Nonhamiltonian Graphs with Given Toughness

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Abstract

In 1973, Chvátal introduced the concept of toughness $\tau$ of a graph and constructed an infinite class of nonhamiltonian graphs with $\tau = \frac{3}{2}$. Later Thomassen found nonhamiltonian graphs with $\tau > \frac{3}{2}$, and Enomoto et al. constructed nonhamiltonian graphs with $\tau = 2 - \epsilon$ for each positive $\epsilon$. The last result in this direction is due to Bauer, Broersma and Veldman, which states that for each positive $\epsilon$, there exists a nonhamiltonian graph with $\tau \geq \frac{9}{4} - \epsilon$. In this paper we prove that for each rational number $t$ with $0 < t < \frac{9}{4}$, there exists a nonhamiltonian graph with $\tau = t$.

Key words: Hamilton cycle, toughness.

1 Introduction

Only finite undirected graphs without loops or multiple edges are considered. The set of vertices of a graph $G$ is denoted by $V(G)$ and the set of edges by $E(G)$. The order and the independence number of $G$ is denoted by $n$ and $\alpha$, respectively. For $S$ a subset of $V(G)$, we denote by $G\setminus S$ the maximum subgraph of $G$ with vertex set $V(G)\setminus S$. The neighborhood of a vertex $x \in V(G)$ is denoted by $N(x)$. A graph $G$ is hamiltonian if $G$ contains a Hamilton cycle, i.e. a cycle of length $n$. A good reference for any undefined terms is [5].

The concept of toughness of a graph was introduced in 1973 by Chvátal [6]. Let $\omega(G)$ denote the number of components of a graph $G$. A graph $G$ is $t$-tough if $|S| \geq t\omega(G\setminus S)$ for every subset $S$ of the vertex set $V(G)$ with $\omega(G\setminus S) > 1$. The toughness of $G$, denoted $\tau(G)$, is the maximum value of $t$ for which $G$ is $t$-tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). By the definition, toughness $\tau$ is a rational number. Since then significant progress has been made toward understanding the relationship between the toughness of a graph and its cycle structure. Much of the research on this subject have been inspired by the following conjecture due to Chvátal [6].

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Conjecture 1. There exists a finite constant $t_0$ such that every $t_0$-tough graph is hamiltonian.

In [6], Chvátal constructed an infinite family of nonhamiltonian graphs with $\tau = \frac{3}{2}$, and then Thomassen [4, p.132] found nonhamiltonian graphs with $\tau > \frac{3}{2}$. Later Enomoto et al. [7] have found nonhamiltonian graphs with $\tau = 2 - \epsilon$ for each positive $\epsilon$. The last result in this direction is due to Bauer, Broersma and Veldman [2] inspired by special constructions introduced in [1] and [3].

Theorem A. For each positive $\epsilon > 0$, there exists a nonhamiltonian graph with $\frac{9}{4} - \epsilon < \tau < \frac{9}{4}$.

In view of Theorem A, the following problem seems quite reasonable.

Problem. Is there a nonhamiltonian graph $G$ with $\tau(G) = t$ for a given rational number $t$ with $0 < t < \frac{9}{4}$?

In this paper we prove the following.

Theorem 1. For each rational number $t$ with $0 < t < \frac{9}{4}$, there exists a nonhamiltonian graph $G$ with $\tau(G) = t$.

2 Preliminaries

To prove Theorem 1, we need the following graph constructions.

Definition 1. Let $L^{(1)}$ be a graph obtained from $C_8 = w_1w_2...w_8w_1$ by adding the edges $w_2w_4, w_4w_6, w_6w_8$ and $w_2w_8$. Put $x = w_1$ and $y = w_5$. This is the well-known building block $L$ used to obtain $(\frac{9}{4} - \epsilon)$-tough nonhamiltonian graphs (see [2], Figure 1).

In this paper we will use a number of additional modified building blocks.

Definition 2. Let $L^{(2)}$ be the graph obtained from $L^{(1)}$ by deleting the edges $w_1w_2, w_2w_8$ and identifying $w_2$ with $w_8$.

