Stokes flow in a channel with permeable boundaries

An exact solution for Stokes flow in a channel with arbitrarily large wall permeability

G. Herschlag,¹ J.-G. Liu,¹,² and A. T. Layton¹
¹) Mathematics Department, Duke University
²) Physics Department, Duke University.

(Dated: 14 November 2014)

We derive an exact solution for Stokes flow in an in a channel with permeable walls. We assume that at the channel walls, the normal component of the fluid velocity is described by Darcy’s law and the tangential component of the fluid velocity is described by the no slip condition. The pressure exterior to the channel is assumed to be constant. Although this problem has been well studied, typical studies assume that the permeability of the wall is small relative to other non-dimensional parameters; this work relaxes this assumption and explores a regime in parameter space that has not yet been well studied. A consequence of this relaxation is that transverse velocity is no longer necessarily small when compared with the axial velocity. We use our result to explore how existing asymptotic theories break down in the limit of large permeability.

Keywords: Filtration; permeable boundaries; Stokes flow

I. INTRODUCTION

There is a great deal of interest in the analysis and simulation of fluid flow along permeable tubes, in large part owing to the multitude of applications. In engineering, examples include ultrafiltration membrane systems used in water treatment. In biology, examples include glomerular filtration in the kidney, dialysis machines, and capillary transport. Given the wide range of applications, much effort has gone into changing the assumption on the normal component of the fluid velocity at the wall is assumed to be a known constant a priori and independent of position. Berman’s result was later extended to a cylindrical geometry by Yuan and Finkelstein, again with constant normal velocity at the walls, which was further expounded upon by Terrill and Shrestha. In membrane transport, the normal fluid velocity is frequently driven by hydrostatic pressure gradient. The classical assumption is that the outer and inner channel pressures have the same gradient, leading to the assumption of a constant normal velocity. In practice, however, it is well known that this is not typically the case, and much work has gone into changing the assumption on the pressure exterior of the channel to be a constant along the axial direction, rather than having a similar gradient. Enforcing Darcy’s law at the boundaries, this change in assumptions causes the normal wall velocity to change as a function of the hydrostatic pressure. A notable example of this is provided by Galowin, Fletcher, and De-Santis, in which the flow profile for a semi-infinite pipe with a closed end wall is considered with normal velocity described by Darcy’s law, and the external pressure assumed to be constant. This work was later expanded by Granger, Dodds, and Midoux who assume one permeable and one impermeable wall, along with parabolic inflow conditions at a certain point within the pipe. Indeed, there have been a host of works dealing with this topic. More recently, Haldenwang has studied the problem in more detail; nonetheless, this work still considers a channel or pipe of finite length with prescribed inflow and outflow conditions; the inflow condition in these last set of papers is matched with the analytic solution by Berman in which the normal velocities at the wall are matched. In the work of Tilton et al., the authors relaxed the assumption of a finite length pipe and found a self similar profile in a pipe of arbitrary length.

The works mentioned above assume either small permeability or a small ratio between transverse and axial flow velocities. With these assumptions, leading order asymptotic expansions result in parabolic profiles and predictions at low vanishing Reynolds numbers typically are equivalent to the leading order asymptotic term. We may ask then about the error between the predicted parabolic profile and the actual flow profile as a function of increasing permeability; in other words we ask for a rigorous region in which we may assume the above asymptotic expansions to be valid. To perform this study, we examine vanishingly small Reynolds numbers as it has proven difficult to find a closed form solution to this problem for the Navier-Stokes equations.

Indeed the choice of small Reynolds number is well within the confines of many applications in this field and, in particular, many of the biological applications listed above occur at negligible Reynolds number. In Pozrikidis, the authors have analyzed a similar problem in modeling capillary blood flow in a numerical study, in which the author assumes a region of impermeable pipe feeds into a permeable region and the flow is described...
by Stokes equations. Collecting ducts of rat kidneys also occur at small Reynolds number occurring at most on the order of $10^{-4}$ (see Refs.\textsuperscript{14} and \textsuperscript{15}). We therefore feel that a strong statement about the breakdown in any asymptotic expansion about small permeability and small Reynolds number will provide a useful mathematical footing for such expansions to be employed.

In answering this question, we have examined a stronger statement by determining an exact solution for channel flow in the Stokes regime, in which at the channel walls, the normal component of the fluid velocity is described by Darcy’s law and the tangential component of the fluid velocity is described by the no slip condition. The pressure exterior to the channel is assumed to be uniform and similarly to Tilton \textit{et al.},\textsuperscript{12} we do not assume the structure of an inlet flow and allow such structure to arise from the equation solution. These assumptions are consistent with previous problem statements in this field.

Finding an exact solution in the Stokes regime is useful for several reasons. In solving such a system we are not only able to rigorously describe the error accumulation as a function of permeability mentioned above, but we are also able to determine a potential profile for inlet conditions that may aid in future numerical studies for biological flows in the Stokes regime. For example, in Pozrikidis\textsuperscript{12}, the author models fluid exchange in capillaries with boundary conditions satisfied by transitions into non-permeable tubes which must be long enough to account for errors in inlet conditions that are assumed to contain parabolic flow profiles. Our solution has the potential of replacing this region with a much simpler and less expensive boundary condition. We also note that if one were to encounter a problem of permeable channel flow in which permeability or transverse velocities were large, the solution to our problem described above would provide a solution to this system at zero Reynolds number, and may be used as the first term in an asymptotic expansion for low Reynolds number (see section\textsuperscript{11}). Although we know of no such application at the moment, we provide a possible physical experiment that we expect our model to predict the dynamics of far more accurately than any other existing models (see section\textsuperscript{11}).

We organize the paper in the following way. In section\textsuperscript{11} we present and solve the system of equations described above. In section\textsuperscript{11} we then demonstrate the break down of existing asymptotic theories for small permeability as the permeability grows. In section\textsuperscript{11} we frame our result in the context of an asymptotic expansion about low Reynolds number, and offer an informed hypothesis for where our Stokes flow solution should accurately predict flow structure of the laminar Navier-Stokes equations. We conclude with a discussion in section\textsuperscript{11} and suggest a physical experiment that may be preformed to validate our analysis.

The solution, presented in section\textsuperscript{11} is found with a non-standard approach and thus we outline our method here. First we non-dimensionalize the system, but keep two free variables in the non-dimensionalization. We next take an ansatz that the pressure will satisfy a Robin boundary condition; keeping the two variables free in the non-dimensionalization allows us to scale this Robin condition in such a way that incompressibility will be satisfied. Had we not kept these two variables free the ansatz would not necessarily lead to a valid solution. The ansatz allows us to decouple the system and we first determine the pressure. We then use the pressure to find the velocity profile and finally use the incompressibility constraint to restrict the two free parameters found in the non-dimensionalization.

\section{Stokes Flow Solution to Channel with Permeable Boundaries}

We examine the Stokes flow equation for a channel with arbitrarily large permeability at the boundaries. We assume that the pressure outside of the channel is a constant $P_\text{b}$, and without loss of generality, set this value to be zero. We let the channel walls to be separated by a distance $2r$. We let $x$ describe the location parallel to the axial position and $y$ describe the location in the transverse direction. The flow in each direction is described by $u(x, y)$ and $v(x, y)$, respectively. We assume that at position $x = x_0$, we know the central pressure gradient $\beta$ and the inner channel pressure at the walls so that

$$p(x_0, \pm r) = P_\text{in}, \quad \beta = p_x(x_0, 0).$$

Without loss of generality we set $x_0 = 0$. We will show below that there is a one-to-one correspondence between $\beta$ and the average flow profile

$$\bar{U} = \frac{1}{2r} \int_{-r}^{r} u(x_0, y)dy.$$  \hspace{1cm} (2)

Although the later is a more standard choice (see for example Tilton \textit{et al.},\textsuperscript{12}) we elect to use the former for mathematical convenience.

