Three-Point Functions of Higher-Spin Supercurrents in 4D $\mathcal{N} = 1$ Superconformal Field Theory

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We develop a general formalism to study the three-point correlation functions of conserved higher-spin supercurrent multiplets $J_{\mu_1 \ldots \mu_r}^{\alpha_1 \ldots \alpha_r}$ in 4D $\mathcal{N} = 1$ superconformal theory. All the constraints imposed by $\mathcal{N} = 1$ superconformal symmetry on the three-point function $\langle J_{\alpha_1 \ldots \alpha_r}^{\mu_1 \ldots \mu_r} J_{\beta_1 \ldots \beta_s}^{\nu_1 \ldots \nu_s} J_{\gamma_1 \ldots \gamma_t}^{\rho_1 \ldots \rho_t} \rangle$ are systematically derived for arbitrary $r$, $s$, $t$, thereby reducing the problem mostly to computational and combinatorial. As an illustrative example, we explicitly work out the allowed tensor structures contained in $\langle J_{\alpha}^{\mu} J_{\beta}^{\nu} J_{\gamma}^{\rho} \rangle$, where $J_{\alpha}^{\mu}$ is the supercurrent. We find that this three-point function depends on two independent tensor structures, though the precise form of the correlator depends on whether $r$ is even or odd. The case $r = 1$ reproduces the three-point function of the ordinary supercurrent derived by Osborn. Additionally, we present the most general structure of mixed correlators of the form $\langle LLJ_{\alpha}^{\mu} J_{\beta}^{\nu} J_{\gamma}^{\rho} \rangle$ and $\langle J_{\alpha_1 \ldots \alpha_r}^{\mu_1 \ldots \mu_r} J_{\beta_1 \ldots \beta_s}^{\nu_1 \ldots \nu_s} L \rangle$, where $L$ is the flavour current multiplet.

1. Introduction

Finding how conformal symmetry constrains correlation functions of local, primary operators is one of the basic and most important questions in conformal field theory (see refs. [1–10] for early results). Among primary fields (corresponding to lowest weight states of the conformal algebra) there is an important class of operators which are conserved currents. They correspond to lowest weight states whose scaling dimension saturates the unitarity bound. The systematic approach to study correlation functions of conserved currents was developed in [11, 12] and later extended to superconformal field theories in various dimensions [14–28]. Several novel approaches to the construction of correlation functions of conserved currents which carry out the calculations in momentum space, using methods such as spinor-helicity variables were studied in [29–35]. The most studied examples of conserved currents in (super)conformal field theory are the energy-momentum tensor and vector currents; their most general three-point functions were determined in [11, 12]. However, (super)conformal field theories can possess higher-spin conserved (super)currents. The general structure of three-point functions of conserved higher-spin, bosonic, vector currents was obtained by Stanev [36] and Zhiboedov [37] see [38] for similar results using the embedding formalisms [39–44] (and [45–48] for supersymmetric extensions). Recently, Ref. [50] proposed super-twistor realisations of $(p,q)$ anti-de Sitter (AdS) superspaces in three dimensions and $\mathcal{N}$-extended AdS superspaces in four dimensions, see [51] for the non-supersymmetric case. These formulations might be useful for studying correlation functions on AdS.

As was proven by Maldacena and Zhiboedov in [52] (see also [53–55]), all correlation functions of higher-spin conserved currents are equal to those of a free theory. The main assumption of the Maldacena-Zhiboedov theorem was that the conformal field theory under consideration possesses a unique conserved current of spin 2, the energy-momentum tensor. However, in [52] it was also shown that if a conformal field theory possesses a conserved fermionic higher-spin current then it has an additional conserved current of spin 2. This implies that the Maldacena-Zhiboedov theorem does not seem to hold in superconformal field theories possessing higher-spin currents.

In supersymmetric theories, the bosonic and fermionic currents form supermultiplets. For example, the energy-momentum tensor together with the fermionic supersymmetry current belong to the supercurrent multiplet [16] (see also [57–61]). On the other hand, an additional conserved current of spin 2 will belong to the supermultiplet of superspin 2 which also includes

\[1\] The analyses of [11, 12] were performed in general dimensions and did not consider parity-violating structures relevant for three-dimensional conformal field theories. These structures were later found in [13].

\[2\] Correlation functions of fermionic operators were discussed in [38, 49].
a higher-spin current of spin $\frac{5}{2}$, see [27] for a similar discussion in three dimensions. In general, a superconformal field theory possessing bosonic higher-spin currents will also possess fermionic ones. As discussed above, this naturally leads to a violation of the main assumption of the Maldacena-Zhiboedov theorem. Thus, correlation functions of higher-spin conserved currents in superconformal field theories do not have to coincide with those in free theory which makes their study particularly interesting.

The study of the most general form of three-point correlation functions of higher-spin conserved currents in four-dimensional (4D) $\mathcal{N} = 1$ superconformal field theory was initiated in [26], where the three-point functions of higher-spin spinor currents $S_{\alpha}(r)$ were obtained. In this paper, we continue our study to explore how $\mathcal{N} = 1$ superconformal symmetry and conservation laws constrain the structure of three-point functions involving the conformal higher-spin current multiplet $J_{\alpha_{1}...\alpha_{n}} = J_{\alpha_{1}...\alpha_{n}} \equiv J_{\alpha(\gamma)}$. This real symmetric and traceless rank-$r$ tensor superfield $J_{\alpha(r)}$ is in one-to-one correspondence with the real symmetric spinor superfield $J_{\alpha_{1}...\alpha_{n}} = J_{\alpha_{1}...\alpha_{n}} \equiv J_{\alpha(r)}$ via

$$J_{\alpha(r)} = (\sigma^a\alpha)_{a_1}...\alpha_r...a_nJ_{\alpha_{1}...\alpha_{n}}.$$  

We will refer to (1.1) as higher-spin supercurrent. The superfield $J_{\alpha(r)}$, which belongs to the representation $(r/2, r/2)$ of the Lorentz group, is primary with conformal weight $(q, q) = (\frac{r+2}{2}, \frac{r+2}{2})$ and scaling dimension $r + 2$. It is subject to the conservation equations

$$D^aJ_{\alpha(r)a_{1}(r-1)} = 0,$$

where $D^a, \bar{D}^a$ are the superspace covariant derivatives. The case $r = 1$ corresponds to the ordinary conformal supercurrent.\[56\] The case $r > 1$ was first described in [62].

The structure of the higher-spin current multiplets, including non-conformal ones, have been a subject of several works. For instance, explicit realisations in terms of free scalar and free vector multiplets are known for $J_{\alpha(r)}$, in Minkowski,\[53-59\] AdS\[70\] and conformally-flat backgrounds.\[71\] Higher-spin supercurrent $J_{\alpha(r)}$ may also be realised in terms of the on-shell, gauge-invariant, chiral field strengths $W_{a(r)}$ obeying $\bar{D}_aW_{a(r)} = 0$ and $D^a\bar{W}_{a(r-1)} = 0$. Its realisation is given by [70, 72, 73]:

$$J_{\alpha(r)} = W_{a(r)}\bar{W}_{a(r)}.$$  

Here $W_{a(r)}$ and $W_{a(r)}$ are the gauge-invariant field strengths describing the free vector, gravitino and linearised supergravity multiplets, respectively. For $r > 3$, the superfields $W_{a(r)}$ are the on-shell gauge-invariant field strengths corresponding to the massless higher-spin gauge multiplets\[74,75\] (see section 6.9 of\[76\] for a review). Setting $r = 1$ in (1.3) gives the supercurrent of the free $\mathcal{N} = 1$ vector multiplet, $J_{a\alpha} = W_{a}\bar{W}_{a}$.

The main aim of this work is to develop a general formalism to study the three-point correlation function of conserved higher-spin supercurrents

$$\langle J_{\alpha(r)}(z_1)J_{\beta(r)}(z_2)J_{\gamma(r)}(z_3) \rangle$$  

in 4D $\mathcal{N} = 1$ superconformal field theory. We use a hybrid approach which combines the group theoretic formalism of Osborn\[15\] with the index-free method based on using auxiliary vectors/spinors. The latter enables us to encode symmetric traceless tensors by polynomials in commuting auxiliary spinors. The approach based on auxiliary polarisation vectors is widely used in the literature, e.g.\[31,32,33,34,40\] (for earlier work see also e.g. Ref. [77] and references therein), where one usually has to work with the spacetime points explicitly when analysing conservation laws. In contrast, by making use of the general formalism of Osborn,\[15\] all information about the three-point function (1.4) is concentrated in two polynomials, both of which are functions of a single superconformally covariant bosonic vector $X$, and the auxiliary commuting spinors. Within our approach, the constraints imposed by conservation laws and symmetries with respect to permutations of superspace points take a simple form; hence, they can be efficiently implemented and solved computationally in Mathematica. Our method is a supersymmetric extension of the generating function formalism recently proposed in [49]. Once the formalism is developed, finding the general solution of the correlation function (1.4) to a large extent becomes a computational and combinatorial (though still quite a difficult) problem. One of the complications involves determining various linear independence relations in the possible set of solutions. This problem will be discussed elsewhere. We believe that our formalism can be generalised to the case of the most general higher-spin conformal current multiplets containing both vector and spinor indices, which are of the form $J_{\alpha(\gamma)(\delta)}$ corresponding to Lorentz type $(m/2, n/2)$, with $m, n > 1$ and $m \neq n$.\[78\] This problem will also be studied elsewhere.

The paper is organised as follows. In section 2 we review the general construction of two- and three-point functions of conserved currents mostly following.\[15\] In section 3 we introduce the formalism aimed at computing the three-point function (1.4) for arbitrary $r_1, r_2, r_3$. In the remaining sections, we demonstrate the efficiency of our formalism by computing several new mixed correlation functions of $J_{\alpha(r)}$ with the supercurrent and the flavour current multiplet $L$. The latter is a real scalar superfield containing the vector current $V_m$. More precisely, in section 4, we consider $\langle J_{\alpha(r)}(z_1)J_{\beta(r)}(z_2)J_{\gamma(r)}(z_3) \rangle$ and find the most general solutions in an explicit form for all values of $r$. In particular, for $r = 1$ we reproduce the three-point function of the supercurrent found previously in [15]. For higher $r$, we construct a generating function which produces all possible linearly dependent solutions. The number of linearly independent tensor structures compatible with all the constraints are then found computationally. We find that the correlator depends on two independent tensor structures, though its precise form depends on the values of $r$ (even or odd). In section 5, we generalise our formalism to the mixed three-point functions involving $J_{\alpha(r)}$, and the flavour current multiplet. We find the general solutions for $\langle L(z_1)J_{\alpha(r)}(z_2) \rangle$ and $\langle J_{\alpha(r)}(z_1)J_{\beta(r)}(z_2)L(z_3) \rangle$, for all values of $r, r_1, r_2$ in an explicit form. In the former case, the correlator $\langle LLJ \rangle$ depends on a
single tensor structure. In the (JL) case, the correlator depends on two tensor structures if \( r_1 < r_2 \); and a single tensor structure if \( r_1 = r_2 \).

### 2. Superconformal Building Blocks

This section contains a concise summary of two- and three-point superconformal building blocks in four-dimensional \( \mathcal{N} = 1 \) superspace, which are essential for our analysis. These superconformal structures were introduced in [14, 15], and later generalised to arbitrary \( \mathcal{N} \) in [16] (see also [19] for a review). Our notation and conventions are those of [76].

#### 2.1. Two-Point Structures

We denote the \( \mathcal{N} = 1 \) Minkowski superspace by \( \mathbb{M}^{1|4} \). It is parametrised by coordinates \( x^a = (\mathcal{z}^a, \theta^a, \bar{\theta}_a) \), where \( a = 0, 1, 2, 3; \; a, \bar{a} = 1, 2 \). Let \( z_1 \) and \( z_2 \) be two different points in superspace. All building blocks for the two- and three-point correlation functions are composed of the two-point structures:

\[
\begin{align*}
\chi_{12}^a &= -\chi_{21}^a = x_{12}^a - x_{21}^a + 2i \theta_a \bar{\sigma}^a \bar{\theta}_1, \quad (2.1a) \\
\theta_{12} &= \theta_1 - \theta_2, \quad (2.1b) \\
\tilde{\theta}_{12} &= \bar{\theta}_1 - \bar{\theta}_2, \\
\text{which are invariants of the Q-supersymmetry transformations.} \\
\chi_{12}^a &= (\sigma_a^\alpha) x_{12}^\alpha, \quad (2.2a) \\
x_{21}^a &= (\sigma_a^\alpha) x_{21}^\alpha, \quad (2.2b) \\
x_{12}^{\bar{a}} &= -x_{21}^{\bar{a}}, \quad (2.2c)
\end{align*}
\]

Note that \( x_{12}^a x_{21}^b = x_{12}^c \delta^c_{ab} \). We sometimes employ matrix-like conventions of [15, 19] where the spinor indices are not explicitly written:

\[
\begin{align*}
\psi = (\psi^a), \quad \bar{\psi} = (\psi^a), \quad \bar{\psi} = (\bar{\psi}_a), \\
\chi = (\chi_a).
\end{align*}
\]

Since \( x^2 \equiv x^a x_a = -\frac{1}{2} \text{tr}(\bar{x} x) \), it follows that \( \bar{x}^{-1} \equiv -x/x^2 \). The notation ‘\( \bar{x}^{-1} x \)’ means that \( \bar{x}^{-1} x \) is antichiral with respect to \( z_1 \) and chiral with respect to \( z_2 \). That is,

\[
D_{2|a} \bar{x}_{12} = 0, \quad (2.4)
\]

where \( D_{2|a} \) and \( D_{1|a} \) are the superspace covariant spinor derivatives acting on the point \( z_1 \). Similarly, \( D_{2|a} \) and \( D_{2|a} \) act on the point \( z_2 \). Explicitly, it holds that

\[
\begin{align*}
\bar{D}_a &= \frac{\partial}{\partial \bar{\theta}^a} + i (\sigma^a)_{ab} \theta^b \frac{\partial}{\partial x^b}, \\
D_a &= -\frac{\partial}{\partial \theta^a} - i \theta^b (\sigma^a)_{ba} \frac{\partial}{\partial x^b}.
\end{align*}
\]

The Q-supersymmetry generators are defined as

\[
\begin{align*}
Q_a &= i \frac{\partial}{\partial \bar{\theta}^a} + (\sigma^a)_{ab} \theta^b \frac{\partial}{\partial x^b}, \\
Q_a &= -i \frac{\partial}{\partial \theta^a} - \theta^b (\sigma^a)_{ba} \frac{\partial}{\partial x^b}
\end{align*}
\]

and thus we have that

\[
\{Q_a, Q_b\} = \{\bar{D}_a, Q_b\} = \{\bar{D}_a, Q_b\} = 0.
\]

Indeed, it can be checked that \( \bar{x}_{12} \) is annihilated by the supercharge operators,

\[
Q_{(1)} \bar{x}_{12} = Q_{(2)} \bar{x}_{12} = Q_{(2)} \bar{x}_{12} = 0.
\]

Let us define the normalised two-point functions,[15]

\[
\chi_{21}^{aa} = \frac{x_{21}^{aa}}{(x_{12}^a)^{1/2}}, \quad \chi_{12}^{aa} = \frac{x_{12}^{aa}}{(x_{12}^a)^{1/2}}.
\]

They satisfy

\[
\chi_{12}^{aa} \chi_{21}^{bb} = \delta^{ab}, \quad \chi_{12}^{aa} \chi_{21}^{bb} = \delta^{ab}.
\]

In accordance with [15], one can construct the vector representation of the inversion tensor in terms of the spinor two-point functions (2.9) as follows:

\[
I_{ab}(x_{12}, x_{21}) = I_{ab}(x_{21}, x_{12}) = \frac{1}{2} \text{tr}(\bar{\sigma}_a \bar{x}_{12} \bar{\sigma}_b \bar{x}_{21}),
\]

\[
I_{ab}(x_{12}, x_{21}) = I_{ab}(x_{21}, x_{12}) = \delta_{ab}.
\]

In the purely bosonic case, \( I_{ab}(x_{12}, x_{21}) \) reduces to the conformal inversion tensor[11]

\[
I_{ab}(x_{12}) = \eta_{ab} - \frac{2}{x_{12}^2} x_{22a} x_{22b},
\]

which played a pivotal role in studying conformal invariance in arbitrary dimensions[11,12].

