Curves containing all points of a finite projective Galois plane

Gregory Duran Cunha

Abstract

In the projective plane $PG(2, q)$ over a finite field of order $q$, a Tallini curve is a plane irreducible (algebraic) curve of (minimum) degree $q + 2$ containing all points of $PG(2, q)$. Such curves were investigated by G. Tallini [8, 9] in 1961, and by Homma and Kim [5] in 2013. Our results concern the automorphism groups, the Weierstrass semigroups, the Hasse-Witt invariants, and quotient curves of the Tallini curves.

Keywords: algebraic curve, finite field, automorphism, Weierstrass semigroup

1 Introduction

A fundamental result on plane irreducible (algebraic) curves defined over a finite field $\mathbb{F}_q$ is the Hasse-Weil bound

$$S_q \leq q + 1 + (n - 1)(n - 2)\sqrt{q}$$

where $n$ is the degree of the curve and $S_q$ is the number of its points lying in the projective plane $PG(2, q)$ of order $q$; see [4, Section 9.6]. Any plane (possibly reducible) curve containing all points of $PG(2, q)$ has degree at least $q + 1$, and if equality holds then the curve splits into the $q + 1$ lines of a pencil in $PG(2, q)$. A complete classification of plane irreducible curves of degree $q + 2$ containing all points of $PG(2, q)$ was given by G. Tallini [8, 9]; see also [1], and [5]. Up to projective transformations in $PG(2, q)$, each such curve $X$ has homogeneous equation of type

$$(aX_0 + bX_1 + cX_2)\varphi_{01} - X_0\varphi_{02} + X_2\varphi_{12} = 0,$$

(1)

where $\varphi_{ij} = X_i^qX_j - X_iX_j^q$ and $a, b, c$ are elements in $\mathbb{F}_q$ such that the cubic equation

$$X^3 - cX^2 - aX - b = 0$$

(2)

is irreducible over $\mathbb{F}_q$. In this paper, the above irreducible curve $X$ of degree $n = q + 2$ is named Tallini curve.

G. Tallini proved that $X$ has no singular points in $PG(2, q)$. Homma and Kim [5, Section 3] extended his result to any point in $PG(2, \mathbb{K})$ where $\mathbb{K}$ is the algebraic closure of $\mathbb{F}_q$. Therefore, $X$ is a plane nonsingular curve of genus $g = \frac{1}{2}(n - 1)(n - 2) = \frac{1}{2}q(q + 1)$.

G. Tallini showed that the automorphism group $G_q$ of $X$ over $\mathbb{F}_q$ contains a Singer cycle, that is, a cyclic subgroup $S$ of $PGL(3, q)$ of order $q^2 + q + 1$ acting on $PG(2, q)$ as a regular permutation group. He also claimed that $G_q$ may be a bit larger but only for some special curves, named harmonic and equianharmonic curves in [8, 9]. More precisely, Homma and Kim proved [5, Theorem 5.4] that if $G_q$ with $q > 2$ is larger than $S$ then $G_q$ is the normalizer of $S$ in $PGL(3, q)$, that is, $G_q = S \rtimes S_3$, the semidirect product of $S$ by a group $C_3$ of order 3.
In this paper we go on with the study of the Tallini curves, also from the function field point of view. We look at the Tallini curves in the projective plane $PG(2, \mathbb{K})$ defined over the algebraic closure $\mathbb{K}$ of $\mathbb{F}_q$. Our Theorem 2.2 shows that up to projective equivalence in $PG(2, \mathbb{K})$, the Tallini curve $\mathcal{X}$ is projectively equivalent to the curve

$$X_1^{q+1}X_2 + X_2^{q+1}X_0 + X_0^{q+1}X_1 = 0. \quad (3)$$

For $q = 2$, $\mathcal{X}_q$ is the famous plane Klein quartic whose automorphism group is isomorphic to $PSL(2, 7)$. We mention that the curve $\mathcal{X}_q$ was first investigated in [7], and we refer to it as the Pellikaan curve. From the proof of Theorem 2.2, the smallest projective plane $PG(2, q^{2^1})$ where this equivalency occurs is in general much larger than $PG(2, q^3)$ as $F_{q^{2^1}}$ turns out to be the smallest overfield of $F_{q^3}$ containing the roots of the equation $X^{q^2+q+1} = (\alpha^q - \alpha)^{q^2+q-2}$ where $\alpha$ is a root of [2]. Here $i$ divides $q^2 + q + 1$ and the automorphism group of $\mathcal{X}$ in $PG(2, \mathbb{K})$ is isomorphic to $S \times C_3$ where $S$ is defined over $F_q$ but $C_3$ is in general defined over $F_{q^{2^1}}$.

