Geology of Symmetric Grounds

Toshimichi Usuba (薄葉季路)

Waseda University (早稲田大学)

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Abstract

- We study set-theoretic geology of symmetric grounds, this is work in progress.
Definability of grounds

Throughout this talk, forcing means a set forcing.
A ground of $V$ is a ground model of $V$.

Fact (Laver, Woodin, Fuchs-Hamkins-Reitz)

There is a first-order formula $\varphi(x, y)$ of set theory such that:

1. For each set $r$, the class $W_r = \{x \mid \varphi(x, r)\}$ is a ground of $V$ with $r \in W_r$ ($W_r = V$ is possible).
2. For every transitive model $M \subseteq V$ of ZFC, if $M$ is a ground of $V$, then there is $r$ with $M = W_r$.

The formula $\varphi$ defines all grounds uniformly.

Remark

The choice of the formula $\varphi$ does not depend on $V$: In any models of ZFC, $\varphi$ defines its grounds by the same way.
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**Remark**

The choice of the formula $\varphi$ does not depend on $V$: In any models of ZFC, $\varphi$ defines its grounds by the same way.
The uniform definability allows us to study the structure of the collection of grounds \( \{ W_r \mid r \in V \} \) in ZFC: e.g.,

- One can define (in ZFC) the intersection of two grounds.
- One can ask (in ZFC) whether \( \forall r \exists s (W_s \subset W_r) \)?
- (In ZFC) What is the intersection of all grounds?

This study is now called *set-theoretic geology*. 
Symmetric models

- **Symmetric model** is a powerful tool for constructing choiceless models.
- Symmetric extensions can be seen as a generalization of generic extensions.

**Definition**

A transitive model $W \subseteq V$ is a symmetric ground if $V$ is a symmetric extension of $W$.

- Every ground is a symmetric ground.
- We want to know the global structure of the symmetric grounds using techniques of set-theoretic geology.
- Symmetric extensions and grounds could be choiceless models.
- In the standard geology, the universe $V$ and all grounds are supposed to satisfy the Axiom of Choice (AC), and AC plays an important role.
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Conventions

- From now on, our base theory is ZF unless otherwise specified.
- A model of ZF(C) means a transitive model of ZF(C) containing all ordinals.
- A ground may be a choiceless model.
Basic properties of symmetric grounds 1

Observation
If $N$ satisfies AC, then every symmetric ground is a ground of $N$.

Observation
If $M$ is a symmetric ground of $N$ and $N$ is of $W$, then $M$ is a symmetric ground of $W$. 
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Fact (Grigorieff(?))

If $M \subseteq N \subseteq M[G]$ are models of ZFC and $M[G]$ is a generic extension of $M$, then $M$ is a ground of $N$, and $N$ is of $M[G]$.

Choiceless version of this fact does not hold: Typical example is the Bristol model.

Observation

Suppose $M \subseteq N \subseteq W$, and $M$ is a symmetric ground of $W$.

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T. Usuba (Waseda Univ.) Geology of Symmetric Grounds Nov. 18. 2019 8 / 20
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In ZF, it is still open if all grounds of $V$ are uniformly definable or not. On the other hand, under a certain assumption, we can define all grounds uniformly.

**Definition**

An uncountable cardinal $\kappa$ is a **Löwenheim-Skolem cardinal** if for every $\alpha \geq \kappa$, $\gamma < \kappa$, and $x \in V_\alpha$, there is $\beta \geq \alpha$ and $X \prec V_\beta$ such that:

1. $x \in X$ and $V_\gamma \subseteq X$.
2. $V_\gamma(X \cap V_\alpha) \subseteq X$.
3. The transitive collapse of $X$ belongs to $V_\kappa$.  

