ON SOME MODEL PROBLEM FOR THE PROPAGATION OF INTERACTING SPECIES IN A SPECIAL ENVIRONMENT

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ABSTRACT. The purpose of this note is to study the existence of a nontrivial solution for an elliptic system which comes from a newly introduced mathematical problem so called Field-Road model. Specifically, it consists of coupled equations set in domains of different dimensions together with some interaction of non classical type. We consider a truncated problem by imposing Dirichlet boundary conditions and an unbounded setting as well.

1. Introduction and notation. It has been observed in diffusion or propagation problems that the topography of the environment is playing a crucial role. A new evolution model of biological invasions in two dimensional environment was introduced in [2] in 2013 where the plane is divided in a line “the road” and its complement “the field”. It is assumed that the population in the field is governed by a logistic growth which leads to a KPP type reaction term \( f \) in the field while a more general reaction \( g \) is set on the road. Moreover, exchanges of populations are assumed to take place between the road and the field. Let us remark that this model is motivated by empirical observations that the roads are not only essential for human beings but also can be useful for other species. One can easily imagine that mosquitoes could be interested in socialising with human beings along a road but many other situations seem to have occurred due to this element of civilisation. Later, new features, such as transport and reaction terms on the road, were taken into consideration in [3] and traveling waves, spreading and extinction was studied in [4]. Afterwards, the original model in a strip with Dirichlet homogeneous
boundary condition on the other part of the boundary of the field was analyzed in [12]. The model was recently generalized to higher dimension in [11]. Specifically, the Fisher-KPP type reaction-diffusion equation is set in an unbounded cylinder in \( \mathbb{R}^{N+1} \) coupled with a diffusion equation on the boundary of the cylinder where the exchanges of populations occur as well and particularly, when \( N = 1 \), this situation models two parallel roads bounding a field described as a strip. The interest of the references above is the population dynamics and propagation phenomena with respect to different diffusion assumption a priori from an evolutional point of view, especially, to understand the effect of the road with fast diffusion on the spreading of species in a homogeneous environment. In this note, we consider an elliptic problem where the living space of our species consists of a field and one or several roads that we will assume to be unidimensional. We will also assume the roads to be straight but several extensions can be addressed very easily with our approach. On each domain - field or road - we will consider nonlinear diffusion equations which include for instance the Fisher-KPP types. The interaction with the populations is modeled by a particular flux condition (see for instance [2, 3, 4, 11, 12] and the set of equation below).

Let \( \Omega_\ell \) be the open set of \( \mathbb{R}^2 \), defined for \( \ell, L > 0 \) as
\[
\Omega_\ell = (-\ell, \ell) \times (0, L).
\]

We denote by \( \Gamma_0 \) the part of the boundary of \( \Omega_\ell \) located on the \( x_1 \)-axis i.e.
\[
\Gamma_0 = (-\ell, \ell) \times \{0\}
\]
and by \( \Gamma_1 \) the rest of the boundary that is to say
\[
\Gamma_1 = \partial \Omega_\ell \setminus \Gamma_0.
\]

When convenient we will identify \( \Gamma_0 \) to \((-\ell, \ell)\). In this setting \( \Omega_\ell \) stands for a field and \( \Gamma_0 \) for a portion of a road.

![Figure 1. The domain \( \Omega_\ell \) for one-road problem](image)

Set
\[
V = \{ v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \Gamma_1 \}.
\]

We would like to find a solution to the problem
\[
\begin{cases}
-D\Delta v = f(v) \text{ in } \Omega_\ell, \\
v = 0 \text{ on } \Gamma_1, \quad D \frac{\partial v}{\partial n} = \mu u - \nu v \text{ on } \Gamma_0, \\
-D' u'' + \mu u = g(u) + \nu v \text{ on } \Gamma_0, \\
u = 0 \text{ on } \partial \Gamma_0 = \{-\ell, \ell\}.
\end{cases}
\]
(n denotes the outward unit normal to \( \Omega \)).

In the weak form we would like to find a couple \((u, v)\) such that

\[
\begin{cases}
(u, v) \in H^1_0(\Gamma_0) \times V, \\
\int_{\Omega} D \nabla v \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu v(x_1, 0) \varphi \, dx_1 = \int_{\Omega} f(v) \varphi \, dx + \int_{\Gamma_0} \mu w \varphi \, dx_1 \quad \forall \varphi \in V, \\
\int_{\Gamma_0} D' u' \psi' + \mu w \psi \, dx_1 = \int_{\Gamma_0} \nu v(x_1, 0) \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0).
\end{cases}
\]

Here we assume that

\[
D, D', m, \mu, \nu \text{ are positive constants,}
\]

such that

\[
m \geq \frac{\nu}{\mu}.
\]

\(f, g\) are Lipschitz continuous functions i.e. such that for some positive constants \(L_f, L_g\) it holds

\[
|f(x) - f(y)| \leq L_f |x - y|, \quad |g(x) - g(y)| \leq L_g |x - y| \quad \forall x, y \in \mathbb{R}.
\]

Note that this implies that for \(\lambda \geq L_f\) (respectively \(\eta \geq L_g\)) the functions

\[
x \rightarrow \lambda x - f(x), \quad \eta x - g(x)
\]

are nondecreasing. In addition we will assume

\[
f(0) = f(1) = 0, \quad f > 0 \text{ on } (0, 1), \quad f \leq 0 \text{ on } (1, +\infty).
\]

\[
g(0) = 0, \quad g(m) \leq 0.
\]

One should remark that under the assumptions above, \((0, 0)\) is a solution to (1.1).

We are interested in finding a nontrivial solution to the problem (1.1). For the notation and the usual properties on Sobolev spaces we refer to [5], [6], [8], [9], [10]. We will also consider some extension in the case of a two-road problem which consists of three coupled equations with two interaction conditions on the upper- and lower- boundaries. Finally, we will address the case of an unbounded setting for the one-road problem.

2. Preliminary results.

**Lemma 2.1.** Suppose that \(w\) is a measurable function on \(\Gamma_0\) such that

\[
0 \leq w \leq m.
\]

Then under the assumptions above the problem

\[
\begin{cases}
v \in V, \\
\int_{\Omega} D \nabla v \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu v(x_1, 0) \varphi(x_1, 0) \, dx_1 = \int_{\Omega} f(v) \varphi \, dx + \int_{\Gamma_0} \mu w \varphi \, dx_1 \quad \forall \varphi \in V,
\end{cases}
\]

possesses a minimal and a maximal solution with values in \([0, \frac{\mu}{\nu} m]\).
which shows the continuity of the mapping $S$. Let us remark first that any nonnegative solution to (2.1) takes its values in $[0, \frac{\mu}{v}m]$. Indeed, if $v$ is solution to (2.1) taking as test function $\varphi = (v - k)^+$, $k = \frac{\mu}{v}m \geq 1$ one gets

$$\int_{\Omega_t} D[\nabla(v - k)^+]^2 \, dx = \int_{\Omega_t} D\nabla(v - k) \cdot \nabla(v - k)^+ \, dx = \int_{\Omega_t} D\nabla v \cdot \nabla(v - k)^+ \, dx$$

$$= \int_{\Omega_t} f(v)(v - k)^+ \, dx + \int_{\Gamma_0} \{\mu w - \nu v(x_1, 0)\}(v - k)^+ \, dx_1$$

$$\leq \int_{\Gamma_0} \{\mu w - \nu v(x_1, 0)\}(v - k)^+ \, dx_1 \leq 0,$$

since on the set where $v \geq k$ one has $v \geq \frac{\mu}{v}m$ and $\mu w - \nu v(x_1, 0) \leq \mu w - \mu m \leq 0$.

Next let us note that 0 is a subsolution to (2.1). Indeed this follows trivially from

$$\int_{\Omega_t} D\nabla 0 \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu 0 \varphi \, dx_1 \leq \int_{\Omega_t} f(0) \varphi \, dx + \int_{\Gamma_0} \mu w \varphi \, dx_1 \quad \forall \varphi \in V, \varphi \geq 0.$$  

On the other hand, for $\varphi \in V, \varphi \geq 0$, one has also for $k = \frac{\mu}{v}m$, since $k \geq 1$

$$\int_{\Omega_t} D\nabla k \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu k \varphi \, dx_1 \geq \int_{\Omega_t} f(k) \varphi \, dx + \int_{\Gamma_0} \mu w \varphi \, dx_1$$

and the function constant equal to $k = \frac{\mu}{v}m$ is a supersolution to (2.1).

For $z \in L^2(\Omega_t)$ we denote by $y = S(z)$ the solution to

$$\begin{cases}
y \in V, \\
\int_{\Omega_t} D\nabla y \cdot \nabla \varphi \, dx + \int_{\Omega_t} \lambda y \varphi \, dx + \int_{\Gamma_0} \nu y(x_1, 0) \varphi \, dx_1 \\
= \int_{\Omega_t} f(z) \varphi \, dx + \int_{\Omega_t} \lambda z \varphi \, dx + \int_{\Gamma_0} \mu w \varphi \, dx_1 \quad \forall \varphi \in V,
\end{cases}$$

where we have chosen $\lambda \geq L_f$. Note that the existence of a unique solution $y$ to the problem above is an immediate consequence of the Lax-Milgram theorem.

