Abstract. We present a new class of strategic games, mixed capability games, as a foundation for studying how different player capabilities impact the dynamics and outcomes of strategic games. We analyze the impact of different player capabilities via a capability transfer function that characterizes the payoff of each player at equilibrium given capabilities for all players in the game. In this paper, we model a player’s capability as the size of the strategy space available to that player. We analyze a mixed capability variant of the Gold and Mines Game recently proposed by Yang et al. [5] and derive its capability transfer function in closed form.

1 Introduction

Player capabilities can significantly impact the dynamics and outcomes of strategic games. Recently, Yang et al. [5] analyzed how different player capabilities affect the social welfare in several congestion games. The research models player strategies as programs in a domain-specific language and models the capability of each player as the size of the programs available to that player. All players in a given game have the same capability, with player capabilities varying across games but not within the same game.

We present mixed capability games as a general framework for studying games in which players have different capabilities, both within the same game and across different games. To capture how game outcomes depend on different player capabilities, we propose analyzing a capability transfer function that precisely quantifies the payoffs of individual players given the capabilities of all players in a game. Section 2 presents the concepts in our framework. Section 3 presents an analysis of a mixed capability game, the Mixed Gold and Mines Game, and derives closed-form expressions for the capability transfer function of this game.

2 Mixed Capability Games and Capability Transfer Function

We model the capability of each player as the size of the strategy space available to that player. We first present formal definitions for pure Nash equilibria of normal-form games [2], then extend the definitions to mixed Nash equilibria.
Definition 1. A mixed capability game is a tuple \( G = (\mathcal{N}, (b_i)_{i \in \mathcal{N}}, (L^j_i)_{i \in \mathcal{N}, 1 \leq j \leq b_i}, (u_i)_{i \in \mathcal{N}}) \) where:

- \( \mathcal{N} = \{1, \ldots, n\} \) is the set of players.
- \( b_i \in \mathbb{Z}^+ \) is the maximal capability of player \( i \).
- \( L^j_i \) is the strategy space of player \( i \) when they have capability \( j \). We also require that the strategy spaces of a player form a hierarchy: \( \forall 1 \leq j < b_i : L^j_i \subseteq L^{j+1}_i \), i.e., a player has more strategies to choose from when they have higher capability.
- \( u_i : L^1_{b_i} \times \cdots \times L^n_{b_n} \rightarrow \mathbb{R} \) is the payoff function that computes the payoff for player \( i \) given the strategies chosen by all players.

A specification of the actual capabilities of players is necessary to determine the outcome of the game.

Definition 2. A capability profile for a mixed capability game is a tuple of integers \( c = (c_1, \ldots, c_n) \) where \( 1 \leq c_i \leq b_i \). A capability profile determines the strategy spaces of the players. Player \( i \) can choose strategies only from \( L^1_{c_i} \).

Given a capability profile \( c = (c_1, \ldots, c_n) \), a strategy profile is a tuple \( s = (s_1, \ldots, s_n) \) where \( s_i \in L^1_{c_i} \) that specifies the strategies chosen by all players. A strategy profile is a pure Nash equilibrium if no player can improve their payoff by unilaterally changing their strategy: \( \forall 1 \leq i \leq n : u_i(s) = \max_{s'_i \in L^1_{c_i}} u_i(s'_i, s_{-i}) \). The notation \( (s'_i, s_{-i}) \) denotes a new strategy profile in which player \( i \) plays strategy \( s'_i \) and all other players play the same strategy as in \( s \).

Definition 3. A capability transfer function of a mixed capability game is a function \( f : [1, b_1] \times \cdots \times [1, b_n] \rightarrow 2^\mathcal{S} \) where \([a, b]\) denotes the integers between \( a \) and \( b \), and \( 2^\mathcal{S} \) is the power set of a set \( \mathcal{S} \). The capability transfer function computes the set of player payoffs at equilibrium for a capability profile. Formally, given a capability profile \( c = (c_1, \ldots, c_n) \), \( f(c) \) is a set such that \((y_1, \ldots, y_n) \in f(c) \) if and only if there is a pure Nash equilibrium \( s = (s_1, \ldots, s_n) \) for which \( s_i \in L^1_{c_i} \) and \( y_i = u_i(s) \).

The capability transfer function contains detailed information about the game’s behavior under varying player capabilities. Example 1 illustrates how to use a capability transfer function to define the higher level concept of a capability-positive game.

