VARIATIONAL STRUCTURE AND TWO DIMENSIONAL JET FLOWS FOR COMPRESSIBLE EULER SYSTEM WITH NON-ZERO VORTICITY

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Abstract. In this paper, we investigate the well-posedness theory of compressible jet flows for two dimensional steady Euler system with non-zero vorticity. One of the key observations is that the stream function formulation for two dimensional compressible steady Euler system with non-zero vorticity enjoys a variational structure, so that the jet problem can be reformulated as a domain variation problem. This allows us to adapt the framework developed by Alt, Caffarelli and Friedman for the one-phase free boundary problems to obtain the existence and uniqueness of smooth solutions to the subsonic jet problem with non-zero vorticity. We also show that there is a critical mass flux, such that as long as the incoming mass flux does not exceed the critical value, the well-posedness theory holds true.

1. Introduction and main results

1.1. Background and motivation. One of the most important problems in fluid dynamics is the study of flows through nozzles and the associated free boundary problems, such as shocks, jets and cavities [28, 10, 12]. Historically lots of works are done in the irrotational (zero vorticity) case. The problem of finding a steady irrotational subsonic flows in a two dimensional infinitely long fixed nozzle was first posed by Bers in [12]. For this problem, properties of smooth solutions were studied by Gilbarg [42] via comparison principles, and the well-posedness theory was first established in [62] when the flux of the flows is less than a critical number, cf. [63, 36, 46] for further generalizations. Compared with the above nozzle problem, the jet problem, where the flow is bounded partially by fixed nozzles and partially by free boundaries, is more challenging. The early investigation of the jet problem relied on hodograph method and complex analysis techniques, which mainly deal with two dimensional incompressible jet flows with restrictive assumptions on the nozzles, cf. [13, 11, 43] and references therein. A major breakthrough for the jet problem was made by Alt, Caffarelli and Friedman in 1980’s: they developed a systematic regularity theory for free boundary

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problems which are equivalent to domain variation problems [1, 5, 39]. With the aid of this general theory, they established the well-posedness for two dimensional and three dimensional axisymmetric jet and cavity problems for steady irrotational incompressible and compressible subsonic flows [4, 9]. Recently, the subsonic jet problem for irrotational flows with different boundary conditions was investigated in [26] by the method of Alt, Caffarelli, and Friedman. An important progress for the existence of subsonic-sonic jet flows and jet flows in bounded domains has been made in [59, 58].

The vorticity plays an important role in understanding fluid dynamics not only mathematically but also physically [54, 10]. For example, when the vorticity appears, the steady water waves (free boundary of the incompressible flows) have many new features comparing with the irrotational flows, see [57] and references therein. A major motivation for us to study compressible jet flows with non-zero vorticity comes from the investigation for the flow pattern with a transonic shock inside and a jet issued from the nozzle [28]. In the past two decades significant progress has been made on the stability of transonic shocks in a nozzle. It was proved in [65, 66] that the transonic shock problem for isentropic irrotational flows in a finite nozzle is in general ill-posed with the prescribed exit pressure. However, when the vorticity is not zero in the downstream subsonic region, transonic shocks under the perturbation of the exit pressure were shown to be stable, see [49, 23, 52, 61] in different settings. In order to study the flow patterns with both transonic shock and jet, it should be necessary to study the compressible jet flows with nonzero vorticity.

For rotational fluids (non-zero vorticity) the main difficulty in the study of the steady subsonic flow is that the associated Euler system is a hyperbolic-elliptic coupled system. However, for the nozzle problem in terms of the stream function the system can be reduced to a single second order quasilinear elliptic equation with memory [64]. This observation and a careful study for the associated quasilinear elliptic equation allow to establish the well-posedness theory for subsonic flows with non-zero small vorticity in an infinitely long fixed nozzle [64], see also [15, 31, 32, 19, 37, 34, 35, 22, 14] and references therein for various generalizations.

There are only very few analytical results concerning the effect of vorticity on jet flows. The well-posedness of the steady incompressible jet flows with small non-zero vorticity was established by Friedman in [40]. In this work he crucially relied on the fact that the steady incompressible Euler system can be reduced to a semilinear elliptic equation for the stream function which has a variational structure, hence the jet problem can be formulated as a domain variation problem, see [25] for recent progress in this direction for incompressible jet flows. As far as the jet problems for compressible flows with non-zero vorticity are concerned, to the authors’ knowledge, there are no
rigorous analytical results available. A major obstacle is that it is unknown whether such problem enjoys a variational structure. In this paper, we give an affirmative answer to this question and prove the existence and uniqueness of the steady compressible jet flows with non-zero vorticity.

1.2. Set-up and main results. Two-dimensional steady isentropic compressible ideal flows are governed by the following Euler system

\[
\begin{cases}
\nabla \cdot (\rho u) = 0, \\
\rho u \cdot \nabla u + \nabla p = 0,
\end{cases}
\]

where \( u = (u_1, u_2) \) denotes the flow velocity, \( \rho \) is the density, and \( p = p(\rho) \) is the pressure of the flow. Suppose that the flow is a polytropic gas, after nondimensionalization, the equation of state can be written as \( p = \frac{\rho}{\gamma} \), where the constant \( \gamma > 1 \) is called the adiabatic exponent. The local sound speed and the Mach number of the flow are defined as

\[
c(\rho) = \sqrt{p'(\rho)} = \rho^{\gamma-1} \quad \text{and} \quad M = \frac{|u|}{c(\rho)},
\]

respectively. The flow is called subsonic if \( M < 1 \), sonic if \( M = 1 \), and supersonic if \( M > 1 \).

Consider a nozzle in \( \mathbb{R}^2 \) bounded by two solid boundaries. For simplicity we assume that the nozzle is symmetric about \( x_1 \)-axis, but the results and the proofs in this paper work for asymmetric nozzles as well. Let

\[
S_0 := \{(x_1, 0) : x_1 \in \mathbb{R}\} \quad \text{and} \quad S_1 := \{(x_1, x_2) : x_1 = \Theta(x_2)\}
\]

be the symmetry axis and the upper solid boundary of the nozzle, respectively, where \( \Theta \in C^{1,\alpha}([1, \bar{H}]) \) for some given \( \bar{H} > 1 \), and satisfies

\[
\Theta(1) = 0 \quad \text{and} \quad \lim_{x_2 \to \bar{H}} \Theta(x_2) = -\infty,
\]

i.e. the mouth of the nozzle is at \( A := (0, 1) \) and the nozzle is asymptotically horizontal with height \( \bar{H} \) at upstream \( x_1 \to -\infty \) (cf. Figure 1). The main goal of this paper is to study the following jet problem.

**Problem 1.** Given the total flux \( Q \) and the Bernoulli function \( B = B(x_2) \) of the flow at upstream as \( x_1 \to -\infty \), find \((\rho, u)\), the free boundary \( \Gamma \) and the outer pressure \( p_e \), which we assume to be constant, such that

(1) The free boundary \( \Gamma \) joins the outlet of the nozzle as a continuous curve and tends asymptotically horizontal at downstream as \( x_1 \to \infty \);
The solution \((\rho, u)\) takes the incoming data at upstream, i.e.,

\[
\frac{|u|^2}{2} + \frac{\rho^{\gamma - 1}}{\gamma - 1} \to B \quad \text{as} \quad x_1 \to -\infty,
\]

and

\[
\int_0^1 \rho u_1(0, x_2) \, dx_2 = Q,
\]

and satisfies the Euler system (1) in the flow region \(\mathcal{O}\) bounded by \(S_0, S_1\) and \(\Gamma\). Moreover, it satisfies

\[
p(\rho) = p_e \quad \text{on} \quad \Gamma \quad \text{and} \quad u \cdot n = 0 \quad \text{on} \quad S_1 \cup \Gamma,
\]

where \(n\) is the unit normal along \(S_1 \cup \Gamma\).

![Figure 1. The jet problem](image)

The main results in this paper can be stated as follows.

**Theorem 1.** Given a nozzle which satisfies (3)–(4). Given Bernoulli function \(B \in C^{1,1}([0, \bar{H}])\) and mass flux \(Q > 0\) at the upstream. Suppose that \(B_* = \min_{x_2 \in [0, \bar{H}]} B(x_2) > 0\). There exists a \(\kappa^* > 0\) small depending on \(\gamma, B_*\) and the nozzle, such that if

\[
B'(0) = B' (\bar{H}) = 0 \quad \text{and} \quad \|B'\|_{C^{0,1}([0, \bar{H}])} = \kappa \leq \kappa^*,
\]

then there exist \(Q_* = \kappa^{1/4}\) and \(\hat{Q} > Q_*\) depending on \(\gamma, B_*\) and the nozzle, such that

(i) (Existence and properties of solutions) for any \(Q \in (Q_*, \hat{Q})\), there is a solution \((\rho, u, \Gamma, p_e)\) which solves the Problem 1. Furthermore, the following properties hold:

(a) (Smooth fit) The free streamline \(\Gamma\) joins the outlet of the nozzle as a \(C^1\) curve.

(b) The flow is globally uniformly subsonic and has negative vertical velocity, i.e.,

\[
\sup_{\mathcal{O}} (|u|^2 - c^2(\rho)) < 0 \quad \text{and} \quad u_2 < 0 \quad \text{in} \quad \mathcal{O}.
\]
(c) (Upstream and downstream asymptotics) Let $H$ be the asymptotic height of the free boundary at downstream $x_1 \to \infty$. Then there exist positive constants $\bar{\rho}$ and $\rho$, which are the upstream and downstream density respectively, and positive functions $\bar{u} \in C^{0,1}([0, \bar{H})), \bar{u} \in C^{0,1}([0, H]),$ which are the upstream and downstream horizontal velocity respectively, such that

$$\| (\rho, u_1, u_2)(x_1, \cdot) - (\bar{\rho}, \bar{u}(\cdot), 0) \|_{C^{0,1}([0, \bar{H}])} \to 0 \quad \text{as} \quad x_1 \to -\infty$$

and

$$\| (\rho, u_1, u_2)(x_1, \cdot) - (\rho, \bar{u}(\cdot), 0) \|_{C^{0,1}([0, H])} \to 0 \quad \text{as} \quad x_1 \to \infty.$$ 

Moreover, $\bar{\rho}, \bar{u}$ are uniquely determined by the Bernoulli function $B$ and the flux $Q$; at downstream $\rho = (\gamma p e)^{1/\gamma}$ and the downstream asymptotics $\rho, u, H$ are uniquely determined by $B$, $Q$ and $p_e$.

(ii) (Uniqueness) the Euler flow which satisfies all properties of part (i) is unique;

(iii) (Critical mass flux) $Q_c$ is the upper critical mass flux for the existence of subsonic jet flow in the following sense: either

$$\sup_{Q} (|u|^2 - c^2(\rho)) \to 0 \quad \text{as} \quad Q \to Q_c,$$

or there is no $\sigma > 0$ such that for all $Q \in (Q_c, Q_c + \sigma)$, there are Euler flows satisfying all properties in part (i) and

$$\sup_{Q \in (Q_c, Q_c + \sigma)} \sup_{\mathcal{S}} (|u|^2 - c^2(\rho)) < 0.$$ 

A few remarks are in order.

**Remark 1.1.** Together with the analysis in [64], all results in this paper work for general equation of states $p = p(\rho)$ with $p'(\rho) > 0$ and $p''(\rho) > 0$.

**Remark 1.2.** Similar arguments also work for two dimensional non-isentropic flows and three dimensional axisymmetric flows [50]. Furthermore, the ideas developed in this paper can also be used to deal with the cavitation problem for fluid with non-zero vorticity [51]. The jet flow with more general upstream data $B$ and $Q$ in the nozzles whose boundaries are graphs of functions of $x_1$ will be studied in [56].

**Remark 1.3.** The far field condition (5) prescribes the Bernoulli function (which is a hyperbolic mode for the steady compressible Euler system) at the upstream. If, instead of (5) and (6), the incoming density and horizontal velocity are prescribed at the upstream, as that has been done for the nozzle flows in [35], we can also prove the existence of compressible jet flows under suitable assumptions for the incoming velocity.
Remark 1.4. The jet flows obtained in this paper can be regarded as a first step for the study on the whole transonic flow pattern with a transonic shock inside a De Laval nozzle and a subsonic jet issued from the nozzle.

Here we give the main ideas for the proof of Theorem 1. First, if the streamlines have simple topological structure, one can reformulate the Euler system in terms of the Bernoulli function and the vorticity (cf. Proposition 2.1). With this equivalent reformulation, one can reduce the Euler system into a single second order quasilinear equation for the stream function with a complicated memory term coming from the vorticity. Moreover, the equation is elliptic if and only if the flow is subsonic (cf. Lemma 2.4). One of our key observations is that this quasilinear elliptic equation with a memory term also enjoys a variational structure as in the irrotational case (cf. Lemma 3.1). As a consequence, we can formulate the subsonic jet problem with non-zero vorticity as a domain variation problem, so that the framework developed by Alt, Caffarelli, and Friedman can be adapted to prove the Lipschitz regularity of the free boundary. The higher regularity is obtained with the aid of the regularity theory developed for the free boundary problem to inhomogeneous elliptic equations by De Silva, Ferrari and Salsa [29, 30]. To solve the jet problem, a crucial step is to show that the solution obtained from the domain variation problem satisfies a monotonicity property, which guarantees the equivalence of the Euler system and the domain variation formulation in terms of the stream function. This is done via comparison principles inspired by the proof in [9]. As long as one has existence and uniqueness of the solution for the jet problem for sufficiently small incoming mass flux, the existence of the critical mass flux can be established by the compactness arguments adapted from [64].

The rest of the paper is organized as follows. In Section 2, the stream function formulations for both the Euler system and the jet problem are established. In Section 3, we give the variational formulation for the jet problem after domain and subsonic truncations. Section 4 is devoted to the study for regularity of the truncated free boundary problem. The monotonicity property of the solution is established in Section 5, which is crucial for the equivalence of the stream function formulation and the Euler system. In order to remove the domain truncations later, we also prove some uniform estimates for the truncated problem in Section 5. In Section 6, the continuous fit, smooth fit and the far field asymptotic behavior of the solution are established. In Section 7 we remove the subsonic truncation and complete the proof for the existence of solutions to the jet problem with the small flux. The uniqueness of the subsonic jet problem is proved in Section 8. Finally, the existence of a critical mass flux is established in Section 9.

In this paper, for convention the repeated indices mean the summation.
2. Stream function formulation and subsonic truncation

In this section, we introduce the stream function formulation to reduce the Euler system into a single second order quasilinear equation, which is elliptic in the subsonic region and becomes singular elliptic at the sonic state. In order to deal with the possible degeneracy of the equation near the sonic state, a subsonic truncation is introduced so that the modified equation is always uniformly elliptic.

2.1. The equation for the stream function. First, motivated by the analysis in [64], one has the following equivalent formulation for the compressible Euler system.

Proposition 2.1. Let $\hat{\Omega} \subset \mathbb{R}^2$ be the domain bounded by two streamlines $S_0$ and

$$\hat{S}_1 = \{(x_1, x_2) | x_1 = \hat{\Theta}(x_2)\},$$

where for some $0 < \hat{H} < \hat{\bar{H}} < \infty$, it holds

$$\lim_{x_2 \to \hat{H}} \hat{\Theta}(x_2) = -\infty \quad \text{and} \quad \lim_{x_2 \to \hat{\bar{H}}} \hat{\Theta}(x_2) = \infty.$$

Suppose $u$ satisfies the slip boundary condition $u \cdot n = 0$ on $\partial \hat{\Omega}$, and

$$u_2 < 0 \text{ in } \hat{\Omega},$$

and the following asymptotic behavior

$$u_1, \rho, \text{ and } \partial_{x_2} u_2 \text{ are bounded, while } u_2, \partial_{x_1} u_2, \text{ and } \partial_{x_2} \rho \to 0, \text{ as } x_1 \to -\infty.$$

Then $(\rho, u)$ satisfies the system (1) in $\hat{\Omega}$ if and only if $(\rho, u)$ satisfies

$$\nabla \cdot (\rho u) = 0,$$

$$u \cdot \nabla B(\rho, u) = 0,$$

$$u \cdot \nabla \left( \frac{\omega}{\rho} \right) = 0,$$

where

$$B(\rho, u) := \frac{|u|^2}{2} + h(\rho), \quad \omega := \partial_{x_1} u_2 - \partial_{x_2} u_1, \quad \text{and} \quad h(\rho) := \frac{\rho^{\gamma - 1}}{\gamma - 1},$$

are the Bernoulli function, the vorticity, and the enthalpy of the flow, respectively.

Proof. It is easy to see that smooth solutions of the Euler system (1) satisfy (14).

On the other hand, it follows from the first and the third equations in (14) that

$$\partial_{x_2} (u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 + \partial_{x_1} h(\rho)) - \partial_{x_1} (u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + \partial_{x_2} h(\rho)) = 0.$$

Therefore, there exists a function $\Phi$ such that

$$\partial_{x_1} \Phi = u_1 \partial_{x_1} u_1 + u_2 \partial_{x_2} u_1 + \partial_{x_1} h(\rho), \quad \partial_{x_2} \Phi = u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + \partial_{x_2} h(\rho).$$
Thus the Bernoulli’s law (the second equation in (14)) is equivalent to
\begin{equation}
(u_1, u_2) \cdot \nabla \Phi = 0.
\end{equation}
This implies that \( \Phi \) is a constant along each streamline, in particular along \( S_0 \) and \( \tilde{S}_1 \).
Note that (13) implies \( \partial_{x_1} \Phi \to 0 \) as \( x_1 \to -\infty \). Hence one has
\begin{equation}
\Phi \to C \text{ as } x_1 \to -\infty.
\end{equation}
On the other hand, it follows from (12) that through each point in \( \tilde{\Omega} \), there is one and only one streamline which satisfies
\[
\begin{cases}
\frac{dx_1}{ds} = u_1(x_1(s), x_2(s)), \\
\frac{dx_2}{ds} = u_2(x_1(s), x_2(s)),
\end{cases}
\]
and can be defined globally in the nozzle. Furthermore, it follows from the continuity equation (the first equation in (14)) that any streamline through some point in \( \tilde{\Omega} \) cannot touch the nozzle wall. Indeed, suppose not, then there exists a streamline through \((x_1^0, x_2^0)\) in \( \tilde{\Omega} \) which passes through \((\tilde{\Theta}(\bar{x}_2), \bar{x}_2)\). Due to the continuity equation and the slip boundary condition along each streamline, one has
\[
0 = \int_{x_1^0}^{\tilde{\Theta}(x_2^0)} (\rho u_2)(s, x_2^0) \, ds.
\]
This contradicts (12). Therefore, one can conclude from (16) that \( \Phi \equiv C \) in the whole domain \( \tilde{\Omega} \). This implies that \( \partial_{x_1} \Phi = \partial_{x_2} \Phi \equiv 0 \) in \( \tilde{\Omega} \), i.e.,
\[
u_1 \partial_{x_1} u_1 + u_{x_2} \partial_{x_2} u_1 + \partial_{x_1} h(\rho) = 0 \quad \text{and} \quad u_1 \partial_{x_1} u_2 + u_2 \partial_{x_2} u_2 + \partial_{x_2} h(\rho) = 0.
\]
hold globally in \( \tilde{\Omega} \). The above two equations together with the continuity equation are exactly the original Euler system (1).

It follows from the continuity equation that there is a stream function \( \psi \) satisfying
\[
\nabla \psi = (-\rho u_2, \rho u_1).
\]
The Bernoulli’s law shows that \( B(\rho, \mathbf{u}) \) is conserved along each streamline. Hence the Bernoulli function and the stream function are functional dependent. Therefore, there exists a function \( \mathcal{B} \) such that
\begin{equation}
\tilde{\Phi}(\rho) + \frac{|\nabla \psi|^2}{2\rho^2} = \mathcal{B}(\psi).
\end{equation}
In order to determine the exact form of \( \mathcal{B} \), we first study the flow density and velocity at upstream where the relation between the stream function and Bernoulli function can be determined easily.
Let us digress for the study on the states with the given Bernoulli constant. For the state with given Bernoulli constant \( s \), the density \( \rho \) and speed \( q \) satisfy
\[
 h(\rho) + \frac{q^2}{2} = s.
\]
Therefore, the speed \( q \) satisfies
\[
 q = q(s, \rho) = \sqrt{2(s - h(\rho))}.
\]
Let
\[
 q_c(s) = \left\{ \frac{2(\gamma - 1)}{\gamma + 1} s \right\}^{\frac{\gamma}{\gamma - 1}}, \quad \varrho^*(s) = \left\{ (\gamma - 1) s \right\}^{\frac{1}{\gamma - 1}},
\]
be the critical density and maximum density respectively, and
\[
 q_c(s) = \frac{2(\gamma - 1)}{\gamma + 1} s \quad \text{and} \quad t_c(s) = \left\{ \frac{2(\gamma - 1)}{\gamma + 1} s \right\}^{\frac{2+\gamma}{\gamma + 1}}
\]
be the critical speed and square of the critical momentum respectively for the states with given Bernoulli constant \( s \). A straightforward computation shows that for given Bernoulli constant \( s \), one has
\[
 q(s, \varrho_c(s)) = q_c(s) = c(\varrho_c(s)),
\]
where \( c \) is the sound speed defined in (2). Furthermore, the flow is subsonic if and only if \( \rho > \varrho_c(s) \), or equivalently \( q(s, \rho) < q_c(s) \).

**Proposition 2.2.** Let \( B \in C^{1,1}([0, \bar{H}]) \) and \( Q > 0 \) be the incoming data at upstream in the sense of (5) and (6). Suppose that the flow satisfies the upstream asymptotic behavior (10). Suppose that \( B_s > 0 \) and \( B \) satisfies (8) with \( \kappa := \| B' \|_{C^{0,1}([0, \bar{H}]}) \). There exists \( \bar{\kappa} = \bar{\kappa}(B_s, \bar{H}, \gamma) > 0 \) sufficiently small, such that if \( \kappa < \bar{\kappa} \) and \( Q \in (Q_s, Q^*) \) with \( Q_s = \kappa^{1/4} \) and \( Q^* = Q^*(B_s, \bar{H}, \gamma) > Q_s \), then the upstream state \((\bar{\rho}, \bar{u})\) in (10) is uniquely determined by \( B \) and \( Q \), and \( \bar{\rho} \to \varrho_c(B_s) \) as \( Q \to Q^* \). Moreover, \((\bar{\rho}, \bar{u})\) satisfies
\[
 \varrho_c(B^*) \leq \bar{\rho} \leq \varrho^*(B_s) - C^{-1} \kappa^{1/2}
\]
and
\[
 \bar{u}'(0) = \bar{u}'(\bar{H}) = 0, \quad C^{-1} \kappa^{1/4} \leq \bar{u} \leq C, \quad \| \bar{u}' \|_{C^{0,1}([0, \bar{H}]}) \leq C \kappa^{3/4}
\]
for some \( C = C(B_s, \gamma, \bar{H}) > 0 \).
Proof. Suppose that the flow satisfies asymptotic behavior (10), then one has
\begin{equation}
\bar{u}(x_2) = \sqrt{2(B(x_2) - h(\bar{\rho}))},
\end{equation}
and
\begin{equation}
Q = \int_0^\bar{H} \bar{\rho} \sqrt{2(B(x_2) - h(\bar{\rho}))} dx_2.
\end{equation}
Note that $\bar{\rho} \mapsto Q(\bar{\rho})$ is strictly monotone decreasing in $(\varrho_c(B^*), \varrho^*(B_*))$, as
\begin{equation}
\frac{d}{d\bar{\rho}} Q(\bar{\rho}) = \frac{d}{d\bar{\rho}} \int_0^\bar{H} \bar{\rho} \sqrt{2(B(x_2) - h(\bar{\rho}))} dx_2 < 0 \quad \text{for} \quad \bar{\rho} \in (\varrho_c(B^*), \varrho^*(B_*)).
\end{equation}
Thus as long as $Q \in (Q(\varrho^*(B_*)), Q^*)$, where $Q^* := Q(\varrho_c(B^*))$, there exists a unique solution $\bar{\rho} \in (\varrho_c(B^*), \varrho^*(B_*))$ to (24) and $\bar{u}$ is uniquely determined through (23). Furthermore, if $Q \rightarrow Q^*$, then $\bar{\rho} \rightarrow \varrho_c(B^*)$.