First we claim that the mapping $S$ is continuous from $L^2(\Omega_t)$ into itself. Indeed setting $y' = S(z')$ one has by subtraction of the equations satisfied by $y$ and $y'$

$$\int_{\Omega_t} D\nabla (y - y') \cdot \nabla \varphi \, dx + \int_{\Omega_t} \lambda (y - y') \varphi \, dx + \int_{\Gamma_0} \nu (y - y') \varphi \, dx_1$$

$$= \int_{\Omega_t} \{f(z) - f(z')\} \varphi \, dx + \int_{\Omega_t} \lambda (z - z') \varphi \, dx \quad \forall \varphi \in V.$$  

(2.3)

Taking $\varphi = y - y'$ one derives easily

$$\lambda \int_{\Omega_t} |y - y'|^2 \, dx \leq \int_{\Omega_t} |f(z) - f(z')| |y - y'| \, dx + \lambda \int_{\Omega_t} |z - z'||y - y'| \, dx$$

$$\leq (L_f + \lambda) \int_{\Omega_t} |z - z'| |y - y'| \, dx.$$  

By the Cauchy-Schwarz inequality we obtain then

$$|S(z) - S(z')|_{2,\Omega_t} \leq \frac{L_f + \lambda}{\lambda} |z - z'|_{2,\Omega_t}$$  

which shows the continuity of the mapping $S$ (in $| \cdot |_{2,\Omega_t}$ denotes the usual $L^2(\Omega_t)$-norm).
We show now that the mapping $S$ is monotone. Indeed suppose that $z \geq z'$ and as above denote by $y'$ the function $S(z')$. Taking $\varphi = -(y - y')$ in (2.3) we get
\[
\int_{\Omega_t} D|\nabla (y - y')|^{2} \, dx + \int_{\Omega_t} \lambda((y - y')^{2}) \, dx + \int_{\Gamma_0} \nu((y - y')^{2}) \, dx_1
\]
\[
= - \int_{\Omega_t} \left[\lambda(z - z') + \{f(z) - f(z')\}\right](y - y')^{2} \, dx \leq 0,
\]
since $-\{f(z) - f(z')\} \leq L_f|z - z'| \leq \lambda(z - z')$ (see (1.3), (1.4)). This shows that $(y - y')^{-1} = 0$ and the monotonicity of the mapping $S$.

We consider now the following sequences (Cf. [1]) :
\[
y_0 = 0, \quad \overline{y}_0 = \frac{\mu}{\nu} m = k, \quad y_n = S(y_{n-1}), \quad \underline{y}_n = S(\overline{y}_{n-1}), \quad n \geq 1.
\]
(2.4)

One has
\[
y_0 = 0 \leq y_1 \leq \cdots \leq y_n \leq \overline{y}_n \leq \cdots \leq \overline{y} \leq \underline{y} = \frac{\mu}{\nu} m = k.
\]
(2.5)

Indeed since $y_1 = S(y_0) = S(0)$ one has for $\forall \varphi \in V, \varphi \geq 0,$
\[
\int_{\Omega_t} D\nabla y_1 \cdot \nabla \varphi \, dx + \int_{\Omega_t} \lambda y_1 \varphi \, dx + \int_{\Gamma_0} \nu y_1(x_1, 0) \varphi \, dx_1
\]
\[
= \int_{\Omega_t} f(y_0) \varphi \, dx + \int_{\Omega_t} \lambda y_0 \varphi \, dx + \int_{\Gamma_0} \nu \varphi \, dx_1
\]
\[
\geq \int_{\Omega_t} D\nabla \overline{y}_0 \cdot \nabla \varphi \, dx + \int_{\Omega_t} \lambda y_0 \varphi \, dx + \int_{\Gamma_0} \nu y_0(x_1, 0) \varphi \, dx_1,
\]
since $y_0$ is a subsolution to (2.1). Using this inequality with $\varphi = (y_1 - y_0)^{-}$ one derives easily
\[
\int_{\Omega_t} D\nabla (y_1 - y_0) \cdot \nabla (y_1 - y_0)^{-} \, dx + \int_{\Omega_t} \lambda(y_1 - y_0)(y_1 - y_0)^{-} \, dx
\]
\[
+ \int_{\Gamma_0} \nu(y_1 - y_0)(y_1 - y_0)^{-} \, dx_1 \geq 0.
\]
Thus it follows that $y_1 \geq y_0$. With a similar proof one gets that $\overline{y} \leq \underline{y}$. Applying $S^{n-1}$ to these inequalities leads to
\[
S^{n-1}(y_0) = y_{n-1} \leq y_n = S^{n-1}(y_1), \quad S^{n-1}(\overline{y}_0) = \overline{y}_{n-1} \geq \overline{y}_n = S^{n-1}(\overline{y}).
\]
Furthermore from $y_0 \leq \underline{y}_0$ one derives by applying $S^n$ to both sides of the inequality
\[
y_n \leq \underline{y}_n.
\]
This completes the proof of (2.5). Then for some functions $\underline{\varphi}, \overline{\varphi}$ in $L^2(\Omega_t)$ one has
\[
y_n \to \underline{\varphi}, \quad \overline{y}_n \to \overline{\varphi} \quad \text{in} \quad L^2(\Omega_t).
\]
Clearly $\underline{\varphi}$ and $\overline{\varphi}$ are fixed point for $S$ and thus (see (2.2)) solutions to (2.1). This completes the proof of the lemma.

We denote by $\lambda_1 = \lambda_1(\Omega_t)$ the first eigenvalue of the Dirichlet problem in $\Omega_t$ and by $\varphi_1$ the corresponding first eigenfunction positive and normalised. More precisely
\((\lambda_1, \varphi_1)\) is such that
\[
\begin{cases}
-\Delta \varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega_\ell, \\
\varphi_1 = 0 & \text{on } \partial \Omega_\ell, \\
\varphi_1(0, \frac{L}{2}) = 1.
\end{cases}
\tag{2.6}
\]

We suppose that for \(s > 0\) small enough one has
\[
\lambda_1 \leq f(s) \frac{D_s}{D_s}.
\tag{2.7}
\]

Let us mention that both the normalization for the principal eigenfunction \(\varphi_1\) and the condition for the principal eigenvalue \(\lambda_1\) are essential and will play crucial roles in the sequel to derive the main results in the present paper. Indeed, the normalization \(\varphi_1(0, \frac{L}{2}) = 1\) can ensure that the subsolution will be bounded away from 0 when passing to the limit \(\ell \to 0\). Therefore, as will be seen in (5.4), the component \(v_\ell\) of the nontrivial solution \((u_\ell, v_\ell)\) is bounded from below by a positive constant in a subdomain of \(\Omega_\ell\) uniformly in \(\ell\), which further implies the nontrivial property of the solution in the unbounded strip as the limit of \((u_\ell, v_\ell)\) as \(\ell \to +\infty\). We do not know if the condition (2.7) could be relaxed.

Then one has:

**Lemma 2.2.** Under the assumptions of the preceding lemma and (2.7), for \(\epsilon > 0\) small enough, the maximal solution \(v\) to (2.1) satisfies
\[
\epsilon \varphi_1 \leq v.
\]

In particular \(v\) is bounded away from 0.

**Proof.** Due to (2.7) one has for \(\epsilon > 0\) small enough
\[
D \lambda_1 \epsilon \varphi_1 \leq f(\epsilon \varphi_1).
\]

This allows us to show that \(\epsilon \varphi_1\) is a subsolution to (2.1). Indeed, for \(\varphi \in V, \varphi \geq 0\) it holds after integration by parts
\[
\int_{\Omega_\ell} D\nabla(\epsilon \varphi_1) \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu \epsilon \varphi_1(x_1, 0) \varphi \, dx_1
\]
\[
= \int_{\Omega_\ell} D\nabla(\epsilon \varphi_1) \cdot \nabla \varphi \, dx = \int_{\Omega_\ell} \nabla \cdot (D\nabla(\epsilon \varphi_1)) - D\Delta(\epsilon \varphi_1) \varphi \, dx
\]
\[
= \int_{\partial \Omega_\ell} D\partial_n(\epsilon \varphi_1) \varphi \, d\sigma + \int_{\Omega_\ell} D\lambda_1(\epsilon \varphi_1) \varphi \, dx
\]
\[
\leq \int_{\Omega_\ell} f(\epsilon \varphi_1) \varphi \, dx + \int_{\Gamma_0} \mu w \varphi \, dx_1.
\]

(\(n\) denotes the outward unit normal to \(\Omega_\ell\), note that \(\partial_n(\epsilon \varphi_1) \leq 0\)). Thus \(\epsilon \varphi_1\) is a positive subsolution to (2.1).

