Abstract. In this paper we give conditions under which a topological semigroup can be embedded algebraically and topologically into a compact topological group. We prove that every feebly compact regular first countable cancellative commutative topological semigroup with open shifts is a topological group, as well as every connected locally compact Hausdorff cancellative commutative topological monoid with open shifts. Finally, we use these results to give sufficient conditions on a topological semigroup that guarantee it to have countable cellularity.

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1. INTRODUCTION

In [16] the author presents some properties that allow us to embed, topologically, a cancellative commutative topological semigroup into a topological group, we take advantage of this results to find conditions under which a cancellative commutative topological semigroup has countable cellularity, as well as, when a topological semigroup is a topological group. The class of topological spaces having countable cellularity is wide, in fact, it contains (among other classes) the class of the σ-compact paratopological groups (see [17] Corollary 2.3), the class of the sequentially compact σ-compact cancellative topological monoids (see [7] Theorem 4.8)) and the class of the subsemigroups of precompact topological groups (see [20] Corollary 3.6)). These results have served us as motivation, thus we try to find similar results in the context of topological semigroups.

Compactness type conditions (compactness and sequential compactness) under which a topological semigroup is a topological group are given in [11] Theorem 2.5.2], [15] Theorem 2.4] and [8] Theorem 6], we use the feeble compactness and local compactness to obtain similar results. The reflection on the class of the regular spaces allows us to disregard the axioms of separation to obtain topological monoids with countable cellularity.
2. PRELIMINARIES

We denote by \( \mathbb{Z} \), \( \mathbb{R} \) and \( \mathbb{N} \), the set of all the integer numbers, real numbers and positive integer numbers, respectively. If \( A \) is a set, \( |A| \) will denote the cardinal of \( A \), \( \aleph_0 = |\mathbb{N}| \). If \( X \) is a topological space and \( x \in X \), \( N_x(X) \) will denote the set of all open neighborhoods of \( x \) in \( X \) or simply \( N_x \) when the space is understood.

A semigroup is a set \( S \neq \emptyset \), endowed with an associative operation. If \( S \) also has neutral element, we say that \( S \) is a monoid. A mapping \( f : S \rightarrow H \) between semigroups is a homomorphism if \( f(xy) = f(x)f(y) \) for all \( x, y \in S \). A semitopological semigroup (monoid) consists of a semigroup (resp. monoid) \( S \) and a topology \( \tau \) on \( S \), such that for all \( a \in S \), the shifts \( x \mapsto ax \) and \( x \mapsto xa \) (noted by \( l_a \) and \( r_a \), respectively) are continuous mappings from \( S \) to itself. We say that a semitopological semigroup has open shifts, if for each \( a \in S \) and for each open set \( U \) in \( S \), we have that \( l_a(U) \) and \( r_a(U) \) are open sets in \( S \). A topological semigroup (monoid) (paratopological group) consists of a semigroup (resp. monoid) (resp. group) \( S \) and a topology \( \tau \) on \( S \), such that the operation of \( S \) is jointly continuous. Like [11] we do not require that semigroups to be Hausdorff. If \( S \) is a paratopological group and if also the mapping \( x \mapsto x^{-1} \) is continuous, we say that \( S \) is a topological group. A congruence on a semigroup \( S \) is an equivalence relation on \( S \), \( \sim \), such that if \( x \sim y \) and \( a \sim b \), then \( xa \sim yb \). If \( S \) is a semitopological semigroup, then we say that \( \sim \) is a closed congruence if \( \sim \) is closed in \( S \times S \). If \( \sim \) is an equivalence relation in a semigroup (monoid) \( S \) and \( \pi : S \rightarrow S/\sim \) is the respective quotient mapping, then \( S/\sim \) is a semigroup (monoid) and \( \pi \) an homomorphism if and only if \( \sim \) is a congruence ([5], Theorem 1).

The axioms of separation \( T_0, T_1, T_2, T_3 \) and regular are defined in accordance with [3]. We denote by \( C_i \) the class of the \( T_i \) spaces, where \( i \in \{0, 1, 2, 3, r\} \).

