Quantum Indeterminacy, Polar Duality, and Symplectic Capacities

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Abstract

The notion of polarity between sets, well-known from convex geometry, is a geometric version of the Fourier transform. We exploit this analogy to propose a new simple definition of quantum indeterminacy, using what we call “ℏ-polar quantum pairs”, which can be viewed as pairs of position-momentum indeterminacy with minimum spread. The existence of such pairs is guaranteed by the usual uncertainty principle, but is at the same time more general. We use recent advances in symplectic topology to show that this quantum indeterminacy can be measured using a particular symplectic capacity related to action and which reduces to area in the case of one degree of freedom. We show in addition that polar quantum pairs are closely related to Hardy’s uncertainty principle about the localization of a function and its Fourier transform.

1 Introduction

There are several reasons to question the universality of the Robertson–Schrödinger (RS) inequalities

\[(\Delta x_j)^2(\Delta p_j)^2 \geq \Delta(x_j, p_j)^2 + \frac{1}{4}ℏ^2\]  \hspace{1cm} (1)

in their textbook interpretation, where the quantities \(\Delta p_j\) and \(\Delta x_j\) are viewed as a measurement of the “spread” of the wavefunction corresponding to the state under consideration, and \(\Delta(x_j, p_j)\) their covariance. First, the

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RS inequalities (1) are not a statement about errors of measurement; they
describe the limitation on preparing microscopic objects but have no direct
relevance to the limitation of accuracy of measuring devices, because of the
occurrence of noise. For instance, Ozawa [17] aims to describe the interplay
between error and disturbance for individual states, while Busch et al. [4]
give a state-independent characterization of measuring devices. Secondly,
the RS inequalities are a rigorous mathematical consequence of the defini-
tions of $\Delta p_j$, $\Delta x_j$, and $\Delta(x_j, p_j)$ as (co)variances. However, as Hilgevoord
and Uffink very pertinently note in [13, 14, 21], standard deviations only
give an adequate measurement of the spread of a wavefunction when the
probability density (here the square of the modulus of the wavefunction)
is Gaussian, or nearly Gaussian. Their remarks open up the way to new
formulations of the uncertainty principle: while Gaussian measurements are
indeed ubiquitous because of the central limit theorem of Bayesian statistical
inference, there are situations where measurements do not lead to Gaussian
distributions, as illustrated in the aforementioned papers of Hilgevoord and
Uffink by several examples.

The discussion above suggest that there should be alternative ways to
measure quantum uncertainty—or, as we prefer to call it—quantum inde-
terminacy. In this Letter we propose one such alternative, which has the
advantage of being conceptually very simple and easy to implement prac-
tically. It is based on the notion of polar dual of a centered convex body,
well-known from convex geometry.

We recall ([5] [19] [20] and [7] [8]) that the symmetric matrix

$$
\Sigma = \begin{pmatrix}
\Delta(x, x) & \Delta(x, p) \\
\Delta(p, x) & \Delta(p, p)
\end{pmatrix}
$$

(2)

where $\Delta(x, x) = (\Delta(x_j, x_k))_{1 \leq j, k \leq n}$, etc. is a quantum covariance matrix if
and only if the self-adjoint complex matrix

$$
\Sigma + \frac{i\hbar}{2} J \text{ is positive semidefinite;}
$$

(3)

here $J = \begin{pmatrix} 0_{n \times n} & I_{n \times n} \\ -I_{n \times n} & 0_{n \times n} \end{pmatrix}$ is the standard symplectic matrix. It follows that
$\Sigma$ is positive-definite, and that $\Delta(x, x)$ and $\Delta(p, p)$ are invertible. The condition (3) is equivalent to the RS inequalities (1). The quantum covariance ellipsoid associated with $\Sigma$ is

$$
\Omega_\Sigma : \frac{1}{2} z^T \Sigma^{-1} z \leq 1.
$$

(4)
We have shown in previous work \[6, 7, 8\], that the RS inequalities \(1\) can be rewritten in canonically invariant form as

\[
c(\Omega \Sigma) \geq \frac{1}{2} h
\]  

