Considering boundary conditions for black hole entropy in loop quantum gravity

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Received 28 February 2007, in final form 16 June 2007
Published 17 July 2007
Online at stacks.iop.org/CQG/24/3837

Abstract

We argue for black hole entropy in loop quantum gravity (LQG) by taking into account the interpretation that there is no other side of the horizon. This gives new values for the Barbero–Immirzi parameter ($\gamma = 0.367 \cdots$ or $0.323 \cdots$) which are larger than those considered before ($\gamma = 0.261 \cdots$ or $0.237 \cdots$). We also discuss its consequences for future experiments.

PACS numbers: 04.70.Bw, 04.30.Db, 04.70.Dy

1. Introduction

Black hole thermodynamics is one of the most exciting arenas for those investigating quantum gravity. In particular, discovery of the microscopic origin of black hole entropy in string theory has attracted much attention [1]. However, its understanding is basically restricted by the perturbative formulation of string theory. As a result, the origin of black hole entropy has been revealed only for BPS black holes.

In this respect, the explanation of the origin of black hole entropy in loop quantum gravity (LQG), which has background-independent formulation, is stimulating [2]. This explanation was attempted based on spin network formalism [3]. In LQG, it has also been reported that we can avoid initial or final singularities in the universe [4] and the singularity in black holes [5]. One of the drawbacks in explaining black hole entropy is that there is a free parameter called the Barbero–Immirzi parameter $\gamma$, which is related to an ambiguity in the choice of canonically conjugate variables [6]. This is why we can adjust the parameter to obtain the relation that black hole entropy $S$ is equal to a quarter of the horizon area $A$, i.e., $S = A/4$. Thus, it is important to obtain a consensus about how to count the number of degrees of freedom.
At present, various possibilities have been argued with respect to this freedom [7–13]. However, there is one possibility that has not been considered so far. For example, general expressions for the area spectrum $A$ are [14, 15]

$$A = 4\pi \gamma \sum \sqrt{2j_i^u(j_i^u + 1) + 2j_i^d(j_i^d + 1) - j_i^t(j_i^t + 1)}.$$  

The sum is the addition of all intersections between a surface and edges. Here, the indices $u, d$ and $t$ mean edges above, below and tangential to the surface, respectively. (We can determine which side is above or below arbitrarily.) So far the number of states of black holes has been discussed based on the simplified area spectrum ($j_i^u = j_i^d := j_i$ and $j_i^t = 0$). Another interesting possibility, however, is to take $j_i^d = 0$ to reflect the absence of the other side of the horizon [16]. It is assumed that a horizon (emerging classical null surface) is a two-dimensional surface that is tangential to the selected edges. In the absence of a direct derivation of the classical regime in LQG it is a legitimate speculation. In this case, the area spectrum becomes

$$A = 4\pi \gamma \sum \sqrt{j_i(j_i + 1)},$$  

since we have $j_i^u = j_i^t := j_i$. This is important since it would affect the number of states (and the resulting Barbero–Immirzi parameter) which we discuss in this paper.

Our motivation using (1.1) is based on the calculation in the entanglement entropy approach [17, 18]. That is, if we express the wavefunction $\Psi$ by a product as

$$\Psi := \Psi_O \Psi_H \Psi_I,$$  

where the indices O, H and I mean outside, at and inside the horizon, respectively, then, we can construct a density matrix. Since the inside of a black hole is inaccessible for an asymptotic observer, one traces over $\Psi_I$ to calculate entropy\(^1\). This means that we can determine black hole entropy independent of the inside of the horizon. This view point sometimes appears as a holographic principle [20] and is expressed in AdS/CFT correspondence [21]. Stimulated by this insight, we proceed by assuming that a black hole area is independent of the spin inside the horizon. (Note also that the tangential edges stay tangential for an asymptotic observer. This is important for (1.2) and is self-consistent with the assumptions mentioned above.) Of course, this implicitly assumes a relation between the area and the entropy. Thus, it is crucial to consider a method to justify its consequences. This is discussed in the final section.

This paper is organized as follows. In section 2, we summarize and reconsider the framework [2] (which we call the ABCK framework) that is necessary in counting the number of states of black holes. In section 3, we determine the number of states. In section 4, we summarize our results and discuss their meaning.

