On validity of perturbative quantization of the breathing mode in the Skyrme model

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Abstract

We present a detailed discussion of the breathing mode quantization in the Skyrme model and demonstrate that the chiral angle of the hedgehog soliton is strongly affected by the breathing motion.

It was already demonstrated that the centrifugal effects seriously change chiral angle (profile function) of a rotating skyrmion [1]. At the same time the breathing mode in the Skyrme model is usually studied under the assumption that its influence is small enough for using perturbative quantization and calculations can be done with the chiral angle of the static soliton [2, 3]. In
this Brief Report we investigate validity of this assumption and show that the breathing motion affects strongly the soliton chiral angle behavior.

We are starting from the standard Lagrangian of the model

$$
\mathcal{L}(U) = -\frac{F_\pi^2}{16} \text{Tr} (L_\mu L^\mu) + \frac{1}{32e_S^2} \text{Tr} ([L_\mu, L_\nu] [L^\mu, L^\nu]) + \frac{1}{16} F_\pi^2 m_\pi^2 \text{Tr}(U + U^+ - 2),
$$

(1)

where $L_\mu = U^+ \frac{\partial U}{\partial x^\mu}$; $U = U(t, \vec{x})$ is the $SU(2)$ chiral field matrix; $F_\pi$ and $m_\pi$ stand for the pion decay constant and pion mass, respectively; $e_S$ is the dimensionless Skyrme constant.

The model has the well-known hedgehog static solution with topological charge $B = 1$

$$
U_0 = \exp [i\vec{\tau} \cdot \hat{x} \theta_0(\tilde{r})],
$$

(2)

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the isotopic Pauli matrices, $\hat{x} = \frac{\vec{x}}{|\vec{x}|}$, $\tilde{r} = e_S F_\pi |\vec{x}|$. The chiral angle $\theta_0(\tilde{r})$ is obtained from the variational principle

$$
\frac{\delta \int \mathcal{L}(U_0) d^3x}{\delta \theta_0(\tilde{r})} = 0
$$

(3)

supplemented by the boundary conditions

$$
\theta_0(\tilde{r})|_{\tilde{r}=0} = \pi,
$$

(4)

$$
\theta_0(\tilde{r})|_{\tilde{r} \to \infty} = 0.
$$

(5)

Eq.(3) is equivalent to the following differential equation

$$
(r^2 + 8 \sin^2 \theta_0) \frac{d^2 \theta_0}{dr^2} + 2r \frac{d\theta_0}{dr} - \left( 1 + 4 \frac{\sin^2 \theta_0}{r^2} - 4 \left( \frac{d\theta_0}{dr} \right)^2 \right) \sin 2\theta_0 - \frac{m_\pi^2}{e_S^2 F_\pi^2} \tilde{r}^2 \sin \theta_0 = 0.
$$

(6)

Now let us consider the breathing and rotating hedgehog ansatz

$$
U = A(t) \exp \left( i\vec{\tau} \cdot \theta(e^{\lambda(t)} \tilde{r}) \right) A^+(t),
$$

(7)

where $A \in SU(2)$, $A = a_0(t) + i\vec{a}(t)$, $a_0^2 + \vec{a}^2 = 1$. The quantities $a_p = (a_0, \vec{a})$, $p = 0, 1, 2, 3$ and the homogeneous scale transformation parameter $\lambda(t)$ are
the quantum collective coordinates which describe rotation and vibration of
the soliton, respectively. In terms of these collective coordinates the La-
grangian of the dynamical system is written as [2]:

\[ L = \int d^3x \mathcal{L}(U) = \frac{1}{2} A(\lambda) \dot{\lambda}^2 - B(\lambda) + \frac{1}{2} C(\lambda) \text{Tr}(\dot{A} \dot{A}^+) \],

(8)

where

\[ A(\lambda) = e^{-3\lambda} Q_2 + e^{-\lambda} Q_4, \quad B(\lambda) = e^{-\lambda} V_2 + e^\lambda V_4 + e^{-3\lambda} V_\pi, \quad C(\lambda) = e^{-3\lambda} I_2 + e^{-\lambda} I_4 \]

(9), (10), (11)

and

\[ Q_2 = \frac{\pi}{e_5^3 F_\pi} \int_0^\infty d\tilde{r} \tilde{r}^4 \left( \frac{d\theta(\tilde{r})}{d\tilde{r}} \right)^2, \]

