Black magic session of concordance: 
Regge mass spectrum from Casson’s invariant

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Recently, there had been a great deal of interest in obtaining and describing of all kinds of knots in links in hydrodynamics, electrodynamics, non Abelian gauge field theories and gravity. Although knots and links are observables of the Chern-Simons (C-S) functional, the dynamical conditions for their generation lie outside of the scope of the C-S theory. The nontriviality of dynamical generation of knotted structures is caused by the fact that the complements of all knots/links, say, in $S^3$ are 3-manifolds which have positive, negative or zero curvature. The ability to curve the ambient space thus far is attributed to masses. The mass theorem of general relativity requires the ambient 3-manifolds to be of non negative curvature. Recently, we established that, in the absence of boundaries, complements of dynamically generated knots/links are represented by 3-manifolds of non negative curvature. This fact opens the possibility to discuss masses in terms of dynamically generated knotted/linked structures. The key tool is the notion of knot/link concordance. The concept of concordance is a specialization of the concept of cobordism to knots and links. The logic of implementation of the concordance concept to physical masses results in new interpretation of Casson’s surgery formula in terms of the Regge trajectories. The latest thoroughly examined Chew-Frautschi (C-F) plots associated with these trajectories demonstrate that the hadron mass spectrum for both mesons and baryons is nicely described by the data on the corresponding C-F plots. The physics behind Casson’s surgery formula is similar but not identical to that described purely phenomenologically by Keith Moffatt in 1990. The developed topological treatment is fully consistent with available rigorous mathematical and experimentally observed results related to physics of hadrons.

Keywords: Chern-Simons functional; Casson invariant; theory of knots and links; theory of knot and link concordance; theory of 3 and 4-manifolds; Chew-Frautschi plots; Regge trajectories.

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1. Introduction

Although the discovered in 2012 Higgs boson is believed to be supplying masses to all known particles, already in 2011 there appeared a remarkable paper by Atiyah et al\(^1\) in which the topological origin of masses of stable particles was discussed. In addition, in 2014, in the paper by Bunyi et al\(^2\) the alternative model topologically generating the glueball mass spectrum was discussed. Much earlier, the Abelian reduction of QCD developed by Faddeev and Niemi\(^3\) and subsequently by many other authors resulted in the Faddeev-Skyrme (F-S) model whose stable configurations are Hopfions\(^4\). Hopfions are knotted/linked stable configurations obtained by minimization of the F-S model functional. Hopfions are believed to be responsible for the glueball mass spectrum of QCD\(^5\). According to Baal and Wipf\(^6\), and also to Langmann and Niemi\(^7\), and Cho\(^8\), Hopfions are believed to be stable vacuum configurations while the "instantons are viewed as configurations that interpolate between different knotted vacuum configurations."\(^7\) Such a vision is not shared by the authors of\(^2\). Instead, they claim that glueball spectrum should be associated with various types of excitations of the Yang-Mills (Y-M) fields. In both cases the glueball spectrum was obtained by completely ignoring the quark masses. Thus in all three cases, just described, the masses were generated topologically but mechanisms of mass generation in all these cases are profoundly different. This is unsatisfactory. Furthermore, while the mass-generating mechanism of \(^1\) requires fundamental reconsideration of the whole existing formalism of quantum fields, the proposal based on the F-S model, in principle, allows to develop known QCD formalism rather substantially\(^9\). This fact leaves us no hope for finding a bridge between the F-S model and that suggested in \(^2\). Naturally, the ultimate judge is the experiment. The authors of both\(^2\) and\(^5\) were comparing their theoretical results against experimental/theoretical results of\(^10\). Unfortunately, the results of\(^10\) were subsequently criticized in\(^11\). Not surprisingly, the existing experimental review papers, e.g. read\(^12\), do not mention at all theoretical results of either\(^2\) or\(^5\).\(^13\) Thus the bulk of experimental work on the glueball mass spectrum was not guided thus far by the predictions of the existing theoretical models just described. On the theoretical side, there are also very serious difficulties with both models. Indeed, the existing rigorous mathematical treatments of the Y-M fields are based on works by Andreas Floer. His contributions are discussed in detail in monographs by Donaldson\(^13\) and by Kronheimer and Mrowka\(^14\). A brief summary of Floer’s ideas is given in our work\(^15\). Incidentally, Floer’s work was inspired by the work of Clifford Taubes on the Casson invariant and gauge theory\(^16\).

In this work we develop a connection between theoretical results on the Casson invariant and experimental results interpreted in the style of Regge-type phenomenology. Our treatment does not involve any string-theoretic formalism yet. It is solely based on known

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\(^1\)That is non vacuum (or ground state).
\(^2\)The results of \(^2\) are the latest in the series of papers by the same authors written before\(^12\) was published.
\(^3\)The latest simplification of Taubes work is developed by Masataka\(^17\).
rigorous mathematics results though. Contrary to the results of Baal and Wipf, who believe that Hopfions are stable vacuum configurations of the Y-M fields, in Floer’s theory knots/links are not the ground states of the Y-M vacua. The ground states are represented by 3-manifolds. This fact eliminates the F-S type models from consideration. In the works by Buniy et al there are no instantons. Therefore, they also fall out of consideration. There is a substantial number of derivations of the Casson’s invariant of various degree of complexity. The purpose of this work is not just to reproduce known derivations of the Casson invariant. We have no intentions to copy this or that mathematical result word-for-word. Instead, in many instances, being guided by physical considerations, some new elements in deriving Casson’s invariant are obtained. This task required us to use a huge amount of facts from knot/link theory scattered in literature. Almost all of these facts have not found yet their place in the knot theory textbooks. Let alone, the literature on knot/link theory aimed at physically educated readers is lacking altogether this type of problematics.

2. Dynamically generated knotted and linked structures.

Review of existing results

The isomorphism between the dynamics of incompressible Euler-type fluids and Maxwellian electrodynamics is well documented. In 1985 Moffatt conjectured that in steady incompressible Euler-type fluid flows the streamlines could have knots/links of all types. In 2000 Etnyre and Ghrist using methods of contact geometry developed the existence-type proof of the Moffatt conjecture. Subsequently, in 2012, a different type of existence-type proof was published by Enciso and Peralta-Salas. Using the isomorphism between the incompressible fluids and Maxwellian electrodynamics the constructive-type proof of Moffatt’s conjecture was developed for Maxwellian electrodynamics in based on methods of contact geometry and topology. The proof of Moffatt conjecture for Maxwellian electrodynamics is opening Pandora’s box of all kinds of puzzles. Indeed, since publication of works by Witten and Atiyah it is known that the observables for both Abelian and non Abelian source-free gauge fields are knotted Wilson loops/links. It was demonstrated that only the non Abelian Chern-Simons (C-S) topological field theory is capable of recognizing the nontrivial knots/links. By “nontrivial” we mean knots other than unknots, Hopf links and torus-type knots/links which require for their description only the linking numbers and writhe(s). Being topological in nature, the C-S functional is not capable of taking into account the boundary conditions. In the meantime the boundary conditions do play an important role in the work by Enciso and Peralta-Salas. In general, the path integral methods become impractical whenever there is a need to take into account the non trivial boundary conditions. E.g. everybody familiar with the path integral methods knows that even such “simple” problem as solving the path integral for

4We were able to find 16 different derivations.
a free particle confined into triangle represents a substantial challenge. The Abelian reduction of the Y-M fields, e.g. that for the F-S model, is making them to be describable by the Abelian-type C-S field theory. This reduction was demonstrated in\textsuperscript{7,8} for the Y-M fields and in\textsuperscript{25,26} for Einsteinian gravity formulated as gauge theory for the Lorentz group. The constructive-type proof of the Moffatt conjecture\textsuperscript{22} underscores the differences between the types of knots and links the C-S field theory can produce and can detect.

Specifically, in \textsuperscript{15} the abelianization procedure was discussed starting with the full non-Abelian C-S gauge field functional. This procedure differs from that, say, described in \textsuperscript{3} by the fact that it uses the Arnol’d inequality\textsuperscript{15,18}. Its use is equivalent to the imposition of the Beltrami condition: $\nabla \times \mathbf{v} = \kappa \mathbf{v}$. Here $\mathbf{v}$ is the Abelian gauge field and $\kappa$ is some nonnegative constant. This condition was used by both Etnyre and Ghrist\textsuperscript{20} and by Enciso and Peralta-Salas\textsuperscript{21}. The account for the boundary conditions in respective papers was done differently though. In\textsuperscript{21} the boundary condition was chosen as: $\mathbf{v}|_\Sigma = \mathbf{w}$. Here $\Sigma$ is embedded oriented analytic surface in $\mathbb{R}^3$ while $\mathbf{w}$ is the vector tangent to $\Sigma$. In\textsuperscript{20} the account was made of the fact that the Beltrami condition admits interpretation in terms of contact geometry and topology\textsuperscript{18}. While the symplectic geometry is used for description of dynamics on even dimensional manifolds (e.g. recall the phase space of classical mechanics), the contact geometry is operating in spaces of odd dimensionality. Clearly, $\mathbb{R}^3$ is such a space. The one point compactification converts it into $S^3$. Known isomorphism between the classical mechanics and the hydrodynamics of Euler-type incompressible fluids\textsuperscript{18} allows us to relate the question about the existence of closed orbits on constant energy surfaces (e.g. on $S^3$) to the solution of Weinstein conjecture. This conjecture was solved by Taubes, the same person who obtained the Casson invariant via gauge theory\textsuperscript{16}. Not surprisingly, his solution also involves use of the Seiberg-Witten and Floer theories\textsuperscript{27}. The Etnyre-Ghrist solution involves uses of special universal template instead. It will be described below. Its use is equivalent to use of the boundary conditions in\textsuperscript{21}. Because of this, both solutions are of existence-type. If the boundary conditions are disregarded, the dynamics of closed orbits on $S^3$ becomes strictly Hamiltonian and all closed orbits for such a case were classified in the paper by Fomenko and Zung\textsuperscript{28}. These results were reobtained in our work\textsuperscript{22} with help of different type of methods. The main result of the Fomenko-Zhung (F-Z) paper can be formulated as

**Theorem 2.1.** a) (Fomenko-Zhung\textsuperscript{28}) Generalized iterated torus knots are precisely all possible links of stable periodic trajectories of integrable systems on $S^3$.

This theorem can be conveniently restated as follows

**Theorem 2.1.b)** Generalized iterated torus knots are knots obtained from trivial knots by toral windings and connected sums operations. These are the only knots/links of stable periodic trajectories of integrable systems on $S^3$.

**Corollary 2.2.** The above Theorem implies that not every link of stable periodic trajectories can be generated by some integrable dynamical system living on $S^3$. For instance,
there are no dynamically generated knots/links containing the figure eight knot. This fact immediately excludes results of Etnyre and Ghrist\textsuperscript{20} and Enciso and Peralta-Salas\textsuperscript{21}.

But these results involve account of the boundary conditions and, because of this, they cannot be immediately compared with those by F-Z! Evidently, the F-Z knots/links are exactly those which are observables for the Abelian version of the C-S functional. In view of the existing abelianization procedures for the Y-M and gravity fields mentioned above, it follows then that the totality of such abelianized fields is described by the F-Z theorem.

The physical content of this corollary will be discussed in this section further below. In the meantime, it is of interest to relate the F-Z results to those by Etnyre and Grist and by Enciso and Peralta-Salas. Notice, in all three cases we are dealing with the Abelian gauge fields! But the presence of boundaries creates all kinds of knots/links out of Abelian fields! How this result should be understood? Notice, that solution of Moffat’s conjecture developed in\textsuperscript{20,21} does not involve uses of traditional tools of knot theory such as Alexander or Jones polynomials, etc. It does not involve uses of knot Floer topology and so on. And yet, the solution of this conjecture implies that all types of knots/links can be generated dynamically. Clearly then, the question arises: How one can be sure that, indeed, all knots and links can be generated? To answer this question we need to discuss briefly work by Birman and Williams\textsuperscript{29} done in 1983. These authors posed and solved the following dynamical problem. They studied periodic orbits in the Lorenz system. This dynamical system emerges as finite-dimensional reduction of the Navier-Stokes equation and is made out of three coupled ordinary differential equations. The system of equations is not of Hamiltonian type (that is, it is dissipative) and exhibits strange attractor, chaos, etc. The study of periodic orbits was greatly facilitated by the template construction. A template $\mathcal{T}$ is a compact branched two-manifold with boundary built from finite number of branch charts. In short, the template is working as some kind of a switch regulating flow. E.g. imagine some bug crawling on the figure 8. Each time it reaches the crossing, it should decide which way to go. This is just the simplest example of finite state automaton. In the same paper the following conjecture was formulated

**Conjecture 2.3.** (Birman-Williams\textsuperscript{29}) There are no universal templates. That is say, there are no dynamical systems whose closed orbits have knots and links of all types

But solution of Moffatt’s conjecture does present a counterexample to the Birman-Williams conjecture! Thus, it should be somehow linked with uses of (perhaps different) templates. Indeed, the paper by Etnyre and Ghrist\textsuperscript{20} does involve use of different (the so called universal) template. By design, such a template can support knots and links of any type. Its discovery has its origin in other works though. These are having physical significance which was left unnoticed. This deficiency is corrected in \textsuperscript{22}. Chronologically,
the discovery of the universal template was made earlier by Ghrist\textsuperscript{30}. Subsequently, other universal templates were constructed. In his paper Ghrist was guided by yet another paper by Birman and Williams,\textsuperscript{31} also done in 1983, whose content was linked with results of Etnyre and Ghrist and Enciso and Peralta-Salas in \textsuperscript{22}. In Birman-Williams paper the authors discussed knotted magnetic field configurations surrounding a piece of wire coiled in the shape of figure 8 knot in which current flows. They demonstrated that these knotted/linked configurations contain knots/links of any type. Ghrist\textsuperscript{30} streamlined this result by designing the universal template explicitly. This result can be connected with that of Enciso and Peralta-Salas. For this purpose one should take into account both the Beltrami and the boundary conditions for static magnetic field configurations. These are: $v|_{\Sigma} = w$ and $\nabla \times v = \kappa v$. Here $\Sigma$ is the surface of the wire coiled into the shape of the figure 8 knot. As it was argued in \textsuperscript{18} and enforced in, \textsuperscript{22} the conditions imposed by Enciso and Peralta-Salas are those used for superconductors. The correspondence between the physics of incompressible Euler fluids and physics of superconductors was discovered by Fr"{o}lich in 1966 but was left unnoticed to our knowledge. It was brought to spotlight in \textsuperscript{18}. Once we know how to generate knots/links of all kinds, even for the Abelian gauge fields, the following set of problems emerge.