Let \( X \) be a topological space, a cellular family in \( X \) is a pairwise disjoint non empty family of non empty open sets in \( X \). The cellularity of a space \( X \) is noted by \( c(X) \) and it is defined by

\[
c(X) = \sup\{|U| : U \text{ is cellular family in } X\} + \aleph_0.
\]

If \( c(X) = \aleph_0 \), we say that \( X \) has countable cellularity or \( X \) has the Souslin property.

If \( X \) is a topological space and \( A \subseteq X \), we will note by \( \text{Int}_X(A) \) and \( \text{Cl}_X(A) \), the interior and the closure of \( A \) in \( X \), or simply \( \text{Int}(A) \) and \( \overline{A} \), respectively, when the space \( X \) is understood. An open set \( U \) in \( X \), is called regular open in \( X \) if \( \text{Int}U = U \). It is easy to prove the regular open ones form a base for a topology in \( X \), \( X \) endowed with this topology, will note by \( X_{sr} \), which we will call semiregularitation of \( X \).

From [10], it is well known that for each \( i \in \{0, 1, 2, 3, r\} \) and every topological space, \( X \), there is a topological space, \( C_i(X) \in C_i \), (unique up to homeomorphism) and a continuous mapping \( \varphi_{(X,C_i)} \) of \( X \) onto \( C_i(X) \), such that given a continuous mapping \( f : X \rightarrow Y \), being \( Y \in C_i \), there exists an
unique continuous mapping \( g: C_i(X) \to Y \), such that \( g \circ \varphi_{(X,C_i)} = f \).

According to [2, Section 2] a SAP-compactification of a semitopological semigroup \( S \) is a pair \((G,f)\) consisting of a compact Hausdorff topological group \( G \) and a continuous homomorphism \( f: S \to G \) such that for each continuous homomorphism \( h: S \to K \), being \( K \) a compact Hausdorff topological group, there is an unique continuous homomorphism \( h^*: G \to K \) such that \( h = h^* \circ f \).

A space \( X \) is \textit{feebly compact} if each locally finite family of open sets in \( X \) is finite. By [8, Theorem 1.1.3], pseudocompactness is equivalent to feeble compactness in the class of the Tychonoff spaces.

A space \( X \) is \textit{locally compact} if each \( x \in X \) has a compact neighborhood.

The following proposition gives us some properties of \( C_i(S) \), when \( S \) is a topological monoid with open shifts.

**Proposition 2.1.** Let \( S \) be a topological monoid with open shifts. Then

i) \( C_r(S) \) is a monoid, \( \varphi_{(S,C_i)} \) is an homomorphism (see [6] Proposition 3.8) and \( C_r(S) = C_0(S_{sr}) \) (see [7] Theorem 2.8). If \( S \) is also cancellative, \( C_r(S) \) is cancellative (see [7] Lemma 4.6).

ii) If \( A \) is an open set in \( S \), then \( C_r(A) = \varphi_{(S,C_i)}(A) \) (see [6] Corollary 5.7 and [13] Lemma 4).

iii) \( c(S) = c(C_i(S)) \), for each \( i \in \{0,1,2,3,r\} \).

iv) \( C_2(S) = S_{sr} \). Moreover, if \( S \) is \( T_2 \), \( C_r(S) = S_{sr} \) (see [7] Corollary 2.7 and [13] Proposition 1).

v) \( \varphi_{(S,C_i)} \) is open for each \( i \in \{0,1,2\} \) (see [7] Proposition 2.1).

vi) If \( S \) is a paratopological group, \( C_r(S) \) is a paratopological group (see [19] Corollary 3.3 and Theorem 3.8), [13] Theorem 2.4).

It is easy to see that if \( X \) is a first countable topological space, then \( X_{sr} \) is first countable. Since \( C_r(S) = C_0(S_{sr}) \) and \( \varphi_{(S,C_i)} \) is open, whenever \( S \) is a topological monoid with open shifts, we have the following corollary.

**Corollary 2.2.** If \( S \) is a first-countable topological monoid with open shifts, then \( C_r(S) \) is first-countable.

