Maximally highly proximal flows

Andy Zucker

February 2019; revised October 2019

Abstract

For \( G \) a Polish group, we consider \( G \)-flows which either contain a comeager orbit or have all orbits meager. We single out a class of flows, the maximally highly proximal (MHP) flows, for which this analysis is particularly nice. In the former case, we provide a complete structure theorem for flows containing comeager orbits, generalizing theorems of Melleray-Nguyen Van Thé-Tsankov and Ben Yaacov-Melleray-Tsankov. In the latter, we show that any minimal MHP flow with all orbits meager has a metrizable factor with all orbits meager, thus “reflecting” complicated dynamical behavior to metrizable flows. We then apply this to obtain a structure theorem for Polish groups whose universal minimal flow is distal.

1 Introduction

Let \( G \) be a Polish group. A \( G \)-flow is a compact Hausdorff space equipped with a continuous (right) \( G \)-action \( X \times G \to X \). If \( X \) and \( Y \) are \( G \)-flows, a map \( \varphi: X \to Y \) is a \( G \)-map if \( \varphi \) is continuous and respects the \( G \)-actions. A subflow of a \( G \)-flow \( X \) is any non-empty closed invariant subspace \( Y \subseteq X \). We say \( X \) is minimal if the only subflow of \( X \) is \( X \) itself. Equivalently, \( X \) is minimal if for every \( x \in X \), the orbit \( x \cdot G \subseteq X \) is dense. Notice that if \( \varphi: X \to Y \) is a \( G \)-map, then the image \( \varphi[X] \subseteq Y \) is a subflow; if \( X \) is minimal, so is \( \varphi[X] \), and if \( Y \) is minimal, then \( \varphi \) is surjective. We often call a surjective \( G \)-map a factor.

By a classical theorem of Ellis, there is a universal minimal flow \( M(G) \); this is a minimal \( G \)-flow which admits a \( G \)-map onto any other minimal \( G \)-flow, and \( M(G) \) is unique up to isomorphism. The study of \( M(G) \) is useful because it captures information about all minimal \( G \)-flows. For instance, if \( M(G) \) is metrizable, then every minimal \( G \)-flow is metrizable, and if \( M(G) \) has a (necessarily unique) comeager orbit, then so does every minimal \( G \)-flow \( [1] \). However, \( M(G) \) is often very complicated; for example, if \( G \) is locally compact, then \( M(G) \) is never metrizable, and all of its orbits are meager. However, there are Polish groups \( G \) for which \( M(G) \) is a singleton, and many others for which \( M(G) \) is metrizable and has a concrete description. See \( [9] \) for several examples of these phenomena.

---

2010 Mathematics Subject Classification. Primary: 37B05; Secondary: 54H20, 03E15.
The author was supported by NSF Grant no. DMS 1803489.
The starting point of this paper is the following theorem, first proved by the author \[12\] in the case that $G$ is non-Archimedean, and then by Ben Yaacov, Melleray, and Tsankov \[7\] for general Polish groups.

**Fact 1.1.** If $G$ is a Polish group and $M(G)$ is metrizable, then $M(G)$ has a comeager orbit.

This theorem along with the structure theorem due to Melleray, Nguyen Van Thé, and Tsankov \[11\] provide a complete understanding of the structure of $M(G)$ when it is metrizable. However, the property of $M(G)$ having a comeager orbit remained less well understood. Indeed, it was only recently shown, by an example of Kwiatkowska \[10\], that the converse of Fact 1.1 does not hold.

The study of $M(G)$ is often undertaken by attempting to understand the Samuel compactification $\text{Sa}(G)$, the Gelfand space of the bounded left uniformly continuous functions on $G$. The group $G$ canonically embeds into $\text{Sa}(G)$, and for any $G$-flow $X$ and any $x \in X$, there is a unique $G$-map $\lambda_x : \text{Sa}(G) \to X$ with $\lambda_x(1_G) = x$. In particular, any minimal subflow of $\text{Sa}(G)$ is isomorphic to $M(G)$. The main technical tool introduced in \[7\] is to view $\text{Sa}(G)$ as a topometric space, a topological space endowed with a possibly finer metric $\partial$ which interacts with the topology in nice ways. Letting $\partial$ denote this finer metric, the authors of \[7\] show that if $M \subseteq \text{Sa}(G)$ is a compact metrizable subspace, then $\partial|_M$ is a compatible metric. When the metrizable $M \subseteq \text{Sa}(G)$ is a minimal subflow, the properties of the metric $\partial|_M$ allow them to show that $M$ has a comeager orbit. However, much remained unclear about this metric, especially when $M(G)$ is non-metrizable. Namely, if $M \subseteq \text{Sa}(G)$ is a minimal subflow, can we define $\partial|_M$ just using the dynamics of $M$?

This paper singles out a class of flows, the maximally highly proximal flows, or MHP flows, which all admit a canonical topometric structure. In particular, $M(G)$ and $\text{Sa}(G)$ are both MHP, and the topometric on $M(G)$ agrees with the metric inherited by any minimal subflow of $\text{Sa}(G)$. Using this topometric structure, we provide a structure theorem for MHP flows with a comeager orbit. Here, a compatibility point is a point in $X$ where the topology and the metric coincide (see Definition 5.1).

**Theorem 5.5.** Let $X$ be an MHP flow. The following are equivalent.

1. $X$ has a compatibility point with dense orbit.

2. The set $Y \subseteq X$ of compatibility points is comeager, Polish, and contains a point with dense orbit.

3. $X$ has a comeager orbit.

4. $X \cong \text{Sa}(H \backslash G)$ for some closed subgroup $H \subseteq G$ (see Section 3.2.1)

In Theorem 7.5, we generalize the main result of \[11\] by considering the case that $X \in \{M(G), \Pi(G), \Pi_s(G)\}$, where $\Pi(G)$ and $\Pi_s(G)$ are the universal minimal proximal flow and
the Furstenberg boundary, respectively. In the first and third case, we show that the closed subgroup \(H\) appearing in item (4) is extremely amenable or amenable, respectively, and in the second case we present a partial result towards showing that \(H\) is strongly amenable.

As an application of Theorem 5.5, we prove the following “reflection” theorem, which shows that complicated dynamical behavior of the group \(G\) already appears in the realm of metrizable flows. Note that in minimal flows, all orbits are either meager or comeager.

**Theorem 8.1.** Let \(X\) be a minimal MHP flow all of whose orbits are meager. Then there is a factor \(\varphi: X \to Y\) so that \(Y\) is metrizable and also has all orbits meager.

This theorem was first suggested in [7], but in private communication with the authors, it was realized that the problem remained open.

As an application of Theorem 8.1, we give a complete characterization of when \(M(G)\) is distal in Theorem 9.2 and Corollary 9.3. The theorem says that if \(M(G)\) is distal, then \(M(G)\) is metrizable. Then using results from [11], the corollary shows that any such \(G\) has a normal, extremely amenable subgroup \(H\) with \(M(G) \cong H\backslash G\).

**Acknowledgements**

I thank Todor Tsankov for many helpful discussions, including the suggestion that Theorem 7.5 is true. Some of the work here builds on work in my Ph.D. thesis, and I thank Clinton Conley for his guidance in its completion. I also thank the referee for many helpful suggestions on an earlier draft.

**Notation**

We will use some non-standard notation. The phrases “non-empty open subset of,” “open neighborhood of,” etc. occur often enough that we introduce some notation for this. If \(X\) is a topological space, then \(A \subseteq_{op} X\) will mean that \(A\) is a non-empty open subset of \(X\). If \(x \in X\), we write \(x \in_{op} A\) or \(A \ni_{op} x\) to mean that \(A \subseteq X\) is an open neighborhood of \(x\). Omitting the “op” subscript does not mean that a given set is not open; it is just an easy way to introduce and/or emphasize open sets.

Other notation is mostly standard. We write \(\omega = \{0, 1, 2, \ldots\}\), and we identify a non-negative integer with the set of its predecessors, i.e. \(n = \{0, \ldots, n - 1\}\). If \(f: X \to Y\) is a function and \(K \subseteq X\), we set \(f[K] := \{f(x) : x \in K\}\). All topological spaces we consider are Hausdorff.

**2 Topometric spaces**

This short section collects the background material on topometric spaces that we will need going forward. Most of the material here can be found in [4] or [6].
Definition 2.1. A compact topometric space is a triple \((X, \tau, \partial)\), where \((X, \tau)\) is a compact Hausdorff space and \(\partial\) is a metric which is lower semi-continuous, meaning that for every \(c \geq 0\), the set \(\{(p, q) \in X^2 : \partial(p, q) \leq c\}\) is \((\tau \times \tau)\)-closed.

Note that the metric need not agree with the underlying topology. As a convention, when discussing a topometric space, topological vocabulary will refer to \(\tau\), while metric vocabulary will refer to \(\partial\).

Fact 2.2. Let \((X, \tau, \partial)\) be a compact topometric space.

1. The metric \(\partial\) is finer than the topology.
2. The metric \(\partial\) is complete.

Remark. One can also define topometric spaces where the underlying topological space is not compact. One then includes item (1) above in the definition.

The following fact will be needed going forward.

Fact 2.3 (Ben Yaacov [4]). Let \((X, \tau, \partial)\) be a compact topometric space. Then if \(K, L \subseteq X\) are closed with \(\partial(K, L) > r\), then there is a continuous, 1-Lipschitz function \(f : X \to [0, 1]\) with \(f[K] = \{0\}\) and \(f[L] = \{r\}\).

If \((X, \tau, \partial)\) is a compact topometric space, \(K \subseteq X\), and \(c > 0\), we define \(K(c) := \{p \in X : \partial(p, K) < c\}\) and \(K[c] := \{p \in X : \partial(p, K) \leq c\}\). If \(K = \{p\}\) for some \(p \in X\), we just write \(p(c)\) or \(p[c]\), respectively.

Definition 2.4 ([6], Def. 1.25). A topometric space \((X, \tau, \partial)\) is called adequate if for every open \(A \subseteq X\) and every \(c > 0\), we have \(A(c)\) open.

We will prove (see Theorem 4.8) that the topometric spaces we consider in this paper are all adequate.

3 Maximally highly proximal flows

Throughout this section, \(G\) will denote a fixed Polish group. We let \(d_G\) denote a compatible left-invariant metric of diameter 1, and for \(c > 0\), we set \(U_c := \{g \in G : d_G(1_G, g) < c\}\). We will frequently and without explicit mention make use of the inclusion \(U_c U_c \subseteq U_{c+\epsilon}\).

