ON SOME SYNTACTIC PROPERTIES OF THE MODALIZED HEYTING CALCULUS

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ABSTRACT. We show that the modalized Heyting calculus \[2\] admits a normal axiomatization. Then we prove that in this calculus the inference rule \(\Box \alpha / \alpha\) is admissible (Proposition 5.6), but the rule \(\Box \alpha \rightarrow \alpha / \alpha\) is not (Proposition 6.1). Finally, we show that this calculus and intuitionistic propositional calculus are assertorically equipollent, which leads to a variant of limited separation property for the modalized Heyting calculus.

1. INTRODUCTION

The modalized intuitionistic calculus \(mHC\) was introduced by Leo Esakia \[2\] as a weakening of the proof-intuitionistic logic, nowadays known as \(KM\); see, e.g., \[11\]. In Section 2 we will give another axiomatization of \(mHC\) and call it \(E\) (after Leo Esakia). The main goal of this reformulation is that \(E\) is a normal axiomatic system (in the sense of \[4\], p. 75), while \(mHC\) is not. The last circumstance leads to the fact that the calculus \(mHC\) does not possess the separation property; for \(E\) the question is open, though a limited version of it is presented in Section 7. The present work has been done in direction of (and with hopes for) answering this question in the affirmative. Thus we will be focusing on syntactic, that is proof-theoretic, properties of \(E\) in the Hilbert-style framework.

2. LANGUAGES AND SYSTEMS

We fix a sentential language, \(\mathcal{L}_a\), based on a countable set \(\text{Var}\) of sentential variables and the assertoric logical connectives: \(\land\) (conjunction), \(\lor\) (disjunction), \(\rightarrow\) (conditional, or implication, or entailment), and \(\neg\) (negation). Unspecified variables of \(\text{Var}\) will be denoted by letters \(p, q, r, \ldots\) and unspecified \(\mathcal{L}_a\text{-formulas}\) by letters \(A, B, C, \ldots\). By adding a unary connective \(\square\) (modality) to \(\mathcal{L}_a\), we obtain language \(\mathcal{L}_m\), unspecified formulas of which (\(\mathcal{L}_m\text{-formulas}\)) will be denoted by letters \(\alpha, \beta, \gamma, \ldots\) Formulas of the form \(\square \alpha\) are called \(\square\text{-formulas}\). For a fixed
variable \( p \in \text{Var} \), we denote
\[
\top := p \rightarrow p.
\]

In Section 7 we will be using the usual operation of replacement of a subformula \( \alpha \) of formula \( \gamma \) by a formula \( \beta \), denoting this operation by
\[
\gamma[\alpha : \beta].
\]

In a natural way, this operation is extended to multiple simultaneous replacement.

From an algebraic viewpoint, each of \( \mathcal{L}_a \) and \( \mathcal{L}_m \) defines a similarity type and so does any of their reductions. For any of these similarity types (or languages) one can define a formula algebra, \( \mathfrak{F} \). Given a formula algebra \( \mathfrak{F} \), a substitution is a homomorphism of \( \mathfrak{F} \) into \( \mathfrak{F} \).

Next we introduce main calculi we will be dealing with. All these calculi have one and the same set of inference rules — (uniform) substitution and modus ponens.

Intuitionistic propositional calculus \( \text{Int} \) is defined by the following axioms divided into the four groups:

\[
\begin{align*}
&\text{(i)} \quad p \rightarrow (q \rightarrow r), \quad (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \\
&\text{(c)} \quad (p \land q) \rightarrow p, \quad (p \land q) \rightarrow q, \quad p \rightarrow (q \rightarrow (p \land q)) \\
&\text{(d)} \quad p \rightarrow (p \lor q), \quad p \rightarrow (q \lor p), \quad (p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow ((p \lor q) \rightarrow r)) \\
&\text{(n)} \quad (p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p), \quad p \rightarrow (\neg p \rightarrow q),
\end{align*}
\]

where \( p, q, r \) are three fixed distinct variables of \( \text{Var} \).

We formulate the modalized Heyting calculus \( \text{E} \) by adding to the axioms (2.1) the following group of formulas:

\[
\begin{align*}
&\text{(m)} \quad \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q), \quad p \rightarrow \Box p, \quad \Box p \rightarrow ((q \rightarrow p) \rightarrow q).
\end{align*}
\]

Next we define the calculi which will play merely auxiliary role in our discussion. The common framework for these calculi is \( \text{Int} \) formulated in the language \( \mathcal{L}_m \), which we denote by \( \text{Int}^\square \). Also, we define:

- \( \text{Kuz} := \text{Int}^\square + \Box p \rightarrow (((q \rightarrow p) \rightarrow q) \rightarrow q) \) (where the last formula is the only modal axiom of \( \text{Kuz} \));
- \( \text{Kuz}^* := \text{Int}^\square + \Box p \rightarrow (q \lor (q \rightarrow p)) \) (with the last formula only modal axiom);
- \( \text{mHC} := \text{Kuz}^* + \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) + p \rightarrow \Box p; \) we divide the modal axioms in two groups:

\[
\begin{align*}
&\text{(m1)} \quad \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q), \quad p \rightarrow \Box p, \\
&\text{(m2)} \quad \Box p \rightarrow (q \lor (q \rightarrow p));
\end{align*}
\]

\[1\] These axioms are specifications of the axiom schemata from [5], § 19.
• **KM** := **mHC** + (□p → p) → p. We note that the first axiom of (m_1) is redundant; cf. [7], p. 88.