In order to obtain the desired lower bound for $\bar{u}$ in terms of $\kappa$ in (22), we will select $Q_* \in [Q(\varrho^*(B_*)), Q^*)$ depending on $\kappa$, such that for $Q \in (Q_*, Q^*)$ the solution $\rho$ satisfies the upper bound in (21). In view of (23) the lower bound for $\bar{u}$ in (22) then follows. For that we observe
\begin{equation*}
Q(\varrho^*(B_*)) = \int_0^\bar{H} \varrho^*(B_*) \sqrt{2(B(x_2) - B_*)} dx_2 \leq \bar{H} \varrho^*(B_*) \sqrt{2\kappa \bar{H}}.
\end{equation*}
Thus $Q(\varrho^*(B_*)) \leq \kappa^{1/4}$ provided $\kappa \leq \frac{1}{4(\bar{H}^{\gamma/2}\varrho^*(B_*))^\gamma}$. Since
\begin{equation*}
Q^* = \int_0^\bar{H} \varrho_c(B^*) \sqrt{2(B(x_2) - \frac{2B_*}{\gamma + 1})} dx_2 \geq \bar{H} \varrho_c(B^*) \sqrt{\frac{2\gamma - 1}{\gamma + 1} B^* - 2\kappa \bar{H}} \geq \bar{H} \varrho_c(B^*) \sqrt{\frac{2\gamma - 1}{\gamma + 1} B^*}
\end{equation*}
provided $\kappa \leq \frac{(\gamma - 1)B_*}{2\bar{H}(\gamma + 1)}$, then there exists $\bar{\kappa} = \bar{\kappa}(B_*, \bar{H}, \gamma) > 0$ such that $Q^* > \kappa^{1/4}$ for any $\kappa \in [0, \bar{\kappa})$. Let $Q_* := \kappa^{1/4}$. We will prove the estimates (21) and (22) for $Q \in (Q_*, Q^*)$. By virtue of (24), one has
\begin{equation*}
Q = \int_0^\bar{H} \bar{\rho} \sqrt{2(B(x_2) - h(\bar{\rho}))} dx_2 \geq \int_0^\bar{H} \bar{\rho} \sqrt{2(B(x_2) - B_* + B_* - h(\bar{\rho}))} dx_2 \leq \bar{\rho} \bar{H} \sqrt{2(\kappa \bar{H} + h(\varrho^*(B_*)) - h(\bar{\rho}))}.
\end{equation*}
Thus
\begin{equation*}
\kappa \bar{H} + h(\varrho^*(B_*)) - h(\bar{\rho}) \geq \frac{Q^2}{2\bar{\rho}^2 \bar{H}^2} \geq \frac{\kappa^{1/2}}{2\varrho^*(B_*)^2 \bar{H}^2}.
\end{equation*}
Therefore, there exists a $\bar{\kappa} \in (0, \bar{\kappa}, \bar{\kappa}_0) = (0, \bar{\kappa})$, such that if $0 < \kappa \leq \bar{\kappa}$, then

$$g^*(B_s) - \bar{\rho} \geq C^{-1}\kappa^{1/2}$$

for some $C = C(B_s, \gamma, \bar{H}) > 0$. Consequently, $\bar{u}$ given in (23) satisfies

$$\bar{u} \geq \sqrt{2(B_s - h(\bar{\rho}))} = \sqrt{2(h(\bar{u}^*) - h(\bar{\rho}))} \geq C^{-1}\kappa^{1/4}$$

for some constant $C = C(B_s, \gamma, \bar{H})$. Furthermore, using

$$\bar{u}' = \frac{B'}{\bar{u}}, \quad \bar{u}'' = \frac{B''}{\bar{u}} - \frac{(B')^2}{\bar{u}^3}$$

together with (8) and the lower bound for $\bar{u}$, one obtains (22).

With Proposition 2.2 at hand, one can express the Bernoulli function $B$ in terms of the stream function $\psi$. More precisely, let $h(\psi, \bar{\rho})$ be the position of the streamline at upstream where the stream function has the value $\psi$, i.e.

$$\psi = \bar{\rho} \int_0^{h(\psi, \bar{\rho})} \bar{u}(s) \, ds.$$ (25)

Then the Bernoulli function at the upstream can be written as

$$B(z) := h(\bar{\rho}) + \frac{\bar{u}^2 h(\bar{\rho})}{2} = B(h(z, \bar{\rho})), \quad z \in [0, Q].$$ (26)

In view of (8) one can extend $B$ to a $C^{1,1}$ function in $\mathbb{R}$ by setting $B(z) = B(Q)$ for $z > Q$ and $B(z) = B(0)$ for $z \leq 0$. Similarly, one can extend $\bar{u}$ to be a $C^{1,1}$ function in $\mathbb{R}$ by setting $\bar{u}(x_2) = \bar{u}(\bar{H})$ for $x_2 > \bar{H}$ and then reflecting evenly about $\{x_2 = 0\}$. Note that such extension is consistent with the definition (26). Clearly, $B(z)$ is bounded from above and below, i.e.,

$$0 < B_s \leq B(z) \leq B^* < \infty,$$ (27)

where $B^*$ is defined in (20).

A straightforward computation gives

$$B'(z) = \frac{\bar{u}'(h(z, \bar{\rho}))}{\bar{\rho}} \quad \text{and} \quad B''(z) = \frac{\bar{u}''(h(z, \bar{\rho}))}{\bar{\rho}}.$$ (28)

Therefore, with the aid of (21) and (22), one has

$$\kappa_0 := \|B'\|_{L^\infty([0, Q])} + \|B''\|_{L^\infty([0, Q])} \leq C\kappa^{3/4}.$$ (29)

**Lemma 2.3.** Let $\rho$ and $\psi$ satisfy the Bernoulli’s law (17). Then $\rho$ can be expressed as a function of $|\nabla \psi|^2$ and $\psi$,

$$\rho = \frac{1}{g(|\nabla \psi|^2, \psi)}, \quad \text{if} \ \rho \in (\varrho_c(B(\psi)), g^*(B(\psi))), \quad \text{for some constant} \ C = C(B_s, \gamma, \bar{H}).$$ (30)
where \( \varrho_c \) and \( \varrho^* \) are functions defined in (18), and \( g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) is a function which is smooth in its first component and \( C^2 \) in its second component.

**Proof.** Set
\[
F(\varrho,z) := 2\varrho^2(B(z) - h(\varrho)).
\]
The straightforward computations give
\[
\partial_{\varrho} F(\varrho,z) = 4\varrho \left( B(z) - \gamma + 1 \right) \frac{\gamma - 1}{2(\gamma - 1)} \varrho^{\gamma-1}.
\]

Now, it is easy to see that with \( \varrho_c \) and \( \varrho^* \) defined in (18) one has
(a) \( \varrho \mapsto F(\varrho,z) \) achieves its maximum at \( \varrho_c(B(z)) \);
(b) \( F(\varrho,z) \geq 0 \) if and only if \( 0 \leq \varrho \leq \varrho^*(B(z)) \);
(c) \( \partial_{\varrho} F(\varrho,z) < 0 \) when \( \rho_c(z) < \varrho \leq \varrho^*(B(z)) \),

Thus by the inverse function theorem, if \( t = F(\varrho,z) \), then there exists a function \( g \), which is smooth in \( t \) and \( C^2 \) in \( z \), such that
\[
\varrho = \frac{1}{g(t,z)}, \quad \varrho \in (\rho_c(B(z)), \rho^*(B(z))] \text{ for } t \in [0, t_c(B(z))],
\]
where \( t_c \) is defined in (19). This together with the Bernoulli's law completes the proof of the lemma. \( \square \)

**Lemma 2.4.** Let \( (\rho, u) \) be a solution to the Euler system (14). Assume \( (\rho, u) \) satisfies (5) and (6). Suppose that \( u_2 < 0 \) in \( \tilde{\Omega} \) and the asymptotic behavior (10). Then the stream function \( \psi \) solves
\[
\nabla \cdot \left( g(|\nabla \psi|^2, \psi) \nabla \psi \right) = \frac{B'(\psi)}{g(|\nabla \psi|^2, \psi)},
\]
where \( g \) is defined in (30) and \( B \) is the Bernoulli function at upstream defined in (26). The equation (34) is elliptic if and only if \( |\nabla \psi|^2 < t_c(\mathcal{B}(\psi)) \) with \( t_c \) defined in (19).

**Proof.** Let \( X(s; x) \) be the streamlines satisfying
\[
\begin{cases}
\frac{dX}{ds} = u(X(s; x)), \\
X(0; x) = x.
\end{cases}
\]
It follows from the third equation in (14) that \( \omega/\rho \) is a constant along each streamline. Hence \( \omega/\rho \) can be determined by the associated data in the upstream as long as the streamlines of the flows have simple topological structure, which is guaranteed by the assumption \( u_2 < 0 \). If \( x \in \{ \psi = z \} \), then
\[
\frac{\omega}{\rho}(x) = \lim_{s \to -\infty} \frac{\omega}{\rho}(X(s; x)) = -\frac{\bar{u}'(h(z, \bar{\rho}))}{\bar{\rho}}.
\]
Expressing the vorticity $\omega$ in terms of the stream function $\psi$ and using (28) one has
\[-\nabla \cdot \left( \frac{\nabla \psi}{\rho} \right) = \omega = -B'(\psi)\rho.\]

In view of (30) the above equation can be rewritten into (34).

The equation (34) can be written in the nondivergence form as follows
\[a^{ij}(\nabla \psi, \psi) \partial_{ij} \psi + \partial_z g(|\nabla \psi|^2, \psi) |\nabla \psi|^2 = \frac{B'(\psi)}{g(|\nabla \psi|^2, \psi)},\]
where the matrix
\[(a^{ij}) = g(|\nabla \psi|^2, \psi) I_2 + 2\partial_t g(|\nabla \psi|^2, \psi) \nabla \psi \otimes \nabla \psi\]
is symmetric with the eigenvalues
\[\beta_0 = g(|\nabla \psi|^2, \psi) \quad \text{and} \quad \beta_1 = g(|\nabla \psi|^2, \psi) + 2\partial_t g(|\nabla \psi|^2, \psi)|\nabla \psi|^2.\]

It follows from (30) and (27) that if $t \in (0, t_c(B(z)))$ with $t_c$ defined in (19), then
\[\frac{1}{g^*(B^*)} = g_s \leq g(t, z) \leq g^* := \frac{1}{g_c(B_s)}.\]

Differentiating the identity $t = F(\frac{1}{g(t,z)}, z)$ gives
\[\partial_t g(t, z) = -\frac{g^2(t,z)}{\partial_e F(\frac{1}{g(t,z)}, z)} \quad \text{for} \quad 0 < t < t_c(B(z)).\]

From this one has
\[\partial_t g(t, z) \geq 0 \quad \text{and} \quad \lim_{t \to t_c(B(z))^+} \partial_t g(t, z) = +\infty.\]

Thus $\beta_0$ has uniform upper and lower bounds depending only on $\gamma$ and $B_s$, and $\beta_1$ has a uniform lower bound but blows up when $|\nabla \psi|^2$ approaches $t_c(B(\psi))$. Therefore, the equation (34) is elliptic as long as $|\nabla \psi|^2 < t_c(B(\psi))$, and is singular when $|\nabla \psi|^2 = t_c(B(\psi))$. This completes the proof of the lemma.

2.2. Reformulation for the jet flows in terms of the stream function. Let
\[\Lambda := \sqrt{2B(H) - 2h(\rho_e)\rho_e}\]
be the constant momentum on the free boundary with $\rho_e := (\gamma p_e)^{1/\gamma}$, where $p_e$ is the pressure on the free boundary determined in Problem 1. Assume (12) and (13) hold. Then Problem 1 is equivalent to the following problem in terms of the stream function $\psi$: 
Problem 2. One looks for a triple \((\psi, \Gamma_\psi, \Lambda)\) satisfying \(\partial_{x_1}\psi > 0\) in \(\{\psi < Q\}\) and

\[
\begin{cases}
\nabla \cdot \left( g(|\nabla \psi|^2, \psi) \nabla \psi \right) - \frac{B'(\psi)}{g(|\nabla \psi|^2, \psi)} = 0 & \text{in } \{\psi < Q\}, \\
\psi = 0 & \text{on } S_0, \\
\psi = Q & \text{on } S_1 \cup \Gamma_\psi, \\
|\nabla \psi| = \Lambda & \text{on } \Gamma_\psi,
\end{cases}
\]

(39)

where the free boundary \(\Gamma_\psi := \partial\{\psi < Q\} \setminus S_1\). Furthermore, the free boundary \(\Gamma_\psi\) and the flow \((\rho, u) = \left(\frac{1}{g(|\nabla \psi|^2, \psi)}, \frac{\nabla \psi^\perp g(|\nabla \psi|^2, \psi)}{g(|\nabla \psi|^2, \psi)}\right)\) are expected to satisfy the following properties:

1. \(\Gamma_\psi\) is the graph for a \(C^1\) function \(x_1 = \Upsilon(x_2), x_2 \in [H, 1]\) for some \(H \in (0, 1)\).
2. the free boundary \(\Gamma_\psi\) fits the given nozzle \(S_1\) at \(A = (0, 1)\) in a \(C^1\) fashion, i.e. \(\Upsilon(1) = \Theta(1)\) and \(\Upsilon'(1) = \Theta'(1)\);
3. for \(x_1\) sufficiently large, the free boundary is also an \(x_1\)-graph, i.e., it can be written as \(x_2 = f(x_1)\) for some function \(f\). Furthermore, at downstream \(x_1 \to \infty\), one has

\[
\lim_{x_1 \to \infty} f(x_1) = H, \quad \lim_{x_1 \to \infty} f'(x_1) = 0 \quad \text{and} \quad \lim_{x_1 \to \infty} (u, \rho) = (\bar{u}, 0, \bar{\rho})
\]

for the nonnegative constant \(\rho\) and function \(u = u(x_2)\) determined by \(\Lambda\), \(Q\), and \(B\).

Remark 2.5. In fact, the condition (8) implies that the vorticity at the upstream of the flow is small. Thus it follows from (29) that the right hand side of the equation in (39) is small.

2.3. Subsonic truncation. One of the major difficulties to solve the equation (34) is that (34) becomes degenerate as the flows approach the sonic state. As in [9] our strategy here is to use a subsonic truncation so that after the truncation the equation (34) is always elliptic.

Let \(\varpi : \mathbb{R} \to [0, 1]\) be a smooth nonincreasing function such that

\[
\varpi(s) = \begin{cases} 
1 & \text{if } s < -1, \\
0 & \text{if } s > -1/2, 
\end{cases}
\]

and \(|\varpi'| \leq 4\).

For \(\epsilon \in (0, 1/4)\), let \(\varpi_\epsilon(s) := \varpi(s/\epsilon)\). We define

\[
ge_\epsilon(t, z) := g(t, z) \varpi_\epsilon(t - t_\epsilon(B(z))) + (1 - \varpi_\epsilon(t - t_\epsilon(B(z))))g^*,
\]

(40)

where \(t_\epsilon(B(z))\) is defined in (19) and \(g^*\) is the upper bound for \(g\) in (35). The properties of \(g_\epsilon\) are summarized in the following lemma.
Lemma 2.6. Let $g$ be the function given in Lemma 2.3 and let $g_{\epsilon}$ be the subsonic truncation of $g$ defined in (40). Then the function $g_{\epsilon}(t,z)$ is smooth in $t$ and $C^2$ in $z$. Furthermore, it satisfies the following properties:

(i) There exist $C_*, C^*>0$ depending on $B_*$ and $\gamma$ such that for all $t > 0$,

\begin{align*}
&\quad C_* \leq g_{\epsilon}(t,z) \leq C^*, \\
&0 < \partial_t g_{\epsilon}(t,z) \leq C^* \epsilon^{-1}, \\
&-C_* \leq g_{\epsilon}(t,z) + \partial_t g_{\epsilon}(t,z)t \leq C^* \epsilon^{-1}.
\end{align*}

(ii) If $0 < t < t_*\epsilon(B(z)) - \epsilon$, then

\begin{equation}
\partial_z g_{\epsilon}(t,z) = \partial_z g(t,z) = -\frac{B'(z)\partial_t g(t,z)}{g(t,z)^2}.
\end{equation}

(iii) There exist $T^*$ and $C > 0$ depending only on $B_*$ and $\gamma$ such that

\begin{equation}
|\partial_z g_{\epsilon}(t,z)| \leq C \partial_t g_{\epsilon}(t,z)|B'(z)|, \quad \text{if} \quad 0 \leq t \leq T^*,
\end{equation}

and

\begin{equation}
|\partial_z g_{\epsilon}(t,z)| = 0 \quad \text{if} \quad t > T^*.
\end{equation}

Proof. It follows from Lemma 2.3 and the definition of $\varpi_\epsilon$ that $g_\epsilon$ is smooth with respect to $t$ and $C^2$ with respect to $z$.

(i). Clearly, (41) follows directly from (35) and (40).

To show (42) we note that if $0 < t \leq t_*\epsilon(B(z)) - \epsilon$ the equation (36) and that

\begin{equation*}
\partial_{e}\mathcal{F}(\frac{1}{g},z) \leq -2(\gamma+1)\epsilon g^{-1} \text{ yield}
\end{equation*}

\begin{equation*}
0 < \partial_t g(t,z) \leq (2(\gamma+1)\epsilon)^{-1}g(t,z)^3.
\end{equation*}

Then (42) follows from the definition for $g_\epsilon(t,z)$ and (41). To show (43) a straightforward computation gives

\begin{equation*}
g_{\epsilon}(t,z) + t\partial_t g_{\epsilon}(t,z) = g_{\epsilon}(t,z) + 2\partial_t g(t,z)\varpi_\epsilon(t-t_*\epsilon(B(z))) + 2(g(t,z) - g^*)\varpi_\epsilon'(t-t_*\epsilon(B(z)))t
\end{equation*}

Since the second and the third terms are nonnegative, then the lower bound in (43) follows from the lower bound for $g_\epsilon$ in (41). Combining (41), (42) and the definition of $\varpi_\epsilon$ gives the upper bound in (43).

(ii). If $t \in (0, t_*\epsilon(B(z)) - \epsilon)$, it follows from the Bernoulli law (17) that

\begin{equation}
\rho^2 h(\rho) + \frac{1}{2}t = \rho^2 B(z).
\end{equation}

A differentiation of (47) with respect to $t$ yields

\begin{equation}
\frac{1}{2} = \frac{\partial_t \rho}{\rho} \left(2\rho^2 B(z) - 2\rho^2 h(\rho) - \rho^3 h'(\rho)\right) \Rightarrow \frac{\partial_t \rho}{\rho} \left(t - \rho^{\gamma+1}\right).
\end{equation}
Furthermore, if one differentiates (47) with respect to \(z\) and uses (28), then one gets
\[
-\rho^2 B'(z) = \frac{\partial_z \rho}{\rho} \left( 2\rho^2 B(z) - 2\rho^2 h(\rho) - \rho^3 h'(\rho) \right) = \frac{\partial_z \rho}{\rho} \left( t - \rho^\gamma + 1 \right). 
\]
Combining (48) and (49) gives
\[
\partial_z \rho = -2\rho^2 \partial_t \rho B'(z).
\]
This together with (30) completes the proof for (44).

(iii). A direct computation gives
\[
\partial_z g_\epsilon(t, z) = \partial_z g(t, z) \varpi_\epsilon(t - t_c(B(z))) + (g^* - g) \varpi'(t - t_c(B(z))) t_c(B(z)) B'(z).
\]
Let \(T^* := \sup_{z \geq 0} t_c(B(z))\). It follows from the definition of \(t_c(B(z))\) in (19) and (27) that if the incoming data is subsonic, then \(T^*\) depends only on \(\gamma\) and \(B^*\). If \(t > T^*\), then by the definition of \(\varpi_\epsilon\), \(\partial_z g_\epsilon = 0\). If \(0 \leq t \leq T^*\), using (44) and the explicit computation on \(t_c\) one obtains (45). \(\Box\)

3. Variational formulation for the free boundary problem

One of the key observations in this paper is that the equation (34) can be written as an Euler-Lagrange equation for a Lagrangian functional, and the jet problem is equivalent to a domain variation problem.

From now on, let \(\Omega\) be the domain bounded by \(S_0\) and \(S_1 \cup (0, \infty) \times \{1\}\). Noticing that \(\Omega\) is unbounded, we thus make a further approximation by considering the problems in a series of truncated domains \(\Omega_{\mu, R} := \Omega \cap \{- \mu < x_1 < R\}\), where \(\mu\) and \(R\) are two large positive numbers. To define the energy functional, we set
\[
G_\epsilon(t, z) := \frac{1}{2} \int_0^t g_\epsilon(\tau, z) d\tau + \frac{1}{\gamma} \left( g_\epsilon(0, z)^{-\gamma} - g_\epsilon(0, Q)^{-\gamma} \right)
\]
and
\[
\Phi_\epsilon(t, z) := -G_\epsilon(t, z) + 2\partial_t G_\epsilon(t, z) t.
\]
Straightforward computations show that
\[
\partial_t \Phi_\epsilon(t, z) = \partial_t g_\epsilon(t, z) t + \frac{1}{2} g_\epsilon(t, z),
\]
is positive by the ellipticity condition (43). Since \(\Phi_\epsilon(0, Q) = -G_\epsilon(0, Q) = 0\), one has \(\Phi_\epsilon(t, Q) > 0\) for \(t > 0\).

Given \(\psi^z \in C(\partial \Omega_{\mu, R})\) with \(0 \leq \psi^z \leq Q\), we consider the minimization problem
\[
\inf_{\psi \in K_{\psi^z}} J^z_{\mu, R, \Lambda}(\psi),
\]
where
\begin{equation}
K_{\psi^\ast} := \{\psi \in H^1(\Omega_{\mu,R}) : \psi = \psi^\ast \text{ on } \partial \Omega_{\mu,R}\}
\end{equation}
and
\begin{equation}
J^\ast_{\mu,R,\Lambda}(\psi) := \int_{\Omega_{\mu,R}} G_\epsilon(|\nabla \psi|^2, \psi) + \lambda^2_\epsilon \chi_{\{\psi < Q\}} \, dx, \quad \lambda_\epsilon := \lambda_\epsilon(\Lambda) := \sqrt{\Phi_\epsilon(\Lambda, Q)}
\end{equation}
with \( \Lambda \) defined in (38).

The existence of minimizers for (52) is proved in Lemma 4.2 for a slightly more general functional. Assuming the existence of minimizers, we now derive the Euler-Lagrange equation for the variational problem (52) in \( \{\psi < Q\} \):

**Lemma 3.1.** Assume that \( \Omega_{\mu,R} \cap \{\psi < Q\} \) is open. Let \( \psi \) be a minimizer of (52). Then \( \psi \) is a weak solution to
\begin{equation}
\nabla \cdot \left( g_\epsilon(|\nabla \psi|^2, \psi) \nabla \psi \right) - \partial_\tau G_\epsilon(|\nabla \psi|^2, \psi) = 0 \quad \text{in} \quad \Omega_{\mu,R} \cap \{\psi < Q\}.
\end{equation}
Furthermore, if \( |\nabla \psi|^2 < t_c(B(\psi)) - \epsilon \), it holds that
\begin{equation}
\partial_\tau G_\epsilon(|\nabla \psi|^2, \psi) = \frac{B'(\psi)}{g(|\nabla \psi|^2, \psi)}.
\end{equation}

**Proof.** Let \( \eta \) be a smooth function compactly supported in \( \Omega_{\mu,R} \cap \{\psi < Q\} \). Direct computations give
\begin{equation}
\frac{d}{d\theta} J^\ast_{\mu,R}(\psi + \theta \eta) \bigg|_{\theta = 0} = \int_{\Omega_{\mu,R}} 2 \partial_\tau G_\epsilon(|\nabla \psi|^2, \psi) \nabla \psi \cdot \nabla \eta + \partial_\tau G_\epsilon(|\nabla \psi|^2, \psi) \eta \, dx.
\end{equation}
By the definition of \( G_\epsilon \) in (50) minimizers of (52) satisfy the equation (55).