First, we must take into account that "knots are determined by their complements" as demonstrated by Gordon and Luecke\textsuperscript{32}. This means the following. Suppose we are having just one knot $K$, e.g. figure 8 knot, (that is not a link!) in $S^3$ (we obtain $S^3$ by the one point compactification of $\mathbb{R}^3$). The complement of $K$ in $S^3$ is 3-manifold $M^3$ with boundary. The Gordon-Luecke theorem is telling us that for knots embedded into $S^3$ there is one-to-one correspondence between knots and 3-manifolds. Notice, however, that this theorem could become invalid as soon as we add yet another knot into $S^3$. In this case we are dealing with links (even though 2 knots are disentangled!). And such links are called boundary links. The concordance is dealing, for example, with such types of situations. But, also, with many other situations of physical interest to be discussed below.

Second, the 3-manifolds (with boundaries) can be subdivided into hyperbolic, flat and spherical types (Seifert fibered). Finer classification of 3-manifolds was developed by Thurston\textsuperscript{33} who conjectured that the interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structures. This, the so called geometrization conjecture, was subsequently proved by Perelman. A quick physically motivated introduction to his work can be found in\textsuperscript{34}. In short, this means that such geometric structure is described in terms of a complete locally homogenous Riemannian metric. In terms of knots and their complements this can be stated as follows. Altogether there are only 3 types of knots: torus knots, satellite knots and hyperbolic knots. If $K \subset S^3$, then $S^3 \setminus K$ has a geometric structure if $K$ is not a satellite knot. It has a hyperbolic structure if, in addition, $K$ is not a torus-type knot.

Third, the essence of Einstein’s equations of general relativity can be summarized as follows:

$$\text{Curvature} = \text{Matter}. \quad (2.1)$$
That is to say, the matter is causing the initially flat space to curve. From arguments just provided it follows that, say, the torus-type knots could be associated with massive particles. This means, in particular, that (complements of) knotted structures created by, say, the abelianized Y-M fields (e.g. read Corollary 2.2) can act as masses. This statement requires some refinements. For one thing, unlike the electromagnetism, masses (without charges) can only attract each other so that we have to choose once and for all between the hyperbolic, flat and spherical 3-manifolds. In addition, since general relativity is 3+1 dimensional theory, if we use knots as masses, then the homotopy moves could be interpreted in terms of time evolution. Thus, we have to deal with 4-dimensional extensions of 3-dimensional theory of knots/links. Ultimately, it is this observation which brings us to theories of cobordism and concordance.

The choice between the hyperbolic and spherical manifolds can be made based on positivity of mass theorem in general relativity. An exhaustive treatment of this topic is given in the book by Choquet-Bruhat. A quick introduction into this subject can be found in lectures by Khuri and Bartnik. Thus, the positivity of mass theorem leaves us with the option of considering only the nonhyperbolic 3-manifolds in calculations involving 3+1 decomposition of gravity. The nonhyperbolic 3-manifolds are originating as complements of nonhyperbolic knots, e.g. torus or iterated torus-type knots. But these are the only knots/links which are dynamically generated in the absence of boundaries (Theorem 2.1)! Only such types of knots/links can be dynamically generated in electromagnetism and hydrodynamics as described in detail in. Furthermore, the Abelian reduction of the Y-M and gravity fields, discussed in works by many authors mentioned above, allows us to apply the results of to these fields practically without change.

Thus, already knotted Maxwellian fields are acting as masses even though photons are surely massless (even though the light rays are being bent by gravity fields). Furthermore, the same knotted fields can possess charges. In fact, based on arguments explained in detail in massive charges in electromagnetism can be reinterpreted in terms of the torus-type superconducting knots. This observation when superimposed with results of the Abelian reduction of both the Y-M and gravity fields removes all difficulties (e.g. read page 97) associated with the description of extended objects and charges in classical gravity and Y-M theories.

3. Recovering Regge mass spectrum from knot/link concordance. Fundamentals

3.1. Concordance. First glimpses

Since space can be curved not only by masses but by knots/links as explained in previous section, it is of interest to investigate the extent to which it is possible to push the correspondence between masses and knots/links. We begin with the "thermodynamic" property of masses, i.e. of their extensivity. That is to say, at the very basic "classical"
level two masses $m_1 > 0$ and $m_2 > 0$ obey the law of mass conservation
\[ m_1 + m_2 = m^T, \]  
(3.1)
where $m^T > 0$ is the total mass. In knot theory the analog of this relation is the operation of taking the connected sum which we would like now briefly describe. Formally speaking, suppose we have, say, two knots $K_1$ and $K_2$ and we would like to combine them together somehow. The result of such an operation is conventionally denoted as $K_1 \# K_2$. The execution of this operation is not as formal as the mass addition though. This is so, because with a given knot $K$ in 3-manifold $M$, $K=(M, K)$ one can associate three other knots: the mirror image knot $mK=(-M,K)$, the reverse $rK=(M,-K)$ and the inverse $-K=rmK=(-M,-K)$ knots. In view of $^{15,22}$, the knot $K$ and its reverse can be associated with particles having opposite charges (electric or magnetic in view of electromagnetic duality) while knots living in 3-manifolds and having opposite orientations will represent the corresponding antiparticles. The electrically neutral species would require us to use the connected sums of the type $K \# rK$. The formalism could be developed in both ways: a) that which recognizes orientations and b), that which is blind to orientations. This is very much the same as to use the comparison between numbers in the ring of integers or to use the mod comparison between numbers. In any case, to make things simpler, we shall only consider oriented knots in oriented 3-manifolds for the time being. Then, the operation of taking the connected sum is relatively easy to define and, to avoid redundancy, we refer to the excellent source $^{41}$, pages 20-22, for detailed description of this operation. Once it is defined, it can be treated algebraically and, at this level of treatment, the intricacies associated with orientation can be temporarily suppressed. Thus we notice that the connected sum operation for knots/links is commutative and associative, i.e.
\[ K_1 \# K_2 = K_2 \# K_1; K_1 \# (K_2 \# K_3) = K_1 \# (K_2 \# K_3). \]  
(3.2)
The commutativity and associativity of the connected sum operation is not sufficient for making this operation as a group operation on the set of all oriented knots $K$. Thus far $K$ is only a monoid since it contains an identity element—the unknot. To convert this monoid into a group requires some black magic thinking. First, we need to introduce the notion of cobordism.

Let $M_1$ and $M_2$ be oriented closed 3-manifolds. By definition, $M_1$ is cobordant to $M_2$ if there is compact oriented smooth 4-manifold $W$ (called cobordism between $M_1$ and $M_2$) such that $\partial W = M_1 \cup (-M_2)$, where $-M_2$ denotes manifold $M_2$ with reverse orientation and $\cup$ denotes the disjoint union. Clearly, the cobordism is an equivalence relation. It is possible now to apply the idea of cobordism to knots. This can be done with several levels of sophistication. For instance, let $K_1$ and $K_2$ be some knots. Following $^{40}$, consider now the connected sum $K_1 \# rK_2$ so that effectively $-K_2 = rmK_2$. In addition, following $^{42}$ we introduce a cone $\hat{k}$ over the knot $K$ as depicted in Fig.1.

Topologically $\hat{k}$ is a disc $D^2$. If embedded into 4-dimensional ball $B^4$, the intersections seen in Fig.1 will disappear. These are only visible in $S^3$. The cone apex represents the
Figure 1: A trefoil knot bounds a non-locally flat disc. In 3 dimensions this nonlocality is depicted as a cone with 3 boundary singularities.

only singularity of $D^2$ for any nontrivial $K$. For trivial knots this singularity disappears. The above peculiar features of embedding of $k$ into 4-ball can be explained with help of the famous Whitney Theorem

Definition 3.1. Two knots $K_1$ and $K_2$ are smoothly concordant if there is a smooth embedding $f: S^1 \times [0,1] \rightarrow S^3 \times [0,1]$ such that $f(S^1 \times 0) = K_1$ and $f(S^3 \times 1) = K_2$.

Concordance is the equivalence relation. It will be denoted as $K_1 \sim K_2$.

Definition 3.2. A knot $K$ is slice if $K \sim U$, where $U$ is unknot. Alternatively, and more formally, a knot being a slice means that it is a boundary of $D^2$ smoothly embedded in $B^4$.

It can be demonstrated, e.g. read p.88 of 44, that $K \# rmK \sim U$. Use of this result allows us to define an inverse for the connected sum operation. Indeed, the inverse for a given knot $K$ is $rmK$. This observation allows us to replace the monoid $K$ with the concordance group $C_3$. Physically, the combination $K \# rmK$ brings together particle and antiparticle. They may annihilate so that the rest mass is zero, that is $U$.

Now it is becoming possible to introduce some concordance invariants. Specifically, following Murasugi 41, we notice that the knot signature $\sigma(K)$ is one of such invariants. By definition,

$$\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$$

(3.3)

and $\sigma(rmK) = -\sigma(K). Therefore \quad \sigma(K \# rmK) = \sigma(K) - \sigma(K) = 0 = \sigma(U)$. Thus, the signature invariant can detect slice knots. Because of this property, it is also a concordance invariant. This means that if K is concordant to K’, then $\sigma(K) = \sigma(K’)$. Equation (3.3) provides a homomorphism from the concordance group $C_3$ into $Z$. More accurately, since the signature is always even, we have to use $\sigma(K)/2$ to map $C_3$ into $Z$. Are there other invariants which possess the additivity property? Are they also concordance invariants?
The answer to both questions is "Yes". Some of them will be introduced whenever it will become appropriate. Incidentally, by itself the Casson invariant—the main focus of this paper—is not a concordance invariant. E.g. read \(^{45}\), page XV. Nevertheless, as we shall demonstrate, to introduce this invariant it is required to use many results from the concordance theory.

In the meantime, we would like to notice the following. It is clear that the slices \(K\# rm K\) can be formally constructed for all knots. This procedure is not without some controversy though since it ignores the existence of the so called amphichiral knots. The figure 8 knot is the first example of the fully amphichiral knot. Such knots are (ambient)isotopic with respect to both its reverse \(r\) and its mirror image \(m\). In addition, there are positive and negative amphichiral knots These are respectively homeomorphic to their mirror images without changing orientation (positive) and with changing the orientation (negative). The figure 8 knot could be identified with the Majorana fermion which is simultaneously particle and antiparticle. Moreover the signature \(\sigma(K_8) = 0\) so that figure 8 knot is formally slice. But it is not! Indeed, the knot is slice if, in addition to its signature being zero, its Alexander polynomial \(\Delta_K(t)\) could be represented in the factorized form, that is \(\Delta_K(t) = f(t)f(t^{-1})\) with \(f(t)\) being some polynomial\(^{42}\). In the case of figure 8 knot its Seifert matrix \(V\) is known to be

\[
V = \begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix}
\]

so that its Alexander polynomial is \(\Delta_{K_8}(t) = \det[V - tV^T] = t^3 - 3t + 1\). Evidently, it is not factorizable. Furthermore, the figure 8 knot is a hyperbolic knot. Hyperbolic knots cannot be dynamically generated as proved in\(^{22}\). This result is consistent with the positive mass theorem in general relativity as discussed already. Therefore, even though mathematically this type of a knot does have right to exist, in the absence of boundaries physically it cannot be spontaneously generated. As result, if we identify the Majorana fermion with figure 8 knot, we should admit that the Majorana fermion does not exist in Nature. This result is valid in 3+1 dimensions. Experimentally, at the moment of this writing, indeed, in 3+1 dimensions the Majorana fermion was not found. The 2+1 dimensional version of Majorana fermion is believed to be very valuable in theory of quantum computing\(^{18}\). The description of their generation in condensed matter physics lies outside of the scope of this paper. For completeness, the current status of Majorana Fermions in various dimensions is discussed in detail in\(^{46}\).

To finish this section, we would like to quote the following

**Theorem 3.3.** (The mirror Theorem\(^{44}\), page 173). Let \(K\) be a knot diagram with zero writhe (that is without curls). Then \(K\) is ambient-isotopic to \(mK\) if and only if it is regularly isotopic to \(mK\).

This theorem could be physically understood as follows. First, we notice that, by definition, the regular isotopy is taking place only in 2 dimensions. Next, following Kauffman,
suppose that our knot is made of rubber tube so that twisting is causing some tension along the tube. The twisting costs some energy associated with creation of tension. Thus to bring knot regular-isotopically to its mirror image requires us to go over the potential barrier associated with the creation of tension during this transformation process. That is knot and its mirror image are sitting in two potential wells separated by the potential barrier.

There is an exact analog of the situation just described in quantum mechanics. It is described in the 3rd volume of Feynman’s lectures in physics\text\textsuperscript{17}, page 8-11. Feynman is considering the ammonia molecule NH\textsubscript{3} as a candidate for the working body for the ammonia maser. The ammonia molecule can be visualized as a tetrahedron whose 3 vertices are being occupied by H’s while the 4th- by N. Consider an "up" configuration. It is a configuration in which all 3 hydrogens are sitting on the top of the mirror while the nitrogen is elevated above the mirror. The mirror image of such a tetrahedron, the"down configuration", is another tetrahedron with N molecule below the mirror plane. It happens, that such an ammonia molecule has an induced dipole moment located at the center of symmetry of ammonia and, of course, of its mirror image. But for the "up" configuration the dipole is looking straight down while for the "down" configuration the dipole is looking straight "up". Following Feynman, let $|1\rangle$ be the quantum state of the "up" configuration and $|2\rangle$ be the quantum state for the "down" configuration. Then, according to Feynman, "it is possible for the nitrogen to push its way through three hydrogens and to flip to another side". Such a move of N through H’s is exact equivalent of regular isotopy described by Theorem 3.3. while the tension in the present case is associated with the energy for flipping of the dipole. As result, we just obtained a two-level quantum mechanical system. Such a system is extensively discussed in\text\textsuperscript{18} in connection with problems of quantum computation.

There is yet another way to interpret just obtained results. It is associated with uses of the Vassiliev invariants and virtual knots. Following Manturov\text\textsuperscript{48} we provide the

**Definition 3.4.** A virtual link diagram is a planar four-valent graph endowed with the following structure: each vertex either has an (over) under crossing or is having crossing which is not resolved (a double point).

