Analytical hotspot shapes and magnetospheric radius from 3D simulations of magnetospheric accretion

A. K. Kulkarni* and M. M. Romanova*

Department of Astronomy, Cornell University, Ithaca, NY 14853, USA

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ABSTRACT

We present an analytical formula for the position and shape of the spots on the surface of accreting magnetized stars in cases where a star has a dipole magnetic field tilted at a small misalignment angle \( \Theta \lesssim 30^\circ \) about the rotational axis, and the magnetosphere is 2.5–5 times the radius of the star. We observed that the azimuthal position of the spots varies significantly when the position of the inner disc varies. In contrast, the polar position of the spots varies only slightly because of the compression of the magnetosphere. The azimuthal width of the spots strongly varies with \( \Theta \): spots have the shape of an arc at larger misalignment angles, and resemble a ring at very small misalignment angles. The polar width of the spots varies only slightly with changes in parameters. The motion of the spots in the azimuthal direction can provide phase-shifts in accreting millisecond pulsars, and the ‘drift’ of the period in Classical T Tauri stars. The position and shape of the spots are determined by three parameters: misalignment angle \( \Theta \); normalized corotation radius, \( r_c/R_* \), and normalized magnetospheric radius, \( r_m/R_* \), where \( R_* \) is the stellar radius. We also use our data to check the formula for the Alfvén radius, where the main dependences are \( r_m \sim (\mu^2/M)^{2/7} \), where \( \mu \) is the magnetic moment of the star, and \( M \) is the accretion rate. We found that the dependence is more gradual, \( r_m \sim (\mu^2/M)^{1/5} \), which can be explained by the compression of the magnetosphere by the disc matter and by the non-dipole shape of the magnetic field lines of the external magnetosphere.

Key words: accretion, accretion discs – MHD – stars: neutron.

1 INTRODUCTION

Magnetospheric accretion occurs in a variety of astrophysical systems. The accreting matter is stopped by the stellar magnetic field roughly at a distance from the star where matter and magnetic stresses become equal. Beyond that point, matter flows around the magnetosphere in a funnel flow and falls near the magnetic poles of the star, forming hotspots (e.g. Ghosh & Lamb 1979).

The light curves observed from accreting magnetized stars are often associated with the hotspots on their surfaces. In many applications it is important to know the exact location and shape of the spots. For example, in millisecond pulsars, pulse profiles are significantly affected by the hotspot location and shape (e.g. Poutanen & Gierliński 2003; Ibragimov & Poutanen 2009; Leahy et al. 2009). The position and shape of the spots is also important for understanding the light curves of Classical T Tauri stars (CTTSSs, e.g. Bouvier et al. 2007; Alencar et al. 2010) and in magnetized accreting white dwarfs (e.g. Warner 1993; Wynn 2000; Hellier 2001).

Three-dimensional (3D) magnetohydrodynamic (MHD) simulations of magnetospheric accretion show that matter accretes to a star in two funnel streams, which form two spots on its surface (Romanova et al. 2003, 2004; Kulkarni & Romanova 2005). Simulations show that the spots have the shape of a ring at small misalignment angles, the shape of an arc at \( \Theta \approx 30^\circ \) and the shape of a bar at very large \( \Theta \). In all the cases, the energy flux is largest in the centre of the spot and gradually decreases outward. The position of the spots does not coincide with the magnetic pole.

The simulations also show that the funnel stream can be dragged by the disc so that the spot forms at higher longitudes on the star’s surface. Alternatively, it can trail the magnetosphere, causing the spot to form at lower longitudes (Romanova et al. 2004). Therefore, the azimuthal position varies with accretion rate. In the case of a small dipole misalignment angle, \( \Theta \lesssim 5^\circ \), the funnel stream may pass a whole cycle about the magnetic pole, so that a spot can move faster or slower than the star (e.g. Kulkarni & Romanova 2008; Bachetti et al. 2010).

The motion of the spots along the stellar surface can produce observable effects such as the phase shifts in light curves, timing noise and intermittency of accreting millisecond pulsars (AMPs; e.g. Papitto et al. 2007; Lamb et al. 2009a,b; Patruno, Wijnands & van der Klis 2009; Poutanen, Ibragimov & Annala 2009). The motion of the spots can possibly explain the drifting periods observed in many CTTSSs (e.g. Rucinski et al. 2008; Siwak et al. 2011). Usually, a simple model is used for the spots, such as a circular spot centred...
at the magnetic pole, with a constant or Gaussian distribution of emitted flux.

More recent numerical simulations show that accreting magnetized stars may also be in an unstable regime of accretion, where matter accretes due to the magnetic Rayleigh–Taylor instability (Kulkarni & Romanova 2008; Romanova, Kulkarni & Lovelace 2008). In this regime, multiple chaotic ‘tongues’ penetrate through the magnetosphere and form spots of chaotic shapes and positions. This regime is favourable when the star rotates slower than the inner disc, though additional factors are important for the onset of instability (e.g. Spruit, Stehle & Papaloizou 1995). In this paper, we only consider the set of parameters at which accretion is stable and the spots have regular shapes.

The goal of this work is to derive a convenient formula for the shape and position of the spots, and the dependences of the parameters of the spots on the parameters of the star and the disc. We concentrate on the cases of relatively small misalignment angles, $\Theta \lesssim 30^\circ$, where the spots have the shape of an arc or a ring.

We also use a set of our 3D MHD simulations to test the standard formula for the Alfvén radius, which is routinely used for calculation of the truncation (magnetospheric) radius, $r_m$. We found that the dependences of $r_m$ on the magnetic moment of the star, $\mu$, and the accretion rate, $M$, are different compared with the standard formula.

In Section 2, we describe our numerical model. In Section 3, we provide an analytical formula for the spots. In Section 4, we investigate the dependence of the parameters of the spots on the parameters of the star and the disc. In Section 5, we derive a formula for the magnetospheric radius and compare it with the standard formula for the Alfvén radius. In Section 6, we provide a brief practical guide on how to find the position of the spots from observation. In Section 7, we show applications to different magnetized stars. In Section 8, we outline the main conclusions from this work. Appendix A describes our reference values. Appendix B shows parameters of the spots for all simulation runs.

2 NUMERICAL MODEL

To obtain the shapes of the spots, we perform global 3D MHD simulations of accretion from the disc on to a magnetized rotating star.

The model we use is the same as in our earlier 3D MHD simulations (e.g. Romanova et al. 2004; Kulkarni & Romanova 2008). The star has a dipole magnetic field, the axis of which makes angle $\Theta$ with the star’s rotational axis. The rotation axes of the star and the accretion disc are aligned. A magnetized star is surrounded by an accretion disc which is cold and dense, and by a hot rarefied corona which is 100 times less dense and 100 times hotter in the fiducial point. The disc and the corona are chosen to be initially in a quasi-equilibrium state, where the gravitational, centrifugal and pressure gradient forces are in balance (Romanova et al. 2002). General relativistic effects, which are important in neutron stars, are modelled using the Paczyński–Wiita potential, $\Phi(r) = GM_*/(r - r_e)$ (Paczyński & Wiita 1980), where $M_*$ is the mass of the star and $r_e = 2GM_*/c^2$ is Schwarzschild radius.¹ Viscosity is modelled using the $\alpha$-model (Novikov & Thorne 1973; Shakura & Sunyaev 1973). It is incorporated only in the disc and controls the accretion rate through the disc. We take a small parameter $\alpha = 0.02$ in all simulation runs.

To model accretion, the MHD equations are solved numerically in three dimensions using a Godunov-type numerical code, written in a ‘cubed-sphere’ coordinate system which rotates with the star (Koldoba et al. 2002). The numerical approach is similar to that described in Powell et al. (1999), where the eight-wave Roe-type approximate Riemann solver is used to calculate flux densities between the cells. The grid resolution is identical to that in Kulkarni & Romanova (2008) and equal to $N_r \times N^2 = 72 \times 31^2$ in each of the six blocks of the cubed sphere. Here, $N_r$ is the number of grids in the radial direction, and $N$ is the number of grids in the angular directions in each of the six sides of the cube.

