Quasi-Thick Codimension 2 Braneworld

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We study a codimension 2 braneworld in the Einstein Gauss-Bonnet gravity. We carefully examine the structure of possible singularities in the system which characterize the braneworld through matching conditions. Consequently, we find that the thickness of the brane can be incorporated as the distributional source, which we dub quasi-thickness. On the basis of our formalism, we analyze the linearized gravity and show the conventional Einstein gravity can be recovered on the brane. In the nonlinear regime, however, we find corrections due to the thickness and the bulk geometry. We also point out a possibility that the thickness plays a role of the dark energy/dark matter in the universe.

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I. INTRODUCTION

The old idea [1] that our universe may be a braneworld embedded in a higher dimensional spacetime is renewed by the recent development in string theory which can be consistently formulated only in 10 dimensions [2]. Instead of 10 dimensions, however, most studies of braneworld cosmology have been devoted to 5-dimensional models [3]. This is due to the difficulty in treating the higher codimension object in a relativistic manner [4]. Nevertheless, it is important to explore the possibility of the higher codimension braneworld [5].

It is the Einstein Gauss-Bonnet gravity that is a natural framework in 6-dimensions. Because a natural extension of Einstein gravity to higher dimensions is the Lovelock gravity and it reduces to the Einstein Gauss-Bonnet gravity in 6-dimensions [6]. Moreover, the Gauss-Bonnet term appears in the low energy effective action in string theory. Therefore, it seems reasonable to construct a codimension 2 braneworld in the Einstein Gauss-Bonnet gravity. In fact, Bostock et al. claimed that 6-dimensional Einstein Gauss-Bonnet gravity leads to a thin braneworld where a conventional Einstein gravity holds [7]. More interestingly, they suggested that the deviation to the conventional Einstein theory can be possible when the thickness of the brane is taken into account and consequently the variation of the deficit angle is allowed. However, they have never given a precise scheme to calculate these corrections. Rather, they commented that the corrections can not be fixed exactly by the bulk equations. Therefore, it is important to construct a viable concrete model for the thick braneworld and clarify if corrections can be obtainable or not.

In this paper, we investigate a codimension 2 braneworld in the Einstein Gauss-Bonnet gravity taking into account the thickness. It is difficult to treat the finite thickness as it is. However, by examining the structure of the singularity in the equations of motion, we find a possibility to treat the thickness within the context of the distributional source. We name it a quasi-thick braneworld. This is the important point in our work which gives a framework to clarify various issues in the codimension 2 braneworld. In the case of the linearized gravity, we clarify the effect of the bulk on the brane by solving the whole set of equations of motion in the bulk with proper considerations of the boundary conditions. It turns out that the conventional Einstein gravity is recovered at the linear level. However, we also show some corrections due to the thickness can be expected at the second order level. The effect of the bulk geometry in the nonlinear regime is also discussed. We conclude that corrections to the conventional Einstein theory can be fully obtainable in the case of the quasi-thick braneworld. We also point out an interesting possibility that the thickness plays a role of the dark energy/dark matter in the universe.

The organization of this paper is as follows. In sec.II, we present a model for thick braneworld and derive necessary equations. The structure of the singularity is carefully examined. Then, we introduce an idea of the thick braneworld within the context of the distributional source, which we dub quasi-thickness. In sec.III, we set the background vacuum spacetime. In sec.IV, we solve the bulk geometry using linear perturbation theory. In sec.V, the effective theory for the codimension 2 braneworld is presented both in the linear and nonlinear regime. The consistency of our formalism is emphasized here. In the final section, we summarize our results and discuss possible applications and extensions.

II. QUASI-THICK BRANEWORLD: A MODEL FOR THICK BRANEWORLD

We consider a codimension 2 braneworld with a positive tension in the 6-dimensional bulk spacetime. The Lovelock gravity is a natural framework in higher dimensions and reduces to the Einstein Gauss-Bonnet gravity in 6-dimensions. Moreover, the Gauss-Bonnet term ubiqu-
uitously appears in the low energy limit of string theory. For this reason, the Gauss-Bonnet (GB) term is introduced in our model which is described by the action

\[ S = \frac{1}{2\kappa^2} \int d^6x \sqrt{-g_{(6)}} [R + \alpha R_{GB}^2] - \int d^4x \sqrt{-g} \sigma \]

+ \int d^4x \sqrt{-g} L_{\text{matter}}, \quad (1)

where \( \kappa \) is the 6-dimensional gravitational constant, \( g_{(6)} \) and \( g_{\mu
u} \) are the 6-dimensional bulk and the our 4-dimensional brane metrics, respectively. Here, \( L_{\text{matter}} \) is the Lagrangian density of the matter on the brane, and \( \sigma \) is the brane tension. The GB term is given by

\[ R_{GB}^2 = R_{abcd} R_{abcd} - 4 R_{ab} R_{ab} + R^2. \quad (2) \]

