AN EXTENSION OF THE CAYLEY-HAMILTON THEOREM TO THE CASE OF SUPERMATRICES

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ABSTRACT

Starting from the expression for the superdeterminant of $(xI - M)$, where $M$ is an arbitrary supermatrix, we propose a definition for the corresponding characteristic polynomial and we prove that each supermatrix satisfies its characteristic equation. Depending upon the factorization properties of the basic polynomials whose ratio defines the above mentioned superdeterminant we are able to construct polynomials of lower degree which are also shown to be annihilated by the supermatrix.

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1.- INTRODUCTION

Given any \( n \times n \) real matrix \( M \), its characteristic polynomial is defined by \( P(x) = \text{det}(xI - M) \), where \( I \) denotes the \( n \times n \) identity matrix and \( x \) is a real variable. In general \( P(x) = x^n + \sum_{k=0}^{n-1} c_k x^k \) is a monic polynomial of degree \( n \). The Cayley-Hamilton theorem asserts that \( P(x = M) = 0 \). That is to say, if we substitute in \( P(x) \) the real variable \( x \) by the matrix \( M \) in all the powers \( x^k(k \neq 0) \), and set \( x^0 = I \), we obtain the matrix zero as the result. The coefficients \( c_k(k \neq 0) \) can be written in terms of \( \text{Tr}(M), \text{Tr}(M^2), \ldots, \text{Tr}(M^{n-1}) \) together with their powers and \( c_0 = \text{det}(M) \). This theorem has recently found interesting applications in 2+1 dimensional Chern-Simons (CS) theories [1]. Pure CS theories are of topological nature and the fundamental degrees of freedom are the traces of group elements constructed as the holonomies (or Wilson lines, or integrated connections) of the gauge connection around oriented closed curves on the manifold. The observables are the expectation values of the Wilson lines which turned out to be realized as the various knot polynomials known to mathematicians [2]. Since CS theories are also exactly soluble and possess a finite number of degrees of freedom [3], another aspect of interest is the reduction of the initially infinite-dimensional phase space to the subspace of the true degrees of freedom. The Cayley-Hamilton theorem has played an important role in the construction of the so called skein relations [4], which are relevant to the calculation of expectation values, and also in the process of reduction of the phase space. To illustrate the basic ideas related to this last point let us consider the simple case of two matrices \( M_1 \) and \( M_2 \) which belong to \( SL(2, R) \). In this case the characteristic polinomial is \( P(x) = x^2 - \text{Tr}(M_1)x + 1 \) and we have the Cayley-Hamilton matrix identity

\[
(M_1)^2 - \text{Tr}(M_1)M_1 + I = 0.
\] (1.1)
Multiplying Eq. (1.1) by $M_2 M_1^{-1}$ and taking the trace we obtain the following non-linear constraint among the traces

$$Tr(M_2 M_1^{-1}) + Tr(M_1 M_2) = Tr(M_1) Tr(M_2).$$  \hspace{1cm} (1.2)

The expression (1.2) finds a very useful application in the discussion of the reduced phase space of the de Sitter gravity in $2 + 1$ dimensions, which is equivalent to the Chern-Simons theory of the group $SO(2, 2)$ [3]. This theory can be more easily described in terms of two copies of the group $SL(2, R)$, which is the spinorial group of $SO(2, 2)$. The gauge invariant degrees of freedom associated to one genus of an arbitrary genus $g$ two-dimensional surface turn out to be traces of any product of powers of two $SL(2, R)$ matrices $M_1$ and $M_2$, which correspond to the holonomies (or integrated connections) of the two basic homotopically distinct trajectories on one genus. Nevertheless, because Chern-Simons theories have a finite number of degrees of freedom, one should be able to reduce this infinite set of traces to a finite one. This task can in fact be accomplished by virtue of the relation (1.2). In other words, $Tr(M_1^{p_1} M_2^{q_1} M_1^{p_2} M_2^{q_2} \cdots M_1^{p_n} M_2^{q_n} \cdots)$, for any $p_i, q_i$ in $Z$, can be shown to be reducible and to be expressed as a function of three traces only: $Tr(M_1), Tr(M_2)$ and $Tr(M_1 M_2)$ [5]. A similar reduction can be performed in the case of $2 + 1$ super de Sitter gravity, which is the Chern-Simons theory of the supergroup $Osp(2|1, C)$ [6]. The novelty here is that one is dealing with supermatrices instead of ordinary matrices. In the particular case considered, a Cayley-Hamilton identity for the supermatrices was obtained in an heuristical way and a relation analogous to (1.2) was derived. This allowed to carry out the reduction of the infinite dimensional phase space in the one-genus sector of the theory, this time in terms of five complex supertraces [7]. We observe that the non-linear constraints among the traces that need to be solved in order to accomplish the reduction of the
phase space, of which Eq.(1.2) is an example, are usually obtained starting from the so called Mandelstam identities [8]. The discussion of the relation between the Cayley-Hamilton and the Mandelstam identities, together with the construction of the latter identities in the case of supermatrices is reported in Ref.[9]. It is important to emphasize also that the use of the Mandelstam identities is of fundamental importance in the formulation of arbitrary gauge theories in terms of Wilson loops variables, which constitute an overcomplete set of degrees of freedom [10].

