MULTIPLE SOLUTIONS FOR PERIODIC PERTURBATIONS OF A DELAYED AUTONOMOUS SYSTEM NEAR AN EQUILIBRIUM

PABLO AMSTER* AND MARIEL PAULA KUNA

IMAS – CONICET, Universidad de Buenos Aires
Departamento de Matemática, Facultad de Ciencias Exactas y Naturales
Ciudad Universitaria - Pabellón I - (C1428EGA), Buenos Aires, Argentina

GONZALO ROBLEDO

Universidad de Chile, Departamento de Matemáticas
Facultad de Ciencias, Casilla 653, Santiago, Chile

(Communicated by Alfonso Ruiz-Herrera)

ABSTRACT. Small non-autonomous perturbations around an equilibrium of a nonlinear delayed system are studied. Under appropriate assumptions, it is shown that the number of $T$-periodic solutions lying inside a bounded domain $\Omega \subset \mathbb{R}^N$ is, generically, at least $|\chi \pm 1| + 1$, where $\chi$ denotes the Euler characteristic of $\Omega$. Moreover, some connections between the associated fixed point operator and the Poincaré operator are explored.

1. Introduction.

1.1. Preliminaries. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. An elementary result from the theory of ODEs establishes that if a smooth function $G: \overline{\Omega} \to \mathbb{R}^N$ is inwardly pointing over $\partial \Omega$, that is

$$\langle G(x), \nu(x) \rangle < 0 \quad x \in \partial \Omega,$$

where $\nu(x)$ denotes the outer normal at $x$, then the solutions of the autonomous system of ordinary differential equations

$$u'(t) = G(u(t))$$

with initial data $u(0) = u_0 \in \overline{\Omega}$ are defined and remain inside $\Omega$ for all $t > 0$.

Now, let us denote the space of $T$-periodic continuous functions as

$$C_T := \{ u \in C(\mathbb{R}, \mathbb{R}^N) : u(t + T) = u(t) \}$$

and, for given $p \in C_T$, consider the non-autonomous system

$$u'(t) = G(u(t)) + p(t).$$

2000 Mathematics Subject Classification. Primary: 34K13, 47H10; Secondary: 47H11.

Key words and phrases. Delay differential systems, multiple periodic solutions, Poincaré operator, fixed points, topological degree.

The first author is supported by projects CONICET PIP 11220130100006CO and UBACyT 2002016010002BA.

* Corresponding author.
If $\overline{\Omega}$ has the fixed point property, then the above system has at least one $T$-periodic orbit, provided that $\|p\|_{\infty}$ is small. This is a straightforward consequence of the fact that the time-dependent vector field $G(x) + p(t)$ is still inwardly pointing for all $t$; hence, the set $\overline{\Omega}$ is invariant for the associated flow and thus the Poincaré operator given by $Pu := u(T)$ is well defined for $u_0 \in \overline{\Omega}$ and satisfies $P(\overline{\Omega}) \subset \overline{\Omega}$.

More generally, observe that, when (1) is assumed, the homotopy defined by $h(x, s) := sG(x) - (1 - s)\nu(x)$ with $s \in [0, 1]$ does not vanish on $\partial \Omega$; whence

$$deg_B(G, \Omega, 0) = deg_B(-\nu, \Omega, 0),$$

where $deg_B$ stands for the Brouwer degree. Thus, it follows from [4] (see also §6 in [9]) that $deg_B(G, \Omega, 0) = (-1)^N\chi(\Omega)$, where $\chi(\Omega)$ denotes the Euler characteristic of $\Omega$.

It is worthy to recall (see e.g. [14]) that if $\overline{\Omega}$ has the fixed point property, then $\chi(\Omega)$ is different from 0. This follows easily in the present setting from the fact that if $\chi(\Omega) = 0$ then one can construct a field $G$ satisfying (1) that does not vanish in $\Omega$ (see [9, §6]). If $\overline{\Omega}$ has the fixed point property, then there exist (non-constant) $T$-periodic solutions of all periods which, in turn, implies that $G$ vanishes, a contradiction. Interestingly, the converse of the result in [14] is not true; that is, one can easily find $\Omega$ with nonzero Euler characteristic such that $\overline{\Omega}$ has not the fixed point property. For such a domain, the Poincaré map has obviously a fixed point (because $G$ vanishes in $\Omega$). This yields the conclusion that a fixed point-free map in $C(\overline{\Omega}, \overline{\Omega})$ cannot belong to the closure of the set of all the Poincaré maps associated to the homotopy class of $-\nu$.

Now suppose, independently of the value of $\chi(\Omega)$, that $G$ vanishes at some point $e \in \Omega$, namely, that $e$ is an equilibrium point of the autonomous system. It is well known that if $M := DG(e)$ is nonsingular, then the degree of $G$ over any small neighbourhood $V$ of $e$ is well defined and coincides with $s(M)$, where

$$s(M) := sgn(det(M)).$$

Thus, if $s(M)$ is different from $(-1)^N\chi(\Omega)$, then the excision property of the degree implies that the system has at least another equilibrium point in $\Omega \setminus \overline{\Omega}$. Furthermore, it follows from Sard’s lemma that, for almost all values $\overline{p}$ in a neighbourhood of $0 \in \mathbb{R}^N$, the mapping $G + \overline{p}$ has at least $\Gamma$ different zeros in $\Omega$, with

$$\Gamma = \Gamma(M) := |\chi(\Omega) - (-1)^N s(M)| + 1.$$  

Thus, one might expect that if $p \in C(\mathbb{R}, \mathbb{R}^N)$ is $T$-periodic and $\|p\|_{\infty}$ is small, then the number of $T$-periodic solutions of the non-autonomous system is generically greater or equal to $\Gamma$. Here, ‘generically’ should be understood in the sense of Baire category, that is, the property is valid for all $p$ (close to the origin) in the space of continuous $T$-periodic functions except for a meager set. It can be shown, indeed, that the fixed point index of the Poincaré map $P$ at $e$ is equal to $(-1)^N s(M)$ and, moreover, a homotopy argument shows that the degree of $P$ over $\Omega$ is equal to $\chi(\Omega)$. Details are omitted because the existence of (generically) at least $\Gamma$ solutions follows from the main theorem of the present paper for the particular case $\tau = 0$.

