PROPERNESS OF LOG $F$-FUNCTIONALS

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Abstract. In this paper, we apply the method developed in [T97] and [TZ00] to prove the properness of log $F$-functional on any conic Kähler-Einstein manifolds. As an application, we give an alternative proof for the openness of the continuity method through conic Kähler-Einstein metrics.

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0. Introduction

It has been very active to study conic Kähler-Einstein metrics in recent years partly because of their use in studying problems in algebraic geometry and Kähler geometry. For example, they provide a continuity method for establishing the existence of Kähler-Einstein metrics on any Fano manifold $M$, that is, a compact Kähler manifold with positive first Chern class $c_1(M) > 0$. Such a continuity is used in the recent solution to Yau-Tian-Donaldson conjecture given independently by Tian and Chen-Donaldson-Sun [Ti12], [CDT13]. The conjecture states that a Fano manifold $M$ admits a Kähler-Einstein metric if and only if $M$ is K-stable as defined in [T97] and reformulated in [Do02]. The K-stability is closely related to the properness of Mabuchi’s $K$-energy, or equivalently, the $F$-functional. It is proved in [T97] that if $M$ admits no non-zero holomorphic vector field, then the existence of Kähler-Einstein metrics on $M$ is equivalent to the properness of $F$-functional or

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Mabuchi’s $K$-energy. The purpose of this paper is to adapt the arguments in [Ti97] as well as [TZ00] to show that similar results still hold for conic Kähler metrics.

Now let us recall some basics on conic Kähler metrics. Let $D$ be a smooth divisor of $M$ with $[D] \in \lambda c_1(M)$ for some $\lambda > 0$ and $S$ be a defining section of $D$. Choose a smooth Kähler metric $\omega_0$ with Kähler class $[\omega_0] = 2 \pi c_1(M)$, then there is a Hermitian metric $H_0$ on $[D]$ whose curvature is $\omega_0$. Following computations in [Au84] and [Di88] (also see [Ti87], [DT92]), Jeffres-Mazzeo-Rubinstein introduced a log $F$-functional on the space of Kähler potentials [JMR11]:

$$\mathcal{H}(M, \omega_0) = \{ \psi \in C^\infty(M) \mid \omega_\psi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi > 0 \}.$$  

This log $F$-functional is an Euler-Lagrange energy of conic Kähler-Einstein metrics with cone angle $2 \pi \beta$ along $D$ and is defined by (also see [LS12])

$$F_{\omega_0, \mu}(\psi) = J_{\omega_0}(\psi) - \frac{1}{V} \int_M \psi \omega_0^n + \frac{1}{\mu} \log \left( \frac{1}{V} \int_M \frac{1}{|S|^{n-\beta}} e^{h_0 - \mu \psi} \omega_0^n \right),$$

where $\mu = 1 - (1 - \beta) \lambda \in (0, 1)$, $V = \int_M \omega_0^n$ and $h_0$ is the Ricci potential of $\omega_0$ defined by

$$Ric(\omega_0) - \omega_0 = \sqrt{-1} \partial \bar{\partial} h_0, \quad \int_M (e^{h_0} - 1) \omega_0^n = 0.$$  

Note that $J_{\omega_0}(\phi)$ is defined by (see [Au84], [Ti87])

$$J_{\omega_0}(\phi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_M \sqrt{-1} \partial \phi \wedge \bar{\partial} \phi \wedge \omega_0^n \wedge \omega_0^{n-i-1}.$$  

The main result of this paper is the following

**Theorem 0.1.** Let $D$ be a smooth divisor of a Fano manifold $M$ with $[D] \in \lambda c_1(M)$ for some $\lambda > 0$ such that there is no non-zero holomorphic which is tangent to $D$ along $D$. Suppose that there exists a conic Kähler-Einstein metric on $M$ with cone angle $2 \pi \beta \in (0, 2 \pi)$ along $D$. Then there exists two uniform constants $\delta$ and $C$ such that

$$F_{\omega_0, \mu}(\psi) \geq \delta I_{\omega_0}(\psi) - C, \quad \forall \psi \in \mathcal{H}(M, \omega_0),$$

where $I_{\omega_0}(\psi) = \frac{1}{V} \int_M \phi (\omega_0^n - \omega_0^n).$

Combined with a result in [JMR11], Theorem 0.1 implies that there exists a conic Kähler-Einstein metric on $M$ along $D$ with cone angle $2 \pi \beta \in (0, 2 \pi)$ if and only if $F_{\omega_0, \mu}(\cdot)$ is proper. This generalizes Tian’s theorem in [Ti97] in the case of smooth Kähler-Einstein manifolds.

As an application of Theorem 0.1 or more precisely, its weaker version Theorem 6.1 in Section 6, we give an alternative proof for the openness of the continuity method through conic Kähler-Einstein metrics. The proof of such an openness was first sketched by Donaldson [Do11] as an application of the $C^{2, \alpha; \beta}$ Schauder estimates Donaldson developed for conic Kähler metrics. Since the space $C^{2, \alpha; \beta}$ will depend on the cone angles $2 \pi \beta$ of metrics, the usual Implicit Function Theorem

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1This condition can be removed if $\lambda \geq 1$ by a result in [Be11], or [SW12].
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could not be applied directly to prove the openness. Instead, Donaldson consider a
family of linear elliptic operators associated to approximated conic metrics to get
a prior Schauder estimates needed for proving the openness. Our proof is to use
the perturbation method first introduced in [Ti12] to approximate conic Kähler-
Einstein metrics by smooth Kähler metrics, then we apply the Implicit Function
Theorem to approximated smooth Kähler metrics and take limit (cf. Section 7).
To assume the limit exists, we need to establish a prior $C^0$ and $C^2$-estimates for
the Kähler potentials associated to those approximated metrics. With these a prior
estimates, we can take the limit to get a weak conic Kähler-Einstein metric. This
metric is in fact in sense of $C^{2,\alpha;\beta}$ Schauder theory by the regularity theorem in
[JMR11].

The proof of Theorem 0.1 is an adaption of that in [Ti97] for smooth Kähler-
Einstein manifolds. In our situation, there are some technical issues we need to make
sure. First we need to show how to smooth singular metrics near the conic Kähler-
Einstein metric. We will use a family of twisted Kähler-Ricci flows with initial values
given by smooth metrics which approximate conic metrics (see Section 5, 6). Then
we shall deal with the local smoothing behavior of these flows as well as the local
convergence of flows when the initial values vary. Note that as a parabolic version of
twisted Kähler-Einstein metric equation, which was first introduced by Song-Tian
[ST12], the twisted Kähler-Ricci flow has been also studied by many people, such
as Collins-Szekelyhihi [CS12], Liu-Zhang [LZ14], Liangmin Shen [Sh14] etc..

The organization of this paper is as follows: In Section 1, we recall some basics
on conic Kähler metrics. In Section 2, we prove the lower bound of log $F$-functional
$F_{\omega_0,\mu}(\cdot)$. In Section 3, we introduce a family of smooth Kähler metrics to approxi-
mate the conic Kähler metrics discussed in Section 2. In Section 4, we introduce a
family of twisted Kähler-Ricci flows to smooth the approximated metrics in Section
3, then in Section 5, we prove the local convergence of these flows. Theorem 0.1
will be proved in Section 6. In Section 7, we apply Theorem 0.1 to give a proof
of the openness for the continuity method through conic Kähler-Einstein metrics
which was first given by Donaldson.

1. Conic Kähler metrics

Let $S$ be a defining function of $D$ and $H_0$ a Hermitian metric on $D$ induced by
$\omega_0$. Then it is easy to see that $|S|^{2\beta} = |S|^{2\beta}_{H_0} \in C^{2,\alpha;\beta}(M)$ for any $\alpha \in (0, 1)$ in
sense of [Do11].

Moreover, one can check that $\omega^* = \omega_0 + \delta \sqrt{-1} \partial \bar{\partial} |S|^{2\beta}$ is a conic Kähler metric
with cone angle $2\pi \beta$ along $D$, as long as the number $\delta$ is sufficiently small (cf.
[Br11]). There is an important property of $\omega^*$ shown in [JMR11] that the bisectional
holomorphic curvature of $\omega^*$ is uniformly bounded from above on $M \setminus D$.