The boundary conditions at the star’s surface amount to the assumption that the infalling matter passes through the surface of the star, so that the dynamics of this matter after it falls on to the star is ignored. At the external boundary, matter inflow is permitted at the disc’s part of the boundary, and outflow at the corona’s part of the boundary.

The simulations are done using dimensionless variables. The dimensionless value of every physical quantity $q$ is defined as $\tilde{q} = q/q_0$, where $q_0$ is the reference value for $q$. Appendix A shows how the reference values are determined, and lists the reference values for three classes of central objects: CTTTs, white dwarfs and neutron stars. Subsequently, we drop the tildes above the dimensionless variables and show dimensionless values everywhere unless otherwise stated. For clarity, we keep the tildes above the dimensionless magnetic moment of the star, $\tilde{\mu}$, and the matter and angular momentum fluxes $\tilde{M}$ and $\tilde{L}$, respectively.

Fig. 1 (left-hand panel) shows an example of funnel streams obtained in a typical 3D MHD simulation run. The right-hand panel shows the energy flux distribution at the surface of the star (the hotspot). One can see that the spot has the shape of an arc, and the centre of the spot (the highest energy flux) is off-set from the magnetic pole. The centre of the spot is also off-set from the $\mu - \Omega$ plane, because the inner disc rotates faster than the magnetosphere, and both the funnel stream and the spot are shifted. In this paper, we provide an analytical formula for the shapes and positions of such spots.

3 ANALYTICAL FORMULA FOR THE SPOTS

For relatively small misalignment angles, $\Theta \lesssim 30^\circ$, the position and shape of the spots can be approximated by analytical formula. Namely, each antipodal spot can be well approximated as a circular arc centred at the magnetic pole, with a Gaussian flux distribution centred some distance away from the magnetic pole, as follows (see Fig. 2, two right-hand panels):

$$F(\phi, \theta) = F_c \exp \left\{ -\left( \frac{\theta - \theta_0}{\theta_1} \right)^2 + \left( \frac{\phi - \phi_0}{\phi_1} \right)^2 \right\}, \quad (1)$$

The spherical polar and azimuthal angles $\theta$ and $\phi$ are measured with respect to the magnetic axis $\mu$. The azimuthal position $\phi = 0$ is defined by the $\mu - \Omega$ plane. The spot is described by the following parameters.

(i) The polar and azimuthal positions of the spot $(\theta_0, \phi_0)$.
(ii) The polar and azimuthal widths of the spot $(\theta_1, \phi_1)$, which are determined to be a half-width of the Gaussian in the polar and azimuthal directions. The half-width is defined as the distance from

¹ Since the simulations include general relativistic effects (approximated using the Paczyński–Wiita potential), strictly speaking they are only applicable to neutron stars. However, previous studies (Kulkarni & Romanova 2005) show that use of the Paczyński–Wiita potential does not affect the shape of the spots significantly. So the results of this work can be applied to other types of accreting systems as well.
Figure 1. Left-hand panel: a 3D view of the funnel flow from the disc to a magnetized star, where the dipole moment $\mu$ is tilted by $\Theta = 20^\circ$ about the rotational axis. One of the density levels is shown in green; sample field lines are shown in red. Right-hand panel: the energy flux distribution on the surface of the star. The circles show the position of the magnetic ($\mu$) and rotational ($\Omega$) axes, respectively. Other parameters are $\tilde{\mu} = 2$ and $r_c = 2$.

Figure 2. Left-hand panel: energy flux distribution on the surface of the star obtained in one of the simulation runs, with parameters $\Theta = 10^\circ$, $\tilde{\mu} = 1.5$, $r_c = 1.8$, at time $t = 8$. Middle panel: normalized flux distribution in the fitted spot, calculated with equation (1). The white line shows the boundary where the flux is $e$-times smaller than in the centre of the spot. The position of the spot in polar, $\theta_0$, and azimuthal, $\phi_0$, directions is shown. Right-hand panel: the same as the middle panel, except that the width of the spot is defined in polar, $\theta_1$, and azimuthal, $\phi_1$, directions.

(iii) The flux emitted from the centre of the spot, $F_c$.

We then apply a least-squares fit of this expression (given by equation 1) to the spots obtained from the MHD simulations. Fig. 2 compares the northern hotspot from the simulation, where misalignment angle $\Theta = 10^\circ$ (left-hand panel), with the fitted spot (right-hand two panels). The fitted spot agrees well with simulations. The rms error (over the star’s surface) of the fitted flux is usually within a few per cent of the central flux of the hotspot.

Equation (1) gives the flux distribution for the northern hotspot. The southern hotspot is generally identical to the northern one for a dipole field, so that parameters describing it are simply $(\pi - \theta_0, \pi + \phi_0)$ and $(\theta_1, \phi_1)$. When using equation (1) with these parameters, care must be taken to ensure that $|\phi - \phi_0| \leq \pi$ (by adding integral multiples of $\pi$ to $\phi$) for northern spots and $|\phi - (\pi + \phi_0)| \leq \pi$ for southern spots.

4 DEPENDENCE OF SPOT PARAMETERS ON PARAMETERS OF THE STAR

We performed a large set of 3D MHD numerical simulations at different misalignment angles of the dipole moment $\Theta$, periods of the star $P$, and its magnetic moment $\tilde{\mu}$. The disc is the same in all simulation runs. Our simulations and analysis show that the location and shape of the spots are determined by the following parameters.

(i) Misalignment angle $\Theta$. We varied this parameter from very small values up to relatively large values: $\Theta = 2^\circ$, $5^\circ$, $10^\circ$, $20^\circ$, $30^\circ$.

(ii) Magnetospheric radius $r_m$, normalized by stellar radius $r_m/R_*$. The magnetospheric radius can be determined in several ways. For example, an equality of matter and magnetic stresses, $\beta_1 = (p + \rho v_i^2)/(B^2/8\pi) = 1$, usually corresponds to the innermost region of the disc, where the disc density decreases sharply (e.g. Romanova et al. 2011). On the other hand, the condition $\beta = 8\pi p / B^2$ often corresponds to the region in the disc where the funnel flow begins (Bessolaz et al. 2008). Our simulations show
Table 1. From left to right: dimensionless period of the star \( P_\ast \), dimensionless corotation radius \( r_c \); corotation radius normalized to stellar radius, \( r_c/R_\ast \); dimensionless period of the star \( P_\ast \), for different types of stars.

| \( P_\ast \) (dim) | \( r_c \) | \( r_c/R_\ast \) | \( P_\ast \) (CTTSs) | \( P_\ast \) (White dwarfs) | \( P_\ast \) (Neutron stars) |
|-------------------|----------|----------------|----------------|----------------|----------------|
| 1.8               | 1.5      | 4.3            | 3.2 d          | 52 s           | 4.0 ms         |
| 2.4               | 1.8      | 5.1            | 4.3 d          | 70 s           | 5.3 ms         |
| 2.8               | 2.0      | 5.7            | 5.0 d          | 81 s           | 6.2 ms         |

that the condition \( \beta = 1 \) corresponds to the radius from which matter flows along a funnel stream to the central regions of the spots. This is why we chose the condition \( \beta = 1 \) to derive \( r_m \). We varied the size of the magnetosphere using the dimensionless magnetic moment of the star \( \mu \) in the range of \( \mu = 0.5–2 \), and obtained the magnetospheric radius in the range of \( r_m/R_\ast \approx 2.4–4.9 \).

(iii) Dimensionless period of the star \( P_\ast \), or the ratio \( r_c/R_\ast \), where \( r_c = [GM_\ast(P_\ast/2\pi)^2]^{1/3} \) is the corotation radius (here \( P_\ast \) is the dimensional period). In our model, we use either dimensionless period \( P_\ast \), or the ratio \( r_c/R_\ast \). We chose three periods for the star: \( P_\ast = 1.8 \), 2.4, 2.8 which correspond to \( r_c/R_\ast = 4.2, 5.1, 5.3 \) (see Table 1 for the conversion between dimensional and dimensionless values).

