Basic cosmology
Dedicated to Halton C. Arp

Ll. Bel

Abstract

Basic cosmology describes the universe as a Robertson-Walker model filled with black-body radiation and no barionic matter, and as observational data it uses only the value of the speed of light, the Hubble and deceleration parameters and the black-body temperature at the present epoch. It predicts the value of the next new parameter in the Hubble law.

The Robertson-Walker model.

Its line-element is:

\[ ds^2 = -dt^2 + \frac{1}{c^2} \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right) \]  

(1)

where \( c \equiv c(t) \) with \( c_0 = 299792458.0 \) m/s is the speed of light at the present epoch, \( (t = 0) \).

\( T \equiv T(t) \), with \( T_0 = 2.74K \) being the temperature of the black-body radiation, the mass density \( \rho \equiv \rho(t) \) and pressure \( P \equiv P(t) \) are:

\[ \rho = \frac{a}{c^4} T^4, \quad P = \frac{1}{3} \rho c^2 \]  

(2)

where \( a = 7.565767 \times 10^{-16} \) J/m\(^3\)/K\(^4\) is the radiation constant. No other mass contributes to the dynamics of the model. I assume that either it can be neglected because of its particular fractal distribution or because its content being highly unreliable now it is advisable to postpone its consideration to a non basic model.

Under the preceding assumptions Einstein’s equations:

\[ R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} - \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^2} (\rho c^2 u_\alpha u_\beta + P(g_{\alpha\beta} + u_\alpha u_\beta)) \]  

(3)

where \( \Lambda \) is the cosmological constant with dimensions T\(^{-2}\), reduce to the following two equations:

\[ Eq_1 \equiv 3\ddot{c}^2 + 3kc^4 - \Lambda c^2 = 8\pi G \rho c^2 \]  

(4)

and:

\[ Eq_2 \equiv 2\ddot{c} - 5\dot{c}^2 - kc^4 + \Lambda c^2 = 8\pi G P \]  

(5)

\( ^* \) e-mail: wtpbedil@lg.ehu.es

\(^1 \) \( f(t) \) being a function of \( t \), \( f(0) \) means \( f(0) \)
Using (2) the following linear combination \( \frac{1}{2}(Eq_{1} - 3Eq_{2}) = 0 \) becomes:

\[
Eq_{3} \equiv \dddot{c} - 3\ddot{c} - 6\dot{c} + \frac{2}{3}\Lambda c^{2} = 0 \tag{6}
\]

an equation that will be a useful substitute of \( Eq_{2} \) once \( \Lambda \) and \( k \) will be known.

Solving the system of algebraic equations (4) and (5) with unknowns \( \Lambda \) and \( k \) we get:

\[
\Lambda = \frac{4\pi G}{c^{2}}(\rho c^{2} + 3P) - \frac{3}{c^{2}}(\dddot{c} - 2\ddot{c}) \tag{7}
\]

and:

\[
k = \frac{4\pi G}{c^{4}}(\rho c^{2} + P) - \frac{3}{c^{4}}(\dddot{c} - \dddot{c}^{2}) \tag{8}
\]

**Hubble’s law.**

Light emitted at time \( t_{e} \) at some point with coordinate \( r_{e} \) propagating in the radial direction and being received at time \( t_{r} \) at a point with radial coordinate \( r_{r} \) travels a proper distance \( L \):

\[
L = \int_{r_{e}}^{r_{r}} \frac{dr}{\sqrt{1 - kr^{2}}}, \quad \text{or} \quad L = -\int_{t_{r}}^{t_{e}} c(t) \, dt, \tag{9}
\]

If \( \delta t_{e} \) is the period of the light emitted at \( r_{e} \) and \( \delta t_{r} \) is the period of the light received at \( r_{r} \) then from the preceding formula it follows that:

\[
0 = c_{r}\delta t_{r} - c_{e}\delta t_{e} \tag{10}
\]

from where it follows that defining the red shift \( z \) by:

\[
z = \frac{\delta t_{r}}{\delta t_{e}} - 1 \tag{11}
\]

we get:

\[
z = \frac{c_{e}}{c_{r}} - 1 \tag{12}
\]

The integral defining \( L \) can be approximated as\(^{2}\)

\[
L = -c_{r}(t_{e} - t_{r}) - \frac{1}{2}\dot{c}_{r}(t_{e} - t_{r})^{2} - \frac{1}{6}\dddot{c}_{r}(t_{e} - t_{r})^{3} \tag{13}
\]

and similarly \( c_{e} \) can be approximated by:

\[
c_{e} = c_{r} + \dot{c}_{r}(t_{e} - t_{r}) + \frac{1}{2}\dddot{c}_{r}(t_{e} - t_{r})^{2} + \frac{1}{6}\dddot{c}_{r}(t_{e} - t_{r})^{3} \tag{14}
\]

