Existence, uniqueness and stability of an inverse problem for two-dimensional convective Brinkman–Forchheimer equations with the integral overdetermination

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Received: 30 May 2022 / Accepted: 28 July 2022 © Tusi Mathematical Research Group (TMRG) 2022

Abstract
In this article, we study an inverse problem for the following convective Brinkman–Forchheimer (CBF) equations:

\[ u_t - \mu \Delta u + (u \cdot \nabla)u + \alpha u + \beta |u|^{r-1}u + \nabla p = F := fg, \quad \nabla \cdot u = 0, \]

in a bounded domain \( \Omega \subset \mathbb{R}^2 \) with smooth boundary \( \partial \Omega \), where \( \alpha, \beta, \mu > 0 \) and \( r \in [1, 3] \). The investigated inverse problem consists of reconstructing the vector-valued velocity function \( u \), the pressure field \( p \) and the scalar function \( f \). For the divergence free initial data \( u_0 \in \mathcal{L}^2(\Omega) \), we prove the existence of a solution to the inverse problem for two-dimensional CBF equations with the integral overdetermination condition, by showing the existence of a unique fixed point for an equivalent operator equation (using an extension of the contraction mapping theorem). Moreover, we establish the uniqueness and Lipschitz stability results of the solution to the inverse problem for 2D CBF equations with \( r \in [1, 3] \).

Keywords Convective Brinkman–Forchheimer equations · Inverse source problem · Integral overdetermination condition · Contraction mapping theorem · Well-posedness

Mathematics Subject Classification 35R30 · 35Q35 · 35Q30

Communicated by Maria Alessandra Ragusa.

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Published online: 14 August 2022
1 Introduction

The convective Brinkman–Forchheimer (CBF) equations describe the motion of incompressible fluid flows in a saturated porous medium (cf. [5]). This article’s primary goal is to discuss the well-posedness of an inverse problem to CBF equations in two dimensions for a divergence free initial data in $L^2(\Omega)$.

1.1 The mathematical model and the direct problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial \Omega$ (at least $C^2$-boundary). The CBF equations are given by

$$u_t - \mu \Delta u + (u \cdot \nabla)u + au + \beta |u|^{r-1}u + \nabla p = F := fg, \text{ in } \Omega \times (0, T),$$

$$\nabla \cdot u = 0, \text{ in } \Omega \times (0, T),$$

with initial condition

$$u = u_0, \text{ in } \Omega \times \{0\},$$

and boundary condition

$$u = 0, \text{ on } \partial \Omega \times [0, T).$$

Here, $u(x, t) \in \mathbb{R}^2$ represents the velocity field at position $x$ and time $t$, $p(x, t) \in \mathbb{R}$ denotes the pressure field and $F(x, t) \in \mathbb{R}^2$ stands for the external forcing. The constant $\mu$ denotes the positive Brinkman coefficient (effective viscosity), the positive constants $\alpha$ and $\beta$ stand for the Darcy coefficient (permeability of porous medium) and the Forchheimer coefficient (proportional to the porosity of the material), respectively (cf. [23]). The absorption exponent $r \in [1, \infty)$ and the cases, $r = 3$ and $r > 3$, are known as the critical exponent and the fast growing nonlinearity, respectively. For $\alpha = \beta = 0$, we obtain the classical 2D Navier–Stokes equations (NSE). Thus, one can consider Eqs. (1)–(4) as a modification of the classical NSE by introducing an absorption term $au + \beta |u|^{r-1}u$ in NSE. Thus, one may consider the model (1)–(4) as NSE with damping. To obtain the uniqueness of the pressure $p$, one can impose the condition $\int_{\Omega} p(x, t) dx = 0$, for $t \in [0, T]$. The model given in (1)–(4) is recognized to be more accurate when the flow velocity is too large for the Darcy’s law to be valid alone, and apart from that, the porosity is not too small, so that we call these types of models as non-Darcy models (cf. [30]). It has been proved in Proposition 1.1, [18] that the critical homogeneous CBF equations have the same scaling as NSE only when $\alpha = 0$ and no scale invariance property for other values of $\alpha$ and $r$.

Let us now discuss some results available in the literature on the global solvability of the system (1)–(4) (direct problem). The existence and uniqueness of weak as well as strong solutions in two- and three-dimensional bounded domains are available in
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[2, 16, 23, 31], etc. For the global solvability results in periodic domains and whole space, one may refer to [7, 18, 42], etc. The Navier–Stokes problem modified by an absorption term $|u|^{-1}u$, for $r > 1$, in bounded domains with compact boundary is considered in [2]. The existence of Leray–Hopf weak solutions, for any dimension $d \geq 2$, and its uniqueness for $d = 2$ is established in [2]. For $d = 2$, $r \in [1, \infty)$ and $d = 3$, $r \in [3, \infty)$ ($2/\mu \geq 1$ for $r = 3$), the existence and uniqueness of global Leray–Hopf weak as well as strong solution is established in [31].

1.2 The inverse problem

Even though the direct problem is important, but it requires the knowledge of physical parameters such as the Brinkman coefficient $\mu$, Darcy coefficient $\alpha$, Forchheimer coefficient $\beta$ and the forcing term $F := fg$. When in addition to the solution of the equation, recovery of some physical properties of the investigated object or the effects of the external sources are needed, we require inverse problems to determine a coefficient or to handle the right hand side of the differential equation arising in mathematical modeling of physical phenomena. In such modeled problems, it is more efficient to consider the inverse source problems. However, posing an inverse problem requires some additional information of the solution besides the given initial and boundary conditions. An additional information on a solution of the inverse problem can be an integral overdetermination condition, which is the case considered in this work. The integral overdetermination condition considered in this paper is given by

$$\int_{\Omega} u(x, t) \cdot \omega(x) dx = \varphi(t), \quad t \in [0, T],$$

where $\varphi(t)$ is the measurement data, which stands for the average velocity on the domain $\Omega$ and $\omega$ is a given quantity, which corresponds to the type of device used to measure the velocity.

Let the vector-valued external forcing $F$ appearing in (1) be represented by

$$F(x, t) := f(t)g(x, t),$$

where the scalar function $f$ is unknown and $g$ is a given vector-valued function. The investigation of the nonlinear inverse problem in this article consists of reconstructing the vector-valued velocity function $u$, the pressure field $p$ and the scalar function $f$ from the system (1)–(4), with the integral overdetermination condition (5), and the given functions $u_0$, $\omega$, $\varphi$ and $g$.

The inverse problems with final overdetermination (cf. [6, 17, 20, 21, 38], etc.) and integral overdetermination (cf. [3, 13, 22, 24, 27, 33, 36] etc.) conditions have been well studied in the literature. The existence and uniqueness of the generalized solution of an inverse problem for the nonlinear non-stationary system of Navier–Stokes equations with integral overdetermination is investigated in [40]. The existence results of an inverse problem to NSE with both the integral as well as final overdetermination data are proved in [36] using Schauder’s fixed point theorem in two and three dimensions. The global well-posedness of an inverse source problem
for parabolic systems is examined in [34]. Under the assumption that the initial data $u_0 \in \mathbb{H}$ and the viscosity constant is sufficiently large, the authors in [14] proved that an inverse problem for 2D NSE with the final overdetermination data is well-posed. To prove the same, they have used Tikhonov’s fixed point theorem. In [15], the authors considered an inverse problem of determining a viscosity coefficient in NSE by observing data in a neighborhood of the boundary. For an extensive study on various inverse problems corresponding to NSE and related models, where one requires to reconstruct the density of external forces or some coefficients of the equations on the basis of final or functional overdetermination, we refer the interested readers to [4, 8–11, 19, 21, 25, 29, 35, 36, 41], etc., and references therein. The existence and uniqueness of an inverse problem for three-dimensional nonlinear equations of Kelvin–Voigt fluids is proved in [24].

Recently, using the contraction mapping principle, a local in time existence and uniqueness result of an inverse problem for Kelvin–Voigt fluid flow equations with memory term and integral overdetermination condition is obtained in [27]. The authors in [3] established the local unique solvability result of an inverse problem for generalized Kelvin–Voigt equation with $p$-Laplacian and damping term with integral overdetermination condition. The well-posedness of an inverse problem for 2D and 3D convective Brinkman–Forchheimer equations with final overdetermination data using Schauder’s fixed point theorem is examined in [26]. The results in [26] are obtained by assuming sufficient smoothness on the given data $(u_0 \in \mathbb{H}^2(\Omega) \cap \mathbb{V})$.

### 1.3 Technical difficulties

We emphasize that, due to a technical difficulty in working with bounded domains, if one follows the method employed in [14, 36, 40], etc. (for the initial data $u_0 \in \mathbb{H}$) may only be applicable for $r \in [1, 3]$. Note that in the case of bounded domains, $P_{\mathbb{H}}(|u|^{r-1}u)$ ($P_{\mathbb{H}}$ is the Helmholtz–Hodge orthogonal projection, see Sect. 2.1, [26]) need not be zero on the boundary, and $P_{\mathbb{H}}$ and $-\Delta$ are not necessarily commuting (for a counter example, see Example 2.19, [37]). Furthermore, while taking the inner product with $-\Delta u$ in (1), $-\Delta u \cdot n|_{\partial \Omega} \neq 0$ in general and the term with pressure will not vanish (see [23]). As a result, the equality ([18])

$$\int_{\Omega} (-\Delta u(x)) \cdot |u(x)|^{r-1}u(x)\,dx$$

$$= \int_{\Omega} |\nabla u(x)|^{r-1}u(x)\,dx + 4 \left( \frac{r-1}{(r+1)^2} \right) \int_{\Omega} |\nabla u(x)|^{\frac{r+1}{2}} \,dx$$

$$= \int_{\Omega} |\nabla u(x)|^2 |u(x)|^{r-1} \,dx + \frac{r-1}{4} \int_{\Omega} |u(x)|^{r-3} |\nabla u(x)|^2 \,dx$$

may not be helpful in bounded domains.

### 1.4 Main results

By a solution of the inverse problem (1)–(5), we mean, a triplet $(u, p, f)$ such that
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and in addition, all the relations (1)–(5) hold. In order to prove the existence and uniqueness of solutions for the above investigated inverse problem, we use the method developed in [36, 40]. Note that the works [36, 40] do not take into account of the stability of the solution. The major goals of this article are

(i) the existence and uniqueness of the solution,
(ii) the stability of the solution in the norm of the corresponding function spaces,

to the inverse problem (1)–(5) under the assumptions $u_0 \in \mathbb{H}$, $g \in C([0, T]; \mathbb{L}^2(\Omega))$, $\varphi \in H^1(0, T)$, using an extension of the contraction mapping theorem. In contrast to the results obtained for CBF equations in [26], the well-posedness of the generalized solution of the inverse problem holds for the initial data $u_0 \in \mathbb{H}$ in two dimensions for $r \in [1, 3]$ and the additional information given in the form of integral overdetermination condition.

