TWO-DIMENSIONAL QCD AND STRINGS

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ABSTRACT

A review is given of recent research on two-dimensional gauge theories, with particular emphasis on the equivalence between these theories and certain string theories with a two-dimensional target space. Some related open problems are discussed.

1. Introduction

There has been a recent renewal of interest in two-dimensional gauge theories. In two dimensions any pure gauge theory is locally trivial and has no propagating modes. However, by either considering the theory on a compact 2-manifold or introducing external Wilson loop sources, a nontrivial character of the theory emerges which is almost topological in nature. These two-dimensional theories have provided an interesting simplified model with which to study certain properties of gauge theories in any dimension.

One aspect of gauge theories which has long been of interest is the connection between a gauge theory in $d$ dimensions and a string theory with $d$-dimensional target space. Speculations about how such a connection might be made have motivated a wide range of research in both QCD and strings. In this talk I will describe some recent work in which this connection is made rigorously for two-dimensional gauge theories. The string theories which will be discussed are of a very special type, being described by covering maps from a string world-sheet onto the two-dimensional target space with a finite number of singularities and no folds.

I will begin by giving a brief review of some of the previous research on two-dimensional gauge theories which is relevant to the work at hand. I will then give a simplified description of the proof that these gauge theories are equivalent to string theories. Following this, I will discuss a variety of recent related work, and then close with a brief description of several outstanding problems in this area.
2. History

Almost 20 years ago, ’t Hooft pointed out that in certain circumstances it is reasonable and interesting to expand the partition function and correlation functions of a $U(N)$ gauge theory in powers of $N$. The guiding principle behind this expansion is that when one represents the gluon propagator by double lines corresponding to the adjoint representation of the gauge group, each Feynman diagram has associated with it a natural genus $g$; it turns out that the power of $N$ associated with a Feynman diagram of genus $g$ is precisely $N^{2g-2}$. As a particularly simple example, ’t Hooft considered the case of 2-dimensional QCD. He included fermionic matter fields in the fundamental representation of the gauge group (quarks), and was able to calculate the masses of mesons in the theory to leading order in $N$ (the planar approximation).

In the following years, further progress was made in understanding the large $N$ meson spectrum, and in connecting the two-dimensional QCD theory to a stringy theory. Bars and Hanson showed that in the large $N$ limit, the result of ’t Hooft for the meson spectrum can also be derived by assuming that the quarks interact by a linear potential; this condition is equivalent to taking the Nambu action for a theory of strings connecting the quarks, and neglecting folds and singular points in the string. A major tool for the study of gauge theories was then developed, namely the Makeenko-Migdal loop equations. The loop equations for 2-dimensional gauge theories were first explicitly written down by Kazakov and Kostov, who used these relations to compute the VEV’s of Wilson loops on the plane in a large $N$ expansion. Similar results were achieved by Bralic using a nonabelian version of Stokes’ theorem. The connection between the Wilson loop VEV’s and physical observables of the two-dimensional gauge theory was made explicit by Strominger, who showed that the Green’s functions for quark bilinears could be explicitly written in terms of an integral over Wilson loops.

In a different thread of research, progress was made in understanding the pure gauge theories in 2 dimensions by exact solution. It was first shown by Migdal that by using the heat kernel action for the gauge theory, one arrives at a theory on the lattice which is invariant under triangulations, and which is equivalent to the usual gauge theory as the triangulation becomes arbitrarily fine. Using this approach, it was shown by Rusakov, Witten, and others that the partition function of the Euclidian gauge theory on a compact Riemann surface could be exactly expressed in terms of the group theory of the gauge group. Explicitly, they showed that the partition function on a manifold $M$ of genus $G$ and area $A$ is given by

$$Z(G, \lambda A, N) = \int [DA^\mu] e^{-\frac{1}{4g^2} \int_M \sqrt{g} \, Tr F_{\mu\nu} F^{\mu\nu}} = \sum_R (\text{dim } R)^{2-2G} e^{-\frac{2\lambda}{N^2} C_2(R)},$$

where the sum is taken over all irreducible representations of the gauge group, with $\text{dim } R$ and $C_2(R)$ being the dimension and quadratic Casimir of the representation $R$. ($\lambda$ is related to the gauge coupling $\tilde{g}$ by $\lambda = \tilde{g}^2 N$. )
3. String Theory

We will now explain how the exact expression Eq. (1) for the gauge theory partition function can be rewritten as a string theory partition function to all orders in $1/N$. The results in this section were originally described in the papers [8-10]. Recently, Kostov has described a similar equivalence for a lattice version of the theory$^{11}$.

