Petals and books: The largest Laplacian spectral gap from 1

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Abstract

We prove that, for any connected graph on \( N \geq 3 \) vertices, the spectral gap from the value 1 with respect to the normalized Laplacian is at most 1/2. Moreover, we show that equality is achieved if and only if the graph is either a petal graph (for \( N \) odd) or a book graph (for \( N \) even). This implies that \( \left(\frac{1}{2}, \frac{3}{2}\right) \) is a maximal gap interval for the normalized Laplacian on connected graphs. This is closely related to the Alon–Boppana bound on regular graphs and a recent result by Kollár and Sarnak on cubic graphs. Our result also provides a sharp bound for the convergence rate of some eigenvalues of the Laplacian on neighborhood graphs.

KEYWORDS
eigenvalue 1, maximal gap interval, neighborhood graph, normalized Laplacian, spectral gaps, spectral graph theory

1 | INTRODUCTION

A spectral gap is the maximal difference between two eigenvalues for linear operators in some class. Here, we consider Laplace operators, more precisely, the normalized Laplacian of a finite graph. Such inequalities were first studied for the Laplace operator \( \Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial(x^i)^2} \) on a connected smooth domain \( \Omega \subset \mathbb{R}^n \) with Dirichlet conditions, that is,

\[
\Delta u = \lambda u \text{ in } \Omega,
\]
\begin{equation}
\frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,
\end{equation}

where a (smooth) solution \( u \) that does not vanish identically is called an eigenfunction, and the corresponding value \( \lambda \) an eigenvalue. We choose the sign in the definition of \( \Delta \) to make it a nonnegative operator. Since \( \Omega \) is connected, the smallest eigenvalue \( \lambda_1 \) is positive, and the famous Faber–Krahn inequality \([10, 15, 16]\) says that among all domains with the same volume, the smallest possible value is realized by a ball of that volume. By a result of Ashbaugh and Benguria \([1]\), the ratio between the first two eigenvalues, \( \frac{\lambda_2}{\lambda_1} \), is the largest for the ball. In either case, equality is assumed precisely for the ball. When we replace the Dirichlet boundary condition (2) by the Neumann boundary condition

\begin{equation}
\frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega,
\end{equation}

then the smallest eigenvalue is \( \lambda_1 = 0 \), with the constants being eigenfunctions, and Weinberger \([19]\) proved that the second smallest eigenvalue \( \lambda_2 \) now is always less than or equal to that of the ball, again with equality only for the ball. While the proofs of such results can be difficult, there is a general pattern here, that the extremal cases occur only for a very particular class of domains, balls in this case. Similarly, for eigenvalue problems for the Laplace–Beltrami operator in Riemannian geometry, often the extremal case is realized by spheres (for the eigenvalue problem in Riemannian geometry, see, e.g., the references given in \([4, 11]\)).

There are also discrete versions of those Laplacians, and naturally, their spectra have also been investigated. For the algebraic graph Laplacian, a systematic analysis of spectral gaps is presented in \([14]\), and these authors have identified many beautiful classes of graphs with a particular structure of their spectra and spectral gaps. Here, we consider another discrete Laplacian, the normalized Laplace operator of a connected, finite, simple graph \( \Gamma = (V, E) \) on \( N \geq 3 \) vertices. For a vertex \( v \in V \), we denote by \( \deg v \) its degree, that is, the number of its neighbors, that is, the other vertices \( w \sim v \) connected to \( v \) by an edge. Then the Laplacian for a function \( f : V \to \mathbb{R} \) is

\begin{equation}
\Delta f(v) = f(v) - \frac{1}{\deg v} \sum_{w \sim v} f(w),
\end{equation}

that is, we subtract from the value of \( f \) at \( v \) the average of the values at its neighbors. This operator generates random walks and diffusion processes on graphs, and it was first systematically studied in \([6]\). Since its spectrum is that of an \( (N \times N) \)-matrix with 1s in the diagonal, the eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \) satisfy

\begin{equation}
\sum_{i=1}^{N} \lambda_i = N.
\end{equation}

The first eigenvalue now is \( \lambda_1 = 0 \), but since \( \Gamma \) is connected, the second eigenvalue \( \lambda_2 \) is positive. There are several inequalities controlling \( \lambda_2 \) from below in terms of properties of the graph (see \([6]\)), and the largest value among all graphs with \( N \) vertices is realized by the complete graph \( K_N \) where \( \lambda_2 = \frac{N}{N-1} \), for all other graphs \( \lambda_2 \leq 1 \) \([6, \text{Lemma 1.7}]\). The largest eigenvalue \( \lambda_N \) is always less than or equal to 2, with equality if and only if \( \Gamma \) is bipartite. And
the gap \(2 - \lambda_N\) quantifies how different \(\Gamma\) is from being bipartite [3]. In fact, the smallest possible value \(\lambda_N = \frac{N}{N-1}\) is again realized only for \(K_N\). For all other graphs, \(\lambda_N \geq \frac{N+1}{N-1}\) [8], and again, the extremal graphs, where equality is realized, can be characterized [12].

Thus, the situation for the spectral gaps at the ends of the spectrum, that is, at 0 and near 2 has been clarified. But we may also ask about gaps in the middle of the spectrum. In fact, besides 0 and 2, also the eigenvalue 1 plays a special role. It arises, in particular, from vertex duplications [2]. The extreme case is given by complete bipartite graphs \(K_{n,m}\) with \(n + m = N\). They have the eigenvalue 1 with multiplicity \(N - 2\). In fact, if 1 occurs with this multiplicity, then by (5), 2 also has to be an eigenvalue, and the graph is bipartite. But like the eigenvalue 2, the eigenvalue 1 need not be present in a graph. Therefore, we can ask about the maximal spectral gap at 1, that is, we can ask what the maximal value of

\[
\varepsilon := \min_i |1 - \lambda_i|
\]
could be. In this paper, we show that for any graph with \(N \geq 3\) vertices, \(\varepsilon \leq \frac{1}{2}\) (for the graph with two vertices, the eigenvalues are 0 and 2, therefore, in this particular case, the gap is 1), and as the title already reveals, we can identify the class of graphs for which the maximal possible value \(\varepsilon = \frac{1}{2}\) is realized. In fact, in those cases, except for the triangle \(K_3\), which only has \(\frac{3}{2}\), both values \(\frac{1}{2}\) and \(\frac{3}{2}\) are eigenvalues.

Our results have fit into a larger picture. They have connections with expander graphs and random walks on graphs, including Alon–Boppana’s theorem on Ramanujan graphs, Kollár–Sarnak’s theorem on the maximal gap interval for cubic graphs [14], as well as Bauer–Jost’s Laplacian on neighborhood graphs [3]. We shall explain these relations in Section 2.

2 | MAIN RESULT

Throughout the paper we fix a connected, finite, simple graph \(\Gamma = (V, E)\) on \(N \geq 3\) vertices. We let \(d\) denote the smallest vertex degree, we let \(C(V)\) denote the space of functions \(f : V \to \mathbb{R}\) and, given a vertex \(v\), we let \(N(v) := \{w \in V : w \sim v\}\) denote the neighborhood of \(v\). We let \(\lambda_1 = 0 < \lambda_2 \leq \cdots \leq \lambda_N\) denote the eigenvalues of the Laplacian in (4), and we let

\[
\varepsilon := \min_i |1 - \lambda_i|
\]
be the spectral gap from 1. Clearly, \(\varepsilon \geq 0\) and this inequality is sharp since, for instance, any graph with duplicate vertices (i.e., vertices that have the same neighbors [17]) has 1 as an eigenvalue. As anticipated in the introduction, here we prove that \(\varepsilon \leq \frac{1}{2}\) and equality is achieved if and only if \(\Gamma\) belongs to one of the following two optimal classes.

**Definition 1** (Petal graph, \(N \geq 3\) odd). Given \(m \geq 1\), the \(m\)-petal graph is the graph on \(N = 2m + 1\) vertices such that (Figure 1):

- \(V = \{x, v_1, ..., v_m, w_1, ..., w_m\}\);
- \(E = \{(x, v_i)\}_{i=1}^m \cup \{(x, w_i)\}_{i=1}^m \cup \{(v_i, w_i)\}_{i=1}^m\).
Petal graphs are also known as *Dutch windmill graphs* or *friendship graphs*. They appear in the famous Friendship Theorem from Erdős, Rényi, and Sós [9], which states that the only finite graphs with the property that every two vertices have exactly one neighbor in common are precisely the petal graphs. In fact, a proof of this result can proceed via spectral methods. The friendship assumption determines the square of the adjacency matrix, and hence the spectrum of that matrix, and one then proceeds by showing that this spectrum implies that the graph in question has to be a petal graph. This is another example, and one relevant for the present paper of how the structure of a graph can be determined from its spectrum.

As shown in [6], for the petal graph on $N = 2m + 1$ vertices, the eigenvalues are $0, \frac{1}{2}$ (with multiplicity $m - 1$), and $\frac{3}{2}$ (with multiplicity $m + 1$). Therefore, $\varepsilon = \frac{1}{2}$ in this case.

In fact, the eigenfunctions are easily constructed. Putting $f(w_i) = -f(v_i)$ and $f(x) = 0$ produces $m$ linearly independent eigenfunctions for the eigenvalue $\frac{3}{2}$. Letting $f(v_i) = f(w_i) = 1$ for all $i$ and $f(x) = -2$ produces another eigenfunction for that eigenvalue. With $f(v_i) = f(w_i)$ for all $i$, $\sum_i f(v_i) = 0 = f(x)$ we get $m - 1$ linearly independent eigenfunctions for the eigenvalue $\frac{1}{2}$. Again, this is a good example of how the structure of a graph and properties of its spectrum are tightly related. Whenever we have two neighboring vertices $v, w$ that have all their other neighbors in common, the function with $f(v) = 1, f(w) = -1, f(z) = 0$ for all other vertices is an eigenfunction, and the eigenvalue depends on the number of those common neighbors. When, as here, there is only one common neighbor for such a pair, the eigenvalue is $3/2$. This eigenvalue will also occur for the book graph to be introduced in a moment, which for an even number of pages is a two-fold cover of the petal graph. But the book graph will be bipartite, and hence have the eigenvalue 2, which the petal graph, not being bipartite, cannot possess.

**Definition 2** (Book graph, $N \geq 4$ even). Given $m \geq 1$, the $m$-book graph is the graph on $N = 2m + 2$ vertices such that (Figure 2):

- $V = \{x, y, v_1, ..., v_m, w_1, ..., w_m\}$;
- $E = \{(x, v_i)\}_{i=1}^m \cup \{(y, w_i)\}_{i=1}^m \cup \{(v_i, w_i)\}_{i=1}^m$.

