THE CONSTRAINTS IN SPHERICALLY SYMMETRIC GENERAL RELATIVITY III

IDENTIFYING THE CONFIGURATION SPACE:

\[ J \neq 0 \]

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Abstract

We continue our examination of the constraints in spherically symmetric general relativity. We extend to general configurations with $J \neq 0$ the analysis of II which treated a moment of time symmetry. We exploit the one parameter family of foliations introduced in I which are linear and homogeneous in the extrinsic curvature to characterize apparent horizons and spatial singularities in the initial data. In particular, we demonstrate that these characterizations do not depend sensitively on the foliation.
1. INTRODUCTION

This is the third paper in a series in which we examine the general features of the constraints in general relativity under the assumption that the spatial geometry is spherically symmetric and possesses just one asymptotically flat region [1,2]. In paper II, we focused on solutions of the constraints which occur when the extrinsic curvature $K_{ab}$ momentarily vanishes (MSCs). As such, we did not need to address the issue of fixing the foliation. In this paper, we extend this analysis to incorporate a non-vanishing extrinsic curvature. Unless one sets out to be difficult, this corresponds to a non-vanishing flow of matter, $J$.

The introduction of extrinsic curvature complicates the analysis substantially. The advantage of having dealt separately with moment of time symmetry configurations in paper II is that we can focus here on the physical feedback on the spatial geometry introduced by extrinsic curvature. The important point is that the solutions of the constraints, as well the relationships between the global measures of the energy and the dimensions of the support of matter which we exploit to characterize horizons and singularities, are not sensitively dependent on the gauge fixing the foliation.

To fix this foliation we implement explicitly one of the gauges parametrized by $\alpha$, linear and homogeneous in the extrinsic curvature, introduced in paper I. These gauges will serve to set in context our understanding of the constraints when the initial data is momentarily static [2]. It was shown in paper I that the allowed values assumed by the parameter correspond to all tangent vectors lying within the superspace lightcone. One of these gauges is the maximal slicing gauge. Another is the polar gauge. Among the attractive properties of all such gauges is that they foliate flat spacetime by flat spatial hypersurfaces. When the momentum constraint is satisfied, the extrinsic curvature is linear in $J$, albeit in a non-local way. In this way the extrinsic curvature of the hypersurface responds directly to the (radial) movement of matter on it. Another possibility, which is motivated by the introduction of the optical scalars as canonical variables on the phase space, is to treat the trace of the extrinsic curvature, $K$, itself as an independent datum along with energy density, $\rho$, and $J$. While this is a legitimate gauge, it is not a usual one unless $K = 0$, for if $K \neq 0$, the extrinsic curvature cannot adjust itself to the movement of matter. For this reason it does not correspond to our physical expectations and, therefore, we do not consider this possibility further here.

As we did in paper II, we will focus again on the identification of the global structures that characterize non-trivial geometries — apparent horizons and singularities.
The identification of apparent horizons is complicated by the fact that these physical landmarks no longer coincide with the extremal surfaces of the spatial geometry as they do at a moment of time symmetry: the movement of matter generates extrinsic curvature, thereby affecting how the spatial geometry is embedded in spacetime which, in turn, determines the lightcone structure on the surface.

We begin in sect. 2 with a discussion of the generic analytic structure of the constraints. We derive a spatial diffeomorphism invariant analytic expression for the behavior of the geometry in the neighborhood of a generic singularity. Generally, the singularities of the three-geometry consistent with the constraints will be more severe than those which are possible at a moment of time symmetry. If, however, the movement of matter is tuned so that the extrinsic curvature vanishes as the singularity is approached, the strength of the singularity will be determined entirely by the quasi-local mass (QLM), exactly as it was at a moment of time symmetry [3]. This tuning corresponds to an integrability condition on the current. If, in addition, the tuning is refined such that the QLM also vanishes as we approach the singularity the curvature singularity disappears and the spatial geometry pinches off in a regular way. This latter integrability condition involving the QLM is completely analogous to the integrability condition we encountered at a moment of time symmetry. Regularity at the singularity is, of course, precisely the condition that the interior be a regular closed universe. If the matter fields carry conserved charges these will, in their turn, have integrability conditions associated with them. Viewed this way, regular closed universes appear to be very special universes [4].

In paper I, we represented the configuration space of the spherically symmetric theory by bounded closed trajectories on the optical scalar plane. In sect. 3 we examine these trajectories in vacuum. We discover that any trajectory that finds itself outside a proper subset of this domain is necessarily singular. While the detailed structure of this good subset depends on the specific gauge choice we make to determine the slicing, we find many features that are independent of the slicing. For example, on the boundary of the good subset we find two unstable fixed points. These correspond to the situation where the exterior spatial geometry neither collapses to a singularity nor expands to be asymptotically flat. Instead it becomes a semi-infinite cylinder of fixed radius.

We follow paper II by establishing global necessary and sufficient conditions for the occurrence of apparent horizons and singularities. These conditions are framed in terms of inequalities which relate some appropriate measure of the material energy content on
a given support to a measure of its volume. The challenge is to identify useful measures in both cases. In analogy to the total material energy $M$ (defined in paper I), we can introduce the total momentum, $P$, corresponding to the integrated material current over the proper spatial volume. The sufficiency criteria for the formation of a future (past) apparent horizon can be cast in a form which is a straightforward generalization of the moment of time symmetry inequality: if the difference (sum) of the material energy and the material momentum exceeds some universal constant times the proper radius, $\ell_0$, of the distribution, the geometry will possess a future (past) apparent horizon. The corresponding constant for singularities is larger but the inequality does not involve $P$. As we found at a MSC a more appropriate measure of the material energy for casting the necessary criteria is the maximum value of the energy density of matter, $\rho_{\text{Max}}$. The obvious generalization is the sum $\rho_{\text{Max}} + J_{\text{Max}}$. However, we find that that the inequality is not symmetrical under interchange of $\rho$ and $J$. If the dominant energy condition is satisfied, however, we can however cast the inequalities in the momentarily static form: if $(\rho_{\text{Max}} + J_{\text{Max}})\ell_0^2 < \text{some constant}$, the distribution of matter does will not possess a singularity for one constant and an apparent horizon for some other smaller constant. These inequalities are new.

In the treatment by Bizon, Malec and Ó Murchadha (BMÓM), and more recently by Malec and Ó Murchadha of the sufficiency conditions, the slicing of spacetime was always assumed to be maximal with $K = 0$ [5,6]. If these inequalities are to be interpreted physically, they should, at least qualitatively, be reproduced in other gauges. We note that we never needed to address this question in paper II because the notion of a moment of time symmetry is gauge independent. We examine the sensitivity of these inequalities on the value of $\alpha$ appearing in the gauge condition. Not surprisingly, we find that the strength of the corresponding inequality does depend on the foliation gauge but not in any significant way so long as we are not close to the lightcone in superspace.

Unfortunately, unlike in our examination of the constraints at a moment of time symmetry where we could fall back on the piecewise-constant density models, we enjoy no such exactly solvable standbys here. Even the analogue of the constant density star proves to be analytically intractable when $J \neq 0$. It is not surprising therefore that it is far more difficult to identify sharp inequalities than it was at a moment of time symmetry. Much of our effort is spent bootstrapping on moment of time symmetry inequalities.

There is one extremely useful exactly solvable model consisting of a moving shell. We exploit this to speculate about the likely form of a possible generalization to $J \neq 0$ of the
lower bound on the binding energy derived by BMÓM.

2. THE CONSTRAINTS

In this section we examine various general features of the constraints when $K_{ab} \neq 0$ in a manner which parallels, wherever applicable, the treatment in paper II. We recall that the constraints are given by

$$K_R [K_R + 2K_L] - \frac{1}{R^2} \left[ 2 (RR')' - R'^2 - 1 \right] = 8\pi \rho$$

(2.1a)

and

$$K'_R + \frac{R'}{R} (K_R - K_L) = 4\pi J,$$

(2.1b)

where the line element on the spatial geometry is parametrized

$$ds^2 = d\ell^2 + R^2 d\Omega^2,$$

(2.1c)

and we have expanded the extrinsic curvature ($n^a$ is the outward pointing unit normal to the two-sphere of fixed $\ell$),

$$K_{ab} = n_a n_b K_L + (g_{ab} - n_a n_b) K_R.$$  

(2.1d)

All derivatives are with respect to the proper radius of the spherical geometry, $\ell$. The spatial geometries we consider consist of a single asymptotically flat region with a regular center, $\ell = 0$. The appropriate boundary condition on the metric at $\ell = 0$ is then

$$R(0) = 0.$$  

(2.2)

We recall that $R'(0) = 1$ if the geometry is regular at this point. We assume that both $\rho$ and $J$ are appropriately bounded functions of $\ell$ on some compact support. A non-singular asymptotically flat solution defined for all $\ell \geq 0$ will not, however, always exist for every specification of $\rho$ and $J$. Our task is to understand what can go wrong.

To solve the constraints classically, we need to implement a foliation gauge. This involves some spatial scalar function of the extrinsic curvature tensor. In a spherically symmetric geometry, this tensor has only two independent components. The foliation
either fixes one of these or relates it functionally to the second. Modulo the gauge, the
momentum constraint can be solved for this other component. The extrinsic curvature is
then completely determined by the sources.

In paper I, we introduced the one-parameter family of gauges, defined for each \( \alpha \) by

\[
K_L + \alpha K_R = 0.
\] (2.3)

We showed that the momentum constraint can be solved uniquely in terms of the radial
flow of matter, \( J \), as follows (Eq.(3.3) of I)

\[
K_R = \frac{4\pi}{R^{1+\alpha}} \int_0^\ell d\ell R^{1+\alpha} J.
\] (2.4)

Whenever \( 0.5 < \alpha < \infty \) the gauge is valid everywhere and displays the correct asymptoti-
cally flat falloff outside the support of \( J \) if the geometry is non-singular. When Eqs.(2.3)
and (2.4) are substituted into Eq.(2.1), we obtain a second order singular non-linear integro-
ODE for \( R \).* Subject to the boundary condition, (2.2), the solution is uniquely determined.
Not only is the extrinsic curvature completely determined by the material sources, so also
is the spatial geometry. There are no independent gravitational degrees of freedom.

We note that in the gauge Eq.(2.3), the spatial geometry does not depend on the
global sign of \( J \).

We saw in paper I that if \( K_R \) is regular at the origin then it must also vanish there.
In fact, in the neighborhood of \( \ell = 0 \), \( R \sim \ell \), so that

\[
K_R \sim 4\pi J(0) \frac{\ell}{2 + \alpha}.
\] (2.5)

To determine the \( n^{th} \) derivative of \( R \) at \( \ell = 0 \) we need to differentiate Eq.(2.1) \( n - 1 \)
times. A consequence of the vanishing of \( K_R(0) \) is that \( J \) will only show up at order five
— two orders behind \( \rho \) (see paper II). The behavior of the metric at the origin is clearly
not sensitive to the current flowing there.

