Matrix superharmonic priors for Bayes estimation under matrix quadratic loss

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Abstract

We investigate Bayes estimation of a normal mean matrix under the matrix quadratic loss, which is viewed as a class of loss functions including the Frobenius loss and quadratic loss for each column. First, we derive an unbiased estimate of risk and show that the Efron–Morris estimator is minimax. Next, we introduce a notion of matrix superharmonicity for matrix-variate functions and show that it has analogous properties with usual superharmonic functions, which may be of independent interest. Then, we show that the generalized Bayes estimator with respect to a matrix superharmonic prior is minimax. We also provide a class of matrix superharmonic priors that include the previously proposed generalization of Stein’s prior. Numerical results demonstrate that matrix superharmonic priors work well for low rank matrices.

1 Introduction

Suppose that we have a matrix observation $X \in \mathbb{R}^{n \times p}$ whose entries are independent normal random variables $X_{ij} \sim N(M_{ij}, 1)$, where $n - p - 1 > 0$ and $M \in \mathbb{R}^{n \times p}$ is an unknown mean matrix. By using the notation of matrix-variate normal distributions, it is expressed as $X \sim N_{n,p}(M, I_n, I_p)$, where $I_n$ is the $n$-dimensional identity matrix. In this setting, we consider estimation of $M$ under the matrix quadratic loss [1, 3, 10, 17]:

$$L(M, \hat{M}) = (\hat{M} - M)^\top(\hat{M} - M),$$

which takes a value in the set of $p \times p$ positive semidefinite matrices. Namely, the risk function of an estimator $\hat{M} = \hat{M}(X)$ is defined as

$$R(M, \hat{M}) = \mathbb{E}_M[L(M, \hat{M}(X))],$$

and an estimator $\hat{M}_1$ is said to dominate another estimator $\hat{M}_2$ if $R(M, \hat{M}_1) \preceq R(M, \hat{M}_2)$ for every $M$, where $\preceq$ is the Löwner order: $A \preceq B$ means that $B - A$ is positive semidefinite. Thus, if $\hat{M}_1$ dominates $\hat{M}_2$, then

$$\mathbb{E}_M[\|\hat{M}_1 - M\|^2] \leq \mathbb{E}_M[\|\hat{M}_2 - M\|^2]$$

for every $M$ and $c \in \mathbb{R}^p$. In particular, each column of $\hat{M}_1$ dominates that of $\hat{M}_2$ as an estimator of the corresponding column of $M$ under quadratic loss. In the context of multivariate linear

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regression, improved estimation of the regression coefficient matrix under the matrix quadratic loss implies improved estimation of mean response for any value of the explanatory variables.

Whereas we adopt the matrix quadratic loss in this study, most existing studies on estimation of a normal mean matrix used the Frobenius loss:

$$l(M, \hat{M}) = \|\hat{M} - M\|_F^2 = \sum_{a=1}^{n} \sum_{i=1}^{p} (\hat{M}_{ai} - M_{ai})^2,$$

which is the trace of the matrix quadratic loss: $l(M, \hat{M}) = \text{tr } L(M, \hat{M})$. When $n - p - 1 > 0$, Efron and Morris defined an estimator

$$\hat{M}_{EM} = X \left( I_p - (n - p - 1)(X^T X)^{-1} \right), \tag{1}$$

which is viewed as a matrix version of the James–Stein estimator. They showed that $\hat{M}_{EM}$ is minimax and dominates the maximum likelihood estimator $\hat{M} = X$ under the Frobenius loss. Note that $\hat{M}_{EM}$ does not change the singular vectors but shrinks the singular values of $X$ towards zero. Motivated from this property, Matsuda and Komaki developed a singular value shrinkage prior

$$\pi_{SVS}(M) = \text{det}(M^T M)^{-(n-p-1)/2}, \tag{2}$$

which is viewed as a matrix version of Stein’s prior for a normal mean vector. They proved that this prior is superharmonic and thus the generalized Bayes estimator with respect to $\pi_{SVS}$ is minimax and dominates the maximum likelihood estimator under the Frobenius loss. Both the Efron–Morris estimator and the generalized Bayes estimator with respect to $\pi_{SVS}$ have large risk reduction when $M$ is close to low rank, because they shrink the singular values separately.

In this study, we investigate Bayes shrinkage estimation of a normal mean matrix under the matrix quadratic loss. First, we derive an unbiased estimate of matrix quadratic risk for general estimators and prove that the Efron–Morris estimator is minimax. Next, we introduce a notion of matrix superharmonicity for matrix-variate functions and show that it has analogous properties with usual superharmonic functions, which may be of independent interest. Then, we show that the generalized Bayes estimator with respect to a matrix superharmonic prior is minimax. For a normal mean vector ($p = 1$), it reduces to the result by Stein on quadratic loss. We also provide a class of matrix superharmonic priors, which includes the singular value shrinkage prior. Finally, we give several numerical results, which demonstrate that matrix superharmonic priors work well for low rank matrices.

2 Estimation under matrix quadratic loss

In this section, we provide preliminary results on estimation of a normal mean matrix under the matrix quadratic loss.

2.1 Definition of minimaxity

Since the matrix quadratic loss is only partially ordered, the definition of minimaxity under this loss is not straightforward. In this paper, we adopt the following definition.
Definition 2.1. An estimator $\hat{M}$ is said to be minimax under the matrix quadratic loss if
\[ \sup_M c^\top R(M, \hat{M})c = \inf_{\hat{M}} \sup_M c^\top R(M, \hat{M})c \]
for every $c \in \mathbb{R}^p$.

Namely, we define $\hat{M}$ to be minimax under the matrix quadratic loss if $\hat{M}c$ is a minimax estimator of $Mc$ under quadratic loss for every $c$, because $c^\top R(M, \hat{M})c = E_M[\|\hat{M}c - Mc\|^2]$. In particular, since $Xc$ is a minimax estimator of $Mc$ for every $c$, the maximum likelihood estimator $\hat{M} = X$ is minimax under the matrix quadratic loss with constant risk $R(M, \hat{M}) = nI_p$. Thus, any estimator that dominates the maximum likelihood estimator is also minimax.

2.2 Unbiased estimate of risk

Here, we derive an unbiased estimate of the matrix quadratic risk. Our derivation is based on Stein’s lemma for matrix-variate normal distributions expressed by a matrix version of divergence.

