Graph Search based Polar Code Design

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Abstract—It is well known that to fulfill their full potential, the design of polar codes must be tailored to their intended decoding algorithm. While for successive cancellation (SC) decoding, information theoretically optimal constructions are available, the code design for other decoding algorithms (such as belief propagation (BP) decoding) can only be optimized using extensive Monte Carlo simulations. We propose to view the design process of polar codes as a graph search problem and thereby approaching it more systematically. Based on this formalism, the design-time complexity can be significantly reduced compared to state-of-the-art Genetic Algorithm (GenAlg) and deep learning-based design algorithms. Moreover, sequences of rate-compatible polar codes can be efficiently found. Finally, we analyze both the complexity of the proposed algorithm and the error-rate performance of the constructed codes.

I. INTRODUCTION

Polar codes, introduced by Arkan, have attracted much interest due to their theoretical capability to achieve the capacity of the Binary Input Discrete Memoryless Channel (BI-DMC) under successive cancellation (SC) decoding [1] and their standardization in the fifth generation mobile telecommunication (5G). In the short blocklength regime, however, the performance of polar codes under SC decoding is not satisfactory. Therefore, alternative decoding algorithms have been proposed to improve the error-rate performance (e.g., successive cancellation list (SCL) decoding [2], automorphism ensemble decoding (AED) [3]), providing soft output (e.g., soft cancellation (SCAN) decoding [4]) or reducing the latency (e.g., belief propagation (BP) decoding [5]). It has been shown that different channels and decoding algorithms require different code designs to achieve the best possible error-rate performance [6]. While polar code design for BI-DMC under SC decoding is well studied, there exists no explicit construction optimized for other decoding algorithms. Consequently, finding suitable code designs for these decoders either is based on sub-optimal approximations, heuristics or requires extensive Monte Carlo simulations. In [7], codes for SCL decoding are designed based on a heuristic, while in [8], designs are hand-crafted based on an information-theoretic analysis of the decoder. Density evolution and its Gaussian approximation have been used in [9] and [10]–[12], respectively, to design polar codes. For iterative BP decoding, Log-likelihood ratio (LLR) evolution has been proposed in [13]. More generally applicable code design algorithms are based on Monte Carlo methods. In [14], the bitwise bit error rate (BER) is used to find reliable synthetic channels and successively generate the code design in a greedy fashion. A similar approach is used in [15], where the actual performance of the codes is simulated instead of the BER.

To allow for a broader search than greedy algorithms, the use of a Genetic Algorithm (GenAlg) has been proposed in [6]. Here, each code design is treated as an individual in a population that evolves over multiple generations using selection, crossover and mutation. Since then, the efficiency of GenAlg has been improved by better crossover algorithms and caching [16].

Further, machine learning methods were applied to polar code design. In [17], the code design is learned via gradient descent through an unrolled BP decoder. More recently, polar codes were learned via reinforcement learning [18]. In [19], a neural network (NN) is trained to predict the frame error rate (FER) performance of polar code designs and then, a projected gradient algorithm is used to find the input to the NN (i.e., a polar code design) that minimizes the FER.

The main contributions of this paper can be summarized as follows:

• We present a new perspective on polar code design as a problem on a graph
• First algorithms to optimize single code designs and rate-compatible reliability sequences are proposed
• We propose the use of confidence intervals as a general method to reduce the complexity of Monte Carlo simulation based code search.

II. PRELIMINARIES

A. Polar Codes

Polar codes, as introduced in [1], are based on the n-fold application of the basic channel transformation, transforming $N = 2^n$ identical channels into $N$ polarized synthetic channels. The subset $\mathcal{A} \subseteq \{0,\ldots,N-1\}$ of synthetic channels with $|\mathcal{A}| = k$ is said to be reliable and carries the information (i.e., information set), while the remaining $N-k$ synthetic channels $\mathcal{A}^c$ are said to be unreliable and thus transmit a frozen 0 (i.e., frozen set). The code $C$ is defined by the encoding rule

$$\mathbf{x} = \mathbf{u} \cdot G_N,$$

$$G_N = \begin{bmatrix} [1 0]^{\otimes n} \\
[1 1] \end{bmatrix},$$

with $\mathbf{u}_\mathcal{A} \in \{0,1\}^k$, $\mathbf{u}_\mathcal{A}^c = \mathbf{0}$. Thus, the code rate is $R = k/N$. The choice of $\mathcal{A}$ is called polar code design and optimal

1 The reliability refers to the information after decoding and, thus, is not a universal code property, but also dependent on the decoder.
solutions are dependent on both the channel and the decoding algorithm [6]. An alternative notation for specifying the sets $\mathcal{A}$ and $\mathcal{A}'$, respectively, is the binary vector $\mathbf{A}$ with

$$A_i = \begin{cases} 1 & \text{if } i \in \mathcal{A} \\ 0 & \text{if } i \in \mathcal{A}' \end{cases}.$$ 

Throughout this paper, we will use $\mathcal{A}$-set, $\mathbf{A}$-vector and code $C$ notation interchangeably.

