FLEXIBILITY OF ENTROPY OF BOUNDARY MAPS FOR SURFACES OF CONSTANT NEGATIVE CURVATURES

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In memory of Tolya

Abstract. Given a closed, oriented, compact surface $S$ of constant negative curvature and genus $g \geq 2$, we study the measure-theoretic entropy of the Bowen-Series boundary map with respect to its smooth invariant measure. We obtain an explicit formula for the entropy that only depends on the perimeter of the $(8g-4)$-sided fundamental polygon of the surface $S$ and its genus. Using this, we analyze how the entropy changes in the Teichmüller space of $S$ and prove the following flexibility result: the measure-theoretic entropy takes all values between 0 and a maximum that is achieved on the surface that admits a regular $(8g-4)$-sided fundamental polygon. We also compare the measure-theoretic entropy to the topological entropy of these maps and show that the smooth invariant measure is not the measure of maximal entropy.

1. Introduction

Any closed, oriented, compact surface $S$ of genus $g \geq 2$ and constant negative curvature can be modeled as $S = \Gamma \setminus \mathbb{D}$, where $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ is the unit disk endowed with hyperbolic metric

$$\frac{2|dz|}{1-|z|^2}$$

and $\Gamma$ is a finitely generated Fuchsian group of the first kind acting freely on $\mathbb{D}$.

Recall that geodesics in this model are half-circles or diameters orthogonal to $S = \partial \mathbb{D}$, the circle at infinity. The geodesic flow $\tilde{\varphi}^t$ on $\mathbb{D}$ is defined as an $\mathbb{R}$-action on the unit tangent bundle $T\mathbb{D}$ that moves a tangent vector along the geodesic defined by this vector with unit speed. The geodesic flow $\varphi^t$ on $\mathbb{D}$ descends to the geodesic flow $\varphi^t$ on the factor $S = \Gamma \setminus \mathbb{D}$ via the canonical projection

$$\pi : T^1 \mathbb{D} \to T^1 S$$

of the unit tangent bundles. The orbits of the geodesic flow $\varphi^t$ are oriented geodesics on $S$.

A surface $S$ of genus $g$ admits an $(8g-4)$-gon fundamental domain $\mathcal{F}$ obtained by cutting it with $2g$ closed geodesics that intersect in pairs ($g$ of them go around the “holes” and another $g$ go around the “waists” of $S$) (see Figure 1).

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Figure 1. Chain of $2g$ geodesics when $g = 2$.

The existence of such a fundamental polygon is an old result attributed to Dehn, Fenchel, Nielsen, and Koebe [18, 13, 5]. Adler and Flatto [3, Appendix A] give a careful proof of existence and properties of $\mathcal{F}$.

We label the sides of $\mathcal{F}$ in a counterclockwise order by numbers $1 \leq i \leq 8g - 4$ and label the vertices of $\mathcal{F}$ by $V_i$ so that side $i$ connects $V_i$ to $V_{i+1}$ (mod $8g - 4$) (this gives us a marking of the polygon).

We denote by $P_i$ and $Q_{i+1}$ the endpoints of the oriented infinite geodesic that extends side $i$ to the circle at infinity $S$. The order of endpoints on $S$ is the following:

$$P_1, Q_1, P_2, Q_2, \ldots, P_{8g-4}, Q_{8g-4}.$$  \hfill (2)

The identification of the sides of $\mathcal{F}$ is given by the side pairing rule

$$\sigma(i) := \begin{cases} 4g - i \mod (8g - 4) & \text{if } i \text{ is odd} \\ 2 - i \mod (8g - 4) & \text{if } i \text{ is even}. \end{cases}$$  \hfill (3)

Let $T_i$ denote the Möbius transformation pairing side $i$ with side $\sigma(i)$.

Figure 2. An irregular polygon with side identifications, genus 2.

