PRIMITIVE PAIRS OF K-GROUPS

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Abstract. In Primitive pairs of p-solvable groups, J. Algebra 324 (2010) 841–859, the author proved a non existence theorem for certain types of amalgams of p-solvable groups in the presence of operator groups acting coprimely on the groups in the amalgam. An application of that work was a new proof of the Solvable Signalizer Functor Theorem. In this article, the p-solvable restriction will be weakened to a K-group hypothesis. An application of this work will be a new proof of the Nonsolvable Signalizer Functor Theorem.

1. Introduction

The results of [1] will be extended from p-solvable groups to K-groups. Just as an application of [1] is a new proof of the Solvable Signalizer Functor theorem [2], an application of this work will be a new proof of the Nonsolvable Signalizer Functor Theorem. This work is also a continuation of [3] and [4] in which the theory of automorphisms of K-groups is developed.

We will assume familiarity with [1]. However, in order to make the statement of the main result self contained, we remind the reader of the relevant definitions from [1]. Throughout, group will mean finite group.

Definition. Let M be a group and p a prime. Then M has characteristic p if $C_M(O_p(M)) \leq O_p(M)$.

Definition. Let G be a group. A weak primitive pair for G is a pair $(M_1, M_2)$ of distinct nontrivial subgroups that satisfy:
- whenever $\{i, j\} = \{1, 2\}$ and $1 \neq K \text{ char } M_i$ with $K \leq M_1 \cap M_2$ then $N_{M_j}(K) = M_1 \cap M_2$.

If p is a prime then the weak primitive pair has characteristic p if in addition:
- for each i, $M_i$ has characteristic p and $O_p(M_i) \leq M_1 \cap M_2$.

Definition. Suppose the group R acts coprimely on the group G and that p is a prime. Then

$$O_p(G; R)$$

is the intersection of all the R-invariant Sylow p-subgroups of G.

Recall that there do exist R-invariant Sylow p-subgroups of G, the set of which is permuted transitively by $C_G(R)$. Moreover, every R-invariant p-subgroup of G is

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contained in an $R$-invariant Sylow $p$-subgroup of $G$. Consequently, $O_p(G; R)$ is the unique maximal $RC_G(R)$-invariant $p$-subgroup of $G$.

The main result of this paper is the following, which is a generalization of [1 Corollary C].

**Theorem A.** Let $G$ be a group and $p$ a prime. The following configuration is impossible:

- $(M_1, M_2)$ is a weak primitive pair of characteristic $p$ for $G$.
- $M_1$ and $M_2$ are $K$-groups.
- For each $i$ there is an elementary abelian group $R_i$ that acts coprimely on $M_i$ and $O_p(M_1; R_1) = O_p(M_2; R_2)$.

As in [1], the proof of Theorem A uses ideas of Meierfrankenfeld and Stellmacher [8, 9, 10]. The Corollaries stated and proved in [1] also remain valid but with the $p$-solvable hypothesis replaced by a $K$-group hypothesis.

Although Theorem A is adequate for the proposed new proof of the Non-solvable Signalizer Functor Theorem, it is almost certain that stronger results hold. Trivially, the restriction that the groups $R_i$ be elementary abelian is superfluous. It is included to enable results from [3] and [4] to be used and hence shorten the presentation. More significantly, Theorem A would be a consequence of the following:

**Conjecture B.** Let $p$ be a prime. For each $p$-group $P \neq 1$ there exists a characteristic subgroup $W(P) \neq 1$ with the property: whenever a group $R$ acts coprimely on the group $G$ with $G$ of characteristic $p$ and $P = O_p(G; R)$ then

$$W(P) \leq G.$$ 

In the case that $p > 3$, the conjecture is known to be true, see [5], the characteristic subgroup in question being Glauberman’s subgroup $K^\infty(P)$, which is defined in [6].

## 2. Preliminaries

**Lemma 2.1** (See [3, Theorem 4.1]). Let $r$ be a prime and $K$ a simple $K$-group whose order is coprime to $r$.

(a) The Sylow $r$-subgroups of $\text{Aut}(K)$ are cyclic.

(b) If $\alpha \in \text{Aut}(K)$ has order $r$ then $C_{\text{Aut}(K)}(C_K(\alpha))$ is an $r$-group.

