Sample Complexity versus Depth: An Information Theoretic Analysis

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Abstract

Deep learning has proven effective across a range of data sets. In light of this, a natural inquiry is: “for what data generating processes can deep learning succeed?” In this work, we study the sample complexity of learning multilayer data generating processes of a sort for which deep neural networks seem to be suited. We develop general and elegant information-theoretic tools that accommodate analysis of any data generating process — shallow or deep, parametric or nonparametric, noiseless or noisy. We then use these tools to characterize the dependence of sample complexity on the depth of multilayer processes. Our results indicate roughly linear dependence on depth. This is in contrast to previous results that suggest exponential or high-order polynomial dependence.

Keywords: deep learning, information theory, rate-distortion theory, sample complexity

1. Introduction

The refrain “success is guaranteed” espoused by some deep learning researchers suggests that, given a large data set, a sufficiently wide and deep neural network trained via stochastic gradient descent will deliver a useful model. Perhaps this statement is not intended to be taken literally, as it is easy to generate data in a manner for which no algorithm can accomplish this by learning from any reasonable number of samples. Yet, neural networks have successfully addressed many complex data sets. This begs the question: “for what data generating processes can neural networks succeed?”

In parametric statistics, the number of parameters typically drives sample complexity. Classical results such as those of Haussler (1992) and Bartlett et al. (1998) bound the sample complexity of a neural network in terms of the number of parameters and depth. More recently, Harvey et al. (2017) established a more interpretable result that suggests that for neural networks with piecewise-linear activation units, the sample complexity grows at most linearly in the number of parameters and depth. However, modern neural networks typically reside in a regime where the number of parameters is very large, often exceeding the number of training data samples. Consequently, sample complexity analyses suitable for classical parametric models become vacuous, failing to explain the success of deep learning.
As an alternative to the number of parameters, researchers have produced sample complexity bounds that depend on the product of norms of realized weight matrices. Bartlett et al. (2017) and Neyshabur et al. (2018), for example, establish sample complexity bounds that scale with the product of spectral norms. Neyshabur et al. (2015) and Golowich et al. (2018) establish similar bounds that instead scale in the product of Frobenius norms. While this line of work provides sample complexity bounds that are independent of neural network width, there is an exponential dependence on depth, which is inconsistent with empirical results.

To avoid this exponential dependence on depth, recent work bounds the sample complexity in terms of so-called data-dependent quantities (Dziugaite and Roy, 2017; Arora et al., 2018; Nagarajan and Kolter, 2018; Wei and Ma, 2019). Among these, the most relevant to our work is (Wei and Ma, 2019), which bounds sample complexity as a function of depth and statistics of trained neural network weights. While difficult to interpret due to dependence on complicated data-dependent statistics, their bound suggests a nonic dependence on depth. The work of Arora et al. (2018) is also related to ours, in that it utilizes concepts of compression, which we generalize and expand upon. While they establish a sample complexity bound that explicitly exhibits quadratic on depth, further dependence may be hidden in additional dependencies on data-dependent statistics.

Our information-theoretic framework generalizes that developed by Haussler et al. (1994). That work provided a basis for understanding the relationship between prediction error and information. In a similar vein, Russo and Zou (2019) introduced tools that establish general relationships between mutual information and error. Using these results, Xu and Raginsky (2017) established upper bounds on the generalization error of learning algorithms with countably infinite hypothesis spaces. We extend these results in several directions to enable analysis of data generating processes related to deep learning. For example, the results of Haussler et al. (1994) do not address noisy observations, and all three aforementioned papers do not accommodate continuous parameter spaces, let alone nonparametric data generating processes. A distinction of our work is that it builds on rate-distortion theory to address these limitations. Another line of work that studies neural networks through an information-theoretic lens involves the information bottleneck, developed by Tishby et al. (2000). The rate-distortion function that we study is equivalent to one defined in that work. However, instead of using it as a basis for optimization methods as do Shwartz-Ziv and Tishby (2017), we develop tools to study sample complexity and arrive at concrete and novel results.

In this paper, we consider contexts in which an agent learns from an iid sequence of predictor-response data pairs. We consider a data generating processes for which deep neural networks may be suited and quantify the number of samples required to arrive at a useful model. These analyses rely on general and elegant information-theoretic results that we introduce to study both parametric and nonparametric models. We use this information-theoretic framework to study a multilayer data generating process obtained by cascading single-layer data generated processes. We establish that, under reasonable assumptions, when each single-layer process obeys a natural sample complexity bound, a sample complexity bound for the multilayer process is obtained, within a logarithmic factor, via scaling that by the number of layers. We view this approach to bounding sample complexity of multilayer data generating processes as well as our foundational information-theoretic tools, which we expect to be useful beyond the scope of this paper, to be the primary contributions of this paper.
2. Data and Predictions

We begin by introducing the structure of data generating processes we consider in our framework and defining our notions of error and sample complexity.

2.1 Data Generating Process

We consider a stochastic process that generates a sequence \((X_t, Y_{t+1}) : t = 0, \ldots, T - 1\) of data pairs. We refer to each \(X_t\) as an input and each \(Y_{t+1}\) as an output. We define these and all other random variables we will consider with respect to a probability space \((\Omega, \mathcal{F}, P)\).

Elements of the sequence \((X_t : t = 0, \ldots, T - 1)\) are independent and identically distributed. Denote the history of data generated through time \(t\) by \(H_t = (X_0, Y_1, \ldots, X_{t-1}, Y_t, X_t)\). Each output is distributed according to \(P(Y_{t+1} \in \cdot | E_t, H_t) = E(\cdot | X_t)\). Here, \(E\) is a random function that specifies a conditional output distribution \(E(\cdot | x)\) for each input \(x\). Initial uncertainty about \(E\) is expressed by the prior distribution \(P(E \in \cdot)\). Note that, conditioned on \(E\), the sequence \((X_t, Y_{t+1}) : t = 0, \ldots, T - 1\) is iid.

2.2 Error

We consider an agent that predicts the next output \(Y_{t+1}\) given the history \(H_t\). We take the agent’s prediction to be the posterior predictive distribution:

\[
P_t = P(Y_{t+1} \in \cdot | H_t).
\]

We denote the target distribution by

\[
P^*_t = P(Y_{t+1} \in \cdot | E_t, X_t).
\]

As only a prescient agent would be able to produce such a prediction, it represents what the agent aims to learn. We assess prediction error in terms of the KL-divergence

\[
d_{KL}(P^*_t \| P_t) = \int P^*_t(dy) \ln \frac{dP^*_t}{dP_t}(y).
\]

This quantifies mismatch between the prediction \(P_t\) and target \(P^*_t\). We explain in Appendix A.1 how this generalizes notions of error such as mean-squared error and cross-entropy loss that are commonly used in the machine learning literature. While there are a plethora of possible predictions that an agent could arrive at, it turns out that the posterior predictive distribution that we consider is an optimal prediction under our notion of error (see Theorem 10 in Appendix A.2).

2.3 Sample Complexity

The focus of this paper is on understanding how many samples an optimal agent requires to produce sufficiently accurate predictions. Let \(\mathcal{R}(T)\) denote the expected cumulative error

\[
\mathcal{R}(T) = \mathbb{E} \left[ \sum_{t=0}^{T-1} d_{KL}(P^*_t \| P_t) \right].
\]
With this notation, the error incurred by an optimal uninformed prediction is given by \( R(1) \). We define the sample complexity \( T_\epsilon \) as the duration required to attain expected average error within some fraction \( \epsilon \geq 0 \) of an optimal uninformed prediction:

\[
T_\epsilon = \min \left\{ T : \frac{R(T)}{T} \leq \epsilon R(1) \right\}.
\]

3. The Rate-Distortion Function and Sample Complexity Bounds

As tools for sample complexity analysis, we define concepts for quantifying uncertainty and information. The entropy \( \mathbb{H}(\mathcal{E}) \) quantifies the agent’s initial degree of uncertainty in terms of the information required to identify \( \mathcal{E} \). We measure information in nats, which are units each equivalent to \( 1/\ln 2 \) bits. For example, if the range of \( \mathcal{E} \) is countable then \( \mathbb{H}(\mathcal{E}) = -\sum_\theta \mathbb{P}(\mathcal{E} = \theta) \ln \mathbb{P}(\mathcal{E} = \theta). \)

Uncertainty at time \( t \) can be expressed in terms of the conditional entropy \( \mathbb{H}(\mathcal{E}|H_t) \), which is the expected number of remaining nats after observing \( H_t \). The mutual information \( \mathbb{I}(\mathcal{E}; H_t) = \mathbb{H}(\mathcal{E}) - \mathbb{H}(\mathcal{E}|H_t) \) quantifies the information about \( \mathcal{E} \) gained from \( H_t \). We provide the following lemma relating error to mutual information.

**Lemma 1 (expected prediction error equals information gain)** For all \( t \in \mathbb{Z}_+ \),

\[
\mathbb{E}[d_{\text{KL}}(P_t^* \parallel \hat{P}_t)] = \mathbb{I}(\mathcal{E}; Y_{t+1}|H_t) \quad \text{and} \quad R(t) = \mathbb{I}(\mathcal{E}; H_t).
\]

**Proof** It is well known that the mutual information \( \mathbb{I}(A; B) \) between random variables \( A \) and \( B \) can be expressed in terms of the expected KL-divergence \( \mathbb{I}(A; B) = \mathbb{E}[d_{\text{KL}}(\mathbb{P}(A \in \cdot|B)\|\mathbb{P}(A \in \cdot))]. \)

It follows that

\[
\mathbb{I}(Y_{t+1}; \mathcal{E}|H_t) = \mathbb{E}[d_{\text{KL}}(\mathbb{P}(Y_{t+1} \in \cdot|\mathcal{E}, H_t) \parallel \mathbb{P}(Y_{t+1} \in \cdot|H_t))]
\]

\[
\overset{(a)}{=} \mathbb{E}[d_{\text{KL}}(\mathbb{P}(Y_{t+1} \in \cdot|\mathcal{E}, X_t) \parallel \mathbb{P}(Y_{t+1} \in \cdot|H_t))]
\]

\[
= \mathbb{E}[d_{\text{KL}}(P_t^* \parallel \hat{P}_t)],
\]

where (a) follows from the fact that \( Y_{t+1} \perp H_t | (\mathcal{E}, X_t) \). We then have

\[
R(T) = \mathbb{E} \left[ \sum_{t=0}^{T-1} d_{\text{KL}}(P_t^* \parallel \hat{P}_t) \right] = \sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \mathcal{E}|H_t) \overset{(a)}{=} \mathbb{I}(\mathcal{E}; H_T),
\]

where (a) follows from the chain rule of mutual information. \( \blacksquare \)

3.1 Environment Proxies and the Rate-Distortion Function

An optimal agent learns from its errors, with each error supplying new information. As such, by Lemma 1, the cumulative error \( R(T) = \mathbb{I}(\mathcal{E}; H_T) \) can be bounded by the expected number \( \mathbb{H}(\mathcal{E}) \) of nats required to identify \( \mathcal{E} \). However, all this information may not be necessary to produce sufficiently accurate predictions. Indeed, accurate predictions can be made even when \( \mathbb{H}(\mathcal{E}) = \infty. \)

A proxy is a random variable \( \tilde{\mathcal{E}} \) for which \( \tilde{\mathcal{E}} \perp H_\infty | \mathcal{E} \), where \( H_\infty = (X_0, Y_1, X_1, Y_2, \ldots) \) represents all data the agent will ever observe. In other words, a proxy \( \tilde{\mathcal{E}} \) can offer an agent information about \( \mathcal{E} \), as well as additional information, so long as that additional information does not
improve predictions. We will denote the set of proxies by $\Theta = \{ \tilde{E} : \tilde{E} \perp H_\infty |\mathcal{E}\}$. While an infinite amount of information must be acquired to identify $\mathcal{E}$ when $H(\mathcal{E}) = \infty$, there can be a proxy $\tilde{E}$ with $H(\tilde{E}) < \infty$ such that learning the proxy suffices to produce accurate predictions. The minimal expected error attainable based on the proxy is achieved by a prediction $\tilde{P}_t = P(Y_{t+1} \in \cdot |\tilde{E}, X_t)$. From here on, we will take $\tilde{P}_t$ to be this prediction, which depends on the proxy denoted by $\tilde{E}$.

