Modified Stieltjes Transform and Generalized Convolutions

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Abstract

Classical Stieltjes Transform is modified in a way to generalize both Stieltjes and Fourier transforms. This transform allows to introduce new classes of commutative and non-commutative generalized convolutions.

Key words: Stieltjes Transform; characteristic function; generalized convolution.

1 Introduction

Let us begin with a definitions of classical and generalized Stieltjes transforms. Although, usual these are transforms given on a set of functions, we will consider more convenient for us case of probability measures or cumulative distribution functions. Namely, let \( \mu \) be a probability measure of Borel subsets of real line \( \mathbb{R}^1 \). Its Stieltjes transform is defined as

\[
S(z) = S(z; \mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{x - z},
\]

where \( \text{Im}(z) \neq 0 \). Surely, the integral converges in this case. Generalized Stieltjes transform is represented by

\[
S_{\gamma}(z) = S_{\gamma}(z; \mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(x - z)^\gamma},
\]
for real $\gamma > 0$. A modification of generalized Stieltjes transform was proposed in [1]. Now we prefer to change this modification, and define the following form of transform:

$$R_{\gamma}(u) = R_{\gamma}(u; \mu) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1 - iux)^{\gamma}}. \quad (1.1)$$

Connection to the generalized Stieltjes transform is obvious. It is convenient for us to use this transform for real values of $u$. It is clear that the limit

$$\lim_{\gamma \to \infty} R_{\gamma}(u/\gamma) = \int_{-\infty}^{\infty} \exp\{iux\}d\mu(x), \quad (1.2)$$

represents Fourier transform (characteristic function) of the measure $\mu$. The uniqueness of a measure recovering from its modified Stieltjes transform follows from corresponding result for generalized Stieltjes transform.

Relation (1.2) gives us limit behavior of modified Stieltjes transform as $\gamma \to \infty$. Another possibility ($\gamma \to 0$) without any normalization gives trivial limit equals to 1. However, more proper approach is to calculate the limit $(R_{\gamma}(u) - 1)/\gamma$ as $\gamma \to 0$. It is easy to see, that

$$\lim_{\gamma \to 0} (R_{\gamma}(u) - 1)/\gamma = \int_{-\infty}^{\infty} \log \frac{1}{1 - iux}d\mu(x). \quad (1.3)$$

If the measure $\mu$ has compact support, it is possible to write series expansion for modified Stieltjes transform:

$$R_{\gamma}(u) = \int_{-\infty}^{\infty} \frac{d\mu(x)}{(1 - iux)^{\gamma}} = \sum_{k=0}^{\infty} (-1)^k i^k \binom{-\gamma}{k} \kappa_k(\mu) x^k,$$

where $\kappa_k(\mu) = \int_{-\infty}^{\infty} x^k d\mu(x)$ is $k$th moment of the measure $\mu$.

Modified Stieltjes transform may be interpreted in terms of characteristic functions. Namely, let us consider gamma distribution with probability density function

$$p(x) = \frac{1}{|\lambda|^\gamma \Gamma(\gamma)} x^{\gamma-1} \exp(-x/\lambda), \quad (1.4)$$

for $x \ast \lambda > 0$, and zero in other cases. Note, that this distribution is ordinary gamma distribution for positive $\lambda$, and its “mirror reflection” on negative semi-axes for negative $\lambda$. Let us now consider $\lambda$ as random variable with
cumulative distribution function $\mu$. In this case, (1.1) gives characteristic function of gamma distribution with such random parameter:

$$
 f(t) = \int_{-\infty}^{\infty} \frac{d\mu(\lambda)}{(1 - it\lambda)^\gamma}.
$$

\[ (1.5) \]

## 2 A family of commutative generalized convolutions

Using modifies Stieltjes transform we can introduce a family of commutative generalized convolutions. Main idea for this is the following. Let $\mu_1$ and $\mu_2$ be two probabilities. Take positive $\gamma$ and consider product of modified Stieltjes transforms of these measures $R_\gamma(u, \mu_1)R_\gamma(u, \mu_2)$. We would like to represent this product as a modified Stieltjes transform of a measure. Typically, the product is not modified Stieltjes transform with the same index $\gamma$. However, it can be represented as modified Stieltjes transform with index $\rho > \gamma$ of a measure $\nu$, which is called generalized (more precisely "$(\gamma, \rho)$") convolution of the measures $\mu_1$ and $\mu_2$. Let us mention that the indexes $\rho$ and $\gamma$ are not arbitrary, however, there are infinitely many suitable pairs of indexes. Clearly, the measure $\nu$, if exists, depends on $\mu_1$, $\mu_2$, and on indexes $\gamma$, $\rho$.

