii}\}_{i=1}^n$ in $\mathfrak{h}$. Denote by $L_i$ the image of $\tilde{L}_i$ in $\mathfrak{h}^*$. Therefore any $n - 1$ elements among $L_1, \ldots, L_n$ form a basis in $\mathfrak{h}^*$, and we have $\sum_i L_i = 0$.

The weight lattice $P \subset \mathfrak{h}^*$ is generated by the elements $L_i$. The pairing between $T$ and $P$ is given by $\langle t, L_i \rangle = t_i$ for $t \in T \subset \mathbb{T}^n$. As simple roots we take

$$\alpha_i = L_i - L_{i+1}, \quad 1 \leq i \leq n - 1.$$ 

The fundamental weights are then given by
Consider the homomorphism $| \cdot |: \mathbb{P} \to \mathbb{Z}$ such that $L_1 \mapsto n - 1$ and $L_i \mapsto -1$ for $1 < i \leq n$. In other words,

$$|a_1 \varpi_1 + \cdots + a_{n-1} \varpi_{n-1}| = \lambda_1 + \cdots + \lambda_{n-1},$$

where $\lambda_{n-i}$ is given by $a_1 + \cdots + a_i$. The image of $Q$ under $| \cdot |$ is $n\mathbb{Z}$, and therefore we can use this homomorphism to identify $\mathbb{P}/n\mathbb{Z}$ with $\mathbb{Z}/n\mathbb{Z}$.

### 4.2. Twisted quantum special unitary groups.

By Proposition A.3, the cohomology group $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T})$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and a cocycle generating this group can be defined by

$$\phi(a, b, c) = \zeta_n^{\omega_n(a, b)c}, \text{ where } \zeta_n = e^{2\pi i/n} \text{ and } \omega_n(a, b) = \left[\frac{a+b}{n}\right] - \left[\frac{a}{n}\right] - \left[\frac{b}{n}\right].$$

Using this generator we identify $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T})$ with the group $\mu_n \subset \mathbb{T}$ of units of order $n$. Therefore, given $\zeta \in \mu_n$, we have a category $\text{Rep}(SU_q(n))^\zeta$ with associativity morphisms defined by multiplication by $\zeta^{a_n([\lambda],[\eta])}\mu^n$ on the tensor product $V_\lambda \otimes V_\eta \otimes V_\nu$ of irreducible $U_q(\mathfrak{g})$-modules with highest weights $\lambda, \eta, \nu$. This agrees with the conventions of Kazhdan and Wenzl [KW93].

It is also convenient to identify $Z(SU(n))$ with the group $\mu_n$. Thus, for $\tau = (\tau_1, \ldots, \tau_{n-1}) \in \mu_n^{n-1}$, we can define a twisting $SU_q^\tau(n)$ of $SU_q(n)$. Its representation category is one of $\text{Rep}(SU_q(n))^\zeta$, and to find $\zeta$ we have to compute the homomorphism $\Theta: Z(SU(n))^{n-1} \to H^3(\mathbb{P}/n\mathbb{Z}; \mathbb{T})$ introduced in Section 2.3. Under our identifications this becomes a homomorphism $\mu_n^{n-1} \to \mu_n$.

**Proposition 4.1.** We have $\Theta(\tau) = \prod_{i=1}^{n-1} \tau_i^{-i}$.

**Proof.** Recall the construction of $\Theta$. We choose a function $f: \mathbb{P} \times \mathbb{P} \to \mathbb{T}$ such that it factors through $\mathbb{P} \times (\mathbb{P}/Q)$ and $f(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle f(\lambda, \mu)$. Then $\Theta(\tau)$ is the cohomology class of $\partial f$ in $H^3(\mathbb{P}/Q; \mathbb{T})$.

Note that $\langle \tau_i, \mu \rangle = \tau_i^{-|\mu|}$, which is immediate for $\mu = L_j$, and define a character $\chi$ of $Q \otimes (\mathbb{P}/Q) = Q \otimes (\mathbb{Z}/n\mathbb{Z})$ by

$$\chi(\alpha_i \otimes k) = \tau_i^k \text{ for } 1 \leq i \leq n - 1 \text{ and } k \in \mathbb{Z}/n\mathbb{Z},$$

so that $f(\lambda + \alpha, \mu) = \chi(\alpha \otimes |\mu|) f(\lambda, \mu)$ for all $\alpha \in Q$. By Proposition A.6, the cohomology class of $\partial f$ depends only on the restriction of $\chi$ to

$$\text{ker}(Q \otimes (\mathbb{Z}/n\mathbb{Z}) \to P \otimes (\mathbb{Z}/n\mathbb{Z})) \cong \text{Tor}_1^\mu(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z},$$

and by varying $\tau$ we get this way an isomorphism $\text{Hom}(\text{Tor}_1^\mu(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}), \mathbb{T}) \cong H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T})$. In order to compute this isomorphism we can use the resolution $n\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ instead of $Q \to P \to \mathbb{Z}/n\mathbb{Z}$. Define a morphism between these resolutions by...
$Z \to P$, $1 \mapsto \infty_{n-1} = -L_n$. By pulling back $\chi$ under this morphism, we get a character $\tilde{\chi}$ of $(n\mathbb{Z}) \otimes (\mathbb{Z}/n\mathbb{Z})$ such that

$$\tilde{\chi}(n \otimes k) = \chi(n \infty_{n-1} \otimes k).$$

We have $n \infty_{n-1} = \sum_{i=1}^{n-1} i \alpha_i$. Therefore

$$\tilde{\chi}(n \otimes k) = \zeta^k, \quad \text{where} \quad \zeta = \prod_{i=1}^{n-1} \tau_i^i.$$

