POLOIDS FROM THE POINTS OF VIEW OF PARTIAL
TRANSFORMATIONS AND CATEGORY THEORY

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ABSTRACT. Monoids and groupoids are examples of poloids. On the one hand, poloids can be regarded as one-sorted categories; on the other hand, poloids can be represented by partial magmas of partial transformations. In this article, poloids are considered from these two points of view.

1. Introduction

While category theory is, in a sense, a mathematical theory of mathematics, there does also exist a mathematical (algebraic) theory of (small) categories. The phrase “categories are just monoids” summarizes this theory in a somewhat cryptic manner. One part of this article is concerned with clarifying this statement, systematically developing definitions of category-related algebraic concepts such as semigroupoids, poloids and groupoids, and deriving results that we recognize from category theory. While no new results are presented, the underlying notion that (small) categories are “just webs of monoids” – or partial magmas generalizing monoids, semigroups, groups etc. – may deserve more systematic attention than it has received.

The other part of the article deals with the link between abstract algebraic structures such as poloids and concrete systems of partial transformations on some set. We obtain systems of partial transformations that satisfy the axioms of poloids as abstract algebraic structures by successively adding constraints on partial magmas of partial transformations; it is also shown that every poloid is isomorphic to such a system of partial transformations. This procedure provides an intuitive interpretation of the poloid axioms, helping to motivate the axioms and making it easier to discover important concepts and results. As is known, this approach has shown its usefulness in the study of semigroups, for example, in the work of Wagner [9].

At a late stage in the preparation of the manuscript, I became aware of related work on constellations [4, 5, 7]. Constellations turned out to generalize poloids in a way that I had not considered, yet had several points of contact with my concepts and results. I have added an Appendix where these matters are discussed.

2. Poloids and transformation magmas

2.1. (Pre)functions, (pre)transformations and magmas.

Definition 1. A (partial) prefunction, $f : X \not\to Y$ is a set $X \subseteq X$ and a rule $\tilde{f}$ that assigns exactly one $\tilde{f}(x) \in Y$ to each $x \in X$; to simplify the notation we may write $\tilde{f}(x)$ as $f(x)$. We call $X$ the domain of $f$, denoted $\text{dom}(f)$. The image of $f$, denoted $\text{im}(f)$, is the set $f(\text{dom}(f)) = \{f(x) \mid x \in X\}$. 
Definition 2. A (partial) function \( f : X \rightarrow Y \) is a prefunction \( f : X \not\Rightarrow Y \) and a set \( \mathcal{Y} \) such that \( \text{im}(f) \subseteq \mathcal{Y} \subseteq Y \). The domain of \( f \), denoted \( \text{dom}(f) \), is the domain of \( f \), and \( \mathcal{Y} \) is called the codomain of \( f \), denoted \( \text{cod}(f) \). The image of \( f \), denoted \( \text{im}(f) \), is defined to be the image of \( f \).

Although this terminology will not be used below, \( X \) may be called the total domain for \( f : X \not\Rightarrow Y \) or \( f : X \rightarrow Y \), and \( Y \) may be called the total codomain for \( f : X \not\Rightarrow Y \) or \( f : X \rightarrow Y \).

A total prefunction \( f : X \Rightarrow Y \) is a prefunction such that \( \text{dom}(f) = X \). A non-empty prefunction \( f \) is a prefunction such that \( \text{dom}(f) = \emptyset \). The restriction of \( f : X \not\Rightarrow Y \) to \( X' \subset X \) is the prefunction \( f|_{X'} : X' \not\Rightarrow Y \) such that \( \text{dom}(f|_{X'}) = \text{dom}(f) \cap X' \) and \( f|_{X'}(x) = f(x) \) for all \( x \in \text{dom}(f|_{X'}) \). A pretransformation on \( X \) is a prefunction \( f : X \not\Rightarrow X \); a total pretransformation on \( X \) is a total prefunction \( f : X \Rightarrow X \). An identity pretransformation \( \text{Id}_{S} \) is a pretransformation on \( X \supseteq S \) such that \( \text{dom}((\text{Id}_{S})) = S \) and \( \text{Id}_{S}(x) = x \) for all \( x \in \text{dom}(\text{Id}_{S}) \).

Similarly, a total function \( f : X \rightarrow Y \) is a function such that \( \text{dom}(f) = X \) and \( \text{cod}(f) = Y \). A non-empty function \( f \) is a function such that \( \text{dom}(f) \neq \emptyset \). The restriction of \( f : X \rightarrow Y \) to \( X' \subset X \) is the function \( f|_{X'} : X' \rightarrow Y \) such that \( \text{dom}(f|_{X'}) = \text{dom}(f) \cap X' \), \( \text{cod}(f|_{X'}) = \text{cod}(f) \) and \( f|_{X'}(x) = f(x) \) for all \( x \in \text{dom}(f|_{X'}) \). A transformation on \( X \) is a function \( f : X \Rightarrow X \); a total transformation on \( X \) is a total function \( f : X \rightarrow X \). An identity transformation \( \text{Id}_{S} \) is a transformation on \( X \supseteq S \) such that \( \text{dom}((\text{Id}_{S})) = \text{cod}((\text{Id}_{S})) = S \) and \( \text{Id}_{S}(x) = x \) for all \( x \in \text{dom}(\text{Id}_{S}) \).

Given a pretransformation \( f \) on \( X \), \( f(x) \) denotes some \( x \in X \) if and only if \( x \in \text{dom}(f) \); \( f(f(x)) \) denotes some \( x \in X \) if and only if \( x, f(x) \in \text{dom}(f) \); etc. We describe such situations by saying that \( f(x) \), \( f(f(x)) \), etc. are defined. Similarly, given a transformation \( f \) on \( X \), \( f(x) \), \( f(f(x)) \), etc. are said to be defined if the corresponding pretransformations \( f(x) \), \( f(f(x)) \), etc. are defined.

Definition 3. A (partial) binary operation on a set \( X \) is a non-empty prefunction

\[ \pi : X \times X \not\Rightarrow X, \quad (x, y) \mapsto xy. \]

A total binary operation on \( X \) is a total prefunction \( \pi : X \times X \Rightarrow X \). A (partial) magma \( P \) is a non-empty set \( \mid P \mid \) equipped with a binary operation on \( \mid P \mid \); a total magma \( P \) is a non-empty set \( \mid P \mid \) equipped with a total binary operation on \( \mid P \mid \). A submagma \( P' \) of a magma \( P \) is a set \( \mid P' \mid \subseteq \mid P \mid \) such that if \( x, y \in \mid P' \mid \) then \( xy \in \mid P' \mid \), with the restriction of \( \pi \) to \( \mid P' \mid \times \mid P' \mid \) as a binary operation. (By an abuse of notation, \( P \) will also denote the set \( \mid P \mid \) henceforth.)

The notion of being defined for expressions involving a pretransformation can be extended in a natural way to expressions involving a binary operation. We say that \( xy \) is defined if and only if \( (x, y) \in \text{dom}(\pi) \); that \( (xy)z \) is defined if and only if \( (x, y), (xy, z) \in \text{dom}(\pi) \); that \( z(xy) \) is defined if and only if \( (x, y), (z, xy) \in \text{dom}(\pi) \); and so on. Thus, if \( (xy)z \) or \( z(xy) \) is defined then \( xy \) is defined.

Remark 1. To avoid tedious repetition of the word “partial”, we speak about (pre)functions and magmas as opposed to total (pre)functions and total magmas rather than partial (pre)functions and partial magmas as opposed to (pre)functions and magmas. Note that a binary operation \( \pi : P \times P \not\Rightarrow P \) can always be regarded as a total binary operation \( \pi^0 : P^0 \times P^0 \Rightarrow P^0 \), where \( P^0 = P \cup \{0\} \) and \( 0x = x0 = 0 \) for each \( x \in P \), considering \( xy \) to be defined if and only if \( xy \neq 0 \). If we let \( P^0 \) represent \( P \) in this way, it becomes a theorem that if \( (xy)z \) or \( z(xy) \) is defined then \( xy \) is defined.
2.2. Semigroupoids, poloids and groupoids. We say that \( x \) precedes \( y \), denoted \( x \prec y \), if and only if \( xy \) or \( yz \) or \( (zx)y \) is defined, and we write \( x \prec y \prec z \) if and only if \( x \prec y \) and \( y \prec z \), meaning that \( xy \) and \( yz \) are defined or \( x(yz) \) is defined or \( (xy)z \) is defined.