Then one argues as in the preceding lemma introducing the sequence defined for \(\epsilon\) small by :
\[
y_0 = \epsilon \varphi_1 \leq \frac{\mu}{\nu} m, \quad \overline{y}_0 = \frac{\mu}{\nu} m, \\
y_n = S(y_{n-1}), \quad \overline{y}_n = S(\overline{y}_{n-1}), \quad n \geq 1.
\]

One has with the same proof as above
\[
y_0 = \epsilon \varphi_1 \leq y_1 \leq \cdots \leq y_n \leq \overline{y}_n \leq \cdots \leq \overline{y}_1 \leq \overline{y}_0 = \frac{\mu}{\nu} m.
\]
The result follows from the fact that \( \overline{y_n} \to \overline{v} \). This completes the proof of the lemma.

One has also:

**Lemma 2.3.** Suppose that

\[
\frac{f(s)}{s} \text{ is decreasing on } (0, +\infty).
\] (2.8)

If \( v_1, v_2 \) are positive solutions to (2.1) corresponding to \( w_1, w_2 \) respectively then

\[
w_1 \leq w_2 \implies v_1 \leq v_2.
\]

In particular (2.1) has a unique positive solution.

**Proof.** Denote by \( \theta \) a smooth function such that

\[
\theta(t) = 0 \quad \forall t \leq 0, \quad \theta(t) = 1 \quad \forall t \geq 1, \quad \theta'(t) \geq 0.
\]

Set \( \theta_\varepsilon(t) = \theta(\frac{t}{\varepsilon}) \). Clearly

\[
v_1 \theta_\varepsilon(v_1 - v_2), \ v_2 \theta_\varepsilon(v_1 - v_2) \in V.
\]

From the equations satisfied by \( v_1, v_2 \) one gets, setting \( \theta_\varepsilon = \theta_\varepsilon(v_1 - v_2) \),

\[
\int_{\Omega} D\nabla v_1 \cdot \nabla(v_2 \theta_\varepsilon) \ dx + \int_{\Gamma_0} \nu v_1(x_1, 0)(v_2 \theta_\varepsilon) \ dx_1
\]

\[
= \int_{\Omega} f(v_1)(v_2 \theta_\varepsilon) \ dx + \int_{\Gamma_0} \mu w_1(v_2 \theta_\varepsilon) \ dx_1,
\]

\[
\int_{\Omega} D\nabla v_2 \cdot \nabla(v_1 \theta_\varepsilon) \ dx + \int_{\Gamma_0} \nu v_2(x_1, 0)(v_1 \theta_\varepsilon) \ dx_1
\]

\[
= \int_{\Omega} f(v_2)(v_1 \theta_\varepsilon) \ dx + \int_{\Gamma_0} \mu w_2(v_1 \theta_\varepsilon) \ dx_1.
\]

By subtraction we obtain

\[
\int_{\Omega} D\{\nabla v_2 \cdot \nabla(v_1 \theta_\varepsilon) - \nabla v_1 \cdot \nabla(v_2 \theta_\varepsilon)\} \ dx
\]

\[
= \int_{\Omega} f(v_2)(v_1 \theta_\varepsilon) - f(v_1)(v_2 \theta_\varepsilon) \ dx + \int_{\Gamma_0} \mu(w_2v_1 - w_1v_2)\theta_\varepsilon(v_1 - v_2) \ dx_1.
\]

Clearly the last integral above is nonnegative so that one has

\[
\int_{\Omega} D\{\nabla v_2 \cdot \nabla(v_1 \theta_\varepsilon) - \nabla v_1 \cdot \nabla(v_2 \theta_\varepsilon)\} \ dx \geq \int_{\Omega} f(v_2)(v_1 \theta_\varepsilon) - f(v_1)(v_2 \theta_\varepsilon) \ dx.
\]
By a simple computation writing $\theta'_e$ for $\theta'_e(v_1 - v_2)$ one derives
\[
\int_{\Omega_t} f(v_2)(v_1 \theta_e) - f(v_1)(v_2 \theta_e) \, dx \\
\leq \int_{\Omega_t} D\{\nabla v_2 \cdot (v_1 \theta_e) - \nabla v_1 \cdot (v_2 \theta_e)\} \, dx \\
= \int_{\Omega_t} D\{\nabla v_2 \cdot (v_1 - v_2)\theta'_e v_1 - \nabla v_1 \cdot (v_1 - v_2)\theta'_e v_2\} \, dx \\
= \int_{\Omega_t} D\{v_1 \nabla v_2 - v_2 \nabla v_1\} \cdot \nabla (v_1 - v_2)\theta'_e \, dx \\
= \int_{\Omega_t} D\{v_1 \nabla v_2 - v_2 \nabla v_2 + v_2 \nabla v_2 - v_2 \nabla v_1\} \cdot \nabla (v_1 - v_2)\theta'_e \, dx \\
= \int_{\Omega_t} D\nabla v_2 \cdot \nabla (v_1 - v_2)(v_1 - v_2)\theta'_e \, dx - \int_{\Omega_t} Dv_2 |\nabla (v_1 - v_2)|^2 \theta'_e \, dx \\
\leq \int_{\Omega_t} D\nabla v_2 \cdot \nabla (v_1 - v_2)(v_1 - v_2)\theta'_e \, dx.
\]

Let us set $\gamma_t(s) = \int_0^s \theta'_e(s) \, ds$ in such a way that the inequality above reads
\[
\int_{\Omega_t} f(v_2)(v_1 \theta_e) - f(v_1)(v_2 \theta_e) \, dx \leq \int_{\Omega_t} D\nabla v_2 \cdot \nabla \gamma_t(v_1 - v_2) \, dx.
\]

From the equation satisfied by $v_2$, since $\gamma_t(v_1 - v_2) \in V$ and $v_2, \gamma_t$ are nonnegative one has
\[
\int_{\Omega_t} D\nabla v_2 \cdot \nabla \gamma_t(v_1 - v_2) \, dx \leq \int_{\Omega_t} f(v_2)\gamma_t(v_1 - v_2) \, dx + \int_{\Gamma_0} \mu w_2 \gamma_t(v_1 - v_2) \, dx_1.
\]

Since for some constant $C$
\[
\gamma_t(s) \leq \int_0^s \theta'_e \left(\frac{s}{\epsilon}\right) \frac{1}{\epsilon} \, ds \leq C\epsilon
\]

the right hand side of the above inequality goes to 0 when $\epsilon \to 0$. Since when $\epsilon \to 0$ one has $\theta_e(v_1 - v_2) \to \chi_{\{v_1 > v_2\}}$ the characteristic function of the set $\{v_1 > v_2\} = \{x \in \Omega | \, v_1(x) > v_2(x)\}$ one gets
\[
\int_{\{v_1 > v_2\}} f(v_2)v_1 - f(v_1)v_2 \, dx \leq 0.
\]

But on the set of integration thanks to (2.8) one has $f(v_2)v_1 - f(v_1)v_2 > 0$ hence the set of integration is necessarily of measure 0, i.e. $v_1 \leq v_2$. This completes the proof of the lemma. \qed

3. The main result.

**Theorem 3.1.** Suppose that (1.2)-(1.6),(2.7),(2.8) hold, then the problem (1.1) admits a nontrivial solution.

**Proof.** As mentioned above it is of course clear that $(0, 0)$ is solution to (1.1). Set
\[
K = \{v \in L^2(\Gamma_0) \mid 0 \leq v \leq m\}.
\]
For $u \in K$, let $\overline{v}$ be the unique positive solution to (2.1) associated with $w = u$. For $\eta \geq L_g$ let $U = T(u)$ be the solution to
\[
\begin{cases}
U \in H^1_0(\Gamma_0), \\
\Gamma_0 D^i U \psi^i + \mu U \psi + \eta U \psi dx_1 \\
= \int_{\Gamma_0} \nu \overline{v}(x_1, 0) \psi + g(u) \psi + \eta u \psi dx_1, \forall \psi \in H^1_0(\Gamma_0).
\end{cases}
\]
(3.1)

The existence of $U$ is a consequence of the Lax-Milgram theorem.

We claim that $T$ is continuous on $K \subset L^2(\Gamma_0)$. Indeed suppose that $u_n \to u$ in $K$. Denote by $\overline{v}_n$ the solution to (2.1) associated with $u_n$ i.e. satisfying
\[
\int_{\Omega_t} D \nabla \overline{v}_n \cdot \nabla \varphi dx + \int_{\Gamma_0} \nu \overline{v}_n(x_1, 0) \varphi dx_1 = \int_{\Omega_t} f(\overline{v}_n) \varphi dx + \int_{\Gamma_0} \mu u_n \varphi dx_1 \forall \varphi \in V.
\]
(3.2)

Since $\overline{v}_n$ and $u_n$ are bounded, taking $\varphi = \overline{v}_n$ in the equality above one gets easily
\[
\int_{\Omega_t} D |\nabla \overline{v}_n|^2 dx + \int_{\Gamma_0} \nu \overline{v}_n(x_1, 0)^2 dx_1 \leq C,
\]
where $C$ is a constant independent of $n$. Thus, up to a subsequence, there exists $v \in V$ such that
\[
\overline{v}_n \rightharpoonup v \text{ in } H^1(\Omega_t), \quad \overline{v}_n \to v \text{ in } L^2(\Omega_t), \quad \overline{v}_n(., 0) \to v(., 0) \text{ in } L^2(\Gamma_0).
\]

Passing to the limit in (3.2) it follows from Lemma 2.3 that $v = \overline{v}$ is the solution to (2.1) corresponding to $w = u$. By uniqueness of the limit one has convergence of the whole sequence and in particular
\[
\overline{v}_n(., 0) \to \overline{v}(., 0) \text{ in } L^2(\Gamma_0).
\]

Passing to the limit in (3.1) written for $u = u_n$ one derives $T(u_n) \to T(u)$ in $L^2(\Gamma_0)$.