We note that **E** differs from **mHC** in that the last m-axiom above is replaced with □p → (q ∨ (q → p)) and that

$$\mathbf{E} = \mathbf{Kuz} + \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) + p \rightarrow \Box p$$

As we will show in Section 3, the calculi **E** and **mHC** generate one and the same logic, that is the same set of derivable formulas.

The following interconnection between **Int** and **Int** is almost obvious.

**Proposition 2.1** (cf. [12], Proposition 2.4). For any L_m-formula α, if **Int** ⊬ α, then there is an L_a-formula A such that α is obtained by substitution from A and **Int** ⊬ A, and conversely.

Given a calculus C and formulas α_1, . . . , α_n, β, by

$$C + \alpha_1, \ldots, \alpha_n \vdash \beta$$

we mean such a deducibility where substitution can be applied only to formulas that are derivable in C. We call such a derivation an C-derivation of β from α_1, . . . , α_n without substitution (w. s.). For derivations with unrestricted use of substitution, we employ a conventional notation,

$$C + \alpha_1, \ldots, \alpha_n \vdash \beta.$$

It is obvious that both relations ⊨ and ⊢ are transitive. Also, to indicate a fragment of C, which can be associated with the groups (i) – (m), we use notation C_i, C_ic, etc.

To illustrate, how we are going to use this notation, we prove that

**Int**_{icd} ⊬ ((q ∨ (q → p)) → p) → (q ∨ (q → p)).

(2.2)

To prepare application of deduction theorem, we prove that

**Int**_{icd} + (q ∨ (q → p)) → p ⊩ q ∨ (q → p).

Indeed, we have:

1. (q ∨ (q → p)) → p (premise)
2. (q → p) ∧ ((q → p) → p) (from (1) by **Int_{icd}-derivation w.s.)
3. (q → p) ∧ p (from (2) by **Int_{ic}-derivation w.s.)
4. q → p (from (3) by **Int_{ic}-derivation w.s.)
5. (q ∨ (q → p)) (from (4) by **Int_{id}-derivation w.s.)

In Section 5 we will introduce one more axiomatic system which will play a “supporting” role for **E**.

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2 It should be clear that this restriction on the substitution rule is imposed for the purpose of the use of deduction theorem.
3. Some deducibilities

**Proposition 3.1.** The following deducibilities hold:

(a) \( \text{Kuz}_{icdm} \vdash \Box p \rightarrow (q \lor (q \rightarrow p)) \)
(b) \( \text{Kuz}_{icdm} \vdash \Box p \rightarrow ((q \rightarrow p) \rightarrow q) \).

*Proof.* To prove (a), we show that

\[
\text{Kuz}_{icd} + \Box p \vdash q \lor (q \rightarrow p). \tag{3.1}
\]

Indeed, let us denote

\[ A := q \lor (q \rightarrow p). \]

Then, we obtain:

1. \( \Box p \) (premise)
2. \( \Box p \rightarrow ((A \rightarrow p) \rightarrow A) \) (axiom instance)
3. \( ((A \rightarrow p) \rightarrow A) \rightarrow A \) (from (1) & (2) by modus ponens)
4. \( (((q \lor (q \rightarrow p)) \rightarrow p) \rightarrow (q \lor (q \rightarrow p))) \rightarrow (q \lor (q \rightarrow p)) \) (the same as (3))
5. \( ((q \lor (q \rightarrow p)) \rightarrow p) \rightarrow (q \lor (q \rightarrow p)) \) (deducibility (2.2))
6. \( q \lor (q \rightarrow p) \) (from (5) & (4) by modus ponens)

Next we prove that

\[
\text{Kuz}_{icd}^* + \Box p, (q \rightarrow p) \vdash q \lor q.
\]

Indeed, we have:

1. \( \Box p, (q \rightarrow p) \rightarrow q \) (premises)
2. \( \Box p \rightarrow (q \lor (q \rightarrow p)) \) (axiom instance)
3. \( q \lor (q \rightarrow p) \) (from (1) & (2) by modus ponens)
4. \( q \rightarrow q \) (derivable in \( \text{Int}_1 \))
5. \( (q \rightarrow q) \land ((q \rightarrow p) \rightarrow q) \) (from (1) & (4) by \( \text{Int}_{ie} \)-derivation w.s.)
6. \( (q \lor (q \rightarrow p)) \rightarrow q \) (from (5) by \( \text{Int}_{id} \)-derivation w.s.)
7. \( q \) (from (3) & (6) by modus ponens)

\[ \square \]

**Corollary 3.2.** For any formula \( \alpha \), the following equivalences hold:

(a) \( \text{Kuz} \vdash \alpha \iff \text{Kuz}^* \vdash \alpha \)
(b) \( \text{E} \vdash \alpha \iff \text{mHC} \vdash \alpha \).

Following terminology in [4], p. 75, \( \text{E} \) is a normal axiomatic system for \( \text{mHC} \). Usually, normalization is the first step toward obtaining the separation property, though this property can be formulated for non-normal calculi as well. As we will see in the next section, the separation property for \( \text{mHC} \) does not hold.
Conjecture 1. The calculus $E$ possesses the separation property; that is, any formula derivable in $E$ is also derivable by using only axioms of the group (i) and those ones in the groups (c) – (m) which correspond to the logical connectives actually appearing in the formula.