Let $h^*$ be a log Ricci potential of $\omega^*$ defined by

\begin{equation}
\sqrt{-1} \partial \bar{\partial} h^* = \text{Ric}(\omega^*) - \mu \omega^* - 2\pi (1 - \beta)[D].
\end{equation}
Then we have

\[(1.2)\quad h^* = h_0 - \mu |S|^{2\beta} - \log \frac{|S|^{2(1-\beta)}(\omega^*)^n}{\omega_0^n} + \text{const},\]

where \(h_0\) is a Ricci potential of \(\omega_0\). A direct computation shows that \(h^* \in C^{\gamma;\beta}(M)\), where \(\gamma = \min\left(\frac{2}{\beta} - 2, 1\right)\).

In general, a Kähler potential of conic Kähler metric is not necessary in \(C^{2,\alpha;\beta}(M)\). But, for a conic Kähler-Einstein metric \(\omega_{CKE} = \omega_\phi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \phi\) with angle \(2\pi \beta\) along \(D\), \(\phi\) should be in \(C^{2,\alpha_0;\beta}(M)\) for some positive number \(\alpha_0 \leq \gamma\). This is because \(\omega_{CKE}\) satisfies a conic Kähler-Einstein metric equation,

\[(1.3)\quad \text{Ric}(\omega) - \mu \omega - 2\pi(1-\beta)[D] = 0.\]

Then \(\phi\) satisfies a non-degenerate complex Monge-Ampère equation,

\[(1.4)\quad (\omega^* + \sqrt{-1} \partial \bar{\partial} \phi)^n = e^{h^* - \mu \phi}(\omega^*)^n.\]

The \(C^{2,\alpha;\beta}\)-regularity theorem established in [JMR11] (also in [GP13]) implies that \(\phi \in C^{2,\alpha_0;\beta}(M)\) for some \(\alpha_0 \leq \gamma\).

For any positive number \(\alpha \leq \alpha_0\), we introduce a space of \(C^{2,\alpha;\beta}\) Kähler potentials by

\[\mathcal{H}^{2,\alpha;\beta}(M, \omega_0) = \{\psi \in C^{2,\alpha;\beta}(M) | \omega_\psi = \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi \text{ is a conic Kähler metric on } M\}.\]

One can show that both functionals \(F_{\omega_0, \mu}(\cdot)\) and \(I_{\omega_0}(\cdot)\) are well-defined on \(\mathcal{H}^{2,\alpha;\beta}(M, \omega_0)\).

\textbf{Lemma 1.1.} For any \(\psi \in \mathcal{H}(M, \omega_0)\), there a sequence of \(\psi_\delta \in \mathcal{H}^{2,\alpha;\beta}(M, \omega_0)\) such that

\[F_{\omega_0, \mu}(\psi) = \lim_{\delta \to 0} F_{\omega_0, \mu}(\psi_\delta)\]

and

\[I_{\omega_0}(\psi) = \lim_{\delta \to 0} I_{\omega_0}(\psi_\delta).\]

\textit{Proof.} In fact, one can choose \(\psi_\delta = \psi + \delta |S|^{2\beta}\) to verify the lemma. \(\Box\)

\section{2. Lower bound of \(F_{\omega_0, \mu}(\cdot)\)}

In this section, we use the continuity method of Ding-Tian in [DT92] (also see [Ti97]) to study the lower bound of \(\log F\)-functional \(F_{\omega_0, \mu}(\cdot)\). This method will be extended to prove Main Theorem 0.1 in this paper. It is worthy to mention that the lower bound of \(F_{\omega_0, \mu}(\cdot)\) can be obtained by using a general theorem of Berndtsson for the uniqueness problem of special Kähler potentials in [Be11]. Berndtsson’ method is based on applications of geodesic theory about Kähler potentials space studied in [Se92], [Do98], [Ch00], etc..
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**Theorem 2.1.** Let $\omega = \omega_{CKE} = \omega_0$ be a conic Kähler-Einstein metric on $M$ with cone angle $2\pi \beta$ along $D$. Then $\phi$ obtains the minimum of $F_{\omega_0, \mu} (\cdot)$ on $\mathcal{H}^{2, \alpha; \beta}(M, \omega)$. In particular,

$$F_{\omega_0, \mu} (\psi) \geq -c(\omega_0, \mu), \ \forall \ \psi \in \mathcal{H}^{2, \alpha; \beta}(M, \omega).$$

**Proof.** For any $\psi \in \mathcal{H}^{2, \alpha; \beta}(M, \omega_0)$, log Ricci potential of $\tilde{\omega} = \omega_0$ is given by

$$h_{\tilde{\omega}} = -\log \frac{\omega_n^\alpha}{(\omega^*)^n} - \mu \psi + h^* + \text{const}.$$

Then $h_{\tilde{\omega}} \in C^{\alpha; \beta}(M)$. We consider the following complex Monge-Ampère equations with a parameter $t \in [0, \mu]$:}

$$\left(\tilde{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi\right)^n = e^{h_{\tilde{\omega}} - t \varphi^*} \tilde{\omega}^n. \tag{2.2}$$

By the assumption, there exists a solution $\varphi_{\mu} = \varphi - \psi + \text{const}$ at $t = \mu = 1 - (1 - \beta)\lambda$.

Note that the kernel of operator $(\Delta_\omega + \mu)$ is zero (cf. [Do11]). Then by the Donaldson’s linear theory for Laplace operators associated to conic metrics, we can apply Implicit Function Theorem to show that there exists a $\delta > 0$ such that (2.2) is solvable in the space $\mathcal{H}^{2, \alpha; \beta}(M, \omega_0)$ on any $t \in (\mu - \delta, \mu]$.

Set

$$E = \{s \in [0, \mu]| (\text{2.2}) \text{ is solvable on } t = s \text{ in } \mathcal{H}^{2, \alpha'; \beta}(M, \omega_0) \text{ for some } \alpha' \leq \alpha \}.$$ 

We want to prove $E = [0, \mu]$. Clearly, $E$ is non-empty since $(\mu - \delta, \mu] \subset E$. On the other hand, it is easy to see that (2.2) are equivalent to Ricci curvature equations,

$$\text{Ric}(\tilde{\omega}_\mu) = t \tilde{\omega}_\mu + (\mu - t)\tilde{\omega} + 2\pi(1 - \beta) |D|, \ t \in [0, \mu]. \tag{2.3}$$

Then

$$\text{Ric}(\tilde{\omega}_{\varphi_t}) > t \tilde{\omega}_{\varphi_t}, \text{ in } M \setminus D. \tag{2.4}$$

Thus the first non-eigenvalue of $\Delta_t$ is strictly bigger than $t$ [JMR11], where $\Delta_t$ is the Laplace operator associated to $\omega_t$ and $\omega_t = \hat{\omega}_{\varphi_t} = \hat{\omega} + \sqrt{-1} \partial \bar{\partial} \varphi_t$. It follows that the kernel of operator $(\Delta_t + t)$ is zero on any $t \in E$. By Implicit Function Theorem, $E$ is an open set. It remains to prove that $E$ is also a closed set. This is related to apriori estimates for solution $\varphi_t$ of (2.2) on $t \in E$ below.

First we deal with the $C^0$-estimate. We may assume that $t \geq \delta$ by the implicit theorem since (2.2) is solvable at $t = 0$ [JMR11]. By a direct computation, we have

$$\frac{d}{dt} (I_\omega(\varphi_t) - J_\omega(\varphi_t)) = -\frac{1}{V} \int_M \varphi_t \Delta_t \varphi_t \omega_t^n = \frac{1}{V} \int_M (\Delta_t \varphi_t + t \varphi_t) \Delta_t \varphi_t \omega_t^n. \tag{2.2}$$