We can show also that $T$ is monotone. Indeed, suppose that $u_1 \geq u_2$ and set $U_i = T(u_i)$, $i = 1, 2$. One has, for $\forall \psi \in H^1_0(\Gamma_0)$,
\[
\int_{\Gamma_0} D^i U_1 \psi^i + \mu U_1 \psi + \eta U_1 \psi dx_1 = \int_{\Gamma_0} \nu \overline{v}_1(x_1, 0) \psi + g(u_1) \psi + \eta u_1 \psi dx_1,
\]
\[
\int_{\Gamma_0} D^i U_2 \psi^i + \mu U_2 \psi + \eta U_2 \psi dx_1 = \int_{\Gamma_0} \nu \overline{v}_2(x_1, 0) \psi + g(u_2) \psi + \eta u_2 \psi dx_1.
\]

By subtraction it comes, for $\forall \psi \in H^1_0(\Gamma_0)$,
\[
\int_{\Gamma_0} D^i (U_1 - U_2) \psi^i + \mu (U_1 - U_2) \psi + \eta (U_1 - U_2) \psi dx_1
\]
\[
= \int_{\Gamma_0} \nu (\overline{v}_1(x_1, 0) - \overline{v}_2(x_1, 0)) \psi + (g(u_1) - g(u_2)) \psi + \eta (u_1 - u_2) \psi dx_1.
\]

Choosing $\psi = -(U_1 - U_2)^-$ and taking into account that, by Lemma 2.3, $\overline{v}_1(x_1, 0) - \overline{v}_2(x_1, 0) \geq 0$ and that for $\eta \geq L_g$, $(g(u_1) - g(u_2)) + \eta (u_1 - u_2) \geq 0$ (Cf. (1.3), (1.4)), one gets
\[
\int_{\Gamma_0} D^i [(U_1 - U_2)^-]^2 + \mu [(U_1 - U_2)^-]^2 + \eta [(U_1 - U_2)^-]^2 dx_1 \leq 0.
\]

Thus $(U_1 - U_2)^- = 0$ and $T(u_1) \geq T(u_2)$. 

Next we assert that $T$ maps $K$ into itself. Indeed, if $U_0 = T(0)$ one has, with an obvious notation for $\tau_0$

$$\int_{\Gamma_0} D'U_0 \psi' + \mu U_0 \psi + \eta U_0 \psi \, dx_1 = \int_{\Gamma_0} \nu \tau_0(x_1, 0) \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0).$$

Taking $\psi = -U_0$ one deduces easily since $\tau_0 > 0$ that $U_0 = T(0) \geq 0$. Similarly if $U_m = T(m)$ one has, with an obvious notation for $\tau_m$

$$\int_{\Gamma_0} D'U_m \psi' + \mu U_m \psi + \eta U_m \psi \, dx_1 = \int_{\Gamma_0} \nu \tau_m(x_1, 0) \psi + g(m) \psi + \eta \nu \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0).$$

Thus choosing $\psi = (U_m - m)^+$ it comes since $g(m) \leq 0$, $\tau_m \leq \frac{\nu}{\tau_0} m$

$$\int_{\Gamma_0} D'\{(U_m - m)^+\}'^2 + (\mu + \eta)\{(U_m - m)^+\}'^2 \, dx_1 = \int_{\Gamma_0} (\nu \tau_m - \nu m)(U_m - m)^+ \, dx_1 \leq 0.$$

From this it follows that $U_m \leq m$. By the monotonicity of $T$ it results that $T$ maps the convex $K$ into itself. But clearly $T(K) \subset C^2(\Gamma_0)$ is relatively compact in $L^2(\Gamma_0)$. Thus, by the Schauder fixed point theorem (see [6], [9], [10]), $T$ has a fixed point in $K$ which leads to a nontrivial solution to (1.1). This completes the proof of the theorem. 

If it is clear at this point that the solution we constructed is non degenerate in $\nu$ it is not clear that the same holds for $u$. In fact we have:

**Proposition 3.1.** Let $(u, v)$ be the solution constructed in Theorem 3.1. One has

$$u \neq 0.$$

**Proof.** Suppose that $u \equiv 0$. Due to the second equation of (1.1) one has $v(x_1, 0) = 0$ and from the first equation of (1.1) we get

$$\int_{\Omega_t} D\nabla v \cdot \nabla \varphi \, dx = \int_{\Omega_t} f(v) \varphi \, dx \quad \forall \varphi \in V. \quad (3.3)$$

Consider then a small ball $B = B_{x_0} \subset \Omega_t$ centered at $x_0 \in \Gamma_0$. Set

$$\tilde{v} = \begin{cases} v \text{ in } \Omega_t \cap B, \\ 0 \text{ in the rest of the ball}. \end{cases}$$

Let $\varphi \in \mathcal{D}(B)$. One has by (3.3),

$$\int_B D\nabla \tilde{v} \cdot \nabla \varphi \, dx = \int_{\Omega_t \cap B} D\nabla v \cdot \nabla \varphi \, dx = \int_{\Omega_t \cap B} f(v) \varphi \, dx = \int_B f(\tilde{v}) \varphi \, dx, \forall \varphi \in \mathcal{D}(B).$$

Thus

$$-D\Delta \tilde{v} = f(\tilde{v}) \text{ in } B.$$

It is clear that $\tilde{v}$ and thus $f(\tilde{v})$ are bounded and one has $f(\tilde{v}) \in L^\infty(B) \subset L^p(B)$ $\forall p$. From the usual regularity theory it follows that $\tilde{v} \in W^{2,p}(B) \subset C^{1,\alpha}(B)$. Since $f(\tilde{v}) \geq 0$, $f(\tilde{v}) \neq 0$ it follows that $\tilde{v} > 0$ in $B$ (see [10]). Hence a contradiction. This shows the impossibility for $u$ to be identical to 0 and this completes the proof of the proposition. \qed
Remark 1. One can easily show (see [7], [8]) that
\[ \lambda_1 = \lambda_1(\Omega_\ell) = \left( \frac{\pi}{2\ell} \right)^2 + \left( \frac{\pi}{L} \right)^2. \]
Thus for a smooth function \( f \) it is clear that (2.7) is satisfied if
\[ \lambda_1 = \lambda_1(\Omega_\ell) < \frac{f'(0)}{D}, \]
i.e. for \( \ell \) and \( L \) large enough.

Note that (2.8) (see also (1.5)) is satisfied in the case of the Fisher equation i.e. for
\[ f(v) = v(1 - v) \]
the Lipschitz character of \( f \) being used only on a finite interval.

4. Some extension. In this section, we would like to extend our results in the case of a so called two-road elliptic problem, for which the corresponding evolution problem was studied in [11] in high dimensional unbounded cylinders. To be more precise, we set
\[ \Gamma_0' = (-\ell, \ell) \times \{ L \}, \quad \Gamma_1 = \partial \Omega_\ell \setminus \{ \Gamma_0 \cup \Gamma_0' \}, \]
\[ V = \{ v \in H^1(\Omega_\ell) \mid v = 0 \text{ on } \Gamma_1 \}, \]

![Figure 2. The domain \( \Omega_\ell \) for two-road problem](image-url)

We consider the problem of finding a \((u, v, w)\) solution to
\[
\begin{align*}
\int_{\Omega_\ell} D'v \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu v(x_1, 0) \varphi \, dx_1 + \int_{\Gamma_0'} \nu v(x_1, L) \varphi \, dx_1 \\
= \int_{\Omega_\ell} f(v) \varphi \, dx + \int_{\Gamma_0} \mu v \varphi \, dx_1 + \int_{\Gamma_0'} \mu' w \varphi \, dx_1 \quad \forall \varphi \in V,
\end{align*}
\]
\[
\begin{align*}
\int_{\Gamma_0} D'u' \psi' + \mu u \psi \, dx_1 \\
= \int_{\Gamma_0} \nu v(x_1, 0) \psi \, dx_1 + \int_{\Gamma_0} g(u) \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0),
\end{align*}
\]
\[
\begin{align*}
\int_{\Gamma_0'} D'' w' \phi' + \mu' w \phi \, dx_1 \\
= \int_{\Gamma_0'} \nu' v(x_1, L) \phi \, dx_1 + \int_{\Gamma_0'} h(w) \phi \, dx_1 \quad \forall \phi \in H^1_0(\Gamma_0').
\end{align*}
\]
Here we assume that
\[ D, D', D'', \mu, \nu, \mu', \nu' \] are positive constants,
f, g, h are Lipschitz continuous functions with Lipschitz constants \( L_f, L_g, L_h \) respectively (Cf. (1.3)), which implies that for \( \lambda \geq L_f, \eta \geq L_g \) and \( \xi \geq L_h \) the functions
\[ x \to \lambda x - f(x), \eta x - g(x), \xi x - h(x) \]
are nondecreasing. We will suppose that $f$ satisfies (1.5) and that
\[ g(0) = 0, \quad h(0) = 0. \]

