SPIN(9) GEOMETRY OF THE OCTONIONIC HOPF FIBRATION

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Abstract. We deal with Riemannian properties of the octonionic Hopf fibration $S^{15} \to S^8$, in terms of the structure given by its symmetry group Spin(9). In particular, we show that any vertical vector field has at least one zero, thus reproving the non-existence of $S^1$ subfibrations. We then discuss Spin(9)-structures from a conformal viewpoint and determine the structure of compact locally conformally parallel Spin(9)-manifolds. Eventually, we give a list of examples of locally conformally parallel Spin(9)-manifolds.

1. Introduction

There are some features that distinguish $S^{15}$ among spheres of arbitrary dimension. For example, $S^{15}$ is the only sphere that admits three homogeneous Einstein metrics (see [Zil82]), and the only one that appears as regular orbit in three cohomogeneity one actions on projective spaces, namely of SU(8), Sp(4) and Spin(9) on $\mathbb{C}P^8$, $\mathbb{H}P^4$ and $\mathbb{O}P^2$ respectively (see [Kol02]). Moreover, according to a famous problem of vector fields on spheres, $S^{15}$ is the lowest dimensional sphere with more than 7 linearly independent vector fields (cf. for example [Hus94]). Finally, it has been shown that the Killing superalgebra of $S^{15}$ is isomorphic to the exceptional compact real Lie algebra $e_8$ (see [FO08]).

All of these features can somehow be traced back to the transitive action of the subgroup $\text{Spin}(9) \subset \text{SO}(16)$ on the octonionic Hopf fibration $S^{15} \to S^8$. This latter has a quite exceptional character: it does not admit any $S^1$-subfibration (see [LV92]), and there is no Hopf fibration over the Cayley projective plane $\mathbb{O}P^2$, although its volume is the quotient of those of the spheres $S^{23}$ and $S^7$, natural candidates to its possible total space and fiber (cf. [Ber 72, page 8]).

The mentioned characterizations of $S^{15}$ and the role of Spin(9) in 16-dimensional Riemannian geometry have been a first motivation for the present paper. In this respect, a first result we get is the following:

**Theorem A.** Any global vector field on $S^{15}$ which is tangent to the fibers of the octonionic Hopf fibration $S^{15} \to S^8$ has at least one zero.

Note that the non-existence of $S^1$-subfibrations follows (cf. results obtained in [LV92] and Corollary [LV92]).

A second motivation for this paper is to complete a general scheme of description for metrics which are locally conformally parallel with respect to the $G$-structures that refer to Riemannian holonomies. We next spin this general scheme. We say that we have a **locally conformally parallel $G$-structure** on a manifold $M$ if one has a Riemannian metric $g$ on $M$, a covering $U = \{U_\alpha\}_{\alpha \in A}$

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of $M$, and for each $\alpha \in A$ a metric $g_\alpha$ defined on $U_\alpha$ which has holonomy contained in $G$ such that the restriction of $g$ to each $U_\alpha$ is conformal to $g_\alpha$:

$$g|_{U_\alpha} = e^{f_\alpha}g_\alpha$$

for some smooth map $f_\alpha$ defined on $U_\alpha$.

Some of the possible cases here are:

- $G = U(n)$, where we have the \textit{locally conformally Kähler metrics};
- $G = \text{Sp}(n) \cdot \text{Sp}(1)$, yielding the \textit{locally conformally quaternion Kähler metrics};
- $G = \text{Spin}(9)$, which is the case we are dealing with.

In any of the cases above, one can show that for each overlapping $U_\alpha, U_\beta$ the functions $f_\alpha, f_\beta$ differ by a constant:

$$f_\alpha - f_\beta = c_{\alpha, \beta} \text{ on } U_\alpha \cap U_\beta.$$

This implies that $df_\alpha = df_\beta$ on $U_\alpha \cap U_\beta \neq \emptyset$, hence defining a global, closed 1-form, usually denoted by $\theta$ and called the \textit{Lee form}. Its metric dual with respect to $g$ is denoted by $B$:

$$B = \theta^\sharp$$

and is called the \textit{Lee vector field}.

The case $G = U(n)$ is extensively studied; see for instance \cite{DO98}.

Choosing $G$ to be $\text{Sp}(n)$ or $\text{Sp}(n) \cdot \text{Sp}(1)$, we get close relations to 3-Sasakian geometry; see \cite{OP97} or the surveys \cite{BG99}, \cite{CP99}. Finally, locally conformally parallel $G_2$ and Spin(7)-structures have been studied in \cite{IPP06}, and they relate to nearly parallel SU(3) and G2 geometries, respectively.

In the case we deal with in this paper, it is a classical result by D. Alekseevsky that holonomy $\text{Spin}(9)$ is only possible on manifolds that are either flat or locally isometric to $\mathbb{O}P^2$ or to the hyperbolic Cayley plane $\mathbb{O}H^2$ (see \cite{Ale68} and \cite{BrGr72}). Still, weakened holonomy conditions have been also considered. In particular, the article \cite{Fri01} points out how, exactly like in the frameworks of structure groups $U(n)$ and $G_2$, one can obtain 16 classes of $\text{Spin}(9)$-structures.

One of these classes consists of structures of \textit{vectorial type} (see \cite{AF06} and \cite[page 148]{Fri01}); we show that this class fits into the locally conformally parallel scheme above (see Remark 6.2).

Besides this Remark, our contribution to the completion of the above general scheme with the case $G = \text{Spin}(9)$ consists in the following Theorems.

\textbf{Theorem B.} Let $M^{16}$ be a compact manifold equipped with a locally, non globally, conformally parallel $\text{Spin}(9)$ metric $g$. Then:

1. The Riemannian universal covering $(\tilde{M}, \tilde{g})$ of $M$ is conformally equivalent to the Euclidean space $\mathbb{R}^{16} \setminus \{0\}$, that is, the Riemannian cone over $S^{15}$, and $M$ is finitely isometrically covered by $S^{15} \times \mathbb{R}$.
2. $M$ is equipped with a canonical 8-dimensional foliation.
3. If all the leaves of $F$ are compact, then $M$ fibers over an orbifold $O^8$ finitely covered by $S^8$ and all fibers are finitely covered by $S^7 \times S^1$.

\textbf{Theorem C.} Let $(M, g)$ be a compact Riemannian manifold. Then $(M, g)$ is locally, non globally, conformally parallel $\text{Spin}(9)$ if and only if the following three properties are satisfied:

1. $M$ is the total space of a fiber bundle

$$M \xrightarrow{\pi} S^1_r$$

where $\pi$ is a Riemannian submersion over the circle of a certain radius $r$.
2. The fibers of $\pi$ are isometric to a 15-dimensional spherical space form $S^{15}/K$, where $K \subset \text{Spin}(9)$. 

(3) The structure group of \( \pi \) is contained in the normalizer \( N_{\text{Spin}(9)}(K) \) of \( K \) in \( \text{Spin}(9) \) (that is, the isometries of \( S^{15}/K \) induced by \( \text{Spin}(9) \)).

2. Preliminaries

Let \( \mathbb{O} \) be the algebra of octonions. The multiplication of \( x = h_1 + h_2e \), \( x' = h_1' + h_2'e \in \mathbb{O} \) is defined through the one in quaternions \( \mathbb{H} \) by the Cayley-Dickson process:

\[
xx' = (h_1h_1' - h_2h_2) + (h_2h_1' + h_1h_2) e,
\]

where \( \overline{h}_1, \overline{h}_2 \) are the conjugates of \( h_1', h_2' \in \mathbb{H} \). The conjugation in \( \mathbb{O} \) is defined by \( \overline{x} = h_1 - h_2e \) and relates with the non-commutativity in \( \mathbb{O} \) by \( xx' = \overline{x} \overline{x} \). The non-associativity of \( \mathbb{O} \) gives rise to the associator

\[
[x, x', x''] = (xx')x'' - x(x'x''),
\]

that vanishes whenever two among \( x, x', x'' \) are equal or conjugate. For a survey on octonions and their applications in geometry, topology and mathematical physics, see [Bae02].