### B. Polar Code Reliability Sequences

Practical applications require a simple change of the code rate whenever the channel conditions vary. For a fixed block-length $N$, the code rate can be changed by moving some indices from $\mathcal{A}'$ to $\mathcal{A}$ or vice-versa. A common way to specify the order of freezing/unfreezing is in form of a reliability sequence $Q$ that lists the indices of the synthetic channels in descending reliability order. To construct a polar code with a desired $k$, the $k$ most reliable (i.e., the first $k$) indices are chosen to be the information set, i.e.,

$$\mathcal{A} = \{i \in Q | j < k\}.$$ 

Examples for reliability sequences are based on the Bhattcharyya parameter [1], $\beta$-expansion [20] and the 5G sequence [21].

Remark: Reliability sequences are in general sub-optimal. In other words, given a channel and decoding algorithm, the optimal code designs $\mathcal{A}_k$ for each $k$ do not necessarily fulfill $\mathcal{A}_{k-1} \subset \mathcal{A}_k$ and hence, do not necessarily form a sequence. Fig. 1 illustrates this property for $N = 32$ and BP decoding. Each black dot corresponds to the optimal code design for the respective code dimension $k$. There is no consecutive sequence of synthetic channels that contains all the best codes. Instead, for some code dimensions, sub-optimal codes (gray nodes) must be included to create a sequence.

### III. POLAR CODE DESIGN ON GRAPHS

#### A. Monte Carlo Simulation Based Code Search

For most polar decoding algorithms besides SC decoding, optimal explicit code constructions are unknown. Hence, one has to select good codes based on their measured performance. The performance is estimated at a pre-defined signal-to-noise-ratio (SNR) using Monte Carlo simulation. With the number of simulated frame errors $N_{FE}$ and trials $N_T$, the accuracy of the simulation can be evaluated by a confidence interval $(\hat{P}_{FE, LB}, \hat{P}_{FE, UB})$ which contains the actual FER $P_{FE}$ of the code with a chosen probability $\gamma$, called the confidence level. The frame errors are independent events, and hence, the number of observed frame errors $N_{FE}$ is binomially distributed. The confidence intervals can be thus computed using the relationship between binomial cumulative distribution and the incomplete beta function [22]. However, according to the central limit theorem for $N_T \to \infty$, the distribution of the observed FER $\hat{P}_{FE} = N_{FE}/N_T$ approaches a normal distribution with mean $\mu = P_{FE}$ and variance

$$\sigma^2 = \frac{P_{FE} \cdot (1 - P_{FE})}{N_T}.$$ 

The confidence interval $(\hat{P}_{FE, LB}, \hat{P}_{FE, UB})$ of a Monte Carlo simulation can be approximated as $(\hat{P}_{FE, LB}, \hat{P}_{FE, UB}) = (\hat{P}_{FE} - \delta, \hat{P}_{FE} + \delta)$ with

$$\delta = \sqrt{\frac{\hat{P}_{FE} \cdot (1 - \hat{P}_{FE})}{N_T}} \cdot Q^{-1}(\alpha),$$  

(1)

where $\alpha = 1 - \gamma$ and $Q^{-1}(\alpha)$ is the inverse of the complementary cumulative distribution function (CCDF) of the standard normal distribution [22]. Note that the approximation becomes inaccurate if $N_{FE}$ or $P_{FE} \cdot N_T$ are too small. Confidence intervals can be used to compare two codes $C_0$ and $C_1$. If $P_{FE, UB}(C_0) < P_{FE, LB}(C_1)$ holds, then the FER of $C_0$ is lower than that of $C_1$ with probability

$$P[P_{FE}(C_0) < P_{FE}(C_1)] > 1 - \frac{(1 - \gamma)^2}{4}.$$ 

Furthermore, if an accurate estimation of the FERs is not required, the computational complexity can be reduced by terminating the Monte Carlo simulations as soon as it is determined which code is better. Algorithm 1 generalizes this to finding the best $L$ codes of a set of codes $\mathcal{L} = \{C_0, C_1, \ldots\}$.

