On the Reliability Function of the Common-Message Broadcast Channel with Variable-Length Feedback

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Abstract

We derive upper and lower bounds on the reliability function for the common-message discrete memoryless broadcast channel with variable-length feedback. We show that the bounds are tight when the broadcast channel is stochastically degraded. For the achievability part, we adapt Yamamoto and Itoh’s coding scheme by controlling the expectation of the maximum of a set of stopping times. For the converse part, we adapt Burnashev’s proof techniques for establishing the reliability functions for (point-to-point) discrete memoryless channels with variable-length feedback and sequential hypothesis testing.

Index Terms

Variable-length feedback, Reliability function, Error exponent, Broadcast channel, Stochastic degradation

I. INTRODUCTION

Shannon [1] showed that noiseless feedback does not increase the capacity of single-user memoryless channels. Despite this seemingly negative result, feedback significantly simplifies coding schemes and improves the performance in terms of the error probability [2]–[6]. Burnashev [7] demonstrated that the reliability function for the discrete memoryless channel (DMC) with feedback improves dramatically when the transmission time is random. This is known as variable-length feedback. In fact, the reliability function of a DMC with variable-length feedback admits a particularly simple expression

\[ E(R) = B_1 \left( 1 - \frac{R}{C} \right) \]

for all rates \( 0 \leq R \leq C \), where \( C \) is the capacity of the DMC and \( B_1 \) is determined by the relative entropy between conditional output distributions of the two most “most distinguishable” channel input symbols [7]. Yamamoto and Itoh [8] proposed a simple and conceptually important two-phase coding scheme that attains the reliability function in (1). Since these reliability function (or error exponent) results are of paramount importance in practical single-user feedback communication systems, we are motivated to extend the results to a simple network scenario—namely, the discrete memoryless broadcast channel (DM-BC) with a common message (also known as the common-message DM-BC) [4], [9], [10]. We provide upper and lower bounds on the reliability function and show that the bounds coincide if the DM-BC is stochastically degraded. In this scenario, the reliability function is dominated by the “worst branch” of the DM-BC.

A. Main Contributions

Our main technical contributions are as follows:

- Firstly, for the achievability part, we generalize Yamamoto and Itoh’s coding scheme [8] so that it is applicable to the DM-BC with a common message and variable-length feedback. In this enhanced scheme, we supplement some new elements to the original arguments in [8]. These include (i) defining an appropriate set of \( K \) stopping times and (ii) proving that the expectation of the maximum of these \( K \) stopping times can be appropriately bounded assuming that the individual stopping times’ expectations and variances are also appropriately bounded. This complication of having to control the maximum of a set of stopping times does not arise in single-user scenarios such as [7], [11], [12].

- Secondly, for the converse part, we adapt and combine proof techniques introduced by Burnashev for two different problems—namely, the reliability function for DMCs with variable-length feedback in [7] and that for sequential hypothesis testing in [11]. This allows us to obtain an upper bound for the reliability function for the common-message DM-BC with variable-length feedback. There is an alternative and more elegant proof technique to establish the converse part of (1) by Berlin et al. [13] but generalizing the technique therein to our setting does not seem to be feasible.

- Thirdly, even though the bounds on the reliability function do not match for general DM-BCs, we identify a particular class of DM-BCs, namely stochastically degraded DM-BCs [14, Sec. 5.6] for which the reliability function is known

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exactly. For the less capable DM-BCs (to be defined formally in Definition 3), even though we only have bounds on the reliability function, from these bounds, we can establish the capacity of such channels with variable-length feedback.

B. Related Works

We summarize some related works in this subsection. In [11], Burnashev extended the ideas in his original paper in DMCs with variable-length feedback [7] to be amenable to the more general problem of sequential hypothesis testing. In particular, he studied the minimum expected number of observations (transmissions) to attain some level of reliability and found the reliability function for large class of single-user channels (beyond DMCs), including the Gaussian channel [11]. Berlin et al. [13] provided a simple converse proof for Burnashev’s reliability function [7]. Their converse proof suggests that a confirmation and a confirmation phase are implicit in any scheme for which the probability of error decreases exponentially fast with (the optimal) exponent given by (1). Under this viewpoint, this converse proof approach is parallel to the Yamamoto and Itoh’s achievability scheme [8]. Nakibo˘glu and Gallager [12] investigated variable-length coding schemes for (not necessarily discrete) memoryless channels with variable-length feedback and with cost constraints and established the reliability function. Their achievability proof is an extension of Yamamoto and Itoh’s [8] and their converse proof uses two bounds on the difference of the conditional entropy random variable similarly to [7] with some extra arguments to account for the average cost constraints. Chen, Williamson, and Wesel [15] proposed a two-phase stop-feedback coding scheme where each phase uses an incremental redundancy scheme achieving Burnashev’s reliability function (1) while maintaining an expansion of the size of the message set that yields a small backoff from capacity. Their coding scheme uses a stop-feedback code [16] for the first-phase and a sequential probability ratio test [17] for the second-phase.

We also mention the work by Shrader and Permuter’18 who studied the feedback capacity of compound channels [19], [20]. The authors considered fixed-length feedback while our focus is on variable-length feedback. Mahajan and Tatikonda [21] considered the variable-length case for the same channel and established inner and outer bounds on the so-called error exponent region. While the common-message DM-BC we study is somewhat similar to the compound channel [19], [20], the techniques we use are different and we establish the exact reliability function for stochastically degraded DM-BCs. Tchamkerten and Telatar, in a series of elegant works [22]–[24], considered conditions in which one can achieve Burnashev’s exponent in (1) universally, i.e., without precise knowledge of the DMC.

Recently, there have also been numerous efforts to establish fundamental limits of single- and multi-user channels with variable-length feedback for non-vanishing error probabilities. See [9], [10], [16], [25], [26] for an incomplete list. However, we are concerned with quantifying the exponential rate of decay of the error probability similarly to (1).

C. Paper Organization

The rest of this paper is structured as follows: In Section II, we provide the problem formulation for the DM-BC with a common message under variable-length feedback with termination. The main results concerning the reliability function, conditions under which the results are tight, and some accompanying discussions are stated in Section III. In Section IV, we provide the achievability proof. The converse proof is provided in Section V. We also explain the novelties of our arguments relative to existing works at the end of the proofs in Sections IV and V. Auxiliary technical results that are not essential to the main arguments are relegated to the appendices.

II. Problem Setting

A. Notational Conventions

We use asymptotic notation such as $O(\cdot)$ in the standard manner; $f(n) = O(g(n))$ holds if and only if the implied constant $\limsup_{n \to \infty} |f(n)/g(n)| < \infty$. Also $f(n) = o(g(n))$ if and only if $\lim_{n \to \infty} |f(n)/g(n)| = 0$. In this paper, we use $\ln x$ to denote the natural logarithm so information units throughout are in nats. The binary entropy function is defined as $h(x) := -x \ln x - (1-x) \ln(1-x)$ for $x \in [0,1]$. We also define the function $(x)_a := x \mathbf{1}\{x \geq a\}$ for $x, a \in \mathbb{R}$. The minimum of two numbers $a$ and $b$ is denoted interchangeably as $\min\{a, b\}$ and $a \wedge b$. As is usual in information theory, $Z_i^j$ denotes the vector $(Z_i, Z_{i+1}, \ldots, Z_j)$.

For any discrete product sample space $\mathcal{Z} \times \mathcal{T}$, a sigma-algebra $\mathcal{F}$ on $\mathcal{Z} \times \mathcal{T}$, two random variables $Z, T$ (not necessary
measurable with respect to $\mathcal{F}$, and two regular conditional probability measures $\mathbb{P}(\cdot|\mathcal{F}), \mathbb{Q}(\cdot|\mathcal{F})$ on $\mathcal{Z} \times \mathcal{T}$, define
\begin{align}
\mathcal{H}(Z|\mathcal{F}) & := - \sum_{z \in \mathcal{Z}} \mathbb{P}(z|\mathcal{F}) \ln \mathbb{P}(z|\mathcal{F}), \\
H(Z) & := \mathcal{H}(Z|\sigma(\emptyset, \mathcal{Z} \times \mathcal{T})), \\
D(\mathbb{P}\|\mathbb{Q}) & := \sum_{(z,t) \in \mathcal{Z} \times \mathcal{T}} \mathbb{P}(z,t|\sigma(\emptyset, \mathcal{Z} \times \mathcal{T})) \ln \frac{\mathbb{P}(z,t|\sigma(\emptyset, \mathcal{Z} \times \mathcal{T}))}{\mathbb{Q}(z,t|\sigma(\emptyset, \mathcal{Z} \times \mathcal{T}))}, \\
\mathcal{I}(Z;T|\mathcal{F}) & := \sum_{(z,t) \in \mathcal{Z} \times \mathcal{T}} \mathbb{P}(z,t|\mathcal{F}) \ln \frac{\mathbb{P}(z,t|\mathcal{F})}{\mathbb{P}(z|\mathcal{F}) \mathbb{P}(t|\mathcal{F})}, \\
I(Z;T) & := \mathcal{I}(Z;T|\sigma(\emptyset, \mathcal{Z} \times \mathcal{T})).
\end{align}
If $\mathcal{F} = \sigma(Y^n)$ for some vector $Y^n$, we write $\sigma(Y^n)$ as $Y^n$ in all above notations (2)–(6) for simplicity [27].

B. Basic Definitions

**Definition 1.** A $(M,N)$-variable-length feedback code with termination (VLFT) for a $K$-user DM-BC $P_{Y_1,Y_2,\ldots,Y_K|X}$ with a common message, where $N$ is a positive real and $M$ is a positive integer, is defined by
- A set of equiprobable messages $W = \{1,2,\ldots,M\}$.
- A sequence of encoders $f_n : W \times \mathcal{Y}_1^{n-1} \times \mathcal{Y}_2^{n-1} \times \cdots \times \mathcal{Y}_K^{n-1} \rightarrow \mathcal{X}$, $n \geq 1$, defining channel inputs
  \[ X_n = f_n(W,Y_1^{n-1},Y_2^{n-1},\ldots,Y_K^{n-1}). \]
- $K$ sequences of decoders $g_n^{(j)} : \mathcal{Y}_j^n \rightarrow W$, $j = 1,2,\ldots,K$, providing the best estimate $\hat{W}$ at time $n$ at the corresponding decoders.
- A stopping random variable $\tau := \max\{\tau_1,\tau_2,\ldots,\tau_K\}$, where for each $j \in \{1,2,\ldots,K\}$, $\tau_j$ is a stopping time of the filtration $\{\sigma(Y_j^n)\}_{n=0}^\infty$. Furthermore, $\tau$ satisfies the following constraint:
  \[ \mathbb{E}(\tau) \leq N. \]

The final decision at decoder $j = 1,2,\ldots,K$ is computed at time $\tau_j$ as follows:
\[ \hat{W}_j = g_n^{(j)}(Y_j^\tau_j). \]

The error probability of a given variable-length coding scheme is defined as
\[ P_e(R,N) := \mathbb{P}\left( \bigcup_{j=1}^K \{ \hat{W}_j \neq W \} \right). \]

The rate of the $(M,N)$-VLFT code (cf. Definition 1) is defined as
\[ R_N := \frac{\ln M}{N}. \]

**Definition 2.** $(R,E) \in \mathbb{R}^2_+$ is an achievable rate-exponent pair if there exists a family of $(M_N,N)$-VLFT codes (for $N \rightarrow \infty$) satisfying
\begin{align}
\liminf_{N \rightarrow \infty} R_N & \geq R, \\
\lim_{N \rightarrow \infty} P_e(R_N,N) = 0, \\
\liminf_{N \rightarrow \infty} - \frac{\ln P_e(R,N)}{N} & \geq E,
\end{align}
where $R_N = N^{-1} \ln M_N$. The reliability function of the DM-BC with VLFT is
\[ E(R) := \sup\{ E : (E,R) \text{ is an ach. rate-exp. pair} \}. \]

In a VLFT code for the DM-BC, the word “termination” is used to indicate that in order to realize the code in a practical setting, one needs to send a reliable end-of-packet signal by a method other than using the transmission channel. In other words, the encoder decides when to stop the transmission of signals [10], [16].

