Classical integrability in the BTZ black hole

Justin R. David and Abhishake Sadhukhan

Centre for High Energy Physics,
Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India.

justin@cts.iisc.ernet.in
abhishake@cts.iisc.ernet.in

Abstract: Using the fact the BTZ black hole is a quotient of $AdS_3$ we show that classical string propagation in the BTZ background is integrable. We construct the flat connection and its monodromy matrix which generates the non-local charges. From examining the general behaviour of the eigen values of the monodromy matrix we determine the set of integral equations which constrain them. These equations imply that each classical solution is characterized by a density function in the complex plane. For classical solutions which correspond to geodesics and winding strings we solve for the eigen values of the monodromy matrix explicitly and show that geodesics correspond to zero density in the complex plane. We solve the integral equations for BMN and magnon like solutions and obtain their dispersion relation. Finally we show that the set of integral equations which constrain the eigen values of the monodromy matrix can be identified with the continuum limit of the Bethe equations of a twisted $SL(2, R)$ spin chain at one loop.
1. Introduction

Studying the behaviour of probes in black holes backgrounds have always revealed useful information about the nature of black holes. The geometry of the black hole is usually understood by studying particle motion or geodesics. The spectrum of scalars, vectors, graviton modes in the black hole background have also revealed useful information. For example for the case of black holes which asymptotes to anti-de Sitter space, the study of the first quasi-normal modes has revealed information about the transport properties of the dual field theory [1]. In certain cases it is also possible to study string propagation in black holes exactly. These arise when the black hole background is an exact conformal field theory. A well studied example of such a background is the case of the 2d black hole found in [2, 3]. String propagation and its implications in this background were studied in [4].

Instances of black hole solutions in which string propagation can be exactly solved are rare in higher dimensions. 3 dimensional Anti-de Sitter gravity can be formulated as a Chern-Simons theory and is topological. One might expect that
spectrum of various excitations about black holes in this theory might be exactly solvable. Indeed the quasi-normal modes for scalars, spinors and vectors in the BTZ black hole has been obtained analytically in [5]. In fact the BTZ black hole background along with the WZW term in the sigma model is an exact conformal field theory. The spectrum of strings in this background has been studied in the Euclidean as well as the Lorentzian case in [3, 4, 5]. Thus apart from black holes in 2d gravity, the BTZ black hole offers another background in which the spectrum of strings can perhaps be understood analytically. In this paper we study the properties of classical strings in the Lorentzian BTZ background without the WZW term and observe that classical propagation of strings in this background is integrable. One can construct an infinite set of non-local conserved charges. These backgrounds are quite generic in string theory, they occur in the near horizon limit of the D1-D5 black hole. It is possible that this structure could help in understanding the spectrum of strings in this background just as in the case of $AdS_5 \times S^5$, see [9, 10] for a comprehensive review.

$AdS_3$ is a $SL(2,R)$ group manifold, string propagation in this background is classically integrable. One can easily show that there exists a flat connection whose monodromy can be used to construct a generating function of the infinite set of non-local charges [11, 12]. Recently integrability along with symmetries of $AdS_3 \times S^3$ has been used to obtain both the dispersion relation as well as the S-matrix of magnons for the D1-D5 system [13, 14, 15]. The BTZ black hole can be obtained as a quotient of $AdS_3$ [16]. Since flatness of a connection is a local concept, the flat connection constructed for the case of $AdS_3$ can be used for the case of the BTZ black hole. However in the construction of the charges or the monodromy one has to impose the quotienting which makes it a BTZ black hole. The property that the sigma model admits a flat connection is probably not true for black holes in higher dimensional Anti-de Sitter spaces. In fact recently it has been shown that string world sheet theory on the $AdS_5$ Schwarzschild black hole, $AdS_5$ soliton as well as the on $AdS_5 \times T^{1,1}$ exhibit chaos [18, 19]. In this respect integrability of the world sheet theory for the BTZ black hole seems to be an exception.

As a warm up exercise in studying integrability of the world sheet theory in the BTZ background, we first examine the sigma model on Lens space. Lens space is obtained as a quotient of $S^3$, since the case of sigma model on $S^3 \times R$ has been studied in detail in [20] it is a good starting point. Using the existence of the flat connection of the sigma model on $S^3 \times R$, we construct the corresponding monodromy matrix for the Lens space implementing the required quotient. We then study the properties of the quasi-momentum constructed from the monodromy matrix in the spectral plane and write down the set of integral equations which constrain the quasi-momentum. To cross check our calculations we compare the solutions obtained from examining these integral equations with an explicit solution of the sigma model and obtain agreement. We then show that the integral equations which determine the quasi-
momentum are the agree with the continuum limit of the Bethe equations of the BDS long-range spin chain [21] with an appropriate twist up to two loops. Quotients of $S^3$ were studied earlier [22] in the context of spin chain description for orbifolds of $\mathcal{N} = 4$ super Yang-Mills but as far as we are aware the twisted version of the integral equations for the sigma model on Lens space has not been obtained before.

We then approach the main topic of the paper. Using the flat connection inherited from $AdS_3$ we implement the quotient required to obtain the monodromy matrix and the quasi-momentum for classical solutions of the sigma model in the $BTZ \times S^1$ background. In the main text we restrict ourselves to the non-extremal BTZ black hole. Most string backgrounds which contain a BTZ black hole also contains at least a single $S^1$ thus we do not loose generality. The trace of the monodromy matrix is the generating function of the non-local conserved charges of the world sheet theory.

We then study the properties of the quasi-momentum on the spectral plane and obtain the integral equations which determine the quasi-momentum. This allows the classification of all solutions of the sigma model on the BTZ black hole in terms of the behaviour of a density defined on the spectral plane. We then verify these equations using two explicit class of solutions, geodesics and winding strings. We solve for the quasi-momentum of these solutions explicitly and show that it is in agreement with the properties of the quasi-momentum determined from general considerations. We see that geodesics correspond to zero density in the spectral plane. Thus a classical solution can be described by a density function on the spectral plane which satisfies a set of integral equations. This feature is reminiscent of the identification of the $1/2$ BPS solutions in $AdS_5 \times S^5$ with a density on the complex plane found by [23].

We then solve for these equations for two cases: the case of a density distribution which are localized delta functions and the situation in which the density is uniform and localized on a line. These two situations correspond to plane wave solutions and magnon solutions for the case of $S^3$. Using this method we obtain dispersion relations satisfied by the plane wave like solution and the magnon like solutions in the BTZ background. Finally we show that the integral equations satisfied by the quasi-momentum are same as the Bethe equations of a twisted version of the long range $SL(2, R)$ spin chain at one loop. In appendix A we study the behaviour of the quasi-momentum for the case of the extremal BTZ black hole.

The organization of the paper is as follows: In the next section we study the case of the Lens space as a warm up exercise. Section 3 contains the main results of this paper. In this section we show that the world sheet theory of the sigma model on $BTZ \times S^1$ is integrable and obtain the generator of the non-local charges. We obtain the general properties of the quasi-momentum and verify this by explicitly solving for the quasi-momentum for two classes of solutions. We recast the equations satisfied by the quasi-momentum in terms of a density and show that to the leading order these are same as that of the Bethe equations of a twisted version of the $SL(2, R)$ spin chain. We also obtain the plane wave and magnon like dispersion relations.
for the case of localized delta function density and constant density localized on a segment respectively. Section 4 contains our conclusions. Appendix A discusses the properties of the quasi-momentum for the case of the extremal BTZ black hole. Appendix B has details of the two classes of solutions for which we explicitly obtain the quasi-momentum.

2. Warm up example: Lens space

To set our notations and conventions we first review the construction of the flat connection and the monodromy matrix for the case of $S^3$. The string sigma model on $R \times S^3$ can be written as

$$S = -\frac{\sqrt{\lambda}}{4\pi} \int d\sigma d\tau \left( \text{Tr} \frac{1}{2} (g^{-1} \partial_a gg^{-1} \partial^a g) + \partial_a X_0 \partial^a X_0 \right), \quad (2.1)$$

where $a = 0, 1 = \tau, \sigma$. The light cone coordinates are defined as

$$\sigma_{\pm} = \frac{1}{2} (\tau \pm \sigma), \quad \partial_{\pm} = \partial_{\tau} \pm \partial_{\sigma}. \quad (2.2)$$

$g$ refers to the $SU(2)$ group element which we parametrize as

$$g = \begin{pmatrix} \cos \theta e^{-i\phi} & i \sin \theta e^{i\psi} \\ i \sin \theta e^{-i\psi} & \cos \theta e^{i\phi} \end{pmatrix}, \quad (2.3)$$

and $X_0$ is the time coordinate. Note that in this parametrization the sigma model action in $(2.1)$ reduces to the action in the following metric of $S^3$

$$ds^2 = d\theta^2 + \cos^2 \theta d\phi^2 + \sin^2 \theta d\psi^2. \quad (2.4)$$

It is convenient to introduce the currents

$$j_a = g^{-1} \partial_a g, \quad (2.5)$$

then the equations of motion are given by

$$\partial_+ j_- + \partial_- j_+ = 0. \quad (2.6)$$

Using the definition of the current given in $(2.3)$ it is easy to show that the following equation also holds

$$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0. \quad (2.7)$$

We can now construct the following connection

$$J_\pm (x) = \frac{j_\pm}{1 \mp x}. \quad (2.8)$$
where $x$ is a complex number. Using the equations of motion (2.6) and (2.7) it can be shown that the above connection satisfies the flatness condition

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0. \tag{2.9}$$

The monodromy matrix is constructed from this flat connection as follows.

$$\Omega(x) = P \exp \left( - \int_0^{2\pi} d\sigma J_\sigma \right), \tag{2.10}$$

$$= P \exp \left( - \frac{1}{2} \int_0^{2\pi} d\sigma \left[ \frac{j_+}{1-x} - \frac{j_-}{1+x} \right] \right).$$

The above path ordered integral is performed at a constant world sheet time. To obtain constants of motion, we need to take the trace of the monodromy matrix. The reason is as follows: consider the world sheet time derivative of the monodromy matrix we obtain

$$\partial_\tau \Omega(x) = - \int_0^{2\pi} d\sigma \Omega(2\pi, \sigma) \partial_\tau J_\sigma \Omega(\sigma, 0), \tag{2.11}$$

$$= - \int_0^{2\pi} \Omega(2\pi, \sigma) (\partial_\sigma J_\tau - J_\tau J_\sigma + J_\sigma J_\tau) |_{\sigma=0} \Omega(\sigma, 0),$$

$$= -(J_\tau(2\pi) \Omega(2\pi, 0) - \Omega(2\pi, 0) J_\tau(0)).$$

where

$$\Omega(\sigma, \sigma') = P \exp \left( - \int_{\sigma'}^{\sigma} d\sigma J_\sigma (\hat{\sigma}) \right). \tag{2.12}$$

In the above equations we have used the flatness condition of $J$. In proceeding from the second line to the third line in (2.11) we have integrated by parts and used the equations

$$\partial_\sigma \Omega(\sigma, \sigma') = - J_\sigma \Omega(\sigma, \sigma'), \quad \partial_{\sigma'} \Omega(\sigma, \sigma') = \Omega(\sigma, \sigma') J_{\sigma'}. \tag{2.13}$$

Therefore from the last line of (2.11) we see that if the connection $J_\tau$ is periodic in $\sigma$ with period $2\pi$ we find the trace of the monodromy matrix

$$\text{Tr}(\Omega(x)), \tag{2.14}$$

is a constant of motion. For the case of $S^3$, since it is the $SU(2)$ group manifold, the eigen values of the monodromy matrix will be of the form $\{e^{ip(x)}, e^{-ip(x)}\}$. Thus we have

$$\text{Tr}(\Omega(x)) = 2 \cos p(x), \tag{2.15}$$

where $p(x)$ is called the quasi-momentum and $x$ the spectral parameter.

We can now proceed to obtain the monodromy matrix for the sigma model on Lens space. Lens space is obtained from $S^3$ with the following identifications

$$\phi \sim \phi + \frac{2\pi}{k_1}, \quad \psi \sim \psi + \frac{2\pi}{k_2} \tag{2.16}$$
Under this identification the $SU(2)$ group element $g$ given in (2.3) is identified as 

$$ g \sim \tilde{A}_{(1,1)}gA_{(1,1)}, $$

where $\tilde{A}_{(n_1,n_2)} = 
\begin{pmatrix}
  e^{-i\pi(n_1k_1 - \frac{n_2}{2})} & 0 \\
  0 & e^{i\pi(n_1k_1 - \frac{n_2}{2})}
\end{pmatrix}$,

and $A_{(n_1,n_2)} = 
\begin{pmatrix}
  e^{-i\pi(n_1k_1 + \frac{n_2}{2})} & 0 \\
  0 & e^{i\pi(n_1k_1 + \frac{n_2}{2})}
\end{pmatrix}$.

