Opening up and control of spectral gaps of the Laplacian in periodic domains

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Abstract. The main result of this work is as follows: for arbitrary pairwise disjoint finite intervals $\alpha_j, \beta_j \subset [0, \infty), j = 1, \ldots, m$ and for arbitrary $n \geq 2$ we construct a family of periodic non-compact domains $\{\Omega^\epsilon \subset \mathbb{R}^n\}_{\epsilon > 0}$ such that the spectrum of the Neumann Laplacian in $\Omega^\epsilon$ has at least $m$ gaps when $\epsilon$ is small enough, moreover the first $m$ gaps tend to the intervals $(\alpha_j, \beta_j)$ as $\epsilon \to 0$. The constructed domain $\Omega^\epsilon$ is obtained by removing from $\mathbb{R}^n$ a system of periodically distributed “trap-like” surfaces.

Keywords: periodic domains, Neumann Laplacian, spectrum, gaps, asymptotic analysis, photonic crystals.

INTRODUCTION

The problem we are going to solve belongs to the spectral theory of periodic self-adjoint differential operators. It is known that usually the spectrum of such operators is a locally finite union of compact intervals called bands. In general the bands may overlap. The open interval $(\alpha, \beta) \subset \mathbb{R}$ is called a gap if it has an empty intersection with the spectrum, but its ends belong to it.

In general the presence of gaps is not guaranteed, for example, the spectrum of the Laplacian in $L_2(\mathbb{R}^d)$ has no gaps: $\sigma(-\Delta_{\mathbb{R}^d}) = [0, \infty)$. Therefore one of the central questions arising here is whether the gaps really exist in concrete situations. This question is motivated by various applications, in particular in connection with photonic crystals attracting much attention in recent years. Photonic crystals are periodic dielectric media in which electromagnetic waves of certain frequencies cannot propagate, which is caused by gaps in the spectrum of the Maxwell operator or related scalar operators.

We refer to paper [27] concerning mathematical problems arising in this field.

The problem of constructing of periodic operators with spectral gaps attracts a lot attention in the last twenty years. Various examples were presented in [11,13,15,18,20,24,37] for periodic divergence type elliptic operators in $\mathbb{R}^n$, in [17] for periodic Schrödinger operators, in [12,14] for Maxwell operators with periodic coefficients in $\mathbb{R}^n$, in [9,10,16,23,34] for Laplace-Beltrami operators on periodic Riemannian manifolds, in [32] for Laplacians in periodic domains in $\mathbb{R}^n$. We refer to overview [19] where these and other related questions are discussed in more details. Also we mention papers [1,2,6,7,31,33,36] devoted to the same problem for the operators posed in unbounded domains with a waveguide geometry (quantum waveguides).

The present paper is devoted to spectral analysis of the Neumann Laplacians in periodic domains. We denote by $\mathcal{H}_n$ the set of all domains $\Omega \subset \mathbb{R}^n$ satisfying the property

$$\exists d = d(\Omega) > 0 : \Omega = \Omega + dk, \forall k \in \mathbb{Z}^n$$

(i.e. $\Omega$ is periodic and the cube $(-d/2, d/2)^n$ is a periodic cell). Let $\Omega \in \mathcal{H}_n$ and $\mathcal{A}$ be the Neumann Laplacian in $\Omega$. Operators of this type occur in various areas of physics. For example in the case $n = 2$ the operator $\mathcal{A}$ governs the propagation of $H$-polarized electro-magnetic waves in a periodic dielectric medium with a perfectly conducting boundary. Below (see Remark 0.4) we discuss an application of our results to the theory of 2D photonic crystals.

The example of periodic domain with gaps in the spectrum of the Neumann Laplacian was presented in [32]. Here the authors considered the Neumann Laplacian in $\mathbb{R}^2$ perforated by $\mathbb{Z}^2$-periodic family of circular holes and proved that the gaps in its spectrum open up when the diameter of holes is close enough to the distance between their centers (the last one is fixed).
In the present work we want not only to construct a new type of periodic domains with gaps in the spectrum of the Neumann Laplacian but also be able to control the edges of these gaps making them close (in some natural sense) to predefined intervals. Let us formulate our main result.

**Theorem 0.1 (Main Theorem).** Let \( L > 0 \) be an arbitrarily large number and let \((\alpha_j, \beta_j) (j = 1, \ldots, m, m \in \mathbb{N})\) be arbitrary intervals satisfying
\[
0 < \alpha_1, \quad \alpha_j < \beta_j < \alpha_{j+1}, \quad j = 1, m - 1, \quad \alpha_m < \beta_m < L. \tag{0.1}
\]

Let \( n \in \mathbb{N} \setminus \{1\} \).

Then one can construct the family of domains \( \{\Omega^\varepsilon \in \mathcal{H}_n, \varepsilon > 0\} \) such that the spectrum of the Neumann Laplacian in \( \Omega^\varepsilon \) (we denote it \( \mathcal{A}^\varepsilon \)) has the following structure in the interval \([0, L]\) when \( \varepsilon \) is small enough:
\[
\sigma(\mathcal{A}^\varepsilon) \cap [0, L] = [0, L] \setminus \left( \bigcup_{j=1}^{m} (\alpha_j^\varepsilon, \beta_j^\varepsilon) \right), \tag{0.2}
\]
where the intervals \((\alpha_j^\varepsilon, \beta_j^\varepsilon)\) satisfy
\[
\forall j = 1, \ldots, m : \lim_{\varepsilon \to 0} \alpha_j^\varepsilon = \alpha_j, \quad \lim_{\varepsilon \to 0} \beta_j^\varepsilon = \beta_j. \tag{0.3}
\]

**Remark 0.1.** In work [8] Y. Colin de Verdière proved (among other results) the following statement: for arbitrary numbers \( 0 = \lambda_1 < \lambda_2 < \cdots < \lambda_m < \infty (m \in \mathbb{N}) \) there exists a bounded domain \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) such that the first \( m \) eigenvalues of the Neumann Laplacian in \( \Omega \) are exactly \( \lambda_1, \ldots, \lambda_m \). Our theorem can be regarded as an analogue of this result for the Neumann Laplacians in non-compact periodic domains.

Some preliminary results towards the proof of Theorem 0.1 were obtained by the author and E. Khruslov in [25] where the case \( m = 1 \) was considered. However the general case \( m \geq 2 \) is much more complicated. Similar results for the Laplace-Beltrami operators on periodic Riemannian manifolds without a boundary and for elliptic operators in the entire space \( \mathbb{R}^n \) were obtained by the author in [23] and [24] correspondingly.

![Fig. 1. The system of screens \( S^\varepsilon_{ij} \). Here \( m = 2 \).](image-url)
Finally we will prove (see Lemma 1.1) that for an arbitrary interval \([0, L]\) the spectrum of the operator \(A^e\) has exactly \(m\) gaps in \([0, L]\) when \(\varepsilon\) is small enough. Moreover when \(\varepsilon \to 0\) these gaps converge to some intervals \((\sigma_j, \mu_j)\) \((j = 1, \ldots, m)\) depending in a special way on the domains \(B_j\) and the numbers \(d_j\) and satisfying

\[
0 < \sigma_1 < \sigma_j < \mu_j < \sigma_{j+1}, \quad j = 1, m-1, \quad \sigma_m < \mu_m.
\]

Finally we will prove (see Lemma 1.1) that for an arbitrary intervals \((\alpha_j, \beta_j), j = 1, \ldots, m\) satisfying (0.1) one can choose \(B_j\) and \(d_j\) in such a way that the equalities

\[
\sigma_j = \alpha_j, \quad \mu_j = \beta_j, \quad j = 1, \ldots, m
\]

hold. For the volumes of the sets \(B_j\) and for the numbers \(d_j\) we will present the exact formulae. It is clear that the main theorem follows directly from Theorem 1.1 and Lemma 1.1.

**Remark 0.2.** The idea how to construct the domain \(\Omega^e\) is close to the idea which was used in [24], where the operator \(- (b^e)^{-1} \text{div}(a^e \nabla)\) in \(\mathbb{R}^n\) was studied. In this work the role of ”traps” is played by the family of thin spherical shells which are \(\varepsilon \mathbb{Z}^n\)-periodically distributed in \(\mathbb{R}^n\) and on which \(a^e(x)\) becomes small as \(\varepsilon \to 0\). A similar idea was also used in [23] where the periodic Laplace-Beltrami operator was studied.

The analysis of the asymptotic behaviour of spectra was carried out in [23, 24] using the methods of the homogenization theory. The idea to use this theory in order to open up the gaps in the spectrum of periodic differential operators was firstly proposed in [47]. Since the proof in [23, 24] is rather cumbersome, in the present work we carry out the analysis using another method (see the next remark). On the other hand the results of [23, 24] helped us to guess the form of the equation (1.5) below whose roots are the limits of the right ends of the spectral bands.

Boundary value problems in domains with ”traps” were also considered in [34], where the authors studied the homogenization of semi-linear parabolic equations and their attractors. Similar homogenization problems were studied in [30].

**Remark 0.3.** Let us briefly describe the scheme of the proof of Theorem 1.1. We enclose the left end (resp. the right end) of the \(k\)-th band between the \(k\)-th eigenvalues of the Neumann and periodic (resp. the antiperiodic and Dirichlet) Laplacians posed on the periodicity cell. We prove that both ends of this enclosure converge to \(\mu_{k-1}\) if \(k = 2, \ldots, m + 1\) and to infinity if \(k > m + 1\) (resp. converge to \(\sigma_k\) if \(k = 1, \ldots, m\) and to infinity if \(k > m\)) as \(\varepsilon \to 0\).

The most difficult part of the proof is the investigation of the asymptotic behaviour of the eigenvalues of the Neumann Laplacian (see Theorem 2.2). To obtain the asymptotics of eigenvalues we will construct convenient approximations for the corresponding eigenfunctions. The analysis of the
eigenvalues of the Dirichlet Laplacian (see Theorem 2.3) is carried out using the same ideas but it is essentially simpler. The analysis of the eigenvalues of the periodic (resp. antiperiodic) Laplacian repeats word-by-word the analysis for the eigenvalues of the Neumann (resp. Dirichlet) Laplacian.

Remark 0.4. The obtained results can be applied in the theory of 2D photonic crystals. Let us introduce the following sets in $\mathbb{R}^2$:

$$\Omega^\varepsilon = \{(x_1, x_2, z) \in \mathbb{R}^3 : x = (x_1, x_2) \in \Omega^\varepsilon, \ z \in \mathbb{R}\}, \quad S^\varepsilon = \mathbb{R}^3 \setminus \Omega^\varepsilon,$$

where $\Omega^\varepsilon \subset \mathbb{R}^2$ is defined above. We suppose that $\Omega^\varepsilon$ is occupied by a dielectric medium whereas the union of the screens $S^\varepsilon$ is occupied by a perfectly conducting material. It is supposed that the electric permittivity and the magnetic permeability of the material occupying $\Omega^\varepsilon$ are equal to 1.