4.1 Obtaining the spots’ parameters from simulations

We derive parameters for the shape and position of the spot in each simulation run. Simulations usually show a non-stationary disc–magnetosphere interaction which leads to variation in the accretion rate and in the spot’s position and shape in each simulation run. However, the parameters of the spot vary around some typical value. Therefore, we choose an interval of time in which the funnel flow is established and the accretion rate has a plateau with relatively low variation, and derive an averaged spot (energy flux distribution) using multiple moments in time from this interval. We approximate the averaged spot with equation (1) and derive the best-fitting parameters: \( (\theta_0, \phi_0), (\theta_1, \phi_1) \) and \( T_c \). We observe that in the majority of our cases, formula (1) describes the shape of the averaged spot well, with typical values of the rms residuals of less than 10 per cent. We also use this time interval to derive time-averaged values of accretion rate \( \tilde{M} \) and magnetospheric radius \( r_m \) (see Tables B1 and B2).

The process of averaging has different accuracy in the cases of small and large \( \theta \). At relatively high misalignment angles, \( \Theta = 20°–30° \), the spot varies only slightly in time, and the averaged spot represents the instantaneous spots at different moments of time with relatively high accuracy. We can roughly estimate the uncertainty in averaging as \( \leq 10 \) per cent. However, in cases of small misalignment angles, \( \Theta < 10° \), in particular when the values of \( \Theta \) are very small (\( \Theta = 2° \) and \( 5° \)), a spot strongly changes its position in time. In some cases, it can rotate several times around the magnetic pole during a chosen interval of time. In these cases, the expected uncertainty in averaging is larger, in particular when the magnetosphere rotates slower than the inner disc (that is, at larger values of \( r_c/r_m \)). In cases of a small \( \Theta \), we can estimate the uncertainty in averaging for the azimuthal position \( \phi_0 \) and the azimuthal length \( \phi_1 \) of the spot to be as high as \( 30–50 \) per cent. However, the uncertainty for the longitudinal position \( \theta_0 \) and the thickness \( \theta_1 \) of the spot are smaller (within 10 per cent).

Tables B1 and B2 list the parameters of the averaged spots obtained in the simulations for different values of \( \Theta, \mu \) and \( P_\ast (r_c/R_\ast) \). They also list the averaged values of accretion rate, \( \tilde{M} \), magnetospheric radius, \( r_m \), and the rms residuals obtained during the fitting of an averaged spot with equation (A2).

Figs B1, B2 and B3 show each of the five parameters of the spot for all simulation runs (symbols) for different misalignment angles \( \Theta \) (different colours) and different periods of the star \( P_\ast \) (different symbols) as a function of \( r_m/R_\ast \). Below, we discuss each parameter and derive useful dependences.

4.2 Polar position of the spot, \( \theta_0 \)

We derive the polar position of the spot using two methods: (1) simple analytical approach for a pure dipole field and (2) numerical simulations.

4.2.1 Polar position of the spot derived from the dipole model

One of the parameters – the polar position of the spot, \( \theta_0 \), – can be derived analytically.

Fig. 3 (left-hand panel) shows a schematic view of a star with a dipole field tilted at angle \( \Theta \). The dipole field line which crosses the inner disc radius \( r = r_m \) is shown. This line also shows the expected position of the spot. The dipole field lines obey the relation \( r/\sin^2\theta = \text{constant} \), where \( \theta \) is measured from the magnetic axis. For simplicity, let us assume that the centre of the spot is in the \( \mu-\Omega \) plane.

Referring to Fig. 3, we have

\[
\sin^2 \left( \frac{\theta_0}{2} - \Theta \right) = \frac{R_\ast}{\sin^2 \theta_0},
\]

which gives us the location of the spot, \( \theta_0 \), as

\[
\theta_0 = \sin^{-1} \left( \sqrt{\frac{R_\ast}{r_m}} \cos \Theta \right).
\]

In this idealized model, the derived angle \( \theta_0 \) shows the approximate position of the spot in the polar direction, \( \theta_0 \approx \theta_s \). The position of the spot depends on truncation radius \( r_m/R_\ast \) and misalignment angle \( \Theta \).

This could be a useful approach in finding the polar position of the spots. However, numerical simulations show that the polar position is different from that derived analytically, because the magnetic field is compressed by the disc. Compression changes the shape of the field lines, affecting the position of the spots on the star.

4.2.2 Polar position of the spots derived from numerical simulations

Tables B1, B2 and Fig. B1 (left-hand panel) show values of \( \theta_0 \) for all simulation runs. It can be seen that the polar position varies within a narrow range of angles, \( 15.5° \lesssim \theta_0 \lesssim 18° \). This result is somewhat unexpected, because according to the simple estimates performed in the previous section, larger values of \( \theta_0 \) are expected for smaller misalignment angles \( \Theta \) and smaller magnetospheres (smaller \( r_m/R_\ast \)), and vice-versa. According to equation (3), if we use extreme parameters of \( \Theta = 2° \) (the smallest misalignment angle) and \( r_m/R_\ast = 2.4 \) (the smallest magnetosphere), then we expect the largest angle to be \( \theta_0 = 40°/2 \). For \( \Theta = 30° \) and the largest magnetosphere, \( r_m/R_\ast = 4.9 \), we expect the smallest angle to be \( \theta_0 = 23° \). Both angles are larger than the angles obtained in our numerical simulations. There is an uncertainty in averaging the shape of the spot and in approximating the averaged spot with equation (1) (see Section 4.1). However, these errors are usually...
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Figure 3. Left-hand panel: the sketch shows a star with a dipole magnetic field, the position of the inner disc $r = r_m$ and the polar angle $\theta_3$ of the spot's position on the star. The magnetic axis $\mu$ of the dipole moment is tilted at an angle $\theta$ about the rotational axis $\Omega$. Right-hand panel: density distribution (colour background) and magnetic field lines (yellow lines) in a typical simulation run ($\Theta = 30^\circ$, $r_c = 1.8$, $\hat{r}_3 = 1.5$, at $t = 15$). The red lines show the dipole field lines at $t = 0$. The external field lines are chosen such that both the yellow and the red lines start at the inner magnetospheric radius $r_m$, which is marked as a black circle.

less that 10 per cent, and are therefore not responsible for this inconsistency.

To understand this phenomenon, we compared the shapes of the magnetospheres obtained in our simulations with the shape of the dipole magnetosphere. An example of such a comparison is shown in Fig. 3 (right panel). Within the inner magnetosphere, the field lines of the simulated magnetosphere (yellow lines) depart only slightly from the dipole field line (shown in red). However, in the outer magnetosphere, the field lines of the modelled magnetosphere strongly depart from the dipole field lines. Matter flowing along the compressed field line forms a spot closer to the magnetic pole (at smaller $\theta_3$), compared with the theoretically predicted position of the spot in case of a dipole field.

Fig. 3 also shows the position of the magnetospheric radius $r_m$, where the $\beta = 1$ line crosses the equatorial plane (see two black circles). It can be seen that the field lines which start at $r_m$ connect the disc with the central parts of the spots.

We also plot slices of density distribution in several cases with ‘extreme’ parameters. Namely, we choose those simulation runs where parameters $\Theta$ and $r_m/R_*$ are very large or very small. Thus, according to equation (3), the spots are expected to be at different polar positions $\theta_3$. Fig. 4 shows that in spite of the difference in parameters, the polar position of the spot $\theta_3$ is approximately the same and corresponds to the narrow range of values observed in Fig. B1. This can be explained by the fact that in cases of a large magnetosphere compression is not very strong, but is sufficient enough to shift the spot position towards the magnetic pole and yields a value of $\theta_3$ which is different from that predicted theoretically in the case of a dipole field. In cases of small magnetospheres, theoretically predicted angle $\theta_3$ is larger than in cases of a large magnetosphere. However, compression of the magnetosphere is stronger, and the spot has a stronger shift towards the magnetic pole, compared with cases of a large magnetosphere. This effect leads to similar angles $\theta_3$ in our simulation set. It is clear that at even smaller (than in our set) magnetospheres, the theoretical dependence will not work, due to an even stronger compression of the magnetosphere. However, the formula may be applicable at very large $r_m/R_*$, where compression is expected to be weaker.

