\( \eta \) and \( \eta' \) mesons in the Dyson-Schwinger approach using a generalization of the Witten-Veneziano relation

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The description of the \( \eta \) and \( \eta' \) mesons in the Dyson-Schwinger approach has relied on the Witten-Veneziano relation. The present paper explores the consequences of using instead its generalization recently proposed by Shore. On the examples of three different model interactions, we find that irrespective of the concrete model dynamics, our Dyson-Schwinger approach is phenomenologically more successful in conjunction with the standard Witten-Veneziano relation than with the proposed generalization valid in all orders in the \( 1/N_c \) expansion.

PACS numbers: 11.10.St, 12.38.-t, 14.40.Aq, 12.38.Lg

I. INTRODUCTION

The Dyson-Schwinger (DS) approach to QCD and its modeling is the chirally well-behaved bound-state approach. Thus, it is the most suitable to QCD and its modeling is the chirally well-behaved one solves the (“gap”) DS equations (DSEs) \( \Sigma \) for the dynamically dressed quark propagators \( S_q \), where \( q \) is the quark flavor (\( g = u, d, s, \ldots \)), \( S_q \) is the free quark propagator, and \( \Sigma_q \) is the quark self-energy. These dressed quark propagator solutions are then employed in Bethe-Salpeter equations for the bound-state vertex \( \Gamma_{qq'} \) of the meson composed of the quark of the flavor \( q \) and antiquark of the flavor \( q' \):

\[
[\Gamma_{qq'}]_{e\ell} = \int [S_q \Gamma_{qq'} S_q]_{gh} [K]_{hf}^{bh},
\]

where \( e, f, g, h \) schematically represent spinor, color and flavor indices, integration is meant over loop momenta, and \( K \) is the interaction kernel. Solving Eq. (1) for \( \Gamma_{qq'} \) also yields \( M_{qq'} \), the mass eigenvalue of the \( qq' \) meson.

To obtain the chiral behavior as in QCD, DS and BS equations must be solved in a consistent approximation. The rainbow-ladder approximation (RLA), where DChSB is well-understood, is still the most usual approximation in phenomenological applications. This also entails that in both DSE and BSE (1) we employ the same effective interaction kernel,

\[
[K(k)]_{ef}^{bh} = i g^2 D_{\mu\nu}^{ab}(k)_{\text{eff}} \left[ \frac{\lambda^a}{2} \right]_{eg} \left[ \frac{\lambda^a}{2} \right]_{hf} \gamma^\mu \gamma^\nu,
\]

so that the quark self energy in the gap DSE is

\[
\Sigma_q(p) = - \int \frac{d^4 \ell}{(2\pi)^4} g^2 D_{\mu\nu}^{ab}(k)_{\text{eff}} \left[ \frac{\lambda^a}{2} \right]_{eg} \left[ \frac{\lambda^a}{2} \right]_{hf} \gamma^\mu \gamma^\nu.
\]

In Eqs. (2) and (3), \( D_{\mu\nu}^{ab}(k)_{\text{eff}} \) is an effective gluon propagator. For example, for renormalization-group improved (RGI) interactions (e.g., in Refs. [7, 8, 9, 10, 11]), it has the form

\[
g^2 D_{\mu\nu}^{ab}(k)_{\text{eff}} = 4\pi\alpha_{\text{eff}}(k^2) D_{\mu\nu}^{ab}(k)_{\text{free}}
\]

where \( D_{\mu\nu}^{ab}(k)_{\text{free}} \) is the free gluon propagator, and \( \alpha_{\text{eff}}(k^2) \) is an effective running coupling. For large space-like momenta (\( k^2 \gg 1 \text{ GeV}^2 \), \( \alpha_{\text{eff}}(k^2) \) approaches the perturbative QCD running coupling \( \alpha_s(k^2) \) known from the QCD renormalization group analysis, although it must be modeled at low momenta.

Concretely, in the present paper we recall and utilize the results obtained i) in Refs. [7, 8] by using the RGI of Jain and Munczek [9, ii) in Ref. [10] by using the RGI gluon condensate-induced interaction [11], and iii) in Refs. [12, 13] by using the separable interaction [14]. In any case, such effective interactions must be modeled at least in the low-energy, nonperturbative regime in order to be phenomenologically successful – which above all means to be sufficiently strong in the low-momentum domain to yield DChSB. In the chiral limit (and close to it), light pseudoscalar (\( P \)) meson \( q\bar{q} \) bound states (\( P = \pi^0, K^0, \eta \)) then simultaneously manifest themselves also as (quasi-)Goldstone bosons of DChSB. This enables one to work with the mesons as explicit \( q\bar{q} \) bound states, while reproducing the results of the Abelian axial anomaly for the light pseudoscalars, i.e., the amplitudes for \( P \to \gamma\gamma \) and \( \gamma^* \to P^0 P^+ P^- \). This is unique among
the bound state approaches – e.g., see Refs. [3, 15, 17] and references therein. Nevertheless, one keeps the advantage of bound-state approaches that from the $q\bar{q}$ substructure one can calculate many important quantities (such as the pion, kaon and $s\bar{s}$ pseudoscalar decay constants: $f_\pi$, $f_K$, and $f_{s\bar{s}}$) which are just parameters in most of other chiral approaches to the light-quark sector. The treatment [3, 8, 10, 18] of the $\eta'$ complex is remarkable in that it is very successful in spite of the limitations of RLA. (Very recently, during the work on the present paper, the first and still simplified DS treatments of $\eta$ and $\eta'$ beyond RLA appeared [19, 20]. However, RLA treatments will probably long retain their usefulness in applications where simple modeling is desirable, as in the calculationally demanding finite-temperature calculations [13].) The RLAs treatments of the $\eta$-$\eta'$ complex at first determined [7, 8, 18] the anomalous $\eta_0$ mass parameter by fitting the empirical $\eta$ and $\eta'$ masses. More recently, the treatment was improved by avoiding this fitting while retaining the phenomenologically successful description [10, 13], namely, the anomalous $\eta_0$ mass was no longer a free parameter but determined from the lattice results (on QCD topological susceptibility) through the Witten-Veneziano (WV) relation [21, 22]. However, Shore achieved [23, 24] what can be considered as a generalization of the WV relation, and the purpose of the present paper is exploring the usage of this generalization in the DS context.

