A note on the homology of $\Sigma_n$, the Schwartz genus, and solving polynomial equations

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Abstract. We calculate a certain homological obstruction introduced by De Concini, Procesi and Salvetti in their study of the Schwartz genus of the fibration $F(\mathbb{C}, n) \to F(\mathbb{C}, n)\Sigma_n$. We show that their obstruction group vanishes in almost all, but not all, the hitherto unknown cases. It follows that if $n$ is not a power of a prime, or twice the power of a prime, then the genus is less than $n$. The case of $n = 2p^k$ where $p$ is an odd prime remains undecided for some $p$ and $k$.

1. Introduction

Let $q : E \to B$ be a covering map (or, more generally, a fibration). By a trivialization of $q$ one means a decomposition of $B$ as a finite union of open subsets $B = \bigcup_{i=1}^t U_i$ such that the restriction of $q$ to $q^{-1}(U_i)$ is the trivial covering for all $1 \leq i \leq t$. The Schwartz genus of $q$ is the minimal $t$ among all such trivializations of $q$. The Schwartz genus is also known as the category of a fibration. In this paper we will refer to the Schwartz genus as, simply, the genus. Let $g(q)$ be our notation for the genus of $q$.

This paper is a contribution to the study of the genus of the quotient map

$q_n : F(\mathbb{C}, n) \to F(\mathbb{C}, n)\Sigma_n$

where $F(\mathbb{C}, n)$ is the configuration space of ordered $n$-tuples of distinct points in the plane. The interest in the genus of this covering map stems from its relationship with polynomial equations: The genus of $q_n$ gives a lower bound on the “topological complexity” of any algorithm for finding the roots of a complex degree-$n$ polynomial $[S87]$. Another way to put it is to say that the Schwartz genus of $q_n$ gives a lower bound on the number of functions needed to write the solutions of a polynomial equation of degree $n$ in terms of the coefficients.

Let us first survey what can be said about the genus of $q_n$ from general considerations. Suppose $q : E \to B$ is a normal covering with group $G$. It is well known that the genus of $q$ is less or equal than $n$ if and only if the classifying map $B \to BG$ factors through the $n-1$-th stage in the Milnor construction for $BG$, 1991 Mathematics Subject Classification. 55R80.

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i.e., through the $n - 1$-dimensional space $G^n/G$. It follows that if $B$ is equivalent to an $n - 1$-dimensional complex, then the genus is at most $n$. In our case, it is known that $F(\mathbb{C}, n)_{\Sigma_n}$ is homotopy equivalent to a CW-complex of dimension $n - 1$ \[\text{[FN62]}, \text{and therefore } g(q_n) \leq n. \] So, the next interesting question that one may ask about the genus of $q_n$ is: For which $n$ does $g(q_n) = n$, and for which values of $n$ does the genus satisfy $g(q_n) < n$?

We build on the recent work of De Concini, Procesi and Salvetti, who launched an investigation into the cohomological obstruction for lowering the genus of $q_n$ below the bound given by dimensional considerations \[\text{[dCPS04]}. \] One of their achievements was that they succeeded to convert the rather inaccessible standard cohomological obstruction into a much more tractable homological obstruction.

We now recall their main result. Let $\text{Br}_n$ be the braid group on $n$ strings. It is well known that the space $F(\mathbb{C}, n)_{\Sigma_n}$ can be identified with the classifying space $B\text{Br}_n$ of the braid group. Furthermore, the map $\rho_n : B\text{Br}_n \to BS_n$, induced by the standard group homomorphism, can serve as a model for the classifying map of $q_n$. Let $M$ be a module over $\Sigma_n$. We consider $M$ as a module over $\text{Br}_n$ by pulling back in the obvious way. Standard obstruction theory implies the following theorem:

**Theorem 1.1.** $g(q_n) < n$ if and only if the induced homomorphism on cohomology

$$\rho_n^* : H^{n-1}(\Sigma_n; M) \to H^{n-1}(\text{Br}_n; M)$$

is zero for all $\Sigma_n$-modules $M$.

De Concini, Procesi and Salvetti went further than this in that they showed that it is enough to test that the corresponding homomorphism on homology is zero for a certain universal $\Sigma_n$-module. In more detail, let $L_n := H^{n-1}(F(\mathbb{C}, n))$ be the top cohomology of the configuration space. It is well known that $L_n \cong \mathbb{Z}((n-1)!$. The action of $\Sigma_n$ on $F(\mathbb{C}, n)$ endows $L_n$ with the structure of a $\Sigma_n$-module. We are interested in the group homology $H_{n-1}(\Sigma_n; L_n)$. It turns out that this group serves as a home for a universal obstruction for the Schwartz genus of $q_n$. The following theorem summarizes, and rephrases slightly, the discussion on page 611 of \[\text{[dCPS04]}\] all the way up to Theorem 3.2.

