Abstract. We discuss discretization of Koenigs nets (conjugate nets with equal Laplace invariants) and of isothermic surfaces. Our discretization is based on the notion of dual quadrilaterals: two planar quadrilaterals are called dual, if their corresponding sides are parallel, and their non-corresponding diagonals are parallel. Discrete Koenigs nets are defined as nets with planar quadrilaterals admitting dual nets. Several novel geometric properties of discrete Koenigs nets are found; in particular, two-dimensional discrete Koenigs nets can be characterized by co-planarity of the intersection points of diagonals of elementary quadrilaterals adjacent to any vertex; this characterization is invariant with respect to projective transformations. Discrete isothermic nets are defined as circular Koenigs nets. This is a new geometric characterization of discrete isothermic surfaces introduced previously as circular nets with factorized cross-ratios.
1 Introduction

This paper is devoted to the discretization of a geometrically important class of two-dimensional conjugate nets, very popular in the classical differential geometry under the name of nets with equal invariants. With a view towards discretization, we prefer to call them Koenigs nets, for the following reason: among various geometric and analytic characterizations, the property of having equal Laplace invariants belongs to the minor part which do not survive by discretization, at least literally. Therefore the term “discrete nets with equal invariants” would be misleading. On the other hand, the French geometer G. Koenigs contributed a lot to the study of their properties [K1, K2], see also [Da, E]. The term “discrete Koenigs nets” will be suggestive and well justified.

Another class of nets, whose discretizations are discussed in the present paper, are isothermic surfaces. Classically, their theory was considered as one of the highest achievements of the local differential geometry, see [Bi, Da, E] and modern studies [CGS, BHPP, KPP, Sch, HL, Bu].

Both classes of nets has been discretized already. Historically, discrete isothermic surfaces happened to be introduced earlier [BP], as circular nets with factorized cross-ratios. An approach to the discretization of Koenigs nets have been proposed in [D2], based on a characterization of smooth Koenigs nets as conjugate nets possessing a so called conic of Koenigs in each tangent plane [K2] (a conic of Koenigs has a second order contact both with the $u_1$ tangent line at the corresponding point of the Laplace transform $f_{−1}$ and with the $u_2$ tangent line of $f$ at the corresponding point of the Laplace transform $f_1$).

In the present paper, we propose a novel definition of discrete Koenigs nets and discrete isothermic surfaces. This definition is based on one of the characterizations of the Koenigs nets and isothermic surfaces, namely on the notion of duality. We believe that it is this definition that lies in the core of the whole theory and leads most directly to various other properties. All discretizations we consider belong to the class of Q-nets, or nets with planar elementary quadrilaterals [DS], which are the fundamental objects of discrete differential geometry (see [BS1, BS2] for a detailed presentation of the current state of discrete differential geometry as well as for historical remarks).

Two planar quadrilaterals are said to be dual, if their corresponding sides are parallel and their non-corresponding diagonals are also parallel. In [PLWBW], this property has been identified as a characterization of pairs of quadrilaterals with parallel sides and with the vanishing mixed area, and it has been observed that the corresponding circular quadrilaterals of dual discrete isothermic surfaces possess this property. These observations stimulated the development presented here. Namely, we study here the geometric and analytic properties of nets all of whose quadrilaterals can be dualized simultaneously.

A net with planar quadrilaterals admitting a dual net is called a discrete Koenigs net. A discrete Koenigs net with all circular quadrilaterals is called a discrete isothermic net.

Discrete surfaces we arrive at are not new. The class of discrete isothermic surfaces turns out to coincide with the original class introduced in [BP], so that we get just a novel characterization of the latter. In the case of discrete Koenigs nets, the history is more complicated: they first appeared in [S1] (see also [S2, S3]) in the context of infinitesimal deformations of surfaces, with exactly the same definition as we use (dual quadrilaterals...
are called antiparallel there); however, the geometric and analytic properties of these nets remained to a large extent unexplored. Recently, this class has been studied in [D3, BS3], but again some of the crucial properties passed unnoticed. The main novel results of the present paper include:

- Definition of discrete Koenigs nets as those admitting dual nets (Definition 10).
- A characterization of discrete Koenigs nets in terms of a closed multiplicative one-form on diagonals, defined through ratios of diagonal segments (Theorem 12). Integrating this closed form, we arrive at the function \( \nu \) defined at the vertices of a discrete Koenigs net. This function is a novel and important ingredient of an analytic description of discrete Koenigs nets. In particular, the function \( \nu \) allows us to find a discrete analog of a Laplace equation with equal invariants (equation (30)), and defines the Moutard representatives of a discrete Koenigs net (Theorem 20).
- A novel projective-geometric characterization of two-dimensional Koenigs nets: intersection points of diagonals of elementary quadrilaterals of such a net form a net with planar quadrilaterals (Theorem 14). Interestingly, the net comprised by the intersection points of diagonals of quadrilaterals of a discrete Koenigs net in the sense of our definition turns out to satisfy the definition of discrete Koenigs nets from [D2].
- A novel definition of discrete isothermic nets as circular nets admitting dual nets (Definition 21).
- A novel understanding of the discrete metric of a discrete isothermic net, as the function \( \nu \) in the circular context (Theorem 28).

2 Koenigs nets and isothermic surfaces

2.1 Definitions and duality

**Definition 1 (Koenigs net)** A map \( f : \mathbb{R}^2 \to \mathbb{R}^N \) is called a Koenigs net, if it satisfies a differential equation

\[
\partial_1 \partial_2 f = (\partial_2 \log \nu) \partial_1 f + (\partial_1 \log \nu) \partial_2 f
\]  

(1)

with some scalar function \( \nu : \mathbb{R}^2 \to \mathbb{R}^* \).

The following characterization of Koenigs nets will be of a fundamental importance for us.

**Theorem 2 (Dual Koenigs net)** A conjugate net \( f : \mathbb{R}^2 \to \mathbb{R}^N \) is a Koenigs net, if and only if there exists a scalar function \( \nu : \mathbb{R}^2 \to \mathbb{R} \) such that the differential one-form \( df^* \) defined by

\[
\partial_1 f^* = \frac{\partial_1 f}{\nu^2}, \quad \partial_2 f^* = -\frac{\partial_2 f}{\nu^2}
\]

(2)

is closed. In this case the map \( f^* : \mathbb{R}^2 \to \mathbb{R}^N \), defined (up to a translation) by the integration of this one-form, is also a Koenigs net, called dual to \( f \).
This follows immediately by cross-differentiating eq. (2). A different way to formulate the latter equations is:

\[
\frac{\partial_1 f^*}{|\partial_1 f^*|} \parallel \frac{\partial_2 f^*}{|\partial_2 f^*|}, \quad \frac{(\partial_1 + \partial_2) f^*}{|\partial_1 + \partial_2 f^*|} \parallel \frac{(\partial_1 - \partial_2) f^*}{|\partial_1 - \partial_2 f^*|},
\]

(3)

**Definition 3 (Isothermic surface)** A curvature line parametrized surface \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^N \) is called an isothermic surface, if its first fundamental form is conformal, possibly upon a re-parametrization of independent variables \( u_i \mapsto \varphi_i(u_i) \) \( (i = 1, 2) \), i.e., if at every point \( u \in \mathbb{R}^2 \) of the definition domain there holds

\[
|\partial_1 f|^2 / |\partial_2 f|^2 = \alpha_1(u_1) / \alpha_2(u_2).
\]

In other words, isothermic surfaces are characterized by the relations

\[
(\partial_1 f, \partial_2 f) \in \text{span}(\partial_1 f, \partial_2 f), \quad (\partial_1 f, \partial_2 f) = 0,
\]

(4)

with some \( s : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \) and with the functions \( \alpha_i \) depending on \( u_i \) only \( (i = 1, 2) \).

Conditions (4) may be equivalently represented as

\[
\partial_1 \partial_2 f = (\partial_2 \log s)\partial_1 f + (\partial_1 \log s)\partial_2 f, \quad (\partial_1 f, \partial_2 f) = 0.
\]

(5)

Comparison with eq. (1) shows that isothermic surfaces are nothing but orthogonal Koenigs nets, the role of the function \( \nu \) being played by the metric \( s \).

In the case of isothermic surfaces the duality is specialized as follows.

**Theorem 4 (Dual isothermic surface)** Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^N \) be an isothermic surface.

Then the \( \mathbb{R}^N \)-valued one-form \( df^* \) defined by

\[
\partial_1 f^* = \frac{\partial_1 f}{|\partial_1 f|^2} = \frac{\partial_1 f}{s^2}, \quad \partial_2 f^* = -\frac{\partial_2 f}{|\partial_2 f|^2} = -\frac{\partial_2 f}{s^2}
\]

(6)

is closed. The surface \( f^* : \mathbb{R}^2 \rightarrow \mathbb{R}^N \), defined (up to a translation) by the integration of this one-form, is isothermic, with

\[
(\partial_1 f^*, \partial_2 f^*) = 0, \quad |\partial_1 f^*|^2 = \alpha_1 s^{-2}, \quad |\partial_2 f^*|^2 = \alpha_2 s^{-2}.
\]

(7)

The surface \( f^* \) is called dual to, or the Christoffel transform of the surface \( f \).

### 2.2 Moutard representatives

Remarkably, the defining property (1) turns out to be invariant under projective transformations of \( \mathbb{R}^N \), so that the notion of Koenigs nets actually belongs to projective geometry.

If one considers the ambient space \( \mathbb{R}^N \) of a Koenigs net as an affine part of \( \mathbb{RP}^N \), then there is an important choice of representatives for \( f \sim (f, 1) \) in the space \( \mathbb{RP}^N \) of homogeneous coordinates, namely

\[
y = \nu^{-1}(f, 1).
\]

(8)

Indeed, a straightforward computation shows that the representatives (8) satisfy the following simple differential equation:

\[
\partial_1 \partial_2 y = qy
\]

(9)
with the scalar function $q = \nu \partial_1 \partial_2 (\nu^{-1})$. Differential equation (9) is known as the Moutard equation. Accordingly, we call a map $y : \mathbb{R}^2 \to \mathbb{R}^{N+1}$ a Moutard net, if it satisfies the Moutard equation (9) with some $q : \mathbb{R}^2 \to \mathbb{R}$.