### 3. EMBEDDING TOPOLOGICAL SEMIGROUPS INTO TOPOLOGICAL GROUPS

Let \( S \) be a cancellative commutative semigroup, \( S \times S \) is a cancellative commutative semigroup, by defining the operation coordinate wise. Let us define in \( S \times S \) the following relation: \((x, y) \in R(a, b)\) if and only if \( xb = ya \). It is not hard to prove that \( R \) is a congruence, hence if \( \pi: S \times S \to (S \times S)/R \) is the respective quotient mapping, the operation induced by \( \pi \) makes of \( (S \times S)/R \) a semigroup. It is easy to prove that \( (S \times S)/R \) is a group, where the equivalence class \( \{\pi((x,x)) : x \in S\} \) is the neutral element, and the inverse of \( \pi((a,b)) \) is \( \pi((b,a)) \). Also, the function \( \iota: S \to (S \times S)/R \) defined by \( \iota(x) = \pi((xa,a)) \), for each \( x \in S \), is an algebraic monomorphism, where \( a \) is a fixed element of \( S \) (\( \iota \) does not depend of the choice of \( a \)). Note
Denote by \( \langle S \rangle \) is a semitopological group containing a semigroup, we will call \( \langle S \rangle \).

Proposition 3.4. Let \( S \) be a cancellative commutative semigroup. Since the reflections are unique up to isomorphims, \((S \times S)\) is a cancellative commutative semigroup with open shifts, then \( \langle S \rangle \) is a Hausdorff topological group and the quotient mapping \( \pi: S \times S \rightarrow \langle S \rangle \) is open. Furthermore, if \( S \) has continuous division, \( \iota: S \rightarrow \iota(S) \) is an homeomorphism and \( \iota(S) \) is open in \( \langle S \rangle \).

Proposition 3.3. Every open subsemigroup of a topological group has continuous division.

Proof. Let \( S \) be an open subsemigroup of a topological group \( G \). Let \( x, y \in S \), since \( G \) is a group, \( x^{-1}xy = y \). Let \( V \) be an open subset of \( S \) containing \( y \), then \( V \) is open in \( G \), the continuity of the operations of \( G \), implies that there are open subsets in \( G \), \( K \) and \( M \), containing \( x \) and \( xy \), respectively, such that \( K^{-1}M \subseteq V \). Let us put \( U = K \cap S \) and \( W = M \cap S \), then \( U \) and \( W \) are open subsets of \( S \) containing \( x \) and \( xy \), respectively. Now, if \( t \in W \) and \( u \in U \), then \( u^{-1}t \in K^{-1}M \subseteq V \), hence \( t \in uV \), therefore \( W \subseteq uV \), for every \( u \in U \). We have proved that \( W \subseteq \bigcap_{u \in U} uV \). For \( xy \) we proceed analogously.

So far we just have embedded, algebraically, semigroups into groups, the following proposition gives us a topological and algebraic embedding.

Proposition 3.4. Let \( S \) be a cancellative commutative semitopological semigroup with open shifts. There exists a topology \( \tau \) in \( \langle S \rangle \), such that \((\langle S \rangle, \tau)\) is a semitopological group containing \( S \) as an open semigroup. Moreover

i) \( S \) is 1-contable if and only if \((\langle S \rangle, \tau)\) is 1-contable.

ii) \((\langle S \rangle, \tau)\) is a partopological group if and only if \( S \) is a topological semigroup.
iii) If $S$ is $T_2$, $(\langle S \rangle, \tau)$ is $T_2$.

iv) If $S$ is Hausdorff and locally compact, $(\langle S \rangle, \tau)$ is a locally compact Hausdorff topological group.

