The canonical partition function for relativistic hadron gases

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Abstract

Particle production in high-energy collisions is often addressed within the framework of the thermal (statistical) model. We present a method to calculate the canonical partition function for the hadron resonance gas with exact conservation of the baryon number, strangeness, electric charge, charmness and bottomness. We derive an analytical expression for the partition function which is represented as series of Bessel functions. Our results can be used directly to analyze particle production yields in elementary and in heavy ion collisions. We also quantify the importance of quantum statistics in the calculations of the light particle multiplicities in the canonical thermal model of the hadron resonance gas.

1 Introduction

The statistical model of the hadron resonance gas was shown to be very successful in addressing particle production in high-energy collisions \textsuperscript{[123456789101112]}. In central nucleus-nucleus collisions the formulation of conservation of quantum numbers (sometimes generically referred to as charges) in the grand canonical (GC) ensemble is well suited \textsuperscript{[234561011]}. However, for small systems like $e^+e^-$ or pp as well as for peripheral nucleus-nucleus collisions only the canonical (C) ensemble gives the correct description of hadron yields \textsuperscript{[1789101112]}. In the GC formulation the conservation of charges is implemented on the average and is controlled by the appropriate chemical potentials whereas in the C-ensemble charges are conserved exactly \textsuperscript{[1]}.

The exact treatment of quantum numbers in statistical mechanics is well established \textsuperscript{[13141516171819]}. It is in general obtained by projection of the partition function on
the desired values of the conserved charges by using group theoretical methods [15,17,18,20,21]. For studies of hadron production one needs in general to account for the exact conservation of five quantum numbers: the baryon number $N$, strangeness $S$, electric charge $Q$, charmness $C$, and bottomness $B$. The conservation is a direct consequence of the $G = U_N(1) \otimes \ldots \otimes U_B(1)$ invariance of strong interactions.

The implementation of an exact conservation of quantum numbers in the canonical partition function for the hadron resonance gas requires integration over the symmetry group [1,15]. This in general leads to numerical problems due to the oscillatory behavior of the integrand. To avoid numerical problems, methods were developed to express the partition function as series of Bessel functions which is well suited for numerical implementations [22,23,24,25]. An explicit analytical expression for the canonical partition function using the above methods was obtained up to now only for three conserved quantum numbers i.e. for $N$, $S$ and $Q$ and employing Boltzmann statistics [24].

In the application of the statistical model to hadron production, e.g. in $e^+e^-$ annihilation, one must employ the formulation of the canonical partition function which accounts for the conservation of all five quantum numbers including charmness and bottomness since hadrons carrying heavy quarks contribute sizably to the overall hadron yields [7,9,10,26]. In addition, for $e^+e^-$ collisions, the data are measured with high precision; see [26] for a recent summary. Consequently, in the statistical model analysis the approximation of particle momentum distributions using Boltzmann statistics is not sufficient [9,27].

The main scope of this paper is the extension of the previous results for the canonical partition function for a hadron gas [24] to include the exact conservation of all quantum numbers carried by the light and heavy quarks [28]. We will formulate the canonical partition function as series of the modified Bessel functions. Our expression is numerically stable and can be used to quantitify thermodynamics independently of the values of the thermal parameters and the initial quantum numbers. Our result accounts for the quantum statistics for bosons. For all fermions the masses are significantly larger than the temperature. Consequently, the implementation of Boltzmann statistics for fermions is a very good approximation. An extension of our results to quantum statistics for all particles including fermions would however be quite straightforward.

As model implementation of the canonical partition function we illustrate its application to the calculations of hadron yields. We quantify the quantum statistics effects for the multiplicity of pions and kaons. We also discuss deviations from the exact quantum statistics results when the Boltzmann approximation is used for different values of the thermal parameters. The hadron resonance gas we are working with is containing all known hadrons, including the multi-strange hyperons up to $S = \pm 3$ and all charmed and bottom hadrons, as listed in the latest compilation by the Particle Data Group [26].

