How to solve three fundamental linear matrix inequalities in the Löwner partial ordering

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Abstract. This paper shows how to solve analytically the three fundamental linear matrix inequalities

\[ AXB \succ C \left( \succ C \right), \quad AXA^* \succ B \left( \succ B \right), \quad AX + (AX)^* \succ B \left( \succ B \right) \]

in the Löwner partial ordering by using ranks, inertias and generalized inverses of matrices.

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1 Introduction

Throughout this paper,

- \( \mathbb{C}^{m \times n} \) stands for the set of all \( m \times n \) complex matrices;
- \( \mathbb{C}^{m}_{\mathbb{H}} \) stands for the sets of all \( m \times m \) complex Hermitian matrices;
- \( \mathbb{C}^{m}_{\mathbb{SH}} \) stand for the sets of all \( m \times m \) complex Hermitian matrices and complex skew-Hermitian matrices;
- the symbols \( A^* \), \( r(A) \) and \( \mathcal{R}(A) \) stand for the conjugate transpose, the rank and the range (column space) of a matrix \( A \in \mathbb{C}^{m \times n} \), respectively;
- \([A, B]\) denotes a row block matrix consisting of \( A \) and \( B \);
- the Moore–Penrose inverse of \( A \in \mathbb{C}^{m \times n} \), denoted by \( A^\dagger \), is defined to be the unique solution \( X \) satisfying the four matrix equations \( AXA = A \), \( XAX = X \), \( (AX)^* = AX \) and \( (XA)^* = XA \);
- the symbols \( E_A \) and \( F_A \) stand for \( E_A = I_m - AA^\dagger \) and \( F_A = I_n - A^\dagger A \), their ranks are given by \( r(E_A) = m - r(A) \) and \( r(F_A) = n - r(A) \);
- \( i_+(A) \) and \( i_-(A) \), called the partial inertia of \( A \in \mathbb{C}^{n}_{\mathbb{H}} \), are defined to be the numbers of the positive and negative eigenvalues of \( A \) counted with multiplicities, respectively, where \( r(A) = i_+(A) + i_-(A) \);
- \( A \succ 0 \) (\( A \succ 0 \)) means that \( A \) is Hermitian positive semi-definite (positive definite);
- two \( A, B \in \mathbb{C}^{m}_{\mathbb{H}} \) are said to satisfy the inequality \( A \succ B \) (\( A \succ B \)) in the Löwner partial ordering if \( A - B \) is Hermitian positive semi-definite (positive definite);
- a positive semi-definite matrix \( A \) of order \( m \) is said to be a contraction if all its eigenvalues are less then or equal to 1, i.e., \( 0 \preceq A \preceq I_m \), to be a strict contraction if all its eigenvalues are less then 1, i.e., \( 0 \preceq A \prec I_m \).

A well-known property of the Moore–Penrose inverse is \( (A^\dagger)^* = (A^*)^\dagger \). In particular \( AA^\dagger = A^\dagger A \) if \( A = A^* \). We shall repeatedly use them in the latter part of this paper. One of the most important applications of generalized inverses is to derive some closed-form formulas for calculating ranks and inertias of matrices, as well as general solutions of matrix equations; see Lemmas 2.1–2.9 below. Results on the Moore–Penrose inverse can be found, e.g., in [3][4][12].

The Löwner partial ordering for matrices, as a natural extension of inequalities for real numbers, is one of the most useful concepts in matrix theory for characterizing relations between two complex Hermitian (real symmetric) matrices of the same size, while a main object of study in core matrix theory is to compare Hermitian matrices in the Löwner partial ordering and to establish various possible matrix inequalities. This subject was extensively studied by many authors, and numerous matrix inequalities in the Löwner partial ordering were established in the literature. In the investigation of the Löwner partial ordering between two Hermitian matrices, a challenging task is to solve matrix inequalities that involve unknown matrices. This topic can generally be stated as follows:

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Problem. For a given matrix-valued function \( \phi(X) \) that satisfies \( \phi(X) = \phi^*(X) \), where \( X \) is a variable matrix, establish necessary and sufficient conditions for the matrix inequality
\[
\phi(X) \succ 0, \quad \phi(X) \succeq 0, \quad \phi(X) \preceq 0, \quad \phi(X) \prec 0
\]
to hold, respectively, and find solutions \( X \) of the matrix inequalities.

A matrix-valued function for complex matrices is a map between matrix spaces, which can generally be written as \( Y = \phi(X) \) for \( Y \in \mathbb{C}^{m \times n} \) and \( X \in \mathbb{C}^{p \times q} \), or briefly, \( f : \mathbb{C}^{m \times n} \to \mathbb{C}^{p \times q} \), where \( \mathbb{C}^{m \times n} \) and \( \mathbb{C}^{p \times q} \) are two two complex matrix spaces. As usual, linear matrix-valued functions as common representatives of various matrix-valued functions are extensively studied from theoretical and applied points of view. When \( \phi(X) \) in (1.1) is a linear matrix-valued function, it is usually called a linear matrix inequality (LMI) in the literature. A systematic work on LMIs and their applications in system and control theory can be found, e.g., in \([5, 20]\). LMIs in the Löwner partial ordering are usually taken as convex constraints to unknown matrices and vectors in mathematical programming and optimization theory.

This paper aims at solving the following three groups of LMIs of fundamental type:
\[
AXB \succeq C \succcurlyeq B, \quad AXA^* \succeq B, \quad AX + (AX)^* \succeq B.
\]
They are the simplest cases of various types of LMIs and are the starting point of many advanced study on complicated LMIs.

Recall that any Hermitian nonnegative definite (positive definite) matrix \( M \) can be written as \( M = UU^* \) for certain (nonsingular) matrix \( U \). Hence, the mechanism of a matrix inequality in the Löwner partial ordering can be explained by certain matrix equation that involves an unknown quadratic term. In fact, any matrix inequality \( \phi(X) \succeq 0 \) positive semi-definiteness (matrix inequality \( \phi(X) \succ 0 \) for positive definiteness) can equivalently be relaxed to
\[
\phi(X) - UU^* = 0
\]
for certain (nonsingular) matrix \( U \). Due to the non-commutativity of matrix algebra, there are no general methods for finding analytical solutions of quadratic matrix equations, so that it is hard to solve for the unknown matrices \( X \) and \( U \) from the equation in (1.3) for a general \( \phi(X) \). However, for the three fundamental LMIs in (1.2)–(1.4), we are able to establish their analytical solutions by using the relaxed matrix equation in (1.5), and ordinary operations of the given matrices and their generalized inverses.

Matrix equations and matrix inequalities in the Löwner partial ordering have been main objects of study in matrix theory and their applications. Many new theories and methods were developed in the investigations of matrix equations and inequalities. In particular, the concept of generalized inverses of matrices was introduced when Penrose considered general solutions of the matrix equations \( AX = B \) and \( AXB = C \), cf. \([19]\). The three matrix equations associated with (1.2)–(1.3) are
\[
AXB = C, \quad AXA^* = B, \quad AX + (AX)^* = B,
\]
which were extensively studied from theoretical and practical points of view, while the three matrix-valued functions
\[
\phi_1(X) = C - AXB, \quad \phi_2(X) = B - AXA^*, \quad \phi_3(X) = B - AX - (AX)^*
\]
associated with (1.2)–(1.4) were recently considered in \([15, 16, 22, 24, 25, 30, 31]\). Because (1.2)–(1.3) are some simplest cases of matrix equations, matrix inequalities and matrix functions, they have been attractive objects of study in matrix theory and applications. In fact, it is remarkable that simply knowing when the LMIs in (1.2)–(1.4) are feasible gives some deep insights into the relations between both sides of the LMIs.

This paper is organized as follows. In Section 2, we give a group of known results on matrix equations, as well as some expansion formulas for calculating (extremal) ranks and inertias of matrices. In Section 3, we solve for the inequality in (1.2), and discuss various algebraic properties of the LMI and its solution. In particular, we shall give a group of closed-form formulas for calculating the extremal ranks and inertias of \( D -AXB \) subject to \( AXB \succ C \), and use the formulas to establish necessary and sufficient conditions for the two-sided matrix inequality \( D \succ AXB \succ C \) to be solvable. In Sections 4 and 5, we establish necessary and sufficient conditions for the LMIs in (1.3) and (1.4) to be feasible, respectively, and derive general solutions in closed-forms of these LMIs. In Section 7, we give a group of formulas for calculating the extremal ranks and inertias of \( A - BX - XB^* \) subject to \( BXB^* = C \), and use the formulas to characterize the existence of Hermitian matrix \( X \) that satisfies \( BX + XB^* \succ A(BX + XB^* \succ A) \) subject to \( BXB^* = C \). Some further research problems are presented in Section 8.
2 Preliminaries

In this section, we present some known or new results on solving matrix equations, as well as formulas for calculating ranks and inertias of matrices, which will be used in the latter part of this paper.

Lemma 2.1 ([14]) Let $A, B \in \mathbb{C}^{m \times n}$ be given. Then, the following hold.

(a) The matrix equation

$$AX = B \quad (2.1)$$

has a Hermitian solution $X \in \mathbb{C}^{n \times m}_H$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $AB^* = BA^*$. In this case, the general Hermitian solution of (2.1) can be written in the following parametric form

$$X = A^\dagger B + (A^\dagger B)^* - A^\dagger BA^\dagger A + F_AWF_A, \quad (2.2)$$

where $W \in \mathbb{C}^{n \times m}_H$ is arbitrary.

(b) The matrix equation

$$AXX^* = B \quad (2.3)$$

has a solution for $XX^*$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, $AB^* \succ 0$ and $r(AB^*) = r(B)$. In this case, the general solution of (2.3) can be written in the following parametric form

$$XX^* = B^*(AB^*)^\dagger B + F_AWW^*F_A, \quad (2.4)$$

where $W \in \mathbb{C}^{n \times n}$ is arbitrary.

Lemma 2.2 ([19]) Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{p \times q}$ and $C \in \mathbb{C}^{m \times q}$ be given. Then, the matrix equation

$$AXB = C \quad (2.5)$$

has a solution if and only if $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, or equivalently, $E_A C = 0$ and $CF_B = 0$. In this case, the general solution of (2.5) can be written in the following parametric forms

$$X = A^\dagger CB^\dagger + W - A^\dagger AWBB^\dagger, \quad (2.6)$$

$$X = A^\dagger CB^\dagger + F_AU_1 + U_2E_B, \quad (2.7)$$

respectively, where $W, U_1, U_2 \in \mathbb{C}^{n \times p}$ are arbitrary.

Lemma 2.3 Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times m}$ be given. Then, the following hold.

(a) ([10]) The matrix equation

$$AXA^* = B \quad (2.8)$$

has a solution $X \in \mathbb{C}^{n \times m}_H$ if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, or equivalently, $AA^\dagger B = B$. In this case, the general Hermitian solution of (2.8) can be written in the following parametric forms

$$X = A^\dagger B(A^\dagger)^* + U - A^\dagger AU A^\dagger A, \quad (2.9)$$

$$X = A^\dagger B(A^\dagger)^* + F_AV + V^*F_A, \quad (2.10)$$

respectively, where $U \in \mathbb{C}^{n \times m}_H$ and $V \in \mathbb{C}^{n \times n}$ are arbitrary.

(b) ([10], [14]) There exists an $X \in \mathbb{C}^{n \times n}$ such that

$$AXX^*A^* = B \quad (2.11)$$

if and only if $B \succ 0$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general solution of (2.11) can be written as

$$XX^* = A^\dagger B(A^\dagger)^* + F_AVB(A^\dagger)^* + A^\dagger BV^*F_A + F_AWW^*F_A, \quad (2.12)$$

where $V \in \mathbb{C}^{n \times m}$ and $W \in \mathbb{C}^{n \times n}$ are arbitrary.

(c) ([1]) Under $A, B \in \mathbb{C}^{m \times m}$, there exists an $X \in \mathbb{C}^{m \times m}$ such that

$$AXX^*A^* = B \quad (2.13)$$

if and only if $B \succ 0$ and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general solution of (2.13) can be written as

$$XX^* = (A^\dagger B^\dagger + F_AV)(A^\dagger B^\dagger + F_AV)^*, \quad (2.14)$$

where $V \in \mathbb{C}^{m \times m}$ is arbitrary.
Lemma 2.4 ([31]) Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m}_H$ be given. Then, the following hold.

(a) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* = B$$

(2.15)

if and only if $E_A BE_A = 0$. In this case, the general solution of (2.15) can be written in the following parametric form

$$X = \frac{1}{2} A^\dagger B (2I_m - AA^\dagger) + VA^* + FW,$$

(2.16)

where both $V \in \mathbb{C}^{n\times n}_H$ and $W \in \mathbb{C}^{n \times m}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* = BB^*$$

(2.17)

if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general solution of (2.17) can be written as

$$X = \frac{1}{2} A^\dagger BB^* + VA^* + FW,$$

(2.18)

where both $V \in \mathbb{C}^{n \times n}_H$ and $W \in \mathbb{C}^{n \times m}$ are arbitrary.

Lemma 2.5 Let $A_1 \in \mathbb{C}^{m \times p}$, $B_1 \in \mathbb{C}^{q \times n}$, $A_2 \in \mathbb{C}^{m \times r}$, $B_2 \in \mathbb{C}^{s \times n}$ and $C \in \mathbb{C}^{m \times n}$ be given. Then, the following hold.

(a) [18] There exist $X \in \mathbb{C}^{p \times q}$ and $Y \in \mathbb{C}^{r \times s}$ such that

$$A_1 X B_1 + A_2 Y B_2 = C$$

(2.19)

if and only if the following four rank equalities

$$r[C, A_1, A_2] = r[A_1, A_2], \quad r \begin{bmatrix} C \\ B_1 \\ B_2 \end{bmatrix} = r \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

(2.20)

and

$$r \begin{bmatrix} C & A_1 \\ B_2 & 0 \end{bmatrix} = r(A_1) + r(B_2), \quad r \begin{bmatrix} C & A_2 \\ B_1 & 0 \end{bmatrix} = r(A_2) + r(B_1)$$

(2.21)

hold, or equivalently,

$$[A_1, A_2] [A_1, A_2]^\dagger C = C, \quad C \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^\dagger = C, \quad E_{A_1} CF_{B_2} = 0, \quad E_{A_2} CF_{B_1} = 0.$$

(2.22)

(b) [21] Under (2.20) and (2.21), the general solutions of (2.19) can be decomposed as

$$X = X_0 + X_1 X_2 + X_3 \quad \text{and} \quad Y = Y_0 - Y_1 Y_2 + Y_3,$$

(2.23)

where $X_0$ and $Y_0$ are a pair of special solutions of (2.19), $X_1$, $X_2$, $X_3$ and $Y_1$, $Y_2$, $Y_3$ are the general solutions of the following four homogeneous matrix equations

$$A_1 X_1 + A_2 Y_1 = 0, \quad X_2 B_1 + Y_2 B_2 = 0, \quad A_1 X_3 B_1 = 0, \quad A_2 Y_3 B_2 = 0.$$

(2.24)

By using generalized inverses of matrices, (2.23) can be written in the following parametric forms

$$X = X_0 + [I_p, 0] F_G W E_H \begin{bmatrix} I_q \\ 0 \end{bmatrix} + F_{A_1} W_1 + W_2 E_{B_1},$$

(2.25)

$$Y = Y_0 - [0, I_r] F_G W E_H \begin{bmatrix} 0 \\ I_s \end{bmatrix} + F_{A_2} W_3 + W_4 E_{B_2},$$

(2.26)

where $G = [A_1, A_2]$, $H = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, the five matrices $W, W_1, W_2, W_3$ and $W_4$ are arbitrary.

Lemmas 2.4 and 2.5 show that general solutions of some simple matrix equations can be written as analytical forms composed by the given matrices and their generalized inverses, as well as arbitrary matrices. These analytical formulas can be easily used to establish various algebraic properties of the solutions of the equations, such as, their ranks, ranges, uniqueness, definiteness, etc.

In order to simplify various matrix expression involving generalized inverse of matrices and arbitrary matrices, we need some formulas for ranks and inertias of matrices. The following is obvious from the definitions of rank and inertia.
Lemma 2.6 Let $A \in \mathbb{C}^{m \times m}$, $B \in \mathbb{C}^{m \times n}$, and $C \in \mathbb{C}_H^n$. Then, the following hold.

(a) $A$ is nonsingular if and only if $r(A) = m$.
(b) $B = 0$ if and only if $r(B) = 0$.
(c) $C \succ 0$ ($C \prec 0$) if and only if $i_+(C) = m$ ($i_-(C) = m$),
(d) $C \succeq 0$ ($C \preceq 0$) if and only if $i_-(C) = 0$ ($i_+(C) = 0$).

Lemma 2.7 Let $S$ be a set consisting of matrices over $\mathbb{C}^{m \times n}$, and let $\mathcal{H}$ be a set consisting of Hermitian matrices over $\mathbb{C}_H^n$. Then, the following hold.

(a) Under $m = n$, $S$ has a nonsingular matrix if and only if $\max_{X \in S} r(X) = m$.
(b) Under $m = n$, all $X \in S$ are nonsingular if and only if $\min_{X \in S} r(X) = m$.
(c) $0 \in S$ if and only if $\min_{X \in S} r(X) = 0$.
(d) $S = \{0\}$ if and only if $\max_{X \in S} r(X) = 0$.
(e) $\mathcal{H}$ has a matrix $X \succ 0$ ($X \prec 0$) if and only if $\max_{X \in \mathcal{H}} i_+(X) = m$ ($\max_{X \in \mathcal{H}} i_-(X) = m$).
(f) All $X \in \mathcal{H}$ satisfy $X \succ 0$ ($X \prec 0$) if and only if $\min_{X \in \mathcal{H}} i_+(X) = m$ ($\min_{X \in \mathcal{H}} i_-(X) = m$).
(g) $\mathcal{H}$ has a matrix $X \succeq 0$ ($X \preceq 0$) if and only if $\min_{X \in \mathcal{H}} i_-(X) = 0$ ($\min_{X \in \mathcal{H}} i_+(X) = 0$).
(h) All $X \in \mathcal{H}$ satisfy $X \succeq 0$ ($X \preceq 0$) if and only if $\max_{X \in \mathcal{H}} i_-(X) = 0$ ($\max_{X \in \mathcal{H}} i_+(X) = 0$).

