Generalized Appell Systems
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Abstract

We give a general approach to infinite dimensional non-Gaussian analysis which generalizes the work [KSWY95]. For given measure we construct a family of biorthogonal systems. We study their properties and their Gel’fand triples that they generate. As an example we consider the measures of Poisson type.
1 Introduction

Non-Gaussian analysis was already introduced in [AKS93] for smooth probability measure on infinite dimensional linear spaces, using biorthogonal decomposition as a natural extension of the chaos decomposition that is well known in Gaussian analysis. This biorthogonal “Appell” system has been constructed for smooth measures by Yu. L. Daletskii [Dal91]. For a detailed description of its use in infinite dimensional analysis and for the proof of the results which were announced in [AKS93] we refer to [ADKS96] which was based on quasi-invariance of the measures and smoothness of the logarithmic derivatives.

Kondratiev et al. [KSWY95] considered the case of non-degenerate measures on the dual of a nuclear space with analytic characteristic functionals and no further conditions such as quasi-invariance of the measure or smoothness of the logarithmic derivative was required. In this case the important example of Poisson noise is now accessible. Again for a given measure $\mu$ with analytic Laplace transform [KSWY95] construct an Appell biorthogonal system $A^\mu$ as a pair $(P^\mu, Q^\mu)$ of Appell polynomials $P^\mu$ and a canonical system of generalized functions $Q^\mu$, properly associated to the measure $\mu$. Hence within this framework they obtained:

- explicit description of the test function space introduced in [ADKS96];
- the test functions space is identical for all measures that they consider;
- characterization theorems for generalized as well as test functions was obtained analogously as in Gaussian analysis, see [KLP+96] for more references;
- extension of the Wick product and the corresponding Wick calculus [KLS96] as well as full description of positive distributions (as measures).

Aim of the present work. As in [KSWY95] we consider the case of non-degenerate measures on the dual of a nuclear space with analytic Laplace transform but instead of the $\mu$-exponential $e_\mu(\cdot, \cdot)$ we use the generalized $\mu$-exponential $e_\alpha^\mu(\cdot, \cdot)$ where $\alpha$ is a holomorphic function $\alpha$ on $\mathcal{N}_C$ which is invertible in a neighborhood of zero, i.e., $\alpha \in \text{Hol}_0(\mathcal{N}_C, \mathcal{N}_C)$. Hence using $e_\alpha^\mu(\cdot, \cdot)$ we construct an generalized Appell orthogonal system $A^{\mu, \alpha}$ as a pair.
\((P^{\mu,\alpha}, Q^{\mu,\alpha})\) of generalized Appell polynomials \(P^{\mu,\alpha}\) and a system of generalized functions \(Q^{\mu,\alpha}\).

**Central results.** In the above framework

- we obtain an explicit description of the test function space introduced in [ADKS96];
- the spaces of test functions turns out to be the same for all \(\alpha \in \text{Hol}_0(\mathcal{N}_C, \mathcal{N}_C)\) and for all measures that we consider;
- characterization theorems for generalized as well as test functions are obtained analogously as in the Gaussian case;
- the spaces of distributions for a fixed measure \(\mu\) are again identical for all function \(\alpha\) in the above conditions;
- the well known Wick product and the corresponding Wick calculus [KLS96] extends rather directly;
- in the important case of Poisson white noise a special choice of \(\alpha\) produces the orthogonal system of Charlier polynomials, see Example 5.2.

### 2 General theory

#### 2.1 Some facts on nuclear triples

We start with a real separable Hilbert space \(\mathcal{H}\) with inner product \((\cdot, \cdot)\) and norm \(|\cdot|\). For a given separable nuclear space \(\mathcal{N}\) densely topologically embedded in \(\mathcal{H}\) we can construct the nuclear triple

\[ \mathcal{N} \subset \mathcal{H} \subset \mathcal{N}'. \]

The dual pairing \(\langle \cdot, \cdot \rangle\) of \(\mathcal{N}'\) and \(\mathcal{N}\) then is realized as an extension of the inner product in \(\mathcal{H}\)

\[ \langle f, \xi \rangle = (f, \xi) \quad f \in \mathcal{H}, \ \xi \in \mathcal{N}. \]

Instead of reproducing the abstract definition of nuclear spaces (see e.g., [Sch71]) we give a complete (and convenient) characterization in terms of projective limits of decreasing chains of Hilbert spaces \(\mathcal{H}_p, \ p \in \mathbb{N}\).
Theorem 2.1 The nuclear Fréchet space $\mathcal{N}$ can be represented as

$$\mathcal{N} = \bigcap_{p \in \mathbb{N}} \mathcal{H}_p,$$

where $\{\mathcal{H}_p, p \in \mathbb{N}\}$ is a family of Hilbert spaces such that for all $p_1, p_2 \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that the embeddings $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_1}$, $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p_2}$ are of Hilbert-Schmidt class. The topology of $\mathcal{N}$ is given by the projective limit topology, i.e., the coarsest topology on $\mathcal{N}$ such that the canonical embeddings $\mathcal{N} \hookrightarrow \mathcal{H}_p$ are continuous for all $p \in \mathbb{N}$.

The Hilbert norms on $\mathcal{H}_p$ are denoted by $|\cdot|_p$. Without loss of generality we always suppose that $\forall p \in \mathbb{N}$, $\forall \xi \in \mathcal{N}$ : $|\xi| \leq |\xi|_p$ and that the system of norms is ordered, i.e., $|\cdot|_p \leq |\cdot|_q$ if $p < q$. By general duality theory the dual space $\mathcal{N}'$ can be written as

$$\mathcal{N}' = \bigcup_{p \in \mathbb{N}} \mathcal{H}_{-p},$$

with inductive limit topology $\tau_{\text{ind}}$ by using the dual family of spaces $\{\mathcal{H}_{-p} := \mathcal{H}'_p, p \in \mathbb{N}\}$. The inductive limit topology (w.r.t. this family) is the finest topology on $\mathcal{N}'$ such that the embeddings $\mathcal{H}_{-p} \hookrightarrow \mathcal{N}'$ are continuous for all $p \in \mathbb{N}$. It is convenient to denote the norm on $\mathcal{H}_{-p}$ by $|\cdot|_{-p}$. Let us mention that in our setting the topology $\tau_{\text{ind}}$ coincides with the Mackey topology $\tau(\mathcal{N}', \mathcal{N})$ and the strong topology $\beta(\mathcal{N}', \mathcal{N})$, see e.g., [HKPS93, Appendix 5].

Further we want to introduce the notion of tensor power of a nuclear space. The simplest way to do this is to start from usual tensor powers $\mathcal{H}_p^\otimes n$, $n \in \mathbb{N}$ of Hilbert spaces. Since there is no danger of confusion we will preserve the notation $|\cdot|_p$ and $|\cdot|_{-p}$ for the norms on $\mathcal{H}_p^\otimes n$ and $\mathcal{H}_{-p}^\otimes n$ respectively. Using the definition

$$\mathcal{N}^\otimes n := \text{pr lim}_{p \in \mathbb{N}} \mathcal{H}_p^\otimes n,$$

one can prove [Sch71] that $\mathcal{N}^\otimes n$ is a nuclear space which is called the $n$-th tensor power of $\mathcal{N}$.

The dual space of $\mathcal{N}^\otimes n$ can be written

$$\mathcal{N}'^\otimes n \equiv \text{ind lim}_{p \in \mathbb{N}} \mathcal{H}_{-p}^\otimes n.$$
We also want to introduce the (Boson or symmetric) Fock space $\Gamma(\mathcal{H})$ of $\mathcal{H}$ by

$$\Gamma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^\otimes_n$$

with the convention $\mathcal{H}^\otimes_0 := \mathbb{C}$ and the Hilbert norm

$$\|\varphi\|_{\Gamma(\mathcal{H})}^2 = \sum_{n=0}^{\infty} n! |\varphi^{(n)}|^2, \varphi = \{\varphi^{(n)} | n \in \mathbb{N}_0\} \in \Gamma(\mathcal{H}).$$

### 2.2 Holomorphy on locally convex spaces

We shall collect some facts from the theory of holomorphic functions in locally convex topological vector spaces $\mathcal{E}$ (over the complex field $\mathbb{C}$), see e.g., [Din81]. Let $\mathcal{L}(\mathcal{E}^n)$ be the space of $n$-linear mappings from $\mathcal{E}^n$ into $\mathbb{C}$ and $\mathcal{L}_s(\mathcal{E}^n)$ the subspace of symmetric $n$-linear forms. Also let $P^n(\mathcal{E})$ denote the $n$-homogeneous polynomials on $\mathcal{E}$. There is a linear bijection $\mathcal{L}_s(\mathcal{E}^n) \ni A \longleftrightarrow \widehat{A} \in P^n(\mathcal{E})$. Now let $\mathcal{U} \subset \mathcal{E}$ be open and consider a function $G : \mathcal{U} \to \mathbb{C}$. $G$ is said to be **holomorphic** if for all $\theta_0 \in \mathcal{U}$ and for all $\theta \in \mathcal{E}$ the mapping from $\mathbb{C}$ to $\mathbb{C}$: $\lambda \mapsto G(\theta_0 + \lambda \theta)$ is holomorphic in some neighborhood of zero in $\mathbb{C}$. If $G$ is holomorphic then there exists for every $\eta \in \mathcal{U}$ a sequence of homogeneous polynomials $\frac{1}{n!}d^nG(\eta)$ such that

$$G(\theta + \eta) = \sum_{n=0}^{\infty} \frac{1}{n!}d^nG(\eta)(\theta)$$

for all $\theta$ from some open neighborhood $\mathcal{V}$ of zero. $G$ is said to be **holomorphic**, if for all $\eta$ in $\mathcal{U}$ there exists an open neighborhood $\mathcal{V}$ of zero such that

$$\sum_{n=0}^{\infty} \frac{1}{n!}d^nG(\eta)(\theta)$$

converges uniformly on $\mathcal{V}$ to a continuous function. Of course, $d^nG(\eta)(\theta)$ is the $n$-th partial derivative of $G$ at $\eta$ in direction $\theta$. We say that $G$ is holomorphic at $\theta_0$ if there is an open set $\mathcal{U}$ containing $\theta_0$ such that $G$ is holomorphic on $\mathcal{U}$. The following Proposition can be found e.g., in [Din81].

**Proposition 2.2** $G$ is holomorphic if and only if it is G-holomorphic and locally bounded.
Let us explicitly consider a function holomorphic at the point $0 \in \mathcal{E} = \mathcal{N}_C$, then

1) there exist $p$ and $\varepsilon > 0$ such that for all $\xi_0 \in \mathcal{N}_C$ with $|\xi_0|_p \leq \varepsilon$ and for all $\xi \in \mathcal{N}_C$ the function of one complex variable $\lambda \mapsto G(\xi_0 + \lambda \xi)$ is holomorphic at $0 \in \mathbb{C}$, and

2) there exists $c > 0$ such that for all $\xi \in \mathcal{N}_C$ with $|\xi|_p \leq \varepsilon$:

As we do not want to discern between different restrictions of one function, we consider germs of holomorphic functions, i.e., we identify $F$ and $G$ if there exists an open neighborhood $U : 0 \in U \subset \mathcal{N}_C$ such that $F(\xi) = G(\xi)$ for all $\xi \in U$. Thus we define $\text{Hol}_0(\mathcal{N}_C)$ as the algebra of germs of functions holomorphic at zero equipped with the inductive topology given by the following family of norms

$$n_{p,l,\infty}(G) = \sup_{|\theta|_p \leq 2^{-l}} |G(\theta)|, \quad p, l \in \mathbb{N}.$$ 

For later use we need the space $\text{Hol}_0(\mathcal{N}_C, \mathcal{N}_C)$ of holomorphic functions from $\mathcal{N}_C$ to $\mathcal{N}_C$. Let $U \subset \mathcal{N}_C$ be open and consider a function $\alpha : U \to \mathcal{N}_C$. $\alpha$ is said to be holomorphic at $0 \in \mathcal{N}_C$ iff

1. it is $G$-holomorphic; i.e., there exist $p$ and $\varepsilon > 0$ such that for all $\xi_0 \in \mathcal{N}_C$ with $|\xi_0|_p \leq \varepsilon$ and for all $\xi \in \mathcal{N}_C$ the function of one complex variable $\lambda \mapsto \alpha(\xi_0 + \lambda \xi)$ is holomorphic at $0 \in \mathbb{C}$;

2. $\alpha$ is locally bounded, i.e., for all $p \in \mathbb{N}$ there exist $C_p > 0$ such that $\forall \eta \in A$ with $|\eta|_p \leq C_p$ then $\forall \eta' \in \mathbb{N}$ there exist $C_{p'}$ such that $\forall \eta \in A$ $|\alpha(\eta)|_{p'} \leq C_{p'}$, where $A$ is an bounded set in $\mathcal{N}_C$.

If $\alpha$ is holomorphic at $0 \in \mathcal{N}_C$, then for every $\eta \in U$ there exists a sequence of homogeneous polynomials $\frac{1}{n!} d^n \alpha(\eta)$ such that

$$\theta \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} d^n \alpha(\eta) (\theta)$$

converges and define a continuous function on some neighborhood of zero.

Let us now introduce spaces of entire functions which will be useful later. Let $\mathcal{E}_{2^{-l}}(\mathcal{H}_{-p,C})$ denote the set of all entire functions on $\mathcal{H}_{-p,C}$ of growth $k \in [1, 2]$ and type $2^{-l}$, $p, l \in \mathbb{Z}$. This is a linear space with norm

$$n_{p,l,k}(\varphi) = \sup_{z \in \mathcal{H}_{-p,C}} |\varphi(z)| \exp \left(-2^{-l}|z|_{-p}^k\right), \quad \varphi \in \mathcal{E}_{2^{-l}}(\mathcal{H}_{-p,C}).$$
The space of entire functions on $\mathcal{N}'_C$ of growth $k$ and minimal type is naturally introduced by

$$
\mathcal{E}_{\min}^k(\mathcal{N}'_C) := \text{pr lim}_{p,l\in\mathbb{N}} \mathcal{E}_{2-i}^k(\mathcal{H}_{-p,C}),
$$

see e.g., [Kon91], [BK95], [Oue91]. We will also need the space of entire functions on $\mathcal{N}_C$ of growth $k$ and finite type:

$$
\mathcal{E}_{\max}^k(\mathcal{N}_C) := \text{ind lim}_{p,l\in\mathbb{N}} \mathcal{E}_{2i}^k(\mathcal{H}_{p,C}).
$$

### 2.3 Measures on linear topological spaces

To introduce probability measures on the vector space $\mathcal{N}'$, we consider $\mathcal{C}_\sigma(\mathcal{N}')$ the $\sigma$-algebra generated by cylinder sets on $\mathcal{N}'$, which coincides with the Borel $\sigma$-algebras $\mathcal{B}_\sigma(\mathcal{N}')$ and $\mathcal{B}_\beta(\mathcal{N}')$ generated by the weak and strong topology on $\mathcal{N}'$, respectively. Thus we will consider this $\sigma$-algebra as the natural $\sigma$-algebra on $\mathcal{N}'$. Detailed definitions of the above notions and proofs of the mentioned relations can be found in e.g., [BK95].

We will restrict our investigations to a special class of measures $\mu$ on $\mathcal{C}_\sigma(\mathcal{N}')$ which satisfy two additional assumptions. The first one concerns some analyticity of the Laplace transformation

$$
l_\mu(\theta) := L_\mu 1(\theta) = \int_{\mathcal{N}'} \exp \langle x, \theta \rangle \, d\mu(x) =: \mathbb{E}_\mu(\exp \langle \cdot, \theta \rangle), \quad \theta \in \mathcal{N}_C.
$$

Here we also have introduced the convenient notion of expectation $\mathbb{E}_\mu$ of a $\mu$-integrable function.

**Assumption 1** The measure $\mu$ has an analytic Laplace transform in a neighborhood of zero. That means there exists an open neighborhood $\mathcal{U} \subset \mathcal{N}_C$ of zero, such that $l_\mu$ is holomorphic on $\mathcal{U}$, i.e., $l_\mu \in \text{Hol}_0(\mathcal{N}_C)$. This class of **analytic measures** is denoted by $\mathcal{M}_a(\mathcal{N}')$.