**Definition 4.** A semigroupoid is a magma \( P \) such that for any \( x \prec y \prec z \in P \), \( (xy)z \) and \( x(yz) \) are defined and \( (xy)z = x(yz) \).

A unit in a magma \( P \) is any \( e \in P \) such that \( ex = x \) for all \( x \) such that \( ex \) is defined and \( xe = x \) for all \( x \) such that \( xe \) is defined.

**Definition 5.** A poloid is a semigroupoid \( P \) such that for any \( x \in P \) there are units \( \epsilon_x, \epsilon_x \in P \) such that \( \epsilon_x, x \) and \( x \epsilon_x \) are defined.

For any \( x \in P \), we have \( \epsilon_x x = x = x \epsilon_x \), meaning that \( \epsilon_x, \epsilon_x \) are units; we may call \( \epsilon_x \) an effective left unit for \( x \) and \( \epsilon_x \) an effective right unit for \( x \).

**Definition 6.** A groupoid is a poloid \( P \) such that for every \( x \in P \) there is a unique \( x^{-1} \in P \) such that \( xx^{-1} \) and \( x^{-1}x \) are defined and units.

**Remark 2.** Recall that groups, monoids and semigroups are total magmas with additional properties. Each kind of total magma can be generalized to a (partial) magma with similar properties, sometimes named by adding the ending “-oid”, as in group/groupoid and semigroup/semigroupoid, so that the process of generalizing to a not necessarily total magma has become known as “oidification”. (See the table below.) However, the terminology is not consistent – for example, a monoid is not a (partial) magma. I prefer “poloid” to the rather clumsy and confusing term “monoidoid”, which suggests some kind of “double oidification”. An important concept should have a short name, and the idea behind the current terminology is that a monoid has a single unit, whereas a poloid may have more than one unit.

| total magma (magma, groupoid) | magma (partial magma, halfgroupoid) |
|-------------------------------|--------------------------------------|
| semigroup                     | semigroupoid                         |
| monoid                        | poloid (monoidoid)                   |
| group                         | groupoid                              |

It should be kept in mind that semigroups, monoids and groups can be generalized to other (partial) magmas than semigroupoids, poloids and groupoids, respectively. For example, if we do not require that if \( x \prec y \prec z \) then \( x(yz) \) and \( (xy)z \) are defined and equal but only that if \( x(yz) \) or \( (xy)z \) is defined then \( x(yz) \) and \( (xy)z \) are defined and equal, we obtain a semigroup generalized to a certain (partial) magma but this is not a semigroupoid as defined here. The specific definitions given in this section are suggested by category theory.

2.3. (Pre)Transformation magmas. Recall that the full transformation monoid \( \mathcal{T}_X \) on a non-empty set \( X \) is the set \( \mathcal{T}_X \) of all total functions \( f : X \to X \), equipped with the total binary operation

\[ \circ : \mathcal{T}_X \times \mathcal{T}_X \to \mathcal{T}_X, \quad (f, g) \mapsto f \circ g, \]

where \( (f \circ g)(x) = f(g(x)) \) for all \( x \in X \). More generally, a transformation semigroup \( \mathcal{F}_X \) is a set of total functions \( f : X \to X \) with \( \circ \) as binary operation and such that \( f, g \in \mathcal{F}_X \) implies \( f \circ g \in \mathcal{F}_X \), and a transformation monoid \( \mathcal{M}_X \) is a transformation semigroup such that \( \text{Id}_X \in \mathcal{M}_X \).
Example 1. Set \( X = \{1, 2\} \), let \( e : X \to X \) be defined by \( e(1) = e(2) = 1 \) and let \( M_X \) be the magma with \( \{e\} \) as underlying set and function composition \( \circ \) as binary operation. Then \( M_X \) is a (trivial) monoid of transformations, but it is not a transformation monoid.

When we generalize from total functions \( X \to X \) to functions \( X \to X \) or pre-functions \( X \Rightarrow X \), \( \mathcal{F}_X \) is generalized from a transformation semigroup to a transformation magma \( \mathcal{F}_X \) or a pretransformation magma \( \mathcal{R}_X \).

Definition 7. Let \( X \) be a non-empty set. A pretransformation magma \( \mathcal{R}_X \) on \( X \) is a set \( \mathcal{R}_X \) of non-empty pretransformations \( f : X \Rightarrow X \), equipped with the binary operation
\[
\circ : \mathcal{R}_X \times \mathcal{R}_X \Rightarrow \mathcal{R}_X, \quad (f, g) \mapsto f \circ g,
\]
where \( \text{dom}(\circ) = \{(f, g) \mid \text{dom}(f) \supseteq \text{im}(g)\} \) and \( f \circ g \) if defined is given by \( \text{dom}(f \circ g) = \text{dom}(g) \) and \( (f \circ g)(x) = f(g(x)) \) for all \( x \in \text{dom}(f \circ g) \).

The full pretransformation magma on \( X \), denoted \( \mathcal{R}_X \), is the pretransformation magma whose underlying set is the set of all non-empty pretransformations of the form \( f : X \Rightarrow X \).

Definition 8. Let \( X \) be a non-empty set. A transformation magma \( \mathcal{F}_X \) on \( X \) is a set \( \mathcal{F}_X \) of non-empty transformations \( f : X \to X \), equipped with the binary operation
\[
\circ : \mathcal{F}_X \times \mathcal{F}_X \Rightarrow \mathcal{F}_X, \quad (f, g) \mapsto f \circ g,
\]
where \( \text{dom}(\circ) = \{(f, g) \mid \text{dom}(f) \supseteq \text{im}(g)\} \) and \( f \circ g \) if defined is given by \( \text{dom}(f \circ g) = \text{dom}(g) \) and \( (f \circ g)(x) = f(g(x)) \) for all \( x \in \text{dom}(f \circ g) \).

The full transformation magma on \( X \), denoted \( \mathcal{F}_X \), is the transformation magma whose underlying set is the set of all non-empty transformations \( f : X \to X \).

A (pre)transformation magma is clearly a magma as described in Definition 3.

The plan in this section, derived from the view that categories are “webs of monoids”, is to construct transformation magmas that relate to poloids in the same way that transformation monoids relate to monoids. As a monoid is an associative magma with a unit, we look for appropriate generalizations of these two notions.

Fact 1. Let \( f, g, h \) be elements of a pretransformation magma. If \( (f \circ g) \circ h \) and \( f \circ (g \circ h) \) are defined then \( (f \circ g) \circ h = f \circ (g \circ h) \).

Proof. We have
\[
\text{dom}((f \circ g) \circ h) = \text{dom}(h) = \text{dom}(g \circ h) = \text{dom}(f \circ (g \circ h)),
\]
and
\[
((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = f(g(h(x))) = f((g \circ h)(x)) = (f \circ (g \circ h))(x)
\]
for all \( x \in \text{dom}(f \circ g) \circ h = \text{dom}(f \circ (g \circ h)) \). \( \square \)

Lemma 1. Let \( f, g \) be elements of a pretransformation magma. If \( f \circ g \) is defined then \( \text{im}(f) \supseteq \text{im}(g) \).

Proof. Since \( \text{dom}(f) \supseteq \text{im}(g) \) by definition, we have \( \text{im}(f) = f(\text{dom}(f)) \supseteq f(\text{im}(g)) = f(g(\text{dom}(g))) = (f \circ g)(\text{dom}(f \circ g)) = \text{im}(f \circ g) \). \( \square \)

Fact 2. Let \( f, g, h \) be elements of a pretransformation magma. If \( f \circ g \) and \( g \circ h \) are defined then \( f \circ (g \circ h) \) and \( f \circ (g \circ h) \) are defined.
Proof. We have $\text{dom}(f \circ g) = \text{dom}(g) \supseteq \text{im}(h)$, so $(f \circ g) \circ h$ is defined. Also, $\text{dom}(f) \supseteq \text{im}(g)$ and by Lemma 1 $\text{im}(g) \supseteq \text{im}(g \circ h)$, so $f \circ (g \circ h)$ is defined. □

**Fact 3.** Let $f, g, h$ be elements of a pretransformation magma. If $(f \circ g) \circ h$ is defined then $f \circ (g \circ h)$ is defined.

