Spacetime alternatives in relativistic particle motion

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Abstract

Hartle’s generalized quantum mechanics formalism is used to examine spacetime coarse grainings, i.e., sets of alternatives defined with respect to a region extended in time as well as space, in the quantum mechanics of a free relativistic particle. For a simple coarse graining and suitable initial conditions, tractable formulas are found for branch wave functions. Despite the nonlocality of the positive-definite version of the Klein-Gordon inner product, which means that nonoverlapping branches are not sufficient to imply decoherence, some initial conditions are found to give decoherence and allow the consistent assignment of probabilities.

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I. INTRODUCTION

One of the reasons why we expect a standard quantum mechanics, described by states on a spacelike surface, to be inadequate to describe quantum gravity is that the notion of “spacelike” should be ill-defined in a theory where the metric itself is behaving quantum mechanically. Standard quantum mechanics makes reference to spacelike surfaces not only in its description of the state of the system “at a moment of time”, but also in the very alternatives for which it makes predictions. A theory which predicts spacetime probabilities, such as the probability that a particle passes through an extended region of spacetime during its trajectory, can thus be thought of as one step on the road towards a quantum theory of gravity. Spacetime alternatives in nonrelativistic quantum mechanics have been considered in the past by Feynman [1], Yamada and Takagi [2], and Hartle [3].

The present work considers spacetime alternatives for the quantum mechanics of a free relativistic particle. This is not meant as a quantum theory of actual relativistic particles (which are described by quantum field theory) but rather as a toy model for quantum cosmology. The relativistic particle is a better analogy to gravity than is the nonrelativistic particle, because it exhibits a single reparametrization invariance, which can be though of as a subset of the diffeomorphism invariance exhibited by general relativity.

We will use Hartle’s generalized quantum mechanics formalism [4]. The three fundamental elements of this theory are the possible histories of the system (“fine-grained histories”), allowable partitions of the histories into classes \( \{c_\alpha\} \) so that each history is contained in exactly one class (“coarse grainings”), and a complex matrix \( D(\alpha, \alpha') \) corresponding to each coarse graining (“decoherence functional”). The decoherence functional must satisfy the following properties:

- “Hermiticity”:
  \[
  D(\alpha', \alpha) = D(\alpha, \alpha')^*; \tag{1.1a}
  \]

- positivity of diagonal elements:
\[
D(\alpha, \alpha) \geq 0; \quad (1.1b)
\]

- normalization:
\[
\sum_{\alpha} \sum_{\alpha'} D(\alpha, \alpha') = 1; \quad (1.1c)
\]

- superposition: If \(\{c_\beta\}\) is a coarse graining constructed by combining classes in \(\{c_\alpha\}\) to form larger classes ("coarser graining"), i.e., \(c_\beta = \bigcup_{\alpha \in \beta} c_\alpha\), the decoherence functional for \(\{c_\beta\}\) can be constructed from the one for \(\{c_\alpha\}\) by
\[
D(\beta, \beta') = \sum_{\alpha \in \beta} \sum_{\alpha' \in \beta'} D(\alpha, \alpha'). \quad (1.1d)
\]

When the decoherence functional is diagonal, or nearly so:
\[
D(\alpha, \alpha') \approx \delta_{\alpha\alpha'} p_\alpha \quad (1.2)
\]

["(medium) decoherence"], then the diagonal elements \(\{p_\alpha\}\) are the probabilities of the alternatives \(\{c_\alpha\}\), and obey classical probability sum rules. When the alternatives do not decohere, quantum mechanical interference prevents the theory from assigning probabilities to them.

In this paper, we calculate the decoherence functionals for certain simple coarse grainings. Depending on the initial conditions, some of these sets of alternatives will decohere and others will not. The cases which exhibit decoherence provide predictions for spacetime alternatives, however contrived.

II. DECOHERENCE FUNCTIONAL AND CLASS OPERATORS

A. General prescription

In constructing a generalized quantum mechanics of the a free relativistic particle, we follow the procedure described in [7] (to which the reader is referred for more details).
The general sum-over-histories recipe is this: for a given class of paths \( c_\alpha \), a class operator \( \langle x''|C_\alpha|x' \rangle \) is constructed via a sum of \( \exp(i \text{ action}) \) over those histories which start at coordinate point \( x' \) (here a point in spacetime), end at another (spacetime) point \( x'' \) and are in the class \( c_\alpha \):

\[
\langle x''|C_\alpha|x' \rangle = \sum_{x'\alpha x''} e^{iS[\text{history}]}.
\]  

(2.1)

The initial and final conditions are expressed in terms of weights (or “probabilities”) \( \{p'_j\} \) and \( \{p''_i\} \), respectively, and wave functions \( \{\psi_j(x')\} \) and \( \{\varphi_i(x'')\} \), respectively. Although we do not presuppose the existence of a Hilbert space of wave functions, it is illustrative to think of the conditions as being described by initial and final “density matrices”

\[
\rho'(x'_1,x'_2) = \sum_j \psi_j(x'_1)p'_j\psi^*_j(x'_2)
\]  

(2.2a)

and

\[
\rho''(x''_1,x''_2) = \sum_j \varphi_i(x''_1)p''_i\varphi^*_i(x''_2),
\]  

(2.2b)

respectively. The initial and final wave functions are attached using a Hermitian but not necessarily positive definite inner product \( \circ \):

\[
\langle \varphi_i|C_\alpha|\psi_j \rangle = \varphi_i(x'') \circ \langle x''|C_\alpha|x' \rangle \circ \psi_j(x');
\]  

(2.3)

and finally the decoherence functional is defined as

\[
D(\alpha,\alpha') = \frac{\sum_{i,j}^p p''_i\langle \varphi_i|C_\alpha|\psi_j \rangle \langle \varphi_i|C_{\alpha'}|\psi_j \rangle^* p'_j}{\sum_{i,j}^p |\varphi_i|C_u|\psi_j \rangle|^2 p'_j},
\]  

(2.4)

where \( C_u \) is the class operator corresponding to the class \( c_u \) of all paths. This construction satisfies all the usual requirements for a decoherence functional with positivity of diagonal elements (1.1b) holding as long as the weights \( \{p'_j\} \) and \( \{p''_i\} \) are non-negative. Note that the

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1By which we mean \( \varphi \circ \psi = (\psi \circ \varphi)^* \).
inner product $\circ$ need not be positive definite to ensure positivity of the decoherence functional. The superposition property (1.1d) holds because the class operators are constructed linearly, and thus satisfy their own superposition property:

$$
\langle x''|C_\beta| x' \rangle = \sum_{\alpha \in \beta} \langle x''|C_\alpha| x' \rangle \quad (2.5a)
$$

$$
\sum_{\alpha} \langle x''|C_\alpha| x' \rangle = \langle x''|C_u| x' \rangle. \quad (2.5b)
$$

In our chosen realization of the free relativistic particle in $(D+1)$-dimensional Minkowski spacetime, the canonical action is written in the form

$$
S_{CAN} = \int_0^1 d\lambda \left( p \cdot \frac{dx}{d\lambda} - N \frac{p^2 + m^2}{2m} \right) \quad (2.6)
$$

[where $p^2 = p \cdot p = p^\mu p_\mu = -(p^0)^2 + p^2$] and the fine-grained histories we sum over are parametrized paths $\{p(\lambda), x(\lambda)\}$ through phase space and multiplier histories $N(\lambda)$. The multiplier $N$ is a quantity which classically (i.e., for the path of least action) defines the relationship between proper time and the arbitrary parameter $\lambda$: $N = \frac{d\tau}{d\lambda}$. Note that the paths are allowed to move forward and backward in the “time” coordinate $x^0$. This set of fine-grained histories is Lorentz invariant, as opposed to a theory which restricts the paths to move forward in time in a given Lorentz frame.

