On the classification of convex lattice polytopes (II)

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Abstract. In 1980, V.I. Arnold studied the classification problem for convex lattice polygons of given area. Since then this problem and its analogues have been studied by several authors, upper bounds for the numbers of non-equivalent \(d\)-dimensional convex lattice polytopes of given volume or fixed number of lattice points have been achieved. In this paper, by introducing and studying the unimodular groups acting on convex lattice polytopes, we obtain a lower bound for the number of non-equivalent \(d\)-dimensional centrally symmetric convex lattice polytopes of given number of lattice points, which is essentially tight.

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1. Introduction

Let \(\{e_1, e_2, \ldots, e_d\}\) be an orthonormal basis of the \(d\)-dimensional Euclidean space \(\mathbb{E}^d\). A convex lattice polytope in \(\mathbb{E}^d\) is the convex hull of a finite subset of the integral lattice \(\mathbb{Z}^d\). As usual, let \(P\) denote a \(d\)-dimensional convex lattice polytope, let \(v(P)\) denote the volume of \(P\), let \(\text{int}(P)\) denote the interior of \(P\), and let \(|P|\) denote the cardinality of \(P \cap \mathbb{Z}^d\). For general references on polytopes and lattice polytopes, we refer to [2], [7], [8], [9] and [17].

Let \(P_1\) and \(P_2\) be \(d\)-dimensional convex lattice polytopes. If there is a unimodular transformation \(\sigma\) (\(\mathbb{Z}^d\)-preserving affinely linear transformation) satisfying \(P_2 = \sigma(P_1)\), then we say \(P_1\) and \(P_2\) are equivalent. For convenience, we write \(P_1 \sim P_2\) for short.

Clearly, the equivalence relation \(\sim\) divides convex lattice polytopes into different classes. Using triangulations, it can be easily shown that \(d! \cdot v(P) \in \mathbb{Z}\) holds for any \(d\)-dimensional convex lattice polytope \(P\). Let \(v(d, m)\) denote the number of different classes of the \(d\)-dimensional convex lattice polytopes \(P\) with \(v(P) = m/d!\), where both \(d\) and \(m\) are positive integers.

Let \(f(d, m)\) and \(g(d, m)\) be functions of positive integers \(d\) and \(m\). In this paper \(f(d, m) \ll g(d, m)\) means that, for fixed positive integer \(d\),

\[f(d, m) \leq c_d \cdot g(d, m)\]

holds for all positive integers \(m\), where \(c_d\) is a suitable constant depending only on \(d\). In 1980, Arnold [11] studied the values of \(v(2, m)\) and proved

\[m^4 \ll v(2, m) \ll m^4 \log m.\]  \(1\)

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In 1992, Bárány and Pach [5] improved Arnold’s upper bound to
\[ \log v(2, m) \ll m^{\frac{1}{3}}. \]  
(2)

Bárány and Vershik [6] generalized (2) to \( d \) dimensions by proving
\[ \log v(d, m) \ll m^{\frac{d-1}{d+1}}. \]  
(3)

Although Arnold at the end of his paper [1] where he proved (1) wrote “In \( \mathbb{Z}^d \), 1/3 is probably replaced by \((d - 1)/(d + 1)\). Proof of the lower bound: let \( x_1^2 + \ldots + x_{d-1}^2 \leq x_d \leq A \)” the problem whether
\[ \log v(d, m) \gg m^{\frac{d-1}{d+1}} \]
is still open.

Let \( v^*(d, m) \) denote the number of different classes of the \( d \)-dimensional centrally symmetric convex lattice polytopes \( P \) with \( v(P) = m/d! \), let \( \kappa(d, w) \) denote the number of different classes of \( d \)-dimensional convex lattice polytopes \( P \) with \( |P| = w \), and let \( \kappa^*(d, w) \) denote the number of different classes of \( d \)-dimensional centrally symmetric convex lattice polytopes \( P \) with \( |P| = w \) and \( \text{int}(P) \cap \mathbb{Z}^d \neq \emptyset \). Then we have \( v^*(d, m) = 0 \) whenever \( m \) is odd and \( \kappa^*(d, w) = 0 \) if \( w \) is even. Therefore in this paper we assume that the \( m \) in \( v^*(d, m) \) is even and the \( w \) in \( \kappa^*(d, w) \) is odd.

