The scalar curvature flow with a flat side

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Abstract

We study the near-the-interface behavior of a compact convex scalar curvature flow with a flat side. Under suitable initial conditions on the flat side, we show that the interface propagates with a finite and nondegenerate speed until the flat side vanishes. Then we get optimal derivative estimates of the pressure-like function, optimal decay estimates of curvatures near the interface, and an Aronson-Bénilan-type curvature lower bound, from which we obtain the Hölder regularity of the ratio of the curvature to the optimal decay rate up to the free boundary. In the end, we obtain the short-time and all-time existence of the solution, smooth up to the interface.

1 Introduction

1.1 The problem of the scalar curvature flow

This paper concerns the regularity of the free boundary problem associated with the scalar curvature flow with a flat side. We begin with the evolution equations governing the scalar curvature flow. Let a compact hypersurface \( \Sigma \) in the \((n+1)\)-dimensional Euclidean space be given. We assume \( n \geq 3 \). Suppose that the body evolves in time by an embedding \( X : \Sigma \times [0, T) \to \mathbb{R}^{n+1} \). We denote the image of \( X \) at time \( t \) by \( M_t \). Suppose that the evolution occurs in the inward normal direction \( N = -\nu \) at each point \( x \) of the hypersurface and that the speed is given by the scalar curvature \( \sigma_2 \). Then we have the following evolution equation:

\[
\frac{\partial X}{\partial t}(x, t) = -\sigma_2(x, t)\nu(x, t) \quad \text{with} \quad X(x, 0) = X_0(x) .
\]  

A scalar curvature flow is a solution to the equation above.

1.2 The history of the research on flows with a flat side

W. Firey [8] first considered the evolution of the Gauss curvature flow of compact surfaces. Hamilton [9] showed that if a Gauss curvature flow initially contains a flat side, then there will be a smaller flat side a little later and it takes some time for the surface to become strictly convex.

P. Daskapoulos with R. Hamilton [3] studied the solvability and regularity of the interface \( \Gamma \) between the Gauss curvature flow and its flat side, by viewing the flow as a free boundary problem. They showed that the solution exists and is smooth up the interface, for a short time.

P. Daskapoulos with K.-A. Lee [5] showed the existence of regular solutions to a certain degenerate parabolic equation of the non-divergence form. Using these results of [5] for the model equation under certain coordinates, they [6] showed that the solution exists smoothly and the interface is smooth for all time until the flat side vanishes.

On the other hand, P. Daskapoulos with R. Hamilton [2] studied the n-dimensional porous medium equation with a flat side. They showed the \( C^\infty \) regularity of the free boundary for a short time, using the regularity of a model degenerate equation which is obtained by changing coordinates.
P. Daskopoulos, R. Hamilton, and K.-A. Lee showed [1] that the square root of the pressure is kept concave by the porous medium equation and the solution exists for a long time up to the free boundary while the free boundary being smooth.

K.-A. Lee and E. Lee [11] [12] considered the evolution of a rotation-invariant surface with a concave side and showed that the long-time existence of a solution, smooth up to the free boundary and the free boundary is smooth.

1.3 The equation of the flow in the local coordinates

We need to find the optimal regularity of the hypersurface near the free boundary where the curvatures become degenerate.

Let us assume that the embedded hypersurface of the scalar curvature flow is given as the graph of a smooth function \( y = f(x(t), t) \) for \( x \in \mathbb{R}^n \). Then the scalar curvature flow is given by

\[
\frac{\partial f}{\partial t} = \sigma_2 \sqrt{1 + |\nabla x f|^2} .
\]  

(1.2)

Now, we evaluate the scalar curvature from the metric \( g_{ij} \), the second fundamental form \( h_{ij} \), and the Weingarten map \( h^i_j \), as in [7]. Since

\[
\frac{\partial X}{\partial x^i} = \left( \nabla_i, \frac{\partial f}{\partial x^i} \right), \quad 1 \leq i \leq n, \quad \text{and} \quad \frac{\partial^2 X}{\partial x^i \partial x^j} = \left( 0, \frac{\partial^2 f}{\partial x^i \partial x^j} \right),
\]

(1.3)

\[
g_{ij} = \left( \frac{\partial X}{\partial x^i}, \frac{\partial X}{\partial x^j} \right) = \delta_{ij} + \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} ,
\]

\[
g^{ij} = \left( \delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla x f|^2} \right), \quad 1 \leq i, j \leq n,
\]

(1.4)

\[
- \nabla^2 = \frac{(-\nabla x f, 1)}{\sqrt{1 + |\nabla x f|^2}}
\]

\[
h_{ij} = \left( - \nabla^2 X, \frac{\partial^2 X}{\partial x^i \partial x^j} \right) = \frac{1}{\sqrt{1 + |\nabla x f|^2}} \frac{\partial^2 f}{\partial x^i \partial x^j} , \quad 1 \leq i, j \leq n ,
\]

(1.5)

\[
h^i_j = g^{ik} h_{kj} = \left( \delta_{ik} - \frac{\partial_i f \partial_k f}{1 + |\nabla x f|^2} \right) \frac{1}{\sqrt{1 + |\nabla x f|^2}} \frac{\partial^2 f}{\partial x^k \partial x^j} , \quad 1 \leq i, j \leq n .
\]

Hence, the mean curvature \( H \) of the flow is

\[
H = \sum_{i=1}^{n} h^i_i = \sum_{i,j=1}^{n} g^{ij} h_{ij} = \sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla x f|^2} \right) \frac{1}{\sqrt{1 + |\nabla x f|^2}} \frac{\partial^2 f}{\partial x^i \partial x^j} ,
\]

(1.6)

and the square sum \( |A|^2 = \sum_i \sigma_i^2 \) of the principal curvatures is

\[
|A|^2 = \sum_{i,j=1}^{n} h^i_i h^j_j = \sum_{i,j=1}^{n} \left( \delta_{ij} - \frac{\partial_i f \partial_j f}{1 + |\nabla x f|^2} \right)^2 \frac{1}{1 + |\nabla x f|^2} \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right)^2 ,
\]

(1.7)

so that the scalar curvature is

\[
\sigma_2 = \frac{1}{2} (H^2 - |A|^2) = \frac{1}{2} \sum_{i,j,k,l=1}^{n} g^{ik} g^{jl} (h_{ki} h_{lj} - h_{kj} h_{li})
\]

\[
= \frac{1}{2(1 + |\nabla x f|^2)} \sum_{i,j,k,l=1}^{n} \left( \delta_{ik} - \frac{\partial_i f \partial_k f}{1 + |\nabla x f|^2} \right) \left( \delta_{jl} - \frac{\partial_j f \partial_l f}{1 + |\nabla x f|^2} \right) \left( \frac{\partial^2 f}{\partial x^k \partial x^l} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^k \partial x^j} \frac{\partial^2 f}{\partial x^i \partial x^l} \right) .
\]

(1.8)

From the formula of the scalar curvature (1.8), we get the following lemma.
Lemma 1.1. The scalar curvature flow is given by the graph of the function $f$, solving

$$f_t = \frac{1}{2\sqrt{1 + |\nabla_x f|^2}} \sum_{i,j,k,l=1}^{n} \left( \delta_{ik} - \frac{f_if_k}{1 + |\nabla_x f|^2} \right) \left( \delta_{jl} - \frac{f_jf_l}{1 + |\nabla_x f|^2} \right) (f_{ki}f_{lj} - f_{kj}f_{li}) \quad \text{(1.9)}$$

We note that the two-dimensional scalar curvature flow is the Gauss curvature flow:

$$f_t = \frac{\det D^2 f}{(1 + |\nabla_x f|^2)^{3/2}} \quad \text{(1.10)}$$

Let us define $g = \sqrt{2I}$ and call it the pressure function. Let $I = 1 + g^2|\nabla_x g|^2$. We can express the evolution equation as the following:

$$g_t = \frac{1}{2\sqrt{1 + g^2|\nabla_x g|^2}} \sum_{i,j,k,l=1}^{n} \left( g(g_{ii}g_{jj} - g_{ij}^2) + g_{,i}^2g_{ii} + g_{,j}^2g_{jj} - 2g_{ii}g_{jj} \right)$$

$$- \frac{1}{2I^{3/2}} g^2 \sum_{i,j,k=1}^{n} g_{i}g_{k} \left( g(g_{ik}g_{jj} - g_{jk}g_{ii}) + g_{j}^2g_{ik} + g_{i}g_{k}g_{jj} - g_{i}g_{j}g_{jk} - g_{j}g_{k}g_{ij} \right)$$

$$- \frac{1}{2I^{3/2}} g^2 \sum_{i,j,k=1}^{n} g_{j}g_{i} \left( g(g_{jj}g_{ii} - g_{ij}g_{ii}) + g_{i}^2g_{jj} + g_{j}g_{i}g_{ii} - g_{j}g_{i}g_{ii} - g_{j}g_{ii}g_{ii} \right) \quad \text{(1.11)}$$

because

$$\sum_{i,j,k,l} g_{i}g_{j}g_{k}g_{l} \left( (g_{ii}g_{jj} - g_{ij}g_{ij})g + g_{ii}g_{ii}g_{ij} + g_{ii}g_{ij}g_{ij} - g_{ij}g_{ii}g_{ii} - g_{ij}g_{ij}g_{ij} \right) = 0 \quad \text{(1.12)}$$

Let $\Gamma_\epsilon(t)$ be the level set $\{(x, g(x, t)) | g = \epsilon \}$ and $e_1 = \nu$ be its outer normal vector. From the fact that $g$ stays constant on the level set, we see that if $e_i$ or $e_k$ is a tangential vector to $\Gamma_\epsilon$, then $\delta_{ik} - g^2 \frac{g_{i}g_{k}}{1 + g^2|\nabla_x g|^2} = \delta_{ik}$. Otherwise, if both $e_i$ and $e_k$ are $\nu$, then $\delta_{ik} - g^2 \frac{g_{i}g_{k}}{1 + g^2|\nabla_x g|^2} = 1 - g^2 \frac{|g_{i}|^2}{1 + g^2|\nabla_x g|^2} = \frac{1}{1 + g^2|\nabla_x g|^2}$. Hence, on $\Gamma_\epsilon$,

$$g_t = \frac{1}{2\sqrt{1 + g^2|\nabla_x g|^2}} \sum_{i,j=2}^{n} g(g_{ii}g_{jj} - g_{ij}^2) + \frac{1}{(1 + g^2g_{i}^2)^{3/2}} \sum_{i=2}^{n} g(g_{ii}g_{11} - g_{i1}^2) + g_{ii}^2g_{i} \quad \text{(1.13)}$$

In particular, on the interface $\Gamma(t)$ of the flat spot where $g = 0$, we have

$$g_t = g_{,i}^2 \Delta_r g = g_{,i}^3 H \quad \text{(1.14)}$$

where $H$ is the mean curvature of the interface $\Gamma(t)$.

Also, we introduce a quantity

$$R_{g,2} = \sum_{i,j=1}^{n} (g_{ii}g_{jj} - g_{ij}^2) \quad \text{(1.15)}$$

where $i$ and $j$ run through all indices $1, \ldots, n$. The convexity of $f$ implies that $R_{f,2} = \sum_{i,j=1}^{n} (f_{ii}f_{jj} - f_{ij}^2) = g^2R_{g,2} + 2gg_{i}^2 \sum_{j \neq i} g_{jj} \geq 0$.

1.4 Our assumptions on the flat side and along the interface

In this work, we impose the same assumptions as in P. Daskapoulos and K.-A. Lee [6], the two-dimensional case. Specifically, we assume the following conditions:
• The hypersurface $\Sigma$ at time $t = 0$ satisfies

$$\Sigma = \Sigma_0 \cup \Sigma_1$$

(1.16)

where $\Sigma_0$ is the flat side and $\Sigma_1$ is the strictly convex part of the hypersurface. The interface between two parts is

$$\Gamma = \Sigma_0 \cap \Sigma_1.$$  

(1.17)

• Because the equation (1.9) is invariant under both rotation and translation, we can assume that $\Sigma_0$ is in the hyperplane $\{x_{n+1} = 0\}$ and $\Sigma_1$ lies above the hyperplane.

• A part of the surface $\Sigma$, which we will also call $\Sigma$, can be considered as the graph of a function

$$x_{n+1} = f(x)$$

(1.18)

on a compact domain $\Omega \subset \mathbb{R}^n$ containing $\Sigma_0$, and we can set $\Omega = \{x \in \mathbb{R}^n; |Df(x)| < \infty\}$.

• The function $f$ vanishes quadratically at $\Gamma$, in other words we assume that at time $t = 0$, $x \in \Gamma$,

$$|Dg(x)| \geq \lambda \quad \text{and} \quad D_{ii}^2g(x) \geq \lambda$$

for some number $\lambda > 0$, and for any tangential direction $2 \leq i \leq n$. This is a nondegeneracy condition for the function $g$.

• A closed disc $D_{\rho_0} = \{x \in \mathbb{R}^n; |x| \leq \rho_0\}$ is contained in the flat spot $\Sigma_0(t)$, whose area should be nonzero for time $0 \leq t \leq T$ and $0 < T < T_c$, where $T_c$ is the time when the area of the flat side shrinks to zero.

• The domain of $f$ is contained in an $n$-dimensional ball $B_R$, $R > 0$.

1.5 Main results

In this paper, we prove the following theorem:

**Theorem 1.2.** Under the assumptions in the subsection 1.4, the pressure-like function $g = \sqrt{2J}$ is smooth in $\Omega(t)$ up to the interface $\Gamma(t)$ on time $0 < t < T$ for all $T < T_c$. In particular the free boundary $\Gamma(t)$ between the strictly convex flow and the flat side will be a smooth curve for all time $0 < t < T_c$.

1.6 Summary

The outline of this paper is as follows. In the section 2 we show that the interface $\Gamma$ moves at a finite and non-degenerate speed. In the section 3 we obtain the gradient estimate of the function $g$ and the curvature estimates. In the section 4 we change the coordinates and get the Hölder regularity of the transformed function. Finally in the section 5 we prove the all-time existence and $C^\infty$ regularity up to the interface, which is our goal.

1.7 Notations

Here are some notations which we will use throughout the paper:

- $I = 1 + g^2|\nabla g|^2$.
- $J = |\nabla g|^2 + g$.
- $R_{g,2} = \sum_{i,j=1}^n (g_{ii}g_{jj} - g_{ij}^2)$.
- $R_{g,2} = \sum_{i,j=2}^n (g_{ii}g_{jj} - g_{ij}^2) + \frac{2}{7} \sum_{i=2}^n (g_{ii}g_{11} - g_{i1}^2)$. 

$4$
is a supersolution (or subsolution, respectively) of the equation \((1.9)\) if and only if
\[
2 \mathrm{Finite \ and \ non-degenerate \ speed \ of \ the \ interface \ of \ the \ flat \ side}
\]
This section deals with the speed of the interface of the flat side. Our goal here is to show that the speed is finite and non-degenerate. We begin with the following lemma:

**Lemma 2.1.** The scaled function with sufficiently small \(\epsilon > 0\)
\[
f_\epsilon(x,t) = \frac{1}{1 + C \epsilon} f((1 + A \epsilon)x, (1 + B \epsilon)t)
\]
is a supersolution (or subsolution, respectively) of the equation \((1.3)\) if and only if
\[
B + C - 4A \geq (C - A) \left( \frac{5|\nabla f_\epsilon|^2}{1 + |\nabla f_\epsilon|^2} - 4 \sum_{i,j} \frac{|\nabla f_\epsilon|^2 \delta_{ij} - f_\epsilon f_{\epsilon,ij}}{(1 + |\nabla f_\epsilon|^2) \delta_{ij} - f_\epsilon f_{\epsilon,ij}} \right).
\]
(or \(\leq\), respectively)

**Proof.** Let \(A, B, C \in \mathbb{R}\) be constants. We scale the function \(f(x,t)\) with those constants by
\[
f_\epsilon(x,t) = \frac{1}{1 + C \epsilon} f((1 + A \epsilon)x, (1 + B \epsilon)t), \quad \epsilon > 0.
\]
Then its partial derivatives of the first order are
\[
f_{\epsilon,i}(x,t) = \frac{1 + B \epsilon}{1 + C \epsilon} f_i((1 + A \epsilon)x, (1 + B \epsilon)t),
\]
\[
f_{\epsilon,i} = \frac{\partial f_\epsilon}{\partial x^i} = \frac{1 + A \epsilon}{1 + C \epsilon} f_i((1 + A \epsilon)x, (1 + B \epsilon)t),
\]
with
\[
|\nabla f_\epsilon|^2 = \left( \frac{1 + A \epsilon}{1 + C \epsilon} \right)^2 |\nabla f((1 + A \epsilon)x, (1 + B \epsilon)t)|^2
\]
We denote \(M = \left( \frac{1 + C \epsilon}{1 + A \epsilon} \right)^2 \). Rearranging the equation of the scalar curvature flow about \(f_\epsilon\), we get the following equation:
\[
\frac{\partial}{\partial t} f_\epsilon = \frac{E(x)}{2 \sqrt{1 + |\nabla f_\epsilon|^2}} (\delta_{ik} - \frac{f_i f_k}{1 + |\nabla f_\epsilon|^2}) (\delta_{jl} - \frac{f_j f_l}{1 + |\nabla f_\epsilon|^2}) (f_{ki} f_{lj} - f_{kj} f_{li}),
\]
where
\[
E(x) = (1 + B \epsilon)(1 + C \epsilon) \left( \frac{1 + |\nabla f_\epsilon|^2}{1 + M |\nabla f_\epsilon|^2} \right)^{5/2} \prod_{i,k} \left( 1 + M |\nabla f_\epsilon|^2 \right) \delta_{ik} - M f_{i,ij} f_{i,k}
\]
\[
\cdot \prod_{j,l} \left( 1 + M |\nabla f_\epsilon|^2 \right) \delta_{jl} - f_{e,j} f_{e,l}.
\]
When \(\epsilon \ll 1\), the factor \(E(x)\) is
\[
E(x) = (1 + (B + C - 4A) \epsilon)(1 - 5(C - A) \epsilon) \frac{|\nabla f_\epsilon|^2}{1 + |\nabla f_\epsilon|^2} \left( 1 + 2(C - A) \epsilon \sum_{i,k} \frac{|\nabla f_\epsilon|^2 \delta_{ik} - f_{e,i} f_{e,k}}{(1 + |\nabla f_\epsilon|^2) \delta_{ik} - f_{e,i} f_{e,k}} \right)
\]
\[
\cdot \left( 1 + 2(C - A) \epsilon \sum_{j,l} \frac{|\nabla f_\epsilon|^2 \delta_{ij} - f_{e,j} f_{e,l}}{(1 + |\nabla f_\epsilon|^2) \delta_{ij} - f_{e,j} f_{e,l}} \right) + o(\epsilon^2)
\]
\[
= 1 + \epsilon \left( B + C - 4A - (C - A) \left( \frac{5|\nabla f_\epsilon|^2}{1 + |\nabla f_\epsilon|^2} - 4 \sum_{i,j} \frac{|\nabla f_\epsilon|^2 \delta_{ij} - f_{e,i} f_{e,j}}{(1 + |\nabla f_\epsilon|^2) \delta_{ij} - f_{e,i} f_{e,j}} \right) + o(\epsilon^2) \right)
\]
\[
\geq 1(\text{or } \leq 1), \text{ if the inequality with } \geq (\text{or } \leq) \text{ in } (2.2) \text{ holds respectively.}
\]
So the evolution equation (2.7) of the scaled function $f_\epsilon(x, t)$ finishes the proof when $\epsilon > 0$ is sufficiently small.