The question of whether a given matrix function is nonnegative definite or positive definite everywhere is ubiquitous in mathematics and applications. Lemma 2.7(e)–(h) show that if certain explicit formulas for calculating the global extremal inertias of a given Hermitian matrix function are established, we can use them, as demonstrated in Sections 2, 3 and 5 below, to derive necessary and sufficient conditions for the Hermitian matrix function to be definite or semi-definite.

Lemma 2.8 Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$. Then, the following hold. The following rank expansion formulas hold

\[
\begin{align*}
 r[A, B] &= r(A) + r(E_AB) = r(B) + r(E_B A), \quad (2.27) \\
r[A, C] &= r(A) + r(CF_A) = r(C) + r(AF_C), \quad (2.28) \\
r[A, B] &= r(B) + r(C) + r(E_B A F_C), \quad (2.29) \\
r[AA^*, B] &= r(A) + r(B), \quad (2.30) \\
r[A, B] &= r(A) + r\left[ 0 \quad E_A B \begin{array}{c} 0 \\ CF_A \quad D - CA^1 B \end{array} \right]. \quad (2.31)
\end{align*}
\]

If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

\[
r[A, B] = r(A) + r(D - CA^1 B). \quad (2.32)
\]

Lemma 2.9 Let $A \in \mathbb{C}_H^n$, $B \in \mathbb{C}^{m \times n}$, $D \in \mathbb{C}_H^n$, and define

\[
M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix}
\]

Then, the partial inertias of $M_1$ and $M_2$ can be expanded as

\[
\begin{align*}
i_\pm(M_1) &= r(B) + i_\pm(E_B AE_B), \quad (2.33) \\
i_\pm(M_2) &= i_\pm(A) + i_\pm\begin{bmatrix} 0 & E_A B \\ B^* E_A & D - B^* A^1 B \end{bmatrix}, \quad (2.34)
\end{align*}
\]

In particular,
Lemma 2.10 Let $A, B \in \mathbb{C}^{m \times n}$ and $P \in \mathbb{C}^{m \times n}$.

(a) If $A \succ B$, then $P^*AP \succ P^*BP$.

(b) $A \succ 0$ if and only if $A^\dagger \succ 0$.

(c) If $I_m - A \succ 0$, then $I_m - PP^\dagger APP^\dagger \succ 0$.

(d) If $I_m - A \succ 0$, then $I_m - PP^\dagger APP^\dagger \succeq 0$.

Proof. Result (a) is obvious from the definition of the nonnegative definiteness of Hermitian matrix. Result (b) is obvious from similarity decomposition of $A$ and the definition of the Moore–Penrose inverse of a matrix. If $A$ is Hermitian, then we can find by Lemma 2.9(c), $*$-congruence transformation and (2.32) that

\[
i_\pm(I_m - PP^\dagger APP^\dagger) = i_\pm\begin{bmatrix} A & APP^\dagger \\ PP^\dagger A & I_m \end{bmatrix} - i_\pm(A) = i_\pm(A - APP^\dagger A) - i_\pm(A)
\]

\[
= i_\pm(I_m) + i_\pm(A - APP^\dagger A) - i_\pm(A)
\]

\[
= i_\pm(I_m) + i_\pm\begin{bmatrix} P^*P & P^*A \\ AP & A \end{bmatrix} - i_\pm(APP^*) - i_\pm(A)
\]

\[
= i_\pm(I_m) + i_\pm\begin{bmatrix} P^*P & P^*A \\ AP & A \end{bmatrix} - i_\pm(APP^*) - i_\pm(A)
\]

\[
= i_\pm(I_m) + i_\pm[P^*(I_m - A)P] - i_\pm(APP^*),
\]

namely

\[
i_+(I_m - PP^\dagger APP^\dagger) = m - r(P) + i_+\left[ P^*(I_m - A)P \right],
\]

\[
i_-(I_m - PP^\dagger APP^\dagger) = i_-\left[ P^*(I_m - A)P \right].
\]

If $A \preceq I_m$, then (2.38) reduces to

\[
i_-(I_m - PP^\dagger APP^\dagger) = i_-\left[ P^*(I_m - A)P \right] = 0.
\]

Hence, (c) follows by Lemma 2.6(d). If $A < I_m$, then $P^*(I_m - A)P \not\succ 0$ and $i_+\left[ P^*(I_m - A)P \right] = r\left[ P^*(I_m - A)P \right] = r(P)$. Thus, (2.38) reduces to

\[
i_+(I_m - PP^\dagger APP^\dagger) = m - r(P) + r(P) = m.
\]

Hence, (d) follows by Lemma 2.6(c).
Lemma 2.11 Let \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{m \times k} \) and \( C \in \mathbb{C}^{l \times n} \) be given. Then the global maximum and minimum ranks of \( A - BXC \) with respect to \( X \in \mathbb{C}^{k \times l} \) are given by
\[
\max_{X \in \mathbb{C}^{k \times l}} r(A - BXC) = \min \left\{ r[A, B], r \left[ \begin{array}{c|c} A & \phi \end{array} \right] \right\}, \tag{2.40}
\]
\[
\min_{X \in \mathbb{C}^{k \times l}} r(A - BXC) = r[A, B] + r \left[ \begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right]. \tag{2.41}
\]
In particular,
\[
\max_{X \in \mathbb{C}^{k \times l}} r(A - BXC) = \min \{ r[A, B], n \}, \tag{2.42}
\]
\[
\min_{X \in \mathbb{C}^{k \times l}} r(A - BXC) = r[A, B] - r(B). \tag{2.43}
\]

Lemma 2.12 (\cite{16, 25}) Let \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{m \times n} \) and \( C \in \mathbb{C}^{p \times m} \) be given. Then, the extremal ranks and inertias of \( A - BXC - (BXC)^* \) are given by
\[
\max_{X \in \mathbb{C}^{p \times m}} r(A - BXC - (BXC)^*) = \min \left\{ r[A, B, C^*], r \left[ \begin{array}{c|c} A & B^* \\ \hline C^* & 0 \end{array} \right] \right\}, \tag{2.44}
\]
\[
\min_{X \in \mathbb{C}^{p \times m}} r(A - BXC - (BXC)^*) = 2r[A, B, C^*] + \max \{ s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+ \}, \tag{2.45}
\]
\[
\max_{X \in \mathbb{C}^{p \times m}} i_{\pm}(A - BXC - (BXC)^*) = \min \left\{ i_{\pm} \left[ \begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right], i_{\pm} \left[ \begin{array}{c|c} A & C^* \\ \hline B^* & 0 \end{array} \right] \right\}, \tag{2.46}
\]
\[
\min_{X \in \mathbb{C}^{p \times m}} i_{\pm}(A - BXC - (BXC)^*) = r[A, B, C^*] + \max \{ s_+, t_+ \}, \tag{2.47}
\]
where
\[
s_\pm = i_{\pm} \left[ \begin{array}{c|c} A & B \\ \hline B^* & 0 \\ \hline C^* & 0 \end{array} \right] - r \left[ \begin{array}{c|c|c} A & B & C^* \\ \hline C & 0 & 0 \end{array} \right], \quad t_\pm = i_{\pm} \left[ \begin{array}{c|c} A & C^* \\ \hline B^* & 0 \\ \hline C & 0 \end{array} \right] - r \left[ \begin{array}{c|c|c} A & B & C^* \\ \hline C & 0 & 0 \end{array} \right]. \tag{2.48}
\]
In particular,
\[
\max_{X \in \mathbb{C}^{p \times m}} r(A - BX - (BX)^*) = \min \{ m, r \left[ \begin{array}{c|c} A & B^* \\ \hline C^* & 0 \end{array} \right] \}, \tag{2.49}
\]
\[
\min_{X \in \mathbb{C}^{p \times m}} r(A - BX - (BX)^*) = r \left[ \begin{array}{c|c} A & B^* \\ \hline C^* & 0 \end{array} \right] - 2r(B), \tag{2.50}
\]
\[
\max_{X \in \mathbb{C}^{p \times m}} i_\pm(A - BX - (BX)^*) = i_{\pm} \left[ \begin{array}{c|c} A & B \\ \hline B^* & 0 \end{array} \right], \tag{2.51}
\]
\[
\min_{X \in \mathbb{C}^{p \times m}} i_\pm(A - BX - (BX)^*) = i_{\pm} \left[ \begin{array}{c|c} A & B \\ \hline B^* & 0 \end{array} \right] - r(B). \tag{2.52}
\]

The matrices \( X \) that satisfy (2.41) (2.51) (namely, the global maximizers and minimizers of the objective rank and inertia functions) are not necessarily unique and their expressions were also given in \cite{16, 25} by using certain simultaneous decomposition of the three given matrices and their generalized inverses.

We also need the following results on the ranks and inertias of the quadratic matrix-valued functions
\[
A \pm (BX + C)(BX + C)^* = A \pm (BX^*B^* + BXC^* + CX^*B^* + CC^*)
\]
and their consequences.

Lemma 2.13 (\cite{27}) Let \( A \in \mathbb{C}^m \) and \( B \in \mathbb{C}^{m \times k} \) and \( C \in \mathbb{C}^{m \times n} \) be given, and let
\[
G_1 = \left[ \begin{array}{c|c|c} A + CC^* & B \\ \hline B^* & 0 \end{array} \right], \quad G_2 = \left[ \begin{array}{c|c|c} A - CC^* & B \\ \hline B^* & 0 \end{array} \right], \quad G_3 = \left[ \begin{array}{c|c|c} A & B \\ \hline B^* & 0 \end{array} \right].
\]
Then, the following hold.

(a) The extremal ranks and inertias of \( \phi_1(X) = A \pm (BX + C)(BX + C)^* \) are given by
\[
\max_{X \in \mathbb{C}^{k \times l}} r[\phi_1(X)] = \min \{ r[A, B, C], r(G_1), r(A) + n \}, \tag{2.52}
\]
\[
\min_{X \in \mathbb{C}^{k \times l}} r[\phi_1(X)] = 2r[A, B, C] + \max \{ h_1, h_2, h_3, h_4 \}, \tag{2.53}
\]
\[
\max_{X \in \mathbb{C}^{k \times l}} i_+[\phi_1(X)] = \min \{ i_+(G_1), i_+(A) + n \}, \tag{2.54}
\]
\[
\max_{X \in \mathbb{C}^{k \times l}} i_-[\phi_1(X)] = \min \{ i_-(G_1), i_-(A) \}, \tag{2.55}
\]
\[
\min_{X \in \mathbb{C}^{k \times l}} i_+[\phi_1(X)] = r[A, B, C] + \max \{ i_+(G_1) - r(G_3), i_+(A) - r(A, B) \}, \tag{2.56}
\]
\[
\min_{X \in \mathbb{C}^{k \times l}} i_-[\phi_1(X)] = r[A, B, C] + \max \{ i_-(G_1) - r(G_3), i_-(A) - r(A, B) - n \}. \tag{2.57}
\]
where

\[ h_1 = r(G_1) - 2r(G_3), \quad h_2 = r(A) - 2r[ A, B ] - n, \]
\[ h_3 = i_-(G_1) - r(G_3) + i_+(A) - r[ A, B], \]
\[ h_4 = i_+(G_1) - r(G_3) + i_-(A) - r[ A, B] - n. \]

(b) The extremal ranks and inertias of \( \phi_2(X) = A - (BX + C)(BX + C)^* \) are given by

\[
\begin{align*}
\max_{X \in \mathbb{C}^{m \times n}} r[ \phi_2(X) ] &= \min \{ r[ A, B, C], \ r(G_2), \ r(A) + n \}, \\
\min_{X \in \mathbb{C}^{m \times n}} r[ \phi_2(X) ] &= 2r[ A, B, C] + \max\{ h_5, \ h_6, \ h_7, \ h_8 \}, \\
\max_{X \in \mathbb{C}^{m \times n}} i_+[ \phi_2(X) ] &= \min \{ i_+(G_2), \ i_+(A) \}, \\
\max_{X \in \mathbb{C}^{m \times n}} i_-[ \phi_2(X) ] &= \min \{ i_-(G_2), \ i_-(A) + n \}, \\
\min_{X \in \mathbb{C}^{m \times n}} i_+[ \phi_2(X) ] &= r[ A, B, C] + \max \{ i_+(G_2) - r(G_3), \ i_+(A) - r[ A, B] - n \}, \\
\min_{X \in \mathbb{C}^{m \times n}} i_-[ \phi_2(X) ] &= r[ A, B, C] + \max \{ i_-(G_2) - r(G_3), \ i_-(A) - r[ A, B] \},
\end{align*}
\]

where

\[ h_5 = r(G_2) - 2r(G_3), \quad h_6 = r(A) - 2r[ A, B ] - n, \]
\[ h_7 = i_+(G_2) - r(G_3) + i_-(A) - r[ A, B], \]
\[ h_8 = i_-(G_2) - r(G_3) + i_+(A) - r[ A, B] - n. \]

When \( C = 0 \), Lemma 2.13 reduces to the following result.

**Corollary 2.14** ([26]) Let \( A \in \mathbb{C}_H^m \) and \( B \in \mathbb{C}^{m \times n} \) be given, and let \( M = \begin{bmatrix} A & B^* \\ B & 0 \end{bmatrix} \). Then, the following hold.

(a) The extremal ranks and partial inertias of \( A \pm BXX^*B^* \) are given by

\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}} r( A + BXX^*B^* ) &= r[ A, B], \\
\min_{X \in \mathbb{C}^{n \times n}} r( A + BXX^*B^* ) &= i_+(A) + r[ A, B ] - i_+(M), \\
\max_{X \in \mathbb{C}^{n \times n}} i_+( A + BXX^*B^* ) &= i_+(M), \\
\max_{X \in \mathbb{C}^{n \times n}} i_-( A + BXX^*B^* ) &= i_-(A), \\
\min_{X \in \mathbb{C}_H^{n \times n}} i_+( A + BXX^*B^* ) &= i_+(A), \\
\min_{X \in \mathbb{C}^{n \times n}} i_-( A + BXX^*B^* ) &= r[ A, B ] - i_+(M),
\end{align*}
\]

and

\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}} r( A - BXX^*B^* ) &= r[ A, B], \\
\min_{X \in \mathbb{C}^{n \times n}} r( A - BXX^*B^* ) &= i_-(A) + r[ A, B ] - i_-(M), \\
\max_{X \in \mathbb{C}^{n \times n}} i_+( A - BXX^*B^* ) &= i_+(M), \\
\max_{X \in \mathbb{C}^{n \times n}} i_-( A - BXX^*B^* ) &= i_-(A), \\
\min_{X \in \mathbb{C}^{n \times n}} i_+( A - BXX^*B^* ) &= i_+(A), \\
\min_{X \in \mathbb{C}^{n \times n}} i_-( A - BXX^*B^* ) &= r[ A, B ] - i_-(M),
\end{align*}
\]

(b) If \( A \succeq 0 \), then

\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}} r( A + BXX^*B^* ) &= r[ A, B], \\
\min_{X \in \mathbb{C}^{n \times n}} r( A + BXX^*B^* ) &= r(A), \\
\end{align*}
\]

\[
\begin{align*}
\max_{X \in \mathbb{C}^{n \times n}} r( A - BXX^*B^* ) &= r[ A, B], \\
\min_{X \in \mathbb{C}^{n \times n}} r( A - BXX^*B^* ) &= r(A).
\end{align*}
\]
3 General solutions $AXB \succeq (\succ, \preceq, \prec) C$ and their properties

A necessary condition for (1.2) to hold is $AXB$ is called a symmetrizer of $AXB$; see [2]. In this section, we derive an analytical presentation for the general solution of the LMI in (1.2) by using the given matrices and their generalized inverses, and establish various algebraic properties of the LMI.

**Theorem 3.1** Let $A \in \mathbb{C}^{n \times p}, B \in \mathbb{C}^{q \times m}$ and $C \in \mathbb{C}^{m \times p}$ be given, and define $M = \begin{bmatrix} E_A & F_B \end{bmatrix}$. Then, the following hold.

(a) There exists an $X \in \mathbb{C}^{p \times q}$ such that

\[ AXB \succeq C \quad (3.1) \]

if and only if

\[ M^*CM \preceq 0 \text{ and } \mathcal{R}(M^*CM) = \mathcal{R}(M^*C). \quad (3.2) \]

In this case, the general solution of (3.1) and the corresponding $AXB$ can be written in the following parametric forms

\[ X = A^\dagger CB^\dagger - A^\dagger CM(M^*CM)^\dagger M^*CB^\dagger + A^\dagger E_MU^*E_MB^\dagger + W - A^\dagger AWBB^\dagger, \]
\[ AXB = C - CM(M^*CM)^\dagger M^*C + E_MUU^*E_M, \quad (3.3) \]

where $U \in \mathbb{C}^{n \times m}$ and $W \in \mathbb{C}^{p \times q}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}^{p \times q}$ such that

\[ AXB \succ C \quad (3.5) \]

if and only if

\[ M^*CM \preceq 0 \text{ and } r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} = m + r[A, B^*]. \quad (3.6) \]

In this case, the general solution of (3.5) can be written as (3.3), in which $U \in \mathbb{C}^{m \times m}$ is any matrix such that $r[CM, E_MU] = m$, and $W \in \mathbb{C}^{p \times q}$ is arbitrary.

