LOₚ-Calculus: A Graphical Language for Linear Optical Quantum Circuits

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Abstract
We introduce the LOₚ-calculus, a graphical language for reasoning about linear optical quantum circuits with so-called vacuum state auxiliary inputs. We present the axiomatics of the language and prove its soundness and completeness: two LOₚ-circuits represent the same quantum process if and only if one can be transformed into the other with the rules of the LOₚ-calculus. We give a confluent and terminating rewrite system to rewrite any polarisation-preserving LOₚ-circuit into a unique triangular normal form, inspired by the universal decomposition of Reck et al. (1994) for linear optical quantum circuits.

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1 Introduction
Quantum computing and information processing promise a variety of advantages over their classical analogues, from the potential for computational speedups (e.g. [30, 18]) to enhanced security and communication (e.g. [7, 20]). By encoding information into the states of physical systems that are quantum rather than classical, one can then process that information by evolving and manipulating the systems according to the laws of quantum mechanics. This opens up the possibility of exploiting non-classical behaviours available to quantum systems in order to process information in radically new and potentially advantageous ways.

The development of quantum technologies has proceeded at pace over the past number of years, with a variety of different physical supports for quantum information being pursued. These include matter-based systems like superconducting circuits, cold atoms, and trapped ions, as well as light-based systems, in which information is encoded in photons. Among
these, photons have a privileged role in the sense that regardless of hardware choice it will eventually be necessary to network quantum processors, and (as the only sensible support for communicating quantum information) some quantum information will need to be treated photonically. Yet, in their own right, photons also offer viable approaches to quantum computation in the noisy intermediate-scale [37] and large-scale fault-tolerant [6] regimes.

The standard unit of quantum information is the quantum bit or qubit, and photons allow for a rich variety of ways to encode qubits. However it is also interesting to note that treating photons as informational units in their own right can be advantageous. A good example is BosonSampling, originally proposed by Aaronson and Arkhipov [1], a computational task that is \#P-hard but which can be efficiently solved by interacting photons in an idealised generic linear-optical circuit in which no qubit encoding need be imposed. At present, along with Random Circuit Sampling [2, 9], this provides one of the two main routes to experimental demonstrations of quantum computational advantage [3, 52, 50, 51], in which quantum devices have been claimed to outperform classical supercomputing capabilities for specific tasks.

The usual semantics for quantum computation stemming from quantum mechanics is based on unitary matrices (or unitary operators in general) over Hilbert spaces. Although this faithfully models the extensional behaviour of a computation, it fails to address several key aspects that are of interest when considering the design and implementation of quantum algorithms. A first limitation is the intensional description of the computation: an algorithm or quantum computation in general consists of modular components that are composed and combined in specific way, and one wants to keep track of this information. One therefore needs a language for coding these. The other important aspect is the need to specify and verify the said code. Indeed, classically simulating a quantum process is a task that is exponentially costly in the size of the system, while running code on physical devices is expensive. If some limited testing techniques are available on quantum systems [26, 40], it is however highly desirable to be able to reason and prove the desired properties of the code upstream, and rely on formal methods. If text-based high-level languages oriented towards formal methods have successfully been proposed in the literature [29, 8, 34], we aim in this paper to explore a lower-level, graphical language, making contact with photonic hardware.

Graphical languages for quantum computation have a long history: since Feyman diagrams [27], graphical languages for representing (low-level) quantum processes have been considered as an answer to the limitations of plain unitary matrices. Quantum circuits – the quantum equivalent to classical, boolean circuits – are an obvious candidate for a graphical language, and indeed, several lines of research took them as their main object of study [29, 20, 53, 13]. Quantum circuits in particular form a natural medium for describing the execution flow of a computation. The main problem with the model of quantum circuits is the lack of a satisfactory equational presentation. If several attempts have been made for various subsets [18, 17, 33, 42], none of them provides a complete presentation.

A recent proposal responding to the shortfalls of quantum circuits as a model is the ZX-calculus [19], which, along with its variants [11, 4, 12], have proved to be particularly useful for reasoning about qubit quantum mechanics, for applications such as quantum circuit optimisation [22, 5], verification [23, 25, 52] and representation e.g. for MBQC patterns [24] or error-correction [24, 21]. However, while ZX-calculus is versatile and provides a welcomed formal semantics for quantum computation, it remains at an abstract level.

There is therefore a clear interest in developing a graphical language for quantum photonic processes, especially linear quantum optics, which is closer to photonic hardware and laboratory operations that are easily implementable in bulk optics, fibres, or in integrated
photonic circuits. This would provide a more formal counterpart to software frameworks that have been proposed for defining and classically simulating such processes to the extent that it is tractable \[36, 31\]. The need for such a formal language is also evidenced, for example, by the appeal to diagrams to concisely illustrate equivalent unitaries in recent work in the Physics literature \[45\]. Following on the trend for graphical quantum languages, the PBS-calculus \[14, 10, 15\] has been proposed as a first step towards an alternative to ZX dedicated to linear quantum optical computation (LOQC). The PBS-calculus makes it possible to reason on a small subset of linear optical components only acting on the polarisation of a photon. While it is enough to describe and analyse non causally-ordered computations, it fails short at expressing other aspects of LOQC typically considered in the Physics community, such as the phase.

Our goal here is to take a more bottom-up approach and to propose a new language which formalises the kinds of diagrammatics that are currently in use in the Physics community. In practice this can find many uses including for the design, optimisation, verification, error-correction, and systematic study of linear optical quantum circuits for quantum information.

Our main contributions are the following.

- A graphical language for LOQC featuring most of the physical apparatuses used in the Physics literature. The language comes equipped with an equational theory that is sound and complete with respect to the standard semantics of LOQC.
- A strongly normalising and globally confluent rewrite system and normal form for the polarisation-preserving fragment, for which we recover the Reck et al. \[46\] decomposition as normal form (modulo 0-angled beam splitters and 0-angled phase shifters) with a novel proof of its uniqueness.

Finally, and maybe more importantly, our language makes it possible to formalise and reason within a common framework on various presentations of LOQC stemming from parallel research paths. Our semantics not only allow us to recover, extend and improve on some key results in LOQC such as the universal decompositions of Reck et al. \[46\] and Clements et al. \[16\], but it also gives a unifying language for the different formalisms from the literature.

The article is structured as follows. In Section 2 we present the syntax and the semantics of the \(LO_v\)-calculus. The equational theory and its soundness are given in Section 3. In Section 4 we present the strongly normalising and globally confluent rewrite system. This allows us to prove the completeness of the \(LO_v\)-calculus in Section 5. Finally, we conclude in Section 6.

2 Linear Optical Quantum Circuits

A linear optical quantum computation \[39, 38\] (LOQC) consists of spatial modes through which photons pass – which may be physically instantiated by optical fibers, waveguides in
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LOₜ-circuit implementing a variational quantum eigensolver [44], an algorithm with applications including calculation of ground-state energies in quantum chemistry.

Figure 2

Two equivalent representations of the same LOₜ-circuit.

integrated circuits, or simply by paths in free space (bulk optics) – and operations that act on the spatial and polarisation degrees of freedom of the photons, including in particular beam splitters (\(\square\)), polarising beam splitters (\(\square\)), phase shifters (\(\square\)), wave plates (\(\square\)), pola-negations (\(\square\)) and finally the vacuum state sources and detectors (\(\square\) and \(\square\)). Their action and the semantics are described in Section 2.2.

2.1 Syntax

In order to formalise linear optical quantum circuits, we use the formalism of PROPs [41]. A PRO is a strict monoidal category whose monoid of objects is freely generated by a single \(X\): the objects are all of the form \(X \otimes \ldots \otimes X\), and simply denoted by \(n\), the number of occurrences of \(X\). PROs are typically represented graphically as circuits: each copy of \(X\) is represented by a wire and morphisms by boxes on wires, so that \(\oplus\) is represented vertically and morphism composition “\(\circ\)” is represented horizontally. For instance, \(D_2 \circ D_1\) represented by \(\square\), and vertically composed as \(D_1 \oplus D_2\), represented by \(\square\). A PROP is the symmetric monoidal analogue of PRO, so it is equipped with a swap \(\square\).

Definition 1. LOₜ is the PROP of LOₜ-circuits generated by

\[
\begin{align*}
\theta & : 0 \to 1 \\
\varphi & : 1 \to 0 \\
\square & : 1 \to 1 \\
\square & : 1 \to 1 \\
\end{align*}
\]

where \(\theta, \varphi \in \mathbb{R}\). When the parameters \(\theta\) and \(\varphi\) are omitted we take them to be equal to \(\pi/4\). We write \(\square\) as a shortcut notation for \(\square\). The tensor of the monoidal structure is denoted with \(\otimes\), and the identity, swap and empty circuit (unit of \(\otimes\)) are denoted as follows: \(\square\) : 1 \(\rightarrow\) 1, \(\square\) : 2 \(\rightarrow\) 2, \(\square\) : 0 \(\rightarrow\) 0.