We now recapitulate a set of orderings of channels [14, Ch. 5].
Definition 3. A DM-BC $P_{Y_1,Y_2,...,Y_K|X}$ is less capable\(^1\) \cite[Sec. 5.6]{14} (with respect to the first channel $P_{Y_1|X}$) if
\[
I(X;Y_1) \leq \min_{1 \leq j \leq K} I(X;Y_j)
\]
for all $P_X$. A DM-BC $P_{Y_1,Y_2,...,Y_K|X}$ is stochastically degraded \cite[Sec. 5.4]{14} (with respect to $P_{Y_1|X}$) if there exists a random variable $\tilde{Y}_1$ such that
\[
\tilde{Y}_1|\{X = x\} \sim P_{Y_1|X}(\cdot|x), \quad \forall \tilde{y}_1 \in \mathcal{Y}_1, \quad \text{and}
\]
\[
X - Y_j - \tilde{Y}_1, \quad \forall j = 2, 3, \ldots, K
\]
A DM-BC $P_{Y_1,Y_2,...,Y_K|X}$ is physically degraded \cite[Sec. 5.4]{14} (with respect to $P_{Y_1|X}$) if
\[
X - Y_j - Y_1
\]
forms a Markov chain for all $j = 2, \ldots, K$.

Clearly, the set of all physically degraded DM-BCs contained in the set of all stochastically degraded DM-BCs which is contained in the set of all less capable DM-BCs. We omit another commonly-encountered set of orderings for DM-BCs, namely less noisy DM-BCs \cite[Sec. 5.6]{14}.

Definition 4. For a DM-BC with a common message and VLFT as in Definition 1 we define for each $1 \leq j \leq K$,
\[
B := \max_{x,x' \in X_{1 \leq j \leq K}} \min_{y \in \mathcal{Y}_j} D(P_{Y_j|x}|\cdot|x),
\]
\[
B_j := \max_{x,x' \in X_{1 \leq j \leq K}} D(P_{Y_j|x}|\cdot|x),
\]
\[
B_{\max} := \max_{1 \leq j \leq K} B_j,
\]
\[
T_j := \max_{x,x' \in X_{1 \leq j \leq K}} \frac{P_{Y_j|x}(y|x)}{P_{Y_j|x}(y|x')},
\]
\[
C := \max_{P_X} \min_{1 \leq j \leq K} I(X;Y_j),
\]
\[
C_j := \max_{P_X} I(X;Y_j),
\]
\[
\overline{C} := \min_{1 \leq j \leq K} \max_{P_X} I(X;Y_j).
\]

III. MAIN RESULTS

We now state bounds on the reliability function of the $K$-user DM-BC channel $P_{Y_1,Y_2,...,Y_K|X}$ with a common message and with VLFT.

Theorem 1. For any $K$-user DM-BC channel $P_{Y_1,Y_2,...,Y_K|X}$ with VLFT (cf. Definition 1) such that $B_{\max} < \infty$,
\[
E(R) \geq B_1 \left(1 - \frac{R}{C_1}\right), \quad \forall R < C_1.
\]
and
\[
E(R) \leq \min_{1 \leq j \leq K} B_j \left(1 - \frac{R}{C_j}\right), \quad \forall R < \overline{C}.
\]

Since the reliability function yields bounds on the capacity of the DM-BC, we immediately obtain the following.

Corollary 1. Under the condition $B_{\max} < \infty$, the capacity of the DM-BC with VLFT, namely $C_{\text{BC-VLFT}}$, satisfies
\[
C \leq C_{\text{BC-VLFT}} \leq \overline{C}.
\]

Although there is, in general, a gap between the upper and lower bounds on the reliability function (and capacity) provided in Theorem 1 (and Corollary 1), under some conditions on the DM-BC, the reliability function (and capacity) is known exactly.

Theorem 2. For a less capable DM-BC with VLFT such that $B_{\max} < \infty$,
\[
B_1 \left(1 - \frac{R}{C_1}\right) \leq E(R) \leq B_1 \left(1 - \frac{R}{C_1}\right), \quad \forall R < C_1.
\]

\(^1\)In the literature \cite[Sec. 5.6]{14}, the term more capable is typically used when $Y_1$ is the “strongest receiver”. However, in our context, $Y_1$ is the “weakest receiver” so we use the (somewhat atypical) term less capable here.
Furthermore, if the DM-BC with VLFT is stochastically degraded (or physically degraded),
\[ E(R) = B_1 \left( 1 - \frac{R}{C_1} \right), \quad \forall R < C_1. \] (31)

**Corollary 2.** Under the condition \( B_{\text{max}} < \infty \), the capacity of any less capable DM-BC with VLFT
\[ C_{\text{BC-VLFT}} = C = C_1 = \overline{C}. \] (32)

**Proof of Theorem 2 and Corollary 2:** For any less capable DM-BC we have \( I(X; Y_1) \leq I(X; Y_j) \) for all \( P_X \) and for all \( j = 2, 3, \ldots, K \). Hence,
\[ C = \max_{P_X} \min_{1 \leq j \leq K} I(X; Y_j), \]
\[ = \max_{P_X} I(X; Y_1) = C_1. \] (33) (34)
Plugging this into (27) establishes the lower bound in (30). For less capable DM-BCs, we also have \( C_1 = \max_{P_X} I(X; Y_1) \leq C_j = \max_{P_X} I(X; Y_j) \) for all \( j = 2, 3, \ldots, K \), hence
\[ \overline{C} := \min_{1 \leq j \leq K} \max_{P_X} I(X; Y_j), \]
\[ = \max_{P_X} I(X; Y_1) = C_1. \] (35) (36)
As a result, for less capable DM-BCs, the capacity is \( C = C_1 = \overline{C} \), establishing (32). Moreover, from (28) in Theorem 1, for all \( R < \overline{C} = C_1 \) (cf. Eqn. (36)),
\[ E(R) \leq \min_{1 \leq j \leq K} B_j \left( 1 - \frac{R}{C_j} \right) \leq B_1 \left( 1 - \frac{R}{C_1} \right). \] (37)
This establishes the upper bound in (30).

For stochastically degraded DM-BCs, there exists a random variable \( \tilde{Y}_1 \) such that \( X - Y_j - \tilde{Y}_1 \) for all \( j = 1, 2, \ldots, K \) and \( P_{Y_1|X} = P_{\tilde{Y}_1|X} \). Therefore, we have
\[ D(P_{Y_1|X}(\cdot|x)|P_{Y_1|X}(\cdot|x')) = D(P_{Y_1|X}(\cdot|x)|P_{\tilde{Y}_1|X}(\cdot|x')). \] (38)
Observe that for any \( x, x' \in \mathcal{X} \) and \( j \in \{2, 3, \ldots, K\} \), we also have
\[ D(P_{Y_1|X}(\cdot|x)|P_{Y_1|X}(\cdot|x')) = \sum_{y_1} P_{Y_1|X}(y_1|x) \ln \frac{P_{Y_1|X}(y_1|x)}{P_{Y_1|X}(y_1|x')}, \]
\[ = \sum_{y_1} \sum_{y_j} P_{Y_1|Y_j}(y_1|y_j|x) \ln \frac{P_{Y_1|Y_j}(y_1|y_j|x)}{P_{\tilde{Y}_1|Y_j}(y_1|y_j|x')} \]
\[ = \sum_{y_1} \sum_{y_j} P_{Y_j|X}(y_j|x) P_{Y_1|Y_j}(y_1|y_j) \ln \frac{P_{Y_1|X}(y_1|x)}{P_{\tilde{Y}_1|X}(y_1|x')} \]
\[ \leq \sum_{y_1} \sum_{y_j} P_{Y_j|X}(y_j|x) P_{Y_1|Y_j}(y_1|y_j) \ln \frac{P_{\tilde{Y}_1|X}(y_1|y_j)}{P_{\tilde{Y}_1|X}(y_1|x')} \]
\[ = \sum_{y_j} P_{Y_j|X}(y_j|x) \ln \frac{P_{Y_1|X}(y_j|x)}{P_{\tilde{Y}_1|X}(y_j|x')} \]
\[ = D(P_{Y_j|X}(\cdot|x)|P_{\tilde{Y}_1|X}(\cdot|x')). \] (39) (40) (41) (42) (43) (44)
Here, (41) follows from the Markov chains \( X - Y_j - \tilde{Y}_1 \) for \( j = 1, 2, \ldots, K \) and (42) follows from the log-sum inequality. It follows that
\[ B = \max_{x, x' \in \mathcal{X}} \min_{1 \leq j \leq K} D(P_{Y_j|X}(\cdot|x)|P_{\tilde{Y}_1|X}(\cdot|x')) \]
\[ = \max_{x, x' \in \mathcal{X}} D(P_{Y_j|X}(\cdot|x)|P_{\tilde{Y}_1|X}(\cdot|x')) = B_1, \] (45) (46)
and hence (31) is established.

A few remarks concerning Theorem 1 are in order.

- There is a gap between the lower and upper bounds for the general DM-BC. One reason that pertains to the achievability part is because each decoder \( j \in \{1, 2, \ldots, K\} \), at time \( n \), only has its own sequence \( Y_j^n \). Thus, it is difficult to establish
an appropriate hypothesis test within the coding scheme by Yamamoto-Itoh [8] such that this hypothesis test works for any possible realization of the other random variables \( \{Y_i^n : i \neq j\} \).

- For the converse, if we use the same hypothesis test for single-user channels with VLFT as in Berlin et al.’s work [13], it is challenging to obtain a useful result. The hypothesis test in [13, Prop. 1] involves the sufficient statistic \( V_n := \ln P_Y(Y^n_1) - \ln P_Y(Y^n_0) \). Because \( X_k \) depends on \((W, Y_1^{k-1}, \ldots, Y_{K-1}^{k-1})\) for each \( k \in \mathbb{N} \) (cf. Eqn. (7)), we cannot simply append \( (Y_1^n, \ldots, Y_K^n) \) to \( Y^n \) in the expression for \( V_n \) and still obtain the desired upper bound as in [13, Prop. 1].

- Moreover, if we directly adapt the key ideas in Burnashev’s converse proof for sequential hypothesis testing in [11, Lemmas 3 and 4], we will only obtain the following almost sure bound for each \( j \in \{1, \ldots, K\} \):

\[
\mathbb{E}[\mathcal{H}(W|Y_j^n) - \mathcal{H}(W|Y_j^{n+1})|Y^n] 
\leq \max_{w, w' \in \mathcal{W}} \sup_{n} \sup_{y_{j,n}} D(P_{Y_j,n|Y_j^{n-1},W}(\cdot|y_{j,n}^{-1}, w))\left\| P_{Y_j,n|Y_j^{n-1},W}(\cdot|y_{j,n}^{-1}, w')\right\|.
\]

This is then insufficient to establish our converse.