This lemma implies the following lemma, from which we see that the propagating speed of the free boundary is finite.

**Lemma 2.2.** There exists a constant $\delta_0 > 0$ and a negative constant $B$ and a positive constant $C$ satisfying

$$-C f(x, t) + x \cdot \nabla f(x, t) + B t f_i(x, t) \geq 0$$

on the set $A_{\delta_0} = \{(x, t); \ 0 < f(x, t) \leq \delta_0, \ 0 \leq t \leq T\}$.

**Proof.** Assuming that $f(x, t) \in C_c(\mathbb{R}^n)$, $f$ is uniformly continuous on its support and for any $0 < \eta < 1$ there exists $0 < \delta_0 \ll \eta < 1$ such that

$$\{x; 0 < f(x, t) \leq \delta_0\} \subset \{x; d(x, \Gamma(t)) \leq \frac{\eta \rho_0}{2}\},$$

$$\{(1 + \epsilon)x; d(x, \Gamma(t)) \leq \frac{\eta \rho_0}{2}\} \subset \{x; d(x, \Gamma(t)) \leq \eta \rho_0\}$$

on $0 \leq t \leq T$, for all $\epsilon \ll \delta_0$. Let $\Gamma(t) = \partial\{x; f(x, t) > 0\}$ be the interface. Consider the scaled function $f_\epsilon$ as in Lemma 2.1 with $A = 1, B = -\delta_0^2$, and $C = 8$. We want to show that $f_\epsilon \geq f$ on $A_{\delta_0}$, so that $\frac{\partial}{\partial t} f_\epsilon = 0$ on $A_{\delta_0}$, which proves the lemma.

The inequality (2.2) in Lemma 2.1 for $f_\epsilon$ to be a supersolution becomes

$$-\delta_0^2 + 4 \geq 7 \left(\frac{5|\nabla f_\epsilon|^2}{1 + |\nabla f_\epsilon|^2} - 4 \sum_{i, j} \frac{|\nabla f_i|^2 \delta_{ij} - f_\epsilon,i f_\epsilon,j}{(1 + |\nabla f_i|^2) \delta_{ij} - f_\epsilon,i f_\epsilon,j}\right).$$

(2.13)

We have lower and upper bounds for the last term of the inequality above;

$$\frac{n(n - 1)}{2} \leq \sum_{i, j} \frac{|\nabla f_i|^2 \delta_{ij} - f_\epsilon,i f_\epsilon,j}{(1 + |\nabla f_i|^2) \delta_{ij} - f_\epsilon,i f_\epsilon,j} - \frac{n(n + 1)}{2}.$$  

(2.14)

Let us prove a claim that on $A_{\delta_0}$ with $A = 1, B = -\delta_0^2$, and $C = 8$,

$$|\nabla f((1 + \epsilon)x, (1 + B\epsilon)t)| \leq \sqrt{\frac{2n(n - 1)}{5} + \frac{3}{35}}.$$  

(2.15)

If the claim (2.15) above holds, we see that

$$|\nabla f_\epsilon(x, t)|^2 \leq |\nabla f((1 + \epsilon)x, (1 + B\epsilon)t)|^2 \leq \frac{2n(n - 1)}{5} + \frac{3}{35},$$

$$\frac{5|\nabla f_\epsilon|^2}{1 + |\nabla f_\epsilon|^2} \leq \frac{3}{7} + \frac{2n(n - 1)}{5} \leq \frac{3}{7} + \frac{4}{1 + |\nabla f_i|^2} \frac{|\nabla f_i|^2 \delta_{ij} - f_\epsilon,i f_\epsilon,j}{(1 + |\nabla f_i|^2) \delta_{ij} - f_\epsilon,i f_\epsilon,j}.$$  

(2.16)

and $f_\epsilon$ becomes a supersolution. To prove the claim, we note that the function $f$ is of the class $C^{1,1}$. When the dimension $n$ of the scalar curvature flow is 2, the flow becomes the Gauss curvature flow which has been considered by [6]; they used the Andrews’s result in [1] that the viscosity solution of the scalar curvature flow has bounded mean curvature $H$. Hence, with the convexity assumption, all principal curvatures are positive and bounded by a uniform constant and the function $f$ should be $C^{1,1}$ as in [1].
For scalar curvature flows of higher dimension $n > 2$, the sum of the squared principal curvatures $|A|^2 = \sum_{i=1}^n \lambda_i^2$, the scalar curvature $\sigma_2$, and the mean curvature $H$ of the flow are all bounded. This proves that every viscosity solution $f$ to the equation of the scalar curvature flow has a uniform $C^{1,1}$ estimate. As $|\nabla f| = 0$ along the interface $\Gamma(t)$, the uniform $C^{1,1}$ estimate of $f$ implies that, for some positive constant $M$ which does not depend on $t$, we have for $(x,t) \in A_{\delta_0}$

$$|\nabla f(x,t)| \leq M d(x,\Gamma(t)). \quad (2.17)$$

Note that the function $f$ is nondecreasing in time $t$, because the scalar curvature $\sigma_2$ is positive for all $t > 0$. The positiveness of $\sigma_2$ is derived from Theorem 1 that if the scalar flow is convex at $t = 0$, then it remains convex for all $t > 0$.

Assume that $f(x,t) \leq \delta_0$. The nonnegativity of $f_t$ implies that $f(x,(1+B\epsilon)t) \leq \delta_0$ for $B = -\delta_0^2$ and $\epsilon > 0$. Then,

$$d(x,\Gamma((1+B\epsilon)t)) \leq \frac{\eta \rho_0}{2} \quad (2.18)$$

and

$$d((1+\epsilon)x,\Gamma((1+B\epsilon)t)) \leq \eta \rho_0. \quad (2.19)$$

Taking $\eta$ small enough, $f_\epsilon$ becomes a supersolution in $A_{\delta_0}$. By applying the comparison principle to the supersolutions $f_\epsilon$ and $f$, we can show that if additionally $f_\epsilon \geq f$ on the parabolic boundary $\partial_p A_{\delta_0} = \{(x,t); f(x,t) = \delta_0, 0 \leq t \leq T\}$, then $f_\epsilon \geq f$ in $A_{\delta_0}$. Now, the only remaining part is to prove that $f_\epsilon \geq f$ on $\partial_p A_{\delta_0}$. From the simple calculation

$$\frac{d}{d\epsilon} f_\epsilon(x,t) = -\frac{C}{(1+C\epsilon)^2}f((1+A\epsilon)x,(1+B\epsilon)t) + \frac{Ax}{1+C\epsilon} \cdot \nabla_x f((1+A\epsilon)x,(1+B\epsilon)t)$$

$$+ \frac{Bt}{1+C\epsilon} f_t((1+A\epsilon)x,(1+B\epsilon)t), \quad (2.20)$$

we have, for $A = 1$ in our assumption,

$$\left. \frac{d}{d\epsilon} f_\epsilon(x,0) \right|_{\epsilon=0} = -C f(x,0) + x \cdot \nabla_x f(x,0). \quad (2.21)$$

As $x \cdot \nabla_x f(x,0) > 0$, for small enough $\delta_0$, $\left. \frac{d}{d\epsilon} f_\epsilon(x,0) \right|_{\epsilon=0} > 0$ on $\{x; f(x,0) \leq \delta_0\}$ so that $f_\epsilon(x,0) \geq f(x,0)$ for small $\epsilon > 0$. When $f(x, t) = \delta_0$, we see that $d(x, \Gamma(t)) \leq \eta \rho_0/2 \leq \eta \rho_0$. Since $f$ is convex in the radius $r$, the radial derivative $f_r$ satisfies $f_r \geq \frac{\delta_0}{\eta \rho_0}$ and it is seen that $r = d(0,x) \geq \rho_0$ and $x \cdot \nabla_x f = r f_r(x,t) \geq \rho_0 \frac{\delta_0}{\eta \rho_0} = \frac{\delta_0}{\eta}$ on $\partial_p A_{\delta_0}$.

Hence, for $f = \delta_0$ on $\partial_p A_{\delta_0}$, if $\eta$ is small enough,

$$\left. \frac{d}{d\epsilon} f_\epsilon(x,t) \right|_{\epsilon=0} = -C f(x,t) + x \cdot \nabla_x f(x,t) + B t f_t(x,t) \geq -C \delta_0 + \frac{\delta_0}{\eta} - \delta_0^2 T |f_t|_{L^\infty} > 0. \quad (2.22)$$

which means that $f_\epsilon \geq f$ on $\partial_p A_{\delta_0}$. \hfill \Box

From the proof of Lemma \ref{lemma}, we see that $f \in C^{1,1}$. Thus we may assume that

$$\Omega(t) = \{x \in \mathbb{R}^n; |Df(x,t)| < \infty\} \quad (2.23)$$

for all time $0 < t < T$. We can show that the free boundary moves with a finite speed when $0 \leq t \leq T$. The radius of the free boundary $\Gamma(t)$ is written by $r = \gamma(\theta, t)$ with $0 \leq \theta < 2\pi$ in polar coordinates.
Theorem 2.3. Under the assumption that $D_{\rho_0} \subset \Sigma_0(T)$, there is a constant $B < 0$ such that

$$
\gamma(\theta, t) \geq e^{-\frac{t-t_0}{|B|t_0}} \gamma(\theta, t_0)
$$

(2.24)

for all $0 < t_0 \leq t \leq T$ and $0 \leq \theta < 2\pi$. In particular, the free boundary moves with a finite speed when $0 \leq t \leq T$.

Proof. From Lemma 2.2, for $0 < t_0 \leq t \leq T$, we get the inequality

$$
0 \geq \frac{Cf(x, t)}{|B|t} - \frac{x}{|B|t} \cdot \nabla f(x, t) + f_t(x, t) \geq \frac{Cf(x, t)}{|B|T} - \frac{x}{|B|t_0} \cdot \nabla f(x, t) + f_t(x, t)
$$

(2.25)

so that

$$
\frac{d}{dt}(e_{\frac{C}{|B|t_0}}(t-t_0) f(e^{-\frac{t-t_0}{|B|t_0}} x, t)) \leq 0
$$

(2.26)

and hence

$$
e_{\frac{C}{|B|t_0}}(t-t_0) f(e^{-\frac{t-t_0}{|B|t_0}} x, t) \leq f(x, t_0) = 0
$$

(2.27)

for $|x| = \gamma(x, t_0)$, which implies the conclusion. \hfill \Box

For small $\epsilon > 0$ the $\epsilon$-level set of the function $f$ moves with a finite speed as well, by the following theorem. Let us express the $\epsilon$-level set by its radius $r = \gamma_\epsilon(\theta, t)$.

Theorem 2.4. Under the assumption that $D_{\rho_0} \subset \Sigma_0(T)$, there is a constant $B < 0$ such that

$$
\gamma_\epsilon(\theta, t) \geq e^{-\frac{t-t_0}{|B|t_0}} \gamma_\epsilon(\theta, t_0)
$$

(2.28)

for all sufficiently small $\epsilon > 0$, $0 < t_0 \leq t \leq T$, and $0 \leq \theta < 2\pi$. In particular, for each $\epsilon > 0$ the level set $r = \gamma_\epsilon(\theta, t)$ moves with a finite speed when $0 \leq t \leq T$.

Proof. Fix $0 \leq \theta < 2\pi$. Let $r_0 = \gamma_\epsilon(\theta, t_0)$ and $x_0$ be the point $(r_0, \theta)$ in polar coordinates. Then $f(x_0, t_0) = \epsilon$ and by the inequality (2.27), we have

$$
f(e^{-\frac{t-t_0}{|B|t_0}} x_0, t) \leq e^{-\frac{t-t_0}{|B|t_0}} f(x_0, t_0) \leq f(x_0, t_0) = \epsilon = f(\gamma_\epsilon(\theta, t_0)),
$$

(2.29)

implying that

$$
e^{-\frac{t-t_0}{|B|t_0}} \gamma_\epsilon(\theta, t_0) \leq \gamma_\epsilon(\theta, t).
$$

(2.30)

\hfill \Box

Lemma 2.5. There exist constants $A > 0$, $B < 0$, $C > 0$, and $D > 0$ such that

$$
-Cf(x, t) + Ax \cdot \nabla f(x, t) + (-D + Bt) f_t(x, t) \leq 0
$$

(2.31)

on $\{f(x, t) \leq 1, \ 0 \leq t \leq T\}$.

Proof. Assume that $g$ is smooth up to the interface $\Gamma(t)$ for time $0 \leq t \leq \tau$ for some $\tau > 0$. Let $t^* = \tau/2$ and $A_{t^*} = \{f(x, t) \leq 1, t^* \leq t \leq T\}$. We want to show that for some negative constants $A$ and $B$, and positive constants $C$ and $D$, and for sufficiently small $\epsilon > 0$,

$$
f_t(x, t) = \frac{1}{1 + C\epsilon} f((1 + A\epsilon)x, (1 + B\epsilon)t - D\epsilon) \leq f(x, t)
$$

(2.32)
on $\mathcal{A}_{t^*}$. We first choose $C = 1$ so that by Lemma 2.1, $f_t$ is a subsolution to the equation for the scalar curvature flow if and only if $B + 1 - 4A \leq 0$. Given $A > 0$ to be determined later, take $B < 0$ such that $B \leq 4A - 1$. Then $f_t$ is a subsolution, especially in $\mathcal{A}_{t^*}$. Therefore, the comparison principle implies that it suffices to show that $f_t \leq f$ on the parabolic boundary of $\mathcal{A}_{t^*}$, where $f(x,t) = 1$, $t^* \leq t \leq T$ or $f(x,t) \leq 1$, $t = t^*$. It is equivalent to show that $\frac{\partial f_t}{\partial c}|_{c=0} = -f(x,t) + Ax \cdot \nabla_x f(x,t) + (-D + Bt)f_t(x,t) \leq 0$.

On $\{f(x,t) = 1, \ t^* \leq t \leq T\}$, we can take $A > 0$ sufficiently small that we have $-f(x,t) + Ax \cdot \nabla_x f(x,t) = -1 + Ax \cdot \nabla_x f(x,t) \leq 0$ since $x \cdot \nabla_x f(x,t) \geq 0$ and $f(\cdot,t)$ is uniformly $C^{1,1}$ in $0 \leq t \leq T$, $f \leq 1$. Secondly, on the set $\{f(x,t) \leq 1, \ t = t^*\}$ and for the pressure-like function $g = \sqrt{2f}$, we see that $-f(x,t) + Ax \cdot \nabla_x f(x,t) + (-D + Bt)f_t(x,t) = g\{g(x,t^*) + Ax \cdot \nabla_x g(x,t^*) + (-D + Bt^*)g_t(x,t^*)\}$. Because $g(x,t^*) > 0$, $B < 0$, and $g_t(x,t^*) \geq 0$, we only need to show that $-g(x,t^*) + 2Ax \cdot \nabla_x g(x,t^*) - 2Dg_t(x,t^*) \leq 0$.

However, we assumed the initial non-degeneracy condition for $f$ and hence for $g$ so that for sufficiently small $t^*$

$$|\nabla_x g(x,t^*)| \geq c \quad \text{and} \quad g_{t^*} \geq c \quad \text{at the interface} \quad \Gamma(t^*)$$

(2.33) for some $c > 0$. For this reason there are some $\rho > 0$ and $c_0 > 0$ satisfying $g_t \geq c_0 > 0$, so that $-g(x,t^*) + 2Ax \cdot \nabla_x g(x,t^*) - 2Dg_t(x,t^*) \leq 0$ on the set $\{0 < g(x,t^*) < \sqrt{2}, \ d(x,\Gamma(t^*)) < \rho\}$ if we take sufficiently small $A > 0$ since $D > 0$. The same estimate holds when $0 \leq t \leq t^*$. Hence $f_t \leq f$ on $\{f(x,t) \leq 1, 0 \leq t \leq T\}$.

**Theorem 2.6.** There exists constants $A > 0$, $B < 0$, and $D > 0$ such that

$$\gamma(\theta, t) \leq c \cdot \frac{\Delta(t-t_0)}{D+|B|t} \gamma(\theta, t_0)$$

(2.34) for $0 < t_0 \leq t < T$. Hence, the interface moves with a non-degenerate speed.