Example 1. Capability-positive games \([5]\) are games in which (i) all players share the same capability; and (ii) social welfare at equilibrium cannot decrease as players become more capable. Such games can be defined using the capability transfer function for that game. A game is capability-positive if \( \max W_b \leq \min W_{b+1} \) where \( W_b \overset{\text{def}}{=} \left\{ \sum_{j \in f(i)} i \in f(b, \ldots, b) \right\} \). Note that \( W_b \) is the set of social welfare at equilibrium defined via the capability transfer function of this game.
We extend the definitions to games without pure Nash equilibria. We consider mixed Nash equilibria in which players act stochastically. All finite games have mixed Nash equilibria\(^1\). Given a capability profile \(c = (c_1, \ldots, c_n)\), the strategy of a player \(i\) is a distribution over possible actions, denoted as \(P(a|s_i)\) where \(a \in L_c^i\). Player \(i\) receives expected payoff \(E[u_i|s]\):

\[
E[u_i|s] = \sum_{a_j \in L_j^s} u_i(a_1, \ldots, a_n)P(a_1|s_1) \cdots P(a_n|s_n)
\]

A strategy profile is a mixed Nash equilibrium if no player can unilaterally change their own distribution to improve their expected payoff. In this case, the capability transfer function is defined as the set of expected payoffs of all mixed Nash equilibria given a capability profile.

**Definition 4.** The capability transfer function of a mixed capability game with mixed Nash equilibria is a function \(f : [1, b_1] \times \cdots \times [1, b_n] \mapsto 2^{\mathbb{R}^n}\). Given a capability profile \(c = (c_1, \ldots, c_n)\), \(f(c)\) is a set such that \((y_1, \ldots, y_n) \in f(c)\) if and only if there is a mixed Nash equilibrium \(s = (s_1, \ldots, s_n)\) for which \(s_i\) defines a distribution over \(L_c^i\) and \(y_i = E[u_i|s]\).

One natural question regarding mixed capability games is whether increasing the capability for one player does not make this player receive less payoff. Formally, let \(f_i(c) \overset{\text{def}}{=} \{y_i \mid (y_1, \ldots, y_n) \in f(c)\}\) denote the set of payoffs of player \(i\) at equilibrium, then the question is whether \(\min f_i(c) \leq f_i(c')\) for each \(i\) where \(c = (c_1, \ldots, c_i, \ldots, c_n)\) and \(c' = (c_1, \ldots, c'_i, \ldots, c_n)\) with \(c'_i > c_i\). **Example 2** shows that this is not necessarily true for Nash equilibria since the player with increased capability may switch to another strategy, which triggers responses of other players that ultimately reduce the payoff of the initial player. Note that in a Stackelberg game \(^4\) where the leader announces their strategy before others simultaneously choose their responses, the capability transfer function is monotonic for the leader.

**Example 2.** Consider a two-player two-action bimatrix game. Player 1 is the row player with two possible capabilities: \(L_1^1 = \{1\}\) and \(L_1^2 = \{1, 2\}\). Player 2, the column player, has one capability: \(L_2^1 = \{1, 2\}\). Their payoff matrices are:

\[
u_1 = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

When player 1 has capability 1, they can only play the first row, and player 2 plays the first column, which gives payoffs 1 and 2 for each player respectively. Therefore, we have \(f(1, 1) = \{(1, 2)\}\) for the capability transfer function. When player 1 is allowed to use full capability, the only Nash equilibrium is (second row, second column), which gives \(f(2, 1) = \{(0, 2)\}\). The capability transfer function decreases for player 1 even though player 1’s capability increases.
Fig. 1: An example MGMG instance. Each dot (resp. cross) is a gold (resp. mine). The dashed lines represent a PNE when $C_A = 1$ and $C_B = 2$ (with $\rho < -\mu < 1$).

3 Mixed Gold and Mines Game

We derive exact expressions for the capability transfer function of an asymmetric version of the alternating ordering Gold and Mines Game, a special case of distance-bounded network congestion games originally proposed by Yang et al. [5]. We name this new game the Mixed Gold and Mines Game (MGMG). Unlike the previous Gold and Mines Game of Yang et al. [5] in which all players in the same game have the same capability, in a single Mixed Gold and Mine Game, players may have different capabilities.

