Relation between $\sin^2 \hat{\theta}_W(m_Z)$ and $\sin^2 \theta_{eff}^{lep}$.

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ABSTRACT

The relation between $\sin^2 \theta_{eff}^{lep}$, frequently employed in LEP analyses, and the $\overline{\text{MS}}$–parameter $\sin^2 \hat{\theta}_W(m_Z)$ is discussed and their difference evaluated by means of an explicit calculation.
It has been emphasized that the $\overline{\text{MS}}$ parameter $\sin^2 \hat{\theta}_W(m_Z)$ provides a very convenient framework to discuss physics at the $Z^0$ peak \[1\] and, as it is well known, it plays a crucial rôle in the analysis of grand unification. On the other hand, the LEP collaborations employ an effective coupling, $\sin^2 \hat{\theta}_{\text{eff}}^{\text{lep}}$ \[2, 3\]. It is a common belief among physicists that these two parameters, although very different conceptually, are very close numerically. However, the reason and extent of this coincidence and the precise conceptual and numerical relation between the two has not been spelled out in the literature. In turn this is a source of considerable confusion among theorists and experimentalists alike. The aim of this report is to clarify these issues.

The effective weak interaction angle employed by the LEP groups is defined by

$$1 - 4 \sin^2 \hat{\theta}_{\text{eff}}^{\text{lep}} = \frac{g_V^f}{g_A^f}$$

where $g_V^f$ and $g_A^f$ are the effective vector and axial couplings in the $Z^0 \to \ell \bar{\ell}$ amplitude at resonance, where $\ell$ denotes a charged lepton \[2, 3\]. In order to establish the connection with $\sin^2 \hat{\theta}_W(m_Z)$, we note that this amplitude is proportional to \[4\]

$$< \ell \bar{\ell} | J_Z^\lambda | 0 > = -\bar{u}_\ell \gamma^\lambda \left[ \frac{1 - \gamma_5}{4} - \hat{k}_\ell(q^2) \hat{s}^2 \right] v_\ell,$$

where $\hat{s}^2$ is an abbreviation for $\sin^2 \hat{\theta}_W(m_Z)$, $v_\ell$ and $\bar{u}_\ell$ are the lepton spinors and $\hat{k}_\ell(q^2)$ is an electroweak form factor. Up to terms of order $O(\alpha)$ we have \[3\]

$$\hat{k}_\ell(q^2) = 1 - \frac{\hat{\epsilon}}{\hat{s}} \left[ A_{\gamma Z}(q^2) - A_{\gamma Z}(0) \right]_{\overline{\text{MS}}} + \frac{\hat{\alpha}}{\pi \hat{s}^2} \hat{c}^2 \log c^2 - \frac{\hat{\alpha}}{4\pi \hat{s}^2} V_\ell(q^2),$$

where $A_{\gamma Z}(q^2)$ is the $\gamma-Z$ mixing self-energy, the subscript $\overline{\text{MS}}$ means that the $\overline{\text{MS}}$ renormalization has been carried out (i.e. the pole terms have been subtracted and the 't–Hooft scale has been set equal to $m_Z$), the superscript $f$ stands for fermionic part, $\hat{\alpha}$ is an abbreviation for $\hat{\alpha}(m_Z) = [127.9 \pm 0.1]^{-1}$ \[5\], $\hat{c}^2 \equiv \cos^2 \hat{\theta}_W(m_Z)$, $c^2 \equiv m_W^2/m_Z^2$ and $V_\ell(q^2)$ is a finite vertex correction. Explicitly,

$$V_\ell(q^2) = \frac{1}{2} f\left( \frac{q^2}{m_W^2} \right) + 4 \hat{c}^2 \ g\left( \frac{q^2}{m_W^2} \right) - \frac{1 - 6 \hat{s}^2 + 8 \hat{s}^4}{4 \hat{c}^2} f\left( \frac{q^2}{m_Z^2} \right),$$

where $f(x)$ and $g(x)$ are defined in Eqs. (6d, 6e) of Ref.\[4\]. We have included the photon self–energy $A_{\gamma\gamma,\overline{\text{MS}}}(q^2)$ in the second term of Eq.(3) because, as it will be explained later, it gives rise to relatively large $O(\alpha^2)$ terms.
It is clear from Eq.(2) that the ratio of the vector and axial vector couplings at resonance is given by \(1 - 4 \hat{k}_\ell(m_Z^2) \hat{s}^2\). We now discuss the various contributions to \(\hat{k}_\ell(m_Z^2)\).