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### All grounds may be uniformly definable

**Fact (U.)**

1. If $\kappa$ is supercompact, then $\kappa$ is LS and a limit of LS cardinals.
2. If AC is forceable, then there are proper class many LS cardinals.
3. The statement “there are proper class many LS cardinals” is forcing absolute.
4. If there are proper class many LS cardinals, then all grounds are uniformly definable.
5. The existence of LS cardinals is independent from ZF.

Hence if

- there are many large cardinals, or
- AC is forceable,

then all grounds are uniformly definable. However, this does not immediately yield the uniform definability of symmetric grounds.
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Characterization of symmetric grounds

Definition

Let $M$ be a model of ZF.

1. For a set $X$, let $M(X)$ be the minimal model of ZF containing $M \cup \{X\}$.

2. $\text{HOD}_M$ is the class of all hereditarily definable sets with parameters from the ordinals and $M$. $\text{HOD}_M$ is a model of ZF containing $M$.

Fact (Grigorieff)

Let $M \subseteq N$ be models of ZF, and $M[G]$ a generic extension of $M$ via the $\mathbb{P}$. If $M \subseteq N \subseteq M[G]$, then $N$ is a symmetric extension of $M$ via $\mathbb{P}$ if and only if there is a set $X \in M[G]$ such that $N = (\text{HOD}_{M(X)})^{M[G]}$. 
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All symmetric grounds may be uniformly definable

**Lemma (essentially Grigorieff)**

For models $M \subseteq N$ of ZF, the following are equivalent:

1. $M$ is a symmetric ground of $N$.
2. There is a generic extension $M[G]$ of $M$ such that $N$ is a ground of $M[G]$.
3. There is a generic extension $M[G]$ of $M$ such that $N \subseteq M[G]$ and $N$ is of the form $M(X)$ for some $X \in N$.

**Theorem (U.)**

If all grounds are uniformly definable (e.g. if there are proper class many Löwenheim-Skolem cardinals), then all symmetric grounds are uniformly definable as well.

If the formula $\varphi(x, y)$ defines all grounds uniformly, then each symmetric ground is of the form $\{x \in V \mid \models_{\text{Col}(\alpha)} \varphi(x, r)\}$ for some $r$ and large $\alpha$. 
All symmetric grounds may be uniformly definable

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AC is forceable

Fact (Blass)
Suppose $M$ satisfies AC. Then AC is forceable over any symmetric extensions of $M$.

Corollary
If $V$ has a symmetric ground satisfying AC, then AC is forceable over $V$. In particular $V$ has a proper class of LS cardinals.

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Mantles

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1. The **mantle** is the intersection of all grounds of $\mathcal{V}$.
2. The **generic mantle** is the intersection of all grounds of all generic extensions of $\mathcal{V}$.
3. The **symmetric mantle** is the intersection of all symmetric grounds.

Assuming that there are proper class many Löwenheim-Skolem cardinals, these are definable transitive classes. Moreover the generic mantle is a transitive model of ZF. It is unknown that the mantle is a model of ZF.

Theorem (U.)
Suppose there are proper class many Löwenheim-Skolem cardinals. Then the symmetric mantle coincides with the generic mantle. In particular the symmetric mantle is a model of ZF.
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The downward directedness of grounds

Fact (U., In ZFC)

For every two grounds $W_0$ and $W_1$, if $W_0$ and $W_1$ satisfy AC, then $W_0$ and $W_1$ have a common ground which satisfies AC.

In ZF, the DDG can fail:

Fact (U.)

It is consistent that AC is forceable, but there are two grounds $W_0$ and $W_1$ such that $W_0$ and $W_1$ have no common ground.

- Assuming $V = L$, let $V[G]$ be a generic extension of $V$ by adding $\omega_1$-many cohen reals. Then $L(\mathbb{R})^{V[G]}$ is a model in which DDG fails.
- On the other hand, in this model, every two grounds has a common symmetric ground.
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Suppose that AC is forceable. Then for every two symmetric grounds $W_0$ and $W_1$, there is a common symmetric ground of $W_0$ and $W_1$.