An equivalent description of analytic measures is given by the following lemma and the proof can be founded in [KSW95].

**Lemma 2.3** The following statements are equivalent

1) $\mu \in \mathcal{M}_a(\mathcal{N}')$;

2) $\exists p_\mu \in \mathbb{N}, \; \exists C > 0 : \left| \int_{\mathcal{N}'} \langle x, \theta \rangle^n d\mu(x) \right| \leq n! C^n |\theta|_{p_\mu}^n, \; \theta \in \mathcal{H}_{p_\mu,C}$;
3) \( \exists p'_\mu \in \mathbb{N}, \ \exists \varepsilon_\mu > 0 : \int_{\mathcal{N}'} \exp(\varepsilon_\mu |x|_{-p'_\mu})d\mu(x) < \infty. \)

For \( \mu \in \mathcal{M}_a(\mathcal{N}') \) the estimate in statement 2 of the above lemma allows to define the moment kernels \( M^n_\mu \in \mathcal{N}^{\hat{\otimes} n} \). This is done by extending the above estimate by a simple polarization argument and applying the kernel theorem. The kernels are determined by

\[
l_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M^n_\mu, \theta \otimes^n \rangle
\]

(2.1)

or equivalently

\[
\langle M^n_\mu, \theta \otimes \theta \otimes ... \otimes \theta \rangle = \left. \frac{\partial^n}{\partial t_1 \partial t_2 ... \partial t_n} l_\mu(t_1 \theta_1 + ... + t_n \theta_n) \right|_{t_1=...=t_n=0}.
\]

Moreover, if \( p > p_\mu \) is such that the embedding \( i_{p,p_\mu} : \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_\mu} \) is Hilbert-Schmidt then

\[
|M^n_\mu|_{-p} \leq \left( nC \| i_{p,p_\mu} \|_{HS} \right)^n \leq n! \left( eC \| i_{p,p_\mu} \|_{HS} \right)^n.
\]

(2.2)

**Definition 2.4** A function \( \varphi : \mathcal{N}' \to \mathbb{C} \) of the form

\[
\varphi(x) = \sum_{n=0}^{N} \langle x^{\otimes n}, \varphi^{(n)} \rangle, \ x \in \mathcal{N}', \ N \in \mathbb{N},
\]

is called a **continuous polynomial** (short \( \varphi \in \mathcal{P}(\mathcal{N}') \)) iff \( \varphi^{(n)} \in \mathcal{N}_\mathbb{C}^{\hat{\otimes} n}, \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

Now we are ready to formulate the second assumption on \( \mu \):

**Assumption 2** For all \( \varphi \in \mathcal{P}(\mathcal{N}') \) with \( \varphi = 0 \) \( \mu \)-almost everywhere we have \( \varphi \equiv 0 \). In the following a measure with this property will be called **non-degenerate**.

**Note:** Assumption 2 is equivalent to:
Let \( \varphi \in \mathcal{P}(\mathcal{N}') \) with \( \int_A \varphi d\mu = 0 \) for all \( A \in \mathcal{C}_\sigma(\mathcal{N}') \) then \( \varphi \equiv 0 \).
A sufficient condition can be obtained by regarding admissible shifts of the measure $\mu$. If $\mu(\cdot + \xi)$ is absolutely continuous with respect to $\mu$ for all $\xi \in \mathcal{N}$, i.e., there exists the Radon-Nikodym derivative
\[
\rho_\mu(\xi, x) = \frac{d\mu(x + \xi)}{d\mu(x)} \in L^1(\mathcal{N}', \mu), \quad x \in \mathcal{N}',
\]
then we say that $\mu$ is $\mathcal{N}$-quasi-invariant see e.g., [GV68], [Sko74]. This is sufficient to ensure Assumption 2, see e.g., [KT91], [BK95].

3 The Appell system

The space $\mathcal{P}(\mathcal{N}')$ may be equipped with various different topologies, but there exists a natural one such that $\mathcal{P}(\mathcal{N}')$ becomes isomorphic to the topological direct sum of tensor powers $\mathcal{N}_c^\otimes n$ see e.g., [Sch71, Chap. II 6.1, Chap. II 7.4]
\[
\mathcal{P}(\mathcal{N}') \simeq \bigoplus_{n=0}^{\infty} \mathcal{N}_c^\otimes n
\]
via
\[
\varphi(x) = \sum_{n=0}^{\infty} \langle x^\otimes n, \varphi^{(n)} \rangle \longleftrightarrow \vec{\varphi} = \{\varphi^{(n)} \mid n \in \mathbb{N}_0\}.
\]
Note that only a finite number of $\varphi^{(n)}$ is a non-zero. The notion of convergence of sequences in this topology on $\mathcal{P}(\mathcal{N}')$ is the following: for $\varphi \in \mathcal{P}(\mathcal{N}')$, such that
\[
\varphi(x) = \sum_{n=0}^{N(\varphi)} \langle x^\otimes n, \varphi^{(n)} \rangle
\]
let $p_n : \mathcal{P}(\mathcal{N}') \to \mathcal{N}_c^\otimes n$ denote the mapping $p_n$ defined by $p_n(\varphi) := \varphi^{(n)}$. A sequence $\{\varphi_j, j \in \mathbb{N}\}$ of smooth polynomials converge to $\varphi \in \mathcal{P}(\mathcal{N}')$ iff the $\mathcal{N}(\varphi_j)$ are bounded and $p_n \varphi_j \to p_n \varphi$ in $\mathcal{N}_c^\otimes n$ for all $n \in \mathbb{N}$.

Now we can introduce the dual space $\mathcal{P}_\mu'(\mathcal{N}')$ of $\mathcal{P}(\mathcal{N}')$ with respect to $L^2(\mu)$. As a result we have constructed the triple
\[
\mathcal{P}(\mathcal{N}') \subset L^2(\mu) \subset \mathcal{P}_\mu'(\mathcal{N}')
\]
The (bilinear) dual pairing $\langle \cdot, \cdot \rangle_\mu$ between $\mathcal{P}_\mu'(\mathcal{N}')$ and $\mathcal{P}(\mathcal{N}')$ is connected to the (sesquilinear) inner product on $L^2(\mu)$ by
\[
\langle \varphi, \psi \rangle_\mu = (\varphi, \overline{\psi})_{L^2(\mu)}, \quad \varphi \in L^2(\mu), \; \psi \in \mathcal{P}(\mathcal{N}').
\]
3.1 \(\text{P}^\mu\)-system

Because of the holomorphy of \(l_\mu\) and since that \(l_\mu(0) = 1\), there exists a neighborhood of zero

\[
U_0 = \left\{ \theta \in \mathbb{N}_C \mid 2^{q_0} |\theta|_{p_0} < 1 \right\}
\]

\(p_0, q_0 \in \mathbb{N}, p_0 \geq p'_\mu, 2^{-q_0} \leq \varepsilon_\mu \ (p'_\mu, \varepsilon_\mu \text{ from Lemma 2.3})\) such that \(l_\mu(\theta) \neq 0\) for \(\theta \in U_0\) and the normalized \(\mu\)-exponential

\[
e_\mu(\theta; z) := \frac{\exp(z, \theta)}{l_\mu(\theta)} \quad \text{for} \ \theta \in U_0, \ z \in \mathbb{N}'_C, \quad (3.1)
\]

is well defined. We use the holomorphy of \(\theta \mapsto e_\mu(\theta; z)\) to expand it in a power series in \(\theta\) similar to the case corresponding to the construction of one dimensional Appell polynomials [Bou76]. We have in analogy to [AKS93, ADKS96]

\[
e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} d^n e_\mu(0, z) (\theta),
\]

where \(d^n e_\mu(0, z)\) is an \(n\)-homogeneous form polynomial. But since \(e_\mu(\theta; z)\) is not only \(G\)-holomorphic but holomorphic we know that \(\theta \mapsto e_\mu(\theta; z)\) is also locally bounded. Thus Cauchy’s inequality for Taylor series [Dim81] may be applied, \(\rho \leq 2^{-q_0}, p \geq p_0\)

\[
\left| \frac{1}{n!} d^n e_\mu(0, z) (\theta) \right| \leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} |e_\mu(\theta; z)| |\theta|^n
\]

\[
\leq \frac{1}{\rho^n} \sup_{|\theta|_p = \rho} \frac{1}{l_\mu(\theta)} \exp\left(\rho |z|_{-p}\right) |\theta|^n \quad (3.2)
\]

if \(z \in \mathcal{H}_{-p,C}\). This inequality extends by polarization [Dim81] to an estimate sufficient for the kernel theorem. Thus we have a representation

\[
d^n e_\mu(0, z) (\theta) = \langle P_\mu^n(z), \theta^\otimes n \rangle,
\]

where \(P_\mu^n(z) \in \mathcal{N}_C^\otimes n\). The kernel theorem really gives a little more: \(P_\mu^n(z) \in \mathcal{H}_{p',C}^\otimes n\) for any \(p' \ (p \geq p_0)\) such that the embedding operator

\[
i_{p', p} : \mathcal{H}_{p',C} \hookrightarrow \mathcal{H}_{p,C}
\]

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is Hilbert-Schmidt. Thus we have
\[ e_\mu(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P^\mu_n(z), \theta^{\otimes n} \rangle \text{ for } \theta \in \mathcal{U}_0, \ z \in \mathcal{N}_C'. \] (3.3)

We will also use the notation
\[ P^\mu_n(\varphi^{(n)})(\cdot) := \langle P^\mu_n(\cdot), \varphi^{(n)} \rangle, \ \varphi^{(n)} \in \mathcal{N}_C^{\otimes n}, \ n \in \mathbb{N}, \]
which is called Appell polynomial. Thus for any measure satisfying Assumption 1 we have defined the \( P^\mu \)-system
\[ P^\mu = \left\{ \langle P^\mu_n(\cdot), \varphi^{(n)} \rangle \mid \varphi^{(n)} \in \mathcal{N}_C^{\otimes n}, \ n \in \mathbb{N}_0 \right\}. \]

The following proposition collects some properties of the polynomials \( P^\mu_n(z) \), (for the proof we refer to [KSWY95]).

**Proposition 3.1** For \( x, y \in \mathcal{N}', \ n \in \mathbb{N} \) the following holds

\[ (P1) \quad P^\mu_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^{\otimes k} \otimes P^\mu_{n-k}(0). \] (3.4)

\[ (P2) \quad x^{\otimes n} = \sum_{k=0}^{n} \binom{n}{k} P^\mu_k(x) \otimes M^\mu_{n-k}. \] (3.5)

\[ (P3) \quad P^\mu_n(x + y) = \sum_{k+l+m=n} \frac{n!}{k! l! m!} P^\mu_k(x) \otimes P^\mu_l(y) \otimes M^\mu_m \]
\[ = \sum_{k=0}^{n} \binom{n}{k} P^\mu_k(x) \otimes y^{\otimes (n-k)}. \] (3.6)

\[ (P4) \text{ Further we observe} \]
\[ \mathbb{E}_\mu(\langle P^\mu_m(\cdot), \varphi^{(m)} \rangle) = 0 \text{ for } m \neq 0, \ \varphi^{(m)} \in \mathcal{N}_C^{\otimes m}. \] (3.7)

\[ (P5) \text{ For all } p > p_0 \text{ such that the embedding } \mathcal{H}_p \hookrightarrow \mathcal{H}_{p_0} \text{ is Hilbert–Schmidt and for all } \varepsilon > 0 \text{ small enough } (\varepsilon \leq (2^{p_0} \| i_{p,p_0} \|_{HS} )^{-1}) \text{ there exists a constant } C_{p,\varepsilon} > 0 \text{ with} \]
\[ |P^\mu_n(z)|_{-p} \leq C_{p,\varepsilon} n! \varepsilon^{-n} e \varepsilon |z|_{-p}, \ \ z \in \mathcal{H}_{-p,\mathbb{C}}. \] (3.8)
The following lemma describes the set of polynomials $\mathcal{P}(\mathcal{N}')$.

**Lemma 3.2** For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation

$$\varphi(x) = \sum_{n=0}^{N} \langle P_{n}(x), \varphi^{(n)} \rangle, \quad \varphi^{(n)} \in \mathcal{N}_{\mathbb{C}}^\otimes n$$

(3.9)

and vice versa, any functional of the form (3.9) is a smooth polynomial.

### 3.2 $\mathcal{Q}^\mu$-system

To give an internal description of the type (3.9) for $\mathcal{P}'_{\mu}(\mathcal{N}')$ we have to construct an appropriate system of generalized functions, the $\mathcal{Q}^\mu$-system. We propose to construct the $\mathcal{Q}^\mu$-system using differential operators.

For $\Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}^\otimes n$ define a differential operator, $D(\Phi^{(n)})$, of order $n$ and constant coefficients $\Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}^\otimes n$, such that, applied to monomials $\langle x^\otimes m, \varphi^{(m)} \rangle$, $\varphi^{(m)} \in \mathcal{N}_{\mathbb{C}}^\otimes m$, $m \in \mathbb{N}$

$$D \left( \Phi^{(n)} \right) \langle x^\otimes m, \varphi^{(m)} \rangle = \begin{cases} 
\frac{m!}{(m-n)!} \langle x^\otimes (m-n), \Phi^{(n)} \rangle, & \text{for } m \geq n \\
0, & \text{for } m < n 
\end{cases}$$

(3.10)

and extend by linearity from the monomials to $\mathcal{P}(\mathcal{N}')$.

**Lemma 3.3** $D(\Phi^{(n)})$ is a continuous linear operator from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}')$.

**Remark** For $\Phi^{(1)} \in \mathcal{N}$ we have the usual Gâteaux derivative as e.g., in white noise analysis [HKPS93]

$$D \left( \Phi^{(1)} \right) \varphi = D_{\Phi^{(1)}} \varphi := \frac{d}{dt} \varphi \left( \cdot + t \Phi^{(1)} \right) \big|_{t=0}$$

for $\varphi \in \mathcal{P}(\mathcal{N}')$. Moreover we have $D((\Phi^{(1)})^\otimes n) = (D_{\Phi^{(1)}})^n$, thus $D((\Phi^{(1)})^\otimes n)$ is a differential operator of order $n$.

In view of Lemma 3.3 it is possible to define the adjoint operator

$$D(\Phi^{(n)})^* : \mathcal{P}'_{\mu}(\mathcal{N}') \rightarrow \mathcal{P}'_{\mu}(\mathcal{N}'), \quad \Phi^{(n)} \in \mathcal{N}_{\mathbb{C}}^\otimes n.$$
Further we introduce the constant function $1 \in L^2(\mu) \subset P'_\mu(\mathcal{N}')$ such that $1(x) \equiv 1$ for all $x \in \mathcal{N}'$, so
\[
\langle 1, \varphi \rangle_\mu = \int_{\mathcal{N}'} \varphi(x) \, d\mu(x) = \mathbb{E}_\mu(\varphi).
\]

Now we are ready to define the $Q^\mu$-system.

**Definition 3.4** For any $\Phi^{(n)} \in \mathcal{N}'^{\otimes n}$ we define a generalized function $Q^\mu_n(\Phi^{(n)}) \in P'_\mu(\mathcal{N}')$ by
\[
Q^\mu_n(\Phi^{(n)}) = D(\Phi^{(n)})^* 1.
\]

We want to introduce an additional formal notation which stresses the linearity of $\Phi^{(n)} \mapsto Q^\mu_n(\Phi^{(n)}) \in P'_\mu(\mathcal{N}')$:
\[
\langle Q^\mu_n, \Phi^{(n)} \rangle := Q^\mu_n(\Phi^{(n)})
\]

**Example 3.5** The simplest non trivial case can be studied using finite dimensional real analysis. We consider the nuclear "triple"
\[
\mathbb{R} \subseteq \mathbb{R} \subseteq \mathbb{R}
\]
where the dual pairing between a "test function" and a "distribution" degenerates to multiplication. On $\mathbb{R}$ we consider a measure $d\mu(x) = \rho(x) \, dx$ where $\rho$ is a positive $C^\infty$-function on $\mathbb{R}$ such that assumptions 1 and 2 are fulfilled. In this setting the adjoint of the differentiation operator is given by
\[
\left( \frac{d}{dx} \right)^* f(x) = - \left( \left( \frac{d}{dx} \right) + \beta(x) \right) f(x), \quad f \in C^\infty(\mathbb{R}),
\]
where $\beta$ is the logarithmic derivative of the measure $\mu$ and given by
\[
\beta = \frac{\rho'}{\rho}.
\]

This enables us to calculate the $Q^\mu$-system. One has
\[
Q^\mu_n(x) = \left( \left( \frac{d}{dx} \right)^* \right)^n 1
= (-1)^n \left( \frac{d}{dx} + \beta(x) \right)^n 1
= (-1)^n \frac{\rho(n)(x)}{\rho(x)}.
\]
where the last equality can be seen by simple induction (for \( \rho \) non smooth this construction produce generalized functions \( Q^n_\mu \) even in this one dimensional case).

