Abstract. We show that a complete flat pseudo-Riemannian homogeneous manifold with non-abelian linear holonomy is of dimension $\geq 14$. Due to an example constructed in a previous article [2], this is a sharp bound. Also, we give a structure theory for the fundamental groups of complete flat pseudo-Riemannian manifolds in dimensions $\leq 6$. Finally, we observe that every finitely generated torsion-free 2-step nilpotent group can be realized as the fundamental group of a complete flat pseudo-Riemannian manifold with abelian linear holonomy.

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1. Introduction

The study of pseudo-Riemannian homogeneous space forms was pioneered by Joseph A. Wolf in the 1960s. In the flat case, he proved that the fundamental group $\Gamma$ of such a space $M$ is 2-step nilpotent. For $M$ with abelian linear holonomy group he derived a representation by unipotent affine transformations [9]. The linear holonomy group $\text{Hol}(\Gamma)$ of $M$ is the group consisting of the linear parts of $\Gamma$. Wolf further proved that for $\dim M \leq 4$ and Lorentz signatures, $\Gamma$ is a group of pure translations, and that $\Gamma$ is free abelian for signatures $(n - 2, 2)$. It was unclear whether or not non-abelian $\Gamma$ could exist for other signatures, until Oliver Baues gave a first example in [1] of a compact flat pseudo-Riemannian homogeneous
space with signature \((3, 3)\) having non-abelian fundamental group and abelian linear holonomy group.

In an article \([2]\) by Oliver Baues and the author, Wolf’s unipotent representations for fundamental groups with abelian \(\text{Hol}(\Gamma)\) were generalized for groups with non-abelian linear holonomy. Also, it was shown that a (possibly incomplete) flat pseudo-Riemannian homogeneous manifold \(M\) with non-abelian linear holonomy group is of dimension \(\dim M \geq 8\). In chapter 2 we review the main results about the algebraic structure of the fundamental and holonomy groups of such \(M\).

It was asserted in \([2]\) that if \(M\) is (geodesically) complete, then \(\dim M \geq 14\) holds. This assertion is proved in chapter 3 of the present article. More precisely, we prove the following:

**Theorem.** If \(M\) is a complete flat homogeneous pseudo-Riemannian manifold such that its fundamental group \(\Gamma\) has non-abelian linear holonomy group, then

\[
\dim M \geq 14
\]

and the signature \((n - s, s)\) of \(M\) satisfies \(n - s \geq s \geq 7\).

This estimate is sharp by an example given in \([2]\), which is repeated in Example 3.12 for the reader’s convenience.

In chapter 4 we give a complete description of the fundamental groups of flat pseudo-Riemannian homogeneous spaces up to dimension 6. Although non-abelian fundamental groups may occur in dimension 6, their holonomy groups are abelian as a consequence of the dimension bound in the above theorem.

Further, we will see in chapter 5 how any finitely generated torsion-free 2-step nilpotent group can be realized as the fundamental group of a complete flat pseudo-Riemannian homogeneous manifold with abelian holonomy:

**Theorem.** Let \(\Gamma\) be a finitely generated torsion-free 2-step nilpotent group of rank \(n\). Then there exists a faithful representation \(\varphi : \Gamma \to \text{Iso}(\mathbb{R}^{2n}_n)\) such that \(M = \mathbb{R}^{2n}_n/\varphi(\Gamma)\) a complete flat pseudo-Riemannian homogeneous manifold \(M\) of signature \((n, n)\) with abelian linear holonomy group.

2. Preliminaries

Let \(\mathbb{R}^n\) be the space \(\mathbb{R}^n\) endowed with a non-degenerate symmetric bilinear form of signature \((n - s, s)\) and \(\text{Iso}(\mathbb{R}^n_s)\) its group of isometries. We assume \(n - s \geq s\) throughout. The number \(s\) is called the Witt index. For a vector space \(V\) endowed with a non-degenerate symmetric bilinear form let \(\text{wi}(V)\) denote its Witt index. Affine maps of \(\mathbb{R}^n\) are written as \(\gamma = (I + A, v)\), where \(I + A\) is the linear part \((I\) the identity matrix), and \(v\) the translation part. Let \(\text{im} A\) denote the image of \(A\).

Let \(M\) denote a complete flat pseudo-Riemannian homogeneous manifold. Then \(M\) is of the form \(M = \mathbb{R}^{2n}_n/\Gamma\) with fundamental group \(\Gamma \subset \text{Iso}(\mathbb{R}^{2n}_n)\). Homogeneity is determined by the action of the centralizer \(Z_{\text{Iso}(\mathbb{R}^{2n}_n)}(\Gamma)\) of \(\Gamma\) in \(\text{Iso}(\mathbb{R}^{2n}_n)\) (see \([8, \text{Theorem 2.4.17}]\)):

**Theorem 2.1.** Let \(\tilde{M} \to M\) be the universal pseudo-Riemannian covering of \(M\) and let \(\Gamma\) be the group of deck transformations. Then \(M\) is homogeneous if and only if \(Z_{\text{Iso}(\mathbb{R}^{2n}_n)}(\Gamma)\) acts transitively on \(\tilde{M}\).

This condition further implies that \(\Gamma\) acts without fixed points on \(\mathbb{R}^{2n}_s\). This constraint on \(\Gamma\) is the main difference to the more general case where \(M\) is not required to be geodesically complete.
Now assume \( \Gamma \subset \text{Iso}(\mathbb{R}^n) \) has transitive centralizer in \( \text{Iso}(\mathbb{R}^n) \). We sum up some properties of \( \Gamma \) for later reference (these are originally due to [9], see also [5, 3] for reference).

**Lemma 2.2.** \( \Gamma \) consists of affine transformations \( \gamma = (I + A, v) \), where \( A^2 = 0 \), \( v \perp \text{im} A \) and \( \text{im} A \) is totally isotropic.

**Lemma 2.3.** For \( \gamma_i = (I + A_i, v_i) \in \Gamma \), \( i = 1, 2, 3 \), we have \( A_1 A_2 v_1 = 0 = A_2 A_1 v_2 \), \( A_1 A_2 A_3 = 0 \) and \( [\gamma_1, \gamma_2] = (I + 2A_1 A_2, 2A_1 v_2) \).

**Lemma 2.4.** If \( \gamma = (I + A, v) \in \Gamma \), then \( \langle Ax, y \rangle = -\langle x, Ay \rangle \), \( \text{im} A = (\text{ker} A)^\perp \), \( \text{ker} A = (\text{im} A)^\perp \) and \( Av = 0 \).

**Theorem 2.5.** \( \Gamma \) is 2-step nilpotent (meaning \( [\Gamma, [\Gamma, \Gamma]] = \{ \text{id} \} \)).

For \( \gamma = (I + A, v) \in \Gamma \), set \( \text{Hol}(\gamma) = I + A \) (the linear component of \( \gamma \)). We write \( A = \log(\text{Hol}(\gamma)) \).

**Definition 2.6.** The linear holonomy group of \( \Gamma \) is \( \text{Hol}(\Gamma) = \{ \text{Hol}(\gamma) \mid \gamma \in \Gamma \} \).

Let \( x \in M \) and \( \gamma \in \pi_1(M, x) \) be a loop. Then \( \text{Hol}(\gamma) \) corresponds to the parallel transport \( \tau_x(\gamma) : T_x M \to T_x M \) in a natural way, see [8, Lemma 3.4.4]. This justifies the naming.

**Proposition 2.7.** The following are equivalent:

1. \( \text{Hol}(\Gamma) \) is abelian.
2. If \( (I + A_1, v_1), (I + A_2, v_2) \in \Gamma \), then \( A_1 A_2 = 0 \).
3. The space \( U_\Gamma = \sum_{\gamma \in \Gamma} \text{im} A \) is totally isotropic.

Those \( \Gamma \) with possibly non-abelian \( \text{Hol}(\Gamma) \) were studied in [2]: If \( \text{Hol}(\Gamma) \) is not abelian, the space \( U_\Gamma \) is not totally isotropic. So we replace \( U_\Gamma \) by the totally isotropic subspace

\[
U_0 = U_\Gamma \cap U_\Gamma^\perp = \sum_{\gamma \in \Gamma} \text{im} A \cap \bigcap_{\gamma \in \Gamma} \text{ker} A.
\]

We can find a Witt basis for \( \mathbb{R}^n \) with respect to \( U_0 \), that is a basis with the following properties: If \( k = \dim U_0 \), there exists a basis for \( \mathbb{R}^n \),

\[
\{ u_1, \ldots, u_k, \ u_1, \ldots, w_{n-2k}, \ u_1^*, \ldots, u_k^* \},
\]

such that \( \{ u_1, \ldots, u_k \} \) is a basis of \( U_0 \), \( \{ w_1, \ldots, w_{n-2k} \} \) is a basis of a non-degenerate subspace \( W \) such that \( U_0^* = U_0 \oplus W \), and \( \{ u_1^*, \ldots, u_k^* \} \) is a basis of a space \( U_0^* \) such that \( \langle u_i, u_j^* \rangle = \delta_{ij} \) (then \( U_0^* \) is called a dual space for \( U_0 \)). Then

\[
\mathbb{R}_s^n = U_0 \oplus W \oplus U_0^*
\]

is called a Witt decomposition of \( \mathbb{R}^n \). Let \( \tilde{I} \) denote the signature matrix representing the restriction of \( \langle , \rangle \) to \( W \) with respect to the chosen basis of \( W \).