Definition 3.1. Let \(X\) be a \(G\)-flow. We say that \(X\) is maximally highly proximal, or MHP, if for every \(A \subseteq_{op} X\), every \(x \in \overline{A}\), and every \(c > 0\), we have \(x \in \text{int}(\overline{A U_c})\).
3.1 Highly proximal extensions

The name MHP comes from the notion of a \textit{highly proximal} extension. If $\varphi: Y \to X$ is a surjective $G$-map, we define the \textit{fiber image} of $B \subseteq_{op} Y$ to be $\varphi_{\text{fib}}(B) := \{x \in X : \varphi^{-1}(\{x\}) \subseteq B\}$. The set $\varphi_{\text{fib}}(B)$ is always open whenever $B \subseteq_{op} Y$, but possibly empty. We call $\varphi$ \textit{highly proximal} if $\varphi_{\text{fib}}(B) \neq \emptyset$ for every $B \subseteq_{op} Y$. The composition of highly proximal maps is also highly proximal. Also notice that if $X$ is minimal and $\varphi: Y \to X$ is highly proximal, then $Y$ is also minimal. More precisely, if $X$ is any $G$-flow and $x \in X$ has dense orbit, then if $\varphi: Y \to X$ is any highly proximal extension, then any $y \in \varphi^{-1}(\{x\})$ also has dense orbit.

To motivate why this notion receives the name “highly proximal,” it is helpful to compare this to the notion of a proximal extension. A $G$-map $\varphi: Y \to X$ is called \textit{proximal} if for any $y_0, y_1 \in Y$ with $\varphi(y_0) = \varphi(y_1)$, we can find a net $g_i$ from $G$ and $z \in Y$ with $\lim y_0 g_i = \lim y_1 g_i = z$. Now suppose that $X$ is minimal and that $\varphi: Y \to X$ is highly proximal. Then $\varphi$ is proximal. To see this, let $y_0, y_1 \in Y$ with $\varphi(y_0) = \varphi(y_1) = x$. Fix any $z \in Y$, and let $\{B_i : i \in I\}$ be a base of neighborhoods of $z$. For each $B_i$, we have $\varphi_{\text{fib}}(B_i) := A_i \neq \emptyset$. By minimality, let $g_i \in G$ be such that $x g_i \in A_i$. Then we see that $\lim y g_i = z$ for any $y \in \varphi^{-1}(\{x\})$, so in particular for $y_0$ and $y_1$. In fact, this is historically the definition of a highly proximal extension.

\textbf{Fact 3.2} (\[13\], p. 733). Let $X$ be a minimal flow. Then the extension $\varphi: Y \to X$ is highly proximal iff for any $x \in X$, there is a net $g_i \in G$ and a point $y \in Y$ with $\varphi^{-1}(\{x g_i\}) \to \{y\}$, i.e. for any $B \ni_{op} y$, we eventually have $\varphi^{-1}(x g_i) \subseteq B_i$.

In the case that $X$ is a minimal flow, Auslander and Glasner \[13\] prove the existence and uniqueness of a \textit{universal highly proximal extension}; this is a highly proximal $G$-map $\pi_X: S_G(X) \to X$ so that for any other highly proximal $\varphi: Y \to X$, there is a $G$-map $\psi: S_G(X) \to Y$ with $\pi_X = \varphi \circ \psi$.

\[
\begin{array}{ccc}
S_G(X) & \xrightarrow{\pi_X} & X \\
\downarrow \psi & & \searrow \varphi \\
Y & &
\end{array}
\]

Such a $\psi$ is necessarily also highly proximal.

The notion of a universal highly proximal extension was generalized to any $G$-flow in \[13\], where an explicit construction is given. We briefly review this construction here, referring to \[13\] for all proofs.

\textbf{Definition 3.3}. Fix a $G$-flow $X$, and write $\text{op}(X) := \{A : A \subseteq_{op} X\}$. A collection $p \subseteq \text{op}(X)$ is called a \textit{near ultrafilter} if:

1. For every $k < \omega$, $A_0, \ldots, A_{k-1} \in p$, and $c > 0$, we have $\bigcap_{i < k} A_i U_c \neq \emptyset$. We call this property the \textit{Near Finite Intersection Property}, or NFIP.
2. $p$ is maximal with respect to satisfying item (1).

Let $S_G(X)$ denote the collection of near ultrafilters on $\text{op}(X)$. For $A \subseteq \text{op} X$, we set $C_A = \{ p \in S_G(X) : A \in p \}$ and $N_A = \{ p \in S_G(X) : A \notin p \}$. We endow $S_G(X)$ with a compact Hausdorff topology given by the base $\{ N_A := A \subseteq \text{op} X \}$. For $p \in S_G(X)$, a base of (not necessarily open) neighborhoods of $p$ is given by $\{ C_{A_U} : A \in p, \epsilon > 0 \}$. The group $G$ acts on $S_G(X)$ in the obvious way, where $A \in pg$ iff $Ag^{-1} \in p$. We also have a canonical $G$-map $\pi_X : S_G(X) \rightarrow X$, where $\pi_X(p) = x$ iff for every $A \ni_x p$, we have $A \in p$.

**Fact 3.4.** $\pi_X : S_G(X) \rightarrow X$ is the universal highly proximal extension of $X$.

In particular, the map $\pi_{S_G(X)} : S_G(S_G(X)) \rightarrow S_G(X)$ is an isomorphism. The construction of the space of near ultrafilters in fact works on any $G$-space, where the underlying space $X$ need not be compact. While in this generality we do not get the map $\pi_X$, we will still refer to the universal highly proximal extension of the $G$-space $X$, and the construction will still be idempotent. A remark that will be useful later is that if $Y \subseteq X$ is a dense $G$-invariant subspace of a $G$-space $X$, then $S_G(X)$ and $S_G(Y)$ coincide.

**Proposition 3.5.** The $G$-flow $X$ is MHP iff the universal highly proximal extension $\pi_X : S_G(X) \rightarrow X$ is an isomorphism.

**Proof.** First let $X$ be any $G$-flow. Fix $p \in S_G(X)$, and set $x = \pi_X(p)$. Then we must have $p \subseteq F_x := \{ A \subseteq \text{op} X : x \in \overline{A} \}$. To see why, if $x \notin \overline{A}$, we can find $B \ni_x x$ and $c > 0$ with $A U_c \cap B U_c = \emptyset$. As $B \in p$ by definition of the map $\pi_X$, we cannot have $A \in p$.

Now suppose the $G$-flow $X$ is MHP. Then for every $x \in X$, we have that $F_x$ has the NFIP, so is a near ultrafilter. It follows that if $p \in S_G(X)$ with $\pi_X(p) = x$, then we in fact have $p = F_x$. In particular, the map $\pi_X$ is injective, hence an isomorphism.

Conversely, suppose $X$ is not MHP. Find some $x \in X$, $B \subseteq \text{op} X$ with $x \in \overline{B}$, and $c > 0$ with $x \notin \text{int}(\overline{BU_c})$. Setting $C = X \setminus \overline{BU_c}$, we have $x \in C$. Notice that $BU_{c/2} \cap CU_{c/2} = \emptyset$, so $B$ and $C$ can never belong to the same near ultrafilter. Set $G_x := \{ A \subseteq \text{op} X : x \in A \}$. Let $p \in S_G(X)$ extend $G_x \cup \{ B \}$, and let $q \in S_G(X)$ extend $G_x \cup \{ C \}$. Then $p \neq q$ and $\pi_X(p) = \pi_X(q) = x$. \qed

### 3.2 Examples of MHP flows

We now collect some examples of MHP flows. Of course, the universal highly proximal extension of any $G$-space is an MHP flow, but it will be useful to have some explicit examples in mind.
3.2.1 Samuel compactifications

Let $H \subseteq G$ be a closed subgroup, and let $H \backslash G$ denote the right coset space. We equip $H \backslash G$ with the metric that it inherits from $G$, which we also denote by $d_G$. Explicitly, if $Hg \in H \backslash G$, the ball of radius $\epsilon > 0$ around $Hg$ is given by $HgU_\epsilon$. Then the Samuel compactification $Sa(H \backslash G)$ is the Gelfand space of the bounded uniformly continuous functions on $H \backslash G$. It is a $G$-flow characterized by the property that for any $G$-flow $Y$ containing a point $y \in Y$ with $y \cdot h = y_0$ for every $h \in H$, then there is a (necessarily unique) $G$-map $\varphi: Sa(H \backslash G) \to Y$ with $\varphi(H) = y$. In the case $H = \{1_G\}$, we often write $yp := \varphi(p)$. We identify $H \backslash G$ with its image under the canonical embedding $i: H \backslash G \hookrightarrow Sa(H \backslash G)$.

To see that $Sa(H \backslash G)$ is MHP, suppose $\psi: X \to Sa(H \backslash G)$ were highly proximal. Using the universal property of $Sa(H \backslash G)$, it is enough to show that $\psi^{-1}(\{H\})$ is a singleton. First note that for any $x \in \psi^{-1}(\{H\})$ and any $A \ni_{op} x$, we have $H \in \psi_{fib}(A)$. In particular, since $\psi_{fib}(A)$ is open, we can for any $\epsilon > 0$ find $Hg \in (H \backslash G) \cap \psi_{fib}(A)$ with $d_G(Hg, H) < \epsilon$. Now if $x \neq y \in X$ satisfied $\psi(x) = \psi(y) = H$, we can find $A \ni_{op} x$, $B \ni_{op} y$, and $\epsilon > 0$ with $A\bar U_\epsilon \cap B\bar U_\epsilon = \emptyset$. This implies that $\psi_{fib}(A)U_\epsilon \cap \psi_{fib}(B)U_\epsilon = \emptyset$, a contradiction as $H$ is a member of this intersection.

In particular, by taking $H = \{1_G\}$, we see that $Sa(G)$ is MHP. We also have that $M(G)$ is MHP. There are two ways of seeing this. One is that $S_G(M(G))$ is a minimal flow mapping onto $M(G)$, so by uniqueness of $M(G)$ we have that $\pi_{M(G)}: S_G(M(G)) \to M(G)$ is an isomorphism. The other way is to note that $M(G)$ is a retract of $Sa(G)$ and observe that retracts of MHP flows are also MHP.

Also notice that since $H \backslash G$ is a dense $G$-invariant subspace of $Sa(H \backslash G)$, then by the remark after Fact 3.4 we have $Sa(H \backslash G) \cong S_G(H \backslash G)$. When viewing $Sa(H \backslash G)$ as a space of near ultrafilters, the following fact will be useful to keep in mind (see [14], Ch. 1).

**Fact 3.6.** If $X$ is a compact space and $f: H \backslash G \to X$ is a uniformly continuous function, then the unique continuous extension $f: Sa(H \backslash G) \to X$ is defined by setting, for $p \in Sa(H \backslash G)$ and $x \in X$, $f(p) = x$ iff $\{f^{-1}(U): U \ni_{op} x\} \subseteq p$. Given $p \in Sa(H \backslash G)$, the existence of an $x \in X$ with this property is an easy consequence of compactness; the uniqueness of such an $x$ requires the uniform continuity of $f$.