Inverting (13) we get:

\[
t_{e} - t_{r} = -\frac{1}{c}L - \frac{1}{2c^{3}}L^{2} + \frac{1}{6}\frac{c\dddot{c} - 3c}{c^{5}}L^{3} \tag{15}
\]

and substitution of (15) in (14) and the result in (12) we get:

\(^{2}\)Dots overhead mean derivatives with respect to \( t \).
\[ z = -\frac{\dot{c}}{c^2}L + \frac{1}{2} \frac{\ddot{c} \dot{c} - \dddot{c}^2}{c^4}L^2 + \frac{1}{6} \left( \frac{4\dddot{c} \dot{c} - 3\dddot{c}^3 - \dddot{c}^2}{c^6} \right) L^3 \]  

(16)

Defining the Hubble function \( H \) and the deceleration parameter \( q \) as usual, and the jerk parameter \( j \)

\[ H = -\frac{\dot{c}}{c}, \quad q = \frac{\dddot{c}}{c^2} - 2, \quad j = 6 - 6\frac{\dddot{c}}{c^2} + \frac{\dddot{c}^2}{c^3} \]  

(17)

we extend with an extra term the well known Hubble formula:

\[ z = \frac{H}{c}L + \frac{1}{2} \frac{H^2}{c^2} (1 + q)L^2 + \frac{1}{6} \frac{H^3}{c^3} (6 + 6q + j)L^3 \]  

(18)

The formulas (7) and (8) can be written:

\[ \Lambda = \bar{\Lambda} + 4\pi G \left( \rho + 3P \right), \text{ where } \bar{\Lambda} = -3H^2q \]  

(19)

and:

\[ k = \bar{k} + 4\pi G \left( \rho + P \right) \text{ where } \bar{k} = -\frac{H^2(1 + q)}{c^2} \]  

(20)

Observational data

The Hubble constant and the deceleration parameter have been measured to be \( H_0 = 72 \text{ km/s/Mpc} \) and \( q_0 = -0.55 \). This is all that is needed with \( c_0 \) and \( T_0 \) to derive the values of the cosmological constant \( \Lambda \) and the curvature constant \( k \) using Eqs. (19) and (20).

For \( t = 0 \) the r-h-s of these two equations are known:

\[ \begin{align*} 
\bar{\Lambda}_0 &= 8.983543533 \times 10^{-36} \text{ s}^{-2}, \\
\bar{k}_0 &= -2.726056423 \times 10^{-53} \text{ m}^{-2} 
\end{align*} \]  

(21)

and:

\[ \begin{align*} 
\rho_0 &= 4.473697482 \times 10^{-31} \text{ kg m}^{-3}, \\
P_0 &= 1.340252927 \times 10^{-14} \text{ kg m}^{-1} \text{ s}^{-2} 
\end{align*} \]  

(22)

and therefore the constants \( \Lambda \) and \( k \) are:

\[ \begin{align*} 
\Lambda &= 8.984293774 \times 10^{-36} \text{ s}^{-2}, \\
k &= -2.725499919 \times 10^{-53} \text{ m}^{-2} 
\end{align*} \]  

(23)

With \( \Lambda \) and \( k \) known, from (6) we get:

\[ \ddot{c} = \frac{1}{3} \left( 3c^4k + 9c^2 - 2c^2\Lambda \right) \]  

(24)

Differentiating now with respect to \( t \) we have:

\[ \dddot{c} = \ddot{c} \left( 9c^2k + \frac{15c^2}{c^2} - \frac{14}{3} \Lambda \right) \]  

(25)

\[ ^4j = a^2\ddot{a}/a^3 \text{ if } a = c_0/c \text{ is the scale factor.} \]
and using now the definitions \((17)\) we find the following convenient expression for the new function:

\[
j = \frac{1}{3} \left( \frac{9H^2 + 9c^2k - 2\Lambda}{H^2} \right)
\]  

(26)

Therefore the predicted observational value of \(j(0)\) is\(^4\)

\[
\begin{align*}
j(0) & = 0.55 \\
\end{align*}
\]  

(27)

**Maximally symmetric models.**\(^5\)

Let us assume that at some value of \(t\) we know the values of \(c, \dot{c}\) and \(\ddot{c}\) or equivalently, from \((17)\), the values of \(c, H\) and \(q\) corresponding to some general function \(c(t)\). These data are sufficient to calculate the Riemann, and Einstein’s tensors of the the line-element \((1)\) at this time \(t\). I call Osculating model, \([3]\), at time \(t\) the Robertson-Walker model with line-element:

\[
ds^2 = -dt^2 + \frac{1}{c^2} \left( \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right)
\]  