For every divisor $d$ of $q^2 + q + 1$, the curve $\mathcal{X}_q$ has a quotient curve $\mathcal{X}_q/C_d$ with respect to a cyclic group $C_d$ of order $d$. In case where $q$ is a square, that is, $q = p^{2i}$ with $p$ prime and $i \geq 1$, the factorization $p^{4i} + p^{2i} + 1 = (p^{2i} + p^i + 1)(p^{2i} - p^i + 1)$ raises the question whether the quotient curve $\mathcal{X}_q/C_d$ with $d = p^{2i} - p^i + 1$ is isomorphic to $\mathcal{X}_p$. The answer is affirmative, see Theorem 3.5.

We also show that $\mathcal{X}_p$ is an ordinary curve, that is, its genus $g = \frac{1}{2}p(p+1)$ coincides with its Hasse-Witt invariant. For this purpose, we prove that no exact differential of $K(\mathcal{X}_p)$ is regular, and then use the properties of the Cartier operator to show that $\mathcal{X}_p$ is ordinary. It should be noticed that this result does not hold true for $q > p$; see [6].

2 Irreducible curves of minimal degree containing all points of $PG(2, q)$

Let $\mathcal{X}$ be an irreducible plane curve of degree $q + 2$ defined over $F_q$ containing all points of $PG(2, q)$. In his paper [3], Tallini proved that such a curve exists and it has equation (1).

Now, we recall some facts from [3]. The polar net of $\mathcal{X}$ is a net of conics and it is easy to see that this net is homaloidal with the three base points, namely $A_0 = (\alpha_0 : 1 : \alpha_0^2)$, $A_1 = (\alpha_1 : 1 : \alpha_1^2)$ and $A_2 = (\alpha_2 : 1 : \alpha_2^2)$ where $\alpha_0, \alpha_1, \alpha_2$ are the three solutions of (2) in $F_q^3$. In particular, the points $A_i$ are conjugate over $F_q^3$. Furthermore, they belong to $\mathcal{X}$ and they are simple points for it. We call each of these three points a base point of $\mathcal{X}$.

Also in [3], Tallini showed that the automorphism group of $\mathcal{X}$ contains a Singer cycle, that is, a cyclic subgroup $S$ of $PGL(3, q)$ of order $q^3 + q + 1$ acting on $PG(2, q)$ as a regular permutation group. Moreover, $S$, viewed as a subgroup of $PGL(3, q^3)$, fixes $A_0, A_1, A_2$.

The following result was proved in [3].

**Theorem 2.1.** The curve $\mathcal{X}$ is non-singular.

**Proof.** If $P$ is a singular point of $\mathcal{X}$, then its orbit under $S = \langle \phi \rangle$ consists of singular points of $\mathcal{X}$. Note that the size of $O$ is 1 or $q^2 + q + 1$. Since the number of singular points of $\mathcal{X}$ is at most $q(q+1)/2$, then $O = \{ P \}$. This yields that every singular point of $\mathcal{X}$ is fixed by $S$. Since $S$ fixes only $A_0, A_1$ and $A_2$, which are simple points, it follows that $\mathcal{X}$ has no singular points. 


Theorem 2.2. Any Tallini curve is projectively equivalent to the Pellikaan curve over $\mathbb{F}_{q^{(q^2+q+1)}}$.

Proof. Let $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{F}_{q^3}$ be the distinct solutions of \(\alpha\), and let $\mathcal{Y}$ be the image of $\mathcal{X}$ under the linear map associated to the non-singular matrix

$$M = \begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 \\
1 & 1 & 1 \\
\alpha_0^2 & \alpha_1^2 & \alpha_2^2
\end{pmatrix}.$$

That is, $\mathcal{Y}$ is the curve given by $G(X_0, X_1, X_2) = 0$, where

$$G = F\left(\sum_{i=0}^{2 \alpha_i X_i}, \sum_{i=0}^{2 X_i}, \sum_{i=0}^{2 \alpha_i^2 X_i}\right),$$

and $\mathcal{X} = v(F)$ is the Tallini curve. A straightforward computation gives

$$G = c_{01}X_0^{q+1}X_1 + c_{02}X_0^{q+1}X_2 + c_{10}X_1^{q+1}X_0 + c_{12}X_1^{q+1}X_2 + c_{20}X_2^{q+1}X_0 + c_{21}X_2^{q+1}X_1,$$

where

$$c_{ij} = (\alpha_i - \alpha_j)^2(\alpha_i^2 - \alpha_i)(\alpha_j - \alpha_i^2), \text{ for } 0 \leq i, j \leq 2.$$