Then the function $\tilde{f}: \mathbb{Z} \times \mathbb{Z} \to \mathbb{T}$ defined by

$$\tilde{f}(a, b) = \zeta^{|a/n|b},$$

factors through $\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$, $\tilde{f}(a + n, b) = \tilde{\chi}(n \otimes b)\tilde{f}(a, b)$ and $(\partial \tilde{f})(a, b, c) = \zeta^{-\omega_n(a, b)c}$. Therefore the class of $\partial \tilde{f}$ in $H^3(\mathbb{Z}/n\mathbb{Z}; \mathbb{T}) = \mu_n$ is $\zeta^{-1}$. \hfill \Box

In Section 2.3 we also introduced a homomorphism $\Upsilon$. In the present case we have $H^2(P/Q; \mathbb{T}) = 0$, so $\Upsilon$ is a homomorphism $\ker \Theta \to H^2(P; \mathbb{T})$.

**Lemma 4.2.** The homomorphism $\Upsilon: \ker \Theta \to H^2(P; \mathbb{T})$ is injective.

**Proof.** Assume $\tau \in \ker \Theta$, so $\prod_{i=1}^{n-1} \tau_i^i = 1$. In this case the character $\chi$ of $Q \otimes (P/Q)$ from the proof of the previous proposition extends to $P \otimes (P/Q)$ by

$$\chi(L_i \otimes \mu) = (\tau_1 \cdots \tau_{i-1})^{-|\mu|} \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad \mu \in P.$$

Therefore if we consider $\chi$ as a function on $P \times P$, we can take it as a function $f$ in that proof. Then $f$ is a 2-cocycle, and by definition, the image of $\tau$ under $\Upsilon$ is the cohomology class of $\tilde{f}$. It is well-known, and also follows from Proposition A.1, that $f$ is a coboundary if and only if $f$ is symmetric. For $1 < i < j \leq n$ we have

$$f(L_i, L_j)f(L_j, L_i) = (\tau_i \cdots \tau_{j-1})^{-1}.$$

So if $f$ is symmetric, then $\tau_2 = \cdots = \tau_{n-1} = 1$, but then also $\tau_1 = 1$. \hfill \Box

Therefore Proposition 2.7 does not give us any nontrivial isomorphisms between the quantum groups $SU_q^r(n)$. On the other hand, the flip map on the Dynkin diagram induces an automorphism of $\mathcal{U}(SU_q(n))$ such that $K_i \mapsto K_{n-i}$ and $E_i \mapsto E_{n-i}$ for $1 \leq i \leq n-1$. On $Z(SU_q(n)) \subset \mathcal{U}(SU_q(n))$ this automorphism is $t \mapsto t^{-1}$. It follows that it induces isomorphisms

$$SU_q^{(\tau_1, \cdots, \tau_{n-1})}(n) \cong SU_q^{(\tau_{n-1}, \cdots, \tau_1)}(n).$$

For $0 < q < 1$, these seem to be the only obvious isomorphisms between the quantum groups $SU_q^r(n)$. 

4.3. Generators and relations.

The $C^*$-algebra $C(SU_q(n))$ is generated by the matrix coefficients $(u_{ij})_{1 \leq i,j \leq n}$ of the natural representation of $SU_q(n)$ on $\mathbb{C}^n$, the fundamental representation with highest weight $\varpi_1$. They satisfy the relations [Dri87] and [Wor88]

\begin{align}
  u_{ij}u_{il} &= qu_{il}u_{ij} \quad (j < l), \\
  u_{ij}u_{kj} &= qu_{kj}u_{ij} \quad (i < k), \tag{4.1} \\
  u_{ij}u_{kl} &= u_{kl}u_{ij} \quad (i > k, j < l), \quad (i < k, j < l), \tag{4.2} \\
  q\det((u_{ij})_{i,j}) &= \sum_{\sigma \in S_n} (-q)^{|\sigma|} u_{1\sigma(1)} \cdots u_{n\sigma(n)} = 1. \tag{4.3}
\end{align}

Here, $|\sigma|$ is the inversion number of the permutation $\sigma$. The involution is defined by

$$u_{ij}^* = (-q)^{j-i} q\det(U_{ij}^\natural),$$

where $U_{ij}^\natural$ is the matrix obtained from $U = (u_{kl})_{k,l}$ by deleting the $i$-th row and $j$-th column.