Proof. Let $x$ be a fixed element in $S$, being $S$ a cancellative commutative semitopological semigroup with open shifts, and put $\mathcal{B} = \{x^{-1}V : V \in N_x^{(S)}\}$. We will prove that $\{gU : U \in \mathcal{B}, g \in \langle S \rangle\}$ is a base for a topology of semitopological group in $\langle S \rangle$, for it we will prove the conditions 1, 3 and 4 given in [12] Page 93. It is easy to prove the conditions 1 and 4, let us see that $t$ of semitopological group in $W$ by $B$ and the operation in $W$ that $(b, d)$ also, $(a, d)$

Let us suppose that $\{sU : U \in \mathcal{U}, s \in \langle S \rangle\}$ is a local base at $g$, for every $g \in \langle S \rangle$. Indeed let $U$ be an open set in $\langle S \rangle$ and $g \in U$, then there is $y \in \langle S \rangle$ such that $g \in y \cdot x^{-1}V \subseteq U$. There are $a, b, s, t \in S$ such that $y = ab^{-1}$ and $g = ts^{-1}$, therefore $ts^{-1} \in ab^{-1}x^{-1}V_x \subseteq U$, so that $tbx \in asV_x \subseteq bxSU$, hence there is $W_x$ open in $S$ satisfying $tbW_x \subseteq asV_x \subseteq bxSU$, therefore $g(x^{-1}W_x) \subseteq U$, this implies that $\{gx^{-1}V : V \in N_x^{S}\}$ is local base at $g$.

Let us see that $S$ is a subspace of $(\langle S \rangle, \tau)$. Indeed, let $U$ be an open set in $\langle S \rangle$ and let $s \in U$ be, then $xs \in xU$, there is $V \in N_x^{(S)}$ such that $sV \subseteq xU$, this is equivalent to saying that $s(x^{-1}V) \subseteq U$, so that $s \in (s(x^{-1}V)) \cap S \subseteq U$, therefore $U$ is open in the topology of subspace of $S$. Reciprocally, let $U$ be an open set in $(\langle S \rangle, \tau)$ and $s \in U \cap S$, we can find $W_x \in N_x^{S}$ and $U_s \in N_x^{S}$, such that $s \in sxW \subseteq U_s \subseteq xW \subseteq xU$, then $U_s \subseteq U$ and therefore $U \cap S$ is open in $S$. From the fact that $\{gx^{-1}V : V \in N_x^{S}\}$ is a local base at $g$, it follows that if $S$ is 1-contable (locally compact), then $(\langle S \rangle, \tau)$ is 1-contable (resp. locally compact) as well.

Let us suppose that $S$ is a topological semigroup and let us prove that $(\langle S \rangle, \tau)$ is a paratopological group, it can be concluded if we prove that the condition 2 of [12] Page 93 holds for $\mathcal{B}$. Indeed, let $V_x \in N_x^{(S)}$, since $x^2 \in xV_x$ and the operation in $S$ is jointly continuous, there exists $W_x \in N_x^{(S)}$ such that $(W_x)^2 \subseteq xV_x$, then $(x^{-1}W_x)^2 \subseteq V_x$ and condition 2 holds, this proves that $(\langle S \rangle, \tau)$ is a paratopological group. Let us suppose that $S$ is $T_2$ and let us see $(\langle S \rangle, \tau)$ is $T_2$, indeed let $y, z \in \langle S \rangle$, $z \neq y$, then there are $a, b, c, d \in S$ such that $z = ab^{-1}$ and $y = cd^{-1}$, so that $ad \neq bc$, by fact that $S$ is $T_2$, we can obtain $V_{ad} \in N_{ad}^{(S)}$ and $V_{bc} \in N_{bc}^{(S)}$. Note that $((bd)^{-1}V_{ad}) \cap ((bd)^{-1}V_{bc}) = \emptyset$, also, $(bd)^{-1}V_{ad} \in N_{y}^{(S)}$ and $(bd)^{-1}V_{bc} \in N_{y}^{(S)}$, that is to say $(\langle S \rangle, \tau)$ is $T_2$. Finally, if $S$ is locally compact and $T_2$, $(\langle S \rangle, \tau)$ is a semitopological
group locally compact and $T_2$, by Elii’s Theorem $(\langle S \rangle, \tau)$ is a topological group.

From the item ii) of the Proposition 3.4 and the Proposition 3.3 we have the following result.

**Corollary 3.5.** If $S$ is a cancellative commutative locally compact Hausdorff semitopological semigroup with open shifts, then $S$ has continuous division.