Virtual knots and links were discovered by Kauffman\text\textsuperscript{49} and are currently under active study.

**Definition 3.5.** (Ref. 50, page 72) Any knot invariant can be extended to knots with double points by means of the Vassiliev skein relation.

It is depicted in Fig. 2

Here $v$ is the knot invariant with values in some Abelian group. The left hand side is the value of $v$ on singular (that is with specific crossing not resolved) knot K while the right hand side is the difference of the values of $v$ on regular knots obtained from K by replacing the double point with a positive and negative crossings respectively.

The physical situation related to Theorem 3.3. is reminiscent to the theory of spontaneous symmetry breaking during the 2nd order phase phase transition. Perhaps, it is possible to
develop some kind of Higgs-type mechanism of spontaneous mass generation related to just described process. In addition, however, there are cases when even the ambient isotopy fails to bring a given knot $K$ into its mirror image $mK$. In such a case we are dealing with chiral knots whose simplest representative is trefoil. To bring the trefoil made of rubber tube to its mirror image requires us to break the rubber tube and to reconnect it back. This will switch an over crossing into under crossing or vice-versa. This process surely costs energy. Physically, it is very much the same thing as breaking a bond when a molecule wants to escape the solid phase and go to the gas phase (and vice-versa). This is typical case of the first order phase transition accompanied by the presence of the latent heat (needed for breaking of chemical bonds). We shall demonstrate below that in the case of physical interpretation of the Casson invariant it is exactly these type of processes that are responsible for generation of the Regge-type mass spectrum.

The trefoil is definitely not a slice knot but the connected sum of the trefoil with its mirror image (known as square knot) is slice, e.g. read\footnote{5}{This result belongs to William Thurston. We would like to thank Morwen Thistlethwaite, (UTK), for detailed explanation of this fact.}, page131. Although physically questionable, mathematically it is easily possible to make the connected sum of two figure eight knots (it will be denoted as $K_8^2$). Such a formation, when combined (via connected sum) with yet another $K_8^2$ does yield a slice\footnote{5}{This result belongs to William Thurston. We would like to thank Morwen Thistlethwaite, (UTK), for detailed explanation of this fact.}, page 203. Notice as well that even though the $K_8$ knot is hyperbolic, $K_8#K_8$ is not\footnote{7}{This information was supplied to us by Morwen Thistlethwaite (UTK).}. Questions remain. 1. Are all fully amphichiral knots hyperbolic? 2. Is the same true for positive and negative amphichiral knots? The problem of designing slice knots/links is further complicated by the following observation. The stevedore knot is known to be slice\footnote{6}{In fact, it is dobly slice knot, e.g. see section 4.4 for definition and read \url{http://katlas.org/wiki/6_1} it is a hyperbolic-type knot\footnote{6}{In fact, it is dobly slice knot, e.g. see section 4.4 for definition and read \url{http://katlas.org/wiki/6_1}}. Is there other (than stevedore) slice knots/links which are not of the form $K#rmK$ and are hyperbolic? Since the research on slice knots is still ongoing, it would be too premature in this work to list all the achievements in this domain of research. We shall restrict ourself by the most conventional slices of the type $K#rmK$. The ribbon knots/links are immediately connected with such slices and will
be discussed below in this work, e.g. see Fig.11 and read comments related to this figure.

In Section 2 we argued that the hyperbolic knots cannot be candidates for physical masses. Fortunately, they also cannot be generated dynamically (in the absence of boundaries). However, if the notion of dark matter does make any sense, and it does\footnote{E.g. read our paper http://arxiv.org/abs/1006.4650}, it might be of some interest to investigate what events/processes in Nature can cause hyperbolic knots/links to come to life. Theoretically, though, this is possible only in the presence of boundaries of special type\footnote{E.g. read http://mathoverflow.net/questions/91760/poincare-dodecahedron-space} as explained in previous section.

3.2. From cosmological models to microcosm and back.

How concordance with observational data brings to life the Casson invariant

Casson’s invariant was defined originally only for the homology sphere 3-manifolds. Following Rolfsen\footnote{E.g. read our paper http://arxiv.org/abs/1006.4650}, let us recall the

**Definition 3.6.** A closed connected (but not necessarily simply connected !) 3-manifold \(M\) is a *homology sphere* if \(H_1(M) = 0\). If, in addition \(\pi_1(M) = 1\), then the manifold is a *homotopy sphere*.

The Poincare’ conjecture (recently proven by Grisha Perelman, e.g. read Ref.34 for a quick introduction) claims that the 3-sphere \(S^3\) is the only manifold for which both \(H_1(M) = 0\) and \(\pi_1(M) = 1\) hold. Clearly, homotopy spheres are homology spheres. Using the Poincare’ duality it can be shown that for homology spheres all homology groups are exactly the same as those for \(S^3\). Suppose now that there is a 3-manifold such that \(H_1(M) = 0\) but \(\pi_1(M) \neq 1\). If such a manifold does exist, it represents a *homology sphere*. Poincare’ was the first who designed the manifold of this type known in literature as dodecahedral space. A pedagogical account of such a space as well as its relevance to cosmological models of Universe is discussed in the paper by Weeks\footnote{E.g. read http://mathoverflow.net/questions/91760/poincare-dodecahedron-space}. Subsequent studies put into question usefulness of such a space as good model of Universe. The dodecahedral space represents the first example of homology sphere space which is not simply connected since its fundamental group is \(\pi_1\) that of binary icosahedron\footnote{E.g. read http://mathoverflow.net/questions/91760/poincare-dodecahedron-space}. Surprisingly, This happens to be a trend. There are many homology spheres which are not simply connected. Such multiply connected spaces are the latest candidates of cosmic topological models of Universe\footnote{E.g. read http://mathoverflow.net/questions/91760/poincare-dodecahedron-space}. After Poincare’s description of the dodecahedral space many other designs of homology spheres came to life. It has become possible to construct a countable infinity of different homology spheres\footnote{E.g. read http://mathoverflow.net/questions/91760/poincare-dodecahedron-space}, some of which will be described shortly below. Thus far the recipes for making homology spheres were purely mathematical. This means that they leave completely outside the question of major physical importance: How these homology spheres can be physically realized in nature? Notice that the standard knot theory...
not contain a spatial scale. This means that knots/links could live both in macro and microcosm and everywhere in between. This fact brings to life many puzzles in knot theory revealing themselves the most in the theory of 4-manifolds. We shall touch this topic a bit further later in the text. In we discussed physical mechanisms leading to formation of knots/links. In the previous section it was explained that in the absence of boundaries only non hyperbolic knots/links can be formed. Complement of every knot/link, say in \( S^3 \), is some 3-manifold. Mathematically, these 3-manifolds are obtainable, for instance, by operation of Dehn surgery which can be reinterpreted in terms of cobordism and, hence, of concordance.

**Definition 3.7.** A knot \( K \subset S^3 \) is said to satisfy property \( P \) if there are no non-trivial surgeries on it yielding a simply-connected manifold.

Thus, whenever the property \( P \) holds, homology spheres are not simply connected 3-manifolds. Incidentally, The unknot does not satisfy property \( P \) since, for example, 1-surgery on it yields \( S^3 \), e.g. read. This result follows from the fact that \( S^3 \) can be made out of the union of two solid tori (the most elementary example of Heegard splitting!). The property \( P \) remained a conjecture till 2004 when it was proved by Kronheimer and Mrowka. It is remarkable that cosmological models of multiconnected (almost) flat Universes are compatible with the property \( P \). Furthermore, multiconnectedness reveals itself microscopically. We would like now to explain how this could happen.

We begin with the observation that the property \( P \) holds by knots \( K \) for which \( \frac{d^2}{dt^2} \Delta_K(t)|_{t=1} \neq 0 \), page 662 (bottom) of Ref.58, and page XV. Here, as before, \( \Delta_K(t) \) is the Alexander polynomial for some knot \( K \). The combination \( \frac{d^2}{dt^2} \Delta_K(t)|_{t=1} \) enters the definition of the Casson invariant as explained below and, since the Casson invariant admits physical interpretation, this means that property \( P \) reveals itself microscopically as well.

To proceed, we have to define several additional concepts e.g. that of the spliced sum. Let \( K_1 \) and \( K_2 \) be oriented knots living in oriented homology spheres \( \Sigma_1 \) and \( \Sigma_2 \) respectively. Let \( M_1 = \Sigma_1 \setminus K_1 \) and \( M_2 = \Sigma_2 \setminus K_2 \) be their complements. Both \( M_1 \) and \( M_2 \) are manifolds with boundary: \( \partial M_1 = \partial M_2 = T^2 \), where \( T^2 \) is a torus. Let \( (m_1,l_1) \) and \( (m_2,l_2) \) be the canonical meridian and longitude respectively on \( \partial M_1 \) and \( \partial M_2 \), then we obtain the following

**Definition 3.8.** The spliced sum (along \( K_1 \) and \( K_2 \)) of \( \Sigma_1 \) and \( \Sigma_2 \) is the operation of creating a new homology sphere \( \Sigma = M_1 \cup_{T^2} M_2 \) via gluing \( M_1 \) and \( M_2 \) via orientation-reversing homeomorphism along their common boundary \( T^2 \) in such a way that \( m_1 \) is glued to \( l_2 \) and \( m_2 \) to \( l_1 \).

The connected sum is a special case of the splicing sum. This will become evident upon reading. The notion of the spliced sum leads us directly to our first example of designing the homology spheres. Following Rolfsen, page 251, we only should replace \( \Sigma_1 \) and \( \Sigma_2 \) by

---

10 Although this operation is not limited to the homology spheres, in this work we shall introduce it only in the context of homology spheres.

11 In the sense of Rolfsen, page 136. See also Appendix.
two copies of $S^3$, while keeping both $K_1$ and $K_2$ nontrivial to obtain the homology sphere \( \Sigma = M_1 \cup_{T^2} M_2 \). In the case of $S^3$ the complements $S^3 \setminus K_i, i = 1, 2$, are solid tori\(^{54,56}\). For such a case, it was rigorously proven that various knots produce countable infinity of (non-simply connected) homology $3$-spheres $\Sigma = M_1 \cup_{T^2} M_2$. Since such a design of homology spheres is mathematically plausible but physically not realizable, we shall describe another design admitting physical interpretation. Doing so gives us the opportunity to describe the first example of magical powers of concordance.

We begin with discussion of the paper by Gordon\(^{59}\). In his paper slight change in the rules for construction of spliced sums are described. It is worth discussing the alternative construction of splicing and its implications in some details.

For $i = 1, 2$, let $K_i$ be a knot in homology sphere $\Sigma_i$ with exterior $M_i$. Furthermore, let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ be a $2 \times 2$ integral matrix with $\det A = -1$ defining orientation-reversing diffeomorphism $h: \partial M_1 \to \partial M_2$ so that the closed $3$-manifold $M_1 \cup_h M_2$ will be denoted as $M(K_1,K_2;A)$ or, better, as $M(K_1,K_2;\alpha,\beta,\gamma,\delta)$. Since the exterior of the unknot $U$ is $(\mathbb{R}^2 \times S^1)^2$ the manifolds of the type $M(U,K;A)$ are exactly those which are obtained by removing the tubular neighborhood of $K$ and sewing it back (possibly differently).

**Remark 3.9.** Just described operation is an example of a **surgery along** $K$ (more on this is given in Appendix).

In particular, if $K$ is knot in $S^3$, then the homology sphere is obtained when $M(U,K;\alpha,\beta,\pm 1, \delta)$. This was the prescription for creating homology spheres discovered by Dehn, e.g. read\(^{54}\), page 246. If $\cong$ denotes the orientation-preserving diffeomorphism, then it follows that $M(U,K;\alpha,\delta,\pm 1, \delta) \cong M(U,K;\alpha,1,1,0) \equiv M(K,\alpha)$. Here $\epsilon = \pm 1$. Incidentally, $M(U,U;A)$ is the lens space $L(\gamma,\epsilon\alpha) \cong L(\gamma,\delta)$. This can be seen from reading of page 99 of Ref.60.

The above results can (and should) be rewritten in terms of knot concordance. To avoid any ambiguity, we shall use Gordon’s notations for this purpose. Thus, we have the following

**Definition 3.10.** Closed $3$-manifolds $M^+$ and $M^-$ are $G$-homology cobordant if there exist a $4$-manifold $W$ such that $\partial W \cong M^+ \cup -M^-$, provided that inclusions $M^\pm \to W$ induce isomorphisms $H_*(M^\pm,G) \to H_*(W,G)$. Under such circumstances a $G$-homology sphere is a closed $3$-manifold $M$ such that $H_*(M,G) \cong H_*(S^3,G)$.

**Definition 3.11.** A proper pair of knots $C = (W,S^1 \times I)$ is such a pair that $\partial C \cong C^+ \cup -C^-$, where $C^+ = (M^+,S^1 \times 0)$ and $C^- = (M^-,S^1 \times 1)$ is a homology cobordism (that is concordance) between knots $C^+$ and $C^-$ if $W$ is homology cobordism between $M^+$ and $M^-$.\(^{12}\)

**Definition 3.12.** A knot cobordism (that is concordance) between knots $C^+$ and $C^-$ in $S^3$ is a proper pair $C = (S^3 \times I,S^1 \times I)$ such that $C^+ \cong (S^3 \times 0,S^1 \times 0)$ and $C^- \cong (S^3 \times 1,S^1 \times 1)$. Gordon proves the following\(^{59}\) (his Theorem 3), page 163,
Theorem 3.13.(a) Let $C_1$ and $C_2$ be two knot concordances (cobordisms) such that $C_1^+ = 0$. Suppose that in addition (in matrix $A$) $\gamma = \pm 1$ and that either a) $C_2^+ = 0$ or b) $\alpha = 0$. Then the 4 manifold $W(C_1, C_2; A)$ is homology cobordism between $S^3$ and $M(C_1^-, C_2^-; A)$ with $\pi_1(W) = 1$. By attaching a copy of $\mathbb{R}^4$ to $W$ along $S^3$ the contractible manifold $N$ is obtained.

This allows us, following Gordon, to restate Theorem 3.13 (a) differently

Theorem 3.13.(b) Suppose that (in matrix $A$) $\gamma = \pm 1$. If $K_1$ is a slice knot, and either a) $K_2$ is also a slice knot, or b) $\alpha = 0$ (in matrix $A$). Then the 3-manifold $M(K_1, K_2; A)$ bounds a contractible manifold $N$.

