Soliton equations in N-dimensions as exact reductions of Self-Dual Yang-Mills equation V. Simplest (2+1)-dimensional soliton equations

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Abstract

Some aspects of the multidimensional soliton geometry are considered. It is shown that some simplest (2+1)-dimensional soliton equations are exact reductions of the Self-Dual Yang-Mills equation or its higher hierarchy.
One of the most interesting and important multidimensional integrable equations is the self-dual Yang-Mills equation (SDYME) \[2, 27\]. This four-dimensional equation can be considered as a particular case of the M-LXX equation.
mensional equation arises in relativity [26, 3] and in field theory [27]. The SDYM equations describe a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space. Integrability for a SDYM connection means that its curvature vanishes on certain two-planes in the tangent space of the Grassmannian. As shown in [4,5], this allows one to characterize SDYM connections in terms of the splitting problem for a transition function in a holomorphic bundle over the Riemann sphere, i.e. the trivialization of the bundle [28, 29].

Recently it has been shown that practically all known soliton equations in 1+1 and 2+1 dimensions may be obtained by reductions of the SDYME [2, 4-5, 18-19, 34-37] (see also the book [1] and references therein). On the other hand, it is well known that almost all known integrable equations may be obtained from some equations of soliton geometry by reductions. These equations are the integrability conditions of the system describing the moving orthogonal trihedral of a curve or surface [8-24, 38-40, 43-44]. For example, in 2+1 dimensions, the role of such geometrical equations play the mM-LXII or M-LXII equations. In [8] and in our previous notes of this series [18-21] we have considered some aspects of the multidimensional soliton geometry (see, also the refs. [38-40, 43-44]). Also we have studied the relation between the multidimensional soliton equations and the Self-Dual Yang-Mills equation. In this note we continue this work.

2 Soliton geometry in \(d = 4\) dimensions

It is well known that exist several equations describing the 4-dimensional curves/"surfaces" in \(n\)-dimensional space. Here we present some of them.

2.1 The M-LXVIII equation

Consider the M-LXVIII equation [8]

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_4
\end{pmatrix}
= A
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_4
\end{pmatrix} + B
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_4
\end{pmatrix}
\tag{1a}
\]

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_2 \\
  \xi_3 \\
  \xi_4 \\
  \vdots \\
  \xi_n
\end{pmatrix}
= C
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_2 \\
  \xi_3 \\
  \xi_4 \\
  \vdots \\
  \xi_n
\end{pmatrix} + D
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_2 \\
  \xi_3 \\
  \xi_4 \\
  \vdots \\
  \xi_n
\end{pmatrix}
\tag{1b}
\]

where \(e_j^2 = 1, (e_i e_j) = \delta_{ij}\) and \(A(\lambda), B(\lambda), C(\lambda), D(\lambda)\) are \((n \times n)\)-matrices, \(\lambda\) is some parameter, \(\xi_i\) are coordinates. This equation describes some four-dimensional curves and/or "surfaces" in \(n\)-dimensional space. It is one of main
equations of the multidimensional soliton geometry and admits several integrable reductions [8]. The compatibility condition of these equations gives the M-LXX equation, which contains several interesting integrable nonlinear evolution equations (NEE). In this note we will present some of these integrable reductions.

2.2 The M-LXXI equation

Consider the M-LXXI equation [8]

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
= A
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]

\[\xi_1 = A \xi_1 \] (2a)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_2
= B
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]

\[\xi_2 = B \xi_2 \] (2b)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_4
= C
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
+ D
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]

\[\xi_4 = C \xi_4 + D \xi_4 \] (2c)

The compatibility condition of these equations gives some NEEs (see, e.g. the refs. [8,18-19]).

2.3 The M-LXI equation

Consider the 4-dimensional M-LXI equation [8]

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_1
= A
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]

\[\xi_1 = A \xi_1 \] (3a)

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_2
= B
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]

\[\xi_2 = B \xi_2 \] (3b)
The compatibility condition of these equations gives the 4-dimensional M-LXII equation

\[
A e_1 - B e_3 + [A, B] = 0 \quad (4a)
\]

\[
A e_1 - C e_4 + [A, C] = 0 \quad (4b)
\]

\[
A e_1 - D e_3 + [A, D] = 0 \quad (4c)
\]

\[
C e_2 - B e_3 + [C, B] = 0 \quad (4d)
\]

\[
D e_2 - B e_4 + [D, B] = 0 \quad (4e)
\]

\[
C e_4 - D e_3 + [C, D] = 0. \quad (4f)
\]

This equation contains many interesting NEEs (see, e.g. the refs. [8,18-19]).

3 The M-LXX equation

From (1) we get the following M-LXX equation [8]

\[
AD e_3 - CB e_4 + B e_2 - D e_1 + [B, D] = 0 \quad (5a)
\]

\[
A e_2 - CA e_4 + [A, D] = 0 \quad (5b)
\]

\[
[A, C] = 0 \quad (5c)
\]

\[
C e_1 - AC e_4 + [C, B] = 0. \quad (5d)
\]

If we choose

\[
A = aI, \quad C = bI, \quad a, b = \text{consts}, \quad (6)
\]

then the M-LXX equation (5) takes the form

\[
aD e_3 - bB e_4 + B e_2 - D e_1 + [B, D] = 0. \quad (7)
\]
4 The SDYME as the particular case of the M-LXX equation

Now we assume that

\[ B = A_1 - \lambda A_3, \quad D = A_2 - \lambda A_4, \quad a = b = \lambda. \]  

So that the M-LXVIII equation (1) takes the form

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_1
= \lambda
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_3
+ (A_1 - \lambda A_3)
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_3,
\]