If \( \rho(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) \) is the Gaussian density, then \( Q^n_\mu \) is related to the \( n \)-th Hermite polynomial:

\[
Q^n_\mu(x) = 2^{-n/2} H_n\left(\frac{x}{\sqrt{2}}\right).
\]

**Definition 3.6** We define the \( Q^n_\mu \)-system in \( P'_\mu(N') \) by

\[
Q^n_\mu = \left\{ Q^n_\mu(\Phi^{(n)}) \mid \Phi^{(n)} \in N^\otimes^n, \ n \in \mathbb{N}_0 \right\},
\]

and the pair \( (P^n_\mu, Q^n_\mu) \) will be called the **Appell system** \( A^n_\mu \) generated by the measure \( \mu \).

We have the following central property of the Appell system \( A^n_\mu \).

**Theorem 3.7** (Biorthogonality w.r.t. \( \mu \))

\[
\langle \langle Q^n_\mu(\Phi^{(n)}), \ P^n_\mu(\varphi^{(m)}) \rangle \rangle_\mu = \delta_{m,n} n! \langle \Phi^{(n)} , \varphi^{(n)} \rangle (3.11)
\]

for \( \Phi^{(n)} \in N^\otimes^n \) and \( \varphi^{(m)} \in N^\otimes^m \).

Now we are going to characterize the space \( P'_\mu(N') \).

**Theorem 3.8** For all \( \Phi \in P'_\mu(N') \) there exists a unique sequence \( \{\Phi^{(n)} \mid n \in \mathbb{N}_0\} \), \( \Phi^{(n)} \in N^\otimes^n \) such that

\[
\Phi = \sum_{n=0}^{\infty} Q^n_\mu(\Phi^{(n)}) \equiv \sum_{n=0}^{\infty} \langle Q^n_\mu, \Phi^{(n)} \rangle (3.12)
\]

and vice versa, every series of the form (3.12) generates a generalized function in \( P'_\mu(N') \).

The proofs of this result can be found in [KSWY95].
4 The triple $(\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_\mu$

4.1 Test functions

On the space $\mathcal{P}(\mathcal{N}')$ we can define a system of norms using the Appell decomposition from Lemma 3.2. Let

$$\varphi(x) = \sum_{n=0}^{N} \langle P^\mu_n(x), \varphi^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$$

be given, then $\varphi^{(n)} \in \mathcal{H}_{p,\mathbb{C}}$ for each $p \geq 0$ ($n \in \mathbb{N}_0$). Thus we may define for any $p, q \in \mathbb{N}$ a Hilbert norm on $\mathcal{P}(\mathcal{N}')$ by

$$\|\varphi\|_{p,q,\mu}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{nq} \|\varphi^{(n)}\|_p^2$$

The completion of $\mathcal{P}(\mathcal{N}')$ w.r.t. $\|\cdot\|_{p,q,\mu}$ is denoted by $(\mathcal{H}_p)^1_{q,\mu}$.

**Definition 4.1** We define

$$(\mathcal{N})^1_\mu := \operatorname{pr} \lim_{p,q \in \mathbb{N}} (\mathcal{H}_p)^1_{q,\mu}$$

This space have the following properties (for the proofs see [KSWY95] and references therein).

**Theorem 4.2** $(\mathcal{N})^1_\mu$ is a nuclear space. The topology $(\mathcal{N})^1_\mu$ is uniquely defined by the topology on $\mathcal{N}$: It does not depend on the choice of the family of norms $\{|\cdot|_\mu\}$.

**Theorem 4.3** There exists $p', q' > 0$ such that for all $p \geq p'$, $q \geq q'$ the topological embedding $(\mathcal{H}_p)^1_{q,\mu} \subset L^2(\mu)$ holds.

**Corollary 4.4** $(\mathcal{N})^1_\mu$ is continuously and densely embedded in $L^2(\mu)$.

**Theorem 4.5** Any test function $\varphi$ in $(\mathcal{N})^1_\mu$ has a uniquely defined extension to $\mathcal{N}'_\mathbb{C}$ as an element of $\mathcal{E}_{\min}^1(\mathcal{N}'_\mathbb{C})$. 

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In this construction one unexpected moment was the following:

**Theorem 4.6** For all measures $\mu \in \mathcal{M}_a(N')$ we have the topological identity

$$(N)^1_\mu = \mathcal{E}^1_{\text{min}}(N').$$

Since this last theorem states that the space of test functions $(N)^1_\mu$ is isomorphic to $\mathcal{E}^1_{\text{min}}(N')$ for all measures $\mu \in \mathcal{M}_a(N')$, we will drop the subscript $\mu$. The test function space $(N)^1$ is the same for all measures $\mu \in \mathcal{M}_a(N')$.

### 4.2 Distributions

The space $(N)^{-1}_\mu$ of distributions corresponding to the space of test functions $(N)^1$ can be viewed as a subspace of $\mathcal{P}'(N')$, since $\mathcal{P}(N') \subset (N)^1$ topologically, i.e.,

$$(N)^{-1}_\mu \subset \mathcal{P}'(N')$$

Let us now introduce the Hilbert subspace $(H_{-p})^{-1}_{-q,\mu}$ of $\mathcal{P}'(N')$ for which the norm

$$\|\Phi\|_{-p,-q,\mu}^2 := \sum_{n=0}^{\infty} 2^{-qn} |\Phi(n)|^2_{-p}$$

is finite. Here we used the canonical representation

$$\Phi = \sum_{n=0}^{\infty} Q^\mu_n (\Phi(n)) \in \mathcal{P}'(N')$$

from Theorem 3.8. The space $(H_{-p})^{-1}_{-q,\mu}$ is the dual space of $(H^p)_{q,\mu}$ with respect to $L^2(\mu)$ (because of the biorthogonality of $P^\mu$- and $Q^\mu$-systems). By the general duality theory

$$(N)^{-1}_\mu = \bigcup_{p,q \in \mathbb{N}} (H_{-p})^{-1}_{-q,\mu}$$

is the dual space of $(N)^1$ with respect to $L^2(\mu)$. So, we have the topological nuclear triple

$$(N)^1 \subset L^2(\mu) \subset (N)^{-1}_\mu.$$
on a test function

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle \in (N)^1$$

is given by

$$\langle \langle \Phi, \varphi \rangle \rangle_\mu = \sum_{n=0}^{\infty} n! \langle \Phi^{(n)}, \varphi^{(n)} \rangle.$$  

**Example 4.7 (Generalized Radon-Nikodym derivative)** We want to define a generalized function $\rho_\mu(z, \cdot) \in (N)^{-1}_C$ with the following property

$$\langle \langle \rho_\mu(z, \cdot), \varphi \rangle \rangle_\mu = \int_{N'} \varphi(x-z) \, d\mu(x), \, \varphi \in (N)^1.$$  

That means we have to establish the continuity of $\rho_\mu(z, \cdot)$. Let $z \in \mathcal{H}_{-p,C}$. If $p' \geq p$ is sufficiently large and $\epsilon > 0$ is small enough, there exists $q \in \mathbb{N}$ and $C > 0$ such that

$$\left| \int_{N'} \varphi(x-z) \, d\mu(x) \right| \leq C \|\varphi\|_{p',q,\mu} \int_{N'} \exp \left( \epsilon |x-z|_{-p'} \right) \, d\mu(x) \leq C \|\varphi\|_{p',q,\mu} \exp \left( \epsilon |z|_{-p'} \right) \int_{N'} \exp \left( \epsilon |x|_{-p'} \right) \, d\mu(x).$$

If $\epsilon$ is chosen sufficiently small the last integral exists. Thus we have in fact $\rho_\mu(z, \cdot) \in (N)^{-1}_\mu$. It is clear that whenever the Radon-Nikodym derivative $\frac{d\mu(x+\xi)}{d\mu(x)}$ exists (e.g., $\xi \in \mathcal{N}$ in case $\mu$ is $\mathcal{N}$-quasi-invariant) it coincides with $\rho_\mu(z, \cdot)$ defined above. We will show that in $(N)^{-1}_\mu$ we have the canonical expansion

$$\rho_\mu(z, \cdot) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} Q_n^\mu(z^{\otimes n}).$$

Since both sides are in $(N)^{-1}_\mu$ it is sufficient to compare their action on a total set from $(N)^1$. For $\varphi^{(n)} \in \mathcal{N}^n_C$ we have

$$\langle \langle \rho_\mu(z, \cdot), \langle P_n^\mu, \varphi^{(n)} \rangle \rangle_\mu = \int_{N'} \langle P_n^\mu(x-z), \varphi^{(n)} \rangle \, d\mu(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \int_{N'} \langle P_k^\mu(x), z^{\otimes (n-k)} \otimes \varphi^{(n)} \rangle \, d\mu(x)$$

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\[ (-1)^n \langle z \otimes^n, \varphi^{(n)} \rangle = \left\langle \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k Q^\mu_k \left( z \otimes^k \right), \langle P^\mu_n, \varphi^{(n)} \rangle \right\rangle \mu, \]

where we have used (3.6), (3.7) and the biorthogonality of \( P^\mu - \) and \( Q^\mu - \) systems. In other words, we have proven that \( \rho^\mu (-z, \cdot) \) is the generating function of the \( Q^\mu - \) system

\[ \rho^\mu (-z, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} Q^\mu_n \left( z \otimes^n \right). \]

### 4.3 Integral transformations

#### 4.3.1 Normalized Laplace transform \( S^\mu \)

We first introduce the Laplace transform of a function \( \varphi \in L^2(\mu) \). The global assumption \( \mu \in \mathcal{M}_a(\mathcal{N}^\mu) \) guarantees the existence of \( p'_\mu \in \mathbb{N}, \epsilon_\mu > 0 \) such that

\[ \int_{\mathcal{N}^\mu} \exp \left( -\epsilon_\mu |x|_p \right) \, d\mu (x) < \infty \]

by Lemma 2.3. Thus \( \exp((\cdot , \theta)) \in L^2(\mu) \) if \( 2 |\theta|_{p'_\mu} < \epsilon_\mu, \theta \in \mathcal{H}_{p'_\mu, \mathbb{C}} \). Then by Cauchy-Schwarz inequality the Laplace transform defined by

\[ L^\mu \varphi (\theta) := \int_{\mathcal{N}^\mu} \varphi (x) \exp \langle x, \theta \rangle \, d\mu (x) \]

is well defined for \( \varphi \in L^2(\mu), \theta \in \mathcal{H}_{p'_\mu, \mathbb{C}} \). Now we are interested to extend this integral transform from \( L^2(\mu) \) to the space of distributions \( (\mathcal{N})_{\mu}^{-1} \).

Since our construction of test functions and distributions spaces is closely related to \( P^\mu - \) and \( Q^\mu - \) systems it is useful to introduce the so called \( S^\mu - \) transform

\[ S^\mu \varphi (\theta) := \frac{L^\mu \varphi (\theta)}{l^\mu (\theta)} = \int_{\mathcal{N}^\mu} \varphi (x) e^\mu (\theta; x) \, d\mu (x). \]

The \( \mu \)-exponential \( e^\mu (\theta; \cdot) \) is not a test function in \( (\mathcal{N})^1 \), see [KSWY93, Example 6], so the definition of the \( S^\mu - \) transform of a distribution \( \Phi \in (\mathcal{N})_{\mu}^{-1} \) must be more careful. Every such \( \Phi \) is of finite order, i.e., \( \exists p, q \in \mathbb{N} \) such that \( \Phi \in (\mathcal{H}_p)_{q, \mu}^{-1} \) and \( e^\mu (\theta; \cdot) \) is in the corresponding dual space \( (\mathcal{H}_p)_{q, \mu}^1 \) if
\[ \theta \in \mathcal{H}_{p,c} \text{ is such that } 2^q |\theta|^2_p < 1. \] Then we can define a consistent extension of \( S_\mu \)-transform.

\[ S_\mu \Phi (\theta) := \langle \langle \Phi, e_\mu (\theta, \cdot) \rangle \rangle_\mu \]

if \( \theta \) is chosen in the above way. The biorthogonality of \( P_\mu \)- and \( Q_\mu \)-system implies

\[ S_\mu \Phi (\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}, \theta^{\otimes n} \rangle, \]

moreover \( S_\mu \Phi \in \text{Hol}_0(\mathcal{N}_C) \), see [KSWY95, Theorem 35].

### 4.3.2 Convolution \( C_\mu \)

We define the convolution of a function \( \varphi \in (\mathcal{N})^1 \) with the measure \( \mu \) by

\[ C_\mu \varphi (y) := \int_{\mathcal{N}'} \varphi (x + y) \, d\mu (x), \quad y \in \mathcal{N}'. \]

For any \( \varphi \in (\mathcal{N})^1 \), \( z \in \mathcal{N}_C' \), the convolution has the representation

\[ C_\mu \varphi (z) = \langle \langle \rho_\mu (-z, \cdot), \varphi \rangle \rangle_\mu. \]

If \( \varphi \in (\mathcal{N})^1 \) has the canonical \( P_\mu \)-decomposition

\[ \varphi = \sum_{n=0}^{\infty} \langle P_n^\mu, \varphi^{(n)} \rangle, \]

then

\[ C_\mu \varphi (z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \varphi^{(n)} \rangle. \]

In Gaussian analysis \( C_\mu \)- and \( S_\mu \)-transform coincide. It is a typical non-Gaussian effect that these two transformations differ from each other.

### 4.4 Characterization theorems

Now we will characterize the spaces of test and generalized functions by the integral transforms introduced in the previous section.

We will start to characterize the space \((\mathcal{N})^1\) in terms of the convolution \( C_\mu \).
Theorem 4.8  The convolution $C_\mu$ is a topological isomorphism from $(\mathcal{N})^1$ on $\mathcal{E}_{\min}^1(\mathcal{N}_C')$.

Remark. Since we have identified $(\mathcal{N})^1$ and $\mathcal{E}_{\min}^1(\mathcal{N}')$ by Theorem 4.6, the above assertion can be restated as follows. We have

$$C_\mu : \mathcal{E}_{\min}^1(\mathcal{N}') \to \mathcal{E}_{\min}^1(\mathcal{N}_C'),$$

as a topological isomorphism.

The next Theorem characterizes distributions from $(\mathcal{N})_{\mu}^{-1}$ in terms of $S_\mu$-transform.

Theorem 4.9  The $S_\mu$-transform is a topological isomorphism from $(\mathcal{N})_{\mu}^{-1}$ on $\text{Hol}_0(\mathcal{N}_C)$.

Detailed proofs of the above theorems can be founded in [KSWY95, Theorems 33, 35].