In this paper we discuss the general construction of Cayley-Hamilton type identities for supermatrices. This is an interesting problem in its own, besides possible applications like: (i) the study of the reduced phase space in Chern-Simons theories defined over a supergroup or (ii) the loop space formulation of any supersymmetric gauge theory. In Section 2 we introduce our notation and we propose a definition of the characteristic and null polynomials for supermatrices, starting from the corresponding superdeterminant. In Section 3 we prove the Cayley-Hamilton theorem for the polynomials previously defined.

2.- THE CHARACTERISTIC AND NULL POLYNOMIALS FOR SUPERMATRICES

We consider a Grassmann algebra \( \Lambda = \Lambda_0 \oplus \Lambda_1 \) over the complex numbers \( \mathbb{C} \), where \( \Lambda_0 \) (\( \Lambda_1 \)) is the even (odd) part of \( \Lambda \). Any element \( a \in \Lambda \) is a sum of the body \( \bar{a} \in \mathbb{C} \) plus the nilpotent element \( s(a) \) called the soul. The ring of polynomials over this Grassmann algebra is denoted by \( \Lambda_0[x] \) and consists of all polynomials \( f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \), where \( a_k \) are even elements of the Grassmann algebra. The Grassmann algebra \( \Lambda \) is generated by an infinite number of odd generators \( \xi^A \). Nevertheless, when dealing with an specific supermatrix we consider only superfunctions of the given supermatrix elements. These elements
will have an expansion in terms of the basis \( \{ \xi^A \} \), which is not relevant for our purposes [11].

A \((p + q) \times (p + q)\) supermatrix is a block matrix of the form

\[
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]  

(2.1)

where \( A, B, C \) and \( D \) are \( p \times p, p \times q, q \times p, q \times q \) matrices respectively. The distinguishing feature with respect to an ordinary matrix is that the matrix elements \( M_{RS} \) \( R = (i, \alpha), \ S = (j, \beta) \) are elements of \( \Lambda \) with the property that \( A_{ij} \ (i, j = 1, \cdots p) \) and \( D_{\alpha\beta} \ (\alpha, \beta = 1, \cdots q) \) are even elements, while \( B_{i\alpha} \) and \( C_{\beta j} \) are odd elements of the algebra. In particular this means that such numbers satisfy

\[
B_{i\alpha}B_{j\beta} = -B_{j\beta}B_{i\alpha}, \quad C_{\alpha i}C_{\beta j} = -C_{\beta j}C_{\alpha i}
\]

\[
(2.2)
\]

while \( A_{ij} \) and \( D_{\alpha\beta} \) commute with everything.

Let us recall that the ordinary matrix addition and the ordinary matrix product of two supermatrices is again a supermatrix. Nevertheless, such concepts as the trace and the determinant need to be redefined, because of the odd component piece of the supermatrix.

