Continuous occurrence theory

A new approach to model continuous phenomena

Abdorrahman Haeri

Abstract

Usually gradual and continuous changes in entities will lead to appear events. But usually it is supposed that an event is occurred at once. In this research an integrated framework called continuous occurrence theory (COT) is presented to investigate respective path leading to occurrence of the events in the real world. For this purpose initially fundamental concepts are defined. Afterwards, the appropriate tools such as “occurrence variables computations”, “occurrence dependency function” and “occurrence model” are introduced and explained in a systematic manner. Indeed, COT provides the possibility to: (a) monitor occurrence of events during time; (b) study background of the events; (c) recognize the relevant issues of each event; and (d) understand how these issues affect on the considered event. The developed framework (COT) provides the necessary context to analyze accurately continual changes of the issues and the relevant events in the various branches of science and business. Finally, typical applications of COT and an applied modeling example of it have been explained and a mathematical programming example is modeled in the occurrence based environment.

Keywords: Continuous occurrence theory (COT); gradual behavior; occurrence variable; occurrence state; occurrence degree; occurrence model

1. Introduction

In the real world, most of the events and phenomena have been occurred based on a gradual process. Indeed, they don’t occur at once. For example consider a train that is moving to the “station A”. The train doesn’t appear in the “station A” at once. Instead, it moves gradually in its path to reach to “station A.” For another example suppose a student that is currently studying at university. His/her graduation as an event doesn’t happen at once in a day
(probably graduation celebration day). Instead, it graduation is a step-by-step procedure that is performed during years. He/she studied different courses, did the related assignments, passed their exams, etc to realize “graduation” event. These examples show essence of the most of the events and their related changes. Therefore, it is obvious that events go through a continuous path during time in order to have been occurred.

Gradual behavior and event occurrence have been investigated in many disciplines and applications. Some of the previous researches proposed approaches and models to better explain event occurrence and gradual behavior. For example, Singer and Willett (1993) used survival analysis approach to investigate event occurrence. In this regard, statistical techniques was used to describe event occurrence during time. In addition, they identified a number of predictors that can be used effectively in hazard-related events. Cassez and Grastien (2013) studied predictability capability of event occurrence in time based systems to minimize time delay between prediction of an event occurrence and its occurrence in real world. This issue is considered in time automata. Verberne et al. (2000) considered approximate cases of diagnostic reasoning so that desired gradual behavior of approximate diagnostic methods have been considered to increase their applicability and flexibility. Kozak (1995) developed a system with gradual transition to increase the capability of discrete event system control. For this purpose, predictors were used to count events occurrence. Spezialetti and Kearns (1988) presented a general method to identify event occurrence in distributed computations case. The proposed method considered characteristic of the considered behavior and the environment that the event can occur in it. Time delay between event recognition and its occurrence is one of the important parameters in the proposed method. Chandra and kumar (2002) presented a methodology to improve failure diagnostic and control of discrete event systems to obtain models explaining complex discrete event systems using event occurrence rules. Overton and Gaucas (1990) present an approach to study event occurrence in fuzzy domain. For this purpose, both event start time and event duration is represented using fuzzy sets. In addition, confidence functions were applied to describe reasoning about event occurrences sequence.

A number of researches applying techniques and models in real application cases related to event occurrence and gradual behavior. For instance, Tardin et al. (2014) examined behavioral events and behavioral sates applying multivariate analysis of variance on dolphins. The results shown that a number of events are related with specified behavioral states. Baryarama et al. (2006) developed a mathematical model to explain HIV/AIDS dynamics
with change in gradual behavior. In this research, gradual behavior change is modeled with a variables by using of a time-related function. Teije and Harmelen (1998) discussed about features of problem solving approaches based on the gradual concepts. They proposed a gradual approach to enrich problem-solving methods to effectively utilize of domain knowledge that can be only useful in some aspects. This approach can help decision makers to obtain acceptable solutions not necessary the best solution. Ding et al. (2015) modeled traffic control agency (TCA) decision behavior to reduce traffic congestion due to planned and unplanned events in order to optimize event traffic management.

Despite of the importance of the gradual behavior and its contribution in event occurrence, there is not any integrated framework to analyze phenomena from this perspective. Therefore, in this paper an appropriate framework called continuous occurrence theory, is presented to explain and evaluate behavior of the events, and to manage controllable events using gradual behavior issue. After introduction, section 2 is presented including basic definitions of the “continuous occurrence theory” and the related examples. An important concept called “occurrence-based combination” is given in section 3. “Occurrence variable” and “occurrence-based dependency” have been described in sections 4 and 5. “Occurrence model” and its particular attributes are stated in section 6. Applications of continuous occurrence theory have been presented in section 7. Finally, an applied example of COT have been presented in section 8 and section 9 is dedicated to the conclusion.
2. Basic definitions

In this section the fundamental definitions of the “continuous occurrence theory” have been presented as follows.

Entity (EN): Each object (tangible or intangible) that is considered for investigation and analysis.

Occurrence State (OS): Indicates different states that can be appeared within an entity. The word “within” is a key term that implicates appearance of an “occurrence state” within an entity during time.

Time (t): All events occur during time. Therefore, time is a basic parameter of the “continuous occurrence theory”. It is necessary to identify and analyze how “occurrence states” occur in an “entity” during time. “During” is an important term declares “occurrence states” happen continuously as the time passes.

Occurrence Degree (OD): Quantitative value that represents appearance magnitude of the considered “occurrence state” in the related “entity”. OD of an OS in an EN in time t is shown with \( OD_{OS, EN, t} \).

The four mentioned parameters (\( EN, OS, OD \) and t) are essential elements of the “continuous occurrence theory”. \( OD_{OS, EN, t} \) requires a four-dimensional space. Therefore, its hyperplanes can be considered to show visual representation of the parameters. A number of valuable three-dimensional hyperplanes can be considered as follows.

- \( (OD, OS, EN) \) represents ODs of OSs in ENs at a “specific time”
- \( (OD, OS, t) \) represents ODs of OSs in a “specific EN” during time
- \( (OD, EN, t) \) represents ODs of a “specific OS” in ENs during time

In addition, two-dimensional planes can be considered such as:

- \( (OD, t) \) represents ODs of a “specific OS” in a “specific EN” during time
- \( (OD, OS) \) represents ODs of OSs in a “specific EN” at a “specific time”
- \( (OD, EN) \) represents ODs of a “specific OS” in ENs at a “specific time”
- \( (EN, OS) \) represents considered ENs and their related OSs.

