LONG-TIME BEHAVIOR FOR FOURTH-ORDER WAVE EQUATIONS WITH STRAIN TERM AND NONLINEAR WEAK DAMPING TERM

CHAO YANG AND YANBING YANG*

College of Mathematical Sciences, Harbin Engineering University
No. 145 Nantong Street, Harbin 150001, China

ABSTRACT. We mainly focus on the asymptotic behavior analysis for certain fourth-order nonlinear wave equations with strain term, nonlinear weak damping term and source term. We establish two theorems on the asymptotic behavior of the solution depending on some conditions related to the relationship among the forced strain term, the nonlinear weak damping term and source terms.

1. Introduction. Consideration herein is a class of fourth-order nonlinear wave equations with strain term, nonlinear weak damping term and source term involved in the initial boundary value problem (IBVP) of the mathematical model as

\[ u_{tt} + \Delta^2 u - \alpha \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) + |u_t|^{r-1}u_t = f(u), \quad (t, x) \in [0, T) \times \Omega, \quad (1) \]

\[ u(x, 0) = u_0, \quad u_t(x, 0) = u_1, \quad x \in \Omega, \quad (2) \]

\[ u = \frac{\partial u}{\partial \nu} = 0 \quad \text{or} \quad u = \Delta u = 0, \quad x \in \partial \Omega, \quad t \geq 0, \quad (3) \]

where \( r \geq 1, \Omega \subset \mathbb{R}^n \) is a bounded domain with a smooth boundary \( \partial \Omega \), and \( \nu \) is the outward unit normal vector on \( \partial \Omega \). Additionally, \( \sigma(u) \) and \( f(u) \) are nonlinear functions whose assumptions will be given later.

The overall goal of our work is to investigate the long-time behavior of fourth-order wave equations, concentrating on the interplay among the forced strain term, nonlinear dissipative term and source terms. The original model of Equation (1) came from the following equation

\[ u_{tt} + u_{xxxx} = a(u_x^2)_x + f(x), \quad a < 0, \quad (4) \]

introduced by An and Peirce [3, 2] to analyze the longitudinal motion of an elasto-plastic bar and describe the coupling between the focus effect of the nonlinearity and the dispersive effect of the microstructure terms. It is shown that for some generalized functions \( \sigma(s) \) including \( \sigma(s) = s^n \), the symmetry reductions and integrability

2020 Mathematics Subject Classification. Primary: 35B40, 35L35, 35D30.
Key words and phrases. Fourth-order wave equation, strain term, weakly dissipative term, asymptotic behavior.

The work was supported by the Heilongjiang Postdoctoral Research Start-up Funding Project (No. LBH-Q20013 and No. LBH-Q20086), the National Natural Science Foundation of China (No. 11801114 and No. 11871017) and the Research Funds for the Central Universities.

* Corresponding author: Yanbing Yang.
were considered in [28], and some results about the existence and non-existence of global solutions were also given in [6].

Considering the interaction between the forced strain terms and the nonlinear source terms, Esquivel-Avila [8] dealt with the following model

$$u_{tt} + \Delta^2 u - \alpha \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) = f(u) \quad (5)$$

and gave the characterizations of finite time blow up, boundedness and convergence to the ground state for the initial boundary value problem of Equation (5) with \(\sigma_i = |u_{x_i}|^{m-2}u_{x_i}\) and \(f = \mu |u|^{r-2}u\). Later, an investigation on the interaction of forced strain terms and nonlinear source terms for the same problem was handled by Liu and Xu [21] to derive a threshold result of global existence and nonexistence for the subcritical initial energy case \(E(0) < d\), as well as a global existence result for the critical initial energy case \(E(0) = d\), hereafter \(E(0)\) and \(d\) represent the total initial energy and the depth of potential well respectively. Subsequently, the result about finite time blow up obtained in [21] was extended by Shen et al. [26] from the subcritical initial energy case to the arbitrarily positive initial energy case.

Meanwhile, some qualitative analysis on the interaction of forced strain terms and some dissipative terms were also performed.

For the model with linear damping effects, the numerical tools always play an influential role in discovering new phenomena [25] when people pay great attention to the theme about the existence of solutions and attractors [22]. It is essential to show the asymptotic behavior and describe its phenomena precisely, like the work [4] dealing with the decay rate of the wave equation with weak internal damping with non-constant delay and nonlinear weights. Lian and Xu [17] proved the asymptotic behavior for the nonlinear wave equation with weak and strong damping terms and logarithmic source term. Cazenave and Han [5] studied asymptotic behavior Schrödinger equation with nonlinear dissipation. For the fourth-order wave equation with strong damping, Yang et al. [29] derived the existence of strong and weak uniform attractors with non-compact external forces. Liu [19] was concerned with the long-time behavior under certain assumptions on the memory kernel and the source term. Di et al. [7] considered the IBVP for the fourth-order wave equation with nonlinear boundary damping and proved the uniform decay. For the evolution system, Nguyen [24] studied the asymptotic behavior of a class of partially dissipative linear hyperbolic systems. Kreulich [15] considered the IVP with nonlinear multivalued and dissipative operator. Mohammed [23] gave asymptotic estimates with a fully nonlinear uniformly elliptic differential operator, a non-decreasing function that satisfies appropriate growth conditions at infinity. In [30], Yang proved the global existence and the asymptotic behavior of solutions with the positive definite initial energy and established the finite time blow up of solutions for the negative initial energy. Then these conclusions were extended by Liu and Xu [20] in the aid of potential well theory to obtain the conditions on the initial data for the global existence and finite time blow up of solutions for the subcritical and critical initial energy case. Some recent results on the further influence of both the forced strain term and linear weak damping term were given in [12], which proves the asymptotic behavior of the solution for the subcritical initial energy and finite time blow up of solutions with arbitrarily positive initial energy.

Additionally, some qualitative analysis on the interaction of forced strain term, nonlinear source term and some dissipative terms were carried out in some literature.
For instance, we see that the finite time blow up and long-time behavior for Equation (1) with \( r = 1 \) associated with the IBVP were displayed in [27]. It is noted that this is only a small sample of the existent work on the qualitative behavior for Equation (1) involving the IBVP to uncover the interplay of forced strain term, nonlinear source term and dissipative term. Fatori et al. [9] and Gomes Tavares et al. [11] respectively considered the long-time behavior of Kirchhoff model with nonlinear strain term and sharp decay rates of Kirchhoff model with memory term. Silva and Ma [13] was concerned with asymptotic stability of a class of plate equations with memory term. Lian et al. [16] got asymptotic behavior for a class of fourth-order wave equations with strong damping term, nonlinear weak damping term, strain term and nonlinear source term in polynomial form.

Motived by the works mentioned above, this paper continues to explore the interplay among the forced strain term \( \sigma_i \), the classical nonlinear weak damping term \(|u_t|^{r-1}|u_t|\) and the nonlinear source term \( f \), mainly revealing how this interaction along with the initial data affect the long-time behavior of the solution. Based on the weak dissipative case, i.e., the linear case \( r = 1 \) and the nonlinear case \( r > 1 \), some effect of the interplay between some high-order nonlinearities on the long-time behavior for the problem (1)-(3) will be revealed in our work.