We recall in particular the decomposition of the real vector space \( \mathbb{O}^2 \) into its octonionic lines

\[
l_m \overset{\text{def}}{=} \{(x, mx) | x \in \mathbb{O}\} \quad \text{or} \quad l_\infty \overset{\text{def}}{=} \{(0, x') | x' \in \mathbb{O}\},
\]

that intersect each other only in \((0,0) \in \mathbb{O}^2\) (cf. Section 4). Here \( m \in S^8 = \mathbb{O} P^1 = \mathbb{O} \cup \{\infty\} \) parametrizes the set of octonionic lines \( l \), whose volume elements \( \nu_l \in \Lambda^8 l \) allow to define the following canonical 8-form on \( \mathbb{O}^2 = \mathbb{R}^{16} \):

\[
\Phi_{\text{Spin}(9)} \overset{\text{def}}{=} \int_{\mathbb{O}P^1} p_l^* \nu_l dl \in \Lambda^8(\mathbb{R}^{16}),
\]

where \( p_l \) denotes the orthogonal projection \( \mathbb{O}^2 \to l \). This definition of \( \Phi_{\text{Spin}(9)} \) is due to M. Berger (cf. [Ber72]). The following statement motivates our choice of notation for the canonical 8-form:

**Proposition 2.1.** [Cor92] Proposition 1.4 at page 170 The subgroup of \( \text{GL}(16, \mathbb{R}) \) preserving \( \Phi_{\text{Spin}(9)} \) is the image of \( \text{Spin}(9) \) under its spin representation into \( \mathbb{R}^{16} \).

As such, one can look at \( \text{Spin}(9) \) as a subgroup of \( \text{SO}(16) \). Accordingly, \( \text{Spin}(9) \)-structures can be considered on 16-dimensional oriented Riemannian manifolds. The following definition collects different approaches that have been used (see [Cor92], [Fri01], [PP12]):

**Definition 2.2.** Let \( M \) be a 16-dimensional oriented Riemannian manifold. A \( \text{Spin}(9) \)-structure on \( M \) is the datum of any of the following equivalent alternatives.

1. A rank 9 vector subbundle \( V^9 \subset \text{End}(TM) \), locally spanned by endomorphisms

   \[
   \{\mathcal{I}_\alpha\}_{\alpha=1,\ldots,9}
   \]

   satisfying

   \[
   \mathcal{I}_\alpha^2 = \text{Id}, \quad \mathcal{I}_\alpha \mathcal{I}_\beta = -\mathcal{I}_\beta \mathcal{I}_\alpha \quad \text{for} \quad \alpha \neq \beta,
   \]

   where \( \mathcal{I}_\alpha^\ast \) denotes the adjoint of \( \mathcal{I}_\alpha \).

2. An 8-form \( \Phi_{\text{Spin}(9)} \in \Lambda^8(M) \) which can be locally written as in [PP12 Table B], for a certain orthonormal local coframe \( \{e^1,\ldots, e^{16}\} \).

3. A reduction \( \mathcal{R} \) of the principal bundle of orthonormal frames on \( M \) from \( \text{SO}(16) \) to \( \text{Spin}(9) \).

**Remark 2.3.** From any of the Definitions 2.2 it follows that admitting a \( \text{Spin}(9) \)-structure depends only on the conformal class of \( M \).
We describe now the rank 9 vector bundle of endomorphisms when \( M \) is the model space \( \mathbb{R}^{16} \). Here \( \mathcal{I}_1, \ldots, \mathcal{I}_9 \) can be chosen as generators of the Clifford algebra \( \text{Cl}(9) \), the endomorphisms’ algebra of its 16-dimensional real representation \( \Delta_9 = \mathbb{R}^{16} = \mathbb{O}^2 \). Accordingly, unit vectors \( v \in S^8 \subset \mathbb{R}^9 \) can be viewed, via the Clifford multiplication, as symmetric endomorphisms acting on pairs \( (x, x') \in \mathbb{O}^2 \) by

\[
(x, x') \mapsto \left( \begin{array}{cc} r & R_u \rho \omega_L \\ R_{\overline{u}} & -r \end{array} \right) \left( \begin{array}{c} x \\ x' \end{array} \right),
\]
defining a similar action on the sphere \( S^8 \) of \( \mathbb{O} \)-vector bundles of quaternions. All of this fails for the octonionic Hopf fibration, due to the irreducibility of \( \text{Sp}(1) \)-structure. An explicit description of a canonical basis \( \mathcal{I}_1, \ldots, \mathcal{I}_9 \) and to the non-associativity of octonions.

In this way, one gets the following symmetric endomorphisms:

\[
\mathcal{I}_1 = \left( \begin{array}{cc} 0 & \text{Id} \\ \text{Id} & 0 \end{array} \right), \quad \mathcal{I}_2 = \left( \begin{array}{cc} 0 & -R_i \\ R_i & 0 \end{array} \right), \quad \ldots, \quad \mathcal{I}_8 = \left( \begin{array}{cc} 0 & -R_h \\ R_h & 0 \end{array} \right), \quad \mathcal{I}_9 = \left( \begin{array}{cc} \text{Id} & 0 \\ 0 & -\text{Id} \end{array} \right),
\]

where \( R_i, \ldots, R_h \) are the right multiplications by \( 7 \) unit octonions \( i, \ldots, h \). The subgroup \( \text{Spin}(9) \subset \text{SO}(16) \) is then characterized as preserving the 9-dimensional \( \mathbb{R}^9 \)-vector space \( V^9 \subset \mathbb{R}^{16} \).

### 3. The Quaternionic Hopf Fibration

It is useful to look at \( \text{Spin}(9) \subset \text{SO}(16) \) as the octonionic analogue of the quaternionic group \( \text{Sp}(2) \cdot \text{Sp}(1) \subset \text{SO}(8) \). A simple aspect of the analogy is given by the symmetry group \( \text{Sp}(2) \cdot \text{Sp}(1) \) and the nine vectors

\[
(0, 1), (0, i), (0, j), (0, k), (0, e), (0, f), (0, g), (0, h) \quad \text{and} \quad (1, 0) \in S^8 \subset \mathbb{O} \times \mathbb{R} = \mathbb{R}^9.
\]

In this case, one gets the following symmetric endomorphisms:

\[
\mathcal{I}_1 = \left( \begin{array}{cc} 0 & \text{Id} \\ \text{Id} & 0 \end{array} \right), \quad \mathcal{I}_2 = \left( \begin{array}{cc} 0 & -R_i \\ R_i & 0 \end{array} \right), \quad \ldots, \quad \mathcal{I}_8 = \left( \begin{array}{cc} 0 & -R_h \\ R_h & 0 \end{array} \right), \quad \mathcal{I}_9 = \left( \begin{array}{cc} \text{Id} & 0 \\ 0 & -\text{Id} \end{array} \right),
\]

where \( R_i, \ldots, R_h \) are the right multiplications by the \( 7 \) unit octonions \( i, \ldots, h \). The subgroup \( \text{Spin}(9) \subset \text{SO}(16) \) is then characterized as preserving the 9-dimensional vector space \( V^9 \subset < \mathcal{I}_1, \ldots, \mathcal{I}_9 > \subset \text{End}(\mathbb{R}^{16}) \).