Contextual diagram (CD): Each EN has the capacity of occurrence of a number of “occurrence states”. In addition, each OS can event in some appropriate ENs. It is useful to define/ recognize (EN, OS) as an ordered pair that constitutes the contextual diagram (CD) to understand the considered problem and to obtain basic information about its ENs and OSs.
For example Fig. 1 shows a CD with five ENs and four OSs so that OS4 can occur in all ENs while OS1 can occur in all ENs except EN2. In addition, OS2 and OS3 can occur in all ENs except EN4.

\[
\begin{array}{ccccc}
\text{EN5} & \times & \times & \times & \times \\
\text{EN4} & \times & \times & & \\
\text{EN3} & \times & \times & \times & \times \\
\text{EN2} & \times & \times & \times & \\
\text{EN1} & \times & \times & \times & \\
\text{OS1} & \text{OS2} & \text{OS3} & \text{OS4}
\end{array}
\]

**Fig. 1.** An example of a contextual diagram

*Surjective contextual diagram (SCD):* a contextual diagram is surjective, if and only if all OSs can occur in all ENs. In this case, “all OSs” and “all ENs” indicate the OSs and ENs in the considered problem domain. Figure 2 shows an example of a surjective contextual diagram including five ENs and four OSs.

\[
\begin{array}{ccccc}
\text{EN5} & \times & \times & \times & \times \\
\text{EN4} & \times & \times & \times & \times \\
\text{EN3} & \times & \times & \times & \\
\text{EN2} & \times & \times & \times & \\
\text{EN1} & \times & \times & \times & \\
\text{OS1} & \text{OS2} & \text{OS3} & \text{OS4}
\end{array}
\]

**Fig. 2.** An example of a surjective contextual diagram

Contextual diagram (CD) presents useful insights about the problem. It specifies which OSs and ENs have been considered in the problem. In addition, CD determines which OSs can be occurred in which ENs.

To show the applicability of the above parameters, an example has been stated. Suppose a soccer game between teams A and B. One of the main events in a soccer game is “goal”. Assume that at 9:05 (9 minute and 5 seconds) result of the match is “0-0” and goalkeeper of “team A” initiates the game. The ball gradually is passed through players of “team A” to come close to gateway of “team B” until that first “goal” of the game is occurred at 9:37. An indicator is used to show goals of the game called goal indicator (GI).
To demonstrate the applicability and usefulness of the “continuous occurrence theory”, it is useful to compare “traditional” approaches versus “continuous occurrence theory” about the “goal” as an occurrence state.

a) In the usual approaches, “goal” event is occurred at once. Therefore, before 9:37 time, value of the GI is zero. At 9:37 time, GI is changed to 1 at once (Fig. 3A).

b) In the “continuous occurrence theory” approach, four basic parameter can be considered as follows.

- Entity: soccer game between teams A and B
- Occurrence state: goal
- Occurrence degree: Goal indicator (GI)
- \( t \): time of the game between [0, 90] minutes

Therefore, GI (that is considered as OD) gradually increases during [9:05-9:37]. GI value depends to the ball position and states of players of teams A and B. Indeed, the nearer the ball to gateway of “team B”, the more increased GI we have until at 9:37 that GI reaches to the maximum possible value that is equal to one (Fig. 3B). Indeed, Fig. 3B shows plane in two-dimensional space including GI and Time indicating OD and \( t \) parameters, respectively in the COT framework. In this case, EN (soccer game) and OS (goal) are supposed as the specified parameters.

Fig. 3. (A) Traditional approach about GI indicator

Fig. 3. (B) Continuous occurrence theory approach about GI indicator
Occurrence series: “Occurrence series” can include occurrence degrees of defined OS and EN at the specified points in time. Occurrence series provide useful information about occurrence degrees and their changes during a continuous time period (that is a set with infinite elements) by a discrete set of occurrence degrees at the specified points of time (that is a set with finite elements). For example assume occurrence degree of a specified OS (OS<sub>i</sub>) in a specified EN (EN<sub>j</sub>) during a given time period (t<sub>1</sub>, t<sub>2</sub>, ..., t<sub>n</sub>). Therefore, (OD<sub>OS<sub>i</sub>,EN<sub>j</sub>,t<sub>1</sub>), OD<sub>OS<sub>i</sub>,EN<sub>j</sub>,t<sub>2</sub>), ..., OD<sub>OS<sub>i</sub>,EN<sub>j</sub>,t<sub>n</sub> ) can be considered as a typical occurrence series. Two main attributes are related to the Occurrence series as follows.

a) α-smooth attribute: An occurrence series is called α-smooth series if and only if:

\[(OD_{OS,i,EN,j,t_k} - OD_{OS,i,EN,j,t_{k-1}}) \leq \alpha \quad 2 \leq k \leq n \quad (1)\]

This attribute will be explained more in section 6 (occurrence model).

b) Surjective attribute: Suppose a specific point in time (T) while t<sub>k-1</sub> ≤ T ≤ t<sub>k</sub> . An occurrence series is called surjective series if and only if occurrence degree at each time like T is between maximum and minimum of two related adjacent occurrence degrees that are sequential terms of the occurrence series. This issue is shown as follows.

If t<sub>k-1</sub> ≤ T ≤ t<sub>k</sub> then:

\[OD_{OS,i,EN,j,t_k} \leq OD_{OS,i,EN,j,T} \leq OD_{OS,i,EN,j,t_{k-1}} \quad (2)\]

As shown above, OD<sub>OS,i,EN,j,t_k</sub> and OD<sub>OS,i,EN,j,t_{k-1}</sub> are two sequential terms of the considered occurrence series.

3. Occurrence-based combination

Depending to the considered problem for analysis, it is probable that there are much data in the COT related issues. For example table 1 shows occurrence degree of a specified OS (OS<sub>i</sub>) in a set of ENs (EN<sub>1</sub>, EN<sub>2</sub>, ..., EN<sub>j</sub>, ..., EN<sub>m</sub>) during a given time period (t<sub>1</sub>, t<sub>2</sub>, ..., t<sub>k</sub>, ..., t<sub>n</sub>). Indeed, each row of table 1 shows an occurrence series.

