Abstract

We model the instantaneous power on a velodrome for individual pursuits, taking into account its straights, circular arcs, and connecting transition curves. The forces opposing the motion are air resistance, rolling resistance, lateral friction and drivetrain resistance. We examine the constant-cadence and constant-power cases, and discuss their results, including changes in the kinetic and potential energy.

1 Introduction

This article is a continuation of research presented by Danek et al. (2020a), Danek et al. (2020b), Bos et al. (2020) and, in particular, by Slawinski et al. (2020). Herein, using a mathematical model, we examine the power required on a velodrome, for an individual pursuit. The opposing forces consist of air resistance, rolling resistance, lateral friction and drivetrain resistance. We consider a velodrome with its straights, circular arcs, and connecting transition curves, whose inclusion — while presenting a certain challenge, and neglected in previous studies (e.g., Slawinski et al., 2020) — increases the empirical adequacy of the model.

We begin this article by expressing mathematically the geometry of both the black line and the inclination of the track. Our expressions are accurate analogies for the common geometry of modern 250-metre velodromes (Mehdi Kordi, pers. comm., 2020). We proceed to formulate an expression for power expended against dissipative forces, which we examine for both the constant-cadence and constant-power cases. For either case, we consider, a posteriori, changes in the kinetic and potential energy. We conclude by discussing the results.

2 Formulation

2.1 Black-line parameterization

To model the required power for an individual pursuit of a cyclist who follows the black line, in a constant aerodynamic position, we define this line by three parameters.
– $L_s$: the half-length of the straight
– $L_t$: the length of the transition curve between the straight and the circular arc
– $L_a$: the half-length of the circular arc

The length of the track is $S = 4(L_s + L_t + L_a)$. In Figure 1, we show a quarter of a black line for $L_s = 19\,\text{m}$, $L_t = 13.5\,\text{m}$ and $L_a = 30\,\text{m}$, which results in $S = 250\,\text{m}$. This curve has continuous derivative up to order two; it is a $C^2$ curve, whose curvature is continuous.

To formulate, in Cartesian coordinates, the curve shown in Figure 1, we consider the following.

– The straight, $y_1 = 0$, $0 \leq x \leq a$, shown in gray, where $a := L_s$.
– The transition, shown in black — following a standard design practice — we take to be an Euler spiral, which can be parameterized by Fresnel integrals,

$$
x_2(\varsigma) = a + \sqrt{\frac{2}{A}} \sqrt{\frac{\pi}{2}} \int_0^{\varsigma} \cos(x^2) \, dx
\quad \text{and} \quad
y_2(\varsigma) = \sqrt{\frac{2}{A}} \sqrt{\frac{\pi}{2}} \int_0^{\varsigma} \sin(x^2) \, dx,
$$

with $A > 0$ to be determined; herein, $\varsigma$ is a curve parameter. Since the arclength differential, $ds$, is such that

$$
ds = \sqrt{x_2'(\varsigma)^2 + y_2'(\varsigma)^2} \, d\varsigma
= \sqrt{\cos^2 \left( \frac{A\varsigma^2}{2} \right) + \sin^2 \left( \frac{A\varsigma^2}{2} \right)} \, d\varsigma
= d\varsigma,
$$

we write the transition curve as

$$(x_2(s), y_2(s)), \quad 0 \leq s \leq b := L_t.$$
The circular arc, shown in gray, whose centre is \((c_1, c_2)\) and whose radius is \(R\), with \(c_1\), \(c_2\) and \(R\) to be determined. Since its arclength is specified to be \(c := L_a\), we may parameterize the quarter circle by

\[ x_3(\theta) = c_1 + R \cos(\theta) \quad (1) \]

and

\[ y_3(\theta) = c_2 + R \sin(\theta), \quad (2) \]

where \(-\theta_0 \leq \theta \leq 0\), for \(\theta_0 := c/R\). The centre of the circle is shown as a black dot in Figure 1.

We wish to connect these three curve segments so that the resulting global curve is continuous along with its first and second derivatives. This ensures that the curvature of the track is also continuous.