**Corollary 3.6.** Every cancellative commutative locally compact connected Hausdorff topological monoid with open shifts is a topological group.

**Proof.** Let $S$ be a cancellative commutative locally compact connected Hausdorff and let $\tau$ be the topology given in the Proposition 3.4. Let $U \in N_{e(S)}$, since $e_S = e_{\langle S \rangle}$ and $(\langle S \rangle, \tau)$ is a topological group, there is $V \in N_{e(S)}$ satisfying $V^{-1} = V$ and $V \subseteq U$, then $\bigcup_{n \in \mathbb{N}} V^n \subseteq \bigcup_{n \in \mathbb{N}} U^n$. But $\bigcup_{n \in \mathbb{N}} V^n$ is an open subgroup of $(\langle S \rangle, \tau)$ and therefore is closed in $S$, the connectedness of $S$ implies that $\bigcup_{n \in \mathbb{N}} V^n = S$ and $S$ is a topological group. ■

The following theorem tells us that every cancellative commutative locally compact Hausdorff topological semigroup with open shifts can be embedded as an open semigroup into the locally compact Hausdorff topological group, $\langle S \rangle^*$. 

**Theorem 3.7.** Let $S$ be a cancellative, commutative topological semigroup with open shifts. If $S$ is Hausdorff and locally compact, then so is $\langle S \rangle^*$. Moreover $\iota: S \rightarrow \iota(S)$ is an homeomorphism and $\iota(S)$ is open in $\langle S \rangle^*$.

**Proof.** Since $S$ is locally compact and Hausdorff topological semigroup, so is $S \times S$. By virtue the Proposition 3.2, $\pi: S \times S \rightarrow \langle S \rangle^*$ is open and $\langle S \rangle^*$ is Hausdorff, hence $\langle S \rangle^*$ is locally compact Hausdorff topological group. From Corollary 3.5 it follows that $S$ has continuous division, therefore the Proposition 3.2 guarantees that $\iota: S \rightarrow \iota(S)$ is a homeomorphism and $\iota(S)$ is open in $\langle S \rangle^*$. ■

It is well known that every pseudocompact Tychonoff topological group can be embedded as a subgroup dense into a compact topological group (see [8, Theorem 2.3.2]). The following theorem presents an analogue result in cancellative commutative topological semigroups, where also of the pseudocompactness, it is required the local compactness.

**Theorem 3.8.** If $S$ is a cancellative commutative locally compact pseudocompact Hausdorff topological semigroups with open shifts, then $S$ is an open dense subsemigroup of $\langle S \rangle^*$ and $\langle S \rangle^*$ is a compact topological group.

**Proof.** Since $S$ is a locally compact pseudocompact space, [4, Theorem 3.10.26] implies that $S \times S$ is pseudocompact. From the fact that $\langle S \rangle^*$ is a continuous image of $S \times S$, we have that $\langle S \rangle^*$ is pseudocompact, therefore the Cech-Stone compactification, $\beta \langle S \rangle^*$, is a topological group containing a $\langle S \rangle^*$ as dense subgroup. [4, Theorem 3.3.9] guarantees that $\langle S \rangle^*$ is an
open subgroup of $\beta S$, therefore it is also closed. By the density of $\langle S \rangle^*$, $\langle S \rangle^* = \beta(S)^*$, that is to say, $\langle S \rangle^*$ is a compact topological group. Since $\langle S \rangle^*$ is compact, $S$ is open in $\langle S \rangle^*$ and $\langle S \rangle^* = SS^{-1}$, there are $s_1, s_2, \ldots, s_n$ in $S$ such that $\langle S \rangle^* = \bigcup_{i=1}^{n} S s_i^{-1}$. $\text{Cl}_{\langle S \rangle^*}(S)$ is a compact Hausdorff cancellative semigroup, then by \cite[Theorem 2.5.2]{1}, $\text{Cl}_{\langle S \rangle^*}(S)$ is a topological group, therefore $s_i^{-1} \in \text{Cl}_{\langle S \rangle^*}(S)$ for every $i \in \{1, 2, 3, \ldots, n\}$, hence $(\text{Cl}_{\langle S \rangle^*}(S))s_i^{-1} = \text{Cl}_{\langle S \rangle^*}(S)$ for every $i \in \{1, 2, 3, \ldots, n\}$. Since each shift in $\langle S \rangle^*$ is a homeomorphism, we have that $\langle S \rangle^* = \text{Cl}_{\langle S \rangle^*}(\bigcup_{i=1}^{n} r_{s_i^{-1}}(S)) = \bigcup_{i=1}^{n} \text{Cl}_{\langle S \rangle^*}(r_{s_i^{-1}}(S)) = \bigcup_{i=1}^{n} r_{s_i^{-1}}(\text{Cl}_{\langle S \rangle^*}(S)) = \bigcup_{i=1}^{n} \text{Cl}_{\langle S \rangle^*}(S)s_i^{-1} = \bigcup_{i=1}^{n} \text{Cl}_{\langle S \rangle^*}(S) = \text{Cl}_{\langle S \rangle^*}(S)$, that is to say, $S$ is dense in $\langle S \rangle^*$.