MGMG is a two-player congestion game parameterized by five numbers ($M \in \mathbb{Z}^+, \rho \in \mathbb{R}, \mu \in \mathbb{R}, C_A \in \mathbb{Z}^+, C_B \in \mathbb{Z}^+$). MGMG has resources arranged as a specific pattern; players use line segments to cover resources to receive payoffs. As in all congestion games, MGMG games always have pure Nash equilibria [3].

Resource Layout: Each MGMG game has $4M$ resources arranged on two lines. Each resource is either a gold site or a mine site. Each line contains $M$ gold sites and $M$ mine sites in alternating order. Resources are placed at distinct horizontal locations $0, 1, \cdots, 4M - 1$. For the resource at location $i$, $y_i = (i + 1) \mod 2$ indicates which line it is placed on, and $t_i = \mathbb{1}_{i \mod 4 \leq 1}$ indicates whether it is a gold site ($t_i = 1$) or a mine site ($t_i = 0$).

Game Objective: Two players maximize their payoff by using line segments to cover the resources. A delay function $r_t(n)$, where $t \in \{m, g\}$ is the resource type and $n \in \{1, 2\}$ is the number of players covering the resource, specifies the payoff for covering a resource. For gold sites, $r_g(1) = 1$ and $r_g(2) = \rho$, where $0 < \rho < 1$ so that if two players cover a resource they receive a smaller payoff. For mine sites, $r_m(n) = \mu < 0$ is a constant penalty.

Strategy Space: Each player $p \in \{A, B\}$ uses a function $f_p : [0, 4M - 1] \mapsto \{0, 1\}$ to specify which line player $p$ covers at each horizontal location. Player $p$ covers the resource at location $i$ if $f_p(i) = y_i = (i + 1) \mod 2$. The strategy space $\mathcal{L}_C$ of a player with capability $C$ contains all functions with no more than $C$ segments:

$$\text{Seg}(f) \overset{\text{def}}{=} \left( \sum_{i=0}^{4M-2} \mathbb{1}_{f(i) \neq f(i+1)} \right) + 1$$

$$\mathcal{L}_C \overset{\text{def}}{=} \{ f : [0, 4M - 1] \mapsto \{0, 1\} \mid \text{Seg}(f) \leq C \}$$

We use $C_A$ and $C_B$ to denote the capabilities of player A and B respectively, and use $f_A(\cdot)$ and $f_B(\cdot)$ for their strategies. Note that Yang et al. [5] shows that
the capability bound has a natural interpretation as the size of programs in a Domain-Specific Language (DSL) describing the strategy space.

Below is our main result:

**Theorem 1.** Given an instance of MGMG parameterized by \((M, \rho, \mu, C_A, C_B)\) that satisfies \(0 < \rho < -\mu < 1\), in a pure Nash equilibrium of this game, the players receive the following payoffs \(u_A\) and \(u_B\):

\[
\begin{align*}
    u_A &= \left\lfloor \frac{C'_A + t - 1}{2} \right\rfloor \rho - \left\lceil \frac{C'_A - t}{2} \right\rceil \mu + \left\lfloor \frac{C'_B - t}{2} \right\rfloor (\rho - 1) + (\mu + 1)M \\
    u_B &= \left\lfloor \frac{C'_B - t}{2} \right\rfloor \rho - \left\lceil \frac{C'_B + t - 1}{2} \right\rceil \mu + \left\lfloor \frac{C'_A + t - 1}{2} \right\rfloor (\rho - 1) + (\mu + 1)M
\end{align*}
\]

where \(C'_A = \min(C_A, 2M + 1)\), \(C'_B = \min(C_B, 2M + 1)\), \(t \in \{0, 1\}\)

Three cases determine the value of \(t\):

- When \(\max(C_A, C_B) \leq 2M\), there are two classes of Nash equilibria distinguished by \(t = 0\) and \(t = 1\).
- When \(\min(C_A, C_B) \leq 2M < \max(C_A, C_B)\), \(t = 0\) if \(C_A \leq 2M\) and \(t = 1\) if \(C_B \leq 2M\).
- When \(\min(C_A, C_B) \geq 2M + 1\), there is one Nash equilibrium. The above formulas give the same payoffs regardless of \(t = 0\) or \(t = 1\).

