Quantum calibration of measuring apparatuses

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By quantum calibration we name an experimental procedure apt to completely characterize an unknown measurement apparatus by comparing it with other calibrated apparatuses. Here we show how to achieve the calibration of an arbitrary measuring apparatus, by using it in conjunction with a “tomographer” in a correlation setup with an input bipartite system. The method is robust to imperfections of the tomographer, and works for practically any input state of the bipartite system.

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The calibration of measuring apparatuses is at the basis of any experiment. Theory and experiment are unavoidably interwoven, and the calibration procedure often needs a detailed knowledge of the inner working of the apparatus, especially at extreme precisions and sensitivities, where a quantum mechanical description is needed. Here, the actual “observable” that is measured depends crucially on the microscopic details of the apparatus, and without knowing them the measurement lacks physical interpretation.

In a quantum mechanical description, the calibration of a measuring apparatus corresponds to the knowledge of its POVM (positive operator-valued measure), which gives the probability \( p(n) \) of any measurement outcome \( n \) for arbitrary input state, via the Born rule

\[
p(n) = \text{Tr}[\rho P_n]. \tag{1}\]

In Eq. (1) \( \rho \) is the density operator of the state on the Hilbert space \( \mathcal{H} \) of the system, and the POVM is given by the set of operators \( \{ P_n \} \) on \( \mathcal{H} \). To ensure that \( p(n) \) is a probability, the POVM must satisfy the positivity and normalization constraints \( P_n \geq 0, \sum_n P_n = I \).

The concept of POVM generalizes the familiar von Neumann observable describing perfect measurements. Here the probability of obtaining outcome \( n \) is given by \( p(n) = |\langle \psi | o_n \rangle|^2 \), \( \{ o_n \} \) denoting a complete orthonormal basis for \( \mathcal{H} \), i.e. with POVM given by the one-dimensional projectors \( P_n = |o_n \rangle \langle o_n| \). The physical interpretation of the measurement is given via a quantization rule that associates a self-adjoint operator \( O \) to a classical observable, \( |o_n \rangle \) being the eigenvector of \( O \) corresponding to its \( n \)th eigenvalue \( o_n \). However, this concept of observable does not cover many practical situations—e.g. phase-estimation, joint measurements of incompatible observables, discrimination among non-orthogonal states, informationally complete measurements, transmission of reference frames—and here the POVM description is needed. But then, in absence of a direct physical interpretation of the measurement, we are faced with the problem of assessing the correct functioning of the measuring apparatus.

Inferring the POVM of an apparatus through the theoretical description of its functioning leads to quite involved derivations, based on different kinds of approximations. A paradigmatic case is that of the photocounter \[\text{11}\], where the number of photons claimed to be detected—usually very uncertain—is typically inferred from the cascading mechanism of the amplification process. The calibration is given essentially in terms of quantum efficiency and dark-current, and mostly saturation effects categorize detectors into the major classes of “linear” and “single-photon”. Even in a very simplified model, a theoretical description accounting for the above features is very involved \[\text{11, 12}\], and the resulting theoretical calibration is exceedingly indirect.

The above scenario raises the following problem: is it possible to calibrate a measuring apparatus—i.e. to determine its POVM—with a purely experimental procedure, e.g. by comparing the apparatus with other (previously calibrated) apparatuses? In this paper we propose a method to determine a POVM experimentally. The method uses the unknown apparatus jointly with a calibrated “tomographer” on a suitably prepared bipartite system, as in Fig. 1 and the calibration results from the analysis of the correlations of outcomes. [A tomographer is an apparatus that measures an observable tunable in a complete set called \textit{quorum}; more details on quantum tomography will be given in the following]. The basic scheme of the method stems on a previous method for the tomographic reconstruction of quantum operations \[\text{13}\], and generalizes a popular calibration scheme \[\text{14, 15}\] designed to determine the quantum efficiency of a photodetector. As it will be shown in the following, there is ample freedom in the choice of both the input bipartite state and the tomographer. The joint measurement must be repeated many times, analyzing the measurement outcomes with a proper tomographic algorithm \[\text{16, 17}\]; the POVM calibration being approached in the limit of infinitely many outcomes. For finite set of data, the reconstructed POVM will be affected by statistical errors, which can be precisely estimated via the tomographic algorithm. The method works for generally infinite-dimensional Hilbert space (yielding a finite number of POVM elements, corresponding to the actually occurred...
outcomes).