Herein, for the linear weak damping case \( r = 1 \), we observe that the exponential decay was derived in [27] for the special forced strain term \( \sigma_i(s) = |s|^{m-2}s \) and nonlinear source term \( f(s) = |s|^{p-2}s \). In the present paper, we like to show that this exponential decay result displayed in [27] also holds under the coming Assumption 1.1.

**Assumption 1.1** (Case I of strain term and source term).

(a) **The forced strain term** \( \sigma_i(i = 1, \ldots, n) \) satisfies

\[
\text{(H1)} \quad \begin{cases}
(i) \sigma_i \in C^1 \text{ and } \sigma_i(0) = \sigma'_i(0) = 0. \\
(ii) \sigma_i(s) \text{ are monotone, and are convex for } s > 0, \text{ concave for } s < 0.
\end{cases}
\]

\[
(iii) |\sigma_i(s)| \leq a_2 |s|^q \text{ and } (q + 1)G_i(s) \leq \sigma_i(s) \text{ for some } a_2 > 0;
\]

\[
(iv) G_i(s) = \int_0^s \sigma_i(\tau)d\tau, \quad 1 \leq i \leq n.
\]

(b) **The nonlinear source term** \( f \) satisfies

\[
\text{(H2)} \quad \begin{cases}
(i) f \in C^1 \text{ and } f(0) = f'(0) = 0. \\
(ii) f(s) \text{ is monotone, and is convex for } s > 0, \text{ concave for } s < 0.
\end{cases}
\]

\[
(iii) |f(s)| \leq a_1 |s|^p \text{ and } (p + 1)F(s) \leq sf(s) \text{ for some } a_1 > 0;
\]

\[
(iv) F(s) = \int_0^s f(\tau)d\tau.
\]

In the case of Assumption 1.1, the following theorem describes this exponential decay result.

**Theorem 1.1** (Exponential decay for \( E(0) < d \) and \( r = 1 \)). Let \( u_0 \in H, u_1 \in L^2(\Omega), E(0) < d \) and Assumption 1.1 with \( q \geq p \) hold. Assume \( u_0 \in G \), then there exist some positive constants \( \tilde{C} \) and \( \tilde{\lambda} \) such that

\[
E(t) \leq \tilde{C}e^{\tilde{\lambda}t}, \quad t \geq 0 \quad \text{for} \quad r = 1.
\]
The forced strain term $r$ > damping case

Assumption 1.2 (Case II of strain term and source term).

(a) The forced strain term $\sigma_i(i = 1, ..., n)$ satisfies

\begin{align*}
(i) & \sigma_i \in C^1 \text{ and } \sigma_i(0) = \sigma_i'(0) = 0. \\
(ii) & \sigma_i(s) \text{ are monotone, and are convex for } s > 0, \text{ concave for } s < 0. \\
(iii) & |\sigma_i(s)| \leq a_2|s|^q \text{ and } (q + 1)G_i(s) = \sigma_i(s) \text{ for some } a_2 > 0; \\
& 1 < q < \infty \text{ if } n = 1, 2, \ 1 < q < \frac{n}{n-2} \text{ if } n \geq 3. \\
(iv) & G_i(s) = \int_0^s \sigma_i(\tau)d\tau, \ 1 \leq i \leq n.
\end{align*}

(b) The nonlinear source term $f$ satisfies

\begin{align*}
(i) & f \in C^1 \text{ and } f(0) = f'(0) = 0. \\
(ii) & f(s) \text{ is monotone, and is convex for } s > 0, \text{ concave for } s < 0. \\
(iii) & |f(s)| \leq a_1|s|^p \text{ and } (p + 1)F(s) = sf(s) \text{ for some } a_1 > 0; \\
& 1 < p < \infty \text{ if } n \leq 4, \ 1 < p < \frac{n}{n-4} \text{ if } n \geq 5. \\
(iv) & F(s) = \int_0^s f(\tau)d\tau.
\end{align*}

Theorem 1.2 (Algebraic decay for $E(0) < d$ and $r > 1$). Let $u_0 \in H, \ u_1 \in L^2(\Omega), E(0) < d$ and Assumption 1.2 with $q = p$ hold. Assume $u_0 \in \mathcal{G}$, then there exits a positive constant $A$ such that

$$E(t) \leq E(0) \left(\frac{r + 1}{2 + A(r - 1)t}\right)^{\frac{2}{r-1}}, \ t \geq 0,$$

where

\begin{align*}
1 < r < \frac{n}{n-4}, & \quad \text{if } n \geq 5, \\
1 < r < \infty, & \quad \text{if } n \leq 4.
\end{align*}

Remark 1. The approach of Theorem 1.1 is based on the classical potential well method along with the property of the Nehari functional $I(u)$ given by (11) later. Since the method applied in Theorem 1.1 in the case $r = 1$ can not be easily paralleled to the case $r > 1$, this enforces us to abandon some ranges of the index on the strain term and source term described in Assumption 1.1. Also, this is why we still consider the long-time behavior for the problem (1)-(3) in the case that $r > 1$ under Assumption 1.2, although the condition on the index $r$ of the weak damping term only increases from $r = 1$ to $r > 1$. Indeed, Assumption 1.2 is a special case of Assumption 1.1, which was recently introduced in [16] to achieve an exponential decay for the problem (1)-(3) with $r \geq 1$ in the presence of the so-called strong dissipative term $\Delta u_t$ under Assumption 1.2 (see Theorem 4.4 in [16]). It is clear that the so-called strong dissipative term $\Delta u_t$ makes the solutions for the classical wave equation decay more dramatically. These promote us to consider the medium case compared to Theorem 1.1 of the present paper and Theorem 4.4 in [16], namely the algebraic decay for the problem (1)-(3) in the case that $r > 1$ under Assumption 1.2 as shown in Theorem 1.2.

The approach of Theorem 1.2 is based on the adapted multiplier method along with some analysis techniques, which is also different from that method applied in Theorem 4.4 in [16]. In fact, a disturbance parameter was designed in the proof
of Theorem 4.4 in [16] to balance the so-called strong dissipative term \( \Delta u_t \) and the nonlinear weak damping term \(|u_t|^{-1}u_t\), and also was the key to over control the forced strain term \( \sigma_i \) and the nonlinear source term \( f \), however this technique is invalid if the so-called strong dissipative term \( \Delta u_t \) vanished and only the weak damping term \(|u_t|^{-1}u_t\) \((r > 1)\) acts as the damping role. So, the comparison of Theorem 1.1 and Theorem 1.2 in our work with Theorem 4.4 in [16] reveals the interaction among the forced strain term \( \sigma_i \), the dissipative terms \(|u_t|^{-1}u_t\), \( \Delta u_t \) and the nonlinear source term \( f \) on the asymptotic behavior for some fourth-order wave equations, which is also the intended goal of our work.