Fluid flux across the channel is assumed to be driven by the pressure difference across the interior and exterior of the channel. In the physical case of filtration Darcy’s law is typically assumed in the normal direction at the walls and a no slip condition is assumed in the tangential direction (see, for example, Refs.\textsuperscript{10} \textsuperscript{12} \textsuperscript{13} \textsuperscript{10} \textsuperscript{12} \textsuperscript{13}). We also use these boundary conditions on the channel walls, using Darcy’s law with coefficient $\kappa = \frac{k}{\mu h}$ in the normal direction, where $k$ is the permeability, $\mu$ is the viscosity, and $h$ is the width of the channel connecting the outer and inner fluids. No slip conditions are used for the tangential component of the boundary. For the two dimensional setting, the Stokes equations may then be written as

$$p_x = \mu \Delta u, \quad p_y = \mu \Delta v,$$

$$0 = u_x + v_y,$$  \hspace{1cm} (3)

$$u(x, \pm r) = 0, \quad v(x, \pm r) = \pm \kappa p,$$  \hspace{1cm} (4)

$$u(x, y) = \bar{U} + \frac{y}{h}, \quad v(x, y) = -\frac{\kappa p}{\mu h} x,$$

$$\bar{U} = \frac{1}{2r} \int_{-r}^{r} u(x_0, y)dy.$$  \hspace{1cm} (2)

Although the later is a more standard choice (see for example Tilton \textit{et al.},\textsuperscript{12}) we elect to use the former for mathematical convenience.

Fluid flux across the channel is assumed to be driven by the pressure difference across the interior and exterior of the channel. In the physical case of filtration Darcy’s law is typically assumed in the normal direction at the walls and a no slip condition is assumed in the tangential direction (see, for example, Refs.\textsuperscript{10} \textsuperscript{12} \textsuperscript{13} \textsuperscript{10} \textsuperscript{12} \textsuperscript{13}). We also use these boundary conditions on the channel walls, using Darcy’s law with coefficient $\kappa = \frac{k}{\mu h}$ in the normal direction, where $k$ is the permeability, $\mu$ is the viscosity, and $h$ is the width of the channel connecting the outer and inner fluids. No slip conditions are used for the tangential component of the boundary. For the two dimensional setting, the Stokes equations may then be written as

$$p_x = \mu \Delta u, \quad p_y = \mu \Delta v,$$  \hspace{1cm} (3)

$$0 = u_x + v_y,$$  \hspace{1cm} (4)

$$u(x, \pm r) = 0, \quad v(x, \pm r) = \pm \kappa p,$$  \hspace{1cm} (5)
where $\mu$ is the dynamic viscosity, and $u$ and $v$ are the velocity components in the $x$ and $y$ directions, respectively. We note, and will show below, that this system of equation matches the zeroth order term in an asymptotic expansion of the Navier-Stokes equations about small Reynolds number (see section IV).

A. The non-dimensional system

To search for solutions of this system, we first non-dimensionalize and identify the important non-dimensional parameter. We rescale by taking

\[
x = \frac{r}{\gamma} \tilde{x}, \quad y = \frac{r}{\gamma} \tilde{y}, \quad p = \frac{\beta r}{\gamma} \tilde{p},
\]

\[
x = \beta r^\gamma\alpha \kappa^{2-\alpha} \mu^{1-\alpha} \tilde{u},
\]

\[
v = \beta r^\gamma \alpha \kappa^{2-\alpha} \mu^{1-\alpha} \tilde{v},
\]

where $\gamma, \alpha \in \mathbb{R}$ with $\gamma > 0$. Substituting the non-dimensional rescaling into the original system and dropping the tildes for convenience, we are left with the new system

\[
p(0, \pm \gamma) = B, \quad p_x(0, 0) = 1,
\]

\[
p_x = A^{2-\alpha} \Delta u, \quad p_y = A^{2-\alpha} \Delta v,
\]

\[0 = u_x + v_y,
\]

\[u(x, \pm \gamma) = 0, \quad v(x, \pm \gamma) = \pm A^{\alpha-1} p,
\]

where $A = \gamma \mu k/r$ and $B = P_{tm} \gamma / \beta r$. We define a special case where $\gamma = 1$ as $A_1 = \mu k/r$ and $B_1 = P_{tm} / \beta r$ which will be used below.

It is not standard to introduce $\gamma$ and $\alpha$ and these values are typically chosen implicitly with the non-dimensionalization, the standard being $\gamma = 1$ and $\alpha = 2$. We note that the reason for leaving $\alpha$ free at the moment is that we will take an ansatz that requires the ability to allow a ratio of arbitrary size between $A^{2-\alpha}$ and $A^{\alpha-1}$. This is possible in all cases so long as $A \neq 1$ and thus we introduce a second scaling parameter $\gamma$ to allow us to move away from the $A = 1$ case. It is possible then that we could instead choose $\alpha$ to begin with and allow $\gamma$ to be used to generate the arbitrary ratio. While this is true, we would have to pick $\alpha$ such that when $\gamma$ was chosen to set $A = 1$, no scaling would be necessary. It is unclear which value of $\alpha$ this should be and below we will show that $\alpha = 1$ is the proper choice, which is different from the standard choice of $\alpha = 2$. As there is no way to determine the proper choice of $\alpha$ a priori, we leave both scalings free before solving the problem.

B. Establishing the ansatz

Equations [912] form a complex and coupled system. We may, however, attempt to decouple the pressure term from the velocity equations by setting $\bar{v} = v + p_y$ and assume $\bar{v}$ vanishes at the boundaries. This assumption requires that pressure satisfy a Robin boundary condition and the equations may be rewritten as

\[p(0, \pm \gamma) = B, \quad p_x(0, 0) = 1,
\]

\[\Delta p = 0, \quad \partial_y p(x, \pm \gamma) = -A^{\alpha-1} p(x, \pm \gamma)
\]

\[p_x = A^{2-\alpha} \Delta u, \quad p_y = A^{2-\alpha} \Delta \bar{v},
\]

\[0 = u_x + \bar{v}_y - p_{yy},
\]

\[u(x, \pm \gamma) = 0, \quad \bar{v}(x, \pm \gamma) = 0,
\]

which appears to be an overdetermined system. We note that a solution to this system will provide a solution to the original set of equations [912]. To see this we note the boundary conditions at the channel walls for $p$ and $v$ can be seen to be equivalent via equations [12] and [4] which imply that

\[v(x, \pm \gamma) = \mp \partial_y p(x, \pm \gamma) = \pm A^{\alpha-1} p(x, \pm \gamma).
\]

The rest of the relationships are straightforward. Therefore, if we find a solution to the new system [912] we have found a solution for the original system [912]. In attempting to solve the new system we note that the equation for pressure is decoupled and thus we may solve it without knowing the velocity profile. After solving for the pressure we may then solve the Poisson equations for the velocity $u$ and the adjusted velocity $\bar{v}$. It is unclear however that equation [16] will be satisfied by this attempt to find a solution. We note that we will find $u$ and $\bar{v}$ to be inversely proportional to $A^{2-\alpha}$. Having left both $\gamma$ and $\alpha$ free, we are free to scale these solutions so that $u_x + \bar{v}_y$ has the possibility to become proportional to $p_{yy}$. Although it is not obvious at the outset that this will be the case, we will show that incompressibility will be satisfied with the correct scaling of $\alpha$ and $\gamma$.