| EN<sub>m</sub> | OD<sub>OS<i>,EN<sub一笑</sub>,t<sub>1</sub></sub> | OD<sub>OS<i>,EN<sub一笑</sub>,t<sub>2</sub></sub> | ... | OD<sub>OS<i>,EN<sub一笑</sub>,t<sub>k</sub></sub> | ... | OD<sub>OS<i>,EN<sub一笑</sub>,t<sub>n</sub></sub> |
|-------------|-----------------|-----------------|-----|-----------------|-----|-----------------|
| ...         | ...             | ...             | ... | ...             | ... | ...             |
| \(\text{EN}_j\) | \(OD_{ES,EN_j,t_1}\) | \(OD_{OS,EN_j,t_2}\) | \[\ldots\] | \(OD_{OS,EN_j,t_k}\) | \[\ldots\] | \(OD_{OS,EN_j,t_n}\) |
|---|---|---|---|---|---|---|
| \[\ldots\] | \[\ldots\] | \[\ldots\] | \[\ldots\] | \[\ldots\] | \[\ldots\] |
| \(\text{EN}_2\) | \(OD_{OS,EN_2,t_1}\) | \(OD_{OS,EN_2,t_2}\) | \[\ldots\] | \(OD_{OS,EN_2,t_k}\) | \[\ldots\] | \(OD_{OS,EN_2,t_n}\) |
| \(\text{EN}_1\) | \(OD_{OS,EN_1,t_1}\) | \(OD_{OS,EN_1,t_2}\) | \[\ldots\] | \(OD_{OS,EN_1,t_k}\) | \[\ldots\] | \(OD_{OS,EN_1,t_n}\)

\[\begin{array}{ccccc}
 t_1 & t_2 & \[\ldots\] & t_k & \[\ldots\] & t_n \\
\end{array}\]

It is desirable to condense data of occurrence degrees (as shown in Table 1) for various aims such as managerial decision making or analytical purposes. “Occurrence-based combination” is an approach provides the possibility to summarize occurrence degrees that can be introduced in two horizontal and vertical situations as follows.

**Horizontal Occurrence-based Combination:** In the horizontal occurrence-based combination (HOC), it is intended to combine (aggregate) values of occurrence degrees for specified \(OS\) and \(EN\) during time. In other words, in this case \(OS\) and \(EN\) are fixed and \(t\) can vary in a range. For example consider rows of table 1 while each row includes occurrence degrees of \(ES_i\) in the \(EN\)s like \(EN_j\) \((j=1, 2, \ldots, m)\) during time. Appropriate HOC of each row is calculated by the arithmetic mean as follows.

\[
HOC_{\text{h}}(OS, EN_j) = \frac{\sum_{k=1}^{n}(OD_{OS,EN_j,t_k})}{n} \tag{3}
\]

If values of occurrence degrees are specified for a continuous time period like \([t_1, t_n]\), HOC of each row is calculated by the integral as follows.

\[
HOC_{\text{i}}(OS, EN_j) = \int_{t_1}^{t_n} \frac{OD_{OS,EN_j,t} dt}{t_n - t_1} \tag{4}
\]

Consider \(xy\)-plane so that \(x\)-axis and \(y\)-axis represent \(OD\) and \(t\), respectively. Therefore, the stated integral in equation (4) is area of the region in the \(xy\)-plane bounded by the \(OD\) curve, the \(x\)-axis, and the vertical lines \(t = t_1\) and \(t = t_2\). Therefore, equation (4) calculates centroid as a geometric center of \((OD, t)\) that is a two-dimensional plane. At whole, equations (3) and (4) show that HOC is as an operator that converts a set of values of the occurrence degrees to a single value.

**Vertical Occurrence-based Combination:** In the vertical occurrence-based combination (VOC), it is intended to combine (aggregate) values of occurrence degrees for specified \(OS\) and \(t\) for different \(EN\)s. In other words, in this case \(OS\) and \(t\) are fixed and \(EN\) varies. For
example consider columns of table 1 while each column indicates occurrence degrees of $ES_i$ in various $ENs$ at the time $t_k$ ($k=1, 2, \ldots, n$). Appropriate VOC of each column is calculated by the arithmetic mean as follows.

\[ VOC_{EN_i}^{t_k} (OS_{j_1}, t_k) = \frac{\sum_{j=1}^{m} (OD_{OS_{j_i}, EN_j})}{m} \]  

(5)

Indeed, equation (5) shows that VOC is as an operator that converts a set of values of the occurrence degrees to a single value. At whole it is obvious that occurrence-based combinations obtain a quantitative value as representative of two-dimensional planes so that horizontal and vertical occurrence-based combinations show representatives of $(OD, t)$ and $(OD, EN)$, respectively. It can be obtained by using different methods such as calculation of centroid (equation 4) or arithmetic mean (equations 3 and 5).

**Consolidated Occurrence-based Combination:** It is useful to combine all values of possible occurrence degrees into a single value. This type of combination consolidates two horizontal and vertical occurrence-based combinations and thus is called consolidated occurrence-based combination (COC). Two approaches can be applied to calculate COC as follows.

a) HOC-VOC: First, values of HOC have been calculated for all rows. Then, VOC of the given HOCs will be obtained as follows.

\[ COC_{EN_i}^{t_k} (OS_{j_1}) = \frac{\sum_{j=1}^{m} HOC_{t_k}^{t_i} (OS_{j_1}, EN_j)}{m} \]  

(6)

b) VOC-HOC: Initially, values of VOC have been calculated for all columns. Afterwards, HOC of the given VOCs will be obtained as follows.

\[ COC_{EN_i}^{t_k} (OS_{j_1}) = \frac{\sum_{i=1}^{n} VOC (OS_{j_1}, t_k)}{n} \]  

(7)

These two approaches have been shown in table 2.

**Table 2.** Horizontal, vertical and consolidated occurrence-based combinations of occurrence degrees

| VOC | $VOC_{EN_i}^{t_k} (OS_{j_1}, t_1)$ | $VOC_{EN_i}^{t_k} (OS_{j_1}, t_2)$ | $\ldots$ | $VOC_{EN_i}^{t_k} (OS_{j_1}, t_k)$ | $\ldots$ | $VOC_{EN_i}^{t_k} (OS_{j_1}, t_n)$ | $COC_{voc-voc} (OS_{j_1}) = \frac{\sum_{i=1}^{m} HOC_{t_k}^{t_i} (OS_{j_i}, EN_j)}{m}$ | $COC_{voc-voc} (OS_{j_1}) = \frac{\sum_{i=1}^{n} VOC_{t_k}^{t_i} (OS_{j_i}, t_k)}{n}$ |
**THEOREM.** Results of the “HOC-VOC” and “VOC-HOC” approaches of COC calculation are the same.