**Theorem 1.2.** (De Concini-Procesi-Salvetti) The Schwartz genus of $q_n$ is less than $n$ if and only if the induced homomorphism on homology

$$\rho_* : H_{n-1}(\text{Br}_n; L_n) \to H_{n-1}(\Sigma_n; L_n)$$

is zero.

The proof of theorem \[\text{[dCPS04]}\] utilizes a specific model for the equivariant topology of $F(\mathbb{C}, n)$ (namely, the authors of \[\text{[dCPS04]}\] use the Salvetti complex). The theorem does not seem to follow solely from the fact that $L_n$ is the top cohomology of $F(\mathbb{C}, n)$.

As a corollary, we have the following theorem.

**Theorem 1.3.** If $H_{n-1}(\Sigma_n; L_n) = \{0\}$ then $g(q_n) < n$.

Our goal in this note is to describe the calculation of the homology groups $H_*(\Sigma_n; L_n)$, with special attention to the group $H_{n-1}(\Sigma_n; L_n)$. Our main result (corollary \[\text{[dCPS04]}\]) is the following:
Theorem 1.4. If $n$ is not a power of a prime, or twice the power of a prime, then $H_i(\Sigma_n; L_n) \cong \{0\}$ for all $i$. Therefore, if $n$ is not a power of a prime or twice the power of a prime, then $g(q_n) < n$.

On the other hand, Vassiliev showed that if $n$ is a prime power then $g(q_n) = n$. This leaves the case $n = 2p^k$ where $p$ is an odd prime. In this case the groups $H_i(\Sigma_{2p^k}; L_{2p^k})$ do not vanish for all $i$. Of course, the question that really interests us is whether the group $H_{2p^k-1}(\Sigma_{2p^k}; L_{2p^k})$ vanishes. It turns out that sometimes it does and some times it does not. We offer the following partial results to make this point:

Theorem 1.5. (1) For all odd primes $p$, $H_{2p-1}(\Sigma_{2p}; L_{2p}) = \{0\}$. Therefore, if $n = 2p$ then $g(q_n) < n$.
(2) $H_{17}(\Sigma_{18}; L_{18}) \neq \{0\}$

Part (2) of the theorem says that the obstruction group does not vanish for $n = 2 \cdot 32$. We are unable to make any conclusions about the genus of $q_{2p^k}$ for those $p$ and $k$ for which $H_{2p^k-1}(\Sigma_{2p^k}; L_{2p^k}) \neq \{0\}$.

Theorem 1.5 is almost implicit in [V92]. However, it is not made explicit there, and the reader will see that there are a couple of technicalities to be sorted out. In particular, we will use a not entirely trivial lemma about Spanier-Whitehead duality, which is probably of some independent interest (lemma [AM99]). The lemma has the following little history: I posted it as a question to Don Davis' discussion list. There were a few responses, and the best solution (the one that used least and proved most) was offered by Goodwillie, whose proof we reproduce in section [G04]. It is archived on the internet.

In any case, the main purpose of this note is to give a concise description of what is known about $H_*(\Sigma_n; L_n)$, and of what we can conclude about the Schwartz genus at the moment. Perhaps the most interesting aspect of the paper is the question that it leaves open: what can one say about the genus of $q_{2p^k}$ when $H_{2p^k-1}(\Sigma_{2p^k}; L_{2p^k})$ does not vanish? In view of Theorem 1.5 this can be reformulated as a question about differentials in the Serre spectral sequence for the homology of the fibration sequence

$$F(\mathbb{C}, 2p^k) \to F(\mathbb{C}, 2p^k)_{\Sigma_{2p^k}} \to B \Sigma_{2p^k}$$

taken with coefficients in $L_n$. More precisely, there is a Serre spectral sequence

$$H_i(\Sigma_n; H_j(F(\mathbb{C}, n)) \otimes L_n) \Rightarrow H_{i+j}(\text{Br}_n; L_n)$$

Taking $n = 2p^k$, $i = 2p^k - 1$, $j = 0$, we see that there is a copy of $H_{2p^k-1}(\Sigma_{2p^k}; L_{2p^k})$ at location $(2p^k - 1, 0)$ of the $E^2$ term of the spectral sequence. Theorem 1.5 can be rephrased as saying that $g(q_{2p^k}) < 2p^k$ if and only if this copy of $H_{2p^k-1}(\Sigma_{2p^k}; L_{2p^k})$ gets wiped out by differentials in the spectral sequence. The author’s guess is that whenever this group is non-zero, it does not get hit by differentials. In other words, we would like to offer the following conjecture.