- Our Lemma 6 is stronger than the corresponding one to prove the converse of (1) in Burnashev [7, Lemma 3] since we do not need to assume that the conditional entropies \( \mathcal{H}(W|Y_j^n) \) for \( j = 1, 2, \ldots, K \) are bounded. Consequently, the construction of submartingales in the proof of Lemma 9 (in the converse proof in Section V) is much simpler.

- We have a tight reliability function result for stochastically degraded DM-BCs in (31). Usually, orderings of the channels (less/more capable, less noisy, stochastically and physically degraded) are used to obtain tight capacity or capacity region results for DM-BCs [14, Secs. 3.4 & 3.6]. Here, in contrast, we use the orderings to establish a tight reliability function result.

### IV. Achievability Proof of Theorem 1

In this section, we provide the achievability proof of Theorem 1. We start with a preliminary lemma.

**Lemma 1** (Expectation of the Maximum of Random Variables). Let \( \{(X_{1L}, X_{2L}, \ldots, X_{KL})\}_{L \geq 1} \) be \( K \) sequences of random variables satisfying

\[
\mathbb{E}[X_{jL}] = L + o(1), \quad \text{and} \quad \text{Var}(X_{jL}) = o(1), \quad j = 1, 2, \ldots, K,
\]

as \( L \to \infty \). Then, as \( L \to \infty \), we have

\[
\mathbb{E}(\max\{X_{1L}, X_{2L}, \ldots, X_{KL}\}) = L + O(\sqrt{L}).
\]

**Proof:** The proof can be found in Appendix A.

The achievability part of Theorem 1 can be stated succinctly as follows.

**Lemma 2.** If \( B_{\text{max}} < \infty \),

\[
E(R) \geq B \left(1 - \frac{R}{C}\right), \quad \forall R < C.
\]

**Proof:** The achievability proof is an extension of Yamamoto-Itoh’s variable-length coding scheme [8] for the DMC with noiseless variable-length feedback. However, we devise some additional and crucial ingredients to account for the presence of multiple channel outputs and multiple decoded messages. In the coding scheme, the encoder decides whether or not to stop the transmission. We show that for all \( L \in \mathbb{N} \) there exists an \((e^{RL}, L + O(\sqrt{L}))\)-VLFT code with achievable exponent \( B(1 - R/C) \).

Choose \( P_X := \arg \max_{P_X} \min_{1 \leq j \leq K} I(X; Y_j) \) and \( x_c, x_o \in \mathcal{X} \) such that

\[
(x_c, x_o) := \arg \max_{(x, x') \in \mathcal{X} \times \mathcal{X}} 1 \leq j \leq K \sup_{w \in \mathcal{W}} D(P_{Y_j|X}(\cdot|x))\left\| P_{Y_j|X}(\cdot|x')\right\|.
\]

Since we assume that \( B_{\text{max}} < \infty \), we have \( P_{Y_j|X}(y|x) > 0 \) for all \( y \in \mathcal{Y}_j, x \in \mathcal{X} \) for all \( j = 1, 2, \ldots, K \). Fix a non-negative number \( R \) satisfying \( 0 \leq R < C \).

We design a code for each block of \( L \) transmissions as per the Yamamoto-Itoh coding scheme with rate \( R \) [8]. Let this code length \( L \) be divided into two parts, \( \gamma L \) for the message mode and \((1 - \gamma) L \) for the control mode. In the message mode, one of \( M = \lfloor e^{LR} \rfloor \) messages is transmitted by a random coding scheme with block-length \( \gamma L \) [28], and in the control mode a pair of control signals \((c, e)\) is transmitted by another code with length \((1 - \gamma) L \). The control signal \( c \) is only sent when all the \( K \) receivers correctly decode the message in the message mode.

Now, the variable-length coding scheme for the DM-BC with a common message is created by repeating the length-\( L \) transmission at times \( n \in \{\mu L : \mu = 1, 2, 3, \ldots\} \) and using the same decoding algorithm as in [8] at all the decoders. The decoder \( j \in \{1, 2, \ldots, K\} \) defines a stopping time \( \tau_j \) as follows:
1) If \( n \in \{ \mu L : \mu = 2, 3, 4, \ldots \} \), we define

\[
1\{\tau_j = n\} = \prod_{t=1}^{\mu-1} \left\{ g_n^{(j)} \left( Y_{j,(t-1)L+L}^{(t-1)L+L+1} \right) = e \right\} 1\left\{ g_n^{(j)} \left( Y_{j,(t-1)L+L+1}^{n} \right) = c \right\}; \tag{53}
\]

2) If \( n = L \), we define

\[
1\{\tau_j = n\} = 1\left\{ g_n^{(j)} \left( Y_{\gamma \gamma \gamma L+L}^{L} \right) = c \right\}; \tag{54}
\]

3) Otherwise,

\[
1\{\tau_j = n\} = 1\{\emptyset\}. \tag{55}
\]

In addition, the estimated message at the stopping time \( \tau_j \) has the following form:

\[
\hat{W}_j := g_n^{(j)} \left( Y_{j,(t-1)L}^{n(t-1)L+L} \right), \quad j = 1, 2, \ldots, K. \tag{56}
\]

Since \( Y_j \) for \( j \in \{ 1, 2, \ldots, K \} \) is finite, for each fixed \( n \in \mathbb{Z}_+ \) all the decoding regions at each decoder \( j \) are finite sets, which are Borel sets in \( \mathbb{R}^n \). Combining this fact with the definition of \( \tau_j \), we have \( 1\{\tau_j = n\} \in \sigma(Y_j^n) \) for all \( n \in \mathbb{N} \). Let

\[
q_{L}^{(j)} := P \left( g_n^{(j)}(Y_{\gamma \gamma \gamma L+L}^{L}) = e \right), \quad j = 1, 2, \ldots, K. \tag{57}
\]

By the proposed transmission method, given \( W = w \in \mathcal{W} \) we have that \( Y_{j,(t-1)L+L}^{n(t-1)L+L+1} \) for \( t \in \mathbb{N} \) are independent random vectors. Since the messages in \( \mathcal{W} \) are equiprobable, we obtain

\[
P(\tau_j = n) = \begin{cases} 
\left[ q_{L}^{(j)} \right]^{n-1} \left[ 1 - q_{L}^{(j)} \right], & \text{if } n \in \{ \mu L : \mu = 1, 2, 3, \ldots \}. \\
0, & \text{otherwise}
\end{cases} \tag{58}
\]

Hence, we have

\[
\sum_{n=0}^{\infty} P(\tau_j = n) = \sum_{\mu=1}^{\infty} \left[ q_{L}^{(j)} \right]^{\mu-1} \left[ 1 - q_{L}^{(j)} \right] = 1. \tag{59}
\]

Thus, \( \tau_j \) is a stopping time with respect to \( \{ \sigma(Y_j^n) \}_{n=0}^{\infty} \).

Now, since we use the same decoding algorithm as \[8\] for each repeated transmission block of length \( L \) at each decoder \( j \), it is easy to see that the error probability for the \( j \)-th decoder \( \mathbf{P}_E^{(j)} := P(\hat{W}_j \neq W) \) and \( q_{L}^{(j)} \) can be written as follows \[8\]:

\[
\begin{align*}
\mathbf{P}_E^{(j)} &= \mathbf{P}_{1e}^{(j)} \mathbf{P}_{2ec}, \tag{60} \\
q_{L}^{(j)} &= \mathbf{P}_{1e}^{(j)} (1 - \mathbf{P}_{2ec}) + (1 - \mathbf{P}_{1e}^{(j)}) \mathbf{P}_{2ec}. \tag{61}
\end{align*}
\]

Here, \( \mathbf{P}_{1e}^{(j)}, \mathbf{P}_{2ec}^{(j)} \), and \( \mathbf{P}_{2ec}^{(j)} \), respectively denote the error probability of decoder \( j \) in the message mode, the probability that the message \( e \) is sent at the control mode but the decoder \( j \) decodes the message \( c \), the probability that \( c \) is sent at the control mode but the decoder \( j \) decodes \( e \) \[8, pp. 730\].

Since \( q_{L}^{(j)} \) is the same for all repeated transmissions, each of blocklength \( L \), we have for all \( j = 1, 2, \ldots, K \),

\[
\begin{align*}
\mathbb{E}(\tau_j) &= \sum_{n=0}^{\infty} n P(\tau_j = n) \\
&= \sum_{\mu=1}^{\infty} \mu L \left[ q_{L}^{(j)} \right]^{\mu-1} \left[ 1 - q_{L}^{(j)} \right] \\
&= \frac{L}{1 - q_{L}^{(j)}}. \tag{62}
\end{align*}
\]

In addition, we also have

\[
\text{Var}(\tau_j) = \frac{L^2 q_{L}^{(j)}}{\left[ 1 - q_{L}^{(j)} \right]^2}. \tag{65}
\]

Let \( l := (1 - \gamma)L \). We assign length-\( l \) codewords \( X_c^l = (x_c, x_c, \ldots, x_c) \in \mathcal{X}^l \) and \( X_e^l = (x_e, x_e, \ldots, x_e) \in \mathcal{X}^l \) to control the signals \( c \) and \( e \) respectively. Decoding of the control signal is done as follows. Choose an arbitrarily small \( \delta > 0 \). Let us say the number of output symbols \( y \in \mathcal{Y}_j \) contained in the received sequence \( Y_{ij}^l = y_{ij}^l \) equals to \( l_y \in \{ 1, \ldots, l \} \). We suppress the dependence of \( l_y \) on \( j \) for notational convenience. If every \( l_y \) satisfies the typicality condition

\[
(1 - \delta) P_{Y_{ij} \mid X}(y \mid x_c) \leq \frac{l_y}{l} \leq (1 + \delta) P_{Y_{ij} \mid X}(y \mid x_c), \tag{66}
\]
then $y_j^l$ is decoded to $c$, otherwise to $e$. Then, defining $F(\cdot)$ to be the random coding error exponent for DMCs [28] and $R_{L\gamma} := R/\gamma < \min_{1 \leq j \leq K} I(X; Y_j) = C$ (since $X \sim P_X^L$), it follows from [8] that

$$P_{\text{le}}^{(j)} \leq \exp[-\gamma LF(R_{L\gamma})],$$

(67)

$$P_{2\text{ec}}^{(j)} \leq \exp[-(1 - \gamma) L(f_j(\delta) - o(1))],$$

(68)

where $f_j(\delta) > 0$ for any $\delta > 0$. In (67) and (68) we used the usual notation $a_L \leq b_L$ to mean that $\limsup_{L \to \infty} \frac{1}{L} \log \frac{a_L}{b_L} \leq 0$. Also, by Stein’s lemma,

$$\lim_{L \to \infty} -\frac{\ln P_{2\text{ec}}^{(j)}}{(1 - \gamma)L} = D(P_{Y_j|X}(\cdot|x_e)\|P_{Y_j|X}(\cdot|x_e)).$$

(69)

Moreover from (60) and (67)–(68) we have

$$q_L^{(j)} \leq \exp(-Lc^{(j)}), \quad j = 1, 2, \ldots, K$$

(70)

for some exponent $c^{(j)} > 0$.