From the above equation it is clear that the general boundary conditions for the currents $g^{-1}\partial g$ is given by 

$$ g^{-1}\partial g(\tau, 2\pi) = A_{(n_1,n_2)}^{-1}g^{-1}\partial g(\tau, 0)A_{(n_1,n_2)}, $$

$(n_1,n_2)$ can be thought of as winding numbers. From the expression for the flat connection in terms of the currents given in (2.8) we see that the above boundary conditions induce the same periodicity conditions on the flat connection. We write this down explicitly for future reference

$$ J_{\pm}(\tau, 2\pi) = A_{(n_1,n_2)}^{-1}J_{\pm}(\tau, 0)A_{(n_1,n_2)}. $$

To construct the monodromy matrix we need to consider the following

$$ \hat{\Omega}_{(n_1,n_2)}(x) = A_{(n_1,n_2)}\Omega(x), $$

$$ = A_{(n_1,n_2)}P\exp\left(-\int_{0}^{2\pi} d\sigma J_{\sigma}\right), $$

$$ = A_{(n_1,n_2)}P\exp\left(-\int_{0}^{2\pi} d\sigma \frac{1}{2} \left[ \frac{j_{+}}{1 - x} - \frac{j_{-}}{1 + x} \right] \right), $$

where $\Omega(x)$ is defined in (2.10). Going through the same steps as for the case of $S^3$, it is easy to demonstrate that the $\text{Tr}(\hat{\Omega}_{(n_1,n_2)}(x))$ is independent of worldsheet time. The quasi-momentum in this case can be obtained by

$$ 2\cos p(x) = \text{Tr}(\hat{\Omega}(x)). $$

To proceed further let us define the conserved charges as follows: Let us expand the currents in terms of Pauli matrices as

$$ j_a = g^{-1}\partial_a g = \frac{1}{2i}\sigma^a j^a, $$

where $\sigma^a$ are the Pauli matrices. Similarly the left currents are

$$ l_a = gj_aj^{-1} = \partial_ag^{-1} = \frac{l^a}{2i} \sigma^a. $$
The conserved charges corresponding to the $\sigma^3$ component of these currents are given by

$$Q_R^3 = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma j_0^3 = L - 2J, \quad (2.24)$$

$$Q_L^3 = \frac{\sqrt{\lambda}}{4\pi} \int d\sigma i_0^3 = L. \quad (2.25)$$

The energy of the string solution which is generated by global time translations is given by

$$\Delta = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma \partial_\tau X_0 = \sqrt{\lambda} \kappa. \quad (2.26)$$

In defining these charges we have used the notations of [20]. Finally the Virasoro constraints are given by

$$\text{Tr}(j^2_+ - j^2) = -2\kappa^2. \quad (2.27)$$

### 2.1 Properties of the quasi-momentum

We now study the properties of the quasi-momentum in the complex $x$-plane which is called the spectral plane. From these properties we show that all classical solutions of the sigma model can be characterized in terms of a density on the spectral plane which satisfies certain integral equations.

The quasi-momentum $p(x)$ has poles at $x \neq 1$. This is evident from the same asymptotic analysis as done in [24]. The quotienting by the lens space does not affect this behaviour. This is due to the fact that this behaviour can be obtained by an analysis for which the dependence of the worldsheet $\sigma$ coordinate can be ignored.

From this analysis it can be shown that the quasi-momentum has poles at $x \rightarrow \pm 1$ with the residues given by

$$p(x) = -\frac{\pi \kappa}{x \pm 1} + \ldots, \quad (x \rightarrow \mp 1) \quad (2.28)$$

We now expand the quasi-momentum at zero and at infinity. As $x \rightarrow \infty$, we obtain

$$J_\sigma = -\frac{j_0}{x} - \frac{j_1}{x^2} + \ldots. \quad (2.29)$$

Substituting this expansion in the expression for the quasi-momentum we obtain

$$2 \cos p(x) = \text{Tr}[A_{n_1,n_2} P \exp \int_0^{2\pi} d\sigma (\frac{j_0}{x} + \frac{j_1}{x^2})] \quad (2.30)$$

$$= \text{Tr} \left\{ \left[ \cos \pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) - i \sin \pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) \sigma_3 \right] \right\} \times \left[ 1 - \frac{i \sigma_i}{2x} \int_0^{2\pi} d\sigma j_0^i - \frac{i \sigma_i}{2x^2} \int_0^{2\pi} d\sigma j_1^i - \frac{\sigma_i \sigma_j}{4x^2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' j_0^i(\sigma) j_0^j(\sigma') \right]$$

$$= 2 \cos \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) - \frac{1}{x} \sin \pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) \int_0^{2\pi} d\sigma j_0^3 - \frac{\hat{C}}{x^2} + \ldots,$$

$^1$We will repeat this analysis for the case of the BTZ background in section 3.
where
\[
\hat{C} = \frac{1}{2} \cos \pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' j_0^1(\sigma) j_0^1(\sigma') \int_0^{2\pi} d\sigma' j_0^2(\sigma) j_0^2(\sigma') 
+ \frac{1}{2} \sin \pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) \left[ \int_0^{2\pi} d\sigma j_1^3 + \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' j_0^1(\sigma) j_0^1(\sigma') - \int_0^{2\pi} d\sigma \int_0^{\sigma} d\sigma' j_0^2(\sigma) j_0^2(\sigma') \right].
\] (2.31)

In the above equation we have retained terms to $O(1/x^2)$ to show the first non-local charge explicitly. The leading terms in the expansion as $x \to \infty$ can be obtained from the second equality in (2.30). Thus the quasi-momentum as $x \to \infty$ is given by
\[
p(x) = -\pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) - \frac{2\pi(L - 2J)}{\sqrt{\lambda x}} + ..., \quad (x \to \infty),
\] (2.32)
where we have used (2.24). We now examine the behaviour of the quasi-momentum as $x \to 0$. To do this we can use the following relation
\[
\partial_{\sigma} + j_1 + x j_0 + \cdots = g^{-1} (\partial_{\sigma} + x l_0 + \cdots) g.
\] (2.33)
in the equation for the monodromy matrix to obtain
\[
\hat{\Omega}(x) = A_{n_1,n_1} g^{-1} (2\pi) P (\exp -x \int_0^{2\pi} d\sigma l_0 + ...) g(0).
\] (2.34)
This gives the following expression for the quasi-momentum
\[
2 \cos p(x) = \text{Tr} \left\{ A_{n_1,n_1}^{-1} P \exp \left( -x \int_0^{2\pi} d\sigma l_0 + \cdots \right) \right\},
\] (2.35)
where we have used the equation given in (2.18). Proceeding as before and substituting for the charge using (2.25) we obtain
\[
p(x) = 2\pi m + \pi \left( \frac{n_1}{k_1} - \frac{n_2}{k_2} \right) + \frac{2\pi L}{\sqrt{\lambda}} x + \cdots, \quad (x \to 0).
\] (2.36)
Finally the quasi-momentum satisfies the following jump condition above and below the branch cuts
\[
p(x + i\epsilon) + p(x - i\epsilon) = 2\pi n_k,
\] (2.37)
where $k$ labels the branch cuts and $n_k$ is an integer. The $i\epsilon$ is used to denote the value of the quasi-momentum above and below the branch cut. This condition arises because the monodromy matrix is an unimodular matrix and the quasi-momentum above and below the branch cuts correspond to the two different eigen values. The condition given in (2.37) just ensures the unimodularity of the monodromy matrix.
We will now follow the procedure discussed in [20] to show that given the quasi-momentum satisfying the properties discussed above one can define a density on the spectral plane \( x \), which satisfies a set of integral equations. Thus each classical solution of the sigma model on \( S^3 \) corresponds to a density defined on the spectral plane since each classical solution defines a quasi-momentum. To show this we first define the resolvent given by

\[
G(x) = p(x) + \frac{\pi \kappa}{x - 1} + \frac{\pi \kappa}{x + 1} + \pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right).
\] (2.38)

The above resolvent is written by removing the singularities at \( x \to \pm 1 \). We also subtract \(-\pi \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right)\) so as to make \( G(x) \sim \frac{1}{x} + \cdots \), as \( x \to \infty \). This ensures that \( G(x) \) does not have a pole at \( \infty \). Now since \( G(x) \) is an analytic function without any poles, by using standard complex analysis it can be represented by an integral of density by

\[
G(x) = \int d\xi \frac{\rho(\xi)}{x - \xi},
\] (2.39)

where the integral is along the branch cuts and \( \rho(x) \) is given by

\[
\rho(x) = \frac{1}{2\pi i} \left( G(x + i\epsilon) - G(x - i\epsilon) \right).
\] (2.40)

The shift in \( \epsilon \) is to denote the value of the resolvent above and below the branch cuts. From the asymptotic behaviour of the resolvent at \( x \to \infty \) given in (2.32) we obtain the following normalization conditions on the density

\[
\int dx \rho(x) = \frac{2\pi}{\sqrt{\lambda}} (\Delta + 2J - L).
\] (2.41)

Also from the behaviour of the resolvent at \( x \to 0 \) as given in the equation (2.36) we see that

\[
G(x) = 2\pi (m + \frac{n_1}{k_1}) + \frac{2\pi}{\sqrt{\lambda}} (L - \Delta)x + O(x^2),
\] (2.42)

we get the following equations:

\[
\frac{-1}{2\pi i} \int \frac{G(x) dx}{x} = \int dx \frac{\rho(x)}{x} = -2\pi (m + \frac{n_1}{k_1}),
\] (2.43)

\[
\frac{-1}{2\pi i} \int \frac{G(x) dx}{x^2} = \int dx \frac{\rho(x)}{x^2} = \frac{2\pi (\Delta - L)}{\sqrt{\lambda}}.
\] (2.44)

Finally the unimodular condition given in (2.37) can be be recast in the following integral equation for the density \( \rho(x) \).

\[
G(x + i\epsilon) + G(x - i\epsilon) = 2 \int d\xi \frac{\rho(\xi)}{x - \xi},
\] (2.45)

\[
= \frac{2\pi \kappa}{x - 1} + \frac{2\pi \kappa}{x + 1} + 2\pi (n_k + \frac{n_1}{k_1} + \frac{n_2}{k_2}).
\]
The integration $\int$ means that the integration in the complex plane has been done by excluding and moving the contour around the poles. The integral equations (2.41), (2.43), (2.44) and (2.45) are the conditions satisfied by the density $\rho(x)$. Thus we have shown that given a classical solution, it defines a density on the spectral plane satisfying certain integral equations.

### 2.2 The rotating string

In this section we write down the classical solution for the rotating string and the dispersion relation satisfied by this solution. We then obtain the resolvent for this solution using the condition discussed in the previous subsection and show that the dispersion relation obtained from the resolvent agrees with that obtained from the explicit solution. Thus this comparison serves as a check on the equations satisfied by the density obtained in the previous subsection. A similar analysis for the rotating string in $S^3$ was performed in [20].

To write down the solution corresponding to the rotating string it is convenient to parametrize the group element $g$ as the following

$$
g = \begin{pmatrix} X_1 + iX_2 & i(X_3 + iX_4) \\
i(X_3 - iX_4) & X_1 - iX_2 \end{pmatrix},
$$

(2.46)

together with the constraint

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 = 1.
$$

(2.47)

The ansatz for the rotating string is given by

$$X_1 + iX_2 = \cos \frac{\theta_0}{2} e^{-iw_1\tau + \frac{m_1'}{k_1} \sigma}, \quad X_3 + iX_4 = \sin \frac{\theta_0}{2} e^{-iw_2\tau + \frac{m_2'}{k_2} \sigma},
$$

(2.48)

$$t = \kappa \tau,$$

where $m_1'$, $m_2'$ are integers. Note that this ansatz satisfies the twisted boundary condition

$$g(\tau, \sigma + 2\pi) = \tilde{A}(-m_1', m_2') g(\tau, \sigma) A(-m_1', m_2').
$$

(2.49)

The equations of motion reduce to the following algebraic equation

$$w_1^2 - \left( \frac{m_1'}{k_1} \right)^2 = w_2^2 - \left( \frac{m_2'}{k_1} \right)^2,
$$

(2.50)

while the Virasoro constraints reduce to

$$\kappa^2 = \left( w_1^2 + \left( \frac{m_1'}{k_1} \right)^2 \right) \cos^2 \frac{\theta_0}{2} + \left( w_2^2 + \left( \frac{m_2'}{k_2} \right)^2 \right) \sin^2 \frac{\theta_0}{2} = \kappa^2,
$$

(2.51)

$$\frac{m_1'}{k_1} \cos \frac{\theta_0}{2} + \frac{m_2'}{k_2} \sin \frac{\theta_0}{2} = 0.
$$

(2.52)
The energy $\Delta$ and the angular momentum corresponding to rotation in the $X_1 - X_2$ and $X_3 - X_4$ plane is given by

$$\Delta = \sqrt{\lambda \kappa}, \quad J_1 = \sqrt{\lambda} \cos^2 \frac{\theta_0}{2} w_1, \quad J_2 = \sqrt{\lambda} \sin^2 \frac{\theta_0}{2} w_2. \quad (2.53)$$

$J_1$ and $J_2$ are related to $Q^3_R$, $Q^3_L$ defined in (2.23), (2.24) by the following relations

$$Q^3_R = J_1 - J_2 = L - 2J, \quad Q^3_L = J_1 + J_2 = L. \quad (2.54)$$

Now using the equations of motion (2.50) and the second equation in (2.51) we can eliminate $\omega_2$ to obtain the following equations

$$w_1^2 + \left( \frac{m'_2}{k_2} \right)^2 - \left( \frac{m'_1}{k_1} \right)^2 \left( J_1 - \sqrt{\lambda} w_1 \right)^2 - \left( J_2 w_1 \right)^2 = 0, \quad (2.55)$$

$$J_1 \frac{m'_1}{k_1} + J_2 \frac{m'_2}{k_2} = 0.$$

The last line is just a rewriting of the Virasoro constraints using the definition of $J_1$ and $J_2$. Now a solution to these equations is the following

$$\frac{m'_1}{k_1} = -\frac{m'_2}{k_2}. \quad (2.56)$$

For this solution we can derive a dispersion relation by using the first Virasoro constraint in (2.51). This results in

$$\Delta^2 = (J_1 + J_2)^2 + \lambda \left| \frac{m'_1 m'_2}{k_1 k_2} \right|, \quad (2.57)$$

Now for large $L = J_1 + J_2$ and for

$$\frac{m'_1}{k_1} = -\frac{m'_2}{k_2} + O(1/L), \quad (2.58)$$

then the dispersion relation is basically given by the leading approximation of (2.57). Thus we obtain

$$\Delta = L + \frac{\lambda}{2L} \left| \frac{m'_1 m'_2}{k_1 k_2} \right| + O(1/L^2). \quad (2.59)$$

This relation was obtained for the twisted rotating string earlier by [22].

