Kaluza-Klein Picture of the World
and Global Solution of the Spectral Problem

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Abstract

We suggest that Kaluza-Klein idea may provide the global solution of the spectral problem in hadronic spectroscopy.

1 Introduction

The strong interactions are characterized by multi-particle production. The dynamics of the multi-particle systems with a necessity contains the so called many-body forces. Many-body forces are fundamental forces which take place in the multi-particle systems where the number of particles is greater than two, and they are responsible for the dynamics of the production processes. For example, the three-body forces are responsible for the dynamics of one-particle inclusive reactions; see Ref. [1] and references therein. A description of the many-body forces requires the use of multidimensional spaces. Therefore, seems it would be naturally to formulate the strong interactions theory in a multidimensional space from the beginning.

The idea to use the multidimensional spaces in fundamental physics is not new: famous works of Kaluza and Klein were the first ones where this idea has been elaborated. The original idea of Kaluza and Klein is based on the hypothesis that the input space-time is a \((4 + d)\)-dimensional space \(M_{(4+d)}\) which can be represented as a tensor product of the visible four-dimensional world \(M_4\) with a compact internal \(d\)-dimensional space \(K_d\)

\[ M_{(4+d)} = M_4 \times K_d. \]  

The compact internal space \(K_d\) is space-like one i.e. it has only spatial dimensions which may be considered as extra spatial dimensions of \(M_4\). An especial example of \(M_{(4+d)}\) is a space with the factorizable metric. In according with the tensor product structure of the space \(M_{(4+d)}\) the metric may be chosen in a factorizable form. This means that if \(z^M = \{x^\mu, y^m\}\), \((M = 0, 1, \ldots, 3 + d, \mu = 0, 1, 2, 3, m = 1, 2, \ldots, d)\), are local coordinates on \(M_{(4+d)}\) then the factorizable metric looks like

\[ ds^2 = \mathcal{G}_{MN}(z)dz^Mdz^N = g_{\mu\nu}(x)dx^\mu dx^\nu + \gamma_{mn}(x, y)dy^m dy^n, \]

where \(g_{\mu\nu}(x)\) is the metric on \(M_4\).

In the year 1921, Kaluza proposed a unification of the theory of gravity and the Maxwell theory of electromagnetism in four dimensions starting from the theory of gravity in five dimensions. He assumed that the five-dimensional space \(\mathcal{M}_5\) had to be a product
of a four-dimensional space-time $M_4$ and a circle $S_1$: $M_5 = M_4 \times S_1$. It was shown that the zero mode sector of the Kaluza model is equivalent to the four-dimensional theory which describes the Hilbert-Einstein gravity with a four-dimensional general coordinate transformations and the Maxwell theory of electromagnetism with a gauge transformations.

Recently some models with extra dimensions have been proposed to attack the electroweak quantum instability of the Standard Model known as hierarchy problem between the electroweak and gravity scales. However, it is obviously that the basic idea of the Kaluza-Klein scenario may be applied to any model in Quantum Field Theory. As example, let us consider the simplest case of $(4+d)$-dimensional model of scalar field with the action

$$S = \int d^{4+d}z \sqrt{-G} \left[ \frac{1}{2} \left( \partial_{\mu} \Phi \right)^2 - \frac{m^2}{2} \Phi^2 + \frac{G_{(4+d)}}{4!} \Phi^4 \right],$$

(2)

where $G = \text{det} |G_{MN}|$, $G_{MN}$ is the metric on $M_{(4+d)} = M_4 \times K_d$, $M_4$ is pseudo-Euclidean Minkowski space-time, $K_d$ is a compact internal $d$-dimensional space with the characteristic size $R$. Let $\Delta_{K_d}$ be the Laplace operator on the internal space $K_d$, and $Y_n(y)$ are ortho-normalized eigenfunctions of the Laplace operator

$$\Delta_{K_d} Y_n(y) = -\frac{\lambda_n}{R^2} Y_n(y),$$

(3)

and $n$ is a (multi)index labeling the eigenvalue $\lambda_n$ of the eigenfunction $Y_n(y)$. A $d$-dimensional torus $T^d$ with equal radii $R$ is an especially simple example of the compact internal space of extra dimensions $K_d$. The eigenfunctions and eigenvalues in this special case look like

$$Y_n(y) = \frac{1}{\sqrt{V_d}} \exp \left( i \sum_{m=1}^d n_m y^m / R \right),$$

(4)

$$\lambda_n = |n|^2, \quad |n|^2 = n_1^2 + n_2^2 + \ldots + n_d^2, \quad n = (n_1, n_2, \ldots, n_d), \quad -\infty \leq n_m \leq \infty,$$

where $n_m$ are integer numbers, $V_d = (2\pi R)^d$ is the volume of the torus.