For several reasons, the situation is different for the delayed system

$$u'(t) = g(u(t), u(t - \tau))$$

where, for simplicity, we shall assume that $g : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}^N$ is continuously differentiable. In the first place observe that, due to the delay, the condition that the field $G(x) := g(x, x)$ is inwardly pointing does not necessarily avoid that solutions with
initial data \( x_0 := \phi \in C([-\tau, 0], \Omega) \) may eventually abandon \( \Omega \). However, taking into account that
\[
|u(t_0 - \tau) - u(t_0)| \leq \tau \max_{t \in [t_0 - \tau, t_0]} |u'(t)|,
\]
it follows that the flow-invariance property, now over the set \( C([-\tau, 0], \Omega) \), is retrieved under the stronger assumption
\[
\langle g(x, y), \nu(x) \rangle < 0 \quad (x, y) \in A_\tau(\Omega) \tag{5}
\]
where
\[
A_\tau(\Omega) := \{(x, y) \in \partial \Omega \times \Omega : |y - x| \leq \tau \|g\|_{\infty}\}.
\]

In the second place, the previous considerations regarding the Poincaré map become less obvious, since the latter is now defined not over \( \Omega \) but over the metric space \( C([-\tau, 0], \Omega) \). In connection with this fact, we recall that the characteristic equation for the autonomous linear delayed systems is transcendental (also called quasipolynomial equation), so there exist typically infinitely many complex characteristic values.

Throughout the paper, we shall assume as before that system (4) has an equilibrium point \( e \in \Omega \), that is, such that \( g(e, e) = 0 \). This necessarily occurs when \( \chi(\Omega) \neq 0 \), although this latter condition shall not be imposed.

Denote by \( A, B \in \mathbb{R}^{N \times N} \) the respective matrices \( D_x g(e, e) \) and \( D_y g(e, e) \). Again, if \( A + B \) is nonsingular and \( s(A + B) \) is different from \((-1)^N \chi(\Omega)\), then the system has at least one extra equilibrium point in \( \Omega \): furthermore, the number of equilibria in \( \Omega \) is generically greater or equal to \( \Gamma \). This is readily verified by writing the set of all the functions \( g \in C^1(\Omega \times \Omega, \mathbb{R}^N) \) satisfying (5) as the union of the closed sets
\[
X_n := \left\{ g \in C^1(\Omega \times \Omega, \mathbb{R}^N) : \langle g(x, y), \nu(x) \rangle \leq -\frac{1}{n} \text{ for } (x, y) \in A_\tau(\Omega) \right\}
\]
and noticing that \( X_n \cap \mathcal{C} \) is nowhere dense, where \( \mathcal{C} \) denotes the set of those functions \( g \) such that 0 is a critical value of the corresponding \( G \).

Our goal in this work is to extend the preceding ideas for non-autonomous periodic perturbations of (4), namely the problem
\[
u'(t) = g(u(t), u(t - \tau)) + p(t) \tag{6}
\]
with \( p \in C_T \).

As a basic hypothesis, we shall assume that the linearisation at the equilibrium, that is, the system
\[
u'(t) = Au(t) + Bu(t - \tau) \tag{7}
\]
has no nontrivial \( T \)-periodic solutions. This clearly implies, in particular, the abovementioned condition that \( A + B \) is invertible. From the Floquet theory for DDEs, it is known that the latter condition is also sufficient for nearly all positive values of \( T \) (we recall that, according for example to [5], a property holds nearly everywhere if it is true except for a countable set). For the sake of completeness, this specific consequence of the Floquet theory shall be shown below (see Remark 1).

### 1.2. Main result.

Our main result reads as follows.

**Theorem 1.1.** Let the equilibrium \( e \) and the matrices \( A \) and \( B \) be as before and assume that the linear system (7) has no nontrivial \( T \)-periodic solutions. Then:

(a) There exists \( r > 0 \) such that for any \( p \) belonging to the ball \( B_r(0) \subset C_T \) the non-autonomous problem (6) has at least one \( T \)-periodic solution.
(b) If moreover, \(5\) holds and \(s(A + B) \neq (-1)^N \chi(\Omega)\) with \(s\) defined as in \(2\), then \((6)\) has at least two \(T\)-periodic solutions.

(c) Furthermore, there exists a residual set \(\Sigma_r \subset C_T\) such that if \(p \in \Sigma_r \cap B_r(0)\), then the number of \(T\)-periodic solutions is at least \(\Gamma(A + B)\), where \(\Gamma\) is given by \((3)\).

The next result is an immediate consequence of Theorem 1.1 combined with the preceding comments.

**Corollary 1.** Let \(e, A\) and \(B\) be as before and assume that \(A + B\) is invertible. Then for nearly all \(T > 0\) there exists \(r = r(T) > 0\) such that if \(p \in C_T\) with \(\|p\|_\infty < r\) then the non-autonomous problem \((6)\) has at least one \(T\)-periodic solution. If moreover \((5)\) holds and \(s(A + B) \neq (-1)^N \chi(\Omega)\), then the number of \(T\)-periodic solutions is at least \(2\) and generically \(\Gamma(A + B)\).

For small delays, the condition that \((7)\) has no nontrivial \(T\)-periodic solutions can be formulated explicitly in terms of the matrix \(A + B\):

**Corollary 2.** Let \(e, A\) and \(B\) be as before and assume that \(2k\pi i\) is not an eigenvalue of the matrix \(A + B\) for all \(k \in \mathbb{N}_0\). Then for each \(\tau\) small enough there exists \(r = r(\tau)\) such that the non-autonomous problem \((6)\) has at least one \(T\)-periodic solution for any \(p \in C_T\) with \(\|p\|_\infty < r\). If moreover \((1)\) holds for \(G(x) := g(x, x)\) and \(s(A + B) \neq (-1)^N \chi(\Omega)\), then \((6)\) has at least two \(T\)-periodic solutions and generically \(\Gamma(A + B)\).