Definition 2.2. For a function $g : \mathbb{R}^{n \times p} \to \mathbb{R}^{n \times p}$, its matrix divergence $\tilde{\text{div}} g : \mathbb{R}^{n \times p} \to \mathbb{R}^{p \times p}$ is defined as
\[ (\tilde{\text{div}} g(X))_{ij} = \sum_{a=1}^{n} \frac{\partial}{\partial X_{ai}} g_{aj}(X). \]

Lemma 2.3. Let $X \sim N_{n,p}(M, I_n, I_p)$ and $g : \mathbb{R}^{n \times p} \to \mathbb{R}^{n \times p}$ be a weakly differentiable function. Then,
\[ E_M[(X - M)^\top g(X)] = E_M[\tilde{\text{div}} g(X)]. \]

Proof. By applying Stein’s lemma,
\[ E_M[(X - M)^\top g(X)] = E_M \left[ \sum_{a} (X_{ai} - M_{ai}) g_{aj}(X) \right] = E_M \left[ \sum_{a} \frac{\partial}{\partial X_{ai}} g_{aj}(X) \right] = E_M \left[ (\tilde{\text{div}} g(X))_{ij} \right]. \]

Theorem 2.4. The matrix quadratic risk of an estimator $\hat{M} = X + g(X)$ with a weakly differentiable function $g$ is given by
\[ R(M, \hat{M}) = nI_p + E_M[\tilde{\text{div}} g(X) + (\tilde{\text{div}} g(X))^\top + g(X)^\top g(X)]. \]

Proof. By using Lemma 2.3,
\[ R(M, \hat{M}) = E_M[(X + g(X) - M)^\top (X + g(X) - M)] = E_M[(X - M)^\top (X - M) + (X - M)^\top g(X) + g(X)^\top (X - M) + g(X)^\top g(X)] = nI_p + E_M[\tilde{\text{div}} g(X) + (\tilde{\text{div}} g(X))^\top + g(X)^\top g(X)]. \]
2.3 Minimaxity of Efron–Morris estimator

By using Theorem 2.4, we show that the Efron–Morris estimator (1) is minimax under the matrix quadratic loss.

Theorem 2.5. When \( n - p - 1 > 0 \), the Efron–Morris estimator \( \hat{M}_{EM} \) in (1) is minimax under the matrix quadratic loss.

Proof. Let \( g(X) = -(n - p - 1)X(X^\top X)^{-1} \) so that \( \hat{M}_{EM} = X + g(X) \).

To calculate \( \tilde{\text{div}} \ g(X) \), we use the formula 
\[
\frac{\partial}{\partial X_{ai}} (X(X^\top X)^{-1})_{aj} = \frac{\partial}{\partial X_{ai}} \sum_k X_{ak} ((X^\top X)^{-1})_{kj} \\
= ((X^\top X)^{-1})_{ij} - \sum_{k,l,m} X_{ak} ((X^\top X)^{-1})_{kl} (X^\top E_{ai} X + E_{ai}^\top X)(X^\top X)^{-1}
\]
where \( E_{ai} \in \mathbb{R}^{n \times p} \) is the matrix unit with 1 in the \((a, i)\)-th entry and 0s elsewhere. Then,
\[
\sum_a \frac{\partial}{\partial X_{ai}} (X(X^\top X)^{-1})_{aj} \\
= n((X^\top X)^{-1})_{ij} - \sum_{k,l,m} ((X^\top X)^{-1})_{kl} (\delta_{im} X_{al} + \delta_{il} X_{am})((X^\top X)^{-1})_{mj},
\]
where \( \delta_{ij} \) is the Kronecker delta. Thus,
\[
\tilde{\text{div}} \ g(X) = -(n - p - 1)^2(X^\top X)^{-1}.
\]

Also,
\[
g(X)^\top g(X) = (n - p - 1)^2(X^\top X)^{-1}.
\]

Therefore, from Theorem 2.4
\[
R(M, \hat{M}_{EM}) = nI_p - (n - p - 1)^2E_M[(X^\top X)^{-1}] \preceq nI_p, \tag{3}
\]
which means that \( \hat{M}_{EM} \) is minimax. \( \square \)

Corollary 2.6. The matrix quadratic risk of the Efron–Morris estimator at \( M = O \) is
\[
R(O, \hat{M}_{EM}) = (p + 1)I_p.
\]
Proof. When $M = O$, the matrix $(X^\top X)^{-1}$ follows the inverse Wishart distribution $W^{-1}(n, I_p)$. Thus, from the formula for the mean of the inverse Wishart distribution \(3\),

$$E_{M=O}[(X^\top X)^{-1}] = (n - p - 1)^{-1}I_p.$$  

Therefore, from \(3\),

$$R(O, \hat{M}_{EM}) = nI_p - (n - p - 1)^2E_{M=O}[(X^\top X)^{-1}] = (p + 1)I_p.$$

We will study the matrix quadratic risk of the Efron–Morris estimator numerically in Section 5.

3 Matrix superharmonicity

In this section, we introduce a notion of matrix superharmonicity for matrix-variate functions, which will be used to derive Bayes minimax estimators in the next section.

First, we review the definition of a superharmonic function \(9, Definition 2.3.3\). Let $S_{x,r} = \{x + re \mid e \in \mathbb{R}^n, \|e\| = 1\} \subset \mathbb{R}^n$ be the sphere with center $x$ and radius $r > 0$. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$, let

$$L(f : x, r) = \frac{1}{\Omega_n} \int_{S_{x,r}} f(z) ds(z) = \frac{1}{\Omega_n} \int_{S_{0,1}} f(x + re) ds(e)$$

be the average value of $f$ on $S_{x,r}$, where $ds$ denotes the surface area element and $\Omega_n$ is the surface area of the unit sphere $S_{0,1}$ in $\mathbb{R}^n$. Then, $f$ is said to be superharmonic if it satisfies the following:

1. $f$ is lower semicontinuous;
2. $f \neq \infty$;
3. $L(f : x, r) \leq f(x)$ for every $x \in \mathbb{R}^n$ and $r > 0$.

Now, we introduce matrix superharmonicity. For a matrix-variate function $f : \mathbb{R}^{n \times p} \rightarrow \mathbb{R} \cup \{\infty\}$, we can define its superharmonicity as the superharmonicity of $f \circ \text{vec}^{-1} : \mathbb{R}^{np} \rightarrow \mathbb{R} \cup \{\infty\}$, where $\text{vec} : \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{np}$ is the vectorization operator \(3\). However, such a definition does not take into account the matrix structure. We propose a stronger version of superharmonicity for matrix-variate functions, which we refer to as matrix superharmonicity. For $X \in \mathbb{R}^{n \times p}$ and $\rho \in \mathbb{R}^p$, let $S_{X,\rho} = \{X + e\rho^\top \mid e \in \mathbb{R}^n, \|e\| = 1\}$ and

$$L(f : X, \rho) = \frac{1}{\Omega_n} \int_{S_{0,1}} f(X + e\rho^\top) ds(e)$$

be the average value of $f$ on $S_{X,\rho}$.