It is instructive to also examine the values assumed by the optical scalars in the
neighborhood of \( \ell = 0 \). Recall that [1,7]

* It is possible to rewrite Eqs.(2.1) so that they can be differentiated once to yield a
local third order singular ODE modulo Eqs.(2.3) and (2.4). However, it is not particularly
illuminating to cast them this way.
We can combine Eq.(2.4) of paper II and (2.5) to obtain
\[ \omega \pm \sim 2 - \frac{8\pi}{3} (\rho(0) \mp \frac{3}{2 + \alpha} J(0)) \ell^2. \]  
(2.7)

Suppose that the dominant energy condition (DEC) \( \rho \geq |J| \) is satisfied. If \( \alpha \geq 1 \), then \( \omega \pm \leq 2 \). If, however, \( \alpha < 1 \) this is not the case. This demonstrates explicitly that the inequalities (6.2a) and (6.2b) in paper I cannot generally be relaxed to the \( K = 0 \) value.

We note also that Eq.(2.7) implies

\[ \omega_+ \omega_- \leq (2 - \frac{8\pi}{3} [\rho(0) \ell^2])^2 - \left( \frac{8}{2 + \alpha} \right)^2 \pi^2 J(0)^2 \ell^4 \leq 4, \]

which is consistent with the inequality (5.2) in paper I for all values of \( \alpha \). Note also that the absolute maximum of the product \( \omega_+ \omega_- \) obtains at the boundary values \( \ell = 0 \) and \( \ell = \infty \) and it is also the flat space value. When \( K = 0 \), this is also true of both \( \omega_+ \) and \( \omega_- \). In general, the absolute maximum of neither need occur at these points.

2.1 The Quasi-Local Mass

As we found in paper II in a simpler context, the definition of the quasi-local mass can be exploited to provide an extremely useful first integral of the constraints. We recall that (Eq.(4.7) of paper I):
\[ m = \frac{R}{2} \left( 1 - R'^2 \right) + \frac{1}{2} K_R^2 R^3. \]  
(2.8)

Modulo the constraints (Eq.(4.8') of paper I), and the boundary condition, (2.2),
\[ m = 4\pi \int_0^\ell d\ell R^2 [\rho R' + J R K_R]. \]  
(2.9)

If the geometry is non-singular, \( m \) is positive everywhere, coinciding in the limit \( \ell \to \infty \) with the ADM mass, \( m_\infty \). Eqs.(2.8) and (2.9) are gauge invariant. To exploit Eqs.(2.8) and (2.9) to solve the constraints, we substitute the solution of the momentum constraint (2.4) in the gauge (2.3) into Eqs.(2.8) and (2.9).
Note that the leading spatial derivative in Eq.(2.8) is $R'$. Outside the sources, $m$ is constant while $K_R \sim C/R^{1+\alpha}$. From a functional point of view, Eq.(2.8) is identical to the energy integral in classical mechanics. To exploit this analogy, we recast Eq.(2.8) for all $\ell$ as follows:

$$R'^2 = 1 - \frac{2m}{R} + K^2 R^2.$$  \hspace{1cm} (2.8')

Now, formally at least,

$$\ell = \int_0^R \frac{dR}{\sqrt{1 - \frac{2m}{R} + K^2 R^2}},$$

where $m$ is given by Eq.(2.9).

2.2 The Neighborhood of Singularities

In paper I, as a lemma to the positive quasi-local mass theorem, we proved that when the weak energy condition $\rho \geq 0$ holds, $R'^2 \leq 1$ everywhere in any regular geometry. Thus if $R'^2 > 1$ anywhere the geometry must be singular. Let us suppose that $R'^2 > 1$ at some point. Then, when $K_{ab}$ satisfies Eq.(2.3) and $\alpha > 0.5$, Eq.(2.1b) implies that $R'' < 0$, so that $R'$ is decreasing there. This can only occur by $R'$ falling through $R' = -1$. Once $R'$ falls below this value it will continue decreasing monotonically thereafter. The surface $R' = -1$ in the configuration space therefore acts as a oneway membrane. Suppose that the circumferential radius is $R_0$ when $R' = -1$. We know now that the solution must crash, i.e. $R \to 0$ in a finite proper distance which is less than or equal to $R_0$ from that point. In fact, this is the only way the spatial geometry can become singular.

How do we know that we have covered all possible singularities? We argue that the converse of the lemma holds. In general, $-1 < R' \leq 1$ if and only if the geometry is non-singular. As we will see, a singularity with $R' = -1$ is a result of a very special fine-tuning of the matter distribution.

In the neighborhood of the point $\ell = \ell_S$ at which $R = 0$, Eq.(2.4) implies that

$$K_R \sim \frac{C_\alpha(\ell_S)}{R^{1+\alpha}},$$  \hspace{1cm} (2.10)

where
\[ C_\alpha(\ell) = 4\pi \int_0^\ell d\ell J R^{1+\alpha} \]  

is finite. \( K_R \) will therefore be singular (for any physically acceptable value of \( \alpha \)) if the geometry pinches off unless the current is tuned such that

\[ C_\alpha(\ell_S) = 0. \]  

Now, if \( K_R \neq 0 \), and \( \alpha > 0.5 \), the most singular term in Eq.\( (2.8') \) is the quadratic in \( K_R \). This implies that

\[ R'^2 \sim R^2 K_R^2 \]  

in the neighborhood of \( R = 0 \), or \( R'^2 \sim C_\alpha^2/R^{2\alpha} \). Generically, therefore, \( R'^2 \) diverges. The solution is

\[ R \sim \left( \frac{C_\alpha}{\alpha + 1} \right)^{\frac{1}{\alpha+1}} (\ell_S - \ell)^{\frac{1}{\alpha+1}}. \]  

If \( \alpha > 0.5 \), such spatial singularities are more severe than the strong singularities discussed in paper II which are consistent with the Hamiltonian constraint at a moment of time symmetry. We will refer to the generic kind of singularity driven by extrinsic curvature as a strong \( J \)-type singularity. As \( \alpha \) increases, the power law determining the strength of the singularity increases. Note that the limit \( \alpha \to \infty \) (the polar gauge discussed in I) is extremely singular. This is, however, a gauge artifact reflecting how poor the polar gauge really is.

Unlike the strong singularities occurring in MSCs, at which the scalar curvature \( \mathcal{R} \) remained finite, \( \mathcal{R} \) will generally blow up (as \( K_R^2 \sim 1/(\ell_S \ell)^2 \)). On dimensional grounds, we expect all curvature scalars to blow up as \( 1/(\ell_S - \ell)^2 \) as we approach a singularity unless there is some constraint obstructing them from doing so.

It is important to confirm that \( m \) remains suitably bounded as we approach a strong singularity. We do this by demonstrating that the volume integral \( (2.9) \) is always finite. We note that for suitably bounded \( \rho \) and \( J \),

\[ (\rho R^2 R', J R^3 K_R) \sim (\rho, J)(\ell_S - \ell)^{-\left(\frac{\alpha-2}{\alpha+1}\right)}. \]  

(2.15)
If $\alpha \leq 2$, the integrand itself remains finite. In general, the integral will be finite if the exponent of $(\ell_S - \ell)^{-1}$ is bounded by one. But $(\alpha - 2)/(\alpha + 1) < 1$ for all finite values of $\alpha$ thus guaranteeing that the integrals over $R^2 \rho \rho'$ and $R^3 J K_R$ converge.

It is clear that $m(\ell_S)$ is always finite. Its sign, however, will depend on the details of the current flow. This is obvious from the definition Eq.(2.8). Even if $R^2 > 1$, a sufficiently large value of $K_R$ can render $m$ positive. In particular, unlike the value of $m$ assumed at strong $\rho$-singularities of MSCs which is always negative, the sign can assume either value. Indeed $m$ need never even be negative in a singular geometry. Though $R'$ decreases monotonically, $m$ nonetheless remains positive. There is no conflict with the positive QLM theorem. In our examination of MSCs in paper II, we found that $m$ is positive everywhere except at the origin or in a neighborhood of it if and only if the geometry is non-singular. This is a consequence of the coincidence of the converse of the bounded $R^2$ lemma and the converse of the positive QLM theorem when $K_{ab} = 0$. In the general case, when $K_{ab} \neq 0$, no such coincidence occurs.

What are the implications of the integrability condition, Eq.(2.12)? If Eq.(2.12) is satisfied the strong $J$ singularity is moderated to one which is only strong a la $\rho$. The behavior in the vicinity of the singularity will then be determined by the $m/R$ term in Eq.(2.8') even if the system was originally ‘driven’ towards the singularity by extrinsic curvature. If, in addition,

$$m(\ell_S) = 4\pi \int_0^{\ell_S} d\ell \left[ \rho R^2 R' + J R^3 K_R \right] = 0, \quad (2.16)$$

the singularity will be a weak one with $R'(\ell_S) = -1$. The corresponding bag of gold will be a regular closed universe.

These integrability conditions depend on $\alpha$. If a given function $J$ satisfies Eq.(2.12) with one value of $\alpha$, generally it will not satisfy that condition with any other value. What is missing is a spacetime diffeomorphism invariant statement of the integrability.

If $J$ is positive (or negative) everywhere, $C_\alpha(\ell)$ defined by Eq.(2.11) cannot vanish. Thus, if matter is collapsing or exploding everywhere, all singularities must be strong $J$-type singularities.

This contrasts with the obstruction, $\rho' < 0$, discussed in paper II, prohibiting the formation of any singularity when $K_{ab} = 0$. In general, we note that on performing an integration by parts on the first term, $m$ can be rewritten
\[ m = \frac{4\pi}{3} \rho R^3 + 4\pi \int_0^\ell d\ell R^3 [JK_R - \rho'] . \] (2.17)

The first term is manifestly positive. So is the third if \( \rho' \leq 0 \). If \( J \) is positive (negative) everywhere then so is \( m \) in any \( \alpha \)-gauge. However, the third term appearing on the RHS of Eq.(2.8') may still pull the geometry into a singularity if \( J \) is sufficiently large. The peculiarity of momentarily static configurations with \( \rho' < 0 \) discussed in paper II can clearly be destabilized by the motion of matter.

All regular closed cosmologies simultaneously satisfy two integrability conditions, Eqs.(2.12) and (2.16). There can be no net flow of material from one pole to the other. In particular, \( J \) must change sign between the poles. In addition, Eq.(2.17) tells us that

\[ m(\ell_S) = 4\pi \int_0^{\ell_S} d\ell R^3 [JK_R - \rho'] = 0 . \] (2.18)

In particular, \( JK_R - \rho' \) must change sign between the poles. These conditions will be examined in the closed cosmological context in a subsequent publication [4].

