A computationally efficient method for sensitivity matrix calculation in magnetic induction tomography

Tianrui Chen and Jingbo Guo
Department of Electrical Engineering, Tsinghua University, Beijing 100084, China

1 E-mail: guojb@tsinghua.edu.cn

Abstract. Magnetic induction tomography (MIT) is a method of electromagnetic parameter reconstruction in the object region by measuring the magnetic field and the induced voltage. The governing equations for the forward problem in MIT is derived to apply FEM and the forward problem is solved. In order to solve the inverse problem, a fast algorithm for the calculation of the sensitivity matrix is derived and the numerical simulation is carried out. The conductivity distribution is reconstructed and it is verified that the presented algorithm is computationally efficient and correct.

1. Introduction
Magnetic induction tomography (MIT) [1] is a noncontact and non-destructive imaging technology, which is able to measure conductivity and other parameters. It can be widely applied in biological, medical, industrial, agricultural and other scientific research areas. The theoretical basis of magnetic induction imaging is eddy current in electromagnetic field. The sinusoidal ac current in the exciting coil generates a primary field. The primary field induces eddy currents in the target domain, and the eddy currents then produce a secondary field. The total time-varying field produces induction voltages in the detection coils, and the induction voltages are measured, then the electromagnetic parameters of the measured object are obtained.

The solution process of magnetic induction tomography contains two problems: the forward problem and the inverse problem. Given the distribution of conductivity or other parameters with an excitation current, the forward problem calculates the estimated measurement signals from the sensors. Given the excitation current, results from the forward problem and measured data from detection coils, the inverse problem determines the distribution of the conductivity or other parameters in the field. The methodology of the inverse problem in MIT has been studied by various researchers, such as the convolution and back-projection method [2]. However, the sensitivity based inverse problem solver is the most common procedure in MIT. This paper derives the governing equations from the Maxwell equation for this electromagnetic field problem, and a fast calculation for the sensitivity matrix is established.

2. FEM based numerical solution for the forward problem
The aim of the forward problem in MIT is to estimate the induced voltages on the detection coils when the electromagnetic parameters are known and the excitation coils are excited with given currents. In fact, the forward problem of MIT is substantially a boundary value problem [3]. In this section, the governing equations are derived based on [4] and then FEM is applied to solve the forward problem in MIT.
2.1. Governing equations for the boundary value problem
As shown in figure 1, the whole target domain contains the eddy current region $\Omega_1$ and the non-conducting region $\Omega_2$. The interface between $\Omega_1$ and $\Omega_2$ is $\partial \Omega_1$ and the outer boundary of the whole region is $\partial \Omega$. The current source only exists in the non-conducting region.

![Figure 1. Schematic of the boundary value problem.](image)

We start with the differential form of Maxwell’s equations as below. Considering the sinusoidal steady state situation, the phasor form is used in the following discussion.

\[
\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \tag{1}
\]

\[
\nabla \times \mathbf{E} = -j\omega \mathbf{B} \tag{2}
\]

\[
\nabla \cdot \mathbf{B} = 0 \tag{3}
\]

\[
\nabla \cdot \mathbf{D} = \rho \tag{4}
\]

With the assumption of isotropic materials, we have the constitutive equations

\[
\mathbf{B} = \mu \mathbf{H} \tag{5}
\]

\[
\mathbf{D} = \varepsilon \mathbf{E} \tag{6}
\]

\[
\mathbf{J} = J_s + \sigma \mathbf{E} \tag{7}
\]

As the quasi-static field conditions are satisfied, the displacement current can be neglected. Substituting equation (5), (6) and (7) into equation (1), we have

\[
\nabla \times \left( \frac{1}{\mu} \mathbf{B} \right) = J_s + (\sigma + j\omega\varepsilon) \mathbf{E} \tag{8}
\]

Introducing the magnetic vector potential $\mathbf{A}$

\[
\mathbf{B} = \nabla \times \mathbf{A} \tag{9}
\]

Then the Maxwell’s second equation can be rewritten as

\[
\nabla \times (\mathbf{E} + j\omega \mathbf{A}) = 0 \tag{10}
\]

Introducing the electric scalar potential $U$

\[
\mathbf{E} + j\omega \mathbf{A} = -\nabla U \tag{11}
\]

Substituting equation (9) and (11) into equation (8), we have

\[
\nabla \times \left( \frac{1}{\mu} \nabla \times \mathbf{A} \right) + j\omega \mathbf{A} + \nabla (\gamma U) = J_s \tag{12}
\]

in which

\[
\gamma = \sigma + j\omega\varepsilon \tag{13}
\]

Taking the divergence of equation (12), we have

\[
j\omega \nabla \cdot (\gamma \mathbf{A}) + \nabla \cdot (\gamma \nabla U) = 0 \tag{14}
\]

In the eddy current region $\Omega_1$, the governing equations are as follows.
\[ \nabla \times \left( \frac{1}{\mu} \nabla \times A \right) + j\omega \sigma A + \sigma \nabla U = 0 \]  
\hspace{1cm} (15)

\[ j\omega \nabla \times (\sigma A) + \nabla \times (\sigma \nabla U) = 0 \]  
\hspace{1cm} (16)