**Remark 1.** For the book graph on $N = 2m + 2$ vertices, 0 and 2 are eigenvalues with multiplicity 1, since $\Gamma$ is connected and bipartite. Moreover, one can check that

![The petal graph.](image)
\( \lambda = 1 \pm \frac{1}{2} \) are eigenvalues with multiplicity \( m \) each. In fact, the corresponding eigenfunctions can be constructed as follows:

1. By letting \( \sum_i f(v_i) = 0, f(x) = f(y) = 0, f(w_i) = \mp f(v_i) \), we obtain \( m - 1 \) linearly independent eigenfunctions.
2. By letting \( f(v_i) = -f(w_i) = 1 \) and \( f(x) = -f(y) = 2 \), we obtain one more eigenfunction for \( \frac{1}{2} \) and similarly by letting \( g(v_i) = g(w_i) = -1 \) and \( g(x) = g(y) = 2 \), we obtain one more eigenfunction for \( \frac{3}{2} \).

Hence, also in this case, \( \varepsilon = \frac{1}{2} \).

Our main result is that these, and only these, examples have the largest possible spectral gap \( \varepsilon = \frac{1}{2} \) at 1.

**Theorem 1.** For any connected graph \( \Gamma \) on \( N \geq 3 \) vertices,

\[
\varepsilon \leq \frac{1}{2}.
\]

Moreover, equality is achieved if and only if \( \Gamma \) is either a petal graph (for \( N \) odd) or a book graph (for \( N \) even).

As a consequence of Theorem 1, we can infer that both petal graphs and book graphs are uniquely characterized by their normalized Laplacian spectra. This relates to [7], where it has been proved that, among connected graphs, the petal graphs are uniquely determined by the eigenvalues of the adjacency matrix.

We prove Theorem 1 in Section 3. For completeness, we also state the following result on the value \( \max_{i \neq 1} |\lambda_i - 1| \).

**Proposition 2.** For any connected graph \( \Gamma \) on \( N \geq 2 \) vertices,

\[
\frac{1}{N - 1} \leq \max_{i \neq 1} |\lambda_i - 1| \leq 1.
\]
Moreover, the lower bound is an equality if and only if $\Gamma$ is the complete graph; the upper bound is an equality if and only if $\Gamma$ is bipartite.

**Proof.** By (5), using the fact that $\lambda_1 = 0$, it follows that $\sum_{i=2}^{N} (\lambda_i - 1) = 1$. Thus,

$$\frac{1}{N-1} \leq \frac{1}{N-1} \cdot \left( \sum_{i=2}^{N} |\lambda_i - 1| \right) \leq \max_{i \neq 1} |\lambda_i - 1|$$

and equality holds if and only if $\lambda_2 = \cdots = \lambda_N = \frac{N}{N-1}$, that is, if and only if $\Gamma$ is the complete graph.

The other claim follows from the fact that

$$0 < \lambda_2 \leq \cdots \leq \lambda_N \leq 2$$

and $\lambda_N = 2$ if and only if $\Gamma$ is bipartite. \(\square\)

We also prove the following lemma, which will be needed in the proof of Theorem 1 and which is an interesting result itself, since it allows us to characterize $\varepsilon$ for any graph.

**Lemma 3.** For any graph $\Gamma$,

$$\varepsilon^2 = \min_{f \in \mathcal{C}(V) \setminus \{0\}} \frac{\sum_{v \in V} \frac{1}{\deg w} \left( \sum_{u \in \mathcal{N}(w)} f(u) \right)^2}{\sum_{w \in V} \deg w \cdot f(w)^2}.$$ 

**Proof.** We observe that the values $(1 - \lambda_1)^2, \ldots, (1 - \lambda_N)^2$ are exactly the eigenvalues of the matrix $M := (\text{Id} - D^{-\frac{1}{2}} \Delta D^{-\frac{1}{2}})^2$ whose entries are

$$M_{uv} = \sum_{w \in \mathcal{N}(u) \cap \mathcal{N}(v)} \frac{1}{\deg w \sqrt{\deg u \cdot \deg v}}$$

for $u, v \in V$, where $\text{Id}$ is the $N \times N$ identity matrix and $D = \text{diag}(\deg 1, \ldots, \deg N)$ is the diagonal matrix consisting of degrees. In particular, $\varepsilon^2$ is the smallest eigenvalue of $M$. Therefore, by the Courant–Fischer–Weyl min–max principle, it can be written as

$$\varepsilon^2 = \min_{f \in \mathcal{C}(V) \setminus \{0\}} \frac{\sum_{u, v \in V} \frac{1}{\deg w} \left( \sum_{u \in \mathcal{N}(w)} f(u) \right)^2}{\sum_{w \in V} \deg w \cdot f(w)^2}.$$ 

Now, observe that the numerator can be rewritten as

$$\sum_{u, v \in V} \sum_{w \in \mathcal{N}(u) \cap \mathcal{N}(v)} \frac{f(u) \cdot f(v)}{\deg w \sqrt{\deg u \cdot \deg v}} = \sum_{w \in V} \sum_{u, v \in \mathcal{N}(w)} \frac{f(u) \cdot f(v)}{\deg w \sqrt{\deg u \cdot \deg v}}$$

$$= \sum_{w \in V} \frac{1}{\deg w} \left( \sum_{v \in \mathcal{N}(w)} \frac{f(v)}{\sqrt{\deg v}} \right)^2.$$
It follows that

\[
\varepsilon^2 = \min_{f \in C(V) \setminus \{0\}} \frac{\sum_{w \in V} \frac{1}{\text{deg } w} \left( \sum_{v \in N(w)} f(v) \right)^2}{\sum_{w \in V} f(w)^2} = \min_{f \in C(V) \setminus \{0\}} \frac{\sum_{w \in V} \frac{1}{\text{deg } w} \left( \sum_{v \in N(w)} f(v) \right)^2}{\sum_{w \in V} \text{deg } w \cdot f(w)^2}.
\]

This lemma will play an important role for controlling \( \varepsilon \), because we can derive inequalities on vertex degrees in case \( \varepsilon \geq \frac{1}{2} \), by choosing suitable local functions \( f \).

**Remark 2.** A gap interval with respect to a family \( G \) of simple graphs is an open interval such that there are infinitely many graphs in \( G \) whose Laplacian spectrum does not intersect the interval. Clearly, Theorem 1 trivially implies that, for \( G = \{ \text{connected graphs} \}, \left( \frac{1}{2}, \frac{3}{2} \right) \) is a maximal gap interval. This can be compared to a result by Kollár and Sarnak [14], which states that, for \( G = \{ \text{connected regular graphs of degree 3} \}, \left( \frac{2}{3}, \frac{4}{3} \right) \) is a maximal gap interval. (The original result, Theorem 3 in [14], is formulated in terms of the adjacency matrix, but since these graphs are regular, the statement can be equivalently reformulated in terms of \( \Delta \).) We also refer to the recent work of the application of gap intervals to microwave coplanar waveguide resonators by Kollár et al. [13]. This line of research has an origin in the Alon–Boppana theorem [5, 18], which implies that \( \left( 0, 1 - \frac{2\sqrt{2}}{3} \right) \) is a maximal gap interval for the Laplacian on cubic graphs, and this gap is achieved by Ramanujan graphs.

Combining Theorems 1 and 16, we have the following:

**Theorem 4.** For any connected graph \( \Gamma \) on \( N \geq 3 \) vertices with smallest degree \( d \geq 2 \),

\[
\varepsilon \leq \frac{\sqrt{d - 1}}{d}.
\]

This is rather interesting, and can be compared to Alon–Boppana’s work stating that for any \( d \)-regular Ramanujan graph,

\[
\max_{i \neq 1} |\lambda_i - 1| \leq 2\frac{\sqrt{d - 1}}{d}.
\]

See Proposition 2 for another comparison.
Furthermore, our results provide a sharp bound for the convergence rate of some eigenvalues of the Laplacian on neighborhood graphs [3].

The neighborhood graph $\Gamma^{[\ell]}$ of order $\ell$ of a graph $\Gamma = (V, E)$ is a weighted graph whose edge weight $w_{ij}$ equals the probability that a random walker starting at $i$ reaches $j$ in $\ell$ steps. The neighborhood graphs $\{\Gamma^{[\ell]}\}_{\ell=1}^{\infty}$ encode properties of random walks on $\Gamma$, asymptotic ones if $\ell \to \infty$. We thereby gain a new source of geometric intuition for obtaining eigenvalue estimates. The graph Laplacian $\Delta^{[\ell]}$ on $\Gamma^{[\ell]}$ satisfies $\Delta^{[\ell]}u = (I - (I - \Delta))u$ for $u \in \ell^2(\Gamma)$ (see [3]). By Theorem 1, we have:

**Theorem 5.** For every connected graph $\Gamma$ with at least three vertices, there is some eigenvalue $\lambda^{[\ell]}$ of $\Delta^{[\ell]}$ with

$$|1 - \lambda^{[\ell]}| \leq \frac{1}{2^\ell}.$$  

When $\ell$ is even, the largest eigenvalue of $\Gamma^{[\ell]}$ satisfies

$$1 - \frac{1}{2^\ell} \leq \lambda^{[\ell]}_N \leq 1,$$

and both bounds are sharp.

### 3 | PROOF OF THE MAIN RESULT

This section is dedicated to the proof of Theorem 1 that we split into two main parts. In particular, in Section 3.1 we prove that $\epsilon \leq \frac{1}{2}$ for any connected graph $\Gamma$ on $N \geq 3$ vertices, while in Section 3.2 we prove that $\epsilon = \frac{1}{2}$ if and only if $\Gamma$ is either a petal graph or a book graph.

#### 3.1 | Upper bound

In this section we prove the first claim of Theorem 1, namely.

**Theorem 6.** For any connected graph $\Gamma$ on $N \geq 3$ vertices,

$$\epsilon \leq \frac{1}{2}.$$  

Before proving it by contradiction, we show several properties that a connected graph on $N \geq 3$ vertices with $\epsilon > \frac{1}{2}$ would have. We begin by observing that, by Lemma 3, such a graph should satisfy

$$\sum_{v \in V} \frac{1}{\deg v} \left( \sum_{v \in N(w)} f(v) \right)^2 > \frac{1}{4} \sum_{w \in V} \deg w \cdot f(w)^2, \quad \forall f \in C(V) \setminus \{0\}. \tag{6}$$
Lemma 7. Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices such that \( \varepsilon > \frac{1}{2} \). Then, for any \( v \in V \), there exists \( w \in \mathcal{N}(v) \) such that \( \deg w \leq 3 \).