It is worthy mentioning that if \(\Omega\) is for example a ball, then the condition \(s(A + B) \neq (-1)^N \chi(\Omega)\) implies that the equilibrium is unstable. As we shall see, this can be regarded as a consequence of the fact that the Leray-Schauder index of the fixed point operator defined in the proof of our main theorem is \((-1)^{N+1}\). This connection can be deduced from a version of the Krasnoselskii relatedness principle, which implies that the mentioned index coincides except for a \((-1)^N\) factor with that of the Poincaré operator. As shown in Proposition 1, this implies, in turn, that the equilibrium cannot be stable.

### 1.3. Outline

The paper is organised as follows. In the next section, we prove some basic facts concerning the linearised problem \((7)\); in particular, we give a necessary and sufficient condition in order to ensure that it has no nontrivial \(T\)-periodic solutions. In section 3 we present a proof of Theorem 1.1 by means of an appropriate fixed point operator. In section 4, we give a proof of Corollary 2. In section 5, we make some considerations on the stability of the equilibrium and the indices, on the one hand, of the fixed point operator defined in section 3 and of the Poincaré map, on the other hand. Finally, a simple application of the main results for a singular system is introduced in section 6.

### 2. Linearised system

In this section, we shall prove some basic facts concerning the linear system \((7)\). To this end, let us introduce some notation. For \(k \in \mathbb{N}_0\), define

\[
\lambda_k := \frac{2k\pi}{T}
\]

and

\[
\varphi_k(t) := \cos(\lambda_k t) \quad \psi_k(t) := \sin(\lambda_k t).
\]

It is readily verified that

\[
\varphi_k(t - \tau) = \varphi_k(t)\varphi_k(\tau) + \psi_k(t)\psi_k(\tau).
\]
Corollary 3.

\[ \psi_k(t - \tau) = \psi_k(t)\varphi_k(\tau) - \varphi_k(t)\psi_k(\tau) \]

and

\[ \varphi'_k = -\lambda_k \psi_k, \quad \psi'_k = \lambda_k \varphi_k. \]

For an element \( u \in C_T \), we may consider its Fourier series, namely

\[ u = a_0 + \sum_{k=1}^{\infty} (\varphi_k a_k + \psi_k b_k) \]

in the \( L^2 \) sense, with \( a_k, b_k \in \mathbb{R}^N \). Furthermore, recall that if \( u \) is smooth (e.g., of class \( C^1 \)) then the series converges uniformly to \( u \). Moreover, \( u \) has a locally \( L^2 \) weak derivative if and only if the sequences \( \{ka_k\}, \{kb_k\} \in \ell^2 \) and, in this case, \( u' \) coincides with the term-by-term derivative of the series in the \( L^2 \) sense.

Lemma 2.1. Let \( u \in C_T \) and define

\[ X_k := A + \varphi_k(\tau)B, \quad Y_k := \lambda_k I + \psi_k(\tau)B. \]  

(8)

Then \( u \) is a solution of (7) if and only if

\[ \left( \begin{array}{c} X_k \\ Y_k \\ \end{array} \right) = \lambda_k \left( \begin{array}{c} A \\ \varphi_k(t)B \\ \end{array} \right) \]

(9)

for all \( k \in \mathbb{N}_0 \).

Proof. Observe, in the first place, that \( \lambda_k \to +\infty \); thus, when \( k \) large the matrix \( Y_k = \lambda_k \left( I + \frac{1}{\lambda_k}\psi_k(\tau)B \right) \) is invertible and, moreover, \( \det(Y_k) \simeq \lambda_k^N \). Because \( X_k \) is bounded, using the Schur complement we deduce:

\[ \det \left( \begin{array}{cc} X_k & -Y_k \\ Y_k & X_k \end{array} \right) = \det(Y_k)\det(Y_k + X_kY_k^{-1}X_k) \simeq \lambda_k^{2N} > 0. \]

This implies, in particular, that if (9) is satisfied then the Fourier coefficients \( a_k \) and \( b_k \) may be nontrivial only for finitely many values of \( k \). Hence, \( u \) is of class \( C^\infty \).

Since \( \varphi_k(t), \varphi_k(t-\tau), \psi_k(t) \) and \( \psi_k(t-\tau) \) belong to \( \text{span}\{\varphi_k(t), \psi_k(t)\} \), it follows that \( u \) is a solution of (7) if and only if

\[ (A + B)a_0 = 0 \]

and

\[ \varphi_k(t)a_k + \psi_k(t)b_k = A(\varphi_k(t)a_k + \psi_k(t)b_k) + B(\varphi_k(t-\tau)a_k + \psi_k(t-\tau)b_k) \]

for all \( k > 0 \). The latter identity, in turn, is equivalent to

\[ \lambda_k b_k = [A + \varphi_k(\tau)B]a_k - \psi_k(\tau)Bb_k, \quad -\lambda_k a_k = \psi_k(\tau)Ba_k + [A + \varphi_k(\tau)B]b_k, \]

that is,

\[ X_k a_k - Y_k b_k = Y_k a_k + X_k b_k = 0, \]

because \( X_0 = A + B \) and \( Y_0 = 0 \), we deduce that \( u \) is a solution of (7) if and only if (9) holds for all \( k \in \mathbb{N}_0 \).

Corollary 3. (7) has no nontrivial \( T \)-periodic solutions if and only if

\[ h_k := \det \left( \begin{array}{cc} X_k & -Y_k \\ Y_k & X_k \end{array} \right) \neq 0 \quad \text{for all} \ k \in \mathbb{N}_0. \]

(10)

Remark 1. With respect to \( h_k \), it is interesting to point out that:
1. Because $A + B$ is invertible, for nearly all $T > 0$ condition (10) is satisfied for all $k$. Indeed, it suffices to observe that $h_k$, regarded as a function of $T \in (0, +\infty)$, is an analytic function and, consequently, it has at most a countable number of zeros.