**Proof.** Inequality (1.2) is obviously equivalent to the following linear-quadratic matrix equation

\[ AXB = C + YY^*. \quad (3.7) \]

By Lemma 2.2, this equation is solvable for $X$ if and only if

\[ E_A(C + YY^*) = 0 \text{ and } (C + YY^*)F_B = 0, \quad (3.8) \]

that is,

\[ \begin{bmatrix} E_A \\ F_B \end{bmatrix} YY^* = - \begin{bmatrix} E_A C \\ F_B C \end{bmatrix}. \quad (3.9) \]

By Lemma 2.1b), this quadratic matrix equation is solvable for $YY^*$ if and only if

\[ \begin{bmatrix} E_A \\ F_B \end{bmatrix} C[E_A, F_B] \preceq 0 \text{ and } r\left( \begin{bmatrix} E_A \\ F_B \end{bmatrix} C[E_A, F_B] \right) = r(C[E_A, F_B]), \]

establishing (3.2). Under (3.2), the general solution of (3.7) can be written as

\[ YY^* = -CM(M^*CM)^\dagger M^*C + E_MUU^*E_M, \]
where $U \in \mathbb{C}^{m \times m}$ is arbitrary. Substituting the $YY^*$ into (3.4) gives
\begin{equation}
AXB = C - CM(M^*CM)^\dagger M^* + E_{MM}^{U*}E_{M}.
\end{equation}

By Lemma 2.2, the general solution of (3.10) is
\begin{equation}
U_x^{*-1} = \begin{bmatrix}
C & 0 \\
A & 0
\end{bmatrix}
\end{equation}
establishing (3.3) and (3.4).

It can be seen from (3.10) that (3.5) holds if and only if
\begin{equation}
-CM(M^*CM)^\dagger M^*C + E_{MM}^{U*}E_{M} > 0
\end{equation}
for some $U$. Under (2.2), we have
\begin{equation}
Hence,
\begin{equation}
\min r[-CM(M^*CM)^\dagger M^*C + E_{MM}^{U*}E_{M}] = r[CM, EM_U].
\end{equation}

The following result can be shown similarly.

\textbf{Corollary 3.2} Let $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times m}$ and $C \in \mathbb{C}_m$ be given, and define $M = [E_A, F_B]$. Then, the following hold.

\textbf{(a)} There exists an $X \in \mathbb{C}^{p \times q}$ such that
\begin{equation}
AXB \preceq C
\end{equation}
if and only if
\begin{equation}
M^*CM \succ 0 \quad \text{and} \quad \mathcal{R}(M^*CM) = \mathcal{R}(M^*C).
\end{equation}

In this case, the general solution of (3.15) and the corresponding $AXB$ can be written in the following parameteric forms
\begin{equation}
X = A^tCB^t - A^tCM(M^*CM)^\dagger M^*CB^t - A^tE_{MM}^{U*}E_{M}B^t + W - A^tAWBB^t,
\end{equation}
\begin{equation}
AXB = C - CM(M^*CM)^\dagger M^*C - E_{MM}^{U*}E_{M},
\end{equation}
where $U \in \mathbb{C}^{m \times m}$ and $W \in \mathbb{C}^{p \times q}$ are arbitrary.

\textbf{(b)} There exists an $X \in \mathbb{C}^{p \times q}$ such that
\begin{equation}
AXB \succeq C
\end{equation}
if and only if
\begin{equation}
M^*CM \succ 0 \quad \text{and} \quad r\begin{bmatrix}
A & 0 \\
0 & B
\end{bmatrix} = m + r[A, B^*].
\end{equation}

In this case, the general solution of (3.19) can be written as (3.17), in which $U$ is any matrix such that $r[CM, EM_U] = m$, and $W \in \mathbb{C}^{p \times q}$ is arbitrary.

We next establish some algebraic properties of the fixed parts in (3.3) and (3.17).

\textbf{Corollary 3.3} Let $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times m}$ and $C \in \mathbb{C}_m$ be given, and define
\begin{equation}
\tilde{X} = A^tCB^t - A^tCM(M^*CM)^\dagger M^*CB^t, \quad M = [E_A, F_B], \quad N = \begin{bmatrix}
C & C & A & 0 \\
C & C & 0 & B^* \\
A^* & 0 & 0 & 0 \\
0 & B & 0 & 0
\end{bmatrix}.
\end{equation}
(a) Under the condition that (3.11) has a solution, the $\hat{X}$ in (3.21) satisfies $A\hat{X}B \succ C$, and

\[ i_\pm(A\hat{X}B) = r(A) + r(B) + i_\pm(C) - i_\pm(N), \]
\[ r(\hat{X}) = r(A\hat{X}B) = 2r(A) + 2r(B) + r(C) - r(N), \]
\[ i_+(A\hat{X}B - C) = r(A\hat{X}B - C) = 2r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r(N). \]  

(b) Under the condition that (3.15) has a solution, the $\hat{X}$ in (3.21) satisfies $A\hat{X}B \prec C$, and

\[ i_\pm(A\hat{X}B) = r(A) + r(B) + i_\pm(C) - i_\pm(N), \]
\[ r(\hat{X}) = r(A\hat{X}B) = 2r(A) + 2r(B) + r(C) - r(N), \]
\[ i_-(A\hat{X}B - C) = r(A\hat{X}B - C) = 2r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - r(N). \]

**Proof.** Under the condition that (3.11) has a solution, set $U = W = 0$ in (3.3). Then we see that the $\hat{X}$ in (3.21) is a solution of $AXB \succ C$. Also note from (3.10) that

\[ A\hat{X}B = C - CM(M^*CM)^\dagger M^*C, \quad C - A\hat{X}B = CM(M^*CM)^\dagger M^*C \preceq 0. \]  

In this case, applying (2.37) and (2.32) to (3.28), we obtain

\[ r(\hat{X}) = r[A^1CB^\dagger - A^1CM(M^*CM)^\dagger M^*CB^\dagger] = r\begin{bmatrix} M^*CM & M^*CB^\dagger \\ A^1CM & A^1CB^\dagger \end{bmatrix} - r(M^*CM), \]
\[ i_\pm(A\hat{X}B) = i_\pm[C - CM(M^*CM)^\dagger M^*C] = i_\pm\begin{bmatrix} M^*CM & M^*C \\ CM & C \end{bmatrix} - i_\pm(M^*CM), \]
\[ r(C - A\hat{X}B) = r[CM(M^*CM)^\dagger M^*C] = r\begin{bmatrix} M^*CM & M^*C \\ CM & 0 \end{bmatrix} - r(M^*CM). \]

Applying elementary matrix operations, congruence matrix operations and (2.29), we obtain

\[ r\begin{bmatrix} M^*CM & M^*CB^\dagger \\ A^1CM & A^1CB^\dagger \end{bmatrix} = r\left[ \begin{bmatrix} E_A \\ F_B \end{bmatrix} C[ E_A, F_B, B^\dagger] \right] = r\left[ \begin{bmatrix} I_m \\ F_B \end{bmatrix} C[ E_A, I_m, B^\dagger] \right] = r(C), \]
\[ i_\pm\begin{bmatrix} M^*CM & M^*C \\ CM & C \end{bmatrix} = i_\pm\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} = i_\pm(C), \]
\[ r\begin{bmatrix} M^*CM & M^*C \\ CM & 0 \end{bmatrix} = 2r(M^*C) = 2r\begin{bmatrix} A & 0 & C \\ 0 & B^* & C \end{bmatrix} - 2r(A) - 2r(B), \]
\[ i_\pm(M^*CM) = i_\pm\left[ \begin{bmatrix} E_A \\ F_B \end{bmatrix} C[ E_A, F_B] \right] = i_\pm(N) - r(A) - r(B), \]
\[ r(M^*CM) = r(N) - 2r(A) - 2r(B). \]

Substituting these formulas into (3.29)–(3.31) yields (3.22)–(3.24). Results (b) can be shown similarly. \(\square\)

**Corollary 3.4** Let $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times m}$ and $C \in \mathbb{C}_R^q$ be given, and let $M$ and $N$ be of the forms in (3.21). Also assume that (3.1) is feasible, and define

\[ S_1 = \{ X \in \mathbb{C}^{p \times q} \mid AXB \succeq C \}. \]

Then, the following hold.

(a) The minimal matrices of $AXB$ and $AXB - C$ subject to $X \in S_1$ in the Löwner partial ordering are given by

\[ \min_{\succeq} \{ AXB \mid X \in S_1 \} = C - CM(M^*CM)^\dagger M^*C, \]
\[ \min_{\succeq} \{ AXB - C \mid X \in S_1 \} = -CM(M^*CM)^\dagger M^*C. \]
(b) The extremal ranks and partial inertias of $AXB$ and $AXB - C$ subject to $X \in S_1$ are given by

$$\max_{X \in S_1} r(AXB) = \max_{X \in S_1} i_+(AXB) = r(A) + r(B) - r[A, B^*],$$  \hspace{1cm} (3.40)

$$\min_{X \in S_1} r(AXB) = \min_{X \in S_1} i_+(AXB) = r(A) + r(B) + i_+(C) - i_+(N),$$  \hspace{1cm} (3.41)

$$\max_{X \in S_1} i_-(AXB) = r(A) + r(B) + i_-(C) - i_-(N),$$  \hspace{1cm} (3.42)

$$\min_{X \in S_1} i_-(AXB) = 0,$$  \hspace{1cm} (3.43)

$$\max_{X \in S_1} r(AXB - C) = r(N) - r(A) - r(B) - r[A, B^*],$$  \hspace{1cm} (3.44)

$$\min_{X \in S_1} r(AXB - C) = r \left[ \begin{array}{ccc} A & 0 & C \\ 0 & B^* & C \end{array} \right] - r(A) - r(B).$$  \hspace{1cm} (3.45)

In consequence, the following hold.

(c) There exists an $X \in \mathbb{C}^{p \times q}$ such that $AXB \succ 0$ and $AXB \succeq C$ if and only if $r[A, B^*] = r(A) + r(B) - m$.

(d) There exists an $X \in \mathbb{C}^{p \times q}$ such that $0 \succ AXB \succeq C$ if and only if $C \prec 0$ and $r \left[ \begin{array}{ccc} A & 0 & C \\ 0 & B^* & C \end{array} \right] = r(A) + r(B)$.

(e) There exists an $X \in \mathbb{C}^{p \times q}$ such that $0 \succ AXB \succ C$ if and only if $C \preceq 0$.

(f) There always exists an $X \in \mathbb{C}^{p \times q}$ such that $AXB \succ 0$ and $AXB \succ C$.

Proof. From (3.10), $AXB$ and $AXB - C$ subject to $X \in S_1$ can be written as

$$AXB = C - CM(M^*CM)^\dagger M^*C + E_MUU^*E_M = A\hat{X}B + E_MUU^*E_M,$$

$$AXB - C = -CM(M^*CM)^\dagger M^*C + E_MUU^*E_M = A\hat{X}B - C + E_MUU^*E_M.$$  \hspace{1cm} (3.46)

Hence,

$$AXB \succ C - CM(M^*CM)^\dagger M^*C, \quad AXB - C \succ -CM(M^*CM)^\dagger M^*C$$  \hspace{1cm} (3.47)

hold for any $U \in \mathbb{C}^{m \times m}$, which implies (3.38) and (3.39).

Applying elementary matrix operations, congruence matrix operations and (2.29), we obtain

$$r(E_M) = m - r[E_A, F_B] = m - r(E_A) - r(F_BA) = r(A) + r(B) - r[A, B^*],$$  \hspace{1cm} (3.49)

$$r[E_M, A\hat{X}B] = r(E_M) + r(A\hat{X}BM) = r(E_M) + r[A\hat{X}BE_A, A\hat{X}FB_B] = r(E_M)$$

$$= r(A) + r(B) - r[A, B^*],$$  \hspace{1cm} (3.50)

$$i_+ \left[ \begin{array}{cc} A\hat{X}B & E_M \\ E_M & 0 \end{array} \right] = i_+ \left[ \begin{array}{cc} 0 & E_M \\ E_M & 0 \end{array} \right] = r(E_M) = r(A) + r(B) - r[A, B^*].$$  \hspace{1cm} (3.51)

Applying (2.63)–(2.69) to (3.40) and (3.47) and simplifying by (3.49)–(3.51), we obtain

$$\max_{X \in S_1} r(AXB) = \max_{U \in \mathbb{C}^{m \times m}} r(A\hat{X}B + E_MUU^*E_M) = r[E_M, A\hat{X}B] = r(A) + r(B) - r[A, B^*],$$

$$\min_{X \in S_1} r(AXB) = \min_{U \in \mathbb{C}^{m \times m}} r(A\hat{X}B + E_MUU^*E_M)$$

$$= i_+(A\hat{X}B) + r[E_M, A\hat{X}B] - i_+ \left[ \begin{array}{cc} A\hat{X}B & E_M \\ E_M & 0 \end{array} \right] = r(A) + r(B) + i_+(C) - i_+(N),$$

establishing (3.40)–(3.45). Note from (3.2), that

$$r(AXB - C) = r[-CM(M^*CM)^\dagger M^*C + E_MUU^*E_M]$$

$$= r[-CM(M^*CM)^\dagger M^*C, E_MUU^*E_M] = r[CM, E_MU].$$
Hence, we can find from (2.27), (2.28), (3.34) and (3.36) that
\[
\max_{X \in S_1} r(AXB - C) = \max_{U \in \mathbb{C}^{m \times m}} r( CM, E_M U ) = r( CM, E_M ) = r(M^*CM) + r(E_M) = r(N) - r(A) - r(B) - r[A, B^*],
\]
\[
\min_{X \in S_1} r(AXB - C) = \min_{U \in \mathbb{C}^{m \times m}} r( CM, E_M U ) = r(CM) = r \left[ \begin{array}{cc} A & 0 \\ 0 & B^* \end{array} \right] - r(A) - r(B),
\]
establishing (3.44) and (3.45). Result (b) can be shown similarly. \( \square \)

**Corollary 3.5** Let \( A \in \mathbb{C}^{m \times p} \), \( B \in \mathbb{C}^{q \times m} \) and \( C \in \mathbb{C}_R^{p \times q} \) be given, and let \( M \) and \( N \) be of the forms in (3.21). Also assume that (3.15) is feasible, and define
\[
S_2 = \{ X \in \mathbb{C}^{p \times q} | AXB \preceq C \}. 
\]

Then, the following hold.

(a) The maximal matrices of \( AXB \) and \( AXB - C \) subject to \( X \in S_2 \) in the Löwner partial ordering are given by
\[
\max\{AXB | X \in S_2\} = C - CM(M^*CM)^\dagger M^*C, \\
\max\{AXB - C | X \in S_2\} = -CM(M^*CM)^\dagger M^*C.
\]

(b) The extremal ranks and partial inertias of \( AXB \) and \( AXB - C \) subject to \( X \in S_2 \) are given by
\[
\max_{X \in S_2} r(AXB) = \max_{X \in S_2} i_-(AXB) = r(A) + r(B) - r[A, B^*], \\
\min_{X \in S_2} r(AXB) = \min_{X \in S_2} i_-(AXB) = r(A) + r(B) + i_-(C) - i_-(N), \\
\max_{X \in S_2} i_+(AXB) = r(A) + r(B) + i_+(C) - i_+(N), \\
\min_{X \in S_2} i_+(AXB) = 0, \\
\max_{X \in S_2} r(AXB - C) = r(N) - r(A) - r(B) - r[A, B^*], \\
\min_{X \in S_2} r(AXB - C) = r \left[ \begin{array}{cc} A & 0 \\ 0 & B^* \end{array} \right] - r(A) - r(B).
\]

In consequence,

(c) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( AXB \preceq 0 \) and \( AXB \preceq C \) if and only if \( r[A, B^*] = r(A) + r(B) - m \).

(d) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( 0 \preceq AXB \preceq C \) if and only if \( C \succeq 0 \) and \( r \left[ \begin{array}{cc} A & 0 \\ 0 & B^* \end{array} \right] = r(A) + r(B) \).

(e) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( 0 \preceq AXB \preceq C \) if and only if \( C \succeq 0 \).

(f) There always exists an \( X \in \mathbb{C}^{p \times q} \) such that \( AXB \preceq 0 \) and \( AXB \preceq C \).

In what follows, we give some consequences of Theorem 3.1 for different choice of \( C \) in (1.2).

**Theorem 3.6** Let \( A \in \mathbb{C}^{m \times p} \), \( B \in \mathbb{C}^{p \times m} \) and \( C \in \mathbb{C}_R^{m \times m} \) be given, and assume that \( AXB = C \) is consistent. Then, the following hold.

(a) The general solution of \( AXB \succeq C \) and the corresponding \( AXB \) can be written in the following parametric forms
\[
X = A^\dagger CB^\dagger + A^\dagger E_M UU^*E_M B^\dagger + W - A^\dagger AWBB^\dagger, \\
AXB = C + E_M UU^*E_M,
\]
where \( M = [E_A, F_B]\), and \( U \in \mathbb{C}^{m \times m} \) and \( W \in \mathbb{C}^{p \times q} \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( AXB \succeq C \) if and only if \( r(A) = r(B) = m \). In this case, the general solution of \( AXB \succeq C \) and the corresponding \( AXB \) can be written as
\[
X = A^\dagger CB^\dagger + A^\dagger UB^\dagger + W - A^\dagger AWBB^\dagger, \\
AXB = C + U,
\]
where \( 0 \preceq U \) and \( W \in \mathbb{C}^{p \times q} \) are arbitrary.
Corollary 3.7 Let \( A \in \mathbb{C}^{m \times p} \), \( B \in \mathbb{C}^{q \times m} \) and \( C \in \mathbb{C}^{m \times m} \) be given, and let \( M = [E_A, F_B] \). Then, the following hold.