We obtain the following corollary.

**Corollary 3.9.** The closure of any subsemigroup of a cancellative commutative locally compact pseudocompact Hausdorff topological semigroup with open shifts can be embedded as a dense open subsemigroup into a compact Hausdorff topological group.

**Proof.** Let $S$ be a cancellative commutative locally compact pseudocompact Hausdorff topological semigroup with open shifts and let $K$ be a subsemigroup of $S$. By Theorem 3.8, $S$ is an open subsemigroup of $\langle S \rangle^*$, since $\text{Cl}_S(K) = \text{Cl}_{\langle S \rangle^*}(K) \cap S$, we have that $\text{Cl}_S(K)$ is open in $\text{Cl}_{\langle S \rangle^*}(K)$. Now, $K$ is dense in $\text{Cl}_{\langle S \rangle^*}(K)$ and $K \subseteq \text{Cl}_S(K) \subseteq \text{Cl}_{\langle S \rangle^*}(K)$, this proves that $\text{Cl}_S(K)$ is dense in $\text{Cl}_{\langle S \rangle^*}(K)$, but $\text{Cl}_{\langle S \rangle^*}(K)$ is a compact Hausdorff cancellative topological semigroup, so that it is a topological group by \cite[Theorem 2.5.2]{1}, and so we have finished the proof.

**Proposition 3.10.** If $S$ is a cancellative commutative locally compact pseudocompact Hausdorff topological semigroup with open shifts, then $((\langle S \rangle^*, \cdot, \iota)$ coincides with the SAP-compactification of $S$.

**Proof.** Let $S$ be a cancellative commutative locally compact pseudocompact Hausdorff topological semigroup with open shifts and $f: S \rightarrow G$ a continuous homomorphism, being $G$ a Hausdorff compact topological group. Since $S$ is commutative, $f(S)$ is a commutative subsemigroup of $G$, so that $f(S)$ is a compact commutative cancellative topological semigroup, which is a topological group by \cite[Theorem 2.5.2]{1}. Let us define $f^*: (\langle S \rangle^*, \cdot) \rightarrow \overline{f(S)}$ by $f^*(xy^{-1}) = f(x)(f(y))^{-1}$, $f^*$ is a continuous homomorphism and moreover $f^* \circ \iota = f$, this ends the proof.

It is known that each pseudocompact Tychonoff paratopological group is a topological group (see \cite[Theorem 2.6]{13}). The following theorem gives us a similar result in cancellative commutative topological monoids with open shifts, but instead of group structure we have required the first axiom of countability.

**Theorem 3.11.** Let $S$ be a cancellative commutative feebly compact topological monoid with open shifts satisfying the first axiom of countability. Then
\(C_r(S)\) is a compact metrizable topological group. Moreover, the following statements hold:

i) If \(S\) is \(T_2\), \(S\) is a paratopological group.

ii) If \(S\) is regular, \(S\) is a compact metrizable topological group.

