IRRATIONAL NUMBERS ASSOCIATED TO SEQUENCES WITHOUT GEOMETRIC PROGRESSIONS

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Abstract. Let \( s \) and \( k \) be integers with \( s \geq 2 \) and \( k \geq 2 \). Let \( g_k^{(s)}(n) \) denote the cardinality of the largest subset of the set \( \{1, 2, \ldots, n\} \) that contains no geometric progression of length \( k \) whose common ratio is a power of \( s \). Let \( r_k(\ell) \) denote the cardinality of the largest subset of the set \( \{0, 1, 2, \ldots, \ell - 1\} \) that contains no arithmetic progression of length \( k \). The limit
\[
\lim_{n \to \infty} \frac{g_k^{(s)}(n)}{n} = (s - 1) \left( 1 - \frac{1}{s} \right) \min \left( r_k^{-1}(m) \right)
\]
exists and converges to an irrational number.

1. Maximal subsets without geometric progressions

Let \( \mathbb{N} \) denote the set of positive integers. For every real number \( x \), the integer part of \( x \), denoted \( \lfloor x \rfloor \), is the unique integer \( n \) such that \( n \leq x < n + 1 \).

Let \( s \geq 2 \) be an integer. Every positive integer \( a \) can be written uniquely in the form
\[
a = bs^v
\]
where \( b \) is a positive integer not divisible by \( s \) and \( v \) is a nonnegative integer. If \( G \) is a finite geometric progression of length \( k \) whose common ratio is a power of \( s \), say, \( s^d \), then
\[
G = \{ a \, (s^d)^j : j = 0, 1, \ldots, k - 1 \}. 
\]
Writing \( a \) in the form \( a = bs^v \), we have
\[
(1) \quad G = \{ bs^{v + dj} : j = 0, 1, \ldots, k - 1 \} \subseteq \{ bs^i : i \in \mathbb{N}_0 \}
\]
and so the set of exponents of \( s \) in the finite geometric progression \( G \) is the finite arithmetic progression \( \{ v + dj : j = 0, 1, \ldots, k - 1 \} \). Conversely, if \( P \) is a finite arithmetic progression of \( k \) nonnegative integers and if \( b \) is a positive integer not divisible by \( s \), then \( \{ bs^i : i \in P \} \) is a geometric progression of length \( k \).

Let \( \ell \) and \( k \) be positive integers with \( k \geq 2 \). Let \( r_k(\ell) \) denote the cardinality of the largest subset of the set \( \{0, 1, 2, \ldots, \ell - 1\} \) that contains no arithmetic progression of length \( k \). Note that \( r_k(\ell) = \ell \) for \( \ell = 1, \ldots, k - 1 \), that \( r_k(k) = k - 1 \), and that, for every \( \ell \in \mathbb{N} \), there exists \( \varepsilon_\ell \in \{0, 1\} \) such that
\[
r_k(\ell + 1) = r_k(\ell) + \varepsilon_\ell.
\]

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Thus, the function $r_k : \mathbb{N} \to \mathbb{N}$ is nondecreasing and surjective. This implies that, for every positive integer $m$, the set

$$r_k^{-1}(m) = \{ \ell \in \mathbb{N} : r_k(\ell) = m \}$$

is a nonempty set of consecutive integers, and so

$$\max (r_k^{-1}(m)) + 1 = \min (r_k^{-1}(m + 1))$$

and

$$\min (r_k^{-1}(m)) \geq m$$

for all $m \in \mathbb{N}$.

**Lemma 1.** Let $u_m = \min (r_k^{-1}(m))$ for $m \in \mathbb{N}$. The sequence $(u_m)_{m=1}^{\infty}$ is a strictly increasing sequence of positive integers such that

$$\limsup_{m \to \infty} (u_{m+1} - u_m) = \infty.$$

**Proof.** Identity (2) implies that the sequence $(u_m)_{m=1}^{\infty}$ is strictly increasing. We use Szemerédi’s theorem, which states that $r_k(\ell) = o(\ell)$, to prove that the sequence $(u_m)_{m=1}^{\infty}$ has unbounded gaps.

