Wigner Function and Quantum Kinetic Theory in Curved Space–Time and External Fields

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Abstract

A new definition of the Wigner function for quantum fields coupled to curved space–time and an external Yang–Mills field is studied on the example of a scalar and a Dirac fields. The definition uses the formalism of the tangent bundles and is explicitly covariant and gauge invariant. Derivation of collisionless quantum kinetic equations is carried out for both quantum fields by using the first order formalism of Duffin and Kemmer. The evolution of the Wigner function is governed by the quantum corrected Liouville–Vlasov equation supplemented by the generalized mass–shell constraint. The structure of the quantum corrections is perturbatively found in all adiabatic orders. The lowest order quantum–curvature corrections coincide with the ones found by Winter.

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I. Introduction

The Wigner–function method is well–known as the most effective bridge between quantum and macroscopic regimes. It allows one to derive from a quantum wave equation a Liouville equation for a quantum distribution function and explore the semiclassical limit, providing thus, a powerful ground for kinetic theory. Recently, the Wigner transformation in curved space–time was studied in different approaches. In the present work, a covariant and gauge invariant Wigner function for quantum fields coupled to external gravitational and Yang–Mills fields is defined by using the horizontal lift of the derivative operator in the tangent bundle. The tangent and the cotangent bundles arise naturally in general relativistic kinetic theory and it was Vlasov, who first recognized this. This paper demonstrates, how the formalism of the bundles allows one to derive quantum Liouville equations explicitly in all orders of the perturbation theory.

Sec. II. recalls some features of general relativistic kinetic theory and gives the definition of a covariant Wigner function, proceeding from considering expressions for dynamical observables (such as the current density vector, the stress–energy tensor and so on) of quantum fields. In Sec. II A. the first order formalism of Duffin and Kemmer is generalized to curved space–time and it is shown that a scalar field in the first order formalism is formally equivalent to a Dirac field. This allows one to derive quantum Liouville equations for different quantum fields in a universal fashion. Sec. III. derives quantum kinetic equations and generalized mass–shell constraints for the Wigner functions in terms of a semiclassical expansion. It is shown, by using the method of Ref. 2, how to eliminate the mass–shell constraint to get a transport equation for a reduced function on the classical mass–shell. In Sec. IV some
unsolved problems are mentioned. The Appendices contain some technical details and useful formulae.

Notations and conventions for a curvature tensor and a Yang–Mills field throughout are such that the commutator of two covariant and gauge invariant derivatives is

\[
[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta] X^\mu = R_{\alpha\beta\mu\nu} X^\nu + \frac{ie}{\hbar} F_{\alpha\beta} X^\mu ,
\]

(1.1)

where \(e\) is a gauge charge and \(\hbar\) is Planck’s constant (the signature for a metric is \((+++−−)\)).

II. Field equations and Wigner functions

I will consider both a scalar quantum field and a Dirac field coupled to external gravitational and Yang–Mills fields. Let’s consider first a scalar field. The model is described by the quantum field equation (see, for instance, Ref. [1]):

\[
(g^{\alpha\beta}(x)\tilde{\nabla}_\alpha \tilde{\nabla}_\beta + m^2/\hbar^2 - \xi R(x)) \varphi(x) = V_{\text{int}}[\varphi] ,
\]

(2.1)

the Einstein equations:

\[
R_{\alpha\beta}(x) - \frac{1}{2} R(x) g_{\alpha\beta}(x) = k \langle T_{\alpha\beta}(x) \rangle + k T_{\alpha\beta}^{YM}(x) ,
\]

(2.2)

and the Yang–Mills equations:

\[
\tilde{\nabla}_\alpha F^{\alpha\beta}(x) = e \langle J^\beta(x) \rangle .
\]

(2.3)

Here \(\xi\) is the nonminimal gravitational coupling constant, \(T_{\alpha\beta}^{YM}(x)\) is the Yang–Mills field’s stress–energy tensor and \(\langle J^\alpha(x) \rangle\) and \(\langle T_{\alpha\beta}(x) \rangle\) are the ensemble averaged current density vector and stress–energy tensor of the scalar quantum field \(\varphi(x)\):

\[
\langle J^\alpha(x) \rangle = \left( \frac{i\hbar}{2} \right) \langle \varphi^\dagger(x) \tilde{\nabla}_\alpha \varphi(x) \rangle ,
\]

(2.4)
\begin{equation}
\langle T_{\alpha\beta}(x) \rangle = \left( \frac{i\hbar}{2} \right)^2 \langle \varphi^\dagger(x) \overset{\leftrightarrow}{\nabla}_\alpha \overset{\leftrightarrow}{\nabla}_\beta \varphi(x) \rangle + 
+ \hbar^2 \left( \frac{1}{4} - \xi \right) \langle \varphi^\dagger(x) \varphi(x) \rangle + \langle T^{\text{inf}}_{\alpha\beta}(x) \rangle. \tag{2.5}
\end{equation}

The last term in Eq. (2.5) depends on quantum interactions (the right hand side of Eq. (2.1)), \( \varphi^\dagger(x) \) is a conjugate field and

\begin{equation}
\overset{\leftrightarrow}{\nabla}_\alpha = \nabla_\alpha - \nabla_\alpha,
\end{equation}

the arrows indicate the directions of acting of the derivative operator.

The ensemble averaging means that we have specified in-particles in an asymptotic past (with not necessary vanishing curvature, but admitting a reasonable definition of particles) and a density matrix \( \rho \) based on in-states. Then the ensemble averaging of an operator \( A(x) \) is

\begin{equation}
\langle A(x) \rangle = Sp(\rho A(x))/Sp \rho.
\end{equation}

Generally speaking, expressions like (2.7) diverge and one has to renormalize it. It can be done in terms of a Wigner function by using, for example, a procedure like the BPHZ-procedure in quantum field theory or the dimensional regularization. We will return to this question elsewhere. In this paper all quantum operators throughout are considered to be somehow regularized.

In the classical limit a system of scalar point particles is described by a distribution function \( F(x,p) \) which is a probability density in the phase space. The crucial feature of general relativistic kinetic theory is that in general relativity the structure of the phase space differs from the one in nongravity physics (see, for instance, Ref. 9).

First, the second argument of a distribution function is a (co)tangent vector on a space–time manifold \( \mathcal{M} \), which belongs to the (co)tangent space or the fibre \( T^*_x(\mathcal{M}) \) over point \( x \) (see,
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for instance, Ref. [10]. Coordinate transformations in the manifold,

$$x'^\alpha = x'^\alpha(x),$$  \hspace{1cm} (2.8)

induce transformations in the fibre:

$$p_{\alpha'} = \frac{\partial x^\alpha}{\partial x'^\alpha} p_\alpha.$$  \hspace{1cm} (2.9)

The set of cotangent spaces over all points of the manifold forms the cotangent bundle over $\mathcal{M}$ with $\mathcal{M}$ being a base space:

$$\mathcal{T}^*(\mathcal{M}) = \bigcup_{x \in \mathcal{M}} \mathcal{T}^*_x(\mathcal{M}),$$  \hspace{1cm} (2.10)

and the distribution function $F(x,p)$ is a scalar function on the base space and the cotangent bundle, that means that under the transformations (2.8), (2.9) it behaves as the following:

$$F(x,p) = F'(x'(x), \frac{\partial x}{\partial x'} \cdot p).$$  \hspace{1cm} (2.11)

An invariant volume element on the base space is known (see, for instance, Ref. [11]),

$$dV^{(4)} = (-g(x))^{1/2} \, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$  \hspace{1cm} (2.12)

Analogously, an invariant volume element on the cotangent bundle is

$$d\Pi^{(4)} = (-g(x))^{-1/2} \, dp_0 \wedge dp_1 \wedge dp_2 \wedge dp_3,$$  \hspace{1cm} (2.13)

but the order of integration over the phase space in general relativity is crucial: to overcome an ambiguity, one should first integrate over the momentum space at fixed points $x$ and then over the space–time manifold. In order to define procedures like coarse graining, which needs integration over finite space–time regions of a distribution function at fixed momentum variables $p$, one has to set a way of comparing the energy–momentum vectors of distant
particles.

Secondly, the energy–momentum vector of a classical point particle, $p_\alpha$, is constrained by the mass–shell equation:

$$g^{\alpha\beta}(x) p_\alpha p_\beta - m^2 = 0 , \quad (2.14)$$

where $m$ is the particle’s mass (we suppose that all particles are identical). Moreover, the energy $p_0$ of a particle must be positive. So, the particle number in a phase space volume element is

$$dN = F(x,p) \theta(p_0) \delta(p_0 - m^2) d\Pi^{(4)} dV^{(4)} . \quad (2.15)$$

One can include $\theta$– and $\delta$–functions to the definition of the volume element on the physical sector of the phase space. Then after integration over $p_0$ we will recover the 7–dimensional phase space. But as one will see below, for quantum particles the mass–shell constraint is more complicated and not only expressed by $\delta$–function. It is thus more natural to define the classical distribution function as a distributed function,

$$f_{cl}(x,p) = F(x,p) \theta(p_0) \delta(p_0 - m^2) , \quad (2.16)$$

which has to be recovered from a quantum distribution function in the classical limit. If particles couple to external (selfconsistent) gravitational and electromagnetic fields, the Liouville’s equation which governs the dynamics of the distribution function is then

$$(p^\alpha D_\alpha - eF_{\alpha\beta} p_\alpha \frac{\partial}{\partial p_\alpha}) f_{cl}(x,p) = C[f_{cl}] , \quad (2.17)$$

where

$$D_\alpha = \nabla_\alpha + \Gamma^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\beta} , \quad (2.18)$$

$\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols and $C[f]$ is a collision integral depended on the mechanism of particles interactions. The operator $D_\alpha$ is called the horizontal lift of the derivative operator.
in the cotangent bundle and it was introduced to kinetic theory by Vlasov. It is invariant under the transformations (2.8), (2.9).

The current density vector and the stress–energy tensor are expressed in terms of moments of the distribution function in the momentum space:

\[ \langle J_{\alpha}(x) \rangle = \int \frac{d^4p}{\sqrt{-g(x)}} p_\alpha f_{cl}(x,p) , \]  

(2.19)

\[ \langle T_{\alpha\beta}(x) \rangle = \int \frac{d^4p}{\sqrt{-g(x)}} p_\alpha p_\beta f_{cl}(x,p) + \langle T_{\alpha\beta}^{int}(x) \rangle , \]  

(2.20)

where the second term in Eq. (2.20) depends on particles interactions.

