MONOPOLE ORDER PARAMETER
IN SU(2) LATTICE GAUGE THEORY*

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ABSTRACT

We present the results of the numerical calculation of the probability distribution of the value of the monopole creation operator in SU(2) lattice gluodynamics. We work in the maximal abelian projection. It occurs that at the low temperature, below the deconfinement phase transition the maximum of the distribution is shifted from zero, which means that the effective constraint potential is of the Higgs type. Above the phase transition the minimum of the potential (the maximum of the monopole field distribution) is at the zero value of the monopole field. This is the direct proof of the existence of the abelian monopole condensate in the confinement phase of the gluodynamics, which confirms the dual superconductor model of the confining vacuum.

1. Introduction

The monopole mechanism of the colour confinement is generally accepted by the community. Still there are many open questions. In the lattice gluodynamics it is very important to find the order parameter, constructed from the monopole field, for the deconfinement phase transition. The first candidate is the monopole condensate, which should be nonzero in the confinement phase and vanish at the phase transition. To study the monopole condensate we need the explicit expression for the operator \( \Phi_{\text{mon}}(x) \), which creates the abelian monopole at the point \( x \). The operator \( \Phi_{\text{mon}}(x) \) was found for the compact electrodynamics with the Villain form of the action by Fröhlich and Marchetti [1], and it was studied numerically in refs.[2]. In Section 2 we construct the monopole creation operator for an arbitrary abelian projection of lattice SU(2) gluodynamics. The numerical results presented in Section 3 are obtained for the maximal abelian projection, for this projection many numerical simulations show that the gluodynamic vacuum behaves as the dual superconductor (see [3] and references therein). In refs.[4] the another form of the monopole creation operator was studied, and it was found that its expectation value vanishes in the deconfinement phase; as we discuss at the end of Section 2, our operator is positively defined, therefore our definition of \( \Phi_{\text{mon}}(x) \) differs from that of ref.[4], still our conclusions and that of ref.[4]

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coincide: the monopole condensate exists in the confinement phase of lattice gluodynamics. The analogous claim is done in ref\[5\], where the monopole condensate is calculated on the basis of the percolation properties of the monopole currents.

2. Monopole Creation Operator

First we give the formal construction of the monopole creation operator for the abelian projection of SU(2) gluodynamics. Let us parametrize SU(2) link matrix in the standard way:

\[ U_{11}^{x_{\mu}} = \cos \phi_{x_{\mu}} e^{i\phi_{x_{\mu}}}, \quad U_{12}^{x_{\mu}} = \sin \phi_{x_{\mu}} e^{i\chi_{x_{\mu}}}, \quad U_{22}^{x_{\mu}} = U_{11}^{x_{\mu}}, \quad U_{21}^{x_{\mu}} = -U_{12}^{x_{\mu}}; \quad 0 \leq \phi \leq \pi/2, \quad -\pi < \theta, \chi \leq \pi. \]

The plaquette action in terms of the angles \( \phi, \theta \) and \( \chi \) can be written as follows:

\[ S_P = \frac{1}{2} \text{Tr} U_1 U_2 U_3^+ U_4^+ = S^a + S^n + S^i, \tag{1} \]

where

\[ S^a = \cos \theta_P \cos \phi_1 \cos \phi_2 \cos \phi_3 \cos \phi_4, \tag{2} \]

\( S^n \) and \( S^i \) describe the interaction of the fields \( \theta \) and \( \chi \) and selfinteraction of the field \( \chi \) \[6\]:

\[ \theta_P = \theta_1 + \theta_2 - \theta_3 - \theta_4, \tag{3} \]

here the subscripts \( 1, \ldots, 4 \) correspond to the links of the plaquette: \( 1 \to \{x, x + \hat{\mu}\}, \ldots, 4 \to \{x, x + \hat{\nu}\} \).