Note that $u_1 = 1$. If $\limsup_{m \to \infty} (u_{m+1} - u_m) < \infty$, then there is an integer $c \geq 2$ such that $u_{m+1} - u_m < c$ for all $m \in \mathbb{N}$. It follows that

$$\max (r_k^{-1}(m)) + 1 = \min (r_k^{-1}(m + 1)) = u_{m+1} = \sum_{i=1}^{m} (u_{i+1} - u_i) + u_1 < cm + 1.$$

Thus, $\max (r_k^{-1}(m)) < cm$ and so $r_k(cm) > m$. Equivalently,

$$\frac{r_k(cm)}{cm} > c > 0$$

and

$$\liminf_{\ell \to \infty} \frac{r_k(\ell)}{\ell} \geq c > 0.$$  

This contradicts Szemerédi’s theorem, and completes the proof. \hfill \square

For $k \geq 2$, let $g_k(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length $k$. Rankin [4] introduced this function, and it has been investigated by M. Beiglböck, V. Bergelson, N. Hindman, and D. Strauss [1], by Brown and Gordon [2], and by Riddell [5]. The best upper bound for the function $g_k(n)$ is due to Nathanson and O’Bryant [3].

For $s \geq 2$ and $k \geq 2$, let $g_k^{(s)}(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length $k$ whose common ratio is a power of $s$. We shall prove that the limit

$$\lim_{n \to \infty} \frac{g_k^{(s)}(n)}{n} = (s - 1) \sum_{m=1}^{\infty} \left( \frac{1}{s} \right)^{\min (r_k^{-1}(m))}$$

exists and converges to an irrational number.
2. Maximal geometric progression free sets

**Lemma 2.** If \( k \) and \( s \) are integers with \( k \geq 2 \) and \( s \geq 2 \), then

\[
g_k^{(s)}(n) = \sum_{b \in \mathcal{B}_n} r_k \left( 1 + \left\lfloor \log_s(n/b) \right\rfloor \right).
\]

**Proof.** Let \( n \) be a positive integer, and let

\[\mathcal{B}_n = \{b \in \{1,2,\ldots,n\} : s \text{ does not divide } b\}.
\]

If \( b \in \mathcal{B}_n \) and \( i \in \mathbb{N}_0 \), then \( bs^i \leq n \) if and only if \( 0 \leq i \leq \log_s(n/b) \). We define

\[T(b) = \{t \in \{1,2,\ldots,n\} : t = bs^i \text{ for some } i \in \mathbb{N}_0\} = \{bs^i : i = 0,1,\ldots,\left\lfloor \log_s(n/b) \right\rfloor\}.
\]

Then \( b \in T(b) \), and

\[\{1,2,\ldots,n\} = \bigcup_{b \in \mathcal{B}_n} T(b)
\]

is a partition of \( \{1,2,\ldots,n\} \) into pairwise disjoint nonempty subsets.

If the set \( \{1,2,\ldots,n\} \) contains a finite geometric progression of length \( k \) whose common ratio is a power of \( s \), then, by (1), this geometric progression is a subset of \( T(b) \) for some \( b \in \mathcal{B}_n \), and the set of exponents of \( s \) is a finite arithmetic progression of length \( k \) contained in the set of consecutive integers \( \{0,1,\ldots,\left\lfloor \log_s(n/b) \right\rfloor\} \). It follows that the largest cardinality of a subset of \( T(b) \) that contains no \( k \)-term geometric progression is equal to the largest cardinality of a subset of \( \{0,1,\ldots,\left\lfloor \log_s(n/b) \right\rfloor\} \) that contains no \( k \)-term arithmetic progression. This number is

\[r_k \left( 1 + \left\lfloor \log_s(n/b) \right\rfloor \right).
\]

If \( A_n \) is a subset of \( \{1,2,\ldots,n\} \) of maximum cardinality that contains no \( k \)-term geometric progression whose common ratio is a power of \( s \), then

\[|A_n \cap T(b)| = r_k \left( 1 + \left\lfloor \log_s(n/b) \right\rfloor \right).
\]

Because \( A = \bigcup_{b \in \mathcal{B}_n} T(b) \) is a partition of \( \{1,\ldots,n\} \), it follows that

\[|A_n| = \sum_{b \in \mathcal{B}_n} |A_n \cap T(b)| = \sum_{b \in \mathcal{B}_n} r_k \left( 1 + \left\lfloor \log_s(n/b) \right\rfloor \right).
\]