Definition 3.1. A matrix-variate function $f : \mathbb{R}^{n \times p} \rightarrow \mathbb{R} \cup \{\infty\}$ is said to be matrix superharmonic if it satisfies the following:

1. $f$ is lower semicontinuous;
2. \( f \not\equiv \infty; \)

3. \( L(f : X, \rho) \leq f(X) \) for every \( X \in \mathbb{R}^{n \times p} \) and \( \rho \in \mathbb{R}^p \).

We will provide examples of matrix superharmonic functions in Sections 4.3 and 4.4.

**Proposition 3.2.** If a function \( f : \mathbb{R}^{n \times p} \to \mathbb{R} \cup \{ \infty \} \) is matrix superharmonic, then \( f \circ \text{vec}^{-1} \) is superharmonic.

**Proof.** For every \( X \in \mathbb{R}^{n \times p} \) and \( r > 0 \),

\[
L(f \circ \text{vec}^{-1} : \text{vec}(X), r) = \frac{1}{\Omega_p^{r-p-1}} \int_{S_{0,r}} L(f : X, \rho) d\sigma(\rho).
\]

Since \( L(f : X, \rho) \leq f(X) \) for every \( \rho \) from the matrix superharmonicity of \( f \),

\[
L(f \circ \text{vec}^{-1} : \text{vec}(X), r) \leq f(X).
\]

\( \square \)

The converse of Proposition 3.2 does not hold when \( p \geq 2 \). One counterexample is \( f(X) = \|X\|_F^2 - np \) as we will show in Proposition 4.15.

**Remark 3.3.** When \( n = 1 \), \( S_{X, \rho} = \{X - \rho, X + \rho\} \) and thus the third condition of matrix superharmonicity reduces to midpoint convexity:

\[
L(f : X, \rho) = \frac{f(X + \rho) + f(X - \rho)}{2} \leq f(X),
\]

which is equivalent to usual convexity for a lower semicontinuous function \( f \) on \( \mathbb{R}^p \). On the other hand, the definition of midpoint convexity is not unique for functions defined on discrete space \([15]\). It may be interesting to investigate discrete analogue of matrix superharmonicity and its applications.

We provide a characterization of matrix superharmonicity for \( C^2 \) functions. Recall that a \( C^2 \) function \( f : \mathbb{R}^n \to \mathbb{R} \) is superharmonic if and only if its Laplacian is nonpositive:

\[
\Delta f(x) = \sum_{a=1}^n \frac{\partial^2}{\partial x_a^2} f(x) \leq 0
\]

for every \( x \) \([9, Lemma 2.3.4]\). This property is extended to matrix superharmonic functions by using a matrix version of the Laplacian.

**Definition 3.4.** For a \( C^2 \) function \( f : \mathbb{R}^{n \times p} \to \mathbb{R} \), its matrix Laplacian \( \tilde{\Delta} f : \mathbb{R}^{n \times p} \to \mathbb{R}^{p \times p} \) is defined as

\[
(\tilde{\Delta} f(X))_{ij} = \sum_{a=1}^n \frac{\partial^2}{\partial X_{ai} \partial X_{aj}} f(X).
\]

**Theorem 3.5.** A \( C^2 \) function \( f : \mathbb{R}^{n \times p} \to \mathbb{R} \) is matrix superharmonic if and only if its matrix Laplacian is negative semidefinite \( \tilde{\Delta} f(X) \preceq 0 \) for every \( X \).
Assume that $f$ is matrix superharmonic. Let $e \in \mathbb{R}^n$ be a unit vector. From Taylor's theorem,

$$f(X + e\rho^\top) = f(X) + \sum_{a,i} \frac{\partial f}{\partial X_{ai}}(X)e_a\rho_i + \frac{1}{2} \sum_{a,i,b,j} \frac{\partial^2 f}{\partial X_{ai} \partial X_{bj}}(X)e_a\rho_i e_b\rho_j + o(\|\rho\|^2).$$

Thus,

$$L(f : X, \rho) = \frac{1}{\Omega_n} \int_{S_{0,1}} f(X + e\rho^\top)ds(e) = f(X) + \frac{1}{2n} \sum_{a,i,j} \frac{\partial^2 f}{\partial X_{ai} \partial X_{aj}}(X)\rho_i\rho_j + o(\|\rho\|^2)$$

$$= f(X) + \frac{1}{2n}\rho^\top\Delta \tilde{f}(X)\rho + o(\|\rho\|^2),$$

where we used

$$\frac{1}{\Omega_n} \int_{S_{0,1}} e_a ds(e) = 0, \quad \frac{1}{\Omega_n} \int_{S_{0,1}} e_a e_b ds(e) = \frac{1}{n}\delta_{ab}.$$ 

On the other hand, $L(f : X, \rho) \leq f(X)$ from the matrix superharmonicity of $f$. Therefore, we must have

$$\rho^\top\Delta \tilde{f}(X)\rho \leq 0$$

for every $\rho$. Hence, $\Delta \tilde{f}(X) \preceq 0$ for every $X$.

Conversely, assume that $\Delta \tilde{f}(X) \preceq 0$ for every $X$. For arbitrary $X$ and $\rho$, let $\tilde{f}(e) = f(X + e\rho^\top)$. Then,

$$\Delta \tilde{f}(e) = \sum_a \frac{\partial^2 \tilde{f}}{\partial e_a^2} = \sum_{a,i,j} \frac{\partial}{\partial e_a} \left( \frac{\partial f}{\partial X_{ai}} \right)\rho_i = \sum_{a,i,j} \rho_i \frac{\partial^2 f}{\partial X_{ai} \partial X_{aj}}\rho_j = \rho^\top\Delta \tilde{f}(X + e\rho^\top)\rho \leq 0.$$ 

Let $C_\eta = \{ e \in \mathbb{R}^n \mid \eta < \|e\| < 1 \}$ and $\tilde{g}(e) = \|e\|^{2-n} - 1$. Then, $\tilde{g} \geq 0$ and $\Delta \tilde{g} = 0$ on $C_\eta$. Therefore, from Green’s theorem,

$$\int_{\partial C_\eta} (\tilde{f}D_n\tilde{g} - \tilde{g}D_n\tilde{f}) \, ds(e) = \int_{C_\eta} (\Delta \tilde{g} - \tilde{g}\Delta \tilde{f}) \, de \geq 0,$$

where $\partial C_\eta = S_{0,\eta} \cup S_{0,1}$ is the boundary of $C_\eta$ and $D_n\tilde{f}$ is the directional derivative of $\tilde{f}$ in the direction of the outer normal unit vector. On the other hand, since $D_n\tilde{g} = -(2 - n)\eta^{1-n}$ on $S_{0,\eta}$ and $D_n\tilde{g} = 2 - n$ on $S_{0,1},$

$$\int_{\partial C_\eta} \tilde{f} D_n\tilde{g} ds(e) = -(2 - n)\eta^{1-n} \int_{S_{0,\eta}} \tilde{f} ds(e) + (2 - n) \int_{S_{0,1}} \tilde{f} ds(e) = -(2 - n)\Omega_n L(f : X, \eta\rho) + (2 - n)\Omega_n L(f : X, \rho).$$

Also, since $\tilde{g} = \eta^{2-n} - 1$ on $S_{0,\eta}$ and $\tilde{g} = 0$ on $S_{0,1},$

$$\int_{\partial C_\eta} \tilde{g} D_n\tilde{f} ds(e) = (\eta^{2-n} - 1) \int_{S_{0,\eta}} D_n\tilde{f} ds(e),$$

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which is \(O(\eta)\) as \(\eta \to 0\). Therefore,

\[
L(f : X, \rho) \leq L(f : X, \eta \rho) + O(\eta).
\]

By taking \(\eta \to 0\), we obtain \(L(f : X, \rho) \leq f(X)\). Since \(X\) and \(\rho\) are arbitrary, \(f\) is matrix superharmonic.