Note also that the action is invariant under reparametrizations of the parameter $\lambda$, if $N$ transforms as the derivative of an invariant quantity. Since we only consider reparametrization-invariant coarse grainings as being physically meaningful, we may restrict our sum over histories to those histories which satisfy the “gauge condition” $\frac{dN}{d\lambda} = 0$. In this gauge, we need only integrate over a single $N$, which is the total proper time of the path. The theory will turn out to have a closer correspondence to field theory if we integrate only over positive values of $N$. The class operator is thus defined by

$$
\langle x''|C_\alpha| x' \rangle = \int_0^\infty dN \int_{x''x'''} DxDp \exp \left[ i \int_0^N d\tau \left( p \cdot \frac{dx}{d\tau} - \frac{p^2 + m^2}{2m} \right) \right]. \quad (2.7)
$$

[We only wish to consider coarse grainings which restrict the configuration space path $x(\lambda)$, but it is useful to express the sum over histories in terms of phase space histories because the measure for the path integral is then naturally defined.]
To specify the inner product \( \circ \) we define an “initial” spacelike \( D \)-surface \( \sigma' \) and a “final” spacelike \( D \)-surface \( \sigma'' \) to the future of the initial one, and apply the Klein-Gordon inner product on those surfaces:

\[
\varphi(x') \circ \psi(x') = \int_{\sigma'} dD \Sigma' \varphi_i^*(x') i \hat{\nabla}_\nu \psi_j(x')
\]

(2.8a)

and

\[
\varphi(x'') \circ \psi(x'') = \int_{\sigma''} dD \Sigma'' \varphi_i^*(x'') i \hat{\nabla}_\mu \psi_j(x'').
\]

(2.8b)

(Here \( \leftrightarrow \) is the usual bidirectional derivative:

\[
\varphi \leftrightarrow \nabla_\mu \psi = \varphi \nabla_\mu \psi - \psi \nabla_\mu \varphi.
\]

Thus

\[
\langle \varphi | C_{\alpha} | \psi \rangle = \int_{\sigma''} dD \Sigma'' \int_{\sigma'} dD \Sigma' \varphi_i^*(x'') i \hat{\nabla}_\mu (x''||C_{\alpha}||x') i \hat{\nabla}_\nu \psi_j(x').
\]

(2.9)

Integrating over all paths gives the unrestricted propagator

\[
\langle x''||C_{u}||x' \rangle = 2m \Delta_F(x''-x'),
\]

(2.10)

where

\[
\Delta_F(x''-x') = \int \frac{d^{D+1}p}{(2\pi)^{D+1}} \frac{e^{ip(x''-x')}}{-p^2 + m^2 + i\varepsilon}
\]

(2.11)

is the Feynman propagator, which propagates positive energy solutions forward in time and annihilates negative energy solutions:

\[
\langle x''||C_{u}||x' \rangle \circ e^{-i\omega_p t''} e^{iP \cdot x'} = 2me^{-i\omega_p t''} e^{iP \cdot x''}
\]

(2.12a)

\[
\langle x''||C_{u}||x' \rangle \circ e^{i\omega_p t'} e^{iP \cdot x'} = 0
\]

(2.12b)

assuming \( t'' > t' \) (where \( \omega_p = \sqrt{p^2 + m^2} \)). The restriction of the multiplier \( N \) to positive values has given the advertised correspondence to field theory, as our propagator is the familiar Feynman propagator. This has also led to the bias towards positive energy solutions \( (2.12) \).

\(^2\)We have, of course, treated the class operator \( \langle x''||C_{\alpha}||x' \rangle \) as a “matrix” and not taken its complex conjugate to apply the inner product \( \circ \).
B. Spacetime alternatives

As an example of a simple spacetime coarse graining, we define a spacetime region $R$, and a set of two exclusive and exhaustive alternatives as follows: $c_r$ is the class of paths which at some point enter $R$, and $c_r'$ is the class of paths which never enter it. (See Fig. 1.) If we define

$$
\langle x''N | | C_r | | x'0 \rangle = \int \mathcal{D}x\mathcal{D}p \exp \left[ i \int_0^N d\tau \left( p \cdot \frac{dx}{d\tau} - \frac{p^2 + m^2}{2m} \right) \right],
$$

(2.13)

so that

$$
\langle x'' | | C_r | | x' \rangle = \int_0^\infty dN \langle x''N | | C_r | | x'0 \rangle,
$$

(2.14)

comparing (2.13) to the path integral expression for a nonrelativistic propagator, we can show (see [5] for more details) that $\langle x''N | | C_r | | x'0 \rangle$ obeys a five-dimensional Schrödinger-like equation

$$
\left( -i \frac{\partial}{\partial N} + \frac{-\nabla^2_x + m^2}{2m} - i E_R(x'') \right) \langle x''N | | C_r | | x'0 \rangle = 0
$$

(2.15a)

with initial condition

$$
\langle x''0 | | C_r | | x'0 \rangle = \delta^{D+1}(x''-x')e^{-E_R(x')},
$$

(2.15b)

where we explicitly allow for the possibility that the region $R$ intersects the initial slice $\sigma'$ or the final slice $\sigma''$. Here

$$
E_R(x) = \begin{cases} 
0, & x \notin R \\
\infty, & x \in R 
\end{cases}
$$

(2.16)

is the excluding potential for the region $R$. Note that

$$
e^{-E_R(x)} = \begin{cases} 
1, & x \notin R \\
0, & x \in R 
\end{cases}
$$

(2.17)
(2.13) is equivalent to the homogeneous PDE
\[
\left(-i \frac{\partial}{\partial N} + \frac{-\nabla^2 + m^2}{2m}\right) \langle x''\nu|C\nu||x'0\rangle = 0, \quad x'' \notin R
\] (2.18a)
with boundary condition
\[
\langle x''\nu|C\nu||x'0\rangle = 0, \quad x'' \in \partial R
\] (2.18b)
and initial condition
\[
\langle x''0|C\nu||x'0\rangle = \delta^{D+1}(x''-x')e^{-E_R(x')}. \tag{2.18c}
\]

### III. SOLUTION BY METHOD OF IMAGES

For a sufficiently simple region, we can construct the class operator \( C_\nu \) by the method of images. Let \( n \) be a constant spacelike unit vector \((n \cdot n = 1)\), and \( x_n = n \cdot x \) be the component of \( x \) along \( n \). Then define \( R(n) \) by \( x_n \leq 0 \) (Fig. 2), so that \( e^{-E_R(x_n)} = \Theta(x_n) \) (where \( \Theta \) is the Heavyside step function). If we define the reflection of \( x \) through the plane \( x_n = 0 \) by \( x_c = x - 2x_n n \), \( \langle x''\nu|C\nu||x'0\rangle - \langle x''\nu|C\nu||x'c0\rangle \) satisfies (2.18a) (by the principle of superposition) and (2.18b), and has initial value
\[
\langle x''0|C\nu||x'0\rangle - \langle x''0|C\nu||x'c0\rangle = \delta^{D+1}(x''-x') - \delta^{D+1}(x''-x_c'), \tag{3.1}
\]
which is equal to \( \delta^{D+1}(x''-x') \) for \( x', x'' \notin R(n) \). Thus
\[
\langle x''\nu|C\nu(n)||x'0\rangle = \Theta(x_n')\Theta(x_n'') \left( \langle x''\nu|C\nu||x'0\rangle - \langle x''\nu|C\nu||x'c0\rangle \right) \tag{3.2}
\]
solves (2.18), and yields the class operator
\[
\langle x''|C\nu(n)||x'\rangle = 2mi\Theta(x_n')\Theta(x_n'') \left[ \Delta_F(x''-x') - \Delta_F(x''-x_c') \right]. \tag{3.3}
\]

To avoid confusion, keep in mind that \( x_n \) is just a number, while \( x_c \) is a \((D+1)\)-vector.
IV. DEPENDENCE ON INITIAL AND FINAL TIME SLICES

Since our construction (2.9) of the matrix elements \{\langle \varphi_i | C_\alpha | \psi_j \rangle \} from the class operator \langle x''|C_\alpha||x' \rangle makes explicit reference to a choice of nonintersecting spacelike surfaces \sigma' and \sigma'', those matrix elements and hence the decoherence functional could, in principle, depend on the choice of surfaces, and we would like to determine what, if any, that dependence is.

