Eigenvalues, clique number and walks of signed graphs

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Abstract

A signed graph Σ = (G, σ) is a graph where the function σ assigns either 1 or −1
to each edge of the simple graph G. The adjacency matrix of Σ, denoted by A(Σ), is
defined canonically. In a recent paper, Wang et al. extended the eigenvalue bounds
of Hoffman and Cvetković for the signed graphs. They proposed an open problem
related to the balanced clique number and the largest eigenvalue of a signed graph. We
solve a strengthened version of this open problem. As a byproduct, we give alternate
proofs for some of the known classical bounds for the least eigenvalues of the unsigned
graphs. We extend the Turán’s inequality for the signed graphs. Besides, we study the
Bollobás and Nikiforov conjecture for the signed graphs and show that the conjecture
need not be true for the signed graphs. Nevertheless, the conjecture holds for signed
graphs under some assumptions. Finally, we study some of the relationships between
the number of signed walks and the largest eigenvalue of a signed graph.

Keywords. Signed graph, Eigenvalue, Balanced clique number, Bipartite index, Signed
walk.

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1 Introduction

A signed graph Σ is a pair (G, σ), where G = (V, E) is an undirected graph, called the
underlying graph, and σ : E → {−1, +1} is the sign function. The adjacency matrix of a
signed graph Σ = (G, σ) is a symmetric matrix, denoted by A(Σ) and its (i, j)th entry is
defined as follows:

\[
a_{ij} = \begin{cases} 
\sigma(e_{ij}) & \text{if } v_i \sim v_j, \\
0 & \text{otherwise.}
\end{cases}
\]

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The spectrum and the spectral radius of $\Sigma$ are the spectrum and the spectral radius of $A(\Phi)$ and denoted by $\text{spec}(\Sigma)$ and $\rho(\Sigma)$, respectively. Let $(G, 1)$ and $(G, -1)$ denote the signed graphs with all the edge signs are equal to 1 and $-1$, respectively. For more details about the notion of signed graphs, we refer to [3, 7, 18].

The sign of a cycle (with some orientation) $C = v_1v_2 \ldots v lv_1$, denoted by $\sigma(C)$, is defined as the product of the signs of its edges, that is

$$\sigma(C) = \sigma(e_{12})\sigma(e_{23}) \cdots \sigma(e_{(l-1)l})\sigma(e_{1l}).$$

A cycle $C$ is neutral if $\sigma(C) = 1$, and a signed graph is balanced if all its cycles are neutral. A function from the vertex set of $G$ to the set $\{1, -1\}$ is called a switching function. We say that, two signed graphs $\Sigma_1 = (G, \sigma_1)$ and $\Sigma_2 = (G, \sigma_2)$ are switching equivalent, written as $\Sigma_1 \sim \Sigma_2$, if there is a switching function $\eta: V \to \{1, -1\}$ such that

$$\sigma_2(e_{ij}) = \eta(v_i)^{-1}\sigma_1(e_{ij})\eta(v_j).$$

The switching equivalence of two signed graphs can be defined in the following equivalent way: Two signed graphs $\Sigma_1 = (G, \sigma_1)$ and $\Sigma_2 = (G, \sigma_2)$ are switching equivalent, if there exists a diagonal matrix $D_\eta$ with diagonal entries from $\{1, -1\}$ such that

$$A(\Sigma_2) = D_\eta^{-1}A(\Sigma_1)D_\eta. \quad (1.1)$$

Switching equivalence preserves connectivity and balance.

A complete subgraph of a simple graph $G$ is a called a clique in $G$. The clique number of $G$, denoted by $\omega(G)$, is the number of vertices in a largest clique in $G$. Let $S^+_n = \{x(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum x_i = 1, x_i \geq 0 \text{ for } i = 1, \ldots, n\}$ and $S^\pm_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n |x_i| = 1\}$.

In the seminal paper [10], Motzkin and Straus proved the following result.

**Theorem 1.1** ([10, Theorem 1]). Let $G$ be a graph on $n$ vertices with the clique number $\omega(G)$. If $A(G)$ is the adjacency matrix $G$, then

$$\max_{x \in S^+_n} x^T A(G)x = \frac{\omega(G) - 1}{\omega(G)}.$$

Using this result, Nikiforov came up with a proof of the following theorem, which was conjectured by Edwards and Elphick [5].

**Theorem 1.2** ([11, Theorem 2.1]). Let $G$ be a graph with $n$ vertices and $m$ edges. Let $\lambda_1(G)$ be the largest eigenvalue of $A(G)$, and $\omega(G)$ be the clique number of $G$. Then

$$\lambda^2_1(G) \leq 2m\frac{\omega(G) - 1}{\omega(G)}.$$
The balanced clique number of a signed graph $\Sigma$, denoted by $\omega_b(\Sigma)$, is the maximum order of a balanced complete subgraph [16]. Wang et al. extended the Motzkin-Straus theorem for the signed graphs. The MS-index of a signed graph $\Sigma = (G, \sigma)$, denoted by $\mu(\Sigma)$, is defined as follows [16]:

$$\mu(\Sigma) = \max_{x \in S^+_n} \sum_{i \sim j} \sigma(e_{ij})x_ix_j = \max_{x \in S^+_n} \frac{x^T A(\Sigma)x}{2}.$$ 

**Theorem 1.3 ([16, Theorem 5]).** Given a signed graph $\Sigma$ with vertices $1, 2, \ldots, n$. Let $A(\Sigma)$ be its signed adjacency matrix and $\omega_b(\Sigma)$ be its balanced clique number. Let $S^+_n$ be the simplex in $\mathbb{R}^n$ given by $\sum_{i=1}^n |x_i| = 1$. Then

$$\mu(\Sigma) = \frac{1}{2} \left( \frac{\omega_b(\Sigma) - 1}{\omega_b(\Sigma)} \right).$$

As a consequence of the above theorem, Wang et al. extended Cvetković’s theorem for the signed graphs.