5  Generalized Appell Systems

5.1 Description of the $P^{\mu,\alpha}$-system

Remember that the $\mu$-exponential is the generating function of the $P^{\mu}$-system, i.e., if $\theta \in \mathcal{U}_0 \subset \mathcal{N}_C$ and $z \in \mathcal{N}_C'$, then

$$e_\mu (\theta, z) := \frac{l_\mu (\theta)}{\mu} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu} (z), \theta^\otimes n \rangle, P_n^{\mu} (z) \in \mathcal{N}_C^\otimes n.$$

In view to generalize the Appell system we consider $\alpha \in \text{Hol}_0(\mathcal{N}_C, \mathcal{N}_C)$ an invertible function such that $\alpha(0) = 0$; moreover we have the following decomposition

$$\alpha (\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle \alpha^{(n)} (0), \theta^\otimes n \rangle, \theta \in \mathcal{U}_\alpha \subset \mathcal{N}_C$$

(5.1)

where $\alpha^{(n)} (0) \in \mathcal{N}_C^\otimes n \otimes \mathcal{N}_C$ since $\alpha$ is vector valued. Analogously for the inverse function $\alpha^{-1} =: g_\alpha$, we have

$$g_\alpha (\theta) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle g_\alpha^{(n)} (0), \theta^\otimes n \rangle, \theta \in \mathcal{V}_\alpha \subset \mathcal{N}_C,$$

(5.2)
where \( g_\alpha^{(n)}(0) \in \mathcal{N}_C^{\otimes n} \otimes \mathcal{N}_C \). Now we introduce a new normalized exponential using the function \( \alpha \), i.e.,

\[
e_{\mu}^{\alpha}(\theta; z) := e_{\mu}(\alpha(\theta); z) = \frac{\exp(z, \alpha(\theta))}{l_{\mu}(\alpha(\theta))}, \quad \theta \in \mathcal{U}_\alpha', z \in \mathcal{N}_C'.
\]

Using the same procedure as in Section 3 there exist \( P_{\mu,\alpha}^{n}(z) \in \mathcal{N}_C^{\otimes n} \) called generalized Appell polynomial or \( \alpha \)-polynomial such that

\[
e_{\mu}^{\alpha}(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{\mu,\alpha}^{n}(z), \theta^{\otimes n} \rangle, \quad \theta \in \mathcal{U}_\alpha', z \in \mathcal{N}_C',
\]

which for fixed \( z \in \mathcal{N}_C' \) converges uniformly on some neighborhood of zero on \( \mathcal{N}_C \). Hence we have constructed the \( P_{\mu,\alpha}^{n} \)-system

\[
P_{\mu,\alpha} = \left\{ \langle P_{\mu,\alpha}^{n}(\cdot), \varphi^{(n)}_\alpha \rangle \mid \varphi^{(n)}_\alpha \in \mathcal{N}_C^{\otimes n}, n \in \mathbb{N} \right\}.
\]

In this case the related moments kernels of the measure \( \mu \) are determined by

\[
l_{\mu}(\theta) := l_{\mu}(\alpha(\theta)) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle M_{\mu,\alpha}^{n}(\theta), \theta^{\otimes n} \rangle, \quad \theta \in \mathcal{N}_C, M_{\mu,\alpha}^{n} \in \mathcal{N}_C^{\otimes n}.
\]

Let us collect some properties of the polynomials \( P_{\mu,\alpha}^{n}(z) \).

**Proposition 5.1** For \( z, w \in \mathcal{N}_C, n \in \mathbb{N} \) the following holds

\[
(P_{\alpha} 1) \quad P_{\mu,\alpha}^{n}(z) = \sum_{m=1}^{n} \frac{1}{m!} \langle P_{\mu}^{m}(z), A_{\alpha}^{m} \rangle,
\]

where \( A_{\alpha}^{m} \) are related to the kernels of \( \alpha \) and are given in the proof, see \([5.14]\) below;

\[
(P_{\alpha} 2) \quad z^{\otimes n} = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \langle P_{\mu,\alpha}^{m}(z), B_{\alpha}^{m} \rangle \otimes M_{\mu,\alpha}^{n-k},
\]

where \( B_{\alpha}^{m} \) are related with the kernels of \( g_\alpha \) and are given in the proof, see \([5.13]\) below;

\[
(P_{\alpha} 3) \quad P_{\mu,\alpha}^{n}(z+w) = \sum_{k+i+j=n} \frac{n!}{k!i!j!} P_{\mu,\alpha}^{k}(z) \otimes P_{\mu,\alpha}^{i}(w) \otimes M_{\mu,\alpha}^{j}.
\]
\[(P_{\alpha 4})\]

\[P_{n}^{\mu,\alpha}(z + w) = \sum_{k=0}^{n} \binom{n}{k} P_{k}^{\mu,\alpha}(z) \hat{\otimes} P_{n-k}^{\mu,\alpha}(w). \tag{5.7}\]

\[(P_{\alpha 5})\]

Further, we observe

\[\mathbb{E}_{\mu}(\{P_{m}^{\mu,\alpha}(\cdot), \varphi^{(m)}_{\alpha}\}) = 0 \quad \text{for} \quad m \neq 0, \varphi^{(m)}_{\alpha} \in \mathcal{N}^{s,n}_{\mathbb{C}}. \tag{5.8}\]

\[(P_{\alpha 6})\]

For all \(p' > p\) such that the embedding \(\mathcal{H}_{p'} \to \mathcal{H}_{p}\) is of Hilbert-Schmidt class and for all \(\epsilon > 0\) there exist \(\sigma_{\epsilon} > 0\) such that

\[|P_{n}^{\mu,\alpha}(z)|_{-p'} \leq 2 n! \sigma_{\epsilon}^{-n} \exp(\epsilon |z|_{-p}), \quad z \in \mathcal{H}_{-p'}, \alpha, n \in \mathbb{N}, \tag{5.9}\]

where \(\sigma_{\epsilon}\) is chosen in such a way that \(|\alpha(\theta)| \leq \epsilon\) and \(|\mu(\alpha(\theta))| \geq 1/2\) for \(|\theta|_{p} = \sigma_{\epsilon}^{m}\).

**Proof.** \((P_{\alpha 1})\) Analogously with (3.3) we have

\[e_{\mu}^{\alpha}(\theta; z) := \frac{\exp \langle z, \alpha(\theta) \rangle}{l_{\mu}(\alpha(\theta))} = \sum_{m=0}^{\infty} \frac{1}{m!} \langle P_{m}^{\mu}(z), \alpha(\theta)^{\otimes m} \rangle. \tag{5.10}\]

Using the representation from (5.1) we compute \(\alpha(\theta)^{\otimes m}\):

\[\alpha(\theta)^{\otimes m} = \sum_{l_{1}, \ldots, l_{m}=1}^{\infty} \frac{1}{l_{1}! \cdots l_{m}!} \langle \alpha^{(l_{1})}(0) \otimes \cdots \otimes \alpha^{(l_{m})}(0), \theta^{\otimes (l_{1}+\cdots+l_{m})} \rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \langle A_{n}^{m}, \theta^{\otimes n} \rangle, \tag{5.11}\]

where

\[A_{n}^{m} = \begin{cases} 
\sum_{l_{1}, \ldots, l_{m}=n} \frac{n!}{l_{1}! \cdots l_{m}!} \alpha^{(l_{1})}(0) \otimes \cdots \otimes \alpha^{(l_{m})}(0) & \text{for } n \geq m, \\
0 & \text{for } n < m.
\end{cases} \tag{5.12}\]
Now we introduce (5.11) in (5.10) to obtain

\[ e^{\alpha}_{\mu}(\theta; z) = \sum_{m=0}^{\infty} \frac{1}{m!} \left( \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_{m=0}^{\infty} \frac{1}{m!} \langle P_{m}^{\mu}(z), A_{m}^{\alpha}, \theta^{\otimes n} \rangle \right) \right), \]

By definition

\[ e^{\alpha}_{\mu}(\theta; z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{m=0}^{\infty} \frac{1}{m!} \langle P_{m}^{\mu}(z), A_{m}^{\alpha}, \theta^{\otimes n} \rangle \right), \]

so we conclude that

\[ P_{n}^{\mu, \alpha}(z) = \sum_{m=1}^{n} \frac{1}{m!} \langle P_{m}^{\mu}(z), A_{m}^{\alpha}, \theta^{\otimes n} \rangle. \]

(P_\alpha 2) Since \( \theta = \alpha(g_{\alpha}(\theta)) \) we have

\[ e_{\mu}(\theta, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P_{n}^{\mu, \alpha}(z), g_{\alpha}(\theta)^{\otimes n} \rangle. \]

Having in mind (5.2) we first compute \( g_{\alpha}(\theta)^{\otimes n} \):

\[ g_{\alpha}(\theta)^{\otimes n} = \sum_{l=1}^{\infty} \frac{1}{l!} \langle g_{\alpha}^{(l)}(0), \theta^{\otimes l} \rangle \otimes \cdots \otimes \sum_{l=1}^{\infty} \frac{1}{l!} \langle g_{\alpha}^{(l)}(0), \theta^{\otimes l} \rangle \]

\[ = \sum_{l_1, \ldots, l_n=1}^{\infty} \frac{1}{l_1! \cdots l_n!} \langle g_{\alpha}^{(l_1)}(0) \otimes \cdots \otimes g_{\alpha}^{(l_n)}(0), \theta^{\otimes (l_1+\cdots+l_n)} \rangle \]

\[ = \sum_{m=1}^{\infty} \frac{1}{m!} \langle P_{m}^{\alpha}(\theta)^{\otimes m} \rangle, \]

where

\[ P_{m}^{\alpha} = \begin{cases} \sum_{l_1, \ldots, l_n=1}^{m} \frac{1}{l_1! \cdots l_n!} g_{\alpha}^{(l_1)}(0) \otimes \cdots \otimes g_{\alpha}^{(l_n)}(0) & \text{for } m \geq n \\ 0 & \text{for } m < n \end{cases} \] (5.13)
Hence
\[
e_{\mu}(\theta, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_{\mu}^{n, \alpha}(z), \sum_{m=1}^{\infty} \frac{1}{m!} \langle B_{m}^{n}, \theta^{\otimes m} \rangle \right\rangle
= \sum_{m=1}^{\infty} \frac{1}{m!} \left\langle \sum_{n=0}^{m} \frac{1}{n!} \left\langle P_{n}^{\mu, \alpha}(z), B_{n}^{m} \right\rangle, \theta^{\otimes m} \right\rangle.
\]

On the other hand
\[
e_{\mu}(\theta, z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_{n}^{\mu}(z), \theta^{\otimes n} \right\rangle,
\]
so we conclude that
\[
P_{m}^{\mu}(z) = \sum_{n=1}^{m} \frac{1}{n!} \left\langle P_{n}^{\mu}(z), B_{m}^{n} \right\rangle.
\]

The result follows using property (P2) of the polynomials \( P_{n}^{\mu}(z) \).

(Pa3) Let us start from the equation of the generating functions
\[
e_{\mu}^{\alpha}(\theta, z + w) = e_{\mu}^{\alpha}(\theta, z) e_{\mu}^{\alpha}(\theta, w) l_{\mu}^{\alpha}(\theta).
\]
This implies
\[
\sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_{n}^{\mu, \alpha}(z + w), \theta^{\otimes n} \right\rangle
= \sum_{k,l,m=0}^{\infty} \frac{1}{k!l!m!} \left\langle P_{k}^{\mu, \alpha}(z) \otimes P_{l}^{\mu, \alpha}(w) \otimes M_{m}^{\mu, \alpha}, \theta^{\otimes (k+l+m)} \right\rangle,
\]
from this (Pa3) follows immediately.

(Pa4) We note that
\[
e_{\mu}^{\alpha}(\theta; z + w) = e_{\mu}^{\alpha}(\theta; z) \exp \langle w, \alpha(\theta) \rangle, \quad \theta \in \mathcal{U}_{0} \subset \mathcal{N}_{C}.
\]

Now, since \( l_{\delta_{0}}(\theta) = 1 \), we have the following decomposition
\[
\exp \langle w, \alpha(\theta) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle P_{n}^{\delta_{0}, \alpha}(w), \theta^{\otimes n} \right\rangle,
\]
(5.15)
where for \( \alpha \equiv \text{id} \), \( P_n^{\delta_0, \alpha} (w) = w^{\otimes n} \). The result follows as done in (P\text{a}3).

(P\text{a}5) To see this we use, \( \theta \in \mathcal{N}_\mathbb{C} \),

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \mathbb{E}_\mu \left( \langle P_m^{\mu, \alpha} (\cdot), \theta^{\otimes n} \rangle \right) = \mathbb{E}_\mu \left( e_\mu (\theta; \cdot) \right) = \frac{\mathbb{E}_\mu (\exp \langle \cdot, \alpha (\theta) \rangle)}{l_\mu (\alpha (\theta))} = 1.
\]

Then the polarization identity and a comparison of coefficients give the result.

(P\text{a}6) Using the definition of \( P_m^{\mu, \alpha} \) and Cauchy’s inequality for Taylor series we have

\[
\left| \langle P_n^{\mu, \alpha} (z), \theta^{\otimes n} \rangle \right| = n! \left| \frac{d^n e_\mu (0; z) (\theta)}{d^n \theta_p} \right| \leq n! \frac{1}{\sigma^n} \sup_{|\theta|_p = \sigma} \frac{\exp (|\alpha (\theta)|_p |z|_{-p})}{|l_\mu (\alpha (\theta))|} |\theta|^n \leq 2n! \sigma^{-n} \exp \left( \epsilon |z|_{-p} \right) |\theta|^n.
\]

The result follows by polarization and kernel theorem. \( \blacksquare \)

Let us give a concrete example which furnish good arguments to use the \( P^{\mu, \alpha} \)-system.

**Example 5.2 (Poisson noise)** Let us consider the classical (real) Schwartz triple

\( S (\mathbb{R}) \subset L^2 (\mathbb{R}) \subset S' (\mathbb{R}) \).

The **Poisson white noise measure** \( \pi \) is defined as a probability measure on \( \mathcal{C}_\sigma (S' (\mathbb{R})) \) with Laplace transform

\[
l_\pi (\theta) = \exp \left( \int_\mathbb{R} (\exp \theta (t) - 1) dt \right) = \exp \left[ \langle \exp \theta (\cdot) - 1, 1 \rangle \right], \quad \theta \in \mathcal{S}_\mathbb{C} (\mathbb{R}),
\]

see e.g., [GV68]. It is not hard to see that \( l_\pi \) is a holomorphic function on \( S_\mathbb{C} (\mathbb{R}) \), so assumption 1 is satisfied. But to check Assumption 2, we need additional considerations.

First of all we remark that for any \( \xi \in S (\mathbb{R}) \), \( \xi \neq 0 \) the measure \( \pi \) and \( \pi (\cdot + \xi) \) are orthogonal (see [GGV75] for a detailed analysis). It means that \( \pi \) is not \( S (\mathbb{R}) \)-quasi-invariant and the note after Assumption 2 is not applicable now.

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Let some $\varphi \in \mathcal{P}(S'(\mathbb{R}))$, $\varphi = 0$ $\pi$-a.e. be given. We need to show that then $\varphi \equiv 0$. To this end we will introduce a system of orthogonal polynomials in the space $L^2(S'(\mathbb{R}), \pi)$ which can be constructed in the following way. The mapping

$$\theta (\cdot) \mapsto \alpha (\theta) (\cdot) = \log (1 + \theta (\cdot)) \in S_C (\mathbb{R}), \quad \theta \in S_C (\mathbb{R})$$

is holomorphic on a neighborhood $\mathcal{U} \subset S_C (\mathbb{R})$, $0 \in \mathcal{U}$. Then

$$e^\alpha_\pi (\theta; x) = \exp \left\langle x, \alpha (\theta) \right\rangle = \exp \left[ \left\langle x, \alpha (\theta) \right\rangle - \langle \theta, 1 \rangle \right], \quad \theta \in \mathcal{U}, \ x \in S' (\mathbb{R})$$

is a holomorphic function on $\mathcal{U}$ for any $x \in S' (\mathbb{R})$. The Taylor decomposition and the kernel theorem give

$$e^\alpha_\pi (\theta; x) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle C_n (x), \theta^\otimes n \rangle,$$

where $C_n : S'(\mathbb{R}) \to S'(\mathbb{R})^\otimes n$ are polynomial mappings. For $\varphi^{(n)} \in S_C (\mathbb{R})^\otimes n$, $n \in \mathbb{N}_0$, we define Charlier polynomials

$$x \mapsto C_n \left( \varphi^{(n)}; x \right) := \langle C_n (x), \varphi^{(n)} \rangle \in \mathbb{C}, \ x \in S' (\mathbb{R}).$$

Due to [Ito88], [IK88] we have the following orthogonality property:

$$\forall \varphi^{(n)} \in S_C (\mathbb{R})^\otimes n, \ \forall \psi^{(m)} \in S_C (\mathbb{R})^\otimes m$$

$$\int C_n \left( \varphi^{(n)} \right) C_m \left( \psi^{(m)} \right) d\pi = \delta_{nm} n! \langle \varphi^{(n)}, \psi^{(m)} \rangle.$$

Now the rest is simple. Any continuous polynomial $\varphi$ has a uniquely defined decomposition

$$\varphi (x) = \sum_{n=0}^{N} \langle C_n (x), \varphi^{(n)} \rangle, \quad x \in S' (\mathbb{R}),$$

where $\varphi^{(n)} \in S_C (\mathbb{R})^\otimes n$. If $\varphi = 0$ $\pi$-a.e., then

$$\|\varphi\|^2_{L^2(\pi)} = \sum_{n=0}^{N} n! \langle \varphi^{(n)}, \overline{\varphi^{(n)}} \rangle = 0.$$

Hence $\varphi^{(n)} = 0, n = 0, \ldots, N$, i.e., $\varphi \equiv 0$. So Assumption 2 is satisfied.
Lemma 5.3 For any $\varphi \in \mathcal{P}(\mathcal{N}')$ there exists a unique representation

$$
\varphi(x) = \sum_{n=0}^{N} \langle P_{n}^{\mu,\alpha}(x), \varphi^{(n)}_{\alpha} \rangle , \quad \varphi^{(n)}_{\alpha} \in \mathcal{N}_{C}^{\otimes n}
$$

(5.16)

and vice versa, any functional of the form (5.16) is a smooth polynomial.