**Definition 1.1** [40] Let $r \in [1, 3]$. A triplet $(u, p, f)$ is said to be a weak solution of a nonlinear inverse problem (1)–(5) if

$$
\begin{align*}
&u \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1}(\Omega)), \\
p \in L^{\frac{r+1}{r}}(0, T; L^2(\Omega)), f \in L^2(0, T), \quad \text{and they satisfy the integral identity}
\end{align*}
$$

\[
\frac{d}{dt} \int_\Omega u(x, t) \cdot v(x) dx + \mu \int_\Omega u(x, t) \cdot \nabla v(x) dx \\
+ \int_\Omega (u(x, t) \cdot \nabla) u(x, t) \cdot v(x) dx + \alpha \int_\Omega u(x, t) \cdot v(x) dx \\
+ \beta \int_\Omega |u(x, t)|^{r-1} u(x, t) \cdot v(x) dx = \int_\Omega f(t) g(x, t) \cdot v(x) dx,
\]

along with the initial condition (3) and the integral overdetermination condition (5), for any $v \in \mathbb{V}$.

In this definition, we have incorporated the incompressibility of the fluid velocity and the boundary condition (4) as in Definition 1.1 in the sense that the function $u(\cdot, t) \in \mathbb{V}$ for a.e. $t \in [0, T]$. Let us now state the main result in this work on the well-posedness of solutions of the inverse problem (1)–(5).

**Theorem 1.2** Let $\Omega \subset \mathbb{R}^2$, $u_0 \in \mathbb{H}$, $g \in C([0, T]; \mathbb{L}^2(\Omega))$, $\omega \in H^2(\Omega) \cap \mathbb{V}$, $\varphi \in H^1(0, T)$ and
\[
\left| \int_{\Omega} g(x,t) \cdot \omega(x) \, dx \right| \geq g_0 > 0, \quad (g_0 \equiv \text{constant}), \quad 0 \leq t \leq T,
\]
and in addition, \( \nabla \omega \in \tilde{L}^\infty \) and the compatibility condition
\[
\int_{\Omega} u_0(x) \cdot \omega(x) \, dx = \varphi(0), \tag{7}
\]
be satisfied. Then, under the assumptions of Theorem 3.2 (see below), the following assertions are satisfied for the inverse problem (1)–(5):

(i) There exists a unique weak solution \((u, p, f)\) to the inverse problem (1)–(5).

(ii) Let \((u_i, p_i, f_i)\) \((i = 1, 2)\) be two solutions of the inverse problem (1)–(5) corresponding to the input data \((u_{0i}, \varphi_i, g_i)\) \((i = 1, 2)\). Then, there exists a constant \(C\) such that
\[
\|u_1 - u_2\|_{L^\infty(0,T;\mathbb{H})} + \|u_1 - u_2\|_{L^2(0,T;\mathbb{V})} + \|u_1 - u_2\|_{L^{r+1}(0,T;\tilde{L}^{r+1})} \\
+ \|p_1 - p_2\|_{L^{r+1}(0,T;\tilde{L}^{r+1}(\Omega))} + \|f_1 - f_2\|_{L^2(0,T)} \\
\leq C \left( \|u_{01} - u_{02}\|_{\mathbb{H}} + \sup_{t \in [0,T]} \|g_1 - g_2(t)\|_{L^2} + \|\varphi_1 - \varphi_2\|_{H^1(0,T)} \right), \tag{8}
\]
where \(C\) depends on the input data, \(\mu, \alpha, \beta, r,T\) and \(\Omega\).

The rest of the paper is organized as follows: in Sect. 2, we first state and prove the relation between the solvability of the inverse problem (1)–(5) and an equivalent operator equation of the second kind (Theorem 2.2). In Sect. 3, we prove our main result, that is, Theorem 1.2, by first showing the existence of a solution to the equivalent operator equation using an extension of the contraction mapping theorem and then demonstrating the uniqueness and stability of the solution to the inverse problem. In Appendix 1, we deduce a number of a-priori estimates that are needed to handle the inverse problem (1)–(5).

2 Mathematical formulation

In this section, we state and prove Theorem 2.2, which converts the inverse problem (1)–(5) into an equivalent nonlinear operator equation of the second kind (3) and prove their equivalence. We commence this section by introducing function spaces and some standard notations (cf. Section 2.1, [31, 32]), which will be used throughout the paper.
2.1 Function spaces

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary $\partial \Omega$. Let $C^\infty_0(\Omega; \mathbb{R}^2)$ be the space of all infinitely differentiable functions ($\mathbb{R}^2$-valued) with compact support in $\Omega \subset \mathbb{R}^2$. Let us define

$$\mathcal{V} := \{ u \in C^\infty_0(\Omega, \mathbb{R}^2) : \nabla \cdot u = 0 \},$$

$$\mathcal{H} := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^2(\Omega) = L^2(\Omega; \mathbb{R}^2),$$

$$\mathcal{V} := \text{the closure of } \mathcal{V} \text{ in the Sobolev space } H^1_0(\Omega) = H^1(\Omega_0; \mathbb{R}^2),$$

$$\mathcal{L}^p := \text{the closure of } \mathcal{V} \text{ in the Lebesgue space } L^p(\Omega) = L^p(\Omega; \mathbb{R}^2),$$

for $p \in (2, \infty]$. Then, under some smoothness assumptions on the boundary, we characterize the spaces $\mathcal{H}$, $\mathcal{V}$, $\mathcal{L}^p$ and $\mathcal{L}^\infty$ as

$$\mathcal{H} = \{ u \in L^2(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \},$$

where $n$ is the unit outward drawn normal to $\partial \Omega$, and $u \cdot n|_{\partial \Omega}$ should be understood in the sense of trace in $H^{-1/2} \left( \partial \Omega \right)$ (cf. Theorem 1.2, Chapter 1, [39]), with the norm

$$\| u \|_{\mathcal{H}} := \int_{\Omega} |u(x)|^2 \, dx,$$

$$\mathcal{V} = \{ u \in H^1_0(\Omega) : \nabla \cdot u = 0 \},$$

with the norm $\| u \|_{\mathcal{V}} := \int_{\Omega} |\nabla u(x)|^2 \, dx$ (since $\Omega$ is a bounded smooth domain),

$$\mathcal{L}^p = \{ u \in L^p(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \},$$

with the norm $\| u \|_{\mathcal{L}^p} := \int_{\Omega} |u(x)|^p \, dx$, and

$$\mathcal{L}^\infty = \{ u \in L^\infty(\Omega) : \nabla \cdot u = 0, u \cdot n|_{\partial \Omega} = 0 \},$$

with the norm $\| u \|_{\mathcal{L}^\infty} := \text{ess sup}_{x \in \Omega} |u(x)|$, respectively. Let $(\cdot, \cdot)$ denote the inner product in the Hilbert space $\mathcal{H}$ and $\langle \cdot, \cdot \rangle$ represents the duality pairing between the spaces $\mathcal{V}$ and its dual $\mathcal{V}'$, and $\mathcal{L}^p$ and its dual $\mathcal{L}^{p'}$. The following well-known Ladyzhenskaya inequality (cf. Lemma 1, [28])

$$\| u \|_{L^4} \leq 2^{1/4} \| u \|_{L^2}^{1/2} \| \nabla u \|_{L^2}^{1/2}, \text{ for all } u \in H^1_0(\Omega)$$

is used repeatedly in the paper. Wherever needed, we assume that $p_0 \in L^2_0(\Omega)$, where $L^2_0(\Omega) := \{ p \in L^2(\Omega) : \int_{\Omega} p(x) \, dx = 0 \}$. In the sequel, $C$ denotes a generic constant which may take different values at different places.

Later discussions of the equivalent formulation of an inverse problem (1)–(5) are based on the works [36, 40], in which the authors established a relationship between the solvability of the inverse problem for NSE in two dimensions and an operator equation of the second kind.
2.2 Equivalent formulation

We require the following input data function requirements:

\[ u_0 \in \mathbb{H}, \ g \in C([0, T]; L^2(\Omega)), \ \omega \in H^2(\Omega) \cap \mathbb{V}, \ \nabla \omega \in L^\infty, \ \varphi \in H^1(0, T), \]

and \[ \int_{\Omega} g(x, t) \cdot \omega(x) dx \geq g_0 > 0, \ (g_0 \equiv \text{constant}), \ 0 \leq t \leq T. \]

Under the above assumptions, we derive an operator equation of the second kind for the scalar function \( f \). We define the nonlinear operator \( A : L^2(0, T) \to L^2(0, T) \) by

\[
(Af)(t) := \frac{1}{g_1(t)} \left\{ \int_{\Omega} \left( \mu \nabla u \cdot \nabla \omega + (u \cdot \nabla)u \cdot \omega \right.ight.
\]
\[
+ au \cdot \omega + \beta |u|^{r-1}u \cdot \omega \right) dx + \varphi'(t),
\]

where \( u \) is obtained using the weak solution of the direct problem (1)–(4) and

\[
g_1(t) = \int_{\Omega} g(x, t) \cdot \omega(x) dx. \]

We analyze the following nonlinear operator equation of the second kind over the space \( L^2(0, T) \):

\[
f = Af. \]

The solvability of the inverse problem (1)–(5) is inevitably connected to the fixed points of the operator \( A \). For this purpose, we first consider a closed ball \( D \) in the space \( L^2(0, T) \) with center \( \varphi'/g_1 \) such that

\[
D = \{ f : f \in L^2(0, T) \text{ and } \|f - \varphi'/g_1\|_{L^2(0,T)} \leq a \}. \]

Remark 2.1 Since Eq. (1) is taken in the divergence free space \( \mathbb{V} \), the pressure \( p(\cdot, \cdot) \) does not appear in Eqs. (6) and (1). The pressure \( p(\cdot, \cdot) \) can be recovered from Eq. (1) after reconstructing the pair \((u, f)\).

The relation between the solvability of the inverse problem (1)–(5) and the nonlinear operator equation of the second kind (3) is verified by the following theorem.

**Theorem 2.2** Let \( \Omega \subset \mathbb{R}^2, \ u_0 \in \mathbb{H}, \ g \in C([0, T]; L^2(\Omega)), \ \omega \in H^2(\Omega) \cap \mathbb{V}, \ \varphi \in H^1(0, T) \) and
Then, the following assertions hold true:

(i) If the inverse problem \((1)–(5)\) has a solution, then the operator equation \((3)\) also has a solution.

(ii) If the operator equation \((3)\) has a solution and the compatibility condition \((7)\) is satisfied, then there exists a solution of the inverse problem \((1)–(5)\).

**Proof** Let us first prove (i). Let the inverse problem \((1)–(5)\) has a solution, say \((u, p, f)\). Multiplying both sides of \((1)\) by \(u(x)\), and integrating by parts, we obtain

\[
\frac{d}{dt} \int_{\Omega} u(x, t) \cdot \omega(x) dx + \mu \int_{\Omega} \nabla u(x, t) \cdot \nabla \omega(x) dx \\
+ \int_{\Omega} (u(x, t) \cdot \nabla)u(x, t) \cdot \omega(x) dx + \alpha \int_{\Omega} u(x, t) \cdot \omega(x) dx \\
+ \beta \int_{\Omega} |u(x, t)|^{r-1} u(x, t) \cdot \omega(x) dx = f(t) \int_{\Omega} g(x, t) \cdot \omega(x) dx.
\]

From the definition \((1)\) of the operator \(A\), the integral overdetermination condition \((5)\), and assumption \((2)\), one can easily deduce from \((5)\) that

\[
Af = f.
\]

This implies that the function \(f\) solves the operator equation \((3)\); thereby proving (i).