**Theorem 1.4 ([16, Proposition 5]).** Let $\Sigma$ be a signed graph with $n$ vertices, and balanced clique number $\omega_b(\Sigma)$. Let $\lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_n(\Sigma)$ be the eigenvalues of the adjacency matrix of $\Sigma$. Then

$$\lambda_1(\Sigma) \leq n \left( 1 - \frac{1}{\omega_b(\Sigma)} \right).$$

In the same paper, Wang et al. explained the difficulties in extending Nikiforov’s theorem (Theorem 1.2) for the signed graphs and mentioned this as an open problem. One of the main objectives of this article is to give a proof of a strengthened version of this open problem for the signed graph (see Theorem 3.3). We give an alternate proof for Theorem 1.4 too. As a byproduct, we derive a spectral lower bound for the edge bipartiteness of a simple graph and deduce the upper bounds for the least eigenvalues of unsigned graphs obtained by Constantine [4]. The edge bipartiteness of an unsigned graph $G$ is the least number of edges whose deletion yields a bipartite graph [6]. These results are given in Section 3.

Let $K_n$ denote the complete graph on $n$ vertices. In [4], Bollobás and Nikiforov proposed the following conjecture. If $G$ is a $K_{r+1}$-free graph on at least $r + 1$ vertices with $m$ edges, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $A(G)$, then $\lambda_1^2 + \lambda_2^2 \leq 2m \left( \frac{r-1}{r} \right)$. Recently Lin et al. [9] confirmed this conjecture for the triangle free graphs. We show that the signed graph version of this conjecture need not be true for the signed graphs even for the triangle free graphs. We prove the signed analogue of this conjecture is true for the triangle free graphs whose largest eigenvalue is the spectral radius. This is done in Section 4.

Given a unsigned graph $G$, a $k$-walk is a sequence of vertices $v_1, \ldots, v_k$ of $G$ such that $v_i$ is adjacent to $v_{i+1}$ for all $i = 1, \ldots, k - 1$. The length of a walk is the number of edges it has, counting repeated edges as many times as they appear. So the length of a $k$-walk is $k - 1$. Let $w_r(G)$ be the number of $r$-walks in $G$. Note that, if $G$ is a unsigned graph, then $w_r(G) = e^TA^{r-1}(G)e$, where $e = (1,1,\ldots,1)^T$. Nikiforov proved the following result.
connecting the number of walks in \( G \), chromatic number of \( G \) and the spectral radius of the adjacency matrix \( A(G) \) of \( G \).

**Theorem 1.5** ([13 Theorem 5 & Theorem 14]). Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( A(G) \) be its adjacency matrix and \( \omega(G) \) be its clique number. Let \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \) be the eigenvalues of the adjacency matrix of \( G \). Then for \( r > 0 \) and odd \( q > 0 \),

\[
\frac{w_{q+r}(G)}{w_q(G)} \leq \lambda_1^r(G) \leq w_r(G) \frac{\omega(G) - 1}{\omega(G)}.
\]

In Section 5 we extend the above result for the signed graphs.

## 2 Preliminaries

Next, we collect the needed preliminaries about doubly stochastic matrices. For more details on related knowledge, we refer the reader to [2, 19]. A non-negative square matrix is **doubly stochastic** if the sum of the entries in every row and every column is 1, and it is **doubly substochastic** if the sum of the entries in every row and every column is at most 1. A square matrix is called a **weak-permutation matrix** if every row and every column has at most one non-zero entry and all the non-zero entries (if any) are 1.

For a vector \( x \in \mathbb{R}^n \), let \( x^\downarrow \) denote the vector obtained by rearranging the coordinates of \( x \) in the decreasing order. Given two vectors \( x, y \in \mathbb{R}^n \), we say that \( x \) is **weakly majorised** by \( y \), denote by \( x \prec_w y \) if

\[
\sum_{j=1}^k x_j^\downarrow \leq \sum_{j=1}^k y_j^\downarrow \quad \text{for all } 1 \leq k \leq n.
\]

If \( x \prec_w y \) and \( \sum_{i=1}^n x_i^\downarrow = \sum_{i=1}^n y_i^\downarrow \), then we say that \( x \) is **majorized** by \( y \) and denote it by \( x \prec y \).

**Lemma 2.1** ([19 Theorem 3.9]). Let \( x, y \in \mathbb{R}^n_+ \). Then \( x \prec_w y \) if and only if there exists a doubly substochastic matrix \( A \) such that \( x = Ay \).

For \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( 1 \leq p < \infty \), define \( \|x\|_p = \left\{ \sum_{i=1}^n |x_i|^p \right\}^{\frac{1}{p}} \).

**Theorem 2.2** ([9 Theorem 2.1]). Let \( x = (x_1, x_2, \ldots, x_n) \), \( y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}_+^n \), such that \( x_i \) and \( y_i \) are in non-increasing order. If \( y \prec_w x \), then \( \|y\|_p \leq \|x\|_p \) for any real number \( p > 1 \), where equality holds if and only if \( x = y \).

**Theorem 2.3.** Suppose \( A \in \mathbb{R}^{n \times n} \) is symmetric. Let \( B \in \mathbb{R}^{m \times m} \) be a principal submatrix (obtained by deleting both \( i \)-th row and \( i \)-th column for some values of \( i \)). Suppose \( A \) has eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \) and \( B \) has eigenvalues \( \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m \). Then

\[
\lambda_k \leq \beta_k \leq \lambda_{n-m+k} \quad \text{for } k = 1, \ldots, m.
\]
3 Eigenvalues and Balanced clique number

In this section, we prove an open problem posed by Wang et al. [16], and as a corollary, we obtain some of the known classical bounds for the least eigenvalue of the adjacency matrix of unsigned graphs. We reprove the Motzkin-Strauss theorem for signed graphs and Stanley’s bound for signed graphs using techniques different from Wang et al. and Stanic, respectively. On a final note, we prove Turán’s theorem for signed graphs.

For a signed graph Σ, the frustration index, denoted by \( \epsilon(\Sigma) \), is the minimum number of edges to be deleted such that the resultant signed graph is balanced [1].