In [2, Theorem 4.4] we derived the following representation for \( \Gamma \):

**Theorem 2.8.** Let \( \gamma = (I + A, v) \in \Gamma \) and fix a Witt basis with respect to \( U_0 \). Then the matrix representation of \( A \) in this basis is

\[
A = \begin{pmatrix}
0 & -B^\top \tilde{I} & C \\
0 & 0 & B \\
0 & 0 & 0
\end{pmatrix},
\]

with \( B \in \mathbb{R}^{(n-2k) \times k} \) and \( C \in s_0^k \) (where \( k = \dim U_0 \)). The columns of \( B \) are isotropic and mutually orthogonal with respect to \( \tilde{I} \).
3. The Dimension Bound for Complete Manifolds

In this section we derive further properties of the matrix representation in (2.4).

3.1. Properties of the Matrix Representation. We fix a Witt basis for $U_0$ as in the previous section. Let $\gamma_i \in \Gamma$ with $\gamma_i = (I + A_i, v_i)$, $i = 1, 2$. Then $B_i$ and $C_i$ refer to the respective matrix blocks of $A_i$ in (2.4). Set $[\gamma_1, \gamma_2] = \gamma_3 = (I + A_3, v_3)$.

**Lemma 3.1.** We have $\langle v_3, 2A_1v_2 - 2A_2v_1 \rangle \in U_0$. Further, if $\gamma_3 \neq I$ and $\Gamma$ acts freely, then $v_3 \neq 0$.

**Proof.** By Lemma (2.3), $v_3 = 2A_1v_2 - 2A_2v_1 \in \text{im } A_1$. Because $\Gamma$ is 2-step nilpotent, $\gamma_3$ is central. Again by Lemma (2.3), $v_3 \in \bigcap_{i \in I} \ker A_i$. Hence $v_3 \in U_0$.

If $\Gamma$ acts freely and $\langle v_3 \rangle \neq I$, then $v_3 \neq 0$ because otherwise 0 would be a fixed point for $\gamma_3$. $\square$

**Lemma 3.2.** If $u_1^*, u_2^*$ denote the respective $U_0^*$-components of the translation parts $v_1, v_2$, then $u_1^*, u_2^* \in \ker B_1 \cap \ker B_2$.

**Proof.** Let $v_3 = u_3 + u_3^* + u_3^*$ be the Witt decomposition of $v_3$. By Lemma (3.1), $u_3 = 0$, $u_3^* = 0$. Writing out the equation $v_3 = A_1v_2 - A_2v_1$ with (2.4) it follows that $B_1u_2^* = 0 = B_2u_1^*$. By Lemma (2.4), $B_1u_1^* = 0 = B_2u_2^*$. $\square$

The following rules were already used in [2, Theorem 5.1] to derive the general dimension bound for (possibly incomplete) flat pseudo-Riemannian homogeneous manifolds:

1. **Isotropy rule:** The columns of $B_i$ are isotropic and mutually orthogonal with respect to $\tilde{I}$ (Theorem 2.5).
2. **Crossover rule:** Given $A_1$ and $A_2$, let $b_i^j$ be a column of $B_i$ and $b_i^k$ a column of $B_i$. Then $\langle b_i^j, b_i^k \rangle = -\langle b_i^k, b_i^j \rangle$. In particular, $\langle b_1^i, b_2^i \rangle = 0$, and $\langle b_1^i, b_1^i \rangle \neq 0$. If $\langle b_1^i, b_2^i \rangle \neq 0$ then $b_1^i, b_1^j, b_2^i, b_2^j$ are linearly independent. (The product of $A_1A_2$ contains $-B_1^1IB_2$ as the skew-symmetric upper right block, so its entries are the values $-\langle b_1^i, b_2^j \rangle$.)
3. **Duality rule:** Assume $A_1$ is not central (that is $A_1A_2 \neq 0$ for some $A_2$).

Then $B_2$ contains a column $b_2^j$ and $B_1$ a column $b_1^j$ such that $\langle b_1^j, b_2^j \rangle \neq 0$.

**Lemma 3.3.** Assume $A_1A_2 \neq 0$ and that the columns $b_1^j$ in $B_1$ and $b_2^j$ in $B_2$ satisfy $\langle b_1^j, b_2^j \rangle \neq 0$. The subspace $W$ in (2.3) has a Witt decomposition

$$(3.1) \quad W = W_{ij} \oplus W' \oplus W_{ij}^*,$$

where $W_{ij} = \mathbb{R}b_1^j \oplus \mathbb{R}b_2^j$, $W_{ij}^* = \mathbb{R}b_2^j \oplus \mathbb{R}b_1^j$, $W' \perp W_{ij}$, $W' \perp W_{ij}^*$, and $\langle \cdot, \cdot \rangle$ is non-degenerate on $W'$. Furthermore,

$$(3.2) \quad \text{wi}(W) \geq \text{rk } B_1 \geq 2 \quad \text{and} \quad \dim W \geq \text{rk } B_1 \geq 4.$$

**Proof.** $\mathbb{R}b_1^j \oplus \mathbb{R}b_2^j$ is totally isotropic because $\text{im } B_1$ is. By the crossover rule, $\{b_2^j, b_2^j\}$ is a dual basis to $\{b_1^j, b_1^j\}$ (after scaling, if necessary).

$W$ contains $\text{im } B_1$ as a totally isotropic subspace, so it also contains a dual space. Hence $\text{wi}(W) \geq \text{rk } B_1 \geq \dim W_{ij} \geq 2$ and $\dim W \geq \text{rk } B_1 \geq 2 \dim W_{ij} = 4$. $\square$
3.2. Criteria for Fixed Points. In this subsection, assume the centralizer of \( \Gamma \subset \text{Iso}(\mathbb{R}^n) \) has an open orbit in \( \mathbb{R}^n \), but does not necessarily act transitively.

Remark 3.4. If the centralizer does act transitively on \( \mathbb{R}^n \), then \( \Gamma \) acts freely: Assume \( \gamma, p = p \) for some \( \gamma \in \Gamma, p \in \mathbb{R}^n \). For every \( q \in \mathbb{R}^n \) there is \( z \in \text{Z}_{\text{iso}(\mathbb{R}^n)}(\Gamma) \) such that \( z.p = q \). So \( \gamma, q = \gamma(z.p) = z.(\gamma, p) = z.p = q \) for all \( q \in \mathbb{R}^n \). Hence \( \gamma = I \).

We will deduce some criteria for \( \Gamma \) to have a fixed point, which allows us to exclude such groups \( \Gamma \) as fundamental groups for complete flat pseudo-Riemannian homogeneous manifolds.

Let \( \Gamma, U_0, \gamma_1, \gamma_2, \gamma_3 = [\gamma_1, \gamma_2], A_1, B_1, C_1 \) be as in the previous sections. For any \( v \in \mathbb{R}^n \) let \( v = u + w + u^* \) denote the Witt decomposition with respect to \( U_0 \). From (2.4) we get the following two coordinate expressions which we use repeatedly:

\[
(3.3) \quad A_1 A_2 = \begin{pmatrix} 0 & -B_1^T \tilde{I} & C_1 \\ 0 & 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 & -B_2^T \tilde{I} & C_2 \\ 0 & 0 & B_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -B_1^T \tilde{I} B_2 \\ 0 & 0 & 0 \end{pmatrix}
\]

(note that \( A_3 = 2A_1 A_2 \) as a consequence of Lemma 2.3), and for \( v \in \mathbb{R}_s \)

\[
(3.4) \quad A_1 v = \begin{pmatrix} 0 & -B_1^T \tilde{I} & C_1 \\ 0 & 0 & B_1 \end{pmatrix} \begin{pmatrix} u \\ w \\ u^* \end{pmatrix} = \begin{pmatrix} -B_1^T \tilde{I} w + C_1 u^* \\ B_1 u^* \end{pmatrix}.
\]

In the following we assume that the linear parts of \( \gamma_1, \gamma_2 \) do not commute, that is \( A_1 A_2 \neq 0 \). In particular, \( A_1 v_2 \neq 0 \).

Lemma 3.5. If \( u_3 \in \text{im} B_1^T \tilde{I} B_2 \), then \( \gamma_3 \) has a fixed point.