The Latin indices \( \{a, b, \cdots \} \) and the Greek indices \( \{\mu, \nu, \cdots \} \) are used for tensors defined in the bulk and on the brane, respectively. The 6-dimensional Einstein equation derived by varying the above action with respect to \( g_{(6)} \) takes the form

\[ G_{ab} + \alpha H_{ab} = \kappa^2 T_{ab}, \quad (3) \]

where \( \alpha \) is the GB coupling constant with dimension \([\alpha] = L^2\) and

\[ H_{ab} = -\frac{1}{2} g_{ab} R_{GB}^2 + 2 R_{ab} - 4 R_{ad} R^d_b - 4 R_{de} R_{adbc} + 2 R_{adef} R_{bdef}. \quad (4) \]

is an analogue of the Einstein tensor stemmed from the GB term \([4]\).

We will assume that a 6-dimensional metric has axial symmetry, which reads

\[ ds^2 = dr^2 + g_{\mu
u}(r, x^\mu) dx^\mu dx^\nu + L^2(r, x^\mu) d\theta^2. \quad (5) \]

This assumption corresponds to the \( Z_2 \) symmetry in the Randall-Sundrum braneworld model. Here we have introduced polar coordinates \((r, \theta)\) for the two extra spatial dimensions, where \(0 \leq r < \infty\) and \(0 \leq \theta < 2\pi\). As we locate a four-dimensional brane at \( r = 0\), which is a string like defect, we must take the boundary condition

\[ \lim_{r \to 0} L(r, x^\mu) = 0, \quad \lim_{r \to 0} L'(r, x^\mu) = \text{const.}, \quad (6) \]

where the prime denotes derivatives with respect to \( r \). The first condition realizes the 4-dimensional brane at \( r = 0\). And the second condition allows the existence of the conical singularity. The structure of conical singularity is a 2-dimensional delta function \( \delta(r)/L \). As we will see later, the deficit angle is determined by the tension of the brane.

Possible components of the energy-momentum tensor \( T_{ab} \), which could be balanced with the singular part of Einstein tensor, are given by

\[ T_{ab}(\text{singular}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_{\mu\nu}(\text{singular}) & 0 \\ 0 & 0 & T_{a\theta}(\text{singular}) \end{pmatrix}. \quad (7) \]

Notice that \((r, r)\) component does not appear in the energy-momentum tensor \([7]\), because this component must be balanced with the \((r, r)\) part of Einstein tensor which consists of only the first derivatives with respect to \( r \) and hence there is no chance to have a singularity. On the other hand, \((\theta, \theta)\) component may have a singularity. To determine the structure of the singularity in the energy-momentum tensor, we need to examine the structure of the singularity in the Einstein tensor. The singular part of the Einstein tensor gives the matching conditions. Using the metric \([4]\), the non-linear matching condition for \((\theta, \theta)\) component is written as

\[ -K' + \alpha \left\{ \frac{4}{3} K^\alpha_\beta K^\beta_\gamma K^\gamma_\alpha - 2 K K^\alpha_\beta K^\alpha_\beta + \frac{2}{3} K^3 \right\} \]

\[ + 4 K^\beta_\alpha R^\alpha_\beta - 2 K R \right\} = \kappa^2 T_{\theta\theta} \quad \text{(singular)} \quad (8) \]

where \( K \) is the trace of is the extrinsic curvature, \( K_{\mu\nu} = -1/2 g_{\mu\nu} \). The \((\mu, \nu)\) components of matching conditions read

\[ \delta^\mu_\nu \left( \frac{L'}{L} G^\mu_\nu + \left( \frac{L'}{L} W^\mu_\nu \right)' \right) + \left[ K^\mu_\nu - \delta^\mu_\nu K \right]' \]

\[ - 4\alpha \left\{ K^\mu_\nu \frac{\Box L}{L} + K \frac{L^{\mu|\nu}}{L} - K^\mu_\nu \frac{L^{|\mu|\nu}}{L} \right\}' \]

\[ - K^\alpha_\nu \frac{L^{|\mu|\nu}}{L} - \delta^\mu_\nu \left( \frac{\Box L}{L} - K^\alpha_\beta \frac{L^{|\beta|\alpha}}{L} \right)' \]

\[ + 2\alpha \left\{ K^\mu_\nu K^\beta_\alpha K^\gamma_\mu - 2 K^\mu_\alpha K^\beta_\alpha K^\gamma_\nu + 2 K K^\mu_\alpha K^\nu_\gamma \right. \]