(12)

\[ Q_4 = \frac{8\pi}{e_5^3 F_\pi} \int_0^\infty d\tilde{r} \tilde{r}^2 \left( \frac{d\theta(\tilde{r})}{d\tilde{r}} \right)^2 \sin^2 \theta(\tilde{r}), \]

(13)

\[ V_2 = \frac{\pi F_\pi}{2 e_S} \int_0^\infty d\tilde{r} \tilde{r}^2 \left( \left( \frac{d\theta(\tilde{r})}{d\tilde{r}} \right)^2 + 2 \sin^2 \theta(\tilde{r}) \right), \]

(14)

\[ V_4 = \frac{2\pi F_\pi}{e_S} \int_0^\infty d\tilde{r} \sin^2 \theta(\tilde{r}) \left( 2 \left( \frac{d\theta(\tilde{r})}{d\tilde{r}} \right)^2 + \frac{\sin^2 \theta(\tilde{r})}{\tilde{r}^2} \right), \]

(15)

\[ V_\pi = \frac{2\pi m_\pi^2}{F_\pi e_S^3} \int_0^\infty d\tilde{r} \tilde{r}^2 \sin^2 \frac{\theta}{2}, \]

(16)

\[ I_2 = \frac{4\pi}{3 F_\pi e_S^3} \int_0^\infty d\tilde{r} \tilde{r}^2 \sin^2 \theta(\tilde{r}), \]

(17)

\[ I_4 = \frac{16\pi}{3 F_\pi e_S^3} \int_0^\infty d\tilde{r} \tilde{r}^2 \sin^2 \theta(\tilde{r}) \left( \left( \frac{d\theta(\tilde{r})}{d\tilde{r}} \right)^2 + \frac{\sin^2 \theta(\tilde{r})}{\tilde{r}^2} \right). \]

(18)

(Integrals with the subscripts 2, 4 and \( \pi \) are the contributions of the kinetic
term, the Skyrme term and the symmetry breaking term, respectively.)

In the perturbative quantization approach [2, 3] the integrals (12)-(18)
are estimated by means of substitution of the static solution \( \theta_0(\tilde{r}) \) instead of
\( \theta(\tilde{r}) \).
To take into account the influence of the breathing mode nonperturbatively, one has to determine the chiral angle $\theta(\tilde{r})$ from the variational principle averaged over the quantum state $|nj\rangle$ [5]:

$$\left< nj \left| \frac{\delta L}{\delta \theta(\tilde{r})} \right| nj \right> = 0,$$

where $n$ and $j$ denote quantum numbers of the rotational and breathing excitations, respectively.

Using the following notations

\begin{align*}
\alpha_2^2 &= \frac{1}{e^2 F^2_\pi} \frac{\langle nj|\lambda^2 e^{-3\lambda}|nj\rangle}{\langle nj|e^{-\lambda}|nj\rangle}, \\
\alpha_4^2 &= \frac{4}{e^2 F^2_\pi} \frac{\langle nj|\lambda^2 e^{-\lambda}|nj\rangle}{\langle nj|e^{-\lambda}|nj\rangle}, \\
\beta_4^2 &= \frac{\langle nj|e^{\lambda}|nj\rangle}{\langle nj|e^{-\lambda}|nj\rangle}, \\
\beta_\pi^2 &= \frac{m^2}{e^2 F^2_\pi} \frac{\langle nj|e^{-3\lambda}\text{Tr}(\dot{A}\dot{A}^+)|nj\rangle}{\langle nj|e^{-\lambda}|nj\rangle}, \\
\gamma_2^2 &= \frac{2}{3 e^2 F^2_\pi} \frac{\langle nj|e^{-3\lambda}\text{Tr}(\dot{A}\dot{A}^+)|nj\rangle}{\langle nj|e^{-\lambda}|nj\rangle}, \\
\gamma_4^2 &= \frac{8}{3 e^2 F^2_\pi} \frac{\langle nj|e^{-\lambda}\text{Tr}(\dot{A}\dot{A}^+)|nj\rangle}{\langle nj|e^{-\lambda}|nj\rangle},
\end{align*}

one can rewrite (19) in the form\[1\]

