Tropical representation of Weyl groups associated with certain rational varieties

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Dedicated to Professor Kazuo Okamoto on his sixtieth birthday

Abstract

Starting from certain rational varieties blown-up from \((\mathbb{P}^1)^N\), we construct a tropical, i.e., subtraction-free birational, representation of Weyl groups as a group of pseudo isomorphisms of the varieties. We develop an algebro-geometric framework of \(\tau\)-functions as defining functions of exceptional divisors on the varieties. In the case where the corresponding root system is of affine type, our construction yields a class of (higher order) \(q\)-difference Painlevé equations and its algebraic degree grows quadratically.

1 Introduction

The aim of the present work is to develop the theory of birational representation of Weyl groups associated with algebraic varieties.

At the beginning of the twentieth century, it was discovered by Coble and Kantor, and later by Du Val, that certain types of Cremona transformations act on the configuration space of point sets [Cob29]. Let \(X_{m,n}\) be the configuration space of \(n\) points in general position in the projective space \(\mathbb{P}^{m-1}\). Then, the Weyl group \(W(T_{2,m,n-m})\) corresponding to the Dynkin diagram \(T_{2,m,n-m}\) (see Figure 1) acts birationally on \(X_{m,n}\) and is generated by permutations of \(n\) points and the standard Cremona transformation with respect to each \(m\) points. An algebro-geometric and modern interpretation of this theory is due to Dolgachev and Ortland [DO88]; they showed that the Cremona action of \(W(T_{2,m,n-m})\) induces a pseudo isomorphism, i.e., an isomorphism except for subvarieties of codimension two or higher, between certain rational varieties blown-up from \(\mathbb{P}^{m-1}\) at generic \(n\) points, which they call generalized Del Pezzo varieties. It is worth mentioning that if \((m,n) = (3,9)\), the affine Weyl group of type \(E_8^{(1)}\) appears and its lattice part gives rise to an important discrete dynamical system, i.e., the elliptic-difference Painlevé equation [Sak01]; see also [ORG01, KMNOY03, KMNOY06].

In this paper, starting from a certain rational variety blown-up from \((\mathbb{P}^1)^N\) along appropriate subvarieties that are not only point sets, we construct a birational representation of Weyl groups corresponding to the Dynkin diagram \(T^k_t\). Here \(T^k_t\) refers to the graph given in Figure 1 specified

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by a pair of sequences $k = (k_1, \ldots, k_N), \ell = (\ell_1, \ldots, \ell_N) \in (\mathbb{Z}_{>0})^N$. It is remarkable that $T^k_\ell$ includes all of the simply-laced affine cases $A_n^{(1)}$, $D_n^{(1)}$ and $E_n^{(1)}$, which are relevant to a class of higher-dimensional discrete dynamical systems of Painlevé type. This representation of Weyl groups is *tropical*, i.e., given in terms of subtraction-free birational mappings \cite{Kir01} and, interestingly enough, possesses a geometric framework of $\tau$-functions.

In the next section, we begin with blowing-up $(\mathbb{P}^1)^N$ along certain subvarieties of codimension three. Let $X$ be the rational variety thus obtained. These $X$’s constitute a family. Applying a cohomological technique (cf. \cite{DO88}), we construct the root and coroot lattices of type $T^k_\ell$ included in the Néron-Severi bilattice $N(X) \cong (H^2(X, \mathbb{Z}), H_2(X, \mathbb{Z}))$. The associated Weyl group $W = W(T^k_\ell)$ naturally acts on $N(X)$ as isometries (Lemma \ref{lem:isometry}). We see that this linear action of $W$ on $N(X)$ leads to a birational representation of $W$ on the family of varieties itself as a group of pseudo isomorphisms (Theorem \ref{thm:birational}). An element of $W$ naturally induces an appropriate permutation among the set of exceptional divisors on $X$, as similar to the classical topic: 27 lines (or exceptional curves) on a cubic surface and a Weyl group of type $E_6$. In §3 for the purpose of describing the action of $W$ at the level of defining polynomials of exceptional divisors, we introduce a geometric framework of $\tau$-functions (Definition \ref{def:tau} and Theorems \ref{thm:tau} and \ref{thm:tau2}). One important advantage of our $\tau$-functions is that we can trace the resulting value for any element $w \in W$ by using only the defining polynomials of suitable divisors, although it is generally difficult to compute iterations of rational mappings. In particular, our representation in an affine case provides a discrete dynamical system arising from the lattice part of the affine Weyl group. Such a discrete dynamical system is equipped with a set of commuting discrete time evolutions and its algebraic degree grows in the quadratic order. And it is regarded as a (higher order) $q$-difference Painlevé equation; in §4 the $A_n^{(1)}$ and $D_n^{(1)}$ cases are demonstrated as typical examples. In §5 we briefly indicate (from a soliton-theoretic point of view) some interesting relationships between $\tau$-functions and the character polynomials appearing in representation theory of classical groups, i.e., the Schur functions or the universal characters.
2 Tropical Weyl group actions on rational varieties

In this section, we consider a certain rational variety $X$ blown-up from $(\mathbb{P}^1)^N$ and construct appropriate root and coroot lattices included in the Néron-Severi bilattice $N(X)$. Moreover, the action of the corresponding Weyl group is realized as pseudo isomorphisms of exponents. Hereafter we regard the $\text{su}$ for appropriate root and coroot lattices included in the Néron-Severi bilattice.

2.1 Rational variety and root system

Let $f = (f_1, f_2, \ldots, f_N)$ denote the inhomogeneous coordinates of $(\mathbb{P}^1)^N$ where $N \geq 3$. Fix a pair of sequences $k = (k_1, \ldots, k_N)$ and $\ell = (\ell_1, \ldots, \ell_N)$ of positive integers. Consider the following subvarieties:

$$C^i_n = \{ f_{n-1} = 0, f_n = -u_n^i, f_{n+1} = \infty \}, \quad i = 1, \ldots, k_n,$$

$$C^j_n = \{ f_{n-1} = \infty, f_n = -v_n^{-j}, f_{n+1} = 0 \}, \quad j = 1, \ldots, \ell_n,$$

for $n = 1, 2, \ldots, N$, where $u_n^i$ and $v_n^{-j}$ are nonzero parameters (note: superscripts are indices, not exponents). Hereafter we regard the suffix $n$ of the coordinate $f_n$ as an element of $\mathbb{Z}/N\mathbb{Z}$, namely, $f_{n+N} = f_n$. Let $\epsilon : X \to (\mathbb{P}^1)^N$ be the blowing-up along $(C^i_n, C^j_n)$. Since $X$ is a rational variety, we have

$$H^2(X, \mathbb{Z}) \cong \text{NS}(X) = \bigoplus_{n=1}^N \left( \mathbb{Z}H_n \oplus \mathbb{Z}E_n^i \oplus \mathbb{Z}E_n^j \right),$$

where $\text{NS}(X)$ is the Néron-Severi group of $X$; we denote by $H_n$ the divisor class of hyperplanes $\{ f_n = \text{const.} \}$ and by $E_n^i$ the class of exceptional divisors $\epsilon^{-1}(C^i_n)$. The Poincaré duality guarantees $H_2(X, \mathbb{Z}) \cong (H^2(X, \mathbb{Z}))^*$. We can choose a basis $\{ h_n, e_n^i \}$ of $H_2(X, \mathbb{Z})$, where $h_n$ corresponds to a line of degree $(0, \ldots, 0, 1, 0, \ldots, 0)$ and $e_n^i$ to a line restricted in a fibre $(\cong \mathbb{P}^2)$ of the exceptional divisor $\epsilon^{-1}(C^i_n)$. Thus the intersection pairing $\langle \cdot, \cdot \rangle : H^2(X, \mathbb{Z}) \times H_2(X, \mathbb{Z}) \to \mathbb{Z}$ is defined by $\langle H_m, h_n \rangle = \delta_{m,n}$, $\langle E_m^i, e_n^j \rangle = -\delta_{m,n}\delta_{i,j}$ and otherwise $= 0$.