The paper is organized as follows: in the next Section we derive the analytical expression for the canonical partition function that accounts for the conservation of five quantum numbers. In Section 3 we present a particular numerical implementation to calculate the pion and kaon multiplicities and discuss the role of quantum statistics. We summarize our results in Section 4.
2 The canonical partition function

The appropriate tool to deal in a statistical mechanics framework with a system of quantum numbers \( \vec{X} = (N, S, Q, C, B) \) related with the \( G = U_N(1) \otimes \ldots \otimes U_B(1) \) symmetry group is the canonical partition function \([17]\) \((\hbar = c = 1)\):

\[
Z_{N,S,Q,C,B}(\vec{X}) = \frac{1}{(2\pi)^5} \int d^5\vec{\phi} \, e^{i\vec{X}\vec{\phi}} \exp\left\{ \sum_j g_j V \left( \frac{1}{2\pi} \int d^3p \ln(1 \pm e^{-\sqrt{p^2 + m_j^2} - i\vec{x}_j\vec{\phi}})^\pm \right) \right\} \tag{1}
\]

where the vector \( \vec{X} = (N, S, Q, C, B) \) characterizes the initial quantum numbers of a system related with each of the \( U(1) \) symmetry groups and the \( \pm \) sign refers to fermions and bosons. The exact quantum number conservation is implemented by the integration over the group \( G \) with \( \vec{\phi} = (\phi_N, \phi_S, \phi_Q, \phi_C, \phi_B) \) being an element of \( G \). The vector \( \vec{x}_j = (N_j, S_j, Q_j, C_j, B_j) \) describes the quantum numbers of a particle \( j \) with mass \( m_j \) and the spin-isospin degeneracy factor \( g_j \). The sum in the exponential is taken over all particles and resonances which carry quantum numbers related with an internal symmetry \( G \).

The partition function (1) depends only on two parameters, the temperature \( T \) and the volume \( V \) of the system. However, the integral representation of this partition function is very inconvenient for numerical implementations. This is particularly the case for non-vanishing initial quantum numbers due to the oscillatory nature of the integrand. An additional complication appears due to quantum statistics effects. Clearly, one could simplify the problem and employ the Boltzmann approximation by keeping only the first term of the series expansion of the \( \ln(1 \pm x)^\pm \) in the exponential. However, such an approximation is only valid if the particle mass is significantly larger than the temperature. As the values of temperatures extracted from the fits of hadron abundances obtained in heavy ion and elementary collisions \([1,2,7,9,10]\) are close to the pion mass, the Boltzmann approximation will give rise to deviations from the correct quantum statistics values in particular for light particles like pions or kaons. To calculate the partition function accurately (using quantum statistics) one needs in general to include the whole series of the expansion of the logarithm

\[
\ln(1 \pm x)^\pm = \sum_{k=1}^{\infty} \frac{(\pm 1)^k \vec{x}^k}{k} \tag{2}
\]

in the partition function for all particle species. However, as the fermions are heavy, the error caused by the Boltzmann approximation in this case is very small. Therefore we will use the whole series of the logarithm only for light bosons (mesons without charm or bottom quarks) and apply the usual Boltzmann approximation to fermions and heavy bosons (bosons containing charm or bottom quarks). With this approximation equation (1) becomes:

\[
Z_{N,S,Q,C,B}(\vec{X}) = \frac{1}{(2\pi)^5} \int d^5\vec{\phi} \, e^{i\vec{X}\vec{\phi}} \exp\left\{ \sum_j z_j^1 e^{-i\vec{x}_j\vec{\phi}} + \sum_b \sum_{k=2}^{\infty} z_b^k e^{-ik\vec{x}_b\vec{\phi}} \right\} \tag{3}
\]
with the particle partition function

$$z_j^k = \frac{g_j V}{k(2\pi)^3} \int d^3 p \ e^{-\frac{\sqrt{\vec{p}^2+m_j^2}}{T}}$$

(4)

where \( j \) runs over all particles, while \( b \) runs only over the light bosons.

For charm and bottom hadrons, because of their large masses the term \( \sum_{j_{c/b}} z_{j_{c/b}}^{1} \) is much less than 1. Consequently, we can use the following approximation [7]:

$$\exp \left\{ \sum_{j_{c/b}} z_{j_{c/b}}^{1} e^{-i\vec{x}_j \vec{\phi}} \right\} \approx 1 + \sum_{j_{c/b}} z_{j_{c/b}}^{1} e^{-i\vec{x}_j \vec{\phi}}$$

(5)