Note that $c_{ij} = 0$ whenever $\alpha_j = \alpha_i^q$. Since the Frobenius map acts transitively on $\{\alpha_0, \alpha_1, \alpha_2\}$, it follows that either $(\alpha_0, \alpha_1, \alpha_2) = (\alpha_1^q, \alpha_1^q, \alpha_1)$ or $(\alpha_0, \alpha_1, \alpha_2) = (\alpha_1^q, \alpha_2^q, \alpha_1^q)$. In the former case, we have $c_{10} = c_{02} = c_{21} = 0$ and then

$$G = c_{01}X_0^{q+1}X_1 + c_{12}X_1^{q+1}X_2 + c_{20}X_2^{q+1}X_0,$$

whereas the latter case gives $c_{01} = c_{12} = c_{20} = 0$ and

$$G = c_{02}X_0^{q+1}X_2 + c_{10}X_1^{q+1}X_0 + c_{21}X_2^{q+1}X_1.$$

We prove the result in the case $G$ is given by \(5\), and then case \(6\) will follow analogously. Note that from $(\alpha_0, \alpha_1, \alpha_2) = (\alpha_1^q, \alpha_2^q, \alpha_1)$, equation \(\text{[5]}\) can be written as

$$(\alpha_1 - \alpha_1)X_0^{q+1}X_2 + (\alpha_1^q - \alpha_1)\alpha_1^{q+1}X_0 + (\alpha_1 - \alpha_1^q)X_2^{q+1}X_1 = 0.$$ \(7\)

Finally, one can easily check that the curve given by \(7\) is the image of $X_q$ under the transformation

$$(X_0, X_1, X_2) \mapsto (\mu X_0, \lambda X_1, X_2)$$

where $\mu, \lambda \in \mathbb{F}_{q^{(q^2+q+1)}}$ satisfy

$$\lambda^{q^2+q+1} = \frac{(\alpha_1^q - \alpha_1)^3}{(\alpha_1^q - \alpha_1)^{q^2+q+1}} \text{ and } \mu = \frac{\lambda^{q+1}(\alpha_1^q - \alpha_1^q)}{\alpha_1 - \alpha_1^q}.$$  

This finishes the proof. $\Box$
3 Weierstrass semigroup at a base point

Let $\Sigma = \mathbb{K}(x, y)$, with $xy^{q+1} + x^{q+1} + y = 0$, be the function field of the Pellikaan curve $X_q$ and consider the fundamental triangle

$$O = (0 : 0 : 1), \quad X_\infty = (1 : 0 : 0), \quad Y_\infty = (0 : 1 : 0).$$

The tangent lines to $X_q$ at $O$, $X_\infty$ and $Y_\infty$ are $l_Y = v(Y)$, $l_Z = v(Z)$ and $l_X = v(X)$, respectively. Note that the points in $l_Y \cap X_q$ are $O$ and $X_\infty$ with

$$I(O, l_Y \cap X_q) = q + 1$$

and the points in the intersection $l_Z \cap X_q$ are $X_\infty$ and $Y_\infty$ with

$$I(X_\infty, l_Z \cap X_q) = q + 1$$

and the points in the intersection $l_X \cap X_q$ are $Y_\infty$ and $O$ with

$$I(Y_\infty, l_X \cap X_q) = q + 1$$

and $I(O, l_X \cap X_q) = 1$.

From [3, Theorem 6.42] the principal divisor of $x$ is given by

$$(x) = l_X \cdot X_q - l_Z \cdot X_q$$
$$= I(O, l_X \cap X_q)O + I(Y_\infty, l_X \cap X_q)Y_\infty - I(X_\infty, l_Z \cap X_q)X_\infty - I(Y_\infty, l_Z \cap X_q)Y_\infty$$
$$= (q + 1)O + (q + 1)X_\infty - (q + 1)Y_\infty$$

and the principal divisor of $y$ is given by

$$(y) = l_Y \cdot X_q - l_Z \cdot X_q$$
$$= I(O, l_Y \cap X_q)O + I(X_\infty, l_Y \cap X_q)X_\infty - I(X_\infty, l_Z \cap X_q)X_\infty - I(Y_\infty, l_Z \cap X_q)Y_\infty$$
$$= (q + 1)O + X_\infty - (q + 1)X_\infty - Y_\infty$$
$$= (q + 1)O - qX_\infty - Y_\infty.$$