Next, by the definition of \( G_\epsilon \) in (50) one has
\begin{equation}
\partial_\tau G_\epsilon(t, \tau, z) = \frac{1}{2} \int_0^t \partial_\tau g_\epsilon(\tau, z) \, d\tau - g_\epsilon(0, \tau, z)^{-(\gamma+1)} \partial_\tau g_\epsilon(0, \tau, z).
\end{equation}
This together with (44) gives
\begin{equation}
\partial_\tau G_\epsilon(t, \tau, z) = B'(z) \rho(t, z) - B'(z) \rho(0, z) - g(0, \tau, z)^{-(\gamma+1)} \partial_\tau g(0, \tau, z) \text{ in } \mathcal{R}_\epsilon,
\end{equation}
where \( \mathcal{R}_\epsilon := \{(t, \tau, z) \in [0, \infty) \times [0, Q] : 0 \leq t < t_c(B(z)) - \epsilon\} \). It follows from (48) that \( \partial_\tau \rho(0, \tau, z) = -\frac{1}{2} \rho(0, \tau, z)^{-\gamma} \). Hence from (44) one has
\begin{equation}
\partial_\tau g(0, \tau, z) = -B'(z) \rho(0, \tau, z)^{-\gamma} = -B'(z) g(0, \tau, z)^{\gamma}.
\end{equation}
A substitution of (56) into the expression of \( \partial_\tau G_\epsilon(t, \tau, z) \) gives
\begin{equation}
\partial_\tau G_\epsilon(t, \tau, z) = B'(z) \rho(t, \tau, z) = \frac{B'(z)}{g(t, \tau, z)} \text{ for } (t, \tau, z) \in \mathcal{R}_\epsilon.
\end{equation}
This completes the proof for the lemma. \( \square \)
Next we show that local minimizers of (52) satisfy the free boundary condition in the following domain variation sense.

Lemma 3.2. Let \( \psi \) be a local minimizer of the problem (52). Then

\[
\Phi_\epsilon(\vert \nabla \psi \vert^2, \psi) = \lambda_\epsilon^2 \quad \text{on } \Gamma_\psi
\]

in the sense that

\[
\lim_{s \searrow 0} \int_{\partial \{ \psi < Q - s \}} \left[ \Phi_\epsilon(\vert \nabla \psi \vert^2, \psi) - \lambda_\epsilon^2 \right] (\eta \cdot \nu) \, d\mathcal{H}^1 = 0
\]

for any \( \eta \in C_0^\infty(\Omega_{\mu,R}; \mathbb{R}^2) \), where \( \mathcal{H}^1 \) is the one dimensional Hausdorff measure.

Remark 3.3 (Relation between \( \lambda_\epsilon \) and \( \Lambda \)). If the free boundary \( \Gamma_\psi := \partial \{ \psi < Q \} \cap \Omega_{\mu,R} \) is smooth and \( \psi \) is smooth near \( \Gamma_\psi \), then it follows from the monotonicity of \( t \mapsto \Phi_\epsilon(t, z) \) for each \( z \) and the definition of \( \lambda_\epsilon \) in (54) that

\[
\vert \nabla \psi \vert = \Lambda \quad \text{on } \Gamma_\psi.
\]

Moreover, there is a constant \( C > 0 \) depending only on \( B_* \) and \( \gamma \) such that

\[
\lambda_\epsilon^2 \geq C \Lambda^2.
\]

Indeed, by the definition of \( G_\epsilon \) and \( \Phi_\epsilon \) in (50) and (51), respectively, one has

\[
\lambda_\epsilon^2 = \Phi_\epsilon(\Lambda^2, Q) = -\frac{1}{2} \int_0^{\Lambda^2} g_\epsilon(\tau, Q) \, d\tau + g_\epsilon(\Lambda^2, Q) \Lambda^2 \geq \frac{1}{2} g_\epsilon(\Lambda^2, Q) \Lambda^2 \geq C \Lambda^2.
\]

Proof of Lemma 3.2. Let \( \eta \in C_0^\infty(\Omega_{\mu,R}; \mathbb{R}^2) \) and \( \tau_\vartheta(x) := x + \vartheta \eta(x) \). If \( \vert \vartheta \vert \) is sufficiently small, then \( \tau_\vartheta \) is a diffeomorphism of \( \Omega_{\mu,R} \). Let \( \psi_\vartheta(y) := \psi(\tau_\vartheta^{-1}(y)) \). Since \( \psi_\vartheta \in K_{\psi_\vartheta} \) and \( \psi \) is a minimizer, then

\[
0 \leq J_{\mu,R}^\epsilon(\psi_\vartheta) - J_{\mu,R}^\epsilon(\psi)
\]

\[
= \int_\Omega \left( G_\epsilon(\vert \nabla \psi \nabla \tau_\vartheta \vert^{-1}, \psi) + \lambda_\epsilon^2 \chi_{\{ \psi < Q \}} \right) \det(\nabla \tau_\vartheta)
\]

\[
= \int_\Omega \left( G_\epsilon(\vert \nabla \psi \vert^2, \psi) + \lambda_\epsilon^2 \chi_{\{ \psi < Q \}} \right) \nabla \cdot \eta
\]

\[
= \vartheta \int_\Omega \left( G_\epsilon(\vert \nabla \psi \vert^2, \psi) + \lambda_\epsilon^2 \chi_{\{ \psi < Q \}} \right) \nabla \cdot \eta
\]

\[
- 2 \vartheta \int_\Omega \partial_t G_\epsilon(\vert \nabla \psi \vert^2, \psi) \nabla \psi \nabla \eta \nabla \psi + o(\vartheta).
\]

Dividing \( \vartheta \) on both sides of (57) and passing to the limit \( \vartheta \to 0 \) yield

\[
\int_\Omega \left( G_\epsilon(\vert \nabla \psi \vert^2, \psi) + \lambda_\epsilon^2 \chi_{\{ \psi < Q \}} \right) \nabla \cdot \eta
- 2 \int_\Omega \partial_t G_\epsilon(\vert \nabla \psi \vert^2, \psi) \nabla \psi \nabla \eta \nabla \psi = 0.
\]

(58)
Note that
\[ (G_{\epsilon}(|\nabla \psi|^2, \psi) + \lambda^2 \chi_{\psi<Q}) \nabla \cdot \eta = \nabla \cdot \left( (G_{\epsilon}(|\nabla \psi|^2, \psi) + \lambda^2 \chi_{\psi<Q}) \eta \right) - 2 \partial_t G_{\epsilon} \nabla \psi D^2 \psi \eta - \partial_z G_{\epsilon} \nabla \psi \cdot \eta \]
and
\[ \nabla \psi \nabla \eta \nabla \psi + \nabla \psi \nabla^2 \psi \eta = \nabla (\eta \cdot \nabla \psi) \cdot \nabla \psi. \]

Using the divergence theorem for (58) we have
\[ 0 = \lim_{s \to 0} \int_{\Omega \cap \{\psi<Q\}} (G_{\epsilon}(|\nabla \psi|^2, \psi) + \lambda^2 \chi_{\psi<Q}) \nabla \cdot \eta - 2 \partial_t G_{\epsilon}(|\nabla \psi|^2, \psi) \nabla \psi \nabla \eta \nabla \psi
\]
\[ = \lim_{s \to 0} \int_{\partial \psi<Q-s} \left[ G_{\epsilon}(|\nabla \psi|^2, \psi) - 2 \partial_t G_{\epsilon}(|\nabla \psi|^2, \psi) |\nabla \psi|^2 + \lambda^2 \right] (\eta \cdot \nu) \]
\[ + \int_{\Omega \cap \{\psi<Q\}} \left[ 2 \nabla \cdot \left( \partial_t G_{\epsilon}(|\nabla \psi|^2, \psi) \nabla \psi \right) - \partial_z G_{\epsilon}(|\nabla \psi|^2, \psi) \right] (\nabla \psi \cdot \eta) \]
\[ = \lim_{s \to 0} \int_{\partial \psi<Q-s} \left[ -\Phi_{\epsilon}(|\nabla \psi|^2, \psi) + \lambda^2 \right] (\eta \cdot \nu), \]
where we have used the equation of $\psi$ in (55) in the open set $\{\psi<Q\}$ to get the last equality. This finishes the proof of the lemma. \qed

4. The existence and regularity for the free boundary problem

In this section we study the existence and regularity of the minimizer, as well as the regularity of the free boundary away from the nozzle.

4.1. Existence of minimizers. For the ease of notations in the sequel let
\[ \mathcal{D} := \Omega_{\mu, R}, \quad \mathcal{G}(p, z) := G_{\epsilon}(|p|^2, z), \quad \lambda := \lambda_{\epsilon}, \quad \text{and} \quad \mathcal{J}(\psi) := J_{\mu, R, \Lambda}(\psi). \]
Thus $\mathcal{D}$ is a bounded Lipschitz domain in $\mathbb{R}^2$ and is contained in the infinite strip $\mathbb{R} \times [0, \bar{H}]$, $\mathcal{G} : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ is smooth in $p$ and $C^2$ in $z$ (with further properties summarized in Proposition 4.1 below), and $\lambda$ is a positive constant. The minimization problem (52)–(60) can be rewritten as finding minimizers of
\[ \mathcal{J}(\psi) := \int_{\mathcal{D}} \mathcal{G}(\nabla \psi, \psi) + \lambda^2 \chi_{\psi<Q} \, dx, \]
over the admissible set
\[ \mathcal{K}_{\psi^\sharp} := \{ \psi \in H^1(\mathcal{D}) : \psi = \psi^\sharp \text{ on } \partial \mathcal{D} \}. \]
Here $\psi^\sharp$ is given and assumed to be continuous with $0 \leq \psi^\sharp \leq Q$ on $\partial \mathcal{D}$.

Properties of $\mathcal{G}$ is summarized in the following proposition.

**Proposition 4.1.** Let $G_{\epsilon}$ be defined in (50). Then the following properties hold:
(i) There exist two positive constants $b_*$ and $b^*$, where $b_* = b_*(\gamma, B_*)$ and $b^* = b^*(\gamma, B_*, \epsilon)$, such that

\begin{align}
(61) & \quad b_* |p|^2 \leq p_i \partial_{p_i} G(p, z) \leq b_*^{-1} |p|^2, \\
(62) & \quad b_* |\xi|^2 \leq \xi_i \partial_{p_i}^2 G(p, z) \xi_j \leq b_* |\xi|^2 \text{ for all } \xi \in \mathbb{R}^2.
\end{align}

(ii) One has

\begin{equation}
\partial_z G(p, z) = 0 \text{ in } \{(p, z) : p \in \mathbb{R}^2, z \in (-\infty, 0) \cup [Q, \infty)\}.
\end{equation}

Furthermore, there exists a constant $\delta = C \epsilon^{-1} \kappa_0$, where $\kappa_0$ is the constant in (29) and $C = C(\gamma, B_*) > 0$, such that

\begin{align}
(64) & \quad |\partial_z G(p, z)| \leq \delta (Q - z)_+,
(65) & \quad |e^{-1} \partial_z G(p, z) + |\partial_{zp_i} G(p, z)| + |\partial_{zz} G(p, z)| \leq \delta,
(66) & \quad G(0, Q) = 0, \quad G(p, z) \geq b_* |p|^2 - \delta (Q - z)^2_+.
\end{align}

Proof. (i). In view of the definition of $G_*$ in (50) a straightforward computation yields

\begin{align*}
p_i \partial_{p_i} G(p, z) &= g_e(|p|^2, z)|p|^2, \\
\partial_{p_i} G(p, z) &= g_e(|p|^2, z) \delta_{ij} + 2 \partial_i g_e(|p|^2, z) p_ip_j.
\end{align*}

Thus from (41) and (43) one immediately gets (61)–(62).

(ii). To show (63) we note that

\begin{align*}
\partial_z G(p, z) &= \frac{1}{2} \int_0^{|p|^2} \partial_z g_e(\tau, z) d\tau - (g_e^{-\gamma-1} \partial_z g_e)(0, z) \\
&= \frac{1}{2} \int_0^{|p|^2} \partial_z g_e(\tau, z) d\tau + \frac{B'(z)}{g_e(0, z)}.
\end{align*}

This, together with the estimate for $|\partial_z g_e|$ in (44) and (45), yields that $|\partial_z G(p, z)| \leq C |B'(z)|$. Since $B'(z) = 0$ for $z \in (-\infty, 0) \cup [Q, \infty)$, which follows from (26) and our extension, then (63) holds true.

It follows from the explicit expression for $\partial_z G$, (45), (46) as well as the bound for $B'$ in (29) that

\[|\partial_z G(p, z)| \leq C \kappa_0 (g_e(|p|^2, z) - g_e(0, z)) + C \kappa_0 \leq C \kappa_0\]

where $C = C(\gamma, B_*)$. The estimate for $\partial_{zp_i} G$ follows from the relation $|\partial_{zp_i} G(p, z)p_i| = |\partial_z g_e(|p|^2, z)||p|^2$ and (45). Using the expression for $\partial_z G$ and (56) one has

\[\partial_{zz} G(p, z) = \frac{1}{2} \int_0^{|p|^2} \partial_{zz} g_e(\tau, z) d\tau + \frac{B''(z)}{g_e(0, z)} + (B'(z))^2 g_e(0, z)^{\gamma-2}.
\]

It follows from (46) and (29) that $|\partial_{zz} G(p, z)| \leq C \kappa_0 \epsilon^{-1}$. 

In view of (50), one has \( G(0,Q) = 0 \). Thus as long as the estimates (65) and (63) hold, one immediately has (64) and (66). This completes the proof of the proposition. □

With Proposition 4.1 at hand, the existence of minimizers for the variational problem (52) follows from standard theory for calculus of variations.

Lemma 4.2. Assume \( G \) satisfies (61), (62) and (66). Then the minimization problem (59) over the admissible set (60) has a minimizer.

Proof. First, in view of Proposition 4.1 it is not hard to find a function \( \tilde{\psi} \in \mathcal{K}_{\psi^*} \) with \( J(\tilde{\psi}) < \infty \). Let \( \{\psi_k\} \subset \mathcal{K}_{\psi^*} \) be a minimizing sequence. By (66) one has

\[
\mathbf{b}_\ast \int_D |\nabla \psi_k|^2 \leq J(\psi_k) + \delta Q^2 |D|.
\]

Then it follows from the Poincaré inequality that \( \psi_k - \psi^* \) are bounded in \( L^2(D) \).

Therefore, there exists a subsequence (not relabeled) satisfying

\[
\psi_k \rightarrow \psi \text{ in } L^2(D) \cap L^2(\partial D), \quad \psi_k \rightarrow \psi \text{ a.e. in } D, \quad \nabla \psi_k \rightarrow \nabla \psi \text{ in } L^2(D).
\]

Since \( G \) is convex with respect to \( p \), using Fatou’s lemma one obtains

\[
J(\psi) = \int_D G(\nabla \psi, \psi) + \lambda^2 \chi_{\{\psi < Q\}} \, dx \leq \liminf_{k \to \infty} \int_D G(\nabla \psi_k, \psi_k) + \lambda^2 \chi_{\{\psi_k < Q\}} \, dx = \liminf_{k \to \infty} J(\psi_k).
\]

This means that \( \psi \in \mathcal{K}_{\psi^*} \) is a minimizer. □

4.2. \( L^\infty \) estimate and Hölder regularity. The main goal of this subsection is to establish the \( L^\infty \) bounds and the Hölder regularity for the local minimizers. Firstly, one can show that the minimizers are supersolutions of the elliptic equation

\[
\partial_i (\partial_p G(\nabla \psi, \psi)) - \partial_p G(\nabla \psi, \psi) = 0.
\]

Lemma 4.3. Let \( \psi \) be a minimizer for (59). Then \( \psi \) satisfies

\[
\int_D \partial_p G(\nabla \psi, \psi) \partial_i \zeta + \partial_p G(\nabla \psi, \psi) \zeta \geq 0, \text{ for all } \zeta \geq 0, \quad \zeta \in C_0^\infty(D).
\]

Proof. Since \( \psi \) is a minimizer, for any \( \vartheta > 0 \),

\[
0 \leq \vartheta^{-1} (J(\psi + \vartheta \zeta) - J(\psi)) \leq \vartheta^{-1} \int_D G(\nabla \psi + \vartheta \nabla \zeta, \psi + \vartheta \zeta) - G(\nabla \psi, \psi)
\]

\[
= \vartheta^{-1} \int_D G(\nabla \psi + \vartheta \nabla \zeta, \psi) - G(\nabla \psi, \psi) + \vartheta^{-1} \int_D G(\nabla \psi + \vartheta \nabla \zeta, \psi + \vartheta \zeta) - G(\nabla \psi + \vartheta \nabla \zeta, \psi).
\]
where the second inequality follows from the simple fact $\{\psi + \vartheta \zeta < Q\} \subset \{\psi < Q\}$. Thus taking $\vartheta \to 0$ yields (68).

Next, one has the following $L^\infty$ estimate for the minimizer.

**Lemma 4.4.** If $\psi$ is a minimizer for (59) over $K_{\vartheta \psi}$ defined in (60), then

$$0 \leq \psi \leq Q.$$

**Proof.** Set $\psi^\vartheta := \psi + \vartheta \min\{0, Q - \psi\}$ for $\vartheta > 0$. Since $\psi^\vartheta \leq Q$ on $\partial D$, one has $\psi^\vartheta \in K_{\vartheta \psi}$ and thus $\psi^\vartheta \in K_{\psi\vartheta}$. Since $\{\psi^\vartheta < Q\} = \{\psi < Q\}$ and $G$ is convex with respect to $p$, it follows from the same argument as in Lemma 4.3 that

$$0 \leq \vartheta - 1 (J(\psi^\vartheta) - J(\psi)) \leq \vartheta - 1 (\partial_{p_i} G(\nabla \psi^\vartheta, \psi) - \partial_{p_i} G(\nabla \psi^\vartheta, \psi)) (Q - \psi)$$

Letting $\vartheta \to 0$, there holds:

$$0 \leq \int_{D \cap \{\psi > Q\}} -\partial_{p_i} G(\nabla \psi, \psi) \partial_i \psi + \partial_{p_i} G(\nabla \psi, \psi)(Q - \psi) \leq -b_* \int_{D \cap \{\psi > Q\}} |\nabla \psi|^2.$$

This implies $\psi \leq Q$.

The proof for the lower bound is similar. Set $\tilde{\psi}^\vartheta := \psi - \vartheta \min\{0, \psi\}$ with $\vartheta > 0$. It is straightforward to check that for $\vartheta \in (0, 1)$, $\{\tilde{\psi}^\vartheta < Q\} = \{\psi < Q\}$. Thus

$$0 \leq \vartheta - 1 (J(\tilde{\psi}^\vartheta) - J(\psi)) \leq \vartheta - 1 (\partial_{p_i} G(\nabla \tilde{\psi}^\vartheta, \psi) - \partial_{p_i} G(\nabla \tilde{\psi}^\vartheta, \psi)) (Q - \psi)$$

Letting $\vartheta \to 0$ gives

$$\int_{D \cap \{\psi < 0\}} -\partial_{p_i} G(\nabla \psi, \psi) \partial_i \psi - \partial_{p_i} G(\nabla \psi, \psi) \psi \geq 0.$$ 

It follows from (63) and (61) that the measure of the set $\{\psi < 0\}$ is zero. Hence $\psi \geq 0$. This finishes the proof of the lemma. 

Now we are in position to prove the Hölder regularity for the local minimizers. The proof follows from standard Morrey type estimates.

**Lemma 4.5.** Let $\psi$ be a minimizer for (59) over the admissible set $K_{\psi\vartheta}$. Then $\psi \in C^{\beta, \alpha}_{loc}(D)$ for any $\alpha \in (0, 1)$. Moreover,

$$\|\psi\|_{C^{\beta, \alpha}(K)} \leq C(b_*, Q, K, \delta, \lambda, \alpha) \text{ for any } K \subset D.$$
Proof. Given any $B_r \subset \mathcal{D}$, let $\phi \in H^1(B_r)$ be the solution to
\begin{equation}
\begin{cases}
\partial_i (\partial_p G(\nabla \phi, Q)) = 0 \text{ in } B_r, \\
\phi = \psi \text{ on } \partial B_r.
\end{cases}
\end{equation}
Such a solution can be obtained by minimizing the functional $\int_{B_r} G(\nabla \phi, Q) dx$ among $\phi \in H^1(B_r)$ with $\phi = \psi$ on $\partial B_r$. It follows from the regularity theory for elliptic equations (cf. for example [45]) that
\[ \|\nabla \phi\|_{L^2(B_{r_1})} \leq C(b_*) \|\nabla \phi\|_{L^2(B_{r_2})} \frac{r_1}{r_2} \text{ for any } 0 < r_1 < r_2 < r. \]
Furthermore, applying the maximum principle one has $0 \leq \phi \leq Q$ in $B_r$.

Since $\psi$ is a minimizer, one has
\[ \int_{B_r} G(\nabla \psi, \psi) - G(\nabla \phi, \phi) \leq \lambda^2 \int_{B_r} \left( \chi_{\{ \phi < Q \}} - \chi_{\{ \psi < Q \}} \right). \]
It follows from (62) and (65) that
\[ G(\nabla \psi, \psi) - G(\nabla \phi, \phi) = (G(\nabla \psi, \psi) - G(\nabla \psi, Q)) + (G(\nabla \psi, Q) - G(\nabla \phi, Q)) + (G(\nabla \phi, Q) - G(\nabla \phi, \phi)) \geq -\delta(Q - \psi) + \partial_p G(\nabla \phi, Q) \partial_i (\psi - \phi) + b_* |\nabla \psi - \nabla \phi|^2 - \delta(Q - \phi). \]
Multiplying the equation in (69) with $\psi - \phi$ and integrating by parts we obtain
\[ \int_{B_r} \partial_p G(\nabla \phi, Q) \partial_i (\psi - \phi) = 0. \]
Combining the above estimates together one has
\[ b_* \int_{B_r} |\nabla \psi - \nabla \phi|^2 \leq \int_{B_r} \delta(Q - \phi) + \delta(Q - \psi) + \lambda^2 (\chi_{\{ \phi < Q \}} - \chi_{\{ \psi < Q \}}). \]
Since $\int_{B_r} \chi_{\{ \phi < Q \}} \leq |B_r|$, this together with the simple relation $1 - \chi_{\{ \psi < Q \}} = \chi_{\{ \psi = Q \}}$ gives
\[ b_* \int_{B_r} |\nabla \psi - \nabla \phi|^2 \leq 2\delta Q + \lambda^2 \chi_{\{ \psi = Q \}} \leq C(\delta, Q, \lambda) r^2. \]
The desired Hölder regularity then follows from the standard Morrey type estimates (cf. [41, the proof for Theorem 2.2 in Chap. III]).

4.3. Lipschitz regularity and nondegeneracy. In this subsection we establish the (optimal) Lipschitz regularity of the solution and the so-called nondegeneracy property, which play an important role to get the measure theoretic properties of the free boundary. We start with the following comparison principle for the associated elliptic equation.
Lemma 4.6. Let $\psi \in H^1(D)$ be a supersolution of the equation (67) in the sense of (68) and $\phi \in H^1(D)$ be a solution of (67) in the following sense

\[(70) \quad \int_D \partial_p G(\nabla \phi, \phi) \partial \zeta + \partial_z G(\nabla \phi, \phi) \zeta = 0, \text{ for all } \zeta \in C_0^\infty(D).\]

Assume that $\phi \leq \psi$ on $\partial D$. Then $\phi \leq \psi$ in $D$ as long as $\delta$ in Proposition 4.1 is sufficiently small depending on $b^*$ and $\bar{H}$.