**Proof.**

$$COC_{\text{VOC-HOC}}(\text{OS}_i) = \sum_{j=1}^{m} HOC_{t_j}(\text{OS}_i, EN_j) = \sum_{j=1}^{m} \frac{\sum_{k=1}^{n} (OD_{OS, EN_j,t_k})}{m} = \frac{\sum_{j=1}^{m} \sum_{k=1}^{n} (OD_{OS, EN_j,t_k})}{mn} = \frac{\sum_{k=1}^{n} \sum_{j=1}^{m} (OD_{OS, EN_j,t_k})}{mn} = COC_{\text{VOC-VOC}}(\text{OS}_i)$$

(8)

4. **Occurrence variables and their computations**

To understand concept of occurrence variables, two basic issues should be considered as follows.

a) Occurrence states can be related to the variables. Suppose the condition that “a variable is equal to a specific value” as an occurrence state. Based on the COT, this occurrence state has an occurrence degree.

b) Each variable can have various values in different conditions. Therefore, there are different occurrence states representing various conditions that the variable is equal to the different values. In this case, the considered variable is called basic variable (BV).
Based on the above issues, consider a typical OS like OS\(_i\) stating that: “a basic variable such as BV\(_i\) is equal to a specified quantitative value such as \(a_i\)”. In this regard, an ordered pair can be defined including \(a_i\) as the first element and \(OD\) of \(OS\(_i\)\) as the second element as follows.

\[
(a_i, OD_{OS_i, EN_j, T_k}) = (a_i, OD_{BV_i=a_i, EN_j, T_k}) \quad (9)
\]

A set of the ordered pairs (as stated in the equation (9)) constitutes the occurrence variable (OV).

**Example of OV:** “Company A” is an advisory organization that performs a consulting project which has various conditions from the “revenue” perspective as the considered basic variable. Suppose that progress percentage of the project is 40% while “company A” received only 25% of the total project cost \((R_1)\) yet. Therefore, three “occurrence states” related to the “revenue” of the “company A” can be explained as follows:

a) A number of tasks (equivalent to 25% of the all project tasks) have been performed and appropriate cost \((0.25R_1)\) has been paid. In this state, \(OD\(_1\)\) is equal to the maximum possible value that is 1. Therefore, ordered pair \((0.25R_1, 1)\) represents this occurrence state.

b) Some tasks (equivalent to 15% \((40\%-25\%)\) of the all project tasks) have been performed completely, but the payment \((0.15R_1)\) has not been done. Since that the tasks have been done and the “company A” is waiting to get paid, the \(OD\(_2\)\) is less than 1 \((OD\(_1\))\), maybe is equal to 0.7. Therefore, ordered pair \((0.15R_1, 0.7)\) represents this occurrence state.

c) Other tasks (equivalent to 60% \((100\%-40\%)\) of the all project tasks) have not been performed and surly the payment \((0.60R_1)\) has not been done. But in this case there is a project in progress that its remained tasks \((60\% \text{ of all tasks})\) will be performed based on the signed contract and then “company A” will be paid based on the 0.60\(R_1\) in the future. Therefore, \(OD\(_3\)\) is less than \(OD\(_2\)\) and maybe is equal 0.5. Therefore, ordered pair \((0.60R_1, 0.5)\) represents this occurrence state.

At whole, it is specified that “revenue” of the “company A” is an “occurrence variable” and is shown as follows.

\[
\text{Revenue}_{\text{Company(A)}} = \{(0.25R_1,1),(0.15R_1,0.7),(0.60R_1,0.5)\} \quad (10)
\]
This example demonstrates that classical variables can have only one specific value at each time. Instead, each occurrence variable can have various values with different occurrence degrees. This is the main distinction between “occurrence” and “classical” variables.

*Computations of occurrence variables:*

Similar to classical variables, occurrence variables can have its appropriate arithmetic. For describing computations of OVs, suppose two occurrence variables as follows.

\[
OV_1 = \{(a_1, \text{od}_{a_1}), \ldots, (a_i, \text{od}_{a_i}), \ldots, (a_n, \text{od}_{a_n})\}
\]

\[
OV_2 = \{(b_1, \text{od}_{b_1}), \ldots, (b_j, \text{od}_{b_j}), \ldots, (b_m, \text{od}_{b_m})\}
\]

Main arithmetic operators of occurrence variables computations are explained sequentially as follows.

*Occurrence addition:* Suppose that it is intended to calculate addition of two occurrence variables. To calculate addition of two OVs, first the set of all ordered pairs of them are considered in descending order based on their ODs. Therefore, the ordered pairs with common ODs have been specified since that they have been placed sequentially. Each two ordered pairs with common ODs is equivalent to an ordered pair with the common OD while its first element is equal to the addition of two considered first elements. For example if \((x, \text{od}_1)\) and \((y, \text{od}_1)\) are two ordered pairs with common OD then, the addition of them is equal to \((x+y, \text{od}_1)\). This procedure is performed for all ordered pairs of \(OV_1\) and \(OV_2\) with common ODs. For instance, consider that “company A” has two projects that their “occurrence revenue” variables are shown with \(OV_1\) and \(OV_2\). Total revenue of the “company A” is indicated by summation of \(OV_1\) and \(OV_2\) \((OV_1 + OV_2)\) as follows.

\[
OV_1 + OV_2 = ((25, 1), (15, 0.7), (40, 0.5), (40, 0.3))
\]

\[
OV_1 = ((50, 1), (40, 0.7), (30, 0.5))
\]

\[
\rightarrow OV_1 + OV_2 = ((25, 1), (50, 1), (15, 0.7), (40, 0.7), (60, 0.5), (30, 0.5), (40, 0.3))
\]

As stated above, “occurrence addition” is shown with \(\oplus\) symbol.
Occurrence multiplication: To calculate occurrence multiplication of two OV's, first all possible 2-combinations of ordered pairs of two OV's have been considered. Then, occurrence multiplication of two ordered pairs is equivalent to an ordered pair so that its first/second element is equal to the multiplication of first/second elements of the considered ordered pairs. This procedure is shown in below.

\[
OV_1 \times OV_2 = \\
\{(a_1 b_1, od_{a_1} od_{b_1}), ..., (a_1 b_j, od_{a_1} od_{b_j}), ..., (a_1 b_m, od_{a_1} od_{b_m}) \} \\
\vdots \\
\{(a_n b_1, od_{a_n} od_{b_1}), ..., (a_n b_j, od_{a_n} od_{b_j}), ..., (a_n b_m, od_{a_n} od_{b_m}) \} \\
\]

As stated above, “occurrence multiplication” is shown with \( \times \) symbol. In addition, multiplication of an OV by a constant value, \( c \), is considered as multiplication of two OV's so that first OV is equal to the constant value, \( c \), while its OD is equal to one as this: \{\( (c, 1) \)\}. Therefore, this case of multiplication is calculated as follows.