To \(\mathcal{O}(\hat{\alpha})\) the fermionic contribution to the real part of Eq. (3) can be written in the form

\[-\hat{c} \frac{\text{Re} A^f_{\gamma Z}(m_Z^2)}{m_Z^2} = \frac{\hat{c}^2}{\hat{s}^2} \sum_i \left( \frac{Q_i C_i}{4} - s^2 Q_i^2 \right) \frac{\text{Re} \Pi^V(m_Z^2, m_i, m_i)}{m_Z^2}, (5)\]

where \(Q_i, C_i, m_i\) are the charge, third component of weak isospin (with eigenvalues \(\pm 1\)), and mass of the \(i\)-th fermion, the summation includes the color degree of freedom, \(\Pi^V\) is the vacuum polarization function involving vector currents, and henceforth the \(\overline{\text{MS}}\) renormalization is not indicated explicitly. For the leptons we set \(m_i = 0\) and find that the contribution to Eq.(5) is

\[(\hat{\alpha}/\pi \hat{s}^2)(5/12)(1 - 4\hat{s}^2) = 3.2 \times 10^{-4} [6].\]

In this calculation and henceforth we employ \(\hat{s}^2 = 0.2323\), which corresponds to the central values \(m_t = 162\) GeV and \(m_H = 300\) GeV in the global fit of Ref.[3], and \(\hat{\alpha} = (127.9)^{-1}\).

For the first five quark flavors we again set \(m_i = 0\) and, including \(\mathcal{O}(\hat{\alpha} \hat{s}^2)\) corrections, obtain a contribution [5] \((\hat{\alpha}/\pi \hat{s}^2)(7/12 - 11\hat{s}^2/9) [5/3 + (\hat{\alpha}_s/\pi)(55/12 - 4\zeta(3))] = 5.32 \times 10^{-3}\), where we have used \(\hat{\alpha}_s = \hat{\alpha}_s(m_Z) = 0.118\) and \(\zeta(3) = 1.20206\ldots\)

The top quark contribution to Eq.(3) is of the form [4]

\[-\hat{c} \frac{\text{Re} A^{(\text{top})}_{\gamma Z}(m_Z^2)}{m_Z^2} = -\frac{\hat{\alpha}}{6\pi \hat{s}^2} \left(1 - \frac{8}{3} \hat{s}^2\right) \left[1 + \frac{15 \hat{\alpha}_s}{4 \pi}\right] \log \xi_t - \frac{15 \hat{\alpha}_s}{4 \pi} + D(1/\xi_t), (6)\]

where \(\xi_t \equiv m_t^2/m_Z^2\) and \(D(1/\xi_t)\) represents small terms that decouple in the limit \(\xi_t \to \infty\). For the current range 120 GeV \(< m_t < 200\) GeV [3], \(D(1/\xi_t)\) varies from \(1.0 \times 10^{-4}\) to \(3 \times 10^{-5}\) and is of the same order of magnitude as neglected two-loop contributions \(\sim (\hat{\alpha}/\pi \hat{s}^2)^2 \approx 10^{-4}\) to Eq.(3).

According to the Marciano–Rosner [M–R] convention [4], adopted also in Ref.[3], the first term in Eq.(3) is subtracted in the evaluation of \(\text{Re} A^{(\text{top})}_{\gamma Z}(m_Z^2)/m_Z^2\). This is part of the \(\overline{\text{MS}}\) renormalization prescription of these authors, the idea being that contributions from particles of mass \(m > m_Z\) that do not decouple in the limit \(m \to \infty\) are subtracted from this particular amplitude and absorbed in the definition of \(\sin^2 \hat{\theta}_W(m_Z)\). The aim of the prescription is to make the value of \(\sin^2 \hat{\theta}_W(m_Z)\), as extracted from the on–resonance asymmetries, very insensitive to heavy particles of mass \(m > m_Z\). We reach the conclusion that
when the M–R prescription is applied, the top quark contribution to Eq.(5) is very small.