Proof. If $(f \circ g) \circ h$ is defined so that $f \circ g$ is defined then $\text{dom}(g) = \text{dom}(f \circ g) \supseteq \text{im}(h)$. Thus, $g \circ h$ is defined so Fact 2 implies that $f \circ (g \circ h)$ is defined. □

The implication in the opposite direction does not hold.

**Example 2.** Let $f, g, h$ be pretransformations on $\{1, 2\}$; specifically, $f = h = \text{ld}_{\{1\}}$ and $g = \text{ld}_{\{1,2\}}$. Then, $\text{dom}(g) \supseteq \text{im}(h)$ and $\text{im}(g \circ h) = \{1\}$, so $\text{dom}(f) \supseteq \text{im}(g \circ h)$. Hence, $f \circ (g \circ h)$ is defined, but we do not have $\text{dom}(f) \supseteq \text{im}(g)$, so $f \circ g$ is not defined and hence $(f \circ g) \circ h$ is not defined.

So, somewhat surprisingly, pretransformation magmas do not have a two-sided notion of associativeness. We need the notion of a transformation magma and an additional assumption to derive the complement of Fact 3.

**Definition 9.** A transformation semigroupoid $\mathcal{S}_X$ on $X$ is a transformation magma $\mathcal{S}_X$ such that if $\text{dom}(f) \supseteq \text{im}(g)$ for some $f, g \in \mathcal{S}_X$ then $\text{dom}(f) = \text{cod}(g)$.

Of course, if $\text{dom}(f) = \text{cod}(g)$ then $\text{dom}(f) \supseteq \text{im}(g)$. Thus, in a transformation semigroupoid $f \circ g$ is defined if and only if $\text{dom}(f) = \text{cod}(g)$.

If $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are defined then $\text{cod}((f \circ g) \circ h) = \text{cod}(f \circ g) = \text{cod}(f \circ (g \circ h))$, so Fact 1 holds for transformation magmas as well. It is also clear that the proofs of Facts 2 and 3 apply to transformation magmas as well. Thus, we can use Facts 1–3 also when dealing with transformation magmas. On the other hand, Example 2 applies to transformation magmas as well, but not to transformation semigroupoids.

**Fact 4.** Let $f, g, h$ be elements of a transformation semigroupoid. If $f \circ (g \circ h)$ is defined then $(f \circ g) \circ h$ is defined.

Proof. If $f \circ (g \circ h)$ is defined then $\text{dom}(f) = \text{cod}(g \circ h) = \text{cod}(g)$. Thus, $f \circ g$ is defined, and as $g \circ h$ is defined as well Fact 2 implies that $(f \circ g) \circ h$ is defined. □

**Theorem 1.** A transformation semigroupoid is a semigroupoid.

Proof. By Facts 2, 3 and 4, if $f \prec g \prec h$ then $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are defined, and by Fact 1 this implies that $(f \circ g) \circ h = f \circ (g \circ h)$. □

Poloids are semigroupoids with effective left and right units. Such units can be added to transformation semigroupoids in a quite natural way.

**Definition 10.** A transformation poloid $\mathcal{P}_X$ is a transformation semigroupoid $\mathcal{S}_X$ such that if $f \in \mathcal{S}_X$ then $\text{ld}_{\text{dom}(f)}, \text{ld}_{\text{cod}(f)} \in \mathcal{S}_X$.

**Fact 5.** Let $\mathcal{P}_X$ be a transformation poloid. For any $f \in \mathcal{P}_X$, $\text{ld}_{\text{dom}(f)}$ and $\text{ld}_{\text{cod}(f)}$ are units.

Proof. If $f, g \in \mathcal{P}_X$ and $\text{ld}_{\text{dom}(f)} \circ g$ is defined then

$$\text{dom}(\text{ld}_{\text{dom}(f)} \circ g) = \text{dom}(g),$$

$$\text{cod}(\text{ld}_{\text{dom}(f)} \circ g) = \text{cod}(\text{ld}_{\text{dom}(f)}) = \text{dom}(\text{ld}_{\text{dom}(f)}) = \text{cod}(g),$$
and $\text{Id}_{\text{dom}(f)}(g(x)) = g(x)$ for all $x \in \text{dom}(g)$. Hence, $\text{Id}_{\text{dom}(f)} \circ g = g$.

Also, if $f, h \in \mathcal{P}_X$ and $h \circ \text{Id}_{\text{dom}(f)}$ is defined then

\[
\text{dom}(h) = \text{cod}(\text{Id}_{\text{dom}(f)}) = \text{dom}(\text{Id}_{\text{dom}(f)}) = \text{dom}(h \circ \text{Id}_{\text{dom}(f)}),
\]

\[
\text{cod}(h) = \text{cod}(h \circ \text{Id}_{\text{dom}(f)}),
\]

and $h(\text{Id}_{\text{dom}(f)}(x)) = h(x)$ for all $x \in \text{dom}(\text{Id}_{\text{dom}(f)}) = \text{dom}(h)$, so $h \circ \text{Id}_{\text{dom}(f)} = h$.

We have thus shown that $\text{Id}_{\text{dom}(f)}$ is a unit.

It is shown similarly that if $\text{Id}_{\text{cod}(f)} \circ g$ is defined then $\text{Id}_{\text{cod}(f)} \circ g = g$, and if $h \circ \text{Id}_{\text{cod}(f)}$ is defined then $h \circ \text{Id}_{\text{cod}(f)} = h$, so $\text{Id}_{\text{cod}(f)}$ is a unit as well.

Fact 6. Let $\mathcal{P}_X$ be a transformation poloid. For any $f \in \mathcal{P}_X$, $f \circ \text{Id}_{\text{dom}(f)}$ and $\text{Id}_{\text{cod}(f)} \circ f$ are defined.

Proof. We have $\text{dom}(f) = \text{dom}(\text{Id}_{\text{dom}(f)}) = \text{cod}(\text{Id}_{\text{dom}(f)})$ and $\text{dom}(\text{Id}_{\text{cod}(f)}) = \text{cod}(f)$. □

Theorem 2. A transformation poloid is a poloid.

Proof. Immediate from Facts 5 and 6. □

Remark 3. We have considered two requirements for $f \circ g$ or $f \circ g$ being defined, namely that $\text{dom}(f) \supseteq \text{im}(g)$ or that $\text{dom}(f) = \text{cod}(g)$. Other definitions are common in the literature. Instead of requiring that $\text{dom}(f) \supseteq \text{im}(g)$, it is often required that $\text{dom}(f) \cap \text{im}(g) \neq \emptyset$, and instead of requiring that $\text{dom}(f) = \text{cod}(g)$, it is sometimes required that $\text{dom}(f) = \text{im}(g)$.

Of these alternative definitions, the first one tends to be too weak for present purposes, while the second one tends to be too restrictive.

For example, if we stipulate that $f \circ g$ is defined if and only if $\text{dom}(f) \cap \text{im}(g) \neq \emptyset$ then $f \circ g$ and $g \circ h$ being defined does not imply that $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are defined, contrary to Fact 2.

Also, if we stipulate that $f \circ g$ is defined if and only if $\text{dom}(f) = \text{im}(g)$ and let $f, g$ be total transformations on $X$, then $f \circ g$ is defined only if $g$ is surjective so that $\text{im}(g) = X$. Thus, $\mathcal{F}_X = \{ f \mid f : X \to X \}$ is not a monoid under this function composition. As monoids are poloids, this anomaly suggests that the condition $\text{dom}(f) = \text{im}(g)$ for $f \circ g$ to be defined is not appropriate in the context of poloids.

On the other hand, stipulating that $f \circ g$ is defined if and only if $\text{dom}(f) \supseteq \text{im}(g)$ does not give a fully associative binary operation (Example 2). This is a fatal flaw for many purposes, including representing poloids as magmas of transformations.