In addition to the ansatz of a Robin boundary condition for pressure (and that the resulting equations will remain incompressible), we also assume an axisymmetric flow profile so that that $v$ is odd in $y$ and $u$ and $p$ are even in $y$.

For the remainder of this section we show that the new problem statement allows us to generate solutions for the Stokes equations for all values of $A$ except for $A = 1$. There is one exception to this statement in which solutions are defined for all values of $A$ at a special value of $\gamma$. We will show that the special choice of $\gamma$ results in setting $\alpha = 1$.

C. Solving the system

We begin by solving for $p$ using normal mode analysis. Due to the symmetry assumption that $p$ is even in $y$, we may write a discrete Fourier expansion for the pressure as

\[p(x, y) = \sum_{n=0}^{\infty} \hat{p}_n(x) \cos(\lambda_n y),
\]
Stokes flow in a channel with permeable boundaries

with eigenvalues satisfying
\[ \lambda_n \sin(\lambda_n \gamma) = A^{n-1} \cos(\lambda_n \gamma). \]  
(20)

We note, and will later use, that equation (20) implies
\[ \pi(n + 1/2)/\gamma > \lambda_n > \pi n/\gamma, \]  
(21)

for \( A > 0. \)

Plugging into the equation [4] and solving, we find
\[ p(x, y) = \sum_{n=0}^{\infty} (c_n \sinh(\lambda_n x) + d_n \cosh(\lambda_n x)) \cos(\lambda_n y). \]  
(22)

We do not, however, know the values of the \( c_n \) or \( d_n. \) We do know however that we must have
\[ p_x(0, 0) = \sum_{n=0}^{\infty} c_n = 1 \]  
(23)

\[ p(0, \pm \gamma) = \sum_{n=0}^{\infty} d_n \cos(\lambda_n \gamma) = B \]  
(24)

We note that in the case that \( \kappa = A = 0, \) we have \( \lambda_0 = 0 \) and the solution for pressure will change to
\[ p(x, y) = c_0 x + \sum_{n=1}^{\infty} c_n \sinh(\lambda_n x) \cos(\lambda_n y) \]  
\[ + d_0 + \sum_{n=1}^{\infty} d_n \cosh(\lambda_n x) \cos(\lambda_n y). \]  
(26)

The typical assumption that leads to Poiseuille flow is that \( c_n = 0 \) for \( n > 0 \) which ensures a linear growth in the pressure as \( x \) goes to \( \pm \infty. \) Indeed, we can see that as \( \kappa \) approaches 0 from above, \( \lambda_n \) approaches \( \pi n/\gamma \) from the right.

Ignoring the \( \cosh \) terms for the moment, it is physically reasonable to assume that for small values of \( \kappa, \) the pressure will be linear in \( x \) about a neighborhood of \( x, \) which will ensure that the solution will approach Poiseuille within this neighborhood. The only way to ensure this is to set \( c_n = 0 \) for \( n > 0 \) as otherwise the pressure would grow exponentially in a uniformly bounded region for all values of \( \kappa. \) With \( c_0 \neq 0, \) however, a Taylor expansion of \( \sinh \) about zero (assuming small \( \lambda_0 \)) shows that the leading order term for the pressure will be linear and dominant, so long as \( x \in (-1/\lambda_0, 1/\lambda_0). \) When redimensionalizing the system, we expect for \( p_x = \beta \) when \( \kappa = 0, \) and we require \( p_x(0, 0) = \beta \) for \( \kappa > 0. \)

For the even component of the pressure a similar argument to the one above may be made. We again consider the case of vanishing permeability in which \( \kappa \) goes to zero. We expect the pressure to approach a constant and the flow to vanish. This is again, only possible about a neighborhood of \( x = 0 \) (ignoring the odd term in this case) if \( d_n = 0 \) for all \( n > 0. \) Furthermore, we require that in redimensionalizing the system, the pressure at the channel walls be equal to \( P_{tm}. \) Thus we find an expression for the non-dimensional pressure to be
\[ p(x, y) = \begin{cases} 
\frac{1}{\lambda_0} \sinh(\lambda_0 x) \cos(\lambda_0 y) + B \cos(\lambda_0 y) ) & \lambda_0 > 0, \\
\frac{1}{\lambda_0} & \lambda_0 = 0.
\end{cases} \]  
(27)

We note that the exponential growth in the pressure is well known and we compare growth coefficients to earlier work in section III A.

We now have a simple equation for \( u \) and \( \bar{v} \) for \( \kappa \neq 0 \)
\[ \left( \cosh(\lambda_0 x) + \frac{B \lambda_0 \sinh(\lambda_0 x)}{\cos(\lambda_0 y)} \right) = A^{2-\alpha} u, \]  
(28)

\[ - \left( \sinh(\lambda_0 x) + \frac{B \lambda_0 \cosh(\lambda_0 x)}{\cos(\lambda_0 y)} \right) = A^{2-\alpha} \bar{v}. \]  
(29)

where \( \bar{v} = v + p_y \) with boundary conditions located at \( \pm \gamma \) and given in equation [17]. We use normal mode analysis to solve both equations below.

1. Solution for \( u \)

As we have assumed \( u \) to be even in \( y, \) we plug in a Fourier series of the form
\[ \sum_{n=0}^{\infty} \bar{u}_n(x) \cos(\omega_n y), \quad \omega_n = \left(\frac{1}{2} + n\right) \pi/\gamma, \]  
(30)

into equation [28] to solve the Poisson equation. Solving the orthogonal mode equations gives
\[ \bar{u}_n(x) = -\frac{d_n \left( \cosh(\lambda_0 x) + \frac{B \lambda_0 \sinh(\lambda_0 x)}{\cos(\lambda_0 y)} \right)}{A^{2-\alpha} (\omega_n^2 - \lambda_0^2)} \]  
\[ + \frac{a_n \cosh(\omega_n x) + b_n \sinh(\omega_n x)}{\omega_n^2 - \lambda_0^2}, \]  
(31)

where the \( a_n \)’s and \( b_n \)’s are unknowns and are solutions to the homogenous flow equations where \( \beta = P_{tm} = 0. \) The coefficients \( d_n \) arise from Fourier expanding \( \cos(\lambda_0 y) \) and are found to be
\[ d_n = \frac{\int_{-\gamma}^{\gamma} \cos(\omega_n y) \cos(\lambda_0 y) dy}{\int_{-\gamma}^{\gamma} \cos(\omega_n y)^2 dy} = \frac{2(-1)^n \omega_n \cos(\lambda_0 \gamma)}{\gamma (\omega_n^2 - \lambda_0^2)}. \]  
(32)

We expect that there is no flow when the pressure gradient \( \beta \) and the pressure across the channel \( B (\propto P_{tm}) \) go to zero meaning that the homogenous solutions are zero and \( a_n = b_n = 0 \) for all \( n \) in this limit. The non-imensional parameter \( A \) does not depend either on \( \beta \) or \( P_{tm} \) and thus we set \( a_n = b_n = 0. \)