For $1 \leq n \leq q + 1$, the above relations gives

$$\left(\frac{y}{x^n}\right) = (q + 1 - n)O + (n(q + 1) - q)X_\infty - (nq + 1)Y_\infty,$$

hence the divisor of poles of $y/x^n$ is

$$\left(\frac{y}{x^n}\right)_\infty = (nq + 1)Y_\infty, \quad \text{for } 1 \leq n \leq q + 1.$$

**Theorem 3.1.** The Weierstrass semigroup at a base point of $X_q$ is the semigroup generated by $q + 1, 2q + 1, \ldots, (q + 1)q + 1$.

**Proof.** We may assume that the base point is $Y_\infty$. From the above discussion, $q + 1, 2q + 1, \ldots, (q + 1)q + 1$ belong to the Weierstrass semigroup $H(Y_\infty)$. Let $H = (q + 1, 2q + 1, \ldots, (q + 1)q + 1)$ be the semigroup generated by $q + 1, 2q + 1, \ldots, (q + 1)q + 1$. Since $H \subseteq H(Y_\infty)$ and the number of gaps in $H$ is $q(q + 1)/2$, which is the number of gaps in $H(Y_\infty)$, the assertion follows. \qed
Theorem 3.2. If the function field of $X_q$ is given by $\mathbb{K}(x,y)$, with $xy^{q+1} + x^{q+1} + y = 0$, then the divisor of the differential $dx$ is

$$(dx) = (q^2 + 2q)Y_\infty - (q + 2)X_\infty.$$ 

Proof. The curve $X_q$ is constituted by two points in the infinity $X_\infty$, $Y_\infty$ and the affine points $P = (a : b : 1)$, with $ab^{q+1} + a^{q+1} + b = 0$. The tangent line to $X_q$ at an affine point $P = (a, b)$ is not vertical, in fact, suppose by contradiction that $X = 0$ is the tangent to $X_q = \mathbf{v}(F)$ at $P$. Then $\frac{\partial F}{\partial Y} = 0$ in the point $P = (a, b)$. It means that $ab^q + 1 = 0$ and hence $a \neq 0$. On the other hand, $ab^{q+1} + a^{q+1} + b = 0$ becomes $-b + a^{q+1} + b = 0$, and therefore $a = 0$, a contradiction. Thus, a primitive parametrization of $X_q$ at $P$ is given by

$$x(t) = a + t,$$
$$y(t) = b_0 + b_1t + \cdots, \quad b_0 = b.$$ 

Therefore, $\text{ord}_P dx = 0$, for all affine point $P$ in $X_q$. So it follows that $(dx) = nX_\infty + mY_\infty$ with $n + m = 2g - 2$, where $g = q(q + 1)/2$ is the genus of the curve $X_q$. Since $(x) = qY_\infty + O - (q + 1)X_\infty$ and $q + 1$ is not divisible by $p$, $\text{ord}_{X_\infty} dx = -(q + 2)$, that is, $n = -(q + 2)$. From $n + m = 2g - 2$ it follows that $m = q^2 + 2q$. \qed

4 The automorphism group

Again, let $\mathbb{K}(X_q) = \mathbb{K}(x,y)$, with $xy^{q+1} + x^{q+1} + y = 0$, be the function field of the Pellikaan curve $X_q$ and let $\lambda \in \mathbb{K}$ be a primitive $(q^2 + q + 1)$-root of unity and define the linear collineations

$$\sigma : (X_0, X_1, X_2) \mapsto (X_0, \lambda X_1, \lambda^{q+1} X_2).$$

and

$$\tau : (X_0, X_1, X_2) \mapsto (X_2, X_0, X_1).$$

It is straightforward to see that the automorphism group of $X_q$ in $PG(2, \mathbb{K})$ contains $S \rtimes C_3$ as a subgroup, where $S = \langle \sigma \rangle$ and $C_3 = \langle \tau \rangle$.

Observe that $S$ has order $q^2 + q + 1$. Therefore, going back to the original equation (1) of $X$, $S$ becomes a Singer cycle of $PG(2, q)$.

To show that $S \rtimes C_3$ is actually the whole automorphism group of $X_q$ over $\mathbb{K}$, we need some lemmas.