The paper is organized as follows: in the next section, we recapitulate the procedures and results of our previous treatments [3, 10, 13] relying on the WV relation (17), and present in Table I also their extension to the scheme of the four decay constants (and two mixing angles) of $\eta$ and $\eta'$. In Section III, we expose the usage of the pertinent Shore’s equations [23, 24] in the context of DS approach. The last section concludes after giving the results of solving the pertinent equations.

II. $\eta$-$\eta'$ MASS MATRIX FROM WITTEN-VENEZIANO RELATION

All $q\bar{q}'$ model masses $M_{q\bar{q}'}$ ($q, q' = u, d, s$) used in the present paper, and corresponding $q\bar{q}'$ bound-state amplitudes, were obtained in Refs. [7, 8, 10, 12, 13, 25] in RLA, i.e., with an interaction kernel which (irrespective of how one models the dynamics) cannot possibly capture the effects of the non-Abelian, gluon axial anomaly. Thus, when we form the $\eta$-$\eta'$ mass matrix

$$\hat{M}_{\eta\eta}^2 = \begin{bmatrix} M_{ss}^2 & M_{s\bar{s}}^2 \\ M_{s\bar{s}}^2 & M_{\bar{s}\bar{s}}^2 \end{bmatrix},$$

(5)

in this case in the octet-singlet basis $\eta_8-\eta_0$ of the (broken) flavor-SU(3) states of isospin zero,

$$\eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}), \quad \eta_0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}),$$

(6)

this matrix [5], consisting of our calculated $q\bar{q}$ masses,

$$M_{s\bar{s}}^2 \equiv \langle \eta_8 | \hat{M}_{\eta\eta}^2 | \eta_8 \rangle = \frac{2}{3} (M_{ss}^2 + \frac{1}{2} M_{u\bar{u}}^2),$$

(7)

$$M_{s\bar{s}}^2 \equiv \langle \eta_0 | \hat{M}_{\eta\eta}^2 | \eta_0 \rangle = M_{\bar{s}\bar{s}}^2 = \sqrt{2} \left( M_{u\bar{u}}^2 - M_{ss}^2 \right) < 0,$$

(8)

$$M_{\bar{s}\bar{s}}^2 \equiv \langle \eta_0 | \hat{M}_{\eta\eta}^2 | \eta_0 \rangle = \frac{2}{3} \left( 2 M_{s\bar{s}}^2 + M_{u\bar{u}}^2 \right),$$

(9)

is purely non-anomalous (NA), vanishing in the chiral limit. In the isospin limit, to which we adhere throughout, the pion is strictly decoupled from the gluon anomaly and $M_{u\bar{u}} = M_{d\bar{d}}$ is exactly our model pion mass $M_\pi$. Also the unphysical $s\bar{s}$ quasi-Goldstone’s mass $M_{s\bar{s}}$ results from RLA BSE and does not include the contribution from the gluon anomaly. This is consistent with the fact that due to the Dashen-Gell-Mann-Oakes-Renner (DGMOR) relation, it is in a good approximation [7, 8, 10, 13] given by

$$M_{s\bar{s}}^2 = 2M_K^2 - M_\tau^2,$$

(10)

e.g., by the kaon and pion masses protected from the anomaly by strangeness and/or isospin.

In our previous DS studies [7, 8, 10, 12, 13, 25], to which we refer for all model details, the phenomenology of the non-anomalous sector was successfully reproduced, e.g., $f_\pi, f_K$, as well as the empirical masses $M_\pi$ and $M_K$ (see the upper part of Table I), yielding a strongly non-diagonal $M_{\eta\eta}^2 \equiv 5$. Its diagonalization leads to the eigenstates known as the nonstrange-strange (NS-S) basis,

$$\eta_{NS} = \frac{1}{\sqrt{2}} (u\bar{u} + d\bar{d}), \quad \eta_s = s\bar{s},$$

(11)

and to $M_{\eta\eta}^2 = \text{diag}[M_{s\bar{s}}^2, M_{s\bar{s}}^2]$. In contrast to these mass-squared eigenvalues, the experimental masses are such that $(M_{\eta'}^2)_{\text{exp}} < (M_\eta^2)_{\text{exp}}$, and $\eta'$ is too heavy, $\langle M_{\eta'} \rangle_{\text{exp}} = 958$ MeV, to be considered even as the $s\bar{s}$ quasi-Goldstone boson. This is the well-known $U_A(1)$ problem, resolved by the fact that the complete $\eta$-$\eta'$ mass matrix $M_2$ must contain the anomalous (A) part $M_2^A$. That is, $M_2 = M_{\eta\eta}^2 + M_2^A$.