2. It can be shown that the function $h_k$ is nonnegative; in particular, its zeros have even multiplicity. The proof is straightforward when $A$ and $B$ commute, since in this case

$$\det \left( \begin{array}{cc} X_k & -Y_k \\ Y_k & X_k \end{array} \right) = \det(X_k^2 + Y_k^2).$$

The conclusion then follows, because for any pair of square real matrices $X, Y$ such that $XY = YX$ it is verified that

$$\det(X^2 + Y^2) = \det[(X + iY)(X - iY)] = \det(X + iY)\det(X - iY) \geq 0.$$ 

A proof for the non-commutative case is given below in section 3, step 3.

Remark 2. As said in the proof of Lemma 2.1, there exists an integer $k_0 > 0$ such that if $u$ is a $T$-periodic solution of (7) then $a_k = b_k = 0$ for $k > k_0$. This implies that $u$ is a (vector) trigonometric polynomial. Incidentally, note that, as the family $\{\varphi_k, \psi_k\}$ is uniformly bounded, $k_0$ may be chosen independent of $\tau$.

In other words, if we consider the linear operator $L : C_T \cap C^1 \to C_T$ defined by $Lu(t) := u'(t) - Au(t) - Bu(t - \tau)$, then $\ker(L) \subset \text{span}\{\varphi_k, \psi_k\}_{0 \leq k \leq k_0}$. Moreover, $\text{Im}(L)$ consists of all $v = \tilde{a}_0 + \sum_{k > 0} (\varphi_k \tilde{a}_k + \psi_k \tilde{b}_k)$ with $\tilde{a}_0 \in \text{Im}(A + B)$ and $(\tilde{a}_k, \tilde{b}_k) \in \text{Im}(M_k)$ for $k > 0$, where $M_k$ is the matrix defined in (9).

Indeed, if $u = a_0 + \sum_{k > 0} (\varphi_k a_k + \psi_k b_k) \in C_T \cap C^1$ satisfies $Lu = v$ then by the uniqueness of the Fourier expansion,

$$(A + B)a_0 = \tilde{a}_0, \quad M_k \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} \tilde{a}_k \\ \tilde{b}_k \end{pmatrix}.$$ 

Conversely, take $v \in C_T$ as before and assume that the latter identities are verified. Recall that $M_k$ is invertible for $k$ large and, furthermore, it follows that $M_k^{-1} = \frac{1}{\pi_k} N_k$, where $N_k \to \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ as $k \to \infty$. Thus, $\|a_k, b_k\| = O \left( \frac{1}{\pi} \|\tilde{a}_k, \tilde{b}_k\| \right)$ and hence $u$ has a weak derivative $u' \in L^2_{\text{loc}}$. A straightforward computation shows that $u'(t) = Au(t) + u(t - \tau) + v(t)$ in the weak sense and, consequently, $u \in C_T \cap C^1$ and satisfies $Lu = v$.

By Lemma 2.1, we also note that $u \in C_T \cap C^1$ belongs to $\ker(L)$ if and only if $a_0 \in \ker(A + B)$ and $(a_k, b_k) \in \ker(M_k)$ for $k > 0$. Thus, the rank-nullity theorem yields a direct proof of the well-known fact that $L$ is a zero-index Fredholm operator. Moreover, it is verified that $(a_k, b_k) \in \ker(M_k)$ if and only if $(-b_k, a_k) \in \ker(M_k)$, a fact that will be of relevance in the proofs of our results.

3. Proof of the main theorem. For convenience, a little of extra notation shall be introduced. For a function $u \in C_T$, let us write

$$\mathcal{I}u(t) := \int_0^t u(s) \, ds, \quad \overline{u} := \frac{1}{T} \mathcal{I}u(T).$$

Moreover, denote by $\mathcal{N}$ the Nemitskii operator associated to the problem, namely
$$\mathcal{N}u(t) := g(u(t), u(t - \tau)).$$

Without loss of generality we may assume $e = 0$ and fix $T > 0$ such that (7) has no nontrivial $T$-periodic solutions. For simplicity, we shall assume from the beginning that all the assumptions are satisfied; it shall be easy for the reader to deduce the existence of one solution near the equilibrium when (5) is not satisfied.

Define the open bounded set $U = \{u \in C_T : u(t) \in \Omega \text{ for all } t\}$ and the compact operator $K : \overline{U} \to C_T$ defined by
$$Ku(t) := \pi - t\mathcal{N}u + \mathcal{I}\mathcal{N}u(t) - \mathcal{I}\mathcal{N}u.$$

We shall prove that the Leray-Schauder degree of $I - K$ is equal to $(-1)^N\chi(\Omega)$ over $U$ and to $s(A + B)$ over $B_\rho(0)$ for small values of $\rho > 0$.

To this end, let us proceed in several steps:

1. Let $K_0u := \pi - \frac{T}{2}\mathcal{N}u$ and define, for $s \in [0, 1]$, the operator given by $K_s := sK + (1 - s)K_0$. We claim that $K_s$ has no fixed points on $\partial U$. Indeed, for $s > 0$ it is clear that $u \in \overline{U}$ is a fixed point of $K_s$ if and only if $u'(t) = s\mathcal{N}u(t)$, that is:
$$u'(t) = sg(u(t), u(t - \tau)).$$

Suppose there exists $t_0$ such that $u(t_0) \in \partial \Omega$, then we deduce, as before,
$$|u(t_0 - \tau) - u(t_0)| \leq \tau \max_{t \in [t_0 - \tau, t_0]} |u'(t)| \leq \tau \|g\|_{\infty}$$
and by (5) we obtain
$$0 = \langle u'(t_0), \nu(u(t_0)) \rangle = s\langle g(u(t_0), u(t_0 - \tau)), \nu(u(t_0)) \rangle < 0,$$
a contradiction. On the other hand, we observe that the range of $K_0$ is contained in the set of constant functions, which can be identified with $\mathbb{R}^N$; thus, the Leray-Schauder degree of $I - K_0$ can be computed as the Brouwer degree of its restriction to $\overline{U} \cap \mathbb{R}^N = \overline{\Omega}$.