Definition 3. Let $L^{(3)}$ be the graph obtained from $L^{(1)}$ by adding a new vertex $w_9$ and the edges $w_4w_9, w_5w_9$.

Definition 4. Let $L^{(4)}$ be the graph obtained from the triangle $w_1w_2w_3w_1$ by adding the vertices $w_4, w_5$ and the edges $w_1w_4, w_3w_5$. Put $x = w_4$ and $y = w_5$. 
Definition 5. For each \( L \in \{ L^{(1)}, L^{(2)} \} \), define the graph \( G(L, x, y, l, m) \) \((l, m \in \mathbb{N})\) as follows. Take \( m \) disjoint copies \( L_1, L_2, ..., L_m \) of \( L \), with \( x_i, y_i \) the vertices in \( L_i \) corresponding to the vertices \( x \) and \( y \) in \( L \) \((i = 1, 2, ..., m)\). Let \( F_m \) be the graph obtained from \( L_1 \cup ... \cup L_m \) by adding all possible edges between pairs of vertices in \( x_1, ..., x_m, y_1, ..., y_m \). Let \( T = K_l \) and let \( G(L, x, y, l, m) \) be the join \( T \vee F_m \) of \( T \) and \( F_m \).

The following can be checked easily.

Claim 1. The vertices \( x \) and \( y \) are not connected by a Hamilton path of \( L^{(i)} \) \((i = 1, 2, 3)\).

The proof of the following result occurs in [1], which we repeat here for convenience.

Claim 2. Let \( H \) be a graph and \( x, y \) two vertices of \( H \) which are not connected by a Hamilton path of \( H \). If \( m \geq 2l + 1 \) then \( G(H, x, y, l, m) \) is non-hamiltonian.

Proof. Suppose \( G(H, x, y, l, m) \) contains a Hamilton cycle \( C \). The intersection of \( C \) and \( F_m \) consists of a collection \( \mathcal{R} \) of at most \( l \) disjoint paths, together containing all vertices in \( F_m \). Since \( m \geq 2l + 1 \), there is a subgraph \( H_{i_0} \) in \( F_m \) such that no endvertex of a path of \( \mathcal{R} \) lies in \( H_{i_0} \). Hence the intersection of \( C \) and \( H_{i_0} \) is a path with endvertices \( x_{i_0} \) and \( y_{i_0} \) that contains all vertices of \( H_{i_0} \). This contradicts the fact that \( H_{i_0} \) is a copy of the graph \( H \) without a Hamilton path between \( x \) and \( y \). Claim 1 is proved.

3 Proof of Theorem 1

Let \( t \) be a rational number with \( 0 < t < \frac{9}{4} \) and let \( t = \frac{a}{b} \) for some integers \( a, b \).

Case 1. \( 0 < \frac{a}{b} < 1 \).

Let \( K_{a,b} \) be the complete bipartite graph \( G = (V_1, V_2; E) \) with vertex classes \( V_1 \) and \( V_2 \) of order \( a \) and \( b \), respectively. Since \( \frac{a}{b} < 1 \), we have \( \alpha(G) = b > (a + b)/2 \) and therefore, \( K_{a,b} \) is a nonhamiltonian graph. Clearly, \( \tau \leq |V_1|/\omega(G\setminus V_1) = a/b \). Choose \( S \subseteq V(G) \) such that \( \tau(K_{a,b}) = |S|/\omega(G\setminus S) \). Put \( S \cap V_i = S_i \) and \( |S_i| = s_i \) \((i = 1, 2)\). If \( V_i \setminus S \neq \emptyset \) \((i = 1, 2)\) then clearly \( \omega(G\setminus S) = 1 \), which is impossible by the definition. Hence \( V_i \setminus S = \emptyset \) for some \( i \in \{1, 2\} \), i.e. \( V_i \subseteq S \).

Case 1.1. \( i = 2 \).