We can also construct higher-spin extensions of the above operators, which act on the space of symmetric traceless tensors of arbitrary rank. Specifically, we define

\[
I_{a_1 \ldots a_k}(x_{12}) := \bar{x}_{12}^{a_1} \ldots \bar{x}_{12}^{a_k},
\]

along with its inverse

\[
I_{a_1 \ldots a_k}^{(b_1 \ldots b_m)}(x_{12}) := \bar{x}_{12}^{a_1} \ldots \bar{x}_{12}^{a_k}. \]

Due to the properties (2.2b) and (2.2c), it holds that

\[
I_{a_1 \ldots a_k}(x_{12}) = (-1)^k \epsilon_{a_1 \ldots a_k} \ldots \epsilon_{a_1 \ldots a_k} \bar{I}_{a_1 \ldots a_k}^{(b_1 \ldots b_m)}(x_{12}).
\]

As a generalisation of (2.10), we have that

\[
I_{a_1 \ldots a_k}^{(b_1 \ldots b_m)}(x_{12}) I_{a_1 \ldots a_k}^{(b_1 \ldots b_m)}(x_{12}) = \delta_{a_1 \ldots a_k}^{b_1 \ldots b_m}.
\]

\[
I_{a_1 \ldots a_k}^{(b_1 \ldots b_m)}(x_{12}) I_{a_1 \ldots a_k}^{(b_1 \ldots b_m)}(x_{12}) = \delta_{a_1 \ldots a_k}^{b_1 \ldots b_m}.
\]
Since all superconformal transformations may be generated by combining inversions with ordinary supersymmetry, the operators (2.13) and (2.14) play a crucial role in constructing correlation functions for primary operators with arbitrary spin.

We also note several useful differential identities:

\[
D_{(1)\alpha}(x_{12})^{\beta} = 4i\delta_\alpha^\mu \tilde{\theta}_\mu^{12}, \quad \tilde{D}_{(1)\alpha}(x_{12})^{\beta} = 4i\delta_\alpha^\mu \theta_\mu^{12}, \quad (2.17a)
\]

\[
D_{(1)\alpha}(x_{12})^\beta = \frac{1}{x_{12}}(\tilde{\theta}_{12}^{-1})_{\alpha\beta}, \quad \tilde{D}_{(1)\alpha}(x_{12})^\beta = \frac{1}{x_{12}}(\tilde{\theta}_{12}^{-1})_{\alpha\beta} \theta_\mu^{12}. \quad (2.17b)
\]

Here and throughout, we assume that the superspace points are not coincident, \(z_1 \neq z_2\).

Consider a tensor superfield \(\mathcal{O}^A(z)\) transforming in a representation \(T\) of the Lorentz group with respect to the index \(A\). Such a superfield is called primary if its infinitesimal superconformal transformation law reads

\[
\delta \mathcal{O}^A(z) = -\xi \mathcal{O}^A(z) + (\tilde{\sigma}^{\alpha\beta}(z) M_{\alpha\beta} + \tilde{\sigma}^{\alpha\beta}(z) \tilde{M}_{\alpha\beta}) A \mathcal{O}^A(z) - 2(\tilde{q} \sigma(z) + \tilde{q} \tilde{q} \sigma(z)) \mathcal{O}^A(z). \quad (2.18)
\]

In the above, \(\xi\) is the superconformal Killing vector,

\[
\xi = \tilde{\xi} = \xi^a \partial_a + \xi^\alpha \theta_a + \tilde{\xi}_\alpha \tilde{\theta}^a. \quad (2.19)
\]

The superfield parameters \(\tilde{\sigma}^{\alpha\beta}(z)\) correspond to the ‘local’ Lorentz and scale transformations: they are expressed in terms of \(\xi^\mu = (\xi^a, \xi^\alpha, \tilde{\xi}_\alpha)\); see [15] for details. The weights \(q\) and \(\tilde{q}\) are such that \((q + \tilde{q})\) is the scale dimension and \((q - \tilde{q})\) is proportional to the \(U(1)\) charge of the superfield \(\mathcal{O}^A\).

Following the general formalism of [14–16], the two-point function of a primary superfield \(\mathcal{O}^A\) with its conjugate \(\tilde{\mathcal{O}}^A\) is given by

\[
(\mathcal{O}^A(z_1) \tilde{\mathcal{O}}^A(z_2)) = C_0 \frac{I^{AB}(x_{12}) I_{AB}(x_{12})}{(x_{12})^2}. \quad (2.20)
\]

2.2. Three-Point Structures

Given three superspace points \(z_1, z_2\), and \(z_3\), we have the following three-point structures \(z_1, z_2\), and \(z_3\), with \(Z_1 = (X^{(a)}, \Theta^{(a)}, \tilde{\Theta}^{(a)})\) (see [14, 15] for details):

\[
X_1 = \tilde{x}_{13}^{-1} \tilde{x}_{23}^{-1}, \quad \Theta_1 = i\left(\tilde{x}_{13}^{-1} \tilde{x}_{12} - \tilde{x}_{23}^{-1} \tilde{x}_{12}\right), \quad \Theta_1 = i\left(\tilde{x}_{23}^{-1} \tilde{x}_{12} - \tilde{x}_{23}^{-1} \tilde{x}_{12}\right). \quad (2.22a)
\]

\[
X_2 = \tilde{x}_{13}^{-1} \tilde{x}_{23}^{-1}, \quad \Theta_2 = i\left(\tilde{x}_{13}^{-1} \tilde{x}_{23} - \tilde{x}_{23}^{-1} \tilde{x}_{23}\right), \quad \Theta_2 = i\left(\tilde{x}_{13}^{-1} \tilde{x}_{23} - \tilde{x}_{23}^{-1} \tilde{x}_{23}\right). \quad (2.22b)
\]

\[
X_3 = \tilde{x}_{13}^{-1} \tilde{x}_{23}^{-1}, \quad \Theta_3 = i\left(\tilde{x}_{13}^{-1} \tilde{x}_{23} - \tilde{x}_{23}^{-1} \tilde{x}_{23}\right), \quad \Theta_3 = i\left(\tilde{x}_{13}^{-1} \tilde{x}_{23} - \tilde{x}_{23}^{-1} \tilde{x}_{23}\right). \quad (2.22c)
\]

Since (2.22b) and (2.22c) are obtained through cyclic permutations of superspace points, it suffices to study the properties of (2.22a). Let us also define

\[
X_0 = X_1^{(a)} - x_{12}^{-1} x_{12}^{-1}. \quad (2.23)
\]

Similar relations hold for \(X_2, X_3\).

We list several properties of \(Z\)’s which will be useful later

\[
X_1^2 = \frac{x_{13}^2 x_{21}^2 x_{12}^2}{x_{13} x_{21} x_{12}}, \quad \tilde{X}_1^2 = \frac{x_{12}^2 x_{13} x_{21}}{x_{13}^2 x_{21}}. \quad (2.24a)
\]

\[
\tilde{X}_{aa} = X_{aa} + iP_{aa}, \quad p_{aa} = -4\Theta_{a} \tilde{\Theta}_a. \quad (2.24b)
\]

\[
\frac{1}{X^2} - \frac{1}{X^{2k}} - 2ik \frac{(P \cdot X)}{X^{2k+2}} - \frac{k^2 (k-1)}{2} \frac{P^2}{X^{2k+2}}. \quad (2.24c)
\]

where, throughout the paper, we adopt the notation \(X^k \equiv (X^3)^{k/2}\). In particular, we see that \(\tilde{X}\) is not an independent variable for it can be expressed in terms of \(X, \Theta, \tilde{\Theta}\). Variables \(Z\) with different labels are related to each other via the identities

\[
\tilde{x}_{13} x_{12} x_{13} = -x_{13}^{-1} \tilde{x}_{12}^{-1} = -x_{12}^{-1} \tilde{x}_{13}^{-1} = -x_{13}^{-1} \tilde{x}_{12}^{-1}. \quad (2.25a)
\]

\[
\frac{x_{13}^2}{x_{12}^2} x_{13} x_{13} = -x_{13}^{-1} \tilde{x}_{12}^{-1} = -x_{13}^{-1} \tilde{x}_{12}^{-1} = \tilde{x}_{12}^{-1} \tilde{x}_{13}^{-1} \tilde{x}_{13}^{-1} = \tilde{x}_{12}^{-1} \tilde{x}_{13}^{-1}. \quad (2.25b)
\]

It is also convenient to define the normalised three-point building block \(\tilde{X}_{aa}\)

\[
\tilde{X}_{aa} = \frac{X_{aa}}{(X^3)^{1/2}}. \quad (2.26)
\]
We then construct the higher-spin operator
\[ I_{\alpha(a_l)\beta(b_l)}(X) = \hat{X}_{\alpha(a_l)} \cdots \hat{X}_{\alpha(a_l)} \]  
(2.27)
along with its inverse
\[ \bar{I}_{\alpha(a_l)\beta(b_l)}(X) = \hat{X}^\dagger_{\alpha(a_l)} \cdots \hat{X}^\dagger_{\alpha(a_l)} \]  
(2.28)
with properties
\[ I_{\alpha(a_l)\beta(b_l)}(X) I_{\gamma(\ell_l)\delta(\ell_l)}(X) = (-1)^{\delta_{\alpha \gamma}} \delta_{\beta \delta}, \]  
(2.29a)
\[ \bar{I}_{\alpha(a_l)\beta(b_l)}(X) I_{\gamma(\ell_l)\delta(\ell_l)}(X) = (-1)^{\delta_{\beta \gamma}} \delta_{\alpha \delta}. \]  
(2.29b)

The normalised, higher-spin operator for \( X \) can also be defined in a similar way:
\[ I_{\alpha(a_l)\beta(b_l)}(\hat{X}) = \hat{X}_{\alpha(a_l)} \cdots \hat{X}_{\alpha(a_l)}. \]  
(2.30)

In the vector representation, we can also define
\[ I_{\alpha\beta}(\hat{X}, X) = -\frac{1}{2} \text{tr}\left( \hat{\sigma}_\alpha \hat{X} \hat{\sigma}_\beta \hat{X} \right). \]  
(2.31)

Various primary superfields, including conserved current multiplets, are subject to certain differential constraints. These need to be taken into account when constraining correlation functions. For three-point functions, the action of covariant spinor derivatives on an arbitrary function \( t(X, \theta, \bar{\theta}) \) can be simplified using these useful differential identities:[15]

\[ D_{(1)\mu} t(X, \theta, \bar{\theta}) = -i x^{i} t(x_{(1)\mu} \bar{\theta}) \]  
(2.32a)
\[ D_{(1)\mu} t(X, \theta, \bar{\theta}) = -i x^{i} t(x_{(1)\mu} \theta) \]  
(2.32b)
\[ D_{(2)\mu} t(X, \theta, \bar{\theta}) = i x^{i} t(x_{(2)\mu} \bar{\theta}) \]  
(2.32c)
\[ D_{(2)\mu} t(X, \theta, \bar{\theta}) = i x^{i} t(x_{(2)\mu} \theta) \]  
(2.32d)

where, for \( (X, \theta, \bar{\theta}) \rightarrow (X, \theta, \bar{\theta}) \); \( D_{(1)}, \bar{D}_{(1)} \rightarrow D, \bar{D} \) and \( Q_{(j)}, \bar{Q}_{(j)} \rightarrow Q, \bar{Q} \), we define the conformally covariant operators
\[ D_\lambda = \frac{\partial}{\partial X^a} D_a, \]  
\[ Q_\lambda = \frac{\partial}{\partial X^a} Q_a, \]  
where
\[ D_\lambda = (\partial/\partial X^a, D_a, D^a), \]  
\[ Q_\lambda = (\partial/\partial X^a, Q_a, Q^a). \]  
(2.33)

We can also derive these anti-commutation relations
\[ \{D^a, \bar{D}^a\} = 2i (\bar{\sigma}^a)^{ab} \frac{\partial}{\partial X^b}, \]  
\[ \{Q^a, \bar{Q}^a\} = -2i (\bar{\sigma}^a)^{ab} \frac{\partial}{\partial X^b}. \]  
(2.34)

Let \( \Phi^{\alpha_1, \psi^{\alpha_2}} \) and \( \Pi^{\alpha_3, \psi^{\alpha_4}} \) be primary superfields with weights \( (q_{1, \bar{q}_1}, q_{2, \bar{q}_2}) \) and \( (q_{1, \bar{q}_1}, q_{2, \bar{q}_2}) \) respectively. Then, the three-point correlation function may be constructed using the general expression:[14–16]

\[ \langle \Phi^{\alpha_1}(z_1) \psi^{\alpha_2}(z_2) \Pi^{\alpha_3}(z_3) \rangle = \frac{I^{\alpha_1, \beta_1}(X_1)}{I^{\alpha_2, \beta_2}(X_2)} \frac{I^{\alpha_3, \beta_3}(X_3)}{I^{\beta_1, \beta_2}(X_2)} H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(X_1, \theta, \bar{\theta}), \]  
(2.35)
where the functional form of the tensor \( H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2} \) is highly constrained by the superconformal symmetry as follows:

(i) It possesses the homogeneity property
\[ H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(\lambda \hat{X}, \theta, \bar{\theta}) = \lambda^{2a} H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2}(X, \theta, \bar{\theta}), \]  
(2.36)
This condition guarantees that the correlation function has the correct transformation law under the superconformal group. By construction, Equation (2.35) has the correct transformation properties at the points \( z_1 \) and \( z_2 \). The above homogeneity property implies that it also transforms correctly at \( z_3 \). The tensor \( H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2} \) has dimension \( (a + \bar{a}) \).