2.3 No strong \( J \) singularities in the Euclidean Theory

The singularity structure we have investigated has one important consequence for Euclidean general relativity. If the sign of the quadratic term in \( K_R \) appearing in Eq.(2.8) had been negative, instead of facilitating the occurrence of singularities it would have presented an obstacle to their occurrence. Any non-vanishing extrinsic curvature would therefore tend to stabilize the spatial geometry against singularity formation. We note that there is precisely such a sign switch in the Hamiltonian constraint of Euclidean general relativity. The Bianchi identities there tell us that the solutions of the constraints represent all possible configurations the system may assume as it is evolved with respect to Euclidean time. This suggests that gravitational instantons will tend to be more regular than their Lorenzian counterparts. In fact, the most singular Euclidean geometries will occur when the geometry is momentarily static. In a tunneling Euclidean four-geometry, such three-geometries correspond to the initial and final hypersurfaces of the Lorentzian spacetimes between which it interpolates. If these hypersurfaces are themselves non-singular, \( i.e. \) do not involve Planck scale structures, then Planck Scale physics does not enter the semi-classical description of tunneling between them. This would appear to validate the application of the semi-classical approximation.
3 SOLUTIONS OF THE CONSTRAINTS AS TRAJECTORIES ON THE 
\((\omega_+, \omega_-)\) PLANE

In paper I we found that a very useful representation of the phase space was provided 
by the representation of solutions to the constraints as trajectories on the optical scalar 
plane. We have [1,7]

\[
\begin{align*}
(\omega_+)' &= -8\pi R(\rho - J) - \frac{1}{4R} \left[ 2\omega_+^2 - 4 - 4\omega_+KR - \omega_+\omega_- \right], \\
(\omega_-)' &= -8\pi R(\rho + J) - \frac{1}{4R} \left[ 2\omega_-^2 - 4 + 4\omega_-KR - \omega_+\omega_- \right].
\end{align*}
\]  

(3.1a, b)

3.1 Non-Singular Geometries off the support of matter

In this section, we will focus on the behavior of trajectories outside the support of 
matter. We will suppose that the interior solution is regular. This is a more useful exercise 
than it might appear at first sight. This is because, as we have seen, the behavior of 
trajectories depends non-locally on the sources, \(\rho\) and \(J\). In particular, the appearance of 
singularities does not depend sensitively on the values of \(\rho\) and \(J\) in the immediate vicinity 
of the singularity. In addition, as we will see, the behavior of vacuum trajectories upon 
entering into a shell of matter is described in a simple way.

Let us first recall briefly the case the momentarily static solution outside matter. We 
set \(J = 0\) everywhere. We found that

\[
R'^2 = 1 - \frac{2m_\infty}{R}.
\]  

(3.1)

where

\[
m_\infty = 4\pi \int_0^{\ell_0} d\ell R'^2 \rho R'.
\]  

(3.2)

If \(m_\infty \leq 0\) and \(R'(\ell_0) \leq 0\), the vacuum geometry will be singular. If \(R'(\ell_0) > 0\), the 
vacuum geometry is non-singular but the positive quasi-local mass theorem tells us that 
the interior must harbor a geometrical singularity.

If \(m_\infty > 0\), there is no way the geometry can be singular. If \(R'(\ell_0) > 0\), the solution 
grows monotonically. If \(R'(\ell_0) \leq 0\), \(R\) will decrease until an apparent horizon forms at 
\(R = 2m_\infty\) and thereafter increase.
When \( J \neq 0 \) the solution is considerably less simple. Let us rewrite Eq.(2.8) and (2.9) as

\[
R'^2 = 1 - V(R),
\]

where the potential \( V(R) \) is given by

\[
V(R) = \frac{2m_\infty}{R} - \frac{C_\alpha^2}{R^{2\alpha}},
\]

(3.3)

(3.4)

(3.5)

and \( C_\alpha = C_\alpha(\ell_0) \) is given by Eq.(2.11). We suppose that \( 0.5 < \alpha < \infty \).

If \( m_\infty \) is negative and \( R'(\ell_0) \leq 0 \) then, as before, the potential is monotonic and unbounded from below, and the geometry will be singular. It will generally be a strong \( J \)-singularity unless \( J \) is tuned such that \( C_\alpha(\ell_0) = 0 \). If \( R'(\ell_0) > 0 \), the vacuum geometry is non-singular but the positive quasi-local mass theorem tells us that the interior must be singular.

What is much more interesting is the case \( m_\infty > 0 \). Unlike the case of a MSC, a positive \( m_\infty \) does not guarantee a non-singular exterior geometry.

If \( m_\infty \) is positive, the potential possesses a maximum at the point

\[
R_c = \left( \frac{\alpha C_\alpha^2}{m_\infty} \right)^{\frac{1}{2\alpha - 1}}.
\]

(3.6)

The value assumed by the potential at this point is

\[
V(R_c) = (2 - \alpha^{-1}) \frac{m_\infty}{R_c}.
\]

(3.7)

There are two possibilities we need to consider:

If \( V(R_c) < 1 \), the geometry will be singular outside if and only if \( R'(\ell_0) < 0 \).

If \( V(R_c) \geq 1 \), the nature of the geometry will depend not on \( R'(\ell_0) \) but on the relative values of \( R(\ell_0) \) and \( R_c \). If \( R(\ell_0) < R_c \) the geometry will always be singular. If \( R(\ell_0) > R_c \), however, there is no way the geometry can be singular. The qualitative dependence on the sign of \( R'(\ell_0) \) is the same as that for \( m_\infty > 0 \) when \( J = 0 \).

The condition \( V(R_c) < 1 \) is equivalent to the inequality
\[ m_{\infty}^{2\alpha} < (2\alpha - 1) \left( \frac{\alpha}{2\alpha - 1} \right)^{2\alpha} \frac{C_{\alpha}^2}{2\alpha}. \] (3.8)

Typically Eq.(3.8) will hold for a given \( m_{\infty} \) whenever \( C_{\alpha} \) is large which corresponds, roughly speaking, to a large material current. However, because \( m_{\infty} \) itself also involves \( J \) this criterion is not very precise. We need to isolate the dependence of \( m_{\infty} \) on \( K_R \). The inequality can then be cast as an inequality between \( R' \) and \( RK_R \). We recall that \( C_{\alpha} = R^\alpha RK_R \). Now Eq.(3.8) can be cast in the form

\[ \left( \frac{1}{2} \right)^{\alpha} (1 - R'^2 + R^2 K_R^2)^{\frac{\alpha}{2}} < (2\alpha - 1)^{1/2} \left( \frac{\alpha}{2\alpha - 1} \right)^{\alpha} |RK_R|. \] (3.9)

We can also represent the inequality, \( R < R_c \), as the exterior of the ellipse

\[ R'^2 + (2\alpha - 1) R^2 K_R^2 = 1 \] (3.10)

on the \((R', RK_R)\) plane. The beauty about Eqs.(3.9) and (3.10) is that when they are cast in terms of the optical scalar variables, \( \omega_+ \) and \( \omega_- \) they are independent of \( R \).

Let us examine Eq.(3.9) in greater detail. We first cast it in the form

\[ f(RK_R) \leq R'^2, \] (3.11)

where we define

\[ f(x) = x^2 - 2(2\alpha - 1)^{1/2} \left( \frac{\alpha}{2\alpha - 1} \right) |x|^{1/\alpha} + 1. \] (3.12)

We note that \( f(x) \) is positive everywhere. In particular, \( f(0) = 1 \) and

\[ f(\pm 1/\sqrt{2\alpha - 1}) = 0 = f'(\pm 1/\sqrt{2\alpha - 1}). \] (3.13)

In the neighborhood of the points \( x = \pm 1/\sqrt{2\alpha - 1} \),

\[ f^{1/2}(x) \sim \sqrt{\frac{2\alpha - 1}{\alpha}} \left| x \mp 1/\sqrt{2\alpha - 1} \right|. \] (3.14)

As \( x \to \infty \), \( f^{1/2}(x) \sim x \). The two branches of the function \( R' = \pm f(RK_R)^{1/2} \) correspond to the boundary \( V(R_c) = 1 \). They can be represented on the \((\omega_+, \omega_-)\) plane (see fig.(3.1)) as the union of arc segments \( \{C'P, PP', \ P'D\} \) and \( \{C''P, PQ, QP', \ P'D'\} \), where the
coordinates of the points $\mathcal{P}$ and $\mathcal{P}'$, corresponding to the two minima of $f$, are given respectively by $(-2/\sqrt{2\alpha - 1}, 2\sqrt{2\alpha - 1})$ and $(2/\sqrt{2\alpha - 1}, -2\sqrt{2\alpha - 1})$. Fig.(3.1) corresponds to $\alpha = 2$.

The ellipse defined by Eq.(3.10) is also represented on fig.(3.1) for $\alpha = 2$. We note that for each $\alpha$, the points $\mathcal{P}$ and $\mathcal{P}'$ both lie on this ellipse. The inequality, $R < R_c$, is represented by the region on the phase plane outside the ellipse.

What is this figure telling us? There is a wedgelike region $\Omega_0$, bounded by the arc segments, $CQ, QD$ indicated on fig.(3.1) which determines the maximum excursion a vacuum trajectory can make from its point of departure, $P = (2, 2)$, and still return home. This is a disjoint union of two regions, one in which $V(R_c) > 1$ and $R > R_c$, the other in which $V(R_c) \leq 1$ and $R' > 0$.

Any trajectory which lies outside $\Omega_0$ on exiting the support of matter is necessarily singular. This region, likewise, decomposed into a disjoint union, one in which $V(R_c) < 1$ and $R' < 0$, the other in which $V(R_c) \geq 1$ and $R < R_c$. We note that these considerations did not rely on any energy condition, dominant, weak or otherwise.

When the DEC holds, we note that $\Omega_0$ reduces to a proper subset of the domain, $\Omega$, introduced in paper I to which all non-singular trajectories are confined. For $\alpha = 2$, $\Omega$ is given by the square, $|\omega_\pm| \leq 2$. The region, $\Omega - \Omega_0$ is rendered forbidden outside the support of matter. In particular, we note that the barriers $\omega_\pm = -2$ are completely out of bounds. There always exists, however, a suitable $\rho$ and $J$, which can be added within the region $\Omega$ so as to render the trajectory straying into this region non-singular. To see this, consider the addition of a shell with a source four-vector given by Eq.(4.1) at $\ell = \ell_0$. Both $\omega_+$ and $\omega_-$ will suffer a discontinuity at the shell. The discontinuity $(\Delta \omega_\pm)$ is given by integrating Eqs.(3.1a) or (3.1b) across the surface:

$$\Delta \omega_\pm = -8\pi R(\ell_0)(\sigma \mp j).$$

By a suitable choice of $\sigma$ and $j$ it is always possible to raise or reduce one or the other of $\omega_\pm$ while leaving the other unchanged. In particular, as the arrow on the point $Q$ indicates the value of $\omega_-$ can be reduced in such a way that the trajectory is delivered back to safety albeit by flirting dangerously close to the singular point $Q$.

What is the physical significance of the points $\mathcal{P}$ and $\mathcal{P}'$? These are both fixed points of $\omega_+, \omega_-$ and $R$ outside matter:
\[
\omega_+ = 0 \quad \omega_- = 0 \quad R' = 0.
\]

(3.16)

As a result, \(R'' = 0\) and all higher derivatives vanish at these points. If we exit matter at any one of these two points, the spatial geometry degenerates into an semi-infinite cylinder, \(S^2 \times R_+\) outside. They are clearly unstable fixed points. Under any small perturbation, the vacuum trajectories terminating on either of these points will find themselves either returned to the origin, \(P\) or consigned to singular oblivion.