**Lemma 2.2** (See [3, Theorem 4.1]). Let $R$ be a group of prime order $r$ that acts nontrivially and coprimely on the $K$-group $G$. Set $K = F^*(G)$ and suppose that $K$ is simple. Let $p$ be a prime and set $P = O_p(G; R)$. The following table lists all the cases in which $P \neq 1$.

| $K$   | $\text{Out}(K)$ | $M(K)$  | $C_K(R)$ | $P$                           |
|-------|-----------------|---------|----------|-------------------------------|
| $L_2(2^r)$ | $r$            | $1$     | $3 : 2$  | $|2^r + 1|_p$                 |
| $L_2(3^r)$ | $2 \times r$   | $2$     | $2^2 : 3$ | $p = 2, P = O_2(C_K(R)) \cong 2^2$ |
| $U_3(2^r)$ | $3 \times 2 \times r$ | $3$     | $3^2 : Q_8$ | $p = 3, P = O_3(C_K(A)) \cong 3^2$ |
| $Sz(2^r)$ | $r$            | $1(2^2$ for $Sz(2^3))$ | $5 : 4$ | $|2^r + 2^{(r-1)/2} + 1|_p$ |

where $\epsilon = 1$ if $r \equiv \pm 1 \mod 8$ and $-1$ if $r \equiv \pm 3 \mod 8$; also an integer on its own stands for the cyclic group of that order.

Moreover, $P$ is abelian and a Sylow $p$-subgroup of $K$. 
Lemma 2.3. Let $r$ be a prime and $R$ an elementary abelian $r$-group that acts coprimely on the quasisimple $K$-group $G$. Let $p$ be a prime and suppose that $A$ is an elementary abelian $p$-group with $1 \neq A \leq O_p(G; R)$. Suppose that $V$ is a faithful irreducible $\text{GF}(p)[RG]$-module. Then

$$|V/C_V(A)| > |A|^2.$$ 

Proof. Set $\overline{G} = G/Z(G)$, so $\overline{G}$ is simple. By Coprime Action, $C_{\overline{G}}(R) = C_G(R)$ so $O_p(G; R) \leq O_p(\overline{G}; R)$. Now $V$ is irreducible so $O_p(G) = 1$, whence $1 \neq \overline{A} \leq O_p(\overline{G}; R)$. Then $\overline{G}$ is one of the groups listed in Lemma 2.2. A result of Guralnick and Malle [7] completes the proof. In fact, only Propositions 4.1, 4.3 and 4.6 of [7] are needed. \hfill \Box

3. The subgroups $O_p(G; R)$

The aim of this section is to prove the following:

Theorem 3.1. Let $r$ be a prime and $R$ an elementary abelian $r$-group. Assume that $R$ acts coprimely on the $K$-group $G$ and that $p$ is a prime. Set $P = O_p(G; R)$.

(a) $P$ acts trivially on $\text{comp}_{\text{sol}}(G)$.

(b) If $L \in \text{comp}_{\text{sol}}(G)$ and $P$ acts nontrivially on $L/\text{sol}(L)$ then $L/\text{sol}(L)$ is isomorphic to one of $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $Sz(2^r)$.

(c) $P \leq O_{\text{sol},E}(G)$ and $P' \leq \text{sol}(G)$.

(d) Assume $\text{sol}(G) = 1$ and $G = \langle P^G \rangle \neq 1$. Then $G$ is the direct product of nonabelian simple groups, each of which is isomorphic to one of $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $Sz(2^r)$. Moreover, $P$ is abelian and a Sylow $p$-subgroup of $G$.

Recall from [3] that a sol-component of $G$ is a perfect subnormal subgroup of $G$ that maps onto a component of $G/\text{sol}(G)$. An $R$-component of $G$ is the subgroup generated by an orbit of $R$ on $\text{comp}(G)$. The sets of sol-components and $R$-components of $G$ are denoted by $\text{comp}_{\text{sol}}(G)$ and $\text{comp}_R(G)$ respectively. We will use the results generated in [3] §6 of [3] that describe the structure of $R$-simple groups and also the results of [4] §5 to analyze $O_p(G; R)$.