Proof. Let \( f \) be such that \( f(v) := 1 \) and \( f(u) := 0 \) for all \( u \neq v \). Then, (6) implies

\[
\sum_{w \in \mathcal{N}(v)} \frac{1}{\deg w} > \frac{1}{4} \deg v.
\]

Now, if \( \deg w \geq 4 \) for all \( w \in \mathcal{N}(v) \), then

\[
\sum_{w \in \mathcal{N}(v)} \frac{1}{\deg w} \leq \sum_{w \in \mathcal{N}(v)} \frac{1}{4} = \frac{1}{4} \deg v,
\]

which is a contradiction. \( \square \)

Given two vertices \( u \) and \( v \), we denote by \( \mathcal{N}(u) \triangle \mathcal{N}(v) \) the symmetric difference of \( \mathcal{N}(u) \) and \( \mathcal{N}(v) \), that is, the set of vertices that are neighbors of either \( u \) or \( v \).

Lemma 8. Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices such that \( \varepsilon > \frac{1}{2} \). If \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \), then \( \mathcal{N}(u) \triangle \mathcal{N}(v) \neq \emptyset \) and

\[
\sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \left( \frac{1}{\deg w} - \frac{1}{4} \right) > \frac{1}{2}.
\]  

(7)

Proof. Let \( f \) be such that \( f(v) := 1, f(u) := -1 \), and \( f(w) := 0 \) for all \( w \in V \setminus \{u, v\} \). Then, by (6),

\[
\sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \frac{1}{\deg w} > \frac{1}{4} (\deg v + \deg u),
\]

which also implies \( \mathcal{N}(u) \triangle \mathcal{N}(v) \neq \emptyset \). Moreover, since

\[
\deg v + \deg u = |\mathcal{N}(u) \triangle \mathcal{N}(v)| + 2|\mathcal{N}(u) \cap \mathcal{N}(v)|,
\]

the above inequality can be rewritten as

\[
\sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \left( \frac{1}{\deg w} - \frac{1}{4} \right) > \frac{1}{4} \cdot 2|\mathcal{N}(u) \cap \mathcal{N}(v)|.
\]

Since \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \), we have that \( |\mathcal{N}(u) \cap \mathcal{N}(v)| \geq 1 \), implying that

\[
\sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \left( \frac{1}{\deg w} - \frac{1}{4} \right) > \frac{1}{2}.
\]  

\( \square \)
Lemma 9. Let $\Gamma$ be a connected graph on $N \geq 3$ vertices such that $\varepsilon > \frac{1}{2}$.

1. If $\deg u = \deg w = 1$, $u \neq w$, then there is no vertex $v$ with $u \sim v \sim w$.
2. There are no distinct vertices $u \sim v \sim w$ with

$$1 \leq \deg u \leq 2, \quad 1 \leq \deg v \leq 2, \quad \text{and} \quad 1 \leq \deg w \leq 2.$$ 

Proof. The first statement is a direct consequence of Lemma 8.

We now prove the second statement. Assume the contrary, that is, there exist three vertices $u \sim v \sim w$ with $\{\deg u, \deg v, \deg w\} \subseteq \{1, 2\}$. We only need to check the following two cases.

Case 1: $\deg u = 1$, $\deg v = 2$, and $\deg w = 2$.

Since $\mathcal{N}(u) \cap \mathcal{N}(w) \neq \emptyset$, by (7), the vertex $w_1 \in \mathcal{N}(w) \setminus \{v\}$ has degree 1. Hence, $\Gamma$ is a path of length 3 (Figure 3). But this is a contradiction, since $\varepsilon = \frac{1}{2}$ for the path of length 3, which is also the 1-book graph.

Case 2: $\deg u = 2$, $\deg v = 2$, and $\deg w = 2$.

Since $\mathcal{N}(u) \cap \mathcal{N}(w) \neq \emptyset$, by (7), the vertices $w_1 \in \mathcal{N}(w) \setminus \{v\}$ and $u_1 \in \mathcal{N}(u) \setminus \{v\}$ have degree 1. Hence, $\Gamma$ is a path of length 4. But this is also a contradiction, since $\varepsilon < \frac{1}{2}$ in this case.

Lemma 10. Let $\Gamma$ be a connected graph on $N \geq 3$ vertices such that $\varepsilon > \frac{1}{2}$. Then, there are no vertices $u \sim v \sim w$ with

$$2 \leq \deg u \leq 3, \quad 2 \leq \deg v \leq 3, \quad \text{and} \quad 2 \leq \deg w \leq 3.$$ 

Proof. Assume the contrary, that is, there exist three vertices $u \sim v \sim w$ with $\{\deg u, \deg v, \deg w\} \subseteq \{2, 3\}$. We need to check five cases.

Case 1: $\deg u = 2$, $\deg v = 3$, and $\deg w = 2$.

By applying (7) three times, we have that:

- There exists $w_1 \in \mathcal{N}(u) \Delta \mathcal{N}(w)$ with $\deg w_1 = 1$, and without loss of generality, we may assume that $w_1 \sim w$;
- There exists $v_1 \in \mathcal{N}(v) \Delta \mathcal{N}(w_1) = \mathcal{N}(v) \setminus \{w\}$ with $\deg v_1 = 1$;
- There exists $u_1 \in \mathcal{N}(u) \Delta \mathcal{N}(v_1) = \mathcal{N}(u) \setminus \{v\}$ with $\deg u_1 = 1$.

Hence, $\Gamma$ is the graph in Figure 4.

By letting $f(u_1) := f(w_1) := -1, f(v) := 1$, and $f(v_1) := f(u) := f(w) := 0$, from (6) we derive a contradiction.

Case 2: $\deg u = 2$, $\deg v = 2$, and $\deg w = 3$.

If there is only one vertex $x$ in $\mathcal{N}(u) \Delta \mathcal{N}(w)$, then by (7), it must have degree 1.
Also, by construction, it is clear that $x \sim w$. Moreover, by (7), the only vertex $y \in \mathcal{N}(w) \setminus \{v, x\}$ has degree 1. But this is a contradiction, since we must also have $y \sim u$.

Otherwise, there are three vertices in $\mathcal{N}(u) \Delta \mathcal{N}(w)$ and at least two of them have degree 2. In this case, we reduce to Case 1.

**Case 3:** $\deg u = 2$, $\deg v = 3$, and $\deg w = 3$.

If $\deg x \geq 2$ for all $x \in \mathcal{N}(u) \Delta \mathcal{N}(w)$, then by (7), there are at least two vertices $x$ and $y$ in $\mathcal{N}(u) \Delta \mathcal{N}(w)$ with degree 2, and therefore we reduce to Case 1.

There are two subcases left:

(a) If there exists $u_1 \sim u$ with $\deg u_1 = 1$, then by (7) there exists $v_1 \sim v$ with $\deg v_1 = 1$ and there exists $w_1 \sim w$ with $\deg w_1 = 1$. Therefore, $\Gamma$ has the same local structure as the graph in Figure 5. By letting $f(u_1) = f(w_1) = -1, f(v) = 1$, and $f|_{V \setminus \{v, u_1, w_1\}} = 0$, from (6) we get $1 > \frac{1}{4}(1 + 3 + 1)$, which is a contradiction.

(b) If there exists $w_1 \sim w$ with $\deg w_1 = 1$, then there exists $v_1 \sim v$ with $\deg v_1 = 1$ and there exists $u_1 \sim u$ with $\deg u_1 = 1$. Hence, as in the previous subcase, $\Gamma$ has the same local structure as the graph in Figure 5, which brings to a contradiction.

**Case 4:** $\deg u = 3$, $\deg v = 2$, and $\deg w = 3$.

If $\deg x \geq 2$ for all $x \in \mathcal{N}(u) \Delta \mathcal{N}(w)$, then by (7) there are at least two vertices $x$ and $y$ in $\mathcal{N}(u) \Delta \mathcal{N}(w)$ of degree 2, and thus we reduce to Case 1.

Otherwise, there exists $u_1 \sim u$ with $\deg u_1 = 1$. Applying Lemma 8 to the vertex pair $u_1$ and $v$, we derive a contradiction.

**Case 5:** $\deg u = 3$, $\deg v = 3$, and $\deg w = 3$.

Similar to the above cases, by repeatedly applying (7) we can see that there exist $v_1 \sim v, u_1 \sim u$, and $w_1 \sim w$ with $\deg v_1 = \deg u_1 = \deg w_1 = 1$. Therefore, $\Gamma$ has the same local structure as the graph in Figure 6. Similarly to Case 3, also in this case we derive a contradiction.

We now fix the following notations. Given a vertex $v$, we let $\square$
Lemma 11. Let $\Gamma$ be a connected graph on $N \geq 3$ vertices such that $\varepsilon > \frac{1}{2}$. Then, for each $v \in V$, $|N_{2,3}(v)| \leq 1$.

Proof. Fix $v \in V$ and suppose the contrary, that is, $|N_{2,3}(v)| \geq 2$.

Case 1: $N(v) = N_{2,4}(v) \cup N_{2,3}(v)$.
In this case, for any two vertices $v_i, v_j \in N_{2,3}(v)$,

$$\min_{x \in N(v) \setminus N_{2,3}(v)} \deg x = 1,$$

because otherwise, we can reduce to Lemma 10. Therefore, except for at most one vertex in $N_{2,3}(v)$, any other vertex in $N_{2,3}(v)$ is adjacent to a vertex of degree 1. Now, let $f(x) = -1$ if $\deg x = 1$ and $x$ is adjacent to some vertex in $N_{2,3}(v)$, let $f(v) := 1$ and let $f(y) := 0$ otherwise. Then, by (6) we obtain

$$\frac{1}{2} + \frac{1}{4} |N_{2,4}(v)| \geq \sum_{w \in V} \frac{1}{\deg w} \left( \sum_{v \in N(w)} f(v) \right)^2$$

$$> \frac{1}{4} \sum_{w \in V} \deg w \cdot f(w)^2$$

$$\geq \frac{1}{4} (\deg v + |N_{2,3}(v)| - 1)$$

$$\geq \frac{1}{4} (|N_{2,4}(v)| + 3),$$

which is a contradiction.

Case 2: There exists $w \in N(v)$ with $\deg w = 1$. Then, by Lemma 9,

$$N(v) \setminus \{w\} = N_{2,4}(v) \cup N_{2,3}(v).$$

Similarly to the previous case, this implies that
\[
1 + \frac{1}{4} |\mathcal{N}_{\geq 4}(v)| \geq \sum_{w \in V} \frac{1}{\deg w} \left( \sum_{v \in \mathcal{N}(w)} f(v) \right)^2 \\
> \frac{1}{4} \sum_{w \in V} \deg w \cdot f(w)^2 \\
\geq \frac{1}{4} \left( \deg v + |\mathcal{N}_{2,3}(v)| \right) \\
\geq \frac{1}{4} \left( |\mathcal{N}_{\geq 4}(v)| + 4 \right),
\]

which is a contradiction. \hfill \square

**Lemma 12.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices such that \( \varepsilon > \frac{1}{2} \). If \( w \sim v \) and \( \deg w = 1 \), then \( \deg v \leq 2 \).