**Remark 1.** As usual, in this paper centrally symmetric convex lattice polytopes are those centered at lattice points. In this sense, the unit cube \( \{ x \in \mathbb{E}^d : 0 \leq x_i \leq 1 \} \) is not a centrally symmetric convex lattice polytope, though it is a convex lattice polytope and is centrally symmetric.

Recently, Liu and Zong [14] studies Arnold’s problem for the centrally symmetric lattice polygons and the classification problem for convex lattice polytopes of given cardinality by proving
\[ m^{\frac{1}{3}} \ll \log v^*(2, m) \ll m^{\frac{1}{3}}, \]
\[ w^{\frac{1}{3}} \ll \log \kappa(2, w) \ll w^{\frac{1}{3}}, \]
\[ w^{\frac{1}{3}} \ll \log \kappa^*(2, w) \ll w^{\frac{1}{3}}, \]
\[ \kappa(d, w) = \infty, \text{ if } w \geq d + 1 \geq 4, \]
\[ \log \kappa'(d, w) \ll w^{\frac{d-1}{d+1}} \]  
(4)

and
\[ \log \kappa^*(d, w) \ll w^{\frac{d-1}{d+1}}. \]  
(5)

In Section 2 of this paper we introduce and study unimodular groups acting on convex lattice polytopes. In particular, the orders of these groups are estimated. In Section 3, by applying the results obtained in Section 2, we prove the following result:

**Theorem 1.** Let \( \kappa^*(d, w) \) denote the number of different classes of \( d \)-dimensional centrally symmetric convex lattice polytopes \( P \) with \( |P| = w \), then
\[ \log \kappa^*(d, w) \gg w^{\frac{d-1}{d+1}}. \]
This theorem, together with (4) and (5), produces the following consequences (when
\(w\) is even, to deduce Theorem 3 needs a little extra care since it is no longer symmetric):

**Theorem 2.** Let \(\kappa^*(d, w)\) denote the number of different classes of \(d\)-dimensional centrally symmetric convex lattice polytopes \(P\) with \(|P| = w\), then
\[
w^{\frac{d-1}{d}} \ll \log \kappa^*(d, w) \ll w^{\frac{d-1}{d}}.
\]

**Theorem 3.** Let \(\kappa'(d, w)\) denote the number of different classes of \(d\)-dimensional convex lattice polytopes \(P\) with \(|P| = w\) and \(\text{int}(P) \cap \mathbb{Z}^d \neq \emptyset\), then
\[
w^{\frac{d-1}{d}} \ll \log \kappa'(d, w) \ll w^{\frac{d-1}{d}}.
\]

### 2. Unimodular groups of convex lattice polytopes

In this section we introduce and study unimodular groups acting on convex lattice polytopes. In particular, Lemma 3 will be essential for our proof of Theorem 1.

As usual, a unimodular transformation \(\sigma(x)\) of \(E^d\) is an \(\mathbb{Z}^d\)-preserving affinely linear transformation, i.e.,
\[
\sigma(x) = xU + v,
\]
where \(U\) is a \(d \times d\) integral matrix satisfying \(|\det(U)| = 1\) and \(v\) is an integral vector. In particular, if \(U\) also satisfies \(UU' = I\), where \(U'\) is the transpose of \(U\) and \(I\) is the \(d \times d\) unit matrix, we call \(\sigma(x)\) an orthogonal unimodular transformation. It is known in linear algebra that an orthogonal unimodular keeps the Euclidean distances unchanged.

Let \(\sigma_1\) and \(\sigma_2\) be two unimodular transformations in \(E^d\). It is known in linear algebra that both \(\sigma_1 \cdot \sigma_2\) and \(\sigma_1^{-1}\) are unimodular transformations. Therefore, all unimodular transformations in \(E^d\) form a multiplicative group. We denote it by \(G_d\). Similarly, all orthogonal unimodular transformations in \(E^d\) form a subgroup of \(G_d\). We denote it by \(G'_d\).

The group \(G_d\) is different from \(\text{GL}(d, \mathbb{Z})\). But, they are closely related. In fact, we have
\[
G_d \cong \left\{ \begin{pmatrix} A & O \\ V & 1 \end{pmatrix} : A \in \text{GL}(d, \mathbb{Z}), \ V \in \mathbb{Z}^d, \ O = (0, 0, \ldots, 0)' \right\} \subseteq \text{GL}(d + 1, \mathbb{Z}).
\]

Let \(P\) be a convex lattice polytope in \(E^d\), and let \(\mathcal{P}_d\) denote the family of all \(d\)-dimensional convex lattice polytopes. We define
\[
\sigma(P) = \{\sigma(x) : x \in P\}.
\]
Clearly, \(\sigma(P)\) is a convex lattice polytope as well. Then we define
\[
G(P) = \{\sigma \in G_d : \sigma(P) = P\}
\]
and
\[
G'(P) = \{\sigma \in G'_d : \sigma(P) = P\}.
\]
Both $G(P)$ and $G'(P)$ are finite subgroups of $G_d$, and $G'(P)$ is a subgroup of $G(P)$. We call $G(P)$ the unimodular group of $P$ and call $G'(P)$ the orthogonal unimodular group of $P$.