Proof of Theorem 3.1(a). Set $\overline{G} = G/\text{sol}(G)$. By Coprime Action, $C_{\overline{G}}(R) = C_G(R)$ so $\overline{P} \leq O_p(\overline{G}; R)$. Lemma 2.2] implies that the map $\text{comp}_{\text{sol}}(G) \rightarrow \text{comp}(\overline{G})$ defined by $K \mapsto \overline{K}$ is a bijection. Hence we may assume that $\text{sol}(G) = 1$. Apply [4] Lemmas 7.6 and 7.3. \hfill \Box

Lemma 3.2. Assume the hypotheses of Theorem 3.1. Suppose that $K \in \text{comp}_R(G)$, that $[K, P] \neq 1$ and that $Z(K) = 1$. Then:

(a) $P \cap K = O_p(K; R) \neq 1$.

(b) Let $R_\infty = \ker(R \rightarrow \text{Sym}(\text{comp}(K)))$. Then $R_\infty$ acts nontrivially on each component of $K$; $O_p(K; R) = O_p(K; R_\infty)$ and $C_K(R_\infty)$ normalizes $P$.

(c) Each component of $K$ is isomorphic to one of $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $Sz(2^r)$.

(d) $O_p(K; R)$ is an abelian Sylow $p$-subgroup of $K$.

(e) $P = C_P(K) \times O_p(K; R)$.

Proof. (a). Since $[K, P] \neq 1$, [4] Lemma 4.1 implies that $P \cap K \neq 1$. This is in fact a consequence of Lemma 2.1(b). Clearly $P \cap K \leq O_p(K; R)$. Note that $C_G(R)$ acts
on \text{comp}_P(G)$ by conjugation. Let $K_1, \ldots, K_n$ be the distinct $C_G(R)$-conjugates of $K$. Set $Q = O_p(K_1; R) \times \cdots \times O_p(K_n; R)$. For each $i$, $C_G(R) \cap N_G(K_i)$ normalizes $O_p(K_i; R)$. It follows that $Q$ is $RC_G(R)$-invariant. Then $O_p(K; R) \leq Q \cap K \leq P \cap K$, whence $O_p(K; R) = P \cap K$.

(b). \[\text{Lemma 6.6}] implies that the $RC_K(R)$-invariant overdiagonal subgroups of $K$ are $R$-simple. Since $O_p(K; R) \neq 1$ is not $R$-simple, it follows that $K$ possesses an $RC_K(R)$-invariant underdiagonal subgroup so \[\text{Lemma 6.8(c)] implies that} R_\infty acts nontrivially on each component of $K$. \[\text{Lemma 5.5(b)] implies} O_p(K; R) = O_p(K; R_\infty)$. Let $N_1, \ldots, N_m$ be the components of $K$. They are permuted transitively by $R$ and $K = N_1 \times \cdots \times N_m$. For each $i$ let $\pi_i : K \rightarrow N_i$ be the projection map. Now $O_p(K; R_\infty) = O_p(N_1; R_\infty) \times \cdots \times O_p(N_m; R_\infty)$ so using \[\text{Lemma 5.5(c)] we obtain}\[
\begin{align*}
[P, C_K(R_\infty)] \leq O_p(N_1; R_\infty) \times \cdots \times O_p(N_m; R_\infty) = O_p(K; R_\infty) \leq P.
\end{align*}
Thus $C_K(R_\infty)$ normalizes $P$.

(c), (d). We have shown that $O_p(K; R) = O_p(N_1; R_\infty) \times \cdots \times O_p(N_m; R_\infty)$. \[\text{Lemma 2.2}] implies that for all $i$, $N_i$ is isomorphic to $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $S_3(2^r)$ and that $O_p(N_i; R_\infty)$ is an abelian Sylow $p$-subgroup of $N_i$. Thus $O_p(K; R)$ is an abelian Sylow $p$-subgroup of $K$.