3.2.2 Fraïssé expansion classes

This example will not be needed in later sections and assumes some familiarity with Fraïssé theory and expansion classes (see [9] or [12]). Suppose $L$ is a countable language and $G = \text{Aut}(K)$ for some Fraïssé $L$-structure $K = \text{Flim}(\mathcal{K})$ with underlying set $\omega$. Let $\text{Fin}(K)$ denote the collection of finite substructures of $K$. Let $\mathcal{K}^*$ be a reasonable precompact expansion of $\mathcal{K}$ in a countable language $L^* \supseteq L$. Let $X_{L^*}$ denote the space of $L^*$-structures on $\omega$ endowed with the logic topology. We can endow $X_{L^*}$ with a continuous $G$-action, where for a structure $x \in X_{L^*}$, a relational symbol $R \in L^*$ of arity $n$, points $a_0, \ldots, a_{n-1} \in \omega$, and
Let \( A \in \text{Fin}(K) \) and an expansion \( A^* \in K^* \), a typical basic clopen neighborhood of \( X_{K^*} \) is given by
\[
N_{A^*} = \{ K^* \in X_{K^*} : K^*|_A = A^* \}.
\]

**Proposition 3.7.** Suppose \( K^* \) has the amalgamation property \((AP)\). Then \( X_{K^*} \) is MHP.

**Proof.** For \( A \in \text{Fin}(K) \), write \( U_A \subset G \) for the pointwise stabilizer of \( A \). Then \( U_A \subset G \) is a clopen subgroup and a typical basic open neighborhood of \( 1_G \in G \). Let \( W \subset X_{K^*} \) be open. It suffices to show that \( \overline{W U_A} \) is clopen. To that end, we will show that for any \( B \in \text{Fin}(K) \) with \( A \subset B \) and any expansion \( B^* \in K^* \), we have \( \overline{N_{B^*} U_A} = N_{A^*} \), where \( A^* \) is the expansion of \( A \) inherited from \( B^* \). The left-to-right inclusion is clear. For the other way, suppose \( C \in \text{Fin}(K) \) is finite and \( C^* \) is an expansion so that \( N_{C^*} \subset N_{A^*} \). By shrinking \( N_{C^*} \) if necessary, we may assume that \( A^* \subset C^* \). Using the AP in \( K^* \), we can find \( D \in \text{Fin}(K) \) and an expansion \( D^* \) so that \( C^* \subset D^* \) and \( f[B^*] \subset D^* \) for some \( f \in \text{Emb}(B^*, D^*) \) with \( f|_A = 1_A \). If \( g \in G \) satisfies \( g|_B = f \), then \( N_{B^*} \cap N_{C^*} \) is non-empty as desired.

We can provide a converse result as follows. Recall that a \( G \)-flow \( X \) is topologically transitive if for every \( A, B \subsetop X \), there is \( g \in G \) with \( Ag \cap B \neq \emptyset \).

**Proposition 3.8.** Suppose \( X \) is a metrizable MHP \( G \)-flow. Then there is an reasonable, precompact expansion class \( K^* \) with the AP so that \( X \cong X_{K^*} \). If \( X \) is also topologically transitive, then we can take the class \( K^* \) to be Fraïssé.

**Proof.** Let \( A \in \text{Fin}(K) \). Then if \( W \subset X \) is open, the equality \( \overline{W \cdot U_A \cdot U_A} = \overline{W \cdot U_A} \) and MHP show that \( \overline{W \cdot U_A} \) is clopen. Call a clopen set \( Y \subset X \) \( U_A \)-clopen if \( Y \cdot U_A = Y \); the collection \( \mathcal{B}(A) \) of \( U_A \)-clopen sets forms an algebra.

Suppose \( \mathcal{B}(A) \) were infinite. Then we could find \( \{ Y_n : n < \omega \} \) a collection of pairwise disjoint members of \( \mathcal{B}(A) \). For \( S \subset \omega \), write \( Y_S = \bigcup_{n \in S} Y_n \). Then if \( y \in Y_S \), we have \( y \in \text{int}(\overline{Y_S \cdot U_A}) = \text{int}(\overline{Y_S}) \), i.e. the set \( Y_S \) is clopen. It follows that for \( S, T \subset \omega \) disjoint, we have \( Y_S \cap Y_T = \emptyset \). It follows that if \( p_n \in Y_n \) for each \( n < \omega \), then \( \{ p_n : n < \omega \} \) is isomorphic to \( \mathbb{N} \), contradicting our assumption that \( X \) is metrizable. Hence \( \mathcal{B}(A) \) is finite, hence atomic. Let \( \text{Atoms}(A) \subset \mathcal{B}(A) \) denote the atoms.

To each \( A \in \text{Fin}(K) \), we can view \( \text{Atoms}(A) \) as a set of “expansions” of \( A \). Suppose \( B \in \text{Fin}(K), Z \in \text{Atoms}(B) \), and \( f : A \to B \) is an embedding. We need to determine which
expansion of $A$ is induced by $f$ when we expand $B$ using $Z$. We do this as follows: first find $g \in G$ with $g|_A = f$. We will argue that $Zg$ is contained in some $U_A$-atom, and that this does not depend on the $g$ we chose. So suppose $W$ is $U_A$-clopen. By choice of $g$, We have $g^{-1}U_Bg \subseteq U_A$, so $Wg^{-1}U_B = Wg^{-1}$. This shows that $Wg^{-1}$ is $U_B$-clopen. Therefore if $Zg \cap W \neq \emptyset$, then $Z \cap Wg^{-1} \neq \emptyset$, so $Z \subseteq Wg^{-1}$ and $Zg \subseteq W$. It follows that $Zg$ is contained in some $U_A$-atom, say $Y$. If $h \in G$ also satisfies $h|_A = f$, then $g^{-1}h \in U_A$, so $Zg(g^{-1}h) \subseteq Y$ as well. Therefore if $B^Z$ is the corresponding expansion of $B$, we declare that $A^Y$ is the expansion that $A$ inherits from $B^Z$ along the map $f: A \to B$. All of this can be coded by adding countably many new relational symbols to $L$, producing a language $L^* \supseteq L$ and a reasonable precompact expansion class $\mathcal{K}^*$ of $\mathcal{K}$.

For each $A \in \text{Fin}(K)$, the set $\text{Atoms}(A)$ is a finite clopen partition of the space $X$. If $x \in X$, it follows that $x \in Y$ for exactly one $U_A$-atom for each $A \in \text{Fin}(K)$, giving rise to a surjective $G$-map $\varphi: X \to X_{K^*}$. If $x \neq y \in X$, then by continuity of the action, we can find $V \ni_{op} x, W \ni_{op} y$, and $A \in \text{Fin}(K)$ with $\overline{VA} \cap \overline{WA} = \emptyset$, showing that $\varphi$ is injective, hence an isomorphism.

To show that this expansion class has the AP, suppose we have $A, B, C \in \text{Fin}(K)$ with $A \subseteq B$ and $A \subseteq C$. Let $Y_A, Y_B, Y_C \subseteq X$ be clopen atomic sets for $U_A, U_B, U_C$, respectively, with $Y_B \subseteq Y_A$ and $Y_C \subseteq Y_A$. Since $Y_A$ is a $U_A$-atom, the action of $U_A$ on $Y_A$ is topologically transitive, so we can find $g \in U_A$ with $Y_Cg \cap Y_B \neq \emptyset$. We can then find some suitably large finite $D \subseteq K$ so that for some $U_D$-atom $Y_D$ we have $Y_D \subseteq Y_Cg \cap Y_B$. By enlarging $D$ more if needed, we can assume that $B \subseteq D$ and $g^{-1}[C] \subseteq D$. It follows that $i_B: B^{Y_B} \to D^{Y_B}$ and $(g^{-1})|_C: C^{Y_C} \to D^{Y_B}$ amalgamate the maps $i_A: A^{Y_A} \to B^{Y_B}$ and $i_A: A^{Y_A} \to C^{Y_C}$.

A similar argument shows that if $X$ is topologically transitive, then the expansion $\mathcal{K}^*$ that we constructed above will have the joint embedding property (JEP) as well.

In the case that $X$ is topologically transitive, MHP, but not necessarily metrizable, two important cases emerge. Either for every finite $A \subseteq K$, the algebra of $U_A$-clopen sets is atomic, or this fails for some $A$; the equivalent conditions of Theorem 5.5 correspond to the first case.

4 Topometrics on MHP flows

For the rest of the section, fix an MHP flow $(X, \tau)$, where $\tau$ is the compact topology on $X$. Our goal is to endow $X$ with a topometric structure. This has been done in the case of $\text{Sa}(G)$ in [7], where they use the following definition. Before stating the definition, we note that if $f: G \to [0, 1]$ is left-uniformly continuous, we can continuously extend it to $\text{Sa}(G)$, and we will also use $f: \text{Sa}(G) \to [0, 1]$ to denote this extension.

**Definition 4.1.** Given $p, q \in \text{Sa}(G)$, we set

$$\partial(p, q) = \sup(|f(p) - f(q)| : f: G \to [0, 1] \text{ 1-Lipschitz}).$$
Notice that if \( f : G \to [0, 1] \) is 1-Lipschitz and we continuously extend to \( \text{Sa}(G) \), then \( f \) has the following property, which we define more generally.

**Definition 4.2.** Let \( X \) be a \( G \)-flow. A function \( f \in C(X, [0, 1]) \) is called orbit Lipschitz if whenever \( x \in X \) and \( g \in G \), we have

\[
|f(x) - f(xg)| \leq d_G(1_G, g).
\]

We write \( C_{\text{OL}}(X, [0, 1]) \) for the collection of orbit Lipschitz functions.

Eventually, we will show that the analogue of Definition 4.1 with 1-Lipschitz replaced by orbit Lipschitz provides the MHP flow \( X \) with a topometric structure. The problem is that a priori, we do not know whether \( X \) has any non-constant orbit Lipschitz functions. Therefore we start with an entirely different definition of the topometric structure, then use Fact 2.3 to produce an ample supply of continuous Lipschitz functions, which will turn out to be precisely the orbit Lipschitz functions.

**Definition 4.3.** Given \( x, y \in X \) and \( c \geq 0 \), we define \( \partial(x, y) \leq c \) iff any of the following four equivalent items hold.

1. Whenever \( A \subseteq_{\text{op}} X \) with \( x \in \overline{A} \) and \( \epsilon > 0 \), we have \( y \in \text{int}(\overline{AU_{c+\epsilon}}) \).

2. Whenever \( A \subseteq_{\text{op}} X \) with \( x \in \overline{A} \) and \( \epsilon > 0 \), we have \( y \in \overline{AU_{c+\epsilon}} \).

3. Whenever \( A \ni_{\text{op}} x \) and \( \epsilon > 0 \), we have \( y \in \text{int}(\overline{AU_{c+\epsilon}}) \).

