Buchdahl’s Stability Bound in Eddington-inspired Born-Infeld Gravity

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Abstract

We give the Buchdahl’s stability bound in Eddington inspired Born-Infeld (EiBI) gravity. We show that this bound is equation-of-state (EoS)-dependent instead of the EoS-independent feature in general relativity. In addition, to avoid the potential pathologies in EiBI, a Hagedorn-like EoS at the center of a compact star is inevitable, which is similar to the Hagedorn temperature in string theory.

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Despite general relativity (GR) being the most successful theory to describe gravity, there have been a number of modified theories of GR to resolve various problems, such as singularities, local energy problems and incompatibilities with our quantum world. In particular, motivated by Born-Infeld electrodynamics [1], Vollick (2004) [2] first considered a Born-Infeld-like action for gravity in the Palatini formalism. Later on, Bañados and Ferreira (2010) [3] showed that this action can be regarded as the Eddington action if the matter action is absent [4]. Furthermore, when matter *solely* couples to metric, the highly nonlinear coupling between matter and gravity could avoid the cosmological singularity in the radiation dominated era, resulting in significant deviations from GR in the extremely high density environment. Since then, this model has aroused much interest, dubbed Eddington-inspired Born-Infeld (EiBI) gravity.

It is known that when we apply a modified theory of gravity to a stellar sphere, the junction conditions must be dealt with carefully. For example, in \( f(R) \) gravity the matching of first and second fundamental forms [5] is not enough to merge two spacetime regions since the derivative terms of curvatures in the field equations play important roles [6, 7]. In contrast to the higher derivative theory, EiBI is equivalent to GR in vacuum, so that the junction conditions are the same as those in GR. This advantage makes EiBI an interesting modification of GR. Nevertheless, when applying EiBI to compact stars, it has been found that there are some pathologies, such as the surface singularities [8] and anomalies associated with phase transitions [9]. However, the surface singularities can be prevented if the equations of state (EoS) is modified by the geodesic deviation (Jacobi) equation near the surface of the star [10]. This aspect is quite similar to the EoSs in \( f(R) \) theories, in which more stringent junction conditions are needed [11, 12].

In GR, the Buchdahl’s stability bound is that the radius of a stable stellar sphere cannot be smaller than 9/8 of its gravitational radius [13]. This inequality is independent of the EoS contained in the sphere, which is a generic feature of GR. For EiBI gravity, it is not clear if there is a similar Buchdahl’s bound. However, it is expected that the bound in GR should be different from that in EiBI gravity. To address this issue, we start from the EiBI action [2, 3] with the geometric unit of \( G = c = 1 \), given by

\[
S_{\text{EiBI}}[g, \Gamma] = \frac{2}{8\pi\kappa} \int d^4x \left[ \sqrt{-\det(g + \kappa \mathcal{R}(\Gamma))} - \lambda \sqrt{-\det(g)} \right] + S_M[g, \Psi],
\]

where \( g \) and \( \mathcal{R} \) correspond to the metric tensor and Ricci curvature based on the connection
Γ with the matrix elements $g_{\mu\nu}$ and $\mathcal{R}_{\mu\nu}$, respectively, and $S_M[g, \Psi]$ is the matter action, in which the generic matter field $\Psi$ couples only to the metric tensor $g$. Note that the cosmological constant is defined as $\Lambda = (\lambda - 1)/\kappa$ in this model. However, when it comes to compact stars, it is reasonable to set $\lambda = 1$ such that $\Lambda = 0$. Under the Palatini formalism, varying the action with respect to $g$ and $\Gamma$ independently gives the equations of motion of EiBI gravity as follows:

$$q_{\mu\nu} = g_{\mu\nu} + \kappa \mathcal{R}_{\mu\nu},$$

$$q^{\mu\nu} = \tau (g^{\mu\nu} - 8\pi \kappa T^{\mu\nu}),$$

where $\tau = \sqrt{|g|}/|q|$ with $|\bullet|$ denoting the absolute value of the determinant of the matrix, $T^{\mu\nu}$ is the physical energy momentum tensor, and $q^{\mu\nu} \equiv q^{-1}_{\mu\nu}$. Here, the auxiliary metric is used for raising/lowering index in the geometric sector, while the metric tensor is for the matter sector. As a result, it follows that

$$\delta^{\mu\nu} = \kappa \mathcal{R}^{\mu\nu} = q^{\mu\lambda} g_{\lambda\nu} = \tau (\delta^{\mu\nu} - 8\pi \kappa T^{\mu\nu}).$$