The limit of an increasing sequence of superharmonic functions is also superharmonic \([9, \text{Theorem 2.4.8}]\). Matrix superharmonic functions have a similar property.

**Lemma 3.6.** Let \(f_1 \leq f_2 \leq \ldots\) be an increasing sequence of matrix superharmonic functions and assume that \(f = \lim_{k \to \infty} f_k \neq \infty\). Then, \(f\) is also matrix superharmonic.

**Proof.** Since each \(f_k\) is lower semicontinuous, their supremum \(f\) is also lower semicontinuous. Also, from Lemma 2.2.10 of [9],

\[
\int_{S_{0,1}} f(X + e\rho^\top)ds(e) = \lim_{k \to \infty} \int_{S_{0,1}} f_k(X + e\rho^\top)ds(e).
\]

Therefore,

\[
L(f : X, \rho) = \lim_{k \to \infty} L(f_k : X, \rho) \leq \lim_{k \to \infty} f_k(X) = f(X),
\]

where we used the matrix superharmonicity of each \(f_k\). \(\square\)

### 4 Bayes estimation with matrix superharmonic prior

In this section, we investigate Bayes shrinkage estimation under the matrix quadratic loss.

#### 4.1 Uniqueness of Bayes estimator

Let

\[
m_\pi(X) = \int p(X \mid M)\pi(M)\,dM
\]

be the marginal distribution of \(X \sim N_{n,p}(M, I_n, I_p)\) with prior \(\pi(M)\).

Similarly to the Frobenius loss, the (generalized) Bayes estimator under the matrix quadratic loss is uniquely given by the posterior mean, even though the matrix quadratic loss is only partially ordered.

**Definition 4.1.** For a function \(f : \mathbb{R}^{n \times p} \to \mathbb{R}\), its matrix gradient \(\tilde{\nabla} f : \mathbb{R}^{n \times p} \to \mathbb{R}^{n \times p}\) is defined as

\[
(\tilde{\nabla} f(X))_{ai} = \frac{\partial}{\partial X_{ai}} f(X).
\]

**Lemma 4.2.** If \(m_\pi(X) < \infty\) for every \(X\), then the (generalized) Bayes estimator of \(M\) with respect to a prior \(\pi(M)\) under the matrix quadratic loss is uniquely given by the posterior mean:

\[
\hat{M}^\pi(X) = E_\pi[M \mid X] = \frac{\int M p(X \mid M)\pi(M)\,dM}{\int p(X \mid M)\pi(M)\,dM} = X + \tilde{\nabla} \log m_\pi(X).
\]
Proof. For each estimator $\hat{M} = \hat{M}(X)$, its posterior risk is decomposed as

$$E_\pi[(\hat{M}(X) - M)\top(\hat{M}(X) - M) \mid X]$$

$$= E_\pi[(E_\pi[M \mid X] - M)\top(E_\pi[M \mid X] - M)] + D\top D,$$

where $D = \hat{M}(X) - E_\pi[M \mid X]$. Thus, the posterior risk is uniquely minimized under the Lowner order by taking $D = O$, which means that $\hat{M}(X) = E_\pi[M \mid X]$.

4.2 Sufficient condition for minimaxity

Now, we provide a sufficient condition for minimaxity of (generalized) Bayes estimators under the matrix quadratic loss.

Theorem 4.3. If $\sqrt{m_\pi(X)}$ is matrix superharmonic, then the generalized Bayes estimator $\hat{M}_\pi(X) = X + \tilde{\nabla} \log m_\pi(X)$ with respect to $\pi(M)$ is minimax under the matrix quadratic loss.

Proof. From Theorem 2.4, the matrix quadratic risk of the generalized Bayes estimator $\hat{M}_\pi$ is given by

$$R(M, \hat{M}_\pi) = nI_p + \int \nabla \log m_\pi(X) + \nabla \log m_\pi(X)\top \nabla \log m_\pi(X)\,

$$

Since $\sqrt{m_\pi(X)}$ is matrix superharmonic and $C^2$ by definition, $\tilde{\Delta} \sqrt{m_\pi(X)} \preceq O$ for every $X$ from Theorem 3.5. Therefore, $R(M, \hat{M}_\pi) \preceq nI_p$ and thus $\hat{M}_\pi$ is minimax.

We also provide another sufficient condition based on the matrix superharmonicity of prior itself.

Lemma 4.4. If $\pi(M)$ is matrix superharmonic and $m_\pi(X) < \infty$ for every $X$, then $m_\pi(X)$ is also matrix superharmonic.

Proof. Let $\phi(Z)$ be the probability density of $Z \sim N_{n,p}(O, I_n, I_p)$. Then,

$$m_\pi(Z) = \int \phi(Z - M)\pi(M)dM = \int \phi(A)\pi(Z - A)dA.$$

Thus, for every $X \in \mathbb{R}^{n\times p}$ and $\rho \in \mathbb{R}^p$,

$$L(m_\pi : X, \rho) = \frac{1}{\Omega_n} \int_{S_{0,1}} \left\{ \int \phi(A)\pi(X + e\rho\top - A)dA \right\} ds(e)$$

$$= \frac{1}{\Omega_n} \int \phi(A) \left\{ \int_{S_{0,1}} \pi(X + e\rho\top - A)ds(e) \right\} dA$$

$$= \int \phi(A) L(\pi : X - A, \rho)dA$$

$$\leq \int \phi(A)\pi(X - A)dA$$

$$= m_\pi(X),$$

where the second equation follows from Fubini’s theorem and the inequality follows from the matrix superharmonicity of $\pi$. Therefore, $m_\pi$ is matrix superharmonic.
Lemma 4.5. If \( f : \mathbb{R}^{n \times p} \to \mathbb{R} \) is matrix superharmonic and \( \phi : \mathbb{R} \to \mathbb{R} \) is monotone increasing and concave in the range of \( f \), then \( \phi \circ f \) is also matrix superharmonic. In particular, if \( f \) is matrix superharmonic and non-negative, then \( \sqrt{f} \) is also matrix superharmonic.