These ten compositions \( \mathcal{I}_\alpha \mathcal{I}_\beta \), for \( \alpha < \beta \), yield complex structures on \( \mathbb{R}^8 = \mathbb{H}^2 \), and a basis of the Lie algebra \( \mathfrak{sp}(2) \). In particular, the sum of squares of their Kähler forms \( \omega_{\alpha \beta} \) gives (cf. [PP12] page 329):

\[
\sum_{1 \leq \alpha < \beta \leq 5} \omega_{\alpha \beta}^2 = -2 \Omega_L,
\]

where \( \Omega_L \) is the left quaternionic 4-form in \( \mathbb{R}^8 \), defined as usual by

\[
\Omega_L \overset{\text{def}}{=} \omega_{L_1}^2 + \omega_{L_2}^2 + \omega_{L_3}^2,
\]

in terms of the Kähler forms \( \omega \) of the left multiplications \( L_i, L_j \) and \( L_k \).
Thus, on a Riemannian manifold $M^8$, the datum of a $\text{Sp}(2) \cdot \text{Sp}(1)$-structure can be given through two different approaches. One can simply fix the usual rank 3 vector subbundle $Q^3$ of skew-symmetric elements in $\text{End}(TM)$, whose local generators can be denoted by $I, J, K$. In the model space $\mathbb{R}^8$, the subgroup of rotations commuting with the standard complex structures $I, J, K$ is $\text{Sp}(2) \subset \text{SO}(8)$, and the second factor $\text{Sp}(1)$ of the reduced structure group here works as the double covering of $\text{SO}(3)$, allowing to change the admissible hypercomplex structure.

Since both factors of $\text{Sp}(2) \cdot \text{Sp}(1)$ are double coverings of rotation groups - namely of $\text{SO}(5)$ and $\text{SO}(3)$, respectively - one can reverse the role of the two factors. Accordingly, one can follow a different approach to fix a $\text{Sp}(2) \cdot \text{Sp}(1)$ reduction of the structure group on a Riemannian $M^8$.

This second approach is what can be called a quaternionic Hopf structure (cf. [PP12 page 327]), and consists of a vector subbundle $V^5 \subset \text{End}(TM)$ of symmetric elements, whose local bases of sections $\mathcal{I}_a \in \Gamma(V^5)$ ($a = 1, \ldots, 5$) satisfy relations (2.2). On the model space $\mathbb{R}^8$, the subgroup of rotations commuting with the standard $\mathcal{I}_1, \ldots, \mathcal{I}_5$ is the diagonal $\text{Sp}(1)$ subgroup of $\text{SO}(8)$, and now it is the left factor of $\text{Sp}(2) \cdot \text{Sp}(1)$ to allow admissible five dimensional rotations in the choice of bases of sections in $V^5$. As already recalled, the quaternionic 4-form of $\mathbb{H}^2 \cong \mathbb{R}^8$ can be easily written according to both the mentioned approaches.

In Section 5 we will deal with locally conformally parallel $\text{Spin}(9)$-structures. It will be useful to have in mind some known facts for their corresponding 8-dimensional analogues, Riemannian manifolds $M^8$ whose metric is locally conformally related to metrics with holonomy $\text{Spin}(2) \cdot \text{Sp}(1)$.

We rephrase here some of these facts in terms of the rank 5 vector bundle $V^5 \subset \text{End}(TM)$.

As mentioned in the Introduction, a quaternion Hermitian manifold $(M^8, g)$ is called locally conformally quaternion Kähler (or lcqK, briefly) if $g_{U} = e^{f_U} g_{U}'$ with local quaternion Kähler metrics $g_{U}'$, defined over open neighborhoods $U$ covering $M$. The Lee form $\theta$, locally $\theta_{U} = df_U$, allows to characterize globally the lcqK condition (cf. [OP97 page 643]):

$$d\Omega_L = \theta \wedge \Omega_L, \quad d\theta = 0.$$  

The Levi-Civita connections of local quaternion Kähler metrics $g_{U}'$ glue together to the Weyl connection $D$, defined on tangent vector fields $X,Y$ as

$$D_X Y = \nabla_X Y - \frac{1}{2} \{ \theta(X) Y + \theta(Y) X - g(X, Y) B \},$$

where $\nabla$ is the Levi-Civita connection of $g$ and $B = \theta^2$ is the Lee vector field. Then the lcqK condition can be viewed as an example of Einstein-Weyl structure, i.e. the datum of the conformal class $[g]$ of metrics together with the torsion-free connection $D$ satisfying the Einstein condition and preserving both the conformal class $[g]$ and the vector bundle $V^5 \rightarrow M^8$, that is, satisfying $Dg = \theta \otimes g$ and $DV^5 \subset V^5$.

Abundant examples exist in the subclass of 8-dimensional compact locally conformally hyperkähler manifolds: for instance any product $S^1 \times S^3$ of a compact 3-Sasakian 7-dimensional manifold $S$ with a circle, where the former can be chosen having any second Betti number $b_2(S)$ (see [BGMR08]).

However, lcqK metrics on compact $M^8$ are either globally conformally quaternion Kähler or locally conformally quaternion Kähler with the local quaternion Kähler metrics of vanishing scalar curvature (OP97 page 645), so that the locally quaternion Kähler metrics $g_{U}'$ are necessarily Ricci flat.

Note that this does not imply the existence of a global hypercomplex structure on $M^8$, even on the open neighborhood where the local hyperkähler metrics $g_{U}'$ are defined.

In the following, we see how a locally conformally quaternion Kähler manifold $M^8$ can be described by looking at the vector bundle $V^5 \rightarrow M^8$, and by using the vector fields $\mathcal{I}_aB, \ldots, \mathcal{I}_5B$ on $M^8$. 

Lemma 3.1. Let $M^8$ be a compact manifold equipped with a locally, non globally, lcqK metric $g$. Let $B$ be its Lee vector field and $V^5 \subset \text{End}(TM)$ the vector bundle defining the $\text{Sp}(2) \cdot \text{Sp}(1)$-structure, locally spanned by $I_1, \ldots, I_5$. Then the local vector fields $I_1B, \ldots, I_5B$ are orthonormal and $B$ belongs to their 5-dimensional distribution $VB$. The orthogonal complement $(VB)^\perp$ is integrable.

Proof. Consider on $M$ the distribution $F$ spanned by the Lee vector field $B$ and its transformation under the (local) compatible almost complex structures. As already mentioned, the whole $\text{Sp}(2) \cdot \text{Sp}(1)$-structure can be given either by a rank 3 vector subbundle $Q^3$ of skew-symmetric elements in $\text{End}(TM)$ (whose local generators are compatible almost complex structures usually denoted by $I, J, K$), or by a vector subbundle $V^5 \subset \text{End}(TM)$ of symmetric elements, whose local generators we denote here by $I_1, \ldots, I_5$.

To prove the statement, there are now two possibilities. The first one is to refer to the work [OP97], and to rephrase the integrability of $F$, a consequence of Frobenius Theorem in [OP97, page 645], in terms of the vector bundle $V^5$. The geometric interplay between the foliation $F$ and the distribution $VB$, locally spanned by the vector fields $I_1B, \ldots, I_5B$, follows from a computation that can be performed in the model space $\mathbb{R}^8$. This gives rise to the situation described in the statement. The same computation shows that none of the $I_\alpha B$ is in general perpendicular to $B$, and that the orthogonal complement $(VB)^\perp$ is locally spanned by $IB, JB, KB$.

A second way to prove the statement is by a straightforward computation. This will be essentially done later in the proof of Theorem B, and more precisely of its statement (2). Although this latter refers to the Spin(9) case, the same computations, if limited to the choices $1, \ldots, 5$, prove the present statement. □

The following Proposition gives now a more complete description of lcqK manifolds in terms of the vector bundle $V^5$. Again, its proof follows from results in [OP97] (see in particular Theorem 3.8 at page 649).

Proposition 3.2. Let $M^8$ be a compact manifold equipped with a locally, non globally, lcqK metric $g$, with the same notation as in Lemma 3.1.