4. Whenever \( A \ni_{\text{op}} x \) and \( \epsilon > 0 \), we have \( y \in \overline{AU_{c+\epsilon}} \).

**Remark.** The directions (1) \( \Rightarrow \) (2) \( \Rightarrow \) (4) as well as (1) \( \Rightarrow \) (3) \( \Rightarrow \) (4) are clear. Suppose (4) holds, and let \( A \subseteq_{\text{op}} X \) with \( x \in \overline{A} \). Also fix \( \epsilon > 0 \). Then as \( X \) is MHP, we have \( x \in \text{int}(\overline{AU}) \). By (4), we have \( y \in \text{int}(\overline{AU}) \overline{U_{c+\epsilon}} \subseteq \overline{AU_{c+2\epsilon}} \). Using MHP once more, we obtain \( y \in \text{int}(\overline{AU_{c+3\epsilon}}) \), showing that (1) holds.

**Proposition 4.4.** The function \( \partial \) from Definition 4.3 is a topometric on \( X \).

**Proof.** Suppose \( x, y \in X \) have \( \partial(x, y) = 0 \). If \( A \ni_{\text{op}} x \), we can find \( B \ni_{\text{op}} x \) and \( \epsilon > 0 \) with \( \overline{BU_{\epsilon}} \subseteq A \). So in particular \( y \in A \), so \( x = y \).

Suppose \( \partial(x, y) \leq c \) for some \( c \geq 0 \) towards showing that \( \partial(y, x) \leq c \). Let \( B \ni_{\text{op}} y \) and \( \epsilon > 0 \). Notice that if \( A \ni_{\text{op}} x \), then \( AU_{c+\epsilon} \cap B \neq \emptyset \). So also \( A \cap \overline{BU_{\epsilon}} \neq \emptyset \). It follows that \( x \in \overline{BU_{\epsilon}} \).

Now suppose \( \partial(x, y) \leq c \) and \( \partial(y, z) \leq d \). Fix \( A \ni_{\text{op}} x \) and \( \epsilon > 0 \). Then \( AU_{c+\epsilon} \) is open with \( y \in \overline{AU_{c+\epsilon}} \). We then have \( z \in \overline{AU_{c+d+2\epsilon}} \), showing that \( \partial(x, z) \leq c + d \) as desired.

Having shown that \( \partial \) is a metric on \( X \), we now show that it is \( \tau \)-lsc. Fix \( c \geq 0 \), and let \( x_i \to x \) and \( y_i \to y \) be nets with \( \partial(x_i, y_i) \leq c \). Let \( A \ni_{\text{op}} x \), and fix \( \epsilon > 0 \). Then for a tail of \( x_i \), we also have \( x_i \in A \), implying that \( y_i \in \overline{AU_{c+\epsilon}} \). So \( y \in \overline{AU_{c+\epsilon}} \), and by MHP, \( y \in \text{int}(\overline{AU_{c+2\epsilon}}) \). It follows that \( \partial(x, y) \leq c \). \( \blacksquare \)
Remark. If $G$ is a discrete group and $X$ is an MHP $G$-flow, then $\partial$ is just the discrete metric on $X$. If $G$ is locally compact and $c \geq 0$ is small enough so that $U_{c+\epsilon} \subseteq G$ is precompact for some $\epsilon > 0$, then given an MHP $G$-flow $X$ and $x, y \in X$, we have $\partial(x, y) \leq c$ iff there is $g \in G$ with $d(g, 1_G) \leq c$ and $xg = y$. Hence topometric structures on MHP flows are most interesting when $G$ is not locally compact.

Remark. Suppose $H \subseteq G$ is a closed subgroup, and form $\text{Sa}(H \backslash G)$. On the orbit $H \backslash G \subseteq \text{Sa}(H \backslash G)$, the metric $\partial$ coincides with the metric $d$. In particular, this is true for $G \subseteq \text{Sa}(G)$. This will be easiest to see by using Corollary 4.7.

Remark. Suppose $K = \text{Flim}(K)$ is a Fraïssé structure with $G = \text{Aut}(K)$. Write $K = \bigcup_{n \geq 1} A_n$ as an increasing union of finite structures. A compatible left-invariant metric $d$ on $G$ is given by $d(g, h) \leq 1/n$ iff $g|_{A_n} = h|_{A_n}$. Write $V_n = \{ g \in G : g|_{A_n} = \text{id}_{A_n} \}$, and notice that for any suitably small $\epsilon > 0$, we have $V_n = U_{1/n+\epsilon}$.

Now suppose $X$ is an MHP $G$-flow. As in the discussion before Proposition 3.8, let $B_n$ be the Boolean algebra of $V_n$-clopen subsets of $X$. As we make no metrizability assumption here, $B_n$ may be infinite. However, if $\mathcal{Y} \subseteq B$ and we set $Y = \bigcup \mathcal{Y}$, we see that for $y \in Y$, we have $y \in \text{int}(Y \cdot V_n) = \text{int}(\overline{Y})$. In particular, $\overline{Y} \subseteq B_n$. Setting $\bigvee \mathcal{Y} = \overline{Y}$, we see that $B_n$ is a complete Boolean algebra. Let $X_n = \text{St}(B_n)$ be the Stone space. Then $X \cong \lim X_n$, and given $x = (x_n)_n$ and $y = (y_n)_n$ in $X$, we have that $\partial(x, y) \leq 1/n$ iff $x_n = y_n$. To see this, first suppose $x_n \neq y_n$, and find some $A \in B_n$ with $x \in A$ and $y \notin A$. But since $A = AU_{1/n+\epsilon} = \overline{AU_{1/n+\epsilon}}$, we have $\partial(x, y) > 1/n$ by item (4) of Definition 4.3. In the other direction, suppose $x_n = y_n$. Then if $A \ni x$, we have $\text{int}(\overline{AU_{1/n+\epsilon}}) \subseteq B_n$, hence $y \in \text{int}(\overline{AU_{1/n+\epsilon}})$. Therefore $\partial(x, y) \leq 1/n$ by item (3) of Definition 4.3.

We next investigate how this topometric structure interacts with the $G$-flow structure. Not only is this a canonical topometric to place on an MHP flow $X$, but it will also behave well when comparing different MHP flows. When discussing multiple MHP flows $X, Y, \text{etc.}$, we write $\partial_X, \partial_Y, \text{etc.}$ to refer to the topometric structure on each flow.

**Proposition 4.5.** Let $X$ and $Y$ be MHP flows endowed with the topometric structure from Definition 4.3.

1. If $x \in X$ and $g \in G$, then $\partial(x, xg) \leq d_G(1_G, g)$.

2. For each $g \in G$, the map $\rho_g : (X, \partial) \to (X, \partial)$ given by $\rho_g(x) = xg$ is uniformly continuous.

3. If $\varphi : X \to Y$ is a $G$-map, then $\varphi$ is metrically non-expansive, i.e. for any $x, y \in X$, we have $\partial_Y(\varphi(x), \varphi(y)) \leq \partial_X(x, y)$.

**Proof.** For item (1), write $c = d_G(1_G, g)$. Then for any $\epsilon > 0$, we have $g \in U_{c+\epsilon}$. Hence if $A \ni x$, we have $xg \in AU_{c+\epsilon}$.
For item (2), fix $c > 0$. Find $d > 0$ so that $g^{-1}U_d g \subseteq U_c$. Now suppose $x, y \in X$ satisfy $\partial(x, y) < d$. Let $A \ni op x$, and fix $\epsilon > 0$. Then $Ag^{-1} \ni op x$, so $y \in Ag^{-1}U_d$. It follows that $yg \in AU_c$, so $\partial(y, x) \leq c$.

For item (3), write $c = \partial X(x, y)$, and let $B \ni op \phi(x)$. Then $x \in \phi^{-1}(B)$, so we have $y \in \phi^{-1}(B)U_{c+\epsilon} = \phi^{-1}(BU_{c+\epsilon}) \subseteq \phi^{-1}(BU_{c+\epsilon})$ for any $\epsilon > 0$. So $\phi(y) \in BU_{c+\epsilon}$ as desired. \(\Box\)

**Remark.** Notice that item (3) shows that if $M \subseteq Sa(G)$ is a minimal subflow, then the topometric structure computed internally in $M$ is the same as the topometric structure inherited from $Sa(G)$. This is because $M$ is a retract of $Sa(G)$.

Denote by $C_L(X, [0, 1])$ the collection of continuous, 1-Lipschitz functions from $X$ to $[0, 1]$. The next proposition along with Fact 2.3 will give us Corollary 4.7, the analogue of Definition 4.1 for any MHP flow.

**Proposition 4.6.** $C_L(X, [0, 1]) = C_{OL}(X, [0, 1])$

**Proof.** First suppose $f \in C_L(X, [0, 1])$. Then since for any $p \in X$ and $g \in G$, we have $\partial(p, pg) \leq d(1_G, g)$, we see that $f \in C_{OL}(X, [0, 1])$.

Now suppose $f \in C_{OL}(X, [0, 1])$, and fix $p, q \in X$. Suppose $\partial(p, q) \leq c$, and let $\epsilon > 0$. Find $A \ni op p$ so that $|f(p') - f(p)| < \epsilon$ for $p' \in A$. Then $q \in AU_{c+\epsilon}$, so find $p_i \in A$ and $g_i \in U_{c+\epsilon}$ with $p_i g_i \rightarrow q$. As $f$ is orbit Lipschitz, we have $|f(p_i g_i) - f(p)| < c + 2\epsilon$. As $\epsilon > 0$ is arbitrary, we have $|f(q) - f(p)| \leq c$ as desired. \(\Box\)

**Corollary 4.7.** Let $x, y \in X$. Then $\partial(x, y) = \sup \{|f(x) - f(y)| : f \in C_{OL}(X, [0, 1])\}$.

We end the section by proving that the topometric space $(X, \tau, \partial)$ is adequate. For the proof, it will be easier to work with closed sets rather than open sets. Given $K \subseteq X$ and $c > 0$, we write $K(-c) := X \setminus ((X \setminus K)(c)) = \{x \in X : x(c) \subseteq K\}$. So a topometric space $(X, \tau, \partial)$ is adequate if for every closed $K \subseteq X$ and every $c > 0$, we have $K(-c)$ closed.

**Theorem 4.8.** The topometric space $(X, \tau, \partial)$ is adequate.

**Proof.** Fix $K \subseteq X$ closed. We show that the set $K(-c)$ is also closed. Write $K = \bigcap_i K_i$ with each $K_i$ a regular closed set. Then $K(-c) = \bigcap_i K_i(-c)$. So it suffices to prove the theorem in the case that $K$ is regular closed (we will only need this at the very end). For such $K$, we will show that

$$K(-c) = \bigcap_{\epsilon > 0} \left(X \setminus \text{int} \left((X \setminus KU_{\epsilon}) \cup r \right) \right).$$

Suppose $p \in X$ is not in the left hand side. Then there is $q \in X \setminus K$ with $\partial(p, q) < c$. Given $r$ with $\partial(p, q) < r < c$, then for every $A \ni op q$, we have $p \in \text{int}(AU_r)$. Now for some suitably
small $\epsilon > 0$, we have $q \in X \setminus KU_\epsilon$. Taking $A = X \setminus KU_\epsilon$, we see that $p$ is not in the right hand side.