Notice that in general the polygon $\mathcal{F}$, whose sides are geodesic segments, need not be regular, but the sides $i$ and $\sigma(i)$ must have equal length, and the angles at

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1 The points $P_i$, $Q_i$ in this paper and [7, 11, 2] are denoted by $a_i$, $b_{i-1}$, respectively, in [3].
The entropy of the boundary map is given by Theorem 1.

\[ h \] from Fenchel–Nielsen Theorem there exists an orientation-preserving homeomorphism \( g \) on the space of polygons is as follows:

\[ f_p(x) = T_i x \quad \text{if} \quad x \in [P_i, P_{i+1}). \] (4)

The map is Markov with respect to the partition (2), and it admits a smooth ergodic measure \( \mu_p \) (see [7, Theorem 1.2]). Adler and Flatto [3] gave a thorough analysis of these maps, their two-dimensional geometric extensions and applications to the symbolic coding of the geodesic flow on \( \Gamma \setminus \mathbb{D} \). In [2, 11], the measure \( \mu_p \) is described as a two-step projection of the invariant Liouville measure \( m \) for the geodesic flow.

We can now state our first main result:

**Theorem 1.** The entropy of the boundary map is given by

\[ h_{\mu_p}(f_p) = \frac{\pi^2(2g-2)}{\text{Perimeter}(\mathcal{F})} = \frac{\pi}{2} \cdot \frac{\text{Area}(\mathcal{F})}{\text{Perimeter}(\mathcal{F})}. \] (5)

Let \( S = \Gamma \setminus \mathbb{D} \) be any compact surface of genus \( g \geq 2 \), and \( S_0 = \Gamma_{\text{reg}} \setminus \mathbb{D} \) be a special genus \( g \) surface that admits a regular \((8g-4)\)-sided fundamental region \( \mathcal{F}_{\text{reg}} \). By Fenchel–Nielsen Theorem there exists an orientation-preserving homeomorphism \( h \) from \( \mathbb{D} \) onto \( \mathbb{D} \) such that \( \Gamma = h \circ \Gamma_{\text{reg}} \circ h^{-1} \) and the sides of the fundamental polygon \( \mathcal{F} \) for \( \Gamma \) belong to geodesics \( P_i Q_{i+1} \), where \( P_i = h(P_i), Q_{i+1} = h(Q_{i+1}) \) and \( P_i Q_{i+1} \) are the extensions of the sides of \( \mathcal{F}_{\text{reg}} \). The map \( h|_{\mathcal{S}} \) is a homeomorphism of \( \mathcal{S} \) preserving the order of the points \( \{P_i\} \cup \{Q_i\} \).

The space \( \mathcal{T}(S) \) can be thought of as any of the following:

1. the space of Riemann surface structures on \( S \) modulo conformal maps isomorphic to the identity. [9, Section 1].
2. the space of marked Fuchsian groups \( \Gamma \) such that \( \pi_1(S) \xrightarrow{\sim} \Gamma \) and \( S \) is orientation-preserving homeomorphic to \( \Gamma \setminus \mathbb{D} \). [10, Definition 2.1.1]
3. the space of all marked canonical hyperbolic \((8g-4)\)-gons in the unit disk \( \mathbb{D} \) such that side \( i \) and side \( \sigma(i) \) have equal length and the internal angles at vertices \( i \) and \( \sigma(i) + 1 \) sum to \( \pi \), up to an isometry of \( \mathbb{D} \). (The topology on the space of polygons is as follows: \( P_i \to P \) if and only if the lengths of all sides converge and the measures of all angles converge.)