Recall that $R(1) = E[\text{KL}(P^*_0 \parallel P_0)]$ is the error incurred by an optimal uninformed prediction. For a prediction conditioned on a proxy $\tilde{E}$, the error becomes $E[\text{KL}(P^*_t \parallel \tilde{P}_t)]$. Let $\Theta_\epsilon = \{ \tilde{E} : \tilde{E} \in \Theta : E[\text{KL}(P^*_t \parallel \tilde{P}_t)] \leq \epsilon R(1) \}$ denote the set of proxies that produce predictions that err by no more than a fraction $\epsilon$ of the error incurred by an uninformed prediction.

The mutual information $I(\mathcal{E} ; \tilde{E})$ is the expected number of nats about $\mathcal{E}$ required to identify the proxy $\tilde{E}$. The rate-distortion function $H_\epsilon(\mathcal{E}) = \inf_{\tilde{E} \in \Theta_\epsilon} I(\mathcal{E} ; \tilde{E})$, characterizes the minimal expected number of nats that must be acquired to identify a proxy in $\Theta_\epsilon$. Perhaps more intuitively, this can be thought of as the amount of information about the environment required to make $\epsilon$-accurate predictions. Even when $H(\mathcal{E})$ is infinite and $\epsilon$ is small, $H_\epsilon(\mathcal{E})$ can be a tractable finite number.

3.2 Sample Complexity Bounds

We characterize fundamental limits of performance by establishing upper and lower bounds on the sample complexity attained by an optimal agent. These bounds are very general, applying to any data generating process. The result bounds sample complexity in terms of the rate-distortion function. As such, for any particular data generating process, sample complexity bounds can be produced by characterizing the associated rate-distortion function.

**Theorem 2 (optimal sample complexity)** For all $\epsilon \geq 0$,

$$\frac{H_\epsilon(\mathcal{E})}{\epsilon R(1)} \leq T_\epsilon \leq \inf_{\delta \in [0, \epsilon]} \left[ \frac{H_{\epsilon - \delta}(\mathcal{E})}{\delta R(1)} \right] \leq \left[ \frac{2H_{\epsilon/2}(\mathcal{E})}{\epsilon R(1)} \right].$$

**Proof** For a proof refer to section B.2 of the appendix.

Note that the sample complexity $T_\epsilon$ of an optimal agent depends on conditional distributions $P_t = P(Y_{t+1} \in \cdot |H_t)$ which may be complicated and difficult to analyze. The power of Theorem 2 is that it allows us to abstract this away and instead characterize the sample complexity in terms of the rate-distortion function, which is often much simpler.

4. Multilayer Environments

We will study the rate-distortion function of an environment $\mathcal{E}$. Until now we considered a sequence $((X_t, Y_{t+1}) : t = 0, 1, \ldots, T - 1)$ of input-output pairs and characterized sample complexity in
terms of the number $T$ of data points required to attain some level of error. As we established, sample complexity is governed by the rate distortion function $\mathcal{H}_\epsilon(\mathcal{E})$. In this section, we will focus on characterizing this rate-distortion function, and for this it suffices to restrict attention to a representative input-output pair rather than a sequence. We will denote this representative pair by $(X, Y)$. The input $X$ is distributed $\mathbb{P}(X \in \cdot) \sim \mathcal{N}(0, I)$, while the conditional distribution of the output is $\mathbb{P}(Y \in \cdot | \mathcal{E}, X) = \mathcal{E}(\cdot | X)$.

For environments we consider in this section, both $X$ and $Y$ take values in $\mathbb{R}^d$, and $Y = f(X) + W$ for a random function $f$ and random vector $W \sim \mathcal{N}(0, \sigma^2 I_d)$. We assume $X$, $f$, and $W$ are independent. The environment is produced by composing $K$ independent and identically distributed random functions: $f = f_K \circ \cdots \circ f_1$. In this sense, the environment is multilayer, with each $k$th layer represented by a function $f_k$. We denote inputs and outputs of these functions by $U_0 = X$ and $U_k = f_k(U_{k-1})$ for $k = 1, \ldots, K$. Hence, $Y = U_K + W$. We assume that, for all $k$, $U_k$ is distributed standard Gaussian in $d$-dimensions. Figure 1 illustrates the structure of such an environment.

Figure 1: A multilayer environment.

Our analysis will relate the rate-distortion function $\mathcal{H}_\epsilon(\mathcal{E})$ of the multilayer environment to that of $K$ single-layer environments. Each such single-layer environment, which we denote by $\mathcal{E}_k$, takes the form $\mathcal{E}_k(\cdot | u) \sim \mathcal{N}(f_k(u), \sigma^2 I_d)$. In other words, conditioned on $\mathcal{E}_k$ and the input $U_{k-1}$, the output of $\mathcal{E}_k$ is distributed according to $U_k + W$.

To frame our results, we define a class of proxies that decompose independently across layers. Recall that an environment proxy of an environment $\mathcal{E}$ is a random variable $\tilde{\mathcal{E}}$ for which $\tilde{\mathcal{E}} \perp H_\infty | \mathcal{E}$. Similarly, an environment proxy of $\mathcal{E}_k$ is a random variable $\tilde{\mathcal{E}}_k$ for which $\tilde{\mathcal{E}}_k \perp H_\infty | \mathcal{E}_k$. This definition allows for dependence between the proxies across layers even though we have assumed environments to be iid across layers. To restrict attention to independent single-layer proxies, we define a multilayer proxy to be a tuple $\tilde{\mathcal{E}} = (\tilde{\mathcal{E}}_1, \ldots, \tilde{\mathcal{E}}_K)$ such that $\tilde{\mathcal{E}}_k \perp (\mathcal{E}_{-k}, \tilde{\mathcal{E}}_{-k}, H_\infty) | \mathcal{E}_k$, where $\mathcal{E}_{-k}$ and $\tilde{\mathcal{E}}_{-k}$ denote tuples of single-layer environments and proxies, with the $k$th omitted.

4.1 A Prototypical Layer

Let us provide as a motivating example a prototypical single-layer environment fits the aforementioned framework and assumptions. Let $g$ be a random function that maps $\mathbb{R}^d$ to $\mathbb{R}^d$. We will characterize the distribution of $g$ through specifying a generative process. First, fix a set of $N$ sign activation units, each taking the form

$$\phi_n(u) = \text{sign}(a_n^T u + b_n),$$

for fixed vectors $a_1, \ldots, a_N$ and scalars $b_1, \ldots, b_N$. Then, we take each $i$th component of the function $g$ to be a linear combination

$$g_i(u) = \sum_{n=1}^N \theta_{i,n} \phi_n(u),$$
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with random coefficients $\theta_{i,1}, \ldots, \theta_{i,n}$. Assume that, for some fixed $M \leq N$, the coefficients are $M$-sparse with probability one. Let each nonzero coefficient be sampled independently from $\mathcal{N}(0,1/M)$. It is easy to see that, for each $u \in \mathbb{R}^d$, $g(u)$ is distributed a standard Gaussian. Hence, as posited by our formulation, $U_k$ is distributed standard Gaussian.

Now suppose that $f_1, \ldots, f_K$ are independent and identically distributed according to $\mathbb{P}(g \in \cdot)$. It is easy to see that, for all $k$, $U_k$ is distributed standard Gaussian. Further, as we establish in Appendix C, the associated single-layer environment $E_k$ is given by the following result:

**Theorem 3 (sparse regression)** For all $\epsilon \geq 0$, the rate-distortion function of our prototypical single-layer environment with input/output dimension $d$, $N$ activation units, and $M$-sparse coefficients satisfies

$$H_s(E_k) = O\left(dM \ln \left(\frac{N}{M\sigma^2\epsilon \ln (1 + 1/\epsilon^2)}\right)\right).$$

This bound on the single-layer rate-distortion function is typical of sparse linear regression models. The logarithmic dependence allows for the number $N$ of activation units to be exponentially large while maintaining tractable sample complexity. Such sparsity is consistent with empirical observations of trained neural network weights (LeCun et al. 1989; Hassibi and Stork 1992; Han et al., 2015). As established by Jones (1992) and Barron (1993), even in the absence of such sparsity, under weaker regularity conditions, sparse weights yield low-error approximations. Desirable properties of our “prototypical layer,” including the sort of rate-distortion bound characterized by Theorem 3, should extend to such cases.

### 4.2 Information Propagation

In order to frame results of this section, we begin by defining some notation and terminology.

For brevity, in the remainder of the paper, for $i \geq j$, let $f_{i:j}$ denote $(f_i, f_{i-1}, \ldots, f_j)$, $E_{i:j}$ denote $(E_i, E_{i-1}, \ldots, E_j)$ and $\hat{E}_{i:j}$ denote $(\hat{E}_i, \hat{E}_{i-1}, \ldots, \hat{E}_j)$. We say a multilayer proxy $\hat{E} = \hat{E}_{K:1}$ is an $\epsilon$-minimal multilayer proxy for a multilayer environment $E = E_{K:1}$ if, for each $k$, $\hat{E}_k$ attains the minimal rate $H_s(E_k)$ among proxies with distortion $I(E_k; U_k + W|\hat{E}_k, U_{k-1}) \leq \epsilon I(E_k; U_k + W|U_{k-1})$. We say a multilayer environment $E = E_{K:1}$ is stable with respect to a multilayer proxy $\hat{E} = \hat{E}_{K:1}$ if, for all $k$,

$$I(E_k; U_k + W|f_{K:k+1}, \hat{E}_k, U_{k-1}) \leq I(E_k; U_k + W|\hat{E}_k, U_{k-1}).$$

To interpret the notion of stability we have defined, let us focus on the special case of $K = 2$ and $k = 1$. Figure 2 illustrates alternative information structures associated with this special case. In each diagram, an agent is uncertain about $f_1$, and its partial information about $f_1$ is represented by a proxy $\hat{E}_1$. Suppose the agent learns from observing an input-output data pair. On the left, the observation is of the output from $f_1$, corrupted by noise. On the right, the output of $f_1$ is passed through a known realization of $f_2$ before being corrupted by noise and observed. In this context, stability implies that, on average over $X$, $\hat{E}_1$, $f_1$, $f_2$, and $W$, what is observed on the left is more informative than the right. In other words, the less processed observation $U_1 + W$ is more informative that the more processed $U_2 + W$. This is natural: the less processed signal more directly informs the agent about what it wants to learn. The concept of stability extends to any $K$ and $k \leq K$.

