Loss of Causality in Discretized Light-Cone Quantisation

T. Heinzl\textsuperscript{a}, H. Kröger\textsuperscript{b} and N. Scheu\textsuperscript{c}

\textsuperscript{a} Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, D-07743 Jena, Germany
\textsuperscript{b} Département de Physique, Université Laval, Québec, Québec G1K 7P4, Canada
\textsuperscript{c} Institut für Theoretische Physik, Universität Linz, A-4040 Linz, Austria

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We demonstrate that front form quantisation with periodicity in a compact light-like direction ("discretized light-cone quantisation") violates microcausality.

INTRODUCTION

Microcausality is a fundamental postulate in relativistic quantum field theory. Mathematically, it states that two local operators $O_1(x)$ and $O_2(y)$ at space-time positions $x$ and $y$ must commute if $x - y$ is a space-like distance. Physically, it means that signals cannot be transmitted faster than with the velocity of light, $c$. Imposing the requirement of microcausality eliminates a large class of possible quantisation schemes. For instance, quantising scalar fields with anti-commutators violates microcausality, enforcing quantisation of scalars in terms of commutators. Along these lines, the celebrated spin-statistics theorem \cite{1} is established.

In this letter we demonstrate that the popular method of front form (FF) or light-cone (LC) quantisation\cite{2} leads to a breakdown of microcausality when space-time is compactified to a cylinder with the periodic direction being chosen as light-like. As the momenta conjugate to this direction become discrete, the method is usually referred to as discretized light-cone quantisation (DLCQ) \cite{3}. Its range of applicability has recently been extended to include M-theory \cite{5}.

We will be concerned with a massive scalar field in $d$ space-time dimensions. After some general remarks we will actually specialise to $d = 2$. Our notations and conventions are

\begin{align}
  x^\pm &= x^0 \pm x^{d-1}, \quad k^\pm = k^0 \pm k^{d-1}, \quad (1) \\
  x \cdot k &= g_{\mu\nu} x^\mu k^\nu, \quad g_{++} = g_{--} = 1/2. \quad (2)
\end{align}

FF quantisation (for a recent review see \cite{3}) amounts to prescribing field commutators on the quantisation surface $x^+ = 0$. This is a hyperplane tangent to the LC which in $d = 2$ collapses to a light ray.

It has been shown by a number of authors \cite{1} \cite{2}, that quantisation on just one light-like surface is ambiguous. Knowledge of initial conditions on two quantisation surfaces, say $x^+ = 0$ and $x^- = 0$, is necessary in order to have a well-posed (characteristic) initial-value problem \cite{2}. As a result, the characteristic initial values uniquely determine solutions to the Klein-Gordon equation for all $x^\pm$ larger than the initial ones.

In 1994, however, Heinzl and Werner \cite{2} were able to show that the introduction of periodic boundary conditions (pBC) in $x^-$ direction (assumed to be compact) uniquely determines the fields on the second quantisation surface, $x^- = 0$, in terms of the fields on the first quantisation surface, $x^+ = 0$. An infinite-volume formulation of this problem, however, is rather involved and in general will require the use of distribution theory \cite{10,11}. It will not be addressed in this letter. We will rather restrict to the finite-volume case and show that prescribing light-like BC—though solving the initial-value problem—is in conflict with microcausality. The results presented here are built upon the Ph.D. thesis \cite{12} where the loss of causality in DLCQ has been reported for the first time. Recently, this finding has been confirmed by other researchers \cite{14}.

VIOLATION OF MICROCAUSALITY

A. Generalities

We start with a free scalar field $\phi$ in $d$ space-time dimensions. The commutator of two free scalar fields is of course known for all times (i.e. everywhere in Minkowski space), $[\phi(x), \phi(0)] = i \Delta(x)$. $\Delta(x)$ denotes the Pauli-Jordan \cite{13} or Schwinger \cite{14} function,

$$
\Delta(x) = \frac{1}{i} \int \frac{d^d k}{(2\pi)^{d-1}} \delta(k^2 - m^2) \sgn(k^0) \ e^{-ik\cdot x}. \quad (3)
$$

As the sign of $k^0$ does not change under proper orthochronous Lorentz transformations, $\Lambda \in L^+_+, \text{if } k^2$ is spacelike, $\Delta$ is a Lorentz invariant function. The sign function in addition guarantees that $\Delta$ is antisymmetric, $\Delta(x) = \Delta(-x)$, as is necessary for a commutator.