The basic invariant under similarity transformations for supermatrices is the supertrace, defined by

\[
Str(M) = Tr(A) - Tr(D),
\]

(2.3)

where the trace \((Tr)\) over the even matrices is the standard one. An important property of the above definition is the cyclic identity \( Str(M_1M_2) = Str(M_2M_1) \), for arbitrary supermatrices, which is just a consequence of the relative minus sign in (2.3). The generalization of the determinant, called the superdeterminant \((Sdet)\), is obtained from (2.3) by defining

\[
Sdet(M) = \exp \ Str(lnM),
\]

(2.4)
which leads to the the following equivalent expressions for the superdeterminant [12]

\[ S\text{det}(M) = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}. \]  

(2.5)

All the matrices involved now are even in the Grassmann algebra and the determinant \( (\det) \) has its usual meaning. The superdeterminant inherits the basic property \( S\text{det}(M_1M_2) = S\text{det}(M_2M_1) \) and requires \( \det D \neq 0 \) and \( \det A \neq 0 \) in order to be defined. An explicit demonstration of the equality of the alternative ways (2.5) of calculating \( S\text{det}(M) \) is given in Ref. [13].

In order to proceed we introduce \( a(x) = \det(xI - A) \) and \( d(x) = \det(xI - D) \), which are the characteristic polynomials of the even matrices \( A \) and \( D \).

Starting from the two alternative expressions (2.5) of calculating the superdeterminant we find it convenient to state the following:

**Lemma 2.1** For any \((p + q) \times (p + q)\) supermatrix \( M \), the characteristic function \( h(x) = S\text{det}(xI - M) \) can be written as

\[ h(x) = \frac{\tilde{F}(x)}{\tilde{G}(x)} = \frac{F(x)}{G(x)}, \]  

(2.6)

where the basic polynomials \( \tilde{F}, \tilde{G}, F \) and \( G \) are given by

\[ \tilde{F}(x) = \det(d(x)(xI - A) - B\text{adj}(xI - D)C), \quad \tilde{G}(x) = (d(x))^{p+1}, \]  

(2.7a)

\[ F(x) = (a(x))^{q+1}, \quad G(x) = \det(a(x)(xI - D) - C\text{adj}(xI - A)B). \]  

(2.7b)

**Proof.** The above expressions are directly obtained from Eqs.(2.5) using the relation \((xI - F)^{-1} = [\det(xI - F)]^{-1}\text{adj}(xI - F)\) valid for any even matrix \( F \). Notice that \( \tilde{F} \) is expressed in terms of the determinant of a \( p \times p \) even matrix, while \( G(x) \) is the determinant of a \( q \times q \) even matrix.
In order to motivate the basic idea of our definition for the characteristic polynomial of a supermatrix let us consider the simple case of a block-diagonal supermatrix $M$ (i.e. $B = 0, C = 0$). Here $h(x) = a(x)/d(x)$ and clearly the characteristic polynomial is $P(x) = a(x)d(x)$, which is the product of the numerator and the denominator of the corresponding superdeterminant. In fact we have

$$P(M) = \begin{pmatrix} a(A) & 0 \\ 0 & a(D) \end{pmatrix} \begin{pmatrix} d(A) & 0 \\ 0 & d(D) \end{pmatrix} \equiv 0$$

(2.8)

because $a(A) = 0, d(D) = 0$. In the general case where $h(x)$ is given by Eq.(2.6), the numerator of the superdeterminant is $\tilde{F}$ ($F$) while the denominator is $\tilde{G}$ ($G$), which motivates the following:

**Definition 2.1** For an arbitrary $(p + q) \times (p + q)$ supermatrix $M$ we define the characteristic polynomial

$$\mathcal{P}(x) = \tilde{F}(x)G(x) = F(x)\tilde{G}(x),$$

(2.9)

where the basic polynomials $\tilde{F}, \tilde{G}, F, G$ are given in Eqs.(2.7). For notational simplicity we will not necessarily write explicitly the $x$-dependence on many of the polynomials considered in the sequel.