Let \( \Sigma = (G, \sigma) \) be a signed graph. An edge \( e \) of \( \Sigma \) is positive (resp. negative) if \( \sigma(e) = 1 \) (resp. \( \sigma(e) = -1 \)). For a signed graph \( \Sigma \), let \( m^+(\Sigma) \) denote the number of positive edges in \( \Sigma \), and \( m^-(\Sigma) \) denote the number of negative edges in \( \Sigma \).

Lemma 3.1. Let \( \Sigma \) be a signed graph with \( n \) vertices, \( m \) edges and frustration index \( \epsilon(\Sigma) \). If \( \Sigma \) and \( \Sigma' \) are switching equivalent, then \( m^+(\Sigma') \leq m - \epsilon(\Sigma) \).

Proof. Let \( H \) be the signed subgraph of \( \Sigma' \) consisting of all the positive edges of \( \Sigma' \). Then \( H \) is balanced, and hence \( \epsilon(\Sigma') \leq m^-(\Sigma') \). Thus \( m^+(\Sigma') \leq m - \epsilon(\Sigma) \), as \( \epsilon(\Sigma) = \epsilon(\Sigma') \).

Essentially Lemma 3.1 says that the maximum number of positive edges in the whole switching equivalence class of \( \Sigma = (G, \sigma) \) is bounded above by \( m - \epsilon(\Sigma) \).

Next, we prove a lemma which is crucial in the proofs of main theorems.

Lemma 3.2. Let \( \Sigma = (G, \sigma) \) be a signed graph with the balanced clique number \( \omega_b(\Sigma) \). If \( \Sigma \) and \( \Sigma' \) are switching equivalent, then \( \omega_b(\Sigma) = \omega_b(\Sigma') \).

Proof. Since switching in signed graphs preserves the balance of cycles, the balance of cliques must also be preserved.

In [16], Wang et al. proved the Motzkin-Strauss theorem and Wilf’s theorem for signed graphs. In the same paper extending Theorem 1.2 for the signed graphs is mentioned as an open problem. Next, we prove a stronger version of this open problem.

Theorem 3.3. Let \( \Sigma = (G, \sigma) \) be a signed graph with \( n \) vertices and \( m \) edges. If \( \lambda_1(\Sigma) \) is the largest eigenvalue of the adjacency matrix \( A(\Sigma) \), then

\[
\lambda_2^2(\Sigma) \leq 2(m - \epsilon(\Sigma)) \left(1 - \frac{1}{\omega_b(\Sigma)}\right),
\]

where \( \omega_b(\Sigma) \) and \( \epsilon(\Sigma) \) are the balanced clique number and frustration index of \( \Sigma \), respectively.

Proof. Let \( x = (x_1, \ldots, x_n) \) be an eigenvector corresponding to the eigenvalue \( \lambda_1(\Sigma) \) with \( \|x\|_2 = 1 \). Define \( y = (|x_1|, \ldots, |x_n|) \). Consider the switching function \( \eta : V(G) \to \{1, -1\} \) defined as follows:

\[
\eta(v_i) = \begin{cases} 
\text{sgn } x_i & \text{if } x_i \neq 0, \\
1 & \text{otherwise.}
\end{cases}
\]
Now, switch the signed graph $\Sigma = (G, \sigma)$ to $\Sigma' = (G, \sigma')$ by using the switching function $\eta$ as defined above. Then $A(\Sigma') = D_\eta^{-1} A(\Sigma) D_\eta$. Note that $\lambda_1(\Sigma) = \lambda_1(\Sigma')$ and $\omega_b(\Sigma) = \omega_b(\Sigma')$.

Now,

$$\lambda_1(\Sigma) = x^T A(\Sigma) x$$

$$= y^T D_\eta^{-1} A(\Sigma) D_\eta y$$

$$= y^T A(\Sigma') y$$

$$= 2 \sum_{i \sim j} \sigma'_{ij} y_i y_j$$

[$\sigma'_{ij}$ is the $(i, j)^{th}$ entry of $A(\Sigma')$]

$$= \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i y_j - \sum_{i \sim j, \sigma'_{ij} = -1} 2y_i y_j.$$ 

As all the entries of vector $y$ are non-negative, we have

$$\lambda_1(\Sigma) \leq \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i y_j,$$  

(3.1)

and hence

$$\lambda_1(\Sigma)^2 \leq \left( \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i y_j \right)^2.$$

Let $\Sigma'_+\, be the signed subgraph of $\Sigma'$ consisting of all the positive edges of $\Sigma'$. Let $m^+$ be the number of edges in $\Sigma'_+$, and $\omega(\Sigma'_+)$ be the clique number of $\Sigma'_+$. By the Cauchy-Schwartz inequality, we have

$$\left( \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i y_j \right)^2 \leq 2m^+ \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i^2 y_j^2.$$ 

Since all the edges of the signed subgraph $\Sigma'_+$ are positive and the vector $y$ is a unit eigenvector (that is, $\|y\|_2 = 1$), by Theorem 1.1, we have

$$\sum_{i \sim j, \sigma'_{ij} = 1} 2y_i^2 y_j^2 \leq \left( 1 - \frac{1}{\omega(\Sigma'_+)} \right).$$

Thus

$$\lambda_1(\Sigma)^2 \leq 2m^+ \left( 1 - \frac{1}{\omega(\Sigma'_+)} \right).$$

Since $\Sigma'_+$ is the subgraph of $\Sigma'$ containing all positive edges, by Lemma 3.1, we have $m^+ \leq m - \epsilon(\Sigma)$ and $\omega(\Sigma'_+) \leq \omega_b(\Sigma)$. Thus

$$\lambda_1^2(\Sigma) \leq 2(m - \epsilon(\Sigma)) \left( 1 - \frac{1}{\omega_b(\Sigma)} \right).$$
Next, we prove some of the known classical bounds on the least eigenvalue of the adjacency matrix of unsigned graphs using Theorem 3.3. The edge bipartiteness of an unsigned graph \( G \) is the least number of edges whose deletion yields a bipartite graph \( [6] \). The following is a result of Favaron et al. \([7]\), and the result was stated in a different form. We rewrite the bound in terms of edge bipartiteness.

