Partial-Adaptive Submodular Maximization

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The goal of a typical adaptive sequential decision making problem is to design an interactive policy that selects a group of items sequentially, based on some partial observations, to maximize the expected utility. It has been shown that the utility functions of many real-world applications, including pooled-based active learning and adaptive influence maximization, satisfy the property of adaptive submodularity. However, most of existing studies on adaptive submodular maximization focus on the fully adaptive setting, i.e., one must wait for the feedback from all past selections before making the next selection. Although this approach can take full advantage of feedback from the past to make informed decisions, it may take a longer time to complete the selection process as compared with the non-adaptive solution where all selections are made in advance before any observations take place. In this paper, we explore the problem of partial-adaptive submodular maximization where one is allowed to make multiple selections in a batch simultaneously and observe their realizations together. Our approach enjoys the benefits of adaptivity while reducing the time spent on waiting for the observations from past selections. To the best of our knowledge, no results are known for partial-adaptive policies for the non-monotone adaptive submodular maximization problem. We study this problem under both cardinality constraint and knapsack constraints, and develop effective and efficient solutions for both cases. We also analyze the batch query complexity, i.e., the number of batches a policy takes to complete the selection process, of our policy under some additional assumptions.

1. Introduction

Adaptive sequential decision making, where one adaptively makes a sequence of selections based on the stochastic observations collected from the past selections, is at the heart of many machine learning and artificial intelligence tasks. For example, in experimental design, a practitioner aims to perform a series of tests in order maximize the amount of “information” that can be obtained to yield valid and objective conclusions. It has been shown that in many real-world applications, including pool-based active learning [Golovin and Krause (2011)], sensor selection [Asadpour and Nazerzadeh (2016)], and adaptive viral marketing [Tang and Yuan (2020)], the utility function is adaptive submodular. Adaptive submodularity a notion that generalizes the notion of submodularity from sets to
The goal of adaptive submodular maximization is to design an interactive policy that adaptively selects a group of items, where each selection is based on the feedback from the past selections, to maximize an adaptive submodular function subject to some practical constraints. Although this problem has been extensively studied in the literature, most of existing studies focus on the fully adaptive setting where every selection must be made after observing the feedback from all past selections. This fully adaptive approach can take full advantage of feedback from the past to make informed decisions, however, as a tradeoff, it may take a longer time to complete the selection process as compared with the non-adaptive solution where all selections are made in advance before any observations take place. This is especially true when the process of collecting the observations from past selections is time consuming. In this paper, we study the problem of partial-adaptive submodular maximization where one is allowed to make multiple selections simultaneously and observe their realizations together. Our setting generalizes both non-adaptive setting and fully adaptive setting. As compared with the fully adaptive strategy, our approach enjoys the benefits of adaptivity while using fewer number of batches. To the best of our knowledge, no results are known for partial-adaptive policies for the non-monotone adaptive submodular maximization problem. We next summarize the main contributions made in this paper.

- We first study the partial-adaptive submodular maximization problem subject to a cardinality constraint. We develop a partial-adaptive greedy policy that achieves a $\alpha/e$ approximation ratio against the optimal fully adaptive policy where $\alpha$ is the degree of adaptivity of our policy. One can balance the performance/adaptivity tradeoff through adjusting the value of $\alpha$. In particular, if we set $\alpha = 1$, our policy reduces to a non-adaptive policy, and if we set $\alpha = 1$, our policy reduces to a fully adaptive policy.

- For the partial-adaptive submodular maximization problem subject to a knapsack constraint, we develop a sampling based partial-adaptive policy that achieves an approximation ratio of $\min\{1/2, \alpha\} 6+\min\{1, 2\alpha\}$ with respect to the optimal fully adaptive policy.

- We theoretically analyze the batch query complexity of our policy and show that if the utility function is policywise submodular, a stronger assumption than adaptive submodularity, then the above sampling based partial-adaptive policy takes at most $O(\log n \log B c_{\min})$ number of batches to achieve a constant approximation ratio where $B$ is the budget constraint and $c_{\min}$ is the cost of the cheapest item. It was worth noting that if we consider a
cardinality constraint $k$, then $O(\log n \log \frac{B}{c_{\min}})$ is upper bounded by $O(\log n \log k)$ which is polylogarithmic.

2. Related Work

Maximizing a submodular function subject to various practical constraints has been extensively studied in the literature Nemhauser et al. (1978), Badanidiyuru and Vondrák (2014), Buchbinder et al. (2014). However, the classic notation of submodularity is not capable of capturing the interactive nature of many adaptive sequential decision making problems. Recently, Golovin and Krause (2011) extends this notation from sets to policies by proposing the concept of adaptive submodularity. Intuitively, it states that if a function is adaptive submodular, then the expected marginal benefit of an item never increases as we collect more observations from past selections. They show that if a function is adaptive monotone and adaptive submodular, then a simple adaptive greedy policy achieves a $(1 - 1/e)$ approximation ratio. For maximizing a non-monotone adaptive submodular function, Tang (2021a) show that a randomized greedy policy achieves a $\frac{1}{e}$ approximation ratio. Unlike most of existing studies on adaptive submodular maximization which is conducted under the fully adaptive setting, our focus is on partial-adaptive (a.k.a batch-mode) setting.

Recently, there are a few results that are related to partial-adaptive submodular maximization. Chen and Krause (2013) propose a policy that selects batches of fixed size $r$, and they show that their policy achieves a bounded approximation ratio compare to the optimal policy which is restricted to selecting batches of fixed size $r$. However, their approximation ratio becomes arbitrarily bad with respect to the optimal fully adaptive policy. In the context of adaptive viral marketing, Yuan and Tang (2017) develop a partial-adaptive seeding policy that achieves a bounded approximation ratio against the optimal fully adaptive seeding policy. Our study is similar to theirs in that both studies introduce a controlling parameter to balance the performance/adaptibility tradeoff. However, their results can not be extended to solve a general adaptive submodular maximization problem. Moreover, unlike the batch-mode setting, their model allows for observing a partial realization of an item during the selection process. Tang and Yuan (2020) extends the above study by developing a new partial-adaptive seeding policy subject to a general knapsack constraint. Their design is general enough to be applied to other adaptive submodular maximization problems. Very recently, Esfandiari et al. (2021) study the batch-mode monotone adaptive
submodular optimization problem and they develop an efficient semi adaptive policy that achieves an almost tight $1 - 1/e - \epsilon$ approximation ratio. To the best of our knowledge, all existing studies are focusing on maximizing a monotone adaptive submodular function. We are the first to study the non-monotone partial-adaptive submodular maximization problem subject to both cardinality and knapsack constraints. We also provide a rigorous analysis of the batch query complexity of our policy.

3. Preliminaries and Problem Formulation

3.1. Items and States

The input of our problem is a set $E$ of $n$ items (e.g., tests in experimental design). Each items $e \in E$ has a random state $\Phi(e) \in O$ where $O$ is a set of all possible states. Let $\phi(e) \in O$ denote a realization of $\Phi(e)$. Thus, a realization $\phi$ is a mapping function that maps items to states: $\phi : E \rightarrow O$. In the example of experimental design, the item $e$ may represent a test, such as the temperature, and $\Phi(e)$ is the result of the test, such as, $38^\circ C$. There is a known prior probability distribution $p(\phi) = \Pr(\Phi = \phi)$ over realizations $\phi$. When realizations are independent, the distribution $p$ completely factorizes. However, in many real-world applications such as active learning, the realizations of items may depend on each other. For any subset of items $S \subseteq E$, let $\psi : S \rightarrow O$ denote a partial realization and $\text{dom}(\psi) = S$ is called the domain of $\psi$. Consider a partial realization $\psi$ and a realization $\phi$, we say $\phi$ is consistent with $\psi$, denoted $\phi \sim \psi$, if they are equal everywhere in $\text{dom}(\psi)$. Moreover, consider two partial realizations $\psi$ and $\psi'$, we say that $\psi$ is a subrealization of $\psi'$, and denoted by $\psi \subseteq \psi'$, if $\text{dom}(\psi) \subseteq \text{dom}(\psi')$ and they agree everywhere in $\text{dom}(\psi)$. Let $p(\phi | \psi)$ represent the conditional distribution over realizations conditional on a partial realization $\psi$: $p(\phi | \psi) = \Pr[\Phi = \phi | \Phi \sim \psi]$. In addition, there is an additive cost function $c(S) = \sum_{e \in S} c(e)$ for any $S \subseteq E$.

3.2. Policies and Problem Formulation

A policy is a function $\pi$ that maps a set of partial realizations to some distribution of $E$: $\pi : 2^{E \times O} \rightarrow \mathcal{P}(E)$, specifying which item to select next. By following a given policy, we can select items adaptively based on our observations made so far. We next introduce two additional notions related to policies Golovin and Krause (2011).

**Definition 1 (Policy Concatenation).** Given two policies $\pi$ and $\pi'$, let $\pi @ \pi'$ denote a policy that runs $\pi$ first, and then runs $\pi'$, ignoring the observation obtained from running $\pi$. 

**Golovin and Krause (2011).**
**Definition 2 (Level-\(t\)-Truncation of a Policy).** Given a policy \(\pi\), we define its level-\(t\)-truncation \(\pi_t\) as a policy that runs \(\pi\) until it selects \(t\) items.

There is a utility function \(f: 2^{E \times O} \to \mathbb{R}_{\geq 0}\) which is defined over items and states. Let \(E(\pi, \phi)\) denote the subset of items selected by \(\pi\) under realization \(\phi\). The expected utility \(f_{avg}(\pi)\) of a policy \(\pi\) can be written as

\[
f_{avg}(\pi) = \mathbb{E}_{\Phi \sim p, \Pi}[f(E(\pi, \Phi), \Phi)]
\]

where the expectation is taken over possible realizations and the internal randomness of the policy.