The construction (3) of \( \mathcal{T}(S) \) by varying “marked” fundamental polygons is less common than others. Schnitz Schaller [16] considers canonical \( 4g \)-gons following the earlier work [8, 19, 17], but the canonical \( 8g-4 \)-gons may be considered as well. In fact, the dimension of \( \mathcal{T}(S) \) is calculated using \( 8g-4 \)-gons rather easily by the following heuristic argument. The lengths of the identified pairs of sides are given by \( 4g-2 \) real parameters; \( 2g-1 \) real parameters represent the angles since four angles at each vertex are determined by one real parameter. The dimension of the space of isometries of \( \mathbb{D} \) is 3, so we remain with \( (4g-2) + (2g-1) - 3 = 6g-6 \) parameters.
A few years ago, Anatole Katok suggested a new area of research—or at very least, a new viewpoint—called the “flexibility program,” which can be vaguely formulated as follows: under properly understood general restrictions, within a fixed class of smooth dynamical systems, dynamical invariants take arbitrary values. Taking this point of view, it is natural to ask how the measure-theoretic entropy \( h_{\mu_P}(f_P) \) changes in \( T(S) \). We have:

**Theorem 2** (Maximum and flexibility of entropy).

(i) Among all surfaces in \( T(S) \), the maximum value of the entropy \( h_{\mu_P}(f_P) \) is achieved on the surface for which \( F \) is regular and is equal to

\[
H(g) = h_{\mu_P}^{\text{reg}}(f_P^{\text{reg}}) = \frac{\pi^2(2g-2)}{(8g-4) \cosh^{-1}(1 + 2 \cos \frac{\pi}{4g-2})}.
\]

(ii) For any value \( h \in (0, H(g)] \) there exists \( F \in T(S) \) such that \( h_{\mu_P}(f_P) = h \).

The paper is organized as follows. In Section 2 we prove Theorem 1. The natural extension \( F_P \) of \( f_P \) and the “geometric map” \( F_G \) of [2] are used in the proof. In Section 3 we prove Theorem 2 by invoking the Isoareal Inequality and using Fenchel-Nielsen coordinates in the Teichmüller space. In Section 4 we compute the topological entropy of the boundary map \( f_P \) and show that the smooth invariant measure \( \mu_P \) is not the measure of maximal entropy. In Appendix A we provide some computational tools for genus 2.

2. Proof of Theorem 1

Adler and Flatto [3] introduced the “rectilinear map” defined by

\[
F_P(u, w) = (T_i u, T_i w) \quad \text{if } w \in [P_i, P_{i+1})
\]

and showed the existence of an invariant domain \( \Omega_P \subset \mathbb{S} \times \mathbb{S} \) such that \( F_P \) restricted to \( \Omega_P \) is a two-dimensional geometric realization of the natural extension map of \( f_P \).\(^2\) The set \( \Omega_P \) is bounded away from the diagonal \( \Delta = \{ (w, w) : w \in \mathbb{S} \} \) and has a finite rectangular structure.

It is a standard computation that the smooth measure

\[
d\nu = \frac{|du||dw|}{|u - w|^2}
\]
is preserved by Möbius transformations applied to unit circle variables \( u \) and \( w \). Hence \( F_P \) preserves the smooth probability measure

\[
d\nu_P = \frac{d\nu}{\int_{\Omega_P} d\nu}.
\]

The boundary map \( f_P \) is a factor of \( F_P \) (projecting on the second coordinate), so one can obtain its smooth invariant probability measure \( \mu_P \) as a projection.

The geodesic flow on \( S \) can be realized as a special flow over a cross-section that is parametrized by \( \Omega_P \), and the first return map to this cross-section acts exactly as \( F_P : \Omega_P \to \Omega_P \). Using this realization, we have from [2, Proposition 10.1] that

\[
h_{\mu_P}(f_P) = h_{\mu_P}(F_P) = \frac{\pi^2(2g-2)}{\int_{\Omega_P} d\nu}.
\]