Conjecture 1.6. $g(q_n) = n$ for all $n$ for which $H_{n-1}(\Sigma_n; L_n)$ is non-trivial.

There also is the larger question of determining $g(q_n)$ precisely, rather than just saying whether $g(q_n) < n$. The author hopes that someone will take up these challenges.

Organization of the paper: Our starting point for studying $H_*(\Sigma_n; L_n)$ is the identification of $L_n$ with the homology of a certain familiar space of partitions.
We prove this identification in section 2, proposition 2.1. We then prove our main results about $H_\ast(\Sigma_n; L_n)$ in section 3. The reader will see that our way to get at $H_\ast(\Sigma_n; L_n)$ is somewhat roundabout, in that we first consider the homology of $\Sigma_n$ with coefficients in the module $L_n \otimes \mathbb{Z}[-1]$ where $\mathbb{Z}[-1]$ is the sign representation of $\Sigma_n$. Then we relate $H_\ast(\Sigma_n; L_n)$ with $H_\ast(\Sigma_n; L_n \otimes \mathbb{Z}[-1])$ using homotopy theory in a slightly sneaky way. Finally, in section 4 we do some calculations in the non-vanishing case and prove theorem 1.5.

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2. Relation with the poset of partitions

In this section we recall that $H^{n-1}(F(\mathbb{C}, n))$, also known in this paper as $L_n$, is isomorphic, as a $\Sigma_n$-module, to the homology of the familiar space of partitions. Let us recall the definition of the space of partitions. Let $n = \{1, \ldots, n\}$. Let $\Lambda$ be the poset (or category) of partitions (equivalence relations) on $n$ ordered by refinements, where we adopt the convention that $\lambda_1 \leq \lambda_2$ if $\lambda_2$ is a refinement of $\lambda_1$. Clearly, $\Lambda$ has an initial and a final object. Let $\Lambda^i$, $\Lambda^f$, and $\Lambda^i_f$ be the posets obtained from $\Lambda$ by deleting the initial object, the final object, and both the initial and final object respectively. For $n \geq 2$, let $K_n$ be the unreduced suspension of the geometric realization of $\Lambda^i_f$. $K_n$ is homeomorphic to the join $S0 \ast |\Lambda^i_f|$. It is well-known that $K_n$ is homotopy equivalent to a wedge of $(n-1)!$ spheres of dimension $n - 2$. We define $K_1$ to be the empty set. We will also need the following generalization of $K_n$: let $\lambda$ be a partition of $n$ (and assume $\lambda$ is not the final partition). Consider the category of partitions of $n$ that are refinements of $\lambda$. Again, this category has an initial and a final object and we let $K_\lambda$ be the unreduced suspension of the geometric realization of the category obtained by removing the initial and final object. It is easy to see that $K_\lambda$ is homeomorphic to the join $K_{\lambda_1} \ast K_{\lambda_2} \ast \cdots \ast K_{\lambda^i}$, where $\lambda_1, \lambda_2, \ldots, \lambda^i$ are the components of $\lambda$ (so $i = |c(\lambda)|$, where $c(\lambda)$ is the set of components of $\lambda$). In particular, $K_\lambda$ is easily seen to be equivalent to a wedge sum of spheres of dimension $n - |c(\lambda)| - 1$.

Clearly, the symmetric group $\Sigma_n$ acts on $K_n$, thus making $H_{n-2}(K_n)$ (the only non-trivial reduced homology group of $K_n$) into a $\Sigma_n$-module. The purpose of this section is to prove the following proposition:

**Proposition 2.1.** There is an isomorphism of $\Sigma_n$-modules

$$L_n \cong H_{n-2}(K_n)$$

The proposition is folklore knowledge, but we are not aware of a precise reference, although it can easily be deduced from various facts scattered in the literature. Another reason we give our proof here is that we would like to have a reference to lemma 2.2 and remark 2.3 below.