In this paper, we first study the problem of partial-adaptive submodular maximization subject to a cardinality constraint \(k\).

\[
\max_{\pi}\{f_{avg}(\pi) \mid |E(\pi, \phi)| \leq k \text{ for all realizations } \phi\}
\]

Then we generalize this study to consider a knapsack constraint \(B\).

\[
\max_{\pi}\{f_{avg}(\pi) \mid c(E(\pi, \phi)) \leq B \text{ for all realizations } \phi\}
\]

**3.3. Adaptive Submodularity and Adaptive Monotonicity**

We next introduce some additional notations that are used in our proofs.

**Definition 3 (Conditional Expected Marginal Utility of an Item).** Given a utility function \(f: 2^{E \times O} \to \mathbb{R}_{\geq 0}\), the conditional expected marginal utility \(\Delta(e \mid S, \psi)\) of an item \(e\) on top of a group of items \(S \subseteq E\) conditional on a partial realization \(\psi\) is defined as follows

\[
\Delta(e \mid S, \psi) = \mathbb{E}_{\Phi}[f(S \cup \{e\}, \Phi) - f(S, \Phi) \mid \Phi \sim \psi]
\]

where the expectation is taken over \(\Phi\) with respect to \(p(\phi \mid \psi) = \Pr(\Phi = \phi \mid \Phi \sim \psi)\).

**Definition 4 (Conditional Expected Marginal Utility of a Policy).** Given a utility function \(f: 2^{E \times O} \to \mathbb{R}_{\geq 0}\), the conditional expected marginal utility \(\Delta(\pi \mid S, \psi)\) of a policy \(\pi\) on top of a group of items \(S \subseteq E\) conditional on a partial realization \(\psi\) is

\[
\Delta(\pi \mid S, \psi) = \mathbb{E}_{\Phi, \Pi}[f(S \cup E(\pi, \Phi), \Phi) - f(S, \Phi) \mid \Phi \sim \psi]
\]

where the expectation is taken over \(\Phi\) with respect to \(p(\phi \mid \psi) = \Pr(\Phi = \phi \mid \Phi \sim \psi)\) and the random output of \(\pi\).
Now we are ready to introduce the notations of adaptive submodularity and adaptive monotone (Golovin and Krause 2011). Intuitively, adaptive submodularity is a generalization of the classic notation of submodularity from sets to policies. This condition states that the expected marginal benefit of an item never increases as we collect more observations from past selections.

**Definition 5 (Adaptive Submodularity and Adaptive Monotonicity).** A function \( f : 2^{E \times O} \rightarrow \mathbb{R}_{\geq 0} \) is submodular adaptive if for any two partial realizations \( \psi \) and \( \psi' \) such that \( \psi \subseteq \psi' \), the following holds for each \( e \in E \setminus \text{dom}(\psi') \):

\[
\Delta(e | \text{dom}(\psi), \psi) \geq \Delta(e | \text{dom}(\psi'), \psi')
\]

Moreover, we say a utility function \( f : 2^{E \times O} \rightarrow \mathbb{R}_{\geq 0} \) is adaptive monotone (Golovin and Krause 2011) if for any partial realization \( \psi \) and any \( e \in E \setminus \text{dom}(\psi) \):

\[
\Delta(e | \text{dom}(\psi), \psi) \geq 0.
\]

### 4. Cardinality Constraint

We first study the problem of partial-adaptive submodular maximization subject to a cardinality constraint \( k \). It has been shown that if the utility function is adaptive submodular, then a fully adaptive greedy policy can achieve a \( 1/e \) approximation ratio against the optimal fully adaptive policy (Tang 2021a). However, one weakness about a fully adaptive policy is that one must wait for the observations from all past selections before making a new selection. To this end, we develop a *Partial-Adaptive Greedy Policy* \( \pi^p \) that allows to make multiple selections simultaneously within a single batch. We show that \( \pi^p \) achieves a \( \alpha/e \) approximation ratio with respect to the optimal fully adaptive policy where \( \alpha \in [0, 1] \) is called *degree of adaptivity*. One can adjust the value of \( \alpha \) to balance the performance/adaptivity tradeoff.

#### 4.1. Algorithm Design

We next explain the design of \( \pi^p \) (a detailed implementation of \( \pi^p \) is listed in Algorithm 2). We first add a set \( V \) of \( 2k - 1 \) dummy items to the ground set, such that \( \Delta(e | \text{dom}(\psi), \psi) = 0 \) for any \( e \in V \) and any partial realization \( \psi \). Let \( E' = E \cup V \) denote the expanded ground set. These dummy items are added to avoid selecting any item with negative marginal utility. Note that we can safely remove all these dummy items from the solution without affecting its utility. For any partial realization \( \psi \) and any subset of items \( S \subseteq \text{dom}(\psi) \), define
$M(S, \psi)$ as a set of $k$ items that have the largest marginal utility on top of $S$ conditional on $\psi$, i.e.,

$$M(S, \psi) \in \arg \max_{V \subseteq E'} \sum_{e \in E'} \Delta(e \mid S, \psi)$$

For any iteration $t \in [k]$ of $\pi^p$, let $S_t$ denote the first $t$ items selected by $\pi^p$, let $b[t]$ denote the batch index of the $t$-th item selected by $\pi^p$, i.e., the $t$-th item is selected in batch $b[t]$, and let $S_q$ denote the set of selected items from batch $q$. Let $\psi_q$ denote the partial realization of the first $q$ batches of items selected by $\pi^p$, i.e., $\text{dom}(\psi_q) = \cup_{i \in [1,q]} S[i]$. Set the initial solution $S_0 = \emptyset$ and the initial partial realization $\psi_0 = \emptyset$.

- Starting with the first iteration $t = 1$.
- In each iteration $t$, we compare $\sum_{e \in M(S_{t-1}, \psi_{b[t-1]-1})} \Delta(e \mid S_{t-1}, \psi_{b[t-1]-1})$ with $\alpha \cdot \sum_{e \in M(\text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})} \Delta(e \mid \text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})$, then decide whether to start a new batch or not based on the result of the comparison as follows.

  - If

  $$\sum_{e \in M(S_{t-1}, \psi_{b[t-1]-1})} \Delta(e \mid S_{t-1}, \psi_{b[t-1]-1}) \geq \alpha \cdot \sum_{e \in M(\text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})} \Delta(e \mid \text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})$$

  then $\pi^p$ chooses to stay with the current batch, i.e., $b[t] = b[t-1]$. It samples an item $e_t$ uniformly at random from $M(S_{t-1}, \psi_{b[t]-1})$, which is identical to $M(S_{t-1}, \psi_{b[t]-1})$ due to $b[t] = b[t-1]$, and updates the solution $S_t$ using $S_{t-1} \cup \{e_t\}$. Move to the next iteration $t = t + 1$.

  - Otherwise, $\pi^p$ starts a new batch, i.e., $b[t] = b[t-1] + 1$, and observe the partial realization $\Phi(e)$ of all items $e$ from the previous batch $S_{b[t-1]}$. Then it updates the observation $\psi_{b[t]-1}$ using $\psi_{b[t]-1} \cup \{(e, \Phi(e)) \mid e \in S_{b[t-1]}\}$. Note that $S_{t-1} = \text{dom}(\psi_{b[t]-1})$ in this case. At last, it samples an item $e_t$ uniformly at random from $M(S_{t-1}, \psi_{b[t]-1})$ and updates the solution $S_t$ using $S_{t-1} \cup \{e_t\}$. Move to the next iteration $t = t + 1$.

- The above process iterates until $\pi^p$ selects $k$ items (which may include some dummy items).

**Remark:** Note that $\sum_{e \in M(\text{dom}(\psi_{b[t]-1}), \psi_{b[t]-1})} \Delta(e \mid \text{dom}(\psi_{b[t]-1}), \psi_{b[t]-1})$ in (1) is an upper bound of $\sum_{e \in M(S_{t-1}, \psi_{b[t]-1})} \Delta(e \mid S_{t-1}, \psi_{b[t]-1})$ due to $\text{dom}(\psi_{b[t]-1}) \subseteq S_{t-1}$ and $f : 2^{E \times O} \to \mathbb{R}_{\geq 0}$ is adaptive submodular. Intuitively, satisfying (1) ensures that the expected gain of each iteration is sufficiently large to achieve a constant approximation ratio. Unlike
some other criteria proposed in previous studies (Tang and Yuan 2020, Esfandiari et al. 2021), evaluating (4) is relatively easy since it does not involve the calculation of the expectation of the maximum of \( n \) random variables. Under our framework, one can adjust the degree of adaptivity \( \alpha \in [0, 1] \) to balance the performance/adaptivity tradeoff. In particular, choosing a smaller \( \alpha \) makes it easier to satisfy (4) and hence leads to fewer number of batches but poorer performance. Consider an extreme case when \( \alpha = 0 \), \( \pi^p \) takes only one batch to complete the selection process, i.e., \( \pi^p \) reduces to a non-adaptive policy in this case. At the other extreme, if we set \( \alpha = 1 \), then our policy becomes fully adaptive.

**Algorithm 1 Partial-Adaptive Greedy Policy \( \pi^p \)**

1. \( t = 1; b[0] = 1; \psi_0 = \emptyset; S_0 = \emptyset; \forall i \in [n], S_{[i]} = \emptyset. \)

2. **while** \( t \leq k \) **do**

3. \( \text{let } M(S_{t-1}, \psi_{[t-1]-1}) \leftarrow \arg\max_{|V| = k} \sum_{e \in E'} \Delta(e | S_{t-1}, \psi_{[t-1]-1}); \)

4. \( \text{if } \sum_{e \in M(S_{t-1}, \psi_{[t-1]-1})} \Delta(e | S_{t-1}, \psi_{[t-1]-1}) \geq \alpha \cdot \sum_{e \in M(\psi_{[t-1]-1}); \pi^p \text{ reduces to a non-adaptive policy in this case. At the other extreme, if we set } \alpha = 1, \text{ then our policy becomes fully adaptive.} \)

4.2. Performance Analysis

We next analyze the performance of \( \pi^p \) against the optimal fully adaptive strategy. The following main theorem shows that \( \pi^p \) with degree of adaptivity \( \alpha \) achieves an approximation ratio of \( \alpha/e \).
Theorem 1. If $f : 2^{E \times O} \to \mathbb{R}_{\geq 0}$ is adaptive submodular, then the Partial-Adaptive Greedy Policy $\pi^p$ with degree of adaptivity $\alpha$ achieves a $\alpha/e$ approximation ratio in expectation.