Since \( \tau = (s_1 + b)/(a - s_1) \geq b/a \), we have \( s_1 = 0 \), i.e. \( S = V_2 \) and \( \tau = b/a \), contradicting that fact that \( \tau \leq a/b \).
Case 1.2. $i = 1$.

Since $\tau = (s_2 + a) / (b - s_2) \geq a/b$, we have $s_2 = 0$, implying that $S = V_1$ and $\tau = a/b$.

Case 2. $\frac{a}{b} = 1$.

Let $G$ be a graph obtained from $C_6 = x_1x_2...x_6x_1$ by adding a new vertex $x_7$ and the edges $x_1x_7, x_4x_7, x_2x_6$. Clearly, $G$ is not hamiltonian and $\tau(G) = 1$.

Case 3. $1 < \frac{a}{b} < \frac{3}{2}$.

Case 3.1. $\frac{a}{b} < \frac{3}{2} - \frac{1}{b}$.

Let $V_1, V_2, V_3$ be pairwise disjoint sets of vertices with

$$V_1 = \{x_1, x_2, ..., x_{a-b+1}\}, \ V_2 = \{y_1, y_2, ..., y_b\}, \ V_3 = \{z_1, z_2, ..., z_b\}.$$ 

Join each $x_i$ to all the other vertices and each $z_i$ to every other $z_j$ as well as to the vertex $y_i$ with the same subscript $i$. Call the resulting graph $H$. To determine the toughness of $H$, choose $W \subset V(H)$ such that $\tau(H) = |W|/\omega(H \setminus W)$. Put $m = |W \cap V_3|$. Clearly, $W$ is a minimal set whose removal from $H$ results in a graph with $\omega(H \setminus W)$ components. As $W$ is a cutset, we have $V_1 \subset W$ and $m \geq 1$. From the minimality of $W$ we easily conclude that $V_2 \cap W = \emptyset$ and $m \leq b - 1$. Then we have $|W| = m + a - b + 1$ and $\omega(H \setminus W) = m + 1$. Hence

$$\tau(H) = \frac{|W|}{\omega(H \setminus W)} = \min_{1 \leq m \leq b-1} \frac{m + a - b + 1}{m + 1} = \frac{a}{b}.$$ 

To see that $H$ is nonhamiltonian, let us assume the contrary, i.e. let $C$ be a Hamilton cycle in $H$. Denote by $F$ the set of edges of $C$ having at least one endvertex in $V_2$. Since $V_2$ is independent, we have $|F| = 2|V_2|$. On the other hand, there are at most $2|V_1|$ edges in $F$ having one endvertex in $V_1$ and at most $|V_3|$ edges in $F$ having one endvertex in $V_3$. Thus

$$2b = 2|V_2| = |F| \leq 2|V_1| + |V_3| = 2(a - b + 1) + b = 2a - b + 2.$$ 

But this is equivalent to $a/b \geq 3/2 - 1/b$, contradicting the hypothesis.

Case 3.2. $\frac{a}{b} \geq \frac{3}{2} - \frac{1}{b}$.

By choosing $q \in N$ sufficiently large with

$$\frac{a}{b} = \frac{aq}{bq} < \frac{3}{2} - \frac{1}{bq},$$

we can argue as in Case 3.1.

Case 4. $\frac{a}{b} = \frac{3}{2}$.

An example of a nonhamiltonian graph with $\tau = 3/2$ is obtained when in the Petersen graph, each vertex is replaced by a triangle.
Case 5. \( \frac{3}{2} < \frac{9}{b} < \frac{7}{4} \).