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_2
= \lambda
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_4
+ (A_2 - \lambda A_4)
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\xi_4.
\]

From (7) we obtain the SDYME

\[ A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0 \]  

\[ A_{4\xi_3} - A_{3\xi_4} + [A_4, A_3] = 0 \]  

\[ A_{1\xi_4} - A_{4\xi_1} + [A_1, A_4] = A_{2\xi_3} - A_{3\xi_2} + [A_2, A_3] \]

or

\[ F_{\xi_1\xi_2} = 0 \]

\[ F_{\xi_3\xi_4} = 0 \]

\[ F_{\xi_1\xi_3} - F_{\xi_3\xi_2} = 0. \]

Here

\[ F_{\xi_i\xi_k} = A_k\xi_i - A_i\xi_k + [A_k, A_i]. \]

The SDYME (10) on a connection \( A \) are the self-duality conditions on the curvature under the Hodge star operation

\[ F = *F \]

or in index notation

\[ F_{\mu\nu} = \epsilon_{\mu\nu\rho\delta} F^{\rho\delta} \]

where * is the Hodge operator, \( \epsilon_{\mu\nu\rho\delta} \) stands for the completely antisymmetric tensor in four dimensions with the convention: \( \epsilon_{1234} = 1 \). The SDYME is integrable by the Inverse Scattering Transform (IST) method (see, e.g. [2,27]). The Lax representation (LR) of the SDYME has the form [27, 30]

\[ \Phi_{\xi_1} - \lambda \Phi_{\xi_3} = (A_1 - \lambda A_3)\Phi \]

\[ \Phi_{\xi_2} - \lambda \Phi_{\xi_4} = (A_2 - \lambda A_4)\Phi. \]
5 The M-LXVII equation as the particular case of the M-LXVIII equation

In this section we consider the 3-dimensional curves/"surfaces". Let variables in the M-LXVIII equation (1) are independent of $\xi_3$. Then we obtain the following M-LXVII equation [8]

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_1} = B
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}
\]

(16a)

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_2} = C
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_4} + D
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}
\]

(16b)

In this case we get the 3-dimensional M-LXX equation

\[-bB\xi_4 + B\xi_2 - D\xi_1 + [B, D] = 0\]

(17)

and the 3-dimensional SDYME

\[A_2\xi_1 - A_1\xi_2 + [A_2, A_1] = 0\]

(18a)

\[-A_3\xi_4 + [A_4, A_3] = 0\]

(18b)

\[A_1\xi_4 - A_4\xi_1 + [A_1, A_4] = -A_3\xi_2 + [A_2, A_3].\]

(18c)

We note that for the 3-dimensional SDYME (18) the corresponding M-LXVII equation has the form

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_1} = (A_1 - \lambda A_3)
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}
\]

(19a)

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_2} = \lambda
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_4} + (A_2 - \lambda A_4)
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}
\]

(19b)

So that the curve or "surface" corresponding to the equation (19) is the integrable.
6 The Zakharov equation as the exact reduction of the SDYME

Now let us we consider the gauge condition

$$A_4 = 0. \quad (20)$$

Then the 3-dimensional SDYME takes the form

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0 \quad (21a)$$

$$A_{3\xi_4} = 0 \quad (21b)$$

$$A_{1\xi_4} = -A_{3\xi_2} + [A_2, A_3]. \quad (21c)$$

If we take

$$A_3 = \text{const} \quad (22)$$

then the 3-dimensional SDYME has the form

$$A_{2\xi_1} - A_{1\xi_2} + [A_2, A_1] = 0 \quad (23a)$$

$$A_{1\xi_4} = [A_2, A_3]. \quad (23b)$$

Now we consider the case $n = 3$. And we take the following representations of the connections $A_1, A_2, A_3$

$$A_1 = \begin{pmatrix}
0 & i(\phi - r^2 \bar{\phi}) & (\phi + r^2 \bar{\phi}) \\
-i(\phi - r^2 \bar{\phi}) & 0 & 0 \\
-(\phi + r^2 \bar{\phi}) & 0 & 0
\end{pmatrix} \quad (24a)$$

$$A_2 = \begin{pmatrix}
0 & (\phi + r^2 \bar{\phi})_y & -i(\phi - r^2 \bar{\phi})_y \\
-(\phi + r^2 \bar{\phi})_y & 0 & -v \\
i(\phi + r^2 \bar{\phi})_y & v & 0
\end{pmatrix} \quad (24b)$$

$$A_3 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix} \quad (24c)$$

and let

$$\xi_1 = x, \quad \xi_2 = t, \quad \xi_4 = y. \quad (25)$$

So in our case the equation (23) takes the form

$$A_{2x} - A_{1t} + [A_2, A_1] = 0 \quad (26a)$$

$$A_{1y} = [A_2, A_3] \quad (26b)$$

or in elements

$$i\phi_t = \phi_{xy} + v\phi \quad (27a)$$

$$v_x = 2r^2 \partial_y |\phi|^2. \quad (27b)$$
It is the Zakharov equation (ZE) [49]. We note that in this case the corresponding M-LXVII equation (19) looks like

$$
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}_x = (A_1 - \lambda A_3)
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}_t = \lambda
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix} + A_2
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}.
$$