(e). For each $i$ set $G_i = P N_i$. \[\text{Theorem 3.1(a)] implies} G_i$ is a subgroup. Set $G_i = G_i / \text{sol}(G_i)$. Then $F^*(G_i) = N_i$ is simple. (b) and Coprime Action imply that $\overline{T} \leq O_p(G_i; R_\infty)$ so \[\text{Lemma 2.2}] implies that $\overline{T} \leq N_i$. Then $P \leq \text{sol}(G_i) N_i \leq C_{G_i}(N_i) \leq C_{G_i}(N_i)$. In particular, $P$ induces inner automorphisms on $N_i$. Then $P$ induces inner automorphisms on $N_1 \times \cdots \times N_m = K$ and so $P \leq C_G(K) \times K$. The projection of $P$ into $K$ is $RC_K(R)$-invariant and hence is contained in $O_p(K; R)$, which by (a) is contained in $P$. We deduce that $P$ projects onto $O_p(K; R)$ and then that $P = C_p(K) = O_p(K; R)$.

\[\text{Proof of Theorem 3.1(b).}] Let $\overline{G} = G / \text{sol}(G)$. As in the proof of (a), $\overline{T} \leq O_p(G; R)$. Note that $P \leq N_G(L)$ by (a) and also that $L \cap \text{sol}(G) = Z(L)$. Since $L$ is perfect and $[L, P] \neq 1$ it follows from the Three Subgroups Lemma that $[L, P] \neq Z(L)$. Then $\overline{L}, \overline{P} \neq 1$. Hence we may assume that $\text{sol}(G) = 1$. Set $K = \{ L^R \} \in \text{comp}_R(G)$. Then $Z(K) = 1$ and $[K, P] \neq 1$. Apply \[\text{Lemma 5.5(c)]}.\]

\[\text{Proof of Theorem 3.1(d).}] Since $G = \langle P^G \rangle$, (a) implies that every $R$-component of $G$ is normal in $G$. As $\text{sol}(G) = 1$ we may choose $K \in \text{comp}_R(G)$. Note that $Z(K) \leq \text{sol}(G) = 1$. \[\text{Lemma 3.2(c)] implies} P = C_p(K) \times (P \cap K)$, then $P \cap K$ is an abelian Sylow $p$-subgroup of $K$ and that each component of $K$ is isomorphic to $L_2(2^r)$, $L_2(3^r)$, $U_3(2^r)$ or $S_3(2^r)$. Now $K \leq G = \langle P^G \rangle$ so $P \leq C_G(K) \times K \leq G$ whence $G = C_{G}(K) \times K$; $P = (P \cap C_G(K)) \times (P \cap K)$; $P \cap C_G(K) = O_p(C_G(K); R)$ and $C_G(K) = \langle (P \cap C_G(K))^{C_G(K)} \rangle$. Apply induction.

\[\text{Proof of Theorem 3.1(c)]}. As previously, we may assume that $\text{sol}(G) = 1$. Set $G_0 = \langle P^G \rangle \leq G$ and suppose that $G_0 = G$. By induction we have $P \leq E(G_0) \leq E(G)$ and $P' \leq \text{sol}(G_0) \leq \text{sol}(G) = 1$. Hence we may assume that $\langle P^G \rangle = G$. Apply (d).

4. PROOF OF THE MAIN THEOREM

Throughout this section, $r$ is a prime. The reader is referred to \[\text{for the definition of nearly quadratic 2F-offender}].
Theorem 4.1. Suppose that $R$ is an elementary abelian $r$-group that acts coprimely on the $K$-group $G$, that $p$ is a prime and that $V$ is a faithful $GF(p)|RG|$-module. Then $O_p(G; R)$ contains no nearly quadratic $2F$-offender for $RG$ on $V$.

Proof. Assume false and let $A \leq O_p(G; R)$ be a counterexample. We may suppose that $|G| + |A| + \dim(V) + |R|$ has been minimized. A standard reduction, see the proof of [1 Theorem B], shows that $RG$ is irreducible on $V$. In particular,

$$O_p(G) = 1.$$ 

Set $P = O_p(G; R)$. Now $\langle A^{RG} \rangle \leq RG$ so $O_p(\langle A^{RG} \rangle) = 1$. Using the minimality of $|G|$ we obtain

$$G = \langle P^G \rangle = \langle A^{RG} \rangle.$$ 

We claim that if $1 < B < A$ then

$$|C_V(B)/C_V(A)| < |A/B|^2.$$ 

Indeed, since we are studying a counterexample we have $|V/C_V(A)| \leq |A|^2$. On the other hand, the minimality of $|A|$ implies $|V/C_V(B)| > |B|^2$. Then (2) follows on division.