**Proof.** For time $0 < t_0 < t < T$, the inequality (2.31) implies that

$$0 \leq \frac{Cf(x,t)}{D+|B|t} - \frac{Ax}{D+|B|t} \cdot \nabla_x f(x,t) + f_t(x,t)$$

(2.35)

by which we obtain

$$\frac{d}{dt}(e^{\frac{Cf(x,t)}{D+|B|t_0}}f(e^{-\frac{\Delta(t-t_0)}{D+|B|t} x, t})) \geq 0.$$  (2.36)

Hence we have

$$e^{\frac{Cf(x,t)}{D+|B|t_0}}f(e^{-\frac{\Delta(t-t_0)}{D+|B|t} x, t}) \geq f(x,t_0),$$

(2.37)

and for sufficiently large $D > 0$

$$f(e^{-\frac{\Delta(t-t_0)}{D+|B|t} x, t}) \geq f(x,t_0).$$

(2.38)

From the last inequality and the monotonicity of $f$ in the radius, the conclusion follows. \qed
3 Derivative estimates

3.1 Evolution equations of derivatives of $g$

For simplicity, we write $1 + g^2 |\nabla g|^2$ as 1. Let us consider the following linear operator, $L[w]$, which will occur at the equations of $g_m$ and $g_{mp}$: set

$$L[w] = \sum_{i,j=1}^{n} a_{ij} w_{ij} + \sum_{i=1}^{n} b_{i} w_{i}, \quad (3.1)$$

where

$$a_{ii} = \frac{1}{T^{3/2}} \left( \sum_{j \neq i} (g_{ij}^2 + gg_{jj} + g^3 (g_{kk}^2 g_{ij} - g_{ji} g_{k} g_{k})) - g^3 (g_{i}^2 g_{k k} - g_{ik} g_{k}) \right), \quad (3.2)$$

and

$$a_{ij} = -\frac{1}{T^{3/2}} (g_{ij}^2 + gg_{ij} + g^3 (g_{kj}^2 g_{ij} - g_{ik} g_{k} g_{j})),$$

for $i \neq j$.

Also, define

$$b_{i} = \frac{1}{2T^{5/2}} \left( 4I (g_{ij}^2 - g_{ij}) + 6 g^3 g_{ik} g_{ij} (g_{k} g_{ij} - g_{j} g_{k}) + 6 g^5 g_{ik} g_{ij} (g_{k} g_{ij} - g_{j} g_{k}) \right), \quad (3.3)$$

and

$$c = \frac{1}{2T^{3/2}} \left( I (g_{ij}^2 - g_{ij}) - 6 g^5 g_{ik} g_{ij} (g_{k} g_{ij} - g_{j} g_{k}) - 6 g^7 (g_{j}^2 g_{k k} - g_{jk} g_{kj}) \right). \quad (3.4)$$

Lemma 3.1. For $1 \leq m \leq n$, $(g_m)_t$ is given by

$$(g_m)_t = L[g_m] + c g_m. \quad (3.5)$$

Proof. This equation (3.5) can be obtained by a direct calculation. \hfill \Box

In the local coordinates where $g_{11} = g_\nu$, $g_{ii} = 0$ for $i \neq 1$ and $g_{ij} = 0$ for $i \neq j$, we have

$$a_{11} = \frac{1}{T^{3/2}} g \sum_{j \neq 1} g_{jj}, \quad a_{ij} = 0, \quad \text{for } i \neq j, \quad (3.6)$$

$$a_{ii} = \frac{1}{T^{3/2}} (g_{ii}^2 + gg_{11} + I g \sum_{j \neq i} g_{jj}), \quad \text{for } i \neq 1,$$

$$b_{1} = \frac{1}{2T^{5/2}} \left( -I g^3 g_{11} R_{s,2} + (4I g_{1} - 6 g^2 g_{11} (g_{1}^2 + gg_{11}) + 2I g^3 g_{11}) \sum_{j \neq 1} g_{jj} \right), \quad (3.7)$$

$$b_{i} = 0, \quad \text{for } i \neq 0,$$

and

$$c = \frac{1}{2T^{3/2}} (I R_{s,2} - 6 gg_{11}^2 (g_{1}^2 + gg_{11}) \sum_{j \neq 1} g_{jj}). \quad (3.8)$$

which simplifies the evolution of $g_m$ as follows:
\[(g_1)_t = \frac{1}{I^{3/2}} \sum_{j \neq 1} g_{jj} g_{j11} + \sum_{i \neq 1} \left( g_i^2 + g_i g_{i11} + Ig \sum_{j \neq 1} g_{ij} \right) g_{i1i} \]
\[+ \frac{1}{2I^{5/2}} \left( -Ig^3 g_1 R_{g,2} + (4Ig_1 - 6g_1^2 g_1^2 + 2g_1^3 g_1 g_{11}) \sum_{j \neq 1} g_{jj} \right) g_{11} \]
\[+ \frac{1}{2I^{5/2}} (IR_{g,2} - 6gg_1^2 (g_1^2 + g_1 g_{11}) \sum_{j \neq 1} g_{jj}) g_1, \]
\[(3.9)\]

and for \(m \neq 1\)
\[(g_m)_t = \frac{1}{I^{3/2}} \sum_{j \neq 1} g_{jj} g_{m11} + \sum_{i \neq 1} \left( g_i^2 + g_i g_{i11} + Ig \sum_{j \neq 1, i} g_{ij} \right) g_{mi1} \]
\[+ \frac{1}{I^{3/2}} (IR_{g,2} - 6gg_1^2 (g_1^2 + g_1 g_{11}) \sum_{j \neq 1} g_{jj}) g_1. \]
\[(3.10)\]

**Lemma 3.2.** For \(1 \leq m, p \leq n, \ (g_{mp})_t \) is given by
\[(g_{mp})_t = \sum_{i,j=1}^{n} a_{ij} g_{mpi} + \sum_{i,j,k,l=1}^{n} b_{mp,ij,kl} g_{mi1j} g_{jkl1} + \sum_{i,j,k=1}^{n} c_{mp,ijk} g_{ijk} + d_{mp}, \]
\[(3.11)\]
with the fourth derivatives
\[\sum_{i,j=1}^{n} a_{ij} g_{mpi} = \frac{1}{I^{3/2}} \sum_{i,j=1}^{n} (Ig_{ij} - g_3 g_i g_{j1} + g_2) g_{mi1}, \]
\[- \frac{1}{I^{3/2}} \sum_{i,j=1}^{n} (Ig_{ij} + g_3 g_i g_{j1} + g_2) g_{mi1}, \]

the terms which are quadratic in third derivatives
\[\sum_{i,j,k,l=1}^{n} b_{mp,ij,kl} g_{mi1j} g_{jkl1} = \frac{1}{I^{3/2}} \sum_{i=1}^{n} g_{mi1i} (Ig \sum_{k=1}^{n} g_{k1k} - g_3 \sum_{k,l=1}^{n} g_{k1k} g_{kl} - Ig \sum_{i,j=1}^{n} g_{mi1j}) \]
\[+ \frac{1}{I^{3/2}} g_3 \sum_{i,j=1}^{n} g_{mi1j} g_{j11} g_{j11} - \frac{1}{I^{3/2}} g_3 \sum_{i,j=1}^{n} g_{mi1j} g_{j11} g_{j11}, \]

the terms which are linear in third derivatives
\[\sum_{i,j,k=1}^{n} c_{mp,ijk} g_{ijk} = \frac{1}{I^{3/2}} \sum_{i=1}^{n} \left( 4I \sum_{j=1}^{n} (g_i g_{jj} - g_j g_{ij}) - Ig^2 \sum_{j,k=1}^{n} (g_{i} g_{kk} g_{jj} - g_{kj} g_{ij}^2) + 4g_3 g_{i} g_{kk} g_{jj} - g_{kj} g_{ij}^2 \right) g_{j11} \]
\[- \frac{3}{I^{3/2}} g_3^2 \sum_{j=1}^{n} (Ig_{jj} + g_3 g_3 g_j g_{j11} + g_2 g_3 \sum_{j,k,l=1}^{n} (g_i g_{ik} g_{kk} g_{jj} - g_{kj} g_{kk} g_{jj}) g_{mi1} \]
\[- \frac{3}{I^{3/2}} g_3^2 \sum_{j=1}^{n} \left( 2g_{j11} + \sum_{k=1}^{n} (2g_3 g_{kk} g_{jj} g_{kk} - g_3 g_{j11} g_{kk} g_{kk} - g_3 g_{kk} g_{jj} g_{kk}) \right) g_{mi1} \]
\[+ \frac{1}{I^{3/2}} \sum_{j=1}^{n} \left( 2g_{j11} + \sum_{k=1}^{n} (2g_3 g_{kk} g_{jj} g_{kk} - g_3 g_{j11} g_{kk} g_{kk} - g_3 g_{kk} g_{jj} g_{kk}) \right) g_{mi1} \]
\[+ \frac{3}{I^{3/2}} g_3^2 \sum_{i,j=1}^{n} (Ig_{ij} + g_3 g_j + g_2 \sum_{k=1}^{n} (g_i g_{kk} g_{kk} - g_i g_{kk} g_{kk} - g_i g_{kk} g_{kk}) g_{mi1} \]
\[+ \frac{1}{I^{3/2}} \sum_{i,j=1}^{n} (g_i g_{pi} + g_i g_{pi}) g_{mi1} \]

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\[- \frac{1}{T_3/2} g^3 \sum_{k=1}^{n} \sum_{i,j=1}^{n} (g_{ij}g_{pi} + g_{ik}g_{pj}) + \frac{1}{T_3/2} g^3 \sum_{l=1}^{n} \sum_{i,j,k=1}^{n} \left((g_{ij}g_{ik} + g_{ij}g_{jk})g_{pk} + g_{k}(g_{jk}g_{pi} + g_{ik}g_{pj})\right) + \frac{3}{T_3/2} g \sum_{i,j=1}^{n} \left(Ig_{ij} + g_i g_j + g^3 \sum_{k=1}^{n} (g_{ij}g_{jk} - g_{ik}g_{jk} - g_{ki}g_{jk})\right) + \frac{3}{T_3/2} g \sum_{i,j,k=1}^{n} \left(Ig_{ij} + g_i g_j + g^3 \sum_{k=1}^{n} (2g_{ij}g_{jj} - 3g_{ij}g_{jk})\right)\]

and the terms involving only first-order and second-order derivatives

\[
d_{mp} = \frac{1}{2T_3/2} \left( I \sum_{i,j=1}^{n} (g_{ij}g_{jj} - g_{ij}^2) \right) g_{mp} + \frac{1}{2T_3/2} \sum_{i=1}^{n} g_{mi} \left( I \sum_{j=1}^{n} (g_{ij}g_{pi} - g_{ij}g_{pj}) \right) + \frac{1}{2T_3/2} \sum_{j,k=1}^{n} g_{kj} \left( I \sum_{i=1}^{n} (g_{ij}g_{jj} - g_{ij}g_{ij}) + 4g_{ij}(g_{ij} - g_{ij})g_{jk} \right) - 2g^2 \sum_{j,k=1}^{n} \left( g_{ij}(g_{kk} - g_{ij}^2) + 4g_{jk}(g_{kk} - g_{ij}) \right) g_{kp} \]

\[
- 6g^3 \sum_{j,k=1}^{n} \left( g_{ij}(g_{kk}g_{jj} - g_{ij}^2) + 4g_{jk}(g_{kk} - g_{ij}) \right) g_{pl} \]

\[
+ 6g^4 \sum_{j,k,l=1}^{n} \left( g_{ij}(g_{kk}g_{jj} - g_{ij}^2) + 4g_{jk}(g_{kk} - g_{ij}) \right) g_{kp} + 6g_{ij}(g_{ij} - g_{ij})g_{pl} + 6g_{ij}(g_{ij} - g_{ij})g_{pk} \]

\[
g^2 (-6 \sum_{j,k=1}^{n} (g_{ij}^2 - g_{ij}g_{jk})g_{pi} + 8(g_{ij}g_{jj} - g_{ij}g_{ij})g_{pk} - 12g_{i}(g_{jj}g_{kk}g_{pj} - g_{kk}g_{ij}g_{pj})) \]

\[
- \frac{5}{2T_3/2} g^2 \sum_{i=1}^{n} g_{mi} \sum_{q=1}^{n} g_{qmp} \left( I \sum_{j=1}^{n} (g_{ij}g_{jj} - g_{ij}g_{ij}) \right) + g^2 \sum_{j,k=1}^{n} \left( g_{ij}(g_{kk} - g_{ij}) + 4g_{jk}(g_{kk} - g_{ij}) \right) g_{pl} \]

\[
- 2g^2 \sum_{j,k=1}^{n} \left( g_{ij}(g_{kk}g_{jj} - g_{ij}^2) + 4g_{jk}(g_{kk} - g_{ij}) \right) g_{kp} \]

\[
+ \frac{1}{2T_3/2} \sum_{i=1}^{n} g_{mi} g_{pj} \sum_{j,k=1}^{n} \left( 8g_{jk}(g_{ij} - g_{ij}) - 2g(g_{ij}g_{kk}g_{jj} - g_{ij}g_{jj}) + 4g_{jk}(g_{kk}g_{jj} - g_{ij}g_{ij}) \right) g_{lp} \]

\[
+ \frac{1}{2T_3/2} \sum_{i=1}^{n} g_{mi} g_{jp} \sum_{j,k=1}^{n} \left( 8g_{jk}(g_{ij} - g_{ij}) - 2g(g_{ij}g_{kk}g_{jj} - g_{ij}g_{jj}) + 4g_{jk}(g_{kk}g_{jj} - g_{ij}g_{ij}) \right) g_{lp} \]

\[
- 4g^2 \sum_{k=1}^{n} g_{kk}g_{jj} - g_{ij}g_{ij}) \]
If the level set afterwards. What we want to obtain is a gradient estimate from above.

To begin with, we assume that

\[ T \text{ is of class } C^{2+\beta} \text{ for some } 0 < \beta < 2 \text{ for time } 0 \leq t < T < T_c, \]

\[ g \text{ is smooth up to the interface for a finite amount of time}, \]

\[ g \text{ has first-order and second-order derivatives having the non-degeneracy condition at the initial time } t = 0: \] there is a positive number \( \lambda \) such that for the derivative \( |\nabla g(x)| \) and the second-order tangential derivatives \( |\nabla^2_{\tau} g(x)| \) of \( g \), \( |\nabla g(x)| \geq \lambda \) and \( |\nabla^2_{\tau} g(x)| \geq \lambda \) \( \forall x \in \Gamma \),

\[ f \text{ satisfies the non-degeneracy condition } \max_{x \in \Omega(t)} f(\cdot, t) \geq 2 \text{ on } 0 \leq t \leq T < T_c \text{ so that } g \text{ also meets the condition } \max_{x \in \Omega(t)} g(\cdot, t) \geq 2 \text{ for } \Omega(t) = \{ x \in \mathbb{R}^n; |\nabla f(x, t)| < \infty \}. \]

We approximate \( f \) by a decreasing sequence of positive, strictly convex and smooth functions \( f_\epsilon \) solving the equation \( (1.9) \) of scalar curvature flow. We set \( g_\epsilon = \sqrt{2f_\epsilon} \) and denote it simply by \( g \) afterwards. What we want to obtain is a gradient estimate from above.

**Lemma 3.3.** If the level set \( \Gamma_\epsilon(t) \) of \( g \) is convex, then there exists a constant \( C > 0 \) such that

\[ |\nabla g| \leq C, \text{ on } 0 \leq g(\cdot, t) \leq 1, \ 0 \leq t \leq T. \]
Proof. Let us set $X = \frac{1}{2}|\nabla g|^2 = \frac{1}{2} \sum_i g_i^2$. Then $X = \frac{1}{2}g_{\nu}^2$ for the normal vector $\nu$ to the level set of $g$. Suppose that at each time $0 \leq t < T$, $X$ attains an interior maximum at $P_0 = (x_0, t)$ so that $X(x_0, t) = \sup \{X(x, t) ; x \in \Sigma, 0 \leq g(x, t) \leq 1 \}$. Rotating the coordinates, we can make $g_1 = g_\nu > 0$ and $g_i = 0$ for $2 \leq i \leq n$ at $P_0$. As

$$X_1 = g_1g_{i1} + \sum_{i \geq 2} g_i g_{i1} = g_1 g_{i1} = 0, \quad X_i = g_1 g_{ii} + \sum_{j \geq 2} g_j g_{ji} = g_1 g_{ii} = 0, \quad i \geq 2,$$  

(3.14)

we have $g_{i1} = g_{ii} = 0$ for $2 \leq i \leq n$ at $P_0$. So the nonzero second-order derivatives of $g$ at $P_0$ are $g_{ij}$ for $2 \leq i, j \leq n$. In particular, $g_{ii} \geq 0$, $2 \leq i \leq n$ by convexity of level sets of $g$.

Now, let us look at the second-order derivatives of $X$. At $P_0$, we have

$$X_{i1} = g_1 g_{i11} + g_{i1}^2 + \sum_{i \geq 2} (g_i g_{i11} + g_{i1}^2) = g_1 g_{i11} \leq 0,$$

$$X_{ii} = g_1 g_{ii1} + g_{ii}^2 + \sum_{j \geq 2} (g_j g_{ji1} + g_{ji}^2) = g_1 g_{ii1} + \sum_{j \geq 2} g_{ji}^2 \leq 0, \quad 2 \leq i \leq n,$$

(3.15)

and

$$X_{ij} = g_1 g_{ij1} + g_1 (g_{i1}) + \sum_{k \geq 2} (g_k g_{kij} + g_{kij}) = g_1 g_{ij1} + \sum_{k \geq 2} g_{kij} \leq 0, \quad 1 \leq i, j \leq n,$$

so that $g_1 g_{i11} \leq 0$ and $g_1 g_{ii1} \leq 0$.