(5)

where \(c(\Omega \Sigma)\) is the symplectic capacity of the covariance ellipsoid. The number \(c(\Omega \Sigma)\), which is a measure of uncertainty, has the dimension of an area; it is defined by a symplectic “non-squeezing” property \[11, 15, 18\]: we have \(c(\Omega \Sigma) = \pi R^2\) where \(R\) is the radius of the largest phase space ball that can be sent inside \(\Omega \Sigma\) using a symplectomorphism (linear, or not). The formulation \(5\) of the RS inequalities is invariant under arbitrary symplectomorphisms, whereas the RS inequalities themselves are only invariant under linear or affine symplectomorphisms. The following statement \[9\] makes inequality \(5\) more intuitive: the intersection of \(\Omega \Sigma\) by any symplectic plane is an ellipse with area at least \(\frac{1}{2} h\) (a symplectic plane is obtained by applying a linear symplectic transformation to any of the planes of conjugate variables \(x_j, p_j\)).

2 Quantum Indeterminacy: \(n = 1\)

Making a great number of simultaneous position and momentum observations on a one-dimensional quantum system, we find they are contained in intervals \([x_0 - a, x_0 + a], a > 0\), and \([p_0 - b, p_0 + b], b > 0\), respectively. Set now \(X = [-a, a]\) and define its “\(h\)-polar dual” \(X^h\) of \(X\) as being the interval of all real numbers \(p\) such that \(px \leq h\); we have \(X^h = [-h/a, h/a]\). We will say that \((X, P)\) is a \(h\)-polar quantum pair if the inclusion \(X^h \subset P\) holds; this relation is reflexive because it is equivalent to \(P^h \subset X\): if \((X, P)\) is a quantum pair, so is \((P, X)\). The notion \(h\)-polarity is a generalization of the notion of spreading used in the Robertson–Schrödinger inequalities. Assume in fact that

\[
(\Delta x)^2 (\Delta p)^2 \geq (\Delta x, p)^2 + \frac{1}{4} h^2
\]

and consider the associated covariance ellipse

\[
\Omega_\Sigma : \frac{(\Delta p)^2}{2D} x^2 + \frac{\Delta (x, p)}{D} px + \frac{(\Delta x)^2}{2D} p^2 \leq 1
\]  

(6)

where we have set \(D = (\Delta x)^2 (\Delta p)^2 - (\Delta x, p)^2\). The projections of \(\Omega_\Sigma\) on the \(x\) and \(p\) axes are the intervals \(X = [-\sqrt{2} \Delta x, \sqrt{2} \Delta x]\) and \(P = [-\sqrt{2} \Delta p, \sqrt{2} \Delta p]\). The \(h\)-polar dual \(X^h\) is the interval \([-h/\sqrt{2} \Delta x, h/\sqrt{2} \Delta x]\).
and it is contained in $P$ if and only if $h/\sqrt{2}\Delta x \leq \sqrt{2}\Delta p$, which is equivalent to Heisenberg’s inequality $\Delta x\Delta p \geq \frac{1}{2}h$.

There is also an interesting analytic motivation for the introduction of $h$-polar dual pairs. Consider a square integrable non-zero function $\psi$ and its Fourier transform $\hat{\psi}$. It is a “folk theorem” that $\psi$ and $\hat{\psi}$ cannot be simultaneously arbitrarily sharply localized. This trade-off between a function and its Fourier transform, which is related to Heisenberg’s uncertainty principle, was rigorously stated by Hardy [12] in 1932: if there exists a constant $C > 0$ such that

$$|\psi(x)| \leq Ce^{-x^2/4\sigma_X^2}, \quad |\hat{\psi}(p)| \leq Ce^{-p^2/4\sigma_P^2},$$

then we must have $\sigma_X\sigma_P \geq \frac{1}{2}h$ and:

(i) if $\sigma_X\sigma_P = \frac{1}{2}h$ the function $\psi$ must be a Gaussian $\psi(x) = ke^{-x^2/4\sigma_X^2}$ for some complex constant $k$;

(ii) if $\sigma_X\sigma_P > \frac{1}{2}h$ then $\psi$ is a finite linear combination of Hermite functions. Hardy’s condition is equivalent to saying that the intervals $X = [-\sqrt{2}\sigma_X, \sqrt{2}\sigma_X]$ and $P = [-\sqrt{2}\sigma_P, \sqrt{2}\sigma_P]$ form a dual pair.