In order to find generators and relations of $\mathbb{C}[SU_q^+(n)]$, we will use the embedding of the algebra $\mathbb{C}[SU_q^+(n)]$ into $\mathbb{C}[SU_q(n)] \rtimes_{\text{Ad}\psi} \hat{T}_r$ described in Theorem 3.1. Recall that $\psi: \hat{T}_r \to T/Z(SU(n)) = T/\mu_n$ is the homomorphism such that $\langle \tilde{\psi}(\chi), \alpha_i \rangle = \chi(\tau_i)$, where $\tilde{\psi}(\chi)$ is a lift of $\psi(\chi)$ to $T$. Hence

$$\tilde{\psi}(\chi) = (z, z\chi(\tau_1)^{-1}, \ldots, z\chi(\tau_1 \cdots \tau_{n-1})^{-1}) \in T \subset \mathbb{T}^n,$$

where $z \in \mathbb{T}$ is a number such that $z^n = \prod_{i=1}^{n-1} \chi(\tau_i)^{-i}$. It follows that

$$(\text{Ad}_\psi(\chi))(u_{ij}) = \left( \prod_{1 \leq p < i} \chi(\tau_p) \right) \left( \prod_{1 \leq p < j} \chi(\tau_p)^{-1} \right) u_{ij}. \tag{4.4}$$

Now, the algebra $\mathbb{C}[SU_q^+(n)]$ is generated by matrix coefficients of the fundamental representation of $SU_q^+(n)$ with highest weight $\varpi_1$. Under the embedding $\mathbb{C}[SU_q^+(n)] \hookrightarrow \mathbb{C}[SU_q(n)] \rtimes_{\text{Ad}\psi} \hat{T}_r$, these matrix coefficients correspond to $v_{ij} = u_{ij}u_{\text{nat}}$, where $\chi_{\text{nat}} \in \hat{T}_r$ is the character determined by the natural representation of $SU_q(n)$ on $\mathbb{C}^n$, so $\chi_{\text{nat}}(\tau_i) = \tau_i$. From (4.1)–(4.3) we then get the following relations:

\begin{align}
  v_{ij}v_{il} &= \left( \prod_{j \leq p < l} \tau_p^{-1} \right) q v_{il} v_{ij} \quad (j < l), \quad v_{ij}v_{kj} = \left( \prod_{i \leq p < k} \tau_p \right) q v_{kj} v_{ij} \quad (i < k), \tag{4.5} \\
  v_{ij}v_{kl} &= \left( \prod_{k \leq p < i} \tau_p^{-1} \right) \left( \prod_{j \leq p < l} \tau_p^{-1} \right) v_{kl} v_{ij} \quad (i > k, j < l), \tag{4.6} \\
  \left( \prod_{j \leq p < l} \tau_p \right) v_{ij}v_{kl} - \left( \prod_{i \leq p < k} \tau_p \right) v_{kl}v_{ij} &= (q - q^{-1}) v_{il} v_{kj} \quad (i < k, j < l). \tag{4.7}
\end{align}
\[ \sum_{\sigma \in S_n} r^{m(\sigma)}(-q)^{||\sigma||} v_{1\sigma(1)} \cdots v_{n\sigma(n)} = 1, \quad (4.8) \]

where \( m(\sigma) = (m(\sigma)_1, \ldots, m(\sigma)_{n-1}) \) is the multi-index given by \( m(\sigma)_i = \sum_{k=2}^n (k-1)m_i(k,\sigma(k)) \) and

\[
m_i^{(k,j)} = \begin{cases} 
1, & \text{if } k \leq i < j, \\
-1, & \text{if } j \leq i < k, \\
0, & \text{otherwise}.
\end{cases}
\]

**Proposition 4.3.** For any \( \tau \in \mu_n^{-1} \), the algebra \( \mathbb{C}[SU_q^\tau(n)] \) is a universal algebra generated by elements \( v_{ij} \) satisfying relations (4.5)–(4.8).

**Proof.** We already know that relations (4.5)–(4.8) are satisfied in \( \mathbb{C}[SU_q^\tau(n)] \), so we just have to show that there are no other relations. Let \( \mathcal{A} \) be a universal algebra generated by elements \( w_{ij} \) satisfying relations (4.5)–(4.8). We can define an action of \( \hat{T}_\tau \) on \( \mathcal{A} \) by (4.4). Then in \( \mathcal{A} \times \hat{T}_\tau \) the elements \( w_{ij}u_{\chi_{\text{nat}}}^{-1} \) satisfy the defining relations of \( \mathbb{C}[SU_q(n)] \), so we have a homomorphism \( \mathbb{C}[SU_q^\tau(n)] \to \mathcal{A} \times \hat{T}_\tau \) mapping \( u_{ij} \) into \( w_{ij}u_{\chi_{\text{nat}}}^{-1} \).

It extends to a homomorphism \( \mathbb{C}[SU_q^\tau(n)] \to \mathcal{A} \times \hat{T}_\tau \) that is identity on the group algebra of \( \hat{T}_\tau \). Restricting to \( \mathbb{C}[SU_q^\tau(n)] \subset \mathbb{C}[SU_q(n)] \times \hat{T}_\tau \), we get a homomorphism \( \mathbb{C}[SU_q^\tau(n)] \to \mathcal{A} \) mapping \( v_{ij} \) into \( w_{ij} \).

The involution on \( \mathbb{C}[SU_q^\tau(n)] \) is determined by requiring the invertible matrix \( (v_{ij})_{i,j} \) to be unitary. An explicit formula can be easily found using that for \( \mathbb{C}[SU_q(n)] \).