Proof. The representation from Definition 2.4 and equation (5.16) can be transformed into one another using (5.4) and (5.5). ■

5.2 Description of the $Q^{\mu,\alpha}$-system

5.2.1 Using $S^{\mu}$-transform

By assumption we know that $\alpha$ is invertible with inverse given by $g_{\alpha}$ and $\alpha(\theta) \in \mathcal{V}_{\alpha} \subset \mathcal{N}_{C}, \forall \theta \in \mathcal{U}_{\alpha}$. For given $\Phi^{(n)}_{\alpha} \in \mathcal{N}_{C}^{\otimes n}$ we define a generalized function $Q_{\mu,\alpha}^{\mu,\alpha}(\Phi^{(n)}_{\alpha})$ via the $S^{\mu}$-transform

$$
S_{\mu}(Q_{\mu,\alpha}^{\mu,\alpha}(\Phi^{(n)}_{\alpha}))(\theta) := \langle \Phi^{(n)}_{\alpha}, g_{\alpha}(\theta) \otimes \nabla \rangle , \quad \theta \in \mathcal{V}_{\alpha}.
$$

(5.17)

5.2.2 Using differential operators

Using the kernels $g^{(n)}_{\alpha}(0)$ of $g_{\alpha}$, see (5.2), we define a differential operator (of infinite order) from $\mathcal{P}(\mathcal{N}')$ to $\mathcal{P}(\mathcal{N}') \otimes \mathcal{N}_{C}$ as follows

$$
G_{\alpha} = \sum_{n=0}^{\infty} \frac{1}{n!} \langle g^{(n)}_{\alpha}(0), \nabla^{\otimes n} \rangle ,
$$

such that, if $\varphi \in \mathcal{P}(\mathcal{N}')$ and $\xi \in \mathcal{N}'_{C}$ we have

$$
G_{\alpha}^{\xi}(\varphi)(x) := \langle \xi, G_{\alpha}(\varphi)(x) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \xi, g^{(n)}_{\alpha}(0), \nabla^{\otimes n} \varphi(x) \rangle , \quad x \in \mathcal{N}',
$$

i.e., $G_{\alpha}^{\xi} : \mathcal{P}(\mathcal{N}') \to \mathcal{P}(\mathcal{N}')$ and formally $G_{\alpha} := g_{\alpha}(\nabla)$.

Let us state the following useful lemma.

Lemma 5.4 For all $\xi \in \mathcal{N}'_{C}$, $x \in \mathcal{N}'$ and $\theta \in \mathcal{N}_{C}$ we have

$$
\langle \xi, g_{\alpha}(\nabla) \rangle (\exp \langle x, \theta \rangle) = \langle \xi, g_{\alpha}(\theta) \rangle \exp \langle x, \theta \rangle .
$$
Proof. Using the representation given in (5.2) we have

\[
\langle \xi, g_\alpha (\nabla) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle g_\alpha^{(n)} (0), \nabla^{\otimes n} \rangle, \quad g_\alpha^{(n)} (0) = \langle g_\alpha^{(n)} (0), \xi \rangle \in \mathcal{N}_C^{\otimes n}.
\]

For simplicity we put \( g_\alpha^{(n)} (0) \equiv \Psi(n) \). At first we apply the operator to some monomial. For given \( \theta \in \mathcal{N}_C, m \geq n \)

\[
\langle \Psi(n), \nabla^{\otimes n} \rangle \langle x, \theta \rangle^m = \sum_{m=n}^{\infty} \frac{m (m-1) \cdots (m-n+1)}{m!} \langle \Psi(n) \otimes x^{\otimes (m-n)}, \theta^{\otimes m} \rangle
\]

\[
= \sum_{m=n}^{\infty} \frac{m (m-1) \cdots (m-n+1)}{m!} \langle \Psi(n) \otimes x^{\otimes (m-n)}, \theta^{\otimes m} \rangle
\]

\[
= \langle \Psi(n), \theta^{\otimes n} \rangle \sum_{m=n}^{\infty} \frac{1}{(m-n)!} \langle x, \theta \rangle^{m-n}
\]

\[
= \langle \Psi(n), \theta^{\otimes n} \rangle \exp \langle x, \theta \rangle.
\]

Therefore

\[
\langle \xi, g_\alpha (\nabla) \rangle \left( \exp \langle x, \theta \rangle \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Psi(n), \nabla^{\otimes n} \rangle \exp \langle x, \theta \rangle
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \langle \Psi(n), \theta^{\otimes n} \rangle \exp \langle x, \theta \rangle
\]

\[
= \langle \xi, g_\alpha (\theta) \rangle \left( \exp \langle x, \theta \rangle \right).
\]

\[\blacksquare\]

**Theorem 5.5** Under the above conditions the \( Q_n^{\mu,\alpha} (\xi^{\otimes n}) \) are given by

\[
Q_n^{\mu,\alpha} (\xi^{\otimes n}) (\cdot) = \left( \langle \xi, g_\alpha (\nabla) \rangle^{\ast n} 1 \right) (\cdot).
\]  \hspace{1cm} (5.18)
Proof. Applying the $S_\mu$-transform to the r.h.s of (5.18) we have

\[
S_\mu \langle \langle \xi, g_\alpha (\nabla) \rangle \rangle \rangle n 1 \rangle (\theta) = \langle \langle \langle 1, \langle \xi, g_\alpha (\nabla) \rangle \rangle \rangle n \rangle e_\mu (\theta, \cdot) \rangle \rangle \mu
= \frac{1}{l_\mu (\theta)} \int_{\mathcal{N}'} \langle \xi, g_\alpha (\nabla) \rangle \rangle n \exp (x, \theta) \, d\mu (x)
= \frac{\langle \xi, g_\alpha (\theta) \rangle \rangle n}{l_\mu (\theta)} \int_{\mathcal{N}'} \exp (x, \theta) \, d\mu (x)
= \langle \xi, g_\alpha (\theta) \rangle \rangle n.
\]  

(5.19)

On the other hand the $S_\mu$-transform of the l.h.s. (5.18), by (5.17), is the same as (5.19) which prove the result.

Example 5.6 As an illustration of $G_\alpha$ we use again the Poisson measure $\pi$ (see Example 5.3) and $\alpha (\theta) (\cdot) = \log (1 + \theta (\cdot))$, $\theta \in S (\mathbb{R})$. For this choice we have

\[
g_\alpha (\theta) (\cdot) = \exp \theta (\cdot) - 1 = \sum_{n=1}^{\infty} \frac{\theta^n (\cdot)}{n!}.
\]

On the other hand, from (5.2) we have

\[
g_\alpha (\theta) (\cdot) = \sum_{n=1}^{\infty} \frac{1}{n!} \langle g_\alpha^{(n)} (0), \theta^{\otimes n} \rangle (\cdot),
\]

so we conclude that

\[
g_\alpha^{(n)} (0) = \delta (t_1 - t) \cdots \delta (t_n - t).
\]

We introduce the notation of functional derivative (see [IK88]),

\[
\nabla_{\delta t} (\theta) = \frac{\delta}{\delta \theta (t)}, \quad \theta \in S (\mathbb{R}), t \in \mathbb{R}.
\]

With this, we easily see that for $\nabla_h = \langle \nabla, h \rangle$ we have

\[
(\exp (\nabla_h f) (\cdot) = f (\cdot + h), \quad f \in \mathcal{P} (S' (\mathbb{R})), h \in S (\mathbb{R}).
\]

Hence

\[
(g_\alpha (\nabla_{\delta t}) (\theta)) (f (\cdot)) = \left( \exp \left( \frac{\delta}{\delta \theta (t)} \right) \right) f (\cdot) = f (\cdot + \delta_t) - f (\cdot)
\]

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and if $\xi \in S_C(\mathbb{R})$ we have
\[
\langle g_\alpha (\nabla_{\delta t}) \xi \rangle f (\cdot) = \int_{\mathbb{R}} [f (\cdot + \delta t) - f (\cdot)] \xi (t) \, dt.
\]
Therefore if $f \in \mathcal{P}(S'(\mathbb{R}))$ then
\[
G_\alpha : f (\cdot) \mapsto -\langle f (\cdot + \delta t) - f (\cdot) \rangle.
\]
This mapping can be considered as a ”gradient” operator on the Poisson space $(S'(\mathbb{R}), \mathcal{B}(S'(\mathbb{R})), \pi)$.

**Definition 5.7** We define the $Q^{\mu,\alpha}$-system in $\mathcal{P}'_\mu(N')$ by
\[
Q^{\mu,\alpha} = \left\{ Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha) \mid \Phi^{(n)}_\alpha \in N^{\otimes n}_C, \ n \in \mathbb{N}_0 \right\},
\]
and the pair $(P^{\mu,\alpha}, Q^{\mu,\alpha})$ will be called the **generalized Appell system** $A^{\mu,\alpha}$ generated by the measure $\mu$ and given mapping $\alpha \in \text{Hol}_0(N_C,N_C)$.

Now we are going to discuss the central property of the generalized Appell system $A^{\mu,\alpha}$.

**Theorem 5.8** (Biorthogonality of $Q^{\mu,\alpha}$ and $P^{\mu,\alpha}$ w.r.t. $\mu$)
\[
\langle Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha), P^{\mu,\alpha}_{m} (\varphi^{(m)}_\alpha) \rangle_\mu = \delta_{nm} n! \langle \Phi^{(n)}_\alpha, \varphi^{(n)}_\alpha \rangle,
\]
for $\Phi^{(n)}_\alpha \in N^{\otimes n}_C$ and $\varphi^{(m)}_\alpha \in N^{\otimes m}_C$.

**Proof.** By definition of $S_\mu$ we have
\[
S_\mu \left( Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha) \right) (\theta) := \langle Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha), e_\mu (\theta, \cdot) \rangle_\mu
\]
if we substitute $\theta \mapsto \alpha(\eta)$, then we obtain
\[
S_\mu \left( Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha) \right) (\alpha (\eta)) = \langle Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha), e_\mu (\alpha (\eta), \cdot) \rangle_\mu
\] \[
= \sum_{m=0}^{\infty} \frac{1}{m!} \langle Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha), P^{\mu,\alpha}_{m} (\cdot, \eta^{\otimes m}) \rangle_\mu.
\]
Substituting of $\theta$ by $\alpha(\eta)$ in (5.17) give us
\[
S_\mu \left( Q^{\mu,\alpha}_{n} (\Phi^{(n)}_\alpha) \right) (\alpha (\eta)) = \langle \Phi^{(n)}_\alpha, \eta^{\otimes n} \rangle.
\]
Then a comparison of coefficients and the polarization identity give the desired result. $\blacksquare$

Now we characterize the space $\mathcal{P}'_\mu(N')$. 

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Theorem 5.9 For all $\Phi \in \mathcal{P}_\mu'(\mathcal{N}')$ there exists a unique sequence $\{\Phi_n^{(n)} \mid n \in \mathbb{N}_0\}$, $\Phi_n^{(n)} \in \mathcal{N}_C^{\hat{\otimes} n}$ such that
\[
\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_n^{(n)}) \equiv \sum_{n=0}^{\infty} \langle Q_n^{\mu,\alpha}, \Phi_n^{(n)} \rangle
\] (5.21)
and vice versa, every series of the form (5.21) generates a generalized function in $\mathcal{P}_\mu'(\mathcal{N}')$.

Proof. For $\Phi \in \mathcal{P}_\mu'(\mathcal{N}')$ we can uniquely define $\Phi_n^{(n)} \in \mathcal{N}_C^{\hat{\otimes} n}$ by
\[
\langle \Phi_n^{(n)}, \varphi_n^{(n)} \rangle := \frac{1}{n!} \langle \Phi, \langle P_n^{\mu,\alpha}, \varphi_n^{(n)} \rangle \rangle_{\mu}, \quad \varphi_n^{(n)} \in \mathcal{N}_C^{\hat{\otimes} n},
\]
which is well defined since $\langle P_n^{\mu,\alpha}, \varphi_n^{(n)} \rangle \in \mathcal{P}(\mathcal{N}')$. The continuity of $\varphi_n^{(n)} \mapsto \langle \Phi_n^{(n)}, \varphi_n^{(n)} \rangle$ follows from the continuity of $\varphi \mapsto \langle \langle \Phi, \varphi \rangle \rangle_{\mu}, \varphi \in \mathcal{P}(\mathcal{N}')$. This implies that
\[
\varphi \mapsto \sum_{n=0}^{\infty} n! \langle \Phi_n^{(n)}, \varphi_n^{(n)} \rangle
\]
is continuous on $\mathcal{P}(\mathcal{N}')$. This defines a generalized function in $\mathcal{P}_\mu'(\mathcal{N}')$, which we denote by
\[
\sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_n^{(n)}).
\]
In view of Theorem 5.8 it is easy to see that
\[
\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_n^{(n)}).
\]
To see the converse consider a series of the form (5.21) and $\varphi \in \mathcal{P}(\mathcal{N}')$. Then there exists $\varphi_n^{(n)} \in \mathcal{N}_C^{\hat{\otimes} n}$, $n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that we have the representation
\[
\varphi = \sum_{n=0}^{N} P_n^{\mu,\alpha} (\varphi_n^{(n)}).
\]
So we have
\[
\left\langle \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_n^{(n)}), \varphi \right\rangle_{\mu} = \sum_{n=0}^{N} n! \langle \Phi_n^{(n)}, \varphi_n^{(n)} \rangle,
\]
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because of Theorem 5.8. The continuity of
\[ \varphi \mapsto \left\langle \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} \langle \Phi^{(n)}_\alpha, \varphi \rangle \right\rangle_{\mu} \]
follows because \( \varphi^{(n)}_\alpha \mapsto \langle \Phi^{(n)}_\alpha, \varphi^{(n)}_\alpha \rangle \) is continuous for all \( n \in \mathbb{N} \). □

6 Test functions on a linear space with measure

6.1 Test functions spaces

We will construct the test function space \( (\mathcal{N})_{\mu,\alpha}^1 \) using \( P^{\mu,\alpha} \)-system and study some properties. On the space \( P(\mathcal{N}') \) we can define a system of norms using the representation from (5.16)

\[ \varphi (\cdot) = \sum_{n=0}^{N} \langle P_n^{\mu,\alpha} (\cdot), \varphi^{(n)}_\alpha \rangle, \]

with \( \varphi^{(n)}_\alpha \in \mathcal{H}_{\mu,\alpha}^{\otimes n} \) for each \( p > 0 \ (n \in \mathbb{N}) \). Thus we may define for any \( p, q \in \mathbb{N} \)
a Hilbert norm on \( P(\mathcal{N}') \) by

\[ \| \varphi \|_{p,q,\mu,\alpha}^2 = \sum_{n=0}^{N} (n!)^2 2^{nq} |\varphi^{(n)}_\alpha|^2_p < \infty \]

The completion of \( P(\mathcal{N}') \) w.r.t. \( \| \|_{p,q,\mu,\alpha}^2 \) is called \( (\mathcal{H}_p)_{q,\mu,\alpha}^1 \).