Proof. For every \( X \in \mathbb{R}^{n \times p} \) and \( \rho \in \mathbb{R}^p \),

\[
L(\phi \circ f : X, \rho) = \frac{1}{\Omega_n} \int_{S_{0,1}} \phi(f(X + e\rho^\top))ds(e)
\]

\[
\leq \phi \left( \frac{1}{\Omega_n} \int_{S_{0,1}} f(X + e\rho^\top)ds(e) \right)
\]

\[
= \phi(L(f : X, \rho))
\]

\[
\leq \phi(f(X)),
\]

where the first inequality follows from Jensen’s inequality and the second inequality follows from the matrix superharmonicity of \( f \) and the monotonicity of \( \phi \). Therefore, \( \phi \circ f \) is matrix superharmonic.

Theorem 4.6. Let \( \pi(M) \) be a superharmonic prior with \( m_\pi(X) < \infty \) for every \( X \). Then, the generalized Bayes estimator \( \hat{M}^\pi(X) = X + \tilde{\nabla} \log m_\pi(X) \) with respect to \( \pi(M) \) is minimax under the matrix quadratic loss.

Proof. From Lemma 4.4 and Lemma 4.5, \( \sqrt{m_\pi(X)} \) is matrix superharmonic. Therefore, from Theorem 4.3, \( \hat{M}^\pi \) is minimax.

When \( p = 1 \), Theorem 4.3 and Theorem 4.6 reduce to the classical results by Stein [16] on minimax Bayes estimators of a normal mean vector.

Remark 4.7. Whereas matrix superharmonic priors provide minimax Bayes estimators of any linear combination \( Mc \) simultaneously, there may be cases where we are only interested in linear combination with nonnegative coefficients \( c \in \mathbb{R}_+^p \). In such cases, only the copositivity \([2]\) of \( -\tilde{\Delta} m_\pi(X) \) suffices. Thus, it may be interesting to develop another version of matrix superharmonicity based on copositivity.

Remark 4.8. Recently, superharmonic priors have been found to give minimax predictive densities under the Kullback–Leibler loss \([11, 7]\). It is an interesting future problem to investigate properties of matrix superharmonic priors in predictive density estimation under some analogue of matrix quadratic loss.

4.3 Matrix superharmonic priors

Here, we provide a class of matrix superharmonic priors, which includes the previously proposed generalization \([2]\) of Stein’s prior.

Let

\[
\pi_{\alpha,\beta}(M) = \det(M^\top M + \beta I_p)^{-\frac{(\alpha+n+p-1)}{2}},
\]

where \(-n - p + 1 \leq \alpha \leq -2p \) and \( \beta \geq 0 \). When \( \alpha = -2p \) and \( \beta = 0 \), \( \pi_{\alpha,\beta}(M) \) coincides with the singular value shrinkage prior \([2]\). Note that \( \pi_{\alpha,\beta}(M) \) with \( \alpha > 0 \) and \( \beta > 0 \) is the...
so-called matrix t-distribution with \( \alpha \) degrees of freedom \([8]\). When \( p = 1 \), \( \pi_{\alpha,\beta}(M) \) reduces to the \( n \)-dimensional (improper) multivariate t-prior \([5]\)

\[
\pi_{\alpha,\beta}(\mu) = (\|\mu\|^2 + \beta)^{-(\alpha+n)/2},
\]

and it is superharmonic when \(-n \leq \alpha \leq -2\) and \( \beta \geq 0 \). This result is extended to general \( p \) as follows.

**Theorem 4.9.** If \(-n - p + 1 \leq \alpha \leq -2p \) and \( \beta \geq 0 \), then the prior \( \pi_{\alpha,\beta}(M) \) in \([9]\) is matrix superharmonic.

**Proof.** In the following, the subscripts \( a, b, \ldots \) run from 1 to \( n \) and the subscripts \( i, j, \ldots \) run from 1 to \( p \). We denote the \((i, j)\)-th entry of \( S^{-1} \) by \( S_{ij} \) and the Kronecker delta by \( \delta_{ij} \).

First, assume that \( \beta > 0 \). Let \( S = M^\top M + \beta I_p \succ O \). Since

\[
\frac{\partial S_{kl}}{\partial M_{ai}} = \delta_{ik} M_{al} + \delta_{il} M_{ak},
\]

(6)

Also,

\[
\frac{\partial}{\partial S_{ij}} \det S = S^{ij} \det S.
\]

Thus,

\[
\frac{\partial}{\partial M_{ai}} \det S = \sum_{k,l} \frac{\partial S_{kl}}{\partial M_{ai}} \frac{\partial}{\partial S_{kl}} \det S = 2 \sum_k M_{ak} S^{ik} \det S.
\]

(7)

Therefore,

\[
\frac{\partial^2}{\partial M_{ai} \partial M_{aj}} \det S
\]

\[
= 2 \left( S^{ij} - \sum_{k,l} M_{ak} M_{al} S^{ij} S^{kl} + \sum_{k,l} M_{ak} M_{al} S^{ik} S^{jl} \right) \det S,
\]

(8)

where we used

\[
\frac{\partial S^{ik}}{\partial M_{aj}} = -S^{il} S^{km} \frac{\partial S_{tm}}{\partial M_{aj}} = -\sum_l M_{al} S^{ij} S^{kl} - \sum_l M_{al} S^{il} S^{jk},
\]

which is obtained by differentiating \( \sum_k S_{jk} S^{ik} = \delta_{ij} \) and using (6).

Now,

\[
(\tilde{\Delta}(\det S)^{-(\alpha+n+p-1)/2})_{ab} = \sum_a \frac{\partial^2}{\partial M_{ai} \partial M_{aj}} (\det S)^{-(\alpha+n+p-1)/2}
\]

\[
= \frac{\alpha + n + p - 1}{2} (\det S)^{-(\alpha+n+p-1)/2} \sum_a (A_{aij} + B_{aij}),
\]

(9)
where
\[ A_{aij} = \frac{\alpha + n + p + 1}{2} (\det S)^{-2} \left( \frac{\partial}{\partial M_{ai}} \det S \right) \left( \frac{\partial}{\partial M_{aj}} \det S \right), \]
\[ B_{aij} = -(\det S)^{-1} \frac{\partial^2}{\partial M_{ai} \partial M_{aj}} \det S. \]

By using (7) and \[ \sum_{a} M_{ak} M_{al} = S_{kl} - \beta \delta_{kl}, \]

\[ \sum_{a} A_{aij} = 2(\alpha + n + p + 1) \sum_{a} \left( \sum_{k} M_{ak} S_{ik} \right) \left( \sum_{l} M_{al} S_{jl} \right) = 2(\alpha + n + p + 1) \left( S_{ij} - \beta \sum_{k} S_{ik} S_{jk} \right). \]