\[ - K^2 K^\mu_\nu - 2 K^\beta_\alpha R^\mu_\nu \right. \]

\[ - 2 K^\mu_\alpha R^\nu_\gamma - K^\gamma_\alpha K^\nu_\beta + K^\mu_\alpha R^\nu_\beta \]

\[ + \delta^\mu_\nu \left\{ \frac{2}{3} K^\beta_\alpha K^\gamma_\gamma - K^\beta_\alpha K^\gamma_\gamma + \frac{1}{3} K^3 \right. \]

\[ + 2 K^\gamma_\alpha R^\beta_\beta - K R \right\} = \kappa^2 T_{\mu\nu} \quad \text{(singular)} \quad (9) \]

where \( | \) denotes the 4-dimensional covariant derivative and \( W^\mu_\nu \) is defined as

\[ W^\mu_\nu = K^\mu_\alpha K^\nu_\alpha - K K^\mu_\nu + \frac{1}{2} \delta^\mu_\nu (K^2 - K^\alpha_\beta K^\beta_\alpha) \quad (10) \]

In general, \( K_{\mu\nu} \) and \( L'/L \) could have discontinuities at \( r = 0 \), namely, \( \lim_{r \to 0} K_{\mu\nu}(r = \epsilon, x^\mu) \neq K_{\mu\nu}(r = 0, x^\mu) \) and the same for \( L'/L \). For this case, two kinds of singular structures exist in Eqs. \([8] \) and \([9] \) under the boundary condition \([6] \). One is a 2-dimensional delta function singularity \( \delta(r)/2\pi L \) like \( K^\mu_\nu/L, L'/L, \) etc. The other is a 1-dimensional singularity \( \delta(r) \) such as \( K^\mu_\nu \), which is less singular. These kinds of singularities require the same kinds of singular contribution in the energy-momentum tensor. The origin of two kinds of singularities in the energy-momentum tensor can be understood by the fol-
where considering the "internal structure" of the braneworld.

\[ T_{ab}(r = \epsilon, x^\mu) = \frac{1}{\epsilon^2} T_{ab}^{(0)}(r = 0, x^\mu) + \frac{1}{\epsilon} T_{ab}^{(1)}(r = 0, x^\mu) \]  

(11)

where \( T_{ab} \) is the energy-momentum tensor of smooth matter, and \( T_{ab}^{(0)} \) and \( T_{ab}^{(1)} \) denote the Taylor coefficients. Here, \( \epsilon \) represents the thickness of the matter distribution. In the conventional thin limit, only the first term is usually taken. We propose to incorporate the second term to take into account the thickness of braneworld. Thus the possible energy-momentum tensor takes the form

\[ T_{ab}(\text{singular}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & T_{\mu\nu} \frac{\delta \epsilon(r)}{2\pi \epsilon} + S_{\mu\nu} \delta(r) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  

(12)

where \( T_{\mu\nu} \) represents the conventional brane matter, while \( S_{\mu\nu} \) and \( S_{\theta\theta} \) describe the extra matter which mimics the thickness of the braneworld. Note that \( T_{\theta\theta} \) can not be allowed due to the structure of Eq. (8). It should be stressed that this ansatz completely makes sense in the axially symmetric spacetime. In this way, the thickness can be treated within the context of a distributional matter, which we dub quasi-thickness. In a sense, we are considering the "internal structure" of the braneworld.

III. VACUUM BRANEWORLD

The linear analysis is a first step to understand the effective theory on the braneworld. As a background spacetime, let us first consider the vacuum braneworld. Later, we will consider the fluctuations around this vacuum solution. Then, we can proceed to the discussion of the nonlinear dynamics of the codimension 2 braneworld.

Because of the symmetry of the vacuum, the metric can be expressed as

\[ ds^2 = dr^2 + a^2(r) \eta_{\mu\nu} dx^\mu dx^\nu + b^2(r) d\theta^2 \]  

(13)

where \( a(r) \) and \( b(r) \) depend only on \( r \). The energy-momentum tensor \( T^{ab} \) of the vacuum brane can be characterized by the tension \( \sigma \):

\[ T^{ab}(\text{singular}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sigma \delta_{\mu\nu} \frac{\delta \epsilon(r)}{2\pi \epsilon} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]  

(14)

Here, we have introduced only the 2-dimensional distribution. This seems a reasonable assumption because the vacuum can not have an internal structure.