This theorem was proved subsequently by many authors in various settings. For the purposes of this work it is of interest to restate the same theorem, still differently, in the context of Freedman’s theorem on Fake Balls\textsuperscript{43}, page 83.

Theorem 3.13 (c) (Freedman). Every homology sphere 3-manifold $\Sigma$ bounds a contractible 4-manifold $W$ - a fake 4-ball.

Corollary 3.14. Having some knot (representing a particle) in physical vacuum (created by the slice knot(s) in $S^3$) changes the Euclidean space $S^3 = \mathbb{R}^3 + \text{point at infinity}$ into homology sphere $\Sigma$. The properties of vacuum are determined by the presence of fake contractible 4-manifold. Since all smooth 4-manifolds are symplectic, they are automatically Kähler. Therefore such manifolds, at least locally, can be looked upon as the Calabi-Yau manifolds (alternatively, as the K3 surfaces). This fact was noticed by Fintushel and Stern\textsuperscript{61} (e.g. read their Corollary 1.4.) whose work is summarized in\textsuperscript{43,62}. Alternatively, the observed existence of homology spheres modelling our Universe is reflection of the fact that the physical vacuum state is filled with particles and antiparticles in equal amounts. This is in contrast with results of knot theory where, in addition to ”physical” slices $K\# r mK$, other slices had been discovered (e.g. read previous subsection).

Based on just described information, we shall proceed in a way alternative (but equivalent) to that developed by Fintushel and Stern. By studying implications of Freedman’s Fake Balls theorem we would like to reveal yet another magical properties of concordance while trying to make our presentation as physical as possible.

Following\textsuperscript{56}, page 73, we notice that any knot $K \subset S^3$ can be analytically described as an intersection of the sphere $S^3 = \{(z, w) \in |z|^2 + |w|^2 = 1\}$ with 4-ball $B^4 = \{(z, w) \in |z|^2 + |w|^2 \leq 1\}$. Since every knot/link bounds a Seifert surface $F_K$\textsuperscript{13} so that $K = F_K \cap S^3$, by looking at Fig.1 we can associate the apex of the cone $k$ with the center of $B^4$ while the points $x \in K$ and $x \in F_K$ are to be treated as points inside the ball $B^4$. Next, we assume that there exist some function $f(x)$ which we consider as the Morse function for which

13E.g.see Fig.10.
Figure 3: Hopf link as intersection of 2-surfaces in 4-space

$f: F_K \rightarrow \mathbb{R}$, as usual. The function $f(x)$ can be selected based on the requirement $f(x) \neq 0 \ \forall x \in F_K$. To take advantage of such defined function it is convenient to change the rules a little bit. Specifically, the extra (time) dimension can be introduced now via $S^3_t = \{(z, w) \in |z|^2 + |w|^2 = t^2\}$ so that the intersection $F_K \cap S^3_t$ at a point $p \in F_K \cap S^3_t$ is transverse. This means that $T_p F_K \oplus T_p S^3_t$ spans $T_p \mathbb{B}^4$. By definition, critical points $p^i$ of the function $f(x)$ are points at which $\partial f/\partial x(p^i) = 0$. If there are such points, then the standard protocol of Morse theory can be applied. Since $F_K$ is compact such designed Morse function will contain only finite number of critical points which will be assumed to be non degenerate (that is these are well separated points). In the degenerate case, one should use the Morse-Bott version of Morse theory. For a fixed $t$ each intersection $F_K \cap S^3_t$ is a curve in 3-space which is becoming a 2 dimensional surface $S_K$ in $\mathbb{B}^4$. If there is a link, say, made of two knots $K_1$ and $K_2$, then it is appropriate to talk about the intersection of the surface $S_{K_1}$ with $S_{K_2}$, i.e. $S_{K_1} \cdot S_{K_2}$. The details of this construction are beautifully explained in [43], chapter 4. The result $S_{K_1} \cdot S_{K_2}$ can be understood on example of a Hopf link as depicted in Fig.3.

Using this picture, it follows that

$$S_{K_1} \cdot S_{K_2} = \text{lk}(K_1, K_2). \quad (3.4)$$

This result is unexpected in the following sense. In standard textbooks on knot theory [41, 44] we learned that the linking number is a 3-dimensional object. Now we just found that the linking in 3-dimensions can be equivalently described in terms of intersecting surfaces “living” in 4 dimensions. This fact allows us to define the intersection form [43] $Q(\alpha, \beta)$ for links, say, $\alpha$ and $\beta$ as follows

$$Q(\alpha, \beta) = S_\alpha \cdot S_\beta. \quad (3.5a)$$

It is an integral matrix since its entries are integers. The matrix $Q(\alpha, \beta)$ is invertible over integers. If $V$ is a (Seifert) matrix of linking coefficients associated with Seifert surface, then the intersection form can be represented as (Ref.56, page 90)

$$Q(\alpha, \beta) = V^T - V. \quad (3.5b)$$
The black magic of concordance is revealed now in the following (absolutely magical)

**Theorem 3.15.** (Ref.56, page 72) The intersection form (3.5) is unimodular, that is \( \det Q(\alpha, \beta) = \pm 1 \), if and only if \( \partial W \) is the disjoint union of integral homology 3-spheres.

This result admits simple physical interpretation. Indeed, since the 4-manifold is contractible, the linking matrix \( Q(\alpha, \beta) \) should represent the unlinks/unknots. That is going down to 3 dimensions we have to deal only with the links/knots living in \( \Sigma \) whose mutual linking numbers are zero. These are exactly the conditions, given without explanation, under which the Casson invariant was defined in\(^{55,56,63}\). Superficially, such a requirement is looking too simple to be physically interesting. This happens not to be the case. Below we shall provide evidence that such an assumption is too simplistic.

At this point we came very close to defining the Casson invariant. To do so still requires from us to make few steps. This is caused by the fact that all mathematics literature on Casson’s invariant, beginning with its first detailed exposition\(^{45}\), defines this invariant purely axiomatically. Surely, we could do the same here. In such a case the unlinks/unknots condition as part of the axiomatic package would look to the nonexpert as completely mysterious. This circumstance, in part, explains why the Casson invariant escaped physical interpretation thus far.

Going back to our discussion, we need to introduce some facts about the Conway and Alexander polynomials. Although these are obtainable by many authors in many ways, we prefer to work with the description given in\(^{44,51}\) and\(^{64}\). It is based on the presentation of Seifert surfaces for knots/links as discs with attached bands/ribbons\(^{14}\). The mathematical legitimacy of such a presentation of Seifert surfaces is nicely explained in the book by Massey\(^{65}\). In any case, from these sources we find that: a) for connected sum of knots \( K_1 \# K_2 \) the associated with them Alexander polynomial \( \Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t) \cdot \Delta_{K_1}(t) \). The symbol \( = \) means “up to a factor” \( \pm t^n, n \in \mathbb{Z} \); b) the Alexander polynomial \( \Delta_K(t) \) is ”blind” with respect to \( r \) and \( m \) operations defined in section 3.1. Mathematically, this blindness is reflected in uses of the mod 2-type calculations. Physically, this resembles the difference between the magnets (where both the direction and the orientation with respect to some axis matters) and the nematic liquid crystals, where only orientation (the angle with respect to some axis) matters. The classical (Ising-type) spin naturally admits mod 2 (or \( \mathbb{Z}_2 \)) description. In the book by Adams\(^{66}\), on page page 213, it is stated that ”In the case of Ising model, the resulting partition function yields a knot invariant known as the Arf invariant.” The Arf invariant is derivable from both the Alexander and Conway polynomials and in some references (listed below) it is (mistakenly) identified with the Casson invariant. Surprisingly, the relationship between the Arf invariant and the Ising model is not discussed any further in\(^{66}\). Instead, it was established much more recently, in 2012, in\(^{67}\). This description is surely not the only way of obtaining the Arf invariant. Many other ways exist. For example, it can be also obtained with help of the Jones

\(^{14}\)E.g. see Fig.10
Figure 4: Links participating in skein relation

In this work we shall not explore this possibility, though, for reasons which will become obvious upon reading.

Thus we begin with the Conway invariant $\nabla_K(z)$. It is defined with help of the skein relation

$$\nabla_K(z) - \nabla_{\bar{K}}(z) = z \nabla_L(z)$$

applied to three knots/links $K$, $\bar{K}$ and $L$ as depicted in Fig.4.

By definition, $\nabla_U(z) = 1$ (the normalization condition), where, $U$ is the unknot, as before. The Alexander polynomial $\Delta_K(t)$ is obtained from the Conway polynomial by simple replacement $z \rightarrow (t^{1/2} - t^{-1/2})$, e.g. see Ref41, page 110, resulting in

$$\Delta_K(t) - \Delta_{\bar{K}}(t) = (t^{1/2} - t^{-1/2}) \Delta_L(t).$$

The normalization condition is given now, page 64, for any (oriented) knot $K$ by $\Delta_K(1) = \pm 1$. Incidentally, this normalization condition is just a convenience/convention used in knot theory. In physical applications we are free to choose other normalizations as long as they are non negative. Furthermore, formally,

$$\nabla_K(z) = a_0(K) + a_1(K)z + a_2(K)z^2 + \cdots. \quad (3.7)$$

Here $a_n(K) \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$. Each of $a_n(K)$'s is (finite-type Vassiliev) invariant of $K$ of order $n$. At the same time, in mod 2-type calculations, each $a_n(K)$ is the concordance invariant. Following Cohran, we notice that for the $u$-component link the expansion (3.7) should be rewritten as

$$\nabla_K(z) = z^{u-1}(a_0(K) + a_2(K)z^2 + \cdots + a_{2n}(K)z^{2n}). \quad (3.8)$$

Hoste demonstrated that $a_0$ depends only on the pairwise linking numbers of the components of $L$ while Murakami demonstrated in accord with Cohran that, with accuracy up to mod 2 the coefficient $a_2$ is the Arf invariant of $L$. For knots this can be seen by using the analogous result obtained by Kauffman. He demonstrated that for each knot $K$ ($u = 1$) the Arf($K$) ≡ $a_2(K)$ mod 2. For knots, $a_0(K) = 1$.

The Arf($K$) invariant can take only two values 0 and 1 dividing all knots into two classes: those which can be reduced to the unknot $U$ (Arf($U$) = 0) and those which can be
reduced to the trefoil knot (Arf(K)=1). More on this will be said in the next section.

Just like the signature σ(K), the Arf(K) is invariant of concordance since it vanishes on slice knots. This means that it behaves like the signature with respect to the operation of taking the connected sum (3.3). While the signature σ(K) provided us with the mapping (a homomorphism) σ(K) → Z (or C_3 → Z), the Arf(K) ∈ Z_2 = \{0,1\} is providing us with a homomorphism C_3 → Z_2.

The information just described is sufficient formally for introduction of the Casson invariant. It is given by the following

**Theorem 3.16.** (Ref.50, page 88) The Casson invariant coincides with the second coefficient of the Conway polynomial (i.e. with a_2(K)).

**Remark 3.17.** There is no mention of mod 2 in this theorem. In addition, this definition of the Casson invariant formally is not in accord with those given in 45, 55, 63, 73. Last but not the least, taken mod 2 the coefficient a_2(K) is coinciding with the Arf(K) invariant which is a concordance invariant while we mentioned already that the Casson invariant is not the concordance invariant, e.g. read Ref.45 page XV (bottom).

Thus, now we have to discuss other definitions of Casson’s invariant and how they are related to that given in 50. This discussion will be helpful for us in a number of ways.

We begin with the following observations. Let V be 2n × 2n Seifert matrix for some knot/link K in some basis with n = 2g + u − 1, where g is the genus of Seifert’s surface. Using V the Alexander polynomial Δ_K(t) can be defined as 74, page 112,

$$\Delta_K(t) = \det(V^T - tV).$$  (3.9)

Following Kauffman 44, page 200, it is convenient to introduce the potential function of K as

$$\Omega_K(t) = \det(t^{-1}V - tv^T) = \pm t^{2n} \det(V^T - t^2V) = \pm t^{2n} \Delta_K(t^2).$$  (3.10)

When applied to three knots/links K, K and L depicted in Fig.4, the skein relation for Ω_K(t) is given by

$$\Omega_K(t) - \Omega_K(t^{-1}) = (t - t^{-1})\Omega_L(t)$$  (3.11)

Comparison between eq.s (3.6a) and (3.11) yields

$$\Omega_K(t) = \nabla_K(t - t^{-1}).$$  (3.12)

At the same time, we already have an equation (3.6b). Therefore,

$$\Delta_K(t) = \Omega_K(\sqrt{t})$$  (3.13)

By comparing (3.12) and (3.13) we obtain,

$$\Delta_K(t) = \nabla_K(\sqrt{t} - \sqrt{t^{-1}}).$$  (3.14)
In addition, it is important to notice that \( \Delta_K(t) = \Delta_K(t^{-1}) \). Indeed,
\[
\Delta_K(t) = \det(V^T - tV) = (-t)^{2n} \det(-t^{-1}V^T + V) = \det(-t^{-1}V^T + V)^T = \det(V^T - t^{-1}V)
\]
From this property it follows that
\[
\Delta_K(t) = b_{-n}(K)t^{-n} + b_{-(n-1)}(K)t^{-(n-1)} + \cdots + b_{(n-1)}(K)t^{(n-1)} + b_n(K)t^n
\]
implying \( b_{-n}(K) = b_n(K), b_{-(n-1)}(K) = b_{-(n-1)}(K), \ldots, b_{-1} = b_1 \).