To do so, let us consider the connection between the straight and the Euler spiral. Herein, \(x_2(0) = a\) and \(y_2(0) = 0\), so the spiral connects continuously to the end of the straight at \((a, 0)\). Also, at \((a, 0)\),

\[ \frac{dy}{dx} = \frac{y_2'(0)}{x_2'(0)} \quad (3) \]

which matches the derivative of the straight line. Furthermore, the second derivatives match, since

\[ \frac{d^2y}{dx^2} = \frac{y_2''(0)x_2'(0) - y_2'(0)x_2''(0)}{(x_2'(0))^2} = 0, \]

which follows, for any \(A > 0\), from

\[ x_2'(\varsigma) = \cos \left( \frac{A \varsigma^2}{2} \right), \quad y_2'(\varsigma) = \sin \left( \frac{A \varsigma^2}{2} \right) \quad (3) \]

and

\[ x_2''(\varsigma) = -A \varsigma \sin \left( \frac{A \varsigma^2}{2} \right), \quad y_2''(\varsigma) = A \varsigma \cos \left( \frac{A \varsigma^2}{2} \right). \]

Let us consider the connection between the Euler spiral and the arc of the circle. In order that these connect continuously,

\[ (x_2(b), y_2(b)) = (x_3(-\theta_0), y_3(-\theta_0)), \]

we require

\[ x_2(b) = c_1 + R \cos(\theta_0) \iff c_1 = x_2(b) - R \cos \left( \frac{c}{R} \right) \quad (4) \]

and

\[ y_2(b) = c_2 - R \sin(\theta_0) \iff c_2 = y_2(b) + R \sin \left( \frac{c}{R} \right). \quad (5) \]

For the tangents to connect continuously, we invoke expression (3) to write

\[ (x_2'(b), y_2'(b)) = \left( \cos \left( \frac{A b^2}{2} \right), \sin \left( \frac{A b^2}{2} \right) \right). \]

Following expressions (1) and (2), we obtain

\[ (x_3'(-\theta_0), y_3'(-\theta_0)) = \left( R \sin(\theta_0), R \cos(\theta_0) \right), \]

respectively. Matching the unit tangent vectors results in

\[ \cos \left( \frac{A b^2}{2} \right) = \sin \left( \frac{c}{R} \right), \quad \sin \left( \frac{A b^2}{2} \right) = \cos \left( \frac{c}{R} \right). \quad (6) \]
For the second derivative, it is equivalent — and easier — to match the curvature. For the Euler spiral,

\[
\kappa_2(s) = \frac{x'_2(s)y''_2(s) - y'_2(s)x''_2(s)}{\left( (x'_2(s))^2 + (y'_2(s))^2 \right)^{\frac{3}{2}}}
\]

\[
= As \cos^2 \left( \frac{As^2}{2} \right) + As \sin^2 \left( \frac{As^2}{2} \right)
\]

\[
= As,
\]

which is indeed the defining characteristic of an Euler spiral: the curvature grows linearly in the arclength. Hence, to match the curvature of the circle at the connection, we require

\[
Ab = \frac{1}{R} \iff A = \frac{1}{bR}.
\]

Substituting this value of \( A \) in equations (6), we obtain

\[
\cos \left( \frac{b}{2R} \right) = \sin \left( \frac{c}{R} \right), \quad \sin \left( \frac{b}{2R} \right) = \cos \left( \frac{c}{R} \right)
\]

\[
\iff \frac{b}{2R} = \frac{\pi}{2} - \frac{c}{R}
\]

\[
\iff R = \frac{b + 2c}{\pi}.
\]

It follows that

\[
A = \frac{1}{bR} = \frac{\pi}{b(b + 2c)};
\]

hence, the continuity condition stated in expressions (4) and (5) determines the centre of the circle, \((c_1, c_2)\).

For the case shown in Figure 1, the numerical values are \( A = 3.1661 \times 10^{-3} \text{ m}^{-2}, \quad R = 23.3958 \text{ m}, \quad c_1 = 25.7313 \text{ m} \) and \( c_2 = 23.7194 \text{ m} \). The complete track — with its centre at the origin, \((0,0)\) — is shown in Figure 2. The corresponding curvature is shown in Figure 3. Note that the curvature transitions linearly from the constant value of straight, \( \kappa = 0 \), to the constant value of the circular arc, \( \kappa = 1/R \).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Black line of 250-metre track}
\end{figure}
2.2 Track-inclination angle

There are many possibilities to model the track inclination angle. We choose a trigonometric formula in terms of arclength, which is a good analogy of an actual 250-metre velodrome. The minimum inclination of 13° corresponds to the midpoint of the straight, and the maximum of 44° to the apex of the circular arc. For a track of length $S$,

$$\theta(s) = 28.5 - 15.5 \cos \left( \frac{4\pi}{S} s \right);$$  \hspace{1cm} (7)

$s = 0$ refers to the midpoint of the lower straight, in Figure 2, and the track is oriented in the counterclockwise direction. Figure 4 shows this inclination for $S = 250 \text{ m}$.