In MGMG, when one player’s capability increases, their own payoff increases by \(\rho\) or \(-\mu\), but their opponent’s payoff decreases by \(\rho - 1\). If both players get the same capability increment, the social welfare (i.e., the sum of their payoffs) can increase, decrease, or stay the same, depending on the sign of \(2\rho - \mu - 1\). If both players have the same capability \(C\), the social welfare is \(u_A + u_B = (2\rho - \mu - 1)(\min(C, 2M + 1) - 1) + 2(\mu + 1)M\), which confirms Theorem 16 of Yang et al. \([5]\) up to a constant bias because in MGMG we remove the last gold site on each line to simplify our analysis.

In the following two sections, we first present three lemmas that characterize the Nash equilibria in MGMG, and then derive the above results based on these lemmas.

### 3.1 Characteristics of Nash equilibria

We introduce some notation:

- A pair \((f_A, f_B)\) denotes a strategy profile, i.e., the strategies of both players.
- Given a strategy profile, \(u_A(f_A, f_B)\) and \(u_B(f_A, f_B)\) are the payoffs of individual players.
Given a strategy \( f(\cdot) \), \( G(f)/M(f) \) and \( \#g(f)/\#m(f) \) denote the locations and numbers of gold and mine sites covered by the strategy:

\[
G(f) \overset{\text{def}}{=} \{4i \mid 0 \leq i < M \text{ and } f(4i) = 1\} \\
\cup \{4i + 1 \mid 0 \leq i < M \text{ and } f(4i + 1) = 0\}
\]

\[
M(f) \overset{\text{def}}{=} \{4i + 2 \mid 0 \leq i < M \text{ and } f(4i + 2) = 1\} \\
\cup \{4i + 3 \mid 0 \leq i < M \text{ and } f(4i + 3) = 0\}
\]

\[
\#g(f) \overset{\text{def}}{=} |G(f)|
\]

\[
\#m(f) \overset{\text{def}}{=} |M(f)|
\]

– A strategy \( f_p(\cdot) \) for player \( p \), discontinuity points (DPs) are the locations where \( f_p(\cdot) \) changes the line that \( p \) covers. We also differentiate between upward discontinuity points (UDPs, denoted by \( D^\uparrow(f) \)) and downward discontinuity points (DDPs, denoted by \( D^\downarrow(f) \)):

\[
D^\uparrow(f) \overset{\text{def}}{=} \{i \mid 0 \leq i \leq 4M - 2 \text{ and } f(i) = 0 \text{ and } f(i + 1) = 1\}
\]

\[
D^\downarrow(f) \overset{\text{def}}{=} \{i \mid 0 \leq i \leq 4M - 2 \text{ and } f(i) = 1 \text{ and } f(i + 1) = 0\}
\]

Note that \( \text{Seg}(f) = |D^\uparrow(f)| + |D^\downarrow(f)| + 1 \).

– A strategy \( f(\cdot) \) is a perfect cover for resources located between \([a, b]\) if all gold sites are covered and all mine sites are avoided: \([a, b] \setminus G(f) = \emptyset \) and \([a, b] \cap M(f) = \emptyset \). We also call the resources \([a, b]\) perfectly covered in this case, and imperfectly covered otherwise. Here \([a, b]\) denotes all integers in the interval: \([a, b]\) = \( \{i \mid a \leq i \leq b\} \). Note that to perfectly cover resources \([4i, 4j - 1]\) for \( i < j \), one needs \( 2(j - i) + 1 \) segments.

– Strict strategy spaces use exactly the given number of segments:

\[
\tilde{L}_1 \overset{\text{def}}{=} L_1
\]

\[
\tilde{L}_C \overset{\text{def}}{=} L_C - L_{C-1}
\]

\[
= \{f : [0, 4M - 1] \mapsto \{0, 1\} \mid \text{Seg}(f) = C\}
\]

– A strategy profile \((f_A, f_B)\) is a complete-gold-coverage for a MGMG if both players cover all gold sites together, i.e., \( |G(f_A) \cup G(f_B)| = 2M \).

First, we show that DPs only occur at certain locations:

**Lemma 1.** Let \( f(\cdot) \) be a best response of a player given the other player’s strategy. Upward discontinuity points in \( f(\cdot) \) occur only at neighboring mine sites, and downward discontinuity points occur only at neighboring gold sites:

\[
\forall i \in D^\uparrow(f) : i \mod 4 = 2
\]

\[
\forall i \in D^\downarrow(f) : i \mod 4 = 0
\]

**Proof.** We consider cases in a local region for different values of \( f(4k) \) and \( f(4k+4) \).
Fig. 2: Cases of a $f(4k) = 0$ and $f(4k + 4) = 1$ in a local region with one DP. The numbers are locations of resources modulo 4. Dashed lines indicate a local part of the strategy.