Claim 3. For \( l \geq 2 \) and \( m \geq 1 \),

\[
\tau(G(L(2), x, y, l, m)) = \frac{l + 3m}{1 + 2m}.
\]

Proof. Let \( G = G(L(2), x, y, l, m) \) for some \( l \geq 2 \) and \( m \geq 1 \). Choose \( S \subseteq V(G) \) such that \( \omega(G \setminus S) > 1 \) and \( \tau(G) = |S|/\omega(G \setminus S) \). Obviously, \( V(T) \subseteq S \). Define \( S_i = S \cap V(L_i) \), \( s_i = |S_i| \), and let \( \omega_i \) be the number of components of \( L_i \setminus S_i \) that contain neither \( x_i \) nor \( y_i \) (\( i = 1, ..., m \)). Then

\[
\tau(G) = \frac{l + \sum_{i=1}^{m} s_i}{c + \sum_{i=1}^{m} \omega_i} \geq \frac{l + \sum_{i=1}^{m} s_i}{1 + \sum_{i=1}^{m} \omega_i},
\]

where \( c = 0 \) if \( x_i, y_i \in S \) for all \( i \in \{1, ..., m\} \) and \( c = 1 \) otherwise. It is easy to see that

\[
\omega_i \leq 2, \quad s_i \geq \frac{3}{2} \omega_i \quad (i = 1, ..., m).
\]

Then

\[
\tau \geq \frac{l + \frac{3}{2} \sum_{i=1}^{m} \omega_i}{1 + \sum_{i=1}^{m} \omega_i} = \frac{l - \frac{3}{2}}{1 + \sum_{i=1}^{m} \omega_i} + \frac{3}{2} \geq \frac{l - \frac{3}{2}}{1 + 2m} + \frac{3}{2} = \frac{l + 3m}{1 + 2m}.
\]

Set \( U = V(T) \cup U_1 \cup ... \cup U_m \), where \( U_i \) is the set of vertices of \( L_i \) having degree at least 4 in \( L_i \) (\( i = 1, ..., m \)). The proof of Claim 3 is completed by observing that

\[
\tau(G) \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l + 3m}{2m + 1}.
\]

Case 5.1. \( b = 2k + 1 \) for some integer \( k \).

Consider the graph \( G(L(2), x, y, a - \frac{3}{2}(b - 1), \frac{b - 1}{2}) \).

Case 5.1.1. \( \frac{a}{b} \leq \frac{7}{4} - \frac{a}{2b} \).

By the hypothesis,

\[
m = \frac{b - 1}{2} \geq 2(a - \frac{3}{2}(b - 1)) + 1 = 2l + 1,
\]

implying by Claim 2 that \( G \) is not hamiltonian. Clearly \( b \geq 3 \), implying that \( m = (b - 1)/2 \geq 1 \).

Case 5.1.1.1. \( \frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b} \).

By the hypothesis, \( l = a - \frac{3}{2}(b - 1) \geq 2 \). By Claim 3, \( \tau(G) = \frac{a}{b} \).

Case 5.1.1.2. \( \frac{a}{b} < \frac{3}{2} + \frac{1}{2b} \).
By choosing a sufficiently large integer \( q \) with
\[
\frac{a}{b} = \frac{aq}{bq} \geq \frac{3}{2} + \frac{1}{2bq},
\]
we can argue as in Case 5.1.1.

**Case 5.1.2.** \( \frac{a}{b} > \frac{7}{4} - \frac{9}{4b} \).

By choosing a sufficiently large integer \( q \) with
\[
\frac{a}{b} = \frac{aq}{bq} \leq \frac{7}{4} - \frac{9}{4bq},
\]
we can argue as in Case 5.1.1.

**Case 5.2.** \( b = 2k \) for some integer \( k \).

Consider the graph \( G' \) obtained from \( G(L^{(2)}, x, y, l, m) \) by replacing \( L_m \) with \( L^{(3)} \).

**Claim 4.** For \( l \geq 2 \) and \( m \geq 1 \),
\[
\tau(G') = \frac{l + 3m + 1}{2(m + 1)}.
\]

**Proof.** Choose \( S \subseteq V(G') \) such that \( \omega(G' \setminus S) < 1 \) and \( \tau(G') = |S|/\omega(G' \setminus S) \).