Consequently, from (64), (65), and (70) we obtain for all $j$

$$E(t_j) = L + o(1),$$

(71)

$$\text{Var}(t_j) = o(1).$$

(72)

From (71), (72), and Lemma 1 we obtain that

$$E(\tau) = L + O(\sqrt{L}).$$

(73)

Now, since for each $j = 1, 2, \ldots, K$, $P_{E}^{(j)}$ is kept the same for all repeated transmission blocks of length $L$, we have

$$P_e(R, L + O(\sqrt{L})) \leq \sum_{j=1}^{K} P_{E}^{(j)}.$$  

(74)

Moreover, it is easy to see from (60), (67)–(68), and (73) that $P_{E}^{(j)} \to 0$ for all $j = 1, 2, \ldots, K$ as $L \to \infty$ if $0 \leq R_{L\gamma} = R/\gamma < C$ and $0 \leq \gamma < 1$. Combining these requirements and (74), we have $P_e(R, L + O(\sqrt{L})) \to 0$ as $L \to \infty$ if we choose $1 > \gamma > R/C$. Now, since $\gamma > R/C$, a feasible value of $\gamma$ that we can choose is

$$\gamma = \frac{R}{C - \varepsilon},$$

(75)

where $\varepsilon > 0$ is chosen small enough so that $\gamma$ remains smaller than 1. It follows that for any $R \in [0, C)$, we have

$$\liminf_{L \to \infty} -\frac{\ln P_e(R, L + O(\sqrt{L}))}{L + O(\sqrt{L})} \geq \liminf_{L \to \infty} -\frac{\ln \left( \sum_{j=1}^{K} P_{E}^{(j)} \right)}{L + O(\sqrt{L})}$$

(76)

$$\geq \liminf_{L \to \infty} \left\{ \min_{1 \leq j \leq K} \left[ -\frac{\ln \left( \sum_{j=1}^{K} P_{E}^{(j)} \right)}{L + O(\sqrt{L})} \right] \right\}$$

(77)

$$= \min_{1 \leq j \leq K} \left( \liminf_{L \to \infty} -\frac{\ln P_{E}^{(j)}}{L} \right)$$

(78)

$$\geq \min_{1 \leq j \leq K} \left( \liminf_{L \to \infty} -\frac{\ln P_{2\text{ec}}^{(j)}}{L} \right)$$

(79)

$$= \min_{1 \leq j \leq K} D(P_{Y_j|X}(\cdot|x_e)\|P_{Y_j|X}(\cdot|x_e)) \left( 1 - \frac{R}{C - \varepsilon} \right)$$

(80)

$$= B \left( 1 - \frac{R}{C - \varepsilon} \right),$$

(81)

where (78) follows from the facts that $K$ is a constant and that $\liminf_{L \to \infty} \min_{j} \{a_{jL}\} = \min_{j} \liminf_{L \to \infty} \{a_{jL}\}$ for any family of sequences $\{a_{jL}\}$; (79) follows from (60); and (80) follows from (69) and (75).

This means that $(R, B(1 - R/(C - \varepsilon)))$ is an achievable rate-exponent pair for any $0 \leq R < C$. By the arbitrariness of $\varepsilon > 0$, we obtain

$$E(R) \geq B \left( 1 - \frac{R}{C} \right).$$  

(82)
Finally, for any $N \in \mathbb{R}_+$ choose $L = \lfloor N - O(\sqrt{N}) \rfloor$ such that $L + O(\sqrt{L}) \leq N$. By using the $(\lceil e^{RL}\rceil, L + O(\sqrt{L}))$-VLFT code constructed above, we conclude that there exists an $(\lceil e^{(N-O(\sqrt{N}))R}\rceil, N)$-VLFT code such that (51) holds.

We remark that for the proof of Lemma 2, we extended Yamamoto and Itoh’s coding scheme [8] for the DM-BC with a common message and VLFT. In the proof, we supplemented some new elements to the original argument in [8]. These include defining appropriate stopping times $\{\tau_1, \tau_2, \ldots, \tau_K\}$ and proving that the expectation of the maximum of these $K$ stopping times with expectations and variances respectively bounded by $L + o(1)$ and $o(1)$ is $L + O(\sqrt{L})$ (cf. Lemma 1).

V. CONVERSE PROOF OF THEOREM 1

In this section, we provide the converse proof of Theorem 1. We start with a few preliminary lemmas. At the end of the proof (after the proof of Lemma 9), we discuss the novelities in our converse proof vis-à-vis Burnashev’s works in [7] and [11].

**Lemma 3.** Under the condition that $\mathbb{P}(\tau < \infty) = 1$ (cf. Definition 1), the following inequalities hold

$$\mathbb{E}[\mathcal{H}(W|Y_j^n)] \leq h_{e}(R_N, N) + \mathbb{P}(R_N, N) \ln(M - 1),$$

for each $1 \leq j \leq K$ and $N$ sufficiently large.

**Proof:** The proof of this Lemma is essentially the same as [11, Lemma 1]. For completeness and compatibility in the notations, we provide the complete proof in Appendix B. Note that the error event here is different from [11, Lemma 1]. It is the union of error events of individual branches of the DM-BC, i.e., $\cup_{j=1}^K \{W_j \neq W\}$.

**Lemma 4.** For any $n \geq 0$ the following inequalities hold almost surely (cf. Definition 4)

$$\mathbb{E}[\mathcal{H}(W|Y_j^n) - \mathcal{H}(W|Y_j^{n+1})|Y_j^n] \leq C_j,$$

$1 \leq j \leq K$.

**Proof:** Observe that

$$\begin{align*}
\mathbb{E}[\mathcal{H}(W|Y_1^n) - \mathcal{H}(W|Y_1^{n+1})|Y_1^n] &= \mathbb{E}[\mathcal{H}(W|Y_1^n) - \mathcal{H}(W|Y_1^{n+1})|Y_1^n] \\
&= \mathbb{E}[\mathcal{I}(W; Y_1, n+1|Y_1^n)] \\
&= \mathbb{I}(W; Y_1, n+1|Y_1^n) \\
&\leq \mathbb{I}(X_{n+1}; Y_1, n+1|Y_1^n) + \sum_{x \in \mathcal{X}} \mathbb{I}(W; Y_1, n+1|X_{n+1} = x, Y_1^n) \mathbb{P}(X_{n+1} = x|Y_1^n).
\end{align*}$$

Now, for any fixed $Y_1^n = y_1^n$, the (random) mutual information in the sum can be expressed as

$$\begin{align*}
\mathbb{I}(W; Y_1, n+1|X_{n+1} = x, Y_1^n = y_1^n) &= I(W; Y_1, n+1|X_{n+1} = x, Y_1^n = y_1^n) \\
&= \sum_{w \in \mathcal{W}, y \in \mathcal{Y}_1} \mathbb{P}(W = w, Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n) \\
&\times \ln \frac{\mathbb{P}(W = w, Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n)}{\mathbb{P}(W = w, Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n) \mathbb{P}(Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n)}.
\end{align*}$$

Since $(W, Y_1^n, Y_2^n, \ldots, Y_K^n) - X_{n+1} - (Y_1, n+1, Y_2, n+1, \ldots, Y_K, n+1)$ forms a Markov chain, we obviously also have the following Markov chain:

$$\begin{align*}
(W, Y_1^n) - X_{n+1} - Y_1, n+1.
\end{align*}$$

Hence, we have

$$\begin{align*}
\mathbb{P}(W = w, Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n) &= \mathbb{P}(W = w, Y_1^n = y_1^n, W = w|Y_1, n+1 = y, X_{n+1} = x) \\
&= \mathbb{P}(W = w|X_{n+1} = x, Y_1^n = y_1^n) \mathbb{P}(Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n, W = w) \\
&= \mathbb{P}(W = w|X_{n+1} = x, Y_1^n = y_1^n) \mathbb{P}(Y_1, n+1 = y|X_{n+1} = x) \\
&= \mathbb{P}(W = w|X_{n+1} = x, Y_1^n = y_1^n) \mathbb{P}(Y_1, n+1 = y|X_{n+1} = x, Y_1^n = y_1^n).
\end{align*}$$

From (91) we obtain

$$\begin{align*}
\mathbb{I}(W; Y_1, n+1|X_{n+1} = x, Y_1^n = y_1^n) &= 0, \quad \forall (x, y_1^n) \in \mathcal{X} \times \mathcal{Y}_1^n.
\end{align*}$$

It follows from (89) that

$$\mathbb{E}[\mathcal{H}(W|Y_j^n) - \mathcal{H}(W|Y_j^{n+1})|Y_j^n] \leq \mathbb{I}(X_{n+1}; Y_1, n+1|Y_1^n) \leq C_1, \quad a.s.$$
A completely analogous argument goes through to yield the corresponding upper bounds for $j = 2, 3, \ldots, K$. ■

We remark that in the above proof, we need to use some additional arguments involving the Markov chain in (92) to show that Lemma 4 holds in the (general DM-BC) case where $X_{n+1}$ is a function of $W$ and all $Y^n_j$ for $j = 1, 2, \ldots, K$. In the DMC, $X_{n+1}$ is a function of $W$ and only $Y^n_1$.

The following lemma is a restatement of [7, Lemma 7].

**Lemma 5.** For arbitrary non-negative numbers $p_l, f_i, \beta_i$ where $l = 1, 2, \ldots, L$ and $i = 1, 2, \ldots, N$, we have the following inequality

$$\sum_{l=1}^{L} p_l \ln \left( \frac{\sum_{i=1}^{N} f_i}{\sum_{i=1}^{N} \beta_i} \right) \leq \max_i \sum_{l=1}^{L} p_l \ln \frac{f_i}{\beta_i}.$$

(100)

**Lemma 6.** For any $n \geq 0$ the following inequalities hold almost surely (cf. Definition 4)

$$\mathbb{E}[\ln \mathcal{H}(W|Y^n_j) - \ln \mathcal{H}(W|Y^n_{j+1})|Y^n_j] \leq B_j, \quad 1 \leq j \leq K.$$

(101)

**Proof:** The proof is based on Burnashev’s arguments in [7] and [11] with some modifications to account for the fact that at each transmission time $n + 1$, the transmitted signal $X_{n+1}$ is a function of $W$ and all $Y^n_1, Y^n_2, \ldots, Y^n_K$. We can assume that $P_{Y^n_j|X}(y_j|x) > 0$ for all $x \in \mathcal{X}, y_j \in \mathcal{Y}_j$ and all $j = 1, 2, \ldots, K$, otherwise the inequalities (101) trivially hold since $B_j = \infty$.