Note that

$$\Delta_t \varphi_t = -t \varphi_t - \varphi_t$$

by differentiating $\int_M e^{h_{\tilde{\omega}} - t \varphi_t} \tilde{\omega}^n = V$. By the fact that the first non-eigenvalue of $\Delta_t$ is strictly bigger than $t$, we get

$$\frac{d}{dt} (I_\omega(\varphi_t) - J_\omega(\varphi_t)) \geq 0.$$
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This means that $I_{\bar{\omega}}(\varphi_t) - J_{\bar{\omega}}(\varphi_t)$ is increasing in $t$. Thus

$$I_{\bar{\omega}}(\varphi_t) \leq (n + 1) I_{\bar{\omega}}(\varphi_0) \leq C.$$ 

By using the Green formula [JMR11], we derive

$$\text{osc}(\varphi_t) \leq C.$$ 

To get the $C^2$-estimate, we rewrite (2.2) as

$$\frac{\omega^* + \sqrt{-1} \partial \bar{\partial} \varphi^*}{n} = e^{h^* - t \varphi^* - (\mu - t)(\psi - \delta(\mathcal{S}_{\beta}))(\omega^*)}.$$ 

where $\varphi^* = \varphi - \delta |\mathcal{S}|^\beta + \psi$ and $h^*$ is the log Ricci potential of $\omega^*$ as in (1.1). Since $\text{Ric}(\omega^*_0) > 0$, by the Chern-Lu inequality [Cher68], [Lu68], we have

$$\Delta_t \log \text{tr}_\omega (\omega^*) \geq -a(\omega^*) \text{tr}_\omega (\omega^*),$$

in $M \setminus D$,

where $a = a(\omega^*)$ is a uniform constant which depends only on the upper bound of bisectional holomorphic curvature of $\omega^*$, and so it depends only on $\omega_0$ and the divisor $D$. Set

$$u = \log \text{tr}_\omega (\omega^*) - (a + 1) \varphi^*.$$ 

Then there exists a uniform constant $C = C(\sup_M \phi, \sup_M \psi)$ such that

$$\Delta_t u \geq e^{u - C(a + 1)} - n(a + 1).$$ 

By the maximum principle as in [JMR11], it follows

$$\omega_t \geq C^{-1} \omega^*.$$ 

By (2.2), we also get

$$\omega_t \leq C \omega^*.$$ 

Once (2.6) and (2.8) hold, we can apply the $C^{2,\alpha';\beta}$-regularity theorem in [JMR11] to show

$$\|\varphi\|_{C^{2,\alpha';\beta}(M)} \leq C$$ 

for some $\alpha' \leq \alpha$. Thus $\varphi_t \in \mathcal{H}^{2,\alpha';\beta}(M, \omega_0)$. This implies that $E$ is a closed set and so $E = [0, \mu]$.

By a direct computation as in [DT92], we have

$$t \left( J_{\bar{\omega}}(\varphi_\mu) - \frac{1}{V} \int_M \varphi_\mu \hat{\omega}^n \right) = - \int_0^t (I - J)_{\bar{\omega}}(\varphi_\varsigma) ds \leq 0, \forall t \leq \mu.$$ 

Note that $\int_M e^{h^* - \mu \varphi^*} \hat{\omega}^n = V$. Thus

$$F_{\omega,\mu}(\varphi_\mu) = J_{\bar{\omega}}(\varphi_\mu) - \frac{1}{V} \int_M \varphi_\mu \hat{\omega}^n \leq 0.$$ 

By the cocycle condition of log $F$-functional, it follows

$$F_{\omega,\mu}(\psi - \phi) = -F_{\omega,\mu}(\varphi_\mu) \geq 0.$$ 

Again by the cocycle condition, we get

$$F_{\omega_0,\mu}(\psi) = F_{\omega,\mu}(\psi - \phi) + F_{\omega_0,\mu}(\phi) \geq F_{\omega_0,\mu}(\phi).$$ 

Hence we prove that $F_{\omega_0,\mu}(\cdot)$ takes the minimum at $\phi$. □
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**Corollary 2.2.** Suppose that there exists a conic Kähler-Einstein metric on $M$ with cone angle $2\pi \beta$ along $D$. Then

$$F_{\omega_0, \mu}(\psi) \geq -c(\omega_0, \mu), \quad \forall \psi \in \mathcal{U}_0 \subset \mathcal{U}_\alpha \subset \mathcal{U}_\omega \mathcal{F}(M, \omega_0).$$

3. APPROXIMATION OF CONIC KÄHLER METRICS

In this section, we construct approximated smooth Kähler potentials of solution $\varphi_t$ of (2.2) on each $t \in (0, \mu)$ by solving certain complex Monge-Ampère equation. First we shall smooth the conic metric $\hat{\omega} = \omega_0$. Note

$$\hat{\omega}^n = f_0 \omega^n_0,$$

where $f_0 = g\frac{1}{|S|^2}$ for some $L^\infty$-function $g$. In particular, $f_0$ is a $L^p$-function.

Take a family of smooth functions $f_\delta$ with $\int_M f_\delta \omega_0^n = \int_M \omega_0^n$ such that $f_\delta$ converge to $f_0$ in $L^p$ as $\delta \to 0$. Then by the Yau’s solution to Calabi’s problem, there are a family of Kähler potentials $\Psi_\delta$, which solve equations $(\delta > 0)$,

$$(\omega_0 + \sqrt{-1} \partial \bar{\partial} \Psi_\delta)^n = f_\delta \omega^n_0.$$

By the Kolodziej’s Hölder estimate [Kol08], $\Psi_\delta$ converge to $\psi$ in the $C^n$-norm modulo constants as $\delta \to 0$. For simplicity, we set $\omega_\delta = \omega_0 + \sqrt{-1} \partial \bar{\partial} \Psi_\delta$.

We modify (2.3) by a family of Ricci curvature equations with parameter $\delta \in (0, \delta_0]$ for each $t \in [0, \mu]$,

$$\text{Ric}(\omega_\delta^\beta) = t \omega_\delta^\beta + (\mu - t) \omega_\delta + (1 - \beta) \eta_\delta,$$

where $\omega_\delta^\beta = \omega_\delta + \sqrt{-1} \partial \bar{\partial} \varphi_\delta$ and $\eta_\delta = \lambda \omega_0 + \sqrt{-1} \partial \bar{\partial} \log(\delta + |S|^2)$. (3.1) are in fact a family of twisted Kähler-Einstein metric equations associated to positive $(1,1)$-forms $\Omega = (\mu - t) \omega_\delta + (1 - \beta) \eta_\delta$ [ST12]. One can check that (3.1) are equivalent to the following complex Monge-Ampère equations,

$$(\omega_\delta + \sqrt{-1} \partial \bar{\partial} \varphi_\delta)^n = e^{h_\delta - t \varphi_\delta} \omega_\delta^n,$$

where $h_\delta$ are twisted Ricci potentials of $\omega_\delta$ defined by

$$(3.3) \quad \sqrt{-1} \partial \bar{\partial} h_\delta = \text{Ric}(\omega_\delta) - \mu \omega_\delta - (1 - \beta) \eta_\delta.$$

We shall study the solutions of (3.2) and their convergence as $\delta \to 0$.

Rewrite (3.1) as

$$(3.4) \quad \text{Ric}(\omega_\delta^\beta) = t \omega_\delta^\beta + (\mu - t) \omega_0 + (1 - \beta) \eta_\delta + (\mu - t) \sqrt{-1} \partial \bar{\partial} \Psi_\delta.$$

Then equations (3.4) are equivalent to

$$(3.5) \quad (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_\delta)^n = \frac{1}{(\delta + |S|^2)^{1 - \beta}} e^{h_0 - t \varphi_0 - (\mu - t) \Psi_0} \omega_0^n, \quad t \in (0, \mu],$$

where $\varphi_\delta = \varphi_t, \delta = \varphi_t + \Psi_\delta$.

As in [Ti12], for a fixed $\delta > 0$, we define a family of twisted F-functionals with parameter $t \in (0, \mu]$ as follows,

$$F_{t, \delta}(\varphi) = J_{\omega_0}(\varphi) - \frac{1}{V} \int_M \varphi_0^n - \frac{1}{t} \log \left( \frac{1}{V} \int_M e^{h_\delta - t \varphi_\delta} \omega_\delta^n \right),$$

where $V$ is the volume of $M$.
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where

$$\hat{h}_\delta = h_0 - (1 - \beta) \log(\delta + |S|^2) + (t - \mu)\Psi_\delta + C_\delta, \quad \int_M (e^{\hat{h}_\delta} - 1)\omega_0^n = 0.$$  

Then all $F_{t,\delta}(\cdot)$ are proper for any $t \in (0, \mu)$, $\delta \in (0, \delta_0]$ since log $F$-functionals $F_{\omega_{t,\delta}}(\cdot)$ defined in (3.5) are proper for any $t \in (0, \mu)$. The latter follows from a result in [LS12] by using the fact that $F_{\omega_{t,\mu}}(\cdot)$ is bounded from below according to Theorem 2.1. By the Green formula, we get

$$\text{osc}_M \hat{\varphi}_{t,\delta} \leq C(I_{\omega_0}(\hat{\varphi}_{t,\delta}) + 1) \leq C', \quad \forall \ t \in (0, \mu),$$  

where the constants $C, C'$ depend only on $t$. Note that all higher order estimates for solutions $\hat{\varphi}_{t,\delta}$ depend only on $\delta$ and their $C^0$-norm. Thus by using the continuity method as in the proof of Theorem 2.5 in [T12], (3.8), and so (3.9) are solvable on any $t \in (0, \mu)$, $\delta \in (0, \delta_0]$.