Example 2. An example of a linear optical quantum circuit using all of the connectives presented in Definition 1 is shown in Figure 2.
\[
\begin{bmatrix}
0 \\
-1
\end{bmatrix} = 
\begin{bmatrix}
-1 \\
0
\end{bmatrix} = 
\begin{bmatrix}
\vdots \\
\vdots
\end{bmatrix} = 0
\]
\[
\begin{bmatrix}
\theta
\end{bmatrix} = [c_p] \rightarrow \cos(\theta)|c_p \rangle + i \sin(\theta)|c_{1-p} \rangle
\]\n\[
\begin{bmatrix}
-\theta
\end{bmatrix} = 
\begin{bmatrix}
|V_0 \rangle \rightarrow \cos(\theta)|V_0 \rangle + i \sin(\theta)|H_0 \rangle \\
|H_0 \rangle \rightarrow \cos(\theta)|H_0 \rangle + i \sin(\theta)|V_0 \rangle
\end{bmatrix}
\]
\[
\begin{bmatrix}
\emptyset
\end{bmatrix} = 
\begin{bmatrix}
|c_0 \rangle \rightarrow e^{i\phi}|c_0 \rangle
\end{bmatrix}
\]

Table 1 Semantics of LO\_v-circuits.

\textbf{Remark 7.} The axioms of PROPs guarantee that linear optical quantum circuits are defined up to deformations: Figure 3 shows two equivalent circuits under the equations of PROPs.

Among the generators, the beam splitters and phase shifters are known to preserve the polarisation of the photons, as a consequence, we define a polarisation-preserving sub-PRO of LO\_v as follows.

\textbf{Definition 4.} LO\_PP is the PRO of polarisation-preserving circuits generated by beam splitters \(\begin{bmatrix}\theta\end{bmatrix}\) and phase shifters \(\begin{bmatrix}\emptyset\end{bmatrix}\).

Notice that we define polarisation-preserving circuits as a PRO rather than a PROP, thus they do not include swaps.

\subsection*{2.2 Single-Photon Semantics}

We will characterise photons by their spatial and polarisation modes. Spatial modes refer to position, and polarisation can be horizontal (H) or vertical (V). Note that the quantum formalism admits (normalised complex) superpositions of both spatial and polarisation modes. For any \(n \in \mathbb{N}\), let \(M_n = \{V, H\} \times [n]\), where \([n] = \{0, \ldots, n-1\}\), be the set of states (spatial and polarisation modes). The elements of \(M_n\) are denoted \(c_p\) with \(c \in \{V, H\}\) and \(p \in [n]\). The state space of a single photon is \(\mathbb{C}^{M_n} = \text{span}(\{|V_i\rangle, |H_i\rangle | i \in [n]\})\). Notice that \(\mathbb{C}^{M_0} = \mathbb{C}^0 = \{0\}\) is the Hilbert space of dimension 0. For instance, on 2 spatial modes (i.e. 2 wires), there are four possible basis states: \(H_0, H_1, V_0, V_1\). Indeed, a photon can be on one of the two wires, while in the horizontal or vertical polarisation. The state space is then a 4-dimensional Hilbert space. The semantics of a LO\_v-circuit is defined as follows.

\textbf{Definition 5.} For any LO\_v-circuit \(\mathcal{D} : n \rightarrow m\), let \([\mathcal{D}] : \mathbb{C}^{M_n} \rightarrow \mathbb{C}^{M_m}\) be the linear map inductively defined by Table 3 and by \([D_2 \circ D_1]\) = \([D_2]\) \(\circ\) \([D_1]\), \([D_1 \oplus D_2]\) = \([D_1]\) \(\oplus\) \([D_2]\), where for all \(f \in \mathbb{C}^{M_n} \rightarrow \mathbb{C}^{M_m}\) and \(g \in \mathbb{C}^{M_n'} \rightarrow \mathbb{C}^{M_{m'}}\), \((f \oplus g)(\|c_k\|) = f(\|c_k\|)\) if \(k < n\) and \(S_{n,m'}(g(\|c_{k-n}\|))\) if \(k \geq n\), with \(S_{n,m'} : \mathbb{C}^{M_{m'}} \rightarrow \mathbb{C}^{M_{m'+n}} = |c_k\rangle \rightarrow |c_{k+m}|\) a shift of the positions by \(m\).

\textbf{Example 6.} The negation inverts polarisation: \(\|\begin{bmatrix}\theta\end{bmatrix}\| : |V_0\rangle \mapsto |H_0\rangle\) and \(|H_0\rangle \mapsto |V_0\rangle\).

\textbf{Remark 7.} The semantics of the circuits is sound with respect to the axioms of PROPs. In other words two circuits that are equal up to deformation have the same semantics. More formally, \([\|\| : \text{LO}_v \rightarrow (\text{Hilb}, \oplus, 0)\) is a monoidal functor where \(\text{Hilb}\) is the category of state spaces \(\mathbb{C}^{M_n}\) and linear maps.

\footnote{There are many possible conventions for beam splitters. We have chosen this one as it is a symmetric operation with good composition properties (see Figure 9). The convention for the wave plate has been chosen for similar reasons (see Figure 12).}
Remark 8. All the generators of the LO$_v$-circuits are photon preserving, even the vacuum state sources (△) and detectors (□). Indeed the vacuum state source produces no photons, whereas the semantics of the detector corresponds to a postselection on the case where no photons are detected.

Definition 9. For any LO$_{PP}$-circuit $D : n \rightarrow n$, we define $[D]_{PP} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ as the unique linear map such that $[\iota] \circ \iota = [\iota]_{PP}$ where $\iota : \mathbb{C}^n \rightarrow \mathbb{C}^{Mn} = |k\rangle \mapsto |H_k\rangle$.

For instance $[\gamma \phi \chi]_{PP} = \begin{pmatrix} \cos(\theta) & i \sin(\theta) \\ i \sin(\theta) & \cos(\theta) \end{pmatrix}$.

Polarisation-preserving circuits are universal for unitary transformations, this is a direct consequence of the result of Reck et al. [46]. Unitary transformations can actually be uniquely represented by LO$_{PP}$-circuits, as illustrated by the following two cases on 2 and 3 modes, the general case being proved in Section 4.

Lemma 10. For any unitary $2 \times 2$ matrix $U$, there exist unique $\beta_1, \alpha_1 \in [0, \pi)$ and $\beta_2, \beta_3 \in [0, 2\pi)$ such that $[D]_{PP} = U$, and $\alpha_1 \in \{0, \frac{\pi}{2}\} \Rightarrow \beta_1 = 0$.

Proof. The proof is given in Appendix A.5.

Lemma 11. For any unitary $3 \times 3$ matrix $U$, there exist unique angles $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in [0, \pi)$ and $\beta_4, \beta_5, \beta_6 \in [0, 2\pi)$ such that $[D]_{PP} = U$ where $\forall i \in \{1, 2, 3\}, \alpha_i \in \{0, \frac{\pi}{2}\} \Rightarrow \beta_i = 0$, and where $\alpha_2 = 0 \Rightarrow \alpha_1 = 0$.

Proof. The existence of such a canonical form is shown in [46]. The uniqueness can then be derived by analysing the possible cases (See Appendix A.4).

LO$_v$-circuits are more expressive than LO$_{PP}$-ones, they not only act on the polarisation but the use of detectors and sources allow the representation of non-unitary evolutions: For any LO$_v$-circuit $D : n \rightarrow m$, $[D]$ is sub-unitary$^2$.

LO$_v$-circuits are actually universal for sub-unitary transformations:

Theorem 12 (Universality of LO$_v$). For every sub-unitary map $U : \mathbb{C}^{Mn} \rightarrow \mathbb{C}^{Mm}$ (i.e. such that $U^\dagger U \subseteq I$) there exists a diagram $D : n \rightarrow m$ s.t. $[D] = U$.

Proof. The proof given in Appendix A.13 relies on the normal forms developed in Section 5.

3 Equational Theory

Two distinct LO$_v$-circuits may represent the same quantum evolution: for instance, composing two negations is equivalent to the identity. In order to characterise equivalences of LO$_v$-circuits, we introduce a set of equations, shown in Figure 4. They capture basic properties of LO$_v$-circuits, such as: detectors and sources essentially absorbing the other generators (Equations 9 to 12); parameters forming a monoid (Equations 1 and 2); and various

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$^2$ $U$ is sub-unitary (see for instance [47]) iff $U^\dagger U \subseteq I$, where $\subseteq$ is the Löwner partial order, i.e. $I - U^\dagger U$ is a positive semi-definite.
\[ \phi_2 \phi_1 = \phi_1 \phi_2 \] (1)

\[ 0 = 0 \] (2)

\[ \theta_0 = \theta_0 \] (3)

\[ \alpha_1 \alpha_2 \alpha_3 = \alpha_3 \alpha_2 \alpha_1 \] (4)

\[ \beta_1 = \beta_2 \] (5)

\[ \gamma_1 \gamma_2 = \gamma_2 \gamma_1 \] (6)

\[ \theta \theta = \theta \theta \] (7)

\[ \theta_0 \theta_0 = \theta_0 \theta_0 \] (8)

\[ \phi_\pi \phi_\pi = \phi_\pi \phi_\pi \] (9)

\[ \alpha_1 \alpha_2 \alpha_3 = \alpha_3 \alpha_2 \alpha_1 \] (10)

\[ \alpha_1 \alpha_2 \alpha_3 = \alpha_3 \alpha_2 \alpha_1 \] (11)

\[ \beta_1 = \beta_2 \] (12)

\[ \gamma_1 \gamma_2 = \gamma_2 \gamma_1 \] (13)

\[ \theta \theta = \theta \theta \] (14)

\[ \theta_0 \theta_0 = \theta_0 \theta_0 \] (15)

\[ \theta_0 \theta_0 = \theta_0 \theta_0 \] (16)

\[ \theta_0 \theta_0 = \theta_0 \theta_0 \] (17)

\[ \theta \theta = \theta \theta \] (18)

**Figure 4** Axioms of the LO\(_v\)-calculus. The equations are valid for arbitrary parameters \(\varphi, \varphi_i, \theta, \theta_i \in \mathbb{R}\). In Equation (18), the angles on the left-hand side can take any value while the right-hand side is given by Lemma 11 (where \(U\) is the \([1]_{pp}\)-semantics of the left-hand side of the equation).

commutation properties (Equations (15), (16)). Notice that there are two equations acting on 3 modes: Equation (6) and Equation (18). Equation (6) is a variant of the Yang-Baxter Equation [35], whereas Equation (18) is a property of decompositions into Euler angles. Indeed, in 3-dimensional space, the two sides of this equation correspond to two distinct decompositions in elementary rotations.