However, $M_2^A$ is inaccessible to RLA which yields our Goldstone pseudoscalars. In Refs. [7, 8, 10, 12, 13], $M_2^A$ was extracted from lattice data through the WV relation [the second equality in Eq. (17)]. The purpose of the present paper, instead, is to approach $\eta$ and $\eta'$ through Shore’s [23, 24] recent generalization of that relation.

Before that, however, we review the usage of the WV relation in Refs. [7, 8, 10, 12, 13]. The expansion in the large number of colors, $N_c$, indicates that the leading approximation in that expansion describes the bulk of main features of QCD. The gluon anomaly is suppressed as $1/N_c$ and can be viewed as a perturbation in the large $N_c$ expansion. In the SU(3) limit [compare Eqs. 

$\ldots$
and [13], it is coupled only to the singlet combination \( \eta_0 \); only the \( \eta_0 \) mass receives, from the gluon anomaly, a contribution which, unlike quasi-Goldstone masses \( M_{qg} \)'s comprising \( M^2_{NN} \), does not vanish in the chiral limit. As discussed in Refs. [7, 11], in the present bound-state context it is thus meaningful to include the effect of the gluon anomaly just on the level of a mass shift for the \( \eta_0 \) as the lowest-order effect, and retain the \( q\bar{q} \) bound-state amplitudes and the corresponding mass eigenvalues \( M_{qq} \) as calculated by solving DSEs and BSEs with kernels in RLA.

References [7, 8, 10, 12, 13] thus break the \( U_A(1) \) symmetry, and avoid the \( U_A(1) \) problem, by shifting the \( \eta_0 \) (squared) mass by an amount denoted by \( 3\beta \) (in the notation of Refs. [8, 10]). The complete mass matrix \( M^2 = M^2_{NN} + M^2_\Lambda \) then contains the anomalous part

\[
\tilde{M}^2_\Lambda = \text{diag}[0, 3\beta],
\]

where the anomalous \( \eta_0 \) mass shift \( 3\beta \) is related to the topological susceptibility of the vacuum, but in the present approach must be treated as a parameter to be determined outside of our RLA model, i.e., fixed by phenomenology or taken from the lattice calculations [26]. (The possibility of employing an additional microscopic model for the gluon anomaly contribution, such as the one of Ref. [27], is presently not considered.)

The SU(3) flavor symmetry breaking and its interplay with the gluon anomaly modifies [10] \( M^2_\Lambda \) [12] to

\[
\tilde{M}^2_\Lambda = \beta \begin{bmatrix}
\frac{1}{2} (1 - X)^2 & \frac{\sqrt{3}}{2} (1 - X)(2 + X) \\
\frac{\sqrt{3}}{2} (1 - X)(2 + X) & \frac{1}{2} (2 + X)^2
\end{bmatrix},
\]

where \( X \) is the flavor symmetry breaking parameter. It is most often estimated as \( X = f_\pi / f_{K^*} \sim 0.7 - 0.8 \) (see, e.g., Refs. [8, 10, 28, 29], although there are some other [8], of course related, estimates of \( X \)). Presently we also adopt \( X = f_\pi / f_{K^*} \), which means that \( X \) is a calculated quantity in our approach. The employed models achieved good agreement with phenomenology [8, 10, 12], e.g., fitted the experimental value of \( M^2_{\eta'} + M^2_{\eta} \) for \( \beta \) around 0.26 – 0.28 GeV\(^2\). The anomalous contribution \( \tilde{M}^2_\Lambda \) then brings the complete \( M^2 \) rather close to a diagonal form for all considered models [8, 10, 12]; that is, to diagonalize \( M^2 \), only a relatively small rotation (\( |\theta| \sim 13^\circ \pm 2^\circ \)) of the \( \eta_\pi-\eta_\theta \) basis states,

\[
\eta = \cos \theta \eta_\theta - \sin \theta \eta_0, \quad \eta' = \sin \theta \eta_\theta + \cos \theta \eta_0,
\]

is needed to align them with the mass eigenstates, i.e., with the physical \( \eta \) and \( \eta' \). In contrast to this, the \( \eta-\eta' \) mass matrix in the NS-S basis [11],

\[
\tilde{M}^2 = \begin{bmatrix}
M^2_{\eta NS} & M^2_{\eta_\theta NS} \\
M^2_{\eta_\theta NS} & M^2_{\eta_\theta}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
M^2_\eta + 2\beta \sqrt{2} \beta X & \sqrt{2} \beta X \\
\sqrt{2} \beta X & M^2_{\eta_\theta} + \beta X^2
\end{bmatrix} \phi^{-1} \begin{bmatrix}
M^2_\eta & 0 \\
0 & M^2_{\eta_\theta}
\end{bmatrix} \phi,
\]

is then strongly off-diagonal. The indicated diagonalization, given by

\[
\eta = \cos \phi \eta_{NS} - \sin \phi \eta_\theta, \quad \eta' = \sin \phi \eta_{NS} + \cos \phi \eta_\theta,
\]

is thus achieved for a large NS-S state-mixing angle \( \phi \sim 42^\circ \pm 2^\circ \). Of course, this is again in agreement with phenomenological requirements [8, 10], since \( \phi \) is fixed to the \( \eta_\pi-\eta_\theta \) state-mixing angle \( \theta \) by the relation \( \phi = \theta + \arctan \sqrt{2} = \theta + 54.74^\circ \). The masses are

\[
M^2_\eta = \frac{1}{2} \left[ M^2_{\eta NS} + M^2_{\eta_\theta} - \sqrt{(M^2_{\eta NS} - M^2_{\eta_\theta})^2 + 8\beta^2 X^2} \right],
\]

\[
M^2_{\eta'} = \frac{1}{2} \left[ M^2_{\eta NS} + M^2_{\eta_\theta} + \sqrt{(M^2_{\eta NS} - M^2_{\eta_\theta})^2 + 8\beta^2 X^2} \right].
\]