Next we improve higher order estimates for solutions $\hat{\varphi}_{t,\delta}$ to show that they are independent of $\delta > 0$. Let’s introduce a family of smooth Kähler potentials $\Phi_\delta^\beta$ ($\delta > 0$) constructed by Guenancia-Paun in [GP13]. Such $\Phi_\delta^\beta$ have property:

1) $\Phi_\delta^\beta$ converge to $\Phi_0^\beta = k|S|^{2\beta}$ as $\delta \to 0$ in sense of Hölder-norm.

2) Let

$$h_{\kappa_\delta} = - \log \frac{\kappa_\delta^n}{\omega_0^n} - \Phi_\delta^\beta + h_0$$

be Ricci potentials of $\kappa_\delta = \omega_0 + \sqrt{-1}\partial\bar{\partial}\Phi_\delta^\beta$. Then $\frac{1}{\delta + |S|^2}e^{h_{\kappa_\delta}}$ is uniformly bounded on $\delta$.

3) The bisectional holomorphic curvatures $R_{\delta\bar{i}\bar{j}j\bar{j}}$ of $\kappa_\delta$ satisfy: for any Kähler metric $\omega_{\phi + \Phi_\delta^\beta} = \kappa_\delta + \sqrt{-1}\partial\bar{\partial}\phi$, it holds

$$\sum_{i<j}(1 + \phi_{\bar{i}i}) + \frac{1}{1 + \phi_{\bar{j}j}} - 2)R_{\delta\bar{i}\bar{j}j\bar{j}} - C_0 \text{tr}_{\kappa_\delta}(\omega_{\phi + \Phi_\delta^\beta}) \Delta_{\omega_{\phi + \Phi_\delta^\beta}} \Phi_\delta^\beta$$

$$\quad + \Delta_{\kappa_\delta} \log \left(\frac{\kappa_\delta^n}{\omega_0^n} \times (\delta + |S|^2)^{1-\beta}\right)$$

$$\quad \leq C \sum_{i<j}(1 + \phi_{\bar{i}i}) + \frac{1 + \phi_{\bar{j}j}}{1 + \phi_{\bar{i}i}} + C \text{tr}_{\kappa_\delta}(\omega_{\phi + \Phi_\delta^\beta}) \times \text{tr}_{\omega_{\phi + \Phi_\delta^\beta}}(\kappa_\delta) + C,$$

where $C_0$ and $C$ are two uniform constants.

The following is about uniform apriori $C^2$-estimate for $\varphi = \varphi_{t,\delta}$.

**Lemma 3.1.** For any $t \in (0, \mu), \delta \in (0, \delta_0]$, it holds

$$C^{-1}\kappa_\delta \leq \omega_{\varphi_{t,\delta}} \leq C\kappa_\delta .$$

Here $C$ is a uniform constant which depends only on the metric $\hat{\omega}$ and $t$.

**Proof.** Let $\bar{\varphi} = \bar{\varphi}_\delta = \hat{\varphi}_\delta - \Phi_\delta^\beta$. Then (3.8) are equivalent to

$$(\kappa_\delta + \sqrt{-1}\partial\bar{\partial}\bar{\varphi})^n = \frac{1}{(\delta + |S|^2)^{1-\beta}} e^{h_{\kappa_\delta} + \bar{\varphi} - (\mu - t)\Psi_\delta - (1-t)\Phi_\delta^\beta} \kappa_\delta^n, \quad t \in (0, \mu_0).$$
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Following Yau’s $C^2$-estimate in [Yau78], we have

$$-\Delta_{\omega_{\delta}^\phi} \log \text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta) \leq \frac{1}{\text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta)} \sum_{i<j} \left( \frac{1 + \bar{\phi}_{ii}}{1 + \bar{\phi}_{jj}} + \frac{1 + \bar{\phi}_{jj}}{1 + \bar{\phi}_{ii}} - 2 \right) R_{\delta i j i j}$$

$$+ \frac{1}{\text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta)} \Delta_{\kappa_{\delta}}(t\bar{\phi} + (\mu - t)\Psi_{\delta} + (1 - t)\Phi_{\delta}^\beta - \bar{h}_{\kappa_{\delta}}).$$

On the other hand, by

$$B\kappa_{\delta} + \sqrt{-1}\partial\bar{\partial}\Psi_{\delta} \geq 0,$$

it is easy to see

$$\Delta_{\omega_{\phi_{\delta}}^\delta} \Psi_{\delta} \geq \frac{\Delta_{\kappa_{\delta}} \Psi_{\delta}}{\text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta)} - n\text{tr}_{\omega_{\phi_{\delta}}^\delta}(\kappa_{\delta}).$$

Using the fact

$$\Delta_{\kappa_{\delta}} h_0 \geq -A,$$

we get

$$\frac{1}{\text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta)} \Delta_{\kappa_{\delta}}(t\bar{\phi} + (\mu - t)\Psi_{\delta} + (1 - t)\Phi_{\delta}^\beta - \bar{h}_{\kappa_{\delta}}) - \Delta_{\omega_{\phi_{\delta}}^\delta} \Psi_{\delta}$$

$$\leq \frac{1}{\text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta)} \Delta_{\kappa_{\delta}} \log \left( \frac{\kappa_{\delta}^\beta}{\omega_{\delta}^\phi} \right) (\delta + |S'|_{0}^{1-\beta}) + \frac{n(1 - t) + A}{\text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta)} + t + n\text{tr}_{\omega_{\phi_{\delta}}^\delta}(\kappa_{\delta}).$$

Thus by the Guenancia-Paun inequality (3.7) for metrics $\omega_{\phi_{\delta}}^\delta$, we deduce from (3.10),

$$-\Delta_{\omega_{\phi_{\delta}}^\delta} (\log \text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta) + C_0 \Phi_{\delta}^\beta - \Psi_{\delta}) \leq C'\text{tr}_{\omega_{\phi_{\delta}}^\delta}(\kappa_{\delta}) + C'.$$

Let

$$u = \log \text{tr}_{\kappa_{\delta}}(\omega_{\phi_{\delta}}^\delta) + C_0 \Phi_{\delta}^\beta - \Psi_{\delta} - (C' + 1)\bar{\phi}_{\delta}.$$

Then

$$\Delta_{\omega_{\phi_{\delta}}^\delta} u \geq \text{tr}_{\omega_{\phi_{\delta}}^\delta}(\kappa_{\delta}) - C''.$$

By the maximum principle, it follows

$$\omega_{\phi_{\delta}}^\delta = \kappa_{\delta} + \sqrt{-1}\partial\bar{\partial}\bar{\phi}_{\delta} \geq C^{-1}\kappa_{\delta}.$$

By (3.9), we can also get

$$\omega_{\phi_{\delta}}^\delta \leq C\kappa_{\delta}.$$

\begin{theorem}
For any $t \in (0, \mu)$, it holds

$$\lim_{\delta} \phi_{t,\delta} = \phi_t$$

in sense of Hölder-norm.
\end{theorem}
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Proof. First we claim that $\varphi_{t, \delta}$ converges to a $C^{2, \alpha; \beta}$-solution of (2.2) as $\delta \to 0$. In fact, by the Kolodziej’s Hölder estimate, we see that $\hat{\varphi}_{t, \delta}$ converge to a Hölder continuous solution $\varphi'$ of following complex Monge-Ampère equation in the current sense,

$$\left(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi'\right)^n = \frac{1}{|S|^{2-2\beta}} e^{h_0 - t\psi' + (t-\mu)\psi} \omega_0^n. \quad (3.14)$$

Clearly, (3.14) is nothing, just (2.2). Since $\omega^* = \omega_0 + \sqrt{-1} \partial \bar{\partial} \Phi_0$ is equivalent to $\hat{\omega}$, by Lemma 3.1, we get

$$C^{-1} \hat{\omega} \leq \omega' \leq C \hat{\omega}, \text{ in } M \setminus D, \quad (3.15)$$

where $C$ is a uniform constant. Note that (3.14) implies that $\varphi' - \psi$ is a solution of (2.2). Thus by the $C^{2, \alpha; \beta}$ regularity theorem, $\varphi' - \psi$ is a $C^{2, \alpha; \beta}$-solution of (2.2). This proves the claim.