**Remark 4.4.** The relations in \( \mathbb{C}[SU_q^\tau(n)] \) cannot be obtained using the FRT-approach, since the categories \( \text{Rep}(SU_q^\tau(n)) \) are typically not braided. More precisely, \( \text{Rep}(SU_q^\tau(n))^\xi \) has a braiding if and only if either \( \zeta = 1 \) or \( n \) is even and \( \zeta = -1 \). This statement is already implicit in [KW93], and it can be proved as follows. If \( \zeta = 1 \) or \( n \) is even and \( \zeta = -1 \), then a braiding indeed exists, see e.g. [Pin07]. Conversely, suppose we have a braiding. In other words, there exists an \( R \)-matrix \( \mathcal{R} \) for \( \mathcal{U}(SU_q(n)), \Delta_q, \Phi \), where \( \Phi = \zeta^{\omega_n(\cdot,|\cdot|)}|\cdot| \). Recall that this means that \( \mathcal{R} \) is an invertible element in \( \mathcal{U}(SU_q(n) \times SU_q(n)) \) such that \( \Delta_q^{\text{op}} = \mathcal{R} \Delta_q \mathcal{R}^{-1} \) and

\[
(\Delta_q \otimes \iota)(\mathcal{R}) = \Phi_{312} \mathcal{R}_{13} \Phi^{-1}_{132} \mathcal{R}_{23} \Phi, \quad (\iota \otimes \Delta_q)(\mathcal{R}) = \Phi^{-1}_{231} \mathcal{R}_{13} \Phi \mathcal{R}_{213} \mathcal{R}_{12} \Phi^{-1}.
\]

Since \( \Phi \) is central and symmetric in the first two variables, the last two identities can be written as

\[
(\Delta_q^{\text{op}} \otimes \iota)(\mathcal{R}) = \mathcal{R}_{23} \mathcal{R}_{13} \Phi, \quad (\iota \otimes \Delta_q)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12} \Phi^{-1}.
\]

On the other hand, we know that \( \text{Rep}(SU_q(n)) \) is braided, so there exists an element \( \mathcal{R}_q \) satisfying the above properties with \( \Phi \) replaced by 1. Consider the element \( F = \mathcal{R}_q^{-1} \mathcal{R} \).

Then \( F \) is invariant, meaning that it commutes with the image of \( \Delta_q \). Furthermore, we have
This implies that $\text{Rep}(SU_q(n))$ is monoidally equivalent to $\text{Rep}(SU_q(n))^{\Phi_{321}}$. Since the cocycle $\Phi_{321}$ on the dual of the center is cohomologous to the cocycle $\zeta^{2\omega_n([\lambda],[\eta])^{|\nu|}}$, this means that $\text{Rep}(SU_q(n))$ is monoidally equivalent to $\text{Rep}(SU_q(n))^2$. By the Kazhdan–Wenzl classification this is the case only if $\zeta^2 = 1$.

**Appendix A. Cocycles on abelian groups.**

Let $\Gamma$ be a discrete abelian group. As is common in operator algebra, we denote the generators of the group algebra $\mathbb{Z}[\Gamma]$ by $\lambda_\gamma$ ($\gamma \in \Gamma$). Let $(C_*(\Gamma), d)$ be the nonnormalized bar-resolution of the $\mathbb{Z}[\Gamma]$-module $\mathbb{Z}$, so $C_n(\Gamma)$ ($n \geq 0$) is the free $\mathbb{Z}[\Gamma]$-module with basis consisting of $n$-tuples of elements in $\Gamma$, written as $[\gamma_1|\cdots|\gamma_n]$, and the differential $d: C_n(\Gamma) \to C_{n-1}(\Gamma)$ is defined by

$$d[\gamma_1|\cdots|\gamma_n] = \lambda_{\gamma_1}[\gamma_2|\cdots|\gamma_n] + \sum_{i=1}^{n-1} (-1)^i[\gamma_1|\cdots|\gamma_i + \gamma_i+1|\cdots|\gamma_n] + (-1)^n[\gamma_1|\cdots|\gamma_{n-1}].$$

Let $M$ be a commutative group endowed with the trivial $\Gamma$-module structure. The group cohomology $H^*(\Gamma; M)$ can be computed from the standard complex induced by the bar-resolution. Concretely, we have a cochain complex

$$C^*(\Gamma; M) = \text{Hom}_{\mathbb{Z}[\Gamma]}(C_*(\Gamma), M) = \text{Map}(\Gamma^*, M),$$

endowed with the boundary map $\partial: C^n(\Gamma; M) \to C^{n+1}(\Gamma; M)$ defined by

$$(\partial \phi)(\gamma_1, \ldots, \gamma_{n+1}) = \phi(\gamma_2, \ldots, \gamma_{n+1}) - \phi(\gamma_1 + \gamma_2, \gamma_3, \ldots, \gamma_{n+1}) + \cdots$$

$$+ (-1)^n \phi(\gamma_1, \ldots, \gamma_{n-1}, \gamma_n + \gamma_{n+1}) + (-1)^{n+1} \phi(\gamma_1, \ldots, \gamma_n).$$

By $M$-valued cocycles on $\Gamma$ we mean cocycles in $(C^*(\Gamma; M), \partial)$. We will consider only $\mathbb{T}$-valued cocycles, but with minor modifications everything what we say remains true for cocycles with values in any divisible group $M$.