The radius of this cylinder is fixed by the value of the ADM mass. We note that Eq.(3.8) implies

\[
R_0 = \left(2 - \frac{1}{\alpha}\right)m_\infty. \tag{3.17}
\]

We have sketched the exterior behavior explicitly for \(\alpha = 2\) on fig.(3.1). How sensitively dependent is this picture on the gauge parameter, \(\alpha\)?

If \(\alpha\) is reduced below two, the points \(\mathcal{P}\) and \(\mathcal{P}'\) slide out along the \(R' = 0\) diagonal in opposite directions reaching infinity at \(\alpha = 0.5\) — the superspace lightcone value. If \(\alpha > 2\), \(\mathcal{P}\) and \(\mathcal{P}'\) converge on the \(R' = 0\) diagonal, coinciding asymptotically on the \(K_R = 0\) axis as \(\alpha \to \infty\), the polar gauge value (discussed in paper I). We see explicitly how polar gauge imitates a moment of time symmetry in a very singular way.

When \(\alpha = 1\), Eq.(3.9) reads \(R'^2 > (RK_R \mp 1)^2\), where the \(\mp\) correspond respectively to \(|K_R| = \pm K_R\). These are two cones with apices at the points \(\mathcal{P}\) and \(\mathcal{P}'\) given by \(RK_R = \pm 1\), \(R' = 0\). \(\Omega_0\) is now simply a square, whereas \(\Omega\) is some more complicated figure.

We note, however, that while the partition of the \((\omega_+, \omega_-)\) plane depends qualitatively on \(\alpha\), topologically it is identical to the partition illustrated in fig.(3.1) for \(\alpha = 2\).

In the vacuum region the trajectory is determined by two constants, \(m_\infty\) and \(C_\alpha\), which are related to the optical scalars via

\[
\omega_+ \omega_- = 4 - \frac{8m_\infty}{R}, \tag{3.18}
\]

and

\[
\omega_+ - \omega_- = 4RK_R = \frac{4C_\alpha}{R^\alpha}. \tag{3.19}
\]
There are two possible solutions

\[ \omega_\pm = \frac{2C_\alpha}{R^\alpha} \left( \pm 1 + \sqrt{1 + \frac{R^{2\alpha-1}(R - 2m_\infty)}{C_\alpha^2}} \right), \]

(3.20a)

and

\[ \omega_+ + \omega_- = \frac{4C_\alpha}{R^\alpha} \sqrt{1 + \frac{R^{2\alpha-1}(R - 2m_\infty)}{C_\alpha^2}}. \]

\[ \omega_+ + \omega_- = -\frac{4C_\alpha}{R^\alpha} \sqrt{1 + \frac{R^{2\alpha-1}(R - 2m_\infty)}{C_\alpha^2}}. \]

(3.20b)

From these equations it is easy to analyse the behaviour of a trajectory as it approaches a singularity. Let us assume that \( C_\alpha > 0 \). Since we approach a singularity we can assume that \( R \) is small and positive and that \( 4R' = \omega_+ + \omega_- \) is negative. This means that the trajectory is given by Eq.(3.21b). Further, since \( \alpha > 0.5 \) the second term in the square root is much less than 1 and we can use the Taylor expansion to get

\[ \omega_+ \sim \frac{2m_\infty R^{\alpha-1}}{C_\alpha}, \]

\[ \omega_- \sim \frac{-4C_\alpha}{R^\alpha}. \]

(3.21)

Thus we see again the same structure that we described in Section 2 whereby the singularities we get when \( C_\alpha \neq 0 \) are stronger than in the MSC configuration.

**4 THE SPHERICALLY SYMMETRIC SHELL WHEN J ≠ 0**

Clearly we cannot solve Eq.(2.4) exactly. Furthermore, if \( J \neq 0 \), even the uniform current/density model becomes non-trivial. The only model we will solve exactly is the shell. The dynamics of moving shells is a subject which has received extensive study. Our focus of interest will, however, be restricted to an examination of the constraints and the identification of constraints on the sources avoiding singularities. From one point of view, we have already essentially solved the problem in our examination of the exterior solution. This is because all of the interesting physics occurs in this exterior region.

In paper II, we examined the corresponding MSC. A very rich configuration space is revealed when we relax \( J = 0 \). Let
\[ \rho = \sigma \delta(\ell - \ell_0) \]
\[ J = j \delta(\ell - \ell_0) , \]  
(4.1)

where

\[ \sigma = (\sigma_0^2 + j^2)^{1/2} . \]

In this form, \( \sigma_0 \) is the rest mass of the shell. If \( \sigma_0 \) is real we satisfy the DEC.

Inside the shell, the space is flat so that \( R = \ell \). As was the case at a moment of time symmetry the material energy \( M \) coincides with its Newtonian value

\[ M = 4\pi \sigma \ell_0^2 , \]  
(4.2)

and is unaffected by the motion of the shell.

In any \( \alpha \)-gauge, the momentum constraint implies

\[ K_R = \begin{cases} 0 & \ell < \ell_0 \\ 4\pi \left( \frac{\ell}{R} \right)^{1+\alpha} j & \ell \geq \ell_0 . \end{cases} \]  
(4.3)

In particular,

\[ K_R(\ell_0+) = 4\pi j \]  
(4.4)

is independent of \( \alpha \). \( K_R \) is finite everywhere so long as \( R \) remains bounded from below outside the shell. We can now integrate Eq.(2.1) across \( \ell = \ell_0 \) to determine the discontinuity in \( R' \) at the shell

\[ \Delta R' = -4\pi \sigma \ell_0 . \]  
(4.5)

This discontinuity is independent of \( J \). Eq.(4.5) implies that on the outer surface of the shell,

\[ R'(\ell_0+) = 1 - 4\pi \sigma \ell_0 = 1 - \frac{M}{\ell_0} . \]  
(4.6)

If \( \sigma_0 > 0 \), \( R'(\ell_0+) \) is bounded above by one. The ADM mass is now given by

\[ m_\infty = m(\ell_0+) = \frac{\ell_0}{2} \left( 1 - R'^2(\ell_0+) \right) + \frac{1}{2} \ell_0^3 K_R^2(\ell_0+) . \]  
(4.7)
We now substitute Eqs.(4.4) and (4.6) into Eq.(4.7). Exploiting Eq.(4.2), we can express $m_\infty$ in terms of $M$, $\ell_0$, $\sigma$ and $j$ as follows,

$$m_\infty = M - \frac{M^2}{2\ell_0}(1 - v^2), \quad (4.8)$$

where we introduce the notation

$$v = \frac{j}{\sigma}.$$ 

We note that

$$1 - v^2 = \frac{\sigma_0^2}{\sigma_0^2 + j^2}$$

is manifestly positive if the DEC holds. We note that $\alpha$ does not appear in Eq.(4.8). Eq.(4.8) generalizes Eq.(7.6) of paper II. The binding energy defined in I,

$$-E_B = M - m = \frac{M^2}{2\ell_0}(1 - v^2), \quad (4.8')$$

is diminished below its Newtonian value by the motion of the shell. It is, however, still negative whenever matter satisfies the DEC, consistent with our hopes and allaying our fears. If $\sigma_0 = 0$ corresponding to a null shell which saturates the DEC (moving either inward or outward), $E_B = 0$.

The divergence of outward bound future (past) directed null geodesics at the surface of the shell is given by

$$(\Theta_\pm)_0 = \frac{2}{R} \left(1 - 4\pi\ell_0\sigma(1 \mp v))\right). \quad (4.9)$$

We see that $(\Theta_+)_0 < 0$ and hence that a future horizon must form at some point outside the shell whenever

$$4\pi\sigma\ell_0(1 - v) \geq 1. \quad (4.10)$$

If $v = 1$, corresponding to a null outward moving shell, no future horizon can form.* As one would expect it is easier to form an horizon when $v$ is negative.

If $\Delta R' \leq -2$ the geometry will be singular. This reads

* This does not, however, mean that the geometry cannot be singular.
\[ \frac{M}{2\ell_0} \geq 1, \]

which is independent of \( v \).

Let us interpret the exterior solution we examined in sect.3 in terms of the parameters of the shell model. We note that \( m_\infty \) is given by Eq.(4.8) and

\[ C_\alpha = 4\pi \ell_0^{1+\alpha} j = \ell_0^{\alpha-1} M v. \tag{4.11} \]

If \( m_\infty \) is negative, the positive mass theorem tells us that the geometry must be singular. We note that \( m_\infty \leq 0 \) in Eq.(4.8) implies

\[ \frac{M}{2\ell_0} \geq \frac{1}{1 - v^2}. \tag{4.12} \]

The potential \( V(R) \) is monotonic and unbounded from below. We note that Eq.(4.12) implies that \( R'(\ell_0+) < -1 \) (see Eq.(4.6)).

If \( m_\infty \) is positive, \( V(R) \) possesses a maximum at the point \( R_c \) given by Eq.(3.6). There are two factors which determine the nature of the exterior geometry. In sect.3 we saw that the condition \( V(R_c) < 1 \) can be cast in the form (3.8). Substituting Eq.(4.8) for \( m_\infty \) and (4.11) for \( C_\alpha \), we get

\[ \left( 1 - \frac{y}{2} (1 - v^2) \right)^{2\alpha} \leq \left( 2\alpha - 1 \right) \left( \frac{\alpha}{2\alpha - 1} \right)^{2\alpha} v^2 y^{2(1-\alpha)}, \tag{4.13} \]

where \( y := M/\ell_0 \). This relationship clearly depends on the choice of \( \alpha \). When \( \alpha = 1 \), the condition \( V(R_c) \leq 1 \) is simple:

\[ \frac{M}{2\ell_0} > \frac{1}{1 + |v|}. \]

If, in addition, \( R'(\ell_0+) < 0 \) the geometry will be singular. Eq.(4.6) then implies

\[ \frac{M}{\ell_0} \geq 1. \tag{4.14} \]

When the DEC is satisfied all such geometries are singular.

† It will be a strong \( J \)-singularity unless \( v = 0 \) which is the only way that \( C \) can vanish. The simplest model with a non-trivially vanishing \( C \) consists of two shells moving in opposite directions.
We also note that $\ell_0 < R_c$ is equivalent to

$$\frac{M}{2\ell_0} > \frac{1}{1 + (2\alpha - 1)v^2}.$$ 

When $\alpha = 1$,

$$\frac{M}{2\ell_0} > \frac{1}{1 + v^2}.$$ 

But, when the DEC holds,

$$\frac{1}{1 + v^2} \geq \frac{1}{1 + |v|}.$$ 

There are therefore no geometries which simultaneouly satisfy $V(R_c) \geq 1$ and $\ell_0 < R_c$ when the DEC is satisfied.* Thus all geometries with $V(R_c) \geq 1$ are non-singular when the DEC holds. We illustrate the situation in fig.(4.1). We are now in a position to conclude that

$$M \leq \frac{2\ell_0}{1 + |v|}. \quad (4.15)$$

in any non-singular shell geometry.