The propagation of electromagnetic waves in $\Omega^\varepsilon$ is governed by the Maxwell operator $\mathcal{M}^\varepsilon$ (below by $E$ and $H$ we denote the electric and magnetic fields, $U = (E, H)$)

$$\mathcal{M}^\varepsilon U = (-i \text{curl}H, i \text{curl}E)$$

subject to the conditions

$$\text{div} E = \text{div} H = 0 \text{ in } \Omega^\varepsilon, \quad E_r = 0, \ H_r = 0 \text{ on } S^\varepsilon.$$

Here $E_r$ and $H_r$ are the tangential and normal components of $E$ and $H$, correspondingly. We are interested only on the waves propagated along the plane $z = 0$, i.e. when $E, H$ depends on $x_1, x_2$ only.

It is known that if the medium is periodic in two directions and homogeneous with respect to the third one (so-called 2D crystals) then the analysis of the Maxwell operator reduces to the analysis of scalar elliptic operators. Let us formulate this statement more precisely. We denote

$$J = \{(E, H) : \text{div} E = \text{div} H = 0 \text{ in } \Omega^\varepsilon, \ E_r = 0, \ H_r = 0 \text{ on } S^\varepsilon\},$$

$$J_E = \{(E, H) \in J : E_1 = E_2 = H_3 = 0\}, \quad J_H = \{(E, H) \in J : H_1 = H_2 = E_3 = 0\}.$$

The elements of the subspaces $J_E$ and $J_H$ are usually called $E$- and $H$-polarized waves. The subspaces $J_E$ and $J_H$ are $L_2$-orthogonal and each $U \in J$ can be represented in unique way as $U = U_E + U_H$, where $U_E \in J_E$, $U_H \in J_H$. Moreover $J_E$ and $J_H$ are invariant subspaces of $\mathcal{M}^\varepsilon$. Thus $\sigma(\mathcal{M}^\varepsilon)$ is a union of $\sigma(\mathcal{M}^\varepsilon|_{J_E})$ ($E$-subspectrum) and $\sigma(\mathcal{M}^\varepsilon|_{J_H})$ ($H$-subspectrum).

We denote by $\mathcal{A}^\varepsilon_0$ and $\mathcal{A}^\varepsilon$ the Dirichlet and the Neumann Laplacians in $\Omega^\varepsilon$, correspondingly. It can be easily shown on a formal level of rigour (see, e.g. [21]) that $\omega \in \sigma(\mathcal{M}^\varepsilon|_{J_E}) \iff \omega^2 \in \sigma(\mathcal{A}^\varepsilon_0)$ and $\omega \in \sigma(\mathcal{M}^\varepsilon|_{J_H}) \iff \omega^2 \in \sigma(\mathcal{A}^\varepsilon)$. Using Friedrichs type inequalities one can easily prove (see [25] Lemma 3.1) that $(\mathcal{A}^\varepsilon_0 u, u)_{L_2(\Omega^\varepsilon)} \geq a\varepsilon^{-2}||u||_{L_2(\Omega^\varepsilon)}^2, \ \forall u \in \text{dom}(\mathcal{A}^\varepsilon_0)$ (here $a > 0$ is a constant) and therefore

$$\min\{\lambda : \lambda \in \sigma(\mathcal{A}^\varepsilon_0)\} \to \infty, \quad \text{as } \varepsilon \to 0.$$  

(0.5)

Then using Theorem 1.1 Lemma 1.1 and (0.5) we conclude that for an arbitrarily large $L > 0$ the Maxwell operator $\mathcal{M}^\varepsilon$ has $2m$ gaps in $[-L, L]$ when $\varepsilon$ is small enough and as $\varepsilon \to 0$ these gaps converge to intervals $\pm\sqrt{\sigma_j}$, which can be controlled via a suitable choice of $B_j$ and $d_j$.

1. Construction of the family $(\Omega^\varepsilon)_\varepsilon$ and main results

Let $\varepsilon > 0$ be a small parameter and let $n \in \mathbb{N} \setminus \{1\}$. Let $B_j, \ j = 1, \ldots, m$ be arbitrary open domains with Lipschitz boundaries satisfying the following conditions:

(b1) $B_j \cap B_k = \emptyset$ for $j \neq k$,

(b2) $\bigcup_{j=1}^m \overline{B_j} \subset Y$, where

$$Y = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| < 1/2, \ \forall i\},$$
(b₃) for any \( j = 1, \ldots, m \) the boundary of \( \partial B_j \) has a flat subset, namely
\[
\exists \tilde{x} \in \partial B_j^\varepsilon, \exists r_j > 0 : \text{the set } B_j^\varepsilon(\tilde{x}) \cap \partial B_j \text{ belongs to a } (n - 1)\text{-dimensional hyperplain.}
\]
Here \( B_r(x) \) we denote the ball with the center at the point \( x \) and the radius \( r \).

For \( j = 1, \ldots, m \) we denote
- \( D_j^\varepsilon = \{ x \in \partial B_j^\varepsilon : |x - \tilde{x}| < d_j^\varepsilon \} \), where \( d_j^\varepsilon \) is defined by the following formula:
\[
d_j^\varepsilon = \begin{cases} 
  d_j e^{\frac{\tilde{x}^2}{2}}, & n > 2, \\
  e^{-1} \exp \left( -\frac{1}{d_j^\varepsilon} \right), & n = 2.
\end{cases}
\]
Here \( d_j, j = 1, \ldots, m \) are positive constants. It is supposed that \( \varepsilon \) is small enough so that \( d_j^\varepsilon < r_j \).
- \( S_j^\varepsilon = \partial B_j \setminus \left( \bigcup_{j=1}^{m} D_j^\varepsilon \right) \).

Finally we set
\[
\Omega^\varepsilon = \mathbb{R}^n \setminus \bigcup_{j=1}^{m} S_j^\varepsilon, \text{ where } S_j^\varepsilon = \varepsilon(S_j^\varepsilon + i).
\]

Let us define precisely the Neumann Laplacian \( \mathcal{A}^\varepsilon \) in \( \Omega^\varepsilon \). We denote by \( \eta^\varepsilon[u, v] \) the sesquilinear form in \( L_2(\Omega^\varepsilon) \) which is defined by the formula
\[
\eta^\varepsilon[u, v] = \int_{\Omega^\varepsilon} \langle \nabla u, \nabla v \rangle \, dx \tag{1.1}
\]
and the definitional domain \( \text{dom}(\eta^\varepsilon) = H^1(\Omega^\varepsilon) \). Here \( \langle \nabla u, \nabla v \rangle = \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_k} \). The form \( \eta^\varepsilon[u, v] \) is densely defined closed and positive. Then (see, e.g., [22, Chapter 6, Theorem 2.1]) there exists the unique self-adjoint and positive operator \( \mathcal{A}^\varepsilon \) associated with the form \( \eta^\varepsilon \), i.e.
\[
(\mathcal{A}^\varepsilon u, v)_{L_2(\Omega^\varepsilon)} = \eta^\varepsilon[u, v], \quad \forall u \in \text{dom}(\mathcal{A}^\varepsilon), \quad \forall v \in \text{dom}(\eta^\varepsilon). \tag{1.2}
\]

We denote by \( \sigma(\mathcal{A}^\varepsilon) \) the spectrum of \( \mathcal{A}^\varepsilon \). To describe the behaviour of \( \sigma(\mathcal{A}^\varepsilon) \) as \( \varepsilon \to 0 \) we need some additional notations.

In the case \( n > 2 \) we denote by \( \kappa \) the capacity of the disc
\[
T = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : |x| < 1, \ x_n = 0 \}
\]
Recall (see, e.g., [28]) that it is defined by
\[
\kappa = \inf_{w} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx,
\]
where the infimum is taken over smooth and compactly supported in \( \mathbb{R}^n \) functions equal to 1 on \( T \).

We set (below \( j = 1, \ldots, m \))
\[
\sigma_j = \begin{cases} 
  \frac{\kappa d_j^{n-2}}{4 b_j}, & n > 2, \\
  \frac{\pi d_j}{2 b_j}, & n = 2.
\end{cases} \tag{1.3}
\]
where $b_j$ is the volume of the domain $B_j$. We assume that the numbers $d_j$ and $b_j$ are such that

$$\sigma_j < \sigma_{j+1}, \ j = 1, \ldots, m - 1.$$  

(1.4)

Let us consider the following equation (with unknown $\lambda \in \mathbb{C}$):

$$1 + \sum_{j=1}^{m} \frac{\sigma_j b_j}{(1 - \sum_{i=1}^{m} b_i)(\sigma_j - \lambda)} = 0.$$  

(1.5)

It is easy to show (see [23, Subsect. 3.2]) that if (1.4) holds then equation (1.5) has exactly $m$ roots, they are real and interlace with $\sigma_j$. We denote them $\mu_j, \ j = 1, \ldots, m$ supposing that they are renumbered in the increasing order, i.e.

$$\sigma_j < \mu_j < \sigma_{j+1}, \ j = 1, \ldots, m - 1, \ \sigma_m < \mu_m < \infty.$$  

(1.6)

Now we can formulate the main result on the behaviour of $\sigma(\mathcal{A}^\varepsilon)$ as $\varepsilon \to 0$.

**Theorem 1.1.** Let $L$ be an arbitrary number satisfying $L > \mu_m$. Then the spectrum $\sigma(\mathcal{A}^\varepsilon)$ of the operator $\mathcal{A}^\varepsilon$ has the following structure in $[0, L]$ when $\varepsilon$ is small enough:

$$\sigma(\mathcal{A}^\varepsilon) \cap [0, L] = [0, L] \setminus \left( \bigcup_{j=1}^{m} (\sigma_j^\varepsilon, \mu_j^\varepsilon) \right),$$  

(1.7)

where the intervals $(\sigma_j^\varepsilon, \mu_j^\varepsilon)$ satisfy

$$\forall j = 1, \ldots, m : \ \lim_{\varepsilon \to 0} \sigma_j^\varepsilon = \sigma_j, \ \lim_{\varepsilon \to 0} \mu_j^\varepsilon = \mu_j.$$  

(1.8)

Theorem 1.1 shows that $\sigma(\mathcal{A}^\varepsilon)$ has exactly $m$ gaps when $\varepsilon$ is small enough and when $\varepsilon \to 0$ these gaps converge to the intervals $(\sigma_j, \mu_j)$. Now, our goal is to find such numbers $d_j$ and domains $B_j$ that the corresponding intervals $(\sigma_j, \mu_j)$ coincide with the predefined ones.

We use the notations $d = (d_1, \ldots, d_m)$, $b = (b_1, \ldots, b_m)$, $\sigma = (\sigma_1, \ldots, \sigma_m)$, $\mu = (\mu_1, \ldots, \mu_m)$. Let

$$\mathcal{L} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m, \ (d, b) \xrightarrow{\mathcal{L}} (\sigma, \mu)$$

be the map with the definitional domain

$$\text{dom}(\mathcal{L}) = \left\{(d, b) \in \mathbb{R}^m \times \mathbb{R}^m : d_j > 0, b_j > 0, \sum_{j=1}^{m} b_j < 1 \text{ and (1.4) holds} \right\}$$

and acting according to formulae (1.3), (1.5), (1.6) (i.e. $\sigma_j$ are defined by (1.3) and $\mu_j$ are the roots of equation (1.5) renumbered according to (1.6)).