We note that the exact formalization of the notion of “partial function” is important. A “partial function” $f$ is often defined as being equipped only with a domain and an image (range), and then there are only three reasonable ways of composing “partial transformations”: $f \circ g$ is defined if and only if $\text{dom}(f) \cap \text{im}(g) \neq \emptyset$ or $\text{dom}(f) \supseteq \text{im}(g)$ or $\text{dom}(f) = \text{im}(g)$. But according to Definition 2, “partial functions” have codomains of their own, so we can stipulate that $f \circ g$ is defined if and only if $\text{dom}(f) = \text{cod}(g)$, and this turns out to be just what we need when specializing magmas of “partial transformations” to semigroupoids and poloids.

3. Poloids and categories

3.1. Elementary properties of abstract poloids. Recall that a poloid $P$ is a magma satisfying the following conditions:

(P1). For any $x \prec y \prec z \in P$, $(xy)z$ and $x(yz)$ are defined and $(xy)z = x(yz)$.

(P2). For any $x \in P$ there are units $\varepsilon_x, \varepsilon_x \in P$ such that $\varepsilon_x x$ and $x \varepsilon_x$ are defined.

Let us derive some elementary properties of poloids as abstract algebraic structures.
Proposition 1. Let $P$ be a poloid and $e \in P$ a unit. Then $ee$ is defined and $ee = e$.

Proof. Let $e \in P$ be an effective left unit for the unit $e$. Then, $ee$ is defined and $e = ee = e$, implying the assertion. □

By Proposition 1 every unit is an effective left and right unit for itself.

Proposition 2. Let $P$ be a poloid. If $e_x$ and $e'_x$ are effective left units for $x \in P$ then $e_x = e'_x$, and if $e_x$ and $e'_x$ are effective right units for $x \in P$ then $e_x = e'_x$.

Proof. By assumption, $e_x x$ and $e'_x x$ are defined and equal to $x$, so $e_x (e'_x x)$ is defined. Thus, $(e_x e'_x) x$ is defined, so $e_x e'_x$ is defined. As $e_x$ and $e'_x$ are units, this implies $e_x = e_x e_x = e'_x$. The uniqueness of the effective right unit for $x$ is proved in the same way. □

Note that if $xy$ is defined then $(e_x x) y$ is defined so $e_x (xy) = (e_x x) y = xy = e_y (xy)$, so by Proposition 2 we have $e_x = e_y$. A similar argument shows that $e_y = e_x$.

Also note that in a groupoid, where $xx^{-1}$ and $x^{-1}x$ are defined and units, we have $x (x^{-1} x) = x$, $x^{-1} (xx^{-1}) = x^{-1}$, $(xx^{-1}) x = x$, and $(x^{-1} x) x^{-1} = x^{-1}$, where the four left-hand sides are defined. Thus, by Proposition 2 we have $xx^{-1} = e_x = e_{x^{-1}}$, and $x^{-1} x = e_x = e_{x^{-1}}$.

Proposition 3. Every poloid $P$ can be equipped with surjective functions

$$s : P \rightarrow E, \quad x \mapsto e_x,$$

$$t : P \rightarrow E, \quad x \mapsto e_x,$$

where $E$ is the set of all units in $P$ and $s(e) = t(e) = e$ for all $e \in E$.

Proof. Immediate from (P2), Proposition 1 and Proposition 2. □

Proposition 4. Let $P$ be a poloid. For any $x, y \in P$, $xy$ is defined if and only if $e_x = e_y$.

Proof. If $xy$ is defined then $(xe_x) y$ is defined, so $e_x y$ is defined and as $e_x$ is a unit we have $e_x y = y = e_y y$, so $e_x = e_y$ by Proposition 2. Conversely, if $e_x = e_y$ then $e_x y$ is defined, and as $xe_x$ is defined, $(xe_x) y = xy$ is defined. □

A total poloid is a poloid $P$ whose binary operation $\pi$ is a total function.

Proposition 5. A total poloid has only one unit.

Proof. For any pair $e, e' \in P$ of units, $ee'$ is defined so $e = ee' = e'$. □

Proposition 6. A poloid with only one unit is a monoid.

Proof. Let $P$ be a poloid. By assumption, there is a unique unit $e \in P$ such that $e = e_x = e_y$ for any $x, y \in P$. Therefore, it follows from Proposition 4 that $xy$ and $yz$ are defined for any $x, y, z \in P$. Hence, $(xy) z$ and $x(yz)$ are defined and equal for any $x, y, z \in P$. Also, $x = e_x x = xe_x$ for any $x \in P$ implies $x = ex = xe$ for any $x \in P$.

A poloid can thus be regarded as a generalized monoid, and also as a generalized groupoid; in fact, poloids generalize groups via monoids and via groupoids.

Proposition 7. A groupoid with only one unit is a group.

Proof. A monoid with inverses is a group. □
3.2. Subpoloids, poloid homomorphisms and poloid actions. Recall that a submonoid of a monoid \( M \) is a monoid \( M' \) such that \( M' \) is a submagma of \( M \) and the unit in \( M' \) is the unit in \( M \). Subpoloids can be defined similarly.

**Definition 11.** A subpoloid of a poloid \( P \) is a poloid \( P' \) such that \( P' \) is a submagma of \( P \) and every unit in \( P' \) is a unit in \( P \).

Homomorphisms and actions of poloids similarly generalize homomorphisms and actions of monoids.

**Definition 12.** Let \( P \) and \( Q \) be poloids. A poloid homomorphism from \( P \) to \( Q \) is a total function \( \phi : P \rightarrow Q \) such that

1. if \( x, y \in P \) and \( xy \) is defined then \( \phi(x)\phi(y) \) is defined and \( \phi(xy) = \phi(x)\phi(y) \);
2. if \( e \) is a unit in \( P \) then \( \phi(e) \) is a unit in \( Q \).

A poloid isomorphism is a poloid homomorphism \( \phi \) such that the inverse function \( \phi^{-1} \) exists and is a poloid homomorphism.

Note that \( \phi(x) = \phi(\varepsilon_x x) = \phi(\varepsilon_x)\phi(x) \) by (1) and \( \phi(\varepsilon_x) \) is a unit by (2) in Definition 12. Using Proposition 2 we have \( \phi(\varepsilon_x) = \varepsilon_{\phi(x)} \). Dually, \( \phi(\varepsilon_x) = \varepsilon_{\phi(x)} \).

Let \( P \) be a poloid, let \( Q \) be a magma and assume that there exists a total function \( \phi : P \rightarrow Q \) satisfying (1) and (2) in Definition 12 and also such that (1') if \( \phi(x)\phi(y) \) is defined then \( xy \) is defined. It is easy to verify that then \( \phi(P) \) is a magma, (P1) is satisfied in \( \phi(P) \), and if \( x' = \phi(x) \in \phi(P) \) then \( \phi(\varepsilon_x) \phi(\varepsilon_x) \) is an effective left (right) unit for \( x' \), so \( \phi(P) \) is a poloid.

**Definition 13.** A poloid action of a poloid \( P \) on a set \( X \) is a total function

\[
\alpha : P \rightarrow \alpha(P) \subseteq \mathcal{F}_X
\]

which is a poloid homomorphism such that if \( e \in P \) is a unit then \( \alpha(e) \in \alpha(P) \) is an identity transformation \( \text{Id}_{\text{dom}(\alpha(e))} \) on \( X \).

A prefunction poloid action of a poloid \( P \) on \( X \) is similarly a total function

\[
\alpha : P \rightarrow \alpha(P) \subseteq \mathcal{F}_X
\]

which is a poloid homomorphism such that if \( e \in P \) is a unit then \( \alpha(e) \in \alpha(P) \) is an identity pretransformation \( \text{Id}_{\text{dom}(\alpha(e))} \) on \( X \).

A poloid action \( \alpha \) thus assigns to each \( x \in P \) a non-empty transformation

\[
\alpha(x) : X \rightarrow X, \quad t \mapsto \alpha(x)(t)
\]

such that if \( xy \) is defined then \( \alpha(x) \circ \alpha(y) \) is defined and \( \alpha(xy) = \alpha(x) \circ \alpha(y) \), and for each unit \( e \in P \) its image \( \alpha(e) \) is a unit in \( \alpha(P) \) such that \( \alpha(e)(t) = t \) for each \( t \in \text{dom}(\alpha(e)) \).