The rest of the present paper is organized as follows. Some notations and preliminaries are introduced in Section 2. The asymptotic behavior takes center stage in Section 3 which is made up of two subsections. In Subsection 3.1, the exponential decay for \( r = 1 \) under Assumption 1.1 is established in the framework of the potential well. In Subsection 3.2, the proof of algebraic decay for \( r > 1 \) under Assumption 1.2 is carried out by utilizing the adapted multiply method.

2. Setup. This section presents some notations and preliminary results for the main results of the present paper.

2.1. Notations. Throughout the present paper, \( L^p(\Omega) \) denotes the usual space of \( L^p \)-functions on \( \Omega \) endowed with the norm \( \|u\|_{L^p(\Omega)} = \|u\|_p \), \( \|u\|_{L^2(\Omega)} = \|u\| \) and the inner product \( (u, v) = \int_{\Omega} uv \, dx \). An auxiliary space \( H := \left\{ u \in H^2(\Omega) \cap H^1_0(\Omega) \mid \frac{\partial u}{\partial n} = 0 \text{ or } \Delta u = 0 \text{ on } \partial \Omega \right\} \)
eq 0 under Assumption 1.1 is established in the framework of \( \Delta u \) and the nonlinear source term \( f \) in Section 3 which is made up of two subsections. In Subsection 3.1, the exponential decay for \( r = 1 \) under Assumption 1.1 is established in the framework of the potential well. In Subsection 3.2, the proof of algebraic decay for \( r > 1 \) under Assumption 1.2 is carried out by utilizing the adapted multiply method.

Lemma 2.1. \([18, 1] \|\Delta u\| \) is equivalent to \( \|u\|_{2, 2} \) for \( u \in H \).

Corollary 1. For any \( u \in H \), \( \|u\|_H \) is equivalent to \( \|u\|_{2, 2} \). Let \( k_2 \) be the optimal constant such that \( k_2\|u\|_H \geq \|u\| \).

Corollary 2. \([1] \) Let \( p \) and \( q \) be defined in Assumption 1.1. Then
\[
\begin{align*}
(i) & \ H \leftrightarrow L^{p+1}(\Omega) \text{ compactly and } \|u\|_{p+1} \leq C_1\|u\|_H, \\
(ii) & \ H \leftrightarrow W^{1, q+1}(\Omega) \text{ compactly and } \|u\|_{1, q+1} \leq C_2\|u\|_H,
\end{align*}
\]
where \( C_1 \) and \( C_2 \) are the corresponding best constants.

The definition of a solution to the problem (1)-(3) in weak sense is displayed as follows.

Definition 2.2 (Weak solution). Function \( u(t) \) is called a weak solution to the problem (1)-(3) on \( \Omega \times [0, T) \) if
\[
u(t, x) \in C(H, [0, T]) \cap C^1(L^2(\Omega), [0, T]) \cap C^2(H^{-1}, [0, T])
\]
with \( u_t \in L^{r+1}(\Omega \times (0, T)) \) satisfying
\[
\begin{align*}
(u_t, w) + (\Delta u, \Delta w) + \alpha(\nabla u, \nabla w) + \left(|u_t|^{r-1}u_t, w\right) &= (f, w) + \sum_{i=1}^{n} \left(\sigma_i(u_{x_i}), w_{x_i}\right), \quad w \in H, \quad 0 < t < T
\end{align*}
\]
with $u_0 \in H$ and $u_1 \in L^2(\Omega)$, hereafter we denote the duality pairing between $H^{-1}$ and $H$ by $\langle \cdot, \cdot \rangle$.

Additionally, the existence and uniqueness of local solution can be established by combining some arguments of [8, 16, 18, 10].

**Theorem 2.3** (Local existence). Let $u_0 \in H$, $u_1 \in L^2(\Omega)$ and Assumption 1.1 hold. Then problem (1)-(3) admits a unique local solution

$$u(x, t) \in C \left( H, [0, T) \right) \cap C^1 \left( L^2(\Omega), [0, T) \right) \cap C^2 \left( H^{-1}, [0, T) \right)$$

with

$$u_t \in L^{r+1} \left( \Omega \times (0, T) \right)$$

for some small enough $T > 0$.

We also remark that $C$ are various positive constants that may vary from line to line.

### 2.2. Potential well.

This part mainly provides the nature of the potential energy functional $J(u(t))$ and the Nehari functional $I(u(t))$, highlights their relationships with the problem (1)-(3) and extracts the depth of the potential well, i.e., the so-called mountain pass level.

First for problem (1)-(3) we introduce the total energy functional

$$E(t) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|u\|^2_H - \sum_{i=1}^n \int_\Omega G_i(u_{x_i}) dx - \int_\Omega F(u) dx, \quad (9)$$

the potential energy functional

$$J(u) = \frac{1}{2} \|u\|^2_H - \sum_{i=1}^n \int_\Omega G_i(u_{x_i}) dx - \int_\Omega F(u) dx, \quad (10)$$

the Nehari functional

$$I(u) = \|u\|^2_H - \sum_{i=1}^n \int_\Omega u_{x_i} \sigma_i(u_{x_i}) dx - \int_\Omega uf(u) dx \quad (11)$$

and the potential well (stable set)

$$G := \{ u \in H \mid I(u) > 0 \} \cup \{0\}. \quad (12)$$

Obviously, a combination of (10), (11) and Assumption 1.1 implies the following properties.

**Lemma 2.4.** Assume Assumption 1.1 with $q \geq p$ holds, then

$$J(u) \geq \frac{p-1}{2(p+1)} \|u\|^2_H + \frac{1}{p+1} I(u). \quad (13)$$

**Remark 2.** Lemma 2.4 along with (9) and (10) enjoys

$$E(t) = \frac{1}{2} \|u_t(t)\|^2 + J(u(t)) \geq \frac{1}{2} \|u_t(t)\|^2 + \left( \frac{1}{2} - \frac{1}{p+1} \right) \|u(t)\|^2_H + \frac{1}{p+1} I(u(t)). \quad (14)$$

Similar to that in [21], a relationship between $J(u)$ and $I(u)$ can be viewed as follows.

**Lemma 2.5.** [21] Let Assumption 1.1 hold. Then for any $u \in H$, $\|u\|_H \neq 0$, it possesses
(i) \( \lim_{\lambda \to 0} J(\lambda u) = 0, \lim_{\lambda \to +\infty} J(\lambda u) = -\infty; \)

(ii) There exists a unique \( \lambda^* = \lambda^*(u) \) on the interval \( 0 < \lambda < +\infty \) such that
\[
\frac{d}{d\lambda} J(\lambda u) \bigg|_{\lambda = \lambda^*} = 0;
\]

(iii) \( J(\lambda u) \) is increasing on the interval \( 0 \leq \lambda \leq \lambda^* \), is decreasing on the interval \( \lambda^* < \lambda < +\infty \) and takes the maximum at \( \lambda = \lambda^* \);

(iv) \( I(\lambda u) > 0 \) for \( 0 < \lambda < \lambda^* \), \( I(\lambda u) < 0 \) for \( \lambda^* < \lambda < +\infty \) and \( I(\lambda^* u) = 0 \).