**Corollary 3.4 ([7, Theorem 2.17]).** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( A(G) \) be its adjacency matrix and \( \omega(G) \) be its clique number. Let \( \epsilon_b(G) \) denote the bipartite index of the graph \( G \). Let \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \) be the eigenvalues of \( A(G) \). Then

\[
\lambda_2^2(G) \leq m - \epsilon_b(G).
\]

**Proof.** Let \( \Sigma = (G, -1) \) be the signed graph with all negative edges on \( G \). Then, by Theorem 3.3, we get

\[
\lambda_1^2(\Sigma) = \lambda_2^2(G) \leq 2(m - \epsilon(\Sigma)) \left( 1 - \frac{1}{\omega_b(\Sigma)} \right).
\]

In the signed graph \( \Sigma = (G, -1) \), a triangle is not balanced, so \( \omega_b(\Sigma) = 2 \). Note that \( \Sigma \) is balanced if and only if \( G \) is bipartite, thus \( \epsilon(\Sigma) = \epsilon_b(G) \). Substituting these in the expression above, gives the desired result.

From Corollary 3.4, we can also prove two other classical bounds on \( \lambda_n(G) \) for a graph \( G \).

**Corollary 3.5 ([4, Proposition 2]).** Let \( G \) be a graph with \( n \) vertices and \( m \) edges. Let \( A(G) \) be its adjacency matrix and \( \omega(G) \) be its clique number. Let \( \lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) \) be the eigenvalues of \( A(G) \).

1. If \( n = 2k \), then \( |\lambda_n(G)| \leq k \).
2. If \( n = 2k + 1 \), then \( |\lambda_n(G)| \leq \sqrt{k(k+1)} \).

**Proof.** From Corollary 3.4 we have

\[
\lambda_2^2(G) \leq m - \epsilon_b(G).
\]

By the definition of \( \epsilon_b(G) \), \( m - \epsilon_b(G) \) is the number of edges in a maximal bipartite subgraph of the graph \( G \). Let \( V_1 \) and \( V_2 \) be a bipartition of the vertex set of a maximal bipartite subgraph obtained by removing \( \epsilon_b(G) \) edges. Then

\[
|\lambda_n(G)| \leq \sqrt{m - \epsilon_b(G)} \leq \sqrt{|V_1||V_2|}.
\]

Now we split in two cases,

1. if \( |V_1| + |V_2| = n = 2k \), then by AM-GM inequality, \( \sqrt{|V_1||V_2|} \leq k \).
2. if \( |V_1| + |V_2| = n = 2k + 1 \), then let \( |V_1| = k + a \), \( |V_2| = k + 1 - a \), where \( a \) is an integer. Then \( \sqrt{|V_1||V_2|} = \sqrt{(k+a)(k+1-a)} = \sqrt{k(k+1) + a - a^2} \leq \sqrt{k(k+1)} \), as \( a \) is an integer.
The following theorem extends Turán’s inequality for the signed graphs in terms of the balanced clique number and frustration index.

**Theorem 3.6.** Let $\Sigma$ be a signed graph with $n$ vertices, $m$ edges, balanced clique number $\omega_b(\Sigma)$ and the frustration index $\epsilon(\Sigma)$. Then

$$m \leq \epsilon(\Sigma) + \frac{n^2}{2} \left( 1 - \frac{1}{\omega_b(\Sigma)} \right).$$

**Proof.** Since $\epsilon(\Sigma)$ is the frustration index, so there exists set $S$ of $\epsilon(\Sigma)$ edges such that the signed graph $\Sigma \setminus S$ is balanced. Let $\eta$ be the switching function which switches $\Sigma \setminus S$ to all positive signed graph. Let $\omega$ be the clique number of $\Sigma \setminus S$. By Lemma 3.1 and Turán’s theorem for unsigned graphs, we have

$$m^+(\Sigma \setminus S) \leq m - \epsilon(\Sigma) \leq \frac{n^2}{2} \left( 1 - \frac{1}{\omega} \right).$$

As $\omega \leq \omega_b(\Sigma)$, we get

$$m \leq \epsilon(\Sigma) + \frac{n^2}{2} \left( 1 - \frac{1}{\omega_b(\Sigma)} \right). \quad \square$$

For a simple graph $G$ on $m$ edges, Stanley [15] established that $\rho(A(G)) \leq \frac{1}{2}(-1 + \sqrt{1 + 8m})$. Later Nikiforov [12] proved the following improved bound

$$\rho(A(G)) \leq \sqrt{2 \left( 1 - \left\lfloor \frac{1}{2} + \sqrt{2m + \frac{1}{4}} \right\rfloor^{-1} \right)m}.$$

We extend Nikiforov’s bound for the signed graphs in the next theorem. The proof idea is similar to that of Nikiforov [12]. For a real number $x$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to $x$.

**Theorem 3.7.** Let $\Sigma = (G, \sigma)$ be a signed graph with $n$ vertices, $m$ edges, balanced clique number $\omega_b(\Sigma)$ and frustration index $\epsilon(\Sigma)$. Then

$$\lambda_1(\Sigma) \leq \sqrt{2(m - \epsilon(\Sigma)) \left( 1 - \left\lfloor \frac{1}{2} + \sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}} \right\rfloor^{-1} \right)}.$$

**Proof.** Let $\Omega$ be a balanced complete subgraph of $\Sigma$ with $\omega_b(\Sigma)$ vertices. Let $\eta'$ be a switching function on $\Omega$ which switches all the edges of $\Omega$ to positive. Define the switching function $\eta$ on $\Sigma$ as follows:

$$\eta(v_i) = \begin{cases} 
\eta'(v_i) & \text{if } v_i \in \Omega, \\
1 & \text{otherwise}
\end{cases}$$
Let $\Sigma'$ be the signed graph obtained from $\Sigma$ by applying the switching $\eta$. Then, by Lemma 3.1,

\[
\left( \frac{\omega_b(\Sigma)}{2} \right) \leq m^+(\Sigma') \leq m - \epsilon(\Sigma).
\]

By solving the above expression for $\omega_b(\Sigma)$, we get

\[
\omega_b(\Sigma) \leq \frac{1}{2} + \sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}}.
\]

The result follows by noting that $\omega_b(\Sigma)$ is a positive number and hence

\[
\omega_b(\Sigma) \leq \left\lfloor \frac{1}{2} + \sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}} \right\rfloor.
\]

Substituting this version in Theorem 3.3 proves the result.