By defining $\mathcal{R} \equiv \mathcal{R}^{\mu}_{\mu}$ and $T \equiv T^{\mu}_{\mu}$, the field equations can be recast into the GR-like field ones with the Einstein tensor built with the auxiliary metric [14], given by

$$G^{\mu\nu}[q] = \mathcal{R}^{\mu\nu} - \frac{1}{2} \delta^{\mu\nu} \mathcal{R} = 8\pi \left(\tau T^{\mu\nu} + \mathcal{P} \delta^{\mu\nu}\right) \equiv 8\pi \mathcal{T}^{\mu\nu},$$

where $\mathcal{P} = (\tau - 1)/(8\pi \kappa) - \tau T/2$ is the isotropic pressure and $\mathcal{T}^{\mu\nu}$ is defined as the “auxiliary” energy momentum tensor. Furthermore, by taking the determinant of Eq. (4), we can express $\tau$ solely in terms of $T^{\mu\nu}$ as

$$\tau = \left[\det(\delta^{\mu\nu} - 8\pi \kappa T^{\mu\nu})\right]^{-\frac{1}{2}}.$$

If we model the self-gravitating sphere by a perfect fluid with $T^{\mu\nu} = (\rho + p)u^\mu u_\nu + p \delta^{\mu\nu}$, where $\rho$, $p$ and $u^\mu$ denote the energy density, pressure and four-velocity of the fluid with $u^\mu u_\mu = -1$, respectively, we can further write $\tau$ in terms of $\rho$ and $p$ as

$$\tau = \left[\left(1 + 8\pi \kappa \rho\right)(1 - 8\pi \kappa p)^3\right]^{\frac{1}{2}} \equiv \frac{1}{ab^2},$$

where $a \equiv \sqrt{1 + 8\pi \kappa \rho}$ and $b \equiv \sqrt{1 - 8\pi \kappa p}$ are required to be positive real number. In addition,
we can define the “auxiliary” density $\tilde{\rho}$ and pressure $\tilde{p}$ in terms of $a$ and $b$ through

\[ T^0_0 = \frac{-a^2 + 3b^2 - 2ab}{16\pi kab^3} \equiv -\tilde{\rho}, \] (8)

\[ T^i_j = \frac{a^2 + b^2 - 2ab}{16\pi kab^3} \equiv \tilde{p}, \] (9)

respectively.

For a spherically symmetric and static spacetime, the physical and auxiliary metrics are given by

\[ g_{\mu\nu} dx^\mu dx^\nu = -F^2(r) dt^2 + G^2(r) dr^2 + H^2(r) d\Omega^2; \]

\[ q_{\mu\nu} dx^\mu dx^\nu = -A^2(r) dt^2 + B^2(r) dr^2 + r^2 d\Omega^2, \]

respectively, where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

For a perfect fluid, $T^{\mu\nu} = (\rho + p) u^\mu u^\nu + pg^{\mu\nu}$, the relations between two sets of the metrics $g_{\mu\nu}$ and $q_{\mu\nu}$ via Eq.(3) are given by

\[ F^2 = A^2 ab^{-3}; \quad G^2 = B^2/ab; \quad H^2 = r^2/ab. \] (10)

Once the auxiliary metric is solved, the physical one can be obtained immediately, and vice versa. We note that these two metrics are identical to those in the absence of matter, i.e. vacuum, in which $a = b = 1$.