For each $i = 1, 2, \ldots, M$ and $y \in \mathcal{Y}_i$, define

$$p_i := \mathbb{P}(W = i|Y^n_1),$$

(102)

$$p_i(y) := \mathbb{P}(W = i|Y^n_1, Y_{1,n+1} = y),$$

(103)

$$p(y|W = i) := \mathbb{P}(Y_{1,n+1} = y|Y^n_1, W = i),$$

(104)

$$p(y|W \neq i) := \mathbb{P}(Y_{1,n+1} = y|Y^n_1, W \neq i),$$

(105)

$$p(y) := \mathbb{P}(Y_{1,n+1} = y|Y^n_1).$$

(106)

We may assume without loss of generality that $p_i \neq 1$ for all $i \in \mathcal{W} = \{1, 2, \ldots, M\}$. Otherwise, again the inequalities in (101) trivially hold. Using Lemma 5 and the definitions in (102)–(106) we have

$$\mathbb{E} \left[ \ln \mathcal{H}(W|Y^n_1) - \ln \mathcal{H}(W|Y^n_{i+1}) | Y^n_1 \right] = \sum_{y \in \mathcal{Y}_i} p(y) \ln \left[ \frac{-\sum_{i=1}^{M} p_i \ln p_i}{-\sum_{i=1}^{M} p_i(y) \ln p_i(y)} \right]$$

(107)

$$\leq \max_i \left\{ \sum_{y \in \mathcal{Y}_i} p(y) \ln \left[ \frac{-p_i \ln p_i}{-p_i(y) \ln p_i(y)} \right] \right\}$$

(108)

Define

$$F_i := \sum_{y \in \mathcal{Y}_i} p(y) \ln \left[ \frac{-p_i \ln p_i}{-p_i(y) \ln p_i(y)} \right]$$

(109)

It is easy to see that

$$p(y) = p_i p(y|W = i) + (1 - p_i) p(y|W \neq i),$$

(110)

$$p_i(y) = \frac{p_i p(y|W = i)}{p(y)},$$

(111)

and

$$p(y|W = i) = \mathbb{P}(Y_{1,n+1} = y|Y^n_1, W = i)$$

$$= \sum_{x \in \mathcal{X}} \mathbb{P}(X_{n+1} = x|W = i, Y^n_1) \mathbb{P}(Y_{1,n+1} = y|X_{n+1} = x, W = i, Y^n_1)$$

(112)

$$= \sum_{x \in \mathcal{X}} \mathbb{P}(X_{n+1} = x|W = i, Y^n_1) \mathbb{P}(Y_{1,n+1} = y|X_{n+1} = x)$$

(113)

$$= : \sum_{x \in \mathcal{X}} \alpha_{ix} P_{Y_1|X}(y|x).$$

(114)

Here, (114) follows from the Markov chain $(W, X^n_1, X^n_2, \ldots, X^n_K) - X_{n+1} - (Y_{1,n+1}, Y_{2,n+1}, \ldots, Y_{K,n+1})$ and (115) follows from the invariance (stationarity) of the distribution $\mathbb{P}(Y_{1,n+1} = y|X_{n+1} = x)$ in $n$, which is derived from the invariance of
the distribution $P(Y_{1,n+1} = y_1, Y_{2,n+1} = y_2, \ldots, Y_{K,n+1} = y_K | X_{n+1} = x)$ in $n$. Similarly, we have
\begin{equation}
p(y|W \neq i) = P(Y_{1,n+1} = y|Y_{1}^n, W \neq i) = \sum_{x \in X} P(X_{n+1} = x|W \neq i, Y_{1}^n) P(Y_{1,n+1} = y|X_{n+1} = x, W \neq i, Y_{1}^n) \tag{116}
\end{equation}
\begin{equation}
= \sum_{x \in X} P(X_{n+1} = x|W \neq i, Y_{1}^n) P(Y_{1,n+1} = y|X_{n+1} = x)
\end{equation}
\begin{equation}
= \sum_{x \in X} \beta_{ix} P_{Y_{1}|X}(y|x). \tag{119}
\end{equation}

It is easy to see that for each fixed message $i \in W = \{1, \ldots, M\}$ we have
\begin{equation}
\sum_{x \in X} \alpha_{ix} = \sum_{x \in X} \beta_{ix} = 1, \quad \alpha_{ix} \geq 0, \beta_{ix} \geq 0. \tag{120}
\end{equation}

Observe that $F_i$ is a function of variables $p_i, \{\alpha_{ix}\}$ and $\{\beta_{ix}\}$. For the purpose of finding an upper bound on $\max_i \{F_i\}$ in (108), we can consider only the constraints in (120) and find the maximization of $F_i$ over this convex set since other constraints that define the feasible set will only make $F_i$ smaller. With this consideration, let us consider the maximization of $F_i$ over $\{\beta_{ix}\}$ with the assumption that $\sum_{x \in X} \beta_{ix} = 1$ and $\beta_{ix} \geq 0$. Fix an arbitrary $x' \in X$, then we have $\beta_{ix'} = 1 - \sum_{x \in X \setminus \{x'\}} \beta_{ix}$. We readily obtain that the derivatives of $F_i$ for any $x \in X \setminus \{x'\}$ are
\begin{equation}
\frac{d^2 F_i}{d \beta_{ix}^2} = \frac{\partial^2 F_i}{\partial \beta_{ix}^2} + \frac{\partial^2 F_i}{\partial \beta_{ix'}^2} - 2 \frac{\partial^2 F_i}{\partial \beta_{ix} \partial \beta_{ix'}}, \tag{121}
\end{equation}
\begin{equation}
\frac{\partial^2 F_i}{\partial \beta_{ix} \partial \beta_{ix'}} = (1 - p_i)^2 \sum_{y \in \mathcal{Y}_i} \frac{\partial^2 F_i}{\partial p(y)^2} P_{Y_{1}|X}(y|x) P_{Y_{1}|X}(y|x'), \tag{122}
\end{equation}
\begin{equation}
\frac{\partial^2 F_i}{\partial p(y)^2} = \frac{1}{p(y)} \left[ 1 - \left( \frac{p(y)}{p_i p(y|W = i)} \right)^{-1} + \left( \frac{p(y)}{p_i p(y|W = i)} \right)^{-2} \right] > 0. \tag{123}
\end{equation}

Hence, from (121) to (123) we obtain
\begin{equation}
\frac{d^2 F_i}{d \beta_{ix}^2} = (1 - p_i)^2 \sum_{y \in \mathcal{Y}_i} \frac{\partial^2 F_i}{\partial p(y)^2} \left( P_{Y_{1}|X}(y|x) - P_{Y_{1}|X}(y|x') \right)^2 \geq 0, \tag{124}
\end{equation}
for any $x \in X \setminus \{x'\}$.

If for all $x \in X \setminus \{x'\}$ we have $D(P_{Y_{1}|X}(\cdot|x)||P_{Y_{1}|X}(\cdot|x')) = 0$, it follows that
\begin{equation}
p(y|W = i) = \sum_{x \in X} \alpha_{ix} P_{Y_{1}|X}(y|x)
\end{equation}
\begin{equation}
= \sum_{x \in X} \alpha_{ix} P_{Y_{1}|X}(y|x')
\end{equation}
\begin{equation}
= \sum_{x \in X \setminus \{x'\}} \alpha_{ix} P_{Y_{1}|X}(y|x') + \alpha_{ix'} P_{Y_{1}|X}(y|x')
\end{equation}
\begin{equation}
= (1 - \alpha_{ix'}) P_{Y_{1}|X}(y|x') + \alpha_{ix'} P_{Y_{1}|X}(y|x')
\end{equation}
\begin{equation}
= (1 - \alpha_{ix'}) P_{Y_{1}|X}(y|x) + \alpha_{ix'} P_{Y_{1}|X}(y|x)
\end{equation}
\begin{equation}
= P_{Y_{1}|X}(y|x), \tag{130}
\end{equation}
for any $i \in W$ and $y \in \mathcal{Y}_1$. In combination with the fact that the message is uniformly distributed on the message set $W$, we obtain
\begin{equation}
p(y|W \neq i) = P_{Y_{1}|X}(y|x). \tag{131}
\end{equation}
Hence, it is easy to show that
\begin{equation}
p(y) = P_{Y_{1}|X}(y|x), \tag{132}
\end{equation}
\begin{equation}
p_i(y) = p_i, \tag{133}
\end{equation}
for all $i \in W$ and $y \in \mathcal{Y}_1$. Therefore, we have
\begin{equation}
\mathbb{E} \left[ \ln \mathcal{H}(W|Y_1^n) - \ln \mathcal{H}(W|Y_1^{n+1}) | Y_1^n \right] = 0. \tag{134}
\end{equation}

Now, we treat the remaining case where the relative entropy is positive. For any $x \in X$ there always exists an $x' \in X \setminus \{x\}$ such that $D(P_{Y_{1}|X}(\cdot|x)||P_{Y_{1}|X}(\cdot|x')) > 0$. By choosing $x'$ as a fixed symbol satisfying the preceding condition, (124)
becomes a strict inequality. Therefore, $\beta_{ix}$ must be zero or one. Consequently, for all fixed $i \in \mathcal{W}$, all the values of $\beta_{ix}$ for all $x \in \mathcal{X}$ except for one are zero.

Similarly, for any $x \in \mathcal{X} \setminus \{x'\}$ such that $D(P_{Y_i|X}(\cdot|x)||P_{Y_i|X}(\cdot|x')) > 0$, we have

$$\frac{\partial^2 F_i}{\partial \alpha_{ix}^2} = \sum_{y \in \mathcal{Y}_i} (P_{Y_i|X}(y|x) - P_{Y_i|X}(y|x'))^2 \frac{(p(y) - p_i p(y|W = i))^2}{p_i p(y|W = i)^2} \times \left[ 1 - \left( \ln \frac{p(y)}{p_i p(y|W = i)} \right)^{-1} + \left( \ln \frac{p(y)}{p_i p(y|W = i)} \right)^{-2} \right] > 0. \quad (135)$$

Consequently, either $\alpha_{ix} = 0$ or $\alpha_{ix} = 1, x \in \mathcal{X}$.

From (108), (110), and (111) together with above results, we obtain

$$\mathbb{E} \left[ H(W|Y_1^n) - H(W|Y_1^{n+1}|Y_1^n) \right] \leq \max \left\{ 0, \max_{x,x'} \max_{\eta} \left\{ \sum_{y \in \mathcal{Y}_i} p(y) \ln \frac{\eta \ln \eta}{f(y) \ln f(y)} \right\} \right\}, \quad (136)$$

where $\eta \in \{p_1, p_2, \ldots, p_M\}$, $(x, x') \in \mathcal{X}^2$ and

$$p(y) = \eta P_{Y_i|X}(y|x) + (1-\eta)P_{Y_i|X}(y|x'), \quad (137)$$

$$f(y) = \eta \frac{P_{Y_i|X}(y|x)}{p(y)}. \quad (138)$$

We see from (137) and (138) that

$$\sum_{y \in \mathcal{Y}_i} p(y) \ln \frac{\eta \ln \eta}{f(y) \ln f(y)} = \sum_{y \in \mathcal{Y}_i} p(y) \ln \left[ \frac{\eta p(y)}{P_{Y_i|X}(y|x) P_{Y_i|X}(y|x')p(y)} \right] + \sum_{y \in \mathcal{Y}_i} p(y) \ln \left[ \frac{P_{Y_i|X}(y|x') \ln \eta}{p(y) \ln f(y)} \right]. \quad (139)$$

Note that

$$\frac{P_{Y_i|X}(y|x')}{p(y)} = \frac{1 - f(y)}{1 - \eta}. \quad (140)$$

It follows that

$$\ln \left[ \frac{P_{Y_i|X}(y|x') \ln \eta}{p(y) \ln f(y)} \right] = \ln \left[ \frac{(1 - f(y)) \ln \eta}{(1 - \eta) \ln f(y)} \right] \quad (141)$$

$$= \ln(1 - f(y)) - \ln(- \ln f(y)) - \ln(1 - \eta) - \ln(- \ln \eta). \quad (142)$$