Lemma 4.1. If $Q$ is a point in $X_q$ and $t_Q$ is its tangent line, then

$$I(Q, X_q \cap t_Q) = \left\{ \begin{array}{ll} q + 1, & \text{if } Q \in \{X_\infty, Y_\infty, O\} \\ 2, & \text{if } Q \notin \{X_\infty, Y_\infty, O\}. \end{array} \right.$$ 

Proof. We have this result for the points in the fundamental triangle, see section 3. Suppose $Q = (a, b)$ with $ab^{q+1} + a^{q+1} + b = 0$ and $ab \neq 0$. The tangent line $t_Q$ to $X_q$ at $Q$ is given by

$$T(X, Y) = \frac{b}{a} X + \frac{a^{q+1}}{b} Y + ab^{q+1} = 0.$$ 

A primitive parametrization of $X_q$ at $Q$ is given by

$$x(t) = a + t,$$
$$y(t) = b - \frac{b^2}{a^{q+2}} t - \frac{b^{q+3}}{a^{2q+3}} t^2 - \frac{b^{2q+4}}{a^{3q+4}} t^3 - \cdots.$$ 

Therefore $T(x(t), y(t)) = -(b/a)^{q+2} t^2 + \cdots$ has order 2, that is, $I(Q, X_q \cap t_Q) = 2$. \qed
Lemma 4.2. Every automorphism in $\text{Aut}(\mathcal{X}_q)$ preserves the triangle $\{X_\infty, Y_\infty, O\}$.

Proof. Let $\alpha \in \text{Aut}(\mathcal{X}_q)$ and suppose $\alpha : P \mapsto Q$ with $P \in \{X_\infty, Y_\infty, O\}$ and $Q \notin \{X_\infty, Y_\infty, O\}$. Consider the lines $t_Q$ and $l_Q$ given by

$$
t_Q : T(X, Y) = 0
$$

$$
l_Q : L(X, Y) = 0
$$

such that $t_Q$ is the tangent line to $\mathcal{X}_q$ at $Q$ and $l_Q$ is a secant line through $Q$. Consider the curve $C$ of degree $q-1$ given by

$$
T(X, Y)L(X, Y)^{q-2} = 0.
$$

By the Lemma 4.1, $I(Q, \mathcal{X}_q \cap C) = I(Q, \mathcal{X}_q \cap t_Q) + (q-2)I(Q, \mathcal{X}_q \cap l_Q) = 2 + (q-2) = q$.

Observe that $W := \mathcal{X}_q \cdot C$ is a canonical divisor such that $L(W - qQ) \neq L(W - (q+1)Q)$. Thus, by Riemann-Roch Theorem, $\ell((q+1)Q) = \ell(qQ)$. Hence $q + 1$ is a gap number at $Q$, but this is a contradiction as $q+1$ is a non-gap at $P$.

Theorem 4.3. $\text{Aut}(\mathcal{X}_q) = S \rtimes C_3$.

Proof. Let $\alpha \in \text{Aut}(\mathcal{X}_q)$. Since $\mathcal{X}_q$ is non-singular, $\alpha$ can be represented as a matrix $A$ in $PGL(3, \mathbb{K})$. By the Lemma 4.2, $\alpha$ preserves the fundamental triangle. First, suppose that $\alpha$ fixes all vertices of the fundamental triangle, then

$$
A = \begin{pmatrix}
\xi & 0 & 0 \\
0 & \eta & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

Since $\alpha$ preserves $\mathcal{X}_q$,

$$
\xi \eta^{q+1}xy^{q+1} + \xi^{q+1}x^{q+1} + \eta y = 0
$$

in $\mathbb{K}(\mathcal{X}_q) = \mathbb{K}(x, y)$. Hence $\eta = \xi^{-1}$ and $\xi^{q+1} = 1$. Therefore $\alpha \in S$. Now, suppose that $\alpha$ fixes no vertices of the fundamental triangle. In that case, $\alpha = \tau$ or $\alpha = \tau^2$. To complete the proof we only need to show that the case when $\alpha$ fixes only one point in the fundamental triangle does not happen. Suppose that $\alpha$ only fixes the origin, thus $\alpha$ interchanges $X_\infty$ and $Y_\infty$. Hence,

$$
A = \begin{pmatrix}
0 & \xi & 0 \\
\eta & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

where $\xi, \eta \in \mathbb{K}$. Since $\alpha$ preserves $\mathcal{X}_q$,

$$
\xi \eta^{q+1}x^{q+1} + \xi^{q+1}y^{q+1} + \eta x = 0
$$

in $\mathbb{K}(x, y)$. It means that, there exists $c \neq 0$ in $\mathbb{K}$ such that

$$
\xi \eta^{q+1}X^{q+1} + \xi^{q+1}Y^{q+1} + \eta X = c(XY^{q+1} + X^{q+1} + Y),
$$

which is a contradiction. The cases when $\alpha$ fixes only $X_\infty$ or $Y_\infty$ are analogues. \qed