**Lemma 1.1.** The map $\mathcal{L}$ maps $\text{dom}(\mathcal{L})$ onto the set

$$\mathcal{G} = \left\{(\sigma, \mu) \in \mathbb{R}^m \times \mathbb{R}^m : \sigma_j < \mu_j < \sigma_{j+1}, \ j = 1, \ldots, m - 1, \ \sigma_m < \mu_m < \infty \right\}.$$
Moreover \( L \) is one-to-one and the inverse map \( L^{-1} \) is given by the following formulae:

\[
d_j = \begin{cases} 
\frac{4\sigma_j \rho_j}{\kappa (1 + \sum_{i=1}^m \rho_i)} & , \quad n > 2, \\
\frac{2\sigma_j \rho_j}{\pi (1 + \sum_{i=1}^m \rho_i)} & , \quad n = 2, \\
\frac{\rho_j}{1 + \sum_{i=1}^m \rho_i} & 
\end{cases}
\quad (1.9)
\]

where

\[
\rho_j = \frac{\mu_j - \sigma_j}{\sigma_j} \prod_{i=1, i\neq j}^m \left( \frac{\mu_i - \sigma_j}{\sigma_j - \sigma_i} \right).
\quad (1.11)
\]

**Proof.** Let \((\sigma, \mu)\) be an arbitrary element of \( \mathcal{G} \). We have to show that

\[
\exists! (d, b) \in \text{dom}(L) \text{ such that } \forall j = 1, \ldots, m \begin{cases} 
(1.3) \text{ holds}, \\
(1.5) \text{ holds with } \lambda = \mu_j,
\end{cases}
\]

moreover, this \((d, b)\) is defined by formulae \((1.9)-(1.11)\).

At first we find \( b_1, \ldots, b_m \). Let us consider the following system of \( m \) linear equations with respect to unknowns \( \rho_1, \ldots, \rho_m \):

\[
1 + \sum_{j=1}^m \frac{\sigma_j \rho_j}{\sigma_j - \mu_j} = 0, \quad j = 1, \ldots, m.
\quad (1.12)
\]

It is proved in [23] Lemma 4.1 that this system has the unique solution which is defined by formula \((1.11)\). Therefore in view of \((1.5)\) in order to find \( b_j \) we need to solve the following system:

\[
b_j (1 - \sum_{i=1}^m b_i)^{-1} = \rho_j, \quad j = 1, \ldots, m.
\]

It is clear that it has the unique solution \( b_1, \ldots, b_m \) which is defined by \((1.10)\). Since \((\sigma, \mu) \in \mathcal{G} \) then

\[
\forall j : \mu_j > \sigma_j; \quad \forall i \neq j : \text{ sign}(\mu_i - \sigma_j) = \text{ sign}(\sigma_i - \sigma_j) \neq 0
\]

and hence \( \rho_j > 0 \). Therefore \( b_j > 0 \) and \( \sum_{j=1}^m b_j < 1 \).

Finally knowing \( b_j \) we express \( d_j \) from \((1.3)\) and obtain the formula \((1.9)\). The lemma is proved. \( \Box \)

Now, Theorem 0.1 follows from directly from Theorem 1.1 and Lemma 1.1. Indeed, let \((\alpha_j, \beta_j), \quad j = 1, \ldots, m \) be arbitrary intervals satisfying \((0.1)\) (and therefore by Lemma 1.1 \((\alpha, \beta) \in \text{image}(L)\)). We define the numbers \( d_j, b_j \) by formulae \((1.9)-(1.10)\) with \( \alpha_j, \beta_j \) instead of \( \sigma_j, \mu_j \). For the obtained numbers \( b_j \) we construct the domains \( B_1, \ldots, B_m \) satisfying \((b_1) - (b_2)\) and such that

\[
|B_j| = b_j \quad \text{for } j = 1, \ldots, m
\quad (1.13)
\]

(it is easy to do, see example below for one of possible constructions). Finally using the domains \( B_1, \ldots, B_m \) and the numbers \( d_1, \ldots, d_m \) we construct the family of periodic domains \( \{\Omega^e\}_e \). In view of Theorem 1.1 the corresponding family of Neumann Laplacians \( \{\mathcal{A}^e\}_e \) satisfies \((0.2)-(0.3)\).
Example 1.1. Let $b_j$, $j = 1, \ldots, m$ be arbitrary numbers satisfying

$$b_j > 0, \ j = 1, \ldots, m \text{ and } \sum_{j=1}^{m} b_j < 1.$$ 

We present one of the possible choices of the domains $B_j$ satisfying $(b_1) - (b_3)$ and $(1.13)$.

We denote:

$$l = \left(\frac{1}{2} + \frac{1}{2} \sum_{i=1}^{m} b_i\right)^{1/n}, \quad \hat{l} = \frac{1}{2(n-1)^{l-1}} \left(1 - \sum_{i=1}^{m} b_i\right), \quad l_j = \frac{b_j}{|B|}.$$ 

Finally we define the domains $B_j$, $j = 1, \ldots, m$ by the following formula:

$$B_j = \left\{x \in \mathbb{R}^n : x_1 \in \left(-\frac{l}{2} + (j-1)\hat{l} + \sum_{i=1}^{j-1} l_i, -\frac{l}{2} + (j-1)\hat{l} + \sum_{i=1}^{j} l_i\right), \ |x_1| < \frac{l}{2}, \ k = 2, \ldots, n\right\}$$

It is easy to show that these domains satisfy conditions $(b_1) - (b_3)$ and $(1.13)$.

2. Proof of Theorem 1.1

2.1. Preliminaries. We present the proof of Theorem 1.1 for the case $n \geq 3$ only. For the case $n = 2$ the proof is repeated word-by-word with some small modifications.

In what follows by $C, C_1$... we denote generic constants that do not depend on $\varepsilon$.

Let $B$ be an open domain in $\mathbb{R}^n$. By $(u)_B$ we denote the mean value of the function $v(x)$ over the domain $B$, i.e.

$$(u)_B = \frac{1}{|B|} \int_B u(x)dx.$$ 

Here by $|B|$ we denote the volume of the domain $B$.

If $\Sigma \subset \mathbb{R}^n$ is a $(n-1)$-dimensional surface then the Euclidean metrics in $\mathbb{R}^n$ induces on $\Sigma$ the Riemannian metrics and measure. We denote by $ds$ the density of this measure. Again by $(u)_\Sigma$ we denote the mean value of the function $u$ over $\Sigma$, i.e.

$$(u)_\Sigma = \frac{1}{|\Sigma|} \int_{\Sigma} u ds,$$ 

where $|\Sigma| = \int_{\Sigma} ds$.

We introduce the following sets:

$$Y^\varepsilon = Y \setminus \bigcup_{j=1}^{m} S_j^\varepsilon.$$ 

By $A^\varepsilon$ we denote the Neumann Laplacian in $\varepsilon^{-1}\Omega^\varepsilon$. It is clear that

$$\sigma(A^\varepsilon) = \varepsilon^{-2} \sigma(A^\varepsilon).$$ 

(2.1)

It is more convenient to deal with the operator $A^\varepsilon$ since the external boundary of its period cell is fixed (it coincides with $\partial Y$).

In view of the periodicity of $A^\varepsilon$ the analysis of its spectrum $\sigma(A^\varepsilon)$ reduces to the analysis of the spectrum of the Laplace operator on $Y^\varepsilon$ with the Neumann boundary conditions on $\bigcup_{j=1}^{m} S_j^\varepsilon$ and quasi-periodic boundary (or $\theta$-periodic) boundary conditions on $\partial Y$. Namely, let

$$T^n = \{\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{C}^n : |\theta_k| = 1, \ \forall k\}.$$
For $\theta \in \mathbb{T}^m$ we introduce the functional space $H^1_0(Y^\theta)$ consisting of functions from $H^1(Y^\theta)$ that satisfy the following condition on $\partial Y$:
\[ \forall k = 1, n : \quad u(x + e_k) = \theta_k u(x) \quad \text{for} \quad x = (x_1, x_2, \ldots, -1/2, \ldots, x_n), \quad (2.2) \]

where $e_k = (0, 0, \ldots, 1, \ldots, 0)$.

By $\eta^{\theta, \varepsilon}$ we denote the sesquilinear form defined by formula (1.1) (with $Y^\theta$ instead of $\Omega$) and the definititional domain $H^1_0(Y^\theta)$. We define the operator $A^{\theta, \varepsilon}$ as the operator acting in $L_2(Y^\theta)$ and associated with the form $\eta^{\theta, \varepsilon}$, i.e.
\[ (A^{\theta, \varepsilon} u, v)_{L_2(Y^\theta)} = \eta^{\theta, \varepsilon} [u, v], \quad \forall u \in \text{dom}(A^{\theta, \varepsilon}), \; \forall v \in \text{dom}(\eta^{\theta, \varepsilon}). \]

The functions from $\text{dom}(A^{\theta, \varepsilon})$ satisfy the Neumann boundary conditions on $\bigcup_{j=1}^m S^\varepsilon_j$, condition (2.2) on $\partial Y$ and the condition
\[ \forall k = 1, n : \quad \frac{\partial u}{\partial x_k}(x + e_k) = \theta_k \frac{\partial u}{\partial x_k}(x) \quad \text{for} \quad x = (x_1, x_2, \ldots, -1/2, \ldots, x_n). \quad (2.3) \]

The operator $A^{\theta, \varepsilon}$ has purely discrete spectrum. We denote by $\{\lambda^{\theta, \varepsilon}_k\}_{k \in \mathbb{N}}$ the sequence of eigenvalues of $A^{\theta, \varepsilon}$ written in the increasing order and repeated according to their multiplicity.

The Floquet-Bloch theory (see, e.g., [5, 26, 33]) establishes the following relationship between the spectra of the operators $A^\varepsilon$ and $A^{\theta, \varepsilon}$:
\[ \sigma(A^\varepsilon) = \bigcup_{k=1}^\infty L_k, \quad \text{where} \quad L_k = \bigcup_{\theta \in \mathbb{T}^m} \{\lambda^{\theta, \varepsilon}_k\}. \quad (2.4) \]

The sets $L_k$ are compact intervals.

Also we need the Laplace operators on $Y^\varepsilon$ with the Neumann boundary conditions on $\bigcup_{j=1}^m S^\varepsilon_j$ and either Neumann or Dirichlet boundary conditions on $\partial Y$. Namely, we denote by $\eta^{N, \varepsilon}$ (resp. $\eta^{D, \varepsilon}$) the sesquilinear form in $L_2(Y^\varepsilon)$ defined by formula (1.1) (with $Y^\varepsilon$ instead of $\Omega^\varepsilon$) and the definitional domain $H^1_0(Y^\varepsilon)$ (resp. $\tilde{H}^1_0(Y^\varepsilon) = \{u \in H^1(Y^\varepsilon) : u = 0 \; \text{on} \; \partial Y\}$). Then by $A^{N, \varepsilon}$ (resp. $A^{D, \varepsilon}$) we denote the operator associated with the form $\eta^{N, \varepsilon}$ (resp. $\eta^{D, \varepsilon}$), i.e.
\[ (A^{\ast, \varepsilon} u, v)_{L_2(Y^\varepsilon)} = \eta^{\ast, \varepsilon} [u, v], \quad \forall u \in \text{dom}(A^{\ast, \varepsilon}), \; \forall v \in \text{dom}(\eta^{\ast, \varepsilon}), \]

where $\ast = N$ (resp. $D$).