Since $\Gamma_0$ and $\Gamma'_0$ are playing exactly identical roles there is no loss of generality in assuming for instance
\[ \frac{\mu}{\nu} \geq \frac{\mu'}{\nu'}. \]

Then for
\[ m \geq \frac{\nu}{\mu}, \quad m' = \frac{\nu'}{\mu'} m, \]
we will assume
\[ g(m) \leq 0, \quad h(m') \leq 0. \quad (4.2) \]

One should notice the following properties
\[ \frac{\mu}{\nu} m \geq 1, \quad \frac{\mu'}{\nu'} m' = \frac{\mu}{\nu} m, \]
\[ m' = \frac{\nu'}{\mu'} m \geq \frac{\nu'}{\mu'}. \]

Then with small variants we can reproduce the results we had in the preceding sections. First we have

**Lemma 4.1.** Suppose that $\tilde{u}, \tilde{w}$ are measurable functions on $\Gamma_0$ and $\Gamma'_0$ respectively such that
\[ 0 \leq \tilde{u} \leq m, \quad 0 \leq \tilde{w} \leq m'. \]

Then under the assumptions above the problem
\[
\begin{cases}
\int_{\Omega} D\nabla v \cdot \nabla \varphi \ dx + \int_{\Gamma_0} \nu v(x_1, 0) \varphi(x_1, 0) \ dx_1 \\
+ \int_{\Gamma'_0} \nu' v(x_1, L) \varphi(x_1, L) \ dx_1 = \int_{\Omega} f(v) \varphi \ dx \\
+ \int_{\Gamma_0} \mu \tilde{w} \varphi(x_1, 0) \ dx_1 + \int_{\Gamma'_0} \mu' \tilde{w} \varphi(x_1, L) \ dx_1 \forall \varphi \in V
\end{cases}
\]
possesses a minimal and a maximal solution with values in $[0, \frac{\nu}{\mu} m]$.

**Proof.** Let us remark first that any nonnegative solution to (4.3) takes its values in $[0, \frac{\nu}{\mu} m]$. Indeed if $v$ is solution to (4.3) taking as test function $\varphi = (v - k)^+$, $k = \frac{\nu}{\mu} m \geq 1$ one gets
\[
\begin{align*}
\int_{\Omega} D|\nabla (v - k)^+|^2 \ dx \\
= \int_{\Omega} D\nabla (v - k) \cdot \nabla (v - k)^+ \ dx \\
= \int_{\Omega} f(v)(v - k)^+ \ dx + \int_{\Gamma_0} \{\mu \tilde{u} - \nu v(x_1, 0)\}(v - k)^+ \ dx_1 \\
+ \int_{\Gamma'_0} \{\mu' \tilde{w} - \nu' v(x_1, L)\}(v - k)^+ \ dx_1 \\
\leq \int_{\Gamma_0} \{\mu \tilde{u} - \nu v(x_1, 0)\}(v - k)^+ \ dx_1 + \int_{\Gamma'_0} \{\mu' \tilde{w} - \nu' v(x_1, L)\}(v - k)^+ \ dx_1 \\
\leq 0,
\end{align*}
\]
since on the set where $v \geq k = \frac{\nu}{\mu} m = \frac{\nu'}{\mu'} m'$ one has $\{\mu \tilde{u} - \nu v(x_1, 0)\} \leq \{\mu \tilde{u} - \mu m\} \leq 0$ and $\{\mu' \tilde{w} - \nu' v(x_1, L)\} \leq \{\mu' \tilde{w} - \mu' m'\} \leq 0$. 

Next we note that 0 is a subsolution to (4.3). Indeed \( \forall \varphi \in V, \varphi \geq 0 \), one has

\[
\int_{\Omega} D\nabla 0 \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu 0 \varphi \, dx_1 + \int_{\Gamma'_0} \nu' 0 \varphi \, dx_1 \\
\leq \int_{\Omega} f(0) \varphi \, dx + \int_{\Gamma_0} \mu \tilde{u} \varphi \, dx_1 + \int_{\Gamma'_0} \mu' \tilde{w} \varphi \, dx_1.
\]

On the other hand, \( k = \frac{\mu}{\nu} m \) is a supersolution since for \( \varphi \in V, \varphi \geq 0 \),

\[
\int_{\Omega} D\nabla k \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu k \varphi \, dx_1 + \int_{\Gamma'_0} \nu' k \varphi \, dx_1 \\
\geq \int_{\Omega} f(k) \varphi \, dx + \int_{\Gamma_0} \mu \tilde{u} \varphi \, dx_1 + \int_{\Gamma'_0} \mu' \tilde{w} \varphi \, dx_1.
\]

For \( z \in L^2(\Omega_\ell) \) we denote by \( y = S(z) \) the solution to

\[
\begin{aligned}
\begin{cases}
y \in V, \\
\int_{\Omega} D\nabla y \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \lambda y \varphi \, dx_1 + \int_{\Gamma'_0} \nu y(x_1, 0) \varphi \, dx_1 + \int_{\Gamma_0} \nu' y(x_1, L) \varphi \, dx_1 \\
= \int_{\Omega} f(z) \varphi \, dx + \int_{\Omega} \lambda \varphi \, dx + \int_{\Gamma_0} \mu \tilde{u} \varphi \, dx_1 + \int_{\Gamma'_0} \mu' \tilde{w} \varphi \, dx_1 \quad \forall \varphi \in V,
\end{cases}
\end{aligned}
\]

where \( \lambda \geq L_f \). The existence of a unique solution \( y \) to the problem above follows from the Lax-Milgram theorem. Then reproducing the arguments of Lemma 2.1 it is easy to show that \( S \) is continuous and monotone. Introducing the sequence defined in (2.4) one concludes as in the Lemma 2.1 to the existence of a minimal and a maximal solution \( \underline{v} \) and \( \overline{v} \).

Then one has:

**Lemma 4.2.** Under the assumptions of the preceding lemma and (2.7), for \( \epsilon > 0 \) small enough, the maximal solution \( \overline{v} \) to (4.3) satisfies

\[
\epsilon \varphi_1 \leq \overline{v}.
\]

In particular \( \overline{v} \) is bounded away from 0.

**Proof.** Due to (2.7) one has for \( \epsilon > 0 \) small enough

\[
D\lambda_1 \epsilon \varphi_1 \leq f(\epsilon \varphi_1).
\]

Then for \( \varphi \in V, \varphi \geq 0 \) it holds after integration by parts

\[
\int_{\Omega_\ell} D\nabla (\epsilon \varphi_1) \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu \epsilon \varphi_1(x_1, 0) \varphi \, dx_1 + \int_{\Gamma'_0} \nu' \epsilon \varphi_1(x_1, L) \varphi \, dx_1
\]

\[
= \int_{\Omega_\ell} D\nabla (\epsilon \varphi_1) \cdot \nabla \varphi \, dx = \int_{\Omega_\ell} \nabla \cdot (D\nabla (\epsilon \varphi_1)) \varphi \, dx
\]

\[
= \int_{\partial \Omega_\ell} D\partial_n (\epsilon \varphi_1) \varphi\, d\sigma + \int_{\Omega_\ell} D\lambda_1 (\epsilon \varphi_1) \varphi \, dx
\]

\[
\leq \int_{\Omega_\ell} f(\epsilon \varphi_1) \varphi \, dx + \int_{\Gamma_0} \mu \tilde{u} \varphi \, dx_1 + \int_{\Gamma'_0} \mu' \tilde{w} \varphi \, dx_1.
\]

(n denotes the outward unit normal to \( \Omega_\ell \), note that \( \partial_n (\epsilon \varphi_1) \leq 0 \)). Thus \( \epsilon \varphi_1 \) is a positive subsolution to (4.3) and one concludes as in the proof of Lemma 2.2.

By analogy to Lemma 2.3 one has:
Lemma 4.3. Suppose that \( f \) satisfies (2.8). If \( v_1, v_2 \) are positive solutions to (4.3) corresponding to \((u_1, w_1)\) and \((u_2, w_2)\) respectively then
\[
u_1 \leq u_2 \quad \text{and} \quad v_1 \leq w_2 \quad \text{imply} \quad v_1 \leq v_2.
\]
In particular, (4.3) has a unique positive solution.