**Definition 13** (LO\(_v\)-calculus). Two LO\(_v\)-circuits \(D_1, D_2\) are equivalent according to the rules of the LO\(_v\)-calculus, denoted \(\text{LO}_v \vdash D_1 = D_2\), if one can transform \(D_1\) into \(D_2\) using the equations given in Figure 4. More precisely, \(\text{LO}_v \vdash \cdot = \cdot\) is defined as the smallest congruence which satisfies the equations of Figure 4 in addition to the axioms of PROP.

**Proposition 14** (Soundness). For any two LO\(_v\)-circuits \(D_1\) and \(D_2\), if \(\text{LO}_v \vdash D_1 = D_2\) then \([D_1] = [D_2]\).

**Proof.** Since semantic equality is a congruence it suffices to check that for every equation of Figure 4 both sides have the same semantics, which follows from Definition 5 and Lemma 11. ▷
Proposition 15. The rules of the LO-v-calculus imply that the parameters are 2π-periodic, i.e. for any $\theta, \varphi \in \mathbb{R}$:

\[
\text{LO}_v \vdash \theta = \theta + 2\pi \\
\text{LO}_v \vdash \varphi = \varphi + 2\pi
\]

Proof. The proof is given in Appendix A.1.

We now state one of our main results: the completeness of the LO-v-calculus.

Theorem 16 (Completeness). For any two LO-v-circuits $D_1$ and $D_2$, if $\|D_1\| = |D_2|$ then $\text{LO}_v \vdash D_1 = D_2$.

The proof of Theorem 16 is given in Section 5. As a step towards proving the theorem, we first consider the fragment of the LOPP-circuits.

4 Polarisation-Preserving Circuits

This section gives a universal normal form for any LOPP-circuit. We prove the uniqueness of that form by introducing a strongly normalising and confluent polarisation-preserving rewrite system: PPRS.

Definition 17. The rewrite system PPRS is defined on LOPP-circuits with the rules of Figure 5.

Lemma 18. If $D_1$ rewrites to $D_2$ using the PPRS rewrite system then $\text{LO}_v \vdash D_1 = D_2$.

Proof. The proof is given in Appendix A.10.

Theorem 19. The rewrite system PPRS is strongly normalising.

Proof. The proof is done by defining a lexicographic order on six distinct values: numbers of beam splitters of various angle ranges, count of specific patterns, numbers and positions of phase shifters. The order is shown to be decreasing with respect to the rewrite rules of PPRS. The complete proof is given in Appendix A.6.

As PPRS is terminating, we can therefore derive the existence of normal forms. The next step is to show that these normal forms are unique: this is derived from Theorem 20.

Theorem 20. PPRS is globally confluent.

Proof. PPRS is locally confluent. Indeed, one can show by case analysis that the non-trivial peaks all use at most three wires. Each peak can be closed since for any polarisation-preserving LO-v-circuit of size $n \in \{1, 2, 3\}$, PPRS terminates to a specific unique normal form: when $n = 1$, a simple phase-shift; when $n = 2$, the form shown in Lemma 10; when $n = 3$, the form shown in Lemma 11. See Appendix A.7 for details. Finally, using Theorem 19, global confluence is deduced from Newman’s lemma [49].

Definition 21. A PPRS triangular normal form is a circuit with a triangular shape similar to Figure 1a, but with all 0-angled generators replaced with identities and with additional conditions on the angles, as described in Figure 6.

Figure 7 shows an example: the figure on the left is the “full” circuit with 0-angled beam splitters while on the right is the corresponding PPRS triangular normal form.
\[ \psi \rightarrow \psi \mod 2\pi \] (28)

\[ \phi_2 \rightarrow \phi_1 + \phi_2 \] (30)

\[ 0 \rightarrow \] (31)

\[ \theta_1 \rightarrow \theta_1 \] (32)

\[ \theta_1 \rightarrow \theta_1 \] (33)

\[ \theta_2 \rightarrow \theta_2 \] (34)

\[ \theta_3 \rightarrow \theta_3 \] (35)

\[ \phi_0 \rightarrow \phi_0 \] (36)

\[ \phi_0 \rightarrow \phi_0 \] (37)

\[ \phi_0 \rightarrow \phi_0 \] (38)

**Figure 5** Rewriting rules of PPRS. \( \psi \in \mathbb{R} \setminus [0, 2\pi) \), \( \varphi, \varphi_1, \varphi_2 \in (0, 2\pi) \), \( \varphi_0, \theta_4 \in [\pi, 2\pi) \), \( \theta, \theta_0, \theta_1, \theta_2, \theta_3 \in (0, \pi) \), and \( \theta_0 \neq \frac{\pi}{2} \). \( \phi^* \) denotes either \( \phi \) or \( \phi^* \). In Rules (37) and (38), the angles on the left-hand side can take any value while the right-hand side is given by Lemma 11 and Lemma 10 respectively.

**Lemma 22.** Any irreducible LOPP-circuit is a PPRS triangular normal form.

**Proof.** This property can be proven by induction. First, we explicit properties of any irreducible circuit that can be directly deduced from the PPRS rules of Figure 5. Then, we give two more properties characterising the PPRS triangular normal forms. By induction, we prove that any irreducible circuit respects those two properties, so that any irreducible circuit is a PPRS triangular normal form. The initialisation is deduced by Lemma 11 and Lemma 10 respectively. See Appendix A.8 for more details.

**Theorem 23.** Any LOPP-circuit, with the rules of PPRS, converges to a unique PPRS triangular normal form.

**Proof.** PPRS is globally confluent and terminating: normal forms are unique. From Lemma 22, PPRS triangular normal forms are the only irreducible forms. Therefore, any polarisation-preserving circuit terminates to such a unique normal form.

**Remark 24.** By using Equation (18) (together with Equations 1 and 42) and by adding 0-angled beam splitters if necessary, one can turn any circuit in PPRS triangular normal form into a circuit in the rectangular form of [16] shown in Figure 1b. A schematic example of such a transformation is shown in Appendix A.3.

We can now prove the completeness of the polarisation-preserving fragment.
For any LOPP-circuits $C_1, C_2$ such that $\llbracket C_1 \rrbracket_{pp} = \llbracket C_2 \rrbracket_{pp}$, their normal forms are equal, i.e., $N_1 = N_2$, where $N_1$ (resp. $N_2$) is the unique normal form of $C_1$ (resp. $C_2$) given by Theorem 23.

Proof. As the rewrite system preserves the semantics, it is sufficient to prove that $[N_1]_{pp} = [N_2]_{pp} \Rightarrow N_1 = N_2$. First, we can show by induction that $[N]_{pp} = [I_n]_{pp} \Rightarrow N = I_n$. Indeed, to have the semantics of identity, we can show the upper beam splitter and phase shifters are necessarily 0-angled. The proof follows from the induction property, details are given in Appendix A.9. Let $P$ be an inverse circuit of $N_1$ and $N_2$, that is, a polarisation-preserving circuit such that $[P]_{pp} = [N_1]_{pp}$. The existence of such a circuit follows from [46]. As $[N_1 P]_{pp} = [P N_2]_{pp} = [I_n]_{pp}$, the term $N_1 P N_2$ can both be reduced to $N_1$ (by reducing $P N_2$ first) and $N_2$ (by reducing $N_1 P$ first). By Theorem 23, $N_1 = N_2$.

Proposition 26 (Universality and uniqueness in the polarisation-preserving fragment). For any unitary $U : \mathbb{C}^n \rightarrow \mathbb{C}^n$, there exists a unique circuit $T$ in PPRS triangular normal form such that $[T]_{pp} = U$.

Proof. This follows directly from [46], Theorems 23 and 25 and the fact that all PPRS triangular normal forms are irreducible.

5 Completeness of the LO$\nu$-Calculus

To prove the completeness of the LO$\nu$-Calculus (Theorem 16), we introduce the following notion of normal form.

Definition 27 (Normal form). A circuit in normal form $N : n \rightarrow m$ is a circuit of the form shown in Figure 8, where $T$ is a PPRS triangular normal form (Definition 21). If $n' = m' = 0$, then $N$ is said to be in pure normal form.
Lemma 28 (Uniqueness of the pure normal form). If two circuits $N_1$ and $N_2$ in pure normal form are such that $[N_1] = [N_2]$, then $N_1 = N_2$.

Proof. Let $T_1$ (resp. $T_2$) be the LO$\text{PP}$-circuit associated with $N_1$ (resp $N_2$) as in Figure 8. Notice that $[T_i]_{\text{pp}} \circ \mu = \mu \circ [N_i]$ where $\mu : \mathbb{C}^{M_n} \to \mathbb{C}^{2n}$ is the isomorphism $|V_k\rangle \mapsto |2k\rangle$ and $|H_k\rangle \mapsto |2k+1\rangle$. Thus $[N_1] = [N_2]$ implies $[T_1]_{\text{pp}} = [T_2]_{\text{pp}}$ so that the result follows from Theorem 23.