The goal of kinetic theory is to replace the field equation (2.1) by a kinetic equation for a quantum distribution function and express the expectation values (2.4), (2.5) in terms of this function, so that in the classical limit (which has to be defined properly) we will restore all the set of Eqs. (2.16)–(2.20). The most convenient method for exploring the classical limit is the Wigner–function one. In the special relativistic quantum field theory the (one-particle) Wigner function is the Fourier transform of a Green function with respect to the difference of its coordinates:

\[ f(x,p) = (\pi\hbar)^{-4} \int d^4s e^{-2is^\alpha p_\alpha/\hbar} \langle \varphi(x_1) \varphi^\dagger(x_2) \rangle , \]  

(2.21)

where

\[ s^\alpha = \frac{1}{2}(x_1^\alpha - x_2^\alpha) , \]  

(2.22)

\[ x^\alpha = \frac{1}{2}(x_1^\alpha + x_2^\alpha) . \]  

(2.23)

In general relativity the situation becomes ambiguous because the definitions of the difference of coordinates (2.22) and a middle point (2.23) are not invariant under transformations of coordinates and there is no natural covariant generalization of the Fourier
transformation. The first attempt to generalize the definition (2.21) to general relativity
and to evaluate the lowest order curvature corrections to the Vlasov equation was made by
Winter. His definition is based on considering a geodesic connecting two points $x_1$ and
$x_2$. For evaluating the quantum corrections to the Vlasov equation in Winter’s method one
has to solve the geodesic and the geodesic deviation equations, which is a very complicated
problem even in the lowest adiabatic order.

A different approach is based on introducing Riemann normal coordinates centered
at any point in the vicinity of $x_1$ and $x_2$, on defining the difference of coordinates and the
middle point by Eqs. (2.22),(2.23) in this coordinate system and on applying a certain dif-
ferential operator on the Wigner function, in order to remove the dependence on the choice
of a center of Riemann coordinates. This method is also covariant and is equivalent to the
Winter’s one, but it saves much work while deriving the quantum corrections to the Vlasov
equation.

An alternative approach was proposed in Ref. [14], which uses the formalism of the
tangent bundles. This approach is explicitly covariant and also gauge invariant from the
beginning and allows one to evaluate the quantum corrections to the kinetic equation up to
any adiabatic order with few efforts. I will recall the main idea of Ref. [14].

Consider first the expression for the current density vector (2.4). Let me rewrite this
expression in the following way:

$$
\langle J_\alpha(x) \rangle = \frac{i\hbar}{2} \int_{T_x(M)} d^4y \left( \frac{\partial}{\partial y^\alpha} \delta^4(y) \right) \langle Tr \Phi(x,-y)\Phi^\dagger(x,y) \rangle, \quad (2.24)
$$

where $y^\alpha$ is a tangent vector belonging to the tangent space $T_x(M)$ at a point $x$, $\delta^4(y)$ is the
4-dimensional $\delta$-function, the trace is taken in the gauge group’s representation space and

\[8\]
\( \Phi(x, -y) \) and \( \Phi^\dagger(x, y) \) are the following series in the operators \( \tilde{\nabla}_\alpha \) acting to the fields \( \varphi(x) \) and \( \varphi^\dagger(x) \) at a point \( x \):

\[
\Phi(x, -y) = \left( 1 - y^\alpha \tilde{\nabla}_\alpha + \frac{1}{2!} y^\alpha y^\beta \tilde{\nabla}_\alpha \tilde{\nabla}_\beta - \frac{1}{3!} y^\alpha y^\beta y^\gamma \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\nabla}_\gamma + \ldots \right) \varphi(x) \quad (2.25)
\]

\[
\Phi^\dagger(x, y) = \left( 1 + y^\alpha \tilde{\nabla}_\alpha + \frac{1}{2!} y^\alpha y^\beta \tilde{\nabla}_\alpha \tilde{\nabla}_\beta + \frac{1}{3!} y^\alpha y^\beta y^\gamma \tilde{\nabla}_\alpha \tilde{\nabla}_\beta \tilde{\nabla}_\gamma + \ldots \right) \varphi^\dagger(x) \quad (2.26)
\]

The \( \delta \)-function in Eq. (2.24) can be represented by the standard way in a covariant manner, if one integrates over the cotangent space \( T^*_x(M) \) at a point \( x \):

\[
\delta^4(y) = (\pi \hbar)^{-4} \int_{T^*_x(M)} d^4 p e^{-2i y^\alpha p_\alpha / \hbar} . \quad (2.27)
\]

Substituting the last expression into Eq. (2.24) gives the representation for the current density vector as an integral over the momentum space:

\[
\langle J_\alpha(x) \rangle = \int \frac{d^4 p}{\sqrt{-g(x)}} p_\alpha Tr f(x, p) , \quad (2.28)
\]

with the scalar function \( f(x, p) \) being

\[
f(x, p) = (\pi \hbar)^{-4} \sqrt{-g(x)} \int d^4 y e^{-2i y^\alpha p_\alpha / \hbar} \langle \Phi(x, -y) \Phi^\dagger(x, y) \rangle . \quad (2.29)
\]

The expression (2.28) for the current density vector looks like the expression (2.19) in the kinetic theory, this indicates that we are on the right path. To be more sure, I will also represent the stress–energy tensor (2.5) in terms of the function (2.29). It can be done by the same way as for the current density vector by noting that an item

\[
\left( \frac{i \hbar}{2} \right)^n \langle \varphi^\dagger(x) \tilde{\nabla}^\dagger_\alpha_1 \tilde{\nabla}^\dagger_\alpha_2 \ldots \tilde{\nabla}^\dagger_\alpha_n \varphi(x) \rangle , \quad (2.30)
\]

where the indices are symmetrized with \( \frac{1}{n!} \)–multiplier, can be represented by the same steps (2.24)–(2.27) as the following:

\[
\int \frac{d^4 p}{\sqrt{-g(x)}} p_{\alpha_1} p_{\alpha_2} \ldots p_{\alpha_n} Tr f(x, p) , \quad (2.31)
\]
with the same function (2.29). After that, we can write the noninteracting part of the stress–energy tensor (2.3) as an integral over the momentum space:

$$\langle T_{\alpha\beta}(x) \rangle = \int \frac{d^4 p}{\sqrt{-g(x)}} p_\alpha p_\beta Trf(x, p) +$$

$$+ \hbar^2 \left( \xi R_{\alpha\beta}(x) + \left( \frac{1}{4} - \xi \right) (\nabla_\alpha \nabla_\beta - g_{\alpha\beta}(x) \Box) \right) \int \frac{d^4 p}{\sqrt{-g(x)}} Trf(x, p) + \langle T_{\alpha\beta}^{\text{int}}(x) \rangle . (2.32)$$

If one compares the expressions (2.28) and (2.32) with (2.19), (2.20), one can guess that in the classical limit the function $Trf(x, p)$ will tend to the classical distribution function (2.16).

The classical limit for the nonrelativistic Wigner function is usually explored from its transport equation (see, for instance, Ref. 15). So now, the next step is to evaluate the general relativistic quantum kinetic equation. But the definition (2.29) for a general relativistic Wigner function is still useless because of the expansions (2.25), (2.26). The covariant derivative operator $\tilde{\nabla}_\alpha$ doesn’t annihilate the vector $y^\beta$ and one has to work a little harder to collect the series in Eqs. (2.25), (2.26) into a convenient operator, as it was in nongravity physics (compare with Ref. 13). To do that, let me introduce the horizontal lift of the covariant derivative on the tangent bundle:

$$\hat{\nabla}_\alpha = \tilde{\nabla}_\alpha - \Gamma^\beta_{\alpha\gamma} y^\gamma \frac{\partial}{\partial y^\beta} . (2.33)$$

(compare with Eq. (2.18)). It can be easily shown that the operator $\hat{\nabla}_\alpha$ does annihilate $y^\beta$ and so, we can rewrite the n–th term of the series (2.25), (2.26) in the following way:

$$\frac{1}{n!} y^{\alpha_1} \cdots y^{\alpha_n} \hat{\nabla}_{\alpha_1} \cdots \hat{\nabla}_{\alpha_n} = \frac{1}{n!} \left( y^\alpha \hat{\nabla}_\alpha \right)^n . (2.34)$$

After that we get compact expressions:

$$\Phi(x, -y) = \exp(-y^\alpha \hat{\nabla}_\alpha) \varphi(x) , (2.35)$$

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\[ \Phi^\dagger(x,y) = \exp(y^\alpha \hat{\nabla}_\alpha) \varphi^\dagger(x) . \] (2.36)

Eqs. (2.29),(2.35) and (2.36) give us the definition of a covariant and gauge invariant Wigner function. As one will see below, this definition is very convenient for evaluating the quantum kinetic equation, but it looks rather formal if one wishes to compute the Wigner function explicitly for a concrete system. It is shown in Appendix A that the Wigner function (2.29) can be expressed in terms of base space quantities only, and conjunctions with other approaches are discussed.

Let’s now consider a Dirac field \( \psi(x) \) coupled to external gravitational and Yang–Mills fields and satisfying the generalized Dirac equation (see, for instance, Ref. 6):

\[ \left( i \hbar \gamma^\alpha(x) \hat{\nabla}_\alpha - m \right) \psi(x) = V_{\text{int}}[\psi] , \] (2.37)

where \( \gamma^\alpha(x) \) are the Dirac matrices in curved space–time obeying the (anti)commutative rule:

\[ \gamma^\alpha(x) \gamma^\beta(x) + \gamma^\beta(x) \gamma^\alpha(x) = 2g^{\alpha\beta}(x) , \] (2.38)

and the acting of the derivative operator to a Dirac field is determined by the Fock–Ivanenko connection \( \Omega_\alpha(x) \):

\[ \hat{\nabla}_\alpha \psi(x) = \frac{\partial}{\partial x^\alpha} \psi(x) - \Omega_\alpha(x) \psi(x) + \frac{ie}{\hbar} A_\alpha(x) \psi(x) \] (2.39)

\( (A_\alpha(x) \) is the Yang–Mills potential).

A Dirac conjugate field \( \bar{\psi}(x) = \psi^\dagger(x) \gamma(x) \), with \( \gamma(x) \) commuting with all \( \gamma^{\dagger \alpha}(x) \), satisfies the equation:

\[ \bar{\psi}(x) \left( -i \hbar \gamma^\alpha \hat{\nabla}_\alpha - m \right) = \bar{\psi}(x) , \] (2.40)

with

\[ \hat{\nabla}_\alpha \bar{\psi}(x) = \frac{\partial}{\partial x^\alpha} \bar{\psi}(x) + \psi(x) \Omega_\alpha(x) - \frac{ie}{\hbar} \bar{\psi}(x) A_\alpha(x) . \] (2.41)
A covariant and gauge invariant Wigner function for a Dirac field can be defined by the same trick as for the scalar field. It is the following (matrix) function (the sign minus appears here because of the Fermi–statistics):

$$N(x,p) = -(\pi\hbar)^{-4} \sqrt{-g(x)} \int d^4 y \, e^{-2i\gamma^a p_\alpha / \hbar} \langle \Psi(x,-y) \bar{\Psi}(x,y) \rangle , \quad (2.42)$$

where the functions $\Psi(x,-y)$ and $\bar{\Psi}(x,y)$ are related to the Dirac field $\psi(x)$ and the conjugate field $\bar{\psi}(x)$ in the same way as for a scalar field (Eqs. (2.35), (2.36) and (2.33)):

$$\Psi(x,-y) = \exp(-g^a \hat{\nabla}_a) \psi(x) , \quad (2.43)$$

$$\bar{\Psi}(x,y) = \exp(g^a \hat{\nabla}_a) \bar{\psi}(x) . \quad (2.44)$$

In Ref. [14] the noninteracting part of the stress–energy tensor, the current density vector and the spin tensor of a Dirac field are expressed by using the generalized Gordon decomposition (see, for instance, Ref. [17]), in terms of the independent components of the Wigner function (2.42):

$$A(x,p) = m^{-1} Tr N(x,p) , \quad (2.45)$$

$$A^{\alpha\beta}(x,p) = \frac{1}{2} im^{-1} Tr \left( \gamma^{[\alpha}(x) \gamma^{\beta]}(x) N(x,p) \right) , \quad (2.46)$$

$$B(x,p) = -im^{-1} Tr \left( \gamma^5 N(x,p) \right) , \quad (2.47)$$

where $\gamma^5 = -i/4! \sqrt{-g(x)} \varepsilon_{\alpha\beta\mu\nu} \gamma^\alpha(x) \gamma^\beta(x) \gamma^\mu(x) \gamma^\nu(x)$.