3.1. Statement of Main Result

In the string theory which we describe here, the partition function is given by a sum over topologically distinct maps from a two-dimensional string world sheet $\mathcal{N}$ of any genus $g$ to the target space $\mathcal{M}$. Essentially, the string partition function is written

$$Z = \int_{\Sigma(\mathcal{M})} d\nu W(\nu),$$

(2)

where $\Sigma(\mathcal{M})$ is a set of covering maps $\nu: \mathcal{N} \to \mathcal{M}$, which are allowed to have singularities at a finite set of points in $\mathcal{M}$. The specific types of singular points allowed in the maps in $\Sigma(\mathcal{M})$ depend upon the choice of gauge group and the genus $G$. The weight $W(\nu)$ associated with a certain string map $\nu$ is given by

$$W(\nu) = \pm \frac{N^{2-2g}}{|S_\nu|} e^{-\frac{n\lambda A}{2}},$$

(3)

where $g$ is the genus of $\mathcal{N}$, $n$ is the degree of the map $\nu$, and $|S_\nu|$ is the symmetry factor of the map $\nu$ (the number of diffeomorphisms $\pi$ of $\mathcal{N}$ which satisfy $\nu\pi = \nu$). In general, the manifold $\mathcal{N}$ need not be connected; the genus of a disconnected manifold is defined so that the Euler characteristic is additive for disjoint unions of connected manifolds. Apart from the sign, which depends upon the types of singularity points in the map $\nu$, this is a very natural weight for a string theory. The power of $N$ is just the usual power of the string coupling $1/N$, the exponent of $n\lambda A/2$ is just the usual Nambu action, and the symmetry factor is the usual one associated with Feynman diagrams in any field theory. The unusual feature about this string theory is that the sum over string maps is restricted to the set of maps $\Sigma(\mathcal{M})$. In particular, the strings are not allowed to have folds other than at the finite number of singular points. Of course, the gauge theory we are studying has no propagating degrees of freedom; allowing folds into the string theory would clearly violate this characteristic, so the absence of folds is in some sense not surprising.

3.2. Outline of Proof – Simple Case

We will now give a brief description of the essential features in a proof of Eq. (2); we will also describe in more detail the set of allowed maps $\Sigma(\mathcal{M})$. We will assume for the remainder of this section that the gauge group is $SU(N)$. For now, we will also restrict attention to the torus $G = 1$, where the contribution from the dimension terms vanishes in Eq. (1). We will return later to the corrections due to these dimension terms.

In order to write an asymptotic expansion of Eq. (2) in powers of $1/N$, we would like to proceed by computing the quadratic Casimir of each representation $R$.
of $SU(N)$ as a function of $N$, and thus writing an asymptotic expansion separately for each term in the sum over representations. Because as $N$ varies, the set of representations of $SU(N)$ itself changes, we must find a way of implementing this procedure which is well-defined. The theory of Young tableaux gives us such an approach. We can associate each representation $R$ with a certain Young tableau, which we also denote by $R$. Given a Young tableau $R$ with $n$ boxes, the quadratic Casimir of the representation is given by

$$C_2(R) = nN + \frac{n(n-1)\chi_R(T_n)}{\chi_R(1)} - \frac{n^2}{N},$$

(4)

where $\chi_R(T_n)$ and $\chi_R(1) = d_R$ are characters of the symmetric group $S_n$ in the representation associated with the Young tableau $R$. We denote by $T_n$ the conjugacy class of elements in the symmetric group which have one cycle of length 2 and $n-2$ cycles of length 1.

