An algebraic approach to the study of multipartite entanglement

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Abstract. A simple algebraic approach to the study of multipartite entanglement for pure states is introduced together with a class of suitable functionals able to detect entanglement. On this basis, some known results are reproduced. Indeed, by investigating the properties of the introduced functionals, it is shown that a subset of such class is strictly connected to the purity. Moreover, a direct and basic solution to the problem of the simultaneous maximization of three appropriate functionals for three-qubit states is provided, confirming that the simultaneous maximization of the entanglement for all possible bipartitions is compatible only with the structure of GHZ-states.
1. Introduction

Entanglement plays an important role in applications of quantum mechanics to nanotechnologies, especially in the field of quantum information and communication. The concept of entanglement is mathematical, since it corresponds, in the case of pure states, to the impossibility of writing the quantum state describing a compound physical system as a simple product of states related to the single subsystems. Nevertheless, this mathematical property of the state has interesting physical implications, such as the presence of non classical correlations between quantum systems, even when they are quite far from each other (non locality).

In spite of the strong interest and deep studies developed on this subject, detection and classification of entanglement in multipartite systems are unsolved problems up to date. Indeed, it has not yet been defined a universal quantity able to measure the entanglement level to be associated to a generic pure or even mixed state. (The concept of non entangled state is translated into factorizability for pure states and separability — simple or generalized — for mixed states, though the latter concept includes the former as a special case.) Besides, there are properties that any measurement of entanglement should possess. In particular, it must be discriminant (vanishing iff the state is separable), not increasing under LOCC (local operations and classical communication) and convex (it must not increase when two or more states are combined in a mixture) [1].

Though a general answer to measurement and detection of entanglement is absent, if the system is composed by a few of low dimensional subsystems it is possible to provide conditions for the presence of entanglement, but rarely these conditions are both necessary and sufficient. In 1996, Peres [2] introduced a sufficient condition for the detection of bipartite entanglement that can be written in terms of a functional addressed as negativity, subsequently studied by Horodecki et al, who have proven that this condition is necessary and sufficient for low-dimensional systems [3, 4]. Another important functional is the concurrence introduced by Wootters in [5, 6], which is a true measure of entanglement, both for pure and mixed states, but unfortunately applicable only to systems which are couples of two level systems (i.e., a couple of qubits).

The study of multipartite entanglement is more complex [7, 8, 9], not only from the computational point of view, but even at a conceptual level, since for instance it is neither immediate nor intuitive to understand what is a state with a maximum level of multipartite entanglement. In 2008, Sabin et al. [10], tried to extend the negativity to the tripartite case, succeeding in finding a functional that detects the tripartite entanglement when applied to pure states. However, this functional gives only a clue about the presence of entanglement in mixed states, indeed in this case the condition given is neither necessary nor sufficient. Nevertheless, its effectiveness has been shown in the study of simple physical systems [11]. Another interesting quantity, introduced for detecting entanglement of pure states of three qubits, is the three-tangle [12], which is based on the concurrence and whose validity has been criticized [13]. Recently, a
classification of entangled states for three-qubit states has been given on the basis of some invariant quantities [14]. Moreover, attempts to apply well known results of algebra or geometry to study multipartite entanglement have been made. In [15], for instance, Mäkelä et al associated to every pure state of N-qubits a polynomial, characterizing completely the state, capable not only to detect factorizability of the state but even the number of separable qubits. Instead, in [16], Miyake used the hyperdeterminants and the theory of Segre variety to give a classification of the entanglement of pure states in tripartite systems. A very useful tool to study multipartite entanglement in mixed states is given by the necessary condition for separability expressed by Huber et al [17], which exploits a correlation function defined through replicas of the states under scrutiny and operators of partial swapping, and which has been exploited to reveal thermal tripartite correlations in spin-star systems [18].

Another important quantity that has been used to reveal entanglement in several cases (bipartite and multipartite systems), is the purity (strictly connected to the linear entropy [19]) of the reduced density matrix, which in passing is strictly connected with the concept of mixedness [20]: the more two systems are entangled, the less pure is the reduced state describing each one of the two systems.