(a) The inequality \( AXB \succ -CC^* \) (3.69) is always feasible; the general solution of (3.69) and the corresponding \( AXB \) can be written in the following parametric forms

\[
X = A^tCB^t - A^tE_ME_M^*B^t + W - A^tAWBB^t, \tag{3.65}
\]

\[
AXB = C - E_ME_M^*, \tag{3.66}
\]

where \( U \in \mathbb{C}^{m \times m} \) and \( W \in \mathbb{C}^{p \times q} \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( AXB \succ -CC^* \) (3.70) if and only if \( r \left[ \begin{array}{ccc} A & 0 & C \\ 0 & B^* & C \end{array} \right] = m + r[A, B^*]. \) In this case, the general solution of (3.72) can be written as (3.70), in which \( U \in \mathbb{C}^{m \times m} \) is any matrix such that \( r[CC^*M, E_MU] = m \), and \( W \in \mathbb{C}^{p \times q} \) is arbitrary.

(c) The inequality \( AXB \preceq CC^* \) (3.73) is always feasible; the general solution of (3.73) and the corresponding \( AXB \) can be written in the following parametric forms

\[
X = A^tCC^*B^t - A^tC(M^*C)^t(M^*C)C^*B^t - A^tE_ME_M^*E_M^*B^t + W - A^tAWBB^t, \tag{3.74}
\]

\[
AXB = CC^* - C(M^*C)^t(M^*C)C^* + E_ME_M^*E_M^*, \tag{3.75}
\]

where \( U \in \mathbb{C}^{m \times m} \) and \( W \in \mathbb{C}^{p \times q} \) are arbitrary.

(d) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( AXB \prec CC^* \) (3.76) if and only if \( r \left[ \begin{array}{ccc} A & 0 & C \\ 0 & B^* & C \end{array} \right] = m + r[A, B^*]. \) In this case, the general solution of (3.74) can be written as (3.76), in which \( U \) is any matrix such that \( r[CC^*M, E_MU] = m \), and \( W \in \mathbb{C}^{p \times q} \) is arbitrary.

Corollary 3.8 Let \( A \in \mathbb{C}^{m \times p} \), \( B \in \mathbb{C}^{q \times m} \) and \( C \in \mathbb{C}^{m \times m} \) be given, and let \( M = [E_A, F_B] \). Then, the following hold.

(a) There exists an \( X \in \mathbb{C}^{p \times q} \) such that \( AXB \succ CC^* \) (3.77) if and only if

\[
\mathcal{R}(C) \subseteq \mathcal{R}(A) \text{ and } \mathcal{R}(C) \subseteq \mathcal{R}(B^*). \tag{3.78}
\]

In this case, the general solution of (3.77) and the corresponding \( AXB \) can be written in the following parametric forms

\[
X = A^tCC^*B^t + A^tE_ME_M^*E_M^*B^t + W - A^tAWBB^t, \tag{3.79}
\]

\[
AXB = CC^* + E_ME_M^*E_M^*, \tag{3.80}
\]

where \( U \in \mathbb{C}^{m \times m} \) and \( W \in \mathbb{C}^{p \times q} \) are arbitrary.
(b) There exists an $X \in \mathbb{C}^{p \times q}$ such that
\[
AXB \succ CC^* \tag{3.81}
\]
if and only if $r(A) = r(B) = m$. In this case, the general solution of (3.81) can be written as (3.74), in which $U \in \mathbb{C}^{q \times q}$ is any matrix with $r(E_M U) = m$, and $W \in \mathbb{C}^{p \times q}$ is arbitrary.

(c) There exists an $X \in \mathbb{C}^{p \times q}$ such that
\[
AXB \preceq -CC^* \tag{3.82}
\]
if and only if (3.78) holds. In this case, the general solution of (3.82) can be written in the following parametric forms
\[
X = -A^\dagger CC^* B^\dagger - A^\dagger E_M U^* E_M B^\dagger + W - A^\dagger AWBB^\dagger, \tag{3.83}
\]
\[
AXB = -CC^* - E_M U^* E_M, \tag{3.84}
\]
where $U \in \mathbb{C}^{m \times m}$ and $W \in \mathbb{C}^{p \times q}$ are arbitrary.

(d) There exists an $X \in \mathbb{C}^{p \times q}$ such that
\[
AXB \prec -CC^* \tag{3.85}
\]
if and only if $r(A) = r(B) = m$. In this case, the general solution of (3.85) can be written as (3.88), in which $U \in \mathbb{C}^{m \times m}$ is any matrix with $r(E_M U) = m$, and $W \in \mathbb{C}^{p \times q}$ is arbitrary.

**Corollary 3.9** Let $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times m}$ be given, and let $M = [E_A, F_B]$. Then, the following hold.

(a) The general solution of
\[
AXB \succ 0
\]
and the corresponding $AXB$ can be written in the following parametric forms
\[
X = A^\dagger E_M U^* E_M B^\dagger + W - A^\dagger AWBB^\dagger, \tag{3.87}
\]
\[
AXB = E_M U^* E_M, \tag{3.88}
\]
where $U \in \mathbb{C}^{m \times m}$ and $W \in \mathbb{C}^{p \times q}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}^{p \times q}$ such that
\[
AXB \succ 0 \tag{3.89}
\]
if and only if $r(A) = r(B) = m$. In this case, the general solution of (3.89) can be written as (3.87), in which $U \in \mathbb{C}^{m \times m}$ is any matrix such that $r(E_M U) = m$, and $W \in \mathbb{C}^{p \times q}$ is arbitrary.

We next establish a group of formulas for calculating the ranks and inertias of $AXB - D$ subject to (3.1), and use the results obtained to derive necessary and sufficient conditions for the following two-sides inequality
\[
D \succ AXB \succ C \tag{3.90}
\]
and their variations to hold.

**Corollary 3.10** Let $A \in \mathbb{C}^{m \times p}$, $B \in \mathbb{C}^{q \times m}$ and $C, D \in \mathbb{C}_H^m$ be given, and let $S_1$ be of the forms in (3.37), and define
\[
K_1 = \begin{bmatrix} C & C & C & A & 0 \\ C & C & C & 0 & B^* \\ C & C & C - D & 0 & 0 \\ A^* & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} D & D & A & 0 \\ D & D & 0 & B^* \\ A^* & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} A & 0 & D \\ 0 & B^* & D \end{bmatrix}. \tag{3.91}
\]

Then, the extremal ranks and partial inertias of $AXB - D$ subject to $X \in S_1$ are given by
\[
\max_{X \in S_1} r(AXB - D) = r(K_3) - r[A, B^*], \tag{3.92}
\]
\[
\min_{X \in S_1} r(AXB - D) = i_+(K_1) + r(K_3) - r(K_2), \tag{3.93}
\]
\[
\max_{X \in S_1} i_+(AXB - D) = i_+(K_2) - r[A, B^*], \tag{3.94}
\]
\[
\max_{X \in S_1} i_-(AXB - D) = i_-(K_1) - i_-(K_2), \tag{3.95}
\]
\[
\min_{X \in S_1} i_+(AXB - D) = i_+(K_1) - i_+(K_2), \tag{3.96}
\]
\[
\min_{X \in S_1} i_-(AXB - D) = r(K_3) - i_-(K_2). \tag{3.97}
\]

In consequence, the following hold.
(a) There exists an $X \in \mathbb{C}^{p \times q}$ such that $AXB \succ D$ and $AXB \succeq C$ if and only if $i_-(K_2) = r[A, B^*] + m$.

(b) There exists an $X \in \mathbb{C}^{p \times q}$ such that $D \succ AXB \succeq C$ if and only if $D \succ C$ and $i_-(K_1) = i_-(K_2) + m$.

(c) There exists an $X \in \mathbb{C}^{p \times q}$ such that $D \succ AXB \succeq C$ if and only if $D \succ C$ and $i_+(K_1) = i_+(K_2)$.

(d) There exists an $X \in \mathbb{C}^{p \times q}$ such that $AXB \succ C$ and $AXB \succeq D$ if and only if $r(K_3) = i_-(K_2)$.

**Proof.** From (3.10), $AXB - D$ subject to $X \in S_1$ can be written as

$$AXB - D = C - D - CM(M^*CM)^\dagger M^*C + E_MUU^*E_M = A\hat{X}B - D + E_MUU^*E_M.$$  \hspace{1cm} (3.98)

Applying elementary matrix operations, congruence matrix operations and (2.29), we obtain

$$r(E_M) = m - r[E_A, F_B] = m - r(E_A) - r(F_BA)$$

$$= r(A) + r(B) - r[A, B^*],$$ \hspace{1cm} (3.99)

$$r[E_M, A\hat{X}B - D] = r(E_M) + r[(A\hat{X}B - D)M] = r(E_M) + r[(B^*\hat{X}B^* - D)E_A, (A\hat{X}B - D)F_B]$$

$$= r(E_M) + r[DE_A, DF_B] = r[A + D, 0 - D]$$

$$= (K_3) - r[A, B^*],$$ \hspace{1cm} (3.100)

$$i_\pm \left[ \begin{array}{c} A\hat{X}B - D \\ E_M \\ 0 \end{array} \right] = i_\pm [M(A\hat{X}B - D)M] + r(E_M) = i_\mp (MDM) + r(E_M)$$

$$= i_\mp (K_2) - r[A, B^*],$$ \hspace{1cm} (3.101)

$$i_\pm (A\hat{X}B - D) = i_\pm [C - D - CM(M^*CM)^\dagger M^*C] = i_\pm \left[ \begin{array}{c} M^*CM \\ CM \\ C - D \end{array} \right] - i_\mp (M^*CM)$$

$$= i_\pm \left[ \begin{array}{cccc} C & C & C & A \\ C & C & C & 0 \\ C & C & C & 0 \end{array} \right]$$

$$- i_\mp \left[ \begin{array}{cccc} C & C & A \\ C & C & 0 & B^* \\ A^* & 0 & 0 & 0 \\ 0 & B & 0 & 0 \end{array} \right]$$

$$= i_\pm (K_1) - i_\mp (K_2).$$ \hspace{1cm} (3.102)

Applying (2.64)–(2.69) to (3.98) and simplifying by (3.99)–(3.102), we obtain

$$\max_{X \in S_1} r(AXB - D) = \max_{U \in \mathbb{C}^{m \times m}} r(A\hat{X}B - D + E_MUU^*E_M) = r[E_M, A\hat{X}B - D]$$

$$= r(K_2) - r[A, B^*],$$

$$\min_{X \in S_1} r(AXB - D) = \min_{U \in \mathbb{C}^{m \times m}} r(A\hat{X}B - D + E_MUU^*E_M)$$

$$= i_+(A\hat{X}B - D) + r[E_M, A\hat{X}B - D] - i_+ \left[ \begin{array}{cc} A\hat{X}B - D & E_M \\ E_M & 0 \end{array} \right]$$

$$= i_+(K_1) + r(K_3) - r(K_2),$$

$$\max_{X \in S_1} i_+(AXB - D) = \max_{U \in \mathbb{C}^{m \times m}} i_+(A\hat{X}B - D + E_MUU^*E_M)$$

$$= i_+(K_2) - r[A, B^*],$$

$$\max_{X \in S_1} i_- (AXB - D) = \max_{U \in \mathbb{C}^{m \times m}} i_- (A\hat{X}B - D + E_MUU^*E_M)$$

$$= i_- (K_1) - i_- (K_2),$$

$$\min_{X \in S_1} i_+ (AXB - D) = \min_{U \in \mathbb{C}^{m \times m}} i_+ (A\hat{X}B - D + E_MUU^*E_M)$$

$$= i_+(K_1) - i_+(K_2),$$

$$\min_{X \in S_1} i_- (AXB - D) = \min_{U \in \mathbb{C}^{m \times m}} i_- (A\hat{X}B - D + E_MUU^*E_M)$$

$$= r[E_M, A\hat{X}B - D] - i_+ \left[ \begin{array}{cc} A\hat{X}B - D & E_M \\ E_M & 0 \end{array} \right]$$

$$= r(K_3) - i_-(K_2),$$

as required for (3.92)–(3.97).
4 General Hermitian solution of the LMI $AXA^* \succ (\succ, \preceq, \preceq) B$ and its properties

The LMIs in (1.3) are the simplest case of all LMIs with symmetric pattern. Due to the importance of matrix inequalities in the Löwner partial ordering, any contribution on this type of LMIs is valuable from both theoretical and practical points of view. Some previous work on solvability and general solutions of (1.3) and their applications in system and control theory were given in [20] by using SVDs of matrices. In a recent paper [24], necessary and sufficient conditions for the LMIs in (1.3) to hold were obtained by using some expansion formulas for the inertia of the matrix function $B - AXA^*$, while general Hermitian solution of $AXA^* \preceq B$ was established in [28]. In this section, we reconsider (1.3) and give a group of complete conclusions on Hermitian solutions of the LMIs and their algebraic properties.

Theorem 4.1 Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}_H^n$ be given, and let $N = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}$. Then,

(a) The following statements are equivalent:

(i) There exists an $X \in \mathbb{C}_H^n$ such that

$$AXA^* \succ B.$$ (4.1)

(ii) $E_AB \preceq 0$ and $\mathcal{R}(E_AB) = \mathcal{R}(E_B)$.

(iii) $i_+(N) = r(A)$ and $i_-(N) = r(A, B)$.

In this case, the general Hermitian solution of (4.1) and the corresponding $AXA^*$ can be written in the following parametric forms

$$X = A^\dagger B(A^\dagger)^* - A^\dagger BE_A(E_AB)\dagger E_AB(A^\dagger)^* + UU^* + W - A^\dagger AW A^\dagger A,$$ (4.2)

$$AXA^* = B - BE_A(E_{AB})\dagger E_FB + AUU^* A^*,$$ (4.3)

where $U \in \mathbb{C}^{n \times n}$ and $W \in \mathbb{C}_H^n$ are arbitrary.

(b) There exists an $X \in \mathbb{C}_H^n$ such that

$$AXA^* \succeq B.$$ (4.4)

if and only if

$$E_AB \succeq 0 \text{ and } r(E_AB) = r(E_A).$$ (4.5)

In this case, the general Hermitian solution of (4.1) can be written as (4.2), in which $U$ is any matrix such that $r[BE_A, AU] = m$, say, $U = I_n$, and $W \in \mathbb{C}_H^n$ is arbitrary.

(c) [28] The following statements are equivalent:

(i) There exists an $X \in \mathbb{C}_H^n$ such that

$$AXA^* \preceq B.$$ (4.6)

(ii) $E_AB \preceq 0$ and $\mathcal{R}(E_AB) = \mathcal{R}(E_B)$.

(iii) $i_+(N) = r(A, B)$ and $i_-(N) = r(B)$.

In this case, the general Hermitian solution of (4.6) and the corresponding $AXA^*$ can be written in the following parametric forms

$$X = A^\dagger B(A^\dagger)^* - A^\dagger BE_A(E_AB)\dagger E_FB - UU^* A^* - W - A^\dagger AW A^\dagger A,$$ (4.7)

$$AXA^* = B - BE_A(E_AB)\dagger E_FB - AUU^* A^*,$$ (4.8)

where $U \in \mathbb{C}^{n \times n}$ and $W \in \mathbb{C}_H^n$ are arbitrary.

(d) There exists an $X \in \mathbb{C}_H^n$ such that

$$AXA^* \prec B.$$ (4.9)

if and only if [15] holds. In this case, the general Hermitian solution of (4.9) can be written as (4.7), in which $U$ is any matrix such that $r[BE_A, AU] = m$, say, $U = I_n$, and $W \in \mathbb{C}_H^n$ is arbitrary.
Proof. Inequality (4.1) can be relaxed to the following quadratic matrix equation

$$AXA^* = B + YY^*.$$  (4.10)

By Lemma 2.3(a), (4.10) is solvable for $X$ if and only if $E_A(B + YY^*) = 0$, that is,

$$E_AYY^* = -E_A B.$$  (4.11)

By Lemma 2.1(b), (4.11) is solvable for $YY^*$ if and only if $E_A BE_A \preceq 0$ and $r(E_A BE_A) = r(E_A B)$, establishing the equivalence (i) and (ii) in (a). The equivalence (ii) and (iii) in (a) follows from (2.33) and $i_-(E_A BE_A) \preceq r(E_A BE_A) \preceq r(E_A B)$. In this case, the general solution of (4.11) can be written as

$$YY^* = -BE_A(E_A^2 B)^\dagger E_A B + AA^\dagger UU^* A A^\dagger,$$

where $U$ is an arbitrary matrix. Substituting the $YY^*$ into (4.10) gives

$$AXA^* = B - BE_A(E_A^2 B)^\dagger E_A B + AA^\dagger UU^* A A^\dagger.$$  (4.12)

By Lemma 2.3(a), the general Hermitian solution of (4.12) can be written as

$$X = A^\dagger B(A^\dagger)^* - A^\dagger BE_A(E_A^2 B)^\dagger E_A B(A^\dagger)^* + A^\dagger UU^*(A^\dagger)^* + W - A^\dagger AW A^\dagger A,$$

where $U \in \mathbb{C}^{n \times m}$ and $W \in \mathbb{C}_H^m$ are arbitrary. Replacing $A^\dagger UU^*(A^\dagger)^*$ with $UU^*$ gives (4.12), which is also the general solution of (4.11).

It can be seen from (4.3) that (4.1) holds if and only if

$$-BE_A(E_A^2 B)^\dagger E_A B + A U U^* A^* \succeq 0$$  (4.13)

for some $U$. Under (ii) in (a), we have

$$r(-BE_A(E_A^2 B)^\dagger E_A B + A U U^* A^*) = r(-BE_A(E_A^2 B)^\dagger E_A B, A U U^* A^*) = r(BE_A, A U).$$

Hence,

$$\max_U r(-BE_A(E_A^2 B)^\dagger E_A B + A U U^* A^*) = \max_U r(BE_A, A) = r(E_A^2 B) + r(A),$$

so that (4.4) holds if and only if $r(E_A^2 B) + r(A) = m$. Thus (b) follows. Results (c) and (d) can be shown similarly.

Concerning the constant term in (4.2), we have the consequence.