Inserting now Eq. (5) into Eq. (3) we obtain:

$$Z_{N,S,Q,C,B}(\vec{X}) \approx \frac{1}{(2\pi)^5} \int d^5 \vec{\phi} \ e^{i\vec{X} \vec{\phi}} e^f(\vec{\phi})$$

$$+ \sum_{j_{c}} z_{j_{c}}^{1} \frac{1}{(2\pi)^5} \int d^5 \vec{\phi} \ e^{i(\vec{X}-\vec{x}_{j_{c}}) \vec{\phi}} e^f(\vec{\phi})$$

$$+ \sum_{j_{b}/\bar{C}_{j_{b}}=0} z_{j_{b}}^{1} \frac{1}{(2\pi)^5} \int d^5 \vec{\phi} \ e^{i(\vec{X}-\vec{x}_{j_{b}}) \vec{\phi}} e^f(\vec{\phi})$$

$$+ \sum_{j_{c}} \sum_{\bar{C}_{j_{b}}=0} z_{j_{c}}^{1} z_{j_{b}}^{1} \frac{1}{(2\pi)^5} \int d^5 \vec{\phi} \ e^{i(\vec{X}-\vec{x}_{j_{c}}-\vec{x}_{j_{b}}) \vec{\phi}} e^f(\vec{\phi})$$

(6)

with

$$f(\vec{\phi}) = \sum_{j} z_{j}^{1} e^{-i\vec{x}_j \vec{\phi}} + \sum_{b} \sum_{k=2}^{\infty} z_{b}^{k} e^{-ik\vec{x}_b \vec{\phi}}.$$  

(7)

The index \( j \) in Eq. (7) runs over all hadrons except those which carry heavy flavors whereas in Eq. (6) \( j_{c} \) runs over all charm and \( j_{b} \) over all bottom hadrons.

From Eq. (6) it is transparent that the integrals over \( \phi_C \) and \( \phi_B \) related with the charm and bottom quantum numbers contribute to the partition function as Kronecker delta functions. Consequently,

$$Z_{N,S,Q,C,B}(\vec{X}) \approx \frac{1}{(2\pi)^3} \int d^3 \vec{\phi} \ e^{i\vec{X} \vec{\phi}} e^f(\vec{\phi}) \delta_{C,0} \delta_{B,0}$$

$$+ \sum_{j_{c}} z_{j_{c}}^{1} \frac{1}{(2\pi)^3} \int d^3 \vec{\phi} \ e^{i(\vec{X}-\vec{x}_{j_{c}}) \vec{\phi}} e^f(\vec{\phi}) \delta_{C,j_{c}} \delta_{B,j_{c}}$$

$$+ \sum_{j_{b}/\bar{C}_{j_{b}}=0} z_{j_{b}}^{1} \frac{1}{(2\pi)^3} \int d^3 \vec{\phi} \ e^{i(\vec{X}-\vec{x}_{j_{b}}) \vec{\phi}} e^f(\vec{\phi}) \delta_{C,0} \delta_{B,j_{b}}$$

$$+ \sum_{j_{c}} \sum_{\bar{C}_{j_{b}}=0} z_{j_{c}}^{1} z_{j_{b}}^{1} \frac{1}{(2\pi)^3} \int d^3 \vec{\phi} \ e^{i(\vec{X}-\vec{x}_{j_{c}}-\vec{x}_{j_{b}}) \vec{\phi}} e^f(\vec{\phi}) \delta_{C,j_{c}} \delta_{B,j_{c}+j_{b}}$$

(8)

The particle partition function \( z_{j_{c/b}}^{1} \) is \( \mathcal{O}(10^{-4}) \) for charm particles and \( \mathcal{O}(10^{-13}) \) for bottom particles.
where $\vec{X}$ and $\vec{x}_j$ are now three-dimensional vectors composed of the baryon number, the strangeness and the electric charge, while the charmness $C$ and bottomness $B$ appear only through the Kronecker functions.

With the approximation (5) for charm and bottom contributions to the generating functional we have reduced the five dimensional integrations to three dimensional ones in the canonical partition function. The integrals to compute are of the following generic form:

$$I_{N,S,Q} = \frac{1}{(2\pi)^3} \int_0^{2\pi} d^3\vec{\phi} \ e^{i\vec{X} \cdot \vec{\phi}} e^{i\vec{X} \cdot \vec{\phi}} \ e^{i\vec{X} \cdot \vec{\phi}}$$

and correspond to the canonical partition function with the conservation of three quantum numbers that accounts for the quantum statistics of bosons.