With the auxiliary metric at hand, we can also solve the auxiliary Einstein equations in Eq.(5) as GR. First, we give an Ansatz $B^2 \equiv \left[1 - 2m(r)/r\right]^{-1}$, where $m(r)$ represents the auxiliary mass within the radial distance $r$. The $tt$ and $rr$–components lead to

\[ m'(r) = 4\pi r^2 \tilde{\rho} \] (11)

and

\[ \frac{A'}{A} = \frac{m + 4\pi r^3 \tilde{\rho}}{r(r - 2m)}, \] (12)

respectively, where the “$'$” denotes the derivative with respect to the coordinate $r$. Together with $\tilde{\nabla}_\alpha T^\alpha_\beta = 0$ ($\tilde{\nabla}$ denotes the covariant derivative associated with $q_{\alpha\beta}$), we find

\[ \frac{1}{\tilde{\rho} + \tilde{p}} \frac{dp}{dr} = \frac{m + 4\pi r^3 \tilde{p}}{r(r - 2m)}, \] (13)

which is the Tolman-Oppenheimer-Volkoff (TOV) equation for auxiliary quantities, as expected. With the help of Eq.(13), the derivative of $\tilde{p}(\rho, p)$ with respect to $r$ can be written
in terms of $dp/dr$, resulting in the modified TOV equation from Eq.(13) in EiBI gravity, given by

$$-8\pi \kappa \left[ \frac{(a^2 - b^2)(b^2/c_s^2 + 3a^2) + 4a^2b^2}{4a^2b^2(a^2 - b^2)} \right] \frac{dp}{dr} = \frac{m + 4\pi r^3 \tilde{p}}{r(r - 2m)},$$

(14)

where $c_s^2 \equiv dp/d\rho$ with $c_s$ the sound speed. This equation indicates that a negative value of $\kappa$ seems not possible to support an equilibrium stellar structure [9]. Consequently, we will only focus on the case of $\kappa > 0$ throughout the study.

By taking the derivative of $\tilde{p}$ in Eq.(12) and substituting the expression into Eq.(13), we find

$$\frac{d}{dr} \left[ \frac{A}{r} \sqrt{1 - \frac{2m}{r}} \right] = \frac{A}{\sqrt{1 - \frac{2m}{r}}} \left( \frac{m}{r^3} \right),$$

(15)

which characterizes how the average auxiliary density changes in accordance with the auxiliary potentials, and plays the key role in proving the Buchdahl’s stability bound. Naively, we expect the same Buchdahl’s stability bound as that in GR due to the GR-like field equations of the auxiliary quantities. However, in order to examine the bound, we need to express these auxiliary quantities in terms of the physical ones. Due to the highly nonlinear coupling of matter to gravity in EiBI, the EoS clearly affects the determination of the bound.

For a star submerged in the Schwarzschild vacuum, we use the Darmois-Israel matching conditions [5] since EiBI is equivalent to GR in vacuum due to $B^{-2}(r) = A^2(r) = G^{-2}(r) = F^2(r) = 1 - 2M/r$, where $M$ is the Schwarzschild mass in the spherically static spacetime. As the auxiliary mass $m(r)$ is quite different from the physical one with the mass function appearing in the physical metric inside the interior of the star, it should match $m(r_s) = M$ at radius $r_s$ with the condition $p(r_s) = 0$. This suggests that we should take another Ansatz $G^2 \equiv \left[1 - 2M(r)/r\right]^{-1}$ together with $G^2 = B^2/ab$, where $M(r)$ stands for the physical mass, to obtain

$$M = m + \frac{1}{2} (1 - ab)(r - 2m),$$

(16)

or equivalently

$$m = M + \frac{1}{2ab} (ab - 1)(r - 2M).$$

The “effective” density associated with $M(r)$ can be defined by $\rho_{\text{eff}} \equiv M'(r)/4\pi r^2$ in analogy to Eq.(11). Subsequently, we derive

$$\rho_{\text{eff}} = ab\tilde{\rho} + \frac{1 - ab}{8\pi r^2} + \frac{\kappa}{2r} \left(1 - \frac{2m}{r}\right) \left(\frac{a^2c_s^2 - b^2}{ab}\right) \frac{d\rho}{dr}.$$

(17)
Instead of the auxiliary density $\tilde{\rho}$, $\rho_{\text{eff}}$ is the one in EiBI, corresponding to the physical density $\rho$ in GR. Therefore, the non-increasing monotonicity of $\rho_{\text{eff}}$ must be assumed to show the Buchdahl’s stability bound. On the other hand, it is worth noting that the second term of $(1 - ab)/8\pi r^2$ in Eq. (17) is potentially divergent at $r \to 0$ unless $ab = 1$. To avoid the singularity of $\rho_{\text{eff}}$ in this model, we must have $ab = 1$ at the center of the star as $r \to 0$.