Proof. We have \( C_3 = -B_1^T \tilde{I} B_2 \) by Lemma 2.3 and (2.4). By Lemma 3.1 \( v_3 = u_3 \in U_0 \). If there exists \( u^* \in U_0^* \) such that \( C_3 u^* = u_3 \), then \( \gamma_3(-u^*) = (I + A_3, v_3)(-u^*) = -u^* - C_3 u^* + u_3 = -u^* \). So \( -u^* \) is fixed by \( \gamma_3 \).

Lemma 3.6. If \( \text{rk} B_1^T \tilde{I} B_2 = \text{rk} B_1 \) and the \( \Gamma \)-action is free, then \( u_1^* \neq 0, u_2^* \neq 0 \).

Proof. From (3.3) we get \( u_3 = -B_1^T \tilde{I} w_2 + C_1 u_2^* \). Also, \( \text{im} B_1^T \tilde{I} B_2 \subset \text{im} B_1^T \). But by our rank assumption, \( \text{im} B_1^T \tilde{I} B_2 = \text{im} B_1^T \).

So, if \( u_2^* = 0 \), then \( u_3 \in \text{im} B_1^T = \text{im} B_1^T \tilde{I} B_2 \), which implies the existence of a fixed point by Lemma 3.5. So \( u_2^* \neq 0 \) if the action is free. Using \( v_3 = A_1 v_2 = -A_2 v_1 \), we can conclude \( u_1^* \neq 0 \) in a similar manner.

Corollary 3.7. If \( \dim U_0 = 2 \), then the \( \Gamma \)-action is not free.

Proof. By Lemma 3.5 \( 2 \leq \text{rk} B_1 \leq \dim U_0 = 2 \), so \( B_1 \) is of full rank. Now \( A_1 v_1 = 0 \) and (3.4) imply \( u_1^* = 0 \), so by Lemma 3.0 the \( \Gamma \)-action is not free.

Lemma 3.8. If \( \dim U_0 = 3 \) and \( \dim(\text{im} B_1 + \text{im} B_2) \leq 5 \), then \( \gamma_3 \) has a fixed point.

Proof. By Lemma 3.5 \( \text{rk} B_1, \text{rk} B_2 \geq 2 \). We distinguish two cases:

(i) Assume \( \text{rk} B_1 = 2 \) (or \( \text{rk} B_2 = 2 \)). Because \( C_3 = -B_1^T \tilde{I} B_2 \neq 0 \) is skew, it is also of rank 2. Then \( \text{im} B_1^T \tilde{I} B_2 = \text{im} B_1^T \). \( \ker B_1 \) is a 1-dimensional subspace due to \( \dim U_0^* = 3 \). Because \( u_1^*, u_2^* \in \ker B_1 \), we have \( u_1^* = \lambda u_2^* \) for some number \( \lambda \neq 0 \).
From (3.4) and $A_1 v_1 = 0$ we get
\[
\lambda u_3 = -B_1^T \tilde{I} \lambda w_2 + C_1 \lambda u_2 = -B_1^T \tilde{I} \lambda w_2 + C_1 u_1^*,
\]
\[
0 = -B_1^T \tilde{I} w_1 + C_1 u_1^*.
\]
So $\lambda u_3 = \lambda u_3 - 0 = B_1^T \tilde{I} (w_1 - \lambda w_2)$. In other words, $u_3 \in \text{im} B_1^T = \text{im} B_1^T \tilde{I} B_2$, and $\gamma_3$ has a fixed point by Lemma 3.5.

(ii) Assume $\text{rk} B_1 = \text{rk} B_2 = 3$. As $[A_1, A_2] \neq 0$, the duality rule and the crossover rule imply the existence of a pair of columns $b_1^i, b_1^j$ in $B_1$ and a pair of columns $b_2^i, b_2^j$ in $B_2$ such that $\alpha = \langle b_1^i, b_2^j \rangle = -\langle b_1^i, b_2^j \rangle \neq 0$. For simplicity say $i = 1$, $j = 2$. As $\text{rk} B_1 = 3$, the column $b_1^i$ is linearly independent of $b_1^j, b_2^i$, and these columns span the totally isotropic subspace $\text{im} B_1$ of $W$.

- Assume $b_1^3 \in \text{im} B_1$ (or $b_1^3 \in \text{im} B_2$). Then $b_1^3$ is a multiple of $b_1^j$. In fact, let $b_1^3 = \lambda_1 b_1^i + \lambda_2 b_1^j + \lambda_3 b_1^k$. Then $\langle b_1^3, b_1^i \rangle = 0$ because $\text{im} B_1$ is totally isotropic. Since $\text{im} B_2$ is totally isotropic and by the crossover rule,
  \[
  0 = \langle b_1^3, b_1^j \rangle = \lambda_1 \langle b_1^i, b_1^j \rangle + \lambda_2 \langle b_1^j, b_1^j \rangle + \lambda_3 \langle b_1^k, b_1^j \rangle = \lambda_2 \alpha - \lambda_3 (b_2^3, b_1^j).
  \]
  Because $\alpha \neq 0$, this implies $\lambda_2 = 0$ and in the same way $\lambda_1 = 0$. So $b_1^3 = \lambda_3 b_1^j$. Now $b_1^3 \perp b_1^j, b_2^j$ for all $i, j$. We have $u_3^* = 0$ because $B_2 u_2^* = 0$ and $B_2$ is of maximal rank. Then $\langle b_1^i, w_2 \rangle = \langle b_2^i, w_2 \rangle = 0$, because $0 = B_1^T \tilde{I} w_2 + C_2 u_2^* = B_2^T \tilde{I} w_2$. Hence $C_3$ and $u_3$ take the form

\[
C_3 = -B_1^T \tilde{I} B_2 = \begin{pmatrix}
0 & -2 & 0 \\
\alpha & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

$u_3 = -B_1^T \tilde{I} w_2 = \begin{pmatrix}
-\langle b_1^1, w_2 \rangle \\
-\langle b_2^1, w_2 \rangle \\
0
\end{pmatrix}.$

It follows that $u_3 \in \text{im} C_3$, so in this case $\gamma_3$ has a fixed point by Lemma 3.5.

- Assume $b_1^3 \notin \text{im} B_1$ and $b_1^j \notin \text{im} B_2$. This means $b_1^3$ and $b_1^j$ are linearly independent. If $b_1^3 \perp \text{im} B_1$, then $b_1^j \perp \text{im} B_2$ by the crossover rule. With respect to the Witt decomposition $W = W_{12} \oplus W' \oplus W_{12}$ (Lemma 3.3), this means $b_1^j, b_2^j$ span a 2-dimensional subspace of $(W_{12} \oplus W_{12})^\perp = W'$. But then $\dim(\text{im} B_1 + \text{im} B_2) = 6$, contradicting the lemma's assumption that this dimension should be $\leq 5$.

So $b_1^3 \not\perp \text{im} B_1$ and $b_1^j \not\perp \text{im} B_2$ hold. Because further $b_1^j \perp \text{im} B_1$, $b_2^3 \perp \text{im} B_2$ and $\dim(\text{im} B_1 + \text{im} B_2) \leq 5$, there exists a $b \in W'$ (with $W'$ from the Witt decomposition above) such that

\[
b_1^j = \lambda_1 b_1^j + \lambda_2 b_1^j + \lambda_3 b,
\]
\[
b_2^3 = \mu_1 b_1^3 + \mu_2 b_2^3 + \mu_3 b.
\]

Because $B_1, B_2$ are of maximal rank, we have $u_1^* = 0 = u_2^*$ as a consequence of $A_i v_i = 0$. Then

\[
0 = B_1^T \tilde{I} w_2 = \begin{pmatrix}
\langle b_1^1, w_2 \rangle \\
\langle b_2^1, w_2 \rangle \\
\langle b_3^2, w_2 \rangle
\end{pmatrix}.
\]
and this implies $\langle b, w_2 \rangle = 0$. Put $\xi = \langle b_1^1, w_2 \rangle$, $\eta = \langle b_1^2, w_2 \rangle$. Then

$$u_3 = -B_1^\top \tilde{I} w_2 = - \begin{pmatrix} \xi \\ \eta \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix}$$

and (recall $\alpha = \langle b_1^1, b_1^2 \rangle = -\langle b_1^2, b_1^1 \rangle$)

$$C_3 = B_2^\top \tilde{I} B_1 = \begin{pmatrix} 0 & -\alpha & -\lambda_2 \alpha \\ \alpha & 0 & \lambda_1 \alpha \\ \lambda_2 \alpha & -\lambda_1 \alpha & 0 \end{pmatrix}.$$ 

So

$$C_3 \cdot \frac{1}{\alpha} \begin{pmatrix} -\eta \\ \xi \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix} = - \begin{pmatrix} \xi \\ \eta \\ \lambda_1 \xi + \lambda_2 \eta \end{pmatrix} = u_3.$$ 

By Lemma $3.5$, $\gamma_3$ has a fixed point.

So in any case $\gamma_3$ has a fixed point. \qed

**Lemma 3.9.** If $\dim U_0 = 4$ and $\text{rk } B_1^\top \tilde{I} B_2 = \text{rk } B_1 = \text{rk } B_2$, then $\gamma_3$ has a fixed point.