### Table 1

The combinations of quantum numbers ($N=$baryon number, $S=$strangeness, $Q=$electric charge) for hadrons and the corresponding notation for the sum of particle partition functions for each hadron class, $Z_{\text{hadr}}$. The correspondence of $Z_{\text{hadr}}$ to the index $n_j$ in Eq. 13 (see text) is also given.

| Quantum numbers | $Z_{\text{hadr}}$ | Index $n_j$ |
|-----------------|-------------------|-------------|
| $N=0 \ S=0 \ Q=0$ | $Z_0$ | - |
| $N=0 \ S=1 \ Q=0$ | $Z_{K^0}$ | - |
| $N=1 \ S=0 \ Q=0$ | $Z_0$ | - |
| $N=0 \ S=0 \ Q=1$ | $Z_{\pi^\mp}$ | - |
| $N=1 \ S=0 \ Q=1$ | $Z_p$ | $n_1$ |
| $N=1 \ S=0 \ Q=-1$ | $Z_{\Delta^\mp}$ | $n_2$ |
| $N=1 \ S=0 \ Q=2$ | $Z_{\Delta^{++}}$ | $n_3$ |
| $N=0 \ S=1 \ Q=1$ | $Z_{K^\pm}$ | $n_4$ |
| $N=1 \ S=-1 \ Q=0$ | $Z_{\Lambda}$ | $n_5$ |
| $N=1 \ S=-1 \ Q=1$ | $Z_{\Sigma^+}$ | $n_6$ |
| $N=1 \ S=-1 \ Q=-1$ | $Z_{\Sigma^-}$ | $n_7$ |
| $N=1 \ S=-2 \ Q=0$ | $Z_{\Xi^0}$ | $n_8$ |
| $N=1 \ S=-2 \ Q=-1$ | $Z_{\Xi^\mp}$ | $n_9$ |
| $N=1 \ S=-3 \ Q=-1$ | $Z_{\Omega^\mp}$ | $n_{10}$ |

The integral representation of $I_{N,S,Q}$ is not convenient for numerical analysis as the integrand is a strongly oscillatory function, particularly for large initial quantum numbers $N$, $S$ or $Q$. Following the methods described in Refs. [22,23] and [24] we express $I_{N,S,Q}$ as a series of Bessel-functions. First, we observe that, in the argument of the exponential function in Eq. (9), the particles appear pairwise with their anti-particles. Second, the contributions of all particles in the sum can be grouped into 14 categories defined by their quantum numbers, see Table 1 for definitions. For instance $Z_{K^0}$ describes the sum over all partition functions $z_j$ of particles $j$ with $N=0$, $S=1$ and $Q=0$. Consequently, we rewrite
Applying the relation (9) as follows:

\[
I_{N,S,Q} = \exp(Z_0) \frac{1}{2\pi} \int_0^{2\pi} d\phi_N e^{iN\phi_N} \exp[Z_n(e^{i\phi_N} + e^{-i\phi_N})] \\
\frac{1}{2\pi} \int_0^{2\pi} d\phi_S e^{iS\phi_S} \exp[Z_{K'}(e^{i\phi_S} + e^{-i\phi_S})] \\
\frac{1}{2\pi} \int_0^{2\pi} d\phi_Q e^{iQ\phi_Q} \exp[Z_{\pi\pm}(e^{i\phi_Q} + e^{-i\phi_Q})] \\
\exp[Z_p(e^{i(\phi_N+\phi_Q)} + e^{-i(\phi_N+\phi_Q)})] \\
\exp[Z_{\Delta^+}(e^{i(\phi_N-\phi_Q)} + e^{-i(\phi_N-\phi_Q)})] \\
\exp[Z_{\Delta^+}(e^{i(\phi_N+2\phi_Q)} + e^{-i(\phi_N+2\phi_Q)})] \\
\exp[Z_{K^\pm}(e^{i(\phi_S+\phi_Q)} + e^{-i(\phi_S+\phi_Q)})] \\
\exp[Z_A(e^{i(\phi_N-\phi_S)} + e^{-i(\phi_N-\phi_S)})] \\
\exp[Z_{\Sigma^+}(e^{i(\phi_N-\phi_S+\phi_Q)} + e^{-i(\phi_N-\phi_S+\phi_Q)})] \\
\exp[Z_{\Sigma^-}(e^{i(\phi_N-\phi_S-\phi_Q)} + e^{-i(\phi_N-\phi_S-\phi_Q)})] \\
\exp[Z_{\Xi}(e^{i(\phi_N-2\phi_S-\phi_Q)} + e^{-i(\phi_N-2\phi_S-\phi_Q)})] \\
\exp[Z_{\Omega}(e^{i(\phi_N-3\phi_S-\phi_Q)} + e^{-i(\phi_N-3\phi_S-\phi_Q)})] \\
\exp \left[ \sum_{k=2}^{\infty} Z^k_{\pi\pm}(e^{ik\phi_Q} + e^{-ik\phi_Q}) \right] \\
\exp \left[ \sum_{h=2}^{\infty} Z^h_{K'}(e^{ih\phi_S} + e^{-ih\phi_S}) \right] \\
\exp \left[ \sum_{l=2}^{\infty} Z^l_{K^\pm}(e^{il(\phi_S+\phi_Q)} + e^{-il(\phi_S+\phi_Q)}) \right].
\]