We are going to see that everything can be generalized to the case of an arbitrary number $n$ of degrees of freedom.

3 A Geometric Fourier Transform

Let $X$ be a convex body in $\mathbb{R}^n$. If $X$ is centrally symmetric (i.e. $X = -X$), the $h$-polar set of $X$ is by definition

$$X^h = \{p \in \mathbb{R}^n_p : p^T x \leq h \text{ for all } x \in X\};$$

when $h = 1$ it is the usual polar set $X^o$ familiar from convex geometry. Notice that the $h$-polar transformation reverses inclusions: if $X \subset Y$ then $Y^h \subset X^h$ and if $X$ is convex then $(X^h)^h = X$. We have

$$B^n(R)^h = B^n(h/R).$$

($B^n(R)$ the ball $|x| \leq R$). In fact, given $p$ in $B^n(R)^h$ choose $x$ and $p$ colinear and $|x| = R$. Then $p^T x = |p|R \leq h$ hence $|p| \leq h/R$. If conversely $|p| \leq h/R$ then $p^T x \leq |x||p| \leq h$ for all $x$ such that $|x| \leq R$, hence our claim. It is also easily verified that for every invertible $n \times n$ matrix $L$ we have

$$(LX)^h = (L^T)^{-1}X^h;$$

in particular if $X$ is scaled up then $X^h$ is scaled down: $(\lambda X)^h = \lambda^{-1}X^h$ for every $\lambda > 0$. The $h$-polar set of an ellipsoid is again an ellipsoid: let
\( B^n_A(R) : x^T A x \leq R^2 \) where \( A \) is a positive definite and symmetric matrix; then
\[
B^n_A(R) = B^n_{A^{-1}}(\hbar R).
\] (11)

(it suffices to notice that \( B^n_A(R) \) is the image of \( B^n(\hbar R) \) by the linear automorphism \( x \mapsto A^{-1/2} x \) and to use formula (10) and the equality (9)).

Let us introduce the following definition and terminology:

\textbf{Definition 1} Let \( X \) and \( P \) be two symmetric convex bodies in \( \mathbb{R}^n \). We will say that \((X,P)\) is a \( \hbar \)-polar quantum pair if \( X \hbar \subset P \). As in the case \( n = 1 \) this relation is reflexive: \((X,P)\) is a \( \hbar \)-polar quantum pair if and only if \((P,X)\) is.

Here is one simple example when \( n = 2 \) (it can easily be generalized to arbitrary dimension \( n \)). Assume that numerous position measurements are all allocated, after the elimination of outliers, in a disk \( D(x_0,R_x) : |x - x_0| \leq R_x \) and that, similarly, momentum measurements lead to a disk \( D(p_0,R_p) : |p - p_0| \leq R_p \). The \( \hbar \)-polar dual \( X^\hbar \) of \( X = D(0,R_p) \) is the disk \( D(0,R_x)^\hbar : p_1^2 + p_2^2 \leq \hbar / R_x^2 \) and the condition \( D(0,R_x)^\hbar \subset D(p_0,R_p) \) is equivalent to \( R_x R_p \geq \hbar / \pi \). If we assume that the probability distributions on the clouds \( D(x_0,R_x) \) and \( D(p_0,R_p) \) are uniform, an easy calculation yields the variances \( \sigma_{x_1}^2 = \sigma_{x_2}^2 = \pi R_x^2 / 4 \) and \( \sigma_{p_1}^2 = \sigma_{p_2}^2 = \pi R_p^2 / 4 \) and we thus have \( \sigma_{x_1} \sigma_{p_1} \geq \pi \hbar / 4 \) and \( \sigma_{x_2} \sigma_{p_2} \geq \pi \hbar / 4 \). In the “minimum indeterminacy case” \( X^\hbar = P \) we have \( \sigma_{x_1} \sigma_{p_1} = \sigma_{x_2} \sigma_{p_2} = \pi \hbar / 4 \) and this value exceeds the theoretical value \( \frac{\hbar}{2} \) predicted by Heisenberg’s relations by approximately 50%.