On the other hand, according to the proof in Theorem 2.1, it is easy to see that $C^{2, \alpha; \beta}$-solution of (2.2) as a twisted Kähler-Einstein metric is unique. Thus $\varphi' - \psi$ must be $\varphi_t$. The theorem is proved. \hfill \Box

4. Smoothing of twisted Ricci potentials

Define a Log Ricci potential $h_t$ of $\hat{\omega}_{\varphi_t}$ of solution of (2.2) on $t$ by

$$\sqrt{-1} \partial \bar{\partial} h_t = \text{Ric}(\hat{\omega}_{\varphi_t}) - \mu \hat{\omega}_{\varphi_t} - 2\pi (1 - \beta) [D], \quad \int_M e^{h_t} \hat{\omega}_{\varphi_t}^n = V,$

and define a twisted Ricci potential of $\omega_{\varphi_{t, \delta}} = \omega_{\delta} + \sqrt{-1} \partial \bar{\partial} \varphi_{t, \delta}$ of solution of (3.2) on $t$ by

$$\sqrt{-1} \partial \bar{\partial} h_{t, \delta} = \text{Ric}(\omega_{\varphi_{t, \delta}}) - \mu \omega_{\varphi_{t, \delta}} - (1 - \beta) \eta_{\delta}, \quad \int_M e^{h_{t, \delta}} (\omega_{\varphi_{t, \delta}})^n = V.$$ \hfill (4.1)

Then it is easy to see

$$h_t = -(\mu - t) \varphi_t + \text{const},$$

and

$$h_{t, \delta} = -(\mu - t) \varphi_{t, \delta} + \text{const}.$$ \hfill (4.2)

Thus by Theorem 3.2, we have

Lemma 4.1. For any $t \in (0, \mu)$, it holds

$$\lim_{\delta \to 0} h_{t, \delta} = h_t$$

in sense of Hölder-norm.

To smooth $h_{t, \delta}$ for each fixed $\delta \in (0, \delta_0)$, we introduce the following twisted Kähler-Ricci flow,

$$\frac{\partial}{\partial s} \omega_{\delta}^\psi = -\text{Ric}(\omega_{\delta}^\psi) + \mu \omega_{\delta}^\psi + (1 - \beta) \eta_{\delta},$$

$$\omega_{\delta}^\psi(0, 0) = \omega_{\psi_{0, \delta}} = \omega_{\psi_{0, \delta}}|_{s=0} = \omega_{\phi_{t, \delta}}.$$
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Clearly, the twisted Ricci potential $h_{t,s,\delta}$ of $\omega^\delta_{\psi_{s,\delta}}$ is given by

$$h_{t,s,\delta} = -\frac{\partial}{\partial s}\psi_{s,\delta} + \text{const.}$$

In particular,

$$h_{t,\delta} = -\frac{\partial}{\partial s}\psi_{s,\delta}|_{s=0} + \text{const.}$$

(4.1) reduces to a complex Monge-Ampère flow,

$$\partial_{s}\tilde{\psi}_{s,\delta} = \log\left(\frac{(\omega^\delta_{\varphi_{t,\delta}}+\tilde{\psi})^n}{(\omega^\delta_{\varphi_{t,\delta}})^n} + \mu \tilde{\psi} - h_{t,\delta}\right),$$

(4.2)

Here $h_{t,\delta}$ can be normalized so that $h_{t,\delta} = -(\mu - t)\varphi_{t,\delta}$. Then $h_{t,\delta} = -\frac{\partial}{\partial s}\psi_{s,\delta}|_{s=0}$.

Similarly to Kähler-Ricci flow in [Ti97], applying the maximum principle to (4.2), we have following estimates (also see [LZ14]).

**Lemma 4.2.**

1) \(|\frac{\partial}{\partial s}\tilde{\psi}_{s,\delta}|^2 + 8|\nabla'\frac{\partial}{\partial s}\tilde{\psi}_{s,\delta}|^2 \leq e^{2\mu s}((\mu - t)^2 \|\varphi_{t,\delta}\|_{C^0(M)})^n.

2) $\Delta'(-\frac{\partial}{\partial s}\tilde{\psi}_{s,\delta}) \geq e^{\mu s}h_{t,\delta}.$

Here $\Delta'$, $\Delta$ are Laplace operators associated to metrics $\omega^\delta_{\psi_{s,\delta}}$, $\omega^\delta_{\varphi_{t,\delta}}$, respectively.

**Lemma 4.3.** Let $v = v_{t,s,\delta}$ be a normalization of $h_{t,s,\delta}$ by adding a suitable constant such that

$$\int_M v(\omega^\delta_{\psi_{s,\delta}})^n = 0.$$  

Let $\gamma = \frac{1}{2+8n}$. Then there exists a small number $\epsilon > 0$ such that for any $t$ and $\varphi_{t,\delta}$ satisfying

$$((\mu - t)^{1+\frac{\gamma}{2}} \|\varphi_{t,\delta}\|_{C^0(M)} \leq \epsilon,$$

we have

$$\|v\|_{C^{1+\frac{\gamma}{2}}(M)} \leq C((\mu - t)^{\frac{\gamma}{2}}$$

and

$$\text{osc}_M \tilde{\psi}_{s,\delta} \leq C(\mu - t)^{\frac{\gamma}{2}}, \forall s \in [(\mu - t)^{2\gamma}, 1],$$

provided that for any $s \in [(\mu - t)^{2\gamma}, 1]$ the first non-zero eigenvalue

$$\lambda_1 \geq \lambda_0 > 0$$

of Laplace operator $\Delta'$ associated to the metric $\omega^\delta_{\psi_{s,\delta}}$ and the following condition holds: there exists a constant $a > 0$ such that for any $x_0 \in M$ and $0 < r < 1$,

$$\text{vol}(B_r(x_0)) \geq ar^{2n}$$

with respect to $\omega^\delta_{\psi_{s,\delta}}$. Here $C = C(a, \lambda_0)$ denotes a uniform constant depending only on the constants $a$ and $\lambda_0$. 

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Proof. Lemma 4.3 can be proved following the argument of smoothing lemma in [Ti97] (also see Proposition 4.1 in [CTZ05]). In fact, under the conditions (4.5) and (4.6), using the estimates 1) and 2) in Lemma 4.1, we get

$$|v| \leq C(a, \lambda_0)(1 + \|h_t,\delta\|_{C^0(M)})(\mu - t)^{\frac{3}{4(n+1)}}, \forall \ s \in [(\mu - t)^{2\gamma}, 1].$$

On the other hand, again by the estimate 1) in Lemma 4.1, we have

$$\|\nabla' v\|_{C^0(M)}^2 \leq \frac{1}{s} e^2 \|h_t,\delta\|_{C^0(M)}^2, \forall \ s > 0.$$

Combining these two relations, we derive

$$\|v_s\|_{C^1(M)}^2 \leq C(a, \lambda_0)(1 + \|h_t,\delta\|_{C^0(M)})(\mu - t)^{\frac{3}{8(n+1)}}, \forall \ s \in [(\mu - t)^{2\gamma}, 1].$$

Note

$$(\mu - t)\varphi_t,\delta = h_t,\delta.$$

Thus under the assumption (4.3), we get (4.4) immediately.

By the estimate 1) in Lemma 4.2, we have

$$|\frac{\partial}{\partial s} \tilde{\psi}| \leq e \|h_t,\delta\|_{C^0(M)}, \forall \ s \leq 1.$$

Note

$$\tilde{\psi} = \left( \int_0^{(\mu - t)^{2\gamma}} + \int_s^{(\mu - t)^{2\gamma}} \right) \left( \frac{\partial}{\partial s} \tilde{\psi} \right).$$

Then by the assumption (4.3), we obtain

$$\text{osc}_M \tilde{\psi} \leq (\mu - t)^{2\gamma} \sup_{s \in [0,1]} \|\frac{\partial}{\partial s} \tilde{\psi}\|_{C^0(M)}$$

$$\quad + \sup_{s \in [(\mu - t)^{2\gamma}, 1]} \|\frac{\partial}{\partial s} \tilde{\psi}\|_{C^0(M)}$$

$$(4.8) \leq C \epsilon (\mu - t)^{2\gamma}.$$

□

5. Convergence of twisted Kähler-Ricci flows

In this section, we deal with the local convergence of flows (4.1). First, similarly to Lemma 3.1, we have

Lemma 5.1. For any $t \in (0, \mu), \delta \in (0, \delta_0]$, it holds

$$C^{-1} \kappa_\delta \leq \omega^\delta_{\psi,\delta} \leq C \kappa_\delta.$$ (5.1)

Here $C$ is a uniform constant depends only on metrics $\tilde{\omega}, \omega^*$ and norms of $\|\psi_{s,\delta}\|_{C^0(M)}$, $\|\frac{\partial}{\partial s} \psi_{s,\delta}\|_{C^0(M)}$.  