In this paper we reproduce some known results about entanglement detection by introducing a simple algebraic approach to establish whether a pure state of a multipartite system is entangled with respect to a given bipartition. Through this analysis it is possible to determine whether a given pure state is completely separable, separable or totally entangled. In the last case one can infer the presence of genuine multipartite entanglement. On the basis of this approach, we naturally introduce a class of functionals which includes, among others, quantities traceable back to the purity and the linear entropy, strictly related to the so called concurrence vectors, presented in [21] by Akhtarshenas and studied by Mintert et al in [22, 23].

The paper is organized as follows. In the next section we describe our approach to the study of multipartite entanglement and give the definition of a relevant class functionals, proving the ability of such quantities to detect entanglement in pure states. In section 3 we show that such quantities are strictly connected with purity and mixedness. In section 4 we show, by exploiting our simple and algebraic approach, that in the case of three qubits, simultaneous maximization of the three relevant functionals is possible only for states equivalent (i.e., equal up to a local and unitary transformation) to the GHZ state. Finally, in the last section, we give some comments and concluding remarks.

2. Factorizability conditions

Each pure state of a bipartite system can be written in the form:

$$|\phi\rangle = \sum_{i,j} a_{ij} |ij\rangle,$$  \hspace{1cm} (1)
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where we take \{\ket{ij}\} as a standard basis in the Hilbert space \(\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2\). This state is said factorizable if it can be written in the form:

\[
(\sum_i \alpha_i \ket{i})(\sum_j \beta_j \ket{j}).
\] (2)

This means that the state is factorizable iff \(a_{ij} = \alpha_i \beta_j, \forall i, j\). So, introducing the matrix of the probability amplitudes \(A = (a_{ij})\), the state \(\ket{\phi}\) is factorizable iff columns (respectively rows) span a 1-dimensional subspace, i.e. \(\text{rank} A = 1\).

In general, this means \(\det A = 0\) and in the case of a two qubits system, in [24] it is proven that this condition is equivalent to the Peres-Horodecki criterion. The condition here expressed can be easily extended to investigate the presence of entanglement in every pure states of multipartite systems. We can summarize this result for bipartite systems with the following theorem.

**Theorem 1.** A pure state of a bipartite \(m \times n\) system is factorizable iff the rank of the corresponding matrix of the probability amplitudes is unity.

Now we can extend this result to tripartite systems and we can generalize it to every multipartite system. Consider a \(m \times n \times p\) Hilbert space. Each pure state \(\ket{\phi}\) living in it can be written in the form:

\[
\ket{\phi} = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \sum_{k=0}^{p-1} a_{ijk} \ket{ijk},
\] (3)

where \(\{\ket{ijk}\}\) is the standard product basis for the Hilbert space related to the whole system. Again, the state is factorizable with respect, for instance, to the first component, iff:

\[
a_{ijk} = \alpha_i \beta_{jk} \quad \forall i, j, k.
\] (4)

This time the matrix of the probability amplitudes is a cubic one and the separability condition of the first component can be traced back to the proportionality of the layers in the direction of \(i\). For a better visualization of this statement we can write two layers in the \(i\)-direction:

\[
\begin{pmatrix}
\alpha_r \beta_{11} & \cdots & \alpha_r \beta_{1p} \\
\vdots & \ddots & \vdots \\
\alpha_r \beta_{n1} & \cdots & \alpha_r \beta_{np}
\end{pmatrix},
\]

\[
\begin{pmatrix}
\alpha_s \beta_{11} & \cdots & \alpha_s \beta_{1p} \\
\vdots & \ddots & \vdots \\
\alpha_s \beta_{n1} & \cdots & \alpha_s \beta_{np}
\end{pmatrix},
\]

which are the \(r^{th}\) and \(s^{th}\) layers.

Starting from these considerations, we can prove the following theorem.

**Theorem 2.** A pure state \(\ket{\phi}\) in a \(m \times n \times p\) Hilbert space is factorizable iff one of the following is true:

- \(a_{ijk}a_{ij'k'} - a_{ij'k}a_{i'jk} = 0\), \(\forall i, j, k, i', j', k'\);
- \(a_{ijk}a_{i'j'k'} - a_{i'jk}a_{ij'k} = 0\), \(\forall i, j, k, i', j', k'\).
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- $a_{ijk}a_{i'j'k'} - a_{i'j'k}a_{ijk} = 0$, $\forall i, j, k, i', j', k'$.