Let $N$ be a minimal $R$-invariant normal subgroup of $G$ chosen minimal subject to $[N, A] \neq 1$. Note that $N$ exists because $A \not\subseteq O_p(Z(G)) = 1$. Suppose that $N \neq O^p(N)$. Then $[O^p(N), A] = 1$ and (1) implies that $O^p(N) \leq Z(G)$. In particular, $N$ is nilpotent. Then $O_p(N) \leq O_p(G) = 1$ whence $N = O^p(N)$, a contradiction. We deduce that $N = O^p(N)$.

We claim that $G = PN$. Assume false and set $G_0 = PN$. Let $B = A \cap O_p(G_0)$. Now $[O_p(G_0), N] \leq O_p(N) \leq O_p(G) = 1$ so using the minimality of $|G|$ and the fact that $[N, A] \neq 1$, we have $1 < B < A$. Set

$$U = C_V(O_p(G_0)).$$

Now $N = O^p(N)$ and $[O_p(G_0), N] = 1$ so the $P \times Q$-Lemma implies that $N$ is faithful on $U$. As $G_0 = PN$ it follows that $C_{G_0}(U)$ is a $p$-group and then that

$$C_{G_0}(U) = O_p(G_0).$$

The minimality of $|G|$ implies that $|A/B|^2 < |U/C_U(A)|$. Then (2) yields

$$|C_V(B)/C_V(A)| < |U/C_U(A)|.$$ 

However $U \leq C_V(B)$ so $|U/C_U(A)| \leq |C_V(B)/C_V(A)|$, a contradiction. We deduce that $G = PN$.

Suppose that $[\text{sol}(G), A] \neq 1$. Then we could have chosen $N$ with $N \leq \text{sol}(G)$, whence $G = P \text{sol}(G)$ and $G$ is solvable. But then [1 Theorem B] supplies a contradiction. Thus $[\text{sol}(G), A] = 1$ and then (1) forces

$$\text{sol}(G) = Z(G).$$

Set $\overline{G} = G/Z(G)$. Then $\overline{P} = O_p(\overline{G}; R), \text{sol}(\overline{G}) = 1$ and $\overline{G} = \langle \overline{P^G} \rangle$. Theorem 3.3(d) implies that $\overline{G}$ is the direct product of simple groups and that $\overline{P}$ is an abelian Sylow $p$-subgroup of $\overline{G}$. Choosing $N$ to be an $R$-component of $G$ it follows that $\overline{G}$ is $R$-simple. Let $R_\infty = \ker(R \rightarrow \text{Sym}(\text{comp}(G)))$. Lemma 3.2(b) implies that $\overline{P} = O_p(\overline{G}; R_\infty)$. Then as $\overline{G} = G/Z(G)$ we have $P = O_p(G; R_\infty)$ and the minimality of $|R|$ forces $R = R_\infty$. In particular, $\overline{G}$ is simple. We have $\overline{G} = \overline{G'}$ so $G = Z(G)G' = \langle P^G \rangle$. As $Z(G)$ is a $p'$-group we see that $P \leq G'$ whence $G = G'$.
and $G$ is quasisimple. Lemma 2.3 implies that $|V/C_V(A)| > |A|^2$, contrary to $A$ being a $2F$-offender for $RG$ on $V$. The proof is complete. \[\square\]

**Lemma 4.2.** Suppose $R$ is an elementary abelian $r$-group that acts coprimely on the $K$-group $G$. Suppose that $F^*(G) = O_p(G)$ for some prime $p$. Let $V$ be an elementary abelian normal subgroup of $O_p(G;R)$. Then either:

- $V \leq O_p(G)$, or
- there exists $A \leq O_p(G;R)$ such that $A$ acts nontrivially and nearly quadratically on $V$ and

$$|V/C_V(A)| \leq |A/C_A(V)|^2.$$ 

**Proof.** Set $P = O_p(G;R)$ and $G_0 = \langle V^{RG} \rangle \leq G$. Then $F^*(G_0) = O_p(G_0) \leq O_p(G)$ and $P \cap G_0 = O_p(G_0;R)$. Hence we may suppose that $G = G_0$. Then

$$G = \langle V^{RG} \rangle = \langle P^G \rangle.$$ 

Set

$$\overline{G} = G/O_p(G).$$

**Claim 1.** $P \in \text{Syl}_p(G)$ and $\overline{V}$ is normal in every Sylow $p$-subgroup of $\overline{G}$ in which it is contained.

