Dedekind Multiplication Semimodules

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Abstract
The aim of this paper is to introduce the concept of Dedekind semimodules and study the related concepts, such as the class of D-semimodules, and Dedekind multiplication semimodules. And thus study the concept of the embedding of a semimodule in another semimodule.

Keywords: Semirings, semimodules, invertible subsemimodules, Dedekind semirings, Dedekind semimodules, multiplication semimodules.

Introduction
In ring theory, an ideal I of a commutative ring with identity R is said to be invertible if I' = R where I' = \{ x ∈ R_S : xI ⊆ R \} and R_S is the total quotient ring of R. The concept of an invertible submodule was introduced by Naoum and Al-Alwan [1] as a generalization of the concept of an invertible ideal.

A semiring is a non-empty set R together with two binary operations addition (+) and multiplication (·) such that (R, +) is a commutative monoid with identity element 0; (R, ·) is a monoid with identity element 1 ≠ 0; r0 = 0r = 0 for all r ∈ R; a(b + c) = ab + ac and (b + c)a = ba + ca for every a, b, c ∈ R. We say that R is a commutative semiring if the monoid (R, ·) is commutative. Let (M, +) be an additive abelian monoid with additive identity 0_M. Then M is called an R-semimodule if there exists a scalar multiplication R × M → M denoted by (r, m) ↦ rm, such that (rr')m = r(r'm); r(m + m') = rm + r'm; (r + r')m = rm + r'm; 1m = m and r0_M = 0_M = 0m for all r, r' ∈ R and all m, m' ∈ M.

Throughout this paper R will denote a commutative semiring with identity, M is unitary (left) R-semimodule. This paper consists four sections. Section I is devoted to introducing the concept of invertible subsemimodules of semimodules as a generalization of the concept of an invertible ideal in semiring. We will also find out some properties of this invertible subsemimodules. A non-zero

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semimodule \( M \) is a Dedekind semimodule if each non-zero subsemimodule of \( M \) is invertible.

Section 2 argues multiplication semimodules. We show that every multiplicatively cancellative multiplication semimodule is finitely generated.

Section 3 discusses Dedekind multiplication semimodules. We show that if \( M \) is a faithful multiplication \( R \)-semimodule, then \( M \) is a Dedekind semimodule if \( R \) is a Dedekind semiring.

Let \( A \) and \( B \) be \( R \)-semimodules, and \( H = \text{Hom}_R(A, B) \). Here’s a question that shows: when does \( H \) contain a monomorphism? If \( H \) contains a monomorphism we say that \( A \) is embeds in \( B \).

It was proved by Low and Smith [2] that if \( A \) is a torsionless multiplication \( R \)-module then \( A \) embeds in \( R \) if and only if \( \exists \beta \in A^* = \text{Hom}_R(A, R) \) such that \( \text{ann}(R\beta) = \text{ann}(A^*) \).

Indeed if \( A \) is not a multiplication semimodule then this condition is not sufficient see Remark 3.2.

Here the importance of the invertible subsemimodules in obtaining the sufficient condition for the existence of a monomorphism.

In the last section we establish that if \( A \) is any semimodule, with \( \bigcap_{\beta \in H} \ker \beta = \{0\} \) and \( T_H \subseteq T_B \), and if there is a cyclic invertible subsemimodule \( Rf \) in \( H \), then \( f \) is a monomorphism.

1. Invertible Subsemimodules and Invertible Ideals

In this section we introduce the concept of invertible subsemimodule of a semimodule as a kind of generalization of the concept of invertible ideal in semiring.

**Remark (1.1):** Let \( R \) be a commutative semiring with identity 1. A set \( S \subseteq R \) is said to be a multiplicatively closed set of \( R \) provided that if \( a, b \in S \), then \( ab \in S \). The localization of \( R \) at \( S \) (\( R_S \)) is defined in the following way:

First define the equivalence relation \( \sim \) on \( R \times S \) by \( (a, b) \sim (c, d) \) if and only if \( \text{sad} = \text{sbc} \) for some \( s \in S \). Then put \( R_S \) the set of all equivalence classes of \( R \times S \) and define addition and multiplication on \( R_S \) respectively by \( [a, b] + [c, d] = [ad + bc, bd] \) and \( [a, b] \cdot [c, d] = [ac, bd] \), where \( [a, b] \) also denoted by \( a/b \), we mean the equivalence class of \( (a, b) \). It is, then, easy to see that \( R_S \) with the mentioned operations of addition and multiplication on \( R_S \) above is a semiring [3, 4].

**Definition (1.2):** In Remark 1.1, if \( S \) is the set of all not zero-divisors of \( R \). Then, the total quotient semiring \( Q(R) \) of the semiring \( R \) is defined as the localization of \( R \) at \( S \). Note that \( Q(R) \) is also an \( R \)-semimodule. If \( R \) is a semidomain one can define the semifield of fractions \( F(R) \) of \( R \) as the localization of \( R \) at \( R - \{0\} \) [5, 6].

**Definition (1.3):** Let \( M \) be an \( R \)-semimodule. In Remark 1.1, if \( S \) is the set of all not zero-divisors of \( R \), and \( T = T_M = \{ s \in S | sm = 0 \text{ for some } m \in M \text{ implies } m = 0 \} \). The total quotient semiring \( Q_T(R) \) of the semiring \( R \) is defined as the localization of \( R \) at \( T \). Note that \( Q_T(R) \) is also an \( R \)-semimodule.

Consider \( R = \mathbb{N} \) and \( M = Q^+ / \mathbb{N} \). Then \( T = \{1\} \) and so \( Q_T(R) = \{ \frac{1}{n} : n \in \mathbb{N} \} \).

Similar to that in modules see [1], we give the following remark.

**Remark (1.4):** Let \( M \) be an \( R \)-semimodule and let \( N \) be a non-zero subsemimodule of \( M \). Suppose that \( N' = \{ x \in Q_T(R) | xN \subseteq M \} \). Then \( N' \) is an \( R \)-subsemimodule of \( Q_T(R) \) and \( R \subseteq N' \), and \( N'N \subseteq M \).

**Definition (1.5):** Let \( M \) be an \( R \)-semimodule. A **subactive** subsemimodule (or \( k \)-subsemimodule) \( N \) is a subsemimodule of \( M \) such that if \( x, x + y \in N \), then \( y \in N \). A **prime** subsemimodule of \( M \) is a proper subsemimodule \( P \) of \( M \) in which \( x \in P \) or \( rM \subseteq P \) whenever \( rx \in P \). We define \( k \)-ideals and prime ideals of a semiring \( R \) in a analogous manner [5].

**Remark (1.6):** Let \( M \) be an \( R \)-semimodule, we say that \( M \) is a torsion-free semimodule whenever \( r \in R \) and \( m \in M \) with \( rm = 0 \) implies that either \( m = 0 \) or \( r = 0 \). If \( N \) is a subsemimodule of \( M \), then \( [N : M] = \{ r \in R : rM \subseteq N \} \) and \( \text{ann}(M) = [0 : M] = \{ r \in R : rM = 0 \} \) are \( k \)-ideals of \( R \), [5].