The spectra of the operators $A^{N, \varepsilon}$ and $A^{D, \varepsilon}$ are purely discrete. We denote by $\{\lambda^{N, \varepsilon}_k\}_{k \in \mathbb{N}}$ (resp. $\{\lambda^{D, \varepsilon}_k\}_{k \in \mathbb{N}}$) the sequence of eigenvalues of $A^{N, \varepsilon}$ (resp. $A^{D, \varepsilon}$) written in the increasing order and repeated according to their multiplicity.

Using the min-max principle (see, e.g., [35]) and the enclosure $H^1(Y^\varepsilon) \supset H^1_0(Y^\varepsilon) \supset \tilde{H}^1_0(Y^\varepsilon)$ one can easily prove the inequality
\[ \forall k \in \mathbb{N}, \; \forall \theta \in \mathbb{T}^m : \quad \lambda^{N, \varepsilon}_k \leq \lambda^{\theta, \varepsilon}_k \leq \lambda^{D, \varepsilon}_k. \quad (2.5) \]

2.2. Number-by-number convergence of eigenvalues of the Dirichlet, Neumann and $\theta$-periodic Laplacians. We denote
\[ B_{m+1} = Y \setminus \bigcup_{j=1}^m B_j. \]

By $\Delta_{B_j}$, $j = 1, \ldots, m$ we denote the operator which acts in $L_2(B_j)$ and is defined by the operation $\Delta$ and the Neumann boundary conditions on $\partial B_j$. By $\Delta^{N}_{B_{m+1}}$ (resp. $\Delta^{D}_{B_{m+1}}, \Delta^{\theta}_{B_{m+1}}$) we denote the operator which
acts in $L_2(B_{m+1})$ and is defined by the operation $\Lambda$, the Neumann boundary conditions on $\bigcup_{j=1}^m \partial B_j$ and the Neumann (resp. Dirichlet, $\theta$-periodic) boundary conditions on $\partial Y$. Finally, we introduce the operators $A^N, A^D, A^\theta$ which act in $\bigoplus_{j=1}^m L_2(B_j)$ and are defined by the following formulae:

$$A_N = \begin{pmatrix} \Delta B_1 & 0 & \cdots & 0 \\ 0 & \Delta B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta B_m \end{pmatrix}, \quad A_D = \begin{pmatrix} \Delta B_1 & 0 & \cdots & 0 \\ 0 & \Delta B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta B_m \end{pmatrix}, \quad A_\theta = \begin{pmatrix} \Delta B_1 & 0 & \cdots & 0 \\ 0 & \Delta B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta B_m \end{pmatrix}.$$  

We denote by $\{\lambda^N_k\}_{k \in \mathbb{N}}$ (resp. $\{\lambda^D_k\}_{k \in \mathbb{N}}, \{\lambda^\theta_k\}_{k \in \mathbb{N}}$) the sequence of eigenvalues of $A^N$ (resp. $A^D, A^\theta$) written in the increasing order and repeated according to their multiplicity. It is clear that

$$\lambda_1^N = \lambda_2^N = \cdots = \lambda_{m+1}^N = 0, \quad \lambda_{m+2}^N > 0,$$

(2.6)

$$\lambda_1^D = \lambda_2^D = \cdots = \lambda_{m+1}^D = 0, \quad \lambda_{m+2}^D > 0,$$

(2.7)

$$\lambda_1^\theta = \lambda_2^\theta = \cdots = \lambda_{m+1}^\theta = 0, \quad \lambda_{m+2}^\theta > 0 \text{ if } \theta = (1, 1, \ldots, 1),$$

(2.8)

$$\lambda_1^\theta = \lambda_2^\theta = \cdots = \lambda_{m+1}^\theta = 0, \quad \lambda_{m+2}^\theta > 0 \text{ if } \theta \neq (1, 1, \ldots, 1).$$

(2.9)

**Theorem 2.1.** For each $k \in \mathbb{N}$ one has

$$\lim_{\varepsilon \to 0} \lambda^N_k = \lambda^N_k,$$

(2.10)

$$\lim_{\varepsilon \to 0} \lambda^D_k = \lambda^D_k,$$

(2.11)

$$\lim_{\varepsilon \to 0} \lambda^\theta_k = \lambda^\theta_k \quad (\forall \theta \in T^m).$$

(2.12)

For the case $m = 1$ Theorem 2.1 was proved in [25]. For $m > 1$ the proof is similar.

2.3. **Asymptotics of the first $m$ non-zero eigenvalues of the Neumann Laplacian.** We get more complete information about the behaviour of $\lambda^N_{k,\varepsilon}, k = 2, \ldots, m+1$ (it is clear that $\lambda^N_1 = 0$).

**Theorem 2.2.** For $k = 2, \ldots, m+1$ one has

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} \lambda^N_{k,\varepsilon} = \mu_{k-1}.$$  

(2.13)

**Proof.** Let $u^\varepsilon_k, k \in \mathbb{N}$ be the eigenfunctions corresponding to $\lambda^N_{k,\varepsilon}$ and satisfying the conditions

$$(u^\varepsilon_k, u^\varepsilon_l)_{L_2(Y^\varepsilon)} = \delta kl,$$

(2.14)

$u^\varepsilon_k$ are real functions

(here $\delta kl$ is the Kronecker delta). It is clear that $u^\varepsilon_1 = \pm 1$.

By the min-max principle (see, e.g. [35]) we get

$$\forall k \in \mathbb{N} : \lambda^N_k = \min_{u \in H(u^\varepsilon_1, \ldots, u^\varepsilon_{k-1})} \frac{\|\nabla u\|^2_{L_2(Y^\varepsilon)}}{\|u\|^2_{L_2(Y^\varepsilon)}},$$

(2.15)

where

$$H(u^\varepsilon_1, \ldots, u^\varepsilon_{k-1}) = \{u \in H^1(Y^\varepsilon) : (u, u^\varepsilon_l)_{L_2(Y^\varepsilon)}, l = 1, \ldots, k-1\}.$$  

(2.16)

Using the Cauchy inequality we get the estimate

$$\sum_{j=1}^{m+1} \left( (u^\varepsilon_k)_{B_j} \right)^2 \leq C.$$
and therefore there exist a subsequence (for convenience still denoted by \( \varepsilon \)) and numbers \( s_j^k \in \mathbb{R}, k \in \mathbb{N}, j = 1, \ldots, m + 1 \) such that
\[
\lim_{\varepsilon \to 0} \langle u^\varepsilon \rangle_{B_j} = s_j^k.
\] (2.17)

We denote \( s^k = (s_1^k, \ldots, s_{m+1}^k) \in \mathbb{R}^{m+1} \).

During the proof we will use the function \( F : \mathbb{R}^{m+1} \to [0, \infty) \) defined by the formula (below \( s = (s_1, \ldots, s_{m+1}) \in \mathbb{R}^{m+1} \))
\[
F(s) = \sum_{j=1}^{m} \sigma_j b_j (s_{m+1} - s_j)^2.
\] (2.18)

Also we will use the function \( U^e : Y^e \times \mathbb{R}^{m+1} \to \mathbb{R} \) which is defined by the following formula (below \( x \in Y^e, s = (s_1, \ldots, s_{m+1}) \in \mathbb{R}^{m+1} \)):
\[
U^e(x, s) = \begin{cases} 
    s_j - \frac{s_j - s_{m+1}}{2 \psi \left( \mathcal{R}_j^{-1} \left( \frac{x - \tilde{x}_j}{d_j^e} \right) \right) \varphi \left( \frac{|x - \tilde{x}_j|}{r} \right)} & \text{if } x \in B_j, j = 1, \ldots, m, \\
    s_{m+1} + \sum_{i=1}^{m} \frac{s_i - s_{m+1}}{2} \psi \left( \mathcal{R}_j^{-1} \left( \frac{x - \tilde{x}_j}{d_i^e} \right) \right) \varphi \left( \frac{|x - \tilde{x}_j|}{r} \right) & \text{if } x \in B_{m+1}.
\end{cases}
\] (2.19)

Here \( \varphi : \mathbb{R} \to \mathbb{R} \) is a twice-continuously differentiable function satisfying
\[
\varphi(\rho) = 1 \text{ as } \rho \leq 1/4 \text{ and } \varphi(\rho) = 0 \text{ as } \rho \geq 1/2,
\] (2.20)
r is an arbitrary positive constant satisfying the conditions
\[
r < \min_{j=1,m} r_j, \quad B_r(\tilde{x}_j) \cap \left( \partial Y \bigcup_{i \neq j} B_i \right) = \emptyset.
\] (2.21)

\( \mathcal{R}_j : \mathbb{R}^n \to \mathbb{R}^n (j = 1, \ldots, m) \) is the operator of rotation mapping the disc \( T \) onto a set which is parallel to a flat part of \( \partial B_j \) containing \( \tilde{x}_j \); \( \psi \) is a solution of the following problem:
\[
\Delta \psi = 0 \text{ in } \mathbb{R}^n \setminus \overline{T},
\] (2.22)
\[
\psi = 1 \text{ in } \partial T,
\] (2.23)
\[
\psi(x) = o(1) \text{ as } |x| \to \infty.
\] (2.24)
(recall that \( T = \{ x \in \mathbb{R}^n : |x| < 1, x_n = 0 \} \), obviously \( \partial T = \overline{T} \)). It is well-known that problem (2.22)-(2.24) has the unique solution \( \psi(x) \) satisfying
\[
\int_{\mathbb{R}^n \setminus T} |
\nabla \psi|^2 \, dx < \infty.
\]

Using a standard regularity theory it is easy to prove that \( \psi(x) \) has the following properties:
\[
\psi \in C^\infty(\mathbb{R}^n \setminus \overline{T}),
\] (2.25)
\[
\psi(x_1, x_2, \ldots, x_{n-1}, 0) = \psi(x_1, x_2, \ldots, x_{n-1}, -x_n).
\] (2.26)

These properties imply:
\[
\frac{\partial \psi}{\partial x_n} = 0 \text{ in } \{ x \in \mathbb{R}^n : x_n = 0 \} \setminus \overline{T},
\] (2.27)
\[
\frac{\partial \psi}{\partial x_n}_{|x_n=0} + \frac{\partial \psi}{\partial x_n}_{|x_n=-0} = 0 \text{ in } T.
\] (2.28)

Furthermore the function \( \psi(x) \) satisfies the estimate (see, e.g., \cite{30} Lemma 2.4):
\[
|D^a \psi(x)| \leq C|x|^{2-n-a} \text{ for } |x| > 2, |a| = 0, 1, 2.
\] (2.29)
And finally one has the following equality:

\[ \kappa = \int_{\mathbb{R}^n \setminus T} |\nabla \psi|^2 \, dx. \]  

(2.30)

It is easy to see that for \( \varepsilon \) small enough (namely, when \( \max_j d_j^e < r/4 \)) and for any \( s \in \mathbb{R}^{m+1} \)

\( U^e(\cdot, s) \in \text{dom}(A^{N,e}) \) due to (2.20), (2.21), (2.23), (2.25), (2.27), (2.28).

