MULTIPlicative SUMmATIONS INTO ALGEBRAICALLY CLOSED FIELDS

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Abstract. In this paper, extending our earlier program, we derive maximal canonical extensions for multiplicative summations into algebraically closed fields. We show that there is a well-defined analogue to minimal polynomials for a series algebraic over a ring of series, the “scalar polynomial”. When that ring is the domain of a summation $\mathcal{S}$, we derive the related concepts of the $\mathcal{S}$-minimal polynomial for a series, which is mapped by $\mathcal{S}$ to a scalar polynomial. When the scalar polynomial for a series has the form $(t - a)^n$, $a$ is the unique value to which the series can be mapped by an extension of the original summation.

1. Introduction

This paper is a sequel to [3], and we follow the notations and conventions established therein. Let $R$ be a commutative unital ring with $0 \neq 1$, and let $E$ be a commutative unital $R$-algebra with $0 \neq 1$. Let $D$ be a set and $\mathcal{S} : D \to E$ a map.

Definition 1.1. The tuple $(R, E, D, \mathcal{S})$ is a summation on $R$ to $E$ if it satisfies the following axioms:

(I) We have $R[\sigma] \subseteq D \subseteq R[[\sigma]]$;

(II) The set $D$ is an $R$-module, and the map $\mathcal{S}$ is an $R$-module homomorphism;

(III) We have $\mathcal{S}(1) = 1$;

(IV) We have $(1 - \sigma)D \subseteq D$, and the morphism $\mathcal{S}$ factors through $D/(1 - \sigma)D$.

Axiom (IV) gives us the following commutative diagram of $R$-modules:

\[
\begin{array}{ccc}
D & \xrightarrow{\mathcal{S}} & E \\
\downarrow & & \downarrow \\
D/(1 - \sigma) & \xrightarrow{\tilde{\mathcal{S}}} & E
\end{array}
\]

We write $(D, \mathcal{S})$ or simply $\mathcal{S}$ for the summation $(R, E, D, \mathcal{S})$ when no confusion results. We write $\mathcal{S}(R, E)$ for the set of all summations on $R$ to $E$. If $X$ is a power series in the variable $\sigma$, we write $(X)_n$ for the $n$th coefficient of $X$, or $X_n$ if no confusion arises. We typically use capital letters for series over $R$, and lower-case letters for scalars in $E$.

Our definition of summations leaves open the possibility that the underlying ring $R$ and the codomain $E$ are not the same. Classically, this freedom allows us to sum a rational series into $\mathbb{R}$; telescoping also allows us to sum an integral series into $\mathbb{Q}$. But we do not even demand that $R$ is a subring of $E$. If it is not, then the map $x \mapsto \mathcal{S}(x + 0 + 0 + \cdots)$ is not injective. We call a summation proper if $x \mapsto \mathcal{S}(x + 0 + 0 + \cdots)$ is injective. The following example illustrates this distinction.

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Example 1.2. We define $A : R[\sigma] \to R$ by $A : X(\sigma) \mapsto X(1)$. This is the unique minimal summation over $R$ with values in $R$; it is proper.

Given a ring homomorphism $f : R \to E$, the composition $f \circ A$ is a summation over $R$ with values in $E$, which by slight overloading of notation we can call $A : R[\sigma] \to E$. It is proper if and only if $f$ is injective.

For instance, $A : Z[\sigma] \to Z/2Z$ is a summation that maps $1 + 2 + 3$ to 0, and is not proper.

Recall that the fulfillment of a summation $S \in S(R, E)$ is its unique maximal canonical extension. We write $\text{Reg}(E)$ for the nonzero divisors of $E$.

Definition 1.3. For any summation $(D, S) \in S(R, E)$, the telescopic extension $(TD, TS)$ of $(D, S)$ is defined as follows. For $X \in R[[\sigma]]$, we say $X \in TD$ if there exists $A \in D$, $F \in R[\sigma]$, $f \in \text{Reg}(E)$, and $x \in E$ such that $A = FX$, $S(F) = f$, and $S(A) = fx$. We define $TS : TD \to E$,

$$TS : X \mapsto x \text{ if X is as above.}$$

Theorem 1.4. The functor of summations

$$T : S(R, E) \to S(R, E)$$

is an idempotent extension map. Moreover, if $R$ is an integral domain, then $TS$ is the fulfillment of $S$.

The tuple $(R, E, D, S)$ is a multiplicative summation on $R$ to $E$ if it satisfies axioms (I), (III), (IV) from Section 1, and the following strengthened version of axiom (II):

(II') The set $D$ is an $R$-algebra, and the map $S$ is an $R$-algebra homomorphism;

In this context, axiom (IV) gives us the following commutative diagram of $R$-algebras:

$$\begin{array}{ccc}
D & \xrightarrow{\delta} & E \\
\downarrow & & \downarrow \\
D/(1-\sigma) & \xrightarrow{\tilde{\delta}} & E
\end{array}$$

Definition 1.5. If a summation has an extension which is multiplicative, we call it weakly multiplicative.

We considered this property in detail previously [3]. In particular, we show that it is possible for a summation to preserve all products that exist in its domain, and yet not be weakly multiplicative [3][Example 3.7]. However, if a summation $S$ is weakly multiplicative, then it has a unique minimal multiplicative extension $MS$. We write $\text{wMS}(R, E)$ for the set of weakly multiplicative summations on $R$ to $E$; organized by inclusion, $\text{wMS}(R, E)$ is the full subcategory of $S(R, E)$ with objects the weakly multiplicative summations.

Definition 1.6. For a multiplicative summation $(D, S) \in \text{MS}(R, E)$, the rational extension $(QD, QS)$ of $(D, S)$ is defined as follows. For $X \in R[[\sigma]]$, we say $X \in QD$ if there exists $A, B \in D$, $b \in \text{Reg}(E)$, and $x \in E$ such that $A = BX$, $S(B) = b$, and $S(A) = bx$. We define $QS : QD \to E$,

$$QS : X \mapsto x \text{ if X is as above.}$$

We extend this definition to weakly multiplicative summations $(D, S) \in \text{wMS}(R, E)$ by setting $(QD, QS) := (QMD, QMS)$. 

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Repeating (essentially) the proof of Theorem 1.4 we obtain the following theorem [3][Theorem 5.3].

**Theorem 1.7.** The functor of summations

\[ Q : \text{wMS} (R, E) \to \text{MS} (R, E) \]

is an idempotent extension map which subsumes \( M, T, M_T, \) and \( T_M. \)

If \( \mathcal{S} = Q\mathcal{S}, \) we say that \( \mathcal{S} \) is rationally closed.

**Proposition 1.8.** Let \( E \) be a field. For any multiplicative summation \( (D, \mathcal{S}) \in \text{MS} (R, E), \) the image of \( QD \) under \( Q\mathcal{S} \) is the field of fractions of \( \mathcal{S}(D). \)

In [3] we posed the following question.

**Question 1.9.** What is the multiplicative fulfillment of a weakly multiplicative summation?

In this paper, we answer this question for multiplicative summations whose codomains are algebraically closed fields (see Theorems 4.5 and 4.7).

The remainder of this paper is organized as follows. Fix a summation \( \mathcal{S}. \) In Section 2 we define the scalar polynomial of a series \( X \in R[[\sigma]], \) and and show that the roots of this scalar polynomial determine all images of \( X \) under extensions of \( \mathcal{S}. \) In Section 3 we make a more careful study of the absolutely \( \mathcal{S}-\)algebraic series, which are precisely those series contained in every multiplicative extension of \( \mathcal{S} \) (see Proposition 3.2). In Section 4 we use the theory of absolutely \( \mathcal{S}-\)algebraic series to characterize the multiplicative fulfillment of \( \mathcal{S}. \) Finally, in Section 5 we discuss some directions for future work.

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2. THE ALGEBRA OF SUMMATIONS

Throughout the remainder of this paper, we assume our summations are multiplicative, and when we speak of extensions, fulfillments, and so forth, we mean multiplicative extensions, multiplicative fulfillments, and so forth. We also assume that the codomains of our (multiplicative) summations are algebraic closed fields unless otherwise noted. This is not an onerous assumption; if \( E \) is a field and \( \overline{E} \) is the algebraic closure of \( E, \) the inclusion map \( \iota : E \hookrightarrow \overline{E} \) induces injections

\[
\begin{align*}
\text{MS} (R, E) & \hookrightarrow \text{MS} (R, \overline{E}), \\
\text{wMS} (R, E) & \hookrightarrow \text{wMS} (R, \overline{E}), \\
\mathcal{S} (R, E) & \hookrightarrow \mathcal{S} (R, \overline{E})
\end{align*}
\]

via \( \mathcal{S} \mapsto \iota \circ \mathcal{S}. \)

A series polynomial is simply a polynomial \( P(T) = \sum_{k=0}^{n} P_k T^k \) with coefficients in \( R[[\sigma]]. \) If \( P(T) \in D(T), \) we write \( \mathcal{S}(P)(t) \) for the polynomial \( \sum_{k=0}^{n} \mathcal{S}(P_k)t^k \in E[t]. \)

**Definition 2.1.** For a series \( X \in R[[\sigma]] \) and a summation \( (R, E, D, \mathcal{S}), \) let

\[ A(X; \mathcal{S}) := \{ P(T) : P(T) \in D[T], \mathcal{S}(P)(t) \neq 0, P(X) = 0 \} \]

A polynomial \( P(T) \) in \( A(X; \mathcal{S}) \) for which \( \mathcal{S}(P)(t) \) has minimal degree will be called a \( \mathcal{S}-\)minimal polynomial for \( X. \) If \( A(X; \mathcal{S}) \) is nonempty, we say that the \( \mathcal{S}-\)degree of \( X \) is the smallest degree of a \( \mathcal{S}-\)minimal polynomial for \( X. \)
Example 2.2. The series

\[ X = 2 - \frac{\sigma}{4} - \frac{\sigma^2}{64} - \frac{\sigma^3}{512} - \frac{5\sigma^4}{16384} - \cdots, \]

obtained as the Taylor series of \( \sqrt{4-\sigma} \), satisfies \( X^2 - (4-\sigma) = 0 \), but satisfies no nontrivial linear equation with coefficients in \( \mathbb{C}[\sigma] \). Thus the polynomial \( P(T) = T^2 - (4-\sigma) \) is minimal and \( \mathfrak{A} \)-minimal for \( X \) over \( \mathbb{C}[\sigma] \), and \( \mathfrak{A}(P)(t) = t^2 - 3 \).