The following result, which is proved in Appendix C, provides a sufficient condition for stability. This condition pertains the Lipschitz constants $L_1, \ldots, L_K$, of the functions $f_1, \ldots, f_K$. Note that
by Lipschitz constant, we mean the smallest possible for each realized function. As such, each $L_k$ depends on the random function $f_k$ and is therefore a random variable.

**Theorem 4 (Lipschitz continuous layers)** For each $k$, let $L_k$ be the Lipschitz constant of $f_k$. Let $\hat{E}_{k:1}$ be a multilayer proxy such that for each $k$, $U_k$ conditioned on $(\hat{E}_k, U_{k-1})$ is isotropic Gaussian. If, for all $k$, $\mathbb{E}[L_k^2] \leq 1$ then the environment is stable with respect to $\hat{E}_{K:1}$.

Since Lipschitz constants are commonly used in machine learning to characterize smoothness of functions, this result relates stability to the associated literature. In general, assuming that a function being learned has a particular fixed Lipschitz constant is very restrictive because that requires the associated degree of smoothness to apply over all functions that can be realized and over all inputs to the function. Theorem 4 is less restrictive because it treats each Lipschitz constant $L_k$ as a random variable that can depend on $f_k$ and only constrains its second moment $\mathbb{E}[L_k^2]$. Hence, the theorem asserts that an environment can be stable (w.r.t a multilayer proxy) even when realizations of $f_k$ can have arbitrarily large Lipschitz constants. However, the theorem does require this level of smoothness to apply over all inputs and is therefore overly restrictive in that way. Indeed, stability is less restrictive, instead imposing a weaker notion of regularity that applies on average across inputs. The following result establishes that stability of a multilayer proxy is satisfied for another model where, for each realization $f_k$, the Lipschitz constant $L_k$ is infinite.

**Theorem 5 (multivariate threshold layers)** For all $d \in \mathbb{Z}_{+}$, let $(\Theta_k : k = 1, \ldots, K)$ be an sequence of iid matrices in $\mathbb{R}^{d \times d}$ and $(b_k : k = 1, \ldots, K)$ be a fixed sequence of vectors in $\mathbb{R}^d$. Let the elements of $\Theta_k$ be distributed independent $\mathcal{N}(0, \frac{1}{d})$. Let each $k$th layer of the multilayer environment $E$ be governed by the function $f_k(u) = \Theta_k \text{sign}(u + b_k)$. Then, for sufficiently small $\epsilon > 0$, there exists an $\epsilon$-minimal multilayer proxy for which the multilayer environment is stable.

This result is proved in Appendix C. Each function $f_k$ characterized by Theorem 5 represents a threshold activation unit with a random bias and scaling. The Lipschitz constant of $f_k$ is infinite because the sign function switches abruptly from $-1$ to $1$. Yet, this model is stable with respect to an $\epsilon$-minimal multilayer proxy. Intuitively, this is because the sign function is very smooth – in fact, flat – across almost all inputs and only changes abruptly at a single input. Stability allows for such models that are smooth on average instead of uniformly across inputs.

### 4.3 From Single to Multiple Layers

We now consider multilayer environments of the kind described at the beginning of Section 4. In particular, we establish a bound on the rate-distortion function that applies to environments that are stable with respect to an $\epsilon$-minimal multilayer proxy. Before arriving at this result (Theorem 8) we cover two others that serve as stepping stones. We begin with a result on how informativeness of an observed output depends on available information about the input and early layers.
Lemma 6 (informativeness versus available information) Let $\tilde{\mathcal{E}}_{K:1}$ be a multilayer proxy. Then,

$$\mathbb{I}(U_K + W; \mathcal{E}_k|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, U_0) \leq \mathbb{I}(U_K + W; \mathcal{E}_k|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}, U_{k-1}).$$

Proof

$$\mathbb{I}(\mathcal{E}_k; U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, U_0)$$

$$= \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}}_{k-1:1}, U_0, U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}) - \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}}_{k-1:1}, U_0|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k})$$

$$= \mathbb{I}(\mathcal{E}_k; U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}) + \mathbb{I}(\mathcal{E}_k; \tilde{\mathcal{E}}_{k-1:1}, U_0|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}, U_K + W)$$

$$\leq \mathbb{I}(\mathcal{E}_k; U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}) + \mathbb{I}(\mathcal{E}_k; \mathcal{E}_{k-1:1}, U_0|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}, U_K + W)$$

$$= \mathbb{I}(\mathcal{E}_k; \mathcal{E}_{k-1:1}, U_0, U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k})$$

$$\equiv \mathbb{I}(\mathcal{E}_j; U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}, \mathcal{E}_{k-1:1}, X)$$

$$\equiv \mathbb{I}(\mathcal{E}_j; U_K + W|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}, U_{k-1}),$$

where (a) follows from the fact that $\mathcal{E}_k \perp \tilde{\mathcal{E}}_{k-1:1}|(U_0, U_K + W, \mathcal{E}_{k-1:1})$ and the data processing inequality, (b) follows from the fact that $\mathbb{I}(\mathcal{E}_k; \mathcal{E}_{k-1:1}, U_0|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}) = 0$, and (c) follows from the fact that $U_K + W \perp (\mathcal{E}_{k-1:1}, U_0)|U_{k-1}$. 

Lemma 6 states that we learn more information about $\mathcal{E}_k$ when we are given the true input $U_{k-1}$ than when we are given $(U_0, \tilde{\mathcal{E}}_{k-1:1})$ and have to infer $U_{k-1}$. This is intuitive as we should be able to recover more about $\mathcal{E}_k$ when we observe its input exactly. If the multilayer environment is stable with respect to an $\epsilon$-minimal multilayer proxy, this result implies the following relation between distortion induced by a multilayered environment and that associated with each layer.

Corollary 7 (multilayer distortion bound) If $\mathcal{E}$ is a multilayer environment that is stable with respect to an $\epsilon$-minimal multilayer proxy $\tilde{\mathcal{E}}_{K:1}$, then

$$\mathbb{I}(U_K + W; \mathcal{E}_{K:1}|\tilde{\mathcal{E}}_{K:1}, U_0) \leq \sum_{k=1}^{K} \mathbb{I}(U_k + W; \mathcal{E}_k|\tilde{\mathcal{E}}_k, U_{k-1}).$$

Proof We have that

$$\mathbb{I}(U_K + W; \mathcal{E}_{K:1}|\tilde{\mathcal{E}}_{K:1}, U_0) \equiv \sum_{k=1}^{K} \mathbb{I}(U_K + W; \mathcal{E}_k|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{K:1}, U_0)$$

$$\equiv \sum_{k=1}^{K} \mathbb{I}(U_K + W; \mathcal{E}_k|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k:1}, U_0)$$

$$\equiv \sum_{k=1}^{K} \mathbb{I}(U_K + W; \mathcal{E}_k|\mathcal{E}_{K:k+1}, \tilde{\mathcal{E}}_{k}, U_{k-1}),$$

$$\equiv \sum_{k=1}^{K} \mathbb{I}(U_k + W; \mathcal{E}_k|\tilde{\mathcal{E}}_k, U_{k-1}),$$
where (a) follows from the chain rule of mutual information, (b) follows from the fact that \((U_K + W) \perp \tilde{E}_{K;k+1}(U_0,\tilde{E}_{K;k+1})\), (c) follows from Lemma 6 and (d) follows from stability.

Corollary 7 states that if \(\tilde{E}_{K;1}\) holds then we can bound the distortion of a proxy \(\tilde{E}_{K;1}\) of a multilayered process by the sum of the distortions incurred by each single layer process. This suggests that if each element of a multilayer proxy achieves distortion \(\epsilon/K\) in its single-layer environment, the multilayer proxy achieves distortion \(\epsilon\) in the multilayer environment. Our next result establishes our main result, which follows from this fact.

**Theorem 8 (multilayer rate-distortion bound)** If multilayer environment \(E\) of depth \(K\) is stable with respect to an \(\frac{\epsilon}{K}\)-minimal multilayer proxy, then

\[
\mathbb{H}_\epsilon(E) \leq \sum_{k=1}^{K} \mathbb{H}_{\epsilon/K}(E_k).
\]

**Proof** We begin by bounding \(\mathcal{R}(1)\):

\[
\mathcal{R}(1) = \mathbb{I}(U_K + W; \tilde{E}_{K;1}|U_0) \\
= \mathbb{I}(U_K + W; \tilde{E}_K|\tilde{E}_{K-1;1}, U_0) + \mathbb{I}(Y + W; \tilde{E}_{K-1;1}|U_0) \\
\overset{(a)}{=} \mathbb{I}(U_K + W; \tilde{E}_K|U_{K-1}) \\
\overset{(b)}{=} \mathbb{I}(U_k + W; \tilde{E}_k|U_{k-1}) \\
= \mathcal{R}_k(1),
\]

for all \(k\), where (a) follows from the fact that mutual information is non-negative and the fact that \(U_K \perp (\tilde{E}_{K-1;1}, X)|U_{K-1}\) and (b) follows from the fact that the \(f_k\) and \(U_{k-1}\) are iid.

From the assumption that there exists a \(\frac{\epsilon}{K}\)-minimal multilayer proxy \(\tilde{E}_{K;1}\), we have that for each \(k\), if \(\tilde{E}_k\) attains the optimal rate \(\mathbb{H}_{\epsilon/K}(E_k)\) subject to distortion \(\mathbb{I}(U_k + W; \tilde{E}_k|U_{k-1}) \leq \mathcal{R}_k(1)\epsilon/K\), then we have the following upper bound on the distortion of the multilayer process:

\[
\mathbb{I}(U_K + W; \tilde{E}_{K;1}|U_0) \overset{(a)}{\leq} \sum_{k=1}^{K} \mathbb{I}(U_k + W; \tilde{E}_k|U_{k-1}) \\
\overset{(b)}{\leq} \frac{\epsilon}{K} \sum_{k=1}^{K} \mathcal{R}_k(1) \\
= \mathcal{R}_K(1)\epsilon \\
\overset{(c)}{\leq} \mathcal{R}(1)\epsilon,
\]

where (a) follows from corollary 7, (b) follows from the fact that \(\tilde{E}_k\) achieves distortion level \(\mathcal{R}_k(1)\epsilon/K\), and (c) follows from the inequality established above involving \(\mathcal{R}(1)\). This means that \((\tilde{E}_1, \ldots, \tilde{E}_K)\) attains distortion level \(\mathcal{R}(1)\epsilon\) for the multilayer process.
Consequently,

\[ \mathbb{H}_e(E_{K:1}) \leq \sum_{k=1}^{K} \mathbb{H}(E_k; \tilde{E}_k) = \sum_{k=1}^{K} \mathbb{H}_{e/K}(E_k). \]

Recall the prototypical single layer environment introduced in section 4.1. Now, consider a multilayer environment which is the result of composing \( K \) prototypical single-layer environments. As a consequence of Theorem 8, we have the following rate-distortion bound for the multilayer environment (proof provided in Appendix C).