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**Proof.** Let $\tilde{\psi}_s = \tilde{\psi}_{s,\delta} = \psi_{s,\delta} + \Psi_\delta - \Phi_\delta^\beta$. Then by (4.2), $\tilde{\psi}$ satisfies the following complex Monge-Ampère flow,

$$
\frac{\partial}{\partial s} \tilde{\psi} = \log \frac{(\kappa_\delta + \delta \tilde{\partial} \tilde{\partial} \tilde{\psi})^n}{\kappa_\delta^n} + \mu \tilde{\psi} - \tilde{h}_\kappa,
$$

(5.2)

where $\tilde{\psi}_{0,\delta} = \tilde{\psi}_s(0, \cdot) = \varphi_{t,\delta} + \Psi_\delta - \Phi_\delta^\beta$.

Following the estimate of (3.10), for the parabolic equation (5.2), we get Yau’s $C^2$-estimate,

$$
\left(\frac{\partial}{\partial s} - \Delta^\delta_{\omega_{\tilde{\psi}_s}}\right) \log \tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s})
\leq \frac{1}{\tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s})} \sum_{i<j} \left(1 + \tilde{\psi}_{ii} + 1 + \tilde{\psi}_{jj} - 2\right) R_{\delta ii,jj} + \frac{1}{\tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s})} \Delta_{\kappa_\delta}(\mu \tilde{\psi}_s - \tilde{h}_\kappa)
$$

On the other hand, by (3.11), we have

$$
\frac{1}{\tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s})} \Delta_{\kappa_\delta}(\mu \tilde{\psi}_s - \tilde{h}_\kappa)
\leq \frac{1}{\tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s})} \Delta_{\kappa_\delta} \log \left(\frac{\omega_0^n}{\omega^\delta_{\tilde{\psi}_s}}\right) \leq \frac{A}{\tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s})} + \mu.
$$

By the Guenancia-Paun inequality (3.7), it follows

$$
\left(\frac{\partial}{\partial s} - \Delta^\delta_{\omega_{\tilde{\psi}_s}}\right) \left(\log \tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s}) + C_0 \Phi_\delta^\beta\right)
\leq C_0' \tr_{\omega^\delta_{\tilde{\psi}_s}}(\kappa_\delta) + C_0',
$$

where $C_0$ and $C_0'$ are two uniform constants depending only on metrics $\tilde{\omega}$ and $\omega^\delta$. Hence by choosing a large number $B$, we deduce

$$
\left(\frac{\partial}{\partial s} - \Delta^\delta_{\omega_{\tilde{\psi}_s}}\right) \left(\log \tr_{\kappa_\delta}(\omega^\delta_{\tilde{\psi}_s}) + C_0 \Phi_\delta^\beta - B \tilde{\psi}_s\right)
\leq -\tr_{\omega^\delta_{\tilde{\psi}_s}}(\kappa_\delta) + C_0''.
$$

Now we can apply the maximum principle to see that there exists a uniform constant $C$, which depends only on $\tilde{\omega}, \omega^\delta, \|\psi_{s,\delta}\|_{C^0(M)}$ and $\|\frac{\partial}{\partial s} \psi_{s,\delta}\|_{C^0(M)}$, such that

$$\omega^\delta_{\tilde{\psi}_{s,\delta}} = \kappa_\delta + \sqrt{-1} \tilde{\partial} \tilde{\partial}^\delta \tilde{\psi}_{s,\delta} \geq C^{-1} \kappa_\delta.$$

By (5.2), we also obtain

$$\kappa_\delta \geq C' \omega^\delta_{\tilde{\psi}_{s,\delta}},$$

where $C' = C'(\tilde{\omega}, \omega^\delta, \|\psi_{s,\delta}\|_{C^0(M)}, \|\frac{\partial}{\partial s} \psi_{s,\delta}\|_{C^0(M)})$. \qed

**Theorem 5.2.** For any $s \in (0,1]$, $\omega^\delta_{s,\delta}$ converge to a conic Kähler metric $\omega_{\tilde{\psi}_{s,\delta}} = \tilde{\omega} + \sqrt{-1} \tilde{\partial} \tilde{\partial}^\delta_{s,\delta}$ in sense of $C^{2,\alpha;\beta}$ Kähler potentials.

**Proof.** By the estimate 1) in Lemma 4.2, we have

$$
\|\psi_{s,\delta}\|_{C^0(M)}, \|\frac{\partial}{\partial s} \psi_{s,\delta}\|_{C^0(M)} \leq e(\mu - t)\|\varphi_{t,\delta}\|_{C^0(M)}.
$$
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Then by Lemma 5.1, we see that there exists a uniform constant $C$, which depends only on $\omega, \varphi$, such that

$$C^{-1} \kappa \delta \leq \omega_{\psi, \delta} \leq C \kappa \delta, \quad \forall \delta \in (0, \delta_0).$$

Thus the Sobolev constant associated to $\omega_{\psi, \delta}$ is uniformly bounded above as same as the metric $\kappa \delta$ (cf. [LZ14]). Derivativing (5.2) on $s$, we have

$$(\partial_{ss} - \Delta_{\psi, \delta}) \dot{\psi} = \mu \dot{\psi},$$

where $\dot{\psi} = \dot{\psi}_{s, \delta} = \partial_s \psi_{s, \delta}$. Hence the standard Moser iteration method for the parabolic equation implies that there exist a positive number $\alpha$ and a uniform constant $C$ such that

$$\sup_{x, y \in M} |\dot{\psi}(x) - \dot{\psi}(y)| \leq C.$$ 

As a consequece, $\dot{\psi}$ converges to a Hölder continuous function $f$ associated to the metric $\omega$ as $\delta \to 0$. Namely,

$$\sup_{x, y \in M} |f(x) - f(y)| \leq C.$$

On the other hand, by the Kolodziej’s Hölder estimate, $\dot{\psi}$ are uniformly Hölder continuous functions, so they converge to a Hölder continuous function $\hat{\phi}_t = \hat{\phi}_{t, s}$ as $\delta \to 0$. Moreover, $\hat{\phi}_t$ is a current solution of following complex Monge-Ampère equation,

$$(\hat{\omega} + \sqrt{-1} \partial \bar{\partial} \hat{\phi})^n = e^{f + h - \mu \hat{\phi}}^n.$$

By the regularity theorem in [JMR11], it follows that $\hat{\phi}_t$ is a $C^{2, \alpha'; \beta}$-solution. Hence $\hat{\omega}_{\hat{\phi}_t}$ is a conic Kähler metric.

Lemma 5.3. For any $t$ and $\varphi_t$ satisfying

$$(\mu - t)^{1+\frac{2}{\gamma}} \|\varphi_t\|_{C^0(M)} \leq e,$$

it holds

$$(5.6) \quad \text{osc}_M (\hat{\phi}_{t, s} - \varphi_t) \leq C (\mu - t)^{\frac{2}{\gamma}}, \quad \forall s \in (0, 1].$$

Proof. In (4.8), we in fact prove

$$\text{osc}_M \dot{\psi}_{s, \delta} \leq C (\mu - t)^{\frac{2}{\gamma}}, \quad \forall s \in (0, 1].$$

Then (5.6) follows immediately from Theorem 3.2 and Theorem 5.2 by taking $\delta \to 0$. □

Lemma 5.4. Let $\tilde{v} = \tilde{v}_{t, s}$ be a normalization of $h \tilde{\omega}_{\varphi_{t, s}}$ by adding a suitable constant such that

$$\int_M \tilde{v}(\tilde{\omega}_{\varphi_{t, s}})^n = 0.$$ 

Then for any $t$ and $\varphi_t$ satisfying (5.3), it holds

$$(5.7) \quad \|\tilde{v}_{t, s}\|_{C^{2, \alpha'}(M)} \leq C (1 - t)^{\frac{2}{\gamma}}, \quad \forall s \in [(\mu - t)^{2\gamma}, 1].$$

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Proof. We claim: for the metric $\hat{\omega}_{\phi_{t,s}}$, it holds

$$\frac{1}{2} \hat{\omega} \leq \hat{\omega}_{\phi_{t,s}} \leq 2 \hat{\omega}. \quad (5.8)$$

By the above claim together Theorem 5.2, we see that there exists a small $\delta_0$ such that the conditions (4.6) and (4.7) in Lemma 4.3 are satisfied for metrics $\omega^\delta_{\psi_{t,s}}$ with $\delta \in (0, \delta_0]$. Note that (4.3) also holds by Theorem 3.2. Then by Lemma 5.1 it follows that (4.4) holds for $v = v_{t,s,\delta}$ as $\delta \to 0$, we get (5.7).