2. Solution for \( \bar{v} \)

As we have assumed \( u \) to be even in \( y, \) we plug in a Fourier series of the form
\[ \sum_{n=0}^{\infty} \bar{u}_n(x) \sin(\bar{\omega}_n y), \quad \bar{\omega}_n = n \pi/\gamma, \]  
(33)
into equation [29] to solve the Poisson equation. Solving the orthogonal mode equations gives

\[
\hat{v}_n(x) = \frac{d_n \left( \sinh(\lambda_0 x) + \frac{B \lambda_0}{\cos(\lambda_0 \gamma)} \cosh(\lambda_0 x) \right)}{A^{2-\alpha}(\tilde{\omega}_n^2 - \lambda_0^2)} + a_n^\alpha \sinh(\tilde{\omega}_n x) + b_n^\alpha \sinh(\tilde{\omega}_n x),
\]

(34)

where the \(a_n^\alpha\)'s and \(b_n^\alpha\)'s are unknowns that are solutions to the homogenous flow equations where \(\beta = \bar{P}_{in} = 0\). The coefficients \(d_n\) arise from Fourier expanding \(\cos(\lambda_0 y)\) and are found to be

\[
\tilde{d}_n = \frac{\int_{-\gamma}^{\gamma} \sin(\tilde{\omega}_n y) \sin(\lambda_0 y) dy}{\int_{-\gamma}^{\gamma} \sin(\tilde{\omega}_n y)^2 dy} = \frac{2(-1)^{n+1} \tilde{\omega}_n \sin(\lambda_0 \gamma)}{\gamma(\tilde{\omega}_n^2 + \lambda_0^2)},
\]

Therefore in order for the incompressibility criteria to be satisfied, we need

\[
A^{2-\alpha} = \sum_{n=0}^{\infty} \left( \frac{d_n}{\tilde{\omega}_n^2 - \lambda_0^2} \cos(\tilde{\omega}_n y) - \frac{\tilde{d}_n \tilde{\omega}_n}{\lambda_0 \tilde{\omega}_n^2 - \lambda_0^2} \cos(\lambda_0 y) \right),
\]

(39)

which will only have a solution if the righthand side of the equation has no dependence on \(y\). Indeed it does not and we show this in proposition [4] in appendix [A]. In the proposition we also simplify equation [39] to read

\[
A^{2-\alpha} = \frac{1}{2} \left( \frac{\gamma}{\sin(\lambda_0 \gamma) \cos(\lambda_0 \gamma)} - \frac{1}{\lambda_0} \right) \equiv C(\lambda_0),
\]

(40)

where we have used equation [A7]. We can further simplify this expression by dividing equation [40] with equation [20] to eliminate \(\alpha\) and relate \(A\) to \(\lambda_0\). This leads to the expression

\[
A = \frac{1}{2} \left( \frac{\gamma}{\cos^2(\lambda_0 \gamma)} - \frac{\sin(\lambda_0 \gamma)}{\lambda_0 \cos(\lambda_0 \gamma)} \right).
\]

(41)

The righthand side of this equation is positive and increasing for \(\lambda_0 \in (0, \pi(2\gamma)^{-1})\) and has range \((0, \infty)\) (see proposition [4] in the appendix and figure [1]). Therefore there is a one-to-one correspondence between \(\lambda_0 \in (0, \pi(2\gamma)^{-1})\) and \(A \in (0, \infty)\). To find the necessary value of \(\alpha\) to enforce incompressibility, we substitute the above expression for \(A\) into equation [20] which leads to an expression for \(\alpha\):

\[
\alpha = \frac{\log(\lambda_0 \tan(\lambda_0 \gamma))}{\log(A)} + 1
\]

(42)

3. Enforcing incompressibility

To enforce that equations [31] and [32] provide a solution to the original non-dimensional equations [9] and [11] we must check that the flow field is divergence free given a proper choice of \(\alpha\) and \(\gamma\). Taking partial derivatives, we find

\[
u_y = \frac{\lambda_0 \sinh(\lambda_0 x) + \frac{B \lambda_0}{\cos(\lambda_0 \gamma)} \cosh(\lambda_0 x)}{A^{2-\alpha}} \sum_{n=0}^{\infty} \frac{d_n}{\tilde{\omega}_n^2 - \lambda_0^2} \cosh(\tilde{\omega}_n x) \cos(\lambda_0 y) + \frac{\lambda_0 \sinh(\lambda_0 x) + \frac{B \lambda_0}{\cos(\lambda_0 \gamma)} \cosh(\lambda_0 x)}{A^{2-\alpha}} \sum_{n=0}^{\infty} \frac{\tilde{d}_n}{\tilde{\omega}_n^2 - \lambda_0^2} \sinh(\tilde{\omega}_n x) \cos(\lambda_0 y),
\]

Similarly as for \(u\), we set \(a_n^\alpha = b_n^\alpha = 0\) for all \(n\).

![FIG. 1. \(A_1\) and \(\gamma\) as functions of \(\lambda\). Note that there is a one-to-one correspondence between \(A_1\) and \(\lambda\) and between \(\gamma\) and \(\lambda\).](image-url)

Although we have found a solution to the original non-dimensional problem (equations [9],[11]), there is a potential singularity in \(\alpha\) for \(A = 1\). This should not be surprising as we lose the ability to scale the left hand side of equation [10] in this case. We next simplify \(\alpha\) to eliminate the potential singularity.
4. Simplifying $\alpha$

We begin by letting $\lambda = \lambda_0 \gamma$ so that $\lambda \in (0, \pi/2)$, and will use $A_1 = A/\gamma = \mu k / r$, as it is defined above. Equation 20 may then be rewritten as

$$\lambda \sin(\lambda) = A_1^{\alpha-1} \gamma^\alpha \cos(\lambda), \quad (43)$$

equation 41 as

$$A_1 = \frac{1}{2} \left( \frac{1}{\cos^2(\lambda)} - \frac{\sin(\lambda)}{\lambda \cos(\lambda)} \right), \quad (44)$$

and equation 42 as

$$\alpha = \frac{\log(\lambda \tan(\lambda)) + \log(A_1)}{\log(\gamma) + \log(A_1)}, \quad (45)$$

from which we can derive an equation for $\gamma$ in terms of $\alpha$ as

$$\gamma = (\lambda \tan(\lambda) A_1^{-\alpha})^{1/\alpha}. \quad (46)$$

Equation 44 demonstrates that the rate of pressure gain per unit channel length will be independent of $\gamma$, as expected. Equation 45 demonstrates that there will be an essential singularity in $\alpha$ located at $\gamma = 1/A_1$ (corresponding to $A = 1$). A natural way to handle this issue is to set

$$\alpha = 1, \quad (47)$$

which fixes

$$\gamma = \lambda \tan(\lambda), \quad (48)$$

and makes the singularity irrelevant. We denote this special value of $\gamma$

$$\gamma = \lambda \tan(\lambda). \quad (49)$$

We remark again that it is not known a priori that $\alpha = 1$ leads to a convenient non-dimensionalization and have thus left it free until this point. We further remark that we could have chosen $\alpha$ to be any other fixed value and obtain the expression form equation 40, however this equation also makes it clear that $\alpha = 1$ is the aesthetically pleasing choice; this choice would not have been obvious had we chosen $\alpha$ from the start. We plot $A_1$ and $\gamma$ verses $\lambda$ in figure 1.