Definition 6.1 We define

\[ (\mathcal{N})_{\mu,\alpha}^1 := \text{pr lim } (\mathcal{H}_p)_{q,\mu,\alpha}^1 \]

Theorem 6.2 \( (\mathcal{N})_{\mu,\alpha}^1 \) is a nuclear space. The topology in \( (\mathcal{N})_{\mu,\alpha}^1 \) is uniquely defined by the topology on \( \mathcal{N} \). It does not depend on the choice of the family of norms \( \{|\cdot|_p\} \).
Proof. Nuclearity of \((N)_{\mu,\alpha}^1\) follows essentially from that of \(N\). For fixed \(p, q\) choose \(p'\) such that the embedding
\[
i_{p', p} : \mathcal{H}_{p'} \hookrightarrow \mathcal{H}_p
\]
is Hilbert-Schmidt and consider the embedding
\[
I_{p', q', p, q, \alpha} : (\mathcal{H}_{p'})^1_{q', \mu, \alpha} \hookrightarrow (\mathcal{H}_p)^1_{q, \mu, \alpha}.
\]
Then \(I_{p', q', p, q, \alpha}\) is induced by
\[
I_{p', q', p, q, \alpha}(\varphi) = \sum_{n=0}^{\infty} \langle P_{n}^{\mu, \alpha}, i_{p', p}^\otimes n \varphi \rangle \quad \text{for} \quad \varphi = \sum_{n=0}^{\infty} \langle P_{n}^{\mu, \alpha}, \varphi^{(n)} \rangle \in (\mathcal{H}_{p'})^1_{q', \mu, \alpha}.
\]
Its Hilbert-Schmidt norm, for a given orthonormal basis of \((\mathcal{H}_{p'})^1_{q', \mu, \alpha}\), can be estimate by
\[
\|I_{p', q', p, q, \alpha}\|_{HS}^2 = \sum_{n=0}^{\infty} 2^{n(q-q')} \|i_{p', p}\|_{HS}^{2n}
\]
which is finite for a suitably chosen \(q'\).

To prove the independence of the family of norms, let us assume that we are given two different systems of Hilbert norms \(\|\cdot\|_p\) and \(\|\cdot\|'_{k}\), such that they induce the same topology on \(N\). For fixed \(k\) and \(l\) we have to estimate \(\|\cdot\|'_{k, l, \mu, \alpha}\) by \(\|\cdot\|_{p, q, \mu, \alpha}\) for some \(p, q\) (and vice versa which is completely analogous). But for all \(f \in \mathcal{N}\) we have \(|f|'_{k} \leq C |f|_p\) for some constant \(C\) and some \(p\), since \(|\cdot|'_{k}\) has to be continuous with respect to the projective limit topology on \(\mathcal{N}\). That means that the injection \(i\) from \(\mathcal{H}_p\) into the completion \(\mathcal{K}_k\) of \(N\) with respect to \(|\cdot|'_{k}\) is a mapping bounded by \(C\). We denote by \(i\) also its linear extension from \(\mathcal{H}_{p,C}\) into \(\mathcal{K}_{k,C}\). It follows that \(i\otimes n\) is bounded by \(C^n\) from \(\mathcal{H}_{p,C}^\otimes n\) into \(\mathcal{K}_{k,C}^\otimes n\). Now we choose \(q\) such that \(2^{\frac{q-l}{2}} \geq C\). Then
\[
\|\cdot\|'_{k, l, \mu, \alpha} = \sum_{n=0}^{\infty} (n!)^2 2^{nl} |\cdot|_{k}^2
\]
\[
\leq \sum_{n=0}^{\infty} (n!)^2 2^{nl} C^{2n} |\cdot|_{p}^2
\]
\[
\leq \|\cdot\|_{p, q, \mu, \alpha}
\]
which is exactly what we need.
Lemma 6.3  There exist $p, C, K > 0$ such that for all $n \in \mathbb{N}_0$

$$\int |P_{n}^{\mu,\alpha}(z)|^2_p d\mu(z) \leq 4(n!)^2 C^n K.$$  
(6.1)

Proof. We can use the estimate (5.9) and Lemma 2.3 to conclude the result.  

Theorem 6.4  There exists $p', q' > 0$ such that for all $p \geq p', q \geq q'$ the topological embedding $(\mathcal{H}_p)^1_{q,\mu,\alpha} \subset L^2(\mu)$ holds.

Proof. Elements of the space $(\mathcal{N})^1_{\mu,\alpha}$ are defined as series convergent in the given topology. Now we need the convergence of the series in $L^2(\mu)$. Choose $q'$ such that $C > 2q'$ ($C$ from estimate (6.1)). Let us take an arbitrary

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi^{(n)}_{\alpha} \rangle \in P(\mathcal{N}).$$

For $p > p'$ ($p'$ from the Lemma 6.3) and $q > q'$ the following estimates hold

$$\| \varphi \|_{L^2(\mu)} \leq \sum_{n=0}^{\infty} \left\| \langle P_n^{\mu,\alpha}, \varphi^{(n)}_{\alpha} \rangle \right\|_{L^2(\mu)}$$

$$\leq \sum_{n=0}^{\infty} \left| \varphi^{(n)}_{\alpha} \right|_p \left\| P_n^{\mu,\alpha} \right\|_{L^2(\mu)}$$

$$\leq 2K^{1/2} \sum_{n=0}^{\infty} n! 2^{nq/2} \left| \varphi^{(n)}_{\alpha} \right|_p (C2^{-q})^{n/2}$$

$$\leq 2K^{1/2} \left( \sum_{n=0}^{\infty} (C2^{-q})^n \right)^{1/2} \left( \sum_{n=0}^{\infty} (n!)^2 2^{nq} \left| \varphi^{(n)}_{\alpha} \right|_p^2 \right)^{1/2}$$

$$= 2K^{1/2} (1 - C2^{-q})^{-1/2} \left\| \varphi \right\|_{p, q, \mu, \alpha}.$$

Taking the closure the inequality extends to the whole space $(\mathcal{H}_p)^1_{q,\mu,\alpha}$.  

Corollary 6.5  $(\mathcal{N})^1_{\mu,\alpha}$ is continuously and densely embedded in $L^2(\mu)$.
6.2 Description of test functions

**Proposition 6.6** Any test function \( \varphi \) in \((\mathcal{N})_{\mu,\alpha}^1 \) has a uniquely defined extension to \( \mathcal{N}'_\mathbb{C} \) as an element of \( \mathcal{E}_{\min}^1(\mathcal{N}'_\mathbb{C}) \).

**Proof.** Any element \( \varphi \) in \((\mathcal{N})_{\mu,\alpha}^1 \) is defined as a series of the following type

\[
\varphi = \sum_{n=0}^{\infty} \langle P_{\mu,\alpha}^n, \varphi_{\alpha}^{(n)} \rangle, \quad \varphi_{\alpha}^{(n)} \in \mathcal{N}'_\mathbb{C},
\]

such that

\[
\|\varphi\|_{p,q,\mu,\alpha}^2 = \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_{\alpha}^{(n)}|_p^2 < \infty
\]

for each \( p, q \in \mathbb{N} \). So we need to show the convergence of the series

\[
\sum_{n=0}^{\infty} \langle P_{\mu,\alpha}^n (z), \varphi_{\alpha}^{(n)} \rangle, \quad z \in \mathcal{H}_{-p,\mathbb{C}}
\]

to an entire function in \( z \). Let \( \epsilon > 0 \) and \( \sigma_\epsilon > 0 \) as in (P_\alpha 6) of Proposition 5.1. We use (5.9) and estimate as follows

\[
\sum_{n=0}^{\infty} \langle P_{\mu,\alpha}^n (z), \varphi_{\alpha}^{(n)} \rangle \\
\leq \sum_{n=0}^{\infty} |P_{\mu,\alpha}^n (z)|_{-p} |\varphi_{\alpha}^{(n)}|_p \\
\leq 2 \sum_{n=0}^{\infty} n! |\varphi_{\alpha}^{(n)}|_p \sigma_\epsilon^{-n} \\
\leq 2 \exp \left( \epsilon |z|_{-p'} \right) \left( \sum_{n=0}^{\infty} (n!)^2 2^{nq} |\varphi_{\alpha}^{(n)}|_p^2 \right)^{1/2} \left( \sum_{n=0}^{\infty} 2^{-nq} \sigma_\epsilon^{-2n} \right)^{1/2} \\
\leq 2 \|\varphi\|_{p,q,\mu,\alpha} (1 - 2^{-q} \sigma_\epsilon^{-2})^{-1/2} \exp \left( \epsilon |z|_{-p'} \right),
\]

if \( 2^q > \sigma_\epsilon^{-2} \) and \( p' \) is such that \( \mathcal{H}_p \hookrightarrow \mathcal{H}_{p'} \) is Hilbert-Schmidt. That means the series

\[
\sum_{n=0}^{\infty} \langle P_{\mu,\alpha}^n (z), \varphi_{\alpha}^{(n)} \rangle
\]

is convergent.
converges uniformly and absolutely in any neighborhood of zero of any space $\mathcal{H}_{-p,C}$. Since each term $\langle P_n^{\mu,\alpha}(z), \varphi^{(n)} \rangle$ is entire in $z$ the uniform convergence implies that

$$z \mapsto \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}(z), \varphi^{(n)} \rangle$$

is entire on each $\mathcal{H}_{-p,C}$ and hence on $\mathcal{N}_C'$. This complete the proof. ■

The following corollary gives an explicit estimate on the growth of test functions and is a consequence of the above Proposition.

**Corollary 6.7** For all $p > p'$ such that the embedding $\mathcal{H}_p \hookrightarrow \mathcal{H}_{p'}$ is of the Hilbert-Schmidt class and for all $\epsilon > 0$ there exists $\sigma_\epsilon$ ($\sigma_\epsilon$ from Proposition 5.1), such that for $p \in \mathbb{N}$ we obtain the following bound

$$|\varphi(z)| \leq C \|\varphi\|_{p,q,\mu,\alpha} \exp(\epsilon |z|_{p'}) , \varphi \in (\mathcal{N})^1_{\mu,\alpha} , z \in \mathcal{H}_{-p,C},$$

where $2\epsilon > \sigma_\epsilon^{-2}$ and

$$C = 2 \left(1 - 2^{-q}\sigma_\epsilon^{-2}\right)^{-1/2}.$$

**Remark 6.8** Proposition 6.6 states

$$(\mathcal{N})^1_{\mu,\alpha} \subseteq \mathcal{E}_{\min}^1 (\mathcal{N})$$

as sets, where

$$\mathcal{E}_{\min}^1 (\mathcal{N}) = \{ \varphi|_{\mathcal{N}} : \varphi \in \mathcal{E}_{\min}^1 (\mathcal{N}_C') \}.$$ 

Now we are going to show that the converse also holds.

**Theorem 6.9** For all functions $\alpha \in \text{Hol}_0(\mathcal{N}_C, \mathcal{N}_C)$, as in Subsection 5.1, and for all measure $\mu \in \mathcal{M}_a(\mathcal{N})$, we have the topological identity

$$(\mathcal{N})^1_{\mu,\alpha} = \mathcal{E}_{\min}^1 (\mathcal{N}).$$

**Proof.** Let $\varphi(z) \in \mathcal{E}_{\min}^1 (\mathcal{N})$ be given such that

$$\varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n}, \psi^{(n)} \rangle,$$
with
\[ \| \varphi \|_{p,q,1}^2 = \sum_{n=0}^{\infty} (n!)^2 \sum_{q=0}^{\infty} |\psi(n)|^2_p < \infty \]
for each \( p, q \in \mathbb{N} \). So we have
\[ |\psi(n)|_p \leq (n!)^{-1/2} \| \varphi \|_{p,q,1} \cdot \]
On the other hand, we can use (5.3) to evaluate \( \varphi(z) \) as
\[ \varphi(z) = \sum_{n=0}^{\infty} \langle z^{\otimes n} , \psi(n) \rangle \]
\[ = \sum_{n=0}^{\infty} \left\langle \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \left( \langle P_{m}^{\mu,\alpha}(z), B_{k}^{m} \rangle \right) M_{n-k}^{\mu} \psi(n) \right\rangle \]
\[ = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} \frac{1}{m!} \left( \langle P_{m}^{\mu,\alpha}(z), B_{k}^{m} \rangle, (M_{n-k}^{\mu}, \psi(n)) \right) \mathcal{H}^\otimes(n-k) \]
\[ = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} \left( \langle P_{m}^{\mu,\alpha}(z), B_{k+m}^{m} \rangle, (M_{n-k}^{\mu}, \psi(n+m)) \right) \mathcal{H}^\otimes(n-k) \]
\[ = \sum_{m=0}^{\infty} \left( \langle P_{m}^{\mu,\alpha}(z), \varphi^{(m)} \rangle \right), \]
such that, if
\[ \varphi(z) = \sum_{m=0}^{\infty} \langle P_{m}^{\mu,\alpha}(z), \varphi^{(m)} \rangle, \]
then we conclude that
\[ \varphi^{(m)} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n+m}{k+m} \frac{1}{m!} \langle B_{k+m}^{m}, (M_{n-k}^{\mu}, \psi(n+m)) \rangle \mathcal{H}^\otimes(n-k) \].

Now for \( p \in \mathbb{N} \) we need estimate \( |\varphi^{(n)}|_p \) by \( \| \cdot \|_{p,q,1} \) since the nuclear topology given by the norms \( \| \cdot \|_{p,q,1} \), is equivalent to the projective topology induced
by the norms \( n_{p,j,k} \) (see [KSWY95]). Now we estimate \( \varphi^{(m)}_\alpha \) as follows

\[
|\varphi^{(m)}_\alpha|_p \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} |B_{k+m,\mu}^{(n+m)}(M_{n-k}^{\mu}, \psi^{(n+m)})|_p
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} |B_{k+m,\mu}^{(n+m)}|_p \big| \psi^{(n+m)} \big|_p .
\]

Let us, at first, estimate the norm

\[
|B_{k+m,\mu}^{m}|_{-p,p} := |B_{k+m,\mu}^{m}|_p .
\]

To do this we choose \( p > p_\mu \) such that \( \|i_{p,p_\mu}\|_{HS} \) is finite and define

\[
D_{\alpha,\epsilon} := \sup_{|\theta|_p = \epsilon} |g_\alpha(\theta)|_p \quad \text{and} \quad \tilde{\epsilon} := \frac{\epsilon}{\epsilon \|i_{p,p_\mu}\|_{HS}} .
\]

So, with this

\[
|B_{k+m,\mu}^{m}|_{-p,p} \leq \sum_{l_1,\ldots,l_n=m} \frac{m!}{l_1! \cdots l_n!} |g^{(l_1)}(0)|_{-p,p} \cdots |g^{(l_n)}(0)|_{-p,p}
\]

\[
\leq \sum_{l_1,\ldots,l_n=m} \frac{m!}{l_1! \cdots l_n!} D_{\alpha,\epsilon}^m \tilde{\epsilon}^m
\]

\[
\leq m! D_{\alpha,\epsilon}^m \tilde{\epsilon}^m .
\]

that means

\[
|B_{k+m,\mu}^{m}|_{-p,p} \leq (k + m)! D_{\alpha,\epsilon}^m 2^{k+m} \tilde{\epsilon}^{-(k+m)} .
\]

Now let \( q \in \mathbb{N} \) such that \( 2^{q/2} > K_p \) \( (K_p := eC \|i_{p,p_\mu}\|_{HS} \) as in (2.2) \) and such that \( 2/(\tilde{\epsilon} K_p) < 1 \), then we obtain

\[
|\varphi^{(m)}_\alpha|_p \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{k!} (m + k)! D_{\alpha,\epsilon}^m 2^{k+m} \tilde{\epsilon}^{m} \frac{2^{-(n+m)q/2}}{(n+m)!}
\]

\[
\leq \|\varphi\|_{p,q,1} \frac{2^{-mq/2}}{m!} D_{\alpha,\epsilon}^m \sum_{n=0}^{\infty} \left( \frac{2}{\tilde{\epsilon} K_p} \right)^n \sum_{k=0}^{n} \left( \frac{2}{\tilde{\epsilon} K_p} \right)^k
\]

\[
\leq \|\varphi\|_{p,q,1} \frac{2^{-mq/2}m}{m!} \tilde{\epsilon}^m D_{\alpha,\epsilon}^m \left( 1 - 2^{-q/2} K_p \right)^{-1} \frac{\tilde{\epsilon} K_p}{\tilde{\epsilon} K_p - 2}
\]

\[
\equiv L_{p,q,\alpha,\tilde{\epsilon}} \frac{2^{-mq/2}m}{m!} \tilde{\epsilon}^m D_{\alpha,\epsilon}^m \|\varphi\|_{p,q,1} .
\]

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For \( q' < q \) such that \( 2^{2\epsilon - 2}2^{(q' - q)}D_{\alpha,\epsilon} < 1 \) this follows the following estimate

\[
\|\varphi\|_{p,q',\mu,\alpha}^2 \leq \sum_{m=0}^{\infty} (m!)^2 2^{mp'} |\varphi^{(m)}|_p^2 \leq \|\varphi\|_{p,q,1}^2 L_{p,q,\alpha,\epsilon}^2 \sum_{m=0}^{\infty} \left( 2^{2\epsilon - 2}2^{(q' - q)}D_{\alpha,\epsilon} \right)^m < \infty.
\]

This complete the proof. ■

Since we now have proved that the space of test functions \( (N)^1\) is isomorphic to \( S_{\min}(N') \), for all measures \( \mu \in M(a(N')) \) and for all holomorphic invertible function \( \alpha \in Ho_0(N_C, N_C) \), such that \( \alpha(0) = 0 \), we will now drop the subscript \( \mu, \alpha \). The test function space \( (N)^1 \) is the same for all measures and functions \( \alpha \) in the above conditions.