The rest of this section is devoted to proving Theorem 1. We first present two technical lemmas. Recall that for any iteration $t \in [k]$, $S_{t-1}$ represents the first $t-1$ selected items, $\psi_{b[t-1]}$ represents the partial realization of all items selected from the first $b[t] - 1$ batches, and $M(S, \psi) \in \arg\max_{V \subseteq E' | |V| = k} \sum_{b \in E} \Delta(e | S, \psi)$.

**Lemma 1.** For each iteration $t \in [k]$, $\sum_{e \in M(S_{t-1}, \psi_{b[t-1]})} \Delta(e | S_{t-1}, \psi_{b[t-1]}) \geq \alpha \cdot \sum_{e \in M(\text{dom}(\psi_{b[t-1]}), \psi_{b[t-1]})} \Delta(e | \text{dom}(\psi_{b[t-1]}), \psi_{b[t-1]})$.

**Proof:** We prove this lemma in two cases.

- If $b[t] = b[t-1]$, i.e., the $t-1$-th item and the $t$-th item are selected within the same batch, then according to the design of $\pi^p$, it must be the case that $\sum_{e \in M(S_{t-1}, \psi_{b[t-1]})} \Delta(e | S_{t-1}, \psi_{b[t-1]}) \geq \alpha \cdot \sum_{e \in M(\text{dom}(\psi_{b[t-1]}), \psi_{b[t-1]})} \Delta(e | \text{dom}(\psi_{b[t-1]}), \psi_{b[t-1]})$. This together with $b[t] = b[t-1]$ implies that $\sum_{e \in M(S_{t-1}, \psi_{b[t-1]})} \Delta(e | S_{t-1}, \psi_{b[t-1]}) \geq \alpha \cdot \sum_{e \in M(\text{dom}(\psi_{b[t-1]}), \psi_{b[t-1]})} \Delta(e | \text{dom}(\psi_{b[t-1]}), \psi_{b[t-1]})$.

- Otherwise, if $b[t] = b[t-1] + 1$, i.e., the $t-1$-th item and the $t$-th item are selected within different batches, then we have $\text{dom}(\psi_{b[t]}) = S_{t-1}$. Thus, $\sum_{e \in M(S_{t-1}, \psi_{b[t]})} \Delta(e | S_{t-1}, \psi_{b[t]}) = \sum_{e \in M(\text{dom}(\psi_{b[t]}), \psi_{b[t]})} \Delta(e | \text{dom}(\psi_{b[t]}), \psi_{b[t]}) \geq \alpha \cdot \sum_{e \in M(\text{dom}(\psi_{b[t]}), \psi_{b[t]})} \Delta(e | \text{dom}(\psi_{b[t]}), \psi_{b[t]})$ where the inequality is due to $\alpha \leq 1$.

This finishes the proof of this lemma.

The next lemma shows that for any iteration $t \in [k]$, the sum of expected marginal benefits of all items from $M(\text{dom}(\psi_{b[t]}), \psi_{b[t]})$ is sufficiently high. This will be used later to lower bound the expected gain of each iteration of our policy.

**Lemma 2.** Let $\pi^*$ denote an optimal fully adaptive policy. In each iteration $t \in [k]$, $\sum_{e \in M(\text{dom}(\psi_{b[t]}), \psi_{b[t]})} \Delta(e | \text{dom}(\psi_{b[t]}), \psi_{b[t]}) \geq \Delta(\pi^* | S_{t-1}, \psi_{b[t-1]})$.

**Proof:** Observe that for any $S_{t-1}$ and $\psi_{b[t-1]}$, the marginal utility of the optimal policy $\pi^*$ on top of $S_{t-1}$ conditional on $\psi_{b[t-1]}$ can be represented as

$$\Delta(\pi^* | S_{t-1}, \psi_{b[t-1]}) = \mathbb{E}_{\psi \sim \psi_{b[t-1]}} \left[ \Delta(\pi^* | S_{t-1}, \psi_{b[t-1]} \cup e \in S_{t-1} \setminus \text{dom}(\psi_{b[t-1]})) \Delta(e, \Phi(e)) \right]$$
In the above equation, $\psi_{b|t-1} \cup e \in S_{t-1} \backslash \{\text{dom}(\psi_{b|t-1})(e, \Phi(e)) \}$ represents a random realization of $S_{t-1}$ conditional on $\psi_{b|t-1}$. For convenience, let $\Phi(S_{t-1})$ denote $\psi_{b|t-1} \cup e \in S_{t-1} \backslash \{\text{dom}(\psi_{b|t-1})(e, \Phi(e)) \}$ for short. Then we have

$$
\Delta(\pi^* \mid S_{t-1}, \psi_{b|t-1}) = \mathbb{E}_{\Phi \sim \psi_{b|t-1}}[\Delta(\pi^* \mid S_{t-1}, \Phi(S_{t-1}))]
$$

$$
\leq \mathbb{E}_{\Phi \sim \psi_{b|t-1}}[\Delta(e \mid S_{t-1}, \Phi(S_{t-1}))]
$$

$$
\leq \mathbb{E}_{\Phi \sim \psi_{b|t-1}}[\sum_{e \in M(S_{t-1}, \Phi(S_{t-1}))} \Delta(e \mid \{\psi_{b|t-1}, \psi_{b|t-1}\})]
$$

$$
\leq \mathbb{E}_{\Phi \sim \psi_{b|t-1}}[\sum_{e \in M(\{\psi_{b|t-1}, \psi_{b|t-1}\}, \{\psi_{b|t-1}, \psi_{b|t-1}\})} \Delta(e \mid \{\psi_{b|t-1}, \psi_{b|t-1}\})]
$$

$$
= \sum_{e \in M(\{\psi_{b|t-1}, \psi_{b|t-1}\}, \{\psi_{b|t-1}, \psi_{b|t-1}\})} \Delta(e \mid \{\psi_{b|t-1}, \psi_{b|t-1}\})
$$

The first inequality is due to $f : 2^E \times O \to \mathbb{R}_{\geq 0}$ is adaptive submodular and Lemma 1 in [Gotovos et al. 2015]. The second inequality is due to $f : 2^E \times O \to \mathbb{R}_{\geq 0}$ is adaptive submodular and $\psi_{b|t-1} \subseteq \Phi(S_{t-1})$. □

Now we are ready to prove the main theorem. Let the random variable $S_{t-1}$ denote the first $t - 1$ items selected by $\pi^*$ and let the random variable $\Psi_{b|t-1}$ denote the partial realization of the first $b[t] - 1$ batches of items selected by $\pi^*$. For any $t \in [k]$, we next bound the expected marginal gain of the $t$-th iteration of $\pi^*$,

$$
f_{av}(\pi^*_t) - f_{av}(\pi^*_t) = \mathbb{E}_{S_{t-1}, \Psi_{b|t-1}}[\mathbb{E}_{e_t}[\Delta(e_t \mid S_{t-1}, \Psi_{b|t-1})]]
$$

$$
= \frac{1}{k} \mathbb{E}_{S_{t-1}, \Psi_{b|t-1}}[\sum_{e \in M(S_{t-1}, \Psi_{b|t-1})} \Delta(e \mid S_{t-1}, \Psi_{b|t-1})]
$$

$$
\geq \frac{1}{k} \mathbb{E}_{S_{t-1}, \Psi_{b|t-1}}[\sum_{e \in M(\Psi_{b|t-1}, \Psi_{b|t-1})} \Delta(e \mid \{\Psi_{b|t-1}, \Psi_{b|t-1}\})]
$$

$$
= \frac{\alpha}{k} \mathbb{E}_{S_{t-1}, \Psi_{b|t-1}}[\sum_{e \in M(\{\Psi_{b|t-1}, \Psi_{b|t-1}\})} \Delta(e \mid \{\Psi_{b|t-1}, \Psi_{b|t-1}\})]
$$

$$
\geq \frac{\alpha}{k} \mathbb{E}_{S_{t-1}, \Psi_{b|t-1}}[\Delta(\pi^* \mid S_{t-1}, \Psi_{b|t-1})]
$$

$$
= \frac{\alpha}{k} (f_{av}(\pi^* @ \pi^*_0) - f_{av}(\pi^*_0))
$$

$$
\geq \frac{\alpha}{k} (1 - \frac{1}{k})^{t-1} f_{av}(\pi^*) - f_{av}(\pi^*_0)
$$

The second equality is due to the fact that at each round $t \in [k]$, $\pi^*$ adds an item uniformly at random from $M(S_{t-1}, \psi_{b|t-1})$ to the solution. The first inequality is due to Lemma [1]
The second inequality is due to Lemma 2. The last inequality is due to Lemma 1 in (Tang 2021a) where they show that $f_{avg}(\pi^* \oplus \pi^p_{t-1}) \geq (1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*)$.