4. Algebraic background

Below we consider Heyting algebras in the signature: $\land$ (greatest lower bound), $\lor$ (least upper bound), $\rightarrow$ (relative pseudocomplementation), $\neg$ (pseudocomplementation), and 1 (unit), as well as their expansions by a unary operation $\square$ (modality). We call the latter algebras $\square$-enhanced Heyting algebras.

Definition 4.1 (modal Heyting algebra, Kuz-algebra, $E$-algebra). A $\square$-enhanced Heyting algebra is a modal Heyting algebra if the following identities hold:

(a) $\square 1 = 1$
(b) $\square(x \land y) = \square x \land \square y$.

The latter algebra is a Kuz-algebra if in addition the next identity is valid:

(c) $\square x \leq y \lor (y \rightarrow x)$.

And the latter in turn is an $E$-algebra if in addition to (a) – (c) the following identity is true as well:

(d) $x \leq \square x$.

As usual, we employ the notation

$\mathcal{A} \models \alpha$

to indicate that a formula $\alpha$ is valid (in a usual sense) in an algebra $\mathcal{A}$ (for all types of algebras used in this paper).

Proposition 4.2. For any formula $\alpha$,

(a) $\text{Kuz} \vdash \alpha \iff \text{Kuz}^* \vdash \alpha \iff \mathcal{A} \models \alpha$, for any Kuz-algebra $\mathcal{A}$;
(b) $\text{E} \vdash \alpha \iff \text{mHC} \vdash \alpha \iff \mathcal{A} \models \alpha$, for any $E$-algebra $\mathcal{A}$.

Now we are ready to demonstrate that the separation property (as it is expressed in Conjecture 1) does not hold for $\text{mHC}$.

Indeed, let us take the formula $\alpha_0 = \square p \rightarrow ((q \rightarrow p) \rightarrow q) \rightarrow q)$. This formula contains only two connectives and it is not derivable in the calculus $\text{mHC}_{im}$ based on the axioms of the groups (i) and (m).

The last claim becomes clear if we consider a 3-element $\square$-enhanced Heyting algebra with $\square x = 1$, since the above formula is invalid in this algebra but all (i)-axioms and (m)-axioms are. Hence $\text{mHC}_{im} \nvdash \alpha_0$. However, according to Proposition 3.1 this formula is derivable in $\text{mHC}$. 
5. Admissibility of the rule $\Box\alpha/\alpha$

To explain the task of this section we need to introduce another player — logic system $\textbf{K4.Grz}$ defined in [2].

\[
\textbf{K4.Grz} := \text{Int}^0 + \neg p \rightarrow p + \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q) + \Box p \rightarrow \Box \Box p + \Box((\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow \Box p + \frac{\alpha}{\Box\alpha}.
\]

We aim to prove that the rule $\frac{\Box\alpha}{\alpha}$ is admissible in both $\textbf{K4.Grz}$ and $\text{mHC}$, as well as, according to Proposition 4.2, in $\textbf{E}$.

Since we shall work with the algebraic semantics of $\textbf{K4.Grz}$, we start with it.

**Definition 5.1 (K4.Grz-algebra).** Let $\mathfrak{A} = (\mathcal{A}, \land, \lor, \neg, 1, \Box)$ be a Boolean algebra with a unary operation $\Box$. $\mathfrak{A}$ is a $\textbf{K4.Grz}$-algebra if it is a modal algebra (that is the identities (a) and (b) of Definition 4.1 hold), in which the following identities are valid:

\[
\begin{align*}
\text{(a\textsuperscript{*})} & \quad \Box x \leq \Box \Box x \\
\text{(b\textsuperscript{*})} & \quad \Box(\neg \Box(\neg x \lor \Box x) \lor x) \leq \Box x.
\end{align*}
\]

It is obvious that $\textbf{K4.Grz} \vdash \alpha \iff \mathfrak{A} \models \alpha$, for any $\textbf{K4.Grz}$-algebra $\mathfrak{A}$. (5.1)

**Definition 5.2 (doubling, doubleton).** Let $\mathfrak{A}$ be an algebra of similarity type $\langle \land, \lor, \neg, 1, \Box \rangle$ be a Boolean algebra with a unary operation $\Box$. $\mathfrak{A}$ is a $\textbf{K4.Grz}$-algebra if it is a modal algebra (that is the identities (a) and (b) of Definition 4.1 hold), in which the following identities are valid:

\[
\begin{align*}
\text{(a\textsuperscript{*})} & \quad \Box x \leq \Box \Box x \\
\text{(b\textsuperscript{*})} & \quad \Box(\neg \Box(\neg x \lor \Box x) \lor x) \leq \Box x.
\end{align*}
\]

Working with expansions of Boolean algebras, we will be using the following notation:

\[
x \Rightarrow y := \neg x \lor y.
\]

Thus the above condition (b\textsuperscript{*}) can be rewritten as follows:

\[
\Box(\Box(x \Rightarrow \Box x) \Rightarrow x) \leq \Box x.
\]

**Proposition 5.3.** The variety of $\textbf{K4.Grz}$-algebras is closed under doubling.

**Proof.** Let $\mathfrak{B}$ be the doubleton of a $\textbf{K4.Grz}$-algebra $\mathfrak{A}$. Let us take two elements, $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$, of $|\mathfrak{B}|$. First we notice that $\Box(1, 1) = (1, 1)$.