The invariant trace of the mass matrix [15], together with Eq. [10], gives the first equality in

\[
\beta (2 + X^2) = M^2_\eta + M^2_{\eta'} - 2M^2_K = \frac{6}{f_\pi^2} \chi_{YM}.
\]

The second equality is the Witten-Veneziano (WV) relation [21, 22] between the \( \eta, \eta' \) and kaon masses and \( \chi_{YM} \), the topological susceptibility of the pure gauge, Yang-Mills theory. Thus, \( \beta \) does not need to be a free parameter, but can be determined from lattice results on \( \chi_{YM} \), so that no fitting parameters are introduced. For the three models [9, 11, 14] utilized in our treatments [7, 8, 10, 12] of \( \eta \) and \( \eta' \), the bare quark mass parameters and the interaction parameters were fixed already in the non-anomalous model, by requiring the good pion and kaon phenomenology. (See the \( \pi \) and \( K \) masses and decay constants in the uppermost part of Table [II]) Then, following Refs. [10, 13] in adopting the central value of the weighted average of the recent lattice results on Yang-Mills topological susceptibility [30, 31, 52],

\[
\chi_{YM} = (175.7 \pm 1.5 \text{MeV})^4,
\]

we have obtained the good descriptions of the \( \eta-\eta' \) phenomenology [7, 8, 10, 12], exemplified by the first three columns (one for each DS models used) of the middle part of Table [II], giving the predictions for the \( \eta \) and \( \eta' \) masses and for the NS-S mixing angle \( \phi \).

The lowest part of the table, below the second horizontal dividing line, contains the results on the quantities \( \langle \theta_\pi, \theta_\theta, \text{etc.} \rangle \) defined in the scheme with four \( \eta \) and \( \eta' \) decay constants and two mixing angles, introduced and explained in the following Section [III] Table [II] also compares these results of ours (in the first three columns) with the corresponding results of Shore’s approach [23, 24], in which the experimental values of the meson masses \( M_\pi, M_K, M_{\eta'}, \) and \( f_K \) (in contrast to our \( q\bar{q} \) bound-state model predictions for these quantities) are used as inputs enabling the calculation of various decay constants in the \( \eta-\eta' \) complex and the two mixing angles \( \theta_\pi \) and \( \theta_\theta \) (corresponding to \( \phi = 38.24^\circ \) in our approach).
III. USAGE OF SHORE’S EQUATIONS IN DS APPROACH

The WV relation was derived in the lowest-order approximation in the large $N_c$ expansion. However, considerations by Shore [23, 24] contain what amounts to the generalization of the WV relation, which is valid to all orders in $1/N_c$. Among the relations he derived through the inclusion of the gluon anomaly in DGMOR relations, the following are pertinent for the present paper:

\[
(f^0_{\eta'})^2 M^2_{\eta'} + (f^0_{\bar{\eta}})^2 M^2_{\bar{\eta}} = \frac{1}{3} (f^2_{\pi} M^2_{\pi} + 2 f^2_K M^2_K) + 6A, \tag{19}
\]

\[
f^0_{\eta'} f^0_{\bar{\eta}} M^2_{\eta'} + f^0_{\bar{\eta}} f^0_{\eta} M^2_{\bar{\eta}} = \frac{2\sqrt{2}}{3} (f^2_{\pi} M^2_{\pi} - f^2_K M^2_K), \tag{20}
\]

\[
(f^8_{\eta'})^2 M^2_{\eta'} + (f^8_{\bar{\eta}})^2 M^2_{\bar{\eta}} = \frac{1}{3} (f^2_{\pi} M^2_{\pi} - 4 f^2_K M^2_K), \tag{21}
\]

where $A$ is the full QCD topological charge parameter, and $f^0_{\eta'}, f^0_{\bar{\eta}}, f^8_{\eta'}, f^8_{\bar{\eta}}$ are the four decay constants [33, 34, 35] associated with the two isoscalar pseudoscalars $\eta$ and $\eta'$. The nonperturbative parameter $A$ is related to the QCD topological susceptibility, quark condensates and quark masses [23, 24]. At large $N_c$, it should be well-approximated by the topological susceptibility, $A \approx \chi$. More precisely, it reduces to the YM topological susceptibility in the large $N_c$ limit: $A = \chi_{\text{YM}} + \mathcal{O}(1/N_c)$, but at present it is not known better than that, as there are still no lattice data on this nonperturbative QCD parameter. Therefore, in his own phenomenological analysis, Shore himself had to approximate $A$ by a value of $\chi_{\text{YM}}$ [23, 24]. In that sense, because of this crucial assumption based on the lowest-order $1/N_c$ approximation, even his analysis was not (and, because of the lack of the corresponding lattice data, could not be) carried out numerically consistently in the orders of $N_c$, even though his formulas are valid in all orders in the $1/N_c$ expansion.