We will show below that the projections \( X \) and \( P \) on position and momentum spaces of the quantum covariance ellipsoid form a quantum pair. Put differently, the RS inequalities (1) imply “\( \hbar \)-polar indeterminacy”.

\section{Symplectic Capacities}

Let us return briefly to the case \( n = 1 \): the \( \hbar \)-polar dual” of the interval \( X = [-a,a] \) is \( X^\hbar = [-\hbar/a,\hbar/a] \) hence, if \((X,P)\) is a \( \hbar \)-polar quantum pair of intervals we have
\[
\text{Area}(X \times P) \geq \text{Area}(X \times X^\hbar) = 4 \hbar.
\] (12)

The generalization of this inequality to the case of arbitrary \( n \) is not straightforward: a first educated guess seems to suggest that the word “area” could simply be replaced by the word “volume” in higher dimensions. But it is not
so; we will need the very subtle notion of symplectic capacity to extend \([12]\); our proof will rely on a recent mathematical result due to Artstein-Avidan, Karasev, and Ostrover \([3]\). Let us first recall the general notion of symplectic capacity \([15, 18]\), reviewed in \([8]\): it is a mapping \(c\) associating to every subset \(\Omega\) of phase space a number \(c(\Omega)\) having the following properties:

- **Monotonicity:** If \(\Omega \subset \Omega'\) then \(c(\Omega) \leq c(\Omega')\);
- **Conformality:** For every real scalar \(\lambda\) we have \(c(\lambda \Omega) = \lambda^2 c(\Omega)\);
- **Symplectic invariance:** We have \(c(f(\Omega)) = c(\Omega)\) for every symplectomorphism \(f\);
- **Normalization:** We have

\[
   c(B^{2n}(R)) = \pi R^2 = c(Z_j^{2n}(R)) \tag{13}
\]

where \(B^{2n}(R)\) is the ball \(|z| \leq R\) and \(Z_j^{2n}(R)\) the cylinder \(x_j^2 + p_j^2 \leq R^2\).

We are assuming that the phase space \(\mathbb{R}^{2n}\) is equipped with the standard symplectic form \(\sigma(z, z') = (z')^T Jz\). Property (13) is often dubbed the “principle of the symplectic camel” \([8, 9]\); it is equivalent to Gromov’s non-squeezing theorem \([11]\). There are infinitely many symplectic capacities, but all agree on ellipsoids. In the case of one degree of freedom all symplectic capacities on the phase plane are identical to area on connected and simply connected surfaces. The smallest (resp. the largest) symplectic capacity \(c_{\min}\) (resp. \(c_{\max}\)) are defined by

\[
   c_{\min}(\Omega) = \sup_f \{ \pi R^2 : f(B^{2n}(R)) \subset \Omega \}
\]

\[
   c_{\max}(\Omega) = \inf_f \{ \pi R^2 : f(\Omega) \subset Z_j^{2n}(R) \}
\]

where \(f\) ranges over the group \(\text{Symp}(n)\) of all symplectomorphisms of \((\mathbb{R}^{2n}, \sigma)\).