\begin{align*}
\left( \tilde{r}^2 (1 - \alpha_2^2 \tilde{r}^2) + \left( 8 \beta_4^2 - 2(\alpha_4^2 + \gamma_4^2) \tilde{r}^2 \right) \sin^2 \theta \right) \frac{d^2 \theta}{dr^2} \\
+ \left( 2 \tilde{r} (1 - 2 \alpha_2^2 \tilde{r}^2) - 4(\alpha_4^2 + \gamma_4^2) \tilde{r} \sin^2 \theta \right) \frac{d\theta}{dr} \\
+ \left( 4 \beta_4^2 - (\alpha_4^2 + \gamma_4^2) \tilde{r}^2 \right) \left( \frac{d\theta}{dr} \right)^2 \sin 2\theta \\
- \left( 1 - \gamma_2^2 \tilde{r}^2 + \left( \frac{4 \beta_4^2}{\tilde{r}^2} - 2 \gamma_4^2 \right) \sin^2 \theta \right) \sin 2\theta - \beta_\pi^2 \tilde{r}^2 \sin \theta = 0.
\end{align*}

\[1\] Eq.(26) can also be obtained from the minimization condition on the total energy, while the solution of Eq.(6) minimizes only the skyrmion static energy.
It should be supplemented by the boundary conditions

\[ \theta(\tilde{r})|_{\tilde{r}=0} = \pi, \quad (27) \]
\[ \theta(\tilde{r})|_{\tilde{r} \to \infty} = 0. \quad (28) \]

The main difference between the static case and the quantum breathing one consists in the factors of the second order derivative terms in Eqs. (6) and (26): contrary to (6), in Eq. (26) this factor contains the term proportional to \( \tilde{r}^4 \). The latter arises from the breathing motion kinetic energy. This difference changes the asymptotic behavior of the chiral angle drastically.

Let us compare the asymptotic solutions of the two equations at large \( \tilde{r} \). First, Eq. (6) is asymptotically reduced to

\[ \tilde{r}^2 \frac{d^2 \theta^\infty_0}{d\tilde{r}^2} + 2\tilde{r} \frac{d\theta^\infty_0}{d\tilde{r}} - (2 + \mu^2_\pi \tilde{r}^2)\theta^\infty_0 = 0, \quad (29) \]

where \( \mu^2_\pi = \frac{m^2_\pi}{e_\pi^2 F^2_\pi} \) and \( \theta^\infty_0(\tilde{r}) = \lim_{r \to \infty} \theta_0(\tilde{r}) \ll 1 \). The general solution of Eq. (29) is

\[ \theta^\infty_0 = C_1 \left( \frac{\mu_\pi}{r} - \frac{1}{\tilde{r}^2} \right) e^{\mu_\pi \tilde{r}} + C_2 \left( \frac{\mu_\pi}{r} + \frac{1}{\tilde{r}^2} \right) e^{-\mu_\pi \tilde{r}}. \quad (30) \]

In order to satisfy the boundary condition (5) the coefficient \( C_1 \) in (30) must be zero which can be realized by appropriate choice of \( \theta'(\tilde{r})|_{\tilde{r}=0} \).

Next, the asymptotic form of Eq. (26) is

\[ \tilde{r}^2 \frac{d^2 \theta^\infty}{d\tilde{r}^2} + 4\tilde{r} \frac{d\theta^\infty}{d\tilde{r}} - \nu \theta^\infty = 0 \quad (31) \]

with \( \nu = \frac{2\gamma^2_2 - \beta^2_\pi}{\alpha^2_2} \), and the general solution is expressed as

\[ \theta^\infty(\tilde{r}) = \begin{cases} 
C_1 \tilde{r}^{\kappa_1} + C_2 \tilde{r}^{\kappa_2}, & \text{if } \nu > -\frac{9}{4}, \\
C_1 \tilde{r}^{-3/2} \ln \tilde{r} + C_2 \tilde{r}^{-3/2}, & \text{if } \nu = -\frac{9}{4}, \\
C_1 \tilde{r}^{-3/2} \sin (\kappa_0 \ln \tilde{r}) + C_2 \tilde{r}^{-3/2} \cos (\kappa_0 \ln \tilde{r}), & \text{if } \nu < -\frac{9}{4}.
\end{cases} \quad (32) \]
where
\[ \kappa_{1,2} = \frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 + \nu}, \quad \kappa_0 = \sqrt{|\nu| - \left(\frac{3}{2}\right)^2}. \] (33)