**Proof.** By assumption

$$\text{im } B_1^\top \tilde{I} B_2 = \text{im } B_1^\top = \text{im } B_2^\top.$$ 

First, assume $u_1^* = \lambda u_2^*$ for some number $\lambda \neq 0$. Writing out $A_1 v_2 = v_3$ and $A_1 v_1 = 0$ with (3.3), we get

$$\lambda u_3 = -B_1^\top \tilde{I} \lambda w_2 + C_1 \lambda u_2^* = -B_1^\top \tilde{I} \lambda w_2 + C_1 u_1^*,$$

$$0 = -B_1^\top \tilde{I} w_1 + C_1 u_1^*.$$ 

So

$$\lambda u_3 = \lambda u_3 - 0 = B_1^\top \tilde{I} (w_1 - \lambda w_2).$$

In other words, $u_3 \in \text{im } B_1^\top = \text{im } B_1^\top \tilde{I} B_2$, and $\gamma_3$ has a fixed point by Lemma $3.5$.

Now, assume $u_1^*$ and $u_2^*$ are linearly independent. Lemma $3.2$ can be reformulated as

$$\text{im } B_1^\top = \text{im } B_2^\top \subseteq \ker u_1^* \cap \ker u_2^*.$$ 

ker $u_1^*$, ker $u_2^*$ are 3-dimensional subspaces of the 4-dimensional space $U_0^*$, and their intersection is of dimension 2 (because $u_1^*, u_2^*$ are linearly independent). By Lemma $3.3$ $\text{rk } B_1 \geq 2$, so it follows that

$$\text{im } B_1^\top = \text{im } B_2^\top = \ker u_1^* \cap \ker u_2^*.$$ 

With $A_1 v_1 = 0$ and (3.4) we conclude $C_1 u_1^* = b$ for some $b \in \text{im } B_1^\top$. Thus, by the skew-symmetry of $C_1$,

$$0 = (u_2^T C_1 u_1^*)^T = -u_1^T C_1 u_2^*.$$ 

So $C_1 u_2^* \in \ker u_1^*$. In the same way $C_2 u_1^* \in \ker u_2^*$. But $u_3 = C_1 u_2^* + b_1 = -C_2 u_1^* + b_2$ for some $b_1, b_2 \in \text{im } B_1^\top$. Hence

$$u_1^T u_3 = u_1^T C_1 u_2^* + u_1^T b_1 = 0,$$

$$u_2^T u_3 = -u_2^T C_2 u_1^* + u_2^T b_2 = 0.$$
So \( u_3 \in \ker u_1^T \cap \ker u_2^T = \text{im} B_1^T = \text{im} B_1^T \bar{I} B_2 \). With Lemma 3.3 we conclude that there exists a fixed point for \( \gamma_3 \).

### 3.3. The Dimension Bound

Let \( \Gamma, \gamma_1, \gamma_2, \gamma_3 = [\gamma_1, \gamma_2] \) be as in the previous subsection, let \( \mathbb{R}^n = U_0 \oplus W \oplus U'_n \) be the Witt decomposition (2.3), and let \( A_i, B_i, C_i \) refer to the matrix representation (2.4) of \( \gamma_i \). We will assume that the linear parts \( A_1, A_2 \) of \( \gamma_1, \gamma_2 \) do not commute, that is, \( \text{Hol}(\Gamma) \) is not abelian.

**Theorem 3.10.** Let \( \Gamma \subset \text{Iso}(\mathbb{R}^n) \) and assume the centralizer \( Z_{\text{Iso}(\mathbb{R}^n)}(\Gamma) \) acts transitively on \( \mathbb{R}^n \). If \( \text{Hol}(\Gamma) \) is non-abelian, then

\[
\begin{align*}
&\quad s \geq 7 \quad \text{and} \quad n \geq 14. \\
\end{align*}
\]

As Example 3.12 shows, this is a sharp lower bound.

**Proof.** We will show \( s \geq 7 \), then it follows immediately from \( n - s \geq s \) that

\[
\begin{align*}
&\quad n \geq 2s \geq 14. \\
\end{align*}
\]

If the centralizer is transitive, then \( \Gamma \) acts freely. From Corollary 3.7, we know that \( \dim U_0 \geq 3 \). By Lemma 3.3, \( \text{wi}(W) \geq 2 \), and if \( \dim U_0 \geq 5 \), then

\[
\begin{align*}
&\quad s = \dim U_0 + \text{wi}(W) \geq 5 + 2 = 7, \\
\end{align*}
\]

and we are done. So let \( 2 < \dim U_0 < 5 \).

(i) First, let \( \dim U_0 = 4 \). Assume \( \text{rk} B_1 = \text{rk} B_2 = 2 \). Because \( C_3 = -B_1^T \bar{I} B_2 \neq 0 \) is skew, it is of rank 2. So \( \text{rk} B_1 = \text{rk} B_2 = 2 = \text{rk} B_1^T \bar{I} B_2 \). By Lemma 3.3, the action of \( \Gamma \) is not free.

Now assume \( \text{rk} B_1 \geq 3 \). It follows from Lemma 3.3, that \( \text{wi}(W) \geq 3 \) and \( \dim W \geq 6 \), so once more

\[
\begin{align*}
&\quad s = \dim U_0 + \text{wi}(W) \geq 4 + 3 = 7. \\
\end{align*}
\]

So the theorem holds for \( \dim U_0 = 4 \).

(ii) Let \( \dim U_0 = 3 \). If \( \dim(\text{im} B_1 + \text{im} B_2) \leq 5 \), there exists a fixed point by Lemma 3.8 so \( \Gamma \) would not act freely. So let \( \dim(\text{im} B_1 + \text{im} B_2) = 6 \). As \([A_1, A_2] \neq 0 \), the crossover rule implies the existence of a pair of columns \( b_1^i, b_2^i \) in \( B_1 \) and a pair of columns \( b_2^i, b_2^j \) in \( B_2 \) such that \( \alpha = \langle b_1^i, b_2^j \rangle = -\langle b_2^j, b_1^i \rangle \neq 0 \). For simplicity say \( i = 1, j = 2 \). The columns \( b_1^1, b_2^1, b_2^2 \) span the totally isotropic subspace \( \text{im} B_1 \) of \( W \), and \( b_1^2, b_2^1, b_2^2 \) span \( \text{im} B_2 \). We have a Witt decomposition with respect to \( W_1 = \mathbb{R} b_1^1 \oplus \mathbb{R} b_2^1 \) (Lemma 3.3),

\[
W = W_1 \oplus W' \oplus W_{12}^*,
\]

where \( W_{12}^* = \mathbb{R} b_2^2 \oplus \mathbb{R} b_2^3 \). Because \( b_1^3 \perp \text{im} B_1 \) and \( b_2^3 \perp \text{im} B_2 \),

\[
\begin{align*}
&\quad b_1^3 = \lambda_1 b_1^1 + \lambda_2 b_2^1 + b', \\
&\quad b_2^3 = \mu_1 b_2^1 + \mu_2 b_2^2 + b'',
\end{align*}
\]

where \( b', b'' \in W' \) are linearly independent because \( \dim(\text{im} B_1 + \text{im} B_2) = 6 \). From \( 0 = \langle b_1^1, b_1^2 \rangle \) it follows that \( \langle b', b' \rangle = 0 \), and similarly \( \langle b'', b'' \rangle = 0 \). The crossover rule then implies

\[
\begin{align*}
&\lambda_1 \langle b_2^2, b_1^1 \rangle = -\langle b_2^3, b_2^1 \rangle = -\mu_1 \langle b_2^1, b_2^1 \rangle = \mu_1 \langle b_2^3, b_2^1 \rangle, \\
&\lambda_2 \langle b_2^1, b_2^1 \rangle = -\langle b_2^3, b_1^1 \rangle = -\mu_2 \langle b_2^3, b_1^1 \rangle = \mu_2 \langle b_2^1, b_2^1 \rangle.
\end{align*}
\]
As the inner products are $\neq 0$, it follows that $\lambda_1 = \mu_1$, $\lambda_2 = \mu_2$. Then, by the duality rule,
\[
0 = \langle b_1^3, b_2^3 \rangle = (\lambda_1 \mu_2 - \lambda_2 \mu_1) \langle b_2^2, b_1^1 \rangle + \langle b', b'' \rangle = \langle b', b'' \rangle.
\]
So $b'$ and $b''$ span a 2-dimensional totally isotropic subspace in the non-degenerate space $W'$, so this subspace has a 2-dimensional dual in $W'$ and $\dim W' \geq 4$, $\text{wi}(W') \geq 2$, follows. Hence
\[
\text{wi}(W) = \dim W_{12} + \text{wi}(W') \geq 2 + 2 = 4,
\]
and again
\[
s = \dim U_0 + \text{wi}(W) \geq 3 + 4 = 7.
\]
In any case $s \geq 7$ and $n \geq 14$. \qed

**Corollary 3.11.** If $M$ is a complete flat homogeneous pseudo-Riemannian manifold such that its fundamental group $\Gamma$ has non-abelian linear holonomy group $\text{Hol}(\Gamma)$, then
\[
\dim M \geq 14
\]
and the signature $(n - s, s)$ of $M$ satisfies $n - s \geq s \geq 7$.