**Proposition (1.7):** Let \( M \) be a non-zero \( R \)-semimodule, and let \( T \) be the set defined as in Definition 1.3, then \( T \) has the following properties:

1) \( T \cap \text{ann}(M) \) is the empty set.
2) \( T \) is a multiplicative subset of \( S \) and \( 1 \in T \).
3) \( M \) is torsion-free then \( T = S \).

**Proof:** For (1) from the definition of \( T \) we have \( T \cap \text{ann}(M) = \emptyset \). For (2) first observe that \( 1 \in T \). Let \( s_1, s_2 \in T \), and \( s_1s_2m = 0 \) for some \( m \in M \), then since \( s_1, s_2 \in T \), then \( s_2m = 0 \) and hence \( m = 0 \), therefore \( s_1s_2 \in T \) Thus \( T \) a multiplicative subset of \( S \). For (3) from definition of \( T \), then \( T \subseteq S \). Now, assume that \( M \) is torsion-free. Let \( s \in S \) and \( sm = 0 \) for some \( m \in M \), since \( M \) is torsion-free then \( m = 0 \), and hence \( s \in T \). Thus \( S \subseteq T \). This completes the proof.
Definition (1.8): [4] A subset \( I \) of the total quotient semiring \( \mathbb{Q}(R) \) of \( R \) is called fractional ideal of a semiring \( R \), if the following hold:

1. \( I \) is an \( R \)-subsemimodule of \( \mathbb{Q}(R) \), that is, if \( a, b \in I \) and \( r \in R \), then \( a + b \in I \) and \( ra \in I \).

2. There exists a not zero-divisor element \( d \in R \) such that \( dl \subseteq R \).

   Let \( I \) be two fractional ideals of a semiring \( R \). Then \( I = \{ a_1b_1 + a_2b_2 + \cdots + a_nb_n : a_i \in I, b_i \in J, \forall i, 1 \leq i \leq n, n \in \mathbb{N} \} \).

   By \( \text{Frac}(R) \), we mean the set of all nonzero fractional ideals of a semiring \( R \). It is easy to check that \( \text{Frac}(R) \) equipped with the above multiplication of fractional ideals is an abelian monoid [4].

   It is clear that each ideal \( I \) of \( R \) is fractional ideal of a semiring \( R \) since (1) and (2) holds for \( d = 1, 1I \subseteq R \).

Definition (1.9): [4] Let \( I \) be a fractional ideal of a semiring \( R \), then \( I \) is called invertible if there exists a fractional ideal \( J \) of \( R \) such that \( IJ = R \). Note that \( I \) is unique and will be denoted that by \( I^{-1} \).

   The set of all invertible fractional ideals of \( R \) is an abelian group.

Example (1.10): Let \( \mathbb{N} \) be the set of all non-negative integers. Clearly \( \mathbb{Q}^+ \) its semifield of fractions.

   Let \( n \) be a positive integer. The set \( I = \{ \frac{m}{n} : m \in \mathbb{N} \} \) is a fractional ideal of \( \mathbb{N} \). It is clear I as an \( \mathbb{N} \)-subsemimodule of \( \mathbb{Q}^+ \) is generated by \( \frac{1}{n} \) and \( nl \subseteq N \). While \( J =< \frac{1}{2^n} > \), where \( n \) runs over all positive integers. Since there is no positive integer \( d \) such that \( df \subseteq \mathbb{N} \), \( J \) is not a fractional ideal of \( \mathbb{N} \).

   Let \( R \) be a semidomain, \( \text{Frac}(R) \) its semifield of fractions, \( A \) and \( B \) \( R \)-subsemimodules of \( \text{Frac}(R) \). Then the residual quotient of \( A \) by \( B \) is defined as \( [A : B] = \{ x \in \text{Frac}(R) : xB \subseteq A \} \), see [6].

Proposition (1.11): Let \( R \) be a semidomain, \( A \) and \( B \) some fractional ideals of \( R \). Then the following statements hold:

1. \( [AB : A] A = AB \).
2. \( [R : A] \) is a fractional ideal of \( R \).
3. If \( A \) is invertible, then \( A^{-1} = [R : A] \).
4. If \( A \) is an invertible ideal of \( R \), then \( A \) is finitely generated.

Proof: (1): Suppose that \( t \in AB \), then \( t = \sum_{i=1}^{n} a_ib_i \), where \( a_i \in A, b_i \in B, \forall i \). Now \( b_iA \subseteq AB \), so \( b_i \in [AB : A] \), \( \forall i \). Therefore \( t \in [AB : A]A \), and \( AB \subseteq [AB : A]A \).

   By similar way we prove that \( [AB : A]A \subseteq AB \). Thus \( [AB : A]A = AB \).

(2): \( R \) is fractional and \( A \) an \( R \)-semimodule, \( 1 \) is a common denominator of \( R \). Choose a non-zero \( t \) in \( A \cap R \). Clearly, for any \( x \in [R : A] \), then \( xt \in R \). Therefore, \( t \) is a common denominator of \( [R : A] \) and hence \( [R : A] \) is fractional.

(3): In the formula, \( [AB : A]A = AB \), put \( AB = R \).

(4): Let \( A \) be an invertible ideal of \( R \). So, there is a fractional ideal \( B \) of \( R \) such that \( AB = R \). This implies that \( 1 = \sum_{i=1}^{n} x_iy_i \), for some \( x_1, x_2, \cdots, x_n \in A \) and \( y_1, y_2, \cdots, y_n \in B \). Clearly, the set \( \{ x_i \}_{i=1}^{n} \) generates \( A \) in \( R \).

   Now we can give our definition of invertible subsemimodule, as in modules theory [1].

Definition (1.12): Let \( M \) be a non-zero \( R \)-semimodule and \( N \) be a subsemimodule of \( M \). If \( N'N = M \), then we say that \( N \) is an invertible subsemimodule of \( M \). Note that if \( N \) is invertible then \( N \neq 0 \). It is clear that \( M \) is invertible in \( M \).

   The following proposition is useful for testing the invertibility of subsemimodules.

Proposition (1.13): Let \( M \) be a non-zero \( R \)-semimodule.

1) A non-zero subsemimodule \( N \) of \( M \) is invertible of \( M \) iff \( \forall m \in M, \exists \sum_{i=1}^{k} \omega_i n_i \leq k \) such that \( m = \sum_{i=1}^{k} \omega_i n_i \).

2) If \( N \) is invertible subsemimodule in \( M \), then \( \forall m \in M, \exists t \in T \) such that \( tm \in N \).