This completes the proof. \( \square \)

3. Construction of an irrational number

**Lemma 3.** Let \( s \) be an integer with \( s \geq 2 \). Let \( x \) and \( y \) be real numbers with \( x < y \). The number of integers \( n \) such that \( x < n \leq y \) and \( s \) does not divide \( n \) is

\[
\left( \frac{s-1}{s} \right)(y-x) + O(1).
\]

**Proof.** For every real number \( x \), the interval \((x,x+s]\) contains exactly \( s \) integers. These integers are consecutive, so \((x,x+s]\) contains exactly \( s-1 \) integers not divisible by \( s \). Let \( x \) and \( y \) be real numbers with \( x < y \), and let

\[h = \left[ \frac{y-x}{s} \right].
\]

Then

\[x + hs \leq y < x + (h+1)s.
\]
The interval \((x, x + hs]\) contains exactly \((s - 1)h\) integers not divisible by \(s\), and the interval \((x, x + (h + 1)s]\) contains exactly \((s - 1)(h + 1)\) integers not divisible by \(s\). If \(N\) denote the number of integers in the interval \([x, y]\) that are not divisible by \(s\), then

\[(s - 1)h \leq N \leq (s - 1)(h + 1)\]

and so

\[
\frac{y - x}{s} - 1 < h \leq \frac{N}{s - 1} \leq h + 1 \leq \frac{y - x}{s} + 1.
\]

Equivalently,

\[
\left(\frac{s - 1}{s}\right)(y - x) - (s - 1) < N \leq \left(\frac{s - 1}{s}\right)(y - x) + s - 1.
\]

This completes the proof. \(\square\)

**Theorem 1.** Let \(k\) and \(s\) be integers with \(k \geq 2\) and \(s \geq 2\). The limit

\[
\lim_{n \to \infty} \frac{g_k^{(s)}(n)}{n} = (s - 1) \sum_{m=1}^{\infty} \left(\frac{1}{s}\right)^{\min\left(r_k^{-1}(m)\right)}
\]

exists and converges to an irrational number.

**Proof.** For every positive integer \(b\) we have

\[1 + \lfloor \log_s(n/b) \rfloor = \ell\]

if and only if

\[
\frac{n}{s^\ell} < b \leq \frac{sn}{s^\ell}.
\]

By Lemma 3, the number of integers in this interval that are also in \(B_n\), that is, are not divisible by \(s\), is

\[
\left(\frac{s - 1}{s}\right) \frac{(s - 1)n}{s^\ell} + O(1) = \frac{n(s - 1)^2}{s^{\ell+1}} + O(1).
\]

Because \(1 \in B_n\), we have

\[L = L(n) = \max \{1 + \lfloor \log_s(n/b) \rfloor : b \in B_n\} = 1 + \lfloor \log_s n \rfloor.\]
Also, if $\ell \leq L$, then $r_k(\ell) \leq \ell \leq L$. By Lemma 2,

$$g_k^{(s)}(n) = \sum_{b \in B_n} r_k(1 + \lfloor \log_s(n/b) \rfloor)$$

$$= \sum_{\ell=1}^L r_k(\ell) \times \lvert \{b \in B_n : \ell = 1 + \lfloor \log_s(n/b) \rfloor \} \rvert$$

$$= \sum_{\ell=1}^L r_k(\ell) \left( \frac{n(s-1)^2}{s^\ell + 1} + O(1) \right)$$

$$= \frac{n(s-1)^2}{s} \sum_{\ell=1}^L \frac{r_k(\ell)}{s^\ell} + O \left( \sum_{\ell=1}^L r_k(\ell) \right)$$

$$= \frac{n(s-1)^2}{s} \sum_{\ell=1}^L \frac{r_k(\ell)}{s^\ell} + O(L^2)$$

$$= n \left( \frac{(s-1)^2}{s} \sum_{\ell=1}^L \frac{r_k(\ell)}{s^\ell} + O \left( \frac{\log^2 n}{n} \right) \right)$$

$$= n \left( \frac{(s-1)^2}{s} \sum_{\ell=1}^L \frac{r_k(\ell)}{s^\ell} + o(1) \right).$$
Let \( M = M(n) = r_k(L(n)) \). We have