And also the depth of the potential well is defined as
\[
d = \inf_{u \in \mathcal{N}} J(u), \tag{15}
\]

where
\[
\mathcal{N} = \left\{ u \in H \setminus \{0\} \mid I(u) = 0 \right\}.
\]

Likewise, Lemma 2.7 in [16] can formulate the value of the depth of potential well \( d \) as

**Lemma 2.6 (Depth of potential well).** Let Assumption 1.1 with \( q \geq p \) hold, then
\[
d = \left( \frac{1}{2} - \frac{1}{p+1} \right) \bar{\gamma}^2, \tag{16}
\]

where \( \bar{\gamma} \) is the unique real root of equation
\[
h(\gamma) := a_1 C_1^{p+1} \gamma^{p-1} + a_2 C_2^{q+1} \gamma^{q-1} = 1. \tag{17}
\]

To the end, a property for the total energy formulated as (9) reads.

**Lemma 2.7 (Non-increasing total energy).** Let \( u(t) \) be a solution to problem (1)-(3) with \( u_0 \in H \) and \( u_1 \in L^2(\Omega) \), then \( E(t) \) is a non-increasing function with respect to time \( t \).

**Proof of Lemma 2.7.** Multiplying Equation (1) by \( u_t \) and integrating over \( \Omega \times [s, t) \) yields
\[
E(t) + \int_s^t \| u_\tau \|_{H^1}^2 d\tau = E(s), \tag{18}
\]

which gives the conclusion. \( \square \)

3. **Asymptotic behavior for \( E(0) < d \).** This section focuses on the asymptotic behavior of the solution to the problem (1)-(3) for the subcritical initial energy \( E(0) < d \). Before we proceed to the main result about the long-time behavior, we first present the detailed statement of the global existence result for subcritical initial energy \( E(0) < d \) in the following Lemma 3.1, which further enables us to explore the long-time behavior.

**Lemma 3.1 (Global existence for \( E(0) < d \)).** Let \( u_0 \in H \), \( u_1 \in L^2(\Omega) \), \( E(0) < d \) and Assumption 1.1 with \( q \geq p \) hold. Assume \( u_0 \in \mathcal{G} \), then problem (1)-(3) with \( r \geq 1 \) admits a global weak solution
\[
u(t) \in L^\infty \left( \mathbb{R}^+; H \right), \quad u_t(t) \in L^\infty \left( \mathbb{R}^+; L^2(\Omega) \right)
\]

and also \( u(t) \in \mathcal{G} \).
Remark 3. Very recently the existence of the global solution to Equation (1) with the strong dissipative term \( \Delta u_t \) associated with the corresponding IBVP was established in [16] (see Theorem 4.2 thereof). We note that the method used in the proof of Theorem 4.2 in [16] was based on the bounded principle in the framework of the potential well, which can be easily paralleled to Lemma 3.1. So the detailed proof of Lemma 3.1 is omitted here.

The following lemma will draw the relationship between some functionals that will be used to prove the main results.

Lemma 3.2. Let \( u_0 \in H, \ u_1 \in L^2(\Omega), \ E(0) < d \) and Assumption 1.1 with \( q \geq p \) hold. Assume \( u_0 \in G \), then

\[
\|u(t)\|_H^2 \leq \frac{2(p+1)}{p-1} E(t) \leq \frac{2(p+1)}{p-1} E(0) \tag{19}
\]

and

\[
\|u(t)\|_H^2 \leq \frac{1}{\beta} I(u), \tag{20}
\]

where \( \beta = 1 - \frac{\beta}{p+1} \) with

\[
\beta := a_1 C_\beta^{p+1} \left( \frac{2(p+1)}{p-1} E(0) \right)^{(p-1)/2} + a_2 C_\beta^{q+1} \left( \frac{2(p+1)}{p-1} E(0) \right)^{(q-1)/2}. \tag{21}
\]

Remark 4. From (16), it is noted that \( E(0) < d \) allows \( \beta < 1 \).

Proof of Lemma 3.2. Notice that Lemma 3.1 implies \( I(u(t)) > 0 \) for all \( t \in [0,T) \), which along with Lemma 2.7 and (14) confirms (19). Recalling Assumption 1.1 and Corollary 2 tells

\[
\sum_{i=1}^{n} \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx + \int_{\Omega} u f(u) \, dx \\
\leq a_1 \|u\|_{p+1}^{p+1} + a_2 \sum_{i=1}^{n} \|u_{x_i}\|_{q+1}^{q+1} \tag{22}
\]

\[
\leq a_1 C_\beta^{p+1} \|u\|_H^{p+1} + a_2 C_\beta^{q+1} \|u\|_H^{q+1}.
\]

Then combining Assumption 1.1 with \( q \geq p \), (22) and (19) yield that

\[
\sum_{i=1}^{n} \int_{\Omega} G_i(u_{x_i}) \, dx + \int_{\Omega} F(u) \, dx \\
\leq \frac{1}{q+1} \sum_{i=1}^{n} \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx + \frac{1}{p+1} \int_{\Omega} u f(u) \, dx \\
\leq \frac{1}{p+1} \left( \sum_{i=1}^{n} \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) \, dx + \int_{\Omega} u f(u) \, dx \right) \\
\leq \frac{1}{p+1} \left( a_1 C_\beta^{p+1} \|u\|_H^{p-1} + a_2 C_\beta^{q+1} \|u\|_H^{q-1} \right) \|u\|_H^2
\]
\[
\leq \frac{1}{p+1} \left( a_1 C_1^{p+1} \left( \frac{2(p+1)}{p-1} E(0) \right)^{(p-1)/2} + a_2 C_2^{p+1} \left( \frac{2(p+1)}{p-1} E(0) \right)^{(q-1)/2} \right) \|u\|_{H^p}^2
\]

which together with (11) gives (20). Hence the proof of this lemma is completed. □

Now we turn to the main results of the present paper.

3.1. **Exponential decay for** $E(0) < d$ **and** $r = 1$. This subsection offers the proof of the exponential decay of the solution to problem (1)-(3) with $r = 1$ for the subcritical initial energy $E(0) < d$ under Assumption 1.1 (Theorem 1.1) by means of Lemma 3.2.