For the sake of computation, it is also convenient to introduce the integer homology
\[ H_n(\Gamma) = H_n(\Gamma; \mathbb{Z}), \] which is given as the homology of the complex \( C_* (\Gamma; \mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}[\Gamma]} C_*(\Gamma) \). Since the action of \( \Gamma \) on \( \mathbb{T} \) is trivial, we have \( C^* (\Gamma; \mathbb{T}) = \operatorname{Hom}_{\mathbb{Z}[\Gamma]} (C_*(\Gamma), \mathbb{T}) = \operatorname{Hom} (C_*(\Gamma; \mathbb{Z}), \mathbb{T}) \). Moreover, the injectivity of \( \mathbb{T} \) as a \( \mathbb{Z} \)-module implies that any character of \( H_n(\Gamma; \mathbb{Z}) \) can be lifted to a character of \( C_n(\Gamma; \mathbb{Z}) \). It follows that the groups \( H^n(\Gamma; \mathbb{T}) \) and \( H_n(\Gamma) \) are Pontryagin dual to each other. This is a particular case of the Universal Coefficient Theorem.

A map \( \phi: \Gamma^n \to \mathbb{T} \) \((n \geq 1)\) is called an \( n \)-character on \( \Gamma \) if it is a character in every variable, so it is defined by a character on \( \Gamma^{\otimes n} \) (unless specified otherwise, all tensor products in this appendix are over \( \mathbb{Z} \)). It is easy to see that every \( n \)-character is a \( \mathbb{T} \)-valued cocycle. An \( n \)-character \( \phi \) is called alternating if \( \phi(\gamma_1, \ldots, \gamma_n) = 1 \) as long as \( \gamma_i = \gamma_{i+1} \) for some \( i \); then \( \phi(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)}) = \phi(\gamma_1, \ldots, \gamma_n)^{\text{sgn}(\sigma)} \) for any \( \sigma \in S_n \). In other words, an \( n \)-character is alternating if it factors through the exterior power \( \Lambda^n \Gamma \), which is the quotient of \( \Gamma^{\otimes n} \) by the subgroup generated by elements \( \gamma_1 \otimes \cdots \otimes \gamma_n \) such that \( \gamma_i = \gamma_{i+1} \) for some \( i \). It will sometimes be convenient to view \( \Lambda^n \Gamma \) as a subgroup of \( \Gamma^{\otimes n} \) via the embedding

\[ \gamma_1 \wedge \cdots \wedge \gamma_n \mapsto \sum_{\sigma \in S_n} \text{sgn}(\sigma) \gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(n)}. \]

We will also consider \( \Lambda^n \Gamma \) as a subgroup of \( H_n(\Gamma) \). The embedding \( \Lambda^* \Gamma \hookrightarrow H_*(\Gamma) \) is constructed using the canonical isomorphism \( \Gamma \cong H_1(\Gamma) \) and the Pontryagin product on \( H_*(\Gamma) \), see [Bro94, Theorem V.6.4]. On the chain level the latter product can be defined using the shuffle product, so that \( \gamma_1 \wedge \cdots \wedge \gamma_n \) is identified with the homology class of the cycle

\[ \sum_{\sigma \in S_n} \text{sgn}(\sigma) (1 \otimes [\gamma_{\sigma(1)} | \cdots | \gamma_{\sigma(n)}]) \in C_n(\Gamma; \mathbb{Z}). \]

For free abelian groups we have \( \Lambda^* \Gamma = H_*(\Gamma) \). By duality we get the following description of cocycles.

**Proposition A.1.** If \( \Gamma \) is free abelian, then for every \( n \geq 1 \) we have:

1. any \( \mathbb{T} \)-valued \( n \)-cocycle on \( \Gamma \) is cohomologous to an alternating \( n \)-character;
2. an \( n \)-character is a coboundary if and only if it vanishes on \( \Lambda^n \Gamma \subseteq \Gamma^{\otimes n} \); in particular, an alternating \( n \)-character is a coboundary if and only its order divides \( n! \).

**Proof.** The value of an \( n \)-cocycle \( \phi \) on \( \gamma_1 \wedge \cdots \wedge \gamma_n \in H_n(\Gamma) \) is

\[ \langle \phi, \gamma_1 \wedge \cdots \wedge \gamma_n \rangle = \prod_{\sigma \in S_n} \phi(\gamma_{\sigma(1)}, \ldots, \gamma_{\sigma(n)})^{\text{sgn}(\sigma)}. \]

This immediately implies (ii), since if \( \phi \) is an \( n \)-character, then the above product is exactly the value of \( \phi \) on \( \gamma_1 \wedge \cdots \wedge \gamma_n \) considered as an element of \( \Gamma^{\otimes n} \).

Turning to (i), assume \( \psi \) is an \( n \)-cocycle. It defines a character \( \chi \) of \( H_n(\Gamma) = \Lambda^n \Gamma \).
Let $\phi$ be a character of $\bigwedge^n \Gamma$ such that $\phi^{n!} = \chi$. Then $\phi$ is an alternating $n$-character, and $\phi$ is cohomologous to $\psi$, since both cocycles $\phi$ and $\psi$ define the same character $\chi$ of $H_n(\Gamma) = \bigwedge^n \Gamma$.