**Theorem 9** If \( E \) is a multilayer environment which consists of prototypical (from section 4.1) single-layer environments \( E_k \) for \( k \in \{1, \ldots, K\} \) each with input and output dimensions \( d \), \( N \) activation units and \( M \)-sparse coefficients, then

\[ \mathbb{H}_e(E_k) = O \left( dKM \ln \left( \frac{NK}{M\sigma^2 \epsilon \ln(1 + \frac{1}{\sigma^2})} \right) \right). \]

For a multilayer environment \( E = (E_1, \ldots, E_K) \) comprised of such single layers, Theorem 9 states that

\[ \mathbb{H}_e(E) = O \left( dKM \ln \left( \frac{NK}{M\sigma^2 \epsilon \ln(1 + \frac{1}{\sigma^2})} \right) \right). \]

A sample complexity bound follows from Theorem 13:

\[ T_\epsilon \leq \left\lceil \frac{2\mathbb{H}_{e/2}(E)}{\epsilon R(1)} \right\rceil = O \left( \frac{KM}{\epsilon \ln(1 + \frac{1}{\sigma^2})} \ln \left( \frac{KN}{M\sigma^2 \epsilon \ln(1 + \frac{1}{\sigma^2})} \right) \right). \]

Inspecting this expression, we see that the sample complexity scales with the depth \( K \) at a rate no faster than \( K \log K \).

**5. Closing Remarks**

We have studied multilayer environments of a sort for which deep neural networks seem to be suited and established results suggesting that sample complexity grows roughly linearly with depth. This is in contrast to previous results that suggest exponential or high-order polynomial dependence. To arrive at our results, we develop general and elegant information-theoretic tools that accommodate analysis of any data generating process and ought to be useful beyond the scope of this paper.

Our analysis extend to variations of the multilayer environments we have considered. For example, if the agent observes \( \arg \max_i Y_i \) rather than \( Y \) itself, we obtain a variant associated with classification instead of regression. The data processing inequality implies that the rate-distortion function for this classification environment obeys the same bound as the regression environment we have studied. This translates to a sample complexity bound as well.
Notably omitted from this paper is any analysis of practical algorithms for estimating the environment. We have assumed perfect Bayesian inference, while in practice, particular neural network architectures are used together with stochastic gradient descent. Whether practical algorithms of this sort can achieve sample complexity bounds similar to what we have established for our multilayer environments remains an interesting subject for future research.

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Appendix A.

A.1 Relating KL-divergence to Common Notions of Error

Example 1 (mean-squared error) Fix \( \mu \in \mathbb{R} \) and \( \sigma^2 \in \mathbb{R}_{++} \). Let \( \mathbb{P}(Y_{t+1} \in \mathcal{E}, X_t) \sim \mathcal{N}(\mu, \sigma^2) \). Consider a point prediction \( \hat{\mu}_t \) that is determined by \( H_t \) and a distributional prediction \( P_t \sim \mathcal{N}(\hat{\mu}_t, \sigma^2) \). Then,

\[
d_{KL}(P_t^* || P_t) = \frac{E[(\mu - \hat{\mu}_t)^2 | \mathcal{E}, H_t]}{2\sigma^2} = \frac{E[(Y_{t+1} - \hat{\mu}_t)^2 | \mathcal{E}, H_t] - E[(Y_{t+1} - \mu)^2 | \mathcal{E}, H_t]}{2\sigma^2}.
\]

Hence, KL-divergence grows monotonically the squared error \( E[(Y_{t+1} - \hat{\mu}_t)^2 | \mathcal{E}, H_t] \). However, while the minimal squared error \( E[(Y_{t+1} - \mu)^2 | \mathcal{E}, H_t] = \sigma^2 \) that is attainable with full knowledge of the environment remains positive, the minimal KL-divergence, which is delivered by \( P_t \sim \mathcal{N}(\mu, \sigma^2) \), is zero.

Now consider a distributional prediction \( P_t \sim \mathcal{N}(\hat{\mu}_t, \hat{\sigma}_t^2) \), based on a variance estimate \( \hat{\sigma}_t^2 \neq \sigma^2 \). Then,

\[
d_{KL}(P_t^* || P_t) = \frac{E[(\mu - \hat{\mu}_t)^2 | \mathcal{E}, H_t]}{2\hat{\sigma}_t^2} = 1 + \frac{1}{2} \left( \frac{\sigma^2}{\hat{\sigma}_t^2} - 1 - \ln \frac{\hat{\sigma}_t^2}{\sigma^2} \right).
\]

Consider optimizing the choice of \( \hat{\sigma}_t^2 \) given \( H_t \):

\[
\min_{\hat{\sigma}_t^2} E[d_{KL}(P_t^* || P_t)]|H_t|.
\]

The minimum is attained by

\[
\hat{\sigma}_t^2 = \sigma^2 + \mathbb{E}[(\mu - \mathbb{E}[\mu | H_t])^2 | H_t] + \mathbb{E}[(\mathbb{E}[\mu | H_t] - \hat{\mu}_t)^2 | H_t],
\]

which differs from \( \sigma^2 \). While \( \sigma^2 \) characterizes aleatoric uncertainty, the incremental variance \( \hat{\sigma}_t^2 - \sigma^2 \) accounts for epistemic uncertainty and bias.

Example 2 (cross-entropy loss) Suppose the set \( \mathcal{Y} \) of responses is finite. Then,

\[
d_{KL}(P_t^* || P_t) = \sum_{y \in \mathcal{Y}} P_t^*(y) \ln \frac{P_t^*(y)}{P_t(y)} = \sum_{y \in \mathcal{Y}} P_t^*(y) \ln P_t^*(y) - \sum_{y \in \mathcal{Y}} P_t^*(y) \ln P_t(y).
\]

The first term does not depend on \( P_t \), so minimizing KL-divergence is equivalent to minimizing the final term,

\[
-\sum_{y \in \mathcal{Y}} P_t^*(y) \ln P_t(y) = -E[\ln(P_t(Y_{t+1})|\mathcal{E}, P_t, X_t)],
\]

which is exactly the expected cross-entropy loss of \( P_t \), as is commonly used to assess classifiers.

A.2 Proof of Optimal Predictions

Recall that an agent is characterized by a function \( \pi \), which generates predictions \( P_t = \pi(H_t, Z_t) \), where \( Z_t \) represents algorithmic randomness. The following result establishes that the conditional distribution \( P_t = \mathbb{P}(Y_{t+1} \in \cdot | H_t) \) offers an optimal prediction under our notion of error.
**Theorem 10 (optimal prediction)** For all $t \geq 0$,

$$
\mathbb{E}[d_{KL}(P_t^\star \| \hat{P}_t) \mid H_t] = \inf_\pi \mathbb{E}[d_{KL}(P_t^\star \| P_t) \mid H_t],
$$

where $P_t = \pi(H_t, Z_t)$.

**Proof** Let $\hat{P}_t = \mathbb{P}(Y_{t+1} \in \cdot | H_t)$. By Gibbs' inequality,

$$
\inf_{\hat{P}_t} d_{KL}(\hat{P}_t \| P_t) = d_{KL}(\hat{P}_t \| \hat{P}_t) = 0.
$$

Let $P_t^\star = \mathbb{P}(Y_{t+1} \in \cdot | \mathcal{E}, X_t)$. Then, for all $P_t$,

$$
d_{KL}(P_t^\star \| P_t) = \mathbb{E} \left[ \ln \frac{dP_t^\star}{d\hat{P}_t}(Y_{t+1}) \mid \mathcal{E}, H_t \right]
= \mathbb{E} \left[ \ln dP_t^\star(Y_{t+1}) \mid \mathcal{E}, H_t \right] - \mathbb{E} \left[ \ln dP_t(Y_{t+1}) \mid \mathcal{E}, H_t \right]
= \mathbb{E} \left[ \ln dP_t^\star(Y_{t+1}) \mid \mathcal{E}, H_t \right] - \mathbb{E} \left[ \ln \hat{P}_t(Y_{t+1}) \mid \mathcal{E}, H_t \right]
+ \mathbb{E} \left[ \ln \frac{d\hat{P}_t}{dP_t}(Y_{t+1}) \mid \mathcal{E}, H_t \right]
= \mathbb{E} \left[ \ln \frac{dP_t^\star}{d\hat{P}_t}(Y_{t+1}) \mid \mathcal{E}, H_t \right] + \mathbb{E} \left[ \ln \frac{d\hat{P}_t}{dP_t}(Y_{t+1}) \mid \mathcal{E}, H_t \right].
$$

It follows that

$$
\inf_\pi \mathbb{E}[d_{KL}(P_t^\star \| P_t) \mid H_t] = \inf_\pi \mathbb{E} \left[ d_{KL}(P_t^\star \| \hat{P}_t) + \mathbb{E} \left[ \ln \frac{d\hat{P}_t}{dP_t}(Y_{t+1}) \mid \mathcal{E}, H_t \right] \mid H_t \right]
= \mathbb{E}[d_{KL}(P_t^\star \| \hat{P}_t) \mid H_t] + \inf_\pi \mathbb{E}[d_{KL}(\hat{P}_t \| P_t) \mid H_t]
= \mathbb{E}[d_{KL}(P_t^\star \| \hat{P}_t) \mid H_t] + \mathbb{E}[d_{KL}(\hat{P}_t \| P_t) \mid H_t]
= \mathbb{E}[d_{KL}(P_t^\star \| \hat{P}_t) \mid H_t].
$$

**Appendix B.**

**B.1 Proof of Results Pertaining to Error and Mutual Information**

The agent’s ability to predict tends to improve as it learns from experience. This is formalized by the following result, which establishes that expected prediction errors are monotonically nonincreasing.

**Lemma 11 (expected prediction error is monotonically nonincreasing)** For all $t \in \mathbb{Z}_+$,

$$
\mathbb{E}[d_{KL}(P_t^\star \| \hat{P}_t)] \geq \mathbb{E}[d_{KL}(P_{t+1}^\star \| \hat{P}_{t+1})].
$$
Proof We have
\[ \mathbb{E}[d_{KL}(P^*_t || \hat{P}_t)] \stackrel{(a)}{=} \mathbb{I}(E; Y_{t+2} | H_{t+1}) \]
\[ = \mathbb{I}(E; Y_{t+2} | H_{t-1}, Y_t, X_t, Y_{t+1}, X_{t+1}) \]
\[ \leq \mathbb{I}(E; Y_{t+2} | H_{t-1}, Y_t, X_{t+1}) \]
\[ = \mathbb{I}(E; Y_{t+1} | H_t) \]
\[ \leq \mathbb{I}(E; Y_{t+1} | H_t - 1, Y_t, X_t) \]
\[ = \mathbb{I}(E; Y_{t+1} | H_t) \],
where \( (a) \) follows from Lemma 1, \( (b) \) follows from the fact that conditioning cannot increase mutual information, \( (c) \) follows from the fact that \((X_t, Y_{t+1})\) and \((X_{t+1}, Y_{t+2})\) are independent and identically distributed conditioned on \((H_{t-1}, Y_t)\), and \( (d) \) follows from the equivalence between mutual information and expected KL-divergence.