Theorems 4.3 and 2.2 have the following corollary.
Corollary 4.4. The automorphism group of $X$ is $S \times C_3$, where $S$ is defined over $\mathbb{F}_q$ but $C_3$ is defined over $\mathbb{F}_{q^i}$ with $i = 3(q^2 + q + 1)$.

Remark 4.5. By a result of Cossidente and Siciliano, see [2], and [4, Theorem 11.110], if a plane nonsingular curve $C$ of $PG(2, q)$ of degree $q + 2$ has an automorphism group $S \times C_3$ where $S$ is a Singer cycle of $PG(2, q)$ then $C$ is projectively equivalent to the Pellikaan curve $X_q$ where equivalency is meant in $PG(3, q^2)$. It should be noted that the authors in [2] claimed their result was valid under a weaker condition, namely when the automorphism group of $C$ contains $S$. But this turns out incorrect by Theorem 2.2.

5 Quotient curves

Suppose that $q$ is a square, say $q = p^{2i}$, $i \geq 1$. Thus $q^2 + q + 1 = (p^{2i} + p^i + 1)(p^{2i} - p^i + 1)$. Let $\lambda$ be a primitive $(q^2 + q + 1)$-root of unity in $K$. A straightforward computation shows that $\alpha$ defined by

$$\alpha(x) = \lambda x, \quad \alpha(y) = \lambda^{q+1}y,$$

is a $K$-automorphism of $K(X_q)$ of order $q^2 + q + 1$, where $K(X_q) = K(x, y)$ with $xy^{q+1} + x^{q+1} + y = 0$, is the function field of $X_q$.

Let $h = \alpha^{p^i+p^i+1}$. The group $H = \langle h \rangle$ has order $p^{2i} - p^i + 1$. The next results provide equations for the quotient curve $X_{q}/H$ and one of those equations turns out to be

$$XY^{p^i+1} + X^{p^i+1} + Y = 0,$$

which is indeed the equation of the Tallini curve $X_{p^i}$.

Proposition 5.1. The quotient curve $X_{q}/H$ is isomorphic to the curve given by the equation

$$X^{p^{2i}+p^i+1}Y^{q+1} + Y + 1 = 0.$$

Proof. Take $\xi, \eta$ from $K(x, y) = K(X_q)$ given by

$$\xi = x^{p^{2i}+p^i+1}, \quad \eta = x^{-(q+1)}y.$$ 

Clearly $h(\xi) = \xi$ and $h(\eta) = \eta$, then $K(\xi, \eta) \subset K(x, y)^H$. Note that $K(x, y) = K(\xi, \eta)(x)$ and $T^{p^{2i}+p^i+1} - \xi$ is a polynomial in $K(\xi, \eta)[T]$ which has $x$ as a root. Thus $[K(x, y) : K(\xi, \eta)] \leq p^{2i} - p^i + 1$. Note that $[K(x, y) : K(x, y)^H] = \text{ord}(H) = p^{2i} - p^i + 1$, hence $[K(x, y)^H : K(\xi, \eta)] = 1$, therefore $K(x, y)^H = K(\xi, \eta)$.

Finally, since $xy^{q+1} + x^{q+1} + y = 0$ and $\eta = x^{-(q+1)}y$ we get

$$x^{2q+2}y^{q+1} + x^{q+1} + x^{q+1}\eta = 0.$$ 

Thus, $x^{2q+1}y^{q+1} + 1 + \eta = 0$. Since $\xi = x^{p^{2i}+p^i+1}$ and $q^2 + q + 1 = (p^{2i} + p^i + 1)(p^{2i} - p^i + 1)$ we get

$$\xi^{p^{2i}+p^i+1}y^{q+1} + \eta + 1 = 0.$$ 

Proposition 5.2. The quotient curve $X_{q}/H$ is isomorphic to the curve given by the equation

$$X^{p^{2i}+p^i+1} + Y^{p^i+1} + Y^{p^i} = 0.$$ 

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Proof. By the previous proposition, $K(X_q/H) = K(x, y)$, with