It is easy to see that both $G_d$ and $G'_d$ act on $P_d$, $G(P)$ is the stabilizer of $P$ in $G_d$, and $G'(P)$ is the stabilizer of $P$ in $G'_d$. Therefore, we have

**Lemma 1.** If $P \in P_d$ and $\sigma \in G_d$, then we have

$$G(\sigma(P)) = \sigma G(P) \sigma^{-1}.$$  

If $\tau \in G'_d$, then we have

$$G'(\tau(P)) = \tau G'(P) \tau^{-1}.$$  

**Lemma 2.** Let $O_d$ denote the multiplicative group of orthogonal unimodular transformations of $\mathbb{E}^d$ which keep the origin fixed. Then, we have

$$|O_d| = 2^d \cdot d!!.$$  

If $\sigma \in O_d$, then we have

$$\sigma(e_i) \in \{ \pm e_1, \pm e_2, \ldots, \pm e_d \},$$  

by which one can easily deduce the lemma. In fact, $O_d$ is the multiplicative group of the $d \times d$ orthogonal integral matrices.

**Corollary 1.** For any $d$-dimensional centrally symmetric convex lattice polytope $P$, $|G'(P)|$ is a divisor of $2^d \cdot d!!$.

We have two basic problems about the unimodular groups of convex lattice polytopes.

**Problem 1.** Determine the values of

$$\max_{P \in \mathbb{E}_d} \{|G(P)|\}.$$  

**Remark 2.** It was proved by Minkowski [15] that, there is a constant $c_d$ depends only on $d$,

$$|G| \leq c_d$$  

holds for any finite subgroup $G$ of $\text{GL}(d, \mathbb{Z})$. According to [10] and [12], in an unpublished manuscript W. Feit proved (based on an unfinished manuscript of B. Weisfeiler) that, if $d > 10$ and $G$ is a finite subgroup of $\text{GL}(d, \mathbb{Z})$, then

$$|G| \leq 2^d \cdot d!!.$$  

Feit’s result implies that, when $d > 10$,

$$|G(C)| \leq 2^d \cdot d!!$$  

holds for all $d$-dimensional centrally symmetric lattice polytopes $C$.  

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Problem 2. Is it true that, for any finite subgroup $G$ of $G_d$, there is a $P \in \mathbb{P}_d$ such that $G(P) \cong G$? Is it true that, for any finite subgroup $G$ of $GL(d, \mathbb{Z})$, there is a $d$-dimensional centrally symmetric convex lattice polytope $P$ satisfying $G(P) \cong G$?

These problems show the close relation between $d$-dimensional convex lattice polytopes and the finite subgroups of $GL(d, \mathbb{Z})$ and $GL(d + 1, \mathbb{Z})$.

Let $m$ be a positive integer and let $\rho$ be a real number satisfying $1 \leq \rho \leq \infty$. We define

$$P_{d,m,\rho} = \text{conv} \left\{ z \in \mathbb{Z}^d : \left( \sum_{i=1}^{d} |z_i|^\rho \right)^{1/\rho} \leq m \right\},$$

where $\left( \sum |z_i|^{\infty} \right)^{1/\infty} = \max\{|z_i|\}$. One can easily verify that $P_{d,m,\rho}$ is a $d$-dimensional centrally symmetric convex lattice polytope. In particular, $P_{d,m,1}$ is a lattice cross-polytope and $P_{d,m,\infty}$ is a lattice cube.

Lemma 3. When $d$ and $m$ are positive integers and $\rho$ is a positive number satisfying $1 \leq \rho \leq \infty$, we have

$$G(P_{d,m,\rho}) = G'(P_{d,m,\rho}) = O_d$$

and

$$|G(P_{d,m,\rho})| = |G'(P_{d,m,\rho})| = 2^d \cdot d!.$$

Proof. First of all, since $P_{d,m,\rho}$ is centrally symmetric and centered at the origin, we have

$$\sigma(o) = o$$

for all $\sigma \in G(P_{d,m,\rho})$.