\(^2\)In [11], the authors showed that \( \Omega_P \) is also the global attractor of \( F_P : \mathbb{S} \times \mathbb{S} \setminus \Delta \to \mathbb{S} \times \mathbb{S} \setminus \Delta \).
But by the Gauss–Bonnet formula, $\text{Area}(\mathcal{F}) = 2\pi(2g - 2)$, hence

$$h_{\mu_{\mathcal{F}}} \left( f_{\mathcal{F}} \right) = \frac{\pi}{2} \cdot \frac{\text{Area}(\mathcal{F})}{\int_{\Omega_{\mathcal{F}}} d\nu}.$$  \hfill (9)

To prove Theorem 1, it remains only to show that $\int_{\Omega_{\mathcal{F}}} d\nu$ is equal to the (hyperbolic) perimeter of $\mathcal{F}$. For that, we use another map, also introduced by Adler–Flatto in [3], called the “curvilinear map” (or “geometric map” in [2]). It is defined on the set

$$\Omega_G := \{ (u, w) : uw \text{ intersects } \mathcal{F} \} \subset \mathbb{S} \times \mathbb{S} \setminus \Delta$$

and given by

$$F_G(u, w) = (T_i u, T_i w) \quad \text{if } uw \text{ exits } \mathcal{F} \text{ through side } i.$$  

There is a key correspondence between $\Omega_G$ and $\Omega_{\mathcal{F}}$:

**Proposition 3 ([3, Theorem 5.1]).** The map $\Phi : \Omega_G \rightarrow \Omega_{\mathcal{F}}$ given by

$$\Phi = \begin{cases} 
\text{Id} & \text{on } \Omega_G \cap \Omega_{\mathcal{F}} \\
T_{\sigma(i)-1}^{-1} T_i & \text{on } B_i,
\end{cases}$$

where $B_i = \{ (u, w) \in \Omega_G \setminus \Omega_{\mathcal{F}} : w \in [P_i, P_{i+1}] \}$, is bijective.

Since $\Phi$ acts by fractional linear transformations, which preserve the measure $\nu$, we have that

$$\int_{\Omega_{\mathcal{F}}} d\nu = \int_{\Omega_G} d\nu.$$ \hfill (10)

Having proved (10), we now want to show that $\int_{\Omega_{\mathcal{F}}} d\nu$ is equal to the perimeter of $\mathcal{F}$.

**Lemma 4.** For any oriented geodesic segment $s$ on $\mathbb{D}$,

$$\int_{\Psi^+(s)} d\nu = \text{length}(s),$$

where $\Psi^+(s)$ is the set of oriented geodesics intersecting $s$ with the oriented angle at the intersection between 0 and $\pi$.  

**Figure 3.** Bulges $B_i$ of $\Omega_G$ (left) are mapped to corners of $\Omega_{\mathcal{F}}$ (right)
The proof involves expressing the Liouville measure in a coordinate system based on movement along geodesics. See [6, Appendix A3] for details.

Recall that the domain of the geometric map \( F \) is the set \( \Omega = \bigcup_{i=1}^{8g-4} G_i \), where 
\[
G_i = \{ (u,w) : uw \text{ exits } F \text{ through side } i \} = \Psi_+(\text{side } i).
\]

Thus from Lemma 4 we immediately get
\[
\int_{\Omega} d\nu = \sum_{i=1}^{8g-4} \int_{G_i} d\nu = \sum_{i=1}^{8g-4} \text{length(side } i) = \text{Perimeter}(F). \tag{11}
\]

Combining (11) with (10), one can replace \( \int_{\Omega} P d\nu \) by the perimeter of \( F \) in the denominator of (9); this completes the proof of Theorem 1.