#### B. The Graph of Polar Code Designs

To relate different polar code designs to each other, we propose to use a (directed) graph. Each polar code design (i.e., $\mathbf{A}$-vector) corresponds to a vertex. Two codes $\mathcal{A}$ and $\mathcal{A}'$
differing exactly by one frozen/unfrozen bit are connected by an edge, and the edge label indicates the bit position in which they differ, i.e.,

\[ A \xleftarrow{j} A' \iff A' = A \cup \{j\}. \]

Note that this is identical to the Hasse diagram of all information sets ordered by inclusion. We define the partial order

\[ A \prec A' \iff A \subset A' \]

that can also compare codes not directly neighboring, but connected via a chain of edges. This notion of order is motivated by the fact that the FERs of two codes \( A \) and \( A' \) with \( A \prec A' \) at identical \( E_s / N_0 \) fulfill \( P_{FE}(A) \leq P_{FE}(A') \), as the decoder of \( A \) has access to more a priori information (additional frozen bits) than the decoder of \( A' \). Therefore, the graph implies some local “smoothness” of the FER in the neighborhood around each code.

In Fig. 2 an excerpt of the graph for \( N = 8 \) is shown. Note that we implicitly assume increasing code dimensions from left to right and, thus, the direction of the edges is omitted for readability. Fig. 3 shows the complete graph for all polar codes with blocklength \( N = 4 \).

### C. Optimization of a Single Polar Code Design

A first algorithm to traverse the graph in order to find an optimized, single code design is the bit swapping algorithm shown in Algorithm 2. Starting from any code design \( C_0 \) with the desired code dimension \( k \) (e.g., using \( \beta \)-expansion), information and frozen bits are alternately exchanged. The algorithm keeps a list of the \( L \) best candidates and estimates the performance of its left neighbors using Algorithm 1. Then, the right neighbors of these codes are simulated. This way, the algorithm “zig-zags” through the graph between \( k \) and \( k - 1 \), until no more progress is made. An example for a single iteration of the algorithm is illustrated in Fig. 4 for \( N = 8 \).

### Algorithm 2: Optimization of a single code design

**Input**: Start code \( C_0 \), list size \( L \), confidence level \( \gamma \), \( E_s / N_0 \)  
**Output**: List \( \mathcal{L}^* \) of \( L \) best codes  
1. \( \mathcal{L} \leftarrow \{ C_0 \} \);  
2. while no further improvement do  
3. \( \mathcal{L} \leftarrow \bigcup_{C \in \mathcal{L}} \) (left neighbors of \( C \));  
4. \( \mathcal{L} \leftarrow \) Algorithm 1(\( \mathcal{L}, L, \gamma, E_s / N_0 \));  
5. \( \mathcal{L} \leftarrow \bigcup_{C \in \mathcal{L}} \) (right neighbors of \( C \));  
6. \( \mathcal{L} \leftarrow \) Algorithm 1(\( \mathcal{L}, L, \gamma, E_s / N_0 \));  
7. end  
8. \( \mathcal{L}^* \leftarrow \mathcal{L} \);  

### D. Optimizing a Bit Reliability Sequence

A similar approach to Algorithm 2 can be used to optimize a rate-compatible sequence of codes. This procedure is listed in Algorithm 3. Starting from a list of good codes that was found using Algorithm 2 for some starting code dimension \( k_{\text{start}} \), the algorithm develops sequences of neighboring codes outwards...
to \( k = 0 \) and \( k = N \). In each step, the best \( L \) sequences \( S \) are kept based on a path metric

\[
\tau(S) = \sum_{C \in S} \log P_{FE}(C) + c,
\]

where \( P_{FE,\text{best},k}(C) \) is the FER of the best found code for the same code dimension as \( C \). This path metric can be interpreted as the error-rate loss of the codes in the sequence versus the best codes that are possible for each \( k \). This way, the algorithm aims at finding a good compromise of decently performing codes under the constraint that they form a sequence. This constraint is enforced by lines 9 and 15, where the currently found paths are augmented by appending (or pre-pending, respectively) only neighboring codes in the currently simulated batch \( \mathcal{L}_k \). If multiple codes neighbor the last code \( S_{\text{last}} \) (or first code \( S_{\text{first}} \)) in the sequence \( S \), the sequence is duplicated for each option. Likewise, the work-list of codes to simulate in the next step includes all codes neighboring \( S_{\text{last}} \) (line 7) and \( S_{\text{first}} \) (line 13), respectively. For a list size of \( L = 1 \) and starting code dimension \( k_{\text{start}} = 0 \), the algorithm degenerates to the greedy procedure presented in [15]. Fig. 5 illustrates Algorithm 3 for \( N = 8 \) and \( k_{\text{start}} = 4 \). The bit reliability sequence \( Q \) can be extracted as the sequence of edge labels on the path from \( k = 0 \) to \( k = N \); in this example \( Q = [7, 6, 3, 5, 4, 1, 2, 0] \).