We proceed to prove (ii). Let the operator equation \((3)\) has a solution, say \(f \in L^2(0, T)\). Upon substituting \(f\) into \((1)\), we use the system \((1)–(4)\) to find a unique weak solution \((u(\cdot), p(\cdot))\) of the direct problem \((1)–(4)\). We claim that the function \(u(\cdot)\) satisfies the integral overdetermination condition \((5)\). By inserting \(v(x) = \omega(x)\) in \((6)\), and using assumption \((2)\), we arrive at

\[
\frac{d}{dt} \int_{\Omega} u(x, t) \cdot \omega(x) dx + \mu \int_{\Omega} \nabla u(x, t) \cdot \nabla \omega(x) dx \\
+ \int_{\Omega} (u(x, t) \cdot \nabla)u(x, t) \cdot \omega(x) dx + \alpha \int_{\Omega} u(x, t) \cdot \omega(x) dx \\
+ \beta \int_{\Omega} |u(x, t)|^{r-1} u(x, t) \cdot \omega(x) dx = f(t)g_1(t).
\]
Subtracting Eq. (7) from Eq. (6), we find

\[ \phi'(t) + \mu \int_{\Omega} \nabla u(x, t) \cdot \nabla \omega(x) \, dx + \int_{\Omega} (u(x, t) \cdot \nabla) u(x, t) \cdot \omega(x) \, dx + a \int_{\Omega} u(x, t) \cdot \omega(x) \, dx + \beta \int_{\Omega} |u(x, t)|^{r-1} u(x, t) \cdot \omega(x) \, dx = f(t)g_1(t). \]  

(15)

Integrating Eq. (8) with respect to time \( t \) from 0 to \( t \), and then using the compatibility condition (7), we obtain

\[ \frac{d}{dt} \int_{\Omega} u(x, t) \cdot \omega(x) \, dx - \phi'(t) = 0. \]

(16)

Integrating Eq. (8) with respect to time \( t \) from 0 to \( t \), and then using the compatibility condition (7), we obtain

\[ \int_{\Omega} u(x, t) \cdot \omega(x) \, dx = \phi(t), \]

which is the integral overdetermination condition (5). Having obtained this, we can say that the triplet \((u, p, f)\) is a weak solution of the inverse problem (1)–(5) and this proves the second assertion of the theorem.

\[ \square \]

3 Proof of Theorem 1.2

The energy estimates obtained in Appendix 1 allow us to establish the existence and uniqueness of solution to the inverse problem (1)–(5) as well as the stability of the solution obtained. For proving the existence of a solution to the inverse problem (1)–(5), using Theorem 2.2, it suffices to show that the nonlinear operator \( \mathcal{A} \) has a fixed point in \( \mathcal{D} \) and this follows from an application of an extension of the contraction mapping theorem. Subsequently, arguments for the existence and uniqueness of the solution of the inverse problem (1)–(5) are based on the works [36, 40], where the existence and uniqueness of a solution of the inverse problem for 2D NSE has been investigated, by exploiting the following extension of the contraction mapping theorem.

**Theorem 3.1** (Theorem 2.4, [1]) *Let \((\mathcal{X}, d)\) be a complete metric space and let \( \mathcal{A} : \mathcal{X} \to \mathcal{X} \) be a mapping such that for some positive integer \( k \), \( \mathcal{A}^k \) is a contraction on \( \mathcal{X} \). Then, \( \mathcal{A} \) has a unique fixed point.*

3.1 Existence

The solvability of the inverse problem (1)–(5) is inevitably connected to the fixed points of the operator \( \mathcal{A} \). Now, we demonstrate that the operator \( \mathcal{A} \) has a unique fixed point. The following theorem provides sufficient conditions under which the operator \( \mathcal{A} \) maps the closed ball \( \mathcal{D} \) into itself.
Theorem 3.2 Let $\Omega \subset \mathbb{R}^2$, $u_0 \in H$, $g \in C([0,T];L^2(\Omega))$, $\omega \in H^2(\Omega) \cap \mathcal{V}$, $\nabla \omega \in L^\infty$, $\varphi \in H^1(0,T)$ and

$$\left| \int_{\Omega} g(x,t) \cdot \omega(x) dx \right| \geq g_0 > 0, \ (g_0 \equiv \text{constant}), \ 0 \leq t \leq T.$$

(i) For $r \in (2,3]$, if

$$m_1 < a,$$

where

$$m_1 = \frac{C}{g_0} \left\{ (\| \Delta \omega \|_{H^1} + \| \omega \|_{\mathcal{V}}) \left( \| u_0 \|_{\mathcal{V}} + T^{1/2} \tilde{a} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right) 
+ \| \nabla \omega \|_{L^\infty} \left( \| u_0 \|_{\mathcal{V}}^2 + \tilde{a}^2 \sup_{t \in [0,T]} \| g(t) \|_{L^2}^2 \right)^{1/2} 
+ T^{3/2} \| \omega \|_{H^2}^2 \left( \| u_0 \|_{\mathcal{V}} + T^{1/2} \tilde{a} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right) \right\}^{1/2},$$

$$\tilde{a} = a + \| \varphi' / g_1 \|_{L^2(0,T)},$$

and $a$ is the radius of the closed ball $D$.

(ii) For $r \in [1,2]$, if

$$m_2 < a,$$

where

$$m_2 = \frac{CT^{1/2}}{g_0} \left\{ (\| \Delta \omega \|_{H^1} + \| \omega \|_{\mathcal{V}}) \left( \| u_0 \|_{\mathcal{V}} + T^{1/2} \tilde{a} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right) 
+ \| \nabla \omega \|_{L^\infty} \left( \| u_0 \|_{\mathcal{V}}^2 + \tilde{a}^2 \sup_{t \in [0,T]} \| g(t) \|_{L^2}^2 \right) 
+ \| \omega \|_{H^2} \left( \| u_0 \|_{\mathcal{V}} + T^{1/2} \tilde{a} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right)^{1/2} \right\}^{1/2}.$$  

Then, the operator $A$ maps the closed ball $D$ into itself.

Proof Let $f$ be an arbitrary fixed function in $D$, then by the definition of the ball $D$, we deduce that

$$\| f \|_{L^2(0,T)} \leq \tilde{a}.$$  

Case I: $r \in (2,3]$. The norm of the function $Af - \varphi'/g_1$ can be estimated as follows:
Using (13) in (17), we deduce that

\[
\| Af - \varphi' / g_1 \|_{L^2(0,T)} = \left( \int_0^T | Af - \varphi' / g_1 |^2 \, dt \right)^{\frac{1}{2}} 
\leq \frac{1}{g_0} \left[ \int_0^T \left( \mu \| \Delta \varphi \|_{\mathcal{H}} \| u(t) \|_{\mathcal{H}} + \| \nabla \varphi \|_{\mathcal{L}^\infty} \| u(t) \|_{\mathcal{L}^2}^2 \right)^{\frac{1}{2}} \right].
\]  

(22)

We estimate \( \int_0^T ||u(t)||^2 \, dt \) using interpolation and Hölder’s inequalities as

\[
\int_0^T ||u(t)||^2 \, dt \leq \int_0^T \left( ||u(t)||^{\frac{2}{r-1}} ||u(t)||^{\frac{2}{2(r-1)}} \right) \, dt 
\leq \int_0^T \left( ||u(t)||^{\frac{4}{r-1}} ||u(t)||^{\frac{2}{2(r-1)}} \right) \, dt 
\leq T^{\frac{1}{r-1}} \sup_{t \in [0,T]} ||u(t)||^{\frac{4}{r-1}} \left( \int_0^T ||u(t)||^{\frac{4}{r-1}} \, dt \right)^{\frac{1}{2}}.
\]  

(23)

Substituting the estimate (15) in (14), we obtain

\[
\| Af - \varphi' / g_1 \|_{L^2(0,T)} 
\leq \frac{C}{g_0} \left[ T \left\{ (\| \Delta \varphi \|_{\mathcal{H}} + \| \varphi \|_{\mathcal{H}}) \sup_{t \in [0,T]} ||u(t)||_{\mathcal{H}} + \| \nabla \varphi \|_{\mathcal{L}^\infty} \sup_{t \in [0,T]} ||u(t)||_{\mathcal{L}^2} \right\}^2 
+ T^{\frac{1}{r-1}} ||\varphi||_{\mathcal{H}}^2 \sup_{t \in [0,T]} ||u(t)||_{\mathcal{H}}^{\frac{4}{r-1}} \left( \int_0^T ||u(t)||^{\frac{4}{r-1}} \, dt \right)^{\frac{1}{2}} \right].
\]  

(24)

With reference to (52) and (53), it can be seen from (16) that

\[
\| Af - \varphi' / g_1 \|_{L^2(0,T)} 
\leq \frac{C}{g_0} \left[ T \left\{ (\| \Delta \varphi \|_{\mathcal{H}} + \| \varphi \|_{\mathcal{H}}) \left( ||u_0||_{\mathcal{H}} + T^{1/2} \sup_{t \in [0,T]} ||g(t)||_{L^2} ||f||_{L^2(0,T)} \right) \right.
+ \| \nabla \varphi \|_{\mathcal{L}^\infty} \left( ||u_0||_{\mathcal{H}}^2 + \left\{ \frac{1}{\alpha} \sup_{t \in [0,T]} ||g(t)||_{L^2} ||f||_{L^2(0,T)} \right\} \right)^2 
+ T^{\frac{1}{r-1}} ||\varphi||_{\mathcal{H}}^2 \left( ||u_0||_{\mathcal{H}} + T^{1/2} \sup_{t \in [0,T]} ||g(t)||_{L^2} ||f||_{L^2(0,T)} \right) \right]
\times \left\{ \frac{1}{2\beta} \left( ||u_0||_{\mathcal{H}}^2 + \left\{ \frac{1}{\alpha} \sup_{t \in [0,T]} ||g(t)||_{L^2} ||f||_{L^2(0,T)} \right\} \right)^{\frac{2(\frac{1}{r-1})}{2(r-1)}} \right]^\frac{1}{2}.
\]  

(25)

Using (13) in (17), we deduce that
\[ \| Af - \varphi' / g_1 \|_{L^2(0,T)} \leq m_1, \]  

(26)

where \( m_1 \) is defined in (10). One can take \( T \) sufficiently small such that \( m_1 < a \) (if needed \( a \) can be chosen sufficiently large so that \( a > \| u_0 \|_{H} \)). For the case \( r = 3 \), apart from smallness of \( T \), one may need to restrict the input data also. Using this relation, the estimate (18) immediately implies that the nonlinear operator \( A \) brings the ball \( \mathcal{D} \) into itself.