We next prove

$$f_{avg}(\pi^p_t) \geq f_{avg}(\pi^p_{t-1}) + \frac{\alpha}{k}((1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*) - f_{avg}(\pi^p_{t-1})) \tag{12}$$

by induction on $t$. For $t = 0$, $f_{avg}(\pi^p_0) \geq 0 \geq \frac{\alpha}{k} ((1 - \frac{1}{k})^{0-1} f_{avg}(\pi^*)$. Assume Eq. (12) is true for $t' < t$, let us prove it for $t$.

$$f_{avg}(\pi^p_t) \geq f_{avg}(\pi^p_{t-1}) + \frac{\alpha}{k} ((1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*) - f_{avg}(\pi^p_{t-1}))$$

$$= (1 - \frac{\alpha}{k}) f_{avg}(\pi^p_{t-1}) + \frac{\alpha}{k} ((1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*)$$

$$\geq (1 - \frac{\alpha}{k}) \cdot (\alpha (t-1)/k) \cdot (1 - 1/k)^{t-2} \cdot \alpha f_{avg}(\pi^*) + \frac{\alpha}{k} ((1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*)$$

$$\geq (1 - 1/k) \cdot (\alpha (t-1)/k) \cdot (1 - 1/k)^{t-2} \cdot \alpha f_{avg}(\pi^*) + \frac{\alpha}{k} ((1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*)$$

$$= \frac{\alpha t}{k} (1 - \frac{1}{k})^{t-1} f_{avg}(\pi^*)$$

The first inequality is due to (11), the second inequality is due to the inductive assumption. When $t = k$, we have $f_{avg}(\pi^p_k) \geq (1 - 1/k)^k \cdot \alpha f_{avg}(\pi^*) \geq (\alpha / e) f_{avg}(\pi^*)$. This finishes the proof of the main theorem.

**Remark:** When the utility function $f : 2^E \times O \rightarrow \mathbb{R}_{\geq 0}$ is adaptive submodular and adaptive monotone, Golovin and Krause (2011) show that $f_{avg}(\pi^* \oplus \pi^p_{t-1}) \geq f_{avg}(\pi^*)$ for all $t \in [k]$. Thus, for all $t \in [k]$,

$$f_{avg}(\pi^p_t) - f_{avg}(\pi^p_{t-1}) \geq \frac{\alpha}{k} (f_{avg}(\pi^* \oplus \pi^p_{t-1}) - f_{avg}(\pi^p_{t-1}))$$

$$\geq \frac{\alpha}{k} (f_{avg}(\pi^*) - f_{avg}(\pi^p_{t-1})) \tag{13}$$

The first inequality is due to (10). Through induction on $t$, we have $f_{avg}(\pi^p) \geq (1 - e^{-\alpha}) f_{avg}(\pi^*)$.

**Theorem 2.** If $f : 2^E \times O \rightarrow \mathbb{R}_{\geq 0}$ is adaptive submodular and adaptive monotone, then the Partial-Adaptive Greedy Policy $\pi^p$ achieves a $1 - e^{-\alpha}$ approximation ratio in expectation.

5. Knapsack Constraint

In this section, we study our problem subject to a knapsack constraint $B$. In (Tang 2021b, Amanatidis et al. 2020), they develop a fully adaptive policy that achieves a bounded approximation ratio against the optimal fully adaptive policy. We extend their design
by developing a partial-adaptive policy which allows to select multiple items in a single batch. Our policy with degree of adaptivity $\alpha$ achieves an approximation ratio of $\frac{1}{6+4/\alpha}$ with respect to the optimal fully adaptive policy. Again, one can balance the performance/adaptivity tradeoff by adjusting the degree of adaptivity $\alpha$. In addition, we provide a rigorous analysis of the batch query complexity of our policy under some additional assumptions.

5.1. Algorithm Design

We first construct two candidate policies: the first policy always picks the best singleton $o$ that maximizes the expected utility, i.e., $o \in \arg \max_{e \in E} \mathbb{E}_{\Phi \sim \rho}[f(\{e\}, \Phi)]$ and the second candidate is a sampling based “density-greedy” policy $\pi^k$. Our final policy picks one from the above two candidates to execute such that $\{o\}$ is picked with probability $\frac{3+1/\alpha}{3+2/\alpha}$ and $\pi^k$ is picked with probability $\frac{3+1/\alpha}{3+2/\alpha}$. In the rest of this paper, let $f(o)$ denote $\mathbb{E}_{\Phi \sim \rho}[f(\{o\}, \Phi)]$ for short.

We next explain the idea of Partial-Adaptive Density-Greedy Policy $\pi^k$ (the second candidate policy). $\pi^k$ first selects a random subset $R$ which is obtained by including each item $e \in E$ independently with probability 1/2. Then we run a “density-greedy” algorithm only on $R$. We first introduce some notations. For each iteration $t \in [n]$, let $b[t]$ denote the batch index of $t$, i.e., the $t$-th item selected in batch $b[t]$, for convenience, we define $b[0] = 1$. Let $S_{t-1}$ denote the first $t - 1$ items selected by $\pi^k$, and $\psi_{b[t-1]-1}$ represent the partial realization of the first $b[t-1]-1$ batches of selected items. Set the initial solution $S_0 = \emptyset$ and the initial partial realization $\psi_0 = \emptyset$.

- Starting from iteration $t = 1$ and batch $b[t] = 1$.
- In each iteration $t$, let $e'$ be the item that has the largest benefit-cost ratio on top of $S_{t-1}$ conditional on $\psi_{b[t-1]-1}$ from $R \setminus S_{t-1}$, i.e.,

$$e' \leftarrow \arg \max_{e \in R \setminus S_{t-1}} \frac{\Delta(e \mid S_{t-1}, \psi_{b[t-1]-1})}{c(e)} \quad (14)$$

Let $e''$ be the item that has the largest benefit-cost ratio on top of $\text{dom}(\psi_{b[t-1]-1})$ conditional on $\psi_{b[t-1]-1}$ from $R \setminus \text{dom}(\psi_{b[t-1]-1})$, i.e.,

$$e'' \leftarrow \arg \max_{e \in R \setminus \text{dom}(\psi_{b[t-1]-1})} \frac{\Delta(e \mid \text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})}{c(e)} \quad (15)$$

It will become clear later that $e''$ stores the first selected item, if any, from the $b[t-1]$-th batch. Note that $\text{dom}(\psi_{b[t-1]-1}) \subseteq S_{t-1}$.

Compare $\frac{\Delta(e'[S_{t-1}, \psi_{b[t-1]-1}])}{c(e')}$ with $\alpha \cdot \frac{\Delta(e''[\text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1}])}{c(e'')}$. 

\[ \text{...} \]
Algorithm 2 Partial-Adaptive Density-Greedy Policy $\pi^k$

1. $t = 1; b[0] = 1; \psi_0 = \emptyset; S_0 = \emptyset; \forall i \in [n], S_{[i]} = \emptyset.$

2. for $i \in E$ do
   3. let $r \sim \text{Bernoulli}(\frac{1}{2})$
   4. if $r = 1$ then
      5. add $i$ to $R$
   6. while $R \setminus S_{t-1} \neq \emptyset$ do
      7. let $e' \leftarrow \arg\max_{e \in R \setminus S_{t-1}} \frac{\Delta(e|S_{t-1}, \psi_{b[t-1]-1})}{c(e)}$
      8. if $t = 1$ then
         9. $e'' \leftarrow e'$
         10. if $\frac{\Delta(e'|S_{t-1}, \psi_{b[t-1]-1})}{c(e')}$ \(\geq\) $\alpha \cdot \frac{\Delta(e''|\text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})}{c(e'')}$ then
            11. \{stay in the current batch\}
         12. if $c(e') + c(S_{t-1}) \leq B$ then
            13. $b[t] = b[t-1] + 1; e_t \leftarrow e'; S_t = S_{t-1} \cup \{e_t\}; S_{[b[t]]} = S_{[b[t]]} \cup \{e_t\}; t = t + 1$
         14. else
            15. break
      16. else
            17. \{start a new batch\}
            18. $b[t] = b[t-1] + 1$
            19. observe $\{(e, \Phi(e)) \mid e \in S_{[b[t-1]]}\}; \psi_{b[t]-1} = \psi_{b[t-1]-1} \cup \{(e, \Phi(e)) \mid e \in S_{[b[t-1]]}\}$
            20. let $e'' \leftarrow \arg\max_{e \in R \setminus \text{dom}(\psi_{b[t]-1})} \frac{\Delta(e|\text{dom}(\psi_{b[t]-1}), \psi_{b[t]-1})}{c(e)}$
            21. if $\frac{\Delta(e''|\text{dom}(\psi_{b[t]-1}), \psi_{b[t]-1})}{c(e'')}$ \(>\) $0$ then
               22. if $c(e'') + c(S_{t-1}) \leq B$ then
                  23. $e_t \leftarrow e''; S_t = S_{t-1} \cup \{e_t\}; S_{[b[t]]} = S_{[b[t]]} \cup \{e_t\}; F = F \setminus \{e_t\}; t = t + 1$
               24. else
                  25. break
            26. else
               27. break

\[\frac{\Delta(e'|S_{t-1}, \psi_{b[t-1]-1})}{c(e')} \geq \alpha \cdot \frac{\Delta(e''|\text{dom}(\psi_{b[t-1]-1}), \psi_{b[t-1]-1})}{c(e'')},\] and adding $e'$ to the solution does not violate the budget constraint, then stay in the current batch, i.e., $b[t] = b[t-1]$, add $e'$ to the solution, i.e, $S_t = S_{t-1} \cup \{e'\}$. Move to the next iteration, i.e., $t = t + 1$;
— if \( \Delta(e'|S_{t-1}, \psi_{[t-1]-1}) \geq \alpha \cdot \Delta(e'|\text{dom}(\psi_{[t-1]-1}), \psi_{[t-1]-1}) \) and adding \( e' \) to the solution violates the budget constraint, then terminate;
— if \( \Delta(e'|S_{t-1}, \psi_{[t-1]-1}) < \alpha \cdot \Delta(e'|\text{dom}(\psi_{[t-1]-1}), \psi_{[t-1]-1}) \), then start a new batch, i.e., \( b[t] = b[t-1] + 1 \), observe the partial realization of all items selected so far, i.e., \( \psi_{[t]-1} = \psi_{b[t]-1} \cup \{(e, \Phi(e)) \mid e \in S_{[b[t]-1]}\} \). If \( \max_{e \in R' \cap \text{dom}(\psi_{[t]-1})} \Delta(e|\text{dom}(\psi_{[t]-1}), \psi_{[t]-1}) > 0 \) and adding \( \arg \max_{e \in R' \cap \text{dom}(\psi_{[t]-1})} \Delta(e|\text{dom}(\psi_{[t]-1}), \psi_{[t]-1}) \) to the solution does not violate the budget constraint, then add \( \arg \max_{e \in R' \cap \text{dom}(\psi_{[t]-1})} \Delta(e|\text{dom}(\psi_{[t]-1}), \psi_{[t]-1}) \) to the solution, and move to the next iteration, i.e., \( t = t + 1 \); otherwise, terminate.