Given the metric (13), the Einstein Gauss-Bonnet equations (3), off the brane, in the bulk are written as

\[ 3 \frac{a''}{a} + 3 \left( \frac{a'}{a} \right)^2 + 3 \frac{b'}{b} \frac{a''}{a} + \frac{b''}{b} = 0 \]  

(15)

\[ 6 \left( \frac{a'}{a} \right)^2 + 4 \frac{a'^3}{a^3} b - 12 \alpha \left[ \left( \frac{a'}{a} \right)^4 + \frac{a'^6}{a^3} b \right] = 0 \]  

(16)

\[ 4 \frac{a''}{a} + 6 \left( \frac{a'}{a} \right)^2 - 12 \alpha \left[ \left( \frac{a'}{a} \right)^4 + \frac{a'^6}{a^3} b \right] = 0 \]  

(17)

The solution is obtained as

\[ ds^2 = dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu + c^2 r^2 d\theta^2 \]  

(18)

where \( c \) is a constant of integration. For \( c \neq 1 \), we have a conical singularity at \( r = 0 \). This can be seen in the following way: First, we specify the boundary condition at \( r = 0 \) by following the standard procedure, \( b'(r = 0) = 1 \). Then, the 2-dimensional curvature is calculated as

\[ \left. \frac{b''}{b} \right|_{r=0} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \frac{b'(r = \epsilon) - b'(r = 0)}{\epsilon} \right) = (c - 1) \frac{\delta(r)}{b} \]  

(19)

Now, the matching condition (12) determines the deficit angle in terms of the brane tension as

\[ 1 - c = \kappa^2 \frac{\delta}{2\pi} \]  

(20)

Note that \( c < 1 \), because the tension \( \sigma \) is positive. This result gives the insight that the deficit might depend on \( x^\mu \) when the inhomogeneous matter exists on the brane.

IV. LINEAR PERTURBATION ANALYSIS

Having considered a background solution and found that the deficit angle seems to depend on the brane matter, we now consider a perturbation around this solution to examine the behavior of the deficit angle fluctuations.

It is possible to classify perturbations using properties under the 4-dimensional coordinate transformation. Let us discuss the scalar, vector, and tensor perturbations, separately.

A. Scalar Perturbation

Although we take the Gaussian normal coordinate system for discussing the effective theory on the brane, the following gauge is convenient for the calculation in the case of the scalar perturbations:

\[ ds^2 = (1 + \delta \varphi + 2\Psi) dr^2 + (1 + \delta \varphi + 2\Phi) \eta_{\mu\nu} dx^\mu dx^\nu + r^2 (1 - 3\delta \varphi) d\theta^2 \]  

(21)
where we absorbed the background value of the deficit angle into the definition of \( \theta \), i.e. \( 0 \leq \theta < 2\pi c \). In linearized Einstein Gauss-Bonnet gravity, we do not require the GB term in the flat bulk background because it consists of a quadratic form of curvature. Then, the 6-dimensional Einstein equation becomes

\[
\frac{1}{2} \Box \delta \varphi + \frac{1}{2} \delta \varphi'' - \frac{7}{2r} \delta \varphi' + \Box \Psi - \frac{1}{r} \Psi' + 4 \Phi = 0 , \tag{22}
\]

\[
\left[ -3 \Phi' + \frac{3}{2} \delta \varphi + \frac{1}{r} \Psi \right]_{\mu} = 0 , \tag{23}
\]

\[
\frac{3}{2} \Box \delta \varphi + \frac{3}{2} \delta \varphi'' + \frac{3}{2r} \delta \varphi' + \frac{1}{r} \Psi' - \frac{4}{r} \Phi' = 0 , \tag{24}
\]

\[
\left[ \Psi + 2 \Phi \right]_{,\mu\nu} = 0 , \tag{25}
\]

\[
\frac{1}{2} \Box \delta \varphi + \frac{1}{2} \delta \varphi'' + \frac{1}{2} r \delta \varphi' + \Box \Phi + \Phi'' + \frac{1}{r} \Phi' = 0 . \tag{26}
\]

Integrating Eqs. (23) and (25), we can write \( \Psi \) and \( \delta \varphi \) in terms of \( \Phi \),

\[
\Psi = -2 \Phi , \tag{27}
\]

\[
\delta \varphi = \frac{3}{2 r} \Phi' + \Phi + \frac{r}{2} d(r) , \tag{28}
\]

where \( d(r) \) is a constant of integration and depends only on \( r \). Here, we imposed the regularity at \( x^\mu \to \infty \). After eliminating \( \Box \delta \varphi \) using Eqs. (24) and (26), putting Eqs. (27) and (28) in the resulting equation, we find the equation of motion for \( \Phi \),

\[
\Phi'' + \frac{3}{r} \Phi' + \Box \Phi = 0 . \tag{29}
\]

After applying the same procedure to Eq. (22) and (25), we obtain

\[
\Phi'' + \frac{3}{r} \Phi' + \Box \Phi + \frac{2}{3r} [d(r) r']' = 0 . \tag{30}
\]

Comparing the above Eq. (30) with Eq. (25), we get

\[
d(r) r = \text{const.} \equiv C_0 . \tag{31}
\]

This constant is related to the mass of the system determined from the given energy-momentum tensor \( \Theta \).