**Case II:** \( r \in [1, 2] \). From (14), the norm of the function \( Af - \varphi' / g_1 \) can be estimated as follows:

\[
\begin{align*}
\| Af - \varphi' / g_1 \|_{L^2(0,T)} & \leq \frac{C}{g_0} \left\{ \left( \| \Delta \omega \|_{H} + \| \omega \|_{H} \right) \sup_{t \in [0,T]} \| u(t) \|_{H} + \| \nabla \omega \|_{L^\infty} \sup_{t \in [0,T]} \| u(t) \|_{H} \right\}^2 \\
& + |\Omega|^{2-r} \| \omega \|_{H^2}^2 \int_0^T \| u(t) \|_{H}^{2r} dt \right\}^{\frac{1}{2}} \\
& \leq \frac{CT^{1/2}}{g_0} \left\{ \left( \| \Delta \omega \|_{H} + \| \omega \|_{H} \right) \sup_{t \in [0,T]} \| u(t) \|_{H} + \| \nabla \omega \|_{L^\infty} \sup_{t \in [0,T]} \| u(t) \|_{H} \right\} \\
& + \| \omega \|_{H^2} \sup_{t \in [0,T]} \| u(t) \|_{H} \right\},
\end{align*}
\]

(27)

where \(|\Omega|\) is the Lebesgue measure of \( \Omega \). Substituting the estimates (52) and (53) in the inequality (19), we get

\[
\begin{align*}
\| Af - \varphi' / g_1 \|_{L^2(0,T)} & \leq \frac{CT^{1/2}}{g_0} \left\{ \left( \| \Delta \omega \|_{H} + \| \omega \|_{H} \right) \left( \| u_0 \|_{H} + T^{1/2} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \| f \|_{L^2(0,T)} \right) \right. \\
& + \| \nabla \omega \|_{L^\infty} \left( \| u_0 \|_{H} + \frac{1}{a} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \| f \|_{L^2(0,T)}^2 \right) \right. \\
& + \left. \left( \| u_0 \|_{H} + T^{1/2} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \| f \|_{L^2(0,T)} \right)^r \right\}.
\end{align*}
\]

(28)

Using the relation (13) in (20) results to

\[
\| Af - \varphi' / g_1 \|_{L^2(0,T)} \leq m_2, \]

(29)

where \( m_2 \) is defined in (12). One can choose \( T \) sufficiently small so that \( m_2 < a \). Considering this relationship, the estimate (21) implies that the nonlinear operator \( A \) brings the ball \( \mathcal{D} \) into itself.

As a result, for \( r \in [1, 3] \), the nonlinear operator \( A \) maps the ball \( \mathcal{D} \) into itself, which completes the proof. \( \square \)
The next theorem establishes the existence of a solution to the inverse problem (1)–(5).

**Theorem 3.3** Let $\Omega \subset \mathbb{R}^2$, $u_0 \in \mathbb{H}$, $g \in C([0, T]; \mathbb{L}^2(\Omega))$, $\omega \in \mathbb{H}^2(\Omega) \cap \mathbb{V}$, $\varphi \in \mathbb{H}^1(0, T)$ and

$$
\int_{\Omega} g(x, t) \cdot \omega(x) \, dx \geq g_0 > 0, \quad (g_0 \equiv \text{constant}), \quad 0 \leq t \leq T.
$$

Also assume that the nonlinear operator $\mathcal{A}$ maps the ball $D$ into itself. Then, there exists a positive integer $k$ such that the operator $\mathcal{A}^k$ is a contraction mapping in the ball $D$.

**Proof** Let both of the functions $f_1$ and $f_2$ belong to the ball $D$. Let $u_1$ and $u_2$ be the functions corresponding to the coefficients $f_1$ and $f_2$, respectively. By the definition of the operator $\mathcal{A}$, we have

$$
\|\mathcal{A}f_1 - \mathcal{A}f_2\|_{L^2(0, T)}^2 = \int_0^T |\mathcal{A}f_1 - \mathcal{A}f_2|^2 \, dt
\leq \frac{1}{g_0^2} \int_0^T \left( \mu \|u_1(t) - u_2(t)\|_H \|\Delta \omega\|_H + \|\nabla \omega\|_{L^\infty} \|u_1(t) - u_2(t)\|_H \ight.
\times \left( \|u_1(t)\|_H + \|u_2(t)\|_H \right) + \alpha \|\omega\|_H \|u_1(t) - u_2(t)\|_H
\left. + \beta \|\omega\|_{L^\infty} \|u_1(t)\|_H \|u_2(t)\|_H \right)^2 \, dt
\leq \frac{C}{g_0^2} \left[ T \sup_{r \in [0, T]} \|u_1(t) - u_2(t)\|_H \right]^2 \left\{ \|\Delta \omega\|_H + \|\omega\|_H \ight.
\times \left( \sup_{r \in [0, T]} \|u_1(t)\|_H + \sup_{r \in [0, T]} \|u_2(t)\|_H \right) \right)^2
\left. + \|\nabla \omega\|_{L^\infty} \int_0^T \left\| u_1(t) \|_H^{-1} u_1(t) - u_2(t) \|_H^{-1} u_2(t) \right\|^2_{L^1} \, dt \right].
$$

(30) Let us define $h(u) := |u|^{-1} u$. Then, by applying Taylor’s formula (Theorem 7.9.1, [12]), we obtain (cf. [32])
\[
\int_0^T \left\| |\mathbf{u}_1|^{r-1}\mathbf{u}_1 - |\mathbf{u}_2|^{r-1}\mathbf{u}_2 \right\|^2_{L^1} \, dt
\]

\[
= \int_0^T \left\| \int_0^1 h'(\theta \mathbf{u}_1 + (1 - \theta)\mathbf{u}_2) d\theta (\mathbf{u}_1 - \mathbf{u}_2) \right\|^2_{L^1} \, dt
\]

\[
\leq \int_0^T \sup_{0 < \theta < 1} \left\| (\mathbf{u}_1(t) - \mathbf{u}_2(t)) |\mathbf{u}_1(t) + (1 - \theta)\mathbf{u}_2(t)|^{r-1} + (r - 1)(\mathbf{u}_1(t) + (1 - \theta)\mathbf{u}_2(t)) \right\|^2_{L^1} \, dt
\]

\[
\times \left( (\mathbf{u}_1(t) + (1 - \theta)\mathbf{u}_2(t)) \cdot (\mathbf{u}_1(t) - \mathbf{u}_2(t)) \right) \right\|^2_{L^1} \, dt
\]

(31)

\[
\leq C \int_0^T \sup_{0 < \theta < 1} \|\mathbf{u}_1(t) + (1 - \theta)\mathbf{u}_2(t)\|^{2(r-1)}_{H^r} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{L^1} \, dt
\]

\[
\leq C \int_0^T \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{H^r} \left( \|\mathbf{u}_1(t)\|^{2(r-1)}_{L^{2(r-1)}} + \|\mathbf{u}_2(t)\|^{2(r-1)}_{L^{2(r-1)}} \right) \, dt
\]

\[
\leq C \sup_{t \in [0,T]} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{H^r} \left( \sup_{t \in [0,T]} \|\mathbf{u}_1(t)\|^{2(r-2)}_H \int_0^T \|\mathbf{u}_1(t)\|^2_V d\tau \right)
\]

(32)

\[
\leq C T^{3-r} \sup_{t \in [0,T]} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{H^r} \left( \sup_{t \in [0,T]} \|\mathbf{u}_1(t)\|^{2(r-2)}_H \left( \int_0^T \|\mathbf{u}_1(t)\|^2_V d\tau \right)^{r-2} \right)
\]

Substituting the relations (58) and (24) in (22), we arrive at

\[
\text{Case I: } r \in (2, 3). \text{ An application of Gagliardo–Nirenberg’s and Hölder’s inequalities in (23) yields}
\]

\[
\int_0^T \left\| |\mathbf{u}_1|^{r-1}\mathbf{u}_1 - |\mathbf{u}_2|^{r-1}\mathbf{u}_2 \right\|^2_{L^1} \, dt
\]

\[
\leq C \sup_{t \in [0,T]} \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{H^r} \left( \sup_{t \in [0,T]} \|\mathbf{u}_1(t)\|^2_2 \int_0^T \|\mathbf{u}_1(t)\|^2_V d\tau \right)
\]

(32)
The relation (25) simplifies to
\[
\|A f_1 - A f_2\|_{L^2(0,T)}^2 \leq \frac{C}{g_0} \left( \sup_{t \in [0,T]} \|g(t)\|_{L^2} \|f_1 - f_2\|_{L^2(0,T)}^2 \right) \exp \left( \frac{2}{\mu} \int_0^T \|u_2(t)\|_\psi^2 \, dt \right)
\times \left( T \left\{ \|\Delta \omega\|_{H^1} + \|\omega\|_{H^1} + \|\nabla \omega\|_{H^\infty} \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^2} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^2} \right) \right\}^2 + T^{3-r} \|\omega\|_{H^2}^2 \left\{ \sup_{t \in [0,T]} \|u_1(t)\|_{H^2}^2 \left( \int_0^T \|u_1(t)\|_\psi^2 \, dt \right)^{r-2} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^2}^2 \left( \int_0^T \|u_2(t)\|_\psi^2 \, dt \right)^{r-2} \right\} \right]
\]
\[
= m_3 \|f_1 - f_2\|_{L^2(0,T)}^2,
\]
where
\[
m_3 = \frac{C T^{3-r}}{g_0} \sup_{t \in [0,T]} \|g(t)\|_{L^2} \exp \left( \int_0^T \|u_2(t)\|_\psi^2 \, dt \right)
\times \left( T^{r-2} \left\{ \|\nabla \omega\|_{H^\infty} \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^2} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^2} \right) + \|\Delta \omega\|_{H^1} \right\}^2 + \left\{ \sup_{t \in [0,T]} \|u_1(t)\|_{H^2}^2 \left( \int_0^T \|u_1(t)\|_\psi^2 \, dt \right)^{r-2} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^2}^2 \left( \int_0^T \|u_2(t)\|_\psi^2 \, dt \right)^{r-2} \right\} \right]
\]
Using the energy estimates (52) and (53), the quantity \( m_3 \) can be bounded by
\[
m_3 \leq \frac{C T^{3-r}}{g_0} \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \exp \left\{ \|u_0\|_{H^2}^2 + \frac{1}{\alpha} \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \right\}
\times \left( T^{r-2} \left\{ \|\nabla \omega\|_{H^\infty} \left( 2 \|u_0\|_{H^1} + T^{1/2} \sup_{t \in [0,T]} \|g(t)\|_{L^2} \right)^2 + \|\omega\|_{H^1} + \|\Delta \omega\|_{H^1} \right\}^2 + \left\{ 2 \|u_0\|_{H^2}^2 + \frac{1}{\alpha} \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \right\} \right]^{r-1}
\]
Since the functions \( f_1 \) and \( f_2 \) lie within the ball \( D \), a combination of the inequalities (26) and (27) gives the estimate
\[
\|A f_1 - A f_2\|_{L^2(0,T)} \leq \left( m_4 T^{3-r} \right)^{1/2} \|f_1 - f_2\|_{L^2(0,T)}, \tag{36}
\]

where
\[
m_4 = \frac{C}{g_0^2} \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \exp \left\{ \|u_0\|_{\mathcal{H}}^2 + \tilde{a}^2 \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \right\} \times \left[ T^{r-2} \left\{ \|\nabla \omega\|_{\mathcal{H}}^2 \left( \|u_0\|_{\mathcal{H}} + T^{1/2} \tilde{a} \sup_{t \in [0,T]} \|g(t)\|_{L^2} \right) + \|\Delta \omega\|_{\mathcal{H}} + \|\omega\|_{\mathcal{H}} \right\}^2 + \|\omega\|_{\mathcal{H}}^2 \left( \|u_0\|_{\mathcal{H}}^2 + \tilde{a}^2 \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \right)^{r-1} \right]^{r-1}
\]

and \( \tilde{a} = a + \|\varphi'/g_1\|_{L^2(0,T)}, \)

and \( a \) is the radius of the ball \( \mathcal{D} \). Note that \( m_4 \) is expressed only in terms of the input data.