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure1.png}
\caption{Experimental setup to determine the POVM of the unknown measurement apparatus A. The apparatus A is used jointly with a “tomographer” on a bipartite system prepared in a pre-determined state $R$. The tomographer measures an observable $B^{(k)}$ from the quorum $\{B^{(k)}\}$, yielding result $m$, whereas the unknown apparatus gives outcome $n$. The joint outcomes $(n,m)$ are then processed using a tomographic algorithm, to finally obtain the POVM $\{P_n\}$ of A.}
\end{figure}

The following simple example illustrates how the procedure works. Suppose we know that the apparatus measures an observable, but we don’t know which one, and denote it by the orthonormal basis $\{|o_i}\}$. We can use the maximally entangled input state $|\Psi\rangle = \sum_i^d |i\rangle|i\rangle/\sqrt{d}$ in the space $\mathcal{H} \otimes \mathcal{T}$, $\mathcal{T}$ denoting the space of the quantum system impinging into the tomographer. The state can be equivalently written as $|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |o_i\rangle|o_i^*\rangle (|o_i^*\rangle$ denotes the vector with the complex conjugated coefficients of $|o_i\rangle$ with respect to the basis $\{|i\}\}$). Then, the outcome $n$ of the unknown measuring apparatus conditions the state $\rho_n = |o_n^*\rangle\langle o_n^*|$ at the tomographer, and the POVM can be recovered using state reconstruction.

We now present the general quantum calibration procedure. Let’s fix one observable at the tomographer, and denote it by $\{|b_n\}\}$. Upon denoting by $\{P_n\}$ the POVM of our measuring apparatus that we want to calibrate, the Born rule $\{1\}$ predicts that the outcome $(n,m)$ of the joint measurement will occur with probability

$$p(n,m) = \text{Tr} [(P_n \otimes |b_m\rangle\langle b_m|) R],$$

(2)

where $R$ is the joint state of the two quantum systems, and we remind that the POVM of the joint measurement is given by the tensor product of the individual POVM’s. Upon rewriting the joint probability in terms of the conditional probability $p(m|n)$ via Bayes’ rule, we conveniently introduce the state $\rho_n$ at the tomographer conditioned by outcome $n$ at the unknown measuring apparatus, namely

$$p(n,m) = p(n) p(m|n) = p(n) \text{Tr} [\rho_n |b_m\rangle\langle b_m|].$$

(3)

Upon evaluating the trace in Eq. (2) in two steps, i.e.

$$p(n,m) = \text{Tr} [\langle b_m| \text{Tr}_1 [(P_n \otimes 1) R] |b_m\rangle],$$

(4)

and by equating Eqs. (3) and (4) for any possible vector $|b_n\rangle$ (i.e., any possible observable), we have $\rho_n p(n) = \text{Tr}_1 [(P_n \otimes 1) R]$, namely

$$\rho_n = \frac{\text{Tr}_1 [(P_n \otimes 1) R]}{\text{Tr} [(P_n \otimes 1) R]}, \quad p(n) = \text{Tr} [(P_n \otimes 1) R].$$

(5)

The POVM element $P_n$ can be recovered from the conditioned state $\rho_n$ as follows

$$P_n = p(n) \mathcal{R}^{-1}(\rho_n),$$

(6)

by inverting the map

$$\mathcal{R}(X) = \text{Tr}_1 [(X \otimes 1) R],$$

(7)

$X$ denoting an operator on $\mathcal{H}$. The map $\mathcal{R}$ depends only on the input state $R$, which then must be known. Hence we need a pre-calibration stage in which we previously determine the joint state $R$ (this can be done via a joint quantum tomography with two equal tomographers on the input state $R$). Invertibility of the map $\mathcal{R}$ corresponds to a so-called faithful state $\{13\}$.