**Proof of proposition 2.1.** Consider $F(\mathbb{C}, n)$ as a subspace $F(\mathbb{C}, n) \subset \mathbb{C}^n \subset S^{2n}$. Let $\Delta^n S2$ be the complement of $F(\mathbb{C}, n)$ in $S^{2n}$. More generally, for a pointed space $X$, let $\Delta^n X$ be the “fat diagonal” in $X^{\times n}$. It follows from duality, together with the fact that $\Sigma_n$ acts trivially on $H_{2n}(S^{2n})$ that the top cohomology of $F(\mathbb{C}, n)$ is isomorphic, as a $\Sigma_n$-module, to the bottom homology of $\Delta^n S2$. This bottom
homology occurs in dimension $n$ of $\Delta^n S2$. Thus, there are isomorphisms of $\Sigma_n$-modules

$$L_n \cong H^{n-1}(F(C, n)) \cong \text{H}_n(\Delta^n S2)$$

The space $\Delta^n S2$, or more generally $\Delta^n X$, can in turn be thought of as a homotopy colimit over the category $\Lambda^f$. Indeed, for an object $\lambda$ of $\Lambda$, let $c(\lambda)$ be the set of components of $\lambda$. It is easy to see that for a pointed space $X$, diagonal inclusion defines a functor from $\Lambda$ to spaces given on objects by $\lambda \mapsto X^{\wedge c(\lambda)}$. It is easy to see that for a well-pointed $X$ there are equivalences

$$\Delta^n X \cong \text{colim}_{\lambda \in \Lambda^f} X^{\wedge c(\lambda)} \simeq \text{hocolim}_{\lambda \in \Lambda^f} X^{\wedge c(\lambda)}$$

where we take the colimit and homotopy colimit in the based category. Taking $X = S2$, we obtain that

$$\Delta^n S2 \cong \text{colim}_{\lambda \in \Lambda^f} S^{2c(\lambda)} \simeq \text{hocolim}_{\lambda \in \Lambda^f} S^{2c(\lambda)}$$

We will now introduce a filtration of the functor $\Delta^n X$, with properties given in the following lemma:

**Lemma 2.2.** There exist functors (from spaces to spaces with an action of $\Sigma_n$) $\Delta^n_1 X$ and $\Sigma_n$-equivariant natural transformation

$$* = \Delta^n_0 X \to \Delta^n_1 X \to \Delta^n_2 X \to \cdots \to \Delta^n_{n-1} X = \Delta^n X$$

where the homotopy cofiber of the map $\Delta^n_{i-1} X \to \Delta^n_i X$ is $\Sigma_n$-equivalent to

$$\bigvee_{|\lambda| = n-i} K_{\lambda} \wedge X^{\wedge c(\lambda)}$$

**Remark 2.3.** If we apply $\Sigma^\infty$ to the filtration above then the subquotients become homogeneous functors of $X$. It follows that the previous lemma is giving a model for the Taylor tower of $\Sigma^\infty \Delta^n X$. In particular, the $j$-th homogeneous layer of this functor is

$$\bigvee_{|\lambda| = j} \Sigma^\infty K_{\lambda} \wedge X^{\wedge j}$$

Let $X$ be $k$-connected, $k \geq 1$. It follows from the lemma there is a $\Sigma_n$-equivariant map

$$\Delta^n X \xrightarrow{\sim} \Delta^n_{n-1} X \to \Delta^n_{n-1} X/\Delta^n_{n-2} X \xrightarrow{\sim} K_n \wedge X$$

which is $n + 2k - 1$ connected. In particular, if $X$ is 1-connected the map is $n + 1$-connected. Taking $X = S2$ we see that there is an $n + 1$-connected map $\Delta^n S2 \to S2 \wedge K_n$. In particular, it induces an isomorphism on $\pi_n(\text{--})$ which by Hurewicz theorem is the same as $\text{H}_n(\text{--})$. Thus, we have an isomorphism of $\Sigma_n$-modules $L_n \cong \text{H}_{n-2}(K_n)$. This completes the proof of proposition 2.1. □

**Proof of Lemma 2.2.** We will filter the category $\Lambda$ by the number of components. For $1 \leq i \leq j \leq n$ let $\Lambda^i_j$ be the full subcategory of $\Lambda$ consisting of partitions $\lambda$ such that $i \leq |c(\lambda)| \leq j$. Thus $\Lambda^j_j = \Lambda_1^1$, and we have a system of subcategories

$$\Lambda_{n-1}^n \hookrightarrow \Lambda_{n-2}^n \hookrightarrow \cdots \hookrightarrow \Lambda_1^n$$

Notice that at each stage the poset is enlarged by throwing in minimal objects, or to put it differently, there are morphisms from objects of $\Lambda_j^n \setminus \Lambda_j^{n+1}$ to objects of $\Lambda_j^{n-1}$, but not the other way around.
We define
\[ \Delta^n_i X := \mathrm{colim}_{\Lambda_n^{i-1}} X^{\wedge c(\lambda)} \]