Proof. Denote by \( \theta \) a smooth function such that
\[
\theta(t) = 0 \quad \forall t \leq 0, \quad \theta(t) = 1 \quad \forall t \geq 1, \quad \theta'(t) \geq 0.
\]
Set \( \theta \) \( \equiv \theta(\frac{t}{t}) \). Clearly
\[
v_1 \theta_\epsilon(v_1 - v_2), \quad v_2 \theta_\epsilon(v_1 - v_2) \in V.
\]
From the equations satisfied by \( v_1, v_2 \) one gets, setting \( \theta_\epsilon = \theta_\epsilon(v_1 - v_2) \),
\[
\int_{\Omega_\epsilon} D\nabla v_1 \cdot \nabla (v_2 \theta_\epsilon) \, dx + \int_{\Gamma_0} \nu v_1(x_1, 0)(v_2 \theta_\epsilon) \, dx_1 + \int_{\Gamma_0} \nu' v_1(x_1, L)(v_2 \theta_\epsilon) \, dx_1
= \int_{\Omega_\epsilon} f(v_1)(v_2 \theta_\epsilon) \, dx + \int_{\Gamma_0} \mu u_1(v_2 \theta_\epsilon) \, dx_1 + \int_{\Gamma_0} \mu' w_1(v_2 \theta_\epsilon) \, dx_1,
\]
\[
\int_{\Omega_\epsilon} D\nabla v_2 \cdot \nabla (v_1 \theta_\epsilon) \, dx + \int_{\Gamma_0} \nu v_2(x_1, 0)(v_1 \theta_\epsilon) \, dx_1 + \int_{\Gamma_0} \nu' v_2(x_1, L)(v_1 \theta_\epsilon) \, dx_1
= \int_{\Omega_\epsilon} f(v_2)(v_1 \theta_\epsilon) \, dx + \int_{\Gamma_0} \mu u_2(v_1 \theta_\epsilon) \, dx_1 + \int_{\Gamma_0} \mu' w_2(v_1 \theta_\epsilon) \, dx_1.
\]
By subtraction we obtain
\[
\int_{\Omega_\epsilon} D\{\nabla v_2 \cdot \nabla (v_1 \theta_\epsilon) - \nabla v_1 \cdot \nabla (v_2 \theta_\epsilon)\} \, dx = \int_{\Omega_\epsilon} f(v_2)(v_1 \theta_\epsilon) - f(v_1)(v_2 \theta_\epsilon) \, dx
+ \int_{\Gamma_0} \mu(u_2 v_1 - u_1 v_2) \theta_\epsilon(v_1 - v_2) \, dx_1 + \int_{\Gamma_0} \mu' w_2(v_1 \theta_\epsilon) - \mu'(v_2(v_1 \theta_\epsilon) - w(v_1 \theta_\epsilon)) \, dx_1.
\]
Clearly the last two integrals above are nonnegative so that one has
\[
\int_{\Omega_\epsilon} D\{\nabla v_2 \cdot \nabla (v_1 \theta_\epsilon) - \nabla v_1 \cdot \nabla (v_2 \theta_\epsilon)\} \, dx \geq \int_{\Omega_\epsilon} f(v_2)(v_1 \theta_\epsilon) - f(v_1)(v_2 \theta_\epsilon) \, dx.
\]
Then the rest of the proof is like in Lemma 2.3. \( \Box \)

Then we can show:

Theorem 4.4. Under the assumptions above the problem (4.1) admits a nontrivial solution.

Proof. It is of course clear that \((0, 0, 0)\) is solution to (4.1). Set
\[
K = \{ u \in L^2(\Gamma_0) \mid 0 \leq u \leq m \}, \quad K' = \{ w \in L^2(\Gamma_0') \mid 0 \leq w \leq m' \}.
\]
For \((u, w) \in K \times K'\), let \( \pi \) be the unique positive solution to (4.3) associated with \((\bar{u}, \bar{w}) = (u, w)\). For \( \eta \geq L_g, \xi \geq L_h \), let \((U, W) = T(u, W)\) be the solution to
\[
\begin{aligned}
(U, W) &\in H^1_0(\Gamma_0) \times H^1_0(\Gamma_0'), \\
\int_{\Gamma_0} D'U' \psi' + \mu U \psi + \eta U \psi \, dx_1 &= \int_{\Gamma_0} \nu \pi(x_1, 0) \psi + g(u) \psi + nh \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0), \\
\int_{\Gamma_0} D''W' \phi' + \mu' W \phi + \xi W \phi \, dx_1 &= \int_{\Gamma_0} \nu \pi(x_1, L) \phi + h(w) \phi + \xi w \phi \, dx_1 \quad \forall \phi \in H^1_0(\Gamma_0).
\end{aligned}
\]
The existence of \((U, W)\) is a consequence of the Lax-Milgram theorem.
We show as in Theorem 3.1 that $T$ is continuous on $K \times K' \subset L^2(\Gamma_0) \times L^2(\Gamma'_0)$. Indeed suppose that $u_n \to u$ in $K$ and $w_n \to w$ in $K'$. Denote by $\tau_n$ the solution to (4.3) associated with $(u_n, w_n)$ i.e. satisfying

$$\int_{\Omega_\ell} D\nabla \tau_n \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu \tau_n(x_1, 0) \varphi \, dx_1 + \int_{\Gamma'_0} \nu' \tau_n(x_1, L) \varphi \, dx_1 = \int_{\Omega_\ell} f(\tau_n) \varphi \, dx + \int_{\Gamma_0} \mu u_n \varphi \, dx_1 + \int_{\Gamma'_0} \mu' w_n \varphi \, dx_1 \quad \forall \varphi \in V.$$  

(4.4)

Since $\tau_n$, $u_n$ and $w_n$ are bounded, taking $\varphi = \tau_n$ in the equality above one gets easily

$$\int_{\Omega_\ell} D|\nabla \tau_n|^2 \, dx + \int_{\Gamma_0} \nu \tau_n(x_1, 0)^2 \, dx_1 + \int_{\Gamma'_0} \nu' \tau_n(x_1, L)^2 \, dx_1 \leq C,$$

where $C$ is a constant independent of $n$. Thus, up to a subsequence, there exists $v \in V$ such that

$$\tau_n \rightharpoonup v \text{ in } H^1(\Omega_\ell), \quad \tau_n \rightharpoonup v \text{ in } L^2(\Omega_\ell),$$

$$\tau_n(\cdot, 0) \to v(\cdot, 0) \text{ in } L^2(\Gamma_0), \quad \tau_n(\cdot, L) \to v(\cdot, L) \text{ in } L^2(\Gamma'_0).$$

Passing to the limit in (4.4) one derives as in Theorem 3.1 that $T(u_n, w_n) \to T(u, w)$ in $L^2(\Gamma_0) \times L^2(\Gamma'_0)$.

We can show also that $T$ is monotone. Indeed, suppose that $(u_1, w_1) \geq (u_2, w_2)$ in the sense that $u_1 \geq u_2$ and $w_1 \geq w_2$ and set $(U_i, W_i) = T(u_i, w_i)$, $i = 1, 2$. First, for $U_1$ one has

$$\int_{\Gamma_0} D'U_1 \psi' + \mu U_1 \psi + \eta U_1 \psi \, dx_1 = \int_{\Gamma_0} \nu \tau_1(x_1, 0) \psi + g(u_1) \psi + \eta u_1 \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0),$$

$$\int_{\Gamma_0} D'U_2 \psi' + \mu U_2 \psi + \eta U_2 \psi \, dx_1 = \int_{\Gamma_0} \nu \tau_2(x_1, 0) \psi + g(u_2) \psi + \eta u_2 \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0).$$

By subtraction it comes

$$\int_{\Gamma_0} D'(U_1 - U_2) \psi' + \mu (U_1 - U_2) \psi + \eta (U_1 - U_2) \psi \, dx_1$$

$$= \int_{\Gamma_0} \nu(\tau_1(x_1, 0) - \tau_2(x_1, 0)) \psi + (g(u_1) - g(u_2)) \psi + \eta (u_1 - u_2) \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0).$$

Choosing $\psi = -(U_1 - U_2)^-$ and taking into account that, by Lemma 4.3, $\tau_1(x_1, 0) - \tau_2(x_1, 0) \geq 0$ and that for $\eta \geq L_g$, $(g(u_1) - g(u_2)) + \eta (u_1 - u_2) \geq 0$ (Cf. (1.3), (1.4)), one gets

$$\int_{\Gamma_0} D'|{(U_1 - U_2)^-}'|^2 + \mu |{(U_1 - U_2)^-}'|^2 + \eta |{(U_1 - U_2)^-}'|^2 \, dx_1 \leq 0.$$

Thus $(U_1 - U_2)^- = 0$ and $U_1 \geq U_2$. Similarly one shows that $W_1 \geq W_2$.