Furthermore, for $u(t) \equiv u \in \overline{\Omega}$ it is clear that $K_0u = u - \frac{T}{2}G(u)$, which does not vanish on $\partial \Omega = \partial U \cap \mathbb{R}^N$. By the homotopy invariance of the degree, we conclude that
$$\text{deg}(I - K, U, 0) = \text{deg}\left(\frac{T}{2}G, \Omega, 0\right) = (-1)^N\chi(\Omega).$$

2. Let $K_L$ be the operator associated to the linearised problem, defined by
$$K_Lu(t) := \pi - t\mathcal{N}_Lu + \mathcal{I}\mathcal{N}_Lu(t) - \mathcal{I}\mathcal{N}_L,$$
with $\mathcal{N}_Lu(t) := Au(t) + Bu(t - \tau)$. As before, it is seen that $K_Lu = u$ if and only if $u$ is a solution of (7); hence, it follows from the assumptions that $K_L$ has no nontrivial fixed points.

Furthermore, the degree of $I - K_L$ coincides with the degree of $I - K$ on $B_\rho(0)$ when $\rho$ is small. This is a well-known fact but, for the reader’s convenience, a simple proof is sketched as follows.

Since the degree is locally constant, we may assume that $g$ is of class $C^2$ near $(0, 0)$, then for some $C > 0$,
$$\|Kv - K_Lv\|_{\infty} \leq C\|\mathcal{N}v - \mathcal{N}_Lv\|_{\infty} = o(\rho).$$
Because $K_L$ is compact, it is verified that, for some $\theta > 0$,
$$\|v - K_Lv\|_{\infty} \geq \theta \rho.$$
for all \( v \in \partial B_{\rho}(0) \). Indeed, due to linearity, it suffices to prove the claim for \( \rho = 1 \). By contradiction, suppose there exists a sequence \( \{v_n\} \subset \partial B_1(0) \) such that \( \|v_n - K Lv_n\|_\infty \to 0 \), then passing to a subsequence we may assume that \( \{K Lv_n\} \) converges to some \( v \). Then \( v_n \to v \) which, in turn, implies that \( \|v\|_\infty = 1 \) and \( v = K Lv \), a contradiction. It follows that if \( \rho > 0 \) is small then \( sK + (1 - s)K_L \) has no fixed points on \( \partial B_{\rho}(0) \) for \( s \in [0, 1] \) because

\[
\|v - sK v - (1 - s)K Lv\|_\infty \geq \|v - KLv\|_\infty - \|K Lv - K Lv\|_\infty \geq \theta \rho - o(\rho) > 0
\]

for \( v \in \partial B_{\rho}(0) \). Thus, the degree of \( I - K \) is well defined and coincides with the degree of \( I - K_L \) over \( B_{\rho}(0) \).

3. Claim: \( \text{deg}(I - K_L, B_{\rho}(0), 0) = s(A + B) \).

Indeed, for \( u \) as before it is seen by direct computation that

\[
u - K_L u = \tilde{a}_0 + \sum_{k \geq 1} (\varphi_k \tilde{a}_k + \psi_k \tilde{b}_k)
\]

where

\[
\tilde{a}_0 = M_0 a_0
\]

and

\[
\begin{pmatrix}
\tilde{a}_k \\
\tilde{b}_k
\end{pmatrix} = M_k \begin{pmatrix}
a_k \\
b_k
\end{pmatrix}
\]

with

\[
M_0 := \frac{T}{2}(A + B) \quad \text{and} \quad M_k := \frac{1}{\lambda_k} \begin{pmatrix}
Y_k & X_k \\
-X_k & Y_k
\end{pmatrix} \quad \text{for } k > 0.
\]

Hence, the degree coincides with the sign of the determinant of the block matrix

\[
\begin{pmatrix}
M_0 & 0 & 0 & \ldots & 0 \\
0 & M_1 & 0 & \ldots & 0 \\
0 & 0 & M_2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & M_J
\end{pmatrix}
\]

for \( J \) sufficiently large. Thus, the proof follows in a straightforward manner from the fact that \( \det(M_k) > 0 \) for all \( k > 0 \). We remark that the latter property holds even when \( A \) and \( B \) do not commute (see Remark 1).

Indeed, identifying the pairs \((a, b) \in \mathbb{R}^N \times \mathbb{R}^N\) with vectors \( a + ib \in \mathbb{C}^N \), a matrix of the form

\[
\begin{pmatrix}
X & -Y \\
Y & X
\end{pmatrix}
\]

may be called a \( \mathbb{C} \)-linear matrix. Thus, we need to prove that if \( M \) is an arbitrary invertible \( \mathbb{C} \)-linear matrix, then the algebraic multiplicity of each eigenvalue \( \sigma < 0 \) of \( M \) is even. It is known that this value can be computed as the dimension of the kernel of the matrix \((M - \sigma I)^m\), where \( m \) is the minimum integer such that \( \ker(M - \sigma I)^m = \ker(M - \sigma I)^{m+1} \). Now observe that the set of \( \mathbb{C} \)-linear matrices is a subring of \( \mathbb{R}^{2N \times 2N} \); thus, \((M - \sigma I)^m\) is again a \( \mathbb{C} \)-linear matrix. In particular, \((a, b) \in \ker(M - \sigma I)^m\) if and only if \((-b, a) \in \ker(M - \sigma I)^m\). Let us show that the dimension of the kernel of a matrix \( M \in \mathbb{R}^{2N \times 2N} \) with this latter property is even. Assume \( 0 \neq (a, b) \in \ker(M) \), then \((-b, a)\) is linearly independent with \((a, b)\) and also belongs to \( \ker(M) \). Inductively, suppose that \( \{(a_j, b_j), (-b_j, a_j)\}_{1 \leq j \leq k} \) is a linearly independent subset of \( \ker(M) \) and \((a, b) \notin \text{span}\{(a_j, b_j), (-b_j, a_j)\} \). We claim that \((-b, a) \notin \text{span}\{(a_j, b_j), (-b_j, a_j)\} \); thus, a basis of \( \ker(M) \) can
be constructed by adding successively pairs of linearly independent vectors, which yields the desired result. In order to prove the claim, write

\[ (-b, a) = -r(a, b) + \sum_{j=1}^k [r_j(a_j, b_j) + s_j(-b_j, a_j)], \]

then, by simple computation,

\[ (1 + r^2)(a, b) = \sum_{j=1}^k [(rr_j + s_j)(a_j, b_j) - (r_j - rs_j)(-b_j, a_j)]. \]

Thus, \( rr_j + s_j = r_j - rs_j = 0 \) for all \( j \) and we conclude that \( r_j = s_j = 0 \) for all \( j \) and \((-b, a) = r(a, b),\) a contradiction.