We prove (5.8) by contradiction. If (5.8) is not true, then there exists a $\psi \in \mathcal{H}^{2,\alpha;\beta}(M,\omega_0)$ such that the solution $\phi_t$ of (2.2) on $t$ satisfies (5.5) and the $C^{2,\alpha;\beta}$-norm of Kähler potential $\tilde{\phi}_{t,s}$ in Theorem 5.2 satisfies

$$\|\tilde{\phi}_{t,s}\|_{C^{2,\alpha;\beta}(M)} \geq A_0, \quad (5.9)$$

where $A_0$ is a positive number. On the other hand, from the proof of Theorem 5.2 the $C^{2,\alpha;\beta}$-norm of $\tilde{\phi}_{t,s}$ depends on $\phi_t$ continuously. Thus we may also assume that

$$\|\tilde{\phi}_{t,s}\|_{C^{2,\alpha;\beta}(M)} \leq 2A_0$$

and

$$\frac{1}{4} \hat{\omega} \leq \hat{\omega}_{\tilde{\phi}_{t,s}} \leq 4 \hat{\omega}. \quad (5.10)$$

Once (5.10) holds, we can use the above argument again to conclude that there exists a small $\delta_0$ such that (4.4) holds for $v = v_{t,s,\delta}$ as $\delta \in (0, \delta_0]$. Taking the limit of $v_{t,s,\delta}$ as $\delta \to 0$, we get (5.7) for $\hat{\omega}_{\tilde{\phi}_{t,s}}$. Applying Implicit Function Theorem to (5.4), we obtain

$$\|\tilde{\phi}_{t,s}\|_{C^{2,\alpha;\beta}(M)} \leq C(\epsilon) \to 0, \text{ as } \epsilon \to 0.$$ \hspace{1cm} But this is impossible by (5.9). The claim is proved.\hfill \Box

6. Properness of $F_{\omega_0,\mu}(\cdot)$

By using the estimates at last section, we can improve Theorem 2.1 to

Theorem 6.1. Suppose that there exists a conic Kähler-Einstein metric $\omega = \omega_{\text{CKE}}$ on $M$ along $D$ with cone angle $2\pi \beta \in (0, 2\pi)$. Then there exists two uniform constants $\delta$ and $C$ such that

$$F_{\omega_0,\mu}(\psi) \geq \delta I(\psi)_{\omega_{\text{CKE}}} - C, \quad \forall \psi \in \mathcal{H}^{2,\alpha;\beta}(M,\omega_0). \quad (6.1)$$

Proof. First, by the first relation in (2.9), we get an identity

$$F_{\omega,\mu}(\psi - \phi) = F_{\omega,\mu}(-\varphi_\mu) = \frac{1}{\mu} \int_0^\mu (I - J) \omega s(\varphi_\mu)ds.$$ \hspace{1cm} (15)
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Then as in [TZ00], [CTZ05], we obtain

$$F_{\omega, \mu}(\psi - \phi) \geq \frac{1}{\mu} (\mu - t) (I - J) \omega (\varphi_t)$$

$$\geq \frac{1}{n\mu} (\mu - t) J \omega (\varphi_t)$$

$$\geq \frac{1}{n\mu} (\mu - t) J \omega (\varphi_\mu) - \frac{1}{n\mu} (\mu - t) \text{osc}_M (\varphi_t - \varphi_\mu)$$

(6.2)

$$\geq \frac{1}{n\mu(n+1)} (\mu - t) I_\omega (\varphi_\mu) - \frac{1}{n\mu} (\mu - t) \text{osc}_M (\varphi_t - \varphi_\mu).$$

Next, for a small $\epsilon$, we choose a $t$ such that

$$\mu - t \geq (\mu - t)^{1 + \frac{2}{n}} \| \varphi_t \|_{C^0(M)} = \epsilon. \quad (6.3)$$

Without loss of generality, we may assume that the above inequality can be obtained, otherwise $\| \varphi_t \|_{C^0(M)}$ is uniformly bounded and the situation will be simple.

Then by Theorem 5.2 and Lemma 5.3, there exists a $C^{2, \alpha'}$ Kähler potential $\tilde{\varphi}_t$ such that

$$\text{osc}_M (\varphi_t - \varphi_\mu) \leq \text{osc}_M (\tilde{\varphi}_t - \varphi_\mu) + \text{osc}_M (\tilde{\varphi}_t - \varphi_t)$$

$$\leq \text{osc}_M (\tilde{\varphi}_t - \varphi_\mu) + C \epsilon (\mu - t)^{\frac{2}{n}}. \quad (6.4)$$

On the other hand, by Lemma 5.4, we can apply Implicity Function Theorem to (5.4) to get

$$\text{osc}_M (\tilde{\varphi}_t - \varphi_\mu) \leq C(\epsilon) \to 0, \text{ as } \epsilon \to 0.$$

Thus

(6.4)

$$\text{osc}_M (\varphi_t - \varphi_\mu) \leq C(\epsilon).$$

Combining (6.2) and (6.4), we have

$$F_{\omega, \mu}(\psi - \phi) \geq \frac{1}{n\mu(n+1)} (\mu - t) I(\varphi_\mu) - C. \quad (6.5)$$

Note

$$\| \varphi_t \|_{C^0(M)} \leq \text{osc}_M (\varphi_t).$$

Then by (6.4), we get

$$\| \varphi_t \|_{C^0(M)} \leq \text{osc}_M (\varphi_\mu) + 1. \quad (6.6)$$

In a special case, we assume that the Kähler potential $\psi$ satisfies

$$\text{osc}_M \psi \leq I_{\omega_0}(\psi) + C_0, \quad (6.7)$$

where $C_0$ is a uniform constant. Then by the relation (6.3) and (6.6), a simple computation shows

$$F_{\omega, \mu}(\psi - \phi) \geq \delta I_{\omega_0}(\varphi_\mu) \frac{1}{\text{osc}_M} - C'$$

$$\geq \delta I_{\omega_0}(\psi) \frac{1}{\text{osc}_M} - C', \quad (6.8)$$
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where $\delta, C' > 0$ are two uniform constants which depending only on the choice of $\epsilon$ in (6.3). Using the cocycle condition in (2.10), we derive immediately,

$$F_{\omega, \mu}(\psi) \geq \delta I_{\omega, \mu}(\psi)^{\frac{1}{1+\theta}} - C''.$$  

In general case, we can use a trick in [TZ00] to derive (6.9) for $\psi$. In fact, we can first apply (6.9) for solutions $\varphi_t$ with $t \geq \epsilon_0 > 0$ to get an estimate for $\text{osc}_M(\varphi_t - \varphi_\mu)$, then by the relation in (6.2) we obtain (6.9) for $\psi$.

End of proof of Theorem 1.1. Theorem 1.1 is an improvement of Theorem 6.1. By Lemma 1.1 we suffice to obtain the estimate (1.2) in Theorem 1.1 for Kähler potentials in $\mathcal{H}^{2,\alpha,\beta}(M, \omega_0)$. It was observed by Phong-Song-Strum-Weinkove that (1.2) can come from (6.1) in case of Kähler-Einstein metric [PSS07]. In fact, as in [TZ00], by (6.1) for solutions $\varphi_t$ with $t \geq \epsilon_0 > 0$, they further show that there exists a $t_0$ with $\mu - t_0 \geq \delta_0 > 0$ (where $\mu = 1$) for some uniform constant $\delta_0$ such that

$$\text{osc}_M(\varphi_{t_0} - \varphi_\mu) \leq A,$$

where $A$ is a uniform constant which depends only on the Kähler-Einstein metric. We show that such choice of $t_0$ can be done similarly in our case of conic Kähler-Einstein metric $\omega_0$ as follows.

