Analysis of $8 \times 8$ and $9 \times 9$ optical flow patches by nudged elastic band

Qingli Yin $^{1,3}$ and Wen Wang $^2$

$^1$School of Thermal Engineering, Shandong Jianzhu University, Jinan 250101 P. R. China
$^2$School of Science, Shandong Jianzhu University, Jinan 250101 P. R. China
$^3$Email: yinql69@163.com

Abstract. We build a Morse function by a density estimator from optical flow sampled data, by using the Morse function and the nudged elastic band method we establish one-dimensional cell complexes, we study $8\times8$ and $9\times9$ optical flow patches to discover their topological features from the cell complexes. We discover that spaces of $8\times8$ and $9\times9$ patches have same topology as a circle, which spreads some known results to larger optical flow patches.

1. Introduction

Optical flow calculation is a main topic in computer vision, and it has a lot of applications, such as, in motion estimation and video compression ([1, 2, 3 and 4]). Nudged elastic band (NEB) is a valid technique to search the lowest energy path among a given two images, which arises from computational chemistry, and it is also efficient way for investigating nonlinear dimensional data. The authors of [5] analyzed $3\times3$ optical flow patches by nudged elastic band, they proved found that $3\times3$ optical flow patches has same topology as that of a circle. The authors of papers [6, 7] study optical flow $4\times4$, $5\times5$, $6\times6$, and $7\times7$-patches, and they show the same result as $3\times3$ optical flow patches.

Now, we make use of NEB method to optical flow $8\times8$ and $9\times9$-patches and investigate topological features of $8\times8$ and $9\times9$-patches. We apply the cell complexes constructed from the Morse function and the NEB to show that $8\times8$ and $9\times9$-patches have subspaces with the topology of a circle. Our spaces of $8\times8$ and $9\times9$-patches are very different topological spaces from $n\times n$-patches for $n = 3, 4, 5, 6, 7$, our results will be useful in the area of image compression [8].

2. Data sets of $8\times8$, $9\times9$ patches

Our spaces $M_8$ and $M_9$ of $8\times8$ and $9\times9$-patches are created from the database [2], one flow field sequence is displayed in figure 1. The spaces $M_8$ and $M_9$ are sets of optical flow $8\times8$ and $9\times9$ high contrast patches of as constructed in [8, 9 and 10].
An $8 \times 8$ optical flow patch is written as

$$(u_i, v_i) \quad (u_{i+1}, v_{i+1}) \quad (u_{i+2}, v_{i+2})$$

$$(u_i, v_i) \quad (u_{i+1}, v_{i+1}) \quad (u_{i+2}, v_{i+2})$$

$$(u_i, v_i) \quad (u_{i+1}, v_{i+1}) \quad (u_{i+2}, v_{i+2})$$

$$(u_i, v_i) \quad (u_{i+1}, v_{i+1}) \quad (u_{i+2}, v_{i+2})$$

where $\mu$ represents horizontal orientation of optical flow, $v$ represents the vertical orientation. A $8 \times 8$ patch is taken as a vector $x = (u_1, K, u_{64}, v_1, K, v_{64}) \in R^{128}$. Similarly, one $9 \times 9$ optical flow patch is denoted as

$$\begin{pmatrix}
(u_i, v_i) & (u_{i+1}, v_{i+1}) & (u_{i+2}, v_{i+2}) \\
(u_i, v_i) & (u_{i+1}, v_{i+1}) & (u_{i+2}, v_{i+2}) \\
(u_i, v_i) & (u_{i+1}, v_{i+1}) & (u_{i+2}, v_{i+2}) \\
(u_i, v_i) & (u_{i+1}, v_{i+1}) & (u_{i+2}, v_{i+2}) \\
\end{pmatrix}$$

every $9 \times 9$ patch of $M_9$ is considered as a vector $x = (u_1, K, u_{64}, v_1, K, v_{64}) \in R^{162}$. For a space $X$, $z \in X$ and a positive integer n, we let $\rho_n(z) = |z - z_n|$, here $z_n$ being the $n$-th nearest neighbor of point $z$. Given $n$, we list points of $X$ in descending density, and select points of $X$ with top $t$ percent densities denoted by $X(n, t)$. In this note, we study subsets $M_9(200, 30)$ and $M_9(200, 30)$.

![Figure 1](image-url)

**Figure 1.** One flow field sequence of the database. The left image is horizontal movement, the right is vertical movement.

### 3. Basic knowledge

We briefly describe several basic concepts used in this paper: nudged elastic band; CW-complexes; Morse theory.