In accordance with the assumption made, the operator \( A \) maps the ball \( \mathcal{D} \) into itself which makes it possible to define for any positive integer \( k \), the \( k \)th degree of the operator \( A \). Let this operator be denoted by the symbol \( A^k \). Using the standard technique of mathematical induction on \( k \), the inequality (28) validates the estimate
\[
\left\| A^k f_1 - A^k f_2 \right\|_{L^2(0,T)} \leq \left( \frac{m_4^k T^{(3-r)k}}{k!} \right)^{1/2} \|f_1 - f_2\|_{L^2(0,T)}, \tag{37}
\]

It is visible that
\[
\left( \frac{m_4^k T^{(3-r)k}}{k!} \right)^{1/2} \to 0 \quad \text{as} \quad k \to \infty.
\]

This implies that there exists a positive integer \( k_0 \) such that
\[
\left( \frac{m_4^{k_0} T^{(3-r)k_0}}{k_0!} \right)^{1/2} \leq 1.
\]

Due to estimate (29), one can conclude that the operator \( A^{k_0} \) is a contraction mapping in the ball \( \mathcal{D} \).

**Case II:** \( r \in [1, 2] \). Applying Hölder’s inequality in (23), we get
\[
\int_0^T \left\| u_1^{r-1} - u_2^{r-1} \right\|_{L^1}^2 \, dt \\
\leq C \sup_{t \in [0,T]} \|u_1(t) - u_2(t)\|_{\mathcal{H}}^2 \int_0^T |\Omega|^{2-r} \left( \|u_1(t)\|_{\mathcal{H}}^{2(r-1)} + \|u_2(t)\|_{\mathcal{H}}^{2(r-1)} \right) dt \tag{38}
\]
\[
\leq CT \sup_{t \in [0,T]} \|u_1(t) - u_2(t)\|_{\mathcal{H}}^2 \left( \sup_{t \in [0,T]} \|u_1(t)\|_{\mathcal{H}}^{2(r-1)} + \sup_{t \in [0,T]} \|u_2(t)\|_{\mathcal{H}}^{2(r-1)} \right),
\]

where \( |\Omega| \) is the Lebesgue measure of \( \Omega \). Substitution of the relations (58) and (30) in (22) results to
\[ \|A^1 - A^2\|_{L^2(0,T)}^2 \leq \frac{CT}{g_0} \left( \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \|f_1 - f_2\|_{L^2(0,T)}^2 \right) \exp \left( \frac{2}{\mu} \int_0^T \|u_2(t)\|^2 dt \right) \times \left\{ \|\Delta \omega\|_{H^1} + \|\omega\|_{H^1} + \|\nabla \omega\|_{L^\infty} \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^1} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^1} \right) \right. \\
+ \left. \|\omega\|_{H^2} \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^1} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^1} \right)^{-1} \right\}^2. \]

From the above estimate, it follows that
\[ \|A^1 - A^2\|_{L^2(0,T)}^2 \leq m_5 \|f_1 - f_2\|_{L^2(0,T)}^2, \tag{39} \]

where
\[ m_5 = \frac{CT}{g_0} \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \exp \left( \int_0^T \|u_2(t)\|^2 dt \right) \times \left\{ \|\Delta \omega\|_{H^1} + \|\omega\|_{H^1} + \|\nabla \omega\|_{L^\infty} \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^1} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^1} \right) \right. \\
+ \left. \|\omega\|_{H^2} \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^1} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^1} \right)^{-1} \right\}^2. \]

Using the estimates (52) and (53), the quantity \(m_5\) can be bounded as
\[ m_5 \leq \frac{CT}{g_0} \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \exp \left\{ \|u_0\|_{H^1}^2 + 1 \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \|f_2\|_{L^2(0,T)}^2 \right\} \times \left\{ \|\Delta \omega\|_{H^1} + \|\omega\|_{H^1} + \|\nabla \omega\|_{L^\infty} \left( 2 \|u_0\|_{H^1} + T^{1/2} \sup_{t \in [0,T]} \|g(t)\|_{L^2} \right) \right. \\
+ \left. \left( \|f_1\|_{L^2(0,T)} + \|f_2\|_{L^2(0,T)} \right) \right\} + \|\omega\|_{H^2} \left( 2 \|u_0\|_{H^1} \right)^{-1} \left\}^2. \tag{40} \]

A combination of the inequalities (31) and (32) gives the following estimate:
\[ \|A^1 - A^2\|_{L^2(0,T)} \leq (m_6 T)^{1/2} \|f_1 - f_2\|_{L^2(0,T)}, \tag{41} \]

where
and \( a \) is the radius of the ball \( D \). The quantity \( m_6 \) is also expressed in terms of the input data only.

Using the similar arguments made in the case for \( r \in (2, 3] \), we can define the operator \( A^k \). Using the standard technique of mathematical induction on \( k \), the inequality (33) validates the estimate

\[
\left\| A^k f_1 - A^k f_2 \right\|_{L^2(0,T)} \leq \left( \frac{m_6^k T^k}{k!} \right)^{1/2} \left\| f_1 - f_2 \right\|_{L^2(0,T)}. \tag{42}
\]

It is clear that

\[
\left( \frac{m_6^k T^k}{k!} \right)^{1/2} \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore, there exists a positive integer \( \tilde{k}_0 \) such that

\[
\left( \frac{m_6^{\tilde{k}_0} T^{\tilde{k}_0}}{\tilde{k}_0!} \right)^{1/2} < 1.
\]

From the estimate (34), it is immediate that the operator \( A^{\tilde{k}_0} \) is a contraction mapping in the ball \( D \).

Let us now prove the existence of a solution \((u, p, f)\) to the inverse problem (1)–(5).

**Proof of Theorem 1.2 (i)-Existence** Since the operator \( A^k \) is a contraction mapping in the ball \( D \), an application of Theorem 3.1 yields the operator \( A \) has a unique fixed point. Hence, the inverse problem (1)–(5) has a solution, which completes the proof.

**3.2 Uniqueness**

Let us now prove the uniqueness of the solution obtained in Theorem 1.2 (i).

**Proof of Theorem 1.2 (i)-Uniqueness** Assume on the contrary that there are two distinct solutions, \((u_1, p_1, f_1)\) and \((u_2, p_2, f_2)\) of the inverse problem (1)–(5) such that both \( f_1 \) and \( f_2 \) lie within the ball \( D \).
We wish to prove that \( f_1 \) does not coincide with \( f_2 \) a.e. on \([0, T]\). Indeed, if \( f_1 \) is equal to \( f_2 \) a.e. on \([0, T]\), then \((u_1,p_1)\) will coincide with \((u_2,p_2)\) a.e. in \( \Omega \times [0,T] \), which is in accordance with the uniqueness theorem for the direct problem.

We begin by analyzing the first triplet \((u_1,p_1,f_1)\). Using similar arguments as in the proof of Theorem 2.2, we can conclude that the function \( f_1 \) represents a solution to Eq. (3). Similar arguments also prove that the function \( f_2 \) also solves the same Eq. (3). But we have just proved the uniqueness of the solution to Eq. (3) in \( D \). Thus, we have arrived at the conclusion that the assumption about the existence of two distinct solutions \((u_i,p_i,f_i)\) \((i = 1,2)\) does not hold. Hence, the inverse problem (1)–(5) has a unique solution, which completes the proof. \( \square \)

We will now provide an example (motivated from [40]) to show that the class of functions satisfying the conditions of part (i) of the Theorem 1.2 is not empty.

**Example** Let \( \Omega \subset \mathbb{R}^2, u_0 \in \mathbb{H}, g = g(x), g \in L^2(\Omega), \omega \in \mathbb{H}^2(\Omega) \cap \mathbb{V}, \forall \omega \in L^\infty, \varphi \in H^1(0,T) \) and

\[
\left| \int_{\Omega} g(x) \cdot \omega(x) dx \right| = g_0 > 0.
\]

If we consider

\[
\int_{\Omega} u_0(x) \cdot \omega(x) dx \equiv \varphi(t),
\]

that is, \( \varphi \equiv \text{constant} \), and if we set \( a = 1 \), it is clear to observe that \( \tilde{a} = 1 \), since \( \tilde{a} = a + \|g'/g_1\|_{L^2(0,T)} \).

**Case 1:** \( r \in (2,3] \). The inequality (9) has the following form:

\[
\frac{C}{g_0} \left[ T \left( \|\Delta \omega\|_{\mathbb{H}} + \|\omega\|_{\mathbb{H}} \right) \left( \|u_0\|_{\mathbb{H}} + T^{1/2} \sup_{t \in [0,T]} \|g(t)\|_{L^2} \right) \right.
\]

\[
+ \|\nabla \omega\|_{L^\infty}^2 \left( \|u_0\|_{\mathbb{H}}^2 + \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \right) \right]^{\frac{1}{2}}
\]

\[
+ T^{\frac{3}{r\nu}} \left( \|u_0\|_{\mathbb{H}} + T^{1/2} \sup_{t \in [0,T]} \|g(t)\|_{L^2} \right)^\frac{4}{r-1}
\]

\[
\left. \times \left( \|u_0\|_{\mathbb{H}}^2 + \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 \right) \right]^{\frac{1}{r-1}} < 1.
\]

Clearly, as \( T \to 0^+ \), the left-hand side of (35) approaches zero for \( r \in (2,3] \). As a result, there exists a time \( T' > 0 \) such that for any \( T \in (0,T'] \), the estimate (35) holds true. Based on this reasoning, it appears reasonable to investigate the inverse problem (1)–(5), with \( T \in (0,T'] \) and \( a = 1 \), the radius of the ball \( D \).

For \( r = 3 \), the condition becomes \( \frac{C}{g_0} \|\omega\|_{\mathbb{H}} \|u_0\|_{\mathbb{H}} \left( \|u_0\|_{\mathbb{H}}^2 + \|g(0)\|_{L^2}^2 \right)^{1/2} < 1 \).