4.3 Azimuthal position of the spot, $\phi_0$

Fig. B1 (right-hand panel) shows the azimuthal position of the spots for all simulation runs. The figure shows a wide range of angles $\phi_0$. We noticed that some correlation can be seen between groups of runs with the same period of the star $P_*$ (which are marked with the same symbol in the figure). To better demonstrate this apparent correlation, we divided all data into three groups corresponding to three periods of the star, and obtained much clearer dependences of $\phi_0$ on $r_m/R_*$ (see Fig. 5, left-hand panel). The least-square approximation shows linear correlations:

$$\phi_0 = 84^\circ - 17 \frac{r_m}{R_*} \quad \text{for} \quad P_* = 1.8 \left( \frac{r_c}{R_*} = 4.3 \right)$$

$$\phi_0 = 206^\circ - 39 \frac{r_m}{R_*} \quad \text{for} \quad P_* = 2.4 \left( \frac{r_c}{R_*} = 5.1 \right)$$

$$\phi_0 = 300^\circ - 57 \frac{r_m}{R_*} \quad \text{for} \quad P_* = 2.8 \left( \frac{r_c}{R_*} = 5.7 \right).$$

It can be seen that angle $\phi_0$ varies more rapidly in the case of a slower rotating star ($P_* = 2.8$), and more slowly in cases of a more rapidly rotating star ($P_* = 1.8$). These dependences can be interpreted as an azimuthal phase-shift that varies with accretion rate: smaller values of $r_m/R_*$ correspond to higher accretion rates and stronger shifts $\phi_0$.

One can see that the linear fit is very approximate, and there is a significant deviation of points from the approximating lines, particularly in the case of $P_* = 1.8$. These deviations are probably connected with an uncertainty which arises when we obtain the averaged spot (see Section 4.1). The uncertainty is larger in cases of smaller misalignment angles ($\Theta \lesssim 10^\circ$) and larger ratios $r_c/r_m$. In addition, the deviation may be connected with the fact that we changed both the period of the star and $\Theta$.

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2 Fig. B1 (left-hand panel) shows that for groups of points having same parameters $\Theta$ and $P_*$, the scatter is usually less than $1^\circ$, and is only a fraction of a degree for many groups.
The plots show that the polar angle $\theta_0$ (the angle between solid lines) is approximately the same for different parameters of the star. Top and bottom panels show cases of large ($\Theta = 30^\circ$) and small ($\Theta = 5^\circ$) misalignment angles. The left- and right-hand panels show cases of small ($\tilde{\mu} = 0.5$) and large ($\tilde{\mu} = 1.8$) magnetic moments of the star. Light-grey line shows the $\beta = 1$ line, and the yellow dots show the position of the magnetospheric radius $r_m$. The dipole field lines, which cross the magnetospheric radius are shown in cyan. The dashed black lines show the theoretically predicted polar position of the spot, $\theta_s$ (see equation 3).

Figure 5. Left-hand panel: dependence of $\phi_0$ on the magnetospheric radius $r_m/R_*$ for different periods of the star $P_*$. The case of $\Theta = 2^\circ$ is removed from the set because the spot changes its position, and its averaged preferred position has a large degree of uncertainty. Right-hand panel: dependence of the azimuthal position of the spot $\phi_0$ on the ratio of magnetospheric to corotation radius $r_m/r_c$. The least-square fit does not take into account the $\Theta = 2^\circ$ case (green circles).

The azimuthal position of the spot depends on the relative velocity of the inner disc with respect to the magnetosphere. Hence, the dependence of $\phi_0$ on the ratio $r_m/r_c$ is expected. We combined all the simulation runs and were able to obtain a good correlation between $\phi_0$ and $r_m/r_c$ (see Fig. 5, right-hand panel). The least-square fit shows a linear dependence:

$$\phi_0 = 148^\circ - 135^\circ \frac{r_m}{r_c}$$

Therefore, if a value of $r_m/r_c$ is known, then the approximate azimuthal position of the spot is known as well. The linear fit describes the distribution of points relatively well. The points with the largest deviation correspond to the cases of smallest misalignment angles ($\Theta = 2^\circ$ and $5^\circ$), where the measurement error is largest (see Section 4.1). To decrease this error, we excluded the points corresponding to $\Theta = 2^\circ$ from the set (but kept them in the plot as green circles). In Fig. 5 we show the fit which excludes only the cases where $\Theta = 2^\circ$. Note that when we keep all the points, the slope of the fitting line changes by about $10^\circ$. However, when we remove both $\Theta = 2^\circ$ and $5^\circ$ from the set, the slope changes by only $1^\circ$.

Fig. 6 demonstrates how the spot position $\phi_0$ varies with $r_m/R_*$. For a fixed magnetic moment of the star, variation of $r_m/R_*$ is associated with variation in accretion rate. This plot demonstrates that at larger accretion rates (smaller $r_m/R_*$) the azimuthal shift of
the spot $\phi_0$ is larger, compared with the cases of smaller accretion rates.

### 4.4 Polar width of the spot, $\theta_1$

Fig. B2 (left-hand panel) shows that the polar width of the spots varies within a relatively narrow interval, $7^\circ \lesssim \theta_1 \lesssim 10^\circ$, and is $\theta_1 = 7^\circ - 8^\circ$ for most models. In a few cases, it is larger: $\theta_1 = 9^\circ$ and $\theta = 10^\circ$ in one simulation run. There is a weak correlation between $\theta_1$ and $r_m/R_c$, with $\theta_1$ being smaller in the cases of larger $r_m/R_c$.

The scatter of points within the same group of simulation runs (same $\Theta$ and $P_*$) is smaller than $1^\circ$, and is only a fraction of a degree for many groups. These estimates give us the upper limit for possible error associated with spot averaging and finding parameter $\theta_1$ from the simulations. In reality, we take parameter $\theta_1$ for different sizes of the magnetosphere $r_m/R_c$ in each group, so that the scatter in $\theta_1$ mainly reflects the variation of $\theta_1$ with magnetospheric size. In this case, the measurement error is even smaller.

### 4.5 Azimuthal width of the spot, $\phi_1$

Fig. B2 (right-hand panel) shows the azimuthal width of the spots $\phi_1$ for all simulation runs. This parameter is also scattered significantly, like $\phi_0$. However, some ordered variation can be seen if we consider the subsets of runs with the same misalignment angle $\Theta$ (marked by different colours). The points with a large $\Theta$ are located at the bottom part of the plot, where $\phi_1 \approx 100^\circ$ and the spots have the shape of an arc. The points with smallest $\Theta$ are located towards larger $\phi_1$, and have the shape of a ring.

Taking into account this (expected) dependence of $\phi_1$ on $\Theta$, we separate all the runs into subsets with a different $\Theta$. We notice that the correlations become stronger when we plot $\phi_1$ as a function of $r_m/R_c.$ Fig. 7 (left-hand panel) shows subsets of data for each $\Theta$ and a linear approximation for each subset. At large misalignment angles, $\Theta = 20^\circ - 30^\circ$, the azimuthal width does not vary much with $r_m/R_c$, and $\phi_1 \approx 90^\circ - 100^\circ$. However, $\phi_1$ strongly increases with $r_m/R_c$ for small $\Theta$. One can see that the linear fit is more accurate in the cases of larger misalignment angles, $\Theta = 20^\circ$ and $30^\circ$, where the deviation of points is small. Deviation from the linear fits increases for cases of a smaller $\Theta$, because the measurement error in those cases is larger (see Section 4.1).

The lines which approximate sets of data have different slopes, and thus, the dependences on $\Theta$ are different. To derive these dependences, we choose several values of $r_m/R_c$ and take the values of $\phi_1$ that correspond to linear fits at different $\Theta$. We plot these values in Fig. 7 (right-hand panel). The plot illustrates the dependence of $\phi_1$ on $\Theta$. The figure shows that for $\Theta \gtrsim 15^\circ$, the width of the spots is relatively small for all values of $r_m/R_c$. However, for $\Theta \lesssim 15^\circ$, the shape varies from an arc for smaller $r_m/R_c$ to a ring for larger $r_m/R_c$. For a star with the same corotation radius $r_c$, larger/smaller values of $r_m/R_c$ correspond to a smaller/larger accretion rate.