#### 3.1. Nudged elastic band

The NEB is a valid technique to search the lowest energy path among two images. An elastic band having $N + 1$ images is defined by $[W_0, W_1, ..., W_N]$, $W_0$ and $W_N$ are first and last images. The other $N - 1$ images are revised according to an optimum algorithm [11]. The following formula is the total force on an image:

$$F_i = F_i^{S} \mathbf{v} - \nabla E(W_i).$$

The NEB uses a majorization algorithm to alter each image in light of the total force in the above formula for disclosing the lowest energy path. Readers may refer to [11, 12 and 13] for details of NEB.
3.2. CW-complexes
A $k$ -dimension unite disk \( \{ x \in \mathbb{R}^k \mid \| x \| \leq 1 \} \) is called a $k$-cell. One defines a CW-complex using follows [14]. The points of $X$ (i.e. 0-cell) is zero-skeleton $X^{(0)}$. One-skeleton $X^{(1)}$ is obtained via attaching the endpoints of one-cells to the zero-skeleton and so on.

3.3. Morse theory
A Morse function $g : M \rightarrow \mathbb{R}$ be smooth having non-degenerate critical points $t_1, \ldots, t_m \in M$ such that (here $M$ being a compact $d$ -dimension manifold):

\[
b_0 < g(t_1) < b_1 < g(t_2) < \ldots < b_{k-1} < g(t_m) < b_m.
\]

Each critical point $t_i$ has index $\beta_i$. Hence a smallest point possess index 0, a greatest point possess index $d$, but a saddle point possess index from 1 to $d - 1 [15]$. Set $M_b = g^{-1}([-\infty, b])$, the sublevel set of $b \in \mathbb{R}$, then each $M_{b_i}$ is homotopically identical to a CW-complex having one $\beta_i$-cell to critical point $t_i$.

4. Main computing steps
The following main steps of computing come from [5], please read the paper [5] for full details.

4.1. Denseness estimator
For $Y \subset \mathbb{R}^n$, suppose $\Phi_{x, \sigma} : \mathbb{R}^n \rightarrow [0, \infty)$ being the probability density of a normal distribution centered at $x \in Y$, we utilize the differential density estimator $g(y) = \| Y \|^{-1} \sum_{x \in Y} \Phi_{x, \sigma}(y)$ to approximate the unknown density.

4.2. 0-cells
We define mean shift function $p(y) : \mathbb{R}^n \rightarrow \mathbb{R}^n$,

\[
p(y) = \frac{\sum_{x \in Y} \Phi_{x, \sigma}(y) x}{\sum_{x \in Y} \Phi_{x, \sigma}(y)}.
\]

We arbitrarily pick first point $y_0 \in Y$, and iteratively generate a sequence \( \{ y_0, y_1, \ldots \} \) by $y_{n+1} = p(y_n)$. The sequence \( \{ y_n \} \) converges to a local maximum of $g$ [16]. We apply single-link clustering to collect the limits points of \( \{ y_n \} \), and pick the densest point from every cluster to be a zero-cell.

4.3. 1-cells
Given two zero-cells, a 1-cell is formed between them if we discover a convergent band using NEB method. Supposed a set $Y \subset \mathbb{R}^n$ from unknowing probability density $f : \mathbb{R}^n \rightarrow [0, \infty)$. To super-level sets $Y^{\beta} = f^{-1}([\beta, \infty))$, we build CW-complex models $Z^{\beta}$ to approach $Y^{\beta}$.

Firstly, we construct a differential density function to approximate the unconscious density from the sampled data. Secondly, we search local maxima of the estimation density function to find zero-cells. Finally, we look for the convergent bands by NEB from initial bands, and we receive 1-cells. Therefore, we could find the topological properties of the sampled data $Y$ by making cell complexes $Z^{\beta}$.
5. Results

We choose two classes subsets in $M_\alpha$ ($n=8, 9$): (1) random subsets $M_\alpha(15000)$ of $M_\alpha$ having 15000 vectors; (2) core subsets $M_\alpha(200,30)$. To draw a pretty projected figure, we do DCT to these data. Such as, we do DCT for $\mu$-part and $\nu$-part of $M_\alpha(200,30)$ respectively, thus points of $M_\alpha(200,30)$ are changed into points of $\mathbb{R}^{126}$. Our results are not affected by DCT, because the topology of a space is not changed under DCT.