**Remark 5.** In [11], the authors introduced and investigated dynamical properties of boundary maps \( f_A \) defined for arbitrary partitions \( A = \{A_i\}_{i=1}^{8g-4} \), with \( A_i \in (P_i, Q_i) \) for all \( 1 \leq i \leq 8g-4 \), satisfying the so-called “short cycle property” \( f_A(T_i A_i) = f_A(T_{i-1} A_i) \). Additional results from [2] allow us to prove the same relationship (5) for the entropy of \( f_A \):
\[
h_{\mu_A}(f_A) = \frac{\pi^2(2g - 2)}{\text{Perimeter}(F)} = \frac{\pi}{2} \cdot \frac{\text{Area}(F)}{\text{Perimeter}(F)} = h_{\mu_P}(f_P).
\]

In other words, the entropy remains unchanged for all boundary maps \( f_A \) defined using a partition \( A = \{A_i\}_{i=1}^{8g-4} \) satisfying the short cycle property.

## 3. PROOF OF THEOREM 2

To prove that Theorem 2(i) follows from Theorem 1, we only need to show that for each genus \( g \) the perimeter of \( F \) in \( \mathcal{T}(S) \) is minimized on the regular polygon.

**Theorem 6** (Isoareal Inequality). Among all hyperbolic polygons with a given area and number of sides, a regular polygon has the smallest perimeter.

**Proof.** For a hyperbolic \( n \)-gon \( P_n \), the following inequality
\[
\text{Perimeter}(P_n)^2 \geq 4d_n \text{Area}(P_n), \quad d_n = n \tan\left(\frac{\text{Area}(P_n)}{2n}\right)
\]

is given in [14, Theorem 1.2(a)], which also states that equality is achieved on a regular polygon. Both isoperimetric and isoareal inequalities follow. In our setting, \( \text{Area}(F) = 2\pi(2g - 2) \) is constant in \( \mathcal{T}(S) \), so \( d_n \) is also constant, and thus the perimeter of \( F \) is minimal when \( F \) is a regular polygon.

The expression for the maximum value \( H(g) \) in (6) comes directly from (5), with
\[
cosh^{-1}\left(1 + 2 \cos\left(\frac{\pi}{4g-2}\right)\right)
\]
being the length of a single side of the regular \((8g-4)\)-gon. This completes the proof of Theorem 2(i).

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3Thank you to Alena Erchenko for providing this reference.
Proof of Theorem 2(ii). The Teichmüller space $\mathcal{T}(S)$ is $6g - 6$ dimensional. Fenchel–Nielsen coordinates use a decomposition of $S$ into $2g - 2$ pairs of pants using $3g - 3$ non-intersecting closed geodesics whose lengths can be manipulated independently (these lengths form $3g - 3$ of the $6g - 6$ coordinates). We take one of these geodesics to also be a geodesic from the chain described in Section 1 that corresponds to one entire side of $\mathcal{F}$ (this shared geodesic is on the far right in both parts of Figure 4). Since the length of this side (one of the Fenchel–Nielsen coordinates) can be made arbitrarily large, the perimeter of $\mathcal{F}$ can also be made arbitrarily large, which by (5) means that $h_{\mu_p}(f_{\mathcal{P}})$ can be made arbitrarily small.

![Figure 4](image_url)

**Figure 4.** Chain of $2g$ geodesics on $S$ forming the sides of $\mathcal{F}$ (top) and decomposition of $S$ into $2g - 2$ pairs of pants by $3g - 3$ non-intersecting geodesics (bottom) for $g = 3$.

By the continuity of the Fenchel-Nielsen coordinates, if $\Gamma \to \Gamma'$ in $\mathcal{T}(S)$, then, by Fenchel-Nielsen Theorem, $\Gamma = h \circ \Gamma' \circ h^{-1}$ for some orientation preserving homeomorphism $h : \mathbb{D} \to \mathbb{D}$, and $h|_S \to \text{Id}$ as circle homeomorphism, i.e. $d(h(x), x) \to 0$ for $x \in S$. Therefore for the endpoints of the geodesics $P_i Q_{i+1}$ containing the sides of the fundamental polygon $\mathcal{F}$ and the geodesics $P'_i Q'_{i+1}$ containing the sides of the fundamental polygon $\mathcal{F}'$, we have $P_i \to P'_i$ and $Q_{i+1} \to Q'_{i+1}$. It follows that the vertices of $\mathcal{F}$ will tend to the vertices of $\mathcal{F}'$, and hence $\text{Perimeter}(\mathcal{F}) \to \text{Perimeter}(\mathcal{F}')$, i.e. the perimeter of $\mathcal{F}$ varies continuously within $\mathcal{T}(S)$. From Theorem 1 we conclude the continuity of the entropy $h_{\mu_p}(f_{\mathcal{P}})$ within $\mathcal{T}(S)$. By the Intermediate Value Theorem, $h_{\mu_p}(f_{\mathcal{P}})$ must take on all values between 0 and its maximum. 