(ii) If any of the superfields \( \Phi, \psi \) and \( \Pi \) obey differential equations (e.g., conservation laws for conserved current multiplets), then \( H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2} \) is constrained by certain differential equations too. The latter may be derived using (2.32).

(iii) If any (or all) of the superfields \( \Phi, \psi \) and \( \Pi \) coincide, then \( H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2} \) obeys additional constraints, the so-called “point-switch symmetries”. These are consequences of the symmetry under permutations of superspace points. As an example,
\[ \langle \Phi^{\alpha_1}(z_1) \psi^{\alpha_2}(z_2) \Pi^{\alpha_3}(z_3) \rangle = (-1)^{\epsilon(\Phi)} \langle \Phi^{\alpha_1}(z_2) \psi^{\alpha_2}(z_3) \Pi^{\alpha_3}(z_1) \rangle, \]  
(2.37)
where \( \epsilon(\Phi) \) denotes the Grassmann parity of \( \Phi^{\alpha} \). Note that under permutations of any two superspace points, the three-point building blocks transform as
\[ X_{1a} \rightarrow -X_{1a}, \]  
\[ \Theta_{1a} \rightarrow -\Theta_{1a}, \]  
\[ \bar{X}_{1a} \rightarrow -\bar{X}_{1a}, \]  
\[ \bar{\Theta}_{1a} \rightarrow -\bar{\Theta}_{1a}. \]  
(2.38a)
\[ X_{1a} \rightarrow -X_{1a}, \]  
\[ \Theta_{1a} \rightarrow -\Theta_{1a}, \]  
\[ \bar{X}_{1a} \rightarrow -\bar{X}_{1a}, \]  
\[ \bar{\Theta}_{1a} \rightarrow -\bar{\Theta}_{1a}. \]  
(2.38b)
\[ X_{1a} \rightarrow -X_{1a}, \]  
\[ \Theta_{1a} \rightarrow -\Theta_{1a}, \]  
\[ \bar{X}_{1a} \rightarrow -\bar{X}_{1a}, \]  
\[ \bar{\Theta}_{1a} \rightarrow -\bar{\Theta}_{1a}. \]  
(2.38c)

The above conditions fix the functional form of \( H_{\beta_1, \beta_2}^{\alpha_1, \alpha_2} \) (and, therefore, the three-point function under consideration) up to a few arbitrary constants.

A few comments are in order regarding the three-point functions of conserved current multiplets. It is worth pointing out that, depending on the exact way in which one constructs the general expression (2.35), it can be impractical to impose conservation equations on one of the three superfields due to a lack of useful identities such as (2.32). To illustrate this, let us go back
to Equation (2.35):
\[ (\Phi^{A_1}(z_1) \Psi^{A_1}(z_2) \Pi^{A_1}(z_3)) = \frac{I^{A_1;B_1}(x_{11}, x_{11}) I^{A_1;B_1}(x_{22}, x_{22})}{(x_{11}^2)^6 (x_{11}^2)^6 (x_{22}^2)^6} \cdot H_{B_1B_1}(X_1, \Theta_1, \Theta_1). \] 

All information about this correlation function is encoded in the tensor \( H \); however, this particular formulation prevents us from imposing conservation on \( \Pi \) in a straightforward way. A way out is to rearrange the correlator with \( \Pi \), say, at the second point:

\[ (\Psi^{A_1}(z_2) \Pi^{A_1}(z_3) \Phi^{A_1}(z_1)) = \frac{I^{A_1;B_1}(x_{22}, x_{11}) I^{A_1;B_1}(x_{11}, x_{33})}{(x_{22}^2)^6 (x_{11}^2)^6 (x_{33}^2)^6} \cdot \tilde{H}_{B_1B_1}(X_1, \Theta_1, \Theta_1). \] 

Here, all information about the correlator is now encoded in the tensor \( \tilde{H} \), which is a completely different solution compared to \( H \). Thus, we require a simple equation relating the tensors \( H \) and \( \tilde{H} \), which corresponds to different representations of the same correlation function. Indeed, once \( \tilde{H} \) is obtained, we can then easily impose conservation on \( \Pi \) as if it were located at the “second point”, with the aid of identities analogous to (2.32c) and (2.32d). However, as we will see in the next sections, this transformation proves to be complicated for correlators of higher-spin primary operators due to the proliferation of tensor/spinor indices. In section 3, we will develop an index-free approach to study the correlator involving three insertions of the higher-spin supercurrent \( J_{\mu \nu \rho \sigma} \) and derive the explicit formula relating \( H \) and \( \tilde{H} \).

3. General Formalism

In this section we will develop the general formalism to derive all the necessary constraints on the three-point function containing three insertions of the higher-spin supercurrent \( J_{[\mu \nu \rho \sigma]} \sim J_{\mu \nu \rho \sigma} \). The ansatz for the correlator consistent with the expression (2.35) is given by

\[ \langle J_{\mu \nu \rho \sigma}(x_1) J_{\rho \sigma}(x_2) J_{\mu \nu}(x_1) \rangle = \frac{1}{k_t} I^{\mu \nu}(x_1) I^{\rho \sigma}(x_2) I^{\mu \nu}(x_1) \cdot \tilde{H}_{\rho \sigma}(X_1, \Theta_1, \Theta_1), \] 

with \( k_t := (x_{11} x_{33})^{r-s} (x_{11} x_{33})^{r-s} \). We recall that \( J_{\mu \nu \rho \sigma} \) is primary real with weights \( (q, q) = (r/2, r/2) \) and scaling dimension \( r + 2 \). We first assume that all the operators are of different spins; hence, we will not impose any point-switch symmetries. The correlator (3.1) is thus subject to the following constraints:

(i) Homogeneity: The tensor \( H_{\mu \nu \rho \sigma}(x_1) J_{\rho \sigma}(x_2) J_{\mu \nu}(x_1) \) has the scaling property

\[ H_{\mu \nu \rho \sigma}(x_1) J_{\rho \sigma}(x_2) J_{\mu \nu}(x_1) = (\Lambda \bar{\Lambda})^{r-2} H_{\mu \nu \rho \sigma}(x_1) J_{\rho \sigma}(x_2) J_{\mu \nu}(x_1), \]

and, hence, its dimension is \( r_1 - (r_1 + r_2 + 2) \). This ensures that the correlator transforms correctly under scale transformations.

(ii) Conservation: The conservation of \( J_{\mu \nu \rho \sigma} \) at \( z_1 \) and \( z_2 \) imply

\[ D_{i}^\rho (J_{\mu \nu \rho \sigma}(z_1) J_{\rho \sigma}(z_2) J_{\mu \nu}(z_1)) = 0, \]

\[ \tilde{D}_{i}^\rho (J_{\mu \nu \rho \sigma}(z_1) J_{\rho \sigma}(z_2) J_{\mu \nu}(z_1)) = 0, \]

\[ D_{i}^\rho (J_{\mu \nu \rho \sigma}(z_1) J_{\rho \sigma}(z_2) J_{\mu \nu}(z_1)) = 0, \]

\[ \tilde{D}_{i}^\rho (J_{\mu \nu \rho \sigma}(z_1) J_{\rho \sigma}(z_2) J_{\mu \nu}(z_1)) = 0. \]

With the use of identities (2.32), these requirements are translated to the following differential constraints on \( H \):

\[ D^\rho H_{\mu \nu \rho \sigma}(x_1, x_2, x_1, x_2) = 0, \]

\[ \tilde{D}^\rho H_{\mu \nu \rho \sigma}(x_1, x_2, x_1, x_2) = 0, \]

\[ \tilde{Q}^\rho H_{\mu \nu \rho \sigma}(x_1, x_2, x_1, x_2) = 0, \]

\[ Q^\rho H_{\mu \nu \rho \sigma}(x_1, x_2, x_1, x_2) = 0. \]

There are further constraints arising from the conservation at \( z_1 \):

\[ D_{i}^\rho (J_{\mu \nu \rho \sigma}(z_1) J_{\rho \sigma}(z_2) J_{\mu \nu}(z_1)) = 0, \]

\[ \tilde{D}_{i}^\rho (J_{\mu \nu \rho \sigma}(z_1) J_{\rho \sigma}(z_2) J_{\mu \nu}(z_1)) = 0. \]

These constraints are non-trivial to impose as there are no identities analogous to (2.32) which allow the spinor derivatives acting on \( z_1 \) to pass through the prefactor of (3.1). We thus employ the procedure outlined at the end of Subsection 2.2, where in Equation (2.40) we reformulated the ansatz in terms of \( \tilde{H} \) and with \( \Pi \) at the second point. The details of this will be illustrated in the next subsection.

(iii) Reality: Since the higher-spin supercurrent \( J_{\mu \nu \rho \sigma} \) is a real superfield, the reality condition on the correlator leads to the following constraint on the tensor \( H \):

\[ H_{\mu \nu \rho \sigma}(x_1, x_2, x_1, x_2) = \pi H_{\mu \nu \rho \sigma}(x_1, x_2, x_1, x_2)(X, \Theta, \bar{\Theta}), \]

where \( \pi H(X, \Theta, \bar{\Theta}) \) is the conjugate of \( H(X, \Theta, \bar{\Theta}) \) (with indices suppressed). More precisely, we introduce the following definitions. Suppose \( H(X, \Theta, \bar{\Theta}) \) is composed out of a finite basis of linearly independent tensor structures \( T_i(X, \Theta, \bar{\Theta}) \), that is \( H(X, \Theta, \bar{\Theta}) = \sum_i c_i T_i(X, \Theta, \bar{\Theta}) \), with \( c_i \) constant complex parameters. Define the conjugate of \( H \) by

\[ \pi H(X, \Theta, \bar{\Theta}) = \sum_i \bar{c}_i T_i(X, \Theta, \bar{\Theta}). \]
The problem of computing higher-spin correlator (3.1) is thus reduced to determining the most general form of \( H(X, \Theta, \tilde{\Theta}) \), subject to the above constraints. The first observation is that since \( H \) is Grassmann even, it must be an even function of \( \Theta \) and \( \tilde{\Theta} \). We can then write a general expansion for \( H \) as follows (to recall, \( P_{ab} = -4 \Theta_a \tilde{\Theta}_b \)):

\[
H(X, \Theta, \tilde{\Theta}) \equiv F(X) - \frac{1}{2} P^{ab} G_{ab}(X) \\
+ A^{(1)}(X) \Theta^2 + A^{(2)}(X) \tilde{\Theta}^2 + A^{(3)}(X) \Theta \tilde{\Theta}^2.
\]  

(3.9)

Now, constraints (3.4a) and (3.4d) from conservation imply the following conditions

\[
\partial^2 X H(X, \Theta, \tilde{\Theta}) = 0, \quad Q^2 H(X, \Theta, \tilde{\Theta}) = 0,
\]  

(3.10)

which lead to

\[
A^{(1)}(X) = A^{(2)}(X) = A^{(3)}(X) = 0.
\]  

(3.11)

As a result, the general solution for \( H(X, \Theta, \tilde{\Theta}) \) can always be presented in the form

\[
H_{X}^{(r_1), \beta (r_2), \gamma (r_3)}(X, \Theta, \tilde{\Theta}) = \frac{1}{2} P^{ab} G_{ab}(X) \\
- \frac{1}{2} P^{ab} G_{ab}(X, \Theta, \tilde{\Theta}) [X, \Theta, \tilde{\Theta}].
\]  

(3.12)

Our task is to find the appropriate tensorial structures for \( F \) and \( G \), consistent with homogeneity, conservation and reality constraints. This is a technically challenging problem due to the proliferation of spinor indices. It is advantageous to develop an index-free approach which allows us to systematically write the ansatz for \( H \), which is an extension of the non-supersymmetric generating formalism recently proposed in [49]. In this work, we provide a formalism to efficiently impose all the required (differential) constraints on the three-point function of the conserved higher-spin supercurrent (3.1). These constraints can then be efficiently implemented and solved computationally by Mathematica.

### 3.1. Generating Function

We begin by introducing a method that allows us to encode symmetric traceless tensors by polynomials obtained by contracting the tensor with sets of commuting auxiliary spinors. Let \((u^a, \tilde{u}^a)\) be a set of commuting auxiliary spinors satisfying, by construction, \(u^2 = \tilde{u}^2 = 0\); \(u^2 = \tilde{u}^2 = 0\). This is, in fact, equivalent to defining a null vector \( U^a \)

\[
U^a = -\frac{1}{2} (\bar{\sigma}^a)_{ab} U^b, \quad U_{ab} = u_a \tilde{u}_b, \quad U^a = u_a \tilde{u}_a,
\]  

(3.13)

such that \( U^2 = U^a U_a = 0 \). Extending this construction, we define

\[
U^{(r_1), \alpha (r_2)} = u^\alpha_1 \ldots u^\alpha_{r_1} \tilde{u}^{\alpha_1} \ldots \tilde{u}^{\alpha_{r_1}},
\]  

(3.14a)

\[
V^{(r_1), \beta (r_2)} = v^\beta_1 \ldots v^\beta_{r_2} \tilde{v}^{\beta_1} \ldots \tilde{v}^{\beta_{r_2}},
\]  

(3.14b)

\[
W^{(r_1), \gamma (r_2)} = w^{\gamma_1} \ldots w^{\gamma_{r_2}} \tilde{w}^{\gamma_1} \ldots \tilde{w}^{\gamma_{r_2}}.
\]  

(3.14c)

Here the spinors satisfy \( u^a = \tilde{u}^a = 0 \), \( v^a = \tilde{v}^a = 0 \), \( w^a = \tilde{w}^a = 0 \). We then represent \( H^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}(X, \Theta, \tilde{\Theta}) \) in terms of the following generating polynomial

\[
H(X, \Theta, \tilde{\Theta}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = U^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)} H_{X}^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}(X, \Theta, \tilde{\Theta}).
\]  

(3.15)

The tensor \( H \) can be extracted from the generating polynomial by acting on it with partial derivatives

\[
H_{X}^{(r_1), \beta (r_2), \gamma (r_3)}(X, \Theta, \tilde{\Theta}) = \frac{\partial}{\partial U^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}} \frac{\partial}{\partial V^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}} \frac{\partial}{\partial W^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}} H(X, \Theta, \tilde{\Theta}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) .
\]  

(3.16)

In the above, the partial derivative operators are defined by

\[
\frac{\partial}{\partial U^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}} = \frac{1}{r_1 !} \left( \frac{\partial}{\partial u^a_1 \tilde{u}_a_1} \right), \quad \frac{\partial}{\partial V^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}} = \frac{1}{r_2 !} \left( \frac{\partial}{\partial v^a_1 \tilde{v}_a_1} \right), \quad \frac{\partial}{\partial W^{(r_1), \alpha (r_2), \beta (r_3), \gamma (r_4)}} = \frac{1}{r_3 !} \left( \frac{\partial}{\partial w^a_1 \tilde{w}_a_1} \right).
\]  