Let us establish some properties of \( U^e(x, s) \) for fixed \( s \in \mathbb{R}^{m+1} \). Using (2.29), (2.30) we obtain:

\[ \|\nabla U^e(\cdot, s)\|_{L^2(Y^e)}^2 - \frac{1}{4} \kappa d_n^{n-2} \varepsilon^2 \sum_{j=1}^m (s_{m+1} - s_j)^2 = \varepsilon^2 F(s) \quad (\varepsilon \to 0), \]  

(2.31)

\[ \|\Delta U^e(\cdot, s)\|_{L^2(Y^e)}^2 \leq C \varepsilon^4. \]  

(2.32)

Since \( [U^e(\cdot, s) - s]_{\text{int}} = 0 \) on \( \partial B \setminus \{ x : |x - \bar{x}| \leq r/2 \} \), \( j = 1, \ldots, m \) (here \( \cdot \) denotes the value of the function when we approach \( \partial B_j \) from inside of \( B_j \)) and \( U^e(\cdot, s) - s_{m+1} = 0 \) on \( \partial Y \) we have the following Friedrichs inequalities

\[ \|U^e(\cdot, s) - s_j\|_{L^2(B_j)}^2 \leq C\|\nabla U^e(\cdot, s)\|_{L^2(B_j)}^2 \leq C_1 \varepsilon^2, \quad j = 1, \ldots, m + 1 \]

and therefore

\[ \forall j = 1, \ldots, m + 1 : \ U^e(\cdot, s) \to s_j \text{ strongly in } L_2(B_j). \]  

(2.33)

By \( (\cdot, \cdot)_B \) we denote the following scalar product in \( \mathbb{R}^{m+1} \):

\[ (s, t)_B = \sum_{j=1}^{m+1} s_j t_j |B_j|, \quad s, t \in \mathbb{R}^{m+1}. \]

It follows from (2.33) that

\[ \|U^e(\cdot, s)\|_{L^2(Y^e)}^2 \sim (s, s)_B \text{ as } \varepsilon \to 0. \]  

(2.34)

**Lemma 2.1.** One has for \( k = 1, \ldots, m + 1 \):

\[ \lim_{\varepsilon \to 0} \lambda^N_{k,e} \leq C \varepsilon^2. \]  

(2.35)

**Proof.** We prove this lemma by induction. For \( k = 1 \) (2.35) is obvious (namely, \( \lambda^N_{1,e} = 0 \)). Now, let

\[ \lim_{\varepsilon \to 0} \lambda^N_{k,e} \leq C \varepsilon^2 \text{ for } k \leq k' - 1 \]  

(2.36)

and let us prove (2.36) for \( k = k' \).

One has the following Poincaré inequalities:

\[ \|u_k^e - \langle u_k^e \rangle_{B_j}\|_{L^2(B_j)}^2 \leq C\|\nabla u_k^e\|_{L^2(Y^e)}^2 = C \lambda^N_k, \quad j = 1, \ldots, m + 1, \quad k = 1, 2, 3 \ldots \]

and therefore due to (2.17), (2.36) one has

\[ \forall l = 1, \ldots, k' - 1, \forall j = 1, \ldots, m + 1 : \ u_l^e \to s_l^j \text{ strongly in } L_2(B_j). \]  

(2.37)

Now, let \( \hat{s} \in \mathbb{R}^{m+1} \setminus \{0\} \) be an arbitrary vector satisfying:

\[ (\hat{s}, s_l^j)_B = 0, \quad l = 1, \ldots, k' - 1. \]  

(2.38)

The choice of such a vector is always possible whenever \( k' \leq m + 1 \). We denote

\[ \hat{u}^e(x) = U^e(x, \hat{s}) - w^e(x, \hat{s}) \]
where
\[ w^e(x, \hat{s}) = \sum_{l=1}^{k-1} (U(\cdot, \hat{s}), u^e_l(\cdot))_{L_2(Y^e)} u^e_l(x). \]

In view of (2.33), (2.37), (2.38) we obtain
\[ (U^e(\cdot, \hat{s}), u^e_l(\cdot))_{L_2(Y^e)} \to_{\varepsilon \to 0} (\hat{s}, s^l)_B = 0. \]

Using (2.36) and (2.39) one has
\[ \|w^e(\cdot, \hat{s})\|^2_{L_2(Y^e)} = \sum_{l=1}^{k-1} (U^e(\cdot, \hat{s}), u^e_l(\cdot))^2_{L_2(Y^e)} \to_0 0, \]
(2.40)
\[ \varepsilon^{-2}\|\nabla w^e(\cdot, \hat{s})\|^2_{L_2(Y^e)} = \varepsilon^{-2} \sum_{l=1}^{k-1} \lambda^N_{k, l} (U^e(\cdot, \hat{s}), u^e_l(\cdot))^2_{L_2(Y^e)} \leq C \sum_{l=1}^{k-1} (U^e(\cdot, \hat{s}), u^e_l(\cdot))^2_{L_2(Y^e)} \to_0 0. \]
(2.41)

Obviously \( \hat{\nu}^e \in H(u^e_1, \ldots, u^e_{k-1}) \). Then using (2.15), (2.31), (2.34), (2.40), (2.41) we obtain

\[ \lambda^N_{k, l} \leq \frac{\|\nabla \hat{\nu}^e\|^2_{L_2(Y^e)}}{\|\hat{\nu}^e\|^2_{L_2(Y^e)}} \sim \frac{\|\nabla U^e(\cdot, \hat{s})\|^2_{L_2(Y^e)}}{\|U^e(\cdot, \hat{s})\|^2_{L_2(Y^e)}} \sim \frac{F(\hat{s})\varepsilon^2}{(\hat{s}, \hat{s})_B} \leq C \varepsilon^2. \]

The lemma is proved.

\[ \square \]

**Lemma 2.2.** One has for \( k = 1, \ldots, m + 1 \):
\[ \lim_{\varepsilon \to 0} \varepsilon^{-2} \lambda^N_{k, l} = F(s^l). \]
(2.42)

**Proof.** Using the Poincare inequality and Lemma 2.1 we obtain the following estimates:
\[ \|u^e_k - u^e_l\|_{L_2(B_j)}^2 \leq C \varepsilon^2, \quad j, k = 1, \ldots, m + 1 \]
and therefore in view of (2.17)
\[ \forall k = 1, \ldots, m + 1, \forall j = 1, \ldots, m + 1 : u^e_k(\cdot, s) \to s^l_j \text{ strongly in } L_2(B_j). \]
(2.43)

Using (2.43) we get from (2.14)
\[ (s^k, s^l) = \delta_{kl}. \]
(2.44)

We construct an approximation \( u^e_k \in H(u^e_1, \ldots, u^e_{k-1}) \) for the eigenfunction \( u^e_k \) by the formula
\[ u^e_k(x) = U^e(x, s^k) - w^e_k(x), \]
(2.45)

where
\[ w^e_k(x) = \sum_{l=1}^{k-1} (U^e(\cdot, s^k), u^e_l(\cdot))_{L_2(Y^e)} u^e_l(x). \]

Using (2.31), (2.32), (2.34) and (2.44) we obtain
\[ \|\nabla U^e(\cdot, s^k)\|^2_{L_2(Y^e)} \sim \varepsilon^2 F(s^k), \quad \|U^e(\cdot, s^k)\|^2_{L_2(Y^e)} \sim (s^k, s^l)_B = 1, \quad \|\Delta U^e(\cdot, s^k)\|^2_{L_2(Y^e)} \leq C \varepsilon^2. \]
(2.46)

The functions \( w^e_k(x) \) brings vanishingly small contribution to \( u^e_k \). Namely, using (2.33), (2.35), (2.43), (2.44) we get
\[ \|w^e_k\|^2_{L_2(Y^e)} + \varepsilon^{-2}\|\nabla w^e_k\|^2_{L_2(Y^e)} + \varepsilon^{-4}\|\Delta w^e_k\|^2_{L_2(Y^e)} \leq C \sum_{l=1}^{k-1} (U^e(\cdot, s^k), u^e_l(\cdot))_{L_2(Y^e)} \to_0 C \sum_{l=1}^{k-1} (s^k, s^l)_B^2 = 0. \]
(2.47)
Now let us estimate the difference
\[ \delta_k^\varepsilon = u_k^\varepsilon - u_k. \]
Taking into account (2.33), (2.43), (2.47) we get
\[ \|\delta_k^\varepsilon\|_{L^2(Y)} \leq 3 \left( \sum_{j=1}^{m+1} \|u_k^\varepsilon - s_j^\varepsilon\|_{L^2(B_j')}^2 + \sum_{j=1}^{m+1} \|s_j - U^\varepsilon(\cdot, s_j^\varepsilon)\|_{L^2(B_j')}^2 + \|w_k^\varepsilon\|_{L^2(Y)}^2 \right) \to 0. \] (2.49)
Since \( u_k^\varepsilon \in H(u_k^1, \ldots, u_k^{k-1}) \) we get
\[ \|\nabla u_k^\varepsilon\|_{L^2(Y)}^2 \leq \frac{\|\nabla u_k\|_{L^2(Y)}^2}{\|u_k\|_{L^2(Y)}^2}. \] (2.50)
Plugging (2.48) into (2.50) and integrating by parts we obtain
\[ \|\nabla \delta_k^\varepsilon\|_{L^2(Y)}^2 \leq 2\|\Delta u_k^\varepsilon, \delta_k^\varepsilon\|_{L^2(Y)} + \|\nabla u_k^\varepsilon\|_{L^2(Y)}^2 \left( \|u_k^\varepsilon\|_{L^2(Y)}^2 - 1 \right). \]
and then in view of (2.46), (2.47), (2.49) we conclude that
\[ \lim_{\varepsilon \to 0} \varepsilon^{-2}\|\nabla \delta_k^\varepsilon\|_{L^2(Y)}^2 = 0. \] (2.51)
Finally using (2.46), (2.47), (2.51) we obtain
\[ \varepsilon^{-2} A_{k,\varepsilon}^{N,\varepsilon} = \varepsilon^{-2}\|\nabla u_k^\varepsilon\|_{L^2(Y)}^2 \sim \varepsilon^{-2}\|\nabla u_k\|_{L^2(Y)}^2 \sim \varepsilon^{-2}\|\nabla U^\varepsilon(\cdot, s)^2\|_{L^2(Y)}^2 \sim F(s^\varepsilon) \quad (\varepsilon \to 0). \] (2.52)
The lemma is proved. 

\[ \square \]

**Lemma 2.3.** The vectors \( s^k, k = 1, \ldots, m + 1 \) satisfy the following inequalities:
\[ F(s^k) \leq F(s) \text{ for any } s \in H(s^1, \ldots, s^{k-1}), \]
where
\[ H(s^1, \ldots, s^{k-1}) = \{ s \in \mathbb{R}^{m+1} : (s, s)_B = 1, (s, s)_B = 0, \ l = 1, k-1 \}. \]