We will now obtain the dispersion relation given in (2.59) by constructing the resolvent which solves the equations (2.41), (2.43), (2.44) and (2.45). Following the analysis in [20] we rescale $x$ by $x \rightarrow \frac{x}{4\kappa}$ in the equations determining the resolvent. Thus they reduce to

$$\int dx \rho(x) = \frac{J}{\Delta} + \frac{\Delta - L}{2\Delta}. \quad (2.60)$$
\[-1 \frac{2\pi i}{2\pi i} \int \frac{G(x)dx}{x} = -2\pi (m + \frac{n_1}{k_1}),\]
\[-\lambda \frac{8\pi^2 \Delta}{2\pi i x^2} \int \frac{G(x)dx}{2\pi i x^2} = \Delta - L,\]

\[G(x + i0) + G(x - i0) = 2 \int d\xi \frac{\rho(\xi)}{x - \xi},\]

\[= \frac{1}{2} \left( \frac{1}{x - 1} + \frac{1}{x + 1} \right) + 2\pi \left( n + \frac{m_1}{k_1} + \frac{m_2}{k_2} \right).\]

Note that these equations are the same as the ones obtained by [20] but with the following replacements \(^2\)

\[n \to -(n + \frac{n_1}{k_1} + \frac{n_2}{k_2}), \quad m \to -(m + \frac{n_1}{k_1}). \quad (2.61)\]

where the variables on the left hand side refers to the variables used in [20].

The form of the resolvent which satisfies these is given by

\[G(x) = \frac{1}{4} \left( \frac{1}{x - a} + \frac{1}{x + a} \right) + \frac{1}{4} \left( \frac{(1 + \epsilon)^{-1/2}}{x - a} + \frac{(1 - \epsilon)^{-1/2}}{x + a} \right) \sqrt{Ax^2 + Bx + C} \]

\[+\pi \left( n + \frac{m_1}{k_1} + \frac{m_2}{k_2} \right). \quad (2.62)\]

Following the same steps as in [20] it is possible to solve for the constants \(A, B, C, \epsilon\) and obtain the dispersion relation

\[\Delta = L + \frac{\lambda}{2L} \left| (n - m + \frac{n_2}{k_2})(m + \frac{n_1}{k_1}) \right| + ... \quad (2.63)\]

We can now compare with the equation (2.53) and see that the anomalous dimension has the same form. That is it is a product of a fractions of \(m'_1/k_1\) and \(m'_2/k_2\). This agreement of obtaining the dispersion relation from the actual solution as well as from the resolvent provides a check on the equations (2.41), (2.43), (2.44) and (2.45) which determine the density corresponding to each classical solution in the Lens space.

2.3 The relation with the \(SU(2)\) spin chain

In this section we show that the equations which determine the density can be obtained from the continuum limit of the Bethe equations of a twisted version of the long range BDS spin chain introduced by [21]. The twisted spin chain we consider satisfies the following Bethe equations

\[\left( \frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})} \right)^L \exp \left[ 2\pi i \left( \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) \right] = \prod_{j=1, j\neq k}^J \frac{u_k - u_j + i}{u_k - u_j - i}. \quad (2.64)\]

\(^2\)See equations (5.39) in that paper.
where \( x \) as a function of \( u \) is given by

\[
x(u) = \frac{u}{2} + \frac{u}{2}\sqrt{1 - \frac{2g^2}{u^2}}, \quad u(x) = x + \frac{g^2}{2x}.
\]  
(2.65)

The \( x \)'s are the Bethe roots and related to the momentum of the spin wave excitations of the chain. The cyclicity constraint is given by

\[
\prod_{k=1}^{J} \left( \frac{x(u_k + \frac{i}{2})}{x(u_k - \frac{i}{2})} \right) = \exp \left( -2\pi i (m + \frac{n_2}{k_2}) \right),
\]  
(2.66)

and the energy of the spin chain is given by

\[
D = 2g^2 \sum_{i=1}^{J} \left( \frac{i}{x(u + \frac{i}{2})} - \frac{i}{x(u - \frac{i}{2})} \right).
\]  
(2.67)

These equations define the twisted version of the long range \( SU(2) \) chain. On setting the twists \( n_1 = n_2 = 0 \), the twisted version of the spin chain reduces to that studied in [21]. We will show that the continuum limit of these equations reduces to that satisfied by resolvent of the sigma model.

To take the continuum limit we perform the following scaling \( u_k \to Lu_k \) with \( L \to \infty \). After taking logarithm on both sides of the equation (2.64) we obtain

\[
L \ln \left( \frac{1 + \frac{i}{2Lu_k}}{1 - \frac{i}{2Lu_k}} \right) + 2\pi i \left( m + \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) + 2\pi in = \sum_{j \neq k} \ln \left( \frac{1 + \frac{i}{L(u_k - u_j)}}{1 - \frac{i}{L(u_k - u_j)}} \right).
\]  
(2.68)

Approximating the sum by an integral and expanding to \( O(1/L) \) we obtain

\[
\frac{1}{u} + 2\pi \left( n + \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) = 2 \int \frac{dv \tilde{\rho}(v)}{u - v},
\]  
(2.69)

where we have introduced \( \tilde{\rho} \), the density of spins which satisfy the normalization given by

\[
\int du \tilde{\rho}(u) = \frac{J}{L}.
\]  
(2.70)

The cyclicity constraint reduces to

\[
\int du \frac{\tilde{\rho}(u)}{u} = -2\pi (m + \frac{n_1}{k_1}),
\]  
(2.71)

and energy of the spin chain is then given by the expression

\[
D = \frac{2g^2}{L} \int du \frac{\tilde{\rho}(u)}{u^2} + O(g^4).
\]  
(2.72)

Here again we have approximated the sum by an integral and also kept the leading term in \( g \)
Now to compare these equations with that of the resolvent given in (2.41), (2.43), (2.44) and (2.45) we first need to identify the change of variables from the spectral parameter $u$ to $x$ in the sigma model. This was identified in [21] for the $SU(2)$ case and we use it for the Lens space. Let us redefine $u$ to be

$$u = x + \frac{g'^2}{x}, \quad g' = \frac{g}{L}.$$  

(2.73)

Then the normalization of the density becomes

$$\int dx \left( 1 - \frac{g'^2}{x^2} \right) \tilde{\rho}(u(x)) = \frac{J}{L}. \quad (2.74)$$

To rewrite the Bethe equations in terms of the variable $x$ we need to keep track of higher powers of $g$ in the LHS of the Bethe equations. One can show that

$$L \ln \left( \frac{x(Lu + \frac{i}{2})}{x(Lu - \frac{i}{2})} \right) = \frac{1}{x} + \frac{g'^2}{x^3} + \frac{g'^4}{x^4} \sim \frac{x}{x^2 - g'^2}. \quad (2.75)$$

On substituting this in the equation (2.68) we obtain

$$\frac{x}{x^2 - g'^2} + 2\pi \left( n + \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) = 2\int dy \left( 1 - \frac{g'^2}{y^2} \right) \frac{\tilde{\rho}(u(x))}{(x - y)(1 - \frac{g'^2}{xy})}, \quad (2.76)$$

$$= 2\int \frac{\tilde{\rho}(u(y))}{(x - y)} - \frac{2g'^2}{x} \int \frac{\tilde{\rho}(y)}{y^2} + O(g'^4).$$

The cyclicity constraint in the variable $x$ reduces to

$$\int \frac{\tilde{\rho}(u(x))}{x} = -2\pi (m + \frac{n_2}{k_2}), \quad (2.77)$$

where we have used (2.73). Now to $O(g^2)$ the energy of the spin chain given in (2.72) in terms of the variable $x$ can be written as

$$D = \frac{2g'^2}{L} \int dx \frac{\tilde{\rho}(u(x))}{x^2} + O(g^4). \quad (2.78)$$

Thus we have written the Bethe equations, the normalization condition of the density and the energy of the spin chain in terms of the parameter $x$.

We will now show the correspondence of the Bethe equations and the equations determining the resolvent of the sigma model. Consider the difference of (2.41) and (2.44). We obtain

$$\int dx \rho(x) (1 - \frac{1}{x^2}) = \frac{4\pi}{\sqrt{\lambda}} J, \quad (2.79)$$

On performing the rescaling

$$x \to \frac{4\pi L}{\sqrt{\lambda}} x, \quad (2.80)$$
the equation in (2.79) reduces to
\[ \int dx \rho(x) \left( 1 - \left( \frac{\sqrt{\lambda}}{4\pi L} \right)^2 \frac{1}{x^2} \right) = \frac{J}{L}. \] (2.81)

We see that this equation is the same as the normalization condition (2.74) of the spin chain on the identification
\[ \frac{\sqrt{\lambda}}{4\pi L} = g', \quad \rho(x) = \tilde{\rho}(u(x)). \] (2.82)

We can now examine the equation (2.45). We first rewrite (2.44) using the rescaling in (2.80) as
\[ \Delta = 1 + 2g^2 \int dx \frac{\rho(x)}{x^2}. \] (2.83)

Performing the rescaling in (2.80) in (2.45) and using the above equation we obtain
\[ - \int d\xi \frac{\rho(\xi)}{x - \xi} = \frac{x}{x^2 - g^2} - 2g^2 \int \frac{\rho(y)}{y^2} + 2\pi \left( n + \frac{n_1}{k_1} + \frac{n_2}{k_2} \right) + O(g^4). \] (2.84)

Now comparing (2.84) and (2.76) we see that they agree to $O(g^2)$. Finally the expression for the energy of the spin chain is identical to the equation given in (2.78) reduces to the equation given in (2.83) on identifying the energy $D = \Delta - L$. Thus to $O(g^4)$ we have shown that the equations of the spin chain is identical to that satisfied by the resolvent. This completes our proof that to two loops, the equations determining the resolvent of the Lens space sigma model are the same as that of the twisted long range spin chain.

3. The BTZ black hole

To set up notations and conventions we first review the construction of the BTZ black hole as a quotient of the $AdS_3$ hyperboloid given in [13]. We will restrict our attention to the non-extremal black hole, the extremal case is discussed in appendix A. We first define the BTZ background as follows. Consider the hyperboloid defined by
\[ -u^2 - v^2 + x^2 + y^2 = -1. \] (3.1)

The BTZ black hole is constructed by the following parametrization of the hyperboloid

**Region I** \( r_+ < r \)
\[ u = \sqrt{A(r)} \cosh \tilde{\phi}(t, \phi), \quad x = \sqrt{A(r)} \sinh \tilde{\phi}(t, \phi), \]
\[ y = \sqrt{B(r)} \cosh \tilde{t}(t, \phi), \quad v = \sqrt{B(r)} \sin \tilde{t}(t, \phi). \] (3.2)
Region II  \( r_- < r < r_+ \)

\[
\begin{align*}
    u &= \sqrt{A(r)} \cosh \tilde{\phi}(t, \phi), \quad x = \sqrt{A(r)} \sinh \tilde{\phi}(t, \phi), \\
    y &= -\sqrt{-B(r)} \sinh \tilde{t}(t, \phi), \quad v = -\sqrt{-B(r)} \cosh \tilde{t}(t, \phi).
\end{align*}
\]

Region II  \( 0 < r < r_- \)

\[
\begin{align*}
    u &= \sqrt{-A(r)} \cosh \tilde{\phi}(t, \phi), \quad x = \sqrt{-A(r)} \sinh \tilde{\phi}(t, \phi), \\
    y &= -\sqrt{-B(r)} \sinh \tilde{t}(t, \phi), \quad v = -\sqrt{-B(r)} \cosh \tilde{t}(t, \phi),
\end{align*}
\]

where

\[
A(r) = \frac{r^{2} - r_-^{2}}{r_+^{2} - r_-^{2}}, \quad B(r) = \frac{r^{2} - r_+^{2}}{r_+^{2} - r_-^{2}},
\]

and

\[
\tilde{t} = r_+ t - r_- \phi, \quad \tilde{\phi} = -r_- t + r_+ \phi.
\]

In the coordinates \( r, t, \phi \) one obtains the metric

\[
ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2,
\]

\[
N^2(r) = -M + r^2 + \frac{j^2}{4r^2},
\]

\[
N^\phi(r) = -\frac{j}{2r^2}.
\]

The mass \( M \) and the angular momentum \( j \) of the BTZ black hole are related to \( r_+ \) and \( r_- \) by

\[
r_+^2 + r_-^2 = M, \quad r_+ r_- = \frac{j}{2}
\]

The metric is a solution of the equations of motion of the action

\[
I = \frac{1}{2\pi} \int d^3 x \sqrt{g} (R + 2).
\]

Note that we have set the radius of \( AdS_3 \) to be unity. Further more note that in all the regions we have the following constraint.

\[
A(r) - B(r) = 1.
\]

The BTZ solution as a quotient of the \( AdS_3 \) which is a \( SL(2, R) \) group manifold. This is seen as follows: let us parametrize the \( SL(2, R) \) group element as

\[
g = \begin{pmatrix} u + x & y + v \\ y - v & u - x \end{pmatrix},
\]
with the constraint given in (3.1). On rewriting the group element \( g \) in terms of the variables \( r, \tilde{\phi}, \tilde{t} \), the global coordinates of the BTZ metric we obtain the following parametrization of the \( SL(2, R) \) group element in the various regions.