Example 2.3. The series

\[ Y = 1 + \frac{\sigma}{2} + \frac{3\sigma^2}{8} + \frac{5\sigma^3}{16} + \frac{35\sigma^4}{128} + \cdots, \]

obtained as the Taylor series of \( (1-\sigma)^{-1/2} \), satisfies \( (1-\sigma)Y^2 - 1 = 0 \), but satisfies no nontrivial linear equation with coefficients in \( \mathbb{C}[\sigma] \). The polynomial \( Q(T) = (1-\sigma)T^2 - 1 \) is minimal and \( \mathfrak{A} \)-minimal for \( Y \) over \( \mathbb{C}[\sigma] \), and \( \mathfrak{A}(Q)(t) = -1 \) is constant.

Proposition 2.4. If \( R \) is an integral domain, and \( X \in R[[\sigma]] \) is nonzero, then no monomial is an element of \( A(X; \mathcal{G}) \).

Proof. Let \( A_j \) and \( X_k \) be the first nonzero coefficients of \( A \) and \( X \) respectively. We see \( (AX^n)_{j+k n} \neq 0 \).

Example 2.5 below shows Proposition 2.4 need not hold if \( R \) has zero divisors.

Given a summation \( (R, E, D, \mathcal{G}) \) and a series \( X \in R[[\sigma]] \), it is possible that a polynomial \( P(T) \in D[T] \) satisfies \( P(X) = 0 \) for some series \( X \in R[[\sigma]] \), but that \( \mathcal{G}(P)(t) = 0 \). In this case, the polynomial relation \( P(X) = 0 \) implies nothing about the possible values of \( X \) in an extension of \( \mathcal{G} \). This circumstance may even be universal among polynomial \( P(T) \in D[T] \) with \( P(X) = 0 \), as Example 2.6 shows. Such an \( X \) may be formally algebraic over \( D \), but still “behaves like a transcendental series” in that \( X \) may consistently be assigned any value in \( E \) in an appropriate extension of \( \mathcal{G} \) (see Theorem 2.18).

Example 2.5. Let \( U, V \in \mathbb{C}[[\sigma]] \) be invertible (i.e. \( U_0 \) and \( V_0 \) are nonzero) and algebraically independent over \( \mathbb{C}[\sigma] \), and let \( D := \mathbb{C}[\sigma][U, V] \). We see that \( D \) is the ring of all series of the form \( \sum a_{ij} A_{ij} U^j V^j \) where \( A_{ij} \in \mathbb{C}[\sigma] \) and the double sum is finitely supported. Let \( \mathcal{G}(U) = \mathcal{G}(V) = 0 \); then \( \mathcal{G}(\sum A_{ij} U^j V^j) = \mathfrak{A}(A_{00}) \). If we let \( X := UV^{-1} \), then \( VX - U = 0 \), and \( P(X) = 0 \) implies \( \mathcal{G}(P)(t) = 0 \).

However, the next proposition says that this cannot happen for \( (R[\sigma], \mathfrak{A}) \) if addition is proper (that is, if \( x \mapsto \mathfrak{A}(x + 0 + 0 + \cdots) \) is injective).

Proposition 2.6. For any ring \( R \), if \( \mathfrak{A} \) is proper, \( X \in R[[\sigma]] \), \( P \in R[\sigma][T] \), \( P(X) = 0 \), and \( P \neq 0 \), then there exists \( P' \in R[\sigma][T] \) such that \( \mathfrak{A}(P')(t) \neq 0 \) and \( P'(X) = 0 \).

Proof. If \( x \mapsto \mathfrak{A}(x + 0 + 0 + \cdots) \) is injective, then for \( P \in R[\sigma] \), we have \( \mathfrak{A}(P)(t) = 0 \) if and only if \( 1 - \sigma \) divides \( P \).

Suppose now that \( \mathfrak{A} \) is proper, \( P(X) = 0 \), and \( P \neq 0 \). If \( \mathfrak{A}(P)(t) \neq 0 \), then we are done. Otherwise, by properness, the term \( 1 - \sigma \) divides \( P(T) \). By induction the power of \( 1 - \sigma \) dividing \( P(T) \), there exists \( n > 0 \) and \( P'(T) \in R[\sigma][T] \) such that

\[ P(T) = (1 - \sigma)^n P'(T) = (1 - \sigma)^n \sum_k P_k' T^k \]

and \( \mathfrak{A}(P')(t) \neq 0 \). By assumption,

\[ 0 = P(X) = (1 - \sigma)^n \sum_k P_k' X^k, \]
and \( \mathfrak{A}(P'_k) \neq 0 \) for some \( k \). But multiplication by \((1 - \sigma)\) leaves the first nonzero term of a series unchanged; so \( \sum_k P'_k X^k = 0 \) as claimed.

**Example 2.7.** The Grandi series

\[
G_{-1} := \sum_{n=0}^{\infty} (-1)^n \sigma^n = 1 - \sigma + \sigma^2 - \sigma^3 + \cdots = 1 - 1 + 1 - 1 + \cdots
\]

can (infelicitously) be telescoped with shift 2, via the equation \((1 - \sigma^2)G_{-1} - (1 - \sigma) = 0\). Summing the coefficients of the polynomial \((1 - \sigma^2)T - (1 - \sigma)\) yields \( 0t - 0 \), which places no constraints on the sum of the Grandi series. However, dividing our polynomial by \(1 - \sigma\), we obtain \((1 + \sigma)T - 1\) which sums to \( 2t - 1 \), so the Grandi series should sum to \( \frac{1}{2} \) (as it does using many classical summation methods).

The next example illustrates why we stipulate that \( \mathfrak{A} \) is proper in Proposition 2.6.

**Example 2.8.** Let \( \mathfrak{A}_2 : \mathbb{Z}[\sigma] \to \mathbb{Z}/2\mathbb{Z} \), and let \( X := 2G_{-1} = 2 - 2 + 2 - 2 + \cdots \); this has minimal polynomial \( P_X(T) := (1 + \sigma)T - 2 \). But \( \mathfrak{A}_2(P_X)(t) = 0t + 0 \), the null polynomial, and so \( \mathfrak{A}_2(P_X)(t) = 0 \) for all \( t \in \mathbb{Z}/2\mathbb{Z} \).

In general, \( \mathcal{S} \)-minimal polynomials need not be minimal, and minimal polynomials need not be \( \mathcal{S} \)-minimal.

**Example 2.9.** Let \( R = \mathbb{C}[\epsilon]/\epsilon^2 \mathbb{C}[\epsilon] \) be the dual numbers over \( \mathbb{C} \), let \( E = \mathbb{C} \), and let \( \mathcal{S} \in \text{MS}(R, \mathbb{C}) \) be the summation with domain

\[
D := \{ X + \epsilon Y : \ X, \ Y \in R[\sigma] \},
\]

which is defined by \( \mathcal{S} : X + \epsilon Y \mapsto \mathfrak{A}(X) \). Finally, let \( Y := \frac{1}{1 - \sigma} = \epsilon + \epsilon + \epsilon + \cdots \), let \( F(T) = \epsilon T \), and let \( G(T) = T^2 + \epsilon H(T) \) where \( H(T) \in R[T] \) is arbitrary. Then \( F(T) \) is a minimal polynomial for \( Y \) over \( D \), but as \( \mathcal{S}(F)(t) = 0 \), \( F(T) \) is not a \( \mathcal{S} \)-minimal polynomial for \( Y \). On the other hand, \( G(T) \) is a \( \mathcal{S} \)-minimal polynomial for \( Y \), but not a minimal polynomial. By varying \( H(T) \), we also see \( \mathcal{S} \)-minimal polynomials need not be unique even up to scalars or degree.

In the last example, the polynomial \( F(T) \) failed to be \( \mathcal{S} \)-minimal because \( \mathcal{S} \) mapped it to the null polynomial, which is treated specially in the definition. (Proposition 2.6 above shows that this cannot happen for \( \mathfrak{A} \).) A minimal polynomial can also fail to be \( \mathcal{S} \)-minimal because \( \mathcal{S} \) lets another polynomial “jump the queue”.

**Example 2.10.** Let

\[
X := 1 + \frac{\sigma}{4} + \frac{9\sigma^2}{64} + \frac{49\sigma^3}{512} + \frac{1165\sigma^4}{16384} + \cdots
\]

be the unique real series satisfying \((1 - \sigma)X^3 + X - 2 = 0 \), and let

\[
A := X^2 = 1 + \frac{\sigma}{2} + \frac{11\sigma^2}{32} + \frac{67\sigma^3}{256} + \frac{1719\sigma^4}{8192} + \cdots.
\]

Let \( D := \mathbb{R}[\sigma][A] \) and let \( \mathcal{S} \) take finitely supported sums \( \sum_j P_j A^j \) to \( \sum_j 4^j P_j(1) \). Then \( T^2 - A \) is minimal for \( X \) over \( D \), because no linear polynomial with coefficients in \( D \) annihilates \( X \). However, \((1 - \sigma)T^3 + T - 2 = 0 \) is \( \mathcal{S} \)-minimal, because its image is the linear polynomial \( t - 2 \), and any lower-degree polynomial annihilating \( X \) would be constant, hence necessarily the null polynomial.
Although $\mathcal{S}$-minimal polynomials are not unique, their images under $\mathcal{S}$ are, up to a constant: if $P_1(T)$ and $P_2(T)$ are $\mathcal{S}$-minimal polynomials for $X$, then $\mathcal{S}(P_1)(t) = a \cdot \mathcal{S}(P_2)(t)$ for some unit $a \in E^\times$. The roots of such images are the possible sums for $X$ in an extension of $\mathcal{S}$ (see Theorem 2.18). For convenience, we pick a canonical representative.