**Theorem 5 (Koenigs nets = Moutard nets in homogeneous coordinates)** For a Koenigs net $f : \mathbb{R}^2 \to \mathbb{R}^N$, the lift $\tilde{f}$ is a Moutard net. Conversely, given a Moutard net $y : \mathbb{R}^2 \to \mathbb{R}^{N+1}$ with a non-vanishing last component $\nu^{-1} : \mathbb{R}^2 \to \mathbb{R}^+$, define $f : \mathbb{R}^2 \to \mathbb{R}^N$ by eq. (8), then $f$ is a Koenigs net.

More generally, for a given Moutard net $y$ in $\mathbb{R}^{N+1}$, it is not difficult to figure out the condition for a scalar function $\nu : \mathbb{R}^2 \to \mathbb{R}^+$, under which $\tilde{f} = \nu y$ satisfies equation of the Laplace type: $\nu^{-1}$ has to be a solution of the same Moutard equation (9) (not necessarily one of the components of the vector $y$), and then there holds

$$\partial_1 \partial_2 \tilde{f} = (\partial_2 \log \nu) \partial_1 \tilde{f} + (\partial_1 \log \nu) \partial_2 \tilde{f}.$$  

Of course, Moutard nets can be considered also on their own rights, i.e., one does not have to regard the ambient space $\mathbb{R}^{N+1}$ of a Moutard net as the space of homogeneous coordinates for $\mathbb{P}^N$. Nevertheless, such an interpretation is useful in the most cases.

In application to isothermic surfaces, the construction of Moutard representatives can be performed within the projective model of Möbius geometry. Recall that, although conditions (4) are formulated in Euclidean terms, they are invariant not only with respect to Euclidean motions and dilations in $\mathbb{R}^N$, but also with respect to the inversion $f \mapsto f/(f, f)$. Therefore, the notion of isothermic surfaces belongs to Möbius differential geometry.

Recall (see, e.g., [HJ] or [BG]) that the basic space of the projective model of Möbius geometry in $\mathbb{R}^N$ is the projectivization $\mathbb{P}(\mathbb{R}^{N+1,1})$ of the Minkowski space $\mathbb{R}^{N+1,1}$. The latter is the space spanned by $N+2$ linearly independent vectors $e_1, \ldots, e_{N+2}$ and equipped with the Minkowski scalar product

$$\langle e_i, e_j \rangle = \begin{cases} 
1, & i = j \in \{1, \ldots, N+1\}, \\
-1, & i = j = N + 2, \\
0, & i \neq j.
\end{cases}$$

It is convenient to introduce two isotropic vectors $e_0 = \frac{1}{2}(e_{N+2} - e_{N+1})$, $e_\infty = \frac{1}{2}(e_{N+2} + e_{N+1})$, satisfying $\langle e_0, e_\infty \rangle = -\frac{1}{2}$.

A point $f \in \mathbb{R}^N$ is modelled in the space $\mathbb{P}(\mathbb{R}^{N+1,1})$ by the element with homogeneous coordinates $\hat{f} = f + e_0 + |f|^2 e_\infty$. Thus, points $f \in \mathbb{R}^N \cup \{\infty\}$ are in a one-to-one correspondence with points of the projectivized light cone $\mathbb{P}(\mathbb{L}^{N+1,1})$, where

$$\mathbb{L}^{N+1,1} = \{ \xi \in \mathbb{R}^{N+1,1} : \langle \xi, \xi \rangle = 0 \}.$$  

A surface $f : \mathbb{R}^2 \to \mathbb{R}^N$ is curvature lines parametrized, if and only if its lift $\hat{f} : \mathbb{R}^2 \to \mathbb{L}^{N+1,1}$ into the light cone is a conjugate net. In particular, eqs. (5) are equivalent to

$$\partial_1 \partial_2 \hat{f} = (\partial_2 \log s) \partial_1 \hat{f} + (\partial_1 \log s) \partial_2 \hat{f}.$$  

Thus, the following result by Darboux [Da] holds:
Theorem 6 (Isothermic surfaces = Moutard nets in the light cone) For an isothermic surface \( f : \mathbb{R}^2 \to \mathbb{R}^N \), with the conformal metric \( s : \mathbb{R}^2 \to \mathbb{R}^+ \), define its lift \( y : \mathbb{R}^2 \to \mathbb{L}^{N+1,1} \) into the light cone by
\[
y = s^{-1}(f + e_0 + |f|^2 e_\infty).
\] (11)
Then \( y \) satisfies the Moutard equation (9) with \( q = s \partial_1 \partial_2 (s^{-1}) \).

Conversely, given a Moutard net \( y : \mathbb{R}^2 \to \mathbb{L}^{N+1,1} \) in the light cone, define \( s : \mathbb{R}^2 \to \mathbb{R}^* \) and \( f : \mathbb{R}^2 \to \mathbb{R}^N \) by eq. (11), so that \( s^{-1} \) is the \( e_0 \)-component, and \( s^{-1} f \) is the \( \mathbb{R}^N \)-part of \( y \) in the basis \( e_1, \ldots, e_N, e_0, e_\infty \). Then \( f \) is an isothermic surface, and the definition (4) holds with the functions \( \alpha_i = (\partial_i y, \partial_i y) \) depending on \( u_i \) only.

Note that in the second part of the theorem we can always assume that \( s : \mathbb{R}^2 \to \mathbb{R}^+ \), changing \( y \) to \(-y\), if necessary.

3 Discrete Koenigs and Moutard nets

3.1 Notion of dual quadrilaterals

Definition 7 (Dual quadrilaterals, see [S1, S2, S3, PLWBW]) Two quadrilaterals \((A, B, C, D)\) and \((A^*, B^*, C^*, D^*)\) in a plane are called dual, if their corresponding sides are parallel:
\[
(A^* B^*) \parallel (AB), \quad (B^* C^*) \parallel (BC), \quad (C^* D^*) \parallel (CD), \quad (D^* A^*) \parallel (DA),
\] (12)
and the non-corresponding diagonals are parallel:
\[
(A^* C^*) \parallel (BD), \quad (B^* D^*) \parallel (AC).
\] (13)

![Figure 1: Dual quadrilaterals](image)

Lemma 8 (Existence and uniqueness of dual quadrilateral) For any planar quadrilateral \((A, B, C, D)\) a dual one exists and is unique up to scaling and translation.
Proof. Uniqueness of the form of the dual quadrilateral can be argued as follows. Denote the intersection point of the diagonals of \((A, B, C, D)\) by \(M = (AC) \cap (BD)\). Take an arbitrary point \(M^*\) in the plane as the designated intersection point of the diagonals of the dual quadrilateral. Draw two lines \(\ell_1\) and \(\ell_2\) through \(M^*\) parallel to \((AC)\) and \((BD)\), respectively, and choose an arbitrary point on \(\ell_2\) to be \(A^*\). Then the rest of construction is unique: draw the line through \(A^*\) parallel to \((AB)\), its intersection point with \(\ell_1\) will be \(B^*\); draw the line through \(B^*\) parallel to \((BC)\), its intersection point with \(\ell_2\) will be \(C^*\); draw the line through \(C^*\) parallel to \((CD)\), its intersection point with \(\ell_1\) will be \(D^*\).

It remains to see that this construction closes, namely that the line through \(D^*\) parallel to \((DA)\) intersects \(\ell_2\) at \(A^*\). Clearly, this property does not depend on the initial choice of \(A^*\) on \(\ell_2\), since this choice only affects the scaling of the dual picture. Therefore, it is enough to demonstrate the closing property for some choice of \(A^*\), or, in other words, to show the existence of one dual quadrilateral. This can be done as follows.

Denote by \(e_1\) and \(e_2\) some vectors along the diagonals, and introduce the coefficients \(\alpha, \ldots, \delta\) by

\[
\begin{align*}
\overrightarrow{MA} &= \alpha e_1, & \overrightarrow{MB} &= \beta e_2, & \overrightarrow{MC} &= \gamma e_1, & \overrightarrow{MD} &= \delta e_2, \\
\end{align*}
\]

so that

\[
\begin{align*}
\overrightarrow{AB} &= \beta e_2 - \alpha e_1, & \overrightarrow{BC} &= \gamma e_1 - \beta e_2, \\
\overrightarrow{CD} &= \delta e_2 - \gamma e_1, & \overrightarrow{DA} &= \alpha e_1 - \delta e_2.
\end{align*}
\]

Construct a quadrilateral \((A^*, B^*, C^*, D^*)\) by setting

\[
\begin{align*}
\overrightarrow{M^*A^*} &= -\frac{e_2}{\alpha}, & \overrightarrow{M^*B^*} &= -\frac{e_1}{\beta}, & \overrightarrow{M^*C^*} &= -\frac{e_2}{\gamma}, & \overrightarrow{M^*D^*} &= -\frac{e_1}{\delta}.
\end{align*}
\]

Its diagonals are parallel to the non-corresponding diagonals of the original quadrilateral, by construction. The corresponding sides are parallel as well:

\[
\begin{align*}
\overrightarrow{A^*B^*} &= -\frac{1}{\beta} e_1 + \frac{1}{\alpha} e_2 = \frac{1}{\alpha \beta} \overrightarrow{AB}, \\
\overrightarrow{B^*C^*} &= -\frac{1}{\gamma} e_2 + \frac{1}{\beta} e_1 = \frac{1}{\beta \gamma} \overrightarrow{BC}, \\
\overrightarrow{C^*D^*} &= -\frac{1}{\delta} e_1 + \frac{1}{\gamma} e_2 = \frac{1}{\gamma \delta} \overrightarrow{CD}, \\
\overrightarrow{D^*A^*} &= -\frac{1}{\alpha} e_2 + \frac{1}{\delta} e_1 = \frac{1}{\delta \alpha} \overrightarrow{DA}.
\end{align*}
\]

Thus, the quadrilateral \((A^*, B^*, C^*, D^*)\) is dual to \((A, B, C, D)\). \(\square\)

Note that the quantities \(\alpha, \ldots, \delta\) in eq. (14) are not well defined by the geometry of the quadrilateral \((A, B, C, D)\), since they depend on the choice of the vectors \(e_1, e_2\). Well defined are their ratios, which can be viewed also as ratios of the directed lengths of the corresponding segments of diagonals, say \(\gamma : \alpha = l(M, C) : l(M, A)\) and \(\delta : \beta = l(M, D) : l(M, B)\). It is natural to associate these ratios with directed diagonals:

**Definition 9 (Ratio of diagonal segments)** Given a quadrilateral \((A, B, C, D)\), with the intersection point of diagonals \(M = (AC) \cap (BD)\), we set

\[
q(\overrightarrow{AC}) = \frac{l(M, C)}{l(M, A)}, \quad q(\overrightarrow{BD}) = \frac{l(M, D)}{l(M, B)}.
\]
Changing the direction of a diagonal corresponds to inverting the associated quantity \( q \).