To reduce the multidimensional theory to the effective four-dimensional one we write a harmonic expansion for the multidimensional field $\Phi(z)$

$$\Phi(z) = \Phi(x, y) = \sum_n \phi^{(n)}(x) Y_n(y).$$

(5)

The coefficients $\phi^{(n)}(x)$ of the harmonic expansion (5) are called Kaluza-Klein (KK) excitations or KK modes, and they usually include the zero-mode $\phi^{(0)}(x)$, corresponding to $n = 0$ and the eigenvalue $\lambda_0 = 0$. Substitution of the KK mode expansion into action (2) and integration over the internal space $K_d$ gives

$$S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left( \partial_{\mu} \phi^{(0)} \right)^2 - \frac{m^2}{2} \left( \phi^{(0)} \right)^2 + \frac{g}{4!} \left( \phi^{(0)} \right)^4 + \sum_{n \neq 0} \left\{ \frac{1}{2} \left( \partial_{\mu} \phi^{(n)} \right) \left( \partial^{\mu} \phi^{(n)} \right)^* - \frac{m_n^2}{2} \phi^{(n)} \phi^{(n)*} \right\} + \frac{g}{4!} \left( \phi^{(0)} \right)^2 \sum_{n \neq 0} \phi^{(n)} \phi^{(n)*} \right\} + \ldots.$$  

(6)

For the masses of the KK modes one obtains

$$m_n^2 = m^2 + \frac{\lambda_n}{R^2},$$

(7)
and the coupling constant \( g \) of the four-dimensional theory is related to the coupling constant \( G_{(4+d)} \) of the initial multidimensional theory by the equation

\[
g = \frac{G_{(4+d)}}{V_d}, \tag{8}
\]

where \( V_d \) is the volume of the compact internal space of extra dimensions \( K_d \). The fundamental coupling constant \( G_{(4+d)} \) has dimension \([\text{mass}]^{-d}\). So, the four-dimensional coupling constant \( g \) is dimensionless one as it should be. Eqs. (7,8) represent the basic relations of Kaluza-Klein scenario. Similar relations take place for other types of multidimensional quantum field theoretical models. From four-dimensional point of view we can interpret each KK mode as a particle with the mass \( m_n \) given by Eq. (7). We see that in according with Kaluza-Klein scenario any multidimensional field contains an infinite set of KK modes, i.e. an infinite set of four-dimensional particles with increasing masses, which is called the Kaluza-Klein tower. Therefore, an experimental observation of series KK excitations with a characteristic spectrum of the form (7) would be an evidence of the existence of extra dimensions. So far the KK partners of the particles of the Standard Model have not been observed. In the Kaluza-Klein scenario this fact can be explained by a microscopic small size \( R \) of extra dimensions \((R < 10^{-17} \text{ cm})\); in that case the KK excitations may be produced only at super-high energies of the scale \( E \sim 1/R > 1 \text{ TeV} \). Below this scale only homogeneous zero modes with \( n = 0 \) are accessible ones for an observation in recent high energy experiments. That is why, there is a hope to search the KK excitations at the future LHC and other colliders.

We have calculated early \cite{2}

\[
\frac{1}{R} = 41.481 \text{ MeV}, \tag{9}
\]

or

\[
R = 24.1 \text{ GeV}^{-1} = 4.75 \times 10^{-13} \text{ cm}. \tag{10}
\]

If we relate the strong interaction scale with the pion mass

\[
G_{(4+d)} \sim \frac{10}{[m_\pi]^d}, \tag{11}
\]

then

\[
g \sim \frac{10}{(2\pi m_\pi R)^d}, \tag{12}
\]

end

\[
g(d = 1) \sim 0.5.
\]

On the other hand

\[
geff \equiv g_{\pi NN} \exp(-m_\pi R) \sim 0.5, \quad \left(g_{\pi NN}^2/4\pi = 14.6\right). \tag{13}
\]