Hence, the evolution of $X$ at $P_0$ is given by

$$X_t = \left( \frac{g}{\sqrt{T}} \sum_{i=2}^{n} g_{ii} + \frac{g}{T^{3/2}} \sum_{i=2}^{n} g_{i1}^2 + \frac{g_{ii}^2}{T^{3/2}} \right) g_1 g_{i11}$$

$$+ \frac{g_{ii}^2}{T^{3/2}} \sum_{i=2}^{n} g_{i1}^2 + \frac{g}{\sqrt{T}} \sum_{i,j=2}^{n} g_{i1} g_{ii1} - \frac{g}{\sqrt{T}} \sum_{i,j=2}^{n} g_{ij} g_{ij1}$$

$$- 3 g g_1^6 \sum_{i=2}^{n} g_{i1} + \frac{g_{i1}^2}{2 T^{3/2}} \sum_{i,j=2}^{n} (g_{ii1} g_{jj} - g_{i1} g_{j1}^2)$$

(3.16)

which is written as

$$X_t = \left( \frac{g g_{1H}}{T^{3/2}} + \frac{g g_{H}}{\sqrt{T}} + \frac{g_{i1}^2}{T^{3/2}} \right) X_{ii} + \sum_{i=2}^{n} \left( \frac{g g_{1H}}{\sqrt{T}} + \frac{g_{i1}^2}{T^{3/2}} \right) X_{ij}$$

$$+ \frac{-3g g_{i1}^6}{2 T^{3/2}} g_{1H} + \frac{1}{2 T^{3/2}} g_{ii1}^2 \frac{g_{i1}^2}{T^{3/2}} + \frac{-n}{T^{3/2}} g_{ij}^2$$

$$- \sum_{i=2}^{n} \left( \frac{g g_{1H}}{\sqrt{T}} + \frac{g_{i1}^2}{T^{3/2}} \right) g_{i1}^2 + \sum_{i,j=2}^{n} \frac{g g_{i1}}{\sqrt{T}} \sum_{k \geq 2} g_{kij}$$

(3.17)

where $H_c = \frac{1}{g_1} \sum_{i=2}^{n} g_{ii}$ is the mean curvature of the level set $\Gamma_c(t)$ at $P_0$.

The level set $\Gamma_c(t)$ of the convex function $f$ is also convex. We can use local coordinates at $P_0$ such that $g_{ij} = 0$ for $i \neq j$ with $g_{1} = g_{\nu} > 0$ and $g_i = 0$ for $2 \leq i \leq n$. Then, since $X_i = 0$ and $X_{ii} \leq 0$ for all $1 \leq i \leq n$ at $P_0$,

$$X_t = a_{11} X_{i1} + \sum_{i=2}^{n} a_{ii} X_{ii} + \frac{g_{i1}^2}{2 T^{3/2}} (g_{ii1}^2 - \sum_{i=2}^{n} g_{ii1}^2) + \frac{-3g g_{i1}^6}{2 T^{3/2}} g_{1H} + \frac{-n}{T^{3/2}} g_{ij}^2 \leq \frac{g_{i1}^4}{2 T^{3/2}} H_c^2$$

(3.18)

where $a_{11} = \frac{g g_{1H}}{T^{3/2}} + \frac{g g_{H}}{\sqrt{T}} + \frac{g_{i1}^2}{T^{3/2}} \geq 0$ and $a_{ii} = \frac{g}{\sqrt{T}} \sum_{j \neq i} g_{jj} + \frac{g_{i1}^2}{T^{3/2}} \geq 0$ for $2 \leq i \leq n$. 

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In the previous sections, we proved that the speed of the level set is non-degenerate and finite:

$$
\gamma(\theta, t_0) e^{-\frac{t-t_0}{\eta_0}} \leq \gamma(\theta, t) \leq \gamma(\theta, t_0) e^{-\frac{A(t-t_0)}{D_B(B^T)}}
$$

which implies that

$$
\frac{\gamma(\theta, t_0)(e^{-\frac{t-t_0}{\eta_0}} - 1)}{t - t_0} \leq \frac{\gamma(t) - \gamma(t_0)}{t - t_0} \leq \frac{\gamma(t_0)(e^{-\frac{A(t-t_0)}{D_B(B^T)}} - 1)}{t - t_0},
$$

$$
\gamma(t_0) \frac{-A}{|B|^0} \leq \gamma'(t_0) \leq \gamma(t_0) \frac{-A}{D + |B|^T}.
$$

On the other hand, on the level set $\Gamma_e(t)$,

$$
g_r \gamma'(t) + g_t = 0, \quad g_1 = g_r D x_1^r = g_r \frac{x_1}{r} \Longrightarrow \frac{g_1}{g_t} = \frac{x_1}{r} \gamma'(t),
$$

$$
0 < C_1 = \frac{|B|^0}{R^2} \rho \leq \frac{|B|^0}{\gamma^2(x_0)} \leq \frac{D + |B|^T}{A\gamma^2(x_0)} x_1 \leq \frac{D + |B|^T}{A\rho^2} R = C_2
$$

since the initial flat spot $\Sigma_0(0)$ is contained in the ball $B_R$. So we can say that $\frac{g_1}{g_t}$ is of order 1 and we write $\frac{g_1}{g_t} \sim 1$.

Since the level set $\Gamma_e(t)$ is convex by assumption, it holds that at $P_0$

$$
\sum_{i,j=2}^{n} (g_{ii} g_{jj} - g_{ii}^2) = 2 \sum_{2 \leq i < j \leq n} g_{ij} g_{jj} \geq 0
$$

and

$$
C_1 \leq \frac{g_t}{g_1} = \frac{1}{2 g_1 \sqrt{1}} \sum_{i,j=2}^{n} (g_{ii} g_{jj} - g_{ii}^2) + \frac{1}{g_1 \frac{3}{2}} \sum_{i=2}^{n} \left( g(g_{ii} g_{11} - g_{11}^2) + g_{11}^2 g_{ii} \right) \leq C_2.
$$

At $P_0$, $g_{11} = g_{11} = 0$ so we have

$$
C_1 \leq \frac{g_t}{g_1} = \frac{g}{2 g_1 \sqrt{1}} \sum_{i,j=2}^{n} (g_{ii} g_{jj} - g_{ii}^2) + \frac{g_1^2}{1 \frac{3}{2}} H_e \leq C_2,
$$

and

$$
X_t \leq \frac{g_1^4}{2 \frac{3}{2}} \frac{H_e^2}{ \frac{3}{2}} \left( \frac{g_1^2}{1 \frac{3}{2}} H_e \right)^{2} \leq \frac{1}{2} C_2^{2} \frac{3}{2} = \frac{1}{2} C_2^{2} (1 + g^2 g_1^2)^{3/2}.
$$

If $g^2 g_1^2 \leq 1$, then

$$
X_t \leq \sqrt{2} C_2^{2}
$$

and

$$
X(t) \leq X(0) + t \sqrt{2} C_2^{2} \leq X(0) + T \sqrt{2} C_2^{2}.
$$

Otherwise, if $g^2 g_1^2 \geq 1$, then at $P_0$ on $\Gamma_e(t)$

$$
X_t \leq \sqrt{2} C_2^{2} g_1^3 \frac{\gamma_1}{2} = \sqrt{2} C_2^{2} e^3 (2X)^{3/2} = 4 C_2^{2} e^3 X^{3/2}.
$$

Hence, $X(t)$ is bounded above by the solution $Y(t)$ to the ODE $Y_t = 4 C_2^{2} e^3 Y^{3/2}$, $Y(0) = X(0)$:

$$
X(t) \leq Y(t) = \frac{1}{(X(0) - 2 C_2^{2} e^3 t)^2} \leq Y(T) = \frac{1}{(X(0) - 2 C_2^{2} e^3 T)^2}
$$

for time $0 \leq t \leq T$.

Taking $\epsilon > 0$ sufficiently small that $2 C_2^{2} e^3 T \leq \frac{1}{2} X(0)$, we have

$$
X(t) \leq \frac{4}{(X(0))^2}
$$

for time $0 \leq t \leq T$. Hence, we have proved the lemma. \qed
Lemma 3.4. If the level set $\Gamma_\epsilon(t)$ of $g$ is convex, then there exists a constant $C > 0$ such that
\[ |\nabla g| \geq C, \quad \text{on } 0 \leq g(\cdot, t) \leq 1, \quad 0 \leq t \leq T. \tag{3.31} \]

Proof. Let us define a quantity $X = Y - k x \cdot \nabla f = x \cdot \nabla g - k x \cdot \nabla f$, where $k$ is a positive constant that will be determined later. Suppose that at each time $0 \leq t < T$, $X$ attains an interior minimum at $P_0 = (x_0, t)$ so that $X(x_0, t) = \inf \{X(x, t); \ x \in \Sigma, \ 0 \leq g(x, t) \leq 1\}$. We make $x_1 > 0$ and $x_i = 0$ for $i = 2, \ldots, n$.

At $P_0$ on $\Gamma_\epsilon$, we choose local coordinates such that the normal derivative is $g_\nu = g_1$ and the tangential derivatives to the level set $\Gamma_\epsilon$ vanish, in other words, $g_i = 0$ for $i = 2, \ldots, n$, and furthermore $g_{ij} = 0$ for all $i \neq j$. Then we have $\sum_{i,j=2}^n (g_i g_{jj} - g_{ij}^2) = \sum_{i,j=2}^n g_i g_{jj} - \sum_{i=1}^n g_{ii}^2 = \sum_{i,j=2, i \neq j} g_i g_{jj}$ and $\sum_{i=2}^n (g_{ii} g_{11} - g_{i1}^2) = \sum_{i=2}^n g_{ii} g_{11}$.

Then the inequality (3.34) becomes at $P_0$,
\[ C_1 \leq \frac{g_t}{g_1} = \frac{1}{2g_1 \sqrt{T}} g \sum_{i,j=2}^n g_i g_{jj} + \frac{1}{g_1 T^{5/2}} \sum_{i=2}^n (g_{ii} g_{11} + g_{ii}^2) \leq C_2 \tag{3.32} \]
\[ C_1 \leq \frac{g_t}{g_1} = \frac{1}{2g_1 \sqrt{T}} g \bar{R}_{g,2} + \frac{1}{T^{3/2}} g \sum_{i=2}^n g_{ii} \leq C_2, \tag{3.33} \]
where
\[ \bar{R}_{g,2} = \sum_{i,j=2}^n (g_i g_{jj} - g_{ij}^2) + \frac{2}{T} \sum_{i=2}^n (g_{ii} g_{11} - g_{i1}^2). \tag{3.34} \]

At $P_0$, $x \cdot \nabla f = g_\nu \cdot \nabla g = g_1 x_1 + x_1 g_1 = g_1 x_1 g_1$ in our local coordinates where $g_i = 0$ for $i = 2, \ldots, n$. Also, we have $X = x_1 g_1 - k x_1 g_1$ and the first-order derivatives of $X$ are
\[ X_1 = g_1 + x_1 g_{11} - k(x_1 g_{11} + 2 g_{11}) = 0, \]
\[ X_i = g_i + x_1 g_{1i} - k(g_i x_1 g_1 + g g_i + g x_1 g_1) = x_1 g_{1i} - k x_1 g_{1i} = 0 \] for $i = 2, \ldots, n$, and its second-order derivatives at $P_0$ are
\[ X_{11} = x_1 g_{111} + 2 g_{11} - k(3 x_1 g_{111} + 2 g_{11}) + k x_1 g_{11}, \]
\[ X_{ii} = x_1 g_{ii} + 2 g_{ii} - k(2 x_1 g_{ii} + 2 g_{ii}) + k x_1 g_{ii}, \ i = 2, \ldots, n, \tag{3.35} \]
\[ X_{ij} = x_1 g_{ij} + 2 g_{ij} - k(g_{ij} x_1 g_1 + g g_{ij} + g x_1 g_1), \ i, j = 2, \ldots, n. \tag{3.36} \]

Then the evolution equation of $X$ at the point $P_0$ can be written as
\[ X_t = \left( \sum_{i=2}^n g_{ii} x_1 g_{1i} \right) \sum_{i=2}^n (X_{ii} - 2 g_{ii} + k x_1 g_{1i} + g g_{ii}) + \frac{x_1 g_{1i}}{2 \sqrt{T}} \frac{g_{11}}{T^{5/2}} \bar{R}_{g,2} + \frac{k x_1 g_{1i}}{2 \sqrt{T} T^{5/2}} \bar{R}_{g,2} \]
\[ + \sum_{i=2}^n g_{ii} \left( \frac{g_{ii}}{T^{5/2}} \right) \left( X_{11} - 2 g_{11} + 3 x_1 g_{111} + 2 k g_{11}^2 + 2 k g g_{11} \right) \]
\[ + \left( \frac{2 g_{ii}}{T^{5/2}} \right) \left( X_1 - g_{1i} + k g_1 x_1 + k g_{1i} \right) - \left( \frac{1}{2 \sqrt{T}} \bar{R}_{g,2} + \frac{1}{T^{3/2}} g \sum_{i=2}^n g_{ii} \right) k x_1 g_{1i} \]
\[ - \left( \frac{1}{2 T^{1/2}} g \bar{R}_{g,2} + \frac{1}{T^{3/2}} g \sum_{i=2}^n g_{ii} \right) x_1 g g_{11} (g_{11}^2 + g g_{11}) \]
\[ + \left( \frac{1}{2 T^{1/2}} g \bar{R}_{g,2} + \frac{1}{T^{3/2}} g \sum_{i=2}^n g_{ii} \right) k x_{11} g_{1i} (g_{11}^2 + g g_{11}) \]
\[ - \frac{2 x_1 g g_{11} (g_{11}^2 + g g_{11})^2}{T^{5/2}} \sum_{i=2}^n g_{ii} + \frac{2 k x_{11} g_{1i} (g_{11}^2 + g g_{11})^2}{T^{5/2}} \sum_{i=2}^n g_{ii} \] \[ \tag{3.37} \]
which implies that

\begin{equation}
X_t \geq kC_1 x_1 g_1^2 + kC_1 g g_1 + \frac{k x_1 g_1}{I^{3/2}} (g_1^2 + 3 g g_1) \sum_{i=2}^{n} g_i + \frac{k g}{I^{3/2}} (3g_i^2 + g g_1) \sum_{i=2}^{n} g_i \\
- 3C_2 g_1 - C_2 k x_1 g_1^2 - \frac{3C_2}{I} x_1 g g_1 (g_1^2 + g g_1) + \frac{x_1 g_1}{2\sqrt{I}} n_{g,2} \\
+ \frac{C_1}{I} k x_1 g_1^2 (g_1^2 + g g_1) + \frac{2 k x_1 g_1^2}{I^{3/2}} (g_1^2 + g g_1)^2 \sum_{i=2}^{n} g_i
\end{equation}

(3.38)

On the other hand, we have at \( P_0 \)

\[ 0 = X_1 = x_1 (1 - k g) g_1 + g_1 - k (g_1^2 x_1 + g g_1) \]  

(3.39)

so that by letting \( g \leq \frac{1}{2k} \) it holds that

\[ g_1 = \frac{-g_1 + k (g_1^2 x_1 + g g_1)}{x_1 (1 - k g)} \leq \frac{2k (C^2 R + C)}{\rho} \]  

(3.40)

and that

\[ X_t \geq -k(C_2 - C_1) C X - \frac{3C_2}{\rho} X - 3C_2 C^3 X - 3C_2 C \frac{2k (C^2 R + C)}{\rho} X \]

\[ - \left( k(C_2 - C_1) C + \frac{3C_2}{\rho} + (3C_2 - C_1) C^3 + (3C_2 - C_1) C \frac{2k (C^2 R + C)}{\rho} \right) k \beta C \]  

(3.41)

Hence, by Gronwall’s inequality, we obtain

\[ \min_{x \in M} X(x, t) \geq e^{-\alpha t} \left( \min_{x \in M} X(x, 0) - k \beta t \right) \geq \frac{1}{2} e^{-\alpha T} \min_{x \in M} X(x, 0) =: C', \]  

(3.42)

if we take \( k \leq \frac{\min_{x \in M} X(x, 0)}{2 \beta t} \). Then \( Y = x_1 g_1 \geq X \geq C' \) and \( |\nabla g| = g_1 \geq \frac{C'}{\rho} \). We have obtained the lower bound of \( |\nabla g| \).

3.3 Second-order derivative estimates

The purpose of this section is to obtain the optimal bound of the second-order tangential derivatives of \( g \) which gives the optimal decay rates of the second-order derivatives of \( f \). We get estimates on second-order derivatives of \( g \) first, under the assumptions that we have given in the subsection 1.4. We also find a lower bound of \( R_{g,2} \), which is similar to the Aronson-Bénilan inequality \( \Delta u \geq -\frac{\rho}{T} \) for the porous medium equation.

**Lemma 3.5.** Under the assumptions in the subsection 1.4, there exists a constant \( c > 0 \) for which

\[ c \leq g_t \leq c^{-1}. \]  

(3.43)

**Proof.** We have

\[ g_r \dot{\gamma}(\theta, t) + g_t = 0 \]  

(3.44)

so that

\[ g_t = -g_r \dot{\gamma}(\theta, t). \]  

(3.45)

On the other hand, \( g_r \) is bounded, because

\[ g_r = \nabla_x g \cdot \partial_r x = \frac{vg_i}{x_i} \]  

(3.46)

where \( v \) are bounded by Lemma 3.3 and 3.4 and \( x_i \) are bounded by \( R > 0 \) in the subsection 1.4. Also, \( \dot{\gamma}(\theta, t) \) is bounded by two negative constants. From these bounds we get the result. \( \square \)
Lemma 3.6. Under the assumptions in the subsection I.4, there exists a constant $C > 0$ such that
\[
C \leq \frac{1}{2g\sqrt{1 + |\nabla f|^2}} \sum_{i,j=1}^{n} (f_{ii}f_{jj} - f_{ij}^2) \leq C^{-1}. \tag{3.47}
\]

Proof. Recall that, in the coordinates where $g_i = g_1 \delta_{1i}$ so that $f_i = f_1 \delta_{1i}$,
\[
f_t = \frac{1}{\sqrt{1 + |\nabla f|^2}} (1 - \frac{f_j^2}{1 + |\nabla f|^2}) \sum_{i=2}^{n} (f_{ii}f_{jj} - f_{ij}^2) + \frac{1}{2\sqrt{1 + |\nabla f|^2}} \sum_{i,j=2}^{n} (f_{ii}f_{jj} - f_{ij}^2) \tag{3.48}
\]
we have
\[
\frac{1}{2(1 + |\nabla f|^2)^{3/2}} \sum_{i,j=1}^{n} (f_{ii}f_{jj} - f_{ij}^2) \leq f_t \leq \frac{1}{2\sqrt{1 + |\nabla f|^2}} \sum_{i,j=1}^{n} (f_{ii}f_{jj} - f_{ij}^2). \tag{3.49}
\]
Since $f_t = gg_t$, we get the desired result as $g$ tends to 0 near the interface. \hfill \Box

Lemma 3.7. Let us assume the conditions in the subsection I.4. Then there exists a constant $C > 0$ satisfying
\[
0 < \sum_{i \neq j} g_{ii} \leq C.
\]
for sufficiently small value of $g$ near the interface.