While the present bound-state DS approach clearly cannot improve on the consistency aspect, it offers the possibility of a phenomenological analysis entirely different from Shore’s. Namely, in addition to $A \approx \chi_{\text{YM}}$, Shore used the experimentally known quantities (pion, kaon, $\eta$ and $\eta'$ masses, as well as the pion and kaon decay constants) as inputs in Eqs. (19)–(21) to obtain the $\eta$ and $\eta'$ decay constants $f^0_{\eta'}, f^0_{\bar{\eta}}, f^8_{\eta'}, f^8_{\bar{\eta}}$. On the other hand, the predicting power of our bound-state DS approach is much larger: not only are pion and kaon masses and decay constants calculated quantities, predicted from the $q\bar{q}$ substructure, but once we formulate the incorporation of Shore’s generalization within the bound-state DS approach, it will become obvious that also these four $\eta$ and $\eta'$ decay constants and their masses $M_\eta$ and $M_{\bar{\eta}}$ come out as pure predictions. Such a phenomenological analysis, complementary to Shores, motivates us to formulate and perform the treatment based on Shore’s generalization, instead of the original WV relation (or fitting the anomalous $\theta_0$ mass shift) as in our earlier references [7, 8, 10, 13, 18].

Adding Eqs. (19) and (21), one gets the relation

\[
(f^0_{\eta'})^2 M^2_{\eta'} + (f^0_{\bar{\eta}})^2 M^2_{\bar{\eta}} + (f^8_{\eta'})^2 M^2_{\eta'} + (f^8_{\bar{\eta}})^2 M^2_{\bar{\eta}} = 6A \tag{22}
\]

which is the analogue of the standard WV formula [17], to which it reduces in the large $N_c$ limit where $A \to \chi_{\text{YM}}$, the $f^0_{\eta'}, f^0_{\bar{\eta}}, f^8_{\eta'}, f^8_{\bar{\eta}} \to f_\pi$, and the limit of vanishing subdominant decay constants (since $\eta$ and $\eta'$ are dominantly $\eta_8$ and $\eta_0$, respectively), i.e., $f^0_{\eta'}, f^0_{\bar{\eta}}, f^8_{\eta'}, f^8_{\bar{\eta}} \to 0$. However, we will need to use not just this single equation, but the three equations (19)–(21) from Shore’s generalization.

These four $\eta$ and $\eta'$ decay constants are often parameterized in terms of two decay constants, $f_8$ and $f_0$, and two mixing angles, $\theta_8$ and $\theta_0$:

\[
f^8_\eta = \cos \theta_8 f_8, \quad f^0_\eta = - \sin \theta_0 f_0, \tag{23}
\]

| FROM | Ref. & WV | Ref. & WV | Shore [23, 24] | Experiment |
|------|---------|---------|----------------|-------------|
| $M_\eta$ | 137.3 | 135.0 | 140.0 | $(138.0)_{\text{isospin}}^{\text{average}}$ |
| $M_K$ | 495.7 | 494.9 | 495.0 | $(495.7)_{\text{average}}$ |
| $M_K^* | 700.7 | 722.1 | 684.8 | |
| $f_\pi$ | 93.1 | 92.9 | 92.0 | 92.4 ± 0.3 |
| $f_K$ | 113.4 | 111.5 | 110.1 | 113.0 ± 1.0 |
| $f_{K^*}$ | 135.0 | 132.9 | 119.1 | |
| $M_\eta$ | 568.2 | 577.1 | 542.3 | 547.75 ± 0.12 |
| $M_\eta'$ | 920.4 | 932.0 | 932.6 | 957.78 ± 0.14 |
| $\phi$ | 41.42° | 39.56° | 40.75° | (38.24°) |
| $\theta_0$ | −2.86° | −5.12° | −6.80° | −12.3° |
| $\theta_8$ | −22.59° | −24.14° | −20.58° | −20.1° |
| $f_0$ | 108.8 | 107.9 | 101.8 | 106.6 |
| $f_8$ | 122.6 | 121.1 | 110.7 | 104.8 |
| $f_0'$ | 5.4 | 6.6 | 12.1 | 22.8 |
| $f_8'$ | 108.7 | 107.5 | 101.1 | 104.2 |
| $f_{K^*}^0$ | 113.2 | 110.5 | 103.7 | 98.4 |
| $f_{K^*}^8$ | −47.4 | −49.5 | −38.9 | −37.6 |
This is the so-called two-angle mixing scheme, which shows explicitly that it is inconsistent to assume that the mixing of the decay constants follows the pattern \( f_{8} = \sin \theta_{8} f_{S}, \quad f_{0} = \cos \theta_{0} f_{0} \). \( 24 \)

The advantage of our model is that, as we shall see, we are able to calculate the \( f_{8} \) and \( f_{0} \) parts of the physical decay constants \( 23, 24 \) from the \( qg \) substructure. However, we cannot keep the full generality of Shore’s approach, which allows for the mixing with the gluonic pseudoscalar operators, and therefore employs the definition \( 23, 24 \) of the decay constants which, in general, due to the gluonic contribution, differs from the following standard definition through the matrix elements of the axial currents \( A^{a\mu}(x) \):

\[
\{0|A^{a\mu}(x)|P(p)\} = i f_{a}^{p} p^{\mu} e^{-ip\cdot x}, \quad a = 8, 0; \quad P = \eta, \eta'. \quad (25)
\]

Nevertheless, Shore’s definition \( 23, 24 \) coincides with the above standard one in the non-singlet channel, where there cannot be any admixture of the pseudoscalar gluonic component. Similarly, since our BS solutions (from Refs. \([7, 8, 10, 13]\)) are the pure \( qg \) states, without any gluonic components, using Shore’s definition would not help us calculate the gluon anomaly influence on the decay constants. We thus employ the standard definitions \( 25 \), also used by, e.g., Gasser, Leutwyler, and Kaiser \( 33, 34, 35 \), as well as by Feldmann, Kroll, and Stech (FKS) \( 28, 29, 37 \).