**Corollary 4.2** Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}_H^m$ be given, and let

$$\hat{X} = A^\dagger B(A^\dagger)^* - A^\dagger BE_A(E_A^2 B)^\dagger E_A B(A^\dagger)^*.$$  (4.14)

Then, the following hold.

(a) Under the condition that (4.1) is feasible, $\hat{X}$ is a Hermitian solution of (4.1), and

$$i_+(\hat{X}) = i_+(A \hat{X} A^*) = i_+(B),$$  (4.15)

$$i_-(\hat{X}) = i_-(A \hat{X} A^*) = r(A) + i_-(B) - r[A, B],$$  (4.16)

$$r(\hat{X}) = r(A \hat{X} A^*) = r(A) + r(B) - r[A, B],$$  (4.17)

$$i_-(B - A \hat{X} A^*) = r(B - A \hat{X} A^*) = r(B) - r(A \hat{X} A^*) = r[A, B] - r(A).$$  (4.18)

(b) Under the condition that (4.6) is feasible, $\hat{X}$ is a Hermitian solution of (4.6), and

$$i_+(\hat{X}) = i_+(A \hat{X} A^*) = r(A) + i_+(B) - r[A, B],$$  (4.19)

$$i_-(\hat{X}) = i_-(A \hat{X} A^*) = i_-(B),$$  (4.20)

$$r(\hat{X}) = r(A \hat{X} A^*) = r(A) + r(B) - r[A, B],$$  (4.21)

$$i_+(B - A \hat{X} A^*) = r(B - A \hat{X} A^*) = r(B) - r(A \hat{X} A^*) = r[A, B] - r(A).$$  (4.22)
Proof. Under the condition that (4.14) has a solution, set \( U = W = 0 \) in (4.2), we see that \( \tilde{X} \) in (4.14) is a Hermitian solution of \( AXA^* \geq B \). In this case, applying (2.37) to (4.14) and simplifying by congruence matrix operations, we obtain

\[
i_{\pm}(\tilde{X}) = i_{\pm}[A^t(BA)^* - A^tBEA(EA^*BEA)^tEAB(A^*)^*] =
\begin{bmatrix}
E_BE_A & E_A(BA)^* \\
E_A(BA)^* & A^tBEA
\end{bmatrix} - i_{\pm}(E_BE_A) = i_{\pm}(B) - i_{\pm}(E_BE_A),
\]

(4.23)

In consequence,

\[
i_{+}(\tilde{X}) = i_{+}(AX^*A^*) = i_{+}(B),
i_{-}(\tilde{X}) = i_{-}(AX^*A^*) = i_{-}(E_BE_A) = i_{-}(B) - r(E_BE_A) = i_{-}(B) + r(A) - r[A, B],
\]

establishing (4.15), (4.16) and (4.17). Applying (2.37) and simplifying by congruence matrix operations, we obtain

\[
i_{\pm}(B - A\tilde{X}A^*) = i_{\pm}[BEA(EA^*BEA)^tEAB]
\begin{bmatrix}
-E_BE_A & E_AB \\
E_AB & 0
\end{bmatrix} - i_{\pm}(E_BE_A) = r(E_BE_A) - r[E_BE_A].
\]

(4.25)

In consequence,

\[
i_{+}(B - A\tilde{X}A^*) = r(E_BE_A) - r(E_BE_A) = r(E_BE_A) - r(E_BE_A) = 0,
i_{-}(B - A\tilde{X}A^*) = r(E_BE_A) - r(E_BE_A) = r[A, B] - r(A),
\]

establishing (4.18). Result (b) can be shown similarly. \( \Box \)

**Corollary 4.3** Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^n \) be given. Then, the following hold.

(a) Under the condition that (4.11) is feasible, define

\[
S_1 = \{ X \in \mathbb{C}^n \mid AXA^* \geq B \}.
\]

(4.26)

Then, the minimal matrices of \( AXA^* \) and \( AXA^* - B \) subject to \( X \in S_1 \) in the Löwner partial ordering are given by

\[
\min_{B \geq 0}\{AXA^* \mid X \in S_1\} = B - BEA(EA^*BEA)^tEAB,
\]

(4.27)

\[
\min_{B \geq 0}\{AXA^* - B \mid X \in S_1\} = -BEA(EA^*BEA)^tEAB,
\]

(4.28)

while the extremal ranks and partial inertias of \( AXA^* \) and \( AXA^* - B \) subject to \( X \in S_1 \) are given by

\[
\max_{X \in S_1} r(AXA^*) = \max_{X \in S_1} i_{+}(AXA^*) = r(A),
\]

(4.29)

\[
\min_{X \in S_1} r(AXA^*) = \min_{X \in S_1} i_{+}(AXA^*) = i_{+}(B),
\]

(4.30)

\[
\max_{X \in S_1} i_{-}(AXA^*) = r(A) + i_{-}(B) - r[A, B],
\]

(4.31)

\[
\min_{X \in S_1} i_{-}(AXA^*) = 0,
\]

(4.32)

\[
\max_{X \in S_1} r(AXA^* - B) = r[A, B],
\]

(4.33)

\[
\min_{X \in S_1} r(AXA^* - B) = r[A, B] - r(A).
\]

(4.34)
Theorem 4.5

Let \( B \) hold.

Then, the maximal matrices of \( AXA^* \) and \( AXA^* - B \) subject to \( X \in S_2 \) in the Löwner partial ordering are given by

\[
\max \{ AXA^* \mid X \in S_2 \} = B - BEA(E_ABE_A)^\dagger E_AB, \tag{4.36}
\]
\[
\max \{ AXA^* - B \mid X \in S_2 \} = -BEA(E_ABE_A)^\dagger E_AB, \tag{4.37}
\]

while the extremal ranks and partial inertias of \( AXA^* \) and \( AXA^* - B \) subject to \( X \in S_2 \) are given by

\[
\max r(AXA^*) = \max i_-(AXA^*) = r(A), \tag{4.38}
\]
\[
\min r(AXA^*) = \min i_-(AXA^*) = i_-(B), \tag{4.39}
\]
\[
\max i_+(AXA^*) = r(A) + i_+(B) - r(A, B), \tag{4.40}
\]
\[
\min i_+(AXA^*) = 0, \tag{4.41}
\]
\[
\max r(AXA^* - B) = r(A, B), \tag{4.42}
\]
\[
\min r(AXA^* - B) = r(A, B) - r(A). \tag{4.43}
\]

Corollary 4.4 Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}_H^n \) be given, and assume that \( AXA^* = B \) is consistent. Then, the following hold.

(a) The general Hermitian solution of \( AXA^* \succ B \) and the corresponding \( AXA^* \) can be written in the following parametric forms

\[
X = A^\dagger B(A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.44}
\]
\[
AXA^* = B + AUU^* A^*, \tag{4.45}
\]

where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}_H^n \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}_H^n \) such that \( AXA^* \succ B \) if and only if \( r(A) = m \). In this case, the general Hermitian solution \( AXA^* \succ B \) and the corresponding \( AXA^* \) can be written in the following parametric forms

\[
X = A^\dagger B(A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.46}
\]
\[
AXA^* = B + AUU^* A^*, \tag{4.47}
\]

where \( U \in \mathbb{C}^{n \times n} \) is any matrix such that \( r(AU) = m \) and \( W \in \mathbb{C}_H^n \) is arbitrary.

(c) The general Hermitian solution of \( AXA^* \preceq B \) and the corresponding \( AXA^* \) can be written in the following parametric forms

\[
X = A^\dagger B(A^\dagger)^* - UU^* + W - A^\dagger AWA^\dagger A, \tag{4.48}
\]
\[
AXA^* = B - AUU^* A^*, \tag{4.49}
\]

where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}_H^n \) are arbitrary.

(d) There exists an \( X \in \mathbb{C}_H^n \) such that \( AXA^* \prec B \) if and only if \( r(A) = m \). In this case, the general Hermitian solution of \( AXA^* \prec B \) and the corresponding \( AXA^* \) can be written in the following parametric forms

\[
X = A^\dagger B(A^\dagger)^* - UU^* + W - A^\dagger AWA^\dagger A, \tag{4.50}
\]
\[
AXA^* = B - AUU^* A^*, \tag{4.51}
\]

where \( U \in \mathbb{C}^{n \times n} \) is any matrix such that \( r(AU) = m \) and and \( W \in \mathbb{C}_H^n \) is arbitrary.

Theorem 4.5 Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{m \times m} \) be given. Then, the following hold.

(a) The inequality

\[
AXA^* \succ -BB^* \tag{4.52}
\]

is always feasible; the general Hermitian solution of \( AXA^* \prec -BB^* \) and the corresponding \( AXA^* \) can be written in the following parametric forms

\[
X = A^\dagger B(E_AB) (E_AB)^\dagger (A^\dagger)^* + UU^* + W - A^\dagger AWA^\dagger A, \tag{4.53}
\]
\[
AXA^* = B(E_AB) (E_AB)^\dagger (A^\dagger)^* - BB^* + AUU^* A^*, \tag{4.54}
\]

where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}_H^n \) are arbitrary.
Corollary 4.7 Let \( A \in \mathbb{C}^{n \times n} \) such that
\[
AXA^* \succ -BB^*
\] (4.55)
if and only if \( r[ A, B ] = m \). In this case, the general Hermitian solution of (4.55) can be written as (4.53), in which \( U \) is any matrix such that \( r( AU ) = r( A ) \), say, \( U = I_n \), and \( W \in \mathbb{C}^n_H \) is arbitrary.

(c) The inequality
\[
AXA^* \preceq BB^*
\] (4.56)
is always feasible; the general Hermitian solution of (4.56) and the corresponding \( AXA^* \) can be written in the following parametric forms
\[
X = A^1BB^*(A^1)^* - A^1B(E_A B)^1(E_A B)B^*(A^1)^* - UU^* + W - A^1AWA^1A,
\] (4.57)
\[
AXA^* = BB^* - B(E_A B)^1(E_A B)B^* - AUU^*A^*,
\] (4.58)
where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^n_H \) are arbitrary.

(d) There exists an \( X \in \mathbb{C}^{n \times n} \) such that
\[
AXA^* \prec BB^*
\] (4.59)
if and only if \( r[ A, B ] = m \). In this case, the general Hermitian solution of (4.59) can be written as (4.57), in which \( U \) is any matrix such that \( r( AU ) = r( A ) \), say, \( U = I_n \), and \( W \in \mathbb{C}^n_H \) is arbitrary.

Corollary 4.6 Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{m \times m} \) be given. Then, the following hold.

(a) There exists an \( X \in \mathbb{C}^{n \times n} \) such that
\[
AXA^* \succ BB^*
\] (4.60)
if and only if \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \). In this case, the general Hermitian solution and the corresponding \( AXA^* \) can be written in the following parametric forms
\[
X = A^1BB^*(A^1)^* + UU^* + W - A^1AWA^1A,
\] (4.61)
\[
AXA^* = BB^* + AUU^*A^*,
\] (4.62)
where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^n_H \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{n \times n} \) such that
\[
AXA^* \succ BB^*
\] (4.63)
if and only if \( r(A) = m \). In this case, the general Hermitian solution of (4.63) can be written as (4.61), in which \( U \in \mathbb{C}^{n \times n} \) is any matrix such that \( r( AU ) = m \), and \( W \in \mathbb{C}^n_H \) is arbitrary.

(c) There exists an \( X \in \mathbb{C}^n_H \) such that
\[
AXA^* \preceq -BB^*
\] (4.64)
if and only if \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \). In this case, the general Hermitian solution of (4.64) and the corresponding \( AXA^* \) can be written in the following parametric forms
\[
X = -A^1BB^*(A^1)^* - UU^* + W - A^1AWA^1A,
\] (4.65)
\[
AXA^* = -BB^* - AUU^*A^*,
\] (4.66)
where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^n_H \) are arbitrary.

(d) There exists an \( X \in \mathbb{C}^n_H \) such that
\[
AXA^* \prec -BB^*
\] (4.67)
if and only if \( r(A) = m \). In this case, the general Hermitian solution of (4.67) can be written as (4.65), in which \( U \in \mathbb{C}^{n \times n} \) is any matrix such that \( r( AU ) = m \), and \( W \in \mathbb{C}^n_H \) is arbitrary.

Corollary 4.7 (24) Let \( A \in \mathbb{C}^{m \times n} \) be given. Then, the following hold.

(a) The general solution of \( AXA^* \succ 0 \) and the corresponding \( AXA^* \) can be written as can be written in the following parametric forms
\[
X = UU^* + W - A^1AWA^1A,
\] (4.68)
\[
AXA^* = -AUU^*A^*,
\] (4.69)
where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^n_H \) are arbitrary.
(b) There exists an \( X \in \mathbb{C}^{n \times n} \) such that \( AXA^* \succeq 0 \) if and only if \( r(A) = m \). In this case, the general Hermitian solution of \( AXA^* \succeq 0 \) can be written as (4.68), in which \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^{n \times n} \) are arbitrary.

(c) The general Hermitian solution of \( AXA^* \preceq 0 \) and the corresponding \( AXA^* \) can be written in the following parametric forms

\[
X = -UU^* + W - A^\dagger AW A^\dagger A, \tag{4.70}
\]
\[
AXA^* = -AUU^* A^*, \tag{4.71}
\]

where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^{n \times n} \) are arbitrary.

(d) There exists an \( X \in \mathbb{C}^{n \times n} \) such that \( AXA^* \asymp 0 \) if and only if \( r(A) = m \). In this case, the general Hermitian solution of \( AXA^* \asymp 0 \) can be written as (4.70), in which \( U \in \mathbb{C}^{n \times n} \) is any matrix such that \( r(AU) = m \), and \( W \in \mathbb{C}^{n \times n} \) is arbitrary.

**Theorem 4.8** Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{m \times m} \) be given. Then, the following hold.

(a) There exists an \( X \in \mathbb{C}^{n \times n} \) such that

\[
AXX^* A^* \asymp BB^* \tag{4.72}
\]

if and only if \( \mathcal{R}(B) \subseteq \mathcal{R}(A) \). In this case, a solution of (4.72) and the corresponding \( AXX^* A^* \) can be written in the following parametric forms

\[
XX^* = [A^\dagger (BB^* + AUU^* A^*)^{1/2} + FAW] [A^\dagger (BB^* + AUU^* A^*)^{1/2} + FAW]^*, \tag{4.73}
\]
\[
AXX^* A^* = BB^* + AUU^* A^*, \tag{4.74}
\]

where \( U \in \mathbb{C}^{m \times m} \) and \( W \in \mathbb{C}^{n \times n} \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{n \times n} \) such that

\[
AXX^* A^* \asymp BB^* \tag{4.75}
\]

if and only if \( r(A) = m \). In this case, a solution of (4.75) can be written as

\[
XX^* = [A^\dagger (BB^* + UU^*)^{1/2} + FAW] [A^\dagger (BB^* + UU^*)^{1/2} + FAW]^*, \tag{4.76}
\]
\[
AXX^* A^* = BB^* + UU^*, \tag{4.77}
\]

where \( U \in \mathbb{C}^{m \times m} \) is any matrix with \( r(U) = m \), and \( W \in \mathbb{C}^{n \times n} \) is arbitrary.

An application to partitioned matrices is given below.

**Corollary 4.9** Let

\[
\phi(X) = \begin{bmatrix} AXA^* & B \\ B^* & CC^* \end{bmatrix}. \tag{4.78}
\]

where \( A \in \mathbb{C}^{m \times n} \), \( B \in \mathbb{C}^{m \times p} \) and \( C \in \mathbb{C}^{p \times p} \) are given. Then, the following hold.

(a) There exists an \( X \in \mathbb{C}^{n \times n} \) such that \( \phi(X) \succeq 0 \) if and only if

\[
\mathcal{R}(B) \subseteq \mathcal{R}(A) \quad \text{and} \quad \mathcal{R}(B^*) \subseteq \mathcal{R}(C). \tag{4.79}
\]

In this case, the general solution of \( \phi(X) \succeq 0 \) can be written in the following parametric form

\[
X = A^\dagger B(CC^*)^{-1} B^* (A^\dagger)^* + UU^* + W - A^\dagger AW A^\dagger A, \tag{4.80}
\]

where \( U \in \mathbb{C}^{n \times n} \) and \( W \in \mathbb{C}^{n \times n} \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{n \times n} \) such that \( \phi(X) \succeq 0 \) in (4.78) if and only if

\[
r(A) = m \quad \text{and} \quad r(C) = p. \tag{4.81}
\]

In this case, the general solution of \( \phi(X) \succeq 0 \) can be written in the following parametric form

\[
X = A^\dagger B(CC^*)^{-1} B^* (A^\dagger)^* + UU^* + W - A^\dagger AW A^\dagger A, \tag{4.82}
\]

where \( U \in \mathbb{C}^{n \times n} \) is any matrix such \( r(AU) = m \) and \( W \in \mathbb{C}^{n \times n} \) is arbitrary.
Theorem 4.10 Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times m}$ be given. Then, the following hold.

(a) There exists an $X \in \mathbb{C}^{n \times n}$ such that both $AX \neq 0$ and

$$AXX^*A^* \preceq BB^*$$

(4.85)

if and only if

$$\mathcal{R}(A) \cap \mathcal{R}(B) \neq \{0\}.$$  (4.86)

In this case, a solution of (4.85) and the corresponding $AXX^*A^*$ can be written in the following parametric forms

$$XX^* = \left[ A^1(BF_B, V F_B, B^*)^{1/2} + FA W \right] A^1(BF_B, V F_B, B^*)^{1/2} + FA W]^*,$$  (4.87)

$$AXX^*A^* = BF_B, V F_B, B^*,$$  (4.88)

where $B_1 = E_A B, V$ is any matrix satisfying $0 \prec V \preceq I_m$, and $W \in \mathbb{C}^{n \times m}$ is arbitrary. The rank of (4.88) is

$$\max_{AXX^*A^* \preceq BB^*} r(AXX^*A^*) = r(A) + r(B) - r[A, B].$$  (4.89)

(b) There exists an $X \in \mathbb{C}^{n \times n}$ such that $AX \neq 0$ and

$$AXX^*A^* \prec BB^*$$

(4.90)

if and only if

$$A \neq 0 \text{ and } r(B) = m.$$  (4.91)

In this case, a solution of (4.90) can be written as (4.87), in which $V$ is any matrix satisfying $0 \prec V \preceq I_m$, and $W \in \mathbb{C}^{n \times m}$ is arbitrary.