From the general form (3.12), we find that

\[
H(X, \Theta, \tilde{\Theta}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) - \frac{1}{2} P^{ab} G_{ab}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}).
\]  

(3.18)

Hence, it suffices to work with polynomials \( F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \) and \( G_{ab}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \), which are functions of a single, bosonic covariant building block \( X \), and the auxiliary commuting spinors \( u, \tilde{u}, v, \tilde{v}, w, \tilde{w} \). To ensure that (3.15) is satisfied, the polynomials \( F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \) and \( G_{ab}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \) must be homogeneous of degree \( r_1 \) in \( u, \tilde{u} \), degree \( r_2 \) in \( v, \tilde{v} \) and degree \( r_3 \) in \( w, \tilde{w} \) (equivalently, we can view them as homogeneous polynomials of degree \( r_1 + r_2 + r_3 \), respectively). Next, we define the invariants which serve as the building blocks for constructing solutions for \( F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \) and \( G_{ab}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \). Since \( H \) has a scale dimension \( d = r_1 + r_2 + 2 \), we can then write

\[
F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \frac{1}{X^{d}} F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}).
\]  

(3.19)

Here the polynomial \( F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \) is now homogeneous of degree 0 in \( X \) and is constructed out of scalar combinations of \( X \), along with the auxiliary spinors \( U, V \) and \( W \) with the appropriate homogeneity. We choose the following basis for the invariants:

\[
UV = -\frac{1}{2} (u \cdot v)(\tilde{u} \cdot \tilde{v}), \quad UW = -\frac{1}{2} (u \cdot w)(\tilde{u} \cdot \tilde{w}),
\]  

(3.20a)

\[
VW = -\frac{1}{2} (v \cdot w)(\tilde{v} \cdot \tilde{w}).
\]  

(3.20b)
the homogeneity constraint on independent and dependent terms will be clear later. Due to the homogeneity constraints on $F$, here $p_i$ and $q_j$ are non-negative integers which solve the following linear system of equations:

$$p_1 + p_2 + p_3 = q_1 + q_2 + q_3 + 1 = r_1,$$

$$p_1 + p_2 + p_3 = q_1 + q_3 + q_4 + 1 = r_2,$$

$$p_1 + p_3 + p_5 = q_2 + q_3 + q_5 + 1 = r_3.$$  

In constructing a basis of invariants for $G_{aa}(X; u, v, w, w)$, let us write

$$G_{aa}(X; u, v, w, w) = \frac{1}{X^{1-g}} G_{aa}(\hat{X}; u, v, w, w).$$  

(3.23)

Now, we observe that $P^a G_a(\hat{X}; u, v, w, w)$ is constructed out of scalar combinations of $\hat{X}$, along with the auxiliary spinors $U$, $V$ and $W$ with the appropriate homogeneity. We can then construct

$$P^a G_a(\hat{X}; u, v, w, w) = P^a \left[ U_a G_a(\hat{X}; u, v, w, w) + V_a G_a(\hat{X}; u, v, w, w) + W_a G_a(\hat{X}; u, v, w, w) + Z_{1a} G_a(\hat{X}; u, v, w, w) + Z_{2a} G_a(\hat{X}; u, v, v, w, w) + Z_{3a} G_a(\hat{X}; u, v, v, v, w, w) \right].$$  

(3.24a)

where we have defined

$$Z_{1a} = e_{abcd} \hat{X}^a U^b V^c W^d,$$

$$Z_{2a} = e_{abcd} \hat{X}^a V^b W^c,$$

$$Z_{3a} = e_{abcd} \hat{X}^b U^c W^d,$$

(3.24b)

We may express the expansion (3.24a) in spinor notation using $P^a G_a(\hat{X}; u, v, w, w) = \frac{1}{2} \sum_j (\sigma^a)_{ij} G_a(\hat{X}; u, v, w, w)$. This gives

$$G_{aa}(\hat{X}; u, v, w, w) = \sum_{j=1}^8 \kappa_{j,aa} G_a(\hat{X}; u, v, w, w).$$  

(3.25a)

where the $\kappa_{j,aa}$ structures take the form

$$\kappa_{1,aa} = U_a = u_a \delta_{a}^a,$$

$$\kappa_{2,aa} = \hat{X}_a = \hat{X}_a,$$

$$\kappa_{3,aa} = V_a = v_a \delta_{a}^a,$$

$$\kappa_{4,aa} = W_a = w_a \delta_{a}^a,$$

$$\kappa_{5,aa} = Z_{1,aa} = \frac{i}{2} \left[ \hat{X}_a v_a u^b (u \cdot v) - \hat{X}_a v_a u^b (\bar{u} \cdot \bar{v}) \right],$$

$$\kappa_{6,aa} = Z_{2,aa} = \frac{i}{2} \left[ \hat{X}_a v_a u^b (v \cdot w) - \hat{X}_a u_a v^b (\bar{v} \cdot \bar{w}) \right],$$

$$\kappa_{7,aa} = Z_{3,aa} = \frac{i}{2} \left[ \hat{X}_a u_a v^b (v \cdot w) - \hat{X}_a u_a v^b (\bar{v} \cdot \bar{w}) \right],$$

$$\kappa_{8,aa} = Z_{4,aa} = \frac{i}{2} \left[ w_a \bar{u}_a (u \cdot v) (\bar{v} \cdot \bar{w}) - u_a \bar{w}_a (v \cdot w) (\bar{v} \cdot \bar{w}) \right].$$  

(3.25b)

It should be noted, however, that not all of the $\kappa$-structures are linearly independent. For instance, we have that

$$v_a \bar{v}_a J + (V W) Z_{1,aa} + (U V) Z_{2,aa} - Z_{4,aa} (V \hat{X}) = 0.$$  

(3.26)

As will be shown in section 4, there will be more linear dependence relations. Now that $G_a(\hat{X}; u, v, w, w)$ are scalars, they can be constructed in an analogous way as in $F_1$ and $F_2$:

$$G_a(\hat{X}; u, v, w, w) = \sum_{j=1}^8 B_j(p_j)(U V)^{i(a}(U W)^{b)}(U \hat{X})^{c(a}(V \hat{X})^{d)}(W \hat{X})^{e)}.$$

(3.27)

Let us now recast the conservation and reality constraints (ii) and (iii) in terms of $F$ and $G$. Since $D_\alpha = -\partial / \partial \hat{\Theta}^\alpha$, the first differential constraint (3.4a) simply turns into

$$\epsilon^{\alpha \beta \gamma \delta} \frac{\partial}{\partial \hat{\Theta}^\alpha} G_{a+b}(X; u, v, w, w) = 0.$$  

(3.28)
In the second equation (3.4b), note that $D_{a} = \partial / \partial \Theta^a - 2i \bar{\Theta}^a \partial \Theta^a$, with $\partial \Theta^a = (\sigma^a)_{\bar{a}} \partial / \partial X^a$. Thus, this gives rise to two conditions:

$$\begin{align*}
\partial a \frac{\partial}{\partial u} G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon a \frac{\partial}{\partial v} G_{av}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) - i \partial a F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0.
\end{align*}$$

(3.29a, b)

In a similar fashion, it is not hard to see that (3.4c) is equivalent to

$$\begin{align*}
\partial a \frac{\partial}{\partial u} G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon a \frac{\partial}{\partial v} G_{av}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) - i \partial a F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0,
\end{align*}$$

(3.30a, b)

while the last equation (3.4d) implies that

$$\begin{align*}
\epsilon a \frac{\partial}{\partial v} G_{av}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0.
\end{align*}$$

(3.31)

We also have a nice consistency check which follows directly from Equations (3.30a) and (3.30b):

$$\begin{align*}
\Box F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) - i \partial a G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\Box = - \frac{1}{2} \partial a \partial a.
\end{align*}$$

(3.32)

As for the reality constraint, note that (3.8) allows us to express the conjugate of $H$ as

$$\begin{align*}
\tilde{H}(X, \Theta, \bar{\Theta}; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{a}) - \frac{1}{2} P^{a} G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{b}), \\
\tilde{H}(X, \Theta, \bar{\Theta}; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{a}) + \frac{1}{2} P^{a} G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{b}).
\end{align*}$$

(3.33)

Here $a_i$ and $b_i$ are constant complex parameters of the linearly independent tensor structures in $F(X; u, \bar{u}, v, \bar{v}, w, \bar{w})$ and $G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w})$, respectively. With the definitions (2.24b) and the fact that $\Theta P^2 = \bar{\Theta} P^2 = 0$, we can apply Taylor’s expansion on the right-hand side of (3.33) and arrive at

$$\begin{align*}
\tilde{H}(X, \Theta, \bar{\Theta}; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{a}) - \frac{1}{2} P^{a} \left[ i \partial a F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{a}) + G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{b}) \right] \\
&- \frac{1}{2} P^{a} \left[ \Box F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{a}) + i \partial a G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{b}) \right].
\end{align*}$$

(3.34)

The reality constraint reads

$$\begin{align*}
H(X, \Theta, \bar{\Theta}; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= \tilde{H}(X, \Theta, \bar{\Theta}; u, \bar{u}, v, \bar{v}, w, \bar{w}) \\
&= \tilde{H}(X, \Theta, \bar{\Theta}; u, \bar{u}, v, \bar{v}, w, \bar{w}).
\end{align*}$$

(3.35)

Upon substituting (3.34) into the right-hand side of (3.35), it appears that we obtain three conditions:

$$\begin{align*}
F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; a_i) &= F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{a}_i), \\
G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; b_i) &= G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{b}_i) \\
&+ i \partial a F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; a_i),
\end{align*}$$

(3.36a, b, c)

However, it is not hard to see that Equation (3.36c) follows automatically as a result of imposing (3.36b) and (3.32). Therefore, the reality constraint only demands that the parameters $a_i$ be real and

$$\begin{align*}
G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; b_i) &= G_{aa}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; \bar{b}_i) \\
&+ i \partial a F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}; a_i),
\end{align*}$$

(3.37)

which gives a set of equations relating the imaginary parts of $b_i$ to $a_i$.

### 3.2. Imposing Conservation on $z_3$

We now turn to analysing the conservation condition at $z_3$. In this particular case, we rewrite our correlator as

$$\begin{align*}
\langle J_{\mu(1)}(z_1) J'_{\nu(3)}(z_2) J''_{\rho(5)}(z_3) \rangle &= \langle J'_{\mu(3)}(z_2) J''_{\nu(1)}(z_3) J_{\rho(5)}(z_1) \rangle \\
&= \frac{1}{k_2} I_{\mu(1)}(z_1) I_{\nu(1)}(z_3) I_{\rho(5)}(z_2) I_{\mu(3)}(z_2) I_{\nu(1)}(z_3) I_{\rho(5)}(z_1),
\end{align*}$$

(3.38)

with $k_2 := \langle x_{12} x_{31} \rangle ^{+2} \langle x_1 x_3 \rangle ^{-2}$. Making use of the following relations:

$$\begin{align*}
\tilde{I}_{\mu(1)}(z_1) I_{\nu(1)}(z_3) I_{\rho(5)}(z_2) &= \tilde{I}_{\rho(5)}(z_2) I_{\nu(1)}(z_3) I_{\rho(1)}(z_1), \\
I_{\rho(5)}(z_2) I_{\nu(1)}(z_3) I_{\rho(5)}(z_1) &= \tilde{I}_{\rho(5)}(z_2) I_{\nu(1)}(z_3) I_{\rho(1)}(z_1).
\end{align*}$$

(3.39a, b)

These parameters correspond to $A(p_i), A'(p_i), B_i(p_i), B_i'(p_i)$ in Equations (2.21) and (2.27).
\[
\hat{h}_{\alpha(\beta)}(x_{\alpha}, t) \hat{h}_{\beta(\alpha)}(x_{\beta}, t) \equiv \left( -1 \right)^{l_{\alpha} + l_{\beta}} \frac{(\lambda \lambda)^{l_{\alpha} + l_{\beta}}}{(x_{\alpha} x_{\beta})^{l_{\alpha} + l_{\beta}}} \hat{h}_{\alpha(\beta)}(x_{\alpha}) \hat{h}_{\beta(\alpha)}(x_{\beta}) \times \hat{h}_{\gamma(\delta)}(x_{\gamma}) \hat{h}_{\delta(\gamma)}(x_{\delta}) \times \hat{h}_{\lambda(\mu)}(x_{\lambda}) \hat{h}_{\mu(\lambda)}(x_{\mu})
\]

(3.40a)

where

\[
\lambda \lambda = \frac{1}{(X_{\alpha} X_{\beta} X_{\gamma} X_{\delta})^{1/2}}.
\]

(3.40b)

Equation (3.40a) is quite impractical to work with due to the presence of both two- and three-point functions. It would be desirable to obtain a simpler relation which only involves the three-point building blocks \(X, \Theta, \tilde{\Theta}\). Indeed, this procedure can be done via the following steps. First, the analogues of relations (2.11) allow us to define

\[
I_{ab}^{(1)}(x_{ab}, x_{\gamma}) \equiv \frac{1}{2} \left( \hat{X}^{(a)} \cdot \hat{X}^{(b)} \cdot \hat{X}^{(\gamma)} \right)
\]

(3.41a)

\[
\hat{I}_{ab}^{(1)}(x_{ab}, x_{\gamma}) \equiv \frac{1}{2} \left( \hat{X}^{(a)} \cdot \hat{X}^{(b)} \cdot \hat{X}^{(\gamma)} \right)
\]

(3.41b)

The operator \(I_{ab}^{(1)}\) is just an orthogonal matrix with determinant \(-1\), that is it satisfies

\[
I_{ab}^{(1)} \eta^{cd} = \eta_{ab},
\]

(3.42a)

\[
I_{ab}^{(1)} I_{cd}^{(1)} \delta^{bc} = - \varepsilon_{abcd}.
\]

(3.42b)

With the above definitions, the transformations (2.25) may be expressed as

\[
I_{ab}^{(1)} X^{b}_{\gamma} = \hat{I}_{ab}^{(1)} X^{b}_{\gamma} = \lambda \lambda X^{b}_{\gamma}.
\]

(3.43a)

\[
I_{ab}^{(1)} P^{b}_{\gamma} = \lambda \lambda P^{b}_{\gamma},
\]

(3.43b)

\[
\hat{I}_{ab}^{(1)} (X, X) \equiv \frac{1}{2} \left( X^{\alpha} \hat{X}^{(a)} \cdot \hat{X}^{(b)} \right) \hat{X}_{\alpha}.
\]

(3.43c)

In the higher-spin case, from (3.41) and (3.43c), one can show that

\[
\hat{I}_{\alpha(\beta)}^{(\gamma)}(x_{\alpha}) \hat{I}_{\delta(\gamma)}^{(\delta)}(x_{\delta}) = \frac{1}{2} \left( \hat{X}^{(\alpha)} \hat{X}^{(\beta)} \hat{X}^{(\gamma)} \right) \hat{X}_{\alpha},
\]

as well as

\[
\hat{I}_{\alpha(\beta)}^{(\gamma)}(X_{\alpha}) \hat{I}_{\delta(\gamma)}^{(\delta)}(X_{\delta}) = \left( -1 \right)^{k} \left( \hat{X}^{(\alpha)} \hat{X}^{(\beta)} \hat{X}^{(\gamma)} \right) \hat{X}_{\alpha},
\]