At the end of this section, I would like to note that a coupling to an external scalar field $\phi(x)$ can be described formally by changing the mass terms in Eqs. (2.1) and (2.37), (2.40):

$$m \mapsto m + \phi(x) . \quad (2.48)$$
Such coupling doesn’t influence the definition of the Wigner functions but does influence the kinetic equation.

A. The covariant Duffin–Kemmer formalism

To derive the quantum kinetic equation it will be convenient for us to describe the scalar field within the first order formalism, in which the field equation looks formally like the Dirac equation. In special relativity this type of formalism was proposed by Duffin\textsuperscript{18} and developed by Kemmer\textsuperscript{19}. I will extend it to general relativity. Let me introduce the new fields:

\[
\begin{align*}
\textbf{u}_\alpha(x) &= i\hbar\mu^{-1/2}\hat{\nabla}_\alpha \varphi(x) \quad ; \quad \textbf{u}_4(x) = \mu^{1/2} \varphi(x), \\
\end{align*}
\] (2.49)

where \(\mu\) is an arbitrary parameter of the mass dimension. Uniting the fields (2.49) into one five–component field:

\[
\textbf{u}(x) = \begin{pmatrix} \textbf{u}_\alpha(x) \\ \textbf{u}_4(x) \end{pmatrix},
\] (2.50)

and defining the \(5 \times 5\)–matrices:

\[
\begin{align*}
\Gamma^\alpha(x) &= \begin{pmatrix} 0 & \delta^\alpha_\nu \\ - & - & - & - \\ g^{\alpha\beta}(x) & 0 \end{pmatrix}, \\
M(x) &= \begin{pmatrix} \mu \delta^\beta_\nu & 0 \\ - & - & - \\ O & m(x) \end{pmatrix},
\end{align*}
\] (2.51, 2.52)

with

\[
m(x) = \mu^{-1}(m^2 - \xi h^2 R(x)),
\] (2.53)
one can rewrite the equation (2.1) in a form which is formally the same as for a Dirac field (2.37) (with a matrix variable mass):

\[(i\hbar \Gamma^\alpha(x) \tilde{\nabla}_\alpha - M(x)) u(x) = V'_{\text{int}}[u], \tag{2.54} \]

the only difference is that the matrices $\Gamma^\alpha(x)$ obey the Duffin–Kemmer commutative rule rather than the Dirac one (2.38):

\[\Gamma^\alpha(x)\Gamma^\nu(x)\Gamma^\beta(x) + \Gamma^\beta(x)\Gamma^\nu(x)\Gamma^\alpha(x) = g^\nu\alpha(x)\Gamma^\beta(x) + g^\nu\beta(x)\Gamma^\alpha(x). \tag{2.55} \]

The covariant differentiation in Eq. (2.54) is defined by the connection $\Lambda_\alpha(x)$:

\[\Lambda_\alpha(x) = \begin{pmatrix} \Gamma^{\beta}_{\alpha\nu}(x) & O \\ - & - \\ O & 0 \end{pmatrix}, \tag{2.56} \]

so that (compare with Eq. (2.39))

\[\tilde{\nabla}_\alpha u(x) = \frac{\partial}{\partial x^\alpha} u(x) - \Lambda_\alpha(x)u(x) + \frac{ie}{\hbar}A_\alpha(x)u(x). \tag{2.57} \]

If we define the matrix $\Gamma(x)$, which commutes with all $\Gamma^\dagger\alpha(x)$:

\[\Gamma(x) = \begin{pmatrix} g^{\alpha\beta}(x) & O \\ - & - \\ O & 1 \end{pmatrix}, \tag{2.58} \]

then the conjugate field

\[\bar{u}(x) = u^\dagger(x)\Gamma(x) = \left(u^{\dagger\alpha}(x), u^{\dagger}_{4}(x)\right) \tag{2.59} \]

satisfies the following equation:

\[\bar{u}(x) \left(-i\hbar \Gamma^\alpha(x) \tilde{\nabla}_\alpha - M(x)\right) = V'_{\text{int}}[u]. \tag{2.60} \]
It can be checked that the covariant derivative operator annihilates the $\Gamma$–matrices (2.51) and (2.58) as well as the Dirac $\gamma$–matrices:

$$\hat{\nabla}_\alpha \Gamma^\beta(x) = \frac{\partial}{\partial x^\alpha} \Gamma^\beta(x) - \left[ \Lambda_\alpha(x), \Gamma^\beta(x) \right] + \Gamma^\beta_\alpha\nu(x) \Gamma^\nu(x) = 0 \ ,$$

(2.61)

$$\hat{\nabla}_\alpha \Gamma(x) = \frac{\partial}{\partial x^\alpha} \Gamma(x) - \Lambda^\dagger_\alpha(x) \Gamma(x) - \Gamma(x) \Lambda^\alpha(x) = 0 \ .$$

(2.62)

The analogy between the scalar field in the first order formalism and the Dirac field will show itself more explicitly if one looks at the expressions for the commutator of two covariant derivatives acting to each field. If one denotes:

$$u_s(x) = \begin{cases} u(x) & \text{for } s = 1 \text{ (the scalar field)} \\ \Psi(x) & \text{for } s = 1/2 \text{ (the Dirac field)} \end{cases}$$

(2.63)

and

$$\Gamma_s^\alpha(x) = \begin{cases} \Gamma^\alpha(x) & \text{for } s = 1 \\ \gamma^\alpha(x) & \text{for } s = 1/2 \end{cases}$$

(2.64)

then it can be obtained easily from Eqs. (2.39),(2.57) that

$$\left[ \hat{\nabla}_\alpha, \hat{\nabla}_\beta \right] u_s(x) = A_{s\alpha\beta}(x) u_s(x) \ ,$$

(2.65)

with

$$A_{s\alpha\beta}(x) = s^2 R_{\alpha\beta\mu\nu}(x) \Gamma^\mu_s(x) \Gamma^\nu_s(x) + \frac{ie}{\hbar} F_{\alpha\beta}(x) \ .$$

(2.66)

For the conjugate fields one gets:

$$\left[ \hat{\nabla}_\alpha, \hat{\nabla}_\beta \right] \bar{u}_s(x) = -\bar{u}_s(x) A_{s\alpha\beta}(x) \ .$$

(2.67)

Let me now define the unified Wigner function:

$$N_s(x,p) = \epsilon_s (\pi\hbar)^{-4} \sqrt{-g(x)} \int d^4 y \ e^{-2ig^\alpha p_\alpha/\hbar} \langle U_s(x,-y) \bar{U}_s(x,y) \rangle \ ,$$

(2.68)
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with $U_s(x,-y)$ and $\bar{U}_s(x,y)$ being related to $u_s(x)$ and $\bar{u}_s(x)$ respectively by means of formulae analogous to (2.35),(2.36) and

$$\epsilon_s = \begin{cases} 1 & \text{for } s = 1 \\ -1 & \text{for } s = 1/2 \end{cases}.$$

Then for $s = 1/2$ one will get the Wigner function of the Dirac field (2.42) immediately.

In order to extract the Wigner function of the scalar field (2.29) from (2.68), one should introduce the projection operators:

$$P_1 = \frac{1}{3} \left( \Gamma^\alpha(x) \Gamma_\alpha(x) - 1 \right) = \begin{pmatrix} O & O \\ - & - \\ O & 1 \end{pmatrix}, \quad (2.70)$$

$$P_2 = \frac{1}{3} \left( 4 - \Gamma^\alpha(x) \Gamma_\alpha(x) \right) = \begin{pmatrix} \delta_{\nu}^\beta & 0 \\ - & - \\ 0 & 0 \end{pmatrix}, \quad (2.71)$$

with the following properties:

$$P_1 + P_2 = 1 \ ; \ P_1 P_2 = 0 \ ; \ P_1^2 = P_1, \quad (2.72)$$

$$P_1 \Gamma^\alpha(x) = \Gamma^\alpha(x) P_2 \ ; \ P_2 \Gamma^\alpha(x) = \Gamma^\alpha(x) P_1, \quad (2.73)$$

$$\Gamma^\alpha(x) \Gamma^\beta(x) P_1 = g^{\alpha\beta}(x) P_1.$$

Then

$$f(x,p) = \mu^{-1} \text{tr} \left( P_1 N_1(x,p) \right). \quad (2.75)$$

In conclusion, it is worth noting that the Duffin–Kemmer formalism can be introduced for a vector field also. The evaluation of a quantum kinetic equation in this case was carried out in Ref. [1].
III. Quantum kinetic equations and mass–shell constraints

A. The first order equation for the unified Wigner function

If one takes into account the formal analogy between the Dirac field and the scalar field in the first order formalism, one can guess that the equations which govern the dynamics of the Wigner functions, look formally similar for both fields. These equations can be extracted from the equation for the unified Wigner function (2.68). Before proceeding to the derivation of this equation, I will prove the following identity (see also Ref. 20):

\[ \tilde{D}_\alpha N_s(x, p) = \epsilon_s (\pi \hbar)^{-1} \sqrt{-g(x)} \int d^4 y e^{-2iy^\sigma p_\sigma / \hbar} \hat{\nabla}_\alpha \langle U_s(x, -y) U_s(x, y) \rangle, \]  

(3.1)

where \( \tilde{D}_\alpha \) is the horizontal lift of the derivative operator in the cotangent bundle (Eq. (2.18) with the minimal gauge invariant extension) and \( \hat{\nabla}_\alpha \) is the horizontal lift of the derivative operator in the tangent bundle (2.33).

The identity (3.1) can be easily proved if one works in the Riemann normal coordinates centered at point \( x_1 \). In this coordinate system both the operator \( \tilde{D}_\alpha \) and the operator \( \hat{\nabla}_\alpha \) coincide with the gauge invariant extension of the partial derivative operator \( \tilde{\partial}_\alpha \) and the identity (3.1) is trivial. The proof will be completed if one takes into account the invariance of the derivative operators \( \tilde{D}_\alpha \) and \( \hat{\nabla}_\alpha \) and of the volume element in Eq. (3.1).

