1 EXTENDED ABSTRACT

Deep learning systems have steadily advanced the state of the art in a wide variety of benchmarks, demonstrating impressive performance in tasks ranging from image classification [Taigman et al., 2014; Zhai et al., 2021], language processing [Devlin et al., 2019; Brown et al., 2020], open-ended environments [Silver et al., 2016; Arulkumaran et al., 2019], to coding [Chen et al., 2021].

A central aspect that enables the success of these systems is the ability to train deep models instead of wide shallow ones [He et al., 2016]. Intuitively, a neural network is decomposed into hierarchical representations from raw data to high-level, more abstract features. While training deep neural networks repetitively achieves superior performance against their shallow counterparts, an understanding of the role of depth in representation learning is still lacking.

Contributions. We suggest a new perspective on understanding the role of depth in deep learning. We hypothesize that SGD training of overparameterized neural networks exhibits an implicit bias that favors solutions of minimal effective depth. Namely, SGD trains neural networks for which the top several layers are redundant. To evaluate the redundancy of layers, we revisit the recently discovered phenomenon of neural collapse [Papyan et al., 2020; Han et al., 2021]. Informally, (among other conditions) neural collapse identifies training dynamics in which training deep overparameterized neural networks for classification, the penultimate layer’s feature embeddings of samples belonging to the same class tend to concentrate around their class means. Specifically, within-class variance normalized by the intra-class-covariance tends to decrease during training, especially at the terminal stage when training proceeds beyond perfectly fitting the training labels. Hence, the classification is essentially done already at the penultimate layer. In this spirit, to evaluate redundancy we measure the degree of neural collapse in the various layers.

2 EXPERIMENTS

We experimentally analyze the presence of neural collapse in the various layers of a trained network. We also show that, as expected intuitively, neural collapse is strongly correlated with classification performance.

2.1 SETUP

Method. We consider k-class classification problems and train a multilayered neural network $h = c \circ f = q \circ g_L \circ \cdots \circ g_1 : \mathbb{R}^n \rightarrow \mathbb{R}^C$ on some balanced training data $S = \bigcup_{c=1}^C S_c = \bigcup_{c=1}^C \{(x_i, y_i)\}_{i=1}^n$. The model is trained using cross-entropy loss minimization between its logits and the one-hot encodings of the labels. Here, $g_1, \ldots, g_L$ are the various hidden layers of the network and $q$ is its top linear layer. As a second stage, the embedding performance of each sub-architecture $f_i = g_i \circ \cdots \circ g_1(x)$ ($i \in \{1, \ldots, L\}$) is evaluated by training a new auxiliary linear classifier $\tilde{q}$ on top of $f$ and evaluating its test accuracy.

Neural collapse. To evaluate neural collapse, we follow the process suggested by Galanti et al. [2022]. For a feature map $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ and two distributions $P_1, P_2$ over $\mathbb{R}^n$, we define their class-distance normalized variance (CDNV) to be $V_f(P_1, P_2) = \frac{\text{Var}(f_{P_1}) + \text{Var}(f_{P_2})}{2\|f_{P_1} - f_{P_2}\|_2^2}$. Here, $f \circ P$ stands for the distribution of $f(x)$ for $x \sim P$ and Var($P$) is the variance of $P$. The definition of Galanti et al. [2022] for neural collapse asserts that when training $h = c \circ f$ for classification of $S$, we have $\lim_{t \rightarrow \infty} \text{Avg}_{j \neq \hat{c}}[V_f(S_j, S_{\hat{c}})] = 0$ (where $t$ is the training iteration). As shown by Galanti et al. [2022], features with lower CDNVs are typically associated with a higher degree of linear separability. Therefore, if we encounter neural collapse in a certain layer, we treat the higher layers as redundant since they are replaceable by a linear classifier.

Hyperparameters. The optimizations of $h$ and $\tilde{q}$ were carried using SGD with batch size 128, momentum 0.9 and weight decay $5e-4$. 

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**2.2 RESULTS**

**Minimal depth in MLPs.** To study the bias towards learning solutions of minimal depth, we trained a set of fully connected neural networks with $2^i$ hidden layers ($i \in [4]$) of width 100 for 500 epochs on CIFAR10. Each model was trained with learning rate scheduling with an initial learning rate 0.1, decayed twice at epochs 150 and 225. In Fig. 1, we plot the results of this experiment. In each plot, we consider a neural network of a different depth. The $i$’th line stands for the CDNV at train time of the $i$’th hidden layer of the neural network. We observe that in the 8 and 16 hidden layers network, the fifth and higher layers enjoy neural collapse. Therefore, these networks are effectively of depth 4, which is the minimal depth that allows perfect interpolation of the training data as observed in Fig. 1(e).

**Implicit depth pruning in deep networks.** We study the presence of neural collapse within the layers of a ResNet18 trained to (perfectly) fit the training data. During training, we evaluate the CDNV on the train and test data along with the layer’s performance (see ’Method’). The CDNV is computed over each layer following a group of residual blocks. To train each model, we used learning rate scheduling with an initial learning rate 0.1, decayed three times at epochs 60, 120, and 160. The results are summarized in Fig. 2. As can be seen, the few top-most layers of the neural network tend to collapse. In addition, in Fig. 2(c), we observe that collapsed layers tend to be redundant in the sense that their performance already matches that of the full network. In that sense, we argue that SGD implicitly prunes the top layers of the trained neural network, since it tends to select weights for which the top layers are replaceable by a linear classifier.

**Figure 1:** Within-class feature variation collapse with MLPs trained on CIFAR10. In (a-c) we plot the CDNV on the train data when varying the number of hidden layer in the network (plotted in lin-log scale). Each line stands for a different layer within the network. In (e) we plot the train accuracy rates of the various architecture.

**Figure 2:** Within-class feature variation collapse with ResNet18. We plot (row 1) the CDNV over the train data, (row 2) the CDNV over the test data and (row 3) the embedding performances (see ’Method’) along with the train and test accuracy rates. In each column we present the results on a different datasets.
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