Introduce the root lattice $Q$ and coroot lattice $\check{Q}$ as follows:

$$Q = \bigoplus_{n=1}^N \bigoplus_{-\ell_n+1 \leq k_n \leq \ell_n} \mathbb{Z} \alpha_n^i \subset H^2(X, \mathbb{Z}) \quad \text{and} \quad \check{Q} = \bigoplus_{n=1}^N \bigoplus_{-\ell_n+1 \leq k_n \leq \ell_n} \mathbb{Z} \check{\alpha}_n^i \subset H_2(X, \mathbb{Z}),$$

where

$$\alpha_n^0 = H_n - E_n^1 - E_n^{-1}, \quad \check{\alpha}_n^0 = h_{n-1} + h_{n+1} - e_n^1 - e_n^{-1},$$

$$\alpha_n^i = E_n^i - E_n^{i+1}, \quad \check{\alpha}_n^i = e_n^i - e_n^{i+1} \quad (i = 1, \ldots, k_n - 1),$$

$$\alpha_n^{-j} = E_n^{-j} - E_n^{-j-1}, \quad \check{\alpha}_n^{-j} = e_n^{-j} - e_n^{-j-1} \quad (j = 1, \ldots, \ell_n - 1).$$

For instance, we have $\langle \alpha_n^i, \check{\alpha}_n^j \rangle = -2$, $\langle \alpha_n^0, \check{\alpha}_{n \pm 1}^0 \rangle = \langle \alpha_n^0, \check{\alpha}_{n \pm 1}^0 \rangle = 1$, etc. The Dynkin diagram of the canonical root root basis forms $T^k$: 

3
The simple reflection \( s_n^i \) associated with a root \( \alpha_n^i \) naturally acts on the Néron-Severi bilattice \( N(X) \equiv (H^2(X, \mathbb{Z}), H_2(X, \mathbb{Z})) \) as

\[
\begin{align*}
    s_n^i(\Lambda) &= \Lambda + \langle \Lambda, \alpha_n^i \rangle \alpha_n^i, \quad \Lambda \in H^2(X, \mathbb{Z}), \\
    s_n^i(\lambda) &= \lambda + \langle \alpha_n^i, \lambda \rangle \alpha_n^i, \quad \lambda \in H_2(X, \mathbb{Z}).
\end{align*}
\]

One can easily check that these reflections indeed satisfy the fundamental relations (see [Kac90]) of the Weyl group \( W = W(T^k) = \langle s_n^i \rangle \); for instance, we have \( (s_n^i)^2 = \text{id} \) and \( s_n^0 s_{n+1} s_n^0 = s_n^0 s_{n+1} s_n^0 \).

The half of the anti-canonical class \( -\frac{1}{2}K_X = \sum_{n=1}^N (H_n - \sum_{j=1}^{\ell_n} E_n^j) \in H^2(X, \mathbb{Z}) \) can be decomposed in two ways

\[
-\frac{1}{2}K_X = \sum_{n=1}^N D_n^0 = \sum_{n=1}^N D_n^\infty,
\]

where \( D_n^0 = H_n - \sum_{i=1}^{\ell_n} E_n^i - \sum_{j=1}^{\ell_n} E_n^{-j} \) and \( D_n^\infty = H_n - \sum_{i=1}^{\ell_n} E_n^i - \sum_{j=1}^{\ell_n} E_n^{-j} \). Note that divisor classes \( D_n^0 \) and \( D_n^\infty \) are effective and are represented by the strict transforms of hyperplanes \( \{ f_n = 0 \} \) and \( \{ f_n = \infty \} \), respectively. In parallel, we shall formally define an element \( -\frac{1}{2}k_X \in H_2(X, \mathbb{Z}) \) by

\[
-\frac{1}{2}k_X = \sum_{n=1}^N \left( 2h_n - \sum_{i=1}^{\ell_n} e_n^i - \sum_{j=1}^{\ell_n} e_n^{-j} \right) = \sum_{n=1}^N d_n^0 = \sum_{n=1}^N d_n^\infty,
\]

where \( d_n^0 = h_{n-1} + h_{n+1} - \sum_{i=1}^{\ell_n} e_{n+1}^i - \sum_{j=1}^{\ell_n} e_{n-1}^{-j} \) and \( d_n^\infty = h_{n-1} + h_{n+1} - \sum_{i=1}^{\ell_n} e_{n+1}^i - \sum_{j=1}^{\ell_n} e_{n-1}^{-j} \). We see that \( Q \subset \{ d_n^0, d_n^\infty \}^\perp \) and \( \hat{Q} \subset \{ D_n^0, D_n^\infty \}^\perp \). Hence we have the

**Lemma 2.1.** All elements \( w \in W \subset \text{Aut}(N(X)) \) leave the intersection pairing \( \langle , \rangle \), \( \frac{1}{2}K_X \) and \( \frac{1}{2}k_X \) invariant.

### 2.2 Birational representation of Weyl groups

We summarize below the linear action of generators \( s_n^i \) on the basis of \( H^2(X, \mathbb{Z}) \):

\[
\begin{align*}
    s_n^0(H_{n \pm 1}) &= H_{n \pm 1} + H_n - E_n^1 - E_n^{-1}, \\
    s_n^0(E_n^\pm 1) &= H_n - E_n^{\pm 1}, \\
    s_n^i(E_n^{[i, i+1]}) &= E_n^{[i, i+1]} \quad \text{for} \quad 1 \leq i \leq k_n - 1, \\
    s_n^{-j}(E_n^{-[j, j-1]}) &= E_n^{-[j, j-1]} \quad \text{for} \quad 1 \leq j \leq \ell_n - 1.
\end{align*}
\]
Now, let us extend the above linear action of the Weyl group \( W = \langle s_i \rangle \) to the level of birational transformations on the rational variety \( X \).

To this end, we first introduce the multiplicative root variables \( a_n^i \in \mathbb{C}^\times \) attached to the canonical roots \( \alpha_n^i \) and fix the action of \( s_i^j \) on them as

\[
    s_n^j(a_n^i) = \frac{1}{a_n^i}, \quad s_n^j(a_n^{i\pm 1}) = a_n^{i\pm 1}, \quad s_n^j(a_n^0) = a_n^0.
\]  

(2.2)

Using the root variables, we fix the parameterization of subvarieties \( C_n^i \) as follows:

\[
    u_n^i = u_n = \frac{a_n^0 \prod_{j=1}^{\ell_n-1} (a_n^{-j})^{1-j/\ell_n} u_n^{1/(k_n+\ell_n)}}{(\prod_{j=1}^{\ell_n-1} (a_n^{1-j})^{1-j/\ell_n}) u_n^{1/(k_n+\ell_n)}}, \quad v_n^{-1} = v_n = \frac{a_n^0 \prod_{j=1}^{\ell_n-1} (a_n^{-j})^{1-j/\ell_n} u_n^{1/(k_n+\ell_n)}}{(\prod_{j=1}^{\ell_n-1} (a_n^{1-j})^{1-j/\ell_n}) u_n^{1/(k_n+\ell_n)}},
\]  

(2.3)

and \( u_n^i = a_n^{-1} u_n^{-1}, v_n^{-j} = a_n^{-j+1} v_n^{-j+1} \) for \( i, j \geq 2 \). Namely, a projective equivalence class of the arrangement of subvarieties \( \{ C_n^i \} \) in \( \mathbb{P}^1 \) can be identified with a point of the Cartan subalgebra associated with the Dynkin diagram \( T_k^1 \). The rational varieties \( X \)'s under consideration constitute a family parameterized by the multiplicative root variables \( a = (a_n^i) \); so we shall write clearly as \( X = X_a \).

Next, for each \( w \in W \subset \text{Aut}(H^2(X, \mathbb{Z})) \), there is a birational mapping \( \text{cr}(w) : X_a \to X_{w(a)} \) such that \( w = \text{cr}(w)^* \) (the pullback). Hereafter we will intentionally omit to write clearly as \( \text{cr}(w) \), and write shortly as \( w \) instead. Let \( \mathbb{K}(f) \) be the field of rational functions in \( f = (f_1, f_2, \ldots, f_N) \), where the coefficient field \( \mathbb{K} = \mathbb{C}(a^{1/p}) \) is generated by a suitable fractional power \( (a_n^i)^{1/p} \) of \( a_n^i \). We then have the following birational representation of \( W \) over \( \mathbb{K}(f) \), acting on the family \( X = \bigcup_a X_a \) of varieties as pseudo isomorphisms.