\[
c \times OV_1 = \{(c, 1)\} \times\{(a_1, od_{a_1}), ..., (a_i, od_{a_i}), ..., (a_n, od_{a_n})\} = \{(c \times a_1, 1 \times od_{a_1}), ..., (c \times a_i, 1 \times od_{a_i}), ..., (c \times a_n, 1 \times od_{a_n})\} = \{(ca_1, od_{a_1}), ..., (ca_i, od_{a_i}), ..., (ca_n, od_{a_n})\} \]

Occurrence subtraction: Occurrence subtraction that is shown with \( \ominus \) symbol is combination of two mentioned occurrence operators. To calculate \( OV_1 \ominus OV_2 \) first, \( OV_2 \) is multiplied by \(-1\) as a constant value. Then, the obtained result is added to \( OV_1 \) in below.

\[
OV_1 \ominus OV_2 = OV_1 \oplus (-1 \times OV_2) \]

For example suppose that \( OV_1 \) and \( OV_2 \) presents “occurrence revenue” and “occurrence cost” of “company A”, respectively. Therefore, its “occurrence profit” is shown with \( OV_1 \ominus OV_2 \) (“occurrence revenue” - “occurrence cost”).
Occurrence inversion: To calculate inversion of an OV initially, first element of each ordered pair of OV is inversed and the second element (occurrence degree) is remained without any change. Inversion of an OV is shown with $\frac{1}{e}$ symbol.

Occurrence division: Occurrence division that is shown with $\frac{1}{e}$ symbol, is combination of “occurrence multiplication” and “occurrence inversion” operators. To calculate $O V_1 e O V_2$ first, inversion of $O V_2$ is obtained. Then, $O V_1$ is multiplied by the inversed $O V_2$ as follows.

$$O V_1 e O V_2 = O V_1 x \left(\frac{1}{e} O V_2\right)$$  \hspace{1cm} (16)

Occurrence summarization: It is known that occurrence variables include various values with different occurrence degrees. It is desirable to summarize the values into one quantitative value. The summarization can be performed in different ways such as weighted summation (WS) approach that converts each OV to the weighted sum of the basic variable (BV) values while weight coefficients are related OD as follows.

$$WS(O V) = e \sum_{i=1}^{N} \left\{ (a_i, OD_{OV,EN,i}) \right\} = e \sum_{i=1}^{N} \left\{ (a_i, OD_{BV,OD,EN,i}) \right\} = \sum_{i=1}^{N} \left( a_i \times OD_{BV,OD,EN,i} \right)$$ \hspace{1cm} (17)

Symbol $e \sum$ in the above equation is the “occurrence sigma” and indicates “weighted summation” operator. For example WS of Revenue$_{Company(A)}$ as an occurrence variable is shown as follows.

$$WS\left(\text{Revenue}_{\text{Company(A)}}\right) = e \sum \{(0.25R_1,1), (0.15R_1,0.7), (0.60R_1,0.5)\}$$
$$= (0.25R_1 \times 1) + (0.15R_1 \times 0.7) + (0.60R_1 \times 0.5) = 0.655R_1$$  \hspace{1cm} (18)

Occurrence cut: Maybe it is required to consider some pairs of an occurrence variable so that the related occurrence degree of basic variable is greater than or equal to a specified threshold like $\alpha$. This subset of pairs is called “occurrence $\alpha$-cut” of the occurrence variable. For example some $\alpha$-cuts of revenue of company A (as an occurrence variable) are shown as follows.

$$\text{Revenue}_{\text{Company(A)}} = \{(0.25R_1,1), (0.15R_1,0.7), (0.60R_1,0.5)\}$$

$$\text{Revenue}_{1.0\text{-cut}} = \{(0.25R_1,1)\}$$

$$\text{Revenue}_{0.5\text{-cut}} = \{(0.25R_1,1)\}$$
\[ \text{Revenue}_{0.7-\text{cut}} = \{(0.25R_1, 1), (0.15R_1, 0.7)\} \]
\[ \text{Revenue}_{0.6-\text{cut}} = \{(0.25R_1, 1), (0.15R_1, 0.7)\} \]

5. Occurrence-based dependency

Occurrence states and occurrence variables are two main issues in the continuous occurrence theory. It is probable that OSs and OVs are dependent to themselves and one another. Three cases of dependency can be investigated in the “continuous occurrence theory” as follows.

a) Dependency between OSs: Suppose two occurrence states called \( OS_i \) (in \( EN_1 \)) and \( OS_j \) (in \( EN_2 \)). \( OS_i \) is dependent to \( OS_j \), if and only if changes in \( OD_{OS_i, EN_1, t_1} \) (when other conditions are fixed), will lead to change in \( OD_{OS_j, EN_2, t_2} \) as follows.

\[
\Delta OD_{OS_i, EN_1, t_1} = f_{OS_i, OS_j} (\Delta OD_{OS_j, EN_2, t_2})
\] (19)

In the above equation, \( f_{OS_i, OS_j} \) is the dependency occurrence function (DOF) explains how \( OS_j \) affects on \( OS_i \). If \( EN_1 \) and \( EN_2 \) are the same entity, it is called “endogenous dependency” stating dependency between “occurrence states” of an entity. Otherwise it is called “exogenous dependency” indicating dependency between “occurrence states” of two different entities.

b) Dependency between OVs: Suppose two “occurrence variables” called \( OV_1 \) and \( OV_2 \). \( OV_1 \) is dependent to \( OV_2 \), if and only if changes in \( OV_2 \) (in the case that other conditions are fixed), will lead to change in \( OV_1 \) as follows.

\[
\Delta OV_1 = f_{OV_1, OV_2} (\Delta OV_2)
\] (20)

In the above equation, \( f_{OV_1, OV_2} \) is the “dependency occurrence function” explains how \( OV_2 \) affects on \( OV_1 \).

c) Dependency between OV and OS: \( OV_1 \) as an “occurrence variable” is dependent to \( OS_i \) as an occurrence state, if and only if changes in \( OD_{OS_i, EN_1, t_1} \) (in the case that other conditions are fixed), will lead to change in \( OV_1 \) as follows.

\[
\Delta OV_1 = f_{OV_1, OS_i} (\Delta OD_{OS_i, EN_1, t_1})
\] (21)
In the above equation, $f_{OVi, OSi}$ is the “dependency occurrence function” explains how $OS_i$ affects on $OV_i$. In addition, $OS_i$ as an “occurrence state” is dependent to $OOVi$ as an “occurrence variable”, if and only if changes in $OV_i$, (in the case that other conditions are steady), will lead to change in $OD_{OS,EN,t}$ as follows.

$$\Delta OD_{OS,EN,t} = f_{OSi,OV1}(\Delta OV1) \quad (22)$$

In the above equation, $f_{OSi,OV1}$ is the “dependency occurrence function” explains how $OV_1$ affects on $OS_i$. Each dependency case can be existed in two generic conditions. If changes in an OS or OV will lead to changes in the dependent OV/OS at the same time, it is called “prompt dependency”. In addition, if changes in an OS or OV will lead to changes in the dependent OV/OS by a time interval, it is called “lagged dependency”.