On the other hand, from (8),
\[ \sum_{a} B_{aij} = -2(n - p + 1) S_{ij} - 2\beta S_{ij} \sum_{k} S_{kk} + 2\beta \sum_{k} S_{ik} S_{jk}. \]

Hence,
\[ \sum_{a} (A_{aij} + B_{aij}) = 2(\alpha + 2p) S_{ij} - 2(\alpha + n + p) \beta \sum_{k} S_{ik} S_{jk} - 2\beta S_{ij} \sum_{k} S_{kk}. \]

Substituting this expression into (9) gives
\[ \tilde{\Delta}(\det S)^{-\frac{(\alpha + n + p - 1)}{2}} = \frac{\alpha + n + p - 1}{2} (\det S)^{-\frac{(\alpha + n + p - 1)}{2}} \times \left( 2(\alpha + 2p) S^{-1} - 2(\alpha + n + p) \beta (S^{-1})^2 - 2\beta \text{tr}(S^{-1}) S^{-1} \right), \quad (10) \]

which is negative semidefinite from \(-n - p + 1 \leq \alpha \leq -2p, \beta > 0\) and \(S^{-1} \succ O\). Therefore, by Theorem 3.5, \( \pi_{\alpha, \beta}(M) = (\det S)^{-\frac{(\alpha + n + p - 1)}{2}} \) is matrix superharmonic.

Next, assume that \( \beta = 0 \). Let
\[ \pi^{(k)}(M) = \det \left( M^\top M + k^{-1} I_p \right)^{-\frac{(\alpha + n + p - 1)}{2}}, \quad k = 1, 2, \ldots. \]

Then, each \( \pi^{(k)} \) is matrix superharmonic from the above discussion. Also, \( \pi^{(1)} \leq \pi^{(2)} \leq \cdots \) and \( \lim_{k \to \infty} \pi^{(k)}(M) = \pi_{\alpha, \beta}(M) \) for every \( M \). Therefore, from Lemma 3.6, \( \pi_{\alpha, \beta}(M) \) is also matrix superharmonic.

**Proposition 4.10.** If \(-n - p + 1 \leq \alpha \leq -2p \) and \( \beta \geq 0 \), then the marginal density \( m_\pi(X) \) in (4) with \( \pi = \pi_{\alpha, \beta} \) in (3) is finite for every \( X \).

**Proof.** We use the fact that \( m_\pi(X) \) in (4) is interpreted as the expected value of \( \pi(M) \) with respect to \( M \sim N_{n,p}(X, I_n, I_p) \).

When \( \beta > 0 \), since \( \pi_{\alpha, \beta}(M) \leq \pi_{\alpha, \beta}(O) = \beta^{-p(\alpha + n + p - 1)/2}, \)
\[ m_\pi(X) \leq \beta^{-p(\alpha + n + p - 1)/2} \]
for every $X$.

When $\beta = 0$,

$$m_\pi(X) = E \left[ (\det S)^{-(\alpha+n+p-1)/2} \right],$$

where $S = M^\top M$ has a noncentral Wishart distribution $S \sim W_p(n, I_p, X^\top X)$ from Theorem 3.5.1 in [8]. Therefore, by using Theorem 3.5.6 of [8],

$$m_\pi(X) = Cetr \left( -\frac{1}{2} X^\top X \right) {}_1F_1 \left( -\frac{\alpha + p - 1}{2}; \frac{n}{2}; -\frac{1}{2} X^\top X \right),$$

where $C = 2^{-p(\alpha+n+p-1)/2} \Gamma_p(-(\alpha + p - 1)/2)/\Gamma_p(n/2)$, $\Gamma_p$ is the multivariate Gamma function and ${}_1F_1$ is the hypergeometric function of a matrix argument [8]. Thus, $m_\pi(X)$ is finite for every $X$.

From Theorem 4.6, Theorem 4.9 and Proposition 4.10, we obtain the following.

**Theorem 4.11.** If $-n - p + 1 \leq \alpha \leq -2p$ and $\beta \geq 0$, then the generalized Bayes estimator with respect to the prior $\pi_{\alpha,\beta}(M)$ in (5) is minimax under the matrix quadratic loss.

By taking $\alpha = -2p$ and $\beta = 0$ in Theorem 4.9 and Theorem 4.11, we obtain the following result on the singular value shrinkage prior (2).

**Corollary 4.12.** When $n - p - 1 > 0$, the singular value shrinkage prior $\pi_{\text{SVS}}(M)$ in (2) is matrix superharmonic. Also, the generalized Bayes estimator with respect to $\pi_{\text{SVS}}(M)$ is minimax under the matrix quadratic loss.

In particular, the matrix superharmonicity of $\pi_{\text{SVS}}$ is strongly concentrated on the space of low rank matrices, which has measure zero, in the same way as the Laplacian of Stein’s prior $\pi(\mu) = \|\mu\|^{2-n}$ becomes a Dirac delta function at the origin.

**Corollary 4.13.** If $M$ has full-rank, then $\tilde{\Delta}_{\pi_{\text{SVS}}}(M) = O$.

**Proof.** When $\alpha = -2p$, $\tilde{\Delta}(\det S)^{-(\alpha+n+p-1)/2}$ in (11) is $O(\beta)$ at a full-rank $M$. \qed

**Remark 4.14.** The Efron–Morris estimator (11) can be viewed as a pseudo-Bayes estimator $M = X + \nabla \log m(X)$ [6] with the pseudo-marginal $m(X) = \pi_{\text{SVS}}(X) = \det(X^\top X)^{-(\alpha+n+p-1)/2}$. Combining such a pseudo-Bayes interpretation with Theorem 4.9 and Theorem 4.11 with $\beta = 0$, it follows that the estimator $M = X(I_p - c(X^\top X)^{-1})$ is minimax for $0 \leq c \leq 2(n - p - 1)$.

### 4.4 Stein-type priors

We further investigate matrix superharmonicity of another types of shrinkage priors.

Since Stein’s prior $\pi(\mu) = \|\mu\|^{2-n}$ for $\mu \in \mathbb{R}^n$ is superharmonic [10], the prior

$$\pi_S(M) = \|M\|_{\text{F}}^{2-np} = \|\text{vec}(M)\|^{2-np}$$

is also superharmonic. More generally, the shrinkage prior $\pi(M) = \|M\|^{-c}$ with $c \geq 0$ is superharmonic if and only if $0 \leq c \leq np - 2$ [6]. However, the range of $c$ for matrix superharmonicity is narrower. In particular, Stein’s prior $\pi_S(M)$ in (11) is not matrix superharmonic.
Proposition 4.15. The prior $\pi(M) = \|M\|_F^{-c}$ with $c \geq 0$ is matrix superharmonic if and only if $0 \leq c \leq n - 2$.