**Theorem 2.2.** The birational transformations \( s_n^j \) defined by

\[
    s_n^0(f_{n-1}) = (a_n^0)^{k_{n-1}/(k_n+\ell_n)} f_{n-1} + 1/v_n, \quad \frac{u_n + 1/v_n}{f_n + u_n},
\]  

(2.4a)

and \( s_n^j(f_n) = f_n \) (otherwise), together with (2.2), realize the Weyl group \( W = W(T_k^1) \) over the field \( \mathbb{K}(f) \). Moreover, each \( s_n^j \) maps \( X_a \) to \( X_{s_n^j(a)} \) as a pseudo isomorphism of rational varieties and actually induces the linear action (2.1) on \( N(X) \).

**Proof.** First we notice that \( s_n^0 : u_n \leftrightarrow 1/v_n, s_n^i : u_n^i \leftrightarrow u_n^{i+1} \) (\( 1 \leq i \leq k_n - 1 \)) and \( s_n^{-j} : v_n^{-j} \leftrightarrow v_n^{-j-1} \) (\( 1 \leq j \leq \ell_n - 1 \)). It is easy to verify by direct computation that \( \langle s_n^j \rangle \) satisfies the fundamental relations of \( W(T_k^1) \).

For \( i \neq 0 \), \( s_n^i \) is trivial and acts as an isomorphism \( X_a \to X_{s_n^i(a)} \), so we have only to check that \( s_n^0 \) actually induces a pseudo isomorphism. The points of indeterminacy of the rational map \( s_n^0 : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N \) are given by the following subvarieties:

\[
    I_1(a) = \{ f_n = 0, f_{n-1} = 0 \}, \quad I_2(a) = \{ f_n = 0, f_{n-1} = 1 \}, \quad I_3(a) = \{ f_n = 0, f_{n+1} = 0 \}, \quad I_4(a) = \{ f_n = 0, f_{n-1} = 1 \},
\]

of codimension two. Since the same argument is also valid for the other \( I_n \) (\( n = 2, 3, 4 \)), in what follows we shall only treat \( I_1 \). We see that \( s_n^0 \) maps \( I_1 \) to \( I_3 \) generically, except for \( C_n^1 = I_1 \cap I_2 \);
furthermore, \( C_1^n \) equals the inverse image \((s_0^n)^{-1}(\{f_n = -1/v_n\})\). By the blowing-up \( \epsilon : X_a \to (\mathbb{P}^1)^N \), the indeterminacy at \( C_1^n \) is resolved and \( s_0^n \) is extended holomorphically to \( \epsilon^{-1}(C_1^n) \). Thus we have proved that \( s_0^n \) acts on \( X_a \) as a pseudo isomorphism, i.e., an isomorphism except for subvarieties of codimension two or higher.

\[ \square \]

Remark 2.3. Our representation given in Theorem 2.2 above is obviously tropical and hence admits a combinatorial counterpart via the ultra-discretization \([TTMS96]\):

\[
a \times b \to a + b, \quad a/b \to a - b, \quad a + b \to \min(a, b).\]

3 \( \tau \)-Functions

An element of the Weyl group \( W = W(T^F) \) naturally induces a permutation among the exceptional divisors on the rational variety \( X \). In this section, we introduce a geometric framework of \( \tau \)-functions, which describes the above permutations and therefore governs our tropical representation of \( W \).

3.1 Representation over the field of \( \tau \)-variables

We set

\[
\theta_n^0 = k_n + \ell_n - 1 \quad \text{and} \quad \theta_n^\infty = k_n - 1 + \ell_n + 1,
\]

which are equal to the numbers of centers of the blowing-up included in hyperplanes \( \{f_n = 0\} \) and \( \{f_n = \infty\} \), respectively. From now on, we assume for simplicity that

\[
k_n - 1 \ell_n + 1 = \ell_n - 1 \ell_n + 1.
\]

In this case, (2.4) can be equivalently rewritten into

\[
\begin{align*}
    s_n^0(f_{n-1}) &= f_{n-1} \frac{v_n \omega_1 f_n + v_n - 1 + \omega_1}{u_n - \omega_1 f_n + u_n - 1 - \omega_1}, \\
    s_n^0(f_{n+1}) &= f_{n+1} \frac{u_n - \omega_1 f_n + u_n - 1 + \omega_1}{v_n \omega_1 f_n + v_n - 1 + \omega_1},
\end{align*}
\]

where a rational number \( \omega_n \), which is not a suffix, is given by \( \omega_n = \theta_n^0 / (\theta_n^0 + \theta_n^\infty) \). Now we consider the decomposition of variables

\[
f_n = \frac{x_{n+1}}{x_n},
\]

and take a birational transformation

\[
s_n^0(x_n) = x_n \frac{v_n \omega_1 x_{n+1} + v_n - 1 + \omega_1 x_n - 1}{u_n - \omega_1 x_{n+1} + u_n - 1 - \omega_1 x_n - 1}.
\]

One can easily verify that (3.3) with (2.2) also defines a birational representation of \( W \) on the field \( \mathbb{K}(x) \). Moreover, the representation (3.1) at the level of the \( f \)-variables is of course derived from (3.3) via (3.2). Introducing new variables \( \tau_n^j \) attached to the centers \( C_n^j \) of the blowing-up, we shall consider a further decomposition of variables in terms of the \( \tau \)-variables

\[
x_n = \frac{\xi_n}{\eta_n}, \quad \xi_n = \prod_{j=1}^{k_n} \tau_n^j, \quad \eta_n = \prod_{j=1}^{\ell_n} \tau_n^j
\]

(3.4)
Theorem 3.1. Define the birational transformations $s^i_n$ by

$$s^0_n(\tau^1_n) = \frac{v_n^{\omega_n} \xi_n^{0} \eta_{n-1} + v_n^{-1+\omega_n} \xi_{n-1}}{\tau_n^{1}}, \quad (3.5a)$$

$$s^0_n(\tau^{-1}_n) = \frac{u_n^{-\omega_n} \xi_{n+1} \eta_{n-1} + u_n^{1-\omega_n} \xi_{n-1}}{\tau_n^{1}}, \quad (3.5b)$$

$$s^i_n(\tau^{i+1}_n) = \tau_n^{i+1}, \quad (1 \leq i \leq k_n - 1), \quad (3.5c)$$

$$s^i_n(\tau^{j-1}_n) = \tau_n^{j-1}, \quad (1 \leq j \leq \ell_n - 1), \quad (3.5d)$$

$$s^i_n(\tau^j_m) = \tau^j_m, \quad (\text{otherwise}), \quad (3.5e)$$

where $\omega_n = \theta_n^0/(\theta_n^0 + \theta_n^{\infty})$, $\xi_n = \prod_{i=1}^{k_n} \tau^i_n$ and $\eta_n = \prod_{j=1}^{\ell_n} \tau^j_n$. Then (3.5) with (2.2) realizes the Weyl group $W(T^1_k)$ over the field $L = \mathbb{K}(\tau)$ of rational functions in indeterminates $\tau^i_n$.

We remark that, conversely, both realizations in terms of $f$-variables and $x$-variables are deduced from Theorem 3.1 through

$$f_n = \frac{\xi_n^{0+1} \eta_{n-1}}{\xi_{n-1} \eta_{n+1}}, \quad \xi_n = \prod_{i=1}^{k_n} \tau^i_n, \quad \eta_n = \prod_{j=1}^{\ell_n} \tau^j_n \quad (3.6)$$

and through (3.4), respectively.

3.2 $\tau$-Function and its geometric meaning

By means of Theorem 3.1 we now introduce the concept of $\tau$-functions; cf. [KMNOY03, Tsu06]. Let us consider the Weyl group orbit $M = W([E^i_n] \subset H^2(X, \mathbb{Z})$ of the classes of exceptional divisors. Note that $\dim |A| = 0$ for any $\Lambda \in M$.

Definition 3.2. The $\tau$-function $\tau : M \to L$ is defined by

(i) $\tau(E^i_n) = \tau^i_n$,

(ii) $\tau(w.A) = w.\tau(A)$ for any $\Lambda \in M$ and $w \in W$.

Here we suppose that an element $w \in W$ acts on a rational function $R(a, \tau) \in L$ as $w.R(a, \tau) = R(a.w, \tau.w)$, that is, $w$ acts on the independent variables from the right. By applying the Weyl group action given in Theorem 3.1 we can determine inductively the explicit formula of $\tau(\Lambda)$ as a rational function in $\tau^i_n$.

