A FUBINI RULE FOR $\infty$-COENDS

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Abstract. We prove a Fubini rule for $\infty$-co/ends of $\infty$-functors $F : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$. This allows to lay down “integration rules”, similar to those in classical co/end calculus, also in the setting of $\infty$-categories.

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1. Introduction

In [Lur17, §5.2.1] (we shorten the reference to this source to “HA” from now on, and similarly we call simply “T” the reference [Lur09]) the author introduces the definition of twisted arrow $\infty$-category of an $\infty$-category; in [GHN15] this paves the way to the definition of co/end for a $\infty$-functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{D}$. Here we briefly recall how this construction works.

Definition 1.1. Let $\epsilon : \Delta \to \Delta$ be the functor $[n] \mapsto [n] \ast [n]^p$. The edgewise subdivision $\text{esd}(X)$ of a simplicial set $S$ is defined to be the composite $\epsilon^* S$. If $\mathcal{C}$ is an $\infty$-category, we define $\text{Tw}(\mathcal{C})$ to be the simplicial set $\epsilon^* \mathcal{C}$. The $n$-simplices of $\text{Tw}(\mathcal{C})$ are, in particular, determined as

$$\text{Tw}(\mathcal{C})_n \cong \text{sSet}([n], \text{Tw}(\mathcal{C})) = \text{sSet}([n] \ast [n]^p, \mathcal{C}).$$

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Remark 1.2. In dimension 0 and 1, the \( n \)-simplices of \( \text{Tw}(\mathcal{C}) \) correspond respectively to edges \( f \) of \( \mathcal{C} \) and to commutative squares
\[
\begin{array}{ccc}
\downarrow s & \downarrow g \\
\downarrow f & \downarrow t
\end{array}
\]
The canonical natural transformations given by the embedding of \([n], [n]^o\) in the join entail that there is a well-defined projection map \( \Sigma_\mathcal{C} : \text{Tw}(\mathcal{C}) \to \mathcal{C}^o \times \mathcal{C} \). Note that from HA.5.2.1.11 we deduce that \( \Sigma_\mathcal{C} \) is the right fibration HA.5.2.1.3 (this entails that if \( \mathcal{C} \) is an \( \infty \)-category, then \( \text{Tw}(\mathcal{C}) \) is also an \( \infty \)-category) classified by \( \text{Map} : \mathcal{C}^o \times \mathcal{C} \to \mathcal{S} \).

If the \( \infty \)-category \( \mathcal{C} \) is of the form \( \mathcal{N}(A) \) for some category \( A \), \( \text{Tw}(\mathcal{C}) \) corresponds to the nerve of the classical twisted arrow category of \( A \), as defined in [ML98, IX.6.3].

**Definition 1.3.** Let \( F : \mathcal{C}^o \times \mathcal{C} \to \mathcal{D} \) be a functor; when it exists, the **end** of \( F \) is the limit
\[
\int^\mathcal{C} F := \lim_{\text{Tw}(\mathcal{C})} (F \cdot \Sigma)
\]
Dually, when it exists, the **coend** of \( F \) is the colimit
\[
\int_\mathcal{C} F := \colim_{\text{Tw}(\mathcal{C})} (F \cdot \Sigma)
\]
It is clear that a sufficient condition for \( \int^\mathcal{C} F \) to exists is that \( \mathcal{D} \) is cocomplete, and dually a sufficient condition for \( \int_\mathcal{C} F \) to exist is that \( \mathcal{D} \) is complete.

[GHN15] employs this notation to prove [ibi, Thm. 1.1] that

**Theorem 1.4.** Suppose \( F : \mathcal{C} \to \mathcal{C}_{\text{at}_{\infty}} \) is a functor of \( \infty \)-categories,

1. LC1) if \( \mathcal{E} \to \mathcal{C} \) is a cocartesian fibration associated to \( F \). Then \( \mathcal{E} \) is the lax colimit\(^1\) of the functor \( F \).
2. LC2) if \( \mathcal{E} \to \mathcal{C} \) is a cartesian fibration associated to \( F \). Then \( \mathcal{E} \) is the oplax colimit of the functor \( F \).

\(^1\)The lax colimit of \( F : \mathcal{C} \to \mathcal{C}_{\text{at}_{\infty}} \) is defined by the coend
\[
\int^\mathcal{C} \mathcal{C}_{/C} \times F(C)
\]
Dually, the oplax colimit of \( F \) is defined by the coend
\[
\int_\mathcal{C} \mathcal{C}_{/C} \times F(C)
\]
where in both cases the weights are the slice \( \infty \)-categories of T.1.2.9.2 and T.1.2.9.5.
Lemma 1.5. Let $\mathcal{C}$ be a small $\infty$-category, and $\mathcal{D}$ be a presentable $\infty$-category; then $\mathcal{D}$ is tensored and cotensored over $\mathcal{S} = N(Kan)$. This entails that there is a two-variable adjunction

\[
\mathcal{D}^o \times \mathcal{D} \xrightarrow{\text{Map}} \mathcal{S} \cong \mathcal{S} \times \mathcal{D} \xrightarrow{\otimes} \mathcal{D} \cong \mathcal{S} \times \mathcal{D} \xrightarrow{\odot} \mathcal{D}
\]

such that

\[
\mathcal{D}(X \odot D, D') \cong \mathcal{S}(X, \mathcal{D}(D, D')) \cong \mathcal{D}(D, X \odot D).
\]