Therefore, using this result the canonical form for the Alexander polynomial is known to be
\[
\Delta_K(t) = \tilde{a}_0(K) + \tilde{a}_1(K)(t + t^{-1}) + \cdots + \tilde{a}_n(K)(t^n + t^{-n}). \quad (3.15a)
\]
In view of equations (3.6b) and (3.8), in the case of knots we formally have instead
\[
\Delta_K(t) = \tilde{a}_0(K) + \tilde{a}_2(K)(\sqrt{t} - \sqrt{t^{-1}})^2 + \cdots + \tilde{a}_{2n}(K)(\sqrt{t} - \sqrt{t^{-1}})^{2n} \quad (3.15b)
\]
\[
\equiv b_0(K) + b_1(K)(t + t^{-1}) + \cdots + b_n(K)(t^n + t^{-n}). \quad (3.15c)
\]
Now we are in the position to refine Theorem 3.16 and to introduce the Casson invariant correctly. For this purpose we have to rewrite the expansion for the Alexander polynomial in (3.15b) in the form used for the Conway polynomial, e.g. (3.8) (with \( u = 1 \)). That this is possible can be seen from general results presented in [74], page 113. Since this is possible, we formally obtain:
\[
a_2(K) = \frac{1}{2} \frac{d^2}{dz^2} \Delta_K(z) |_{z=0} \equiv \frac{1}{2} \Delta_K''(1), \quad (3.16)
\]
because \( z = 0 \) is the same as \( t = 1 \). Thus, we need to solve the following problem: given the Conway-like expansion for the Alexander polynomial for the knot \( K \),
\[
\Delta_K(z) = a_0(K) + a_2(K)z^2 + \cdots + a_{2n}(K)z^{2n}, \quad (3.17)
\]
find coefficients \( a_0(K), a_2(K), \ldots, a_{2n}(K) \) in terms of the coefficients \( b_0(K), b_1(K), b_2(K), \ldots, b_n(K) \) in (3.15c). This was almost done in [44], pages 205-206. However, Ref.44 contains some misprints causing us to redo calculations. Clearly, we are only interested in calculating \( b_1(K) \).

In view of equations (3.11)-(3.14) we introduce the following identification: \( z = \sqrt{t} - \sqrt{t^{-1}} \).

With help of such an identification we obtain:
\[
t + t^{-1} = z^2 + 2, \\
t^2 + t^{-2} = z^4 + 4z^2 + 2, \\
t^3 + t^{-3} = z^6 + 6z^4 + 9z^2 + 2, \\
\ldots \\
t^n + t^{-n} = z^{2n} + 2nz^{2n-2} + \cdots + 2.
\]
With help of these results we obtain:

\[ a_2(K) = b_1(K) + 4b_2(K) + 9b_3(K) + \cdots + n^2b_n(K) \equiv \sum_{i=1}^{n} i^2 b_i(K). \]  

(3.18)

We would like to test just obtained results using known results for the trefoil knot. In this case the Conway polynomial is given by \( \nabla K(z) = 1 + z^2 \) while the Alexander polynomial is given by \( \Delta K(t) = 1 + (\sqrt{t} - \sqrt{t^{-1}})^2 = t - 1 + t^{-1} \). Accordingly, \( a_2(K) = 1 \), in view of the definition (3.16). Thus, we just reobtained well known standard results for the trefoil knot.

Now we are ready to define the Casson invariant of homology spheres. Let \( M \) be an oriented homology 3-sphere (e.g. complement of the standard slice knot in \( S^3 \)). It is described by the integer-valued invariant \( \lambda(M) \) — the Casson invariant. Notice that for slice knots/links \( \lambda(M) \neq 0 \). Therefore it is not a concordance invariant (in accord with page XV of Ref.45.). It possesses the following properties:

1) For \( S^3 \), that is when \( \pi_1(M) = 1 \), \( \lambda(S^3) = 0 \). Incidentally, \( \pi_1(M(K_1, \alpha)) = 1 \) only when \( \alpha = 0 \) (e.g. read Remark 3.9 and Ref.59, page 154).

2) \( \lambda(-M) = -\lambda(M) \).

3) If \( K_2 \) is knot in \( M \) (e.g. read Theorem 3.13 (b)) and \( K_1 \) is slice knot, then \( M(K_1, K_2; \alpha, \beta, \gamma, \delta) \) is homology sphere if \( |\gamma| = 1 \). Since \( \det A = 1 \) we obtain \( \alpha \delta - \beta \gamma = \alpha \delta - \pm \beta = -1 \) that is \( \beta = \pm (\alpha \delta + 1) \). Therefore, we can choose \( \alpha \) and \( \delta \) as independent variables along with the sign \( \epsilon = \pm 1 \). This allows us to relabel \( M(K_1, K_2; \alpha, \beta, \gamma, \delta) = M(K_1, K_2; -1, 0, \pm 1, 1) \) and introduce the notation

\[ M(K_1, K_2; -1, 0, \pm 1, 1) \simeq ((K_1 \# K_2)_{\pm 1}; M_1 \# M_2) \equiv (\bar{K}_{\pm 1}; \bar{M}). \]  

(3.19)

Taking into account the Remark 3.9, consider a knot \( \bar{K} \) in a homology sphere \( \bar{M} \) so that the symbol \( (\bar{K}, \bar{M}) \) will denote an oriented homology sphere obtained by performing \( \pm/n \) Dehn surgery on \( M \) along \( \bar{K}, n \in \mathbb{Z} \). Then, the difference (the derivative) is defined as

\[ \lambda'(\bar{K}, \bar{M}) = \lambda(\bar{K}_{n+1}; \bar{M}) - \lambda(\bar{K}_n; \bar{M}). \]  

(3.20)

Because thus defined discrete ”derivative” is \( n \)-independent (this crucial property will be used essentially below), by applying the induction we obtain:

4) \( \lambda(\bar{K}_n; \bar{M}) = \lambda(\bar{M}) + n\lambda'(\bar{K}, \bar{M}). \)  

(3.21)

implying, \( \lambda(\bar{K}_0; \bar{M}) = \lambda(\bar{M}). \)

If in addition we require that

5) \( \lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2), \)  

(3.22)
then in view of (3.19) we obtain:

$$\lambda(((K_1 \# K_2)\# M_1 \# M_2)) = \lambda((K_1)_n; M_1) + \lambda((K_2)_n; M_2).$$  (3.23)

By using equations (3.19)-(3.22) we obtain

$$\lambda(M'(K_1,K_2; -1,1)) = \lambda(M_1) + \epsilon \lambda'(K_1; M_1) + \lambda(M_2) + \epsilon \lambda'(K_2; M_2).$$

To connect the obtained results with the content of Theorem 3.16 the following observations are helpful. Following Ref.44 we notice that for coefficients of the Conway polynomial defined by equation (3.7) the following recursion formula takes place

$$a_{n+1}(K) = a_{n+1}(\bar{K}) = a_n(L)$$  (3.24)

where, as before, $K$, $\bar{K}$ and $L$ are the same as depicted in Fig.4. Therefore

$$a_0(K) = \begin{cases} 1 & \text{if } K \text{ has one component} \\ 0 & \text{if } K \text{ has more than 1 component} \end{cases}$$  (3.25)

and

$$a_1(K) = \begin{cases} lk(K) & \text{if } K \text{ has two components} \\ 0 & \text{otherwise} \end{cases}$$  (3.26)

In the simplest case we obtain,

$$a_2(K) = a_2(\bar{K}) = lk(L).$$  (3.27)

Equivalently, just obtained results can be rewritten in terms of the second derivatives of Alexander polynomial with help of equation (3.16). Since $a_2(K) \mod 2$ is Arf (K) invariant and we already noticed that Arf (K) is the concordance invariant, this implies that in $S^3$ we should have

$$a_2(K_1 \# K_2) = a_2(K_1) + a_2(K_2).$$  (3.28)

Now it remains to compare (3.22) with this result. This causes us to check/to prove whether or not

$$\lambda'(K_1 \# K_2; M_1 \# M_2) = \lambda'(K_1; M_1) + \lambda'(K_2; M_2).$$  (3.29)

If this, indeed, can be proven, then using equations (3.16),(3.28) and (3.29) it is possible to make an identification.

$$\lambda'(K; M) = \frac{1}{2} \Delta_{K,M}(1).$$  (3.30)

This result was obtained for the first time by Casson\textsuperscript{45,56}. Equivalently, using definition of the discrete derivative (3.20) and taking into account that such a derivative is $n$-independent, equation (3.30) can be equivalently rewritten in the form

$$\lambda(K_n; M) = \lambda(M) + \frac{n}{2} \Delta_{K,M}(1).$$  (3.31)
This relation (not an equation!) admits physical interpretation to be discussed in the next subsection.

3.3. Regge mass spectrum from the Casson invariant (a hint)

In our previous work\textsuperscript{15} we took advantage of the fact that the path integral for pure Y-M gauge fields, when treated nonperturbatively, is reducible to the topological C-S field theory. By applying the Abelian reduction it is converted into the hydrodynamics/ideal magnetohydrodynamics-type model functional whose detailed study is presented in the companion work\textsuperscript{22}. Connections of the C-S model with the Kontsevich invariants/integrals and the Vassiliev-type invariants, including $a_2(K)$, is discussed in detail in\textsuperscript{50,75}. For reasons explained in the previous subsection, we cannot use these results. Furthermore, we also cannot use these results because treatments of the Casson invariant presented in\textsuperscript{50,75} are intrinsically perturbative. They are distinctly different from the instanton (Floer)-type nonperturbative treatments presented in\textsuperscript{16,17}. We would like to remind to our readers that the basics of instanton (Floer) approach, as explained in our work,\textsuperscript{15} is done in full compliance with much more comprehensive treatments\textsuperscript{13,14}. In view of the (Mirror) Theorem 3.3. uses of istantons in the present case are as essential as their uses in analogous situations, e.g. in chemical reactions\textsuperscript{76}. In this work we only present needed arguments for such instanton-type treatment. It will be developed in the future publications. In this work the results are presented at the rigorous mathematical level inspired by known phenomenological results.

We begin the description of these results with a gentle reminder of the results of nonrelativistic scattering theory. In it, the scattering amplitude is expected to possess the (Regge-type) poles in the complex angular momentum $J$ plane. The pole equation $J = \alpha(E)$ relates $J$ to the energy $E$. The function $\alpha(E)$ is called the Regge trajectory. In the relativistic case, the energy parameter $E$ is replaced by the appropriate Mandelstam variable, say, $s$. Ref.\textsuperscript{77} is describing the basic steps for building the dual resonance models using the Regge phenomenology as an input. The first push towards development of dual resonance models was given by Veneziano\textsuperscript{78} in 1968. In that year he postulated that the meson-meson scattering amplitudes can be described in terms of combination of Euler’s Beta functions symmetric with respect to permutation of its arguments. These are the Mandelstam variables $s, t, u$. Search for a model reproducing these amplitudes resulted in all kinds of string models as is well known. The output of these relativistic models is scattering amplitudes possessing the Regge-type poles. In connection with this result several questions emerge. The first among them is this: How string models are related to quantum chromodynamics (QCD)? That is to say, can string models be derived from the QCD or, vice-versa, can QCD be derived from the string models? Since this is not a review paper, we mention only the very latest works relevant to this question. In particular, we begin with\textsuperscript{79,80}. In Ref.\textsuperscript{79} the authors consider results originating from the diquark model. Quoting them, ”At a crude level, the idea is that pairs of quarks form bound states which can be treated as (confined)
particles...It is plausible that baryons with large values of the angular momentum $J$ form extended bar-like structures, with quarks pushed to the extremities by centrifugal forces. The bar-like structures are made of electric flux tubes, or strings. The results of the bar-like structures were carefully analyzed in Ref. 80. It happens that the comparison between theory and experiment (including computer simulations) can be made by using the Chew-Frautschi (C-F) formula

$$M^2 = a + bJ.$$  (3.32)

Here $a$ and $b$ are some constants, $M$ is the hadron (baryon or meson) mass and the angular momentum $J = n$, $n = 0, 1, 2, ...$ The above formula is basically the same thing as the Regge trajectory: $\alpha(t) = \alpha(0) + \alpha' t$, $\alpha' > 0$. Results of Ref. 80 indicate that the C-F formula compares exceptionally well with experimental data for both baryons and mesons. From the theoretical side, the interest lies in obtaining the values of parameters $a$ and $b$. As demonstrated in Ref.80, in the limit of zero quark masses, the diquark model compares very well with the C-F formula and, therefore, with the experiment. The results obtained in Ref.79 were reproduced and further improved in Refs. 81, 82 with the purpose of obtaining the values of $a$ and $b$ theoretically. Contrary to the initial expectations, experimental data convincingly suggest that the slope parameter $b$ is the same for both baryons and mesons, even if the quark masses are not vanishingly small! The slope was calculated in Refs. 81, 82 and was fitted to the C-F formula with reasonable degree of success. The value of intercept $a$ was calculated in Ref. 83, where entirely different model (the conventional bosonic string model) was used and, accordingly, the entirely different calculations (more traditional, string-theoretic) were done, in dimensions 4 and higher. The obtained results are much less satisfactory though. Notice, the obtained results do not provide an answer to the question posed above. Moreover, the results of sections 1 and 2 are also not helpful. The second question is the following: To what extent the C-F formula confirms or denies the existence of instantons in QCD?

To our knowledge, at the moment it is possible, in principle, to give answers both ways. For instance, if one believes that strings models can incorporate gravity (not renormalizable by the conventional methods), then the QCD should be derivable from the string models. However, the abelianization procedures discussed in section 2 indicate that all gauge fields: electromagnetic, Yang-Mills and gravity, admit the same type of treatment, e.g. outlined in our works 15, 22, 84. And, if this is the case, then the QCD is reduced to the C-S topological field theory and its Abelian version is apparently sufficient. That is to say, in such a case the problem of recovering of string model from QCD was to a large extent solved already by Nambu 85. In our paper 86 some practical applications (other than in high energy physics) of this line of thought were discussed in detail. It is being hoped, that physical insight coming from fields other than high energy physics could be helpful for resolving problems of high energy physics.

Notice also that dynamically generated nonhyperbolic knots and links are observables in the C-S field theory. In fact, without asking the question about how these knots/links were created, for the C-S model all knots and links are observables 87. The fermionic effects
can be modelled in such an environment purely topologically. Since the black holes can be described in terms of elementary particles, it appears that no fields other than gauge fields (Abelian or not) are needed. The effects of charges can be included consistently into such formalism.

Now we are in the position to compare the C-F formula (3.32) with the Casson surgery formula (3.31). Clearly, they formally coincide. The question remains: What is the physical content of the Casson Surgery formula? The answer is provided in the rest of this paper.

4. Physics behind the Casson surgery formula

4.1. Physical content of the Dehn surgery.

From Moffatt intuition to Hempel, Rolfsen and Kirby proofs

The Casson surgery formula (3.31) involves the notion of Dehn surgery, e.g. the symbol \((\bar{K}_n; \bar{M})\) defined in section 3 denotes an oriented homology sphere obtained by performing \(\pm/n\) Dehn surgery on \(\bar{M}\) along \(\bar{K}\), \(n \in \mathbb{Z}\). Based on this information, the question emerges: Since the C-F and Casson surgery formula look identical, could this be just a coincidence or, could it be that the Casson surgery formula indeed carries some hidden physics in it? We would like to demonstrate that, indeed, this is the case. For this we have to demonstrate that the elementary Dehn surgery can be looked upon (equivalent to) as an operation of crossing change depicted in Fig.5.