2. Consideration of the ABCK framework

Here, we briefly introduce and consider the ABCK framework in [2]. One usually considers the event horizon, which is determined after the complete evolution of spacetime, when one describes black hole thermodynamics. Thus, it would be too restrictive to establish a thermodynamical situation in which the system is isolated. To explore this idea appropriately, the isolated horizon (IH) is defined in the ABCK framework. The main difficulty in defining IH is to establish the surface gravity or black hole thermodynamics when, in general, there

\(^1\) See [19] which argued for yet another possibility of the degrees-of-freedom counting using entanglement considerations in loop quantum gravity.
is no global Killing field. For details, see [22]. Because of the requirement at IH, we can reduce the $SU(2)$ connection to the $U(1)$ connection. Using the curvature $F_{ab}$ of the $U(1)$ connection, we can express the boundary condition between IH and the bulk as

$$F_{ab} = -\frac{2\pi \gamma}{A} \sum_{i} r_i,$$

where $A$ is the area of IH. $\Sigma_{i}^{ab}$ is related to a triad density $E_{i}^{ab}$ as

$$E_{i}^{ab} = \gamma \eta_{i}^{abc} \Sigma_{i}^{bc},$$

where $\eta_{i}^{abc}$ is the Levi-Civita 3-form density. $\Sigma_{i}^{ab}$ is its pull back to IH and $r_i$ is the unit normal. Equation (2.1) plays an important role in determining the condition (iv) below.

Usually, we consider the Hilbert space using the spin network in LQG. When there is IH, we decompose the Hilbert space as the tensor product of that in IH $H_{IH}$ and that in the bulk $H_{\Sigma}$, i.e., $H_{IH} \otimes H_{\Sigma}$.

First, we consider $H_{\Sigma}$. Using edges having spin $(j_1, j_2, \ldots, j_n)$ which pierce IH, we can write $H_{\Sigma}$ as the orthogonal sum

$$H_{\Sigma} = \bigoplus_{j_i, m_i} H_{\Sigma}^{j_i, m_i},$$

where $m_i$ takes the value $-j_i, -j_i + 1, \ldots, j_i$. This is related to the flux operator eigenvalue $e_{i}^{m_i}$ that is normal to IH ($s'$ is the part of IH that has only one intersection between the edge with spin $j_i$).

$$e_{i}^{m_i} = 8\pi \gamma m_i.$$ (2.4)

Then, when we argue the possibility (1.1), it is natural to restrict as $m_i > 0$ since $m_i < 0$ corresponds to the flux operator eigenvalue inside the horizon. Basically, constraints in the bulk do not affect the number counting, as shown in [2]. The area operator eigenvalue $A_j$ should satisfy

(i) $$A_j = 4\pi \gamma \sum_i \sqrt{j_i(j_i + 1)} \leq A.$$ (2.5)

Next, we consider $H_{IH}$. We have, in general, difficulty in constructing $H_{IH}$. However, if we fix the horizon area $A$ as

$$A = 4\pi \gamma k,$$ (2.6)

where $k$ is a natural number, which is called the level of the Chern–Simons theory, we can construct $H_{IH}$ using a function which is invariant under the diffeomorphism and the $Z_k$ gauge transformation, i.e., a ‘quantized’ $U(1)$ gauge transformation. In addition to this condition, it is required that

(ii) we should fix an ordering $(j_1, j_2, \ldots, j_n).$ (2.7)

At IH, we do not consider the scalar constraint, since the lapse function disappears. As a result, $H_{IH}$ is written as an orthogonal sum similar to (2.3) by eigenstates $\Psi_b$ of the holonomy operator $\hat{h}_i$, i.e.,

$$\hat{h}_i \Psi_b = e^{2\pi \gamma b_i} \Psi_b.$$ (2.8)

From the quantum Gauss–Bonnet theorem that guarantees that IH is $S^2$, we require

(iii) $$\sum_{i=1}^{n} b_i = 0 \mod k.$$ (2.9)

2 Though we consider $A$ not an interval $[A - \delta A, A + \delta A]$, this does not affect the final results. Because of expression 3.4, $S := \ln W := \ln \left( \frac{d N}{\delta A} \right)$ is equal to $\ln N$ for $A \to \infty$. 
Finally, we should consider the quantization of the boundary condition between \( \Sigma \) and the bulk (2.1). Since only the exponential version \( \exp(i\hat{F}) \) is well defined on \( \Sigma \), we consider

\[
(\exp(i\hat{F}) \otimes 1)\Psi = \left( 1 \otimes \exp \left( -\frac{2\pi i}{\Lambda} \sum r \right) \right) \Psi,
\]

(2.10)

where \( \Psi \) expresses the state in \( \Sigma \). From this, we have

(iv) \( b_i = -2m_i \mod k \).

(2.11)

All we need to consider in number counting is (i)–(iv).