1. There exists a metric in the conformal class of $g$ whose Lee form $\theta$ is parallel.
2. On each integral manifold $N^7$ of $\ker(\theta)$, the distribution $(VB)^\perp$, orthogonal in $M$ to $VB$, is integrable and its leaves are 3-dimensional spherical space forms. The distribution on $M$ spanned by $(VB)^\perp$ and $B$ is the 4-dimensional vertical foliation $F$, whose leaves are lcqK (generally non primary) Hopf surfaces.
3. The leaf space $M/F$, when a manifold or an orbifold, carries a projected positive self-dual Einstein metric.

4. Spin(9) and the octonionic Hopf fibration

For any $(x, y) \in S^{15} \subset \mathbb{O}^2 = \mathbb{R}^{16}$, we denote by

$$B \overset{\text{def}}{=} (x, y) \overset{\text{def}}{=} (x_1, \ldots, x_8, y_1, \ldots, y_8)$$

the (outward) unit normal vector field of $S^{15}$ in $\mathbb{R}^{16}$. Here and in the following, we are identifying the tangent spaces $T_{(x,y)}(\mathbb{R}^{16})$ with $\mathbb{R}^{16}$. 
Through the involutions \( \mathcal{I}_1, \ldots, \mathcal{I}_9 \) one gets then the following sections of \( T(\mathbb{R}^{16})_{\epsilon^{15}} \) of length one:

\[
\begin{align*}
\mathcal{I}_1 B &= (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8), \\
\mathcal{I}_2 B &= (y_2, -y_1, -y_4, y_3, -y_6, y_5, y_8, -y_7, -x_2, x_1, x_4, -x_3, x_6, -x_5, -x_8, x_7), \\
\mathcal{I}_3 B &= (y_3, y_4, -y_1, -y_2, -y_7, -y_8, y_5, y_6, -x_3, -x_4, x_1, x_2, x_7, x_8, -x_5, -x_6), \\
\mathcal{I}_4 B &= (y_4, -y_3, y_2, -y_1, -y_8, y_7, -y_6, y_5, -x_4, x_3, -x_2, x_1, x_8, -x_7, x_6, -x_5), \\
\mathcal{I}_5 B &= (y_5, y_6, y_7, y_8, -y_1, -y_2, -y_3, -y_4, -x_5, -x_6, -x_7, -x_8, x_1, x_2, x_3, x_4), \\
\mathcal{I}_6 B &= (y_6, -y_5, y_8, -y_7, y_2, -y_1, y_4, -y_3, -x_6, x_5, -x_8, x_7, -x_2, x_1, -x_4, x_3), \\
\mathcal{I}_7 B &= (y_7, y_8, -y_6, y_5, y_3, -y_4, -y_1, y_2, -x_7, x_8, x_5, -x_6, -x_3, x_4, x_1, -x_2), \\
\mathcal{I}_8 B &= (y_8, y_7, -y_6, -y_5, y_4, y_3, -y_2, -y_1, -x_8, x_7, x_6, x_5, -x_4, -x_3, x_2, x_1), \\
\mathcal{I}_9 B &= (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, -y_1, -y_2, -y_3, -y_4, -y_5, -y_6, -y_7, -y_8).
\end{align*}
\]

(4.1)

As mentioned, \( \text{Spin}(9) \subset \text{SO}(16) \) is the group of symmetries of the octonionic Hopf fibration. This latter is defined by looking at the decomposition of \( \mathbb{O}^2 \) into the octonionic lines

\[
l_m \overset{\text{def}}{=} \{ (x, mx) | x \in \mathbb{O} \} \quad \text{or} \quad l_\infty \overset{\text{def}}{=} \{ (0, y) | y \in \mathbb{O} \},
\]

mentioned in Section 2. One has to be careful that the octonionic line through \((0, 0)\) and \((x, y) \in \mathbb{O}^2\) is not \(\{ (xo, yo) | o \in \mathbb{O} \} \). This latter in fact is not even an octonionic line, the correct line being instead \(l_{y^{-1}} = \{ (o, (yx^{-1})o) | o \in \mathbb{O} \} \) if \(x \neq 0\), and \(l_\infty\) if \(x = 0\). In this way the fibration

\[
\mathbb{O}^2 \setminus 0 \to S^8 = \{ m \in \mathbb{O} \} \cup \{ \infty \}
\]

is obtained, with fibers \( \mathbb{O} \setminus 0 \), and the intersection with the unit sphere \( S^{15} \subset \mathbb{O}^2 \) provides the octonionic Hopf fibration

\[
S^{15} \to S^8, \quad \text{or as homogeneous fibration} \quad \frac{\text{Spin}(9)}{\text{Spin}(7)} \overset{\text{Spin}(8)}{\sim} \text{Spin}(9).
\]

Denote by \( VB \) the 9-dimensional span of \( \mathcal{I}_1 B, \ldots, \mathcal{I}_9 B \):

\[
\text{VB} \overset{\text{def}}{=} < \mathcal{I}_1 B, \ldots, \mathcal{I}_9 B >,
\]

and note that 9-planes of \( VB \) are generally not tangent to \( S^{15} \).

**Proof of Theorem A.** First, note that \( VB \) is invariant under \( \text{Spin}(9) \): this is clear for the unit normal \( B = (x, y) \), since \( \text{Spin}(9) \subset \text{SO}(16) \), and on the other hand the nine endomorphisms \( \mathcal{I}_\alpha \) are rotating under the \( \text{Spin}(9) \) action inside their vector space \( V^B \subset \text{End}(\mathbb{R}^{16}) \).

Next, \( VB \) contains \( B \). In fact:

\[
B = \lambda_1 \mathcal{I}_1 B + \lambda_2 \mathcal{I}_2 B + \cdots + \lambda_9 \mathcal{I}_9 B,
\]

where the coefficients \( \lambda_\alpha \) can be computed from \( 111 \) in terms of the inner products (here all the arrows denote vectors in \( \mathbb{R}^8 \))

\[
\bar{x} = (x_1, \ldots, x_8), \quad \bar{y} = (y_1, \ldots, y_8) \in \mathbb{R}^8
\]

and of the right translations \( R_1, \ldots, R_9 \) as follows:

\[
\lambda_1 = 2 \bar{x} \cdot \bar{y}, \quad \lambda_2 = -2 \bar{x} \cdot \bar{R}_1 y, \ldots, \quad \lambda_9 = -2 \bar{x} \cdot \bar{R}_8 y, \quad \lambda_9 = |\bar{x}|^2 - |\bar{y}|^2.
\]

In particular, at points with \( \bar{x} = 0 \), that is on the octonionic line \( l_\infty \), the vector fields \( \mathcal{I}_1 B, \ldots, \mathcal{I}_8 B \) are orthogonal to the unit sphere \( S^8_\infty \subset l_\infty \). This latter is the fiber of the Hopf fibration \( S^{15} \to S^8 \) over the north pole \((0, \ldots, 0, 1) \in S^8 \), and the mentioned orthogonality of this fiber \( S^7 \) is immediate from \( 111 \) for \( \mathcal{I}_1 B, \ldots, \mathcal{I}_8 B \). Also, for these points, we have \( \mathcal{I}_9 B = B \).
so $\mathcal{I}B$ is orthogonal to $S^2_{15}$. Now, the invariance under Spin(9) of the octonionic Hopf fibration shows that all its fibers are characterized as orthogonal in $\mathbb{R}^{16}$ to the vector fields $\mathcal{I}X, \ldots, \mathcal{I}B$.