Note that
\[
(A, B, C, D) \text{ convex } \iff q(\overrightarrow{AC}) < 0 \text{ and } q(\overrightarrow{BD}) < 0.
\]  

(18)

3.2 Notion of discrete Koenigs nets

In dealing with discrete nets \( f : \mathbb{Z}^m \to \mathbb{R}^N \) we will use the usual notations:
\[
\tau_i f(u) = f(u + e_i), \quad \delta_i f(u) = f(u + e_i) - f(u),
\]
where \( e_i \) stands for the unit vector of the \( i \)th coordinate direction. Moreover, we often abbreviate \( f(u), \tau_i f(u), \tau_i \tau_j f(u) \) to \( f, f_i, f_{ij} \), respectively. The following definition is the fundamental one for the present paper.

**Definition 10 (Discrete Koenigs net)** A Q-net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) is called a discrete Koenigs net, if it admits a dual net, i.e., a Q-net \( f^* : \mathbb{Z}^m \to \mathbb{R}^N \) such that all elementary quadrilaterals of the net \( f^* \) are dual to the corresponding quadrilaterals of \( f \):
\[
\delta_1 f^* \parallel \delta_1 f, \quad \delta_2 f^* \parallel \delta_2 f, \\
\delta_1 f^* - f^* \parallel f_1 - f_2, \quad f_1^* - f_2^* \parallel f_12 - f.
\]  

(19)

This definition can be seen as a discretization of conditions (3).

In order to understand restrictions imposed on a Q-net by this definition, we start with the following construction. Each lattice \( \mathbb{Z}^m \) is bi-partite: one can color its vertices black and white so that each edge connects a black vertex with a white one (for instance, one can call vertices \( u = (u_1, \ldots, u_m) \) with an even value of \( |u| = u_1 + \ldots + u_m \) black and those with an odd value of \( |u| \) white). Each elementary quadrilateral has a black diagonal (the one connecting two black vertices) and a white one. One can introduce the **black graph** \( \mathbb{Z}^m_{\text{even}} \) with the set of vertices consisting of the white vertices of \( \mathbb{Z}^m \) and the set of edges consisting of black diagonals of all elementary quadrilaterals of \( \mathbb{Z}^m \), and the analogous **white graph** \( \mathbb{Z}^m_{\text{odd}} \). The geometry of the elementary quadrilaterals of a Q-net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) induces, according to Definition 10 the quantities \( q \) (ratios of directed lengths of diagonal segments) on all directed diagonals, white and black.

**Definition 11 (Multiplicative one-form)** Given a graph \( G \) with the set of vertices \( V \) and with the set of directed edges \( \vec{E} \), the function \( q : \vec{E} \to \mathbb{R}^* \) is called a multiplicative one-form on \( G \), if for any directed edge \( e \in \vec{E} \) there holds \( q(\overrightarrow{e}) = 1/q(\overleftarrow{e}) \). Such a form is called closed, if for any cycle of directed edges the product of values of \( q \) along this cycle is equal to one.

Thus, any Q-net yields a multiplicative one-form \( q \) (or, better, two multiplicative one-forms) on both the black and the white graphs of \( \mathbb{Z}^m \).

**Theorem 12 (Algebraic characterization of discrete Koenigs nets)** A Q-net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) is a Koenigs net, if and only if the multiplicative one-form \( q \) is closed on both \( \mathbb{Z}^m_{\text{even}} \) and \( \mathbb{Z}^m_{\text{odd}} \).
Proof. For a given Q-net, one can try to construct a dual net, applying Lemma 8, starting with an arbitrary quadrilateral. It is easy to realize that obstructions in extending this construction to the whole net may appear when running along closed chains of elementary quadrilaterals in which any two subsequent quadrilaterals share an edge.

$m = 2$. The basic example of a closed chain of quadrilaterals in this case is given by four elementary quadrilaterals attached to a (black, say) vertex $f$. Let the diagonals of each quadrilateral be divided by their intersection point in the relations $\gamma_k : \alpha_k$ and $\delta_k : \beta_k$ ($k = 1, \ldots, 4$), as on Fig. 2. The dual quadrilaterals are determined up to scaling factors $\lambda_k$ ($k = 1, \ldots, 4$), say. Matching the edge shared by the dual quadrilaterals 1 and 2, we find the relation between their scaling factors:

$$\frac{\lambda_1}{\alpha_1 \delta_1} = \frac{\lambda_2}{\alpha_2 \beta_2} \iff \frac{\lambda_1}{\lambda_2} = \frac{\alpha_1 \delta_1}{\alpha_2 \beta_2}.$$ 

Similarly, we find:

$$\frac{\lambda_2}{\lambda_3} = \frac{\alpha_2 \delta_2}{\alpha_3 \beta_3}, \quad \frac{\lambda_3}{\lambda_4} = \frac{\alpha_3 \delta_3}{\alpha_4 \beta_4}, \quad \frac{\lambda_4}{\lambda_1} = \frac{\alpha_4 \delta_4}{\alpha_1 \beta_1}.$$ 

All four edges adjacent to $f$ can be matched, if and only if the cyclic product of expressions for the quotients of scaling factors is equal to one. This condition reads:

$$\frac{\alpha_1 \delta_1}{\alpha_2 \beta_2} \cdot \frac{\alpha_2 \delta_2}{\alpha_3 \beta_3} \cdot \frac{\alpha_3 \delta_3}{\alpha_4 \beta_4} \cdot \frac{\alpha_4 \delta_4}{\alpha_1 \beta_1} = 1,$$

or

$$\frac{\delta_1}{\beta_1} \cdot \frac{\delta_2}{\beta_2} \cdot \frac{\delta_3}{\beta_3} \cdot \frac{\delta_4}{\beta_4} = 1. \quad (20)$$
This is nothing but the closeness condition of the form $q$ for an elementary quadrilateral of the white graph. All other white and black cycles are products of elementary ones, therefore (20) for all elementary white and black cycles are necessary and sufficient for the closeness of the form $q$. But it is easy to see that if the closeness condition is fulfilled for all white and black cycles, then no closed chain of quadrilaterals can lead to an obstruction by the construction of the dual net.

$m = 3$. In this case the most elementary closed chain of quadrilaterals is given by three faces of any elementary hexahedron of the net, sharing a (black, for definiteness) vertex $f$, see Fig. 3. The further arguments are completely analogous to the two-dimensional case.

Figure 3: Three quadrilaterals around a vertex of a three-dimensional net

Matching the edges shared by the dual quadrilaterals 1 and 2, by the dual quadrilaterals 2 and 3, and by the dual quadrilaterals 3 and 1, we find the relations between their scaling factors:

$$\frac{\lambda_1}{\lambda_2} = \frac{\alpha_1 \delta_1}{\alpha_2 \beta_2}, \quad \frac{\lambda_2}{\lambda_3} = \frac{\alpha_2 \delta_2}{\alpha_3 \beta_3}, \quad \frac{\lambda_3}{\lambda_1} = \frac{\alpha_3 \delta_3}{\alpha_1 \beta_1}.$$

All three edges adjacent to $f$ can be matched simultaneously, if and only if the cyclic product of expressions for the quotients of scaling factors is equal to one, which condition after cancellations reads:

$$\frac{\delta_1}{\beta_1} \cdot \frac{\delta_2}{\beta_2} \cdot \frac{\delta_3}{\beta_3} = 1.$$

This is nothing but the closeness condition for the elementary cycle of the white graph of the lattice $\mathbb{Z}^3$, which is a triangle. All cycles of the white and of the black graphs (including those encountered in the $m = 2$ case, i.e., the squares of the two-dimensional slices of the white and the black graphs of $\mathbb{Z}^3$) are products of elementary triangles. Again, closeness condition for all white and black cycles guarantees that no closed chain of quadrilaterals leads to an obstruction.
\( m \geq 4 \). Also in this case any white or black cycle is a product of elementary triangles, as for \( m = 3 \), therefore no additional conditions appear. \( \square \)

### 3.3 Geometric characterization of two-dimensional discrete Koenigs nets

The definition of discrete Koenigs nets obviously belongs to affine geometry, since it relies on the notion of parallelism. It turns out however that the class of discrete Koenigs nets is projectively invariant (it has been pointed out already in [S1, S2]). The proof of the corresponding projectively invariant characterizations will rely on the generalized Menelaus theorem [Bo, BN] which has a similar flavor: its conditions are of affine-geometric nature, while its conclusions are projectively invariant.