**Region I: \( r > r_+ \)**

In the region outside the horizon \( g \) can be written as

\[
 g = \begin{pmatrix} 0 & -e^{\frac{1}{2}(\tilde{\phi}+\tilde{t})} \\ e^{-\frac{1}{2}(\tilde{\phi}+\tilde{t})} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{A} & -\sqrt{B} \\ -\sqrt{B} & \sqrt{A} \end{pmatrix} \begin{pmatrix} 0 & e^{\frac{1}{2}(\tilde{t}-\tilde{\phi})} \\ -e^{-\frac{1}{2}(\tilde{t}-\tilde{\phi})} & 0 \end{pmatrix}. \quad (3.12)
\]

**Region II \( r_- < r < r_+ \)**

In the region between the inner and outer horizon \( g \) is given by

\[
 g = \begin{pmatrix} 0 & -e^{\frac{1}{2}(\tilde{\phi}+\tilde{t})} \\ e^{-\frac{1}{2}(\tilde{\phi}+\tilde{t})} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{A} & -\sqrt{B} \\ -\sqrt{B} & \sqrt{A} \end{pmatrix} \begin{pmatrix} 0 & e^{\frac{1}{2}(\tilde{t}-\tilde{\phi})} \\ -e^{-\frac{1}{2}(\tilde{t}-\tilde{\phi})} & 0 \end{pmatrix}. \quad (3.13)
\]

**Region III \( 0 < r < r_- \)**

Finally in the region inside the inner horizon \( g \) is given by

\[
 g = \begin{pmatrix} 0 & -e^{\frac{1}{2}(\tilde{\phi}+\tilde{t})} \\ e^{-\frac{1}{2}(\tilde{\phi}+\tilde{t})} & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{A} & -\sqrt{B} \\ \sqrt{B} & \sqrt{A} \end{pmatrix} \begin{pmatrix} 0 & e^{\frac{1}{2}(\tilde{t}-\tilde{\phi})} \\ -e^{-\frac{1}{2}(\tilde{t}-\tilde{\phi})} & 0 \end{pmatrix}. \quad (3.14)
\]

The quotienting of the group element arises because of the identification

\[
 \phi \sim \phi + 2\pi. \quad (3.15)
\]

Under this identification and using (3.12), (3.13) and (3.14) it can be seen that in all regions, the above identifications acts as

\[
 g \sim \tilde{A}(1) g A(1), \quad (3.16)
\]

where

\[
 \tilde{A}(k) = \begin{pmatrix} e^{(r_+-r_-)\pi k} & 0 \\ 0 & e^{-(r_+-r_-)\pi k} \end{pmatrix}, \quad A(k) = \begin{pmatrix} e^{(r_-+r_+)\pi k} & 0 \\ 0 & e^{-(r_-+r_+)\pi k} \end{pmatrix}. \quad (3.17)
\]

We have now cast the quotienting of the \( SL(2, R) \) group manifold which results in the BTZ black hole on similar lines to that of obtaining the Lens space from the \( SU(2) \) group manifold discussed in the previous section.

We will be interested in classical solutions only in the physical region I, that is the region outside the horizon. We will obtain classical solutions in the global coordinates \( r, t, \phi \). Note that in these coordinates any in falling geodesic approaches the horizon asymptotically. As we will see subsequently, this will be the situation for the solutions we discuss in this paper. Using the parametrization of the \( SL(2, R) \)
group element given in (3.12), (3.13) and (3.14) the sigma model for the string propagating in BTZ times a $S^1$ is given by

$$S = -\frac{\lambda}{2} \int d^2\sigma \left( \frac{1}{2} \text{Tr}(g^{-1} \partial_a g g^{-1} \partial^a g^{-1}) + \partial_a Z \partial^a Z \right),$$

(3.18)

where $Z$ is the coordinate along the $S^1$ and $\lambda$ is the coupling of the sigma model. The equations of motion of the sigma model are given by

$$\partial^a (g^{-1} \partial_a g) = 0, \quad \partial^a \partial_a Z = 0.$$  

(3.19)

As before let us define $j_a = g^{-1} \partial_a g$, then the Virasoro constraints are given by

$$\frac{1}{2} \text{Tr}(j_\tau j_\tau) + \partial_\tau Z \partial_\tau Z = 0.$$  

(3.20)

The sum of the above two equations results in the Hamiltonian constraint

$$\frac{1}{2} \text{Tr}(j_\tau j_\tau + j_\sigma j_\sigma + (\partial_\tau Z)^2 + (\partial_\sigma Z)^2 = 0.$$  

(3.21)

Note that $j_a$ is an $SL(2, R)$ current. Let us chose the generators of $SL(2, R)$ as

$$t^1 = \sigma^1, \quad t^2 = i\sigma^2, \quad t^3 = \sigma^3.$$  

(3.22)

Then we see that in equation (3.21) the only negative definite quantity is the term $-(\text{Tr}(j_\tau^2 + j_\sigma^2))$, the remaining terms are positive definite. Thus for the Virasoro constraint to be satisfied we must have either $j_\tau^2 \neq 0$ or $j_\sigma^2 \neq 0$. This condition is important to note for the following reason: unlike for the case of the $SU(2)$ as done in [20], it is not possible for us to restrict the class of solutions for with only one component of the charge say $\sigma^3$ is turned on. We will elaborate on this in the next subsection. There are two consequences of the quotienting given in (3.16). The first one is that the sigma model admits solutions which have the twisted boundary condition on the currents of the sigma model.

$$j_a(\tau, \sigma + 2\pi) = A_{(k)}^{-1} j_a(\tau, \sigma) A_{(k)}.$$  

(3.23)

Thus $k$ labels the winding number of the solution. The other is that in the quantum theory the charge corresponding to global shifts in $\phi$ quantized. We will not be dealing with the quantum theory so this condition is not relevant for the discussion in this paper. Now going through the same logic as in the case of the lens space we see that the sigma model on the BTZ space is integrable and the the conserved charges can be extracted from the which monodromy matrix given by

$$\hat{\Omega}_k(x) = A_{(k)} P \exp \left[ - \int_{0}^{2\pi} d\sigma \frac{1}{2} \left( \frac{j_+}{1-x} - \frac{j_-}{1+x} \right) \right].$$  

(3.24)
Since the monodromy matrix is an $SL(2,R)$ group element its eigen values are of the form $\{\exp(ip(x)), \exp(-ip(x))\}$. $p(x)$ is called the quasi-momentum and it characterizes the monodromy matrix. We also have the relation

$$\text{Tr}(\Omega(x)) = 2 \cos p(x).$$

(3.25)

We will see that corresponding to each classical solution there exists a density which characterizes this solution in the spectral plane. To proceed further we will define the global charges which will play a special role in our analysis. The energy which corresponds to the global translations in time $t$ is given by

$$E = \frac{\lambda}{4} \left[ \int_0^{2\pi} d\sigma \left( (r_+ - r_-)\text{Tr}(\partial_0 gg^{-1}\sigma^3) - (r_+ + r_-)\text{Tr}(g^{-1}\partial_0 g\sigma^3) \right) \right].$$

(3.26)

The spin which corresponds to the global translations in $\phi$ is given by

$$S = \frac{\lambda}{4} \left[ \int_0^{2\pi} d\sigma \left( (r_+ - r_-)\text{Tr}(\partial_0 gg^{-1}\sigma^3) + (r_+ + r_-)\text{Tr}(g^{-1}\partial_0 g\sigma^3) \right) \right].$$

(3.27)

Thus the following combinations of charges have a simple relation in terms of the right and left currents.

$$E + S = \frac{\lambda}{2} (r_+ - r_-) \int_0^{2\pi} d\sigma \text{Tr}(\partial_0 gg^{-1}\sigma^3),$$

(3.28)

$$E - S = -\frac{\lambda}{2} (r_+ + r_-) \int_0^{2\pi} d\sigma \text{Tr}(g^{-1}\partial_0 g\sigma^3).$$

(3.29)

Global translations in the coordinate $Z$ leads to the charge $\hat{J}$ which is given by

$$\hat{J} = \lambda \int d^2\sigma \partial_0 Z,$$

(3.30)

We will choose a gauge in which

$$Z = \frac{\hat{J}}{2\pi \lambda} \tau + \hat{m} \sigma,$$

(3.31)

where $\hat{m}$ is the winding number, this is the same gauge chosen in [24]. The Virasoro constraints then reduce to

$$\text{Tr}(j^2_{\pm}) = 2 \left( \frac{\hat{J}}{2\pi \lambda} \pm \hat{m} \right).$$

(3.32)

Before we begin our analysis of the quasi-momentum we mention that the sigma model on $AdS_3 \times S^1$ was studied earlier with a parametrization of the global $AdS_3$ coordinates by [24]. For this parametrization the right and left charges corresponding to the $t^2 = i\sigma^2$ generator, the compact direction played an important role.

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3See equation (3.13) of the paper.
3.1 Properties of the quasi-momentum

In this subsection we discuss the properties of the quasi-momentum in the spectral plane. This will enable us to demonstrate that every classical solution of the sigma model is characterized by a density in the complex $x$ plane. The properties of the quasi-momentum obtained from general considerations will be confirmed in the next subsection by explicitly solving for the quasi-momentum for two class of solutions. To discuss the properties of the quasi-momentum we need to be careful of two sectors of the theory. The sector with winding zero $k = 0$ and the sector with $k \neq 0$. We start with the zero winding sector.

**Sector** $k = 0$

Just as in the case of Lens space we will see that the quasi-momentum has a pole at $x = \pm 1$. Let us first examine the case $x \to 1$. In this limit, we see that the monodromy matrix becomes

$$\hat{\Omega}(x) \to P \left[ \exp \left( -\int_{0}^{2\pi} d\sigma \frac{1}{2} \frac{j_{+}}{1-x} \right) \right], \quad x \to 1, \quad (3.33)$$

For this path ordered exponential to make sense, the integrability condition of the first order equation satisfied by $\hat{\Omega}$ implies that

$$\partial_{-}j_{+} = 0. \quad (3.34)$$

Then from the equation of motion (2.6) and the equation (2.7) we obtain

$$\partial_{+}j_{-} = 0, \quad [j_{+}, j_{-}] = 0. \quad (3.35)$$

A simple solution to these equations is that $j_{+}, j_{-}$ approach a constant as $x \to 1$ and we can choose $j_{\pm}$ to point along the same direction in the group space. To satisfy the Virasoro constraint in (3.32) it is convenient to choose choose $j_{+}$ to be along $i\sigma^{2}$. Thus we obtain

$$j_{+} = i \left( \frac{1}{2\pi \lambda} \hat{j} + \hat{m} \right) \sigma^{2}, \quad x \to +1 \quad (3.36)$$

where we have chosen the positive square root. Note that this form for $j_{+}$ also satisfies the condition $\partial_{-}j_{+} = 0$. This means that the monodromy matrix reduces to

$$\Omega \sim \exp \left( -i \frac{j}{2\pi \lambda} \frac{\hat{j} + \hat{m}}{1-x} \pi \sigma^{2} \right), \quad x \to 1. \quad (3.37)$$

Thus we conclude that the quasi-momentum at $x \to 1$ is given by

$$p \sim \pi \frac{j}{2\pi \lambda} + \frac{\hat{m}}{x-1}. \quad (3.38)$$
A similar analysis as \( x \to -1 \) gives the following behaviour of the quasi-momentum

\[
p \sim \pi \frac{\hat{J}}{x} - \hat{m}.
\]

Thus we have shown that the quasi-momentum has poles at \( x = \pm 1 \) with residues determined by the charge \( \hat{J} \) and the winding number \( \hat{m} \). Now let us examine the behaviour of the monodromy matrix as \( x \to \infty \). At \( x \to \infty \) the expansion of \( J_\sigma \) is given in (2.29). Substituting this expansion in the monodromy matrix we obtain

\[
2 \cos p(x) = 2 + \frac{1}{2x^2} \int_0^{2\pi} d\sigma d\sigma' \text{Tr}(j_\tau(\sigma)j_\tau(\sigma')) + \cdots,
\]

where we have defined

\[
Q_R^i = \int_0^{2\pi} d\sigma \text{Tr}(g^{-1} \partial_{\tau} g t^i).
\]

Let us denote the invariant

\[
Q_R^2 = (Q_R^1)^2 - (Q_R^2)^2 + (Q_R^3)^2.
\]

Then from the equation (3.40) we have

\[
p(x) \to \frac{i}{2x} \sqrt{Q_R^2}, \quad x \to \infty.
\]

At this point it is worthwhile to point out a difference in the analysis of [20] and [24]. There they assume that among the three global charges only one of them contribute \(^4\), here we retain the dependence on all the three charges. As we will see subsequently that geodesics of the BTZ background in general carry all the three global charges and it is necessary to retain their dependence to discuss all solutions. Now we can examine the behaviour as \( x \to 0 \). Using the relation in (2.33) we can write the monodromy matrix as

\[
\Omega(x) = g^{-1}(2\pi)P \exp \left(-x \int_0^{2\pi} \partial_{\tau}gg^{-1}\right) g(0).
\]

We then use the fact that \( g(2\pi) = g(0) \) in the untwisted sector to obtain

\[
2 \cos p(x) = 2 + \frac{x^2}{4} \left((Q_L^1)^2 - (Q_L^2)^2 + (Q_L^3)^2\right),
\]

where

\[
Q_L^i = \int_0^{2\pi} d\sigma \text{Tr}(t^i \partial_{\tau} gg^{-1}).
\]

\(^4\)See equations (4.38) and (3.27) of the respective papers.
Thus the behaviour of the quasi-momentum as $x \to 0$ is given by

$$p(x) \to 2\pi m + i\frac{x}{2}\sqrt{Q_L^2}, x \to 0,$$

where

$$Q_L^2 = (Q_L^1)^2 - (Q_L^2)^2 + (Q_L^3)^2.$$  \hfill (3.48)

Here again we have retained the dependence on all the three components of the global charges. Across branch cuts the quasi-momentum satisfies the equation

$$p(x + i\epsilon) + p(x - i\epsilon) = 2\pi n_l,$$

where $n_l$ is an integer for the $l$th cut. The reason for this is the same as in the case of the $SU(2)$, across branch cut the quasi-momentum takes the two possible different eigen values of the monodromy matrix. The condition in (3.49) just arises from the unimodularity of the monodromy matrix.