Utilizing now the identities (B2),(B8) from Appendix B, we will get, after integrating once by parts:

\[ \tilde{D}_\alpha N_s(x, p) = \frac{2i}{\hbar} \epsilon_s p_\alpha N_s(x, p) + \epsilon_s (\pi \hbar)^{-1} \sqrt{-g(x)} \int d^4 y e^{-2iy^\sigma p_\sigma / \hbar} \left( e^{-y^\nu \hat{\nabla}_\nu u_s(x)} \hat{\nabla}_\alpha \langle U_s(x, y) - G_{\alpha\alpha}(x, y) \rangle \right), \]  

(3.2)

\[ + \epsilon_s (\pi \hbar)^{-1} \sqrt{-g(x)} \int d^4 y e^{-2iy^\sigma p_\sigma / \hbar} \langle e^{-y^\nu \hat{\nabla}_\nu u_s(x)} \hat{\nabla}_\alpha \langle U_s(x, y) - G_{\alpha\alpha}(x, y) \rangle \rangle. \]
where

\[ G_{s\alpha}(x, y) = U_s(x, -y) \left( \hat{G}_\alpha(x, y) \bar{U}_s(x, y) \right) - \left( \hat{G}_\alpha(x, -y) U_s(x, -y) \right) \bar{U}_s(x, y) + 2 \left( \hat{H}_\alpha(x, -y) U_s(x, -y) \right) \bar{U}_s(x, y) . \] (3.3)

So far, the equality (3.2) is an identity. In order to obtain an equation, let me multiply it by \( i\bar{\hbar}/2 \Gamma_s^\alpha(x) \) (with a summation over \( \alpha \)). Taking into account the property (2.61) and using the field equations (2.37) or (2.54) together with the identity (B5), one will get:

\[ \hat{M}(x, -\frac{i\bar{\hbar}}{2} \frac{\partial}{\partial p}) N_s(x, p) = \Gamma_s^\alpha(x) \left( p_\alpha + \frac{i\hbar}{2} \tilde{D}_\alpha \right) N_s(x, p) + \]
\[ + \frac{i\hbar}{2} \Gamma_s^\alpha(x) \epsilon_s (\pi \hbar)^{-4} \sqrt{-g(x)} \int d^4 y e^{-2i\gamma'^{\nu} p_\nu / \bar{\hbar}} \langle G_{s\alpha}(x, y) \rangle + C_s^{col} , \] (3.4)

where \( \hat{M}(x, -\frac{i\bar{\hbar}}{2} \frac{\partial}{\partial p}) \) is given by Eq. (B6) with the substitution:

\[ y^\alpha \mapsto \frac{i\hbar}{2} \frac{\partial}{\partial p_\alpha} , \] (3.5)

and \( C_s^{col} \) is a "collision" term which depends on the interaction terms in Eqs. (2.37) and (2.54). I will not discuss this term in this paper but return to this question elsewhere. In the rest of the paper I will omit the "collision" term and deal only with collisionless kinetic equations.

The equation (3.4) will be a kinetic equation if one also expresses the second term on the right–hand side in terms of the function \( N_s(x, p) \). It can be done by using the results of Appendix C. Substituting Eq. (C12) into the integrand in Eq. (3.4), using the identity (3.1) once again and integrating by parts, one will get the following (collisionless) equation:

\[ \hat{M}(x, -\frac{i\hbar}{2} \frac{\partial}{\partial p}) N_s(x, p) = \Gamma_s^\alpha(x) \left( \frac{i\hbar}{2} \tilde{R}_1^\nu \tilde{D}_\nu + \tilde{R}_2^\nu p_\nu \right) N_s(x, p) + \]
\[ + \frac{i\hbar}{2} \Gamma_s^\alpha(x) \tilde{A}_{1\alpha} N_s(x, p) + \frac{i\hbar}{2} \Gamma_s^\alpha(x) N_s(x, p) \tilde{A}_{2\alpha} . \] (3.6)
Here $\hat{R}_{1,2\alpha}$ and $\hat{A}_{s1,2\alpha}$ are represented by the infinite series in the momentum derivatives:

$$\hat{R}_{1,2\alpha} = \delta_{\alpha}^{\nu} + \sum_{k=2}^{\infty} \hat{R}_{k\alpha}^{1,2\beta} \hat{R}_{\beta}^{(k)\nu}, \quad (3.7)$$

$$\hat{A}_{s1,2\alpha} = \sum_{k=2}^{\infty} \hat{K}_{k\alpha}^{3,4\beta} \hat{A}_{k\beta}^{(k-1)} , \quad (3.8)$$

with

$$\hat{R}_{k\alpha}^{1,2\beta} = \delta_{\alpha}^{\nu} C_{k}^{\nu} + \sum_{n=1}^{\infty} \sum_{k_{1}=2}^{\infty} \cdots \sum_{k_{n}=2}^{\infty} C_{k}^{i_{n}} R_{k\alpha}^{(k_{n})\beta_{n}} \cdots R_{\beta_{2}}^{(k_{1})\beta} \quad (3.9)$$

and $C_{k_{1}\ldots k_{n}}$ being numerical coefficients (their explicit expressions are presented in Appendix C). The operators $\hat{R}_{\alpha}^{(k)\beta}$ and $\hat{A}_{s\alpha}^{(k)}$ in Eqs. (3.7)–(3.9) are defined as follows:

$$\hat{R}_{\alpha}^{(k)\beta} = \left( \frac{i\hbar}{2} \right)^{k} R_{\nu_{1}\alpha\nu_{2}\nu_{3}\ldots \nu_{k}} \partial^{k} / \partial p_{\nu_{1}} \cdots p_{\nu_{k}} , \quad (3.10)$$

$$\hat{A}_{s\alpha}^{(k)} = \left( \frac{i\hbar}{2} \right)^{k} A_{s\alpha \nu_{1}\nu_{2}\ldots \nu_{k}} \partial^{k} / \partial p_{\nu_{1}} \cdots p_{\nu_{k}} , \quad (3.11)$$

with $A_{s\alpha}$ being defined by (2.66).

The equation (3.6) together with the definitions (3.7)–(3.11) give us the quantum kinetic equation for the unified Wigner function (2.68) in the collisionless approximation. But this equation is not yet in a useful form for exploring quantum kinetic properties. Firstly, the infinite series in Eqs. (3.7)–(3.9) are formal as long as nothing is known about their convergence. The situation here is quite similar to the one arisen in the nonrelativistic case (see, for instance, Ref. 15). One has to assume that quantum interference is suppressed somehow and that the external fields are "slowly varying" at the effective Compton scale (compare also with the discussion in Ref. 2). Then Eqs. (3.7)–(3.9) can be approximated by the first few terms and Eq. (3.6) will be a differential equation of a finite order. On the other hand, in some cases the infinite series in the momentum derivatives can be collected into integrals and the kinetic equation will be an integro–differential equation.
system is in a highly coherent state then some selective summation of infinite subseries could provide a reasonable approximation. The problem is a serious one but its discussion will be reserved for further study (for the recent discussion see Ref. 22). It is worth noting that similar expansions arise in the description of the extended bodies in general relativity.

Secondly, the equation (3.6) doesn’t look like a transport equation. It will do so if we “square” it (compare with the evaluation of the transport equation for 1/2-spin particles in Ref. 13). For this purpose let me write Eq. (3.6) in compact notations:

\[ \hat{M}N_s(x, p) = \hat{K}_s N_s(x, p), \] (3.12)

where by \( \hat{K}_s \) I have denoted the operator acting on the function \( N_s(x, p) \) on the right-hand side of Eq. (3.6).

Later on, the scalar field and the Dirac field will be treated separately.

B. The scalar field case

An equation for the Wigner function of the scalar field (2.29) can be obtained by the means of the prescription (2.75). Multiplying the equation (3.12) from the left by the projection operators (2.70) and (2.71), one gets, by using the properties (2.73) and the explicit matrix form of Eq. (3.6), the following two equations:

\[ \mathcal{P}_1 \hat{M}N_1(x, p) = \hat{K}_1 \mathcal{P}_2 N_1(x, p), \] (3.13)

\[ \mathcal{P}_2 \hat{M}N_1(x, p) = \hat{K}_1 \mathcal{P}_1 N_1(x, p). \] (3.14)

Taking the structures of the matrices (2.52) and (2.70)(2.71) into account, one can easily obtain

\[ \mathcal{P}_1 \hat{M} = \mu^{-1} \hat{m} \mathcal{P}_1, \] (3.15)
\[ \mathcal{P}_2 \dot{M} = \mu \mathcal{P}_2 , \] (3.16)

where the operator \( \hat{m} \) is given by

\[ \hat{m}(x, -\frac{i\hbar}{2} \frac{\partial}{\partial p}) = m^2 - \xi \hbar^2 \hat{R}(x, -\frac{i\hbar}{2} \frac{\partial}{\partial p}) . \] (3.17)

Using (3.15) and (3.16) in (3.13) and (3.14) leads to the following equation:

\[ \left( m^2 - \xi \hbar^2 \hat{R}(x, -\frac{i\hbar}{2} \frac{\partial}{\partial p}) \right) P_1 N_1(x, p) = \hat{K}_1 \hat{K}_1 P_1 N_1(x, p) . \] (3.18)

Substituting the explicit form of the operator \( \hat{K}_1 \) from Eq. (3.6) into Eq. (3.18) and taking of the property (2.74) into account, one can obtain the equation for the Wigner function (2.29) of the scalar field by making use of the prescription (2.75). The result is:

\[ \left( m^2 - \xi \hbar^2 \hat{R}(x, -\frac{i\hbar}{2} \frac{\partial}{\partial p}) \right) f(x, p) = \left( \hat{K}^\alpha \hat{K}_\alpha + \frac{i\hbar}{2} \hat{R}_1 \hat{K}^\alpha \right) f(x, p) , \] (3.19)

where \( \hat{K}_\alpha \) is the following operator:

\[ \hat{K}_\alpha f(x, p) = \left( \frac{i\hbar}{2} \hat{R}_1^\nu \tilde{D}_\nu + \hat{R}_2^\nu p_\nu - \frac{e}{2} \hat{F}_1 \right) f(x, p) - \frac{e}{2} f(x, p) \hat{F}_2 \alpha , \] (3.20)

and the operators \( \hat{R}_1 \) and \( \hat{F}_2 \) are defined by Eqs. (3.8)–(3.11), replacing:

\[ A_{\alpha\nu} \rightarrow \begin{cases} R_{\alpha\nu} & \text{for } \hat{R}_1 \alpha \\ F_{\alpha\nu} & \text{for } \hat{F}_2 \alpha \end{cases} . \] (3.21)

Eq. (3.19) contains, in fact, two equations. To see that, one should remember that the Wigner function is an hermitian (matrix) function in view of the definition (2.29). Separating the hermitian and the antihermitian parts of Eq. (3.19), one will get two independent equations for the only function \( f(x, p) \), which have to be satisfied simultaneously. Here I will only write out the explicit form of the classical terms and the first quantum corrections of these equations.
The first quantum corrections due to the Yang–Mills field are multiplied by \( \hbar \) while the ones due to the gravitational field are multiplied by \( \hbar^2 \). It is worth noting that to get the terms of the order of \( \hbar^n \) in both equations, one has to expand Eq. (3.19) up to the terms of the order of \( \hbar^{n+1} \), because the antihermitian part of it includes the overall multiplier \( i\hbar \) (we suppose that quantum interference effects have been quenched by averaging, so that the Wigner function itself is independent on \( \hbar \), see also the discussion after Eq. (3.11)). Then two equations to the single function will be consistent up to the terms of the order of \( \hbar^n \) (see the discussion below).