- $f(4k) = 0$, $f(4k + 4) = 1$: Figure 2 shows the cases with one DP. The payoffs of covered gold sites are denoted as $r_0$ and $r_1$, which can be 1 or 0 depending on the opponent’s strategy. Clearly, the payoff is maximized only when the DP is at location 2 modulo 4. It can be verified that using more DPs while maintaining $f(4k) = 0$ and $f(4k + 4) = 1$ does not improve payoff.
- $f(4k) = 1$, $f(4k + 4) = 0$: Similarly, the best response in this case has one DDP at location 0.
- $f(4k) = f(4k + 4) = 0$: The best response should have no DP. If there are DPs, there should be one UDP and one DDP to cover one gold site and no mine site, but moving the DDP rightward to also cover the gold at $4k + 4$ gives better payoff with the same number of segments.
- $f(4k) = f(4k + 4) = 1$: The best response should either have no DP (covering two gold sites and one mine site) or two DPs (covering three gold sites and no mine site) at locations 0 and 2 modulo 4.

Now we show that the number of gold and mine sites covered by an optimal strategy is fairly predictable, i.e., it only depends on $f(0)$ and $\text{Seg}(f)$:

**Lemma 2.** If $f(\cdot)$ is a strategy that conforms to Lemma 1, then

$$
\# g(f) = M + \left\lfloor \frac{\text{Seg}(f) + f(0) - 1}{2} \right\rfloor
$$

$$
\# m(f) = M - \left\lfloor \frac{\text{Seg}(f) - f(0)}{2} \right\rfloor
$$

**Proof.** We define a series of strategies $\{f_i\}$. Let $f_0 = f$ and define $f_{i+1}$ the strategy obtained by removing the last DP of $f_i$, i.e., $f_{i+1}(x) = f_i(\min(x, x_i))$ where $x_i = \max(D^i(f_i) \cup D^i(f_i))$. Lemma 1 implies that each DDP adds an extra gold site and each UDP avoids a mine site, which means either $\# g(f_i) - \# g(f_{i+1}) = 1$ or $\# m(f_{i+1}) - \# m(f_i) = 1$, depending on whether the last DP of $f_i$ is DDP or UDP. It follows that $\# g(f_i) = \# g(f_j) + |D^i(f_i)| - |D^i(f_j)|$ and $\# m(f_i) = \# m(f_j) - |D^i(f_i)| + |D^i(f_j)|$ for any pair $i, j$.

We first assume $f(0) = 1$. In this case, since UDPs and DDPs are interleaving, we have $|D^i(f)| = \left\lfloor \frac{\text{Seg}(f)}{2} \right\rfloor$ and $|D^i(f)| = \left\lfloor \frac{\text{Seg}(f) - 1}{2} \right\rfloor$. Let $c = \text{Seg}(f)$. 

\[\begin{array}{c|c|c|c|c}
0 & 1 & 2 & 3 & 0 \\
\hline
\bullet & - & - & - & \times \\
\hline
\bullet & - & - & \times & \times \\
\hline
\bullet & - & \times & \times & \times \\
\hline
\bullet & \times & \times & \times & \times \\
\end{array}\]

(a) Upward at 0. Payoff is $\mu + r_1$. 

(b) Upward at 1. Payoff is $\mu + r_0 + r_1$. 

(c) Upward at 2. Payoff is $r_0 + r_1$, which is the best. 

(d) Upward at 3. Payoff is $\mu + r_0 + r_1$. 

We also know that \( \#g(f_c) = \#m(f_c) = M \) since \( f_c \) covers exactly one line. Therefore, \( \#g(f) = \#g(f_0) = \#g(f_c) + |D^1(f_0)| - |D^1(f_c)| = M + \left\lfloor \frac{\text{Seg}(f)}{2} \right\rfloor \) and \( \#m(f) = M - \left\lfloor \frac{\text{Seg}(f)-1}{2} \right\rfloor \). A similar analysis for the case \( f(0) = 0 \) gives \( \#g(f) = M + \left\lfloor \frac{\text{Seg}(f)-1}{2} \right\rfloor \) and \( \#m(f) = M - \left\lfloor \frac{\text{Seg}(f)}{2} \right\rfloor \). Lemma 2 summarizes these results compactly.