Proof. Let $\eta := (\phi - \psi)^+$. Then $\eta = 0$ on $\partial D$. Substituting $\zeta = \eta$ in (68) and using (70) one has

\[\int_D (\partial_p G(\nabla \phi, \phi) - \partial_p G(\nabla \psi, \psi)) \partial \eta + \int_D (\partial_z G(\nabla \phi, \phi) - \partial_z G(\nabla \psi, \psi)) \zeta \eta \leq 0.\]

The convexity of $G$ in (62) and the triangle inequality yield

\[\int_D (\partial_p G(\nabla \phi, \phi) - \partial_p G(\nabla \psi, \psi)) \partial \eta \geq b_* \int_D |\nabla \eta|^2 + \int_D (\partial_p G(\nabla \psi, \phi) - \partial_p G(\nabla \psi, \psi)) \partial \eta,\]

where the last integral in the above inequality can be estimated from (65) as

\[\int_D (\partial_p G(\nabla \psi, \phi) - \partial_p G(\nabla \psi, \psi)) \partial \eta = \int_D \int_0^1 \partial_{z \psi} G(\nabla \psi, \phi + s(\psi - \phi)) \partial \eta \, ds \, (\psi - \phi) \geq -\delta \int_D |\nabla \eta| \eta.\]

Similarly, using (65) and the triangle inequality one has

\[\int_D (\partial_z G(\nabla \phi, \phi) - \partial_z G(\nabla \psi, \psi)) \eta \geq -\delta \int_D |\nabla \eta| + |\eta|^2.\]

Combining the above estimates together yields

\[b_* \int_D |\nabla \eta|^2 \leq 2\delta \int_D (|\eta \nabla \eta| + \eta^2).\]

Applying Cauchy-Schwarz and Poincaré inequalities to $\eta(x_1, \cdot)$ for each $x_1$ (notice that $\eta = 0$ on $\partial D$ and $D \subset \mathbb{R} \times [0, \bar{H}]$), one has

\[b_* \int_D |\nabla \eta|^2 \leq \delta \int_D |\nabla \eta|^2 + C\delta \bar{H}^2 \int_D |\nabla \eta|^2,\]

where $C > 0$ is a universal constant. If $\delta$ is sufficiently small depending on $b_*$ and $\bar{H}$, then necessarily $\nabla \eta = 0$ in $D$, which implies that $\eta = 0$ in $D$. This completes the proof of the lemma. \qed

In the next two lemmas we show that a local minimizer $\psi$ has the linear growth at the free boundary. For notational convenience, from now on, denote

\[(71) \quad \psi^* := Q - \psi \quad \text{and} \quad \tilde{G}(p, z) := G(-p, Q - z).\]
If \( \psi \) is a minimizer for (59), then \( \psi^* \) satisfies
\[
\partial_x, (\partial_p \tilde{G}(\nabla \psi^*, \psi^*)) - \partial_x \tilde{G}(\nabla \psi^*, \psi^*) \geq 0 \text{ in } D,
\]
and
\[
\partial_x, (\partial_p \tilde{G}(\nabla \psi^*, \psi^*)) - \partial_x \tilde{G}(\nabla \psi^*, \psi^*) = 0 \quad \text{in } \{ y \in D : \psi^*(y) > 0 \}.
\]
Define \( \psi^*_{x,r,a}(x) := a\psi^*(\bar{x} + rx)/r \) with \( a, r > 0 \). A straightforward computation shows that \( \psi^*_{x,r,a} \) is a solution to
\[
\partial_x, (\partial_p \tilde{G}_{r,a}(\nabla \phi, \phi)) - \partial_x \tilde{G}_{r,a}(\nabla \phi, \phi) = 0,
\]
where
\[
\tilde{G}_{r,a}(p,z) := a^2 \tilde{G}(a^{-1}p, ra^{-1}z).
\]
The properties of \( G \) in Proposition 4.1 can be translated into the properties of \( \tilde{G}_{r,a} \) in an obvious fashion: \( \tilde{G}_{r,a} \) satisfies (61) and (62) with the same constants \( b_* \) and \( b^* \), \( \text{supp}(\partial_x \tilde{G}_{r,a}) \subset \mathbb{R} \times [0, aQ/r) \), and
\[
|\partial_x \tilde{G}_{r,a}(p,z)| \leq ra\delta, \quad |\partial_p \tilde{G}_{r,a}| \leq r\delta, \quad |\partial_{zz} \tilde{G}_{r,a}| \leq r^2\delta.
\]
Therefore, solutions to the rescaled equation (73) enjoys the comparison principle as well, provided \( \delta \) is sufficiently small depending on \( a, r, b_* \), and \( \tilde{H} \).

**Lemma 4.7.** Let \( \psi \) be a local minimizer of (59). Let \( \bar{x} \in \{ \psi > 0 \} \) satisfy \( \text{dist}(\bar{x}, \Gamma_\psi) \leq \min \{ 1, \frac{1}{4} \text{ dist}(\bar{x}, \partial D) \} \) where \( \psi^* \) is defined in (71). Then if \( \delta = \delta(b_*) \) is sufficiently small, there exists \( C = C(b_*) > 0 \) such that
\[
\psi^*(\bar{x}) \leq C \text{ dist}(\bar{x}, \Gamma_\psi).
\]

**Proof.** Let \( m_0 := \psi^*(\bar{x}) > 0 \) and \( r := \text{dist}(\bar{x}, \Gamma_\psi) \in (0,1) \). Let \( \psi^*_{\bar{x},r}(x) := \psi^*(\bar{x} + rx)/r \). Then \( \psi^*_{\bar{x},r} \) satisfies
\[
\mathcal{G}(\nabla \psi^*_{\bar{x},r}, \psi^*_{\bar{x},r}) \geq 0 \text{ in } D_{\bar{x},r} \quad \text{and} \quad \mathcal{G}(\nabla \psi^*_{\bar{x},r}, \psi^*_{\bar{x},r}) = 0 \in \{ \psi^*_{\bar{x},r} > 0 \},
\]
where \( D_{\bar{x},r} := \{ \frac{x-\bar{x}}{r} | x \in D \} \) and
\[
\mathcal{G}(p,z) := \partial_x, (\partial_p \tilde{G}(p,z)) - \partial_x \tilde{G}(p,z) \quad \text{with} \quad \tilde{G}(p,z) := \tilde{G}_{r,1}(p,z).
\]
By the scaling property, without loss of generality, we assume that \( r = 1 \) and \( \bar{x} = 0 \) after the translation of the coordinates. The aim is to show that there exists a constant \( C \) depending only on \( b_* \) such that \( m_0 \leq C \lambda \). The proof is divided into four steps.

**Step 1. Construction of the barrier function.** For any given \( y \in \Gamma_\psi \cap \overline{B_1} \), let \( \phi \) be the solution of
\[
\begin{cases}
\mathcal{G}(\nabla \phi, \phi) = 0 & \text{in } B_2(y), \\
\phi = \psi^* & \text{on } \partial B_2(y).
\end{cases}
\]
By the comparison principle (cf. Lemma 4.6), if $\delta = \delta(b_*)$ is small enough one has $\phi \geq \psi^*$ in $B_2(y)$. In particular, $\phi(0) \geq \psi^*(0) = m_0$. The interior Hölder estimate for (76) yields that for some $C_0 = C_0(b_*) > 0$,

$$\phi(x) \geq C_0 m_0, \quad \text{for any } x \in B_1(y).$$

We claim that $\phi$ actually satisfies

$$\phi(x) \geq \frac{1}{2} C_0 m_0 (2 - |x - y|) \quad \text{for any } x \in B_2(y).$$

Indeed, by virtue of (77), it suffices to prove (78) in $B_2(y) \setminus B_1(y)$. Set

$$\Phi_0(x) := C \left( e^{-\mu|x-y|^2} - e^{-4\mu} \right).$$

One can choose $C_\mu > 0$ such that $\phi \geq \Phi_0$ on $\partial(B_2(y) \setminus B_1(y))$. A straightforward computation gives

$$\mathcal{G}(\nabla \Phi_0, \Phi_0) = \sum_{i,j} \partial_{p,p_i} \mathcal{G}(\nabla \Phi_0, \Phi_0) \partial_{ij} \Phi_0 + \partial_{p,z} \mathcal{G}(\nabla \Phi_0, \Phi_0) \partial_i \Phi_0 + \partial_z \mathcal{G}(\nabla \Phi_0, \Phi_0)$$

$$\geq \sum_{i,j} \partial_{p,p_i} \mathcal{G}(\nabla \Phi_0, \Phi_0) \left( 2\mu C_\mu e^{-\mu|x-y|^2} (2\mu(x_i - y_i)(x_j - y_j) - \delta_{ij}) \right)$$

$$- \delta(2\mu C_\mu e^{-\mu|x-y|^2}) - \partial_z \mathcal{G}(\nabla \Phi_0, \Phi_0).$$

It follows from (64) and (74) that

$$\partial_z \mathcal{G}(\nabla \Phi_0, \Phi_0) \leq -\delta|\Phi_0| \leq -\delta C_\mu e^{-\mu|x-y|^2}.$$

Therefore, if $\mu$ is chosen to be sufficiently large and $\delta$ is sufficiently small depending on $b_*$, then $\mathcal{G}(\nabla \Phi_0, \Phi_0) \geq 0$ in $B_2(y) \setminus B_1(y)$. By the comparison principle (Lemma 4.6), $\phi \geq \Phi_0$ in $B_2(y) \setminus B_1(y)$. This proves the claim (78).

**Step 2. Gradient estimate.** We claim that $\delta = \delta(b_*)$ is sufficiently small, then

$$\int_{B_2(y)} |
abla (\phi - \psi^*)|^2 \leq C(b_*) \int_{B_2(y)} \lambda^2 \chi_{\{\psi^*=0\}}.$$

Indeed, the minimality of $\psi^*$ yields

$$\int_{B_2(y)} \mathcal{G}(\nabla \psi^*, \psi^*) - \mathcal{G}(\nabla \phi, \phi) \leq \lambda^2 \int_{B_2(y)} \chi_{\{\phi>0\}} - \chi_{\{\psi^*>0\}}.$$

Noting that $\phi > 0$ in $B_2(y)$ and arguing similarly as in Lemma 4.5 one has

$$b_* \int_{B_2(y)} |
abla (\phi - \psi^*)|^2 \leq 2\delta \int_{B_2(y)} (\phi - \psi^*) + \int_{B_2(y)} \lambda^2 \chi_{\{\psi^*=0\}}.$$

By Hölder and Poincaré inequalities, the first term on the right-hand side of (80) can be absorbed by the left-hand side of (80) when $\delta = \delta(b_*)$ is sufficiently small. Thus one has (79).
Step 3. Poincaré type estimate. We claim the following Poincaré type estimate

\[ C_0 m_0 |S|^{1/2} \leq C \| \nabla (\phi - \psi^*) \|_{L^2(B_2(y))} \quad \text{with} \quad S := \{ x \in B_2(y) : \psi^*(x) = 0 \}. \]

The proof is similar as [1, Lemma 3.2] or [5, Lemma 2.2]. For the completeness, we provide the details here. For \( w \in B_1(y) \), consider a transformation \( A_w \) from \( B_2(y) \) to itself which fixes \( \partial B_2(y) \) and maps \( w \) to \( y \), for instance,

\[ A_w^{-1}(x) = 2 - \frac{|x-y|}{2}(w-y) + x. \]

Set \( \psi^*_w(x) := \psi^*(A_w^{-1}(x)) \) and \( \phi_w(x) := \phi(A_w^{-1}(x)) \). Given a direction \( \xi \in S^1 \), define

\[ r_\xi := \inf \{ r : 1/4 \leq r \leq 2, \psi^*_w(y + r\xi) = 0 \}. \]

Hence

\[ \phi_w(y + r_\xi \xi) = \int_2^{r_\xi} \frac{d}{dr}(\phi_w - \psi^*_w)(y + r\xi)dr \leq \int_{r_\xi}^2 |\nabla(\phi_w - \psi^*_w)| dr \]

\[ \leq \sqrt{2 - r_\xi} \left( \int_{r_\xi}^2 |\nabla(\phi_w - \psi^*_w)|^2 dr \right)^{1/2}. \]

On the other hand, by (78), one has

\[ \phi_w(y + r_\xi \xi) \geq \frac{1}{2} C_0 m_0 \left( 2 - \frac{2 - r_\xi}{2}(w - y) + r_\xi \xi \right) \geq \frac{1}{4} C_0 m_0 (2 - r_\xi). \]

Combining (82) and (83) together gives

\[ (C_0 m_0)^2 (2 - r_\xi) \leq 16 \int_{r_\xi}^2 |\nabla(\phi_w - \psi^*_w)|^2 dr. \]

An integration of \( \xi \) over \( S^1 \) yields

\[ (C_0 m_0)^2 \int_{B_2(y) \setminus B_{1/2}(w)} \chi_{\{\psi^* = 0\}} \leq C \int_{B_2(y)} |\nabla(\phi - \psi^*)|^2. \]

A further integration over \( w \in B_1(y) \) yields

\[ (C_0 m_0)^2 |S| \leq C \int_{B_2(y)} |\nabla(\phi - \psi^*)|^2. \]

This is the desired estimate (81).

Step 4. Conclusion. Combining (81) and (79) from Steps 2 and 3 gives

\[ m_0 |S|^{1/2} \leq C \lambda |S|^{1/2} \]

for some \( C = C(b_*) \). If \( |S| > 0 \), then \( m_0 \leq C \lambda \), which is the desired estimate. Otherwise, \( \psi^* = \phi \) a.e. in \( B_2(y) \). By the interior regularity theory for elliptic equations ([44]), \( \psi^* \) and \( \phi \) are continuous. Thus \( \psi^* = \phi \) pointwise in \( B_2(y) \). This, however, contradicts with the fact that \( y \) is a free boundary point. \( \square \)
The next lemma investigates the nondegeneracy of the free boundary, whose proof is similar to that for Lemma 4.7 and is based on the minimality and suitable barrier functions.

**Lemma 4.8.** Let $\psi$ be a minimizer for (59) and let $\psi^* := Q - \psi$. Assume that $\delta = \delta(b_\ast)$ is sufficiently small. Then for any $p > 1$ and any $0 < r < 1$, there exists a constant $c_r > 0$ such that for any $B_R(\bar{x}) \subset D$ with $R \leq 1$, if

$$
\frac{1}{R} \left( \frac{1}{|B_R(\bar{x})|} \int_{B_R(\bar{x})} |\psi^*|^p \right)^{1/p} \leq c_r \lambda,
$$

then $\psi^* = 0$ in $B_r R(\bar{x})$.

**Proof.** As in Lemma 4.7, we assume that $\bar{x} = 0$ and $R = 1$. By the $L^\infty$ estimate for the subsolution of (72) (cf. [44, Theorem 8.17] with $f = g = 0$), one has for any $r \in (0, 1)$,

$$M_r := \frac{1}{\sqrt{r}} \sup_{B_r} |\psi^*| \leq C \|\psi^*\|_{L^p(B_1)} \leq C C_r \lambda,$$

where $C = C(p, b_\ast)$. Thus it suffices to show that if $M_r / \lambda$ is sufficiently small, then $\psi^* = 0$ in $B_r$. This is proved in three steps.

**Step 1. Upper bound of the energy.** We claim that for some $\tilde{C} = \tilde{C}(b_\ast) > 0$,

$$
\int_{B_r} \tilde{\mathcal{G}}(\nabla \psi^*, \psi^*) \leq \tilde{C} M_r \int_{\partial B_r} \psi^*,
$$

where $\tilde{\mathcal{G}}$ is defined in (75). The proof is based on the construction of a suitable energy competitor. Let

$$
\Phi_0(x) := \begin{cases} 
\sqrt{r} M_r \frac{e^{-\mu r^2} - e^{-\mu |x|^2}}{e^{-\mu r^2} - e^{-\mu r}} & \text{for } x \in B_{\sqrt{r}} \setminus B_r, \\
0 & \text{for } x \in B_r.
\end{cases}
$$

Similar as the proof for Lemma 4.7, for $\mu = \mu(b_\ast)$ sufficiently large and $\delta = \delta(b_\ast)$ small, there is $C_1 = C_1(b_\ast) > 0$ such that

$$
\partial_{x_i} (\partial_{p_j} \tilde{\mathcal{G}}(\nabla \Phi_0, \Phi_0)) \leq -C_1 \sqrt{r} M_r e^{-\mu |x|^2} \text{ in } B_{\sqrt{r}} \setminus B_r.
$$

Let $\phi := \min \{\psi^*, \Phi_0\}$. Note that $\phi = \psi^*$ on $\partial B_{\sqrt{r}}$, $\phi \equiv 0$ in $B_r$, and $\{x \in B_{\sqrt{r}} : \phi(x) \geq 0\} \subset \{x \in B_{\sqrt{r}} : \psi^*(x) \geq 0\}$. Since $\psi^*$ is a local energy minimizer, one has

$$
\int_{B_r} \tilde{\mathcal{G}}(\nabla \psi^*, \psi^*) + \lambda^2 \chi_{\{\psi^* > 0\}} \leq \int_{B_{\sqrt{r}} \setminus B_r} \tilde{\mathcal{G}}(\nabla \phi, \phi) - \tilde{\mathcal{G}}(\nabla \psi^*, \psi^*).
$$
By the convexity of $p \mapsto \mathcal{G}(p, z)$ one has
\[
\int_{B_{r} \setminus B_r} \mathcal{G}(\nabla \phi, \phi) - \mathcal{G}(\nabla \psi^*, \psi^*)
\]
(87) \leq \int_{B_{r} \setminus B_r} \left[ -\partial_p \mathcal{G}(\nabla \phi, \phi) \partial_i (\psi^* - \phi) - (\psi^* - \phi) \int_0^1 \partial_z \mathcal{G}(\nabla \psi^*, \phi + s(\psi^* - \phi)) ds \right]
\leq \int_{(B_{r} \setminus B_{r'}) \cap \{\psi^* > \Phi_0\}} \left[ -\partial_p \mathcal{G}(\nabla \Phi_0, \Phi_0) \partial_i (\psi^* - \Phi_0) + \delta M_r \sqrt{r}(\psi^* - \Phi_0) \right],
\]
where the last inequality follows from the bound for $\partial_z \mathcal{G}$ in (64) and $0 \leq \phi + s(\psi^* - \phi) \leq M_r \sqrt{r}$ for $s \in [0, 1]$. Multiplying (85) by $(\psi^* - \Phi_0)_+$ and integrating by parts (noting that $\psi^* - \Phi_0 = \psi^*$ on $\partial B_r$ and $(\psi^* - \Phi_0)_+ = 0$ on $\partial B_{r'}$) one has
\[
\int_{(B_{r} \setminus B_{r'}) \cap \{\psi^* > \Phi_0\}} -\partial_p \mathcal{G}(\nabla \Phi_0, \Phi_0) \partial_i (\psi^* - \Phi_0) dx
\leq -\int_{\partial B_r} \psi^* \partial_p \mathcal{G}(\nabla \Phi_0, \Phi_0) \cdot \nu_i - \int_{(B_{r} \setminus B_{r'}) \cap \{\psi^* > \Phi_0\}} C_1 \sqrt{r} M_r e^{-\mu |\nabla|r^2}(\psi^* - \Phi_0).
\]
This, together with (86) and (87), yields that if $\delta = \delta(b_*)$ is sufficiently small, then
\[
\int_{B_r} \mathcal{G}(\nabla \psi^*, \psi^*) + \lambda^2 \chi_{\{\psi^* > 0\}} \leq -\int_{\partial B_r} \psi^* \partial_p \mathcal{G}(\nabla \Phi_0, \Phi_0) \nu_i.
\]
In view of the expression for $\mathcal{G}$ and $|\nabla \Phi_0|$ on $\partial B_r$ as well as (61), there is $\tilde{C} = \tilde{C}(b_*)$ such that
\[
-\partial_p \mathcal{G}(\nabla \Phi_0, \Phi_0) \nu_i \leq \tilde{C} M_r.
\]
Combining the above two estimates gives the desired estimate (84).

Step 2. Lower bound of the energy. We claim that there is a constant $C = C(b_*, r) > 0$ such that
\[
(88) \quad \int_{B_r} \psi^* \leq \frac{C}{\lambda} \left( \frac{M_r}{\lambda} + 1 \right) \int_{B_r} \mathcal{G}(\nabla \psi^*, \psi^*) + \lambda^2 \chi_{\{\psi^* > 0\}}.
\]
First, by the trace estimate there exists $C = C(r) > 0$ such that
\[
\int_{\partial B_r} \psi^* \leq C \left( \int_{B_r} \psi^* + \int_{B_r} |\nabla \psi^*| \right).
\]
The Cauchy-Schwarz inequality and the fact $0 \leq \psi^* \leq M_r$ in $B_r$ yield
\[
\int_{B_r} |\nabla \psi^*| \leq \int_{B_r} \frac{1}{\lambda} |\nabla \psi^*|^2 + \lambda \chi_{\{\psi^* > 0\}} \quad \text{and} \quad \int_{B_r} \psi^* \leq M_r \int_{B_r} \chi_{\{\psi^* > 0\}}.
\]
It follows from the above two estimates that
\[
\int_{\partial B_r} \psi^* \leq \frac{C}{\lambda} \left( \frac{M_r}{\lambda} + 1 \right) \left[ \int_{B_r} |\nabla \psi^*|^2 + \lambda^2 \chi_{\{\psi^* > 0\}} \right].
\]
The estimate (66), together with Poincaré inequality, gives that if \( \delta \) is sufficiently small depending on \( b^* \), then
\[
\int_{B_r} |\nabla \psi|^2 \leq \frac{2}{b^*} \int_{B_r} \tilde{\mathcal{G}}(\nabla \psi^*, \psi^*).
\]
Combining all the above estimates together yields (88).

**Step 3. Conclusion.** From (84) and (88) there is a constant \( C = C(b^*, r) \) such that
\[
\hat{B}_r \tilde{\mathcal{G}}^*(\nabla \psi^*, \psi^*) + \lambda^2 \chi_{\{\psi^* > 0\}} \leq CMr(\lambda^2 \chi_{\{\psi^* > 0\}} + 1)\hat{B}_r \tilde{\mathcal{G}}(\nabla \psi^*, \psi^*) \]
Taking \( Mr/\lambda \) sufficiently small gives
\[
\hat{B}_r \tilde{\mathcal{G}}(\nabla \psi^*, \psi^*) + \lambda^2 \chi_{\{\psi^* > 0\}} = 0.
\]
This implies \( \psi^* \equiv 0 \) in \( B_r \). Hence the proof of the lemma is complete. □

Lemma 4.7 together with the elliptic estimates away from the free boundary yields the Lipschitz regularity of the local minimizers.

**Proposition 4.9.** Let \( \psi \) be a minimizer of the functional (59). If \( \delta = \delta(b^*) \) is sufficiently small, then \( \psi \in C^{0,1} \) in \( D \). Moreover, for any connected domain \( K \subset D \) containing a free boundary point, the Lipschitz constant of \( \psi \) in \( K \) is estimated by \( C\lambda \), where \( C \) depends on \( b^*, \delta, \Lambda, \) and \( D \).

**Proof.** (i). Let \( x \in \{ \psi < Q \} \cap D \). If \( \text{dist}(x, \Gamma_\psi) \geq \frac{1}{4} \text{dist}(x, \partial D) \), then the standard interior elliptic estimate gives \( \psi \in C^{1,\alpha} \). If \( d(x) := \text{dist}(x, \Gamma_\psi) \leq \min\{1, \text{dist}(x, \partial D)/4\} \), then let
\[
\psi^{x,*}(y) := \frac{Q - \psi(x + d(x)y)}{d(x)}.
\]
By Lemma 4.7, there is a constant \( C = C(b^*) > 0 \) such that
\[
0 \leq \psi^{x,*}(y) \leq C\lambda \quad \text{for any } y \in B_1.
\]
Note that \( \psi^{x,*} \) satisfies (73) in \( B_1 \) (with \( r = d(x) \) and \( a = 1 \)). By virtue of (62), \( \partial_t \psi^{x,*} \) solves a uniformly elliptic equation of divergence form and thus is \( C^{0,\alpha} \) by the De Giorgi-Nash-Moser estimate ([44]). Hence one has
\[
|\nabla \psi(x)| = |\nabla \psi^{x,*}(0)| \leq C \left( \|\psi^{x,*}\|_{L^\infty(B_1)} + \|\partial_z \tilde{\mathcal{G}}\|_{L^\infty(B_1)} \right) \leq C\lambda.
\]
Here the upper bound for \( |\partial_z \tilde{\mathcal{G}}| \) in (74) has been used to get the last inequality.