The following result relates the expected error of a prediction based on proxy \( \tilde{E} \) to a particular conditional mutual information.

Lemma 12 (proxy error equals response information) For all \( \tilde{E} \in \Theta \),
\[ \mathbb{E}[d_{KL}(P^*_t || \tilde{P}_t)] = \mathbb{I}(E; Y_1 | \tilde{E}, X_0). \]

Proof As in the proof of Lemma 1, by using the same relation between expected KL-divergence and mutual information, we have
\[ \mathbb{I}(E; Y_1 | \tilde{E}, X_0) = \mathbb{E}[d_{KL}(P(Y_1 \in \cdot | E, \tilde{E}, X_0) || P(Y_1 \in \cdot | \tilde{E}, X_0))] \]
\[ \stackrel{(a)}{=} \mathbb{E}[d_{KL}(P(Y_1 \in \cdot | E, X_0) || P(Y_1 \in \cdot | \tilde{E}, X_0))] \]
\[ = \mathbb{E}[d_{KL}(P^*_t || \tilde{P}_t)] \],
where \( (a) \) follows from the fact that \((X_0, Y_1) \perp \tilde{E} | E). \]

B.2 Proof of Results Pertaining to Sample Complexity Bounds
The following result brackets the cumulative error of optimal predictions.

Theorem 13 (optimal cumulative error) For all \( \tilde{E} \in \Theta \),
\[ \sup_{\epsilon \geq 0} \min \{ \mathbb{H}_\epsilon(\mathcal{E}), \epsilon \mathcal{R}(1)T \} \leq \mathcal{R}(T) \leq \inf_{\epsilon \geq 0} \{ \mathbb{H}_\epsilon(\mathcal{E}) + \epsilon \mathcal{R}(1)T \}. \]

Proof We begin by establishing the upper bound. By Lemma 1, for each \( t \),
\[ \mathbb{E} \left[ d_{KL}(P^*_t || \hat{P}_t) \right] = \mathbb{I}(Y_{t+1}; E | H_t) \]
\[ = \mathbb{I}(Y_{t+1}; E, \tilde{E} | H_t) \]
\[ \stackrel{(a)}{=} \mathbb{I}(Y_{t+1}; \tilde{E} | H_t) + \mathbb{I}(Y_{t+1}; E | \tilde{E}, H_t) \]
\[ \leq \mathbb{I}(Y_{t+1}; \tilde{E} | H_t) + \mathbb{I}(Y_{t+1}; E | \tilde{E}, X_t), \]
where (a) and (b) follow from the chain rule and data processing inequality, respectively. Another
application of the chain rule and data processing inequality gives us
\[
\sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; \tilde{E} | H_t) = \mathbb{I}(H_T; \tilde{E}) \leq \mathbb{I}(E; \tilde{E}).
\]  
(2)

If \( \tilde{E} \in \Theta_e \) then
\[
\mathbb{I}(Y_{t+1}; E | \tilde{E}, H_t) \leq \mathbb{I}(Y_{t+1}; E | \tilde{E}, X_t) = \mathbb{E}[d_{KL}(P^*_t \| \tilde{P}_t)] \leq \epsilon \mathbb{I}(E; Y_1 | X_0),
\]
and it follows that
\[
\sum_{t=0}^{T-1} \mathbb{I}(Y_{t+1}; E | \tilde{E}, H_t) \leq \epsilon \mathbb{I}(E; Y_1 | X_0)T = \epsilon \mathcal{R}(1)T.
\]  
(3)

It follows from (1), (2), and (3), that
\[
\mathcal{R}(T) \leq \mathbb{I}(E; \tilde{E}) + \frac{T}{2} \ln \left( 1 + \frac{\epsilon \mathcal{R}(1)}{\sigma^2} \right).
\]  
(4)

Since (4) holds for all \( \epsilon \geq 0 \) and \( \tilde{E} \in \Theta_e \), the result follows.

Next, we establish the lower bound. Fix \( T \in \mathbb{Z}^+ \). Let \( \tilde{E} = (\tilde{H}_{T-2}, \tilde{Y}_{T-1}) \) be independent from but distributed identically with \( (H_{T-2}, Y_{T-1}) \), conditioned on \( E \). In other words, \( \tilde{E} \perp (H_{T-2}, Y_{T-1}) | E \) and \( \mathbb{P}(\tilde{E} \in \cdot | E) = \mathbb{P}((H_{T-2}, Y_{T-1}) \in \cdot | E) \). This implies that
\[
\mathbb{P}((E, \tilde{H}_{T-2}, \tilde{Y}_{T-1}, X_{T-1}, Y_T) \in \cdot) = \mathbb{P}((E, H_{T-2}, Y_{T-1}, X_{T-1}, Y_T) \in \cdot),
\]
and therefore, \( \mathbb{I}(E; Y_T | H_{T-1}) = \mathbb{I}(E; Y_T | \tilde{E}, X_{T-1}) \).

Fix \( \epsilon \geq 0 \). If \( \mathcal{R}(T) < \mathbb{I}_e(E) \) then \( \tilde{E} \notin \Theta_e \) and
\[
\mathcal{R}(T) = \mathbb{I}(E; H_T)
\]  
(a)
\[
\leq \sum_{t=0}^{T-1} \mathbb{I}(E; Y_{t+1} | H_t)
\]  
(b)
\[
\geq \mathbb{I}(E; Y_T | H_{T-1})T
\]  
(c)
\[
= \mathbb{I}(E; Y_T | \tilde{E}, X_{T-1})T
\]  
(d)
\[
= \mathbb{E}[d_{KL}(P^*_T \| \tilde{P}_T)]T
\]  
(d)
\[
> \epsilon \mathcal{R}(1)T,
\]  
(e)
where (a) follows from Lemma 1, (b) follows from the chain rule of mutual information, (c) follows
from Lemma 11 (d) follows from Lemma 12, and (e) follows from the fact that \( \tilde{E} \notin \Theta_e \). Therefore,
\[
\mathcal{R}(T) \geq \min\{\mathbb{I}_e(E), \epsilon \mathcal{R}(1)T\}.
\]

Since this holds for any \( \epsilon \geq 0 \), the result follows.
This upper bound is intuitive. An uninformed agent incurs error $\mathcal{R}(1)$. Knowledge of a proxy $\hat{E} \in \Theta_\epsilon$ enables an agent to limit prediction error to $\epsilon \mathcal{R}(1)$. Getting to that level of prediction error requires $\mathcal{H}_\epsilon(\hat{E})$ nats, and therefore, that much cumulative error. Hence, $\mathcal{R}(T) \leq \mathcal{H}_\epsilon(\hat{E}) + \epsilon \mathcal{R}(1)T$.

To motivate the lower bound, note that an agent requires $\mathcal{H}_\epsilon(\hat{E})$ nats to attain per timestep error within $\epsilon \mathcal{R}(1)$. Obtaining those nats requires cumulative error at least $\mathcal{H}_\epsilon(\hat{E})$. So the agent either incurs that or, alternatively, $\epsilon \mathcal{R}(1)$ error per timestep.

We now provide a proof for Theorem 2 from the main text. It follows almost as a direct consequence of Theorem 13.

**Theorem 2 (optimal sample complexity)** For all $\epsilon \geq 0$,

$$\frac{\mathcal{H}_\epsilon(\hat{E})}{\epsilon \mathcal{R}(1)} \leq T_\epsilon \leq \inf_{\delta \in [0, \epsilon]} \left[ \frac{\mathcal{H}_{\epsilon-\delta}(\hat{E})}{\delta \mathcal{R}(1)} \right] \leq \left\lceil \frac{2 \mathcal{H}_\epsilon(\hat{E})}{\epsilon \mathcal{R}(1)} \right\rceil$$.

**Proof** We begin by showing the upper bound. Fix $\epsilon \geq 0$ and $\delta \in [0, 1]$. Let

$$T = \left\lceil \frac{\mathcal{H}_{\epsilon-\delta}(\hat{E})}{\delta \mathcal{R}(1)} \right\rceil,$$

so that $\mathcal{H}_{\epsilon-\delta}(\hat{E}) \leq \delta \mathcal{R}(1)T$. It follows from the upper bound of Theorem 13 that

$$\mathcal{R}(T) \leq \mathcal{H}_{\epsilon-\delta}(\hat{E}) + (\epsilon - \delta)\mathcal{R}(1)T \leq \delta \mathcal{R}(1)T + (\epsilon - \delta)\mathcal{R}(1)T = \epsilon \mathcal{R}(1)T.$$

Since $T_\epsilon = \min\{T : \mathcal{R}(T) \leq \epsilon \mathcal{R}(1)T\}$, it follows that $T \geq T_\epsilon$. The result follows.

We now show the lower bound. Fix $\epsilon \geq 0$. By the definition of $T_\epsilon$, we have

$$\mathcal{R}(T_\epsilon) \leq \epsilon T_\epsilon \mathcal{R}(1).$$

In the proof of the lower bound in Theorem 13, we show that for all $\epsilon \geq 0$, $\mathcal{R}(T) < \mathcal{H}_\epsilon(\hat{E}) \implies \mathcal{R}(T) > \epsilon \mathcal{R}(1)T$. Therefore, using the contrapositive and the above definition of $T_\epsilon$, we have that $\mathcal{H}_\epsilon(\hat{E}) \leq \epsilon \mathcal{R}(1)T_\epsilon$ and therefore

$$\mathcal{H}_\epsilon(\hat{E}) \leq \mathcal{R}(T_\epsilon) \leq \epsilon T_\epsilon \mathcal{R}(1).$$

The result follows.

**Appendix C. Proofs of Results Pertaining to Multilayer Environments**

We establish a bound on the rate-distortion function for the sort of single-layer environment described in Section 4.1.

**Theorem 3 (sparse regression)** For all $\epsilon \geq 0$, the rate-distortion function of our prototypical single-layer environment with input/output dimension $d$, $N$ activation units, and $M$-sparse coefficients satisfies

$$\mathcal{H}_\epsilon(\mathcal{E}_k) = O \left( dM \ln \left( \frac{N}{M\sigma^2 \epsilon \ln(1 + \frac{1}{\sigma^2})} \right) \right) .$$
**Proof** Let $\mathcal{E} = (S, \theta)$ where $\theta = (\theta_{i,n} : 1 \leq i \leq d, 1 \leq n \leq N)$ and $S$ specifies which coefficients among $\theta$ are nonzero. We first determine an expression for $R(1)$.

$$R(1) = \mathbb{I}(g(U) + W; \mathcal{E}|U)$$
$$= h(g(U) + W|U) - h(g(U) + W|\mathcal{E}, U)$$
$$= h(g(U) + W|U) - h(W)$$
$$= \frac{d}{2} \ln \left( 2\pi e (\sigma^2 + 1) \right) - \frac{d}{2} \ln \left( 2\pi e \sigma^2 \right)$$
$$= \frac{d}{2} \ln \left( 1 + \frac{1}{\sigma^2} \right),$$

where (a) follows from the fact that for all realizations $u$ of the input $U$, $g_i(u) = \sum_{n=1}^{N} \theta_{i,n} \phi_n(u)$ is distributed standard Gaussian and independent of $W$.