Observe that for a given surface \sigma with normal vector \textbf{u}, the Klein-Gordon inner product (2.8) on that surface depends only on the values on \sigma of the wave function \psi and its first normal derivative \textbf{u} \cdot \nabla \psi. Thus the construction of the decoherence functional (2.4) depends only on the values on \sigma'' of \varphi_i(x'') and \textbf{u}'' \cdot \nabla \varphi_i(x'') and the values on \sigma' of \psi_j(x') and \textbf{u}' \cdot \nabla \psi_j(x'). To discuss the behavior of the decoherence functional under changes of \sigma' or \sigma'', we need to define how the wave functions \varphi and \psi vary off of those surfaces, and we do so by requiring them to satisfy the Klein-Gordon equation.

Now we can consider how \langle \varphi_i | C_\alpha || x' \rangle = \varphi_i(x'') \circ \langle x''|C_\alpha||x' \rangle varies under changes of \sigma''. As a consequence of (2.15a) the class operator \langle x''|C_\alpha||x' \rangle will satisfy the following (for any region \textit{R}):

\[
\left( \frac{-\nabla^2_{x''} + m^2}{2m} \right) \langle x''|C_\alpha||x' \rangle = 0, \quad x' \neq x'' \notin \textit{R} \quad (4.1a)
\]
\[
\langle x''|C_\alpha||x' \rangle = 0, \quad x'' \in \textit{R}. \quad (4.1b)
\]

We assume here, as throughout this work, that the surfaces \sigma' and \sigma'' do not intersect one another, so that \( x' \neq x'' \) holds as far as we're concerned. Thus \langle x''|C_\alpha||x' \rangle satisfies the Klein-Gordon equation on \( x'' \) everywhere except on the boundary \( \partial \textit{R} \). Since the final wave functions \{\varphi_i\} are taken to be solutions to the Klein-Gordon equation, the usual demonstration of invariance of the Klein-Gordon inner product tells us that we can deform the surface \sigma'' without changing \langle \varphi_i | C_\alpha || x' \rangle so long as its intersection \sigma'' \cap \partial \textit{R} with the boundary of \textit{R} stays fixed. Examining the behavior of the sum-over-histories construction (2.7) under the substitutions \( u = N - \tau, y(u) = x(N - u) \) and \( k = -p \), we see that the class operator is symmetric under the interchange of ends of the path (\( \langle x''|C_\alpha||x' \rangle = \langle x'||C_\alpha||x'' \rangle \))
so long as the class $c_\alpha$ does not distinguish one end of the path from the other. The class $c_r$ is such a class. Thus $\langle x''|C_r|x'\rangle$ must satisfy the analogous properties to (4.1) with respect to the other argument $x'$. Thus changes of $\sigma'$ which leave $\sigma' \cap \partial R$ unchanged will not change $\langle x''|C_r|\psi_j \rangle$ either. Since $\langle x''|C_r|x'\rangle + \langle x''|C_u|x'\rangle = 2mi\Delta_F(x'' - x')$ by (2.5b), and the Feynman propagator satisfies the Klein-Gordon equation on its (nonvanishing) argument, $\langle x''|C_r|x'\rangle$ will satisfy the equation whenever $\langle x''|C_r|x'\rangle$ does, and all elements of the decoherence functional will be unchanged under any change of $\sigma'$ and $\sigma''$ which leaves their intersection with $\partial R$ unchanged. (Fig. 3)

This argument has previously been used [5] to show that the decoherence functional is independent of the choice of nonintersecting surfaces so long as $\sigma'$ lies completely to the past and $\sigma''$ completely to the future of $R$. The nature of the region $R(n)$ defined in Sec. III prevents us from choosing initial and final spacelike surfaces which do not intersect $R(n)$. What we can do without changing the decoherence functional is generate the D-surface $\sigma$ from the $(D - 1)$-surface $\sigma \cap \partial R(n)$ via curves everywhere tangent to $n$. (Fig. 4) Then $n$ will lie in the surface at all points, and $n^\mu d^2\Sigma_\mu = 0$. This will later prove crucial.

V. OUR CHOSEN SET OF ALTERNATIVES

We can take advantage of the fact that for a given normal vector $n$, the regions $R(n)$ ($n \cdot x \leq 0$) and $R(-n)$ ($-n \cdot x \leq 0$) are on opposite sides of the same boundary $x_n = 0$. (Fig. 4) Loosely calling $R(n)$ the “left” side and $R(-n)$ the “right” side of the “wall” $x_n = 0$, we can define a set of alternatives by the answers to the two questions “does the particle ever enter $R(n)$ ($x_n \leq 0$)?” and “does the particle ever enter $R(-n)$ ($x_n \geq 0$)?”

4 An example of a class which does distinguish one end of the class from the other is one which refers to the first time in its trajectory that a particle crosses a surface or enters a region.

5 Note that if $1 + 1$ dimensions, this allows us to choose our surface to be a surface of constant time in the reference frame where $n^0 = 0$. 

10
\(\mathcal{C}_{(n)} \cap \mathcal{C}_{(-n)}\), corresponding to both answers being “no”, is empty. The three nontrivial alternatives are: 
\(c_l = \mathcal{C}_{(n)} \cap \mathcal{C}_{(-n)} = \mathcal{C}_{(n)}\), in which the particle is on the left side of the wall throughout its entire trajectory; 
\(c_r = \mathcal{C}_{(n)} \cap \mathcal{C}_{(-n)} = \mathcal{C}_{(n)}\), in which the particle is always on the right side; and 
\(c_b = \mathcal{C}_{(n)} \cap \mathcal{C}_{(-n)}\), in which the particle spends some time on each side of the wall, and crosses it in between. This set of three alternatives, illustrated in Fig. 6, is exhaustive and mutually exclusive, and is thus a suitable coarse graining. The class operators for \(c_l\) and \(c_r\) were calculated in Sec. III, and are given by

\[
\langle x''| |\mathcal{C}_{l}| |x' \rangle = \langle x''| |\mathcal{C}_{(n)}| |x' \rangle = 2mi\Theta(-x'_n)\Theta(-x''_n) [\Delta_F(x''-x') - \Delta_F(x''-x'_c)] \tag{5.1a}
\]

\[
\langle x''| |\mathcal{C}_{r}| |x' \rangle = \langle x''| |\mathcal{C}_{(n)}| |x' \rangle = 2mi\Theta(x'_n)\Theta(x''_n) [\Delta_F(x''-x') - \Delta_F(x''-x'_c)], \tag{5.1b}
\]

where we have used the fact that \(x_{-n} = -n \cdot x = -x_n\) [and also that \(x_c\) is defined the same way with respect to \(n\) and \(-n\): \(x_c = x - 2nx_n = x + 2nx_{-n} = x - 2(-n)x_{-n}\)]. The class operator for \(c_b\) can be calculated from the superposition law (2.5b):

\[
\langle x''| |\mathcal{C}_{b}| |x' \rangle = \langle x''| |\mathcal{C}_{u}| |x' \rangle - \langle x''| |\mathcal{C}_{l}| |x' \rangle - \langle x''| |\mathcal{C}_{r}| |x' \rangle = 2mi\{[\Theta(x'_n)\Theta(-x''_n) + \Theta(x'_n)\Theta(x''_n)]\Delta_F(x''-x') \tag{5.1c}

- \Theta(x'_n)\Theta(x''_n) + \Theta(x'_n)\Theta(x''_n)\Delta_F(x''-x'_c)\}\.

VI. PROPERTIES FOR CERTAIN INITIAL AND FINAL CONDITIONS

A. Pure initial state

If we specialize to a pure initial state \(\psi(x')\), it becomes useful to define the branch wave function
\( \psi_\alpha(x'') = \frac{1}{2m} \langle x''||C_\alpha||\psi \rangle = \frac{1}{2m} \langle x''||C_\alpha||x' \rangle \circ \psi(x'), \) \hspace{1cm} (6.1)

so that the decoherence functional (2.4) has elements

\[ D(\alpha, \alpha') = \frac{\psi_\alpha' \circ \rho'' \circ \psi_\alpha}{\psi^+ \circ \rho'' \circ \psi^+}. \] \hspace{1cm} (6.2)

Here \( \psi^+ \) is the positive energy part of \( \psi \) [see (2.12)]:

\[ \psi^+(x'') = i \Delta_F(x''-x') \circ \psi(x') = \frac{1}{2m} \langle x''||C_\alpha||x' \rangle, \] \hspace{1cm} (6.3)

and is the branch wave function corresponding to the class \( c_u \) of all paths. The superposition property for class operators (2.5) and the definition of the branch wave function (6.1) imply an analogous superposition law for branch wave functions:

\[ \psi_\beta(x'') = \sum_{\alpha \in \beta} \psi_\alpha(x'') \] \hspace{1cm} (6.4a)

\[ \sum_\alpha \psi_\alpha(x'') = \psi^+(x''). \] \hspace{1cm} (6.4b)

We postpone for the moment discussion of the final condition \( \rho'' \).