When $a(x)$ and $d(x)$ have a common factor $f(x)$ in the block-diagonal case,

$$a(x) = f(x)a_1(x), \quad d(x) = f(x)d_1(x),$$

(2.10)

the characteristic polynomial is given by $P(x) = f(x)a_1(x)d_1(x)$, which is a polynomial of lower degree than the product $a(x)d(x)$. Motivated by this fact together with the work of Ref. [14], we have realized that there are some cases in which we can construct null polynomials of lower degree than $\mathcal{P}(x)$, according to the factorization properties of the basic polynomials $\tilde{F}, \tilde{G}, F, G$. At this point it is important
to observe that we do not have a unique factorization theorem for polynomials defined over a Grassmann algebra. This can be seen, for example, from the identity 
\[ x^2 = (x + \zeta \alpha)(x - \zeta \alpha), \]
where \( \alpha \) is an even Grassmann with \( \alpha^2 = 0 \) and \( \zeta \) is an arbitrary complex number. The construction of the null polynomials of lower degree starts from finding the divisors of maximum degree of the pairs \( \tilde{F}, \tilde{G}, (F, G) \) which we denote by \( R(S) \) respectively. This means that one is able to write

\[
\tilde{F} = R\tilde{f}, \quad \tilde{G} = R\tilde{g},
\]

\[
F = Sf, \quad G = Sg,
\]

where all polynomials are monic and also \( \tilde{f}, \tilde{g}, f, g \) are of least degree by construction. They must satisfy

\[
\frac{\tilde{f}}{\tilde{g}} = \frac{f}{g},
\]

because of Eq. (2.6) and the expressions in (2.11) might be not unique. Let us observe that in the case of polynomials over the complex numbers Eq. (2.12) would imply at most \( \tilde{f} = \lambda f, \tilde{g} = \lambda g \) with \( \lambda \) being a constant. Since we are considering polynomials over a Grassmann algebra this is not necessarily true as can be seen again in the above mentioned identity \( x/(x - \zeta \alpha) = (x + \zeta \alpha)/x \), which we have rewritten in a convenient way. The above discussion leads to the following:

**Definition 2.2** Given an arbitrary \((p + q) \times (p + q)\) supermatrix \( M \), with a characteristic function \( h(x) \) such that \( \tilde{F}, \tilde{G} \) have a common factor \( R \) (\( \tilde{F} = R\tilde{f}, \tilde{G} = R\tilde{g} \)) and \( F, G \) have a common factor \( S \) (\( F = Sf, G = Sg \)), where \( \tilde{f}/\tilde{g} = f/g \), we define a null polynomial of \( M \) by

\[
P(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x).
\]

The above polynomial is clearly of lower degree than \( P(x) \), which is just a particular case of the null polynomials (2.13) when \( R = S = 1 \). We will concentrate mostly on Def. (2.2) in the sequel.
3. THE CAYLEY-HAMILTON THEOREM FOR SUPERMATRICES

In this section we prove that the polynomial given in Def.(2.2) does in fact annihilate the supermatrix $M$. The first step of our strategy to prove the Cayley-Hamilton theorem for supermatrices is based on one of the standard methods to prove such theorem for ordinary matrices [15]. We briefly recall such procedure and emphasize that it is independent of the matrix considered being a standard matrix or a supermatrix.

**Lemma 3.1.** Let $M$, $(xI - M)$ and $N(x)$ be $(p+q) \times (p+q)$ supermatrices where $M$ is independent of $x \in \Lambda_0$, with $N(x)$ being a polynomial supermatrix of degree $(n-1)$, $N(x) = N_0 x^{n-1} + N_1 x^{n-2} + \ldots + N_{n-1} x^0$, (where each $N_k (k = 0, \ldots, n-1)$ is a $(p+q) \times (p+q)$ supermatrix independent of $x$) such that

$$(xI - M)N(x) = P(x)I,$$  \hspace{1cm} (3.1)

where $P(x) = p_0 x^n + p_1 x^{n-1} + \ldots + p_n x^0$ is a numerical polynomial of degree $n \in \Lambda_0[x]$, then $P(M) = p_0 M^n + p_1 M^{n-1} + \ldots + p_n I \equiv 0$.

**Proof.** The proof follows by comparing the independent powers of $x$ in Eq. (3.1) and then explicitly computing $P(M)$ [15].