If we define Fourier transformation,

\[
\Phi(r, x^\mu) = \int d^4 p e^{ip \cdot x} \Phi(r, p) , \tag{32}
\]

Then the solution of Eq. (25) is

\[
\Phi(r, p) = \frac{A(q)}{r} J_1(q r) , \tag{33}
\]

where \( q^2 = -p^\mu p_\mu \) and we have imposed the regularity at the origin. The amplitude \( A(q) \) should be determined by the matching condition. The solution is given by

\[
ds^2 = (1 + \delta g_{rr}) dr^2 + (\eta_{\mu\nu} + \delta g_{\mu\nu}) dx^\mu dx^\nu + (r^2 + \delta g_{\theta\theta}) d\theta^2 , \tag{34}
\]

where

\[
\delta g_{rr} = \frac{C_0}{2} - \frac{3}{2} Z_1(r, x^\mu) - \frac{3}{2} Z_2(r, x^\mu) , \tag{35}
\]

\[
\delta g_{\mu\nu} = \left( \frac{C_0}{2} + \frac{3}{2} Z_1(r, x^\mu) - \frac{3}{2} Z_2(r, x^\mu) \right) \eta_{\mu\nu} , \tag{36}
\]

\[
\delta g_{\theta\theta} = r^2 \left( -\frac{3}{2} C_0 - \frac{3}{2} Z_1(r, x^\mu) + \frac{9}{2} Z_2(r, x^\mu) \right) . \tag{37}
\]

Here, we have defined

\[
Z_1(r, x^\mu) = \int d^4 p e^{ip \cdot x} A(q) J_1(q r) , \tag{38}
\]

\[
Z_2(r, x^\mu) = \int d^4 p e^{ip \cdot x} A(q) q J_2(q r) . \tag{39}
\]

In order to impose the matching conditions, we must move to the Gaussian normal coordinate system

\[
ds^2 = dr^2 + (\eta_{\mu\nu} + \delta \bar{g}_{\mu\nu}) dx^\mu dx^\nu + (1 + \delta \bar{g}_{\theta\theta}) d\theta^2 , \tag{40}
\]

where the axial symmetry is taken into account. This can be achieved by the gauge transformations associated with the infinitesimal coordinate transformations \( x^\mu \to x^\mu - \xi^\mu \):

\[
\delta \bar{g}_{\mu\nu} = \delta g_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} , \tag{41}
\]

\[
\delta \bar{g}_{\theta\theta} = \delta g_{\theta\theta} + 2 c^2 r \xi_r . \tag{42}
\]

Note that the location of the brane is not necessarily at constant \( r \). However, there exist residual gauge transformations

\[
\xi_r = \chi(x^\mu) , \tag{45}
\]

\[
\xi_\mu = \chi_\mu r . \tag{46}
\]

Using this residual gauge, one can adjust the brane position to be located at the constant \( r \) in the new coordinate system. Thus, the distortion function \( \chi(x^\mu) \) takes into account the fact that the brane is not necessarily located at the constant \( r \) in the coordinate system which we used to solve equations in the bulk. The final result is written by

\[
ds^2 = dr^2 + (1 + h_{\mu\nu}) \eta_{\mu\nu} dx^\mu dx^\nu + L^2 d\theta^2 , \tag{47}
\]

where

\[
h_{\mu\nu} = \frac{C_0}{2} + \frac{3}{2} Z_1(r, x^\mu) - \frac{3}{2} Z_2(r, x^\mu)
+ \int_0^r dr \int_0^r dr \delta g_{rr,\nu} - 2 \chi_{,\mu} r , \tag{48}
\]

\[
L^2 = c^2 r^2 \left[ 1 - \frac{3}{2} C_0 - \frac{3}{2} Z_1(r, x^\mu) + \frac{9}{2} Z_2(r, x^\mu) \right]
- c^2 r \int_0^r dr \delta g_{rr} + 2 c^2 r \chi , \tag{49}
\]
and we have reintroduced the background deficit, $c$, explicitly. Now, the brane is located at $r = 0$ in this coordinate system.

### B. Vector perturbations

Now, we move on to vector perturbations which turn out to be irrelevant. As to the vector perturbations, we choose the gauge

$$ ds^2 = dr^2 + (\eta_{\mu\nu} + F_{\mu\nu} + F_{\nu\mu}) \, dx^\mu \, dx^\nu + c^2 r^2 \, d\theta^2 \, , $$

where $F^\mu \_\mu = 0$. Note that there exist residual gauge transformations

$$ F_{\mu} = f_{\mu}(x^\mu) \, , \quad \eta^\mu \_, \mu = 0 \, , $$

where the parameter $\eta_{\mu}$ depends only on $x^\mu$.