(c) Under the condition $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, there always exists an $X \in \mathbb{C}^{n \times n}$ such that both $AX \neq 0$ and

$$AXX^*A^* \preceq BB^*$$

(4.92)

and a solution of (4.92) and the corresponding $AXX^*A^*$ can be written in the following parametric forms

$$XX^* = \left[ A^1(BVB^*)^{1/2} + FA W \right] A^1(BVB^*)^{1/2} + FA W]^*,$$  (4.93)

$$AXX^*A^* = BVB^*,$$  (4.94)

where $V$ is any matrix satisfying $0 \prec V \preceq I_m$, and $W \in \mathbb{C}^{n \times m}$ is arbitrary.

(d) Under the condition $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, there exists an $X \in \mathbb{C}^{n \times n}$ such that $AX \neq 0$ and

$$AXX^*A^* \prec BB^*$$

(4.95)

if and only if $r(B) = m$. In this case, a solution of (4.95) can be written as (4.93), in which $V$ is any matrix satisfying $0 \prec V \preceq I_m$, and $W \in \mathbb{C}^{n \times m}$ is arbitrary.

Proof. It can be seen from Lemma (2.9g) that if there exists an $X$ such that $AX \neq 0$ and (4.86) hold, then $\mathcal{R}(AX) \subseteq \mathcal{R}(B)$, which obviously implies that (4.86) holds. On the other hand, it can be derived from $E_ABFE_{AB} = 0$ that

$$AA^1BF_{E_{AB}} = BF_{E_{AB}},$$  (4.96)
and from \((2.27)\) and \((2.28)\) that
\[
r(BF_{E_A B}) = r\left[ \begin{bmatrix} B \\ E_A B \end{bmatrix} - r(E_A B) = r(A) + r(B) - r[A, B] = \dim(\mathcal{H}(A) \cap \mathcal{H}(B)). \tag{4.97} \right.
\]
Hence if \((4.85)\) holds, then \(BF_{E_A B} \neq 0\) and \(\mathcal{H}(BF_{E_A B}) = \mathcal{H}(A) \cap \mathcal{H}(B)\) by \((4.30)\) and \((4.37)\). In this case,
\[
AA^\dagger BF_{E_A B} V F_{E_A B} B^* = BF_{E_A B} V F_{E_A B} B^*.
\]
Thus we can derive from \((4.87)\) and Lemma \((2.10)\) (c) that
\[
BB^* - AX X^* A^* = BB^* - BF_{E_A B} V F_{E_A B} B^* = B(I_m - F_{E_A B} V F_{E_A B}) B^* > 0,
\]
that is, \((4.87)\) is a solution of \((4.85)\). The two conditions in \((4.91)\) are obvious under the condition that both \(AX \neq 0\) and \((4.90)\) hold. Conversely, if \((4.91)\) holds, we can derive from \((4.90)\) and Lemma \((2.13)\) (d) that \(I_m - F_{E_A B} V F_{E_A B} > 0\) and
\[
BB^* - AX X^* A^* = BB^* - BF_{E_A B} V F_{E_A B} B^* = B(I_m - F_{E_A B} V F_{E_A B}) B^* > 0.
\]
Results (c) and (d) are direct consequences of (a) and (b). \(\square\)

A direct consequence of Corollary \((3.10)\) is given below.

**Corollary 4.11** Let \(A \in \mathbb{C}^{m \times n}\) and \(B, C \in \mathbb{C}^m_H\) be given, and let \(S_1\) be of the form in \((4.26)\), and define
\[
K_1 = \begin{bmatrix} B & B & A \\ B & B & C \\ A^* & 0 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}.
\tag{4.98}
\]
Then, the extremal ranks and partial inertias of \(AX A^* - C\) subject to \(X \in S_1\) are given by
\[
\max_{X \in S_1} r(AX A^* - C) = r[A, C], \tag{4.99}
\]
\[
\min_{X \in S_1} r(AX A^* - C) = i_+(K_1) - r(K_2) + r[A, C], \tag{4.100}
\]
\[
\max_{X \in S_1} i_+(AX A^* - C) = i_-(K_2), \tag{4.101}
\]
\[
\max_{X \in S_1} i_-(AX A^* - C) = i_-(K_1) - i_-(K_2), \tag{4.102}
\]
\[
\min_{X \in S_1} i_+(AX A^* - C) = i_+(K_1) - i_+(K_2), \tag{4.103}
\]
\[
\min_{X \in S_1} i_-(AX A^* - C) = r[A, C] - i_-(K_2). \tag{4.104}
\]
In consequence, the following hold.

(a) There exists an \(X \in \mathbb{C}^m_H\) such that \(AX A^* \succ B\) and \(AX A^* \succ C\) if and only if \(i_-(K_2) = m\).

(b) There exists an \(X \in \mathbb{C}^m_H\) such that \(C \succ AX A^* \succeq B\) if and only if \(C \succeq B\) and \(i_-(K_1) = i_-(K_2) + m\).

(c) There exists an \(X \in \mathbb{C}^m_H\) such that \(C \succ AX A^* \succeq B\) if and only if \(D \succeq C\) and \(i_+(K_1) = i_+(K_2)\).

(d) There exists an \(X \in \mathbb{C}^m_H\) such that \(AX A^* \succ B\) and \(AX A^* \succeq C\) if and only if \(i_-(K_2) = r[A, C]\).

### 5 General solution of \(AX + (AX)^* \succeq (\succ, \preceq, \prec) B\) and its properties

The inequality in \((1.4)\) was approached in \((25)\) by using a relation method and their general solutions were given analytically. In this section, we reconsider this inequality and give some new conclusions on algebraic properties of its solution.

**Theorem 5.1** Let \(A \in \mathbb{C}^{m \times n}\) and \(B \in \mathbb{C}^m_H\) be given, and let \(M = \begin{bmatrix} B & A \\ A^* & 0 \end{bmatrix}\). Then, the following hold.

(a) \((25)\) The following statements are equivalent:

(i) There exists an \(X \in \mathbb{C}^n\) such that
\[
AX + (AX)^* \succeq B. \tag{5.1}
\]

(ii) \(E_A B E_A \preceq 0\).

(iii) \(i_+(M) = r(A)\).
In this case, the general solution of (5.1) and the corresponding \( AX + (AX)^* \) can be written in the following parametric forms

\[
X = \frac{1}{2} A^\dagger B \hat{A} + \frac{1}{2} A^\dagger (AU + J^\frac{1}{2}) (AU + J^\frac{1}{2})^* \hat{A} + VA^* + FAW, \tag{5.2}
\]

\[
AX + (AX)^* = B + (AU + J^\frac{1}{2}) (AU + J^\frac{1}{2})^*, \tag{5.3}
\]

where \( J = -E_A BE_A, \hat{A} = 2I_m - AA^\dagger \), and \( U, W \in \mathbb{C}^{n \times m} \) and \( V \in \mathbb{C}^{\mathbb{R}} \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{n \times m} \) such that

\[
AX + (AX)^* \succ B \tag{5.4}
\]

if and only if

\[
E_A BE_A \preceq 0 \quad \text{and} \quad \mathcal{R}(E_A) = \mathcal{R}(E_A), \tag{5.5}
\]

or equivalently, \( i_\cdot (M) = m \). In this case, the general solution of (5.1) can be written as (5.2), in which \( U \) is any matrix such that \( r(AU + J^\frac{1}{2}) = m \), say \( U = A^* \), \( V \in \mathbb{C}^{n \times m} \) and \( W \in \mathbb{C}^{n \times m} \) are arbitrary.

(c) Under (a), let

\[
S_1 = \{ X \in \mathbb{C}^{n \times m} \mid AX + (AX)^* \succ B \}. \tag{5.6}
\]

Then, the extremal ranks and partial inertias of \( AX + (AX)^* \) and \( AX + (AX)^* - B \) subject to \( X \in S_1 \) are given by

\[
\max_{X \in S_1} r[AX + (AX)^*] = \min \{ 2r(A), r[A, B] \}, \tag{5.7}
\]

\[
\min_{X \in S_1} r[AX + (AX)^*] = \max \{ 2r(A) + 2r[A, B] - 2r(M), r(B) - m, i_+(B) + r(A) + r[A, B] - r(M), i_-(B) + r(A) + r[A, B] - r(M) - m \}, \tag{5.8}
\]

\[
\max_{X \in S_1} i_+ [AX + (AX)^*] = r(A), \tag{5.9}
\]

\[
\min_{X \in S_1} i_- [AX + (AX)^*] = \min \{ r(A), i_-(B) \}, \tag{5.10}
\]

\[
\max_{X \in S_1} i_+ [AX + (AX)^*] = \max \{ r[A, B] + r(A) - r(M), i_+(B) \}, \tag{5.11}
\]

\[
\min_{X \in S_1} i_- [AX + (AX)^*] = r[A, B] + r(A) - r(M), \tag{5.12}
\]

\[
\max_{X \in S_1} r[AX + (AX)^* - B] = r(M) - r(A), \tag{5.13}
\]

\[
\min_{X \in S_1} r[AX + (AX)^* - B] = r(M) - 2r(A). \tag{5.14}
\]

In consequence,

(d) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( AX + (AX)^* \succeq 0 \) and \( AX + (AX)^* \succeq B \) if and only if \( r(A) = m \).

(e) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( 0 \succeq AX + (AX)^* \succeq B \) if and only if \( r(A) = m \) and \( B \preceq 0 \).

(f) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( AX + (AX)^* \succeq 0 \) and \( AX + (AX)^* \succeq B \) if and only if \( r(N) = r[A, B] + r(A) \).

(g) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( 0 \succeq AX + (AX)^* \succeq B \) if and only if \( B \preceq 0 \).

**Proof.** Inequality (5.1) can be relaxed to the following quadratic matrix equation

\[
AX + (AX)^* = B + YY^*, \tag{5.15}
\]

where \( Y \in \mathbb{C}^{m \times m} \). From Lemma 2.4(a), there exists an \( X \) that satisfies (5.15) if and only if \( YY^* \) satisfies \( E_A(B + YY^*) E_A = 0 \), that is,

\[
E_A YY^* E_A = -E_A BE_A = J. \tag{5.16}
\]

Further by Lemma 2.3(c), there exists a \( YY^* \) that satisfies (5.16) if and only if (ii) of (a) holds, in which case, the general solution of (5.16) can be written as

\[
YY^* = (AU + J^\frac{1}{2})(AU + J^\frac{1}{2})^*, \tag{5.17}
\]
where \( U \in \mathbb{C}^{n \times m} \) is arbitrary. Substituting this \( YY^* \) into (5.15) gives
\[
AX + (AX)^* = B + (AU + J^\Delta)(AU + J^\Delta)^*.
\]
(5.18)

Applying Lemma 2.4(a) to this equation, we obtain (5.2).

Setting (5.13) equal to \( m \) gives \( r(M) - r(A) = m \), i.e., \( r(E_ABE_A) = r(E_A) \) by (2.33), which is further equivalent to (5.5). The equivalence of \( i_-(M) = m \) and (5.5) follows from (2.33) and \( i_-(E_ABE_A) \leq r(E_ABE_A) \leq r(E_A) \).

Applying Lemma 2.13(a) to (5.18), we obtain
\[
\max_{U \in \mathbb{C}^{n \times m}} r[B + (AU + J^\Delta)(AU + J^\Delta)^*] = \min \left\{ r[A, B, J^\Delta], \frac{r(B) + m}{r(B) + m} \right\},
\]
(5.19)

\[
\min_{U \in \mathbb{C}^{n \times m}} r[B + (AU + J^\Delta)(AU + J^\Delta)^*] = 2r[A, B, J^\Delta] + \max \{ h_1, h_2, h_3, h_4 \},
\]
(5.20)

\[
\max_{U \in \mathbb{C}^{n \times m}} i_+[B + (AU + J^\Delta)(AU + J^\Delta)^*] = \min \left\{ i_+[B + J A A^* 0], i_+(B) + m \right\},
\]
(5.21)

\[
\max_{U \in \mathbb{C}^{n \times m}} i_-[B + (AU + J^\Delta)(AU + J^\Delta)^*] = \min \left\{ i_-[B + J A A^* 0], i_-(B) \right\},
\]
(5.22)

\[
\min_{U \in \mathbb{C}^{n \times m}} i_+[B + (AU + J^\Delta)(AU + J^\Delta)^*] = r[A, B, J^\Delta] + \max \left\{ i_+[B + J A A^* 0] - r[B A A^* 0 0], i_+(B) - r[A, B] \right\},
\]
(5.23)

\[
\min_{U \in \mathbb{C}^{n \times m}} i_-[B + (AU + J^\Delta)(AU + J^\Delta)^*] = r[A, B, J^\Delta] + \max \left\{ i_-[B + J A A^* 0] - r[B A A^* 0 0], i_-(B) - r[A, B] - m \right\},
\]
(5.24)

where
\[
h_1 = r[B + J A A^* 0] - 2r[B A A^* 0 0],
\]
\[
h_2 = r(B) - 2r[A, B] - m,
\]
\[
h_3 = i_-[B + J A A^* 0] - r[B A A^* 0 0] + i_+(B) - r[A, B],
\]
\[
h_4 = i_+[B + J A A^* 0] - r[B A A^* 0 0] + i_-(B) - r[A, B] - m.
\]

Simplifying the ranks and partial inertias of the block matrices in (5.19)–(5.24) gives
\[
r[A, B, J^\Delta] = r[A, B, J] = r[A, B, E_ABE_A] = r[A, B],
\]
(5.25)

\[
i_\pm[B + J A A^* 0] = i_\pm[B - E_ABE_A A^* 0] = i_\pm[0 A A^* 0] = r(A),
\]
(5.26)

\[
r[B A J^\Delta A^* 0] = r[B A A^* 0 0] = r[B A E_ABE_A A^* 0 0] = r[B A E_ABE_A A^* 0 0] = r[B A E_ABE_A A^* 0 0] = r(A).
\]
(5.27)

Substituting (5.25)–(5.27) into (5.19)–(5.24) gives (5.7)–(5.12). It can be seen from (5.18) that
\[
r[A, B, J^\Delta] = r(AU + J^\Delta).
\]
(5.28)

Hence, we derive from (2.42) and (2.33) that
\[
\max_{U \in \mathbb{C}^{n \times m}} r[A, B, J^\Delta] = \max_{U \in \mathbb{C}^{n \times m}} r[A, J^\Delta] = r[A, E_ABE_A] = r(A) + r(E_ABE_A) \quad \text{(by (2.27))}
\]
\[
\min_{U \in \mathbb{C}^{n \times m}} r[A, B, J^\Delta] = \min_{U \in \mathbb{C}^{n \times m}} r[A, J^\Delta] = r[A, E_ABE_A] - r(A) = r(E_ABE_A) \quad \text{(by (2.27)),}
\]

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establishing (5.13) and (5.14).

The following results can be shown similarly.

**Theorem 5.2** Let $A \in \mathbb{C}^{n \times n}$ and $B, C \in \mathbb{C}^{m \times n}$ be given, and let $M = \begin{bmatrix} B & A^* \\ A^* & 0 \end{bmatrix}$. Then, the following hold.

(a) Let $\mathcal{C}$ be a parametric forms.

(b) There exists an $X \in \mathbb{C}^{n \times m}$ such that $A^* (AX)^* < B$.

(c) Under (a), let $\mathcal{S}_2 = \{ X \in \mathbb{C}^{n \times m} | AX + (AX)^* \preceq B \}$. Then, the extremal ranks and partial inertias of $AX + (AX)^*$ and $AX + (AX)^* - B$ subject to $X \in \mathcal{S}_2$ are given by

\[
\max_{X \in \mathcal{S}_2} r[ AX + (AX)^* ] = \min \{ 2r(A), \ r[A, B] \},
\]

\[
\min_{X \in \mathcal{S}_2} r[ AX + (AX)^* ] = \max \{ 2r(A) + 2r[A, B] - 2r(N), \ r(B) - m, \ i_+(B) + r(A) + r[A, B] - r(N) - m, \ i_-(B) + r(A) + r[A, B] - r(N) \},
\]

\[
\max_{X \in \mathcal{S}_2} i_+ [ AX + (AX)^* ] = \min \{ r(A), \ i_+(B) \},
\]

\[
\min_{X \in \mathcal{S}_2} i_- [ AX + (AX)^* ] = r(A),
\]

\[
\max_{X \in \mathcal{S}_2} i_+ [ AX + (AX)^* ] = r(A),
\]

\[
\min_{X \in \mathcal{S}_2} i_- [ AX + (AX)^* ] = r[A, B] + r(A) - r(N),
\]

\[
\min_{X \in \mathcal{S}_2} i_- [ AX + (AX)^* ] = r[A, B] + r(A) - r(N),
\]

\[
\max_{X \in \mathcal{S}_2} r[ AX + (AX)^* - B ] = r(N) - r(A),
\]

\[
\min_{X \in \mathcal{S}_2} r[ AX + (AX)^* - B ] = r(N) - 2r(A).
\]

In consequence,

(d) There exists an $X \in \mathbb{C}^{n \times m}$ such that $0 \prec AX + (AX)^* \preceq B$ if and only if $r(A) = m$ and $B \succ 0$.

(e) There exists an $X \in \mathbb{C}^{n \times m}$ such that $AX + (AX)^* \prec 0$ and $AX + (AX)^* \preceq B$ if and only if $r(A) = m$.

(f) There exists an $X \in \mathbb{C}^{n \times m}$ such that $AX + (AX)^* \preceq 0$ and $AX + (AX)^* \preceq B$ if and only if $r(N) = r[A, B] + r(A)$.