**Proof of Theorem 1.1.** Multiplying (18) with $r = 1$ by $e^{\tilde{\alpha}t} (\tilde{\alpha} > 0)$ transpires

\[
\frac{d}{dt} \left( e^{\tilde{\alpha}t} E(t) \right) + e^{\tilde{\alpha}t} \|u_t\|^2 = \tilde{\alpha} e^{\tilde{\alpha}t} E(t), \quad 0 \leq t < +\infty
\]

and

\[
e^{\tilde{\alpha}t} E(t) + \int_0^t e^{\tilde{\alpha}\tau} \|u_\tau\|^2 d\tau = E(0) + \tilde{\alpha} \int_0^t e^{\tilde{\alpha}\tau} E(\tau) d\tau, \quad 0 \leq t < +\infty.
\]

It is noted that testing Equation (8) by $u(x,t)$ yields

\[
\langle u_{tt}, u \rangle + \|u\|_{H^p}^2 + \int_\Omega |u_t|^{r-1} u_t u dx = \sum_{i=1}^n \int_\Omega u_{x_i} \sigma_i(u_{x_i}) dx + \int_\Omega u f(u) dx,
\]

which along with $r = 1$ and (11) gives

\[
I(u) = \|u_t\|^2 - \frac{d}{dt} \left( (u_t, u) + \frac{1}{2} \|u\|^2 \right), \quad 0 \leq t < +\infty,
\]

and thereby

\[
\int_0^t e^{\tilde{\alpha}\tau} E(\tau) d\tau
\]

\[
= \int_0^t e^{\tilde{\alpha}\tau} \left( \frac{1}{2} \|u_\tau\|^2 + \frac{1}{2} \|u\|^2 \right) - \int_0^t \int_\Omega G_i(u_{x_i}) dx - \int_\Omega F(u) dx d\tau
\]

\[
\leq \int_0^t e^{\tilde{\alpha}\tau} \left( \frac{1}{2} \|u_\tau\|^2 + \left( 1 + \frac{\beta}{p+1} \right) \|u\|^2 \right) d\tau
\]

\[
\leq \int_0^t e^{\tilde{\alpha}\tau} \left( \frac{1}{2} \|u_\tau\|^2 + C(p, \beta) I(u) \right) d\tau
\]

\[
= \left( \frac{1}{2} + C(p, \beta) \right) \int_0^t e^{\tilde{\alpha}\tau} \|u_\tau\|^2 d\tau
\]

\[- C(p, \beta) \int_0^t e^{\tilde{\alpha}\tau} \frac{d}{d\tau} \left( (u_\tau, u) + \frac{1}{2} \|u\|^2 \right) d\tau, \quad 0 \leq t < +\infty
\]

with

\[
C(p, \beta) := \frac{1}{\beta} \left( 1 + \frac{\beta}{p+1} \right),
\]
where both (23) and (20) have been used.

The next step is to estimate the last term on the right-hand side of (27). Indeed from the Cauchy-Schwarz inequality, there appears

\[
- \int_0^t e^{\tilde{\alpha} \tau} \frac{d}{d\tau} \left( (u_\tau, u) + \frac{1}{2} \|u\|^2 \right) d\tau
= (u_1, u_0) + \frac{1}{2} \|u_0\|^2 - e^{\tilde{\alpha} t} \left( (u_t, u) + \frac{1}{2} \|u\|^2 \right)
+ \tilde{\alpha} \int_0^t e^{\tilde{\alpha} \tau} \left( (u_\tau, u) + \frac{1}{2} \|u\|^2 \right) d\tau
\leq \frac{1}{2} \left( \|u_1\|^2 + 2 \|u_0\|^2 \right) + \frac{1}{2} e^{\tilde{\alpha} t} \left( \|u_t\|^2 + 2 \|u\|^2 \right)
+ \frac{\tilde{\alpha}}{2} \int_0^t e^{\tilde{\alpha} \tau} \left( \|u_\tau\|^2 + 2 \|u\|^2 \right) d\tau,
\]

\(0 \leq t < +\infty.\) (28)

And an application of both (19) and Corollary 1 implies that there exists a positive constant \(C\) such that

\[
\frac{1}{2} \left( \|u_t\|^2 + 2 \|u_0\|^2 \right) \leq CE(t), \quad 0 \leq t < +\infty.
\]

Thus, substitution of (27), (28) and (29) into (24) reveals that there exist constant \(C_0\) and \(\tilde{C}_1\) such that

\[
e^{\tilde{\alpha} t} E(t) + \int_0^t e^{\tilde{\alpha} \tau} \|u_\tau\|^2 d\tau
\leq E(0) + \tilde{\alpha} \left( \frac{1}{2} + C(p, \beta) \right) \int_0^t e^{\tilde{\alpha} \tau} \|u_\tau\|^2 d\tau
+ \frac{\tilde{\alpha}}{2} C(p, \beta) \left( \|u_1\|^2 + 2 \|u_0\|^2 \right)
+ \frac{\tilde{\alpha}}{2} e^{\tilde{\alpha} t} C(p, \beta) \left( \|u_t\|^2 + 2 \|u\|^2 \right)
+ \frac{\tilde{\alpha}^2}{2} C(p, \beta) \int_0^t e^{\tilde{\alpha} \tau} \left( \|u_\tau\|^2 + 2 \|u\|^2 \right) d\tau
\leq C_0 E(0) + \tilde{\alpha} \left( \frac{1}{2} + C(p, \beta) \right) \int_0^t e^{\tilde{\alpha} \tau} \|u_\tau\|^2 d\tau
+ \tilde{\alpha} \tilde{C}_1 e^{\tilde{\alpha} t} E(t) + \tilde{\alpha}^2 \tilde{C}_1 \int_0^t e^{\tilde{\alpha} \tau} E(\tau) d\tau,
\]

\(0 \leq t < +\infty.\) (30)

Take \(\tilde{\alpha}\) such that

\[0 < \tilde{\alpha} < \min \left\{ \frac{1}{2 \tilde{C}_1}, \frac{1}{\frac{1}{2} + C(p, \beta)} \right\}.\]

Then (30) gives

\[e^{\tilde{\alpha} t} E(t) \leq 2C_0 E(0) + 2\tilde{\alpha}^2 \tilde{C}_1 \int_0^t e^{\tilde{\alpha} \tau} E(\tau) d\tau,\]

which together with the Gronwall inequality shows

\[E(t) \leq 2C_0 E(0) e^{-\tilde{\lambda} t}, \quad 0 \leq t < +\infty,\]

where

\[\tilde{\lambda} = \tilde{\alpha}(1 - 2\tilde{C}_1 \tilde{\alpha}) > 0.\]
Therefore, we complete the proof of Theorem 1.1.

3.2. **Algebraic decay for \( E(0) < d \) and \( r > 1 \).** This subsection is to prove the algebraic decay of the solution to problem (1)-(3) with \( r > 1 \) for the subcritical initial energy \( E(0) < d \) under Assumption 1.2, i.e., Theorem 1.2. The proof of Theorem 1.2 is performed on the following lemma and some analysis techniques via the multiplier method.