Taking into account the above simplifications, we rescale $x$ and $y$ by $\gamma^{-1}$ and write the full non-dimensional solution in terms of $B_1$ (defined above as $B_1 = P_{tm}/\beta r$, $\lambda$, $x$ and $y$ as

$$p(x, y) = \left( \tan(\lambda) \sinh(\lambda x) + \frac{B_1 \lambda \tan(\lambda)}{\cos(\lambda)} \cosh(\lambda y) \right) \cos(\lambda y), \quad (50)$$

$$u(x, y) = \frac{32 \lambda^2 \sin(\lambda)}{\pi^3 (\lambda \sec^2(\lambda) - \tan(\lambda))} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (1 + 2n)}{(1 + 2n)^2 - \left(\frac{2 \lambda}{\gamma}\right)^2} \cos \left( \left(\frac{1 + 2n}{2}\right) \gamma y \right), \quad (51)$$

$$v(x, y) = \frac{4 \lambda^2 \cos(\lambda) \sin^2(\lambda)}{\pi^3 (\lambda - \sin(\lambda) \cos(\lambda))} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(n^2 - \left(\frac{\lambda}{\gamma}\right)^2) \gamma} \sin(\pi ny) \sin(\lambda y). \quad (52)$$

The domain of the rescaled $y$ is now $(-1, 1)$. We remark again that we have a bijection between $A_1$ and $\lambda$ through a transcendental equation so that knowing $A_1$ will provide us with the correct choice for $\lambda$. This solution represents a closed form solution for Stokes flow through a channel with uniformly permeable walls but having arbitrarily large permeability. This is the major result of the paper in that all other results presented below are simple corollaries that arise from it. We demonstrate the streamlines of this solution for $\lambda = \pi/4$ with $B_1 = 0$ in figure 2.

As a tool for the prediction for fluid flow, our result is exact within the Stokes regime, however there are two caveats. The first is that in nature and physical application, permeability (i.e. $A_1$ or $\lambda$) is found to be small and there has been a great deal of asymptotic work done in the realistic parameter regime. While our work contributes to this body of work it is not clear how significant the contribution will be. We do note, however, that below we will present a physical set up that will theoretically display arbitrarily large permeabilities. Next, we note that our solution contains exponential increase in the axial direction for both the velocity and pressure profiles.

Finally we remark on the relationship between the average flow velocity through $x = 0$ and the point gradient
noting that as permeability decreases the behavior of $\lambda$ as a function of $A_1$ which may be found by noting that as permeability decreases $A_1$, $\bar{\gamma}$ and $\lambda$ all go to zero. This may be shown via the definition of $A_1$ and equations \ref{A1} and \ref{gamma}. In this limit we can approximate $A_1$ and $\bar{\gamma}$ by Taylor expanding equations \ref{A1} and \ref{gamma}

$$A_1 = \frac{\lambda^2}{3} + O(\lambda^4) \Rightarrow \lambda \approx \sqrt[3]{3A_1},$$

(55)

$$\bar{\gamma} = \lambda^2 + O(\lambda^4) \Rightarrow \bar{\gamma} \approx 3A_1.$$  

(56)

### A. Axial pressure profile

We first compare the pressure profile along the axial direction with several other studies in the literature. We note that in the limit of small permeability, $\lambda \to 0$, and thus the pressure profile appears to be constant in $y$ as the leading order of $\cos(\lambda y)$ is 1. This is what is found in Haldenwang

2, Karode

10, Tilton et al.\textsc{12}, Regirer\textsc{16} to leading order. Each of these works shows a similar profile in $x$ to the pressure profile we have presented, namely a linear combination of a hyperbolic sine and cosine. The scale of the arguments of these hyperbolic trig functions presented in Karode\textsc{10} and Tilton et al.\textsc{12} is $3A_1$ which we have shown is roughly $\lambda$ in the limit of small permeability \textsc{55}. The non-dimensional approximation for the pressure profile is in agreement in Karode\textsc{10} and Tilton et al.\textsc{12} and is given as

$$P(x, y) = \sqrt{3A_1} \sinh(\sqrt{3A_1}x) + 3A_1 B_1 \cosh(\sqrt{3A_1}x),$$

(57)

where we will use a capital $P$ to denote the asymptotic approximation. In Regirer\textsc{16}, a solution for pipe rather than channel flow is performed, however we have confirmed that an identical result is found when the methodology is applied to channel flow. To see this quickly we note that Regirer\textsc{16} performs an asymptotic expansion about small Reynolds number in which the permeability is assumed to be small and we discuss this further below. We also note that Haldenwang

2 develops the solution of Regirer\textsc{16} before extending the work to non-zero transverse Reynolds number and therefore we have a similar result for all four works.

It is easy to see that equation \ref{57} approximates equation \ref{55} in the limit of small permeability through use of equation \ref{55} and appropriately Taylor expanding $\tan(\lambda)$ and $\cos(\lambda)$ about $\lambda = 0$. We may ask now, how well the prediction for axial pressure increase is approximated in the limit of small permeability noting that we will have exponential increase like $\exp(3A_1)$ rather than $\exp(\sqrt{3A_1})$. It is natural then examine the difference between $\lambda$ and $\sqrt{3A_1}$ in order to obtain how the exponential rate increase will be changed with $\lambda$. We plot the relative error given as

$$\frac{\lambda - \sqrt{3A_1}}{\lambda},$$

(58)

in figure\textsc{8}. We may numerically determine the linear behavior of the relative error about $A_1 = 0$ and find that

\[\text{FIG. 2. Streamlines of channel flow for the rescaled solution presented in equations \ref{51} and \ref{52} in the domain } (x, y) \in [-2, 2] \times [-1, 1]; \text{ in this system } \lambda = \pi/4 \text{ (corresponding to } A_1 = 0.36338) \text{ and } B_1 = 0.\]
for small \( A_1 \), we have that the relative error is roughly 1.124\( A_1 \). The error in the asymptotic limit for large \( x \) may then be approximated as the difference between exponentials and we thus have that the relative error between the asymptotic approximation and the actual solution at zero Reynolds number is given as

\[
\frac{|p(x, y) - P(x, y)|}{|p(x, y)|} \leq C_1|1 - \exp(-1.124A_1x)| \quad (59)
\]

for \( x \) large and \( A_1 \) small. This estimate is novel. We note that there is a great deal of interest in the length of a pipe over which a solution may be valid. Our result suggests that the existing asymptotic expansion about small Reynolds number will only be valid over axial domains that have length significantly less than \((-1.124A_1)^{-1}\) in non-dimensional units, however we note the caveat that the Reynolds number may not continue to be small within this regime. We note furthermore that the magnitude of the velocity in \( v \) and \( u \) along the axial direction is directly related to that of the pressure, and thus a similar statement may be made for the relative error in these profiles.

### B. Transverse velocity profile

We note that in the existing works found in the literature, leading order behavior in the profile for \( u(0, x) \) is found to be parabolic and a polynomial expression is found for the leading order behavior for \( v(0, x) \). We note that the solution in \( u \) described in equation (51) may be seen to be parabolic in \( y \) by Taylor expanding each Fourier coefficient about \( \lambda = 0 \) and noting that the leading order \( (O(\lambda^0)) \), the coefficients are that of \( F(x)(1-y^2) \) in the chosen basis. Similarly we may find that the leading order profile in \( v \) may be written as \( G(x)(y^2 - 3y) \) which may be found by noting that the two terms (the sum and \( \sin(\lambda y) \)) may be found in the leading order to be

\[
v(x, y) \approx G(x)\lambda(y - y^3) + 2G(x)\lambda y = \lambda(3y - y^3). \quad (60)
\]

We further remark that this is precisely the leading order expression found in Berman and Tilton et al. [12].

Having verified the leading order behavior in the limit of small permeability, we may now ask how the asymptotic approximation breaks down in the transverse velocity profiles \( u(\cdot, y) \) and \( v(\cdot, y) \). We note that both the solution presented in the present work and the solutions to the asymptotic approximations in the literature provide self similar profiles so that we may concern ourselves purely with the prediction of the shape in the profile.