Let $v$ be a primitive integral vector in $\mathbb{Z}^d$ and let $P$ be a centrally symmetric convex lattice polytope in $\mathbb{E}^d$. We define

$$L(P,v) = \{ zv : z \in \mathbb{Z} \} \cap P$$

and

$$\ell(P) = \max_v \{|L(P,v)|\},$$

where the maximum is over all primitive integral vectors in $\mathbb{Z}^d$. Recall that $\{e_1, e_2, \ldots, e_d\}$ is an orthonormal basis of $\mathbb{E}^d$. We consider two cases as follows:

Case 1. $\rho < \infty$. Notice that

$$\sum_{i=1}^{d} |mv_i|^\rho = m^\rho \sum_{i=1}^{d} |v_i|^\rho,$$

it can be easily deduce that

$$\ell(P_{d,m,\rho}) = 2m + 1$$

and

$$|L(P_{d,m,\rho},v)| = 2m + 1.$$
holds if and only if \( \mathbf{v} = \pm \mathbf{e}_i \) for some index \( i \). Thus, for any \( \sigma \in G(P_{d,m,\rho}) \), we have
\[
\{ \sigma(\mathbf{e}_1), \sigma(\mathbf{e}_2), \ldots, \sigma(\mathbf{e}_d) \} \subset \{ \pm \mathbf{e}_1, \pm \mathbf{e}_2, \ldots, \pm \mathbf{e}_d \}.
\] (7)

**Case 2.** \( \rho = \infty \). In this case \( P_{d,m,\infty} \) is a \( d \)-dimensional cube. It has \( 2d \) facets \( \pm F_1, \pm F_2, \ldots, \pm F_d \), each is a \((d-1)\)-dimensional cube. The centers of the facets are \( \pm m\mathbf{e}_1, \pm m\mathbf{e}_2, \ldots, \pm m\mathbf{e}_d \). If \( \sigma \in G(P_{d,m,\infty}) \), we have
\[
\{ \sigma(F_1), \sigma(F_2), \ldots, \sigma(F_d) \} \subset \{ \pm F_1, \pm F_2, \ldots, \pm F_d \},
\]
\[
\{ \sigma(m\mathbf{e}_1), \sigma(m\mathbf{e}_2), \ldots, \sigma(m\mathbf{e}_d) \} \subset \{ \pm m\mathbf{e}_1, \pm m\mathbf{e}_2, \ldots, \pm m\mathbf{e}_d \}
\]
and therefore
\[
\{ \sigma(\mathbf{e}_1), \sigma(\mathbf{e}_2), \ldots, \sigma(\mathbf{e}_d) \} \subset \{ \pm \mathbf{e}_1, \pm \mathbf{e}_2, \ldots, \pm \mathbf{e}_d \}.
\] (8)

Assume that the unimodular transformation \( \sigma \) is defined by
\[
\sigma(\mathbf{x}) = \mathbf{x}U + \mathbf{b}.
\]
It follows by (6) that \( \mathbf{b} = \mathbf{o} \). In both cases, since \( U \) is nonsingular, by (7) and (8) we get
\[
\sum_{j=1}^{d} |u_{ij}| = 1, \quad i = 1, 2, \ldots, d
\]
and
\[
\sum_{i=1}^{d} |u_{ij}| = 1, \quad j = 1, 2, \ldots, d.
\]
Thus, we obtain
\[
G(P_{d,m,\rho}) \subseteq O_d.
\] (9)

On the other hand, it is easy to verify that
\[
O_d \subseteq G'(P_{d,m,\rho}).
\] (10)

As a conclusion of (9) and (10) we get
\[
O_d \subseteq G'(P_{d,m,\rho}) \subseteq G(P_{d,m,\rho}) \subseteq O_d
\]
and finally
\[
G(P_{d,m,\rho}) = G'(P_{d,m,\rho}) = O_d.
\]

The second assertion of the lemma follows from Lemma 2. The lemma is proved.
\( \square \)

**Remark 3.** Let \( S_d \) denote the \( d \)-dimensional lattice simplex with vertices \( \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_d \) and \( \mathbf{o} \). Then we have \( |G'(S_d)| = d! \) and \( |G(S_d)| = (d+1)! \). This example shows that \( |G'(P)| \) and \( |G(P)| \) can be different.
3. Proof of Theorem 1

In this section we study the classification problem for convex lattice polytopes of given cardinality. In particular, Theorem 1 will be proved.