The dimension bound in Corollary 3.11 is sharp, as the following example from [2] shows:

**Example 3.12.** Let $\Gamma \subset \text{Iso}(\mathbb{R}^{14})$ be the group generated by
\[
\gamma_1 = \begin{pmatrix} I_5 & -B_1^1 \bar{I} & C_1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} I_5 & -B_2^1 \bar{I} & C_2 \end{pmatrix},
\]
in the basis representation (2.4). Here,
\[
B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad u_1^* = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix},
\]
\[
B_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad u_2^* = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
\]
and $\bar{I} = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. Their commutator is
\[
\gamma_3 = [\gamma_1, \gamma_2] = \begin{pmatrix} I_5 & 0 & C_3 \\ 0 & I_4 & 0 \\ 0 & 0 & I_5 \end{pmatrix}, \quad u_3^* = \begin{pmatrix} u_3 \\ 0 \\ 0 \end{pmatrix},
\]
with

\[ C_3 = \begin{pmatrix} 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{pmatrix}. \]

The group \( \Gamma \) is isomorphic to a discrete Heisenberg group, and the linear parts of \( \gamma_1, \gamma_2 \) do not commute. In [2] Example 6.4 it was shown that \( \Gamma \) has transitive centralizer in \( \text{Iso}(\mathbb{R}^{14}_3) \) and acts properly discontinuously and freely on \( \mathbb{R}^{14}_3 \). Hence \( M = \mathbb{R}^{14}/\Gamma \) is a complete flat pseudo-Riemannian homogeneous manifold of dimension 14 with non-abelian linear holonomy.

4. Low Dimensions

In this section, we determine the structure of the fundamental groups of complete flat pseudo-Riemannian homogeneous spaces \( M \) of dimensions \( \leq 6 \) and of those with signature \((n-2,2)\). The signatures \((n,0),(n-1,1)\) and \((n-2,2)\) were already studied by Wolf [8, Corollary 3.7.13]. In particular, he derived the following:

**Proposition 4.1 (Wolf).** If \( M \) is a complete homogeneous flat Riemannian or Lorentzian manifold, then the fundamental group of \( M \) is an abelian group consisting of pure translations.

Let \( \Gamma \subset \text{Iso}(\mathbb{R}^n) \) denote the fundamental group of \( M \) and \( G \subset \text{Iso}(\mathbb{R}^n) \) its real Zariski closure with Lie algebra \( g \). Let \( U_\Gamma \) be as in Proposition 2.7.

We start by collecting some general facts about \( \Gamma \) and \( G \).

**Remark 4.2.** \( G \subset \text{Iso}(\mathbb{R}^n) \) is unipotent, hence simply connected. Then \( \Gamma \) is finitely generated and torsion-free by [7, Theorem 2.10], as it is a discrete subgroup of \( G \). Further, \( \text{rk} \Gamma = \dim G \).

The fundamental theorem for finitely generated abelian groups states:

**Lemma 4.3.** If \( \Gamma \) is abelian and torsion-free, then \( \Gamma \) is free abelian.

By [4] Theorem 5.1.6], there exists a Malcev basis of \( G \) which generates \( \Gamma \). We shall call it a Malcev basis of \( \Gamma \).

**Lemma 4.4.** Let \( \gamma_1, \ldots, \gamma_k \) denote a Malcev basis of \( \Gamma \). If \( M \) is complete, then the translation parts \( v_1, \ldots, v_k \) of the \( \gamma_i = (I + A_i, v_i) \) are linearly independent.

**Proof.** \( \Gamma \) has transitive centralizer in \( \text{Iso}(\mathbb{R}^n) \). By continuity, so does \( G \). Hence \( G \) acts freely on \( \mathbb{R}^n \). Then the orbit map \( G \to \mathbb{R}^n, g \mapsto g.0 \), at the point 0 is a diffeomorphism onto the orbit \( G.0 \). Because \( G \) acts by affine transformations, \( G.0 \) is the span of the translation parts of the \( \gamma_i \). So

\[ k = \text{rk} \Gamma = \dim G = \dim G.0 = \dim \text{span}\{v_1, \ldots, v_k\}. \]

So the \( v_i \) are linearly independent. \( \Box \)

4.1. **Signature \((n-2,2)\).** As always, we assume \( n-2 \geq 2 \).

**Proposition 4.5 (Wolf).** Let \( M = \mathbb{R}^n_2/\Gamma \) be a flat pseudo-Riemannian homogeneous manifold. Then \( \Gamma \) is a free abelian group. In particular, the fundamental group of every flat pseudo-Riemannian homogeneous manifold \( M \) of dimension \( \dim M \leq 5 \) is free abelian.
Proof: It follows from Corollary 3.11 that $\Gamma$ has abelian holonomy. Consequently, if $\gamma = (I + A, v) \in \Gamma$ such that $A \neq 0$, then

$$A = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in a Witt basis with respect to $U_\Gamma$. Here, $C \neq 0$ is a skew-symmetric $2 \times 2$-matrix, so we have $\text{rk} A = 2$. Because $A \subseteq U_\Gamma$ and both these spaces are totally isotropic, we have $\dim \text{im} A = \dim U_\Gamma = 2$. Because $Av = 0$ we get $v \in \ker A = (\text{im} A)^{\perp} = U_\Gamma^{\perp}$. But then $Bv = 0$ for any $(I + B, w) \in \Gamma$, and as also $BA = 0$, it follows that $[(I + B, w), (I + A, v)] = (I + 2BA, 2Bv) = (I, 0)$. Hence $\Gamma$ is abelian. It is free abelian by Lemma 4.4.

In the remainder of this section, the group $\Gamma$ is always abelian, so the space $U_\Gamma = \sum_A \text{im} A$ is totally isotropic (in particular, $U_0 = U_\Gamma$). We fix a Witt decomposition with respect to $U_\Gamma$,

$$R^n_2 = U_\Gamma \oplus W \oplus U_\Gamma^*$$

and any $v \in R^n_2$ decomposes into $v = u + w + u^*$ with $u \in U_\Gamma$, $w \in W$, $u^* \in U_\Gamma^*$.

Remark 4.6. As seen in the proof of Proposition 4.7 if $\dim U_\Gamma = 2$, then $U_\Gamma = \text{im} A$ for any $\gamma = (I + A, v)$ with $A \neq 0$.

We can give a more precise description of the elements of $\Gamma$:

**Proposition 4.7.** Let $M = R^3_2/\Gamma$ be a complete flat pseudo-Riemannian homogeneous manifold. Then:

1. $\Gamma$ is generated by elements $\gamma_i = (I + A_i, v_i)$, $i = 1, \ldots, k$, with linearly independent translation parts $v_1, \ldots, v_k$.
2. If there exists $(I + A, v) \in \Gamma$ with $A \neq 0$, then in a Witt basis with respect to $U_\Gamma$,

   $$\gamma_i = (I + A_i, v_i) = \left( \begin{array}{ccc} I_2 & 0 & C_i \\ 0 & I_{n-4} & 0 \\ 0 & 0 & I_2 \end{array} \right) \left( \begin{array}{c} u_i \\ w_i \\ 0 \end{array} \right)$$

   with $C_i = \left( \begin{array}{cc} 0 & c_i \\ -c_i & 0 \end{array} \right)$, $c_i \in R$, $u_i \in R^2$, $w_i \in R^{n-4}$.
3. $\sum \lambda_i u_i = 0$ implies $\sum \lambda_i C_i = 0$ (equivalently $\sum \lambda_i A_i = 0$) for all $\lambda_1, \ldots, \lambda_k \in R$.

Proof. We know from Proposition 4.4 that $\Gamma$ is free abelian. Let $\gamma_1, \ldots, \gamma_k$ denote a minimal set of generators with $\gamma_i = (I + A_i, v_i)$.

1. Lemma 4.4
2. If $A \neq 0$ exists, then $U_\Gamma = \text{im} A$ is a 2-dimensional totally isotropic subspace. The matrix representation (4.1) is known from the proof of Proposition 4.5. As $\Gamma$ is abelian, we have $A_i v_j = 0$ for all $i, j$. So $v_j \in \bigcap_i \ker A_i = U_\Gamma^\perp$ for all $j$.
3. Assume $\sum \lambda_i u_i = 0$ and set $C = \sum \lambda_i C_i$. Then $\sum \lambda_i (A_i, v_i) = (A, u)$, where $u \in U_\Gamma$. If $A \neq 0$, then $G$ would have a fixed point (see Corollary 4.7). So $A = 0$, which implies $C = 0$.