From (138), we have

$$\sum_{y \in \mathcal{Y}_i} p(y) f(y) = \sum_{y \in \mathcal{Y}_i} \eta P_{Y_i|X}(y|x) = \eta. \quad (143)$$

Combining with the fact that the function $x \mapsto \ln(1-x) - \ln(-\ln x)$ is concave on $(0, 1)$ [11, pp. 424], we obtain the following almost surely

$$\sum_{y \in \mathcal{Y}_i} p(y) \left[ \ln(1 - f(y)) - \ln(- \ln f(y)) \right] \leq \ln(1 - \eta) - \ln(- \ln \eta). \quad (144)$$

Note that $p(y)$ and $\eta$ are random because they depend on $Y_1^n$ which is also random (cf. Eqns. (102) and (103)). This means that

$$\sum_{y \in \mathcal{Y}_i} p(y) \ln \left[ \frac{P_{Y_i|X}(y|x') \ln \eta}{p(y) \ln f(y)} \right] \leq 0. \quad (145)$$
In addition, observe that
\[
p(y) \ln \left[ \frac{p^2(y)}{P_{Y_1|X}(y|x)P_{Y_1|X}(y|x')} \right]
= (\eta P_{Y_1|X}(y|x) + (1 - \eta)P_{Y_1|X}(y|x')) \ln \left[ \frac{\left(\eta P_{Y_1|X}(y|x) + (1 - \eta)P_{Y_1|X}(y|x')\right)^2}{P_{Y_1|X}(y|x)P_{Y_1|X}(y|x')} \right]
\leq (\eta P_{Y_1|X}(y|x) + (1 - \eta)P_{Y_1|X}(y|x')) \ln \left[ \frac{\eta P_{Y_1|X}(y|x)}{P_{Y_1|X}(y|x')} + (1 - \eta)\ln \frac{P_{Y_1|X}(y|x')}{P_{Y_1|X}(y|x)} \right]
\leq \max \left\{ 0, \eta P_{Y_1|X}(y|x) \ln \left[ \frac{P_{Y_1|X}(y|x)}{P_{Y_1|X}(y|x')} \right] + (1 - \eta)P_{Y_1|X}(y|x') \ln \left[ \frac{P_{Y_1|X}(y|x')}{P_{Y_1|X}(y|x)} \right] \right\}.  
\]  
(146)
(147)
(148)
(149)
(150)
(151)

Here, note that the inequality in (150) can be removed if \( \ln \left[ \eta P_{Y_1|X}(y|x) + (1 - \eta)\ln \frac{P_{Y_1|X}(y|x')}{P_{Y_1|X}(y|x)} \right] \leq 0 \). Inequality (151) follows from the convexity of the function \( x \mapsto x \ln x \) for \( x > 0 \).

Hence, we obtain
\[
\sum_{y \in \mathcal{Y}_1} p(y) \ln \left[ \frac{p^2(y)}{P_{Y_1|X}(y|x)P_{Y_1|X}(y|x')} \right]
\leq \max \left\{ 0, \eta \sum_{y \in \mathcal{Y}_1} P_{Y_1|X}(y|x) \ln \left[ \frac{P_{Y_1|X}(y|x)}{P_{Y_1|X}(y|x')} \right] + (1 - \eta) \sum_{y \in \mathcal{Y}_1} P_{Y_1|X}(y|x') \ln \left[ \frac{P_{Y_1|X}(y|x')}{P_{Y_1|X}(y|x)} \right] \right\}.
\]  
(152)
(153)

From (136), (145), and (153) we have (101) for \( j = 1 \). We obtain the inequalities for \( j \in \{2, 3, \ldots, K\} \) analogously. \( \blacksquare \)

Lemma 7. For any \( n \geq 0 \) and \( y \in \mathcal{Y}_j \) the following inequalities hold almost surely (cf. Definition 4)
\[
\ln \mathcal{H}(W|Y^n_j) - \ln \mathcal{H}(W|Y^n_{j+1}) \big| Y^n_j, Y^n_{j, n+1} = y \leq \ln T_j,
\]  
(154)

for all \( j = 1, 2, \ldots, K \). The conditioning on the random variable \( Y^n_j \) and the event \( \{Y^n_{j, n+1} = y\} \) means that the inequalities (154) hold almost surely \( Y^n_j \) (i.e., for all realizations of \( Y^n_j \)) for a fixed realization of \( Y^n_{j, n+1} = y \).

Proof: This proof is on Burnashev’s argument in [7] with some additional arguments in the corresponding optimization problem to account for the fact that the transmitted signal at time \( n+1 \), i.e. \( X_{n+1} \), depends on \( W \) and all \( Y^n_1, \ldots, Y^n_{K-1} \). Note the inequality [7, pp. 264]
\[
\frac{1}{K} \sum_{i=1}^K \alpha_i \beta_i \geq \min_i \alpha_i \beta_i, \quad \alpha_i, \beta_i \geq 0.
\]  
(155)

Using the same notation as in Lemma 6 and the fact that the function \( x \mapsto -x \ln x \) is concave, we have for any \( y \in \mathcal{Y}_1 \) that
\[
\psi(y) := \frac{\mathcal{H}(W|Y^n_1)Y^n_1}{\mathcal{H}(W|Y^n_1)} \big| Y^n_1, \{Y^n_{1, n+1} = y\}
= -\sum_{i=1}^M p_i(y) \ln p_i(y) - \sum_{i=1}^M p_i \ln p_i
\geq \min_i \left[ \frac{p_i(y) \ln p_i(y)}{p_i \ln p_i} \right].
\]  
(156)
(157)
(158)

It follows that
\[
-\ln \psi(y) = \ln \mathcal{H}(W|Y^n_1) - \ln \mathcal{H}(W|Y^n_{n+1}) \big| Y^n_1, \{Y^n_{1, n+1} = y\}
\leq \ln \left\{ \max_i \left[ \frac{p_i \ln p_i}{p_i(y) \ln p_i(y)} \right] \right\}.
\]  
(159)
(160)
Similarly to the argument in the proof of Lemma 6, we first disregard all other constraints and consider the optimization (maximization) problem in the \( \{ \ldots \} \) in (160) subject to the constraints

\[
\sum_{x \in \mathcal{X}} \alpha_{ix} = 1, \quad (161) \\
\sum_{x \in \mathcal{X}} \beta_{ix} = 1, \quad (162) \\
\alpha_{ix} \geq 0, \quad (163) \\
\beta_{ix} \geq 0. \quad (164)
\]

Note that we have

\[
p_i(y) = \frac{p_i \sum_{x \in \mathcal{X}} \alpha_{ix} P_{Y_i|X}(y|x)}{p_i \sum_{x \in \mathcal{X}} \alpha_{ix} P_{Y_i|X}(y|x) + (1 - p_i) \sum_{x \in \mathcal{X}} \beta_{ix} P_{Y_i|X}(y|x)}. \quad (165)
\]

Define

\[
\chi_{x,x',\eta} := \frac{\eta P_{Y_i|X}(y|x)}{\eta P_{Y_i|X}(y|x) + (1 - \eta) P_{Y_i|X}(y|x')}, \quad (166)
\]

and

\[
A_{x,x',\eta} := \frac{\eta \ln \eta}{\chi_{x,x',\eta} \ln \chi_{x,x',\eta}}. \quad (167)
\]

Using the same arguments as Lemma 6, we can show that

\[
\frac{p_i \ln p_i}{p_i(y) \ln p_i(y)} \leq \max \left\{ 0, \max_{0 \leq \eta \leq 1} \max_{x,x' \in \mathcal{X}} A_{x,x',\eta} \right\}. \quad (168)
\]

Now, if \( P_{Y_i|X}(y|x') \geq P_{Y_i|X}(y|x) \), we have

\[
\max_{0 \leq \eta \leq 1} A_{x,x',\eta} = \frac{P_{Y_i|X}(y|x')}{P_{Y_i|X}(y|x)}. \quad (169)
\]

If \( P_{Y_i|X}(y|x') < P_{Y_i|X}(y|x) \), then by using the fact that for any \( 0 \leq x \leq 1 \) and \( 1 \leq a \leq 1/x \) we have

\[
\frac{x \ln x}{(ax) \ln(ax)} \leq \frac{1 - x}{1 - ax}, \quad (170)
\]

we obtain

\[
\max_{0 \leq \eta \leq 1} A_{x,x',\eta} \leq \max_{0 \leq \eta \leq 1} \frac{1 - \eta}{1 - \chi_{x,x',\eta}} = \max_{0 \leq \eta \leq 1} \frac{\eta P_{Y_i|X}(y|x) + (1 - \eta) P_{Y_i|X}(y|x')}{P_{Y_i|X}(y|x')} = \frac{P_{Y_i|X}(y|x)}{P_{Y_i|X}(y|x')}. \quad (171)
\]

Consequently, the conclusion of the lemma in (101) follows by combining (160), (168), and (173).

\[\text{Lemma 8.} \text{ The following inequalities for each } 1 \leq j \leq K \text{ hold almost surely}
\]

\[
\mathbb{E} \left[ (\ln \mathcal{H}(W|Y^n) - \ln \mathcal{H}(W|Y^{n+1})_{a}|Y^n) \right] \leq \varphi(a) \quad (174)
\]

where

\[
\varphi(a) := \max_{1 \leq j \leq K} (\ln T_j)_{a}. \quad (175)
\]

\[\text{Under the condition } B_{\max} < \infty, \varphi(a) = 0 \text{ for a sufficiently large.}
\]

\[\text{Proof:} \text{ From Lemma 7 we know that for any } n \geq 0 \text{ and } y \in \mathcal{Y}_1 \text{ we have the following inequalities}
\]

\[
\ln \mathcal{H}(W|Y^n) - \ln \mathcal{H}(W|Y^{n+1})_{a}\bigg|Y^n, \{Y_{1,n+1} = y_1\} \leq \ln T_1. \quad (176)
\]

Since \( \ln T_1 \) is non-negative and using the fact that if \( x \leq y \) and \( y \geq 0 \) we have \( (x)_{a} \leq (y)_{a} \) for any \( a \in \mathbb{R} \), we obtain

\[
(\ln \mathcal{H}(W|Y^n) - \ln \mathcal{H}(W|Y^{n+1})_{a}\bigg|Y^n, \{Y_{1,n+1} = y_1\} \leq (\ln T_1)_{a}. \quad (177)
\]
where the conditioning on \( \{Y_{1,n+1} = y_1\} \) in (178) means that \( Y_{1,n+1} \) in the term \( \ln \mathcal{H}(W|Y_1^{n+1}) \) takes on the value \( y_1 \). Similarly, for the other \( j = 2, \ldots, K \), we have

\[
E \left[ \left( \ln \mathcal{H}(W|Y_j^n) - \ln \mathcal{H}(W|Y_j^{n+1}) \right)_a \right]_{Y_j^n} \leq (\ln T_j)_a.
\]

(181)

Recall the definition of \( \varphi \) in (175). We note that since \( B_{\max} < \infty \), we have \( P_{Y_j|X}(y|x) > 0 \) for all \( x \in X \) and \( y \in Y_j \) for all \( j = 1, 2, \ldots, K \). It follows that \( T_j < \infty \) for all \( j = 1, 2, \ldots, K \) and so \( \varphi(a) \) is 0 for \( a \) sufficiently large. This concludes the proof of the lemma.