Proof. Let us define a quantity
\[
X := \sum_{i,j=1}^{n} (g_{ij}^2 - g_{ij}g_{ji}) + \Delta f = \sum_{i,j=1}^{n} (g_{ij}^2 - g_{ij}g_{ji}) + \sum_{i=1}^{n} (g_{ii}^2 + gg_{ii}). \tag{3.50}
\]
We work at an interior point $P_0$ of the domain, where $X$ takes its supremum. At $P_0$, we take the coordinate system where $g_i = 0$ for $i \neq 1$ and $e_1 = \nu$, the normal direction to $\Gamma_e$, and $g_{ij} = 0$ for $i \neq j$. We assume that $P_0$ is on a level set $\Gamma_e$. Then we have
\[
X = g_{11}^2 \sum_{i=2}^{n} g_{ii} + (g_{11}^2 + gg_{11} + g \sum_{i=2}^{n} g_{ii}). \tag{3.51}
\]
In Lemma 3.3 and 3.4, we have shown $|\nabla g| = g_1$ is bounded from above and below; there exist a constant $c > 0$ such that $0 < c \leq g_1 \leq c^{-1}$ on $g > 0$, $0 \leq t \leq T$. Also, $\Delta f = g_{11}^2 + g \sum_{i=1}^{n} g_{ii} = g_{11}^2 + gg_{11} + g \sum_{i=2}^{n} g_{ii}$ is bounded as $f \in C^{1,1}$. Hence, an upper bound of $X$ will give an upper bound of the tangential laplacian $\DeltaT g = \sum_{i=2}^{n} g_{ii}$.

Differentiating $X$ with respect to space variables at $P_0$, we get for all $1 \leq k \leq n$
\[
0 = X_k = \sum_{i,j=1}^{n} (g_j^2 g_{ik} - g_1 g_{jk}g_{ij} + 2g_1 g_{i1}g_{jk} - 2g_1 g_{ij}g_{ik}) + \sum_{i=1}^{n} (gg_{iik} + 2g_1 g_{i1k} + g_{1i1k}). \tag{3.52}
\]
We have, at $P_0$,
\[
0 = X_1 = gg_{111} + (g_{11}^2 + g) \sum_{i \neq 1} g_{ii} + 2g_1 g_{11} \sum_{i \neq 1} g_{ii} + 3g_1 g_{11} + g_1 \sum_{i \neq 1} g_{ii}, \tag{3.53}
\]
\[
0 = X_k = g_{11}^2 \sum_{i=1}^{n} g_{iik} - g_{11}^2 g_{11k} + \sum_{i=1}^{n} (gg_{iik}) = gg_{11k} + (g_{11}^2 + g) \sum_{i \neq 1} g_{iik},
\]

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and then
\[ \sum_{i \neq j} g_{ii} = -\frac{1}{g + g_1^2} (gg_{111} + 2g_1g_{i1} \sum_{i \neq j} g_{ii} + 3g_1g_{i1} + \sum_{i \neq j} g_{i1}), \]  
(3.54)

\[ \sum_{i \neq j} g_{ii} = -\frac{1}{g + g_1^2} gg_{11k}, \text{ for } k \neq 1. \]

Differentiating \( X \) with respect to time, we get \( X_t \) as follows. We use the indices \( m, p \) instead of \( i, j \) in order to avoid any possible confusions. The right side of the evolution equation of \( X \) will contain spatial derivatives of \( g \) of order no greater than 4, because the equation (3.11) is a second-order equation. Using an index \( 2 \leq m \leq n \), we can simply write the evolution of \( X \) as the following:

\[ X_t = g_1^2 \sum_{m \neq 1} (g_{mm})_t + 2g_1(g_1)_t \sum_{m \neq 1} g_{mm} + (g(g_{11}))_t + g \sum_{m \neq 1} (g_{mm})_t + 2g_1(g_1)_t + gg_{111} + gt \sum_{m \neq 1} g_{mm}. \]
(3.55)

Now, let us consider the second-order derivatives of \( X \). They are

\[ X_{kl} = \sum_{i,j=1}^{n} (g_{j}^2 g_{ikl} - g_{i}g_{j}g_{ijkl}) \]
\[ + \sum_{i,j=1}^{n} (2g_{j}g_{ij}g_{ik} - 2g_{i}g_{jl}g_{ij} + 2g_{j}g_{jk}g_{i1} - 2g_{j}g_{ik}g_{jl} + 2g_{j}g_{i1}g_{jkl} - 2g_{j}g_{i1}g_{ikl} + 2g_{j}g_{ik}g_{j} - 2g_{j}g_{i}g_{ij}) \]
\[ + \sum_{i=1}^{n} (g_{g_{ik1} + g_{k}g_{i1} + g_{g_{i1k} + g_{ik1} + g_{ik}g_{i1} + g_{ik}g_{i1}}) \]
(3.56)

At \( P_0 \) under the coordinates \( g_{k} = 0 \) for \( k \neq 1 \) and \( g_{ij} = 0 \) for \( i \neq j \),

\[ X_{kl} = g_1^2 \sum_{i=2}^{n} g_{ikl} + 2g_1g_{il} \sum_{i=1}^{n} g_{ikl} + 2g_1g_{ik} \sum_{i=1}^{n} g_{iil} + 2g_1 \sum_{i=1}^{n} g_{ii}g_{ikl} + 2g_{k}g_{ik} \sum_{i=1}^{n} g_{ii} \]
\[ + (g \sum_{i=1}^{n} g_{iikl} + 2g_1g_{ikl} + g_{k} \sum_{i=1}^{n} g_{iik} + g_{i} \sum_{i=1}^{n} g_{iik} + g_{kl} \sum_{i=1}^{n} g_{iil} + 2g_{k}g_{ikk}) \]
(3.57)

In particular,

\[ X_{11} = g_1^2 \sum_{i=2}^{n} g_{i111} + 4g_1g_{i1} \sum_{i=1}^{n} g_{i1} + 2g_1(\sum_{i=2}^{n} g_{ii} - 2g_{11})g_{111} + 2g_{11}^2 \sum_{i=2}^{n} g_{ii} \]
\[ + (g \sum_{i=1}^{n} g_{i111} + 2g_1g_{i111} + 2g_1 \sum_{i=1}^{n} g_{i11} + g_{11} \sum_{i=1}^{n} g_{ii} + 2g_{11}^2), \]
(3.58)

and for \( k \neq 1 \)

\[ X_{kk} = g_1^2 \sum_{i=2}^{n} g_{i1kk} + 2g_1(\sum_{i=1}^{n} g_{ii} - g_{kk})g_{1kk} + 2g_{kk}^2 \sum_{i=1}^{n} g_{ii} \]
\[ + (g \sum_{i=1}^{n} g_{i1kk} + 2g_1g_{i1kk} + g_{kk} \sum_{i=1}^{n} g_{ii} + 2g_{kk}^2), \]
(3.59)
For \( k \neq 1 \)
\[
X_{1k} = g_1^2 \sum_{i=2}^{n} g_{i1k} + 2g_1g_{11} \sum_{i=1}^{n} g_{iik} + 2g_1(\sum_{i=1,k}^{n} g_{ii} - g_{11})g_{11k} \\
+ (g \sum_{i=1}^{n} g_{i1k} + 2g_1g_{11k} + g_1 \sum_{i=1}^{n} g_{iik}),
\]  
(3.60)

and for \( 2 \leq k, l \leq n \) such that \( k \neq l \)
\[
X_{kl} = g_1^2 \sum_{i=2}^{n} g_{iikl} + 2g_1 \sum_{i \neq 1,k,l} g_{iigkl} + (g \sum_{i=1}^{n} g_{iikl} + 2g_1g_{1kl}).
\]  
(3.61)

In our coordinate system at \( P_0 \), the quantity \( J = |\nabla g|^2 + g \) becomes \( J = g_2^2 + g \). Then the term \( \frac{1}{g} \) goes to \(+\infty\) as \( g \) goes to \( 0^+ \) and \( \Delta f = g_2^2 + gg_{11} + g \sum_{i \neq 1} g_{ii} \) is bounded so that \( 0 \leq f_{11} = gg_{11} + g_1^2 \leq C \) and \( 0 \leq f_{ii} = gg_{ii} \leq C \) for \( i \neq 1 \). We have the relations \( 0 = X_m = \sum_{i \neq 1} (g_1^2 + g)g_{iim} + gg_{11m} + g_{11}(1 + 2g_{11}) \sum_{i \neq 1} g_{ii} + 3g_{11}, \) so that
\[
J \sum_{i \neq 1} g_{ii} = -(gg_{111} + (g_1 + 2g_{11}) \sum_{i \neq 1} g_{ii} + 3g_{11}),
\]  
(3.62)

\[
J \sum_{i \neq 1} g_{ik} = -gg_{11k}, \text{ for } k \neq 1.
\]

Then we have slightly modified relations as well.
\[
g_{11}g_{i1} \sum_{i \neq 1} g_{ii}g_{111} = -Jg_{11g1} \sum_{j \neq 1} g_{jj} \sum_{i \neq 1} g_{ii} - g_{11g1} \sum_{j \neq 1} g_{jj}((g_1 + 2g_{11}) \sum_{i \neq 1} g_{ii} + 3g_{11}),
\]  
(3.63)

\[
g_{11}g_{111} = -Jg_1 \sum_{i \neq 1} g_{ii} - g_1((g_1 + 2g_{11}) \sum_{i \neq 1} g_{ii} + 3g_{11}).
\]

We use the finite and nondegenerate speed of the level set of \( g \) to bound the term \( \left( \sum_{i \neq 1} g_{ii}^2 \right) \) above. Previously, we obtained the result that for some sufficiently small constant \( 0 < C < 1 \),
\[
C \leq \frac{g_i}{g_1} = \frac{1}{2g_1 \sqrt{T} g} \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) + \frac{1}{g_1 \sqrt{T} / 2}(gg_{11} \sum_{i \neq 1} g_{ii} + g_1^2 \sum_{i \neq 1} g_{ii}) \leq C^{-1}.
\]  
(3.64)

and we also see that, by the boundedness of \( g_1 \),
\[
C \leq \frac{1}{2 \sqrt{T} g} \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) + \frac{1}{g_1 \sqrt{T} / 2}(gg_{11} \sum_{i \neq 1} g_{ii} + g_1^2 \sum_{i \neq 1} g_{ii}) \leq C^{-1}.
\]  
(3.65)

This also implies that
\[
\frac{1}{2 \sqrt{T} g} R_{g,2} + \frac{1}{g_1 \sqrt{T} / 2} g_{11}^2 \sum_{i \neq 1} g_{ii} \leq C^{-1},
\]  
(3.66)

\[
\frac{1}{2 \sqrt{T} g} R_{g,2} + \frac{1}{g_1 \sqrt{T} / 2} g_{11}^2 \sum_{i \neq 1} g_{ii} \leq C^{-1},
\]  
(3.67)

and
\[
0 \leq \frac{1}{g_1 \sqrt{T} / 2}(gg_{11} + g_1^2) \sum_{i \neq 1} g_{ii} \leq C^{-1}.
\]  
(3.68)
Since $\Delta f$ is bounded and $X = g_1^2 \sum_{i \neq 1} g_{ii} + \Delta f$ in our coordinates at $P_0$, without loss of generality we may assume that

$$C^{-1} \leq \frac{1}{2f^{3/2}} g_1^2 \sum_{i \neq 1} g_{ii} \leq X \leq 2g_1^2 \sum_{i \neq 1} g_{ii},$$

(3.69)

because, if $\frac{1}{2f^{3/2}} g_1^2 \sum_{i \neq 1} g_{ii} \leq C^{-1}$, then $X$ is bounded above and we get the upper bound of $\sum_{i \neq 1} g_{ii}$, which is our goal.

So we have

$$0 \leq \frac{1}{f^{3/2}} g_1^2 \sum_{i \neq 1} g_{ii} \leq C^{-1} - \frac{1}{2 \sqrt{f}} g \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) - \frac{1}{f^{3/2}} g g_{11} \sum_{i \neq 1} g_{ii}$$

and

$$\frac{1}{2 \sqrt{f}} g \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) + \frac{1}{f^{3/2}} g g_{11} \sum_{i \neq 1} g_{ii} \leq C^{-1} - \frac{1}{f^{3/2}} g_1^2 \sum_{i \neq 1} g_{ii} \leq - \frac{1}{2f^{3/2}} g_1^2 \sum_{i \neq 1} g_{ii}.$$  

(3.70)

(3.71)

Hence, we get

$$0 \leq \frac{1}{2f^{3/2}} g_1^2 \sum_{i \neq 1} g_{ii} \leq C^{-1} - \frac{1}{2 \sqrt{f}} g \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) - \frac{1}{f^{3/2}} g g_{11} \sum_{i \neq 1} g_{ii},$$

(3.72)

and

$$0 \leq \frac{g_1^2}{g} \sum_{i \neq 1} g_{ii} \leq -I \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) - 2g_{11} \sum_{i \neq 1} g_{ii} = -IR_{g,2} + 2g_1^2 g_{11} \sum_{i \neq 1} g_{ii}.$$  

(3.73)

In particular, since the level set of $g$ is convex, we get

$$2g_{11} \sum_{i \neq 1} g_{ii} \leq -I \sum_{2 \leq i,j} (g_{ii}g_{jj} - g_{ij}^2) \leq 0,$$

(3.74)

$$IR_{g,2} \leq 2g_1^2 g_{11} \sum_{i \neq 1} g_{ii} \leq 0$$

(3.75)

so that $g_{11} \leq 0$ and $R_{g,2} \leq 0$.

As $g$ tends to zero, $LX$ at $P_0$ becomes

$$LX := X_t - a_{ij} D_{ij} X = -(31J - 20gI) \frac{1}{20Jf^{3/2}} g_1^3 g_{11}^2 - \frac{1}{f^{3/2}} (2gJ^2 + g^2(4J - Ig)) \sum_{i \neq 1} g_{ii}^2$$

$$- \frac{1}{J \sqrt{f}} g \sum_{i,j,k \neq 1} (J g_{ijk})^2 - \frac{1}{J \sqrt{f}} g \sum_{i,j,k \neq 1} (J g_{ijk})^2 - \frac{2}{f^{3/2}} J g \sum_{i,j \neq 1, i \neq j} g_{ij}^2 - \frac{2}{f^{3/2}} J g \sum_{i,j \neq 1, i \neq j} g_{ij}^2$$

$$- \frac{1}{20Jf^{3/2}} g \sum_{i \neq 1} (g_{ii} - \frac{20}{I} g_1 g g_{ii})^2 + (g_{11} - \frac{5}{J} g_1 g^2 R_{g,2})^2 + (g_{11} - \frac{10}{J} g_1 g g_{11} g_{jj})^2$$

$$+ (g_{11} + \frac{90}{I} g_1 (g_{11} + g_{11}^2) g \sum_{j \neq 1} g_{ij})^2 + (g_{11} + \frac{20}{J} g_1 g \sum_{i \neq 1} g_{ii})^2 + (g_{11} - \frac{60}{J} g_1 g g_{11} g_{ij})^2$$

$$+ (g_{11} + 20g \sum_{i \neq 1} g_{ii})^2 + (g_{11} + 60g g_{11})^2 + (g_{11} - \frac{40}{J} g_1 g g_{11} \sum_{i \neq 1} g_{ii})^2$$

$$- \frac{1}{8f^{3/2}} J g \sum_{i \neq 1} (g_{ii} + \frac{8}{I} g_1 g g_{ii})^2 + (g_{ii} - \frac{24}{J} g_1 g_{ii})^2 + (g_{ii} + \frac{24}{J} g g_{ii} g_{ii})^2$$