**Algorithm 3:** Rate-compatible polar code sequence optimization

| Input | List of start codes \( \mathcal{L}_{k\text{start}}, k_{\text{start}} \), list size \( L \), confidence level \( \gamma \), \( E_b/N_0 \) |
|-------|---------------------------------------------------------------|
| Output| Best sequence \( S^* \) |

1. \( k_{\text{min}} \leftarrow k_{\text{start}} \), \( k_{\text{max}} \leftarrow k_{\text{start}} \);
2. \( \mathcal{L}_{k\text{start}} \leftarrow \) Algorithm 1(\( \mathcal{L}_{k\text{start}}, L, \gamma, E_b/N_0 \));
3. \( S_{\text{paths}} \leftarrow \{ |C| \mid C \in \mathcal{L}_k \} \);
4. While \( k_{\text{min}} > 0 \) or \( k_{\text{max}} < N \) do
5.   If \( k_{\text{min}} < N \) then
6.     \( k_{\text{max}} \leftarrow k_{\text{max}} + 1 \);
7.     \( \mathcal{L}_{k\text{max}} \leftarrow \bigcup_{S \in S_{\text{paths}}} \{ \text{right neighbors of } S_{\text{last}} \} \);
8.     \( \mathcal{L}_{k\text{max}} \leftarrow \) Algorithm 1(\( \mathcal{L}_{k\text{max}}, L, \gamma, E_b/N_0 \));
9.     Augment \( S \in S_{\text{paths}} \) using codes \( C \in \mathcal{L}_{k\text{max}} \);
10.  End
11.  If \( k_{\text{min}} > 0 \) then
12.     \( k_{\text{min}} \leftarrow k_{\text{min}} - 1 \);
13.     \( \mathcal{L}_{k\text{min}} \leftarrow \bigcup_{S \in S_{\text{paths}}} \{ \text{left neighbors of } S_{\text{first}} \} \);
14.     \( \mathcal{L}_{k\text{min}} \leftarrow \) Algorithm 1(\( \mathcal{L}_{k\text{min}}, L, \gamma, E_b/N_0 \));
15.     Augment \( S \in S_{\text{paths}} \) using codes \( C \in \mathcal{L}_{k\text{min}} \);
16.  End
17.  Prune \( S_{\text{paths}} \) to best \( L \) paths w.r.t. \( \tau(S) \) from (2);
18. End
19. \( S^* \leftarrow \arg \min_{S \in S_{\text{paths}}} \tau(S) \);

**IV. RESULTS**

**A. Single Code Design Optimization**

We evaluate Algorithm 2 for designing polar codes for the AWGN channel and BP decoding. For more information on BP, we refer the interested reader to [23]. We compare the proposed method to optimizations using the deep learning approach from [19] and the GenAlg proposed in [6] with the complexity reduction improvements from [16]. For the deep learning based method, the NN consists of three dense layers with 128 neurons each and it is trained for 100 epochs per design algorithm iteration. The GenAlg uses a population size of 50.

First, we design \((128,64)\) polar codes for \( N_{t,\text{max}} = 100 \) BP decoding iterations at an SNR \( E_b/N_0 = 3 \) dB. The graph search algorithm uses a list size \( L = 4 \) and \( \gamma = 0.8 \). We notice that all algorithms converge to the identical, presumably globally optimal code design with the same FER performance.

Therefore, to compare the algorithms quantitatively, we record the total number of frames transmitted in the Monte Carlo simulation. As all algorithms are incremental and intermediate solutions can be taken at any step in the optimization progress, we plot the mean and median FER performance of the best codes from 11 independent runs of each optimizer in Fig. 6. We can see that the NN-based method has the largest design complexity as it requires a large data-set until the projected gradient method can start to produce gains. The GenAlg starts off the fastest, however, then converges more slowly than the proposed graph search, which needs the least complexity to reliably converge to the optimal code design.

Next, we design longer \((512,128)\) codes for \( N_{t,\text{max}} = 20 \) BP
The proposed methods are very general and can be easily applied to other decoding algorithms such as SCAN, belief propagation list (BPL) and AED. In particular, the graph can be altered such that the resulting codes follow desired properties such as the partial order of synthetic channels.

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V. Conclusion

In this paper, we introduced a new perspective on polar code design as a search on a graph. This makes it possible to systematically optimize a single code design and also find reliability sequences for rate-compatible polar codes. To this end, we proposed two algorithms for traversing the graph and showed that they provide lower computational complexity than other Monte Carlo simulation based design methods and can result in better code designs with respect to the error-rate performance.

The figure (Fig. 8) shows the performance of rate-compatible polar code sequences with various k-values for different N values. The graph search algorithm produces a sequence with much smoother transitions from one code rate to another, i.e., more predictable performance when the rate is changed, while the curves for the traditional code designs are very jagged.