We now turn to the more complicated case of finite abelian groups and concentrate on 3-cocycles. In this case $\bigwedge^3 \Gamma$ is a proper subgroup of $H_3(\Gamma)$: as follows from Proposition A.3 below, the quotient $H_3(\Gamma) / \bigwedge^3 \Gamma$ is (noncanonically) isomorphic to $\Gamma \oplus (\Gamma \wedge \Gamma)$. Correspondingly, not every third cohomology class can be represented by a 3-character. Additional 3-cocycles can be obtained by the following construction.

**Lemma A.2.** Assume $\Gamma = \Gamma_1 / \Gamma_0$ for some abelian groups $\Gamma_1$ and $\Gamma_0$. Suppose $f : \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ is a function such that

$$f(\alpha, \beta + \gamma) = f(\alpha, \beta) \quad \text{and} \quad f(\alpha + \gamma, \beta) = \chi(\gamma \otimes \beta)f(\alpha, \beta)$$

for all $\alpha, \beta \in \Gamma_1$ and $\gamma \in \Gamma_0$, where $\chi$ is a character of $\Gamma_0 \otimes \Gamma$. Then the function

$$(\partial f)(\alpha, \beta, \gamma) = f(\beta, \gamma)f(\alpha + \beta, \gamma)^{-1}f(\alpha, \beta + \gamma)f(\alpha, \beta)^{-1}$$

on $\Gamma_1^3$ is $\Gamma_0^3$-invariant, hence it defines a $\mathbb{T}$-valued 3-cocycle on $\Gamma$.

**Proof.** This is a straightforward computation. $\square$

In order to describe explicitly generators of $H^3(\Gamma; \mathbb{T})$, let us introduce some notation. For natural numbers $n_1, \ldots, n_k$, denote by $(n_1, \ldots, n_k)$ their greatest common divisor. For $n \in \mathbb{N}$, denote by $\chi_n$ the character of $\mathbb{Z}/n\mathbb{Z}$ defined by $\chi_n(1) = e^{2\pi i / n}$. Finally, for integers $a$ and $b$ and a natural number $n$, put

$$\omega_n(a, b) = \left\lfloor \frac{a + b}{n} \right\rfloor - \left\lfloor \frac{a}{n} \right\rfloor - \left\lfloor \frac{b}{n} \right\rfloor.$$ 

Note that $\omega_n$ is a well-defined function on $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ with values 0 or 1.

**Proposition A.3.** Assume $\Gamma = \bigoplus_{i=1}^m \mathbb{Z}/n_i\mathbb{Z}$ for some $n_i \geq 1$. Then

$$H^3(\Gamma; \mathbb{T}) \cong \bigoplus_i \mathbb{Z}/n_i\mathbb{Z} \oplus \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j)\mathbb{Z} \oplus \bigoplus_{i<j<k} \mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}.$$ 

Explicitly, generators $\phi_i$ of $\mathbb{Z}/n_i\mathbb{Z}$, $\phi_{ij}$ of $\mathbb{Z}/(n_i, n_j)\mathbb{Z}$ and $\phi_{ijk}$ of $\mathbb{Z}/(n_i, n_j, n_k)\mathbb{Z}$ can be defined by

$$\phi_i(a, b, c) = \chi_{n_i}(\omega_{n_i}(a_i, b_i)c_i), \quad \phi_{ij}(a, b, c) = \chi_{n_j}(\omega_{n_j}(a_i, b_i)c_j), \quad \phi_{ijk}(a, b, c) = \chi_{(n_i, n_j, n_k)}(a_i b_j c_k).$$

**Proof.** Recall first how to compute the homology of finite cyclic groups. Consider the group $\mathbb{Z}/n\mathbb{Z}$. Then there is a free resolution $(P_*, d)$ of the $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module $\mathbb{Z}$ such
that $P_k$ is generated by one basis element $e_k$, and

$$de_{2k+1} = \lambda_1 e_{2k} - e_{2k} \quad \text{and} \quad de_{2k+2} = \sum_{a \in \mathbb{Z}/n\mathbb{Z}} \lambda_a e_{2k+1} \quad \text{for} \quad k \geq 0.$$  

The morphism $P_0 \to \mathbb{Z}$ is given by $e_0 \mapsto 1$. Using this resolution we get

$$H_{2k+1}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \quad \text{and} \quad H_{2k+2}(\mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for} \quad k \geq 0.$$  

Turning to the proof of the proposition, the first statement is equivalent to

$$H_3(\Gamma) \cong \bigoplus_i \mathbb{Z}/n_i \mathbb{Z} \oplus \bigoplus_{i<j} \mathbb{Z}/(n_i, n_j) \mathbb{Z} \oplus \bigoplus_{i<j<k} \mathbb{Z}/(n_i, n_j, n_k) \mathbb{Z}.$$  

This, in turn, is proved by induction on $m$ using the isomorphisms

$$H_1(\Gamma) \cong \Gamma, \quad H_2(\Gamma) \cong \Gamma \wedge \Gamma,$$  

which are valid for any abelian group $\Gamma$, and the Künneth formula, which gives that $H_3(\Gamma \oplus \mathbb{Z}/n\mathbb{Z})$ is isomorphic to

$$H_3(\Gamma) \oplus (H_2(\Gamma) \otimes H_1(\mathbb{Z}/n\mathbb{Z})) \oplus H_3(\mathbb{Z}/n\mathbb{Z}) \oplus \text{Tor}_1^\mathbb{Z}(H_1(\Gamma), H_1(\mathbb{Z}/n\mathbb{Z})).$$  