Now suppose $p \in X$ is not in the right hand side as witnessed by $\epsilon > 0$ and $r < c$. In particular, we have $p \in (X \setminus KU_\epsilon)U_r$. Let $A \ni_p p$. Then $A \cap (X \setminus KU_\epsilon)U_r \neq \emptyset$. It follows that $AU_r \cap (X \setminus KU_\epsilon) \neq \emptyset$. Therefore we have

$$(X \setminus KU_\epsilon) \cap \left( \bigcap \{AU_r : A \ni_p p \} \right) \neq \emptyset.$$  

Fix some $q$ from this set. It follows that $\partial(p, q) \leq r < c$. To see that $q \notin K$, notice that for any $x \in K$, we have $x \in \text{int}(K)$, so we have $x \in \text{int}(\text{int}(K)U_\epsilon) = \text{int}(KU_\epsilon)$.

---

### 5 Comeager orbits in MHP flows

We continue with most of the notation of the previous section. In particular, $G$ is a Polish group, and $(X, \tau, \partial)$ is an MHP $G$-flow endowed with the topometric structure from Definition 4.3. In this section, we undertake a deeper study of the interaction between the topology $\tau$ and the metric $\partial$, connecting this to various properties that the $G$-flow $X$ might enjoy.

The main theorem is Theorem 5.5, which gives a complete characterization of when an MHP flow has a comeager orbit.

**Definition 5.1.** Let $p \in X$. We say that $\partial$ is compatible at $p$ or that $p$ is a compatibility point if for every $c > 0$, we have $p \in \text{int}(\text{int}(K)U_\epsilon)$.

Compatibility points are precisely the points in $X$ where the topologies given by $\tau$ and $\partial$ coincide. This is a notion which has been studied in the context of continuous logic, especially in regards to type spaces and the omitting types theorem (see [5], Ch. 12). We can now generalize one of the key theorems from [7]. We will repeatedly use the fact that if $A, B \subset_p X$ with $A \cap B = \emptyset$, then $\text{int}(A) \cap \text{int}(B) = \emptyset$.

**Lemma 5.2.** Suppose $x, y \in X$ satisfy $\partial(x, y) > 2c$. Then there are $A \ni_p x$ and $B \ni_p y$ with $\overline{AU_c} \cap \overline{BU_c} = \emptyset$.

**Proof.** We can find $A \ni_p x$ and $\epsilon > 0$ with $y \notin \overline{AU_{2c+2\epsilon}}$. Setting $B = X \setminus \overline{AU_{2c+2\epsilon}}$, we have that $\overline{AU_c} \cap \overline{BU_c} = \emptyset$ as desired. Indeed if $x \in \overline{AU_c} \cap \overline{BU_c}$, then by MHP $x \in \text{int}(\overline{AU_{c+\epsilon}}) \cap \text{int}(\overline{BU_{c+\epsilon}})$, contradicting that $AU_{c+\epsilon} \cap BU_{c+\epsilon} = \emptyset$.

**Theorem 5.3.** $(X, \tau)$ is metrizable iff $\partial$ is a compatible metric for $\tau$, i.e. iff $\partial$ is compatible at every point in $X$. Furthermore, if $(X, \tau)$ is not metrizable, then $X$ embeds a copy of $\beta\omega$, the space of ultrafilters on $\omega$. 

---

13
Proof. One direction is clear, so suppose \( \partial \) generates a strictly finer topology than \( \tau \). In particular, \((X, \partial)\) is not compact, so find \( c > 0 \) and an infinite \( Y \subseteq X \) with \( \partial(x, y) > 2c \) for any \( x \neq y \in Y \).

We will inductively define infinite \( Y_n \subseteq Y \), \( x_n \in Y_n \), and \( A_n \ni \varphi x \) for each \( n < \omega \). We will ensure that the following all hold.

1. \( Y_{n+1} \subseteq Y_n \) for every \( n < \omega \).
2. \( Y_n \cap A_k \cap \partial U_c = \emptyset \) for every \( k < n < \omega \).
3. \( A_n \cap A_k \cap \partial U_c = \emptyset \) for each \( k < n < \omega \)

Set \( Y_0 = Y \). Suppose \( Y_0, \ldots, Y_n, x_0, \ldots, x_{n-1}, \) and \( A_0 \ldots A_{n-1} \) have been chosen. Pick \( x \neq y \in Y_n \), and use Lemma 5.2 to find \( A \ni \varphi x \) and \( B \ni \varphi y \) with \( A \cap B \cap \partial U_c = \emptyset \). We also demand by shrinking \( A \) and \( B \) if needed that \( A \cap A_k \cap \partial U_c = \emptyset \) and \( B \cap A_k \cap \partial U_c = \emptyset \) for every \( k < n \); this is possible by item (2). Now at least one of \( Y_n \setminus A \cap \partial U_c \) or \( Y_n \setminus B \cap \partial U_c \) is infinite, without loss of generality the former. Set \( Y_{n+1} = Y_n \setminus A \cap \partial U_c \), \( x_n = x \), and \( A_n = A \).

Having completed the inductive construction, define \( \varphi : \beta \omega \to X \) to be the continuous extension of the map \( \varphi(n) = x_n \). We show that \( \varphi \) is injective. If \( S \subseteq \omega \), set \( A_S = \bigcup_{n \in S} A_n \). It is enough to show that if \( S, T \subseteq \omega \) with \( S \cap T = \emptyset \), then \( A_S \cap A_T = \emptyset \). To see why this is, note that \( A_S \subseteq \text{int}(A_S \cap \partial U_c/2) \), likewise for \( A_T \), and that \( A_S \cap \partial U_c/2 \cap A_T \cap \partial U_c/2 = \emptyset \).

Next we investigate what happens when some, but not all, points in \( X \) are compatibility points. We remind the reader that the topometric space \((X, \tau, \partial)\) was proven in Theorem 4.8 to be adequate.

Lemma 5.4.

1. Let \( Y \subseteq X \) denote the set of compatibility points. Then \( Y \) is \( G \)-invariant, \( \partial \)-closed, and topologically \( G_\delta \).

2. Suppose \( x \in X \) is not a compatibility point. Then there is \( c > 0 \) so that \( \text{int}(x(c)) = \emptyset \).

Proof.

1. That \( Y \) is \( G \)-invariant follows from item (2) of Proposition 4.5. To show \( Y \) is \( \partial \)-closed, let \( y_n \uparrow \partial y \), and fix \( c > 0 \). Then for some \( n < \omega \), we have \( y_n \cap \partial U_c \subseteq y(c) \). By assumption, \( y_n \in \text{int}(y_n(c)/2) \), so in particular, we have \( \text{int}(y(c)) \neq \emptyset \). Using adequacy, we have \( y \in (\text{int}(y(c)))(c) \subseteq \text{int}(y(2c)) \). Lastly, to show that \( Y \) is \( G_\delta \), let \( Y_c = \bigcup_{y \in Y} \text{int}(y(c)) \). Then \( Y_c \subseteq X \) is open with \( Y = \bigcap_{c > 0} Y_c \).

2. Suppose \( x \in X \) is a point with \( \text{int}(x(c)) \neq \emptyset \) for every \( c > 0 \). Then \( x \) is a compatibility point, as by adequacy, we have \( x \in (\text{int}(x(c)))(c) \subseteq \text{int}(x(2c)) \). \( \square \)
Theorem 5.5. The following are equivalent.

1. $X$ has a compatibility point with dense orbit.

2. The set $Y \subseteq X$ of compatibility points is comeager, Polish, and $G$ acts on $Y$ topologically transitively.

3. $X$ has a comeager orbit.

4. $X \cong \text{Sa}(H \setminus G)$ for some closed subgroup $H \subseteq G$.

Proof.

(1) $\Rightarrow$ (2) Letting $Y \subseteq X$ denote the set of compatibility points, item (1) of Lemma 5.4 shows us that $Y$ is $G$-invariant, $\partial$-closed, and $G_\delta$. By (1), $Y \subseteq X$ is dense. As $(Y, \tau)$ and $(Y, \partial)$ are homeomorphic, we see that $(Y, \tau)$ is separable and that $\partial$ is a compatible complete metric, hence $Y$ is Polish. As $Y$ contains a dense orbit, the action of $G$ on $Y$ is topologically transitive.

(2) $\Rightarrow$ (3) It is enough to show that $Y$ has a comeager orbit. We mostly follow the proof from [7], with a few differences to adapt to our more general setting. Using a criterion due to Rosendal (see [7] for a proof of the criterion), we need to show that for every $\epsilon > 0$ and every $A \subseteq Y$, there is $B \subseteq Y$ so that the local action of $U_\epsilon$ on $B$ is topologically transitive. To that end, let $B \subseteq Y$ be any open set of $\partial$-diameter less than $\epsilon$; any $A \subseteq Y$ will contain such a $B$ since $(Y, \partial)$ and $(Y, \tau)$ are homeomorphic. Fix $C_0, C_1 \subseteq \text{op} Y$. If $p_0 \in C_0$ and $p_1 \in C_1$, then $\partial(p_0, p_1) < \epsilon$. So $p_1 \in C_0 U_\epsilon$. In particular, $C_0 U_\epsilon \cap C_1 \neq \emptyset$ as desired.

(3) $\Rightarrow$ (4). Let $Z \subseteq X$ denote the comeager orbit, and pick $p \in Z$. Let $H = \text{Stab}(p)$. By the Effros theorem, we have that $Z \cong H \setminus G$ as $G$-spaces. So also $S_G(Z) \cong S_G(H \setminus G) \cong \text{Sa}(H \setminus G)$. But since $Z \subseteq X$ is dense, we have $S_G(Z) \cong S_G(X) \cong X$.

(4) $\Rightarrow$ (1) We will make use of Fact 3.6. For each $\epsilon > 0$, we have that $C_{HU_\epsilon} \subseteq \text{Sa}(H \setminus G)$ is a neighborhood of $H$. Let $f: \text{Sa}(H \setminus G) \to [0, 1]$ be continuous and orbit-Lipschitz. In particular, $f|_{H \setminus G}$ is 1-Lipschitz. So if $p \in C_{HU_\epsilon}$, we have $|f(p) - f(H)| \leq \epsilon$. Therefore $C_{HU_\epsilon} \subseteq H(2\epsilon)$, so $H$ is a compatibility point in $\text{Sa}(H \setminus G)$.