Fig. 8 demonstrates the shape of the spots in the case of the same magnetic moment $\mu$ and period $P_*$ of the star (approximately the
same value of \( r_m/r_c \), but a different misalignment angle of the dipole \( \Theta \). As we can see, the shapes of the spots vary from a ring (at small \( \Theta \)) to an arc (at large \( \Theta \)).

### 4.6 Energy flux in the centre of the spots, \( F_c \)

From our simulations and fits of the observed spots with equation (1), we obtain the maximum energy flux in the centre of the spot \( F_c \) (in dimensionless units). The dimensional flux \( F_c \), which is used in equation (1), can be derived from equation (A2). It can be seen that the main dimensionless variable in equation (A2) is the ratio \( \tilde{F}_c = F_c/\tilde{\mu}^2 \). This is why we put this variable into Tables B1 and B2 and use it in Fig. B3.

Fig. B3 shows that the values of \( \tilde{F}_c \) are mainly confined to the interval \( 0.5 \leq \tilde{F}_c \leq 4 \). It is larger, \( \tilde{F}_c \approx 6.5 - 8.5 \), in a few points corresponding to a smaller magnetosphere, \( r_m/R_\ast \approx 2.8 - 3.2 \). The fluxes are about two times smaller in cases of a small \( \Theta \) (see red line in Fig. B3).

### 5 MAGNETOSPHERIC RADIUS

Magnetospheric radius \( r_m \) (which corresponds to the boundary between the magnetosphere and the disc) is often derived from the condition that it is proportional to the Alfvén radius \( r_A \), derived for spherical accretion under the assumption that the magnetic pressure in the magnetosphere equals to the ram pressure of free-falling matter (e.g. Lamb, Pethick & Pines 1973):

\[
\frac{r_m}{r_A} \approx \mu^2 M^2 / (2GM^2M^2)^{1/7},
\]

where \( k_A \) is the dimensionless coefficient of the order of unity. This formula has been tested a few times. In these tests, the inner disc radius has been derived in two different ways. First, it has been derived from the simulations using conditions \( \beta = 1 \) or \( \beta = 1 \). Second, it has been derived using formula 8 for \( r_A \), where values such as \( \mu \) and \( M \) were taken from the simulations. Long, Romanova & Lovelace (2005) performed such comparisons in three models with slightly different parameters \( \mu \) and the condition \( \beta = 1 \), and obtained an average approximate value of \( k_A \approx 0.5 \). Bessolaz et al. (2008) used a variety of definitions for the inner disc radius, such as \( \beta = 1 \) and \( \beta = 1 \), and obtained \( k_A \approx 0.4 \) in the case of the \( \beta = 1 \) condition (see also Zanni & Ferreira 2013). However, none of this work studied the dependence of \( r_m \) on \( \mu \) or \( M \).

We have a sufficiently large set of simulations, so these dependences can be tested here. As the first step, we take the formula for the Alfvén radius (equation 8), substitute in our dimensional units (see Section A) and derive the magnetospheric radius in a dimensionless form:

\[
\frac{r_m}{R_\ast} = \frac{k_A}{0.35} \left( \frac{\tilde{\mu}^2}{M} \right)^{2/7}. \tag{9}
\]

We can test this formula in our simulations, where the dimensionless magnetic moment \( \tilde{\mu} \) is the main initial parameter in each simulation run, and the dimensionless accretion rate \( \tilde{M} \) is the output from each simulation run. We also derived the ratio \( r_m/R_\ast \) from the condition that \( \beta = 1 \) in each simulation run. Fig. 9 shows the resulting dependence of \( r_m/R_\ast \) on \( \tilde{\mu}^2/M \). The dependence is well approximated by the power law in the following form:

\[
\frac{r_m}{R_\ast} \approx 2.2 \left( \frac{\tilde{\mu}^2}{M} \right)^{1/5}. \tag{10}
\]

It should be noted that our measurement of the averaged value of \( r_m \) does not depend on the misalignment angle \( \Theta \) (see Section 4.1), so that the measurement error is approximately the same for different \( \Theta \). This is why the deviation of different points from the linear fit is not high. This is different from the plots where parameters of the spots are present. We can see that the power obtained from the simulations, \( a \approx 0.2 = 1/5 \) is smaller than the power given by the formula for the Alfvén radius, \( a = 2/7 \approx 0.29 \). We suggest that the compression of the magnetosphere is probably responsible for different dependence of \( r_m \) on \( \tilde{\mu}^2/M \). Namely, when the magnetosphere is compressed, the magnetospheric radius varies slower with magnetic moment and accretion rate.
From the comparison of equations (10) and (9) we can derive coefficient $k_\lambda$, which is not a constant anymore:

$$k_\lambda \approx 0.77 \left( \frac{\mu^2}{M} \right)^{-0.086}.$$  \hspace{1cm} (11)

The dependence of $k_\lambda$ on $\mu^2/M$ is weak, so that for the range of $2 \lesssim \mu^2/M \lesssim 50$ used in our set, we obtain $k_\lambda$ in the range of $0.55 \lesssim k_\lambda \lesssim 0.72$. These values of parameter $k_\lambda$ are in agreement with the values derived earlier in axisymmetric simulations (e.g. Long et al. 2005).

Next, we use equations (10) and (A1) to derive the ratio $r_m/R_\ast$ through dimensional parameters:

$$\frac{r_m}{R_\ast} \approx 1.06 \left( \frac{\mu^4}{M^2 G M_\ast R_\ast^2} \right)^{1/10}.$$  \hspace{1cm} (12)

This formula can be used if the magnetic moment of the star and the other parameters are known.

We should note that the dependence derived in equation (10) is relevant for the magnetospheric radii $2.5 \lesssim r_m/R_\ast \lesssim 5$ investigated in this work. A separate analysis should be done for larger and smaller radii. We anticipate that at larger $r_m/R_\ast$ compression will be less important and the power in equation (12) will be larger than 0.2, while at smaller $r_m/R_\ast$ compression will be more important and the power will be smaller. Special simulation runs should be performed to investigate this dependence at larger and smaller $r_m/R_\ast$.

6 SOME PRACTICAL CONSIDERATIONS

The above analysis shows that the shapes and positions of the spots are determined by three parameters: (1) the misalignment angle of the dipole moment $\Theta$, (2) the normalized corotation radius $r_c/R_\ast$, and (3) the normalized magnetospheric radius $r_m/R_\ast$. Observations of magnetized stars are usually complex, and most of the parameters of the star and the inner disc are not known, or can be estimated only approximately. Below, we discuss possible ways to derive these parameters.

(i) It is often the case that the period of the star $P_\ast$ is one of the best-known parameters. If mass $M_\ast$ and radius $R_\ast$ of the star are known, then the dimensionless period $P_\ast$ and the ratio $r_c/R_\ast$ can be derived.

(ii) Many evolved magnetized stars are expected to be in the rotational equilibrium state, where a star’s spin varies around the value corresponding to rotational equilibrium. In these cases, the magnetospheric radius can be derived from condition $r_m = kr_c$ (where $k \approx 1$). If a star is not in rotational equilibrium, then this condition cannot be used.

(iii) One of the most important parameters, $\phi_0$, which stands for the azimuthal shift of the spot, depends only on the ratio $r_m/r_c$ (see equation 5). If a star is in the rotational equilibrium state, then the value $r_m/r_c$ can be taken from numerical simulations. Axisymmetric simulations performed by Long et al. (2005) show that $r_m/r_c \approx 0.7$–0.8, while Zanni & Ferreira (2013) found $r_m/r_c \approx 0.6$. This ratio is smaller, when a larger amount of angular momentum flows into the corona along inflated or partially inflated field lines. This ratio can also vary in time. Therefore, the above numbers are approximate and can be used only as estimates.