The equation of motion for $F_{\mu}$ becomes

$$ (F_{\mu\nu} + F_{\nu\mu})'' + \frac{1}{r} \left( (F_{\mu\nu} + F_{\nu\mu})' \right)' = 0 \, . $$

The regular solution of this equation is

$$ F_{\mu} = f_{\mu}(x^\mu) \, . $$

This solution can be eliminated using the residual gauge transformations. Hence, there exist no vector perturbations.

### C. Tensor perturbations

Finally, we consider tensor perturbations. They are characterized by a metric

$$ ds^2 = dr^2 + (\eta_{\mu\nu} + h_{\mu\nu}^{TT}) \, dx^\mu \, dx^\nu + c^2 r^2 \, d\theta^2 \, , $$

where $h_{\mu\nu}^{TT}$ satisfies transverse and traceless conditions. Tensor perturbations remain invariant under a coordinate transformation. The equation of motion for $h_{\mu\nu}^{TT}$ gives

$$ h_{\mu\nu}^{TT}'' + \frac{1}{r} h_{\mu\nu}^{TT} + \Box h_{\mu\nu}^{TT} = 0 \, . $$

The solution becomes

$$ h_{\mu\nu}^{TT}(r,x^\mu) = \int d^4 p \ e^{ipx} \varepsilon_{\mu\nu}(q) J_0(qr) \, , $$

where $\varepsilon_{\mu\nu}$ should be determined by the matching condition.

### V. EFFECTIVE THEORY

#### A. Linear Regime

Since we have solved the bulk geometry, we next discuss the matching conditions. Using the energy-momentum tensor, matching conditions Eqs. and reduce to

$$ - \delta K' = \kappa^2 S^\theta_\theta(r) \, , $$

$$ \delta K''_{\nu} - \delta^\mu_\nu \delta K' = \kappa^2 S^\mu_\nu \delta(r) \, , $$

$$ \delta^\mu_\nu \delta L'' = 4\alpha L'' \delta G^\mu_\nu = \frac{\kappa^2}{2\pi} T^\mu_\nu \delta(r) \, , $$

where we dropped all the remaining terms in Eqs. and because they do not contribute at first order in perturbation.

Using Eq. and, the deficit angle can be obtained from

$$ \lim_{r \to 0} L'(r = \epsilon) = \lim_{r \to 0} \left[ b'(r = \epsilon) + \delta L'(r = \epsilon) \right] = c(1 - C_0) \, . $$

Note that the deficit angle is perturbed on the brane at the linear level, but it is constant. From Eqs. and , the extrinsic curvature in this limit gives

$$ \lim_{r \to 0} K_{\mu\nu}(r = \epsilon, x^\mu) = \lim_{r \to 0} \delta K_{\mu\nu}(r = \epsilon, x^\mu) $$

$$ = -\frac{1}{2} \lim_{r \to 0} h_{\mu\nu}'(r = \epsilon) = \chi_{\mu\nu} \, (61) $$

Notice that the background value of $K_{\mu\nu}$ is zero and the tensor perturbations do not contribute to the extrinsic curvature.

From this result, we see that the extrinsic curvature is nonzero near the brane, if the distortion field $\chi(x^\mu)$ exists. In the coordinate system where we have solved the equations of motion in the bulk, the location of the brane is specified by $\chi(x^\mu)$ and the resulting shape of the braneworld becomes the deformed cylinder in which the distortion is represented by $\chi(x^\mu)$ (see Fig.1). At first sight, this deformed cylinder looks like a 5-dimensional hypersurface. However, at the location of the brane ($r = \epsilon$ in the Gaussian normal gauge), the circumference radius of the cylinder vanishes because of $L(r = \epsilon, x^\mu) = 0$ there. Hence, the deformed cylinder is the 4-dimensional braneworld.

Now, we must specify $K_{\mu\nu}(r = 0, x^\mu)$ which is not known a priori. Without losing the generality, we can write

$$ K_{\mu\nu}(r = 0, x^\mu) = \chi_{0,\mu\nu}(x^\mu) + \rho_{\mu\nu}(x^\mu) \, , \quad \rho_{\mu\nu} = 0 \, . (62) $$

where we decomposed the extrinsic curvature at $r = 0$ into the traceless part $p_{\mu\nu}(x^\mu)$ and the trace part $\Box \chi_0(x^\mu)$. As the field $\chi(x^\mu)$ represents the location of the brane $r = \epsilon$, it is natural to regard $\chi_0(x^\mu)$ as the location of $r = 0$. Hence, $\psi(x^\mu) \equiv \chi(x^\mu) - \chi_0(x^\mu)$ can be interpreted as the thickness of the braneworld.