In the non-conducting region \( \Omega_2 \),

\[ \nabla \times \left( \frac{1}{\mu} \nabla \times A \right) = J_s \]  
\hspace{1cm} (17)

According to the field theory, a vector field’s divergence, as well as curl, needs to be defined in order to determine it uniquely. Here we use the Coulomb gauge \( \nabla \cdot A = 0 \) and substitute the well-known vector identity \( \nabla \times \nabla \times M = \nabla (\nabla \cdot M) - \nabla^2 M \) into equation (12), then

\[ \nabla^2 A + \nabla \ln \mu \times \nabla \times A - \mu \gamma (j\omega A + \nabla U) = J_s \]  
\hspace{1cm} (18)

If we concentrate on the conductivity and permittivity, and assume that the variation of permeability to be pretty small, which is true in terms of most bio tissues, the second term on the left end of equation (12) can be neglected. Besides, the primary magnetic field is far greater than the secondary field, thus the gradient of electric potential can also be neglected. The simplified version of (18) is as follows.

\[ \frac{1}{\mu} \nabla^2 A = j\omega \gamma A - J_s \]  
\hspace{1cm} (19)

2.2. The FEM solver for the forward problem

The basic thoughts of finite element method is that it converts the original differential equations into the variation of a functional equivalently, then uses field discretization and local interpolation to transfer the original problem into a matrix equation problem to get the answer.

Using equation (19) as an example, the equivalent variation problem is as follows.

\[ J(A) = \frac{1}{2} \int \left( \frac{1}{\mu} \nabla^2 A + j\omega \gamma A^2 - 2J_s A \right) dv = \min \]  
\hspace{1cm} (20)

It is hard to solve equation (20) directly because the integration cannot be intuitively calculated. We divide the whole region into finite small elements, and set a local interpolation function in every element, then we have

\[ A = \sum_{i=1}^{n_e} \left( N_i^e A_i \right) \]  
\hspace{1cm} (21)

in which the index \( i \) indicates nodes in an element, \( N_i^e \) is the \( i \)th node’s shape function of the element \( e \). Substituting equation (21) into equation (20), we get the multivariate function form of the original problem. Let all the partial derivative equal to zero and solve the matrix equation, then the answer is achieved. Other field quantities and potentials can be obtained through the basic equations.

3. Fast Calculation of sensitivity matrix

If a function \( U = F(\gamma) \) can be established from the forward problem, we are expecting to get \( \gamma = F^{-1}(U) \) from it so that we can solve the inverse problem. However, considering the ill-conditioning of the inverse problem, this inverse function can hardly be calculated directly. Take the Taylor expansion of \( U = F(\gamma) \) at point \( \gamma = \gamma_0 \), we get

\[ U = F(\gamma) \approx F(\gamma_0) + \frac{\partial F}{\partial \gamma} \left( \gamma - \gamma_0 \right) + O\left( \| \gamma - \gamma_0 \|_2^2 \right) \]  
\hspace{1cm} (22)

We call \( S = \frac{\partial F}{\partial \gamma} \) the sensitivity matrix.
Hence $\Delta U = S \Delta y$.

The straightforward way of calculating the sensitivity matrix is to add a small disturbance to every voxel and record the voltage change in every coil. However, this is such a time consuming process that can hardly be used in practice. The calculation of sensitivity matrix needs some simplification.

Let us carry on two measurements on the same testing region [5], with index 1 and 2. In the first measurement, coil 1 is excited and in the second measurement coil 2 is excited. For two pairs of electric field intensity and magnetic field intensity generated by sources in finite space, we have

$$\int (J_{s1} \cdot E_2 - J_{s2} \cdot E_1 + (\gamma_1 - \gamma_2) E_1 \cdot E_2 + j \omega (\mu_1 - \mu_2) H_1 \cdot H_2) \, dv = 0$$

(25)

It is readily seen that

$$\int J_{s1} \cdot E_2 \, dv - \int J_{s2} \cdot E_1 \, dv = -\int (\gamma_1 - \gamma_2) E_1 \cdot E_2 \, dv + \int j \omega (\mu_1 - \mu_2) H_1 \cdot H_2 \, dv$$

(26)

The first term on the left end of equation (26) is the integral of $E$ along the path of coil 1 in the second measurement, which equals to the induced voltage on coil 1 multiplied by $I_1$. And the second term has the same physical meaning. Hence

$$\Delta Z = - \int \delta y \frac{E_1 \cdot E_2}{I_1 I_2} \, dv + \int j \omega \delta \mu \frac{H_1 \cdot H_2}{I_1 I_2} \, dv$$

(27)

Using the discrete form, the integral turns into sum, neglecting the permeability term, and we can calculate the element in the sensitivity matrix as follows.

$$S_{i-j,k} = \frac{\partial V_{ij}}{\partial \gamma_k} = - \frac{1}{I_{i} I_{j}} \left( \int_{\Omega_k} E_i \cdot E_j \, dv \right)$$

(28)

in which $i - j$ is the index of excitation-detection coil pair, and $k$ is the index of the finite element. Accordingly, $E_i$ and $E_j$ are the electric field intensity when the $i$th and $j$th coil are excited respectively.