Obviously, \( V(T) \subseteq S \). Define \( S_i = S \cap V(L_i) \), \( s_i = |S_i| \), and let \( \omega_i \) be the number of components of \( L_i \setminus S_i \) that contain neither \( x_i \) nor \( y_i \) (\( i = 1, \ldots, m \)). Since \( s_i \geq \frac{3}{2}\omega_i \) (\( i = 1, \ldots, m - 1 \)) and \( s_m \geq \frac{4}{3}\omega_m \), we have
\[
\tau(G') \geq \frac{l + \sum_{i=1}^{m} s_i}{c + \sum_{i=1}^{m} \omega_i} = \frac{l + \frac{3}{2} \sum_{i=1}^{m-1} \omega_i + \frac{4}{3}\omega_m}{1 + \sum_{i=1}^{m} \omega_i} = \frac{l - \frac{1}{6}\omega_m + \frac{3}{2}}{1 + \sum_{i=1}^{m} \omega_i},
\]
where \( c = 0 \) if \( x_i, y_i \in S \) for all \( i \in \{1, \ldots, m\} \) and \( c = 1 \) otherwise. Observing also that \( \omega_i \leq 2 \) (\( i = 1, \ldots, m - 1 \)) and \( \omega_m \leq 3 \), we obtain
\[
(l - 2) \sum_{i=1}^{m} \omega_i + \frac{1}{3}(m + 1)\omega_m \leq (l - 2)(2m + 1) + (m + 1) \leq 2l(m + 1).
\]

But this is equivalent to
\[
\frac{l - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^{m} \omega_i} + \frac{3}{2} \geq \frac{l - 2}{2(m + 1)} + \frac{3}{2},
\]
implying that
\[
\tau(G') \geq \frac{l - 2}{2(m + 1)} + \frac{3}{2} = \frac{l + 3m + 1}{2(m + 1)}.
\]

Set \( U = V(T) \cup U_1 \cup \ldots \cup U_m \), where \( U_i \) is the set of vertices of \( L_i \) having degree at least 4 in \( L_i \) (\( i = 1, \ldots, m \)). The proof of Claim 4 is completed by observing that
\[
\tau(G') \leq \frac{|U|}{\omega(G' \setminus U)} = \frac{l + 3m + 1}{2(m + 1)}. \quad \blacksquare
\]
Consider the graph $G'$ with $m = \frac{b}{2} - 1$ and $l = a - \frac{3}{2}b + 2$. Clearly $m = \frac{b}{2} - 1 \geq 1$ and $l = a - \frac{3}{2}b + 2 \geq 2$. By Claim 4, $\tau(G') = \frac{a}{b}$.

**Case 5.2.1.** $\frac{a}{b} \leq \frac{7}{4} - \frac{3}{b}$.

By the hypothesis, $m \geq 2l + 1$, and by Claim 2, $G'$ is not hamiltonian.

**Case 5.2.2.** $\frac{a}{b} > \frac{7}{4} - \frac{3}{b}$.

By choosing a sufficiently large $q$ with

$$\frac{a}{b} = \frac{aq}{bq} < \frac{7}{4} - \frac{3}{b},$$

we can argue as in Case 5.2.1.

**Case 6.** $\frac{7}{4} - \epsilon < \frac{a}{b} \leq 2$.

Let $m = m_1 + m_2 \geq 2l + 1$ and let $G''$ be the graph obtained from $G(L^{(1)}, x, y, l, m)$ by replacing $L_i$ with $L^{(2)}$ ($i = m_1 + 1, m_1 + 2, ..., m$). By Claim 2, $G''$ is not hamiltonian.

**Claim 5.** For $l \geq 2$, $m \geq 1$ and $m_2 \geq l - 2$,

$$\tau(G'') = \frac{l + 3m_2}{2m_2 + 1}.$$

**Proof.** Choose $S \subseteq V(G'')$ such that

$$\omega(G'' \setminus S) > 1, \quad \tau(G'') = |S|/\omega(G'' \setminus S).$$

Obviously, $V(T) \subseteq S$. Define $S_i = S \cap V(L_i)$, $s_i = |S_i|$, and let $\omega_i$ be the number of components of $L_i \setminus S_i$ that contain neither $x_i$ nor $y_i$ ($i = 1, ..., m$). Since $s_i \geq 2\omega_i$ ($i = 1, ..., m_1$) and $s_i \geq \frac{1}{2}\omega_i$ ($i = m_1 + 1, ..., m$), we have