Now let’s shift our attention from one player’s strategy to a strategy profile.

**Lemma 3.** If \( \rho < -\mu < 1 \) and \((f_A, f_B)\) is a pure Nash equilibrium when players are limited to the strict strategy spaces \( \hat{L}_{CA} \) and \( \hat{L}_{CB} \), then \((f_A, f_B)\) is a complete-gold-coverage.

**Proof.** We prove this statement in two steps. We first show that for any player, their payoff is maximized when they cover as many unoccupied gold sites as possible. Then we show that complete-gold-coverage is always feasible.

Let \( T(f_A, f_B) \) be the total number of gold sites covered by a strategy profile: \( T(f_A, f_B) \equiv |G(f_A) \cup G(f_B)| \).

Without loss of generality, we focus on player A. We show that if there is a strategy \( f'_A \) such that \( T(f_A, f_B) < T(f'_A, f_B) \) where \( \{f_A, f'_A\} \subseteq \hat{L}_{CA} \), then \((f_A, f_B)\) is not a Nash equilibrium because A can get better payoff by switching to \( f'_A \). Note that the number of gold sites covered by both players in the strategy profile \((f_A, f_B)\) is \( \#g(f_A) + \#g(f_B) - T(f_A, f_B) \), while the number of gold sites covered by A exclusively is \( T(f_A, f_B) - \#g(f_B) \), which implies:

\[
\begin{align*}
\ u_A(f_A, f_B) &= (T(f_A, f_B) - \#g(f_B)) \cdot 1 \\
& \quad + (\#g(f_A) + \#g(f_B) - T(f_A, f_B)) \cdot \rho + \#m(f_A) \cdot \mu
\end{align*}
\]

Substituting the results of Lemma 2 into the above:

\[
\begin{align*}
\ u_A(f_A, f_B) &= \rho \left[ \frac{C_A + f_A(0) - 1}{2} \right] + (-\mu) \left[ \frac{C_A - f_A(0)}{2} \right] + (1 - \rho) T(f_A, f_B) \\
& \quad + (\rho - 1) \#g(f_B) + (\rho + \mu) M
\end{align*}
\]

Let \( h(f) \equiv \rho \left[ \frac{C_A + f(0) - 1}{2} \right] + (-\mu) \left[ \frac{C_A - f(0)}{2} \right] \) be the first two terms. One can verify that \( h(f'_A) - h(f_A) \in \{0, \rho + \mu, -\rho - \mu\} \) for all possible values of \( C_A, f_A(0), \) and \( f'_A(0) \). Note \( \rho < -\mu \) implies \( \rho + \mu < 0 \). Thus \( h(f'_A) - h(f_A) \geq \rho + \mu \).

\[
\begin{align*}
\ u_A(f'_A, f_B) - u_A(f_A, f_B) &= h(f'_A) - h(f_A) + (T(f'_A, f_B) - T(f_A, f_B)) (1 - \rho) \\
& \geq (\rho + \mu) + (1 - \rho) \\
& > 0
\end{align*}
\]

Therefore, A first maximizes \( T(f_A, f_B) \) and then maximizes \( h(f_A) \) in their best response. The maximum possible value of \( T(f_A, f_B) \) is \( 2M \) which is achieved when \((f_A, f_B)\) is a complete-gold-coverage.
Next we show that complete-gold-coverage is always feasible. We assume $C_A \geq C_B$ WLOG. For any strategy $f_B(\cdot)$ played by player B that conforms to Lemma 1, we show that there exists $f_A \in \mathcal{L}_{C_A}$ such that $T(f_A, f_B) = 2M$.

If $C_A \geq 2M$, then A can cover all gold sites trivially. Now we consider the case $1 \leq C_A \leq 2M - 1$. We first construct a strategy $f_A'(\cdot)$ that may or may not use all the segments. For $0 \leq k < M$, we set $f_A'(4k) = 1 - f_B(4k)$ and $f_A'(4k + 3) = 1 - f_B(4k + 3)$. When $f_A'(4k) = f_A'(4k + 3)$, we use the same value for $f_A'(4k + 1)$ and $f_A'(4k + 2)$; otherwise we add one discontinuity point at $f_A'(4k)$ or $f_A'(4k + 2)$ according to Lemma 1. Note that Lemma 1 also implies $f_A'(4k - 1) = f_A'(4k)$. It is easy to verify that $T(f_A', f_B) = 2M$ and $\text{Seg}(f_A') \leq \text{Seg}(f_B) \leq C_A$. We then derive $f_A$ from $f_A'$ using Algorithm 1 so that $\text{Seg}(f_A) = C_A$.