Proof: The proof of (1) is an immediate consequence of the Definition 1.12. For (2) Since \( N'N = M \), then \( \forall m \in M, \exists \sum_{i=1}^{k} \omega_i n_i \leq k \) such that \( m = \sum_{i=1}^{k} \omega_i n_i \), where \( r_i \in R, \ t_i \in T \). Put \( t = t_1t_2\cdots t_k \), and \( q_i = r_i \sum_{i=1}^{k} \omega_i t_i \), \( 1 \leq i \leq k \), then \( tm = \sum_{i=1}^{k} q_i n_i \in N \).

   As a special case of Proposition 1.13 we obtain.

Corollary (1.14): A non-zero cyclic subsemimodule \( R_n \) of \( M \) is invertible in \( M \) iff \( \forall m \in M, \exists t \in T, \ r \in R \) such that \( tm = rn, r \) depends on \( m \).

Proposition (1.15): If \( N \) is a non-zero invertible subsemimodule of \( R \)-semimodule \( M \). Then \( M = \sum_{\phi \in H} \phi(N) \), where the sum is taken over all \( \phi \in H = \text{Hom}(N, M) \).
Proof: Since $N'N = M$. Hence each element of $N'$ can be thought of as an $R$-homomorphism in $\text{Hom}(N,M)$. In fact, $\forall m \in M$, $m = \sum_{i=1}^{k} q_{i} n_{i}$, where $q_{i} \in N'$, $n_{i} \in N, 1 \leq i \leq k$. i.e. $m = \sum_{i=1}^{k} q_{i}(n_{i})$, where if $q \in N'$, then $q(n) = qn, \forall n \in N$. This completes the proof.

Definition (1.10): A non-zero R-semimodule $M$ is called a Dedekind semimodule (or D semimodule), if each non-zero subsemimodule of $M$ is invertible in $M$, and $M$ is called a $D_{1}$ semimodule if each non-zero cyclic subsemimodule of $M$ is invertible in $M$. It is clear that every D semimodule is a $D_{1}$ semimodule.

Example (1.17): Here some examples to explain invertible subsemimodules and D semimodules:

1) Let $R = \mathbb{Z}_{4}$ as a semiring, and let $I = R2 = \{0, 2, 4, 6\}$. So $T = T_{1} = \{1, 3, 5, 7\}$. Let $H = R4$. $H' = \{x \in \mathbb{Q}(R) | xH \subseteq \mathbb{I}\}$. It is easy to check that $\mathbb{Q}(R) = R$, and hence $H' = R$. Then $H' = H \neq I$. Thus $H$ is not invertible in $I$.

2) Let $N$ be the semiring of non-negative integer numbers and $0 \neq a \in N$. Let $I = aN$, since the set $S$ of all not zero-divisors of $N$ is $N - \{0\}$, hence $T = T_{1} = \{s \in N - \{0\} | sa \neq 0\} = N - \{0\}$. Therefore, $(aN)' = I' = \{x \in \mathbb{Q}^{+} | x(aN) \subseteq N\} = \frac{1}{a} N$, where $\mathbb{Q}^{+}$ is the semifield of non-negative rational numbers. Then it is clear that $I' = I^{-1}$. Since $I$ is an invertible ideal in $N$, we have $I^{-1} = I'N = N$, and $I$ is an invertible subsemimodule. Now let $H = 4N$ be a subsemimodule of the $N$-semimodule $2N$. Then $H = \{x \in \mathbb{Q}^{+} | x(4N) \subseteq 2N\}$.

One can check that $H' = \frac{a}{2} N$, therefore $H'H = (\frac{1}{2} N)(4N) = 2N$, i.e., $4N$ is an invertible subsemimodule in $2N$.

3) Consider $\mathbb{Q}^{+}$ as an $N$-semimodule. Suppose that $N$ be a non-zero subsemimodule of $\mathbb{Q}^{+}$. Since $\mathbb{Q}^{+}$ is torsion-free, then $T = S = N - \{0\}$, and $Q_{T}(R) = Q(R) = \mathbb{Q}^{+}$. Thus $N' = (\mathbb{Q}^{+})_{\mathbb{Q}(\mathbb{Q}^{+})(\mathbb{Q}^{+})}$. It is clear that $N' = \mathbb{Q}^{+}$, and we obtain $\mathbb{Q}^{+}N = \mathbb{Q}^{+}$, hence $\mathbb{Q}^{+}$ is a Dedekind $N$-semimodule.

4) Consider $Z_{n}$ as a $Z$-semimodule, where $n$ is any positive integer $> 1$, which is not prime number. Let $N$ be a non-zero proper subsemimodule of $Z_{n}$. Now $T = \{\frac{m}{m} \in Z | \gcd(m, n) = 1\}$. $Q_{T}(Z) = \{\frac{n}{m} \in Z | \gcd(m, n) = 1\}$. Hence it is clear that, $N' = \{x \in Q_{T}(Z) | xN \subseteq Z_{n}\} = Q_{T}(Z)$. Therefore $N'N = Q_{T}(Z)N = N \neq Z_{n}$. Hence $N$ is not an invertible subsemimodule in $Z_{n}$. While, if $n$ is a prime number, then $Z_{n}$ is simple semimodule; $Z_{n}$ has no non-zero proper subsemimodule, hence $Z_{n}$ is a Dedekind $N$-semimodule.

5) Let $p$ be a prime number, and let $N_{(p)}$ be the set of $\mathbb{N}^{\mathbb{N}}$-rational numbers of the form $m/n$, with $m$ and $n$ are in $\mathbb{N}$ and $n$ is not divisible by $p$. Then $N_{(p)}$ is a subsemigroup of $Q^{+}$. $N_{p} = Q^{+}/N_{(p)}$ is a $N$-semimodule. It is known that each proper non-zero subsemigroup of $N_{p}$ is cyclic of the form $N_{p}$. Note that since each element of $f(N_{p})$, where $f \in \text{Hom}(N_{p})$, is of order less than or equal to $p^n$. Thus $N_{p} = \sum f \in \text{Hom}(N_{p}, N_{p})$. Hence by Proposition 1.15, we have $N_{p}$ has no proper invertible subsemimodule.

Lemma (1.18): Let $M_{1}$ and $M_{2}$ be torsion-free $R$-semimodules and $f$ be an $R$-epimorphism from $M_{1}$ to $M_{2}$. If $N$ is an invertible subsemimodule of $M_{1}$ then $f(N)$ is an invertible subsemimodule of $M_{2}$.

Proof: Suppose $N$ is invertible subsemimodule in $M_{1}$. Then $N'N = M_{1}$, $N' = \{x \in Q_{T}(R) | xN \subseteq M_{1}\}$. If $x \in N'$ then $xN \subseteq M_{1}$ and so $xf(N) = f(xN) \subseteq M_{2}$.

So $N' \subseteq (f(N))' = \{x \in Q_{T}(R) | xf(N) \subseteq M_{2}\}$.

Take $m \in M_{2}$. Let $m' \in M_{1}$ be such that $f(m') = m$.