**Definition 2.11.** If $\mathcal{S}(P)(t) = 0$ for every $\mathcal{S}$-minimal polynomial for $X$, we define the scalar polynomial $s_X(t)$ for $X$ over $\mathcal{S}$ to be $s_X(t) := 0$. (Note that this may be true either vacuously, when $X$ is transcendental over $D$, or nonvacuously, as in Example 2.5.) Otherwise, we define the scalar polynomial $s_X(t)$ for $X$ over $\mathcal{S}$ to be the unique monic polynomial in $E[t]$ which is a nonzero scalar multiple of $\mathcal{S}(P)(t)$ for some (equivalently, for every) $P(T)$ that is $\mathcal{S}$-minimal for $X$. We define the scalar degree of $X$ to be the degree of its scalar polynomial.

Note that even if $R$ is a field, there may be series $X \in R[[\sigma]]$ for which the degree of $s_X(t)$ is less than the degree of every $\mathcal{S}$-minimal polynomial $P(T)$ for $X$.

**Example 2.12.** Let $R = E = C$ and let $Y$ be the Taylor series of

$$\frac{1 + \sqrt{1 - 4\sigma + 4\sigma^2}}{2 - 2\sigma};$$

that is,

$$Y = 1 - \sigma - 2\sigma^2 - 5\sigma^3 - 13\sigma^4 - 36\sigma^5 - 104\sigma^6 - \ldots. $$

Then $(\sigma - 1)T^2 + T - (\sigma + \sigma^2) \in C[\sigma][T]$ is an $A$-minimal polynomial for $Y$, and thus $s_Y(t) = t - 2$ is the scalar polynomial for $Y$. But $Y$ is not a rational function of $\sigma$, so does not have a $A$-minimal polynomial of degree 1.

**Definition 2.13.** We partition the series in $R[[\sigma]]$ based on their scalar polynomials as follows:

- A series $X \in R[[\sigma]]$ is multiplicatively $\mathcal{S}$-transcendental if its scalar polynomial $s_X(t)$ is 0. We write $T(\mathcal{S}) \subseteq R[[\sigma]]$ for the set of $\mathcal{S}$-transcendental series;
- A series $X \in R[[\sigma]]$ is multiplicatively $\mathcal{S}$-algebraic if its scalar polynomial $s_X(t)$ is nonconstant. We write $A(\mathcal{S}) \subseteq R[[\sigma]]$ for the set of $\mathcal{S}$-algebraic series;
- A series $X \in R[[\sigma]]$ is multiplicatively $\mathcal{S}$-infinite if its scalar polynomial $s_X(t)$ is 1. We write $I(\mathcal{S}) \subseteq R[[\sigma]]$ for the set of $\mathcal{S}$-infinite series.

Continuing our standard practice, we suppress the adverb “multiplicatively” when discussing $\mathcal{S}$-transcendental, $\mathcal{S}$-algebraic, and $\mathcal{S}$-infinite series where this does not result in ambiguity.

We observe that every summation $\mathcal{S} \in MS(R, E)$, the set of formal series $R[[\sigma]]$ may be partitioned as

$$R[[\sigma]] = T(\mathcal{S}) \cup A(\mathcal{S}) \cup I(\mathcal{S}).$$

It is easy to see that each transcendental series is $\mathcal{S}$-transcendental. On the other hand, each $\mathcal{S}$-algebraic series and each $\mathcal{S}$-infinite series is algebraic. We note (see Example 2.3) that multiplicatively $\mathcal{S}$-infinite series need not be $\mathcal{S}$-infinite under the definition given in [4], which pertains only to linear equations $AX + B = 0$ with $A, B \in D, \mathcal{S}(A) = 0$.

**Proposition 2.14.** Let $X \in R[[\sigma]]$ be any series, and let $Q(T) \in D[T]$ be a polynomial such that $Q(X) = 0$. Then $s_X(t)$ divides $\mathcal{S}(Q)(t)$.

**Proof.** Let $P(t) \in D[t]$ be a $\mathcal{S}$-minimal polynomial for $X$, and let $Q(t) \in D[t]$ be any polynomial with $Q(X) = 0$. The proposition is immediate by applying the Euclidean algorithm to $\mathcal{S}(P)(t)$ and $\mathcal{S}(Q)(t)$. \(\square\)
Corollary 2.15. Suppose \((D', \mathcal{G}')\) extends \((D, \mathcal{G})\), and let \(X \in R[[\sigma]]\) be any series. If \(s_X(t)\) is the scalar polynomial for \(X\) over \(D\), and \(s'_X(t)\) is the scalar polynomial for \(X\) over \(D'\), then \(s'_X(t)\) divides \(s_X(t)\).

Let \(P(T) \in D[T]\) be a polynomial of degree \(m\). Following [3][Chapter 7.3], if \(m \neq \infty\), we define the reflected polynomial of \(P(T)\) to be
\[
P^R(T) := T^m P(T^{-1}).
\]
Otherwise \(P(T) = 0\), and we define \(P^R(T) := 0\).

Corollary 2.16. Let \(U \in R[[\sigma]]\) be a unit, and suppose \(P(T) \in D[T]\) is a \(\mathcal{G}\)-minimal polynomial for \(U\). Then \(s_{U^{-1}}(T)\) divides \(\mathcal{G}(P^R(T))\).

Write
\[
Z(X; \mathcal{G}) := \{ x \in E : s_X(x) = 0 \}
\]
for the set of zeroes of the scalar polynomial \(s_X(t)\) of \(X\). Note that this is also the set of common zeroes of the sums of the polynomials in \(A(X; \mathcal{G})\).

Corollary 2.17. For every summation \(\mathcal{G}' \in MS (R, E)\) extending \(\mathcal{G}\) and every \(X \in R[[\sigma]]\), we have
\[
Z(X; \mathcal{G}) \supseteq Z(X; \mathcal{G}').
\]
For \(X \in R[[\sigma]]\) and \(x \in E\), write \(\mathcal{G}_{X,x} : D[X] \to E\) for the summation given by
\[
\mathcal{G}_{X,x} : P(X) \mapsto \mathcal{G}(P)(x)\text{ if }P(T) \in D[T],
\]
whenever this summation is well-defined. Clearly \(\mathcal{G}_{X,x}\) is a multiplicative summation whenever it is well-defined.

Theorem 2.18. Fix a multiplicative summation \((D, \mathcal{G}) \in MS (R, E)\).

(a) Let \(X \in \mathcal{T}(\mathcal{G})\). Then for every \(x \in E\), there exists a multiplicative summation \(\mathcal{G}'\) extending \(\mathcal{G}\) such that \(\mathcal{G}'(X) = x\); moreover, \(\mathcal{G}_{X,x} : D[X] \to E\) is the unique minimal multiplicative extension of \(\mathcal{G}\) which maps \(X\) to \(x\).

(b) Let \(X \in \mathcal{A}(\mathcal{G})\). Then for every \(x \in E\) a root of \(s_X(t)\), there exists a multiplicative summation \(\mathcal{G}'\) extending \(\mathcal{G}\) such that \(\mathcal{G}'(X) = x\); moreover, \(\mathcal{G}_{X,x} : D[X] \to E\) is the unique minimal multiplicative extension of \(\mathcal{G}\) which maps \(X\) to \(x\). Conversely, if \(\mathcal{G}'\) is a multiplicative extension of \(\mathcal{G}\) for which \(\mathcal{G}'(X)\) is defined, then \(\mathcal{G}'(X)\) is a root of \(s_X(t)\).

(c) Let \(X \in \mathcal{I}(\mathcal{G})\). Then \(X\) is not in the domain of any multiplicative extension of \(\mathcal{G}\).

Proof. (a) Let \(X\) be \(\mathcal{G}\)-transcendental, and let \(x \in E\) be arbitrary. We claim
\[
\mathcal{G}_{X,x} : D[X] \to E
\]
is well-defined; indeed, if \(A(X) = B(X)\) then \(\mathcal{G}(A-B)(t) = 0\) by the definition of \(\mathcal{G}\)-transcendence, and in particular \(\mathcal{G}(A-B)(x) = 0\). It is clear that \(\mathcal{G}_{X,x}\) is a multiplicative summation, and moreover that \(\mathcal{G}_{X,x}\) is minimal among summations \(\mathcal{G}'\) extending \(\mathcal{G}\) for which \(\mathcal{G}'(X) = x\).

(b) Let \(X\) be \(\mathcal{G}\)-algebraic, and let \(x\) be a root of \(s_X(t)\) in \(E\). The proof that \(\mathcal{G}_{X,x}\) is well-defined is similar to the one given in [a] above. Suppose that \(A(X) = B(X)\) for \(A(T), B(T) \in D[T]\). Then \(\mathcal{G}(A-B)(t) = 0\), and so \(s_X(t)\) divides \(\mathcal{G}(A-B)(t)\); say \(\mathcal{G}(A-B)(t) = s_X(t) \cdot r(t) \in E[t]\). Then
\[
\mathcal{G}(A-B)(x) = s_X(x) \cdot r(x) = 0,
\]
and so \(\mathcal{G}_{X,x}\) is well-defined.
Conversely, suppose \( \mathcal{S}' \) is a multiplicative extension of \( \mathcal{S} \) for which \( \mathcal{S}'(X) \) is defined. Choose \( P(T) \in D[T] \) a \( \mathcal{S} \)-minimal polynomial, and write \( \mathcal{S}(P(t)) = a \cdot s_X(t) \) with \( a \in E^\times \). Then

\[
a \cdot s_X(\mathcal{S}'(X)) = \mathcal{S}'(X) = \mathcal{S}(P(X)) = 0,
\]

and so \( \mathcal{S}'(X) \) is a root of \( s_X(t) \) as desired.