Let us now recast these properties of the quasi-momentum in terms of the resolvent. We can define the resolvent as

$$G(x) = \rho\left(\frac{j}{2\pi\lambda} + \hat{m}\right) - \rho\left(\frac{j}{2\pi\lambda} - \hat{m}\right).$$  \hfill (3.50)

This ensures that the resolvent is a function in the complex plane without poles. Thus using standard complex analysis one can write the resolvent as

$$G(x) = \int d\xi \rho(x)\frac{x}{x - \xi},$$  \hfill (3.51)

where the integral is along the cuts. In fact $\rho$ is given by

$$\rho(x) = \frac{1}{2\pi i} \left(G(x + i\epsilon) - G(x - i\epsilon)\right).$$  \hfill (3.52)

Now from the behaviour of the quasi-momentum at $x \to \infty$ we obtain

$$\int d\xi \rho(\xi) = -\frac{j}{\lambda} + \frac{i}{2}\sqrt{Q_R^2}.$$  \hfill (3.53)

and from the behaviour of the quasi-momentum at $x \to 0$ we obtain the following conditions on the density

$$\frac{1}{2\pi i} \int dx \frac{G(x)}{x} = -\int d\xi \frac{\rho(\xi)}{\xi} = 2\pi(m + \hat{m}),$$  \hfill (3.54)

$$\frac{1}{2\pi i} \int dx \frac{G(x)}{x^2} = -\int d\xi \frac{\rho(\xi)}{\xi^2} = \frac{j}{\lambda} + \frac{i}{2}\sqrt{Q_L^2}.$$  \hfill (3.55)
Finally the condition in (3.49) for the resolvent across branch cuts gives rise to

\[ G(x + i \epsilon) + G(x - i \epsilon) = 2 \int d\xi \frac{\rho(\xi)}{x - \xi}, \]

\[ = -\frac{2\pi (i \lambda \hat{m})}{x - 1} - \frac{2\pi (i \lambda \hat{m})}{x + 1} + 2\pi n_{\lambda}. \]

Equations (3.53), (3.54), (3.55) and (3.55) show that give a classical solution in the \( k = 0 \) sector, it determines a density in the spectral plane \( x \).

**Sector** \( k \neq 0 \)

Let us now repeat the analysis for the twisted sectors. The behaviour of the monodromy matrix as \( x \to \pm 1 \) is same as in the case of \( k = 0 \). From the earlier analysis it is easy to see that the presence of the matrix \( A_k \) for the \( k \neq 0 \) does not affect the singular behaviour of the quasi-momentum as \( x \to \pm 1 \). Thus we have

\[ p \to \pi \frac{i}{2\pi \lambda} \pm \hat{m}, \quad x \to \pm 1 \]  

(3.57)

As \( x \to \infty \) we can use the expansion given in (2.29) to obtain the following expression for the quasi-momentum

\[ 2 \cos p(x) = \text{Tr} \left( A_{(k)} P \exp \int_0^{2\pi} \left( \frac{d\sigma j_0}{x} + \frac{j_1}{x^2} \ldots \right) \right) \]

\[ = \text{Tr} \left[ \left( \cosh \pi k (r_+ + r_-) + \sigma^3 \sinh \pi k (r_+ + r_-) \right) \times \left( 1 + \frac{i^i}{x} \int_0^{2\pi} d\sigma j_0^i + \frac{i^i x^i}{x^2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' j_0^i(\sigma)j_0^i(\sigma') + \frac{i^i}{x^2} \int_0^{2\pi} d\sigma j_1^i(\sigma) \right) \right] \]

\[ = 2 \cosh \pi k (r_+ + r_-) + \frac{1}{x} \sinh \pi k (r_+ + r_-) \text{Tr} \left( \int_0^{2\pi} d\sigma j_0^3(\sigma) + \frac{C}{x^2} + \ldots \right), \]

\[ = 2 \cosh \pi k (r_+ + r_-) - \frac{1}{x} \sinh \pi k (r_+ + r_-) \frac{2(E - S)}{\lambda(r_+ + r_-)} + \frac{C}{x^2} + \ldots. \]

In the last line we have used the definition of the global charge given in (3.29). Here we have expanded the currents as

\[ g^{-1} \partial_a g = t^i j_a^i. \]

(3.59)

The first non-local charge \( C \) is given by

\[ C = 2 \cosh \pi (r_+ + r_-) \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' j_0^i(\sigma)j_0^i(\sigma') \]

\[ -2 \sinh \pi (r_+ + r_-) \left( \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' j_0^1(\sigma)j_0^2(\sigma') - \int_0^{2\pi} d\sigma j_1^3(\sigma) \right), \]

(3.60)
where \( j_0^1 j_0^1 = j_0^2 j_0^2 + j_0^3 j_0^3 \). Thus the leading behaviour of the quasi-momentum is determined by the global charges \( E, S \) and is given by

\[
p(x) \sim i\pi k (r_+ + r_-) - i \frac{1}{x} \frac{E - S}{\lambda (r_+ + r_-)} \tag{3.61}
\]

At this point it is relevant to point out the difference in behaviour for the \( k = 0 \) case. From (3.43) one see that the all the three components of the global charges determine the \( O(1/x) \) term unlike the case above. Therefore setting \( k = 0 \) in the expression (3.61) does not reduce to (3.43) unless we restrict to the situation in which charges corresponding to the \( \sigma^3 \) direction is turned on and others are set to zero. Note that the two leading terms in quasi-momentum is purely imaginary as \( x \to \infty \). Now let us examine the limit \( x \to 0 \), again using (2.33) we get

\[
\Omega(x) = A(k) g^{-1}(2\pi) P \exp \left( - \int_0^{2\pi} d\sigma (x \partial_+ gg^{-1} + x^2 \partial_\sigma gg^{-1}) \right) g(0), \tag{3.62}
\]

where we have retained the \( O(x^2) \) term also. Taking the trace and using the fact

\[
g(2\pi) = \tilde{A}(k) g(0) A(k), \tag{3.63}
\]

we obtain

\[
2 \cos p(x) = \text{Tr} (\tilde{A}^{-1}_{(k)} \exp \left( - \int_0^{2\pi} d\sigma (x \partial_+ gg^{-1} + x^2 \partial_\sigma gg^{-1}) \right)), \tag{3.64}
\]

\[
= \text{Tr} \left[ \left( \cosh \pi k (r_+ - r_-) - \sigma^3 \sinh \pi k (r_+ - r_-) \right) \times \left( 1 - x t_i \int_0^{2\pi} d\sigma l_{i0}^t + x^2 t_i t_j \int_0^{2\pi} d\sigma l_{ij0}^t (\sigma) - x^2 t_i \int_0^{2\pi} d\sigma l_{i1}^t \right) \right],
\]

\[
= 2 \cosh \pi k (r_+ - r_-) + 2x \sinh \pi k (r_+ - r_-) \frac{E + S}{\lambda (r_+ - r_-)} + x^2 Q + \cdots,
\]

where we have used (3.28) to obtain the last line. Here we have expanded the currents as

\[
\partial_a gg^{-1} = t_i l_{ia}^t, \tag{3.65}
\]

For completeness we write down the non-local charge obtained at \( O(x^2) \) which is given by

\[
Q = 2 \cosh \pi k (r_+ + r_-) \int_0^{2\pi} d\sigma \int_0^\sigma d\sigma' l_{ij0}^t (\sigma) l_{ij0}^t (\sigma') \tag{3.66}
\]

\[
+ 2 \sinh \pi k (r_+ + r_-) \left( \int_0^{2\pi} d\sigma \int_0^\sigma d\sigma' l_{ij0}^t (\sigma) l_{ij0}^t (\sigma') + \int_0^{2\pi} d\sigma l_{ij1}^t \right)
\]

Thus the leading behaviour of the quasi-momentum at \( x \to 0 \) is given by

\[
p(x) \sim 2\pi m + i\pi k (r_+ - r_-) + ix \frac{E + S}{\lambda (r_+ - r_-)}. \tag{3.67}
\]
Again we see that the quasi-momentum is purely imaginary as \( x \to 0 \). Furthermore, unlike the case of \( k = 0 \), see equation (3.47), the leading behaviour depends on only the third component of the global charge. The unimodularity of the monodromy matrix imposes the following condition on the quasi-momentum across branch cuts.

\[
p(x + i\epsilon) + p(x - i\epsilon) = 2\pi n_l,
\]

where \( n_l \) refers to an integer corresponding to the \( l \)th branch cut.

Let us now use all these information and obtain the conditions on the resolvent. We define the resolvent as

\[
G(x) = p(x) - \pi \frac{j}{2\pi \lambda + \hat{m}} \frac{x}{x - 1} - \pi \frac{j}{2\pi \lambda - \hat{m}} \frac{x}{x + 1} - i\pi k(r_+ + r_-).
\]

This ensures that the resolvent is a function on the complex plane without any poles and falls off as \( 1/x \) for large values of the spectral parameter. By standard complex analysis we can write the resolvent in terms of a density function by

\[
G(x) = \int d\xi \frac{\rho(x)}{x - \xi}.
\]

here the integral is along the various cuts. Then from the behaviour of the quasi-momentum at \( x \to \infty \) given in (3.61) we obtain

\[
\int d\xi \rho(\xi) = -\frac{j}{\lambda} - i\frac{E - S}{\lambda(r_+ + r_-)}.
\]

From the behaviour of the quasi-momentum at \( x \to 0 \) given in (3.67) we obtain the following two equations

\[
\frac{1}{2\pi i} \int dx \frac{G(x)}{x} = -\int d\xi \frac{\rho(x)}{\xi} = 2\pi (\hat{m} + m - ikr_-),
\]

\[
\frac{1}{2\pi i} \int dx \frac{G(x)}{x^2} = -\int d\xi \frac{\rho(\xi)}{\xi^2} = \frac{j}{\lambda} + i\frac{E + S}{\lambda(r_+ - r_-)}.
\]

Finally we have to use the unimodularity of the monodromy matrix in (3.68) to get

\[
G(x + i\epsilon) + G(x - i\epsilon) = 2\int d\xi \frac{\rho(\xi)}{x - \xi},
\]

\[
= -\frac{2\pi (\frac{j}{2\pi \lambda + \hat{m}})}{x - 1} - \frac{2\pi (\frac{j}{2\pi \lambda - \hat{m}})}{x + 1} + 2\pi n_l - 2\pi ik(r_+ + r_-).
\]

Thus we have completed the proof that a classical solution of of the sigma model in the BTZ background corresponds to a density function in the complex plane. The density is determined by solving the above set of integral equations.
3.2 Classical Solutions and quasi-momentum

In this subsection we study two examples of classical solutions of the BTZ sigma model and evaluate the quasi-momentum explicitly. We will show that the behaviour of the quasi-momentum agrees with the general discussion in the previous subsection for these examples.

To obtain classical solutions it is convenient to write the BTZ sigma model in terms of the coordinates of the $\text{AdS}_3$ hyperboloid $u, v, x, y$ and impose the constraint given in (3.1). Then the sigma model action is given by

$$ S = -\frac{\lambda}{2} \int d\tau d\sigma \left[ -\partial_a u \partial^a u - \partial_a v \partial^a v + \partial_a x \partial^a x + \partial_a y \partial^a y - \Lambda (-u^2 - v^2 + x^2 + y^2 + 1) + \partial_a Z \partial^a Z \right], $$

where $\Lambda$ is the Lagrange multiplier. The equations of motion are given by

$$ -\partial^a \partial_a u = \Lambda u, \quad -\partial^a \partial_a v = \Lambda v, \quad \partial^a \partial_a x = -\Lambda x, \quad \partial^a \partial_a y = -\Lambda y, \quad \partial^a \partial_a Z = 0. $$

The Virasoro constraints in these variables are given by

$$ -\partial_+ u \partial_+ v - \partial_+ x \partial_+ y + \partial_+ y \partial_+ y + \partial_+ Z \partial_+ Z = 0. $$

Since we are interested in closed string solutions, all solutions are subject to the periodicity conditions

$$ r(\tau, \sigma + 2\pi) = r(\tau, \sigma), \quad t(\tau, \sigma + 2\pi) = t(\tau, \sigma), \quad \phi(\tau, \sigma + 2\pi) = \phi(\tau, \sigma) + 2\pi k $$

where $k$ is the winding number. With these preliminaries we are ready to discuss the classical solutions. Our solutions fall into two classes, the geodesics and the winding strings.

**Geodesics**

We first consider solutions which are independent of the world sheet variable $\sigma$. We will show that these solutions correspond to geodesics in the BTZ background. For this we consider the following ansatz.

$$ u + x = a(\tau) \exp(f(\tau)), \quad u - x = a(\tau) \exp(-f(\tau)), \quad y + v = b(\tau) \exp(g(\tau)), \quad y - v = b(\tau) \exp(-g(\tau)). $$

Substituting this ansatz in the action given in (3.1) we obtain the following Lagrangian

$$ \frac{1}{2} \left( (\dot{a})^2 - (\dot{b})^2 - (\dot{f})^2 a^2 + (\dot{g})^2 b^2 - \Lambda (-a^2 + b^2 + 1) \right). $$. 