Under this smallness assumption on the data, one can obtain the existence and uniqueness of the inverse problem (1)–(5) for any \( T \in (0,T'] \).
Case II: $r \in [1, 2]$. The inequality (11) has the following form:

$$\frac{CT^{1/2}}{g_0} \left\{ (\| \Delta \omega \|_{H} + \| \omega \|_{H}) \left( \| u_0 \|_{H} + T^{1/2} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right) \right.$$  
$$+ \| \nabla \omega \|_{L^2} \left( \| u_0 \|_{H}^2 + \sup_{t \in [0,T]} \| g(t) \|_{L^2}^2 \right)$$  
$$+ \omega \|_{H} \left( \| u_0 \|_{H} + T^{1/2} \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right)^\prime \right\} < 1. \tag{44}$$

Clearly, as $T \to 0^+$, the left-hand side of (36) approaches zero. Thus, there exists a time $T' > 0$ such that for any $T \in (0, T']$, the estimate (36) holds true. Based on this reasoning, it seems acceptable to investigate the inverse problem (1)–(5), with $T \in (0, T']$ and $a = 1$, the radius of the ball $\mathcal{D}$.

Hence, for $r \in [1, 3]$, it is apparent that the inverse problem with these input data satisfies the conditions of part (i) of Theorem 1.2.

### 3.3 Stability

To get a result on the stability, we first provide some supporting Lemmas. Let $(u_i, p_i, f_i) (i = 1, 2)$ be the solutions of inverse problem (1)–(5) corresponding to the given data $(u_{0i}, \varphi_i, g_i) (i = 1, 2)$ and set

$$u := u_1 - u_2, \quad p := p_1 - p_2, \quad f := f_1 - f_2,$$

$$u_0 := u_{01} - u_{02}, \quad \varphi := \varphi_1 - \varphi_2, \quad g := g_1 - g_2.$$

The following lemma establishes the stability of the velocity field $u(\cdot)$ of the solution of the inverse problem.

**Lemma 3.4** Let $u_{0i} \in H$, $g_i \in C([0,T]; L^2(\Omega))$ and $f_i \in L^2(0,T)$, for $i = 1, 2$. Then, the following estimate holds:

$$\sup_{t \in [0,T]} \| u(t) \|_{L^2}^2 + \mu \int_0^T \| u(t) \|_{L^2}^2 \, dt + \frac{\beta}{2r-2} \int_0^T \| u(t) \|_{L^{r+1}}^{r+1} \, dt$$  
$$\leq C \left( \| u_0 \|_{L^2}^2 + \sup_{t \in [0,T]} \| f(t) \|_{L^2}^2 \right),$$

where $C$ depends on the input data, $\mu, \alpha, \beta, r, T$ and $\Omega$.

**Proof** Subtracting Eq. (1) for $(u_i, p_i, f_i) (i = 1, 2)$, we deduce that

$$u_i(t) - \mu \Delta u_i(t) + (u_i(t) \cdot \nabla)u_i(t) + (u_i(t) \cdot \nabla)u_2(t) + \alpha u_i(t)$$  
$$+ \beta (|u_1(t)|^{r-1}u_1(t) - |u_2(t)|^{r-1}u_2(t)) + \nabla p(t) = f(t)g_1(t) + f_2(t)g(t), \tag{46}$$

for a.e. $t \in [0,T]$ in $\mathcal{V}'$. Taking the inner product with $u(\cdot)$ in Eq. (38), we obtain
Using Hölder’s and Young’s inequalities, we estimate \(|fg_1 + f_2 g, u)| as
\[
|fg_1 + f_2 g, u) | \leq (|f| \|g_1\|_{L^2} + |f_2| \|g\|_{L^2}) \|u\|_{H^1} \\
\leq \frac{1}{\alpha} (|f|^2 \|g_1\|_{L^2} + |f_2|^2 \|g\|_{L^2}^2) + \frac{\alpha}{2} \|u\|_{H^1}^2.
\]

Making use of Hölder’s, Ladyzhenskaya’s, and Young’s inequalities, we have
\[
|((u \cdot \nabla)u_2, u) | \leq \|u\|_{L^4}^2 \|\nabla u_2\|_{H^1} \\
\leq \sqrt{2} \|u\|_{H^1} \|u\|_{H^1} \|u_2\|_{V} \leq \frac{\mu}{2} \|u\|_{V}^2 + \frac{1}{\mu} \|u\|_{H^1}^2 \|u_2\|_{V}^2.
\]

An estimate similar to (64) gives
\[
\beta (|u_1|^r - 1 - |u_2|^r - 1, u) \geq \frac{\beta}{2} \|u_1|^{r-1} u\|_{H^1}^2 + \frac{\beta}{2} \|u_2|^{r-1} u\|_{H^1}^2 \\
\geq \frac{\beta}{2(r-1)} \|u\|_{L^{r+1}}^{r+1}.
\]

Substituting (40)–(42) in (39), and then integrating from 0 to \(t\), we obtain
\[
\|u(t)\|_{H^1}^2 + \mu \int_0^t \|u(s)\|_{L^6}^2 \|\nabla u_2\|_{L^6}^2 ds + \alpha \int_0^t \|u(s)\|_{H^1}^2 ds + \frac{\beta}{2(r-2)} \int_0^t \|u(s)\|_{L^{r+1}}^{r+1} ds \\
\leq \|u_0\|_{H^1}^2 + \frac{2}{\mu} \int_0^t \|u_2(s)\|_{H^1}^2 ds \\
+ \frac{2}{\alpha} \int_0^t (|f(s)|^2 \|g_1(s)\|_{L^2}^2 + |f_2(s)|^2 \|g_2(s)\|_{L^2}^2) ds,
\]

for all \(t \in [0, T]\). Applying Gronwall’s inequality in (43), it can be seen that
\[
\|u(t)\|_{H^1}^2 \leq \exp \left( \frac{2}{\mu} \int_0^T \|u_2(t)\|_{L^6}^2 dt \right) \left\{ \|u_0\|_{H^1}^2 \\
+ \frac{2}{\alpha} \left( \sup_{t \in [0,T]} \|g_1(t)\|_{L^2}^2 + \sup_{t \in [0,T]} \|g_2(t)\|_{L^2}^2 \right) \right\},
\]

for all \(t \in [0, T]\). Thus, from (43), it is immediate that
\[
\sup_{t \in [0, T]} \|u(t)\|^2_H + \mu \int_0^T \|u(t)\|^2_{\mathcal{V}} \, dt + \frac{\beta}{2^{r-2}} \int_0^T \|u(t)\|^{r+1}_{L^{r+1}} \, dt \leq \exp \left( \frac{2}{\mu} \int_0^T \|u_2(t)\|^2 \, dt \right) \left\{ \|u_0\|^2_H + \frac{2}{\alpha} \left( \sup_{t \in [0, T]} \|g_1(t)\|^2_{L^2(0,T)} + \sup_{t \in [0, T]} \|g(t)\|^2_{L^2(0,T)} \right) \right\} 
+ \frac{2}{\alpha} \left( \sup_{t \in [0, T]} \|g_1(t)\|^2_{L^2(0,T)} + \sup_{t \in [0, T]} \|g(t)\|^2_{L^2(0,T)} \right) \right\} 
\leq \exp \left\{ \frac{1}{\mu^2} \left( \|u_{02}\|^2_H + \sup_{t \in [0, T]} \|g_{02}(t)\|^2_{L^2(0,T)} \right) \right\} \left\{ \|u_0\|^2_H + \sup_{t \in [0, T]} \|g(t)\|^2_{L^2(0,T)} \right\} \right\} 
\leq C(\mu, \alpha, \beta, r, \|u_{02}\|_{L^2}, \|f_2\|_{L^2}, \|g_2\|_{L^2}, T) 
\times \left( \|u_0\|^2_H + \|f\|^2_{L^2(0,T)} + \sup_{t \in [0, T]} \|g(t)\|^2_{L^2} \right), 
\]
and (37) follows. \qed

The next lemma establishes the stability of the scalar function \( f \) of the solution of the inverse problem.

**Lemma 3.5** Let \( u_0 \in H, \omega \in H^2(\Omega) \cap \mathcal{V}, \nabla \omega \in \tilde{L}^\infty, g_i \in C([0, T]; L^2(\Omega)), \phi_i \in H^1(0, T), and f_i \in L^2(0, T), for i = 1, 2. Then, the following estimate holds:

\[
\|f\|_{L^2(0,T)} \leq C(\|u_0\|_H + \|\omega\|_{H^1(0,T)} + \sup_{t \in [0, T]} \|g(t)\|_{L^2}), \tag{52}
\]

where \( C \) depends on the input data, \( \mu, \alpha, \beta, r, T and \Omega. \)

**Proof** Taking the inner product with \( \omega(x) \) in Eq. (38), we obtain

\[
\int_{\Omega} u_1 \cdot \omega \, dx - \int_{\Omega} \mu \Delta u \cdot \omega \, dx + \int_{\Omega} (u_1 \cdot \nabla) u \cdot \omega \, dx + \int_{\Omega} (u \cdot \nabla) u_2 \cdot \omega \, dx + \alpha \int_{\Omega} u \cdot \omega \, dx + \beta \int_{\Omega} (|u_1|^{r-1} u_1 - |u_2|^{r-1} u_2) \cdot \omega \, dx \tag{53}
\]

From the above equation, we have the following estimate:

\[
\int_{\Omega} f g_1 \cdot \omega \, dx + \int_{\Omega} f_2 g \cdot \omega \, dx.
\]
\[ \|f(t)\|^2_{L^2(0,T)} \leq 2\|f_2(t)\|^2_{L^2} \|\varphi(t)\|^2_{H_0^1} + C \left[ \|\varphi'(t)\|^2 + \|u(t)\|^2 \|\Delta \varphi\|^2_{H_0^1} \right] + 2 \sup_{t \in [0,T]} \|u(t)\|^2_{H_0^1} + \|\omega\|^2_{L^2} \left( \|u_1(t)\|^2_{H_0^1} + \|u_2(t)\|^2_{H_0^1} \right) + |\varphi(t)|^2 \] (54)

Integrating the estimate (46) from 0 to T, we deduce that

\[ g_0^2 \|f\|^2_{L^2(0,T)} \leq 2 \sup_{t \in [0,T]} \|g(t)\|^2_{L^2} \|\varphi\|^2_{H_0^1} + \|\Delta \varphi\|^2_{H_0^1} + \|\Delta u\|^2_{H_0^1} + \|\Delta \varphi\|^2_{H_0^1} + \|\Delta u\|^2_{H_0^1} \left( \sup_{t \in [0,T]} \|u_1(t)\|^2 + \sup_{t \in [0,T]} \|u_2(t)\|^2 \right) \] (55)

**Case I:** \( r \in (2, 3) \). Using the estimate (24) in (47), one can deduce that

\[ g_0^2 \|f\|^2_{L^2(0,T)} \leq 2 \sup_{t \in [0,T]} \|g(t)\|^2_{L^2} \|\varphi\|^2_{H_0^1} + \|\Delta \varphi\|^2_{H_0^1} + \|\Delta u\|^2_{H_0^1} + \|\Delta \varphi\|^2_{H_0^1} \left( \sup_{t \in [0,T]} \|u_1(t)\|^2 + \sup_{t \in [0,T]} \|u_2(t)\|^2 \right) \] (56)

Substituting the estimates (52), (53), and (37) in (48), we arrive at

\[ \|f\|^2_{L^2(0,T)} \leq C \left( \|u_0\|^2_{H_0^1} + \|\varphi\|^2_{H_0^1} + \sup_{t \in [0,T]} \|g(t)\|^2_{L^2} \right), \]

and finally we get

\[ \|f\|^2_{L^2(0,T)} \leq C \left( \|u_0\|^2_{H_0^1} + \|\varphi\|^2_{H_0^1} + \sup_{t \in [0,T]} \|g(t)\|^2_{L^2} \right), \]

which is the estimate (44).