From (20), (22) and (28) we get the following LR of the ZE

$$
\Phi_x = (A_1 - \lambda A_3)\Phi
$$

$$
\Phi_t - \lambda \Phi_y = A_2 \Phi
$$

where $A_i$ are $\text{so}(3)$ or $\text{so}(2, 1)$ matrices. It is convenient to use the well known isomorphism $\text{so}(3) \simeq \text{su}(2)$ or $\text{so}(2, 1) \simeq \text{su}(1, 1)$ and to rewrite the equations (29) in terms of $2 \times 2$ matrices. As result we have the following standard LR of the ZE (27)

$$
\Psi_x = U \Psi
$$

$$
\Psi_t = \lambda \Psi_y + V \Psi
$$

where

$$
U = \frac{i\lambda}{2} \sigma_3 + G,
G = \begin{pmatrix}
  0 & \phi \\
  -r^2\phi & 0 \\
\end{pmatrix}
$$

$$
V = i\sigma_3(\frac{1}{2}vI + G_y).
$$

We note that in the (1+1)-dimensional case, i.e. when $y = x$ instead of the equations (28) and (27) we obtain the (1+1)-dimensional M-LXVII equation

$$
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}_x = (A_1 - \lambda A_3)
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}_t = [\lambda(A_1 - \lambda A_3) + A_2]
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
\end{pmatrix}
$$

and the nonlinear Schrodinger equation

$$
 i\phi_t = \phi_{xx} + 2r^4|\phi|^2\phi = 0
$$

At last we note that if $n > 3$ then we get the $n$-component generalisation of the Zakharov equation

$$
 i\phi_{jt} = \phi_{jxy} + v\phi_j = 0
$$

$$
v_x = 2r^2(\sum_{k=1}^{n} |\phi_k|^2)_y
$$
Integrable spin systems in 2+1 dimensions also can be considered as exact reductions of the SDYME. As example we show that Myrzakulov I (M-I) equation is the reduction of the SDYME. Consider the following gauge condition

\[ A_1 = A_2 = 0. \]  

(35)

In this case the 3-dimensional SDYME takes the form

\[ -A_3 \xi_4 + [A_4, A_3] = 0 \]  

(36a)

\[ -A_4 \xi_1 = -A_3 \xi_2 \]  

(36b)

or in terms of \( x, y, t \)

\[ -A_y + [A_4, A_3] = 0 \]  

(37a)

\[ -A_{4x} = -A_{3t}. \]  

(37b)

Let the connections \( A_4, A_3 \) have the forms

\[
A_3 = \begin{pmatrix}
0 & rS_1 & -irS_2 \\
-rS_1 & 0 & S_3 \\
irS_2 & -S_3 & 0
\end{pmatrix}
\]  

(38a)

\[
A_4 = \begin{pmatrix}
0 & -ir[2iS_3S_2y - 2iS_2S_3y + iuS_1] & -r[2S_3S_1y - 2S_1S_3y - uS_2] \\
ir[2iS_3S_2y - 2iS_2S_3y + iuS_1] & 0 & -[ir^2(S^+S_3^y - S^-S_y^+) - uS_3] \\
r[2S_3S_1y - 2S_1S_3y - uS_2] & ir^2(S^+S_3^y - S^-S_y^+) - uS_3 & 0
\end{pmatrix}
\]  

(38b)

Substituting (38) into (37) we get the M-I equation

\[ iS_t = ([S, S_y] + 2iuS)_x \]  

(39a)

\[ u_x = -\frac{1}{2i} tr(S[S_x, S_y]) \]  

(39b)

where

\[ S = \begin{pmatrix}
S_3 & S^- \\
S^+ & -S_3
\end{pmatrix}, \quad S^\pm = S_1 \pm iS_2. \]  

(40)

For the M-I equation the corresponding geometrical equation (19) looks like

\[
\begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}_x = -\lambda A_3 \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}
\]  

(41a)

\[
\begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}_t = \lambda \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}_y - \lambda A_4 \begin{pmatrix} e'_1 \\ e'_2 \\ e'_3 \end{pmatrix}
\]  

(41b)
and called the M-LXVI equation. As known [8], the M-LXVII and M-LXVI
equations (41), (28) and (19) are some integrable (2+1)-dimensional extensions
of the Serret-Frenet equation (SFE)

\[
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix} = 
\begin{pmatrix}
0 & k & 0 \\
-\beta k & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix}
\]

(42)
or the Codazzi-Mainardi equation (CME) for the surfaces.

The LR of the M-I equation follows from the LR of the SDYME and has the
form

\[
\Phi'_x = -\lambda A_3 \Phi'
\]

(43a)

\[
\Phi'_t = \lambda \Phi'_y - \lambda A_4 \Phi'.
\]

(43b)

As for ZE, we can rewrite the LR of the M-I equation in the standart form, in
terms of 2×2 - matrices

\[
\Psi'_x = U'\Psi'
\]

(44a)

\[
\Psi'_t = \lambda \Psi'_y + V'\Psi'
\]

(44b)

where

\[
U' = \frac{i\lambda}{2} S
\]

(45a)

\[
V' = \frac{\lambda}{4} ([S, S_y] + 2iuS).
\]

(45b)

As above for the ZE (27), in the 1+1 dimensions (y = x) instead of the
equations (41) and (39), we get the (1+1)-dimensional M-LXVI equation

\[
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix} = -\lambda A_3
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix}
\]

(46a)

\[
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix} = -(\lambda^2 A_3 + \lambda A_4)
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix}
\]

(46b)

and the Landau-Lifshitz equation

\[
iS_t = \frac{1}{2} [S, S_{xx}].
\]

(47a)

8 Gauge equivalence

It is well known that the ZE (33) and the M-I equation (39) are gauge and
Lakshmanan equivalent (G-equivalent and L-equivalent) to each other. In our
case this fact realize by the transformation

\[
\begin{pmatrix}
e'_1 \\
e'_2 \\
e'_3
\end{pmatrix} = G
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\]

(48)
or
\[ \Phi' = h^{-1} \Phi \] (49a)
or
\[ \Psi' = h^{-1} \Psi \] (50)
where \( h \) is the solution of the equations (29) or (30) as \( \lambda = 0 \).