These two equations are

\[
\left( m^2 - p^\alpha p_\alpha \right) f(x, p) = \frac{ie\hbar}{4} p^\nu \frac{\partial}{\partial p_\alpha} [F_{\alpha\nu}, f(x, p)] - \frac{\hbar^2}{4} \tilde{D}_\alpha \tilde{D}_\alpha f(x, p) + \\
+ \hbar^2 \left( \frac{\xi - \frac{1}{4}}{3}R - \frac{1}{12} R_{\alpha\nu\beta\mu} p^\mu p^\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} - \frac{1}{4} R^\mu p^\mu \frac{\partial}{\partial p_\nu} \right) f(x, p),
\tag{3.22}
\]

\[
p^\alpha \tilde{D}_\alpha f(x, p) + e p^\nu \frac{\partial}{\partial p_\alpha} \{F_{\alpha\nu}, f(x, p)\} = \frac{ie\hbar}{8} \frac{\partial}{\partial p_\alpha} \left[ F_{\nu\alpha}, \tilde{D}_\nu f(x, p) \right] - \\
- \frac{ie\hbar}{8} p^\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} [F_{\nu\alpha\beta}, f(x, p)] - \frac{ie^2\hbar}{16} \frac{\partial^2}{\partial p_\alpha \partial p_\beta} \left[ F_{\nu\alpha} F_{\beta}^{\nu}, f(x, p) \right] + \\
+ \hbar^2 \left( \frac{1}{6} R_{\beta\mu\alpha}^\nu p^\mu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} \tilde{D}_\nu - \frac{1}{24} R_{\alpha\nu\beta\sigma} p^\mu p^\nu \frac{\partial^3}{\partial p_\alpha \partial p_\beta \partial p_\sigma} + \right. \\
+ \left. \frac{1}{12} R_{\beta\mu\alpha}^\nu \frac{\partial}{\partial p_\alpha} \tilde{D}_\nu - \frac{1}{24} R_{\alpha\beta\nu\sigma} p^\mu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} + \frac{1}{2} \left( \xi - \frac{1}{4} \right) R \frac{\partial}{\partial p_\alpha} \right) f(x, p).
\tag{3.23}
\]

The \([,]\)–bracket stand here for the commutator and the \(\{,\}\)–bracket – for the anticommutator of two matrices in the gauge group’s representation space. Remember also that the operator \( \tilde{D}_\alpha \) is defined as following:

\[
\tilde{D}_\alpha f(x, p) = \left( \partial_\alpha + \frac{\gamma_\gamma}{4} p_\gamma \frac{\partial}{\partial p_\beta} \right) f(x, p) + \frac{ie}{\hbar} [A_\alpha, f(x, p)].
\tag{3.24}
\]

Eqs. (3.22) and (3.23) were first obtained by Winter through a very different method. The first equation gives the quantum corrections to the mass–shell constraint of classical parti-
cles \( (2.14) \) and the second equation is the quantum Liouville–Vlasov equation. These two equations must be satisfied simultaneously. It could lead to inconsistency with the classical limit if we were not able to reduce them to a single equation.

Indeed, in the naive classical limit \((\hbar \rightarrow 0)\) the Wigner function satisfies two classical equations:

\[
\Omega f(x, p) = 0, \quad (3.25)
\]

\[
\hat{L} f(x, p) = 0, \quad (3.26)
\]

where

\[
\Omega = p^\alpha p_\alpha - m^2 \quad (3.27)
\]

and \(\hat{L}\) is the classical Liouville operator, which acts on the left hand side of Eq. \((3.23)\) (it coincides with the left hand side of Eq. \((2.17)\) if the gauge group is \(U(1)\)). One can easily check that the operator \(\hat{L}\) annihilates \(\Omega\) and therefore the function

\[
f_{cl}(x, p) = F_{cl}(x, p) \delta(\Omega) \quad (3.28)
\]

will satisfy Eq. \((3.26)\) if \(F_{cl}(x, p)\) satisfies the same equation but on the mass–shell only. It means that the following equation holds:

\[
\hat{L} F_{cl}(x, p) = \Omega \Delta(x, p), \quad (3.29)
\]

where \(\Delta(x, p)\) is an arbitrary function, which is nonsingular on the mass–shell.

We see that in the classical limit one is able to reduce two equations for the Wigner function (which is a distributed function in the relativistic case) to a single equation which coincides with the Vlasov equation for the 7-dimensional distribution function on the mass–shell (more exactly, the sum of the distribution functions for particles and antiparticles,
compare with Ref. 13. The same situation can be expected in each order in $\hbar$, because otherwise one wouldn’t get the right semiclassical limit.

To show that Eqs. (3.22) and (3.23) lead to a single equation up to the terms of the next adiabatic order, I will use the method of Ref. 2. Let me look for a solution to Eq. (3.22) of the form:

$$f(x, p) = F_0(x, p) \delta(\Omega) + F_1(x, p) \delta'(\Omega) + F_2(x, p) \delta''(\Omega),$$

(3.30)

where $F_0(x, p)$ is the quantum corrected distribution function on the mass–shell:

$$F_0(x, p) = F_{cl}(x, p) + F_{qu}(x, p),$$

(3.31)

and $F_1(x, p)$ and $F_2(x, p)$ have purely quantum origin. It is worth noting that if one has been interested in all terms in the adiabatic expansion then, as it is mentioned in Ref. 2, one should include the term $\hbar(\xi - 1/6)R(x)$ in $\Omega$ to get an improved asymptotic approximation for the Wigner function. But as, for the moment, we are only interested in the lowest adiabatic order, it is more convenient to consider that this term also contributes to the off mass–shell part of the Wigner function.

Using now the property:

$$\Omega \delta^{(n)}(\Omega) = -n \delta^{(n-1)}(\Omega),$$

(3.32)

and treating all derivatives of the $\delta$–function as linearly independent, one can get easily from Eq. (3.22) the first quantum corrections to the classical distribution function off the mass–shell:

$$F_1(x, p) = \hat{\Pi} F_{cl}(x, p),$$

(3.33)

$$F_2(x, p) = \frac{1}{2} \left[ \hat{\Pi}, \Omega \right] F_{cl}(x, p) = -\frac{\hbar^2}{3} R_{\mu\nu} p^\mu p^\nu F_{cl}(x, p),$$

(3.34)
where $\tilde{\Pi}$ is the operator acting on the right hand side of Eq. (3.22).

The substitution of the trial solution (3.30) with (3.31), (3.33) and (3.34) into Eq. (3.23) gives, after omitting the higher adiabatic order terms,

$$
\delta(\Omega) \hat{L} \left( F_{cl}(x,p) + F_{qu}(x,p) \right) + \delta'(\Omega) \hat{L} \tilde{\Pi} F_{cl}(x,p) + \frac{1}{2} \delta''(\Omega) \hat{L} \left[ \tilde{\Pi}, \Omega \right] F_{cl}(x,p) = \\
= \delta(\Omega) \hat{A} F_{cl}(x,p) + \delta'(\Omega) \left[ \hat{A}, \Omega \right] F_{cl}(x,p) + \frac{1}{2} \delta''(\Omega) \left[ \left[ \hat{A}, \Omega \right], \Omega \right] F_{cl}(x,p),
$$

(3.35)

where $\hat{A}$ is the operator acting on the right hand side of Eq. (3.23) (the operators $\tilde{\Pi}$ and $\hat{A}$ are written out explicitly in Appendix D).

Eq. (3.35) seems to give rise to inconsistency of the method, but hopefully, the structures of the operators in Eqs. (3.22) and (3.23) are so fine tuned that it leads to the cancelling of the terms which multiply the derivatives of $\delta$–function. Indeed, as it is shown in Appendix D, the operators $\tilde{\Pi}$ and $\hat{A}$ satisfy the following identity:

$$
\left[ \hat{A}, \tilde{\Pi} \right] = \left[ \hat{A}, \Omega \right].
$$

(3.36)

Using also the fact that the operator $\hat{L}$ annihilates $\Omega$ and, therefore,

$$
\left[ \hat{L}, \tilde{\Pi}, \Omega \right] = \left[ \left[ \hat{L}, \tilde{\Pi} \right], \Omega \right],
$$

(3.37)

one can transform Eq. (3.35) into:

$$
\delta(\Omega) \hat{L} \left( F_{cl}(x,p) + F_{qu}(x,p) \right) + \delta'(\Omega) \tilde{\Pi} \hat{L} F_{cl}(x,p) + \\
+ \frac{1}{2} \delta''(\Omega) \left[ \tilde{\Pi}, \Omega \right] \hat{L} F_{cl}(x,p) = \delta(\Omega) \hat{A} F_{cl}(x,p).
$$

(3.38)

Remember now that the classical distribution function $F_{cl}(x,p)$ satisfies Eq. (3.23). This leads, after using the property (3.32), to the equation for evaluating the first quantum corrections to the classical distribution function on the mass–shell:

$$
\delta(\Omega) \hat{L} F_{qu}(x,p) = \delta(\Omega) \left( \hat{A} F_{cl}(x,p) + \tilde{\Pi} \Delta(x,p) \right).
$$

(3.39)
Eqs. (3.30), (3.31), (3.33), (3.34) and (3.39) solve, in principle, the task of finding the lowest order quantum corrections to the classical distribution function (3.28) of scalar particles due to the coupling to external gravitational and Yang–Mills fields. As it was emphasized in Ref. [4], some terms of the higher order must be added, they give the right trace anomaly of the stress–energy tensor.

C. The Dirac field case

An equation for the Wigner function of the Dirac field (2.42) can be obtained by squaring Eq. (3.12) for \( s = 1/2 \). Remembering that in this case the operator \( \hat{M} \) coincides with the field mass \( m \), one easily obtains:

\[
m^2 N(x,p) = \hat{K}_{1/2} \hat{K}_{1/2} N(x,p).
\]

(3.40)

Utilizing the algebra of the Dirac matrices (2.38), one can deduce from Eq. (3.40) the equations for the independent components of the matrix Wigner function (2.45)–(2.47). It happens that the equations form a coupled system of equations and other components of the Wigner function (2.42) (a vector and a pseudovector functions) defined by:

\[
A^\alpha(x,p) = Tr (\gamma^\alpha N(x,p)) ,
\]

(3.41)

\[
B^\alpha(x,p) = Tr (\gamma^5 \gamma^\alpha N(x,p)) ,
\]

(3.42)

are expressed in terms of the functions (2.45)–(2.47) and their derivatives only.