**Theorem 13 (Generalized Menelaus theorem)** Let \( P_1, \ldots, P_{n+1} \) be \( n + 1 \) points in general position in \( \mathbb{R}^n \), so that the affine space through the points \( P_i \) is \( n \)-dimensional. Let \( P_{i,i+1} \) be some points on the lines \( (P_iP_{i+1}) \) (indices are read modulo \( n+1 \)). The \( n+1 \) points \( P_{i,i+1} \) lie in an \( (n-1) \)-dimensional affine subspace, if and only if the following relation for the ratios of the directed lengths holds:

\[
\prod_{i=1}^{n+1} \frac{l(P_i, P_{i,i+1})}{l(P_{i,i+1}, P_{i+1})} = (-1)^{n+1}.
\]

**Proof.** The points \( P_{i,i+1} \) lie in an \( (n-1) \)-dimensional affine subspace, if there is a non-trivial linear dependence

\[
\sum_{i=1}^{n+1} \mu_i P_{i,i+1} = 0 \quad \text{with} \quad \sum_{i=1}^{n+1} \mu_i = 0.
\]

Substituting \( P_{i,i+1} = (1 - \xi_i)P_i + \xi_i P_{i+1} \), and taking into account the general position condition, which can be read as linear independence of the vectors \( \overrightarrow{P_iP_i} \), we come to a homogeneous system of \( n + 1 \) linear equations for \( n + 1 \) coefficients \( \mu_i \):

\[
\xi_i \mu_i + (1 - \xi_{i+1}) \mu_{i+1} = 0, \quad i = 1, \ldots, n+1
\]

(where indices are understood modulo \( n+1 \)). Clearly it admits a non-trivial solution if and only if

\[
\prod_{i=1}^{n+1} \frac{\xi_i}{1 - \xi_i} = \prod_{i=1}^{n+1} \frac{l(P_i, P_{i,i+1})}{l(P_{i,i+1}, P_{i+1})} = (-1)^{n+1}.
\]

(Menelaus theorem corresponds to \( n = 2 \).) \( \square \)

In the following considerations, we use the negative indices \(-1, -2\) to denote the downward shifts \( \tau_1^{-1}, \tau_2^{-1} \). Consider four elementary quadrilaterals of a Q-net adjacent to the point \( f = f(u) \), i.e., the quadrilaterals \( (f, f_i, f_{ij}, f_j) \) with \( (i, j) \in \{(\pm 1, \pm 2)\} \). We assume that the vertex \( f \) is non-planar, i.e., that there is no plane containing these four quadrilaterals (or, what is the same, there is no plane containing \( f \) and its four neighbors \( f_i, i \in \{\pm 1, \pm 2\} \)). Recall that we always assume that the dimension of the ambient space is \( N \geq 3 \).
Theorem 14 (Discrete 2d Koenigs nets; characterization in terms of intersection points of diagonals) A two-dimensional Q-net \( f : \mathbb{Z}^2 \to \mathbb{R}^N \) with non-planar vertices is a discrete Koenigs net, if and only if for every point \( f = f(u) \) the intersection points of diagonals of the four quadrilaterals adjacent to \( f \) lie in a two-dimensional plane.

Proof. This is an immediate consequence of eq. (20) and the \( n = 3 \) case of the generalized Menelaus theorem (Theorem 13). \( \square \)

Remark. Thus, intersection points of diagonals of elementary quadrilaterals of a two-dimensional Koenigs net comprise a Q-net. Such Q-nets are not generic; it turns out that they can be characterized as discrete Koenigs nets in the sense of [D2].

Theorem 15 (Discrete 2d Koenigs nets; characterization in terms of vertices)

1) Let \( f : \mathbb{Z}^2 \to \mathbb{R}^N \) be the a Q-net in the space of dimension \( N \geq 4 \). Then \( f \) is a discrete Koenigs net, if and only if for every \( u \in \mathbb{Z}^2 \) the five points \( f \) and \( f_{\pm 1, \pm 2} \) lie in a three-dimensional subspace \( V = V(u) \subset \mathbb{R}^N \), not containing some (and then any) of the four points \( f_{\pm 1}, f_{\pm 2} \).

2) Let \( f : \mathbb{Z}^2 \to \mathbb{R}^3 \) be a Q-net in the space of dimension \( N = 3 \). Then \( f \) is a discrete Koenigs net, if and only if for every \( u \in \mathbb{Z}^2 \) the three planes

\[
\Pi^{(\text{up})} = (ff_{12} f_{-1,2}), \quad \Pi^{(\text{down})} = (ff_{1,-2} f_{-1,-2}), \quad \Pi^{(1)} = (ff_1 f_{-1})
\]

have a common line \( \ell^{(1)} \), or, equivalently, the three planes

\[
\Pi^{(\text{left})} = (ff_{-1,2} f_{-1,-2}), \quad \Pi^{(\text{right})} = (ff_{1,2} f_{1,-2}), \quad \Pi^{(2)} = (ff_2 f_{-2})
\]

have a common line \( \ell^{(2)} \).

Proof. 1) If the net \( f \) satisfies the property of Theorem 14, then the space \( V \) through \( f \) and \( f_{\pm 1, \pm 2} \) is clearly three-dimensional. Conversely, let this space be three-dimensional. The four quadrilaterals \( (f, f_i, f_{ij}, f_j) \) lie in a four-dimensional space through \( f, f_{\pm 1}, f_{\pm 2} \). The intersection points of their diagonals lie in the intersection of \( V \) with the three-dimensional space through \( f_{\pm 1}, f_{\pm 2} \). The intersection of two three-dimensional subspaces of a four-dimensional space is generically a plane.

2) Let \( M_{ij} \) denote intersection point of diagonals of the quadrilateral \( (f, f_i, f_{ij}, f_j) \), with \( (i, j) \in \{ (\pm 1, \pm 2) \} \). Co-planarity of the four points \( M_{ij} \) is equivalent to the statement that the lines \( (M_{1,2} M_{-1,2}) \) and \( (M_{1,-2} M_{-1,-2}) \) intersect. These two lines lie in the planes \( (f_1 f_2 f_{-1}), (f_1 f_2 f_{-1}) \), respectively, therefore their intersection point has to belong to the intersection of these planes, i.e., to the line \( (f_1 f_{-1}) \). Thus, coplanarity of the points \( M_{ij} \) is equivalent to the fact that three lines \( (M_{1,2} M_{-1,2}), (M_{1,-2} M_{-1,-2}), \) and \( (f_1 f_{-1}) \) have a common point \( L^{(1)} \), see Fig. 4. Now the planes \( \Pi^{(\text{up})}, \Pi^{(\text{down})} \) and \( \Pi^{(1)} \) can be viewed as the planes through the point \( f \) and the lines \( (M_{1,2} M_{-1,2}), (M_{1,-2} M_{-1,-2}), \) and \( (f_1 f_{-1}) \), respectively. Therefore their intersection is the line \( \ell^{(1)} \) through \( f \) and \( L^{(1)} \). \( \square \)

Remark 1. It is not difficult to see that in the dimension \( N \geq 4 \) the property formulated in part 1) of Theorem 13 automatically yields the property formulated in part 2). Indeed, for \( N \geq 4 \) all nine points \( f, f_{\pm 1}, f_{\pm 2} \) and \( f_{\pm 1, \pm 2} \) lie generically in a four-dimensional subspace of \( \mathbb{R}^P \). In this subspace one can consider, along with the
three-dimensional subspace $V$, the three-dimensional subspaces $V^{(\text{up})}$ containing the two quadrilaterals $(f, f_1, f_{12}, f_2)$, $(f, f_{-1}, f_{-1,2}, f_2)$, and $V^{(\text{down})}$ containing the quadrilaterals $(f, f_1, f_{1,-2}, f_{-2})$, $(f, f_{-1}, f_{-1,-2}, f_{-2})$. Obviously, one has:

$$
\Pi^{(\text{up})} = V^{(\text{up})} \cap V, \quad \Pi^{(\text{down})} = V^{(\text{down})} \cap V, \quad \Pi^{(1)} = V^{(\text{up})} \cap V^{(\text{down})}.
$$

Generically, three three-dimensional subspaces $V$, $V^{(\text{up})}$ and $V^{(\text{down})}$ of a four-dimensional space intersect along a line $\ell^{(1)}$.

**Remark 2.** The equivalence of two conditions in part 2) of Theorem 15 follows, of course, from the fact that in the notion of discrete Koenigs nets there is no asymmetry between the coordinate directions 1 and 2. However, it might be worthwhile to give an additional illustration of this equivalence. For this aim, consider a central projection of the whole picture from the point $f$ to some plane not containing $f$. In this projection, the planarity of elementary quadrilaterals $(f, f_i, f_{ij}, f_j)$ turns into collinearity of the triples of points $f_i$, $f_j$ and $f_{ij}$. The traces of the planes $\Pi^{(\text{up})}$, $\Pi^{(\text{down})}$ and $\Pi^{(1)}$ on the projection plane are the lines $(f_{12}f_{-1,2})$, $(f_{1,-2}f_{-1,-2})$, and $(f_{1}f_{-1})$, respectively, and the first version of the condition of part 2) of Theorem 15 turns into the requirement for these three lines to meet in a point. Similarly, the traces of the planes $\Pi^{(\text{left})}$, $\Pi^{(\text{right})}$ and $\Pi^{(2)}$ on the projection plane are the lines $(f_{-1,2}f_{-1,-2})$, $(f_{1,2}f_{1,-2})$, and $(f_{2}f_{-2})$, respectively. The requirement for the latter three lines to meet in a point is equivalent to the previous one – this is the statement of the famous Desargues theorem, see Fig. 5.

### 3.4 Geometric characterization of three-dimensional discrete Koenigs nets

**Theorem 16 (Discrete 3d Koenigs nets; characterization in terms of intersection points of diagonals)** A three-dimensional Q-net $f : \mathbb{Z}^d \to \mathbb{R}^N$ is a discrete Koenigs net, if and only if for every point $f = f(u)$ and for every elementary hexahedron with a
vertex $f$, the intersection points of diagonals of the three hexahedron faces adjacent to $f$ are collinear.

**Proof.** This is nothing but the re-formulation of eq. (21) in terms of Menelaus theorem ($n = 2$ case of Theorem 13). □

**Theorem 17 (Discrete 3d Koenigs nets; characterization in terms of vertices)**

A $Q$-net $f : \mathbb{Z}^3 \to \mathbb{R}^N$ is a discrete Koenigs net, if and only if for every elementary hexahedron of the net its four white vertices are co-planar, or its four black vertices are co-planar (each one of these conditions implies another one).

**Proof.** Consider an elementary hexahedron with the vertices $f, f_i, f_{ij}, f_{123}$. Denote the intersection points of diagonals of the quadrilaterals $(f, f_i, f_{ij}, f_j)$ by $M_{ij}$, and the intersection points of diagonals of the quadrilaterals $(f_k, f_{ik}, f_{123}, f_{jk})$ by $Q_{ij}$. Clearly, if the points $M_{ij}$ are collinear, then the four points $f$ and $f_{ij}$ (the black ones) are co-planar. We show next that the co-planarity of the four black points yield the co-planarity of the four white points, as well.