Proof. First, assume that $0 \leq c \leq n - 2$. Let $\pi_\beta(M) = (\|M\|_F^2 + \beta)^{-c/2}$ with $\beta > 0$. Then, $\pi_\beta(M)$ is $C^2$ and its matrix Laplacian is

$$\tilde{\Delta}_{\pi_\beta}(M) = -c(\|M\|_F^2 + \beta)^{-c/2}(n(\|M\|_F^2 + \beta)I_p - (c + 2)M^TM) \preceq 0,$$

where we used $n(\|M\|_F^2 + \beta)I_p - (c + 2)M^TM \succeq 0$ from $n \geq c + 2$ and $\|M\|_F^2 I_p \succeq M^TM$. Therefore, from Theorem 3.14, $\pi_\beta(M)$ is matrix superharmonic for every $\beta > 0$. Then, $\pi^{(k)}(M) = (\|M\|_F^2 + k^{-1})^{-c/2}$ is an increasing sequence of matrix superharmonic functions with $\lim_{k \to \infty} \pi^{(k)}(M) = \pi(M)$. Thus, from Lemma 3.6, $\pi(M)$ is also matrix superharmonic.

Next, assume that $c > n - 2$. Consider $M$ and $\rho$ defined by

$$M_{ai} = \begin{cases} 1 & (i = 1) \\ 0 & (2 \leq i \leq p) \end{cases}, \quad \rho_i = \begin{cases} 1 & (i = 1) \\ 0 & (2 \leq i \leq p) \end{cases}.$$

Then,

$$L(\pi : M, \rho) = \frac{1}{\Omega_n} \int_{S_{0,1}} \pi(M + c\rho^\top)ds(e) = \frac{1}{\Omega_n} \int_{S_{0,1}} g(e)ds(e),$$

where the function $g : \mathbb{R}^n \to \mathbb{R}$ is given by $g(e) = \|1 + e\|^{-c}$ with the all-one vector $1 = (1, \ldots, 1)^\top \in \mathbb{R}^n$. Since $c > n - 2$, $\Delta g(e) = c(c - n + 2)\|1 + e\|^{-c-2} > 0$. Thus, by Green’s theorem,

$$\frac{1}{\Omega_n} \int_{S_{0,1}} g(e)ds(e) > g(0) = \pi(M).$$

Hence, we have $L(\pi : M, \rho) > \pi(M)$. Therefore, $\pi(M)$ is not matrix superharmonic.

Corollary 4.16. When $p \geq 2$, Stein’s prior $\pi_S(M) = \|M\|_F^{2-np}$ is not matrix superharmonic.

[1] showed that the column-wise shrinkage estimator of James–Stein type

$$\hat{M} = XD, \quad D = \text{diag}(d_1, \ldots, d_p), \quad d_i = 1 - \frac{c}{\sum a_i X_{ai}^2}$$

is minimax under the matrix quadratic loss when $0 \leq c \leq 2(n-2)/p$. Note that this estimator can be viewed as a pseudo-Bayes estimator $\hat{M} = X + \nabla \log m(X)$ [2] with the pseudo-marginal $m(X) = \prod_i (\sum a_i X_{ai}^2)^{-c/2}$. Their result is understood from the viewpoint of matrix superharmonicity as follows.

Proposition 4.17. The prior $\pi(M) = \prod_i (\sum a_i M_{ai}^2)^{-c/2}$ with $c \geq 0$ is matrix superharmonic if and only if $0 \leq c \leq (n - 2)/p$.

Proof. First, assume that $0 \leq c \leq (n - 2)/p$. Let $\pi_\beta(M) = \prod_i (\sum a_i M_{ai}^2 + \beta)^{-c/2}$ with $\beta > 0$. Then, $\pi_\beta(M)$ is $C^2$ and its matrix Laplacian is

$$\tilde{\Delta}_{\pi_\beta}(M) = c\pi_\beta(M)(cA - (n - 2)B - n\beta C),$$

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where $A, B, C$ are $p \times p$ positive semidefinite matrices with entries

$$A_{ij} = \left( \sum_{a} M_{ai}^2 + \beta \right)^{-1} \left( \sum_{a} M_{aj}^2 + \beta \right)^{-1} \left( \sum_{a} M_{ai}M_{aj} \right),$$

$$B_{ij} = \delta_{ij} A_{ij}, \quad C_{ij} = \delta_{ij} \left( \sum_{a} M_{ai}^2 + \beta \right)^{-2}.$$

Let $S = B^{-1/2}AB^{-1/2} \preceq O$. Then, all diagonal entries of $S$ are one and thus $S \preceq (\text{tr}S)I_p = pI_p$. Thus, $A = B^{1/2}SB^{1/2} \preceq pB$. Therefore, from $c \leq (n-2)/p$, 

$$\bar{\Delta}_n \pi(M) \leq c(cp - n + 2) \pi(M) B \preceq O.$$ 

Hence, from Theorem 3.5, $\pi_\beta(M)$ is matrix superharmonic for every $\beta > 0$. Then, $\pi^{(k)}(M) = \prod_i (\sum_a M_{ai}^2 + k^{-1})^{-e/2}$ is an increasing sequence of matrix superharmonic functions with $\lim_{k \to \infty} \pi^{(k)}(M) = \pi(M)$. Thus, from Lemma 3.6, $\pi(M)$ is also matrix superharmonic.

Next, assume that $c > (n-2)/p$. Suppose that all entries of $M$ and $\rho$ are one. Then,

$$L(\pi : M, \rho) = \frac{1}{\Omega_n} \int_{S_{0,1}} \pi(M + c\rho^T) ds(e) = \frac{1}{\Omega_n} \int_{S_{0,1}} g(e) ds(e),$$

where the function $g : \mathbb{R}^n \to \mathbb{R}$ is given by $g(e) = \|1 + e\|^{-cp}$ with the all-one vector $1 = (1, \ldots, 1)^\top \in \mathbb{R}^n$. Since $c > (n-2)/p$, $\Delta g(e) = cp(c - n + 2)\|1 + e\|^{-cp-2} > 0$. Thus, by Green's theorem,

$$\frac{1}{\Omega_n} \int_{S_{0,1}} g(e) ds(e) > g(0) = \pi(M).$$

Hence, we have $L(\pi : M, \rho) > \pi(M)$. Therefore, $\pi(M)$ is not matrix superharmonic.

\section{Numerical results}

In this section, we present several numerical results on the matrix quadratic risk of shrinkage estimators. We denote the $i$-th singular value of $M$ by $\sigma_i$. Note that singular values are in descending order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$.