Later, we will discover that the (weaker) bound,

$$M \leq 2\ell_0, \quad (4.16)$$

holds in a non-singular geometry regardless of $J$ and as a result also of the DEC. We recall that the shell saturated this condition when $v = 0$ (the MSC result).

It is useful to recall the moment of time symmetry analysis. It was conjectured by ADM and subsequently proven by BMÓM that [8] (also see ref.[2])

$$M - m \geq M^2/2\ell_0. \quad (4.17)$$

The conjecture was motivated by the fact that in Newtonian gravity, the configuration that minimized the binding energy for a given total $M$ is the shell. If (4.17) holds, then Eq.(4.16) follows by the positivity of the quasilocal mass in any non-singular geometry. It is important that the bound (4.16) is saturated by a shell.

* Equality $\ell_0 = R_c$ gives $R' = 0$ outside so that the exterior of the shell is a cylinder of radius $\ell_0$. This is singular.
When $v \neq 0$, Eq.(4.15) is stronger than (4.16). When $J \neq 0$, we would expect the analogue of Eq.(4.17) to imply a bound on $M$ at least as good as (4.16).

What is this analogue? If we were to take Eq.(4.8$'$) at face value, we would conjecture that

$$M - m \geq \frac{M^2 - P^2}{2\ell_0},$$

(4.18)

where $P = 4\pi \int jR^2 dl = 4\pi j\ell_0^2$. The positivity of $m$, however, would now imply that

$$M(1 - P^2/M^2) \leq 2\ell_0,$$

(4.19)

which is considerably weaker than Eq.(4.16). The problem is that in the shell, when $v$ is large the only solutions with $m$ simultaneously small and positive are all singular (see fig.(4.1). To be more precise, if we substitute the inequality Eq.(4.15) into Eq.(4.8$'$) we get

$$m \geq M|v|.$$ 

Hence $m$ is bounded from below if $v$ is non-vanishing.

This suggests that we can do better than (4.16). In fact, Eq.(4.15) suggests that

$$M + |P| < 2\ell.$$
At a moment of time symmetry, there is a remarkable similarity between the signal for the presence of an apparent horizon, $R' = 0$ and that for the presence of a singularity, $R = 0$. In paper II, this meant that the techniques which were good for analyzing apparent horizons were almost always also good for singularities, and the effort required almost identical. In general, however, the signal for an apparent horizon will involve the extrinsic curvature of the spatial hypersurface and we need to distinguish between future and past horizons. It is this feature which complicates the analysis of apparent horizons. It is remarkable that the non-triviality of the momentum constraint and its coupling to the Hamiltonian constraint does not present a serious obstacle.

5 SUFFICIENCY

In paper II, we demonstrated that sufficiency conditions for the presence of trapped surfaces and singularities at a moment of time symmetry could be cast in terms of inequalities of the form, if $M >$ some constant times $\ell_0$, the geometry must contain a trapped surface for one constant and a singularity for some other constant. In this section, we generalize the inequalities of this form to general initial data.

As we will see, the natural generalizations of $M$ which arise are the quantities $M \pm P$, where

$$P := 4\pi \int_0^\ell d\ell R^2 J,$$

is a measure of the total current. We note that when the DEC is satisfied, then

$$M \pm P = 4\pi \int_0^\ell d\ell R^2 (\rho \pm J)$$

is positive. A few years ago, Bizon, Malec and Ó Murchadha demonstrated that if

$$M - P > \frac{7}{6}\ell,$$

on a maximal slice, assuming only that $\rho \geq 0$, the spatial geometry must contain a future trapped surface [5,6]. This generalizes the inequality which is valid at a moment of time symmetry. They also showed that the numerical coefficient appearing on the RHS is
sharp. They did this by constructing a solution with $M - P \geq (7/6 - \epsilon)\ell$ but without any trapped surface. This solution notably did not satisfy the DEC. More recently, Malec and Ó Murchadha were able to prove that if the DEC holds, the improved bound,

$$M - P > \ell,$$  \hspace{1cm} (5.4)

holds [7]. They did this exploiting in a striking way their reformulation of the constraints in terms of the optical scalar variables. The inequality (5.4) is particularly impressive because it coincides with the MSC result when $P = 0$.

Unfortunately, neither of the inequalities (5.3) and (5.4) involves spacetime scalars on the LHS so it is not clear what invariant significance they possess. Does a change of foliation change these results? To examine this question, in this section we will examine the issue within the framework of the one parameter family $\alpha$ foliations. We will inequalities similar to (5.3) and (5.4) which are not tied to maximal slicing but are valid in any valid $\alpha$-gauge.

5.1 Trapped Surfaces: Weak Energy, $\alpha$ - Gauge

Let us first assume that only the weak energy condition is satisfied but instead of considering only $\alpha = 2$ as BMÓM did in their derivation of Eq.(5.3), we will suppose that $0.5 < \alpha < \infty$. When $\alpha = 2$, Eq.(5.3) is satisfied. We will prove that, in general,

$$M - P \geq f(\alpha)\ell,$$  \hspace{1cm} (5.5)

where

$$f(\alpha) := 1 + \frac{1}{2} \frac{(1 - \alpha)^2}{2\alpha - 1},$$  \hspace{1cm} (5.6)

reproducing the bound Eq.(5.3) when $\alpha = 2$. However, the minimum of $f(\alpha)$ is assumed when $\alpha = 1$ where we reproduce Eq.(5.4). Curiously, the gauge providing the best bound when we do not assume dominant energy is not maximal slicing. The likely reason for this is that in this gauge, $K_R = P/R^2$.

The original proof by BMÓM exploited conformal coordinates. Our approach eschews tying ourselves to any particular spatial coordinate. Not only is the end result independent of the spatial coordinate, it is clear that the coordinate invariant approach is not only more transparent but also more efficient.
When Eq.(2.3) holds we can rewrite Eq.(2.1a) in the form

\[
4\pi \rho R^2 + \partial_\ell (RR') = \frac{1}{2} \left( 1 + (R')^2 \right) + \frac{1 - 2\alpha}{2} R^2 K_R^2.
\] (5.7)

We integrate from \( \ell = 0 \) up to the surface value \( \ell \):

\[
M + RR' = \Gamma + \frac{1 - 2\alpha}{2} \int_0^\ell d\ell R^2 K_R^2,
\] (5.8)

where \( \Gamma \) is defined (as in paper II) by

\[
\Gamma := \frac{1}{2} \int_0^{\ell} d\ell \left[ 1 + (R')^2 \right].
\] (5.9)

We now eliminate \( R' \) in the surface term in favor of the optical scalar \( \omega_+ \) and \( K_R \) using the defining relation (2.6a). The vanishing of \( \omega_+ \) signals that the geometry possesses an apparent horizon. Let us assume that the surface is not trapped so that \( \omega_+ > 0 \).

To eliminate the \( K_R \) dependence on the boundary which comes along with the replacement of \( R' \) by \( \omega_+ \) in Eq.(2.6a), we note that we can integrate the momentum constraint, Eq.(I.3.2) to obtain*

\[
R^2 K_R = P + (1 - \alpha) \int_0^\ell d\ell RR' K_R.
\] (5.10)

Substituting Eq.(2.6a) and (5.10) into (5.8) we now obtain

\[
M - P + 2\omega_+ = \Gamma + \frac{1}{2} (1 - 2\alpha) \int_0^\ell d\ell R^2 K_R^2 - (1 - \alpha) \int_0^\ell d\ell RR' K_R.
\] (5.11)

When \( \alpha > 0.5 \), the second term on the RHS is manifestly negative. As such we could discard it to cast (5.11) as an inequality. However, it is clear that we can do better by first completing the square in the sum of the second and third terms before discarding:

\[
\frac{1}{2} (1 - 2\alpha) \int_0^\ell d\ell R^2 K_R^2 - (1 - \alpha) \int_0^\ell d\ell RR' K_R
\]

\[
= \frac{1}{2} (1 - 2\alpha) \int_0^\ell d\ell \left( RK_R + \frac{1 - \alpha}{1 - 2\alpha} R' \right)^2 + \frac{1}{2} \frac{(1 - \alpha)^2}{2\alpha - 1} \int_0^\ell d\ell R'^2
\]

\[
\leq \frac{1}{2} \frac{(1 - \alpha)^2}{2\alpha - 1} \ell.
\] (5.12)

* The privileged role of the gauge with \( \alpha = 1 \) is evident.
On the last line, we have used the fact that when \( \rho \) is positive and \( \alpha \geq 0.5 \), \( R^2 \leq 1 \). In addition, under these conditions, we obtain the same upper bound on \( \Gamma \),

\[
\Gamma \leq \ell . \tag{5.13}
\]
as we obtained at a MSC. We conclude that if the spherical surface is not trapped, then

\[
M - P < 1 + \frac{1}{2} \frac{(1 - \alpha)^2}{2\alpha - 1} \ell . \tag{5.14}
\]

Thus if the surface is trapped or the interior contains a trapped surface Eq.(5.5) holds.

Let us now examine some extreme cases:

We note that \( f \) diverges as we approach the minisuperspace lightcone, \( \alpha \to 0.5 \) and \( \alpha \to \infty \). While it is tempting to interpret this as a signal of the breakdown of the gauge on the lightcone, it is also clear that the discarded negative term blows up at these two values.

Let us consider the two extreme distributions saturating the DEC everywhere, \( P = \pm M \) which are respectively the cases of a radially outward and a radially inward moving null fluid. In the former case, Eq.(5.14) becomes a vacuous statement — even though we do expect it to be more difficult (if not impossible) to form an apparent horizon. In the later case, we have that if \( 2M \geq f(\alpha)\ell \), the geometry will possess a trapped surface. It is twice as easy to form an apparent horizon with an inflowing null fluid as it is with a stationary fluid.

It is also possible to tighten the sufficiency condition in the same way we did for MSCs when \( \rho' \leq 0 \) if, in addition, \( J \) has a fixed sign.

We note that in same way we did when \( K_{ab} = 0 \), when \( \rho' \leq 0 \) we can replace Eq.(5.13) by

\[
\Gamma \leq \ell - \frac{M}{3} - \frac{8\pi}{R} \int_0^\ell d\ell R^3 K_R - \int_0^\ell d\ell R^2 K_R^2 . \tag{5.15}
\]

If \( J \) is positive (negative) everywhere, the inequality still holds when the third term on the RHS is dropped. The (negative) last term on the RHS can now be added to the (negative) term of the same form in Eq.(5.11) before the completion of the square. We get

\[
\frac{4M}{3} - P < \left[ 1 + \frac{(1 - \alpha)^2}{2(2\alpha + 1)} \right] \ell . \tag{5.16}
\]
As before, this is minimized when \( \alpha = 1 \) and when \( P = 0 \) again reproduces the result at a moment of time symmetry. We note that both the LHS and the RHS have been improved. When \( \alpha = 0.5 \), unlike Eq.(5.14) the RHS of Eq.(5.16) does not diverge. From one point of view, Eq.(5.18) is not very satisfactory — we have broken the symmetry between \( J \) and \( \rho \). However, on the other hand this asymmetry permits us to write down a non-vacuous sufficiency condition when the the spatially averaged DEC is saturated with \( P = M \). Whereas Eq.(5.14) is vacuous under these conditions, Eq.(5.18) provides the non-trivial statement: suppose \( \rho' \leq 0 \) and the motion of matter is outward and null, then if

\[
M > 3 \left[ 1 + \frac{(1 - \alpha)^2}{2(2\alpha + 1)} \right] \ell ,
\]

the spatial geometry will possess an apparent horizon. When \( \alpha = 1 \), this value of \( M \) is three times larger than that of a corresponding stationary distribution of matter.