\[\text{This construction was for the first time introduced in [6], p. 216, for a similar purpose we employ it here. In [9] the present construction had slightly been generalized, which was later used [10] to prove a property similar to Proposition 5.7.}\]
Next we show that

$$\Box (\bar{x} \land y) = (\Box \bar{x} \land \Box y),$$

that is

$$\Box (x_1 \land y_1, x_2 \land y_2) = \Box (x_1, x_2) \land \Box (y_1, y_2).$$

Let us denote

$$\Box (x_1 \land y_1, x_2 \land y_2) = (\Box x_1 \land z_1)$$
$$\Box (x_1, x_2) = (\Box x_1, z_2)$$
$$\Box (y_1, y_2) = (\Box y_1, z_3).$$

Thus we have to show that

$$z_1 = z_2 \land z_3.$$  \hspace{1cm} (5.2)

If $x_1 \land y_1 \neq 1$, then $z_1 = 0$ and either $z_2 = 0$ or $z_3 = 0$. Thus we have \hspace{1cm} (5.2) true. If $x_1 \land y_1 = 1$, then both $x_1 = 1$ and $y_1 = 1$. Therefore, $z_1 = z_2 = z_3 = 1$ and hence \hspace{1cm} (5.2) is true again.

Next we prove that

$$\Box \bar{x} \leq \Box \Box \bar{x},$$

that is

$$\Box (x_1, x_2) = (\Box x_1, z_2) \leq \Box (\Box x_1, z_2).$$

For this, denoting

$$\Box (x_1, z_2) = (\Box x_1, z_4),$$

we have to show that

$$z_2 \leq z_4.$$

We have to consider the two cases: $x_1 \neq 1$ and $x_1 = 1$. In case $x_1 \neq 1$, $z_2 = 0$. In case $x_1 = 1$, $\Box x_1 = 1$ and hence $z_4 = 1$.

Thus it remains to check that

$$\Box (\Box (\bar{x} \Rightarrow \Box \bar{x}) \Rightarrow \bar{x}) \leq \Box \bar{x},$$

that is

$$\Box (\Box ((x_1, x_2) \Rightarrow \Box (x_1, x_2)) \Rightarrow (x_1, x_2)) \leq \Box (x_1, x_2).$$

In terms of notation introduced above, we have to show that

$$\Box (((x_1 \Rightarrow \Box x_1, x_2 \Rightarrow z_1) \Rightarrow (x_1, x_2)) \leq (\Box x_1, z_1).$$

We denote:

$$\Box (x_1 \Rightarrow \Box x_1, x_2 \Rightarrow z_1) = (\Box x_1 \Rightarrow \Box x_1, z_5)$$
$$\Box (\Box (x_1) \Rightarrow x_1, z_6) \Rightarrow z_2) = (\Box (\Box (x_1) \Rightarrow x_1, z_6).$$

To complete the proof we need to show that

$$z_6 \leq z_1.$$
Indeed, assume first that $\Box(x_1 \Rightarrow \Box x_1) \leq x_1$. Then $\Box x_1 = 1$ and hence $x_1 = 1$. The latter means that $z_1 = 1$. On the other hand, if $\Box(x_1 \Rightarrow \Box x_1) \not\leq x_1$, then $z_6 = 0$. 

**Corollary 5.4.** The inference rule $\Box \alpha / \alpha$ (weakening) is admissible in $\text{K4.Grz}$. 

**Proof.** Suppose $\text{K4.Grz} \not\vdash \alpha$. This means that $\alpha$ can be refuted in some $\text{K4.Grz}$-algebra $\mathfrak{A}$. Suppose a refuting valuation is $p \mapsto a, \ldots, q \mapsto b$, where $p, \ldots, q$ are all variables occurring in $\alpha$. Let an algebra $\mathfrak{B}$ be obtained by the doubling of $\mathfrak{A}$. According to Proposition 5.3 $\mathfrak{B}$ is also a $\text{K4.Grz}$-algebra. It is clear that $\Box \alpha$ is refuted in $\mathfrak{B}$ by the valuation $p \mapsto (a, 1), \ldots, q \mapsto (b, 1)$. Hence $\text{K4.Grz} \not\vdash \Box \alpha$. 

The transfer of the weakening rule from $\text{K4.Grz}$ onto $\text{mHC}$, and thereby (Proposition 4.2) onto $\text{E}$, can be conducted through Esakia’s embedding theorem of $\text{mHC}$ into $\text{K4.Grz}$. This embedding employs an extension of the Gödel-McKinsey-Tarski translation (mapping $t$ below) to modal language with a subsequent splitting (mapping $s$ below), which had for the first time been used in [10] and since then became common place.

First, we expand the language $\mathcal{L}_m$ by adding another unary modality $\bigcirc$ thus obtaining bimodal language $\mathcal{L}_b$. We denote the set of formulas of the first language by $\mathcal{F}_m$ and that of the second by $\mathcal{F}_b$. Next we define the two mappings, $t : \mathcal{F}_m \rightarrow \mathcal{F}_b$ and $s : \mathcal{F}_b \rightarrow \mathcal{F}_m$ as follows.

- $t(p) = \bigcirc p$ if $p \in \text{Var}$;
- $t(\alpha \land \beta) = t(\alpha) \land t(\beta)$;
- $t(\alpha \lor \beta) = t(\alpha) \lor t(\beta)$;
- $t(\alpha \rightarrow \beta) = \bigcirc (t(\alpha) \rightarrow t(\beta))$;
- $t(\neg \alpha) = \bigcirc \neg t(\alpha)$;
- $t(\Box \alpha) = \bigcirc \Box t(\alpha)$.