**Case II:** \( r \in [1, 2] \). Substituting the estimate (30) in (47), we obtain
\[
\begin{align*}
g_0^2 \|f\|_{L^2(0,T)}^2 \\
\leq 2 \sup_{t \in [0,T]} \|g(t)\|_{L^2}^2 + \|f\|_{H^1}^2 + C \left[ \|\phi\|_{H^1(0,T)}^2 + T \sup_{t \in [0,T]} \|u(t)\|_{H^1}^2 \right] \\
\times \left\{ \left| \Delta \omega \right|_{H^1}^2 + \left| \nabla \omega \right|_{L^2}^2 \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^1}^2 + \sup_{t \in [0,T]} \|u_2(t)\|_{H^1}^2 \right) \right\} \\
+ CT \|\omega\|_{H^1}^2 \sup_{t \in [0,T]} \|u(t)\|_{H^1}^2 \left( \sup_{t \in [0,T]} \|u_1(t)\|_{H^1}^{2(r-1)} + \sup_{t \in [0,T]} \|u_2(t)\|_{H^1}^{2(r-1)} \right). \\
\end{align*}
\]  

(57)

Using the estimates (52), (53), and (37) in (49), one can easily deduce the estimate (44), which completes the proof. \(\square\)

**Proof of part (ii) of Theorem 1.2** Stability of the pressure field \(p\): Equation (38) can be used directly to demonstrate the stability of the pressure field \(p\). Taking divergence on both sides of Eq. (38), we deduce that

\[
-\Delta p = -\nabla \cdot (fg_1 + f_2 g) + \nabla \cdot \left[ (u_1 \cdot \nabla) u + (u \cdot \nabla) u_2 \right] \\
+ \beta \nabla \cdot \left( |u_1|^{r-1} u_1 - |u_2|^{r-1} u_2 \right),
\]

in the weak sense. The above equation yields

\[
p = (-\Delta)^{-1} \left\{ \nabla \cdot \left[ - (fg_1 + f_2 g) + \nabla \cdot (u_1 \otimes u) \right] \\
+ \nabla \cdot (u \otimes u_2) + \beta \left( |u_1|^{r-1} u_1 - |u_2|^{r-1} u_2 \right) \right\}. 
\]

(58)

Taking \(L^2\)-norm on both sides in Eq. (50), and then using elliptic regularity (Cattabriga’s regularity theorem), Hölder’s inequality and Taylor’s formula, we have

\[
\|p\|_{L^2} = \|(-\Delta)^{-1} \left\{ \nabla \cdot \left[ - (fg_1 + f_2 g) + \nabla \cdot (u_1 \otimes u) \right] \\
+ \nabla \cdot (u \otimes u_2) + \beta \left( |u_1|^{r-1} u_1 - |u_2|^{r-1} u_2 \right) \right\}\|_{L^2} \\
\leq C \|\nabla \cdot \left[ - (fg_1 + f_2 g) + \nabla \cdot (u_1 \otimes u) \right] \\
+ \nabla \cdot (u \otimes u_2) + \beta \left( |u_1|^{r-1} u_1 - |u_2|^{r-1} u_2 \right) \|_{H^{1-2}} \\
\leq C \|fg_1 + f_2 g\|_{V'} + C \|u_1 \otimes u\|_{H^1} + C \|u \otimes u_2\|_{H^1} \\
+ C \beta \|u_1|^{r-1} u_1 - |u_2|^{r-1} u_2\|_{V'} \\
\leq C \|fg_1 + f_2 g\|_{L^2} + C \|u_1 \otimes u\|_{L^4} + C \|u \otimes u_2\|_{L^4} \\
+ C \|u_1|^{r-1} u_1 - |u_2|^{r-1} u_2\|_{L^{4+1}} \\
\leq C \|fg_1\|_{L^2} + C \|f_2\|_{L^2} \|g\|_{L^2} + C \|u_1\|_{L^4} + C \|u_2\|_{L^4} \|u\|_{L^4} \\
+ C \|u_1 - u_2\|_{L^{4+1}} \left( \|u_1\|_{L^{4+1}} + \|u_2\|_{L^{4+1}} \right),
\]

(59)

where we have used the embedding \(V \subset L^{r+1} \subset H \equiv H' \subset L^{\frac{r+1}{r}} \subset V'\). Taking the \((r+1)^{th}\) power on both sides in (51) and then integrating the resulting inequality from 0 to \(T\), followed by applying Hölder’s inequality, we deduce that
\[ \int_0^T \| p(t) \|_{L^2}^{\frac{r+1}{r}} \, dt \leq C \sup_{t \in [0,T]} \| g_1(t) \|_{L^2}^{\frac{r+1}{r}} \int_0^T |f(t)|^{\frac{r+1}{r}} \, dt + C \sup_{t \in [0,T]} \| g(t) \|_{L^2}^{\frac{r+1}{r}} \int_0^T |f_2(t)|^{\frac{r+1}{r}} \, dt \]

\[ + C \int_0^T \| u_1(t) \|_{L^r}^{\frac{r+1}{r}} \| u(t) \|_{L^4}^{\frac{r+1}{r}} \, dt + C \int_0^T \| u_2(t) \|_{L^r}^{\frac{r+1}{r}} \| u(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \]

\[ + C \int_0^T \| u_1(t) \|_{L^r}^{\frac{r+1}{r}} \| u(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \]

\[ + C \int_0^T \| u_2(t) \|_{L^r}^{\frac{r+1}{r}} \| u(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \]

\[ \leq CT \sup_{t \in [0,T]} \| g_1(t) \|_{L^2}^{\frac{r+1}{r}} \| f \|_{L^2((0,T))} + CT \sup_{t \in [0,T]} \| g(t) \|_{L^2}^{\frac{r+1}{r}} \| f_2 \|_{L^2((0,T))} \]

\[ + CT \left( \int_0^T \| u(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \right)^{\frac{r+1}{r}} \left( \int_0^T \| u_1(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \right)^{\frac{r+1}{r}} \]

\[ + CT \left( \int_0^T \| u(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \right)^{\frac{r+1}{r}} \left( \int_0^T \| u_2(t) \|_{L^4}^{\frac{r+1}{r}} \, dt \right)^{\frac{r+1}{r}} \]

\[ + C \left( \int_0^T \| u(t) \|_{L^r}^{\frac{r+1}{r}} \, dt \right)^{\frac{1}{r}} \left( \int_0^T \| u_1(t) \|_{L^r}^{\frac{r+1}{r}} \, dt \right)^{\frac{r+1}{r}} \]

\[ + C \left( \int_0^T \| u(t) \|_{L^r}^{\frac{r+1}{r}} \, dt \right)^{\frac{1}{r}} \left( \int_0^T \| u_2(t) \|_{L^r}^{\frac{r+1}{r}} \, dt \right)^{\frac{r+1}{r}} . \]

Substituting the energy estimates (53), (37) and (44) in the above estimate, we arrive at

\[ \| p \|_{L^{\frac{r+1}{r}}((0,T); L^2(\Omega))} \leq C \left( \| u_0 \|_{H^1} + \sup_{t \in [0,T]} \| g(t) \|_{L^2} + \| f \|_{L^2} \right) \]

\[ \leq C \left( \| u_0 \|_{H^1} + \| \varphi \|_{H^1((0,T))} + \sup_{t \in [0,T]} \| g(t) \|_{L^2} \right), \]

where \( C \) depends on the input data, \( \mu, \alpha, \beta, r, T \) and \( \Omega \).

We can see that the solution depends continuously on the data from the stability estimate of the pressure field \( p \) and Lemmas 3.4 and 3.5. We infer the Lipschitz stability of the solution \((u, p, f)\) as

\[ \| u_1 - u_2 \|_{L^\infty((0,T); H^1)} + \| u_1 - u_2 \|_{L^2((0,T); V)} + \| u_1 - u_2 \|_{L^{r+1}(0,T; L^{r+1})} \]

\[ + \| p_1 - p_2 \|_{L^{\frac{r+1}{r}}((0,T); L^2(\Omega))} + \| f_1 - f_2 \|_{L^2((0,T))} \]

\[ \leq C \left( \| u_{01} - u_{02} \|_{H^1} + \sup_{t \in [0,T]} \| (g_1 - g_2)(t) \|_{L^2} + \| \varphi_1 - \varphi_2 \|_{H^1((0,T))} \right), \]

which completes the proof of part (ii) of Theorem 1.2. \( \square \)
Appendix 1: Energy estimates

Since the existence and uniqueness of weak solutions for the system (1)–(4) are known, we derive a number of a-priori estimates for the solutions. To obtain the energy estimates of the solutions of CBF equations (1)–(4), we assume that \( u_0 \in \mathbb{H}, g \in C([0, T]; L^2(\Omega)) \) and \( f \in L^2(0, T) \). The next lemma provides the usual energy estimates for the CBF equations (1)–(4).

**Lemma A.1** Let \((u(\cdot), p(\cdot))\) be the unique weak solution of the CBF equations (1)–(4) and \( u_0 \in \mathbb{H} \). Then, the following estimate holds:

\[
\sup_{t \in [0, T]} \|u(t)\|_{\mathbb{H}} \leq \|u_0\|_{\mathbb{H}} + T^{1/2} \sup_{t \in [0, T]} \|g(t)\|_{L^2} \|f\|_{L^2(0, T)},
\]

(60)

and

\[
\sup_{t \in [0, T]} \|u(t)\|^2_{\mathbb{H}} + 2\mu \int_0^T \|u(t)\|^2_{\mathcal{V}} dt + \alpha \int_0^T \|u(t)\|^2_{\mathcal{V}'} dt + 2\beta \int_0^T \|u(t)\|_{L^{r+1}}^{r+1} dt \leq \|u_0\|^2_{\mathbb{H}} + \frac{1}{\alpha} \sup_{t \in [0, T]} \|g(t)\|^2_{L^2} \|f\|^2_{L^2(0, T)}.
\]

(61)

**Proof** The existence of a unique weak solution to the system (1)–(4) guarantees that \( u \in L^\infty(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap L^{r+1}(0, T; L^{r+1}) \). Taking the inner product with \( u(\cdot) \) to Eq. (1) and using the fact that \( \langle (u \cdot \nabla)u, u \rangle = 0 \) and \( \langle \nabla p, u \rangle = 0 \) to obtain

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{\mathbb{H}} + \mu \|u(t)\|^2_{\mathcal{V}} + \alpha \|u(t)\|^2_{\mathcal{V}'} + \beta \|u(t)\|_{L^{r+1}}^{r+1} = \langle f(t)g(t), u(t) \rangle,
\]

(62)

for a.e. \( t \in [0, T] \). Simplifying (54), we get

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|^2_{\mathbb{H}} \leq \|f(t)\|_{L^2} \|g(t)\|_{L^2} \|u(t)\|_{\mathbb{H}},
\]

which further reduces to

\[
\frac{d}{dt} \|u(t)\|_{\mathbb{H}} \leq |f(t)| \|g(t)\|_{L^2} \|u(t)\|_{\mathbb{H}}.
\]

(63)

Integrating the inequality (55) from 0 to \( t \), and taking supremum over 0 to \( T \) on both sides, one can easily deduce (52).