(ii). If \( K \subset D \) is connected and \( K \) contains a free boundary point \( x \), then it follows from the Harnack inequality (Theorem 8.17 in [44]) and the connectedness of \( K \) that
\[
Q - \psi \leq C\lambda \quad \text{in } K.
\]
for some $C$ depending on $b_*$, $\delta$, $K$ and $\mathcal{D}$. Given $x \in K$, either $x$ is closer to the free boundary or $x$ is closer to $\partial K$. Similar arguments as in (i) give that $|\nabla \psi(x)| \leq C\lambda$, where $C$ depends on $b_*$, $b^*$, $\delta$, $K$, and $\mathcal{D}$. \hfill \Box

**Remark 4.10.** By the boundary estimate for the elliptic equations, $\psi$ is Lipschitz up to the $C^{1,\alpha}$ portion $\Sigma \subset \partial \mathcal{D}$ as long as the boundary data $\psi^\sharp \in C^{0,1}(\mathcal{D} \cup \Sigma)$. Moreover, if $K$ is a subset of $\overline{\mathcal{D}}$ with $K \cap \partial \mathcal{D}$ being $C^{1,\alpha}$, $K \cap \mathcal{D}$ is connected, and $K$ contains a free boundary point, then $|\nabla \psi| \leq C\lambda$ in $K$.

4.4. Measure theoretic property and higher regularity of the free boundary.

From the Lipschitz regularity (Proposition 4.9) and nondegeneracy of the solutions (Lemma 4.8), one has the following measure theoretic properties of the free boundary.

**Proposition 4.11.** Let $\psi$ be a minimizer for $J(\psi)$ in (59). Assume that $\delta = \delta(b_*)$ is sufficiently small. Then one has

1. $\mathcal{H}^1(\Gamma_\psi \cap K) < \infty$ for any $K \subset \mathcal{D}$;
2. There is a Borel measure $\varsigma_\psi$ such that
   
   \[ \partial_x_i(\partial_{p_i} G(\nabla \psi, \psi)) - \partial_z G(\nabla \psi, \psi) = \varsigma_\psi \mathcal{H}^{n-1}[\Gamma_\psi]. \]
3. For any $K \subset \mathcal{D}$, there exist positive constants $C_\delta$, $C^\diamond$ such that for every ball $B_r(x) \subset K$ with $x \in \partial \{ \psi < Q \}$
   
   \[ C_\delta \leq \varsigma_\psi \leq C^\diamond, \quad C^\diamond r^{n-1} \leq \mathcal{H}^{n-1}(B_r(x) \cap \Gamma_\psi) \leq C^\diamond r^{n-1}. \]

The proof is the same as that for [5, Theorem 3.2], so we omit the details here.

In order to get the regularity of the free boundary, the following strengthened estimate for $|\nabla \psi|$ is needed.

**Lemma 4.12.** Let $\psi$ be a local minimizer. Then for $K \subset \mathcal{D}$, there exist $\alpha \in (0, 1)$ and $C = C(b_*, b^*, K) > 0$ such that for any ball $B_r(x) \subset K$ with $x \in \partial \{ \psi < Q \}$ one has

\[ \sup_{B_r(x)} |\nabla \psi(x)| \leq \Lambda + Cr^\alpha, \]

where $\Lambda$ is defined in (38).

**Proof.** The proof is similar as [5, Theorem 4.1]. It follows from the equation of $\psi$ and the methods in [44, Chapter 15] that $w := |\nabla \psi|^2$ satisfies

\[ \mathcal{L}w := \partial_i(a^{ij} \partial_j w) + \partial_i(b^i w) - b^j \partial_j w - cw \geq 0 \]

where $\mathcal{L}$ is uniformly elliptic with

\[ a^{ij} = \partial_{p_i p_j} G(\nabla \psi, \psi), \quad b^i = \partial_{p_i z} G(\nabla \psi, \psi), \quad \text{and} \quad c = \partial_{zz} G(\nabla \psi, \psi). \]
For \( \vartheta \geq 0 \), let
\[
W_\vartheta := \begin{cases} 
(\|\nabla \psi\|^2 - \Lambda^2 - \vartheta)\,^+, & \text{in } \{\psi < Q\}, \\
0, & \text{in } \{\psi = Q\}.
\end{cases}
\]
Denote \( W_\vartheta^*(r) := \sup_{B_r(x)} W_\vartheta \). Then \( W_\vartheta^*(r) - W_\vartheta \) satisfies
\[
\begin{cases} 
\mathcal{L}(W_\vartheta^*(r) - W_\vartheta) \leq 0 & \text{in } B_r(x), \\
W_\vartheta^*(r) - W_\vartheta = W_\vartheta^*(r) & \text{in } B_r(x) \cap \{\psi = Q\}.
\end{cases}
\]
Using [44, Theorem 8.17] with \( p \in [1, \infty) \) one has
\[
\inf_{B_{r/2}(x)} [W_\vartheta^*(r) - W_\vartheta] \geq C r^{-\frac{2}{p}} \|W_\vartheta^* - W_\vartheta\|_{L^p(B_r(x))} \geq CW_\vartheta^*(r).
\]
Here in the last inequality the positive density property of \( B_r(x) \cap \{\psi = Q\} \) (cf. Proposition 4.11(iii)) has been used. Taking \( \vartheta \to 0 \) and a rearrangement of the inequality yield
\[
W_0^*(r/2) = \sup_{B_{r/2}(x)} W_0 \leq (1 - C) W_0^*(r).
\]
It follows from the iteration lemma [41] that \( W_0^*(s) \leq Cs^\alpha \) for some \( C > 0 \) and \( \alpha \in (0, 1) \).

Similar arguments as [5, Theorem 4.2] give the following gradient estimate.

**Proposition 4.13.** Let \( \psi \) be a local minimum and that \( K \in \mathcal{D} \). Then for any \( B_r(x) \subset K \) with \( x \in \Gamma_\psi \), one has
\[
\int_{B_r(x) \cap \{\psi < Q\}} (\Lambda^2 - \|\nabla \psi\|^2)^+ \leq C |\ln r|^{-1}.
\]

**Proof.** Without loss of generality, assume that \( x \) is the origin. For any \( \eta \in C^\infty_0(\mathcal{D}) \), \( \eta \geq 0 \) and \( \vartheta > 0 \), the function \( \psi_{\vartheta, \eta} := \min\{\psi + \vartheta \eta, Q\} \) is admissible, so that \( J(\psi) \leq J(\psi_{\vartheta, \eta}) \). Thus
\[
\int_{\{Q - \vartheta \eta < \psi < Q\}} \lambda^2 \leq \int_{\mathcal{D}} \mathcal{G}(\nabla \psi_{\vartheta, \eta}, \psi_{\vartheta, \eta}) - \mathcal{G}(\nabla \psi, \psi).
\]

The integrand on the right hand side can be estimated as
\[
\begin{align*}
\mathcal{G}(\nabla \psi_{\vartheta, \eta}, \nabla \psi) - \mathcal{G}(\nabla \psi, \nabla \psi) \\
= & \mathcal{G}(\nabla \psi_{\vartheta, \eta}, \psi) - \mathcal{G}(\nabla \psi, \psi) + \mathcal{G}(\nabla \psi_{\vartheta, \eta}, \psi_{\vartheta, \eta}) - \mathcal{G}(\nabla \psi_{\vartheta, \eta}, \psi) \\
= & \partial_j \mathcal{G}(\nabla \psi, \psi) \partial_i (\psi_{\vartheta, \eta} - \psi) + (\psi_{\vartheta, \eta} - \psi) \int_0^1 \partial_j \mathcal{G}(\nabla \psi_{\vartheta, \eta}, \psi + s(\psi_{\vartheta, \eta} - \psi)) ds \\
+ & \int_0^1 d\tau \int_0^1 \partial_i (\psi_{\vartheta, \eta} - \psi) \partial_j (\psi_{\vartheta, \eta} + s(\psi_{\vartheta, \eta} - \psi), \psi) \partial_j (\psi_{\vartheta, \eta} + s(\psi_{\vartheta, \eta} - \psi)) ds.
\end{align*}
\]
In the set \( \{ \psi + \partial \eta < Q \} \) one has \( \psi_{\theta,*} - \psi = \partial \eta \). Hence by (62), it holds that

\[
\int_{\{ \psi + \partial \eta < Q \}} \int_{0}^{1} d\tau \int_{0}^{\tau} \partial_{i}(\psi_{\theta,*} - \psi) \partial_{p_{p_{j}}} G(\nabla \psi + s \nabla (\psi_{\theta,*} - \psi), \psi) \partial_{j}(\psi_{\theta,*} - \psi) d\tau d\eta 
\leq b^{*} \partial^{2} \int_{\{ \psi + \partial \eta < Q \}} |\nabla \eta|^{2}.
\]

In the set \( \{ \psi + \partial \eta \geq Q \} \) one has \( \psi_{\theta,*} = Q \), thus

\[
\int_{\{ \psi + \partial \eta \geq Q \}} \int_{0}^{1} d\tau \int_{0}^{\tau} \partial_{i}(\psi_{\theta,*} - \psi) \partial_{p_{p_{j}}} G(\nabla \psi + s \partial_{j}(\psi_{\theta,*} - \psi), \psi) \nabla (\psi_{\theta,*} - \psi) d\tau d\eta = \int_{\{ \psi + \partial \eta \geq Q \}} \partial_{i} \psi \partial_{p_{p_{j}}} G((1 - s) \nabla \psi, \psi) \partial_{j} \psi d\tau d\eta.
\]

On the other hand,

\[
G(0, \psi) = G(\nabla \psi, \psi) - \partial_{p_{i}} G(\nabla \psi, \psi) \partial_{i} \psi + \int_{0}^{1} d\tau \int_{0}^{\tau} \partial_{i} \psi \partial_{p_{p_{j}}} G((1 - s) \nabla \psi, \psi) \partial_{j} \psi d\tau d\eta,
\]

and \( G(0, \psi) = -(Q - \psi) \int_{0}^{1} \partial_{i} G(0, \psi + s(Q - \psi)) ds \), which follows from \( G(0, Q) = 0 \). Thus

\[
\int_{\{ \psi + \partial \eta \geq Q \}} \int_{0}^{1} d\tau \int_{0}^{\tau} \partial_{i} \psi \partial_{p_{p_{j}}} G((1 - s) \nabla \psi, \psi) \partial_{j} \psi d\tau d\eta = \int_{\{ \psi + \partial \eta \geq Q \}} \partial_{p_{i}} G(\nabla \psi, \psi) \partial_{i} \psi - G(\nabla \psi, \psi) - \int_{\{ \psi + \partial \eta \geq Q \}} (Q - \psi) \int_{0}^{1} \partial_{j} G(0, \psi + s(Q - \psi)) ds.
\]

Combining the above inequalities and using the \( L^{\infty} \) bound for \( \partial_{2} G \) yield

\[
\int_{\Omega} G(\nabla \psi_{\theta,*}, \psi_{\theta,*} - \psi) - G(\nabla \psi, \psi) 
\leq b^{*} \partial^{2} \int_{\{ \psi + \partial \eta \geq Q \}} |\nabla \eta|^{2} + \int_{\{ \psi + \partial \eta \geq Q \}} \partial_{p_{i}} G(\nabla \psi, \psi) \partial_{i} \psi - \partial_{2} G(\nabla \psi, \psi),
\]

\[
+ \partial \eta \int_{\partial D} \eta + \int_{D} \partial_{p_{i}} G(\nabla \psi, \psi) \partial_{i}(\psi_{\theta,*} - \psi).
\]

Using the equation (67) in \( D \cap \{ \psi < Q \} \) and that \( \psi_{\theta,*} - \psi = 0 \) in \( \{ \psi = Q \} \), one has

\[
\int_{D} \partial_{p_{i}} G(\nabla \psi, \psi) \partial_{i}(\psi_{\theta,*} - \psi) = - \int_{D} \partial_{2} G(\nabla \psi, \psi)(\psi_{\theta,*} - \psi) \leq \partial \eta \int_{\partial D} \eta.
\]

Recall that \( \Phi(\nabla \psi, \psi) := \partial_{p_{i}} G(\nabla \psi, \psi) \partial_{i} \psi - G(\nabla \psi, \psi) \) and \( \Phi(\Lambda, Q) = \lambda^{2} \). Thus the above inequality together with (89) gives

\[
\int_{\{ \psi + \partial \eta \geq Q \}} \Phi(\Lambda, Q) \leq \int_{\{ \psi + \partial \eta \geq Q \}} \Phi(\nabla \psi, \psi) + b^{*} \partial^{2} \int_{\{ \psi + \partial \eta \geq Q \}} |\nabla \eta|^{2} + 2 \partial \eta \int_{\partial D} \eta.
\]
Now for $0 < r < \hat{r} < R$, let
\[
\eta(x) := 1 \text{ in } B_r, \quad \eta(x) := \frac{\ln(\hat{r}/|x|)}{\ln(\hat{r})} \text{ in } B_{\hat{r}} \setminus B_r, \quad \eta(x) := 0 \text{ in } \mathbb{R}^2 \setminus B_{\hat{r}}.
\]
Let $\vartheta = C r$, where $C = C(\lambda, b^*) > 0$ is such that $Q - \psi \leq C r$ in $B_r$. Then one has
\[
\int_{B_r \cap \{\psi < Q\}} (\Phi(\Lambda, Q) - \Phi(\nabla \psi, \psi))^+ \leq \int_{B_{\hat{r}} \cap \{\psi + \vartheta \eta \geq Q\}} (\Phi(\nabla \psi, \psi) - \Phi(\Lambda, Q))^+ + C r^2 \left(\frac{\ln(\hat{r}/r)}{r}\right)^{-1} + C r \hat{r}^2,
\]
for some $C = C(\lambda, b^*, \delta) > 0$. Since $|\partial_p \Phi| + |\partial_z \Phi| \leq C$ for some $C = C(b^*, b_*, \delta)$, it holds that
\[
(\Phi(\nabla \psi, \psi) - \Phi(\Lambda, Q))^+ \leq C \left(\frac{\hat{r}}{r}\right)^{\alpha} + C r.
\]
Applying of Lemma 4.12 to the first term on the right side of the above expression and using $Q - \psi \leq \vartheta \eta$ one has
\[
(\Phi(\nabla \psi, \psi) - \Phi(\Lambda, Q))^+ \leq C \left(\frac{\hat{r}}{r}\right)^2 \left(\frac{\hat{r}}{r}\right)^{\alpha} + C \frac{\ln(\hat{r}/r)}{r} + C \hat{r}^2.
\]
Taking $R = \sqrt{\hat{r}}$ and $\hat{r} = r^s$ with some $s \in \left(\frac{2}{2 + \frac{\alpha}{2}}, 1\right)$ we obtain the desired estimate. □

An immediate consequence of Proposition 4.13 is the following corollary.

**Corollary 4.14.** Let $\psi$ be a local minimizer to (59). Then every blow up limit of $\psi$ at $\bar{x} \in \Gamma_\psi \cap D$ is a half plane solution with slope $\lambda$ in a neighborhood of the origin.

The proof is the same as in [5, Corollary 4.4], which we do not repeat here. With Corollary 4.14 at hand, one can make use of the improvement of flatness arguments, which is rather standard now, to show that the free boundary is locally a $C^{1, \alpha}$ graph for some $\alpha \in (0, 1)$. For the proof we refer to [5, Section 5] for the case of minimizers of the energy $\int_D G(\nabla \psi)$, and [29, 30] for a different argument which does not rely on energy minimality and also allows to deal with the inhomogeneous terms.

Higher regularity of the free boundary follows from [47, Theorem 2]. For the completeness, we state the result here and omit the proof.

**Proposition 4.15.** The free boundary $\Gamma_\psi$ is locally $C^{k+1, \alpha}$ if $G(p, z)$ is $C^{k, \alpha}$ in its components, and it is locally real analytic if $G(p, z)$ is real analytic.

**Remark 4.16.** The results in this section hold for general variational problem as long as $G$ satisfies the structure conditions in Proposition 4.1.
5. Fine Properties for the Free Boundary Problem

The major property of the solutions obtained in this section is that streamlines, in particular the free boundary, are graphs. This is one of the key ingredients to justify the equivalence between Problem 1 and Problem 2.

The key step towards establishing the graph property for the streamlines is to show that the stream function is monotone in the $x_1$ variable. For that we take a specific boundary value on $\partial \Omega_{\mu,R}$ for the truncated problem. It should be noted that due to the uniqueness of the problem after removing the truncation (cf. Section 8), the solution to the subsonic jet problem does not depend on the boundary data which one takes for the truncated problems.

In order to state the dependence of solutions on the parameters clearly and make use of the special structure of the equation (55), we resume to use the notations introduced in Section 3.

![Figure 2. The truncated domain $\Omega_{\mu,R}$.

To describe the boundary datum on $\partial \Omega_{\mu,R}$, some notations are needed. Let $b_\mu := \Theta^{-1}(-\mu)$. Choose a point $(-\mu, b'_\mu)$ with $1 < b'_\mu < b_\mu$ and $k_\mu := b_\mu - b'_\mu$ small such that $k_\mu^2 < Q$. Let $s \in (\frac{1}{2}, 1)$ be a fixed constant and $\tilde{H} = \tilde{H}(\Lambda)$ be such that $\frac{\Lambda}{1-s} \tilde{H}^{1-s} = Q$. Define

$$
\psi^\dagger(x_2) := \begin{cases} 
\min\left( \frac{\Lambda}{1-s} x_2^{1-s}, Q \right) & \text{if } \tilde{H} < 1, \\
Q x_2^{1-s} & \text{if } \tilde{H} \geq 1.
\end{cases}
$$
Set

\[
\psi_{\mu,R}^\pm(x_1, x_2) := \begin{cases} 
0 & \text{if } x_1 = -\mu, \ 0 < x_2 < b_\mu', \\
Q \left( \frac{x_2 - b_\mu'}{k_\mu} \right)^{1+s} & \text{if } x_1 = -\mu, \ b_\mu' \leq x_2 \leq b_\mu, \\
\psi^+(x_2) & \text{if } x_1 = R, \ 0 < x_2 < 1, \\
0 & \text{if } x_2 = 0, \\
Q & \text{if } (x_1, x_2) \in S_1 \cup ([0, R] \times \{1\}).
\end{cases}
\]

(90)

Note that \( \psi_{\mu,R}^\pm \) is continuous and it satisfies \( 0 \leq \psi_{\mu,R}^\pm \leq Q \).

**Lemma 5.1.** Assume \( \|B'\|_{L^\infty} \leq \kappa_0 \). If \( \kappa_0 = \kappa_0(\gamma, B_\gamma) \) is sufficiently small, then \( \psi_{\mu,R}^\pm(-\mu, \cdot) \) is a subsolution to (55) and \( \psi_{\mu,R}^\pm(R, \cdot) \) is a supersolution to (55) in \( \Omega_{\mu,R} \).

**Proof.** Let \( \varphi(x_2) := Q \left( \frac{x_2 - b_\mu'}{k_\mu} \right)^{1+s} \). Note that \( \psi_{\mu,R}^\pm(\mu, x_2) = \varphi(x_2) \) when \( x_2 \in [b_\mu', b_\mu] \) and \( \psi_{\mu,R}^\pm(\mu, x_2) = 0 \) when \( x_2 \in [0, b_\mu] \). We claim that \( \varphi \) is a subsolution to (55) in \( [b_\mu', b_\mu] \), i.e.

\[
(\gamma(|\varphi'|^2, \varphi)\varphi)' - \partial_z G_\gamma(|\varphi'|^2, \varphi) > 0 \quad \text{in } [b_\mu', b_\mu].
\]

(91)

Since \( \psi_{\mu,R}^\pm(-\mu, \cdot) \in H^2_{\mathrm{loc}}(\mathbb{R}) \) and since 0 is a solution to (55), we can conclude from (91) that \( \psi_{\mu,R}^\pm(-\mu, \cdot) \) is a subsolution to (55) in \( \mathbb{R} \times [0, b_\mu] \supseteq \Omega_{\mu,R} \). Hence it remains to verify (91). A straightforward computation gives

\[
(\gamma(|\varphi'|^2, \varphi)\varphi)' = \left( \gamma + 2\partial_t \gamma(\varphi')^2 \right) \varphi'' + \partial_z \gamma(\varphi')^2.
\]

It follows from Lemma 2.6, (65) and the bound on \( B' \) in (29) that

\[
|\partial_z G_\gamma| \leq C\kappa_0, \quad |\partial_x g_\gamma| \leq C\kappa_0 \partial_t g_\gamma, \quad g_\gamma + 2\partial_t g_\gamma |\varphi'|^2 \geq C > 0,
\]

where \( C \) depends only on \( B_\gamma \) and \( \gamma \). Therefore, if \( \kappa_0 = \kappa_0(\gamma, B_\gamma) \) is sufficiently small, the function \( \varphi \) satisfies (91) in \( [b_\mu', b_\mu] \).

Similarly, one can show that \( \psi_{\mu,R}^\pm(R, \cdot) \) is a supersolution. \( \square \)

The next lemma shows that the minimizers for (52) in \( K_{\psi_{\mu,R}} \) must stay between \( \psi_{\mu,R}^\pm(-\mu, \cdot) \) and \( \psi_{\mu,R}^\pm(R, \cdot) \). This in particular implies that for a minimizer \( \psi \), \( \partial_x \psi \) is nonnegative at the two sides \( x_1 = -\mu \) and \( x_1 = R \) of \( \Omega_{\mu,R} \).
Lemma 5.2. Let $\psi$ be a minimizer for the truncated problem (52) in $\Omega_{\mu,R}$ with the boundary value $\psi^\dagger_{\mu,R}$ constructed in (90). Assume that $u$ satisfies (29). Then if $\kappa_0 = \kappa_0(\gamma, B_\star)$ is sufficiently small, we have

$$
\psi^\dagger_{\mu,R}(-\mu, x_2) < \psi(x_1, x_2) < \psi^\dagger_{\mu,R}(R, x_2) \text{ for all } (x_1, x_2) \in \Omega_{\mu,R}.
$$

Proof. The proof is divided into two steps.

Step 1. Lower bound. Let $D := \Omega_{\mu,R} \cap \{b_\mu' < x_2 < b_\mu\}$. By Lemma 5.1, $\psi^\dagger_{\mu,R}(-\mu, \cdot) \leq \psi$ on $\partial D$. Furthermore, it follows from (91) that $\psi^\dagger_{\mu,R}(-\mu, \cdot)$ is a strict subsolution in $D$ if $\kappa_0$ is sufficiently small. Thus by the strong maximum principle (cf. [55]), $\psi^\dagger_{\mu,R}(-\mu, x_2) < \psi(x_1, x_2)$ in $D$. Since $\psi^\dagger_{\mu,R}(-\mu, x_2) \equiv 0$ if $x_2 \in [0, b_\mu']$, one has $\psi^\dagger_{\mu,R}(-\mu, x_2) < \psi(x_1, x_2)$ in $\Omega_{\mu,R}$.

Step 2. Upper bound. Denote $\Psi^{(\tau)} = \Psi^{(\tau)}(x_2) := \psi^\dagger_{\mu,R}(R, x_2) + \tau$ for $\tau > 0$. It follows from similar calculations for the proof of Lemma 5.1 that $\Psi^{(\tau)}$ is a supersolution if $\kappa_0$ is sufficiently small. Since $\psi$ is bounded, then we have $\psi < \Psi^{(\tau)}$ provided that $\tau$ is large enough. Let

$$
\tau_* := \inf\{\tau : \psi \leq \Psi^{(\tau)}\}.
$$

be the first time when the graph of $\psi$ touches that of $\Psi^{(\tau)}$. We claim $\tau_* = 0$. Suppose that $\tau_* > 0$. Clearly the equality $\psi = \Psi^{(\tau_*)}$ cannot be attained along $x_2 = 0$ or $x_1 = R$. Since $\Psi^{(\tau_*)} \equiv Q + \tau$ for $x_2 \geq 1$, then the equality cannot be attained at $x_1 = -\mu$ or the nozzle boundary $S_1$ either. Now one needs to consider the following two cases depending on the height $\bar{H}$:

Case 1. $\bar{H} \geq 1$. This means that the free boundary is empty. Since one has $\Psi^{(\tau_*)} > Q$ along $x_2 = 1$, the graph of $\Psi^{(\tau_*)}$ has to touch the graph of $\psi$ at an interior point. This contradicts with the strong comparison principle in [55].