(iv) The misalignment angle of the dipole’s magnetic moment $\Theta$, is known approximately in some cases. For example, in a few CTTSs, the magnetic field distribution on the surface of the star has been derived from polarimetric observations and then approximated with the set of tilted magnetic moments of a different order. It was found that the misalignment angle of the dipole component is usually small, $\Theta \approx 10^\circ$–$20^\circ$ (e.g. Donati et al. 2008). In some cases, the misalignment angle can be obtained from the shapes of observed light curves and comparisons with those obtained in 3D MHD numerical simulations (Romanova et al. 2004). For example, for some probable inclination angle of $i \approx 45^\circ$, a nearly symmetric light curves with one peak per period yield a relatively small misalignment angles, $\Theta \lesssim 3^\circ$, while light curves with two peaks per period yield a larger $\Theta$.

7 APPLICATIONS TO DIFFERENT MAGNETIZED STARS

The results of our simulations can be applied to magnetized stars with magnetospheric radii $r_m \approx (2$–$5)R_\ast$. Knowledge of the shapes and positions of the spots will allow for accurate calculation of the light curves from rotating spots and for comparison with observations. In addition, the motion of the spots on the stellar surface may help us understand the observed phase shifts in different stars. Below, we discuss applications of our results to several classes of magnetized stars.

(1) AMPs. The magnetic field of AMPs is relatively weak, $10^9$–$10^{10}$ G, and the sizes of their magnetospheres may be within the range investigated in this paper. One of the important results obtained from our research is the dependence of the phase shift of the spots, $\phi_0$, on the magnetospheric radius $r_m$. This result is particularly important for observations. For example, in millisecond pulsars, it may lead to phase shifts in light curves (Papitto et al. 2007; Patruno et al. 2009) and to timing noise (e.g. Poutanen et al. 2009). Papitto et al. (2007) found a correlation between the pulse phase shifts and the X-ray flux in millisecond pulsar XTE J1814–338 (see also Patruno et al. 2009). The authors argued that the observed phase shifts were due to movements of the hotspot in response to variation in accretion rate. This phenomenon can be explained by the correlation between hotspot longitude and the location of the magnetospheric radius, as shown in this paper (see equations 4–6). Namely, the variation in accretion rate leads to the variation in magnetospheric radius $r_m$ and to a shift in the azimuthal position of the spot, $\phi_0$. When the accretion rate increases, $r_m$ decreases and the spot shift is stronger in the direction of disc rotation. This is indeed the correlation observed from this source (see fig. 1 of Papitto et al. 2007). A similar trend is also observed in the case of another AMP, XTE J1807–294 (Riggio et al. 2008).

(2) Cataclysmic variables. Most of the magnetic cataclysmic variables have larger magnetospheres compared with those investigated in this paper. The intermediate polars (IPs), which clear the
gap in the inner disc and have regular pulsations, usually show periods of a few hundred seconds and their magnetospheric radii are usually larger than \( \sim 10 R_\star \) (e.g. Warner 1993; Wynn 2000). The results of our paper cannot be directly applied to those stars. However, a few IPs have much smaller periods (and therefore smaller magnetospheres), including the prototype IP DQ Her with a period of 71 s, or twice this value (Walker 1956; Bloemen et al. 2010). This period is close to the periods corresponding to our simulations (see Table 1), and our results can be applied to this star. In another example, AE Aqr, a star rotates rapidly with a period of 33 s. However, this CV is probably in the propeller regime of accretion, where periods of accretion alternate with periods of ejection (e.g. Wynn, King & Horne 1997). We do not include the cases corresponding to the propeller regime in our model, due to strong variation or absence of spots.

(3) CTTSs. Magnetospheres of CTTSs are usually within a few stellar radii, so that the results of our paper can be applied to these stars. It is often the case that no precise period is observed in these stars. Instead, the period derived at different times of observation is different, and therefore varies with time (e.g. Rucinski et al. 2008). This phenomenon can be explained by hotspots which move in the azimuthal direction in response to the accretion rate. For the investigation of these movements, formulae (4)–(6) can be used. In CTTSs (and possibly in other stars) the magnetic field may have higher order multipolar components which channel matter near the star, and therefore the shape of the spots may be more complex (e.g. Long, Romanova & Lovelace 2008; Long et al. 2011; Romanova et al. 2011). In this case, equation (1) cannot be used.

We anticipate that equation (1) could be applicable to stars with much larger or smaller magnetospheres. The formula reflects the fact that (1) in the funnel stream, most of the energy flows in the interior parts of the stream and decreases outward in both the azimuthal and meridional directions; (2) matter flows along the field lines of the closed magnetosphere towards the magnetic pole, and hence has a crescent shape around this pole. However, this suggestion should be checked in future simulations, in particular for cases of very large magnetospheres, where the azimuthal motion of the funnel streams and the spots are expected to be smaller.

8 CONCLUSIONS AND DISCUSSION

In this work, we derived a useful analytical formula (equation 1) that provides an approximation for arc-shaped and ring-shaped spots which form in accreting magnetized stars. Multiple 3D MHD simulation runs were performed to find the positions of the spots \((\theta_0, \phi_0)\), their shapes \((\theta_1, \phi_1)\) and their dependences on the period of the star \(P_\star\), the misalignment angle of the dipole \(\Theta\) and the ratio of the magnetospheric radius to the radius of the star, \(r_\star/R_\star\). The main conclusions are as follows.

(1) The polar positions of the spots \(\theta_0\) vary within the narrow range of \(15.5 \lesssim \theta_0 \lesssim 12.8\). These angles are different from the polar angles obtained with the simple analytical formula (see equation 3), where \(\theta_0\) has larger values and varies considerably with \(r_\star/R_\star\) and the dipole misalignment angle \(\Theta\). We noticed that in all our numerical models the external closed magnetosphere is compressed and deforms significantly from the dipole shape, which leads to the misplacement of the spot.

(2) The azimuthal positions of the spots \(\phi_0\) strongly vary with the magnetospheric radius \(r_\star/R_\star\) (accretion rate). They increase, when \(r_\star\) decreases. For the same interval of \(r_\star/R_\star\), \(\phi_0\) varies more rapidly for stars with larger periods.

(3) An important correlation has been found for the dependence of \(\phi_0\) on the ratio \(r_\star/R_\star\), which is valid for all simulation runs.

(4) The polar widths of the spots \(\theta_1\) vary within the narrow interval of \((7^\circ \lesssim \theta_1 \lesssim 9^\circ)\), and the majority of the spots have a width of \((7^\circ \sim 8^\circ)\).

(5) The azimuthal widths of the spots \(\phi_1\) strongly depend on the dipole misalignment angle \(\Theta\). They vary from \(\phi_1 = 90^\circ \sim 100^\circ\) for \(\Theta = 30^\circ\) (arc-shaped spots) to \(\phi_1 \approx 600^\circ\) for \(\Theta = 2^\circ\) (ring-shaped spots).

(6) We also used our data to check the formula for the Alfvén radius, where the main dependence is \(r_\text{m} \sim (\mu^2/M_\star)^{3/7}\). We found that the dependence is more gradual, \(r_\text{m} \sim (\mu^2/M_\star)^{1/3}\), which can be explained by the compression of the magnetosphere by the disc matter and by the non-dipole shape of the magnetic field lines of the external magnetosphere.

Once these parameters are known, the analytical formula given by equation (1) can be used to model the energy flux distribution in the spot. The light curves can be calculated using a separate numerical program. In this program, the radiation model, the anisotropy of the radiation, and the inclination angle of the system can be taken into account. For neutron stars, this code should take into account relativistic light-bending, Doppler-shift and other relativistic effects (see, e.g. Kulkarni & Romanova 2005).

Here, we present results for a relatively narrow interval of the ratio \(0.6 \lesssim r_\text{m}/R_\star \lesssim 1.1\), which corresponds to ordered accretion through two funnel streams. The smallest values correspond to a boundary between stable and unstable regimes of accretion, where \(r_\text{m}/R_\star \sim 0.6\) accretion proceeds in the equatorial plane through the magnetic Rayleigh–Taylor instability. The largest value, \(r_\text{m}/R_\star \approx 1.1\), corresponds to the propeller regime, in which the funnel accretion is forbidden by the centrifugal barrier (e.g. Illarionov & Sunyaev 1975; Lovelace, Romanova & Bisnovatyi-Kogan 1999).