$$x^{p^{2i}+p^i+1}y^{q+1} + y + 1 = 0,$$

that is,

$$x^{p^{2i}+p^i+1} + \frac{1}{y^{q+1}} + \frac{1}{y^{q+1}} = 0.$$

Putting $\xi = x$ and $\eta = 1/y$ gives

$$\xi^{p^{2i}+p^i+1} + \eta^{q+1} + \eta^{q+1} = 0.$$

Dividing by $\eta^{p^{2i}+p^i+1}$ and using $q = p^2$ gives

$$\frac{\xi^{p^{2i}+p^i+1}}{\eta^{p^{2i}+p^i+1}} + \frac{1}{\eta^{p^i+1}} + \frac{1}{\eta^{p^i}} = 0.$$

Replacing $u = \xi/\eta$ and $v = 1/\eta$ gives

$$u^{p^{2i}+p^i+1} + v^{p^i+1} + v^{p^i} = 0.$$

Theorem 5.3. The quotient curve $X_q/H$ is isomorphic to the curve $X_{p^i}$ given by the equation

$$XY^{p^i+1} + X^{p^i+1} + Y = 0.$$

Proof. Consider the function field $K(X_{p^i}) = K(x, y)$, with $xy^{p^i+1} + x^{p^i+1} + y = 0$. We have that

$$x(y^{p^i+1} + x^{p^i}) + y = 0.$$

Raising to the $p^i$-th power gives

$$x^{p^i}(y^{p^i+1} + x^{p^i})^{p^i} + y^{p^i} = 0.$$

Multiplying by $x$ gives

$$x^{p^i+1}(y^{p^i+1} + x^{p^i})^{p^i} + xy^{p^i} = 0.$$

This also can be written as,

$$x^{p^{2i}+p^i+1} + (-xy^{p^i} - 1)^{p^i}(-xy^{p^i}) = 0.$$

Putting $u = x$ and $v = -xy^{p^i} - 1$ gives

$$u^{p^{2i}+p^i+1} + v^{p^i+1} + v^{p^i} = 0.$$

Note that $K(u, v) = K(x, y^{p^i}) \subset K(x, y)$. Since $xy^{p^i+1} + x^{p^i+1} + y = 0$,

$$y = -\frac{x^{p^i+1}}{xy^{p^i} + 1},$$

that is, $y$ belongs to $K(x, y^{p^i})$. Therefore, $K(u, v) = K(x, y)$.
6 The Hasse-Witt invariant

In this section $q = p$ is a prime number. Let $\Sigma = \mathbb{K}(x, y)$, with $xy^{p+1} + x^{p+1} + y = 0$, be the function field of the Pellikaan curve $X_p$, $g$ its genus and $\gamma$ its Hasse-Witt invariant. The partial derivative of $F(X, Y) = XY^{p+1} + X^{p+1} + Y$ with respect to $Y$ is $F_Y(X, Y) = XY^p + 1$.

Consider $\Delta_\Sigma = \{u \, dx \mid u \in \Sigma\}$ the differential module of $\Sigma$ and $C : \Delta_\Sigma^{(1)} \to \Delta_\Sigma^{(1)}$ the Cartier operator defined on the space of holomorphic differentials

$$\Delta_\Sigma^{(1)} = \{w \in \Delta_\Sigma \mid (w) \geq 0\}.$$

**Theorem 6.1.** The Hasse-Witt invariant of $X_p$ is equal to its genus.

**Proof.** Let $w$ be an exact differential in $\Delta_\Sigma^{(1)}$, that is, $C(w) = 0$. Then $w$ can be written in the form

$$w = (u_1^p + u_2^px + \cdots + u_{p-2}^p x^{p-2})dx.$$

From $\mathfrak{B}$, a basis for the $\mathbb{K}$-vector space $\Delta_\Sigma^{(1)}$ is given by

$$\mathfrak{B} = \left\{ \frac{x^iy^j}{F_Y(x, y)} \, dx \mid 0 \leq i + j \leq p - 1 \right\}.$$