(3.44b)

As a result, Equation (3.40a) can be rewritten as follows

\[
\hat{I}_{\alpha(\beta)}^{(\gamma)}(x_{\alpha}) \hat{I}_{\delta(\gamma)}^{(\delta)}(x_{\delta}) = \left( -1 \right)^{l_{\alpha} + l_{\beta} + l_{\gamma} + l_{\delta}} \left( \hat{X}^{(\alpha)} \hat{X}^{(\beta)} \hat{X}^{(\gamma)} \hat{X}^{(\delta)} \right) \hat{X}_{\alpha},
\]

(3.45)

The second step is to compute the last line of (3.45), that is:

\[
I_{\mu(\nu)}^{(1)}(x_{\mu}, t) \hat{I}_{\nu(\mu)}^{(1)}(x_{\nu}, t) \hat{I}_{\gamma(\delta)}^{(1)}(x_{\gamma}) \hat{I}_{\delta(\gamma)}^{(1)}(x_{\delta}) = \left( -1 \right)^{l_{\mu} + l_{\nu} + l_{\gamma} + l_{\delta}} \left( \hat{X}^{(\mu)} \hat{X}^{(\nu)} \hat{X}^{(\gamma)} \hat{X}^{(\delta)} \right) \hat{X}_{\mu},
\]

(3.46)

\[
= \left( -1 \right)^{l_{\mu} + l_{\nu} + l_{\gamma} + l_{\delta}} \left( \hat{X}^{(\mu)} \hat{X}^{(\nu)} \hat{X}^{(\gamma)} \hat{X}^{(\delta)} \right) \hat{X}_{\mu},
\]

(3.47)

The same argument also applies to \(G(X)\). We find that

\[
\hat{I}_{\alpha(\beta)}^{(\gamma)}(x_{\alpha}) \hat{I}_{\delta(\gamma)}^{(\delta)}(x_{\delta}) = \left( -1 \right)^{l_{\alpha} + l_{\beta} + l_{\gamma} + l_{\delta}} \left( \hat{X}^{(\alpha)} \hat{X}^{(\beta)} \hat{X}^{(\gamma)} \hat{X}^{(\delta)} \right) \hat{X}_{\alpha},
\]

(3.48)
Equation (3.45) then turns into
\[
\tilde{F}_{\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3)}(X, \Theta_1, \Theta_2) = \left( \frac{-1}{X_1, X_2} \right) \left( \tilde{\sigma}_{\rho(\gamma_1)} \right) \left( \tilde{\sigma}_{\rho(\gamma_2)} \right) \left( \tilde{\sigma}_{\rho(\gamma_3)} \right) \times \left( \tilde{X}_1 \right) \times \left( X_1 \right)
\]
where we have defined
\[
\tilde{F}(X; u, v, w, \bar{u}, \bar{v}, \bar{w}) = \frac{(-1)^{\gamma}}{r_2 \gamma_{1r_2}} \left( \frac{1}{X X_{r_2} \gamma_{1r_2}} \right)
\]
\[
\times \left[ F_1(X; w, \bar{u}, s, 3, 5, \bar{v}) - F_2(X; w, \bar{v}, s, 3, 5, \bar{v}) \right],
\]
(3.53a)

\[
\tilde{C}_{\alpha \beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{(-1)^{\gamma}}{r_2 \gamma_{1r_2}} \left( \frac{1}{X X_{r_2} \gamma_{1r_2}} \right)
\]
\[
\times \left( 2i(r_1 - r_3 + 1) \left( \frac{1}{u \alpha \beta} X_{\alpha \beta} \right) - \left( \frac{1}{u \alpha \beta} X_{\alpha \beta} \right) \right)
\]
\[
\times \left[ F_1(X; w, \bar{v}, s, 3, \bar{v}) - F_2(X; w, \bar{v}, s, 3, \bar{v}) \right]
\]
\[
-2ir_2 \left( \frac{1}{u \alpha \beta} \frac{1}{d X^2} \right)
\]
\[
\left[ F_1(X; w, \bar{u}, s, 3, \bar{v}) - F_2(X; w, \bar{u}, s, 3, \bar{v}) \right]
\]
\[
+ \frac{i}{r_2} \left( \frac{1}{u \alpha \beta} \frac{1}{d X^2} \right)
\]
\[
\times \left[ G_{1 \beta \beta}(X; w, \bar{v}, s, 3, v) - G_{2 \beta \beta}(X; w, \bar{v}, s, 3, v) \right]
\]
(3.53b)

In addition, the final result can be presented in a more compact way once we contract all indices with the auxiliary commuting spinors. We define
\[
\tilde{H}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \left( \frac{X}{X} \right)^{d} \left( \frac{1}{\tilde{F}_{\rho(\gamma_1)\rho(\gamma_2)\rho(\gamma_3)}}(X) \right)
\]
\[
+ \frac{1}{8} \left( \tilde{\sigma}_{\rho(\gamma_1)} \right) \left( \tilde{\sigma}_{\rho(\gamma_2)} \right) \left( \tilde{\sigma}_{\rho(\gamma_3)} \right) \times \left( \tilde{X}_1 \right) \times \left( X_1 \right)
\]
(3.51)

After some lengthy computations and recalling that \( \tilde{H}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) has scale dimension \( r_1 - (r_2 + r_3 + 2) \), one can present it in the form
\[
\tilde{H}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{-1}{2} \left( \frac{1}{u X_{\alpha \beta}} \right) \left( \frac{1}{w X_{\alpha \beta}} \right)
\]
\[
+ \frac{1}{8} \left( \tilde{\sigma}_{\rho(\gamma_1)} \right) \left( \tilde{\sigma}_{\rho(\gamma_2)} \right) \left( \tilde{\sigma}_{\rho(\gamma_3)} \right) \times \left( \tilde{X}_1 \right) \times \left( X_1 \right)
\]
(3.52)

where \( \tilde{F}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) vanishes due to conservation conditions on the first two points. This means that
\[
\tilde{F}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{-1}{2} \left( \frac{1}{u X_{\alpha \beta}} \right) \left( \frac{1}{w X_{\alpha \beta}} \right)
\]
(3.55)

Having derived the full expression for \( \tilde{H}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) which appears in the correlator
\[
\langle \mathcal{J}^{\rho(\gamma_1)}_{r_1}(X_1) \rangle^{\rho(\gamma_1)}_{r_1}(X_2) \rangle \langle \mathcal{J}^{\rho(\gamma_2)}_{r_2}(X_3) \rangle^{\rho(\gamma_2)}_{r_2}(X_4) \rangle
\]
The general solution for the tensor \( H \) can always be presented in the form

\[
\begin{aligned}
H_{\psi(h_{\mu}),\ell_{\nu}(\hat{\mu}),\ell_{\nu}(\hat{\nu})}(X, \Theta, \hat{\Theta}) \\
= F_{\psi(h_{\mu}),\ell_{\nu}(\hat{\mu}),\ell_{\nu}(\hat{\nu})}(X) \\
- \frac{1}{2} p^a G_{a(h_{\mu}),a(\hat{\mu}),\ell_{\nu}(\hat{\nu})}(X).
\end{aligned}
\]  

We then associate to \( H_{\psi(h_{\mu}),\ell_{\nu}(\hat{\mu}),\ell_{\nu}(\hat{\nu})}(X, P) \) the polynomial

\[
\begin{aligned}
H(X, P; u, \bar{u}, v, \bar{v}, w, \bar{w}) = F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) - \frac{1}{2} p^a G_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}).
\end{aligned}
\]  

and to \( \tilde{H}_{\psi(h_{\mu}),\ell_{\nu}(\hat{\mu}),\ell_{\nu}(\hat{\nu})}(X, P) \) the polynomial

\[
\begin{aligned}
\tilde{H}(X, P; u, \bar{u}, v, \bar{v}, w, \bar{w}) = F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) - \frac{1}{2} p^a \tilde{G}_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}).
\end{aligned}
\]  

Our formalism essentially allows us to express \( \tilde{F} \) and \( \tilde{G}_{a} \) in terms of only \( F, G_{a} \) and their vector derivatives (see Equations (3.53a) and (3.53b) for the explicit relations). This is useful since the polynomials \( F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) and \( G_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) are functions of a single, bosonic covariant building block \( X \) (and the auxiliary commuting spinors). When written in terms of \( F, G_{a} \), \( \tilde{F} \) and \( \tilde{G}_{a} \), the constraints due to conservation laws at all points take a simple form. More precisely, all the required constraints are:

1. **Homogeneity:**

\[
F(\Lambda \Lambda X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = (\Lambda \Lambda)^{1-\gamma_{\gamma}+\gamma_{\gamma}} F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}).
\]

2. **Conservation conditions at all points:**

\[
\begin{aligned}
\epsilon^{\alpha \beta} \frac{\partial}{\partial u^\alpha} G_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon^{\alpha \beta} \frac{\partial}{\partial \bar{u}^\alpha} \tilde{G}_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon^{\alpha \beta} \frac{\partial}{\partial v^\alpha} G_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon^{\alpha \beta} \frac{\partial}{\partial \bar{v}^\alpha} \tilde{G}_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon^{\alpha \beta} \frac{\partial}{\partial w^\alpha} G_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0, \\
\epsilon^{\alpha \beta} \frac{\partial}{\partial \bar{w}^\alpha} \tilde{G}_{a}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) &= 0.
\end{aligned}
\]
The above differential constraints lead to consistency conditions

\[ \Box F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = 0, \quad (3.65a) \]

\[ \Box \bar{F}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = 0. \quad (3.65b) \]

(3) Reality: \( a_1 \) are real and

\[ G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}); \quad b \]

\[ \text{and} \]

\[ G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{i}{2} \partial_{a\bar{b}} F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}). \quad (3.66) \]

Here \( a \) and \( b \) are constant parameters of the linearly independent tensor structures in \( F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) and \( G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}), \) respectively.

In the remaining sections we will apply our formalism to specific examples. We will discussed mixed correlation functions of \( J_{(\mu,\nu,\rho)} \) with the supercurrent and the flavour current multiplets.

4. Correlator \( \langle J_{(\mu,\nu,\rho)}(z_1)J_{(\rho,\sigma)}(z_2)J_{(\tau)}(z_3) \rangle \)

In this section we compute a mixed three-point function containing the higher-spin supercurrent \( J_{(\mu,\nu,\rho)} \) and the supercurrent \( J_{(\tau)} \). In accordance with (3.59), the ansatz for this correlator has the form

\[ \langle J_{(\mu,\nu,\rho)}(z_1)J_{(\rho,\sigma)}(z_2)J_{(\tau)}(z_3) \rangle \]

\[ = \frac{1}{(x_{12}x_{13}x_{23})^d} I_{(\mu,\nu,\rho)}(x_{12})I_{(\rho,\sigma)}(x_{13})I_{(\tau)}(x_{23}) \]

\[ \times H_{(\mu,\nu,\rho,\sigma,\tau)}(x_{123}) = F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}), \quad (4.1) \]

where \( H_{(\mu,\nu,\rho,\sigma,\tau)} \) has dimension \(-r+2\). The case \( r = 1 \) corresponds to the three-point function of the ordinary \( N = 1 \) supercurrent, which was first derived by Osborn.\(^{[15]}\) In subsection 4.1, we will rederive this result using the procedure outlined in section 3, thus providing a nice check of our formalism.

We shall now find the general expansion for the vector \( G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \):

\[ G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X^2} G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}). \quad (4.8) \]

We can now decompose \( G \) further using the structures defined in (3.25):

\[ G_{ab}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \sum_{i=1}^{8} \mathcal{K}_i G_i(X; u, \bar{u}, v, \bar{v}, w, \bar{w}). \quad (4.9) \]

Here it should be noted that \( G_i \) now are polynomials which are homogeneous of degree 0 in \( X \), degree 2 in \( U \), and degree 1 in both \( V \) and \( W \). We find that:

\[ \mathcal{K}_1 \text{ structures:} \]

\[ \mathcal{K}_1 G_{1}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = u_\alpha \bar{u}_\alpha \left[ b_1(V \bar{X})(W \bar{X}) + b_2(WV) \right]. \quad (4.10a) \]
**K**_2_ structures:

\[ \mathcal{K}_{2,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \]

\[ = \tilde{X}_a \left[ b_1(\tilde{U} \tilde{X})(\tilde{V} \tilde{X})(\tilde{W} \tilde{X}) + b_4(U \tilde{X})(V \tilde{X})(W \tilde{X}) \right] + b_7(\tilde{V} \tilde{X})(UW) + b_8(\tilde{W} \tilde{X})(UV) + b_7J]. \]  

\[ (4.10b) \]

**K**_3_ structures:

\[ \mathcal{K}_{3,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \nu TV \left[ b_4(\tilde{U} \tilde{W}) + b_8(\tilde{U} \tilde{X})(\tilde{W} \tilde{X}) \right]. \]

\[ (4.10c) \]

**K**_4_ structures:

\[ \mathcal{K}_{4,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \nu TV \left[ b_{10}(UV) + b_{11}(U \tilde{X})(\tilde{V} \tilde{X}) \right]. \]

\[ (4.10d) \]

**K**_5_ structure:

\[ \mathcal{K}_{5,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \tilde{Z}_{1,a} \left[ b_{12}(W \tilde{X}) \right]. \]

\[ (4.10e) \]

**K**_6_ structure:

\[ \mathcal{K}_{6,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \tilde{Z}_{2,a} \left[ b_{13}(V \tilde{X}) \right]. \]

\[ (4.10f) \]

**K**_7_ structure:

\[ \mathcal{K}_{7,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \tilde{Z}_{3,a} \left[ b_{14}(U \tilde{X}) \right]. \]

\[ (4.10g) \]

**K**_8_ structure:

\[ \mathcal{K}_{8,a} \mathcal{G}_a(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = b_{15} \tilde{Z}_{4,a}. \]

\[ (4.10h) \]

It is worth noting that not all of the \( \mathcal{K} \)-structures are linearly independent. In particular, we see that

\[ \tilde{X}_aJ + (W \tilde{X}) \tilde{Z}_{1,a} - (V \tilde{X}) \tilde{Z}_{2,a} + (U \tilde{X}) \tilde{Z}_{3,a} - \tilde{Z}_{4,a} = 0. \]  

\[ (4.11) \]

Thus, we choose to construct a linearly independent basis of polynomial structures for the vector \( \mathcal{G}_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) \) by removing the \( b_7 \) structure. As a result, we can express

\[ \mathcal{G}_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \mathcal{G}_{2,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) + \mathcal{G}_{2,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}). \]

\[ (4.12a) \]

\[ \mathcal{G}_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \frac{1}{X} \sum_{i=1}^{3} \mathcal{K}_{i,a} \mathcal{G}_i(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}). \]

\[ (4.12b) \]

\[ \mathcal{G}_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \frac{1}{X} \sum_{i=3}^{5} \mathcal{K}_{i,a} \mathcal{G}_i(\tilde{X}; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}). \]

\[ (4.12c) \]

We also note that all \( a_i \) and \( b_i \) are still assumed to be complex coefficients.