**Lemma 3.3** ([14]). Let \( y(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) be a non-increasing function and assume that there exist two constants \( r > 1 \) and \( A > 0 \) such that

\[
\int_0^\infty y(s) \frac{r+1}{r-1} ds \leq \frac{1}{A} y(0) \frac{r+1}{r-1} y(t), \quad t \in \mathbb{R}^+,
\]

then

\[
y(t) \leq y(0) \left( \frac{r+1}{2+A(r-1)} \right)^{\frac{1}{r-1}}, \quad t \geq 0 \quad \text{for} \quad r > 1.
\]

**Remark 5.** Note that Assumption 1.1 covers Assumption 1.2, thus Theorem 3.1 and Lemma 3.2 both hold under Assumption 1.2.

**Proof of Theorem 1.2.** Multiplying Equation (1) by \( E(t)^\frac{r+1}{2} u(t) \) and integrating over \( \Omega \times [S, T] \subset [0, +\infty) \) shows

\[
0 = \int_S^T E(t)^\frac{r+1}{2} \left( \int \nabla u \cdot \nabla u dx + ||u||_H^2 + \int |u|^{-1} u_t u dx \right) dt
\]

\[
= \int_S^T E(t)^\frac{r+1}{2} \left( \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) dx + \int \Omega u f(u) dx \right) dt.
\]

In conjunction with

\[
\int_S^T E(t)^\frac{r+1}{2} \int_{\Omega} u_t u dx dt
\]

\[
= \int_{\Omega} E(t)^\frac{r+1}{2} u_t dx |_S^T - \int_S^T E(t)^\frac{r+1}{2} ||u_t||_2^2 dt
\]

\[
= \frac{r-1}{2} \int_S^T E(t)^\frac{r+1}{2} E'(t) \int_{\Omega} u_t u dx dt,
\]

we arrive at

\[
0 = -\int_S^T E(t)^\frac{r+1}{2} \left( ||u_t||^2 - ||u||_H^2 - \int \nabla u \cdot \nabla u dx \right) dt
\]

\[
+ \int_S^T E(t)^\frac{r+1}{2} u_t u dx |_S^T - \frac{r-1}{2} \int_S^T E(t)^\frac{r+1}{2} E'(t) \int_{\Omega} u_t u dx dt
\]

\[
- \int_S^T E(t)^\frac{r+1}{2} \left( \sum_{i=1}^n \int_{\Omega} u_{x_i} \sigma_i(u_{x_i}) dx + \int \Omega u f(u) dx \right) dt
\]

\[
= \int_S^T E(t)^\frac{r+1}{2} \left( ||u_t||^2 + ||u||_H^2 - 2 \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) dx - 2 \int_{\Omega} F(u) dx \right) dt
\]

\[
+ 2 \int_S^T E(t)^\frac{r+1}{2} \left( \sum_{i=1}^n \int_{\Omega} G_i(u_{x_i}) dx + \int \Omega F(u) dx \right) dt.
\]
where

\[ E \]

with

\[ q \]

It is noted that Assumption 1.2 with \( q = p \) allows

\[
2 \int_S E(t)^{-\frac{r-1}{2}} \left( \sum_{i=1}^n \int_G G_i(u_{x_i})dx + \int F(u)dx \right) dt
= \int_S E(t)^{-\frac{r-1}{2}} \left( \frac{2}{q+1} \sum_{i=1}^n \int u_{x_i} \sigma_i(u_{x_i})dx + \frac{2}{p+1} \int f(u)dx \right) dt
= \frac{2}{p+1} \int_S E(t)^{-\frac{r-1}{2}} \left( \sum_{i=1}^n \int u_{x_i} \sigma_i(u_{x_i})dx + \int f(u)dx \right) dt.
\]

Thus, an insertion of (33) into (32) yields

\[
0 = \int_S E(t)^{-\frac{r-1}{2}} \left( 2E(t) - \frac{p-1}{p+1} \left( \sum_{i=1}^n \int u_{x_i} \sigma_i(u_{x_i})dx + \int f(u)dx \right) \right) dt
- 2 \int_S E(t)^{-\frac{r-1}{2}} ||u_t||^2 dt + \int_S E(t)^{-\frac{r-1}{2}} \int |u_t|^{r-1} u_t u d x d t
+ \int_S E(t)^{-\frac{r-1}{2}} u u_t d x \left| T_s - \frac{r-1}{2} \right| \int_S E(t)^{-\frac{r-1}{2}} E(t) E'(t) u u_t d x d t,
\]

which along with (22), \( q = p \) and (19) gives

\[
0 \geq \int_S E(t)^{-\frac{r-1}{2}} \left( 2E(t) - \frac{p-1}{p+1} \left( a_1 C_{t^p+1} + a_2 C_{2^p+1} \right) ||u||_{H}^{p+1} \right) dt
- 2 \int_S E(t)^{-\frac{r-1}{2}} ||u_t||^2 dt + \int_S E(t)^{-\frac{r-1}{2}} \int |u_t|^{r-1} u_t u d x d t
+ \int_S E(t)^{-\frac{r-1}{2}} u u_t d x \left| T_s - \frac{r-1}{2} \right| \int_S E(t)^{-\frac{r-1}{2}} E(t) E'(t) u u_t d x d t
\geq 2(1 - \hat{\beta}) \int_S E(t)^{-\frac{r-1}{2}} dt - 2 \int_S E(t)^{-\frac{r-1}{2}} ||u_t||^2 dt + \int_S E(t)^{-\frac{r-1}{2}} u u_t d x d t
- \frac{r-1}{2} \int_S E(t)^{-\frac{r-1}{2}} E'(t) u u_t d x d t + \int_S E(t)^{-\frac{r-1}{2}} ||u_t|^{r-1} u_t u d x d t,
\]

with

\[
\hat{\beta} := \left( a_1 C_{t^p+1} + a_2 C_{2^p+1} \right) \left( \frac{2(p+1)}{p-1} E(0) \right)^{(p-1)/2} < 1,
\]

where \( E(0) = d \), (16) and \( q = p \) have been used.
The following proof of this theorem is approached on the control of the last four terms on the right-hand side of (34). First, for the second term
\[ \int_S^T \int_\Omega E \frac{r+1}{k} |u_t|^2 \, dx \, dt, \]
from the following Young’s inequality
\[ XY \leq \frac{\delta^k}{k} X^k + \frac{\delta^{-q}}{q} Y^q, \quad X, Y \geq 0, \quad \delta > 0, \quad \frac{1}{k} + \frac{1}{q} = 1 \] (35)
with \( k = \frac{r+1}{r-1} \), \( s = \frac{r+1}{2} \), \( \delta = \delta_1 \) and (18), it transpires
\[ \int_S^T \int_\Omega E \left( |u_t|^2 \right) \, dx \, dt \]
\[ \leq C(\delta_1) \int_S^T \int_\Omega \left| E(t) \right|^{\frac{r+1}{k}} \, dx \, dt + \tilde{\delta}_1 \int_S^T \left| u_t \right|^{r+1} \, dt \]
\[ \leq C(\delta_1) m(\Omega) \int_S^T \int_\Omega \left( |u_t|^2 \right) \, dx \, dt + \tilde{\delta}_1 \left( E(S) - E(T) \right) \]
\[ \leq C(\delta_1) m(\Omega) \int_S^T \int_\Omega E \left( |u_t|^2 \right) \, dx \, dt + \tilde{\delta}_1 E(S), \]
where
\[ C(\delta_1) = \frac{r - 1}{r + 1} \delta_1^{r+1}, \quad \tilde{\delta}_1 = \frac{r + 1}{r + 1} \delta_1^{r+1} \]
and \( m(\Omega) \) denotes the Lebesgue measure of the bounded domain \( \Omega \).

