Quantum computational tensor network on string-net condensate

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String-net condensate is a new class of materials which exhibits quantum topological order. In order to answer the important question, “how useful is string-net condensate in quantum information processing?” we consider the simplest example of string-net condensate, namely the \( Z_2 \) gauge string-net condensate on the two-dimensional hexagonal lattice, and investigate possibilities of universal measurement-based quantum computation on it by using the framework of quantum computational tensor network. We show that universal measurement-based quantum computation is possible by coupling two correlation space wires with a physical two-body interaction. We also show that universal measurement-based quantum computation is possible solely with single-qubit measurements if we consider a slightly modified version of the \( Z_2 \) gauge string-net condensate on the two-dimensional hexagonal lattice. These results suggest that even the simplest example of string-net condensate is equipped with the correlation space that has the capacity for universal quantum computation.

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I. INTRODUCTION

Exploiting the full potential of quantum many-body states for quantum information processing is one of the most central research subjects in today’s quantum information science and condensed matter physics. Plenty of theoretical and experimental researches over the last few years have contributed to the important cross-fertilization between these two fields. In particular, the seminal work by Raussendorf and Briegel about the one-way quantum computation [1] has offered the clear view to the role of quantum many-body states played in quantum computation. Once the special many-body state, which is called the cluster state, is prepared, universal quantum computation can be performed by adaptively measuring each qubit. Recently, the one-way quantum computation was generalized to more abstract and beautiful framework, so-called quantum computational tensor network [2-4]. This framework, which cleverly uses the matrix product representation [5,6] (or, more generally, the tensor-network representation [7,8]) of quantum many-body states, has established the novel concept of quantum computation, namely virtual quantum computation in the correlation space. The exploration of resource many-body states has been thus replaced with the exploration of correlation spaces which have full capacity for universal quantum computation [4,21].

On the other hand, in quantum many-body physics, such as condensed matter physics, statistical physics, and nuclear physics, orders and phase transitions lie at the heart of main research topics. Traditionally, phase transitions have been studied in the framework of Landau’s symmetry-breaking theory [22]. However, a new type of order, so-called quantum topological order, has been discovered in several many-body systems, such as the fractional quantum Hall system [23], and it has been pointed out that the quantum topological order slips through the framework of Landau’s symmetry-breaking theory.

String-net condensate is a new class of materials introduced in Ref. [24] to describe such a quantum topological order. As is pointed out in Ref. [25], it is of great worth to explore applications of such a new type of condensate to technologies, such as a hard disk drive, a liquid crystal display, and semiconducting devices. In particular, it is very important to answer the question, “how useful is string-net condensate in quantum information processing?” Indeed, the famous scheme of the fault-tolerant quantum memory with the Kitaev’s toric code state [26,27], which is the \( Z_2 \) gauge string-net condensate on the two-dimensional square lattice, would be the first great example of an application of string-net condensate to quantum information technology.

In this paper, by using the framework of quantum computational tensor network [24], we study how useful is string-net condensate in quantum computation. For this purpose, we consider the simplest example of string-net condensate, namely the \( Z_2 \) gauge string-net condensate on the two-dimensional square lattice. As is shown in Ref. [28], Kitaev’s toric code state on the two-dimensional square lattice is not a universal resource state in the sense that the measurement-based quantum computation solely with single-qubit measurements on that state can be classically simulated. Their result can be generalized to the two-dimensional hexagonal lattice [29]. Therefore, the \( Z_2 \) gauge string-net condensate on the two-dimensional hexagonal lattice is neither a universal resource state in the strict sense (here, the strict definition of the universality is that universal measurement-based quantum computation can be performed solely with single-qudit measurements.) However, we show that universal measurement-based quantum computation is possible on the \( Z_2 \) gauge string-net condensate on the two-dimensional hexagonal lattice, by coupling two correlation space wires with a physical two-body interaction.
Considering the fact that string-net condensate is central in condensed-matter physics and the fact that many examples of resource states which require physical two-body interactions have greatly contributed to the recent development of measurement-based quantum computation \cite{4, 10, 12, 13, 21, 33, 34}, it is still very important to investigate possibilities of measurement-based quantum computation with the help of physical two-body interactions. We also show that a slightly modified version of the $Z_2$ gauge string-net condensate on the two-dimensional hexagonal lattice is a universal resource state in the strict sense. Our results imply that even the simplest example of string-net condensate is equipped with the correlation space that has the capacity for universal quantum computation.