A crucial assumption in proving the Buchdahl’s stability bound in GR is the monotonically non-increasing property of the physical density $\rho$. However, is the monotonically non-increasing $\rho$ enough to have the same monotonic behavior of $\tilde{\rho}(\rho, p)$ and $\rho_{\text{eff}}(\rho, p)$? To answer this question, let us examine the differential relation between $\tilde{\rho}$ and $\rho$:

$$\frac{d\tilde{\rho}}{dr} = \left[\frac{3a^2(a^2 - b^2)c_s^2 + (3b^2 + a^2)b^2}{4a^3b^5}\right]\frac{d\rho}{dr}. \tag{18}$$

The term in the numerator with $24\pi\kappa(\rho + p)a^2c_s^2 + (3b^2 + a^2)b^2 > 0$ is required to guarantee the non-increasing monotonicity of $\tilde{\rho}$ once $\rho$ is so. For a positive $\kappa$, the positivity is true if the null energy condition holds, i.e. $\rho + p \geq 0$.

For the effective density, taking the derivative of Eq. (17) with respect to $r$ gives

$$\frac{d\rho_{\text{eff}}}{dr} = ab\frac{d\tilde{\rho}}{dr} + \frac{(ab - 1)}{4\pi r^3} + \kappa \left[\frac{1}{2r}\left(1 - \frac{2m}{r}\right)\rho'' + \left(\frac{2m}{r^3} - 8\pi \tilde{\rho}\right)\rho' + \mathcal{O}(\kappa)\left(\frac{a^2c_s^2 - b^2}{ab}\right)\right]. \tag{19}$$

Since the derivative terms associated with $a$ and $b$ will pick out one more order of $\kappa$, we collect them as $\mathcal{O}(\kappa)$ terms. Clearly, the first term in Eq. (19) is negative if $\rho$ is monotonically non-increasing according to Eq. (18). However, the assumption of the monotonically non-increasing $\rho$ cannot lead to the same monotonicity of $\rho_{\text{eff}}$ due to the potentially positive terms in Eq. (19). To elaborate it, we consider two situations: (i) $ab \geq 1$ ($\rho - p \geq 8\pi\kappa pp$) yields $a^2c_s^2 \leq b^2$, in the GR case with $\kappa \to 0$ resulting in $\rho - p \geq 0$ ($c_s^2 \leq 1$), which is just the requirement of the causal energy condition with the positive density and pressure; and (ii) $ab \leq 1$ ($\rho - p \leq 8\pi\kappa pp$) gives $a^2c_s^2 \geq b^2$, which violates the causality condition if $\kappa \to 0$, but it is allowed for any finite value of $\kappa$ within $a > 0, b > 0$. In these two cases, the net effect can offset each other only if the terms in the square bracket of Eq. (19) are positive, i.e.

$$\frac{1}{2r}\left(1 - \frac{2m}{r}\right)\rho'' + \left(\frac{2m}{r^3} - 8\pi \tilde{\rho}\right)\rho' > 0. \tag{20}$$

To $\mathcal{O}(\kappa)$, we are obliged to impose this condition as an additional assumption, if we want to interpret this effective density as the physical one corresponding to GR, i.e. the
density contributing to the physical mass $M(r)$. Since the profile of $\rho$ depends on the EoS contained inside the star as well as the modified TOV equation in Eq. (14), the assumption of the monotonically non-increasing $\rho_{\text{eff}}$ limits the possible classes of the EoS in the EiBI theory. Note that the three densities tend to be the same in the GR limit of $\kappa \to 0$.

Now, we show the corresponding Buchdahl’s stability bound in the EiBI theory based on the assumptions made above. To begin with, we can prove that if $\rho_{\text{eff}}$ is a monotonically non-increasing function, then

$$M(r) \geq \frac{r^3}{r_s^3} M,$$

(21)

where $r_s$ is the radius of the star at which $p(r_s) = 0$ and $M \equiv M(r_s)$.