It is clear that there are \( \Sigma_n \)-maps \( \Delta_{n-1}^n X \to \Delta^n_i X \) induced by inclusions of categories. It is not hard to see that in this case colimit is equivalent to homotopy colimit, so \( \Delta^n_i X \simeq \mathrm{hocolim}_{\Lambda_n^{i-1}} X^{\wedge c(\lambda)} \). It remains to analyze the homotopy cofiber of the map \( \Delta_{n-1}^n X \to \Delta^n_i X \). It is easy to see that \( \Delta_{n-1}^n X \) can be thought of as a homotopy colimit over the category \( \Lambda_n^{n-i} \) rather than \( \Lambda_n^{n-i+1} \) where one extends the functor \( \lambda \mapsto X^{\wedge c(\lambda)} \) to have value \( \lambda \mapsto * \) if \( c(\lambda) = n-i \). It follows that the cofiber is equivalent to the homotopy colimit \( \mathrm{hocolim}_{\Lambda_n^{n-i}} G(\lambda) \), where the functor \( G : \Lambda_n^{n-i} \to \text{Spaces} \) is defined by

\[ G(\lambda) = \begin{cases} X^{\wedge c(\lambda)} & \text{if } |c(\lambda)| = n-i \\ * & \text{otherwise} \end{cases} \]

It remains to analyze the homotopy colimit of \( G \). Consider again the poset \( \Lambda_n^n \). Its minimal elements are partitions \( \lambda \) which have exactly \( n-i \) components. For a partition \( \lambda \), let \( \Lambda_{\leq \lambda} \) the the poset of partitions greater (finer) or equal to \( \lambda \). Then the poset \( \Lambda_n^{n-i} \) can be thought of as a union

\[ \Lambda_n^{n-i} = \bigcup_{\{\lambda | |c(\lambda)| = n-i \}} \Lambda_{\leq \lambda} \]

It follows that any homotopy colimit over \( \Lambda_n^{n-i} \) can be written as a homotopy colimit over the category of non-empty collections of minimal elements of \( \Lambda_n^{n-i} \), where to each such collection \( \{\lambda_i\} \) one associates the homotopy colimit over the intersection \( \bigcap_i \Lambda_{\leq \lambda_i} \). In particular, for our functor \( G \), we can write

\[ \mathrm{hocolim}_{\Lambda_n^{n-i}} G(\lambda) = \mathrm{hocolim}_{\lambda_1, \ldots, \lambda_k} \mathrm{hocolim}_{\lambda \in \bigcap_{i=1}^k \Lambda_{\geq \lambda_i}} G(\lambda) \]

where the outer homotopy colimit is over finite (non-empty) collections \( \lambda_1, \ldots, \lambda_k \) of distinct minimal elements of \( \Lambda_n^{n-i} \) (i.e., partitions with \( n-i \) components). Now observe that for any such collection

\[ \bigcap_{i=1}^k \Lambda_{\geq \lambda_i} = \Lambda_{\geq \bigwedge_{i=1}^k \lambda_i} \]

where by \( \bigwedge_{i=1}^k \lambda_i \) we mean the coarsest common refinement of \( \lambda_1, \ldots, \lambda_k \). Note, moreover, that if \( k > 1 \) then \( \bigwedge_{i=1}^k \lambda_i \) has more than \( n-i \) components, so \( G(\lambda) = * \) for all \( \lambda \in \Lambda_{n-i}^{n-i} \). It follows that whenever \( k > 1 \),

\[ \mathrm{hocolim}_{\lambda \in \Lambda_{n-i}^{n-i}} \Lambda_{\geq \lambda_i} G(\lambda) = \mathrm{hocolim}_{\lambda \in \bigwedge_{i=1}^k \Lambda_{\geq \lambda_i}} * = * \]

(because we are taking pointed homotopy colimit). It follows, finally, that

\[ \mathrm{hocolim}_{\lambda \in \Lambda_{n-i}^{n-i}} G(\lambda) \simeq \bigvee_{\{\lambda | |c(\lambda)| = n-i \}} \mathrm{hocolim}_{\Delta \in \Lambda_{\geq \lambda}} G(\Delta) \]