**Proof.** For the sake of simplicity and without loss of generality, let us consider the factorizability condition for the first qubit.

If the state $|\phi\rangle$ is factorizable with respect to such first qubit, using the factorizability condition in equation (4) we have:

$$\alpha_i \beta_j \alpha_{i'} \beta_{j'} \alpha_{i''} \beta_{j''} = 0.$$  

(5)

Vice versa, if we suppose $a_{ijk}a_{i'j'k'} - a_{i'j'k}a_{ijk} = 0$, $\forall i, j, k, i', j', k'$ then, fixing $i, j, k$ such that $a_{ijk} \neq 0$, it turns out that:

$$a_{i'j'k'} = \frac{a_{i'j}k}{a_{ijk}} a_{i'j'k'} = \alpha_{i'} \beta_{j'} k', \quad \forall i', j', k',$$

where we take $\alpha_{i'} = \frac{a_{i'j}k}{a_{ijk}}$ and $\beta_{j'} k' = a_{i'j'k'}$, which is the factorizability condition of the first component. \hfill \Box

Using this theorem we can define a class of functionals able to detect entanglement in each pure state of a tripartite system.

Let $f : \mathbb{C} \to \mathbb{R}$ be a positive function such that $\forall x \in \mathbb{C} f(x) \geq 0$ and $f(x) = 0$ iff $x = 0$, then we have the following:

**Theorem 3.** A pure state $|\phi\rangle$ in a $m \times n \times p$ Hilbert space is factorizable iff at least one of the following is true:

- $M_{1}^{(f)} := \sum_{i,j,k,i',j',k'} f(a_{ijk}a_{i'j'k'} - a_{i'j'k}a_{ijk}) = 0$;
- $M_{2}^{(f)} := \sum_{i,j,k,i',j',k'} f(a_{ijk}a_{i'j'k'} - a_{i'j'k}a_{ijk}) = 0$;
- $M_{3}^{(f)} := \sum_{i,j,k,i',j',k'} f(a_{ijk}a_{i'j'k'} - a_{i'j'k}a_{ijk}) = 0$.

The proof of this result follows immediately from the theorem [2] and from the properties of $f$.

We can make some observation about this. Firstly, if $M_{k}^{(f)} = 0$ then the corresponding state is factorizable with respect to the $k^{th}$ component, so if two of these quantities are zero also the third has to be equal to zero (considering that, for pure states, if two of the three subsystems are separable, then the third one is separable as well). This statement can be easily proven considering the factorizability condition in equation (4) for two of the three components. Secondly, if all the three $M_{k}^{(f)}$ are equal to zero then the state is completely factorizable — i.e., it is the product of three states of the three subsystems — and if each one of them is different from zero the state is genuinely tripartite entangled.

Theorem [3] can be extended to the multipartite case considering all the possible bipartitions of the system. For instance, in the case of a quadripartite system we have seven conditions given by:

- $M_{1}^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l}a_{ijkl}) = 0$;
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• $M_2^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l'}a_{ijkl}) = 0$;
• $M_3^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l'}a_{ijkl}) = 0$;
• $M_4^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l'}a_{ijkl}) = 0$;
• $M_1^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l'}a_{ijkl}) = 0$;
• $M_1^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l'}a_{ijkl}) = 0$;
• $M_2^{(f)} := \sum_{i,j,k,l,i',j',k',l'} f(a_{ijkl}a_{i'j'k'l'} - a_{i'j'k'l'}a_{ijkl}) = 0$.

where $M_k^{(f)}$ functionals are related to separation of the $k$-th subsystem when the other three are considered as a whole, while $M_k^{(f)}$'s are related to the bipartition obtained by considering the couple $k - j$ as a whole and the other two parts as the second subsystem of the bipartition.