$$\tau(G'') \geq \frac{l + \sum_{i=1}^{m_1} s_i + \sum_{i=m_1+1}^{m} s_i}{c + \sum_{i=1}^{m_1} \omega_i} \geq \frac{l + 2 \sum_{i=1}^{m_1} \omega_i + \frac{3}{2} \sum_{i=m_1+1}^{m} \omega_i}{1 + \sum_{i=1}^{m_1} \omega_i},$$

$$= \frac{l + \frac{1}{2} \sum_{i=1}^{m_1} \omega_i - \frac{3}{2} + \frac{3}{2} (1 + \sum_{i=1}^{m_1} \omega_i)}{1 + \sum_{i=1}^{m_1} \omega_i} = \frac{2l + \sum_{i=1}^{m_1} \omega_i - 3}{2(1 + \sum_{i=1}^{m_1} \omega_i)} + \frac{3}{2},$$

where $c = 0$ if $x_i, y_i \in S$ for all $i \in \{1, ..., m\}$ and $c = 1$ otherwise. Observing also that $\omega_i \leq 2$ ($i = 1, ..., m$), we obtain

$$(2l - 3) \sum_{i=m_1+1}^{m} \omega_i - (2m_2 - 2l + 4) \sum_{i=1}^{m_1} \omega_i \leq 4lm_2 - 6m_2.$$

But this is equivalent to

$$\frac{2l + \sum_{i=1}^{m_1} \omega_i - 3}{2(1 + \sum_{i=1}^{m_1} \omega_i)} + \frac{3}{2} \geq \frac{2l - 3}{2(2m_2 + 1)} + \frac{3}{2}.$$
implying that
\[ \tau(G'') \geq \frac{2l - 3}{2(2m_2 + 1)} + \frac{3}{2} = \frac{l + 3m_2}{2m_2 + 1}. \]

Set \( U = V(T) \cup U_1 \cup \ldots \cup U_m \), where \( U_i \) is the set of vertices of \( L_i \) having degree at least 4 in \( L_i \) \( (i = 1, \ldots, m) \). The proof of Claim 5 is completed by observing that
\[ \tau(G'') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l + 3m_2}{2m_2 + 1}. \]

**Case 6.1.** \( b = 2k + 1 \) for some integer \( k \).
Consider the graph \( G'' \) with \( m_2 = \frac{b - 1}{2} \) and \( l = a - \frac{3}{2}(b - 1) \).

**Case 6.1.1.** \( \frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b} \).
Since \( \frac{a}{b} \leq 2 \), we have
\[ m_2 = \frac{b - 1}{2} \geq a - \frac{3}{2}(b - 1) - 2 = l - 2. \]
Next, since \( \frac{a}{b} \geq \frac{3}{2} + \frac{1}{2b} \), we have \( l = a - \frac{3}{2}(b - 1) \geq 2 \). By Claim 5, \( \tau(G'') = \frac{a}{b} \).

**Case 6.1.2.** \( \frac{a}{b} < \frac{3}{2} + \frac{1}{2b} \).
By choosing a sufficiently large integer \( q \) with
\[ \frac{aq}{bq} = \frac{a}{b} \geq \frac{3}{2} + \frac{1}{2bq}, \]
we can argue as in Case 6.1.

**Case 6.2.** \( b = 2k \) for some integer \( k \).
Consider the graph \( G''' \) obtained from \( G'' \) by replacing \( L_m \) with \( L^{(3)} \).