First, as illustrated by Figure 1, for a positive integer \( r \) we define

\[
K_{d,r} = \left\{ x \in \mathbb{E}^d : x_d \geq 0, x_d + \sum_{i=1}^{d-1} x_i^2 \leq r^2 \right\},
\]

\[
B_{d,r} = \left\{ x \in \mathbb{E}^d : x_d = 0, \sum_{i=1}^{d-1} x_i^2 \leq r^2 \right\}
\]

and

\[
P_{d,r} = \text{conv} \left\{ K_{d,r} \cap \mathbb{Z}^d \right\}.
\]

It is easy to compute that

\[
v(K_{d,r}) = \int_0^{r^2} \frac{\pi \frac{d+1}{2}}{\Gamma(d+1)} \cdot (r^2 - x)^{\frac{d+1}{2}} dx = c_1(d) \cdot r^{d+1}
\]

and

\[
s'(K_{d,r}) = \int_0^{r^2} \frac{(d-1)\pi \frac{d-1}{2}}{\Gamma(d+1)} \cdot (r^2 - x)^{\frac{d-2}{2}} dx = (d-1)c_1(d) \cdot r^d,
\]

where \( s'(K_{d,r}) \) denotes the surface area of \( K_{d,r} \) without the base and \( c_1(d) \) is a constant that depends only on \( d \).

Next, we define

\[
C_{d,r}^1 = \left\{ x \in \mathbb{E}^d : 0 \leq x_d \leq r^2; \sum_{i=1}^{d-1} x_i^2 \leq r^2 \right\}
\]
\[ C^2_{d,r} = \left\{ x \in \mathbb{E}^d : -1 \leq x_1 \leq 1, \, 0 \leq x_d \leq r^2, \, x_d + \sum_{i=2}^{d-1} x_i^2 \leq r^2 \right\} \]

and their intersection

\[ C_{d,r} = C^1_{d,r} \cap C^2_{d,r}. \]

In fact, \( C^1_{d,r} \) is a cylinder over a base \( B_{d,r} \) and \( C^2_{d,r} \) is a cylinder over a base

\[ \left\{ x \in \mathbb{E}^d : x_d \geq 0, \, x_1 = 0, \, \sum_{i=2}^{d-1} x_i^2 + x_d \leq r^2 \right\}, \]

as shown in Figure 2.
Notice that (as illustrated in Figure 3), if

\[ x^2 = \sum_{i=2}^{d-1} x_i^2 \]  

and

\[ (0, x_2, x_3, \ldots, x_{d-1}, 0) \in C_{d,r}, \]

then we have

\[ (x_1, x_2, \ldots, x_{d-1}, x_d) \in C_{d,r} \]

provided \( |x_1| \leq \sqrt{r^2 - x^2} \) and \( 0 \leq x_d \leq r^2 - x^2 \). In fact, all the points satisfying both (13) and (14) together form a \((d-2)\)-dimensional sphere of radius \( x \) which has area measure

\[ \frac{(d-2)\pi^{d-2}}{\Gamma\left(\frac{d}{2}\right)} \cdot x^{d-3}. \]

Thus, we get

\[
v(C_{d,r}) = \int_0^r \frac{2(d-2)\pi^{d-2}}{\Gamma\left(\frac{d}{2}\right)} \cdot x^{d-3} (r^2 - x^2) \frac{d}{2} dx \]

\[ = \frac{2(d-2)\pi^{d-2}}{\Gamma\left(\frac{d}{2}\right)} \cdot r^{d+1} \int_0^\frac{\pi}{2} \sin^{d-3} \theta \cos^d \theta d\theta \]

\[ = \frac{2(d-2)\pi^{d-2}}{\Gamma\left(\frac{d}{2}\right)} \cdot r^{d+1} \int_0^\frac{\pi}{2} \left(\sin^{d-3} \theta - 2 \sin^{d-1} \theta + \sin^{d+1} \theta\right) d\theta \]

\[ = c_2(d) \cdot r^{d+1} \quad (15) \]

and

\[
s' (C_{d,r}) \leq s' (C_{d,r}^2) + s' (C_{d,r}^2) \leq c_3(d) \cdot r^d, \]

where \( c_2(d) \) and \( c_3(d) \) are constants depending only on \( d \). It follows by \( K_{d,r} \subset C_{d,r} \) that

\[ c_1(d) < c_2(d). \]