Suppose that the four black points $f, f_{ij}$ lie in a plane $\Pi_0$. Let $\Pi_1$ be the plane through the three points $f_1, f_2, f_3$. Set $\ell = \Pi_0 \cap \Pi_1$. Then the intersection points $M_{ij}$ of diagonals of the quadrilaterals $(f, f_i, f_{ij}, f_j)$ belong to $\ell$. Denote by $O_{ij}$ intersection points of the lines $(f_{ik}, f_{jk}) \subset \Pi_0$ with $\ell$. Then the three lines $(f_{k}O_{ij}) \subset \Pi_1$ intersect in one point, which is clearly $f_{123} \in \Pi_1$, so that the four points $f_i, f_{123}$ are co-planar. This claim is nothing but the classical Pappus theorem illustrated on Fig. 5. This incidence theorem of projective geometry is not to be confused with another Pappus theorem, the latter being a particular case of the Pascal hexagon theorem, when a conic section degenerates into a pair of lines. The former characterizes a quadrilateral set of points on a line $\ell$ which can be defined as consisting of intersection points of this line with the six lines connecting all pairs among four points in some plane containing $\ell$. Quadrilateral sets admit several equivalent
Figure 6: Pappus theorem

classifications: a multi-ratio of such a set is equal to 1; in other words, the points of a quadrilateral set always build three point pairs of a projective involutive self-map of $\ell$.

Now we can finish the proof of Theorem 17 as follows. Suppose that the black vertices of an elementary hexahedron of a Q-net are co-planar. Then also the white vertices of this hexahedron are co-planar. Then the intersection points of diagonals of all six faces of the hexahedron are collinear (they belong to the common line of the “black” and the “white” planes). According to the characterization of Theorem 16, the net is Koenigs. □

Remark. The characterizations of Theorems 15, 17 coincide with the definitions of B-quadrilateral nets in [D3] and of discrete Moutard nets in [BS3]. Thus, the point we make here is a new property of these nets, fixed as Definition 10 and put in the base of the whole theory. A novel derivation and understanding of the Moutard property of discrete Koenigs nets will be given below, in Sect. 3.6.

3.5 Dual discrete Koenigs nets

We start with the following statement which is a direct consequence of the algebraic characterization of discrete Koenigs nets given in Theorem 12. Indeed, in our local setting, due to the simple-connectedness of the underlying graphs, the closeness of the multiplicative one-form $q$ is equivalent to its exactness:

**Corollary 18 (Function $\nu$ for a discrete Koenigs net)** A Q-net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is a discrete Koenigs net, if and only if there exists a real-valued function $\nu : \mathbb{Z}^m \to \mathbb{R}^*$ with the following property: for every elementary quadrilateral $(f, f_i, f_{ij}, f_j)$ there holds:

\[
\frac{\nu_{ij}}{\nu} = q(f f_{ij}) = \frac{l(M, f_{ij})}{l(M, f)} \quad \text{and} \quad \frac{\nu_i}{\nu} = q(f_i f_j) = \frac{l(M, f_j)}{l(M, f_i)},
\]

(22)

where $M = (f f_{ij}) \cap (f_i f_j)$ is the intersection point of diagonals.
On both the black and the white graphs of $\mathbb{Z}^m$ such a function $\nu$ is defined up to a multiplicative constant. This freedom is fixed by prescribing values of $\nu$ arbitrarily at one black and at one white point.

Eq. (22) is equivalent to

\[
\frac{1}{\nu_{ij}} M f_{ij} = \frac{1}{\nu_i} M f_i, \quad \frac{1}{\nu_j} M f_j = \frac{1}{\nu_j} M f_j,
\]

which can be re-written also as

\[
\frac{f_{ij}}{\nu_{ij}} - \frac{f}{\nu} = \left( \frac{1}{\nu_{ij}} - \frac{1}{\nu} \right) M, \quad \frac{f_i}{\nu_i} - \frac{f_j}{\nu_j} = \left( \frac{1}{\nu_i} - \frac{1}{\nu_j} \right) M.
\]

There follows:

\[
\left( \frac{1}{\nu_j} - \frac{1}{\nu_i} \right) \left( \frac{f_{ij}}{\nu_{ij}} - \frac{f}{\nu} \right) = \left( \frac{1}{\nu_{ij}} - \frac{1}{\nu} \right) \left( \frac{f_j}{\nu_j} - \frac{f_i}{\nu_i} \right).
\]

This formula can be used for an elegant representation of the dual Koenigs net for $f$.

**Theorem 19 (Dual Koenigs net)** Let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a discrete Koenigs net, and let $\nu : \mathbb{Z}^m \to \mathbb{R}^*$ be the function defined by the property (22). Then the $\mathbb{R}^N$-valued discrete one-form $\delta f^*$ defined by

\[
\delta_i f^* = \frac{\delta f}{\nu_{ii}}
\]

is closed. Its integration defines (up to a translation) the dual Koenigs net $f^* : \mathbb{Z}^m \to \mathbb{R}^N$.

**Proof.** Eq. (25) can be equivalently re-written as

\[
\frac{f_{ij} - f_i}{\nu_{ij} \nu_i} + \frac{f_j - f_i}{\nu_i \nu_j} = \frac{f_{ij} - f_j}{\nu_{ij} \nu_j} + \frac{f_j - f_i}{\nu_i \nu_j}.
\]

This is equivalent to the closeness of the discrete form $\delta f^*$. Note that eq. (26) says that the corresponding sides of elementary quadrilaterals of the nets $f$ and $f^*$ are parallel. It remains to show that the non-corresponding diagonals of elementary quadrilaterals of $f$ and $f^*$ are also parallel, so that these quadrilaterals are dual in the sense of Definitions 7.

For this aim we demonstrate the following two formulas:

\[
f_{ij}^* - f^* = a_{ij} \frac{f_j - f_i}{\nu_i \nu_j}, \quad f_i^* - f_i^* = \frac{1}{a_{ij}} \frac{f_{ij} - f}{\nu_{ij} \nu_i},
\]

where

\[
a_{ij} = \frac{\left( \frac{1}{\nu_{ij}} - \frac{1}{\nu} \right)}{\left( \frac{1}{\nu_j} - \frac{1}{\nu_i} \right)}.
\]

Indeed, upon using eqs. (25) and (29) we find:

\[
f_{ij}^* - f^* = (f_{ij}^* - f_i^*) + (f_i^* - f^*) = \frac{f_{ij} - f_i}{\nu_i \nu_{ij}} + \frac{f_j - f_i}{\nu_i \nu_j} = \frac{1}{\nu_i} \left( \frac{f_{ij}}{\nu_{ij} - \frac{f}{\nu}} \right) - \frac{f_i}{\nu_i} \left( \frac{1}{\nu_{ij}} - \frac{1}{\nu} \right) = a_{ij} \frac{f_j - f_i}{\nu_i \nu_j}.
\]
and, similarly,
\[
\begin{align*}
  f^*_j - f^*_i &= (f^*_{ij} - f^*_i) - (f^*_j - f^*_j) \\
  &= \frac{f_{ij} - f_i}{\nu_i \nu_{ij}} - \frac{f_{ij} - f_j}{\nu_j \nu_{ij}} \\
  &= \frac{1}{a_{ij} \nu_{ij}} \left( \frac{f_{ij} - f}{\nu} \right) - \frac{f_{ij} - f}{a_{ij} \nu_{ij}} \left( \frac{1}{\nu} \right) = \frac{1}{a_{ij} \nu_{ij}} \left( f_{ij} - f - \frac{f_{ij} - f}{\nu} \right).
\end{align*}
\]

Theorem 19 is completely proven. □

For future reference, we note here that after some manipulations formula (25) can be transformed into
\[
\delta_i \delta_j f = \frac{\nu_j \nu_{ij} - \nu \nu_i}{\nu(\nu_i - \nu_j)} \delta_i f + \frac{\nu_i \nu_{ij} - \nu \nu_j}{\nu(\nu_j - \nu_i)} \delta_j f.
\] (30)

3.6 Moutard representative of a discrete Koenigs net

Constructions of the previous subsection (functions \(\nu\) and \(a_{ij}\) for a given Koenigs net) can be used also in a different spirit.

**Theorem 20 (Discrete Koenigs nets = discrete Moutard nets in homogeneous coordinates)** A Q-net \(f: \mathbb{Z}^m \rightarrow \mathbb{R}^N\) is a discrete Koenigs net, if and only if there exists a function \(\nu: \mathbb{Z}^m \rightarrow \mathbb{R}^*\) such that the points \(y: \mathbb{Z}^m \rightarrow \mathbb{R}^{N+1}\),
\[
y = \nu^{-1}(f, 1),
\] (31)

satisfy the Moutard equation with minus signs
\[
\tau_i \tau_j y - y = a_{ij} (\tau_j y - \tau_i y) \tag{32}
\]
with \(a_{ij} \in \mathbb{R}\) given by eq. (29). The net \(y = \nu^{-1}(f, 1)\), considered as a special lift of \(f\) to the space of homogeneous coordinates for \(\mathbb{R}P^N\), will be called the Moutard representative of the discrete Koenigs net \(f\).

**Proof.** First let \(f: \mathbb{Z}^m \rightarrow \mathbb{R}^N\) be a discrete Koenigs net. Define the function \(\nu: \mathbb{Z}^m \rightarrow \mathbb{R}^*,\) according to Corollary 18. Then eq. (24) holds, with \(M\) being the intersection point of diagonals of the quadrilateral \((f, f_i, f_{ij}, f_j)\). Denoting \(y = \nu^{-1}(f, 1)\), we immediately arrive at eq. (32) with the coefficients \(a_{ij}\) defined by eq. (29).

Note that the quantities \(a_{ij}\) are naturally assigned to elementary quadrilaterals of \(\mathbb{Z}^m\) parallel to the coordinate plane \(B_{ij}\).

Conversely, given a solution \(y: \mathbb{Z}^m \rightarrow \mathbb{R}^{N+1}\) of the Moutard equation (32) in \(\mathbb{R}^{N+1}\), define \(\nu: \mathbb{Z}^m \rightarrow \mathbb{R}\) and \(f: \mathbb{Z}^m \rightarrow \mathbb{R}^N\) by \(y = \nu^{-1}(f, 1)\). In other words, let \(\nu^{-1}\) denote the last component of \(y\), and let \(f\) be the vector in \(\mathbb{R}^N\) obtained by multiplying the first \(N\) components of \(y\) by \(\nu\). Then, inverting the previous arguments, it is easy to show that \(f\) is a discrete Koenigs net. Indeed, one finds immediately expression (29) for the coefficient \(a_{ij}\) of the Moutard equation, then from
\[
y_{ij} - y = a_{ij} (y_j - y_i)
\]
there follows eq. (25). This allows to define the point \( M \) by eq. (24). The latter equation is equivalent to (23), therefore \( M \) is nothing but the intersection point of diagonals of \((f, f_i, f_{ij}, f_j)\). There holds eq. (22), so by Corollary [18] \( f \) is a Koenigs net. □

In the context of discrete integrable systems the discrete Moutard equation (32) has been introduced in [DJM], its importance for discrete differential geometry has been re-iterated in [NSch], based on the fact that this equation expresses the permutability properties of the so called Moutard transformation for the differential Moutard equation [M, Bi, GT, NSch]. The role played by the discrete Moutard equation in the discrete differential geometry turns out to be manifold. In particular, the so called Lelieuvre representation of discrete asymptotic nets involves discrete Moutard nets in \( \mathbb{R}^3 \) [KP, D1]. For the multidimensional consistency of discrete Moutard nets, which lies in the basis of the transformation theory for discrete Koenigs nets, the reader is referred to [BS1, BS3, D3].