Next we assert that $T$ maps $K \times K'$ into itself. Indeed, if $(U_0, W_0) = T(0, 0)$ one has, with an obvious notation for $\tau_0$

$$\int_{\Gamma_0} D'U_0 \psi' + \mu U_0 \psi + \eta U_0 \psi \, dx_1 = \int_{\Gamma_0} \nu \tau_0(x_1, 0) \psi \, dx_1 \quad \forall \psi \in H^1_0(\Gamma_0).$$
\[
\int_{\Gamma_0} D''W_0^*\phi' + \mu W_0 \phi + \xi W_0 \phi \, dx_1 = \int_{\Gamma_0} \nu' \varpi_0(x_1,L) \phi \, dx_1 \quad \forall \phi \in H_0^1(\Gamma_0').
\]
Taking \( \psi = -U_0^-, \phi = -W_0^- \), one deduces easily since \( \varpi_0(x_1,0) \geq 0 \) and \( \varpi_0(x_1,L) \geq 0 \) that \( U_0 \geq 0 \) and \( W_0 \geq 0 \). Similarly, if \((U_1, W_1) = T(m, m')\) one has, with an obvious notation for \( \nu_1 \)

\[
\begin{align*}
\int_{\Gamma_0} D'U_1^* \psi' + \mu U_1 \psi + \eta U_1 \psi \, dx_1 &= \int_{\Gamma_0} \nu' \varpi_1(x_1,0)\psi + g(m)\psi + \eta m \psi \, dx_1, \forall \psi \in H_0^1(\Gamma_0), \\
\int_{\Gamma_0} D'W_1^* \phi' + \mu W_1 \phi + \xi W_1 \phi \, dx_1 &= \int_{\Gamma_0} \nu' \varpi_1(x_1,L)\phi + h(m')\phi + \xi m' \phi \, dx_1, \forall \phi \in H_0^1(\Gamma_0').
\end{align*}
\]

Thus choosing \( \psi = (U_1 - m)^+ \) and \( \phi = (W_1 - m')^+ \), due to (4.2) and \( \nu_1 \leq \frac{\mu}{\nu} m = \frac{\mu'}{\nu'} m' \) it comes

\[
\begin{align*}
\int_{\Gamma_0} D'|\{(U_1 - m)^+\}|^2 + (\mu + \eta)\{(U_1 - m)^+\}^2 \, dx_1 &\leq \int_{\Gamma_0} (\nu' \varpi_1(x_1,0) - \mu m)(U_1 - m)^+ \, dx_1 \leq 0. \\
\int_{\Gamma_0} D'|\{(W_1 - m')^+\}|^2 + (\mu' + \xi)\{(W_1 - m')^+\}^2 \, dx_1 &\leq \int_{\Gamma_0} (\nu' \varpi_1(x_1,L) - \mu' m')(W_1 - m')^+ \, dx_1 \leq 0.
\end{align*}
\]

From this it follows that \( U_1 \leq m, W_1 \leq m' \). By the monotonicity of \( T \) it results that \( T \) maps the convex \( K \times K' \) into itself. But clearly \( T(K \times K') \subset C^{\frac{1}{2}}(\Gamma_0) \times C^{\frac{1}{2}}(\Gamma_0') \) is relatively compact in \( L^2(\Gamma_0) \times L^2(\Gamma_0') \). Thus, by the Schauder fixed point theorem (see [6], [9], [10]), \( T \) has a fixed point in \( K \times K' \) which leads to a nontrivial solution \((u, v, w)\) to (4.1). This completes the proof of the theorem. \( \square \)

**Remark 2.** One can show as in Proposition 3.1 that \( u, w \) are also non degenerate in the sense that

\[ u \neq 0, \quad w \neq 0. \]

5. **The case of an unbounded domain.** The goal of this section is to show that when \( \ell = +\infty \) it remains possible to define and find a nontrivial solution to problem (1.1), for which we refer to [3] and [12] for relevant study of the evolution problem in half space and in an unbounded strip, respectively. Let us introduce some notation. For convenience we will denote by \( V_\ell \) the space \( V \) defined in Section 1. Similarly we will indicate the dependence in \( \ell \) for \( \Gamma_0 \), i.e.

\[ \Gamma_0 = \Gamma_0^\ell = (-\ell, \ell) \times \{0\}. \]

When convenient we will set \( I_\ell = (-\ell, \ell) \). In addition, we set

\[ \Omega_\infty = \mathbb{R} \times (0, L), \quad \Gamma_0^\infty = \mathbb{R} \times \{0\}, \quad \Gamma_1^\infty = \mathbb{R} \times \{L\}, \]

\[ V_\infty = \{ v \in H^1_0(\Omega_\infty) \mid v = 0 \text{ on } \Gamma_1^\infty \}, \]

where

\[ H^1_0(\Omega_\infty) = \{ v \mid v \in H^1(\Omega_\infty) \forall \ell_0 > 0 \}. \]

Then we have
**Theorem 5.1.** Suppose that (1.2), (1.3), (1.5), (1.6), (2.8) hold and that
\[ \frac{f(s)}{D_s} > \left( \frac{\pi}{L} \right)^2, \] (5.1)
then under the assumptions above there exists \((u, v)\) nontrivial solution to
\[
\begin{cases}
(u, v) \in H^1_0(\Omega_0^\infty) \times V_\infty, \\
\int_{\Omega_0} Dv \cdot \nabla \varphi \, dx + \int_{\Gamma_0} \nu v(x_1, 0) \varphi \, dx_1 \\
= \int_{\Omega_0} f(v) \varphi \, dx + \int_{\Gamma_0} \mu u \varphi \, dx_1 \quad \forall \varphi \in V_\infty, \ \forall \ell_0,
\int_{\Gamma_0} D'u' \psi' + \mu u \psi \, dx_1 \\
= \int_{\Gamma_0} \nu v(x_1, 0) \psi \, dx_1 + \int_{\Gamma_0} g(u) \psi \, dx_1 \quad \forall \psi \in H^1_0(I_0), \ \forall \ell_0.
\end{cases}
\] (5.2)
(We identify \(\Gamma_0^\infty\) with \(\mathbb{R}\). Recall that \(I_\ell = (-\ell, \ell)\)).

**Proof.** Let \((u_\ell, v_\ell)\) be a solution to (1.1). We can find such a solution for every \(\ell\) sufficiently large such that (2.7), i.e., \(\lambda(\Omega_\ell) \leq \frac{f(s)}{D_s}\) holds, thanks to Theorem 3.1 and (5.1). Note that \(\lambda(\Omega_\ell) = \left( \frac{\pi}{2\ell} \right)^2 + \left( \frac{\pi}{L} \right)^2\) (see [7]). One notices that for \(\ell' \geq \ell\) there holds
\[
\Omega_\ell \subset \Omega_{\ell'}, \ H^1_0(\Omega_\ell) \subset H^1_0(\Omega_{\ell'}),
\]
(we suppose the functions of \(H^1_0(\Omega_\ell)\) extended by 0 outside \(\Omega_\ell\)). By definition of \(\lambda_1 = \lambda_1(\Omega_\ell)\) one has
\[
\lambda_1(\Omega_\ell) = \inf_{H^1_0(\Omega_\ell) \setminus \{0\}} \frac{\int_{\Omega_\ell} |\nabla v|^2 \, dx}{\int_{\Omega_\ell} v^2 \, dx},
\]
and thus clearly
\[
\lambda_1(\Omega_{\ell'}) \leq \lambda_1(\Omega_\ell) \quad \forall \ell' \geq \ell.
\]
Let us assume for some \(\ell_1 > 0\) (Cf. (2.7))
\[
\lambda_1(\Omega_{\ell_1}) \leq \frac{f(s)}{D_s} \quad \text{for } s > 0 \text{ small enough.} \quad (5.3)
\]
Then for any \(\ell \geq \ell_1\) one has for \(s > 0\) small enough
\[
\lambda_1(\Omega_\ell) \leq \frac{f(s)}{D_s}
\]
Moreover, since it is easy to see that \(\varphi_1\) defined in (2.6) is given by
\[
\varphi_1 = \sin \frac{\pi}{2\ell}(x_1 + \ell) \sin \frac{\pi}{L} x_2,
\]
one has \(0 \leq \varphi_1 \leq 1\) and if (5.3) holds one has
\[
D\lambda_1(\Omega_\ell) \epsilon \varphi_1 \leq f(\epsilon \varphi_1)
\]
for \(\epsilon > 0\) small enough independently of \(\ell \geq \ell_1\). We suppose from now on that this \(\epsilon\) is fixed such that if \((u_\ell, v_\ell)\) is a solution to (1.1) constructed as in Theorem 3.1 one has
\[
\epsilon \varphi_1 \leq v_\ell
\]
and in particular for every \(\ell \geq \ell_1\)
\[
\epsilon(\sin \frac{\pi}{4})^2 \leq \epsilon \varphi_1 \leq v_\ell \quad \text{a.e. } x \in (-\frac{\ell}{2}, \frac{\ell}{2}) \times (\frac{L}{4}, \frac{3L}{4}).
\] (5.4)
One should also notice that independently of \(\ell\) one has
\[
0 \leq u_\ell \leq m, \quad \epsilon \varphi_1 \leq v_\ell \leq \frac{\mu}{\nu} m.
\] (5.5)
We assume from now on $\ell \geq \ell_1$ and for $\ell_0 \leq \ell - 1$ we define $\rho$ by

$$
\rho = \rho(x_1) = \begin{cases} 
1 & \text{on } I_{\ell_0}, \\
x_1 + \ell_0 + 1 & \text{on } (-\ell_0 - 1, -\ell_0), \\
-x_1 + \ell_0 + 1 & \text{on } (\ell_0, \ell_0 + 1), \\
0 & \text{outside } I_{\ell_0+1},
\end{cases}
$$

whose graph is depicted below.