3. Number counting

Here, we consider number counting based on the ABCK framework. If we use (iii) and (iv), we obtain

(iii)’ \( \sum_{i=1}^{n} m_i = n' \frac{k}{2} \)

(3.1)

In [8], it was shown that this condition is irrelevant in number counting. Thus, we perform number counting concentrating only on (i) and (ii) below. There are two opinions in number counting. The one adopted in the original paper [2, 7, 8] counts the surface freedom \( (b_1, b_2, \ldots, b_n) \). The second counts the freedom for both \( j \) and \( m \) [10, 11, 13]. Here, for simplicity, we base our argument mainly on the second possibility. We can perform this calculation in a manner quite analogous to [13]. The only point for which care must be taken is that we should include the condition \( m_i > 0 \) (or equivalently \( b_i < 0 \)). Thus, each \( j_i \) has freedom \( j_i \) for the integer and the \( j_i + 1/2 \) way for the half-integer. They are summarized as \( \left[ \frac{2j_i + 1}{2} \right] \), where \( \left[ \cdot \right] \) is the integer part. We define \( N(a) := \frac{\Lambda}{2\pi \gamma} \), which is the number of states accounting for the entropy, as

\[
N(a) := \left\{ (j_1, \ldots, j_n) | 0 \neq j_i \in \frac{N}{2}, \sum j_i(j_i + 1) \leq k = a \right\}.
\]

(3.2)

Then, we obtain the recursion relation

\[
N(a) = N \left( a - \frac{\sqrt{3}}{4} \right) - 1 + \left\{ N \left( a - \frac{\sqrt{2}}{2} \right) - 1 \right\}
+ \cdots + \left[ \frac{2j_i + 1}{2} \right] \left\{ N \left( a - \frac{\sqrt{j_i(j_i + 1)}}{2} \right) - 1 \right\} + \cdots + \left[ \sqrt{16a^2 + 1} \right].
\]

(3.3)

The factor \( \left[ \frac{2j_i + 1}{2} \right] \) in this formula comes directly from the freedom of \( j_i \). If we use the relation

\[
N(a) = Ce^{\frac{\gamma_M}{\Lambda}} \gamma_M,
\]

(3.4)

where \( C \) is a constant, we obtain

\[
1 = \sum_{j_i = \frac{N}{2}} \left[ \frac{2j_i + 1}{2} \right] \exp(-\pi \gamma_M \sqrt{j_i(j_i + 1)})
\]

(3.5)

by plugging (3.4) into (3.3) and taking the limit \( \Lambda \to \infty \). Then if we require \( S = \Lambda/4 \), we have \( \gamma = \gamma_M = 0.367 \cdots \). Quite analogously, if we count the surface freedom, we obtain

\[
1 = \sum_{j = \frac{N}{2}} \exp(-\pi \gamma_M \sqrt{j(j + 1)}),
\]

(3.6)

where we have \( \gamma_M = 0.323 \cdots \). Importantly, these are larger values compared with those of [7–13].
4. Conclusion and discussion

In this paper, we have considered condition (1.1) to derive the number of states of black holes in the ABCK framework. As a result, we obtained two values of the Barbero–Immirzi parameter according to how the freedom is counted. Importantly, these are larger values compared with previous results [7–13].

What are the consequences of this result? The first one is related to cosmology [23]. If we assume an isotropic and homogeneous universe, we can write the effective inverse cube of the scale factor as

\[(a^{-3})_{\text{eff}} = a^{-3} f(q)^{2/(1-2l)},\]

where

\[f(q) = 3q^{-1/l} (q + 1)^{l+1} - |q - 1|^{l+1},\]

and where \(q = a^2/a^2_\ast\) \(a^2_\ast = \gamma \ell P_j/3\). \(j\) and \(l\) are the spin and the ambiguity in quantization.

We can argue for an inflationary scenario using the following scalar field effective action:

\[H_M = \frac{(a^{-3})_{\text{eff}}}{2} p_\phi^2 + a^3 V(\phi).\]

Thus, the result certainly depends on the value of the Barbero–Immirzi parameter. This would affect the thermal fluctuation in the universe, and might be seen by, e.g., PLANCK.

As other possibilities, we can also discuss particle velocity and its dependence on the Barbero–Immirzi parameter [24], or discuss the evaporation process of black holes [25]. Thus, the value of the Barbero–Immirzi parameter should also be certified in future experiments.

Acknowledgments

We would like to thank Hidefumi Nomura, Lee Smolin and Olaf Dreyer for useful discussion. The numerical calculations were carried out on the Altix3700 BX2 at YITP, Kyoto University. This work was partially supported by the 21st Century COE Program (Holistic Research and Education Center for Physics Self-Organization Systems) at Waseda University.

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