**Proof.** Set $G_1 = P\text{O}_{p,p'}(G)$. Then $F^*(G_1) = O_p(G_1)$ and $P \in \text{Syl}_p(G_1)$, since $O_p(G) \leq P$. Set $\overline{V} = O_{p'}(\overline{G})$. Now $\overline{G_1} = \overline{P}^{\overline{V}}$ and $\overline{V} \leq \overline{P}$. In particular, $C_{\overline{G}}(\overline{V})$ acts transitively on the set of Sylow $p$-subgroups of $\overline{G_1}$ that contain $\overline{V}$. Consequently, $\overline{V}$ is normal in every Sylow $p$-subgroup of $\overline{G_1}$ in which it is contained. Hence if $G = G_1$ then the claim holds.

Suppose $G_1 \neq G$. Since $G_1$ is $C_G(R)$-invariant it follows that $P = O_p(G_1;R)$. By induction, we may suppose that $V \leq O_p(G_1)$. Then $V$ centralizes $O_{p'}(G/O_p(G))$.

As $G = \langle V^{RG} \rangle$ it follows that $G$ centralizes $\text{sol}(G/O_p(G))$, whence $\text{sol}(G) = Z(\overline{G})$.

Note that $Z(\overline{G})$ is a $p'$-group and that $\overline{P} = O_p(G;R)$.

Put $G^* = \overline{G}/\text{sol}(\overline{G})$. Then $\text{sol}(G^*) = 1$ and $G^* = \langle P^{*G^*} \rangle$. As $\text{sol}(\overline{G}) = Z(\overline{G})$ it follows from Coprime Action that $P^* = O_p(G^*;R)$. Theorem 3.1(d) implies that $P^*$ is an abelian Sylow $p$-subgroup of $G^*$. Since $\text{sol}(\overline{G})$ is a $p'$-subgroup it follows that $\overline{P}$ is an abelian Sylow $p$-subgroup of $\overline{G}$. In particular, $\overline{V}$ is normal in every Sylow $p$-subgroup of $\overline{G}$ in which it is contained. As $O_p(G) \leq P$ we have $P \in \text{Syl}_p(G)$ and the claim is established. \[\square\]

Choose $\overline{G}_2$ minimal subject to

$$\overline{V} \leq \overline{G}_2 \leq \overline{G} \quad \text{and} \quad \overline{V} \not\leq O_p(\overline{G}_2).$$

Choose $\overline{Q} \in \text{Syl}_p(\overline{G})$ with $\overline{V} \leq \overline{Q} \cap \overline{G}_2 \leq \text{Syl}_p(\overline{G}_2)$. Claim [I] implies $\overline{V} \leq \overline{P}$ and $\overline{V} \leq \overline{Q}$. Hence there exists $\overline{P} \in N_{\overline{G}}(\overline{V})$ with $\overline{Q}^{\overline{P}} = \overline{P}$. Replacing $\overline{G}_2$ with $\overline{G}_2^{*\overline{P}}$, we may suppose that

$$\overline{P} \cap \overline{G}_2 \leq \text{Syl}_p(\overline{G}_2).$$

Let $G_2$ be the full inverse image of $\overline{G}_2$ in $G$ and let $G_3 = \langle V^{G_2} \rangle \leq G_2$. In particular, $V \not\leq O_p(G_3)$. Now $P \cap G_2 \in \text{Syl}_p(G_2)$ whence $P \cap G_3 \in \text{Syl}_p(G_3)$.