Now, assume that $X$ is a vertical vector field of $S^{15} \to S^8$. By the previous characterization we have the following orthogonality relations in $\mathbb{R}^{16}$:

\[ \langle X, \mathcal{I}_\alpha B \rangle = 0, \quad \text{for } \alpha = 1, \ldots, 9, \]

and it follows that $\langle \mathcal{I}_\alpha X, B \rangle = 0$. But from the definition of a Spin(9)-structure we see that if $\alpha \neq \beta$, then $\langle \mathcal{I}_\alpha X, \mathcal{I}_\beta X \rangle = 0$. Thus, if $X$ is a nowhere zero vertical vector field, we would obtain in this way 9 pairwise orthogonal vector fields $\mathcal{I}X, \ldots, \mathcal{I}B$, all tangent to $S^{15}$. But $S^{15}$ is known to admit at most 8 linearly independent vector fields by the classical Hurwitz-Radon-Adams result (see for example [Hus94] or [PP13]). Thus $X$ cannot be vertical and nowhere zero, and Theorem A is proved. \hfill $\Box$

One gets as a consequence the following alternative proof of a result in [AV92]:

**Corollary 4.1.** The octonionic Hopf fibration $S^{15} \to S^8$ does not admit any $S^1$ subfibration.

**Proof.** In fact, any $S^1$ subfibration would give rise to a real line subbundle $L \subset T_{\text{vert}}(S^{15})$ of the vertical subbundle of $T(S^{15})$. Such line bundle $L$ is necessarily trivial, due to the vanishing of its first Stiefel-Whitney class $w_1(L) \in H^1(S^{15}; \mathbb{Z}_2) = 0$. It follows that $L$ would admit a nowhere zero section, thus a global vertical nowhere zero vector field. \hfill $\Box$

5. **Locally conformally parallel Spin(9) manifolds**

**Definition 5.1.** A Riemannian manifold $(M^{16}, g)$ is locally conformally parallel Spin(9) (LCP, briefly) if over open neighbourhoods \{U\} covering $M$ the restriction $g|_U$ of the metric $g$ is conformal to a (local) metric $g'_U$ having holonomy contained in Spin(9).

The conformality relations $g|_U = e^{f_U} g'_U$ give rise to a Lee form $\theta$, locally defined as $\theta|_U := df|_U$. Next, recall that Spin(9) is characterized as the subgroup of GL(16, $\mathbb{R}$) that preserves the 8-form $\Phi_{\text{Spin}(9)}$ (cf. the already quoted [Cor92], page 170]). Thus, a Spin(9)-structure on $M^{16}$ is equivalent to the datum of a $\Phi \in \Lambda^8(M)$, which can be locally written as in [PP12] Table B] and, under the LCP hypothesis, on each $U$ the metric $g'_U$ defines a similar 8-form $\Phi'_U$ parallel with respect to the Levi-Civita connection of $g'_U$. It follows that the restriction of $\Phi$ to $U$ satisfies

\[ \Phi|_U = e^{f_U} \Phi'_U, \]

henceforth one has

\[ df = \theta \wedge \Phi. \]

Moreover, the Levi-Civita connections of the local parallel metrics $g'_U$ glue together to the global Weyl connection $D$ on $M$:

\[ DXY = \nabla_X Y - \frac{1}{2} \{\theta(X)Y + \theta(Y)X - g(X, Y)B\}, \]

where $\nabla$ is the Levi-Civita connection of $g$. Recall that, since the metrics $g'_U$ are assumed to have holonomy contained in Spin(9), they are Einstein metrics. Thus the conditions $DV \subset V$, $Dg = \theta \otimes g$, $d\theta = 0$ and $g'_U$ Einstein, insures that the conformal class $[g]$ defines a closed Einstein-Weyl manifold $(M, [g], D)$.

We will next give the

**Proof of Theorem B.** First, recall that $g$ defines a closed Einstein-Weyl structure on a compact manifold, with Lee form $\theta$ non exact (but closed). Then the following properties hold (cf. [Gau95], page 10, Theorem 3]): (a) its Weyl connection $D$ is Ricci-flat; (b) one can choose, in the conformal...
class \([g]\), a metric \(g_0\) (unique up to homotheties) such that its Lee form \(\theta_0\) is parallel with respect to the Levi-Civita connection \(\nabla^{g_0}\) of \(g_0\).

Thus, to prove our statement we can assume, without loss of generality, that the Lee form \(\theta\) of \(g\) is parallel with respect to the Levi-Civita connection \(\nabla^g\). Henceforth also its Lee field \(B = \theta^i\) is parallel and as a consequence, by de Rham decomposition theorem, the universal covering \((\tilde{M}, \tilde{g})\) is reducible:

\[
(\tilde{M}, \tilde{g}) = (\mathbb{R}, dt^2) \times (\tilde{N}, g_{\tilde{S}}), \quad \tilde{N} \text{ complete and simply connected.}
\]

With respect to this decomposition we have that the pull-back of \(\theta\) is \(\tilde{\theta} = dt\). The diffeomorphism \(\mathbb{R} \times \tilde{N} \to \mathbb{R}^+ \times \tilde{N}\) given by

\[(t, x) \mapsto (s = e^t, x)\]

shows that \((\tilde{M}, \tilde{g})\) is globally conformal, with conformity factor \(\frac{1}{2}\), to the so-called \textit{metric cone}

\[C(\tilde{N}) = (\tilde{M}, ds^2 + s^2 g_{\tilde{N}})\]

Using the classical D. Alekseevsky theorem ([Alek88 Corollary 1 at page 98]) we see that the Ricci-flatness of the local metrics (as mentioned, consequence of their holonomy contained in Spin(9) and of the Theorem of Gauduchon on closed Weyl structures), insures their flatness, so that the cone \(C(\tilde{N})\) is flat. We can use then the relation between the curvature operator \(R\) of the warped product \(C(\tilde{N}) = \mathbb{R}^+ \times_{\tilde{g}} \tilde{N}\) and the curvature operator \(R^\tilde{N}\) of its fiber \(\tilde{N}\):

\[0 = R_{VW}^\tilde{N} = R_{VW}^\mathbb{R} + \frac{4}{s^2} (g_{\tilde{N}}(V, Z) W - g_{\tilde{N}}(W, Z)V)\]

(see for example [ON83 page 210]) to recognize that \(\tilde{N}\), being complete, is the sphere \(S^{15}\). All of this insures that the universal covering of \(M\) is conformally equivalent to the cone \(C(S^{15})\) and, since the Lee vector field \(B\) is parallel, that \(M\) is locally isometric, up to homotheties, to \(S^{15} \times \mathbb{R}\). This proves statement 1.

We now prove statement 2. Denote by \(\Theta\) the codimension 1 foliation on \(M\) defined by the equation \(\theta = 0\), with \(\theta = B^t\), and note that the parallelism of \(\theta\) insures that \(\Theta\) is a totally geodesic foliation, so that the Levi-Civita connection on any leaf \(T = T^{15}\) is just the restriction of \(\nabla^g\). Next, consider the vector bundle \(V = V^0 \subset \text{End}(TM)\) given by the Spin(9)-structure, locally spanned by \(\mathcal{I}_1, \ldots, \mathcal{I}_9\), and the corresponding distribution

\[V_B \overset{\text{def}}{=} \langle \mathcal{I}_1 B, \ldots, \mathcal{I}_9 B \rangle \subset T(M)\]

generated by the orthonormal vector fields \(\mathcal{I}_1 B, \ldots, \mathcal{I}_9 B\). Then \(V_B\) contains the Lee vector field \(B\), as seen in the proof of Theorem A.

We now show that the 8-dimensional distribution

\[\mathcal{F} \overset{\text{def}}{=} \langle \mathcal{I}_1 B, \ldots, \mathcal{I}_9 B \rangle \perp B \rangle = (V_B)^\perp \perp B \rangle\]

is integrable.