Since invertible maps are a dense set, then almost any quantum state $R$ is faithful. Of course, when approaching a state corresponding to a non-invertible map, some information on the POVM $\{P_n\}$ will be lost, corresponding to increasingly large statistical errors for some matrix elements of the operators $P_n$ (inverting a linear map is clearly equivalent to inverting an operator: for the reader who prefers operators to maps, an explicit connection between operators and maps is given in Ref. $\{13\}$).

Once the inverse map $\mathcal{R}^{-1}$ has been calculated, we use quantum tomography in order to recover $\rho_n$. Shortly quantum tomography is a method that allows us to estimate the ensemble average of an arbitrary (complex) operator $X$ by measuring a set of observables $\{B^{(k)}\}$, called quorum, which span the space of operators of the system (for recent reviews on quantum tomography, see Refs. $\{10\}$ $\{18\}$). Typical examples of quorums are the three Pauli matrices $\sigma_x$, $\sigma_y$, $\sigma_z$ for a qubit, or the set of quadratures $X_\phi = \frac{1}{2} (a_+ e^{i\phi} + a_- e^{-i\phi})$ for a single mode of the radiation field with annihilation and creation operators $a$ and $a^\dagger$, respectively, $\phi \in [0, \pi]$ playing the role of the observable label within the quorum $\{X_\phi\}$ (where $X_\phi$ is measured by a homodyne detector at phase $\phi$ relative to the local oscillator $\{16\}$). In short, the generic operator $X$ is expanded as $X = \sum_k \text{Tr} [X C^{(k)}] B^{(k)}$, $\{C^{(k)}\}$ denoting a dual set of the quorum $\{B^{(k)}\}$ (the sum being replaced by an integral for continuous $k$), the quantity $\text{Tr} [X C^{(k)}]$ being evaluated analytically. Notice that one can remove the effects of noise at the tomographer if the noise map $\mathcal{N}$ is invertible, by writing $X = \sum_k \text{Tr} [\mathcal{N}^{-1}(X) C^{(k)}] B^{(k)}$.

For the tomographic reconstruction we can either: $a)$ average over the quorum, e.g. estimate $\langle X \rangle$ via the ensemble averages of the quorum observable as $\langle X \rangle = \sum_k \text{Tr} [\mathcal{N}^{-1}(X) C^{(k)}] B^{(k)}$. 
\[
\sum_k \text{Tr}[X C^{(k)\dagger} B^{(k)}] \quad (\text{the estimation of the density matrix element } \rho_{ij} \text{ corresponding to } X = |j\rangle\langle i|); \ b) \text{ we can use the maximum likelihood approach.} \]

In this case, the estimated POVM elements \( P_n \) will maximize the probability \( \text{Tr}[(P_n \otimes |b^{(k)}_m\rangle\langle b^{(k)}_m|) R] \) in the joint measurement on the pre-determined input state \( R \) of getting outcome \( n \) on the unknown measuring apparatus and \( m \) for the \( k \)th observable \( B^{(k)} \) of the quorum. Equivalently, one can maximize the logarithm of this quantity and consider simultaneously all the \( N \) joint measurement outcomes, corresponding to maximizing the likelihood functional

\[
L \{P_n\} = \sum_{i=1}^N \log \text{Tr}\left[(P_n \otimes |b^{(k)}_m\rangle\langle b^{(k)}_m|) R\right], \quad (8)
\]

under the constraints \( P_n \geq 0 \) and \( \sum_n P_n = 1 \). Other prior knowledge about \( P_n \) can be easily incorporated by adding further constraints. Moreover, we can account for a known source of noise \( \mathcal{N} \) at the tomographer, by replacing the projector \( |b^{(k)}_m\rangle\langle b^{(k)}_m| \) in Eq. (8) with \( \mathcal{N}(|b^{(k)}_m\rangle\langle b^{(k)}_m|) \).