Case 2. $\bar{H} < 1$. It follows from the strong comparison principle that the graphs of $\Psi^{(\tau_*)}$ and $\psi$ touch at a free boundary point $\bar{x} = (\bar{x}_1, \bar{x}_2)$. Note that necessarily $\bar{x}_2 < 1$ since $\Psi^{(\tau_*)} > Q$ along $x_2 = 1$. Then by the boundary Hopf lemma (cf. [55])

$$
\Lambda = |\nabla \psi(\bar{x})| > |\nabla \Psi^{(\tau_*)}(\bar{x}_2)| = \Lambda(\bar{x}_2)^{-s} > \Lambda,
$$

which is a contradiction.

Therefore, one has $\tau_* = 0$. This implies that $\psi(x_1, x_2) \leq \psi(R, x_2)$. Hence the proof of the lemma is completed. \hfill \square

The next proposition concerns the uniqueness of the minimizer and the monotonicity of the minimizer with respect to $x_1$.

Proposition 5.3. Under the same assumptions as in Lemma 5.2, $\psi$ is the unique minimizer to (52) over $K^\dagger_{\psi^\dagger_{\mu,R}}$. Furthermore, $\partial_{x_1}\psi \geq 0$ in $\Omega_{\mu,R}$.
Proof. (i). Nonnegativity of $\partial_{x_1}\psi$. For $k > 0$ small, denote
\[
\psi^{(k)}(x_1, x_2) := \psi(x_1 - k, x_2), \quad \text{for } (x_1, x_2) \in \Omega^{k}_{\mu,R} := \{(x_1 + k, x_2) : (x_1, x_2) \in \Omega_{\mu,R}\}.
\]
Since $\psi$ is a minimizer for $J^{*}_{\mu,R}A$ over $K^\ast_{\psi^{*}_{\mu,R}}$, then $\psi^{(k)}$ is a minimizer for $J^{*,k}_{\mu,R}A$ in $K_{\psi^{*,(k)}_{\mu,R}}$.

Here $J^{*,k}_{\mu,R}A$ is defined by performing the corresponding translation for $J^{*}_{\mu,R}A$. For ease of notations, we denote $J^{*,k}_{\mu,R}A$ and $J^{*,k}_{\mu,R}A$ by $J^k$ and $J$, respectively, in the rest of the proof. Below we drop the dependence on $\epsilon, \mu, R, A$ in the energy functional and write $K := K^\ast_{\psi^\ast_{\mu,R}}$, $K^k := K_{\psi^{*,(k)}_{\mu,R}}$ for simplicity. By Lemma 5.2, one has
\[
\min\{\psi^{(k)}(k), \psi\} \in K^k \quad \text{and} \quad \max\{\psi^{(k)}(k), \psi\} \in K.
\]
Since the energy functional $J$ depends only on $\nabla \psi$ and $\psi$, it is easy to verify that
\[
J^k(\psi^{(k)}) + J(\psi) = J^k(\min\{\psi^{(k)}(k), \psi\}) + J(\max\{\psi^{(k)}(k), \psi\}).
\]

This, together with the minimality of $\psi^{(k)}$ and $\psi$, gives
\[
J(\psi) = J(\max\{\psi^{(k)}(k), \psi\}).
\]

Our aim is to show that $\psi^{(k)} < \psi$ in $\Omega_{\mu,R} \cap \Omega^{(k)}_{\mu,R}$. By Lemma 5.2 and the continuity of $\psi$, one has $\psi^{(k)} < \psi$ near $\{x_1 = -\mu + k\} \cap \Omega_{\mu,R}$. If the strict inequality were not true in the connected component of $\Omega_{\mu,R} \cap \{\psi < Q\}$ containing $\{x_1 = -\mu\}$, then one would find a ball $B \subseteq \Omega_{\mu,R}$ and $\bar{x} \in \partial B$ with $\psi(\bar{x}) < Q$ such that $\psi^{(k)} < \psi$ in $B$, $\psi^{(k)}(\bar{x}) = \psi(\bar{x})$.

By the boundary Hopf lemma (cf. [55]), one has
\[
\lim_{x \to \bar{x}, x \in B} \partial_\nu(\psi^{(k)} - \psi)(x) > 0,
\]
where $\nu$ is the outer unit normal of $\partial B$ at $\bar{x}$. Thus there exists a curve through $\bar{x}$ such that along this curve $\max\{\psi^{(k)}(k), \psi\}$ is not $C^1$ at $\bar{x}$. On the other side, since $\max\{\psi^{(k)}(k), \psi\}$ is a minimizer by (92) and $\max\{\psi^{(k)}(k), \psi\}(\bar{x}) < Q$, then by the interior regularity for the minimizers, $\max\{\psi^{(k)}(k), \psi\}$ is $C^{1,\alpha}$ around $\bar{x}$. This is however a contradiction.

Note that every component $\{\psi < Q\}$ has to touch $\partial \Omega_{\mu,R}$ (otherwise it would violate the strong maximum principle). Moreover, from the construction of the boundary datum $\psi^{*,k}_{\mu,R}$ one has that $\{\psi < Q\} \cap \partial \Omega_{\mu,R}$ is a connected arc. Hence $\{\psi < Q\} \cap \Omega_{\mu,R}$ consists only one component. Consequently, $\psi^{(k)} < \psi$ in $\Omega_{\mu,R} \cap \{\psi < Q\}$. Taking $k \to 0+$ yields $\partial_{x_1} \psi \geq 0$.

(ii). Uniqueness. Assume that $\psi_1$ and $\psi_2$ are two minimizers with the same boundary data. Let $\psi^{(k)}_1(x_1, x_2) := \psi_1(x_1 - k, x_2)$. The same argument as above gives that $\psi^{(k)}_1 \leq \psi_2$ in $\Omega_{\mu,R}$. Taking $k \to 0$ yields $\psi_1 \leq \psi_2$. Similarly, it holds that $\psi_2 \leq \psi_1$. Thus one has $\psi_1 = \psi_2$. \qed
An important consequence of the property $\partial_x \psi \geq 0$ obtained from Proposition 5.3 is that, the free boundary (if not empty) is an $x_2$-graph, which is proved in Proposition 5.6 below. More precisely, Let

$$\Gamma_{\mu,R,A} := \partial \{ \psi < Q \} \cap \{(x_1, x_2) : (x_1, x_2) \in (-\mu, R) \times (0,1)\}$$

denote those free boundary points which lie strictly below $\{x_2 = 1\}$. Note that if $\tilde{H} \geq 1$, then $\Gamma_{\mu,R,A} = \emptyset$.

Before we prove that the free boundary is a graph, let us digress for the proof of the following non-oscillation property.

**Lemma 5.4 (Non-oscillation).** Let $D$ be a domain in $\Omega_{\mu,R} \cap \{ \psi < Q \}$ bounded by two disjoint arcs $\Gamma_1$ and $\Gamma'_1$ of the free boundary and by $\{x_1 = k_1\}$ and $\{x_1 = k_2\}$. Suppose that $\Gamma_1$ ($\Gamma'_1$) lies in $\{k_1 < x_1 < k_2\}$ with endpoints $(k_1, l_1)$ and $(k_2, l_2)$ ($\Gamma'_1$) and $(k_2, l'_2)$). Suppose that $\text{dist}((0,1), D) \geq c_0 > 0$. Then

$$|k_1 - k_2| \leq C \max\{|l_1 - l'_1|, |l_2 - l'_2|\}$$

for some $C$ depending on $B_\ast, \gamma, \epsilon$ and $c_0$.

**Proof.** The proof is inspired by [9, Lemma 4.4]. Integrating the equation (55) in $D$ gives

$$\int_{\Gamma_1 \cup \Gamma_1'} g_\epsilon(|\nabla \psi|^2, \psi) \partial_x \psi + \int_{\partial D \cap \{(x_1 = k_1) \cup \{x_1 = k_2\}\}} g_\epsilon(|\nabla \psi|^2, \psi) \partial \nu \psi$$

$$= \int_D \partial_x G_\epsilon(|\nabla \psi|^2, \psi) dx.$$ 

Noting that $\partial_x \psi = \Lambda$ along $\Gamma_1$ and $\Gamma'_1$, and $g_\epsilon \geq C_\ast(B_\ast, \gamma)$ by (41), one has

$$\int_{\Gamma_1 \cup \Gamma_1'} g_\epsilon(|\nabla \psi|^2, \psi) \partial_x \psi \geq C_\ast \Lambda (\mathcal{H}^1(\Gamma_1) + \mathcal{H}^1(\Gamma'_1)).$$

Figure 3.
Applying the interior Lipschitz regularity (Theorem 4.9) and the $C^{1, \alpha}$ estimate for $\psi$ (cf. Remark 4.10) yields

$$\int_{\partial K \cap \{(x_1 = k_1) \cup \{(x_1 = k_2)\}} g_\epsilon(|\nabla \psi|^2, \psi) \partial_\nu \psi \leq C \Lambda \max\{|l_1 - l'_1|, |l_2 - l'_2|\},$$

where $C$ depends on the $\gamma, B_\ast, \epsilon$ and also $c_0$. It follows from (64) and Remark 4.10 that

$$|\partial_2 G_\epsilon(|\nabla \psi|^2, \psi)| \leq C \kappa_0(Q - \psi) \leq C \kappa_0 \Lambda \max\{|l_1 - l'_1|, |l_2 - l'_2|\},$$

where $C$ depends on $\gamma, B_\ast, \epsilon$ and $c_0$. Combining the above estimates together and noticing that $|k_1 - k_2| \leq \mathcal{H}^1(\Gamma_1) + \mathcal{H}^1(\Gamma'_1)$ give the desired inequality (94).

The following result on the uniqueness of the Cauchy problem for the elliptic equations, plays a crucial role in the proof for the graph property of the free boundary as well as the study of the various blowup limits.

**Lemma 5.5.** Let $D$ be a bounded connected smooth domain in $\mathbb{R}^2$, and let $\Sigma$ be a nonempty open set of $\partial D$. Let $\psi, \tilde{\psi} \in W^{1,2}_{\text{loc}}(D)$ be two solutions to the Cauchy problem

$$\begin{cases}
\partial_{x_1}(\partial_{x_1} G(\nabla \psi, \psi)) - W(\nabla \psi, \psi) = 0 & \text{in } D, \\
\psi = Q, \partial_\nu \psi = \Lambda & \text{on } \Sigma,
\end{cases}$$

where $Q$ and $\Lambda$ are constants. Assume that $G$ is uniformly elliptic, cf. (61)–(62) and that $\partial_2 G, \partial_3 W, \partial_2 W \in L^\infty$. Then $\tilde{\psi} = \psi$.

**Proof.** The proof relies on the unique continuation property for the uniformly elliptic equations with Lipschitz coefficients. First, it follows from the boundary regularity for the elliptic equation ([5, Lemma 1.7]) that $\psi, \tilde{\psi} \in C^{1, \alpha}(D \cup \Sigma)$. Let $\varphi := \tilde{\psi} - \psi$. Then $\varphi$ solves

$$\partial_i (a^{ij} \partial_j \varphi) = W \varphi + b^i \partial_i \varphi - \partial_i (b^i \varphi),$$

where

$$a^{ij}(x) := \int_0^1 (dt) \partial_{p,p_j} G(\nabla \tilde{\psi}(x) + t \nabla (\psi - \tilde{\psi})(x), \tilde{\psi}(x) + t(\psi - \tilde{\psi})(x))dt,$$

$$b^i(x) := \int_0^1 (dt) \partial_{2p} G(\nabla \tilde{\psi}(x) + t \nabla (\psi - \tilde{\psi})(x), \tilde{\psi}(x) + t(\psi - \tilde{\psi})(x))dt,$$

$$\tilde{b}_i(x) := \int_0^1 (dt) \partial_{2p} W(\nabla \tilde{\psi}(x) + t \nabla (\psi - \tilde{\psi})(x), \tilde{\psi}(x) + t(\psi - \tilde{\psi})(x))dt,$$

$$W(x) := \int_0^1 (dt) \partial_2 W(\nabla \tilde{\psi}(x) + t \nabla (\psi - \tilde{\psi})(x), \tilde{\psi}(x) + t(\psi - \tilde{\psi})(x))dt$$

with the Cauchy data $\varphi = \partial_\nu \varphi = 0$ on $\Sigma$. The Cauchy data allow to extend $\varphi$ into a solution in a neighborhood $U \supset \Sigma$ by setting $\varphi = 0$ outside of $U \cap D$. It follows from
the assumptions on $\mathcal{G}$ and $\mathcal{W}$ that $b^i$, $\hat{b}^i$ and $W \in L^\infty$, and $A(x)$ is uniformly elliptic and Lipschitz continuous. Thus by the unique continuation property, cf. for example [48, Theorem 1], one has $\varphi \equiv 0$ in $U \cup D$. This means that $\psi = \tilde{\psi}$. \hfill \Box

Now we are ready to show that the free boundary is a graph.

**Proposition 5.6.** Let $\Gamma_{\mu,R,A}$ be as in (93). If $\Gamma_{\mu,R,A} \neq \emptyset$, then it can be represented as a graph of a function of $x_2$, i.e., there exists a function $\Upsilon_{\mu,R,A}$ and $l_{\mu,R,A} \geq \tilde{H}$ such that

$$\Gamma_{\mu,R,A} = \{(x_1, x_2) : x_1 = \Upsilon_{\mu,R,A}(x_2), \ x_2 \in (l_{\mu,R,A}, 1)\}.$$

Furthermore, $\Upsilon_{\mu,R,A}$ is continuous, $-\mu < \Upsilon_{\mu,R,A} \leq R$ and $\lim_{x_2 \to 1^-} \Upsilon_{\mu,R,A}(x_2)$ exists.

**Proof.** It follows from Proposition 5.3 that $\partial_x \psi \geq 0$. Hence there is a function $\Upsilon_{\mu,R,A}(x_2)$ with values in $[-\mu, R]$ such that for any $x_2 \in (0, 1)$,

$$\psi(x_1, x_2) < Q \text{ if and only if } x_1 < \Upsilon_{\mu,R,A}(x_2).$$

Since $\psi(-\mu, x_2) = 0$ for $x_2 \in [0, b'_\mu]$ with $b'_\mu > 1$, then $\Upsilon_{\mu,R,A}(x_2) > -\mu$.

It follows from Lemma 5.2 that one has

$$\psi(x_1, x_2) \leq \psi_{0,R}(R, x_2) < Q \text{ in } \mathbb{R} \times [0, \tilde{H}).$$

This implies that there is no free boundary points in the strip $\mathbb{R} \times [0, \tilde{H})$. Thus there is a constant $l_{\mu,R,A} \geq \tilde{H}$ such that the free boundary is a graph $x_1 = \Upsilon_{\mu,R,A}(x_2)$ for $x_2 \in (l_{\mu,R,A}, 1)$.

Finally, one can also show $\Upsilon_{\mu,R,A}$ is continuous. Clearly, $\Upsilon_{\mu,R,A}$ is lower semi-continuous. Suppose there is a jump point $\bar{x}_2$ such that

$$\lim_{x_2 \to \bar{x}_2} \Upsilon_{\mu,R,A}(x_2) = \Upsilon_{\mu,R,A}(\bar{x}_2) + \delta$$

for some $\delta > 0$. Let $\bar{x}_1 := \Upsilon_{\mu,R,A}(\bar{x}_2)$. Since $\partial_x \psi \geq 0$, the line segment $[\bar{x}_1, \bar{x}_1 + \delta] \times \{\bar{x}_2\}$ belongs to the free boundary. Therefore, one can find an upper or lower neighborhood $U$ of $[\bar{x}_1, \bar{x}_1 + \delta]$, such that $\psi$ solves the quasilinear equation (55) in $U$, and $\psi = Q$, $\partial_x \psi = \Lambda$ on $[\bar{x}_1, \bar{x}_1 + \delta]$, where $\nu$ is the inner unit normal. Note the Cauchy problem

$$\begin{cases}
\nabla \cdot (g_e(\|\nabla \psi\|^2, \psi) \nabla \psi) - \partial_z G_e(\|\nabla \psi\|^2, \psi) = 0 & \text{in } \mathbb{R} \times [0, \bar{x}_2], \\
\psi = Q, \quad \partial_x \psi = \Lambda & \text{at } x_2 = \bar{x}_2
\end{cases}$$

(95)

admits a one dimensional solution $\psi_{1d}(x) = \psi_{1d}(x_2)$, which is the solution of ODE

$$\begin{cases}
\frac{d}{dx_2} (g_e(\|\psi_{1d}\|^2, \psi_{1d}) \psi_{1d}) - \partial_z G_e(\|\psi_{1d}\|^2, \psi_{1d}) = 0 & \text{in } [0, \bar{x}_2], \\
\psi_{1d} = Q, \quad \psi'_{1d} = \Lambda & \text{at } x_2 = \bar{x}_2.
\end{cases}$$
By the uniqueness of the Cauchy problem (cf. Lemma 5.5), one has $\psi = \psi_{1d}$ in $\Omega_{\mu,R}$. This is however impossible. Thus $\Upsilon_{\mu,R,\Lambda}$ is continuous. The existence of $\lim_{x_2 \to 1-} \Upsilon_{\mu,R,\Lambda}(x_2)$ follows from Lemma 5.4.

6. Continuous fit, smooth fit and the far-field behavior

In the first part of this section, we show the continuous fit property: given a nozzle and incoming data $(\bar{\rho}, \bar{u})$ at upstream, there exists a momentum $\Lambda > 0$ along the free boundary, such that the associated free boundary is nonempty and it matches the outlet of the nozzle. Subsequently, the free boundary join the nozzle wall in a continuous differentiable fashion via a blow-up argument. Finally, we remove the truncation of the domain by letting $\mu, R \to +\infty$ and study the asymptotic behavior of the jet at downstream $x_1 \to \infty$. The downstream asymptotic behavior plays an important role in proving the equivalence between Problem 1 and Problem 2.

6.1. Continuous fit. In this subsection, $\mu, R$ and $\epsilon$ are fixed. To emphasize the dependence on the parameter $\Lambda$, we let $\psi_{\Lambda} := \psi_{\mu,R,\Lambda}$ denote the minimizer of the truncated problem (52), and the free boundary $\Gamma_{\Lambda} := \Gamma_{\mu,R,\Lambda}$ defined in (93) with

$$\Gamma_{\Lambda} = \{(x_1, x_2) | x_1 = \Upsilon_{\Lambda}(x_2) := \Upsilon_{\mu,R,\Lambda}(x_2) \text{ for } x_2 \in (l_{\Lambda}, 1) \text{ with } l_{\Lambda} := l_{\mu,R,\Lambda}\}.$$

The following lemma shows that $\psi_{\Lambda}$ and $\Upsilon_{\Lambda}$ depend on $\Lambda$ continuously.

Lemma 6.1. If $\Lambda_n \to \Lambda$, then $\psi_{\Lambda_n} \to \psi_{\Lambda}$ uniformly in $\Omega_{\mu,R}$ and $\Upsilon_{\Lambda_n}(x_2) \to \Upsilon_{\Lambda}(x_2)$ for each $x_2 \in (l_{\Lambda}, 1]$.

Proof. The proof consists of two steps.

Step 1. Convergence of $\psi_{\Lambda_n}$. This is similar to [39], but for completeness we sketch the proof here. For any subsequence $\psi_{\Lambda_j} \to \psi_{\Lambda}$ weakly in $W^{1,2}$ and strongly in $C^{0,\alpha}_{\text{loc}}$, one has $\partial\{\psi_{\Lambda_j} < Q\} \to \partial\{\psi_{\Lambda} < Q\}$ locally in the Hausdorff distance, $\chi_{\{\psi_{\Lambda_j} < Q\}} \to \chi_{\{\psi_{\Lambda} < Q\}}$ in $L^1_{\text{loc}}$ and $\nabla \psi_{\Lambda_j} \to \nabla \psi_{\Lambda}$ a.e. ([39, Lemma 3.6]). Thus $\psi_{\Lambda}$ is a minimizer for $J_{\mu,R}(\psi)$ over $K_{\psi_{\mu,R}}$. Then it follows from the uniqueness of the minimizer (cf. Proposition 5.3) that $\psi_{\Lambda_n} \to \psi_{\Lambda}$.

Step 2. Convergence of $\Upsilon_{\Lambda_n}$. For each $x_2 \in (l_{\Lambda}, 1)$, it follows from the uniform convergence of $\psi_{\Lambda_n}$ to $\psi_{\Lambda}$ and Proposition 4.9 that $\Upsilon_{\Lambda_n}(x_2) \to \Upsilon_{\Lambda}(x_2)$. It remains to show $\Upsilon_{\Lambda_n} \to \Upsilon_{\Lambda}$ at $x_2 = 1$, which will be shown by contradiction. Assume that $\Upsilon_{\Lambda_n}(1) \to \Upsilon_{\Lambda}(1) + \iota$ with $\iota \neq 0$. Now there are three cases to be analyzed in detail.

Case (i): $\iota < 0$. First, we claim that $\psi_{\Lambda}$ solves

$$\begin{cases}
\nabla \cdot (g(|\nabla \psi|^2, \psi) \nabla \psi) - \partial_z G(|\nabla \psi|^2, \psi) = 0 & \text{in } D_{1,\sigma_0}, \\
\psi = Q, \lim_{x_2 \to 1-} \partial_{x_2} \psi = \Lambda & \text{on } \overline{D_{1,\sigma_0}} \cap \{x_2 = 1\},
\end{cases}$$

in $D_{1,\sigma_0}$, on $\overline{D_{1,\sigma_0}} \cap \{x_2 = 1\}$,
where $D_{i,\sigma_0} := (\Upsilon_{\Lambda}(1) + \frac{2}{3} i, \Upsilon_{\Lambda}(1) + \frac{1}{3} i) \times (1 - \sigma_0, 1)$ for some small $\sigma_0 > 0$. The proof for (96) is given in Step 3. Assume that (96) holds, then it follows from Lemma 5.5 (cf. also the proof for Proposition 5.6) that $\psi_1$ must be the one dimensional solution, i.e. $\psi_1(x) = \psi_{1d}(x)$. This is however impossible, since it contradicts the fact that $\psi_1$ solves the jet problem, for which the domain depends on $x_1$ as well.

**Case (ii):** $i > 0$ and $\Upsilon_{\Lambda}(1) < 0$. Similarly as for case (i), $\psi_1$ solves

$$
\begin{cases}
\nabla \cdot (g(|\nabla \psi|^2, \psi) \nabla \psi) - \partial_2 G(|\nabla \psi|^2, \psi) = 0 & \text{in } D_{i,\sigma_0}^+ \\
\psi = Q, \lim_{x_2 \to -1} \partial_2 \psi = \Lambda & \text{on } \overline{D_{i,\sigma_0}^+ \cap \{x_2 = 1\}},
\end{cases}
$$

where $D_{i,\sigma_0}^+ := (\Upsilon_{\Lambda}(1) + \frac{1}{3} i < x_1 < \Upsilon_{\Lambda}(1) + \frac{3}{4} i) \times (1, 1 + \sigma_0)$, $i := \min\{i, -\Upsilon_{\Lambda}(1)\}$, and $\sigma_0 > 0$ is a small constant depending on $i$ and the nozzle $S_1$. Using Lemma 5.5 again, we have $\psi_1$ is a one dimensional solution. This leads to a contradiction.