As a corollary of above the theorem we get the following result of Stanić.

**Corollary 3.8 (\cite[Theorem 4.2]{14}).** Let $\Sigma = (G, \sigma)$ be a signed graph with $n$ vertices, $m$ edges and frustration index $\epsilon(\Sigma)$. Let $\lambda_1(\Sigma)$ be the largest eigenvalue of the adjacency matrix $A(\Sigma)$. Then $\lambda_1(\Sigma) \leq \sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}} - \frac{1}{2}$.

**Proof.** From the proof of the above theorem, we have

\[
\omega_b(\Sigma) \leq \frac{1}{2} + \sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}}.
\]

Substituting the above bound in Theorem 3.3 we get

\[
\lambda_1(\Sigma) \leq \sqrt{2(m - \epsilon(\Sigma)) \left(1 - \frac{1}{\omega_b(\Sigma)}\right)}
\]

\[
\leq \sqrt{2(m - \epsilon(\Sigma)) \left(\frac{1}{2} + \frac{1}{\sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}}}\right)}
\]

\[
= \frac{1}{2} + \sqrt{2(m - \epsilon(\Sigma)) + \frac{1}{4}}.
\]

\[
\square
\]

Let us recall the definition of the MS-index of a signed graph $\Sigma = (G, \sigma)$: $\mu(\Sigma) = \max_{x \in S^+_n} \sum_{i \sim j} \sigma(e_{ij})x_ix_j = \max_{x \in S^+_n} \frac{x^T A(\Sigma)x}{2}$. Next, we give an alternate proof for \cite[Theorem 4.3]{14}.

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Theorem 3.9 ([16], Theorem 5]). Given a signed graph $\Sigma$ with vertices $1, 2, \ldots, n$. Let $A(\Sigma)$ be its signed adjacency matrix and $\omega_b(\Sigma)$ be its balanced clique number. Let $S_n^\pm$ be the simplex in $\mathbb{R}^n$ given by $\sum_{i=1}^n |x_i| = 1$. Then

$$\mu(\Sigma) = \frac{1}{2} \frac{\omega_b(\Sigma) - 1}{\omega_b(\Sigma)}.$$  

Proof. Let $x = (x_1, \ldots, x_n) \in S_n^\pm$ be a vector such that $\mu(\Sigma) = \sum_{i \sim j} x_i x_j$. Define $y = (|x_1|, \ldots, |x_n|)$. Consider the switching function $\eta : V(G) \to \{1, -1\}$ defined as follows:

$$\eta(v_i) = \begin{cases} 
\text{sgn} \ x_i & \text{if } x_i \neq 0, \\
1 & \text{otherwise.}
\end{cases}$$

Now, switch the signed graph $\Sigma$ to $\Sigma'$ by using the switching function $\eta$ as defined above. Then $A(\Sigma') = D^{-1}_\eta A(\Sigma) D_\eta$. Note that $\mu(\Sigma) = \mu(\Sigma')$ and $\omega_b(\Sigma) = \omega_b(\Sigma')$. Now,

$$2\mu(\Sigma) = x^T A(\Sigma) x = y^T D^{-1}_\eta A(\Sigma) D_\eta y = y^T A(\Sigma') y = 2 \sum_{i \sim j} \sigma'_{ij} y_i y_j \quad \text{[} \sigma'_{ij} \text{is the } (i, j)\text{th entry of } A(\Sigma')\text{]}$$

$$= \sum_{i \sim j, \ \sigma'_{ij} = 1} 2y_i y_j - \sum_{i \sim j, \ \sigma'_{ij} = -1} 2y_i y_j.$$  

As all the entries of the vector $y$ are non-negative, we have

$$2\mu(\Sigma) \leq \sum_{i \sim j, \ \sigma'_{ij} = 1} 2y_i y_j.$$  

Let $\Sigma'_+\!$ be the signed subgraph of $\Sigma'$ consisting of all the positive edges of $\Sigma'$. Let $m^+$ be the number of edges in $\Sigma'_+$, and $\omega(\Sigma'_+\!)$ be the clique number of $\Sigma'_+\!$. By Theorem 1.1, we have

$$\sum_{i \sim j, \ \sigma'_{ij} = 1} 2y_i y_j \leq 2\mu(\Sigma') = \frac{\omega(\Sigma'_+) - 1}{\omega(\Sigma'_+) \!}.$$  

Since $\omega(\Sigma'_+) \leq \omega_b(\Sigma)$, we have

$$\mu(\Sigma) \leq \frac{1}{2} \frac{\omega_b(\Sigma) - 1}{\omega_b(\Sigma)}.$$  

Now, let $\Omega$ be a balanced complete subgraph of $\Sigma$ with $\omega_b(\Sigma)$ vertices. Let $\eta'$ be a switching function on $\Omega$ which switches all the edges of $\Omega$ to positive. Define the switching function $\eta$ on $\Sigma$ as follows:
\[ \eta(v_i) = \begin{cases} \eta'(v_i) & \text{if } v_i \in \Omega, \\ 1 & \text{otherwise.} \end{cases} \]

Let \( \Sigma'' \) be the signed graph obtained from \( \Sigma \) by applying the switching function \( \eta \). Let \( z = (z_1, z_2, \ldots, z_n)^T \) be the vector given by
\[ z_i = \begin{cases} \frac{1}{\omega_b(\Sigma)} & \text{if } v_i \in \Omega, \\ 0 & \text{otherwise.} \end{cases} \]

Then, we have
\[ z^T A(\Sigma') z = \frac{\omega_b(\Sigma) - 1}{\omega_b(\Sigma)} \leq 2\mu(\Sigma') = 2\mu(\Sigma), \]
and thus we have the result. \( \square \)

## 4 Bollobás and Nikiforov’s conjecture: Signed version

In this section, we prove Lin et al.’s result for the signed graph with an additional restriction, and we also give a counterexample for the result when the restriction is dropped. This example also shows that the conjecture posed by Nikiforov and Bollobás does not hold for signed graphs in general. Then we derive a bound for the largest eigenvalue of a signed graph in terms of the number of edges and the number of triangles in it.