Lemma 29. For any circuit $D$ without vacuum state sources or detectors there exists a circuit in pure normal form $N$ such that $\text{LO}_\gamma \vdash D = N$.

Proof. The proof is given in Appendix A.11.

Completeness for circuits without vacuum state sources or detectors follows directly from Lemmas 28 and 29.

Proposition 30. Given any two circuits $D_1$ and $D_2$ without any $0$ or $0$, if $[D_1] = [D_2]$ then $\text{LO}_\gamma \vdash D_1 = D_2$.

Proof. By Lemma 29, there exist two circuits in pure normal form $N_1$ and $N_2$ such that $\text{LO}_\gamma \vdash D_1 = N_1$ and $\text{LO}_\gamma \vdash D_2 = N_2$. By Proposition 14, one has $[N_1] = [D_1] = [D_2] = [N_2]$, so that by Lemma 28, $N_1 = N_2$. The result follows by transitivity.

Proof of Theorem 16

We now have the required material to to finish the proof of Theorem 16. Let $D_1, D_2 : n \to m$ be any two LO$\gamma$-circuits such that $[D_1] = [D_2]$. By deformation, we can write them as

\[
\begin{align*}
n \{ \begin{array}{c} \ldots \ \vdots \ 1 \ \vdots \ \ldots \\
\end{array} \} & \ D_1' \ \{ \begin{array}{c} \text{m} \\
\end{array} \\
\} \ 
\text{n'} \ {\begin{array}{c} \begin{array}{c} \begin{array}{c} 1 \ \\
\end{array} \ \\
\end{array} \ \\
\end{array}} \ & \text{m'} \ 
\end{align*}
\]

and

\[
\begin{align*}
n \{ \begin{array}{c} \ldots \ \vdots \ 1 \ \vdots \ \ldots \\
\end{array} \} & \ D_2' \ \{ \begin{array}{c} \text{m} \\
\end{array} \\
\} \ 
\text{n''} \ {\begin{array}{c} \begin{array}{c} \begin{array}{c} 1 \ \\
\end{array} \ \\
\end{array} \ \\
\end{array}} \ & \text{m''} 
\end{align*}
\]

where $D_1', D_2'$ do not contain $0$ or $0$. Up to using Equation (8), we can assume that $n'' = n'$. Since circuits without vacuum state sources and detectors necessarily have the same number of input wires as of output wires, this implies that $m'' = m'$. By Lemma 29, we can put $D_1'$ and $D_2'$ in pure normal form. Then by using Equations (9)–(14), we get two
circuits in normal form

\[
D_{1}^{\text{NF}} = \begin{pmatrix}
\vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \star & \star & \star & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \star & \star & \star & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots
\end{pmatrix}
\]

and

\[
D_{2}^{\text{NF}} = \begin{pmatrix}
\vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \star & \star & \star & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \star & \star & \star & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots
\end{pmatrix}
\]

with \( T_1 \) and \( T_2 \) in PPRS triangular normal form.

\([D_1] = [D_2]\) implies that \( \pi \circ [T_1]_{pp} \circ \iota = \pi \circ [T_2]_{pp} \circ \iota \) where \( \iota : \mathbb{C}^{2n} \to \mathbb{C}^{2n+n'} \) is the injection \( |k\rangle \mapsto |k\rangle \) and \( \pi : \mathbb{C}^{2m+m'} \to \mathbb{C}^{2m} \) is the projector s.t. \( \pi|k\rangle = |k\rangle \) when \( k < 2m \) and \( \pi|k\rangle = 0 \) otherwise. Thus there exists two unitaries \( Q, Q' \) s.t. \([T_2]_{pp} = (I \oplus Q') \circ [T_1]_{pp} \circ (I \oplus Q)\) (see Lemma 36 in Appendix A.12).

By Proposition 26 there exist two circuits \( T_{\text{in}} \) and \( T_{\text{out}} \) in PPRS triangular normal form such that \([T_{\text{in}}]_{pp} = Q \) and \([T_{\text{out}}]_{pp} = Q'\). By Equations (9), (12), (48), and (49), we can make \( T_{\text{in}} \) and \( T_{\text{out}} \) appear, turning \( D_{1}^{\text{NF}} \) into

\[
\begin{pmatrix}
\vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \star & \star & \star & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\cdots & \star & \star & \star & \cdots \\
\vdots & \ddots & \ddots & \ddots & \vdots
\end{pmatrix}
\]

Since by construction, the middle part has the same single-photon semantics as \( T_2 \), by Proposition 30 we can transform it into \( T_2 \) using the axioms of the \( \text{LO}_v \)-calculus, which means transforming \( D_{1}^{\text{NF}} \) into \( D_{2}^{\text{NF}} \). The result follows by transitivity.

\[\square\]

6 Conclusion

In this paper, we presented the \( \text{LO}_v \)-calculus, a graphical language for LOQC capturing most of the components typically considered in the Physics community for linear optical quantum circuits. The language comes equipped with a sound and complete semantics, and we discussed how it provides a unifying framework for many of the existing approaches in the literature. We explained how several existing results can be ported in the \( \text{LO}_v \) framework.

An obvious direction for future work is to extend the language to allow for sources and detectors of a non-zero number of photons. A more exploratory research avenue is to add support for features such as squeezed states or continuous variables.

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A Proofs

A.1 $2\pi$-Periodicity: Proof of Proposition 15

We actually prove a stronger version of the $2\pi$-periodicity for the phase shifter:

$$\phi \equiv \phi \mod 2\pi$$

as follows:

$$\phi(39) = 0$$

Then, the equality of Proposition 15 follows straightforwardly:

$$\phi(39) = \phi \mod 2\pi(39) = \phi + 2\pi$$

To prove the $2\pi$-periodicity for the beam splitter, we proceed as follows:

$$\theta(40) = 0$$

where $\varepsilon = \begin{cases} 0 & \text{if } \theta \mod 2\pi \in [0, \pi) \\ 1 & \text{if } \theta \mod 2\pi \in [\pi, 2\pi) \end{cases}$

Finally, the $2\pi$-periodicity for the wave plate follows from that for the beam splitter as follows:

$$\theta(43) = \theta + 2\pi$$
Remark 31. Note that we could also prove the following stronger equations, for any \( k \in \mathbb{Z} \), with the same sequence of rewrite steps, that is, in a bounded number of steps:

\[
\begin{align*}
\phi & = \phi + 2k\pi \\
\theta & = \theta + 2k\pi \\
\overline{\theta} & = \overline{\theta + 2k\pi}
\end{align*}
\]

A.2 Useful Consequences of the Axioms

Lemma 32. The equations of Figure 9 are consequences of the axioms of the LO_\nu-calculus.

To prove Equation (40), we have:

![Diagram](image)

To prove Equation (41), we have:

![Diagram](image)

To prove Equation (44), we have (cf. [14], Appendix D):
Figure 9 Useful consequences of the axioms of the LOₜ-calculus. In Equation (40), the angles on the left-hand side can take any value, and the right-hand side is given by Lemma 10. Explicit expressions of α₁, β₁, β₂ and β₃ in terms of θ₁, θ₂ and ϕ₁ are given in Appendix A.5.
Equation (45) is a direct consequence of Equations (1) and (15).

To prove Equation (42), we have:

To prove Equation (43), we have:
Equation (46) is a direct consequence of Equations (9) and (10).
Equation (47) is a direct consequence of Equations (12) and (13).
Equation (48) is a direct consequence of Equations (10), (11), (17), and (46).
Equation (49) is a direct consequence of Equations (13), (14), (17), and (47).
To prove Equation (50), we have:

To prove Equation (51), we have:
Equation (52) is proved in [14] (Appendix B.2.2.1) as Equation (17). Equation (53) is a direct consequence of Equations (17), (51), and (52).

To prove Equation (54), we have:

To prove Equation (55), we have:
To prove Equation (56), we need the following auxiliary equations, which are consequences of Equations (4), (5), (6) and (7):

\begin{align}
\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} & = 0 \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} & = -\frac{1}{\rho} \frac{\partial P}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} & = -\frac{1}{\rho} \frac{\partial P}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} & = -\frac{1}{\rho} \frac{\partial P}{\partial z} + \nu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)
\end{align}

Equations (59) and (60) are proved in [14] (Appendix B.2.2.1) as Equations (25) and (24) respectively. Equations (57) and (58) are direct consequences of Equations (4) and (44).

To prove Equation (61), we have:
to prove Equation (62), we have:

and to prove Equation (63), we have
The last step is by mere deformation of the circuit, by exchanging the two PBS. To prove Equation (64), we have:

Now we can prove Equation (56):

Now we can prove Equation (56):
A.3 Triangular to Rectangular Form

If necessary, we add 0-angled beam splitters in the PPRS triangular normal form to obtain a triangular shape, as in Figure 1a. Then, for example with 7 spatial modes, we proceed as follows:\(^3\)

---

\(^3\) Here we only show how the beam splitters move along the process. We interpret Equation 18 as sliding one of the beam splitters through the two others while changing the parameters and adding some phase shifts. Before and after each move it may be necessary to manipulate the phase shifters with the help of Equations 1, 42 and 44. Beam splitters represented in red are just to be moved, and beam splitters represented in blue have just been moved.
This leads us to a rectangular form of \([16]\) (see Figure 1b).