Then $m' = x_{1}n_{1} + \cdots + x_{k}n_{k}$ for some $k \in \mathbb{N}, x_{i} \in N$ and $n_{i} \in N$.

Then $m = f(m') = x_{1}f(n_{1}) + \cdots + x_{k}f(n_{k})$, and therefore $M_{2} = N'f(N) \subseteq (f(N))'f(N) \subseteq M_{2}$. Thus $f(N)$ is an invertible subsemimodule in $M_{2}$. 

Corollary (1.19): Every homomorphic image of a Dedekind semimodule is again Dedekind.

Remark (1.20): If $N$ is a non-zero proper direct summand of an $R$-semimodule $M$, then $N$ is not invertible subsemimodule in $M$. 

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Proof: Let $N$ be invertible subsemimodule in $M$; thus $N'N = M$, where $N' = \{ x \in Q_T(R) \mid xN \subseteq M \}$, and $T = \{ s \in S \mid sm = 0 \text{ for some } m \in M \text{ implies } m = 0 \}$. Since $N$ is a direct summand of $M$, i.e. there is a subsemimodule $K$ of $M$ such that $M = N \oplus K$. If $0 \neq k \in K$, since $N$ is invertible in $M$, then by Proposition 1.13, $\exists t \in T$ with $tk \in N$, but $tk \in K$, hence $tk \in N \cap K = \emptyset$, and since $t \in T$, then $k = 0$, which is a contradiction, then $N$ is not invertible in $M$. 

**Corollary (1.21):** It easy checked that if $M = N \oplus K$, and $N$ is an invertible subsemimodule in $M$, then $M = N$. 

**Proposition (1.22):** Let $R$ be a semiring and $I$ be a non-zero ideal of $R$, then $I$ is an invertible ideal in $R$ if and only if $I$ is an invertible $R$-subsemimodule in $R$. 

**Proof:** Let $S$ be the set of all not zero-divisors of $R$. Then $T = T_I = \{ s \in S \mid sa = 0 \text{ for some } a \in I \text{ implies } a = 0 \}$. So that $T = S$. Thus $Q_T(R)$ is the total quotient semiring $Q(R)$. Hence $I' = I^{-1}$, i.e. $I'I = I^{-1}I$, and so $I$ is an invertible ideal in $R$ if and only if $I$ is invertible $R$-subsemimodule in $R$. 

A semiring $R$ is said to be a *Dedekind semidomain* if every non-zero ideal of $R$ is invertible in $R$ [6]. According to the equivalent conditions explained on page 143 in Narkiewicz’ book [7], a Dedekind domain is a domain in which non-zero fractional ideals form a group under multiplication. Inspired by this, we give the following definition: We define a semidomain $R$ to be a Dedekind semidomain if every non-zero fractional ideal of $R$ is invertible. Hence $R$ is a Dedekind semidomain if and only if $\text{Frac}(R)$ is an abelian group. 

**Corollary (1.23):** Let $R$ be a semiring. Then 

1) $R$ is Dedekind $R$-semimodule if and only if $R$ is a Dedekind semidomain. 
2) $R$ is a D$_1$ semimodule if and only if $R$ is a semidomain, i.e. each non-zero principal ideal of $R$ is invertible as a subsemimodule in $R$ if and only if it is generated by not a zero divisor. 

The following remark shows that $D_1$ semimodule may not be D semimodule. 

**Remark (1.24):** Let $R$ be a semidomain, and $R_1$ the polynomial semiring $R[x, y]$ in two independent variables $x$ and $y$. Then $R_1$ is a semidomain. By Corollary 1.21, $R_1$ is a D$_1$ semimodule. But if we take the ideal $I$ generated by $x$ and $y$, it is clear that $I$ is not invertible subsemimodule of $R_1$. Thus $R_1$ is not a D $R_1$ -semimodule. 

Next, we defined the notion of "essential" subsemimodule. In Golan book’s [8], it was proposed the following definitions. An $R$-monomorphism $f : M \rightarrow M'$ of $R$-semimodules is essential if for any $R$-homomorphism $g : M' \rightarrow M''$, $g \circ f$ is a monomorphism implies that $g$ is a monomorphism. 

A subsemimodule $N$ of an $R$-semimodule $M$ is essential (or large) in $M$ if the inclusion mapping $i_N : N \hookrightarrow M$ is an essential $R$-monomorphism. Note that $f : M \rightarrow M'$ is an essential $R$-homomorphism if and only if $f(M)$ is a large subsemimodule of $M'$ [8]. 

Another way for defining the notion of ”essential” is proposed in [9] as follows. A subsemimodule $N$ of $M$ is said to be semi-essential in $M$, written as $N \triangleleft_\text{ ess} M$, if for every subsemimodule $H$ of $M : N \cap H = 0 \Rightarrow H = 0$. A monomorphism $f : M \rightarrow M'$ of $R$-semimodules is said to be semi-essential if: $f(M) \triangleleft_\text{ ess} M'$. 

In [9], we have the following characterization of semi-essential subsemimodules. 

**Lemma (1.25):** A subsemimodule $N$ of an $R$-semimodule $M$ is a semi-essential if and only if for each $0 \neq m \in M$, there exists $r \in R$ such that $0 \neq rm \in N$. 

**Lemma (1.26):** Every invertible subsemimodule of $M$ is a semi-essential subsemimodule of $M$. 

**Proof:** Let $N$ be invertible subsemimodule of $M$. Let $0 \neq m \in M$. By Proposition 1.13, $\exists t \in T$ such that $0 \neq tm \in N$ and hence $N$ is essential. 

**Proposition (1.27):** Let $M$ be a $D_1$ semimodule. Then $\text{ann}(Rm) = \text{ann}(M)$, for each $0 \neq m \in M$. 

**Proof:** It is clear that $\text{ann}(M) \subseteq \text{ann}(Rm)$, so it is enough to show that $\text{ann}(Rm) \subseteq \text{ann}(M)$. Let $r \in \text{ann}(Rm)$, then $rm = 0$. Let $a \in M$. Since $M$ is a $D_1$ semimodule; then $Rm$ is invertible in $M$, and hence by Corollary 1.14, $\exists t \in T, s \in R$ such that $ta = sm$. Thus $tra = rms = 0$. Hence $ra = 0$, and $\text{ann}(Rm) \subseteq \text{ann}(M)$. This completes the proof. 

From now on, we will put $\text{End}_R(M)$, for the semiring of endomorphisms of $R$-semimodule $M$. 

**Lemma (1.28):** Let $M$ be a non-zero $R$-semimodule and $f \in \text{End}_R(M)$. If ker$f$ contains an
invertible subsemimodule of $M$ then $f = 0$. Therefore if $M$ is a $D_1$ semimodule then every non-zero element of $\text{End}_R(M)$ is a monomorphism.