So, \( R \) has a clear physical meaning: the size \( R \) just corresponds to the scale of distances where the strong Yukawa forces in strength come down to the electromagnetic ones. Moreover,

\[
M \sim R^{-1} \left(M_{\text{Pl}}/R^{-1}\right)^{2/(d+2)} |_{d=6} \sim 5 \text{ TeV}. \tag{14}
\]

Mass scale \( M \) is just the scale accepted in the Standard Model, and this is an interesting observation as well.
2 Peculiarities of Kaluza-Klein excitations

From the formula for the masses of the KK modes

\[ m_n = \sqrt{m^2 + \frac{n^2}{R^2}} \]

one obtains

\[ m_n = m + \delta m_n, \quad \delta m_n = \frac{n^2}{2mR^2}, \quad n << mR, \quad (15) \]

and this just corresponds to the spectrum of potential box with the size which is equal to the size of internal compact extra space. In other case we have

\[ m_n = n\omega + \delta m_n, \quad \delta m_n = \frac{m\alpha^2}{2n}, \quad \omega \equiv \frac{1}{R}, \quad \alpha^2 \equiv mR, \quad \alpha^2 << n, \quad (16) \]

and here we come to the (quasi)oscillator (quasi, because \( n \) instead of \( n + 1/2 \) for one-dimensional case) and (quasi)Coulomb (quasi, because \( 1/n \) instead of \( 1/n^2 \) and \( \alpha^2 = mR \) instead of \( \alpha^2 = (1/137)^2 \) spectra). Clearly, we can neglect the (quasi)Coulomb contribution in the region \( n >> \alpha^2 \equiv mR \).

It is a very remarkable fact that KK modes of relativistic origin, being made with a quantization of finite moving in the space of extra dimensions, interpolate the non-relativistic spectrum of a potential box and the oscillator spectrum.

The spectrum of two (\( a \) and \( b \))-particle compound system is defined in fundamental (input) theory by the formula

\[ M_{n}^{ab} = m_a + m_b + \delta m_{ab}^n(m_a, m_b, G_{4+d}). \quad (17) \]

The goal of the fundamental theory is to calculate \( \delta m_{ab}^n(m_a, m_b, G_{4+d}) \). We have not the solution of that problem in strong interaction theory because this is significantly non-perturbative problem. However, in the framework of Kaluza-Klein approach we can rewrite the above formula in an equivalent form

\[ M_{n}^{ab} = m_{a,n} + m_{b,n} + \delta m_{ab,n}(m_{a,n}, m_{b,n}, g), \quad (18) \]

where \( m_{a,n}, m_{b,n} \) are KK modes of particles \( a \) and \( b \), and we can calculate using four-dimensional perturbation theory for quantity \( \delta m_{ab,n}(m_{a,n}, m_{b,n}, g) \). Moreover, because \( \delta m_{ab,n}(m_{a,n}, m_{b,n}, g) << m_{a(b),n} \), we can put with a high accuracy

\[ M_{ab}^n \approx m_{a,n} + m_{b,n}, \quad (19) \]

and this fact allows one to formulate a global solution of the spectral problem in hadronic spectroscopy.

3 On global solution of the spectral problem

According to Kaluza and Klein we suggest that the input (fundamental) space-time \( \mathcal{M}_{(4+d)} \) is represented as

\[ \mathcal{M}_{(4+d)} = M_4 \times \mathcal{K}_d. \]
Let $\lambda_n$ are characteristic numbers of the Laplace operator on $\mathcal{K}_d$ with a characteristic size $R_{\mathcal{K}}$

$$\Delta_{\mathcal{K}_d} Y_n(y) = -\frac{\lambda_n}{R_{\mathcal{K}}^2} Y_n(y).$$

Let $\lambda_{\mathcal{K}}$ be the set of all characteristic numbers of the Laplace operator

$$\lambda_{\mathcal{K}} \equiv \left\{ \lambda_n : n \in \mathbb{Z}^d \equiv \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \right\}. \quad (20)$$