**Proof.** Assume the contrary, that is, \( w \sim v \), \( \deg w = 1 \), and \( \deg v \geq 3 \). Then, similarly to the proof of Lemma 11, we have that for any \( u \sim v \), there exists \( u_1 \sim u \) with \( \deg u_1 = 1 \). Therefore,

\[
1 \geq \sum_{w \in V} \frac{1}{\deg w} \left( \sum_{v \in \mathcal{N}(w)} f(v) \right)^2 \\
> \frac{1}{4} \sum_{w \in V} \deg w \cdot f(w)^2 \\
\geq \frac{1}{4} \left( \deg v + |\mathcal{N}(v)| - 1 \right) \\
\geq \frac{1}{4} \left( 2\deg v - 1 \right) > 1,
\]

which is a contradiction. \hfill \square

**Lemma 13.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices such that \( \varepsilon > \frac{1}{2} \). Then, there exist three vertices \( u \sim v \sim w \) with \( \deg u \leq 3 \) and \( \deg w \leq 3 \).

**Proof.** By Lemma 7, there exists \( u_1 \sim u_2 \) with \( \deg u_1 \leq 3 \) and \( \deg u_2 \leq 3 \). If \( v \sim u_2 \), then by (7) there exists \( w \in \mathcal{N}(u_1) \Delta \mathcal{N}(v) \) with \( \deg w \leq 3 \).

If \( w \in \mathcal{N}(u_1) \setminus \mathcal{N}(v) \), then we have \( w \sim u_1 \sim u_2 \) with \( \deg w \leq 3 \) and \( \deg u_2 \leq 3 \); while if \( w \in \mathcal{N}(v) \setminus \mathcal{N}(u_1) \), then we have \( w \sim v \sim u_2 \) with \( \deg w \leq 3 \) and \( \deg u_2 \leq 3 \). \hfill \square

**Lemma 14.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices such that \( \varepsilon > \frac{1}{2} \). Then, there exist three vertices \( u \sim v \sim w \) such that \( \deg u = 1 \), \( \deg v = 2 \), and \( \deg w \in \{2, 3\} \).

**Proof.** Fix \( u \sim v \sim w \) with \( \deg u \leq 3 \) and \( \deg w \leq 3 \) as in Lemma 13. By Lemma 9, at most one between \( u \) and \( w \) has degree 1, and by Lemma 11, at most one between \( u \) and \( w \)
has degree 2 or 3. Therefore, we may assume that \( \deg u = 1 \) and \( \deg w \in \{2, 3\} \). By Lemma 12, this implies that \( \deg v = 2 \).

We are now finally able to prove Theorem 6.

**Proof of Theorem 6.** Assume by contradiction that there exists a connected graph \( \Gamma \) on \( N \geq 3 \) vertices such that \( \varepsilon > \frac{1}{2} \). Then, by Lemma 14, there exists \( u \sim v \sim w \) such that \( \deg u = 1, \deg v = 2, \) and \( \deg w \in \{2, 3\} \). By (7), there exists a vertex \( w_1 \in N(w) \setminus \{v\} \) satisfying \( \deg w_1 = 1 \), and by Lemma 12, \( \deg w = 2 \). Therefore \( \Gamma \) is the path of length 3 in Figure 3. But as we know this is a contradiction, since \( \varepsilon = \frac{1}{2} \) for the path of length 3, which is also the 1-book graph.

\[ \square \]

### 3.2 Optimal cases

In Section 3.1, we proved that \( \varepsilon \leq \frac{1}{2} \) for any connected graph \( \Gamma \) on \( N \geq 3 \) vertices. Hence, to prove Theorem 1, it is left to prove that \( \varepsilon = \frac{1}{2} \) if and only if \( \Gamma \) is either a petal graph or a book graph. We dedicate this section to the proof of this claim, that we further split into three parts:

1. In Section 3.2.1 we prove that, if the smallest vertex degree \( d \) is greater than or equal to 3, then \( \varepsilon < \frac{1}{2} \). In fact, we prove an even stronger result, since we show that, in this case,

\[ \varepsilon \leq \frac{\sqrt{d - 1}}{d} < \frac{1}{2}. \]

2. In Section 3.2.2 we show that, if \( d = 2 \), then \( \varepsilon = \frac{1}{2} \) if and only if \( \Gamma \) is either a petal graph or a book graph.

3. In Section 3.2.3 we show that, if \( d = 1 \) and \( \Gamma \) is not the 1-book graph, then \( \varepsilon < \frac{1}{2} \).

Before, we observe that, by Lemma 3 and Theorem 6, if a connected graph \( \Gamma \) on \( N \geq 3 \) vertices is such that \( \varepsilon = \frac{1}{2} \), then

\[ \sum_{w \in V} \frac{1}{\deg w} \left( \sum_{v \in N(w)} f(v) \right)^2 \geq \frac{1}{4} \sum_{w \in V} \deg w \cdot f(w)^2, \quad \forall f \in C(V) \setminus \{0\}. \]  

Moreover, with the same proof as Lemma 8, one can prove the following:

**Lemma 15.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices such that \( \varepsilon = \frac{1}{2} \). If \( N(u) \cap N(v) \neq \emptyset \), then \( N(u) \cup N(v) \neq \emptyset \) and

\[ \sum_{w \in N(u) \cup N(v)} \left( \frac{1}{\deg w} - \frac{1}{4} \right) \geq \frac{1}{2}. \]

\[ (9) \]
3.2.1 | The case $d \geq 3$

In this section we prove the following

**Theorem 16.** For any connected graph $\Gamma$ on $N \geq 3$ vertices with smallest degree $d \geq 3$,

$$\varepsilon \leq \frac{\sqrt{d - 1}}{d} < \frac{1}{2}.$$  

Before that theorem, we shall prove several preliminary results.

**Lemma 17.** Let $\Gamma$ be a connected graph on $N \geq 3$ vertices with smallest degree $d \geq 3$ and $\varepsilon > \frac{\sqrt{d - 1}}{d}$. Then,

$$\sum_{w \in V} \frac{1}{\deg(w)} \left( \sum_{v \in N(w)} f(v) \right)^2 > \frac{d - 1}{d^2} \sum_{w \in V} \deg(w) \cdot f(w)^2, \quad \forall f \in C(V) \setminus \{0\}. \quad (10)$$

Moreover, if $N(u) \cap N(v) \neq \emptyset$, then $N(u) \triangle N(v) \neq \emptyset$ and

$$\sum_{w \in N(u) \triangle N(v)} \left( \frac{1}{\deg(w)} - \frac{d - 1}{d^2} \right) > 2 \cdot \frac{d - 1}{d^2}. \quad (11)$$

**Proof.** The first claim follows directly from Lemma 3, while the second claim can be proved as Lemma 8. \qed

**Lemma 18.** Let $\Gamma$ be a connected graph on $N \geq 3$ vertices with smallest degree $d \geq 3$ and $\varepsilon > \frac{\sqrt{d - 1}}{d}$. If $N(u) \cap N(v) \neq \emptyset$, then

$$|\{w \in N(u) \triangle N(v) : \deg(w) \in \{d, d + 1\}\}| \geq 2d - 1.$$  

**Proof.** Assume the contrary, then

$$\sum_{w \in N(u) \triangle N(v)} \left( \frac{1}{\deg(w)} - \frac{d - 1}{d^2} \right) \leq \sum_{w \in N(u) \triangle N(v)} \left( \frac{1}{\deg(w)} - \frac{d - 1}{d^2} \right)_{\deg(w) \in \{d, d + 1\}}$$

$$< (2d - 2) \left( \frac{1}{d} - \frac{d - 1}{d^2} \right) = 2 \cdot \frac{d - 1}{d^2},$$

which contradicts (11). \qed

**Lemma 19.** Let $\Gamma$ be a connected graph on $N \geq 3$ vertices with smallest degree $d \geq 3$ and $\varepsilon > \frac{\sqrt{d - 1}}{d}$. If $N(u) \cap N(v) \neq \emptyset$ and
\[ |\{ w \in \mathcal{N}(u) \triangle \mathcal{N}(v) : \deg w \in [d, d + 1]\}| \leq 2d, \]

then \[ |\{ w \in \mathcal{N}(u) \triangle \mathcal{N}(v) : \deg w = d\}| \geq 2d - 3. \]

Proof. If not, then

\[
\sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \left( \frac{1}{\deg w} - \frac{d - 1}{d^2} \right) \\
\leq \frac{1}{\deg w = d} \sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \left( \frac{1}{\deg w} - \frac{d - 1}{d^2} \right) + \frac{1}{\deg w = d + 1} \sum_{w \in \mathcal{N}(u) \triangle \mathcal{N}(v)} \left( \frac{1}{\deg w} - \frac{d - 1}{d^2} \right) \\
< (2d - 4) \left( \frac{1}{d} - \frac{d - 1}{d^2} \right) + 2d \left( \frac{1}{d + 1} - \frac{d - 1}{d^2} \right) \\
= \frac{2d - 4}{d^2} + \frac{2}{(d + 1) d^2} < \frac{2d - 4}{d^2} + \frac{2}{d^2} = 2 \cdot \frac{d - 1}{d^2},
\]

which contradicts (11). \[ \square \]

Lemma 20. Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d \geq 3 \) and \( \varepsilon > \frac{\sqrt{d-1}}{d} \). Then, there exist \( u \) and \( v \) with \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \) and \( \deg u, \deg v \in [d, d + 1] \).

Proof. Fix three vertices \( u' \sim w' \sim v' \). By Lemma 18, we either have

\[ |\{ w \in \mathcal{N}(u') \setminus \mathcal{N}(v') : \deg w \in [d, d + 1]\}| \geq d \]

or

\[ |\{ w \in \mathcal{N}(v') \setminus \mathcal{N}(u') : \deg w \in [d, d + 1]\}| \geq d. \]

Without loss of generality, we may assume that there are two vertices \( u \) and \( v \) in \( \mathcal{N}(u') \setminus \mathcal{N}(v') \) with \( \deg u, \deg v \in [d, d + 1] \). This implies that \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \), hence \( u \) and \( v \) satisfy the claim. \[ \square \]

Lemma 21. Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d \geq 3 \) and \( \varepsilon > \frac{\sqrt{d-1}}{d} \). Then, there exist \( u \) and \( v \) with \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \) and \( \deg u = \deg v = d \).