9 A nonisospectral case and the breaking solutions of the SDYME

Usually for the SDYME (10) the spectral parameter \( \lambda = \text{constant} \). But in general it satisfies the following set of nonlinear equations

\[ \lambda \xi_1 = \lambda \lambda \xi_3 \] (51a)
\[ \lambda \xi_2 = \lambda \lambda \xi_4. \] (51b)

These equations have the following solutions

\[ \lambda = \frac{m_1 \xi_3 + n_3}{n_4 - n_1 \xi_1} \] (52a)
\[ \lambda = \frac{m_1 \xi_4 + m_3}{n_4 - m_1 \xi_2}. \] (52b)

So that the general solution of the set (51) has the form

\[ \lambda = \frac{n_1 \xi_3 + n_3 + m_1 \xi_4}{n_4 - n_1 \xi_1 - m_1 \xi_2} \] (53)

where \( m_i, n_i = \text{constants} \). The corresponding solution of the SDYME (10) is called the breaking (overlapping) solutions. In the case (18), i.e. for the ZE and the M-I equation the set of equations (21) takes the form

\[ \lambda_x = 0 \] (54a)
\[ \lambda_t = \lambda \lambda_y \] (54b)

and the solution (9) has the form

\[ \lambda = \frac{n_1 y + n_3}{n_4 - n_1 t}. \] (55)

10 Hierarchy of the M-LXVIII and Self-Dual Yang-Mills equations

The higher hierarchy of the M-LXVIII equation for the SDYME case we write in the form

\[
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3 \\
    \vdots \\
    e_n
\end{pmatrix}_{\xi_1} = \sum_{i=0}^{k_1} A_i \lambda_i \\
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3 \\
    \vdots \\
    e_n
\end{pmatrix}_{\xi_3} = \sum_{i=0}^{k_2} B_i \lambda_i
\begin{pmatrix}
    e_1 \\
    e_2 \\
    e_3 \\
    \vdots \\
    e_n
\end{pmatrix}_t
\] (56a)
The compatibility condition of these equations yields the higher hierarchy SDYME. As example, we consider the 3-dimensional case, work the notation (25) and \( n = 3 \). Then instead of (56) we obtain the M-LXVII equation in the form

\[
\begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_2} = \sum_{i=0}^{k_3} C_i \lambda^i \begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}_{\xi_1} + \sum_{i=0}^{k_4} D_i \lambda^i \begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n \\
\end{pmatrix}. \quad (56b)
\]

It is remarkable that using the equation (57) we can show that some known (2+1)-dimensional soliton equations are exact reductions of the higher hierarchy of the SDYME. Here we present some of them.

### 10.1 The (2+1)-dimensional mKdV equation

In the equation (57) we assume that

\[ k_2 = 1, k_3 = 2, k_4 = 2, C_2 = 1, \quad C_1 = C_0 = 0. \quad (58) \]

Thus in this case the M-LXVII equation looks like

\[
\begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\end{pmatrix}_x = \sum_{i=0}^{k_2} B_i \lambda^i \begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\end{pmatrix} \quad (57a)
\]

\[
\begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\end{pmatrix}_t = \sum_{i=0}^{k_3} C_i \lambda^i \begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\end{pmatrix}_t + \sum_{i=0}^{k_4} D_i \lambda^i \begin{pmatrix}
  e_1 \\
e_2 \\
e_3 \\
\end{pmatrix}. \quad (57b)
\]

where

\[
A_1 = \begin{pmatrix}
  0 & -(q + p) & i(q - p) \\
  q + p & 0 & 0 \\
  -i(q - p) & 0 & 0 \\
\end{pmatrix} \quad (60)
\]

and \( A_3, D_k \) are some matrices \cite{8}. Then the complex functions \( q, p \) satisfy the (2+1)-dimensional complex mKdV equation

\[
q_t + q_{xxx} - (qv_1)_x - qv_2 = 0, \quad (61a)
\]

\[
v_{1x} = 2E(\bar{q}q)_y \quad (61b)
\]


If $p = q$ is real, we get the following (2+1)-dimensional mKdV equation

$$q_t + q_{xyy} - (qv_1)_x = 0,$$

(62a)

$$v_{1x} = 2E(q^2)_y = 4Eqq_y.$$  

(62b)

10.2 The (2+1)-dimensional derivative NLSE

In (57) now we put

$$k_2 = k_3 = k_4 = 2; C_2 = 2c, C_1 = C_0 = 0, \quad B_2 = -cA_3, \quad B_2 = 2cA_1.$$  

(63)

So that the M-LXVII equation takes the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = (A_3\lambda^2 + A_1\lambda) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x$$

(64a)

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = \lambda^2 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_y$$

(64b)

where $A_1$ is given by (60). Then the complex functions $q, p$ satisfy the (2+1)-dimensional derivative NLSE

$$iq_t = q_{xy} - 2ic(vq)_x$$

(65a)

$$-ip_t = p_{xy} + 2ic(vq)_x$$

(65b)

$$v_x = 2(pq)_y$$

(65c)

which is the Strachan equation [45].