Applying the relation:

\[
\exp \left[ \frac{x}{2} \left( t + \frac{1}{t} \right) \right] = \sum_{m=-\infty}^{\infty} t^m I_m(x) \tag{11}
\]

and the integral representation of the Bessel-function of order \( h \):

\[
I_h(x) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \exp(x \cos \phi) \exp(-ih\phi), \tag{12}
\]

the group integrals in Eq. (10) can be done explicitly, yielding:

\[
I_{N,S,Q} = \exp(Z_0) \prod_{j=1}^{10} \left[ \sum_{n_j=\infty}^{\infty} I_{n_j}(2Z_{hadr}) \right] \prod_{k=2}^{\infty} \left[ \sum_{m_k=\infty}^{\infty} I_{m_k}(2Z^k_{\pi\pm}) \right] \prod_{h=2}^{\infty} \left[ \sum_{m_h=\infty}^{\infty} I_{m_h}(2Z^h_{K'}) \right] \\
\prod_{l=2}^{\infty} \left[ \sum_{m_l=\infty}^{\infty} I_{m_l}(2Z^l_{K^\pm}) \right] I_{-\nu_1}(2Z_{K'}) I_{-\nu_2}(2Z_{n}) I_{-\nu_3}(2Z_{\pi\pm}), \tag{13}
\]

with
\[
\nu_1 = Q + n_1 - n_2 + 2n_3 + n_4 + n_6 - n_7 - n_9 - n_{10} + \sum k m_k + \sum l m_l
\]
\[
\nu_2 = S + n_4 - n_5 - n_6 - n_7 - 2n_8 - 2n_9 - 3n_{10} + \sum h m_h + \sum l m_l
\]
\[
\nu_3 = N + n_1 + n_2 + n_3 + n_5 + n_6 + n_7 + n_8 + n_9 + n_{10},
\]

and with the specific \( Z_{\text{hadr}} \) pertaining to each index \( n_j \) according to Table 1. The above equation is a generalization of that for a canonical partition function with exact conservation of baryon number, electric charge and strangeness [24] to the case where quantum statistics for bosons is explicitly included.

To obtain the partition function conserving five quantum numbers we use Eq. (9) to transform Eq. (8) into:

\[
Z_{N,S,Q,C,B}(\vec{X}) \approx I_{N,S,Q} \delta_{C,0} \delta_{B,0} + \sum_{j} c_j \zeta_{1j} I_{N+S+jc,Q+jc} \delta_{C,c} \delta_{B,0} + \sum_{j} \sum_{c_j} \zeta_{1j} \sum_{c_{jb}} \zeta_{1jb} I_{N+S+jc+S+jb,Q+jc+jb} \delta_{C,c} \delta_{B,b} + \sum_{j} \sum_{c_{jb}} \zeta_{1jb} \sum_{c_{jbc}} \zeta_{1jbc} I_{N+S+jc+S+jb+S+jc+jb} \delta_{C,c} \delta_{B,b} \delta_{C,b} \delta_{B,0}.
\]

This expression, together with (13) is our final result for the partition function that accounts for exact conservation of baryon number, electric charge, strangeness, charmness and bottomness. The partition function (14), contrary to its integral representation (1), is free from oscillations and is numerically stable independent of the values of the thermal parameters or the values of initial quantum numbers.