(c) Let \( X \) be \( \mathcal{S} \)-infinite, and choose a polynomial \( P(T) \in D[T] \) with \( P(X) = 0 \) and \( \mathcal{S}(P(t)) = p \in E^\times \). Suppose by way of contradiction that \( \mathcal{S}' \) is an extension of \( \mathcal{S} \) with \( X \) in its domain. Then

\[
0 \neq p = (\mathcal{S}'P)(\mathcal{S}'X) = \mathcal{S}'(P(X)) = 0,
\]

which is a contradiction.

\[\square\]

**Corollary 2.19.** If \( \mathcal{S} \in MS(R, E) \) is a multiplicative summation, for any \( X \in R[[\sigma]] \) we have

\[
Z(X; \mathcal{S}) = \{ x \in E : x = \mathcal{S}'(X) \text{ for some } \mathcal{S}' \supseteq \mathcal{S} \},
\]

and:

(a) the series \( X \) is \( \mathcal{S} \)-transcendental if and only if \( Z(X; \mathcal{S}) = E \)

(b) the series \( X \) is \( \mathcal{S} \)-algebraic if and only if \( Z(X; \mathcal{S}) \) is finite and nonempty;

(c) the series \( X \) is \( \mathcal{S} \)-infinite if and only if \( Z(X; \mathcal{S}) = \emptyset \).

**Observation 2.20.** As stated above, we are implicitly assuming \( E \) to be algebraically closed. Theorem 2.18 does not require this hypothesis; but without it, the corollary is weaker. While \([\text{a}]\) holds unchanged, “if and only if” is replaced by “if” in \([\text{b}]\) and by “only if” in \([\text{c}]\).

**Example 2.21.** Let \( X \) and \( (D, \mathcal{S}) \) be as in Example 2.3. As shown, \( X \) is algebraic over \( D \), with minimal polynomial \( VX - U \), so \( X \) is algebraic; but \( s_X(t) = 0 \), so \( Z(X; \mathcal{S}) = \mathbb{C} \).

**Example 2.22.** Let \( X \) and \( (D, \mathcal{S}) \) be as in Example 2.8. As shown, \( X \) is algebraic over \( D \), with minimal polynomial \((1+\sigma)X - 2 \), so \( X \) is algebraic; but \( s_X(t) = 0 \), so \( Z(X; \mathcal{S}) = \mathbb{Z}/2\mathbb{Z} \).

**Example 2.23.** Consider the summation \((\mathbb{Q}[\sigma], \mathbb{A}_\mathbb{Q}) \in MS(\mathbb{Q}, \mathbb{Q})\). The Taylor series \( X \) for \( \sqrt{1+4\sigma} \) is

\[
X = 1 + 2\sigma - 2\sigma^2 + 4\sigma^3 - 10\sigma^4 + 28\sigma^5 - \cdots.
\]

The series \( X \) satisfies \( X^2 - (1 + \sigma) = 0 \), and is thus \( \mathbb{A}_\mathbb{Q} \)-algebraic; but \( Z(X; \mathbb{A}_\mathbb{Q}) \) is empty.

In [3], we showed that two sums can be consistent without being multiplicatively compatible. In the same spirit we now construct a series \( Y \) which is summed to a unique value by every summation compatible with \( \mathbb{A} \), but which nonetheless is not in the multiplicative fulfillment of \( \mathbb{A} \).

**Example 2.24.** Let \( Y \) be as in Example 2.12 and define \( Y' := \frac{1}{1-\sigma} - Y \). We see that \( Y \) and \( Y' \) have the same scalar polynomial \( s_Y(t) = s_{Y'}(t) = t - 2 \); therefore, by Theorem 2.18 every summation compatible with \( \mathbb{A} \) that sums \( Y \) sums it to 2, and the same holds for \( Y' \). Moreover, the summations \( \mathbb{A}_{Y,2} \) and \( \mathbb{A}_{Y',2} \) are examples of such summations, so these assertions are nonvacuous. But now if \( Y \) and \( Y' \) were in the multiplicative fulfillment of \( \mathbb{A} \), then

\[
Y + Y' = \frac{1}{1-\sigma} = \sum_{j=0}^{\infty} \sigma^j
\]

would also be in the multiplicative fulfillment of \( \mathbb{A} \), which is absurd.
Example 2.24 motivates the following definition.

**Definition 2.25.** A $G$-algebraic series $X \in A(G)$ is absolutely $G$-algebraic if it is $G'$-algebraic for every $G'$ extending $G$. We write $B(G)$ for the set of absolutely $G$-algebraic series; obviously, $B(G) \subseteq A(G)$.

**Example 2.26.** The series $Y'$ of Example 2.24 is $U$-algebraic, but $U_{Y,2}$-infinite; therefore it is not absolutely $U$-algebraic. However, as shown in [4], any series that is telescopic over $U$ is telescopic to the same value over any summation whatsoever; thus, for instance, over any field in which $0 \neq 2$, the Grandi series $G_{-1}$ (Example 2.7) is absolutely $U$-algebraic.

We could define absolutely $G$-transcendental series and absolutely $G$-infinite series similarly, but these definitions would be superfluous; no $G$-transcendental series is absolutely $G$-transcendental, and every $G$-infinite series is absolutely $G$-infinite.

**Proposition 2.27.** Let $(D', G')$ be a multiplicative extension of $(D, G)$. Then

(a) $T(G) \supseteq T(G')$,

(b) $B(G) \subseteq B(G')$,

(c) $I(G) \subseteq I(G')$.

**Proof.** Suppose $X \in T(G')$. Then for every polynomial $P(T) \in D'[T]$ with $P(X) = 0$, we have $G'(P)(t) = 0$. *A fortiori*, for every polynomial $P(T) \in D[T]$ with $P(X) = 0$, we have $G(P)(t) = 0$. Thus $T(G) \supseteq T(G')$.

Suppose $X \in B(G)$. Then $X$ is $G'$-algebraic for every extension $G''$ of $G$. But every extension of $G'$ is also an extension of $G$, and so we see $X \in B(G')$ as desired.

Finally, suppose $X \in I(G)$. Then there exists a polynomial $P(T) \in D[T]$ with $P(X) = 0$, and $G(P)(t) = a \in E^\times$. Then viewing $P(T)$ as an element of $D'[T]$, we see that $X \in I(G')$. \qed

In classical analysis, the convergence or divergence of a series $X$ is unaffected by changing finitely many terms of $X$. Within our axiomatic paradigm for summing series, something analogous holds true.

**Definition 2.28.** Two series $X$ and $Y$ in $R[[\sigma]]$ are tail-equivalent if there exist $m, n \in \mathbb{N}$ such that $\lambda^m(X) = \lambda^n(Y)$, where

$$\lambda : \sum_{n=0}^{\infty} A_n \sigma^n \mapsto \sum_{n=0}^{\infty} A_{n+1} \sigma^n$$

is the left-shift operator. We call the tail-equivalence class $[X]$ of $X$ the tail of $X$.

Informally, two series are tail-equivalent if they can be made equal by removing a finite set of terms from each; moreover, tail-equivalence is the symmetric closure of the relation

$$\exists n \in \mathbb{N}, F \in R[\sigma] : X = F + \sigma^n Y.$$ 

This formulation is useful because the operator $\lambda$ does not have an internal representation in $R[[\sigma]]$.

**Theorem 2.29.** If $P(T)$ is a $G$-minimal polynomial for $X$, and $F := \sum_{j=0}^{n-1} X_j \sigma^j$, then $P'(T) := P(F + \sigma^n T)$ is a $G$-minimal polynomial for $\lambda^n(X)$. Conversely, if $Q(T) = \sum_{j=0}^{m} Q_j T^j$ is a $G$-minimal polynomial for $\lambda^m(X)$, then $Q'(T) := \sum_{j=0}^{m} \sigma^{n(j)} Q_j (T - F)^j$ is a $G$-minimal polynomial for $X$. 

Proof. For ease of notation, write \( Y := \lambda^n(X) \). Clearly
\[
F + \sigma^n Y = \sum_{j=0}^{n-1} X_j \sigma^j + \sum_{j=n}^\infty X_j \sigma^j = X,
\]
so \( P'(Y) = P(F + \sigma^n Y) = P(X) = 0 \). Conversely,
\[
Q'(X) = \sum_{j=0}^{m} \sigma^{n(m-j)} Q_j (X - F)^j = \sigma^{nm} Q(Y) = \sigma^{nm} Q(Y) = 0.
\]
Now write \( \mathcal{G}(P)(t) = a \cdot s_X(t) \) with \( a \in \mathbb{E}^x \), and \( \mathcal{G}(Q)(t) = b \cdot s_Y(t) \) with \( b \in \mathbb{E}^x \). Then \( \mathcal{G}(P')(t) = a \cdot s_X(t') + t \) divides \( s_Y(t) \) and \( \mathcal{G}(Q')(t) = b \cdot s_Y(t' - x') \) divides \( s_X(t) \). We conclude that \( s_X(t') \) divides \( s_Y(t) \), which in turn divides \( s_X(t' + x') \), and so \( s_X(t') + x' = s_Y(t) \) as desired. Now as \( P'(Y) = 0 \) and \( \mathcal{G}(P')(t) \) is a nonzero multiple of \( s_Y(t) \), we conclude that \( P'(T) \) is a \( \mathcal{G} \)-minimal polynomial for \( Y \), and analogously \( Q'(T) \) is a \( \mathcal{G} \)-minimal polynomial for \( X \).

Corollary 2.30. Let \( X \) and \( Y \) be tail-equivalent series.

(a) If \( X \) is \( \mathcal{G} \)-transcendental, then \( Y \) is also \( \mathcal{G} \)-transcendental.
(b) If \( X \) is (absolutely) \( \mathcal{G} \)-algebraic, then \( Y \) is also (absolutely) \( \mathcal{G} \)-algebraic, and \( X \) and \( Y \) have both the same \( \mathcal{G} \)-degree and the same scalar degree.
(c) If \( X \) is \( \mathcal{G} \)-infinite, then \( Y \) is also \( \mathcal{G} \)-infinite.

Thus, it makes sense to call tails (absolutely) \( \mathcal{G} \)-algebraic, \( \mathcal{G} \)-transcendental, or \( \mathcal{G} \)-infinite, and, in the first case, to refer to the \( \mathcal{G} \)-degree or scalar degree of a tail.