**Proof.** Let \( s^k \in H(s^1, \ldots, s^{k-1}) \) be an arbitrary vector satisfying
\[ F(s^k) \leq F(s) \text{ for any } s \in H(s^1, \ldots, s^{k-1}) \]
(i.e. \( s^k \) is a minimizer of \( F(s) \) on \( H(s^1, \ldots, s^{k-1}) \)). Since \( s^k \in H(s^1, \ldots, s^{k-1}) \) then
\[ F(s^k) \leq F(s^k). \] (2.53)

Using the min-max principle we get the inequality
\[ \varepsilon^{-2} A_{k,\varepsilon}^{N,\varepsilon} \leq \frac{\varepsilon^{-2}\|\nabla U(\cdot, s^k) - \nabla u_k^\varepsilon\|_{L^2(Y)}^2}{\|U^\varepsilon(\cdot, s^\varepsilon) - \tilde{w}_k^\varepsilon\|_{L^2(Y)}^2}, \] (2.54)
where \( \tilde{w}_k^\varepsilon(x) = \sum_{l=1}^{k-1} (U_k^\varepsilon(\cdot, s^k), u_l^\varepsilon(\cdot))_{L^2(B_l')} u_l^\varepsilon(x) \). Using the same arguments as in Lemmas 2.1, 2.2 one can easily prove that
\[ \varepsilon^{-2}\|\nabla U(\cdot, s^k)\|_{L^2(Y)}^2 \sim F(s^k), \quad \|U^\varepsilon(\cdot, s^k)\|_{L^2(Y)}^2 \sim (s^k, s^k)_B = 1, \]
\[ \lim_{\varepsilon \to 0} \varepsilon^{-2}\|\nabla u_k^\varepsilon\|_{L^2(Y)}^2 + \|\tilde{w}_k^\varepsilon\|_{L^2(Y)}^2 = 0 \]
and therefore (2.54) implies
\[ \lim_{\varepsilon \to 0} \varepsilon^{-2} A_{k,\varepsilon}^{N,\varepsilon} \leq F(s^k). \] (2.55)
It follows from (2.42), (2.53), (2.55) that
\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} \mathcal{A}_k^{N,E} = F(S_k).
\]
The lemma is proved. \(\square\)

It is more convenient to work with the usual scalar product in \(\mathbb{R}^{m+1}\) (instead of the product \((\cdot, \cdot)_B\)). In this connection we reformulate Lemmas 2.2 2.3 We introduce the function \(\mathcal{F} : \mathbb{R}^{m+1} \to [0, \infty)\) by
\[
\mathcal{F}(q) = \sum_{j=1}^{m} \sigma_j b_j \left( \frac{q_m+1}{\sqrt{|B_m+1|}} - \frac{q_j}{\sqrt{|B_j|}} \right)^2.
\]
We also introduce the vectors \(q^k \in \mathbb{R}^{m+1}, k = 1, \ldots, m + 1\) by the formula
\[
q_j^k = s_j^k \sqrt{|B_j|}, \quad j = 1, \ldots, m + 1.
\]

Then, obviously, Lemmas 2.2 2.3 can be reformulated in the following way:

**Corollary 2.1.** One has for \(k = 1, \ldots, m + 1:\)
\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} \mathcal{A}_k^{N,E} = \mathcal{F}(q^k) \leq \mathcal{F}(q) \text{ for any } q \in \mathcal{H}(q^1, \ldots, q^{k-1}),
\]
where
\[
\mathcal{H}(q^1, \ldots, q^{k-1}) = \{ q \in \mathbb{R}^{m+1} : (q, q) = 1, (q, q^l) = 0, l = 1, k - 1 \}.
\]
Here \((\cdot, \cdot)\) is the usual scalar product in \(\mathbb{R}^{m+1}\), i.e. \((q, s) = \sum_{j=1}^{m+1} q_j s_j\).

It is clear that \(q^1\) is either \((\sqrt{|B_1|}, \sqrt{|B_2|}, \ldots, \sqrt{|B_{m+1}|})\) or \(- (\sqrt{|B_1|}, \sqrt{|B_2|}, \ldots, \sqrt{|B_{m+1}|})\).

Let us denote by \(E_\kappa\) (here \(\kappa > 0\) is a parameter) the \((m-1)\)-dimensional ellipsoid which is a cross-section of the elliptic cylinder \(\mathcal{F}(q) = \kappa\) by the hyperplane \(\{ q \in \mathbb{R}^{m+1} : (q, q^1) = 0 \}:
\[
E_\kappa = \{ q \in \mathbb{R}^{m+1} : \mathcal{F}(q) = \kappa, (q, q^1) = 0 \}.
\]
We denote by \(h_1(\kappa) \geq h_2(\kappa) \geq \cdots \geq h_m(\kappa)\) the half-axes of this ellipsoid (recall that there is some orthogonal change of variables \(q \mapsto \tilde{q}\) such that \(E_\kappa\) has the following form in coordinates \(x\):
\[
\tilde{q}_{m+1} = 0, \quad \sum_{j=1}^{m} \left( \frac{\tilde{q}_j}{h_j(\kappa)} \right)^2 = 1.
\]

By \(S\) we denote the \((m-1)\)-dimensional unit sphere which is a cross-section of the \(m\)-dimensional sphere \(\{ q \in \mathbb{R}^{m+1} : (q, q) = 1 \}\) by the plane \(\{ q \in \mathbb{R}^{m+1} : (q, q^1) = 0 \}\).

Let \(\kappa_1, \kappa_2, \ldots, \kappa_m\) be the numbers satisfying
\[
h_k(\kappa_k) = 1.
\]
We also set \(\kappa_0 = 0\). It is clear that \(\kappa_j \leq \kappa_{j+1}\). Later we will prove (see the end of the proof of Lemma 2.5) that if \(\sigma_j < \sigma_{j+1}\) \((j = 1, \ldots, m-1)\) then
\[
\kappa_j \leq \kappa_{j+1}, \quad j = 0, \ldots, m - 1.
\]

The ellipsoids \(E_{\kappa_k}\), \(k = 1, \ldots, m\) touch the sphere \(S\). Taking into account (2.57) it is easy to show that they touch \(S\) only in two points \(\tilde{q}_k^k\) and \(\tilde{q}_k^l\) which are symmetric to each other with respect to the origin and satisfies the following properties:
\[
\forall k, l = 1, m : (\tilde{q}_k^k, \tilde{q}_l^l) = \delta_{kl},
\]
\[
\forall k = 1, m, \forall \kappa \in (\kappa_{k-1}, \kappa_k) : E_\kappa \cap \{ q \in \mathbb{R}^{m+1} : (q, q) = 1, (q, q^l) = 0, l = 1, k - 1 \} = \emptyset,
\]
\[
\forall k = 1, m : \mathcal{F}(\tilde{q}_k^k) = \kappa_k.
\]
The following lemma follows easily from Corollary 2.1 and (2.57) – (2.60).

**Lemma 2.4.** One has for \( k = 2, \ldots, m + 1 \)

\[
q^k = \bar{q}_{-1}^k \quad \text{or} \quad q^k = \bar{q}_{+1}^k \quad (\text{and therefore} \quad \lim_{\varepsilon \to 0} \varepsilon^{-2} \lambda_k^{N,\varepsilon} = \kappa_{k-1}).
\]

Now, let us make a change of variables \( q \mapsto p \):

\[
p_j = \sqrt{c_j |B_j|} \left( \frac{q_{m+1}}{\sqrt{|B_{m+1}|}} - \frac{q_j}{\sqrt{|B_j|}} \right), \quad j = 1, \ldots, m, \quad p_{m+1} = \sum_{i=1}^{m+1} q_i \sqrt{|B_i|}.
\]

Simple calculations shows that

- the plane \( \{ q \in \mathbb{R}^{m+1} : (q, q_1) = 0 \} \) onto the plane \( \{ p \in \mathbb{R}^{m+1} : p_{m+1} = 0 \} \),
- the ellipsoid \( \kappa \) onto the sphere

\[
S_{\kappa} = \{ p \in \mathbb{R}^{m+1} : \sum_{i=1}^m p_i^2 = \kappa, \; p_{m+1} = 0 \},
\]
- the sphere \( S \) onto the ellipsoid

\[
E = \left\{ p \in \mathbb{R}^{m+1} : \sum_{i=1}^m p_i^2 \frac{1}{\sigma_i} - \sum_{i,j=1, i \neq j} p_i p_j \frac{|B_i||B_j|}{\sigma_i \sigma_j} = 1, \; p_{m+1} = 0 \right\}.
\]

As any linear non-degenerate map \( f \) preserves tangency points, i.e. \( S_{\kappa} \) touches \( E \) in the points \( \hat{p}_k = f(\bar{q}_k^k) \). We denote \( h_1 \leq h_2 \leq \cdots \leq h_m \) the semiaxes of the ellipsoid \( E \). It is clear that \( S_{\kappa} \) touches \( E \) iff for some \( j \) one has \( \kappa = h_j^2 \). Therefore using Lemma 2.4 we get for \( k = 2, \ldots, m + 1 \)

\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} \lambda_k^{N,\varepsilon} = h_k^2.
\]

Thus in order to complete the proof of Theorem 2.2 it remains to prove the following lemma.

**Lemma 2.5.** One has for \( k = 1, \ldots, m \):

\[
\mu_k = h_k^2, \quad (2.61)
\]

**Proof.** It is well-known that the numbers \( h_k^2 \) are the roots of the equation

\[
\det(M - \lambda^{-1}I) = 0, \quad \lambda \in \mathbb{C} \quad \text{is unknown number}, \quad (2.62)
\]

where \( I \) is the identity \( m \times m \) matrix, and the matrix \( M = \{ M_{ij} \}_{i,j=1}^m \) is defined by

\[
M_{ij} = \begin{cases} 
\frac{1}{\sigma_i} |B_i| & \text{if } i = j \quad \text{and} \\
\frac{-|B_i||B_j|}{\sigma_i \sigma_j} & \text{if } i \neq j.
\end{cases}
\]

We denote by \( M(i_1, i_2, \ldots, i_k) \) the minor of the matrix \( M \) which is on the intersection of \( i_1 \)-th, \( i_2 \)-th, \ldots, \( i_k \)-th rows and the columns with the same indexes. One has the following formula (see [29]):

\[
\det(M - \lambda^{-1}I) = \sum_{k=0}^m \lambda^{k-m} E_k(M), \quad (2.63)
\]

where

\[
E_0 = (-1)^m, \quad E_k(M) = (-1)^{m-k} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} M(i_1, i_2, \ldots, i_k). \quad (2.64)
\]
Let us prove that for \( k = 1, \ldots, m \)

\[
M(i_1, i_2, \ldots, i_k) = \frac{1 - |B_{i_1}| - |B_{i_2}| - \cdots - |B_{i_k}|}{\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_k}}. \tag{2.65}
\]



We carry out the proof by induction. For \( k = 1 \) and \( k = 2 \), (2.65) can be easily proved via direct calculations. Now, suppose that (2.65) is valid for \( k = l - 1, l - 2 \) and let us prove it for \( k = l \).