It is well known that $\Delta(x)$ obeys microcausality. This is the statement that $\Delta(x)$ has to vanish for $x$ space-like, i.e. $x^2 < 0$. A very elegant argument to see this is due to Gasiorowicz \cite{14}. For $x^2 < 0$, there is a Lorentz transformation $\Lambda \in L^+_+$ that takes $x$ to $-x$, thus, by invariance, $\Delta(x) = \Delta(-x)$, for $x^2 < 0$. Therefore, outside the LC, $\Delta$ is both symmetric and antisymmetric in $x$ and must vanish. The argument does not work for $d = 2$, as the
regions \( x^1 > 0 \) and \( x^1 < 0 \) are disconnected. Nevertheless, causality also holds in \( d = 2 \), as one can see upon evaluating (3) for this case,

\[
\Delta(x) = -\frac{1}{2} \text{sgn}(x^0) \theta(x^2) J_0(m \sqrt{x^2}) , \\
= -\frac{1}{4} [\text{sgn}(x^+) + \text{sgn}(x^-)] J_0(n \sqrt{x^2}) ,
\]

which indeed vanishes outside the LC (here \( J_0 \) denotes the Bessel function). The restrictions of (4) to \( x^0 = 0 \) and \( x^+ = 0 \) yield the canonical commutators of the two quantisation schemes \([6]\).

In (4) we have given both the IF and FF versions of \( \Delta \) which are, of course, simply related by the coordinate transformation \([4]\). These two forms can actually be represented as one dimensional integrals by performing the energy integrations over \( k^0 \) and \( k^- \), respectively,

\[
\text{IF: } \Delta(x) = -\int \frac{dk^1}{2 \pi \omega_k} \sin(\omega_k x^0 - k^1 x^1) , \\
\text{FF: } \Delta(x) = -\int_0^\infty \frac{dk^+}{2 \pi k^+} \sin(k^- x^+ / 2 + k^+ x^- / 2) .
\]

The on-shell values of the energies are given by \( \omega_k = (k^2 + m^2)^{1/2} \) and \( k^- = m^2 / k^+ \). Note the restriction of the integration in (5) which is due to the positivity of the longitudinal momentum \( k^+ \). Both representations \([4]\) and \([5]\) can be integrated and yield \([4]\). As a cross check we note that \([4]\) and \([5]\) are still related by the coordinate transformation \([4]\) applied to the on-shell momenta,

\[
k^\pm = \omega_k \pm k^1 = \sqrt{k_1^2 + m^2} \pm k^1 .
\]

This makes the positivity of \( k^+ \) explicit and entails that the integration measures are related by the singular transformation \( dk^1 / \omega_k = dk^+ / k^+ \). Let us now investigate how \([4]\) and \([5]\) get modified in a finite volume. To this end we restrict the spatial coordinates, \(-L \leq x^1, x^- \leq L\), and impose pBC for the field \( \phi \).

The conjugate momenta become discrete, \( k_1^1 = \pi n / L \) and \( k_n^- = 2 \pi n / L \), respectively. The finite volume representations are defined by replacing the integrals \([4]\) and \([5]\) by the discrete sums,

\[
\Delta_{IF}(x) \equiv -\sum_{n=-N}^{N} \frac{1}{2 \omega_n L} \sin(\omega_n x^0 - n \pi x^1 / L) , \\
\Delta_{FF}(x) \equiv -\sum_{n=1}^{N} \frac{1}{2 \pi n} \sin(k^-_n x^+ / 2 + n \pi x^- / L) ,
\]

where the limit \( N \to \infty \) is understood. The on-shell energies for discrete momenta are defined as \( \omega_n = (n^2 / 2 + m^2)^{1/2} \) and \( k^-_n = m^2 L / 2 \pi n \).