- We can find a generic extension $V[G]$ of $V$ such that AC holds in $V[G]$, and $W_0$ and $W_1$ are grounds of $V[G]$.

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If AC holds, then for every ground $W$ of $V$, there is a ground $M$ of $V$ such that $M$ satisfies AC, $M \subseteq W$, and $W$ is of the form $M(X)$ for some $X$.

- Hence we can find two grounds $W'_0$ and $W'_1$ of $V[G]$ and sets $X_0, X_1$ such that $W'_0(X_0) = W_0$ and $W'_1(X_1) = W_1$.
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Suppose that AC is forceable. Then for every two symmetric grounds $W_0$ and $W_1$, there is a common symmetric ground of $W_0$ and $W_1$.

- We can find a generic extension $V[G]$ of $V$ such that AC holds in $V[G]$, and $W_0$ and $W_1$ are grounds of $V[G]$.

**Fact (U.)**

If AC holds, then for every ground $W$ of $V$, there is a ground $M$ of $V$ such that $M$ satisfies AC, $M \subseteq W$, and $W$ is of the form $M(X)$ for some $X$.

- Hence we can find two grounds $W'_0$ and $W'_1$ of $V[G]$ and sets $X_0$, $X_1$ such that $W'_0(X_0) = W_0$ and $W'_1(X_1) = W_1$.
- By DDG in ZFC, there is a common ground $W$ of $W'_0$ and $W'_1$.
- One can check that $W$ is a common symmetric ground of $W_0$ and $W_1$. 

T. Usuba (Waseda Univ.)

Geology of Symmetric Grounds

Nov. 18. 2019
Question

1. Without any assumptions, are all symmetric grounds uniformly definable?
2. Without any assumptions, are all grounds uniformly definable?
3. Without any assumptions, are all symmetric grounds downward directed?
4. Is the mantle a model of ZF?
5. Is it possible that the symmetric mantle is different from the mantle?
6. Is it possible that the symmetric mantle does not satisfy AC?
7. What is the geology of some special model? For instance:
   1. Does the Bristol model have a proper (symmetric) ground?
   2. Are all grounds and symmetric grounds of the Bristol model uniformly definable?
   3. What are the symmetric mantle and the mantle of the Bristol model?
Case Study: Cohen forcing extension of $L$

Let $c$ be a Cohen real over $L$. Since $L[c]$ satisfies AC, there are proper class many LS cardinals, and all grounds and symmetric grounds are uniformly definable. Obviously $L$ is the mantle and the symmetric mantle of $L[c]$, and $L[c]$ is a generic extension of its mantle.

**Fact (Grigorieff)**

Let $M, N$ be models of ZF, $M[G]$ a generic extension of $M$, and suppose $M \subseteq N \subseteq M[G]$. Then $N$ is a ground of $M[G]$ if and only if $N = M(X)$ for some set $X \in M[G]$.

- By Grigorieff’s result, for each set $X \in L[c]$, $L(X)$ is a ground of $L[c]$.
- Moreover, every symmetric ground of $L[c]$ is of the form $L(X)$ for some set $X \in L[c]$. 

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Case Study: Cohen forcing extension of $L$

**Fact**

1. $L[c]$ has $2^\omega$ many grounds satisfying AC.
2. In ZFC, $V$ has the minimum ground if and only if $V$ has set many grounds satisfying AC.

**Proposition (Karagila)**

$L[c]$ has proper class many grounds.

- Let $M$ be the Bristol model. AC is never forceable over $M$.
- For each $\alpha \in \text{ON}$, $L(M_\alpha)$ is a ground of $L[c]$, and AC is forceable over $L(M_\alpha)$.
- Hence $\{L(M_\alpha) \mid \alpha \in \text{ON}\}$ is a proper class collection of grounds; otherwise, $M$ is of the form $L(M_\alpha)$ for some $\alpha$, so AC can be forceable over $M$. 
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Thank you for your attention!