A CHARACTERIZATION OF MAXIMAL IDEALS IN THE FRÉCHET ALGEBRAS OF HOLOMORPHIC FUNCTIONS \( F_p \) \((1 < p < \infty)\)

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Abstract. The space \( F_p \) \((1 < p < \infty)\) consists of all holomorphic functions \( f \) on the open unit disk \( D \) for which \( \lim_{r \to 1} (1 - r)^{1/p} \log^+ M(r, f) = 0 \), where \( M(r, f) = \max_{|z| \leq r} |f(z)| \) with \( 0 < r < 1 \). Stoll [5, Theorem 3.2] proved that the space \( F_p \) with the topology given by the family of seminorms \( \{ \| \cdot \|_{q,c} \} \) \( c > 0 \) defined for \( f \in F_q \) as \( \|f\|_{q,c} := \sum_{n=0}^{\infty} |a_n| \exp(-cn^{1/(q+1)}) < \infty \), is a countably normed Fréchet algebra.

Notice that for each \( p > 1 \), \( F_p \) is the Fréchet envelope of the Privalov space \( N_p \). In this paper we study the structure of maximal ideals in the algebras \( F_p \) \((1 < p < \infty)\).

1. Introduction, Preliminaries and Results

Let \( D \) denote the open unit disk in the complex plane and let \( T \) denote the boundary of \( D \). Let \( L^q(T) \) \((0 < q \leq \infty)\) be the familiar Lebesgue space on the unit circle \( T \).

The Privalov class \( N_p \) \((1 < p < \infty)\) is defined as the set of all holomorphic functions \( f \) on \( D \) such that
\[
\sup_{0 < r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \frac{d\theta}{2\pi} < +\infty
\]
holds, where \( \log^+ |a| = \max\{\log |a|, 0\} \). These classes were firstly considered by Privalov in [1, pages 93–10], where \( N_p \) is denoted as \( A_q \).

Notice that for \( p = 1 \), the condition (1) defines the Nevanlinna class \( N \) of holomorphic functions in \( D \). Recall that the Smirnov class \( N^+ \) is the set of all functions \( f \) holomorphic on \( D \) such that
\[
\lim_{r \to 1} \int_0^{2\pi} |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < +\infty
\]
where \( f^* \) is the boundary function of \( f \) on \( T \); that is,
\[
f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})
\]
is the radial limit of \( f \) which exists for almost every \( e^{i\theta} \in T \). We denote by \( H^q \) \((0 < q \leq \infty)\) the classical Hardy space on \( D \).

It is known (see [2, 3, 4]) that the following inclusion relations hold:
\[
N^r \subset N^p \quad (r > p), \quad \bigcup_{q > 0} H^q \subset \bigcap_{p > 1} N^p, \quad \text{and} \quad \bigcup_{p > 1} N^p \subset N^+ \subset N,
\]
where the above containment relations are proper.

The study of the spaces \( N^p \) \((1 < p < \infty)\) was continued in 1977 by M. Stoll [5] (with the notation \((\log^+ H)^a\) in [5]). Further, the topological and functional properties of these spaces have been studied by several authors (see [2, 6, 7, 8] and [9–23]).
M. Stoll [5, Theorem 4.2] proved that for each $p > 1$ the space $N^p$ (with the notation $(\log^+ H)^\alpha$ in [5]) equipped with the topology given by the metric $d_p$ defined by

$$d_p(f, g) = \left( \int_0^{2\pi} \left( \log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in N^p,$$

becomes an $F$-algebra, that is, $N^p$ is an $F$-space (a complete metrizable topological vector space with the invariant metric) in which multiplication is continuous.

Recall that the function $d_1 = d$ defined on the Smirnov class $N^+$ by (5) with $p = 1$ induces the metric topology on $N^+$. N. Yanagihara [24] showed that under this topology, $N^+$ is an $F$-space.

In connection with the spaces $N^p (1 < p < \infty)$, Stoll [5] (see also [6] and [18, Section 3]) also studied the spaces $F^q (0 < q < \infty)$ (with the notation $F_1^/q$ in [5]), consisting of those functions $f$ holomorphic on $D$ such that

$$\lim_{r \to 1} (1 - r)^{1/q} \log^+ M_\infty(r, f) = 0,$$

where

$$M_\infty(r, f) = \max_{|z| \leq r} |f(z)| \quad (0 < r < 1).$$

Here, as always in the sequel, we will need some Stoll’s results concerning the spaces $F^q$ only with $1 < q < \infty$, and hence, we will assume that $q = p > 1$ be any fixed number.

**Theorem 1** (see [5, Theorem 2.2]). Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a holomorphic function on $D$. Then the following statements are equivalent:

(a) $f \in F^p$;

(b) there exists a sequence $\{c_n\}_n$ of positive real numbers with $c_n \to 0$ such that

$$|a_n| \leq \exp \left( c_n n^{1/(p+1)} \right), \quad n = 0, 1, 2, \ldots;$$

(c) for any $c > 0$,

$$\|f\|_{p,c} := \sum_{n=0}^{\infty} |a_n| \exp \left( -cn^{1/(p+1)} \right) < \infty.$$  

**Remark 2.** Notice that in view of Theorem 1 ((a)$\iff$(c)), by (10) it is well defined the family of seminorms $\{\| \cdot \|_{p,c}\}_{c>0}$ on $F^p$.