The next step is to impose differential constraints arising from the conservation on the first two points, Equations (3.64a)–(3.64b). This can be computed quickly using Mathematica. We find that only 4 independent complex coefficients remain, which we choose to be \( a_1, a_2, a_3, b_1 \). Reality condition implies that

\[ a_1, a_2, a_3, b_1 \] are all real; while \( b_i = \bar{b}_i + i a_i / 2 \), thus leaving us with 4 real coefficients \( a_1, a_2, a_3, b_1 \).

In this particular case, the correlator is completely symmetric under point switches 1 ↔ 2 and 2 ↔ 3 (hence, it is also symmetric under 1 ↔ 3). We must then impose further constraints due to these symmetries. The 1 ↔ 2 symmetry demands that

\[ H(X, P; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = H(-X, P; v, \tilde{v}, u, \tilde{u}, w, \tilde{w}), \]

\[ (4.13) \]

which translates to the following constraints

\[ F(X, u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = F(-X, v, \tilde{v}, u, \tilde{u}, w, \tilde{w}), \]

\[ (4.14a) \]

\[ G_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = G_{s,a}(-X; v, \tilde{v}, u, \tilde{u}, w, \tilde{w}) \]

\[ + i \partial_{\bar{a}_a} F(-X; v, \tilde{v}, u, \tilde{u}, w, \tilde{w}). \]

\[ (4.14b) \]

Imposing (4.14) leads to \( a_1 = a_2 = 0 \) and, hence, at this stage we end up with two real parameters: \( a_3 \) and \( b_1 \). The relations between coefficients are given by

\[ a_1 = a_2 = a_3 = a_4 = 0, \]

\[ b_a = \frac{1}{4}(2a_5 + b_1), \quad b_1 = -3b_1, \quad b_5 = -2a_5 - \frac{3}{2} b_1, \]

\[ b_6 = 2a_5 + b_1, \]

\[ b_8 = \frac{1}{4}(2a_5 + b_1), \quad b_9 = b_1, \quad b_{10} = \frac{1}{4}(-2a_5 + b_1), \]

\[ b_{11} = -\frac{1}{2} b_1, \]

\[ b_{12} = -b_{13} = b_{14} = 2i a_5, \quad b_{15} = -\frac{3}{2} i a_5. \]

As for the 2 ↔ 3 symmetry, it is not hard to show that the required constraints can be expressed in terms of (3.53a) and (3.53b) as follows:

\[ \tilde{F}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \tilde{F}(-X; v, \tilde{v}, u, \tilde{u}, w, \tilde{w}), \]

\[ (4.16a) \]

\[ \tilde{G}_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = \tilde{G}_{s,a}(-X; v, \tilde{v}, u, \tilde{u}, w, \tilde{w}) \]

\[ + i \partial_{\bar{a}_a} \tilde{F}(-X; v, \tilde{v}, u, \tilde{u}, w, \tilde{w}). \]

\[ (4.16b) \]

It can be verified that the 2 ↔ 3 symmetry and conservation conditions at \( z_j \) (described by Equations (3.64g)–(3.64i)) are automatically satisfied for the choice of coefficients (4.15). Our result here is thus in agreement with [15]. Indeed, our real parameters \( a_3 \) and \( b_1 \) are related to those in [15] by

\[ a_3 = -A, \quad b_1 = 2(2C - A). \]

\[ (4.17) \]

### 4.2. Analysis for \( r = 2 \)

We construct a generating function

\[ H(X, P; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) = U^{(8)(a)(2)} V^{(8)} W^{(8)} \]

\[ H_{(2)(a)(2)}(X, P) \]

\[ = F(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}) - \frac{1}{2} p^{(2)} G_{s,a}(X; u, \tilde{u}, v, \tilde{v}, w, \tilde{w}). \]

\[ (4.18) \]
The general expansion for the polynomial \( F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) is thus
\[
F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X^2} F(\bar{X}; \bar{u}, \bar{v}, \bar{w}, \bar{w}). \tag{4.19}
\]

Here the polynomial \( F \) is homogeneous of degree 0 in \( X \), degree 2 in \( u, \bar{u} \) and degree 1 in \( v, \bar{v}, w, \bar{w} \) (equivalently, we can say that it is of degree 2 in \( U \) and degree 1 in \( V \) and \( W \)). It is not difficult to construct all possible polynomial structures for \( F \) in terms of our basis structures described in (3.20). Our expansion gives 6 independent structures:
\[
P(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = (W\bar{X}) [a_1(\bar{U}\bar{X})^2(V\bar{X}) + a_2(UV)]\]
\[+ (UW)[a_1(U\bar{X})(V\bar{X}) + a_4(UV)]\]
\[+ (V\bar{X})[a_1(\bar{U}\bar{X})^2(V\bar{X}) + a_5(U\bar{X})J]. \tag{4.20}\]

We write \( F(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = F_1(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) + F_2(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \), where
\[
F_1(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X^2} [a_1(W\bar{X})(U\bar{X})^2(V\bar{X}) + a_2(W\bar{X})(U\bar{X})(V\bar{X})] + a_4(UV)(U\bar{X}) \]
\[+ a_5(U\bar{X})J]. \tag{4.21a}
\]
\[
F_2(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{d_6}{X^4}(U\bar{X})J. \tag{4.21b}
\]

We shall now find the general expansion for the vector \( G_{\alpha\beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \)
\[
G_{\alpha\beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X^2} G_{\alpha\beta}(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}). \tag{4.22}
\]
and decompose \( G \) further using the structures defined in (3.25):
\[
G_{\alpha\beta}(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \sum_{i=1}^{8} \kappa_{\alpha\beta} G_i(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}). \tag{4.23}
\]

In particular, we have that
\[\kappa_1 \text{ structures:}\]
\[
\kappa_{1,\alpha} G_1(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = u_{\alpha,\beta} b_1(U\bar{X})(V\bar{X})(\bar{U}\bar{X}) + b_2(U\bar{X})(UV) \]
\[+ b_3(V\bar{X})(U\bar{X}) + a_5(U\bar{X})J]. \tag{4.24a}
\]

\[\kappa_2 \text{ structures:}\]
\[
\kappa_{2,\alpha} G_2(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \bar{X}_{\alpha,\beta} [b_4(U\bar{X})^2(V\bar{X})(W\bar{X}) + b_5(U\bar{X})^2(VW) \]
\[+ b_6(U\bar{X})(V\bar{X})(U\bar{X}) + b_7(U\bar{X})(W\bar{X})(UV) \]
\[+ b_8(U\bar{X})(U\bar{X}) + c_{\alpha,\beta} J]. \tag{4.24b}\]

As in the previous subsection, we know that not all of the above \( \kappa \)-structures are linearly independent. In addition to the linear dependence relation (4.11), we also have the following relation
\[
u_{\alpha,\beta} J + (U\bar{X}) Z_{1,\alpha\beta} - (U\bar{X})(U\bar{X}) Z_{2,\alpha\beta} = 0. \tag{4.25}
\]

Hence, we may choose to construct a linearly independent basis of polynomial structures for the vector \( G_{\alpha\beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) by removing the \( c_1 \) and \( c_2 \) structures. This then allows us to write
\[
G_{\alpha\beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = G_{1,\alpha\beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) + G_{2,\alpha\beta}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}). \tag{4.26a}
\]

\[\kappa_3 \text{ structures:}\]
\[
\kappa_{3,\alpha} G_3(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X^2} \sum_{i=1}^{4} \kappa_{3,\alpha} G_i(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}). \tag{4.26b}
\]

\[\kappa_4 \text{ structures:}\]
\[
\kappa_{4,\alpha} G_4(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = 1 \frac{1}{X^2} \sum_{i=3}^{8} \kappa_{4,\alpha} G_i(\bar{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}). \tag{4.26c}\]

We also note that all \( a_i \) and \( b_i \) are still assumed to be complex and hence, initially we have 25 complex parameters.

The next step is to impose differential constraints arising from the conservation on the first two points, Equations (3.64a)–(3.64f). This can be computed relatively quickly using Mathematica. These conditions fix the correlation function up to 4 independent complex coefficients, which we choose to be \( a_1, a_5, a_6, b_1 \).
Specifically, the relations between the coefficients are given by:

\[ a_2 = -\frac{1}{6}a_1 - a_3, \quad a_3 = \frac{1}{2}a_1 + a_5, \quad a_4 = -\frac{1}{6}a_1, \]
\[ b_2 = -\frac{i}{4}(a_1 - 4a_5 + 4ia_6 + iib_1), \quad b_1 = \frac{1}{4}(2ia_3 + 2a_6 + b_1), \]
\[ b_3 = \frac{i}{12}(2a_1 - 6a_5 + 6ia_6 + 3ib_1), \]
\[ b_5 = -2i(a_1 - ib_1), \quad b_1 = i(a_1 - 3a_5 + i(3a_5 + b_1)), \]
\[ b_7 = \frac{i}{2}(a_1 - 6a_5 + 6ia_6 + 4ib_1), \]
\[ b_9 = -\frac{i}{2}(a_1 - 6a_5 + 6ia_6 + 2ib_1), \quad b_9 = \frac{1}{12}(ia_1 + 3b_1), \]
\[ b_{10} = \frac{i}{4}(2a_1 - 2ia_5 - ib_1), \quad b_{11} = \frac{b_1}{2}, \]
\[ b_{12} = -\frac{i}{12}(4a_1 + 6a_5 - 6ia_6 + 3ib_1), \quad b_{13} = \frac{3i}{2}a_1 - b_1, \]
\[ b_{14} = -\frac{a_1}{2} + 3i(ia_5 + a_6), \quad b_{15} = \frac{1}{12}(a_1 - 6i(ia_5 + a_6)), \]
\[ b_{16} = \frac{3}{2}(a_1 - 2i(ia_5 + a_6)), \quad b_{17} = \frac{1}{4}(a_1 - 2i(ia_5 + a_6)), \]
\[ b_{18} = -a_1 + 3i(ia_5 + a_6), \]
\[ b_{19} = \frac{2}{3}a_1 - 2i(ia_5 + a_6). \]  

We then express

\[ F(X; \bar{u}, \bar{v}, \bar{w}, \bar{w}_1) = F_1(X; \bar{u}, \bar{v}, \bar{w}, \bar{w}_1) + F_2(X; \bar{u}, \bar{v}, \bar{w}_1, \bar{w}_2, \bar{w}_1, \bar{w}_3). \]  

As for the general expansion for \( G_{\alpha \beta} (\tilde{X}; \bar{u}, \bar{v}, \bar{w}, \bar{w}_1) \), we write

\[ G_{\alpha \beta} (\tilde{X}; \bar{u}, \bar{v}, \bar{w}, \bar{w}_1) = \sum_{r=1}^{\infty} \mathcal{K}_{1,r} G_r (\tilde{X}; \bar{u}, \bar{v}, \bar{w}, \bar{w}_1). \]  

4.3. Analysis for General \( r > 2 \)

We now complete the analysis for arbitrary \( r > 2 \). The general expansion for the polynomial \( F(X; u, \bar{u}, v, \bar{v}, \bar{w}, \bar{w}_1) \) is thus

\[ F(X; u, \bar{u}, v, \bar{v}, \bar{w}, \bar{w}_1) = \frac{1}{X^2} F(\tilde{X}; u, \bar{u}, v, \bar{v}, \bar{w}, \bar{w}_1). \]  

Our polynomial \( F(\tilde{X}; \bar{u}, \bar{v}, \bar{w}, \bar{w}_1) \) is homogeneous of degree 0 in \( X \), degree \( r \) in \( U \) and degree 1 in \( W \) and \( W_1 \). This gives 6 independent structures, just like the case \( r = 2 \):
\( \mathcal{K}_6 \) structures:
\[
\mathcal{K}_{6,\alpha}(\hat{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = Z_{1,\alpha} \left[ b_{16}(U\bar{X})^{-1}(V\bar{X}) + b_{17}(U\bar{X})^{-2}(UV) \right],
\]
(4.32f)
\( \mathcal{K}_7 \) structure:
\[
\mathcal{K}_{7,\alpha}(\hat{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = Z_{1,\alpha} \left[ b_{18}(U\bar{X})^{-1} \right],
\]
(4.32g)
\( \mathcal{K}_8 \) structure:
\[
\mathcal{K}_{8,\alpha}(\hat{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}) = Z_{1,\alpha} \left[ b_{29}(U\bar{X})^{-1} \right].
\]
(4.32h)

Here we note the additional \( \mathcal{K}_1 \) structure containing the \((U\bar{X})^{-3}\) term, which exists for \( r \geq 3 \). Due to the linear dependence relations (4.11) and (4.25), we may choose to construct a linearly independent basis of polynomial structures for \( \mathcal{G}_{1,\alpha}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) \) by removing the \( c_i \) and \( c_2 \) structures. As a result, we can express
\[
\mathcal{G}_{1,\alpha}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = G_{1,\alpha}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) + G_{2,\alpha}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}),
\]
(4.33a)
\[
G_{1,\alpha}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X_{r+1}} \sum_{i=1}^{4} \mathcal{K}_{i,\alpha}(\hat{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}),
\]
(4.33b)
\[
G_{2,\alpha}(X; u, \bar{u}, v, \bar{v}, w, \bar{w}) = \frac{1}{X_{r+1}} \sum_{i=5}^{8} \mathcal{K}_{i,\alpha}(\hat{X}; u, \bar{u}, v, \bar{v}, w, \bar{w}).
\]
(4.33c)

Since \( a_i \) and \( b_i \) are initially assumed to be complex, we have 26 complex parameters.