3.7 Continuous limit

In order for a Q-net to admit a continuous limit, all its quadrilaterals should be of a reasonable shape. Anyway, they should be convex. As mentioned in subsection 3.2, diagonals of convex quadrilaterals carry negative quantities \( q \) (ratios of segments of diagonals). Theorem 12 shows that a discrete Koenigs net cannot consist of convex quadrilaterals (and thus cannot admit a continuous limit) for \( m \geq 3 \). However, there are no obstructions in case \( m = 2 \). This is in a good agreement with the existence of two-dimensional smooth Koenigs nets only. Eq. (22) shows that in case \( m = 2 \) with all convex quadrilaterals we can assume, without losing generality, that the sign of \( \nu(u) \) at \( u = (u_1, u_2) \in \mathbb{Z}^2 \) is either \((-1)^{u_1}\) or \((-1)^{u_2}\). Clearly, such a wildly oscillating function cannot have a well-behaved continuous limit. However, upon re-defining

\[
\nu(u) \leftrightarrow (-1)^{u_1} \nu(u), \quad \text{resp.} \quad \nu(u) \leftrightarrow (-1)^{u_2} \nu(u)
\]  

(33)

we get a positive function, which turns out to be a proper discrete analog of the function \( \nu \) for smooth Koenigs nets. Note that this re-definition is equivalent to changing eq. (22) to

\[
\frac{\nu_{12}}{\nu} = \frac{l(f_{12}, M)}{l(M, f)} , \quad \frac{\nu_2}{\nu_1} = \frac{l(f_2, M)}{l(M, f_1)} .
\]  

(34)

We mention also that eq. (30) with the re-defined \( \nu \) changes its shape into

\[
\delta_1 \delta_2 f = \frac{\nu_2 \nu_{12} - \nu_1 \nu}{\nu (\nu_1 + \nu_2)} \delta_1 f + \frac{\nu_1 \nu_{12} - \nu_2 \nu}{\nu (\nu_1 + \nu_2)} \delta_2 f ,
\]  

(35)

with eq. (1) as a continuous limit. Likewise, formulas (26) turn into

\[
\delta_1 f^* = \frac{\delta_1 f}{\nu \nu_1} , \quad \delta_2 f^* = \frac{\delta_2 f}{\nu \nu_2} ,
\]  

(36)

where the second re-definition of \( \nu \) in eq. (33) has been used, for definiteness (the first one would result in changing signs of both fractions).
For the Moutard representative \( y : \mathbb{Z}^2 \to \mathbb{R}^{N+1} \) of a two-dimensional discrete Koenigs net the change (33) leads to

\[
y(u) \mapsto (-1)^{u_1} y(u), \quad \text{resp.} \quad y(u) \mapsto (-1)^{u_2} y(u).
\]

(37)

These points satisfy the Moutard equation with the plus signs:

\[
\tau_1 \tau_2 y + y = a(\tau_1 y + \tau_2 y),
\]

(38)

or, equivalently,

\[
\delta_1 \delta_2 f = \frac{1}{2} q(\tau_1 f + \tau_2 f),
\]

(39)

with some \( a = 1 + \frac{1}{2} q : \mathbb{Z}^2 \to \mathbb{R} \). Clearly, the latter equation has eq. (9) as continuous limit.

4 Discrete isothermic nets

4.1 Notion of a discrete isothermic net

Definition 21 (Discrete isothermic net) A discrete isothermic net is a circular Koenigs net, i.e., a circular net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) admitting a dual net \( f^\ast : \mathbb{Z}^m \to \mathbb{R}^N \) in the sense of Definition 10.

We can use characterizations of Koenigs net derived in Sect. 3 in order to find characterizations of discrete isothermic nets. For this aim, we use the fact that for a circular net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) its lift \( \hat{f} = f + e_0 + |f|^2 e_\infty \) into the light cone \( \mathbb{L}^{N+1,1} \) satisfies the same equation of the Laplace type as the net \( f \) itself. In particular, a circular net \( f \) in \( \mathbb{R}^N \) is discrete Koenigs, if and only if \( \hat{f} \) is a discrete Koenigs net in \( \mathbb{R}^{N+1,1} \).

Projectively invariant characterizations of Koenigs nets \( \hat{f} \) in \( \mathbb{R}^{N+1,1} \) immediately translate into Möbius-geometric characterizations of isothermic nets \( f \) in \( \mathbb{R}^N \). Thereby conditions like “points \( \hat{f} \) lie in a \( d \)-dimensional space” should be understood as “vectors \( \hat{f} \) span a \( (d+1) \)-dimensional linear subspace”, and this is translated as “points \( f \) belong to a \( (d−1) \)-dimensional sphere”.

Translating in this fashion Theorem 15 applied to a two-dimensional Koenigs net \( \hat{f} \) in \( \mathbb{R}^{N+1,1} \), into the language of Möbius geometry in \( \mathbb{R}^N \), we come to the following statement.

Theorem 22

1) (Central spheres for a discrete isothermic surface) A two-dimensional circular net \( f : \mathbb{Z}^2 \to \mathbb{R}^N \) not lying in a two-sphere is discrete isothermic, if and only if for every \( u \in \mathbb{Z}^2 \) the five points \( f \) and \( f_{\pm 1, \pm 2} \) lie on a two-sphere not containing some (and then any) of the four points \( f_{\pm 1}, f_{\pm 2} \).

2) (Discrete isothermic net on a sphere) A two-dimensional circular net \( f : \mathbb{Z}^2 \to S^2 \subset \mathbb{R}^N \) in a two-sphere is discrete isothermic, if and only if for every \( u \in \mathbb{Z}^2 \) the three circles through \( f \),

\[
C^{(\text{up})} = \text{circle}(f, f_{12}, f_{-12}), \quad C^{(\text{down})} = \text{circle}(f, f_{1-2}, f_{-1-2}), \quad C^{(1)} = \text{circle}(f, f_1, f_{-1}),
\]
have one additional point in common, or, equivalently, the three circles through $f$,

$$C^{\text{(left)}} = \text{circle}(f, f_{-1,2}, f_{-1, -2}), \quad C^{\text{(right)}} = \text{circle}(f, f_{1,2}, f_{1, -2}),$$

$$C^{(2)} = \text{circle}(f, f_2, f_{-2}),$$

have one additional point in common.

Figure 7: Four circles of a generic discrete isothermic surface, with a central sphere.

Figure 8: Four circles of a planar (or spherical) discrete isothermic net.

The cases 1), 2) of Theorem 22 are illustrated on Figs. 7, 8, respectively.

Similarly, translating Theorem 17, applied to a multidimensional Koenigs net $\hat{f}$ in $\mathbb{R}^{N+1,1}$, into the language of Möbius-geometric properties of the net $f$ in $\mathbb{R}^N$, we get the following statement.
Theorem 23 (Multidimensional discrete isothermic nets) A circular net \( f : \mathbb{Z}^m \rightarrow \mathbb{R}^N \) is discrete isothermic, if and only if for any elementary hexahedron of the net its four white vertices are concircular, and its four black vertices are concircular (each one of these conditions implies another one).

4.2 Cross-ratio characterization of discrete isothermic nets

Another characterization of discrete isothermic surfaces can be given in terms of the cross-ratios. Recall that for any four concircular points \( a, b, c, d \in \mathbb{R}^N \) their (real-valued) cross-ratio is defined by

\[
q(a, b, c, d) = (a - b)(b - c)^{-1}(c - d)(d - a)^{-1},
\]

with the Clifford multiplication in the Clifford algebra \( \mathcal{C}(\mathbb{R}^N) \). The Clifford product of \( x, y \in \mathbb{R}^N \) satisfies \( xy + yx = -2(x, y) \), and the inverse element of \( x \in \mathbb{R}^N \) in the Clifford algebra is given by \( x^{-1} = -x/|x|^2 \). Alternatively, one can identify the plane of the quadrilateral \( (a, b, c, d) \) with the complex plane \( \mathbb{C} \), and then multiplication in eq. (40) can be interpreted as the complex multiplication. An important property of the cross-ratio is its invariance under M"obius transformations.

For discrete isothermic surfaces Theorem 22 yields the following characterization.

Theorem 24 (Cross-ratios of four adjacent quadrilaterals) A two-dimensional circular net \( f : \mathbb{Z}^2 \rightarrow \mathbb{R}^N \) is a discrete isothermic surface, if and only if the cross-ratios \( q = q(f, f_1, f_2, f_3) \) of its elementary quadrilaterals satisfy the following condition:

\[
q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2}.
\]

Here, as usual, the negative indices \(-i\) denote the backward shifts \( \tau_i^{-1} \), so that, e.g., \( q_{-1} = q(f_{-1}, f_2, f_{-1,2}) \), see Fig. 9.

Proof. Perform a M"obius transformation sending \( f \rightarrow \infty \). Under such a transformation, the four adjacent circles through \( f \) turn into four straight lines \( (f_{\pm 1}, f_{\pm 2}) \), containing the corresponding points \( f_{\pm 1, \pm 2} \). The cross-ratios turn into ratios of directed lengths, e.g.,

\[
q(f, f_1, f_{1,2}, f_2) = -\frac{l(f_1, f_{1,2})}{l(f_{1,2}, f_2)}.
\]

If the affine space through the points \( f_{\pm 1}, f_{\pm 2} \) is three-dimensional, then, according to part 1) of Theorem 22, the four points \( f_{\pm 1, \pm 2} \) lie in a plane (a sphere through \( f = \infty \)). Generalized Menelaus theorem (Theorem 13) provides us with the following necessary and sufficient condition for this, which reads:

\[
\frac{l(f_2, f_{1,2})}{l(f_{1,2}, f_1)} \cdot \frac{l(f_1, f_{1,-2})}{l(f_{1,-2}, f_{-2})} \cdot \frac{l(f_{-2}, f_{-1,-2})}{l(f_{-1,-2}, f_{-1})} \cdot \frac{l(f_{-1}, f_{-1,2})}{l(f_{-1,2}, f_2)} = 1.
\]

This is equivalent to eq. (41) with \( f = \infty \).