It remains to point out that if for each minimal \( \lambda, \Lambda_{\geq \lambda} \) is a category with an initial object, and \( G \big| \Lambda_{\geq \lambda} \) is a functor which takes value \( X \) on the initial object and the value \( * \) on all other objects. It follows easily that

\[ \mathrm{hocolim}_{\Delta \in \Lambda_{\geq \lambda}} G(\Delta) \simeq X \wedge \Lambda_{\geq \lambda} \]
for every partition \( \lambda \) with \( n - i \) components. So

\[
\hocolim_{\lambda \in \mathcal{A}_n} G(\lambda) \simeq \bigvee_{\{\lambda || e(\lambda) = n-i\}} X \wedge K_\lambda
\]

This completes the proof of the lemma.

\[\square\]

3. Proof of the main results

We think of \( K_n \) as a topological realization of the module \( L_n \). Since the reduced homology of \( K_n \) is concentrated in dimension \( n - 2 \), it follows from proposition 2.1 and a Serre spectral sequence argument that

\[
H_*(\Sigma_n; L_n) \cong H_{*+n-2}( (K_n)_h \Sigma_n )
\]

where by \((K_n)_h \Sigma_n\) we mean the reduced Borel construction: \((K_n)_h \Sigma_n = K_n \wedge \Sigma_n \wedge \Sigma_n \). In particular, \( H_{n-1}(\Sigma_n; L_n) \cong H_{2n-3}( (K_n)_h \Sigma_n ) \).

We will want to relate \( H_*(\Sigma_n; L_n) \) with \( H_*(\Sigma_n; L_n \otimes \mathbb{Z}[-1]) \) where \( \mathbb{Z}[-1] \) is the sign representation of \( \Sigma_n \). Our topological realization of the module \( L_n \otimes \mathbb{Z}[-1] \) is the space \( K_n \wedge S^n \) with the diagonal action of \( \Sigma_n \). Obviously, the homology of this space is concentrated in dimension \( 2n - 2 \), and its non-trivial homology gives the representation \( L_n \otimes \mathbb{Z}[-1] \) since the action of \( \Sigma_n \) on \( H_n(S^n) \) gives the sign representation. We will also have an occasion to use the desuspension of this space, which is space \( K_n \wedge S^{n-1} \), where \( S^{n-1} \) has an action of \( \Sigma_n \) via the standard action on \( \mathbb{R}^{n-1} \). This space realizes the module \( L_n \otimes \mathbb{Z}[-1] \) in dimension \( 2n - 3 \).

The following theorem is part of [AD01] Theorem 1.1:

**Theorem 3.1.** If \( n \) is not a power of a prime, then

\[
(K_n \wedge S^n)_h \Sigma_n \simeq *
\]

(and therefore also \((K_n \wedge S^{n-1})_h \Sigma_n \simeq *\)). Moreover, if \( n = p^k \) with \( k > 0 \), then the homology of this space is all \( p \)-torsion.

Next, there is the following theorem, which relates \((K_n)_h \Sigma_n\) and \((K_n \wedge S^n)_h \Sigma_n\).

**Theorem 3.2.** For all \( n \), there is a cofibration sequence

\[
(K_\mathbb{Z} \wedge S^\mathbb{Z})_h \Sigma_n \rightarrow (K_n)_h \Sigma_n \rightarrow (K_n \wedge S^{n-1})_h \Sigma_n
\]

(where \( K_\mathbb{Z} \) is understood to be a point if \( n \) is odd).

The following corollary lists some immediate consequences of theorems 3.1 and 3.2 put together.

**Corollary 3.3.** (1) Unless \( n \) is either a power of a prime or twice the power of a prime, the space \((K_n)_h \Sigma_n\) is contractible, and therefore all the homology groups \( H_*(\Sigma_n; L_n) \) vanish.

(2) If \( n = 2^k \), with \( k \geq 1 \), then there is a long exact sequence of homology groups

\[
\cdots \rightarrow H_i(\Sigma_\mathbb{Z}; L_\mathbb{Z}) \rightarrow H_i(\Sigma_n; L_n) \rightarrow H_{i-n+1}(\Sigma_n; L_n \otimes \mathbb{Z}[-1]) \rightarrow \]

\[
H_{i-1}(\Sigma_\mathbb{Z}; L_\mathbb{Z}) \rightarrow \cdots
\]

(3) For every odd prime \( p \) there are isomorphisms

\[
H_*(\Sigma_p; L_p^h) \cong H_{*+p^k-1}(\Sigma_p; L_p^h \otimes \mathbb{Z}[-1])
\]

In particular, \( H_i(\Sigma_p; L_p^h) \cong \{0\} \) for \( i < p^k - 1 \).
(4) For every odd prime \( p \) there are isomorphisms

\[
H_*(\Sigma_p^k; L_p^k \otimes \mathbb{Z}[-1]) \to H_*(\Sigma_{2p}^k; L_{2p}^k)
\]