**Proof.** By definition, $M(r) = \int_{0}^{r} 4\pi \xi^2 \rho_{\text{eff}}(\xi) d\xi$. Using the mean-valued theorem, there exists $\bar{r} \in (0, r)$, such that

$$\rho_{\text{eff}}(\bar{r}) = \frac{\int_{0}^{r} 4\pi \xi^2 \rho_{\text{eff}}(\xi) d\xi}{\int_{0}^{r} 4\pi \xi^2 d\xi} \equiv \bar{\rho}_{\text{eff}}(r),$$

resulting in

$$M(r) = \frac{4\pi r^3}{3} \bar{\rho}_{\text{eff}}(r) \geq \frac{r^3}{r_s^3} \left( \frac{4\pi r_s^3}{3} \bar{\rho}_{\text{eff}}(r_s) \right) = \frac{r^3}{r_s^3} M,$$

where $\bar{\rho}_{\text{eff}}(r) \geq \bar{\rho}_{\text{eff}}(r_s)$ by the non-increasing monotonicity and $M(r_s) = M$.

Together with Eq. (16), we readily obtain

$$m(r) + \frac{1}{2} (1 - ab) [r - 2m(r)] \geq \frac{r^3}{r_s^3} M.$$

(22)

On the other hand, if $\bar{\rho}$ is a monotonically non-increasing function, then

$$M(r) + \frac{1}{2ab} (ab - 1) [r - 2M(r)] \geq \frac{r^3}{r_s^3} M,$$

(23)

where we have taken the fact that $m(r_s) = M(r_s) \equiv M$ at the surface $(a = b = 1)$ of the star.

From the inequalities in Eqs. (22) and (23), we get the dual relations between the two mass functions. Without forming a black hole, we must have $r - 2m > 0$ as well as $r - 2M > 0$ throughout the interior of the star. The signs of the extra terms in the inequalities, which are absent in GR due to $a = b = 1$, depend on that of $(ab - 1)$. If $ab > 1$, the inequality in Eq. (22) is stronger than that in Eq. (23), and vice versa. When proving the Buchdahl’s stability bound, we will use the stricter assumption depending on $ab$.

Furthermore, if $\bar{\rho}$ is a monotonically non-increasing function, then

$$\frac{m'}{r^3} \leq 0.$$  

(24)
\[ \left( \frac{m}{r^3} \right)' = \frac{m'}{r^3} - \frac{3m}{r^4} = \frac{4\pi}{r} \left[ \tilde{\rho}(r) - \tilde{\rho}(\bar{r}) \right] \leq 0, \]

where \( \bar{r} \in (0, r) \). Here, we have used the mean-valued theorem and non-increasing monotonicity for \( \tilde{\rho} \) to achieve the inequality. \( \square \)

**Theorem.** If (i) both \( \tilde{\rho} \) and \( \rho_{\text{eff}} \) are finite and monotonically non-increasing functions and (ii) \( A^2 \) and \( B^2 \) are positive definite, the Buchdahl’s stability bound in EiBI gravity for \( ab > 1 \) is given by

\[ r_s \left( 1 - \frac{1}{2} g - \frac{1}{2} g^2 \right) \geq \frac{9}{4} \mathcal{M}, \tag{25} \]

where

\[ g \equiv \frac{\mathcal{M}}{r_s^3} \int_0^{r_s} \frac{\sqrt{ab} - 1}{\sqrt{1 - \frac{2m(r)}{r}}} r dr. \tag{26} \]

**Proof.** With Eq. (24) and \( A(B) > 0 \), integrating Eq. (15) from \( r \) to \( r_s \), one gets

\[ \frac{A'(r_s)}{r_s} \sqrt{1 - \frac{2m(r_s)}{r_s}} - \frac{A'(r)}{r} \sqrt{1 - \frac{2m}{r}} \leq 0. \]