We begin with the profile in \( u \) which is proportional to the profile

\[
u(\cdot, y) \propto \sum_{n=0}^{\infty} \frac{(-1)^n+1(1+2n)}{(1+2n)^2 - (\frac{2A_1\lambda}{\pi})^2} \cos\left(\frac{\pi y}{2} + \pi n y\right)\quad (61)
\]

We may then examine the relative error between our predicted profile and a parabolic profile global error in the following sense. First, we set \( B = 1/u(0,0) \) and scale \( u \) by \( B \) so that the profiles agree at \( y = 0 \). Next, we calculate the relative error between \( (1-y^2) \) and \( B \times u(0, y) \) via

\[
\epsilon_p(\lambda) = \frac{||(1-y^2) - B \times u(\cdot, y)||_p}{||B \times u(\cdot, y)||_p}, \quad (62)
\]

where \( || \cdot ||_p \) is the \( L^p([0,1]) \) norm; we consider the error for \( p \in \{2, \infty\} \). We calculate this error numerically and visualize it in figure [3]. In the \( L^\infty \) norm we find that the relative error increases to over 5% in the limit of large permeability. In the \( L^2 \) norm we find that the relative error increases to over 0.3% in the limit of large permeability. We plot the profile at large permeability and compare it with the parabolic profile in [3]. We also note that our error bounds could be even tighter if we redefine the relative error to be

\[
\epsilon_p^L(\lambda) = \inf_B \frac{||(1-y^2) - B \times u(0, y)||_p}{||B \times u(0, y)||_p}. \quad (63)
\]

The result is that we have shown the behavior of how the break down in the theory of small permeability breaks down once permeability can no longer considered to be small. Although the error is small, the parabolic profile does break down in this limit.

Next we wish to examine the profile in \( v \) by analyzing the self similar profile \( v(\cdot, y) \). In order to show the difference between the asymptotic approximation and our solution, we again consider the relative error between the asymptotic profile and our profile, rescaling so that the incoming velocity is equivalent in each case; this is to say we compare \((3y - y^3)/3\) with \(v(\cdot, y)/v(\cdot, 1)\). We then analyze the relative error and plot the results in figure [5]. We find that The error achieves a maximum not in the

**FIG. 3.** We plot the relative error between the exponential scaling \( \lambda \) and \( \sqrt{A_1} \). The later result is a prediction of the asymptotic theory and is seen to be the first order term in a Taylor expansion for \( \lambda \) about \( \lambda = 0 \).
FIG. 4. The relative error defined by equation (62) is plotted as a function of $\lambda$ for the transverse profiles of $u$ and $v$ (top). The flow profiles in $u$ are compared between parabolic flow and the present work for $\lambda = 0.9999 \times \pi/2$; the error is plotted for comparison (bottom).

limit of large permeability but at an intermediate permeability corresponding to $\lambda = 1.0105$ or $A_1 = 0.98$. The maximum relative error is 0.8% and 8% respectively in the two and infinity norms which is more notable than the error in the profile for $u$.

C. Axial flow exhaustion and crossflow reversal

Axial flow exhaustion occurs when the axial velocity profile changes sign. Physically this occurs because flow is either being injected or suctioned out of the channel with a high enough pressure difference or small enough average axial flow (i.e. large $B_1$) to break down axial flow across the channel at a point, corresponding to $u(x_{AFE}, y) = 0$. Crossflow reversal is described by a change of sign in the pressure, and hence velocity profile $v$, in the axial direction. Physically this is described by the channel walls transitioning from fluid suction to fluid injection or vice-versa. In this section we compare the predictions of where, and whether axial flow exhaustion or crossflow reversal occurs by comparing the leading order asymptotic analysis of Tilton et al. with our model. Axial exhaustion will take place when there is a change of sign in the $u$ profile along $x$ and find that the location of axial flow exhaustion, denoted $x_{AFE}$, will occur at

$$\frac{1}{\sqrt{A_1 B_1}} \arctanh \left( -\frac{1}{B_1 \sqrt{A_1}} \right), \quad (64)$$
$$\frac{1}{B_1 \lambda} \arctanh \left( -\cos(\lambda) \right), \quad (65)$$

for the asymptotic theory and the present work respectively. We can also determine the location of crossflow reversal by finding the location of the sign change in either $v$ or $p$ in the axial direction. We denote the location of crossflow reversal as $x_{CFR}$ and find

$$\frac{1}{\sqrt{A_1 B_1}} \arctanh \left( -B_1 \sqrt{A_1} \right), \quad (66)$$
$$\frac{1}{B_1 \lambda} \arctanh \left( -\frac{B_1}{\cos(\lambda)} \right), \quad (67)$$

in the asymptotic and current theory respectively. As is noted in Haldenwang, the two regimes are mutually exclusive since the arguments of the arctanh function for $x_{AFE}$ and $x_{CFR}$ are reciprocals in both theories. We may therefore describe the predicted transition location in parameter space as

$$\left| \frac{1}{B_1 \sqrt{A_1}} \right| = 1, \quad \left| \frac{\cos(\lambda)}{B_1 \lambda} \right| = 1 \quad (68)$$

in the respective theories, and note that we may re-express this relationship as a prediction for the magnitude of $B_1$ as a function of $\lambda$. We can then examine the relative error in the prediction for the required magnitude of $B_1$, as a function of $\lambda$, in which a transition will occur. We then analyze the relative error in the transition value $B_1$ between where axial flow exhaustion occurs and where it does not, is given as

$$\left| \frac{\cos(\lambda)/\lambda - (3A_1)^{-1/2}}{\cos(\lambda)/\lambda} \right|^2. \quad (69)$$

We find that the relative error for the cut off condition on $B_1$ grows with $\lambda$ and that for permeability the error grows to be over 25% (see figure 3). Finally we examine the error in the prediction for the location of $x_{AFE}$ and $x_{CFR}$. We note that as $B_1$ approaches the transition value, that the location of $x_{AFE}$ and $x_{CFR}$ goes to infinity in both the asymptotic and our prediction. To make a demonstrative comparison we therefore fix a value of $B_1 = -1/2$ and compare the predictions of $x_{AFE}$ and $x_{CFR}$ in figure 4. We find that for large values of $\lambda$ the error is of order 1 near the transition regions which further demonstrates the break down of the asymptotic prediction. We remark that when $\lambda$ is small we expect that $B_1$ must be large in order for axial flow exhaustion to occur which corresponds either to large values of transmembrane pressure $P_{lm}$ or small values of the pressure gradient $\beta$ or, equivalently, average axial velocity profile $U$. 