**Proof of theorem 3.2.** The theorem is almost proved in [AM99, Section 4.2], but not quite. What is proved there is that there is a kind of dual cofibration sequence

\[
[D(K_{\frac{n}{2}} \wedge S_{\frac{n}{2}}^n)]_{h\Sigma_n} \leftarrow [D(K_n)]_{h\Sigma_n} \leftarrow [D(K_n \wedge S^{n-1})]_{h\Sigma_n}
\]

Where \( D(-) \) denotes the Spanier-Whitehead dual of a spectrum. The proof of the dual cofibration sequence uses calculus of functors and the EHP sequence. Here by the EHP sequence we mean the sequence of functors \( X \to \Omega \Sigma X \to \Omega \Sigma X \wedge 2 \). This sequence is a fibration sequence in a stable range, and is an actual fibration sequence if \( X \) is an odd-dimensional sphere. Passing to layers in the Goodwillie tower gives the dual cofibration sequence (details are given in [op. cit.]). We have a slightly interesting situation here: there is in fact a sequence of \( \Sigma_n \)-equivariant maps

\[
\Sigma_{n+} \wedge \Sigma_{\frac{n}{2}} D(K_{\frac{n}{2}} \wedge S_{\frac{n}{2}}^n) \leftarrow D(K_n) \leftarrow D(K_n \wedge S^{n-1})
\]

where the composite map is null-homotopic, but which is not equivalent to a cofibration sequence. However, upon passing to homotopy orbits, the sequence becomes equivalent to a cofibration sequence. We would like to conclude that the Spanier-Whitehead dual sequence has the same property. It is easy to see that the proof boils down to the following lemma, which may be of some independent interest:

**Lemma 3.4.** Let \( G \) be a finite group (actually, \( G \) can be any Lie group whose adjoint representation is orientable) and let \( E \) be a finite spectrum with an action of \( G \). Suppose that \( E_{hG} \simeq \ast \). Then \( D(E)_{hG} \simeq \ast \).

**Remark 3.5.** The above lemma is trivially true if, for instance, \( E \) is itself contractible, or if \( E \) is equivalent to a finite spectrum with a free action of a finite group \( G \).

**Remark 3.6.** One nice consequence of the lemma is that if a collection of subgroups of a finite group is ample in the sense of [AD01, Definition 3.6], then, automatically, it is reverse ample in the sense of the same definition.

**Goodwillie’s proof of lemma 3.4** reproduced from [G04]. The key is to start with the case when \( G \) is connected. If the connected group \( G \) acts on the spectrum \( E \) and \( E \) is bounded below then the only way \( E_{hG} \) can be contractible is if \( E \) is itself contractible (The first nontrivial homotopy group of \( E \) must be the same as that of \( E_{hG} \)).

Now let \( G \) be any compact Lie group and embed it as a subgroup of a compact connected Lie group \( U \). If \( G \) acts on \( E \) (a spectrum non-equivariantly equivalent to a finite one), then consider the induced \( U \)-spectrum \( \text{Ind}(E) = U_+ \wedge G E \). The homotopy orbit spectrum \( \text{Ind}(E)_{hU} \) is the same as \( E_{hG} \), so the latter is contractible if and only if \( \text{Ind}(E) \) is contractible.

The same applies to the \( G \)-action on the dual spectrum \( D(E) \). And

\[
D(\text{Ind}(D(E))) = \text{Coind}(E) = \text{Map}^G(U_+, E)
\]

so the question becomes: If the induced \( U \)-spectrum \( \text{Ind}(E) \) is contractible, must the co-induced \( U \)-spectrum \( \text{Coind}(E) \) be contractible, too? The answer is ‘yes’: the two spectra are not quite the same, but they differ by a little twist that doesn’t matter.
for our purposes. Namely, \( \text{Coind}(E) \simeq \text{Ind}(S^{-1}V E) \) where \( V \) is a representation of \( G \). More precisely, \( V \) the difference between the adjoint representation of \( U \) (restricted to \( G \)) and that of \( G \) (this is, essentially, the Wirthm"uller isomorphism). And that means that the homology of \( \text{Coind}(E) \) and of \( \text{Ind}(E) \) are related by a Thom isomorphism as long as the adjoint representation of \( G \) is orientable. So if one is contractible then the other has no homology and therefore (being bounded below) is contractible.

This completes the proof of theorem \(^{32}\).