### A.4 Existence and Uniqueness of the 3-Mode Triangular Form: Proof of Lemma \([11]\)

Let us consider such \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in [0, \pi)\) and \(\beta_4, \beta_5, \beta_6 \in [0, 2\pi)\). We first prove that, assuming that they exist, their values are uniquely determined by \(U\).

Let \(U_1 :=\)

\[
\begin{bmatrix}
\beta_1 \\
\alpha_1
\end{bmatrix}
\]

\(\circ U^\dagger, U_2 :=\)

\[
\begin{bmatrix}
\beta_1 \\
\alpha_1
\end{bmatrix}
\]

\(\circ U^\dagger\) and \(U_3 :=\)

\[
\begin{bmatrix}
\beta_1 \\
\alpha_1
\end{bmatrix}
\]
Thus, we denote by $\beta$ implies that $U$ and $w$. Since $\beta$ and $U$.

If $U = 0$, then this equality implies that $\sin(\alpha) = 0$ and $\sin(\alpha) = 0$ (indeed, if $\cos(\alpha) = 0$ then $\sin(\alpha) = \pm 1$ and conversely, which in both cases prevents the equality from being satisfied). Hence, $\beta$ is the unique angle in $[0, \pi)$ such that $\arg(U_{0, 1}) = \arg(U_{0, 2}) + \frac{\pi}{2}$ mod $\pi$. Then $\alpha$ is the unique angle in $[0, \pi) \setminus \{\frac{\pi}{2}\}$ such that $\tan(\alpha) = -\frac{U_{0, 2}}{U_{0, 1}}$.

If $U_{0, 2} = 0$ and $U_{0, 1} \neq 0$, then $\sin(\alpha) = 0$, which means, since $\alpha \in [0, \pi)$, that $\alpha = 0$. Due to the constraints on the angles, this implies that $\beta = 0$ too.

If $U_{0, 1} = 0$ and $U_{0, 2} \neq 0$, then $\cos(\alpha) = 0$, which means, since $\alpha \in [0, \pi)$, that $\alpha = \frac{\pi}{2}$. Due to the constraints on the angles, this implies that $\beta = 0$ too.

If $U_{0, 1} = U_{0, 2} = 0$, then since $U$ is unitary, it is of the form $U = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$, where $*$ denotes any complex number. Then, regardless of $\alpha$ and $\beta$, $U_1$ is of the same form:

$$U_1 = \begin{pmatrix} e^{i\phi} & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$  Consequently, $U_2 = \begin{pmatrix} e^{i\phi} \cos(\alpha_2) & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$. By \ref{A}, this implies that $\sin(\alpha_2) = 0$, which means, since $\alpha_2 \in [0, \pi)$, that $\alpha_2 = 0$. Due to the constraints on the angles, this implies that $\alpha_1 = \beta_1 = \beta_2 = 0$ too.

Thus, $\alpha_1$ and $\beta_1$, and in turn $U_1$, are uniquely determined given $U$.

Since $(U_1)_{2, 0} = 0$, $U_1$ can be written as $\begin{pmatrix} (U_1)_{0, 0} & * & * \\ (U_1)_{1, 0} & * & * \\ 0 & * & * \end{pmatrix}$. By \ref{A} we have $(U_2)_{1, 0} = 0$, that is, $i e^{i\beta_2} \sin(\alpha_2) (U_1)_{0, 0} + \cos(\alpha_2) (U_1)_{1, 0} = 0$. Since $U_1$ is unitary, $|(U_1)_{0, 0}|^2 + |(U_1)_{1, 0}|^2 = 1$, so that we cannot have $(U_1)_{0, 0} = (U_1)_{1, 0} = 0$. The other cases are similar to those of $\alpha_1$ and $\beta_1$.

\footnote{We denote by $M_{i,j}$ the entry of indices $(i, j)$ of a matrix $M$, the index of the first row and column being 0.}
If \((U_1)_{0,0}, (U_1)_{1,0} \neq 0\), then similarly, the equality implies that \(\cos(\alpha_2) \neq 0\) and \(\sin(\alpha_2) \neq 0\).

Hence, \(\beta_2\) is the unique angle in \([0, \pi]\) such that \(\frac{\omega_{\beta_2}^\dagger(U_1)_{0,0}}{\omega_{\beta_2}^\dagger(U_1)_{1,0}} \in \mathbb{R}\), namely \(\arg((U_1)_{0,0}) + \frac{\pi}{2}\) mod \(\pi\). Then \(\alpha_2\) is the unique angle in \([0, \pi) \setminus \{\frac{\pi}{2}\}\) such that \(\tan(\alpha_2) = -\frac{\omega_{\beta_2}^\dagger(U_1)_{0,0}}{\omega_{\beta_2}^\dagger(U_1)_{1,0}}\).

If \((U_1)_{1,0} = 0\) and \((U_1)_{0,0} \neq 0\), then \(\sin(\alpha_2) = 0\), which means, since \(\alpha_2 \in [0, \pi]\), that \(\alpha_2 = 0\). Due to the constraints on the angles, this implies that \(\beta_2 = 0\) too.

If \((U_1)_{0,0} = 0\) and \((U_1)_{1,0} \neq 0\), then \(\cos(\alpha_2) = 0\), which means, since \(\alpha_2 \in [0, \pi]\), that \(\alpha_2 = \frac{\pi}{2}\). Due to the constraints on the angles, this implies that \(\beta_2 = 0\) too.

Thus, \(\alpha_2\) and \(\beta_2\), and in turn \(U_2\), are also uniquely determined given \(U\).

Furthermore, (A) implies that

If \((U_2)_{1,1}, (U_2)_{2,1} \neq 0\), then \(\beta_3\) is the unique angle in \([0, \pi]\) such that \(\frac{\omega_{\beta_3}^\dagger(U_2)_{1,1}}{\omega_{\beta_3}^\dagger(U_2)_{2,1}} \in \mathbb{R}\), namely \(\arg((U_2)_{1,1}) - \arg((U_2)_{2,1}) + \frac{\pi}{2}\) mod \(\pi\), and \(\alpha_3\) is the unique angle in \([0, \pi]\) such that \(\tan(\alpha_3) = \frac{\omega_{\beta_3}^\dagger(U_2)_{1,1}}{\omega_{\beta_3}^\dagger(U_2)_{2,1}}\).

If \((U_2)_{2,1} = 0\) and \((U_2)_{1,1} \neq 0\) then \(\sin(\alpha_3) = 0\), which means, since \(\alpha_3 \in [0, \pi]\), that \(\alpha_3 = 0\). Due to the constraints on the angles, this implies that \(\beta_3 = 0\) too.

If \((U_2)_{1,1} = 0\) and \((U_2)_{2,1} \neq 0\), then \(\cos(\alpha_3) = 0\), which means, since \(\alpha_3 \in [0, \pi]\), that \(\alpha_3 = \frac{\pi}{2}\). Due to the constraints on the angles, this implies that \(\beta_3 = 0\) too.

Thus, \(\alpha_3\) and \(\beta_3\), and in turn \(U_3\), are also uniquely determined given \(U\).

Finally, since \(U_3 = \begin{pmatrix} e^{-i\beta_4} & 0 & 0 \\ 0 & e^{-i\beta_5} & 0 \\ 0 & 0 & e^{-i\beta_6} \end{pmatrix}\), we necessarily have \(\beta_4 = -\arg((U_3)_{0,0}),\) \(\beta_5 = -\arg((U_3)_{1,1})\) and \(\beta_6 = -\arg((U_3)_{2,2})\). This finishes proving the uniqueness.

Conversely, it is easy to see that the unique possible values given above for \(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5\) and \(\beta_6\) are well defined for any unitary \(U\) and satisfy the desired properties, which proves the existence.

\[\textbf{Remark 33.}\] It is possible to generalise this proof to extend the result to an arbitrary number of modes. This provides an alternative proof of Proposition \([26]\).

### A.5 Existence and Uniqueness of the 2-Mode Triangular Form: Proof of Lemma \([10]\)

Let us consider such \(\beta_1, \alpha_1 \in [0, \pi)\) and \(\beta_2, \beta_3 \in [0, 2\pi)\). We have (with \([-]\) \(_{pp}\) defined in Definition \([9]\))

\[
U = \left[ \begin{array}{cccc}
\alpha_1 & & & \\
& \beta_2 & & \\
& & \beta_3 \\
& & & \end{array} \right]_{pp} = \begin{pmatrix} e^{i(\beta_1+\beta_2)} \cos(\alpha_1) & ie^{i\beta_2} \sin(\alpha_1) \\
\frac{e^{i(\beta_1+\beta_3)} \sin(\alpha_1)}{\cos(\alpha_1)} & e^{i\beta_3} \cos(\alpha_1) \end{pmatrix}
\]

If \(U\) has a null entry, the since it is unitary, it is either diagonal or anti-diagonal. If it is diagonal, then \(\sin(\alpha_1) = 0\), which, since \(\alpha_1 \in [0, \pi)\), implies that \(\alpha_1 = 0\), which by the constraint on \(\beta_1\) and \(\alpha_1\), implies that \(\beta_1 = 0\). Consequently, \(\beta_2 = \arg(U_{0,0})\) and \(\beta_3 = \arg(U_{1,1})\). If \(U\) is anti-diagonal, then \(\cos(\alpha_1) = 0\), which, since \(\alpha_1 \in [0, \pi)\), implies that \(\alpha_1 = \frac{\pi}{2}\). By the constraint on \(\beta_1\) and \(\alpha_1\), implies that \(\beta_1 = 0\). Consequently, \(\beta_2 = \arg(U_{0,0})\) and \(\beta_3 = \arg(U_{1,1})\).