6 More on Samuel compactifications

Given item (4) in Theorem 5.5, let us spend some time to develop a more detailed understanding of the topometric $G$-space $\text{Sa}(H \setminus G)$, which we continue to view as a space of near ultrafilters. We first consider the left completion $\widehat{H \setminus G}$. Notice that if $f: H \setminus G \to X$ is a uniformly continuous function with $X$ a complete uniform space, then $f$ continuously extends to $H \setminus G$. In particular, if $f|_{H \setminus G}$ is 1-Lipschitz. So if $p \in C_{HU_\epsilon}$, we have $|f(p) - f(H)| \leq \epsilon$. Therefore $C_{HU_\epsilon} \subseteq H(2\epsilon)$, so $H$ is a compatibility point in $\text{Sa}(H \setminus G)$.
Fact 6.1. Given \( p \in \text{Sa}(H\backslash G) \), we have \( p \in \widehat{H\backslash G} \) iff for every \( \epsilon > 0 \), there is \( A \subseteq_{op} H\backslash G \) of diameter less than \( \epsilon \) with \( A \in p \).

From the proof of Theorem 5.5, we know that \( H \in \text{Sa}(H\backslash G) \) is a compatibility point. As \( H \) has dense orbit in \( \text{Sa}(H\backslash G) \), and since the topology and the metric coincide on the set of compatibility points, we see that \( H \) has a \( \partial \)-dense orbit in the set of compatibility points. Since \( \partial \) and \( d \) coincide on \( H\backslash G \), we obtain the following.

**Proposition 6.2.** In \( \text{Sa}(H\backslash G) \), the set of compatibility points is precisely \( \widehat{H\backslash G} \).

In particular, by Theorem 5.5, we have that \( H \) is a compatibility point. As \( H \) has dense orbit in \( \text{Sa}(H\backslash G) \), and since the topology and the metric coincide on the set of compatibility points, we see that \( H \) has a \( \partial \)-dense orbit in the set of compatibility points. Since \( \partial \) and \( d \) coincide on \( H\backslash G \), we obtain the following.

**Proposition 6.3.** In \( \text{Sa}(H\backslash G) \), the orbit \( H\backslash G \subseteq \text{Sa}(H\backslash G) \) is comeager.

**Remark.** This proposition is really a statement about topology rather than dynamics. Whenever \((X,d)\) is a Polish metric space and \( S(X) \) is the Samuel compactification of \( X \) with its metric uniformity, then \( X \subseteq S(X) \) is comeager.

We now take some time to understand the canonical \( G \)-map \( \pi : \text{Sa}(G) \to \text{Sa}(H\backslash G) \). To do this, we first need to understand how near ultrafilters on \( H \) interact with those on \( G \). Let \( p \in \text{Sa}(H) \). Then if \( A \in p \) and \( \epsilon > 0 \), we have \( AU_{\epsilon} \subseteq_{op} G \), and the collection \( \{AU_{\epsilon} : A \in p, \epsilon > 0\} \) extends to a unique near ultrafilter in \( \text{Sa}(G) \). This gives rise to an embedding \( i : \text{Sa}(H) \hookrightarrow \text{Sa}(G) \). More explicitly, given \( p \in G \), we set

\[
i(p) = \{B \subseteq_{op} G : B \cap AU_{\epsilon} \neq \emptyset \text{ for every } A \in p, \epsilon > 0\}.
\]

Now given \( p \in \text{Sa}(G) \), we have \( p \in i[\text{Sa}(H)] \) iff \( HU_{\epsilon} \in p \) for every \( \epsilon > 0 \). One direction is clear. For the other, if \( HU_{\epsilon} \in p \) for every \( \epsilon > 0 \), it follows that for every \( A \in p \) and \( \epsilon > 0 \), we have \( AU_{\epsilon} \cap H \neq \emptyset \), and the collection

\[
\{B \subseteq_{op} H : B \cap AU_{\epsilon} \neq \emptyset \text{ for every } A \in p, \epsilon > 0\}
\]

is a near ultrafilter \( q \) on \( H \) satisfying \( i(q) = p \).

From here on out, we will identify \( \text{Sa}(H) \) as a subspace of \( \text{Sa}(G) \) and suppress the embedding \( i \). We now consider the quotient \( \pi : G \to H\backslash G \) and extend it continuously to the respective Samuel compactifications. Given \( p \in \text{Sa}(G) \) and \( q \in \text{Sa}(H\backslash G) \), we have by Fact 3.6 that \( \pi(p) = q \) iff \( \pi^{-1}(AU_{\epsilon}) \in p \) for every \( A \in q \) and \( \epsilon > 0 \). In particular, \( \pi(p) = H \) iff \( HU_{\epsilon} \in p \) for every \( \epsilon > 0 \). We obtain the following.

**Proposition 6.4.** With \( \pi : \text{Sa}(G) \to \text{Sa}(H\backslash G) \) the canonical map, we have \( \pi^{-1}([H]) = \text{Sa}(H) \).
In the next section, we will be particularly interested in minimal MHP flows. Recall that \( S \subseteq G \) is called \textit{syndetic} if there is a finite set \( F \subseteq G \) with \( SF = G \). We have the following folklore fact.

**Fact 6.5** ([2], Ch. 1, Lem. 6). Suppose \( X \) is a \( G \)-flow and \( x \in X \). Then \( x \in X \) belongs to a minimal subflow iff for every \( A \ni_{op} X \), the set \( \{ g \in G : xg \in A \} \) is syndetic.

The following simple proposition gives a combinatorial characterization for when \( \text{Sa}(H\setminus G) \) is minimal.

**Proposition 6.6.** Let \( H \subseteq G \) be a closed subgroup. Then the following are equivalent.

1. \( \text{Sa}(H\setminus G) \) is minimal.
2. For every \( \epsilon > 0 \), the set \( HU_{\epsilon} \subseteq G \) is syndetic.

**Remark.** Compare this to the notion of co-precompactness, where \( H \subseteq G \) is \textit{co-precompact} if \( \text{Sa}(H\setminus G) \cong \widehat{H}\setminus G \), the left completion of \( H\setminus G \). This occurs iff for every \( \epsilon > 0 \), there is a finite \( F \subseteq G \) with \( HFU_{\epsilon} = G \).

**Proof.** First assume \( \text{Sa}(H\setminus G) \) is minimal. Since \( H \in \text{Sa}(H\setminus G) \) is a compatibility point, we have that \( HU_{\epsilon} \subseteq H\setminus G \) is relatively open. Item (2) then follows from minimality.

Conversely, assume item (2) holds. It follows that in \( \text{Sa}(H\setminus G) \), the return times of \( H \) to any open neighborhood of \( H \) are syndetic. Then by Fact 6.5, \( H \in \text{Sa}(H\setminus G) \) belongs to a minimal subflow, and the orbit of \( H \) is dense in \( \text{Sa}(H\setminus G) \). \( \square \)

Also in the next section, we will need to consider two closed subgroups \( H, H' \subseteq G \) and understand when a \( G \)-map \( \varphi : \text{Sa}(H\setminus G) \to \text{Sa}(H'\setminus G) \) can exist.

**Proposition 6.7.** Suppose \( H, H' \subseteq G \) are closed subgroups with both \( \text{Sa}(H\setminus G) \) and \( \text{Sa}(H'\setminus G) \) minimal. Then there is a \( G \)-map \( \varphi : \text{Sa}(H\setminus G) \to \text{Sa}(H'\setminus G) \) iff there is \( g \in G \) with \( H \subseteq g^{-1}H'g \).

**Proof.** For the forward direction, let \( \varphi \) be a \( G \)-map as above. By Proposition 14.1 in [1], we know that \( \varphi \) must preserve the comeager orbit. In particular, \( \varphi(H) = H'g \) for some \( g \in G \). It follows that for every \( h \in H \), we have \( H'gh = H'g \), i.e. that \( H \subseteq g^{-1}H'g \).

For the reverse direction, if \( H \subseteq g^{-1}Hg \) for some \( g \in G \), it follows that \( H \) stabilizes the point \( H'g \in \text{Sa}(H'\setminus G) \). Then the existence of a \( G \)-map \( \varphi \) as above follows from the universal property of \( \text{Sa}(H\setminus G) \). \( \square \)
7 Canonical minimal flows

In this section, we consider the universal minimal flow as well as two other "canonical" minimal flows in the context of Theorem 5.5. These other special flows both deal with the notion of proximality.

**Definition 7.1.** Fix a $G$-flow $X$.

1. We say that $X$ is *proximal* if for any $x, y \in X$, there is a net $g_i \in G$ and $z \in X$ with $x g_i \to z$ and $y g_i \to z$. Equivalently, there is $p \in Sa(G)$ with $x p = y p$.

2. Let $P(X)$ denote the compact space of probability measures on $X$ endowed with the weak*-topology. Then $P(X)$ is also a $G$-flow. We say that $X$ is *strongly proximal* if $P(X)$ is proximal. Equivalently, $X$ is strongly proximal iff $X$ is proximal and for any $\mu \in P(X)$, there is a net $g_i$ from $G$ with $\mu g_i \to \delta_x$ for some $x \in X$, where $\delta_x$ denotes the Dirac measure supported at $x$.

In [8], it is shown that there exist a universal minimal proximal flow, denoted $\Pi(G)$, and a universal minimal strongly proximal flow, denoted $\Pi_s(G)$ and often called the Furstenberg boundary. Here, if $P$ is a property of flows, a universal minimal $P$ flow is a minimal flow with property $P$ which admits a $G$-map onto any other minimal flow with property $P$. Both are unique up to isomorphism.

**Lemma 7.2.** If $X$ is a minimal, proximal $G$-flow, then the only $G$-map from $X$ to $X$ is the identity.

**Proof.** Suppose $\varphi : X \to X$ is a $G$-map. If there is $x \in X$ with $\varphi(x) = x$, then also $\varphi(x p) = x p$ for every $p \in Sa(G)$. As $X$ is minimal, this implies that $\varphi$ is the identity map. Now suppose $\varphi \neq \text{id}_X$. Fix $x \in X$, and find $p \in Sa(G)$ with $x p = \varphi(x)p$. But as $\varphi(x)p = \varphi(x p)$, this is a contradiction since $\varphi$ has no fixed points. \qed

**Lemma 7.3.** The flows $\Pi(G)$ and $\Pi_s(G)$ are both MHP.

**Proof.** Suppose $\varphi : X \to \Pi(G)$ is a non-trivial highly proximal $G$-map. Then it follows that $X$ is also minimal and proximal, so let $\psi : \Pi(G) \to X$ be a $G$-map. It follows that $\psi \circ \varphi : X \to X$ is a non-trivial $G$-map, contradicting Lemma 7.2.