Proof. Fix \( u' \) and \( v' \) that satisfy Lemma 20. Then,

\[ |\mathcal{N}(u') \triangle \mathcal{N}(v')| \leq \deg u' + \deg v' - 2 \leq 2d. \]

By Lemma 19, this implies that

\[ |\{ w \in \mathcal{N}(u') \triangle \mathcal{N}(v') : \deg w = d\}| \geq 2d - 3 \geq 3. \]
Hence, without loss of generality, we can assume that there are two vertices \( u \) and \( v \) in \( \mathcal{N}(u') \setminus \mathcal{N}(v') \) with \( \deg u = \deg v = d \). This proves the claim. \( \square \)

We can now prove Theorem 16.

Proof of Theorem 16. We first observe that the second inequality follows from the fact that the sequence

\[
\left\{ \frac{\sqrt{d-1}}{d} \right\}_d \geq \left\{ \frac{1}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{3}}{4}, \frac{2}{5}, \ldots \right\}
\]

is strictly decreasing. Hence, it is left to show that \( \varepsilon \leq \frac{\sqrt{d-1}}{d} \). Assume, by contradiction, that \( \varepsilon > \frac{\sqrt{d-1}}{d} \). By Lemma 21, there exist \( u \) and \( v \) such that \( \mathcal{N}(u) \cap \mathcal{N}(v) \neq \emptyset \) and \( \deg u = \deg v = d \). This implies that \( |\mathcal{N}(u) \triangle \mathcal{N}(v)| \leq 2d - 2 \), but together with Lemma 18, this brings to a contradiction. \( \square \)

3.2.2 The case \( d = 2 \)

Here we prove the following

**Theorem 22.** For any connected graph \( \Gamma \) on \( N \geq 3 \) vertices with smallest degree \( d = 2 \),

\[ \varepsilon \leq \frac{1}{2}, \]

with equality if and only if \( \Gamma \) is either a petal graph or a book graph.

Also in this section, we first prove several preliminary results. We start by showing that, if \( \Gamma = C_N \) is the cycle graph on \( N \geq 3 \) vertices, then \( \varepsilon = \frac{1}{2} \) if and only if either \( N = 3 \) (in which case \( \Gamma = C_3 \) is the 1-petal graph) or \( N = 6 \) (in which case \( \Gamma = C_6 \) is the 2-book graph).

**Proposition 23.** If \( \Gamma = C_N \) is the cycle graph on \( N \geq 3 \) vertices, then \( \varepsilon = \frac{1}{2} \) if and only if \( N \in \{3, 6\} \).

**Proof.** As shown in [6], for the cycle graph \( C_N \) on \( N \geq 3 \) vertices, the eigenvalues are \( 1 - \cos(2\pi k/N) \), for \( k = 0, \ldots, N - 1 \). We already know that \( \varepsilon = \frac{1}{2} \) if \( N \in \{3, 6\} \) since, in this case, \( \Gamma \) is either the 1-petal graph or the 2-book graph. Therefore, it is left to show that \( \varepsilon < \frac{1}{2} \) for \( N \geq 4, N \neq 6 \). We consider three cases.

1. If \( N = 4 \), then 1 is an eigenvalue, therefore \( \varepsilon = 0 \).
2. If \( N = 5 \), by letting \( k = 1 \) we can see that \( 1 - \cos(2\pi/5) \) is an eigenvalue that has distance \( \cos(2\pi/5) \approx 0.3 \) from 1. Hence, \( \varepsilon < 1/2 \).

3. Let \( N > 6 \) and let

\[
k \in \left\{ \frac{N - 2}{4}, \frac{N - 1}{4}, \frac{N}{4}, \frac{N + 1}{4} \right\}.
\]

Then,

\[
\frac{2k}{N} \leq \frac{2}{N} \cdot \frac{N + 1}{4} = \frac{N + 1}{2N} < \frac{2}{3},
\]

since \( N > 3 \), and

\[
\frac{2k}{N} \geq \frac{2}{N} \cdot \frac{N - 2}{4} = \frac{N - 2}{2N} > \frac{1}{3},
\]

since \( N > 6 \). Hence,

\[
\frac{2k}{N} \in \left( \frac{1}{3}, \frac{2}{3} \right),
\]

implying that

\[
\left| \cos \frac{2\pi k}{N} \right| < \frac{1}{2},
\]

therefore \( \varepsilon < 1/2 \).

\[\square\]

**Proposition 24.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d = 2 \) and \( \varepsilon = 1/2 \). If three vertices are such that \( u \sim v \sim w \) and \( \deg u = \deg v = \deg w = 2 \), then \( \Gamma = C_3 \) or \( C_6 \).

**Proof.** If \( u \sim w \), then clearly \( \Gamma = C_3 \). If \( u \not\sim w \), then (9) applied to \( \mathcal{N}(u) \triangle \mathcal{N}(w) \) implies that the vertices \( u_1 \in \mathcal{N}(u) \setminus \{v\} \) and \( w_1 \in \mathcal{N}(w) \setminus \{v\} \) are such that \( \deg u_1 = \deg w_1 = 2 \). If \( u_1 = w_1 \), then clearly \( \Gamma = C_4 \). Otherwise, we can repeat the process and obtain any cycle graph \( C_N \). By Proposition 23, the claim follows.

\[\square\]

**Lemma 25.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d = 2 \) and \( \varepsilon = 1/2 \). Then, there exist two adjacent vertices \( u \sim v \) such that \( \deg u = \deg v = 2 \).

**Proof.** We illustrate this proof in Figure 7.
Fix \( u \sim v \sim w \) such that \( \deg v = 2 \). Then, by (9), there exists \( w_1 \in \mathcal{N}(u) \triangle \mathcal{N}(w) \) with \( \deg w_1 \in \{2, 3\} \). Without loss of generality, we assume that \( w_1 \sim w \). Again by (9), there are at least two vertices in \( \mathcal{N}(v) \triangle \mathcal{N}(w_1) \) of degree 2. If \( \deg u = 2 \), then we have done. If \( \deg u \geq 3 \), then \( \deg w_1 = 3, \deg u \leq 4 \), and \( \deg x = 2 \) for any \( x \in \mathcal{N}(w_1) \setminus \{w\} \). Now, let \( x_1, x_2 \in \mathcal{N}(w_1) \setminus \{w\} \). By applying (9) to \( x_1 \) and \( x_2 \), we infer that there exists a vertex adjacent to \( x_1 \) of degree 2. Since also \( \deg x_1 = 2 \), this proves the claim. \( \square \)

**Lemma 26.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d = 2 \) and \( \varepsilon = \frac{1}{2} \). For any vertex \( v \) of degree 2, there exists \( u \sim v \) with \( \deg u = 2 \).

**Proof.** Suppose the contrary, then there are \( u \sim v \sim w \) with \( \deg u, \deg w \geq 3 \), and \( \deg v = 2 \).

**Claim 1:** For all \( x \in \mathcal{N}(u) \triangle \mathcal{N}(w) \), \( \deg x \geq 3 \).

If not, without loss of generality we can assume that there exists \( x \in \mathcal{N}(u) \setminus \mathcal{N}(w) \) with \( \deg x = 2 \). This implies that
\[
\mathcal{N}(v) \triangle \mathcal{N}(x) = \{w, y\},
\]
for some vertex \( y \). By the assumption, \( \deg y \geq 2 \) and \( \deg w \geq 3 \). Hence, applying (9) to \( \mathcal{N}(v) \triangle \mathcal{N}(x) \) implies
\[
\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \geq \left(\frac{1}{\deg y} - \frac{1}{4}\right) + \left(\frac{1}{\deg w} - \frac{1}{4}\right) \geq \frac{1}{2},
\]
which is a contradiction.

**Claim 2:** There are at least six vertices in \( \mathcal{N}(u) \triangle \mathcal{N}(w) \) of degree 3, and thus \( \deg u + \deg w \geq 8 \).

If there are at most five vertices in \( \mathcal{N}(u) \triangle \mathcal{N}(w) \) of degree 3, then the other vertices in \( \mathcal{N}(u) \triangle \mathcal{N}(w) \) have a degree at least 4. Applying (9) to \( \mathcal{N}(u) \triangle \mathcal{N}(w) \), we then obtain
\[
5\left(\frac{1}{3} - \frac{1}{4}\right) \geq \sum_{x \in \mathcal{N}(u) \triangle \mathcal{N}(w)} \left(\frac{1}{\deg x} - \frac{1}{4}\right) \geq \frac{1}{2},
\]
which is a contradiction.

**Claim 3:** Assume that there exists \( w_1 \sim w \) with \( \deg w_1 = 3 \). Then, \( x_1, x_2 \in \mathcal{N}(w_1) \setminus \{w\} \) are such that \( \deg x_1 = \deg x_2 = 2 \).

![Figure 7](image-url)  
**Figure 7** The vertices in the proof of Lemma 25.
If not, then without loss of generality we can assume that \( \deg x_1 \geq 3 \). Applying (9) to \( \mathcal{N}(v) \Delta \mathcal{N}(w_1) \), this implies

\[
\frac{5}{12} = \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) \\
\geq \left( \frac{1}{\deg u} - \frac{1}{4} \right) + \left( \frac{1}{\deg x_1} - \frac{1}{4} \right) + \left( \frac{1}{\deg x_2} - \frac{1}{4} \right) \\
= \sum_{x \in \mathcal{N}(v) \Delta \mathcal{N}(w_1)} \left( \frac{1}{\deg x} - \frac{1}{4} \right) \geq \frac{1}{2},
\]

which is a contradiction.

Now, if \( x_1 \sim x_2 \), then by letting \( f(x_1) := -1, f(w_1) := 1 \), and \( f := 0 \) otherwise, (9) brings to a contradiction. Hence, \( x_1 \not\sim x_2 \). By applying (9) again, we can infer that there exist \( y_1 \sim x_1 \) and \( y_2 \sim x_2 \) such that \( \deg y_1 = \deg y_2 = 2 \). If \( y_1 = y_2 \), then we reduce to Proposition 24. If \( y_1 \neq y_2 \), then a similar reasoning implies \( y_1 \not\sim y_2 \). By applying then (9) to \( \mathcal{N}(y_1) \Delta \mathcal{N}(w_1) \), we infer that there exists \( z_1 \sim y_1 \) with \( \deg z_1 = 2 \), and therefore we again reduce to Proposition 24. In any case, we obtain a contradiction.

**Proposition 27.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d = 2 \) and \( \varepsilon = \frac{1}{2} \). If there exists \( u \sim v \sim w \) such that \( \deg u = \deg v = 2 \) and \( \deg w = 3 \), then \( \Gamma \) is the book graph on \( N = 8 \) vertices.