To match with the second fundamental form to the Schwarzschild vacuum, it is required that

\[ A(r_s) = \sqrt{1 - \frac{2\mathcal{M}}{r_s}}, \quad \text{and} \quad A'(r_s) = \frac{\mathcal{M}}{r_s^2} \frac{1}{\sqrt{1 - \frac{2\mathcal{M}}{r_s}}}, \]

leading to

\[ A'(r) \geq \frac{r}{\sqrt{1 - \frac{2m(r)}{r}}} \frac{\mathcal{M}}{r_s^2}, \]

and

\[ A(r_s) - A(0) \geq \frac{\mathcal{M}}{r_s^3} \int_0^{r_s} \frac{r dr}{\sqrt{1 - \frac{2m(r)}{r}}} \equiv I. \]

For \( ab > 1 \), by writing \( m(r) \) in terms of \( M(r) \) we have \( r - 2m = (r - 2M)/ab \), which yields \( M(r) \geq \frac{r^2}{r_s^2} \mathcal{M} \) from Eq. (22). As a result, we obtain that

\[ I = \frac{\mathcal{M}}{r_s^3} \int_0^{r_s} \frac{\sqrt{abr} dr}{\sqrt{1 - \frac{2m(r)}{r}}} \geq \frac{\mathcal{M}}{r_s^3} \int_0^{r_s} \frac{\sqrt{abr} dr}{\sqrt{1 - \frac{2\mathcal{M}}{r}}} \frac{1}{\sqrt{1 - \frac{2\mathcal{M}}{r_s^2} \mathcal{M}}}. \]

\[ = \frac{\mathcal{M}}{r_s^3} \int_0^{r_s} \frac{r dr}{\sqrt{1 - \frac{2\mathcal{M}}{r_s^2} \mathcal{M}}} + \frac{\mathcal{M}}{r_s^3} \int_0^{r_s} \frac{(\sqrt{ab} - 1) r dr}{\sqrt{1 - \frac{2\mathcal{M}}{r_s^2} \mathcal{M}}} \]

\[ = \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{2\mathcal{M}}{r_s}} \right] + g. \]
All of the above imply that
\[
\sqrt{1 - \frac{2M}{r_s}} = A(r_s) \geq A(0) - \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{2M}{r_s}} \right] + g,
\]
solving the inequality with the desired result. \[\square\]

If \( ab > 1 \) throughout the interior of the sphere, \( g \) is positive definite. This means that the lower bound of the stable radius in EiBI gravity is larger than \((9/4)M\). On the other hand, for \( ab < 1 \), we can replace Eq. (23) with \( m(r) \geq \frac{r^3}{r_s^2}M \) in the proof to get the same bound as that in GR.

The intuitive explanation is the EoS switch-on of the “repulsive effect” in EiBI gravity. It becomes clear by expanding the auxiliary quantities in Eqs. (8) and (9) to the leading order of \( \mathcal{O}(\kappa) \), given by
\[
\tilde{\rho} = \rho - \pi \kappa (5\rho^2 - 6\rho p - 3p^2) + \mathcal{O}(\kappa^2),
\]
\[
\tilde{p} = p + \pi \kappa (\rho^2 + 2\rho p + 9p^2) + \mathcal{O}(\kappa^2).
\]
We observe that the repulsive effect with \( \tilde{\rho} < \rho \) and \( \tilde{p} > p \) in EiBI is significant only when \( \rho > \left[(3 + 2\sqrt{3})/5\right]p \approx 1.29p \). Note that we do not consider the case with \( \rho < \left[(3 - 2\sqrt{3})/5\right]p \approx -0.09p \) if the physical density and pressure are both positive inside the star. For \( \tilde{\rho} > \rho \) and \( \tilde{p} > p \), the effect is reduced due to the increased density in \( \tilde{\rho} \) compared to \( \rho \). When including all orders of \( \kappa \), the two cases correspond to \( ab > 1 \) and \( ab < 1 \), respectively. Note that the criteria for the repulsive effect to become significant will also depend on the specific value of \( \kappa \), but the mechanism is the same as we just look at the contributions of \( \mathcal{O}(\kappa) \). In Fig. 1, we plot the changes between the auxiliary and physical densities v.s. pressures in terms of \( ab \), ranging from 0.85 to 1.2 to represent the different classes of EoSs. The borderline near \( ab \approx 1 \) marks \( \tilde{\rho} \approx \rho \), where the repulsive effect switches on/off. Moreover, as we shall see, the critical value of \( ab = 1 \) corresponds to an exotic EoS.