10.3 The M-III$^q$ equation

Now we consider the case

$$k_2 = k_3 = k_4 = 2, C_2 = 2cI, C_1 = 2dI, C_0 = 0, B_2 = -cA_3, B_1 = -dA_3 + 2cA_1, B_0 = dA_1.$$  

(66)

for which the M-LXVII equation has the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = [A_3(c\lambda^2 + d\lambda) + A_1(2c\lambda + d)] \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x$$

(67a)

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = 2(c\lambda^2 + d\lambda) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_y + (D_2\lambda^2 + D_1\lambda + D_0) \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_y$$

(67b)
where $A_1$ is given by (60). Then the complex functions $q, p$ satisfy the (2+1)-
dimensional M-III equation [8]

\[
\begin{align*}
  iq_t &= q_{xy} - 2ic(vq)_x + d^2vq \\
  -ip_t &= p_{xy} + 2ic(vq)_x + d^2vp \\
  v_x &= 2(pq)_y.
\end{align*}
\]  

(68a)  

(68b)  

(68c)

The M-III equation (68) admits two integrable reductions: the Strachan equation (65) as $d = 0$ and the ZE (27) as $c = 0$. If we rewrite the equations (67) in terms of $2 \times 2$ - matrices, we get the following LR of the M-III equation

\[
\begin{align*}
  \Psi_x &= U \Psi \\
  \Psi_t &= 2(c\lambda^2 + d\lambda)\Psi_y + V \Psi
\end{align*}
\]

(69a)  

(69b)

where

\[
U = i[(c\lambda^2 + d\lambda)\sigma_3 + (2c\lambda + d)G], \quad G = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix}
\]

(70a)

\[
V = B_2\lambda^2 + B_1\lambda + B_0.
\]

(70b)

Here

\[
B_0 = \frac{d}{2c}B_1 - \frac{d^2}{4c^2}B_2, \quad B_2 = -2ic^2v\sigma_3,
\]

\[
B_1 = -2icdv\sigma_3 + 2cGy\sigma_3 - 4ic^2vG.
\]

(71)

10.4 The M-XXII equation

Now let

\[
k_2 = k_3 = k_4 = 2, B_2 = A_3, B_1 = A_1, B_0 = \frac{pq}{4}A_3, C_2 = 2I, C_1 = C_0 = 0.
\]

(72)

Then the functions $q, p$ satisfy the following $M - XXII_q$ equations [8]

\[
\begin{align*}
  iq_t + q_{yx} + \frac{i}{2}[(v_1q)_x - v_2q - pqq_y] &= 0 \\
  ip_t - p_{yx} + \frac{i}{2}[(v_1p)_x + v_2p - qpp_y] &= 0 \\
  v_{1x} &= (pq)_y \\
  v_{2x} &= p_{yx}q - pq_{yx}.
\end{align*}
\]

(73a)  

(73b)  

(73c)  

(73d)

This set of equations is the G- and L-equivalent counterpart of the M-XXII_s equation (spin system)[8,25]. The LR of this equation has the form

\[
\begin{align*}
  \Psi_{2x} &= \{-i(\lambda^2 - \frac{pq}{4})\sigma_3 + \lambda Q\}\Psi_2 \\
  \Psi_{2t} &= 2\lambda^2\Psi_{2y} + (\lambda^2B_2 + \lambdaB_1 + B_0)\Psi_2
\end{align*}
\]

(74a)  

(74b)
with
\[
Q = \left( \begin{array}{cc} 0 & q \\ -p & 0 \end{array} \right), \quad B_2 = i \frac{1}{2} v_1 \sigma_3, \quad B_1 = i \sigma_3 Q_y - \frac{1}{2} v_1 Q, \quad B_0 = \frac{1}{4} v_2 - i \frac{1}{8} pq v_1.
\]

Now let us consider the following transformation
\[
q' = q \exp\left[ -\frac{i}{2} \partial_x^{-1} (pq) \right], \quad p' = p \exp\left[ \frac{i}{2} \partial_x^{-1} (pq) \right].
\]

Then the new variables \( p', q' \) satisfy the Strachan equation [45]
\[
i q'_t + q'_{xy} + i (v' q')_x = 0, \quad (77a)
i p'_t - p'_{xy} + i (v' p') = 0, \quad (77b)
v'_x = E(p' q')_y. \quad (77c)
\]

We see that the M-XXII\(q \) equation (73) and the Strachan equation (70) is gauge equivalent to each other. The transformation (76) induces the following transformation of the Jost function \( \Psi_1 \)
\[
\Psi_1 = f^{-1} \Psi_2 \quad (78)
\]
where \( \Psi_1 \) is the solution of the equations (30) as \( d = 0 \) and
\[
f = \exp\left( -\frac{i}{4} \partial_x^{-1} \mid q \mid^2 \sigma_3 \right) = \Psi_1^{-1} \mid_{\lambda = 0} \quad (79)
\]
Then the new Jost function \( \Psi_2 \) satisfies the following set of equations
\[
\Psi_{2x} = \{ -i \lambda^2 \sigma_3 + \lambda Q' \} \Psi_2 \quad (80a)
\Psi_{2t} = 2 \lambda^2 \Psi_{2y} + \{ \lambda^2 B'_2 + \lambda B'_1 + B'_0 \} \Psi_2 \quad (80b)
\]
with
\[
Q = \left( \begin{array}{cc} 0 & q' \\ -p' & 0 \end{array} \right) \quad (81)
\]
and \( B'_j \) are given in [8,25,46]. Note that the Strachan (77), (65) and MXXII\(q \) (73) equations are the simplest (2+1)-dimensional extensions of the following known NLSE
\[
i q_t + q_{xx} + i (pq^2)_x = 0 \quad (82a)
i p_t - p_{xx} + i (qp^2)_x = 0 \quad (82b)
\]
and
\[
i q_t + q_{xx} + ipq q_x = 0 \quad (83a)
i p_t - p_{xx} + ipq p_x = 0 \quad (83b)
\]
respectively. It is well known that these equations are gauge equivalent to each other [47].
11 The M-LXI and M-LXII equations and soliton equations in \( d = 3 \) dimensions