We would now like to insert Eq. (4) into Eq. (3) to form the asymptotic expansion of the partition function. However, we must be careful about which representations we include in the sum. Naively, one might expect that in the asymptotic $1/N$ expansion, it would suffice to include all representations corresponding to Young tableaux with a finite number of boxes. In fact, however, it is necessary to include all representations whose quadratic Casimir is of the form $C_2(T) = nN + O(1)$. In addition to the Young tableaux with a finite number of boxes, we must include another set of representations corresponding to the conjugates of these representations, and “composite” representations arising from tensor products of these two types of representations. To simplify the presentation, we will temporarily assume that the sum in Eq. (3) can be replaced by a sum over representations with a finite number of boxes. We denote this simplified partition function by $Z_Y$. We will return to the correct sum over composite representations shortly; the simplifying assumption of only including Young tableaux with finite boxes corresponds to only considering a single “chiral” sector of that complete theory.

Replacing the sum over representations by a sum over all Young tableaux in each set $Y_n$ of tableaux with a finite number $n$ of boxes, we have

$$Z_Y(1, \lambda A, N) = \sum_n \sum_{R \in Y_n} \exp \left[ -\frac{\lambda A}{2N}(nN + \frac{n(n-1)\chi_R(T_n)}{\chi_R(1)} - \frac{n^2}{N}) \right].$$

(5)

Expanding the exponential and using some elementary identities from the theory of characters of the symmetric group, we have

$$Z_Y(1, \lambda A, N) = \sum_{n,i,t,h} e^{-\frac{\lambda A}{2N}N^2-2g(\lambda A)^{i+t+h}} \frac{(-1)^i n^h(n^2-n)^t}{2^{i+h}} \cdot \sum_{p_1, \ldots, p_i \in T_n} \sum_{s,t \in S_n} \left[ \frac{1}{n!} \delta(p_1 \cdots p_i st s^{-1} t^{-1}) \right],$$

(6)

where $2-2g = -2(t+h) - i$.

We would like to now interpret this expression in terms of a sum over covering maps of the torus. This interpretation follows from a simple theorem which holds...
for any genus $G$. Define $\Sigma(G,n,i)$ to be the set of all topologically distinct covering maps onto a genus $G$ target space of degree $n$ and with $i$ elementary branch point singularities at a fixed set of points $q_j$. Then

$$\sum_{\nu \in \Sigma(G,n,i)} \frac{1}{|S_\nu|} = \sum_{p_1, \ldots, p_i \in T_n} \sum_{s_G,t_G \in S_n} \frac{1}{n!} \delta(p_1 \cdots p_i \prod_{j=1}^{G} s_j t_j s_j^{-1} t_j^{-1}).$$  \hspace{1cm} (7)$$

This theorem can be proven as follows. We can cut the surface along the usual

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Surface with Genus $G = 2$, $i = 4$ Branch Points}
\end{figure}

homotopy generators $a_j, b_j$ to form a $4G$-gon. A covering space with branch points at $q_j$ can be described by choosing a labeling $1, 2, \ldots, n$ of the sheets of the covering space at a point $p$, and determining the permutations on this set of labels which are realized by moving around the homotopy generators and a set of loops $c_j$ which encircle the branch points. Denote the permutations associated with the loops $a_j, b_j$ and $c_j$ by $s_j, t_j$ and $p_j$ respectively. The statement that the branch points are elementary is equivalent to the condition that $p_j$ is an element of the conjugacy class $T_n$ for all $j$. The single condition on the homotopy group $\pi_1(M \setminus \{q_1, \ldots, q_i\})$ is that

$$c_1 \cdots c_i a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_G b_G a_G^{-1} b_G^{-1} = 1.$$  \hspace{1cm} (8)$$

The permutation associated with the cycle on the left hand side of this equation must therefore be the identity permutation. We can now view the sum over all $p_j, s_j, t_j$ in Eq. (7) as a sum over all distinct labeled coverings with branch points at $q_j$, where the $\delta$ function enforces the condition that the association of permutations to homotopy generators be a homeomorphism into the symmetric group. By noting that the number of distinct labelings for a particular topological type of covering map is precisely $n!/|S_\nu|$, we see that the combinatorial factor in the sum over coverings works out correctly, and we have proven Eq. (7).