First, let \(X, Y \in (V_B)^\perp\), so that \(g(X, \mathcal{I}_a B) = g(Y, \mathcal{I}_a B) = 0\) for \(a = 1, \ldots, 9\). Then, in terms of the Weyl connection \(D\), we have

\[g([X, Y], \mathcal{I}_a B) = g(\mathcal{I}_a (DX Y), B) - g(\mathcal{I}_a (DY X), B)\]

Recall now that \(DV\subset V\) gives rise to 1-forms \(a_{a\beta}\) such that \(D\mathcal{I}_a = \sum a_{a\beta} \otimes \mathcal{I}_\beta\). It follows:

\[g(\mathcal{I}_a (DX Y), B) = g(D_X (\mathcal{I}_a Y), B) - g((DX (\mathcal{I}_a Y), B) =
\]

\[= g(D_X (\mathcal{I}_a Y), B) - \sum a_{a\beta}(X)g(\mathcal{I}_\beta Y, B),\]

and since \(g(\mathcal{I}_\beta Y, B) = 0\), we obtain

\[g(\mathcal{I}_a (DX Y), B) = g(D_X (\mathcal{I}_a Y), B)\]

On the other hand, since $\nabla B = 0$ we have also $D_Y B = -\frac{1}{2}Y$ (cf. [DO98 page 37]). Thus, by applying $D$ to the identity $g(\mathcal{I}_aX, B) = 0$, we obtain

$$0 = Y(g(\mathcal{I}_aX, B)) = g(D_Y(\mathcal{I}_aX), B) + g(\mathcal{I}_aX, D_Y B) + \theta(Y)g(\mathcal{I}_aX, B),$$

so that

$$g(D_Y(\mathcal{I}_aX), B) = \frac{1}{2}g(\mathcal{I}_aX, Y).$$

All of this gives

$$g([X, Y], \mathcal{I}_aB) = g(D_X(\mathcal{I}_aY), B) - g(D_Y(\mathcal{I}_aX), B) = \frac{1}{2}g(\mathcal{I}_aX, Y) - \frac{1}{2}g(\mathcal{I}_aX, Y) = 0,$$

so we get that $[X, Y] \in (VB)^\perp$.

Now, to obtain the integrability of $\mathcal{F}$, we must further check that for $X \in (VB)^\perp$ the bracket $[X, B] = D_X B - D_B X = -X - D_B X$ belongs to $\mathcal{F}$. In fact $D_B X = \nabla B X = \frac{1}{2}X \in (VB)^\perp$, and $\nabla_B X \in (VB)^\perp$ is a consequence of $g(X, B) = g(X, I_aB) = 0$. This ends the proof of statement $\mathbf{3}$.

As for statement $\mathbf{3}$, one can use the same argument as in [OP97 Theorem 2.1] to show that $\mathcal{F}$ is a Riemannian totally geodesic foliation and that the leaf space, when a manifold or an orbifold, carries a metric of spherical space form type. $\square$

**Proof of Theorem C.** Our arguments will follow basically the same ideas as in [OV03], and we first show that the locally conformally parallel Spin(9) condition implies on compact manifolds the structure described by properties $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$.

In fact, if $(M, g)$ is compact and locally, non globally, conformally parallel Spin(9), recall from the proof of Theorem $\mathbf{1}$ that its universal covering $(\tilde{M}, \tilde{g})$ is conformally equivalent to the metric cone $C(S^{15}) = \mathbb{R}^{16} \setminus 0$ with conformal factor $\frac{1}{2} = e^{-2t}$, that is, the cone metric $g_{cone}$ is given by

$$g_{cone} = e^{-2t} \tilde{g}.$$ 

Any $\gamma \in \pi_1(M)$ can be thought as a map $\gamma : \tilde{M} \to \tilde{M}$ preserving $\tilde{g}$, and we get:

$$\gamma^*(g_{cone}) = \gamma^*(e^{-2t} \tilde{g}) = (e^{-2t} \circ \gamma) \gamma^*(\tilde{g}) = (e^{-2t} \circ \gamma) \tilde{g} = (e^{-2t} \circ \gamma) e^{2t} g_{cone},$$

showing that $\pi_1$ acts by conformal maps. Moreover, taking differentials of

$$\gamma^*(\Phi_{Spin(9)}) = (e^{-2t} \circ \gamma) e^{2t} \Phi_{Spin(9)}$$

and using $d\Phi_{Spin(9)} = 0$, we see that $\pi_1(M)$ acts by homotheties.

Indeed, the homothety factor $\rho(\gamma)$ of $\gamma \in \pi_1(M)$ defines a homomorphism $\rho : \pi_1(M) \to \mathbb{R}^+$, whose image is a finitely generated subgroup of $\mathbb{R}^+$, thus isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$. The locally conformal flatness of the metric allows to apply the arguments used to prove [GPP06 Corollary 4.7], and to see that the image of $\rho$ is isomorphic to $\mathbb{Z}$.

Next, notice that $K \equiv \ker \rho$ consists of isometries of $C(S^{15})$ that leave the form $\Phi_{Spin(9)}$ invariant, so that in particular $K \subset \pi_1(M)$. Moreover, any isometry of $C(S^{15})$ induces the identity map on the $\mathbb{R}^+$-component (see again [GPP06 Theorem 5.1]), and it leaves the fibers of the projection $C(S^{15}) \to \mathbb{R}^+$ invariant. Since $S^{15}$ is compact and $\pi_1(M)$ acts properly discontinuously and freely on $C(S^{15})$, $K$ is finite and without fixed points on $S^{15}$. It follows:

$$C(S^{15})/K = C(S^{15}/K).$$

Consider now a homothety $\gamma \in \pi_1(M)$ such that $h \equiv \rho(\gamma) \in \mathbb{R}^+$ generates $\text{Im}(\rho)$. Then $\gamma$ is a homothety on $C(S^{15}/K)$, and

$$(5.1) \quad \gamma(s, x) = (h \cdot s, \psi(x)), \quad \text{for } x \in \frac{S^{15}}{K}, s \in \mathbb{R}^+ \text{ and } \psi \in \text{Isom}(\frac{S^{15}}{K}).$$
Thus, for any \( n \in \mathbb{Z} \) we have:

\[
(5.2) \quad \gamma^n(s, x) = (h^n \cdot s, \psi^n(x)).
\]

Consider the projection \( \text{pr} : C(\frac{S^{15}}{K}) \to \mathbb{R}^+ \) on the first factor of the cone. Then formula (5.2) shows that \( \text{pr} \) is equivariant with respect to the actions of \( \langle \gamma \rangle = \mathbb{Z} \) on \( C(\frac{S^{15}}{K}) \) and of \( n \in \mathbb{Z} \) on \( s \in \mathbb{R}^+ \), given by \( h^n \cdot s \). The induced map

\[
(5.3) \quad M = C(\frac{S^{15}}{K}) \xrightarrow{\pi} \mathbb{R}^+ / \mathbb{Z} = S^1
\]

is, up to rescaling the metric on \( S^1 \), the map in (11) in the statement. Then (2) follows. As for (3), observe that \( \psi \) in formula (5.1) comes from an element of \( \text{SO}(16) \) which preserves \( \Phi_{\text{Spin}(9)} \).