6. Occurrence model

As before stated, each OV or OS can be dependent to other OVs and OSs. Therefore, it is suitable to illustrate interrelated OVs and OSs in a comprehensive framework called occurrence model (OM). Each OM can be illustrated in a hierarchical structure including at least two levels. Top level (level 0) of an OM is the main aim of creating and analyzing an occurrence model called “target element”. Other levels (level 1, level 2 and so on) include dependent “occurrence states” and “occurrence variables”. Dependency relationships between levels of an OM have been explained with the appropriate dependency occurrence functions (DOFs). Figure 4 shows a typical OM so that each “element” of the model can be an “occurrence state” or an “occurrence variable” and is called “EL”.
Fig. 4. A typical illustration of an occurrence model (OM)

As shown in Fig. 4, elements in each level (OS or OV) are related to elements in the lower level. Indeed, arrows in the OM indicate DOFs between OVs and OSs of two adjacent levels of the OM model. For example $f_{EL_0,\{EL_{1,1},EL_{1,2},\ldots,EL_{1,m_1}\}}$ is the DOF stating effect of a set of elements in level 1 (EL$_{1,1}$, EL$_{1,2}$, ..., EL$_{1,m_1}$) on EL$_0$. In addition, $f_{EL_{1,2},\{EL_{2,1},EL_{2,2},\ldots,EL_{2,m_2}\}}$ is another DOF stating effect of a set of elements in level 2 (EL$_{2,1}$, EL$_{2,2}$, ..., EL$_{2,m_2}$) on EL$_{1,2}$.

**Smoothly attribute of OM**: It is known that the basic concept of “continuous occurrence theory” is gradual and continuous occurrence of a state in the considered entity. Therefore, it is desirable that OD of each OS, smoothly changes during time. In other words, it is not expected that OD has much fluctuations. This issue is recognized as “smoothly” attribute stating gradual change of OD during time. To investigate smoothly attribute from quantification viewpoint, a threshold like $\alpha$ is considered as upper level of changes in OD. Specifically, an $OD_{OS,\mathrm{EN}_i}$ is called $\alpha$-smooth OD if and only if:
In discrete state

\[ (OD_{OS,EN,t+1} - OD_{OS,EN,t}) \leq \alpha \quad \forall t \]  

(23)

In continuous state

\[ \frac{dOD_{OS,EN,t}}{dt} \leq \alpha \quad \forall t \]

Similarly, \( OV \) as an occurrence variable is called \( \alpha \)-smooth \( OV \) if and only if:

In discrete state

\[ \frac{(OV_{t+1} - OV_t)}{OV_t} \leq \alpha \quad \forall t \]  

(24)

In continuous state

\[ \frac{dOV_t}{dt} \leq \alpha \quad \forall t \]

In the above equation, \( OV_t \) indicates value of \( OV \) at time \( t \). In addition, suppose that \( F \) is a dependency occurrence function (DOF). Each DOF has appropriate arguments (inputs) that are values of elements (OSs or OVs) such as \( (EL_1, EL_2, ..., EL_n) \). \( F(EL_1, EL_2, ..., EL_n) \) is called smooth DOF if and only if when all \( (EL_1, EL_2, ..., EL_n) \) are \( \alpha \)-smooth, then \( F(EL_1, EL_2, ..., EL_n) \) will be \( \alpha \)-smooth too. For example \( f(EL) = EL_t + 0.2 \) is a smooth DOF. Generally, function \( F \) is called \( \alpha \)-smooth DOF if and only if:

In discrete state

\[ (F_{t+1} - F_t) \leq \alpha \quad \forall t \]  

(25)

In continuous state

\[ \frac{dF_t}{dt} \leq \alpha \quad \forall t \]

In the above equations, value of function \( F \) at time \( t \) is shows with \( F_t \). Finally, An OM is called \( \alpha \)-smooth model, if and only if all of its DOFs are \( \alpha \)-smooth. There are fundamental theorems about smoothly DOFs that are stated as follows.

**THEOREM.** If \( F \) is an \( \alpha \)-smooth DOF and \( \beta \geq \alpha \), then \( F \) is \( \beta \)-smooth as well.

**Proof.** In discrete state:

(1): Function \( F \) is \( \alpha \)-smooth \( \Rightarrow F_{t+1} - F_t \leq \alpha \quad \forall t \)

(2): \( \alpha \leq \beta \)

(1) & (2) \( \Rightarrow F_{t+1} - F_t \leq \beta \quad \forall t \)

In continuous state:
(1): Function F is $\alpha$-smooth \[ \frac{dF_t}{dt} \leq \alpha \quad \forall t \]

(2): $\alpha \leq \beta$

(1) & (2) \[ \Rightarrow \frac{dF_t}{dt} \leq \beta \quad \forall t \]

**THEOREM.** If F and G are two $\alpha$-smooth DOFs, then each of its linear combinations is an $\alpha$-smooth as well.

*Proof.* Each linear combination is shown with $H = kF + (1-k)G$ while $0 \leq k \leq 1$.

In discrete state:

(1): Function F is $\alpha$-smooth

\[ \Rightarrow F_{t+1} - F_t \leq \alpha \Rightarrow kF_{t+1} - kF_t \leq k\alpha \quad \forall t \]

(2): Function G is $\alpha$-smooth

\[ \Rightarrow G_{t+1} - G_t \leq \alpha \Rightarrow (1-k)G_{t+1} - (1-k)G_t \leq (1-k)\alpha \quad \forall t \]

(1) & (2)

\[ \Rightarrow kF_{t+1} - kF_t + (1-k)G_{t+1} - (1-k)G_t \leq k\alpha + (1-k)\alpha \quad \forall t \]

\[ \Rightarrow [kF_{t+1} + (1-k)G_{t+1}] - [kF_t + (1-k)G_t] \leq \alpha \Rightarrow H_{t+1} - H_t \leq \alpha \quad \forall t \]

In continuous state:

(1): Function F is $\alpha$-smooth

\[ \Rightarrow \frac{dF_t}{dt} \leq \alpha \Rightarrow k \frac{dF_t}{dt} \leq k\alpha \quad \forall t \]

(2): Function G is $\alpha$-smooth

\[ \Rightarrow \frac{dG_t}{dt} \leq \alpha \Rightarrow (1-k) \frac{dG_t}{dt} \leq (1-k)\alpha \quad \forall t \]

(1) & (2)

\[ \Rightarrow k \frac{dF_t}{dt} + (1-k) \frac{dG_t}{dt} \leq k\alpha + (1-k)\alpha \Rightarrow \frac{dH_t}{dt} \leq \alpha \quad \forall t \]
THEOREM. If \( x \) is \( \alpha \)-smooth, then \( f(x) = cx + b \) is \( c\alpha \)-smooth.