Of course, the analytical formula (equation 1) describing the shape of the spot can be applied only in the cases where the magnetic field of the star has a strong dipolar component. In the case of a more complex field, the higher order component of the field may govern the funnel flow near the star, and the shapes of the spots will be determined by higher order multipoles of the field (e.g. Long et al. 2011; Romanova et al. 2011). This case should be investigated separately.

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REFERENCES

Alencar S. H. P. et al., 2010, A&A, 519, 88
Bachetti M., Romanova M. M., Kulkarni A., Burderi L., di Salvo T., 2010, MNRAS, 403, 1193

\(^4\) Recent simulations on a grid twice as fine show that this boundary occurs at somewhat larger values of \(r_\text{m}/R_\star\) (Blinova et al., in preparation).
We choose fiducial values for stellar mass $M_\ast$, radius $R_\ast$, and equatorial surface magnetic field $B_\ast$; the reference values of all other dynamical quantities are then obtained from these fiducial values.

The unit of distance is chosen to be $R_\odot = 2R_\odot/0.35$. The reference velocity is the Keplerian velocity at $R_\odot$, $v_0 = (GM_\odot/R_\odot^3)^{1/2}$ and $\omega_0 = v_0/R_\odot$ is the reference angular velocity. The reference time is the Keplerian rotation period at $R_\odot$, $P_0 = 2\pi R_\odot/v_0$.

The reference magnetic moment $\mu_0 = \mu/\bar{\mu}$, where $\bar{\mu}$ is the dimensionless magnetic moment of the star. The reference magnetic field is $B_0 = \mu_0/R_\odot^2$. The reference density is $\rho_0 = B_0^2/(\mu_0^2 R_\odot^2)$. The reference accretion rate is $\dot{M}_0 = \rho_0 v_0 R_\odot^2$. Using the relationships for $\rho_0$, $B_0$, and $v_0$, we obtain $M_0 = \mu/\bar{\mu}^2 (\mu_0^2 R_\odot^2)$. The dimensional accretion rate is $\dot{M} = \bar{\mu} M_0$, where $\bar{\mu}$ is the dimensionless accretion rate on to the surface of the star which is obtained from the simulations. Substituting the expression for $M_0$ and using $\mu = B_\ast R_\ast^2$, we obtain

$$\dot{M} = \left(\frac{\bar{\mu}}{\mu}^2\right) \frac{B_\ast^2 R_\ast^5}{(GM_\ast R_\ast^4)^{1/2}}. \tag{A1}$$

Thus, if the stellar parameters $M_\ast$, $R_\ast$, and $B_\ast$ are fixed, then the accretion rate is determined by $\dot{M}/\bar{\mu}^2$. Therefore, variation of parameter $\bar{\mu}$ can be interpreted as variation of the magnetospheric radius of the star, or it can be equally validly interpreted as a parameter that is responsible for variation of the accretion rate.

The reference energy flux is defined as $F_0 = \rho_0 v_0 R_\odot^2$. Using this formula, we get a similar conversion formula for the energy flux:

$$F = \left(\frac{\bar{\mu}}{\mu}^2\right) \frac{B_\ast^2 (GM_\ast)^{1/2}(0.35)^{1/2}}{R_\ast^{1/2}}. \tag{A2}$$

We use a dimensionless parameter $\bar{F} = F/\bar{\mu}^2$ to characterize the flow in the centres of the spots.

**APPENDIX B: PARAMETERS OF SPOTS**

Parameters of the spots for all simulation runs are given in Tables B1 and B2. Simulations were done for five misalignment angles $\Theta = 2^\circ$, $5^\circ$, $10^\circ$, $20^\circ$, $30^\circ$ and three (dimensionless) periods of the star $P_\ast = 1.8$, $2.4$, $2.8$, which correspond to normalized corotation radii of $r_\ast/R_\ast = 4.3$, $5.1$, $5.7$. In each set of runs with the same parameters $\Theta$ and $P_\ast$, we varied the dimensionless magnetic moment of the star $\bar{\mu}$ and observed that the magnetospheric radius $r_\ast$ increases with $\bar{\mu}$. The magnetospheric radius is an important parameter because the funnel flow starts approximately at $r_\ast$. In this paper, we use the dimensionless ratio $r_\ast/R_\ast$ so that results can be applied to different types of stars.
Figure B1. Left-hand panel: the polar coordinates of the spots position \( \theta_0 \) in all simulation runs as a function of the magnetospheric radius normalized to the stellar radius, \( r_m/R_\star \), for different misalignment angles: \( \Theta = 2^\circ \) (red colour), \( \Theta = 5^\circ \) (green colour), \( \Theta = 10^\circ \) (blue colour), \( \Theta = 20^\circ \) (pink colour) and \( \Theta = 30^\circ \) (black colour), and for different periods of the star: \( P_\star = 1.8 \) (thick solid lines, squares), \( P_\star = 2.4 \) (dashed lines, triangles) and \( P_\star = 2.8 \) (thin solid lines, circles). The symbols show results of individual simulation runs, while the lines show interpolation between runs performed for the same parameters \( \Theta \) and \( P_\star \).

Figure B2. Width of the spots in polar (\( \theta_1 \), see left-hand panel) and azimuthal (\( \phi_1 \), see right-hand panel) directions as a function of the magnetospheric radius normalized to the stellar radius, \( r_m/R_\star \). Symbols and lines are the same as in Fig. B1.

Figure B3. Dimensionless function \( \bar{F}_c = \bar{F}_c/\bar{\mu}^2 \), which determines the energy flux in the centres of the spots, shown for all simulation runs. Symbols and lines are the same as in Fig. B1.

Tables B1 and B2 show the polar and azimuthal positions of the spots (\( \theta_0, \phi_0 \)), their polar and azimuthal widths (\( \theta_1, \phi_1 \)) and the ratio \( \bar{F}_c/\bar{\mu}^2 \), which characterizes the dimensionless energy flux in the centres of the spots. If parameters \( \Theta, r_c/R_\star \) and \( r_m/R_\star \) are known, then Tables B1 and B2 provide the positions and shapes of the spots.

Figs B1, B2 and B3 show parameters of the spots for all simulation runs. Fig. B1 shows the polar and azimuthal positions of the spots. Fig. B2 shows the polar and azimuthal widths of the spots. Fig. B3 shows dimensionless energy flux in the centre of the spots. A set of runs with the same \( \Theta \) is marked by the same colour. A set of runs with the same \( P_\star \) is marked by the same symbols. The lines show interpolations between points corresponding to sets of runs with the same \( \Theta \) and \( P_\star \). These lines helped us derive important dependences discussed in the paper (see Section 4).
Table B1. Parameters of the spots for different dipole misalignment angles \( \Theta \), three periods of the star, \( P = 1.8, 2, 2.4 \), and different dimensionless magnetic moments of the star, \( \tilde{\mu} \). Polar and azimuthal angles (\( \theta_0 \) and \( \phi_0 \), respectively) determine the polar and azimuthal positions of the spots. Polar and azimuthal angles (\( \theta_1 \) and \( \phi_1 \), respectively) determine the size of the spots. The dimensionless function \( F_c = F_{c0}/\tilde{\mu}^2 \) determines the energy flux in the centres of the spots. The rms residuals obtained during the fitting of an averaged spot with equation (1) are shown. Table also shows normalized magnetospheric radius \( r_m/R_* \) and dimensionless accretion rate \( \dot{M} \). Cases where accretion becomes unstable are marked and were not included. In a few cases simulations were not performed; they are marked with long dash.