3. Connection with the Purity

In this section we prove that if we take as function $f$ the squared modulus, the relevant functionals, simply denoted as $M_k$, are strictly connected with the purity of the relevant reduced density operators. To this end, let us first of all rewrite $M_k$ by considering only two indexes (this means that the two indexes, say $i$ and $j$, will span multipartite subsystems):

$$M_k = \sum_{i,j,i',j'} |a_{ij}a_{i'j'} - a_{ij'}a_{i'j}|^2. \quad (7)$$

Then, by expanding the modulus, one gets:

$$M_k = \sum_{i,j,i',j'} (a_{ij}a_{i'j'}a_{i'j'} - a_{ij'j'}a_{ij}a_{i'j'}) = 0;$$

$$M_k = 2(1 - p) \quad (8)$$

where

$$p = \sum_{i,j,i',j'} a_{ij}a_{i'j'}a_{i'j'} \sum_{i,j,i',j'} a_{ij}a_{i'j'}a_{i'j'} = \sum_{i,j,i',j'} a_{ij}a_{i'j'}a_{i'j'} \quad (9)$$

is the purity of the density operator associated to any of the two subsystems constituting the bipartition. This makes the functional $M_k$ essentially equal (up to the factor 2) to the linear entropy (mixedness) [20].

Therefore, the quantities $M_k$ are invariant under local unitary transformations and are bounded as follows: $0 \leq M_k \leq 2(D - 1)/D$, where $D$ is the dimension of the smaller among the two Hilbert subspaces associated to the bipartition considered. Moreover, maximization is reached when the whole-system state can be written as a superposition of $D$ states with the same weights:

$$|\psi\rangle = D^{-1/2} \sum_{j=1}^{D} e^{i\chi_j} |j\rangle_k |\Phi_j\rangle_{\bar{k}}, \quad (10)$$
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Figure 1. Matrix of the probability amplitudes related to the W-state. The vertexes of the first layer correspond (from top-part left-side, clockwise) to the coefficients: $a_{000}, a_{001}, a_{011}, a_{010}$; the vertexes of the second layer instead correspond to the coefficients: $a_{100}, a_{101}, a_{111}, a_{110}$.

where $|j\rangle_k$ and $|\Phi_j\rangle_k$ refer to the two subsystems constituting the bipartition, and $\langle j| l \rangle = \langle \Phi_j| \Phi_l \rangle = \delta_{jl}$.

3.1. Some Examples

For a better understanding of the results given in the previous section we apply the previously proven criteria to some pure states. First of all, let us consider the W-state which, as well known, is genuinely tripartite entangled:

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle).$$  \hspace{1cm} (11)

The relevant cubic matrix of probability amplitudes is shown in fig. 1.

A rapid calculation gives:

$$M_1 = M_2 = M_3 = \frac{8}{9}.$$  \hspace{1cm} (12)

and therefore, the state is genuinely tripartite entangled in accordance with theorem 3.

As a second example, let us consider the GHZ-state:

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$  \hspace{1cm} (13)

whose matrix of the probability amplitudes is shown in fig. 2.
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Once again a straightforward calculation leads to:

\[ M_1 = M_2 = M_3 = 1. \] (14)

By the way, we mention that in the following section we prove unity to be the maximum value that these quantities can reach for a three-qubit system. Moreover, it is worth noting that maximization of two \( M_k \) does not imply maximization of the third, as shown by the following example. Indeed, consider the state:

\[ |\psi_{\text{bis}}\rangle = \frac{1}{\sqrt{2}} |000\rangle + \frac{1}{\sqrt{2}} |011\rangle, \] (15)

which is factorizable with respect to the first qubit, being \( M_1 = 0 \), but it is not factorizable with respect to the other two qubits, being \( M_2 = M_3 = 1 \).

Finally, let us consider a state for which it is not evident whether it is entangled or not:

\[ |\varphi\rangle = \frac{1}{2\sqrt{26}} (5 |000\rangle + 3 |010\rangle + 2 |001\rangle + 4 |011\rangle + 7 |101\rangle + |111\rangle), \] (16)

Since it turns out that \( M_1 \approx 0.88, M_2 \approx 0.42 \) and \( M_3 \approx 0.70 \) we can conclude that it is a genuinely tripartite entangled state.

4. Maximization of \( M \)-quantities for a three qubit system

In the second example of section 2, we proved that for the GHZ-state all the \( M_k \)'s reach their maximum value. In this section we prove that all pure states of three qubits for which \( M_k = 1 \forall k \) are equivalent to the GHZ-state, i.e. there exists a local and unitary transformation that maps this state into GHZ-state.