A detailed description of \( \pi^k \) is presented in Algorithm 2.

5.2. Performance Analysis

For ease of analysis, we present an auxiliary policy in Algorithm 3. Unlike Algorithm 2 where \( R \) is sampled at the beginning of the algorithm, we defer this decision in Algorithm 3 that is, we toss a coin of success 1/2 to decide whether or not to add an item to the solution each time after an item is being considered. It is easy to verify that both versions of the algorithm have identical output distributions.

We first provide some useful observations that will be used in the proof of the main theorem. Consider an arbitrary partial realization \( \psi \), let \( W(\psi) = \{e \in E \mid \Delta(e \mid \text{dom}(\psi), \psi) > 0\} \) denote the set of all items whose marginal utility with respect to \( \text{dom}(\psi) \) conditional on \( \psi \) is positive. We number all items \( e \in W(\psi) \) by decreasing ratio \( \frac{\Delta(e|\text{dom}(\psi), \psi)}{c(e)} \), i.e., \( e(1) \in \arg \max_{e \in W(\psi)} \frac{\Delta(e|\text{dom}(\psi), \psi)}{c(e)} \). If \( \sum_{e \in W(\psi)} c(e) \geq B \), let \( l = \min\{i \in \mathbb{N} \mid \sum_{j=1}^{i} c(e(i)) \geq B\} \); otherwise, if \( \sum_{e \in W(\psi)} c(e) < B \), let \( l = |W(\psi)| \). Define \( D(\psi) = \{e(i) \in W(\psi) \mid i \in [l]\} \) as the set containing the first \( l \) items from \( W(\psi) \). Intuitively, \( D(\psi) \) represents a set of best-looking items conditional on \( \psi \).

Consider any \( e \in D(\psi) \), assuming \( e \) is the \( i \)-th item in \( D(\psi) \), let
\[
x(e, \psi) = \min\{1, \frac{B - \sum_{s \in \cup_{j \in [i-1]} \{e(j)\}} c(s)}{c(e)}\}
\]
where \( \cup_{j \in [i-1]} \{e(j)\} \) represents the first \( i - 1 \) items in \( D(\psi) \).

Define
\[
d(\psi) = \sum_{e \in D(\psi)} x(e, \psi) \Delta(e \mid \text{dom}(\psi), \psi)
\]
Algorithm 3 Equivalent $\pi^k$

1: $t = 1; b[0] = 1; \psi_0 = \emptyset; F = E; S_0 = \emptyset; \forall i \in [n], S[i] = \emptyset.$
2: while $F \neq \emptyset$ do
3:    let $e' \leftarrow \arg max_{e \in F} \frac{\Delta(e|S[t-1],\psi_{b[t-1]-1})}{c(e)}$
4:    if $t = 1$ then
5:        $e'' \leftarrow e'$
6:        if $\frac{\Delta(e'|S[t-1],\psi_{b[t-1]-1})}{c(e')} \geq \alpha \cdot \frac{\Delta(e''|\text{dom}(\psi_{b[t-1]-1}),\psi_{b[t-1]-1})}{c(e'')}$ then
7:            {stay in the current batch}
8:            if $c(e') + c(S[t-1]) \leq B$ then
9:                consider $e'$
10:               let $r \sim \text{Bernoulli}(\frac{1}{2})$
11:              if $r = 1$ then
12:                  $b[t] = b[t-1]; e_t \leftarrow e'; S_t = S[t-1] \cup \{e_t\}; S[b[t]] = S[b[t]] \cup \{e_t\}; F = F \setminus \{e_t\}; t = t + 1$
13:              else
14:                  $F = F \setminus \{e'\}$
15:              else
16:                  break
17:            else
18:                {start a new batch}
19:                $b[t] = b[t-1] + 1$
20:               observe $\{(e, \Phi(e)) \mid e \in S[b[t-1]]\}$; update $\psi_{b[t-1]-1} = \psi_{b[t-1]-1} \cup \{(e, \Phi(e)) \mid e \in S[b[t-1]]\}$;
21:               let $z \leftarrow \arg max_{e \in F} \frac{\Delta(e|\text{dom}(\psi_{b[t-1]-1}),\psi_{b[t-1]-1})}{c(e)}$
22:               if $\frac{\Delta(z|\text{dom}(\psi_{b[t-1]-1}),\psi_{b[t-1]-1})}{c(e''')} > 0$ then
23:                   if $c(z) + c(S_t-1) \leq B$ then
24:                       consider $z$
25:                       let $r \sim \text{Bernoulli}(\frac{1}{2})$
26:                       if $r = 1$ then
27:                           $e_t \leftarrow z; S_t = S_{t-1} \cup \{e_t\}; S[b[t]] = S[b[t]] \cup \{e_t\}; F = F \setminus \{e_t\}; e'' \leftarrow z; \{z \text{ is selected as the first item in batch } b[t]\}$
28:                           $t = t + 1$
29:                       else
30:                           $F = F \setminus \{z\}$
31:                       else
32:                           break
33:                   else
34:                       break
35:               else
36:                   break
37:        else
38:            $F = F \setminus \{e''\}$
39:        else
40:            break
41:    else
42:        break
43:  else
44:      break
In analogy to Lemma 1 of (Gotovos et al. 2015),

\[
d(\psi) \geq \Delta(\pi^* | \text{dom}(\psi), \psi)
\]

(16)

We use \(\lambda = (\{S^\lambda_t, \psi^\lambda_{b[t]−1} | t \in [z^\lambda]\}, \psi^\lambda_{b[z^\lambda]}\) to represent a fixed run of \(\pi^k\) where \(S^\lambda_t\) denotes the first \(t\) items selected by \(\pi^k\) under \(\lambda\), \(\psi^\lambda_{b[t]−1}\) denotes the partial realization of first \(b[t]−1\) batches of selected items under \(\lambda\), and \(\psi^\lambda_{b[z^\lambda]}\) denotes the partial realization of all selected items under \(\lambda\), i.e., \(\pi^k\) selects \(z^\lambda\) items under \(\lambda\). Hence, \(\text{dom}(\psi^\lambda_{b[z^\lambda]}) = S^\lambda_{z^\lambda}\). Define \(C(\lambda)\) as those items in \(D(\psi^\lambda_{b[z^\lambda]})\) that have been considered by \(\pi^k\) but not added to the solution because of the coin flips. Let \(U(\lambda)\) denote those items in \(D(\psi^\lambda_{b[z^\lambda]})\) that have not been considered by \(\pi^k\). (16) implies that

\[
d(\psi^\lambda_{b[z^\lambda]}) = \sum_{e \in D(\psi^\lambda_{b[z^\lambda]})} x(e, \psi^\lambda_{b[z^\lambda]}) \Delta(e | S^\lambda_{z^\lambda}, \psi^\lambda_{b[z^\lambda]})
\]

(17)

\[
= \sum_{e \in U(\lambda) \cup C(\lambda)} x(e, \psi^\lambda_{b[z^\lambda]}) \Delta(e | S^\lambda_{z^\lambda}, \psi^\lambda_{b[z^\lambda]})
\]

\[
\geq \Delta(\pi^* | S^\lambda_{z^\lambda}, \psi^\lambda_{b[z^\lambda]})
\]

(18)

Before presenting the main theorem, we provide two technical lemmas.

**Lemma 3.** Let the random variable \(\Lambda\) denote a random run of \(\pi^k\),

\[
f_{\text{avg}}(\pi^k) \geq \frac{1}{2} \mathbb{E}[\sum_{e \in C(\Lambda)} \Delta(e | S^\lambda_{z^\lambda}, \psi^\lambda_{b[z^\lambda]})]
\]

(19)

**Proof:** For each \(e \in E\), let \(H(e) = \{\lambda | e \in C(\lambda)\}\) denote the set of all possible runs of \(\pi^k\) under which \(e\) is being considered and it is among best-looking items. Let \(D(H(e))\) denote the prior probability distribution over \(H(e)\). In addition, let \(H^+(e)\) denote the set of all possible runs of \(\pi^k\) under which \(e\) is being considered and let \(D(H^+(e))\) represent the prior probability distribution over \(H^+(e)\). Thus, \(H(e) \subseteq H^+(e)\). Consider any fixed run \(\lambda \in H^+(e)\), assume \(S^\lambda_e\) contains all items that are selected before \(e\) is being considered and let \(\psi^\lambda_e\) denote the partial realization of \(S^\lambda_e\), i.e., \(\text{dom}(\psi^\lambda_e) = S^\lambda_e\). Note that \(\psi^\lambda_{e}\) is a subrealization of \(\psi^\lambda_{b[z^\lambda]}\), i.e., \(\psi^\lambda_e \subseteq \psi^\lambda_{b[z^\lambda]}\). Then we have

\[
f_{\text{avg}}(\pi^k) = \sum_{e \in E} \mathbb{E}_{\lambda \sim D(H^+(e))}[\text{Pr}[e \text{ is selected given it was considered}]\Delta(e | S^\lambda_e, \psi^\lambda_e)]
\]

(20)

\[
\geq \sum_{e \in E} \mathbb{E}_{\lambda \sim D(H(e))}[\text{Pr}[e \text{ is selected given it was considered}]\Delta(e | S^\lambda_e, \psi^\lambda_e)]
\]