![Graph of the function $\rho(x_1)$](image)

**Figure 3.** The graph of the function $\rho(x_1)$

Clearly $\rho^2 v_\ell = \rho^2(x_1)v_\ell \in V_\ell$ and from the first equation of (1.1) one gets

$$
\int_{\Omega_\ell} D\nabla v_\ell \cdot \nabla (\rho^2 v_\ell) \, dx + \int_{\Gamma_0} \nu \rho^2 v_\ell^2(x_1, 0) \, dx_1 = \int_{\Omega_\ell} f(v_\ell) \rho^2 v_\ell \, dx + \int_{\Gamma_0} \mu u_\ell \rho^2 v_\ell(x_1, 0) \, dx_1.
$$

(5.6)

One should notice that in the integrals over $\Omega_\ell$ one integrates only on $\Omega_{\ell_0+1}$ and for the ones over $\Gamma_0$ on $I_{\ell_0+1}$. Then remark that

$$
\int_{\Omega_\ell} \nabla v_\ell \cdot \nabla (\rho^2 v_\ell) \, dx = \int_{\Omega_\ell} |\nabla v_\ell|^2 \rho^2 + 2\rho v_\ell \nabla v_\ell \cdot \nabla \rho \, dx,
$$

and

$$
\int_{\Omega_\ell} |\nabla (\rho v_\ell)|^2 \, dx = \int_{\Omega_\ell} |\rho \nabla v_\ell + v_\ell \nabla \rho|^2 \, dx = \int_{\Omega_\ell} |\nabla v_\ell|^2 \rho^2 + 2\rho v_\ell \nabla v_\ell \cdot \nabla \rho + v_\ell^2 |\nabla \rho|^2 \, dx.
$$

From this it follows that

$$
\int_{\Omega_\ell} \nabla v_\ell \cdot \nabla (\rho^2 v_\ell) \, dx = \int_{\Omega_\ell} |\nabla (\rho v_\ell)|^2 \, dx - \int_{\Omega_\ell} v_\ell^2 |\nabla \rho|^2 \, dx.
$$

Thus, since the second integral of (5.6) is nonnegative, it comes

$$
D \int_{\Omega_{\ell_0+1}} |\nabla (\rho v_\ell)|^2 \, dx 
\leq D \int_{\Omega_{\ell_0+1}} v_\ell^2 |\nabla \rho|^2 \, dx + \int_{\Omega_{\ell_0+1}} f(v_\ell) \rho^2 v_\ell \, dx + \int_{I_{\ell_0+1}} \mu u_\ell \rho^2 v_\ell(x_1, 0) \, dx_1.
$$

Using the definition of $\rho$ and in particular the fact that $\rho = 1$ on $\Omega_0$ we get easily by (5.5)

$$
\int_{\Omega_0} |\nabla v_\ell|^2 \, dx \leq C
$$

(5.7)
where $C$ is independent of $\ell$. Taking now $\psi = \rho^2 u_\ell$ in the second equation of (1.1) we get
\[
\int_{I_{\ell_0+1}} D' u'_\ell(\rho^2 u_\ell)' + \mu \rho^2 u_\ell^2 \, dx_1 = \int_{I_{\ell_0+1}} \nu \nu_\ell(x_1,0) \rho^2 u_\ell + g(u_\ell) \rho^2 u_\ell \, dx_1.
\]
Arguing as above we derive easily
\[
\int_{I_{\ell_0+1}} u'_\ell(\rho^2 u_\ell)' \, dx_1 = \int_{I_{\ell_0+1}} |(\rho u_\ell)'|^2 \, dx_1 - \int_{I_{\ell_0+1}} u_\ell^2 \rho^2 \, dx_1.
\]
This leads to
\[
\int_{I_{\ell_0+1}} D'|(\rho u_\ell)'|^2 + \mu \rho^2 u_\ell^2 \, dx_1 \leq \int_{I_{\ell_0+1}} D'u_\ell^2 \rho^2 \, dx_1 + \nu \nu_\ell(x_1,0) \rho^2 u_\ell + g(u_\ell) \rho^2 u_\ell \, dx_1.
\]
Integrating only on $I_{\ell_0}$ in the first integral i.e. where $\rho = 1$ we obtain
\[
\int_{I_{\ell_0}} (u_\ell')^2 + u_\ell^2 \, dx_1 \leq C
\]
where $C$ is some other constant independent of $\ell$. It results from (5.7), (5.8) that $(u_\ell, v_\ell)$ is bounded in $H^1(I_{\ell_0}) \times V_{\ell_0}$ independently of $\ell$. Thus there exists a subsequence of $(u_\ell, v_\ell)$ that we will denote by $(u_{n,0}, v_{n,0})$ such that when $n \to \infty$
\[
u_{n,0} \to v^0 \text{ in } V_{\ell_0},
\]
\[
u_{n,0} \to v^0 \text{ in } L^2(I_{\ell_0}),
\]
\[
u_{n,0}(:,0) \to v^0(:,0) \text{ in } L^2(I_{\ell_0}).
\]
Considering the equations
\[
\int_{\Omega_{\ell_0}} D\nabla v_\ell \cdot \nabla \varphi \, dx + \int_{I_{\ell_0}} \nu \nu_\ell(x_1,0) \varphi \, dx_1 = \int_{\Omega_{\ell_0}} f(v_\ell) \varphi \, dx + \int_{I_{\ell_0}} \mu u_\ell \varphi \, dx_1 \forall \varphi \in V_{\ell_0},
\]
\[
\int_{I_{\ell_0}} D'u_\ell' + \mu u_\ell \psi \, dx_1 = \int_{I_{\ell_0}} \nu \nu_\ell(x_1,0) \psi \, dx_1 + \int_{I_{\ell_0}} g(u_\ell) \psi \, dx_1 \forall \psi \in H^1_0(I_{\ell_0}),
\]
with $(u_\ell, v_\ell)$ replaced by $(u_{n,0}, v_{n,0})$, one can pass to the limit in $n$ and see that $(u^0, v^0) \in H^1(I_{\ell_0}) \times V_{\ell_0}$ satisfies
\[
\int_{\Omega_{\ell_0}} D\nabla v^0 \cdot \nabla \varphi \, dx + \int_{I_{\ell_0}} \nu v^0(x_1,0) \varphi \, dx_1 = \int_{\Omega_{\ell_0}} f(v^0) \varphi \, dx + \int_{I_{\ell_0}} \mu u^0 \varphi \, dx_1 \forall \varphi \in V_{\ell_0},
\]
\[
\int_{I_{\ell_0}} D'u^0 + \mu u^0 \psi \, dx_1 = \int_{I_{\ell_0}} \nu v^0(x_1,0) \psi \, dx_1 + \int_{I_{\ell_0}} g(u^0) \psi \, dx_1 \forall \psi \in H^1_0(I_{\ell_0}).
\]
(Note that a function of $V_{\ell_0}$ extended by 0 belongs to $V_\ell$). Clearly -as a subsequence of $(u_\ell, v_\ell)$- the sequence $(u_{n,0}, v_{n,0})$ is bounded in $H^1(I_{\ell_0+1}) \times V_{\ell_0+1}$ independently of $n$ and one can extract a subsequence that we still label by $n$ and denote by $(u_{n,1}, v_{n,1})$ such that
\[
u_{n,1} \to v^1 \text{ in } V_{\ell_0+1},
\]
\[
u_{n,1} \to v^1 \text{ in } L^2(I_{\ell_0+1}),
\]
\[
u_{n,1}(:,0) \to v^1(:,0) \text{ in } L^2(I_{\ell_0+1}).
\]
Note that \((u^1, v^1) = (u^0, v^0)\) on \(I_{\ell_0} \times \Omega_{\ell_0}\). Clearly \((u^1, v^1)\) satisfies
\[
\int_{\Omega_{\ell_0+1}} D\nabla v^1 \cdot \nabla \varphi \, dx + \int_{I_{\ell_0+1}} \nu v^1(x_1, 0) \varphi \, dx_1 \\
= \int_{\Omega_{\ell_0+1}} f(v^1) \varphi \, dx + \int_{I_{\ell_0+1}} \mu u^1 \varphi \, dx_1 \quad \forall \varphi \in V_{\ell_0+1},
\]
\[
\int_{I_{\ell_0+1}} D'u^1 \psi' + \mu u^1 \psi \, dx_1 \\
= \int_{I_{\ell_0+1}} \nu v^1(x_1, 0) \psi \, dx_1 + \int_{I_{\ell_0+1}} g(u^1) \psi \, dx_1 \quad \forall \psi \in H^1_0(I_{\ell_0+1}).
\]

By induction one constructs a sequence \((u_{n,k}, v_{n,k})\) extracted from the preceding, converging toward \((u^k, v^k)\) and satisfying the equations above where we have replaced \(\ell_0 + 1\) by \(\ell_0 + k\). Then using the usual diagonal process it is clear that the sequence \((u_{n,n}, v_{n,n})\) will converge toward a nontrivial solution to (5.2) thanks to (5.4). This completes the proof of the theorem.

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