From this standpoint the basic skein relation for the Vassiliev invariants, Fig.2, can be physically interpreted as follows. Imagine a knot \(K\) made of rope, a flux tube, say. Since in this work we relate knots to particles, different knots/links are obtained from each other by cutting the rope and regluing it back. Each time such an operation is performed, it is associated with the fixed once and for all amount of energy. Thus the basic skein relation for the Vassiliev invariants is graphical illustration of this process. In the case of Casson

\[\text{Figure 5:}\]

\[\text{Figure 5:}\]
surgery formula, the analog of such skein relation is given by the combined use of equations (3.20) and (3.30), that is by

$$\lambda(K_{n+1}; M) - \lambda(K_n; M) = \frac{1}{2} \Delta''_{K,M}(1),$$

(4.1)

provided that we can prove that the elementary Dehn surgery can be replaced by the elementary crossing change. We shall do just this now.

We begin with some excerpts from the paper by Livingston\textsuperscript{91}. Recall that in the Conway polynomial expansion (3.7) each of $a_n(K)$'s is finite-type Vassiliev invariant of $K$ of order $n$. Take into account equations (3.24), (3.25) and (3.26). Then, the skein relation depicted in Fig.2 acquires the following look:

$$\text{lk}(K) - \text{lk}(\bar{K}) = 1.$$  \hspace{1cm} (4.2)

This result is illustrated in Fig.6 on example of the Hopf link.

Notice that in this case $\bar{K}$ is unlinked and, therefore, $\text{lk}(\bar{K}) = 0$. Now, take into account equations (3.17) and (3.27) and apply them to the case of the trefoil knot $K$ as depicted in Fig.7

In such a case we obtain:

$$a_2(K) = \text{lk}(L)$$  \hspace{1cm} (4.3)

since $a_2(\bar{K}) = 0$ (because $\bar{K}$ is unknot) and $L$ is the Hopf link. Using this result in (4.1) while taking into account (3.16), (3.27) and (3.30) we just obtained a very plausible result strongly hinting at the connection between the operation of elementary crossing change and of the elementary Dehn surgery. These suggestive observations we would like now to convert into a solid proof. In view of its fundamental importance for this work, we would like to put the content of this proof into physical frame following ideas of Moffatt\textsuperscript{92}.

In this brief note he said the following: "Any knot or link may be characterized by an ‘energy spectrum’ - a set of positive real numbers determined solely by its topology. The lowest energy provides a possible measure of knot or link complexity.” The energy spectrum is obtained by the operation which Moffatt describes as follows. 1. Surround knot by a
tubular neighborhood (e.g. read the Appendix), that is by the solid torus. The "magnetic" (Abelian or non Abelian) flux circulating inside this torus is carrying some energy (e.g. read\textsuperscript{15}). 2. The flux tube is cut at any section \( \varphi = \text{const} \). 3. Thus formed cylinder is twisted through the angle \( 2\pi h_0 \). 4. After this, the tube is reconnected. In Figure 2 of\textsuperscript{92} Moffatt provides a physical sketch of how this process can actually take place in real world. He observes that "The unknotted tube \( T_0 \) is converted into the knotted tube \( T_K \) by switching a number of crossings." This knotting/unknotting is facilitated by the unknots which appear from nowhere and disappear into nowhere in Moffatt’s paper. He writes: ” Each switch is equivalent to the insertion of a small loop that cancels the field on one side\textsuperscript{16} and makes it to reappear on the other. The change in helicity \( h \) associated with the switches is ..\( h = h_0 - 4 \." From Arnol’d inequality, equation (3.15) of\textsuperscript{15}, it follows that the flux energy is bounded from below by the helicity with equality achieved for the force-free fields\textsuperscript{15,22}. This means that sequential crossing changes produce the energy spectrum \( h = h_0 \pm 4n, \ n = 0, 1, 2, ... \) Clearly, this result is the C-F formula (3.32).

Being armed with these physically motivating arguments, we are ready to bring them into correspondence with rigorous mathematics. The major issue requiring clarification is associated with the insertion /deletion of small loops ”that cancel the field on one side and make it to reappear on the other” . Our readers are encouraged to read the Appendix at this point. With this assumption, the operation of a single crossing change is depicted in Fig.8.

It requires uses of both types of Kirby moves as depicted. The same result was obtained\textsuperscript{16}Of the flux tube (our comment)
The results of previous subsection should be used with some caution. This is so because of several reasons. Our readers should not confuse the physical picture suggested by Moffatt with the actual picture in the context of Casson surgery formula. If one follows Moffatt’s ideas, then one should start with some knot $K$ and apply to it successfully the crossing change operations. If one begins with the unknot $U$, then the tower of excited states is obtained by producing knots of ever increasing complexity. If, instead, one begins with some nontrivial knot then this knot relaxes to the unknot via crossing changes mechanism. In the Casson case physics is different. From the description already presented it follows that
instead of crossing changes one should use the sequence of Dehn twists. To our knowledge, there is no direct physical analog of the Dehn twists, even if one is using some viscous elastic medium. However, the results depicted in Fig.s 8,9 indicate that it is perfectly legitimate to identify the elementary Dehn twist with the crossing change operation. This type of homeomorphism admits physical interpretation analogous to that developed by Moffatt. Analogous but not identical! And, therefore, this new interpretation is much more suitable for the high energy physics applications. If one follows Moffatt’s ideas, then every knot/link relaxes to its ground state -the unknot. In the present case, the Dehn surgery operation on the unknot in $S^3$ results in unknot. By performing the elementary Dehn surgery on the simplest nontrivial knot-trefoil, one is obtaining the dodecahedral Poincare′ space-a homology sphere. This means that relaxation process-from some homology sphere-back to the space in which the trefoil lives will end up in a stable particle associated with the trefoil. This makes perfect sense physically. Indeed, both, in our work$^{22}$ and in$^2$ the trefoil emerges as the stable ground state particle. In addition, recent attempt to build the standard model based on different labelings of the trefoil knot was developed in$^{96}$ (and references therein). Between$^2,22$ and$^{96}$, only$^{22}$ uses rigorous mathematical results consistent with results by Floer and Taubes$^{16}$. Only this development is allowing us to bring ultimately into play the concept of concordance. The result for $\Delta_{K,M}(t)$ entering (3.31) was obtained with help of the Alexander polynomial calculated for a knot K in 3-manifold M. How such a polynomial can be calculated in M if all textbooks on knot theory provide us only with calculations of this polynomial for knots in $S^3$? Suppose we calculate $\Delta_{K,S^3}(t)$. Can this information be used for calculation of $\Delta_{K,M}(t)$? Could it be that $\Delta_{K,M}(t) = \Delta_{K,S^3}(t)$?

To prove or disprove this equality we have to recall the protocol for designing of the Alexander polynomial. It begins as follows. 1. We associate with a given knot/link the Seifert surface. 2. This surface always can be presented as disc with ribbons, e.g. see Fig.10. 3. By known rules, using such a disc with ribbons in a standard way described in any knot theory textbook it is possible to obtain the Seifert matrix $V$ (e.g. see section 3) and, subsequently, with its help-the Alexander polynomial, e.g. see (3.9), or its equivalent, the potential function (3.10). 4. The ribbons attached to the disc could be knotted and linked with each other, and each of them could be twisted an even number of times (since the Seifert surface is orientable). Topologically, the effects of Kirby moves (e.g. read Appendix) exhibit themselves via sliding one of the attachment points of a given ribbon over another ribbon and reconnecting it back to the disc without changing the isotopy of its boundary. The resulting surface is again a disc with ribbons. This sliding move is understood the best in the context of the theory of 4-manifolds$^{43}$ or, alternatively, Ref.97, Chapter 5. For homology sphere 3-manifolds obtained via $\pm 1$ surgery on framed links Habiro$^{98}$ elegantly refined Kirby calculus using theory of 4-manifolds. To such modified disc with ribbons it is permissible to add two additional bands. One of them is untwisted and unknotted while the other could be twisted and/or knotted and can link another ribbons as schematically depicted in Fig.10. Topologically, however, for the links such an addition does not affect
the boundary of the Seifert surface (this boundary is made out of the link in question) so that the Seifert surface with two extra ribbons still represents the same link. All this is beautifully explained in the Kauffman’s book\textsuperscript{44}, pages 196,197. The operation of adding or subtracting these two bands is called stabilization. Two Seifert surfaces are stably equivalent if there is a sequence of stabilizations applied to each of them so that they can be deformed into each other. It is known\textsuperscript{56,64} that any compact orientable 3-manifold M can be obtained via surgery on a link \( \mathcal{L} \) in \( S^3 \). The crossing change operation depicted in Fig.9 which is equivalent to elementary Dehn surgery was used by Hempel\textsuperscript{93} in his proof of the following

**Theorem 4.1.** (Hempel) If M is a compact connected orientable 3-manifold, then there is a collection \( \{ T_1, \ldots, T_k \} \) of mutually exclusive solid tori in M and a collection \( \{ \tilde{T}_1, \ldots, \tilde{T}_k \} \) of mutually exclusive solid tori in \( S^3 \) such that the closure of \( (M \setminus \{ T_1, \ldots, T_k \}) \) is homeomorphic to the closure of \( (S^3 \setminus \{ \tilde{T}_1, \ldots, \tilde{T}_k \}) \). Furthermore, the tori \( \{ \tilde{T}_1, \ldots, \tilde{T}_k \} \) may be chosen in such a way that they are unknotted in \( S^3 \).

**Remark 4.2.** Chronologically this theorem is precursor of Kirby calculus. Generic examples of topologically different links related to each other homeomorphically can be found on page 49 of\textsuperscript{54} or in Fig. 12 of\textsuperscript{22}. Evidently, that the unknottedness of the collection of tori \( \{ \tilde{T}_1, \ldots, \tilde{T}_k \} \) is not essential.

**Remark 4.3.** Since every 3-manifold is obtainable by surgery on some link \( \mathcal{L} \) it is possible (using Theorem 4.1.) to go backward in this process and to reobtain \( S^3 \) with some link \( \mathcal{L}' \) related to \( \mathcal{L} \) via some homeomorphism. Because the elementary surgery can be modelled by the elementary crossing change (Fig.s 8,9) this means that the link \( \mathcal{L}' \) should be simpler than \( \mathcal{L} \). Accordingly, the Seifert surface for such a link should be simpler. It should look either as a) Seifert surface for a collection of unknots, b) Seifert surface for

![Figure 10: A typical Seifert surface for some knot or link](image-url)
collection of trefoils or, c) Sifert surface for collection of unknots and trefoils. E.g. see Fig.8.22 of Ref.66. More on this will be said below, in the next subsection.

Supplied information brings us much closer to proving/disproving that $\Delta_{K,M}(t) = \Delta_{K,S^3}(t)$. To answer this question we need to recall and to use Theorems 3.13.(b) and 3.13 (c) and 13.15. These theorems contain the essence of black magic of concordance in the most concentrated form. They are providing information about how the homology sphere 3- manifold $M$ should look like. It is made of nontrivial knots and slice knots. Recall that the slice knots are representing the vacuum state while the nontrivial knots are representing particles. The nontrivial knots eventually relax to stable particles represented by trefoils. Thus, trefoils are living in $S^3$. The Dehn surgeries can be made on trefoils providing us with the hadron excitation spectrum. Mathematically, such Dehn surgeries convert $S^3$ into the homology 3-sphere. Remarkably, these are exactly the main features defining the Casson invariant defined in subsection 3.2! According to Theorem 3.15 we notice that for homology spheres slice knots are not linked with nontrivial knots. In mathematics such a situation is characterized in terms of either unlinks or boundary links.

**Definition 4.4.** Let $k$ and $l$ be some knots forming a link $k \cup l$, then the unlink is characterized by the condition $lk(k,l) = 0$.

**Definition 4.5.** A link $L = L_1 \cup \cdots \cup L_n$ is a boundary link if there exists an orientable Seifert surface $S$ made of $n$ disjoint components $S = S_1 \cup \cdots \cup S_n$ such that for each $i < n$ we have $\partial S_i = L_i$.

Now we are in the position to formulate yet another

**Theorem 4.6.** (Saveliev$^{56}$, page 94) Let $k \cup l$ be a boundary link in a homology sphere $\Sigma$, and let $\Sigma' = \Sigma + \varepsilon k$, a surgery of $\Sigma$ along $k$ with $\varepsilon = \pm 1$. Then $\Delta_{l \subset \Sigma}(t) = \Delta_{l \subset \Sigma'}(t)$ where $l \subset \Sigma'$ is the image of $l \subset \Sigma$ under the surgery.

**Remark 4.7.** The content of this theorem is just a restatement of Hempel’s theorem in a specific setting. Saveliev’s theorem is illustrated in Fig.s 8,9 superimposed with Fig.12 of Ref.22. Indeed, in Fig.s 8,9 it is sufficient to identify $k$ with the thickened unknot and $l$ with, say, a trefoil.

Using Theorem 4.6, it is possible to prove the following

**Theorem 4.8.** (Saveliev$^{56}$, page 95) Let $k$ be a knot in a homology sphere $\Sigma$. Then there exists a knot $l$ in $S^3$ such that $\Delta_{k \subset \Sigma}(t) = \Delta_{l \subset S^3}(t)$.

**Remark 4.9.** Just stated theorem provides negative answer to the question: ”Could it be that $\Delta_{K,M}(t) = \Delta_{K,S^3}(t)$?” This fact apparently creates some difficulty in calculating of $\Delta_{K,M}(t)$. The situation can be dramatically improved based on the following observations.

Let us try to apply the definition of derivative in (3.20) to the boundary link made out of two components $K$ and $L$. Following$^{56}$ this can be achieved by replacing (3.20) by
the second derivative defined as follows:

\[
\begin{align*}
\lambda(K_{m+1}, L_{n+1}; M) - \lambda(K_m, L_{n+1}; M) - \lambda(K_{m+1}, L_n; M) + \lambda(K_m, L_n; M) \\
&= \lambda'(K, L_{n+1}; M) - \lambda'(K, L_n; M) \\
&= \lambda'(K_{m+1}, L; M) - \lambda'(K_m, L; M) \equiv \lambda''(K, L; M).
\end{align*}
\]

If, following Casson, we require that \( \lambda''(K, L; M) = 0 \), this would imply that \( \lambda'(K_{m+1}, L; M) = \lambda'(K_m, L; M) \) and \( \lambda'(K, L_n; M) = \lambda'(K, L_{n+1}; M) \), which can be formulated as

**Corollary 4.10.** Although the Theorem 3.13 (b) does not mention about the mutual alignment of slices and nontrivial knots, just obtained result requires slices and nontrivial knots to be arranged as boundary links. Only under such circumstances equation (3.31) is valid.