By the first relation in (2.9) together with the equation (2.2), we have

$$F_{\omega, \mu}(\varphi_t - \varphi_\mu) = F_{\tilde{\omega}, \mu}(\varphi_t) - F_{\omega, \mu}(\varphi_\mu)$$

$$= -\frac{1}{\mu} \int_0^t (I - J) (\varphi_s) ds + \frac{1}{\mu} \int_0^\mu (I - J) (\varphi_\mu) ds - \frac{1}{\mu} \log(\frac{1}{V}) \int_M e^{(t-\mu)\varphi_t} \hat{\omega}^n_{\varphi_t}$$

$$\leq -\frac{1}{\mu} \int_0^t (I - J) (\varphi_s) ds + \frac{1}{\mu} \int_0^\mu (I - J) (\varphi_\mu) ds + \frac{\mu - t}{\mu V} \int_M \varphi_t \hat{\omega}^n_{\varphi_t}.$$

Note that the first relation in (2.9) is equivalent to

$$-\frac{1}{V} \int_M \varphi_t \hat{\omega}^n_{\varphi_t} = (I - J) (\varphi_t) - \frac{1}{t} \int_0^t (I - J) (\varphi_s) ds.$$

It follows

$$F_{\omega, \mu}(\varphi_t - \varphi_\mu) \leq \frac{1}{\mu} \int_0^\mu (I - J) (\varphi_\mu) ds - \mu - \frac{t}{\mu} (I - J) (\varphi_t)$$

$$\leq \frac{\mu - t}{\mu} [(I - J) (\varphi_\mu) - (I - J) (\varphi_t)]$$

$$(6.10)$$

On the other hand, by the Green formula in [JMR11], (6.10) holds for $\varphi_t - \varphi_\mu$ whenever $t \geq \epsilon_0 > 0$ since Ricci curvature of $\hat{\omega}_{\varphi_t} = \omega + \sqrt{-1} \partial \bar{\partial} (\varphi_t - \varphi_\mu)$ is strictly positive. Then applying (6.3) for $\varphi_t - \varphi_\mu$, we see that there exist two constants $A_0, C > 0$ such that

$$F_{\omega, \mu}(\varphi_t - \varphi_\mu) \geq A_0 I_{\omega}(\varphi_t - \varphi_\mu)^\frac{1}{1+\theta} - C, \ \forall \ t \geq \epsilon_0.$$
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Combining this with (6.10), we derive

\[
I_\omega(\varphi_t - \varphi_\mu) \leq C' \left[ A_0 - \frac{\mu - t}{t} I_\omega(\varphi_t - \varphi_\mu)^{1 - \frac{1}{n+\eta}} \right] \leq C'.
\]

Case 1: For any $t \in [\frac{\mu}{2}, \mu]$, it holds

\[
(\mu - t) I_\omega(\varphi_t - \varphi_\mu)^{1 - \frac{1}{n+\eta}} < \frac{A_0\mu}{2n}.
\]

Then we can choose $t = t_0 = \frac{\mu}{2}$ so that $I_\omega(\varphi_{t_0} - \varphi_\mu)$, and also $\text{osc}_M(\varphi_{t_0} - \varphi_\mu)$ is uniformly bounded. Thus by the relation in (6.12), we get (0.2). The proof is finished.

Case 2: There exists a $t_0 \in [\frac{\mu}{2}, \mu]$ such that

\[
(\mu - t_0) I_\omega(\varphi_{t_0} - \varphi_\mu)^{1 - \frac{1}{n+\eta}} = \frac{A_0\mu}{2n}.
\]

By the above choice of $t_0$, from (6.11) it is easy to see that $\text{osc}_M(\varphi_{t_0} - \varphi_\mu)$ is uniformly bounded. Again by (6.12), we get $\mu - t_0 \geq \delta_0 > 0$ for some uniform constant $\delta_0$. The theorem is proved.

There is another way to get (0.2) by using the Donaldson’s openness theorem, Theorem 7.1 in next section. This is observed in [LS12].

7. A New Proof of Donaldson’s Openness Theorem

In this section, we apply Theorem 6.1 to prove the following Donaldson’s openness theorem.

**Theorem 7.1.** Let $D$ be a smooth divisor of a Fano manifold $M$ with $[D] \in \lambda c_1(M)$ for some $\lambda > 0$ such that there is no non-zero holomorphic field which is tangent to $D$ along $D$. Suppose that there exists a conic Kähler-Einstein metric on $M$ with cone angle $2\pi \beta_0 \in (0, 2\pi)$ along $D$. Then for any $\beta$ close to $\beta_0$ there exists a conic Kähler-Einstein metric with cone angle $2\pi \beta$.

**Proof.** Let $\mu_0 = 1 - \lambda(1 - \beta_0)$. Then $F_{\mu_0, t_0}(\cdot)$ is proper by Theorem 6.1. Thus twisted $F$-functionals $F_{\mu_0, \lambda}^{t_0}(\cdot)$ defined by (3.6) are all proper for any $\delta \in (0, \delta_0]$. By the argument in Section 3, it follows that there exists a solution of (3.2) on $t = \mu_0$ for any $\delta \in (0, \delta_0]$. Hence, for a fixed $\delta = \frac{\delta_0}{2}$, we apply Implicit Function Theorem to see that there exists an $\epsilon_0$ such that (3.2) is solvable for any $\mu \in [\mu_0, \mu_0 + \epsilon_0]$. Note that the twisted Ricci potential $h_\delta$ of $\omega_\delta$ in (3.2) satisfies (3.3) with $\beta = 1 - \frac{1}{\lambda} - \frac{1}{\mu}$. This means that there exists a twisted Kähler-Einstein metric associated to positive $(1,1)$-form $\Omega = (1 - \beta)\eta_{\lambda, \delta}$ for any $\mu \in [\mu_0, \mu_0 + \epsilon_0]$. By a result of X. Zhang and X. W. Zhang [ZhZ13], the twisted $F$-functionals $F_{\mu, t_0}(\cdot)$ is proper for any $\mu \in [\mu_0, \mu_0 + \epsilon_0]$. In fact, they prove a version of Theorem 6.1 on a Fano manifold which admits a twisted Kähler-Einstein metric.

By a direct computation, it is easy to see that for any $\delta \in (0, \delta_0]$ it holds

\[
|F_{\mu, \delta}(\psi) - F_{\mu, \delta}(\psi)| \leq C(\|\Psi_\delta\|_{C^0(M)}, \|\Psi_{\delta_0}\|_{C^0(M)}) \leq C, \forall \psi \in H(M, \omega_0).
\]
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This implies that $F_{\mu, \delta}(\cdot)$ are all proper for any $\delta \in (0, \delta_0)$ and $\mu \in [\mu_0, \mu_0 + \epsilon_0)$. Thus by the argument in Section 3, we see that there exists a solution $\varphi_{\mu, \delta}$ of (3.2) on $t = \mu \in [\mu_0, \mu_0 + \epsilon_0)$ for any $\delta \in (0, \delta_0)$. Moreover,

$$\text{osc}_M \varphi_{\mu, \delta} \leq C,$$

where $C$ is a uniform constant independent of $\mu$ and $\delta$. On the other hand, the estimate (3.3) in Lemma 3.1 also holds for metrics $\omega^\delta_{\varphi_{\mu, \delta}}$. By taking a sequence $\delta_i \to 0$, $\varphi_{\mu, \delta_i}$ converge to a Hölder continuous function $\varphi_\infty - \psi$ which satisfies the weak conic Kähler-Einstein metric equation,

$$\text{Ric}(\omega_\varphi) = \mu \omega_\varphi + 2\pi (1 - \beta)[D], \quad \mu \in [\mu_0, \mu_0 + \epsilon_0)$$

with property:

$$C^{-1} \omega_\varphi \leq \omega_\varphi \leq C \omega, \quad \text{in } M \setminus D$$

for some uniform positive number $C$. By the regularity theorem in [JMR11], $\omega_\varphi_\infty$ is a conic Kähler-Einstein metric in sense of $C^{2,\alpha;\beta}$ Kähler potentials. The proof of Theorem 7.1 is completed.

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