11.1 The M-LXI equation

The M-LXI equation in \( d = 3 \) dimensions has the form

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_n
\end{pmatrix}
= A
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]  
\tag{84a}

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_n
\end{pmatrix}
= B
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]  
\tag{84b}

\[
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\begin{pmatrix}
  \xi_1 \\
  \xi_2 \\
  \xi_3 \\
  \vdots \\
  \xi_n
\end{pmatrix}
= C
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3 \\
  \vdots \\
  e_n
\end{pmatrix}
\]  
\tag{84c}

11.2 The M-LXII equation

The compatibility condition of the equations (84) gives the 3-dimensional M-LXII equation

\[
A\xi_2 - B\xi_1 + [A,B] = 0
\]  
\tag{85a}

\[
A\xi_3 - C\xi_1 + [A,C] = 0
\]  
\tag{85b}

\[
C\xi_2 - B\xi_3 + [C,B] = 0.
\]  
\tag{85c}

This equation is the particular case of the Bogomolny equation (BE) [1]

\[
\Psi_{\xi_3} + [\Psi, C] + A\xi_2 - B\xi_1 + [A,B] = 0
\]  
\tag{86a}

\[
\Psi_{\xi_2} + [\Psi, B] + A\xi_3 - C\xi_1 + [A,C] = 0
\]  
\tag{86b}

\[
\Psi_{\xi_1} + [\Psi, A] + C\xi_2 - B\xi_3 + [C,B] = 0.
\]  
\tag{86c}

In fact, from hence as \( \Psi = 0 \) we obtain the M-LXII equation (85). As well known that the BE (86) is integrable (see, e.g. the book [1]). As the particular case of the integrable BE (86), the M-LXII equation is also integrable. The corresponding LR has the form

\[
\Phi_{\xi_1} - \lambda \Phi_{\xi_3} = [-iC - \lambda(A + iB)]\Phi
\]  
\tag{87a}

\[
\Phi_{\xi_2} - \lambda \Phi_{\xi_4} = [A - iB - \lambda iC]\Phi.
\]  
\tag{87b}
Let us consider the case \( n = 3, \xi_1 = x, \xi_2 = y, \xi_3 = t \). Then the mM-LXI and mM-LXII equations take the form

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_x = A
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\tag{88a}
\]

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_y = B
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\tag{88b}
\]

\[
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}_t = C
\begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix}
\tag{88c}
\]

and

\[
A_y - B_x + [A, B] = 0
\tag{89a}
\]

\[
A_t - C_x + [A, C] = 0
\tag{89b}
\]

\[
C_y - B_t + [C, B] = 0
\tag{89c}
\]

where

\[
A = \begin{pmatrix}
0 & k & -\sigma \\
-\beta k & 0 & \tau \\
\beta \sigma & -\tau & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & m_3 & -m_2 \\
-\beta m_3 & 0 & m_1 \\
\beta m_2 & -m_1 & 0
\end{pmatrix}
\]

\[
D = \begin{pmatrix}
0 & \omega_3 & -\omega_2 \\
-\beta \omega_3 & 0 & \omega_1 \\
\beta \omega_2 & -\omega_1 & 0
\end{pmatrix}
\tag{90}
\]

The mM-LXII equation contains many (and perhaps all?) known soliton equations (see, e.g. the refs.\[8,18-24,38-40,43-44\]). We note that in the case \( \sigma = 0 \) (89) is called the M-LXII equation. Some examples as follows.

### 11.2.1 The Ishimori and DS equations

In this section, we obtain the Ishimori (IE) and DS equations from the M-LXI and M-LXII (\( \sigma = 0 \)) equations as some exact reductions. The IE reads as

\[
S_t = S \land (S_{xx} + \alpha^2 S_{yy}) + u_x S_y + y S_x
\tag{91a}
\]

\[
u_{xx} - \alpha^2 u_{yy} = -2\alpha^2 S \cdot (S_x \land S_y).
\tag{91b}
\]

We take the following identification

\[
S = e_1.
\tag{92}
\]

In this case we have

\[
m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2} M_{Ish}^1 u]
\tag{93a}
\]
$$m_2 = -\frac{1}{2\alpha^2 k} M_2^{Isb} u$$  \hspace{1cm} (93b)

$$m_3 = \partial_x^{-1} [k_y + \tau \frac{1}{2\alpha^2 k} M_2^{Isb} u]$$  \hspace{1cm} (93c)

$$M_2^{IE} u = u_{xx} - \alpha^2 u_{yy}$$  \hspace{1cm} (94)

and

$$\omega_1 = \frac{1}{k} [-\omega_2 x + \tau \omega_3]$$  \hspace{1cm} (95a)

$$\omega_2 = -k_x - \alpha^2 (m_3y + m_2m_1) + im_2 u_x$$  \hspace{1cm} (95b)

$$\omega_3 = -k\tau + \alpha^2 (m_2y - m_3m_1) + iku_y + im_3 u_x.$$  \hspace{1cm} (95c)

Now let us introduce two complex functions \(q, p\), according to the formulae

$$q = a_1 e^{ib_1}, \quad p = a_2 e^{ib_2}.$$  \hspace{1cm} (96)