5.2 Trapped Surfaces: Dominant Energy, \( \alpha \) - Gauge

When the DEC holds, our experience suggests that the appropriate variables are the optical scalars. The optical scalar which marks the presence of a future trapped surface is \( \omega_+ \). Remarkably, only the constraint (3.1a) determining the spatial derivative of \( \omega_+ \) will play a role in the determination of the inequality. Let us first recast Eq.(3.1a) as an equation for the spatial derivative of \( R\omega_+ \):

\[
(R\omega_+)' = -8\pi R^2 (\rho - J) + \frac{1}{4} \left[ 2\omega_+\omega_- + 4 + 4RK\omega_+ - \omega_+^2 \right].
\]  

(5.18)

This equation can be integrated up to give

\[
R\omega_+ = -2(M - P) + 2\Gamma_+ ,
\]

(5.19)

where

\[
\Gamma_+ := \frac{1}{8} \int_0^\ell d\ell \left[ 2\omega_+\omega_- + 4 + 4RK\omega_+ - \omega_+^2 \right]
\]

(5.20)

are the natural optical scalar generalizations of \( \Gamma \). In particular, when \( K_{ab} = 0 \), \( \Gamma_+ = \Gamma \). In general, in the gauge (2.3),

\[
RK = (2 - \alpha)RK_R = (2 - \alpha)\frac{1}{4}(\omega_+ - \omega_-),
\]

(5.21)
so that

\[ \Gamma_+ = \frac{1}{8} \int_0^\ell d\ell \left[ 4 + \alpha \omega_+ \omega_- + (1 - \alpha) \omega_+^2 \right]. \tag{5.22} \]

Let us examine two special cases. The case examined by MÓM was \( \alpha = 2 \). Now

\[ \Gamma_+ = \frac{1}{8} \int_0^\ell d\ell \left[ 4 + 2 \omega_+ \omega_- - \omega_-^2 \right]. \tag{5.23} \]

We note that \( 0 \leq (\omega_+ - \omega_-)^2 \) implies

\[ 2 \omega_+ \omega_- - \omega_-^2 \leq \omega_-^2. \tag{5.24} \]

We now exploit the inequality (2.19) to obtain \( \Gamma_+ \leq \ell \). If the surface is not future trapped then \( \omega_+ > 0 \) in Eq.(5.19) reproducing Eq.(5.4). If \( \alpha = 1 \), then*

\[ \Gamma_+ = \frac{1}{8} \int_0^\ell d\ell [4 + \omega_+ \omega_-] \leq \ell, \tag{5.25} \]

using Eq.(2.22), which is the same as the \( \alpha = 2 \) value. The \( \alpha = 1 \) bound does not improve even though the energy condition is more stringent. This suggests that this bound is sharp.

What if \( \alpha \) is not one of these two special values? In general, the bound on both \( \omega_+ \) and \( \omega_- \) depends on \( K \). This makes it less obvious how to bound \( \Gamma_+ \) for any \( \alpha \) other than these two values. What we can do is bootstrap on Eq.(5.21) to turn this into a bound which is independent of \( K \). Let \( \kappa_R = \text{Max}|RK_R| \). We note that Eq.(5.21) implies

\[ \kappa_R \leq \frac{1}{4} (\text{Max} |\omega_+| + \text{Max} |\omega_-|). \tag{5.26} \]

We know from Eqs.(6.4a & b) in paper I that \( \text{Max} |\omega_+| = \text{Max} |\omega_-| \leq \kappa + \sqrt{|\kappa|^2 + 4} \). Hence

\[ \leq \frac{1}{2}(\kappa + (|\kappa|^2 + 4)^{1/2}) \]

\[ = \frac{1}{2} \left( |2 - \alpha| \kappa_R + (|2 - \alpha|^2 \kappa_R^2 + 4)^{1/2} \right). \tag{5.26'} \]

It is straightforward to invert Eq.(5.26') to obtain the bound

\[ \kappa_R \leq \frac{1}{\sqrt{1 - |2 - \alpha|}}. \tag{5.27} \]

* Note that in this gauge, \( \Gamma_+ = \Gamma_- \).
This bound unfortunately is valid only for $1 < \alpha < 3$. Eq.(5.21), in turn, implies the bounds on $\omega_+$ and $\omega_-$,

$$\omega_+^2 \leq \frac{4}{1 - |2 - \alpha|}.$$  \hspace{1cm} (5.28)

We note that when $\alpha = 2$ we reproduce the bounds, Eq.(2.19). What is remarkable is that these bounds are independent of $J$. Note that our knowledge of the bound on $R'^2$ does not help (nor should it be expected to help) to improve these inequalities.

We now return to Eq.(5.22). We introduce a parameter $b$, and we rewrite

$$\Gamma_+ = \frac{1}{8} \int_0^\ell d\ell \left[ 4 + (\alpha + (1 - \alpha)b)\omega_+\omega_- + (1 - \alpha)(\omega_+^2 - b\omega_+\omega_-) \right].$$  \hspace{1cm} (5.29)

We now complete the square on the last term and use an obvious modification of Eq.(5.24) to obtain the inequality

$$\Gamma_+ \leq \frac{1}{8} \int_0^\ell d\ell \left[ 4 + (\alpha + (1 - \alpha)b)\omega_+\omega_- + (\alpha - 1)\frac{b^2}{4}\omega_-^2 \right].$$  \hspace{1cm} (5.30)

We can now exploit Eqs.(2.22) and (5.28) to obtain,

$$\Gamma_+ \leq \begin{cases} \frac{1}{2} \left[ 1 + \alpha + (1 - \alpha)b + \frac{b^2}{4} \right] \ell, & 1 \leq \alpha \leq 2 \\ \frac{1}{2} \left[ 1 + \alpha + (1 - \alpha)b + \frac{2(3 - \alpha)}{2 - \alpha} \right] \ell, & 2 \leq \alpha \leq 3. \end{cases}$$  \hspace{1cm} (5.31)

The idea is to find the $b$ which minimizes the RHS. For fixed $\alpha$, the RHS is minimized when

$$b = \begin{cases} \frac{2(\alpha - 1)}{2 - \alpha} & 1 \leq \alpha \leq 2 \\ \frac{2(3 - \alpha)}{2} & 2 \leq \alpha \leq 3. \end{cases}$$  \hspace{1cm} (5.32)

The corresponding values of $\Gamma_+$ are

$$\Gamma_+ \leq \begin{cases} \frac{1}{2} \alpha(3 - \alpha)\ell, & 1 \leq \alpha \leq 2 \\ \frac{1}{2} \left[ \alpha^2 - 3\alpha + 4 \right] \ell, & 2 \leq \alpha \leq 3. \end{cases}$$  \hspace{1cm} (5.33)

These bounds reproduce the optimal values obtained when $\alpha = 1$ and $\alpha = 2$. In the neighborhood of $\alpha = 2$, this bound is an improvement over the bound (5.5) which does not assume the DEC. Even though the bound (5.27) diverges both as $\alpha \to 1_+$ and as $\alpha \to 3_-$, the bounds (5.14) at these limit points nonetheless are finite, and in the former case as we
have just seen even coincides with its optimal value there. It is not clear how to extend this technique outside the range $\alpha = 2 \pm 1$.

5.3 Singularities

It is not obvious how to import the DEC into the statement of a sufficiency condition for singularities. What we have is an obvious generalization of the moment of time symmetry result: We recall that, in general,

$$M + RR' = \Gamma + \frac{1 - 2\alpha}{2} \int_0^\ell d\ell R^2 K^2_R. \quad (5.34)$$

We proceed exactly as for a moment of time symmetry. $\Gamma$ is always bounded by one whenever $\rho$ is positive. Furthermore, $R' \leq 1$ so that $R(\ell) \leq \ell$ everywhere on a non-singular geometry and $R' \geq -1$. The surface term is therefore bounded from below by $-\ell$. Finally, the second term is negative whenever $\alpha \geq 0.5$ — The $K_{ab}$ dependence is trivially handled. Thus we get

$$M \leq 2\ell, \quad (5.35)$$

independent of the value of $\alpha \geq 0.5$ which is exactly the result at a moment of time symmetry.

As at a moment of time symmetry, if we place constraints on the sources it is possible to tighten the inequality. We note that when $\rho' \leq 0$ and $J$ is positive (or negative) everywhere, then Eq.(5.15) can be truncated even more brutally, $\Gamma \leq \ell - M/3$. We get

$$M \leq \frac{3}{2} \ell. \quad (5.36)$$

Unlike the moment of time symmetry discussion we cannot claim that this represents a universal bound when $\rho' < 0$ and $J$ is positive (negative). The reason is that the geometry can still turn singular if $J$ is large enough.

We note that there is no obvious way of introducing $P$ into either Eq.(5.35) or Eq.(5.36). The singularity condition is not symmetrical in $M$ and $P$.

6 NECESSITY

We noted in paper II, in our examination of a moment of time symmetry, that the necessary conditions we were able to formulate with respect to $M$ and $\ell$ were extremely
weak. If \( J \neq 0 \) even these conditions appear to be beyond our reach. What one can do is provide generalizations of the necessary conditions which were formulated with respect to the variables, \( \rho_{\text{Max}} \) and \( \ell_0 \). These inequalities assumed the form

\[
\rho_{\text{Max}} \ell^2 < \text{constant}.
\] (6.1)

Typically, we would expect \( |J_{\text{Max}}| \) and \( \alpha \) to enter into this description. We would expect that by appealing to the DEC the inequalities should simplify. Crucial to the derivation of Eq.(6.1) are two simple Sobolev inequalities of the form

\[
S \int_0^{\ell_1} d\ell \, R^2 \leq \int_0^{\ell_1} d\ell \, R'^2,
\] (6.2)

where \( S \) depends on the boundary conditions satisfied by \( R \). In general \( R(0) = 0 \). At the first trapped surface, \( R'(\ell_1) = 0 \) and \( S = \pi^2/4\ell_1^2 \). At a singularity, \( R(\ell_1) = 0 \) and \( S = \pi^2/\ell_1^2 \). At a singularity, we found that \( R \) tends to zero like \( R \sim (\ell - \ell_1)^{2/3} \) so that \( R' \) diverges like \( (\ell - \ell_1)^{-1/3} \). Even though \( R' \) diverges so that the integrand on the RHS of Eq.(5.9) diverges, the integral itself remains finite. When \( J \neq 0 \), however, \( R \) diverges more strongly, \( R \sim (\ell - \ell_1)^{1/1+\alpha} \) (see Eq.(2.14)) so that \( R' \sim (\ell - \ell_1)^{-\alpha/1+\alpha} \). Thus the integral on the RHS of Eq.(5.9) will only exist if \( \alpha < 1 \) — outside the range found to provide the best sufficiency results in Sect.5.1. Thus, whereas we found that we could optimize the inequalities of necessity at a moment of time symmetry by weighting \( R'^2 \) by an appropriate power of \( R \), a non-trivial weighting will be essential when \( J \neq 0 \) at least in the case of singularities.