To show that $\Pi_s(G)$ is MHP, suppose $\varphi : X \to \Pi_s(G)$ is a non-trivial highly proximal $G$-map. As a highly proximal extension of a minimal proximal flow, $X$ is proximal. Now suppose $\mu \in P(X)$. We can find a net $g_i \in G$ so that $\varphi \mu g_i \to \delta_p$ for some $p \in \Pi_s(G)$. We may assume that $\mu g_i \to \nu$ for some $\nu \in P(X)$ supported on $\varphi^{-1} \{\{p\}\}$. Then since $\varphi$ is highly proximal and $X$ is minimal, we can use Fact 3.2 and find another net $h_j \in G$ so that $\varphi^{-1}(\{p\})h_j$ shrinks down to some point $x \in X$. Hence $\nu h_j \to \delta_x$, showing that $X$ is strongly proximal. Now a similar argument to the proximal case shows that $\varphi$ must be an isomorphism. \qed
We can use \( M(G) \) to create a particularly nice representation of \( \Pi_s(G) \). Form the \( G \)-flow \( P(M(G)) \), and let \( A \subseteq P(M(G)) \) be a minimal affine subflow of \( P(M(G)) \), i.e. a subflow which is closed under convex combinations and minimal with this property. Then \( A \) is strongly proximal, and \( \overline{ex(A)} \), the closure of the extreme points of \( A \), is the unique minimal subflow of \( A \). We then obtain \( \overline{ex(A)} \cong \Pi_s(G) \). More details can be found in chapter 3 of [8].

From this characterization of \( \Pi_s(G) \), it follows that a topological group \( G \) is amenable iff \( G \) admits no nontrivial minimal strongly proximal actions. As for proximal actions, we call \( G \) strongly amenable if \( G \) admits no nontrivial minimal proximal actions. In particular, every strongly amenable group is amenable.

**Lemma 7.4.** Let \( X \) be a proximal \( G \)-flow, and let \( H \subseteq G \) be a closed subgroup with \( \Sa(H \setminus G) \) minimal. Then \( H \) acts proximally on \( X \).

**Proof.** Let \( x, y \in X \). As \( X \) is a proximal \( G \)-flow, find \( p \in \Sa(G) \) with \( xp = yp \). Since \( \Sa(H \setminus G) \) is minimal, we can find \( q \in \Sa(G) \) with \( pq \in \Sa(H) \). Then \( xpq = ypq \), showing that \( H \) acts proximally on \( X \). \( \square \)

The following provides a generalization of Theorem 1.2 from [11].

**Theorem 7.5.** Fix a minimal MHP flow \( X \) with a comeager orbit.

1. \( X \cong M(G) \) iff \( X \cong \Sa(H \setminus G) \) for some extremely amenable closed subgroup \( H \subseteq G \).
2. \( X \cong \Pi_s(G) \) iff \( X \cong \Sa(H \setminus G) \) for some maximal amenable subgroup \( H \subseteq G \).
3. If \( X \cong \Sa(H \setminus G) \) for some strongly amenable closed subgroup \( H \subseteq G \) and \( X \) is proximal, then \( X \cong \Pi(G) \).

**Proof.** (1) First assume \( X \cong M(G) \), and let \( H \subseteq G \) be the closed subgroup given by item (4) of Theorem 5.5. Fix a minimal subflow \( M \subseteq \Sa(G) \), and consider the canonical map \( \pi: \Sa(G) \rightarrow \Sa(H \setminus G) \). Then \( \pi \) is surjective, and \( \pi|_M \) is an isomorphism. Since by Proposition 6.4 we have \( \pi^{-1}(\{H\}) = \Sa(H) \), it follows that \( M \cap \Sa(H) \) is a singleton and an \( H \)-flow. As any minimal subflow of \( \Sa(H) \) is isomorphic to \( M(H) \), we see that \( H \) is extremely amenable.

Conversely, suppose \( H \subseteq G \) is an extremely amenable closed subgroup of \( G \) with \( \Sa(H \setminus G) \) minimal. Then \( M(G) \) must have an \( H \)-fixed point. It follows that there is a \( G \)-map \( \varphi: \Sa(H \setminus G) \rightarrow M(G) \). As we assumed that \( \Sa(H \setminus G) \) was minimal, it follows that \( \varphi \) is an isomorphism.

(2) We break the argument into the following parts.

- If \( X \cong \Pi_s(G) \), then \( X \cong \Sa(H \setminus G) \) with \( H \subseteq G \) a closed amenable subgroup.
- If \( X \cong \Sa(H' \setminus G) \) with \( H' \subseteq G \) a closed amenable subgroup, then \( X \) maps onto any strongly proximal flow.
From these two items, the theorem follows, since if \( H \not\subseteq H' \) are both closed amenable subgroups of \( G \), then by Proposition 6.7 we have a non-trivial factor map \( \Sa(H \setminus G) \to \Sa(H' \setminus G) \). If we had \( \Pi_s(G) \cong \Sa(H \setminus G) \), then the second item would allow us to build a non-trivial \( G \)-map from \( \Pi_s(G) \) to itself, contradicting Lemma 7.2. Conversely, if \( X \cong \Sa(H' \setminus G) \) for \( H' \subseteq G \) a maximal amenable subgroup, then using the second item we obtain a map \( \Sa(H' \setminus G) \to \Pi_s(G) \). By the first item, we have \( \Pi_s(G) \cong \Sa(H \setminus G) \) for some closed amenable subgroup \( H \subseteq G \). By Proposition 6.7 we must have \( H \subseteq g^{-1}H'g \), so in fact \( H = g^{-1}H'g \) as \( H \) was assumed maximal. It follows that \( \Sa(H \setminus G) \cong \Sa(H' \setminus G) = \Pi_s(G) \).

To prove the first item, suppose \( X \cong \Pi_s(G) \cong \Sa(H \setminus G) \). Let \( M \subseteq \Sa(G) \) be a minimal subgroup, and let \( A \subseteq P(M) \) be a minimal affine subgroup. Then \( X \cong \overline{ex(A)} \), the unique minimal subgroup of \( A \). Now letting \( \pi : \Sa(G) \to \Sa(H \setminus G) \) be the canonical map, we have the affine extension \( \pi_* : P(\Sa(G)) \to P(\Sa(H \setminus G)) \) to the spaces of measures. Identifying each \( p \in \Sa(H \setminus G) \) with the Dirac measure \( \delta_p \), we have that \( \Sa(H \setminus G) \) is the unique minimal subgroup of \( P(\Sa(H \setminus G)) \). It follows that \( \pi_*|_{\overline{ex(A)}} : \overline{ex(A)} \to \Sa(H \setminus G) \) is an isomorphism. However, we also have \( \pi_*^{-1}(\{H\}) = P(\Sa(H)) \), so \( P(\Sa(H)) \cap \overline{ex(A)} \) is a singleton and an \( H \)-flow, i.e. an \( H \)-invariant measure on \( \Sa(H) \). Hence \( H \) is amenable.

To prove the second item, we assume \( X \cong \Sa(H' \setminus G) \) with \( H' \subseteq G \) a closed amenable subgroup. On \( P(\Pi_s(G)) \), \( H' \) acts proximally by Lemma 7.4, hence \( H' \) acts strongly proximally on \( \Pi_s(G) \). Since \( H' \) is amenable, it follows that \( \Pi_s(G) \) has an \( H' \)-fixed point, so there is a \( G \)-map from \( \Sa(H' \setminus G) \) to \( \Pi_s(G) \).

(3) As for the third item, we assume that \( X \cong \Sa(H \setminus G) \) is proximal and that \( H \subseteq G \) is strongly amenable. By Lemma 7.4 \( H \) acts proximally on \( \Pi(G) \). As \( H \) is strongly amenable, \( \Pi(G) \) has an \( H \)-fixed point, so there is a \( G \)-map from \( \Sa(H \setminus G) \) to \( \Pi(G) \). As \( \Sa(H \setminus G) \) was assumed proximal, we have \( \Sa(H \setminus G) \cong \Pi(G) \).

\( \square \)

Remark. When considering the Furstenberg boundary or the universal minimal proximal flow of locally compact groups, we note that if \( \Sa(H \setminus G) \) is minimal, then in fact \( \Sa(H \setminus G) = H \setminus G \), i.e. that \( H \) is a cocompact subgroup of \( G \). This is because \( H \setminus G \subseteq \Sa(H \setminus G) \) is comeager, but also \( F_\sigma \), being an orbit of a locally compact group action. So \( \Sa(H \setminus G) \setminus (H \setminus G) \) is \( G_\delta \), and if it were non-empty, then by minimality it would be dense, a contradiction.

The following question addresses whether item (3) in Theorem 7.5 can be strengthened to have the same form as items (1) and (2).

**Question 7.6.** Suppose \( \Pi(G) \cong \Sa(H \setminus G) \) for some closed subgroup \( H \subseteq G \). Then must \( H \) be strongly amenable?

### 8 Reflecting meager orbits

The main theorem of this section is the following “reflection” theorem.
Theorem 8.1. Let $X$ be a minimal MHP flow all of whose orbits are meager. Then there is a factor $\varphi \colon X \to Y$ so that $Y$ is metrizable and also has all orbits meager.

Therefore in addition to the notation of the previous sections, we assume that $X$ is minimal and does not have a comeager orbit.

The metrizable factor of $X$ that we produce will be a space of uniformly continuous functions from $G$ (with its left-invariant metric uniformity) to a compact metric space. If $Y$ is a compact metric space, then $Y^G$ is a compact space when endowed with the product topology. The group $G$ acts on $Y^G$ by shift, where for $y \in Y^G$ and $g, h \in G$, we have $y \cdot g(h) = y(gh)$. Now suppose $y \in Y^G$ is uniformly continuous. Then $y \cdot G$ is a uniformly equi-continuous family, and furthermore, the space $y \cdot G$ is metrizable. To see why the last claim is true, note that pointwise convergence of a net of uniformly equi-continuous functions is determined by pointwise convergence on some countable dense subset of $G$.

In order to obtain factors of $X$, we use functions which arise from $X$ in the following way. Suppose $f \colon X \to Y$ is continuous, and fix $x \in X$. Then we obtain a uniformly continuous function $f_x \colon G \to Y$ via $f_x(g) = f(xg)$. Then notice that $f_x \cdot g = f_x g$, and if $x_i \to y$, then $f_{x_i} \to f_y$. It follows that the map $x \to f_x$ is a surjective $G$-map of $X$ onto $\overline{f_x \cdot G}$.

We now turn towards the proof of the theorem. Our first task is to provide a “global” version of item (2) from Lemma 8.2. This doesn’t require minimality.

Lemma 8.2. Suppose $Z$ is an MHP flow with no comeager orbit. Then there is some $c > 0$ and $A \subseteq_{op} Z$ with $x[c]$ nowhere dense for every $x \in A$.