**Claim 2.** For each $M$ with $V \leq M < G_3$, it follows that $V \leq O_p(M)$.


Proof. If $M < G_2$ then this follows from the choice of $G_2$, so suppose $M = G_2$. Then $G_2 = O_2(G)M$. Since $V \leq M$ and $O_2(G)$ normalizes $V$ we have $V^{G_2} \leq M$ so $G_3 \leq M$, a contradiction. The claim is established. □

Since $V \not\leq O_2(G_3)$, Claim 2 and Wielandt’s Maximizer Lemma imply that $V$ is contained in a unique maximal subgroup $M$ of $G_3$. Then $V \leq O_2(M)$. Now $O_2(G_3) \leq P$ so $O_2(G_3)$ normalizes $V$. Apply [1] Lemma 3.1 to get the required subgroup $A$. □

Given a group $M$ and a prime $p$, the subgroup $Y_M$ is defined by

$$Y_M = \langle \Omega_1(Z(S)) \mid S \in \text{Syl}_p(M) \rangle.$$  
Clearly, $Y_M \text{ char } M$. If in addition $M$ has characteristic $p$ then $Y_M \leq \Omega_1(Z(O_2(M)))$ and so $Y_M$ may be regarded as a $\text{GF}(p)[M]$-module. A well-known property of $Y_M$ is that $O_2(M/C_M(Y_M)) = 1$, see [1].

We are now in a position to prove Theorem A, following closely arguments from [1]. Theorem A follows readily from the following slightly stronger result.

**Theorem 4.3.** Suppose that $M$ and $S$ are subgroups of the group $G$, that $R_m$ and $R_s$ are elementary abelian groups that act coprimely on $M$ and $S$ respectively. Assume that:

- $M$ and $S$ have characteristic $p$ for some prime $p$.
- $O_2(M; R_m) = O_2(S; R_s)$.
- $C_S(Y_M) \leq M$.
- $M$ and $S$ are $K$-groups.

Set

$$P = O_2(M)O_2(S).$$

Then the following hold:

(a) If $O_2(M)$ is abelian then $J(P) = J(O_2(M))$.

(b) $J(P) = J(O_2(S))$.

Proof. Note that $O_2(M) \leq O_2(M; R_m) = O_2(S; R_s) \leq S$ and similarly $O_2(S) \leq M$. Then $O_2(M)$ and $O_2(S)$ normalize each other and $P$ is a subgroup.

Claim 1. Suppose $V$ is an elementary abelian characteristic $p$-subgroup of $M$ with $O_2(M/C_M(V)) = 1$. If $A \leq O_2(M; R_m)$ acts nontrivially and nearly quadratically on $V$ then

$$|A/C_A(V)|^2 < |V/C_V(A)|.$$

Moreover $[V, J(P)] = 1$.

Proof. Set $M = M/C_M(V)$, so $O_2(M) = 1$. If the inequality is violated then $A$ is a nearly quadratic $2F$-offender for $M$ on $V$. But $B \leq O_2(M; R_m)$ so Theorem 4.4 supplies a contradiction.

Suppose that $[V, J(P)] \neq 1$. Thompson’s Replacement Theorem, see [1] Theorem 2.4, implies there exists $A \in \mathcal{A}(P)$ with $A$ acting nontrivially and quadratically on $V$. Since $A \in \mathcal{A}(P)$ we have $|V/C_V(A)| \leq |A/C_A(V)|$, which contradicts the inequality. □

Proof of (a). Assume that $O_2(M)$ is abelian and put $V = \Omega_1(O_2(M))$. By Co-prime Action, $O^3(C_M(V)) \leq C_M(O_2(M)) = O_2(M)$ whence $C_M(V) = O_2(M)$ and $O_2(M/C_M(V)) = 1$. By Claim 1 $J(P) \leq C_M(V) = O_2(M) \leq P$ whence $J(P) = J(O_2(M))$. □
Put
\[ W = \langle Y^\text{Aut}(S) \rangle. \]

**Claim 2.** \( Y_M \trianglelefteq O_p(S) \) and \( W \) is elementary abelian.

**Proof.** Put \( V = Y_M \). Then \( O_p(M/C_M(V)) = 1 \) by [1, Lemma 2.2]. Now \( V \trianglelefteq O_p(M; R_m) = O_p(S; R_s) \). Lemma [1,2] with \( S \) in the role of \( G \), and Claim [1,2] imply \( V \trianglelefteq O_p(S) \). Since \( O_p(S) \trianglelefteq O_p(S; R_s) \), we have \( V \trianglelefteq O_p(S) \).