Conversely, every group of the form described in the previous proposition defines a homogeneous space:
Proposition 4.8. Let $U$ be a 2-dimensional totally isotropic subspace of $\mathbb{R}^2_2$, and let $\Gamma \subset \text{Iso}(\mathbb{R}^2_2)$ be a subgroup generated by affine transformations $\gamma_1, \ldots, \gamma_k$ of the form (4.1) with linearly independent translation parts. Further, assume that $\sum_i \lambda_i w_i = 0$ implies $\sum_i \lambda_i C_i = 0$ (equivalently $\sum_i \lambda_i A_i = 0$) for all $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$. Then $\mathbb{R}^2_2/\Gamma$ is a complete flat pseudo-Riemannian homogeneous manifold.

Proof. (i) From the matrix form (4.1) it follows that $\Gamma$ is free abelian, and the linear independence of the translation parts implies that it is a discrete subgroup of $\text{Iso}(\mathbb{R}^2_2)$.

(ii) We check that the centralizer of $\Gamma$ in $\text{Iso}(\mathbb{R}^2_2)$ acts transitively: Let $\text{iso}(\mathbb{R}^2_2)$ denote the Lie algebra of $\text{Iso}(\mathbb{R}^2_2)$. In the given Witt basis, the following are elements of $\text{iso}(\mathbb{R}^2_2)$:

$$S = \begin{pmatrix} 0 & -B^T & 0 \\ 0 & 0 & B \\ 0 & 0 & 0 \end{pmatrix}, \quad x, z \in \mathbb{R}^2, y \in \mathbb{R}^{n-2}.$$ 

Now assume arbitrary $x, y, z$ are given. We will show that we can determine $B$ such that $S$ centralises $\log(\Gamma)$. Writing out the commutator equation $[S, (A_i, v_i)]$ blockwise, we see that $[S, (A_i, v_i)] = 0$ is equivalent to $-B^T w_i = C_i z$.

For simplicity, assume that $w_1, \ldots, w_j$ form a maximal linearly independent subset of $w_1, \ldots, w_k$ ($j \leq k$). As $-B^T$ is a $2 \times (n-2)$-matrix, the linear system

$$-B^T w_1 = C_1 z$$

$$\vdots$$

$$-B^T w_j = C_j z$$

consists of $2j$ linearly independent equations and $2(n-2)$ variables (the entries of $B$). As $\dim W = n - 2 \geq j$, this system is always solvable.

So $S$ can be determined such that it commutes with $\gamma_1, \ldots, \gamma_j$. It remains to check that $S$ also commutes with $\gamma_{j+1}, \ldots, \gamma_k$. By assumption, each $w_l$ ($l > j$) is a linear combination $w_l = \sum_{i=1}^j \lambda_i w_i$. Now $w_l - \sum_{i=1}^j \lambda_i w_i = 0$ implies $C_l - \sum_{i=1}^j \lambda_i C_i = 0$. But this means

$$-B^T w_l = \sum_{i=1}^j \lambda_i (-B^T w_i) = (\sum_{i=1}^j \lambda_i C_i) z = C_l z,$$

so $[(A_l, w_l), S] = 0$.

The elements $\exp(S)$ generate a unipotent subgroup of the centralizer of $\Gamma$, so its open orbit at 0 is closed by [3 Proposition 4.10]. As $x, y, z$ can be chosen arbitrarily, its tangent space at 0 is $\mathbb{R}^2_2$. Hence the orbit of the centralizer at 0 is open and closed, and therefore it is all of $\mathbb{R}^2_2$. Consequently, $\Gamma$ has transitive centralizer.

(iii) Because the centralizer is transitive, the action free everywhere. It follows from [2 Proposition 7.2] that $\Gamma$ acts properly discontinuously.
Now $\mathbb{R}^n_2/\Gamma$ is a complete homogeneous manifold due to the transitive action of the centralizer on $\mathbb{R}^n_2$. □

4.2. Dimension $\leq 5$.

**Proposition 4.9** (Wolf). Let $M = \mathbb{R}^n_2/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension $\leq 4$. Then $\Gamma$ is a free abelian group consisting of pure translations.

For a proof, see [5, Corollary 3.7.11].

**Proposition 4.10.** Let $M = \mathbb{R}^5_2/\Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 5. Then $\Gamma$ is a free abelian group. Depending on the signature of $M$, we have the following possibilities:

1. Signature $(5,0)$ or $(4,1)$: $\Gamma$ is a group of pure translations.
2. Signature $(3,2)$: $\Gamma$ is either a group of pure translations, or there exists $\gamma_1 = (I + A_1, v_1) \in \Gamma$ with $A_1 \neq 0$. In the latter case, $\text{rk} \Gamma \leq 3$, and if $\gamma_1, \ldots, \gamma_k$ ($k = 1, 2, 3$) are generators of $\Gamma$, then $v_1, \ldots, v_k$ are linearly independent, and $w_i = \frac{c_i}{c_1}v_1$ in the notation of (4.7) ($i = 1, \ldots, k$).

**Proof.** $\Gamma$ is free abelian by Proposition 4.5. The statement for signatures $(5,0)$ and $(4,1)$ follows from Proposition 4.1.

Let the signature be $(3,2)$ and assume $\Gamma$ is not a group of pure translations. Then $U^\perp = \text{im} A$ is 2-dimensional (where $(I + A, v) \in \Gamma$, $A \neq 0$). By Lemma 4.4, the translation parts of the generators of $\Gamma$ are linearly independent elements of $U^\perp$, which is 3-dimensional. So $\text{rk} \Gamma \leq 3$. Now, $U^\perp = U^\perp \oplus W$ with $\dim W = 1$. So the $W$-components of the translation parts are multiples of each other, and it follows from part (c) of Proposition 4.7 that $w_1 \neq 0$ and $w_i = \frac{c_i}{c_1}w_1$. □

4.3. Dimension 6. In dimension 6, both abelian and non-abelian $\Gamma$ exist.

We introduce the following notation: For $x \in \mathbb{R}^3$, let

$$T(x) = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}.$$ \tag{4.2}

Then for any $y \in \mathbb{R}^3$,

$$T(x)y = x \times y,$$

where $\times$ denotes the vector cross product on $\mathbb{R}^3$.

**Lemma 4.11.** Let $\Gamma \in \text{Iso}(\mathbb{R}^6_3)$ be a group with transitive centralizer in $\text{Iso}(\mathbb{R}^6_3)$. An element $X \in \log(\Gamma)$ has the form

$$X = \left( \begin{array}{cc} 0 & C \\ 0 & 0 \end{array} \right), \left( \begin{array}{c} u \\ u^* \end{array} \right)$$ \tag{4.2}

with respect to the Witt decomposition $\mathbb{R}^6_3 = U \oplus U^\perp$. Furthermore,

$$C = \alpha_X T(u^*)$$

for some $\alpha_X \in \mathbb{R}$. If $[X_1, X_2] \neq 0$ for $X_1, X_2 \in \log(\Gamma)$, then $\alpha_{X_1} = \alpha_{X_2} \neq 0$.

**Proof.** The holonomy is abelian by Corollary 3.11 so (4.2) follows.

For $X \in \log(\Gamma)$ we have $Cu^* = 0$, that is, for any $\alpha \in \mathbb{R}$,

$$Cu^* = \alpha u^* \times u^* = 0.$$
If $X$ is non-central, then $C \neq 0$ and $u^* \neq 0$. Now let $x, y \in \mathbb{R}^3$ such that $u^*, x, y$ form a basis of $\mathbb{R}^3$. Because $C$ is skew,

$$u^* C x = -u^* C^T x = -(Cu^*)^T x = 0.$$ 

Also,

$$x^T C x = -x^T C^T x \quad \text{and} \quad x^T C x = (x^T C x)^T = x^T C^T x,$$

hence $x^T C x = 0$. So $Cx$ is perpendicular to the span of $x, u^*$ in the Euclidean sense. This means there is a $\alpha \in \mathbb{R}$ such that

$$Cx = \alpha u^* \times x.$$ 

In the same way we get $Cy = \beta u^* \times y$ for some $\beta \in \mathbb{R}$. As neither $x$ nor $y$ is in the kernel of $C$ (which is spanned by $u^*$), $\alpha, \beta \neq 0$.

As $y$ is not in the span of $u^*, x$, we have

$$0 \neq x^T C y = \beta x^T (u^* \times y)$$

$$= -y^T C x = -\alpha y^T (u^* \times x) = -\alpha x^T (y \times u^*) = \alpha x^T (u^* \times y),$$

where the last line uses standard identities for the vector product. So $\alpha = \beta$, and $C$ and $\alpha T(u^*)$ coincide on a basis of $\mathbb{R}^3$.

Now assume $[X_1, X_2] \neq 0$. Then

$$\alpha_2 u_2^* \times u_1^* = C_2 u_1^* = -C_1 u_2^* = -\alpha_1 u_1^* \times u_2^* = \alpha_1 u_2^* \times u_1^*,$$

and this expression is $\neq 0$ because $C_1 u_2^*$ is the translation part of $(\frac{1}{2}[X_1, X_2]) \neq 0$. So $\alpha_1 = \alpha_2$.