**Case (iii):** $i > 0$ and $\Upsilon_{\Lambda}(1) > 0$. In this case we will use the non-oscillation lemma (cf. Lemma 5.4) to get contradiction. First, note that the non-oscillation lemma also holds if one of the arcs $\Gamma_1$ or $\Gamma_1'$ is on the horizontal boundary $(0, R) \times \{1\}$, and

$$\lim_{x_2 \to 1-} \partial_2 \psi \geq \Lambda \text{ along } \Gamma_1 \text{ (or } \Gamma_1').$$

Next, denote $I_i := (\Upsilon_{\Lambda}(1) + \frac{i}{2}, \Upsilon_{\Lambda}(1) + \frac{3i}{4}) \times \{1\}$. By considering the variation $\tau_\vartheta(x) = x + \vartheta \eta(x)$ in the proof for Lemma 3.2, where $\vartheta \geq 0$ and the diffeomorphism $\eta$ satisfies $\eta \cdot e_2 \leq 0$ on $I_i$ and $\eta = 0$ on $\partial \Omega_{\mu,R} \setminus (\Upsilon_{\Lambda}(1), \Upsilon_{\Lambda}(1) + i) \times \{1\}$, and using similar computations as in the proof for Lemma 3.2, we have

$$\lim_{x_2 \to 1-} \partial_2 \psi_n \geq \Lambda \text{ on } I_i,$$

provided that $n$ is sufficiently large. Since $\Upsilon_{\Lambda_n}(1) \to \Upsilon_{\Lambda}(1) + i$ for $i > 0$, it follows from Theorem 5.6 that given any neighborhood of $I_i$, there is a sufficiently large $n$ such that the graph of the function $x_1 = \Upsilon_{\Lambda_n}(x_2)$ crosses the neighborhood. Applying Lemma 5.4 to the region bounded by $x_1 = \Upsilon_{\Lambda} + \frac{i}{2}, x_1 = \Upsilon_{\Lambda}(1) + \frac{3i}{4}, I_i$ and $x_1 = \Upsilon_{\Lambda_n}(x_2)$ then leads to a contradiction. The proof for case (iii) is complete.

**Step 3. Proof for (96).** It follows from Lemma 5.4 that there exists $\sigma_0 = \sigma_0(i) > 0$, such that for any open subset $\mathcal{U} \subseteq D_{i,\sigma_0}$ one has $\psi_1 < Q$ in $\mathcal{U}$ for sufficiently large $n$. Thus as $n \to \infty$ the limit $\psi_1$ satisfies the equation in (96) in $D_{i,\sigma_0}$. The Dirichlet boundary condition $\psi_1 = Q$ follows from $\partial_{x_1} \psi_1 \geq 0$ and the uniform convergence of $\psi_{\Lambda_n}$ to $\psi_1$ in $D_{i,\sigma_0}$. Thus one needs only to show that the Neumann condition holds.

(a). We first claim that $\lim_{x_2 \to 1-} \partial_2 \psi_1 \geq \Lambda$. For simplicity, denote $\psi_n := \psi_{\Lambda_n}$ and $\psi := \psi_1$. Fix $x^{(0)} \in \{x_2 = 1\} \cap \overline{D_{i,\sigma_0}}$ and $r > 0$ such that $B_r(x^{(0)}) \cap \{x_2 < 1\} \subset D_{i,\sigma_0}$.
For any nonnegative function \( \eta \in C_0^\infty(B_r(x^{(0)})) \), one has
\[
\Lambda_n g(\Lambda_n^2, Q) \int_{\partial \{ \psi < Q \}} \eta = \int_{\{ \psi < Q \}} g(\nabla \psi_n^2, \psi_n) \nabla \psi_n \cdot \nabla \eta + \partial_2 G(\nabla \psi_n^2, \psi_n) \eta \rightarrow \int_{\{ \psi < Q \}} g(\nabla \psi^2, \psi) \nabla \psi \cdot \nabla \eta + \partial_2 G(\nabla \psi^2, \psi) \eta.
\]
where \( \chi_{\{ \psi < Q \}} \rightarrow \chi_{\{ \psi < Q \}} \) in \( L^1(B_r(x^{(0)})) \) and \( \psi_n \rightarrow \psi \) a.e. have been used. It follows from the lower semi-continuity of the BV-norm that
\[
\int_{\partial \{ \psi < Q \}} \eta \leq \liminf_{n \to \infty} \int_{\partial \{ \psi < Q \}} \eta.
\]
Combining the above two inequalities together we obtain
\[
Ag(\Lambda^2, Q) \int_{\partial \{ \psi < Q \}} \eta \leq \int_{\{ \psi < Q \}} g(\nabla \psi^2, \psi) \nabla \psi \cdot \nabla \eta + \partial_2 G(\nabla \psi^2, \psi) \eta.
\]
Since \( B_r(x^{(0)}) \cap \partial \{ \psi < Q \} = B_r(x^{(0)}) \cap \{ x_2 = 1 \} \), the solution \( \psi \) is \( C^1 \) up to the boundary. Therefore, an integration by parts gives
\[
\int_{\{ \psi < Q \}} g(\nabla \psi^2, \psi) \nabla \psi \cdot \nabla \eta + \partial_2 G(\nabla \psi^2, \psi) \eta = \int_{\partial \{ \psi < Q \}} \partial_{x_2} \psi g(\partial_{x_2} \psi^2, Q) \eta.
\]
Thus
\[
\Lambda g(\Lambda^2, Q) \int_{\partial \{ \psi < Q \}} \eta \leq \int_{\partial \{ \psi < Q \}} \partial_{x_2} \psi g(\partial_{x_2} \psi^2, Q) \eta.
\]
Since \( \eta \) is arbitrary and \( s \mapsto sg(s^2, Q) \) is monotone increasing for \( s \geq 0 \), we have
\[
\partial_{x_2} \psi \geq \Lambda \quad \text{on} \quad \partial \{ \psi < Q \} \cap B_r(x^{(0)}).
\]

(b). The arguments for \( \lim_{x_2 \to 1} - \partial_{x_2} \psi \leq \Lambda \) on \( \overline{D_{r,\sigma_0}} \cap \{ x_2 = 1 \} \) are similar as those in [39, Theorem 6.1 in Chapter 3]. We only outline the proof here for completeness. For each \( x^{(0)} \in \overline{D_{r,\sigma_0}} \cap \{ x_2 = 1 \} \) and \( r > 0 \) sufficiently small, by the uniform convergence of \( \psi_n \) to \( \psi \) as well as the convergence of the free boundaries, one can find a sequence of domains \( \{ K_n \} \) such that: (i) \( \psi_n < Q \) in \( K_n \cap B_r(x^{(0)}) \), and \( \partial D_n \cap B_r(x^{(0)}) \) is given by a \( C^{1,\alpha} \) graph, which converges to the free boundary \( \partial \{ \psi < Q \} \cap B_r(x^{(0)}) = \{ x_2 = 1 \} \cap B_r(x^{(0)}) \) in \( C^{1,\alpha} \) as \( n \to \infty \); (ii) there is a sequence of points \( \{ x^{(n)} \} \subset \partial K_n \cap B_r(x^{(0)}) \), such that \( \psi_n(x^{(n)}) = Q \) for each \( n \) and \( x^{(n)} \to x^{(0)} \) as \( n \to \infty \). Then let \( \phi_n \) be the solution to the Dirichlet problem
\[
\begin{cases}
\nabla \cdot (g(\nabla \phi_n^2, \phi_n) \nabla \phi_n) - \partial_2 G(\nabla \phi_n^2, \phi_n) = 0 & \text{in } K_n \cap B_r(x^{(0)}), \\
\phi_n = Q & \text{on } \partial D_n \cap B_r(x^{(0)}), \\
\phi_n = \psi_n & \text{on } \partial B_r(x^{(0)}) \cap \partial K_n.
\end{cases}
\]
By the comparison principle, \( \phi_n \geq \psi_n \) in \( K_n \cap B_r(x^{(0)}) \) and thus \( \partial_{\nu^{(n)}} \psi_n(x^{(n)}) \leq \partial_{\nu^{(n)}} \phi_n(x^{(n)}) = \Lambda_n \), where \( \nu^{(n)} := \nu(x^{(n)}) \) is the unit outer normal of \( K_n \) at \( x^{(n)} \).
the boundary estimate for elliptic equations one has \( \phi_n \to \psi \) in \( C^{1,\alpha}(B_{r/2}(x^{(0)}) \cap \overline{K_n}) \). This gives \( \lim_{n \to \infty} \partial_{\nu(n)} \phi_n(x^{(n)}) = \partial_{x_2} \psi(x^{(0)}) \leq \Lambda = \lim_{n \to \infty} \Lambda_n \). The proof for (96) is thus complete. □

The continuous dependence on the parameter \( \Lambda \) together with the nondegeneracy property of the solutions yields the following lemma:

**Lemma 6.2.**

(i) If \( \Lambda \) is large, then the free boundary \( \Gamma_\Lambda \) is nonempty and satisfies \( \Upsilon_\Lambda(1) < 0 \).

(ii) If \( \Lambda \) is small, then \( \Upsilon_\Lambda(1) > 0 \).

**Proof.** The proof is inspired by [9, Lemma 5.2].

(i). If \( \Gamma_\Lambda \neq \emptyset \), then we claim that \( \Lambda \) cannot be too large provided that \( \Upsilon_\Lambda(1) > 0 \). Since \( \Gamma_\Lambda \) connects \( (\Upsilon_\Lambda(1), 1) \) to \( (R, l_\Lambda) \) (note that \( l_\Lambda < 1 \) since \( \Gamma_\Lambda \neq \emptyset \)), there exist \( x \in \Gamma_\Lambda \) and \( r \in (0, 1) \) independent of \( \Lambda \) such that \( B_r(x) \subset \Omega_{\mu,R} \cap \{ x_2 > \sigma_0/2 \} \), or \( B_r(x) \subset \{ x_1 > 0, x_2 > \sigma_0/2 \} \) for some small \( \sigma_0 > 0 \). In either case it follows from the nondegeneracy lemma (Lemma 4.8) that

\[
\frac{Q}{r} \geq \frac{1}{r} \left( \frac{1}{|B_r(x)|} \int_{B_r(x)} |Q - \psi|^2 \right)^{1/2} \geq C \Lambda \ 	ext{Remark 3.3} \geq \tilde{C} \Lambda.
\]

Here \( \tilde{C} > 0 \) depends only on \( \gamma, r, \) and \( B_* \) defined in (20). Since \( Q \) and \( r \) are fixed, there is a contradiction if \( \Lambda \) is large.

Next if \( \Gamma_\Lambda = \emptyset \) (i.e. \( l_\Lambda \geq 1 \)), then one can derive a contradiction by taking any \( x^{(0)} = (x_1, 1) \) with \( x_1 > 0 \) and using the linear nondegeneracy for nonnegative solutions along flat boundaries.

(ii). Suppose that \( \Upsilon_\Lambda(1) \leq 0 \), we will prove that \( \Lambda \) cannot be too small. Similar to that in [9], for \( \vartheta, \sigma > 0 \), denote

\[
\eta_\vartheta(x_1) := \begin{cases} \vartheta \exp \left( \frac{(x_1 - R)^2}{2(x_1 - R)^2 - \sigma^2} \right) & \text{if } |x_1 - R| < \sigma, \\
0 & \text{otherwise}. \end{cases}
\]

One can increase \( \vartheta \) till the first time \( \vartheta_0 \) such that the graph of \( x_2 = \eta_{\vartheta_0}(x_1) \) touches the free boundary at some \( x^{(0)} \in \Gamma_\Lambda \). Let \( K := \Omega_{\mu,R} \cap \{ (x_1, x_2) : x_2 < \eta_{\vartheta_0}(x_1) \} \) and \( \phi \) be a solution to the following problem

\[
\begin{aligned}
\nabla \cdot (g(|\nabla \phi|^2, \phi) \nabla \phi) - \partial_z G(\nabla \phi, \phi) &= 0 \quad \text{in } K, \\
\phi &= Q \quad \text{on } \{ x_2 = \eta_{\vartheta_0}(x_1) \}, \\
\phi &= 0 \quad \text{on } \{ x_2 = 0 \}.
\end{aligned}
\]

It follows from Lemma 4.6 that \( \phi \geq \psi_\Lambda \) in \( K \). Hence

\[
\partial_v \phi(x^{(0)}) \leq \partial_v \psi_\Lambda(x^{(0)}) = \Lambda.
\]
On the other side, using the linear asymptotics of $\phi$ at $x(0)$, which can be obtained by solving the problem for the blow up limit at $x(0)$, one has

$$\partial_\nu \phi(x(0)) \geq CQ$$

for some $C > 0$ independent of $\Lambda$. Combining the above two inequalities together gives $Q/\Lambda \leq 1/C$. This leads to a contradiction if $\Lambda$ is small.

As a consequence of Lemmas 6.1 and 6.2, one has the following continuous fit property for the truncated problem:

**Corollary 6.3.** There exists $\Lambda = \Lambda(\mu, R) > 0$ such that $\Upsilon_\Lambda(1) = 0$. Furthermore, $C^{-1}Q \leq \Lambda(\mu, R) \leq CQ$ for positive constant $C$ depending on $B_\ast$, $\gamma$, and $\bar{H}$, but independent of $\mu$ and $R$.

6.2. **Smooth fit.** With the continuous fit property at hand, one can argue along the same lines as [9, Theorem 6.1] to conclude that the free boundary $\Gamma_\psi$ fits the outlet of the nozzle $S_1$ in a $C^1$ fashion.

**Proposition 6.4.** Let $\psi$ be a solution obtained in Corollary 6.3. Then $S_1 \cup \Gamma_\psi$ is $C^1$ in a $\{\psi < Q\}$-neighbourhood of the point $A = (0, 1)$, and $\nabla \psi$ is continuous in a $\{\psi < Q\}$ neighborhood of $A$.

**Proof.** Let $\psi$ be a solution to the free boundary problem. Suppose that $\{x_m\}$ is a sequence of points in $\{\psi < Q\}$ which converges to $A$. Define the scaled functions

$$\psi_{r,x_m} := Q - \frac{Q - \psi(x_m + rx)}{r}, \quad r \in (0, 1).$$

Then $\psi_{r,x_m}$ satisfies the quasilinear equation in $\{\psi_{r,x_m} < Q\}$

$$\nabla \cdot \left( g_\epsilon(|\nabla \psi_{r,x_m}|^2, Q - r(Q - \psi_{r,x_m})) \nabla \psi_{r,x_m} \right) - r \partial_z G_\epsilon(|\nabla \psi_{r,x_m}|^2, Q - r(Q - \psi_{r,x_m})) = 0.$$

Since $\partial_z G_\epsilon \in L^\infty$, then any blow up limit along a subsequence $\psi_{r_m,x_m}$ satisfies

$$\nabla \cdot \left( g_\epsilon(|\nabla \psi|^2, Q) \right) = 0. \quad (98)$$

Note that (98) is the same as the governing equation for the irrotational flows. Thus with this scaling property at hand, one can use the compactness arguments in [9, Theorem 6.1] together with the unique continuation property for the limiting equation (98) to conclude Proposition 6.4. \qed
6.3. **Asymptotic behaviour of the jet.** The main goal of this subsection is to remove the truncations of the domain by letting \( \mu, R \to \infty \), which is a consequence of Corollary 6.3 and the uniform estimates (with respect to \( \mu \) and \( R \)) of the solutions in Section 4 and Section 6.1. Hence one gets a solution in \( \Omega \), which is the domain bounded by \( S_0 \) and \( S_1 \cup ([0, \infty) \times \{1\}) \). Moreover, the solution inherits the properties of solutions in the truncated domain. The convergence and the properties of the limit solution are summarized in the following proposition.

**Proposition 6.5.** Given a nozzle \( S_1 \) satisfying (3) and \( B \) satisfies (8), let \( \psi_{\mu,R,\Lambda} \) be a minimizer of the problem

\[
\inf_{\psi \in K_{\psi,\mu,R}} J^{\mu,R,\Lambda}(\psi)
\]

with \( \psi_{\mu,R} \) defined in (90). Then for any \( \mu_j, R_j \to \infty \), there is a subsequence (still labeled by \( \mu_j \) and \( R_j \)) such that \( \Lambda_j := \Lambda(\mu_j, R_j) \to \Lambda_\infty \) for some \( \Lambda_\infty \in (0, \infty) \) and \( \psi_{\mu_j,R_j,\Lambda_j} \to \psi_\infty \) in \( C^0_{\text{loc}}(\Omega) \) for any \( \alpha \in (0,1) \). Furthermore, the following properties hold:

(i) The function \( \psi_\infty \) is a local minimizer for

\[
J^{\ast}(\psi) := \int_D G_\varepsilon(|\nabla \psi|^2, \psi) + \lambda^{2}_{\varepsilon,\infty} \chi_{\{\psi < Q\}}, \quad \lambda^{\ast}_{\varepsilon,\infty} := \sqrt{\Phi_\varepsilon(\Lambda_\infty, Q)},
\]

i.e. \( J^{\ast}(\psi_\infty) \leq J^{\ast}(\tilde{\psi}) \) for all \( \tilde{\psi} = \psi_\infty \) outside any bounded domain \( D \).

(ii) \( \partial_{x_1} \psi_\infty \geq 0 \) in \( \Omega \).

(iii) The free boundary \( \Gamma_{\psi_\infty} := \partial\{\psi_\infty < Q\} \setminus S_1 \) is given by the graph \( x_1 = \Upsilon(x_2) \) for some function \( \Upsilon \), where \( \Upsilon \) is \( C^{1,\alpha} \) as long as it is finite.

(iv) \( \lim_{x_2 \to 1^-} \Upsilon(x_2) = 0 \).

(v) There is an \( l \in (0,1) \) such that \( \Upsilon(x_2) \) is finite if and only if \( x_2 \in (l, 1] \), and \( \lim_{x_2 \to l^+} \Upsilon(x_2) = \infty \). Furthermore, there exists an \( \tilde{R} > 0 \) sufficiently large, such that \( \Gamma_{\psi_\infty} \cap \{x_1 > \tilde{R}\} = \{(x_1, f(x_1)) : \tilde{R} < x_1 < \infty\} \) for some \( C^{1,\alpha} \) function \( f \) and \( \lim_{x_1 \to +\infty} f'(x_1) = 0 \).

**Proof.** The proof for (i)–(iii) is based on standard compactness arguments (cf. step 1 in the proof for Lemma 6.1), the regularity of the solution in Section 4 as well as the properties of \( \psi_n \) in Section 5 and Section 6.1. We will not repeat it here.

(iv) The continuous fit property follows from Proposition 5.6, Lemma 6.1 and Corollary 6.3.

(v) Let \( I := \{x_2 \in (0,1) : 0 < \Upsilon(x_2) < \infty\} \). Since \( \Upsilon \) is continuous, \( I \) is open. For each \( \bar{x}_2 \in I \) one has \( \Upsilon(x_2) < \infty \) for all \( x_2 \in (\bar{x}_2, 1) \) ([9, Lemma 5.4]). It follows from Proposition 5.6 that \( l \) is open. Let \( \bar{l} := \inf\{x_2 : x_2 \in I\} \). We claim that

\[
\Upsilon(x_2) = \infty \quad \text{for all } 0 < x_2 \leq \bar{l}.
\]
Indeed, if \( \Upsilon(l) < \infty \), there must be an \( \hat{x}_2 < l \) such that \( \Upsilon(\hat{x}_2) < \infty \). Otherwise, \( \psi \) would solve the following Cauchy problem
\[
\begin{aligned}
\begin{cases}
\nabla \cdot (g_\epsilon(|\nabla \psi|^2, \psi) \nabla \psi) - \partial_x G_\epsilon(|\nabla \psi|^2, \psi) = 0 & \text{in } (\Upsilon(l), \infty) \times (0, l), \\
\psi = Q, \quad \partial_{x_2} \psi = \Lambda & \text{on } (\Upsilon(l), \infty) \times \{l\}.
\end{cases}
\end{aligned}
\]
It follows from Lemma 5.5 that \( \psi = \psi_{1d} \) is a one dimensional solution and thus the free boundary is the horizontal line \( \mathbb{R} \times \{l\} \). This is a contradiction. The fact that \( \Upsilon(\hat{x}_2) < \infty \) implies that \( \Upsilon(x_2) < \infty \) for all \( x_2 \in (\hat{x}_2, 1) \) including \( l \). Thus \( l \) is an interior point of \( I \), which is a contradiction. Hence \( \Upsilon(x_2) < \infty \) for all \( x_2 \in (l, 1) \) and \( \Upsilon(x_2) = \infty \) for all \( x_2 \in (0, l] \). Since \( \lim_{x_2 \to l^+} \Upsilon(x_2) = \infty \), the flatness condition (cf. [5]) is satisfied when \( x_1 > \hat{R} \) for sufficiently large \( \hat{R} \). This implies that the free boundary is an \( x_1 \)-graph \( x_2 = f(x_1) \) for \( x_1 > \hat{R} \). In addition, one has \( f'(x_1) \to 0 \) as \( x_1 \to +\infty \). \( \square \)

The next proposition is about the asymptotic behavior of the solution \( \psi_\infty \) at \( x_1 = \pm \infty \).

**Proposition 6.6.** Let \( \psi_\infty \) be the solution obtained in Proposition 6.5. Then as \( x_1 \to -\infty, \psi_\infty \to \bar{\psi}, \) where
\[
\bar{\psi}(x_2) := \bar{\rho} \int_0^{x_2} \bar{u}(s)ds, \quad x_2 \in [0, \bar{H}]
\]
is the stream function associated with the incoming flow \( \bar{\rho}, \bar{u} \); as \( x_1 \to +\infty, \psi_\infty \to \hat{\psi} \), where \( \hat{\psi} \) is the unique one dimensional solution to the free boundary problem with the constant momentum \( \Lambda_\infty \). In particular, the asymptotic height at downstream is uniquely determined by \( \Lambda_\infty, \bar{\rho}, \bar{u} \) and \( \bar{H} \).

**Proof.** (i). Downstream asymptotic behavior. Let \( \psi^{(-n)}(x_1, x_2) := \psi_\infty(x_1 + n, x_2), \) \( n \in \mathbb{N} \). By the \( C^{1,\alpha} \) regularity of the free boundary and the boundary regularity for elliptic equations, up to a subsequence \( \psi^{(-n)} \) converges to \( \hat{\psi} \) which solves
\[
\begin{aligned}
\begin{cases}
\nabla \cdot (g_\epsilon(|\nabla \hat{\psi}|^2, \hat{\psi}) \nabla \hat{\psi}) - \partial_x G_\epsilon(|\nabla \hat{\psi}|^2, \hat{\psi}) = 0 & \text{in } \mathbb{R} \times (0, l), \\
\hat{\psi} = Q, \quad \partial_{x_2} \hat{\psi} = \Lambda_\infty & \text{on } \mathbb{R} \times \{l\}.
\end{cases}
\end{aligned}
\]
For given \( Q \) and \( \Lambda_\infty > 0 \), there is a unique one dimensional solution \( \psi_{1d} \) to the above problem. Furthermore, since \( \hat{\psi}(x_1, 0) = 0 \), the height \( l \) is uniquely determined by \( Q \) and \( \Lambda_\infty \). Indeed, for \( k > 0 \), consider the variational problem
\[
\inf_{K_k} \int_0^k G_\epsilon(|\varphi'|^2, \varphi)
\]
where \( K_k = \{ \varphi \in C^{0,1}([0, k]; \mathbb{R}) : \varphi(0) = 0, \varphi(k) = Q \} \). There exists a unique minimizer \( \hat{\varphi}_k \) by the direct method in the calculus of variations. Moreover, by the boundary
Hopf lemma $\Lambda(k) := \lim_{x \to k^-} \tilde{\psi}'_k(x)$ is strictly monotone decreasing in $k$. Thus there exists a unique $l$ such that $\Lambda(l) = \Lambda_\infty$. By the uniqueness of the problem for the downstream blowup limit, one can thus conclude that $\psi_\infty \to \psi$ as $x_1 \to +\infty$, and the asymptotic height is uniquely determined.

(ii). Upstream asymptotic behavior. Since the nozzle is asymptotically horizontal with the height $\bar{H}$, the limits of $\psi_\infty(x_1 - n, x_2)$ solve the Dirichlet problem in the infinite strip

$$\begin{cases}
\nabla \cdot (g_e(|\nabla \psi|^2, \psi) \nabla \psi) - \partial_z G_\epsilon(|\nabla \psi|^2, \psi) = 0 & \text{in } \mathbb{R} \times (0, \bar{H}), \\
\psi = Q & \text{on } \mathbb{R} \times \{\bar{H}\}, \\
0 < \psi < Q & \text{in } \mathbb{R} \times (0, \bar{H}).
\end{cases}$$

Furthermore, it follows from the energy estimate (cf. [64, Lemma 1]) that the problem (100) has a unique solution. Clearly $\tilde{\psi}$ defined in (99) satisfies (100). Therefore, $\psi^{(n)} \to \tilde{\psi}$. This proves the asymptotic behavior of the flows in the upstream and thus finishes the proof of the proposition.