In [3], Bollobás and Nikiforov proposed the following conjecture.

**Conjecture 4.1.** If \( G \) is a \( K_{r+1} \)-free graph of order at least \( r+1 \) with \( m \) edges, and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \) are the eigenvalues of \( A(G) \), then \( \lambda_1^2 + \lambda_2^2 \leq \frac{2}{r-1}m. \)

Recently Lin et al. [9] confirmed this conjecture for the triangle-free graphs.

**Theorem 4.1 ([9, Theorem 1.2]).** Let \( G \) be a triangle-free graph on \( n \) vertices and \( m \) edges with \( n \geq 3 \). Then \( \lambda_1^2 + \lambda_2^2 \leq m. \)

The following example shows that the above theorem, as well as the conjecture of Bollobás and Nikiforov, need not be true signed graphs. Let \( C_5 \) denote the cycle on 5 vertices with \( V(C_5) = \{v_1, v_2, v_3, v_4, v_5\} \). Consider the signed graph \( \Sigma = (C_5, \sigma) \) with \( \sigma \) assigns \(-1\) to the edge between the vertices \( v_1 \) and \( v_2 \), and 1 to the remaining edges. Then
\[
A(\Sigma) = \begin{bmatrix}
0 & -1 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0
\end{bmatrix}.
\]

The spectrum of \( A(\Sigma) \) is \( \{1.618, 1.618, -0.618, -0.618, -2\} \). It is easy to see that \( \lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) = 5.2358 > 5. \)

Nevertheless, we have the following result for the signed graph.
**Theorem 4.2.** Let $\Sigma$ be a balanced triangle-free signed graph with $n$ vertices and $m$ edges. If $\lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_{n-1}(\Sigma) \geq \lambda_n(\Sigma)$ are the eigenvalues of $A(\Sigma)$ and $\lambda_1(\Sigma) \geq |\lambda_n(\Sigma)|$, then

$$\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) \leq m.$$ 

We need the following result in the proof of the above theorem.

**Lemma 4.3.** Let $\Sigma = (G, \sigma)$ be a balanced triangle-free signed graph with $n$ vertices. Let $\lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_n(\Sigma)$ be the eigenvalues of the adjacency matrix of $\Sigma$. Then

$$\lambda_2(\Sigma) \geq 0.$$ 

**Proof.** Since $\Sigma$ is a balanced triangle-free signed graph, either $\Sigma$ is a signed graph with an induced negative triangle or its underlying graph $G$ is triangle-free.

If $\Sigma$ has an induced negative triangle, then its spectrum is $\{1, 1, -2\}$. By Theorem 2.3, we have $\lambda_2(\Sigma) \geq 0$.

If $\Sigma$ is triangle free signed graph, then it has an induced path on 3 vertices whose spectrum is $\{\sqrt{2}, 0, -\sqrt{2}\}$. Now by the Theorem 2.3, we have $\lambda_2(\Sigma) \geq 0$.

Thus in either of the cases, we have the result. $\square$

**Proof of Theorem 4.2.** Let the ordered triple $(n^+, n^-, n^0)$ denote the inertia of $\Sigma$, where $n^+$, $n^-$ and $n^0$ are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of $A(\Sigma)$, respectively.

Let $\lambda_1(\Sigma) \geq |\lambda_n(\Sigma)|$. Set $S^+ = \lambda_1^2 + \cdots + \lambda_n^2$, and $S^- = \lambda_{n-n^-+1}^2 + \cdots + \lambda_n^2$. Suppose $\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) > m$. Since $S^+ + S^- = 2m$, we have

$$\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) > m = \frac{S^+ + S^-}{2}$$

and we get,

$$\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) \geq 2(\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma)) - S^+ > S^- \geq 0,$$

$$\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) > S^-.$$ 

Now we construct two $n^-$-vectors $x$ and $y$ such that $x = (\lambda_1^2(\Sigma), \lambda_2^2(\Sigma), 0, \ldots, 0)^T$, $y = (\lambda_n^2(\Sigma), \lambda_{n-1}^2(\Sigma), \ldots, \lambda_{n-n^-+1}^2(\Sigma))^T$. Since $\lambda_1^2(\Sigma) + \lambda_2^2(\Sigma) > S^-$, we have $y \prec_w x$ and $x \neq y$. Set $p = \frac{3}{2}$, by Theorem 2.2 we have

$$\|x\|_{3/2}^{3/2} > \|y\|_{3/2}^{3/2},$$

that is,

$$\lambda_1^3(\Sigma) + \lambda_2^3(\Sigma) > |\lambda_n^3(\Sigma)| + |\lambda_{n-1}^3(\Sigma)| + \cdots + |\lambda_{n-n^-+1}^3(\Sigma)|.$$
Since we know that in a balanced triangle free signed graph,
\[ \sum_{i=1}^{n} \lambda_{i}^{3}(\Sigma) = 6t_{s}(\Sigma) = 6(t^{+}(\Sigma) - t^{-}(\Sigma)) < 0. \]

This implies that
\[ 6t_{s}(\Sigma) = \sum_{i=1}^{n} \lambda_{i}^{3}(\Sigma) > \lambda_{1}^{3}(\Sigma) + \lambda_{2}^{3}(\Sigma) + \lambda_{n}^{3}(\Sigma) + \lambda_{n-1}^{3}(\Sigma) + \cdots + \lambda_{n-n+1}^{3}(\Sigma) > 0. \]

This is a contradiction. Thus we have our desired result. \(\square\)

Let \(t_{s}(\Sigma)\) denote the number of positive triangles minus the number of negative triangles in \(\Sigma\). The technique used in the proof of Theorem 4.2 is similar to that of Lin et al. [9]. This technique works if the assumption of balanced triangle free is replaced by the assumption that the number of positive triangles is less than the number of negative triangles, that is \(t_{s}(\Sigma) \geq 0\). From this we can conclude that in a given signed graph \(\Sigma\), either \(\lambda_{1}(\Sigma)^{2} \leq m\) or \(\lambda_{n}(\Sigma)^{2} \leq m\), as either \(t_{s}(\Sigma) \leq 0\) or \(t_{s}(-\Sigma) \leq 0\) is always true.