(g) There exists an $X \in \mathbb{C}^{n \times m}$ such that $0 \preceq AX + (AX)^* \preceq B$ if and only if $B \succeq 0$. 

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Theorem 5.1 established identifying conditions for the LMI in (1.2) to be solvable, and gave general expression of the matrix $X$ satisfying (1.2). In particular, the general solutions in (5.2) and (5.30) are represented in closed-forms by using generalized inverse of the given matrices and arbitrary matrices. Hence, they can be directly used to deal with various problems on the inequality in (1.2) and its properties. In what follows, we present some consequences of Theorem 5.1 when $A$ and $B$ satisfy some more conditions.

**Corollary 5.3** Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times m}$ be given, and assume that there exists an $X \in \mathbb{C}^{n \times m}$ such that $AX + (AX)^* = B$. Then, the following hold.

(a) The general solution $X \in \mathbb{C}^{n \times m}$ of

$$AX + (AX)^* \succ B$$

and the corresponding $AX + (AX)^*$ can be written in the following parametric forms

$$X = \frac{1}{2} A^t B (2I_m - AA^t) + UU^* A^* + VA^* + FAW,$$

$$AX + (AX)^* = B + 2AUU^* A^*,$$

where $U \in \mathbb{C}^{n \times n}$, $W \in \mathbb{C}^{n \times m}$ and $V \in \mathbb{C}^{n \times n}_{SH}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* \succ B$$

if and only if $r(A) = m$. In this case, the general solution can be written as (5.44), in which $U$ is any matrix such that $r(AU) = m$, and $W \in \mathbb{C}^{n \times m}$ and $V \in \mathbb{C}^{n \times n}_{SH}$ are arbitrary.

(c) The general solution $X \in \mathbb{C}^{n \times m}$ of

$$AX + (AX)^* \preceq B$$

and the corresponding $AX + (AX)^*$ can be written in the following parametric forms

$$X = \frac{1}{2} A^t B (2I_m - AA^t) - UU^* A^* + VA^* + FAW,$$

$$AX + (AX)^* = B - 2AUU^* A^*,$$

where $U \in \mathbb{C}^{n \times n}$, $W \in \mathbb{C}^{n \times m}$ and $V \in \mathbb{C}^{n \times n}_{SH}$ are arbitrary.

(d) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* \prec B$$

if and only if $r(A) = m$. In this case, the general solution of (5.50) can be written as (5.48), in which $U$ is any matrix such that $r(AU) = m$, and $W \in \mathbb{C}^{n \times m}$ and $V \in \mathbb{C}^{n \times n}_{SH}$ are arbitrary.

**Corollary 5.4** Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{m \times k}$ be given. Then, the following hold.

(a) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* \succeq BB^*$$

if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general solution and the corresponding $AX + (AX)^*$ can be written as

$$X = \frac{1}{2} A^t BB^* + UU^* A^* + VA^* + FAW,$$

$$AX + (AX)^* = BB^* + 2UU^* A^*,$$

where $U \in \mathbb{C}^{n \times n}$, $V \in \mathbb{C}^{n \times m}$ and $W \in \mathbb{C}^{n \times m}$ are arbitrary.

(b) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* \succ BB^*$$

if and only if both $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $r(A) = m$. In this case, the general solution can be written as (5.52), in which $U$ is any matrix with $r(AU) = m$, and $V \in \mathbb{C}^{n \times m}$ and $W \in \mathbb{C}^{n \times m}$ are arbitrary.

(c) There exists an $X \in \mathbb{C}^{n \times m}$ such that

$$AX + (AX)^* \preceq -BB^*$$

if and only if $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. In this case, the general solution and the corresponding $AX + (AX)^*$ can be written as

$$X = -\frac{1}{2} A^t BB^* - UU^* A^* + VA^* + FAW,$$

$$AX + (AX)^* = -BB^* - 2UU^* A^*,$$

where $U \in \mathbb{C}^{n \times n}$, $V \in \mathbb{C}^{n \times m}$ and $W \in \mathbb{C}^{n \times m}$ are arbitrary.
Corollary 5.6  Let \( B \in \mathbb{C}^{n \times n} \) be given. Then, the following hold.

(a) The general solution \( X \in \mathbb{C}^{n \times n} \) of
\[ X + X^* \succ BB^* \] (5.59)
and the corresponding \( X + X^* \) can be written as
\[ X = \frac{1}{2} BB^* + UU^* + V - V^*, \] (5.60)
\[ X + X^* = BB^* + 2UU^*, \] (5.61)
where \( U, V \in \mathbb{C}^{n \times n} \) are arbitrary.

(b) The general solution \( X \in \mathbb{C}^{n \times n} \) of
\[ X + X^* \succeq BB^* \] (5.62)
can be written as (5.60), in which \( U, V \in \mathbb{C}^{n \times n} \) are arbitrary with \( r(U) = n \).

(c) The general solution \( X \in \mathbb{C}^{n \times n} \) of
\[ X + X^* \preceq BB^* \] (5.63)
and the corresponding \( X + X^* \) can be written as
\[ X = \frac{1}{2} BB^* - UU^* + V - V^*, \] (5.64)
\[ X + X^* = BB^* - 2UU^*, \] (5.65)
where \( U, V \in \mathbb{C}^{n \times n} \) are arbitrary.

(d) The general solution \( X \in \mathbb{C}^{n \times n} \) of the inequality
\[ X + X^* \prec BB^* \] (5.66)
can be written as (5.64), in which \( U, V \in \mathbb{C}^{n \times n} \) are arbitrary with \( r(U) = n \).

Corollary 5.6  Let \( A \in \mathbb{C}^{m \times n} \) be given. Then, the following hold.

(a) The general solution \( X \in \mathbb{C}^{n \times m} \) of
\[ AX + (AX)^* \succ 0 \] (5.67)
and the corresponding \( AX + (AX)^* \) can be written as
\[ X = UU^* A^* + VA^* + F_A W, \] (5.68)
\[ AX + (AX)^* = 2AUU^* A^* \] (5.69)
where \( U \in \mathbb{C}^{n \times n}, V \in \mathbb{C}^{m}_\text{SH} \) and \( W \in \mathbb{C}^{n \times m} \) are arbitrary.

(b) There exists an \( X \in \mathbb{C}^{n \times m} \) such that
\[ AX + (AX)^* \succ 0 \] (5.70)
if and only if \( r(A) = m \). In this case, the general solution can be written as in (5.68), in which \( U \in \mathbb{C}^{n \times n} \) is any matrix with \( r(AU) = m \), \( V \in \mathbb{C}^{m}_\text{SH} \) and \( W \in \mathbb{C}^{n \times m} \) are arbitrary.

(c) The general solution \( X \in \mathbb{C}^{n \times m} \) of
\[ AX + (AX)^* \preceq 0 \] (5.71)
and the corresponding \( AX + (AX)^* \) can be written as
\[ X = -UU^* A^* + VA^* + F_A W, \] (5.72)
\[ AX + (AX)^* = -2AUU^* A^* \] (5.73)
where \( U \in \mathbb{C}^{n \times n}, V \in \mathbb{C}^{m}_\text{SH} \) and \( W \in \mathbb{C}^{n \times m} \) are arbitrary.
Corollary 5.7 Let $A, B \in \mathbb{C}^{m \times n}$ be given. Then, there always exists an $X \in \mathbb{C}^{n \times m}$ that satisfies
\begin{equation}
(A + B)X + X^*(A + B)^* \succ AB + BA^*. \tag{5.76}
\end{equation}
The general solution and the corresponding $(A + B)X + X^*(A + B)^*$ can be written as
\begin{align*}
X &= \frac{1}{2}[(A + B)^* + (UU^* + V - V^*)(A + B)^* + F_{(A + B)}W] \\
&= (A + B)X + X^*(A + B)^* = (A + B)(A + B)^* + (A + B)UU^*(A + B)^* \\
&\text{where } U, V, W \in \mathbb{C}^{n \times n} \text{ are arbitrary.}
\end{align*}
In particular, there exists an $X \in \mathbb{C}^{m \times m}$ such that
\begin{equation}
(A + B)X + X^*(A + B)^* \succ AB + BA \tag{5.79}
\end{equation}
if and only if $r(A + B) = m$.

We next establish a group of formulas for calculating the ranks and inertias of $AX + (AX)^* - C$ subject to (5.1), and use the results obtained to derive necessary and sufficient conditions for the following two-side LMI
\begin{equation}
C \succ AX + (AX)^* \succ B \tag{5.80}
\end{equation}
and their variations to hold.

Theorem 5.8 Let $A \in \mathbb{C}^{m \times n}$ and $B, C \in \mathbb{C}^{n \times m}$ be given, $S_1$ be as given in (5.10), and let
\begin{align*}
N &= \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}, \\
K_1 &= \begin{bmatrix} B & C & A \\ A^* & 0 & 0 \\ 0 & A^* & 0 \end{bmatrix}, \\
K_2 &= \begin{bmatrix} B & C & A \\ A^* & 0 & 0 \\ 0 & A^* & 0 \end{bmatrix}.
\end{align*}
Then, the extremal ranks and partial inertias of $AX + (AX)^* - C$ subject to $X \in S_1$ are given by
\begin{align*}
\max_{X \in S_1} r[AX + (AX)^* - C] &= \min \{ r(K_2) - r(A), \ r(N) \}, \\
\min_{X \in S_1} r[AX + (AX)^* - C] &= \max \{ t_1, \ t_2, \ t_3, \ t_4 \}, \\
\max_{X \in S_1} i_+ [AX + (AX)^* - C] &= i_+(N), \\
\max_{X \in S_1} i_- [AX + (AX)^* - C] &= \min \{ i_-(B - C), \ i_+(N) \}, \\
\min_{X \in S_1} i_+ [AX + (AX)^* - C] &= \max \{ r(K_2) + i_+(N) - r(K_1), \ r(K_2) + i_-(B - C) - r(A, B - C) - r(A), \ r[K_2 + i_+(B - C) - r(A)] \}, \\
\min_{X \in S_1} i_- [AX + (AX)^* - C] &= \max \{ r(K_2) + i_+(N) - r(K_1), \ r(K_2) + i_-(B - C) - r(A, B - C) - r(A) - m, \ r[K_2 + i_+(B - C) - r(A)] \},
\end{align*}
where
\begin{align*}
t_1 &= 2r(K_2) + r(N) - 2r(K_1), \\
t_2 &= 2r(K_2) + r(B - C) - 2r(A, B - C) - 2r(A) - m, \\
t_3 &= 2r(K_2) + i_+(N) + i_+(B - C) - r(A) - r(A, B - C) - r(K_1), \\
t_4 &= 2r(K_2) + i_-(N) + i_-(B - C) - r(A) - r(A, B - C) - r(K_1) - m.
\end{align*}
In consequence,
(a) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( C \succ AX + (AX)^* \succeq B \) if and only if \( i_+(N) \succeq m \) and \( C \succ B \).
(b) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( AX + (AX)^* \succeq B \) and \( AX + (AX)^* \succeq C \) if and only if \( i_-(N) \succeq m \).
(c) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( AX + (AX)^* \succeq B \) and \( AX + (AX)^* \succeq C \) if and only if
\[
r(K_2) + i_+(N) - r(K_1) = 0 \quad \text{and} \quad r(K_2) + i_-(B - C) = r[A, B - C] + r(A) + m.\]
(d) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( C \succeq AX + (AX)^* \succeq B \) if and only if
\[
C \succeq B, \quad i_-(N) = r(A) \quad \text{and} \quad r(K_2) = r[A, B - C] + r(A).\]

**Proof.** Note from (5.93) that
\[
AX + (AX)^* - C = B - C + (AU + J_{\frac{1}{2}})(AU + J_{\frac{1}{2}})^*.
\]
Applying Lemma 2.13(a) to (5.88), we obtain
\[
\max_{U \in \mathbb{C}^{n \times m}} r[B - C + (AU + J_{\frac{1}{2}})(AU + J_{\frac{1}{2}})^*] = \min_{U \in \mathbb{C}^{n \times m}} \left\{ r[A, B - C, J_{\frac{1}{2}}], r \begin{bmatrix} B - C + J_A A^* & A \\ A^* & 0 \end{bmatrix}, r(B - C) + m \right\},
\]
which is minimized when \( U = r[A - B, C, J_{\frac{1}{2}}] + \max \left\{ i_+ \begin{bmatrix} B - C + J_A A^* & A \\ A^* & 0 \end{bmatrix} - r \begin{bmatrix} B - C & A^* & J_{\frac{1}{2}} \\ A & 0 \end{bmatrix}, i_+(B - C) - r[A, B - C] \right\}.
\]

where
\[
h_1 = r \begin{bmatrix} B - C + J_A A^* & A \\ A^* & 0 \end{bmatrix} - 2r \begin{bmatrix} B - C & A^* & J_{\frac{1}{2}} \\ A & 0 \end{bmatrix},
\]
\[
h_2 = r(B - C) - 2r[A, B - C] - m,
\]
\[
h_3 = i_+ \begin{bmatrix} B - C + J_A A^* & A \\ A & 0 \end{bmatrix} - r \begin{bmatrix} B - C & A^* & J_{\frac{1}{2}} \\ A & 0 \end{bmatrix} + i_+(B - C) - r[A, B - C],
\]
\[
h_4 = i_+ \begin{bmatrix} B - C + J_A A^* & A \\ A & 0 \end{bmatrix} - r \begin{bmatrix} B - C & A^* & J_{\frac{1}{2}} \\ A & 0 \end{bmatrix} + i_-(B - C) - r[A, B - C] - m.
\]

Simplifying the ranks and partial inertias of the block matrices in (5.89) -- (5.94) gives
\[
r[A, B - C, J_{\frac{1}{2}}] = r[A, B - C, J] = r[A, B - C, E_A BE_A] = r[A, B - C, BE_A]
\]
\[
= r \begin{bmatrix} A & B - C & B \\ 0 & \text{A} & \text{A}^* \end{bmatrix} - r(A) = r \begin{bmatrix} B & C & A \\ \text{A} & \text{A}^* & \text{A} \end{bmatrix} - r(A),
\]
\[
i_{\pm} \begin{bmatrix} B - C + J_A A^* & A \\ A^* & 0 \end{bmatrix} = i_{\pm} \begin{bmatrix} B - C - E_A BE_A & A \\ \text{A} & \text{A}^* \end{bmatrix} = i_{\pm} \begin{bmatrix} C & A \\ \text{A}^* & 0 \end{bmatrix},
\]
\[
r \begin{bmatrix} B - C & A & J_{\frac{1}{2}} \\ \text{A} & 0 \end{bmatrix} = r \begin{bmatrix} B - C & A & E_A BE_A \\ \text{A} & 0 \end{bmatrix} = r \begin{bmatrix} B - C & A & B \\ \text{A} & 0 \end{bmatrix} = r \begin{bmatrix} B - C & A & BE_A \\ \text{A} & 0 \end{bmatrix} - r(A) = r(K_1) - r(A).
\]

Substituting (5.95) -- (5.97) into (5.89) -- (5.94) gives (5.88) -- (5.87).

The following result can be shown similarly.
Theorem 5.9 Let \( A \in \mathbb{C}^{m \times n} \) and \( B, C \in \mathbb{C}_H^n \) be given, \( S_2 \) be as given in (5.8), and let

\[
N = \begin{bmatrix} C & A \\ A^* & 0 \end{bmatrix}, \quad K_1 = \begin{bmatrix} B & C & A \\ A^* & 0 & 0 \\ 0 & A^* & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} B & C & A \\ A^* & A^* & 0 \end{bmatrix}.
\]

(5.98)

Then, the extremal ranks and partial inertias of \( AX + (AX)^* - C \) subject to \( X \in S_2 \) are given by

\[
\begin{align*}
\max_{X \in S_2} r[AX + (AX)^* - C] &= \min \{ r(K_2) - r(A), \ r(N) \}, \\
\min_{X \in S_2} r[AX + (AX)^* - C] &= \max \{ t_1, \ t_2, \ t_4, \ t_4 \}, \\
\max_{X \in S_2} i_{+}[AX + (AX)^* - C] &= \min \{ i_{+}(B - C), \ i_{-}(N) \}, \\
\max_{X \in S_2} i_{-}[AX + (AX)^* - C] &= i_{+}(N), \\
\min_{X \in S_2} i_{-}[AX + (AX)^* - C] &= \max \{ r(K_2) + i_{-}(N) - r(K_1), \ r(K_2) + i_{+}(B - C) - r[A, B - C] - r(A) - m \}, \\
\min_{X \in S_2} i_{-}[AX + (AX)^* - C] &= \max \{ r(K_2) + i_{+}(N) - r(K_1), \ r(K_2) + i_{-}(B - C) - r[A, B - C] - r(A) \},
\end{align*}
\]

where

\[
\begin{align*}
t_1 &= 2r(K_2) + r(N) - 2r(K_1), \\
t_2 &= 2r(K_2) + r(B - C) - 2r[A, B - C] - 2r(A) - m, \\
t_3 &= 2r(K_2) + i_{-}(N) + i_{-}(B - C) - r(A) - r[A, B - C] - r(K_1), \\
t_4 &= 2r(K_2) + i_{+}(N) + i_{-}(B - C) - r(A) - r[A, B - C] - r(K_1) - m.
\end{align*}
\]

In consequence, the following hold.

(a) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( C \prec AX + (AX)^* \prec B \) if and only if \( i_{-}(N) \geq m \) and \( B \succ C \).

(b) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( AX + (AX)^* \prec B \) and \( AX + (AX)^* \prec C \) if and only if \( i_{+}(N) \geq m \).