The “singularity avoidance” feature of this model \[14\] relies on the fact that as \( b \to 0 \), i.e. \( 8\pi \kappa p \to 1 \), the auxiliary energy density (pressure) diverges but with a finite physical energy density (pressure). In this regard, \( \kappa \) can be taken as a cutoff scale near the Planck one. However, as noted previously, if we assume \( b \neq 0 \) inside the star, \( \rho_{\text{eff}} \) is still potentially divergent due to \( (1 - ab)/8\pi r^2 \) as \( r \to 0 \). The remedy to cure the pathology is to set \( ab = 1 \), or equivalently
\[
p = \frac{\rho}{1 + 8\pi \kappa \rho}, \tag{27}
\]
around $r = 0$. Physically, this leads to an exotic EoS controlled by $\kappa$ near the center ($r \lesssim \sqrt{\kappa}$) of a star regardless of the real matter contents. For this exotic EoS, the physical pressure $p$ is bounded by $1/8\pi\kappa$. On the other hand, there is no bound on the physical density $\rho$ as $\rho \to \infty$ if $p \to 1/8\pi\kappa$. In other words, the cutoff of the physical pressure does not prevent the divergence of the physical density as shown in Fig. 2. Remarkably, this situation is in close analogy to the Hagedorn temperature [16], in which the energy and entropy diverge but with a fixed and finite (Hagedorn) temperature. This Hagedorn-like EoS [17] manifests somewhat a deep connection of EiBI with string theory [18], which deserves further investigations. However, the discussions above are only at the classical level. The pressure near the cutoff scale may signal a breakdown of EiBI or Hagedorn-like phase transition. Whether this divergence of $\rho$ really occurs during the gravitational collapse requires a real understanding of EiBI as well as its quantized version.

The Buchdahl’s stability bound in EiBI is larger than $(9/4)M$ of GR due to the repulsive effect if $ab > 1$ holds throughout a star. To elaborate on this statement, we see that $g$ in Eq. (26) can be computed by the mean-valued theorem such that

$$r_s \geq \left[ 2 + \frac{\bar{a}ar{b}/2}{1 + \sqrt{\bar{a}\bar{b}}} \right]M,$$

where $\bar{a} = \bar{a}(r_s) \equiv a(\bar{r})$ and $\bar{b} = \bar{b}(r_s) \equiv b(\bar{r})$, respectively, for $\bar{r} \in (0, r_s)$. By the expansion
FIG. 2. Exotic EoS behavior in the $\rho - p$ plane, which is analogous to the effect of the Hagedorn temperature.

in terms of the order of $\kappa$, we have $r_s/2M \geq 9/8 + (3\pi/8)\kappa(\bar{\rho} - \bar{p}) + O(\kappa^2)$. This inequality provides us with a direct way to constrain $\kappa$ just by examining the radii and masses of the most compact spherical objects in the sky. For instance, the typical density $\bar{\rho} \sim 10^{18}$ kg/m$^3$, radius $r_s \sim 12$ km and gravitational radius $2M \sim 6$ km with $M \sim 2M_\odot \sim 3$ km of a neutron star (NS) [19, 20] yield the bound of $\kappa \lesssim 10^9$ m$^2$. Even though this gives the constraint on $\kappa$ in the similar order of magnitude as those in Refs. [21, 22], the bound on $\kappa$ can be further improved to one more order if a NS with the same radius but mass around $3M_\odot$ can be pinned down in the future.

On the other hand, if we believe EiBI gravity is a viable theory down to the cutoff scale $\kappa$ and the observed minimally stable radius of a spherical compact object is found to be $(9/4)M$, it indicates that somewhat unusual EoS ($ab < 1$) is contained inside the compact object. It sheds new light on how we can probe the EoS contained inside a compact star just by examining the smallest stable radius of it. However, the Buchdahl’s stability bound will be modified further as that in GR if an anisotropic fluid is considered. Such a situation merits further studies in the future.
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