We can eliminate the constraint by the following substitution

\[ a(\tau) = \cosh \gamma(\tau), \quad b(\tau) = \sinh \gamma(\tau). \]  

(3.81)

The equations of motion then can be obtained from the Lagrangian

\[ L = \frac{1}{2} \left( -\dot{\gamma}^2 - \dot{f}^2 \cosh^2 \gamma + (\dot{g})^2 \sinh^2 \gamma \right) \]  

(3.82)

From here we see that there are two constants of motion given by

\[ \dot{f} \cosh^2 \gamma = c_1, \quad \dot{g} \sinh^2 \gamma = c_2. \]  

(3.83)

These constants are related to the charges \( E \) and \( S \) as follows

\[ E - S = -(r_+ + r_-)2\pi \lambda (c_1 + c_2), \]
\[ E + S = (r_+ - r_-)2\pi \lambda (c_1 - c_2). \]  

(3.84)

Substituting these constants of motion we obtain the following equation of motion for \( \gamma \)

\[ \ddot{\gamma} = -\frac{\sinh \gamma c_1^2}{\cosh^3 \gamma} + \frac{\cosh \gamma c_2^2}{\sinh^3 \gamma}. \]  

(3.85)

The Virasoro constraints can be used to integrate this equation to quadratures. This is given by

\[ (\dot{\gamma})^2 + \frac{c_1^2}{\cosh^2 \gamma} - \frac{c_2^2}{\sinh^2 \gamma} + \left( \frac{\dot{J}}{2\pi \lambda} \pm \dot{m} \right)^2 = 0. \]  

(3.86)

The two equations imply either \( \dot{J} = 0 \) or \( \dot{m} = 0 \). Let us assume \( \dot{m} = 0 \) therefore we obtain

\[ (\dot{\gamma})^2 + \frac{c_1^2}{\cosh^2 \gamma} - \frac{c_2^2}{\sinh^2 \gamma} + \left( \frac{\dot{J}}{2\pi \lambda} \right)^2 = 0. \]  

(3.87)

It is clear from the above equation that there are always initial conditions which allow for real solutions. We will assume the initial condition that at \( \tau = 0 \) we have \( \gamma_0 \) satisfying

\[ \frac{c_1^2}{\cosh^2 \gamma_0} - \frac{c_2^2}{\sinh^2 \gamma_0} + (\frac{\dot{J}}{2\pi \lambda})^2 = 0, \]  

(3.88)

and \( \dot{\gamma} = 0 \). We can also assume that \( \tau = 0 \), the coordinates \( t = 0, \phi = 0 \). From the structure of the potential in (3.87) we see that the geodesic always falls into the horizon. One can explicitly integrate the equation for \( \gamma \). In Appendix B we show that the equation (3.87) corresponds to the equation of a geodesic in the BTZ background.

Let us now evaluate the monodromy matrix for this solution to verify the general properties discussed in the earlier section. Note that this solution is in the \( k = 0 \)
sector, since there is no winding. Evaluating the current $g^{-1}\partial_\pm g$ on this solution we obtain
\begin{equation}
g^{-1}\partial_\pm g = \begin{pmatrix}
\hat{\gamma} - \frac{\sinh \gamma}{\cosh \gamma} c_1 - \frac{\cosh \gamma}{\sinh \gamma} c_2 & (\hat{\gamma} + \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2) e^{f-g} \\
-c_1^2 + c_2^2 (\hat{\gamma} - \frac{\sinh \gamma}{\cosh \gamma} c_1 - \frac{\cosh \gamma}{\sinh \gamma} c_2) e^{f-g} & -(c_1 + c_2).
\end{pmatrix} \tag{3.89}
\end{equation}

Since the currents do not depend on the world sheet coordinate $\sigma$ it is easy to perform the path ordered integral for the monodromy matrix. We obtain
\begin{equation}
\Omega = \exp \left[ -\frac{2\pi x}{1-x^2} (\hat{\gamma} - \frac{\sinh \gamma}{\cosh \gamma} c_1 - \frac{\cosh \gamma}{\sinh \gamma} c_2) e^{f-g} \right], \tag{3.90}
\end{equation}

where the 3-vector $\theta$ is given by
\begin{equation}
\vec{\theta} = \begin{pmatrix}
-\hat{\gamma} \cosh(g - f) + \left( \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2 \right) \sinh(g - f)
\hat{\gamma} \sinh(g - f) + \left( \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2 \right) \cosh(g - f)
-\hat{\gamma}(c_1 + c_2)
\end{pmatrix}. \tag{3.91}
\end{equation}

Let us evaluate the (modulus)$^2$ of this three vector, it is given by
\begin{equation}
\theta^2 = \left( \frac{2\pi x}{x^2 - 1} \right)^2 \left( \hat{\gamma}^2 - \frac{c_1^2}{\cosh^2 \gamma} + \frac{c_2^2}{\sinh^2 \gamma} \right), \tag{3.92}
\end{equation}

In the second line we have used the Virasoro constraint given in (3.87). From this property it is easy to evaluate the eigen values of this monodromy matrix. They are given by are given by $\exp(\pm i\theta)$. This results in the following values of quasi-momentum on this solution
\begin{equation}
p = \frac{j}{\lambda x^2 - 1}, \tag{3.93}
\end{equation}

where we have taken the positive square root. Thus the behaviour of the quasi-momentum for $x \to \pm 1$ is given by
\begin{equation}
p \to \frac{\hat{j}}{2\lambda} \frac{1}{x \mp 1}, \quad x \to \pm 1, \tag{3.94}
\end{equation}

One can see that the the solution for the quasi-momentum for geodesics satisfy the equations (3.38) and (3.39). We also see from the formula for the resolvent in (3.93) we see that $G(x) = 0$ for this case, this implies the density vanishes. The charges $Q_R$ and $Q_L$ for the geodesic solutions are given by
\begin{align*}
Q_R^1 &= 4\pi \left( \hat{\gamma} \cosh(g - f) + \left( \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2 \right) \sinh(g - f) \right), \tag{3.95} \\
Q_R^2 &= 4\pi \left( \hat{\gamma} \sinh(g - f) + \left( \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2 \right) \cosh(g - f) \right), \\
Q_R^3 &= 4\pi (c_1 + c_2),
\end{align*}
Similarly the charges left charges are given by

\[ Q^1_L = 4\pi \left( \dot{\gamma} \cosh(g + f) - \left( \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2 \right) \sinh(g - f) \right), \quad (3.96) \]
\[ Q^2_L = 4\pi \left( \dot{\gamma} \sinh(g + f) - \left( \frac{\sinh \gamma}{\cosh \gamma} c_1 + \frac{\cosh \gamma}{\sinh \gamma} c_2 \right) \cosh(g + f) \right), \]
\[ Q^3_L = 4\pi (c_1 - c_2). \]

One can evaluate the norms of these charges and show

\[ Q^2_R = Q^2_L = -\frac{4\hat{J}^2}{\lambda^2}. \quad (3.97) \]

This ensures that the equations (3.53) and (3.54) are satisfied for zero density. Since the form for the quasi-momentum given in (3.93) does not have a branch cut, the equation (3.49) does not arise. One can do a similar analysis for the situation in which \( \hat{J} = 0 \) and \( \hat{m} \neq 0 \) and show that all the equations obtained for the quasi-momentum from general considerations hold. At this point we mention that these solutions in general have all the components of the charge turned on unlike the solutions considered in [20] and [24] in which classical solutions carried only one component of the charge. Note that geodesics in the BTZ background can be thought of as a vacuum since the corresponding density in the spectral plane vanishes.

**Winding strings**

In this section we discuss classical strings which wind around the \( \phi \) direction and the corresponding quasi-momentum. For this we start with the ansatz

\[ u + x = a(\tau)e^{(f(\tau) + \nu_1\sigma)}, \quad u - x = a(\tau)e^{-(f(\tau) + \nu_1\sigma)}, \quad (3.98) \]
\[ y + v = b(\tau)e^{(g(\tau) + \nu_2\sigma)}, \quad y - v = b(\tau)e^{-(g(\tau) + \nu_2\sigma)}. \]

From this ansatz we see that

\[ t = \frac{r_+(g(\tau) + \nu_2\sigma) + r_-(f(\tau) + \nu_1\sigma)}{r_+^2 - r_-^2}, \quad \phi = \frac{r_-(g(\tau) + \nu_2\sigma) + r_+(f(\tau) + \nu_1\sigma)}{r_+^2 - r_-^2}. \quad (3.99) \]

The periodicity conditions \( t(\tau, \sigma + 2\pi) = t(\tau, \sigma) \) and \( \phi(\tau, \sigma + 2\pi) = \phi + 2\pi k \) gives rise to the following conditions on \( \nu_2, \nu_2 \).

\[ \nu_1 = r_+ k, \quad \nu_2 = -r_- k. \quad (3.100) \]

Now substituting this ansatz in the action (3.75) we obtain the following Lagrangian

\[ L = \frac{1}{2} \left( (\dot{a})^2 - (\dot{\dot{f}})^2a^2 + (\dot{b})^2 + (\dot{g})^2b^2 + \nu_1^2a^2 - \nu_2^2b^2 \right) - \Lambda(-a^2 + b^2 + 1). \quad (3.101) \]
Again, one can eliminate the constraint by the substitution
\[ a(\tau) = \cosh \gamma(\tau), \quad b(\tau) = \sinh \gamma(\tau). \] (3.102)
The equations of motion can then be obtained from the Lagrangian
\[ L = \frac{1}{2} \left( -\dot{\gamma}^2 - (\dot{\mathcal{J}})^2 \cosh^2 \gamma + (\dot{\mathcal{J}})^2 \sinh^2 \gamma + \nu_1 \cosh^2 \gamma - \gamma_0^2 \sinh^2 \gamma \right). \] (3.103)
As before we have two constants of motion given by
\[ \dot{j} \cosh^2 \gamma = c_1, \quad \dot{g} \sinh^2 \gamma = c_2. \] (3.104)
These constants are related to the charges \( E \) and \( S \) by the equations in (3.84). We can reduce the equations of motion in \( \gamma \) to quadratures using the Virasoro constraints. This is given by
\[ (\dot{\gamma})^2 + \frac{c_1^2}{\cosh^2 \gamma} - \frac{c_2^2}{\sinh^2 \gamma} + \nu_1 \cosh^2 \gamma - \nu_2 \sinh^2 \gamma + (\dot{\mathcal{J}}^2 + \dot{m}^2) = 0, \] (3.105)
\[ c_1 \nu_1 + c_2 \nu_2 + \dot{\mathcal{J}} \dot{m} = 0. \]
Note that \( \nu_1, \nu_2 \) are fixed once the sector label \( k \) is given by (3.100). It is clear that given the constants of motion \( c_1, c_2, J \) and the sector label \( k \), one can determine \( \dot{m} \) from the second of the above equations. The shape of the potential in the first equation shows that solutions always fall into the horizon. This is because given the constants \( c_1, c_2, k, \dot{J} \), on can always find \( \gamma_0 \) for which
\[ \frac{c_1^2}{\cosh^2 \gamma_0} - \frac{c_2^2}{\sinh^2 \gamma_0} + \nu_1 \cosh^2 \gamma_0 - \nu_2 \sinh^2 \gamma_0 + (\dot{\mathcal{J}}^2 + \dot{m}^2) = 0, \] (3.106)
We use this \( \gamma_0 \) as the initial condition, this of course has \( \dot{\gamma} = 0 \) and let the string evolve and fall into the horizon. The initial condition for the other co-ordinates are \( t(0, 0) = 0 \), and \( \phi(0, 0) = 0 \). With these initial condition, it is certainly possible to integrate the equations of motion to obtain the trajectory of the string. We have obtained the trajectory of \( \gamma \) in appendix B.
Without knowing the explicit form of the solution, it is possible to obtain the monodromy matrix. We will demonstrate this below. Let us define the matrix
\[ \Omega(x, \sigma) = \exp \left[ -\int_0^\sigma d\sigma \frac{1}{2} \left( \frac{j_+}{1-x} - \frac{j_-}{1+x} \right) \right]. \] (3.107)
Note the quasi-momentum is evaluating the following expression.
\[ 2 \cos p(x) = \text{Tr} (A_{(k)} \Omega(x)). \] (3.108)
\( \Omega(x) \) satisfies the differential equation
\[ \partial_\sigma \Omega(x, \sigma) = M \Omega(x, \sigma), \] (3.109)
where the entries of $M$ is given by

$$
M_{11} = \frac{1}{1-x^2} \left( x(c_1 + c_2) + \nu_1 \cosh^2 \gamma_0 + \nu_2 \sinh^2 \gamma_0 \right),
$$

$$
M_{22} = -M_{11},
$$

$$
M_{12} = \frac{1}{1-x^2} \left[ \left( \frac{\cosh \gamma_0 \cosh \gamma_0}{\sinh \gamma_0} + \frac{\sinh \gamma_0 \cosh \gamma_0}{\sinh \gamma_0} \right) x + (\nu_1 + \nu_2) \cosh \gamma_0 \sinh \gamma_0 \right] e^{(\nu_2-n_1)\sigma},
$$

$$
M_{21} = \frac{1}{1-x^2} \left[ \left( \frac{\cosh \gamma_0 \cosh \gamma_0}{\sinh \gamma_0} + \frac{\sinh \gamma_0 \cosh \gamma_0}{\sinh \gamma_0} \right) x + (\nu_1 + \nu_2) \cosh \gamma \sinh \gamma \right] e^{-(\nu_2-n_1)\sigma}.
$$

To un-clutter the equations let us define

$$\tilde{A} = M_{11}, \quad \tilde{B} = M_{22}.$$

It is easy to integrate the differential equation in (3.109) with the initial conditions $\Omega(x, 0) = 1$. This results in

$$\Omega(x, \sigma) = \left( \nu_2 - \nu_1 \pm \sqrt{(\nu_2 - \nu_1)^2 - 4(\tilde{B}^2 - \tilde{A}^2 - (\nu_2 - \nu_1)\tilde{A})} \right).$$