- $s(p) = p$ if $p \in \text{Var}$;
- $s$ commutes with the connectives of $\mathcal{L}_m$;
- $s(\bigcirc a) = s(a) \land \Box s(a)$, where $a \in \mathcal{F}_b$.

**Proposition 5.5** (Esakia’s embedding theorem [2], Corollary 21). For any formula $\alpha \in \mathcal{F}_m$,

$$\text{mHC} \vdash \alpha \iff \text{K4.Grz} \vdash s \circ t(\alpha).$$

**Proposition 5.6.** The inference rule $\Box \alpha / \alpha$ is admissible in $\text{mHC}$ and hence in $\text{E}$. 

**Proof.** Let $\text{mHC} \vdash \Box \alpha$. Then, by virtue of Proposition 5.5, $\text{K4.Grz} \vdash s \circ t(\Box \alpha)$, that is $\text{K4.Grz} \vdash \Box t(\alpha) \land \Box \Box t(\alpha)$. The later implies that
K4.Grz ⊢ ◻t(α) and hence, by virtue of Corollary 5.4, K4.Grz ⊢ t(α). Applying Proposition 5.5 one more time, we obtain that mHC ⊢ α; according to Proposition 4.2, the deducibility E ⊢ α is also true. □

Proposition 5.7. There is a continuum of normal extensions of E which are closed under the weakening rule.

Proof. There is a continuum of normal extensions of KM, including KM itself, which are closed under the weakening rule; cf. [10], Theorem 3. Since KM is a normal extension of E, this property is true for extensions of E as well. □

Conjecture 2. There is a proper normal extension of E which is properly included in KM and in which the weakening rule is admissible.

6. The inference rule ◻α → α/α

We note that

E ⊂ KM,

for in any nontrivial ◻-enhanced Heyting algebra with ◻x = x the axioms of E are valid but the formula (◻p → p) → p is not.

Also, it is seen that the rule

\[
\frac{◻α \rightarrow α}{α} \quad (Löb rule)
\]

is not just admissible in KM but derivable in it. The question arises, whether, by adding the Löb rule to E, we receive KM or not? As we will see below, the former is the case.

Proposition 6.1. For any formula α, the following conditions are equivalent.

(a) KM ⊢ α;
(b) E + (Löb rule) ⊢ α;
(c) E + ◻(◻p → p) → ◻p ⊢ α.

Proof. To prove the equivalence of (a) and (b), it suffices to show that

\[
E_{icm} ⊢ ◻((◻p → p) → (◻p → p)).
\]

We prove the last deducibility algebraically. For this, we observe that the following is true in any modal Heyting algebra with x ≤ ◻x:

\[
◻((◻x → x) → x) ∧ (◻x → x) = ◻(((◻x → x) → x) ∧ (◻x → x)) ∧ (◻x → x) = ◻((◻x → x) ∧ (◻x → x)) ∧ (◻x → x) = ◻x ∧ (◻x → x) \leq x.
\]
Similarly, to prove the equivalence of (b) and (c), we show algebraically that

\[ \mathcal{E}_{	ext{icm}} \vdash \Box(\Box p \to p) \to \Box p \to (\Box p \to \Box p). \]

Indeed, in any modal Heyting algebra with \( x \leq \Box x \), we obtain:

\[
\Box(\Box x \to x) \land \Box(\Box x \to x) = \Box((\Box(\Box x \to x) \to x) \land \Box(\Box x \to x)) \land \Box(\Box x \to x) = \Box(\Box x \to x) \land \Box(\Box x \to x) = \Box x \land \Box(\Box x \to x) \leq \Box x \land (\Box x \to x) \leq \Box x.
\]

\[ \square x. \]

\[ \Box \]

\section{7. Assertoric Equivalence of \( E \) and \( \text{Int} \)}

In this section we aim to prove Proposition 7.9. Although this proposition follows from a similar proposition for logic KM, (cf. [12], Proposition 4.2) in view of Conjecture [11], it was desirable to obtain the property in question in a direct way.

Let us denote

\[ P(p, q) := (((p \to q) \to p) \to p) \]

(\text{Peirce law})

\textbf{Lemma 7.1.} \( \text{Int}_i \vdash (P(p, q) \to q) \to q \).

\textit{Proof.} Our proof is semantical and uses Kripke semantics for \( \text{Int} \); see, e.g., [1]. Assume that in some intuitionistic Kripke model \((W, \preceq, \models)\) the formula \( (P(p, q) \to q) \to q \) is refuted at a point \( a \in W \). That is, \( \models \) forces \( P(p, q) \to q \) to be true at \( a \) and \( q \) to be false at \( a \). But this implies that \( P(p, q) \) is also false at \( a \). The latter is only possible when there is a point/world \( b \in W \) such that \( a \preceq b \), where, that is at \( b \), \( \models \) forces \( (p \to q) \to p \) to be true and \( p \) to be false. The latter implies that \( p \to q \) is also false at \( b \). Thus there is \( c \in W \) such that \( b \preceq c \) where \( p \) is true and \( q \) is false. We note that, according to a well-known property, the formula \( P(p, q) \to q \), being true at \( a \), is also true at any \( x \in W \) such that \( a \preceq x \). Applying this property, we obtain that \( P(p, q) \to q \) is true at \( c \). Since \( p \) is true at \( c \), \( P(p, q) \) is true at \( c \) as well. The latter implies that \( q \) is true at \( c \). A contradiction. Thus \( \text{Int}_i \vdash (P(p, q) \to q) \to q \) and, in view of the separation property for \( \text{Int} \), (see, e.g., [3]) \( \text{Int}_i \vdash (P(p, q) \to q) \to q \) is true as well.