The partition function (14) can be used to describe thermodynamical properties of the hadron resonance gas under constraints of the exact conservation of all relevant quantum numbers. In particular, from Eq. (14) we obtain the multiplicity \( \langle n_j \rangle \) for hadron species \( j \) by introducing a fugacity parameter \( \lambda_j \) which multiplies the particle partition function \( z_j \) and by differentiating:

\[
\langle n_j \rangle = \left. \frac{\partial \ln Z_{N,S,Q,C,B}}{\partial \lambda_j} \right|_{\lambda_j=1}.
\]

For bosons, e.g. for \( \pi^\pm \), one obtains from Eqs. (14) and (15),

\[
\langle n_{\pi^\pm} \rangle = \left. \frac{\partial \ln Z}{\partial \lambda_{\pi^\pm}} \right|_{\lambda_{\pi^\pm}=1} = \sum_{k=1}^{\infty} k z_{\pi^\pm} \frac{Z_{N,S,Q,C,B}(\vec{X} - k \vec{r}_{\pi^\pm})}{Z_{N,S,Q,C,B}(\vec{X})}
\]

whereas for fermions there is only one term contributing \( (k=1) \) because fermions are well approximated by Boltzmann statistics.
3 Numerical results

We have applied the above obtained analytical expression for the canonical partition function (14) to quantify particle production in $e^+e^-$ annihilations at LEP energies [9,29]. We will not repeat the discussion given there but focus, in the following on an illustration of the importance of quantum statistics in the calculation of multiplicities of light bosons.

![Graph showing the deviation of the $\pi^+$ and $K^+$ yields from the quantum statistical value caused by the use of Boltzmann statistics as a function of the index $k$ ($h$, $l$) (see Eq. 2).]

Figure 1. The deviation of the $\pi^+$ and $K^+$ yields from the quantum statistical value caused by the use of Boltzmann statistics as a function of the index $k$ ($h$, $l$) (see Eq. 2).

Fig. 1 shows relative deviations of pion and kaon multiplicities from their quantum statistics values with increasing numbers of terms $k$ in the expansion (2). The calculations were performed for $T = 157$ MeV and $V = 32$ fm$^3$, values that are relevant to freezeout conditions in $e^+e^-$ annihilation. It is clear from this figure that the Boltzmann approximation is by far not sufficient to reproduce the quantum statistics results. The pion yield under Boltzmann approximation deviates by more than 7% from the exact quantum statistics result. For kaons this difference is only 1%. While for kaons such an error is comparable to the 1 $\sigma$ error of the data in $e^+e^-$ collisions, for pions the deviation is significantly larger than the error in the data [26]. This underlines the importance of using quantum statistics for the calculation of multiplicities of light mesons. For pions, several terms are needed in the expansion (16) to achieve a precision well below 1%. It is also clear from Fig. 1 that the deviations depend on the mass of the particles and decrease quickly with increasing mass. For protons, the lightest fermions, the corresponding deviation from Fermi-Dirac statistics is below 0.1% already with only the first term, substantiating the applicability of the Boltzmann approximation for all fermions.

In Fig. 2 we illustrate the relative error of the calculated multiplicity of pions and kaons using the partition function in the Boltzmann approximation as a function of $VT^3$. Deviations from the exact quantum statistics values are seen to increase with $VT^3$. This is due to the contribution of higher order terms in the expansion (16), which are suppressed for small $VT^3$. For light bosons the largest deviations from the quantum statistics appear for large values of $VT^3$ i.e. when the system approaches GC thermodynamics.
Figure 2. Relative deviations of the pion and kaon multiplicities calculated using Boltzmann statistics from their quantum statistics values as a function of $VT^3$.

4 Conclusions

We have presented a method to calculate the canonical partition function for the hadron resonance gas that accounts for exact conservation of baryon number, electric charge, strangeness, charmness and bottomness. We have taken into account quantum statistics for light bosons and applied the Boltzmann approximation for fermions and heavier bosons. The results obtained here are an extension of previous studies which were restricted to the conservation of only three quantum numbers within the Boltzmann approximation [24]. Our analytical expression for the partition function, which is represented as a series of Bessel functions, is stable in numerical implementations. It can be used for any value of the initial quantum numbers of the system and for arbitrary thermal parameters. As an application of our results we have discussed the importance of quantum statistics in the calculations of pion and kaon multiplicities. The canonical partition function which we have derived can be used to analyze different thermodynamical properties of the hadron resonance gas with the constraint of exact quantum number conservation. It can be also used to describe particle production in elementary and in heavy ion collisions within the statistical thermal model.

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