Alternatively below we shall present a geometric (and $a$ priori) characterization of our $\tau$-functions. Let us first prepare some notations; let $\zeta = (\zeta^0_1 : \zeta^\infty_1, \ldots, \zeta^0_N : \zeta^\infty_N)$ be the homogeneous coordinates of $(\mathbb{P}^1)^N$ such that

$$f_n = \frac{\zeta^0_n}{\zeta^\infty_n}. \quad (3.7)$$

For a multiple index $m = (m^0_1, m^\infty_1, \ldots, m^0_N, m^\infty_N) \in (\mathbb{Z}_{\geq 0})^{2N}$, we set $\zeta^m = \prod_n (\zeta^{0}_n)^{m^0_n}(\zeta^{\infty}_n)^{m^\infty_n}$. An element $\Lambda = \sum d_n H_n - \sum \mu'_n E^i_n \in M$ corresponds to a unique hypersurface of degree $d = (d_1, \ldots, d_N)$ passing through $C'_n$ with multiplicity $\mu'_n$. Let

$$\Phi^m(\zeta) = \sum_m A_m \zeta^m \in \mathbb{K}[\zeta] \quad (3.8)$$

and through (3.4), respectively.
be the defining polynomial of the hypersurface, where \( A_m \in \mathbb{K} \) is a monic monomial in multiplicative root variables. Normalize \( \Phi_\Lambda \) as its coefficients satisfy the following condition:

\[
\prod_m A_m^{(1/\theta)^m} = 1
\]  

(3.9)

with \((1/\theta)^m = \prod_n (1/\theta_n)^{m_n} (1/\theta_\infty)^{m_\infty}\). A few examples of the normalized defining polynomial \( \Phi_\Lambda(\zeta) \) (\( \Lambda \in \mathcal{M} \)) are

\[
\Phi_{E_1} = 1,
\]

\[
\Phi_{H_n-E_1} = u_n^{-\omega_0} \xi_n f_n + u_n,
\]

\[
\Phi_{H_n+H_{n+1}-E_1-E_n-E_{n+1}} = u_n^{\omega_0(\omega_{n+1}-1)} \xi_n^{\omega_n \omega_{n+1} \omega_{n+1}} f_n f_{n+1} + u_n f_n f_{n+1} + \frac{1}{v_n v_{n+1}}.
\]

Now assume that

\[
\xi_0^n = \xi_{n+1}^n \eta_n^{-1}, \quad \xi_\infty^n = \xi_{n-1}^n \eta_{n+1}.
\]  

(3.10)

We then have an expression of the \( \tau \)-function \( \tau(\Lambda) \) in terms of the normalized defining polynomial \( \Phi_\Lambda(\zeta) \) of the corresponding divisor \( \Lambda \).

**Theorem 3.3.** For any \( \Lambda = \sum_n d_n H_n - \sum_{i,j} \mu_{ij}^m E_i j \in \mathcal{M} \), we have the equality

\[
\tau(\Lambda) \prod_{n,j} (\tau^n_{ij})^{\mu_n} = \Phi_\Lambda(\zeta).
\]  

(3.11)

Hence, one can in principle trace the resulting value with respect to any iteration of Weyl group actions by means of the defining polynomials of appropriate divisors. We see from (3.11) that \( \tau(\Lambda) \) is a Laurent polynomial in the indeterminates \( \tau^n_{ij} \), though, from the definition of \( \tau \)-functions, we can only state that \( \tau(\Lambda) \) is rational. This regularity is an interesting feature of our \( \tau \)-functions and should be relevant to the theory of infinite-dimensional integrable systems such as the KP hierarchy, UC hierarchy, etc.; see §5. Note that the \( \tau \)-function can be considered as a counterpart of height functions in the sense that each of the original inhomogeneous coordinates \( f = (f_1, f_2, \ldots, f_N) \) of \((\mathbb{P}^1)^N\) is recovered as a ratio of \( \tau \)-functions; see (3.6).

**Proof of Theorem 3.3** We will prove the theorem by induction. If \( \Lambda = H_n - E_n^{\pm 1} = s_n^0(E_n^{\pm 1}) \in \mathcal{M} \), then (3.11) follows immediately from (3.5a) and (3.5b). We assume that (3.11) is true for \( \Lambda \in \mathcal{M} \).

Then it is enough to verify for \( \Lambda' = w(\Lambda) \) such that \( w \in \{s_i^0\} \) is a generator of \( \mathcal{W} = W(T_n^\mathbb{Z}) \). The action of \( s_i^0 (i \neq 0) \) is just a permutation of \( \tau \)-variables, so we need only to concentrate upon the nontrivial case \( w = s_n^0 \).

For \( \Lambda = \sum_n d_n H_n - \sum_{i,j} \mu_{ij}^m E_i j \in \mathcal{M} \), let us introduce a polynomial \( \varphi_\Lambda(f) \in \mathbb{K}[f] \) in the inhomogeneous coordinates \( f = (f_1, \ldots, f_N) \) of degree \( d = (d_1, \ldots, d_N) \) defined by \( \varphi_\Lambda(f) = \Phi_\Lambda(\zeta) \prod_n (\xi_n)^{-d_n} \).

**Claim.** We have

\[
s_n^0(\varphi_\Lambda(f)) = c(f_n + u_n)^{-d_{n-1} + \mu_n} f_n + 1/v_n)^{-d_{n+1} + \mu_n} \varphi_{n,\Lambda}(f),
\]  

(3.12)

where the normalizing constant \( c \) is given by \( c = (u_n^{-d_{n-1} - \mu_n} v_n^{-d_{n+1} + \mu_n})^{\omega_n} \).
To prove this, first we notice that \( s_0^n \cdot \Lambda = \Lambda + \langle \Lambda, d_0 \rangle a_0 = \Lambda + (d_{n-1} + d_{n+1} - \mu_n^{-1} - \mu_n^{-1}) \alpha_0 \), accordingly the degree \( d' = (d'_1, \ldots, d'_n) \) of \( \varphi_{\alpha_0, \Lambda}(f) \) reads \( d'_n = d_{n-1} + d_n + d_{n+1} - \mu_n^{-1} - \mu_n^{-1} \) and \( d'_i = d_i \) \( (i \neq n) \). The multiplicity of \( C^1_n = \{ f_{n-1} = 0, f_n = -u_n, f_{n+1} = \infty \} \) on the hypersurface \( \{ \varphi_\Lambda(f) = 0 \} \subset (\mathbb{P}^1)^n \) is given by \( \text{ord}_{C^1_n}(\varphi_\Lambda) = \mu_n^{-1} \). Hence the defining polynomial \( \varphi_\Lambda(f) \) is of the form
\[
\varphi_\Lambda(f) = \sum_{i \geq 0} \psi_i f_{-d_{n-1}+i},
\]
where \( \psi_i = \psi_i(f) \) is homogeneous of degree \( i \) in \( (f_{n-1}, f_n + u_n, f_{n+1}) \). In the same way, from \( \text{ord}_{C^1_n}(\varphi_\Lambda) = \mu_n^{-1} \), we can also describe it as \( \varphi_\Lambda = \sum_{i \geq 0} \bar{\psi}_i f_{-d_{n-1}+i} \) where \( \bar{\psi}_i \) is homogeneous of degree \( i \) in \( (1/f_{n-1}, f_n, 1/v_n, f_{n+1}) \). By applying (3.11), we see that \( s_n^0(\varphi_\Lambda) \times (f_n + u_n d_{n-1}^{-1} \mu_n^{-1} \phi_n + 1/v_n) d_{n-1}^{-1} \mu_n^{-1} \) is a polynomial of degree at most \( d' \) and therefore equals \( (\text{const}) \times \varphi_{\alpha_0, \Lambda} \). Here we used implicitly the fact that \( s_n^0 \) is a pseudo isomorphism (Theorem 2.2) and transforms \( \{ \varphi_\Lambda(f) = 0 \} \) generically into \( \{ \varphi_{\alpha_0, \Lambda}(f) = 0 \} \). Since (3.11) keeps the normalization (3.9) invariant, we can immediately calculate the constant and thus arrive at (3.12).