In the following theorem, we give a different bound for the largest eigenvalue of a signed graph in terms of the total number of triangles present in the signed graph whenever the number of positive triangles is greater than the number of negative triangles in the signed graph.

**Theorem 4.4.** Let \(\Sigma\) be a signed graph with \(n\) vertices, \(m\) edges. Let \(\lambda_{1}(\Sigma) \geq \lambda_{2}(\Sigma) \geq \cdots \geq \lambda_{n-1}(\Sigma) \geq \lambda_{n}(\Sigma)\) be the eigenvalues of \(A(\Sigma)\). If \(t_{s}(\Sigma) \geq 0\), then
\[ \lambda_{2}(\Sigma) \leq m + \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}}. \]

**Proof.** Let the ordered triple \((n^{+}, n^{-}, n^{0})\) denote the inertia of \(\Sigma\), where \(n^{+}, n^{-}\) and \(n^{0}\) are the numbers (counting multiplicities) of positive, negative and zero eigenvalues of \(A(\Sigma)\), respectively.

Let \(S^{+} = \lambda_{1}^{2} + \cdots + \lambda_{n}^{2}\) and \(S^{-} = \lambda_{n-n+1}^{2} + \cdots + \lambda_{n}^{2}\). Suppose \(\lambda_{2}^{2}(\Sigma) > m + \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}}\). Since \(S^{+} + S^{-} = 2m\), we have
\[ \lambda_{2}^{2}(\Sigma) > m + \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}} = \frac{S^{+} + S^{-}}{2} + \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}} \]
and we get,
\[ \lambda_{2}^{2}(\Sigma) - \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}} \geq 2\left(\lambda_{1}^{2}(\Sigma) - \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}}\right) - S^{+} > S^{-} \geq 0, \]
\[ \lambda_{2}^{2}(\Sigma) - \left(6t_{s}(\Sigma)\right)^{\frac{3}{2}} > S^{-}. \]
Now we construct two \((n^+ + 1)\)-vectors \(x\) and \(y\) such that 
\[ x = (\lambda_1^2(\Sigma), 0, 0, \ldots, 0)^T, \quad y = (\lambda_n^2(\Sigma), \lambda_{n-1}^2(\Sigma), \ldots, \lambda_{n-n-1}^2(\Sigma), \left(6t_s(\Sigma)\right)^{\frac{4}{3}})^T. \]
Since \(\lambda_1^2(\Sigma) - \left(6t_s(\Sigma)\right)^{\frac{4}{3}} > S^-\), we have \(y \preceq_w x\) and \(x \neq y\). Set \(p = \frac{3}{2}\), by Theorem 2.2 we have
\[
\|x\|_{4/3} > \|y\|_{4/3},
\]
that is,
\[
\lambda_1^3(\Sigma) > |\lambda_n^3(\Sigma)| + |\lambda_{n-1}^3(\Sigma)| + \cdots + |\lambda_{n-n-1}^3(\Sigma)| + 6t_s(\Sigma).
\]
Since we know that in a signed graph,
\[
\sum_{i=1}^{n} \lambda_i^3(\Sigma) = 6t_s(\Sigma).
\]
This implies that
\[
6t_s(\Sigma) = \sum_{i=1}^{n} \lambda_i^3(\Sigma) > \lambda_1^3(\Sigma) + \lambda_n^3(\Sigma) + \lambda_{n-1}^3(\Sigma) + \cdots + \lambda_{n-n-1}^3(\Sigma) > 6t_s(\Sigma).
\]
This is a contradiction. Thus we have our desired result. \(\square\)

Note that Theorem 4.4 in case of unsigned graphs, simplifies to the inequality
\[
\lambda_1^2(G) \leq m + \left(6t(G)\right)^{\frac{4}{3}},
\]
which equivalent to the Theorem 1.2 in case of triangle free graphs, but it is different when the graph is not triangle free.

5 Signed walks and largest eigenvalue

A walk in a signed graph is positive if the number of its negative edges is even (including the repeating edges); otherwise, it is negative. In the same way we decide whether a cycle in a signed graph is positive or negative. Let \(w_k^+(i, j)\) denote the number of positive walks of length \(k - 1\) starting at \(i\) and terminating at \(j\). Define \(w_k^+(i) = w_k^+(i, i)\) for each \(1 \leq i \leq n\). Similarly \(w_k^-(i, j)\) and \(w_k^-(i)\) are defined in terms of the negative walks. Let \(w^+_k(\Sigma)\) and \(w^-_k(\Sigma)\) denote the number of positive walks and negative walks in signed graph \(\Sigma\) of length \(r - 1\), respectively.

For a signed graph \(\Sigma = (G, \sigma)\), the \((i, j)\)-th entry of \(A(\Sigma)^{r-1}\) is the difference between the number of positive and negative walks of length \(r - 1\) between the vertices \(v_i\) and \(v_j\) in \(\Sigma\). Hence the expression \(e^T A(\Sigma)^{r-1}e\) is the difference between the number of positive and negative walks of length \(r - 1\) in \(\Sigma\). Define \(w_k(\Sigma) = e^T A(\Sigma)^{r-1}e\).
The total number of positive and negative walks in Σ need not be switching invariant. We define the \( r \)-frustration index of signed graph Σ, denoted by \( \epsilon_r(\Sigma) \), to be the minimum number of negative walks of length \( r - 1 \) in the switching equivalence class of Σ. That is,

\[
\epsilon_r(\Sigma) = \min_{\Sigma' \sim \Sigma} w_r^- (\Sigma).
\]

Note that, \( \epsilon_2(\Sigma) = \epsilon(\Sigma) \), and thus \( \epsilon_r(\Sigma) \) generalizes notion of frustration index to signed walks of higher lengths.