Our simple and basic approach makes more transparent the proof of this statement in comparison with the proof given in [25].
Consider a three qubits pure state $|\psi\rangle$ for which it is valid the hypothesis $M_k = 1 \forall k$. Since for this state the $M_k$ reach their maximum, we can write the state in the following form:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_k |\phi_0\rangle_{\bar{k}} + \frac{1}{\sqrt{2}} e^{i \chi} |1\rangle_k |\phi_1\rangle_{\bar{k}},$$  \hspace{1cm} (17)$$

where $\bar{k}$ refers to the two indexes different from $k$, $\{|0\rangle, |1\rangle\}$ is the standard basis of the $k^{th}$ qubit and $|\phi_0\rangle$ and $|\phi_1\rangle$ are such that $\langle \phi_a | \phi_b \rangle = \delta_{ab}$ for $a, b = 0, 1$, simply applying a Schmidt’s decomposition.

By the way, we note that in the case of two qubits, maximization of the unique $M$-functional leads to a Bell-like structure.

In the tripartite case, because of the simultaneous maximization of the three $M_k$’s, we have that this structure is valid for each of the three qubits.

Moreover, this structure is left unchanged by any unitary and local transformation.

Let us consider this structure with respect to the first qubit and, applying again a Schmidt’s decomposition, we can convert the state $|\phi_0\rangle_{23}$ in $(\cos \theta_1 |0\rangle_{23} + e^{i \chi_1} \sin \theta_1 |1\rangle_{23})$, getting:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_1 (\cos \theta_1 |00\rangle_{23} + e^{i \chi_1} \sin \theta_1 |11\rangle_{23})$$

$$+ \frac{1}{\sqrt{2}} e^{i \chi} |1\rangle_1 [|0\rangle_2 (a |0\rangle_3 + b |1\rangle_3) + |1\rangle_2 (c |0\rangle_3 + d |1\rangle_3)],$$  \hspace{1cm} (18)$$

where $\{|0\rangle_l, |1\rangle_l\}$ is a basis, not necessarily standard, for the $l^{th}$ qubit, and $a, b, c, d$ are complex numbers and every coefficient satisfies the normalization and orthogonality condition between the two states $|\phi_0\rangle_{23}$ and $|\phi_1\rangle_{23}$.

Now we can decompose the state $|\psi\rangle$ with respect to the second qubit, getting:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_2 [\cos \theta_1 |00\rangle_{13} + e^{i \chi} |1\rangle_1 (a |0\rangle_3 + b |1\rangle_3)]$$

$$+ \frac{1}{\sqrt{2}} |1\rangle_2 [e^{i \chi_1} \sin \theta_1 |01\rangle_{13} + e^{i \chi} |1\rangle_1 (c |0\rangle_3 + d |1\rangle_3)].$$  \hspace{1cm} (19)$$

The orthogonality conditions involves the orthogonality between the two states $(a |0\rangle_3 + b |1\rangle_3)$ and $(c |0\rangle_3 + d |1\rangle_3)$, which implies $a = \alpha \cos \theta_2$, $b = \alpha e^{i \chi_2} \sin \theta_2$, $c = \beta e^{-i \chi_2} \sin \theta_2$ and $d = \beta \cos \theta_2$.

In addition, exploiting the normalization constrain with respect to the first qubit, we find $\alpha = \cos \theta_3$ and $\beta = e^{i \chi_3} \sin \theta_3$, so that we can rewrite the state in equation (18) as:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_1 |\phi_0\rangle_{23} + \frac{1}{\sqrt{2}} e^{i \chi} |1\rangle_1 |\phi_1\rangle_{23},$$  \hspace{1cm} (20)$$

where
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\[ |\phi_0\rangle_{23} = (\cos \theta_1 |00\rangle_{23} + e^{ix_1} \sin \theta_1 |11\rangle_{23}) \]

and

\[ |\phi_1\rangle_{23} = \cos \theta_3 |0\rangle_2 (\cos \theta_2 |0\rangle_3 + e^{ix_2} \sin \theta_2 |1\rangle_3 \\
+ e^{ix_3} \sin \theta_3 |1\rangle_2 (e^{-ix_2} \sin \theta_2 |0\rangle_3 - \cos \theta_2 |1\rangle_3). \]

Summarizing, we impose the normalization condition of \(|\phi_0\rangle\) e \(|\phi_1\rangle\) with respect to the first qubit and the orthogonality condition with respect to the second qubit. We now have to enforce: 1) The orthogonality condition with respect to the first qubit; 2) The normalization condition with respect to the second qubit; 3) The orthogonality condition with respect to the third qubit; 4) The normalization condition with respect to the third qubit.

