Electronic Structure of Single- and Multiple-shell Carbon Fullerenes

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We study the electronic states of giant single-shell and the recently discovered nested multi-shell carbon fullerenes within the tight-binding approximation. We use two different approaches, one based on iterations and the other on symmetry, to obtain the $\pi$-state energy spectra of large fullerene cages: $C_{240}$, $C_{540}$, $C_{960}$, $C_{1500}$, $C_{2160}$ and $C_{2940}$. Our iteration technique reduces the dimensionality of the problem by more than one order of magnitude (factors of $\sim 12$ and $20$), while the symmetry-based approach reduces it by a factor of $10$. We also find formulae for the highest occupied and lowest unoccupied molecular orbital (HOMO and LUMO) energies of $C_{60+n^2}$ fullerenes as a function of $n$, demonstrating a tendency towards metallic regime for increasing $n$. For multi-shell fullerenes, we analytically obtain the eigenvalues of the intershell interaction.

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than one order of magnitude (factors of ~ either 12 or 20 depending on the chosen initial state). For instance, it reduces \( N \) by about a factor of 20, from 2940 to 154, for the \( C_{2940} \) fullerene molecule. Furthermore, it can provide some precise analytic results for the eigenvalues. By using a different method based only on symmetry arguments, we can reduce \( N \) by a factor of 10. These two approaches give the same results for the energy eigenvalues. However, several distinct features of the first method that make it very convenient include the following: it requires the diagonalization of a fewer number of matrices, each one of them smaller in size, and every one of them with all their elements real.

Here we focus on a simple model of the electronic states of the giant single-shell fullerenes, and multiple-shell carbon “onions” with the number of shells up to seven. The strategy used in this work includes two parts. First, we treat each \( C_{60-n^2} \) fullerene as a \( 60-n^2 \)-site nearest-neighbor tight-binding model. Each site corresponds to one radially directed carbon \( \pi \) orbital. Since the \( \pi \) orbitals provide the dominant contribution to the conducting properties of the molecules, the \( \sigma \) orbitals (being mainly responsible for the elastic properties) are not taken into account. In spite of the slight differences between bond lengths and structure curvature in these molecules, we have assumed a uniform hopping amplitude \( t \) for all the fullerenes under consideration. This assertion is motivated by the fact that results in \( C_{60} \) changed insignificantly by considering the two different hopping integrals. These icosahedrally-symmetric \( C_{60-n^2} \) molecules will be studied for \( n = 1, 2, \ldots, 7 \) (i.e., \( C_{60}, C_{240}, C_{540}, C_{960}, C_{1500}, C_{2160} \) and \( C_{2940} \)).

In the last part of this work, we use an intermolecular tight-binding model to account for the weak intershell interaction between the fullerene layers and analytically obtain its eigenvalues. It is important to stress that the simple tight-binding approximation yields reasonable quantitative results for the energy levels of each individual fullerene, especially for those near the Fermi level. Furthermore, the intermolecular tight-binding model has also been applied to the electronic states of solid \( C_{60} \).

**Single-shell giant fullerenes.**—In the description of the electronic states of \( C_{60-n^2} \) molecules, we employ a tight-binding Hamiltonian (Hückel theory as, for instance, used for fullerenes in Refs. 10-12) written as

\[
H = -\sum_{<ij>} t c_i^+ c_j.
\]

Admittedly, this is a simplified Hamiltonian: the “Ising model” for the electronic structure. However, its understanding is a convenient stepping stone to the study of more complex models. Also, simple models are more accessible for analytical or semi-analytical approaches. Much more realistic and complicated tight-binding Hamiltonians can be found, for instance, in Ref. 13 and papers cited therein. For carbon \( C_{60} \), we have obtained closed-form expressions of the eigenvalues and eigenfunctions for both \( \pi \) and \( \sigma \) states as well as the Green function and local density of states by using recursion and path-integral methods. Here we focus on the giant fullerenes.
To obtain the energy spectrum, we use two entirely different approaches, each one with its advantages. The first approach we use here is our generalization of the recursion method\(^1\), which we call the “generalized block recursion method”. Because of limited space, here we present only a brief outline of this approach. We start from two initial states, \( |u_0 > \) and \( |v_0 > \), formed as

\[
|u_0 > = \frac{1}{\sqrt{10}} \sum_{j=1}^{5} |j > + \mathcal{P} |j' >
\]

and

\[
|v_0 > = \frac{1}{\sqrt{12}} \sum_{j=1}^{6} (-1)^{j+1} |j > + \mathcal{P} |j' >
\]