By comparing the two figures one observes a striking difference. \( \Delta_{IF} \) is a smooth and regular function, while \( \Delta_{FF} \) looks ‘noisy’ and irregular. Furthermore, for fixed \( 0 < x^0 < L \), \( \Delta_{IF} \) has compact support inside the LC, \(-L < x^1 < x^0 \) and \( m^2 L / 2 \pi n \) (and in the periodic copies of this interval). Outside the LC \( \Delta_{IF} \) shows tiny oscillations around the value zero, which vanish in the limit \( N \to \infty \). The oscillations are due to Gibbs’ phenomenon (the Fourier se-
ries does not converge uniformly in the vicinity of points where the limiting function makes jumps). Physically, what happens is that we have point sources located at positions $x^1 = 2Ln$. These ‘emit’ spherical waves which do not interfere unless $x^0 \geq L$. For $x^0 > L$ (not shown here) we have an interference phenomenon so that $\Delta_{IF}$ no longer vanishes outside the LC, which is a straightforward consequence of periodicity.

The situation concerning $\Delta_{FF}$ is different. Numerically, one sees that despite the irregular shape the sum $\sum_{n=0}^{\infty} B_n$ converges to a periodic function. The most important observation, however, is that $\Delta_{IF}$ does not vanish outside the LC, i.e. for $x^- < 0$, $x^+ > 0$ as in Fig. 2. This a clear violation of microcausality. We have numerical evidence for a corresponding behavior in $d = 3, 4$ space-time dimensions.

C. Analytical Results

Let us try to get an analytical understanding of the numerical results beginning with $\Delta_{IF}$. A straightforward application of the Poisson resummation formula yields

$$\Delta_{IF}(x) = \sum_{n} \Delta(x^0, x^1 + 2Ln), \quad (10)$$

with the continuum $\Delta$ from [3]. This result is exactly what we see in Fig. 1, a periodic array of (nearly) smooth functions with support inside the LC (and its periodic copies). It should be stressed that $\Delta_{IF}$ is causal even for finite extension $L$, i.e. without the infinite-volume limit being performed.

Let us now analyze $\Delta_{FF}$. First note the rather weak localization properties of $\Delta$ in the LC direction $x^-$. For positive LC time $x^+$, $\Delta$ vanishes outside the LC, i.e. for $x^- < 0$, and decays slowly for $x^+ > 0$, asymptotically like $(x^-)^{-\frac{1}{4}}$. The integrand in [3], denoted by $I(k^+)$, oscillates rapidly for small $k^+$ such that the zero mode, i.e. the limit $I(k^+ = 0)$, is not defined. It turns out that this makes the application of Poisson resummation impossible. Because the latter cannot be used, let us consider the following alternative which leads to an analytic and close approximation of $\Delta_{FF}$. For this purpose we rewrite [3] in terms of dimensionless variables (cf. Fig. 2),

$$\Delta_{FF}(v, w) = \frac{1}{2\pi} \sin(w/2\pi n + 2\pi n v), \quad (11)$$

with $v \equiv x^-/2L$, $w \equiv n^2L^2(x^+/2L)$. If we expand [11] in powers of $w$, the sum over $n$ can actually be performed using summation formulae 1.4.43.1/2 from [16]. With the restriction $-1 \leq v \leq 1$, the result is

$$\Delta_{FF}(v, w) = -\frac{1}{2} \sum_{n=0}^{\infty} \frac{w^n}{n!(n+1)!} \text{sgn}(n+1)(-v)B_{n+1}(|v|), \quad (12)$$

where $B_n$ denotes the $n$th Bernoulli polynomial [16]. The series [12], being a power series instead of a Fourier series, converges rather rapidly as a function of $w$. In addition, the limit is approached uniformly (no Gibbs phenomenon). This is obvious from Fig. 3 where we compare the resummed expression [12] with the Fourier representation [11] (for $w = 5$). The agreement is quite impressive which is to be expected as we have summed the first 20 terms in [12].

Fig. 3: Comparison of the Fourier representation [11] (for $N = 40$) with the result of Bernoulli resummation [12], for $w = 5$, $-1 \leq v \leq 1$, $N = 20$.

Numerically, one finds convergence if the number $N$ of terms summed over is of the order of $w/2$. This is due to the fact that the amplitude of the Bernoulli polynomials $B_n$ decreases rapidly with $n$. Thus, for $w > 2$, i.e. $x^- \lesssim 4/m^2L$, the first two terms in the expansion [12] are already a rather good approximation so that we can write,

$$\Delta_{FF}(v, w) \approx -\frac{1}{4} \text{sgn}(v) + \frac{v}{2} - \frac{w}{4}(v^2 - |v| + 1/6). \quad (13)$$

This result provides an analytical check that $\Delta_{FF}$ does not vanish outside the LC ($-1 < v < 0$), and thus, that causality is violated.