Applying \( s_n^0 \) to both sides of (3.11) and using the claim above, we obtain
\[
\begin{align*}
(\text{LHS}) & \quad \xrightarrow{s_n^0} \tau(\Lambda') \prod_{i,j} (\tau_n^{\mu_i})^{d'_{ij}} \times \left( \frac{s_n^0(\tau_n^{-1})}{\tau_n} \right)^{\mu_n^{-1}} \\
(\text{RHS}) & \quad = \varphi_\Lambda(f)^{d'} \prod_j (\zeta_j^\infty)^{d'_{ij}} \times \left( \frac{s_n^0(\tau_n^{-1})}{\tau_n} \right)^{\mu_n^{-1}} \frac{(s_n^0(\tau_n^{-1}))}{\tau_n} \left( (\tau_n^{-1})^{d_{n-1}} - d_{n-1} + d_{n+1} \mu_n^{-1} \mu_n^{-1} \right).
\end{align*}
\]
Here we have used the formulae \( f_n + u_n = u_n^{d_{n-1}} s_n^0(\tau_n^{-1}) \tau_n^1 / \zeta_j^\infty \) and \( f_n + 1/v_n = v_n^{-d_{n-1}} s_n^0(\tau_n^{-1}) \tau_n^{-1} / \zeta_j^\infty \). Thus, we get
\[
\Phi_{\Lambda'}(\zeta) = \varphi_\Lambda(f)^{d'} \prod_j (\zeta_j^\infty)^{d'_{ij}} = \tau(\Lambda') \prod_{i,j} (\tau_n^{\mu_i})^{d'_{ij}} \times (\tau_n^{-1})^{d_{n-1} + d_{n+1} - \mu_n^{-1} \mu_n^{-1}},
\]
which is exactly (3.11) for \( \Lambda' = s_n^0 \cdot \Lambda \). The proof of the theorem is complete.

Remark 3.4. Let us extendly apply (3.11) to the effective divisor classes \( D_0^0 = H_n - \sum_{i=1}^{k-1} E_{i+1}^i - \sum_{j=1}^{k-1} E_{n-1}^{-j} \) and \( D_\infty^0 = H_n - \sum_{i=1}^{k-1} E_{i-1}^i - \sum_{j=1}^{k+1} E_{n+1}^{-j} \), which correspond to the hyperplanes \( \{ \zeta_n = 0 \} \) and \( \{ \zeta_n^\infty = 0 \} \), respectively; thus, we obtain
\[
\begin{align*}
\tau(D_n^0 \xi_{n+1}) \eta_{n-1} = \xi_n^0 & \quad \text{and} \quad \tau(D_\infty^0 \xi_{n-1}) \eta_{n+1} = \xi_n^\infty.
\end{align*}
\]
Because \( D_0^0 \) and \( D_\infty^0 \) are invariant under the action of Weyl group \( W = W(T_{\Lambda'}^\Lambda) \), we can define \( \tau(D_n^0) = \tau(D_\infty^0) \equiv 1 \), which affirms the assumption (3.10).
4 Examples of affine case and \(q\)-Painlevé equations

We assume that the Dynkin diagram \(T_k^A\) is of affine type. Then the lattice part of the affine Weyl group \(W(T_k^A)\) provides an interesting class of discrete dynamical systems of Painlevé type. This is based on the following facts:

(i) **Existence of commuting time evolutions** For each root \(\alpha \in Q\), we have the translation \(t_\alpha\) acting on the Néron-Severi bilattice \(N(X) \cong (H^2(X, \mathbb{Z}), H_2(X, \mathbb{Z}))\) by the Kac formula [Kac90, §6.5]:

\[
\begin{align*}
t_\alpha(\Lambda) &= \Lambda - \langle \Lambda, \tilde{\delta} \rangle \alpha + \left( \langle \Lambda, \tilde{\alpha} \rangle - \frac{1}{2} \langle \alpha, \tilde{\alpha} \rangle \langle \Lambda, \tilde{\delta} \rangle \right) \delta, \\
t_\alpha(\lambda) &= \lambda - \langle \delta, \lambda \rangle \tilde{\alpha} + \left( \langle \alpha, \tilde{\lambda} \rangle - \frac{1}{2} \langle \alpha, \tilde{\alpha} \rangle \langle \delta, \lambda \rangle \right) \tilde{\delta}
\end{align*}
\]

for \(\Lambda \in H^2(X, \mathbb{Z})\) and \(\lambda \in H_2(X, \mathbb{Z})\). Here \(\delta = -\frac{1}{2}K_X\) and \(\tilde{\delta} = -\frac{1}{2}K_X\) are \(W(T_k^A)\)-invariants (Lemma 2.1) and play the role of the null vector and its dual, respectively. The additivity property of translations \(t_\alpha t_\beta = t_{\alpha+\beta}\) holds. This implies that the discrete dynamical system

\[
t_\alpha(f_i) = F_{\alpha,i}(a, f) \quad (i = 1, \ldots, N)
\]

corresponding to a root \(\alpha \in Q\) has a set of commuting discrete time evolutions \(\{t_\beta | \beta \in Q\}\), where \(F_{\alpha,i}(a, f) \in \mathbb{K}(f)\) is a rational function explicitly determined by Theorem 2.2; cf. [NY98].

(ii) **Quadratic degree growth** From (i), we have

\[
(t_\alpha)^n(H_i) = t_{n\alpha}(H_i)
\]

\[
= H_i - n\langle H_i, \tilde{\delta} \rangle \alpha + \left( n\langle H_i, \tilde{\alpha} \rangle - \frac{n^2}{2} \langle \alpha, \tilde{\alpha} \rangle \langle H_i, \tilde{\delta} \rangle \right) \delta.
\]

Therefore the value of pairing \(\langle (t_\alpha)^n(H_i), h_j \rangle\) is in the order \(n^2\), that is, the degree of \((t_\alpha)^n(f_i)\) with respect to \(f_i\) is at most \(n^2\).

Generally, a quadratic degree growth asserts integrability of the dynamical system under consideration; in [Tak01], it was proved by a similar technique as above that the degree growth of every second order difference Painlevé equation (in Sakai’s classification [Sak01]) is quadratic. Also, as shown in [Tak04], a certain class of integrable higher dimensional dynamics defined on generalized Del Pezzo varieties has a quadratic degree growth. More elementarily, the degree of the \(n\)-th iteration of an addition formula of an elliptic curve grows quadratically as a rational mapping; see, e.g., [Sil86].

In this section, we shall demonstrate the \(A^{(1)}_r\) and \(D^{(1)}_r\) cases as examples.

4.1 The case of \(A^{(1)}_{N-1}\)

Let us consider the case where \(k = \ell = (1, 1, \ldots, 1) \in (\mathbb{Z}_{>0})^N\) so that the corresponding Dynkin diagram \(T_k^A\) is of type \(A^{(1)}_{N-1}\). We have, from Theorems 2.2 and 3.1 a tropical representation of affine Weyl group \(W(A^{(1)}_{N-1})\) as pseudo isomorphisms of a rational variety \(X_\ell\). Moreover, we can extend \(W(A^{(1)}_{N-1})\) to \(W(A^{(1)}_{N-1}) \times W(A^{(1)}_1)\) in the following manner.
First we reformulate the parameters \( u_n \) and \( v_n \) as follows:
\[
    u_n = \frac{a_n b_1}{b_0}, \quad v_n = \frac{a_n b_0}{b_1},
\]
where \( a_n \) and \( b_i \) (\( i = 0, 1 \)) are nonzero parameters such that
\[
    \prod_{n=1}^{N} a_n = (b_0 b_1)^N = q^N. \tag{4.4}
\]

Here \( a_n \) and \( b_i \) will play roles of multiplicative root variables of \( A_{N-1}^{(1)} \) and \( A_1^{(1)} \), respectively.