4. **Existence of two solutions for small \( p.\)**

From the previous steps and the fact that the degree is locally constant we deduce that

\[ \deg(I - K, U, \hat{p}) = (-1)^N \chi(\Omega), \quad \deg(I - K, B_\rho(0), \hat{p}) = s(A + B) \]

when \( ||\hat{p}||_\infty \) is small. Now the excision property of the Leray-Schauder degree implies

\[ \deg(I - K, B_\rho(0), \hat{p}) = s(A + B) \neq 0, \]

and

\[ \deg(I - K, U \setminus B_\rho(0), \hat{p}) = (-1)^N \chi(\Omega) - s(A + B) \neq 0. \]

Thus, there exists \( \hat{r} > 0 \) such that the equation \( (I - K)u = \hat{p} \) has at least two solutions for \( ||\hat{p}||_\infty < \hat{r}. \) Finally, for each \( p \in C_T \) define

\[ \hat{p}(t) := \mathcal{I}p(t) - \mathcal{I}p - tp, \]

then clearly \( ||\hat{p}||_\infty \leq c||p||_\infty \) for some \( c > 0. \) The result is then deduced from the fact that if \( u - Ku = \hat{p}, \) then \( u \) is a \( T \)-periodic solution of (6).

\[ u'(t) = g(u(t), u(t - \tau)) + p(t). \]

5. **Genericity.**

The last part of the proof follows as a consequence of the following particular case of the Sard-Smale Theorem [12]:

**Theorem 3.1.** Let \( \mathcal{F} : X \rightarrow Y \) be a \( C^1 \) Fredholm map of index 0 between Banach manifolds, i.e. such that \( D\mathcal{F}(x) : T_xX \rightarrow T_{\mathcal{F}(x)}Y \) is a Fredholm operator of index 0 for every \( x \in X. \) Then the set of regular values of \( \mathcal{F} \) is residual in \( Y. \)

At this point, we notice that the argument is a bit subtle: when applied to \( \mathcal{F} := I - K, \) the Sard-Smale Theorem implies the existence of a residual set \( \Sigma \subset C_T \) such that the mapping \( \mathcal{F} - \hat{p} \) has at least \( \Gamma - 1 \) zeros in \( U \setminus B_\rho(0) \) for \( \hat{p} \in \Sigma \cap B_\rho(0). \)

Indeed, it is readily seen that \( K \) is of class \( C^1 \) and \( DK(u) \) is compact for all \( u. \) Thus, \( \mathcal{F} = I - K \) is a zero-index Fredholm operator. If \( \hat{p} \) is a regular value, that is, \( D\mathcal{F}(u) \) is surjective for every preimage \( u \in \mathcal{F}^{-1}(\hat{p}) \) then, since the index is 0, it is also injective and from the open mapping theorem we conclude that \( D\mathcal{F}(u) \) is an isomorphism. Recall that \( \mathcal{F} \neq \hat{p} \) on \( \partial U \) and hence, by compactness of \( K, \) the set \( \mathcal{F}^{-1}(\hat{p}) \cap U = \mathcal{F}^{-1}(\hat{p}) \cap \overline{U} \) is compact. Next, for each preimage \( u \in \mathcal{F}^{-1}(\hat{p}) \cap (U \setminus B_\rho(0)) \) we may take a small neighbourhood \( N_u \) such that \( \mathcal{F} : N_u \rightarrow \mathcal{F}(N_u) \) is a diffeomorphism.
Lemma 4.1. shall make use of the following lemmas: 

The existence of nontrivial solutions of the non-delayed case, thus giving the explicit sufficient condition for the non-

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Proof. The result follows by direct computation, or from Lemma 2.1 with \( \tau = 0 \).

For example, when \( M \) is triangularizable (or, equivalently, when all its eigenvalues are real), 1 is not an eigenvalue of the system \( u'(t) = Mu(t) \) if and only if \( M \) is
nonsingular; in this particular case, the conclusion follows directly, because the system uncouples and the result is obviously true for a scalar equation.

**Lemma 4.2.** Assume that 1 is not a Floquet multiplier of the linear ODE system \( u'(t) = (A + B)u(t) \). Then the DDE system (7) has no nontrivial \( T \)-periodic solutions, provided that \( \tau \) is small.

*Proof.* Suppose that \( u_n \in C_T \) is a nontrivial solution for \( \tau_n \to 0 \). Without loss of generality, it may be assumed that \( ||u_n||_\infty = 1 \) and hence \( ||u'_n||_\infty \leq C \) for some constant \( C \). Thus, we may assume that \( u_n \) converges uniformly to some \( u \in C_T \) with \( ||u||_\infty = 1 \). Because \( ||u_n(t - \tau_n) - u_n(t)|| \leq C\tau_n \to 0 \), it becomes clear that \( u'_n \) converges uniformly to \( (A + B)u \) which, in turn, implies \( u' = (A + B)u \), a contradiction. \( \square \)

**Remark 4.** A more direct proof of Lemma 4.2 follows just by considering Remark 2 and Lemma 4.1. Indeed, in the context of Lemma 2.1 it suffices to check that \( h_k \neq 0 \) only for a finite number of values of \( k \). By continuity, this is true for small \( \tau \), because \( \det([A + B]^2 + \lambda^2 I) \neq 0 \) for all \( k \). However, the previous proof has an interest in its own because it can be extended in a straightforward manner to the non-autonomous case.