Another interesting symplectic capacity is the Hofer–Zehnder capacity \(c_{\text{HZ}}\) \([15, 18]\). It has the following property: if \(\Omega\) is a compact and convex set then

\[
   c_{\text{HZ}}(\Omega) = \oint_{\gamma_{\min}} pdx \tag{15}
\]

where \(\gamma_{\min}\) is the shortest periodic Hamiltonian orbit on the boundary \(\partial \Omega\), viewed as the hypersurface of constant energy of some Hamiltonian function \(H\) (which has not to be of any particular type, e.g. “kinetic energy plus potential”). The orientation of \(\gamma_{\min}\) is chosen so that \(c_{\text{HZ}}(\Omega) \geq 0\). For formula
to be unambiguous we have to show that the action integral is inde-
pendent of the Hamiltonian function $H$. The argument goes as follows (for
a detailed proof see [8]): assume that there exist two Hamiltonian functions
$H$ and $K$ for which $\partial \Omega$ is an energy hypersurface. The vector fields $\nabla_z H$
and $\nabla_z K$ are both normal to $\partial \Omega$, hence the Hamiltonian fields $X_H = J \nabla_z H$
and $X_K = J \nabla_z H$ are proportional and thus have the same trajectories (up
to a reparametrization); in particular they have the same periodic orbits.
We have of course
$$c_{\min}(\Omega) \leq c_{HZ}(\Omega) \leq c_{\max}(\Omega) \quad (16)$$
for every subset $\Omega$ of $\mathbb{R}^{2n}$.
We emphasize that symplectic capacities have nothing to with the notion
of volume. They have the dimension of an area in view of the conformality
axiom.

5 Measuring Quantum Indeterminacy

Let us state the main result:

**Theorem 2** Let $(X, P)$ be a $h$-polar quantum pair. We have
$$c_{\max}(X \times P) = c_{HZ}(X \times P) \geq 4h. \quad (17)$$
with equality if $X^h = P$.

The proof of formula (17) follows by monotonicity from the fact that we have
$$c_{\max}(X \times X^h) = c_{HZ}(X \times X^h) = 4h. \quad (18)$$
The proof of the latter is highly non-trivial and is based on a careful study
of certain Minkowski billiard trajectories and requires the topological ma-
chinery developed in Artstein-Avidan et al. in [1,2,3]. We note that the
equality $c_{HZ}(X \times X^h) = 4h$ was proven in an earlier version of [3]; in a re-
vised version it is shown that for any pair $(X, P)$ of centrally convex bodies
(polar or not) one has the equality
$$c_{\max}(X \times P) = c_{HZ}(X \times P) = 4h \max\{\lambda P^h \subset X\}. \quad (19)$$
One immediate consequence of property (17) is that when $(X, P)$ is a
quantum pair not only is the area of the projection of the product $X \times P$
on any of the conjugate planes $x_j, p_j$ always at least $4h$, but in addition there
is no way to deform $X \times P$ using symplectomorphisms to make the area of
such a projection decrease below the value $4h$. 

7
6 RS Inequalities and $\hbar$-Polar Quantum Pairs

We will need the following characterization \[16\] of the orthogonal projections $X$ and $P$ of an ellipsoid $\Omega_{\Sigma}$: they are the $n$-dimensional ellipses

$$X : \frac{1}{2}x^TA^{-1}x \leq 1 \quad , \quad P : \frac{1}{2}p^TB^{-1}p \leq 1 \quad (20)$$

where

$$A = (I_n, 0_n)\Sigma(I_n, 0_n)^T, \quad B = (0_n, I_n)\Sigma(0_n, I_n)^T \quad (21)$$

are symmetric positive definite $n \times n$ matrices.

**Theorem 3** Let $X$ and $P$ be the orthogonal projections of the quantum covariance ellipsoid $\Omega_{\Sigma}$ on the spaces $\mathbb{R}^n_x$ and $\mathbb{R}^n_p$, respectively. Then $(X, P)$ is a $\hbar$-polar quantum pair, and hence $c_{\text{HZ}}(X \times P) \geq 4\hbar$.