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\[ + (g_{ii} + 2g_1 g_2 R_{g,2})^2 + (g_{ii} + 4 J g^2 \sum_{j \neq 1} g_{jj})^2 + (g_{ii} - 4 J g^2 g^2_1 \sum_{j \neq 1} g_{jj})^2 + (g_{ii} + 16 J g_1 g_2)^2 \\
+ (g_{ii} + 8 J g_1 g_2 g_1 \sum_{j \neq 1} g_{jj})^2 + (g_{ii} - 16 J g_1 \sum_{j \neq 1} g_{jj} g_{ii})^2 + (g_{ii} - 16 J g_1 g_1 g_{ii})^2 \\
+ (g_{ii} + 24 J g_1 g_{ii})^2 + (g_{ii} - 8 J g_1 g_2 g_1 g_{ii})^2 + (g_{ii} + 16 J g_2 \sum_{j \neq 1} g_{jj} g_{ii})^2 + (g_{ii} - 16 J g_1 g_2 g_2 g_{ii})^2 \\
+ \left( \frac{1}{f^{3/2}} O(1) + \frac{n - 1}{8 f^{1/2}} J O(g) + \frac{1}{2 f^{1/2}} (g) + \frac{5}{2 f^{1/2}} O(g^2) \right) + \frac{1}{2 f^{1/2}} ((J + 2 g_1^2) \sum_{i \neq 1} g_{ii} + O(1)) R_{g,2} \\
+ \left( \frac{4}{f^{1/2}} O(1) g_{ii} + \frac{1}{J f^{3/2}} (g_1^2 (g_1 g_{ii}) (180 + 72(n - 1)) - 2 J^2 (g^3 \sum_{i \neq 1} g_{ii}) g_{ii} - \frac{3}{f^{3/2}} (g_1 \sum_{j \neq i} g_{jj}) g_{ii} \right) \\
+ \left( \frac{2}{f^{1/2}} O(1) \sum_{i \neq 1} g_{ii} + \frac{1}{2 f^{1/2}} O(1) (g \sum_{i \neq 1} g_{ii}^2) + \frac{1}{2 f^{1/2}} O(1) (g^3 \sum_{i \neq 1} g_{ii}^4) \\
- \frac{1}{f^{3/2}} g_1^2 (4 + J) (-g_{ii}) \sum_{i \neq 1} g_{ii} - \frac{4}{f^{1/2}} g_1^2 (-g_{ii}) \sum_{i \neq 1} g_{ii}^2 - \frac{1}{2 f^{1/2}} (g_1 \sum_{i \neq 1} g_{ii}) \sum_{i \neq 1} g_{ii}^2 \\
- \frac{1}{f^{3/2}} f_{ii} (2 \sum_{i \neq 1} g_{ii}^2 + (\sum_{i \neq 1} g_{ii})^2) - \frac{2}{2 f^{1/2}} g_1^2 (\sum_{i \neq 1} g_{ii} - \frac{1}{2 f^{1/2}} (g_1 \sum_{i \neq 1} g_{ii}^2 + 2 \sum_{i \neq 1} g_{ii}^2) \right) \\
+ \left( \left( \frac{1}{f^{3/2}} O(1) (\sum_{i \neq 1} g_{ii})^2 + \frac{2}{f^{1/2}} g (\sum_{i \neq 1} g_{ii}^3) + \frac{1}{2 f^{1/2}} O(1) \sum_{i \neq 1} g_{ii}^2 \\
+ \frac{2}{f^{1/2}} (I g_1 g_{ii} - J) \sum_{i \neq 1} g_{ii}^3 - \frac{2}{f^{1/2}} g (\sum_{i \neq 1} g_{ii}^4) - \frac{2}{f^{1/2}} J \sum_{i \neq 1} g_{ii} \sum_{j \neq 1} g_{jj} \right) \\
+ \left( \frac{1}{f^{3/2}} O(1) g_{ii} (\sum_{i \neq 1} g_{ii} + \frac{2}{f^{1/2}} J g_{ii} (\sum_{i \neq 1} g_{ii}^2 + \frac{4}{f^{1/2}} g_1^2 (\sum_{i \neq 1} g_{ii})^2) \right) \\
+ \left( \left( \frac{1}{f^{1/2}} O(1) g_{ii} (\sum_{i \neq 1} g_{ii}^2 + \frac{1}{f^{1/2}} O(1) (\sum_{i \neq 1} g_{ii})^2) - \frac{1}{f^{3/2}} g g_{ii} g_{ii} (\sum_{j \neq 1} g_{ii})^3 \\
+ \left( \frac{1}{f^{3/2}} O(1) \sum_{i \neq 1} g_{ii} \sum_{j \neq i} g_{jj} + \frac{32}{f^{1/2}} g^2 g g_{ii} (\sum_{i \neq 1} g_{ii}^4) + \frac{4}{f^{1/2}} O(1) \sum_{i \neq 1} g_{ii}^3 - \frac{1}{f^{3/2}} g g_{ii} g_{ii} (\sum_{j \neq 1} g_{ii})^3 \right) \leq C \right) \\
\text{for some uniform constant } C > 0, \text{ hence } X_i \text{ satisfies} \\
X_i \leq LX_i \leq C \tag{3.76} \]

at its maximum point } \text{ at each time, so } X \text{ is bounded. Consequently, } \sum_{i \neq 1} g_{ii} \text{ is bounded above.} \]

From the inequality \[\text{[3.77]}, \text{ it holds that for some constant } C > 0 \]
\[\sum_{i,j} (f_{ii} f_{jj} - f_{ij}^2) = 2 \sum_{i \neq 1} (f_{ii} f_{ii} - f_{ii}^2) + \sum_{i, j \neq i} (f_{ii} f_{jj} - f_{ij}^2) \geq C g \tag{3.77} \]
which yields
\[\left( \sum_{i \neq 1} f_{ii} \right)^2 + 2 f_{ii} \sum_{i \neq 1} f_{ii} = (\sum_{i \neq 1} f_{ii} + 2 f_{ii}) g \sum_{i \neq 1} g_{ii} \geq C g. \tag{3.78} \]
As $\Delta f$ is bounded, there exists some constant $\tilde{C} > 0$ such that
\begin{equation}
\sum_{i \neq 1} g_{ii} \geq \tilde{C}, \tag{3.79}
\end{equation}
which implies the following corollary.

**Corollary 3.7.1.** Let us assume the conditions in the subsection 1.4. Then there exists a constant $C > 0$ satisfying
\begin{equation}
C \leq \sum_{i \neq 1} g_{ii} \leq C^{-1}. \tag{3.80}
\end{equation}
for sufficiently small value of $g$ near the interface.

We can also examine the bounds for the second-order derivatives of $f$.

**Corollary 3.7.2.** Let us assume the conditions in the subsection 1.4. Then there exists a constant $C > 0$ satisfying
\begin{equation}
C \leq f_{11}, \frac{1}{g} \sum_{i \neq 1} f_{ii} \leq C^{-1} \text{ and } \sqrt{\frac{1}{g} \sum_{i \neq 1} f_{11}^2} \leq C^{-1}. \tag{3.81}
\end{equation}
for sufficiently small value of $g$ near the interface.

**Proof.** From the inequality (3.80), we see that
\begin{equation}
C \leq \frac{1}{g} \sum_{i \neq 1} f_{ii} \leq C^{-1}. \tag{3.82}
\end{equation}
With the convexity of $f$ and the boundedness $0 \leq f_{11} \leq C^{-1}$, and the inequality (3.78) implies
\begin{equation}
\sum_{i \neq 1} f_{11}^2 \leq \sum_{i \neq 1} f_{ii} \leq C^{-1} g. \tag{3.83}
\end{equation}
Thus
\begin{equation}
\sqrt{\frac{1}{g} \sum_{i \neq 1} f_{11}^2} \leq C^{-1}. \tag{3.84}
\end{equation}
Since $C g^2 \leq \left( \sum_{i \neq 1} f_{ii} \right)^2 \leq C^{-1} g^2$ because of the inequality (3.82) as $g$ tends to 0 near the interface, we see that $f_{11} \geq C$ from the inequality (3.78).}

Let $P = (x, y, t) \in \Omega(t) = \{ x \in \mathbb{R}^n; |\nabla f| < \infty \}$. Then we can define the set $\Omega_P(t) = \{ (x, y, t) \in \Omega(t); f(x, y, t) \leq f(P) \}$ and the level curve $\Gamma_P(t) = \partial \Omega_P(t)$ for $P$. Note that the interface is $\Gamma(t) = \partial \Omega(t)$. Let $\nu(P)$ be the normal vector to $\Gamma_P(t)$ at $P$ and $P$ be the position vector of $P$ with respect to the origin $O$. Then, using the condition that the disk $B(0, \rho_0)$ is always contained in the interior of the interface $\Gamma(t)$, we get the following result.

**Lemma 3.8.** Let us assume the conditions in the subsection 1.4. Then we have
\begin{equation}
P \cdot \nu(P) \geq \rho_0, \text{ for any } P \in \Omega(t), \ 0 \leq t \leq T. \tag{3.85}
\end{equation}
Let $P_0 = (x, y, t_0)$ be an arbitrary point on the interface $\Gamma(t_0)$ with $0 < t_0 \leq T$. We define its position vector $P_0$ and normalize it with $n_0 = \frac{P_0}{|P_0|}$. Then we obtain the following.

**Lemma 3.9.** Let us assume the conditions in the subsection 1.4. Then there exists positive constants $\eta$ and $\gamma$ such that
\begin{equation}
n_0 \cdot \nu(P) \geq \gamma, \text{ for any } P \in \Omega(t), \ |P - P_0| \leq \eta, \ 0 \leq t \leq t_0. \tag{3.86}
\end{equation}
Proof. Note that \(|\mathbf{P}| \leq C\) for some constant \(C > 0\) which depends on initial data. With the previous lemma, it follows that

\[
\frac{1}{|\mathbf{P}|} \mathbf{P} : \nu(P) \geq C^{-1}\rho_0.
\] (3.87)

Setting \(\gamma = C^{-1}\rho_0/2\) and choosing \(\eta\) sufficiently small depending on \(\rho_0\) and \(\gamma\), we get the desired result.

Consequently, the following result holds for the derivatives of \(g\) and \(f\) in the direction \(n_0 = \frac{\mathbf{P}_0}{|\mathbf{P}_0|}\) for any point \(P_0 \in \Gamma(t)\) on the interface.

**Lemma 3.10.** There exist positive constants \(C\) and \(\eta\) depending only on initial data and the constant \(\rho_0\), satisfying

\[
C \leq g_{n_0} \leq C^{-1},
\] (3.88)

\[
C \leq f_{n_0n_0} \leq C^{-1},
\] (3.89)

for all \(P \in \Omega(t)\), \(|P - P_0| \leq \eta, 0 \leq t \leq t_0\).

**Proof.** Let \(\tau(P)\) be the unit vector in the direction of the tangential projection of \(n_0\) and \(\theta\) be the angle between \(n_0\) and the outward normal \(\nu(P)\), for \(P = (x, y, t) \in \Omega(t)\). Then we get expressions

\[
g_{n_0} = \sin \theta \tau + \cos \theta \nu,
\] (3.90)

\[
f_{n_0n_0} = \cos^2 \theta f_{\nu\nu} + 2 \cos \theta \sin \theta f_{\nu\tau} + \sin^2 \theta f_{\tau\tau}.
\] (3.91)

By Lemma 3.9

\[
\cos \theta = n_0 \cdot \nu \geq \gamma > 0
\] (3.92)

for any \(P \in \Omega(t)\) with \(|P - P_0| \leq \eta\). Hence, the desired result follows by the following bounds from Corollary 3.7.2:

\[
C \leq f_{\nu\nu}, \quad \frac{1}{g} f_{\tau\tau} \leq C^{-1} \text{ and } \frac{1}{\sqrt{g}} |f_{\tau\nu}| \leq C^{-1}
\] (3.93)

provided that \(\eta\) is sufficiently small.

Let us consider \(R_{g,2}\) now. In our coordinates where \(g_i = 0\) for \(i \neq 1\) and \(g_{ij} = 0\) for \(i \neq j\), we have

\[
R_{g,2} = \left( \sum_{i \neq 1} g_i \right)^2 - \left( \sum_{i \neq 1} g_i^2 \right) + 2g_{11} \sum_{i \neq 1} g_i
\]

\[
= \left( \sum_{i \neq 1} g_i + 2g_{11} \right) \sum_{j \neq 1} g_{jj} - \sum_{i \neq 1} g_{ii}^2.
\] (3.94)

First, from the control of the speed of the level set, we have

\[
2g_{11} \sum_{i \neq 1} g_i \leq -I \sum_{2 \leq i, j} \left( g_{ii}g_{jj} - g_{ij}^2 \right) \leq -I \left( \sum_{i \neq 1} g_{ii} \right)^2,
\] (3.95)

implying that

\[
R_{g,2} = \left( \sum_{i \neq 1} g_i \right)^2 + 2g_{11} \sum_{i \neq 1} g_i - \sum_{i \neq 1} g_{ii}^2
\]

\[
\leq (1 - I) \left( \sum_{i \neq 1} g_i \right)^2 - \sum_{i \neq 1} g_{ii}^2 = -g^2g_{11} \left( \sum_{i \neq 1} g_{ii} \right)^2 - \sum_{i \neq 1} g_{ii}^2
\] (3.96)

\[
\leq -g^2C^4.
\]

Then a natural question arises. Does \(R_{g,2}\) a uniform lower bound? The answer is yes.
Lemma 3.11. Let us assume the conditions in the subsection \[ \text{Lemma 3.4} \] Then there exists a uniform constant \( C > 0 \) satisfying

\[
R_{g,2} \geq -C
\]  

for any sufficiently small value of \( g \) near the interface.

Proof. To analyze \( R_{g,2} \), we define a quantity

\[
X = \frac{2 \sum_{i \neq 1} (g_{11} g_{ii} - g_{i1}^2)}{\sum_{i,j} (g_i^2 g_{jj} + g_j^2 g_{ii} - 2 g_i g_j g_{ij})} + \exp \left(b \sum_i g_i^2\right)
\]

(3.98)
on \( \Omega(t), \ 0 \leq t \leq T \).

Suppose that \( X \) gets minimum at an interior point \( P_0 = P_0(t) \) of \( \Omega(t) \). At \( P_0 \), we choose a coordinate system where \( g_i = 0 \) for \( i \neq 1 \) and \( g_{ij} = 0 \) for \( i \neq j \). Then

\[
X = \frac{g_{11}}{g_1^2} + \exp \left(b g_1^2\right),
\]

(3.99)
and we take \( b > 0 \) sufficiently large that \( X > 0 \) at time \( t = 0 \). We will later determine the value of \( b \) more precisely.

The quantity \( X \) evolves in time as

\[
X_t = \frac{1}{g_1^2} \sum_{k \neq 1} g_{kk} \left( g_{11} \sum_{i \neq 1} g_{ii} + \sum_{i \neq 1} g_{ii} g_{11} - \frac{\sum_{i \neq 1} g_{11} g_{ii}}{g_1^2 \sum_{k \neq 1} g_{kk}} \left( g_{11}^2 \sum_{i \neq 1} g_{ii} + 2 g_1 \sum_{i \neq 1} g_{ii} g_{11} \right) \right)
+ 2 b g_1 g_{11} \exp \left( b g_1^2 \right)
\]

\[
= \frac{1}{g_1^2} g_{11} + \left( 2 b g_1 \exp \left( b g_1^2 \right) - \frac{2 g_{11}}{g_1^2} \right) g_{11},
\]

(3.100)

At \( P_0 \)

\[
0 = X_1 = \frac{1}{g_1^2} g_{111} + \left( -\frac{2 \sum_{i \neq 1} g_{111} g_{ii}}{2 g_1^2 \sum_{k \neq 1} g_{kk}} + \frac{1}{g_1^2 \sum_{k \neq 1} g_{kk}} g_{11} \right) \sum_{i \neq 1} g_{ii1} + \left( 2 b g_1 \exp \left( b g_1^2 \right) - \frac{2 \sum_{i \neq 1} g_{111} g_{ii}}{g_1^2 \sum_{k \neq 1} g_{kk}} \right) g_{111}
\]

\[
= \frac{1}{g_1^2} g_{111} + \left( 2 b g_1 \exp \left( b g_1^2 \right) - \frac{2 g_{11}}{g_1^2} \right) g_{111}
\]

(3.101)

and for \( m \neq 1 \)

\[
0 = X_m = \frac{1}{g_1^2} g_{111m} + \left( -\frac{2 \sum_{i \neq 1} g_{111m} g_{ii}}{2 g_1^2 \sum_{k \neq 1} g_{kk}} + \frac{1}{g_1^2 \sum_{k \neq 1} g_{kk}} g_{111} \right) \sum_{i \neq 1} g_{im} = \frac{1}{g_1^2} g_{111m}
\]

(3.102)
so that \( g_{111} = -\left( -2 b g_1^3 \exp \left( b g_1^2 \right) + \frac{2 g_{11}}{g_1} \right) g_{111} \) and \( g_{111m} = 0 \) for \( m \neq 1 \).

Accordingly,

\[
X_{11} = \frac{1}{g_1^2} g_{11111} + \left( 2 b g_1 \exp \left( b g_1^2 \right) - \frac{6 g_{11}}{g_1} \right) g_{111} + \left( 4 b^2 g_1^2 + 2b \exp \left( b g_1^2 \right) + \frac{6 g_{11}}{g_1^2} \right) g_{1111}
\]

(3.103)
and for $m \neq 1$

$$X_{mm} = \frac{1}{g_1}g_{mm11} - \frac{2}{g_1^2} \sum_{k \neq 1} \sum_{i \neq 1} g_{ik}^2 g_{mk} g_{im1}$$

$$+ \frac{2bg_1g_{mm11}}{g_1} \exp(bg_1^2) - \frac{2g_{11}}{g_1}g_{mm11} + \frac{4g_{11}}{g_1^2} \sum_{k \neq 1} g_{kk} g_{mm} g_{mm1}$$

$$+ \frac{2bg_{mm}^2}{g_1} \exp(bg_1^2) - \frac{2g_{11}}{g_1} \sum_{i \neq 1} g_{ii} g_{mm}^2 - g_{mm}^3.$$  \hfill (3.104)

Let us define an operator

$$LX := X_t - \frac{1}{f^2} \sum_{j \neq 1}^2 g_{i1}X_t - \frac{1}{f^2} \sum_{m \neq 1} \sum_{j \neq 1} (I_g \sum_{j \neq 1, m} g_{jj} + g_{i1} + g_1^2) X_{mm}$$

$$- \frac{2}{f^2} \sum_{j \neq 1}^2 g_{i1}X_t - \frac{3}{f^2} \sum_{j \neq 1} g_{jj} X_1$$

$$- \frac{1}{2f^2} \left( - I_g^2 (g_{j1} R_{2,2} + 4g_{i1} g_{11} \sum_{j \neq 1} g_{jj}) - 6g_{11}^2 \sum_{j \neq 1} g_{jj} + 6g_{11}^2 g_{i1} \sum_{j \neq 1} g_{jj} \right) X_1$$

$$+ \frac{3}{f^2} (I_g \sum_{j \neq 1} g_{jj} + g_{i1} + g_1^2) g_{j1}^2 g_{11} X_1 - \frac{3}{f^2} \sum_{j \neq 1} g_{jj} \sum_{j \neq 1} g_{jj} + g_{i1} (2g_{j1}^2 + I) g_{11} X_1$$

$$+ \frac{3}{f^2} (I_g \sum_{j \neq 1} g_{jj} + g_{i1} + g_1^2) g_{j1}^2 g_{11} X_1 - \frac{3}{f^2} \sum_{j \neq 1} g_{jj} \sum_{j \neq 1} g_{jj} + g_{i1} (2g_{j1}^2 + I) g_{11} X_1$$

$$- \frac{3}{f^2} g_{11} g_{11} X_1 - \frac{3}{f^2} \sum_{j \neq 1} g_{jj} g_{j1}^2 g_{11} X_1 + \frac{3}{f^2} \sum_{j \neq 1} g_{jj} g_{j1}^2 g_{11} X_1 + \frac{3}{f^2} \sum_{j \neq 1} g_{jj} g_{j1}^2 g_{11} X_1$$

$$- \left( 2bg_1^2 \exp(bg_1^2) - \frac{2g_{11}}{g_1} \right) \frac{1}{f^2} \sum_{j \neq 1} g_{jj} X_1 + \frac{2}{f^2} \sum_{m \neq 1} g_{11}^2 \sum_{n \neq 1} X_{mm}^2.$$  \hfill (3.105)