**Proposition 4.12.** Let $M = \mathbb{R}^n / \Gamma$ be a complete homogeneous flat pseudo-Riemannian manifold of dimension 6, and assume $\Gamma$ is abelian. Then $\Gamma$ is free abelian. Depending on the signature of $M$, we have the following possibilities:

1. Signature $(6, 0)$ or $(5, 1)$: $\Gamma$ is a group of pure translations.
2. Signature $(4, 2)$: $\Gamma$ is either a group of pure translations, or $\Gamma$ contains elements $\gamma = (I + A, v)$ with $A \neq 0$ subject to the constraints of Proposition 4.7. Further, $\text{rk} \Gamma \leq 4$.
3. Signature $(3, 3)$: If $\dim U_\Gamma < 3$, then $\Gamma$ is one of the groups that may appear for signature $(4, 2)$. There is no abelian $\Gamma$ with $\dim U_\Gamma = 3$.

**Proof:** $\Gamma$ is free abelian by Lemma 4.3. The statement for signatures $(6, 0)$ and $(5, 1)$ follows from Proposition 4.1.

If the signature is $(4, 2)$ and $\Gamma$ is not a group of pure translations, then the statement follows from Proposition 4.1. In this case, $U_\Gamma^\perp$ contains the linearly independent translation parts and is of dimension 4. So $\text{rk} \Gamma \leq 4$.

Consider signature $(3, 3)$. If $\dim U_\Gamma = 0$ or $= 2$, then $\Gamma$ is a group as in the case for signature $(4, 2)$. Otherwise, $\dim U_\Gamma = 3$. We show that in the latter case the centralizer of $\Gamma$ does not act with open orbit: Any $\gamma \in \Gamma$ can be written as

$$\gamma = (I + A, v) = \begin{pmatrix} I_3 & C \\ 0 & I_3 \end{pmatrix} \cdot \begin{pmatrix} u \\ u^* \end{pmatrix},$$

where $C \in \mathfrak{so}_3$ and $u, u^* \in \mathbb{R}^3$. In fact, we have $\mathbb{R}^3 = U_\Gamma \oplus U_\Gamma^*$ and $U_\Gamma^\perp = U_\Gamma$.

We will show that $u^* = 0$:

\footnote{That is, with respect to the canonical positive definite inner product on $\mathbb{R}^3$.}
(i) Because \( \text{rk} C = 2 \) for every \( C \in \mathfrak{so}_3 \), \( C \neq 0 \), but \( U_\Gamma = \sum \text{im} A \) is 3-dimensional, there exist \( \gamma_1, \gamma_2 \in \Gamma \) such that the skew matrices \( C_1 \) and \( C_2 \) are linearly independent. So, for every \( u^* \in U_\Gamma^* \), there is an element \( \gamma = (I + A, v) \) such that \( Au^* \neq 0 \).

(ii) \( \Gamma \) abelian implies \( A_1u^*_2 = 0 \) for every \( \gamma_1, \gamma_2 \in \Gamma \). With (i), this implies \( u^*_2 = 0 \). So the translation part of every \( \gamma = (I + A, v) \in \Gamma \) is an element \( v = u \in U_\Gamma \).

Step (ii) implies \( C_1 = \alpha_1 T(u_1^*) = 0 \) by Lemma \ref{lemma} but \( C_1 \neq 0 \) was required in step (i). Contradiction; so \( \Gamma \) cannot be abelian.

\[ \Box \]

**Proposition 4.13.** Let \( M = \mathbb{R}^6/\Gamma \) be a complete homogeneous flat pseudo-Riemannian manifold of dimension 6, and assume \( \Gamma \) is non-abelian. Then the signature of \( M \) is \((3,3)\), and \( \Gamma \) is one of the following:

1. \( \Gamma = \Lambda \times \Theta \), where \( \Lambda \) is a discrete Heisenberg group and \( \Theta \) a discrete group of pure translations in \( U_\Gamma \). Then \( 3 \leq \text{rk} \Gamma = 3 + \text{rk} \Theta \leq 5 \).
2. \( \Gamma \) is discrete group of rank 6 with center \( Z(\Gamma) = [\Gamma, \Gamma] \) of rank 3. In this case, \( M \) is compact.

\[ \text{Proof.} \] If the signature was anything but \((3,3)\) or \( \dim U_0 < 3 \), then \( \Gamma \) would have to be abelian due Proposition \ref{prop} The holonomy is abelian by Corollary \ref{cor}.

For the following it is more convenient to work with the real Zariski closure \( G \) of \( \Gamma \) and its Lie algebra \( \mathfrak{g} \). As \( \mathfrak{g} \) is a 2-step nilpotent, \( \mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}(\mathfrak{g}) \), where \( \mathfrak{v} \) is a vector subspace of \( \mathfrak{g} \) of dimension \( \geq 2 \) spanned by non-central elements. Set \( \mathfrak{v}_\Gamma = \mathfrak{v} \cap \log(\Gamma) \). We proceed in four steps:

(i) Assume there are \( X_i = (A_i, v_i) \in \mathfrak{v} \), \( \lambda_i \in \mathbb{R} \), \( v_i = u_i + u_i^* \) (for \( i = 1, \ldots, m \)), such that \( \sum \lambda_i u^*_i = 0 \). Then \( \sum \lambda_i X_i = (\sum \lambda_i A_i, \sum \lambda_i v_i) = (A, u) \in \mathfrak{v} \), where \( u \in U_\Gamma \). For all \( (A', v') \in \mathfrak{g} \), the commutator with \( (A, u) \) is \( [(A', v'), (A, u)] = (0, 2A' u) = (0, 0) \). Thus \( (A, u) \in \mathfrak{v} \cap \mathfrak{z}(\mathfrak{g}) = \{0\} \).

So if \( X_1, \ldots, X_m, u \) are linearly independent, then \( u_1^*, \ldots, u_m^* \in U_\Gamma^* \) are linearly independent (and by Lemma \ref{lemma} the \( C_1, \ldots, C_m \) are too). But \( \dim U_0^* = 3 \), so \( \dim \mathfrak{v} \leq 3 \).

(ii) If \( Z \in \mathfrak{z}(\mathfrak{g}) \), then \( C_2 = 0 \) and \( u_2^* = 0 \): As \( Z \) commutes with \( X_1, X_2 \), we have \( C_2 u_1^* = 0 = C_2 u_2^* \). By step (i), \( u_1^*, u_2^* \) are linearly independent. So \( \text{rk} C_2 < 2 \), which implies \( C_2 = 0 \) because \( C_2 \) is a skew 3 \times 3-matrix. Also, \( C_1 u_2^* = 0 = C_2 u_2^* \), so \( u_2^* = \ker C_1 \cap \ker C_2 = \{0\} \). So \( \exp(Z) = (I, u_2^*) \) is a translation by \( u_2^* \in \Gamma \).

(iii) Assume \( \dim \mathfrak{v} = 2 \). Let \( \mathfrak{v} \) be spanned by \( X_1, X_2 \), and \( Z_{12} = [X_1, X_2] \) is a pure translation by an element of \( U_\Gamma \). The elements \( X_1, X_2, Z_{12} \) span a Heisenberg algebra \( \mathfrak{h}_3 \) contained in \( \mathfrak{g} \). If \( \dim \mathfrak{g} > 3 \), then \( \mathfrak{z}(\mathfrak{g}) = \mathbb{R} Z_{12} \oplus t \), where according to step (ii) \( t \) is a subalgebra of pure translations by elements of \( U_\Gamma \). So \( \mathfrak{g} = \mathfrak{h}_3 \oplus t \) with \( 0 \leq \dim t < \dim U_\Gamma = 3 \). This gives part (a) of the proposition.