The converse part of Theorem 1 can be stated succinctly as follows.

\textbf{Lemma 9.} The reliability function for a DM-BC with common message and VLFT satisfies

\[
E(R) \leq \min_{1 \leq j \leq K} B_j \left( 1 - \frac{R}{C_j} \right), \quad \forall R < \overline{C}.
\]

(182)

\textbf{Proof:} The proof is similar to Burnashev’s arguments in [7] and [11]. There are some subtle differences, hence for completeness, we provide the entire proof. Here, a combination of [7] and [11] makes the proof that the sequences \( \xi_n^{(j)} \) (as defined in (183) in the following) are submartingales simpler. It is enough to show that (182) holds for \( \mathbb{P}(\tau < \infty) = 1 \) and \( B_{\max} < \infty \). Now, as in Burnashev’s arguments [11], we consider the \( K \) random sequences

\[
\xi_n^{(j)} := \begin{cases} 
C_j^{-1} \mathcal{H}(W|Y_j^n) + n, & \text{if } \mathcal{H}(W|Y_j^n) \geq A_j, \\
B_j^{-1} \ln \mathcal{H}(W|Y_j^n) + b + n, & \text{if } \mathcal{H}(W|Y_j^n) \leq A_j.
\end{cases}
\]

(183)

where \( A_j \) is the largest positive root of the following equation in \( x \):

\[
x = \frac{\ln x}{C_j} + b.
\]

(184)

For \( b \) sufficiently large, we will show that the \( K \) sequences \( \xi_n^{(j)} \) respectively form submartingales with respect to the filtrations \( \{\sigma(Y_j^n)\}_{n=0}^{\infty} \) for \( j = 1, 2, \ldots, K \). Note that when \( b \) sufficiently large, (184) can be shown to have two distinct positive roots \( a_j, A_j \) and that \( A_j/a_j \) can be made arbitrarily large by increasing \( b \) [7, pp. 256].

Indeed, first we suppose that \( \mathcal{H}(W|Y_1^n) \leq A_1 \). Then, we obtain

\[
E \left[ \xi_n^{(1)} - \xi_{n+1}^{(1)} | Y_1^n \right] = -1 + E \left[ B_1^{-1} \ln \mathcal{H}(W|Y_1^n) + b - (B_1^{-1} \ln \mathcal{H}(W|Y_1^{n+1}) + b) 1\{\mathcal{H}(W|Y_1^{n+1}) \leq A_1 \} \right]
\]

\[
- C_1^{-1} \mathcal{H}(W|Y_1^{n+1}) 1\{\mathcal{H}(W|Y_1^{n+1}) > A_1 \} Y_1^n
\]

\[
\leq -1 + B_1^{-1} E \left[ \ln \mathcal{H}(W|Y_1^n) - \ln \mathcal{H}(W|Y_1^{n+1}) | Y_1^n \right]
\]

\[
\leq -1 + B_1^{-1} \times B_1 = 0.
\]

(185)

(186)

(187)

Here, (186) follows from the fact that \( x/C_1 \geq (\ln x)/B_1 + b \) for \( x \geq A_1 \) and (187) follows from Lemma 6.

Now, suppose that \( \mathcal{H}(W|Y_1^n) > A_1 \). Let \( a_1 \) be the smaller of the two positive roots of (184). Then, for \( b \) sufficiently large
we obtain
\[
\mathbb{E} \left[ \xi_n^{(1)} - \xi_{n+1}^{(1)} \mid Y_n \right] = -1 + C_1^{-1} \mathbb{E} \left[ H(W|Y_n^n) - H(W|Y_n^{n+1}) | Y_n \right] \\
+ \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] \\
\leq -1 + C_1^{-1} C_1 + \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] \\
= \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] \\
+ \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{A_1 < H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] \\
\leq \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] (198)
\]

(189)
\[
= \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] \\
+ \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{A_1 < H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] (199)
\]

(200)
\[
\leq \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] + \mathbb{E} \left[ (C_1^{-1} H(W|Y_n^{n+1}) - B_1^{-1} \ln H(W|Y_n^{n+1}) - b) \mathbf{1}\{A_1 < H(W|Y_n^{n+1}) \leq A_1\} | Y_n \right] (201)
\]

Here, (188) follows from Lemma 4, (202) follows from the fact that \( C_1^{-1} H(W|Y_n^{n+1}) \leq B_1^{-1} \ln H(W|Y_n^{n+1}) + b \) when \( a_1 < H(W|Y_n^{n+1}) \leq A_1 \), (203) follows from the fact that if \( H(W|Y_n^{n+1}) \leq a_1 \) and \( H(W|Y_n^n) > A_1 \) we have \( C_1^{-1} H(W|Y_n^{n+1}) - b \leq C_1^{-1} a_1 - b = B_1^{-1} \ln a_1 \leq B_1^{-1} \ln a_1 = B_1^{-1} \ln H(W|Y_n^{n+1}) \), (194) follows from the assumption that \( H(W|Y_n^n) > A_1 \), and (196), (197) follow from the Lemma 8 and the fact that \( A_1/a_1 \) can be made arbitrarily large by increasing \( b \). The above arguments leading to (197) and (197) together with (187) confirm that \( \xi_n^{(1)} \) forms a submartingale with respect to the filtration \( \{\sigma(Y_n^n)\}_{n=0}^{\infty} \). A completely analogous argument goes through for \( j = 2, 3, \ldots, K \).

Now, since we know that \( \xi_0^{(1)} = \mathbb{E}[\xi_0^{(1)}] \leq \mathbb{E}[\xi_{n+1}^{(1)}] \leq \limsup_{n \to \infty} \mathbb{E}[\xi_n^{(1)}] \), it follows that for \( N \) sufficiently large we have
\[
C_1^{-1} \ln M = \xi_0^{(1)} (199)
\]
\[
\leq \limsup_{n \to \infty} \mathbb{E}[\xi_n^{(1)}] \\
\leq C_1^{-1} \limsup_{n \to \infty} \mathbb{E} \left[ H(W|Y_1^{n+1}) | \{H(W|Y_1^{n+1}) \geq A_1\} \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) | \{H(W|Y_1^{n+1}) \leq A_1\} \right] + b (200)
\]
\[
\leq C_1^{-1} \limsup_{n \to \infty} \mathbb{E} \left[ H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (201)
\]
\[
\leq C_1^{-1} \limsup_{n \to \infty} \mathbb{E} \left[ H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (202)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (203)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (204)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (205)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (206)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + b (207)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + O(1) (208)
\]
\[
\leq C_1^{-1} \left[ 1 + \mathbb{P}(R_N, N) \ln M \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + \mathbb{E} \left[ \ln H(W|Y_1^{n+1}) \right] + O(1) (209)
\]

Here, (199) follows from (183) and \( H(W|Y_1^{n}) = H(W) = \ln M \), (201) follows from (183) and (198), (203) follows from the fact that for any random variable \( G, \mathbb{E}[\ln G] \{G \leq g\} \leq \ln \mathbb{E}[G] \) for \( g \geq 1 \) (which is assured by taking \( b \) sufficiently large so \( A_1 \) eventually becomes larger than 1), (205) follows from Lemma 3, (207) follows from the fact that \( -x \ln x \leq 1/e \) for \( 0 \leq x \leq 1 \), and (208) follows from the fact that \( B_1 < \infty \).
Therefore, we obtain
\begin{align}
\ln M &\leq 1 + \Pr_e(R_N, N) \ln M + C_1\mathbb{E} \{ \tau_1 \} + C_1 B_1^{-1} \ln \Pr_e(R_N, N) + C_1 B_1^{-1} \ln (\ln M - \ln \Pr_e(R_N, N)) + O(1) \\
&\leq 1 + \Pr_e(R_N, N) \ln M + C_1 N + C_1 B_1^{-1} \ln \Pr_e(R_N, N) + C_1 B_1^{-1} \ln (\ln M - \ln \Pr_e(R_N, N)) + O(1).
\end{align}
(210)
(211)
A similar bound holds for the other branches indexed by \( j = 2, \ldots, K \). It follows that for all \( j = 1, 2, \ldots, K \), we have
\begin{equation}
E(R) \leq \lim_{N \to \infty} \inf - \frac{\ln \Pr_e(R_N, N)}{N} \leq \lim_{N \to \infty} \sup \frac{\ln \Pr_e(R_N, N)}{N} \leq \lim_{N \to \infty} \sup B_j \left( 1 - \frac{R_N}{C_j} \right),
\end{equation}
(212)
(213)
(214)
\begin{equation}
= B_j \left( 1 - \frac{\lim \inf_{N \to \infty} R_N}{C_j} \right),
\end{equation}
(215)
\begin{equation}
\leq B_j \left( 1 - \frac{R}{C_j} \right)
\end{equation}
(216)
for all \( R < C_j \). Therefore, we finally obtain (182) as desired.

Let us now say a few words about the novelties in the converse proof vis-à-vis Burnashev’s works in [7] and [11]. In the original work on DMCs with variable-length feedback by Burnashev [7], he proved Lemma 6 for the case \( K = 1 \) under the assumption that \( \mathcal{H}(W|Y^n) \) is bounded. Hence, the construction of submartingales in Lemma 9 was more complicated. More specifically, Burnashev needed to make use of [7, Lemma 5], and the constructed submartingale is a combination of two other submartingales in [7, Eqn. (4.20)]. This is meant to account for the constraint concerning the boundedness of \( \mathcal{H}(W|Y^n) \). In a later work for the related problem of sequential hypothesis testing [11], Burnashev proved a lemma similar to Lemma 6 under no constraints on \( \mathcal{H}(W|Y^n) \). However, as we pointed out in the remark in (47), this direct proof does not lead to the desired result for our setting in which \( K \geq 2 \). We need to adapt and combine the two different proof techniques in [7] and [11] to prove Lemma 6.