**Remark 4.** The definition of a poloid homomorphism given here implies the usual definition of a monoid homomorphism. The definition of a monoid action obtained from Definition 13 is also the usual one. Specifically, a monoid action \( \alpha \) of \( M \) on a set \( X \) is a function

\[
\alpha : M \rightarrow \alpha(M) \subseteq \mathcal{F}_X
\]

such that \( \alpha(xy)(t) = \alpha(x) \circ \alpha(y)(t) \) and \( \alpha(e)(t) = t \) for all \( x, y \in M \) and all \( t \in X \). Denoting \( \alpha(x)(t) \) by \( x \cdot t \), this is rendered as \( (xy) \cdot t = x \cdot (y \cdot t) \) and \( e \cdot t = t \). Note that \( \alpha(e) = \text{Id}_X \) is a unit in \( \mathcal{F}_X \) and thus in \( \alpha(M) \).
3.3. Poloids as transformation poloids. Recall that every transformation poloid is, indeed, a poloid. Up to isomorphism, there are, in fact, no other poloids.

Lemma 2. For any poloid $P$, there is a prefunction poloid action

$$\mu : P \to \mu(P) \subseteq \mathcal{F}_P,$$

$$x \mapsto \mu(x)$$

of $P$ on $P$ such that $\mu$ is a poloid isomorphism.

Proof. Set $\mu(x) = \left(\overline{\mu(x)}, \text{dom}(\mu(x))\right)$, where $\overline{\mu(x)}(t) = xt$ for all $t \in \text{dom}(\mu(x))$ and $\text{dom}(\mu(x)) = \{t \mid xt \text{ defined}\}$. Then $\mu(x)$ is a prefunction $P \not\rightarrow P$, and $\mu(x)$ is non-empty for each $x \in P$ since $x\varepsilon_x$ is defined for each $x \in P$.

Furthermore, $\overline{\mu(x)}(\varepsilon_x) = xx = x$ for any $x \in P$, and also $\overline{\mu(y)}(\varepsilon_x) = y\varepsilon_x = y$ for any $y \in P$ such that $y\varepsilon_x$ is defined since $\varepsilon_x$ is a unit. Hence, if $x \neq y$ and $y\varepsilon_x$ is defined then $\overline{\mu(x)}(\varepsilon_x) \neq \overline{\mu(y)}(\varepsilon_x)$, so $\mu(x) \neq \mu(y)$; if $x \neq y$ and $y\varepsilon_x$ is not defined then $\text{dom}(\alpha(x)) \neq \text{dom}(\alpha(y))$ since $\varepsilon_x$ is defined. Thus, $\mu$ is a bijection.

For any fixed $x, y \in P$ such that $xy$ is defined, $(xy)t$ is defined if and only if $t \in P$ is such that $yt$ is defined. Thus, $\text{im}(\mu(y)) = \{yt \mid yt \text{ defined}\} = \{yt \mid x(y)t \text{ defined}\} \subseteq \{t \mid xt \text{ defined}\} = \text{dom}(\mu(x))$ and $\{t \mid (xy)t \text{ defined}\} = \{t \mid yt \text{ defined}\}$, so if $xy$ is defined then $\mu(x) \circ \mu(y)$ is defined and $\text{dom}(\mu(xy)) = \text{dom}(\mu(y)) = \text{dom}(\mu(x) \circ \mu(y))$.

Also, if $xy$ is defined and $t \in \text{dom}(\mu(xy)) = \text{dom}(\mu(y))$, meaning that $yt$ is defined, then $(xy)t$ and $x(y)t$ are defined and equal, and as $(xy)t = \overline{\mu(xy)}(t)$ for all $t \in \text{dom}(\mu(xy))$ and $x(y)t = \overline{\mu(x)}(\mu(y))(t)$ for all $t \in \text{dom}(\mu(x) \circ \mu(y))$, this implies that if $xy$ is defined then $\mu(xy) = \mu(x) \circ \mu(y)$.

Conversely, if $\mu(x) \circ \mu(y)$ is defined so that $\{t \mid yt \text{ defined}\} = \text{dom}(\mu(y)) = \text{dom}(\mu(x) \circ \mu(y)) = \{t \mid x(y)t \text{ defined}\}$, then $yt$ defined implies $xt(yt)$ defined for any fixed $x, y \in P$. But this implication does not hold if $xy$ is not defined; then, $x(y\varepsilon_y)$ is not defined although $y\varepsilon_y$ is defined. Hence, if $\mu(x) \circ \mu(y)$ is defined then $xy$ must be defined. Therefore, $\mu(x) \circ \mu(y) = \mu(xy) \in \mu(P)$, so $\mu(P)$ is a magma with $\circ$ as binary operation.

Let $e \in P$ be a unit. If $\mu(e) \circ \mu(x)$ is defined then $ex$ is defined so $\mu(x) = \mu(ex) = \mu(e) \circ \mu(x)$, and if $\mu(x) \circ \mu(e)$ is defined then $xe$ is defined so $\mu(x) = \mu(xe) = \mu(x) \circ \mu(e)$. Thus, $\mu(x) \in \mu(P)$ is a unit.

Conversely, if $f' = \mu(f) \in \mu(P)$ is a unit and $fx$ is defined then $\mu(f) \circ \mu(x)$ is defined and $\mu(fx) = \mu(f) \circ \mu(x) = \mu(x)$, so $fx = x$ since $\mu$ is injective. Similarly, if $f(\mu(x)) \in \mu(P)$ is a unit and $xf$ is defined then $xf = x$. Hence, $f \in P$ is a unit.

Thus, we have shown that $\mu$ satisfies the conditions labeled (1), (1’) and (2) in Section 3.2, so $\mu(P)$ is a poloid. Also, (1) and (2) in Definition 12 are satisfied by both $\mu$ and $\mu^{-1}$, so $\mu : P \to \mu(P)$ is a poloid isomorphism.

The observation that if $e \in P$ is a unit then $\mu(e)(t) = et = t$ for all $t \in \text{dom}(\mu(e))$, so that $\mu(e)$ is an identity pretransformation $\text{Id}_{\text{dom}(\mu(e))}$, completes the proof. \(\square\)

Lemma 3. For any poloid $P$ and function $\mu$ defined as in Lemma 2, there is a total function $\tau : \mu(P) \to \tau(\mu(P)) \subseteq \mathcal{F}_P$ such that

1. $\tau$ is bijective;
2. $\mu(x) \circ \mu(y)$ is defined if and only if $\tau(\mu(x)) \circ \tau(\mu(y))$ is defined;
3. if $\mu(x) \circ \mu(y)$ is defined then $\tau(\mu(x) \circ \mu(y)) = \tau(\mu(x)) \circ \tau(\mu(y))$;
4. if $e \in P$ is a unit then $\tau(\mu(e)) \in \mathcal{F}_P$ is a unit and identity transformation $\text{Id}_{\text{dom}(\tau(\mu(e)))}$. 