The solutions to the forward problem are the potentials rather than field intensities, so we substitute equation (11) into equation (28) and get

$$S = \frac{\partial V_{ij}}{\partial \gamma_k} = - \int_{\Omega_k} \frac{j \omega A_i + \nabla U_i}{I_i} \cdot \frac{j \omega A_j + \nabla U_j}{I_j} \, dv$$

(29)

If we take into account the electro quasi-static field condition, which leads to $E \approx -j \omega A$, and substitute equation (21) into equation (29), then equation (29) can be simplified as follows.

$$S = \frac{\partial V_{ij}}{\partial \sigma_k} = - \frac{\omega^2}{I_i I_j} \{A_i^T \left[ \int_{\Omega_k} N_i^T \{N_j \} \, dv \right] \{A_j^T \}}$$

(30)

So far, we have obtained a fast calculation for sensitivity matrix in MIT. This procedure only takes two solutions of the forward problems to get a row of the sensitivity matrix, which significantly improves the efficiency, in contrast with the empirical approach.

4. Numerical simulation

In order to verify the validity of the algorithm presented above, we run a numerical simulation to see whether the conductivity in the object region can be correctly reconstructed.

The geometrical and electromagnetic parameters are described as follows. The whole object region is a cylinder with height 240mm and radius 180mm. The eight excitation and detection coils are...
perpendicular to the X-Y plane and uniformly distributed around the cylinder. The distance from the center of the coils to the center of the cylinder is 200mm. The object to be detected is a sphere with radius 50mm in the center of the cylinder. The model of the numerical simulation is shown in figure 2.

The material of the coils is set to copper. To simulate the situation close to real bio tissues, the background cylinder is set with conductivity 1.2S/m, relative permittivity 80 and relative permeability 1, according to Gabriel S et al [6]. The sphere is set with conductivity 1.4S/m, relative permittivity 80 and relative permeability 1. The current source is set to 1A with frequency 10MHz according to Ramli S et al [7]. We use finite element software to solve the forward problem and write a program to obtain the sensitivity matrix.

![Figure 2. Model of the numerical simulation.](image)

As for inverse problem, we apply a Tikhonov regularized [8] Newton one step error reconstruction [9] method to reconstruct the conductivity distribution on the X-Y plane. We use a color image to demonstrate the reconstructed conductivity distribution (relative value), which is shown in figure 3.

![Figure 3. Color image of the reconstructed distribution of conductivity.](image)

As shown above, there is a circular area in the center of figure 3, the radius of which is about 1/3.5 of the outer circle boundary’ radius. The actual ratio between the radius of the inner object and the outer boundary is 50/180, which approximatively equals to 1/3.5. This means the reconstruction of conductivity distribution is effective and the calculation of sensitivity matrix is correct.

5. Conclusions
This paper derives the governing equations for the forward problem in MIT, applies FEM to solve the forward problem and presents a fast calculation of sensitivity matrix. A numerical simulation is carried out and verifies the correctness of the formulation presented. Compared with the cumbersome of the empirical approach, the advantage of the algorithm is the computational efficiency. Fast and still with
sufficient precision, this approach of calculating the sensitivity matrix can be appropriately used in MIT to act as a bridge between the forward problem and the inverse problem.

Acknowledgement
This work was supported by the National Key R&D Program of China (Grant No. 2017YFC0805803).

References
[1] Al-Zeibak S and Saunders N H 1993 A feasibility study of in vivo electromagnetic imaging Physics in Medicine & Biology 38 151-60
[2] Korzhenevskii A V and Cherepenin V A 1997 Magnetic induction tomography Journal of Communications technology and electronics 42 506-12
[3] Merwa R, Hollaus K, Brandstätter B and Scharfetter H 2003 Numerical solution of the general 3D eddy current problem for magnetic induction tomography (spectroscopy) Physiological Measurement 24 545-54
[4] Bras N B, Martins R C, Serra A C and Ribeiro A L 2010 A Fast Forward Problem Solver for the Reconstruction of Biological Maps in Magnetic Induction Tomography IEEE Transactions on Magnetics 46 1193-202
[5] Mortarelli J R 1980 A generalization of the Geselowitz relationship useful in impedance plethysmographic field calculations IEEE Transactions on Biomedical Engineering 27 665-7
[6] Gabriel S, Lau R W and Gabriel C 1996 The dielectric properties of biological tissues: II. measurements in the frequency range 10 Hz to 20 GHz Physics in Medicine & Biology 41 2251-69
[7] Griffiths H, Stewart W R and Gough W 1999 Magnetic induction tomography a measuring system for biological tissues Annals of the New York Academy of Sciences 873 335-45
[8] Calvetti D, Morigi S, Reichel L and Sgallari F 2000 Tikhonov regularization and the L-curve for large discrete ill-posed problems Journal of Computational & Applied Mathematics 123 423-46
[9] Cheney M, Isaacson D, Newell J C Simske S and Goble J 1990 NOSER: An algorithm for solving the inverse conductivity problem International Journal of Imaging Systems & Technology 2 66-75