**Corollary 6.10** \( (N)^1 \) is an algebra under pointwise multiplication.

**Corollary 6.11** \( (N)^1 \) admits ‘scaling’, i.e., for \( \lambda \in \mathbb{C} \) the scaling operator \( \sigma_\lambda : (N)^1 \rightarrow (N)^1 \) defined by \( \sigma_\lambda \varphi(x) := \varphi(\lambda x), \varphi \in (N)^1, x \in N' \) is well-defined.

**Corollary 6.12** For all \( z \in N'_C \) the space \( (N)^1 \) is invariant under the shift operator \( \tau_z : \varphi \mapsto \varphi(\cdot + z) \).

## 7 Distributions

In this section we will introduce and study the space \( (N)^{-1}_{\mu,\alpha} \) of distributions corresponding to the space of test functions \( (N)^1_{\mu,\alpha} \). The goal is to prove that, for a fixed measure \( \mu \) and for all function \( \alpha \), as in the subsection 5.3, the space \( (N)^{-1}_{\mu,\alpha} = (N)_\mu^{-1} \), see Theorem 7.3 below.

Since \( P(N') \subset (N)^1 \) the space \( (N)^{-1}_{\mu,\alpha} \) can be viewed as a subspace of \( P'_\mu(N') \), i.e.,

\[ (N)^{-1}_{\mu,\alpha} \subset P'_\mu(N') \]

Let us now introduce the Hilbert subspace \( (H_{-p})^{-1}_{q,\mu,\alpha} \) of \( P'_\mu(N') \) for which the norm

\[
\|\Phi\|_{-p,q,\mu,\alpha}^2 := \sum_{n=0}^{\infty} 2^{-qn} |\Phi^{(n)}_{\alpha}|_{-p}^2
\]
is finite. Here we used the canonical representation
\[ \Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_\alpha^{(n)}) \in \mathcal{P}_\mu'(\mathcal{N}') \]
from Theorem 5.9. The space \((\mathcal{H}_{-p})^{-1}_{q,\mu,\alpha}\) is the dual space of \((\mathcal{H}_p)_q^{1,\mu,\alpha}\) with respect to \(L^2(\mu)\) (because of the biorthogonality of \(P_{\mu,\alpha}\)- and \(Q_{\mu,\alpha}\)-systems). By general duality theory
\[ (\mathcal{N})^{-1}_{\mu,\alpha} = \bigcup_{p,q \in \mathbb{N}} (\mathcal{H}_{-p})^{-1}_{q,\mu,\alpha} \]
is the dual space of \((\mathcal{N})^1\) with respect to \(L^2(\mu)\). As noted in Section 2 there exists a natural topology on co-nuclear spaces (which coincide with the inductive limit topology). We will consider \((\mathcal{N})^{-1}_{\mu,\alpha}\) as a topological vector space with this topology. So we have the nuclear triple
\[ (\mathcal{N})^1 \subset L^2(\mu) \subset (\mathcal{N})^{-1}_{\mu,\alpha} . \]
The action of a distribution
\[ \Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_\alpha^{(n)}) \in (\mathcal{N})^{-1}_{\mu,\alpha} \]
on a test function
\[ \varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi_\alpha^{(n)} \rangle \in (\mathcal{N})^1 \]
is given by
\[ \langle \langle \Phi, \varphi \rangle \rangle_\mu = \sum_{n=0}^{\infty} n! \langle \Phi_\alpha^{(n)}, \varphi_\alpha^{(n)} \rangle . \]

For a more detailed characterization of the singularity of distributions in \((\mathcal{N})^{-1}_{\mu,\alpha}\) we will introduce some subspaces in this distribution space. For \(\beta \in [0, 1]\) we define
\[ (\mathcal{H}_{-p})^{-\beta}_{q,\mu,\alpha} := \left\{ \Phi \in \mathcal{P}_\mu'(\mathcal{N}') \mid \sum_{n=0}^{\infty} (n!)^{1-\beta} 2^{-nq} |\Phi_\alpha^{(n)}|_{-p}^2 < \infty \right\} \]
for \(\Phi = \sum_{n=0}^{\infty} Q_n^{\mu,\alpha} (\Phi_\alpha^{(n)}) \).
and
\[(N)_{\mu,\alpha}^{-\beta} = \bigcup_{p,q \in \mathbb{N}} (\mathcal{H}_{-p})_{-q,\mu,\alpha}^{-\beta} .\]

It is clear that the singularity increases with increasing \(\beta\):
\[(N)_{\mu,\alpha}^{-1} \subset (N)_{\mu,\alpha}^{-\beta_1} \subset (N)_{\mu,\alpha}^{-\beta_2} \subset (N)_{\mu,\alpha}^{-1}\]
if \(\beta_1 \leq \beta_2\). We will also consider \((N)_{\mu,\alpha}^{-\beta}\) as equipped with the natural topology.

**Example 7.1 (Generalized Radon-Nikodym derivative)**

We want to define a generalized function \(\rho_{\mu}^{\alpha}(z, \cdot) \in (N)_{\mu,\alpha}^{-1}\), \(z \in N'\), with the following property
\[
\langle \langle \rho_{\mu}^{\alpha}(z, \cdot), \varphi \rangle \rangle_{\mu} = \int_{N'} \varphi(x-z) \, d\mu(x), \quad \varphi \in (N)^1 .
\]

That means we have to establish the continuity of \(\rho_{\mu}^{\alpha}(z, \cdot)\). Let \(z \in \mathcal{H}_{-p,C}\).

If \(p \geq p'\) is sufficiently large and \(\epsilon > 0\) small enough, Corollary 6.7 applies, i.e., \(\exists q \in \mathbb{N}\) and \(C > 0\) such that
\[
\left| \int_{N'} \varphi(x-z) \, d\mu(x) \right| \leq C \|\varphi\|_{p,q,\mu,\alpha} \int_{N'} \exp \left( \epsilon |x-z|_{-p'} \right) \, d\mu(x)
\]
\[
\leq C \|\varphi\|_{p,q,\mu,\alpha} \exp \left( \epsilon |z|_{-p'} \right) \int_{N'} \exp \left( \epsilon |x|_{-p'} \right) \, d\mu(x) .
\]

If \(\epsilon\) is chosen sufficiently small the last integral exists (Lemma 2.3-3). Thus we have in fact \(\rho_{\mu}^{\alpha}(z, \cdot) \in (N)^{-1}_{\mu,\alpha}\). It is clear that whenever the Radon-Nikodym derivative \(\frac{d\mu(x+\xi)}{d\mu(x)}\) exists (e.g., \(\xi \in \mathcal{N}\) in case \(\mu\) is \(\mathcal{N}\)-quasi-invariant) it coincides with \(\rho_{\mu}^{\alpha}(\xi, \cdot)\) defined above. We will show that in \((N)^{-1}_{\mu,\alpha}\) we have the canonical expansion
\[
\rho_{\mu}^{\alpha}(z, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \langle Q_{\mu,\alpha}^{\mu,\alpha}(\cdot), P_{\mu,\alpha}^{\mu,\alpha}(-z) \rangle
\]
where \(P_{\mu,\alpha}^{\mu,\alpha}(-z)\) is defined in (5.13). It is easy to see that the r.h.s. defines an element in \((N)^{-1}_{\mu,\alpha}\). Since both sides are in \((N)^{-1}_{\mu,\alpha}\) it is sufficient to
compare their action on a total set from \((\mathcal{N})^1\). For \(\varphi^{(n)}_\alpha \in \mathcal{N}_C^{\otimes n}\) we have

\[
\left\langle \left\langle \rho^{\alpha}_{\mu} (z, \cdot) , \left\langle P^{\mu,\alpha}_n (\cdot), \varphi^{(n)}_\alpha \right \rangle \right \rangle_{\mu}
\]
\[
= \left\langle \left\langle \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \left\langle Q^{\mu,\alpha}_k (\cdot), P^{\delta_0,\alpha}_k (-z) \right \rangle , \left\langle P^{\mu,\alpha}_n (\cdot), \varphi^{(n)}_\alpha \right \rangle \right \rangle_{\mu}
\]
\[
= \left\langle P^{\delta_0,\alpha}_n (-z), \varphi^{(n)}_\alpha \right \rangle,
\]
where we have used the biorthogonality property of the \(Q^{\mu,\alpha}\)- and \(P^{\mu,\alpha}\)-systems. On the other hand

\[
\left\langle \left\langle \rho^{\alpha}_{\mu} (z, \cdot) , \left\langle P^{\mu,\alpha}_n (\cdot), \varphi^{(n)}_\alpha \right \rangle \right \rangle_{\mu}
\]
\[
= \int_{\mathcal{N}} \left\langle P^{\mu,\alpha}_n (x - z), \varphi^{(n)}_\alpha \right \rangle d\mu (x)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} \int_{\mathcal{N}} \left\langle P^{\mu,\alpha}_k (x) \otimes P^{\delta_0,\alpha}_n (\cdot), \varphi^{(n)}_\alpha \right \rangle d\mu (x)
\]
\[
= \sum_{k=0}^{n} \binom{n}{k} E_{\mu} \left( \left\langle P^{\mu,\alpha}_k (\cdot) \otimes P^{\delta_0,\alpha}_n (\cdot), \varphi^{(n)}_\alpha \right \rangle \right)
\]
\[
= \left\langle P^{\delta_0,\alpha}_n (-z), \varphi^{(n)}_\alpha \right \rangle,
\]
where we made use of the relation (5.8). This had to be shown. In other words, we have proven that \(\rho^{\alpha}_{\mu} (z, \cdot)\) is the generating function of the \(Q^{\mu,\alpha}\)-system.

\[
\rho^{\alpha}_{\mu} (-z, \cdot) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle Q^{\mu,\alpha}_n (\cdot), P^{\delta_0,\alpha}_n (z) \right \rangle.
\]

**Example 7.2 (Delta function)** For \(z \in \mathcal{N}_C\) we define a distribution by the following \(Q^{\mu,\alpha}\)-decomposition:

\[
\delta_z = \sum_{n=0}^{\infty} \frac{1}{n!} Q^{\mu,\alpha}_n (P^{\mu,\alpha}_n (z)).
\]

If \(p \in \mathbb{N}\) is large enough and \(\epsilon > 0\) sufficiently small there exists \(\sigma_\epsilon > 0\) according to (5.9) such that

\[
\|\delta_z\|_{p,-q,\mu,\alpha}^2 = \sum_{n=0}^{\infty} (n!)^{-2} 2^{-nq} |P^{\mu,\alpha}_n (z)|^2.
\]
\[ \leq 4 \exp \left( 2\epsilon |z|_{-p} \right) \sum_{n=0}^{\infty} \sigma_n^{-2n} 2^{-nq}, \quad z \in \mathcal{H}_{-p, \mathbb{C}}, \]

which is finite for sufficiently large \( q \in \mathbb{N} \). Thus \( \delta_z \in (\mathcal{N})_{\mu, \alpha}^{-1} \).

For

\[ \varphi = \sum_{n=0}^{\infty} \langle P_{n}^{\mu, \alpha}, \varphi^{(n)} \rangle \in (\mathcal{N})^{1} \]

the action of \( \delta_z \) is given by

\[ \langle \langle \delta_z, \varphi \rangle \rangle_{\mu} = \sum_{n=0}^{\infty} \langle P_{n}^{\mu, \alpha}(z), \varphi^{(n)} \rangle = \varphi(z) \]

because of the biorthogonality property, see Theorem 5.8 pag. 29. This means that \( \delta_z \) (in particular for \( z \) real) plays the role of a “\( \delta \)-function” (evaluation map) in the calculus we discuss.

**Theorem 7.3** For a fixed measure \( \mu \) and for all function \( \alpha \), as in subsection 5.1, we have

\[ (\mathcal{N})_{\mu, \alpha}^{-1} = (\mathcal{N})_{\mu}^{-1}, \]

i.e., the space of distributions is the same for all functions \( \alpha \) in the above conditions.

**Proof.** Let \( \Phi \in (\mathcal{N})_{\mu, \alpha}^{-1} \) be given, then by Theorem 5.9 there exists generalized kernels \( \Phi^{(n)}_{\alpha} \in \mathcal{N}_{\mathbb{C}}^{\otimes n} \) such that \( \Phi \) has the following representation

\[ \Phi = \sum_{n=0}^{\infty} \langle Q_{n}^{\mu, \alpha}, \Phi^{(n)}_{\alpha} \rangle. \]

Now we use the definition of \( Q_{n}^{\mu, \alpha} \) given in (5.17) to obtain

\[ S_{\mu} \hat{\Phi}(\theta) = \sum_{n=0}^{\infty} \langle \Phi^{(n)}_{\alpha}, g_{\alpha}(\theta)^{\otimes n} \rangle = S_{\mu} \hat{\Phi}(g_{\alpha}(\theta)), \quad \theta \in \mathcal{N}_{\mathbb{C}}, \]  

where

\[ \hat{\Phi} = \sum_{n=0}^{\infty} \langle Q_{n}^{\mu}, \Phi^{(n)}_{\alpha} \rangle \in (\mathcal{N})_{\mu}^{-1}. \]
Hence by characterization Theorem 4.9 \( S_\mu \Phi \in \text{Hol}_0(\mathcal{N}_\mathcal{C}) \). But from (7.1) we see that
\[
S_\mu \Phi = \left( S_\mu \hat{\Phi} \right) \circ g_\alpha \in \text{Hol}_0(\mathcal{N}_\mathcal{C}),
\]
since this is the composition of two holomorphic functions (see [Din81]), again by the characterization Theorem 4.9 we conclude that \( \Phi \in ( \mathcal{N} )^{-1}_\mu \). Hence \( ( \mathcal{N} )^{-1}_\mu,\alpha \subseteq ( \mathcal{N} )^{-1}_\mu \).