From the existence of these isomorphisms it is clear that

\[\begin{align*}
V \odot (W \odot D) & \cong W \odot (V \odot D) \cong (V \times W) \odot D, \\
V \odot (W \odot D) & \cong W \odot (V \odot D) \cong (V \times W) \odot D
\end{align*}\]

(1.1) (1.2)

2. The Fubini Rule

Lemma 2.1. Let $F : \mathcal{C}^o \times \mathcal{C} \rightarrow \mathcal{D}$ be a $\infty$-functor and $\mathcal{C}, \mathcal{D}$ $\infty$-categories as in the assumptions of Lemma 1.5. Then

- $F \mapsto \int_C F$ is functorial, and it is a left adjoint;
- $F \mapsto \int_C F$ is functorial, and it is a right adjoint.

Proof. We only prove the first statement for coends; the other one is dual.

Since $\int_C F = \text{colim}_{\text{Tw}(\mathcal{C})}(F \cdot \Sigma)$ acts on $F$ as a composition of $\infty$-functors, it is clearly functorial; then in the diagram

\[
\int_C : [\mathcal{C}^o \times \mathcal{C}, \mathcal{D}] \xrightarrow{\Sigma^*} \text{colim}_{\text{Tw}(\mathcal{C})} \xrightarrow{\text{colim}_{\text{Tw}(\mathcal{C})}} \mathcal{D}
\]

the composition $\int_C = \text{colim}_{\text{Tw}(\mathcal{C})} \cdot \Sigma^*$ is a left adjoint because it is a composition of left adjoints ($c = t^*$ is the constant functor inverse image of the terminal map $\text{Tw}(\mathcal{C}) \rightarrow *$). Dually, the left adjoint to the end functor $\int_C$ is given by $\text{Lan}_c \cdot c(D)$. \qed

Loosely speaking, the Fubini rule for co/ends asserts that when the domain of a functor $F : \mathcal{A}^o \times \mathcal{A} \rightarrow \mathcal{D}$ results as a product $(\mathcal{C} \times \mathcal{E})^o \times (\mathcal{C} \times \mathcal{E})$, then the co/ends of $F$ can be computed as “iterated integrals”

\[\begin{align*}
\int_{(C,E)} F & \cong \int \int \int_{C,E} F \\
\int_{(C,E)} F & \cong \int \int \int_{C,E} F
\end{align*}\]

(2.1) (2.2)

These identifications hide a slight abuse of notation, that is worth to make explicit in order to avoid confusion: thanks to Lemma 2.1 the three objects of (2.1) can be
thought as images of $F$ along certain functors, and the Fubini rule asserts that they are linked by canonical isomorphisms; we can easily turn these functors into having the same type by means of the cartesian closed structure of $sSet$:

\[(2.3)\]
\[
\begin{align*}
\int^C &= \left[\mathcal{C}^0 \times \mathcal{C}^0 \times \mathcal{E}^0 \times \mathcal{E}, \mathcal{D} \right] \cong \left[\mathcal{C}^0 \times \mathcal{C}, \left[\mathcal{C}^0 \times \mathcal{E}, \mathcal{D} \right] \mathcal{E} \right] \rightarrow \mathcal{D}
\end{align*}
\]

\[(2.4)\]
\[
\begin{align*}
\int^E &= \left[\mathcal{C}^0 \times \mathcal{C} \times \mathcal{E}^0 \times \mathcal{E}, \mathcal{D} \right] \cong \left[\mathcal{E}^0 \times \mathcal{E}, \left[\mathcal{E}^0 \times \mathcal{C}, \mathcal{D} \right] \mathcal{C} \right] \rightarrow \mathcal{D}
\end{align*}
\]

\[(2.5)\]
\[
\int^{(C,E)} = \left[\mathcal{C}^0 \times \mathcal{C} \times \mathcal{E}^0 \times \mathcal{E}, \mathcal{D} \right] \cong \left[\left(\mathcal{C} \times \mathcal{E} \right)^0 \times \left(\mathcal{C} \times \mathcal{E} \right), \mathcal{D} \right] \rightarrow \mathcal{D}.
\]

(of course, we can provide similar definitions for the iterated end functor).

Once that this has been clarified, we can deduce the isomorphisms (2.1) and (2.2) from the fact that the three functors $\int^C$, $\int^E$, $\int^{(C,E)}$ have right adjoints isomorphic to each other, and hence they must be isomorphic themselves.