Let \(a_j, b_j\) have the forms

$$a_1^2 = \frac{1}{4} k^2 + \frac{|\alpha|^2}{4} (m_3^2 + m_2^2) - \frac{1}{2} \alpha_R km_3 - \frac{1}{2} \alpha_I km_2$$  \hspace{1cm} (97a)

$$b_1 = \partial_x^{-1} \left\{-\frac{\gamma_1}{2ia_1} - (\bar{A} - A + D - \bar{D})\right\}$$  \hspace{1cm} (97b)

$$a_2^2 = \frac{1}{4} k^2 + \frac{|\alpha|^2}{4} (m_3^2 + m_2^2) + \frac{1}{2} \alpha_R km_3 - \frac{1}{2} \alpha_I km_2$$  \hspace{1cm} (97c)

$$b_2 = \partial_x^{-1} \left\{-\frac{\gamma_2}{2ia_2} - (\bar{A} - A + D - \bar{D})\right\}$$  \hspace{1cm} (97d)

where

$$\gamma_1 = i \left\{ \frac{1}{2} k^2 \tau + \frac{|\alpha|^2}{2} (m_3 km_1 + m_2 k_y) - \right.$$  

$$\left. \frac{1}{2} \alpha_R (k^2 m_1 + m_3 k\tau + m_2 k_x + \frac{1}{2} \alpha_I [k(2k_y - m_3x) - k_x m_3]) \right\}.$$  \hspace{1cm} (98a)

$$\gamma_2 = -i \left\{ \frac{1}{2} k^2 \tau + \frac{|\alpha|^2}{2} (m_3 km_1 + m_2 k_y) + \right.$$  

$$\left. \frac{1}{2} \alpha_R (k^2 m_1 + m_3 k\tau + m_2 k_x + \frac{1}{2} \alpha_I [k(2k_y - m_3x) - k_x m_3]) \right\}.$$  \hspace{1cm} (98b)

Here \(\alpha = \alpha_R + i\alpha_I\). In this case, \(q, p\) satisfy the following DS equation

$$iq_t + q_{xx} + \alpha^2 q_{yy} + vq = 0$$  \hspace{1cm} (99a)

$$-ip_t + p_{xx} + \alpha^2 p_{yy} + vp = 0$$  \hspace{1cm} (99b)

$$v_{xx} - \alpha^2 v_{yy} + 2[(pq)_{xx} + \alpha^2 (pq)_{yy}] = 0.$$  \hspace{1cm} (99c)

It is means that the IE (91) and the DS equation (99) are L-equivalent [44] to each other. As well known that these equations are G-equivalent to each other [48]. A few comments are in order.

i) From these results, we get the Ishimori I and DS-I equations as \(\alpha_R = 1, \alpha_I = 0\)
ii) the Ishimori II and DS-II equations as $\alpha_R = 0, \alpha_I = 1$.

iii) For DS-II equation we have

$$pq = |q|^2 = |p|^2$$

iv) at the same time, for the DS-I equation we obtain

$$pq \neq |q|^2 \neq |p|^2$$

$$|q|^2 = |p|^2 - km_3$$

$$pq = (pq)_R + i(pq)_I$$

so that $pq$ is the complex quantity.

### 11.2.2 The KP and M-X equations

The $(2+1)$-dimensional mM-LXI equation in plane has the form \[8\]

\[
\begin{align*}
\left(\begin{array}{c}
e_1 \\
e_2 
\end{array}\right)_x &= A_p \left(\begin{array}{c}
e_1 \\
e_2 
\end{array}\right), \\
\left(\begin{array}{c}
e_1 \\
e_2 
\end{array}\right)_y &= B_p \left(\begin{array}{c}
e_1 \\
e_2 
\end{array}\right), \\
\left(\begin{array}{c}
e_1 \\
e_2 
\end{array}\right)_t &= D_p \left(\begin{array}{c}
e_1 \\
e_2 
\end{array}\right)
\end{align*}
\]

where

\[
A_p = \begin{pmatrix}
0 & k \\
-\beta k & 0
\end{pmatrix}, \\
B_p = \begin{pmatrix}
0 & m_3 \\
-\beta m_3 & 0
\end{pmatrix}, \\
D_p = \begin{pmatrix}
0 & \omega_3 \\
-\beta \omega_3 & 0
\end{pmatrix}.
\]

In the plane case the mM-LXII equation takes the following simple form

$$k_y = m_{3x}$$ \hspace{1cm} (106a)

$$k_t = \omega_{3x}$$ \hspace{1cm} (106b)

$$m_{3t} = \omega_{3y}.$$ \hspace{1cm} (106c)

Hence we get

$$m_3 = \partial_x^{-1} k_y.$$ \hspace{1cm} (107)

The NEE has the form (106b). Many $(2+1)$-dimensional integrable equations such as the Kadomtsev-Petviashvili, Novikov-Veselov (NV), mNV, KNV, $(2+1)$-KdV, mKdV equations are the integrable reductions of the M-LXII equation (106). For example, let us show that the KP and mKP equations are exact reductions of the mM-LXII equation (106). Consider the M-X equation \[8\]

$$S_t = \frac{\omega_3}{k} S_x$$

where

$$\omega_3 = -k_{xx} - 3k^2 - 3\alpha^2 \partial_x^{-1} m_{3y},$$

(109)
If we put $S = e_1$ then from (106) we obtain the L-equivalent counterpart of the M-X equation which is the KP equation

$$k_t + 6kk_x + k_{xxx} + 3\alpha^2 m_{3y} = 0$$  \quad (110a)$$

$$m_{3x} = k_y.$$  \quad (110b)

As known the LR of this equation is given by

$$\alpha \psi_y + \psi_{xx} + k \psi = 0$$  \quad (111a)$$\psi_t + 4\psi_{xxx} + 6k\psi_x + 3(k_x - \alpha m_3)\psi = 0.$$  \quad (111b)