Conversely, let \( \Psi \in ( \mathcal{N} )^{-1}_\mu,\alpha \) be given, i.e.,
\[
\Psi = \sum_{n=0}^{\infty} \left\langle Q_\mu, \Psi(n) \right\rangle, \quad \Psi(n) \in \mathcal{N}_\mathcal{C}^{\otimes n}.
\]
We want to prove that \( \Psi \in ( \mathcal{N} )^{-1}_\mu,\alpha \). Due to (5.17) and the definition of \( ( \mathcal{N} )^{-1}_\mu \) it is sufficient to show that
\[
S_\mu \Psi(\theta) = \sum_{n=0}^{\infty} \left\langle \hat{\Psi}_\alpha(n), g_\alpha(\theta)^{\otimes n} \right\rangle, \quad \theta \in \mathcal{N}_\mathcal{C},
\]
where \( \hat{\Psi}_\alpha(n) \) satisfy, for \( p, q \in \mathbb{N} \)
\[
\sum_{n=0}^{\infty} 2^{-nq} \left| \hat{\Psi}_\alpha(n) \right|_{-p}^2 < \infty.
\]
On the other hand, for a given \( \theta \in \mathcal{N}_\mathcal{C} \)
\[
S_\mu \Psi(\theta) = \sum_{n=0}^{\infty} \left\langle \Psi(n), \theta^{\otimes n} \right\rangle =: G(\theta)
\]
and, consequently \( G \in \text{Hol}_0(\mathcal{N}_\mathcal{C}) \). But we can write
\[
G(\theta) = G(\alpha \circ g_\alpha(\theta)) = \hat{G}(g_\alpha(\theta)),
\]
where \( \hat{G} = G \circ \alpha \), with \( G \circ \alpha \in \text{Hol}_0(\mathcal{N}_\mathcal{C}) \). Therefore
\[
\hat{G}(g_\alpha(\theta)) = \sum_{n=0}^{\infty} \left\langle \hat{G}_\alpha(n), g_\alpha(\theta)^{\otimes n} \right\rangle,
\]
where the coefficients \( \hat{G}_\alpha(n) \) verify
\[
\sum_{n=0}^{\infty} 2^{-nq} \left| \hat{G}_\alpha(n) \right|_{-p}^2 < \infty.
\]
Therefore with \( \hat{\Psi}_\alpha(n) = \hat{G}_\alpha(n) \) follows the result, i.e., \( \Psi \in ( \mathcal{N} )^{-1}_{\mu,\alpha} \). \( \blacksquare \)
8 The Wick product

Here we give the natural generalization of the **Wick multiplication** in the present setting.

**Definition 8.1** Let $\Phi, \Psi \in (\mathcal{N})^{-1}_\mu$. Then we define the Wick product $\Phi \diamond \Psi$ by

$$S_\mu (\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi.$$  

This is well defined because $\text{Hol}_0(\mathcal{N}_\mathbb{C})$ is an algebra and thus by characterization theorem there exists an element in $(\mathcal{N})^{-1}_\mu \Phi \diamond \Psi$ such that

$$S_\mu (\Phi \diamond \Psi) = S_\mu \Phi \cdot S_\mu \Psi.$$  

From this it follows

$$Q_{\mu,\alpha}^\mu (\Phi^{(n)}_\alpha) \diamond Q_{\mu,\alpha}^\mu (\Psi^{(m)}_\alpha) = Q_{\mu,\alpha}^{\mu,\alpha} (\Phi^{(n)}_\alpha \otimes \Psi^{(m)}_\alpha),$$  

$\Phi^{(n)}_\alpha \in \mathcal{N}_{\mathbb{C}}^{\otimes n}$ and $\Psi^{(m)}_\alpha \in \mathcal{N}_{\mathbb{C}}^{\otimes m}$. So in terms of $Q_{\mu,\alpha}$-decomposition

$$\Phi = \sum_{n=0}^{\infty} Q_{\mu,\alpha}^\mu (\Phi^{(n)}_\alpha) \quad \text{and} \quad \Psi = \sum_{m=0}^{\infty} Q_{\mu,\alpha}^\mu (\Psi^{(m)}_\alpha)$$  

the Wick product is given by

$$\Phi \diamond \Psi = \sum_{n=0}^{\infty} Q_{\mu,\alpha}^\mu (\Xi^{(n)}_\alpha),$$  

where

$$\Xi^{(n)}_\alpha = \sum_{k=0}^{n} \Phi^{(k)}_\alpha \otimes \Psi^{(n-k)}_\alpha.$$  

This allows for a concrete norm estimate.

**Proposition 8.2** The Wick product is continuous on $(\mathcal{N})^{-1}_\mu$. In particular the following estimate holds for $\Phi \in (\mathcal{H}_{-p_1})^{-1}_{-q_1, \mu, \alpha}$, $\Psi \in (\mathcal{H}_{-p_2})^{-1}_{-q_2, \mu, \alpha}$ and $p = \max(p_1, p_2)$, $q = q_1 + q_2 + 1$

$$\|\Phi \diamond \Psi\|_{-p, -q, \mu, \alpha} \leq \|\Phi\|_{-p_1, -q_1, \mu, \alpha} \|\Psi\|_{-p_2, -q_2, \mu, \alpha}.$$  

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Proof. We can estimate as follows
\[
\|\Phi \circ \Psi\|^2_{p,-q,\mu,\alpha} = \sum_{n=0}^{\infty} 2^{-nq} |\Xi(n)|^2
\]
\[
= \sum_{n=0}^{\infty} 2^{-nq} \left( \sum_{k=0}^{n} |\Phi^{(k)}_{\alpha}|^2 |\Psi^{(n-k)}_{\alpha}|^2 \right)^2
\]
\[
\leq \sum_{n=0}^{\infty} 2^{-nq} (n+1) \sum_{k=0}^{n} |\Phi^{(k)}_{\alpha}|^2 |\Psi^{(n-k)}_{\alpha}|^2
\]
\[
\leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} 2^{-nq_1} |\Phi^{(k)}_{\alpha}|^2 2^{-nq_2} |\Psi^{(n-k)}_{\alpha}|^2
\]
\[
\leq \left( \sum_{n=0}^{\infty} 2^{-nq_1} |\Phi^{(n)}_{\alpha}|^2 \right) \left( \sum_{n=0}^{\infty} 2^{-nq_2} |\Psi^{(n)}_{\alpha}|^2 \right)
\]
\[
= \|\Phi\|^2_{-p_1,-q_1,\mu,\alpha} \|\Psi\|^2_{-p_2,-q_2, \mu, \alpha}.
\]

Similar to the Gaussian case the special properties of the space \((\mathcal{N})^{-1}_\mu\) allow the definition of Wick analytic functions under very general assumptions. This has proven to be of some relevance to solve equations e.g., of the type \(\Phi \circ X = \Psi\) for \(X \in (\mathcal{N})^{-1}_\mu\). See [KLS96] for the Gaussian case.

Proposition 8.3 For any \(n \in \mathbb{N}\) and any \(\alpha\) as in Subsection 5.2 we have \(Q_n^{\mu,\alpha} = (Q_1^{\mu,\alpha})^\odot n\).

Proof. Let \(\Phi^{(1)} \in \mathcal{N}_C\) be given. Thus, if \(\theta \in \mathcal{N}_C\), follows
\[
S_\mu \left[ (Q_n^{\mu,\alpha} \left( \Phi^{(1)} \right) )^\odot n \right] (\theta) = \left< \Phi^{(1)}, g_\alpha(\theta) \right>^n
\]
\[
= \left< (\Phi^{(1)})^\odot n, (g_\alpha(\theta))^\odot n \right>
\]
\[
= S_\mu \left[ Q_n^{\mu,\alpha} \left( (\Phi^{(1)})^\odot n \right) \right] (\theta).
\]

Theorem 8.4 Let \(F : \mathbb{C} \to \mathbb{C}\) be analytic in a neighborhood of the point \(z_0 = \mathbb{E}(\Phi)\), \(\Phi \in (\mathcal{N})^{-1}_\mu\). Then \(F^\circ (\Phi)\) defined by \(S_\mu \left( F^\circ (\Phi) \right) = F(S_\mu \Phi)\) exists in \((\mathcal{N})^{-1}_\mu\).
Proof. By Theorems 7.3 and 1.9 we have $S_{\mu} \Phi \in \text{Hol}_0(\mathcal{N}_C)$. Then $F(S_{\mu} \Phi) \in \text{Hol}_0(\mathcal{N}_C)$ since the composition of two analytic functions is also analytic. Again by the above mentioned theorems we find that $F^\circ (\Phi)$ exists in $(\mathcal{N})_{\mu}^{-1}$. ■

Remark 8.5 If $F(z)$ have the following representation

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

then the Wick series

$$\sum_{n=0}^{\infty} a_n (\Phi - z_0)^{\circ n}$$

(where $\Psi^{\circ n} = \Psi \circ \cdots \circ \Psi$ n-times) converges in $(\mathcal{N})_{\mu}^{-1}$ and

$$F^\circ (\Phi) = \sum_{n=0}^{\infty} a_n (\Phi - z_0)^{\circ n}$$

holds.

Example 8.6 The above mentioned equation $\Phi \circ X = \Psi$ can be solved if $E_{\mu} (\Phi) = S_{\mu} \Phi (0) \neq 0$. That implies $(S_{\mu} \Phi)^{-1} \in \text{Hol}_0(\mathcal{N}_C)$. Thus

$$\Phi^{\circ (-1)} = S_{\mu}^{-1} ((S_{\mu} \Phi)^{-1}) \in (\mathcal{N})_{\mu}^{-1}.$$

Then $X = \Phi^{\circ (-1)} \circ \Psi$ is the solution in $(\mathcal{N})_{\mu}^{-1}$. For more instructive examples we refer the reader to Section 5 of [KLS96].

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9 Change of measure

Suppose we are given two measures \( \mu, \tilde{\mu} \in \mathcal{M}_a(N') \) both satisfying Assumption \( \text{??} \). Let a distribution \( \tilde{\Phi} \in (N')^{-1} \) be given. Since the test function space \( (N)^1 \) is invariant under changes of measure in view of Theorem 6.9, the continuous mapping

\[
\varphi \mapsto \langle \tilde{\Phi}, \varphi \rangle_{\tilde{\mu}}, \quad \varphi \in (N)^1,
\]

can also be represented as a distribution \( \widetilde{\Phi} \in (N)_{\tilde{\mu}}^{-1} \). So we have the implicit relation

\[
\tilde{\Phi} \in (N)_{\tilde{\mu}}^{-1} \iff \Phi \in (N)_{\mu}^{-1},
\]

defined by

\[
\langle \Phi, \varphi \rangle_{\mu} = \langle \tilde{\Phi}, \varphi \rangle_{\tilde{\mu}}.
\]

This section provides formulas which make this relation more explicit in terms of re-decomposition of the \( Q^{\mu,\alpha} \)-system. First we need an explicit relation of the corresponding \( P^{\mu,\alpha} \)-system.

**Lemma 9.1** Let \( \mu, \tilde{\mu} \in \mathcal{M}_a(N') \) be given, then

\[
P_n^{\mu,\alpha}(x) = \sum_{k+m+l=n} \frac{n!}{k!m!l!} P_k^{\tilde{\mu},\alpha}(x) \otimes P_m^{\mu,\alpha}(0) \otimes M_l^{\tilde{\mu},\alpha}. \tag{9.1}
\]

**Proof.** Expanding each factor in the formula

\[
e^\alpha_{\mu}(\theta; x) = e^\alpha_{\tilde{\mu}}(\theta; x) l_{\mu}^{\alpha-1}(\theta) l_{\tilde{\mu}}^{\alpha}(\theta),
\]

we obtain

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \langle P_n^{\mu,\alpha}(x), \theta^{\otimes n} \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \langle P_k^{\tilde{\mu},\alpha}(x), \theta^{\otimes k} \rangle \sum_{m=0}^{\infty} \frac{1}{m!} \langle P_m^{\mu,\alpha}(0), \theta^{\otimes m} \rangle \sum_{l=0}^{\infty} \frac{1}{l!} \langle M_l^{\tilde{\mu},\alpha}, \theta^{\otimes l} \rangle
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \sum_{k+m+l=n} \frac{n!}{k!m!l!} P_k^{\tilde{\mu},\alpha}(x) \otimes P_m^{\mu,\alpha}(0) \otimes M_l^{\tilde{\mu},\alpha}, \theta^{\otimes n} \right).
\]

A comparison of coefficients gives the above result. \( \blacksquare \)

An immediate consequence is the next reordering lemma.
Lemma 9.2 Let $\varphi \in (\mathcal{N})^1$ be given. Then $\varphi$ has the representation in $\mathbb{P}^{\mu,\alpha}$-system as well as $\mathbb{P}^{\tilde{\mu},\alpha}$-system:

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi^{(n)} \rangle = \sum_{n=0}^{\infty} \langle P_n^{\tilde{\mu},\alpha}, \tilde{\varphi}_\alpha^{(n)} \rangle,$$

where $\varphi^{(n)}_\alpha, \tilde{\varphi}^{(n)}_\alpha \in \mathcal{N}_C^n$ for all $n \in \mathbb{N}_0$ and the following formula holds

$$\tilde{\varphi}^{(n)}_\alpha = \sum_{n,m,l=0}^{\infty} \frac{(n + m + l)!}{n!m!l!} \langle P_k^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi^{(n+m+l)}_\alpha \rangle_{\mathcal{H}^{(m+l)}}. \quad (9.2)$$

Proof. We use the relation (9.1) to obtain

$$\varphi = \sum_{n=0}^{\infty} \langle P_n^{\mu,\alpha}, \varphi^{(n)}_\alpha \rangle$$

$$= \sum_{n=0}^{\infty} \left( \sum_{k+m+l=n} \frac{n!}{k!m!l!} \langle \tilde{P}_k^{\mu,\alpha}(x) \hat{\otimes} P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi^{(n)}_\alpha \rangle \right)$$

$$= \sum_{k,m,l=0}^{\infty} \left( \frac{(k + m + l)!}{k!m!l!} \langle \tilde{P}_k^{\mu,\alpha}(x), (P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi^{(k+m+l)}_\alpha \rangle_{\mathcal{H}^{(m+l)}} \right)$$

$$= \sum_{k=0}^{\infty} \left( \tilde{P}_k^{\mu,\alpha}(x), \sum_{m,l=0}^{\infty} \frac{(k + m + l)!}{k!m!l!} \langle P_m^{\mu,\alpha}(0) \hat{\otimes} M_l^{\tilde{\mu},\alpha}, \varphi^{(k+m+l)}_\alpha \rangle_{\mathcal{H}^{(m+l)}} \right).$$

Then a comparison of coefficients give the result. 

Now we may prove the announced theorem.

Theorem 9.3 Let $\tilde{\Phi}$ be a generalized function with representation

$$\tilde{\Phi} = \sum_{n=0}^{\infty} \langle Q_n^{\mu,\alpha}, \tilde{\Phi}^{(n)}_\alpha \rangle.$$

Then

$$\Phi = \sum_{n=0}^{\infty} \langle Q_n^{\mu,\alpha}, \Phi^{(n)}_\alpha \rangle,$$

defined by

$$\langle \Phi, \varphi \rangle_\mu = \langle \tilde{\Phi}, \varphi \rangle_{\tilde{\mu}}, \varphi \in (\mathcal{N})^1,$$
is in \((\mathcal{N})_\mu^{-1}\) and the following relation holds

\[
\Phi^{(n)} = \sum_{k+m+l=n} \frac{1}{m!l!} \tilde{\Phi}_\alpha^{(k)} \otimes P_m^{\mu,\alpha}(0) \otimes M_l^{\mu,\alpha}.
\]

**Proof.** We can insert formula (9.2) in the formula

\[
\sum_{n=0}^\infty n! \langle \Phi^{(n)}_\alpha, \varphi^{(n)}_\alpha \rangle
\]

\[
= \sum_{k=0}^\infty k! \langle \tilde{\Phi}_\alpha^{(k)}, \tilde{\varphi}_\alpha^{(k)} \rangle
\]

\[
= \sum_{k=0}^\infty k! \left( \sum_{m,l=0}^\infty \frac{(k + m + l)!}{k!m!l!} \langle P_m^{\mu,\alpha}(0) \otimes M_l^{\mu,\alpha}, \varphi^{(k+m+l)}_\alpha \rangle_{\mathcal{H}^{k+m+l}} \right)
\]

\[
= \sum_{k,m,l=0}^\infty \frac{(k + m + l)!}{m!l!} \langle \tilde{\Phi}_\alpha^{(k)} \otimes P_m^{\mu,\alpha}(0) \otimes M_l^{\mu,\alpha}, \varphi^{(k+m+l)}_\alpha \rangle
\]

\[
= \sum_{n=0}^\infty n! \left( \sum_{k+m+l=n} \frac{1}{m!l!} \tilde{\Phi}_\alpha^{(k)} \otimes P_m^{\mu,\alpha}(0) \otimes M_l^{\mu,\alpha}, \varphi^{(n)}_\alpha \right),
\]

and compare coefficients again. \(\blacksquare\)

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