3. Absolutely \( \mathcal{G} \)-Algebraic Series

Let us take a closer look at the set \( \mathcal{B}(\mathcal{G}) \) of absolutely \( \mathcal{G} \)-algebraic series. We begin this section with a corrected version of Proposition 2.5i from [1].

Theorem 3.1. For every summation \( \mathcal{G} \in \text{MS}(\mathbb{R}, \mathbb{E}) \), the set \( \mathcal{B}(\mathcal{G}) \subseteq \mathbb{R}[[\sigma]] \) is an \( \mathbb{R} \)-algebra.

Proof. Clearly 0 and 1 are absolutely \( \mathcal{G} \)-algebraic series. Now let \( X \) and \( Y \) be any absolutely \( \mathcal{G} \)-algebraic series, and let \( \mathcal{G}' \in \text{MS}(\mathbb{R}, \mathbb{E}) \) be any extension of \( \mathcal{G} \). We consider the set \( Z(X + Y; \mathcal{G}') \), defined as in [4]. Let \( x \) be a root of the scalar polynomial \( s'_X(t) \) for \( X \) over \( \mathcal{G}' \); by Theorem 2.18, \( \mathcal{G}' \) has an extension \( \mathcal{G}'' \in \text{MS}(\mathbb{R}, \mathbb{E}) \) such that \( \mathcal{G}''(X) = x \). Now as \( y \) is absolutely \( \mathcal{G} \)-algebraic, we may choose a root \( y \) of the scalar polynomial \( s'_Y(t) \) for \( Y \) over \( \mathcal{G}'' \); applying Theorem 2.18 again, we may choose a summation \( \mathcal{G}''' \in \text{MS}(\mathbb{R}, \mathbb{E}) \) extending \( \mathcal{G}'' \) for which \( \mathcal{G}'''(Y) = y \). Then
\[
\mathcal{G}'''(X + Y) = x + y \in \mathbb{E},
\]
and so \( Z(X + Y; \mathcal{G}') \) is nonempty. On the other hand, since every summation summing \( X + Y \) may be extended to sum \( X \) and \( Y \) as well, it is clear that
\[
Z(X + Y; \mathcal{G}') \subseteq \{ x + y : x \in Z(X; \mathcal{G}), y \in Z(Y; \mathcal{G}) \}.
\]
Thus, as both of these latter sets are finite, we see that \( Z(X + Y; \mathcal{G}') \) is finite as well. Then by Corollary 2.19, the series \( X + Y \) is \( \mathcal{G}' \)-algebraic for every \( \mathcal{G}' \in \text{MS}(\mathbb{R}, \mathbb{E}) \) extending \( \mathcal{G} \), and so \( X + Y \) is absolutely \( \mathcal{G} \)-algebraic. A completely similar argument shows that \( XY \) is absolutely \( \mathcal{G} \)-algebraic.

We also have the following proposition, which follows from Theorem 2.18.
Proposition 3.2. We have
\[ \mathbb{B}(\mathcal{S}) = \bigcap \{ D' \subseteq \mathbb{R}[[\sigma]] : (D, \mathcal{S}) \subseteq (D', \mathcal{S}') \} . \]

Theorem 3.1 and Proposition 3.2 both suggest that the absolutely \( \mathcal{S} \)-algebraic series are worthy of further study. Definition 2.13 classifies a series by means of its scalar polynomial as either \( \mathcal{S} \)-transcendental, \( \mathcal{S} \)-algebraic, and \( \mathcal{S} \)-infinite. Determining whether a \( \mathcal{S} \)-algebraic series is absolutely \( \mathcal{S} \)-algebraic is a much subtler affair. We will require several intermediate lemmas before we can provide our answer (Theorem 3.10).

Lemma 3.3. Let \( m \) and \( n \) be positive integers, and let \( \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1} \) be indeterminates over \( \mathbb{Z} \). There are unique polynomials
\[ P_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}) \in \mathbb{Z}[\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}][T] \]
and
\[ Q_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}) \in \mathbb{Z}[\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}][T] \]
satisfying the following pair of conditions:
(1) \( P_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}) \) and \( Q_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}) \) are monic of degree \( mn \) as polynomials in \( T \).
(2) For any commutative ring \( R' \) and any elements \( X, Y, A_0, \ldots, A_{m-1}, B_0, B_{n-1} \in R' \) such that
\[ X^m + \sum_{k<m} A_k X^k = Y^n + \sum_{k<n} B_k Y^k = 0, \]
we have
\[ P_{m,n}(X + Y; A_0, \ldots, A_{m-1}, B_0, B_{n-1}) = 0 \]
and
\[ Q_{m,n}(XY; A_0, \ldots, A_{m-1}, B_0, B_{n-1}) = 0 \]
as elements of \( R \).

Proof. Let \( R = \mathbb{Z}[\alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}] \) where \( \alpha_i \) and \( \beta_j \) are indeterminate, let
\[ f(T) := T^m + \sum_{k<m} \alpha_k T^k, \]
and let
\[ g(T) := T^n + \sum_{k<n} \beta_k T^k. \]
Now take \( S := R[\chi, \upsilon]/(f(\chi), g(\upsilon)) \); \( S \) is an integral domain. We let \( P_{m,n}(T) \) be the minimal polynomial for \( \chi + \upsilon \) over \( R \), and let \( Q_{m,n}(T) \) be the minimal polynomial for \( \chi \upsilon \) over \( R \). More explicitly, let \( S' \) be the splitting field for \( f(T) \) and \( g(T) \) over \( S \), let \( \chi =: \chi_1, \chi_2, \ldots, \chi_m \) be the roots of \( f(T) \), and let \( \upsilon =: \upsilon_1, \upsilon_2, \ldots, \upsilon_n \) be the roots of \( g(T) \). Then
\[ P_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}) = \prod_{i=1}^{m} \prod_{j=1}^{n} (T - \chi_i - \upsilon_j), \]
and
\[ Q_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1}) = \prod_{i=1}^{m} \prod_{j=1}^{n} (T - \chi_i \upsilon_j). \]
This shows that \( P_{m,n}(T) \) and \( Q_{m,n}(T) \) are both of degree \( mn \) in \( T \).
Now by construction, $S$ satisfies the following universal property: for any commutative ring $R'$ that contains elements $X, Y, A_0, \ldots, A_{m-1}, B_0, \ldots, B_{n-1}$ satisfying condition \(^2\) of Lemma \(3.3\) there is a unique homomorphism $\varphi : S \to R'$ such that
\[
\varphi : v \mapsto X, \chi \mapsto Y, \alpha_i \mapsto A_i, \beta_j \mapsto B_j.
\]
Now as $\varphi(P_{m,n}(\chi + v)) = 0$ and $\varphi(Q_{m,n}(\chi v)) = 0$, the polynomials $P_{m,n}$ and $Q_{m,n}$ satisfy properties \(^1\) and \(^2\) of Lemma \(3.3\). On the other hand, as monic minimal polynomials in an integral domain are unique, and by the uniqueness of commutative rings satisfying universal properties, $P_{m,n}$ and $Q_{m,n}$ are the only polynomials with the asserted properties.

The polynomials $P_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1})$ and $Q_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1})$ are effectively computable \(^8\) [Section 59].

**Lemma 3.4.** Let $P_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1})$ and $Q_{m,n}(T; \alpha_0, \ldots, \alpha_{m-1}, \beta_0, \ldots, \beta_{n-1})$ be as in Lemma \(3.3\). Then
\[
P_{m,n}(T; 0, \ldots, 0) = Q_{m,n}(T; 0, \ldots, 0) = T^{mn}.
\]

**Proof.** We follow the notation in the proof of Lemma \(3.3\). Note that $\chi_1, \ldots, \chi_m, v_1, \ldots, v_n$ are algebraically independent over $\mathbb{Z}$, and so by the universal mapping property of polynomial rings, there exists a map $\varphi : S' \to \mathbb{Z}$ with $X_i \mapsto 0$ for all $1 \leq i \leq m$, and $Y_j \mapsto 0$ for all $1 \leq j \leq n$. But by the theory of symmetric polynomials, we see that $X_1, \ldots, X_m \mapsto 0$ if and only if $\alpha_0, \ldots, \alpha_{m-1} \mapsto 0$, and $v_1, \ldots, v_n \mapsto 0$ if and only if $\beta_0, \ldots, \beta_{n-1} \mapsto 0$. Then
\[
P_{m,n}(T; 0, \ldots, 0) = \prod_{i=1}^{m} \prod_{j=1}^{n} (T - 0 - 0) = T^{mn},
\]
and
\[
Q_{m,n}(T; 0, \ldots, 0) = \prod_{i=1}^{m} \prod_{j=1}^{n} (T - 0 \cdot 0) = T^{mn},
\]
as desired. \(\square\)

If $X$ and $Y$ are $\mathcal{G}$-infinite, then intuitively, it makes sense for their product to be $\mathcal{G}$-infinite as well. This is true (Corollary \(1.6\)), but we first prove a weaker claim.

**Lemma 3.5.** If $X$ and $Y$ are $\mathcal{G}$-infinite, then $s_{XY}(T) = t^\ell$ for some $\ell \geq 0$. Equivalently, $\mathbb{Z}(XY; \mathcal{G}) \subseteq \{0\}$.