**Proof:** Let $N \subseteq \ker f$ is invertible in $M$. Then by Proposition 1.13, $\forall m \in M, \exists t \in T, and n \in N$ such that $tm = n$. So $0 = f(n) = tf(m)$; but $t \in T$ hence $f(m) = 0$ and $f = 0$.

Now assume that $M$ is a $D_1$ semimodule and $0 \neq f \in \text{End}_R(M)$. Let $0 \neq k \in \ker f$, then $Rk$ invertible in $M$ and subset of $\ker f$ from above; we have $f = 0$, which is a contradiction, then $\ker f = 0$, and $f$ is a monomorphism.

For any $R$-semimodule $M$, there exists an obvious semiring monomorphism: $\phi : R/\text{ann}(M) \to \text{End}_R(M)$. Hence one may think of as a subsemiring of $\text{End}_R(M)$. So we have:

**Corollary (1.29):** If $M$ is a $D_1$ semimodule, then $R/\text{ann}(M)$ is a semidomain and thus $\text{ann}(M)$ is a prime ideal.

As a special case, we record the following.

**Corollary (1.30):** If a semiring $R$ is a $D_1$ $R$-semimodule. Then $R$ is a semidomain.

### 2. Multiplication Semimodules

In this section we study multiplication semimodules. We begin with following definition:

**Definition (2.1):** Let $R$ be a semiring and $M$ an $R$-semimodule. Then $M$ is said to be a multiplication semimodule if for all subsemimodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. In this case it is easy to show that $N = [N : M]M$. For instance, all cyclic $R$-semimodule are multiplication $R$-semimodule [10, Example 2].

Note that, if $I$ is an ideal of $R$, then the set $IM$ consisting of all finite sums of elements $r_im_i$ with $r_i \in R$ and $m_i \in M$ is a subsemimodule of $M$.

**Example (2.2):** Let $R$ be a multiplicatively idempotent semiring. Then all ideals of $R$ are multiplication $R$-semimodule [11].

An element $r$ of a semiring $R$ is multiplicatively-cancellable (abbreviated as MC), if $rx = rwy$ implies $x = y$ for all $x, y \in R$. Each non-zero element in a semidomain is a MC element.

**Theorem (2.3):** Let $R$ be a semiring. An ideal $I$ of $R$ is invertible if and only if it is a multiplication $R$-semimodule which contains an MC element of $R$, see [11].

**Proposition (2.4):** Let $R$ be a semiring. An $R$-semimodule $M$ is a multiplication semimodule if and only if for each $m \in M$ there exists an ideal $I$ of $R$ such that $I_m = IM$.

**Proof:** The necessity is clear. For the sufficiency, assume that for each $m \in M$ there exists an ideal $I$ of $R$ such that $I_m = IM$.

**Theorem (2.5):** Let $M$ be a multiplication semimodule over a semiring $R$. If $N$ is a finitely generated subsemimodule of $M$, then there exists a finitely generated ideal $I$ of $R$ such that $N = IM$.

**Proof:** Suppose that $N = <x_1, x_2, \ldots, x_n>$. Since $M$ is a multiplication, we have $N = [N : M]M$. So, there exists $a_{i, j} \in [N : M]$ and $y_{i, j} \in M$ such that $x_i = a_{i, 1}y_{i, 1} + \cdots + a_{i, r}y_{i, r}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, r$. Let $I$ be an ideal of $R$ generated by $\{a_{1, r}, \ldots, a_{n, r}\}$. It is easy to see that $I \subseteq [N : M]$ and $IM \subseteq [N : M]M$. On the other hand, since for every $i, x_i \in IM$, we must have $N \subseteq IM$. Hence $N \subseteq IM \subseteq [N : M]M \subseteq N$. Thus $N = IM$ and $I$ is finitely generated.

The following shows that every homomorphic image of a multiplication semimodule is again multiplication [11].

**Theorem (2.6):** Let $M$ and $N$ $R$-semimodules and $f : M \to N$ a surjective $R$-homomorphism. If $M$ is a multiplication $R$-semimodule, then $N$ is a multiplication $R$-semimodule.

A semiring $R$ is called yoked if for all $a, b \in R$, there exists an element $t \in R$ such that $a + t = b$ or $b + t = a$ [8, p. 49]. A semiring is entire if $ab = 0$ implies that $a = 0$ or $b = 0$ [8, p. 4].

An $R$-semimodule $M$ is called multiplicatively cancellative (or simply MC) if for any $r, r' \in R$ and $0 \neq m \in M$, $r_m = r'm$ implies $r = r'$ [11]. For example every ideal of a semidomain $R$ is an MC $R$-semimodule.

Note that if $M$ is an MC $R$-semimodule, then $M$ is a faithful semimodule. Let $rM = \{0\}$ for some $r \in R$. If $0 \neq m \in M$, then $r_m = 0 m = 0$. Hence $r = 0$. Thus $M$ is faithful.

An element $m$ of an $R$-semimodule $M$ is called cancellable if $m + m' = m + m''$ implies that $m' = m''$. The semimodule $M$ is cancellative if and only if every element of $M$ is cancellable [8, P. 172].

**Lemma (2.7):** Let $R$ be a yoked entire semiring and $M$ a cancellative faithful
Multiplication $R$-semimodule. Then $M$ is an $MC$ semimodule.

**Theorem (2.8):**[11] Let $R$ be a yoked semidomain and $M$ a cancellative torsion-free $R$-semimodule. Then $M$ is an $MC$ semimodule.

**Lemma (2.9):**[11] Let $M$ be an $R$-semimodule and $\theta(M) = \sum_{m \in M} [Rm : M]$. If $M$ is a multiplication $R$-semimodule, then $M = \theta(M)M$.

**Theorem (2.10):**[11] Let $R$ be a semiring and $M$ an $MC$ multiplication $R$-semimodule. Then $M$ is finitely generated.

By Lemma 2.7, we have the following result.

**Corollary (2.11):** Let $R$ be an entire yoked semiring and $M$ a cancellative faithful multiplication $R$-semimodule. Then $M$ is finitely generated.

The next theorems give a characterization of $MC$ multiplication semimodules, for the proof see[11].

**Theorem (2.12):** If $M$ is an $MC$ multiplication $R$-semimodule. Then $M$ is a projective $R$-semimodule.

**Theorem (2.13):** Let $R$ be a semidomain. If $M$ is an $MC$ multiplication $R$-semimodule, then $M$ is a torsion-free semimodule.

**Theorem (2.14):** Let $R$ be a semidomain. If $M$ is an $MC$ multiplication $R$-semimodule, then $M$ is isomorphic to an invertible ideal in $R$.

### 3. Dedekind Multiplication Semimodules

From Remark 2.3 we can say that a semiring $R$ is a Dedekind semidomain iff each non-zero ideal in $R$ is a multiplication ideal which contains a not zero-divisor. In this section we study Dedekind multiplication semimodules. We begin with the following.