### 3.2 Capability Transfer Function of MGMG

Recall that in the proof of Lemma 3, we have shown that given a strategy profile $(f_A, f_B)$ that is a complete-gold-coverage, A’s payoff is

$$ u_A(f_A, f_B) = \rho \left[ \frac{\text{Seg}(f_A) + f_A(0) - 1}{2} \right] + (-\mu) \left[ \frac{\text{Seg}(f_A) - f_A(0)}{2} \right] + (\rho - 1) \#_g(f_B) + (\mu - \rho + 2)M $$

For two strategy profiles $(f_A, f_B)$ and $(f_A', f_B)$ that are both complete-gold-coverage, we make the following two observations that can be verified using the above expansion of $u_A(f_A, f_B)$:

1. If $\text{Seg}(f_A) < \text{Seg}(f_A')$ and $\min(-\mu, \rho) > 0$, then $u_A(f_A, f_B) < u_A(f_A', f_B)$.
2. If $f_A(0) = 1$, $f_A'(0) = 0$, $\text{Seg}(f_A) = \text{Seg}(f_A')$, and $\rho + \mu < 0$, then $u_A(f_A, f_B) \leq u_A(f_A', f_B)$.

In other words, the best strategy $f_p^*(\cdot)$ of player $p$ given the strategy $f_o(\cdot)$ of the other player satisfies:

1. $(f_p^*, f_o)$ is a complete-gold-coverage.
2. $f_p^*(\cdot)$ uses the full capability of player $p$ up to $2M + 1$ line segments, i.e., $\text{Seg}(f_p^*) = \min(C_p, 2M + 1)$.
3. If there is a strategy that starts at line 0 (i.e., $f_p(0) = 0$) and satisfies both of the above constraints, then player $p$ plays such a strategy.

Next we derive the capability transfer function for the different cases. We assume $C_B \leq C_A$ WLOG:

- $2M + 1 \leq C_B \leq C_A$: All resources are perfectly covered by both players. They receive the same payoff of $2M \rho$. 
Algorithm 1 Modify a strategy to use more segments

Require: Game scale $M \geq 2$
Require: Player capability $C_A$ such that $C_A \leq 2M - 1$
Require: A strategy $f_A^1(\cdot)$ that conforms to Lemma 1 such that $\text{Seg}(f_A^1) \leq C_A$ and the last four resources are imperfectly covered.
Ensure: A strategy $f_A^{1'}(\cdot)$ that conforms to Lemma 1 such that $\text{Seg}(f_A) = C_A$, $G(f_A') \subseteq G(f_A)$, and $f_A(0) = f_A'(0)$.