(c) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( AX + (AX)^* \prec B \) and \( AX + (AX)^* \prec C \) if and only if

\[
\begin{align*}
r(K_2) + i_{-}(N) - r(K_1) &= 0 \quad \text{and} \quad r(K_2) + i_{+}(B - C) = r[A, B - C] + r(A) + m.
\end{align*}
\]

(d) There exists an \( X \in \mathbb{C}^{n \times m} \) such that \( C \sim AX + (AX)^* \sim B \) if and only if

\[
C \sim B, \ i_{+}(N) = r(A) \quad \text{and} \quad r(K_2) = r[A, B - C] + r(A).
\]

6 The extremal ranks and inertias of \( A - BX - XB^* \) subject to \( BXB^* = C \)

We first establish in this section a group of formulas for calculating the extremal ranks and inertias of \( A - BX - XB^* \) subject to \( BXB^* = C \), and use the formulas to characterize the existence of Hermitian matrix \( X \) satisfying the following inequalities

\[
BX + XB^* \succ (\succ, \prec, \prec) A \quad \text{s.t.} \quad BXB^* = C
\]

in the Löwner partial ordering.

Theorem 6.1 Let \( A, C \in \mathbb{C}_H^n \) and \( B \in \mathbb{C}^{m \times n} \) be given, and assume that \( BXB^* = C \) has a Hermitian solution. Also let

\[
M_1 = \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} BA & B^2 \\ B^* & 0 \end{bmatrix}, \quad M_3 = BAB^* - BC - CB^*,
\]

(6.2)

\[
S = \{ X \in \mathbb{C}_H^n \mid BXB^* = C \}.
\]

(6.3)

Then, the following hold.

(a) The maximal rank of \( A - BX - XB^* \) subject to \( X \in S \) is

\[
\max_{X \in S} r(A - BX - XB^*) = \min \{ m + r(B^2, C - BA) - r(B), \ r(M_1), \ 2m - 2r(B) + r(M_3) \}.
\]

(6.4)
(b) The minimal rank of $A - BX - XB^*$ subject to $X \in \mathcal{S}$ is

$$
\min_{X \in \mathcal{S}} (A - BX - XB^*) = \max\{s_1, s_2, s_3, s_4\},
$$

where

$$
s_1 = 2r[B^2, C - BA] + r(M_1) - 2r(M_2),
$$

$$
s_2 = 2r[B^2, C - BA] + r(M_3) - 2r[B^2, BAB^* - CB^*],
$$

$$
s_3 = 2r[B^2, C - BA] - i_+(M_1) - r(M_2) + i_-(M_3) - r[B^2, CB^* - BAB^*],
$$

$$
s_4 = 2r[B^2, C - BA] - i_-(M_1) - r(M_2) + i_+(M_3) - r[B^2, CB^* - BAB^*].
$$

(c) The maximal partial inertia of $A - BX - XB^*$ subject to $X \in \mathcal{S}$ is

$$
\max_{X \in \mathcal{S}} i_\pm(A - BX - XB^*) = \min\{i_\pm(M_1), m - r(B) + i_\pm(M_3)\}.
$$

(d) The minimal partial inertia of $A - BX - XB^*$ subject to $X \in \mathcal{S}$ is

$$
\min_{X \in \mathcal{S}} i_\pm(A - BX - XB^*) = r[B^2, C - BA] + \max\{i_\pm(M_1) - r(M_2), i_\pm(M_3) - r[B^2, CB^* - BAB^*]\}.
$$

Proof. From Lemma 2.3(b), the general Hermitian solution of $BXB^* = C$ can be expressed as

$$
X = B^\dagger C(B^\dagger)^* + F_B V + V^* F_B,
$$

where the matrix $V$ is arbitrary. Substituting it into $A - BX - XB^*$ yields

$$
A - BX - XB^* = A - C(B^\dagger)^* - B^\dagger C - BV^* F_B - F_B V B^*.
$$

Define

$$
\phi(V) = G - BV^* F_B - F_B V B^*.
$$

where $G = A - C(B^\dagger)^* - B^\dagger C$. Applying (6.14) to (6.10) yields

$$
\max_{V \in \mathbb{C}^{m \times m}} r[\phi(V)] = \min\left\{r[G, B, F_B], r\left[G \begin{bmatrix} B & 0 \end{bmatrix} \right], r\left[G \begin{bmatrix} F_B & 0 \end{bmatrix} \right]\right\},
$$

$$
\min_{V \in \mathbb{C}^{m \times m}} r[\phi(V)] = 2r[G, B, F_B] + \max\{s_+ + s_-, t_+ + t_-, s_+ + t_-, s_- + t_+\},
$$

$$
\max_{V \in \mathbb{C}^{m \times m}} i_\pm[\phi(V)] = \min\left\{i_\pm\left[G \begin{bmatrix} B & 0 \end{bmatrix} \right], i_\pm\left[G \begin{bmatrix} F_B & 0 \end{bmatrix} \right]\right\},
$$

$$
\min_{V \in \mathbb{C}^{m \times m}} i_\pm[\phi(V)] = r[G, B, F_B] + \max\{s_\pm, t_\pm\},
$$

where

$$
s_\pm = i_\pm\left[G \begin{bmatrix} B & 0 \end{bmatrix} \right] - r\left[G \begin{bmatrix} B & 0 \end{bmatrix} \right],
$$

$$
t_\pm = i_\pm\left[G \begin{bmatrix} F_B & 0 \end{bmatrix} \right] - r\left[G \begin{bmatrix} F_B & 0 \end{bmatrix} \right].
$$
Applying (2.27) and (2.28), and simplifying by elementary matrix operations and congruence matrix operations, we obtain

\[
\begin{align*}
    r[G, B, F_B] &= r \begin{bmatrix} A - C(B^\dagger)^* - B^\dagger C & B & I_m \\ 0 & B & 0 \end{bmatrix} - r(B) \\
    &= r \begin{bmatrix} -BA + C & 0 & I_m \\ 0 & -B^2 & 0 \end{bmatrix} - r(B) = m + r[B^2, C - BA] - r(B), \\
    r[G B B^* 0 0 F_B] &= r \begin{bmatrix} A B & I_m \\ B^* 0 0 & B \\ 0 0 0 & B \end{bmatrix} - r(B) \\
    &= r \begin{bmatrix} 0 0 & I_m & 0 \\ B^* 0 0 & B^* & 0 \\ -BA + C & -B^2 & 0 \end{bmatrix} - r(B) = m + r[B A & B^2 \end{bmatrix} - r(B), \\
    r[G B F_B 0 0 B^*] &= r \begin{bmatrix} A - B^* C & B & I_m \\ I_m 0 0 & 0 \\ 0 0 B & 0 \end{bmatrix} - 2r(B) \\
    &= r \begin{bmatrix} 0 0 & I_m & 0 \\ I_m 0 0 & 0 \\ -BA + C & -B^2 & 0 \end{bmatrix} - 2r(B) \\
    &= r \begin{bmatrix} 0 0 & I_m & 0 \\ 0 0 0 & 0 \\ 0 -B^2 & 0 \end{bmatrix} - 2r(B) = 2m + r[B^2, CB^* - BAB^*] - 2r(B), \\
    i_{\pm} \begin{bmatrix} G B \\ B^* 0 \end{bmatrix} &= i_{\pm} \begin{bmatrix} A - C(B^\dagger)^* - B^\dagger C & B \\ B^* & 0 \end{bmatrix} = i_{\pm} \begin{bmatrix} A B \\ B^* \end{bmatrix}, \\
    i_{\pm} \begin{bmatrix} G F_B \\ B^* 0 \end{bmatrix} &= r(F_B) + i_{\pm}(BGB^*) = m - r(B) + i_{\pm}(BAB^* - BC - CB^*). 
\end{align*}
\]

Hence,

\[
\begin{align*}
    s_{\pm} &= i_{\pm} \begin{bmatrix} A B \\ B^* 0 \end{bmatrix} - r \begin{bmatrix} BA & B^2 \\ B^* & 0 \end{bmatrix} - m + r(B), \\
    t_{\pm} &= i_{\pm}(BAB^* - BC - CB^*) - r[B^2, BAB^* - CB^*] - m + r(B). 
\end{align*}
\]

Substituting (6.15)–(6.21) into (6.11)–(6.14) yields (6.4)–(6.7).

**Corollary 6.2** Let \( A, C \in \mathbb{C}^m_{\mathbb{H}} \) and \( B \in \mathbb{C}^{m \times m} \) be given, and assume that \( BXB^* = C \) and \( BX + (BX)^* = A \) are consistent, respectively. Also let \( S(X) \) be as given in (6.3). Then, the following hold.

\[
\begin{align*}
    \max_{X \in S}(A - BX - XB^*) &= \min \{ m + r[B^2, C - BA] - r(B), 2r(B), 2m - 2r(B) + r(BAB^* - BC - CB^*) \}, \\
    \min_{X \in S}(A - BX - XB^*) &= \max \{ s_1, s_2, s_3, s_4 \}, \\
    \max_{X \in S}(A - BX - XB^*) &= \min \{ r(B), m - r(B) + i_{\pm}(BAB^* - BC - CB^*) \}, \\
    \min_{X \in S}(A - BX - XB^*) &= r[B^2, C - BA] + \max(-r(B^2), i_{\pm}(BAB^* - BC - CB^*) - r[B^2, CB^* - BAB^*]), \\
    \end{align*}
\]

where

\[
\begin{align*}
    s_1 &= 2r[B^2, C - BA] - 2r(B^2), \\
    s_2 &= 2r[B^2, C - BA] + r(BAB^* - BC - CB^*) - 2r[B^2, CB^* - BAB^*], \\
    s_3 &= 2r[B^2, C - BA] - r(B^2) + i_{-}(BAB^* - BC - CB^*) - r[B^2, CB^* - BAB^*], \\
    s_4 &= 2r[B^2, C - BA] - r(B_2) + i_{+}(BAB^* - BC - CB^*) - r[B^2, CB^* - BAB^*]. 
\end{align*}
\]

**Corollary 6.3** The pair of matrix equations

\[
BX + XB^* = A, \quad BXB^* = C
\]

(6.26)
Proof. Suppose first that the two equations in (6.26) have a common solution. This implies that
\[
\mathcal{R}(C) \subseteq \mathcal{R}(B), \quad \begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = 2r(B), \quad BC + CB^* = BAB^*, \quad \mathcal{R}(C - BA) \subseteq \mathcal{R}(B^2).
\] (6.27)

In that case, the general common Hermitian solution of (6.26) can be expressed as
\[
X = X_0 + [F_A, 0]F_GUE_H \begin{bmatrix} I_m & 0 \end{bmatrix} + [0, I_m]F_GUE_H \begin{bmatrix} 0 \\ E_B \end{bmatrix} + F_BSFB,
\] (6.28)

where \(X_0\) is a special common Hermitian solution of (6.26), \(G = [F_A, -A], H = \begin{bmatrix} B \\ E_B \end{bmatrix}\), \(U\) and \(S\) are arbitrary. In particular, (6.26) has a unique common solution if and only if \(B\) is nonsingular and \(BC + CB^* = BAB^*\). In the case, the unique common solution is \(X = B^{-1}C(B^{-1})^*\).

Proof. Suppose first that the two equations in (6.26) have a common solution. This implies that \(BX + (BX)^* = A\) and \(BXB^* = C\) are consistent, respectively. In this case, setting (6.22) equal to zero leads to (6.27).

We next show that under (6.27), the two equations in (6.26) have a common solution and their general common solution can be written as (6.28). Substituting (6.8) into the first equation in (6.26) yields
\[
F_BVB^* + BV^*F_B = A - C(B^1)^* - B^1C.
\] (6.29)

Solving for \(V\) in (6.29) by Lemma 6.1, we obtain the general solution
\[
V = V_0 + [I_m, 0]F_GWE_H \begin{bmatrix} I_m & 0 \end{bmatrix} - [0, I_m]E_HW^*F_G \begin{bmatrix} 0 \\ I_m \end{bmatrix} + B^1BW_1 + W_2F_B,
\]

where \(V_0\) is a special solution of (6.23), \(G = [F_B, B], H = \begin{bmatrix} B^* \\ F_B \end{bmatrix}\), \(W, W_1\) and \(W_2\) are arbitrary. Substituting this \(V\) into (6.8) yields
\[
X = B^1C(B^1)^* + F_BV + V^*F_B
\]
\[
= B^1C(B^1)^* + F_BV_0 + V_0^*F_B + [I_m, 0]F_GWE_H \begin{bmatrix} I_m & 0 \end{bmatrix} + F_AS_2E_B + [0, I_m]F_GWE_H \begin{bmatrix} 0 \\ I_m \end{bmatrix} + F_AT_1E_B,
\]

which can simply be written in the form of (6.28).

Proof. Let \(A, C \in \mathbb{C}^{m \times m}\) and \(B \in \mathbb{C}^{m \times m}\) be given, and assume that \(BXB^* = C\) is consistent. Also let \(M_1, M_2\) and \(M_3\) be of the forms in (6.2). Then, the following hold.

(a) There exists an \(X \in \mathbb{C}_H^m\) such that
\[
BX + XB^* \preceq A \quad \text{and} \quad BXB^* = C
\] (6.30)

if and only if
\[
\begin{align*}
& r[B^2, C - BA] + i_-(M_1) - r(M_2) = 0, \\
& r[B^2, C - BA] + i_-(M_3) - r[B^2, CB^* - BAB^*] = 0.
\end{align*}
\] (6.31)

(b) There exists an \(X \in \mathbb{C}_H^m\) such that
\[
BX + XB^* \prec A \quad \text{and} \quad BXB^* = C
\] (6.32)

if and only if
\[
i_+(M_1) \geq m, \quad \text{and} \quad i_+(BAB^* - BC - CB^*) = r(B).
\] (6.33)

(c) There exists an \(X \in \mathbb{C}_H^m\) such that
\[
BX + XB^* \succeq A \quad \text{and} \quad BXB^* = C
\] (6.34)

if and only if
\[
\begin{align*}
& r[B^2, C - BA] + i_+(M_1) - r(M_2) = 0, \\
& r[B^2, C - BA] + i_+(M_3) - r[B^2, CB^* - BAB^*] = 0.
\end{align*}
\] (6.35)

(d) There exists an \(X \in \mathbb{C}_H^m\) such that
\[
BX + XB^* \succ A \quad \text{and} \quad BXB^* = C
\] (6.36)

if and only if
\[
i_-(M_1) \equiv m \quad \text{and} \quad i_-(BAB^* - BC - CB^*) = r(B).
\] (6.37)
7 Concluding remarks

In the previous sections, we showed that the three LMIs of fundamental types in (1.2)–(1.4) can equivalently be converted to some quadratic matrix equations. Through the quadratic matrix equations and a variety of known results on linear and quadratic matrix equations, we established necessary and sufficient conditions for these LMIs to be feasible and obtained general solutions of these LMIs. Since the results obtained in the previous sections are represented in closed form by using the given matrices and their generalized inverses, they can be easily used to approach various problems related to these basic LMIs in matrix theory and applications. In particular, they can be used to solve mathematical programming and optimization problems subject to LMIs in the (1.2)–(1.4).

Based on the results in the previous sections, it is not hard to establish analytical solutions of the following constrained LMIs:

(a) $AXB \succ (\succ, \prec, \prec) C$ subject to $PX = Q$ and/or $XR = S$;
(b) $AXA^* \succ (\succ, \prec, \prec) B$ subject to $PX = Q$ and $X = X^*$, or $PXP^* = Q$ and $X = X^*$;
(c) $AX + (AX)^* \succ (\succ, \prec, \prec) C$ subject to $PX = Q$.

The results obtained will sufficiently meet people’s curiosity about analytical solutions of LMIs. In addition, the work in this paper will also motivate finding possible analytical solutions of some general LMIs, such as,

(d) $AX + YB \succ (\succ, \prec, \prec) C$;
(e) $AXA^* + BYB^* \succ (\succ, \prec, \prec) C$;
(f) $AXA^* \succ (\succ, \prec, \prec) B$ and $CXC^* \succ (\succ, \prec, \prec) D$;
(g) $AXB + (AXB)^* \succ (\succ, \prec, \prec) C$,

which are equivalent to the following linear-quadratic matrix equations:

(d1) $AX + YB = C \pm UU^*$;
(e1) $AXA^* + BYB^* = C \pm UU^*$;
(f1) $AXA^* = B \pm UU^*$ and $CXC^* = D \pm VV^*$;
(g1) $AXB + (AXB)^* = C \pm UU^*$.

A special case of (g) for $C \succ 0$ was solved in [32].

In system and control theory, minimizing or maximizing the rank of a variable matrix over a set defined by matrix inequalities in the Löwner partial ordering is referred to as a rank minimization or maximization problem, and is denoted collectively by RMPs. The RMP now is known to be NP-hard in general case, and a satisfactory characterization of the solution set of a general RMP is currently not available. Notice from the results in this paper that for some types of matrix inequality in the Löwner partial ordering, their general solutions can be written in closed form by using the given matrices and their generalized inverses in the inequalities. Hence, it is expected that the results in this paper can used to solve certain RMPs. These further developments are beyond the scope of the present paper and will be the subjects of separate studies.

After a half century’s development of the theory of generalized inverses of matrices, people now are widely using generalized inverses of matrices to solve a huge amount of problems in matrix theory and applications. In particular, one can utilize them to represent solutions of matrix equations and inequalities.

Since linear algebra is a successful theory with essential applications in most scientific fields, the methods and results in matrix theory are prototypes of many concepts and content in other advanced branches of mathematics. In particular, matrix equations and matrix inequalities in the Löwner partial ordering, as well as generalized inverses of matrices were sufficiently extended to their counterparts for operators in a Hilbert space, or elements in a ring with involution, and their algebraic properties were extensively studied in the literature. In most cases, the conclusions on the complex matrices and their counterparts in general algebraic settings are analogous. Also, note that the results in this paper are derived from ordinary algebraic operations of the given matrices and their generalized inverses. Hence, it is no doubt that most of the conclusions in this paper can trivially be extended to the corresponding equations and inequalities for linear operators on a Hilbert space or elements in a ring with involution.
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