In the standard case the polynomial matrix $N(x)$ is just given by $N(x) = adj(xI - M) = det(xI - M)(xI - M)^{-1}$, and $P(x) = det(xI - M)$. In the case of a supermatrix we do not have an obvious generalization either of the matrix $adj(xI - M)$ or of $det(xI - M)$. Nevertheless, following the analogy as close as possible we define

$$N(x) = P(x)(xI - M)^{-1},$$  \hspace{1cm} (3.2)

where $P(x)$ is the polynomial introduced in Def.(2.2) of the previous section. The challenge now is to prove that $N(x)$, which trivially satisfies the Eq. (3.1), is indeed
a polynomial matrix. In this way we would have proved that \( P(M) = 0 \), according to Lemma 3.1. To this end we consider the following:

**Lemma 3.2.** Let \( M \) and \((xI - M)\) be \((p + q) \times (p + q)\) supermatrices, \( x \in \Lambda_0 \), then

\[
(xI - M)_{ij}^{-1} = -\frac{1}{F} \frac{\partial \tilde{F}}{\partial A_{ji}}, \quad (xI - M)_{i\alpha}^{-1} = \frac{1}{G} \frac{\partial \tilde{F}}{\partial C_{i\alpha}}, \quad (xI - M)_{\alpha j}^{-1} = \frac{1}{F} \frac{\partial \tilde{F}}{\partial B_{\alpha j}}, \quad (xI - M)_{\alpha \beta}^{-1} = -\frac{1}{G} \frac{\partial \tilde{F}}{\partial D_{\alpha \beta}},
\]

where \( A_{ij}, B_{j\alpha}, C_{\alpha j} \) and \( D_{\alpha \beta} \) are the entries of the supermatrix \( M \) defined in Eq. (2.1) and \( \tilde{F}, G \), are the polynomials given in Eqs. (2.7). The derivative with respect to an odd Grassmann number is a left derivative defined such that \( \delta \tilde{F} \equiv \delta B_{j\alpha} \frac{\partial \tilde{F}}{\partial B_{j\alpha}} \).

**Proof.** The first step is to calculate \((xI - M)^{-1}\) in block form, with the results

\[
(xI - M)_{11}^{-1} = ((xI - A) - B(xI - D)^{-1}C)^{-1}, \quad (xI - M)_{12}^{-1} = -(xI - A)^{-1}B((xI - D) - C(xI - A)^{-1}B)^{-1}, \quad (xI - M)_{21}^{-1} = -(xI - D)^{-1}C((xI - A) - B(xI - D)^{-1}C)^{-1}, \quad (xI - M)_{22}^{-1} = ((xI - D) - C(xI - A)^{-1}B)^{-1},
\]

where the subindices 11, 12, 21 and 22 denote the corresponding \( p \times p, p \times q, q \times p, \) and \( q \times q \) blocks. Let us concentrate now in the 11 block. Rewriting all the inverse matrices in Eq.(3.4a) in terms of their adjoints together with the corresponding determinants we obtain

\[
(xI - M)_{11}^{-1} = \frac{d}{F} \text{adj}((xI - A)d - B\text{adj}(xI - D)C).
\]

Using the basic property

\[
\delta \text{det}Q = Tr(\text{adj}Q \delta Q),
\]
valid for any even matrix \( Q \), we calculate the change of \( \tilde{F} \) with respect to \( A_{ij} \), keeping constant all other entries, obtaining
\[
\delta \tilde{F} = -d \left[ \text{adj}((xI - A)d - Badj(xI - D)C)_{ij} \right] \delta A_{ji}, \tag{3.7}
\]
which can be written as
\[
\frac{\partial \tilde{F}}{\partial A_{ji}} = -d \left[ \text{adj}((xI - A)d - Badj(xI - D)C)_{ij} \right]. \tag{3.8}
\]
The comparison of Eq.(3.8) with Eq. (3.5) completes the proof of the first relation in Eq. (3.3a). The corresponding proof for the remaining Eqs. (3.3) is performed following a similar procedure.