There is one-to-one correspondence

$$\mathcal{K} \leftrightarrow (R_{\mathcal{K}}, \lambda_{\mathcal{K}}).$$

Let us consider a compound hadronic system $h$ which may decay into some channel

$$h \to a + b + \cdots + c. \quad (21)$$

We introduce the spectral mass function of the given channel by the formula

$$M_{h}^{a\cdots c}(R_{\mathcal{K}}, \lambda_{n_a}, \lambda_{n_b}, \cdots, \lambda_{n_c}) = \sqrt{m_a^2 + \frac{\lambda_{n_a}}{R_{\mathcal{K}}^2}} + \sqrt{m_b^2 + \frac{\lambda_{n_b}}{R_{\mathcal{K}}^2}} + \cdots + \sqrt{m_c^2 + \frac{\lambda_{n_c}}{R_{\mathcal{K}}^2}}. \quad (22)$$

Now we build the Kaluza-Klein tower:

$$t_{h}^{a\cdots c}(\mathcal{K}) = t_{h}^{a\cdots c}(R_{\mathcal{K}}, \lambda_{\mathcal{K}}) \equiv \left\{ M_{h}^{a\cdots c}(R_{\mathcal{K}}, \lambda_{n_a}, \lambda_{n_b}, \cdots, \lambda_{n_c}) : \lambda_{n_i} \in \lambda_{\mathcal{K}} \right\}, \quad (23)$$

$$(i = a, b, \ldots, c).$$

After that we build the Kaluza-Klein town as a union of the Kaluza-Klein towers corresponding to all possible decay channels of the hadronic system $h$

$$T_h(\mathcal{K}) = T_h(R_{\mathcal{K}}, \lambda_{\mathcal{K}}) \equiv \bigcup_{\{ab\cdots c\}} t_{h}^{a\cdots c}(R_{\mathcal{K}}, \lambda_{\mathcal{K}}). \quad (24)$$

We state:

$$M_h \in T_h(\mathcal{K}). \quad (25)$$

Let $\mathcal{H}$ be the set of all possible physical hadronic states. We build the hadronic Kaluza-Klein country by the formula

$$C_{\mathcal{H}}(\mathcal{K}) \equiv \bigcup_{h \in \mathcal{H}} T_h(\mathcal{K}). \quad (26)$$

The whole spectrum of all possible physical hadronic states we denote $M_{\mathcal{H}}$

$$M_{\mathcal{H}} \equiv \left\{ M_h : h \in \mathcal{H} \right\}. \quad (27)$$

We state:

$$M_{\mathcal{H}} \in C_{\mathcal{H}}(\mathcal{K}). \quad (28)$$

The formulae (25) and (28) provide the global solution of the spectral problem in hadronic spectroscopy.
Here we have to make some clarifying remarks. First of all, in the construction of the global solution among all possible decay channels of the hadronic system \( h \) there have to be taken into account only those channels which contain the fundamental particles and their different multi-particle compound systems in the final states, as it should be. An appearance of non-zero KK modes of the fundamental particles and their compound systems in the final states of the decay channels is forbidden by the construction. For example, the decay channel

\[
h \rightarrow a^* + b + \cdots + c,
\]

where \( a^* \) is a non-zero KK mode of the fundamental particle \( a \), cannot be used in the construction. The decay channel

\[
h \rightarrow A + b + \cdots + c,
\]

where \( A \) is some multi-particle compound system which may decay into some channel with the fundamental particles \( a_i (i = 1, 2, \ldots k) \) in the final state

\[
A \rightarrow a_1 + a_2 + \cdots + a_k,
\]

is admissible one by the construction. But the decay channel

\[
h \rightarrow A^* + b + \cdots + c,
\]

where \( A^* \) denote some non-zero KK mode of \( A \), is forbidden. In other words, the underlying physical principle in the construction of the global solution was the principle of non-observability of non-zero KK modes of the fundamental particles and their compound systems. According to that principle non-zero KK modes of the fundamental particles may manifest themselves only virtually during an interaction, for example, when they are staying in a compound system. Non-zero KK modes of the fundamental particles living in a compound system define the main properties of a compound system such as the mass and the life time of the system. As mentioned above, an interaction of KK modes is weak, therefore we can calculate with a high accuracy the mass of a compound system as a simple sum of the masses of KK modes. Moreover, weakly interacting KK modes result very narrow widths of the compound states, and this phenomenon is observed at the recent experiments. If we say about the broad peaks in the hadronic spectra we interpret them as an envelope of the narrow peaks predicted by Kaluza-Klein scenario. Really, the dynamics of the compound systems decays is physically transparent: Non-zero KK modes of the constituents make a transition to zero KK modes, and we observe zero KK modes as the decay products.