Now, consider a proxy $\hat{\mathcal{E}} = (S, \hat{\theta})$. Let $\hat{\theta}$ be defined that for each $(i, n)$ such that $\theta_{i,n}$ is nonzero, $\theta_{i,n} = \hat{\theta}_{i,n} + Z_{i,n}$ where $Z_{i,n}$ is independent of $\hat{\mathcal{E}}_{i,n}$ and distributed $\mathcal{N} \left( 0, \frac{\sigma^2}{M} \left( e^{2R(1)\epsilon} - 1 \right) \right)$. For $(i, n)$ such that $\theta_{i,n}$ is zero, let $\hat{\theta}_{i,n} = 0$. Then we have the following:

$$\mathbb{I}(g(U) + W; \mathcal{E}|\hat{\mathcal{E}}, U) = h(g(U) + W|\hat{\mathcal{E}}, U) - h(g(U) + W|\mathcal{E}, U)$$
$$= h(g(U) + W|\hat{\mathcal{E}}, U) - h(W)$$
$$\leq \mathbb{E} \left[ \frac{d}{2} \ln \left( \frac{1}{d} \sum_{n=1}^{N} \phi_n^2(U) \right) \right] - h(W)$$
$$\overset{(a)}{=} \mathbb{E} \left[ \frac{d}{2} \ln \left( \frac{1}{d} \sum_{n=1}^{N} \phi_n^2(U) \right) \right]$$
$$\overset{(b)}{=} \mathbb{E} \left[ \frac{d}{2} \ln \left( 1 + \frac{dM}{d^2} \mathbb{E} \left[ \sum_{i=1}^{d} \sum_{n=1}^{N} Z_{i,n}^2 \right] \right) \right]$$
$$\overset{(c)}{=} \frac{d}{2} \ln \left( 1 + \frac{dM}{d^2} \mathbb{E} \left[ \sum_{i=1}^{d} \sum_{n=1}^{N} Z_{i,n}^2 \right] \right)$$
$$\overset{(d)}{=} R(1) \epsilon,$

where (a) follows from maximum differential entropy of multivariate Gaussian, (b) follows from the fact that $Z \perp \hat{\theta}$, (c) follows from the fact that $\phi_n^2(U) = 1$ with probability one, and (d) follows from the fact that $Z_{i,n} \sim \mathcal{N} \left( 0, \frac{\sigma^2}{M} \left( e^{2R(1)\epsilon} - 1 \right) \right)$. 


Therefore, \( \tilde{E} \) satisfies the distortion constraint and we have the following upper bound on the rate of \( \tilde{E} \):

\[
\mathbb{H}(\tilde{E}; \mathbb{E}) = \mathbb{H}((S, \theta); (S, \tilde{\theta}))
= \mathbb{H}(S) + \mathbb{H}(\theta; \tilde{\theta}|S)
\leq (a) \leq dM \ln \frac{eN}{M} + \mathbb{H}(\theta; \tilde{\theta}|S)
= dM \ln \frac{eN}{M} + \frac{dM}{2} \ln \left( \frac{1}{\mathbb{V}[Z_{i,n}]} \right)
= dM \ln \frac{eN}{M} + \frac{dM}{2} \ln \left( \frac{1}{\sigma^2 \left( e^{\frac{2R(1)\epsilon}{d} - 1} \right)} \right)
\leq dM \ln \frac{eN}{M} + \frac{dM}{2} \ln \left( \frac{d}{2\sigma^2 R(1)\epsilon} \right)
= dM \ln \frac{eN}{M} + \frac{dM}{2} \ln \left( \frac{1}{\sigma^2 \epsilon \ln \left( 1 + \frac{1}{\sigma^2} \right)} \right)
\]

where \((a)\) follows from the fact that the coefficients are \(M\)-sparse and the fact that \( \left( \frac{N}{M} \right) \leq \left( \frac{eN}{M} \right)^M \).

The result follows. \(\blacksquare\)

The following lemma is crucial for proving Theorems 4 and 5.

**Lemma 14** For all \( K \in \mathbb{Z}_+ \), let \( (f_1, \ldots, f_K) \) be a sequence of random functions. For \( j \geq i \) let \( f_{j:i} \) denote \( (f_j, f_{j-1}, \ldots, f_i) \). Then for all \( k \in \{1, \ldots, K\} \), if \( \tilde{E}_k \) is a proxy such that with probability 1,

\[
\mathbb{E} \left[ \left\| U_K - \mathbb{E} \left[ U_K \mid \tilde{E}_k, f_{K:k+1}, U_{k-1} \right] \right\|_2^2 \mid \tilde{E}_k, U_{k-1} \right] \leq \mathbb{E} \left[ \left\| U_k - \mathbb{E} [U_k \mid \tilde{E}_k, U_{k-1}] \right\|_2^2 \mid \tilde{E}_k, U_{k-1} \right],
\]

and if \( U_k \) is isotropic Gaussian conditioned on \((U_{k-1}, \tilde{E}_k)\), then

\[
\mathbb{I} (\tilde{E}_k; U_K + W \mid f_{K:k+1}, \tilde{E}_k, U_{k-1}) \leq \mathbb{I} (U_k + W; \tilde{E}_k, U_{k-1})
\]

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Proof  We provide a proof by contradiction. Assume that $\mathbb{I}(U_K + W; \xi_k| f_{K:k+1}, \tilde{\xi}_k, U_{k-1}) > \mathbb{I}(U_k + W; \xi_k| \tilde{\xi}_k, U_{k-1})$ Then, we have that

$$
\begin{align*}
&\mathbb{E} \left[ d \ln \left( \sigma^2 + \frac{\mathbb{E} \left[ \|U_K - \mathbb{E}[U_K| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}]\|^2 \| \tilde{\xi}_k, U_{k-1} \| \right]}{d} \right) \right] \\
&\geq \mathbb{E} \left[ d \ln \left( \sigma^2 + \frac{\mathbb{E} \left[ \|U_K - \mathbb{E}[U_K| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}]\|^2 \| \tilde{\xi}_k, f_{K:k+1}, U_{k-1} \| \right]}{d} \right) \right] \\
&\geq 2h(U_K - \mathbb{E}[U_K| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}] + W| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}) - d \ln(2\pi e) \\
&= 2h(U_K + W| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}) - d \ln(2\pi e) \\
&= 2\mathbb{I}(U_k + W; \xi_k| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}) - h(W) - d \ln(2\pi e) \\
&> 2\mathbb{I}(U_k + W; \xi_k| \tilde{\xi}_k, U_{k-1}) - h(W) - d \ln(2\pi e) \\
&= 2h(U_k + W| \tilde{\xi}_k, U_{k-1}) - d \ln(2\pi e) \\
&= 2h(U_k - \mathbb{E}[U_k| \tilde{\xi}_k, X] + W| \tilde{\xi}_k, X) - d \ln(2\pi e) \\
&\geq 0
\end{align*}
$$

where (a) follows from Jensen’s inequality, (b) follows from maximum differential entropy of an isotropic Gaussian random variable, (c) follows from the assumption, and (d) follows from the assumption that conditioned on $(X, \tilde{\xi})$, $f_1(X)$ is and isotropic Gaussian with probability 1.

However, this is a contradiction since we assumed with probability 1 that

$$
\mathbb{E} \left[ \|U_K - \mathbb{E}[U_K| \tilde{\xi}_k, f_{K:k+1}, U_{k-1}]\|^2 \| \tilde{\xi}_k, U_{k-1} \| \right] \leq \mathbb{E} \left[ \|U_k - \mathbb{E}[U_k| \tilde{\xi}_k, U_{k-1}]\|^2 \| \tilde{\xi}_k, U_{k-1} \| \right].
$$

We now provide a proof for Theorem 4 which shows that multilayer environments consisting of single-layer functions satisfying a particular Lipschitz property are stable with respect to multilayer proxies that satisfy a particular single-layer Gaussian condition.

**Theorem 4 (Lipschitz continuous layers)** For each $k$, let $L_k$ be the Lipschitz constant of $f_k$. Let $\tilde{\xi}_{K:1}$ be a multilayer proxy such that for each $k$, $U_k$ conditioned on $(\tilde{\xi}_k, U_{k-1})$ is isotropic Gaussian. If, for all $k$, $\mathbb{E}[L_k^2] \leq 1$ then the environment is stable with respect to $\tilde{\xi}_{K:1}$. 

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**Sample Complexity versus Depth**

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Proof If \( K = 1 \), then the stated inequality holds trivially with equality. Fix \( K \in \mathbb{Z}_{++} \geq 2 \). Then for any \( k \in \{1, \ldots, K\} \), we have the following:

\[
\begin{align*}
\mathbb{E} \left[ \left( U_K - \mathbb{E}[U_K|\tilde{E}_k, f_{K:k+1}, U_{k-1}] \right)^2 | \tilde{E}_k, U_{k-1} \right] \\
&\leq \mathbb{E} \left[ \left( U_K - f_K(\mathbb{E}[U_{K-1}|\tilde{E}_k, f_{K:k+1}, U_{k-1}]) \right)^2 | \tilde{E}_k, U_{k-1} \right] \\
&\leq \mathbb{E} \left[ L^2_K \left( U_{K-1} - \mathbb{E}[U_{K-1}|\tilde{E}_k, f_{K:k+1}, U_{k-1}] \right)^2 | \tilde{E}_k, U_{k-1} \right] \\
&= \mathbb{E} \left[ L^2_K \right] \mathbb{E} \left[ \left( U_{K-1} - \mathbb{E}[U_{K-1}|\tilde{E}_k, f_{K:k+1}, U_{k-1}] \right)^2 | \tilde{E}_k, U_{k-1} \right] \\
&\leq \left( \prod_{j=k+1}^{K} \mathbb{E} \left[ L^2_j \right] \right) \mathbb{E} \left[ \left( U_k - \mathbb{E}[U_k|\tilde{E}_k, U_{k-1}] \right)^2 | \tilde{E}_k, U_{k-1} \right] , \\
&\leq \mathbb{E} \left[ \left( U_k - \mathbb{E}[U_k|\tilde{E}_k, U_{k-1}] \right)^2 | \tilde{E}_k, U_{k-1} \right] ,
\end{align*}
\]

where \((a)\) follows from the fact that the conditional mean minimizes mean squared error, \((b)\) follows from the fact that \( f_K \) is \( L_K \)-Lipschitz, \((c)\) follows from repeatedly applying \((a)\) and \((b)\), and \((d)\) follows from the fact that for each \( k \in \{1, \ldots, K\} \), \( \mathbb{E}[L^2_k] \leq 1 \).

The result follows from Lemma 14.

Before establishing Theorem 5, we provide the following result which derives an \( \epsilon \)-minimal multilayer proxy for this threshold multilayer environment. Note that in the result below, we use \( \mathcal{R}_k(1) \) to denote \( I(U_k + W; \tilde{E}_k|U_{k-1}) \).