**Proof.** Let \(S\) be a commutative cancellative topological monoid with open shifts and put \(G = \langle S \rangle\). From Proposition 3.4 we have that there is a topology \(\tau\), such that \((G, \tau)\) is a paratopological group containing \(S\) as an open monoid. It follows from Proposition 2.1 that \(C_r(G)\) is a regular paratopological group containing \(C_r(S)\). So [11 Corollary 5] implies that \(C_r(G)\) is Tychonoff, therefore so is \(C_r(S)\). Since \(C_r(S)\) is feebly compact and Tychonoff, it is pseudocompact. Then \(C_r(S)\) is a pseudocompact subspace of the regular first-countable paratopological group \(C_r(G)\), following [9 Corollary 4.18], we have that \(C_r(S)\) is metrizable and compact. By Proposition 2.1 i), \(C_r(S)\) is cancelative, therefore [1 Theorem 2.5.2] implies that \(C_r(S)\) is a topological group. Now, if \(S\) is \(T_2\), \(C_r(S) = S_{sr}\), but \(S\) and \(S_{sr}\) coincide algebraically, thus \(S\) is a paratopological group. If \(S\) is regular, then \(S = C_r(S)\), this ends the proof. \(\blacksquare\)

By [8 Example 2.7.10], there is a feebly compact Hausdorff 2-countable paratopological group that fails to be a compact topological group, therefore the regularity in ii) Theorem 3.11 cannot be weakened to the Hausdorff separation property.

**Example 3.12.** Let \(\omega_1\) be the first non countable ordinal, the space \([0, \omega_1)\) of ordinal numbers strictly less than \(\omega_1\) with its order topology is regular first-countable feebly compact space, but \([0, \omega_1)\) is not compact. Then we can see the importance of algebraic structure in the Theorem 3.11.

### 4. CELLULARITY OF TOPOLOGICAL SEMIGROUPS

Finally, we present some results about the cellularity of topological semigroups.

**Theorem 4.1.** Let \(S\) be a cancellative, commutative Hausdorff locally compact \(\sigma\)-compact topological semigroup with open shifts. Then \(S\) has countable cellularity.

**Proof.** Since \(S \times S\) is locally compact and \(\sigma\)-compact, from Proposition 3.2 we have that \(\langle S \rangle^*\) is locally compact, Hausdorff, \(\sigma\)-compact topological group, which has countable cellularity following [17 Corollary 2.3]. Given that \(S\) is open in \(\langle S \rangle^*\), then \(c(S) = c(\langle S \rangle^*) = \aleph_0\). \(\blacksquare\)

In the next corollary we give an analogous result to that of the proposition 4.1, but without considering axioms of separation.

**Corollary 4.2.** Every \(\sigma\)-compact locally compact cancellative commutative topological monoid with open shifts has countable cellularity.
Proof. Let $S$ be a $\sigma$-compact locally compact cancellative commutative topological monoid with open shifts. By Proposition 2.1, $\varphi(S, C_2)$ is open, therefore $C_2(S)$ is locally compact Hausdorff topological semigroup with open shifts, this implies that $C_2(S)$ is regular, so that $C_2(S) = C_r(S)$. Since $C_r(S)$ is cancellative, we can apply the Theorem 4.1 and the Proposition 2.1 to conclude that $c(S) = c(C_r(S)) = \aleph_0$.

**Corollary 4.3.** Every subsemigroup of a commutative cancellative locally compact pseudocompact Hausdorff topological semigroup with open shifts has countable cellularity.

Proof. Let $S$ be a commutative cancellative locally compact pseudocompact Hausdorff topological semigroup with open shifts and let $K$ be a subsemigroup of $S$. By Corollary 3.9, there exists a compact Hausdorff topological group, $G$, containing $Cl_S(K)$ as an open semitopological group, therefore $c(K) \leq c(Cl_S(K)) \leq c(G) = \aleph_0$.

Since the compact topological groups has countable cellularity and $c(S) = c(C_r(S))$ for every topological monoid with open shifts, the Theorem 3.11 implies the following corollary.

**Corollary 4.4.** Let $S$ be a cancellative commutative feebly compact topological monoid with open shifts satisfying the first axiom of countability. Then $S$ has countable cellularity.

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