Stokes flow in a channel with permeable boundaries
IV. Stokes Flow in the Context of an
Asymptotic Expansion about Low Reynolds Number

The Navier-Stokes equations, for our physical set up under the assumption of laminar flow, may be written as

\[ p(x_0, \pm r) = \mathcal{B}, \quad p_x(x_0, 0) = 1, \]
\[ \rho(uu_x + vv_y) = -p_x + A\Delta u, \]
\[ \rho(uu_x + vv_y) = -p_y + A\Delta v, \]
\[ 0 = u_x + v_y, \]
\[ 0(x, \pm r) = 0, \quad v(x, \pm r) = \pm \kappa p, \]

where \( p \) is the density of the fluid. With this, we non-dimensionalize the problem in the same way as above, with \( \alpha = 1 \), we obtain the non-dimensional set of equations (dropping tildes)

\[ p(x_0, \pm r) = \mathcal{B}, \quad p_x(x_0, 0) = 1, \]
\[ \text{Re}(uu_x + vv_y) = -p_x + A\Delta u, \]
\[ \text{Re}(uv_x + vv_y) = -p_y + A\Delta v, \]
\[ 0 = u_x + v_y, \]
\[ 0(x, \pm r) = 0, \quad v(x, \pm r) = \pm p, \]

where the Reynolds number is defined as \( \text{Re} = \rho \kappa^2 \beta r / \gamma^2 \). We may consider an asymptotic expansion about low Reynolds number as is done by Bernales and Haldenwang, Regirer and others, however we do not assume that the permeability is negligible in the zeroth order term. The resulting zeroth order term is identical to the system of equations (80-12). It is possible to seek to determine higher order terms in such an asymptotic expansion, however we have the added difficulty in that there is no separation of scales between pressure and velocity. At each step in the expansion we will have to solve the system of equations

\[ p(x_0, \pm r) = \mathcal{B}, \quad p_x(x_0, 0) = 1, \]
\[ u_{n-1}u_x + u_nu_{n-1}x + v_{n-1}u_y + v_nu_{n-1}y = -p_{n_x} + A\Delta u_n, \]
\[ u_{n-1}v_x + u_nv_{n-1}x + v_{n-1}v_y + v_nv_{n-1}y = -p_{n_y} + A\Delta v_n, \]
\[ 0 = u_{n_x} + v_{n_y}, \]
\[ u_n(x, \pm r) = 0, \quad v_n(x, \pm r) = \pm p_n, \]

where we have taken the expansion to be written as

\[ u \sim u_0 + \text{Re}u_1 + \ldots \]
\[ v \sim v_0 + \text{Re}v_1 + \ldots \]
\[ p \sim p_0 + \text{Re}p_1 + \ldots \]

As \( x \) becomes large in magnitude we expect the velocity terms to grow exponentially and inertial terms begin to dominate. Because of this we caution that the zeroth order term (the Stokes flow in equations 50-52) is not a good physical model for the infinite channel. However work by Brady shows that inlet profiles eventually become irrelevant for low Reynolds number flows. Therefore it is reasonable to conjecture that error away from viscous regimes in the pipe will be irrelevant about a neighborhood in which viscous forces dominate the flow for particular parameters. Further study is required to determine the size of these neighborhoods and their dependency on the choice of non dimensional parameters \( A_1, B_1 \) and \( \text{Re} \).
V. DISCUSSION

We have considered Stokes flow through an infinite channel with permeable walls, such that fluid may be driven by pressure differences across the channel wall to enter or exit the channel. The normal fluid component is given by Darcy’s law, whereas the tangential component is assumed to be zero (i.e., no-slip boundary condition). The novel element to our work is that the permeability may be of arbitrary magnitude. Such a solution allows us to directly and analytically test the break down of existing asymptotic theories at small Reynolds number and we have demonstrated how the theory breaks down in terms of the axial magnitude of the pressure, the transverse velocity profiles, and as a predictive tool for axial flow exhaustion and crossflow reversal. We note that our solution is equivalent to examining the case in which transverse velocity is no longer negligible when compared to the axial velocity profile. We have also contextualized our work in terms of an asymptotic expansion about small Reynolds number. Although we have not attempted to determine the higher order corrections, we note that if such corrections could be made, the theory of $\lambda$ would be extended to a far broader region of the non-dimensionalized parameter space.

In addition to acting as a tool with which to compare the asymptotic analysis, our solution also extends the known analytic results for a wider class of parameters than has been originally explored. The utility of such an exploration is, as of yet, to be seen, as most values for $\lambda_1$ found in nature may be considered to be significantly less than one. We do however note that our result may be useful in setting inlet and outlet boundary conditions for numerical studies. In the numerical study by Pozrikidis, the author studies fluid loss in capillaries. Boundary conditions are set by proposing a parabolic profile and then allowing a region of pipe with zero permeability to transition into a region with non-zero permeability. It was shown that the pressure profile spikes upon this transition and thus in a study involving the concentration of a secondary passive scalar we may see unphysical loss at these regions. Furthermore, there is expense involved in setting these boundary conditions. As a potential remedy, our flow profiles may act as inlet/outlet profiles that do not cause unrealistic spikes in the pressure profiles and may also reduce the number of grid points needed at a boundary.

Next, we propose a theoretical framework where the permeability may not be a small value with the following thought experiment. We imagine the channel wall to be a series of coupled, lubricated cylinders with small spacing as pictured in figure 6. A pressure difference across the channel would then cause flow across the membrane which would rotate the cylinders; a frictional coefficient on the cylinders would determine the relationship between the pressure and the outward velocity that would be significantly larger than if the cylinders were fixed due to the fact that we circumvent the frictional restrictions of the zero slip condition. The practicality, realizability and utility of such system are all left as open questions in the present work.

ACKNOWLEDGEMENTS

This research was supported by the National Institutes of Health, National Institute of Diabetes and Digestive and Kidney Diseases, grant DK089066, to A. Layton; by the National Science Foundation, grant DMS1263995, to A. Layton; Research Training Groups grant DMS0943760 to the Mathematics Department at Duke University, and Research Network in the Mathematical Sciences grant DMS1107444 to KI-Net. We would like to acknowledge Thomas P. Witelski for fruitful discussions and generously editing and critiquing our manuscript.

Appendix A

Proposition 1. The quantity

$$\sum_{n=0}^{\infty} \left( \frac{d_{n}}{\omega_n^2 - \lambda_0^2} \cos(\omega_n y) - \frac{\bar{d}_{n}\bar{\omega}_n}{\lambda_0(\bar{\omega}_n^2 - \lambda_0^2)} \cos(\bar{\omega}_n y) \right),$$

(A1)

from equation 32, is independent of $y$ for all values of $\lambda_0 \in (0, \pi/2)$, given by equation 27.

To prove proposition A we first let $F(\lambda_0, y) = \sum_{n=0}^{\infty} \left( \frac{d_{n}}{\omega_n^2 - \lambda_0^2} \cos(\omega_n y) - \frac{\bar{d}_{n}\bar{\omega}_n}{\lambda_0(\bar{\omega}_n^2 - \lambda_0^2)} \cos(\bar{\omega}_n y) \right)$.

(A2)

We then claim that $F(\lambda_0, y) = C(\lambda_0) \cos(\lambda_0 y)$ which will demonstrate the loss of $y$ dependence. To show this we demonstrate the equivalence of the Fourier modes by finding the correct scaling $C(\lambda_0)$. This is equivalent to showing

$$\langle \cos(k\pi y), F(\lambda_0, y) \rangle = C(\lambda_0) \langle \cos(k\pi y), \cos(\lambda_0 y) \rangle,$$

(A3)

for $k \in \mathbb{N}$, with the inner product defined as usual to be

$$\langle f(y), g(y) \rangle = \int_{-\gamma}^{\gamma} f(y)g(y)dy.$$

(A4)
For equation $\text{A3}$ to be true, the $k = 0$ case implies that we must have

$$
\sum_{n=0}^{\infty} \frac{32\gamma^2 \cos(\lambda_0 \gamma)}{\pi^4 \left(1 + 2k^2 - \left(\frac{2\lambda_0 \gamma}{\pi}\right)^2\right)^2} = C(\lambda_0) \frac{\sin(\lambda_0 \gamma)}{\lambda_0 \gamma},
$$

which we have derived by integrating equation $\text{A3}$. We simplify the left hand side via the identity

$$
\sum_{n=0}^{\infty} \frac{1}{(1 + 2k^2 - x^2)^2} = \frac{\pi^2 \sec^2 \left(\frac{\pi x}{2}\right)}{16x^2} - \frac{\pi \tan \left(\frac{\pi x}{2}\right)}{8x^3},
$$

and then solve for $C(\lambda_0)$, which leads to the condition

$$
C(\lambda_0) = \frac{1}{2} \left(\frac{\gamma}{\sin(\lambda_0 \gamma) \cos(\lambda_0 \gamma) \lambda_0} - \frac{1}{\lambda_0^5}\right). \tag{A7}
$$