In \( |u_0 > \) (\( |v_0 > \), \( j > \) denotes the \( \pi \) orbitals on a pentagon (hexagon), and \( |j' > \) denotes those on the opposite (i.e., \( j \) and \( j' \) are antipodes) pentagon (hexagon). Also \( \mathcal{P} \) stands for parity, with the value +1 or −1. Notice that these two pentagon (hexagon) rings satisfy the prerequisite that a fivefold (threefold) symmetry axis passes through their centers. Further states are then iteratively generated by the general recurrence relation

\[
H |f^\alpha_m > = a^\alpha_m |f^\alpha_m > + b^\beta_m |f^\beta_{m+1} > + b^\gamma_m |f^\gamma_{m-1} >,
\]

where \( m \) designates states which are \( m \) hops away from the initial state. The superscript designates various suitable states. On the left-hand side of Eq. (3), at most three terms are nonzero for each application of \( H \) on \( |f^\alpha_m > \). Among them, \( \beta \) and \( \gamma \) can each take at most two values among their several possible ones. The process terminates after a finite number of iterations. The representation of \( H \) thus becomes a “block-tridiagonal” form. However, the dimensionality of each block is not necessarily the same. The main operational difference between this approach and the standard recursion method is that additional indices (\( \alpha, \beta \) and \( \gamma \)) are needed to fully specify the generated states, producing very simple values (i.e., \( \pm 1, \pm 2 \) and \( \sqrt{2} \)) for the coefficients \( a' \)'s and \( b' \)'s. Furthermore, the standard recursion method is extremely cumbersome to apply for \( C_{60} \cdot n^2 \) (for \( n > 1 \)), while very convenient for only \( n = 1 \).

The advantage of this approach lies in that it reduces the task of diagonalizing the extremely large original Hamiltonian matrix for \( C_{60} \cdot n^2 \) to the diagonalization of four much smaller matrices and provides precise analytic results for some of the eigenvalues (e.g., \( \pm 1, \pm 2, -3 \)). For instance, in order to obtain the whole energy spectrum of \( C_{240} \), we only need to diagonalize two (due to \( \mathcal{P} = \pm 1 \)) \( 14 \times 14 \) (constructed from \( |u_0 > \)) and two \( 18 \times 18 \) (constructed from \( |v_0 > \)) matrices instead of a \( 240 \times 240 \) matrix. Thus, the dimensionality has been reduced by (factors of \( \sim 12 \) and 20) more than one order of magnitude. In general, for \( C_{60} \cdot n^2 \) the dimensionality is reduced from \( 60n^2 \) to \( 3n^2 + n \) \((5n^2 - n)\) starting from the initial state \( |u_0 > \) \( (|v_0 > \)). The relative size reduction is

\[
3
\]
\[
(3n^2+n)/60n^2 = 1/20 + 1/60n \simeq 1/20 \ ( (5n^2-n)/60n^2 = 1/12 - 1/60n \simeq 1/12) \text{ for } |u_0 > (|v_0 >). \text{ Therefore, the dimensionality of the original giant fullerene problem can be easily reduced by more than one order of magnitude.}
\]

The LUMO (HOMO) energy is solved from the matrix constructed from \(|u_0 > (|v_0 >)\) with negative parity.