1: $k \leftarrow 0$
2: $f_A \leftarrow f_A^1$
3: while $C_A - \text{Seg}(f_A) \geq 2$ do
4: \hspace{1em} $\triangleright$ When entering the loop, all resources in $[4, 4k-1]$ are perfectly covered and $f_A(4k) = 1$ when $k \geq 1$. Perfectly covering $[4, 4k-1]$ requires $\text{Seg}(f_A) \geq 2(k - 1) + 1$. We also have $\text{Seg}(f_A) \leq C_A - 2 \leq 2M - 3$ due to the loop condition, thus $2k - 1 \leq \text{Seg}(f_A) \leq 2M - 3$ which means $k \leq M - 1$. When $k \geq 1$, each iteration modifies $f_A(\cdot)$ to perfectly cover $[4k, 4k + 3]$ using no more than two new segments.
5: \hspace{1em} if $f_A(4k + 3) = 0$ then
6: \hspace{2em} $\triangleright$ Lemma 1 ensures $f_A(4k + i) = 0$ for $2 \leq i \leq 5$.
7: \hspace{2em} $f_A(4k + 3) \leftarrow 1$
8: \hspace{2em} if $k + 1 < M$ then
9: \hspace{3em} $f_A(4k + 4) \leftarrow 1$
10: \hspace{2em} end if
11: \hspace{1em} else if $k > 0$ or $f_A(4k) = 1$ then
12: \hspace{2em} $\triangleright$ We now have $f_A(4k) = f_A(4k + 3) = 1$.
13: \hspace{2em} $f_A(4k + 1) \leftarrow 0$
14: \hspace{2em} $f_A(4k + 2) \leftarrow 0$
15: \hspace{2em} end if
16: \hspace{1em} $k \leftarrow k + 1$
17: end while
18: if $C_A - \text{Seg}(f_A) = 1$ then
19: \hspace{1em} $\triangleright$ We have $\text{Seg}(f_A) = C_A - 1 < 2M - 1$ in this case. Thus the resources $[4, 4M - 1]$ are imperfectly covered. Due to our requirement on $f_A(\cdot)$ and the way we construct $f_A(\cdot)$, the last four resources are imperfectly covered.
20: \hspace{1em} if $f_A(4M - 1) = 0$ then
21: \hspace{2em} $f_A(4M - 1) \leftarrow 1$
22: \hspace{1em} else if $f_A(4M - 2) = 1$ then
23: \hspace{2em} $\triangleright$ Lemma 1 implies $f_A(4M - i) = 1$ for $1 \leq i \leq 4$
24: \hspace{2em} $f_A(4M - 3) \leftarrow 0$
25: \hspace{2em} $f_A(4M - 2) \leftarrow 0$
26: \hspace{2em} $f_A(4M - 1) \leftarrow 0$
27: \hspace{1em} else
28: \hspace{2em} $\triangleright$ Now $f_A(4M - 2) = 0$ and $f_A(4M - 1) = 1$. Since the last four resources are imperfectly covered, we have $f_A(4M - i) = 0$ for $2 \leq i \leq 6$.
29: \hspace{2em} $f_A(4M - 5) \leftarrow 1$
30: \hspace{2em} $f_A(4M - 4) \leftarrow 1$
31: \hspace{2em} $f_A(4M - 1) \leftarrow 0$
32: \hspace{1em} end if
33: end if
34: return $f_A$
– $C_B < 2M + 1 \leq C_A$: A perfectly covers all resources. B starts at $f_B(0) = 0$ and uses all their capability.

\[
\# g(f_B) = M + \left\lfloor \frac{C_B - 1}{2} \right\rfloor
\]
\[
u_A = 2M + (\rho - 1) \# g(f_B)
\]
\[
= (\rho + 1)M + (\rho - 1) \left\lfloor \frac{C_B - 1}{2} \right\rfloor
\]
\[
u_B = \left( M + \left\lfloor \frac{C_B - 1}{2} \right\rfloor \right) \rho + \left( M - \left\lfloor \frac{C_B}{2} \right\rfloor \right) \mu
\]
\[
= \left\lfloor \frac{C_B - 1}{2} \right\rfloor \rho - \left\lfloor \frac{C_B}{2} \right\rfloor \mu + (\rho + \mu)M
\]

– $C_B \leq C_A < 2M + 1$: Let B first play an arbitrary strategy $f_B(\cdot)$. If $f_B(0) = 0$, A will set $f_A(0) = 1$ to ensure a complete-gold-coverage; otherwise if $f_B(0) = 1$, A will set $f_A(0) = 0$ due to the second observation noted above. A can derive one of their best response $f_A(\cdot)$ according to the proof of Lemma 3. Following a similar reasoning from B’s perspective, it can be shown that $f_B(\cdot)$ is also a best response of B given A’s strategy $f_A(\cdot)$. Therefore, $(f_A, f_B)$ is a Nash equilibrium. There are two different classes of Nash equilibria: one with $f_A(0) = 0$ and $f_B(0) = 1$, and the other with $f_A'(0) = 1$ and $f_B'(0) = 0$. Let $f_A(0) = t$ and $f_B(0) = 1 - t$. We have:

\[
u_A = \left\lfloor \frac{C_A + t - 1}{2} \right\rfloor \rho - \left\lfloor \frac{C_A - t}{2} \right\rfloor \mu + \left\lfloor \frac{C_B - t}{2} \right\rfloor (\rho - 1) + (\mu + 1)M
\]
\[
u_B = \left\lfloor \frac{C_B - t}{2} \right\rfloor \rho - \left\lfloor \frac{C_B + t - 1}{2} \right\rfloor \mu + \left\lfloor \frac{C_A + t - 1}{2} \right\rfloor (\rho - 1) + (\mu + 1)M
\]

$t \in \{0, 1\}$

Theorem 1 summarizes the results of these three cases.
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