For convenience, we write \( W = \{(x_1, x_2, \ldots, x_d) : |x_i| \leq \frac{1}{2} \} \) and \( B = \{(x_1, x_2, \ldots, x_d) : \sum x_i^2 \leq 1 \} \). By convexity we have

\[
v(C_{d,r}) - \frac{1}{2} \sqrt{d} \cdot s'(C_{d,r}) \leq v(W + C_{d,r} \cap \mathbb{Z}^d) \leq v(C_{d,r} + \frac{1}{2} \sqrt{d} B). \]

Then, by (15) and (16) we get

\[
v(C_{d,r}) - \frac{1}{2} \sqrt{d} \cdot c_3(d) \cdot r^d \leq |C_{d,r} \cap \mathbb{Z}^d| \leq v(C_{d,r}) + O(r^d) \]

and therefore

\[ |C_{d,r} \cap \mathbb{Z}^d| \sim v(C_{d,r}). \quad (18) \]

Similarly, by (11) and (12) we get

\[ |P_{d,r}| \sim v(K_{d,r}). \quad (19) \]
Next, we define
\[ Q_{d,r} = (\text{int} \ (C^1_{d,r}) \cap C^2_{d,r}) \cup B_{d,r}, \]  
\[ H_{d,r} = \text{conv} \{ z \in Q_{d,r} \cap \mathbb{Z}^d : z_1 \leq 0 \} \]  
(20)
and
\[ H'_{d,r} = \text{conv} \{ z \in K_{d,r} \cap \mathbb{Z}^d : z_1 \leq 0 \}. \]  
(22)
By (18) and (19), one can deduce
\[ |H_{d,r}| \sim \frac{1}{2} \cdot |C_{d,r} \cap \mathbb{Z}^d| \sim \frac{1}{2} \cdot v(C_{d,r}) = \frac{c_2(d)}{2} \cdot r^{d+1} \]  
(23)
and
\[ |H'_{d,r}| \sim \frac{1}{2} \cdot |P_{d,r}| \sim \frac{1}{2} \cdot v(K_{d,r}) = \frac{c_1(d)}{2} \cdot r^{d+1}. \]  
(24)

**Remark 4.** Let \( L(x) \) denote the line defined by \( \{ x + \lambda e_d : \lambda \in \mathbb{R} \} \). When \( z \) is a lattice point on the boundary of \( B_{d,r} \), we have
\[ |L(z) \cap Q_{d,r} \cap \mathbb{Z}^d| = 1. \]

Now, we introduce a technical lemma which is useful in the proof of Theorem 1.

**Lemma 4.** When \( r \) is a sufficiently large integer, for any integer \( k \) satisfying \( 0 \leq k \leq 3r|B_{d,r+1} \cap \mathbb{Z}^d| \), there is a convex lattice polytope \( P \) that satisfies
\[ H'_{d,r} \subseteq P \subseteq H_{d,r} \]  
and
\[ |P| = |H'_{d,r}| + k. \]

**Proof.** It is well-known that
\[ |B_{d,r+1} \cap \mathbb{Z}^d| = \frac{\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot (r + 1)^{d-1} + O((r + 1)^{d-2}). \]
\[ = \frac{\pi^{\frac{d+1}{2}}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot r^{d-1} + O(r^{d-2}). \]
By (23), (24) and (17), when \( r \) is sufficiently large, we get
\[ |H_{d,r}| - |H'_{d,r}| \geq \frac{1}{4} \cdot (c_2(d) - c_1(d)) \cdot r^{d+1} \]
\[ \geq c_4(d) \cdot r^2 \cdot |B_{d,r+1} \cap \mathbb{Z}^d| \]
\[ \geq 3r|B_{d,r+1} \cap \mathbb{Z}^d|, \]  
(25)
where \( c_4(d) \) is a constant that depends only on \( d \).