### 11.2.3 The Zakharov and M-IX equations

Now we find the connection between the Myrzakulov IX (M-IX) equation and the curves (the M-LXI equation). The M-IX equation reads as

$$S_t = S \wedge M_1S + A_2S_x + A_1S_y$$  \quad (112a)$$M_2u = 2\alpha^2 S(S_x \wedge S_y)$$  \quad (112b)

where $\alpha, b, a =$ consts and

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b - a) \frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b) \frac{\partial^2}{\partial x^2},$$

$$M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(2a + 1) \frac{\partial^2}{\partial x \partial y} + 4a(a + 1) \frac{\partial^2}{\partial x^2},$$

$$A_1 = i\{\alpha(2b + 1)uy - 2(2ab + a + b)ux\},$$

$$A_2 = i\{4\alpha^{-1}(2a^2b + a^2 + 2ab + b)ux - 2(2ab + a + b)uy\}.$$  \quad (113)

The M-IX equation was introduced in [8] and is integrable. It admits several integrable reductions: 1) the Ishimori equation as $a = b = -\frac{1}{2}$; 2) the M-VIII equation as $a = b = -1$ and so on [8]. In this case we have

$$m_1 = \partial_x^{-1}[\tau_y - \frac{\beta}{2\alpha^2} M_2u]$$  \quad (114a)$$m_2 = -\frac{1}{2\alpha^2 k} M_2u$$  \quad (114b)$$m_3 = \partial_x^{-1}[k_y + \frac{\tau}{2\alpha^2 k} M_2u]$$  \quad (114c)

and

$$\omega_1 = \frac{1}{k}[\omega_{2x} + \tau \omega_3],$$  \quad (115a)$$\omega_2 = -4(a^2 - 2ab - b)k_x - 4\alpha(b - a)k_y - \alpha^2(m_{3y} + m_{2m1}) + m_2 A_1$$  \quad (115b)$$\omega_3 = -4(a^2 - 2ab - b)k\tau - 4\alpha(b - a)km_1 + \alpha^2(m_{2y} - m_{3m1}) + k A_2 + m_3 A_1.$$  \quad (115c)
Functions $q, p$ are given by (96) with

$$a_1^2 = \frac{|a|^2}{|b|^2} a_1^2 = \frac{|a|^2}{|b|^2} \left\{ (l + 1)^2 k^2 + \frac{|\alpha|^2}{4} (m_3^2 + m_2^2) - (l + 1) \alpha_R k m_3 - (l + 1) \alpha_I k m_2 \right\}$$

$$b_1 = \frac{\gamma_1}{2 i a_1^2} \{ (A - A - D - D) + (\bar{A} - A + D - D) \}$$

(116a)

$$a_2^2 = \frac{|b|^2}{|a|^2} a_2^2 = \frac{|b|^2}{|a|^2} \left\{ (l + 1)^2 k^2 + \frac{|\alpha|^2}{4} (m_3^2 + m_2^2) - (l + 1) \alpha_R k m_3 + l \alpha_I k m_2 \right\}$$

(116b)

$$b_2 = \frac{\gamma_2}{2 i a_2^2} \{ (A - A + D - D) - (\bar{A} - A + D - D) \}$$

(116c)

where

$$\gamma_1 = i \{ 2 (l + 1)^2 k^2 \tau + \frac{|\alpha|^2}{2} (m_3 k m_1 + m_3 k y) \}$$

(117a)

$$(l + 1) \alpha_R [k^2 m_1 + m_3 k \tau + m_2 k_x] + (l + 1) \alpha_I [k (2 k_y - m_3 x) - k_x m_3]$$

$$\gamma_2 = -i \{ 2 (l + 1)^2 k^2 \tau + \frac{|\alpha|^2}{2} (m_3 k m_1 + m_3 k y) \}$$

(117b)

$$l \alpha_R (k^2 m_1 + m_3 k \tau + m_2 k_x) - l \alpha_I [k (2 k_y - m_3 x) - k_x m_3]$$

Here $\alpha = \alpha_R + i \alpha_I$. In this case, $q, p$ satisfy the following other Zakharov equation [49]

$$i q_t + M_1 q + v q = 0$$

(118a)

$$i p_t - M_1 p - v p = 0$$

(118b)

$$M_2 v = -2 M_1 (p q)$$

(118c)

which is integrable and admits several particular cases.

As well known the M-IX equation admits several reductions: 1) the M-IXA equation as $\alpha_R = 1, \alpha_I = 0$; 2) the M-IXB equation as $\alpha_R = 0, \alpha_I = 1$; 3) the M-VIII equation as $a = b = 1$ 4) the IE $a = b = -\frac{1}{2}$ and so on. The corresponding versions of the ZE (118), we obtain as the corresponding values of the parameter $\alpha$.

12 Conclusion

In this paper we analyzed the M-LXVIII equation. We have found the some integrable reductions of this equation. Also we have shown that the Zakharov and its spin counterpart the M-I equation are exact reductions of the SDYME. The higher hierarchy of the SDYME was introduced. Using this hierarchy it was shown that several simplest soliton equations in 2+1 dimensions such as $m$KdV, derivative NLS and M-III equations and so on are also its exact reductions.

Finally we would like ask you, dear colleague, if you know, have or will have any results on multidimensional soliton equations, soliton geometry and the Yang-Mills equations, please, inform me (R.M.) or send me a hard copy of your papers. Also any comments, proposals and questions are welcome.
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14 Tasks

Task-1: Please find the breaking solutions (instantons, monopols, dions and so on ) of the SDYME.
Task-2: Please find the breaking solutions of the Bogomolny equation.
Task-3: Please find the breaking solutions of the Ishimori and DS equations.
Task-4: Please find the breaking solutions of the M-IX and Zakharov equations.
Task-5: Please consider the above presented results from twistor point of view.

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