A Bound on \( K_R \)

To form a necessary condition for singularities it is important to possess some control over \( K_R \) in a manner which does not require the geometry to be regular. In particular, we cannot exploit Eq.(5.27) which is only true in regular initial data. It is, however, simple to obtain a bound on \( K_R \) by \( |J_{\text{Max}}| \) without making any assumptions about the regularity of the geometry. We have that

\[
|K_R| \leq \frac{4\pi}{R^{1+\alpha}} \int_0^{\ell_1} d\ell R^{1+\alpha} |J|.
\] (6.3)

In general,
\[ |K_R| \leq \frac{4\pi |J_{\text{Max}}|}{R^{1+\alpha}} \int_0^{\ell_0} d\ell R^{1+\alpha}. \]  

(6.4)

This is the result we will exploit below. There are some interesting related inequalities. Suppose that the DEC holds, and \( \alpha = 1 \). We obtain

\[ |K_R| \leq \frac{M}{R^2}. \]  

(6.5)

This inequality in turn implies that the proper spatial average of \( |K_R| \) is bounded by the product, \( \rho_{\text{Max}} \ell_1 \):

\[ < |K_R| > \leq 4\pi \rho_{\text{Max}} \ell_1, \]  

(6.7)

a pretty result, even if we have not found an application for it.

A Bound on \( R' \)

We will also require a bound on \( R' \) which does not require the geometry to be globally regular. To obtain this bound, we note that in any \( \alpha \)-gauge, the Hamiltonian constraint Eq.(2.1a) reads

\[ RR'' = \frac{1}{2}(1 - R'^2) + \frac{1}{2}R^2(1 - 2\alpha)K_R^2 - 4\pi R^2 \rho. \]

At the origin, we have \( R' = 1 \). At a singularity we have \( R' < 0 \) whereas at a globally regular solution we have \( R' \to 1 \) at infinity. If \( R' \) has an interior maximum then \( R'' \) vanishes there. Hence at that point we have

\[ 1 - R'^2 = R^2(2\alpha - 1)K_R^2 + 8\pi R^2 \rho. \]

Thus, if we have a standard \( \alpha \)-slice, \( i.e., \alpha > 0.5 \) and if the source satisfies the weak energy condition, \( \rho \geq 0 \) we must have

\[ 1 - R'^2 \leq 0, \]

and therefore at the maximum of \( R' \) we must have \( R' \leq 1 \). Therefore this is a global bound independent of whether the slice is regular or not.

6.1 Singularities
The most naive generalization of Eq.(6.1) would be an inequality treating $\rho_{\text{Max}}$ and $|J_{\text{Max}}|$ symmetrically, of the form: if

$$\left(\rho_{\text{Max}} + |J_{\text{Max}}|\right)\ell_0^2 < c, \quad (6.8)$$

for some constant $c$, the geometry is regular. However, our experience examining the approach to singularities suggests that this is too optimistic. The natural inequality we obtain involves not $J$ but its square, assuming the form: if

$$\left(\rho_{\text{Max}}\ell_0^2 + c_1(\sqrt{|J_{\text{Max}}|\ell_0^2})^2 \right) < c_2, \quad (6.9)$$

where $c_1$ and $c_2$ are two constants, the geometry is regular. Even if matter satisfies the DEC, once we foliate extrinsically the symmetry is broken. The value of $J$ plays a more significant role than the value of $\rho$. This is consistent with our findings in Sect.2 in our examination of the generic behavior of the metric in the neighborhood of a singularity in an $\alpha$-foliation of spacetime. The optical scalar variables suggest that a more judicious gauge involving some mix of intrinsic and extrinsic variables might restore the symmetry between $\rho$ and $J$ we have broken with the $\alpha$-parametrized gauges.

We note that Eq.(2.1) implies

$$\frac{1}{2}(1 + R^2) = (RR')' + 4\pi \rho R^2 + \frac{1}{2}(2\alpha - 1)R^2 K_R^2. \quad (6.10)$$

The last term is manifestly positive. Suppose that the geometry is singular at $\ell = \ell_1$. We cannot simply integrate Eq.(6.10) and discard the boundary term. First of all, $R^2$ is not integrable on the interval $[0, \ell_1]$ and, secondly, the surface term $RR'$ does not vanish at the singularity unless $\alpha < 1$.

What we need to do is multiply Eq.(6.10) by some (positive) power of $R$ before integration. The relevant power of $R$ will generally depend on the value of $\alpha$. To restore the divergence appearing in Eq.(6.10) we need to perform an integration by parts. We now integrate up to $\ell_1$:

$$\frac{1}{2} \int_0^{\ell_1} d\ell R^a(1 + (2a + 1)R^2) = R^{1+a}R' \bigg|^{\ell_1}_0 + 4\pi \int_0^{\ell_1} d\ell \rho R^{2+a} + \frac{1}{2}(2\alpha - 1) \int_0^{\ell_1} R^{2+a} K_R^2. \quad (6.11)$$

To discard the boundary term, we require $R^{1+a}R'$ to vanish at the singularity. This implies that
This choice of $a$ simultaneously bounds the integral over $R^a R'^2$.

We also will need to place a bound on the last term on the RHS of Eq.(6.11). We exploit Eq.(6.4) to bound $K_R$. The problem is that this bound involves the positive power of $R$, $R^{1+\alpha}$ in the denominator which is difficult to control. We obtain the bound,

$$
\int_0^{\ell_1} d\ell R^{2+a} K_R^2 \leq (4\pi)^2 J_{\text{Max}}^2 \int_0^{\ell_1} d\ell R^{a-2\alpha} \left( \int_0^\ell d\ell R^{1+\alpha} \right)^2,
$$

(6.13)
on the term quadratic in $K_R$. If the weighting term is chosen such that

$$
a \geq 2\alpha,
$$

(6.14)
the denominator problem is solved. Fortunately, such values are consistent with Eq.(6.12) for all physically acceptable values of $\alpha$. The RHS of Eq.(6.11) is clearly simplest when

$$
a = 2\alpha.
$$

(6.14')
This is the value we will henceforth adopt for $a$. The expression is still not very useful as it stands. A remarkable fact, however, is that we can bound it by an integral over $R^{2(1+\alpha)}$. To understand why this is important, note that the integral over $R'^2$ appearing on the LHS of Eq.(6.11) can be cast in the form

$$
\int_0^{\ell_1} d\ell R^{2a} R'^2 = \frac{1}{(\alpha + 1)^2} \int_0^{\ell_1} d\ell (R^{\alpha+1})' R^{\alpha+1}.
$$

(6.15)
The Sobolev inequality can be exploited to place a bound on the integral over the function $R^{2(1+\alpha)}$:

$$
S_0 \int_0^{\ell_1} d\ell R^{2(\alpha+1)} \leq \frac{1}{(\alpha + 1)^2} \int_0^{\ell_1} d\ell (R^{\alpha+1})'^2,
$$

(6.16)
where the constant $S_0 = \pi^2 / \ell_1^2$ is the Sobolev constant which is relevant for functions which vanish at both $\ell = 0$ and $\ell = \ell_1$.

We now prove the existence of a bound of the form

$$
\int_0^{\ell_1} d\ell \left( \int_0^\ell d\ell R^{1+\alpha} \right)^2 \leq C \int_0^{\ell_1} d\ell R^{2(\alpha+1)},
$$

(6.17)
for some appropriate constant $C$. A crude bound is provided by the positivity of the covariance for any power $R^n$: (Holder Inequality),

$$< R^n >^2 \leq < R^{2n} >,$$  \hspace{1cm} (6.18)

which implies

$$\left( \int_0^\ell d\ell R^n \right)^2 \leq \ell \int_0^\ell d\ell R^{2n},$$ \hspace{1cm} (6.19)

so that

$$\int_0^{\ell_1} d\ell \left( \int_0^\ell d\ell R^{1+\alpha} \right)^2 \leq \frac{\ell_1^2}{2} \int_0^{\ell_1} d\ell R^{2(1+\alpha)}.$$ \hspace{1cm} (6.20)

We can, however, do better. Let

$$G(\ell) := \int_0^\ell d\ell R^n.$$ \hspace{1cm} (6.21)

Now $G(0) = 0$ and $G'(\ell_1) = 0$, for all $n \geq 0$. We apply the Sobolev inequality to $G$ with the appropriate constant

$$\int_0^{\ell_1} d\ell G_n(\ell)^2 \leq \left( \frac{2\ell_1}{\pi} \right)^2 \int_0^{\ell_1} d\ell R^{2n}.$$ \hspace{1cm} (6.22)

so that

$$\int_0^{\ell_1} d\ell \left( \int_0^\ell d\ell R^{2(1+\alpha)} \right)^2 \leq \left( \frac{2\ell_1}{\pi} \right)^2 \int_0^{\ell_1} d\ell R^{2(1+\alpha)}.$$ \hspace{1cm} (6.23)

This is better by a factor of $\pi^2/8$ than the estimate (6.20). The end result is the bound

$$\int_0^{\ell_1} R^{2(1+\alpha)} K_R^2 \leq 64 J_{\text{Max}}^2 \ell_1^2 \int_0^{\ell_1} R^{2(1+\alpha)},$$ \hspace{1cm} (6.24)

on the third term on the RHS of Eq.(6.11). We can now write

$$1 \leq 2 \left[ 4\pi \rho_{\text{Max}} + 32(2\alpha - 1) J_{\text{Max}}^2 \ell_1^2 - \left( \frac{\pi}{\ell_1} \right)^2 \frac{1 + 4\alpha}{2(1 + \alpha)^2} \int_0^{\ell_1} d\ell R^{2(1+\alpha)} / \int_0^{\ell_1} d\ell R^{2\alpha} \right].$$ \hspace{1cm} (6.25)
In paper II, we proved that the ratio of integrals appearing on the RHS can be bounded as follows (Eq.(II 6.3.16)) \((a = 2\alpha)\)

\[
\frac{\int_0^{\ell_1} R^{2+a} d\ell}{\int_0^{\ell_1} R^a d\ell} \leq \frac{1 + a}{3 + a} \ell_1^2 ,
\]

which implies

\[
\frac{1}{2} + \alpha + \frac{1}{2} \frac{4\alpha}{(1 + \alpha)^2} \pi^2 \leq 4\pi \rho_{\text{Max}} \ell_1^2 + 32(2\alpha - 1) J_{\text{Max}}^2 \ell_1^4 .
\]

If \(\alpha = 1,\)

\[
\frac{5\pi}{32} \left[ 1 + \frac{4}{3\pi^2} \right] \leq \rho_{\text{Max}} \ell_1^2 + \frac{8}{\pi} (J_{\text{Max}} \ell_1^2)^2 .
\]

In Eq.(6.27), it does not make much sense to claim that one value of \(\alpha\) provides a better bound than another value. Not only is the inequality invalidated if the quadratic in \(J_{\text{Max}}\) is dropped, \(J_{\text{Max}}\) plays a more decisive role than \(\rho_{\text{Max}}\) in the inequality (6.27), appearing as it does through its square in contrast to \(\rho\) which appears linearly. The MS inequality does not generalize in the obvious linear way.