**Claim 6.** For \( l \geq 2, m \geq 1 \) and \( m_2 \geq l - 2 \),
\[ \tau(G''') = \frac{l + 3m_2 + 1}{2(2m_2 + 1)}. \]

**Proof.** Choose \( S \subseteq V(G''') \) such that
\[ \omega(G''' \setminus S) > 1, \quad \tau(G''') = |S|/\omega(G''' \setminus S) \]
Obviously, \( V(T) \subseteq S \). Define \( S_i = S \cap V(L_i), \ s_i = |S_i|, \) and let \( \omega_i \) be the number of components of \( L_i \setminus S_i \) that contain neither \( x_i \) nor \( y_i \) \( (i = 1, \ldots, m) \). Since \( s_i \geq 2\omega_i \) \( (i = 1, \ldots, m_1) \), \( s_i \geq \frac{3}{2}\omega_i \) \( (i = m_1 + 1, \ldots, m - 1) \) and \( s_m \geq \frac{4}{3}\omega_m \), we have
\[ \tau(G''') \geq \frac{l + \sum_{i=m_1+1}^{m-1} s_i + s_m}{c + \sum_{i=1}^{m} \omega_i} \]
\[ \geq \frac{l + 2 \sum_{i=1}^{m_1} \omega_i + \frac{3}{2} \sum_{i=m_1+1}^{m-1} \omega_i + \frac{4}{3}\omega_m}{1 + \sum_{i=1}^{m} \omega_i} \]

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By Claim 1, \( \omega \) is the set of vertices of \( G \) also that \( \omega_i \leq 2 \) for all \( i \in \{1, \ldots, m\} \) and \( c = 1 \) otherwise. Observing also that \( \omega_i \leq 2 \) for all \( i \in \{1, \ldots, m\} \) and \( \omega_m \leq 3 \), we obtain

\[
(l - 2) \sum_{i=m_1+1}^{m} \omega_i + \frac{1}{3}(m_2 + 1)\omega_m - (m_2 - l + 3)\sum_{i=1}^{m_1} \omega_i \leq l + 2lm_2 + 2.
\]

But this is equivalent to

\[
\frac{l + \frac{1}{3}\sum_{i=m_1}^{m} \omega_i - \frac{1}{6}\omega_m}{1 + \sum_{i=1}^{m} \omega_i} + \frac{3}{2} \geq \frac{l - 2}{2(m_2 + 1)} \cdot \frac{3}{2},
\]

implying that

\[
\tau(G'') \geq \frac{l - 2}{2(m_2 + 1)} + \frac{3}{2} = \frac{l + 3m_2 + 1}{2(m_2 + 1)}.
\]

Set \( U = V(T) \cup U_1 \cup \ldots \cup U_{m_i} \), where \( U_i \) is the set of vertices of \( L_i \) having degree at least 4 in \( L_i \) (\( i = 1, \ldots, m \)). The proof of Claim 6 is completed by observing that

\[
\tau(G'') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l + 3m_2 + 1}{2(m_2 + 1)}.
\]

Consider the graph \( G''' \) with \( m_2 = \frac{b}{2} - 1 \) and \( l = a - \frac{3}{2}b + 2 \).

**Case 6.2.1.** \( \frac{a}{b} \leq 2 - \frac{1}{b} \).

By the hypothesis, \( m_2 = \frac{b}{2} - 1 \geq (a - \frac{3}{2}b + 2) - 2 = l - 2 \). Next, since \( \frac{a}{b} > \frac{3}{4} - \epsilon > \frac{3}{2} \), we have \( l = \frac{3}{2}b + 2 \geq 2 \). By Claim 6, \( \tau(G''') = \frac{a}{b} \).

**Case 6.2.2.** \( \frac{a}{b} > 2 - \frac{1}{b} \).

By choosing a sufficiently large integer \( q \) with \( \frac{a}{b} = \frac{aq}{bq} \leq 2 - \frac{1}{bq} \), we can argue as in Case 6.2.1.