The explicit form of the equations is very unwieldy in general and here I will write out the equations with the first quantum corrections only (of the order of \( \hbar \)). The general form of the higher terms in expansions like (3.19) for the Dirac field can be found in Ref. [7]. It is only worth noting that quantum corrections of two types arise. The corrections of the first
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type are similar to and coincide, up to the terms of the order of \( \hbar^4 \), with the ones in Eq. (3.22),(3.23) when \( \xi = 1/4 \). The corrections of the second type reflect the coupling of the spin and/or the isospin of quantum particles to external gravitational and Yang–Mills fields and they mix different components of the Wigner function (2.45)–(2.47).

As in the scalar field case, Eq. (3.40) leads to two systems of equations: the constraint equations, which describe the quantum corrections to the classical mass–shell constraint (2.14), and the quantum transport equations governing the evolution of the Wigner function. These equations with the lowest order quantum corrections are

\[
\begin{align*}
(\Omega + \hat{\Pi})A(x,p) &= \frac{\hbar}{2} \{ F_{\mu\nu}, A^{\mu\nu}(x,p) \} \\
(\Omega + \hat{\Pi})B(x,p) &= \frac{\hbar}{2} \{ \tilde{F}_{\mu\nu}, A^{\mu\nu}(x,p) \} \\
(\Omega + \hat{\Pi})A_{\mu\nu}(x,p) &= \frac{\hbar}{2} \{ F_{\mu\nu}, A(x,p) \} - \frac{\hbar}{2} \{ \tilde{F}_{\mu\nu}, B(x,p) \},
\end{align*}
\]

(3.43)

\[
\begin{align*}
(\hat{\mathcal{L}} - \hat{\Lambda})A(x,p) &= -\hat{S}_{\mu\nu} A^{\mu\nu}(x,p) \\
(\hat{\mathcal{L}} - \hat{\Lambda})B(x,p) &= -\hat{S}_{\mu\nu} A^{\mu\nu}(x,p) \\
(\hat{\mathcal{L}} - \hat{\Lambda})A_{\mu\nu}(x,p) &= -\frac{1}{2} \hat{S}_{\mu\nu} A(x,p) + \frac{1}{2} \hat{S}_{\mu\nu} B(x,p),
\end{align*}
\]

(3.44)

where \( \Omega \) and the operators \( \hat{\Pi}, \hat{\mathcal{L}} \) and \( \hat{\Lambda} \) are defined as in the previous section (they are written out in Appendix D, only the terms of the order of \( \hbar \) of \( \hat{\Pi} \) and \( \hat{\mathcal{L}} \) are kept in Eqs. (3.43),(3.44) ) and \( \hat{S}_{\mu\nu} \) is the following operator:

\[
\hat{S}_{\mu\nu} A(x,p) = \frac{ie}{2} [F_{\mu\nu}, A(x,p)] + \frac{\hbar}{2} R_{\alpha\beta\mu\nu} p^\beta \frac{\partial}{\partial p_\alpha} A(x,p) + \frac{\hbar}{4} \frac{\partial}{\partial p_\alpha} \{ F_{\mu\nu\alpha}, A(x,p) \}. 
\]

(3.45)

The tilde over tensors indicates dual tensors, for example, \( \tilde{F}_{\mu\nu} = \frac{1}{2} \sqrt{-g(x)} \varepsilon_{\mu\alpha\beta} F^{\alpha\beta} \).

Eqs. (3.43)–(3.45) were obtained for the case of the \( U(1) \)–group (an external electromagnetic field ) in Ref. 14. It is interesting to compare these equations with the heuristic equation proposed by Israel\textsuperscript{26} for describing particles with internal spin in external gravi-
tational and electromagnetic fields. He introduces the extended phase space by adding to it the spin angular momentum $s^{\mu \nu}$ of particles, and the distribution function $N(x,p,s)$ is a function on this space. One can show that the transport equation to the function $N(x,p,s)$ leads, after integrations over $s^{\mu \nu}$, to the equations (3.43)–(3.45), if one identifies the functions $A(x,p)$, $B(x,p)$ and $A^{\mu \nu}(x,p)$ with the corresponding moments of the function $N(x,p,s)$ in the $s^{\mu \nu}$–space. The correspondence here is quite similar to the one which arose in the case of colored particles in a Yang–Mills field (see, for instance, Ref. [27]).

IV. Final remarks

As a conclusion, I would like to mention a few further developments related to the present work, which have not been considered in the paper. (1) This paper didn’t specify any regularization procedure as well as it didn’t discuss in detail the question of the renormalization of the statistical averaged operators. This question is especially important when one wishes to describe the particles creation due to quantum interactions or external fields. This process could be described in kinetic theory, possibly by a source–term in the Liouville equation and could lead to Boltzmann entropy generation (different approaches to defining a nonequilibrium entropy for quantum fields in a cosmological setting have been proposed in Refs. [28, 30], see also Ref. [3]). (2) Specification of particles interactions will give an explicit expression for the collision term in Eq. (3.4). As it is shown in Ref. [31], a satisfactory approximation is provided only by the two–loop order of the perturbation theory and higher. The theory becomes very involved in this approximation and one needs some physical hypothesis to make further progress. (3) In Ref. [32] the quantum kinetic equation for the Wigner function of a scalar field has been solved explicitly in the lowest adiabatic
order in a Robertson–Walker space–time, in Ref. 33 – in a class of manifolds admitting a Killing vector ( for instance, static space–times ) and in Ref. 34 – a conformal Killing vector ( the later class includes, in particular, a Robertson–Walker space–times ). Ref. 3 considers also a few other physical systems permitting an explicit analysis. In each case the quantum–curvature corrections to the local–equilibrium distribution are expressed by local geometrical quantities and a few momentum derivatives of the classical distribution function. This fact extremely simplifies the Einstein–Vlasov problem, reducing it to an analysis of differential equations of a finite order ( such analysis by using the moments method was carried out for the Friedmann cosmology in Ref. 35 ). A natural question arises: is this property general or is it only a lucky chance and how do space–time symmetries show themselves in the Wigner function? The better understanding of the structure of the relativistic phase space could possibly clarify this tie.

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Appendix A  The Wigner function as expressed on the base space

The definition of a covariant and gauge invariant Wigner function (2.29) is not very useful if one wishes to compute it explicitly or explore its ultraviolet behavior, because the fields Φ\((x, -y)\) and \(Φ^\dagger(x, y)\) have been defined on the tangent space rather than on the physical space–time. Fortunately, one can express these fields in terms of fields on the base space, though in general the expressions also include infinite series in the derivatives.

Consider first the field \(Φ(x, -y)\). Let me rewrite expression (2.35) in the following form:

\[
Φ(x, -y) = \left( \exp(-y^α \hat{∇}_α) \exp(y^α \partial_α) \right) \left( \exp(-y^α \partial_α) \varphi(x) \right).
\]

The second cofactor in (A1) is obviously \(\varphi(x - y)\) for any numerous value of \(y^α\) in each fixed coordinate system. The first cofactor is in general a differential operator acting to \(\varphi(x - y)\). It can be written more explicitly by using the Campbell–Baker–Hausdorff formula (see, for instance, Ref. 36):

\[
\exp(\hat{A}) \exp(\hat{B}) = \exp \left( \sum_{n=1}^{∞} \hat{C}_n \right),
\]

(A2)

with \(\hat{C}_n\) being found by the sequence:

\[
\hat{C}_1 = \hat{A} + \hat{B},
\]

\[
\hat{C}_2 = \frac{1}{2} \left[ \hat{A}, \hat{B} \right],
\]

\[
\hat{C}_n = \frac{1}{n!} \left[ \hat{A}, \left[ \hat{A}, \left[ \hat{A}, \left[ \hat{A}, \left[ \hat{A}, \hat{B} \right] \right] \right] \right] \right] - \frac{1}{n} \sum_{m=2}^{n-1} \frac{1}{m!} \sum_{k_1} \cdots \sum_{k_m} k_1 δ_{n,k_1+\ldots+k_m} \left[ \hat{C}_{k_m}, \left[ \cdots, \left[ \hat{C}_{k_2}, \hat{C}_{k_1} \right] \right] \right].
\]

(A3)
If one puts first, $\hat{B} = y^\alpha \hat{\nabla}_\alpha$ and $\hat{A} = -y^\alpha \hat{\nabla}_\alpha = -\hat{B} - \frac{ie}{\hbar} y^\alpha A_\alpha(x)$, then it follows from Eqs. (A2), (A3) that
\[
Z(x, -y) = \exp(-y^\alpha \hat{\nabla}_\alpha) \exp(y^\alpha \hat{\nabla}_\alpha) = \exp \left( \frac{ie}{\hbar} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} y^{\alpha_0} \cdots y^{\alpha_n} A_{\alpha_0;\alpha_1 \cdots \alpha_n} - \frac{e^2}{12\hbar^2} y^\alpha y^\beta y^\sigma [A_{\alpha;\beta}, A_\sigma] + \ldots \right). \tag{A4}
\]
This expression can be written in a compact form by using Feynman’s path ordering operator $P$:
\[
Z(x, -y) = P \exp \left( -\frac{ie}{\hbar} \int_{-1}^{1} dt \sum_{n=0}^{\infty} \frac{t^n}{n!} y^{\alpha_0} \cdots y^{\alpha_n} A_{\alpha_0;\alpha_1 \cdots \alpha_n} \right). \tag{A5}
\]
In the same way, with $\hat{B} = y^\alpha \partial_\alpha$ and $\hat{A} = -y^\alpha \hat{\nabla}_\alpha = -\hat{B} + \Gamma^\beta_{\alpha\sigma} y^\alpha y^\sigma \frac{\partial}{\partial y^\beta}$ Eqs. (A2), (A3) give:
\[
\hat{P}(x, -y) = \exp(-y^\alpha \hat{\nabla}_\alpha) \exp(y^\alpha \partial_\alpha) = \exp \left( y^\alpha y^\beta \Gamma^\sigma_{\alpha\beta} \left( \frac{1}{2} \partial_\sigma + \frac{\partial}{\partial y^\sigma} \right) - y^\alpha y^\beta y^\gamma \Gamma^\sigma_{\alpha\beta;\gamma} \left( \frac{1}{3} \partial_\sigma + \frac{1}{2} \frac{\partial}{\partial y^\sigma} \right) + \ldots \right). \tag{A6}
\]
If one uses the Riemann normal coordinates\cite{1} centered at point $x^\alpha$ (RNC), the operator $\hat{P}(x, -y)$ vanishes. Then the field $\Phi(x, -y)$ differs from $\varphi(x - y)$ only by a phase, which is (A5).