Because of equivalence between elementary Dehn surgeries and crossing changes (Fig.s 8, 9), it is essential now to investigate how the key equation (3.30) should be modified to account for this equivalence. We begin with (3.30) in which \( M = S^3 \). Next, we look at the derivative (3.20) which, in view of Fig.s 8, 9, can be presented as follows

\[
\lambda'(K, S^3) = \lambda(K; S^3 + \mathcal{O}) - \lambda(K; S^3).
\]

where \( \mathcal{O} \) is the unknot concordant to slice knot. Next, the property of \( n \)-independence of the derivative can be restated in view of equation (4.4) as follows

\[
\lambda'(K, S^3) = \lambda(K; S^3 + \mathcal{O}_1 + \mathcal{O}_2) - \lambda(K; S^3 + \mathcal{O}_1) = \lambda(K; S^3 + \mathcal{O}_1) - \lambda(K; S^3)
\]

Evidently, this result implies

\[
\lambda(K; S^3 + \mathcal{O}_1 + \cdots + \mathcal{O}_n) = \lambda(K; S^3) + \frac{n}{2} \Delta_{K,S^3}(1).
\]

By design, equation (4.8) is equivalent to the equation (3.31). Just obtained results require additional streamlining. This is subject matter of the next subsection.

### 4.3. Fine structure of the physical vacuum. Magical role of concordance

Up to now our readers were left under impression that only slice knots/links represent the physical vacuum. However, it is known that every ribbon knot/link is slice knot/link but the converse remains only as conjecture. In fact, there are many examples of slice knots/links which are are ribbons. An example of ribbon and slice knots is depicted in Fig.11

To make a ribbon is easy. Indeed, for any knot \( K \) and its mirror image \( mK \) the connected sum \( K \# mK \) is a ribbon. Recall from section 3 that Arf (K) is the concordance invariant. This means that if K is slice, then Arf (K) = 0. In view of equations (3.16),
(3.31b) and (3.32) and also taking into account the content of Theorem 3.13 (b) the result \( \text{Arf}(K) = 0 \) when \( K \) is slice knot/link makes perfect physical sense. The C-F formula (3.32) describes the hadron mass spectrum of mesons/baryons starting from the meson/baryon of lowest but nonzero mass. Since slice/ribbon knots/links had been identified with the physical vacuum, we do not need to worry about the mass spectrum coming from such a vacuum. According to Kauffman\textsuperscript{72} \( \text{Arf}(K) \) can acquire only two values: 0 for the vacuum, represented, say, by the slice/ribbon knots, and 1, for all other knots. These other knots can be reduced to the trefoil knot via band pass operation. At the level of Seifert surface, e.g. see Fig.10, these pass band operations preserving the pass class of the boundary of this surface (because it is oriented) are depicted in Fig.12.

Thus, the \( \text{Arf}(K) \) is organizing the set of all knots/links \( K \) into two classes. It generates the equivalence relation on \( K \)-the totality of all knots. In physics such a subdivision reflects the fact that massless particles can never acquire nonzero rest mass. In the reminder of this subsection we shall discuss processes leading to such a subdivision.

We begin with the following question. Is the operation of crossing changes the only mechanism by which knots/links can be unknotted/unlinked? Said alternatively (if we use the surgery using Dehn twists): Is this the only operation connecting a given state (that is 3-manifold, perhaps, associated with framed nontrivial knot) to the vacuum state (that is 3-manifold, perhaps, associated with nontrivial knot without framing)?

Following seminal papers by Levine\textsuperscript{101} and Milnor\textsuperscript{102} it is convenient to introduce the concept of link homotopy.

**Definition 4.11.** Link homotopy is an equivalence relation generated by the ambient isotopy and the crossing change move, when two branches of the crossing belong to the same link component.

Immediately, the question emerges: What equivalence is stronger: surgical or homotopy? To measure the degree of homotopy, Milnor suggested his (now known as Milnor)
invariants of link homotopy. These can be defined as follows. Let the set of subscripts \( \{i_1, \ldots, i_k\} \) be denoted as \( i \). Here all indices are distinct and the label \( k \) refers to the \( k \)-th component of the link. Then the commonly accepted notation for the Milnor \( \bar{\mu} \)-invariant is \( \bar{\mu}_i \). In particular, for the two-component links \( \bar{\mu}_{12} \) is a complete link homotopy invariant. It is just the familiar linking number. For the three-component links the collection made of \( \bar{\mu}_{12}, \bar{\mu}_{23}, \bar{\mu}_{13} \) and \( \bar{\mu}_{123} \bmod \{ \bar{\mu}_{12}, \bar{\mu}_{23}, \bar{\mu}_{13} \} \) is a complete set of link homotopy invariants. Evidently, Milnor’s -invariants generalize the concept of the linking number. Their power can be seen when one is looking, for example, at the Borromeo link Fig.13 b) or Whitehead link, Fig. 13a).

In both cases the linking numbers are zero while Milnor’s numbers are not. Levine demonstrated that only for two and tree component links the homotopy equivalence is identical to the surgical equivalence. This is very deep and nontrivial result. It generalizes Hempel’s result depicted in Fig. 9. Subsequently, it was demonstrated\(^{103,104} \) that the link concordance implies link homotopy. From here we obtain the triangle of equivalence relations for two and three-component links in which the concordance is playing the dominant role. Levine’s results were reobtained independently by Matveev\(^{105} \) who studied surgical versus homotopy equivalence for knots/links in the homology spheres. Matveev’s results are of physical importance as we would like to explain now. To do so, we notice that some (but not all!) results of\(^{105} \) are further refined in\(^{106} \). Christine Lescop published
paper\cite{107} containing considerable amount of details missing in both\cite{105,106}. Thus, using both Matveev’s and Lescop’s results we inject some physics into them. The main result of Lescop can be formulated as

**Theorem 4.12.** Any integral homology sphere with Casson invariant zero can be obtained from $S^3$ by the sequence of $\pm 1$ Dehn surgeries on the boundary link $L$ each component of which $L_i$, $i = 1, \ldots, n$, has a trivial Alexander polynomial. That is $\Delta_{L_i}(t) = 1$.

**Remark 4.13.** In\cite{107} Lescop considers $\frac{1}{2} \Delta''_{K,M}(1)$ as the Casson invariant of $K$.

This result by Lescop can be restated in physical language. For this we need to introduce such technical concepts as "smooth concordance" group $H_{\text{diff}}$, "topological concordance" group $H_{\text{top}}$ and "algebraic concordance" group $H_{\text{alg}}$. The $H_{\text{alg}}$ is defined in\cite{40,42}, $H_{\text{diff}}$ and $H_{\text{top}}$ are defined in\cite{108}. Without going into details, it can be demonstrated that $H_{\text{diff}} \to H_{\text{top}} \to H_{\text{alg}}$. At the level of $H_{\text{top}}$ Michael Freedman proved\cite{109}, Chr.11, paragraph 7B, the following

**Theorem 4.14.** (Friedman) Any knot with trivial Alexander polynomial is slice in topological category.

**Remark 4.15.** All these distinctions originate from a delicate difference in embeddings of a cone $\hat{k}$ (a disc $D^2$), Fig.1., into 4-dimensional ball $B^4$. These are issues associated with the topology of 4-manifolds. If we suppress these distinctions, leaving them to mathematicians, then Theorems 4.12. and 4.14. can be reformulated in the physical language as follows:

*Scattering processes involving stable massive particles are not affected by the vacuum fluctuations.* Alternatively: *Massless particles cannot be described in terms of the C-F plot*

\[17\] The most interesting physically
Thus, the sequence of different homology spheres can be created by either the Dehn surgery on boundary links or the Dehn surgery on a knot $K$ in $S^3$. Matveev demonstrated that $\pm 1$ Dehn surgeries on Borromeo rings in $S^3$ can be also used to create the homology spheres. It is of interest to provide some specifics (without proofs). Instead of Dehn surgeries Matveev considers a combination of crossing changes and Kirby moves (e.g. see Fig.s 8,9) on the boundary links. He describes a bit different type of surgery (as compared with the protocol described in Appendix). It is depicted in Fig.14.

The initial state of the handlebody of genus 2 (with the individual sublink) differs from the final state by cutting it along the separating curve (a disc), twisting it by 360$^\circ$; and regluing it back. Instead of the Dehn surgery protocol, Matveev’s surgery protocol is fully complacent with Milnor’s definition of homotopy transformations and is described as follows. 1. Take a handlebody of genus 2 (along with the sublink/link lying inside) out of $S^3$. 2. Make just described 360$^\circ$ twist (not to be confused with the Dehn twist!). 3. Reglue. 4. Insert such modified handlebody back into $S^3$. The net result produces the effect which is equivalent to that depicted in Fig.s 8,9. Therefore, Matveev’s surgery is equivalent to the more traditional Dehn surgery! What we are gaining if we use Matveev prescription for the surgery instead of the traditional one? It happens that such described modification of surgery is very helpful since it allows an extension to the surgery on Borromeo rings. Matveev distinguishes between the strict Borromeo surgery, when it is performed distinctly on three rings and non strict surgery, when it is performed on lesser than three rings. Each non strict Borromeo surgery can be made out of superposition of strict Borromeo surgery and Kirby moves, Matveev claims. Furthermore, he claims that transformations depicted in Fig.14 can be achieved with help of Borromeo surgeries too! Skipping details which can be found in Matveev’s paper, the net result of strict Borromeo surgery is depicted in Fig.15.

Since the crossing change is equivalent to the Dehn surgery, as Matveev argues, the Borromeo surgeries, depicted in Fig.15, can be used for creation of homology spheres.
However, from\textsuperscript{54} we know that that the Borromeo rings are \textbf{not} boundary links! Thus, it seems like we cannot apply Theorems 4.12 and 4.14 to Borromeo rings. Furthermore, we cannot apply Theorem 3.13(b) as well. Very recently Krushkal\textsuperscript{110} demonstrated (following ideas of Michael Friedman) that, nevertheless, in a certain (4-dimensional) sense the Borromeo rings are slice and, therefore, Theorem 3.13(b) can be actually applied! Because of this recent discovery, we arrive at the same physical conclusions as stated immediately after Remark 4.15.

\textbf{Remark 4.16.} While the description of Fig.14 provides explicitly details of crossing change operation, Fig.15 provides only the initial and the final stages of the so called $\Delta-$move operation\textsuperscript{18}. Details are missing in Matveev’s paper. These details were provided for the first time in the paper by Goussarov\textsuperscript{111}. In literature one can encounter statements that in Habiro’s paper\textsuperscript{112} the same results were obtained. This is not true, however, since the paper by Habiro does not contain the description of $\Delta-$move at the level of handlebody of genus 3. Habiro’s paper is more combinatorial and algebraic while Goussarov’s is more topological.

Combination of Kirby and $\Delta-$moves are discussed in detail in\textsuperscript{98,113}. All these papers lie at the foundation of theory of finite type invariants\textsuperscript{50,75}. Since this theory goes far beyond the homology sphere case, there is no need to discuss it here. This theory uses essentially both the Kontsevich integral and the Chern-Simons functional. These are being used perturbatively. As result, the output is given in the form of Theorem 3.16 of section 3. This is consistent with the definition given by Lescop (Remark 4.13) where $\frac{1}{2}\Delta^\gamma_{K,M}(1)$ was used instead of $\frac{1}{2}\Delta^\gamma_{K,S^3}(1)$ since theory of finite type invariants is not restricted to

\textsuperscript{18}This terminology is taken from the theory of finite type invariants. In knot theory such a move is easily recognizable as the 3rd Reidemeister move.
the homology sphere case. Physically, however, we believe that the homology sphere case discussed in detail in this paper is sufficient. In addition, the existing theories of finite type invariants are all perturbative to our knowledge while, in our opinion, the Floer-style nonperturbative approach\textsuperscript{16,17} is essential for detecting fine details not detectable in perturbative treatments.

4.4. Concordance at the next level. Doubly sliced knots and links

From previous discussion it follows that the notion of concordance and of slice (knot) are synonymous. The equivalence relation of concordance originated in early 1960’s in works of Fox, Kerwaire and Milnor in the context of theory of isolated singularities of 2-spheres in 4-manifolds. Let $\mathcal{K}$ be the set of ambient isotopy classes of knots and let $\mathcal{C}$ the set of (smooth) concordance classes of knots. Since isotopic knots are concordant\textsuperscript{103,104}, there is a natural surjection $\mathcal{K} \rightarrow \mathcal{C}$. As we already demonstrated, the connected sum operation endows $\mathcal{C}$ with the structure of an Abelian group called smooth knot concordance group. The identity element is unknot. Any element concordant to unknot is called slice knot as we already know. We also know that the slice knot is that knot which bounds a disc $D^2$ smoothly embedded in 4-ball $B^4$ (e.g. see Fig.1). Surprisingly, a systematic extension\textsuperscript{114} of the notion of concordance to links was made only in 2012! Since the notion of concordance is related to topology of 4-manifolds, it should not come as total surprise that it is possible to elevate this notion to the next level by introducing the notion of doubly slice knots (and, accordingly, of double concordance). Whether or not doubly sliced knots/links have any relevance to physical reality remains to be investigated. Nevertheless, it is important to be aware of their existence already at this level of presentation. Thus, knot is called slice if it can be realized as the equator of an embedding of $S^2$ into $S^4$. Accordingly, a knot is doubly slice if it can be realized as the equator of an unknotted embedding of $S^2$ into $S^4$ (recall that the "classical" knot $K$ is an embedding of $S^1$ into $S^3$). Understanding when knots are slice or doubly slice, is helpful for our understanding of 4-dimensional topology in more familiar setting of knots in $S^3$. One of the most fascinating aspects of topology of 4-manifolds is the discrepancy between the smooth and topologically locally flat spaces (e.g. read previous subsection). This discrepancy is intrinsic only for 4-manifolds\textsuperscript{43,115}. It is nicely captured in study of slice knots\textsuperscript{116}. Recently, Meier proved the following

\textbf{Theorem 4.17.} (Meier\textsuperscript{117}) \textit{There exists an infinite family of slice knots that are topologically doubly slice, but not smoothly doubly slice.}

This paper is part of Meier PhD thesis\textsuperscript{118}. His proofs involve results from the Heegard Floer homology. A quick introduction to this field of study is given ch-r 9 of . In connection with Theorem 4.17. several questions arise: Is infinite family of slice knots in this theorem made of slice knots identifiable with the physical vacuum? If the answer is "yes", what this non-smoothness means physically? If the answer is "no", what else these slice knots
represent? Providing answers to these questions based on physical considerations might help to resolve the deepest mysteries of 4-manifolds topology, e.g. those discussed in\textsuperscript{43,109}.