Therefore, the procedure to calibrate an unknown measurement apparatus can be summarized in the following steps: i) [pre-calibration] using two tomographers, reconstruct the input joint state \( R \). Check whether \( R \) is faithful; ii) [joint measurements with the unknown apparatus] Replace one tomographer with the unknown detector, and collect \( N \) pairs of outcomes \( \{n, m_i\}, i = 1, \ldots, N \) in a set of joint measurements with randomly selected observable \( B^{(k)} \) in the quorum; iii) [Data analysis] From the experimental data collect the probability \( p(n) \) of the outcome \( n \) at the unknown measurement apparatus, and then estimate the POVM \( \{P_n\} \) using a tomographic strategy—either the averaging or the maximum likelihood. In the first case evaluate the density matrix \( \rho_n \) of the state impinging in the unknown measuring apparatus, and then use Eq. (6) to recover the POVM. In the second case, evaluate the POVM directly by maximizing the likelihood functional \( L \) in Eq. (8) on the given set of experimental data, with the state \( R \) obtained at step i).

In Fig. 2 we present a simulated experiment of the quantum calibration of a photo-counter using homodyne tomography with the averaging strategy. The model of the calibrated detector is given in Fig. 3. Since the resulting POVM is diagonal in the photon-number basis, we limit the reconstruction to the diagonal elements only. As input state \( R \) we use a twin beam state from parametric down-conversion of vacuum, of the form \( \propto \sum_m \xi^m |m\rangle \otimes |m\rangle \), where \( \xi \) is related to the amplification gain and \( |m\rangle \) denotes the eigenstate of the photon number. One can easily check that the twin beam is faithful for all \( \xi \neq 0 \). As typical imperfection of the tomographer, we consider non-unit quantum efficiency \( \eta \) for the homodyne detector (the noise map can be inverted as long as \( \eta > \frac{1}{2} \)). Since we reconstruct only the diagonal part of the POVM, one can easily show that there is no need of knowing the homodyne phase \( \phi \), which, however, must be randomly distributed (the knowledge of \( \phi \) would allow to recover also the off-diagonal elements of the POVM).

In Fig. 4 we present the same calibration, but using the maximum likelihood strategy. The convergence of the maximum-search algorithm is assured by the strict convexity of the likelihood functional \( L \) over the space of diagonal POVM’s (the convergence speed, however,
FIG. 4: The same as in Fig. 2, but using the maximum likelihood method. Here only $5 \times 10^4$ simulated data are used. The error bars are obtained by standard bootstrapping techniques over a virtual repetition of 50 experiments. Notice how the result is statistically less noisy than that in Fig. 2, even for a $10^{-2}$ smaller set of data.

can be practically very slow). In the simulation we used a blend of sequential quadratic programming routines (to perform the constrained maximization) along with expectation-maximization techniques [17]. By comparing Figs. 2 and 4 we can see how the maximum likelihood estimation is more statistically efficient (i.e. fewer data are needed to achieve the same statistical error) than the averaging strategy [19], and, in addition, the maximization of the likelihood recovers all the POVM elements simultaneously. On the other hand, compared to the averaging strategy, the maximum likelihood approach has the drawback of being biased, since one needs to put a cut-off to the Hilbert space dimension of the tomographic reconstruction and/or to the cardinality of the POVM.

Both simulated experiments use realistic parameters and are feasible in the lab with current technology (see, for example, Refs. [20-22, 23]), the major challenge of a real experiment being the matching of modes between photo-counter and homodyne detector, also ensuring that the detected modes are the same of the pre-calibration stage.

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