If \(U\) has no null entry, since \(UU^\dagger = I\), we have \(e^{i(\beta_1+\beta_2)} \cos(\alpha_1)U_{1,0}^\dagger + ie^{i\beta_2} \sin(\alpha_1)U_{1,1}^\dagger = 0\). Hence, \(\beta_1\) is the unique angle in \([0, \pi)\) such that \(\frac{e^{i\beta_1}U_{1,0}^\dagger}{\cos(\alpha_1)} \in \mathbb{R}\), namely \(\arg(U_{1,0}^\dagger) - \arg(U_{1,1}^\dagger) + \frac{\pi}{2}\) mod \(\pi\), and \(\alpha_1 = \frac{\pi}{2}\).
\[ \frac{\pi}{2} \mod \pi. \] Then \( \alpha_1 \) is the unique angle in \([0, \pi)\) such that \( \tan(\alpha_1) = -\frac{\sin(i)}{\cos(i)} \), and since \( \alpha_1 \in (0, \pi) \), we have \( \sin(\alpha_1) > 0 \), so that \( \beta_2 = \arg \frac{U_{n+1}}{U_{n+1}^*} \) and \( \beta_3 = \arg \frac{U_{i}U_{n}}{U_{i}U_{n}^*} \).

Finally, it is easy to see that given any unitary \( U \), the unique possible values given above for \( \beta_1, \alpha_1, \beta_2 \) and \( \beta_3 \) are well defined and satisfy the desired properties.

### A.6 Strong Normalisation: Proof of Lemma 19

To prove that the rewrite system is strongly normalising, given a circuit \( D : n \to n \) composed only of phase shifters and beam splitters, let us consider the tuple \((a, b, c, d, e, f)\), where:

- \( a \) is the number of beam splitters in \( D \) with angle not in \([0, \pi)\)
- \( b \) is the number of beam splitters in \( D \) with angle not in \([0, 2\pi)\)
- \( e = \sum_{i=0}^{n-2} c(i) \times 2^{n-i} \) where \( c(i) \) is the number of beam splitters in \( D \) that act on positions \( i \) and \( i+1 \)
- \( d = (d_j)_{j \in \mathbb{N}} \) is the almost-zero sequence of integers such that for any \( j \geq 0 \),
  - \( d_{2j} \) is the number of phase shifters \( \phi \) of depth \( j \) that are in a pattern \( \begin{array}{cccc} \phi & \phi & \phi & \phi \\ \theta & \theta & \theta & \theta \end{array} \) with
    - either \( \theta = \frac{\pi}{2} \), or \( \theta \in (0, \pi) \setminus \{\frac{\pi}{2}\} \) and there exists a subset \( I \) of \( \{1, ..., k\} \) such that
      \( (\phi + \sum_{i \in I} \phi_i) \mod 2\pi \notin [0, \pi) \)
  - \( d_{2j+1} \) is the number of phase shifters \( \phi \) of depth \( j \) that are in a pattern
    \[ \begin{array}{cccc} \phi & \phi & \phi & \phi \\ \theta & \theta & \theta & \theta \end{array} \]
    where the depth of a phase shifter \( p \) in \( D \) is defined as the maximal number of beam splitters that a photon starting from \( p \) and going to the right can traverse before reaching an output port
- \( c \) is the number of phase shifters in \( D \)
- \( f \) is the number of phase shifters in \( D \) with angle not in \([0, 2\pi)\).

We consider the order on almost-zero sequences \((d_j)_{j \in \mathbb{N}}\) given, for two such sequences, by comparing first the indices of their respective last non-zero terms, then in case of equality, the terms with greatest index with different values. It is well known that this gives a well-order, more precisely of order type \( \omega^\omega \). We consider the lexicographic order on the set of tuples \((a, b, c, d, e, f)\). Since it is the lexicographic order on a finite cartesian product of well-ordered sets, it is itself a well-order. Hence, to prove that the rewrite system is strongly normalising, it suffices to prove that each of the rewrite rules strictly decreases the tuple \((a, b, c, d, e, f)\).

- Rule (28) strictly decreases \( f \) without increasing any component of the tuple.
- Rule (29) strictly decreases \( b \) without increasing \( a \).
- Rule (30) strictly decreases \( e \), it does not change \( a, b, c \) since it does not affect the beam splitters, and it does not increase \( d \). Indeed, if after applying Rule (30), \( \phi_1 + \phi_2 \) is at the left of a sequence of phase shifters that contains a subsequence with sum of angles modulo \( 2\pi \) not in \([0, \pi)\), then this was already the case of \( \phi_1 \) before applying Rule (30).
- Rule (31) does not increase \( a, b, c \) since it does not affects the beam splitters, it does not increase \( d \) since it only removes a phase shifter, and it strictly decreases \( e \).
- Rule (32) does not increase \( a \) or \( b \) since it only removes a beam splitter, and strictly decreases \( c \).
β

(a) For n=1

β

(b) For n=2

β

(c) For n=3

Figure 10 Normal forms of PPRS for $n \in \{1, 2, 3\}$. * means that the phase shifter or beam splitter is replaced by (an) identity wire(s) when the angle is zero. ⋆ represents the identity in the preceding case and also when $\alpha_i = 0$. ⋆ represents the identity in the preceding two cases and also when $\alpha_i = \frac{\pi}{2}$. The $\alpha_i$ are in $[0, \pi)$ as well as the phases with a ⋆, all other phases are in $[0, 2\pi)$.

A.7 Proof of Local Confluence

Lemma 34. For any polarisation-preserving LO$_+$-circuit of size $n \in \{1, 2, 3\}$, PPRS terminates to a unique normal form with the shape shown in Figure 10.

Proof. First, we will show that the normal forms are necessarily of the form given in Figure 10.
In a normal form, because of Rule (28), all phase shifters have angle in \([0, 2\pi)\); because of Rules (29) and (36), all beam splitters have angle in \([0, \pi)\); because of Rules (31) and (32), there is no phase shifters or beam splitters with angle 0; because of Rule (31), there is no phase shifter at the top left of a \(\pi/2\)-angled beam splitter; and because of Rule (35), all phase shifters at the top left of a beam splitter have angle in \([0, \pi)\). Thus, if a normal form is of one of the three forms given in Figure 10, then the conditions on the angles are satisfied.

Because of Rule (30), a normal form cannot contain two consecutive phase shifters. This implies in particular that the normal forms have the claimed shape for \(n = 1\).

Additionally, a normal form also cannot contain two consecutive beam splitters separated only by phase shifters (i.e., a pattern of the form \(\theta_1 \phi_1 \theta_2 \phi_2 \)). Indeed, because of Rule (33), in such a pattern in a normal form, there would not be a phase shifter on the bottom wire, so that the pattern would be reducible by Rule (38). Thus, in the case where \(n = 2\), a normal form contains at most one beam splitter. Because of Rule (33), such a beam splitter does not have any phase shifter at its bottom left, and because of Rule (30), there is at most one phase shifter on each of its other three ports. Because of Rule (34), there is no phase shifter on the bottom right if the angle of the beam splitter is \(\pi\). Moreover, if the normal form does not contain a beam splitter, then because of Rule (30), there is at most one phase shifter on each of the two wires. Thus, in all cases, the normal forms have the claimed shape for \(n = 2\).

In the case where \(n = 3\), since there cannot be two consecutive beam splitters in a normal form, the beam splitters are alternatively between the top two wires and the bottom two wires. Because of Rules (33) and (30), if there is a beam splitter between the top two wires, then one between the bottom two wires, and then again one between the top two wires, those three beam splitter necessarily match the left-hand side of Rule (37). Hence, a normal form contains at most three beam splitters, at most one on the top and two on the bottom. Additionally, if the one on the top is not here, then because of Rule (38) there is only one on the bottom. Finally, Rules (33) and (30) guarantee that the phase shifters are such that the normal form has the shape given in Figure 10c.

Now, we want to show that those are unique forms. One can check that given any circuit of the form given in Figure 10b (resp. Figure 10c), there is a unique way of adding \(0\)-angled phase shifters and beam splitters that gives us a circuit of the form of Lemma 10 (resp. Lemma 11) with the conditions on the angles satisfied. Then the uniqueness for \(n = 2\) and \(n = 3\) follows from Lemma 18 and the uniqueness given by Lemma 10 and Lemma 11 respectively. For \(n = 1\), the proof is straightforward given Lemma 18.

Lemma 35. PPRS is locally confluent.

Proof. First, note that the trivial critical pairs, in which the two rewrite rules are applied to disjoint patterns, can be closed in a straightforward way. Indeed, after doing any of the two transformations involved, the other one can be done independently, and the final result does not depend on which transformation was applied first.

Additionally, the non-trivial critical pairs, shown in Figure 11, all involve at most three (spatial) modes.

Indeed, first, two overlapping patterns necessarily share at least one spatial mode, so that if they both involve at most two modes, then their union involves at most three modes. This
implies that any non-trivial critical pair involving at least four modes must arise from at least one instance of Rule (37).