4. Some calculations in the non-vanishing case

The purpose of this section is to do some calculations of

\[
H_{2p^k-1}(\Sigma_{2p^k}; L_{2p^k}) = H_{4p^k-3}((K_{2p^k})_{h\Sigma_{2p^k}})
\]

for \( k = 1, 2 \), and to prove theorem \(^{36}\).

Let \( n = 2p^k \), with \( p \) and odd prime. By theorems \(^{33}\) and \(^{34}\),

\[
(K_{2p^k})_{h\Sigma_{2p^k}} \simeq (K_{p^k} \wedge S^{p^k})_{h\Sigma_{p^k}}
\]

and so

\[
H_{4p^k-3}((K_{2p^k})_{h\Sigma_{2p^k}}) \cong H_{4p^k-3}((K_{p^k} \wedge S^{p^k})_{h\Sigma_{p^k}}).
\]

The homology groups \( H_*(((K_{p^k} \wedge S^{p^k})_{h\Sigma_{p^k}}; \mathbb{F}_p) \) are calculated in \( \text{AM99} \). What basically happens is this: in the standard simplicial model for \( K_{p^k} \), the set of simplices of dimension \( k-1 \) contains a set isomorphic to \( \Sigma_{p^k}/^0 \Sigma_p, \) and the homology groups \( H_*(((K_{p^k} \wedge S^{p^k})_{h\Sigma_{p^k}}; \mathbb{F}_p) \) are isomorphic to the subquotient of \( H_*(S^{p^k}_{h\Sigma_{p^k}}; \mathbb{F}_p) \) given by the “completely inadmissible” words of length \( k \) in the Dyer-Lashof algebra, shifted up by \( k-1 \) degrees.

For instance, in the case \( k = 1 \)

\[
H_4((K_1 \wedge S^1)_{h\Sigma_1}; \mathbb{F}_p) \cong H_4(S^1_{h\Sigma_p}; \mathbb{F}_p)
\]

The right hand side is, in turn, generated by symbols of the form \( Q^s u \) and \( \beta Q^s u, \)

\( s \geq 1, \) where \( u \) is of degree 1 (\( u \) is a generator of \( H_1(S^1) \), \( Q^s \) is a Dyer-Lashof operation raising degree by \( 2s(p-1) \) and \( \beta \) is the homology B"ockstein, lowering degree by 1. Thus \( H_4((K_1 \wedge S^1)_{h\Sigma_1}; \mathbb{F}_p) \) has a generator \( Q^s u \) in every positive dimension that is 1 modulo \( 2(p-1), \) has a generator \( \beta Q^s \) in every positive dimension that is 0 modulo \( 2(p-1), \) and is zero otherwise. It follows that the integral homology \( H_4((K_1 \wedge S^1)_{h\Sigma_1}) \) is non-zero only in dimensions that are 0 modulo \( 2(p-1). \) In particular \( \{0\} \cong H_{4p-3}((K_{p^2} \wedge S^{p^2})_{h\Sigma_p}; \mathbb{F}_p) \cong H_{2p-1}(\Sigma_{2p}; L_{2p}). \) This proves part (a) of theorem \( \text{AM99}. \)

Now consider the case \( k = 2. \) The homology groups \( H_*(((K_{p^2} \wedge S^{p^2})_{h\Sigma_{p^2}}; \mathbb{F}_p) \) are generated by words of the form \( \beta^i Q^{s_1} \beta^j Q^{s_2} u \) where \( \epsilon_i \in \{0, 1\}, \) \( s_2 \geq 1 \) and the words are “inadmissible” in the Dyer-Lashof algebra, meaning that \( s_1 > ps_2 - \epsilon_2. \)

The dimension of such a word is \( 1 - \epsilon_1 + 2s_1(p-1) - \epsilon_2 + 2s_2(p-1) + 1 = 2(s_1 + s_2)(p-1) - (\epsilon_1 + \epsilon_2) + 2. \) In particular, take \( p = 3 \) and consider the word \( \beta Q^2 Q^1 u. \) It gives an element of \( H_{31}((K_9 \wedge S^9)_{h\Sigma_9}; \mathbb{F}_3) \). Moreover, this element is in the image of the B"ockstein, so it must be a reduction of an integral element. It

\(^{1}\)To be precise, the homology calculated in \( \text{AM99} \) is that of the spectrum \( (D(K_{p^2}) \wedge S^{p^2})_{h\Sigma_{p^2}} \), but the same method of calculation applies to the case that interests us here.
follows that \( \{0\} \neq H_{43}((K_9 \wedge S^9)_{\Sigma^0} \cong H_{17}(\Sigma_{18}; L_{18}). \) This completes the proof of part (b) of theorem 1.5.

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