Suppose that \( W \) is nonabelian. Then there exists \( a \in \text{Aut}(S) \) with \( [V, V^a] \neq 1 \). Also \( [V, V^a, V^a] \subseteq [V \cap V^a, V^a] = 1 \). In particular, \( V^a \) acts nontrivially and quadratically on \( V \). Claim [1,2] implies that
\[ |V^a/C_{V^a}(V)| < |V/C_{V^a}(V^a)|. \]
Moreover, \( [V^a, V] \neq 1 \) whence
\[ |V^{a^{-1}}/C_{V^{a^{-1}}}(V)| < |V/C_{V}(V^{a^{-1}})|. \]
Conjugating the second inequality by \( a \) contradicts the first. \( \square \)

Let
\[ Q = O_p(S \text{ mod } C_S(W)). \]

**Claim 3.** \( P \cap Q \subseteq O_p(S) \).

**Proof.** Choose \( T \) with \( P \cap Q \leq T \in \text{Syl}_p(Q) \). Then \( S = Q N_S(T) = C_S(W) N_S(T) = (M \cap S) R_S(T) \) because \( C_S(W) \leq C_S(Y_M) \leq M \). Now \( P \cap Q \) is normalized by \( M \cap S \) because \( P = O_p(M) O_p(S) \leq M \cap S \), whence \( (P \cap Q)^S = (P \cap Q)^{N_S(T)} \leq T \) and so \( P \cap Q \leq O_p(S) \). \( \square \)

**Claim 4.** \( [W, J(P)] = 1 \).

**Proof.** Assume false. By the Replacement Theorem, there exists \( A \in \mathcal{A}(P) \) such that \( A \) acts nontrivially and quadratically on \( W \).

Suppose that \( A \leq Q \). Claim [3,4] implies that \( A \leq O_p(S) \) whence \( A \leq J(O_p(S)) \). Since \( A \in \mathcal{A}(P) \) it follows that \( J(O_p(S)) \leq J(P) \). By Claim [1,2] \( [Y_M, J(P)] = 1 \) whence \( [Y_M, J(O_p(S))] = 1 \). As \( W = \langle Y^\text{Aut}(S) \rangle \) we obtain \( [W, J(O_p(S))] = 1 \) and then \( [W, A] = 1 \), a contradiction. We deduce that \( A \not\subseteq Q \).

Set
\[ A_0 = A \cap Q. \]

By Claim [3,4] \( A_0 \leq O_p(S) \) and \( C_A(W) \leq O_p(S) \leq Q \) so
\[ C_A(W) = C_{A_0}(W). \]

Let \( q = q(Y_M, O_p(S)) \). The reader is referred to [1] or [4] for the definition of this parameter. Since \( O_p(S) \leq O_p(S; R_s) = O_p(M; R_m) \), Claim [1,4] implies
\[ q > 2. \]

Now \( [Y_M, A_0, A_0] = 1 \) because \( A \) is quadratic on \( W \). Applying [1, Lemma 2.5], we obtain
\[ |A_0/C_{A_0}(W)|^q \leq |W/C_W(A_0)|. \]
As \( A \in \mathcal{A}(P) \) we have \( |A| \geq |W C_A(W)| \) so
\[ |A/C_A(W)| \geq |W/C_W(A)|. \]
Raising (2) to the power \( q \), dividing by (1) and using \( C_A(W) = C_{A_0}(W) \) yields
\[
|A/A_0|^q \geq |W/CW(A)|^{q-1}|CW(A_0)/CW(A)|
\geq |W/CW(A)|^{q-1}
\]
because \( A_0 \leq A \). Then
\[
|A/A_0|^{1+1/(q-1)} \geq |W/CW(A)|.
\]
Since \( q > 2 \) and \( A_0 < A \) we obtain
\[
|A/A_0|^2 > |W/CW(A)|.
\]
Thus \( A/C_A(W) \) is a nearly quadratic 2F-offender for \( S/C_S(W) \) on \( W \). Since \( A \leq O_p(S; R_s) \), Theorem 4.1 supplies a contradiction. Hence \( [W, J(P)] = 1 \).

Proof of Theorem A. Assume false. Two applications of the previous theorem yield
\[
J(O_p(M_i)) = J(O_p(M_1)O_p(M_2)) = J(O_p(M_2)).
\]
Since \( J(O_p(M_i)) \leq M_i \), this gives a contradiction since no nontrivial subgroup of \( M_1 \cap M_2 \) can be normal in both \( M_1 \) and \( M_2 \) by the definition of a weak primitive pair.

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