We can simplify the terms in the square root using the Virasoro constraints to obtain

$$2\lambda_\pm = \nu_2 - \nu_1 \pm \tilde{D},
$$

$$\tilde{D} = \left\{ (\nu_2 - \nu_1)^2 - \frac{4}{(1-x^2)^2} \left( \frac{j^2}{4\pi^2\lambda^2} + m^2 + 2x \frac{j\hat{m}}{2\pi\lambda} \right) + \frac{4}{1-x^2} \left( \frac{c_2^2}{\sinh^2 \gamma_0} - \frac{c_1^2}{\cosh^2 \gamma_0} \right) + 4(\nu_2 - \nu_1)\tilde{A} \right\}^{1/2}.
$$

We can now find the quasi-momentum by evaluating (3.108), after substitution for $\Omega$ and using (3.100) we obtain

$$\cos \rho(x) = \cosh \pi \sqrt{\tilde{D}}.$$

which implies

$$p = i\pi \sqrt{\tilde{D}}.$$

Thus we have explicitly solved for the quasi-momentum for these classical solutions which represent winding strings.
We can now verify the behaviour of the quasi-momentum when the spectral parameter $x$ is taken to be near $\pm 1, 0$ and $\infty$. When $x \to \pm 1$, it can be seen from the expression for $\tilde{D}$ in (3.114), that the behaviour given in (3.57) is reproduced. Now as $x \to \infty$ we see that the leading contributions comes from the terms $(\nu_2 - \nu_1)^2$ and the term proportional to $c_1 + c_2$ in $\tilde{A}$ in the expression for $\tilde{D}$. Going through the analysis we find the following behaviour as $x \to \infty$.

\[
p(x) = i\pi k(r_+ + r_-) + \frac{2\pi}{x}(c_1 + c_2) + \cdots, \quad x \to \infty, \tag{3.117}
\]

\[
= i\pi k(r_+ + r_-) - i\frac{E - S}{\lambda x(r_+ + r_-)} + \cdots.
\]

where we have used (3.84) to obtain the second line in the above equation. Comparing (3.117) with equation (3.61) obtained from general considerations we see that they are in agreement. When $x \to 0$, the leading terms in $\tilde{D}$ are given by

\[
\tilde{D} \sim (\nu_1 - \nu_2)^2 + 4\nu_1 \nu_2 - 4x(c_2 - c_1)(\nu_1 + \nu_2). \tag{3.118}
\]

From this behaviour we see $p$ behaves as

\[
p(x) = i\pi (r_+ - r_-)k + i2\pi x(c_1 - c_2) + \cdots, \quad x \to 0, \tag{3.119}
\]

\[
= i\pi (r_+ - r_-)k + i\frac{E + S}{\lambda (r_+ - r_-)} + \cdots.
\]

We see that the above equation is in agreement with (3.67) obtained from general considerations with $m = 0$. Finally from (3.110) we see that just above and below the branch cuts the quasi-momentum just flips sign, thus the condition (3.68) is obeyed with $n_f = 0$.

Since we have the explicit expression for the quasi-momentum given by (3.110) we can obtain the non-local charges $\mathcal{C}$ and $Q$ defined in (3.60) and (3.66) respectively. Expanding the quasi-momentum to $O(1/x^2)$ as $x \to \infty$ we obtain

\[
\mathcal{C} = 2\pi^2(c_1 + c_2)^2 \cosh \pi k(r_+ + r_-) + \frac{2\pi h}{\nu_2 - \nu_1} \sinh \pi k(r_+ + r_-), \tag{3.120}
\]

where

\[
h = \frac{c_2^2}{\sinh^2 \gamma_0} - \frac{c_1^2}{\cosh^2 \gamma_0} + \nu_2^2 \sinh^2 \gamma_0 - \nu_1^2 \cosh^2 \gamma_0 + \nu_1 \nu_2.
\]

Similarly from the $O(x^2)$ terms in the $x \to 0$ expansion of the quasi-momentum we can read out the charge $Q$ which is given by

\[
Q = 2\pi^2(c_2 - c_1)^2 \cosh \pi k(r_+ + r_-) - \frac{2\pi \delta}{\nu_2 + \nu_1} \sinh \pi k(r_+ + r_-) \tag{3.122}
\]
where
\[ \delta = \frac{c_2^2}{\sinh^2 \gamma_0} - \frac{c_1^2}{\cosh^2 \gamma_0} + \nu_2^2 \cosh^2 \gamma_0 - \nu_1^2 \sinh^2 \gamma_0 + \nu_1 \nu_2. \] (3.123)

Using this example of winding strings we have verified the conditions on the quasi-momentum found in the previous section and also demonstrate that this solution corresponds to a density localized on the branch cuts of the quasi-momentum \( p(x) \) given in (3.110).

### 3.3 BMN and magnon like states

In section 3.2 we have listed out the conditions satisfied by the resolvent in terms of the density \( \rho(x) \). These are given in (3.71), (3.72), (3.73) and (3.74). In this section we find two simple solutions to these equations which resemble the BMN states and magnon states found for classical strings propagating on \( R \times S^3 \).

**BMN like states**

These are solutions to the resolvent obtained when the branch cuts shrink to delta functions. For simplicity in illustrating these solutions let us look at the situation when the winding number on \( S^1 \) given by \( \hat{m} \) vanishes and work in the sector \( k \neq 0 \). Let these delta functions be localized at \( x_s \) with strength \( S_s \). Then from (3.74), these positions satisfy

\[ \frac{1}{x_s} = -\frac{\hat{j}}{2\pi \lambda (n_s + ikr_+ + r_-)} \left( 1 - \sqrt{1 + \frac{4\pi^2 \lambda^2}{\hat{j}^2} (n_s + ikr_+ + r_-)^2} \right). \] (3.124)

The equations (3.71), (3.72) and (3.73) reduces to

\[ \sum_s S_s = -\frac{\hat{j}}{\lambda} + i \frac{E - S}{\lambda(r_+ + r_-)}, \] (3.125)
\[ -\sum_s \frac{S_s}{x_s} = 0, \]
\[ -\sum_s \frac{S_s}{x_s^2} = \frac{\hat{j}}{\lambda} + i \frac{E + S}{\lambda(r_+ - r_-)}. \]

We have set \( m = 0 \) since in this approximation the filling fractions are small, we must also have \( r_- = 0 \) for this approximation to be valid.

Following [20], we parametrize the strength of the delta functions as

\[ S_s = \frac{N_s J}{2\pi \lambda} \left( 1 + \sqrt{1 + \frac{4\pi^2 \lambda^2}{\hat{j}^2} (n_s + ikr_+)^2} \right). \] (3.126)
Just as in [20] we can associate $N_s$ to be the occupation numbers. With this parametrization (3.72) reduces to

$$
\sum_s (n_s + ikr_+) N_s = 0.
$$

(3.127)

Since this is a complex equation and $N_s$ are taken to be real, it results in the following two real equations

$$
\sum_s n_s N_s = 0, \quad \sum_s N_s = 0.
$$

(3.128)

Now adding the first and the third equations in (3.125) we obtain

$$
\hat{J} \lambda \sum_s N_s = -i \frac{E - S}{\lambda (r_+ + r_-)} + i \frac{E + S}{\lambda (r_+ - r_-)} = 0.
$$

(3.129)

Finally from the third equation in (3.125) we obtain

$$
\hat{J} \lambda + i \frac{E + S}{\lambda (r_+ - r_-)} = \sum_s N_s \hat{J} 2\pi \lambda \sqrt{1 + \frac{4\pi^2 \lambda^2}{j^2} (n_s + ikr_+)^2}.
$$

(3.130)

Here we have used the fact that $\sum_s N_s = 0$. Matching the real and imaginary parts of the above dispersion relation we obtain the quantum numbers carried by the states in terms of the occupation numbers $N_s$ and the frequency $n_s$. We call these states BMN-like states, since they have a similar dispersion relation.

**Magnon-like states**

Magnon-like solutions can be obtained from the equations (3.71), (3.72) and (3.73) by following the same procedure as in the case of the sigma model on $S^3$ which was done in [25]. The procedure consists of assuming the momentum of the magnon is given by the LHS of (3.72) can be any number $p$. The density is $\rho(x)$ is that the density is $\rho(x)$ is constant along a contour in the $x$-plane given by $i\rho = 1$ say between $x_1$ and $x_2$. There are no branch cuts for the density function, therefore the equation (3.74) does not play a role. Substituting this ansatz in (3.71), (3.72) and (3.73) we obtain the following equations

$$
-i(x_1 - x_2) = -i \frac{\hat{J}}{\lambda} - \frac{i}{\lambda} \frac{E - S}{r_+ + r_-},
$$

$$
-i \ln \frac{x_1}{x_2} = p,
$$

$$
-i \left( \frac{1}{x_1} - \frac{1}{x_2} \right) = \frac{\hat{J}}{\lambda} + \frac{i}{\lambda} \frac{E + S}{r_+ - r_-}.
$$

(3.131)

\[\text{For the case of magnons on } S^3 \text{ the contour had to be a line between two complex conjugate points to ensure that the momentum of the magnon is real.}\]
As mentioned earlier we have replaced the combination $2\pi (\tilde{m} + m + ikr_-)$ as the momentum. The momentum in this case can be complex and thus $x_1$ and $x_2$ need not be at two complex conjugate points. We can now solve for the $x_1, x_2$ and obtain the following dispersion relations for the quantum numbers $E, S$.

$$Q_+ - 2\hat{J}i = \sqrt{Q_-^2 - 16\lambda^2 \sin^2 \frac{P}{2}},$$  \hspace{1cm} (3.132)

where

$$Q_+ = \frac{E + S}{r_+ - r_-} + \frac{E - S}{r_+ + r_-}, \quad Q_- = \frac{E + S}{r_+ - r_-} - \frac{E - S}{r_+ + r_-}. \hspace{1cm} (3.133)$$

Both equations (3.130) and (3.133) resemble the dispersion relations of the quasi-normal modes found by studying the wave equations in the BTZ black hole [5] due to the presence of the imaginary parts. It will be interesting to find explicit classical solutions obeying these dispersion relations.

### 3.4 Relation with the SL$(2, R)$ spin chain

In this section we show that the system of equations (3.71), (3.72), (3.73) and (3.74) which characterize classical solutions in the twisted sector, of the BTZ sigma model can be obtained from the the continuum limit of a twisted version of the SL$(2, R)$ spin chain. The system we will consider is the twisted long range SL$(2, R)$ chain. The Bethe equations of this model are given by

$$\left(\frac{x(u_k + i\frac{1}{2})}{x(u_k - i\frac{1}{2})}\right)^L \exp(2i\pi k(\tilde{r}_+ + \tilde{r}_-)) = \prod_{j=1, j\neq k}^{M} \frac{u_k - u_j - i}{u_k - u_j + i},$$  \hspace{1cm} (3.134)

where $x$’s label the Bethe roots. Note that the important difference between these Bethe equations and the ones for the SU$(2)$ spin chain given in (2.64) is the inversion of the RHS for the case of the SL$(2, R)$ chain. As in the earlier case, $x$ as a function of $u$ is given by the equation (2.63). The cyclicity constraint in this case is given by

$$\prod_{k=1}^{M} \left(\frac{x(u_k + i\frac{1}{2})}{x(u_k - i\frac{1}{2})}\right) = \exp(2\pi ik\tilde{r}_-).$$  \hspace{1cm} (3.135)

The energy of this spin chain is given by

$$D = 2g^2 \sum_{i=1}^{M} \left(\frac{i}{x(u + \frac{1}{2})} - \frac{i}{x(u - \frac{1}{2})}\right).$$  \hspace{1cm} (3.136)

These set of equations defines the twisted version of the SL$(2, R)$ chain.
Let us first obtain the continuum limit of these equations by performing the scaling $u_k \to Lu_k$. Then taking the logarithm of the Bethe equations (3.134) we obtain

$$L \ln \left( \frac{1 - \frac{i}{2Lu_k}}{1 + \frac{i}{2Lu_k}} \right) - 2\pi i k (\tilde{r}_+ + \tilde{r}_-) + 2\pi in = \sum_{j \neq k} \ln \left( \frac{1 + \frac{i}{L(u_k - u_j)}}{1 - \frac{i}{L(u_k - u_j)}} \right).$$ (3.137)

Approximating the sum by an integral and expanding to $O(1/L)$ we obtain

$$- \frac{1}{u} - 2\pi k (\tilde{r}_+ + \tilde{r}_-) + 2\pi n = 2\int dv \frac{\rho(u)}{u-v},$$ (3.138)

where we have introduced a density for the spin. The density satisfies the normalization condition

$$\int du \tilde{\rho}(u) = \frac{M}{L}.$$ (3.139)

The cyclicity constraint given in (3.135) reduces to

$$\int du \frac{\tilde{\rho}(u)}{u} = -2\pi (\tilde{m} - k\tilde{r}_-).$$ (3.140)

Finally the energy of the spin chain given in (3.136) becomes

$$D = \frac{2g^2}{L} \int du \frac{\tilde{\rho}(u)}{u^2} + O(g^4).$$ (3.141)

To relate these equations to that of the integral equations satisfied by the density in the sigma model case, we need to first identify the relation between the variable $u$ of the spin chain with the spectral parameter of the signal model. This is given by

$$u = x + \frac{g'^2}{x}, \quad \text{where} \quad g' = \frac{g}{L}.$$ (3.142)

Then the normalization condition for the density (3.139) becomes

$$\int dx (1 - \frac{g'^2}{x^2}) \tilde{\rho}(u(x)) = \frac{M}{L}.$$ (3.143)