With these considerations, we find Eqs. and become

$$ \Box(\chi - \chi_0) = -\kappa^2 S^\theta_\theta \, , $$

$$ \chi''_{\mu\nu} - \delta^\mu_\nu \Box \chi - [\chi_{0,\mu\nu} - \delta^\mu_\nu \Box \chi_0] - \rho_{\mu\nu} = \kappa^2 S^\mu_\nu \, (64) $$

$$ \delta_{\mu\nu} \left[ \delta L' - \delta L_0' \right] - 4\alpha \left[ b' - b_0' \right] \delta G^\mu_\nu = \frac{\kappa^2}{2\pi} T^\mu_\nu \, . $$

(65)
Adopting the standard boundary condition $L'(r=0) = b'(r=0) + \delta L'(r=0) = 1$ even in the perturbed spacetime, Eq. \(66\) leads to
\[
G^\mu_\nu = \frac{\kappa^2}{8\pi\alpha(1-c)} T^\mu_\nu + \delta T^\mu_\nu + \frac{cC_0}{4\alpha(1-c)} ,
\]
where the second term of the right hand side of \(66\) is the cosmological constant. The Einstein tensor includes the contribution from the tensor perturbations. Eq. \(66\) proves that Einstein gravity is recovered for the codimension 2 braneworld in the case of the linearized gravity. This fact is consistent with the result in \[7\]. However, this is not the end of our story. We must also solve the conditions \(69\) and \(70\).

The conservation law $T^e_\sigma;\alpha = 0$ for the $e = \mu$ at the linearized level gives the 4-dimensional conservation law,
\[
T^\mu_\nu |_\nu = 0 , \quad S^{\mu\nu}_\nu |_\nu = 0 ,
\]
where $|$ denotes the 4-dimensional covariant derivative. On the other hand, the conservation law for the $e = r$ implies,
\[
\left[\delta K^\sigma_\alpha + b' S^{\theta}_\theta \right] \delta(r) = 0 .
\]
Note that Eq. \(68\) appears to contain an ambiguity, since $K^{\mu\nu}$ must be evaluated at $r = 0$ where it is discontinuous. Thus we define the following quantities \[8\]
\[
\delta K = \lim_{r \to 0} \frac{1}{2} \left[ \delta K(r = \epsilon) + \delta K(r = 0) \right] ,
\]
\[
\tilde{b}' = \lim_{r \to 0} \frac{1}{2} \left[ b'(r = \epsilon) + b'(r = 0) \right] .
\]
Using these quantities, one can deduce
\[
\delta K^\sigma_\alpha + \tilde{b}' S^{\theta}_\theta = 0 .
\]

The conservation law \(71\) gives the constraint on the distributional sources as
\[
\Box(\chi + \chi_0) = -\frac{1 + c}{1 - c} \kappa^2 S^{\theta}_\theta ,
\]
where we used Eq. \(20\). Combining Eq. \(20\) with \(72\), we find
\[
\Box \chi = -\frac{\kappa^2}{1 - c} S^{\theta}_\theta , \quad \Box \chi_0 = -\frac{\kappa^2 c}{1 - c} S^{\theta}_\theta .
\]
The above Eqs. \(69\) tell us that $\chi_0 = \epsilon \chi$. Since $c < 1$, $\psi = \chi - \chi_0 = (1 - c) \chi$ is positive. Hence, it is legitimate to interpret $\psi$ as the effective thickness. In the meantime, comparing Eq. \(63\) with the trace of Eq. \(64\), we find the relation between the components of energy-momentum tensor as
\[
S^{\theta}_\theta = \frac{1}{3} S^{\mu}_\mu .
\]
Hence, once we give the 4-dimensional matter $S^{\mu}_\mu$, the extra component $S^{\theta}_\theta$ can be determined. It should be noted that the effective thickness $\psi$ satisfies
\[
\Box \psi = -\frac{\kappa^2}{3} S^{\mu}_\mu .
\]
This is reminiscent of the radion in the codimension 1 braneworld \[10\]. The remaining quantity $\rho^{\mu\nu}$ can be also obtained from Eq. \(64\).