The orthogonality condition for the first qubit leads to \(e^{ix_1} \cos \theta_1 \cos \theta_2 \cos \theta_3 - e^{i(x_3-x_1)} \sin \theta_1 \cos \theta_2 \sin \theta_3 = 0\), and so we have the condition:

\[ \theta_2 = \frac{\pi}{2} \lor \cos \theta_1 \cos \theta_3 - e^{i(x_3-x_1)} \sin \theta_1 \sin \theta_3 = 0. \] (21)

Rewriting the state \(|\psi\rangle\) as,

\[ |\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_2 [\cos \theta_1 |00\rangle_{13} + e^{ix_1} \cos \theta_3 |11\rangle_1 (\cos \theta_2 |0\rangle_3 + e^{ix_2} \sin \theta_2 |1\rangle_3)] \\
+ \frac{1}{\sqrt{2}} |1\rangle_2 [\sin \theta_1 |01\rangle_{13} \\
+ e^{i(x_1+x_3)} \sin \theta_3 |1\rangle_1 (e^{-ix_2} \sin \theta_2 |0\rangle_3 - \cos \theta_2 |1\rangle_3)], \] (22)

the normalization condition for the second qubit leads to \((\cos^2 \theta_1 + \cos^2 \theta_3 \cos^2 \theta_2 + \cos^2 \theta_3 \sin^2 \theta_2 = 1)\), that is,

\[ \cos \theta_3 = \sin \theta_1, \] (23)

and likewise for the other state (we consider only this case because possible sign difference between sine and cosine can be described by adjusting the phase factors).

Now, we decompose the state with respect to the third qubit:

\[ |\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle_3 |\phi_0'\rangle_{12} + \frac{1}{\sqrt{2}} |1\rangle_3 |\phi_1'\rangle_{12}, \] (24)

where

\[ |\phi_0'\rangle_{12} = (\cos \theta_1 |00\rangle_{12} + e^{ix} \cos \theta_3 \cos \theta_2 |10\rangle_{12} \\
+ e^{i(x_3+x_2)} \sin \theta_3 \sin \theta_2 |11\rangle_{12}) \]
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and

\[ |\phi'_1\rangle_{12} = (e^{i\chi_1} \sin \theta_1 |01\rangle_{12} + e^{i(x_1+x_2)} \cos \theta_3 \sin \theta_2 |10\rangle_{12} \\
- e^{i(x_1+x_2)} \sin \theta_3 \cos \theta_2 |11\rangle_{12}), \]

and we impose the third condition (orthogonality with respect to the third qubit):

\[(e^{i\chi_2} \cos \theta_2 \sin \theta_2 - e^{i\chi_2} \sin^2 \theta_3 \cos \theta_2 \sin \theta_2 = 0), \]

that leads to

\[\theta_3 = \frac{\pi}{4} \lor \theta_2 = 0 \lor \theta_2 = \frac{\pi}{2}, \quad (25)\]

(again we consider only \(\theta_3 = \frac{\pi}{4}\) because possible sign difference between \(\sin \theta_3\) and \(\cos \theta_3\) can be englobed in the phase factors).

Summarizing, we can have three possible situations:

\[\theta_3 = \frac{\pi}{4} \Rightarrow \theta_1 = \frac{\pi}{4};\]

\[\theta_2 = 0;\]

\[\theta_2 = \frac{\pi}{2}.\]

The normalization condition at point 4, \(\cos^2 \theta_1 + \cos^2 \theta_3 \cos^2 \theta_2 + \sin^2 \theta_3 \sin^2 \theta_2 = 1\) is automatically satisfied for \(\theta_3 = \frac{\pi}{4}\) and \(\theta_1 = \frac{\pi}{4}\) as for \(\theta_2 = 0\). Instead, if \(\theta_2 = \frac{\pi}{2}\) then necessarily \(\theta_1 = \theta_3 = \frac{\pi}{4}\).