Again using the relation in (2.75) we can rewrite the Bethe equations (3.138) as

$$- \frac{x}{x^2 - g'^2} - 2\pi k (\tilde{r}_+ + \tilde{r}_-) + 2\pi n = 2\int dy \left(1 - \frac{g'^2}{y^2}\right) \frac{\tilde{\rho}(u(x))}{(x-y)(1 - \frac{g'^2}{xy})}.$$ (3.144)

In the leading order in $g$ this equation reduces to

$$- \frac{1}{x} - 2\pi k (\tilde{r}_+ + \tilde{r}_-) + 2\pi n = 2\int dy \frac{\tilde{\rho}(u(x))}{(x-y)}.$$ (3.145)
The cyclicity constraint also reduces to
\[
\int \frac{\tilde{\rho}(x)}{x} = -2\pi (\hat{m} - k\hat{r}_+), \tag{3.146}
\]
where we have again used (2.135). Finally the energy of the spin chain to the leading order is given by
\[
\frac{D}{L} = 2g^2 \int dx \frac{\tilde{\rho}(x)}{x^2}. \tag{3.147}
\]

We can now compare the Bethe equations of the spin chain to that of the sigma model. First consider the sum of the equations given in (3.71) and (3.73) we see that we obtain
\[
\int d\xi \left( 1 - \frac{1}{\xi^2} \right) \rho(\xi) = \frac{i}{2} \left( -\frac{E - S}{\lambda(r_+ + r_-)} + \frac{E + S}{\lambda(r_+ - r_-)} \right). \tag{3.148}
\]
To make the Bethe equations resemble that of the jump condition across branch cuts satisfied by of the resolvent of the sigma model we first scale the spectral parameter of the sigma model by
\[
x \rightarrow \frac{2\hat{J}}{\lambda} x \tag{3.149}
\]
Substituting this scaling in (3.74) we get
\[
2\int d\xi \frac{\rho(\xi)}{x - \xi} = -\frac{x + \pi\lambda^2 \hat{m}}{x^2 \lambda^2} + 2\pi n - 2\pi ik(r_+ + r_-), \tag{3.150}
\]
\[
= -\frac{1}{x} + 2\pi n - 2\pi ik(r_+ + r_-) + O(\lambda^2),
\]
while the equation in (3.148) reduces to
\[
\int d\xi \left( 1 - \frac{\lambda^2}{4\hat{J}^2\xi^2} \right) \rho(\xi) = -\frac{i}{4\hat{J}} \left( \frac{E - S}{r_+ + r_-} - \frac{E + S}{r_+ - r_-} \right). \tag{3.151}
\]
The equation (3.72) of the sigma model remains invariant under the scaling and is given by
\[
\int dx \frac{\rho(x)}{x} = -2\pi (\hat{m} + m - ikr_-). \tag{3.152}
\]
Finally the equation in (3.73) becomes
\[
-\frac{\lambda^2}{2\hat{J}^2} \int dx \frac{\rho(x)}{x^2} = 1 + i \frac{E + S}{\hat{J}(r_+ - r_-)}. \tag{3.153}
\]

We now see that on identification the equation (3.145) (3.146) and (3.147) and (3.143) of the spin chain are the same as (3.150), (3.152), (3.153) and (3.151) of the
sigma model on the following identifications.

\[ g' = \frac{g}{L} = \frac{\lambda}{2J}, \quad \tilde{\rho}(u(x)) = \rho(x), \quad \tilde{\rho}(u(x)) = \rho(x), \quad (3.154) \]

\[ \tilde{r}_+ \rightarrow \tilde{r}_+, \quad \tilde{r}_- \rightarrow \tilde{r}_-, \quad \tilde{m} + m \rightarrow \tilde{m} \]

\[ \frac{1}{4J} \left( \frac{E - S}{(\tilde{r}_+ + \tilde{r}_-)} - \frac{E + S}{(\tilde{r}_+ - \tilde{r}_-)} \right) = \frac{M}{L}, \]

\[ \frac{D}{L} = -1 + \frac{E + S}{J(\tilde{r}_+ - \tilde{r}_-)}. \]

Note that this identification involves the analytical continuation of the parameters \( r_+, r_- \). This completes our proof that at one loop, the Bethe equations of the twisted \( SL(2, R) \) spin chain agrees with that of the finite gap equations of the sigma model.

### 4. Conclusions

We have shown that the sigma model on \( BTZ \times S^1 \) is integrable using the fact that it is locally \( AdS_3 \). We construct the monodromy matrix of the flat connection and studied the general properties of the quasi-momentum. We have obtained integral equations which constrains the quasi-momentum and shown that classical solutions correspond to a density function on the spectral plane. These integral equations have been shown to agree with the continuum limit of a twisted version of the \( SL(2, R) \) spin chain at one loop. For two class of solutions, the geodesics and the winding strings we have evaluated the corresponding quasi-momentum explicitly. Using this we verify that the its properties agree with that obtained by general consideration. Geodesics correspond to solutions with zero density in the spectral plane.

We solved the integral equations for the density function for the BMN like and magnon like solutions and derived their dispersion relations. It will be interesting to construct explicit solutions of the sigma model corresponding to these BMN like and magnon like solutions. Though the BTZ black hole is a well studied, the propagation of strings in this background has not been studied in great detail. It will be interesting to find the allowed string spectrum in this background. A step in this direction will be to find more classical solutions. Our investigations indicate that integrability will be a useful structure to find and organize the spectrum. This will have implications for the dual conformal theory corresponding to the BTZ background just as quasi-normal modes of the scalar field correspond to poles in the two point function of the dual operator.

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A. The case of the extremal black hole

For completeness we wish to repeat the analysis of the quasi-momentum done in main text of the paper for the extremal BTZ black hole. The identification used in the section 3 to obtain the non-extremal black hole from the \(AdS_3\) hyperboloid cannot be used for the extremal case \[^{16}\]. The metric for the extremal BTZ black hole is given by

\[
ds^2 = -\left(\frac{r^2 - r_0^2}{r^2}\right)^2 dt^2 + \frac{r^2}{(r^2 - r_0^2)} dr^2 + r^2 (d\phi + \frac{r_0^2}{r^2} dt)^2, \tag{A.1}
\]

where \(r_0\) is the location of the horizon. We can show that the matrix is locally \(AdS_3\) by the following change of coordinates. Let us define

\[
w^+ = \frac{e^{2r_0(\phi+t)}}{2r_0}, \quad w^- = \phi - t - \frac{r_0}{r^2 - r_0^2}, \quad z = \frac{e^{r_0(\phi+t)}}{\sqrt{r^2 - r_0^2}}, \tag{A.2}
\]

then the metric in (A.1) reduces to the Poincare metric given by

\[
ds^2 = \frac{dz^2 + dw^+ dw^-}{z^2}. \tag{A.3}
\]

However since \(\phi\) is a periodic coordinate, the Poincare metric is subject to indentifications. To write this as a group action, we identify the \(SL(2, \mathbb{R})\) group element as

\[
g = \begin{pmatrix} \frac{1}{z} & \frac{w^-}{z} \\ \frac{w^+}{z} & z^2 + \frac{w^-}{z} \end{pmatrix}. \tag{A.4}
\]

Then the identification

\[
\phi \sim \phi + 2\pi, \quad \text{acts as} \quad g \sim \tilde{A}_{(1)} g A_{(1)}, \tag{A.5}
\]

where

\[
\tilde{A}_{(k)} = \begin{pmatrix} e^{-2\pi kr_0} & 0 \\ 0 & e^{2\pi kr_0} \end{pmatrix}, \quad A_{(k)} = \begin{pmatrix} 1 & 2\pi k \\ 0 & 1 \end{pmatrix}. \tag{A.6}
\]

Thus the monodromy matrix for the extremal BTZ background is given by \[^{3,24}\] but with \(A_k\) given in the above equation. The conserved charges \(E\) and \(S\) in this
case are given by
\[ E + S = -\lambda \left[ \int_0^{2\pi} d\sigma \left( r_0 \text{Tr}(\partial_0 gg^{-1}\sigma^3) \right) \right], \quad (A.7) \]
\[ E - S = -\frac{\lambda}{2} \left[ \int_0^{2\pi} d\sigma \left( \text{Tr}(g^{-1}\partial_0 g\sigma_+) \right) \right], \quad (A.8) \]
where \( \sigma_+ = \sigma^1 + i\sigma^2 \). The behaviour of the quasi-momentum as \( x \to \pm 1 \) is same as before and is given by (3.57). We now examine its behaviour as \( x \to \infty \).

\[ 2 \cos p(x) = \text{Tr} \left[ A_k P \exp \left( \int_0^{2\pi} \left( d\sigma \frac{j_0}{x} + \frac{j_1}{x^2} \cdot \cdot \cdot \right) \right) \right], \quad (A.9) \]
\[ = \text{Tr} \left( (1 + \pi k \sigma_+)(1 + \int_0^{2\pi} d\sigma \frac{j_0}{x} + \frac{j_1}{x^2} \cdot \cdot \cdot ) \right), \]
\[ = 2 + \frac{\pi k}{x} \text{Tr} \left( \int_0^{2\pi} \sigma_+ j_0 d\sigma \right) + \cdot \cdot \cdot , \]
\[ = 2 - \frac{\pi k(E - S)}{x\lambda}. \quad (A.10) \]

Therefore the behaviour of the quasi-momentum is given by
\[ p(x) \sim \sqrt{\frac{\pi k(E - S)}{x\lambda}} + \cdot \cdot \cdot , \quad x \to \infty. \quad (A.11) \]

Note that the quasi-momentum seems to have a square root branch cut at \( \infty \). At \( x \to 0 \) we have
\[ 2 \cos p(x) = \text{Tr}(A_k^{-1}(1 - x \int_0^{2\pi} d\sigma \partial_0 gg^{-1} + \cdot \cdot \cdot)), \quad (A.12) \]
\[ = \text{Tr} \left[ (\cosh 2\pi kr_0 + \sigma^3 \sinh 2\pi kr_0)(1 - x \int_0^{2\pi} d\sigma \partial_0 gg^{-1} + \cdot \cdot \cdot ) \right], \]
\[ = 2 \cosh 2\pi kr_0 + x \sinh 2\pi kr_0 \frac{E + S}{2\lambda r_0}. \]

The final result for the behaviour of the quasi-momentum at \( x \to 0 \) as
\[ p(x) \sim 2\pi \tilde{m} + i2\pi kr_0 + i\frac{x}{4} \frac{E + S}{\lambda r_0}. \quad (A.13) \]

The fact that the quasi-momentum has a square root branch cut as \( x \to \infty \) possibly implies that to construct the resolvent one has to consider the double cover of the spectral plane for this case. It will be interesting to investigate this further.
B. Trajectories

In this section we integrate the equations (3.86) and (3.105) for both the geodesics and the winding strings. Note that the (3.105) reduces to (3.86) on setting $\nu_1 = \nu_2 = 0$, therefore for generality let us examine (3.105). Solving for $\gamma$ as the function of the world sheet time we obtain the following integral

$$
\int d\tau = \frac{1}{2} \int \frac{dx}{\sqrt{A x^3 + B x^2 + C x + D}}
$$

(B.1)

upon substituting

$$x = \sinh^2 \gamma
$$

(B.2)

The constants $A, B, C, D$ are given by and

$$A = \nu_2^2 - \nu_1^2, \quad B = \nu_2^2 - 2\nu_1^2, \quad C = c_2^2 - c_1^2 - \nu_1^2, \quad D = c_2^2 - (\hat{J}^2 + \hat{m}^2).
$$

(B.3)

Geodesics

We first rewrite the equation (3.87) in terms of the BTZ variables and parameter $r, M, j$ using the relations

$$c_2^2 - c_1^2 = \frac{E^2 - S^2}{r_+^2 - r_-^2}, \quad r_\pm^2 = \frac{M}{2} \pm \frac{\sqrt{M^2 - j^2}}{2}, \quad \sinh^2 \gamma = \frac{r_+^2 - r_-^2}{r_+^2 - r_-^2}
$$

(B.4)

(3.8) and (3.84). Note that we have set $\lambda = 2\pi$. We then obtain the following equation for particle trajectory

$$r^2 r' = -\hat{J}^2 (r^4 - Mr^2 + \frac{\hat{J}^2}{4}) + (E^2 - S^2)r^2 + S^2M + ESj
$$

(B.5)

Now this matches exactly with equation (8) of [26] provided we do the following identifications

$$S = -L, \quad \hat{J} = m
$$

(B.6)

where $m$ is the particle mass and $L$ is the conserved angular momentum in the paper [26]. Hence the geodesics given in the paper will also follow.

Winding strings

The integral (B.1) can be solved analytically to give an answer involving Here will will just outline the procedure for solving the integral. Let us denote the roots of the equations

$$Ax^3 + Bx^2 + Cx + D = A(x - x_1)(x - x_2)(x - x_3) = 0,
$$

(B.7)
as \(x_1, x_2, x_3\). Then the integral can be written as

\[
\int d\tau = \frac{1}{2\sqrt{A}} \int \frac{dx}{\sqrt{(x-x_1)(x-x_2)(x-x_3)}}. \tag{B.8}
\]

After the substitution

\[
z = \frac{x-x_1}{x_2-x_1}, \tag{B.9}
\]

the integral reduces to

\[
\int d\tau = \frac{1}{\sqrt{(x_3-x_1)A}} \int \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad k = \sqrt{\frac{x_2-x_1}{x_3-x_1}}. \tag{B.10}
\]

which can be integrated using Jacobi elliptic functions. Thus we have the following solution for \(\gamma(\tau)\)

\[
\sinh^2 \gamma - x_1 = (x_2-x_1)\text{sn}^2(\sqrt{A(x_1-x_3)}(\tau + c)). \tag{B.11}
\]

In passing we mention there are another case in which the integral reduces to simpler function. The case in which the parameters are chosen so that \(C = D = 0\) the integral can be done in terms of hyperbolic functions.

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