Note only that

$$\text{Tor}_1^\mathbb{Z}(\mathbb{Z}/k\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(k, n)\mathbb{Z} \cong \mathbb{Z}/k\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z}.$$  

Let us check next that the functions $\phi_i$, $\phi_{ij}$ and $\phi_{ijk}$ are indeed 3-cocycles. For $\phi_{ijk}$ this is clear, since it is a 3-character. Concerning $\phi_i$, consider the function

$$f_i(a, b) = \chi_{n_i} \left(- \left\lfloor \frac{a_i}{n_i} \right\rfloor b_i \right)$$  

on $\mathbb{Z}^m \times \mathbb{Z}^m$. It is of the type described in Lemma A.2 for $\Gamma_1 = \mathbb{Z}^m$ and $\Gamma_0 = \bigoplus_{i=1}^m n_i \mathbb{Z}$, so $\phi_i(a, b, c) = (\partial f_i)(a, b, c)$ is a 3-cocycle on $\Gamma$. Similarly, consider the function

$$f_{ij}(a, b) = \chi_{n_j} \left(- \left\lfloor \frac{a_j}{n_j} \right\rfloor b_j \right).$$  

It is again of the type described in Lemma A.2, so $\phi_{ij} = \partial f_{ij}$ is a 3-cocycle.

Our next goal is to construct a ‘dual basis’ in $H_3(\Gamma)$. Let $u_i$ be the generator $1 \in \mathbb{Z}/n_i\mathbb{Z} \subset \Gamma$. Denote by $\theta_{ijk}$ the cycle representing $u_i \wedge u_j \wedge u_k \in \wedge^3 \Gamma \subset H_3(\Gamma)$ obtained by the shuffle product, so

$$\theta_{ijk} = \sum_{\sigma \in S_3} \text{sgn}(\sigma) (1 \otimes [u_{\sigma(i)}|u_{\sigma(j)}|u_{\sigma(k)})].$$
where we consider $S_3$ as the group of permutations of $\{i,j,k\}$.

Consider the $\mathbb{Z}[\mathbb{Z}/n_i\mathbb{Z}]$-resolution $(P^i_n, d)$ of $\mathbb{Z}$ described at the beginning of the proof. Let $e^n_i$ be the basis element of $P^i_n$. We have a chain map $P^i_n \to C_*(\mathbb{Z}/n_i\mathbb{Z})$ of resolutions of $\mathbb{Z}$ defined by

$$e^n_0 \mapsto [0], \quad e^n_1 \mapsto [1], \quad e^n_2 \mapsto \sum_{a \in \mathbb{Z}/n_i\mathbb{Z}} [a|1], \quad e^n_3 \mapsto \sum_{a \in \mathbb{Z}/n_i\mathbb{Z}} [1|a|1], \ldots \quad (A.1)$$

It follows that we have a 3-cycle $\theta_i \in C_3(\Gamma; \mathbb{Z})$ defined by

$$\theta_i = \sum_{a=0}^{n_i-1} 1 \otimes [u_i|au_i|u_i].$$

Finally, consider the $\mathbb{Z}[\mathbb{Z}/n_i\mathbb{Z} \oplus \mathbb{Z}/n_j\mathbb{Z}]$-resolution $P^i_n \otimes P^j_n$ of $\mathbb{Z}$. Using this resolution we get a third homology class represented by

$$\frac{n_j}{(n_i,n_j)} \sum_{a=0}^{n_i-1} 1 \otimes ([au_i|u_i|u_j] - [au_i|u_j|u_i] + [u_j|au_i|u_i])$$

$$+ \frac{n_i}{(n_i,n_j)} \sum_{b=0}^{n_j-1} 1 \otimes ([u_i|bu_j|u_j] - [bu_j|u_i|u_j] + [bu_j|u_i|u_j]).$$

The only nontrivial pairings between the cocycles $\phi_i, \phi_{ij}, \phi_{ijk}$ and the cycles $\theta_i, \theta_{ij}, \theta_{ijk}$ are

$$\langle \phi_i, \theta_i \rangle = \zeta_n, \quad \langle \phi_{ij}, \theta_{ij} \rangle = \zeta_{n_i/n_j}^{n_j/(n_i,n_j)} = \zeta_{(n_i,n_j)}, \quad \langle \phi_{ijk}, \theta_{ijk} \rangle = \zeta_{(n_i,n_j,n_k)},$$

where $\zeta_n = e^{2\pi i/n}$. This implies that these cocycles and cycles are the required generators of the Pontryagin dual groups $H^3(\Gamma; \mathbb{T})$ and $H_3(\Gamma)$.