Proof. Notice by the lower semi-continuity of $\partial$ that $x[c]$ is closed for every $c > 0$. Suppose towards a contradiction that for every $c > 0$, the set $D_c := \{ x \in Z : \text{int}(x[c]) \neq \emptyset \}$ is dense. Using adequacy, we see that for every $c > 0$, we have $D_{c/3} \subseteq E_c := \{ x \in Z : x \in \text{int}(x[c]) \}$, so $E_c$ is also dense. Then $\partial$ is compatible at any point in the comeager set $\bigcap_{c > 0} \bigcup_{x \in E_c} \text{int}(x[c])$. Theorem 5.5 then shows that $Z$ has a comeager orbit, contradicting our assumption. □

Fix $c > 0$ and $A \subseteq_{op} X$ as given by Lemma 8.2. Fix $D \subseteq X$ a countable dense set, and write $[D]^2 = \{ \{p_i, q_i\} : i < \omega \}$. Keeping in mind Corollary 4.7, find $\gamma_i \in C_{OL}(X, [0, 1])$ with $|\gamma_i(p_i) - \gamma_i(q_i)| > \partial(p_i, q_i)/2$. Let $\gamma \colon X \to [0, 1]^\omega$ be the concatenation of the $\gamma_i$. It will be helpful to view $[0, 1]^\omega$ as a topometric space whose metric is given by the uniform distance $d_u$.

Lemma 8.3. Let $B \subseteq [0, 1]^\omega$ be a closed $d_u$-ball of radius $c/4$. Then $\gamma^{-1}(B) \cap A \subseteq X$ is nowhere dense.

Proof. As $\gamma^{-1}(B) \cap A$ is relatively closed in $A$, we show that it has empty interior. Let $W \subseteq A$ be non-empty open. Pick $p \in W \cap D$. Then $p[c]$ is a closed, nowhere dense set, so find $q \in (W \setminus p[c]) \cap D$. Then $p, q \in W$ with $\partial(p, q) > c$. Suppose that $\{ p, q \} = \{ p_k, q_k \}$. Then $|\gamma_k(p) - \gamma_k(q)| > c/2$. In particular, $d_u(\gamma(p), \gamma(q)) > c/2$, so $W \not\subseteq \gamma^{-1}(B)$. □
Lemma 8.4. Suppose $Z$ is a minimal $G$-flow, $x \in Z$, and $S \subseteq G$ is syndetic. Then $x \cdot S \subseteq Z$ is somewhere dense.

Proof. Since $S \subseteq G$ is syndetic, find $g_0, \ldots, g_{k-1} \in G$ with $\bigcup_{i<k} Sg_i = G$. Then $\bigcup_{i<k} (x \cdot S) \cdot g_i = x \cdot G \subseteq Z$ is dense, so $x \cdot Sg_i$ is somewhere dense for some $i < k$. Then by translating, $x \cdot S$ is somewhere dense as well. \qed

Now let $\alpha : X \to [0, 1]$ be a continuous function with $\alpha^{-1}(\{1\}) \neq \emptyset$ and $\alpha[X \setminus A] = \{0\}$. Form the function $\theta = \alpha \times \gamma : X \to [0, 1] \times [0, 1]^\omega$. Pick $p \in X$, and then form $\theta_p : G \to [0, 1] \times [0, 1]^\omega$.

We will show that $\theta_p \cdot G$ has all orbits meager. Towards a contradiction, suppose $\theta_q \in \overline{\theta_p \cdot G}$ belonged to a comeager orbit; as $\{q \in X : \alpha(q) > 3/4\} \subseteq X$ is open, we may assume that $q$ belongs to this set. Let $r > 0$ be small enough so that both $r < c/4$ and for $g \in U_r$, we have $\alpha(qg) > 1/2$. By the Effros theorem, $\theta_q \cdot U$ is a relatively open subset of $\theta_q \cdot G$. By Fact 6.5 it follows that $S := \{g \in G : \theta_q \cdot g \in \theta_q \cdot U\}$ is syndetic, so by Lemma 8.4, $q \cdot S \subseteq X$ is somewhere dense. Furthermore, for $g \in G$, we have

\[ \theta_q \cdot g(1_G) = \theta(qg) = (\alpha(qg), \gamma(qg)). \]

It follows that for $g \in S$, there is $h \in U$ with $\alpha(qg) = \alpha(qh) > 1/2$. Hence $q \cdot S \subseteq A$. However, by item (1) of Proposition 4.5, $\gamma[q \cdot S] = \gamma[q \cdot U]$ lies in a $d_u$-ball of radius $c/4$, contradicting Lemma 8.3.

Question 8.5. Theorem 8.1 shows that for any non-metrizable minimal $G$-flow $X$ all of whose orbits are meager, we have a factor $\varphi : S_G(X) \to Y$ where $Y$ is metrizable and has all orbits meager. Is it necessary to pass to the universal highly proximal extension? More precisely, is there an example of a Polish group $G$ and a minimal $G$-flow $X$ with all meager orbits, but all of whose metrizable factors have a comeager orbit?

9 Distal universal minimal flows

As an application of Theorem 8.1 we prove Theorem 9.2 a characterization of when a Polish group $G$ has distal universal minimal flow.

Definition 9.1. A $G$-flow $X$ is called distal if for any pair of points $x \neq y \in X$ and any net $g_i$ from $G$ with $xg_i \to z \in X$, we have $yg_i \not\to z$.

Theorem 9.2. Let $G$ be a Polish group, and assume that $M(G)$ is distal. Then $M(G)$ is metrizable.
In [11], the authors consider Polish groups which are strongly amenable, groups which admit no non-trivial minimal proximal flows. They prove that if $G$ is strongly amenable and $M(G)$ is metrizable, then $G$ has a closed, normal, extremely amenable subgroup $H$ with $G/H$ compact and $M(G) \cong G/H$. As any group $G$ with $M(G)$ distal is also strongly amenable, we obtain the following corollary.

**Corollary 9.3.** Let $G$ be a Polish group with $M(G)$ distal. Then $G$ has a closed, normal, extremely amenable subgroup $H$ with $G/H$ compact and $M(G) \cong G/H$.

We briefly review some facts about enveloping semigroups and distal flows; see [2] for more detail. To any $G$-flow $X$, we can associate to it the enveloping semigroup $E(X)$. Given $g \in G$, form the function $\rho_g: X \to X$ given by $\rho_g(x) = xg$. Then $E(X)$ is the closure of the set $\{\rho_g : g \in G\}$ in the compact space $X^X$. Each $f \in E(X)$ is a function, and because we take our $G$-flows to be right actions, it will be more convenient to write function application and composition on the right, i.e. for $x \in X$ and $f \in E(X)$, we write $xf$ instead of $f(x)$. Then $E(X)$ becomes a compact left-topological semigroup, in particular a $G$-flow, where $f \cdot g = \rho_g \circ f$. For any $x \in X$, the map $\lambda_x: E(X) \to X$ given by $\lambda_x(f) = xf$ is a $G$-map.

When $X$ is distal, then $E(X)$ is a group. Furthermore, if $X$ is also minimal, then $E(X)$ is a minimal distal system. If $f \in E(X)$, then the left multiplication map $\lambda_f$ is a $G$-flow automorphism. In particular, if $M(G)$ is distal, then $E(M(G)) \cong M(G)$, and for any $p, q \in M(G)$, there is a $G$-flow automorphism $\varphi$ with $\varphi(p) = q$.

In the proof of Theorem 9.2, we will need the following simple proposition.

**Proposition 9.4** ([2], Cor. 7(c)). Let $Y$ be a distal flow, and let $\varphi: Y \to X$ be a factor. Then $X$ is also distal.

We will also need to recall the main result of [13].

**Fact 9.5** ([13], Cor. 3.3). If $X$ is a minimal, metrizable flow with all orbits meager, then the universal highly proximal extension $S_G(X)$ is non-metrizable. In particular, the map $\pi_X: S_G(X) \to X$ is a non-trivial highly proximal extension.

We can now complete the proof of Theorem 9.2. Towards a contradiction, suppose $M(G)$ were distal, but not metrizable. Then by Theorem 5.3 we have $|M(G)| = 2^c$, so in particular $M(G)$ contains more than one orbit. As there is a $G$-flow automorphism bringing any one orbit to any other, we see that $M(G)$ contains all meager orbits. By Theorem 8.1 let $X$ be a minimal metrizable flow with all meager orbits. Then by Fact 9.5, $\pi_X: S_G(X) \to X$ is a non-trivial highly proximal extension of minimal flows, which implies that $S_G(X)$ is not distal. Now let $\varphi: M(G) \to S_G(X)$ be a $G$-map. By Proposition 9.4 we must also have $M(G)$ not distal, completing our contradiction.
References

[1] O. Angel, A. S. Kechris, and R. Lyons. Random orderings and unique ergodicity of automorphism groups. *J. European Math. Society*, **16** (2014), 2059–2095.

[2] J. Auslander, *Minimal Flows and Their Extensions*, North Holland, 1988.

[3] J. Auslander and S. Glasner, Distal and Highly Proximal Extensions of Minimal Flows, *Indiana University Math. J.*, **26**(4), (1977), 731–749.

[4] I. Ben Yaacov, Lipschitz functions on topometric spaces, *Journal of Logic and Analysis*, **5:8**, (2013), 1–21.

[5] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model Theory for Metric Structures*, Model theory with applications to algebra and analysis. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 350, Cambridge Univ. Press, Cambridge, 2008, p. 315–427.

[6] I. Ben Yaacov and J. Melleray, Grey subsets of Polish spaces, *Journal of Symbolic Logic*, **80**(4), (2015), 1379–1397.

[7] I. Ben Yaacov, J. Melleray, and T. Tsankov, Metrizable universal minimal flows of Polish groups have a comeager orbit, *Geom. Func. Anal.*, **27**(1) (2017), 67–77.

[8] S. Glasner, *Proximal Flows*, Lecture Notes in Mathematics, **517**, Springer-Verlag, Berlin-New York, 1976.

[9] A. Kechris, V. Pestov, and S. Todorčević, Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups, *Geometric and Functional Analysis*, **15** (2005), 106–189.

[10] A. Kwiatkowska, Universal minimal flows of generalized Ważewski dendrites, *Journal of Symbolic Logic*, to appear, https://arxiv.org/pdf/1711.07869.pdf

[11] J. Melleray, L. Nguyen Van Thé, and T. Tsankov, Polish groups with metrizable universal minimal flow, *Int. Math. Res. Not. IMRN*, no. 5 (2016), 1285–1307.

[12] A. Zucker, Topological dynamics of automorphism groups, ultrafilter combinatorics, and the Generic Point Problem, *Transactions of the American Mathematical Society*, **368**(9), (2016).

[13] A. Zucker, A direct solution to the Generic Point Problem, *Proc. Amer. Math. Soc.*, **146**(5) (2018), 2143–2148.

[14] A. Zucker, *New directions in the abstract topological dynamics of Polish groups*, Ph.D. thesis, Carnegie Mellon University, 2018.