**Proof.** If \( w \sim u \), by letting \( f(w) := 1, f(u) := -1 \), and \( f := 0 \) otherwise, (8) implies that

\[
\frac{1}{2} + \frac{1}{3} + \frac{1}{\deg w_1} \geq \frac{1}{4}(2 + 3),
\]

therefore the vertex \( w_1 \in \mathcal{N}(w) \setminus \{u, v\} \) has degree 2. Similarly, there exists \( x \sim w_1 \) with \( \deg x = 2 \). By letting now \( f(w) := 2, f(u) := f(v) := -1, f(w_1) := 1, f(x) := -1 \), and \( f := 0 \) otherwise, then (8) implies that

\[
2 \cdot \frac{1}{2} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} \geq \frac{1}{4}(2 + 2 + 3 \cdot 2^2 + 2 + 2),
\]

which is a contradiction. Therefore, \( w \not\sim u \).

Now, if the vertex \( u_1 \in \mathcal{N}(u) \setminus \{v\} \) has degree 2, then we reduce to Proposition 24. Thus, \( \deg u_1 \geq 3 \). Applying (9) to the vertices \( u \) and \( w \), we infer that the two vertices \( w_1, w_2 \in \mathcal{N}(w) \setminus \{v\} \) satisfy \( \deg w_1 = \deg w_2 = 2 \). Again, applying (9) to the vertex pairs \( (v, w_1) \) and \( (v, w_2) \), we infer that the vertices \( x_1 \sim w_1 \) and \( x_2 \sim w_2 \) satisfy \( \deg x_1 = \deg x_2 = 2 \).

If \( \mathcal{N}(x_1) \cap \mathcal{N}(x_2) = \emptyset \), then the vertices \( y_1 \sim x_1 \) and \( y_2 \sim x_2 \) have to satisfy \( \deg y_1, \deg y_2 \geq 3 \). By letting \( f(w) := 1, f(x_1) := f(x_2) := -1 \), and \( f := 0 \) otherwise, (8) implies
\[
\frac{1}{3} + \frac{1}{3} + \frac{1}{2} \geq \frac{1}{4}(3 + 2 + 2),
\]

which is a contradiction. Hence, \( \mathcal{N}(x_1) \cap \mathcal{N}(x_2) \neq \emptyset \) and, similarly, also \( \mathcal{N}(u) \cap \mathcal{N}(x_1) \neq \emptyset \) and \( \mathcal{N}(u) \cap \mathcal{N}(x_2) \neq \emptyset \).

In particular, there exists \( z \) such that \( z \sim u, z \sim x_1 \), and \( z \sim x_2 \). By letting now \( f(w) := 1, f(u) := f(x_1) := f(x_2) := -1, \) and \( f := 0 \) otherwise, by (8) we obtain
\[
\frac{1}{\deg z} \cdot 3^2 \geq \frac{1}{4}(3 + 2 + 2 + 2),
\]

which implies that \( \deg z \leq 4 \).

We now claim that \( \deg z = 3 \). Suppose the contrary, then \( \deg z = 4 \). Let \( z_1 \in \mathcal{N}(z) \setminus \{u, x_1, x_2\} \). If there exists \( z_2 \sim z_1 \) of degree 2, then \( f(z_2) := f(u) := -1, f(z) := 1, \) and \( f := 0 \) otherwise together with (8) gives
\[
\frac{1}{3} + 3 \cdot \frac{1}{2} \geq \frac{1}{4}(2 + 4 + 2),
\]

which is a contradiction. Therefore, for any \( a \in \mathcal{N}(z_1) \), \( \deg a \geq 3 \). By (9) applied to \( z_2 \) and \( v \), we infer that there exist at least three vertices of degree 3 in \( \mathcal{N}(z_1) \), and \( \deg z_1 \geq 4 \). Let \( b_1, b_2 \in \mathcal{N}(z_1) \) be such that \( \deg b_1 = \deg b_2 = 3 \). Then, (9) applied to \( b_1 \) and \( b_2 \) implies that there is at least one vertex of degree 2 in \( \mathcal{N}(b_1) \Delta \mathcal{N}(b_2) \). Without loss of generality, we can assume that there exists \( b' \in \mathcal{N}(b_1) \setminus \mathcal{N}(b_2) \) such that \( \deg b' = 2 \). By Lemma 26, there exists \( c' \sim b' \) with \( \deg c' = 2 \). But this is a contradiction, since we proved that all vertices adjacent to \( b_1 \) must have degree 2.

This proves that \( \deg z = 3 \), which implies that \( \Gamma \) must be the book graph on \( N = 8 \) vertices.

**Lemma 28.** Let \( \Gamma \) be a connected graph on \( N \geq 3 \) vertices with smallest degree \( d = 2 \) and \( \varepsilon = \frac{1}{2} \). If \( \Gamma \) is not in the classes described in Propositions 24 and 27, then there is no vertex of degree 3.

**Proof.** Assume the contrary, that is, there exists a vertex \( u \) with \( \deg u = 3 \). We first claim that, for any \( v \in \mathcal{N}(u) \), \( \deg v \geq 3 \). This is true because, otherwise, there would be a vertex \( v \sim u \) with \( \deg v = 2 \), and by Lemma 26, also a vertex \( w \sim v \) with \( \deg w = 2 \). Hence, \( \Gamma \) would be in the class described in Proposition 27, but we are assuming that this is not the case. This shows that \( \deg v \geq 3 \) for any \( v \in \mathcal{N}(u) \).

Now, we prove that there are no vertices \( v \) and \( w \) such that \( u \sim v \sim w \) and \( \deg w \in \{2, 3\} \). This follows from the fact that, otherwise, applying (9) to \( u \) and \( w \) together with the above remarks implies that there exists \( w_1 \sim w \) with \( \deg w_1 = 2 \), while \( \deg w = 3 \), which is a contradiction.

But on the other hand, for any \( x, y \in \mathcal{N}(u) \), by (9) there exists \( w \in \mathcal{N}(x) \Delta \mathcal{N}(y) \) with \( \deg w \in \{2, 3\} \), which contradicts the above argument.

We are now able to prove Theorem 22.
Proof of Theorem 22. We shall illustrate this proof in Figure 8.

Observe that, if $\Gamma$ is in one of the classes described in Propositions 24 and 27, then the claim follows. Therefore, we can assume that this is not the case. We can then apply Lemma 28 and infer that there is no vertex of degree 3. Also, if all vertices have degree <3, from the fact that $d = 2$ it follows that all vertices have degree 2, but this is not possible, since we are assuming that $\Gamma$ does not satisfy the condition in Proposition 24. Therefore, we can assume that there exists a vertex with degree $\geq 4$.

Moreover, we can also assume that there exists one vertex $u$ with $\deg u \geq 4$ such that $N_2(u) \neq \emptyset$. This can be seen by the fact that, if not, then all vertices of degree 2 would only be adjacent to vertices of degree 2, contradicting the fact that $\Gamma$ is connected.

Now, from the fact that $N_2(u) \neq \emptyset$, we can infer two facts about $\mathcal{N}(u) = N_2(u) \cup N_{\geq 4}(u)$:

1. For any $v \in N_2(u)$, there exists a unique $w \sim v$ with $\deg w = 2$. This follows immediately from Lemma 26 and from the fact that $\deg u \geq 4$.

2. If $N_{\geq 4}(u) \neq \emptyset$, then for any $v \in N_{\geq 4}(u)$ there exists $w \sim v$ with $\deg w = 2$. This can be shown by fixing $v' \in N_2(u)$ and then applying (9) to $\mathcal{N}(v') \Delta \mathcal{N}(v)$.

As a consequence of the first fact, we can write

$$\mathcal{N}_2(u) = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+2}\},$$

where

- For each $i = 1, \ldots, k$, there exists a unique $w_i \in V \setminus \mathcal{N}_2(u)$ such that $v_i \sim w_i$ and $\deg w_i = 2$;

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{Illustration of the proof of Theorem 22.}
\end{figure}
• \( v_{k+1} \sim v_{k+2}, \ldots, v_{k+2l-1} \sim v_{k+2l} \).

Now, for \( i = 1, \ldots, k \), since \( v_i \sim w_i \) and \( \deg v_i = \deg w_i = 2 \), we can infer that \( x_i \in N(w_i) \setminus \{v_i\} \) must have degree \( \geq 4 \), because otherwise, \( \Gamma \) would satisfy the assumption of Proposition 24, but we are assuming that this is not the case. Therefore, in particular, \( w_1, \ldots, w_k \) must be pairwise distinct, while \( x_1, \ldots, x_k \) do not need to be pairwise distinct.

Up to relabeling \( x_1, \ldots, x_k \), we assume that

\[
\begin{align*}
x'_1 &:= x_1 = \cdots = x_{k_1}, \\
x'_2 &:= x_{k_1+1} = \cdots = x_{k_1+k_2}, \\
\vdots \\
x'_{p+q} &:= x_{k_1+\cdots+k_{p+q-1}+1} = \cdots = x_{k_1+\cdots+k_{p+q}} = x_k,
\end{align*}
\]

where

• \( k_1, \ldots, k_{p+q} \) are such that \( k = k_1 + \cdots + k_{p+q} \),
• \( x'_1, \ldots, x'_{p+q} \) are pairwise distinct,
• \( x'_1, \ldots, x'_p \not\in N(u) \) while \( x'_{p+1}, \ldots, x'_{p+q} \in N(u) \),
• \( q \) is such that \( q \leq \deg u - k - 2l \).

Following the above notation, we also relabel the vertices \( v_1, \ldots, v_k \) and \( w_1, \ldots, w_k \) as

\[
\begin{align*}
v_1, \ldots, v_{k_1}, v_{k_1+k_2}, \ldots, v_{k_1+\cdots+k_{p+q}}, \\
w_1, \ldots, w_{k_1}, \ldots, w_{k_1+\cdots+k_{p+q}}.
\end{align*}
\]

Now, we write

\[
\mathcal{N}_{2,4}(u) \setminus \{x'_{p+1}, \ldots, x'_{p+q}\} = \{y_1, \ldots, y_r\}.
\]

Then, as observed above, for each \( i = 1, \ldots, r \) there exists \( y'_i \sim y_i \) with \( \deg y'_i = 2 \). Furthermore, by Lemma 26, for each \( i = 1, \ldots, k \) there also exists \( z_i \sim y'_i \sim y_i \) with \( \deg z_i = \deg y'_i = 2 \). Clearly, \( y'_1, \ldots, y'_r \) are pairwise distinct, while \( z_1, \ldots, z_r \) may have some overlap with \( y'_1, \ldots, y'_r \).