Lemma 15 Let \( \Theta_k \in \mathbb{R}^{d \times d} \) be a random matrix and \( b_k \in \mathbb{R}^d \) be a fixed vector. Let the elements of \( \Theta_k \) distributed iid \( \mathcal{N}(0, \frac{1}{d}) \). Let \( f_k(u) = \Theta_k \text{sign}(u + b_k) \). Then \( \tilde{E}_k;1 \) such that for each \( k \),

\[
\Theta_k = \tilde{E}_k + Z_k \quad \text{where} \quad Z_k \perp \tilde{E}_k \quad \text{and} \quad Z_k \in \mathbb{R}^{d \times d} \quad \text{with elements distributed iid} \quad \mathcal{N} \left( 0, \frac{1}{d} \frac{\mathcal{R}_k(1)_{\epsilon - 1}}{\epsilon} \right)
\]

is an \( \epsilon \)-minimal multilayer proxy.

Proof Firstly, we establish the following representation of \( \mathcal{R}_k(1) \):

\[
\mathcal{R}_k(1) = I(U_k + W; \tilde{E}_k|U_{k-1}) \\
= h(U_k + W|U_{k-1}) - h(U_k + W|\tilde{E}_k, U_{k-1}) \\
= h(\Theta_k 1 + W) - h(W) \\
= \frac{d}{2} \ln \left( 2 \pi e \left( 1 + \sigma^2 \right) \right) - \frac{1}{2} \ln \left( 2 \pi e \sigma^2 \right) \\
= \frac{d}{2} \ln \left( 1 + \frac{1}{\sigma^2} \right).
\]
where $(a)$ follows from the fact that with probability 1, $U_k$ conditioned on $U_{k-1}$ is distributed standard Gaussian.

We now provide a lower bound for the distortion constraint:

$$I(U_k + W; E_k | \hat{E}_k, U_{k-1}) = h(U_k + W | \hat{E}_k, U_{k-1}) - h(U_k + W | E_k, \hat{E}_k, U_{k-1})$$

$$= h(U_k + W | \hat{E}_k, U_{k-1}) - h(W)$$

$$(a) \geq \frac{d}{2} \ln \left( e^{\frac{2}{d} h(W)} + e^{\frac{2}{d} h(U_k | \hat{E}_k, U_{k-1})} \right) - \frac{d}{2} \ln \left( 2\pi e \sigma^2 \right)$$

$$= \frac{d}{2} \ln \left( 1 + \frac{e^{\frac{2}{d} h(U_k | \hat{E}_k, U_{k-1})}}{2\pi e \sigma^2} \right)$$

$$(b) = \frac{d}{2} \ln \left( 1 + \frac{e^{\frac{2}{d} h(\Theta_k | \hat{E}_k)}}{2\pi e \sigma^2} \right),$$

where $(a)$ follows from the entropy power inequality and $(b)$ follows from the fact that $U_k = \Theta_k \text{sign}(U_{k-1} + b_k)$ and so the conditional differential entropy only depends on $\Theta_k | \hat{E}$. Tells us that a necessary condition for $I(U_k + W; E_k | \hat{E}_k, U_{k-1}) \leq \epsilon R_k(1)$ to hold is that

$$h(\Theta_k | \hat{E}_k) \leq \frac{d}{2} \ln \left( 2\pi e \sigma^2 \left( e^{\frac{2}{d} R_k(1) \epsilon} - 1 \right) \right).$$

Now, we lower bound the rate of a proxy that satisfies the distortion constraint:

$$I(E_k; \hat{E}_k) = I(\Theta_k; \hat{E}_k)$$

$$= h(\Theta_k) - h(\Theta_k, b_k | \hat{E}_k)$$

$$= h(\Theta_k) - h(\Theta_k | \hat{E}_k)$$

$$(a) \geq \frac{d^2}{2} \ln \left( 2\pi e \frac{1}{d} \right) - h(\Theta_k | \hat{E}_k)$$

$$\geq \frac{d^2}{2} \ln \left( 2\pi e \frac{1}{d} \right) - \frac{d^2}{2} \ln \left( 2\pi e \sigma^2 \left( e^{\frac{2}{d} R_k(1) \epsilon} - 1 \right) \right)$$

$$= \frac{d^2}{2} \ln \left( \frac{1}{d \sigma^2 \left( e^{\frac{2}{d} R_k(1) \epsilon} - 1 \right)} \right)$$

$$= \frac{d^2}{2} \ln \left( \frac{e^{\frac{2}{d} R_k(1) \epsilon} - 1}{d \left( e^{\frac{2}{d} R_k(1) \epsilon} - 1 \right)} \right),$$

where $(a)$ follows from the above necessary condition. This quantity therefore serves as a lower bound on the rate of any proxy $\hat{E}_k$ that attains distortion at most $R_k(1) \epsilon$. 

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Now, consider \( \tilde{\Theta}_k \) such that \( \Theta_k = \tilde{\Theta}_k + Z_k \) where \( Z_k \perp \tilde{\Theta}_k \) and \( Z_k \in \mathbb{R}^{d \times d} \) with elements distributed iid \( \mathcal{N}\left(0, \frac{\sigma^2 R_k(1) - 1}{d} \right) \). The distortion of this proxy is

\[
I(U_k + W; \mathcal{E}_k|\tilde{\mathcal{E}}_k, U_{k-1}) = h(U_k + W|\tilde{\mathcal{E}}_k, U_{k-1}) - h(U_k + W|\mathcal{E}_k, \tilde{\mathcal{E}}_k, U_{k-1})
\]

\[
= (a) h(Z_{k1} + W|\tilde{\mathcal{E}}_k, U_{k-1}) - h(W)
\]

\[
= (b) h(Z_{k1} + W) - h(W)
\]

\[
= (c) \frac{d}{2} \ln \left( e^{\frac{2}{d} h(W)} + e^{\frac{2}{d} h(Z_{k1})} \right) - \frac{d}{2} \ln \left( 2\pi e \right)
\]

\[
= \mathcal{R}_k(1) \epsilon,
\]

where (a) follows from how \( \tilde{\Theta}_k \) is defined and the fact that \( |\text{sign}(U_{k-1} + b_k)| = 1 \) with probability 1, (b) follows from the fact that \( \tilde{\Theta}_k \perp Z_k \), and (c) follows from the entropy power inequality being met with equality for independent Gaussian random variables.

The Rate of this proxy is:

\[
I(\mathcal{E}_k; \tilde{\mathcal{E}}_k) = I(\Theta_k, b_k; \tilde{\mathcal{E}}_k)
\]

\[
= I(\Theta_k; \tilde{\mathcal{E}}_k)
\]

\[
= h(\Theta_k) - h(\Theta_k|\tilde{\mathcal{E}}_k)
\]

\[
= h(\Theta_k) - h(Z_k|\tilde{\Theta}_k)
\]

\[
= h(\Theta_k) - h(Z_k)
\]

\[
= \frac{d^2}{2} \ln(2\pi e) - \frac{d^2}{2} \ln \left( \frac{e^{\frac{2}{d} R_k(1)|x| - 1}}{d \left( e^{\frac{2}{d} R_k(1)} - 1 \right)} \right)
\]

\[
= \frac{d^2}{2} \ln \left( \frac{e^{\frac{2}{d} R_k(1)} - 1}{d \left( e^{\frac{2}{d} R_k(1)|x|} - 1 \right)} \right).
\]

So we have found a proxy \( \tilde{\Theta}_k \) that achieves the lower bound with equality. It therefore achieves the optimal rate at distortion level \( \mathcal{R}_k(1) \epsilon \).

Lemma 16 For all \( d \in \mathbb{Z}_{++} \), if \( X : \Omega \mapsto \{-1, 1\}^d \) be a random variable with independent components, then for all \( a \in \mathbb{R}^d \) and \( b \in \mathbb{R} \),

\[
\mathbb{V}[\text{sign}(a^\top X + b)] \leq \mathbb{V}[1^\top X].
\]

Proof We provide a proof by induction.

Base Case: \( d = 1 \)
Let \( a, b \in \mathbb{R} \). Then,

\[
\mathbb{V}[\text{sign}(a^\top X + b)] = 1[|a| \geq |b|] \cdot \mathbb{V}[X] \\
\leq \mathbb{V}[X].
\]

**Inductive Step: Assume lemma holds for** \( d = k \)

We assume that for \( d = k \), the statement of the lemma holds. Consider now \( d = k + 1 \). Then, \( X \in \{-1, 1\}^{k+1} \) (note we use \( X_i \) to refer to the \( i \)th component element of \( X \)). We have the following:

\[
\begin{align*}
\mathbb{V}\left[\text{sign}(a^\top X + b)\right] &\overset{(a)}{=} \mathbb{E}\left[\mathbb{V}\left[\text{sign}(a^\top X + b) \mid X_{k+1}\right]\right] + \mathbb{V}\left[\mathbb{E}\left[\text{sign}(a^\top X + b) \mid X_{k+1}\right]\right] \\
&\overset{(b)}{\leq} \sum_{i=1}^{k} \mathbb{V}[X_i] + \mathbb{V}\left[\mathbb{E}\left[\text{sign}(a^\top X + b) \mid X_{k+1}\right]\right] \\
&\overset{(c)}{\leq} \sum_{i=1}^{k} \mathbb{V}[X_i] + \mathbb{V}[X_{k+1}] \\
&= \mathbb{V}[1^\top X],
\end{align*}
\]

where \((a)\) follows from the law of total variance, \((b)\) follows from the inductive hypothesis and \((c)\) follows from the fact that \( |\mathbb{E}[\text{sign}(a^\top X + b) \mid X_{k+1} = 1]| \leq 1 \) and \( |\mathbb{E}[\text{sign}(a^\top X + b) \mid X_{k+1} = -1]| \leq 1 \) while \( X_{k+1} \) takes values in \( \{-1, 1\} \). The result follows.

With these results in place, we provide a proof of Theorem 5

**Theorem 5 (multivariate threshold layers)** For all \( d \in \mathbb{Z}_{++} \), let \((\Theta_k : k = 1, \ldots, K)\) be an sequence of iid matrices in \( \mathbb{R}^{d \times d} \) and \((b_k : k = 1, \ldots, K)\) be a fixed sequence of vectors in \( \mathbb{R}^d \). Let the elements of \( \Theta_k \) be distributed independent \( \mathcal{N}(0, \frac{1}{d}) \). Let each \( k \)th layer of the multilayer environment \( \mathcal{E} \) be governed by the function \( f_k(u) = \Theta_k \text{sign}(u + b_k) \). Then, for sufficiently small \( \epsilon > 0 \), there exists an \( \epsilon \)-minimal multilayer proxy for which the multilayer environment is stable.