3 Instantaneous power

A mathematical model to account for the power required to propel a bicycle is based on (e.g., Danek et al., 2020a)

$$P = F V,$$

where $F$ stands for the magnitude of forces opposing the motion and $V$ for speed. Herein, we model the rider as undergoing instantaneous circular motion, in rotational equilibrium about the line of contact of the tires with the ground. Following Slawinski et al. (2020, Section 2), in accordance with
Figure 5, along the black line of a velodrome, in windless conditions,

\[
P = \frac{1}{1 - \lambda} \left\{ \frac{F_g}{m} (\sin \theta \tan \vartheta + \cos \theta) \cos \theta + C_{sr} \left| \frac{F_g}{m g} \frac{\sin(\theta - \vartheta)}{\cos \vartheta} \right| \sin \theta \right\} v \quad (8b)
\]

\[+ \frac{1}{2} C_d A \rho V^3 \Bigg\}, \quad (8c)
\]

where \( m \) is the mass of the cyclist and the bicycle, \( g \) is the acceleration due to gravity, \( \theta \) is the track-inclination angle, \( \vartheta \) is the bicycle-cyclist lean angle, \( C_{rr} \) is the rolling-resistance coefficient, \( C_{sr} \) is the coefficient of the lateral friction, \( C_d A \) is the air-resistance coefficient, \( \rho \) is the air density, \( \lambda \) is the drivetrain-resistance coefficient. Herein, \( v \) is the speed at which the contact point of the rotating wheels moves along the track, which is equivalent to the black-line speed (Dupek et al., 2020a, Appendix B), and \( V \) is the centre-of-mass speed. Since the lateral friction is a dissipative force, it does negative work, and the work done against it—as well as the power—are positive. For this reason, in expression (8b), we consider the magnitude, \( | | \).

\[\text{Figure 5: Force diagram}\]

In expression (8), we assume the steadiness of effort, which—following an initial acceleration—is consistent with a steady pace of an individual pursuit. Formally, this assumption corresponds to setting the acceleration, \( a \), to zero in Slawinski et al. (2020, expression (1)). Herein, the acceleration refers to the change of the centre-of-mass speed. This speed is nearly constant if the power is constant, which can be viewed as a quantification of the cyclist’s effort. In other words, the force—and hence the power required to accelerate the bicycle-cyclist system—is associated mainly with the change of the centre-of-mass speed, not with the change of the black-line speed.

To gain an insight into expression (8), let us consider a few special cases. If \( \theta = \vartheta = 0 \),

\[P = \frac{C_{rr} m g + \frac{1}{2} C_d A \rho V^2}{1 - \lambda} V, \quad (9)\]

where—as expected for a flat, straight road—\( v \equiv V \). Also, on a velodrome, along the straights, \( \vartheta = 0 \) and expression (8b) becomes

\[(C_{rr} m g \cos^2 \theta + C_{sr} m g \sin^2 \theta) V.\]
If, along the curves, $\vartheta = \theta$, the second summand of expression (8b) is zero, as expected.

Let us return to expression (8). Therein, $\theta$ is given by expression (7). The lean angle is (Slawinski et al., 2020, Appendix A)

$$\vartheta = \arctan \frac{V^2}{gr_{\text{CoM}}}, \quad (10)$$

where $r_{\text{CoM}}$ is the centre-of-mass radius, and — along the curves, at any instant — the centre-of-mass speed is

$$V = v \left( \frac{r_{\text{CoM}}}{R} \right) = v \left( 1 - \frac{h \sin \vartheta}{R} \right), \quad (11)$$

where $R$ is the radius discussed in Section 2.1 and $h$ is the centre-of-mass height. Along the straights, the black-line speed is equivalent to the centre-of-mass speed, $v = V$. As expected, $V = v$ if $h = 0$, $\vartheta = 0$ or $R = \infty$.