In view of Proposition 2.1 and Lemma 2.4, with the regularity and asymptotic behavior of solutions at hand, the equivalence between the stream function formulation and the original Euler system is true as long as $\partial_{x_1} \psi > 0$ inside the flow region. This fact is guaranteed by the following proposition.

**Proposition 6.7.** Let $\psi := \psi_\infty$ be a solution obtained in Proposition 6.5. Then $\partial_{x_1} \psi > 0$ in $\mathcal{O} = \Omega \cap \{\psi < Q\}$.

**Proof.** It follows from Proposition 6.5 (ii) that $\partial_{x_1} \psi \geq 0$ in $\mathcal{O}$. The strict inequality can be proved by the strong maximum principle. Indeed, assume that $\partial_{x_1} \psi(x^{(0)}) = 0$ for some $x^{(0)} \in \Omega \cap \{\psi < Q\}$. Let $V := \partial_{x_1} \psi$. Then $V \geq 0$ in $\mathcal{O}$ and it satisfies

$$\partial_i(a^{ij} \partial_j V) + \partial_i(b^i \partial_i V) - b^i \partial_i V + cV = 0 \text{ in } \mathcal{O},$$

where

$$a^{ij} := \partial_{p,p_j} g_e, \quad b^i := \partial_{p,z} g_e, \quad c := \partial_{zz} G_\epsilon.$$

The strong maximum principle (cf. [44, Theorem 2.2]) gives that $V = 0$ in $\mathcal{O}$, and hence all streamlines are horizontal. This is however a contradiction. Hence $V > 0$ in $\mathcal{O}$ and thus the proof of the proposition is complete.

**Remark 6.8.** To show that this is the unique state at downstream, we make use of the flatness of the free boundary at $x_1 \to +\infty$. More precisely, consider the limits $\psi(x_1 + k_n, x_2) \to \psi_\infty$ along sequences $k_n \to \infty$. Then $\psi_\infty$ is a solution to the Cauchy
problem in the infinite strip

\[
\begin{aligned}
\nabla \cdot (g_\epsilon(|\nabla \psi|^2, \psi)\nabla \psi) - \frac{B'(\psi)}{g_\epsilon(|\nabla \psi|^2, \psi)} &= 0 \text{ in } \mathbb{R} \times (0, H_1), \\
\psi &= 0 \text{ on } \mathbb{R} \times \{0\}, \\
\psi &= Q, \partial_{x_2} \psi = \Lambda \text{ on } \mathbb{R} \times \{H\}.
\end{aligned}
\]

One knows that \(\bar{\psi}(x_2) = \rho \int_0^{x_2} u(s) ds\) is a solution to the Cauchy problem with \(g(|\nabla \bar{\psi}|^2, \bar{\psi}) = \frac{1}{\rho}\). Since the flow is subsonic so that the governing equation is elliptic, by the unique continuation we infer that \(\psi\) is the unique solution to the above Cauchy problem \((101)\).

7. Remove the subsonic truncation when the flux is small

In this section, the subsonic truncation is removed. We show that if \((29)\) holds for sufficiently small \(\kappa^*\) and \(Q\) is suitably small, then one always has \(|\nabla \psi|^2 < t_c(B(\psi)) - \epsilon\), where \(t_c(B(z))\) is defined in \((19)\). The proof is based on the classical Bernstein’s technique.

**Proposition 7.1.** Let \(\psi\) be a limiting solution in Proposition 6.5. Then if the condition \((29)\) holds, then

\[
|\nabla \psi| < t_c(B(\psi)) - \epsilon
\]

as long as \(Q \in (Q^*, \hat{Q})\) for some \(\hat{Q}\) sufficiently small depending on \(B^*, \gamma\) and the nozzle.

**Remark 7.2.** Note that in view of \((19)\), one can fix a truncation parameter \(\epsilon > 0\) small depending only on \(B^*\) and \(\gamma\), such that \(t_c(B(\psi))\) is uniformly bounded from below by a positive constant depending only on \(B^*\) and \(\gamma\).

**Proof of Proposition 7.1.** By the boundary estimate for the elliptic equation, for any \(x^{(0)} \in \mathcal{O} := \Omega \cap \{\psi < Q\}\), one has

\[
\|\nabla \psi\|_{L^\infty(B_1(x^{(0)}) \cap \mathcal{O})} \leq C \left( \|\psi\|_{L^\infty(B_2(x^{(0)}) \cap \mathcal{O})} + \|\partial_2 G_\epsilon\|_{L^\infty(B_2(x^{(0)}) \cap \mathcal{O})} \right),
\]

where \(C\) depends on \(B^*, \gamma\) and the nozzle. Since \(0 \leq \psi \leq Q\) in \(\mathcal{O}\) and

\[
\|\partial_2 G_\epsilon\|_{L^\infty} \leq C \kappa_0,
\]

it follows from Propositions 2.2 and 4.1 and Corollary 6.3 that one has \((102)\) provided that \((29)\) holds with \(\kappa_0\) sufficiently small and \(Q \in (Q^*, \hat{Q})\) with \(\hat{Q}\) sufficiently small. Both \(\kappa_0\) and \(\hat{Q}\) depend on \(B^*, \gamma\) and the nozzle. \(\square\)

In fact, when the flows are subsonic, the downstream states of the flows are uniquely determined by the upstreaming incoming data \(B, Q\) as well as \(\hat{H}\) and \(\Lambda\).
Proposition 7.3. Let \((\psi, \Gamma_\psi, \Lambda)\) be a smooth subsonic triple to the Problem 2 with the given incoming Bernoulli function \(B\) and the mass flux \(Q \in (Q_*, Q^*)\), i.e., both \(\psi\) and \(\Gamma_\psi\) are smooth, and that \(\partial_x \psi > 0\) in the fluid region \(\{\psi < Q\}\). Then the states at downstream, \((\bar{\rho}, \bar{u})\), and the asymptotic height \(\bar{H}\) are uniquely determined by \(B\), \(Q\), \(\bar{H}\) and \(\Lambda\).

Proof. On the free boundary the flow at the downstream satisfies

\begin{equation}
B(\bar{H}) = h(\bar{\rho}) + \frac{\Lambda^2}{2\bar{\rho}^2}.
\end{equation}

For a subsonic solution, one has \(\Lambda^2 < \ell_c(B(\bar{H}))\). Thus \(\bar{\rho} \in (\varrho_c(B(\bar{H})), \varrho^*(B(\bar{H})))\) is determined uniquely by \(B(\bar{H})\) and \(\Lambda\). Furthermore, using \((103)\) and Corollary 6.3 gives

\[\varrho^*(B(\bar{H})) - \bar{\rho} \geq C^{-1}\Lambda^2 \geq C^{-1}\kappa^{1/2},\]

where \(C\) depends only \(B_*, \gamma\), and the nozzle. Note that

\[0 < \varrho^*(B(\bar{H})) - \varrho^*(B_*) \leq C\kappa.\]

Therefore, one has

\begin{equation}
\varrho^*(B_*) - \bar{\rho} > C\kappa^{1/2}.
\end{equation}

Next, let \(\theta(x_2)\) be the position at downstream if one follows along the streamline with asymptotic height \(x_2\) in the upstream, i.e., \(\theta : [0, \bar{H}] \to [0, \bar{H}]\) satisfies

\begin{equation}
\int_0^{\theta(x_2)} \rho u(s)ds = \int_0^{x_2} \bar{\rho}\bar{u}(s)ds \quad \text{and} \quad B(x_2) = \frac{u^2(\theta(x_2))}{2} + h(\bar{\rho}), \quad x_2 \in [0, \bar{H}].
\end{equation}

Therefore, one has

\begin{equation}
u(\theta(x_2)) = \sqrt{2B(x_2) - 2h(\bar{\rho})}.
\end{equation}

It follows from \((104)\) that one has

\[C^{-1}\kappa^{1/4} < u(\theta(x_2)) < q_c(B^*).\]

Using the first equation in \((105)\) yields

\begin{equation}
\begin{cases}
\theta'(x_2) = \frac{\bar{\rho}\bar{u}(x_2)}{\rho u(\theta(x_2))}, \\
\theta(0) = 0.
\end{cases}
\end{equation}

Substituting \((106)\) into \((107)\) yields that \(\theta\) satisfies the Cauchy problem for a standard first order ODE. Therefore, \(\theta\), in particular \(\bar{H} = \theta(1)\), can be uniquely determined. The downstream velocity \(y\) is thus also determined by \((106)\).
Proposition 7.1 together with Proposition 6.7 yields the existence of solutions to the Problem 1. The downstream asymptotic behavior of the solutions follows from Proposition 7.3 and Remark 6.8.

8. Uniqueness

In this section we prove the uniqueness of the solution to the Problem 1.

**Proposition 8.1.** Suppose that \((\psi_i, \Gamma_i, \Lambda_i) \ (i = 1, 2)\) are two uniformly subsonic solutions to Problem 2, then they must be the same.

**Proof.** Let \((\bar{\rho}_i, \bar{u}_i, \bar{H}_i), \ i = 1, 2,\) be the downstream asymptotics. The proof is divided into two steps.

**Step 1.** \(\Lambda_1 = \Lambda_2.\) Without loss of generality, assume that \(\Lambda_1 < \Lambda_2.\) Let \(B(\rho, t) := h(\rho) + \frac{1}{2\rho^2}.\) By Bernoulli’s law, along the free boundary \(\Gamma_1\) and \(\Gamma_2\) one has

\[
B(\bar{\rho}_1, \Lambda_1^2) = B(\bar{\rho}_2, \Lambda_2^2) = B(H).
\]

In the subsonic region \(\rho^{\gamma+1} > t\) one has

\[
\frac{\partial B}{\partial \rho} > 0 \quad \text{and} \quad \frac{\partial B}{\partial t} > 0.
\]

Hence \(B(\rho, t) = s\) determines a unique function \(\rho = \rho(t, s)\) which is a decreasing with respect to \(t.\) Thus if \(\Lambda_1 < \Lambda_2,\) one has \(\rho(\Lambda_1, B(H)) > \rho(\Lambda_2, B(H)),\) which implies \(\rho_1 > \rho_2.\) Let \(\theta_i(x_2) \ (i = 1, 2)\) be defined in (107) associated with \((\rho_i, u_i) \ (i = 1, 2).\) By Bernoulli’s law again one has

\[
B(\rho_1, (\rho_1 u_1)^2(\theta_1(x_2))) = B(\rho_2, (\rho_2 u_2)^2(\theta_2(x_2))) = B(x_2).
\]

Thus \(\rho_1 = \rho_2 \frac{u_1(\theta_1(x_2))}{u_2(\theta_2(x_2))}, \ B(x_2), \ i = 1, 2.\) Since \(\rho_1 > \rho_2\) and \(\rho\) is monotone decreasing in \(t,\) one has

\[
\rho_1 u_1(\theta_1(x_2)) < \rho_2 u_2(\theta_2(x_2)).
\]

In view of (107) this implies

\[
(108) \quad \theta_1'(x_2) > \theta_2'(x_2).
\]

Note that \(\theta_1(0) = \theta_2(0) = 0.\) Integrating (108) on \([0, 1]\) gives \(H_1 = \theta(1) > \theta_2(1) = H_2.\)

Let \(O_i\) be the domain bounded by \(S_0, S_1,\) and \(\Gamma_i.\) Since \(H_1 > H_2\) and that \(S_1 \cup \Gamma_i\) is an \(x_2\)-graph, one can find a \(k \geq 0\) such that the domain \(O_1(k) = \{(x_1, x_2) : (x_1 - k, x_2) \in O_1\}\) contains \(O_2.\) Let \(k_\ast\) be the smallest number such that \(O_1(k_\ast)\) contains \(O_2\) and they touch at some point \((x_1^*, x_2^*) \in \Gamma_2.\) Let \(\psi_i^{(k)} := \psi_i(x_1 - k, x_2) \ (i = 1, 2).\) By the
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comparision principle \( \psi_1^{(k_*)} \leq \psi_2 \) in \( O_2 \). Noting that at the touching point one has \( \psi_1^{(k_*)}(x_1^*, x_2^*) = \psi_2(x_1^*, x_2^*) \), then by the boundary Hopf lemma we have

\[
\Lambda_1 = \frac{\partial \psi_1}{\partial \nu}(x_1^*, x_2^*) > \frac{\partial \psi_2}{\partial \nu}(x_1^*, x_2^*) = \Lambda_2.
\]

This leads to a contradiction. Hence one has \( \Lambda_1 = \Lambda_2 \).

Step 2. \( \psi_1 = \psi_2 \) and \( \Gamma_1 = \Gamma_2 \). Suppose that \( \Gamma_1 \neq \Gamma_2 \). Without loss of generality, assume that there exists an \( \tilde{x}_2 \) such that \( \Upsilon_1(\tilde{x}_2) < \Upsilon_2(\tilde{x}_2) \), where \( \Gamma_i = (\Upsilon_i(x_2), x_2) \). From step 1 and Proposition 7.3 one has \( \bar{H}_1 = \bar{H}_2 \). Thus similar as in step 1 there exists \( \tilde{k}_* \geq 0 \) such that they touch at some point \( (\tilde{x}_1^*, \tilde{x}_2^*) \in \Gamma_2 \), moreover,

\[
\Lambda_1 = \frac{\partial \psi_1^{(k_*)}}{\partial \nu}(\tilde{x}_1^*, \tilde{x}_2^*) > \frac{\partial \psi_2}{\partial \nu}(\tilde{x}_1^*, \tilde{x}_2^*) = \Lambda_2.
\]

This is however a contradiction. Thus one has \( \Gamma_1 = \Gamma_2 \) and thus \( \Omega_1 = \Omega_2 \). Let \( \Gamma := \Gamma_1 \) and \( \Omega := \Omega_1 \). It follows from the energy estimate (cf. [64, Proposition 5]) that the boundary value problem

\[
\begin{align*}
\nabla \cdot (g(|\nabla \psi|^2, \psi)\nabla \psi) - \partial_z G(|\nabla \psi|^2, \psi) &= 0 \quad \text{in } \Omega, \\
\psi &= Q \text{ on } S_1 \cup \Gamma, \quad \psi = 0 \text{ on } S_0
\end{align*}
\]

has a unique uniformly subsonic solution. Hence \( \psi_1 = \psi_2 \).

This completes the proof of the lemma. \( \square \)

9. Existence of Critical Mass Flux

For the given Bernoulli function at the upstream satisfying (8) with \( \kappa^* \) sufficiently small depending on \( B_*, \gamma \) and the nozzle, Proposition 7.1 shows that there exists subsonic jet flows as long as \( Q \in (Q^*, \hat{Q}) \) for some \( \hat{Q} > Q^* \) sufficiently small. In this section, we increase \( \hat{Q} \) as large as possible until some critical mass flux \( Q_c \), such that for \( Q > Q_c \) either the flow fails to be subsonic or the stream function \( \psi \) fail to be a solution in the sense of Problem 2.

Proposition 9.1. Let \( B \) satisfy (8), then there exists a critical mass flux \( Q_c \leq Q^* \) for \( Q^* \) appeared in Proposition 2.2 such that if \( Q \in (Q^*, Q_c) \), there exists a unique solution \( \psi \) to Problem 2, which satisfies

\[
0 < \psi < Q \quad \text{in } \Omega, \quad \text{and } \mathcal{M}(Q) := \sup_{\Omega} \left( |\nabla \psi|^2 - \iota_c(\mathcal{B}(\psi)) \right) < 0,
\]

with the domain \( \Omega \) bounded by \( S_0, S_1, \) and \( \Gamma \). Furthermore, as \( Q \to Q_c \) one has either \( \mathcal{M}(Q) \to 0 \), or there does not exist a \( \sigma > 0 \) such that Problem 2 has solutions for all
\(Q \in (Q_c, Q_c + \sigma)\) and
\[
\sup_{Q \in (Q_c, Q_c + \sigma)} \mathcal{M}(Q) < 0. \tag{110}
\]

**Proof.** The proof for the proposition is inspired by [62, Proposition 6], where the existence of critical flux for subsonic solutions with nonzero vorticity in fixed nozzles was established.

For the given Bernoulli function \(B\) in the upstream satisfying (8) and any \(Q \in (Q^*, Q^*_0)\), one can define \(\bar{\rho}\) and \(\bar{u}(x_2)\). Since \(B(\psi)\) and \(\bar{\rho}\) depend on \(Q\) by definition, in this section, we denote them by \(\bar{\rho}(Q)\) and \(B(\psi; Q)\), respectively.

Let \(\{\epsilon_n\}_{n=1}^{\infty}\) be a strictly decreasing sequence of positive numbers such that \(\epsilon_n \downarrow 0\) and \(\epsilon_1 \leq \epsilon\) which is used for the subsonic truncations in previous sections. Let \(\kappa^*\) be the constant defined in (8). Fix \(\kappa^* > 0\) sufficiently small depending on \(B^*, \gamma\) and the nozzle (thus \(\kappa_0\) given in (29) is small), let \(\psi_n(x; Q)\) be the solution (if it exists) to
\[
\begin{cases}
\nabla \cdot \left( g_{\epsilon_n}(|\nabla \psi|^2, \psi) \nabla \psi \right) - \partial_t G_{\epsilon_n}(|\nabla \psi|^2, \psi) = 0 \text{ in } \{ \psi < Q \}, \\
\psi = 0 \text{ on } S_0, \\
\psi = Q \text{ on } S_1 \cup \Gamma_\psi, \\
|\nabla \psi| = \Lambda \text{ on } \Gamma_\psi,
\end{cases} \tag{111}
\]
such that the free boundary condition \(\Gamma\) joins \(S_1\) as a \(C^1\) smooth curve, where \(g_{\epsilon_n}\) and \(G_{\epsilon_n}\) are defined in (40) and (50), respectively, and moreover, if
\[
|\nabla \psi_n(x; Q)|^2 - t_c(B(\psi_n; Q)) \leq -\epsilon_n, \tag{112}
\]
then \(\psi_n(x; Q)\) solves the problem (39), and it satisfies the far fields behavior claimed in Problem 2 and
\[
0 \leq \psi_n(x; Q) \leq Q.
\]

Set
\[
\mathfrak{S}_n(Q) = \{ \psi_n(x; Q) | \psi_n(x; Q) \text{ solves } (111) \}.
\]

Define
\[
\mathcal{M}_n(Q) = \inf_{\psi_n \in \mathfrak{S}_n(Q)} \sup_S (|\nabla \psi_n(x; Q)|^2 - t_c(B(\psi_n; Q)))
\]
and
\[
\mathfrak{T}_n = \{ s | Q_s \leq s \leq Q^*, \mathcal{M}_n(Q) \leq -\epsilon_n \text{ if } Q \in (Q_s, s) \}.
\]

It follows from Proposition 7.1 that \([Q^*, \hat{Q}] \subset \mathfrak{T}_n\) for sufficiently large \(n\). Hence \(\mathfrak{T}_n\) is not an empty set. Define \(Q_n = \sup \mathfrak{T}_n\).

The sequence \(\{Q_n\}\) has some nice properties.
First, \( \mathcal{M}_n(Q) \) is left continuous for \( Q \in (Q_*, Q_n] \). Let \( \{Q_{n,k}\} \subset (Q_*, Q_n) \) and \( Q_{n,k} \uparrow Q_n \) as \( k \to \infty \). Since \( \mathcal{M}_n(Q_{n,k}) \leq -\epsilon_n \), one has
\[
\|\psi_n(x; Q_{n,k})\|_{C^{0,1}([\mathcal{C}_{n,k}])} \leq C \quad \text{and} \quad \|\psi_n(x; Q_{n,k})\|_{C_2,\infty(\overline{\mathcal{C}_{n,k}} \setminus B_0(A))} \leq C(\delta) \quad \text{for any} \quad \delta > 0,
\]
where \( \mathcal{O}_{n,k} \) is the associated domain bounded by \( S_0, S_1, \) and \( \Gamma, \) and \( B_0(A) \) is the disk centered at \( A = (0, 1) \) with radius \( \delta \). Therefore, there exists a subsequence \( \psi_n(x; Q_{n,k}) \) such that \( \psi_n(x; Q_{n,k}) \to \psi \) and \( \mathcal{O}_{n,k_i} \to \mathcal{O} \) as \( l \to \infty \). Furthermore, \( \psi \) solves (111). Thus \( \mathcal{M}_n(Q) \leq \lim_{l \to \infty} \mathcal{M}_n(Q_{n,k_i}) \leq -\epsilon_n \). It follows from Proposition 8.1 that \( \psi \) is unique. Hence \( \mathcal{M}_n(Q) = \lim_{l \to \infty} \mathcal{M}_n(Q_{n,k_i}) \).

Second, \( Q_n < Q^* \). Suppose not, by the definition of \( Q_n, Q^* \in \mathcal{F}_n \). It follows from the left continuity of \( \mathcal{M}_n, \mathcal{M}_n(Q^*) \leq -\epsilon_n \). Thus \( \psi_n(x; Q) \) satisfies far field behavior. However, it follows from the definition of \( Q^* \) that
\[
\sup_{x \in \overline{\mathcal{O}}} \left( |\nabla \psi_n(x; Q^*)|^2 - t_c(B(\psi_n(x; Q^*))) \right) \geq \sup_{x \in [0, H]} \{ |\rho(Q^*)u(x_2; Q^*)|^2 - t_c(B(x_2)) \} = 0.
\]
Thus \( \mathcal{M}_n(Q^*) \geq 0 \). This leads to a contradiction, which implies \( Q_n < Q^* \).

Finally, it follows from the definition of \( \{Q_n\} \) that \( \{Q_n\} \) is an increasing sequence.

Define \( Q_c := \lim_{n \to \infty} Q_n \). Based on previous properties of \( \{Q_n\}, Q_c \) is well-defined and \( Q_c \leq Q^* \). Note that for any \( Q \in (Q_*, Q_c) \), there exists a \( Q_n > Q \) and thus \( \mathcal{M}_n(Q) \leq -\epsilon_n \). Therefore \( \psi = \psi_n(x; Q) \) solves (111) and
\[
\sup_{\overline{\mathcal{O}}} (|\nabla \psi|^2 - t_c(B(\psi))) = \mathcal{M}_n(Q) \leq -\epsilon_n.
\]
Furthermore, \( \partial_{x_1} \psi_n > 0 \) and the induced flow \((\rho, u)\) satisfies all other properties in part (i) of Theorem 1.

If \( \sup_{Q \in (Q_*, Q_c)} \mathcal{M}(Q) < 0 \), then there exists an \( n \) such that \( \sup_{Q \in (Q_*, Q_n]} \mathcal{M}(Q) < -\epsilon_n \). As the same as the proof for the left continuity of \( \mathcal{M}_n(Q) \) on \( (Q_*, Q_n]\), \( \mathcal{M}_n(Q_c) \leq -\epsilon_n \). Suppose that there exists \( \sigma > 0 \) such that Problem 2 always has a solution \( \psi \) for \( Q \in (Q_c, Q_c + \sigma) \), and
\[
\sup_{Q \in (Q_c, Q_c + \sigma)} \mathcal{M}(Q) = \sup_{Q \in (Q_c, Q_c + \sigma)} \sup_{\overline{\mathcal{O}}} (|\nabla \psi|^2 - t_c(B(\psi))) < 0.
\]
Then there exists \( k > 0 \) such that
\[
\sup_{Q \in (Q_c, Q_c + \sigma)} \mathcal{M}(Q) = \sup_{Q \in (Q_c, Q_c + \sigma)} \sup_{\overline{\mathcal{O}}} (|\nabla \psi|^2 - t_c(B(\psi))) \leq -\epsilon_{n+k}.
\]
This yields that $Q_{n+k} \geq Q_c + \sigma$ and leads to a contradiction. Therefore, either $\mathfrak{M}(Q) \to 0$, or there does not exist $\sigma > 0$ such that Problem 2 has solution for all $Q \in (Q_c, Q_c + \sigma)$ and the associated solutions satisfy (110).

This completes the proof of the proposition. □

Combining all the results in previous sections, Theorem 1 is proved.

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