In the following, we focus on the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_p$ of the matrix quadratic risk $R(M, \hat{M})$ of several estimators. Since $R(M, \hat{M}) = nI_p$ for the maximum likelihood estimator $\hat{M} = X$, an estimator is minimax if and only if $\lambda_1 \leq n$ for every $M$.

First, we compare the generalized Bayes estimators with respect to the singular value shrinkage prior $\pi_{SVS}(M)$ in (2) and Stein’s prior $\pi_S(M)$ in (11) for $n = 5$ and $p = 3$. We employed the numerical method of [14] to compute the generalized Bayes estimators.

Figure 1 plots the three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the matrix quadratic risk with respect to $\sigma_2$ when $\sigma_1 = 10$ and $\sigma_3 = 0$. For $\pi_{SVS}$, all eigenvalues do not exceed $n = 5$, which indicates the minimaxity. More specifically, both $\lambda_1$ and $\lambda_3$ are almost constant with values $\lambda_1 \approx 5$ and $\lambda_3 \approx 4$ respectively, whereas $\lambda_2$ increases from 4 to 5 with $\sigma_2$. These behaviors are understood from the fact that $\pi_{SVS}$ shrinks each singular value separately [14]. For $\pi_S$, $\lambda_1$ is larger than $n = 5$ when $\sigma_2 \leq 8$ and thus the estimator is not minimax. This is compatible with

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Proposition 4.15. However, the sum $\lambda_1 + \lambda_2 + \lambda_3$ of eigenvalues, which is equal to the Frobenius risk $E_M[\|M - \hat{M}\|_F^2] = \text{tr} R(M, \hat{M})$, does not exceed $np = 15$, because $\pi_S$ is superharmonic in usual sense. This is similar to the fact that the James–Stein estimator is not minimax componentwise, even though it is minimax under the quadratic loss for the whole vector [12].

Figure 1: Eigenvalues of matrix quadratic risk of generalized Bayes estimators ($n = 5$, $p = 3$, $\sigma_1 = 10$, $\sigma_3 = 0$). left: $\pi_{\text{SVS}}$. right: $\pi_S$. The dashed line shows $n = 5$.

Figure 2 plots the three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of the matrix quadratic risk with respect to $\sigma_1$ when $\sigma_2 = \sigma_3 = 0$. Thus, the rank of $M$ is one. For $\pi_{\text{SVS}}$, both $\lambda_2$ and $\lambda_3$ are almost constant around 4, whereas $\lambda_1$ increases from 4 to 5 with $\sigma_1$. It indicates that $\pi_{\text{SVS}}$ works particularly well when $M$ has low rank. For $\pi_S$, all eigenvalues are fairly small when $\sigma_1 = 0$, namely $M = 0$. However, $\lambda_1$ increases rapidly with $\sigma_1$ and becomes larger than $n = 5$ when $\sigma_1 \geq 4$.

Figure 2: Eigenvalues of matrix quadratic risk of generalized Bayes estimators ($n = 5$, $p = 3$, $\sigma_2 = \sigma_3 = 0$), left: $\pi_{\text{SVS}}$, right: $\pi_S$. The dashed line shows $n = 5$. Note that the second and third eigenvalues almost overlap in both plots.

Next, we compare the Efron–Morris estimator $\hat{M}_{\text{EM}} = X(I - (n - p - 1)(X^\top X)^{-1})$ and the James–Stein estimator $\hat{M}_{\text{JS}} = (1 - (np - 2)/\|X\|_F^2)X$ in higher dimension. Note that these estimators have almost the same risk with the generalized Bayes estimators with respect to
Figure 3 plots the 20 eigenvalues $\lambda_1 \geq \cdots \geq \lambda_{20}$ of the matrix quadratic risk with respect to $\sigma_1$ when $n = 100$, $p = 20$, $\sigma_i = (6 - i)/5 \cdot \sigma_1$ for $i = 2, \ldots, 5$ and $\sigma_6 = \cdots = \sigma_{20} = 0$. Thus, the rank of $M$ is five. The results are qualitatively the same with Figure 2, with the advantage of the singular value shrinkage more pronounced. For $\hat{M}_{EM}$, all eigenvalues are smaller than $n = 100$. In particular, the bottom 15 eigenvalues are almost constant around 20: $\lambda_6 \approx \cdots \approx \lambda_{20} \approx 20$. On the other hand, for $\hat{M}_{JS}$, the largest eigenvalue $\lambda_1$ grows rapidly with $\sigma_1$ and exceeds $n = 100$ when $\sigma_1 \geq 10$. Other eigenvalues also increase with $\sigma_1$, including $\lambda_6, \ldots, \lambda_{20}$. These results show that the singular value shrinkage works well for low rank matrices.

For the Efron–Morris estimator, the above simulation results suggest that the $i$-th eigenvalue $\lambda_i$ of the matrix quadratic risk depends only on the $i$-th singular value $\sigma_i$ of $M$ approximately: $\lambda_i \approx g_{n,p}(\sigma_i)$ for some function $g_{n,p}$. Finally, we investigate this functional relation $g_{n,p}$ numerically. Figure 4 plots $\lambda_1$ with respect to $\sigma_1$ when $\sigma_2 = \cdots = \sigma_p = 0$ for several values of $n$ and $p$. It indicates that $g_{n,p}(0) \approx p$, which is compatible with $R(O, \hat{M}_{EM}) = (p+1)I_p$ from Corollary 2.6. In addition, Figure 4 implies $g_{n,p}(\sigma) \to n$ as $\sigma \to \infty$. This is understood from the fact that the Efron–Morris estimator $\hat{M}_{EM}$ becomes essentially the same with the maximum likelihood estimator $\hat{M} = X$, which has the constant risk $R(M, \hat{M}) = nI_p$, in a direction of a very large singular value $16, 14$. These properties of $g_{n,p}$ provide a quantification of the advantage of the Efron–Morris estimator over the maximum likelihood estimator when $M$ has low rank. Namely, if $M$ has rank $r < p$, then $\sigma_{r+1} = \cdots = \sigma_p = 0$ and thus $\lambda_{r+1}, \ldots, \lambda_p$ should be close to $g_{n,p}(0) \approx p$. Therefore, the reduction in the Frobenius risk is evaluated as

$$np - tr \ R(M, \hat{M}_{EM}) \geq np - rn - (p - r)p = np \left(1 - \frac{r}{p}\right) \left(1 - \frac{p}{n}\right).$$

Thus, the Efron–Morris estimator attains large risk reduction especially when either $p/r$ or $n/p$ is large.
Figure 4: Largest eigenvalue of matrix quadratic risk of the Efron–Morris estimator ($\sigma_2 = \cdots = \sigma_p = 0$). left: $n = 100$, $p = 10, 20, \ldots, 50$. right: $n = 1000$, $p = 100, 200, \ldots, 500$. The dashed line shows $n$.

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