Surprisingly, the possibility of universal measurement-based quantum computation on string-net condensate was an open problem even if a physical two-body interaction is allowed. The cluster state \cite{1} is not topologically ordered, since some quasi-local unitary operations can change the cluster state into a product state \cite{32–34}. In ordered, since some quasi-local unitary operations can change the cluster state into a product state \cite{32–34}. In Refs. \cite{2, 3}, the measurement-based quantum computation on the slightly modified version of the Kitaev’s toric code state was studied. However, it is not clear whether that resource state is string-net condensate or not. The two-dimensional AKLT state \cite{12} and the resource state proposed in Ref. \cite{10} do not seem to be string-net condensate at least in the $Z$ basis from their tensor-network structures \cite{11, 12}. Furthermore, since these resource states are not qubit states but qudit states, string-net condensate should be more complicated than $Z_2$ gauge type if any. In Ref. \cite{28}, a string-net condensate on a highly non-local graph \cite{4} of Ref. \cite{28}, which can be converted into a cluster state by single-qubit measurements and hence universal, was proposed. However, it was an open problem whether such a strong assumption (i.e., the highly non-local graph) can be weakened or not. Our results answer the question: we can perform universal measurement-based quantum computation on a string-net condensate without considering highly non-local graph structure, if two correlation space wires are coupled with a physical two-body interaction, or if the coefficients of the linear-combination of closed-loop configurations are slightly modified.

II. $Z_2$ GAUGE STRING-NET CONDENSATE

Let us consider the two-dimensional hexagonal lattice where two qubits are placed on each edge \cite{4 (a)}. We define the Hamiltonian \cite{35, 36} $H$ on this lattice by

$$H \equiv - \sum \mathbf{X}_{\mathbf{e}} \mathbf{Z}_{\mathbf{e}} - \sum \mathbf{X}_{\mathbf{v}} \mathbf{Z}_{\mathbf{v}} - \sum \mathbf{Z}_{\mathbf{e}} \mathbf{Z}_{\mathbf{v}},$$

where $X$ and $Z$ are Pauli operators. $p$ indicates a hexagonal plaquette, $v$ indicates a vertex, and $e$ indicates an edge (see Fig. 1 (a)). $S_p$ is the set of 12 qubits in the plaquette $p$, $S_v$ is the set of six qubits in the three edges associated with the vertex $v$, and $S_e$ is the set of two qubits in the edge $e$ (see Fig. 1 (a)). The two-body interaction $-\bigotimes_{e \in S_e} Z_e$ forces two qubits on the edge $e$ to be the up-up state $|0\rangle \otimes |0\rangle$ or the down-down state $|1\rangle \otimes |1\rangle$. Here, $Z|s\rangle = (-1)^s|s\rangle$ ($s \in \{0, 1\}$). If two qubits on an edge are $|1\rangle \otimes |1\rangle$, we consider that the edge is occupied by a string. On the other hand, if the two spins are $|0\rangle \otimes |0\rangle$, the edge is considered to be vacant (see Fig. 1 (b)). The six-body interaction $-\bigotimes_{e \in S_p} Z_e$ forces strings to form closed loops (see Fig. 1 (b)). The twelve-body interaction $-\bigotimes_{e \in S_p} X_e$ works as the kinetic term for such closed loops. Therefore, the ground state $|G\rangle$ of the Hamiltonian $H$ is the equal weight superposition of all closed loop configurations. Such a state is called the string-net condensate \cite{24}. It is known \cite{35, 36} that the ground state $|G\rangle$ has a simple tensor-network representation. As is shown in Fig. 2 (a), the tensor $T$ defined in Fig. 2 (b) is placed on each vertex of the hexagonal lattice, and its virtual legs are connected with three nearest-neighbor tensors. As is shown in Fig. 2 (b), the tensor $T$ has three physical legs, each of which corresponds to a single qubit, and three virtual legs, each of which corresponds to the two-dimensional Hilbert space.

FIG. 1: (Color online.) (a): The two-dimensional hexagonal lattice. Green circles represent qubits. Two qubits are placed on each edge. The red hexagon indicates the set $S_p$ of 12 qubits in the plaquette $p$. The blue tree indicates the set $S_v$ of six qubits in the three edges associated with the vertex $v$. The purple bond indicates the set $S_e$ of two qubits in the edge $e$. (b): An example of the string-net configuration where only closed loops exist. Red bonds represent strings.

In Fig. 3 we illustrate how our measurement-based quantum computation runs on the ground state $|G\rangle$. As is shown in Fig. 3 (a), each horizontal line of plaquettes encodes a logical single-qubit wire, and the Controlled-$Z$ (CZ) gate is implemented by interacting two nearest-neighbor logical single-qubit wires. Figure 3 (b) shows the quantum circuit which corresponds to Fig. 3 (a). Throughout this paper, we will adopt the framework of quantum computational tensor network \cite{2–4}. Readers who are not familiar with that framework can refer to Refs. \cite{2–4, 9–17}.
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FIG. 3: (Color online.) (a): A schematic illustration of our measurement-based quantum computation. (b): The corresponding quantum circuit. The horizontal lines are single-qubit wires, whereas the vertical lines are CZ gates.