The branch wave functions for the classes \( c_l, c_r \) and \( c_b \) can be given in terms of the branch wave functions \( \psi_{\tau(\pm n)} \) by

\[ \psi_l(x'') = \psi_{\tau(-n)}(x'') \] \hspace{1cm} (6.5a)

\[ \psi_r(x'') = \psi_{\tau(n)}(x'') \] \hspace{1cm} (6.5b)

\[ \psi_b(x'') = \psi^+(x'') - \psi_l(x'') - \psi_r(x''). \] \hspace{1cm} (6.5c)

Using (3.3), we write \( \psi_{\tau(\pm n)}(x'') \) as

\[ \psi_{\tau(\pm n)}(x'') = \Theta(\pm x''_n) \int_{\sigma'} d^D\Sigma' \Theta(\pm x'_n) \left[ i \Delta_F(x''-x') - i \Delta_F(x''-x'_c) \right] i \nabla'_{\nu} \psi(x'). \] \hspace{1cm} (6.6)

As described in Sec. 4, we can, without loss of generality, choose \( \sigma' \) to satisfy \( n_\nu d^D\Sigma' = 0 \), which allows us to move the \( \Theta(\pm x'_n) \) to the other side of the \( \nabla'_{\nu} \) [since \( \nabla_\nu \Theta(\pm x'_n) = \pm n_\nu \delta(x'_n) \)], which is orthogonal to \( d^D\Sigma' \) and get

\[ \psi_{\tau(\pm n)}(x'') = \Theta(\pm x''_n) \int_{\sigma'} d^D\Sigma' \left[ i \Delta_F(x''-x') - i \Delta_F(x''-x'_c) \right] i \nabla'_{\nu} \Theta(\pm x'_n) \psi(x'). \] \hspace{1cm} (6.7)
If we change the integration variable from $x'$ to $x'_c$ in the second term of the integral (which we can do because the construction of $\sigma'$ ensures that $x'_c \in \sigma'$ if and only if $x' \in \sigma'$), we obtain

$$\psi_{\sigma(\pm n)}(x'') = \Theta(\pm x''_n) \int_{\sigma'} d^D\Sigma' \nu \Delta_F(x'' - x') i \nabla'_{\nu} [\psi(x') \Theta(\pm x''_n) - \psi(x'_c) \Theta(\mp x'_c)] \quad (6.8)$$

Without an additional restriction on $\psi(x')$, it is quite difficult to proceed any further.

1. Antisymmetric initial state

If we choose our initial state to be an odd function of $x_n$ (which we write as $\chi$ to distinguish it from the generic initial state $\psi$):

$$\chi(x_c) = -\chi(x), \quad (6.9)$$

we have $\chi(x') \Theta(\pm x''_n) - \chi(x'_c) \Theta(\mp x'_c) = \chi(x')$, and (6.8) becomes

$$\chi \tau(\pm n)(x'') = \Theta(\pm x''_n) \int_{\sigma'} d^D\Sigma' \nu \Delta_F(x'' - x') i \nabla'_{\nu} \chi(x')$$

$$= \Theta(\pm x''_n) \chi^+(x''). \quad (6.10)$$

Thus the branch wave functions for this initial state are

$$\chi_l(x'') = \chi \tau(-n)(x'') = \Theta(-x''_n) \chi^+(x'') \quad (6.11a)$$

$$\chi_r(x'') = \chi \tau(n)(x'') = \Theta(x''_n) \chi^+(x'') \quad (6.11b)$$

$$\chi_b(x'') = 0. \quad (6.11c)$$

Note that we can construct a Klein-Gordon state satisfying the antisymmetry property (6.9) throughout all spacetime by taking any Klein-Gordon state $\zeta(x)$ which is not symmetric about $x_n = 0$ and defining $\chi(x) = \frac{1}{2} [\zeta(x) - \zeta(x_c)]$, and note also that both the positive and negative energy parts of $\chi$ have the antisymmetry property as well.
Another technique for simplifying the branch wave functions, used on the nonrelativistic particle by Yamada and Takagi [2] is to choose an initial state which vanishes either in or out of the region \( R \). Since we attach the initial state with the Klein-Gordon inner product, we need to go a step further, and require that both the initial state \( \psi(x') \) and its normal derivative \( u'_\nu \nabla'_\nu \psi(x') \) vanish on the appropriate part of the initial surface. For brevity’s sake, we define the “support” of a wave function to be anywhere where the wave function or its normal derivative is nonvanishing. Thus we want to construct a wave function whose support on the initial surface \( \sigma' \) is confined to (say) the left side of the wall \( (x_n < 0) \). It is always possible to construct a solution to the Klein-Gordon equation \( \psi(x) \) which has an arbitrary value \( f(x') \) and normal derivative \( g(x') \) on a surface \( \sigma' \), but it will in general be necessary to construct it out of both positive and negative energy components.

If we construct an initial state (which we call \( \xi \)) whose support on the surface \( \sigma' \) is confined to the left side of the wall:

\[
\xi(x') = 0 = u' \cdot \nabla' \xi(x') \text{ when } x' \in \sigma \text{ and } x'_n \geq 0
\]

(see Fig. 7), then \( \Theta(x'_n) \xi(x') \) and its normal derivative vanish and (6.7) gives

\[
\xi_r(x'') = \xi_{\pi(n)}(x'') = 0.
\]

(6.13a)

Turning the tables and considering the effect the semi-infinite support property (6.12) has on \( \xi_l = \xi_{\pi(-n)} \), we see that \( \Theta(-x'_n) \xi(x') \) has the same value and normal derivative on \( \sigma' \) as \( \xi \) itself, and we will be able to drop the \( \Theta(-x'_n) \) from (6.7), and obtain

\[
\xi_l(x'') = \xi_{\pi(-n)}(x'') = \Theta(-x''_n) \left[ \xi^+(x'') - \xi^+(x''_c) \right].
\]

(6.13b)

[We have used the easily proved result that \( \Delta_F(x''-x'_c) = \Delta_F(x''_c-x'). \)]

\[^{6}\text{I am indebted to R. S. Tate for pointing this out to me.}\]
ξ₅ can again be found by superposition, and is given by:

\[ ξ_b(x'') = Θ(x''_n)ξ^+(x'') + Θ(-x''_n)ξ^+(x'_c). \] (6.13c)

**B. Future indifference**

In order to evaluate the decoherence functional (6.2) we need to consider the final condition \( ρ'' \). In analogy with our observations that the universe has a preferred time direction, we would like to abandon the time-symmetric construction of (2.4) and choose a condition of future indifference, i.e., a completely unspecified final condition. In most time-symmetric formulations of quantum mechanics, this condition is implemented by replacing the final density matrix with the identity operator, so that

\[ ψ\alpha' \circ ρ'' \circ ψ\alpha \rightarrow ψ\alpha' \circ ψ\alpha, \]

but this cannot be the prescription here, since it is not manifestly positive when \( α = α' \), as our initial construction was.