| \( P \), \( (r_c/R_*) \) | \( \tilde{\mu} \rightarrow \) | 0.5 | 1.0 | 1.5 | 2.0 |
|-------------------------|-----------------|------|------|------|------|
| 1.8 (4.3) | \( \theta_0, \phi_0 \) | 18.2, -8.9 | 17.7, 32 | 17.0, -9.8 | 17.1, -3.9 |
| | \( \theta_1, \phi_1 \) | 9.0, 230 | 8.9, 290 | 8.5, 630 | 7.7, 610 |
| | \( F_c \) | 1.7 | 1.5 | 1.8 | 2.7 |
| | rms residuals | 0.025 | 0.028 | 0.025 | 0.034 |
| | \( r_m/R_* \) | 2.90 | 3.80 | 4.46 | 4.84 |
| | \( \dot{M} \) | 0.069 | 0.070 | 0.073 | 0.098 |
| 2.4 (5.1) | \( \theta_0, \phi_0 \) | 17.4, 130 | 15.9, 75 | 16.1, 6 |
| | \( \theta_1, \phi_1 \) | Unstable | 8.3, 200 | 7.5, 270 |
| | \( F_c \) | 2.3 | 2.8 | 3.8 |
| | rms residuals | 0.029 | 0.056 | 0.046 |
| | \( r_m/R_* \) | 3.66 | 4.4 | 4.88 |
| | \( \dot{M} \) | 0.085 | 0.11 | 0.12 |
| 2.8 (5.7) | \( \theta_0, \phi_0 \) | 16.2, 120 | 15.9, 73 |
| | \( \theta_1, \phi_1 \) | Unstable | Unstable | 7.7, 260 | 7.6, 330 |
| | \( F_c \) | 3.6 | 4.4 |
| | rms residuals | 0.053 | 0.059 |
| | \( r_m/R_* \) | 4.24 | 4.7 |
| | \( \dot{M} \) | 0.12 | 0.15 |
| 2.4 (5.1) | \( \theta_0, \phi_0 \) | 16.8, 71 | 16.6, 17 |
| | \( \theta_1, \phi_1 \) | Unstable | 8.5, 210 | 7.5, 220 |
| | \( F_c \) | 2.5 | 3.8 | 4.3 |
| | rms residuals | 0.035 | 0.048 | 0.053 |
| | \( r_m/R_* \) | 3.66 | 4.31 | 4.82 |
| | \( \dot{M} \) | 0.11 | 0.10 | 0.13 |
| 2.8 (5.7) | \( \theta_0, \phi_0 \) | 15.6, 94 | 16.3, 25 |
| | \( \theta_1, \phi_1 \) | Unstable | Unstable | 7.9, 220 | 7.5, 150 |
| | \( F_c \) | 3.9 | 5.9 |
| | rms residuals | 0.063 | 0.075 |
| | \( r_m/R_* \) | 4.2 | 4.78 |
| | \( \dot{M} \) | 0.143 | 0.145 |
| 2.4 (5.1) | \( \theta_0, \phi_0 \) | 17.0, 54 | 17.9, 2.3 |
| | \( \theta_1, \phi_1 \) | 10.3, 230 | 8.8, 290 | 8.2, 330 | 7.7, 410 |
| | \( F_c \) | 1.3 | 1.7 | 2.1 | 2.7 |
| | rms residuals | 0.02 | 0.024 | 0.029 | 0.03 |
| | \( r_m/R_* \) | 3.0 | 3.83 | 4.46 | 4.86 |
| | \( \dot{M} \) | 0.07 | 0.069 | 0.074 | 0.093 |
| 2.8 (5.7) | \( \theta_0, \phi_0 \) | 15.6, 94 | 16.3, 25 |
| | \( \theta_1, \phi_1 \) | Unstable | Unstable | 7.9, 220 | 7.5, 150 |
| | \( F_c \) | 3.9 | 5.9 |
| | rms residuals | 0.063 | 0.075 |
| | \( r_m/R_* \) | 4.2 | 4.78 |
| | \( \dot{M} \) | 0.143 | 0.145 |

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Table B2. Same as in Table B1 but for $\Theta = 20^\circ$ and $\Theta = 30^\circ$.

| $P_\ast (r_c/R_*)$ | $\bar{\mu}$ → | 0.5 | 1.0 | 1.5 | 2.0 |
|---------------------|----------------|-----|-----|-----|-----|
| \(1.8\ (4.3)\)     | $\theta_0, \phi_0$ | 17.5, 31 | 17.5, 41 | 17.4, 35 | 18.0, 6.8 |
|                     | $\theta_1, \phi_1$ | 8.9, 160 | 8.0, 150 | 7.6, 120 | 7.5, 110 |
|                     | $T_c$             | 2.1 | 3.1 | 4.7 | 4.8 |
|                     | $r_{m}/R_*$       | 2.77 | 3.46 | 3.88 | 4.17 |
|                     | $M$               | 0.07 | 0.075 | 0.091 | 0.10 |
|                     | \(r_{m}/R_*)\)    | \(0.031\) | \(0.044\) | \(0.057\) | \(0.068\) |
| $\Theta = 20^\circ$ | $\theta_0, \phi_0$ | 17.5, 70 | 17.2, 56 | 17.1, 28 |
|                     | $\theta_1, \phi_1$ | Unstable | 7.9, 150 | 7.4, 96 | 6.8, 100 |
|                     | $M$               | 0.086 | 0.099 | 0.132 |
|                     | $T_c$             | 3.3 | 5.9 | 8.2 |
|                     | $r_{m}/R_*$       | 3.58 | 3.94 | 4.29 |
|                     | $\dot{M}$         | 4.33 | 4.13 |
| \(2.4\ (5.1)\)     | $\theta_0, \phi_0$ | 16.9, 69 | 16.9, 51 |
|                     | $\theta_1, \phi_1$ | Unstable | Unstable | 7.4, 120 | 7.0, 98 |
|                     | $M$               | 5.7 | 8.3 |
|                     | $r_{m}/R_*$       | 4.09 | 0.108 |
| \(2.8\ (5.7)\)     | $\theta_0, \phi_0$ | Unstable | Unstable | 7.4, 120 | 7.0, 98 |
|                     | $\theta_1, \phi_1$ | 8.9, 130 | 8.1, 140 | 7.7, 110 | 7.4, 100 |
|                     | $M$               | 0.056 | 0.078 | 0.092 | 0.11 |
|                     | $r_{m}/R_*$       | 2.89 | 3.50 | 3.91 | 4.24 |
|                     | $\dot{M}$         | 4.33 | 4.13 |
| $P_\ast (r_c/R_*)$  | $\bar{\mu}$ → | 0.5 | 1.0 | 1.5 | 2.0 |
| \(1.8\ (4.3)\)     | $\theta_0, \phi_0$ | 17.3, 61 | 16.7, 44 | 16.4, 30 | 17.1, 15 |
|                     | $\theta_1, \phi_1$ | 8.9, 130 | 8.1, 140 | 7.7, 110 | 7.4, 100 |
|                     | $T_c$             | 2.1 | 3.4 | 5.2 | 6.2 |
|                     | $r_{m}/R_*$       | 2.89 | 3.50 | 3.91 | 4.24 |
|                     | $M$               | 0.056 | 0.078 | 0.092 | 0.11 |
| $\Theta = 30^\circ$ | $\theta_0, \phi_0$ | 17.5, 59 | 16.0, 52 | 16.4, 38 |
|                     | $\theta_1, \phi_1$ | Unstable | 7.8, 100 | 7.5, 120 | 7.0, 93 |
|                     | $T_c$             | 4.1 | 5.5 | 9.5 |
|                     | $r_{m}/R_*$       | 3.67 | 4.09 | 4.29 |
|                     | $\dot{M}$         | 0.076 | 0.10 | 0.132 |
| \(2.4\ (5.1)\)     | $\theta_0, \phi_0$ | 16.5, 47 |
|                     | $\theta_1, \phi_1$ | Unstable | Unstable | – | 7.1, 94 |
|                     | $M$               | 8.5 | 8.5 |
| \(2.8\ (5.7)\)     | $\theta_0, \phi_0$ | Unstable | Unstable | – | 7.1, 94 |
|                     | $\theta_1, \phi_1$ | 8.5 | 8.5 |
|                     | $M$               | 4.44 | 0.12 |

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