We can apply the conditions we have found, writing the initial state as:

\[\frac{1}{\sqrt{2}} |0\rangle_1 (\sin \theta_3 |00\rangle_{23} + e^{i\chi_3} \cos \theta_3 |11\rangle_{23})\]

\[+ \frac{1}{\sqrt{2}} e^{ix} |1\rangle_1 (\cos \theta_3 |0\rangle_2 (\cos \theta_2 |0\rangle_3 + e^{ix_2} \sin \theta_2 |1\rangle_3)\]

\[+ e^{ix_3} \sin \theta_3 (e^{-ix_2} \sin \theta_2 |0\rangle_3 - \cos \theta_2 |1\rangle_3)], \quad (27)\]

where parameters vary in the following way:

\[\theta_3 = \frac{\pi}{4} \forall \theta_2;\]

\[\theta_2 = 0 \forall \theta_3;\]

\[\theta_2 = \frac{\pi}{2}. \quad (28)\]

It remains to prove that this state is equivalent to the GHZ under local unitary transformation.

If \(\theta_2 = 0\), condition \([21]\) assures either \(\chi_1 = \chi_3\) or \(\theta_3 = 0\) or \(\theta_3 = \frac{\pi}{2}\). Let us examine these three possibilities.

If \(\theta_2 = 0\) and \(\theta_3 = 0\) the state becomes:

\[\frac{1}{\sqrt{2}} e^{ix_3} |011\rangle + \frac{1}{\sqrt{2}} e^{ix} |100\rangle, \quad (29)\]

and we can easily find a transformation that sends it in the GHZ.
If $\theta_2 = 0$ and $\theta_3 = \frac{\pi}{2}$ then we have the state:

$$\frac{1}{\sqrt{2}} |000\rangle - \frac{1}{\sqrt{2}} e^{i(\chi+\chi_3)} |111\rangle,$$

and even in this case we can easily find the transformation needed.

If, finally, $\theta_2 = 0$ and $\chi_1 = \chi_3$ the state becomes:

$$\frac{1}{\sqrt{2}} |0\rangle_1 (\sin \theta_3 |00\rangle_{23} + e^{i\chi_3} \cos \theta_3 |11\rangle_{23})$$

$$+ \frac{1}{\sqrt{2}} e^{ix} |1\rangle_1 (\cos \theta_3 |00\rangle_{23} - e^{i\chi_3} \sin \theta_3 |11\rangle_{23}),$$

and it is converted in the GHZ by:

$$\sin \theta_3 |0\rangle_1 + e^{ix} \cos \theta_3 |1\rangle_1 \rightarrow |0\rangle$$

$$e^{i\chi_3} \cos \theta_3 |0\rangle_1 - e^{i(\chi+\chi_3)} \sin \theta_3 |1\rangle_1 \rightarrow |1\rangle$$

We have now to examine the first case of eq. (28), which is $\theta_3 = \frac{\pi}{4}$. In this analysis we have to distinguish the two cases $\theta_2 = \frac{\pi}{2}$ and $\theta_2 \neq \frac{\pi}{2}$.

If $\theta_3 = \frac{\pi}{4}$ and $\theta_2 \neq \frac{\pi}{2}$, the state has the form:

$$\frac{1}{\sqrt{2}} |0\rangle_1 \left( \frac{1}{\sqrt{2}} |00\rangle_{23} + \frac{1}{\sqrt{2}} e^{i\chi_3} |11\rangle_{23} \right)$$

$$+ \frac{1}{\sqrt{2}} |1\rangle_1 \left( \frac{1}{\sqrt{2}} |0\rangle_2 (\cos \theta_2 |0\rangle_3 + e^{i\chi_2} \sin \theta_2 |1\rangle_3) \right)$$

$$+ \frac{1}{\sqrt{2}} e^{i\chi_3} |1\rangle_2 (e^{-i\chi_2} \sin \theta_2 |0\rangle_3 - \cos \theta_2 |1\rangle_3),$$