For genus 2, the techniques of Maskit (see Appendix A) allow us to draw the fundamental polygon $\mathcal{F}$ for any values of the Fenchel–Nielsen coordinates. Figure 5 shows how the entropy changes as the single Fenchel–Nielsen coordinate representing the length of the bottom side of $\mathcal{F}$ is varied.
Figure 5. Entropy as a function of a single Fenchel–Nielsen coordinate for \( g = 2 \).

4. Topological entropy

The Variational Principle states that the topological entropy of a continuous dynamical system \( f : X \to X \) on a compact space \( X \) is equal to the supremum of the measure-theoretic entropies over all \( f \)-invariant probability measures on \( X \). For some common transformations the Lebesgue measure or some other natural measure \( \mu \) will satisfy \( h_\mu(f) = h_{\text{top}}(f) \), but this is not guaranteed. Indeed, the boundary map \( f_P : \mathbb{S} \to \mathbb{S} \) provides an example where the smooth invariant measure \( \mu_P \) is very far from the measure of maximum entropy, as we now show.

Proposition 7. In terms of only the genus \( g \), we have that the topological entropy (regardless of whether \( F \) is regular) is

\[
h_{\text{top}}(f_P) = \log(4g - 3 + 2\sqrt{4g^2 - 6g + 2}). \tag{12}
\]

Proof of Proposition 7. The expression in (12) comes from a Markov shift. The partition of \( \mathbb{S} \) consisting of all intervals \([P_i, Q_i]\) and \([Q_i, P_{i+1}]\) is a Markov partition for \( f_P \) (see [7, Lemma 2.5] or [3, Theorem 6.1]), so the topological entropy of \( f_P \) can be given as

\[
h_{\text{top}}(f_P) = \log |\lambda|_{\text{max}},
\]

where \(|\lambda|_{\text{max}}\) is the spectral radius (that is, the eigenvalue with largest absolute value) of the adjacency matrix for the topological Markov chain associated to \( f_P \).

Given a Markov partition \( \{I_1, \ldots, I_n\} \), the adjacency matrix \( M \) is the \( n \times n \) matrix given by

\[
m_{ij} = \begin{cases} 1 & \text{if } f_P(I_i) \supset I_j \\ 0 & \text{otherwise.} \end{cases}
\]

Indexing the intervals \( I_1, \ldots, I_{2(8g-4)} \) for \( f_P \) as

\[
I_{2i-1} := [P_i, Q_i], \quad I_{2i} := [Q_i, P_{i+1}], \quad i = 1, \ldots, 8g-4,
\]

(13)
the relations
\[ T_i(P_i) = Q_{\sigma(i)+1}, \quad T_i(Q_i) = Q_{\sigma(i)+2}, \quad T_i(P_{i+1}) = P_{\sigma(i)-1} \]
from [11, Proposition 2.2] (originally [3, Theorem 3.4]) give us that
\[
\begin{align*}
  f_P(I_{2i-1}) &= T_i(I_{2i-1}) = [Q_{\sigma(i)+1}, Q_{\sigma(i)+2}] = I_{2\sigma(i)+2} \cup I_{2\sigma(i)+3} \\
  f_P(I_{2i}) &= T_i(I_{2i}) = [Q_{\sigma(i)+2}, P_{\sigma(i)-1}] = I_{2\sigma(i)+4} \cup I_{2\sigma(i)+5} \cup \cdots \cup I_{2\sigma(i)-4},
\end{align*}
\]
where the indices in \( I_k \) are mod \( 16g-8 \) (although \( \sigma(i) \) is still calculated mod \( 8g-4 \)).