If the other rewrite step of the critical pair is not an instance of Rule (37), it would involve at most two modes. For the union with the instance of Rule (37) to involve four modes, it must involve exactly two modes and the two patterns must share only one mode. Consequently, their union is composed only of phase shifters. Since in the left-hand side of Rule (37) there is at most one phase shifter on each mode, the union of the two patterns must be a single phase shifter. Moreover, for the two patterns to share not more than one mode, it must be on the top mode of one pattern and on the bottom mode of the other pattern. Namely, since the left-hand side of Rule (37) does not have a phase shifter on the bottom mode, it must be on the top mode of the associated pattern and on the bottom mode of the other pattern. The only rule in which the left-hand side involves two modes and has a phase shifter on the bottom mode is Rule (33), but this phase shifter cannot belong to a pattern corresponding to the left-hand side of Rule (37) since it would be both on the top left and on the bottom left of a beam splitter at the same time, which is not possible.

Hence, any non-trivial critical pair involving at least four modes must arise from two instances of Rule (37). Since the left-hand side of this rule does not have a phase shifter on the bottom mode, the two patterns cannot share only one mode and must share at least

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5 Note that two patterns that overlap only by identities can be considered disjoint.
two modes. Then since their union involves at least four modes, they share exactly two modes. These two modes are the top two of one pattern and the bottom two of the other pattern. There are two generators that act only on the bottom two modes in the left-hand side of Rule 37 and therefore can be in the intersection of the two patterns: the phase shifter labeled with \( \phi_1 \) if present, and the beam splitter labelled with \( \theta_2 \). If the phase shifter labeled with \( \phi_1 \) is in the intersection of the two patterns then it necessarily correspond to the phase shifter labeled with \( \phi_2 \) in the other pattern, but this is not possible since on is on the top right of a beam splitter whereas the other is on the bottom right of a beam splitter. Therefore, the two patterns necessarily overlap by one beam splitter, which is the bottom one in one pattern and one of the two top ones in the other pattern. But in the left-hand side of Rule 37, the bottom beam splitter is connected by its top wires to the bottom of another beam splitter on each side, whereas each of the two top ones is connected by at least one of its top wires to the top wire of another beam splitter, hence the two patterns cannot overlap this way.

Thus, all non-trivial critical pairs involve at most three spatial modes. It follows from Lemma 34 that any critical pair on at most three wires can be closed, which gives us the local confluence. 

A.8 Irreducible Polarisation-Preserving LO\(_v\)-Circuits Are PPRS Triangular Normal Forms

For rigour and clarity purposes, \( \text{BS}^i \) (resp. \( \text{PS}^i \)) represents a beam splitter (resp. phase shifter) on the modes \((i, i + 1)\) (resp. on the mode \(i\)) with a non-zero angle (resp. a non-zero phase), or the identity otherwise. We denote as \( \text{PS-BS}^i \) a \( \text{BS}^i \) with a \( \text{PS}^i \) on its top left which is also the identity when the angle of the beam splitter is in \( \{0, \frac{\pi}{2}\} \). \( C^{(i,j)} \) represents a circuit \( C \) whose every component is between the modes \(i\) and \(j\).

We’ll prove by induction that for any \( n \in \mathbb{N}^* \) any irreducible polarisation-preserving LO\(_v\)-circuit of size \( n \) is a PPRS triangular normal form, given in Definition 21.

First, we can show that an irreducible circuit satisfies all the following properties:

- No phase or angle can be outside of \((0, 2\pi)\). This follows from Rules 28, 29, 31 and 32.
- There are no consecutive phase shifters. This follows from Rule 30.
- Angles for beam splitters and phases on the top left of beam splitters are in \((0, \pi)\). This follows from Rules 35 and 36.
- There is no phase on the top left of a \( \frac{\pi}{2} \) beam splitter. This follows from Rule 34.
- There is no phase on the bottom left of a beam splitter. This follows from Rule 33.
- There aren’t two consecutive beam splitters on the same modes. This follows from Rule 38 and the fact that there is no phase on the bottom left of a beam splitter.

Second, we can show that an irreducible circuit of size \( n \geq 3 \) is a PPRS triangular normal form if:

**(H)** The circuit can be decomposed as \( P^{(1,n-1)} \circ D \) where:

- \( D = \text{PS}^0 \circ \text{PS-BS}^1 \circ \text{PS-BS}^2 \circ \text{PS-BS}^n \)
- \( D \) satisfies \((bs_D(0) \leq 1) \land (\forall k \in [1; n - 2] : bs_D(k) \leq bs_D(k - 1))\) and has the shape of an upper-left anti-diagonal of \( \text{PS-BS}^n \) with one \( \text{PS}^0 \) at the end.

**(H2)** \( P^{(1,n-1)} \) is an irreducible circuit on the \( n - 1 \) other lower modes satisfying **(H)** That implies that \( P^{(1,n-1)} \) is a PPRS triangular normal form.
We'll therefore prove by induction that any irreducible circuit satisfies for \( n \geq 3 \) satisfies (H) and therefore, any irreducible circuit of size \( n \geq 3 \) is a PPRS triangular normal form. The case for \( n \in \{1, 2\} \) follows directly from Lemma 34.

Proof. The case for \( n = 3 \) is directly induced from Lemma 34.

Let's consider an irreducible polarisation-preserving LO\textsubscript{c} circuit \( C \) of size \( n + 1 \).

We can show that there is necessarily at most one upper beam splitter on the first two modes, i.e. \( C \) can't contain the pattern \( BS_1^{(1,n)} \circ PS^{(0)} \circ P_{1^{(1,n)}} \circ BS_1^{(2,n)} \). If such a pattern exists in an irreducible circuit, with our induction hypothesis \( P_{1^{(1,n)}} \) is a PPRS triangular normal form. If there is an upper beam splitter \( BS_1 \) in \( P_{1^{(1,n)}} \), then we could use Rule 37 with \( BS_1^{(2,n)} \) and \( BS_1^{(1,n)} \). If the upper is the identity, then \( BS_1^{(2,n)} \) and \( BS_1^{(0,n)} \) would be consecutive beam splitters. Therefore that pattern can't exist, and we necessarily have one beam splitter at most on the first mode.

If there is no upper beam splitter, \( bs(0) = 0 \) and the first mode is at most one phase shifter. By the induction hypothesis, the \( n \) other wires form a PPRS triangular normal form \( P_{1^{(1,n)}} \) of size \( n \). As \( bs(0) = 0, C = P_{1^{(1,n)}} \circ PS^{(0)} \). Therefore (H) is satisfied.

If there is an upper beam splitter, then \( C = P_1^{(1,n)} \circ P_{1^{(1,n)}} \circ PS^{(0)} \circ BS_1 \circ P_{1^{(1,n)}} \) where \( bs(0) = 1 \) and \( P_1^{(1,n)} \in \{1, 2\} \) is a PPRS triangular normal form by the induction hypothesis.

Thus, with Lemma 34 for any \( n \in \mathbb{N}^+ \), PPRS triangular normal forms of size \( n \) are the only irreducible polarisation-preserving LO\textsubscript{c} circuits of size \( n \).

### A.9 Completeness of the Polarisation-Preserving Fragment

Let \( N : n \rightarrow n \) be a PPRS triangular normal form of size \( n \), and \( I_0 \) the identity circuit with \( n \) identity wires. We'll prove by induction that for any \( n \in \mathbb{N}^+ \): \([N] = [I_0] \Rightarrow N = I_0 \). For \( n = 1 \), \( N \) is Figure 10\( a \), the phase is necessarily zero. Therefore \( N = I_1 \).

Let's consider the case for a circuit of size \( n + 1 \). As we can see in the Figure 10\( a \), there is only at most one beam splitter interacting with the upper mode. Therefore, we know the output of the first spatial mode: \([c_0] \mapsto e^{i \beta_c \circ + \gamma_0 \cos(\alpha_{0,0})} |c_0]\). To have the identity operation, we necessarily have \( \alpha_{0,0} = 0 \). By definition of the normal form, \( \beta_0 = 0 \). Thus, \( \gamma_0 = 0 \), and the first wire is the identity, whereas the other \( n \) wires form a PPRS triangular normal form of size \( n \). The induction hypothesis implies that the \( n \) other wires are necessarily the identity, which concludes the proof of the induction step.

Therefore, for any \( n \in \mathbb{N}^+ \): \([N] = [I_0] \Rightarrow N = I_n \).

### A.10 Soundness of the Rewrite System with Respect to the Equational Theory: Proof of Lemma 18

It suffices to show that for each rule of Figure 5, we can transform the left-hand side into the right-hand side using the axioms of the LO\textsubscript{c}-calculus.

The soundness of Rules 28 and 29 is a direct consequence of Proposition 15. Note that in both cases, transforming the left-hand side into the right-hand side using the equations of Figure 4 only requires a bounded number of rewrite steps (see Remark 31 in Appendix A.1).
The soundness of Rule (30) is a direct consequence of Equation (1).
The soundness of Rule (31) is a direct consequence of Equation (2).
The soundness of Rule (32) is a direct consequence of Equation (3).
The soundness of Rule (33) is a direct consequence of Equations (2), (1) and (42).
The soundness of Rule (34) is a direct consequence of Equation (50), (1) and (2).
To prove the soundness of Rule (35), if \( \psi \in [\pi, 2\pi) \) and \( \theta \in (0, \pi) \), then we have:

\[
\begin{align*}
\theta & \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\q
\( \{ \chi, \zeta, \zeta', \zeta'' \} \) and \( \ell + \ell' = n - 1 \) or \( n - 2 \) depending on the type of \( g \) (if \( k = 0 \) then we take the product \( d_k \circ \cdots \circ d_1 \) to be the identity circuit \( I_n \)).