5. Brief summary and discussion

Some time ago Ranada obtained new nontrivial solutions of the Maxwellian (that is the Abelian gauge) fields without sources which were reinterpreted in our work\textsuperscript{15} as particle-like (monopoles, dyons, etc.). These solutions were obtained by the method of Abelian reduction of the non-Abelian Yang-Mills functional. The developed method involved uses of instanon-type calculations typically used for the non-Abelian gauge fields\textsuperscript{13}. Employing the electric-magnetic duality permits then to replace all charges by the corresponding particle-like solutions of the source-free Abelian gauge fields. Such a replacement has many advantages. In particular, if one believes that the non-Abelian gauge fields are just generalizations of more familiar Maxwellian fields, then one encounters a problem about existence of non-Abelian charges -analogs of (seemingly familiar) macroscopic charges in Maxwell’s electrodynamics. Surprisingly, to introduce the macroscopic charges into non-Abelian fields is a challenging task which (to our knowledge) is still not completed. By treating gravity as gauge theory, the analogous problem emerges in gravity too. It is the problem of description of motion of the extended bodies in general relativity. The difficulties in description of extended objects in both the Y-M and gravity fields are summarized on page 97 of Ref.\textsuperscript{18}. The method of Abelian reduction developed in\textsuperscript{15} and further extended in\textsuperscript{22} allows us to improve the existing thus far situation dramatically by employing for both the Abelian and non-Abelian gauge fields the same methods. These were originally used only for description of non-Abelian monopoles, dyons, etc., by employing source-free non-Abelian gauge fields. Clearly, under such conditions, the problem of relating the source-free non-Abelian configurations of gauge fields to their Abelian counterparts with extended sources was left unexplained.

The newly developed Abelian reduction of the source-free gauge fields resulted in dynamically generating (in the bulk) of all kinds of knot and link structures excluding those made of hyperbolic knots/links. Next logical steps in this development should involve study of likely connections of the obtained knotted/linked structures with the hadron physics and, more broadly, with the Standard Model and gravity. This paper is the first logical step in this direction. To make our presentation self-contained, we would like now to mention some recent (and not so recent) works providing independent support to the results of this paper. The story begins with influential paper by Atiyah and Manton entitled “Skyrmions from instantons”\textsuperscript{119}. The Skyrme model is well studied model for description of baryons\textsuperscript{120}. Whether or not it can be applied to hadrons was studied in\textsuperscript{121} with positive outcome. The latest PhD work by Jennings\textsuperscript{122} convincingly demonstrates how skyrmions can be represented by knots and links. These knots and links happen to be exactly of the type described in detail in our work\textsuperscript{22}. Ref.\textsuperscript{123} contains up to date review of efforts by other researchers.
Connections with string models can be developed in principle by studying Ref.124 as point of departure. In it the problem of accounting for the presence of topological constraints in systems such as entangled polymers, etc., undergoing some dynamics,\textsuperscript{125} is studied. The work uses results of Moffatt,\textsuperscript{92} discussed in this work in section 4.1., superimposed with some results from the theory of reaction-diffusion (birth-death type) processes. The authors of\textsuperscript{124} do notice a connection between dynamics of reaction-diffusion processes and dynamics of spin chains. In Ref.s 126, 127 not only the connections between reaction-diffusion processes and spin chains were investigated, but in addition they were used as an input for building several (topological) string models.

We mentioned already works by Taubes\textsuperscript{16} and Masataka\textsuperscript{17} in which Floer-type approach to instantons (e.g. read Ref.15 for an introduction to this approach) was used. In addition, Lim\textsuperscript{128} demonstrated equivalence between Seiberg-Witten and Casson invariants for homology 3-sphere. Furthermore, Kronheimer and Mrowka using instanton Floer homology\textsuperscript{129} reobtained Alexander polynomial used essentially in this work. Since skyrmions are instantons and since knotted/linked skyrmions describe hadrons, the instanton origin of Alexander polynomial provides needed missing link between baryons and instantons. Details of just noticed correspondences will be investigated in future publications.

Appendix. Basics facts about the Dehn surgery and Kirby calculus

1. Dehn surgery. Let K be some knot in 3-manifold M. We surround K by a regular tubular neighborhood \( N(K) \). Evidently, if the boundary \( \partial N(K) = T \), where \( T \) is the torus \( T \),

then \( N(K) \) the solid torus \( N(K) = S^2 \times D^2 \) while \( \tilde{V}(K) \) is the solid torus without boundary. Following Rolfsen\textsuperscript{54}, suppose, we are given:

a) a 3-manifold \( M \), perhaps with boundary. Let its interior be \( \tilde{M} \);

b) a link \( L = K_1 \cup \cdots \cup K_n \);

c) a disjoint tubular neighborhoods \( N(K_i) \equiv N_i \) of the \( K_i \);

d) a specified simple (that is without self intersection \( s \)) closed curve \( J_i \) on each \( \partial N(K_i) \).

Given these data, we may construct the 3-manifold \( M' \)

\[
M' = (M - (\tilde{N}_1 \cup \cdots \cup \tilde{N}_n)) \cup_h ((N_1 \cup \cdots \cup N_n)),
\]

where \( h \) is a union of homeomorphisms \( h_i : \partial N_i \to \partial N_i \subset M \), each of which take a meridian curve \( \mu_i \) of \( \partial N_i \) into the specified curve \( J_i \).

**Definition A.1.** \( M' \) is said to be the result of a Dehn surgery on \( M \) along the link \( L \) with surgery instructions c) and d).

In the most studied case \( M = \mathbb{S}^3 \) or \( \mathbb{R}^3 \) the surgery instructions are being expressed by assigning a rational number \( r_i \) (could be \( \infty \) sometimes) to each component of \( L_i \).
Definition A.2. Let $N(K) = S^2 \times D^2$, then a specified homeomorphism $h_i : N(K_i) \to N(K_i)$ is called framing of $N(K_i)$.

If $J_i \subset \partial N(K_i)$ is a simple closed curve, then the following are equivalent

a) $J_i$ is a longitude $\lambda_i$ of $N(K_i)$;

b) $J_i$ represents a generator of $H_1(N_i) \cong \pi_1(N_i) \cong \mathbb{Z}$;

c) $J_i$ intersects some meridian of $N_i$ transversely in a single point.

The meridian is intrinsic part of $N_i$ while the longitude $\lambda_i$ involves a choice.

Definition A.3. A preferred framing for $N_i$ is such for which $\lambda_i$ is oriented in the same way as $K_i$ and the meridian $\mu_i$ has linking number $\pm 1$ with $K_i$. In such a case

$$h(\mu_i) = [J_i] = a_i \lambda_i + b_i \mu_i.$$  

Here $a_i$ is determined with accuracy up to a sign (dependent upon orientation of $J_i$) while $b_i = lk(K_i, J_i)$.

Definition A.4. The ratio

$$r_i = b_i / a_i$$

is called surgery coefficient associated with $K_i$. If $a_i = 0$ then, by definition, $b_i = \pm 1$ and $r_i = \infty$. Clearly, in this case the surgery is trivial.

Definition A.5. The surgery is called integral if $a_i = \pm 1$.

Any link $L$ with rational numbers attached to its components determines a surgery yielding closed oriented 3-manifold. All closed oriented 3-manifolds arise in this way. In knot theory it is common practice to draw 3-manifolds by drawing the corresponding link with surgery coefficients $r_i$ placed next to respective components. If $U$ is unknot, then $r = 0$ defines a Lens space $L(0,1)$, while for $r = \pm 1, \pm 1/2, \pm 1/3, ...$ surgery on $U$ yields back $S^3$. The first nontrivial surgery leading to the Poincaré homology sphere is obtained by performing $\pm 1$ surgery on the trefoil knot. The same result can be achieved by performing surgery on many other links which topologically are not the same. This observation lies at the heart of Kirby calculus.

2. Kirby calculus and physics associated with them. We just discussed the fact that the Poincaré homology sphere is obtainable via Dehn surgery on the trefoil knot in $S^3$. Fact of the matter is that all 3-manifolds can be obtained surgically. As long as this is done with links, the end result can be achieved in many ways while the theorem of Gordon and Luecke requires 1-to-1 correspondence between knots and their complements in $S^3$. Therefore, by performing surgery on knots naively, we should expect again 1-to-1 correspondence. This is not the case however. In the previous subsection we noticed that the same Poincaré homology sphere is obtainable either via surgery on the trefoil knot or via surgery on many links as discussed in. Now, if knots are associated with masses, then Gordon and Luecke theorem guarantees 1-to-1 correspondence between masses and 3 manifolds. This theorem guarantees that the mass spectrum is discrete.

At this point we need to recall some facts from Einsteinian theory of gravity. The original formulation by Einstein makes heavy emphasis on study of the Schwarzschild
solution of Einstein equations. These are the field equations without sources in which the mass enters as an adjustable parameter at the end of calculations. If masses are associated with knots, we run into the same problems as in Einsteinian relativity. Specifically, suppose that we have several masses, then the space around given knot, say, trefoil, is made not only of the complement of the trefoil but out of complements of the rest of knots/links existing in the trefoil complement. Details are given in\textsuperscript{22}. Furthermore, following Einsteinian logic, we shall assume that knots/links other than trefoil are closed geodesics in the 3-manifold created by the trefoil knot. In such a case, very much like in Einsteinian gravity, we should assume that one can ignore the finiteness of masses moving on geodesics. To account for finite masses and for extended sizes of particles is always a great challenge\textsuperscript{18}, page 97. In section 2 we discussed how such a challenge can be plausibly resolved. It is also difficult problem in knot theory\textsuperscript{22,130}, where it is also not solved systematically. This is so because Gordon and Luecke theorem is valid only for complements of knots in $S^3$. Presence of other knots makes Gordon and Luecke theorem non applicable. From this observation the following question arises.

**Question A.6.** Suppose in some 3-manifold M (other than $S^3$) there are two knots $K_1$ and $K_2$ so that the associated complements are $M \setminus K_1$ and $M \setminus K_2$. Suppose that there is a homeomorphism $h$: $h(M \setminus K_1) = M \setminus K_2$. Will such a homeomorphism imply that $K_1 = K_2$? (With links this happens all the time. E.g. read Remark 4.2.)

If the answer to the above question is negative, this then would imply that different knots would have the same complement in M. If we are interested in attaching some physics to these statements then, we should only look for situations analogous to $S^3$. This means that all physical processes should be subject to selection rules making such degenerate cases physically forbidden. These selection rules presuppose that the degeneracy just described can be realized in nature. And indeed, it can! This leads to the concept of cosmetic knots. We refer our readers to our paper\textsuperscript{22}, section 7, for further details.

The purpose of Kirby calculus is to establish the equivalence classes of different links yielding the same 3-manifolds surgically. Deep down Kirby calculus should be done using theory of 4-manifolds\textsuperscript{43,97}. Fortunately, for the purposes of this work this path is not needed. In fact, as it was rigorously demonstrated by Rolfsen\textsuperscript{131,132}, the four dimensional approach to Kirby calculus could be entirely avoided. While Kirby calculus use only integral surgery, Rolfsen’s calculus use both the integral and rational surgeries indiscriminately. Kirby calculus, however, are bit more physically suggestive. This is so because of the following. The linking number between two links $L_1$ and $L_2$ can be defined via study of links projection on the arbitrary plane. Using Fig.4 if we identify arrows with different links then we can prescribe $\varepsilon = 1$ to K configuration and $\varepsilon = -1$ to $\bar{K}$ configuration so that

$$lk(L_1, L_2) = \sum_{i} \varepsilon_i$$

where summation is over all crossings on the planar link diagram. In the case of integral surgery we have $[J_i] = \lambda_i + b_i \mu_i$. This means that $J$-curve is making exactly one revolution.
in the direction parallel to $\lambda_i$. This can be illustrated as follows. Make a ribbon link out of $K_i$ and $[J_i]$ as depicted in Fig.16 a)

The self-linking number is determined then by $\text{lk}(K_i, \mu_i)$, e.g. see Fig.16b). Alternatively, we can introduce

**Definition A.6.** The integral framing of the knot $K_i$ corresponds to a choice $\text{lk}(K_i, \mu_i) = b_i = n_i$ where $n_i = 0, \pm 1, \pm 2, \ldots$

An example of integral framing is depicted in Fig.16 b). Being armed with these definitions we are ready to address the main problem studied by Kirby\textsuperscript{120} : How to determine when two differently framed links produce the same 3-manifold?

The answer to this question is given in terms of two (Kirby) moves. By design, they are meant to establish the equivalence relation between links with different framings producing the same 3-manifold. Using results of our previous works\textsuperscript{15,22}, the first Kirby move, depicted in Fig.17 can be interpreted physically as graphical statement illustrating the charge conservation.

This move should be interpreted as follows: It is permissible to add or to delete an unknot with framing $\pm 1$ which does not intersect other components $L_i$ to a given link $L$.

The 2nd Kirby move is depicted in Fig.17 and can be physically reinterpreted in terms of interaction between charges. Mathematically, this move can be interpreted as follows. Let $L_1$ and $L_2$ be two link components framed by integers $n_1$ and $n_2$ respectively and $L'_2$
be a longitude defining the framing of $L_2$ that is $lk(L_2, L'_2) = n_2$. Replace now the pair $L_1 \cup L_2$ by another pair $L_1# \cup L_2$ in which $L_1# = L_1#_b L'_2$ and $b$ is 2 sided band connecting $L_1$ with $L'_2$ and disjoint from another link components. While doing so, the rest of the link $L$ remains unchanged. The framings of all components, except $L_1$, are preserved while the framing of $L_1$ is changed into that for $L_1#$ and is given by $n_1 + n_2 + 2lk(L_1, L_2)$. The computation of $lk(L_1, L_2)$ proceeds in a standard way as described above (provided that both $L_1$ and $L_2$ are oriented links).

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