In the same way, the field $\Phi^\dagger(x, y)$ in RNC equals to $Z^\dagger(x, y)\phi^\dagger(x + y)$ and the Wigner function (2.29) becomes
\[
f(x, p)_{|\text{RNC}} = (\pi\hbar)^{-4} \int d^4 y \ e^{-2iy^\alpha p_\alpha / \hbar} Z(x, -y)\langle \phi(x - y)\phi^\dagger(x + y) \rangle Z^\dagger(x, y) \tag{A7}
\]
In absence of an external Yang–Mills field, $Z(x, -y) = 1$ and the function (A7) coincides with that defined in Ref. 2 (in RNC) within a factor equal to the square root of the Van Vleck–Morett determinant, or the one defined in Ref. 1. In other coordinate systems the Wigner function can be calculated by using Eq. (A6) (see also the discussion in Ref. 2).
In another limiting case, when the space–time is flat, the tangent space coincides with Minkovski space. In this case the function (A5) takes the familiar form:

$$Z(x, -y) = P \exp \left( -\frac{i e}{\hbar} \int_0^1 dt \, y^\alpha A_\alpha (x - ty) \right),$$  \hspace{2cm} (A8)

and the gauge–invariant Wigner function (A7) coincides with that defined in Ref. 25 or in Ref. 27, in the Dirac field case (for the case of the gauge group being $U(1)$, see for instance Ref. 37).

Thus, our definition of the Wigner function in curved space–time is locally equivalent to those of Refs. 1, 2. But for nontrivial topologies, the different definitions of the Wigner function for a system in a highly coherent state could lead to globally different results.

The same analysis can be carried out for the Wigner function of a Dirac field (2.42).

**Appendix B Useful identities**

In this Appendix I will prove a few useful identities, which have been used in deriving the equation for the unified Wigner function (3.6).

Let $\hat{A}$ and $\hat{B}$ be two arbitrary operators. Then the following well known identity holds:

$$[\hat{A}, e^{\hat{B}}] = -\sum_{n=1}^{\infty} \frac{1}{n!} \left[ \hat{B}, \ldots, \left[ \hat{B}, \hat{A} \right] \ldots \right] e^{\hat{B}}. \hspace{2cm} (B1)$$

Putting $\hat{A} = \hat{\nabla}_\alpha$ and $\hat{B} = -y^\nu \hat{\nabla}_\nu$ leads to

$$\hat{\nabla}_\alpha U(x, -y) = e^{-y^\nu \hat{\nabla}_\nu} \hat{\nabla}_\alpha u(x) - \hat{H}_\alpha (x, -y) U(x, -y), \hspace{2cm} (B2)$$

with

$$\hat{H}_\alpha (x, -y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} y^{\nu_1} \cdots y^{\nu_n} \left[ \hat{\nabla}_{\nu_1}, \ldots, \left[ \hat{\nabla}_{\nu_n}, \hat{\nabla}_\alpha \right] \ldots \right]. \hspace{2cm} (B3)$$
The field $U(x, -y)$ is defined by the same formula as in (2.35):

$$U(x, -y) = \exp(-y^\alpha \hat{\nabla}_\alpha) u(x).$$  \hspace{1cm} (B4)

If one puts $\hat{A} = M(x)$, then (B1) with the same $\hat{B}$ gives the following identity:

$$e^{-y^\nu \hat{\nabla}_\nu} M(x)u(x) = M(x, -y)U(x, -y),$$  \hspace{1cm} (B5)

with

$$M(x, -y) = M(x) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} y^{\nu_1} \cdots y^{\nu_n} \hat{\nabla}_{\nu_1} \cdots \hat{\nabla}_{\nu_n} M(x).$$  \hspace{1cm} (B6)

The next identity follows from the formula for the left derivative of the exponent of an operator function $\hat{B}(t)$:

$$\frac{d}{dt} e^{\hat{B}(t)} = \left( \frac{d\hat{B}(t)}{dt} + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \left[ \hat{B}(t), \cdots, \hat{B}(t), \frac{d\hat{B}(t)}{dt} \right] \cdots \right) e^{\hat{B}(t)}.$$  \hspace{1cm} (B7)

Substituting $\hat{B}(y^\alpha) = -y^\alpha \hat{\nabla}_\alpha - \sum_{\nu \neq \alpha} y^\nu \hat{\nabla}_\nu$ (with no summation over $\alpha$) into (B7) gives the identity:

$$\frac{\partial}{\partial y^\alpha} U(x, -y) = -\hat{\nabla}_\alpha U(x, -y) - \hat{G}_\alpha(x, -y) U(x, -y),$$  \hspace{1cm} (B8)

where

$$\hat{G}_\alpha(x, -y) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)!} y^{\nu_1} \cdots y^{\nu_n} \left[ \hat{\nabla}_{\nu_1}, \cdots, \hat{\nabla}_{\nu_n}, \hat{\nabla}_\alpha \right] \cdots.$$  \hspace{1cm} (B9)

The same identity for the conjugate field is obtained by reversing the sign of $y^\alpha$.

**Appendix C Evaluation of $G_s\alpha(x, y)$**

In this Appendix I evaluate expression (3.3) with the operators $\hat{H}$ and $\hat{G}$ being defined in (B3),(B9). I will omit here the subscript $s$ for convenience.
Let me first prove the following recursive formula \( n \geq 2 \):

\[
G_{\alpha}^{(n)} = - A_{\alpha}^{(n-1)} U(x, -y) + R_{\alpha}^{(n)} \frac{\partial}{\partial y^\nu} U(x, -y) - \sum_{k=1}^{n-2} C_{k}^{n-2} R_{\alpha}^{(n-k)} G_{\nu}^{(k)},
\]

(C1)

where \( C_{k}^{n} = n! / (n - k)! \) are the binomial coefficients and

\[
\left\{
\begin{array}{l}
G_{\alpha}^{(n+1)} = y^{\nu_1} \ldots y^{\nu_n} \left[ \hat{\nabla}_{\nu_1}, \ldots, \left[ \hat{\nabla}_{\nu_n}, \hat{\nabla}_{\alpha} \right] \right] U(x, -y) \quad \text{if } n \geq 1 \\
G_{\alpha}^{(1)} = \hat{\nabla}_{\alpha} U(x, -y),
\end{array}
\right.
\]

(C2)

\[
\left\{
\begin{array}{l}
R_{\alpha}^{(n)} = y^{\nu_1} \ldots y^{\nu_n} R_{\nu_1 \alpha \nu_2 ; \nu_3 \ldots \nu_n} \quad \text{if } n \geq 3 \\
R_{\alpha}^{(2)} = y^{\nu_1} y^{\nu_2} R_{\nu_1 \alpha \nu_2},
\end{array}
\right.
\]

(C3)

\[
\left\{
\begin{array}{l}
A_{\alpha}^{(n)} = y^{\nu_1} \ldots y^{\nu_n} A_{\alpha \nu_1 ; \nu_2 \ldots \nu_n} \quad \text{if } n \geq 2 \\
A_{\alpha}^{(1)} = y^{\nu} A_{\alpha \nu},
\end{array}
\right.
\]

(C4)

with \( A_{\alpha \nu} \) being defined by Eq. (2.66).

For \( n = 2 \), the last term in Eq. (C1) disappears and the validity of this formula follows from the definition (2.33) and Eq. (2.65). Suppose that Eq. (C1) holds up to some fixed \( n \). Then, for \( n + 1 \), one can get the equality:

\[
G_{\alpha}^{(n+1)} = - A_{\alpha}^{(n)} U(x, -y) + R_{\alpha}^{(n+1)} \frac{\partial}{\partial y^\nu} U(x, -y) - \\
- R_{\alpha}^{(n)} \hat{\nabla}_{\beta} U(x, -y) - \sum_{k=1}^{n-2} C_{k}^{n-2} \left( R_{\alpha}^{(n-k+1)} G_{\nu}^{(k)} + R_{\alpha}^{(n-k)} G_{\nu}^{(k+1)} \right),
\]

(C5)

The proof will be completed if one takes into account the properties of the binomial coefficients:

\[
C_{k}^{n} + C_{k-1}^{n} = C_{k}^{n+1}
\]

(C6)

\[
C_{n}^{n} = 1.
\]

Let me now define the series:

\[
G_{\alpha}[g] = \sum_{n=2}^{\infty} g_n G_{\alpha}^{(n)},
\]

(C7)
with arbitrary \( g_n \)'s depending on \( n \). A repeated use of the formula (C1) gives:

\[
G_\alpha[g] = -\sum_{k=2}^{\infty} g_k \left( A^{(k-1)}_\alpha - R^{(k)}_\alpha \frac{\partial}{\partial y^\nu} + (k-2) R^{(k-1)}_\alpha \partial \beta \right) U(x,-y) + \\
+ \sum_{n=2}^{\infty} \sum_{k_1=2}^{\infty} \cdots \sum_{k_n=2}^{\infty} \left( -1 \right)^n g_n C^{s_2-2}_{s_1} \cdots C^{s_n-2}_{s_{n-1}} R^{(k_1)}_\alpha \cdots R^{(k_2)}_{\beta_2} \times \\
\times \left( A^{(k_1-1)}_{\beta_2} - R^{(k_2)}_{\beta_2} \frac{\partial}{\partial y^\nu} + (k_1-2) R^{(k_1-1)}_{\beta_2} \partial \beta \right) U(x,-y),
\]

(C8)

where

\[
s_m = k_1 + \ldots + k_m. \quad (C9)
\]

By putting \( g_n = (-1)^n/n! \) and \( g_n = (-1)^n/(n+1)! \), one will get expressions for the acting of the operators (B3) and (B9) to \( U(x,-y) \) and then evaluate the function \( G_\alpha(x,y) \), (3.3).