We illustrate the above vertices in Figure 8 and we observe that

\[
\mathcal{N}(u) = \mathcal{N}_2(u) \sqcup \mathcal{N}_{2,4}(u) \\
= \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+2l}\} \sqcup \{x'_{p+1}, \ldots, x'_{p+q}\} \sqcup \{y_1, \ldots, y_r\},
\]

where the above vertices are all pairwise distinct, therefore

\[
\deg u = k + 2l + q + r.
\]
Now, by letting
\[
\begin{align*}
f(u) &:= 2, \\
f(v_{k+1}) := \cdots := f(v_{k+2l}) := f(w_1) := \cdots := f(w_k) := -1, \\
f(v_1) := \cdots := f(v_k) := 1 \quad \text{if } k \leq 2l, \\
f(v_1) := \cdots := f(v_{2l}) := 1, \\
f(v_{2l+i}) := (-1)^i \quad \text{for } t = 1, \ldots, k - 2l, \text{ if } 2l < k, \\
f(y'_1) := \cdots := f(y'_r) := -1, \\
f := 0 \quad \text{otherwise}
\end{align*}
\]

and
\[
c_{k,l} := \begin{cases} 
(2l - k)^2 & \text{if } 2l \geq k, \\
1 & \text{if } 2l < k \text{ and } k \text{ is odd}, \\
0 & \text{if } 2l < k \text{ and } k \text{ is even},
\end{cases}
\]

from (8) we can infer that
\[
\begin{align*}
\frac{c_{k,l}}{\deg u} + \frac{2k + 2l}{2} + \sum_{i=1}^{p} \frac{k_i^2}{\deg x'_i} + \sum_{i=p+1}^{p+q} \frac{(k_i - 2)^2}{\deg x'_i} + r \cdot \frac{1}{2} + \sum_{i=1}^{r} \frac{1}{2} \\
= \sum_{w \in V} \frac{1}{\deg w} \left( \sum_{v \in \mathcal{N}(w)} f(v)^2 \right) \\
\geq \frac{1}{4} \sum_{w \in V} \deg w \cdot f(w)^2 \\
= \frac{1}{4} (\deg u \cdot 2^2 + 2(2k + 2l) + 2r) \\
= \deg u + k + l + \frac{r}{2} = 2k + 3l + q + \frac{3}{2} r.
\end{align*}
\]

We now use the following known facts:

- \( \deg y_i \geq 4, \) for each \( i = 1, \ldots, r; \)
- \( \deg x'_i \geq \max\{k_i, 4\}, \) for \( i = 1, \ldots, p + q; \)
- \( (k_i - 2)^2 \leq k_i^2 \) (since \( k_i \geq 1), \) for \( i = p + 1, \ldots, p + q. \)

Putting everything together, we obtain that
\[
\begin{align*}
2k + 3l + q + \frac{3}{2} r \\
\leq \frac{c_{k,l}}{\deg u} + k + l + \sum_{i=1}^{p+q} \frac{k_i^2}{\deg x'_i} + r \cdot \frac{1}{2} + \sum_{i=1}^{r} \frac{1}{4} \\
\leq \frac{c_{k,l}}{k + 2l + q + r} + k + l + \sum_{i=1}^{p+q} \frac{k_i^2}{k_i} + \frac{3}{4} r,
\end{align*}
\]
which implies
\[ k + 2l + q + \frac{3}{4}r \leq \frac{c_{k,l}}{k + 2l + q + r} + \sum_{i=1}^{p+q} k_i = k + \frac{c_{k,l}}{k + 2l + q + r}. \]

Therefore,
\[ 2l + q + \frac{3}{4}r \leq \frac{c_{k,l}}{k + 2l + q + r} \leq \begin{cases} \frac{(2l - k)^2}{k + 2l} & \text{if } 2l \geq k, \\ \frac{1}{\deg u} \leq \frac{1}{4} & \text{if } 2l < k, \end{cases} \]

which implies that \( r = q = 0 \), either \( l = 0 \) or \( k = 0 \) (but they cannot be both zero), and \( \deg x'_i = k_i \geq 4 \), for \( i = 1, \ldots, p \).

We then have two cases:

1. If \( k = 0 \), then \( \Gamma \) is a petal graph with \( 2l + 1 \) vertices.
2. If \( l = 0 \), then we can write
\[ \Gamma = (\Gamma_1 \sqcup \cdots \sqcup \Gamma_p) / \{ u_1, \ldots, u_p \}, \]

where for \( i = 1, \ldots, p \), \( \Gamma_i \) as a \( k_i \)-book graph and \( u_i \) as one of its vertices of degree \( k_i \), and the above notation means that \( \Gamma \) is given by the union of \( \Gamma_1, \ldots, \Gamma_p \) by identifying \( u_1, \ldots, u_p \) as one vertex, which is \( u \).

In this case, by letting \( f(x'_i) := 2, f(w_i) := (-1)^i \) for \( i = 1, \ldots, k, f(v_k) := \cdots := f(v_2) := -1, \) and \( f := 0 \) otherwise, by \( (8) \) we can infer that \( \deg u = k_i \) (with the same proof that we used for inferring that \( \deg x'_i = k_i \)). As a consequence, \( p = 1 \), therefore \( \Gamma \) is a book graph on \( N = 2k + 2 \) vertices.

This proves the claim. \( \square \)

### 3.2.3 The case \( d = 1 \)

Given \( N \geq 3 \), we let \( P_N \) denote the path graph on \( N \) vertices, and we observe that \( P_4 \) coincides with the 1-book graph. It is left to prove the following

**Theorem 29.** For any connected graph \( \Gamma \) on \( N \geq 3 \) vertices with smallest degree \( d = 1 \), if \( \Gamma \neq P_4 \) then
\[ \varepsilon < \frac{1}{2}. \]

**Proof.** Suppose the contrary, then there exists a connected graph \( \Gamma \neq P_4 \) on \( N \geq 3 \) vertices with \( \varepsilon = \frac{1}{2} \) and \( d = 1 \). We fix such \( \Gamma = (V, E) \) so that it is the graph with these properties that has the smallest possible order, that is, if \( \Gamma' \neq P_4 \) is a connected graph with
smallest degree 1 and \(3 \leq N' < N\) vertices, then \(\varepsilon(\Gamma') < \frac{1}{2}\). If we show that the existence of \(\Gamma\) brings to a contradiction, then we are done.

Fix \(u \in V\) of degree 1, and let \(v \sim u\) be its only neighbor. Then, by letting \(f(u) := 1\) and \(f := 0\) otherwise, (8) gives \(\frac{1}{\deg v} \geq \frac{1}{4}\), that is, \(\deg v \leq 4\).

Now, assume first that \(\deg v = 4\). Let \(\hat{\Gamma} := \Gamma \setminus \{u\}\), let \(\hat{\varepsilon} := \varepsilon(\hat{\Gamma})\) and let \(f\) be a function on \(V(\hat{\Gamma}) = V \setminus \{u\}\) that is an eigenfunction for \((\text{Id} - \Delta(\hat{\Gamma}))^2\) with eigenvalue \(\hat{\varepsilon}^2\). Then,

\[
\hat{\varepsilon}^2 = \frac{1}{\deg v - 1} \left( \sum_{x \in N(v) \setminus \{u\}} f(x)^2 \right)^2 + \sum_{w \in V \setminus \{u, v\}} \frac{1}{\deg w} \left( \sum_{x \in N(w)} f(x)^2 \right)^2.
\]

(12)

Moreover, since \(\hat{\varepsilon}^2 = (1 - \hat{\lambda})^2\) for some eigenvalue \(\hat{\lambda}\) of \(\hat{\Gamma}\), from (4) it follows that

\[
(1 - \hat{\lambda}) f(v) = \frac{1}{\deg v - 1} \left( \sum_{x \in N(v) \setminus \{u\}} f(x) \right),
\]

therefore

\[
\hat{\varepsilon} \cdot (\deg v - 1) \cdot |f(v)| = \left| \sum_{x \in N(v) \setminus \{u\}} f(x) \right|,
\]

and similarly, for all \(w \in V \setminus \{u, v\},\)

\[
\hat{\varepsilon} \cdot \deg w \cdot |f(w)| = \left| \sum_{x \in N(w)} f(x) \right|.
\]

In particular, without loss of generality, we may assume

\[
\hat{\varepsilon} \cdot (\deg v - 1) \cdot f(v) = \sum_{x \in N(v) \setminus \{u\}} f(x)
\]

and

\[
\hat{\varepsilon} \cdot \deg w \cdot f(w) = \sum_{x \in N(w)} f(x), \quad \forall \ w \in V \setminus \{u, v\},
\]

as the proof can be easily adapted otherwise.

Now, since we are assuming that \(\varepsilon = \varepsilon(\Gamma) = \frac{1}{2}\), we have that, independently of the value of \(f(u) \in \mathbb{R}\) that we choose, (8) implies

\[
\frac{\sum_{w \in V} \frac{1}{\deg w} \left( \sum_{x \in N(w)} f(x)^2 \right)^2}{\sum_{w \in V} \deg w \cdot f(w)^2} \geq \frac{1}{4},
\]
or equivalently,

\[ f(v)^2 + \frac{1}{4} \left( \sum_{x \in N(v)} f(x) \right)^2 + \sum_{w \in V \setminus \{v, u\}} \frac{1}{\deg w} \left( \sum_{x \in N(w)} f(x) \right)^2 \geq \frac{1}{4} \sum_{w \in V} \deg w f(w)^2, \]

which by the above observations can also be rewritten as

\[ f(v)^2 + \frac{1}{4} (f(u) + \xi (\deg v - 1)f(v))^2 + \sum_{w \in V \setminus \{v, u\}} \frac{1}{\deg w} (\xi \cdot \deg w \cdot f(w))^2 \geq \frac{1}{4} \left( f(u)^2 + 4 \cdot f(v)^2 + \sum_{w \in V \setminus \{v, u\}} \deg w f(w)^2 \right). \]

We now consider two cases.

**Case 1.** If \( f(v) = 0 \), then the above inequality becomes

\[ \left( \xi^2 - \frac{1}{4} \right) \left( \sum_{w \in V \setminus \{v, u\}} \deg w f(w)^2 \right) \geq 0. \]

Since \( \xi^2 \leq \frac{1}{4} \) by Theorem 6 and \( f(v) = 0 \) implies \( f_{V \setminus \{v, u\}} \neq 0 \), we deduce that \( \xi^2 = \frac{1}{4} \). But this brings to a contradiction. In fact, by construction, it is clear that \( \hat{\Gamma} \) is a connected graph on \( N - 1 \geq 4 \) vertices, since \( v \) has degree 3 in \( \hat{\Gamma} \). Hence, if \( \hat{\Gamma} \) has vertices of degree 1, then by the assumption on the graphs with less than \( N \) vertices, \( \hat{\Gamma} \) must be \( P_4 \), but this is not possible because of the degree of \( v \). Similarly, if \( \hat{\Gamma} \) does not have vertices of degree 1, then by Theorem 22, \( \hat{\Gamma} \) is either a petal graph or a book graph, and in particular, since \( v \) has degree 3 in \( \hat{\Gamma} \), then \( \hat{\Gamma} \) must be the book graph on 8 vertices. But in this case, one can directly check that \( \epsilon = \epsilon(\hat{\Gamma}) < \frac{1}{2} \), which is a contradiction.