(21)
The first inequality is due to $H(e) \subseteq H^+(e)$. The second equality is due to the probability of $e$ being selected conditional on $e$ has been considered is 1/2. The second inequality is due to $\psi_e^\lambda \leq \psi_{b[z]_t}^\lambda$ for all $e \in E$ and $\lambda \in H(e)$ and $f : 2^{E \times O} \to \mathbb{R}_{\geq 0}$ is adaptive submodular.

\[ \Delta(e \mid S_{z_0}^\lambda, \psi_{b[z]_t}^\lambda) \leq \Delta(e \mid S_{z_0}^\lambda, \psi_{b[z]_t}^\lambda) \]

\[ \sum_{t \in [z_0]} \Delta(e \mid S_{t-1}^\lambda, \psi_{b[t]_t}^\lambda) + f(o) \geq \alpha \cdot \sum_{e \in U(\lambda)} x(e, \psi_{b[z]_t}^\lambda) \Delta(e \mid S_{z_0}^\lambda, \psi_{b[z]_t}^\lambda) \]

\[ \sum_{t \in [z_0]} \Delta(e \mid S_{t-1}^\lambda, \psi_{b[t]_t}^\lambda) + f(o) \geq \alpha \cdot \sum_{e \in U(\lambda)} x(e, \psi_{b[z]_t}^\lambda) \Delta(e \mid S_{z_0}^\lambda, \psi_{b[z]_t}^\lambda) \]

We prove the above inequality in two cases, depending on whether Line 16 or Line 32 has been triggered or not under $\lambda$. For the case when neither Line 16 nor Line 32 has been triggered, there are two possible reasons: either all items from $E$ have been considered or the best-looking item with respect to $S_{z_0}^\lambda$ conditional on $\psi_{b[z]_t}^\lambda$ has negative marginal utility, i.e., Line 33 is triggered. In the former case, (27) follows from the fact that $U(\lambda) = \emptyset$. In the latter case, due to the definition of $U(\lambda)$, it must hold that for all $e \in U(\lambda)$, $\Delta(e \mid S_{z_0}^\lambda, \psi_{b[z]_t}^\lambda) < 0$, i.e., the marginal utility of every item in $U(\lambda)$ is negative. This implies that the RHS of (27) is non-positive, moreover, because $\sum_{t \in [z_0]} \Delta(e \mid S_{t-1}^\lambda, \psi_{b[t]_t}^\lambda) > 0$ which implies that the LHS of (27) is non-negative, we have (27).

We next focus on proving (27) for the case when either Line 16 or Line 32 has been triggered. Let $e_{z_0+1}^\lambda$ denote the item who triggers Line 16 or Line 32 i.e., adding $e_{z_0+1}^\lambda$ to $S_{z_0}^\lambda$ violates the budget constraint. Let $b[\lambda_{z_0+1}] = b[z]_t$ if Line 16 has been triggered and let $b[\lambda_{z_0+1}] = b[z]_t + 1$ if Line 32 has been triggered. We first show that for any $t \in [z_0 + 1]$,
The inequality is due to (27). This together with (33) implies (27).

The first inequality is due to the selection rule of \( \pi^k \) and the definition of \( U(\lambda) \). The second inequality is due to \( f: 2^{E \times O} \to \mathbb{R}_{\geq 0} \) is adaptive submodular and \( \psi_{b[t]}^\lambda \subseteq \psi_{b[z^\lambda]}^\lambda \) for any \( t \in [z^\lambda + 1] \).

Now we are ready to prove (27). Let \( y \in \arg \max_{x \in U(\lambda)} \frac{\Delta(e \mid S_{t-1}^\lambda, \psi_{b[t]}^\lambda)}{c(e)} \), (29) implies that

\[
\sum_{t \in [z^\lambda + 1]} \Delta(e_t^\lambda \mid S_{t-1}^\lambda, \psi_{b[t]}^\lambda) \geq \alpha \cdot \sum_{t \in [z^\lambda + 1]} \frac{c(e_t^\lambda) \Delta(y \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)}{c(y)}
\]

It follows that

\[
\sum_{t \in [z^\lambda + 1]} \Delta(e_t^\lambda \mid S_{t-1}^\lambda, \psi_{b[t]}^\lambda) \geq \alpha \cdot \sum_{t \in [z^\lambda + 1]} \frac{c(e_t^\lambda) \Delta(y \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)}{c(y)}
\]

\[
= \alpha \cdot \left( \sum_{t \in [z^\lambda + 1]} c(e_t^\lambda) \right) \frac{\Delta(y \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)}{c(y)}
\]

\[
\geq \alpha \cdot \left( \sum_{t \in [z^\lambda + 1]} c(e_t^\lambda) \right) \frac{\sum_{e \in U(\lambda)} x(e, \psi_{b[z^\lambda]}^\lambda) \Delta(e \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)}{\sum_{e \in U(\lambda)} x(e, \psi_{b[z^\lambda]}^\lambda) c(e)}
\]

\[
\geq \alpha \cdot \sum_{e \in U(\lambda)} x(e, \psi_{b[z^\lambda]}^\lambda) \Delta(e \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)
\]

The last inequality is due to \( \sum_{t \in [z^\lambda + 1]} c(e_t^\lambda) = \sum_{e \in S_{t}^\lambda} c(e) \geq B \) and \( \sum_{e \in U(\lambda)} x(e, \psi_{b[z^\lambda]}^\lambda) c(e) \leq B \). Recall that \( o \) is the best singleton, we have \( \sum_{t \in [z^\lambda]} \Delta(e \mid S_{t-1}^\lambda, \psi_{b[t]}^\lambda) + f(o) \geq \sum_{t \in [z^\lambda + 1]} \Delta(e_t^\lambda \mid S_{t-1}^\lambda, \psi_{b[t]}^\lambda) \) due to \( f: 2^{E \times O} \to \mathbb{R}_{\geq 0} \) is adaptive submodular. This together with (33) implies (27).

Given (27) in hand, now we are ready to prove the lemma.

\[
f_{avg}(\pi^k) + f(o) = \mathbb{E}_A \left[ \sum_{t \in [z^\lambda]} \Delta(e \mid S_{t-1}^\lambda, \psi_{b[t]}^\lambda) + f(o) \right]
\]

\[
\geq \mathbb{E}_A [\alpha \cdot \sum_{e \in U(\lambda)} x(e, \psi_{b[z^\lambda]}^\lambda) \Delta(e \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)]
\]

\[
= \alpha \cdot \mathbb{E}_A [\sum_{e \in U(\lambda)} x(e, \psi_{b[z^\lambda]}^\lambda) \Delta(e \mid S_{t}^\lambda, \psi_{b[z^\lambda]}^\lambda)]
\]

The inequality is due to (27). □

Now we are ready to present the main theorem.
Theorem 3. If we randomly pick a policy from \( \{ o \} \) and \( \pi^k \) with degree of adaptivity \( \alpha \) to execute such that \( \{ o \} \) is picked with probability \( \frac{1/\alpha}{3+2/\alpha} \) and \( \pi^k \) with degree of adaptivity \( \alpha \) is picked with probability \( \frac{3+1/\alpha}{3+2/\alpha} \), then we can achieve the expected utility of at least \( \frac{1}{6+4/\alpha} f_{avg}(\pi^*) \).

Proof: Lemma 3 and Lemma 4 imply that

\[
2 \times f_{avg}(\pi^k) + \frac{1}{\alpha} \times (f_{avg}(\pi^k) + f(o)) \geq \mathbb{E}_\Lambda \left[ \sum_{e \in C(\Lambda)} \Delta(e \mid S^A_{z,\lambda} \psi^A_{b[z]}) \right] + \mathbb{E}_\Lambda \left[ \sum_{e \in U(\Lambda)} x(e, \psi^A_{b[z]}) \Delta(e \mid S^A_{z,\lambda} \psi^A_{b[z]}) \right]
\]

(37)

\[
= \mathbb{E}_\Lambda \left[ \sum_{e \in C(\Lambda)} \Delta(e \mid S^A_{z,\lambda} \psi^A_{b[z]}) \right] + \sum_{e \in U(\Lambda)} x(e, \psi^A_{b[z]}) \Delta(e \mid S^A_{z,\lambda} \psi^A_{b[z]})
\]

(38)

\[
\geq \mathbb{E}_\Lambda \left[ \sum_{e \in U(\Lambda) \cup C(\Lambda)} x(e, \psi^A_{b[z]}) \Delta(e \mid S^A_{z,\lambda} \psi^A_{b[z]}) \right]
\]

(39)

\[
\geq \mathbb{E}_\Lambda [\Delta(\pi^* \mid S^A_{z,\lambda} \psi^A_{b[z]})]
\]

(40)

\[
= f_{avg}(\pi^* \eta \pi^k) - f_{avg}(\pi^k)
\]

(41)

The second inequality is due to \( x(e, \psi^A_{b[z]}) \leq 1 \) for all \( \lambda \). The third inequality is due to (18).