5.1. $8\times8$ optical flow patches

The sampled data information is illustrated as table 1. For $M_\alpha(15000)$, let the value of standard deviation $\sigma=0.33$, we compute by NEB to gain four 0-cells with densities $3.838\times10^8, 6.909\times10^8, 9.826\times10^8, 2.041\times10^8$ respectively (figure 2), and four 1-cells with densities $3.126\times10^8, 3.557\times10^8, 6.909\times10^8, 8.462\times10^8$, a loop is formed by these cells. Hence for $\beta=3.126\times10^8$, $Z^\beta$ takes shape of a loop (figure 3). If we take $\sigma=0.28$ or $0.30$, we get four 0-cells and four 1-cells with different densities, all these cells compose a loop, but when $\sigma=0.35$, we have three 0-cells and three 1-cells with different densities, all these cells also constitute a loop (figure 4). In figure 2-4, small brown dots, green thick dots, and solid lines correspond to $M_\alpha(15000)$ projected to a plane, 0-cells, and 1-cells. The symbols in figure 5-13 have similar meanings.

| Table 1. Data set information. |
|-------------------------------|
| $M_\alpha(15000)$ | $M_\alpha(200,30)$ |
| number of points in data | 15000 | 15000 |
| dimension $n$ | 126 | 126 |
| standard deviation $\sigma$ | 0.33 | 0.32, 0.33 |

**Figure 2.** $M_\alpha(15000)$ and the four 0-cells for $\sigma=0.33$, projected to a plane.

**Figure 3.** $M_\alpha(15000)$ and the loop for $\sigma=0.33$, projected to a plane.

**Figure 4.** $M_\alpha(15000)$ and the loop for $\sigma=0.35$, projected to a plane.

For $M_\alpha(200,30)$, we let $\sigma=0.32$, we have four 0-cells with densities in $[2.796\times10^8, 2.871\times10^8]$ (figure 5), four 1-cells involving densities from $1.143\times10^8$ to $6.658\times10^8$, a loop is made by these cells (figure 6). If $\sigma=0.33$, we find three 0-cells, and three 1-cells, that constitute a ring (figure 7).
5.2. 9×9 optical flow patches

The data information of 9×9 patches is explained in table 2. For $M_s(15000)$, let the value of $\sigma=0.32$, using NEB we obtain 0-cells and 1-cells, and get four 0-cells with densities $3.29 \times 10^3$, $5.872 \times 10^3$, $7.64 \times 10^3$, $1.606 \times 10^4$ respectively (figure 8), four 1-cells with densities $2.472 \times 10^3$, $2.75 \times 10^3$, $5.838 \times 10^3$, $6.251 \times 10^3$, a loop is comprised by these cells. Thus for $\beta=2.472 \times 10^3$, the cell complexes $Z^\beta$ make up of a loop (figure 9). If we take $\sigma=0.28$ or 0.30, we get four 0-cells and four 1- cells, have the result same as for $\sigma=0.32$. When $\sigma=0.34, 0.35, 0.36$, we discover three 0-cells and three 1-cells for three cases and the three 0-cells, 1-cells of each case form a loop, one case is displayed in figure 10. If we take $\sigma=0.27$ or 0.37, we do not find $M_s(15000)$ has the topology of a circle.

**Table 2.** Data set information.

|                     | $M_s(15000)$ | $M_s(200,30)$ |
|---------------------|--------------|---------------|
| number of points in data | 15000        | 15000         |
| dimension $n$       | 160          | 160           |
| standard deviation $\sigma$ | 0.32, 0.34   | 0.28, 0.30, 0.32 |

Figure 5. $M_s(200,30)$ and the four 0-cells for $\sigma=0.32$, projected to a plane.

Figure 6. $M_s(200,30)$ and the loop for $\sigma=0.32$, projected to a plane.

Figure 7. $M_s(200,30)$ and the loop for $\sigma=0.33$, projected to a plane.

Figure 8. $M_s(15000)$ and the four 0-cells for $\sigma=0.32$, projected to a plane.

Figure 9. $M_s(15000)$ and the loop for $\sigma=0.32$, projected to a plane.