**Case 2.** If \( f(v) \neq 0 \), then the above inequality becomes

\[ \frac{3}{2} \cdot \xi f(u)f(v) + \frac{9}{4} \cdot \xi f(v)^2 + \left( \xi^2 - \frac{1}{4} \right) \left( \sum_{w \in V \setminus \{v, u\}} \deg w f(w)^2 \right) \geq 0. \]

If \( \xi \neq 0 \), we can always derive a contradiction from the above inequality by choosing an appropriate value of \( f(u) \in \mathbb{R} \). Therefore, \( \xi = 0 \), implying that the inequality holds if and only if \( f_{V \setminus \{v, u\}} = 0 \). But in this case, (12) is not satisfied, which is a contradiction.

Hence, \( \deg v = 4 \) always brings to a contradiction, implying that \( \deg v \in \{2, 3\} \), since we already know that \( \deg v \leq 4 \).

Let now \( \check{\Gamma} := \Gamma \setminus \{u, v\} \), let \( \check{\xi} := \epsilon(\check{\Gamma}) \) and let \( f \) be a function on \( V(\check{\Gamma}) = V \setminus \{u, v\} \) that is an eigenfunction for \((\text{Id} - \Delta(\check{\Gamma}))^2\) with eigenvalue \( \check{\xi}^2 \). Then,

\[ \check{\xi}^2 = \frac{\sum_{w \in V \setminus \{v, u\}} f(w)^2}{\sum_{w \in V \setminus \{v, u\}} (\deg w - 1)(\deg w)^2} \geq \frac{1}{4} \sum_{w \in V \setminus \{v, u\}} \deg w f(w)^2. \]
and, similarly to the first part of the proof, without loss of generality we can assume that

\[
\begin{align*}
\sum_{x \in \mathcal{N}(w) \setminus \{u\}} f(x) &= \tilde{\varepsilon}(\deg w - 1)f(w), \quad \forall w \in \mathcal{N}(v) \setminus \{u\}, \\
\sum_{x \in \mathcal{N}(w)} f(x) &= \tilde{\varepsilon}\deg wf(w), \quad \forall w \in V \setminus \mathcal{N}(v).
\end{align*}
\]

In fact, the case in which \(\varepsilon\) is replaced by \(-\tilde{\varepsilon}\) is similar.

Since \(\varepsilon = \varepsilon(\Gamma) = \frac{1}{2}\), (8) holds for any value of \(f(u)\) and \(f(v)\). Therefore,

\[
\begin{align*}
f(v)^2 + \frac{1}{\deg v} \left( f(u) + \sum_{w \in \mathcal{N}(v) \setminus \{u\}} f(w) \right)^2 &+ \sum_{w \in V \setminus \mathcal{N}(v)} \frac{1}{\deg w} (\tilde{\varepsilon}\deg wf(w))^2 \\
&+ \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \frac{1}{\deg w} (f(v) + \tilde{\varepsilon}(\deg w - 1)f(w))^2 \\
&\geq \frac{1}{4} \left( f(u)^2 + \deg vf(v)^2 + \sum_{w \in V \setminus \{v, u\}} \deg wf(w)^2 \right).
\end{align*}
\]

This is equivalent to

\[
\begin{align*}
f(v)^2 + \frac{1}{\deg v} \left( f(u) + \sum_{w \in \mathcal{N}(v) \setminus \{u\}} f(w) \right)^2 &+ \left( \tilde{\varepsilon}^2 - \frac{1}{4} \right) \cdot \left( \sum_{w \in V \setminus \mathcal{N}(v)} \deg wf(w)^2 \right) \\
&+ \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \frac{1}{\deg w} (f(v) + \tilde{\varepsilon}(\deg w - 1)f(w))^2 \\
&\geq \frac{1}{4} \left( f(u)^2 + \deg vf(v)^2 + \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \deg wf(w)^2 \right),
\end{align*}
\]

which can be also rewritten as

\[
\begin{align*}
&\left( 1 + \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \frac{1}{\deg w} - \frac{\deg v}{4} \right) f(v)^2 + \left( \frac{1}{\deg v} - \frac{1}{4} \right) f(u)^2 \\
&+ \frac{2}{\deg v} \cdot f(u) \left( \sum_{w \in \mathcal{N}(v) \setminus \{u\}} f(w) \right) + f(v) \left( \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \frac{2}{\deg w} \cdot \tilde{\varepsilon}(\deg w - 1)f(w) \right) \\
&+ \frac{1}{\deg v} \left( \sum_{w \in \mathcal{N}(v) \setminus \{u\}} f(w) \right)^2 \\
&\geq \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \left( \frac{1}{4} \cdot \deg wf(w)^2 - \frac{1}{\deg w} (\tilde{\varepsilon}(\deg w - 1)f(w))^2 \right) \\
&+ \left( \frac{1}{4} - \tilde{\varepsilon}^2 \right) \sum_{w \in V \setminus \mathcal{N}(v)} \deg wf(w)^2.
\end{align*}
\]
Using the fact that \( \deg v \in \{2, 3\} \), we can finally complete the proof by considering the following three cases.

**Case 1:** \( \bar{\varepsilon}^2 \leq \frac{1}{4} \).

In this case, by letting

\[
 f(u) := - \sum_{w \in \mathcal{N}(v) \setminus \{u\}} f(w)
\]

and \( f(v) := 0 \), the above inequality implies that

\[
 0 \geq \sum_{w \in \mathcal{N}(v) \setminus \{u\}} \left( \frac{1}{4} \cdot \deg w f(w)^2 - \frac{1}{\deg w} (\bar{\varepsilon}(\deg w - 1)f(w))^2 \right) + \left( \frac{1}{4} - \bar{\varepsilon}^2 \right) \cdot \left( \sum_{w \in \mathcal{N}(v)} \deg w f(w)^2 \right).
\]

Since, for \( w \in \mathcal{N}(v) \setminus \{u\} \),

\[
 \frac{1}{4} \cdot \deg w f(w)^2 - \frac{1}{\deg w} (\bar{\varepsilon}(\deg w - 1)f(w))^2 \\
 \geq \frac{f(w)^2}{4} \cdot \left( \deg w - \frac{(\deg w - 1)^2}{\deg w} \right) \\
 = \frac{f(w)^2}{4} \cdot \frac{2\deg w - 1}{\deg w} \geq 0,
\]

the inequality (13) must be an equality, implying that \( f|_{\mathcal{N}(v) \setminus \{u\}} = 0 \) and \( \bar{\varepsilon}^2 = \frac{1}{4} \). Thus, we have two cases:

(a) If \( \bar{\Gamma} \) is connected, then it is either a book graph or a petal graph.

(b) If \( \bar{\Gamma} \) is not connected, then it has two connected components and at least one of them is either a book graph or a petal graph.

In both cases it is possible to find an eigenfunction \( f \) for \( (\text{Id} - \Delta(\bar{\Gamma}))^2 \) with eigenvalue \( \bar{\varepsilon}^2 \), such that \( f|_{\mathcal{N}(v) \setminus \{u\}} \neq 0 \), which contradicts the above implication.

**Case 2:** \( \bar{\varepsilon}^2 > \frac{1}{4} \) and \( \deg v = 2 \).

In this case, \( \bar{\Gamma} = P_2 \) and \( \Gamma = P_4 \), which is a contradiction since we are assuming that \( \bar{\Gamma} \neq P_4 \).

**Case 3:** \( \bar{\varepsilon}^2 > \frac{1}{4} \) and \( \deg v = 3 \).

In this case, since we cannot have \( \bar{\Gamma} = P_2 \) (as shown in the previous case), \( \bar{\Gamma} \) must have exactly two connected components, each of them is either \( P_2 \) or a single vertex. But in any of these cases, one can show that \( \varepsilon = \varepsilon(\bar{\Gamma}) < \frac{1}{2} \), which is a contradiction.

This proves the claim. \( \square \)
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REFERENCES
1. M. S. Ashbaugh and R. D. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Ann. of Math. **135** (1992), no. 3, 601–628.
2. A. Banerjee and J. Jost, *On the spectrum of the normalized graph Laplacian*, Linear Algebra Appl. **428** (2008), no. 11–12, 3015–3022.
3. F. Bauer and J. Jost, *Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian*, Comm. Anal. Geom. **21** (2013), 787–845.
4. I. Chavel, *Eigenvalues in Riemannian geometry*, Academic Press, Amsterdam, Netherlands, 1984.
5. P. Chiu, *Cubic Ramanujan graphs*, Combinatorica. **12** (1992), no. 3, 275–285.
6. F. Chung, *Spectral graph theory*, American Mathematical Society, Providence, Rhode Island, 1997.
7. S. M. Cioabă, W. H. Haemers, J. R. Vermette, and W. Wong, *The graphs with all but two eigenvalues equal to ±1*, J. Algebraic Combin. **41** (2015), no. 3, 887–897.
8. K. Das and S. Sun, *Extremal graph on normalized Laplacian spectral radius and energy*, Electron. J. Linear Algebra. **29** (2016), no. 1, 237–253.
9. P. Erdős, A. Rényi, and V. T. Sós, *On a problem of graph theory*, Studia Sci. Math. Hungar. **1** (1966), 215–235.
10. G. Faber, *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Verlag d. Bayer. Akad. d. Wiss. **1923** (1923), no. 8, 4.
11. J. Jost, *Riemannian geometry and geometric analysis*, Springer, Berlin/Heidelberg, Germany, 2017.
12. J. Jost, R. Mulas, and F. Münch, *Spectral gap of the largest eigenvalue of the normalized graph Laplacian*, Commun. Math. Stat. **10** (2021), 371–381.
13. A. J. Kollár, M. Fitzpatrick, P. Sarnak, and A. A. Houck, *Line-graph lattices: Euclidean and non-Euclidean flat bands, and implementations in circuit quantum electrodynamics*, Comm. Math. Phys. **376** (2020), 1909–1956.
14. A. J. Kollár and P. Sarnak, *Gap sets for the spectra of cubic graphs*, Comm. Amer. Math. Soc. **1** (2021), 1–38.
15. E. Krahn, *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94** (1925), no. 1, 97–100.
16. E. Krahn, *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Mattiesen, Göttingen, 1926.
17. R. Medina, C. Noyer, and O. Raynaud, *Twins vertices in hypergraphs*, Electron. Notes Discrete Math. **27** (2006), 87.
18. A. Nilli, *On the second eigenvalue of a graph*, Discrete Math. **91** (1991), no. 2, 207–210.
19. H. F. Weinberger, *An isoperimetric inequality for the N-dimensional free membrane problem*, J. Ration. Mech. Anal **5** (1956), no. 4, 633–636.

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