We are now in a position to rewrite the partition function $Z_Y$ in the form of Eq. (2). We choose an orientation on $M$, and define a set of covers $\Sigma_+(M)$ to be the set of orientation-preserving covering maps from an oriented Riemann surface $N$ to $M$, which have a finite number of branch point singularities, and in addition a finite number of singularities corresponding to handles and tubes in $N$ which are
contracted to points in $\mathcal{M}$. We define a measure $d\nu$ on the space $\Sigma_+(\mathcal{M})$ by using the positions of the singular points as parameters in each connected component, and giving the position each of singularity a measure proportional to $\lambda dA$. With respect to this measure, each topological map with $i$ branch points carries a factor of $(\lambda A)^i/i!$, where the denominator arises from the indistinguishability of the branch points. Similarly, each map with $h$ contracted handles carries a measure factor of $(n\lambda A)^h/h!$, and each map with $t$ contracted tubes carries a measure factor of $(n(n-1)\lambda A)^t/(2^tt!)$. Each contracted handle carries a symmetry factor of $1/2$.

Combining these weights, we have the result that the asymptotic expansion for the partition function on the torus restricted to Young tableaux with a finite number of boxes can be written

$$Z_Y(1, \lambda A, N) = \int_{\Sigma_+(\mathcal{M})} d\nu W(\nu),$$

where the weight $W(\nu)$ is given by

$$W(\nu) = (-1)^{N^2-2g} \frac{n^2}{|S_\nu|} e^{-\frac{n\lambda A}{2N}}.$$  \hspace{1cm} (10)

3.3. Coupled Theory

As we remarked above, to correctly derive the asymptotic expansion of the complete theory it is necessary to consider all Young tableaux with a quadratic Casimir of leading order $N$. The set of Young tableaux which satisfy this condition are precisely those tableaux containing a finite number of columns with $N - k$ boxes where $k$ is finite, and a finite number of columns with a finite number of boxes. We will call the representations associated with these tableaux composite representations. We write composite representations as $\bar{SR}$ where $S, R$ are Young tableaux with a finite number of boxes. The representation $\bar{SR}$ is constructed by taking the Young tableau with the maximum number of boxes in the tensor product of the representation $R$ with the conjugate of the representation $S$. Although the quadratic Casimir for any representation which is not of the composite type scales as $N^2$ and therefore gives a contribution to the partition function proportional to $e^{-N^2}$, one might question whether it is really acceptable to neglect such representations when constructing the asymptotic $1/N$ expansion. Recent work by Douglas and Kazakov\footnote{12} which will be discussed in more detail later demonstrates that in fact these representations can indeed be neglected except on spheres of area $\lambda A < \pi^2$.

Given Young tableaux $S, R$ with $\bar{n}, n$ boxes respectively, we can compute the quadratic Casimir of the composite representation $\bar{SR}$; we find that $C_2(\bar{SR}) = C_2(R) + C_2(S) + 2n\bar{n}/N$. We can write the complete asymptotic expansion for the partition function for any genus $G$ as a sum over composite representations,

$$Z(G, \lambda A, N) = \sum_{n,\bar{n}} \sum_{R,S \in Y_n, Y_{\bar{n}}} (\dim \bar{SR})^{2-2G} e^{-\frac{\lambda A}{2N}} [C_2(\bar{SR})].$$

(11)

Except for the term $2n\bar{n}/N$, this partition function for genus $G = 1$ factorizes into two components, each equal to the partition function $Z_Y$. We can reproduce
this factorization geometrically by including in the set of allowed maps \( \Sigma(M) \) maps from an oriented Riemann surface \( \mathcal{N} \) which are either orientation-preserving or orientation-reversing maps, relative to a fixed orientation on \( \mathcal{M} \). We refer to these two types of maps as two “chiral sectors” of the string theory. To see how the coupling term proportional to \( \tilde{n} \tilde{n} \) can be incorporated, one simply expands the exponential of the quadratic Casimir in the complete partition function as in Eq. (6).

Just as the terms containing \( h \) and \( t \) were interpreted as arising from handles and tubes in the string world-sheet contracted to points in the target space, we can interpret the contribution from the coupling term as arising from infinitesimal tubes which connect sheets of opposite orientation. The number of ways in which one of these tubes can connect \( n \) sheets of one orientation with \( \tilde{n} \) sheets of the opposite orientation is clearly \( n\tilde{n} \), so the counting is correct. From the sign on the coupling term, we see that each orientation-reversing tube carries a factor of \( -1 \).