**Appendix A**

**Proof of Lemma 1**

**Proof:** We use the same proof technique as in [26, Lemma 8]. In Lemma 1, \( K \) may be greater than or equal to 3, so a direct application of [26, Lemma 8] is cumbersome. However, since we are not seeking tight bounds on the second-order term in the asymptotic expansion of \( \mathbb{E}(\max \{ X_{1L}, X_{2L}, X_{3L}, \ldots, X_{KL} \}) \) as in [26, Lemma 8], it is enough to show that if the following conditions hold
\begin{equation}
\mathbb{E}(X_{jL}) = L + O(\sqrt{L}), \quad j = 1, 2, \quad \text{and}
\end{equation}
(217)
\begin{equation}
\var(X_{jL}) = O(L), \quad j = 1, 2
\end{equation}
(218)
then, we have
\begin{equation}
\mathbb{E}(\max \{ X_{1L}, X_{2L} \}) = L + O(\sqrt{L})
\end{equation}
(219)
\begin{equation}
\var(\max \{ X_{1L}, X_{2L} \}) = O(L)
\end{equation}
(220)
This is because if the desired statement in (50) holds for two sequences of random variables, it will hold for three if \( \var(\max \{ X_1, X_2 \}) = O(L) \) since \( \max \{ X_1, X_2, X_3 \} = \max \{ \max \{ X_1, X_2 \}, X_3 \} \). This argument obviously holds verbatim if we have \( K \) sequences of random variables. Now, observe that
\begin{equation}
\max \{ X_{1L}, X_{2L} \} = \frac{1}{2} \left[ X_{1L} + X_{2L} + |X_{1L} - X_{2L}| \right].
\end{equation}
(221)
Moreover, we have
\begin{equation}
\mathbb{E} \left( |X_{1L} - X_{2L}|^2 \right) + \mathbb{E} \left( |X_{1L} + X_{2L}|^2 \right) = 2[\mathbb{E}(X_{1L}^2) + \mathbb{E}(X_{2L}^2)]
\end{equation}
(222)
\begin{equation}
= 2 \left[ \var(X_{1L}) + (\mathbb{E}[X_{1L}])^2 + \var(X_{2L}) + (\mathbb{E}[X_{2L}])^2 \right]
\end{equation}
(223)
\begin{equation}
= 2 \left[ O(L) + (\mathbb{E}[X_{1L}])^2 + (\mathbb{E}[X_{2L}])^2 \right].
\end{equation}
(224)
In addition, we also have
\begin{equation}
\mathbb{E} \left( |X_{1L} + X_{2L}|^2 \right) \geq (\mathbb{E}[X_{1L} + X_{2L}])^2.
\end{equation}
(225)
Hence, we obtain
\[
(E|X_{1L} - X_{2L}|)^2 \leq E \left( |X_{1L} - X_{2L}|^2 \right) 
\]
\[
\leq 2 \left[ O(L) + (E|X_{1L}|)^2 + (E|X_{2L}|)^2 \right] - (E|X_{1L} + X_{2L}|)^2 \quad (226)
\]
\[
= O(L) + (E[X_{1L}] - E[X_{2L}])^2 \quad (227)
\]
\[
= O(L) + (L + O(\sqrt{L}) - L - O(\sqrt{L}))^2 \quad (228)
\]
\[
= O(L) \quad (229)
\]

It follows from (217), (221), and (230) that
\[
E[\max\{X_{1L}, X_{2L}\}] = \frac{1}{2} E[X_{1L} + X_{2L}] + O(\sqrt{L}) \quad (230)
\]
\[
= L + O(\sqrt{L}). \quad (231)
\]

Now, we estimate the variance as follows:
\[
\text{Var}(\max\{X_{1L}, X_{2L}\}) = E(\max\{X_{1L}, X_{2L}\}) - E[\max\{X_{1L}, X_{2L}\}]^2 \quad (232)
\]
\[
= \frac{1}{2} E(X_{1L} + X_{2L} + |X_{1L} - X_{2L}| - E[X_{1L} + X_{2L} + |X_{1L} - X_{2L}|]) \quad (233)
\]
\[
= \frac{1}{2} E \left[ (X_{1L} - E[X_{1L}]) + X_{2L} - E[X_{2L}] + |X_{2L} - X_{2L}| - E[|X_{2L} - X_{2L}|] \right] \quad (234)
\]
\[
\leq \frac{1}{2} E \left[ (X_{1L} - E[X_{1L}])^2 + (X_{2L} - E[X_{2L}])^2 + |X_{2L} - X_{2L}|^2 \right] \quad (235)
\]
\[
= \frac{1}{2} \left[ \text{Var}(X_{1L}) + \text{Var}(X_{2L}) + \text{Var}(X_{2L}) \right] \quad (236)
\]
\[
\leq \frac{1}{2} \left[ \text{Var}(X_{1L}) + \text{Var}(X_{2L}) + E[|X_{1L} - X_{2L}|^2] \right] \quad (237)
\]
\[
\leq \frac{1}{2} \left[ O(L) + O(L) + O(L) \right] \quad (238)
\]
\[
= O(L). \quad (239)
\]

Here, (236) follows from the Cauchy-Schwarz inequality and (239) follows from (218) and (230). Since \(\text{Var}(\max\{X_{1L}, X_{2L}\}) \geq 0\), we obtain from (240) that
\[
\text{Var}(\max\{X_{1L}, X_{2L}\}) = O(L). \quad (241)
\]

**APPENDIX B**

**PROOF OF LEMMA 3**

**Proof:** We have
\[
E \left[ \mathcal{H}(W|Y^n_{1^\tau_1}) \right] = \sum_{i=1}^n E \left[ \mathcal{H}(W|Y^n_i)|\tau_1 = i \right] P(\tau_1 = i) + E \left[ \mathcal{H}(W|Y^n)|\tau_1 > n \right] P(\tau_1 > n). \quad (242)
\]

Using the fact that \(\mathcal{H}(W|Y^n_1)\) is almost surely bounded by \(\ln M\), we have for two natural numbers \(m < n\) that
\[
|\mathcal{H}(W|Y^n_{1^\tau_1}) - \mathcal{H}(W|Y^n_{1^m})| \leq E \left| \mathcal{H}(W|Y^n_1)|\tau_1 > n \right] P(\tau_1 > n)
\]
\[
+ \sum_{i=m+1}^n E \left[ \mathcal{H}(W|Y^n_i)|\tau_1 = i \right] P(\tau_1 = i) + E \left[ \mathcal{H}(W|Y^n)|\tau_1 > m \right] P(\tau_1 > m) \quad (243)
\]
\[
\leq M \left[ P(\tau_1 > n) + \sum_{i=m+1}^n P(\tau_1 = i) + P(\tau_1 > m) \right] \quad (244)
\]
\[
= 2P(\tau_1 > m) \ln M \to 0, \quad \text{as} \quad m \to \infty, \quad (245)
\]

which yields that \(\lim_{n \to \infty} E \left[ \mathcal{H}(W|Y^n_{1^\tau_1}) \right]\) exists since \(\mathbb{R}\) is complete.

Define the error event
\[
\mathcal{E} := \{ \hat{W}_1 \neq W \}. \quad (246)
\]
By Fano’s inequality we have
\[
H(W|\hat{W}_1, \tau_1 = n) \leq h[\mathbb{P}(\hat{W}_1 \neq W|\tau_1 = n)] + \mathbb{P}(\hat{W}_1 \neq W|\tau_1 = n) \ln(M-1) \\
= h[\mathbb{P}(E|\tau_1 = n)] + \mathbb{P}(E|\tau_1 = n) \ln(M-1).
\] (247)

Hence,
\[
H(W|\hat{W}_1, \tau_1 = n) \leq h[\mathbb{P}(E|\tau_1 = n)] + \mathbb{P}(E|\tau_1 = n) \ln(M-1).
\] (249)

It follows that
\[
\sum_{j=1}^{M} H(W|\hat{W}_1 = j, \tau_1 = n) \mathbb{P}(\hat{W}_1 = j|\tau_1 = n) \leq h[\mathbb{P}(E|\tau_1 = n)] + \mathbb{P}(E|\tau_1 = n) \ln(M-1).
\] (250)

Now, for any random variable $Z$, define an $M$-tuple (vector)
\[
\mathbf{P}_{W|Z=z} := (P_{W|Z}(1|z), P_{W|Z}(2|z), \ldots, P_{W|Z}(M|z)).
\] (251)

In the following, we overload the notation $\mathcal{H}(\mathbf{P})$ to mean the entropy of the probability mass function defined by the vector $\mathbf{P}$. Observe that
\[
H(W|\hat{W}_1 = j, \tau_1 = n) = \mathcal{H}(\mathbf{P}_{W|\hat{W}_1=j,\tau_1=n}),
\] (252)

where $Z$ in (251) is replaced by $(\hat{W}, \tau_1)$ and $z$ by $(j, n)$. Now, we see that
\[
P_{W|\hat{W}_1,\tau_1}(w|j,n) = \sum_{y_j^n} P_{W|Y^n_1,\hat{W}_1,\tau_1}(w|y_j^n,j,n)P_{Y^n_1|\hat{W}_1,\tau_1}(y_j^n|j,n) \\
= \sum_{y_j^n} P_{W|Y^n_1}(w|y^n_j)P_{Y^n_1|\hat{W}_1,\tau_1}(y_j^n|j,n) \\
= \mathbb{E} \left[ P_{W|Y^n_1}(W = w|Y^n_1) \middle| \hat{W}_1 = j, \tau_1 = n \right].
\] (255)

Here, (254) follows the Markov chain $W - Y^n_1 - (\hat{W}_1, 1\{\tau_1 = n\})$.

In vector notation, (255) means that
\[
P_{W|\hat{W}_1=j,\tau_1=n} = \mathbb{E} \left[ \mathbf{P}_{W|Y^n_1} \middle| \hat{W}_1 = j, \tau_1 = n \right],
\] (256)

where the expectation on the right is over the randomness of $Y^n_1$. Using (252) and (256) and Jensen’s inequality noting that $\mathbf{P} \mapsto \mathcal{H}(\mathbf{P})$ is concave, we have
\[
H(W|\hat{W}_1 = j, \tau_1 = n) = \mathcal{H} \left( \mathbb{E} \left[ \mathbf{P}_{W|Y^n_1} \middle| \hat{W}_1 = j, \tau_1 = n \right] \right) \\
\geq \mathbb{E} \left[ \mathcal{H} \left( \mathbf{P}_{W|Y^n_1} \middle| \hat{W}_1 = j, \tau_1 = n \right) \right].
\] (258)

From (250) and (258) we obtain
\[
\sum_{j=1}^{M} \mathbb{E} \left[ \mathcal{H} \left( \mathbf{P}_{W|Y^n_1} \middle| \hat{W}_1 = j, \tau_1 = n \right) \right] \mathbb{P}(\hat{W}_1 = j|\tau_1 = n) \leq h[\mathbb{P}(E|\tau_1 = n)] + \mathbb{P}(E|\tau_1 = n) \ln(M-1).
\] (259)

Hence,
\[
\mathbb{E} \left[ \mathcal{H} \left( \mathbf{P}_{W|Y^n_1} \middle| \tau_1 = n \right) \right] \leq h[\mathbb{P}(E|\tau_1 = n)] + \mathbb{P}(E|\tau_1 = n) \ln(M-1)
\] (260)

It follows that for $N$ sufficiently large
\[
\mathbb{E}[\mathcal{H}(W|Y^n_1)] = \sum_{n=1}^{\infty} \mathbb{E} \left[ \mathcal{H} \left( \mathbf{P}_{W|Y^n_1} \middle| \tau_1 = n \right) \right] \mathbb{P}(\tau_1 = n) \\
\leq \sum_{n=1}^{\infty} \left[ h[\mathbb{P}(E|\tau_1 = n)] + \mathbb{P}(E|\tau_1 = n) \ln(M-1) \right] \mathbb{P}(\tau_1 = n) \\
\leq h(\mathbb{P}(E)) + \mathbb{P}(E) \ln(M-1),
\] (262)

\[
\leq h(\mathbb{P}(E)) + \mathbb{P}(E) \ln(M-1) \\
\] (263)

\[
\leq h(\mathbb{P}_c(R_N, N)) + \mathbb{P}_c(R_N, N) \ln(M-1).
\] (264)

Here, (262) follows from (260), (263) follows from the fact that the function $h(x)$ is concave and (264) follows from the increasing property of the entropy function $h(x)$ for $0 \leq x \leq 1/2$, $E \subset \cup_{j=1}^{K} \{W_j \neq W\}$, and $\mathbb{P}_c(R_N, N) \to 0$ as $N \to \infty$ (so $\mathbb{P}_c(R_N, N) \leq 1/2$ for $N$ sufficiently large). A completely analogous argument applies for $j = 2, 3, \ldots, K$. ■
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