If, on the contrary, the four points \( f_{\pm 1}, f_{\pm 2} \) are co-planar, then, according to part 2) of Theorem 22, both lines \( (f_{-1,2} f_{1,2}) \) and \( (f_{-1,-2} f_{1,-2}) \) meet the line \( (f_{-1} f_1) \) at the same
Figure 9: Four adjacent quadrilaterals of a discrete isothermic surface: the cross-ratios satisfy \( q \cdot q_{-1,-2} = q_{-1} \cdot q_{-2} \)

point \( \ell^{(1)} \). Thus, we are in the situation of Fig. 5, described by the Desargues theorem. Here, we apply the Menelaus theorem twice, to the triangle \( \triangle (f_{-1}, f_2, f_1) \) intersected by the line \( (f_{-1,2}f_1) \), and to the triangle \( \triangle (f_{-1}, f_{-2}, f_1) \) intersected by the line \( (f_{-1,-2}f_1) \):

\[
\frac{l(f_2, f_{12})}{l(f_{12}, f_1)} \cdot \frac{l(f_{-1}, f_{1,2})}{l(f_{1,2}, f_2)} = \frac{l(f_{-1}, \ell^{(1)})}{l(f^{(1)}, f_1)} = \frac{l(f_{-2}, f_{1,2}f_1)}{l(f_{1,2}, f_1)} \cdot \frac{l(f_{-1}, f_{-1,2})}{l(f_{-1,2}, f_{-2})}.
\]

This yields formula (42), again. \( \square \)

For multidimensional discrete isothermic nets Theorem 23 yields a similar characterization.

**Theorem 25 (Cross-ratios of three adjacent quadrilaterals)** A circular net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) is discrete isothermic, if and only if the cross-ratios of its elementary quadrilaterals satisfy the following condition:

\[
q(f, f_i, f_{ij}, f_j) \cdot q(f, f_j, f_{jk}, f_k) \cdot q(f, f_k, f_{ki}, f_i) = 1 \tag{43}
\]

for any triple of different indices \( i, j, k \).

**Proof.** Again, perform a Möbius transformation sending \( f \) to \( \infty \). Under such a transformation, the three adjacent circles through \( f \) turn into three straight lines \( (f_i f_j), (f_j f_k) \) and \( (f_k f_i) \), containing the (white) points \( f_{ij}, f_{jk} \) and \( f_{ki} \), respectively. Conicircularity of these white points with \( f \) means simply that they are collinear. The necessary and sufficient condition for this is given by the Menelaus theorem:

\[
\frac{l(f_j, f_{ij})}{l(f_{ij}, f_i)} \cdot \frac{l(f_k, f_{jk})}{l(f_{jk}, f_j)} \cdot \frac{l(f_i, f_{ki})}{l(f_{ki}, f_k)} = -1. \tag{44}
\]
Since the Möbius-invariant meaning of the ratios of directed lengths is given by the corresponding cross-ratios,
\[ q(f, f_i, f_{ij}, f_j) = -\frac{l(f_i, f_{ij})}{l(f_{ij}, f_j)}, \]
eq. (44) is equivalent to eq. (43).
□

The conclusions of Theorems 24, 25 can be summarized with the help of the following notion:

**Definition 26 (Edge labelling)** A system of real-valued functions \( \alpha_i \) defined on the edges of \( \mathbb{Z}^m \) parallel to the \( i \)-th coordinate axis \((i = 1, \ldots, m)\) is called an edge labelling, if they take equal values on each pair of opposite edges of any elementary quadrilateral.

Thus, both edges \((u, u + e_i)\) and \((u + e_j, u + e_i + e_j)\) of an elementary square of \( \mathbb{Z}^m \) parallel to the coordinate plane \((ij)\) carry the label \( \alpha_i = \alpha_i(u) = \alpha_i(u + e_j)\), and, similarly, both other edges \((u, u + e_j)\) and \((u + e_i, u + e_i + e_j)\) carry the label \( \alpha_j = \alpha_j(u) = \alpha_j(u + e_i)\), see Fig. 10. In this notation, there holds \( \tau_j \alpha_i = \alpha_i \) for \( i \neq j \), so that each function \( \alpha_i(u) \) depends on \( u_i \) only.

![Figure 10: Labelling of edges of a discrete isothermic net](image)

The following theorem is an immediate consequence of Theorems 24, 25.

**Theorem 27 (Factorized cross-ratios)** A circular net \( f : \mathbb{Z}^m \rightarrow \mathbb{R}^N \) is discrete isothermic, if and only if the cross-ratios of its elementary quadrilaterals satisfy
\[ q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j}, \]
(45)
where \( \alpha_i \) \((i = 1, \ldots, m)\) constitute a real-valued labelling of the edges of \( \mathbb{Z}^m \).

Theorem 27 says that our definition of discrete isothermic nets coincides with the original definition from [BP]. In the next subsection we will give a more concrete way of determining the labelling \( \alpha_i \) for a given discrete isothermic net.
4.3 Metric of a discrete isothermic net

Now we turn to a characterization of discrete Koenigs nets given in Corollary 18. Being applied to circular nets, it says that such a net $f$ is Koenigs, if and only if there exists a function $s : \mathbb{Z}^n \rightarrow \mathbb{R}^*$ such that for any circular quadrilateral $(f, f_i, f_{ij}, f_j)$ with the intersection point of diagonals $M$ there holds:

$$\frac{l(M, f_{ij})}{l(M, f)} = \frac{s_{ij}}{s}, \quad \frac{l(M, f_j)}{l(M, f_i)} = \frac{s_j}{s_i}. \quad (46)$$

(Note that the notation $s$ comes to replace $\nu$ which we reserve for general Koenigs nets.)

Theorem 28 (Discrete metric for discrete isothermic nets) For a discrete isothermic net $f$, relations (46) define a function $s : \mathbb{Z}^n \rightarrow \mathbb{R}$ uniquely, up to a black-white re-scaling which can be fixed by prescribing $s$ arbitrarily at one black and at one white point. There exists a labelling $\alpha$ of edges of $\mathbb{Z}^n$ such that

$$|f_i - f|^2 = \alpha_i s s_i \quad (i = 1, \ldots, m). \quad (47)$$

A black-white re-scaling of the function $s$ ($s \mapsto \lambda s$ on black vertices, $s \mapsto \mu s$ on white vertices) results in the re-scaling $\alpha \mapsto (\lambda \mu)^{-1} \alpha$ of the labelling $\alpha$.

Proof. For a circular quadrilateral $(f, f_i, f_{ij}, f_j)$ with the intersection point of diagonals $M$, one has two pairs of similar triangles:

$$\triangle(f, f_i, M) \sim \triangle(f_{ij}, f_{ij}, M), \quad \triangle(f, f_j, M) \sim \triangle(f_i, f_{ij}, M).$$

Hence, there holds:

$$\frac{|M f_{ij}|}{|M f_i|} = \frac{|M f_j|}{|M f|} = \frac{|f_{ij} - f_j|}{|f_i - f|}, \quad \frac{|M f_{ij}|}{|M f_i|} = \frac{|M f_j|}{|M f|} = \frac{|f_{ij} - f_i|}{|f_j - f|}. \quad (48)$$

There follows:

$$\frac{|M f_{ij}|}{|M f_i|} \cdot \frac{|M f_j|}{|M f|} = \frac{|f_{ij} - f_j|^2}{|f_i - f|^2}, \quad \frac{|M f_{ij}|}{|M f_i|} \cdot \frac{|M f_j|}{|M f|} = \frac{|f_{ij} - f_i|^2}{|f_j - f|^2}. \quad (49)$$

This can be written as

$$\frac{l(M, f_{ij})}{l(M, f)} \cdot \frac{l(M, f_j)}{l(M, f_i)} = \frac{|f_{ij} - f_j|^2}{|f_i - f|^2}, \quad \frac{l(M, f_{ij})}{l(M, f_i)} \cdot \frac{l(M, f_j)}{l(M, f)} = \frac{|f_{ij} - f_i|^2}{|f_j - f|^2}. \quad (50)$$

Indeed, contemplating Fig. 11 it is not difficult to realize that the fractions on the left-hand side of each one of the two equations in (50) are either both negative (for an embedded quadrilateral), or both positive (for a non-embedded quadrilateral), so that the replacement of the quotients of lengths in eq. (49) by quotients of directed lengths in eq. (50) is legitimate. Substitute the defining relations (46) of the function $s$ into eq. (50):

$$\frac{s_j s_{ij}}{s s_i} = \frac{|f_{ij} - f_j|^2}{|f_i - f|^2}, \quad \frac{s_i s_{ij}}{s s_j} = \frac{|f_{ij} - f_i|^2}{|f_j - f|^2}. \quad (51)$$
Figure 11: Circular quadrilaterals, an embedded and a non-embedded ones.

But this is equivalent to the claim that the functions

\[ \alpha_i = \frac{|f_i - f|^2}{ss_i} \]  

possess the labelling property, \( \tau_j \alpha_i = \alpha_i \). □

The notations \( \alpha_i \) for edge labellings in Theorems 27 and 28 coincide not without a reason.

**Theorem 29 (Origin of the edge labelling for factorized cross-ratios)** If the edge labelling \( \alpha_i \) for a discrete isothermic net \( f : \mathbb{Z}^m \to \mathbb{R}^N \) is introduced according to eq. (47), then the cross-ratios of its elementary quadrilaterals are factorized as in eq. (45).