In Eqs. \( 28, 29, 30 \), the angles are chosen so \( 28 \) that \( \theta_{8} = \theta_{0} = \theta = 0 \) in the limit of the exact SU(3) flavor symmetry, since only then there are just two decay constants, purely octet \( f_{8}^{0} = f_{S} \) and purely singlet \( f_{0}^{0} = f_{0} \), while the off-diagonal decay constants vanish, \( f_{8}^{r} = 0 = f_{0}^{r} \), in this limit. Otherwise, all four decay constants \( 26 \) are different from zero due to the breaking of the SU(3) flavor symmetry, since this leads to \( \theta \neq 0 \) and gives both \( \eta \) and \( \eta' \) the both components \( \eta_{8} \) and \( \eta_{0} \). In the parameterization \( 23, 24 \), the angles \( \theta_{8} \) and \( \theta_{0} \) differ from \( \theta \) since also \( \{0|A_{8}^{|}_{8}\}|\eta_{0}\rangle \neq 0 \neq \{0|A_{8}^{|}_{8}\}|\eta_{8}\rangle \). Thus, although not \( \eta_{8} \) but \( \eta_{0} \) couples to the gluon anomaly, the octet-singlet constants \( f_{8}^{0} \) and \( f_{0}^{0} \) are influenced by the gluon anomaly through its interplay with the SU(3) flavor symmetry breaking [similarly to the anomalous mass matrix \( 15 \) having nonvanishing 88, 08 and 80 elements when \( X \neq 1 \)].

Equivalently to \( f_{0}^{0}, f_{8}^{0}, f_{9}^{0}, f_{9}^{8} \), defined by Eq. \( 28, 29 \), one has four related but different constants \( f_{NS}^{8}, f_{NS}^{0}, f_{NS}^{8}, f_{NS}^{0} \), and \( f_{0}^{0} \), if instead of octet and singlet axial currents \( a = 8, 0 \) in Eq. \( 25 \), one uses the nonstrange axial currents \( a = NS, S \)

\[
A_{NS}^{a}(x) = \frac{1}{\sqrt{3}} A^{a\mu}(x) + \frac{1}{\sqrt{3}} A^{0\mu}(x) = \frac{1}{2} [\bar{u}(x)\gamma^{\mu}\gamma_{5}u(x) + \bar{d}(x)\gamma^{\mu}\gamma_{5}d(x)], \quad (26)
\]

The relation between the two equivalent sets is thus

\[
[f_{NS}^{8}, f_{NS}^{0}] = \left[ \begin{array}{cc} f_{8}^{8} & f_{8}^{0} \\ f_{0}^{8} & f_{0}^{0} \end{array} \right] = \left[ \begin{array}{cc} \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \sqrt{\frac{2}{3}} \end{array} \right]. \quad (28)
\]

Of course, this other quartet of \( \eta \) and \( \eta' \) decay constants can also be parameterized in terms of other two constants and two other mixing angles:

\[
f_{NS}^{8} = \cos \phi_{NS} f_{NS}, \quad f_{NS}^{0} = -\sin \phi_{NS} f_{S}, \quad (29)
\]

\[
f_{NS}^{8} = \sin \phi_{NS} f_{NS}, \quad f_{NS}^{0} = \cos \phi_{NS} f_{S}, \quad (30)
\]

where \( f_{NS} \) and \( f_{S} \) are given by the matrix elements

\[
\{0|A_{NS}^{a}(x)|\eta_{NS}(p)\} = i f_{NS} p^{\mu} e^{-ip\cdot x}, \quad (31)
\]

while \( \{0|A_{NS}^{a}(x)|\eta_{NS}(p)\} = 0 = \{0|A_{S}^{a}(x)|\eta_{NS}(p)\} \).

In the NS-S basis, it is possible to recover a scheme with a single mixing angle \( \phi \) through the application of the Okubo-Zweig-Iizuka (OZI) rule \( 28, 29, 37 \). For example, \( f_{NS} f_{S} \sin(\phi_{NS} - \phi_{S}) \) differs from zero just by an OZI-suppressed term \( 28 \). Neglecting this term thus implies \( \phi_{NS} = \phi_{S} \). (Refs. \( 28, 29, 37 \) denote \( f_{NS}, f_{S}, \phi_{NS}, \phi_{S} \) by, respectively, \( f_{q}, f_{s}, \phi_{q}, \phi_{s} \).) In general, neglecting the OZI-suppressed terms, i.e., application of the OZI rule, leads to the so-called FKS scheme \( 28, 29, 37 \), which exploits a big practical difference between the (in principle equivalent) parameterizations \( 28, 29 \) and \( 30, 31 \), while \( \theta_{8} \) and \( \theta_{0} \) differ a lot from each other and from the octet-singlet state mixing angle \( \theta \approx (\theta_{8} + \theta_{0})/2 \), the NS-S decay-constant mixing angles are very close to each other and both can be approximated by the state mixing angle: \( \phi_{NS} \approx \phi_{S} \approx \phi \). Therefore one can deal with only this one angle, \( \phi \), and express the physical \( \eta, \eta' \) decay constants as

\[
\left[ \begin{array}{c} f_{NS}^{8} \\ f_{NS}^{0} \end{array} \right] = \left[ \begin{array}{cc} f_{S} \cos \phi - f_{S} \sin \phi \\ f_{S} \sin \phi \end{array} \right] \left[ \begin{array}{c} \frac{1}{\sqrt{3}} \sqrt{\frac{2}{3}} \\ -\frac{1}{\sqrt{3}} \sqrt{\frac{2}{3}} \end{array} \right]. \quad (33)
\]