To see why this fails, construct completely unspecified density matrices for the positive and negative energy sectors of the theory:

\[ ρ_±(x_2, x_1) = \int \frac{d^Dp}{(2\pi)^D2\omega_p} e^{±iω_p(t_2-t_1)}e^{ip\cdot(x_2-x_1)}. \] (6.14)

They have the following property under the Klein-Gordon inner product:

\[ ρ_±(x_2, x_1) \circ ψ(x_1) = ±ψ^±(x_2), \] (6.15)

where \( ψ(x) \) is any solution to the Klein-Gordon equation, and \( ψ^+(x) \) and \( ψ^-(x) \) are its positive and negative energy components, respectively \( [ψ(x) = ψ^+(x) + ψ^-(x)] \). The “identity operator” with respect to this inner product is thus \( ρ_+ - ρ_- \). It is unsuitable for a final condition \( ρ'' \), since some of the weights \( \{p''_i\} \) it implies are negative, in violation of the rules set out in Sec. [IIA]. Instead, we take our condition of future indifference to be

\[ ρ_{fi} = ρ_+ + ρ_-, \] (6.16)
so that
\[
\psi_{\alpha'} \circ \rho_{fi} \circ \psi_{\alpha} = \psi^+_{\alpha'} \circ \psi^+_{\alpha} - \psi^-_{\alpha'} \circ \psi^-_{\alpha}.
\] (6.17)

This is equivalent to the result we would have gotten if we had used the positive definite inner product for Klein-Gordon wave functions, and then chosen the identity as our final density matrix. This inner product is nonlocal in the spacetime coordinate \( x \), so, for example, wave functions which do not overlap can still have a nonvanishing inner product.

Note that we can replace the normalization factor \( \psi^+ \circ \rho'' \circ \psi^+ \) in (6.2) with \( \psi^+ \circ \psi^+ \) if we use the final condition (6.16). It will therefore prove useful to normalize our initial wave function so that
\[
\psi^+ \circ \psi^+ = 1.
\] (6.18)

The decoherence functional is then
\[
D(\alpha, \alpha') = \psi_{\alpha'} \circ \rho_{fi} \circ \psi_{\alpha}.
\] (6.19)

VII. RESULTS

7 Technically speaking, we should not talk about the positive and negative energy components of the branch wave functions \( \{\psi_{\alpha}\} \), since we showed in Sec. [IV] that the class operators (and hence the branch wave functions) are guaranteed to satisfy the Klein Gordon equation only when \( x'' \notin \partial R \), and the branch wave functions are thus not in the space of solutions to the Klein-Gordon equation. However, a more careful analysis (see the Appendix) shows that, defining \( \psi^\pm \) by (6.15), \( \varphi \circ \psi = \varphi^+ \circ \psi^+ + \varphi^- \circ \psi^- = (\varphi^+ + \varphi^-) \circ (\psi^+ + \psi^-) \) (where all inner products are taken on the same surface), even if \( \varphi \) and \( \psi \) are not solutions to the Klein-Gordon equation. The division into positive and negative energy parts is thus well-defined for our purposes.
A. Results for antisymmetric initial state

Using the antisymmetric initial state $\chi$ from Sec. VI A 1, the branch wave functions for the three classes are

$$\chi_l(x'') = \Theta(-x'')\chi^+(x'') \quad (6.11a)$$

$$\chi_r(x'') = \Theta(x'')\chi^+(x'') \quad (6.11b)$$

$$\chi_b(x'') = 0. \quad (6.11c)$$

The elements of the decoherence functional (6.19) are calculated in the Appendix, and found (when the final surface $\sigma''$ is taken to be one of constant time $t''$) to be

$$\begin{pmatrix}
D(l, l) = \frac{1}{2} + \Delta D & D(l, r) = -\Delta D & D(l, b) = 0 \\
D(r, l) = -\Delta D & D(r, r) = \frac{1}{2} + \Delta D & D(r, b) = 0 \\
D(b, l) = 0 & D(b, r) = 0 & D(b, b) = 0
\end{pmatrix} \quad (7.2a)$$

where

$$\Delta D = 2 \int \frac{dk_1n dk_2n d^{D-1}k_\perp}{(2\pi)^2} \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1\omega_2}} \chi^+(k_2)^* \chi^+(k_1) \frac{e^{-i(\omega_1 - \omega_2)t''}}{k_1n - k_2n} \ln \left( \frac{\omega_1 - k_1n}{\omega_2 - k_2n} \right). \quad (7.2b)$$

Aside from $D(l, r) = D(r, l) = -\Delta D = \chi_l \circ \rho_fi \circ \chi_r$, all of the off-diagonal elements vanish (this is true for any final condition, in fact). $D(l, r) = D(r, l)$ generally does not vanish, despite the lack of overlap of the branch wave functions, because of the nonlocality of the positive definite inner product induced by the final condition in section VI B. Note that whenever the alternatives do decohere ($\Delta D \approx 0$), the probabilities are given by $p(l) \approx 1/2 \approx p(r)$, $p(b) = 0$. [Symmetry arguments make it clear that we must have $p(l) = p(r).$] Note also that while the decoherence functional depends on the time $t''$ of the final surface, it is completely independent of the initial surface $\sigma'$.

---

8We use here several pieces of notation defined in the Appendix, namely $v_\perp = v - v_n n$ and $\omega_\perp = \sqrt{k_\perp^2 + m^2}$, and also that $\tilde{\chi}^+$ is the Fourier transform (A7) of the positive energy part of $\chi$. We are also working in a reference frame where $n$ has no time component.
To determine whether or not we have decoherence, we need to consider further properties of the initial condition $\chi$ (or equivalently its Fourier transform $\tilde{\chi}$).

Let $\tilde{\chi}$ be given by a Gaussian wavepacket peaked at $k_0, x_0$ and $t_0$, minus its reflection through $k_n = 0$. That is to say

$$\tilde{\chi}(k) = Ce^{i\omega k t_0} \left( e^{-i k_0 x_0} e^{-(k-k_0)^2/4/(\delta k)^2} - e^{-i k_0 x_0} e^{-(k+k_0)^2/4/(\delta k)^2} \right)$$

where the normalization constant is given by

$$|C|^2 = \frac{1}{2(\delta k \sqrt{2\pi})^D \left[ 1 - e^{-k_n^2/4(\delta k)^2} e^{-x_n^2/2(\delta x)^2} \right]}$$

(7.3b)

with $\delta x \delta k = 1/2$. We then have

$$\Delta D = 2|C|^2 \int \frac{dk_1nk_2ndD-1k_1\omega_1 + \omega_2}{(2\pi)^2} e^{-(k_1-k_0\perp)^2/2(\delta k)^2} e^{-(k_2-k_0\perp)^2/2(\delta k)^2} \sum_{\xi=\pm1} \xi e^{-i\xi k_1 n x_0} e^{i\xi k_2 n x_0}$$

$$\times \sum_{\xi_1=\pm1} \sum_{\xi_2=\pm1} \xi_1 \xi_2 e^{-(k_1-k_0\perp)^2/4(\delta k)^2} e^{-(k_2-k_0\perp)^2/4(\delta k)^2} e^{-i\xi_1 k_1 n x_0} e^{i\xi_2 k_2 n x_0}$$

$$\times e^{-i(\omega_1-\omega_2)(t''-t_0)} e^{-i(k_1-k_2) n x_0} \sum_{\xi=\pm1} \frac{2\xi}{k_1n - \xi k_2n} \ln \left( \frac{\omega_1 - k_1n}{\omega_2 - \xi k_2n} \right),$$

(7.4)

where the final form has been arrived at by changing the variables in the integrals $k_1n \rightarrow \xi_1 k_1n$, $k_2n \rightarrow \xi_2 k_2n$, and then making the substitution $\xi = \xi_1 \xi_2$.

In the limit that $\delta k \rightarrow 0$, we can replace $k_1n$ and $k_2n$ with $k_0n$ and $k_\perp$ with $k_0\perp$ everywhere except in the Gaussian factors and perform the integrals. We can do this because

$$\lim_{k_1n \rightarrow k_0n} \frac{1}{k_1n - k_0n} \ln \left( \frac{\omega_1 - k_1n}{\omega_0 - k_0n} \right) = -\frac{1}{\omega_0}$$

(7.5)

is finite, and we obtain

$$\Delta D = \frac{2|C|^2}{(2\pi)^2} (\delta k \sqrt{4\pi})^D (\delta k \sqrt{2\pi})^{D-1} 2 \left[ -\frac{1}{\omega_0} + \frac{1}{k_0n} \ln \left( \frac{\omega_0 - k_0n}{\omega_0\perp} \right) \right] + O([\delta k]^2)$$

$$= -4 \frac{\delta k}{(2\pi)^{3/2}} \left[ \frac{1}{\omega_0} - \frac{1}{k_0n} \ln \left( \frac{\omega_0 - k_0n}{\omega_0\perp} \right) \right] + O([\delta k]^2).$$

(7.6)
Thus we have approximate decoherence to lowest order in $\delta k$. Note that the first order correction to the decoherence functional is independent of the time $t''$ of the final surface.