Invoking expressions (10) and (11), we neglect the vertical variation of the centre of mass and, hence, assume that the centre-of-mass trajectory is contained in a horizontal plane, where — in accordance with the track geometry — this plane is parallel to the plane that contains the black line. Accounting for the vertical motion of the centre of mass would mean allowing for a nonhorizontal centripetal force and including the work done in raising the centre of mass.

### 4 Numerical examples

#### 4.1 Model-parameter values

For expressions (8), (10) and (11), we consider a velodrome discussed in Section 2, to let $R = 23.3958$ m. For the bicycle-cyclist system, we assume, $h = 1.2$ m, $m = 84$ kg, $C_{dA} = 0.2$ m$^2$, $C_{rr} = 0.002$, $C_{sr} = 0.003$ and $\lambda = 0.02$. For the external conditions, $g = 9.81$ m/s$^2$ and $\rho = 1.225$ kg/m$^3$.

#### 4.2 Constant cadence

Let the black-line speed be constant, $v = 16.7$ m/s, which is tantamount to the constancy of cadence. As discussed in Section 3, the assumption of a constant black-line speed means neglecting the acceleration of the centre of mass.

The lean angle and the centre-of-mass speed, as functions of distance — obtained by numerically and simultaneously solving equations (10) and (11), at each point of a discretized model of the track — are shown in Figures 6 and 7, respectively. The average centre-of-mass speed, per lap is $\overline{V} = 16.3329$ m/s. Changes of $V$, shown in Figure 7, result from the lean angle. Along the straights, $\vartheta = 0 \implies V = v$. Along the curves, since $\vartheta \neq 0$, the centre-of-mass travels along a shorter path; hence, $V < v$. Thus, assuming a constant black-line speed implies a variable centre-of-mass speed and, hence, an acceleration and deceleration, even though $dV/dt$, where $t$ stands for time, is not included explicitly in expression (8). Examining Figure 7, we conclude that $dV/dt \neq 0$ along the transition curves only.

The power — obtained by evaluating expression (8), at each point along the track — is shown in Figure 8. The average power, per lap, is $\overline{P} = 580.5941$ W. Since the black-line speed is constant, this is both the arclength average and the temporal average.
Examining Figure 8, we see the decrease of power required to maintain the same black-line speed along the curve. This is due to both the decrease of the centre-of-mass speed, which results in a smaller value of term (8c), and the decrease of a difference between the track-inclination angle and the lean angle, shown in Figure 9, which results in a smaller value of the second summand of term (8b).

The argument presented in the previous paragraph leads to the following conjecture. The most efficient track is circular with \( \theta = \vartheta \), which would correspond to the dashed line in Figure 9. However, this is not possible, since — according to the regulations of the Union Cycliste Internationale — the inner edge of the track shall consist of two curves connected by two parallel straight lines. Hence, the optimization is constrained by the length of the straights.

Examining Figure 10, where — in accordance with expression (8) — we distinguish among the power used to overcome the air resistance, the rolling resistance and the lateral friction, we can quantify their effects. The first has the most effect; the last has the least effect, and is zero at points for which \( \theta = \vartheta \), which corresponds to the zero crossings in Figure 9.

Let us comment on potential simplifications of a model. If we assume a straight flat course — which is tantamount to neglecting the lean and inclination angles — we obtain, following expression (9), \( \mathcal{P} \approx 610 \text{ W} \). If we consider an oval track but ignore the transitions and assume that the straights are flat and the semicircular segments, whose radius is 23 m, have a constant inclination of 43°, we obtain (Slawinski et al., 2020, expression (13)) \( \mathcal{P} \approx 563 \text{ W} \). In both cases, there is a significant discrepancy with the power obtained from the model discussed herein, \( \mathcal{P} = 573.6080 \text{ W} \).

To conclude this section, let us calculate the work per lap corresponding to the model discussed herein. The work performed during a time interval, \( t_2 - t_1 \), is

\[
W = \int_{t_1}^{t_2} P \, dt = \frac{1}{v} \int_{s_1}^{s_2} P \, v \, ds,
\]

where the black-line speed, \( v \), is constant and, hence, \( ds \) is an arclength distance along the black
Considering the average power per lap, we write
\[ W = \frac{S}{v} \int_0^S P \, ds = \mathcal{P} t_\Theta. \]

Given \( \mathcal{P} = 580.5941 \text{ W} \) and \( t_\Theta = 14.9701 \text{ s} \), we obtain \( W = 8691.5284 \text{ J} \).