**Proof.** Using the explicit form \[2\] of $\Sigma$ Eqs. \[21\] imply that $A = \Delta(x, x)$ and $B = \Delta(p, p)$; applying Eqn. \[11\] with $R = \sqrt{2}$ and replacing $A$ with its inverse conditions \[20\] are thus equivalent to

$$X^\hbar : x^T\Delta(x, x)x \leq \frac{1}{2}\hbar^2 \quad , \quad P : \frac{1}{2}p^T\Delta(p, p)^{-1}p \leq 1. \quad (22)$$

In \[8\] we have proven that we can find an arbitrary invertible $n \times n$ matrix $L$ can be such that

$$L^T\Delta(x, x)L = L^{-1}\Delta(p, p)(L^T)^{-1} = \Lambda \quad (23)$$

where $\Lambda = \text{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n})$, the positive numbers $\lambda_1, ..., \lambda_n$ being the eigenvalues of the product $\Delta(x, x)\Delta(p, p)$ (this is a block-diagonal version of Williamson’s \[22\] symplectic diagonalization theorem). The matrix $M_L = \begin{pmatrix} L^T & 0 \\ 0 & L^{-1} \end{pmatrix}$ is symplectic (because $M_LJM_L^T = J$) hence the condition $c(\Omega_{\Sigma}) \geq \frac{1}{2}\hbar$ is equivalent to $c(\Omega_{\Sigma_L}) \geq \frac{1}{2}\hbar$ where $\Sigma_L = M_L\Sigma M_L^T$ is given by

$$\Sigma_L = \begin{pmatrix} \Lambda \\ L^{-1}\Delta(p, x)L & L^T\Delta(x, p)(L^T)^{-1} \end{pmatrix}.$$ 

Replacing $\Sigma$ with $\Sigma_L$ has the effect of replacing $(X, P)$ with $(X_L, P_L) = ((L^T)^{-1}X, LP)$. In view of the property \[10\] the inclusion $X^\hbar \subset P$ is equivalent to $((L^T)^{-1}X)^\hbar \subset LP$. It is equivalent to prove the Theorem when $\Sigma$ is replaced with $\Sigma_L$ provided that we replace simultaneously $(X, P)$ with $(X_L, P_L)$ in which case Eqn. \[23\] becomes

$$X_L^\hbar : x^T\Lambda x \leq \frac{1}{2}\hbar^2 \quad , \quad P_L : \frac{1}{2}p^T\Lambda^{-1}p \leq 1.$$
The inclusion $X_L^h \subset P_L$ is equivalent to $\lambda_j \geq \hbar^2/4$ for $j = 1, \ldots, n$ which are the Heisenberg inequalities since the diagonal elements of $\Lambda$ are $(\Delta x_1)^2 = (\Delta p_1)^2, \ldots, (\Delta x_n)^2 = (\Delta p_n)^2$. ☐

7 Hardy’s Uncertainty Principle

The discussion of Hardy’s uncertainty principle in Section II can be extended to an arbitrary number of degrees of freedom. In [8] we have proven the following generalization of Hardy’s principle: assume that $\psi$ is square integrable on $\mathbb{R}^n$ and that

$$|\psi(x)| \leq Ce^{-\frac{1}{4}x^T A^{-1} x}, |\hat{\psi}(p)| \leq Ce^{-\frac{1}{4}p^T B^{-1} p}$$

(24)

where $A$ and $B$ are two real positive definite symmetric matrices. We then have $\lambda_j \geq \hbar^2/4$ where the $\lambda_j, j = 1, \ldots, n$, are the eigenvalues of $AB$. It easily follows by an argument similar to that in the proof of Theorem[3] that the ellipsoids

$$X : \frac{1}{2} x^T A^{-1} x \leq 1, \quad P : \frac{1}{2} p^T B^{-1} p \leq 1$$

form a $\hbar$-polar dual pair. This relation between $\hbar$-polar duality and Hardy’s uncertainty principle will be fully developed and generalized to arbitrary exponents in a forthcoming paper [10]. We conjecture that if conditions [24] are replaced with

$$|\psi(x)| \leq Ce^{-\frac{1}{2}||x||_X^2}, \quad |\hat{\psi}(p)| \leq Ce^{-\frac{1}{2}||p||_P^2}$$

(25)

where $||x||_X$ and $||p||_P$ are the Minkowski norms associated with $X$ and $P$ then $(X, P)$ is a $\hbar$-polar quantum pair.

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