Thus, in the complete theory containing a sum over all composite representations, we can define the set of covering maps \( \Sigma(M) \) to be the set of all maps from a (possibly disconnected) oriented world-sheet onto \( \mathcal{M} \) which is locally a covering map at all but a finite number of singular points; the allowed types of singular points consist of \((i)\) elementary branch points, \((h)\) contracted handles, \((t)\) contracted tubes between sheets of identical relative orientation, and \((\tilde{t})\) contracted orientation-reversing tubes. The partition function is then given by an equation of the form of Eq. (2), where the weight \( W(\nu) \) is given by

\[
W(\nu) = (-1)^{(i+\tilde{t})} N^{2g} \exp(-\frac{n\lambda A}{2})/|S_v|.
\]

This completes the description of the asymptotic \( 1/N \) expansion of the \( SU(N) \) gauge theory on the torus as a string theory.

3.4. Example

As a simple example of the equivalence between the gauge theory partition function and the sum over covering maps, let us consider the leading order terms in the partition function of a single chiral sector on the torus. To order \( N^0 \), the partition function \( Z_Y \) is given by

\[
Z_Y(1, \lambda A, N) = \sum_n x^n (\pi_n + O(1/N)) = \prod_i \frac{1}{1 - x^i} + O(1/N),
\]

where \( x = \exp(-\frac{n\lambda A}{2}) \), and \( \pi_n \) is the number of ways of partitioning an integer \( n \) into a sum of integers (the number of distinct Young tableaux with \( n \) boxes). In the complete theory with both chiral sectors, the leading order term is simply given by squaring this expression.

As in any field theory, by taking the logarithm of the partition function, we get a free energy which is given by a sum over connected diagrams. In this case,

\[
W_Y(1, \lambda A, N) = \ln Z_Y(1, \lambda A, N) = \sum_n \omega_n x^n + O(1/N),
\]

where \( \omega_n = \sum_{k|n} k/n \). We expect from the string description of the partition function, that the quantity \( \omega_n \) should be precisely the sum of \( 1/|S_v| \) over all unbranched connected \( n \)-fold covers of the torus. This equality can easily be seen to hold, using
the fact that all such covers have a symmetry group of order \( n \). Thus, we have a verification of the string interpretation in this simple case.

### 3.5. Higher Genus

We will now briefly describe the correction to the simple string theory description which arises when \( G \neq 1 \) from the insertion of the \( 2 - 2G \)th power of the dimension \( \dim R \) for each representation. In terms of the string geometry of the theory, the only change which occurs from these extra terms can be described by including in the set \( \Sigma(M) \) maps containing additional singularities at a set of \( |2 - 2G| \) fixed points in \( M \). When \( G = 0 \) we refer to these points as \( \Omega \)-points; when \( G > 1 \), we call the points \( \Omega^{-1} \)-points. The unusual feature of these points is that they are not allowed to move on the manifold \( M \), and do not carry factors of the area \( A \) as do the other singularities. In addition, the types of singularities allowed at \( \Omega \)-points and \( \Omega^{-1} \)-points are different.

In a single chiral sector of the theory, the singularity structure at an \( \Omega \)-point is extremely simple. The map \( \nu \) can contain a singularity which gives rise to an arbitrary permutation on the sheets of the covering space when one follows a closed loop around the singularity. In the coupled theory, at an \( \Omega \)-point, in each sector an arbitrary permutation of the sheets of the covering space is allowed. However, in addition, the sets of sheets which are connected by the cycles of these permutations can be connected in pairs by orientation-reversing tubes.

The set of singularities allowed at a \( \Omega^{-1} \)-point is very similar to that of an \( \Omega \)-point, but slightly more complicated. Essentially, at an \( \Omega^{-1} \)-point, there may be an arbitrary number of nontrivial singularity points of the type from an \( \Omega \)-point, each carrying a factor of \( -1 \). This description holds both in the complete theory and in a single chiral sector.