**Proof.** Let $X$ and $Y$ be $\mathcal{G}$-infinite. By Corollary \(2.30\) we may assume without loss of generality that $X$ and $Y$ are units. Now let $P(T) = \sum_{k=0}^{m} P_k T^k$ be a $\mathcal{G}$-minimal polynomial for $X$, and let $Q(T) = \sum_{k=0}^{n} Q_k T^k$ be a $\mathcal{G}$-minimal polynomial for $Y$. Suppose first that $P_0 = Q_0 = 1$. In this case, we see that $P^{R}(T)$ and $Q^{R}(T)$ are monic, and $\mathcal{G}(P^{R})(t) = t^n$ and $\mathcal{G}(Q^{R})(t) = t^n$. By Corollary \(2.16\) and Lemma \(3.3\), we conclude $X^{-1}Y^{-1}$ is a root of $Q_{m,n}(T; P_m, P_{m-1}, \ldots, P_1, Q_n, Q_{n-1}, \ldots, Q_1)$. Moreover, since $\mathcal{G}(P_k) = 0$ for $k < m$ and $\mathcal{G}Q_k = 0$ for $k < n$, we may apply Lemma \(3.4\) to conclude
\[
\mathcal{G}(Q_{m,n}(t; P_m, P_{m-1}, \ldots, P_1, Q_n, Q_{n-1}, \ldots, Q_0)) = Q_{m,n}(t; 0, \ldots, 0) = t^{mn}.
\]
Then by Proposition \(2.14\) we see $s_{X^{-1}Y^{-1}}(t)$ divides $t^{mn}$, and so $s_{X^{-1}Y^{-1}}$ is a power of $t$. Thus the sum of a $\mathcal{G}$-minimal polynomial for $X^{-1}Y^{-1}$ has exactly one nonzero term, and so the sum of its reflected polynomial will also only have one nonzero term. By Corollary \(2.16\) we conclude that the scalar polynomial for $XY$ also has only one nonzero term, and the lemma follows in this case.
We now relax the condition that $P_0 = Q_0 = 1$. In any case, $P_0Q_0X^{-1}Y^{-1}$ is a root of the monic polynomial

$$Q_{m,n}(T; P_0^{m-1}P_m, P_0^{-2}P_{m-1}, \ldots, P_1, Q_0^{n-1}Q_n, Q_0^{-2}Q_{n-1}, \ldots, Q_0),$$

and

$$\mathcal{S}(Q_{m,n}(t; P_0^{m-1}P_m, P_0^{-2}P_{m-1}, \ldots, P_1, Q_0^{n-1}Q_n, Q_0^{-2}Q_{n-1}, \ldots, Q_0)) = t^{mn}.$$  

As $X$ and $Y$ are $\mathcal{S}$-infinite, we must have $p := \mathcal{S}(P_0) \neq 0$ and $q := \mathcal{S}(Q_0) \neq 0$. Thus $Z(P_0Q_0X^{-1}Y^{-1}; \mathcal{S}) \subseteq \{0\}$. Now suppose by way of contradiction that $Z(XY; \mathcal{S}) \not\subseteq \{0\}$, and choose $a \neq 0$ an element of $Z(XY; \mathcal{S})$. By Theorem 2.18 and Theorem 1.7 we may choose a rationally closed extension $\mathcal{S}'$ of $\mathcal{S}$ with $\mathcal{S}(XY) = a$. As $\mathcal{S}'$ is rationally closed, we must have $\mathcal{S}'(X^{-1}Y^{-1}) = \frac{1}{a}$, and by multiplicativity $\mathcal{S}'(P_0Q_0X^{-1}Y^{-1}) = \frac{pq}{a} \neq 0$. Then by Corollary 2.19 we have

$$0 \neq \frac{pq}{a} \in Z(P_0Q_0X^{-1}Y^{-1}; \mathcal{S}') \subseteq Z(P_0Q_0X^{-1}Y^{-1}; \mathcal{S}) \subseteq \{0\},$$

which is a contradiction. \hfill \Box

**Definition 3.6.** A series $X$ is **practically $\mathcal{S}$-zero** if it is absolutely $\mathcal{S}$-algebraic and $s_X(t) = t^m$ for some $m > 0$.

By Theorem 2.18, Corollary 2.15, and Definition 2.25, a series $X$ is practically $\mathcal{S}$-zero if and only if every summation $\mathcal{S}'$ extending $\mathcal{S}$ has an extension $\mathcal{S}''$ such that $\mathcal{S}''(X) = 0$. See also Definitions 4.2 and 4.3 below.

**Lemma 3.7.** Let $U$ be a unit of $R[[\sigma]]$. Then $U$ is $\mathcal{S}$-infinite if and only $U^{-1}$ is practically $\mathcal{S}$-zero.

**Proof.** Suppose first that $U$ is $\mathcal{S}$-infinite, and let $P(T) = \sum_{k=0}^{m} P_k T^k$ be a $\mathcal{S}$-minimal polynomial for $U$ with $P_m \neq 0$. By Corollary 2.16, $s_{U^{-1}}(t)$ divides $\mathcal{S}(P^R)(t)$. But $\mathcal{S}(P^R)(t) = \mathcal{S}(P_0)t^m$, and so $U^{-1}$ is either $\mathcal{S}$-infinite or $\mathcal{S}$-algebraic with zero as its only root. Now suppose by way of contradiction that $U^{-1}$ is not practically $\mathcal{S}$-zero. Then there is some extension $\mathcal{S}'$ of $\mathcal{S}$ for which $U^{-1}$ is $\mathcal{S}'$-infinite. Now by Lemma 3.5 we see that $U \cdot U^{-1} = 1$ has a scalar polynomial over $\mathcal{S}'$ of the form $t^m$ for some $m$, which is absurd. Thus $U^{-1}$ is practically $\mathcal{S}$-zero as desired.

On the other hand, suppose that $U^{-1}$ is practically $\mathcal{S}$-zero, and suppose by way of contradiction that $U$ is not $\mathcal{S}$-infinite. Then $U$ is $\mathcal{S}$-algebraic or $\mathcal{S}$-transcendental, and so by Theorem 2.18 we may find $x \in E$ and $\mathcal{S}'$ an extension of $\mathcal{S}$ such that $\mathcal{S}'(U) = x$. On the other hand, as $U^{-1}$ is practically $\mathcal{S}$-zero, there is an extension $\mathcal{S}''$ of $\mathcal{S}'$ such that $\mathcal{S}''(U^{-1}) = 0$. Then

$$1 = \mathcal{S}''(U \cdot U^{-1}) = \mathcal{S}''(U) \cdot \mathcal{S}''(U^{-1}) = x \cdot 0 = 0,$$

which is a contradiction. \hfill \Box

**Example 3.8.** The series $Y$ of Example 2.3 is the reciprocal of the series $X$ of Example 3.13; its $\mathfrak{A}$-minimal polynomial is 1, so it is $\mathfrak{A}$-infinite.

**Proposition 3.9.** Let $U \in R[[\sigma]]$ be a unit. Then

(a) The series $U$ is $\mathcal{S}$-transcendental if and only if $U^{-1}$ is $\mathcal{S}$-transcendental.

(b) The series $U$ is (absolutely) $\mathcal{S}$-algebraic if and only if $U^{-1}$ is either $\mathcal{S}$-infinite or (absolutely) $\mathcal{S}$-algebraic but not practically $\mathcal{S}$-zero.

(c) The series $U$ is $\mathcal{S}$-infinite if and only if $U^{-1}$ is practically $\mathcal{S}$-zero.
Proof. Let \( U \in \mathbb{R}[[\sigma]] \) be a unit. We apply Corollary 2.19. For each \( x \in \mathbb{Z}(U; \mathcal{G}) \), there exists a multiplicative summation \( \mathcal{G}_x \) such that \( \mathcal{G}_x(U) = x \). Replacing \( \mathcal{G}_x \) with its rational closure if necessary, we may take \( \mathcal{G}_x \) to be rationally closed. Then if \( x \neq 0 \), we see \( \mathcal{G}_x(U^{-1}) = x^{-1} \). 

A fortiori, we conclude

\[
\{ x^{-1} : x \in \mathbb{Z}(U; \mathcal{G}), \ x \neq 0 \} \subseteq \mathbb{Z}(U^{-1}; \mathcal{G}).
\]

Suppose first that \( U \) is \( \mathcal{G} \)-transcendental. The field \( E \) is algebraically closed, and so is infinite. Thus \( \mathbb{Z}(U^{-1}; \mathcal{G}) \supseteq E^* \) is infinite, so by Corollary 2.19 the series \( U^{-1} \) is \( \mathcal{G} \)-transcendental. 

By symmetry, if \( U^{-1} \) is \( \mathcal{G} \)-transcendental then \( U \) is \( \mathcal{G} \)-transcendental.

Now [a] is just Lemma 3.7, and [b] follows from [a] [c] and the partition [11]. \( \square \)

At last, we are ready to distinguish which \( \mathcal{G} \)-algebraic series are absolutely \( \mathcal{G} \)-algebraic.

**Theorem 3.10.** Let \( X \) be a \( \mathcal{G} \)-algebraic series. The following are equivalent:

(a) The series \( X \) is absolutely \( \mathcal{G} \)-algebraic.

(b) For some unit \( U \) tail-equivalent to \( X \), we have \( s_{U^{-1}}(0) \neq 0 \).

(c) For every unit \( U \) tail-equivalent to \( X \), we have \( s_{U^{-1}}(0) \neq 0 \).

Proof. [a] \( \Rightarrow \) [b]. Let \( X \) be an absolutely \( \mathcal{G} \)-algebraic series, and let \( U := 1 - \sigma + \sigma^2X \). As \( U \in 1 + \sigma \mathbb{R}[[\sigma]] \subseteq \mathbb{R}[[\sigma]]^* \), we see \( U \) is a unit, which is manifestly tail-equivalent to \( X \). Thus by Corollary 2.30, \( U \) is absolutely \( \mathcal{G} \)-algebraic. But now suppose by way of contradiction that \( s_{U^{-1}}(0) = 0 \). Then by Theorem 2.18 there is a multiplicative summation \( \mathcal{G}' \) extending \( \mathcal{G} \) such that \( \mathcal{G}'(U^{-1}) = 0 \). In particular, \( U^{-1} \) is practically \( \mathcal{G}' \)-zero, and so \( U \) is \( \mathcal{G}' \)-finite by Lemma 3.7. But as \( U \) is absolutely \( \mathcal{G} \)-algebraic, Proposition 2.27 tells us \( U \) is absolutely \( \mathcal{G}' \)-algebraic, and we obtain a contradiction.