**Lemma (3.1):** Let $M$ be a torsion-free $R$-semimodule. If $N$ is an invertible subsemimodule of $M$ and $I$ is an invertible ideal in $R$, then $IN$ is an invertible subsemimodule of $M$.

**Proof:** Suppose $H = IN$. But $N'N = M$, $I^{-1}I = R$, and hence $I^{-1}N'H = (I^{-1}I)N'N = M$. From Proposition 1.7, we have $T_M = S$ and from Proposition 1.22, we have $Q_T(R) = Q(R)$. Hence easy to see that $I^{-1}N' \subseteq H$. By above we have $I^{-1}N' = H'$, and $H$ is invertible.

**Lemma (3.2):** Let $M$ be a non-zero $R$-semimodule and $I$ is invertible ideal in $R$. Then $IM$ is an invertible subsemimodule of $M$.

**Proof:** Suppose $K = IM$. But $I^{-1}I = R$, and hence $I^{-1}K = (I^{-1})IM = (I^{-1}I)M = RM = M$. From Proposition 1.22, we have $Q_T(R) = Q(R)$, thus it follows that $I^{-1} \subseteq K'$. Hence $M = I^{-1}K \subseteq K'K \subseteq M$, so $K'K = M$, and $K$ is invertible.

A subsemimodule $N$ of an $R$-semimodule $M$ is called **invariant** subsemimodule if $f(N) \subseteq N$, $\forall f \in \text{Hom}(M,M)$, [3, 12].

**Definition (3.3):** A semimodule $M$ is said to be **duo** if each subsemimodule of $M$ is invariant, [12].

In [12], we have the following characterization of duo subsemimodules.

**Theorem (3.4):** Let $R$ be a yoked semidomain, and $M$ a torsion-free $R$-semimodule. Then $M$ is duo if and only if for each $R$-endomorphism $f$ of $M$, there exists $r \in R$ such that $f(m) = rm$ for all $m \in M$.

**Remark (3.5):** It is clear that any multiplication semimodule is duo. Hence by Theorem 3.4, if $M$ is a multiplication torsion-free semimodule over a yoked semidomain $R$, then for each $f \in \text{End}_R(M)$, $\exists r \in R$, such that $f(m) = rm$ for all $m \in M$.

**Corollary (3.6):** If $M$ is a torsion-free multiplication semimodule over a yoked semidomain $R$, then there exists an epimorphism of semirings from $R$ onto $\text{End}_R(M)$.

**Proof:** By Remark 3.5, $\forall f \in \text{End}_R(M)$, $\exists r \in R$, such that $f = f_r$ and $f_r(m) = rm$ for all $m \in M$. Hence $\phi: R \rightarrow \text{End}_R(M)$, defined by $\phi(r) = f_r$. It is easily check, that $\phi$ is an epimorphism of semirings.

**Theorem (3.7):** If $M$ is a torsion-free multiplication semimodule over a yoked semidomain $R$, then $\text{End}_R(M) \cong R/\text{ann}(M)$

**Proof:** By Corollary 3.6, $\ker\phi = \{r \in R | \phi(r) = 0\} = \{r \in R | f_r = 0\} = \{r \in R | rm = 0 \ \forall m \in M\} = \text{ann}(M)$. But $\text{End}_R(M) \cong R/\ker\phi$, then $\text{End}_R(M) \cong R/\text{ann}(M)$.

By Lemma 2.7, Theorem 2.13, and Theorem 3.7 we have.

**Theorem (3.8):** If $M$ a cancellative faithful multiplication semimodule over a yoked semidomain $R$. Then $\text{End}_R(M) \cong R$.

The following lemma shows the importance of the faithful multiplication semimodules.

**Lemma (3.9):** Let $M$ be a finitely generated cancellative faithful multiplication semimodule over
a yoked semidomain $R$. If $N = IM$ is an invertible subsemimodule of $M$ for some ideal $I$ of $R$, then $I$ is an invertible ideal in $R$.

**Proof:** Since $N \neq 0$, then $I \neq 0$. By assumption $N'N = M$, hence $M = N'N = N'IM$. It is clear that $N'I$ is an $R$-subsemimodule of $R$. Also, it is easy to see that every element of $N'I$ can be considered as an $R$-endomorphism of $M$. Now, since $M$ is a faithful multiplication semimodule, then by Theorem 3.8, $\text{End}_R(M) \cong R$. Therefore $N'I$ is an ideal in $R$. As in modules see [13], it follows that $N'I = R$. Hence $N' \subseteq I^{-1}$, so $R = N'I \subseteq I^{-1}I \subseteq R$ which implies $I^{-1}I = R$.

**Theorem (3.10):** Let $M$ be a cancellative faithful multiplication $R$-semimodule over a yoked Dedekind semidomain $R$. Then $M$ is a finitely generated Dedekind $R$-semimodule.

**Proof:** Since $M$ is a faithful multiplication semimodule, and $R$ is a semidomain. By Corollary 2.11, we have $M$ is a finitely generated. Now, let $N$ be a non-zero subsemimodule of $M$. Hence there exists a non-zero ideal $I$ in $R$ such that $N = IM$. Since $R$ is a Dedekind semidomain, thus $I$ is invertible in $R$, and by Lemma 3.2, $N$ is invertible.

The following theorem is a converse of above theorem:

**Theorem (3.11):** Let $M$ be a cancellative faithful multiplication semimodule over a yoked semidomain $R$. If $M$ is a Dedekind semimodule, then $R$ is a Dedekind semidomain.

**Proof:** By assumption, $R$ is a semidomain. By Corollary 2.11, we get $M$ is a finitely generated. Assume that $I$ is any non-zero ideal of $R$. Then $IM$ is a non-zero subsemimodule of $M$, hence $IM$ is invertible. From Lemma 3.9, $I$ is an invertible ideal.

A semidomain $R$ is said to be a **Prüfer semidomain** if every non-zero finitely generated ideal of $R$ is invertible in $R$ [6]. Note that $R$ is a Dedekind semidomain if and only if $R$ is a Noetherian (each of its ideals is finitely generated) Prüfer semidomain.

Let $D$ be a Dedekind domain ($D$ is a ring). By Theorem 3.7 in [4], the semiring of ideals $\text{Id}(D)$ of $D$ (the set of all ideals of $D$) is a Prüfer semidomain. By Theorem 3.7 in [4], $\text{Id}(D)$ is subtractive (each of its ideals is subtractive). If $\text{Id}(D)$ is also Noetherian, then $\text{Id}(D)$ is a Dedekind semidomain. Note that the semiring $\text{Id}(D)$ is proper semiring, i.e., it is not a ring. If $D$ is a Dedekind semidomain then the above argument remains true. Note that, each Noetherian Prüfer semidomain is Dedekind.

For a more specific example, we assert that $(\text{Id}(\mathbb{Z}), +, \cdot)$ is a principal ideal semidomain (each of its ideals is principal) [6]. Hence, $\text{Id}(\mathbb{Z})$ is evidently a Dedekind semidomain. Note that the semiring $(\text{Id}(\mathbb{Z}), +, \cdot)$ is isomorphic to the semiring $(\mathbb{N}, \text{gcd}, \cdot)$.