By Equations (44), (5) and (8), \( I_n \) is equivalent to the circuit of the form (B) with \( D' = I_{2n} \), which is indeed polarisation-preserving. It remains to prove that for any circuit \( D \) of the form (B) with \( D' \) polarisation-preserving, any \( g \in \{ \chi, \zeta, \zeta', \zeta'' \} \) and any \( \ell \), the circuit \( D \circ (I_{\ell} \oplus g \oplus I_{\ell'}) \) can be put again in the form (B) with \( D' \) polarisation-preserving.

The generator \( g \) passes through the left part of \( D \) as follows:

\[
\begin{align*}
\text{(65)} & \quad \begin{array}{c}
\text{original circuit} \\
\text{reduced circuit}
\end{array} \\
\text{(66)} & \quad \begin{array}{c}
\text{original circuit} \\
\text{reduced circuit}
\end{array} \\
\text{(67)} & \quad \begin{array}{c}
\text{original circuit} \\
\text{reduced circuit}
\end{array} \\
\text{(68)} & \quad \begin{array}{c}
\text{original circuit} \\
\text{reduced circuit}
\end{array} \\
\text{(69)} & \quad \begin{array}{c}
\text{original circuit} \\
\text{reduced circuit}
\end{array}
\end{align*}
\]

Then, since by Equations (50), (1) and (2), \( \pi = \pi - \pi - \pi \), we can remove the swaps in order to turn the middle part into a polarisation-preserving circuit which finishes the proof.

It remains to prove Equations (65)–(69) using the axioms of the LO\(_v\)-calculus.
To prove Equation (65), we have:

\[
\begin{array}{c}
\text{original circuit} \\
\text{reduced circuit}
\end{array}
\]

To prove Equation (66), we have:
To prove Equation (67), we have:

\[
\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} \\
\text{Diagram 3} & = \text{Diagram 4} \\
\text{Diagram 5} & = \text{Diagram 6}
\end{align*}
\]

To prove Equation (68), we have:

\[
\begin{align*}
\text{Diagram 1} & = \text{Diagram 2} \\
\text{Diagram 3} & = \text{Diagram 4} \\
\text{Diagram 5} & = \text{Diagram 6}
\end{align*}
\]
Given any subspace \( W \), we also define analogous notations for \( \mathcal{H} \) following how to construct a LOCC.

Let \( \langle v, v \rangle \) be a sub-unitary map i.e. a map \( U : \mathbb{C}^m \to \mathbb{C}^m \) such that \( \langle v, v \rangle = (U \otimes \Pi_{\text{in}}) \circ U \circ (I \otimes \Pi_{\text{in}}) \). Note that \( \Pi_{\text{in}} \) is the identity map on \( \mathbb{C}^m \).

We define \( U_0 := U \circ \iota_{\mathcal{H}^0}, U_{00} := \pi_{\mathcal{H}^0} \circ U \circ \iota_{\mathcal{H}^0} \) and \( U_{01} := \pi_{\mathcal{H}^0} \circ U \circ \iota_{\mathcal{H}^0} \).

We also define analogous notations for \( U' \). Let \( U' = (I \otimes \Pi_{\text{out}}) \circ U \circ (I \otimes \Pi_{\text{out}}) \).

**Proof.** We denote \( U_0 := U \circ \iota_{\mathcal{H}^0}, U_{00} := \pi_{\mathcal{H}^0} \circ U \circ \iota_{\mathcal{H}^0} \) and \( U_{01} := \pi_{\mathcal{H}^0} \circ U \circ \iota_{\mathcal{H}^0} \).

We define analogous notations for \( U' \). Note that \( U_0 \) and \( U_{00} \) are isometries. For any \( v, v' \in \mathcal{H}_0^m \), one has \( \langle v, v' \rangle = \langle U_0(v)|U_0(v') \rangle = \langle U_{00}(v)|U_{00}(v') \rangle + \langle U_{01}(v)|U_{01}(v') \rangle \). Similarly, \( \langle v, v' \rangle = \langle U_{00}(v)|U_{00}(v') \rangle + \langle U_{01}(v)|U_{01}(v') \rangle \). Since \( U_{00} = U_{00}' \), this implies that

\[
\forall v, v' \in \mathcal{H}_0^m, \quad \langle U_0(v)|U_0(v') \rangle = \langle U_{00}(v)|U_{00}(v') \rangle. \tag{70}
\]

Let \( v_1, ..., v_d \in \mathcal{H}_0^m \) such that \( U_{00}(v_1), ..., U_{00}(v_d) \) is an orthonormal basis of the image \( U_{00}(\mathcal{H}_0^m) \) of \( U_{00} \). By (70), \( U_{00}'(v_1), ..., U_{00}'(v_d) \) is an orthonormal basis of \( U_{00}'(\mathcal{H}_0^m) \). Let \( \mathcal{H}_0^m \to \mathcal{H}_0^m \) be any unitary map such that \( \forall v \in \{1, ..., d\}, Q_{\text{out}}(U_{01}(v_1)) = U_{00}'(v_1) \).

For any \( v \in \mathcal{H}_0^m \), there exist \( \lambda_1, ..., \lambda_d \in \mathbb{C} \) such that \( U_{01}(v) = \sum_{i=1}^d \lambda_i U_{00}(v_i) \). Then by (70), \( \|U_{01}(v) - \sum_{i=1}^d \lambda_i U_{00}(v_i)\| = \|U_{01}(v) - \sum_{i=1}^d \lambda_i U_{01}(v_1)\| = 0 \), so that \( U_{01}(v) = \sum_{i=1}^d \lambda_i U_{00}(v_i) \). Hence, \( Q_{\text{out}}(U_{01}(v)) = U_{00}'(v) \).

In other words \( v \in \mathcal{H}_0^m, U_{01}(v) = (I \otimes Q_{\text{out}}) \circ U(v) \). Hence, \( U'(\mathcal{H}_0^m) = (I \otimes Q_{\text{out}}) \circ U(\mathcal{H}_0^m) \), so that since \( \mathcal{H}_0^m \) and \( \mathcal{H}_0^m \) are the orthogonal complement of each other and \( U, U' \) are unitary, we also have \( U'(\mathcal{H}_0^m) = \mathcal{H}_1^m \) (which is the orthogonal complement of \( U'(\mathcal{H}_0^m) \)). Let \( w_1, ..., w_k \) be an orthonormal basis of \( \mathcal{H}_1^m \), and for every \( i \in \{1, ..., k\} \), let \( u_i := U_{00}' \circ (I \otimes Q_{\text{out}}) \circ U(w_i) \). The fact that \( U'(\mathcal{H}_1^m) = (I \otimes Q_{\text{out}}) \circ U(\mathcal{H}_1^m) \) implies that \( u_1, ..., u_k \) is also an orthonormal basis of \( \mathcal{H}_1^m \).

Let \( \mathcal{Q}_{\text{in}} : \mathcal{H}_1^m \to \mathcal{H}_1^m \) be the unique unitary map such that for all \( i \in \{1, ..., k\} \), \( \mathcal{Q}_{\text{in}}(w_i) = w_i \). Then \( U' = (I \otimes Q_{\text{out}}) \circ U \circ (I \otimes \mathcal{Q}_{\text{in}}) \), as desired.

**A.13 Universality of LOCC Circuits**

Let \( U : \mathbb{C}^{M_n} \to \mathbb{C}^{M_m} \) be a sub-unitary map i.e. a map \( U \) s.t. \( U^* U \subseteq I_n \). We show in the following how to construct a LOCC circuit \( C \) s.t. \( [C] = U \). First notice that \( V : \mathbb{C}^{2n} \to \mathbb{C}^{2m} = \mu_m \circ U \circ \mu_n \) is also a sub-unitary map, where \( \mu_n : \mathbb{C}^{M_n} \to \mathbb{C}^{2n} \) is such that \( \mu|V_k| = 2k \) and \( \mu|H_k| = |2k + 1| \).
Since $I_n - V^\dagger V$ is semi-definite positive there exists $A : \mathbb{C}^{2n} \to \mathbb{C}^{\ell}$ s.t. $A^\dagger A = I_n - V^\dagger V$. As a consequence the matrix $W : \mathbb{C}^{2n} \to \mathbb{C}^{2m+\ell}$ is an isometry since $W^\dagger W = V^\dagger V + A^\dagger A = I_{2n}$. $W$ can be turned into a unitary matrix by adding columns to $W$, i.e. $\exists B, D$ s.t. $U' : \mathbb{C}^{2m+\ell} \to \mathbb{C}^{2m+\ell} = \begin{pmatrix} V & A \\ B & D \end{pmatrix}$ is unitary. By Proposition 26 there exists a $\text{LO}_{\text{PP}}$-circuit $T$ s.t. $[T]_{\text{pp}} = U'$. Let $C$ be the following $\text{LO}_{\nu}$-circuit:

![Diagram](image)

By construction, $[C] = U$.

**B Other Valid Equations**

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$0$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(71)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\theta_1$};
    \node (b) at (1,0) {$\theta_2$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(72)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(73)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(74)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(75)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(76)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(77)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(78)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(79)

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
    \node (a) at (0,0) {$\pi$};
    \node (b) at (1,0) {$\pi$};
    \draw (a) -- (b);
\end{tikzpicture}
\end{array}
\end{align*}

(80)

**Figure 12** Interesting consequences of the axioms of the $\text{LO}_{\nu}$-calculus.