The entries of \( M \) are therefore
\[
m_{ij} = \begin{cases} 
  1 & \text{if } i \text{ odd, } j \in \{ 2\sigma\left(\frac{i}{2}\right) + 2, 2\sigma\left(\frac{i+1}{2}\right) + 3 \} \mod 16g-8 \\
  1 & \text{if } i \text{ even, } j \in \{ 2\sigma\left(\frac{i}{2}\right) + 4, 2\sigma\left(\frac{i}{2}\right) + 5, \ldots, 2\sigma\left(\frac{i}{2}\right) - 4 \} \mod 16g-8 \\
  0 & \text{otherwise,}
\end{cases}
\]
or, using the definition of \( \sigma \) from (3),
\[
m_{ij} = \begin{cases} 
  1 & \text{if } i \text{ odd, } \frac{i+1}{2} \text{ odd, } j \in \{ 5 - i, 6 - i \} \mod 16g-8 \\
  1 & \text{if } i \text{ odd, } \frac{i+1}{2} \text{ even, } j \in \{ 8g - i + 1, 8g - i + 2 \} \mod 16g-8 \\
  1 & \text{if } i \text{ even, } \frac{i}{2} \text{ even, } j \in \{ 8i - 9 - i, \ldots, 1 - i, -i \} \mod 16g-8 \\
  1 & \text{if } i \text{ even, } \frac{i}{2} \text{ odd, } j \in \{ 8g - i + 4, \ldots, 8g - i - 4 \} \mod 16g-8 \\
  0 & \text{otherwise.}
\end{cases}
\]

The characteristic polynomial of \( M \) is
\[
p(x) = (x - 1)^{8g-2}(x + 1)^{8g-8}(x^2 - (8g - 6)x + 1),
\]
so the eigenvalues of \( M \) are \( +1, -1, \) and
\[
\frac{(8g - 6) \pm \sqrt{(8g - 6)^2 - 4}}{2} = 4g - 3 \pm 2\sqrt{4g^2 - 6g + 2}.
\]
The maximal eigenvalue is thus \( 4g - 3 + 2\sqrt{4g^2 - 6g + 2} \), leading to (12). \( \Box \)

**Corollary 8.** The measure-theoretic entropy of \( f_P \) with respect to its smooth invariant measure \( \mu_P \) is strictly less than the topological entropy of \( f_P \).

**Proof.** From (6), we have that \( H(g) = h_{\mu_P}^\text{reg}(f_P) \), computed in Theorem 2(i), is an increasing function of \( g \), and
\[
\lim_{g \to \infty} H(g) = \lim_{g \to \infty} \frac{\pi^2(2g - 2)}{(8g - 4) \cosh^{-1}(1 + 2 \cos \frac{\pi}{4g-2})} = \frac{\pi^2}{4 \cosh^{-1}(3)} = \frac{\pi^2}{4 \log(3 + 2\sqrt{2})}.
\]
Since \( H(g) \) is increasing, its value for any \( g \) is less than or equal to this limit. From (12), the topological entropy is also increasing with \( g \), so its value for any \( g \geq 2 \) is greater than or equal to
\[
h_{\text{top}}(f_P) \bigg|_{g=2} = \log(5 + 2\sqrt{6}).
\]
Then we have
\[
h_{\mu_P}(f_P) < \frac{\pi^2}{4 \log(3 + 2\sqrt{2})} < 2 < \log(5 + 2\sqrt{6}) \leq h_{\text{top}}(f_P)
\]
for every genus \( g \geq 2 \). \( \Box \)
Figure 6. Topological entropy and maximum measure-theoretic entropy for different genera.