This relation is valid also in our approach, where \( \eta \) and \( \eta' \) are the simple \( \eta_{NS}-\eta_{S} \) mixtures \( 16 \). The FKS relations \( 28, 29, 37 \)

\[
f_{S} = \sqrt{\frac{1}{3} f_{NS}^{2} + \frac{2}{3} f_{S}^{2}}, \quad \theta_{S} = \phi - \arctan \left( \frac{\sqrt{2} f_{S}}{f_{NS}} \right), \quad (34)
\]

\[
f_{0} = \sqrt{\frac{2}{3} f_{NS}^{2} + \frac{1}{3} f_{S}^{2}}, \quad \theta_{0} = \phi - \arctan \left( \frac{\sqrt{2} f_{NS}}{f_{S}} \right), \quad (35)
\]
equivalent to Eq. (33), were also shown to hold in our DS approach.

In our present DS approach, mesons are pure $q\bar{q}$ BS solutions, without any gluonium admixtures, which are prominent possible sources of OZI violations. Therefore, our decay constants are calculated quantities, $f_{NS} = f_{u\bar{u}} = f_{d\bar{d}} = f_\pi$ and $f_S = f_{s\bar{s}}$, in agreement with the OZI rule. Our DS approach is thus naturally compatible with the FKS scheme, and we can use the $\eta$ and $\eta'$ decay constants (33) with our calculated $f_{NS} = f_\pi$ and $f_S = f_{s\bar{s}}$ in Shore's equations (19)-(21).

**IV. RESULTS AND CONCLUSIONS**

All quantities appearing on the right-hand side of Eqs. (19)-(21), namely $M_\pi, M_K, f_\pi$, and $f_K$, are calculated in our DS approach (for the three dynamical models [8, 11, 14]), except the full QCD topological charge parameter $A$. Since it is at present unfortunately not yet known, we follow Shore and approximate it by the Yang-Mills topological susceptibility $\chi_{YM}$.

On the left-hand side of Eqs. (19)-(21), the model results for $f_{NS} = f_\pi$ and $f_S = f_{s\bar{s}}$ and Eq. (33) reduce the unknown part of the four $\eta$ and $\eta'$ decay constants $f_{\eta'0}, f_{\eta'0}^0, f_{\eta'8}$, and $f_{\eta'8}^0$ down to the mixing angle $\phi$. The three Shore's equations (19)-(21) can then be solved for $\phi, M_\eta$ and $M_{\eta'}$, providing us with the upper three lines of Table III.

For each of the three different dynamical models which we used in our previous DS studies (8, 10, 12, 22), these results are displayed for $\chi_{YM} = (175.7\text{ MeV})^4$ as in Refs. (10, 12) and for $\chi_{YM} = (191\text{ MeV})^4$ (31) (adopted by Shore (23, 24)). The lower part of the table, displaying various additional results, is then readily obtained through Eq. (33) and/or the useful relations (24-25) which give $f_\pi, f_0, \theta_0 - \phi$ and $\theta_0 - \phi$ in terms of $f_{NS} = f_\pi$ and $f_S = f_{s\bar{s}}$. Thus, unlike the mixing angles, $f_\pi$ and $f_S$ do not result from solving of Eqs. (19)-(21), but are the calculated predictions of a concrete dynamical DS model, independently of Shore's equations.

For all three quite different (RGI [9, 11] and non-RGI [14]) dynamical models which we used in our previous DS studies (7, 8, 10, 12, 13, 22), the situation with the results turns out to be rather similar. Similar results from various models mean that the usage of Shore's generalization in conjunction with the DS approach does not help one to discriminate between various dynamical models and so draw conclusions on the dynamics. This is not surprising, as it has been established (1, 2, 3, 4, 5, 21) that while a successful reproduction of static properties and other low-energy meson phenomenology requires interaction modeling at low momenta, it is possible to achieve a satisfactory description of low-energy phenomenology for many forms of model interactions as long as their integrated strength at low momenta ($p^2 < 1\text{ GeV}^2$) is sufficient to achieve a realistic DChSB. On the other hand, this (similarity of our results from the very different models) has the advantage that our conclusions further below are not sensitive to the changes of the model dynamics.

The most conspicuous feature of our results is that $\eta$ and $\eta'$ masses are both much too low when the weighted average $\chi_{YM} = (175.7 \pm 1.5\text{ MeV})^4$ of Refs. (30, 31, 32) is used, in contrast to the results from the standard WV relation, displayed in Table I. If we single out just the highest of these values (191 MeV) (31), the masses improve somewhat. However, other results are spoiled – e.g., the mixing angle $\phi$ becomes too high to enable agreement with the experimental results on $\eta, \eta' \rightarrow \gamma \gamma$ decays, which require $\phi \sim 40^\circ$ (10).

When we turn to the lower parts of Tables I and III where the results for the $\eta$ and $\eta'$ decay constants, and the corresponding two mixing angles $\theta_0$ and $\theta_8$, are given, we notice a feature common to all our results, as well as Shore's (also given in Table I). The diagonal ones, $f_{\eta'0}^0$ and $f_{\eta'0}$, are all of the order of $f_\pi$, being larger by some 10% to 30%. The off-diagonal ones, $f_{\eta'8}$, and $f_{\eta'8}^0$, are, on the other hand, in general strongly suppressed. This is expected, as $\eta'$ is mostly singlet, and $\eta$ is mostly octet.