**Case 7.** \( 2 < \frac{a}{b} < \frac{q}{q} \).

**Case 7.1.** \( b = 2k + 1 \) for some integer \( k \).

**Case 7.1.1.** \( \frac{a}{b} \leq \frac{2}{q} - \frac{1}{4b} \).

Take the graph \( G(L^{(1)}, y, a - 2b + 2, \frac{b - 1}{2}) \). Since \( \frac{a}{b} > 2 \), we have \( l = a - 2b + 2 \). Next, the hypothesis \( \frac{a}{b} \leq \frac{a}{q} - \frac{1}{4b} \) is equivalent to

\[
m = \frac{b - 1}{2} \geq 2(a - 2b + 2) + 1 = 2l + 1.
\]

By Claim 1, \( G(L^{(1)}, x, y, a - 2b + 2, \frac{b - 1}{2}) \) is not hamiltonian. The toughness \( \tau(G(L^{(1)}, x, y, a - 2b + 2, \frac{b - 1}{2})) \) can be determined exactly as in proof of Theorem A [2],

\[
\tau(G(L^{(1)}, x, y, a - 2b + 2, \frac{b - 1}{2})) \geq \frac{l + 4m}{2m + 1} = \frac{a}{b}.
\]
Case 7.1.2. \( \frac{a}{b} > \frac{9}{4} - \frac{11}{3b} \).

By choosing a sufficiently large integer \( q \) with

\[
\frac{aq}{bq} = \frac{a}{b} \leq \frac{9}{4} - \frac{11}{4bq},
\]

we can argue as in Case 7.1.1.

Case 7.2. \( b = 2k \) for some positive integer \( k \).

Take the graph \( G''' \) obtained from \( G(L(1), x, y, a−2b+2, b^2) \) by replacing \( L_m \) with \( L(4) \). Since \( \frac{a}{b} > 2 \), we have \( l = a−2b+2 > 2 \). We have also \( m = \frac{b}{2} > 1 \), since \( b \geq 3 \).

Claim 7. For \( l \geq 2 \) and \( m \geq 1 \),

\[
\tau(G''') = \frac{l + 4m - 2}{2m}.
\]

Proof. Choose \( S \subseteq V(G''') \) such that

\[
\omega(G''' \setminus S) > 1, \quad \tau(G''') = |S|/\omega(G''' \setminus S)
\]

Obviously, \( V(T) \subseteq S \). Define \( S_i = S \cap V(L_i) \), \( s_i = |S_i| \), and let \( \omega_i \) be the number of components of \( L_i \setminus S_i \) that contain neither \( x_i \) nor \( y_i \) \( (i = 1, ..., m) \). Since \( s_i \geq 2\omega_i \), we have \( \omega_i \leq 2 \) \( (i = 1, ..., m−1) \) and \( \omega_m \leq 1 \), we have

\[
\tau(G''') = \frac{l + \sum_{i=1}^{m} s_i}{c + \sum_{i=1}^{m} \omega_i} \geq \frac{l + 2 \sum_{i=1}^{m} \omega_i}{1 + \sum_{i=1}^{m} \omega_i} = \frac{l + 4m - 2}{2m}.
\]

where \( c = 0 \) if \( x_i, y_i \in S \) for all \( i \in \{1, ..., m\} \) and \( c = 1 \) otherwise. Set \( U = V(T) \cup U_1 \cup ... \cup U_m \), where \( U_i \) is the set of vertices of \( L_i \) having degree at least 4 in \( L_i \) \( (i = 1, ..., m) \). The proof of Claim 7 is completed by observing that

\[
\tau(G''') \leq \frac{|U|}{\omega(G \setminus U)} = \frac{l + 4m - 2}{2m}.
\]

Case 7.2.1. \( \frac{a}{b} < \frac{9}{4} - \frac{3}{b} \).

By the hypothesis,

\[
m - 1 = \frac{b}{2} - 1 \geq 2(a−2b+2) + 1 = 2l + 1.
\]

By Claim 2, \( G''' \) is not hamiltonian. By Claim 7, \( \tau(G''') = \frac{a}{b} \).

Case 7.2.2. \( \frac{a}{b} > \frac{9}{4} - \frac{3}{b} \).

By choosing a sufficiently large integer \( q \) with

\[
\frac{aq}{bq} = \frac{a}{b} \leq \frac{9}{4} - \frac{3}{3bq},
\]

we can argue as in Case 7.2.1. Theorem 1 is proved.
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