For a generic antisymmetric initial condition $\chi$, (7.2f) has no reason to be small, so the current set of alternatives will probably not decohere. However, consider a coarser graining in which $c_l$ and $c_r$ are combined into a single class $c_o$, consisting of all paths which stay on one side or the other of the wall, and never cross it. We can use the superposition property (1.1d) to construct the decoherence functional from the finer-grained one (7.2a).

$$D(o,o) = D(l,l) + D(l,r) + D(r,l) + D(r,r) = 1, \text{ etc.}$$

The elements of the decoherence functional are given by

$$\begin{pmatrix}
D(o,o) = 1 & D(o,b) = 0 \\
D(b,o) = 0 & D(b,b) = 0
\end{pmatrix}$$

(7.7)

so we have exact decoherence, and probabilities of 1 for $c_o$ and 0 for $c_b$. This corresponds to the definite prediction that for a pure initial state antisymmetric about $x_n = 0$, the particle path will not cross that surface. Since the antisymmetry property holds throughout all spacetime, this result is independent of the choice of initial and final surfaces.

This last result can be seen from another point of view, allowing a slight generalization. Using the superposition property for branch wave functions (6.4a), we can construct

$$\chi_o(x'') = \chi_l(x'') + \chi_r(x'') = \chi^+(x'').$$

(7.8)

Recalling that

$$\chi_b(x'') = 0,$$

(6.11c)

we see that all branch wave functions but one vanish. Examination of (6.2) shows that whenever this is the case, the only nonvanishing element of the decoherence functional will be the diagonal one corresponding to the alternative with the nonvanishing branch wave function, and we will have decoherence, and a definite prediction of that alternative. This will hold for any final condition [except of course for pathological cases when the final condition is inconsistent with the initial condition $(\psi \circ \rho'' \circ \psi = 0)$, in which case the denominator of (6.2) vanishes, and the decoherence functional is ill-defined].
B. Results for initial state with restricted support

With the initial state $\xi$ from Sec. VI A 2, which vanishes, along with its normal derivative, on the surface $\sigma' \cap R(-n)$, we find that the branch wave functions for the three classes are

$$
\xi_l(x'') = \Theta(-x''_n)[\xi^+(x'') - \xi^+(x''_c)] \quad (6.13b)
$$

$$
\xi_r(x'') = 0 \quad (6.13a)
$$

$$
\xi_b(x'') = \Theta(x''_n)\xi^+(x'') + \Theta(-x''_n)\xi^+(x''_c). \quad (6.13c)
$$

Now the wave functions $\xi_l$ and $\xi_b$ overlap, so we do not expect decoherence, even naively, unless we coarse grain so that only one of the branch wave functions is nonvanishing. This amounts to recombining $c_l$ and $c_b$ into $c_r(n)$, so that the decoherence functional is

$$
\left(\begin{array}{c}
D(r(n), r(n)) = 1
D(r(n), \tau(n)) = 0
D(\tau(n), r(n)) = 0
D(\tau(n), \tau(n)) = 0
\end{array}\right) \quad (7.9)
$$

which decoheres, with probabilities of 1 for $c_r(n)$ and 0 for $c_{\tau(n)}$. Here we have a definite prediction that the particle will at some point in its trajectory be found in $R(n)$. This result, however, depends very much on the choice of the initial surface $\sigma'$.

VIII. DISCUSSION

For our simple coarse graining (see Fig. 5), we were able to calculate explicit expressions for the class operators $C_{r(\pm n)}$ and $C_{\tau(\pm n)}$, and hence for $C_l$, $C_r$ and $C_b$.

To calculate branch wave functions for a pure initial state, we chose the state to satisfy special conditions.

- If the wave function $\chi$ was antisymmetric under reflection through $x_n = 0$, the branch wave function $\chi_b$ vanished, while the nonvanishing branches $\chi_l$ and $\chi_r$ had no overlap. This result held no matter what the initial surface $\sigma'$. 

If the wave function $\xi$ and its first normal derivative vanished on that part of the initial surface $\sigma'$ which was outside of $R(n)$, the branch wave function $\xi_r$ vanished, but the other two branches, $\xi_l$ and $\xi_b$, overlapped. This held only for one specific choice of $\sigma'$.

We could not simply take the inner product of branch wave functions to calculate the decoherence functional, since that would have been tantamount to choosing a non-positive-definite final density matrix. Thus even for the initial state $\chi$, the alternatives $c_l$ and $c_r$ did not automatically decohere just because the branch wave functions did not overlap. If we restricted the final surface to be flat, we could calculate explicit expressions for the elements of the decoherence functional. For some choices of initial state, the off-diagonal elements were small, but in general they could be appreciable. Whenever the alternatives did decohere, the probability for each was $1/2$, which we would have predicted on symmetry grounds.

If we coarser grained either example so that only one branch wave function was nonvanishing, we of course found decoherence and a definite prediction (probability 1) of the other alternative, viz.:

- For the initial condition $\chi$, if the alternatives were chosen to be $c_b$ and $c_o = c_l \cup c_r$, we found decoherence for any nonpathological final condition, with probabilities of 0 and 1, respectively. This was a definite prediction that the particle did not cross $x_n = 0$, given an antisymmetric initial condition.

- For the initial condition $\xi$, if the alternatives were chosen to be $c_{\tau(n)} = c_r$ and $c_{\tau(n)} = c_l \cup c_b$, we found decoherence for any nonpathological final condition, with probabilities of 0 and 1, respectively. This was a definite prediction that the particle spent part of its trajectory in $R(n)$, given an initial condition which had no support outside of $R(n)$. This is hardly surprising, and it only holds if we attach the initial wave function on the correct hypersurface.

Finally, let us observe that many of our complications were a result of the fact the region which we considered intersected with our initial and final surfaces. If we had considered
a region $R$ bounded in time, we could have chosen our initial surface to lie to the past and our final surface to the future of it. As was discussed in section 4, this would make the decoherence functional necessarily independent of the choice of surface. It would also have eliminated the complications in the choice of the final condition, since the branch wave functions would have been positive energy solutions to the Klein-Gordon equation. The proof is straightforward: construct an intermediate surface of constant time $t_i$ to future of $R$ but the past of $\sigma''$. (Section 4 always allows us to deform the surface $\sigma''$ so that such a constant-time surface will “fit” in.) By a construction analogous to that of Halliwell and Ortiz [7], the propagation from $\sigma'$ to $\sigma''$ avoiding the region $R$ can be broken up (at the last crossing of $t_i$) into propagation from $\sigma'$ to $t_i$ avoiding $R$ followed by propagation from $t_i$ to $\sigma''$ which does not cross back over $t_i$. (See Fig. 3.) The class operator can thus be written

$$
\langle x''|C_r||x'\rangle = \int d^Dx_i \Delta_{1t_i}(x'',x_i)\langle x_i t_i||C_r||x'\rangle (8.1)
$$

where $\Delta_{1t_i}$ is the Newton-Wigner propagator:

$$
\Delta_{1t_i}(x,x_i) = \int \frac{d^Dp}{(2\pi)^D} e^{ip(\chi-x_i)} e^{-i\omega p(t-t_i)} . \quad (8.2)
$$

Since $\Delta_{1t_i}$ is constructed from positive-energy solutions of the Klein-Gordon operator, the branch wave functions $\psi_r$ and $\psi_\rho$ will each be positive-energy solutions themselves. Thus $\psi_{\alpha'} \circ \rho_- \circ \psi_{\alpha} = 0$, so $\psi_{\alpha'} \circ \rho_{fi} \circ \psi_{\alpha} = \psi_{\alpha'} \circ \rho_+ \circ \psi_{\alpha} = \psi_{\alpha'} \circ \psi_{\alpha}$, and we really do simply calculate the inner product of the branches.