\[1\] B. Nonlinear Regime

Our main point is the formulation of the framework of the quasi-thick braneworld. The importance of this point can be manifest in the next order calculations. Up to the second order, we expect
\[
G^{\mu\nu} = \frac{\kappa^2}{8\pi\alpha(1-c+cC_0)} T^{\mu\nu} + \frac{cC_0 - L'}{4\alpha(1-c+cC_0)} \delta^{\mu\nu}
+ c \left[ \chi^{\mu\alpha,\beta} \chi^{\alpha,\nu} - \Box \chi^{\mu\nu} \right]
+ \frac{1}{2} \delta^{\mu\nu} \left\{ (\Box \chi)^2 - \chi^{\alpha,\beta} \chi^{\beta,\alpha} \right\}
+ \frac{1}{1-c} \left[ \rho^{\mu\alpha} \rho^{\nu\beta} - \delta^{\mu\nu} \delta^{\alpha\beta} \right] ,
\]
where the corrections in the last three lines arises due to \(2\), and carry the information of thickness of the brane. Another correction come from \(L'\) carries the effects of bulk, that is, the deformed deficit angle. This needs the next order analysis. Thus, it turns out that some corrections due to the quasi-thickness can be expected at the second order.

Here, we should point out a possibility that the above corrections to Einstein equations could mimic the observed features of dark energy/dark matter. To give some
flavor, let us extrapolate the effective theory \((76)\) to the nonlinear regime. In the case of cosmology, one can put
\[
S^{\mu\nu} = \begin{pmatrix}
-\rho_D & 0 \\
0 & \rho_D \delta^i_j
\end{pmatrix}.
\]
Then, one finds when the condition
\[
\frac{5\kappa^4}{3(1-c)}\rho_D^2 < \frac{(2) L' - cC_0}{4\alpha(1-c + cC_0)}
\]
is satisfied, these kinds of corrections behave as dark energy. While, when the condition
\[
\frac{(2) L' - cC_0}{4\alpha(1-c + cC_0)} = \frac{11\kappa^4}{9(1-c)}\rho_D^2
\]
is satisfied, the effective energy-momentum tensor due to corrections can be regarded as dark matter on the brane. To verify these expectations, we need to solve the bulk geometry.

In reality, we need to consider the full nonlinear theory to discuss the cosmology. The nonlinear effective equation should read
\[
\lim_{\epsilon \to 0} \left[ \delta^\mu_\nu L' - 4\alpha \left\{ L'G^\mu_\nu + L'W^\mu_\nu \right\} 
- 4\alpha \left\{ K^\mu_\nu \square L + KL^\mu_\nu - K^\mu_\alpha L^{|\alpha|}_\nu - K^\alpha_\nu L^{|\mu|}_\alpha 
- \delta^\mu_\nu \left( K \square L - K^\alpha_\beta L^{\beta}_\alpha \right) \right\} \right] = \frac{\kappa^2}{2\pi} T_{\mu\nu}.
\]
where the left hand side Eq. (80) represents the discontinuity at \( r = 0 \). To fully solve the system consistently, we must also take into account the matching conditions come from the less singular part which determines the boundary conditions completely.

It is also intriguing to see how the gravitational waves on the brane are affected by the bulk geometry in the nonlinear situations such as the radiation from the particle falling into the black hole. For this purpose, we must solve the above non-linear problem.

VI. CONCLUSION

We have studied the codimension 2 Einstein Gauss-Bonnet braneworld in the axially symmetric 6-dimensional spacetime. We have carefully examined the structure of possible singularities in the equations of motion which characterize the braneworld through matching conditions. It turned out that the thickness of the brane can be taken into account within the context of the distribution source, which we dubbed quasi-thickness. In the case of the linearized gravity, we have solved all of the equations of motion in the bulk and shown the conventional Einstein gravity can be recovered on the brane. In this process, all of the necessary boundary conditions are clarified. In the nonlinear regime, we found corrections due to the thickness and the bulk geometry. We stressed that the interplay between the bulk and the brane has been completely determined in the context of the quasi-thick braneworld.

In our formulation of the codimension 2 braneworld, \( T_{\mu\nu} \) can be identified with the ordinary matter. While \( S_{\mu\nu} \) is a kind of dark energy/dark matter. The possibility that the thickness plays a role of the dark sector in the universe is attractive. It would be also interesting to construct a viable particle physics model in this context.

There are many remaining issues to be solved. Once \( T_{\mu\nu} \) and \( S_{\mu\nu} \) are given, we can analyze the nonlinear gravity to reveal the nature of the corrections. This would be very complicated but the analysis is straightforward. Application to cosmology seems to be trivial in the linear regime. This is in contrast to the case of codimension 1 braneworld where the analysis of the cosmological perturbations are very complicated. It is interesting to consider the black holes in the context of the codimension 2 braneworld. The role of the Gregory-Laflamme instability should be clarified and the property of the gravitational waves from the black hole must be analyzed. It is also intriguing to extend our formalism to the higher codimension braneworld.

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