Obviously it is enough to prove (2.65) only for \( i_1 = 1, i_2 = 2, \ldots, i_l = l \). One has

\[
M(1, 2, \ldots, l) = \det \begin{pmatrix}
\frac{1 - B_1}{\sigma_1} & -\frac{|B_2|}{\sigma_1} & \cdots & -\frac{|B_l|}{\sigma_1} \\
0 & \frac{1 - B_2}{\sigma_2} & \cdots & -\frac{|B_l|}{\sigma_2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{|B_1|}{\sigma_1} & -\frac{|B_2|}{\sigma_1} & \cdots & \frac{1 - B_l}{\sigma_1}
\end{pmatrix} + \det \begin{pmatrix}
-\frac{|B_1|}{\sigma_1} & \frac{1 - B_2}{\sigma_1} & \cdots & \frac{|B_l|}{\sigma_1} \\
0 & \frac{1 - B_2}{\sigma_2} & \cdots & \frac{|B_l|}{\sigma_2} \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{|B_1|}{\sigma_1} & -\frac{|B_2|}{\sigma_1} & \cdots & \frac{1 - B_l}{\sigma_1}
\end{pmatrix}.
\]

The third determinant is equal to 0 because its first row is equal to its second one multiplied by \( \sqrt{\frac{\sigma_2|B_1|}{\sigma_1|B_2|}} \).

The second determinant can be written as

\[
\det \begin{pmatrix}
\frac{|B_1|}{\sigma_1} & -\frac{|B_2|}{\sigma_2} & \cdots & -\frac{|B_l|}{\sigma_1} \\
0 & \frac{|B_2|}{\sigma_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\frac{|B_1|}{\sigma_1} & -\frac{|B_2|}{\sigma_1} & \cdots & \frac{1 - B_l}{\sigma_1}
\end{pmatrix} = \frac{1}{\sigma_2} \left( M(1, 3, 4, \ldots, l) - \frac{1}{\sigma_1} M(3, 4, \ldots, l) \right).
\]

Finally using formula (2.65) for \( k = l - 1, l - 2 \) we obtain

\[
M(1, 2, \ldots, l) = \frac{1}{\sigma_1} M(2, 3, \ldots, l) + \frac{1}{\sigma_2} \left( M(1, 3, 4, \ldots, l) - \frac{1}{\sigma_1} M(3, 4, \ldots, l) \right) =
\]

\[
= \frac{1}{\sigma_1} \prod_{j=2, l}^{l} \frac{1 - \sum_{j=1, l, j \neq 2}^{j} |B_j|}{\sigma_j} + \frac{1}{\sigma_2} \left( \frac{1}{\prod_{j=1, l, j \neq 2}^{j} \sigma_j} - \frac{1}{\prod_{j=3, l}^{j} \sigma_j} \right) =
\]

\[
= \frac{1 - \sum_{j=2, l}^{l} |B_j|}{\prod_{j=1, l}^{l} \sigma_j} - \frac{1}{\prod_{j=1, l}^{l} \sigma_j} \sum_{j=3, l}^{j} |B_j| = \frac{1 - \sum_{j=1, l}^{j} |B_j|}{\prod_{j=1, l}^{l} \sigma_j}
\]

and (2.65) is proved.

Now let us consider equation (1.5). Multiplying it by \( |B_{m+1}| \prod_{j=1}^{m} (\sigma_j - \lambda) \) and then grouping the terms with the same exponents of \( \lambda \) we obtain

\[
\sum_{k=0}^{m} \lambda^{m-k} A_k = 0, \tag{2.66}
\]
where

\[ A_k = (-1)^{m-k} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} (|B_{m+1}| + |B_{i_1}| + |B_{i_2}| + \cdots + |B_{i_k}|). \]  \hspace{1cm} (2.67)

Using (2.65) and the equalities

\[ |B_{m+1}| + |B_{i_1}| + |B_{i_2}| + \cdots + |B_{i_k}| = 1 - |B_{j_1}| - |B_{j_2}| - \cdots - |B_{j_{m-k}}|, \quad \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_k} = \frac{\prod_{j=1}^{m} \sigma_j}{\sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_{m-k}}} \]

where \( \{j_1, j_2, \ldots, j_{m-k}\} = \{1, 2, \ldots, m\} \setminus \{i_1, i_2, \ldots, i_k\} \) we can rewrite \( A_k \) as

\[ A_k = (-1)^m \left( \prod_{j=1}^{m} \sigma_j \right) E_{m-k} \]  \hspace{1cm} (2.68)

Finally we divide (2.66) by \((-\lambda)^m \left( \prod_{j=1}^{m} \sigma_j \right) \) and taking into account (2.63)-(2.65), (2.68) we get the equation (2.62). Thus we have just proved that

\[ \lambda \in \mathbb{R} \setminus \left( \{0\} \cup \bigcup_{j=1}^{m} \{\sigma_j\} \right) \text{ is a root of (1.5) } \implies \lambda \text{ is a root of (2.62).} \]  \hspace{1cm} (2.69)

Since (1.5) has exactly \( m \) roots \( \mu_k \in \mathbb{R} \setminus \left( \{0\} \cup \bigcup_{j=1}^{m} \{\sigma_j\} \right) \) and \( \mu_k \neq \mu_l \) for \( k \neq l \) then, obviously, (2.69) implies (2.61). Let us note that (2.69) imply also inequality (2.57).

Lemma 2.5 is proved which ends the proof of Theorem 2.2. \( \square \)

2.4. Asymptotics of the first \( m \) eigenvalues of the Dirichlet Laplacian. We get more complete information about the behaviour of \( \lambda_k^{D,e} \), \( k = 1, \ldots, m \).

**Theorem 2.3.** For \( k = 1, \ldots, m \) one has

\[ \lim_{\varepsilon \to 0} e^{-2} \lambda_k^{D,e} = \sigma_k. \]  \hspace{1cm} (2.70)

**Remark 2.1.** We carry out the proof in the same way as in Theorem 2.2. But in the Dirichlet case the proof is simplifed by the fact that the eigenfunctions corresponding to the first \( m \) eigenvalues converge to 0 in \( B_{m+1} \) (i.e. \( s_{m+1}^k = 0, k = 1, \ldots, m \)). This observation leads to the functional \( F_0 \) having more simple form comparing with the functional \( F \).

**Proof.** Let \( u_k^e, k \in \mathbb{N} \) be the eigenfunctions corresponding to \( \lambda_k^{D,e} \) and satisfying the condition

\[ (u_k^e, u_l^e)_{L^2(Y^e)} = \delta_{kl}, \]

\( u_k^e \) are real functions.

For \( \lambda_k^{D,e} \) the min-max principle looks as follows:

\[ \forall k \in \mathbb{N} : \lambda_k^{D,e} = \min_{u \in H(u_1^e, \ldots, u_{k-1}^e)} \frac{\|\nabla u\|^2_{L^2(Y^e)}}{\|u\|^2_{L^2(Y^e)}}, \]  \hspace{1cm} (2.72)

where

\[ H(u_1^e, \ldots, u_{k-1}^e) = \left\{ u \in H^1(Y^e) : u|_{\partial Y} = 0, (u, u_l^e)_{L^2(Y^e)}, l = 1, \ldots, k-1 \right\}. \]  \hspace{1cm} (2.73)
As in Theorem 2.2 we conclude that there exist a subsequence (still denoted by \(\varepsilon\)) and numbers \(s_j^k \in \mathbb{R},\ k \in \mathbb{N},\ j = 1, \ldots, m + 1\) such that
\[
\lim_{\varepsilon \to 0} \langle u^\varepsilon_k \rangle_{B_j} = s_j^k.
\] (2.74)

We denote \(s^k = (s_1^k, \ldots, s_{m+1}^k) \in \mathbb{R}^{m+1}\). Below we will prove that \(s_{m+1}^k = 0\) whenever \(k \leq m\).

**Lemma 2.6.** One has for \(k = 1, \ldots, m\):
\[
\lim_{\varepsilon \to 0} \lambda_{D,\varepsilon}^k \leq C\varepsilon^2.
\] (2.75)

**Proof.** As in Lemma 2.1 we carry out the proof by induction. For an arbitrary \(s \in \mathbb{R}^{m+1}\) such that \(s_{m+1} = 0\) one has:
\[
\lambda_1^{D,\varepsilon} \leq \frac{\|\nabla U^\varepsilon(\cdot, s)\|_{L^2(Y^s)}^2}{\|U^\varepsilon(\cdot, s)\|_{L^2(Y^s)}^2} \leq C\varepsilon^2
\]
(recall that the function \(U^\varepsilon\) is defined by (2.19)). Here we can use the min-max principle since \(U^\varepsilon(x, s) = 0\) for \(x \in \partial Y\) whenever \(s_{m+1} = 0\). Thus (2.75) is valid for \(k = 1\).

Now, suppose that (2.75) is valid for \(k \leq k' - 1\) and let us prove it for \(k = k'\). One has the following Poincaré (for \(B_j, j = 1, \ldots, m\)) and Friedrichs inequalities (for \(B_{m+1}\)):
\[
j = 1, \ldots, m:\ |u^\varepsilon_k - \langle u^\varepsilon_k \rangle_{B_j}|^2_{L^2(B_j)} \leq C\lambda_{D,\varepsilon}^k,
\]
(2.76)
\[
|u^\varepsilon_k|^2_{L^2(B_{m+1})} \leq C\lambda_{D,\varepsilon}^k.
\] (2.77)

Using (2.76), (2.77) and taking into account the validity of (2.75) for \(k \leq k' - 1\) we get
\[
\forall l = 1, \ldots, k' - 1:\ \left\{ \begin{array}{l}
u^\varepsilon_l \xrightarrow{\varepsilon \to 0} \hat{s}^l_j \text{ strongly in } L^2(B_j),\ j = 1, \ldots, m, \\
u^\varepsilon_l \xrightarrow{\varepsilon \to 0} \hat{s}^l_{m+1} = 0 \text{ strongly in } L^2(B_{m+1}).
\end{array} \right.
\] (2.78)

Let \(\hat{s} \in \mathbb{R}^{m+1} \setminus \{0\}\) be an arbitrary vector satisfying:
\[
\hat{s}_{m+1} = 0 \text{ and } (\hat{s}, \hat{s})_B = 0,\ l = 1, \ldots, k' - 1.
\] (2.79)

The choice of such a vector is always possible whenever \(k' \leq m\). We denote
\[
\hat{u}^\varepsilon(x) = U^\varepsilon(x, \hat{s}) - w^\varepsilon(x, \hat{s}),\ \text{where } w^\varepsilon(x, \hat{s}) = \sum_{l=1}^{k-1} (U^\varepsilon(\cdot, \hat{s}) - U^\varepsilon(\cdot, \hat{s}^l))|_{L^2(Y^s)} u^\varepsilon_l(x).
\]

Obviously \(\hat{u}^\varepsilon \in H(u^\varepsilon_1, \ldots, u^\varepsilon_{k' - 1})\). In the same way as in Lemma 2.1 (see (2.40)–(2.41)) we obtain
\[
\lim_{\varepsilon \to 0} \left(\|w^\varepsilon(\cdot, \hat{s})\|^2_{L^2(Y^s)} + \varepsilon^{-2}\|\nabla w^\varepsilon(\cdot, \hat{s})\|^2_{L^2(Y^s)}\right) = 0.
\] (2.80)