Proof. For any prefunction \( \mu(x) = \left( \overline{\mu(x)}, \text{dom}(\mu(x)) \right) : P \rightrightarrows P, x \in P \), the tuple
\[
/\mu(x)/ = \left( \overline{\mu(x)}, \text{dom}(\mu(x)), \text{dom}(\mu(\varepsilon_x)) \right)
\]
is a function \( P \rightrightarrows P \) for which \( /\mu(x)/ = \overline{\mu(x)} \), \( \text{dom}(/\mu(x)/) = \text{dom}(\mu(x)) \) and \( \text{cod}(/\mu(x)/) = \text{dom}(\mu(\varepsilon_x)) \). In fact, \( \varepsilon_x \) is defined, so \( \mu(\varepsilon_x) \circ \mu(x) \) is defined, so \( \text{cod}(/\mu(x)/) = \text{dom}(\mu(\varepsilon_x)) \supseteq \text{im}(\mu(x)) = \overline{\mu(x)(\text{dom}(\mu(x)))} = /\mu(x)//(\text{dom}(/\mu(x)/)) = \text{im}(/\mu(x)/) \), as required. Thus, there is a total function
\[
\tau : \mathcal{T}_\mathcal{P} \ni \mu(P) \to \tau(\mu(P)) \subseteq \mathcal{T}_\mathcal{P}, \quad \mu(x) \mapsto /\mu(x)/.
\]

It remains to prove (1) – (4). (1) and (2) are obvious. Also, \( \text{dom}(/\mu(x) \circ \mu(y)/) = \text{dom}(\mu(x) \circ \mu(y)) = \text{dom}(\mu(y)) = \text{dom}(/\mu(y)/) = \text{dom}(\mu(x) \circ /\mu(y)/) \) and \( \text{cod}(/\mu(x) \circ \mu(y)/) = \text{cod}(/\mu(x)/ \circ /\mu(y)/) = \text{cod}(/\mu(x)/ \circ \mu(y)) \), so \( /\mu(x) \circ \mu(y)/ = /\mu(x)/ \circ /\mu(y)/ \). Concerning (4), \( \mu(e) \) is a unit and identity pretransformation \( \text{Id}_{\text{dom}(\mu(e))} \) in \( \mathcal{T}_\mathcal{P} \), so it suffices to note that \( \text{cod}(/\mu(e)/) = \text{dom}(\mu(\varepsilon_e)) = \text{dom}(\mu(e)) = \text{dom}(/\mu(e)/) \). \( \square \)

**Theorem 3.** For any poloid \( P \), there is a poloid action
\[
\alpha : P \to \alpha(P) \subseteq \mathcal{T}_\mathcal{P}, \quad x \mapsto \alpha(x)
\]
of \( P \) on \( P \) such that \( \alpha \) is a poloid isomorphism and \( \alpha(P) \) equipped with \( \circ \) is a transformation poloid.

Proof. First set \( \alpha = \tau \circ \mu \) and use Lemmas \( \text{[2]} \) and \( \text{[3]} \) to prove the first part of the theorem. It remains to show that \( \alpha(P) \) is a transformation poloid. Recall that \( \mu \) and \( \tau \) are injective so that \( \alpha \) is injective, and note that \( \text{dom}(\alpha(x)) = \text{dom}(\alpha(x \varepsilon_x)) = \text{dom}(\alpha(x) \circ \alpha(\varepsilon_x)) = \text{dom}(\alpha(\varepsilon_x)) \) and that identity transformations, such as \( \alpha(\varepsilon_x) \) and \( \alpha(\varepsilon_y) \), are determined by their domains. Hence, we have
\[
\text{dom}(\alpha(x)) = \text{cod}(\alpha(y)) \\
\iff \text{dom}(\alpha(\varepsilon_x)) = \text{dom}(\alpha(\varepsilon_y)) \\
\iff \alpha(\varepsilon_x) = \alpha(\varepsilon_y) \\
\iff \varepsilon_x = \varepsilon_y \\
\iff xy \text{ defined} \\
\iff \alpha(x) \circ \alpha(y) \text{ defined.}
\]

Thus the poloid of transformations \( \alpha(P) \) is a transformation semigroupoid by Definition \( \text{[4]} \). Also, if \( \alpha(x) \in \alpha(P) \) then \( \alpha(\varepsilon_x) \), \( \alpha(\varepsilon_x) \in \alpha(P) \), \( \alpha(\varepsilon_x) = \text{Id}_{\text{dom}(\alpha(\varepsilon_x))) = \text{Id}_{\text{cod}(\alpha(x))} \) and \( \alpha(\varepsilon_x) = \text{Id}_{\text{dom}(\alpha(\varepsilon_x))) = \text{Id}_{\text{dom}(\alpha(x))) \), so the transformation semigroupoid \( \alpha(P) \) is a transformation poloid by Definition \( \text{[10]} \). \( \square \)

**Corollary 1.** Any poloid is isomorphic to a transformation poloid.

This is a 'Cayley theorem' for poloids; it generalizes similar isomorphism theorems for groupoids, monoids and groups. Note, though, that \( \alpha(P) \) is not only a poloid of transformations isomorphic to \( P \), but actually a transformation poloid isomorphic to \( P \), so Corollary \( \text{[4]} \) is stronger than a straight-forward generalization of the 'Cayley theorem' as usually stated.
3.4. Categories as poloids. It is no secret that a poloid is the same as a small arrows-only category. In various guises, (P1), (P2) and Propositions 1–4 appear as axioms or theorems in category theory. The two-axiom system proposed here is related to the set of “Gruppoid” axioms given by Brandt [1], and essentially equivalent to axiom systems used by Freyd [2], Hastings [6], and others. By Proposition 3, one can define functions $s : x \mapsto \epsilon_x$ and $t : x \mapsto \epsilon_x$; axiom systems using these two functions but equivalent to the one given here, as used by Freyd and Scedrov [3], currently often serve to define arrows-only categories.

Concepts from category theory can be translated into the the language of poloids and vice versa. For example, an initial object in a category corresponds to some unit $\epsilon \in P$ such that for every unit $e \in P$ there is a unique $x \in P$ such that $ex$ and $xe$ are defined (hence, $ex = x = xe$). More significantly, in the language of category theory a subpoloid is a subcategory, and a poloid homomorphism is a functor.

Looking at categories as “webs of monoids” does lead to some shift of emphasis and perspective, however. In particular, whereas the notion of a category acting on a set is not emphasized in texts on category theory, the corresponding notion of a poloid action is central when regarding categories as poloids. For example, recall that letting a group act on itself we obtain Cayley’s theorem for groups. Similarly, letting a poloid act on itself we have obtained a Cayley theorem for poloids [7], corresponding to Yoneda’s lemma for categories. Poloid actions are also a tool that can be used to define ordinary (small) two-sorted categories in terms of poloids – we let a poloid $P$ act on a set $O$ in a special way, then interpreting the elements of $P$ as morphisms and the elements of $O$ acted on by $P$ as objects.

Applying an algebraic perspective on category theory may thus lead to more than merely a reformulation of category theory, especially as the algebraic structures related to categories are also linked to specific magmas of transformations.

APPENDIX A. CONSTELLATIONS

A constellation [4,7], is defined in [5] as follows:

A [left] constellation is a structure $P$ of signature $(\cdot, D)$ consisting of a class $P$ with a partial binary operation and unary operation $D$ [...] that maps onto the set of projections $E \subseteq P$, so that $E = \{D(x) \mid x \in P\}$, and such that for all $e \in E$, $ee$ exists and equals $e$, and for which, for all $x, y, z \in P$:

(C1) if $x \cdot (y \cdot z)$ exists then so does $(x \cdot y) \cdot z$, and then the two are equal;

(C2) $x \cdot (y \cdot z)$ exists if and only if $x \cdot y$ and $y \cdot z$ exist;

(C3) for each $x \in P$, $D(x)$ is the unique left identity of $x$ in $E$ (i.e. it satisfies $D(x) \cdot x = x$);

(C4) for $a \in P$ and $g \in E$, if $a \cdot g$ exists then it equals $a$.

It turns out that constellations generalize poloids. Recall that by Definition 4 a semigroupoid is a partial magma such that if (a) $x(yz)$ is defined or (b) $(xy)z$ is defined or (c) $xy$ and $yz$ are defined then $x(yz)$ and $(xy)z$ are defined and $x(yz) = (xy)z$. Removing (a), we obtain the following definition.

Definition 14. A right-directed semigroupoid is a magma $P$ such that, for any $x, y, z \in P$, if $(xy)z$ is defined or $xy$ and $yz$ are defined then $(xy)z$ and $x(yz)$ are defined and $x(yz) = (xy)z$. 
The condition in this definition corresponds to conditions (C1) and (C2) in \[5\] except for some non-substantial differences. First, we are defining here the left-right dual of the notion defined by (C1) and (C2). This amounts to a difference in notation only, deriving from the fact that functions are composed from left to right in \[5\] while they are composed from right to left here. Second, it is not necessary to postulate that if \((xy)z\) is defined then \(xy\) and \(yz\) are defined, in accordance with (C2), because, by Definition \[14\] if \((xy)z\) is defined then \(x(yz)\) is defined, so \(xy\) and \(yz\) are defined. Finally, in \[5\] \(P\) is assumed to be a class rather than a set; this difference has to do with set-theoretic considerations that need not concern us here.