Unfortunately, we cannot describe all pairs $\gamma, \rho$ for which corresponding generalized convolution $\nu$ of measures $\mu_1$ and $\mu_2$ exists. However, we shall show, the pairs of the form $n, 2n$ (where $n$ is positive, but not necessarily integer number) possess this property.

**Theorem 2.1.** Let $\mu_1, \mu_2$ be two probability measures on $\sigma$-field Borel subsets of real line. For arbitrary real $n > 0$ there exists 

$"(n, 2n)$" convolution $\nu$ of $\mu_1$ and $\mu_2$. In other words, for real $n > 0$ and measures $\mu_1$ and $\mu_2$ there exists a measure $\nu$ such that

$$
 R_{2n}(u; \nu) = R_n(u; \mu_1)R_n(u; \mu_2).
$$

\[ (2.1) \]

**Proof.** Because convex combination of probability measures is a probability measure again, and each probability on real line can be considered as a limit of sequence of measures concentrated in finite number of points each, it is sufficient to prove the statement for Dirac $\delta$-measures only.

Suppose now that the measures $\mu_1$ and $\mu_2$ are concentrated in points $a$ and $b$ correspondingly. We have to prove that there is a measure $\nu$ depending
on \(a, b\) and \(n\) such that

\[
\int_{-\infty}^{\infty} \frac{d\nu(x)}{(1-ixu)^{2n}} = \frac{1}{(1-iau)^n} \cdot \frac{1}{(1-ibu)^n}.
\]  

(2.2)

Of course, it is enough to find the measure \(\nu\) with compact support. Therefore, we must have

\[
k_m = \sum_{k=0}^{m} \frac{n(n+1) \cdots (n+k-1)}{k!} \cdot \frac{n(n+1) \cdots (n+m-k-1)}{(m-k)!} a^k b^{m-k} \frac{(-2n)^m}{m},
\]

(2.3)

where \(k_m = k_m(\nu)\) is \(m\)th moment of \(\nu\). It remains to show that the left hand side of (2.3) really defines for \(m = 0, 1, \ldots\) moments of a distribution.

Let us denote \(\lambda = a/b\) and suppose that \(|\lambda| < 1\) (the case \(|\lambda| = 1\) may be obtained as a limit case). Then \(k_m\) can be rewrite in the form

\[
k_m = (-1)^m b^m \sum_{k=0}^{m} \binom{m}{k} \frac{(n)_k (n)_{m-k}}{(2n)_m} \lambda^k,
\]

where \((s)_j = s \cdots (s+j-1)\) is Pochhammer symbol. Simple calculations allows us to obtain from previous equality that

\[
k_m = \frac{b^m (n)_m \, _2F_1(-m, n, 1-m-n, a/b)}{(2n)_m}.
\]  

(2.4)

Let us consider a random variable \(X\) having Beta distribution with equal parameters \(n\) and \(n\), that is with probability density function

\[
p_X(x) = (1-x)^{n-1} x^{n-1} 2^{2n-2} \Gamma(n+1/2) / (\sqrt{\pi} \, \Gamma(n)),
\]

for \(x \in (0, 1)\), and zero for \(x \notin (0, 1)\). It is not difficul to calculate that

\[
\mathbb{E}\left(aX + b(1-X)\right)^m = b^m \, _2F_1(-m, n, 2n, 1-a/b),
\]

which coincide with (2.4) for non-negative integer \(m\) and real \(n > 0\). \(\square\)

Theorem 2.1 allows us to define a family of depending on \(n\) generalized convolutions \(\nu = \mu_1 *_{n} \mu_2\), which is equivalent to the relation (2.1). Obviously, this operation is commutative. However, it is not associative, which can be
easily verified by comparing the convolutions \((\delta_1 \star_n \delta_2) \star_n \delta_3\) and \(\delta_1 \star_n (\delta_2 \star_n \delta_3)\), where \(\delta_a\) denotes Dirac measure at point \(a\). It is easy to verify that \(\mu_1 \star_n \mu_2(2A) \xrightarrow{n \to \infty} \mu_1 \ast \mu_2(A)\), where \(\ast\) denotes ordinary convolution of measures.