We also use another approach, which closely follows the method presented in Ref. 12. This alternative and quite different method is based on group theory analysis. For instance, for \(C_{240}\), six \(24 \times 24\) matrices (four of them with complex elements) are diagonalized to obtain the energy spectrum. This method reduces the dimensionality of the problem by a factor of 10. Therefore, it is a desirable and useful goal to be able to first reduce the dimensionality of the problem by more than one order of magnitude, in order to study very large fullerene cages.

The energy levels from the first two methods for various fullerenes are shown in Fig. 1. In Table I, we summarize the following physically important quantities for the giant fullerenes: energies of the HOMO, LUMO and their differences (i.e., the band gaps). As the size of the fullerene grows, it can be seen that the band gap becomes smaller. These results demonstrate the tendency towards metallic regime for very large fullerenes.

The best logarithmic fit formulae of the HOMO and LUMO energies (in units of \(t\)) for \(C_{60}\) molecules with \(n\) up to 7 are

\[
E_{\text{HOMO}} = -0.67402 \cdot n^{-0.72956} \quad (4)
\]

and

\[
E_{\text{LUMO}} = 0.14670 \cdot n^{-1.3772}. \quad (5)
\]

By taking \(t\) equal to the typical value of 2.5 eV, these expressions become \(E_{\text{HOMO}} = -1.68505 \cdot n^{-0.72956} \) eV and \(E_{\text{LUMO}} = 0.36675 \cdot n^{-1.3772} \) eV. Even though these fits are not very accurate for small \(n\), we believe that Eqs. (4) and (5) provide good approximations for larger values of \(n \ (n \geq 8)\) where calculations are very difficult. From them, the band gap can also be readily inferred.

**Multi-shell carbon fullerenes.**—In this section, we study the effects of the interaction between consecutive shells on the electronic states of the fullerene shells. In the stackings of the onion-like structures, the pentagons are aligned along the icosahedral directions of the twelve vertices, and the hexagons are stacked in a manner similar to the AB (or Bernal) structure of graphite. Nevertheless, and as verified in Ref. 8, the interlayer interaction is mainly between adjacent shells. Each single-shell fullerene is now assumed to be in its ground (lowest energy) state and here we focus on its HOMO. We assume an intermolecular hopping amplitude \(V_n\) between the pairs of fullerenes \(C_{60} \cdot n^2\) and \(C_{60} \cdot (n+1)^2\). A tight-binding approximation, \(H_I = -\sum_{n=1}^{N-1} V_n \ (|\phi_n > < \phi_{n+1} | + |\phi_{n+1} > < \phi_n |)\), is then used to model the intershell interaction, where \(|\phi_n >\) denotes the HOMO of
the \( C_{60-n^2} \) molecule and \( N_s \) is the total number of shells. The \( V_n \)'s can be obtained, for example, through an intermolecular resonance integrals calculation. At this moment, such calculations cannot yet be accurately done because of the lack of knowledge about the exact atomic positions. However, the \( V_n \)'s are estimated to be of the order of a few hundred meV. Notice that while the multiplicity of the HOMO for each fullerene is five, the nonzero matrix element only exits between those belonging to the same representation. In other words, molecular states \( |\phi_n> \) and \( |\phi_{n+1}> \) in \( H_I \) must be in the same representation. Thus, for a \( N_s \)-shell carbon "onion", a small \( N_s \times N_s \) matrix (constructed from \( H_I \)), with the \( m \)-th row and \( n \)-th column element \( -V_{n-1}\delta_{m,n-1} - V_n\delta_{m,n+1} \), is sufficient to fully investigate the interaction between the \( 5N_s \) HOMOs. The eigenvalues for \( H_I \) can then be analytically solved. The results for various \( N_s \) are presented in Table II. This model with different hopping amplitudes can be readily applied to the threefold degenerate LUMOs.