First of all, we have to note that (contrary to the static case) the coefficient \( C_1 \) in Eq. (32) cannot be chosen equal zero by appropriate choice of \( \theta'(\tilde{r})|_{\tilde{r}=0} \) due to the following reason. The factor at second order derivative of Eq. (26)
\[ g(\tilde{r}) = \tilde{r}^2(1 - \alpha_2^2\tilde{r}^2) + \left(8\beta_2^2 - 2(\alpha_2^2 + \gamma_1^2)\tilde{r}^2\right)\sin^2 \theta \] (34)
is positive for small \( \tilde{r} \) and becomes negative for large \( \tilde{r} \), so there exists at least one point \( \tilde{r}_1 \), where \( g(\tilde{r}_1) = 0 \). This point is a singular point of Eq. (26) and in order to get regular solution the following condition must be fulfilled
\[ \left. \frac{2\tilde{r}(1 - 2\alpha_2^2\tilde{r}^2) - 4(\alpha_4^2 + \gamma_4^2)\tilde{r}^2\sin^2 \theta}{d\tilde{r}} \right|_{\tilde{r}=\tilde{r}_1} \]
\[ + \left. \left(4\beta_4^2 + (\alpha_4^2 + \gamma_4^2)\tilde{r}^2\right) \left(\frac{d\theta}{d\tilde{r}}\right)^2 \sin 2\theta \right|_{\tilde{r}=\tilde{r}_1} \]
\[ - \left. \left(1 - \gamma_2^2\tilde{r}^2 + \left(\frac{4\beta_2^2}{\tilde{r}_2^2} - \gamma_2^2\right)\sin^2 \theta\right)\sin 2\theta \right|_{\tilde{r}=\tilde{r}_1} = 0. \] (35)

A choice of \( \theta'(\tilde{r})|_{\tilde{r}=0} \) is restricted by the fulfillment of the condition (33). Assuming, in addition, the fulfillment of the condition \( C_1 = 0 \), one obtains an overdetermined boundary-value problem (the differential equation of the second order (26) supplemented by the three conditions (27), (35) and \( C_1 = 0 \)). Such a problem has a solution only for a special choice of some spectral parameter. But in the case under consideration the problem has no such parameter: all coefficients of Eq. (26) are fixed by Eqs. (20-25). Thus regular solution satisfying (27) does not satisfy \( C_1 = 0 \).

It is easily seen that for the case \( \kappa_1 \geq 0 \) (which corresponds to \( \nu \geq 0 \) ) the condition (28) is not fulfilled unless \( C_1 = 0 \). At \( \nu < 0 \) the condition (28) can be fulfilled for any \( C_1 \) and \( C_2 \), but there remain some problems about the functionals \( Q_2, V_\pi, \) and \( I_2 \). If \( \nu \leq -\frac{9}{4} \) they are divergent for arbitrary nontrivial values of \( C_1 \) and \( C_2 \). When \( -\frac{9}{4} < \nu < 0 \), \( C_1 \) has to be zero in order to get finite values of these functionals. So there is no solution for the
case of \( \nu \geq 0 \) (such situation arises, for instance, in the chiral limit \( m_\pi = 0 \) ), when \( \nu < 0 \) the existence of solution is not ruled out, but the integrals \( Q_2 \), \( V_\pi \) and \( I_2 \) should be divergent.

It was shown that the nonperturbative quantization of the breathing and rotating modes gives stable soliton solutions in the nonlinear \( \sigma \)-model \textit{without} the Skyrme term \cite{6}. Their energy and mean square radii are finite in spite of divergency of the functionals \( Q_2 \), \( V_\pi \) and \( I_2 \). One can expect that in the present model solutions with finite energy and size could also exist. But, in any case, the slow asymptotic decreasing (at \( \frac{9}{4} \leq \nu < 0 \)) and oscillating behavior (at \( \nu < -\frac{9}{4} \)) of the chiral angle at large \( \tilde{r} \) contradicts the Yukawa law. This problem inherent in the rotating soliton \( (\gamma_2, \gamma_4 \neq 0) \) as well as in nonrotating one \( (\gamma_2, \gamma_4 = 0) \) what means that the source of these difficulties is in the assumption about homogeneous global breathing which does not describe the soliton external part correctly.

Our main result implies that the nonperturbative quantization of the homogeneous global breathing of the skyrmion gives the solutions different drastically from the static ones. In this case the using of the perturbative approach is problematic. One would also face the similar problems considering the homogeneous global breathing quantization of multiskyrmions, solitons in modified Skyrme model and the global nonspherically-symmetrical time dependent deformation of skyrmions \cite{7}.

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