For the last term \( \int_S^T \int_\Omega E \left( |u_t|^2 \right) \, dx \, dt \), it is inferred that
\[ \left| \int_S^T \int_\Omega E \left( |u_t|^2 \right) \, dx \, dt \right| \]
\[ \leq C(\delta_2) \int_S^T \int_\Omega \left( |u_t|^{r-1} u_t \right) \, dx \, dt \]
\[ \leq C(\delta_2) \int_S^T \int_\Omega \left( |u_t|^{r-1} u_t \right) \, dx \, dt \]
\[ \leq C(\delta_2) \int_S^T \int_\Omega \left( |u_t|^{r+1} \right) \, dx \, dt \]
\[ \leq C(\delta_2) E(S) \frac{r+1}{2} \int_S^T \left( |u_t|^{r+1} \right) \, dx \, dt + \tilde{\delta}_2 E(0) \frac{r+1}{2} \int_S^T \left( |u(t)|^{r+1} \right) \, dx \, dt \]
\[ \leq C(\delta_2) E(S) \frac{r+1}{2} + \tilde{\delta}_2 E(0) \frac{r+1}{2} \int_S^T \left( |u(t)|^{r+1} \right) \, dx \, dt \]
\[ \leq C(\delta_2) E(S) \frac{r+1}{2} + \tilde{\delta}_2 E(0) \frac{r+1}{2} \int_S^T \left( |u(t)|^{r+1} \right) \, dx \, dt \]
\[ \leq C(\delta_2) E(S) \frac{r+1}{2} + \tilde{\delta}_2 E(0) \frac{r+1}{2} C_{r+1} \int_S^T \left( |u(t)|^{r+1} \right) \, dx \, dt \]
\[ \leq C(\delta_2) E(S) \frac{r+1}{2} + \tilde{\delta}_2 E(0) \frac{r+1}{2} C_{r+1} \int_S^T \left( |u(t)|^{r+1} \right) \, dx \, dt, \]
with \( C(\delta_2) = \frac{r}{r+1} \delta_2^{(r+1)} \) and \( \hat{\delta}_2 = \frac{\delta_2}{r+1} \). In the above arguments, the Young’s inequality (35) with \( k = \frac{r+1}{r} \), \( s = r+1 \), \( \delta = \delta_2 \), Lemma 2.7, the Sobolev embedding \( H \hookrightarrow \mathbb{L}^{r+1}(\Omega) \) and Lemma 3.2 have been used.

For the fourth term \( \int_S^T E(t)^{\frac{r-1}{2}} E'(t) \frac{du(t)}{dx} \), a combination of the Cauchy-Schwarz inequality, (14) and Lemma 2.7 reveals

\[
\begin{align*}
\left| \int_S^T E(t)^{\frac{r-1}{2}} E'(t) \frac{du(t)}{dx} \right| & \leq \int_S^T E(t)^{\frac{r-1}{2}} E'(t) \left( \frac{1}{2} ||u||^2 + \frac{1}{2} ||u_t||^2 \right) dt \\
& \leq \int_S^T E(t)^{\frac{r-1}{2}} |E'(t)| \left( \frac{1}{2} ||u||^2 + \frac{1}{2} ||u_t||^2 \right) dt \\
& \leq K \int_S^T E(t)^{\frac{r-1}{2}} |E'(t)| \left( \frac{p-1}{p+1} ||u||^2 + \frac{1}{2} ||u_t||^2 \right) dt \\
& \leq K \int_S^T E(t)^{\frac{r-1}{2}} |E'(t)| dt \\
& = - K \int_S^T E(t)^{\frac{r-1}{2}} E'(t) dt \\
& = - \frac{2K}{r+1} E(t)^{\frac{r+1}{2}} \bigg|_S^T \\
& \leq \frac{2K}{r+1} E(S)^{\frac{r+1}{2}},
\end{align*}
\]

where \( K = \max \left\{ \frac{k_2(p+1)}{p-1}, 1 \right\} \).

To the end, similar to (38), an application of Lemma 2.7 to the third term \( \int_\Omega E(t)^{\frac{r+1}{2}} uu_t dx \) implies

\[
\int_\Omega E(t)^{\frac{r+1}{2}} uu_t dx \bigg|_S^T \leq KE(t)^{\frac{r+1}{2}} \bigg|_S^T \leq KE(S)^{\frac{r+1}{2}}.
\]

(39)

Thus a substitution of (36)-(39) into (34) yields

\[
\begin{align*}
2(1-\beta) & \int_S^T E(t)^{\frac{r+1}{2}} dt \\
\leq & 2C(\delta_1)m(\Omega) \int_S^T E(t)^{\frac{r+1}{2}} dt + 2\hat{\delta}_1 E(\Omega) + C(\delta_2) E(S)^{\frac{r+1}{2}} \\
& + \hat{\delta}_2 E(0)^{\frac{r+1}{2}} C_{r+1} \left( \frac{2(p+1)}{p-1} \right)^{\frac{r+1}{2}} \int_S^T E(t)^{\frac{r+1}{2}} dt \\
& + KE(S)^{\frac{r+1}{2}} + \frac{(r-1)K}{r+1} E(S)^{\frac{r+1}{2}}.
\end{align*}
\]