We shown in the previous section that non-zero KK modes look like the states of a particle in confine potentials. Such particle might be considered as a quasi-particle which cannot be observed without the destroying a confine potential. A quasi-particle becomes a real particle by a transition of a non-zero KK mode to a zero KK mode which is equivalent to the destroying a confine potential, and we observe a zero KK mode i.e. real fundamental particle as a decay product. This consideration justifies the underlying physical principle in the construction of the global solution. In fact, we present here quite a new look on the Kaluza-Klein picture as a whole.

### 4 One Comment

In papers \[2 3 4 5 6 7\] we have verified this global solution on the set of experimental data with two-nucleon system, two-pion system, three-pion system, strange mesons,
charmed and charmed-strange mesons and found out that the solution accurately described the experimentally observed hadronic spectra. At that we have used the simplest form of torus for the internal compact extra space and considered only diagonal elements in Kaluza-Klein towers. In fact, we have established the non-trivial physical principle according to which KK modes of decay products preferably paired up in compound system when they lived on one and the same storey in Kaluza-Klein tower. However, there are an exceptional cases. For example, $\rho$ and $\omega$ mesons appear as non-diagonal elements of the Kaluza-Klein towers:

$$m_{\rho} \in M_{\pi^+ \pi^2}^{\pi^+ \pi^2} = \sqrt{m_{\pi^+}^2 + \frac{n^2}{R^2}} + \sqrt{m_{\pi^2}^2 + \frac{m^2}{R^2}},$$  \hspace{1cm} (33)$$

$$M_{\pi^+ \pi^-}^{\pi^+ \pi^-} (n_{\pi^+} = 12, m_{\pi^-} = 4) = 766.97\text{MeV}, \hspace{1cm} M_{\pi^+ \pi^-}^{\pi^+ \pi^-} (n_{\pi^+} = 13, m_{\pi^-} = 4) = 773.85\text{MeV},$$

$$M_{\pi^0 \pi^0}^{\pi^0 \pi^0} (n = 13, m = 4) = 769.78\text{MeV}, \hspace{1cm} M_{\pi^+ \pi^0}^{\pi^+ \pi^0} (n_{\pi^+} = 13, m_{\pi^0} = 4) = 770.92\text{MeV},$$

and

$$m_{\omega} \in M_{\pi^+ \pi^- \pi^0}^{\pi^+ \pi^- \pi^0} = \sqrt{m_{\pi^+}^2 + \frac{n^2}{R^2}} + \sqrt{m_{\pi^-}^2 + \frac{m^2}{R^2}} + \sqrt{m_{\pi^0}^2 + \frac{k^2}{R^2}},$$  \hspace{1cm} (34)$$

$$M_{\pi^+ \pi^- \pi^0}^{\pi^+ \pi^- \pi^0} (n_{\pi^+} = 5, m_{\pi^-} = 6, k_{\pi^0} = 5) = 782.80\text{MeV}.$$ 

In general, as it follows from the observed hadron spectrum, the non-diagonal elements of the Kaluza-Klein towers are physically suppressed.

5 Conclusion

We would like to especially emphasize that one simple formula with one fundamental constant described 120 experimentally observed hadronic states which distributed as 43 two-nucleon states, 29 two-pion states, 9 three-pion states, 25 strange states and 14 charmed and charmed-strange states. Obviously, this is an impressive fact. New recent experimental data presented at last Xth International Conference on Hadron Spectroscopy HADRON ’03 (August 31–September 6, 2003, Aschaffenburg, Germany) are in excellent agreement with the global solution constructed here, and this is a very impressive fact as well [8].

The architecture of the hadronic Kaluza-Klein towns is unambiguously defined by an internal compact extra space with its geometry and shapes, and we have to learn much more about the geometry and shapes of a compact internal extra space. However, one very important point in Kaluza-Klein picture is established now in a reliable way: The size of the internal compact extra space define the global characteristics of the hadronic spectra while the masses of the constituents are the fundamental parameters of the compound systems which the elements of the global structures being. A knowledge of the true internal compact extra space is a knowledge of the Everything that is the God. Our consideration made above shown that we found out a good approximation to the true internal extra space. In our opinion the experimentally observed hadronic spectra reveal the existence of extra dimensions and confirm the Kaluza-Klein picture of the world, which allow us to construct the global solution of the spectral problem in hadronic spectroscopy.
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