Thus

$$u_1^p + u_2^px + \cdots + u_{p-2}^p x^{p-2} = \frac{u(x, y)}{F_Y(x, y)}$$

where $u(X, Y)$ is a polynomial in $\mathbb{K}[X, Y]$ of degree at most $p - 1$. Let

$$u(x, y) = \sum_{i+j \leq p-1} a_{ij} x^i y^j.$$

Since $xy^{p+1} + x^{p+1} + y = 0$ then

$$\frac{1}{y} = -\frac{xy^p + 1}{x^{p+1}}.$$

Thus we have

$$\frac{u(x, y)}{F_Y(x, y)} = \frac{u(x, y)}{xy^p + 1}$$

$$= \frac{1}{xy^p + 1} \sum_{i+j \leq p-1} a_{ij} x^i y^j$$

$$= \frac{y}{xy^p + 1} \sum_{i+j \leq p-1} a_{ij} x^i \left(\frac{1}{y}\right)^{p-j}$$

$$= \frac{y}{xy^p + 1} \sum_{i+j \leq p-1} a_{ij} x^i \left(\frac{-xy^p + 1}{x^{p+1}}\right)^{p-j}$$

$$= \sum_{i+j \leq p-1} (-1)^{j+1} a_{ij} \frac{y^p}{x^{p+1} + p-j} x^{i+j} (xy^p + 1)^{p-1-j}$$

$$= \sum_{i+j \leq p-1} (-1)^{j+1} a_{ij} \frac{y^p}{x^{p+1} + p-j} x^{i+j} (xy^p + 1)^{p-1-j}.$$
\[
\begin{align*}
&= \sum_{i+j \leq p-1} (-1)^{j+1} a_{ij} \frac{y^p}{x^{p^2+p-2}} \sum_{k=0}^{p-1-j} \binom{p-1-j}{k} x^k y^{kp} \\
&= \sum_{i+j \leq p-1} \sum_{k=0}^{p-1-j} (-1)^{j+1} \binom{p-1-j}{k} a_{ij} (x^{jp+p^2-2} y^{(k+1)p}) x^{i+j+k} \\
&= \sum_{i+j \leq p-1} \sum_{k=0}^{p-1-j} w_{ijk} x^{i+j+k}
\end{align*}
\]

where
\[
w_{ijk} = \left( (-1)^{j+1} \binom{p-1-j}{k} a_{ij} \right)^{1/p} x^{i+j-1} y^{k+1}.
\]

Hence,
\[
u_1^p + u_2^p x + \cdots + u_{p-2}^p x^{p-2} = \sum_{i+j \leq p-1} \sum_{k=0}^{p-1-j} w_{ijk} x^{i+j+k}.
\]

The term of degree \(p-1\) in \(x\) on the right side is
\[
\left( \sum_{i+j \leq p-1} w_{ijk_{0}}^{p} \right) x^{p-1}
\]

where \(k_0 = p-1-j-i\). It means that
\[
\sum_{i+j \leq p-1} w_{ijk_0} = 0 \\
\sum_{i+j \leq p-1} w_{ijk_0} = 0 \\
\sum_{i+j \leq p-1} \left( (-1)^{j+1} \binom{p-1-j}{k_0} a_{ij} \right)^{1/p} x^{i+j-1} y^{k_0+1} = 0 \\
\sum_{i+j \leq p-1} \left( (-1)^{j+1} \binom{p-1-j}{p-1-j-i} a_{ij} \right)^{1/p} x^{i+j-1} y^{p-(i+j)} = 0 \\
\sum_{i+j \leq p-1} \left( (-1)^{j+1} \binom{p-1-j}{p-1-j-i} a_{ij} \right)^{1/p} x^{j} y^{p-(i+j)} = 0.
\]

Since the last equation has degree at most \(p\), it must be equal to zero. Thus all coefficients \(a_{ij}\) are equal to zero, and therefore \(u(x, y) = 0\). This shows that \(w = 0\), that is, the Cartier operator \(C : \Delta_{\Sigma}^{(1)} \to \Delta_{\Sigma}^{(1)}\) has trivial kernel.

Let \(V^0 \) be the space of all \(w \in \Delta_{\Sigma}^{(1)}\) such that \(C^i(w) = 0\) for some \(i \geq 1\). Note that if \(C^i(w) = 0\), then \(C^{i-1}(w) \in \ker(C)\). Hence \(V^0 = \{0\}\). This implies that the Hasse-Witt matrix \((h_{ij})\) over \(\mathbb{K}\) of \(C\) has maximum rank equal to \(g\), and consequently the matrix
\[
M = \left( h_{ij} \right) (h_{ij}^{p}) \cdots (h_{ij}^{p^{r-1}})
\]
has rank \(g\). Therefore, \(\gamma = g\).
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Authors’ addresses:

Gregory Duran Cunha
Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo
Avenida Trabalhador São-carlense, 400
13566-590 - São Carlos - SP (Brazil).
E-mail: gduran@usp.br