(40)
Let
\[
\hat{\alpha} := 2(1 - \hat{\beta}) - 2C(\delta_1)m(\Omega) - \hat{\delta}_2E(0)^{\frac{p+1}{p-1}}C_{r+1}^{\frac{r+1}{p}} \left(\frac{2(p+1)}{p-1}\right)^{\frac{r+1}{p-1}}
\]
\[
= 2 \left(1 - \left(a_1C_1^{\frac{r+1}{p-1}} + a_2C_2^{\frac{r+1}{p-1}}\right) \left(\frac{2(p+1)}{p-1}E(0)^{\frac{p-1}{2}}\right)\right)
\]
\[
- \frac{2(r-1)}{r+1}\delta_1^{-\frac{r+1}{p}}m(\Omega) - \delta_2\left(\frac{r+1}{r+1}E(0)^{\frac{r+1}{p-1}}\right)^{\frac{r+1}{p-1}}\left(\frac{2(p+1)}{p-1}\right)^{\frac{r+1}{p-1}},
\]
which together with Lemma 2.7 reduces (40) to
\[
\hat{\alpha} \int_S^T E(t)^{\frac{r+1}{p-1}} \, dt
\]
\[
\leq 2\delta_1E(S) + \left(C(\delta_2) + \frac{2rK}{r+1}\right)E(S)^{\frac{r+1}{p-1}}
\]
\[
= \left(2\delta_1 + \left(C(\delta_2) + \frac{2rK}{r+1}\right)E(S)^{\frac{r+1}{p-1}}\right)E(S) \leq \left(2\delta_1 + \left(C(\delta_2) + \frac{2rK}{r+1}\right)E(0)^{\frac{r+1}{p-1}}\right)E(S) = \nu E(S),
\]
where
\[
\nu := \frac{4}{r+1}\delta_1^{-\frac{r+1}{p}} + \left(\frac{r}{r+1}\delta_2^{-\frac{r+1}{p}} + \frac{2rK}{r+1}\right)E(0)^{\frac{r+1}{p-1}}.
\]
Then for small enough \(\delta_1\) and sufficiently large \(\delta_2\), (41) becomes
\[
\int_S^T E(t)^{\frac{r+1}{p-1}} \, dt \leq \frac{\nu}{\hat{\alpha}}E(S),
\]
which along with Lemma 3.3 gives (7) with \(\hat{A} = \frac{\hat{\alpha}}{\nu}E(0)^{\frac{r+1}{p-1}}\). Then, we finish the proof of Theorem 1.2.

REFERENCES
[1] R. A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] L. J. An and A. Peirce, A weakly nonlinear analysis of elasto-plastic-microstructure models, 
SIAM J. Appl. Math., 55 (1995), 136–155.
[3] L. J. An and A. Peirce, The effect of microstructure on elastic-plastic models, SIAM J. Appl. 
Math., 54 (1994), 708–730.
[4] V. Barros, C. Nonato and C. Raposo, Global existence and energy decay of solutions for 
a wave equation with non-constant delay and nonlinear weights, Electron. Res. Arch., 28 (2020), 205–220.
[5] T. Cazenave and Z. Han, Asymptotic behavior for a Schrödinger equation with nonlinear 
subcritical dissipation, Discrete Contin. Dyn. Syst., 40 (2020), 4801–4819.
[6] G. W. Chen and Z. J. Yang, Existence and non-existence of global solutions for a class of 
nonlinear wave equations, Math. Meth. Appl. Sci., 23 (2000), 615–631.
[7] H. Di, Y. Shang and J. Yu, Existence and uniform decay estimates for the fourth order wave 
equation with nonlinear boundary damping and interior source, Electron. Res. Arch., 28 (2020), 221–261.
[8] J. A. Esquivel-Avila, Dynamics around the ground state of a nonlinear evolution equation, 
Nonlinear Anal., 63 (2005), 331–343.
[9] L. H. Fatori, M. A. Jorge Silva, T. F. Ma and Z. Yang, Long-time behavior of a class of 
thermoelastic plates with nonlinear strain, J. Differential Equations, 259 (2015), 4831–4862.
[10] V. Georgiev and G. Todorova, Existence of solutions of the wave equation with nonlinear damping and source terms, J. Differential Equations, 109 (1994), 295–308.
[11] E. H. Gomes Tavares, M. A. Jorge Silva and T. F. Ma, Sharp decay rates for a class of nonlinear viscoelastic plate models, Commun. Contemp. Math., 20 (2018), 1750010, 21 pp.
[12] J. Han, R. Xu and Y. Yang, Asymptotic behavior and finite time blow up for damped fourth order nonlinear evolution equation, Asymptotic. Anal., 122 (2021), 349–369.
[13] M. A. Jorge Silva and T. F. Ma, On a viscoelastic plate equation with history setting and perturbation of \( p \)-Laplacian type, IMA J. Appl. Math., 78 (2013), 1130–1146.
[14] V. Komornik, Exact Controllability and Stabilization, The Multiplier Method, Research in Applied Mathematics, Masson, Paris, France, 1994.
[15] J. Kreulich, Asymptotic behavior of evolution systems in arbitrary Banach spaces using general almost periodic splittings, Adv. Nonlinear Anal., 8 (2019), 1–28.
[16] W. Lian, V. D. Rădulescu, R. Xu, Y. Yang and N. Zhao, Global well-posedness for a class of fourth order nonlinear strongly damped wave equations, Adv. Calc. Var., 14 (2021), 589–611.
[17] W. Lian and R. Xu, Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term, Adv. Nonlinear Anal., 9 (2020), 613–632.
[18] J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires, Dunod, Paris, 1969.
[19] Y. Liu, Long-time behavior of a class of viscoelastic plate equations, Electron. Res. Arch., 28 (2020), 311–326.
[20] Y. Liu and R. Xu, A class of fourth order wave equations with dissipative and nonlinear strain terms, J. Differential Equations, 244 (2008), 200–228.
[21] Y. Liu and R. Xu, Fourth order wave equations with nonlinear strain and source terms, J. Math. Anal. Appl., 331 (2007), 585–607.
[22] T. F. Ma and M. L. Pelicer, Attractors for weakly damped beam equations with \( p \)-Laplacian, Discrete Contin. Dyn. Syst., 2013 (2013), 525–534.
[23] A. Mohammed, V. D. Rădulescu and A. Vitolo, Blow-up solutions for fully nonlinear equations: Existence, asymptotic estimates and uniqueness, Adv. Nonlinear Anal., 9 (2020), 39–64.
[24] T. T. Nguyen, Asymptotic limit and decay estimates for a class of dissipative linear hyperbolic systems in several dimensions, Discrete Contin. Dyn. Syst., 39 (2019), 1651–1684.
[25] F. Shakeri and M. Dehghan, A hybrid Legendre tau method for the solution of a class of nonlinear wave equations with nonlinear dissipative terms, Numer. Methods Partial Differential Equations, 27 (2011), 1055–1071.
[26] J. Shen, Y. Yang, S. Chen and R. Xu, Finite time blow up of fourth order wave equations with nonlinear strain and source terms at high energy level, Internat. J. Math., 24 (2013), 1350043, 8 pp.
[27] Y. Wang and Y. Wang, On the initial-boundary problem for fourth order wave equations with damping, strain and source terms, Internat. J. Math., 24 (2013), 1350043, 8 pp.
[28] Z.-Y. Yan, Similarity reduction and integrability for the nonlinear wave equations from EPM model, Commun. Theor. Phys (Beijing), 35 (2001), 647–650.
[29] X.-G. Yang, M. J. D Nascimento and M. L. Pelicer, Uniform attractors for non-autonomous plate equations with \( p \)-Laplacian perturbation and critical nonlinearities, Discrete Contin. Dyn. Syst., 40 (2020), 1937–1961.
[30] Z.-J. Yang, Global existence, asymptotic behavior and blowup of solutions for a class of nonlinear wave equations with dissipative term, J. Differential Equations, 187 (2003), 520–540.

Received July 2021; revised August 2021; early access October 2021.

E-mail address: yangchao@hrbeu.edu.cn
E-mail address: yangyanbing@hrbeu.edu.cn