III. IMPLEMENTATION OF GATES

Let us see Fig. 4 (a). If we perform the Z measurement on site a and the X measurement on sites b and c, the operation \( X^\mu_a Z^\mu_b \otimes Z^\mu_c \) is implemented in the path \( b \rightarrow c \). Here, \( \mu \in \{0, 1\} \) is the measurement result (\( \mu \) corresponds to the result of the eigenvalue +1 and \( \mu \) corresponds to the result of the eigenvalue -1, respectively). In the same way, if we perform the Z measurement on site d and the X measurement on sites e and f, the operation \( X^\mu_a Z^\mu_b \otimes Z^\mu_c \) is implemented in the path \( e \rightarrow f \). Let \( \{0\}_u \otimes |1\>_l \) be the basis of the qubit that flows the upper (lower) path, \( \cdots \rightarrow b \rightarrow c \rightarrow h \rightarrow i \rightarrow \cdots \) \( \cdots \rightarrow e \rightarrow f \rightarrow k \rightarrow l \rightarrow \cdots \). We encode a single logical qubit \( \{0\}_L, \{1\}_L \rangle \) by using these two qubits [37]. Let us define two types of the encoding [38]: the type-I encoding is defined by \( |0\>_L \) \( \leftrightarrow |0\>_u \otimes |0\>_l \) and \( |1\>_L \rangle \) \( \leftrightarrow |1\>_u \otimes |1\>_l \), whereas the type-II encoding is defined by \( |0\>_L \) \( \leftrightarrow |0\>_u \otimes |1\>_l \) and \( |1\>_L \rangle \) \( \leftrightarrow |1\>_u \otimes |0\>_l \). Then, the effect of the operation \( X^\mu_a Z^\mu_b \otimes Z^\mu_c \otimes Z^\mu_d \otimes Z^\mu_e \) is just the change of the encoding type, the logical X operation, or the logical Z operation. This is understood as follows: Let \( E \equiv (X^\mu_a Z^\mu_b \otimes \mu_c) \otimes (X^\mu_d Z^\mu_e \otimes \mu_f) \). Then, by using the facts that \( (X \otimes X)|0\>_L \rangle = |1\>_L \rangle \) and that \( XZ = -ZX \), we obtain

\[ E(\alpha|0\>_L \rangle + \beta|1\>_L \rangle) = \alpha E|0\>_L \rangle + (-1)^r \beta(\alpha X|0\>_L \rangle) \]

where \( r = \mu_c \otimes \mu_e \otimes \mu_f \). Note that \( E|0\>_L \rangle = (X \otimes X)|0\>_L \rangle \) up to a phase factor with some \( v \in \{0, 1\} \), where “I/II” stands for “type-I encoding or type-II encoding”. Therefore,

\[ E(\alpha|0\>_L \rangle + \beta|1\>_L \rangle) = X^\mu_c \alpha E|0\>_L \rangle + (-1)^r \beta|1\>_L \rangle \]

where \( X_L = X \otimes X \) and \( Z_L = Z \otimes I \). Similar result is obtained for \( \alpha|0\>_L \rangle + \beta|1\>_L \rangle \).

In order to perform the logical single-qubit z-rotation \( e^{-i\pi Z X \theta / 2} \), we do the X measurement on site i, k, and l, the Z measurement on site g, and the measurement in the basis \( \{ \frac{1}{\sqrt{2}} (|0\rangle \pm e^{-i\theta}|1\rangle) \} \) on site h. Then, the operation \( [(Z^\mu_a \otimes Z^\mu_b \otimes X^\mu_c) \otimes (Z^\mu_d \otimes Z^\mu_e \otimes X^\mu_f)](e^{-i2\theta / 2} I \otimes I) \) is implemented up to a phase factor. This is logical single-qubit z-rotation up to some logical Pauli byproducts. (If there is the logical X byproduct \( X_L = X \otimes X \) before this z-rotation, \( \theta \) should be replaced with \(-\theta\).)