We observe that the conditions for the existence of \((xI - M)^{-1}\) are the same as those for the existence of \(S\text{det}(xI - M)\) and they are \(\text{det}(xI - A) \neq 0\) and \(\text{det}(xI - D) \neq 0\). Since \(x\) is a generic even Grassmann variable we will assume that this is always the case. By virtue of these assumptions the term \((xI - A - B(xI - D)^{-1}C)^{-1}\), for example, can always be calculated as \((I - (xI - A)^{-1}B(xI - D^{-1})C)^{-1}(xI - A)^{-1}\). The factor on the left can be thought as a series expansion of the form \(1/(1 - z) = 1 + z + z^2 + \cdots\), with \(z = (xI - A)^{-1}B(xI - D)^{-1}C\). Moreover, the series will stop at some power because \(z\) is a matrix with body zero and thus it is nilpotent.

Now we come to the principal result of this paper, which we state as the following:

**Theorem 3.1.** Let \(M\) and \((xI - M)\) be \((p + q) \times (p + q)\) supermatrices, \(x \in \Lambda_0\), then \(N(x) = P(x)(xI - M)^{-1}\), with \(P(x)\) given in Def.(2.2), is a polynomial matrix.

**Proof.** Let us consider the block-element 11 of \(N(x)\) to begin with. According to Lemma (3.2) together with Eq. (2.11), this block can be written as
\[
N_{ij} = -g \frac{\partial \hat{f}}{\partial A_{ji}} - g \frac{\hat{f}}{R} \frac{\partial R}{\partial A_{ji}}. \tag{3.9}
\]
The first term of the RHS is clearly of polynomial character. In order to transform
the second term we make use of the property
\[
\frac{\partial \ln \tilde{G}}{\partial A_{ji}} = 0 = \frac{\partial \ln R}{\partial A_{ji}} + \frac{\partial \ln \tilde{g}}{\partial A_{ji}},
\]
which follows from the factorization \( \tilde{G} = R \tilde{g} \), together with the fact that \( \tilde{G} \) is just a
function of \( D_{\alpha \beta} \), according to Eq. (2.7a). In this way, and using also the Eq.(2.12),
we obtain
\[
N_{ij} = f \frac{\partial \tilde{g}}{\partial A_{ji}} - g \frac{\partial \tilde{f}}{\partial A_{ji}},
\]
which leads to the conclusion that the block-matrix \( N_{ij} \) is indeed polynomial. The
proof for \( N_{\alpha i} \) runs along the same lines, except that now the derivatives are taken
with respect to \( B_{i\alpha} \) and that we have to use \( \frac{\partial \ln \tilde{G}}{\partial B_{i\alpha}} = 0 \), instead of Eq. (3.10). The
remaining terms \( N_{i\alpha} \) and \( N_{\alpha \beta} \) can be dealt with in analogous manner by considering
the derivatives of \( G = Sg \) with respect to \( C_{\alpha i} \) and \( D_{\beta \alpha} \), and by replacing the
condition (3.10) by \( \frac{\partial \ln F}{\partial C_{\alpha i}} = 0 \) and \( \frac{\partial \ln F}{\partial D_{\beta \alpha}} = 0 \) respectively. The results are again of
the form (3.11), the only difference been the variables with respect to which the
derivatives are taken.

Finally, using Theorem (3.1) together with Lemma (3.1) we can state the fol-
lowing extension of the Cayley-Hamilton theorem to the case of supermatrices:

**Theorem 3.2.** (Extended Cayley-Hamilton Theorem) Let \( M \) and \( (xI - M) \)
be \((p+q)\times(p+q)\) supermatrices, \( x \in \Lambda_0 \), with \( S\det(xI - M) = \tilde{F}/\tilde{G} = F/G \), where
the polynomials \( \tilde{F}, \tilde{G}, F \) and \( G \) are given in Eqs.(2.7). Then, for any common factor
\( R \) such that \( \tilde{F} = R\tilde{f}, \tilde{G} = R\tilde{g} \) and \( S \) such that \( F = Sf, G = Sg \), where \( \tilde{f}/\tilde{g} = f/g \),
the polynomial \( P(x) = \tilde{f}(x)g(x) = f(x)\tilde{g}(x) \) annihilates \( M \), i.e. \( P(M) = 0 \).

A less formal presentation of the above results can be found in Refs.[16]. A
detailed version of this work containing many examples is given in Ref. [17].
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