The six constraints originating from the conservation on the first two points, Equations (3.64a)–(3.64f) fix the correlation function up to 4 independent complex coefficients, which we choose to be \( a_1, a_5, a_6, b_1 \). For completeness, here we give the relations between them
\[
a_2 = \frac{r - 1}{2(r + 1)} a_1 - a_5, \quad a_1 = \frac{r - 1}{2} a_1 + a_5,
\]
\[
a_4 = \frac{r - 1}{4(r + 1)} a_1,
\]
\[
b_2 = -\frac{1}{8} [r(a_1 - 4a_5 + 4ia_6) + 2ib_1],
\]
\[
b_3 = \frac{r - 1}{8} [i(r - 2)a_1 + 4(ia_5 + a_6) + 2b_1],
\]
\[
b_4 = -\frac{r - 1}{8(r + 1)} [-i(r + 2)a_1 + 4(r + 1)(ia_5 + a_6) + 6b_1],
\]
\[
b_5 = -\frac{r + 2}{2r} [ira_1 + 2b_1],
\]
\[
b_6 = \frac{1}{4r}(r + 2)(ira_1 - 2b_1) - (r + 1)(ia_5 + a_6),
\]
\[
b_7 = -\frac{i}{4}(r^2 - 2r - 2)a_1 - (r + 1)(ia_5 + a_6) - \frac{r + 2}{2} b_1,
\]
\[
b_8 = -\frac{1}{2} [ia_1 - 2(r + 1)(ia_5 + a_6) - 2b_1],
\]
\[
b_9 = -\frac{r - 1}{8(r + 1)} [ir(r - 1)a_1 + 2(r + 1)b_1],
\]
\[
b_{10} = \frac{1}{8} [i(r - 2)a_1 + 4(ia_5 + a_6) + 2b_1], \quad b_{11} = \frac{b_1}{r},
\]
\[
b_{12} = -\frac{1}{8} \left[ \frac{i}{r + 1} (r^2 + 3r - 2)a_1 + 4(ia_5 + a_6) - 2b_1 \right],
\]
\[
b_{13} = \frac{i}{4}(r + 4)a_1 - \frac{r + 2}{2r} b_1,
\]
\[
b_{14} = -\frac{1}{2} [(r - 1)a_1 + 2(r + 1)(a_5 - ia_6)],
\]
\[
b_{15} = -\frac{r - 1}{4(r + 1)} [(r - 1)a_1 - 2i(r + 1)(ia_5 + a_6)],
\]
\[
b_{16} = \frac{1}{2}(2r - 1)a_1 - i(r + 1)(ia_5 + a_6),
\]
\[
b_{17} = -\frac{r - 1}{4(r + 1)} [(2r - 1)a_1 - 2i(r + 1)(ia_5 + a_6)],
\]
\[
b_{18} = -\frac{r - 1}{2} a_1 + i(r + 1)(ia_5 + a_6),
\]
\[
b_{19} = \frac{1}{4} \left[ \frac{r^2 + 3r - 2}{r + 1} a_1 - i(r + 4)(ia_5 + a_6) \right],
\]
\[
b_{20} = -\frac{(r - 1)(r - 2)}{16(r + 1)} (ira_1 + 2b_1).
\]
(4.34)

We must also impose the reality condition (3.36), from which we obtain that \( a_1, a_5, a_6 \) are all real; while
\[
b_1 = b_1 + \frac{ir}{2} a_1.
\]
(4.35)

Hence, at this stage we have 4 independent real coefficients.

The constraints imposed by the symmetry under permutation of superspace points \( z_2 \leftrightarrow z_1 \) imply that the final form of the solution depends on whether \( r \) is even or odd. More precisely, if \( r \) is even, conditions (4.16a) and (4.16b) require that
\[
a_6 = 0, \quad \bar{b}_1 = 0 \Rightarrow b_1 = \frac{ir}{2} a_1,
\]
(4.36)

so, the independent parameters are \( a_1 \) and \( a_5 \). This is consistent with what we found in the \( r = 2 \) case. On the other hand, if \( r \) is odd, the \( 2 \leftrightarrow 3 \) symmetry implies that
\[
a_1 = a_5 = 0,
\]
(4.37)

so, the independent parameters are \( a_6 \) and \( b_1 \). Thus, in any case the correlator is still determined up to two real parameters, though its explicit structure is different.
5. Correlators Involving Flavour Current Multiplet

We now turn to computing mixed correlators containing the higher-spin supercurrent with the flavour current multiplet. In \( \mathcal{N} = 1 \) superconformal field theory, the latter is described by a primary real scalar superfield \( L \) of weights \( (q, \bar{q}) = (1, 1) \) and dimension 2. It satisfies the conservation law

\[
D^i L = \bar{D}^i L = 0. \tag{5.1}
\]

5.1. Correlator \( \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle \)

By virtue of (2.35), the ansatz for this three-point function takes the form

\[
\langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = \frac{1}{(x_{11} x_{12} x_{13})^2} H_{\alpha_1 \beta_1 \gamma_1 \delta_1}(X_1, \Theta_1, \Theta_1). \tag{5.2}
\]

The tensor \( H_{\alpha_1 \beta_1 \gamma_1 \delta_1} \) is subject to the following constraints:

(i) Homogeneity: It has the scaling property

\[
H_{\alpha_1 \beta_1 \gamma_1 \delta_1}(\lambda \hat{X}, \lambda \Theta, \lambda \bar{\Theta}) = (\lambda \hat{\lambda})^{-2} H_{\alpha_1 \beta_1 \gamma_1 \delta_1}(X, \Theta, \bar{\Theta}),
\]

and, hence, its dimension is \( (r - 2) \).

(ii) Conservation: Conservation laws of \( L \) on the first two points require that

\[
D^i_X \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = 0, \tag{5.4a}
\]

\[
D^i_{\bar{X}} \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = 0, \tag{5.4b}
\]

\[
D^i_X \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = 0, \tag{5.4c}
\]

\[
D^i_{\bar{X}} \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = 0. \tag{5.4d}
\]

With the use of (2.32), these requirements are equivalent to imposing these differential constraints on \( H \):

\[
\bar{D}^i H_{\alpha_1 \beta_1 \gamma_1 \delta_1} = 0, \tag{5.5a}
\]

\[
D^i H_{\alpha_1 \beta_1 \gamma_1 \delta_1} = 0, \tag{5.5b}
\]

\[
\bar{Q}^i H_{\alpha_1 \beta_1 \gamma_1 \delta_1} = 0, \tag{5.5c}
\]

\[
Q^i H_{\alpha_1 \beta_1 \gamma_1 \delta_1} = 0. \tag{5.5d}
\]

Equations (5.5a) and (5.5d) tell us that again, the general solution for \( H \) takes the form

\[
H_{\alpha_1 \beta_1 \gamma_1 \delta_1}(X, \Theta, \bar{\Theta}) = H_{\alpha_1 \beta_1 \gamma_1 \delta_1}(X, P) = F_{\alpha_1 \beta_1}(X) - \frac{1}{2} P^{\gamma \delta} G_{\gamma \delta, \alpha_1 \beta_1}(X). \tag{5.6}
\]

This means that we can adopt a similar procedure as in section 3 to solve for the differential constraints in terms of polynomials \( F(X; u, \bar{u}) \) and \( G_{\gamma \delta, \alpha_1 \beta_1}(X; u, \bar{u}) \):

\[
F(X; u, \bar{u}) = u^{\alpha_1} \cdots u^{\alpha_3} \bar{u}^{\delta_1} \cdots \bar{u}^{\delta_3} F_{\alpha_1 \beta_1}(X), \tag{5.7a}
\]

\[
G_{\gamma \delta, \alpha_1 \beta_1}(X; u, \bar{u}) = u^{\gamma} \cdots u^{\gamma_3} \bar{u}^{\delta} \cdots \bar{u}^{\delta_3} G_{\gamma \delta, \alpha_1 \beta_1}(X). \tag{5.7b}
\]

It is straightforward to check that Equations (5.5b) and (5.5c) both imply

\[
\Box F(X; u, \bar{u}) - i \bar{D}^{\gamma} G_{\gamma \delta, \alpha_1 \beta_1}(X; u, \bar{u}) = 0, \tag{5.8}
\]

which, in fact, is the consistency condition (3.32). We must also take into account the conservation of \( J_{\alpha_1 \beta_1 \gamma_1 \delta_1} \):

\[
\bar{D}^i_{(\bar{b})} \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = 0, \tag{5.9}
\]

\[
\bar{D}^i_{(\bar{b})} \langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = 0. \tag{5.10}
\]

As usual, we can rearrange our correlator as

\[
\langle L(z_1) L(z_2) J_{\alpha_1 \beta_1}(z_3) \rangle = \langle L(z_1) J_{\alpha_1 \beta_1}(z_3) L(z_1) \rangle = \frac{1}{(x_{12} x_{13} x_{14})^2} I_{\alpha_1 \beta_1 \gamma_1 \delta_1}(X_1, \Theta_1, \bar{\Theta}_1) \tag{5.11}
\]

which, after some manipulations, leads to

\[
\tilde{H}_{\alpha_1 \beta_1 \gamma_1 \delta_1}(X_1, \Theta_1) \tag{5.12}
\]

As we shall see soon, the simple expression in (5.12) is due to the fact that none of the structures in \( H_{\alpha_1 \beta_1}(x_3, p_3) \) transforms as a pseudotensor under the actions of \( L_{\alpha_1 \beta_1}(x_3, p_3) \). Next, we can easily expand \( \hat{X} = X + iP \) and then start contracting with auxiliary spinors. This results in

\[
\tilde{H}(X, P; u, \bar{u}) = \tilde{F}(X; u, \bar{u}) - \frac{1}{2} \bar{u}^{\alpha_1} \bar{u}^{\alpha_2} \tilde{G}_{\alpha_1 \alpha_2}(X; u, \bar{u}), \tag{5.13a}
\]

where

\[
\tilde{F}(X; u, \bar{u}) = \frac{1}{X^{2\gamma}} F(X; u, \bar{u}), \tag{5.13b}
\]

\[
\tilde{G}_{\alpha_1 \alpha_2}(X; u, \bar{u}) = \frac{1}{X^{2\gamma}} \partial_{\alpha_1} F(X; u, \bar{u}) - 2i(r - 1) \frac{X_{\alpha_1}}{X^{2r+2}} F(X; u, \bar{u}) \tag{5.13c}
\]

Conservation of \( J_{\alpha_1 \beta_1 \gamma_1 \delta_1} \) at \( z_3 \) then amounts to the following constraints:

\[
e^{\alpha_1 \beta_1} \partial_{\gamma_1 \delta_1} \tilde{G}_{\alpha_1 \alpha_2}(X; u, \bar{u}) = 0. \tag{5.14a}
\]
\[ \frac{\partial}{\partial \tilde{u}^a} \tilde{G}_{aw}(X; u, \tilde{u}) = 0, \]  
\hspace{1cm} (5.14b)\\
\[ \epsilon^{a\beta} \frac{\partial}{\partial \tilde{u}^\beta} \left[ \tilde{G}_{aw}(X; u, \tilde{u}) - i\theta_{aw} \tilde{F}(X; u, \tilde{u}) \right] = 0. \]  
\hspace{1cm} (5.14c)

(iii) Reality: Since both \( L \) and \( f_{\mu}(\Theta) \) are real superfields, the reality condition on the correlator leads to

\[ H(X, P; u, \tilde{u} ; a_1, b_1) = \tilde{H}(X, P; u, \tilde{u} ; \tilde{a}_1, \tilde{b}_1), \]  
\hspace{1cm} (5.15)

which are equivalent to

\[ F(X; u, \tilde{u} ; a_1) = F(X; u, \tilde{u} ; \tilde{a}_1) \Rightarrow a_1, \tilde{a}_1, \text{ real}, \]  
\hspace{1cm} (5.16a)\\
\[ G_{\nu\gamma}(X; u, \tilde{u} ; b_1) = G_{\nu\gamma}(X; u, \tilde{u} ; \tilde{b}_1) + i\delta_{\nu\gamma} F(X; u, \tilde{u} ; a_1), \]  
\hspace{1cm} (5.16b)

(iv) 1 \leftrightarrow 2 symmetry: The correlator is symmetric under \( z_1 \leftrightarrow z_2 \)

\[ H(X, P; u, \tilde{u}) = H(-\tilde{X}, P; u, \tilde{u}), \]  
\hspace{1cm} (5.17)

or, equivalently,

\[ F(X; u, \tilde{u}) = F(-\tilde{X}; u, \tilde{u}), \]  
\hspace{1cm} (5.18a)\\
\[ G_{\nu\gamma}(X; u, \tilde{u}) = G_{\nu\gamma}(-X; u, \tilde{u}) + i\delta_{\nu\gamma} F(-X; u, \tilde{u}). \]  
\hspace{1cm} (5.18b)

We are now ready to construct the general structures for \( F(X; u, \tilde{u}) \) and \( G_{\nu\gamma}(X; u, \tilde{u}) \). In terms of our invariants, there is only one possible structure for \( F \):

\[ F(X; u, \tilde{u}) = \frac{\alpha_1}{X^{1-r}} (U \tilde{X})^r \]  
\[ = -\left( \frac{1}{2} \right)^r \frac{\alpha_1}{X^{1-r}} \tilde{X}^{r}_{(a\bar{a})}. \]  
\hspace{1cm} (5.19)

There are two independent structures for \( G_{\nu\gamma}(X; U) \):

\[ G_{\nu\gamma}(X; u, \tilde{u}) = \frac{1}{X^{1-r}} \left[ b_1 \tilde{X}_{\gamma}(U \tilde{X})^r + b_2 u_a \tilde{u}_{\nu} (U \tilde{X})^{-r} \right] \]  
\[ = -\left( \frac{1}{2} \right)^r \frac{1}{X^{1-r}} \left[ b_1 \tilde{X}_{\gamma} \tilde{X}^{r}_{(a\bar{a})} - 2b_2 u_a \tilde{u}_{\nu} \tilde{X}^{r}_{(a\bar{a})} \right]. \]  
\hspace{1cm} (5.20)

The conservation condition (5.8) gives

\[ b_2 = \frac{r}{2} (b_1 + 2ia_1), \]  
\hspace{1cm} (5.21)

where, at this stage the coefficients \( a_1 \) and \( b_1 \) are still assumed to be complex. Next, upon substituting the explicit expressions of \( F(X; u, \tilde{u}) \) and \( G_{\nu\gamma}(X; u, \tilde{u}) \) into (5.13), one can verify that conservation conditions at \( z_1 \), Equations (5.14), are identically satisfied if we impose (5.21). Reality conditions imply that \( a_1 \) is real, while (5.16b) gives

\[ b_1 - \tilde{b}_1 = -2ia_1. \]  
\hspace{1cm} (5.22)

Finally, the 1 \leftrightarrow 2 symmetry (5.18) imposes the following relations

\[ a_1 = (-1)^r a_1 = \frac{i}{2} b_1 ((-1)^r + 1). \]  
\hspace{1cm} (5.23)

Thus, for odd values of \( r = 2k + 1 \), we have that \( a_1 = 0, \tilde{b}_1 = b_1 \), while for even values of \( r = 2k \), the coefficients are related by \( b_1 = -ia_1 \).