Proof.

In discrete state:

\[
(x_{t+1} - x_t) \leq \alpha \Rightarrow (cx_{t+1} - cx_t) \leq c\alpha \Rightarrow (cx_{t+1} + b - cx_t - b) \leq c\alpha \\
\Rightarrow (cx_{t+1} + b) - (cx_t + b) \leq c\alpha
\]

\[
\Rightarrow f(x_{t+1}) - f(x_t) \leq c\alpha \Rightarrow f(x) \text{ is } c\alpha\text{-smooth} \quad \forall t
\]

In continuous state:

(1): \[
\frac{dx}{dt} \leq \alpha \Rightarrow \frac{dx}{dt} \leq c\alpha \quad \forall t
\]

(2): \[
f(x) = cx + b \Rightarrow \frac{df(x)}{dt} \leq c\frac{dx}{dt} \quad \forall t
\]

(1) & (2) \[
\Rightarrow \frac{df(x)}{dt} \leq c\alpha \Rightarrow f(x) \text{ is } c\alpha\text{-smooth} \quad \forall t
\]

7. Application of COT

There are many phenomenons in real world that requires balanced progress between some events. For example some of the applications of COT are states as follows.

- Consider infrastructures and facilities of the cities. It is required to provide a balance between city’s population and its facilities such as green space, hospitals, fire stations, etc. Therefore, as much as the population growth, it is needed to develop new facilities in accordance to the population growth.

- It is desired to control unemployment rate. Therefore, as much as the human resources is graduated or immigrated to a specific region, it is necessary to create new jobs.

- Children should go to schools when they are seven years old. Therefore, in accordance with growth in number of seven-age children, new capacity in schools should be established.

- Suppose that a retailing company plans to establish new stores. Therefore, it is necessary to in parallel with construct, sell or rent of new stores, some parallel work
flows should be performed such as recruiting new personnel, providing new physical equipment (i.e. cabinets, chairs, desks, etc.) and installing of business solutions such as accounting, sales and CRM modules.

Above examples indicate that there are numerous conditions that an event will be occurred in a time period. In this regard, COT approach enables decision makers and managers to proactively make suitable decisions to fulfill requirements of the customers and stakeholders. This approach differs from the previous reactive approaches that wait until a problem has been occurred; next they attempted to solve the emerged problems and challenges.

8. **Applied example of COT**

In this section an applied example of applying continuous occurrence theory is presented. The considered problem is about building of constructions (in this example schools). COT-related requirement is that schools and their capacity should be proportional with seven years old population of children. This is the main assumption that indicates occurrence-related type of problem and should be considered in the problem modeling. For this purpose, it is necessary to define dimensions of the problem in the COT domain as follows (Table 3).

- **Entity (E):** There are two entities in this example: (1) “construction projects/schools” that are buildings for schools, and (2): Children that should be considered when they need to go to school.

- **Occurrence State (OS):** (1) “project completion” that that provides the possibility to use of buildings for educational purposes, and (2): children age of seven that in this age, children should go to school

- **Occurrence degree (OD):** (1) “project progress” that shows how much of occurrence state (“project completion”) have been emerged in the entity (construction projects), and (2): “Age/7” that shows how much of occurrence state (“Age of seven”) have been emerged in the entity (Children)

- **Time (T):** “time periods” that have been considered for projects monitoring
Table 3. Occurrence related features of the considered problem

| Entity                        | OS               | OD               |
|-------------------------------|------------------|------------------|
| construction projects/ schools| Project completion | Project progress rate (%) |
| Children                     | Age of seven     | Age/7            |

To model the above problem in a mathematical model, parameters, decision variables, constraints and objective function have been defined as follow.

**Indices**

- $i$: Index of projects ($i=1, 2, \ldots, N$)
- $t$: Index of time periods ($t=1, 2, \ldots, T$)
- $k$: Index of children’s age ($k=1, 2, \ldots, 7$)
- $h$: Index of $\alpha$-cut levels ($h=1, 2, \ldots, r$)

**Parameters**

- $HB_i$: Amount of budget needed for human resources wage for one percent progress of the $i$-th project
- $EB_i$: Amount of budget needed for equipment cost for one percent progress of the $i$-th project
- $MB_i$: Amount of budget needed for materials cost for one percent progress of the $i$-th project
- $HBL$: Human resources wage budget limit
- $EBL$: Equipment budget limit
- $MBL$: Materials budget limit
- $C_i$: Capacity of the $i$-th project
- $N_k$: Number of children that are $k$ years old
- $a_h$: Value of $h$-th $\alpha$-cut levels
- $a_h$: Acceptable threshold value for difference between $h$-th $\alpha$-cut levels of “schools capacity” and “children age” occurrence variables

**Decision variables**

- $P_{i,t}$: Progress of the $i$-th project at $t$-th time period
Based on the mentioned assumption of the problem that are related to the continuous occurrence concept, it is necessary to define occurrence variables about of two conditions as follows.

- Progress percentage of constructions projects (building schools)
- Achieving children to school age

Therefore, occurrence-based decision variables are as follows.

**OV Capacity**: Occurrence variable related to the capacity of the projects

\[ OV_{Capacity} = \{ (C_1, P_{1,i}), (C_2, P_{2,i}), \ldots, (C_i, P_{i,i}), \ldots, (C_n, P_{n,i}) \} \]

**OV Children**: Occurrence variable related to the children age

\[ OV_{Children} = \{ (N_1, 1/7), (N_2, 2/7), \ldots, (N_k, k/7), \ldots, (N_7, 7/7) \} \]

The problem can be modeled in two status as follows.

**State 1:**

Objective function and constrains of the model are presented as follows.