**Definition (3.12):** A semimodule $M$ is said to be a **Prüfer semimodule** if every non-zero finitely generated subsemimodule of $M$ is invertible in $M$.

The proof of the above theorem is basically the same as the proof of the last results.

**Theorem (3.13):** Let $M$ be a cancellative faithful multiplication semimodule over a yoked semiring $R$. Then $M$ is a Prüfer semimodule if and only if $R$ is a Prüfer semidomain.

If $M$ is a $D_1$ semimodule, we have the following remark which is special case of above theorem.

**Remark (3.14):** Let $M$ be a cancellative faithful multiplication semimodule over a yoked semiring $R$. Then $M$ is a $D_1$ semimodule if and only if $R$ is a semidomain.

**Proof:** (⇒) By Corollary 1.29, we get $R$ is a semidomain, so each non-zero principal ideal is invertible.

(⇐) Assume that $R$ is a semidomain. Let now $Rm$ be a non-zero cyclic subsemimodule of $M$, $Rm = IM$, for some ideal $I$ of $R$. In this case we can take $I = [Rm:M]$, and hence $Rm = [Rm:M]M$. By Corollary 2.11, we get $M$ is finitely generated, and thus $[Rm:M]$ is a multiplication ideal in $R$ [13]. But $R$ is a semidomain; thus by Theorem 2.3, $[Rm:M]$ is an invertible ideal in $R$. Then by Lemma 3.2, $Rm$ is an invertible subsemimodule of $M$.

**Proposition (3.15):** If $M$ is a faithful multiplication Dedekind $R$-semimodule. Then $M^* = \text{Hom}_R(M, R)$ is also a faithful multiplication Dedekind $R$-semimodule.

**Proof:** Similarly in the proof of Theorem 3.10, $M$ is a f.g. faithful multiplication semimodule. So as in the modules see Corollary (2) of [2], we obtain that $M^*$ is a f.g. faithful multiplication $R$-semimodule. By assumption and using Theorem 3.11, we get $R$ is a Dedekind semidomain. Now $M^*$ is a f.g. faithful multiplication $R$-semimodule over the Dedekind semidomain $R$, then by Theorem 3.10, $M^*$ is a Dedekind $R$-semimodule.

4. **Embedding of Semimodules**

In this section we study "embeddability proplem", thus we look for necessary and (or) sufficient
conditions under which an $R$-semimodule $A$ is isomorphic to a subsemimodule of the $R$-semimodule $B$. Now, put $H = \text{Hom}_R(A, B)$, $H$ is an $R$-semimodule. We start by the following.

**Proposition 4.1:** Let $A$ and $B$ be $R$-semimodules. If there exists a monomorphism $f \in H$, then $\text{ann}(Rf) = \text{ann}(H)$.

**Proof:** It is clear that $\text{ann}(H) \subseteq \text{ann}(Rf)$, so it is enough to show that $\text{ann}(Rf) \subseteq \text{ann}(H)$. Let $r \in \text{ann}(Rf)$, then $0 = rf(a) = f(ra)$. But $f$ is a monomorphism, therefore $ra = 0$, and $r \in \text{ann}(A)$. But it is easily seen that $\text{ann}(A) \subseteq \text{ann}(H)$, thus $\text{ann}(Rf) = \text{ann}(H)$.

**Remark 4.2:** The converse of Proposition 4.1 is not true in general.

**Proof:** Let $A$ be a projective $R$-semimodule with a non-commutative endomorphisms semiring, $E(A)$ (for example, $A$ can be any free semimodule of rank > 1, such as $A = \mathbb{Z} \oplus \mathbb{Z}$ as $\mathbb{Z}$-semimodule). Put $B = A \oplus R$. Then $B^* = A^* \oplus R^* \cong A^* \oplus R$, where $B^* = \text{Hom}(B, R)$ and $A^* = \text{Hom}(A, R)$.

If $\beta$ represents a generator of a semiring $R$ in the last direct sum, hence it is clear that $\text{ann}(RB^*) = \text{ann}(B^*) = 0$. Whereas $B^*$ does not contain any monomorphism. To prove this, let $f \in B^*$ such that $\ker f = 0$. Thus $f(B)$ is a projective ideal of $R$ (since $B$ is projective). And thus by [14], $f(B)$, so also $B$ is a multiplication ideal. By [15], $\text{End}_R(B)$ is commutative. By [16, lemma 2.1], we have $\text{End}_R(A)$ is commutative, which is a contradiction.

Now, let us observe that if there exists a monomorphism $f: A \rightarrow B$, for any $R$-semimodules, $A$ and $B$, then it is clear that $\cap_{g \in H} \ker g = \{0\}$.

The following theorem gives a sufficient condition for the existence of a monomorphism in $H = \text{Hom}(A, B)$, in the case $A$ is a multiplication $R$-semimodule.

**Theorem 4.3:** Let $A$ be a multiplication $R$-semimodule and $B$ any $R$-semimodule such that $\cap_{g \in H} \ker g = \{0\}, \forall g \in H = \text{Hom}(A, B)$. Then for any $f \in H$, then $f$ is a monomorphism if $\text{ann}(Rf) = \text{ann}(H)$.

**Proof:** ($\Rightarrow$) If $f$ is a monomorphism then by Proposition 4.1, we have $\text{ann}(Rf) = \text{ann}(H)$.

($\Leftarrow$) Put $N = \ker f$. There is an ideal $I$ in $R$ such that $N = IA$. So $0 = f(N) = f(IA) = If(A)$, which implies $f \subseteq \text{ann}(Rf)$. Then $IH = \{0\}$, hence $I \subseteq \ker g, \forall g \in H$, and thus $IA = \{0\}$. Therefore $N = \{0\}$ and $f$ is a monomorphism.

As a special case of Theorem 4.3, we give the following, comparison with [2, Lemma 1.1]. We say that an $R$-semimodule $A$ is called torsionless if $\cap_{g \in H} \ker g = \{0\}, \forall g \in A^*$. Let $A$ be a torsionless multiplication $R$-semimodule. Then $A$ is embeddable in $R$ iff $\exists \beta \in A^*$ such that $\text{ann}(RB^*) = \text{ann}(A^*)$.

More generally, we have:

**Corollary 4.4:** Let $A$ be a torsionless multiplication $R$-semimodule. Then $A$ is embeddable in $R^n$ iff $\exists$ a f.g. subsemimodule $N$ of $A^*$, which is generated by a set $\{\beta_1, \beta_2, ..., \beta_n\}$, where $\beta_i \in A^*, 1 \leq i \leq n$ and $\text{ann}(N) = \text{ann}(A^*)$.