### 4.3 Constant power

Let us solve numerically the system of nonlinear equations given by expressions (8), (10) and (11), to find the lean angle as well as both speeds, \( v \) and \( V \), at each point of a discretized model of the track, under the assumption of constant power. In accordance with a discussion in Section 3, such an assumption is more consistent with the steadiness of effort than the assumption of a constant cadence examined in Section 4.2.

As in Section 4.2, we let \( R = 23.3958 \text{ m} \), \( h = 1.2 \text{ m} \), \( m = 84 \text{ kg} \), \( C_d A = 0.2 \text{ m}^2 \), \( C_{rr} = 0.002 \), \( C_{sr} = 0.003 \), \( \lambda = 0.02 \), \( g = 9.81 \text{ m/s}^2 \) and \( \rho = 1.225 \text{ kg/m}^3 \). However, in contrast to Section 4.2, we allow the black-line speed to vary, and set the power to be the average obtained in that section, \( P = 580.5941 \text{ W} \).

Stating expression (11), as
\[ v = V \frac{R}{R - h \sin \theta}, \]
we write expression (8) as

\[
P = \frac{V}{1 - \lambda} \left\{ C_{rr} m g (\sin \theta \tan \vartheta + \cos \theta) \cos \theta + C_{sr} \left| \frac{m g \sin(\theta - \vartheta)}{\cos \vartheta} \right| \sin \theta \right\} \frac{R}{R - h \sin \vartheta} + \frac{1}{2} C_d A \rho V^2,
\]

where, in agreement with expression (10),

\[
\vartheta = \arctan \frac{V^2}{g (R - h \sin \vartheta)},
\]

which — given \( g \), \( R \) and \( h \) — can be solved for \( V \) as a function of \( \vartheta \). Inserting that solution in expression (12), we obtain an equation whose only unknown is \( \vartheta \).

The difference of the lean angle — between the case of a constant cadence and a constant power is so small that there is no need to plot it; Figure 6 illustrates it accurately. The same is true for the difference between the track-inclination angle and the lean angle, illustrated in Figure 9, as well as for the dominant effect of the air resistance, illustrated in Figure 10.

The resulting values of \( V \) are shown in Figure 11. As expected, in view of the dominant effect of the air resistance, a constancy of \( P \) entails small variations in \( V \). In comparison to the case discussed in Section 4.2, the case in question entails lesser accelerations and decelerations of the centre of mass — note the difference of vertical scale between Figures 7 and 11 — but the changes of speed are not limited to the transition curves. Even though such changes are not included explicitly in expression (8), a portion of the given power may be accounted for by \( m V \, dV/dt \), which is associated with accelerations and decelerations. The amount of this portion can be estimated a posteriori.
Since

\[ mV \frac{dV}{dt} = \frac{d}{dt} \left( \frac{1}{2} m V^2 \right), \]

the time integral of the power used for acceleration of the centre of mass is the change of its kinetic energy. Therefore, to include the effect of accelerations, per lap, we need to add the increases in kinetic energy. This is an estimate of the error committed by neglecting accelerations in expression (8) to be quantified, for the constant-cadence and constant-power cases, in Section 5.

The values of \( v \), in accordance with expression (11), are shown in Figure 12. The averages are \( \bar{V} = 16.3316 \text{ m/s} \) and \( \bar{v} = 16.7071 \text{ m/s} \). These averages are similar to the case of the constant black-line speed averages. Notably, maintaining a constant cadence or a constant power results in nearly the same laptime, namely, 14.9701 s and 14.9670 s, respectively.

To conclude this section, let us calculate the corresponding work per lap. The work performed during a time interval, \( t_2 - t_1 \), is

\[ W = \int_{t_1}^{t_2} P \, dt = P \int_{t_1}^{t_2} dt = P (t_2 - t_1) = P \bar{t}, \]

where, for the second equality sign, we use the constancy of \( P \); also, we let the time interval to be a laptime. Thus, given \( \bar{P} = 580.5941 \text{ W} \) and \( \bar{t} = 14.9670 \text{ s} \), we obtain \( W = 8689.7680 \text{ J} \).