Let \( \gamma_{1,n} \) and \( \tilde{\gamma}_{1,n} \) be a pair of vectors defined by
\[
    \gamma_{1,n} = -\frac{K_X}{2} - H_n - E_n^{-1} + E_n^{-1} + E_n^{-1} + E_{n+1}^{-1} \in H^2(X, \mathbb{Z}),
\]
\[
    \tilde{\gamma}_{1,n} = h_n - e_n^{-1} \in H_2(X, \mathbb{Z})
\]
for \( n \in \mathbb{Z}/N\mathbb{Z} \). We see that these are mutually orthogonal \((-2)\)-vectors, namely, \( \langle \gamma_{1,i}, \tilde{\gamma}_{1,j} \rangle = -2 \delta_{i,j} \). Define the action of the reflection \( r_{1,n} \) associated with the pair \( (\gamma_{1,n}, \tilde{\gamma}_{1,n}) \) on \( N(X) \cong (H^2(X, \mathbb{Z}), H_2(X, \mathbb{Z})) \) in the same way as \((2.1)\). Let \( \beta_i = \sum_{n=1}^{N} \gamma_{1,n} \) and \( \tilde{\beta}_i = \sum_{n=1}^{N} \tilde{\gamma}_{1,n} \) and define the associated reflection \( r_i \) by \( r_1 = r_{1,1} \circ \cdots \circ r_{1,N} \). In addition, we take an involution \( \iota : E_n^1 \leftrightarrow E_n^{-1}, e_n^0 \leftrightarrow e_n^{-1} \) of \( N(X) \); let \( (\beta_0, \tilde{\beta}_0) = \iota(\beta_1, \tilde{\beta}_1) \) and define \( r_0 = \iota \circ r_1 \circ \iota \). Then we see that \( \langle r_0, r_1 \rangle \cong W(A_1^{(1)}) \) and \( \iota \) realizes the Dynkin diagram automorphism of \( A_1^{(1)} \). Likewise, we can introduce a diagram automorphism \( \pi \) of \( A_{N-1}^{(1)} \) acting on \( N(X) \) as \( \pi : H_n \mapsto H_{n+1}, E_n^1 \mapsto E_n^{1 \pm 1}, h_n \mapsto h_{n+1}, e_n^1 \mapsto e_n^{1 \pm 1} \). Note that \( (\beta_i, \tilde{\beta}_i) \) \((i = 0, 1)\) is orthogonal to \( (\alpha_n, \tilde{\alpha}_n) \), where \( \alpha_n = \alpha_n^0 = h_n - E_n^{-1} - E_{n+1}^{-1} \) and \( \tilde{\alpha}_n = \tilde{\alpha}_n^0 = h_{n-1} + h_{n+1} - e_n^1 - e_n^{-1} \) are the (co-)root bases of \( A_{N-1}^{(1)} \). Furthermore, we have in fact
\[
    Q \oplus \mathbb{Z} \beta_1 = \left( \bigoplus_{n=1}^{N} \mathbb{Z} \alpha_n \right) \oplus \mathbb{Z} \beta_1 = \{ d_n^0, d_n^\infty \} \subset H^2(X, \mathbb{Z}),
\]
\[
    \hat{Q} \oplus \mathbb{Z} \tilde{\beta}_1 = \left( \bigoplus_{n=1}^{N} \mathbb{Z} \tilde{\alpha}_n \right) \oplus \mathbb{Z} \tilde{\beta}_1 = \{ D_n^0, D_n^\infty \} \subset H_2(X, \mathbb{Z}).
\]

In order to realize \( r_i \) \((i = 0, 1)\) as birational transformations, we give the action of \( r_1 \) on the cohomology basis of \( H^2(X, \mathbb{Z}) \):
\[
    r_1(E_n^1) = E_n^1 + \gamma_{1,n} = \sum_{i\neq n} H_i - \sum_{j=1}^{N} E_j^1 - \sum_{k\neq n, k \pm 1} E_k^{-1},
\]
\[
    r_1(E_n^{-1}) = E_n^{-1} \quad \text{and} \quad r_1(H_n) = H_n + \gamma_{1,n}. \text{For} \Lambda = r_1(E_1^1), \text{the normalized defining polynomial reads}
\]
\[
    \Phi_\Lambda(\zeta) = G_n(\zeta) := \left( 1 + \sum_{j=1}^{N-1} \prod_{i=1}^{j} \frac{u_{n+i}}{v_{n+j}} \right)^{N-1} (u_{n+k})^{-1 + k/N} \zeta_{n+k}. \tag{4.3}
\]

By virtue of Theorem \(3.3\) we thus obtain a birational action of \( r_1 \) on \( \tau \)-variables:
\[
    r_1(\tau_n^1) = \frac{G_n(\zeta)}{\prod_j \tau_j \prod_{k\neq n, k \pm 1} \tau_k}, \quad r_1(\tau_n^{-1}) = \tau_n^{-1}.
\]
The following birational transformations
\[
\{s_n, \pi, r_i, \iota\}
\]
realize the extended affine Weyl group \( W = \tilde{W}(A_{N-1}^{(1)}) \times \tilde{W}(A_1^{(1)}) \):

\[
s_n(\tau_n^1) = \frac{v_n^{1/2} \tau_n^{1} \tau_{n+1}^{1-1} + v_n^{-1/2} \tau_n^{1} \tau_{n+1}^{1-1}}{\tau_n^1}, \quad (4.6a)
\]

\[
s_n(\tau_n^{-1}) = \frac{u_n^{-1/2} \tau_n^{-1} \tau_{n+1}^{1-1} + u_n^{1/2} \tau_n^{-1} \tau_{n+1}^{1-1}}{\tau_n^{-1}}, \quad (4.6b)
\]

\[
r_i(\tau_n^1) = \frac{G_n(\zeta)}{\prod_j \tau_j^1 \prod_{k\neq n \pm 1} \tau_k^1}, \quad (4.6c)
\]

\[
\pi(\tau_n^1) = \tau_{n+1}^1, \quad \iota(\tau_n^1) = \tau_n^{-1}, \quad (4.6d)
\]

and \( r_0 = \iota \circ r_1 \circ \iota \). Consequently, with respect to variables \( f_n = (\tau_n^{1+1} \tau_{n+1}^{-1})/(\tau_n^{1-1} \tau_{n+1}^{-1}) \), we have the birational transformations of the form

\[
s_n(f_{n-1}) = a_n f_{n-1} f_n + 1/v_n, \quad s_n(f_{n+1}) = \frac{f_{n+1}}{a_n} f_n + u_n, \quad r_i(f_n) = \frac{g_{n+1}(f)}{f_{n+1} g_n(f)}, \quad \pi(f_n) = f_{n+1}, \quad \iota(f_n) = 1/f_n,
\]

where \( g_n(f) = (1 + \sum_{j=1}^{N-1} \prod_{i=1}^j u_{n+i}/f_{n+i}) \prod_{k=1}^{N-1} (u_{n+k})^{-1+k/N}. \) This is in fact equivalent to the representation due to Kajiwara et al. [KNY02a, KNY02b], which was originally discovered, in connection with a \( q \)-analogue of the KP hierarchy, without any algebro-geometric setup. The birational action \( f_n \mapsto \bar{f}_n = r_1 \circ \iota(f_n) \) of the translation \( r_1 \circ \iota : (b_1, b_0; a_n) \mapsto (qb_1, b_0/q; a_n) \) is the \( q \)-Painlevé equation of type \( A_{N-1}^{(1)} \):

\[
\bar{f}_n = f_{n+1} \frac{g_{n-1}(f)}{g_{n+1}(f)} \quad (n \in \mathbb{Z}/N\mathbb{Z}).
\]

The family \( X = \bigcup_a X_a \) of rational varieties provides a sort of phase space for the dynamics of \( (4.7) \).

### 4.2 The case of \( D_{N+2}^{(1)} \)

We first need to consider a preliminary situation to derive the tropical representation of \( W(D_{N+2}^{(1)}) \) \((N \geq 3)\); for instance, let \( k = \ell = (2, 1, 1, \ldots, 1, 2, 1) \in (\mathbb{Z}_{>0})^N \). Next we shall ignore the simple reflection \( s_N^{0_N} \) corresponding to the \( N \)-th vertex and, simultaneously, put \( a_N^0 = 1 \) and \( \tau_N^1 = \tau_N^{-1} = 1 \).