To understand the dependence of the decay constants on the topological susceptibility $\chi_{YM}$ (approximating $A$), it is important to note that our $f_0$, which in a full QCD bound-state calculation would be influenced by the gluon anomaly, presently is not, since it is calculated (same as $f_\pi$) from the modeled meson $q\bar{q}$ substructure relying on RLA. In Tables I and II one therefore sees no

| Inputs: | from Ref. [8] | from Ref. [10] | from Ref. [13] |
|---------|---------------|---------------|---------------|
| $\chi_{YM}$ | 175.7 | 191 | 175.7 | 191 | 175.7 | 191 |
| $M_\eta$ | 485.7 | 499.8 | 482.8 | 496.7 | 507.0 | 526.2 |
| $M_{\eta'}$ | 815.8 | 934.1 | 814.8 | 934.9 | 868.7 | 983.2 |
| $\phi$ | 46.11$^\circ$ | 52.01$^\circ$ | 46.07$^\circ$ | 51.85$^\circ$ | 40.86$^\circ$ | 47.28$^\circ$ |
| $\theta_0$ | 1.84$^\circ$ | 7.74$^\circ$ | 1.39$^\circ$ | 7.17$^\circ$ | -6.69$^\circ$ | -0.33$^\circ$ |
| $\theta_8$ | -17.90$^\circ$ | -12.00$^\circ$ | -17.90$^\circ$ | -11.58$^\circ$ | -20.47$^\circ$ | -14.11$^\circ$ |
| $f_\pi$ | 108.8 | 108.8 | 107.9 | 107.9 | 101.8 | 101.8 |
| $f_S$ | 122.6 | 122.6 | 121.1 | 121.1 | 110.7 | 110.7 |
| $f_{\eta'0}$ | -3.5 | -14.7 | -2.6 | -13.5 | 11.9 | 0.6 |
| $f_{\eta'8}$ | 108.8 | 107.9 | 107.9 | 107.1 | 101.1 | 101.8 |
| $f_{\eta'8}^0$ | 116.7 | 119.9 | 115.4 | 118.5 | 103.7 | 107.4 |
| $f_{\eta'8}^0$ | -37.7 | -25.5 | -37.6 | -24.9 | -38.7 | -27.0 |

TABLE II: The results of the three DS models obtained through Shore's equations (19)-(21) for the two values of $\chi_{YM}$ approximating $A$: (175.7 MeV)$^4$ and (191 MeV)$^4$. Columns 1 and 2: The results when the non-anomalous inputs for Eqs. (19)-(21), namely $M_\pi, M_K, f_\pi = f_{s\bar{s}}, f_{s\bar{s}} = f_S$, and $f_K$, are taken from Ref. [8], which uses Jain–Munczek Ansatz interaction (9). Columns 3 and 4: The results for the non-anomalous inputs from Ref. [10] using OPE-inspired interaction nonperturbatively dressed by gluon condensates (11). Columns 5 and 6: The results for the inputs from Ref. [13] using the separable Ansatz interaction (14). All masses and decay constants, as well as $\chi_{YM}^{1/4}$, are in MeV, and angles are in degrees.
that the DS approach with the standard WV relation to construct the complete $\eta$-$\eta'$ mass matrix, leads to the conclusion that the DS approach with the standard WV relation (17) is phenomenologically more successful, yielding the masses closer to the experimental ones. This may seem surprising, but one must be aware that we do not yet have at our disposal the full QCD topological charge parameter $A$, and that we (along with Shore) had to use its lowest $1/N_c$ approximation, $\chi_{YM}$. This in general precludes a consistently improved $1/N_c$ treatment in spite of the usage of Shore’s relations. The problems with inconsistencies in the $1/N_c$ counting may well cause spoiling of results, especially in an approach such as ours, where the $\eta$ and $\eta'$ masses are not inputs, but predicted quantities. Our results thus add a new argument to the motivation for undertaking lattice calculations proposed by Shore [23] and aimed at proper finding the quantity $A$. Also, we should recall from Sections 4 and 11 that the very usage of the RLA assumed that the anomaly is implemented on the level of the anomalous mass only, as a lowest order $1/N_c$ correction [7, 8, 10, 12, 13]. Thus, with respect to the orders in $1/N_c$, using Shore’s generalization in the present formulation of our DS approach may be less consistent than using the standard WV relation, which may well be the cause of its lesser phenomenological success.

In spite of the lesser phenomenological success (than the standard WV relation) in the present context of bound-state DS calculations at zero temperature, the presently exposed usage of Shore’s generalization will likely find its application at finite-temperature calculations in the DS context. Namely, there it may help alleviate the difficulties met due to the usage of the standard WV relation in the DS approach at $T > 0$, as discussed in Ref. [13].

Acknowledgments

D.H. and D.Kl. acknowledge the support of the project No. 119-0982930-1016 of MSES of Croatia. D.Kl. also acknowledges the hospitality and support through senior associateship of International Centre for Theoretical Physics at Trieste, Italy, where the present paper was started. D.Kl. also thanks the LIT of JINR for its hospitality in Dubna, Russia, in August 2007. D.Kc. acknowledges the support of the Croatian MSES project No. 098-0982878-2872. Yu.K. thanks for support from Deutsche Forschungsgemeinschaft (DFG) under grant No. BL 324/3-1, and the work of D.B. was supported by the Polish Ministry of Science and Higher Education under contract No. N N202 0953 33.

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