**Proof.** For a circular quadrilateral \((f, f_i, f_{ij}, f_j)\) one has:

\[ q(f, f_i, f_{ij}, f_j) = \epsilon \frac{|f_i - f| \cdot |f_{ij} - f_j|}{|f_j - f| \cdot |f_{ij} - f_i|}, \]

where \( \epsilon < 0 \) for an embedded quadrilateral and \( \epsilon > 0 \) for a non-embedded one. Thus,

\[ q(f, f_i, f_{ij}, f_j) = \epsilon \frac{|f_i - f|^2}{|f_j - f|^2} \cdot \frac{|f_{ij} - f_j|}{|f_{ij} - f_i|} \cdot \frac{|f_j - f|}{|f_i - f|}. \]

Upon using eqs. (47) and (48), the latter equation can be re-written as

\[ q(f, f_i, f_{ij}, f_j) = \epsilon \frac{\alpha_i s_i}{\alpha_j s_j} \cdot \frac{|M f_j|}{|M f_i|} = \epsilon \frac{\alpha_i s_i}{\alpha_j s_j} \cdot \frac{l(M, f_j)}{l(M, f_i)}, \]

and finally, due to eq. (51), we arrive at

\[ q(f, f_i, f_{ij}, f_j) = \epsilon \frac{\alpha_i s_i}{\alpha_j s_j} \cdot \frac{s_j}{s_i} = \epsilon \frac{\alpha_i}{\alpha_j}. \]
which proves the theorem. □

Theorem 28 as it stands cannot be reversed: existence of a function $s$ satisfying (47) does not yield the Koenigs property. Indeed, from eqs. (47) and (50) one finds:

$$\frac{l(M, f_{ij})}{l(M, f)} \cdot \frac{l(M, f_j)}{l(M, f_i)} = \frac{s_iss_{ij}}{ss_i}, \quad \frac{l(M, f_i)}{l(M, f)} \cdot \frac{l(M, f)}{l(M, f_j)} = \frac{s_is_{ij}}{ss_j},$$

which is equivalent to

$$\frac{l(M, f_{ij})}{l(M, f)} = \pm \frac{s_{ij}}{s}, \quad \frac{l(M, f_i)}{l(M, f_j)} = \pm \frac{s_i}{s_j},$$

(with the same sign $\pm$ in both equations). The latter equation is somewhat weaker than eq. (51), which is necessary and sufficient for the net $f$ to be Koenigs. However, assuming some additional information about $f$, it is possible to force the plus signs in the latter formula. For instance, if it is known that all elementary quadrilaterals of a two-dimensional circular net $f$ are embedded, then property (47) is sufficient to assure that $f$ is Koenigs. Indeed, in this case $\alpha_2/\alpha_1 < 0$, so that eq. (47) yields $s_2/s_1 < 0$ and $s_{12}/s < 0$, and then the plus sign has to be chosen in eq. (54).

4.4 Duality of discrete isothermic nets

Specializing the notion of duality from general Koenigs nets to circular ones, the first essential observation is: the dual net for a discrete isothermic net is discrete isothermic, as well. Indeed, any quadrilateral with sides parallel to the corresponding sides of a circular quadrilateral is, obviously, also circular. A more detailed description of duality for discrete isothermic nets is contained in the following theorem.

**Theorem 30 (Dual discrete isothermic net)** Let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a discrete isothermic net, with the factorized cross-ratios

$$q(f, f_i, f_{ij}, f_j) = \frac{\alpha_i}{\alpha_j}$$

and with the discrete metric $s : \mathbb{Z}^m \to \mathbb{R}^*$. Then the $\mathbb{R}^N$-valued discrete one-form $\delta f^*$ defined by

$$\delta_i f^* = \alpha_i \frac{\delta_i f}{|\delta_i f|^2} = \frac{\delta_i f}{ss_i}, \quad i = 1, \ldots, m,$$

is closed. Its integration defines (up to a translation) a net $f^* : \mathbb{Z}^2 \to \mathbb{R}^N$, called dual to the net $f$, or Christoffel transform of the net $f$. The net $f^*$ is discrete isothermic, with the cross-ratios

$$q(f^*, f^*_i, f_{ij}^*, f_j^*) = \frac{\alpha_i}{\alpha_j}$$

and with the discrete metric $s^* = s^{-1} : \mathbb{Z}^m \to \mathbb{R}^*$. Conversely, if for a given net $f : \mathbb{Z}^m \to \mathbb{R}^N$ there exists an edge labelling $\alpha_i$ such that the discrete one-form

$$\delta_i f^* = \alpha_i \frac{\delta_i f}{|\delta_i f|^2}$$

is closed, then $f$ is a discrete isothermic net, with cross-ratios as in eq. (55).
Proof. The first part of the theorem is a consequence of the general construction of dual
Koenigs nets. To prove to converse part, observe that closeness of the one-form (58) implies
that the quadrilateral \((f, f_i, f_{ij}, f_j)\) is planar. Identifying its plane with \(\mathbb{C}\), we see that the
closeness condition is equivalent to (the complex conjugate of)
\[
\frac{\alpha_i}{f_i - f} - \frac{\alpha_i}{f_{ij} - f_j} = \frac{\alpha_j}{f_j - f} - \frac{\alpha_j}{f_{ij} - f_i}.
\]
Upon clearing denominators the latter equation turns into the cross-ratio equation (55)
(in the generic situation, when \(f_{ij} - f_i - f_j + f \neq 0\)). Thus, the closeness of the form (58)
actually characterizes discrete isothermic nets. \(\square\)

Corollary 31 The non-corresponding diagonals of any elementary quadrilateral of a dis-
crete isothermic net \(f\) and of its dual are related by
\[
f_i^* - f_j^* = (\alpha_i - \alpha_j) \frac{f_{ij} - f}{|f_{ij} - f|^2}, \quad f_{ij} - f = (\alpha_i - \alpha_j) \frac{f_i - f_j}{|f_i - f_j|^2}.
\]

Proof. We put eq. (45) into several equivalent forms; these computations hold not only
in the Clifford algebra \(\mathbb{C}\ell(\mathbb{R}^N)\), but in an arbitrary associative algebra with unit \(A\). Being
written as
\[
\alpha_i (f_{ij} - f_i)(f_i - f)^{-1} = \alpha_j (f_{ij} - f_j)(f_j - f)^{-1},
\]
this equation displays the symmetry with respect to the diagonal flips of an elementary
quadrilateral, expressed as \(f_i \leftrightarrow f_j\) and \(f \leftrightarrow f_{ij}\), respectively (both have to be accompanied
by the change \(\alpha_i \leftrightarrow \alpha_j\)). Writing eq. (60) as
\[
\alpha_i (f_{ij} - f)(f_i - f)^{-1} - \alpha_i = \alpha_j (f_{ij} - f)(f_j - f)^{-1} - \alpha_j,
\]
and dividing from the left by \(f_{ij} - f\), we arrive at the so-called three-leg form of the
cross-ratio equation:
\[
(\alpha_i - \alpha_j)(f_{ij} - f)^{-1} = \alpha_i (f_i - f)^{-1} - \alpha_j (f_j - f)^{-1}.
\]
According to eq. (56), the right-hand side of eq. (61) is equal to \(-(f_i^* - f^*) + (f_j^* - f^*) = f_j^* - f_i^*\). This proves the first equation in (59). The second one is analogous. \(\square\)

4.5 Moutard representatives of discrete isothermic nets

The discrete metric of a discrete isothermic net \(f\) can be used to produce its Moutard
representative, or, better, a Moutard representative of its lift \(\hat{f}\) into the light cone of
\(\mathbb{R}^{N+1,1}\). This leads to a new characterization of discrete isothermic nets which is manifestly
Möbius invariant, since it is given entirely within the formalism of the projective model of
Möbius geometry. The following statement is a discrete analog of Theorem 6.

Theorem 32 (Discrete isothermic nets = discrete Moutard nets in light cone)
If \(f : \mathbb{Z}^m \to \mathbb{R}^N\) is a discrete isothermic net, then its lift \(y = s^{-1} \hat{f} : \mathbb{Z}^m \to \mathbb{L}^{N+1,1}\) to the
light cone of \(\mathbb{R}^{N+1,1}\) satisfies the discrete Moutard equation (32).
Conversely, given a discrete Moutard net \( y : \mathbb{Z}^m \rightarrow \mathbb{L}^{N+1,1} \) in the light cone, let the functions \( s : \mathbb{Z}^m \rightarrow \mathbb{R} \) and \( f : \mathbb{Z}^m \rightarrow \mathbb{R}^N \) be defined by

\[
y = s^{-1}(f + e_0 + |f|^2 e_\infty)
\]  

(so that \( s^{-1} \) is the \( e_0 \)-component, and \( s^{-1}f \) is the \( \mathbb{R}^N \)-part of \( y \) in the basis \( e_1, \ldots, e_N, e_0, e_\infty \)). Then \( f \) is a discrete isothermic net.

**Proof.** This follows from Theorem 20 and the fact that for a circular Koenigs net \( f \) in \( \mathbb{R}^N \) the net \( f = f + e_0 + |f|^2 e_\infty \) is also a Koenigs net in the light cone \( \mathbb{L}^{N+1,1} \subset \mathbb{R}^{N+1,1} \).

Thus, we found an interpretation of discrete isothermic nets as an instance of discrete Moutard nets in a quadric. The edge labelling of a discrete isothermic net \( f \) (which provides the factorization (45) of its cross-ratios) is already encoded in its lift \( y \) to the light cone. Indeed,

\[
\alpha_i = \frac{|f_i - f|^2}{ss_i} = -2\langle y, \tau_i y \rangle,
\]

and it is easy to see that these quantities depend on \( u_i \) only.

### 4.6 Continuous limit

In order to enable the continuous limit to smooth isothermic surfaces, one should start with discrete isothermic surfaces (discrete isothermic nets with \( m = 2 \)) with embedded elementary quadrilaterals. In this case the standard re-definition of the function \( s \), namely \( s(u) \mapsto (-1)^{s^2} s(u) \), assures the positivity of \( s \). It is convenient to change the notation for the labelling, as well: \( \alpha_2 \mapsto -\alpha_2 \). Then formula (47) remains valid as it stands, and for the negative cross-ratios of elementary quadrilaterals we get: \( q(f, f_1, f_12, f_2) = -\alpha_1/\alpha_2 \), with positive labels \( \alpha_1 \) and \( \alpha_2 \). Eq. (56) turns into

\[
\delta_1 f^* = \alpha_1 \frac{\delta_1 f}{\delta_1 |f|^2} = \frac{\delta_1 f}{ss_1}, \quad \delta_2 f^* = -\alpha_2 \frac{\delta_2 f}{\delta_2 |f|^2} = -\frac{\delta_2 f}{ss_2},
\]

which is a direct discrete analogue of eq. (5).

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