![Figure 11: Centre-of-mass speed, \( V \), as a function of the black-line distance, \( s \), for constant power](image1.png)

![Figure 12: Black-line speed, \( v \), as a function of the black-line distance, \( s \), for constant power](image2.png)

### 5 Change of kinetic and potential energy along curves

#### 5.1 Formulation

In Section 4, we calculate the power expended and work done by the cyclist against dissipative forces, namely, rolling friction, lateral friction and air resistance, while neglecting changes in the
cyclist’s kinetic and potential energy. However, even assuming a constant black-line speed together with a black line contained within a horizontal plane, the speed and height of the centre of mass change along the curved segments of the track, resulting in changes in kinetic and potential energy. In this section, we calculate the work the cyclist performs to effect these changes. In the case of a constant black-line speed, discussed in Section 4.2, the work done to increase the system’s mechanical energy can be simply added to the work done against the dissipative forces to find the total work done over a single lap. In the case of constant power, discussed in Section 4.3, however, the work done to increase the system’s mechanical energy cannot be so added, but only estimated approximately. This is because, strictly speaking, including that work contradicts the assumption of constant power. To include — in a rigorous manner — the work done to increase mechanical energy in the constant-power case would require modifying the model for instantaneous power, stated in expression (8). Still, we can use an a posteriori calculation to estimate the relative importance of increasing mechanical energy in that case.

Consider a cyclist following a track whose curvature is $R_1$, with a centre-of-mass speed, $V_1$. We wish to determine the work the cyclist must perform to change the kinetic and potential energy when the radius of curvature and speed change to new values, $R_2$ and $V_2$, respectively. Let us first determine the work required to change the kinetic energy.

Neglecting the kinetic energy of the rotating wheels, the kinetic energy of the system is due to the translational motion of the centre of mass,

$$K = \frac{1}{2} m V^2.$$

To find the work done to change kinetic energy, it is important to note that the cyclist-bicycle system is not purely mechanical, in the sense that in addition to kinetic and potential energy the system also possesses internal energy in the form of the cyclist’s chemical energy stores. When the cyclist speeds up, internal energy is converted into kinetic. However, the converse is not true: when the cyclist slows down, kinetic energy is not converted into internal. It follows that, to determine the work the cyclist does to change kinetic energy, we should consider only the increases. Then, if the centre-of-mass speed increases monotonically from $V_1$ to $V_2$, the work done is simply the increase in kinetic energy,

$$\Delta K = \frac{1}{2} m (V_2^2 - V_1^2).$$

The stipulation of a monotonic increase is needed due to the non-mechanical nature of the system, with the cyclist doing positive work — with an associated decrease in internal energy — when speeding up, but not doing negative work — with an associated increase in internal energy — when slowing down. Put another way, the cyclist’s work to increase speed from $V_1$ to $V_2$ depends not only on the initial and final speeds but also on the intermediate speeds.

Now, let us determine the work required to change the potential energy,

$$U = m g h \cos \vartheta.$$

The lean angle, $\vartheta$, is determined by assuming that, at all times, the system is in rotational equilibrium about the line of contact of the tires with the ground; this assumption yields the implicit condition on $\vartheta$, stated in expression (10). Since the lean angle depends on the centre-of-mass speed, $V$, as well as the radius of curvature of the track, $R$, and therefore the height of the centre of mass, change if either $V$ or $R$ changes. The work done is simply the increase in potential energy $U$ resulting from a monotonic decrease in the lean angle from $\vartheta_1$ to $\vartheta_2$,

$$\Delta U = m g h (\cos \vartheta_2 - \cos \vartheta_1),$$

as can be seen in Figure 6. The same considerations due to the nonmechanical nature of the system apply to the work done to change potential energy as to that done to change kinetic energy. The
cyclist does positive work — with an associated decrease in internal energy — when straightening up, but not negative work — with an associated increase in internal energy — when leaning into the turn. And as before, the cyclist’s work to decrease in the lean angle from \( \vartheta_1 \) to \( \vartheta_2 \) depends not only on the initial and final lean angles but also on the intermediate angles.

Note that the changes in potential energy due to a changing lean angle are different in character from the changes associated with hill climbs, stated as the first term in the numerator of expression (1) of Slawinski et al. (2020). When climbing a hill, a cyclist does work to increase potential energy. However, when descending the hill, at least some of that potential energy is converted into kinetic energy of forward motion. This is not the case with the work the cyclist does to straighten up. When the cyclist leans again, potential energy is not converted into kinetic energy of forward motion.