For convenience, we write \( P_0 = H_{d,r} \) and let \( \overline{P} \) denote the set of the vertices of \( P \). If \( v_0 \in \overline{P} \setminus H'_{d,r} \), we define
\[ P_1 = \text{conv} \{(P_0 \cap \mathbb{Z}^d) \setminus \{v_0\}\}. \]
Inductively, if \( P_i \) has been defined and \( v_i \in P_i \setminus H_{d,r} \), we construct
\[
P_{i+1} = \text{conv} \left\{ (P_i \cap \mathbb{Z}^d) \setminus \{v_i\} \right\}.
\]
Thus, we have constructed a finite sequence of convex lattice polytopes \( P_0, P_1, P_2, \ldots, P_\ell = H'_{d,r} \) which satisfies both
\[
P_0 \supset P_1 \supset \cdots \supset P_{\ell-1} \supset P_\ell = H'_{d,r}
\]
and
\[
|P_i| - |P_{i+1}| = 1, \quad i = 0, 1, 2, \ldots, \ell - 1.
\]
By (25) it follows that
\[
\ell \geq 3r \left| B_{d,r+1} \cap \mathbb{Z}^d \right|.
\]
The assertion is proved. \( \square \)

**Proof of Theorem 1.** First, we recall that
\[
P_{d,r} = \text{conv} \left\{ K_{d,r} \cap \mathbb{Z}^d \right\}. \quad (26)
\]
Let \( w \) be a large odd integer and let \( r \) be the integer satisfying
\[
|P_{d,r}| \leq \frac{w}{2} < |P_{d,r+1}|. \quad (27)
\]
By (11) and (19) we get
\[
r^{d+1} \ll w \ll r^{d+1}. \quad (28)
\]
We write
\[
P_{d,r+1}^1 = \{ x \in P_{d,r+1} : x_d \geq 2r + 1 \}
\]
and
\[
P_{d,r+1}^2 = \{ x \in P_{d,r+1} : x_d \leq 2r \}.
\]
It is easy to see that
\[
|P_{d,r}| = |P_{d,r+1}^1|
\]
and therefore
\[
|P_{d,r+1}^1| - |P_{d,r}| = |P_{d,r+1}^2| \leq (2r + 1) \left| B_{d,r+1} \cap \mathbb{Z}^d \right|. \quad (29)
\]
We write
\[
u = w - 2|P_{d,r}| + \left| B_{d,r} \cap \mathbb{Z}^d \right|. \quad (30)
\]
By (27) and (29) we get
\[
u < 2 (|P_{d,r+1}^1| - |P_{d,r}|) + \left| B_{d,r} \cap \mathbb{Z}^d \right| \leq 5r \left| B_{d,r+1} \cap \mathbb{Z}^d \right|. \quad (31)
\]
Let \( V'_{d,r} \) denote the set of the vertices \( v \) of \( P_{d,r} \) satisfying both \( v_d \neq 0 \) and \( v_1 \geq 1 \), and let \( L(x) \) denote the line \( \{ x + \lambda e_d : \lambda \in \mathbb{R} \} \) as defined in Remark 4. By convexity, for all \( z \in B_{d,r} \cap \mathbb{Z}^d \) with \( z_1 \geq 1 \) we have
\[
|L(z) \cap V'_{d,r}| \leq 1.
\]
Thus we get

\[|V'_{d,r}| < \frac{1}{2} |B_{d,r} \cap \mathbb{Z}^d|\]  \hspace{1cm} (32)

and

\[|V'_{d,r}| \geq \frac{1}{2} \left( |\text{int}(B_{d,r}) \cap \mathbb{Z}^d| - |B_{d-1,r} \cap \mathbb{Z}^d| \right) \geq \frac{1}{3} \left( \frac{\pi \frac{d+1}{2}}{\Gamma\left(\frac{d+1}{2}\right)} \cdot r^{d-1} - \frac{\pi \frac{d}{2}}{\Gamma\left(\frac{d}{2}\right)} \cdot r^{d-2} \right) \gg r^{d-1}. \] \hspace{1cm} (33)

With these preparations, we proceed to construct the expected convex lattice polytopes.

**Step 1.** Let \( v_1, v_2, \ldots, v_j \) be \( j \) points in \( V'_{d,r} \) and define

\[P'_{d,r} = \text{conv} \{P_{d,r} \setminus \{v_1, v_2, \ldots, v_j\}\}. \] \hspace{1cm} (34)

We have

\[|P'_{d,r}| = |P_{d,r}| - j. \] \hspace{1cm} (35)

By (31) and (32) we get

\[\frac{u}{2} + j \leq \frac{5}{2} r |B_{d,r+1} \cap \mathbb{Z}^d| + \frac{1}{2} |B_{d,r} \cap \mathbb{Z}^d| < 3r |B_{d,r+1} \cap \mathbb{Z}^d|. \]

According to Lemma 4, there is a convex lattice polytope \( P \) satisfies both

\[H'_{d,r} \subseteq P \subseteq H_{d,r} \] \hspace{1cm} (36)

and

\[|P| = |H'_{d,r}| + \frac{u}{2} + j. \] \hspace{1cm} (37)