**Proof** From Lemma 15, we know that the \( \tilde{E}_k \) that attains the optimal rate at distortion level \( \mathcal{R}_k(1)\epsilon \) has the following conditional variance:

\[
\begin{align*}
\mathbb{E}\left[\|U_k - \mathbb{E}[U_k \mid \tilde{E}_k, U_{k-1}]\|^2 \mid \tilde{E}_k, U_{k-1}\right] &\overset{(a)}{=} \mathbb{E}\left[\|Z_k \text{sign}(U_{k-1} + b_k)\|^2 \mid \tilde{E}_k, U_{k-1}\right] \\
&\overset{(b)}{=} \mathbb{E}\left[\|Z_k \text{sign}(U_{k-1} + b_k)\|^2 \mid U_{k-1}\right] \\
&\overset{(c)}{=} \sum_{i=1}^{d} \mathbb{E}\left[(Z_{k,i} 1)^2\right] \\
&\overset{(d)}{=} d e^{\frac{2}{\epsilon^2} \mathcal{R}_k(1)\epsilon} - 1 \\
&= d e^{\frac{2}{\epsilon^2} \mathcal{R}_k(1)\epsilon} - 1.
\end{align*}
\]
Let \( \delta = \frac{e^{\frac{1}{2}N_{k}^{(1)}}}{e^{\frac{1}{2}N_{k}^{(1)}} - 1} \). We have the following:

\[
\mathbb{E} \left[ \|U_{K} - \mathbb{E}[U_{K}|E_{K}:k+1, \tilde{E}_{k}, U_{k-1}] \|_{2}^{2} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
= \mathbb{E} \left[ \||\Theta_{K} \left( \text{sign}(U_{K-1} + b_{K}) - \mathbb{E} \left[ \text{sign}(U_{K-1} + b_{K}) | E_{K}:k+1, \tilde{E}_{k}, U_{k-1} \right] \right) \|_{2}^{2} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
\leq \mathbb{E} \left[ \||\text{sign}(U_{K-1} + b_{K}) - \mathbb{E} \left[ \text{sign}(U_{K-1} + b_{K}) | E_{K}:k+1, \tilde{E}_{k}, U_{k-1} \right] \|_{2}^{2} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
= \sum_{i=1}^{d} \mathbb{E} \left[ \text{sign}(U_{K-1,i} + b_{K,i})|E_{K}:k+1, \tilde{E}_{k}, U_{k-1} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
= d \cdot \sum_{i=1}^{d} \mathbb{E} \left[ \text{sign}(U_{K-2,j} + b_{K-1,j})|E_{K}:k+1, \tilde{E}_{k}, U_{k-1} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
\leq d \cdot \sum_{i=1}^{d} \mathbb{E} \left[ \text{sign}(U_{K-2,j} + b_{K-1,j})|E_{K}:k+1, \tilde{E}_{k}, U_{k-1} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
\leq d \cdot \mathbb{E} \left[ \text{sign}(U_{K,i} + b_{K,i})|E_{K}:k+1, \tilde{E}_{k}, U_{k-1} \right] \mathbb{E}[E_{K}:k+1, \tilde{E}_{k}, U_{k-1}]
\]

\[
\leq 4d^{K-k} \mathbb{E} \left[ \mathbb{P}(U_{K,i} + b_{K+1,i} \geq 0 | \tilde{E}_{k}, U_{k-1}) \cdot \mathbb{P}(U_{K,i} + b_{K+1,i} < 0 | \tilde{E}_{k}, U_{k-1}) \right] \tilde{E}_{k}, U_{k-1}
\]

\[
\leq 4d^{K-k} \mathbb{E} \left[ \left( e^{-\frac{\text{sign}(U_{K-1} + b_{K}) + b_{K+1,i}^{2}}{28}} \right) \cdot \left( 1 - e^{-\frac{\text{sign}(U_{K-1} + b_{K}) + b_{K+1,i}^{2}}{28}} \right) \right] \tilde{E}_{k}, U_{k-1}
\]

\[
= 4d^{K-k} \mathbb{E} \left[ \left( e^{-\frac{\text{sign}(U_{K-1} + b_{K}) + b_{K+1,i}^{2}}{28}} \right) \cdot \left( 1 - e^{-\frac{\text{sign}(U_{K-1} + b_{K}) + b_{K+1,i}^{2}}{28}} \right) \right] \tilde{E}_{k}, U_{k-1}
\]

where where (a) follows from the fact that \( \mathbb{E} \left[ \Theta_{K} \Theta_{K}^{T} \right] = I_{d} \), (b) follows from Lemma 16. (c) follows from the iid assumption, (d) follows from repeatedly applying steps (b) and (c), and (e) follows from the fact that conditioned on \( \tilde{E}_{k}, U_{k-1}, U_{K} \) is Gaussian.

For small enough \( \delta \), with probability 1,

\[
4d^{K-k} \cdot \mathbb{E} \left[ \left( e^{-\frac{\text{sign}(U_{K-1} + b_{K}) + b_{K+1,i}^{2}}{28}} \right) \cdot \left( 1 - e^{-\frac{\text{sign}(U_{K-1} + b_{K}) + b_{K+1,i}^{2}}{28}} \right) \right] \tilde{E}_{k}, U_{k-1} \leq d\delta.
\]

The result follows by Lemma 14.
activation units and \(M\)-sparse coefficients, then

\[ \mathbb{H}_e(\mathcal{E}_k) = O\left( dKM \ln \left( \frac{NK}{M\sigma^2 \epsilon \ln(1 + \frac{1}{\sigma})} \right) \right). \]

**Proof** It suffices to show that for sufficiently small \(\epsilon\), \(\mathcal{E}\) is stable w.r.t an \(\mathcal{F}\)-minimal multilayer proxy. Since we know from the proof of Theorem 3 that there exists a proxy \(\hat{\mathcal{E}}_k = (S_k, \hat{\theta}_k)\) such that \(U_k\) is isotropic gaussian conditioned on \((U_{k-1}, \hat{\mathcal{E}}_k)\). Lemma 14 states that it suffices to show that for all \(k \in 1, \ldots, K\)

\[ \mathbb{E}\left[ \|U_k - \mathbb{E}[U_k | \hat{\mathcal{E}}_k, U_{k-1}]\|^2 \right] \leq \mathbb{E}\left[ \|U_k - \mathbb{E}[U_k | \hat{\mathcal{E}}_k, U_{k-1}]\|^2 \right]. \]

We begin by showing the following:

\[ \mathbb{E}\left[ \|U_k - \mathbb{E}[U_k | \hat{\mathcal{E}}_k, U_{k-1}]\|^2 \right] = \mathbb{E}\left[ \|Z_k \text{sign}(U_{k-1} + b_k)\|^2 \right] \]
\[ = \mathbb{E}\left[ \|Z_k \text{sign}(U_{k-1} + b_k)\|^2 \right] \]
\[ = \sum_{i=1}^{d} \mathbb{E}\left[ (Z_{k,i})^2 \right] \]
\[ = d \frac{e^{\frac{2}{\epsilon}} R_k(1) - 1}{e^{\frac{2}{\epsilon}} R_k(1) - 1}. \]

Next, we have the following.

\[ \mathbb{E}\left[ \left( A_{k+1,j}^\top (U_k - \mathbb{E}[U_k | \hat{\mathcal{E}}_k, U_{k-1}]) \right)^2 \right] = \mathbb{E}\left[ \left( A_{k+1,j}^\top Z_k \text{sign}(A_k U_{k-1} + b_k) \right)^2 \right] \]
\[ = \mathbb{E}\left[ \left( A_{k+1,j}^\top Z_k \text{sign}(A_k U_{k-1} + b_k) \right)^2 \right] \]
\[ = \|A_{k+1,j}\|^2 \mathbb{E}\left[ (Z_{k,i})^2 \right] \]
\[ = \|A_{k+1,j}\|^2 \frac{e^{\frac{2}{\epsilon}} R_k(1) - 1}{e^{\frac{2}{\epsilon}} R_k(1) - 1}. \]
Let $\delta = \frac{d^2\varepsilon_k(1)^{-1}}{e^{d^2\varepsilon_k(1)^{-1}}}$. Recall that for all $k$, $A_k, b_k$ are not random. We have the following:

\[
\mathbb{E} \left[ \| U_K - \mathbb{E}[U_K | \mathcal{E}_{K:k+1}, \hat{\mathcal{E}}_k, U_{k-1}] \|_2^2 | \mathcal{E}_{K:k+1}, \hat{\mathcal{E}}_k, U_{k-1} \right] \\
= \mathbb{E} \left[ \left\| \Theta K \left( \text{sign}(A_K U_{K-1} + b_K) - \mathbb{E}[\text{sign}(A_K U_{K-1} + b_K) | \mathcal{E}_{K:k+1}, \hat{\mathcal{E}}_k, U_{k-1}] \right) \right\|_2^2 | \hat{\mathcal{E}}_k, U_{k-1} \right] \\
= \frac{d}{N} \sum_{j=1}^{d} \mathbb{E} \left[ \sum_{n=1}^{N} \mathbb{V} \left[ \text{sign}(A_{K,j} U_{K-1} + b_{K,j}) | \mathcal{E}_{K:k+1} \setminus j, \hat{\mathcal{E}}_k, U_{k-1} \right] | \hat{\mathcal{E}}_k, U_{k-1} \right] \\
\leq \frac{d}{N} \sum_{j=1}^{d} \sum_{n=1}^{N} \mathbb{V} \left[ \text{sign}(A_{K,j} U_{K-1} + b_{K,j}) | \mathcal{E}_{K:k+1} \setminus j, \hat{\mathcal{E}}_k, U_{k-1} \right] | \hat{\mathcal{E}}_k, U_{k-1} \right] \\
\leq d N^{K-k-1} \sum_{n=1}^{N} \mathbb{E} \left[ \text{sign}(A_{k+1,n} U_{k+1,n} + b_{k+1,n}) | \hat{\mathcal{E}}_k, U_{k-1} \right] | \hat{\mathcal{E}}_k, U_{k-1} \right] \\
= d N^{K-k-1}.
\]

where (a) follows from the fact that $\mathbb{E} \left[ \Theta_{k}^{T} \Theta_{k} \right] = \frac{d}{N} I_N$, (b) follows from Lemma 16, (c) follows from the fact that each term does not depend on $j$, (d) follows from repeatedly applying (b) and (c), and (e) follows from the fact that $A_{k+1,j} U_k$ is gaussian conditioned on $(\hat{\mathcal{E}}_k, U_{k-1})$.

For small enough $\delta$, with probability one,

\[
\mathbb{E} \left[ \| U_K - \mathbb{E}[U_K | \mathcal{E}_{K:k+1}, \hat{\mathcal{E}}_k, U_{k-1}] \|_2^2 | \mathcal{E}_{K:k+1}, \hat{\mathcal{E}}_k, U_{k-1} \right] \\
\leq 4 d N^{K-k-1} \sum_{n=1}^{N} \left( 1 - e^{-\frac{(2|A_{k+1,n} U_k|)^2}{2d \delta}} \right) \left( 1 - e^{-\frac{(2|A_{k+1,n} U_k|)^2}{2d \delta}} \right) \\
\leq d \delta \\
= \mathbb{E} \left[ \| U_K - \mathbb{E}[U_K | \hat{\mathcal{E}}_k, U_{k-1}] \|_2^2 | \hat{\mathcal{E}}_k, U_{k-1} \right].
\]
The result follows from Lemma 14 and Theorems 3 and 8