At the minimum point $P_0(t)$ of $X$, $X_{11} \geq 0$ and $X_{mm} \geq 0$ for all $m \neq 1$ so that

$$\frac{1}{f^2} \sum_{i \neq 1} g_{i1} X_1 + \frac{1}{f^2} \sum_{m \neq 1} \sum_{j \neq 1} (I_g \sum_{j \neq 1, m} g_{jj} + g_{i1} + g_1^2) X_{mm} \geq 0$$  \hfill (3.106)

and $X_i = 0$ for all $i$ at the minimum point $P_0 = P_0(t)$ of $X$. So $X_t \geq LX$ at $P_0$. Note that, since $g_{11} \leq 0$ and $R_{g,2} \leq 0$, the part

$$P = \frac{1}{g_1} \frac{1}{2f^2} (IR_{g,2} + 2g_{11}^2 g_{11}^2 \sum_{j \neq 1} g_{jj} + 6g_{11}^2 \sum_{j \neq 1} g_{jj} + 6g_{11}^2 \sum_{j \neq 1} g_{jj} - 10g_{11}^2 \sum_{j \neq 1} g_{jj} + 18g_{11}^2 \sum_{j \neq 1} g_{jj} g_{11})$$

$$+ \frac{1}{g_1} \frac{1}{2f^2} g_{11}^2 (4I_g \sum_{j \neq 1} g_{jj} - I_g^2 R_{g,2} + 4I_g^2 g_{11} \sum_{j \neq 1} g_{jj} - 2(gg_{11} R_{g,2} + 4g_{i1} g_{11} \sum_{j \neq 1} g_{jj} g_{11})$$

$$- 10g_{11}^2 \sum_{j \neq 1} g_{jj} + 18g_{11}^2 \sum_{j \neq 1} g_{jj} g_{11})$$

$$- \frac{5}{2f^2} (4I_g \sum_{j \neq 1} g_{jj} - I_g g_{11} R_{g,2} + 4g_{i1} g_{11} \sum_{j \neq 1} g_{jj} - 6g_{11}^2 \sum_{j \neq 1} g_{jj} + 6g_{11}^2 \sum_{j \neq 1} g_{jj} g_{11} g_{11} g_{11})$$

$$+ \frac{5}{2f^2} g_{11}^2 (-I_g R_{g,2} + 6g_{11}^2 g_{11} \sum_{j \neq 1} g_{jj} + 15) \frac{1}{f^2} \sum_{j \neq 1} g_{jj} g_{11} + \frac{1}{f^2} R_{g,2} g_{11}$$

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\[ \begin{align*}
\frac{1}{T_{5/2}^2} & \left(- 4g_1^2g_{11} \sum_{j \neq 1} g_{jj} + g_1g_2 \left( \sum_{i \neq 1} g_{ii} \right)^2 - g_1^2 \sum_{i \neq 1} g_{ii}^2 \right) + \frac{1}{T_{3/2}^2} b g_1^2 \left( \sum_{i \neq 1} g_{ii} \right)^2 - \sum_{i \neq 1} g_{ii}^2 \left( \sum_{i \neq 1} g_{ii} \right) \exp \left(b g_1^2\right) \\
& - \frac{5}{2T_{7/2}^2} \left(I \left( \sum_{i \neq 1} g_{ii} \right)^2 - I \sum_{i \neq 1} g_{ii}^2 - 6g_2^2g_{11} \sum_{j \neq 1} g_{jj} \right) g_1^2g_{11} \\
& = \frac{1}{g_1^2} \frac{1}{T_{5/2}^2} \left(I R_{g,2} + O(g)\right) g_{11} + \frac{1}{g_1^2} \frac{1}{T_{7/2}^2} \left(I \sum_{j \neq 1} g_{jj} + O(g)\right) - \frac{5}{2T_{7/2}^2} \left(I \sum_{i, j \neq 1, i \neq j} g_{ij}g_{jj} + O(g)\right)g_1^2g_{11} \\
& + \frac{5}{2T_{7/2}^2} g_1^2 \left(- I g R_{g,2} + O(g)\right) + \frac{1}{g_1^2} \frac{1}{T_{7/2}^2} \left(- 4g_2^2g_{11} \sum_{j \neq 1} g_{jj} + O(g)\right) - \frac{5}{2T_{7/2}^2} \left(I \sum_{i, j \neq 1, i \neq j} g_{ij}g_{jj} + O(g)\right)g_1^2g_{11} \\
& + \frac{1}{g_1^2} \frac{1}{T_{5/2}^2} \sum_{j \neq 1} g_{jj}g_{11} + \frac{15}{2T_{7/2}^2} g_1^2 \sum_{j \neq 1} g_{jj} \\
& + \frac{1}{T_{3/2}^2} b g_1^2 \left( \sum_{i \neq 1} g_{ii} \right)^2 - \sum_{i \neq 1} g_{ii}^2 \exp \left(b g_1^2\right) \\
\end{align*} \]

is nonnegative when \( g \) is sufficiently small.

Since we have \( X = \frac{g_1}{g_1^2} + \exp \left(b g_1^2\right) \) and \( X^2 = \frac{g_1}{g_1^2} + \frac{2g_2}{g_1} \exp \left(b g_1^2\right) + \exp \left(2b g_1^2\right) \) at \( P_0 \), we get

\[ L X = \frac{1}{g_1^2} \sum_{k \neq 1} g_{kk} \frac{1}{T_{3/2}^2} 2(gg_{11} + g_1^2) \sum_{i, j \neq 1, i \neq j} g_{ij}^2 + 2 \sum_{i, j \neq 1, i \neq j} g_{ii}g_{jj}^3 \]

\[ + \frac{1}{g_1^2} \sum_{k \neq 1} g_{kk} \frac{1}{T_{3/2}^2} g \left( \sum_{i, j \neq 1, i \neq j} g_{ii}g_{jj}^2 + \sum_{i, j \neq 1, i \neq j} g_{ij}g_{ii}^3 \right) \]

\[ + \frac{1}{4g_1^2 \sqrt{T}} \left( \sum_{i \neq 1} g_{ii} \right)^2 + \frac{1}{3g_1^2} \sum_{k \neq 1} g_{kk} \frac{1}{T_{3/2}^2} (gg_{11} + g_1^2) \sum_{i \neq 1} g_{ii}^2 \]

\[ + \frac{g}{4g_1^2 \sqrt{T}} \left( \sum_{i \neq 1} g_{ii} - \frac{4g_2^2g_{11} \sum_{j \neq 1} g_{jj}}{I} \right)^2 + \left( \sum_{i \neq 1} g_{ii} - \frac{12g_1^2}{I} \right)^2 + \left( \sum_{i \neq 1} g_{ii} - \frac{8b g_1 g_{11}}{I} \exp \left(b g_1^2\right) \right)^2 \]

\[ + \frac{12g_1^2}{I} \sum_{k \neq 1} g_{kk} \frac{1}{T_{3/2}^2} f_{11} \sum_{i \neq 1} \left( g_{ii} + \frac{12}{f_{11}} g_1^2 g_{11} \sum_{j \neq 1} g_{jj} g_{ii} \right)^2 + \left( g_{ii} + \frac{24g_1 g_{11}}{g_1 f_{11}} I g g_{ii} \right)^2 \]

\[ + \left( g_{ii} - \frac{24}{g_1} g_{11} g_{ii} \right)^2 + \left( g_{ii} - \frac{12}{f_{11}} g_1 \sum_{j \neq 1} g_{jj} g_{ii} \right)^2 + \left( g_{ii} - \frac{24g_1 g_{11}}{g_1 f_{11}} I g \sum_{j \neq 1} g_{jj} g_{ii} \right)^2 \]

\[ + \left( g_{ii} + \frac{12}{f_{11}} g_1 \sum_{j \neq 1} g_{jj} g_{ii} \right)^2 + \left( g_{ii} + \frac{36}{f_{11}} g_{11} g_{11} \sum_{j \neq 1} g_{jj} \right)^2 + \left( g_{ii} + \frac{24}{g_1 f_{11}} g_{11}^2 \sum_{j \neq 1} g_{jj} \right)^2 \]

\[ + P \left( \frac{b}{I_{5/2}^2} \left( 2 - 10g_1^2 \right) g_1^4 \sum_{j \neq 1} g_{jj} + O(1) \right) \exp \left(b g_1^2\right) \]

\[ - \frac{1}{I_{5/2}^2} \left( 6b g_1^8 \exp \left(b g_1^2\right) + O(g) \right) X^3 + \frac{1}{I_{5/2}^2} \left( O(1) + O(g) \right) X^2 \]

\[ + \frac{1}{I_{5/2}^2} \left( b^2 O(g) \exp \left(4b g_1^2\right) + b^2 O(g) + b O(g^3) \right) \exp \left(2b g_1^2\right) + \left( O(g) + b^2 O(g^2) + b O(g^3) \right) \exp \left(b g_1^2\right) \right) X^2 \]

\[ + \frac{1}{I_{5/2}^2} \left( O(1) + O(g) + b O(g) + b^2 O(g) \right) \exp \left(b g_1^2\right) \]

\[ + \frac{2}{I_{5/2}^2} \sum_{i \neq 1} \left( I g \sum_{j \neq 1} g_{jj} + f_{11} \right) g_{ii}^2 \sum_{k \neq 1} g_{kk} \sum_{i \neq 1, i \neq k} g_{ii} g_{jj} X + \frac{1}{I_{5/2}^2} \left( - I^2 R_{g,2} + O(g) \right) X \]

\[ + 6b \frac{1}{I_{5/2}^2} \sum_{i \neq 1} \left( I g \sum_{j \neq 1} g_{jj} + f_{11} \right) g_{ii}^8 \exp \left(4b g_1^2\right) + \frac{1}{I_{5/2}^2} \left( O(g) + b^2 O(g) + b O(g) \right) \exp \left(3b g_1^2\right) \]
for sufficiently small value of \( g \)

**Proof.**

Let us assume the conditions in the subsection 1.4. Then there exists a constant \( W \). We use a local coordinate change from \( X \) such that for \( t \) such that for contradiction. This concludes the proof that a uniform lower bound of \( b \) is bounded below, we are done. Otherwise, there exists some time \( t_0 > 0 \) such that \( X(P_0(t_0), t_0) = 0 \) for the first time. As \( t \to t_0^+ \), we have \( X = X(P_0(t), t) \to 0^+ \). So there exists some time \( t_1 \in (0, t_0) \) such that for \( t \in [t_1, t_0) \), as \( g \) becomes arbitrarily small,

\[
X_t \geq \left( \frac{2b}{f_{5/2}} g_1 \sum_{j \neq 1} g_{jj} + O(1) \right) \exp (2bg_1^2)
\]

for sufficiently large \( b > 0 \), implying that

\[
X(X_0(t), t) \geq X_0
\]

where \( X_0 = (X_0(t_1), t_1) \) is the data at the time \( t_1 \). Hence, \( X \) cannot become zero, which is a contradiction. This concludes the proof that a uniform lower bound of \( R_{g,2} \) exists. \[ \square \]

**Corollary 3.11.1.** Let us assume the conditions in the subsection 1.4. Then there exists a constant \( C > 0 \) satisfying

\[
g_{11} \geq -C
\]

for sufficiently small value of \( g \) near the interface.

**Proof.**

\[
2g_{11} \sum_{i \neq 1} g_{ii} = R_{g,2} - \left( \sum_{i \neq 1} g_{ii} \right)^2 + \sum_{i \neq 1} g_{ii}^2
\]

and \( \sum_{i \neq 1} g_{ii} \) is uniformly bounded above and below by the inequality 3.80. \[ \square \]

4 Hölder estimates

4.1 \( C_{s,0}^1 \) estimates

We use a local coordinate change from

\[
(x_1, x_2 \ldots, x_n, g(x_1, x_2, \ldots, x_n, t))
\]
to
\[ h(x_{n+1}, x_2, \ldots, x_n, t), x_2, \ldots, x_n, x_{n+1}). \] (4.2)

Let us compute the evolution of the function \( h \). As done in Daskapoulos and Hamilton [3], the first-order derivatives of \( g \) are give by
\[
g_t = -\frac{h_t}{h_{n+1}}, \quad g_1 = \frac{1}{h_{n+1}}, \quad g_i = -\frac{h_i}{h_{n+1}} \quad \text{for} \quad i \neq 1, \quad (4.3)
\]
and the second-order derivatives of \( g \) are written as
\[
g_{11} = -\frac{1}{h_{n+1}^3}h_{n+1,n+1}, \quad g_{1i} = -\frac{1}{h_{n+1}^3}\left(-\frac{h_i}{h_{n+1}}h_{n+1,n+1} + \frac{1}{h_{n+1}}h_{n+1,i}\right) \quad \text{for} \quad i = 2, \ldots, n,
\]
\[
g_{ii} = -\frac{1}{h_{n+1}^3}\left(\frac{h_i^2}{h_{n+1}^2}h_{n+1,n+1} - 2\frac{h_i}{h_{n+1}}h_{n+1,i} + h_{ii}\right) \quad \text{for} \quad i = 2, \ldots, n, \quad (4.4)
\]
\[
g_{ij} = -\frac{1}{h_{n+1}^3}\left(\frac{h_i}{h_{n+1}}h_{n+1,n+1} - \frac{h_i}{h_{n+1}}h_{n+1,i} - \frac{h_j}{h_{n+1}}h_{n+1,i} + h_{ij}\right) \quad \text{for} \quad i = 2, \ldots, n.
\]

Using the notation
\[ \mathcal{I} = h_{n+1}^2 + x_{n+1}^2 + x_{n+1}^2 \sum_{i=2}^{n} h_i^2, \quad (4.5) \]
the evolution equation of \( h \) is
\[
h_t = -\frac{1}{2\sqrt{\mathcal{I}}}h_{n+1}^2 x_{n+1}\left(\frac{h_i^4}{h_{n+1}^2}h_{n+1,n+1} \sum_{i,j=2}^{n} (h_i^2 h_{jj} + h_j^2 h_{ii} - 2h_i h_j h_{ij})
\right.
\]
\[
- \frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} (h_i^2 h_{n+1,i} + h_j^2 h_{n+1,j} - 2h_i h_j h_{n+1,i,j}) + \frac{2}{h_{n+1}^2} \sum_{i=2}^{n} (h_{n+1,n+1} h_{ii} - h_{n+1,i,i})
\]
\[
- \frac{2}{h_{n+1}^2} \sum_{i,j=2}^{n} (h_i h_{n+1,i,j} h_{j,i} - h_i h_{n+1,i,j} h_{j,i} - h_j h_{n+1,i,j} h_{i,j}) + \frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} (h_{ii} h_{jj} - h_{ij}^2)
\]
\[
+ \frac{1}{\sqrt{\mathcal{I}}} h_{n+1}^2 \left(\sum_{i,j=2}^{n} \frac{h_i^2}{h_{n+1}} h_{ii} - h_{ij} h_{ij}\right) + \left(\sum_{i=2}^{n} h_{ii}\right) (4.6)
\]
\[
+ x_{n+1} \sum_{i=2}^{n} (h_{i} h_{n+1,i} - h_{n+1,i}) + \frac{x_{n+1}}{h_{n+1}^2} \sum_{i,j=2}^{n} h_i h_j (h_{ij} h_{n+1,n+1} - h_{ij} h_{n+1,i} h_{n+1,j})
\]
\[
- x_{n+1} \sum_{i,j=2}^{n} h_i h_k (h_{ij} h_{n+1,i} + k h_{n+1,i} - h_{jk} (h_{i} h_{n+1,j} + h_{j} h_{n+1,i})
\]
\[
- \frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} (h_i^2 h_{n+1,i} - h_{n+1,i}) + \frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} h_i^2 h_{n+1,i} h_{ij} - h_i^2 h_{ij}
\]
\[
+ x_{n+1} \sum_{i,j=2}^{n} (h_i h_j h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j}) + \frac{x_{n+1}}{h_{n+1}^2} \sum_{i,j=2}^{n} h_i h_j (h_{ij} h_{n+1,n+1} - h_{ij} h_{n+1,i} h_{n+1,j})
\]
\[
+ x_{n+1} \sum_{i,j=2}^{n} h_i^2 h_{n+1,i} h_{n+1,j} - h_i^2 h_{n+1,j} + \frac{x_{n+1}}{h_{n+1}^2} \sum_{i,j=2}^{n} h_i^2 h_{n+1,i} h_{n+1,j} - h_i h_j h_{ij}\right).
\]
If we use a local coordinates where \( g_i = 0 \) for \( i = 2, \ldots, n \) and \( g_{ij} \) for \( i \neq j \), then
\[
    h_{n+1} = \frac{1}{g_1} \geq 0, \quad h_i = 0 \quad \text{for} \quad i = 2, \ldots, n,
\]
and the second-order derivatives of \( g \) are written as
\[
    h_{n+1,n+1} = -h_{n+1}^3g_{11} \geq 0, \quad h_{n+1,i} = 0 \quad \text{for} \quad i = 2, \ldots, n, \\
    h_{ii} = -h_{n+1}g_{ii} \leq 0 \quad \text{for} \quad i = 2, \ldots, n, \quad h_{ij} = 0 \quad \text{for} \quad i, j = 2, \ldots, n \quad \text{such that} \quad i \neq j. 
\]
Consequently
\[
    C^{-1} \leq h_{n+1} \leq C, \quad g^2C^{-1} \leq h_{n+1,n+1} \leq C, \quad -C \leq \sum_{i=2}^{n} h_{ii} \leq -C^{-1}, \quad (4.9)
\]
\[
    -C \leq R_{g,2} = \frac{1}{h_{n+1}^2} \sum_{i,j=2}^{n} (h_{ii}h_{jj} - h_{ij}^2) + \frac{2}{h_{n+1}^4} \sum_{i=2}^{n} h_{n+1,n+1}h_{ii} \leq -g^2C^{-1}, \quad (4.10)
\]
and
\[
    C^{-1} \leq g_i^2 \sum_{i=2}^{n} g_{ii} = -\frac{1}{h_{n+1}^3} \sum_{i=2}^{n} h_{ii} \leq C \quad (4.11)
\]
for a uniform constant \( C > 0 \), from Lemma 3.3, 3.4, and Corollary 3.7.1.