(28)

solution of Einstein’s equations:

\[
R_{\alpha\beta} - \frac{1}{2} Rg_{\alpha\beta} - \Lambda g_{\alpha\beta} = 0
\]  

(29)

where \(\Lambda\) and \(k\) are defined in \((19)\) and \((20)\). This is a vacuum solution but it is more than that: it is one of the space-times with maximum symmetry. This meaning that the Riemann tensor is:

\[
R_{\alpha\beta\mu
u} = -\frac{1}{3} \lambda (g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu})
\]  

(30)

and therefore it is invariant under a 10 dimensional group of isometries.

The concept of osculating model allow us to look at the history of the Universe as a continuous unfolding of maximally symmetric space-times while we live at an epoch when our osculating universe is one of de Sitter’s space-time models, type \(dS_\infty\), with:

\[
c = \frac{\lambda_0}{\rho_0} \text{csch} \left( \frac{\lambda_0 t + \text{csch}^{-1} \left( \frac{\rho_0 c_0}{\lambda_0} \right)}{\lambda_0} \right), \quad \lambda_0 = \sqrt{\frac{\Lambda_0}{3}}, \quad \rho_0 = \sqrt{-k_0}
\]  

(31)

and the values of \(\Lambda\) and \(k\) are so close to the values of \(\bar{\Lambda}_0\) and \(\bar{k}_0\) that the functions \(c\) and \(\bar{c}\) are almost undistinguishable in the interval \(t = -1.0\). The Figures 1-4 are in succession the graphs of \(c, \rho, \Lambda\) and \(k\). The units are such that: \(c_0 = 1, H_0 = 1, \rho_0 = 1.4 \times 10^{-4}, \lambda_0 = 1.65, k_0 = -0.45\) and \(8\pi G = 1\).

**Hamiltonian formalism**

The differential equation \((3)\) describes the ensemble \(E(\Lambda, k)\) of Robertson-Walker models \((1)\) when its source is such that:

\(^4\)this value is quite compatible with one of the determinations of \(j_0 = 0.631 \pm 0.290\) mentioned in Table 3 of reference \([5]\).

\(^5\)See \([1]\) for instance.
\[
P = \frac{1}{3} \rho c^2
\]
(32)

It can be derived from the Lagrangian:

\[
L = \frac{1}{2} \dot{c}^2 - \frac{1}{2} \frac{k}{c^2} + \frac{1}{6} \frac{\Lambda}{c^4}
\]
(33)

whose associated Hamiltonian is the constant of motion:

\[
\mathcal{H} = \frac{1}{2} \dot{c}^2 + \frac{1}{2} \frac{k}{c^2} - \frac{1}{6} \frac{\Lambda}{c^4}, \quad \frac{d\mathcal{H}}{dt} = 0
\]
(34)

To each value of \(\mathcal{H}\) corresponds a sub-ensemble of models \(S(\Lambda, k; \mathcal{H})\), and if in particular \(\mathcal{H} = 0\) then \(S(\Lambda, k; 0)\) is the ensemble of maximally symmetric spacetime models. Therefore somehow the value of \(\mathcal{H}\) of a model of the ensemble \(\mathcal{E}(\Lambda, k)\) measures its deviation with respect to the corresponding maximally symmetric model.

Using the data and the units mentioned at the end of the preceding section the value of \(\mathcal{H}\) for the basic universe described in this paper is 0.005.

From (4) and (34) it follows the eventually useful formula:

\[
\rho = 3 \frac{c^4 \mathcal{H}}{4 \pi G}
\]
(35)

Open question

Basic cosmology assumes that light from stars and galaxies propagates freely across the black-body radiation. But since it is known that while there is no direct photon-photon interaction there are indirect interactions through intermediate virtual particles [2], it could be that the black body fluid has an index of refraction; and in this case light would not propagate along null geodesics of the Robertson-Walker model and cosmology would radically change from what now we believe it to be.

References

[1] S. Weinberg, *Gravitation...* Chap. 13, John Wiley & Sons (1972)

[2] D. d’Enterria and G. G. Silveira [arXiv:1305.7142v2 [hep-ph]]

[3] Ll. Bel, [arXiv:gr-qc/0306091v1]

[4] Ll. Bel, [arXiv:gr-qc/9905016v1]v1

[5] Ö. Akarsu, T. Dereli, S. Kumar and L. Xu, [arXiv:1305.5190v3 [gr-qc]]

*Figure 1 (c), Figure 2 (ρ), Figure 3 (Λ), Figure 4 (κ).*

*More on that in [4]*