In sum, then, by defining the set \( \Sigma(M) \) of covering spaces to include covers with \( 2 \Omega \)-point type singularities when \( G = 0 \), and to include \( 2G - 2 \Omega^{-1} \)-point type singularities when \( G > 1 \), we can write the gauge theory partition function for arbitrary genus as Eq. (2); in this general case, \( W(\nu) \) is given by

\[
W(\nu) = (-1)^{i+f+\sum x_j} \frac{N^{2-2g} e^{-\frac{\pi c_4}{\lambda A^2}}}{|S_\nu|},
\]

where \( x_j \) is the number of distinct \( \Omega \)-point type singularities at the \( j \)th \( \Omega^{-1} \)-point. Thus, we have defined a string theory representation of the partition function for the \( SU(N) \) gauge theory on an arbitrary genus Riemann surface.

### 4. Further Results

#### 4.1. Other Gauge Groups

Up to this point, we have restricted attention to the gauge group \( SU(N) \). However, for other gauge groups it is also possible to construct a string interpretation of the gauge theory partition function. In general, for a given gauge group, it is necessary to first determine the set of Young tableaux which correspond to representations whose quadratic Casimir is of order \( N \). By then expressing the quadratic
Casimir and dimension of these representations in terms of characters of the symmetric group, one can ascertain the types of singularity structures which are allowed in the set of string maps $\Sigma(M)$ associated with that particular theory. Generally, the subleading terms in the quadratic Casimir correspond to “mobile” singularity types, which carry factors of the area, and the subleading terms in the dimension correspond to the “static” types of singularities which appear in the $\Omega$-points of the theory.

The simplest example of this general analysis is for the gauge group $U(N)$. For $U(N)$, the expression for the dimension is the same as for $SU(N)$, so the $\Omega$-points are identical in this theory. The quadratic Casimir, however, differs from that for $SU(N)$ in that the final term $-n^2/N$ associated with vanishing $U(1)$ charge is absent. Thus, in the $U(N)$ theory the mobile tube and handle singularities do not occur; on the torus, the only allowed types of singularities are branch points.

A similar analysis of the string theories for gauge groups $SO(N)$ and $Sp(N)$ has been carried out by Naculich, Riggs, and Schnitzer, and by Ramgoolam. They found that for these gauge groups, the string world sheet is not necessarily orientable, and that there are additional singularities corresponding to infinitesimal cross-caps in the string maps. The insertion of a cross-cap singularity essentially corresponds to cutting out an infinitesimal disk and replacing it with a projective plane minus a point; moving along a loop which surrounds this point gives a reversal of orientation. The singularity structure at $\Omega$-points for these gauge groups is remarkably similar to that for $\Omega$-points in the $SU(N)$ theory; however, cross-cap singularities also appear at these $\Omega$-points.

4.2. Wilson Loops

Just as the partition function of any 2D gauge theory on an arbitrary Riemann surface can be written as a weighted sum over closed string maps, it is possible to show that the VEV of any Wilson loop $\gamma$ can be expressed as a weighted sum over open string maps, where the boundary of the string world-sheet is taken to $\gamma$ by the covering map. There are some technical complications with the calculation for Wilson loops with self-intersections; one finds that each of the disconnected regions into which the Wilson loop cuts the manifold $M$ must be associated with some fixed number of $\Omega$-point singularities, even for a Wilson loop on the torus. The details of the calculation of Wilson loop VEV’s in terms of open strings are described in [10].

4.3. Phase Transition

An important related development is the recent demonstration by Douglas and Kazakov of a phase transition in the partition function on the sphere at the point $\lambda A = \pi^2$ (the trivial small area phase was previously observed by Rusakov). They used techniques familiar from matrix models to study which representations contribute to the partition function in the large $N$ regime. In the phase with $\lambda A > \pi^2$, they showed that the set of representations which contribute to the partition function is precisely the set of composite representations. Below the critical value of the area, nonperturbative effects of other representations simplify the partition
function and render invalid the restriction to composite representations. This phase transition is analogous to the phase transition which occurs for the Wilson action at large $N^{15}$.

The significance of these results for the understanding of QCD through a string interpretation is not yet clear. The existence of the phase transition clearly represents an obstacle to expanding the string interpretation to small coupling in certain situations; however, the fact that this phase transition does not occur for higher genus surfaces or for finite values of $N$ gives hope that it is not a fundamental obstacle to progress in this direction.