[b] \( \Rightarrow \) [c]. We proceed by contrapositive. Suppose that \( V \) is a unit tail-equivalent to \( X \) with \( s_{V^{-1}}(0) = 0 \). By Theorem 2.18 there exists a summation \( \mathcal{G}' \) extending \( \mathcal{G} \) such that \( \mathcal{G}'(V^{-1}) = 0 \). A fortiori, we see \( V^{-1} \) is practically \( \mathcal{G}' \)-zero, and so by Lemma 3.7, we see \( V \) is \( \mathcal{G}' \)-finite. Then by Corollary 2.30 if \( U \) is a unit tail-equivalent to \( V \), then \( U \) is \( \mathcal{G}' \)-finite; equivalently, if \( U \) is a unit tail-equivalent to \( X \), then \( U \) is \( \mathcal{G}' \)-finite. Applying Lemma 3.7 again, we see that such a \( U^{-1} \) is practically \( \mathcal{G}' \)-zero. Then by Corollary 2.15 we conclude \( s_{U^{-1}}(0) = 0 \), which proves our claim.

[c] \( \Rightarrow \) [a]. Suppose by contrapositive that \( X \) is not absolutely \( \mathcal{G} \)-algebraic, and let \( U := 1 - \sigma + \sigma^2X \). If \( X \) is \( \mathcal{G} \)-transcendental, then by Corollary 2.30 so is \( U \), and so by Proposition 3.9 \( U^{-1} \) is \( \mathcal{G} \)-transcendental. Then \( s_{U^{-1}}(t) = 0 \), and in particular \( s_{U^{-1}}(0) = 0 \), as desired. Suppose now that \( X \) is neither absolutely \( \mathcal{G} \)-algebraic nor \( \mathcal{G} \)-transcendental. Then \( X \) is \( \mathcal{G}' \)-finite for some extension \( \mathcal{G}' \) of \( \mathcal{G} \); by Corollary 2.30 \( U \) is also \( \mathcal{G}' \)-finite. Now by Corollary 3.7 \( U^{-1} \) is effectively \( \mathcal{G}' \)-zero, so the scalar polynomial for \( U^{-1} \) with respect to \( \mathcal{G}' \) is a positive power of \( t \). Then by Corollary 2.15 that same power of \( t \) divides \( s_{U^{-1}}(t) \), and so \( s_{U^{-1}}(0) = 0 \). Our assertion holds by contrapositive. \( \square \)

**Corollary 3.11.** A unit \( U \) of \( \mathbb{R}[[\sigma]] \) is absolutely \( \mathcal{G} \)-algebraic if and only if \( s_{U^{-1}}(0) \neq 0 \).

We have another sufficient condition for a series \( X \) to be absolutely \( \mathcal{G} \)-algebraic.

**Corollary 3.12.** If \( X \) is a \( \mathcal{G} \)-algebraic series with \( \mathcal{G} \)-degree equal to its scalar degree, then \( X \) is absolutely \( \mathcal{G} \)-algebraic.

Proof. Let \( U = 1 - \sigma + \sigma^2X \); by Corollary 2.30 \( U \) is absolutely \( \mathcal{G} \)-algebraic if and only if \( X \) is absolutely \( \mathcal{G} \)-algebraic. Now suppose that that the \( \mathcal{G} \)-degree of \( U \) equals its scalar degree, and let \( P(t) \) be a \( \mathcal{G} \)-minimal polynomial for \( U \) with degree equal to its \( \mathcal{G} \)-degree. Then \( P^R(U^{-1}) = 0 \), and so \( s_{U^{-1}}(t) \) divides \( \mathcal{G}(P^R)(t) \). But observe that \( P^R(0) \neq 0 \), hence \( s_{U^{-1}}(0) \neq 0 \). Then by Theorem 3.10 our claim follows. \( \square \)
Example 3.13. Let $X \in \mathbb{C}[\sigma]$ be the Taylor series for $\sqrt{1 - \sigma}$, so

$$X := 1 - \frac{\sigma}{2} - \frac{\sigma^2}{8} - \frac{\sigma^3}{16} - \frac{5\sigma^4}{128} - \frac{7\sigma^5}{256} - \ldots.$$ 

We see $X$ has minimal polynomial $T^2 - (1 - \sigma)$ and scalar polynomial $t^2$. It is thus $A$-algebraic; indeed, by Corollary 3.12, $Y$ is absolutely $S$-algebraic. As $Z(X; S) = \{0\}$, we conclude that $X$ is practically $S$-zero.

Although Corollary 3.12 provides a sufficient criterion for verifying that a series is absolutely $S$-algebraic, we have been unable to determine if this condition is also necessary. Thus, we pose the following question.

**Question 3.14.** Suppose $S$ is a multiplicative summation to an algebraically closed field $E$. If $X$ is an absolutely $S$-algebraic series, must it have $S$-degree equal to its scalar degree?

### 4. Univalent Extensions

Recall that $E$ is an algebraically closed field. Although $QS$ is multiplicatively canonical, it is generally not the multiplicative fulfillment of $S$. Consider for instance the series $X$ of Example 3.13. This series was shown to be practically $A$-zero; it follows, by the remarks after Definition 3.6, that $X$ is in the multiplicative fulfillment of $A$. On the other hand, $X$ is not a rational function in $\sigma$, and so is not an element of $QR[\sigma]$. This example is suggestive: if the scalar polynomial of an absolutely $S$-algebraic series $X$ is not linear but still only has one root, this forces $X$ to be summed to a unique value, and places it in the multiplicative fulfillment of $S$. In this section, we follow this intuition to construct the multiplicative fulfillment of $S$.

**Definition 4.1.** If $X$ is $S$-algebraic and $s_X(t) = (t - \rho)^m$ for some $\rho \in E$ and $m > 0$, we say $X$ is $S$-univalent with root $\rho = \rho_X$. If in addition $X$ is absolutely $S$-algebraic, we say $X$ is absolutely $S$-univalent with root $\rho_X$.

A series $X$ is practically $S$-zero exactly if it is absolutely $S$-univalent with root $\rho_X = 0$; thus, Definition 4.1 provides a natural extension of Definition 3.6.

**Example 4.2.** Let $Z$ be the Taylor series of $\frac{3 - \sigma + \sqrt{1 - 6\sigma + 5\sigma^2}}{2}$; thus we have

$$Z = 2 - 2\sigma - \sigma^2 - 3\sigma^3 - 10\sigma^4 - 36\sigma^5 - \ldots.$$ 

We claim $Z$ is $A$-univalent. Indeed, $Z$ has $S$-minimal polynomial

$$P(T) = T^2 - (3 - \sigma)T + (2 - \sigma^2) \in D[t];$$

therefore its scalar polynomial is

$$s_Z(t) = t^2 - 2t + 1 = (t - 1)^2$$

and so $Z$ is $A$-univalent with root $\rho_Z = 1$. Corollary 3.12 confirms that $Z$ is absolutely $S$-univalent.

The series $Y$ described in Example 2.12 furnishes an example of a series which is $S$-univalent but not absolutely $S$-univalent.

Note that if $X$ is absolutely $S$-univalent, then by Corollary 2.15 and Proposition 2.27, it is absolutely $S'$-univalent for every $S' \in MS(R, E)$ extending $S$ and the root of $X$ is invariant under extensions of the summation.
Definition 4.3. For any multiplicative summation \((D, \mathcal{G}) \in \text{MS}(R, E)\), the univalent extension \((UD, U\mathcal{G})\) of \((D, \mathcal{G})\) is defined as follows. For \(X \in R[[\sigma]]\), we say \(X \in UD\) if \(X\) is absolutely \(\mathcal{G}\)-univalent. We define

\[
U\mathcal{G} : UD \to E, \\
U\mathcal{G} : X \mapsto \rho_X \text{ if } X \text{ is as above}.
\]

We extend this definition to weakly multiplicative summations \((D, \mathcal{G}) \in \text{wMS}(R, E)\) by setting \((UD, U\mathcal{G}) := (UMD, U\mathcal{M}\mathcal{G})\).

Lemma 4.4. For any multiplicative summation \((D, \mathcal{G})\), \((UD, U\mathcal{G})\) is a well-defined multiplicative summation. Moreover, the image of \(UD\) under \(U\mathcal{G}\) is a purely inseparable extension of the field of fractions of \(\mathcal{G}(D)\). In particular, if \(E\) is of characteristic \(0\), then the image of \(UD\) under \(U\mathcal{G}\) is the field of fractions of \(\mathcal{G}(D)\).

Proof. The map \(U\mathcal{G}\) is well-defined as a function because \(s_X(t)\) is well-defined for every \(\mathcal{G}\)-algebraic \(X \in R[[\sigma]]\). If \(X\) and \(Y\) are absolutely \(\mathcal{G}\)-univalent, then by Theorem 3.1 we see \(X + Y\) and \(XY\) are absolutely \(\mathcal{G}\)-algebraic. Then as \(E\) is algebraically closed, we see

\[\emptyset \subset Z(X + Y; \mathcal{G}) \subset \{x + y : x \in Z(X; \mathcal{G}), y \in Z(Y; \mathcal{G})\} = \{\rho_X + \rho_Y\},\]

and \(X + Y\) is absolutely \(\mathcal{G}\)-univalent with root \(\rho_X + \rho_Y\). A similar argument holds for \(XY\). Thus \(U\mathcal{G}\) is a multiplicative summation, and the image of \(U\mathcal{G}\) is at least an \(R\)-algebra.

The proof that the image of \(U\mathcal{G}\) is a field proceeds along the same lines as in the proof of Proposition 1.8. Let \(x \neq 0\) be in the image of \(U\mathcal{G}\), and choose \(X \in UD\) with \(U\mathcal{G}(X) = x\). Replacing \(X\) with \(1 - \sigma + \sigma^2X\) if necessary, we may assume that \(X\) is a unit. By assumption \(s_X(0) \neq 0\), then by Theorem 3.10 we see \(X^{-1}\) is absolutely \(\mathcal{G}\)-algebraic. But now let \(P(T) = \sum_{k=0}^{\ell} P_k T^k\) be a \(\mathcal{G}\)-minimal polynomial for \(X\), and suppose \(\mathcal{G}(P(t)) = a \cdot (t - x)^m\) with \(a \in E^x\). Note \(\mathcal{G}(P(R)) = a(-x)^m \cdot t^{\ell-m}(t - x^{-1})^m\), then by Corollary 2.10 we see \(s_{X^{-1}}(t)\) divides \(t^{\ell-m}(t - x^{-1})^m\). Applying Theorem 3.10 again and recalling that \(X\) is absolutely \(\mathcal{G}\)-algebraic, we see that \(s_{X^{-1}}(0) \neq 0\), and so \(X^{-1}\) is absolutely \(\mathcal{G}\)-univalent with root \(x^{-1}\).