For $k > 0$ we are left to verify that

$$
\sum_{n=0}^{\infty} \frac{64(-1)^k \gamma^2 \cos(\lambda_0 \gamma)}{(1 + 2n)^2 - 4k^2} \left(1 + 2n^2 - \left(\frac{2\lambda_0 \gamma}{\pi}\right)^2\right)^2 = C(\lambda_0) \frac{2(-1)^k \lambda_0 \gamma \sin(\lambda_0 \gamma)}{-k^2 \pi^2 + \lambda_0^2 \gamma^2}. \tag{A8}
$$

where we have again used $\text{A3}$ to derive this formula. The sum on the left hand side may be reduced via the identity

$$
\frac{\pi^2}{4} \left(4k^2 - \lambda^2\right) \sec \left(\frac{\pi \lambda}{2}\right)^2 + \pi \left(4k^2 + \lambda^2\right) \tan \left(\frac{\pi \lambda}{2}\right)^2 = \sum_{n=0}^{\infty} \frac{(1 + 2n)^2}{(4k^2 - (1 + 2n)^2)((1 + 2n)^2 - \lambda^2)^2}. \tag{A9}
$$

by substituting $\lambda = \frac{2\lambda_0 \gamma}{\pi}$. This leads to an algebraic expression that we have verified to be valid, however we have omitted the details as it leads to a lengthy reduction. This completes the proof of proposition $\text{A1}$. 

\begin{proposition}
The relationship between $A$ and $\lambda_0$, given in equation $\text{A1}$, 

$$
f(\lambda_0) = \frac{1}{2} \left(\frac{\lambda_0 \gamma}{\sin(\lambda_0 \gamma)} - \frac{\cos(\lambda_0 \gamma)}{\lambda_0 \cos(\lambda_0 \gamma)}\right), \tag{A10}
$$

with $f : (0, \pi(2\gamma)^{-1}) \to (0, \infty)$ is bijective.

To prove this we first note that the function $f$ is continuous in $\lambda_0$. We then need to show that

$$
\lim_{\lambda_0 \to 0} f(\lambda_0) = 0, \quad \lim_{\lambda_0 \to \pi(2\gamma)^{-1}} f(\lambda_0) = \infty, \tag{A11, A12}
$$

and that $f$ is monotonically increasing. The first limit can be seen by noting that $\lim_{\lambda_0 \to 0} \sin(\lambda_0 \gamma) = 0$. In the second limit, the first term of $f$ dominates and is unbounded from above. To show that the function is monotonically increasing we first let $\lambda_0 \gamma = \lambda$ and then show that

$$
\frac{df(\lambda)}{d\lambda} > 0, \tag{A13}
$$

which will be true so long as

$$
2 \sec^2(\lambda) \tan(\lambda) + \tan(\lambda) \lambda^{-2} - \sec^2(\lambda) \lambda^{-1} > 0. \tag{A14}
$$

In the limit as $\lambda \to 0$ is zero for both sides of equation $\text{A15}$. Therefore it suffices to show that

$$
\frac{d(2 \tan(\lambda) \lambda^2 + \sin(\lambda) \cos(\lambda))}{d\lambda} > \frac{d\lambda}{d\lambda},
$$

which is true since

$$
2\lambda \sin(\lambda) \cos(\lambda) + \lambda^2 > \sin^2(\lambda) \cos^2(\lambda), \tag{A17}
$$

for $\lambda \in (0, \pi/2)$. This completes the proof of proposition $\text{A2}$. 

\begin{footnotesize}
1. A. S. Berman, “Laminar flow in channels with porous walls,” J. Appl. Phys. 24, 1232–1235 (1953).
2. S. W. Yuan and A. B. Finkelstein, “Laminar pipe flow with injection and suction through a porous wall,” Trans. Am. Soc. Mech. 78, 719–724 (1978).
\end{footnotesize}
Stokes flow in a channel with permeable boundaries

3R. M. Terrill, “Laminar flow in a uniformly porous channel,” Aeronaut. Q. 15, 299–310 (1964).
4R. Terrill and G. Shrestha, “Laminar flow through parallel and uniformly porous walls of different permeability,” Z. Angew. Math. Phys. 16, 470–482 (1966).
5L. S. Galowin, L. S. Fletcher, and M. J. DeSantis, “Investigation of laminar flow in a porous pipe with variable wall suction,” AIAA J. 12, 1585–1589 (1974).
6J. Granger, J. Dodds, and N. Midoux, “Laminar flow in channels with porous walls,” Chem. Engin. J. 42, 193–204 (1989).
7P. Haldenwang, “Laminar flow in a two-dimensional plane channel with local pressure-dependent crossflow,” Euro. J. Mech. B/Fluids 593, 463–473 (2007).
8P. Haldenwang and P. Guichardon, “Pressure runaway in a 2d plane channel with permeable walls submitted to pressure-dependent suction,” Euro. J. Mech. B/Fluids 30, 177–183 (2011).
9B. Bernales and P. Haldenwang, “Laminar flow analysis in a pipe with locally pressure-dependent leakage through the wall,” Euro. J. Mech. B/Fluids 43, 100–109 (2014).
10S. K. Karode, “Laminar flow in channels with porous walls, revisited,” J. Membrane Sci. 191, 237–241 (2001).
11J. F. Brady, “Flow development in a porous channel and tube,” Phys. Fluids 27, 1061–1067 (1984).
12N. Tilton, D. Martinand, E. Serre, and R. Lueptow, “Incorporating darcy's law for pure solvent flow through porous tubes: asymptotic solution and numerical simulations,” AIChE J. 58, 230–244 (2012).
13C. Pozrikidis, “Stokes flow through a permeable tube,” Arch. of appl. mech. 80, 323–333 (2010).
14J. P. Pennell, F. B. Lacy, and R. L. Jamison, “An in vivo study of the concentrating process in the descending limb of the hene’s loop,” Kidney International 5, 337–347 (1974).
15M. A. Knepper, R. A. Danielson, G. M. Saidel, and R. S. Post, “Quantitative analysis of renal medulary anatomy in rats and rabbits,” Kidney International 12, 313–323 (1977).
16S. A.Regirer, “On the approximate theory of the flow of a viscous incompressible liquid in a tube with permeable walls,” Zhurnal Tekhnicheskoi Fiziki 30, 639–643 (1960).
17K. Damak, A. Ayadi, B. Zeghmaitib, and P. Schmitz, “On physically similar systems; illustrations of the use of dimensional equations,” Desalination 161, 67–77 (2004).
18S. Tsangaris, D. Kondaxakis, and N. Vlachakis, “Exact solution for flow in a porous pipe with unsteady wall suction and/or injection,” Comm. in Nonlinear Sci. and Num. Sim. 12, 1181–1189 (2007).
19A. Pak, T. Mohammadi, S. Hosseinalipour, and V. Allahdini, “Cfd modeling of porous membranes,” Desalination 222, 482–488 (2008).
20I. Borsi, A. Farina, and A. Fasano, “Incompressible laminar flow through hollow fibers: a general study by means of a two-scale approach,” Z. Angew. Math. Phys. 62, 681–706 (2011).