Figure 10. $M_s(15000)$ and the loop for $\sigma=0.34$, projected to a plane.
For $M_s(200,30)$, suppose $\sigma = 0.28$, five 0-cells and five 1-cells are found their densities in $[5.66 \times 10^{22}, 8.217 \times 10^{22}]$ and $[2.448 \times 10^{23}, 1.537 \times 10^{23}]$ respectively (figure 11). When $\sigma = 0.30$ we have four 0-cells and four 1-cells of densities in $[1.58 \times 10^{28}, 1.448 \times 10^{29}]$ and $[5.572 \times 10^7, 3.081 \times 10^8]$ respectively (figure 12). If we pick $\sigma = 0.32$, three 0-cells and three 1-cells are found with densities $5.824 \times 10^{19}, 1.449 \times 10^{14}, 5.148 \times 10^{14}$ and $1.228 \times 10^{14}, 2.477 \times 10^{13}, 3.328 \times 10^{13}$ respectively (figure 13). In each of three cases, the 0-cells and 1-cells form a loop, which means $M_s(200,30)$ has same topology as a circle.

6. Conclusions
In this manuscript we exploit the NEB technique to investigate topological features of optical flow $8 \times 8$ and $9 \times 9$ patches. The fact of $8 \times 8$ and $9 \times 9$ optical flow patches being modelled as a loop is shown by numerical experiments. The biggest merit of cell complexes is its simpleness to analyze high dimensional data. Such as, we need merely utilize four cells to pattern $M_s(200,30)$ as a loop, we could demand thousands or more lazy witness complexes to model $M_s(200,30)$ for a similar result. Although, our results of $8 \times 8$ and $9 \times 9$ optical flow patches are proved same as that of $n \times n$-patches for $n = 3, 4, 5, 6, 7$, our spaces $M_8, M_9$ are very different from $n \times n$-patches ($n = 3, 4, 5, 6, 7$), for example, $M_9$ is a subspace of $R^{162}$, the space of $4 \times 4$-patches is a subspace of $R^{32}$, the results are more useful in image compression than that of $n \times n$-patches for $n = 3, 4, 5, 6, 7$. As the increasing of size of optical flow patches, the computation is getting bigger, hence the NEB method is only suitable for identifying low-dimensional topologies (0-dimension and 1-dimension) of a high-dimensional data set. In the NEB method, standard deviation $\sigma$ is one of the most important parameters, its appropriate values should be carefully selected to get a stable result. This one more time shows that NEB is an efficacious tool for identifying topological properties of high-dimensional nonlinear data.

Acknowledgments
This project supported by the NNSF of China (No. 61471409).

References
[1] Barron J L, Fleet D J and Beauchemin S S 1994 Performance of optical flow techniques Int. J. Computer Vision 12 43-77
[2] Roth S and Black M J 2007 On the spatial statistics of optical flow International Journal of
Computer Vision 74 33-50

[3] Baker S, Scharstein D, Lewis J P, Roth S, Black M J and Szeliski R 2011 A database and evaluation methodology for optical flow Int. J. Computer Vision 92 1-31

[4] Jia K, Wang X and Tang X 2011 Optical flow estimation using learned sparse model Proc. IEEE International Conference on Computer Vision, Barcelona, Spain 2391-2398

[5] Adams H, Atanasov A and Carlsson G 2015 Nudged elastic band in topological data analysis Topological Methods in Nonlinear Analysis 45 247-272

[6] Xia S 2016 Use of a nudged elastic band method in analysis of optical flow Journal of Flow Visualization and Image Processing 23 157-169

[7] Xia S and Liang D 2017 NEB in analysis of optical flow 4 × 4 and 6 × 6-Patches Journal of Physics: Conf. Series 787 012002

[8] Carlsson G, Ishkhanov T, Silva V de and Zomorodian A 2008 On the local behavior of spaces of natural images International Journal Computer Vision 76 1-12

[9] Adams H and Carlsson G 2009 On the nonlinear statistics of range image patches SIAM J. Image Sci. 2 110-117

[10] Xia S 2016 A topological analysis on patches of optical flow J. Nonlinear Sci. Appl. 9 1609-1618

[11] Henkelman G, Uberuaga B and Jónsson H 2000 A climbing image nudged elastic band method for finding saddle points and minimum energy paths Journal Chemical Physics 112 9901-9904

[12] Henkelman G and Jónsson H 2000 Improved tangent estimate in the nudged elastic band method for finding minimum energy paths and saddle points Journal Chemical Physics 113 9978-9985

[13] Jegou H, Douze M and Schmid C 2008 Hamming embedding and weak geometry consistency for large scale image search Proceedings of the 10th European conference on Computer vision, October 304-317

[14] May J P 1999 A concise course in algebraic topology University of Chicago Press, Chicago

[15] Milnor J 1965 Morse theory Princeton University Press, Princeton

[16] Cheng Y 1995 Mean shift, mode seeking, and clustering IEEE Trans. Pattern Anal. 17 790-799