**Step 2.** We construct

\[P' = P \cup P'_{d,r}. \] \hspace{1cm} (38)

Let \( i \) be an integer and let \( H_i \) denote the hyperplane \( \{(x_1, x_2, \ldots, x_d) : x_1 = i\} \). By (34) and (36) it is easy to see that \( P' \cap H_i - i\mathbf{e}_1 \) is a subset of \( P' \cap H_0 \). Therefore the convexity of \( P' \) at both sides of \( H_0 \) guarantees the convexity of \( P' \). In other words, we have

\[P' = \text{conv}\{P \cup P'_{d,r}\}. \]

By (35) and (37) we get

\[|P'| = |P_{d,r}| + \frac{u}{2}. \] \hspace{1cm} (39)

**Step 3.** We define

\[P_{v_1, \ldots, v_j} = P' \cup \{-P'\}. \] \hspace{1cm} (40)
Clearly $P_{v_1,\ldots,v_j}$ is a centrally symmetric convex lattice polytope centered at the origin. By (39) and (30) we get

$$|P_{v_1,\ldots,v_j}| = 2\left(|P_{d,r}| + \frac{u}{2}\right) - |B_{d,r} \cap \mathbb{Z}^d|$$

$$= 2|P_{d,r}| + u - |B_{d,r} \cap \mathbb{Z}^d|$$

$$= 2|P_{d,r}| + w - 2|P_{d,r}| + |B_{d,r} \cap \mathbb{Z}^d| - |B_{d,r} \cap \mathbb{Z}^d|$$

$$= w.$$

**Step 4.** Taking all possible subsets of $V_{d,r}'$, we get $2^{|V_{d,r}'|}$ centrally symmetric convex lattice polytopes of cardinality $w$. For convenience, we enumerate them by $P_1, P_2, \ldots, P_{2^{|V_{d,r}'|}}$ and denote the whole family by $\mathcal{F}$.

Now, we study the equivalence relation among $\mathcal{F}$.

Recalling the definitions of $L(P,v)$ and $\ell(P)$ in the proof of Lemma 3, for any $P_i \in \mathcal{F}$ we have

$$\ell(P_i) = 2r^2 + 1$$

and

$$L(P_i,v) = 2r^2 + 1$$

holds if and only if $v = \pm e_d$. Therefore, if $\sigma(P_i) = P_j$ holds for some unimodular transformation $\sigma$, we have $\sigma(0) = 0$ and $\sigma(e_d) \in \{e_d, -e_d\}$.

Let $H$ be a $(d-1)$-dimensional hyperplane which contains the origin of $\mathbb{E}^d$, but not $e_d$. By projecting $P_i \cap H \cap \mathbb{Z}^d$ onto the plane $\{x \in \mathbb{E}^d : x_d = 0\}$, keeping (20), (21), (26), (34), (36), (38), (40) and Remark 4 in mind, it follows that

$$|P_i \cap H \cap \mathbb{Z}^d| \leq |B_{d,r} \cap \mathbb{Z}^d|,$$

where the equality holds if and only if

$$H = \{x \in \mathbb{E}^d : x_d = 0\}.$$

Thus, we get

$$\sigma(B_{d,r} \cap \mathbb{Z}^d) = B_{d,r} \cap \mathbb{Z}^d$$

and therefore

$$\sigma \in G(\text{conv} \{B_{d,r} \cap \mathbb{Z}^d\}) = G(P_{d-1,r,2}).$$

Consequently, by the $\rho = 2$ case of Lemma 3, we get

$$\kappa^*(d,w) \geq \frac{|\mathcal{F}|}{2|G(P_{d-1,r,2})|} = \frac{2^{|V_{d,r}'|}}{2^d \cdot (d-1)!}.$$

By (33) and (28), we deduce

$$\log \kappa^*(d,w) \gg |V_{d,r}'| \gg r^{d-1} \gg \frac{w}{d-1}.$$

The proof is complete. □
Remark 5. It was proved by Pikhurko [16] that \( v(P) \leq c_d \cdot |P| \) holds for all \( d \)-dimensional lattice polytopes with nonempty interior lattice point, where \( c_d \) is a constant depending only on \( d \). Thus, by Theorem 1 one can deduce

\[
\log \left( \sum_{j=1}^{m} v(d, j) \right) \gg m^{\frac{d-1}{d+1}},
\]

which was proved by Bárány [2] by a different method.

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