\textbf{Corollary 7.2.} \( \text{Int}_i \vdash (P(p, q) \to q) \leftrightarrow q. \)

\textbf{Lemma 7.3.} \( \text{Int}_i \vdash p \to P(q, p). \)
Proof. It is obvious that
\[ \text{Int}_i + \{ p, (q \rightarrow p) \rightarrow q \} \vdash q. \]
\[ \square \]

**Lemma 7.4.** \( \text{Int}_i \vdash (q \rightarrow p) \rightarrow (((r \rightarrow p) \rightarrow r) \rightarrow ((r \rightarrow q) \rightarrow r)) \).

**Proof.** We prove that
\[ \text{Int}_i + \{ q \rightarrow p, (r \rightarrow p) \rightarrow r, r \rightarrow q \} \vdash r. \]
Indeed, we have:

1. \( r \rightarrow q \) (premise)
2. \( q \rightarrow p \) (premise)
3. \( (r \rightarrow q) \rightarrow ((q \rightarrow p) \rightarrow (r \rightarrow p)) \) (deducible in \( \text{Int}_i \))
4. \( r \rightarrow p \) (from (1), (2), (3) by modus ponens twice)
5. \( (r \rightarrow p) \rightarrow r \) (premise)
6. \( r \) (from (4) and (5) by modus ponens)

\[ \square \]

**Corollary 7.5.** Given a formula \( \alpha \), let \( A_\alpha \) be the assertoric formula obtained from \( \alpha \) by deleting of all occurrences of \( \square \) in \( \alpha \). Then \( E \vdash \alpha \) implies \( \text{Int} \vdash A_\alpha \).

**Lemma 7.6.** The following holds:
\[ \text{Int}_i \vdash (p \rightarrow r) \rightarrow (((r \rightarrow p) \rightarrow p) \rightarrow (((r \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow r)) \rightarrow ((q \rightarrow p) \rightarrow (s \rightarrow r))) \).

**Proof.** We prove that
\[ \text{Int}_i + \{ p \rightarrow r, (r \rightarrow p) \rightarrow p, ((r \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow r), q \rightarrow p \} \vdash s \rightarrow r. \]
Indeed, we obtain:

1. \( q \rightarrow p \) (premise)
2. \( (r \rightarrow p) \rightarrow p \) (premise)
3. \( (r \rightarrow q) \rightarrow r \) (from (1), (2), and Lemma 7.4 by modus ponens twice)
4. \( ((r \rightarrow q) \rightarrow r) \rightarrow (s \rightarrow r) \) (premise)
5. \( s \rightarrow r \) (from (3) and (4) by modus ponens)

\[ \square \]

**Definition 7.7** (refined derivation). A derivation in a calculus having substitution rule as a postulated rule of inference is called refined if all substitutions, if any, are applied only to the axioms occurring in the derivation and/or to premises, if the derivation has any premises.
We note that, according to [13] (see also [8]), any derivation in each calculus defined above can be made refined.

**Lemma 7.8.** Let \( D : E + \lambda \vdash \sigma \) be a refined derivation. Then for the formulas \( \square \alpha, \square \beta \) and \( \square(\alpha \rightarrow \beta) \), which occur in \( D \) as antecedents of the axioms of \((m)\) or as the consequent of \( (\square \alpha \rightarrow \square \beta) \) which is an inference of the first axiom of \((m)\), there are corresponding formulas \( \Box \alpha, \Box \beta, \) and \( \Box(\alpha \rightarrow \beta) \) such that with corresponding replacements the following deducibility holds:

\[
\textbf{Int} \quad \square + \lambda[\Box \alpha : \Box \alpha, \cdots \Box \beta : \Box \beta, \cdots \Box(\alpha \rightarrow \beta) : \Box(\alpha \rightarrow \beta) \ldots]
\]

\[
+ \{\Box \alpha \rightarrow P(\Box \beta, \alpha), \ldots\} + \{\Box(\alpha \rightarrow \beta) \rightarrow P(\Box \beta, \alpha \rightarrow \beta), \ldots\}
\]

\( \vdash \sigma[\Box \alpha : \Box \alpha, \cdots \Box \beta : \Box \beta, \cdots \Box(\alpha \rightarrow \beta) : \Box(\alpha \rightarrow \beta) \ldots].\)

**Proof.** Suppose

\( D : \gamma_1, \ldots, \gamma_n, \) \hspace{1cm} (7.2)

where \( \gamma_n = \sigma. \)

Let us denote by \( M(D) \) the set of \( \square \)-formulas which occur in \( D \) as antecedents of the axioms of \((m)\) or as a consequent \( (\square \alpha \rightarrow \square \beta) \) of the first axiom of \((m)\).