The other contributions to $\hat{k}_\ell(m_Z^2)$ in Eq.(3) can be readily obtained from the literature. This form factor is gauge invariant, but several individual components are not. We evaluate them in the ’t Hooft–Feynman gauge, using $m_W = 80.23$ GeV \(\text{[3]}\): (i) the bosonic contributions $-(\hat{c}/\hat{s}) [A_{\gamma 2}^{(b)}(m_Z^2) - A_{\gamma 2}^{(b)}(0)]/m_Z^2$ can be extracted from Ref.\(\text{[3]}\) and amount to $-5.92 \times 10^{-3}$; (ii) $-(\hat{a}/4\pi \hat{s}^2)$ Re $V_t(m_Z^2)$ can be obtained from Eq.(4) of this paper and Eqs.(6d,e) of Ref.\(\text{[4]}\), and gives $+3.32 \times 10^{-3}$; (iii) $(\hat{a}/\pi \hat{s}^2)(\hat{c}^2 \log \hat{c}^2) = -2.11 \times 10^{-3}$; (iv) although two–loop effects have not been fully calculated, we include the $O(\hat{a}^2)$ contribution arising from the product of Im$A_{\gamma 2}(m_Z^2)$ and Im$A_{\gamma 2}(m_Z^2)$ in the second term of Eq.(3). It amounts to $+1.9 \times 10^{-4}$. It is quite sizeable, relative to typical $O(\hat{a}^2)$ contributions, because these imaginary parts involve several additive terms. On the other hand, the large logarithmic $O(\hat{a}^2)$ corrections associated with the running of $\hat{a}$ are already taken into account, in the $\overline{\text{MS}}$ scheme, by employing $\hat{a}$ in the evaluation of $A_{\gamma 2}$; (v) for 120 GeV $\lesssim m_t \lesssim 200$ GeV, the $t - \bar{t}$ threshold contribution \(\text{[3, 4]}\) to Eq.(3) ranges from 1.7 \times 10^{-5} to 2.7 \times 10^{-5} and is therefore negligible; (vi) there are imaginary contributions to $\hat{k}_\ell(m_Z^2)$ arising from $A_{\gamma 2}^{(f)}(m_Z^2)$ and $V_t(m_Z^2)$ and amount to $i \, 1.06 \times 10^{-2}$ and $i \, 0.28 \times 10^{-2}$, respectively.

Combining all the above results we have $\hat{k}_\ell(m_Z^2) = 1 + (0.32 + 5.32 - 5.92 + 3.32 - 2.11 + 0.19) \times 10^{-3} + D(1/\xi_t) + i \,(1.06 + 0.28) \times 10^{-2}$, which to good approximation becomes

$$\hat{k}_\ell(m_Z^2) = 1.0012 + i \, 0.0134. \quad (7)$$

It is clear, on the basis of Eq.(7), that at the one–loop level the ratio of effective vector and axial vector couplings in the $Z \to \ell \ell$ amplitude is a complex number. This is also expected from general principles. On the other hand, the LEP groups interpret both sides of Eq.(3) as real quantities. This can be justified on the grounds that the imaginary component of $\hat{k}_\ell(m_Z^2)$ gives a negligible contribution to the leptonic bare asymmetries and partial widths. For instance, the bare left–right asymmetry is given by $A_{LR}^{\ell} = 2 \text{Re}(g_V/g_A)/[1 + |g_V/g_A|^2]$ and one readily verifies that the inclusion of Im$\hat{k}_\ell(m_Z^2)$ decreases its value by only $-0.02\%$. Similarly, $A_{FB}^{\ell}$ is modified by $\approx -0.03\%$. Therefore we identify

$$\sin^2 \theta_{eff}^{lep} = \hat{s}^2 \text{ Re } \hat{k}_\ell(m_Z^2). \quad (8)$$

Using Eq.(4) we have

$$\sin^2 \theta_{eff}^{lep} - \sin^2 \hat{\theta}_W(m_Z) = 2.8 \times 10^{-4} \approx 3 \times 10^{-4}. \quad (9)$$
The following observations are appropriate at this stage: (a) because the Higgs boson does not contribute at the one–loop level to Eq.(3), the results of Eqs.(7, 9) are independent of \( m_H \); (b) it is clear that the closeness of \( \text{Re} \hat{k}_l(m_Z^2) \) to unity and, correspondingly, the small difference in Eq.(9) are due to the cancellation of significantly larger terms. For instance, the light quark and bosonic contributions to the \( \gamma-Z \) mixing self–energy are of the roughly expected order of magnitude \( \sim \hat{\alpha}/(2\pi \hat{s}^2) \approx 5.4 \times 10^{-3} \), but they largely cancel against each other. On the other hand, the \( \mathcal{O}(\hat{\alpha}) \) contributions to the \( \text{Im} \hat{k}_l(m_Z^2) \) are \( \approx 1\% \), an order of magnitude larger; (c) as the relation between \( \sin^2 \hat{\theta}_W(m_Z) \) and \( \sin^2 \theta_W \equiv 1 - m_W^2/m_Z^2 \) is well–known [5], Eq.(9) determines the connection between the three parameters.