To show that (1), (2) and (3) are necessary conditions for \( M \) to be locally conformally parallel \( \text{Spin}(9) \) manifold, we use a topological argument. Assume that \((M, g)\) is a compact Riemannian manifold satisfying (1), (2) and (3) in Theorem C and consider two open sets \( U_1 \) and \( U_2 \times (S^{15}/K) \) by a transition function \( \psi_{\pi} : S^{15}/K \to S^{15}/K \). This transition function depends on \( \pi \), and is usually called the clumping function of the bundle. Moreover, (3) implies that \( \psi \in N_{\text{Spin}(9)}(K) \) is an isometry of \( S^{15}/K \). Now choose \( h \in \mathbb{R}^+ \), and use \( \psi_{\pi} \), \( h \) to define a homothety \( \gamma_{\pi} \) on \( C(S^{15}/K) \) as in formula (5.1):

\[
\gamma_{\pi}(s, x) \overset{\text{def}}{=} (h \cdot s, \psi_{\pi}(x)), \quad \text{for } x \in \frac{S^{15}}{K} \text{ and } s \in \mathbb{R}^+.
\]

Then, let \( M_{\pi} \) be the locally conformally parallel \( \text{Spin}(9) \) manifold

\[
M_{\pi} \overset{\text{def}}{=} C(\frac{S^{15}}{K}) \langle \gamma_{\pi} \rangle.
\]

Since we already proved the sufficiency of conditions in Theorem C, we know that \( M_{\pi} \) is itself a fiber bundle over \( S^1 \) with the same clumping function \( \psi_{\pi} \). Recall on the other hand that for any Lie group \( G \), the equivalence classes of principal \( G \) bundles over \( S^n \) is in natural bijection with the homotopy group \( \pi_{n-1}(G) \) ([Ste99, Theorem 18.5, page 99]). Thus \( M \) and \( M_{\pi} \) are isomorphic as fiber bundles over \( S^1 \), and in particular they are isometric.

**Remark 5.2.** Using the Galoisian terminology described in [GOPP06, Section 2], the pair

\[
(C(\frac{S^{15}}{K}), \langle \gamma \rangle)
\]

is the minimal presentation of \( M \).

**Remark 5.3.** The fibers of \( \pi \) in Theorem C inherit a 7-Sasakian structure (in the sense of [Dea08]) induced by the foliation \((VB)^{-} \) as in the proof of Theorem B. Indeed, this notion of 7-Sasakian structure on 15-dimensional spherical space forms seems to be the induced counterpart on the leaves of a canonical codimension one foliation on \( M^{16} \). Note that, in accordance with [Dea08], such a 7-Sasakian structure does not involve global vertical vector fields, but only a vertical foliation, whose transverse structure we have here related with the \( \text{Spin}(9) \)-structure of \( M^{16} \).

The following is a different way of stating Theorem C.

**Corollary 5.4.** The set of isometry classes of locally, non globally, conformally parallel \( \text{Spin}(9) \) manifolds is in bijective correspondence with the set of triples

\[
\{(r, K, c_K) | r \in \mathbb{R}^+, K \leq \text{Spin}(9) \text{ finite and free on } S^{15}, c_K \in \pi_0 \left( N_{\text{Spin}(9)}(K) \right) \},
\]

where \( \pi_0 \) stands for the connected component functor.
Remark 5.5. We could also describe the map $\pi : M \to S^1$ in Theorem C as the Albanese map defined as follows. Fix any $x_0 \in M$. For any $x \in M$ and any path $\gamma$ joining $x_0$ and $x$, define:

$$\alpha(x) \overset{\text{def}}{=} \left(\int_\gamma \theta\right) \mod G.$$  

Here $\theta$ is the Lee form of $M$, and $G \subset \mathbb{R}$ is the additive subgroup of “periods of $\theta$”, generated by the integrals $\int_\gamma \theta$ over the generators $\sigma$ of $H_1(M,\mathbb{Z})$.

6. Examples

As a consequence of Theorem B the examples will be in the context of the flat Spin(9)-structure on $\mathbb{R}^{16}$. Recalling the threefold approach to Spin(9)-structures given by Definition 2.2 we refer to the data $V = V^9$, $\Phi_{\text{Spin}(9)}$, $\mathcal{R}$ and Spin(9) as the standard data, and the standard inclusion $SO(16) \subset \text{GL}(16,\mathbb{R})$ can be viewed as equivalent to the choice of the standard basis $\{e_1, \ldots, e_{16}\}$ of $\mathbb{R}^{16}$ as orthonormal. Thus, another way to describe the flat Spin(9)-structure on $\mathbb{R}^{16}$ is the standard structure with respect to the standard basis $\{e_1, \ldots, e_{16}\}$.

Thus, if we choose a different basis $\mathcal{B}$ on $\mathbb{R}^{16}$ that we declare to be orthonormal in a suitable metric $g_\mathcal{B}$, we can talk about the standard structure with respect to $\mathcal{B}$. This means that we are choosing a different inclusion $i : SO(16) \hookrightarrow \text{GL}(16,\mathbb{R})$, but the structure is still standard in the sense that $V$, $\Phi_{\text{Spin}(9)}$ and $\mathcal{R}$ are induced by the standard ones using the inclusion $i$.

Observe that this holds even if the inclusion $i$ depends on the point $x \in \mathbb{R}^{16}$, that is if $\mathcal{B}$ is not a basis on the vector space $\mathbb{R}^{16}$, but a parallelization on the manifold $\mathbb{R}^{16}$. In the same way, on any parallelizable $M^{16}$ with a fixed parallelization $\mathcal{B}$ one can speak of the standard Spin(9)-structure on $M$ associated with $\mathcal{B}$, whose associated objects will be denoted by $V_\mathcal{B}$, $\Phi_\mathcal{B}$, $\mathcal{R}_\mathcal{B}$ and $g_\mathcal{B}$ (see [Par01b] and [Par01a] for details).

Example 6.1. On $\mathbb{R}^{16} \setminus 0$, consider the parallelization $\mathcal{B} \overset{\text{def}}{=} \{ |x| \partial_1, \ldots, |x| \partial_16 \} \subset \mathbb{R}^{16}$ where $\partial_1, \ldots, \partial_{16}$ denotes the derivatives with respect to the standard coordinates. Look at the map

$$p : \mathbb{R}^{16} \setminus 0 \to S^{15} \times S^1, \quad p(x) \overset{\text{def}}{=} (x/|x|, \log |x| \mod 2\pi),$$

and observe that $p$ projects $\mathcal{B}$ to a parallelization $\mathcal{B} \overset{\text{def}}{=} p_*(\mathcal{B})$ on $S^{15} \times S^1$ (see also [Bru92] sections 6 and 7] and [Par03]). Consider the standard Spin(9)-structure $g_\mathcal{B}$ on $S^{15} \times S^1$ associated with $\mathcal{B}$. Then $g_\mathcal{B}$ is locally conformally parallel, since $p$ is a covering map, bundle-like by definition, so that $g_\mathcal{B}$ is locally given by $g_{\tilde{\mathcal{B}}}$, that is to say, by $|x|^{-2}g$, where $g$ is the flat metric on $\mathbb{R}^{16}$.

As observed in Theorem B the flat metric on $\mathbb{R}^{16} \setminus 0$ is the cone metric on $C(S^{15})$. The metric $g_\mathcal{B}$ is instead the cylinder metric on the Riemannian universal covering of $S^{15} \times S^1$.

Remark 6.2. In [Fri01] and [AF06] the class of locally conformally parallel Spin(9)-structures has been identified and studied, under the name of “Spin(9)-structures of vectorial type” (cf. the following Definition 6.3). We outline now a proof that, for Spin(9)-structures, vectorial type is equivalent to locally conformally parallel.

Following [Fri01] and [AF06], one can look at the splitting of the Levi-Civita connection in the principal bundle of orthonormal frames on $M$:

$$\nabla = \nabla^* \oplus \theta$$

where $\nabla^*$ is the connection in the induced bundle of Spin(9)-frames and $\theta$ is its orthogonal complement. Thus, $\theta$ is a 1-form with values in the orthogonal complement $\mathfrak{m}$ defined by the splitting $\mathfrak{so}(16) = \mathfrak{spin}(9) \oplus \mathfrak{m}$ and, under canonical identifications, $\theta$ can be seen as a 1-form with values in $\Lambda^3(V)$. 