**Proof:** ($\Rightarrow$) Assume that $A$ embeds in $R^n$, i.e. $\exists \beta: A \rightarrow R^n$ which is a monomorphism. $\forall i, 1 \leq i \leq n$ define $\beta_i: A \rightarrow R$ as follows $\beta_i = \rho_i \circ \beta$, where $\rho_i \forall i, 1 \leq i \leq n$ is the natural projection of $R^n$ onto the $i$th component. Note, since $\text{Hom}(A, R^n)$ is isomorphic to the direct sum of $n$ copies of $A^* = \text{Hom}(A, R)$.

Therefore $\text{ann}(f(A, R^n)) = \text{ann}(A^*)$ and since $\beta$ is a monomorphism hence, by Proposition 4.1 $\text{ann}(\beta) = \text{ann}(A^*)$. Now, $\text{ann}(\beta) = \cap_{\beta \in H} \text{ann}(\beta) = \text{ann}(N)$. Thus $\text{ann}(N) = \text{ann}(A^*)$.

($\Leftarrow$) Assume that $\exists$ a f.g. subsemimodule $N$ of $A^*$, which is generated by a set $\{\beta_1, \beta_2, ..., \beta_n\}$, and $\text{ann}(N) = \text{ann}(A^*)$. Now let us define an $R$-homomorphism $\beta: A \rightarrow R^n$ as follows $\beta(x) = (\beta_1(x), \beta_2(x), ..., \beta_n(x)), \forall x \in A$. Now since $\text{ann}(f(A, R^n)) = \text{ann}(A^*)$, and by assumption $\text{ann}(A^*) = \text{ann}(N) = \cap_{\beta \in H} \text{ann}(\beta) = \text{ann}(\beta)$. Therefore by using Theorem 4.3, we obtain $\beta$ is a monomorphism in $\text{Hom}(A, R^n)$.

From our main results in this section, is that if $\exists \beta \in A^*$ such that $(RB^*)$ is invertible in $A^*$, and $A$ is torsionless, then $\beta$ is a monomorphism, and hence $A$ embeds in $R$, this means $A$ is isomorphic to an ideal of $R$. But now, let us recall that for any $R$-semimodule $B$, $T_B = \{s \in S | s \beta = 0 \text{ for some } b \in B, \text{ then } b = 0\}$. Hence, for an $R$-semimodule $H = \text{Hom}(A, B)$, $T_B = \{s \in S | s \beta = 0 \text{ for some } b \in H, \text{ then } b = 0\}$.

**Theorem 4.6:** Let $A$ and $B$ be any two $R$-semimodules, with $\cap_{\beta \in H} \ker \beta = \{0\}$, and $T_B \subseteq T_B$. If there exists a cyclic invertible subsemimodule $(RF)$ in $H$, then $f$ is a monomorphism, and hence $A$ embeds in $B$. Moreover, if $\Sigma_{\beta \in H} \beta(A) = B$, then $f(A)$ is invertible subsemimodule in $B$. 

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Proof: By Corollary 1.14 \( \forall \beta \in H, \exists t \in T_H, s \in R \) such that \( tf = sf \). Put \( N = ker f \) and let \( x \in N \), then \( sf(x) = t\beta(x) = 0 \), which implies \( x = 0 \). Thus \( f \) is a monomorphism. Next, by assumption, \( \forall \beta \in B, f_1, f_2, \ldots, f_m \in H \) and \( a_1, a_2, \ldots, a_m \in A \) such that \( b = \sum_{i=1}^{m} f_i(a_i) \). Since \( (Rf) \) is invertible in \( H \), so by Corollary 1.14 \( v \), \( 1 \leq i \leq m, \exists t_i \in R, t_i \in T_H \) such that \( f_i = \frac{t_i}{t_i} f \). Hence \( b = \sum_{i=1}^{m} \frac{t_i}{t_i} f(a_i) \), and by Proposition 1.13, we obtain that \( f(A) \) is invertible in \( B \).

The following two corollaries are special case of Theorem 4.6.

Corollary (4.7): Let \( A \) be a torsionless \( R \)-semimodule. If \( A^* \) contains a cyclic invertible subsemimodule, then \( A \) is isomorphic to an ideal of \( R \). Further if \( \text{trace}(A) = R \), then \( A \) is isomorphic to an invertible ideal, and thus is a faithful multiplication semimodule.

Proof: Since \( T_R = S \), where \( S \) is the set of all non-zero devisors in \( R \), and hence \( T_A^* \subseteq T_R \). Let \( \alpha \in A^* \) such that \( (R\alpha) \) is invertible in \( A^* \). Thus by Theorem 4.6, \( \alpha \) is a monomorphism. Since \( \text{trace}(A) = \sum_{\beta \in A} \beta(A) = R \), again by Theorem 4.6, \( \alpha(A) \) is an invertible subsemimodule of \( R \). Hence by Proposition 1.20, \( \alpha(A) \) is an invertible ideal in \( R \). By Remark 2.3, we obtain \( \alpha(A) \), and hence \( A \) is a faithful multiplication semimodule

Corollary (4.8): Let \( A \) be a torsionless \( R \)-semimodule. If \( A^* \) contains a f.g. invertible subsemimodule \( N \), and \( N \) can be generated by \( n \) elements. Then \( A \) embeds in \( R^n \).

Proof: Let \( \{\beta_1, \beta_2, \ldots, \beta_n\} \) be a set of generators of \( N \). Define \( \beta: A \rightarrow R^n \), as follows, \( \beta(x) = (\beta_1(x), \beta_2(x), \ldots, \beta_n(x)) \), \( \forall x \in A \). Now our aim is to show that \( \beta \) is a monomorphism. Suppose \( N \) is invertible in \( A^* \), then by Proposition 1.13, we have \( \forall \alpha \in A^* \), \( \exists t \in T_A^* \subseteq S \) and \( \exists r_i \in R, 1 \leq i \leq n \) such that \( ta = \sum_{i=1}^{n} r_i \beta_i \). Now, let \( y \in ker \beta = \cap_{i=1}^{n} ker \beta_i \). Thus \( ta(y) = \sum_{i=1}^{n} r_i \beta_i(y) = 0 \), but \( t \in S \), then \( \alpha(y) = 0 \) \( \forall \alpha \in A^* \), i.e. \( y \in \cap_{i=1}^{n} ker \alpha = (0) \). Thus \( ker \beta = (0) \), and \( A \) embeds in \( R^n \).

Theorem (4.9): Let \( M \) be a Dedekind semimodule and let \( m \) be a non-zero element of \( M \). Then \( M \) is isomorphic to the \( R \)-subsemimodule \( (Rm) \) of \( Q(R) \).

Proof: Since \( M \) is a Dedekind semimodule, then \( \forall m_i \in M, \exists z_i \in (Rm) \) such that \( m_i = zm \). Define a homomorphism \( f: (Rm) \rightarrow M \) with \( f(z) = zm \) for each \( z \in (Rm) \). It is clear that \( f \) is an \( R \)-isomorphism.

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