Our final result is that the correlator \( \langle L(z_1) L(z_2) f_{\mu}(\Theta) \rangle \) is fixed up to a single real parameter, though its explicit form depends on the values of \( r \):

- **r odd:**
  \[ H(X, P; u, \tilde{u}) = -\frac{1}{2} \frac{p_{\mu}}{p^\mu} G_{\nu\mu}(X; u, \tilde{u}). \]  
  \hspace{1cm} (5.24a)\\
  with
  \[ G_{\nu\mu}(X; u, \tilde{u}) = \frac{b_1}{X^{1-r}} \left[ \tilde{X}_{\mu} \tilde{X}^{r}_{(\bar{a}\bar{b})} - r u_a \tilde{u}_{\mu} \tilde{X}^{r}_{(\bar{a}\bar{b})} \right]. \]  
  \hspace{1cm} (5.24b)

- **r even:**
  \[ H(X, P; u, \tilde{u}) = F(X; u, \tilde{u}) - \frac{1}{2} \frac{p_{\mu}}{p^\mu} G_{\nu\mu}(X; u, \tilde{u}). \]  
  \hspace{1cm} (5.25a)\\
  with
  \[ F(X; u, \tilde{u}) = \frac{a_1}{X^{1-r}} \tilde{X}^{r}_{(\bar{a}\bar{b})}, \]  
  \hspace{1cm} (5.25b)\\
  \[ G_{\nu\mu}(X; u, \tilde{u}) = -\frac{ia_1}{X^{1-r}} \left[ \tilde{X}_{\mu} \tilde{X}^{r}_{(\bar{a}\bar{b})} + r u_a \tilde{u}_{\mu} \tilde{X}^{r}_{(\bar{a}\bar{b})} \right]. \]  
  \hspace{1cm} (5.25c)

5.2. Correlator \( \langle J_{\mu}(z_1) J_{\nu}(z_2) f_{\rho}(\Theta) \rangle \)

Using the general prescription (2.35), the ansatz for this correlator is given by

\[ \langle J_{\mu}(z_1) J_{\nu}(z_2) f_{\rho}(\Theta) \rangle \]  
\[ = \frac{1}{(X_{z_1}^{1-r} X_{z_2}^{1-r})^{1/2}} I_{\mu}(z_1) I_{\nu}(z_2) \times I_{\rho}(z_1) \]  
\[ \times X_{z_1}^{1-r} X_{z_2}^{1-r} \]  
\[ \times H^{\rho}(z_1) J_{\rho}(z_2) f_{\rho}(\Theta) \]  
\hspace{1cm} (5.26)

The tensor \( H^{\rho}(z_1) J_{\rho}(z_2) f_{\rho}(\Theta) \) is subject to the following constraints:

- **Homogeneity:** It is characterised by the scaling property

\[ H^{\rho}(z_1) J_{\rho}(z_2) f_{\rho}(\Theta) = (\lambda \bar{\lambda})^{-r_1 + r_2 + 2} H^{\rho}(z_1) J_{\rho}(z_2) f_{\rho}(\Theta), \]  
\hspace{1cm} (5.27)

and, hence, its dimension is \(-r_1 - r_2 + 2\).

- **Conservation:** Conservation conditions on the first two points require

\[ \tilde{D}_{\hat{X}}^\dag H^{\rho}(z_1) J_{\rho}(z_1) f_{\rho}(\Theta) = 0, \]  
\hspace{1cm} (5.28a)
As for the structure of $G_{\alpha\beta}(X; u, \bar{u}, v, \bar{v})$, we can write

$$G_{\alpha\beta}(X; u, \bar{u}, v, \bar{v}) = \frac{1}{X_{\alpha}^1 + \bar{X}_{\beta}^1} G_{\alpha\beta}(\bar{X}; u, \bar{u}, v, \bar{v}).$$  

(5.35a)

with

$$G_{\alpha\beta}(\bar{X}; u, \bar{u}, v, \bar{v}) = \bar{X}_{\alpha}^{r_i} \sum_{k=0}^{r_i-1} b_k (U \bar{X})^{r_i-k}(V \bar{X})^{k-1} (UV)^k$$

$$+ u_{\alpha} a_{\alpha} \sum_{k=0}^{r_1-1} C_k (U \bar{X})^{\bar{r}_1-k}(V \bar{X})^{k-1} (UV)^k$$

$$+ v_{\beta} a_{\beta} \sum_{k=0}^{r_2-1} D_k (U \bar{X})^{r_2-k}(V \bar{X})^{k-1} (UV)^k$$

$$+ Z_{1,\alpha\beta} \sum_{k=0}^{r_1-1} E_k (U \bar{X})^{\bar{r}_1-k}(V \bar{X})^{k-1} (UV)^k,$$

where we recalled that $Z_{1,\alpha\beta} = (\sigma^\alpha)_{\alpha\beta} Z_{1,\beta} = (\sigma^\beta)_{\alpha\beta} e_{\alpha\beta\gamma} \dot{X}^\alpha U V^\gamma$. Thus, initially we have $(5r_1 + 3)$ independent, complex parameters. To further constrain these parameters, we impose conservation conditions on the first two points, Equations (3.64a)–(3.64f). Here one can verify that $E_k = 0$, for $k = 0, 1, \ldots, r_1 - 1$, which will certainly simplify our calculation for $\bar{H}$ later as there is no need to split $G_{\alpha\beta}$ into two sectors (tensor and pseudotensor ones). Taking the reality condition into account, we find that, for $k = 0, 1, \ldots, r_1$:

$$b_k = b_k - i(r_1 + r_2 + 1 - k) a_k.$$  

(5.36a)

$$C_k = -\frac{k - r_1}{2(r_1 + r_2 + 1 - k)} b_k + \frac{i}{2}(r_1 - k) a_k.$$  

(5.36b)

$$D_k = -\frac{r_2 - k}{2(r_1 + r_2 + 1 - k)} b_k + \frac{i}{2}(r_2 - k) a_k.$$  

(5.36c)

where the parameters $a_k$ and $b_k$ are real. Furthermore, for $k = 1, 2, \ldots, r_1$, they obey following recursive relations:

$$a_k = -\frac{(r_1 + 1 - k)(r_2 + 1 - k)}{2k(r_1 + r_2 + 1 - k)} a_{k-1},$$

(5.36d)

$$b_k = -\frac{(r_1 + 1 - k)(r_2 + 1 - k)}{2k(r_1 + r_2 + 2 - k)} b_{k-1}.$$  

(5.36e)

The above relationships imply that we are now left with just two real parameters: $a_0$ and $b_0$.

Let us analyse the constraints resulting from the conservation law of $L$ on $z_1$. Rewriting our correlator as

$$\langle J_{(r_1 \beta)(r_1 \gamma)}(z_2) I_{(r_1 \alpha)(r_1 \gamma)}(z_1) \rangle = \frac{1}{(x_{21} x_{21})^2} \langle (x_{21} x_{21}) \dot{I}_{(r_1 \beta)(r_1 \gamma)}(z_1) \dot{I}_{(r_1 \alpha)(r_1 \gamma)}(z_1) \rangle$$

$$\times \hat{F}_{(r_1 \beta)(r_1 \gamma)} a_{(r_1 \alpha)(r_1 \gamma)} \langle X_1, P_1 \rangle,$$  

(5.37)
and making use several identities analogous to (3.47) and (3.48), we obtain
\[
\tilde{H}_{(\hat{\rho},\rho)}^{(\hat{\rho},\rho),\mu} (X, P) = (-XX)^{\hat{\rho}} I_{\rho} (\hat{\rho}) (X) I_{\rho} (\hat{\rho}) (P) \tilde{H}_{(\hat{\rho},\rho)}^{(\hat{\rho},\rho),\mu} (X, P).
\]

(5.38)

Defining the polynomial \( \tilde{H}(X, P; u, \bar{u}, v, \bar{v}) \) by
\[
\tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = \mathbf{U}_{\rho} (\rho) \tilde{H}_{(\hat{\rho},\rho)}^{(\hat{\rho},\rho),\mu} (X, P),
\]
we obtain the relation
\[
\tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = (-1)^{\frac{a}{2}} \left( \frac{X}{r_2} \right)^{\hat{\rho} - r_1} (\hat{\rho}) \left( \frac{uX}{\tilde{\rho}} \right)^{\hat{\rho}} \left( \frac{vX}{\tilde{\rho}} \right)^{\hat{\rho}} H(X^1, P^1; v, \bar{v}, s, \bar{s}).
\]

(5.40)

Here \( H(X^1, P^1; v, \bar{v}, s, \bar{s}) \) is defined by
\[
H(X^1, P^1; v, \bar{v}, s, \bar{s}) = F(X^1; v, \bar{v}, s, \bar{s}) = \left( \frac{1}{2} P_{\tilde{\rho}}^2 + G_{\tilde{\rho}} \right)(X^1; v, \bar{v}, s, \bar{s}).
\]

(5.41)

The conservation law of \( L \) demands that
\[
Q^2 \tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = 0.
\]

(5.42a)

\[
Q^2 \tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = 0.
\]

(5.42b)

Now, a useful observation is that \( Q_a X_{\tilde{\rho}} = Q_a X_{\tilde{\rho}} = 0 \). Hence, in computing (5.42a), it is easier to first express the right-hand side of (5.40) in terms of \( X \) and \( P \) (that is, we replace \( X = X + iP \)). On the other hand, to impose (5.42b), we should first replace \( X = X - iP \) in (5.40). Carrying out this procedure leads to
\[
\tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = \tilde{F}(X; u, \bar{u}, v, \bar{v}) - \frac{1}{2} P_{\tilde{\rho}}^2 \tilde{G}_{\tilde{\rho}}(X; u, \bar{u}, v, \bar{v}),
\]

(5.43a)

where
\[
\tilde{F}(X; u, \bar{u}, v, \bar{v}) = (-1)^{\frac{a}{2}} \left( \frac{X}{r_2} \right)^{\hat{\rho} - r_1} (\hat{\rho}) \left( \frac{uX}{\tilde{\rho}} \right)^{\hat{\rho}} \left( \frac{vX}{\tilde{\rho}} \right)^{\hat{\rho}} F(X; v, \bar{v}, s, \bar{s}),
\]

(5.43b)

\[
\tilde{G}_{\tilde{\rho}}(X; u, \bar{u}, v, \bar{v}) = (-1)^{\frac{a}{2}} \left( \frac{X}{r_2} \right)^{\hat{\rho} - r_1} (\hat{\rho}) \left( \frac{uX}{\tilde{\rho}} \right)^{\hat{\rho}} \times
\]
\[
\times \left\{ 2i(r_2 - 1) (\hat{\rho}) \left( \frac{X}{\tilde{\rho}} \right)^{\hat{\rho}} F(X; v, \bar{v}, s, \bar{s}) - 2i r_2 (\hat{\rho}) \left( \frac{X}{\tilde{\rho}} \right)^{\hat{\rho} - 1} \times \frac{\partial}{\partial s^a} F(X; v, \bar{v}, s, \bar{s}) \right\}.
\]

(5.44а)

In deriving the above expression, the terms proportional to \( P^2 \) identically vanish due to the conservation conditions on the first two points. Hence, now it is simple to see that (5.42a) holds, since \( Q_a X_{\tilde{\rho}} = 0 \) and \( Q_a^2 P_{\tilde{\rho}} = 0 \). In a similar manner, by expanding \( X = X - iP \), we obtain
\[
\tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = \tilde{F}(X; u, \bar{u}, v, \bar{v}) - \frac{1}{2} P_{\tilde{\rho}}^2 \tilde{G}_{\til\rho}(X; u, \bar{u}, v, \bar{v}).
\]

(5.44a)

where
\[
\tilde{F}(X; u, \bar{u}, v, \bar{v}) = (-1)^{\frac{a}{2}} \left( \frac{X}{r_2} \right)^{\hat{\rho} - r_1} (\hat{\rho}) \left( \frac{uX}{\til\rho} \right)^{\hat{\rho}} \left( \frac{vX}{\til\rho} \right)^{\hat{\rho}} F(X; v, \bar{v}, s, \bar{s}).
\]

(5.44b)

\[
\tilde{G}_{\til\rho}(X; u, \bar{u}, v, \bar{v}) = (-1)^{\frac{a}{2}} \left( \frac{X}{r_2} \right)^{\hat{\rho} - r_1} (\hat{\rho}) \left( \frac{uX}{\til\rho} \right)^{\hat{\rho}} \times
\]
\[
\times \left\{ 2i(r_2 - 1) (\hat{\rho}) \left( \frac{X}{\til\rho} \right)^{\hat{\rho}} F(X; v, \bar{v}, s, \bar{s}) + 2i r_2 (\hat{\rho}) \left( \frac{X}{\til\rho} \right)^{\hat{\rho} - 1} \times \frac{\partial}{\partial s^a} F(X; v, \bar{v}, s, \bar{s}) \right\}.
\]

(5.44c)

Without explicitly substituting the expressions for \( F \) and \( G_{\til\rho} \) in the right-hand sides of (5.44b) and (5.44c), one can immediately see that \( Q^2 \tilde{H}(X, P; u, \bar{u}, v, \bar{v}) = 0 \), for \( Q_{\til\rho} X_{\til\rho} = 0 \) and \( Q^2 P_{\til\rho} = 0 \). As a result, conservation on \( z_2 \) automatically holds for the choice of coefficients (5.36).

The three-point function \( \langle J_{\til\rho}(\omega); t_{\til\rho}(\omega)|J_{\til\rho}(\omega); t_{\til\rho}(\omega)|z_2|L(z_2) \rangle \), with \( r_1 < r_2 \) is thus fixed up to two real parameters: \( a_0 \) and \( b_0 \).

5.2.2 Analysis for \( r_1 = r_2 \)

In the case when \( r_1 = r_2 \), the correlator is subject to an extra constraint due to the 1 ↔ 2 symmetry. \( F \) and \( G_{\til\rho} \), this symmetry is equivalent to imposing
\[
F(X; u, \bar{u}, v, \bar{v}) = F(-X; v, \bar{v}, u, \bar{u}),
\]

(5.45a)

\[
G_{\til\rho}(X; u, \bar{u}, v, \bar{v}) = G_{\til\rho}(-X; v, \bar{v}, u, \bar{u}) + i \partial_{\til\rho} F(-X; v, \bar{v}, u, \bar{u}).
\]

(5.45b)

Both conditions imply that \( b_0 = 0 \), for \( k = 0, 1, \ldots, r_1 \). As a result, the correlator
is determined up to a single real parameter, $a_0$. The explicit solution is given by

$$H(X; u, \bar{u}, v, \bar{v}) = F(X; u, \bar{u}, v, \bar{v}) = \frac{1}{2} \sum_{k=0}^{r} a_k (U \bar{X})^{-k} (V \bar{X})^{-k} (UV)^k,$$

(5.47b)

with

$$F(X; u, \bar{u}, v, \bar{v}) = \frac{1}{X^{2r+1}} \sum_{k=0}^{r} a_k (U \bar{X})^{-k} (V \bar{X})^{-k} (UV)^k,$$

(5.47c)

Here the parameters are related by

$$B_k = -i (2r + 1 - k) a_k,$$

(5.48a)

$$C_k = D_k = \frac{i}{2} (r - k) a_k,$$

(5.48b)

where, for $k = 1, 2, \ldots, r$, we have the recursive relation

$$a_k = -\frac{(r + 1 - k)^2}{2k(2r + 1 - k)} a_{k-1}.$$  

(5.48c)

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## Conflict of Interest

The authors declare no conflict of interest.

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