From the relations (4.7)-(4.11) and straightforward calculations, the evolution equation of \( \tilde{h} \), which is either \( h_t \) or \( h_i, \quad i = 2, \ldots, n \), is
\[
    \tilde{h}_t = x_{n+1} \left( -\frac{1}{\sqrt{h}} \frac{1}{h_{n+1}^2} \sum_{i=2}^{n} h_{ii} + o(x_{n+1}^2) \right) \tilde{h}_{n+1,n+1} \\
    + \left( \frac{1}{\sqrt{h}} \frac{1}{h_{n+1}} + o(x_{n+1}) \right) \sum_{i=2}^{n} \tilde{h}_{ii} + \left( -\frac{1}{\sqrt{h}} \frac{1}{h_{n+1}^2} \sum_{i=2}^{n} h_{ii} + o(x_{n+1}) \right) \tilde{h}_{n+1}.
\]
(4.12)

It is of the form
\[
    \tilde{h}_t = x_{n+1}a_{n+1,n+1}\tilde{h}_{n+1,n+1} + \sum_{i=2}^{n} a_i \tilde{h}_{ii} + b_{n+1}\tilde{h}_{n+1}.
\]
(4.13)

We immediately see that the matrix \( (a_{ij}) \) is uniformly elliptic and strictly positive, and \( b_{n+1} > 0 \) is uniformly bounded, for sufficiently small \( g = x_{n+1} \), because all the first-order and second-order derivatives of \( g \) and \( h \) are uniformly bounded.

Thus, with minor changes in the higher dimension to the line of proof of Theorem 3.1. in [5], we obtain the following lemma.

**Lemma 4.1.** There exist a number \( 0 < \alpha < 1 \) so that, for any \( r < \rho \)
\[
    \|h_t\|_{C^2(B_r)} + \sum_{i=2}^{n} \|h_i\|_{C^2(B_r)} \leq C(r, \rho)\|h\|_{C^0(B_1)} \quad (4.14)
\]
with respect to the singular distance function
\[
    s((x_1,t_1),(x_2,t_2)) = |\sqrt{x_{1,n+1} - x_{2,n+1}}| + |(x_{1,2}, \ldots, x_{1,n}) - (x_{2,2}, \ldots, x_{2,n})| + |t_1 - t_2| \quad (4.15)
\]
where \( B_\rho = B_\rho(x_0, t_0) \) is the parabolic box
\[
    \{(x_{n+1},x_2, \ldots, x_n, t) : x_{n+1} \geq 0, |x_{n+1} - x_{0,n+1}| \leq \rho^2, |(x_2, \ldots, x_n) - (x_{0,2}, \ldots, x_{0,n})| \leq \rho, \quad -\rho^2 \leq t - t_0 \leq 0\}. \quad (4.16)
\]
Consequently, \( h_t \) and \( h_i, \ i = 2, \ldots, n \) belong to \( C_\alpha^r(\mathcal{B}_\eta) \) near the free boundary.

For \( h = h_{n+1} \), straightforward calculations show that in a local coordinates and the relations (4.11) with the choice of coordinates \( g_i = 0 \) for \( i = 2, \ldots, n \) and \( g_{ij} \) for \( i \neq j \),

\[
\tilde{h}_t = x_{n+1} + \frac{1}{\sqrt{I}} \frac{1}{h_{n+1}} \sum_{i=2}^{n} h_{ii} + \mathcal{O}(x_{n+1}^2) \tilde{h}_{n+1,n+1} + \left( \frac{1}{\sqrt{I}} \frac{1}{h_{n+1}} + \mathcal{O}(x_{n+1}) \right) \sum_{i=2}^{n} \tilde{h}_{ii} \\
+ \frac{1}{\sqrt{I}} \frac{1}{h_{n+1}} \sum_{i=2}^{n} h_{ii} + \mathcal{O}(x_{n+1}) \tilde{h}_{n+1} - \frac{1}{2\sqrt{I}} h_{n+1}^2 R_{g,2} + \mathcal{O}(x_{n+1}).
\]  

(4.17)

Since the part \( -\frac{1}{2\sqrt{I}} h_{n+1}^2 R_{g,2} + \mathcal{O}(x_{n+1}) \) is uniformly bounded for small \( x_{n+1} \), we can apply the same estimate as \( h_t \) and \( h_i \), \( i = 2, \ldots, n \).

**Lemma 4.2.** There exist a number \( 0 < \alpha < 1 \) and positive constants \( \eta > 0 \) and \( C > 0 \) depending only on the initial data and \( \rho_0 \) such that

\[
\| h_t \|_{C_\alpha^r(\mathcal{B}_\eta)} \leq C, \quad \| h_{n+1} \|_{C_\alpha^r(\mathcal{B}_\eta)} \leq C \quad \text{and} \quad \| h_i \|_{C_\alpha^r(\mathcal{B}_\eta)} \leq C \quad \text{for} \quad i = 2, \ldots, n.
\]

Thus, we see that \( h \in C_s^{1,\alpha}(\mathcal{B}_\eta) \).

### 4.2 \( C_s^{2,\alpha} \) estimates

For the point \( P = (z_{n+1}, z_2, \ldots, z_n) \) on the free boundary \( \Gamma(t) \), let us denote \( z = z_{n+1} \) and \( y = (z_2, \ldots, z_n) \). Let \( C_s^{2,\alpha}(\mathcal{B}_\eta) \) be the space of functions on \( \mathcal{B}_\eta \)

\[
C_s^{2,\alpha}(\mathcal{B}_\eta) = \{ h | h, h_t, h_{n+1}, h_i, x_{n+1} h_{n+1,i}, \sqrt{x_{n+1} h_{n+1,i}}, h_{ij} \in C_s^{\alpha}(\mathcal{C}_\eta), i, j = 2, \ldots, n \},
\]

where the parabolic box \( \mathcal{B}_\eta \) is centered at the point \( (0, y_0, t_0) \in \Gamma(t_0) \). Now, we want to get the \( C_s^{2,\alpha} \) regularity of \( h \) on \( \mathcal{B}_\eta \) from its \( C_s^{1,\alpha} \) regularity and the classical regularity theory for strictly parabolic equations, as done for the Gauss curvature flow in [10].

For \( 0 < \mu < 1 \), let us \( C_\mu \) denote the parabolic cylinder

\[
C_\mu = \{ z^2 + |y|^2 \leq \mu, -\mu^2 \leq t \leq 0 \}.
\]

(4.20)

Let \( h^r \) be the dilated function of \( h \) at a point \( Q^r = (r, y_r, t_r) \in \mathcal{B}_\eta \)

\[
h^r(z, y, t) = \frac{1}{r^2} h(r^2 + r^2 z, y_r + ry, t_r + r^2 t).
\]

(4.21)

Then the evolution of \( h^r \) is as follows.

**Lemma 4.3.** In our coordinate where \( g_i = 0 \) for \( i = 2, \ldots, n \) and \( g_{ij} = 0 \) for \( i \neq j \),

\[
h^r_t = -\frac{\tilde{z}}{(h_{n+1}^r)^2} \frac{1}{\sqrt{I}} \left( \sum_{i=2}^{n} h_{ii}^r \right) h_{n+1,n+1}^r + \frac{1}{h_{n+1}^r} \frac{1}{(r^2 \tilde{z})^2} \left( \sum_{i=2}^{n} h_{ii}^r \right) \sum_{i=2}^{n} h_{ii}^r
\]

\[
- \frac{r^2 \tilde{z}}{2\sqrt{I}} \sum_{i,j=2}^{n} (h_{ii}^r h_{jj}^r - (h_{ij}^r)^2).
\]

(4.22)

**Proof.** Let \( \tilde{z} = 1 + z \). Then

\[
\mathcal{I}_i(r^2 + r^2 z, y_r + ry, t_r + r^2 t) = (h_{n+1}^r)^2 + r^4 (\tilde{z})^2 + r^6 (\tilde{z})^2 \sum_{i=2}^{n} (h_{ii}^r)^2
\]

(4.23)

is uniformly bounded below and above. And for \( i, j = 2, \ldots, n \), at \( (r^2 + r^2 z, y_r + ry, t_r + r^2 t) \), we have

\[
h_{n+1} = h_{n+1}^r, \quad h_i = rh_i^r, \quad h_{n+1,n+1} = \frac{1}{r^2} h_{n+1,n+1}^r, \quad h_{n+1,i} = \frac{1}{r} h_{n+1,i}^r, \quad h_{ij} = h_{ij}^r.
\]

(4.24)

from which we get the result. \( \Box \)
Lemma 4.4. For any $0 < \mu_0 < 1$, there exists a constant $C > 0$ depending on $\mu_0$, $r_0$ and the initial data such that

$$\|h^r\|_{C^\infty(\mathbb{C}_\mu)} \leq C$$

(4.25)

for all $0 < \mu < \mu_0$.

Proof. If $P = (z, y, t) \in \mathbb{C}_\mu$ with $0 < \mu < 1$, then $\varepsilon \geq 1 - \mu^2 > 0$. As the derivatives of $h$ are uniformly bounded, by the relation between the derivatives of $h^r$ and $h$ above, $h^r$ satisfies a uniformly parabolic equation with ellipticity constant independent of $r$.

Hence, from the regularity of solutions to fully nonlinear uniformly parabolic equations (see Wang’s papers [13] and [14]), $\|h^r\|_{C^\infty(\mathbb{C}_\mu)}$ is, up to a uniform constant, bounded by $\|h^r\|_{L^\infty(\mathbb{C}_{\mu_0})}$ for all $0 < \mu < \mu_0 < 1$. Since $h_{n+1}$ is bounded in $B_{\eta}$, $\|h^r\|_{L^\infty(\mathbb{C}_{\mu_0})}$ is uniformly bounded. $\blacksquare$

Hence, with the argument in Daskapoulos and Lee [3], we get the following lemma.

Lemma 4.5. There exists a constant $C > 0$, depending only on the initial data, $\rho_0$ and $\eta$, such that for any two points $P_1 = (z_1, y_1, t_1)$ and $P_2 = (z_2, y_2, t_2)$ in $B_{\frac{3}{2}}$, we have

$$|z_1 h_{n+1,n+1}(P_1) - z_2 h_{n+1,n+1}(P_2)| + \sum_{i=2}^n \sqrt{|z_i h_{n+1,i}'}(P_1) - \sqrt{|z_i h_{n+1,i}'}(P_2)| \leq C s(P_1, P_2)^\alpha$$

(4.26)

for our metric distance $s(P_1, P_2) = \sqrt{|z_1 - z_2| + |y_1 - y_2| + |t_1 - t_2|}$. In other words,

$$x_{n+1} h_{n+1,n+1} \in C^\alpha(B_{\frac{3}{2}}), \sqrt{x_{n+1} h_{n+1,i}} \in C^\alpha(B_{\frac{3}{2}}) \text{ for } i = 2, \ldots, n.$$

(4.27)

Also, we get the Hölder regularity of $h_{ii}$, $i = 2, \ldots, n$.

Lemma 4.6.

$$h_{ii} \in C^\alpha(B_{\frac{3}{2}}) \text{ for } i = 2, \ldots, n.$$  

(4.28)

Proof. In our coordinate where $g_i = 0$ for $i = 2, \ldots, n$ and $g_{ij} = 0$ for $i \neq j$,

$$h_t = \frac{x_{n+1}}{2\sqrt{I}} \left( \sum_{i,j=2}^n h_{ij} h_{jj} - \sum_{i=2}^n h_i^2 \right) + \frac{1}{\sqrt{I}} \left( 1 - \frac{x^2_{n+1}}{I} \right) \left( \frac{1}{h_{n+1}} - \frac{x_{n+1}}{h_{n+1}^2} h_{n+1,n+1} \right) \sum_{i=2}^n h_{ii}$$

(4.29)

$$= -\frac{x_{n+1}}{2\sqrt{I}} \left( \sum_{i,j=2;i\neq j} h_{ij} h_{jj} \right) + \frac{1}{h_{n+1}^2} \frac{3}{2} \left( h_{n+1}^2 + x_{n+1} \sum_{i=2}^n h_i^2 \right) (h_{n+1} - x_{n+1} h_{n+1,n+1}) \sum_{i=2}^n h_{ii}$$

so that

$$\sum_{i=2}^n h_{ii} = \frac{I^{3/2} h_t + O(x_{n+1})}{h_{n+1} + O(x_{n+1})}. \quad (4.30)$$

Near the free boundary, $x_{n+1} = g$ tends to zero. Note that the denominator $h_{n+1} + O(x_{n+1}) \geq c > 0$ on $B_{\eta}$ for some uniform constant $C > 0$, from the estimate of $h_{n+1}$. Also, the numerator $I^{3/2} h_t + O(x_{n+1}) \in C^\alpha(B_{\frac{3}{2}})$, by the Hölder regularity of $h_t$, $h_{n+1}$ and $h_{ii}$, $i = 2, \ldots, n$. $\blacksquare$

Consequently, the Hölder regularity of the derivatives of $h$ of order one and two implies the following theorem.
Theorem 4.7. Assume that \( g = \sqrt{2f} \) is of class \( C^{2+\beta} \) up to the interface \( z = 0 \) for some \( 0 < \beta < 1 \), and satisfies the nondegeneracy condition

\[
|Dg(x)| \geq \lambda \text{ and } \sum_{\tau: \text{ tangential}} D^2_{\tau\tau}g(x) \geq \lambda \text{ at all } x \in \Gamma,
\]

for some positive number \( \lambda > 0 \). Also, assume that a ball is included in the flat side: \( D_{\rho_0} = |x| \leq \rho_0 \subset \Sigma_0(T) \). Then there exist some constants \( 0 < \alpha < 1, 0 < C < \infty \) and \( \eta > 0 \) which depend only on the initial data and \( \rho_0 \), such that for any free boundary point \( P_0 = (x_0, y_0, t_0) \) with \( 0 < \tau < t_0 < T \) satisfying \( n_0 := \frac{\partial}{\partial \nu} = e_1 \), the function \( x = h(z, y, t) = h(x_{n+1}, x_2, \ldots, x_n, t) \) satisfies the Hölder estimate

\[
\|h\|_{C^{2+\alpha}(B_\eta)} \leq C
\]

where \( B_\eta = \{0 \leq z \leq \eta^2, |y - y_0| \leq \eta, t_0 - \eta^2 \leq t \leq t_0\} \).

5 Conclusion: proof of the main theorem

5.1 Short-time existence near the interface

The last results have an immediate consequence, the short-time existence of the \( h \) near a free boundary point.

Lemma 5.1. There exists a unique solution \( h \in C^{2+\alpha}_{s}(B_\eta) \) of the equation \( (4.6) \) for a short time \( T > 0 \) in \( B_\eta \) for some constant \( \eta > 0 \) as in Theorem 4.7.

Proof. The linearized equation \( (4.12) \) of \( Lh = 0 \) \( (4.6) \) satisfies the condition of Theorem 7.1. in \( [3] \) with \( k = 0 \) and minor changes in higher dimension \( n \geq 3 \). Applying the Inverse Function Theorem, we get the existence result for \( h \).

Thus, we get the short-time existence of \( h \) along the interface.

Theorem 5.2. There exists a unique smooth solution \( f \) to the scalar curvature flow for a short time \( T > 0 \).

Proof. Let us cover the interface \( \Gamma = \Gamma(0) \) into balls, each of which is centered at a free boundary point on it. We use a coordinate in each ball such that the free boundary point is 0, the free boundary is flat, and the area inside the flat spot is the upper half plane. Then we have the short time existence of \( h \in C^{2+\alpha}_{s}(B_\eta) \) inside each ball. Since \( \Gamma \) is compact, it can be covered by a finite number of such balls as above. The short-time existence of the functions \( g = \sqrt{2f} \) and \( f \) follows.

5.2 All-time \( C^\infty \) regularity up to the interface

First, we prove a lemma to prove the main theorem.

Lemma 5.3. Let \( g \) be the solution which is smooth up to the interface on \( 0 < t < T \) and \( T < T_c \). Then \( g(x) = g(x, T) \) belongs to the class \( C^{2+\beta}_{s} \) for some \( 0 < \beta < 1 \) and satisfies the non-degeneracy conditions \( |Dg(x)| \geq \lambda \) and \( D^2_{\tau\tau}g(x) \geq \lambda \) for any \( x \in \Gamma \) for some constant \( \lambda > 0 \).

Proof. By the theorem \( [4, 7] \) about the Hölder estimate of \( h \) with the relations \( (4.7), (1.11) \) between the first-order and second-order derivatives of \( g \) and \( h \), the conclusion is immediate.

Finally, we prove our main theorem \( [1, 2] \).

Proof. Proof of Theorem \( [1, 2] \) By the short-time existence theorem \( [5, 2] \) there exists a solution \( g \), smooth up to the interface in \( 0 < t < T \), for a maximal time \( T > 0 \). If \( T < T_c \), then \( g(\cdot, T) \) belongs to the class \( C^{2+\beta}_{s} \) up to the interface \( x_{n+1} = 0 \) for some \( 0 < \beta < 1 \) and it satisfies the degeneracy conditions by the lemma \( [5, 3] \). Then the linearization \( (5.5) \) of the evolution equation
of $g$ satisfies the condition of Theorem 7.1 in [3] with minor changes in higher dimension $n \geq 3$. Applying the Inverse Function Theorem, we get the short-time existence for $g(x, t)$ with the initial data $g(x, T)$, for all $T \leq t < T + T'$ for some $T' > 0$ and it is $C^\infty$ up to the interface. This is contradiction to the condition that $T < T_c$ is the maximal time. Hence, we must have $T = T_c$, the critical time of the flow.

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