Write the Dynkin diagram of type \( D_{N+2}^{(1)} \) as follows:

```
0 1 2 3 ... N-1 N N+2
```

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According to the labels of the diagram above, we introduce the multiplicative root variables \( a = (a_0, \ldots, a_{N+2}) \) and define the action of simple reflections \( s_i \) as \( s_i(a_j) = a_j a_i^{-C_{ij}} \), where \( C_{ij} \) is the Cartan matrix of type \( D_{N+2}^{(1)} \). Set
\[
  u_1 = a_1 a_0^{-1/2} a_{N+2}^{1/2}, \quad v_1 = a_1 a_0^{1/2} a_{N+2}^{-1/2}, \\
  u_n = v_n = a_n \quad (2 \leq n \leq N-2), \\
  u_{N-1} = a_{N-1} a_0^{-1/2} a_{N+1}^{1/2}, \quad v_{N-1} = a_{N-1} a_0^{1/2} a_{N+1}^{-1/2}.
\]
From Theorem 3.1, we thus have the following birational transformations \( s_n \) (0 \leq n \leq N + 2) on the field \( \mathbb{C}(a^{1/4})(\tau) \) of \( \tau \)-variables:
\[
  s_n(\tau_n^1) = \frac{v_n^{1/2} \xi_{n+1} \eta_{n-1} + v_n^{-1/2} \xi_{n-1} \eta_{n+1}}{\tau_n^1}, \\
  s_n(\tau_n^{-1}) = \frac{u_n^{-1/2} \xi_{n+1} \eta_{n-1} + u_n^{1/2} \xi_{n-1} \eta_{n+1}}{\tau_n^1}
\]
for \( 1 \leq n \leq N - 1 \), and
\[
  s_0(\tau_1^{1,2}) = \tau_1^{1,2}, \quad s_{N+2}(\tau_1^{-1,-2}) = \tau_1^{-1,-2}, \quad s_{N}(\tau_{N-1}^{1,2}) = \tau_{N-1}^{1,2}, \quad s_{N+1}(\tau_{N-1}^{-1,-2}) = \tau_{N-1}^{-1,-2},
\]
where
\[
  (\xi_n, \eta_n) = \begin{cases}
    (\tau_n^1 \tau_n^{-1}, \tau_n^{-1} \tau_n^1) & \text{if } n = 1, N - 1, \\
    (\tau_n^1, \tau_n^{-1}) & \text{if } 2 \leq n \leq N - 2, \\
    (1, 1) & \text{if } n = 0, N.
  \end{cases}
\]
When \( N = 3 \) (\( D_5^{(1)} \) case), the above representation in \( \tau \)-variables coincides with that of [TM06 §1]. So, the birational action on \( f \)-variables of a translation part of the Weyl group yields the \( q \)-Painlevé VI equation [JS96, Sak01]. For general \( N \geq 3 \), one may regard it as a (higher order) generalization of the \( q \)-Painlevé VI equation equipped with \( W(D_{N+2}^{(1)}) \) symmetry.

**Remark 4.1.** Via the change of variables (3.6), we also have a tropical representation of \( W(D_{N+2}^{(1)}) \) acting on \( f = (f_1, \ldots, f_N) \). The resulting dynamical system looks naively of \( N \) dimensional. However, (i) if \( N \) is odd, it possesses a conserved quantity \( \prod_{n=1}^{N} f_n \); (ii) if \( N \) is even, it does two ones \( \prod_{n=1}^{N/2} f_{2n} \) and \( \prod_{n=1}^{N/2} f_{2n-1} \). Hence the dimension of the dynamical system is essentially \( N - 1 \) (resp. \( N - 2 \)) if \( N \) is odd (resp. even). For instance, we remember that the \( q \)-Painlevé VI equation (\( D_5^{(1)} \) case) is of second order.

## 5 \( \tau \)-Functions and character polynomials

In this section, for the case of \( A_{N-1}^{(1)} \), we try to interpret from a soliton-theoretic point of view the relationship between our \( \tau \)-functions and character polynomials, i.e., the Schur functions or the universal characters.

### 5.1 Universal characters

For a pair of partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_s) \), the universal character \( S_{[\lambda, \mu]}(x, y) \) is a polynomial in \( (x, y) = (x_1, x_2, \ldots, y_1, y_2, \ldots) \) defined by the determinant formula
of twisted Jacobi-Trudi type \[ [\text{Koi89}] \]:

\[
S_{[\lambda,\mu]}(x, y) = \det \left( \begin{array}{ccc} p_{\lambda_1, i} & p_0(y) & p_{-1}(y) \\ p_{\lambda_2, i} & p_2(x) & p_3(x) \\ p_{\lambda_3, i} & p_0(x) & p_1(x) \end{array} \right)_{1 \leq i \leq r'}, \tag{5.1}
\]

where \( p_n \) is a polynomial defined by the generating function \( \sum_{k \in \mathbb{Z}} p_k(x)z^k = \exp \left( \sum_{n=1}^{\infty} x_n z^n \right) \) and \( p_n(x) = 0 \) for \( n < 0 \). The Schur function \( S_{\lambda}(x) \) (see, e.g., \[Mac95\]) is regarded as a special case of the universal character:

\[
S_{\lambda}(x) = \det(p_{\lambda_i-j+i}(x)) = S_{[\lambda,\emptyset]}(x, y).
\]

If we count the degree of the variables as \( \deg x_n = n \) and \( \deg y_n = -n \), then \( S_{[\lambda,\mu]}(x, y) \) is a weighted homogeneous polynomial of degree \( |\lambda| - |\mu| \), where \( |\lambda| = \lambda_1 + \cdots + \lambda_r \). The universal character \( S_{[\lambda,\mu]} \) describes the irreducible character of a rational representation of the general linear group \( GL(n; \mathbb{C}) \) corresponding to a pair of partitions \( [\lambda,\mu] \), while the Schur function \( S_{\lambda} \) does that of a polynomial representation corresponding to a partition \( \lambda \); see \[Koi89\] for details.

**Example 5.1.** For \( \lambda = (2, 1) \) and \( \mu = (1) \), we have

\[
S_{[\lambda(2, 1),\mu(1)]}(x, y) = \begin{vmatrix} p_1(y) & p_0(y) & p_{-1}(y) \\ p_1(x) & p_2(x) & p_3(x) \\ p_{-1}(x) & p_0(x) & p_1(x) \end{vmatrix} = \begin{pmatrix} x_1^3 - 3x \end{pmatrix} y_1 - x_1^2.
\]

### 5.2 Notations

Let us recall some terminology relevant to the case at hand. A subset \( m \subset \mathbb{Z} \) is said to be a Maya diagram if \( i \in m \) (for \( i \ll 0 \)) and \( i \notin m \) (for \( i \gg 0 \)). Each Maya diagram \( m = \{1, \ldots, m_3, m_2, m_1\} \) corresponds to a unique partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1 \). For a sequence of integers \( \nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbb{Z}^N \), we consider the Maya diagram

\[
m(\nu) = (N\mathbb{Z}_{\leq \nu_1} + 1) \cup (N\mathbb{Z}_{\leq \nu_2} + 2) \cup \cdots \cup (N\mathbb{Z}_{\leq \nu_N} + N),
\]

and denote by \( \lambda(\nu) \) the corresponding partition. We call a partition of the form \( \lambda(\nu) \) an \( N \)-core partition. It is well-known that a partition \( \lambda \) is \( N \)-core if and only if \( \lambda \) has no hook with length of a multiple of \( N \). See \[Nou04\].

**Example 5.2.** If \( N = 3 \) and \( \nu = (2, 0, 3) \) then the resulting partition reads \( \lambda(\nu) = (4, 2, 1, 1) \).

Also, we prepare the following standard notations of \( q \)-analysis \[GR90\].

- **\( q \)-shifted factorials:**

\[
(z; q)_0 = \prod_{i=0}^{\infty} (1 - zq^i), \quad (z; p, q)_\infty = \prod_{i,j=0}^{\infty} (1 - zp^i q^j),
\]

and \( (z_1, \ldots, z_r; q)_\infty = (z_1; q)_\infty \cdots (z_r; q)_\infty \), etc.