However, it was the simplicity of the region $R(n)$ which allowed us to solve the PDE problem analytically in the first place. Solution of (2.18) for finite regions of spacetime cannot be accomplished through straightforward method-of-images or separation-of-variables methods. In the nonrelativistic case, this problem is circumvented for example in the case of a region which extends from $t_1$ to $t_2$ by propagating from $t'$ to $t_1$ with the free propagator, from $t_1$ to $t_2$ with the restricted propagator calculated as though the region existed for all time, and then from $t_2$ to $t''$ with the free propagator. Since our paths are not single-valued in time, we cannot “turn off” the restricting region before and after we reach it, since we have to include in the sum paths which double back into a previous regime.
IX. CONCLUSIONS

Using the generalized quantum mechanics formalism described by Hartle for the quantum mechanics of the relativistic worldline, we have examined one particularly simple coarse graining. For a suitable choice of initial conditions, albeit a more restrictive one than for the nonrelativistic theory, we were able to assign approximate probabilities to some sets of alternatives.

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APPENDIX: CALCULATION OF $\chi_{\alpha'} \circ \rho_{fi} \circ \chi_{\alpha}$

To calculate the elements of the decoherence functional for Sec. VII A, we first expand our notational convention for the branches to include $\chi_{-1} \equiv \chi_l$ and $\chi_{+1} \equiv \chi_r$ so that we can write $\chi_{\lambda}(x) = \Theta(\lambda x_n)\chi(x)$, where $\lambda^2 = 1$. The nonvanishing elements of the decoherence functional are now

$$D(\lambda_1, \lambda_2) = \chi_{\lambda_2} \circ \rho_{fi} \circ \chi_{\lambda_1},$$  \hspace{1cm} (A1)

where the inner product is on the surface $\sigma''$.

If $\psi$ is a solution to the Klein-Gordon equation, we know that $(\rho_+ - \rho_-) \circ \psi = \psi_+ + \psi_- = \psi$. This will not be true for $\chi_{\lambda}$ because it is not a solution. However, for the purposes of the Klein-Gordon inner product on the surface $\sigma''$, we only need the value and normal derivative.
of each function on $\sigma''$. We can thus replace $\chi_\lambda$ by $X_\lambda$, a Klein-Gordon wave function\(^9\) which matches $\chi_\lambda$ and its normal derivative on $\sigma''$. This gives us

$$\chi_{\lambda_2} \circ \chi_{\lambda_1} = \chi_{\lambda_2} \circ X_{\lambda_1} = \chi_{\lambda_2} \circ (\rho_+ - \rho_-) \circ X_{\lambda_1}$$

$$= \chi_{\lambda_2} \circ (\rho_+ - \rho_-) \circ \chi_{\lambda_1} \quad (A2)$$

We thus have

$$D(\lambda_1, \lambda_2) = \chi_{\lambda_2} \circ (\rho_+ + \rho_-) \circ \chi_{\lambda_1}$$

$$= \chi_{\lambda_2} \circ \chi_{\lambda_1} + 2\chi_{\lambda_2} \circ \rho_- \circ \chi_{\lambda_1}. \quad (A3)$$

The first term is simple enough to calculate:

$$\chi_{\lambda_2} \circ \chi_{\lambda_1}$$

$$= \int_{\sigma''} d^{D\Sigma''} \Theta(\lambda_2 x''_n) \chi^+(x'') \ast \hat{\chi''}_\mu \Theta(\lambda_1 x''_n) \chi^+(x''). \quad (A4)$$

again, since we can choose $\sigma''$ to satisfy $n_\mu d^{D\Sigma''} = 0$, we can move the step functions through the derivative to get

$$\chi_{\lambda_2} \circ \chi_{\lambda_1}$$

$$= \int_{\sigma''} d^{D\Sigma''} \Theta(\lambda_2 x''_n) \Theta(\lambda_1 x''_n) \chi^+(x'') \ast \hat{\chi''}_\mu \chi^+(x'')$$

$$= \delta_{\lambda_1 \lambda_2} \int_{\sigma''} d^{D\Sigma''} \Theta(\lambda_1 x''_n) \chi^+(x'') \ast \hat{\chi''}_\mu \chi^+(x''). \quad (A5)$$

The symmetry of $\sigma''$ and antisymmetry of $\chi^+$ tell us that $\chi_l \circ \chi_l = \chi_r \circ \chi_r$, so

$$\chi_{\lambda_2} \circ \chi_{\lambda_1} = \delta_{\lambda_1 \lambda_2} \frac{\chi^+ \circ \chi^+}{2} = \frac{\delta_{\lambda_1 \lambda_2}}{2}. \quad (A6)$$

To calculate the correction term $\chi_{\lambda_2} \circ \rho_- \circ \chi_{\lambda_1}$, we first define the Fourier transform of $\chi^+$ by

\(^9\)It is straightforward to show that such a wave function exists, and is uniquely given by $X_\lambda = (\rho_+ - \rho_-) \circ \chi_\lambda$. 

24
\[ \chi^+(x) = \int \frac{d^Dk}{(2\pi)^{D/2}} e^{ik\cdot x} e^{-i\omega_k t} \tilde{\chi}^+(k). \quad (A7) \]

The inner product of two positive energy states is expressed in terms of the Fourier transform by:

\[ \varphi^+ \circ \psi^+ = \int d^Dk \varphi^+(k)^* \psi^+(k) \quad (A8) \]

so the normalization condition (6.18) is written as

\[ \int d^Dk |\chi^+(k)|^2 = 1 \quad (A9) \]

In a reference frame where \( n \) is has no time component, we can split the spatial part \( v \) of a vector \( v \) into components along \( n \): \( (v_n = n \cdot v = n \cdot v) \) and perpendicular to \( n \): \( (v_\perp = v - v_n n) \). In analogy to \( v_c \) defined in Sec. [I], we define \( v_c = v - 2v_n n = -v_n n + v_\perp \).

\( \tilde{\chi}^+ \) is determined from \( \chi^+ \) by

\[ \tilde{\chi}^+(k) = \sqrt{2\omega_k} e^{i\omega_k t} \int \frac{d^Dx}{(2\pi)^{D/2}} e^{-ik\cdot x} \chi^+(x), \quad (A10) \]

so \( \tilde{\chi} \) obeys an antisymmetry property similar to (6.9):

\[ \tilde{\chi}^+(k_c) = -\tilde{\chi}^+(k) \quad (A11) \]

To proceed any further, we would like an explicit form for the surface \( \sigma'' \). The simplest would be that \( \sigma'' \) is a surface of constant time \( t'' \). However, that condition would not be Lorentz invariant, as it would pick out a reference frame in which the final surface was one of constant time. We know from section [I] that we are only restricted in the choice of \( \sigma'' \) by the form of the \((D - 1)\)-surface \( \sigma'' \cap \partial R(n) \). If we restrict our attention to choices of \( \sigma'' \cap \partial R(n) \) which are flat (a suitably invariant condition), we can always work in a reference frame in which \( \sigma'' \) is a surface of constant time. Since we construct \( \sigma'' \) so that \( n \) lies in it, this is consistent with the assumption that \( n \) has no time component.