5.2 Constant cadence

In the example considered in Section 4.2, assuming a constant cadence, which is equivalent to a constant black-line speed, the sum of increases of the centre-of-mass speed squared over one lap, shown by thick black lines in Figure 13, is \( \sum \Delta V^2 = 42.6148 \text{ m}^2/\text{s}^2 \). This results an increase in kinetic energy of \( \Delta K = \frac{1}{2} m \sum \Delta V^2 = 1789.8216 \text{ J} \).

In the same example, the sum of increases of the centre-of-mass height over one lap is \( \sum h \Delta \cos \vartheta = 0.8390 \text{ m} \), which results an increase in potential energy of \( \Delta U = mgh \sum \Delta \cos \vartheta = 691.3486 \text{ J} \).

Adding these to the work done against dissipative forces calculated in Section 4.2, we find that the total work done by the cyclist over one lap is \( W = 8691.5284 + 1789.8216 + 691.3486 = 11172.6986 \text{ J} \). Therefore, for a lap done at constant cadence, approximately 78% of the work goes into overcoming dissipative forces, and 22% into changing the mechanical energy of the centre of mass. In view of the lap time, the corresponding average power is 746.3343 W.

![Figure 13: Increases of \( V^2 \), as a function of the black-line distance, \( s \), for constant cadence](image)

5.3 Constant power

In the example considered in Section 4.3, we assume a constant instantaneous power needed to overcome dissipative forces, without including changes in mechanical energy. The sum of increases of the centre-of-mass speed squared over one lap, shown by thick black lines in Figure 14, is \( \sum \Delta V^2 = 20.9662 \text{ m}^2/\text{s}^2 \), which results in an increase in kinetic energy of \( \Delta K = \frac{1}{2} m \sum \Delta V^2 = 880.5804 \text{ J} \).
In the same example, the sum of increases of the centre-of-mass height over one lap is \( \sum h \Delta \cos \vartheta = 0.8698 \text{ m} \), which results in an increase in potential energy of

\[
\Delta U = mgh \sum \Delta \cos \vartheta = 716.7249 \text{ J}.
\]

Adding these to the work done against dissipative forces calculated in Section 4.3, we find that the total work done by the cyclist over one lap is \( W = 8689.7680 + 880.5804 + 716.7249 = 10287.0733 \text{ J} \). Therefore, in the present case approximately 84% of the work goes into overcoming dissipative forces, and 16% into changing the mechanical energy of the centre of mass. In view of the lap time, the corresponding average power is 687.3170 W. However, as noted above, including the work done to change the mechanical energy of the centre of mass means that, strictly speaking, the lap is not completed at constant power — only the power needed to overcome dissipative forces is constant.

6 Discussion and conclusions

The mathematical model presented in this article offers the basis for a quantitative study of individual pursuits. The model can be used to predict or retrodict the lap times, from the measurements of power, or to estimate the power from the recorded times. Comparisons of such predictions or retrodictions with the measurements of time, speed, cadence and power along the track offer an insight into the empirical adequacy of a model. Given a satisfactory adequacy and appropriate measurements, the model lends itself to estimating the rolling-resistance, lateral-friction, air-resistance and drivetrain-resistance coefficients.

Presented results allow us to comment on aspects of the velodrome design. As illustrated in Figures 6–8, 11, 12, the transitions — between the straights and the circular arcs — are not smooth for the lean angles, speeds and powers. It might suggest that a commonly used Euler spiral, illustrated in Figure 3, is not the optimal transition curve. Perhaps, the choice of a transition curve should consider such phenomena as the jolt, which is the temporal rate of change of acceleration. It might also suggest the necessity for the lengthening of the transition curve.

Furthermore, an optimal velodrome design would strive to minimize the distance between the zero line and the curve in Figure 9, which is tantamount to optimizing the track inclination to accommodate the lean angle of a rider. The smaller the distance, the smaller the second summand in term (8b). As the distance tends to zero, so does the summand.

These considerations are to be explored more thoroughly in a future work. Also, the inclusion of change of kinetic and potential energy within the model for instantaneous power is a refinement to be considered.
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Conflict of Interest

The authors declare that they have no conflict of interest.

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