We have \(2A\) in the left-hand-side because \(\mathbb{E}X = 1/2\). This generalized convolution may be written through independent random variables \(U\) and \(V\) in the form

\[ W = UX + V(1 - X), \]

where \(X\) is random variable independent of \((U, V)\) and having Beta distribution with parameters \((n, n)\), and the distribution of \(W\) is exactly generalized convolution of distributions of \(U\) and \(V\).

In view of non-associativity of \(\ast_n\)-convolution it does not coincide with K. Urbanik generalized convolution (see, [2]). At the same time, it non-associativity shows that the expression \(\mu_1 \star_n \mu_2 \star_n \mu_3\) has no sense. However, one can define this 3-arguments operation by use stochastic linear combinations, that is linear forms of random variables with random coefficients. Now we define such \(k\)-arguments operation. Namely, let \(U_1, \ldots, U_k\) be independent random variables, and \(X_1, \ldots, X_{n-1}\) be a random vector having Dirichlet distribution with parameters \((a_1, \ldots, a_k) = (n, \ldots, n)\). Define

\[ W = X_1 U_1 + \ldots + X_{k-1} U_{k-1} + (1 - \sum_{j=1}^{k-1} U_k) \]  

(2.5)

The map from vector \(U\) of marginal distributions of \((U_1, \ldots, U_k)\) to the distribution of random variable \(W\) call \(k\)-tuple generalized convolution of the components of \(U\). Clearly, this operation is symmetric with respect to permutations of coordinates of the vector \(U\).

3 Connected family of non-commutative generalized convolutions

Let now \(U_1, \ldots, U_k\) be independent random variables, and \(X_1, \ldots, X_{n-1}\) be a random vector having Dirichlet distribution with parameters \((a_1, \ldots, a_k)\), possible different from each other. Using the relation (2.5) define random variable \(W\). Its distribution will be called non-commutative generalized convolution of marginal distributions of the vector \(U\). In particular case of
$k = 2$ we obtain non-commutative variant of two-tuple generalized convolution, which represents more general case of (1.1).

Let us give a property of this generalized convolution. To do so, let us define $\tilde{\text{beta}}_{A,B}$ distribution over interval $(A, B)$ by its probability density function

$$p_{\alpha, \beta}(x) = \begin{cases} \frac{1}{B(\alpha, \beta)(B-A)^{\alpha+\beta-1}}(x-A)^{\alpha-1}(B-x)^{\beta-1}, & \text{if } A < x < B, \\ 0, & \text{otherwise}, \end{cases}$$

for positive $\alpha, \beta$. Here $B(\alpha, \beta)$ is beta function.

**Theorem 3.1.** Let $W_1, W_2$ be two independent identical distributed random variables having $\tilde{\text{beta}}_{A,B}(n, n)$ distribution, and $\mu_1, \mu_2$ be corresponding probability distributions. Then the measure $\nu = \mu_1 \ast_n \mu_2$ corresponds to $\tilde{\text{beta}}_{A,B}(2n, 2n)$ distribution.

**Proof.** From the proof of Theorem 2.1 that $W_j \overset{d}{=} AX_j + B(1 - X_j)$, where $X_1, X_2$ are independent identically distributed random variables having Beta($n, n$) distribution. The rest of the proof is just simple calculation. \hfill $\Box$

The property given by Theorem 3.1 is very similar to classical stability definition.

**Theorem 3.2.** Let $U_j, j = 1, \ldots, k$ be independent random variables having $\tilde{\text{beta}}$ distribution with parameters $\alpha_j = r_j + 1/2$, $\beta_j = r_j + 1/2$. Let $X_1, \ldots, X_{k-1}$ be a random vector having Dirichlet distribution with parameters $(r_1, \ldots, r_k)$. Then random variable

$$W = X_1 U_1 + \ldots + X_{k-1} U_{k-1} + (1 - \sum_{j=1}^{k-1} X_j) U_k$$

has $\tilde{\text{beta}}$ distribution with parameters $\left(\sum_{j=1}^{k} r_j + 1/2, \sum_{j=1}^{k} r_j + 1/2\right)$.

**Proof.** It is sufficient to calculate modified Stieltjes transform of the distribution of $W$ using some properties of Gauss-hypergeometric function. \hfill $\Box$

This property is also similar to classical stability property, but for the case of $k$-tuple operation.
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References

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