Using Hölder’s and Young’s inequalities, we estimate \( |f g, u| \) as

\[
|f g, u| \leq |f| \|g\| \|u\|_{\mathbb{H}} \leq \frac{1}{2\alpha} |f|^2 \|g\|^2_{L^2} + \frac{\alpha}{2} \|u\|^2_{\mathbb{H}}.
\]

(64)

Substituting (56) in (54), and then integrating it from 0 to \( t \), we find
\[\|u(t)\|_\mathcal{H}^2 + 2\mu \int_0^t \|u(s)\|_{\mathcal{V}}^2 \, ds + \alpha \int_0^t \|u(s)\|_{\mathcal{H}^1}^2 \, ds + 2\beta \int_0^t \|u(s)\|_{L^{t+1}}^2 \, ds\]
\[\leq \|u_0\|_{\mathcal{H}^1}^2 + \frac{1}{\alpha} \int_0^t |f(s)|^2 \|g(s)\|_{L^2}^2 \, ds,\]
for all \(t \in [0, T]\) and (53) follows. \(\square\)

The next lemma establishes stability estimates for the CBF equations (1)–(4).

**Lemma A.2** Let \((u_1(\cdot), p_1(\cdot))\) be the solution of the direct problem (1)–(4) corresponding to the external forcing \(f_1g\) and the initial velocity \(u_{01}\); and \((u_2(\cdot), p_2(\cdot))\) be the solution of the same problem corresponding to the external forcing \(f_2g\) and the initial velocity \(u_{02}\). Then, for \(u_{01}, u_{02} \in \mathbb{H}, f_1, f_2 \in L^2(0, T)\) and \(g \in C([0, T]; \mathbb{L}^2(\Omega))\), the following estimate holds:

\[
\sup_{t \in [0, T]} \|u_1 - u_2\|_{\mathcal{H}^1}^2 + \alpha \int_0^T \|u_1 - u_2\|^2_{\mathcal{V}} \, dt + \frac{\beta}{2r-2} \int_0^T \|u_1 - u_2\|_{L^{r+1}}^2 \, dt \leq \left( \|u_{01} - u_{02}\|_{\mathcal{H}^1}^2 + \frac{1}{\alpha} \sup_{t \in [0, T]} \|g(t)\|_{L^2}^2 \|f_1 - f_2\|_{L^2(0, T)}^2 \right) \exp \left( \frac{2}{\mu} \int_0^T \|u_2(t)\|_{\mathcal{V}}^2 \, dt \right). \tag{65}
\]

Moreover, for \(u_{01} = u_{02}\), we have

\[
\sup_{t \in [0, T]} \|u_1 - u_2\|_{\mathcal{H}^1}^2 \leq \left( \frac{1}{\alpha} \sup_{t \in [0, T]} \|g(t)\|_{L^2}^2 \|f_1 - f_2\|_{L^2(0, T)}^2 \right) \exp \left( \frac{2}{\mu} \int_0^T \|u_2(t)\|_{\mathcal{V}}^2 \, dt \right). \tag{66}
\]

**Proof** It is clear that the pair \(((u_1 - u_2)(\cdot), (p_1 - p_2)(\cdot))\) satisfies the following system:

\[
\begin{aligned}
(u_1 - u_2)_t - \mu \Delta (u_1 - u_2) + \alpha (u_1 - u_2) + \nabla (p_1 - p_2) &= F_k, \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot (u_1 - u_2) &= 0, \quad \text{in } \Omega \times (0, T), \\
\n\nabla \cdot (u_1 - u_2) &= 0, \quad \text{on } \partial \Omega \times [0, T), \\
\n(u_1 - u_2) &= u_{01} - u_{02}, \quad \text{in } \Omega \times \{0\},
\end{aligned}
\tag{67}
\]
in \(\nabla'\), where
\[ F_k = (f_1 - f_2)g - (u_1 \cdot \nabla)u_1 + (u_2 \cdot \nabla)u_2 - \beta (|u_1|^{-1}u_1 - |u_2|^{-1}u_2). \tag{68} \]

Taking the inner product with \((u_1 - u_2)(\cdot)\) to the first equation in (59), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|(u_1 - u_2)(t)\|_\mathcal{H}^2 + \mu \|(u_1 - u_2)(t)\|_\mathcal{V}^2 + \alpha \|(u_1 - u_2)(t)\|_\mathcal{H}^2 \\
= ((f_1 - f_2)(t)g(t), (u_1 - u_2)(t)) - \langle ((u_1 - u_2)(t) \cdot \nabla)u_2(t), (u_1 - u_2)(t) \rangle \\
- \beta \langle |u_1(t)|^{-1}u_1(t) - |u_2(t)|^{-1}u_2(t), (u_1 - u_2)(t) \rangle, \tag{69} \]

for a.e. \(t \in [0, T]\). It is worth emphasizing that, for \(r \geq 1\) (see Sec. 2.4, [31, 32])
\[
\beta (|u_1|^{-1}u_1 - |u_2|^{-1}u_2) \\
\geq \frac{\beta}{2} \|u_1|^{-\frac{r}{2}}(u_1 - u_2)\|_\mathcal{H}^2 + \frac{\beta}{2} \|u_2|^{-\frac{r}{2}}(u_1 - u_2)\|_\mathcal{H}^2. \tag{70} \]

It can be easily seen that
\[
\|u_1 - u_2\|_{\mathcal{L}^{r+1}}^{r+1} = \int_\Omega |u_1(x) - u_2(x)|^{-1} |u_1(x) - u_2(x)|^2 \, dx \\
\leq 2^{r-2} \int_\Omega (|u_1(x)|^{-1} + |u_2(x)|^{-1}) |u_1(x) - u_2(x)|^2 \, dx \\
\leq 2^{r-2} \left( \|u_1|^{-\frac{r}{2}}(u_1 - u_2)\|_\mathcal{H}^2 + \|u_2|^{-\frac{r}{2}}(u_1 - u_2)\|_\mathcal{H}^2 \right). \]

From the above inequality, we have
\[
\frac{2^{2-r} \beta}{2} \|u_1 - u_2\|_{\mathcal{L}^{r+1}}^{r+1} \\
\leq \frac{\beta}{2} \|u_1|^{-\frac{r}{2}}(u_1 - u_2)\|_\mathcal{H}^2 + \frac{\beta}{2} \|u_2|^{-\frac{r}{2}}(u_1 - u_2)\|_\mathcal{H}^2. \tag{71} \]

Thus, from (62) and (63), one can easily deduce that
\[
\beta (|u_1|^{-1}u_1 - |u_2|^{-1}u_2, u_1 - u_2) \geq \frac{\beta}{2^{r-1}} \|u_1 - u_2\|_{\mathcal{L}^{r+1}}^{r+1}. \tag{72} \]

Using the Cauchy–Schwarz and Young’s inequalities, we get
\[
((f_1 - f_2)g, u_1 - u_2) \leq \frac{1}{2\alpha} |f_1 - f_2|^2 \|g\|_{L^2}^2 + \frac{\alpha}{2} \|u_1 - u_2\|_{\mathcal{H}}^2. \tag{73} \]

Making use of Hölder’s, Ladyzhenskaya’s, and Young’s inequalities, we have
\[
\langle ((u_1 - u_2) \cdot \nabla)u_2, u_1 - u_2 \rangle \leq \|u_1 - u_2\|_{L^2}^2 \|u_2\|_{\mathcal{V}} \\
\leq \sqrt{2} \|u_1 - u_2\|_{\mathcal{H}} \|u_1 - u_2\|_{\mathcal{V}} \|u_2\|_{\mathcal{V}} \\
\leq \frac{\mu}{2} \|u_1 - u_2\|_{\mathcal{V}}^2 + \frac{1}{\mu} \|u_1 - u_2\|_{\mathcal{H}}^2 \|u_2\|_{\mathcal{V}}^2. \tag{74} \]

Substituting (64)–(66) in (61), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|(u_1 - u_2)(t)\|_q^2 + \frac{\mu}{2} \|(u_1 - u_2)(t)\|_v^2 + \frac{\alpha}{2} \|(u_1 - u_2)(t)\|_\eta^2 + \frac{\beta}{2r-1} \|(u_1 - u_2)(t)\|_{\xi_{r+1}}^{r+1}
\leq \frac{1}{2\alpha} |(f_1 - f_2)(t)|^2 \|g(t)\|^2_{L_2} + \frac{1}{\mu} \|(u_1 - u_2)(t)\|_\eta^2 \|u_2(t)\|_v^2.
\]

Integrating the above inequality from 0 to \(t\), we deduce

\[
\|(u_1 - u_2)(t)\|_\eta^2 + \mu \int_0^t \|(u_1 - u_2)(s)\|_v^2 \, ds + \alpha \int_0^t \|(u_1 - u_2)(s)\|_\eta^2 \, ds + \frac{\beta}{2r-2} \int_0^t \|(u_1 - u_2)(s)\|_{\xi_{r+1}}^{r+1} \, ds
\leq \|(u_{01} - u_{02})\|_\eta^2 + \frac{1}{\alpha} \int_0^t |(f_1 - f_2)(t)|^2 \|g(s)\|^2_{L_2} \, ds
\]

\[
+ \frac{\beta}{2r-2} \int_0^t \|(u_1 - u_2)(s)\|_{\xi_{r+1}}^{r+1} \, ds
\]

for all \(t \in [0, T]\). Applying Gronwall’s inequality in (67) and then taking supremum over 0 to \(T\) on both sides, we finally have (57). One can easily deduce (58) by taking \(u_{01} = u_{02}\) in (57) and it completes the proof. \(\square\)

Acknowledgements P. Kumar and M. T. Mohan would like to thank the Department of Science and Technology (DST), India for Innovation in Science Pursuit for Inspired Research (INSPIRE) Faculty Award (IFA17-MA110). The authors are supported by INSPIRE Faculty Award Grant IFA17-MA110.

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