\[
\sum_{\ell=1}^{L} \frac{r_k(\ell)}{s^{\ell}} = \sum_{m=1}^{M-1} \sum_{\ell \in r_k^{-1}(m)} \frac{1}{s^{\ell}} + M \sum_{\ell \in r_k^{-1}(m) \cap \{1, \ldots, L\}} \frac{1}{s^{\ell}} \\
= \sum_{m=1}^{M-1} \sum_{\ell = \min(r_k^{-1}(m))}^{\max(r_k^{-1}(m))} \frac{1}{s^{\ell}} + M \sum_{\ell = \min(r_k^{-1}(m))}^{L} \frac{1}{s^{\ell}} \\
= \frac{s}{s-1} \sum_{m=1}^{M-1} m \left( \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))} - \left( \frac{1}{s} \right)^{\max(r_k^{-1}(m)) + 1} \right) \\
+ \frac{s}{s-1} M \left( \left( \frac{1}{s} \right)^{\min(r_k^{-1}(M))} - \left( \frac{1}{s} \right)^{L} \right) \\
= \frac{s}{s-1} \sum_{m=1}^{M-1} m \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))} - \sum_{m=2}^{M-1} \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m+1))} \\
+ \frac{s}{s-1} M \left( \frac{1}{s} \right)^{\min(r_k^{-1}(M))} - \frac{sM}{s-1} \left( \frac{1}{s} \right)^{L} \\
= \frac{s}{s-1} \sum_{m=1}^{M} \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))} + o(1)
\]

because

\[
\frac{sM}{s-1} \left( \frac{1}{s} \right)^{L} \ll \frac{M}{s^{L}} \ll \frac{M}{s^{\min(r_k^{-1}(M))}} \ll \frac{M}{s^{M}}
\]

by inequality (3). Therefore,

\[
\frac{g_k^{(s)}(n)}{n} = \frac{(s-1)^2}{s} \left( \frac{s}{s-1} \sum_{m=1}^{M} \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))} + o(1) \right) + o(1) \\
= \left( s-1 \right) \sum_{m=1}^{M} \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))} + o(1)
\]

and so

\[
\lim_{n \to \infty} \frac{g_k^{(s)}(n)}{n} = \left( s-1 \right) \sum_{m=1}^{\infty} \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))}.
\]

The infinite series converges to a real number \( \theta \in (0, 1) \), and the “decimal digits to base \( s \)” of \( \theta \) are 0 or 1. The number \( \theta \) is rational if and only if these digits are eventually periodic, but Lemma 1 implies that there are unbounded gaps between successive digits equal to 1. Therefore, \( \theta \) is irrational. This completes the proof. \( \square \)
4. Open problems

(1) Let $k$ and $s$ be integers with $k \geq 2$ and $s \geq 2$. Is the number

$$\lim_{n \to \infty} \frac{g_k^{(s)}(n)}{n} = (s - 1) \sum_{m=1}^{\infty} \left( \frac{1}{s} \right)^{\min(r_k^{-1}(m))}$$

transcendental?

(2) Let $u_m = \min(r_k^{-1}(m))$ for $m \in \mathbb{N}$. Prove that the sequence $(u_m)_{m=1}^{\infty}$ is not eventually periodic without using Szemerédi’s theorem.

(3) Let $s$ and $s'$ be integers with $2 \leq s < s'$. Is it true that $g_k^{(s')}(n) \leq g_k^{(s)}(n)$ for all $n \in \mathbb{N}$ and that $g_k^{(s')}(n) < g_k^{(s)}(n)$ for all sufficiently large $n \in \mathbb{N}$?

(4) Let $\mathcal{S}$ be a finite set of integers such that $s \geq 2$ for all $s \in \mathcal{S}$. For $k \geq 2$, let $g_k^{(\mathcal{S})}(n)$ denote the cardinality of the largest subset of the set $\{1, 2, \ldots, n\}$ that contains no geometric progression of length $k$ whose common ratio is a power of $s$ for some $s \in \mathcal{S}$. Does

$$\lim_{n \to \infty} \frac{g_k^{(\mathcal{S})}(n)}{n}$$

exist? If so, can this limit be expressed by an infinite series analogous to (4)?

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