Recall that a locally convex $F$-space is called a Fréchet space, and a Fréchet algebra is a Fréchet space that is an algebra in which multiplication is continuous. Stoll [5] also proved the following result.

**Theorem 3** (see [5, Theorem 3.2]). The space $F^q (0 < q < \infty)$ equipped with the topology given by the family of seminorms $\{\| \cdot \|_{q,c}\}_{c>0}$ defined for $f \in F^q$ as

$$\|f\|_{q,c} := \sum_{n=0}^{\infty} |a_n| \exp \left( -cn^{1/(q+1)} \right) < \infty,$$

is a countably normed Fréchet algebra.

For our purposes, we will need the following result which characterizes the topological dual of the space $F^p$.

**Theorem 4** (see [5, Theorem 3.3]). If $\gamma$ is a continuous linear functional on $F^p$, then there exists a sequence $\{\gamma_n\}_n$ of complex numbers with

$$\gamma_n = O\left( \exp \left( -cn^{1/(p+1)} \right) \right), \quad \text{for some } c > 0,$$
such that
\[ \gamma(f) = \sum_{n=0}^{\infty} a_n \gamma_n, \]  
(12)

where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p \), with convergence being absolute. Conversely, if \( \{\gamma_n\}_n \) is a sequence of complex numbers for which
\[ \gamma_n = O \left( \exp \left( -cn^{1/(p+1)} \right) \right), \]  
(13)

then (13) defines a continuous linear functional on \( F^p \).

Notice that the Privalov space \( N^p \) \((1 < p < \infty)\) is not locally convex (see [6, Theorem 4.2] and [12, Corollary]), and hence, \( N^p \) is properly contained in \( F^p \). Moreover, \( N^p \) is not locally bounded (see [19, Theorem 1.1]). Moreover, Stoll showed ([5, Theorem 4.3]) that for each \( p > 1 \) \( N^p \) is a dense subspace of \( F^p \) and the topology on \( F^p \) defined by the family of seminorms (10) is weaker than the topology on \( N^p \) given by the metric \( d_p \) defined by (5). Furthermore, Eoff showed ([6, Theorem 4.2, the case \( p > 1 \)]) that \( F^p \) is the Fréchet envelope of \( N^p \). For more information on Fréchet envelope, see [25, Theorem 1], [22, Section 1] and [26, Corollary 22.3, p. 210].

Remark 5. For \( p = 1 \), the space \( F_1 \) has been denoted by \( F^+ \) and has been studied by N. Yanagihara in [27, 24]. It was shown in [27, 24] that \( F^+ \) is actually the containing Fréchet space for \( N^+ \), i.e., \( N^+ \) with the initial topology embeds densely into \( F^+ \), under the natural inclusion, and \( F^+ \) and the Smirnov class \( N^+ \) have the same topological duals.

Observe that the space \( F^p \) topologised by the family of seminorms \( \{\| \cdot \|_{p,c}\}_{c>0} \) given by (10) is metrizable by the metric \( \lambda_p \) defined as \( \lambda_p(f, g) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|f-g\|_{n^{1/n^p/(p+1)}}}{1+\|f-g\|_{n^{1/n^p/(p+1)}}} \) with \( f, g \in F^p \).

Since Privalov space \( N^p \) and its Fréchet envelope \( F^p \) \((1 < p < \infty)\) are algebras, they can be also considered as rings with respect to the usual ring’s operations addition and multiplication. Notice that these two operations are continuous on \( N^p \) and \( F^p \) because the spaces \( N^p \) and \( F^p \) become \( F \)-algebras.

Motivated by several results on the ideal structure of some spaces of holomorphic functions given in [28, 2, 12] and [29-35], related investigations for the spaces \( N^p \) \((1 < p < \infty)\) and their Fréchet envelopes were given in [2, 9, 12, 30, 18] and [23]. Note that a survey of these results was given in [37]. The \( N^p \)-analogue of the famous Beurling’s theorem for the Hardy spaces \( H^q \) \((0 < q < \infty)\) [29] was proved in [36]. Moreover, it was proved in [31, Theorem B]) that \( N^p \) \((1 < p < \infty)\) is a ring of Nevanlinna–Smirnov type in the sense of Mortini. The structure of closed weakly dense ideals in \( N^p \) was established in [18]. The ideal structure of \( N^p \) and the multiplicative linear functionals on \( N^p \) were studied in [2] and [23, Theorem 1]. These results are similar to those obtained by Roberts and Stoll [31] for the Smirnov class \( N^+ \).

Motivated by results of Roberts and Stoll given in [31, Section 2] concerning a characterization of multiplicative linear functionals on \( F^+ \) and closed maximal ideals in \( F^+ \), in this paper we prove the analogous results for the spaces \( F^p \) \((1 < p < \infty)\) given by Proposition 5, Proposition 6, Theorem 7 and Theorem 8.