Subject to the condition of \( \sigma'' \) being flat[14], then, we can work in a reference frame where it is to be a surface of constant time, so that

\[ ^{\text{10}}\text{Note that if } D = 1, \sigma'' \cap \partial R(n) \text{ is a point, so this holds trivially.} \]
\[ \varphi \circ \psi = \int d^Dx \varphi(x, t)^* i \partial_t \psi(x, t) \bigg|_{t=t''}. \] (A12)

The definition (6.14) of \( \rho_\perp \) means that

\[ \chi_{\lambda_2} \circ \rho_\perp \circ \chi_{\lambda_1} = \int \frac{dDp}{(2\pi)^{D/2}2\omega_p} \left( e^{ip \cdot x} e^{i\omega_p t} \circ \chi_{\lambda_1} \right) \left( e^{ip \cdot x} e^{i\omega_p t} \circ \chi_{\lambda_2} \right)^*. \] (A13)

Now,

\[ e^{ip \cdot x} e^{i\omega_p t} \circ \chi = \int \frac{dDk}{(2\pi)^{D/2} \sqrt{2\omega_k}} \tilde{\chi}^+(k)(\omega_k - \omega_p) e^{-i(\omega_k + \omega_p)t''} \int d^Dx \Theta(\lambda x_n) e^{i(k-p) \cdot x}; \] (A14)

the integral over \( x_\perp \) gives \((2\pi)^{D-1}\delta^{D-1}(k_\perp - p_\perp)\), and the integral over \( x_n \) gives

\[ \int_{-\infty}^{\infty} dx_n \Theta(\lambda x_n) e^{i(\lambda k_n - p_n)x_n} = \int_{-\infty}^{\infty} \lambda dx_n \Theta(x_n) e^{i\lambda(\lambda k_n - p_n)x_n} = \int_0^1 dx_n e^{i\lambda(\lambda k_n - p_n)x_n} = \frac{i}{\lambda(k_n - p_n) + i\varepsilon} = \frac{i\lambda}{k_n - p_n} + \pi\delta(k_n - p_n). \] (A15)

Substituting into (A14) gives

\[ e^{ip \cdot x} e^{i\omega_p t} \circ \chi = i\lambda \int d(k_n)(2\pi)^{D/2-1} \frac{1}{\sqrt{2\omega_k}} \tilde{\chi}^+(k) \left( \omega_k - \omega_p \right) e^{-i(\omega_k + \omega_p)t''} \] (A16)

with \( k_\perp = p_\perp \). We thus have

\[ \chi_{\lambda_2} \circ \rho_\perp \circ \chi_{\lambda_1} = \lambda_1 \lambda_2 \int \frac{dk_1n dk_2n dp_\perp}{2\omega_1 \omega_2 (2\pi)^2} \tilde{\chi}^+(k_2) \tilde{\chi}^+(k_1) e^{-i(\omega_1 - \omega_2)t''} \times \int_{-\infty}^{\infty} dp_n \left( \frac{\omega_p - \omega_1}{p_n - k_1n} \right) \left( \frac{\omega_p - \omega_2}{p_n - k_2n} \right), \] (A17)

where \( k_{1\perp} = k_{2\perp} = p_\perp \) so that \( \omega_1 = \sqrt{k_{1n}^2 + \omega_\perp^2} \) and \( \omega_2 = \sqrt{k_{2n}^2 + \omega_\perp^2} \) where \( \omega_\perp = \sqrt{p_\perp^2 + m^2} \). The integrand of the \( p_n \) integral,

\[ f(p_n) = \frac{1}{2\omega_p} \left( \frac{\omega_p - \omega_1}{p_n - k_1n} \right) \left( \frac{\omega_p - \omega_2}{p_n - k_2n} \right) \] (A18)

is analytic (since the singularities at \( p_n = k_{1n} \) and \( p_n = k_{2n} \) are removable) except for branch points when \( \omega_p = 0 \), namely at \( p_n = i\omega_\perp \) and \( p_n = -i\omega_\perp \). We can thus deform the integration contour to the one shown in Fig. [3] The contributions from the quarter-circle arcs cancel, and the contributions from the branch cut give
\[
\int_{-\infty}^{\infty} f(p_n) dp_n = \int_{\omega_\perp}^{\infty} \frac{d\kappa}{\sqrt{\kappa^2 - \omega_\perp^2}} \frac{\omega_1 \omega_2 + \omega_\perp^2 - \kappa^2}{(i\kappa - k_{1n})(i\kappa - k_{2n})}. \quad (A19)
\]

With the substitution \( \kappa = \omega_\perp \sec \theta \), this becomes
\[
\int_{\pi/2}^{\pi/2} \cos \theta (\omega_1 \omega_2 + \omega_\perp^2 - \omega_\perp^2 \sec^2 \theta) \frac{d\theta}{(i\omega_\perp - k_{1n} \cos \theta)(i\omega_\perp - k_{2n} \cos \theta)}, \quad (A20)
\]
which can be evaluated to give
\[
\int_{-\infty}^{\infty} f(p_n) dp_n = \int_{0}^{\pi/2} \sec \theta d\theta + \frac{\omega_1 + \omega_2}{k_{1n} - k_{2n}} \ln \left( \frac{\omega_1 - k_{1n}}{\omega_2 - k_{2n}} \right). \quad (A21)
\]

The first term is a constant, and is thus even in \( k_{1n} \). The rest of (A17) is odd in \( k_{1n} \) because of (A11) so the constant term gives no contribution to \( \chi_{\lambda_2} \circ \rho_- \circ \chi_{\lambda_1} \), and
\[
\chi_{\lambda_2} \circ \rho_- \circ \chi_{\lambda_1} = \lambda_1 \lambda_2 \int \frac{dk_{1n}dk_{2n}dD^{-1}k_\perp}{(2\pi)^2} \frac{\omega_1 + \omega_2}{2\sqrt{\omega_1 \omega_2}} \tilde{\chi}^+(k_2) \tilde{\chi}^+(k_1) e^{-i(\omega_1 - \omega_2)t''} \ln \left( \frac{\omega_1 - k_{1n}}{\omega_2 - k_{2n}} \right). \quad (A22)
\]

This gives us (7.2).
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FIGURES

FIG. 1. An example of a spacetime coarse graining. The path on the left never enters the spacetime region \( R \) and is thus in the class \( c_l \). The path on the right spends part of its trajectory in \( R \) and is thus in the class \( c_r \). \((D-1) \) of the \( D \) space dimensions have been suppressed.

FIG. 2. The region \( R(n) \) defined by the unit vector \( n \). \((D-1) \) of the \( D \) space dimensions have been suppressed.

FIG. 3. Varying the surfaces \( \sigma' \) and \( \sigma'' \) on which the inner product \((2,8)\) is imposed does not change the decoherence functional, as long as their intersections with \( \partial R \) are unchanged. \((D-1) \) of the \( D \) space dimensions have been suppressed.

FIG. 4. Generating the surface \( \sigma \) from its intersection with \( \partial R(n) \) by projecting along \( n \). \((D-2) \) of the \( D \) space dimensions have been suppressed.) If \( D = 1 \), \( \sigma \cap \partial R(n) \) is a point and \( \sigma \) generated in this fashion will always be flat. With two or more space dimensions, \( \sigma \) will only be flat if \( \sigma \cap \partial R(n) \) is; if \( \sigma \cap \partial R(n) \) is “wavy”, \( \sigma \) will be translationally invariant along \( n \), resembling a sheet of corrugated metal.

FIG. 5. The regions \( R(n) \) (“left”) and \( R(-n) \) (“right”) defined by the unit vector \( n \), along with their common boundary, the “wall” \( x_n = 0 \). \((D-1) \) of the \( D \) space dimensions have been suppressed.

FIG. 6. The coarse graining described in Sec. \[\text{VI}\]. The three paths shown are representatives of, from left to right: the class \( c_l \) of paths which lie completely to the left of the wall; the class \( c_b \) of paths which spend some time on each side of the wall; and the class \( c_r \) of paths which lie completely to the right of the wall. \((D-1) \) of the \( D \) space dimensions have been suppressed.) Compare Fig. 3 of \[\text{3}\].
FIG. 7. Schematic plot of a wave function $\xi$ whose support on $\sigma'$ is confined to $x_n < 0$. This is a plot of $\xi$ as a function of $x_n$ for fixed $x_\perp$ on the surface $\sigma'$. Note that $u'' \nabla' \xi(x')$ must also vanish on the “right” half of the surface $\sigma$ for $\xi$ to have semi-infinite support as defined in Sec. VIA2. [See (6.12).]

FIG. 8. Dividing up a path which avoids a compact region $R$. The path from $\sigma'$ to the last crossing of the intermediate surface $t_i$ is in the class of paths from $\sigma'$ to $t_i$ which avoid $R$. The path from the last crossing of $t_i$ to $\sigma''$ is in the class of paths from $t_i$ to $\sigma''$ which do not cross back over $t_i$, and can be defined without reference to $R$. ($D - 1$ of the $D$ space dimensions have been suppressed.)

FIG. 9. The contour on which the integral in (A17) is calculated to give (A19). The radius $R$ of the quarter-circle arcs is to be taken to infinity.
Figure 9, Whelan
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