4.4. Equivalence of QCD to Other Theories

Other recent work has shown that due to its simple group-theoretic structure, QCD$_2$ can be related to many other theories of current interest. In its Hamiltonian formulation, pure QCD$_2$ on a cylinder is essentially equivalent to quantum mechanics on the manifold of the gauge group. The theory was recently studied from this perspective by Douglas$^{16}$, and by Minahan and Polychronakos$^{17}$, and shown to be equivalent to a theory of free fermions on the circle. Minahan and Polychronakos showed that by writing the string formulation of the theory in Hamiltonian form, one arrives at the bosonization of the fermion theory. Using the collective field formalism, they related this theory to a $c = 1$ matrix model. It has also been shown by several authors that the introduction of a Wilson loop source in QCD$_2$ on the cylinder gives a theory of interacting fermions with a Sutherland-type interaction$^{17,18}$. In a related work, Caselle et al. related QCD$_2$ on the cylinder to a Kazakov-Migdal model with periodic boundary conditions$^{19}$. Their approach was to explicitly write the heat kernel on the cylinder in terms of the invariant angles of group elements. An interesting result of their analysis is a formula for the grand canonical partition function of QCD$_2$ on the cylinder, containing the correct expressions for the partition function for finite values of $N$.

4.5. Finite $N$ Results

Most of the work described so far connecting gauge theories to string theory has been done in the context of an asymptotic $1/N$ expansion. For the gauge theories in two dimensions, however, the explicit form of the exact solution as a sum over representations makes it possible to consider making such a correspondence for a fixed finite value of $N$. For a finite value of $N$, the sum over representations in the partition function must be restricted to the set of irreducible representations of the group. In the case of $SU(N)$, this corresponds to summing over only Young tableaux with less than $N$ rows. By extending the group-theoretic analysis described here to the finite $N$ case, one finds that the effect of this restriction in the summation can be fairly easily described in the string picture. The essential result is that one restricts to only orientation-preserving string maps, and for any genus one must introduce a new type of static singularity point. Only one such point must be introduced for any genus; at this “projection” point, a singularity corresponding to an arbitrary permutation of the covering sheets can occur. The weight of a singularity associated
with a permutation $\sigma$ is given by

$$P^{(N)}(\sigma) = \sum_{R \in Y_n^{(N)}} \frac{d_R}{n!} N^{n-K_\sigma} \chi_R(\sigma),$$

(15)

where $Y_n^{(N)}$ is the set of Young tableaux with $n$ boxes and less than $N$ rows, and $K_\sigma$ is the number of cycles in the permutation $\sigma$. The details of the construction of the string theory in the finite $N$ $SU(N)$ gauge theory are described in [20].

5. Open Questions

5.1. Meson Spectrum

It should be possible to reproduce the planar approximation to the meson spectrum derived by 't Hooft from the string point of view. The work of Bars and Hanson, and Strominger represent a partial result in this direction. However, using a purely string-theoretic argument based on the work presented here, one would like to rederive this result, and calculate the lower order corrections to this meson spectrum. This problem presents an interesting and nontrivial challenge.

5.2. Action Formulation

The primary reason that we are interested in studying gauge theories in two dimensions is for the insight which they give into the structure of gauge theories in 4 dimensions. Unfortunately, the formulation presented here of the string theory associated to gauge theories in two dimensions is highly dependent upon the unique characteristics of maps from 2-manifolds to other 2-manifolds. In order to extend this work to higher dimensions, one natural approach is to attempt to describe the theory by an action description where there is no restriction on the string maps allowed. In such a formulation, the fact that maps with folds do not contribute to the partition function would hopefully arise naturally from an integration over fermion zero modes in the theory or some other familiar mechanism. We have several clues to the form that such an action formulation of the string theory might take. Clearly, the action will contain in some way the Nambu action. The signs associated with branch points indicate the possible existence of a fermionic structure in the theory. The existence of infinitesimal tubes and handles presents an obstruction to a holomorphic characterization of the string maps, since no holomorphic map takes an entire handle of a Riemann surface into a point. The invariance of the theory under the group of area-preserving diffeomorphisms indicates that the string action should also have this invariance. Finally, the static singularities associated with $\Omega$-points may give some hint as to some global structure in the theory we are looking for: the similarity between the $\Omega$-points for different gauge groups is particularly striking in this regard. Unfortunately, as yet no one has successfully described the string theory presented here with an action formalism of the desired type, or even made clear progress towards such a formalism.
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