Unfortunately, the expression for \( G_\alpha(x,y) \) includes terms with

\[
\left( \frac{\partial}{\partial y^\alpha} U(x,-y) \right) \dot{U}(x,y) \quad \text{or} \quad \left( \nabla_\alpha U(x,-y) \right) \dot{U}(x,y).
\]

In order to obtain an expression including only operators acting to \( U(x,-y)\ddot{U}(x,y) \), one should use the identity (B8) for field \( U \) as well as for the conjugate field. It is easy to get:

\[
\left( \frac{\partial}{\partial y^\alpha} U(x,-y) \right) \ddot{U}(x,y) = \frac{1}{2} \frac{\partial}{\partial y^\alpha} \left( U(x,-y) \dot{U}(x,y) \right) - \frac{1}{2} \dot{\nabla}_\alpha \left( U(x,-y) \dot{U}(x,y) \right) - \\
- \frac{1}{2} U(x,-y) \left( \dot{G}_\alpha(x,y) \ddot{U}(x,y) \right) - \frac{1}{2} \left( \dot{G}_\alpha(x,-y) U(x,-y) \dot{U}(x,y) \right),
\]

(C10)

\[
\left( \nabla_\alpha U(x,-y) \right) \ddot{U}(x,y) = -\frac{1}{2} \frac{\partial}{\partial y^\alpha} \left( U(x,-y) \dot{U}(x,y) \right) + \frac{1}{2} \dot{\nabla}_\alpha \left( U(x,-y) \dot{U}(x,y) \right) + \\
+ \frac{1}{2} U(x,-y) \left( \dot{G}_\alpha(x,y) \ddot{U}(x,y) \right) - \frac{1}{2} \left( \dot{G}_\alpha(x,-y) U(x,-y) \dot{U}(x,y) \right).
\]

(C11)

A repeated use of the identity (C8) gives then, after elaborate calculations, the formal representation for the function \( G_\alpha(x,y) \), (3.3) as an infinite series in operators acting to the product \( \dot{U} \ddot{U} = U(x,-y)\ddot{U}(x,y) \):

\[
G_\alpha(x,y) = \left( K_\alpha^\nu \dot{\nabla}_\nu - K_\alpha^\nu \frac{\partial}{\partial y^\nu} + A_{1\alpha} \right) \dot{U} \ddot{U} + \dot{U} \ddot{U} A_{2\alpha},
\]

(C12)
where
\[ K_{1,2\alpha}^{\nu} = \sum_{k=2}^{\infty} K_{k\alpha}^{1,2\beta} R_{\beta}^{(k)} \nu, \]  
(C13)
\[ A_{1,2\alpha} = \sum_{k=2}^{\infty} K_{k\alpha}^{3,4\beta} A_{\beta}^{(k-1)} , \]  
(C14)
with
\[ K_{k\alpha}^{ij\beta} = \delta_{\alpha}^{\beta} C_{k}^{ij} + \sum_{n=1}^{\infty} \cdots \sum_{k_{n}=2}^{\infty} C_{k_{1}..k_{n}}^{ij} R_{\alpha}^{(k_{1})\beta} \cdots R_{\beta}^{(k_{n})\beta} , \]  
(C15)
and \( R_{\alpha}^{(k)\beta} \) and \( A_{\alpha}^{(k)} \) being defined by (C3),(C4).

The numerical coefficients \( C_{k_{1}..k_{n}}^{ij} \) can explicitly be found in the following way. Let \( A(t) \) be an operator function. Let’s introduce the following generating functionals:
\[ K_{k\alpha}[A] = \sum_{n=1}^{\infty} \sum_{k_{1}=2}^{\infty} \cdots \sum_{k_{n}=2}^{\infty} C_{k_{1}..k_{n}}^{ij} A_{\alpha}^{(k_{1})} \cdots A_{\alpha}^{(k_{n})} , \]  
(C16)
where \( A^{(k)} = d^{k}A(0)/dt^{k} \) are the derivatives of \( A(t) \) at \( t = 0 \).

Then, the coefficients \( C_{k_{1}..k_{n}}^{ij} \) are found if we define these functionals as follows:
\[ K^{1}[A] = -1 + KB , \]  
(C17)
\[ K^{2}[A] = -1 + KC , \]  
(C18)
\[ K^{3}[A] = -2\bar{C}' \bar{C}^{-1} + KC \bar{C}^{-1} (\bar{C} - 1) , \]  
(C19)
\[ K^{4}[A] = -K (C - 1) , \]  
(C20)
where
\[ K = \left(1 - B' B^{-1} + C' \bar{C}^{-1}\right) \left(\frac{1}{2} B B^{-1} + \frac{1}{2} C \bar{C}^{-1}\right)^{-1} . \]  
(C21)
Here \( B = B(1), C = C(1), \bar{B} = B(-1) \) and \( \bar{C} = C(-1) \), where \( B(t) \) and \( C(t) \) are the following functionals:
\[ B(t) = 1 + \sum_{n=1}^{\infty} \sum_{k_{1}=2}^{\infty} \cdots \sum_{k_{n}=2}^{\infty} (-1)^{n} s_{n} \prod_{i=1}^{n} \frac{1}{(k_{i} - 2)!} \frac{1}{s_{i}(s_{i} + 1)} A^{(k_{i})} , \]  
(C22)
\[ C(t) = 1 + \sum_{n=1}^{\infty} \sum_{k_n=2}^{\infty} \cdots \sum_{k_1=2}^{\infty} (-1)^n t^n s_n \prod_{i=1}^{n} \frac{1}{(k_i - 2)! s_i(s_i - 1)} A^{(k_i)}. \] (C23)

\( \bar{B}' \) and \( \bar{C}' \) denote the derivatives at \( t = -1 \).

The functionals \( B(t) \) and \( C(t) \) can be written in a compact form by using Feynman’s path ordering operator \( P \):

\[ B(t) = P \exp \left( \int_0^t dx \, b(x) \right), \] (C24)

\[ C(t) = P \exp \left( \int_0^t dx \, c(x) \right), \] (C25)

with

\[ b(x) = x^{-2} \int_0^x dz \, z^2 A^{(2)}(z), \] (C26)

\[ c(x) = \int_0^x dz \, A^{(2)}(z). \] (C27)

In particular, the coefficients in Eq. (C15) needed only for evaluating the linear terms with respect to the curvature tensor in Eq. (3.6) are

\[ C_k^1 = (-1)^k / k! + \left( 1 - (-1)^k \right) / (k + 1)! , \] (C28)

\[ C_k^2 = (-1)^k / k! - \left( 1 + (-1)^k \right) / (k + 1)! , \] (C29)

\[ C_k^3 = (-1)^k (2k - 1) / k! , \] (C30)

\[ C_k^4 = 1 / k! , \] (C31)

\[ C_{k_1 k_2}^3 = (-1)^{k_1 + k_2 + 1} \frac{2(k_1 + k_2) - 1}{k_1! (k_2 - 2)! (k_1 + k_2)(k_1 + k_2 - 1)} + \]

\[ + (-1)^{k_1 + k_2} \left( 1 + (-1)^{k_2} \right) \frac{1}{k_1! (k_2 + 1)!} + (-1)^{k_1 + k_2} \frac{2(k_1 - 1)}{k_1! k_2!} , \] (C32)

\[ C_{k_1 k_2}^4 = \frac{1}{k_1! (k_2 - 2)! (k_1 + k_2)(k_1 + k_2 - 1)} + \]

\[ + \left( 1 + (-1)^{k_2} \right) \frac{k_2}{k_1! (k_2 + 1)!} . \] (C33)
Appendix D  Check of the consistency of the mass–shell constraint

This Appendix proves the identity (3.36). Let me explicitly write out the operators \( \hat{\mathcal{L}}, \hat{\Pi}, \hat{\Lambda} \) and also \( \Omega \):

\[
\hat{\mathcal{L}} f(x,p) = p^\alpha \tilde{D}_\alpha f(x,p) + \frac{e}{2} p^\nu \frac{\partial}{\partial p_\alpha} \{ F_{\alpha\nu}, f(x,p) \},
\]

(D1)

\[
\hat{\Pi} f(x,p) = i e \hbar \frac{4}{p^\nu} \frac{\partial}{\partial p_\alpha} [ F_{\alpha\nu}, f(x,p) ] - \frac{\hbar^2}{4} \tilde{D}_\alpha \tilde{D}_\alpha f(x,p) + 
+ \hbar^2 \left( \xi - \frac{1}{3} R - \frac{1}{12} R_{\alpha\mu\beta\nu} p^\mu p^\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} - \frac{1}{4} R_{\mu\nu} p^\mu \frac{\partial}{\partial p_\nu} \right) f(x,p),
\]

(D2)

\[
\hat{\Lambda} f(x,p) = i e \hbar \frac{\partial}{8} \frac{\partial}{\partial p_\alpha} [ F_{\alpha\nu} , \tilde{D}_\nu f(x,p) ] - 
- \frac{ie\hbar}{8} p^\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} [ F_{\alpha\beta;\gamma}, f(x,p) ] - \frac{ie^2\hbar}{16} \frac{\partial^2}{\partial p_\alpha \partial p_\beta} [ F_{\alpha\nu} F^\nu_{\beta;\gamma}, f(x,p) ] + 
+ \hbar^2 \left( \frac{1}{6} R^\nu_{\beta\mu\rho} p^\rho \frac{\partial^2}{\partial p_\alpha \partial p_\beta} \tilde{D}_\nu - \frac{1}{24} R_{\alpha\beta\nu;\sigma} p^\mu p^\nu \frac{\partial^3}{\partial p_\alpha \partial p_\beta \partial p_\sigma} + 
+ \frac{1}{12} R^\nu_{\alpha;\beta} p^\nu \frac{\partial^2}{\partial p_\alpha \partial p_\beta} + \frac{1}{2} (\xi - \frac{1}{4}) R_{\alpha\beta} \frac{\partial}{\partial p_\alpha} \right) f(x,p),
\]

(D3)

\[
\Omega = p^\alpha p_\alpha - m^2 .
\]

(D4)

To prove the identity (3.36), one needs an expression for the commutator of two operators \( \tilde{D}_\alpha \) defined by Eq. (3.24). It can be shown that

\[
[ \tilde{D}_\alpha, \tilde{D}_\beta ] = [ \tilde{\nabla}_\alpha, \tilde{\nabla}_\beta ] - R_{\alpha\beta\mu\nu} p^\nu \frac{\partial}{\partial p_\mu},
\]

(D5)

where \( \tilde{\nabla}_\alpha \) is the covariant and gauge invariant derivative operator.

Therefore (compare with the formula (C1) ),

\[
[ \tilde{D}_\alpha, \tilde{D}_\beta ] f(x,p) = \frac{ie}{\hbar} [ F_{\alpha\beta}, f(x,p) ] - R_{\alpha\beta\mu\nu} p^\nu \frac{\partial}{\partial p_\mu} f(x,p)
\]

(D6)
and

\[
p^\alpha \left[ \hat{D}_\alpha, \hat{D}^\nu \hat{D}_\nu \right] f(x,p) = \frac{ie}{\hbar} p^\alpha \left[ F^\nu_{\alpha\nu}, f(x,p) \right] + \frac{ie}{\hbar} p^\alpha \left[ F_{\alpha\nu}, \hat{D}^\nu f(x,p) \right] +
\]

\[
+ \left( 2R_{\alpha\mu\beta\nu} p^\alpha p^\beta \frac{\partial}{\partial p_\mu} \hat{D}^\nu - R_{\alpha\nu} p^\alpha \hat{D}^\nu - R_{\alpha\nu;\beta} p^\alpha p^\beta \frac{\partial}{\partial p_\nu} + R_{\alpha\beta;\nu} p^\alpha p^\beta \frac{\partial}{\partial p_\nu} \right) f(x,p). \quad (D7)
\]

Then, by noting the following property:

\[
\{ \hat{A}, \left[ \hat{A}, f \right] \} - \left[ \hat{A}, \{ \hat{A}, f \} \right] = 0,
\]

one can prove after elaborate calculations that the identity (3.36) is right up to the terms of the next adiabatic order.

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