Finally using (2.72) and taking into account (2.31), (2.34), (2.80) we get
\[
\lambda_{D,\varepsilon}^{k,\varepsilon} \leq \frac{\|\nabla \hat{u}^\varepsilon\|^2_{L^2(Y^s)}}{\|\hat{u}^\varepsilon\|^2_{L^2(Y^s)}} \frac{\|\nabla U^\varepsilon(\cdot, \hat{s})\|^2_{L^2(Y^s)}}{\|U^\varepsilon(\cdot, \hat{s})\|^2_{L^2(Y^s)}} \leq C\varepsilon^2
\]
and (2.75) is proved.

\[\square\]

**Lemma 2.7.** One has for \(k = 1, \ldots, m\):
\[
\lim_{\varepsilon \to 0} \varepsilon^{-2} \lambda_{D,\varepsilon}^k = F(s^k).
\] (2.81)
Proof. Using the Poincaré (for $B_j$, $j = 1, \ldots, m$) and Friedrichs (for $B_{m+1}$) inequalities and taking into account Lemma 2.6, we conclude that

$$
\forall k = 1, \ldots, m : \begin{cases}
\lim_{\epsilon \rightarrow 0} u_{k}^\epsilon \rightarrow s_j^k \text{ strongly in } L_2(B_j), & j = 1, \ldots, m \\
\lim_{\epsilon \rightarrow 0} u_{m+1}^\epsilon = 0 \text{ strongly in } L_2(B_{m+1}).
\end{cases}
$$

As in Lemma 2.2, we construct an approximation $u_k^\epsilon$ for the eigenfunction $u_k^\epsilon$ by the formula

$$
u_k^\epsilon(x) = U_k^\epsilon(x, s_k) - \sum_{l=1}^{k-1} (U_k^\epsilon(\cdot, s_k), u_l^\epsilon(\cdot))_{L_2(Y^\epsilon)} u_l^\epsilon(x).$$

Since $s_{m+1}^k = 0$ then $u_k^\epsilon(x) \in \text{dom}(A_{D,\epsilon})$. Repeating word-by-word the arguments of Lemma 2.2, we conclude that

$$
\epsilon^{-2} \| \nabla u_k^\epsilon \|_{L_2(Y^\epsilon)}^2 \sim \epsilon^{-2} \| \nabla U_k^\epsilon(\cdot, s_k) \|_{L_2(Y^\epsilon)}^2 \sim F(s_k) \quad (\epsilon \rightarrow 0),
$$

while $\delta_k^\epsilon := u_k^\epsilon - u_k^\epsilon$ brings vanishingly small contribution to $u_k^\epsilon$, namely

$$
\lim_{\epsilon \rightarrow 0} \epsilon^{-2} \| \nabla \delta_k^\epsilon \|_{L_2(Y^\epsilon)}^2 = 0.
$$

The lemma is proved.

Lemma 2.8. The vectors $s_k^k$, $k = 1, \ldots, m$ satisfy the following conditions:

$$F(s_k^k) \leq F(s) \text{ for any } s \in H_0(s^1, \ldots, s^{k-1}),$$

where

$$H_0(s^1, \ldots, s^{k-1}) = \{ s \in \mathbb{R}^{m+1} : s_{m+1} = 0, (s, s)_B = 1, (s, s')_B = 0, l = 1, k - 1 \}.$$

Proof. The lemma is proved similarly to Lemma 2.3 taking into account that $s_{m+1}^k = 0, k = 1, m$.

We introduce the function $F_0 : \mathbb{R}^m \rightarrow \mathbb{R}$ by the formula

$$F_0(q) = \sum_{j=1}^{m} \alpha_j q_j^2.$$

We also introduce the vectors $q_j^k \in \mathbb{R}^m$ by the formula

$$q_j^k = s_j^k \sqrt{|B_j|}, \quad j = 1, \ldots, m.$$

Taking into account the equality $s_{m+1}^k = 0$ ($k = 1, \ldots, m$) we can easily reformulate Lemmas 2.7, 2.8.

Corollary 2.2. One has for $k = 1, \ldots, m$:

$$
\lim_{\epsilon \rightarrow 0} \epsilon^{-2} A_{D,\epsilon} = F_0(q^k) \leq F_0(q) \text{ for any } q \in H_0(q^1, \ldots, q^{k-1}),
$$

where

$$H_0(q^1, \ldots, q^{k-1}) = \{ q \in \mathbb{R}^m : (q, q) = 1, (q, q') = 0, l = 1, k - 1 \}.$$

Here $(\cdot, \cdot)$ is the usual scalar product in $\mathbb{R}^m$.

Let us denote by $E_\kappa$ (here $\kappa > 0$ is a parameter) the ($m-1$)-dimensional ellipsoid

$$E_\kappa = \{ q \in \mathbb{R}^m : F_0(q) = \kappa \}.$$

The numbers $\left\{ \sqrt{\frac{\alpha_j}{\kappa}} \right\}_{j=1}^m$ are the semi-axes of $E_\kappa$. 
We denote by $S$ the $(m - 1)$-dimensional sphere

$$S = \{ q \in \mathbb{R}^n : (q, q) = 1 \}$$

It is clear that $E_\kappa$ touches $S$ iff $\kappa = \sigma_k$ for some $k$.
The ellipsoids $E_{\sigma_k}$, $k = 1, \ldots, m$ touch the sphere $S$ in two points $\tilde{q}_k = \pm(0, 0, \ldots, 1, \ldots, 0)$.

Using Corollary 2.2 via the same arguments as in the proof of Theorem 2.2 we obtain the following lemma.

**Lemma 2.9.** One has for $k = 1, \ldots, m$:

$$q^k = \tilde{q}_k \quad \text{or} \quad q^k = -\tilde{q}_k.$$ 

It follows from Corollary 2.2 and Lemma 2.9 that $\lim_{\varepsilon \to 0} \varepsilon^{-2} A_{\kappa, \varepsilon}^{\varepsilon} = \sigma_k$. Theorem 2.3 is proved. 

2.5. **Asymptotics of the first eigenvalues of $\theta$-periodic Laplacian.** To complete the proof of Theorem 1.1 we also need the information about the behaviour of the eigenvalues of the operators $A_{\theta, \varepsilon}$. It turns out that their behaviour is the same as the behaviour of the eigenvalues of either Neumann or Dirichlet Laplacians. Namely, the following theorem is valid.

**Theorem 2.4.** Let

$$\theta_1 = (1, 1, \ldots, 1), \quad \theta_2 = -(1, 1, \ldots, 1).$$

Then for $k = 1, \ldots, m$ one has

$$\theta = \theta_1 : \lim_{\varepsilon \to 0} \varepsilon^{-2} A_{\kappa, \varepsilon}^{\varepsilon} = \mu_k, \quad (2.85)$$

$$\theta = \theta_2 : \lim_{\varepsilon \to 0} \varepsilon^{-2} A_{\kappa, \varepsilon}^{\varepsilon} = \sigma_k. \quad (2.86)$$

**Remark 2.2.** In fact it is possible to prove that (2.86) is valid for an arbitrary $\theta \neq \theta_1$. However for the proof of Theorem 1.1 it is enough to prove (2.86) only for $\theta = \theta_2$.

**Proof.** The proof of (2.85) is carried word-by-word as the proof of Theorem 2.2. Indeed it is easy to see that when proving Theorem 2.2 we have used only the following three facts that are specific for the Neumann boundary conditions:

- $A_{N, \varepsilon}^{\varepsilon} = 0$ and the corresponding eigenspace consists of constants,
- There exists an orthonormal sequence of real eigenfunctions of $A_{N, \varepsilon}$,
- For an arbitrary $s$ the function $U^\varepsilon(\cdot, s)$ belongs to dom($A_{N, \varepsilon}$).

However it is clear that all these properties are valid with $\theta_1$ instead of $N$.

The proof of (2.86) is similarly to the proof of Theorem 2.3. Indeed the proof of Theorem 2.3 uses the following three facts that are specific for the Dirichlet boundary conditions:

- The Friedrichs inequality

$$||u||^2_{L^2(B_{m+1})} \leq C_D ||\nabla u||^2_{L^2(B_{m+1})}, \quad u \in \text{dom}(A_{D, \varepsilon}^{\varepsilon}) \quad (2.87)$$

is valid. Here the constant $C_D > 0$ is independent of $u$.
- There exists an orthonormal sequence of real eigenfunctions of $A_{N, \varepsilon}$,
- For an arbitrary $s \in \{ s \in \mathbb{R}^{m+1} : s^{m+1} = 0 \}$ the function $U^\varepsilon(\cdot, s)$ belongs to dom($A_{D, \varepsilon}^{\varepsilon}$).

Inequality (2.87) with $\theta \neq \theta_1$ instead of $D$ was proved in [25] for the case $m = 1$. In the case $m > 1$ the proof is similar. Obviously, the remaining conditions are also valid for $\theta_2$ instead of $D$. 

□
2.6. **End of the proof of Theorem 1.1** It follows from (2.1) and (2.4) that

$$\sigma(\mathcal{A}^e) = \bigcup_{k=1}^{\infty} [a_k^-(\epsilon), a_k^+(\epsilon)]$$

(2.88)

where the compact intervals \([a_k^-(\epsilon), a_k^+(\epsilon)]\) are defined by

$$[a_k^-(\epsilon), a_k^+(\epsilon)] = \bigcup_{\theta \in \mathbb{T}^n} \left\{ \epsilon^{-2} \lambda_{k,\theta}^N \right\}.$$  

(2.89)

It follows from (2.5) and (2.89) that

$$\epsilon^{-2} \lambda_{k,\theta}^{N,e} \leq a_k^-(\epsilon) \leq \epsilon^{-2} \lambda_{k,\theta}^{D,e},$$

(2.90)

$$\epsilon^{-2} \lambda_{k,\theta}^{D,e} \leq a_k^+(\epsilon) \leq \epsilon^{-2} \lambda_{k,\theta}^{D,e}.$$  

(2.91)

Obviously if \(k = 1\) then the left and right-hand-sides of (2.90) are equal to zero. It follows from (2.13), (2.88) that in the case \(k = 2, \ldots, m + 1\) they both converge to \(\mu_{k-1}\) as \(\epsilon \to 0\), while if \(k > m + 1\) they converge to infinity in view of (2.6), (2.8), (2.10), (2.12). Thus

$$a_k^-(\epsilon) = 0, \quad \lim_{\epsilon \to 0} a_k^+(\epsilon) = \mu_{k-1} \text{ if } 2 \leq k \leq m + 1, \quad \lim_{\epsilon \to 0} a_k^+(\epsilon) = \infty \text{ if } k > m + 1.$$  

(2.92)

Similarly in view of (2.7), (2.9), (2.11), (2.12), (2.20), (2.86), (2.91) one has

$$\lim_{\epsilon \to 0^+} a_k^+(\epsilon) = \sigma_k \text{ if } 1 \leq k \leq m, \quad \lim_{\epsilon \to 0^+} a_k^+(\epsilon) = \infty \text{ if } k > m.$$  

(2.93)

Then (1.7) - (1.8) follow directly from (2.88), (2.92), (2.93). Theorem 1.1 is proved.

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