For each \( \square \alpha \in M(D), \) let

\[ \square \alpha \rightarrow P(\beta_1, \alpha), \ldots, \square \alpha \rightarrow P(\beta_k, \alpha) \] \hspace{1cm} (7.3)

be all instances of the last axiom of \((m)\) occurring in \( D \) that start with \( \square \alpha. \) We define

\[ \square \alpha = \begin{cases} \land_{1 \leq j \leq k} P(\beta_j, \alpha)[\Box \alpha : \tau] & \text{if (7.3) is not empty} \\ \tau & \text{if (7.3) is empty}. \end{cases} \]

It is obvious that, providing that (7.3) is not empty,

\[ \textbf{Int}_{ic} \vdash \square \alpha \rightarrow P(\beta_j, \alpha), \] \hspace{1cm} (7.4)

where \( 1 \leq j \leq k. \) Next we show that

\[ \textbf{Int}_{ic} \vdash (\square \alpha \rightarrow \alpha) \rightarrow \alpha. \] \hspace{1cm} (7.5)

According to Lemma [7.1],

\[ \textbf{Int}_{ic} \vdash (P(\beta_1, \alpha) \rightarrow \alpha) \rightarrow \alpha, \]
\[ \textbf{Int}_{ic} \vdash (P(\beta_2, \alpha) \rightarrow \alpha) \rightarrow \alpha, \]

\[ \ldots \ldots \ldots \ldots \]
\[ \textbf{Int}_{ic} \vdash (P(\beta_k, \alpha) \rightarrow \alpha) \rightarrow \alpha. \]
Then, we obtain:

\[
\text{Int}_{ic} \vdash ((P(\beta_1, \alpha) \land P(\beta_2, \alpha) \rightarrow \alpha) \rightarrow (P(\beta_1, \alpha) \rightarrow (P(\beta_2, \alpha) \rightarrow \alpha)),
\]

\[
\text{Int}_{ic} \vdash ((P(\beta_1, \alpha) \land P(\beta_2, \alpha) \rightarrow \alpha) \rightarrow (P(\beta_1, \alpha) \rightarrow \alpha),
\]

\[
\text{Int}_{ic} \vdash ((P(\beta_1, \alpha) \land P(\beta_2, \alpha) \rightarrow \alpha) \rightarrow \alpha,
\]

\[
\text{Int}_{ic} \vdash (\land_{1 \leq j \leq k} P(\beta_j, \alpha) \rightarrow \alpha).\]

Also, by virtue of Lemma 7.3, we have:

\[
\text{Int}_{ic} \vdash \alpha \rightarrow \Box \alpha. \tag{7.6}
\]

Next we prove that if both \(\Box \alpha, \Box \beta \in M(D)\), then

\[
\text{Int}_{ic} + \Box \alpha \rightarrow P(\Box \beta, \alpha) \vdash (\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta). \tag{7.7}
\]

Indeed, according to Lemma 7.6,

\[
\text{Int}_{ic} \vdash ((\beta \rightarrow \Box \beta) \rightarrow (((\Box \beta \rightarrow \beta) \rightarrow \beta)
\rightarrow (((((\Box \beta \rightarrow \alpha) \rightarrow \Box \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)) \rightarrow (((\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta))))).
\]

By virtue of (7.5) and (7.5),

\[
\text{Int}_{ic} \vdash (((\Box \beta \rightarrow \alpha) \rightarrow \Box \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)) \rightarrow (((\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)),
\]

that is

\[
\text{Int}_{ic} \vdash (\Box \alpha \rightarrow P(\Box \beta, \alpha)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)).
\]

The latter is equivalent to

\[
\text{Int}_{ic} + \Box \alpha \rightarrow P(\Box \beta, \alpha) \vdash (\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta).
\]

Now suppose \(\Box (\alpha \rightarrow \beta) \in M(D)\) as the antecedent of the first axiom of (m) (in which case also both \(\Box \alpha, \Box \beta \in M(D)\)). Then, we prove,

\[
\text{Int}_{ic} + \{\Box \alpha \rightarrow P(\Box \beta, \alpha), \Box (\alpha \rightarrow \beta) \rightarrow P(\Box \beta, \alpha \rightarrow \beta)\} \vdash \Box (\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta). \tag{7.8}
\]

We will be proving that

\[
\text{Int}_{ic} + \{\Box \alpha \rightarrow P(\Box \beta, \alpha), \Box (\alpha \rightarrow \beta) \rightarrow P(\Box \beta, \alpha \rightarrow \beta),
\]

\[
\Box (\alpha \rightarrow \beta) \vdash \Box \alpha \rightarrow \Box \beta.
\]
Indeed, we obtain:

\begin{align*}
(1) & \quad \Box (\alpha \rightarrow \beta) \quad \text{(premise)} \\
(2) & \quad \Box (\alpha \rightarrow \beta) \rightarrow P(\Box \beta, \alpha \rightarrow \beta) \quad \text{(premise)} \\
(3) & \quad ((\Box \beta \rightarrow (\alpha \rightarrow \beta)) \rightarrow \Box \beta) \rightarrow \Box \beta) \quad \text{(from (1) and (2) by modus ponens)} \\
(4) & \quad (\Box \beta \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow (\Box \beta \rightarrow \beta)) \quad \text{(deducible in } \text{Int}_i^\Box) \\
(5) & \quad (\Box \beta \rightarrow \beta) \rightarrow \beta \quad \text{(by virtue of (7.3), since } \Box \beta \in M(D)) \\
(6) & \quad (\Box \beta \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta) \quad \text{(from (4) and (5) deducible in } \text{Int}_i^\Box) \\
(7) & \quad \Box \alpha \rightarrow P(\Box \beta, \alpha) \quad \text{(premise)} \\
(8) & \quad (\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta) \quad \text{(by virtue of (7.1))} \\
(9) & \quad (\Box \beta \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\Box \alpha \rightarrow \Box \beta) \quad \text{(from (6) and (8) deducible in } \text{Int}_i^\Box) \\
(10) & \quad \Box \alpha \rightarrow (\Box \beta \rightarrow (\alpha \rightarrow \beta)) \rightarrow \Box \beta) \quad \text{(from (9) deducible in } \text{Int}_i^\Box) \\
(11) & \quad \Box \alpha \rightarrow \Box \beta \quad \text{(from (3) and (10) deducible in } \text{Int}_i^\Box)
\end{align*}