Recall that we toss a coin of success 1/2 to decide whether or not to add an item to the solution each time after an item is being considered, this design, together with Lemma 1 in (Tang 2021b), implies that \( f_{avg}(\pi^* \eta \pi^k) \geq f_{avg}(\pi^*)/2 \). Combining this with (42),

\[
2 \times f_{avg}(\pi^k) + \frac{1}{\alpha} \times (f_{avg}(\pi^k) + f(o)) \geq \frac{f_{avg}(\pi^*)}{2} - f_{avg}(\pi^k)
\]

(43)

It follows that

\[
(3 + \frac{1}{\alpha}) f_{avg}(\pi^k) + \frac{1}{\alpha} f(o) \geq \frac{f_{avg}(\pi^*)}{2}
\]

(44)

Recall that our final policy randomly picks a policy from \( \{ o \} \) and \( \pi^k \) to execute such that \( \{ o \} \) is picked with probability \( \frac{1/\alpha}{3+2/\alpha} \) and \( \pi^k \) is picked with probability \( \frac{3+1/\alpha}{3+2/\alpha} \), then the expected utility of our final policy is

\[
f_{avg}(\pi^k) \times \frac{3+1/\alpha}{3+2/\alpha} + f(o) \times \frac{1/\alpha}{3+2/\alpha}
\]

(45)

(44) and (45) imply that the expected utility of our final policy is at least

\[
\frac{3+1/\alpha}{3+2/\alpha} f_{avg}(\pi^k) + \frac{1}{6+4/\alpha} f_{avg}(\pi^*)
\]

(46)
5.3. Bounding the Batch Query Complexity

We next analyze the batch query complexity, i.e., the number of batches a policy takes to complete the selection process, of our partial-adaptive policy. We show that if we choose an appropriate $\alpha$, then it takes at most $\min\{O(n), O(\log n \log \frac{B}{c_{\min}})\}$, where $c_{\min} = \min_{e \in E} c(e)$ is the cost of the cheapest item, batches to achieve a constant approximation ratio. Notably, if we consider a cardinality constraint $k$, then the above complexity is upper bounded by $O(\log n \log k)$ which is polylogarithmic. This is in sharp contrast to the case of fully adaptive policy whose batch query complexity is $O(n)$. To establish this result, we first introduce two classes of stochastic functions.

**Definition 6.** [Tang and Yuan (2021)](Policywise Submodularity) A function $f : 2^{E \times O} \rightarrow \mathbb{R}_{\geq 0}$ is policywise submodular with respect to a prior $p(\phi)$ and a knapsack constraint $(c, B)$ if for any two partial realizations $\psi$ and $\psi'$ such that $\psi' \subseteq \psi$ and $c(\text{dom}(\psi)) \leq B$, and any $S \subseteq E$ such that $S \cap \text{dom}(\psi) = \emptyset$, we have

$$\max_{\pi \in \Omega} \Delta(\pi|\text{dom}(\psi'), \psi') \geq \max_{\pi \in \Omega} \Delta(\pi|\text{dom}(\psi), \psi)$$

(47)

where $\Omega = \{\pi | \forall \phi : c(E(\pi, \phi)) \leq B - c(\text{dom}(\psi)), E(\pi, \phi) \subseteq S\}$ denotes the set of feasible policies which are restricted to selecting items only from $S$.

**Definition 7 (Weak Policywise Submodularity).** A function $f : 2^{E \times O} \rightarrow \mathbb{R}_{\geq 0}$ is weak policywise submodular with respect to a prior $p(\phi)$ and a knapsack constraint $(c, B)$ if for any partial realization $\psi$ such that $c(\text{dom}(\psi)) \leq B$,

$$f_{\text{avg}}(\pi^*) \geq \max_{\pi \in \Omega} \Delta(\pi|\text{dom}(\psi), \psi)$$

(48)

where $\pi^*$ represents an optimal policy and $\Omega = \{\pi | \forall \phi : c(E(\pi, \phi)) \leq B - c(\text{dom}(\psi))\}$ denotes the set of feasible policies subject to a budget constraint $B - c(\text{dom}(\psi))$.

Observe that policywise submodularity implies weak policywise submodularity. In Tang and Yuan (2021), it has been shown that policywise submodularity (and hence weak policywise submodularity) can be found in a wide range of real-world applications, including generalized binary search (Golovin and Krause 2011), any applications where items are independent (Asadpour and Nazerzadeh 2016), and adaptive viral marketing (Golovin and Krause 2011). We next show that if a utility function is both adaptive submodular and weak policywise submodular, then our partial-adaptive policy achieves a constant approximation ratio using only $O(\log n \log k)$ number of batches.
To analyze the batch query complexity of $\pi^k$, we focus on the original description of $\pi^k$ (Algorithm 2). That is, instead of tossing a coin each time after an item is being considered to be added to the solution, we first select a random subset $R$ which is obtained by including each item $e \in E$ independently with probability $1/2$. Then we run the “density-greedy” algorithm only on $R$. We first present a technical lemma that extends Lemma 7 in (Esfandiari et al. 2021) to a general knapsack constraint.

**Lemma 5.** Consider an arbitrary batch index $l$ and a partial realization $\psi_l$ observed after the first $l$ batches conditional on $R \subseteq E$ is being sampled at the beginning of $\pi^k$. Let $l^+ = l + \log \frac{1}{1-(1-\alpha)/2} (\frac{n}{\delta})$. Assuming we run $\pi^k$ without budget constraint and $f : 2^{E \times O} \rightarrow \mathbb{R}_{\geq 0}$ is adaptive monotone and adaptive submodular, then

$$\max_{e \in R \setminus \text{dom}(\psi_{l^+})} \frac{\Delta(e | \text{dom}(\psi_{l^+}), \psi_{l^+})}{c(e)} \leq (1 - \frac{1 - \alpha}{2}) \max_{e \in R \setminus \text{dom}(\psi_l)} \frac{\Delta(e | \text{dom}(\psi_l), \psi_l)}{c(e)}$$

with probability at least $1 - \delta$. Note that it could be the case that there may not be $l^+$ batches after selecting all items. If that is the case, just add some empty batches to make it $l^+$ batches.

**Proof:** We closely follow the proof of Lemma 7 in (Esfandiari et al. 2021). For any $l' \geq l$, let the random variable $R_{l'}$ denote the set of items from $R \setminus \text{dom}(\psi_{l'})$ such that $\frac{\Delta(e | \text{dom}(\psi_{l'}), \psi_{l'})}{c(e)} \geq (1 - \frac{1 - \alpha}{2}) \max_{e \in R \setminus \text{dom}(\psi_l)} \frac{\Delta(e | \text{dom}(\psi_l), \psi_l)}{c(e)}$. To prove this lemma, it suffices to show that

$$\mathbb{E}[|R_{l'+1}|] \leq (1 - \frac{1 - \alpha}{2}) \mathbb{E}[|R_{l'}|]$$

This is because (50), together with the facts that $|R_l| \leq n$ and $l^+ = l + \log \frac{1}{1-(1-\alpha)/2} (\frac{n}{\delta})$, implies that $\mathbb{E}[|R_{l^+}|] \leq \delta$. Note that $|R_{l^+}|$ is non-negative, thus we have $|R_{l^+}| = 0$ with probability at least $1 - \delta$, as desired.

We next focus on proving (50). First, because of adaptive submodularity, we have $e \in R_{l'+1}$ implies that $e \in R_{l'}$. Thus, $R_{l'+1} \subseteq R_{l'}$. We next show that for any item $e \in R_{l'}$, we have $e \notin R_{l'+1}$ with probability at least $\frac{1 - \alpha}{2}$.

Because we run $\pi^k$ without budget constraint, if $e \in R_{l'}$ and $e$ is not selected within the $(l' + 1)$-th batch of $\pi^k$, then

$$\mathbb{E}_{\psi_{l'+1}} \left[ \frac{\Delta(e | \text{dom}(\psi_{l'+1}), \psi_{l'})}{c(e)} \right] \leq \alpha \max_{e \in R \setminus \text{dom}(\psi_{l'})} \frac{\Delta(e | \text{dom}(\psi_{l'}), \psi_{l'})}{c(e)}$$
Because \( \max_{e \in R \setminus \text{dom}(\psi_T)} \frac{ \Delta(e | \text{dom}(\Psi_T), \Psi_T)}{c(e)} \leq \max_{e \in R \setminus \text{dom}(\psi)} \frac{ \Delta(e | \text{dom}(\Psi_{T+1}), \Psi_{T+1})}{c(e)} \) due to adaptive submodularity, we have \( \mathbb{E}_{\Psi_{T+1}} [ \frac{ \Delta(e | \text{dom}(\Psi_{T+1}), \Psi_{T+1})}{c(e)} ] \leq \alpha \max_{e \in R \setminus \text{dom}(\psi)} \frac{ \Delta(e | \text{dom}(\psi_t), \psi_t)}{c(e)} \). This, together with the assumption that \( f : 2^E \times O \rightarrow \mathbb{R}_{\geq 0} \) is adaptive monotone, implies that the probability that \( \Delta(e | \text{dom}(\Psi_{T+1}), \Psi_{T+1}) \leq (1 - \frac{1 - \alpha}{2}) \max_{e \in R \setminus \text{dom}(\psi)} \frac{ \Delta(e | \text{dom}(\psi_t), \psi_t)}{c(e)} \) is at least \( \frac{1 - \alpha}{2} \). It follows that \( e \notin R_{T+1} \) with probability at least \( \frac{1 - \alpha}{2} \). This indicates that \( \mathbb{E}[|R_{T+1}|] \leq (1 - \frac{1 - \alpha}{2}) \mathbb{E}[|R_T|] \), as desired. \( \square \)

The following lemma shows that if \( f : 2^E \times O \rightarrow \mathbb{R}_{\geq 0} \) is adaptive monotone and adaptive policywise submodular with respect to a prior \( p(\phi) \) and any knapsack constraints, then it takes at most \( O(\log n \log \frac{B}{\min}) \) batches of \( \pi^k \) to achieve a near optimal solution. Let \( \pi^k(T) \) denote a policy that runs \( \pi^k \) and stops if it makes \( T \) batch queries. In the rest of this section, let \( T = \log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\delta}) \times \log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\min}) \) where \( \delta = \frac{1 - \alpha}{\log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\min})} \).

**Lemma 6.** If \( f : 2^E \times O \rightarrow \mathbb{R}_{\geq 0} \) is adaptive monotone, adaptive submodular, and weak policywise submodular with respect to a prior \( p(\phi) \) and any knapsack constraints, then \( f_{\text{avg}}(\pi^k(T)) \geq (1 - \alpha + \alpha^2)f_{\text{avg}}(\pi^k) - (1 - \alpha)f_{\text{avg}}(\pi^*) \).