If the DEC, the inequality simplifies. For \(\alpha = 1\) we obtain

\[
\frac{5}{6} + \frac{5\pi^2}{8} \leq 32 \left( \rho_{\text{Max}} \ell_1^2 + \frac{\pi}{16} \right)^2 - \frac{\pi^2}{8} ,
\]

or

\[
\frac{1}{8} \left[ \sqrt{\frac{5}{3} + \frac{3}{2} \pi^2} - \frac{\pi}{2} \right] \leq \rho_{\text{Max}} \ell_1^2 .
\]

The LHS \(\sim 5/16\), which is approximately half as good as the moment of time symmetry result.

6.2 Apparent Horizons

We note that Eq.(2.1) implies

\[
\frac{1}{2} (1 + R'^2) = (R \omega_+ - R^2 K_R)' + 4\pi \rho R^2 + \frac{1}{2} (2\alpha - 1) R^2 K_R^2 .
\]

Again the third term on the RHS is manifestly positive. We can integrate Eq.(6.31) up to the first future horizon.
\[ \Gamma = -RK_R|_{\ell_1} + 4\pi \int_0^{\ell_1} d\ell R^2 \rho + \frac{1}{2} (2\alpha - 1) \int_0^{\ell_1} d\ell R^2 K_R^2 , \]  

(6.32)

where \( \Gamma \) is given by Eq.(5.9). We note that Eq.(6.3) places a bound on \( K_R \) in the surface term. Thus

\[ \Gamma \leq 4\pi \left( \rho_{\text{Max}} + |J_{\text{Max}}| \right) \int_0^{\ell_1} d\ell R^2 + \frac{1}{2} (2\alpha - 1) \int_0^{\ell_1} d\ell R^2 K_R^2 . \]  

(6.33)

A linear term in \( J_{\text{Max}} \) appears in the apparent horizon inequality condition which is not present in the singularity inequality. This is a reflection of the different boundary conditions enforced there.

We can exploit a Sobolev inequality to place a bound on the integral over the interval \((0, \ell_1)\) of the quadratic \( R^2 \) by the same integral over the quadratic, \( R'^2 \):

\[ S \int_0^{\ell_1} d\ell R^2 \leq \int_0^{\ell_1} d\ell R'^2 . \]  

(6.34)

The relevant boundary conditions are

\[ R' + RK_R = 0 \]  

(6.35)

at \( \ell = \ell_1 \). The inequality is saturated by the trigonometric function,

\[ R(\ell) = \sin(\gamma \ell) , \]

which also determines the optimal value of \( S = \gamma^2 \). The boundary condition, (6.35) determines \( \gamma \) to be the lowest solution of the transcendental equation,

\[ \tan \gamma \ell_1 = -\frac{\gamma}{K_R} . \]  

(6.36)

We note that

\[ \gamma \leq \frac{\pi}{2\ell_1} \]  

(6.37)

if \( K_R \) is negative with \( \gamma \to \pi/2\ell_1 \) as \( K_R \to 0 \) which is the moment of time symmetry bound and \( \gamma \to \pi/\ell_1 \) as \( K_R \to +\infty \).

When we attempt to bound the right hand side we run into the same problem we faced before with the last term. In addition, however, we must contend with the surface term.
The same weighting we found worked before works again. To restore the divergence appearing in Eq.(6.31) we need to perform an integration by parts. We integrate up to $\ell_1$:

$$\frac{1}{2}\int_0^{\ell_1} d\ell R^{2\alpha}(1+(4\alpha+1)R^2) = -R^{2(1+\alpha)}K_R\bigg|_{\ell_1}$$

$$+ 4\pi \int_0^{\ell_1} d\ell \rho R^{2(1+\alpha)} + \frac{1}{2}(2\alpha - 1) \int_0^{\ell_1} R^{2(1+\alpha)}K_R^2. \quad (6.38)$$

We now exploit Eq.(6.3) to bound the $K_R$ and $K_R^2$ term as follows. For the former,

$$R^{2(1+\alpha)}K_R\bigg|_{\ell_1} \leq 4\pi R^{1+\alpha}J_{\text{Max}} \int_0^{\ell_1} d\ell R^{1+\alpha}. \quad (6.39)$$

The weighting process has broken the symmetry under interchange of $\rho$ and $J$ of the linear terms on the RHS of Eq.(6.38). For the term quadratic in $K_R$, we again have ((6.12) with $a = 2\alpha$)

$$\int_0^{\ell_1} R^{4}K_R^2 \leq (4\pi)^2 J_{\text{Max}}^2 \int_0^{\ell_1} d\ell R^2 \bigg( \int_0^{\ell} d\ell R^2 \bigg)^2. \quad (6.39)$$

We again require a bound on the last term by an integral over $R^{2(1+\alpha)}$. This time, however, the Sobolev constant is that which is relevant for functions which vanish at $\ell = 0$ but satisfy Eq.(6.35) at $\ell = \ell_1$, i.e., $S = \gamma^2$, where $\gamma$ is given by (6.36) and (6.37).

The crude bound we derived before, (6.20), is expected to work better this time. As before, however, we can do better. This time we let

$$H(\ell) := \int_0^{\ell} d\ell R^n / \int_0^{\ell_1} d\ell R^n. \quad (6.40)$$

Now $H(0) = 0$ and $H(\ell_1) = 1$ for all $n$. We apply the Sobolev inequality to $H$ with the appropriate constant

$$\int_0^{\ell_1} d\ell H(\ell)^2 \leq \left( \frac{2\ell_1}{\pi} \right)^2 \int_0^{\ell_1} d\ell R^{2n} / \left( \int_0^{\ell_1} d\ell R^n \right)^2. \quad (6.41)$$

so that Eq.(6.23) holds exactly as before and we again obtain the bound (6.24) for the integral over $K_R^2$. We can now write
\[1 \leq 2 \left[4\pi \rho_{\text{Max}} + 32(2\alpha - 1)J^2_{\text{Max}}\ell_1^2 - \gamma^2 \frac{1 + 4\alpha}{2(1 + \alpha)^2} \right] \int_0^{\ell_1} R^{2(1+\alpha)} / \int_0^{\ell_1} dR^{2\alpha} \]
\[+ 8\pi J_{\text{Max}} \ell_1^{1+\alpha} \int_0^{\ell_1} R^{1+\alpha} / \int_0^{\ell_1} dR^{2\alpha}.
\]

(6.42)

An upper bound on \(\gamma\) in Eq.(6.42) is provided by its \(K_R \rightarrow \infty\) limit, i.e., \(\pi/\ell_1\) and the lower limit is zero. We can again exploit (6.26) to bound the ratio of the integrals in the first term of (6.42). In the second term it is clear that

\[\ell_1^{1+\alpha} \int_0^{\ell_1} R^{1+\alpha} / \int_0^{\ell_1} dR^{2\alpha}\]

is bounded by \(\ell_2^2\) if \(\alpha \leq 1\). Therefore a necessary condition for the appearance of a trapped surface at some proper radius \(\ell_1\) is that

\[4\pi \rho_{\text{Max}}\ell_1^2 + 4\pi \frac{3 + 2\alpha}{1 + 2\alpha} J_{\text{Max}}\ell_1^2 + 32(2\alpha - 1)J^2_{\text{Max}}\ell_1^4 \geq \frac{13 + 2\alpha}{1 + 2\alpha}.
\]

(6.43)

CONCLUSIONS

This paper concludes a series of three papers on the identification of the configuration space in spherically symmetric general relativity. We have attempted to provide a coherent synthesis of two very different ways of looking at the constraints, one in terms of the traditional metric variables, the other in terms of the optical scalar variables. Which description is appropriate depends very much on the details of the problem under consideration.

A very satisfying representation has emerged of regular closed solutions as closed bounded trajectories on the \((\omega_+, \omega_-)\) plane. In this representation, \(R\) plays a secondary role. We have performed the analysis explicitly in vacuum. We will show elsewhere that this plane also provides a very profitable representation of \(\alpha\)-slicings of the Schwarzschild spacetime [9].

We have presented a variety of necessary and sufficient conditions for the presence of apparent horizons and singularities in the initial data. This paper is necessarily more open-ended than either paper I or paper II. It is clear that some of the Sobolev inequalities exploited in sect.6 can be sharpened. Indeed, the professional will consider our approach to functional analysis extremely heuristic. As physicists, however, we are more interested in the fact that such bounds can be established than in squeezing them for better constants.
Where does one go from here? The obvious challenge is to generalize this work to non-spherically symmetric geometries. One needs to bear in mind, however, that our ability to describe the configuration space in considerable detail has relied on features of the spherically symmetric problem which we know do not admit generalizations.

There is still, however, much that needs to be done before we can claim to understand spherical symmetry.

We need first of all to examine the classical evolution. Write down the Einstein equations with respect to the optical scalar variables. Can we cast the theory in Hamiltonian form? If the value of these variables in the analysis of the constraints is anything to go by, one has every reason to expect that they will throw light on the solution of the dynamical Einstein equations, both analytically and numerically. Indeed Rendall has recently exploited these variables to establish a global existence result [10].

A physically interesting question that is extremely relevant is the identification of initial data that potentially might develop apparent horizons. In principle it should be possible to do this exploiting in addition to the constraints the dynamical Einstein equations evaluated on the initial hypersurface. These equations involve the pressure of matter though some equation of state. The scenario which is most susceptible to collapse is pressureless matter. We should be able to exploit this condition to formulate necessary conditions along the lines developed in sect.6. At the other extreme, a stiff equation of state would inhibit collapse. Thus such a scenario might provide a sufficient condition. A successful analysis of this nature has the promise of putting an analytical handle on the physics hinted at in Choptuik’s numerical simulations of the collapse of a massless scalar field [11].

Finally, the bounds on the optical scalars are certain to have profound implications for the canonical quantization of this model for gravity [12]. We hope to examine this problem in a subsequent publication.
Figure Captions

\textbf{fig.(3.1)} Non-singular exterior vacuum solutions on the \((\omega_+, \omega_-)\) plane for \(\alpha = 2\). All non-singular exterior trajectories lie within the ‘wedge’ shaped region, \(\Omega_0\), bounded by the arc segments, \(CQ\) and \(QD\).

\textbf{fig.(4.1)} \(M/2\ell_0\) vs. \(v\) in the shell model. All non-singular geometries lie below the curve \(V(R_c) = 1\).
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