**Proof of Corollary 2.** As a consequence of the preceding lemma, the conclusions of Theorem 1.1 hold for small \( \tau \), provided that the linearisation has no nontrivial \( T \)-periodic solutions for the non-delayed case. Thus, in view of Lemma 4.1, the proof is complete. \( \square \)

5. **Poincaré operator.** In this section, we shall make some considerations regarding the Poincaré operator associated to the system. Let us firstly observe that if \( \chi(\Omega) = 1 \) (for example, if \( \Omega \) is homeomorphic to a ball), then the condition \( s(A + B) \neq (-1)^N \chi(\Omega) \) in Theorem 1.1 simply reads \( (-1)^N \det(A + B) < 0 \). This, in turn, implies that the equilibrium is unstable. Indeed, consider the characteristic function \( h(\lambda) = \det(\lambda I - A - Be^{-\lambda T}) \), then \( h(0) = (-1)^N \det(A + B) < 0 \) and \( h(\lambda) = \lambda^N \) for \( |\lambda| \gg 0 \). In particular, this implies the existence of a characteristic value \( \lambda > 0 \).

We shall show that, in the present context, the instability of the equilibrium when \( (-1)^N \det(A + B) < 0 \) is due to the fact, proved in section 3, that the index of the fixed point operator \( K \) at \( e \) (i.e. the degree of \( I - K \) over small balls around \( e \)) is equal to \( (-1)^{N+1} \). When \( \tau = 0 \), this can be regarded as a direct consequence of the following properties:

1. \( \text{deg}(I - K, B_B(e), 0) \) with \( B_B(e) \subset C_T \) is equal to \( (-1)^N \text{deg}_B(I - P, B_B(e), 0) \) with \( B_B(e) \subset \mathbb{R}^N \), where \( P \) is the Poincaré map.
2. If the equilibrium is stable, then the index of \( P \) is 1.

The first property is a particular case of a relatedness principle due to Krasnoselskii (see [8]). The second property is well-known and can be found for example in [7]. For more details see [10], where sufficient conditions for the validity of the converse statement are also obtained.

Our goal in this section consists in understanding the connections between the instability of the equilibrium and the index of the fixed point operator defined in the proof of the main theorem.
With this aim, let us define the Poincaré operator for the delayed case as follows. Let \( \tau \leq T \) and consider a general autonomous system

\[
    u'(t) = F(u_t)
\]

with \( F : C([-\tau, 0]) \to \mathbb{R}^N \) locally Lipschitz, i.e.: for all \( R > 0 \) there exists a constant \( L \) such that

\[
    \|F(\phi) - F(\psi)\| \leq L\|\phi - \psi\|_{\infty}
\]

for all \( \phi, \psi \in B_R(0) \subset C([-\tau, 0], \mathbb{R}^n) \). The notation \( u_t \) expresses, as usual, the mapping defined by \( u_t(\theta) := u(t + \theta) \) for \( \theta \in [-\tau, 0] \).

Denote by \( \text{dom}(P) \subset C([-\tau, 0]) \) the set of those functions \( \phi \) such that the unique solution \( u = u(\phi) \) of the problem with initial condition \( \phi \) is defined up to \( t = T \), then \( P : \text{dom}(P) \to C([-\tau, 0]) \) is defined by

\[
    P\phi(s) := u(T + s).
\]

Clearly, the \( T \)-periodic solutions of the problem can be identified with the fixed points of \( P \). We shall see that, as in the non-delayed case, if the linearisation has no nontrivial \( T \)-periodic solutions then the index \( i(P) \) of the operator \( P \) at a stable equilibrium is equal to 1.

To this end, assume without loss of generality that \( e = 0 \) and observe that stability implies that \( \text{dom}(P) \) is a neighbourhood of 0. It is worth noticing that, in the general setting, extra conditions are required in order to prove the compactness of \( P \) (see e.g. [6]), so the Leray-Schauder degree may be not well defined; however, it is verified that the stability assumption implies that \( P \) is compact over small neighbourhoods of 0. More precisely:

**Lemma 5.1.** Let \( F \) be as before and assume that for some open \( U \subset C([-\tau, 0]) \) there exists \( R > 0 \) such that if \( \phi \in U \) then the solution \( u \) with initial condition \( \phi \) is defined and satisfies \( |u(t)| < R \) for all \( t \in [0, T] \). Then \( P \) is well defined and compact over \( U \).

**Proof.** Let \( B \subset U \) be bounded and observe, in the first place, that \( P(B) \) is bounded. Moreover, if \( u \) is a solution with initial condition \( \phi \in B \), then

\[
    u(t) = \phi(0) + \int_0^t F(u_s) \, ds.
\]

Enlarging \( R \) if necessary, we may assume \( B \subset B_R(0) \), then \( \|u_s\|_{\infty} < R \) for all \( s \in [0, T] \). Given \( t_1 < t_2 \) in \( [-\tau, 0] \), since \( \tau \leq T \) it is verified that

\[
    |P\phi(t_2) - P\phi(t_1)| \leq \int_{t_1}^{T+t_2} |F(u_s)| \, ds.
\]

Let \( L \) be the Lipschitz constant corresponding to \( R \), then

\[
    |F(\phi)| \leq |F(0)| + L\|\phi\|_{\infty} \leq C + LR,
\]

where \( C := |F(0)| \). Hence \( |P(t_2) - P(t_1)| \leq (C + LR)(t_2 - t_1) \) and the result follows from the Arzelà-Ascoli Theorem. \( \square \)

**Remark 5.** For example, the assumptions of the previous lemma are satisfied if \( F \) has linear growth, that is

\[
    |F(\phi)| \leq \gamma\|\phi\|_{\infty} + \delta.
\]
Furthermore, extra assumptions are required to ensure the non-existence of non-trivial periodic solutions near 0; this is why we shall impose this fact as an extra condition (see Proposition 1 below), which is clearly satisfied for example when the stability is asymptotic. For simplicity, we shall also assume that $F$ is Fréchet differentiable at 0, that is,

$$F(\phi) = D_\phi(0)\phi + R(\phi)$$

with $||R(\phi)||_\infty \leq o(||\phi||_\infty)||\phi||_\infty$. Thus, it is readily verified that the linearisation of $P$ at the origin coincides with the Poincaré operator associated to the linearised system $u'(t) = D_\phi(0)u$.

**Proposition 1.** In the previous setting, assume that 0 is a stable equilibrium of (11) such that its linearisation has no nontrivial $T$-periodic solutions. Then $i(P) = 1$.