If the Marciano–Rosner decoupling convention is not applied, so that in the \( \overline{\text{MS}} \) renormalization one only subtracts poles and sets the 't–Hooft scale equal to \( m_Z \), there is a further contribution to \( \text{Re} \hat{k}_l(m_Z^2) \) arising from the first term in Eq.(6). Using \( \hat{\alpha}_s(m_t) \approx 0.11 \), this amounts to \(-3 \times 10^{-4}, -7 \times 10^{-4}, -1.0 \times 10^{-3} \), for \( m_t = 120, 162 \) and \( 200 \) GeV, respectively. Correspondingly, \( \text{Re} \hat{k}_l(m_Z^2) \) becomes \( 1.0009, 1.0005, 1.0002 \), even closer to unity. As a consequence, although the difference between \( \sin^2 \theta_{\text{lep}}^{\text{eff}} \) and \( \sin^2 \hat{\theta}_W(m_Z) \) depends more on \( m_t \) when the M–R prescription is not applied, it is actually smaller for the current range \( 120 \text{GeV} \leq m_t \leq 200 \) GeV. In fact, we find that it is \( +2 \times 10^{-4} \) for \( 120 \leq m_t \leq 135 \) GeV, \( 1 \times 10^{-4} \) for \( 136 \leq m_t \leq 184 \) GeV, and there is no difference in the fourth decimal for \( 185 \leq m_t \leq 200 \) GeV.

One can obtain a rough consistency check of the order of magnitude of Eq.(9) by comparing the fits of Ref.[3] with the calculations of Ref.[5]. Using the LEP, collider and \( \nu \) data, Ref.[3] finds \( m_t = 162^{+16}_{-17}^{+18}_{-21} \) and \( \sin^2 \theta_{\text{lep}}^{\text{eff}} = 0.2325 \pm 0.0005^{+0.0001}_{-0.0002} \) for constrained \( \hat{\alpha}_s \), and the same value of \( m_t \) but a slightly different central value for \( \sin^2 \theta_{\text{eff}}^{\text{lep}} \) (0.2326), with the same errors, in their unconstrained \( \hat{\alpha}_s \) fit. Their central values assume \( m_H = 300 \) GeV, the first error represents experimental and theoretical uncertainties, while the second reflects changes corresponding to the assumptions \( m_H = 60 \) GeV and \( M_H = 1 \) TeV. According to Eq.(3), the corresponding central values for \( \hat{s}^2 \) should be 0.2322 and 0.2323. On the other hand, from Ref.[3] one finds \( \hat{s}^2 = 0.2323 \) for \( m_t = 162 \) GeV and \( m_H = 300 \) GeV. Thus, the comparison of the conclusions of Ref.[5] with the calculations of Ref.[3] is roughly consistent with Eq.(3). Of course, such rough consistency checks are not a substitute for precise, ab initio calculations, like the one leading to Eq.(9).

In order to avoid possible further sources of confusion, we make two additional comments. (a) Sometimes, consistently with Eq.(1), rapporteurs define \( \sin^2 \theta_{\text{eff}}^{\text{lep}} \) in terms of a bare forward–backward asymmetry \( A_{\text{FB}} \), which is obtained
from the physical asymmetry $A_{FB}^\ell$ after extracting the effect of photon–mediated contributions and other radiative correction effects $^2$. Therefore, we should not attempt to find the numerical relation between $\sin^2\theta_{\text{eff}}^\ell$ and $\sin^2\hat{\theta}_W(m_Z)$ by comparing detailed MSS calculations of the physical asymmetry $A_{FB}^\ell$, as those in Ref. 4, with theoretical expressions for $A_{FB}^{0,\ell}$ expressed in terms of $\sin^2\theta_{\text{eff}}^\ell$. The point is that $A_{FB}^\ell$ contains electroweak effects not contained in $A_{FB}^{0,\ell}$. (b) Rapporteurs often cite the value of $\sin^2\theta_{\text{eff}}^\ell$ as extracted only from the on–resonance asymmetries, while they give the prediction for $m_t$ derived from the complete data base. Current asymmetry results lead to determinations of $\sin^2\theta_{\text{eff}}^\ell$ that are somewhat smaller than the $\sin^2\hat{\theta}_W(m_Z)$ numbers corresponding to the central $m_t$. This, however, is not a contradiction with Eq.(9), because the on–resonance asymmetries represent only a part of the experimental information. This is quite visible in the detailed report of Ref. [3], in which one finds $\sin^2\theta_{\text{eff}}^\ell = 0.2321 \pm 0.0006$ from the on–resonance asymmetries and, as mentioned before, larger values from the global fits.

In summary, we have attempted to clarify the connection between $\sin^2\theta_{\text{eff}}^\ell$ and $\sin^2\hat{\theta}_W(m_Z)$ and obtained the value of their difference by means of a detailed calculation, both with and without the M–R decoupling convention. In view of the prospects for a very accurate determination of the mixing angle from further LEP studies and from $A_{LR}$ at SLAC, and the fact that both definitions are frequently employed, we feel that this clarification is indeed timely.

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