**Objective function**

\[
\begin{align*}
\text{Min } Z_1 &= \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})HB_i) + \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})EB_i) + \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})MB_i) \\
&= \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})HB_i) + \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})EB_i) + \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})MB_i) \\
&= (26)
\end{align*}
\]

**Constraints**

\[
\begin{align*}
\left| WS(OV_{Capacity, \alpha_{h} - \text{cut}}) - WS(OV_{Children, \alpha_{h} - \text{cut}}) \right| &\leq a_h \quad h=1, 2, \ldots, r \\
(27)
\end{align*}
\]

\[
WS(OV_{Capacity, \lambda - \text{cut}}) \geq WS(OV_{Children, \lambda - \text{cut}}) \\
(28)
\]

In this state, objective function (26) considers minimizing project budget (including human resource wage, equipment cost and materials cost). Constraint (27) expresses that the difference between weighted summation (WS) of the \( \alpha \)-cut levels of “schools capacity” and “children age” occurrence variables should not be greater than a specified threshold \( a_h \). This provides the confidence that the difference between project progress and children growth (until they will seven years old and need to go to school) is not greater than a specified limit. In other words, this constraint provides the confidence that trends of the projects progress are in balance with the trends of the children growth. Otherwise, either “lack of capacity” or “unused capacity” challenges have been emerged in the future. Constraint (28) ensures that
capacity of completed projects (schools) is greater than or equal to the number of seven years old children. This means that all seven years old children have the possibility to go to school.

State 2:

Objective function and constrains of the model are presented as follows.

Objective function

\[ \text{Min } Z_2 = WS(OV_{\text{Capacity}} - OV_{\text{Children}}) \]  \hspace{1cm} (29)

Constraints

\[ \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})HB_i) \leq HBL \]  \hspace{1cm} (30)

\[ \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})EB_i) \leq EBL \]  \hspace{1cm} (31)

\[ \sum_{i=1}^{N} \sum_{t=2}^{T} ((P_{i,t} - P_{i,t-1})MB_i) \leq MBL \]  \hspace{1cm} (32)

\[ WS(OV_{\text{Capacity,1-cut}}) \geq WS(OV_{\text{Children,1-cut}}) \]  \hspace{1cm} (33)

Objective function (29) investigates minimizing difference (that is shown with occurrence subtraction) between the occurrence variables of “schools capacity” and “children age”. This provides the possibility to maximize balance between project progress and children growth to children need to schools have been fulfilled and unused capacity have been controlled, simultaneously. Constrains (30), (31) and (32) ensures that amount of different project resources such as human resources, equipment and materials have been used so that their related budget limit have been established, respectively. Constraint (33) is the same as constraint (28) in the first state of modeling.

9. Conclusion

Usually each entity in the real world involves complex changes in its different perspectives that are considered as “occurrence states” in COT. In this regard, COT presents a systematic approach to identify, investigate and assess changes in the occurrence states. In this paper, first gradual behavior was explained that is the basic concept of COT. Then, four basic
definitions: entity, occurrence state, time and occurrence degree had been determined. Afterwards, Occurrence-based combination was stated. To utilize of COT in application, occurrence variable and the related operators of occurrence variables computations including occurrence addition, occurrence multiplication, occurrence subtraction, occurrence inversion, occurrence division, occurrence summarization and occurrence cut were stated. Then, occurrence model and occurrence-based dependency were described. At the end of this paper, some applied areas of COT were investigated. Finally, to demonstrate the applicability and usefulness of the proposed theory (COT) a problem was considered with occurrence related issues and the appropriate optimization models were presented.

COT includes a wide range of applications in various disciplines of science and management that can be classified in two categories as follows.

a) Descriptive application: COT provides the possibility to describe how occurrence states happen during time and which dependencies exist between the OSs.

b) Managerial application: COT enables decision makers to apply suitable adjustments to facilitate occurrence of the considered event based on the manager’s viewpoints. This is related to the application of occurrence model that specifies how dependent OSs and OVs affect on the considered occurrence state. In this regard, “dependency occurrence functions” can be used to control relevant parameters to navigate occurrence states in the desirable direction. The intended direction can be increment or decrement of an “occurrence degree” or “occurrence variable”.

Continuous occurrence theory, a novel approach, have been proposed in this research and their related issues have been explained.

References
Baryarama, F., Mugisha, J. Y. T., & Luboobi, L. S. (2006). A Mathematical Model for the Dynamics of HIV/AIDS with Gradual Behaviour Change. *Computational and Mathematical Methods in Medicine*, 7(1), 15–26.

Cassez, F., & Grastien, A. (2013). Predictability of Event Occurrences in Timed Systems. In V. Braberman & L. Fribourg (Eds.), *Formal Modeling and Analysis of Timed Systems* (pp. 62–76). Springer Berlin Heidelberg.

Chandra, V., & Kumar, R. (2002). A event occurrence rules based compact modeling formalism for a class of discrete event systems (Vol. 1, pp. 724–729 vol.1). Presented at the Proceedings of the 2002 American Control Conference (IEEE Cat. No.CH37301), IEEE.

Ding, N., He, Q., Wu, C., & Fetzer, J. (2015). Modeling Traffic Control Agency Decision Behavior for Multimodal Manual Signal Control Under Event Occurrences. *IEEE Transactions on Intelligent Transportation Systems*, 16(5), 2467–2478.
Kozac, P. (1996). Control of guarded automata counting event occurrences. Presented at the 34th IEEE Conference on Decision and Control, Vol. 1, pp. 907–912.

Overton, K. J., & Gaucas, D. E. (1990). A Fuzzy Representation for Event Occurrence (pp. 472–479). Presented at the Symposium on Visual Communications, Image Processing, and Intelligent Robotics Systems, International Society for Optics and Photonics.

Singer, J. D., & Willett, J. B. (1993). New Methods for Studying Event Occurrence Using Survival Analysis in Early Intervention Research. *Journal of Early Intervention, 17*(3), 322–339.

Spezialetti, M., & Kearns, J. P. (1988). A general approach to recognizing event occurrences in distributed computations (pp. 300–307). Presented at the 8th International Conference on Distributed Computing Systems, IEEE.

Tardin, R. H., Pinto, M. P., Alves, M. A. S., & Simão, S. M. (2014). Behavioral event occurrence differs between behavioral states in Sotalia guianensis (Cetartiodactyla: Delphinidae) dolphins: a multivariate approach. *Zoologia (Curitiba), 31*(1), 1–7.

Teije, A. ten, & Harmelen, F. V. (1998). Characterising Problem Solving Methods by gradual requirements: overcoming the yes/no distinction. Presented at the Proceedings of the Eleventh Workshop on Knowledge Acquisition for Knowledge-Based Systems (KAW’98), Banff, Alberta.

Verberne, A., Harmelen, F., & T, A. (2000). Anytime Diagnostic Reasoning using Approximate Boolean Constraint Propagation. Presented at the Proceedings of the Seventh International Conference on Principles of Knowledge Representation and Reasoning (KR2000), Breckenridge, Colorado, USA.