(remember the (24) assures $\chi_1 = \chi_3$).
In this situation the transformation:

$$
\begin{align*}
|0\rangle_1 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_1 + \frac{1}{\sqrt{2}} e^{i\chi} |\tilde{1}\rangle_1 \\
|1\rangle_1 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_1 - \frac{1}{\sqrt{2}} e^{i\chi} |\tilde{1}\rangle_1 \\
|0\rangle_2 & \rightarrow \cos \frac{\theta_3}{2} |\tilde{0}\rangle_2 + e^{i(\chi_1 - \chi_2)} \sin \frac{\theta_3}{2} |\tilde{1}\rangle_2 \\
|1\rangle_2 & \rightarrow \sin \frac{\theta_3}{2} |\tilde{0}\rangle_2 - e^{i(\chi_1 - \chi_2)} \cos \frac{\theta_3}{2} |\tilde{1}\rangle_2 \\
|0\rangle_3 & \rightarrow \cos \frac{\theta_2}{2} |\tilde{0}\rangle_3 + e^{i\chi_2} \sin \frac{\theta_2}{2} |\tilde{1}\rangle_3 \\
|1\rangle_3 & \rightarrow \sin \frac{\theta_2}{2} |\tilde{0}\rangle_3 - e^{i\chi_2} \cos \frac{\theta_2}{2} |\tilde{1}\rangle_3
\end{align*}
$$

maps the GHZ-state in the one found.

If $\theta_3 = \frac{\pi}{4}$ and $\theta_2 = \frac{\pi}{2}$ then the state reduces to:

$$
\frac{1}{\sqrt{2}} |0\rangle \left( \frac{1}{\sqrt{2}} |00\rangle + e^{i\chi_1} \frac{1}{\sqrt{2}} |11\rangle \right) + \frac{1}{\sqrt{2}} e^{i\chi} |1\rangle \left( e^{i\chi_2} \frac{1}{\sqrt{2}} |01\rangle + e^{i(\chi_3 - \chi_2)} \frac{1}{\sqrt{2}} |10\rangle \right)
$$

and we can apply the transformation:

$$
\begin{align*}
|0\rangle_1 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_1 + \frac{1}{\sqrt{2}} e^{i(\chi - \chi_1 - \chi_2)} |\tilde{1}\rangle_1 \\
|1\rangle_1 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_1 - \frac{1}{\sqrt{2}} e^{i(\chi - \chi_1 - \chi_2)} |\tilde{1}\rangle_1 \\
|0\rangle_2 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_2 + \frac{1}{\sqrt{2}} e^{i(\chi_2 - \chi_3 - \chi_4)} |\tilde{1}\rangle_2 \\
|1\rangle_2 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_2 - \frac{1}{\sqrt{2}} e^{i(\chi_2 - \chi_3 - \chi_4)} |\tilde{1}\rangle_2 \\
|0\rangle_3 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_3 + \frac{1}{\sqrt{2}} e^{i(\chi_1 + \chi_2 - \chi_4)} |\tilde{1}\rangle_3 \\
|1\rangle_3 & \rightarrow \frac{1}{\sqrt{2}} |\tilde{0}\rangle_3 - \frac{1}{\sqrt{2}} e^{i(\chi_1 + \chi_2 - \chi_4)} |\tilde{1}\rangle_3
\end{align*}
$$

that returns the GHZ, and this ends the proof.
5. Discussion

In this paper, by introducing a simple algebraic approach we have reproduced some known results which provide recipes to establish whether a certain multipartite pure state is entangled with respect to a given bipartition. Such an analysis allows to determine whether the pure state describing a system is completely separable, separable or totally entangled. In the last case one can say that such a state possesses genuine multipartite entanglement.

Our treatment naturally led us to introduce a class of functionals which include quantities traceable back to the concepts of purity and linear entropy. Moreover, we have dealt with the problem of the simultaneous maximization of the relevant functionals (purities of the reduced matrices), providing an alternative proof to the known result that in the case of three qubits the only states that maximize all the relevant functionals are the GHZ-state and all the equivalent states (i.e., equal up to local unitary transformations).

This preliminary studies paves the way to the analysis of multipartite entanglement in cases wherein more than three subsystems are involved. In particular, simultaneous maximization of the relevant functionals for $N$-partite systems — with $N \geq 4$ — could provide interesting results and ideas in the study of maximal multipartite entanglement. It is of relevance to stress in addition that other known functionals might be found in the class we have introduced, or, alternatively, new quantities could be introduced and their properties explored.

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