Appendix A. Computational tools for genus 2

The polygon in Figure 2 and the details of Figure 3 were produced using the generators of $\Gamma$ described by Maskit [15] in terms of the six Fenchel–Nielsen coordinates for $g = 2$. In case they will be useful for others, we provide below the relevant information for doing numerical experiments in $T(S)$ for genus 2.

Maskit uses the eight parameters $\alpha, \beta, \gamma, \delta, \sigma, \tau, \rho, \mu$ to define matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ acting on the half-plane. Setting $A = \frac{1}{2} \left( \begin{array}{cc} i & 1 \\ 1 & i \end{array} \right) \tilde{A} \left( \begin{array}{cc} -i & 1 \\ 1 & -i \end{array} \right)$, etc., we get the following matrices acting on the disk:

- $A = \frac{\sinh \alpha}{\sinh \mu} \left( \begin{array}{cc} \coth \alpha \sinh \mu + i & -i \cosh \mu \\ i \cosh \mu & \coth \alpha \sinh \mu - i \end{array} \right)$
- $B = \frac{\sinh \beta}{\cosh \tau} \left( \begin{array}{cc} \cosh \tau \coth \beta + i \sinh \sigma & \cosh \sigma + i \sinh \tau \\ \cosh \sigma - i \sinh \tau & \cosh \tau \coth \beta - i \sinh \sigma \end{array} \right)$
- $C = \left( \begin{array}{cc} \cosh \gamma & i \sinh \gamma \\ -i \sinh \gamma & \cosh \gamma \end{array} \right)$
- $D = \frac{\sinh \delta}{\cosh \rho} \left( \begin{array}{cc} \cosh \rho \coth \delta - i \sinh(\gamma + \sigma) & -\cosh(\gamma + \sigma) - i \sinh \rho \\ -\cosh(\gamma + \sigma) + i \sinh \rho & \cosh \rho \coth \delta + i \sinh(\gamma + \sigma) \end{array} \right)$

where $(\alpha, \beta, \gamma, \sigma, \tau, \rho)$ are the Fenchel–Nielsen coordinates for genus 2 and

- $\mu = \cosh^{-1} \left( \coth \beta \cosh \sigma \cosh \tau + \sinh \sigma \sinh \tau \right)$
- $\delta = \coth^{-1} \left( \frac{\cosh \gamma \cosh \mu - \coth \alpha \sinh \gamma \sinh \mu - \sinh \sigma \sinh \rho}{\cosh \sigma \cosh \rho} \right)$

Let $S_i$ be the transformation for which $P_i$ is the repelling fixed point and $Q_{i+1}$ is the attracting fixed point. That is, the oriented axis of $S_i$ contains side $i$. We have
The side-pairing transformations\footnote{In \cite{katok2017structure, abrams2019adler} the maps $T_i$ are always referred to as “generators,” but there are several generating sets for $\Gamma$. In particular, $S_1, \ldots, S_{12}$ also generate $\Gamma$.} are
\begin{align*}
S_1 &= C^{-1}D^{-1}C \\
S_2 &= AC \\
S_3 &= ABDA^{-1} \\
S_4 &= A^{-1} \\
S_5 &= D^{-1}B^{-1} \\
S_6 &= CA \\
S_7 &= D \\
S_8 &= DA^{-1}C^{-1}D^{-1} \\
S_9 &= B^{-1}D^{-1} \\
S_{10} &= B^{-1}AB \\
S_{11} &= C^{-1}DCB \\
S_{12} &= C^{-1}B^{-1}A^{-1}B.
\end{align*}

The defining relation
\[ ABDA^{-1}C^{-1}D^{-1}CB^{-1} = \text{Id} \]
from \cite{forni2013introduction} is equivalent to \cite[Equation 1.5]{katok2017structure} with $g = 2$.

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