**Proof.** Without loss of generality, we may assume that $P$ is compact on $\mathcal{V}$ for some neighbourhood $V$ of 0. It follows from the assumptions that the index of $P$ is well defined and coincides with the index of its linearisation $P_L$. According to Theorem 13.8 in [1], $\text{deg}(I - P_L, B_\rho(0), 0)$ is equal to $(-1)^\alpha$, where $\alpha$ is the sum of the (finite) algebraic multiplicities of the (finitely many) eigenvalues $\sigma$ of $P_L$ satisfying $\sigma > 1$.

If $\text{deg}(I - P_L, B_\rho(0), 0) = -1$, then $P_L$ has an eigenfunction $\phi$ with eigenvalue $\sigma > 1$. If $u$ is the corresponding solution of the linearised problem with initial condition $u = \phi$ on $[-\tau, 0]$ then $u$ can be extended to $\mathbb{R}$ in a $(T, \sigma)$-periodic fashion, that is, with $u(t + T) = \sigma u(T)$ for all $t$ (see [11]). In particular, $u(t)$ is unbounded for $t > 0$. In other words, 0 is unstable for the linearised problem which, in turn, implies that it cannot be stable for the original problem (see e.g. [3, 13]).

In order to complete the picture for system (4), it would be interesting to prove that, indeed, the index of the Poincaré operator at the equilibrium when the linearisation has no nontrivial solutions is $(-1)^N s(A + B) = (-1)^N i(K)$. Here, we shall simply verify that the claim holds when the delay is small; the analysis of the general case and a version of the Krasnoselskii relatedness principle for delayed systems shall be the subject of a forthcoming paper.

To this end, let us start with a direct computation for the non-delayed case:

**Lemma 5.2.** Let $M \in \mathbb{R}^{N \times N}$ and let $P_M$ be the Poincaré operator associated to the linear ODE system $u'(t) = Mu(t)$ for some fixed $T$. If 1 is not a Floquet multiplier, then

$$\text{deg}_B(I - P_M, V, 0) = (-1)^N s(M)$$

for any neighbourhood $V \subset \mathbb{R}^N$ of the origin.

**Proof.** By definition,

$$(I - P_M)(u) = (I - e^{TM})u.$$  

Write $M$ in its (possibly complex) Jordan form $M = C^{-1}JC$, where $J$ is upper triangular. Then

$$\det(I - e^{TM}) = \det(I - e^{TJ}) = \prod_{j=1}^N (1 - e^{\lambda_j T}),$$

where $\lambda_j$ are the eigenvalues of $M$. Now observe that if $\lambda = a + ib \notin \mathbb{R}$, then

$$(1 - e^{iT})(1 - e^{iT}) = 1 + e^{aT}(e^{aT} - 2 \cos(bT)) > 0.$$
Thus, complex eigenvalues do not affect the sign of \( \det(1-e^{TM}) \), as well as it happens with the sign of \( \det(M) \) because \( \lambda \mathbf{X} = |\lambda|^2 \). The result follows now from the fact that, for \( \lambda \in \mathbb{R} \),
\[
\text{sgn} \left( 1 - e^{\lambda t} \right) = -\text{sgn}(\lambda).
\]
\( \square \)

**Remark 6.** An alternative (somewhat exotic) proof follows from the relatedness principle. Indeed, we may consider the operator \( K_L \) in the proof of Theorem 1.1 with \( A = M \) and \( B = 0 \), then \( \deg_B(I-P,V,0) = (-1)^N \deg(I-K_L,V,0) = (-1)^N s(M) \).

The conclusion for small \( \tau \) is obtained now by a continuity argument. Indeed, fix \( r > 0 \) and \( P_L \) as before. The solutions of (7) with initial value \( \phi \in B_r(0) \) are uniformly bounded; thus, by Gronwall’s lemma we deduce that \( \|P - P_0\| = O(\tau) \), where the operator \( P_0 \) is defined by \( P_0(\phi)(t) = v(T) \), with \( v \) the unique solution of the system \( v'(t) = (A + B)v(t) \) satisfying \( v(0) = \phi(0) \). Moreover, recall that if \( \tau \) is small then \( P_L \) is homotopic to \( P_0 \); thus, the result follows from Lemma 5.2.

6. **Example: a system of DDEs with singularities.** A simple example is presented here in order to illustrate our main results. Let \( 0 \leq J_0 \leq J \neq 0 \) and
\[
g(x,y) := -dx + |y|^2 \left( \sum_{j=1}^{J_0} a_j \frac{x - v_j}{|x - v_j|^{\alpha_j}} + \sum_{j=J_0+1}^{J} a_j \frac{y - v_j}{|y - v_j|^{\alpha_j}} \right)
\]
where \( d, a_j > 0, \alpha_j > 2 \) and \( v_j \in \mathbb{R}^N \setminus \{0\} \) are pairwise different vectors. A simple computation shows that
\[
\langle g(x,x), x \rangle < 0 \quad |x| \gg 0
\]
and
\[
\langle g(x,x), v_j - x \rangle < 0 \quad |x - v_j| \ll 1
\]
for \( j = 1, \ldots, J \). Moreover, \( g(0,0) = 0 \) and
\[
A = D_x g(0,0) = -dI, \quad B = D_y g(0,0) = 0.
\]
Thus, taking \( \Omega := B_R(0) \setminus \bigcup_{j=1}^{J} B_\eta(v_j) \) where \( R > 0 \) and \( \eta \ll 1 \), Corollary 2 applies. Since \( \chi(\Omega) = 1 - J < 1 = (-1)^N s(A + B) \), we conclude that the number of \( T \)-periodic solutions of (6) for small \( \tau \) and \( \|p\|_\infty \) is generically \( J + 1 \).

**Acknowledgments.** The first author wants to thank Professor J. Barmak for his thoughtful comments regarding the fixed point property and the Euler characteristic.

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Received May 2018; revised August 2018.

E-mail address: pamster@dm.uba.ar
E-mail address: mpkuna@dm.uba.ar
E-mail address: grobledo@uchile.cl