Under the action of $\Spin(9)$, the space $\Lambda^4(M) \otimes \Lambda^4(V)$ decomposes as a direct sum of 4 irreducible components:

$$
\Lambda^4(M) \otimes \Lambda^4(V) = P_0 \oplus P_1 \oplus P_2 \oplus P_3,
$$
and, looking at all the possible direct sums, this yields 16 types of $\Spin(9)$-structures. The component $P_0$ identifies with $\Lambda^4(M)$. Thus:

**Definition 6.3.** [AF06] A $\Spin(9)$-structure is of **vectorial type** if $\theta$ lives in $P_0$.

Now, let $(M, g)$ be a Riemannian manifold endowed with a $\Spin(9)$-structure of vectorial type. Let $\theta$ be as above, and let $\Phi$ be its $\Spin(9)$-invariant 8-form. According to Theorems B and C, to give examples of compact locally conformally parallel type" is used for vectorial type. See also [AF06].

**Example 6.5.** According to Theorems B and C, to give examples of compact locally conformally parallel $\Spin(9)$ manifolds, one has to look at finite subgroups of $\Spin(9)$ acting without fixed points on $S^{15}$. The classification of such finite subgroups is not an easy problem, and we limit ourselves to exhibit some of them. They will show however that many finite quotients of $S^{15}$ may appear as fibers in the map of Theorem C.

We describe in particular how $S^{15}$ is acted on “diagonally” and without fixed points by a subgroup $\Sp(1)_\Delta \subset \Spin(9)$. Let $(x = h_1 + h_2 e, x' = h'_1 + h'_2 e) \in \mathbb{O}^2$ and define the following action of $q \in \Sp(1)$ on the first octonionic coordinate $x = h_1 + h_2 e \in \mathbb{O}$:

$$
A_q : h_1 + h_2 e \rightarrow h_1 q + (qh_2) e.
$$

Due to the identity $\tau_1 q \tau_1 = \tau_q \tau_1$, this is a right action $A_q : \mathbb{O} \rightarrow \mathbb{O}$ for each $q \in \Sp(1)$. In the real components of $x$, $A_q$ is represented by a matrix of $SO(8)$, and indeed by a matrix in its diagonal $SO(4) \times SO(4)$ subgroup.

Recall now the Triality Principle for $SO(8)$. In the formulation we need here it can be stated as follows (cf. [GWZ86] page 192) or [DS01] pages 143-145).

**Triality Principle.** Consider the triples $A, B, C \in SO(8)$ such that for any $x, m \in \mathbb{O}$:

$$
C(m) A(x) = B(mx).
$$

If any of $A, B, C$ is given, then the other two exist and are unique up to changing sign for both of them.

Given $A \in SO(8)$ we will call any of such matrices $\pm B, \pm C$ a **triality companion** of $A$.
Going back to the transformation $A_q \in \text{SO}(8)$ defined by any $q \in \text{Sp}(1)$, consider a pair $B_q, C_q \in \text{SO}(8)$ of its triality companions. Thus, for any $x, m \in \mathbb{O}$:

$$C_q(m)A_q(x) = B_q(mx),$$

and define the following right action of $q \in \text{Sp}(1)$ on $\mathbb{O}^2$:

$$R_q : (x = h_1 + h_2e, x' = h'_1 + h'_2e) \to (A_qx, B_qx').$$

Thus, $R_q$ carries octonionic lines to octonionic lines, so that $R_q \in \text{Spin}(9)$. In this way, a “diagonal” subgroup $\text{Sp}(1)_{\Delta} \subset \text{Spin}(9)$ is defined, and $\text{Sp}(1)_{\Delta}$ is indeed a subgroup of the $\text{Spin}(8) \subset \text{Spin}(9)$ defined by triples $(A, B, C) \in \text{SO}(8) \times \text{SO}(8) \times \text{SO}(8)$ obeying to the triality principle.

This action is without fixed points on $S^{15}$: from $A_qx = h_1q + 7h_2e = h_1 + h_2e, q \neq 1$ follows $h_1 = h_2 = 0$, so that the only fixed points of $R_q$ could be on the unit sphere $S^8_q$ of the octonionic line $l_\infty$, on which we are acting by the triality companion $B_q$. Now, if $x' \in S^8_q$ is a fixed point of $B_q$, so is $-x'$ and then $B_q$ has to belong to a $\text{SO}(7)$ subgroup of $\text{SO}(8)$, rotating the equator of $S^8_q$ with respect to the poles $x'$ and $-x'$. But then the triple $(A_q, B_q, C_q)$ belongs to a $\text{Spin}(7)$ subgroup of $\text{Spin}(8)$ and hence any of $A_q, B_q, C_q$ has to belong to a $\text{SO}(7) \subset \text{SO}(8)$ (cf. [Mur89], page 194). Recall on the other hand that any subgroup $\text{SO}(7) \subset \text{SO}(8)$, when acting on the sphere $S^7$, admits a fixed point and it is conjugate with the standard $\text{SO}(7)$ (cf. [Var01], Lemma 4, page 168). This is a contradiction, since $A_q$ has no fixed points.

We can now consider finite subgroups of $\text{Sp}(1)_{\Delta}$. Recall that any finite subgroup of $\text{Sp}(1)$ is isomorphic to either a cyclic group or to the binary dihedral, tetrahedral, octahedral, or icosahedral group (see for instance [Cox91] Section 6.5). In the following, we associate a subgroup of $\text{Sp}(1)_{\Delta}$ with every group in the list of abstract finite subgroups of $\text{Sp}(1)$.

- The cyclic group $C_n = \langle a | a^n = 1 \rangle$, for $n \geq 1$. We can choose as generator $R_a$, where $a = e^{\frac{2\pi i}{n}}$.

- The binary dihedral group $D_n = \langle a, b | a^{2n} = 1, b^2 = a^n, b^{-1}ab = a^{-1} \rangle$, for $n \geq 1$. Choose here as generators $R_a, R_b$ with $a = e^{\frac{2\pi i}{n}}$ and $b = j$.

- The binary tetrahedral group $T = \langle a, b, c | a^2 = b^3 = c^4 = abc \rangle$. Choose now generators $R_a, R_b, R_c$ with $b = \frac{1+i+j+k}{2}, c = \frac{1+i-j+k}{2}$ and $a = bc$.

- The binary octahedral group $O = \langle a, b, c | a^2 = b^3 = c^4 = abc \rangle$. Choose generators $R_a, R_b, R_c$ with $b = \frac{1+i+j+k}{2}, c = \frac{1-i+j-k}{2}$ and $a = bc$.

- The binary icosahedral group $I = \langle a, b, c | a^2 = b^3 = c^5 = abc \rangle$. Let $\varphi \overset{\text{def}}{=} \frac{1+i\sqrt{5}}{2}$ be the golden ratio. Choose generators $R_a, R_b, R_c$ where now $b = \frac{1+i+j+k}{2}, c = \varphi + \varphi^{-1} \frac{1+i+j+k}{2}$ and $a = bc$.

Since any finite subgroup of $\text{Sp}(1)$ is conjugate to one in the previous list, this classifies all locally conformally parallel $\text{Spin}(9)$ manifolds such that $K = \ker \rho$ in Theorem C is contained in $\text{Sp}(1)_{\Delta}$. \qed

**Remark 6.6.** The Lee vector field on a locally conformally parallel $\text{Spin}(9)$ manifold $M$ is never vanishing (see proof of Theorem B). By [Fri01], Proposition 1] this means that $M$ admits a $\text{Spin}(7)_{\Delta}$-structure (in the sense of [Fri01]). Thus, the classification of isometry types of $M$ reduces to the finding of finite subgroups of $\text{Spin}(7)_{\Delta} \subset \text{Spin}(9)$ acting without fixed points on $S^{15}$.

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