We shall need some generalizations of the unit concept. First, a left unit in \(P\) is an element \(\epsilon\) of \(P\) such that \(\epsilon x = x\) for all \(x \in P\) such that \(\epsilon x\) is defined, while a right unit in \(P\) is an element \(\epsilon\) of \(P\) such that \(x\epsilon = x\) for all \(x \in P\) such that \(x\epsilon\) is defined. Also, a local left unit \(\lambda_x\) for \(x \in P\) is an element of \(P\) such that \(\lambda_x x\) is defined and \(\lambda_x x = x\), while a local right unit \(\rho_x\) for \(x \in P\) is an element of \(P\) such that \(x\rho_x\) is defined and \(x\rho_x = x\).

**Definition 15.** A right poloid is a right-directed semigroupoid \(P\) such that for any \(x \in P\) there is a unique left unit \(\varphi_x \in P\) such that \(\varphi_x\) is a local right unit for \(x\).

**Proposition 8.** Let \(P\) be a right poloid. If \(\epsilon \in P\) is a left unit then \(\epsilon\epsilon\) is defined and \(\epsilon\epsilon = \epsilon\).

**Proof.** Let \(\varphi_x \in P\) be a local right unit for the left unit \(\epsilon\). Then \(\epsilon\varphi_x\) is defined and \(\varphi_x = \epsilon\varphi_x = \epsilon\), and this implies the assertion. \(\Box\)

Thus, the left unit \(\epsilon\) is the unique local right unit \(\varphi_x\) for itself.

Disregarding (C1) and (C2), which were incorporated in Definition \[14\] the requirements stated in the definition cited above can be summed up as follows:

\[
(C) \quad \text{For each } x \in P, \text{ there is exactly one } D(x) \in E = \{D(x) \mid x \in P\} \text{ such that } D(x) \cdot x \text{ is defined and } D(x) \cdot x = x, \text{ and every } e \in E \text{ is a right unit in } P \text{ and such that } e \cdot e \text{ is defined and equal to } e.
\]

Using (C), it can be proved as in Proposition \[8\] that if \(f \in P\) is a right unit then \(f \cdot f\) is defined and \(f \cdot f = f\), so \(f\) is the unique local left unit \(D(f)\) for itself. Thus, \(E\) equals the set of right units in \(P\), since conversely every \(e \in E\) is a right unit in \(P\) by (C). As all right units are idempotent, this means that the requirement that all elements of \(E\) are idempotent is redundant, so (C) can be simplified to:

\[
(C^*) \quad \text{For each } x \in P, \text{ there is exactly one right unit } D(x) \in P \text{ such that } D(x) \cdot x \text{ is defined and } D(x) \cdot x = x.
\]

In our terminology, this means, of course, that for any \(x \in P\) there is a unique left unit \(\varphi_x\) in \(P\) such \(\varphi_x\) is a local right unit for \(x\). We conclude that a (small) constellation is just a right poloid; note that \(D\) is just the function \(x \mapsto \varphi_x\). Proposition \[8\] generalizes Proposition \[1\] and there are also natural generalizations of Propositions \[2\]–\[4\] to right poloids.

It should be pointed out that in \[5\] an alternative definition of constellations is also given; this definition is essentially the same as Definition \[15\] here (see Proposition 2.9 in \[5\]). So while the definition of constellations cited above reflects the historical development of that notion, it has been shown here and in \[5\] that a more direct approach can also be used.

Let us also look at the transformation systems corresponding to constellations.
Theorem 4. A pretransformation magma is a right-directed semigroupoid.

Proof. Use Facts 1–3 in Section 2.3. □

A domain pretransformation magma is a pretransformation magma $\mathcal{R}_X$ such that if $f \in \mathcal{R}_X$ then $\text{id}_{\text{dom}(f)} \in \mathcal{R}_X$. Corresponding to Theorem 2 in Section 2.3, we have the following result.

Theorem 5. A domain pretransformation magma is a right poloid.

Proof. In view of Theorem 1 it suffices to show that for any $f \in \mathcal{R}_X$ there is a unique left unit $\varphi_f \in \mathcal{R}_X$ such that $f \circ \varphi_f$ is defined and equal to $f$, namely $\text{id}_{\text{dom}(f)}$.

If $f, g \in \mathcal{R}_X$ and $\text{id}_{\text{dom}(f)} \circ g$ is defined so that $\text{dom}(\text{id}_{\text{dom}(f)}) \supseteq \text{im}(g)$ then

$$\text{dom}(\text{id}_{\text{dom}(f)} \circ g) = \text{dom}(g),$$

$$\text{id}_{\text{dom}(f)} \circ g(x) = \text{id}_{\text{dom}(f)}(g(x)) = g(x)$$

for all $x \in \text{dom}(g)$, meaning that $\text{id}_{\text{dom}(f)} \circ g = g$. Thus, $\text{id}_{\text{dom}(f)}$ is a left unit in $\mathcal{R}_X$.

Also, $\text{dom}(f) = \text{dom}(\text{id}_{\text{dom}(f)}) = \text{im}(\text{id}_{\text{dom}(f)})$, so $f \circ \text{id}_{\text{dom}(f)}$ is defined, and

$$\text{dom}(f \circ \text{id}_{\text{dom}(f)}) = \text{dom}(\text{id}_{\text{dom}(f)}) = \text{dom}(f),$$

$$f \circ \text{id}_{\text{dom}(f)}(x) = f(\text{id}_{\text{dom}(f)}(x)) = f(x)$$

for all $x \in \text{dom}(\text{id}_{\text{dom}(f)}) = \text{dom}(f)$. Thus, $f \circ \text{id}_{\text{dom}(f)}$ is defined and equal to $f$, so $\text{id}_{\text{dom}(f)} \in \mathcal{R}_X$ is a left unit $\varphi_f$ such that $f \circ \varphi_f$ is defined and equal to $f$.

It remains to show that $\text{id}_{\text{dom}(f)}$ is the only such $\varphi_f$. Let $\epsilon \in \mathcal{R}_X$ be a left unit. Then $\text{id}_{\text{dom}(\epsilon)} \in \mathcal{R}_X$ and as $\text{dom}(\epsilon) = \text{dom}(\text{id}_{\text{dom}(\epsilon)}) = \text{im}(\text{id}_{\text{dom}(\epsilon)})$, so that $\epsilon \circ \text{id}_{\text{dom}(\epsilon)}$ is defined, we have $\epsilon \circ \text{id}_{\text{dom}(\epsilon)} = \text{id}_{\text{dom}(\epsilon)}$. On the other hand,

$$\text{dom}(\epsilon \circ \text{id}_{\text{dom}(\epsilon)}) = \text{dom}(\text{id}_{\text{dom}(\epsilon)}) = \text{dom}(\epsilon),$$

$$\epsilon \circ \text{id}_{\text{dom}(\epsilon)}(x) = \epsilon(\text{id}_{\text{dom}(\epsilon)}(x)) = \epsilon(x)$$

for all $x \in \text{dom}(\text{id}_{\text{dom}(\epsilon)}) = \text{dom}(\epsilon)$, so $\epsilon \circ \text{id}_{\text{dom}(\epsilon)} = \epsilon$. Thus $\epsilon = \text{id}_{\text{dom}(\epsilon)}$, so $\varphi_f = \text{id}_{\text{dom}(\varphi_f)} = \text{id}_{\text{dom}(f \circ \varphi_f)} = \text{id}_{\text{dom}(f)}$. □

With Theorem 2 and Corollary 1 in mind, one might expect, given Theorem 5 that conversely every right poloid is isomorphic to some domain pretransformation magma (regarded as a right poloid). Indeed, any poloid can be embedded in a pretransformation magma by Lemma 2 and it can be shown that $\mu(\varepsilon_x) = \text{id}_{\text{dom}(\mu(x))}$, so any poloid can actually be embedded in a domain pretransformation magma. Also, the proof of Lemma 2 uses almost only properties of poloids that they share with right poloids. There is one crucial exception, though: both $\varepsilon_x$ and $\varphi_x$ are local right units, but in addition $\varepsilon_x$ is a unit while $\varphi_x$ is just a left unit. The fact that $\varepsilon_x$ is a unit is used to prove that $x \mapsto \mu(x)$ is injective, and this is not true for all right poloids.