In summary, we use two different approaches to study the \( \pi \)-state energy spectra of fullerene cages up to \( C_{2940} \). We also find formulae for the HOMO and LUMO energies of fullerenes \( C_{60-n^2} \) as a function of \( n \). These approximations might be useful since experimentally obtained carbon "onions" can have \( \sim 70 \) shells (with 294,000 atoms for the 70th shell, these systems are well beyond current computational capabilities). We obtain the gap energy as a function of \( n \) and show the tendency towards metallic regime for very large fullerenes. For multi-shell carbon "onions", we analytically obtain the eigenvalues of the intershell interaction. The HOMO, LUMO, and gap energies, as well as the whole energy spectra, of these molecules are relevant to the several important experimental techniques which probe the spectroscopy of single molecules. For instance, a scanning tunneling microscope (e.g., Ref. 16) can be used to probe the local spectroscopy of fullerenes.

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FIG. 1. The energy eigenvalues (horizontal axis) and their corresponding degrees of degeneracy (vertical axis) for various fullerenes. The energies are in units of carbon-carbon hopping integral $t$. The respective fivefold degenerate HOMO and threefold degenerate LUMO levels are indicated. For $C_{60-n^2}$, there are approximately $60-n^2/4 = 15-n^2$ distinct energy levels.

FIG. 2. The HOMO, LUMO and gap energies (in units of eV) versus $1/n$ for various fullerenes $C_{60-n^2}$.
TABLE I. $N_L = 3n^2 + n$ ($N_H = 5n^2 - n$) is the dimensionality of the matrix constructed from $|u_0 >$ ($|v_0 >$), which provide the LUMO (HOMO) energies. These are significantly smaller than the dimensionality $N = 60n^2$ of the original Hamiltonian. Energies of the HOMO, LUMO and band gaps for $C_{60,n^2}$ fullerenes, with $n = 1, 2, \cdots, 7$. In the HOMO and LUMO columns, the first values are in units of $t$ and those in parentheses are the results obtained by taking $t$ equal to the typical value of 2.5 eV.

| Molecule | $N_L$ | $N_H$ | HOMO | LUMO    | Gap (eV) |
|----------|-------|-------|------|---------|----------|
| $C_{60}$ | 4     | 4     | -0.618033 | 0.138564 | 1.891492 |
| $C_{240}$ | 14    | 18    | -0.436772 | 0.059657 | 1.241072 |
| $C_{540}$ | 30    | 42    | -0.324099 | 0.033671 | 0.894424 |
| $C_{960}$ | 52    | 76    | -0.255010 | 0.022169 | 0.692947 |
| $C_{1500}$ | 80   | 120   | -0.209564 | 0.015971 | 0.563837 |
| $C_{2160}$ | 114  | 174   | -0.177673 | 0.012192 | 0.474662 |
| $C_{2940}$ | 154  | 238   | -0.154140 | 0.009690 | 0.409575 |

TABLE II. The energy eigenvalues of $H_I$ for various $N_s$.

| $N_s$ | Eigenvalues |
|-------|-------------|
| 2     | $\pm V_1$  |
| 3     | 0 and $\pm \sqrt{V_1^2 + V_2^2}$ |
| 4     | $-A \pm B$ and $A \pm B$ |
| 5     | 0, $\pm \sqrt{S - T}$ and $\pm \sqrt{S + T}$ |

Here $A = \sqrt{(V_1 + V_3)^2 + V_2^2/2}$, $B = \sqrt{(V_1 - V_3)^2 + V_2^2/2}$, $S = (V_1^2 + V_2^2 + V_3^2 + V_4^2)/2$, and $T = \sqrt{(V_1^2 + V_2^2 - V_3^2 - V_4^2)^2 + 4V_2^2V_3^2}/2$. 

| Molecule | $N_s$ | Eigenvalues |
|----------|-------|-------------|
| $C_{60}$ | 2     | $\pm V_1$  |
| $C_{240}$ | 3     | 0 and $\pm \sqrt{V_1^2 + V_2^2}$ |
| $C_{540}$ | 4     | $-A \pm B$ and $A \pm B$ |
| $C_{960}$ | 5     | 0, $\pm \sqrt{S - T}$ and $\pm \sqrt{S + T}$ |