**Corollary A.4.** Assume $\Gamma$ is a finite abelian group. Write $\Gamma$ as $\Gamma_1/\Gamma_0$ for a finite rank free abelian group $\Gamma_1$. Then for any $\mathbb{T}$-valued 3-cocycle $\phi$ on $\Gamma$ the following conditions are equivalent:

1. $\phi$ vanishes on $\Lambda^3 \Gamma \subseteq H_3(\Gamma)$;
2. $\phi$ lifts to a coboundary on $\Gamma_1$;
3. $\phi = \partial f$ for a function $f : \Gamma_1 \times \Gamma_1 \to \mathbb{T}$ as in Lemma A.2.
Proof. The equivalence of (i) and (ii) is clear, since a cocycle on $\Gamma_1$ is a coboundary if and only if it vanishes on $H_3(\Gamma_1) = \bigwedge^3 \Gamma_1$. Also, obviously (iii) implies (ii). Therefore the only nontrivial statement is that (i), or (ii), implies (iii). Assume $\phi$ is a cocycle that vanishes on $\bigwedge^3 \Gamma \subset H_3(\Gamma)$. We can identify $\Gamma_1$ with $\mathbb{Z}^m$ in such a way that $\Gamma_0 = \bigoplus_{i=1}^m n_i \mathbb{Z}$ for some $n_i \geq 1$. Then in the notation of the proof of the above proposition the assumption on $\phi$ means that $\phi$ vanishes on the cycles $\theta_{ijk}$, whose homology classes are exactly $u_i \wedge u_j \wedge u_k \in \bigwedge^3 \Gamma \subset H_3(\Gamma)$. It follows that $\phi$ is cohomologous to a product of powers of cocycles $\phi_i$ and $\phi_{ij}$. But the cocycles $\phi_i$ and $\phi_{ij}$ are of the form $\partial f$ with $f: \Gamma_1 \times \Gamma_1 \to T$ as in Lemma A.2. Therefore $\phi$ is cohomologous to a cocycle of the form $\partial f$, hence $\phi$ itself is of the same form. \qed

Since every character of $\bigwedge^3 \Gamma \subset \Gamma \otimes^3$ extends to a 3-character on $\Gamma$, this corollary can also be formulated as follows.

Corollary A.5. With $\Gamma = \Gamma_1/\Gamma_0$ as in the previous corollary, any $T$-valued 3-cocycle $\phi$ on $\Gamma$ can be written as product of a 3-character $\chi$ on $\Gamma$ and a cocycle $\partial f$ with $f: \Gamma_1 \times \Gamma_1 \to T$ as in Lemma A.2. Such a cocycle $\phi$ lifts to a coboundary on $\Gamma_1$ if and only if $\chi$ vanishes on $\bigwedge^3 \Gamma \subset \Gamma \otimes^3$, and in this case $\phi = \partial g$ with $g: \Gamma_1 \times \Gamma_1 \to T$ as in Lemma A.2.

Let us now look more carefully at the construction of cocycles described in Lemma A.2. As Corollary A.4 shows, the class of 3-cocycles obtained by this construction does not depend on the presentation of $\Gamma$ as quotient of a finite rank free abelian group. It is also clear that there is a lot of redundancy in this construction, since the group $H_3(\Gamma)$ can be much smaller than $\Gamma_0 \otimes \Gamma$. The following proposition makes these observations a bit more precise.

Proposition A.6. Assume $\Gamma$ is a finite abelian group, and write $\Gamma$ as $\Gamma_1/\Gamma_0$ for a finite rank free abelian group $\Gamma_1$. Let $f: \Gamma_1 \times \Gamma_1 \to T$ be a function as in Lemma A.2, and $\chi$ be the associated character of $\Gamma_0 \otimes \Gamma$. Then the cohomology class of $\partial f$ in $H^3(\Gamma; T)$ depends only on the restriction of $\chi$ to 

$$\ker(\Gamma_0 \otimes \Gamma \to \Gamma_1 \otimes \Gamma) \cong \text{Tor}_1^T(\Gamma, \Gamma) \cong \Gamma \otimes \Gamma.$$ 

Therefore by varying $\chi$ we get a natural in $\Gamma$ homomorphism

$$\text{Hom}(\text{Tor}_1^T(\Gamma, \Gamma), T) \to H^3(\Gamma; T),$$

whose image is the annihilator of $\bigwedge^3 \Gamma \subset H_3(\Gamma)$.

Proof. It is easy to see that the cohomology class of $\partial f$ depends only on $\chi$, so we have a homomorphism $\text{Hom}(\Gamma_0 \otimes \Gamma, T) \to H^3(\Gamma; T)$. We have to check that if a character $\chi$ of $\Gamma_0 \otimes \Gamma$ vanishes on $\ker(\Gamma_0 \otimes \Gamma \to \Gamma_1 \otimes \Gamma)$, then the image of $\chi$ in $H^3(\Gamma; T)$ is zero. But this is clear, since we can extend $\chi$ to a character $f$ of $\Gamma_1 \otimes \Gamma$, and then $f$, considered as a function on $\Gamma_1 \times \Gamma_1$, is of the type described in Lemma A.2, with associated character $\chi$, and $f$ is a 2-character, so $\partial f = 0$. 

Naturality of the homomorphism $\text{Hom}(\text{Tor}_1^\mathbb{Z}(\Gamma, \Gamma), \mathbb{T}) \to H^3(\Gamma; \mathbb{T})$ in $\Gamma$ is straightforward to check. The statement that its image coincides with the annihilator of $\bigwedge^3 \Gamma \subset H_3(\Gamma)$ follows from Corollary A.4.

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