**Theorem 5.1.** Let \( \Sigma = (G, \sigma) \) be a signed graph with \( n \) vertices and \( m \) edges. Let \( A(\Sigma) \) be its signed adjacency matrix and \( \omega_b(\Sigma) \) be its balanced clique number. Let \( w_r(G) \) denote the total number of walks in \( G \) and \( \epsilon_r(\Sigma) \) be the \( r \)-frustration index of \( \Sigma \). Let \( \lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_n(\Sigma) \) be the eigenvalues of the adjacency matrix of \( \Sigma \). Then

\[
\lambda_r^1(\Sigma) \leq (w_r(G) - \epsilon_r(\Sigma)) \left( 1 - \frac{1}{\omega_b(\Sigma)} \right).
\]

**Proof.** Let \( \Sigma' = (G, \sigma') \) be the signed graph obtained from \( \Sigma \) by applying the switching function \( \eta \) defined in Theorem 3.3. From the equation \( 3.1 \) we have \( \lambda_1(\Sigma) \leq \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i y_j \). Let \( \Sigma'_+ \) be the signed subgraph of \( \Sigma' \) spanned by the set of edges \( \sigma'_{ij} = 1 \). Let the number of \( r \)-walks in the all positive signed subgraph \( \Sigma'_+ \) be \( w_r(\Sigma'_+) \) and its clique number be \( \omega(\Sigma'_+) \). Then \( \lambda_1(\Sigma) \leq \sum_{i \sim j, \sigma'_{ij} = 1} 2y_i y_j \leq \lambda_1(\Sigma'_+) \). Now, By Theorem \( 1.5 \) we have

\[
\lambda_r^1(\Sigma) \leq \lambda_r^1(\Sigma'_+) \leq w_r(\Sigma'_+) \left( 1 - \frac{1}{\omega(\Sigma'_+)} \right).
\]

Since \( \Sigma'_+ \) is all positive signed subgraph of \( \Sigma' \), we have \( w_r(\Sigma'_+) \leq w_r(G) - \epsilon_r(\Sigma) \) and \( \omega_+ \leq \omega_b(\Sigma) \). We have,

\[
\lambda_r^1(\Sigma) \leq (w_r(G) - \epsilon_r(\Sigma)) \left( 1 - \frac{1}{\omega_b(\Sigma)} \right). \quad \square
\]

Here, we note that as a corollary of the above theorem when \( r = 1 \), we obtain Theorem 1.4 of Wang et al [16].

As a corollary of the above theorem, we can also get bounds for the least eigenvalue of the adjacency matrix of signed graphs.

**Corollary 5.2.** Let \( \Sigma = (G, \sigma) \) be a signed graph with \( n \) vertices and \( m \) edges. Let \( A(\Sigma) \) be its signed adjacency matrix and \( \omega_b(\Sigma) \) be its balanced clique number. Let \( w_r(G) \) denote the total number of walks in \( G \) and \( \epsilon_r(\Sigma) \) be the \( r \)-frustration index of \( \Sigma \). Let \( \lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_n(\Sigma) \) be the eigenvalues of the adjacency matrix of \( \Sigma \). Then

\[
|\lambda_n^r(\Sigma)| \leq (w_r(G) - \epsilon_r(\Sigma)) \left( 1 - \frac{1}{\omega_b(-\Sigma)} \right).
\]

**Proof.** In the theorem 5.1 if we replace the signed graph \( \Sigma \) by \( -\Sigma \) and noting that \( \lambda_1(-\Sigma) = |\lambda_n(\Sigma)| \), the result follows. \quad \square
Our next objective is to extend the left-hand side of the inequality of Theorem 1.5. Before that, we give a theorem that will be helpful.

**Theorem 5.3.** Let \( \Sigma \) be a signed graph of order \( n \) with eigenvalues \( \lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_n(\Sigma) \) and \( u_1, \ldots, u_n \) be the corresponding orthogonal unit eigenvectors. For every \( 1 \leq i \leq n \), let \( u_i = (u_{i1}, \ldots, u_{in}) \) and \( c_i = (\sum_{j=1}^{n} u_{ij})^2 \). Then

\[
  w_k(\Sigma) = \sum_{i=1}^{n} c_i \lambda_i^{k-1}(\Sigma).
\]

**Proof.** Since \( A(\Sigma) \) is symmetric matrix, by the definition of \( w_k(\Sigma) \) and the spectral theorem, we have

\[
  w_k(\Sigma) = e^T A(\Sigma)^{k-1} e = (Ue)^T D(\Sigma)(Ue).
\]
It is easy to see that, \( Ue = (\sum_{j=1}^{n} u_{1j}, \ldots, \sum_{j=1}^{n} u_{nj})^T \) and thus, the result follows. \( \square \)

**Theorem 5.4.** Let \( \Sigma = (G, \sigma) \) be a signed graph with \( n \) vertices and \( m \) edges. Let \( A(\Sigma) \) be its signed adjacency matrix. Let \( w_r(\Sigma) \) denote the number of signed \( r \)-walks in \( \Sigma \). Let \( \lambda_1(\Sigma) \geq \lambda_2(\Sigma) \geq \cdots \geq \lambda_n(\Sigma) \) be the eigenvalues of the adjacency matrix of \( \Sigma \). Let \( \rho(\Sigma) \) be the spectral radius of \( A(\Sigma) \). Then for all \( r > 0 \) and odd \( q > 0 \),

\[
  \frac{w_{q+r}(\Sigma)}{w_q(\Sigma)} \leq \rho^r(\Sigma).
\]

**Proof.** From Theorem 5.3 we have

\[
  \frac{w_{q+r}(\Sigma)}{w_q(\Sigma)} = \frac{\sum_{i=1}^{n} c_i \lambda_i^{q+r-1}(\Sigma)}{\sum_{i=1}^{n} c_i \lambda_i^{q-1}(\Sigma)} = \rho^r(\Sigma) \frac{\sum_{i=1}^{n} c_i \left( \frac{\lambda_i(\Sigma)}{\rho(\Sigma)} \right)^{q+r-1}}{\sum_{i=1}^{n} c_i \left( \frac{\lambda_i(\Sigma)}{\rho(\Sigma)} \right)^{q-1}} \leq \rho^r(\Sigma).
\]

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