In order to perform the logical single-qubit x-rotation \( e^{-ix_{L} \theta / 2} \), we do the X measurement on site h, i, k, and l, and the measurement in the basis \( \{ \cos \frac{\theta}{2} |0\rangle + i \sin \frac{\theta}{2} |1\rangle, \sin \frac{\theta}{2} |0\rangle - i \cos \frac{\theta}{2} |1\rangle \} \) on site g. Then, the operation \( [(Z^\mu_a \otimes Z^\mu_b \otimes X^\mu_c) \otimes (Z^\mu_d \otimes Z^\mu_e \otimes X^\mu_f)](e^{-i2\theta / 2} I \otimes I) \) is implemented up to the phase factor. This is logical single-qubit x-rotation up to some logical Pauli byproducts. (If there is the logical Z byproduct \( Z_L = Z \otimes I \) before this x-rotation, \( \theta \) should be replaced with \(-\theta\).)

Let us see Fig. 4 (b). In order to perform the logical CZ gate between the upper logical wire and the lower logical wire, we use the coupling technique [37, 44, 45, 46]. We first apply the physical CZ interaction between sites c and e. We next do the Z measurement on site a, and the X measurement on sites c, d, e, and f. Then, the operation \( (Z^\mu_a \otimes Z^\mu_b \otimes X^\mu_c) \otimes (Z^\mu_d \otimes Z^\mu_e \otimes X^\mu_f)(|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle - |11\rangle|11\rangle) \)
\[|10\rangle\langle 10| - |11\rangle\langle 11|\) up to a phase factor is implemented. If the upper logical wire is in the type-I encoding, this is the logical CZ gate up to some logical Pauli byproducts. If the upper logical wire is in the type-II encoding, this is the logical CZ gate plus \(I_L \otimes Z_L\) up to some logical Pauli byproducts.

Note that these physical CZ interactions can be also done in advance before starting the measurement-based quantum computation. In other words, we can start with the resource state \((G') \equiv \bigotimes_{i,j} CZ_{i,j}|G\rangle\), where physical CZ interactions are periodically applied on appropriate places. In this case, the entire computation can be done solely with only single-qubit measurements, since an unwanted CZ gate can be canceled by doing the identity operation (plus some Pauli byproducts) until we arrive at the next CZ gate which cancels the previous one \([15]\). Furthermore, \(|G'\rangle\) is still string-net condensate, although the coefficient of the linear combination of closed-loop configurations are more complicated: \(|G'\rangle = \sum \xi (-1)^{f(\xi)}|\xi\rangle\), where \(\xi\) is a closed-loop configuration and \(f(\xi)\) is a certain function of \(\xi\).

Finally, let us consider the initialization and the final readout of logical qubits. A virtual leg of the tensor \(T\) is initialized in the computational basis if we do the physical \(Z\) measurement on two physical legs that does not correspond to the virtual leg that we want to initialize. For example, in Fig. 3 (a), if we do the \(Z\) measurement on sites \(h\) and \(g\), the site \(i\) is initialized as \(|\mu_h \otimes \mu_g\rangle\).

Thus the initialization is possible. The final readout in the computational basis is also possible in the similar way \([39]\).

**IV. CONCLUSION**

In this paper, we have shown that the correlation space of the \(Z_2\) gauge string-net condensate on the two-dimensional hexagonal lattice has sufficient capacity for universal measurement-based quantum computation.

Since the Hamiltonian considered in this paper is gapped and frustration-free, our quantum computation can enjoy the energy-gap protection by adiabatically turning off appropriate interactions as in Refs. \([8, 12]\). It would be interesting to investigate whether our model has the robustness of the computational capacity against slight changes of Hamiltonian parameters, the edge-state picture, and the symmetrically protected order \([30]\).

Acknowledgments

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[29] The equation above Eq. (6) in Ref. 28 holds for the hexagonal lattice. Any vertex of a subgraph of the two-dimensional square lattice has degree 1, 2, 3, or 4. They showed that it can be assumed that all vertices have degree 3. Since any vertex of a subgraph of the two-dimensional hexagonal lattice has degree 1, 2, or 3, their result can be applied to the hexagonal lattice. Thus the discussion around Eq. (8) holds for the hexagonal lattice. Equation (11) holds for the hexagonal lattice. Lemma on page 5 also holds for the hexagonal lattice.
Universal single-qubit rotation can be also done without such an encoding. However, in this case, the rerouting technique \cite{2,4} is required. At this stage, we don’t know which is easier: encoding or rerouting. It should depend on specific experimental setups.

Two types of logical encoding are required, since, for example, the operation $I \otimes X$ brings state outside of one logical encoding space.

When a horizontal wire is decoupled, its MPS is given by $A[0] = I$ and $A[1] = Z$ or $A[0] = X$ and $A[1] = XZ$. Thus our resource state is in the class of Ref. \cite{17}, and therefore it is free from the finite-size effect \cite{17}.