RESTORATION OF A CAUSAL PROPAGATOR

With representation [3] we are sampling a continuous function $I(k^+)$ by equidistant points on a momentum grid in $k^+$. For small $k^+$, however, this is a very bad approximation, as $I(k^+)$ is rapidly oscillating there, with a frequency increasing roughly as $1/k^+$. The point $k^+ = 0$ is thus an accumulation point of the Fourier spectrum. In its vicinity, we should actually sample with a momentum resolution $\Delta k^+ \sim 1/n$. In other words, instead of harmonic one should use anharmonic “Fourier” analysis. If we use [7] to introduce new discrete longitudinal momenta,
we can write down the causal, finite-volume commutator in terms of light-cone variables,
\[
\Delta_c(x) = \frac{1}{L} \sum_n \frac{k^+}{(k^+)^2 + m^2} \sin(k_n \cdot x), \tag{15}
\]
where \(k_n \cdot x = k_n^+ x^+ / 2 + k_n^- x^- / 2\). Obviously, the momentum grid in (14) is not equidistant. In particular, for small \(k_n^+\), corresponding to large negative \(k_n^-\), one finds \(\Delta k_n^+ \sim 1/n\). Thus, the small-\(k^+\) region becomes sampled in a reasonable way such that the features of \(I(k^+)\), which guarantee the causality of \(\Delta\), are properly described even on a finite (momentum) lattice (see Fig. 4).

![Fig. 4: The causal commutator (15) in the LC representation as a function of \(\nu = x^- / 2L\), for \(x^+ / 2L = 0.2\), \(mL = 50\), \(N = 50\). It vanishes for \(-1 < \nu < 0\) (up to the unavoidable Gibbs phenomenon).](image-url)

### DISCUSSION

The above analysis shows that one cannot have both, periodicity in \(x^-\) and causality of the commutator \(\Delta(x^+, x^-)\). If one insists on periodicity, one violates causality and vice versa. Consequently, the method of DLCQ in which the field operators are expanded in periodic plane waves, is non-causal. This does not come as too big a surprise: it is well known that DLCQ yields a rather poor representation of the small-\(k^+\) behaviour of observables \([17]\). Using the relation between Pauli-Jordan function and Feynman propagator, \(\pi \Delta(k) = -\text{sgn}(k^+) \text{Im} \Delta_F(k)\), we see that a causality violation also affects the Feynman propagator \(\Delta_F\). Note that the causality of \(\Delta\) can be viewed as a delicate cancellation between particle and anti-particle propagation amplitudes. Therefore it seems that also charge conjugation symmetry is violated by imposing light-like periodicity. Furthermore, with microcausality being at the heart of any dispersion relation, one expects problems also there. We have seen that one can Taylor an ad hoc momentum grid with a special anharmonic resolution which remedies the causality violation of the commutator. It is an open question, however, whether this solves the causality problem of DLCQ in general. We expect the answer to be negative: any causal Green function will have its own peculiar small-\(k^+\)-behaviour and thus will require its own momentum grid which generically will be different from the one introduced above. An ensuing dependence of results on a particular discretisation choice clearly cannot be accepted.

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Fig. 1: $\Delta_{IF}(X, T)$ as a function of $X = x^1/2L$, for $T = x^0/2L = 0.2$, $mL = 1$, $N = 50$. 
Fig. 2: $\Delta_{FP}(v, w)$ as a function of $v = x^-/2L$, for $w = m^2Lx^+/2 = 10000$, $N = 70$. 
Fig. 3: Comparison of the Fourier representation Eq.(11), (for $N = 40$) with the result of Bernoulli resummation Eq.(12), for $w = 5$, $-1 \leq v \leq 1$, $N = 20$. 
Fig. 4: The causal commutator Eq.(15) in the LC representation as a function of $v = x^{-}/2L$, for $x^{+}/2L = 0.2$, $mL = 50$, $N = 50$. It vanishes for $-1 < v < 0$ (up to the unavoidable Gibbs phenomenon).