It remains to show that this field is a totally inseparable extension of the field of fractions of \(\mathcal{G}(D)\).

Let \(X\) be \(\mathcal{G}\)-univalent with root \(\rho_X\). Then there exists a \(\mathcal{G}\)-minimal polynomial \(P(T) \in D[t]\) with \(P(X) = 0\) and \(\mathcal{G}(P(t)) = a \cdot (t - \rho_X)^m\) for some \(a \in E^x\) and \(m > 0\) an integer. If \(E\) is of characteristic 0 then

\[\rho_X = -\frac{\mathcal{G}(P_{m-1})}{m\mathcal{G}(P_m)}\]

is an element of the field of fractions of \(\mathcal{G}(D)\) as desired. Otherwise, let \(p\) prime be the characteristic of \(E\), and write \(m = p^\mu \cdot m'\) with \((p, m') = 1\). Then

\[\rho_X^p = -\frac{\mathcal{G}(P_{p^\mu \cdot m' - 1})}{m'\mathcal{G}(P_m)}\]

and so \(\rho_X\) is a \(p^\mu\)th root of an element of the field of fractions of \(\mathcal{G}(D)\), hence purely inseparable over the field of fractions of \(\mathcal{G}(D)\). The claim follows. \(\square\)

Theorem 4.5. The map of multiplicative summations

\[U : \text{MS}(R, E) \to \text{MS}(R, E)\]

is a well-defined extension map. Moreover, \(U\) is idempotent, multiplicatively canonical, preserves multiplicative compatibility, and subsumes \(T\) and \(Q\).
Proof. We showed in Lemma 4.4 that \( UF \) is a well-defined multiplicative summation. Suppose now that \( X \in D \), and define \( P(T) = T - X \in D[T] \). Clearly \( P(X) = 0 \) and
\[
\mathcal{G}(P)(t) = s_X(t) = t - \mathcal{G}X,
\]
then \( UF(X) = \mathcal{G}(X) \). Hence \( UF \) is well-defined, multiplicative, and an extension of \( \mathcal{G} \).

Suppose \( \mathcal{G}' \) extends \( \mathcal{G} \). By Proposition 2.27 we have \( \mathcal{B}(\mathcal{G}') \supset \mathcal{B}(\mathcal{G}) \), and so by Corollary 2.15 if \( X \) is absolutely \( \mathcal{G} \)-univalent then \( X \) is absolutely \( \mathcal{G}' \)-univalent. Then \( UF(\mathcal{G}') \) extends both \( \mathcal{G}' \) and \( UF \), and so \( UF \) is multiplicatively \( \mathcal{G} \)-canonical.

Now suppose that \( X \in UFD \) with root \( \rho \). As \( UF \mathcal{G} \) is a multiplicatively \( \mathcal{G} \)-canonical extension of \( \mathcal{G} \), Theorem 2.18 tells us that the scalar polynomial \( s_X(t) \) for \( X \) has only one root in \( E \), and a fortiori that \( X \) is \( \mathcal{G} \)-algebraic. Then as \( E \) is algebraically closed, we see \( X \) is \( \mathcal{G} \)-univalent. Again, as \( UF \mathcal{G} \) is multiplicatively canonical, \( X \) must be absolutely \( \mathcal{G} \)-algebraic, as otherwise there would be an extension \( \mathcal{G}' \) of \( \mathcal{G} \) over which \( X \) would be \( \mathcal{G}' \)-infinite. Then \( X \) is absolutely \( \mathcal{G} \)-univalent, and \( U \) is idempotent as desired.

Recall that \( U \) preserves multiplicative compatibility if and only if it preserves extensions. But if \( \mathcal{G}' \) extends \( \mathcal{G} \), and \( X \) is \( \mathcal{G} \)-univalent, then by Corollary 2.15 \( X \) is also \( \mathcal{G}' \)-univalent. Then \( U(\mathcal{G}') \) extends \( UF \mathcal{G} \) as desired.

Finally, if \( X \in QD \) then \( X \in UD \), hence by the idempotence of \( U \) we have
\[
UD = UUD = QUD = UQD,
\]
and so \( U \) subsumes \( Q \). But then as \( Q \) subsumes \( T \), we see \( U \) subsumes \( T \) as well. \( \square \)

Corollary 4.6. If \( X \) and \( Y \) are \( \mathcal{G} \)-infinite, then \( XY \) is \( \mathcal{G} \)-infinite.

Proof. Without loss of generality, assume that \( X \) and \( Y \) are units. By Proposition 3.9 we see that \( X^{-1} \) and \( Y^{-1} \) are practically \( \mathcal{G} \)-zero, and so by Theorem 4.5 \( X^{-1}Y^{-1} \) is also practically \( \mathcal{G} \)-zero. Another application of Proposition 3.9 shows that \( (X^{-1}Y^{-1})^{-1} = XY \) is \( \mathcal{G} \)-infinite, as desired.

Theorem 4.7. For any summation \( (D, \mathcal{G}) \in MS(R, E) \), \( (UD, UF) \) is the multiplicative fulfillment of \( \mathcal{G} \).

Proof. By Theorem 4.5 the univalent extension of a summation is multiplicatively canonical, and so every absolutely \( \mathcal{G} \)-univalent series is contained in the multiplicative fulfillment of \( \mathcal{G} \). But now \( X \) be any series in \( R[[\sigma]] \) that is not absolutely \( \mathcal{G} \)-univalent. We claim \( X \) is not in the domain of the multiplicative fulfillment of \( \mathcal{G} \). Indeed, if \( X \) is not absolutely \( \mathcal{G} \)-univalent, then one of the following must occur:

Case 1: \( X \) is \( \mathcal{G} \)-infinite over \( D \).
In this case, by Theorem 2.18 \( X \) is not in the domain of any extension of \( \mathcal{G} \), and in particular is not in the domain of the multiplicative fulfillment of \( \mathcal{G} \).

Case 2: \( X \) is neither \( \mathcal{G} \)-infinite nor \( \mathcal{G} \)-univalent.
In this case, by Theorem 2.18 we may choose extensions \( \mathcal{G}', \mathcal{G}'' \in MS(R, E) \) of \( \mathcal{G} \) with both \( \mathcal{G}'(X) \) and \( \mathcal{G}''(X) \) defined, but with \( \mathcal{G}'(X) \neq \mathcal{G}''(X) \). Then \( X \) is not in the domain of the multiplicative fulfillment of \( \mathcal{G} \).

Case 3: \( X \) is \( \mathcal{G} \)-univalent but not absolutely \( \mathcal{G} \)-algebraic.
In this case, we may choose an extension \( \mathcal{G}' \in MS(R, E) \) of \( \mathcal{G} \) such that \( X \) is \( \mathcal{G}' \)-infinite, and as in Case 1 conclude that \( X \) is not in the domain of the multiplicative fulfillment of \( \mathcal{G} \).

We conclude \( UD \) is the domain of the multiplicative fulfillment of \( \mathcal{G} \), and so \( (UD, UF) \) is the multiplicative fulfillment of \( (D, \mathcal{G}) \). \( \square \)
We can extend $U$ to a functor on weakly multiplicative summations by defining $U S := UM$. Naturally, Theorems 4.5 and 4.7 still apply in this more general context.

5. Future work

In [3] we followed Hardy’s philosophy [6] of treating the shift operator $\sigma$ as fundamental to the study of series and summations, and we found the fulfillment of summations taking values in integral domains. In this paper, we shifted our attention to multiplicative summations, and we found the multiplicative fulfillment of summations taking values in an algebraically closed fields. Notably, neither of these are fully general results, so it is natural to try to find fulfillments or multiplicative fulfillments for summations taking values in general commutative rings.

However, there are other lines of inquiry to pursue. Almost all of the dozens of classical summation operators are shift-invariant; so this is an obvious property to axiomatize. If the finitely-supported series are considered as polynomials in $\sigma$, then there is a unique distributive product; we can consider any shift-closed vector space of series as an $R[\sigma]$-module. The extension to the Cauchy product of pairs of series is not automatic, but is natural; with this product, series become formal power series.

Most well-known summations are also invariant under uniform dilutions of the form $\delta_m : \sum_n a_n \sigma^n \mapsto \sum_n a_n \sigma^{mn}$. From an analytic perspective, this is a consequence of the Mellin transform [9]. Algebraically, if we want dilutions to have internal multiplicative representations, we are led to the Dirichlet product, defined by

$$\left( \sum_n A_n \sigma^n \right) \star \left( \sum_n B_n \sigma^n \right) = \sum_n \left( \sum_{k \ell = n} A_k B_\ell \right) \sigma^n$$

Summations of this type have been studied under such headings as $\zeta$-function regularization [7] and Ramanujan summation [2]. We speculate that the formal approach we follow in this paper and in [4, 3] may be fruitfully applied in this context as well. Indeed, Nori independently developed an analogue to telescopic summation for series indexed by discrete abelian groups [10]: more generally, the study of series indexed by groups or even monoids would include both shift-invariance summations and dilution-invariant summations as special cases.

In the same way that Euler and Grandi studied specific applications of formal shift-invariance methods without a general theory, Ramanujan [11, Chapter 6] applied formal methods involving dilution to the series $1 + 2 + 3 + 4 + \cdots$, obtaining the value $-1/12$. (Ramanujan did have an analytic theory applying the Euler-Maclaurin summation formula to similar summations [1], ; see also Hardy [6] [Section 13.10 and 13.17]). Considered as a formal power series, however, that series is $\mathbb{R}$-infinite, and thus cannot be in the domain of any shift-invariant summation. This contradiction illustrates a tension between dilution-invariance and shift-invariance. It would be nonetheless be interesting to consider fulfillments within families of summations that are both shift-invariant and dilution-invariant.

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