Now for each \(i, 1 \leq i \leq n\), we define:

\[ \gamma_i^* := \gamma_i[\alpha : \Box \alpha, \ldots, \Box \beta : \Box \beta, \ldots, \Box (\alpha \rightarrow \beta) : \Box (\alpha \rightarrow \beta), \ldots] \]

and further

\[ [\gamma_i^*] := \begin{cases} 
\text{a } \text{Int}_i^\Box\text{-derivation according to (7.1)} & \text{if } \gamma_i \text{ is not an instance of an } \mathbf{m}\text{-axiom} \\
\text{a } \text{Int}_i^\Box\text{-derivation according to (7.6)} & \text{if } \gamma_i \text{ is an instance of the third } \mathbf{m}\text{-axiom} \\
\text{a } \text{Int}_i^\Box\text{-derivation according to (7.8)} & \text{if } \gamma_i \text{ is an instance of the second } \mathbf{m}\text{-axiom} \\
\text{a } \text{Int}_i^\Box\text{-derivation according to (7.9)} & \text{if } \gamma_i \text{ is an instance of the first } \mathbf{m}\text{-axiom.}
\end{cases} \]

Now let us consider

\[ D^* : [\gamma_1^*], \ldots, [\gamma_n^*]. \]

One can see that \( D^* \) supports the deducibility (7.1). \( \square \)

**Proposition 7.9.** The calculi \( \mathbf{E} \) and \( \text{Int} \) are assertorically equipollent; that is for any assertoric formulas \( A \) and \( B \),

\[ \mathbf{E} + A \vdash B \iff \text{Int} + A \vdash B. \]

**Proof.** Suppose \( D : \mathbf{E} + \lambda \vdash B \) is a refined derivation, where \( \lambda \) is the conjunction of instances of \( A \). According to Lemma (7.8)

\[ D^* : \text{Int}^\Box + \lambda[\Box \alpha : \Box \alpha, \ldots, \Box \beta : \Box \beta, \ldots, \Box (\alpha \rightarrow \beta) : \Box (\alpha \rightarrow \beta), \ldots] \]

\[ + \{ \Box \alpha \rightarrow P(\Box \beta, \alpha), \ldots \} + \{ \Box (\alpha \rightarrow \beta) \rightarrow P(\Box \beta, \alpha \rightarrow \beta), \ldots \} \]

\[ \vdash B, \tag{7.9} \]

for some formulas \( \Box \alpha, \ldots, \Box \beta \cdots \Box (\alpha \rightarrow \beta), \ldots \)

We denote

\[ \lambda^* := \lambda[\Box \alpha : \Box \alpha, \ldots, \Box \beta : \Box \beta, \ldots, \Box (\alpha \rightarrow \beta) : \Box (\alpha \rightarrow \beta), \ldots]. \]

Now let

\[ \Box \gamma \rightarrow P(\beta_1, \gamma), \ldots, \Box \gamma \rightarrow P(\beta_m, \gamma) \tag{7.10} \]
be all formulas of the second row of (7.9) which begin with $\Box \gamma$. We note that each $\beta_j$ does not contain $\Box \gamma$. Next, for each such a formula $\Box \gamma$, we define:

$$\Box \gamma := \bigwedge_{1 \leq j \leq m} P(\beta_j, \gamma).$$

We observe that

$$\text{Int}^\Box_{ic} \vdash \Box \gamma \rightarrow P(\beta_1, \gamma)[\Box \gamma: \Box \gamma, \ldots]$$

(7.11)

Let

$$D^* : \gamma_1, \ldots, \gamma_n.$$

Then, we define

$$\gamma_i^* := \gamma_i[\Box \gamma : \Box \gamma, \ldots]$$

and

$$[\gamma_i^*] = \begin{cases} \gamma_i^* & \text{if } \gamma_i \text{ is not one of (7.10)} \\ \text{a derivation supported by (7.11)} & \text{if } \gamma_i \text{ is one of (7.10)} \end{cases}$$

Denoting

$$D^{**} : [\gamma_1^*], \ldots, [\gamma_1^*],$$

we observe that $\text{Int}^\Box + \lambda^{**} \vdash B$, where $\lambda^{**} := \lambda^*[\Box \gamma : \Box \gamma, \ldots]$. Thus we have $\text{Int}^\Box \vdash \lambda^* \rightarrow B$. By virtue of Corollary 7.5, $\text{Int} \vdash A_{\lambda^* \rightarrow B}$. We note that $A_{\lambda^* \rightarrow B} = C \rightarrow B$, where $C$ is conjunction of instances of $A$. Hence $\text{Int} + A \vdash B$.

**Corollary 7.10.** Any assertoric formula is derivable in $E$ if it is also derivable by using only axioms of the group (i) and those ones in the groups (c) – (n) which correspond to the logical connectives actually appearing in the formula.

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