**Proof:** We first analyze the expected utility of \( \pi^k(T) \) conditional on \( R \subseteq E \) is sampled at the beginning of \( \pi^k \). Let \( a \) be the first selected item, i.e., \( a \in \arg \max_{e \in R} \frac{ \Delta(e | \emptyset, \emptyset)}{c(e)} \). Assuming we run \( \pi^k \) without budget constraint, by applying Lemma 5 iteratively \( \log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\min}) \) times we have

\[
\max_{e \notin \text{dom}(\Psi_T)} \frac{ \Delta(e | \text{dom}(\Psi_T), \Psi_T)}{c(e)} \leq (1 - \frac{1 - \alpha}{2}) \log \frac{1}{1 - (1 - \alpha)^2} \frac{B}{\min} \frac{ \Delta(a | \emptyset, \emptyset)}{c(a)} (52)
\]

with probability \( 1 - \delta \times \log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\min}) = \alpha \). Recall that \( T = \log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\delta}) \times \log \frac{1}{1 - (1 - \alpha)^2} (\frac{B}{\min}) \), this indicates that with probability \( \alpha \) the total expected marginal benefit of all item added after the \( T \)-th batch is at most

\[
\frac{(1 - \alpha) \Delta(a | \emptyset, \emptyset)}{B} \times B = (1 - \alpha) \Delta(a | \emptyset, \emptyset) \leq (1 - \alpha) \mathbb{E}[f_{\text{avg}}(\pi^k) | R] (54)
\]

where \( \mathbb{E}[f_{\text{avg}}(\pi^k) | R] \) is the expected utility of \( \pi^k \) conditional on \( R \) is being sampled.

Moreover, because \( f : 2^E \times O \rightarrow \mathbb{R} \) is weak policywise submodular with respect to a prior \( p(\phi) \) and any knapsack constraints, we have

\[
\max_{\pi \in \Omega} \Delta(\pi | \text{dom}(\psi_T), \psi_T) \leq f_{\text{avg}}(\pi^*) (55)
\]
where $\Omega = \{ \pi : \forall \phi: c(E(\pi, \phi)) \leq B - c(\text{dom}(\psi_T)) \}$ denotes the set of all feasible policies subject to a budget constraint $B - c(\text{dom}(\psi_T))$. (55) indicates that for any $\psi_T$, the total expected marginal benefit of all item added after the $T$-th batch is at most $f_{avg}(\pi^*)$. This, together with (54), implies that the total expected marginal benefit of the items added after the $T$-th batch is at most

\[
\alpha (1 - \alpha) \mathbb{E}[f_{avg}(\pi^k) | R] + (1 - \alpha) f_{avg}(\pi^*)
\]

(56)

It follows that the expected utility of $\pi_k(T)$ conditional on $R$ is being sampled is at least

\[
\mathbb{E}[f_{avg}(\pi(T)) | R] \geq f_{avg}(\pi^k) - \alpha (1 - \alpha) \mathbb{E}[f_{avg}(\pi^k) | R] - (1 - \alpha) f_{avg}(\pi^*)
\]

(57)

By taking the expectation of both sides of the above inequality over $R$, now we are ready to bound the expected utility of $\pi_k(T)$ as follows:

\[
f_{avg}(\pi^k(T)) = \sum_{R \subseteq E} \Pr[R \text{ is sampled}] \mathbb{E}[f_{avg}(\pi^k(T)) | R]
\]

\[
\geq \sum_{R \subseteq E} \Pr[R \text{ is sampled}] (f_{avg}(\pi^k) - \alpha (1 - \alpha) \mathbb{E}[f_{avg}(\pi^k) | R] - (1 - \alpha) f_{avg}(\pi^*))
\]

(58)

\[
= f_{avg}(\pi^k) - (1 - \alpha) f_{avg}(\pi^*) - \sum_{R \subseteq E} \Pr[R \text{ is sampled}] (1 - \alpha) \mathbb{E}[f_{avg}(\pi^k) | R]
\]

(59)

\[
= f_{avg}(\pi^k) - (1 - \alpha) f_{avg}(\pi^*) - \alpha (1 - \alpha) f_{avg}(\pi^k)
\]

(60)

\[
= (1 - \alpha + \alpha^2) f_{avg}(\pi^k) - (1 - \alpha) f_{avg}(\pi^*)
\]

(61)

The inequality is due to (57). \(\square\)

**Remark:** Note that adaptive submodularity does not necessarily imply (55). As a result, one main result (Theorem 8) presented in [Esfandiari et al., 2021] does not hold without resorting to weak policywise submodularity.

We next show that if we randomly pick a policy from $\pi^k(T)$ and $\{o\}$ to execute, then we can achieve a constant approximation ratio against the optimal fully adaptive policy.

**Lemma 7.** Assuming $f : 2^{E \times O} \rightarrow \mathbb{R}_{\geq 0}$ is adaptive monotone, adaptive submodular, and adaptive policywise submodular with respect to a prior $p(\phi)$ and any knapsack constraints. If we randomly pick a policy from $\pi^k(T)$ and $\{o\}$ to execute such that $\pi^k(T)$ is picked with probability $\frac{3+1/\alpha}{3+2/\alpha}$ and $\{o\}$ is picked with probability $\frac{1/\alpha}{3+2/\alpha}$, then we can achieve the expected utility of at least

\[
\frac{(1-\alpha+\alpha^2)/(3+1/\alpha) - (1-\alpha)(3+1/\alpha)}{3+2/\alpha} f_{avg}(\pi^*)
\]
Proof: Observe that if $\pi^k(T)$ is picked with probability $\frac{3+1/\alpha}{3+2/\alpha}$ and $\{o\}$ is picked with probability $\frac{1}{3+2/\alpha}$, then the expected utility can be represented as $f_{avg}(\pi^k(T)) \times \frac{3+1/\alpha}{3+2/\alpha} + f(o) \times \frac{1}{3+2/\alpha}$. Then we have

$$f_{avg}(\pi^k(T)) \times \frac{3+1/\alpha}{3+2/\alpha} + f(o) \times \frac{1}{3+2/\alpha} \geq ((1-\alpha+\alpha^2)f_{avg}(\pi^k) - (1-\alpha)f_{avg}(\pi^*)) \times \frac{3+1/\alpha}{3+2/\alpha} + f(o) \times \frac{1}{3+2/\alpha}$$

$$\geq \frac{(1-\alpha + \alpha^2)(3+1/\alpha)f_{avg}(\pi^k) + (1/\alpha)f(o)}{3+2/\alpha} - (1-\alpha)\frac{3+1/\alpha}{3+2/\alpha}f_{avg}(\pi^*)$$

$$\geq \frac{(1-\alpha + \alpha^2)((3+1/\alpha)f_{avg}(\pi^k) + (1/\alpha)f(o))}{3+2/\alpha} - (1-\alpha)\frac{3+1/\alpha}{3+2/\alpha}f_{avg}(\pi^*)$$

$$\geq \frac{(1-\alpha + \alpha^2)f_{avg}(\pi^*)/2}{3+2/\alpha} - (1-\alpha)\frac{3+1/\alpha}{3+2/\alpha}f_{avg}(\pi^*)$$

$$= \frac{(1-\alpha + \alpha^2)/2 - (1-\alpha)(3+1/\alpha)}{3+2/\alpha}f_{avg}(\pi^*)$$

(63)

The first inequality is due to Lemma 6. The forth inequality is due to (44). □

6. Conclusion

In this paper, we study the partial-adaptive submodular maximization problem. Our setting allows to select multiple items in a batch simultaneously and observe their realizations together. We develop effective solutions subject to both cardinality constraint and knapsack constraint. We analyze the batch query complexity of our policy under some additional assumptions about the utility function.

References

Amanatidis, Georgios, Federico Fusco, Philip Lazos, Stefano Leonardi, Rebecca Reiffenhäuser. 2020. Fast adaptive non-monotone submodular maximization subject to a knapsack constraint. Advances in neural information processing systems.

Asadpour, Arash, Hamid Nazerzadeh. 2016. Maximizing stochastic monotone submodular functions. Management Science 62 2374–2391.

Badanidiyuru, Ashwinkumar, Jan Vondrák. 2014. Fast algorithms for maximizing submodular functions. Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms. SIAM, 1497–1514.

Buchbinder, Niv, Moran Feldman, Joseph Naor, Roy Schwartz. 2014. Submodular maximization with cardinality constraints. Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms. SIAM, 1433–1452.
Chen, Yuxin, Andreas Krause. 2013. Near-optimal batch mode active learning and adaptive submodular optimization. *ICML (1)* 28:8–1.

Esfandiari, Hossein, Amin Karbasi, Vahab Mirrokni. 2021. Adaptivity in adaptive submodularity. *COLT*.

Golovin, Daniel, Andreas Krause. 2011. Adaptive submodularity: Theory and applications in active learning and stochastic optimization. *Journal of Artificial Intelligence Research* 42:427–486.

Gotovos, Alkis, Amin Karbasi, Andreas Krause. 2015. Non-monotone adaptive submodular maximization. *Twenty-Fourth International Joint Conference on Artificial Intelligence*.

Nemhauser, George L, Laurence A Wolsey, Marshall L Fisher. 1978. An analysis of approximations for maximizing submodular set functions-i. *Mathematical programming* 14:265–294.

Tang, Shaojie. 2021a. Beyond pointwise submodularity: Non-monotone adaptive submodular maximization in linear time. *Theoretical Computer Science* 850:249–261.

Tang, Shaojie. 2021b. Beyond pointwise submodularity: Non-monotone adaptive submodular maximization subject to knapsack and k-system constraints. *Modelling, Computation and Optimization in Information Systems and Management Sciences*. Springer.

Tang, Shaojie, Jing Yuan. 2020. Influence maximization with partial feedback. *Operations Research Letters* 48:24–28.

Tang, Shaojie, Jing Yuan. 2021. Optimal sampling gaps for adaptive submodular maximization. *arXiv preprint arXiv:2104.01750*.

Yuan, Jing, Shaojie Tang. 2017. No time to observe: adaptive influence maximization with partial feedback. *Proceedings of the 26th International Joint Conference on Artificial Intelligence*. 3908–3914.