- **Elliptic gamma function:**

\[
\Gamma(z; p, q) = \frac{(pqz^{-1}; p, q)_\infty}{(z; p, q)_\infty}.
\]
5.3 τ- Functions in terms of character polynomials

In what follows, we shall only treat the case of $A_{\ell-1}^{(1)}$ (see § 4.1). Introduce the lattice $P(A_{\ell-1}) = \mathbb{Z}^\ell/\mathbb{Z}(e_1 + \cdots + e_\ell)$ of rank $\ell - 1$, called the weight lattice of type $A_{\ell-1}$. Here $\{e_i\}$ refers to the canonical basis of $\mathbb{Z}^\ell$. Note that $P(A_{\ell}) \cong \mathbb{Z}$. For an element $\Lambda \in M = \mathcal{W}(E_{\ell}^+)$ of the Weyl group orbit, we can associate a sequence of integers $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}^N$ and an integer $\kappa \in \mathbb{Z}$ through the formulae

$$v_i - v_{i+1} = \langle \Lambda, \delta_i \rangle \quad (i \neq N), \quad v_N - v_1 + 1 = \langle \Lambda, \delta_N \rangle \quad \text{and} \quad \kappa = \langle \Lambda, \delta_0 \rangle.$$  

Since $\nu \in \mathbb{Z}^N$ is uniquely determined modulo $\mathbb{Z}(e_1 + \cdots + e_N)$, we can regard $(\nu, \kappa)$ as a point of $P(A_{\ell-1}) \times P(A_{1})$. In fact, this is a one-to-one correspondence between $M$ and $P(A_{\ell-1}) \times P(A_{1})$.

Now let us index the $\tau$ functions as

$$\tau(\Lambda) =: \sigma^\nu_\nu, \quad (5.2)$$

For example, we see that $\tau(E_n^i) = \sigma^0_{e_1 + \cdots + e_n}$ and $\tau(E_n^{-i}) = \sigma^1_{e_1 + \cdots + e_n}$. We have in general

$$T_n(\sigma^\nu_\nu) = \sigma^\nu_{\nu e_n} \quad (n \in \mathbb{Z}/N\mathbb{Z}), \quad \bar{T}(\sigma^\nu_\nu) = \sigma^{\nu+1}_\nu,$$

where $T_n = \pi \circ s_{n+1} \circ \cdots \circ s_{n} \circ s_{n}$ and $\bar{T} = r_1 \circ t$ are the translations of affine Weyl groups $\bar{W}(A_{\ell-1}^{(1)})$ and $\bar{W}(A_1^{(1)})$, respectively. As previously seen in § 3, the $\tau$-function $\sigma^\nu_\nu$ is expressed as a rational function in the indeterminates $\tau^1_1$ and $\tau^{-1}_i$ ($i \in \mathbb{Z}/N\mathbb{Z}$) by means of the defining polynomial of the corresponding divisor. However, substituting appropriate special values for $\tau^1_1$ and $\tau^{-1}_i$, we have another expression of the $\tau$-function in terms of the character polynomials.

**Theorem 5.3** (cf. Kajiwara-Noumi-Yamada [KNY02b]). Define

$$F(t) = \frac{(q^{2N} t^{2N}; q^{2N} t^{-2N})_\infty}{(q^2 t^2; q^2)_\infty},$$

$$H_\nu = \prod_{(i,j) \in \lambda(\nu)} (q^{h_{ij}} - q^{-h_{ij}}) q^{(i-j)/2},$$

where $h_{ij}$ is the hook-length, that is, $h_{ij} = \lambda_i + \lambda_j' - i - j + 1$. Under the specialization $a_i = q$, $\tau^1_1 = F(1/b_0)$ and $\tau^{-1}_i = F(b_1)$ for all $i \in \mathbb{Z}/N\mathbb{Z}$, the $\tau$-function $\sigma^\nu_\nu$ is expressed in terms of the Schur function attached to the $N$-core partition $\lambda(\nu)$ as

$$\sigma^\nu_\nu = F(t) H_\nu S_{\lambda(\nu)}(x), \quad (5.3)$$

where $t = q^\kappa/b_0$ and

$$x_n = \frac{t^n + t^{-n}}{n(q^n - q^{-n})} \quad (n = 1, 2, \ldots).$$

This theorem is based on the principle that appropriate reductions of infinite-dimensional integrable systems, such as the KP hierarchy, yield Painlevé-type equations. In fact, the Weyl group actions (4.6) lead us to the bilinear relation among $\tau$-functions $\sigma^\nu_\nu \in \mathbb{E}(\tau)$ of the form

$$q^{N(\nu - |\nu| - 1)} \left( \prod_{j=1}^N \left( \frac{a_{i+j-1}}{q} \right) \right)^{i/N} (t^n - \frac{1}{t^n}) \sigma^\nu_\nu \sigma^\nu_{\nu e_i} = t \sigma^{\nu-1}_{\nu e_i} \sigma^{\nu+1}_{\nu e_i} - \frac{1}{t} \sigma^{\nu+1}_{\nu e_i} \sigma^{\nu-1}_{\nu e_i}, \quad (5.4)$$
where \( t = q^x/b_0 \). The bilinear relation \([5,4]\) above is exactly the similarity reduction of the \( q \)-KP hierarchy. Since the Schur function \( S_f \) is a homogeneous solution of the \( q \)-KP hierarchy and is thus compatible with the similarity reduction, we can immediately verify Theorem 5.3. Moreover, when \( N \) is even, \([5,4]\) can also be obtained, alternatively, from the \( q \)-UC hierarchy by a similarity reduction \([Tsu04, Tsu05a]\). Therefore, the following theorem is a direct consequence of the fact that the universal character \( S_{[l,\Omega]} \) is homogeneous and solves the \( q \)-UC hierarchy.

**Theorem 5.4.** In the case \( N = 2g + 2 \ (g = 1, 2, \ldots) \), define

\[
F(c, t) = \frac{(-c q^3 t^2, -c^{-1} q^3 t^2; q^2, q^4)_\infty (q^{4g+4} t^{4g+4}; q^4, q^{4g+4})_\infty}{(q^4 t^4; q^4, q^4)_\infty} \\
\times \Gamma(-c^{1/2} q^{3/2} t, -c^{-1/2} q^{3/2} t; q, q^2) \Gamma(-q^{2g} t^{2g}; q^{2g}, q^{2g}),
\]

\[
H_v = c_v^{(|(v_{\text{even}}) - |(v_{\text{odd}})|)/2} \prod_{(i,j) \in (v_{\text{odd}})} (q^{2h_{ij}} - q^{2h_{ij}}) \prod_{(i,j) \in (v_{\text{even}})} (q^{2h_{ij}} - q^{2h_{ij}}),
\]

with \( c_v = cq^{2(|v_{\text{even}}| - |v_{\text{odd}}|)} \), \( v_{\text{even}} = (v_2, v_4, \ldots, v_{2g+2}) \), \( v_{\text{odd}} = (v_1, v_3, \ldots, v_{2g+1}) \) and \( c \) being an arbitrary nonzero constant. Under the specialization \( a_{2i+1} = c, a_{2i} = q^2/c \), \( \tau_{2i} \) is \( \tilde{F}(c, 1/b_0) \), \( \tau_{2i-1} \) is \( \tilde{F}(c, b_1) \), \( \tau_{2i+1} = \tilde{F}(c/q^2, 1/b_0) \) and \( \tau_{2i+1} = \tilde{F}(c/q^2, b_1) \) for all \( i \in \mathbb{Z}/(g + 1) \mathbb{Z} \), the \( \tau \)-function \( \sigma_{\chi}^x \) is expressed in terms of the universal character attached to a pair of \((g + 1)\)-core partitions as

\[
\sigma_{\chi}^x = \tilde{F}(c_v, t) H_v S_{[|\lambda_{\text{odd}}|, |\lambda_{\text{even}}|]}(x, y),
\]

where \( t = q^x/b_0 \) and

\[
x_n = \frac{t^{2n} + t^{-2n} - (-c_v)^n (q^n + q^{-n})}{n(q^{2n} - q^{-2n})}, \quad y_n = \frac{t^{2n} + t^{-2n} - (-c_v)^{-n} (q^n + q^{-n})}{n(q^{2n} - q^{-2n})} \quad (n = 1, 2, \ldots).
\]

Taking into account the origin of our realization of Weyl groups, Theorems 5.3 and 5.4 exemplify a quite curious coincidence between algebraic geometry of particular rational varieties and representation theory. Similar results have been established also for the \( D_{5}^{(1)} \) and \( E_{6}^{(1)} \) cases \([Tsu05b, TM06]\) by the use of the \( q \)-UC hierarchy. The authors expect there should exist another geometric understanding of such a relationship between \( \tau \)-functions and character polynomials, without employing the theory of integrable systems.

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