Example 3. The magma defined by the Cayley table below is a right poloid with $x = \varphi_x$ and $y = \varphi_y$, but $\mu(x) = \mu(y)$.

$$\begin{array}{c|cc}
  & x & y \\
  \hline
  x & x & y \\
  y & x & y \\
\end{array}$$
This suggests that we look for an additional condition on right poloids to ensure that \( x \mapsto \mu(x) \) is injective. On finding such a condition, we can prove a weakened converse of Theorem 5 by an argument similar to the proof of Lemma 2.

Adapting a definition in [5], we say that a right poloid such that if \( \varphi_x \varphi_y \) and \( \varphi_y \varphi_x \) are defined then \( \varphi_x = \varphi_y \) is normal. (The poloid in Example 5 is not normal.) This notion is the key to the following three results:

**Theorem 6.** A domain pretransformation magma is a normal right poloid.

*Proof.* In a domain pretransformation magma, \( \varphi_\ell = \text{ld}(f) \). Thus, \( \varphi_\ell \circ \varphi_g \) is defined if and only if \( \text{dom}(\text{ld}(f)) \supseteq \text{im}(\text{ld}(g)) = \text{dom}(\text{ld}(g)) \), or equivalently \( \text{dom}(f) \supseteq \text{dom}(g) \), so if \( \varphi_\ell \circ \varphi_g \) and \( \varphi_g \circ \varphi_\ell \) are defined then \( \text{dom}(f) = \text{dom}(g) \), so \( \text{ld}(f) = \text{ld}(g) \) or equivalently \( \varphi_\ell = \varphi_g \). \( \square \)

**Lemma 4.** In a normal right poloid, the correspondence \( x \mapsto \mu(x) \) is injective.

*Proof.* Assume that \( x \neq y \). If \( \text{dom}(\mu(x)) \neq \text{dom}(\mu(y)) \) then \( \mu(x) \neq \mu(y) \) as required. Otherwise, \( \text{dom}(\mu(x)) = \text{dom}(\mu(y)) \), and as \( x \varphi_x \) and \( y \varphi_y \) are defined we have \( \varphi_x, \varphi_y \in \text{dom}(\mu(x)) = \text{dom}(\mu(y)) \). Thus, \( x \varphi_y \) is defined, so \( (x \varphi_x) \varphi_y \) is defined, so \( x \varphi_x \varphi_y \) is defined. Similarly, \( y \varphi_x \) is defined, so \( \varphi_y \varphi_x \) is defined. Therefore, \( \varphi_x = \varphi_y \), so \( \mu(x)(\varphi_x) = x \) and \( \mu(y)(\varphi_x) = \mu(y)(\varphi_y) = y \), so again \( \mu(x) \neq \mu(y) \). \( \square \)

Using Lemma 4 and proceeding as in the proof of Lemma 2, keeping in mind that \( \varphi_\mu(x) = \mu(\varepsilon_x) = \text{ld}(\mu(x)) \), we obtain the following result:

**Theorem 7.** A normal right poloid can be embedded in a domain pretransformation magma.

Theorems 6 and 7 correspond to Proposition 2.23 in [5].

Let us look at another way of narrowing down the notion of a right poloid so that any right poloid considered can be embedded in a domain pretransformation magma. Consider the relation \( \leq \) on a right poloid \( P \) given by \( x \leq y \) if and only if \( y \varphi_x \) is defined and \( x = y \varphi_x \). The relation \( \leq \) is obviously reflexive, and if \( x \leq y \) and \( y \leq z \) then (a) \( y \varphi_x = (z \varphi_y) \varphi_x \) is defined so that \( z \varphi_y \varphi_x = z \varphi_x \) is defined and (b) \( x = y \varphi_x = (z \varphi_y) \varphi_x = z \varphi_y \varphi_x \) is defined. Hence, \( \leq \) is a preorder, called the natural preorder on \( P \), so \( \leq \) is a partial order if and only if it is antisymmetric. A right poloid such that \( \epsilon \leq \epsilon' \) and \( \epsilon' \leq \epsilon \) implies \( \epsilon = \epsilon' \) for any left units \( \epsilon, \epsilon' \in P \) is said to be unit-posetal.

Recall that for any left unit \( \epsilon \in P \) we have \( \varphi_\epsilon = \epsilon \), so \( \varphi_\epsilon \varphi_x = \varphi_x \). Thus, \( \varphi_x \leq \varphi_y \) if and only \( \varphi_y \varphi_x \) is defined and \( \varphi_x = \varphi_y \varphi_x \), so as \( \varphi_y \) is a left unit we have \( \varphi_x \leq \varphi_y \) if and only \( \varphi_y \varphi_x \) is defined. Hence, we obtain the following results.

**Theorem 8.** A right poloid is unit-posetal if and only if it is normal.

**Theorem 9.** A domain pretransformation magma is a unit-posetal right poloid.

**Theorem 10.** A unit-posetal right poloid can be embedded in a domain pretransformation magma.

If we specialize the concept of a unit-posetal right poloid by adding more requirements, the analogue of Theorem 9 need of course not hold. In particular, the partial order on the left units is not necessarily a semilattice.
Example 4. Set $X = \{1, 2, 3\}$ and $\mathcal{R}_X = \\{\text{Id}_{\{1,2\}}, \text{Id}_{\{2,3\}}\}$ with $f \circ g$ defined as usual when $\text{dom}(f) \supseteq \text{im}(g)$. Then $\mathcal{R}_X$ is a domain pretransformation magma where $\text{Id}_{\{1,2\}} \leq \text{Id}_{\{1,2\}}$ and $\text{Id}_{\{2,3\}} \leq \text{Id}_{\{2,3\}}$, but this partial order is not a semilattice.

More broadly, let $\mathcal{A}$ denote a class of abstract algebraic structures corresponding to a class $\mathcal{C}$ of concrete magmas of correspondences (functions, prefunctions etc.) in the sense that any $c$ in $\mathcal{C}$ belongs to $\mathcal{A}$ when certain operations in $\mathcal{C}$ are interpreted as the operations in $\mathcal{A}$. Note that this does not imply that any $a$ in $\mathcal{A}$ can be embedded in some $c$ in $\mathcal{C}$. In particular, if $\mathcal{A}$ is a class of generalized groups, with axioms merely defining a generalized group operation and (optional) generalized identities and inverses, then the fact that the axioms defining $\mathcal{A}$ are satisfied for any concrete magma $c$ in $\mathcal{C}$ does not provide a strong reason to expect that any $a$ satisfying these axioms can be embedded in some $c$ in $\mathcal{C}$. As we have just seen, the relation between right poloids and domain pretransformation magmas is asymmetrical in this respect. (One-sided) restriction semigroups [4] provide another example of this phenomenon.

Example 5. Let $\mathcal{A}$ be the class of semigroups such that for each $a$ in $\mathcal{A}$ and each $x \in a$ there is a unique local right unit for $x$ in $a$. Let $\mathcal{C}$ be the class of semigroups of functional relations on a given set where the binary operation is composition of relations and such that for each $c$ in $\mathcal{C}$ and each $f \in c$ the functional relation $\text{Id}_{\text{dom}(f)}$ belongs to $c$. Then any $c$ in $\mathcal{C}$ belongs to $\mathcal{A}$, with $\text{Id}_{\text{dom}(f)}$ the local right unit for $f$, but it is not the case that any $a$ in $\mathcal{A}$ can be embedded in some $c$ in $\mathcal{C}$. To ensure embeddability, $\mathcal{A}$ needs to be narrowed down by additional conditions, subject to the restriction that $\mathcal{A}$ remains wide enough to accommodate all $c$ in $\mathcal{C}$.

We have seen that any transformation poloid is a poloid and that those poloids which can be embedded in a transformation poloid are simply all poloids, and a similar elementary symmetry exists for inverse semigroups, but such cases are perhaps best regarded as ideal rather than normal, reflecting the fact that poloids and inverse semigroups are particularly natural algebraic structures.

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