**Theorem 2.2** (Fubini rule for co/ends). Let $F : \mathcal{C}^0 \times \mathcal{C} \times \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{D}$ be a $\infty$-functor. Then the $\infty$-functors $\int^C$, $\int^E$, $\int^{(C,E)}$ of (2.3), (2.4), (2.5) are naturally isomorphic.

In order to prove 2.2 we need a preliminary observation characterizing the right adjoint to $\int^C : \left[\mathcal{C}^0 \times \mathcal{C}, \mathcal{D} \right] \rightarrow \mathcal{D}$.

**Lemma 2.3.** The functor $R = \text{Ran}_\Sigma(c\langle \_ \rangle)$ acts “cotensoring with mapping space”: more precisely, the functor $RD : \mathcal{C}^0 \times \mathcal{C} \rightarrow \mathcal{D}$ is isomorphic to the functor

\[\begin{align*}
(C, C') &\mapsto \text{Map}_\mathcal{E}(C, C') \triangleleft D
\end{align*}\]

Dually, the functor $L = \text{Lan}_\Sigma(c\langle \_ \rangle)$ acts “tensoring with mapping space”: more precisely, the functor $LD : \mathcal{C}^0 \times \mathcal{C} \rightarrow \mathcal{D}$ is isomorphic to the functor

\[\begin{align*}
(C, C') &\mapsto \text{Map}_\mathcal{E}(C, C') \triangleright D
\end{align*}\]

**Proof.** We only prove the first statement for coends; the other one is dual.

Being $c(D)$ the constant functor on $D \in \mathcal{D}$, the pointwise formula for right Kan extensions (see [Cis, 6.4.9] for the fact that “Ran are limits”) yields that the desired limit consists of cotensoring with the slice category $(C, C')/\Sigma$ regarded as a simplicial
set (in the Kan-Quillen model structure); now, if we consider the diagram

\[
\begin{array}{ccc}
\text{Map}_C(C,C') \xrightarrow{\sim} (C,C')/\Sigma & \xrightarrow{} & \text{Tw}(\mathcal{C}) \\
\Delta^0 \xrightarrow{\sim} (\mathcal{C}^\alpha \times \mathcal{C})_{(C,C')/} & \xrightarrow{} & \mathcal{C}^\alpha \times \mathcal{C}
\end{array}
\]

expressing the fiber of \( \Sigma \), i.e. the mapping spaces \( \text{Map}_C(C,C') \), as suitable pullbacks, we can easily see that \( (\mathcal{C}^\alpha \times \mathcal{C})_{(C,C')/} \) is contractible in Kan-Quillen (it has an initial object), hence, in the \( \infty \)-category of spaces, the object \( (C,C')/\Sigma \) exhibits the same universal property of \( \text{Map}_C(C,C') \). Since the functor \( \mathcal{D} \) preserves Kan-Quillen weak equivalences, it turns out that

\[
\text{Ran}_\Sigma(c(D)) \cong (C,C')/\Sigma \mathcal{D} \cong \text{Map}_C(C,C') \mathcal{D},
\]

and this concludes the proof. \( \square \)

**Proof of 2.2.** The Fubini rule now follows from uniqueness of adjoints (T.5.2.1.3, T.5.2.1.4): in diagram

\[
\begin{array}{ccc}
\lambda F \int^C \int^E F \xrightarrow{\sim} \lambda D. \lambda CC'. \lambda EE'. \text{Map}_C(C,C') \mathcal{D} \left( \text{Map}_E(E,E') \mathcal{D} \right) & \xrightarrow{} & \lambda D. \lambda CC'. \lambda EE'. \text{Map}_C(C,C') \mathcal{D} \left( \text{Map}_E(E,E') \mathcal{D} \right) \\
\| & & \| \\
\lambda F \int^E \int^C F \xrightarrow{\sim} \lambda D. \lambda EE'. \lambda CC'. \text{Map}_E(E,E') \mathcal{D} \left( \text{Map}_C(C,C') \mathcal{D} \right) & \xrightarrow{} & \lambda D. \lambda EE'. \lambda CC'. \text{Map}_E(E,E') \mathcal{D} \left( \text{Map}_C(C,C') \mathcal{D} \right) \\
\| & & \| \\
\lambda D. \lambda CEC'C'E' \left( \text{Map}_C(C,C') \times \text{Map}_E(E,E') \right) \mathcal{D} \left( (C,E),(C',E') \right) & \xrightarrow{\sim} & \text{Map}_C \times E \left( (C,E),(C',E') \right) \mathcal{D}
\end{array}
\]

the vertical isomorphisms on the right are justified by (1.2). A completely analogous argument, using (1.1) instead, and the left adjoints given by tensoring with the derived mapping space, gives the Fubini rule for (2.2). \( \square \)

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