(iv) Now assume \( \dim \mathfrak{v} = 3 \). We show that \( \mathfrak{z}(\mathfrak{g}) = [\mathfrak{v}, \mathfrak{v}] \) and \( \dim \mathfrak{z}(\mathfrak{g}) = 3 \): Let \( X_1 = (A_1, v_1), X_2 = (A_2, v_2) \in \mathfrak{v}_\Gamma \) such that \( [X_1, X_2] \neq 0 \). By Lemma \ref{lemma} \( C_1 = \alpha T(u_1^*) \) and \( C_2 = \alpha T(u_2^*) \) for some number \( \alpha \neq 0 \). There exists \( X_3 \in \mathfrak{v}_\Gamma \) such that \( X_1, X_2, X_3 \) form basis of \( \mathfrak{v} \). By step (i), \( u_1^*, u_2^*, u_3^* \) are linearly independent. For \( i = 1, 2 \), \( \ker C_i = \mathbb{R} u_i^* \), and \( u_i^* \) is proportional to neither \( u_1^* \) nor \( u_2^* \). This means \( C_1 u_3^* \neq 0 \neq C_2 u_3^* \), which implies \( [X_1, X_3] \neq 0 \neq [X_2, X_3] \). By Lemma \ref{lemma} \( C_3 = \alpha T(u_3^*) \).
Write $Z_{ij} = [X_i, X_j]$. The non-zero entries of the translation parts of the commutators $Z_{12}$, $Z_{13}$ and $Z_{23}$ are

$$C_1 u_i^* = \alpha u_i^* \times u_j^*, \quad C_1 u_j^* = \alpha u_i^* \times u_j^*, \quad C_2 u_j^* = \alpha u_i^* \times u_j^*.$$ 

Linear independence of $u_i^*, u_j^*, u_k^*$ implies that these are linearly independent. Hence the commutators $Z_{12}, Z_{13}, Z_{23}$ are linearly independent in $\mathfrak{g}(\mathfrak{g})$. Because $\dim \mathfrak{g} = \dim \mathfrak{v} + \dim \mathfrak{g}(\mathfrak{g}) \leq 6$, it follows that $\mathfrak{g}(\mathfrak{g})$ is spanned by these $Z_{ij}$, that is $\mathfrak{g}(\mathfrak{g}) = \langle [v, v] \rangle$. This gives part (b) of the proposition.

\[ \square \]

Remark 4.14. In case (2) of Proposition 4.13 it can be shown that $\Gamma$ is a lattice in a Lie group $H_3 \rtimes \text{Ad} \text{e}_4$, see [5, Section 5.3].

We have a converse statement to Proposition 4.13.

**Proposition 4.15.** Let $\Gamma$ be a subgroup of $\text{Iso}(\mathbb{R}_6^3)$. Then $M = \mathbb{R}_6^3/\Gamma$ is a complete flat pseudo-Riemannian homogeneous manifold if there exists a 3-dimensional totally isotropic subspace $U$ and $\Gamma$ is a group of type (1) or (2) in Proposition 4.13 (with $U_T$ replaced by $U$).

**Proof.** Both cases can be treated simultaneously.

Let $X_1, X_2, X_3 \in \log(\Gamma)$ such that the $\exp(X_i)$ generate $\Gamma$. The number $\alpha \neq 0$ from Lemma 4.11 is necessarily the same for $X_1, X_2, X_3$.

(i) The group $\Gamma$ is discrete because the translation parts of the generators $\exp(X_i)$ and those of the generators of $Z(\Gamma)$ form a linearly independent set.

(ii) We show that the centralizer of $\Gamma$ is transitive. Consider the following elements

$$S = \left( \begin{array}{cc} 0 & -\alpha T(z) \\ 0 & 0 \end{array} \right), \quad \left( \begin{array}{c} x \\ z \end{array} \right) \in \text{iso}(\mathbb{R}_6^3)$$

with $x, z \in \mathbb{R}^3$ arbitrary. Then $[X_i, S] = 0$ for $i = 1, 2, 3$, because

$$C_i z = \alpha u^*_i \times z = -\alpha z \times u^*_i = -\alpha T(z) u^*_i.$$ 

Clearly, $S$ also commutes with any translation by a vector from $U$. So in both cases (1) and (2), $\Gamma$ has a centralizer with an open orbit at 0. The exponentials of the elements of $S$ clearly generate a unipotent subgroup of $\text{Iso}(\mathbb{R}_6^3)$, hence the open orbit is also closed and thus all of $\mathbb{R}_6^3$.

(iii) From the transitivity of the centralizer, it also follows that the action is free and thus properly discontinuous (2 Proposition 7.2).

So $\mathbb{R}_6^3/\Gamma$ is a complete homogeneous manifold.

\[ \square \]

In the situation of Proposition 4.13 it is natural to ask whether the statement can be simplified by claiming that $\Gamma$ is always a subgroup of a group of type (2) in Proposition 4.13. But this is not always the case:

**Example 4.16.** We choose the generators $\gamma_i = (I + A_i, v_i), i = 1, 2$, of a discrete Heisenberg group $\Lambda$ as follows: If we decompose $v_i = u_i + u_i^*$ where $u_i \in U_T$, $u_i^* \in U_T^*$, let $u_1 = 0, u_2 = e_1^*, \alpha = 1$ (with $\alpha$ as in the proof of Proposition 4.13) and $e_1^*$ refers to the $i$th unit vector taken as an element of $U_T^*$. Then $\gamma_3 = [\gamma_1, \gamma_2] = (I, v_3)^2$, where $u_3 = e_3, u_3^* = 0$. Let $\gamma_4 = (I, u_4)$ be the translation by $u_4 = \sqrt{2} e_1 + \sqrt{5} e_2 \in U_T$. Let $\Theta = \langle \gamma_4 \rangle$ and $\Gamma = \Lambda \cdot \Theta \cong \Lambda \times \Theta$. 

\[ \square \]
Assume there exists $X = (A, v)$ of the form (1.2) not commuting with $X_1, X_2$. Then the respective translation parts of $[X_1, X]$ and $[X_2, X]$ are

$$e_1 \times u^* = \begin{pmatrix} 0 \\ -\eta_3 \\ \eta_2 \\ \eta_1 \end{pmatrix}, e_2 \times u^* = \begin{pmatrix} \eta_3 \\ 0 \\ -\eta_1 \end{pmatrix} \in U_\Gamma,$$

where $\eta_i$ are the components of $u^*$, and $\eta_3 \neq 0$ due to the fact that $X$ and the $X_i$ do not commute. If $\Gamma$ could be embedded into into a group of type (2), such $X$ would have to exist. But by construction $u_4$ is not contained in the $\mathbb{Z}$-span of $e_3, e_1 \times u^*, e_2 \times u^*$. So the group generated by $\Gamma$ and $\exp(X)$ is not discrete in $\text{Iso}(\mathbb{R}^6_3)$.

5. **Fundamental Groups of Complete Flat Pseudo-Riemannian Homogeneous Spaces**

In this section we will prove the following:

**Theorem 5.1.** Let $\Gamma$ be a finitely generated torsion-free 2-step nilpotent group of rank $n$. Then there exists a faithful representation $\varphi : \Gamma \to \text{Iso}(\mathbb{R}^{2n}_n)$ such that $M = \mathbb{R}^{2n}_n / \varphi(\Gamma)$ a complete flat pseudo-Riemannian homogeneous manifold $M$ of signature $(n, n)$ with abelian linear holonomy group.

We start with a construction given in [1, Paragraph 5.3.2] to obtain nilpotent Lie groups with flat bi-invariant metrics. Let $\mathfrak{g}$ be a real 2-step nilpotent Lie algebra of finite dimension $n$. Then the semidirect sum $\mathfrak{h} = \mathfrak{g} \oplus \text{ad}^* \mathfrak{g}$ is a 2-step nilpotent Lie algebra with Lie product

$$[(X, \xi), (Y, \eta)] = ([X, Y], \text{ad}^* (X) \eta - \text{ad}^* (Y) \xi),$$

where $X, Y \in \mathfrak{g}, \xi, \eta \in \mathfrak{g}^*$ and $\text{ad}^*$ denotes the coadjoint representation. An invariant inner product on $\mathfrak{h}$ is given by

$$\langle (X, \xi), (Y, \eta) \rangle = \xi(Y) + \eta(X).$$

Its signature is $(n, n)$, as the subspaces $\mathfrak{g}$ and $\mathfrak{g}^*$ are totally isotropic and dual to each other.

If $G$ is a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{g}$, then $H = G \rtimes \text{Ad}^* \mathfrak{g}$ (with $\mathfrak{g}^*$ taken as a vector group) is a simply connected 2-step nilpotent Lie group with Lie algebra $\mathfrak{h}$, and $\langle \cdot, \cdot \rangle$ induces a bi-invariant flat pseudo-Riemannian metric on $H$.

**Remark 5.2.** For any lattice $\Gamma_H \subset H$, the space $H/\Gamma_H$ is a compact flat pseudo-Riemannian homogeneous manifold. In particular, $H$ is complete (see [6, Proposition 9.39]). By [2, Theorem 3.1], $\Gamma_H$ has abelian linear holonomy.

**Proof of Theorem 5.1.** Let $\Gamma$ be a finitely generated torsion-free 2-step nilpotent group. The real Malcev hull $G$ of $\Gamma$ is a 2-step nilpotent simply connected Lie group such that $\Gamma$ is a lattice in $G$. In particular, $\text{rk} \Gamma = \dim G = n$. If $\mathfrak{g}$ is the Lie algebra of $G$, let $H$ be as in the construction above. We identify $G$ with the closed subgroup $G \times \{0\}$ of $H$. As $\Gamma$ is a discrete subgroup of $H$, it follows from the remark above that $M = H/\Gamma$ is a complete flat pseudo-Riemannian homogeneous manifold with abelian linear holonomy.

As $H$ has signature $(n, n)$, the development representation $\varphi$ of the right-multiplication of $G$ gives the representation of $\Gamma$ as isometries of $\mathbb{R}^{2n}_n$. \qed
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