**Proposition 5.** Let \( \lambda \in \mathbb{D} \) and let \( \gamma_\lambda \) be a functional on \( F^p \) defined as
\[ \gamma_\lambda(f) = f(\lambda) \]  
(14)

for every \( f \in F^p \). Then \( \gamma_\lambda \) is a continuous multiplicative linear functional on \( F^p \).
For $\lambda \in \mathbb{D}$, we define
\[
\mathcal{M}_\lambda = \{ f \in F^p : f(\lambda) = 0 \}. \tag{15}
\]

**Proposition 6.** The set $\mathcal{M}_\lambda$ defined by (15) is a closed maximal ideal in $F^p$ for each $\lambda \in \mathbb{D}$.

**Theorem 7.** Let $\gamma$ be a nontrivial multiplicative linear functional on $F^p$. Then there exists $\lambda \in \mathbb{D}$ such that
\[
\gamma(f) = f(\lambda) \tag{16}
\]
for every $f \in F^p$. Moreover, $\gamma$ is a continuous map.

**Theorem 8.** Let $p > 1$ and let $\mathcal{M}$ be a closed maximal ideal in $F^p$. Then there exists $\lambda \in \mathbb{D}$ such that $\mathcal{M} = \mathcal{M}_\lambda$.

**2. Proof of the Results**

*Proof of Proposition 5.* Clearly, for each $\lambda \in \mathbb{D}$, $\gamma_\lambda$ is a multiplicative linear functional on the space $F^p$. In order to show that $\gamma_\lambda$ is a continuous functional on $F^p$, note that for any function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in F^p$ ($z \in \mathbb{D}$) we have
\[
\gamma_\lambda(f) = f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n. \tag{17}
\]
Clearly, the sequence $\{\gamma_n\}$ defined as $\gamma_n = \lambda^n$ $(n = 0, 1, 2 \ldots)$ satisfies the asymptotic condition (11) of Theorem 4. This together with the equality (17) implies that $\gamma_\lambda$ is a continuous functional on $F^p$, and the proof is completed. \qed

*Proof of Proposition 6.* Notice that in view of (15), $\mathcal{M}_\lambda$ is the kernel of the functional $\gamma_\lambda$ defined on $F^p$ by (14). From this and the fact that by Proposition 5, $\gamma_\lambda$ is a continuous multiplicative linear functional on the space $F^p$, we conclude that $\mathcal{M}_\lambda$ is a closed maximal ideal in $F^p$. \qed

*Proof of Theorem 7.* If we take $\gamma(z) = \lambda$, then $\gamma(z - \lambda) = 0$. If we suppose that $\lambda \notin \mathbb{D}$, then $z \mapsto 1/(z - \lambda)$ ($z \in \mathbb{D}$) is a bounded function on the closed unit disk $\overline{\mathbb{D}} : |z| \leq 1$. Therefore, $z \mapsto z - \lambda$ ($z \in \mathbb{D}$) is an invertible element of the algebra $F^p$. If $f$ is any invertible element in $F^p$, then $1 = \gamma(1) = \gamma(f) \gamma(f^{-1})$, and thus, $\gamma(f) \neq 0$. Especially, we have $\gamma(z - \lambda) \neq 0$. A contradiction, and hence, it must be $\lambda \in \mathbb{D}$. Then consider the set
\[
(z - \lambda)F^p := \{(z - \lambda)f(z) : f \in F^p\}. \tag{18}
\]
For each $\lambda \in \mathbb{D}$, let $\mathcal{M}_\lambda$ be a set defined by (15). Then obviously, $(z - \lambda)F^p \subset \mathcal{M}_\lambda$. Moreover, if $f \in \mathcal{M}_\lambda$, then by (6) and (7) easily follows that $f$ can be expressed as a product $f(z) = (z - \lambda)g(z)$ with $g \in F^p$. Therefore,
\[
\mathcal{M}_\lambda = (z - \lambda)F^p, \tag{19}
\]
whence it follows that
\[
\mathcal{M}_\lambda \subseteq \ker \gamma, \tag{20}
\]
where $\ker \gamma$ denotes the kernel of the functional $\gamma$. By Proposition 6, $\mathcal{M}_\lambda$ is a closed maximal ideal in $F^p$. This together with the inclusion relation (20) implies that $\mathcal{M}_\lambda = \ker \gamma$. Moreover, $\gamma(f) = f(\lambda)$ for all $f \in F^p$ and $\gamma$ is continuous on $F^p$ by Proposition 5. This completes the proof of the theorem. \qed

*Proof of Theorem 8.* We proceed as in [32, Theorem 2]. If we set $X = F^p/\mathcal{M}$, then in the terminology of Arens [38], $X$ is complete, metrizable, convex complex topological division algebra. Therefore, by [38], $X \cong \mathbb{C}$. Thus, there exists a multiplicative linear functional $\gamma$
on $F^p$ such that $M = \ker \gamma$. Then by Theorem 7, $M = M_\lambda$ for some $\lambda \in \mathbb{D}$, as asserted.

\[\square\]

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