If we fix the abelian projection, each term \( S^a, S^n \) and \( S^i \) is invariant under the residual \( U(1) \) gauge transformations:

\[ \theta_{x_{\mu}} \to \theta_{x_{\mu}} + \alpha_x - \alpha_{x + \hat{\mu}}, \tag{4} \]
\[ \chi_{x_{\mu}} \to \chi_{x_{\mu}} + \alpha_x + \alpha_{x + \hat{\mu}}. \tag{5} \]

The operator which creates the monopoles at the point \( x \) of the dual lattice is defined as follows:

\[ U(x) = \exp \{\beta[-S(\theta_P, \phi) + S(\theta_P + W_P(x), \phi)]\}, \tag{6} \]

we define the function \( W_P(x) \) below. Substituting eq.\[1\] \[2\] in eq.\[6\] we get

\[ U(x) = \exp \left\{ \sum_P \tilde{\beta} [-\cos(\theta_P) + \cos(\theta_P + W_P(x))] \right\}, \tag{7} \]

where \( \tilde{\beta} = \cos \phi_1 \cos \phi_2 \cos \phi_3 \beta \). Effectively the monopole creation operator shifts all abelian plaquette angles \( \theta_P \).
For the compact electrodynamics with the Villain type of the action the above definition coincides with the definition of Fröhlich and Marchetti [1]. For the general type of the action in compact electrodynamics we can use the described above construction. The proof is given in Appendix A. The gluodynamics in the abelian projection contains the compact gauge field $\theta$ and the charged vector field $\chi$. The action (1) in terms of the fields $\theta$ and $\chi$ is rather nontrivial, and at the moment we have no proof that the above construction of the monopole creation operator is valid in this case. Still for the rather similar Abelian – Higgs model, with the general type of the action, the proof exists, and it is analogous to one given in Appendix A. Moreover the numerical results, presented in the next section, clearly show that the suggested operator is the order parameter for the deconfinement phase transition.

3. Numerical Results

We present the results of the numerical calculations on the lattice $10^3 \cdot 4$, we impose the anti–periodic boundary conditions in space directions, since the construction of the operator $\mathcal{U}$ can be done only in the time slice with the anti–periodic boundary conditions. The periodic boundary conditions are forbidden due to the Gauss law; formally there is no solution of equation (13) in the finite box with the periodic boundary conditions. To see that we have the order parameter for the deconfinement phase transition it is very convenient to study the probability distribution of the operator $\mathcal{U}$. It means that we calculate the expectation value $<\delta(\varphi - \mathcal{U}(x))>$. The quantity like the effective constraint potential,

$$e^{-V_{\text{eff}}(\varphi)} = <\delta(\varphi - \frac{1}{V} \sum_x \mathcal{U}(x))>$$

has more physical meaning than the probability distribution. At the moment we have no enough statistics to calculate $V_{\text{eff}}(\varphi)$, and we present our results for the quantity $V(\varphi)$, defined as:

$$e^{-V(\varphi)} = <\delta(\varphi - \mathcal{U}(x))>.$$  

In Figs. 1(a) and 1(b) we show $V(\varphi)$ for the confinement and the deconfinement phases. It is clearly seen that in the confinement phase the minimum of $V(\varphi)$ is shifted from zero, while in the deconfinement phase the minimum is at zero value of the monopole field $\phi$. We used the positively definite operator $\mathcal{U}(x)$ (7), but in the dual representation the creation operator of the monopole (14) is not positively defined, the sign is loosed since we perform the inverse duality transformation on the infinite lattice. On the finite lattice it is possible to get the non–positive defined operator $\mathcal{U}(x)$, in that case instead of Fig. 1(a) we get the Higgs – type potential. This little bit more complicated calculations are now in progress. Still Figs. 1(a),(b) clearly show that the position of the minimum of $V(\varphi)$ plays the role of the order parameter. On Fig.2 we show the dependence of the position of the minimum, $\varphi_c$, on the temperature, it is seen that $\varphi_c$ vanishes at the point of the phase transition.
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Appendix A

Below we construct the monopole creation operator for the compact electrodynamics with the general type of the action; the similar construction exists for the compact Abelian–Higgs model with the general type of the action. First we perform the duality transformation of the partition function for the 4D lattice compact electrodynamics:

\[
Z = \int_{-\pi}^{+\pi} \mathcal{D}\theta \exp\{-S(d\theta)\},
\]

We use the notations of the calculus of differential forms on the lattice (see also Appendix B). The symbol \(\int \mathcal{D}\theta\) denotes the integral over all link variables \(\theta\). The partition function of the dual theory,

\[
Z^d = \sum_{*n(c_1)\in\mathbb{Z}} \exp\{-*S(d^*n)\},
\]

\[
*S(p) = - \ln \int \mathcal{D}F \exp\{-S(F) + i(F,p)\},
\]

can be represented as the following limit of the partition function for the Abelian–Higgs theory:

\[
Z^d = \lim_{\kappa \to \infty} \int_{-\pi}^{+\pi} \mathcal{D}^*\varphi \int_{-\infty}^{+\infty} \mathcal{D}^*B \sum_{*n(c_1)\in\mathbb{Z}} \exp\{-*S(d^*B/2\pi) - \kappa\|*B - d^*\varphi + 2\pi*n\|^2\},
\]

here \(d^*B/2\pi\) is the kinetic energy of the dual gauge field \(d^*B\) (the analogue of \(\tilde{F}_{\mu\nu}^2\)) and the Higgs field \(\exp\{i*\varphi\}\) carry magnetic charge, since it interacts via the covariant derivative with the dual gauge field \(d^*B\). The Dirac operator,

\[
U^d(x) = e^{i*\varphi} \cdot \exp\{-i(*D_x, *B)\},
\]

\[
\delta^*D_x = *\delta_x
\]
is the gauge invariant monopole creation operator. It creates the cloud of photons and the monopole at the point $x$. In (13) $^*\delta_{x}$ is the lattice $\delta$–function, it equals to unity at the cite $x$ of the dual lattice and is zero at other cites. Note that in the above formulas the radial part of the Higgs field which carry the magnetic charge is fixed to unity.

Coming back to the original partition function (10) we get the expectation value of the monopole creation operator in terms of the fields $\theta$:

$$<U(x)> = \frac{1}{Z} \int_{-\pi}^{+\pi} D\theta \exp \{-S(d\theta + W_P(x))\},$$

$$W_P(x) = 2\pi \delta \Delta^{-1}(D_x - \omega_x),$$

where the Dirac string attached to the monopole [1], is represented by the integer valued 1-form $^*\omega_x$, which satisfies the equation: $^*\delta_{x} = ^*\delta_{x}$.

The partition function (10) for the electrodynamics with the general type of the action can be represented as the sum over the monopole closed currents [9]. It is straightforward to show that the monopole creation operator $U(x)$ creates the not–closed monopole trajectory which starts at the point $x$. This fact shows that $U(x)$ is the monopole creation operator.

### Appendix B

Below we briefly summarize the main notions from the theory of differential forms on the lattice [7]. The advantages of the calculus of differential forms consists in the general character of the expressions obtained. Most of the transformations depend neither on the space–time dimension, nor on the rank of the fields. With minor modifications, the transformations are valid for lattices of any form (triangular, hypercubic, random, etc). A differential form of rank $k$ on the lattice is a function $\phi_k$ defined on $k$–dimensional cells $c_k$ of the lattice, eg the scalar (gauge) field is a 0–form (1–form). The exterior differential operator $d$ is defined as follows:

$$(d\phi)(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} \phi(c_k).$$

Here $\partial c_k$ is the oriented boundary of the $k$-cell $c_k$. Thus the operator $d$ increases the rank of the form by unity; $d\varphi$ is the link variable constructed, as usual, in terms of the site angles $\varphi$, and $dA$ is the plaquette variable constructed from the link variables $A$. The scalar product is defined in the standard way: if $\varphi$ and $\psi$ are $k$-forms, then $(\varphi, \psi) = \sum c_k \varphi(c_k) \psi(c_k)$, where $\sum c_k$ is the sum over all cells $c_k$. To any $k$–form on the $D$–dimensional lattice there corresponds a $(D-k)$–form $^*\Phi(*c_k)$ on the dual lattice, $^*c_k$ being the $(D-k)$–dimensional cell on the dual lattice. The codifferential $\delta = ^*d^*$ satisfies the partial integration rule: $(\varphi, \delta \psi) = (d\varphi, \psi)$. Note that $\delta \Phi(c_k)$ is a $(k-1)$–form and $\delta \Phi(c_0) = 0$. The norm is defined by: $\|a\|^2 = (a, a)$; therefore, $\|B - d\varphi + 2\pi m\|^2$ in (13) implies summation over all links, $\sum_{l(c_1) \in \mathbb{Z}}$ denotes the sum over all configurations of the integers $l$ attached to the links $c_1$. 


The action (13) is invariant under the gauge transformations $B' = B + d\alpha$, $\varphi' = \varphi + \alpha$ due to the well known property $d^2 = \delta^2 = 0$. The lattice Laplacian is defined by: $\Delta = d\delta + \delta d$.

**Figure Captions**

Fig1. $V(\varphi)$ for the confinement (a) and the deconfinement (b) phases.
Fig2. Position of the minimum $\varphi_c$ of $V(\varphi)$ vs. temperature $T$.

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$\varphi_c$ vs. $T$

$T_c$

Fig. 2