Can we be tricked into thinking

that \( w \) is less than \( -1 \)?

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Abstract

Dark energy candidates for which the equation-of-state parameter \( w \) is less than \(-1\) violate the dominant energy condition, and are typically unstable. In scalar-tensor theories of gravity, however, the expansion of the universe can mimic the behavior of general relativity with \( w < -1 \) dark energy, without violating any energy conditions. We examine whether this possibility is phenomenologically viable by studying Brans-Dicke models and characterizing both the naturalness of the models themselves, and additional observational constraints from limits on the time-dependence of Newton’s constant. We find that only highly contrived models would lead observers to measure \( w < -1 \).
I. INTRODUCTION

The last several years have seen a sustained flow of cosmological data, beginning with the observations of type Ia supernovae \([1, 2, 3]\), bolstered by large-scale redshift surveys (e.g. \([4]\)) and measurements of the cosmic microwave background (CMB) from the ground and from balloons \([5, 6, 7, 8]\), culminating in the exquisite full-sky maps of the WMAP satellite \([9]\). These studies have indicated that the expansion of the universe is accelerating (the scale factor obeys \(\ddot{a} > 0\)), and that the total amount of clustered matter in the universe is insufficient to account for the small value of its average spatial curvature.

Both of these features (acceleration and flatness) can be explained in the context of conventional general relativity by invoking a smooth, persistent dark-energy component, \(X\) \([10, 11, 12, 13, 14, 15, 16, 17, 18]\). To be compatible with the observed isotropy and homogeneity of our universe on large scales, the energy-momentum tensor of the dark energy should be that of a perfect fluid,

\[
T_{\mu\nu}^{(X)} = (\rho_X + p_X) U_\mu U_\nu + p_X g_{\mu\nu},
\]

where \(U^\mu\) is the fluid rest-frame four-velocity, \(\rho_X\) is the energy density and \(p_X\) is the pressure. The dark energy must be smoothly distributed in order to escape detection in the dynamics of gravitationally bound systems and large-scale structure. To make the universe accelerate, general relativity implies that the pressure \(p_X\) must be appreciable and negative. From the Friedmann equations

\[
\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2} \tag{2}
\]

and

\[
\frac{\ddot{a}}{a} = \dot{H} + 2H^2 = -\frac{4\pi G}{3} (\rho + 3p), \tag{3}
\]

we find that acceleration only occurs if the total pressure \(p\) (dominated by the dark energy, since matter is pressureless) is less than \(-\rho/3\) (including both matter and dark energy).

Since CMB observations imply that the spatial curvature is small (\(|\kappa/a^2| \ll H^2\)), it is convenient to set \(\kappa = 0\) and consider a two-parameter set of cosmological models, characterized by the density parameter in matter \(\Omega_m \equiv 8\pi G \rho_m / 3H^2\) and the dark energy equation-of-state
parameter $w_X$,

$$w_X = p_X / \rho_X .$$  \hspace{1cm} (4)$$

Through the continuity equation

$$\dot{\rho}_X = -3H(1 + w_X)\rho_X ,$$  \hspace{1cm} (5)$$

the equation-of-state parameter governs the rate at which the dark energy evolves as the universe expands,

$$\frac{d\ln \rho_X}{d\ln a} = -3(1 + w_X).$$  \hspace{1cm} (6)$$

A strictly constant vacuum energy (a cosmological constant) would have $w_X = -1$. Observational constraints are often presented as exclusion contours in the $\Omega_m$-$w_X$ plane.

Of course, it is impossible in principle to directly measure the pressure in a component that is smoothly distributed, since there are no pressure gradients (or at least no readily observable ones). Constraints on $w_X$ actually derive from observations of the behavior of the scale factor $a(t)$, and use the Friedmann equations \(2,3\) to translate these into limits on $w_X$. Different observational methods will be sensitive to different integrated behaviors of the dark energy density; it is nevertheless useful to consider the instantaneous effective equation-of-state parameter $w_{\text{eff}}$ that would be derived from (2,3) in a flat universe dominated by matter and dark energy:

$$w_{\text{eff}} = -(1 + \alpha) \left(1 + \frac{2}{3} \frac{\dot{H}}{H^2}\right) ,$$ \hspace{1cm} (7)$$

where

$$\alpha \equiv \frac{\Omega_m}{\Omega_X} = \frac{\Omega_m}{1 - \Omega_m}$$ \hspace{1cm} (8)$$

is the ratio of energy density in matter to that in dark energy. Current observational bounds \[19, 20, 21, 22\] on $w_{\text{eff}}$ yield

$$-1.48 < w_{\text{eff}} < -0.72$$ \hspace{1cm} (9)$$

at the 95% confidence level.

While the task of identifying a compelling candidate source with equation of state parameter in this region is a formidable challenge for fundamental physics, the portion satisfying
$w_{\text{eff}} < -1$ is particularly troublesome theoretically. (Note that recent data, e.g. [23, 24], seem to suggest, at one or two sigma confidence, a $w_{\text{eff}}$ changing with $z$ to values less than -1 today.) It is only possible to obtain $w < -1$ by violating the dominant energy condition (DEC), which for a perfect fluid can be stated as

$$\rho \geq |p|.$$ \hspace{1cm} (10)

The physical motivation for the DEC is to prevent instability of the vacuum or propagation of energy outside the light cone. Nevertheless, such energy components have been known for some time [25, 26, 27] and their role as possible dark energy candidates was raised by Caldwell [28], who referred to DEC-violating sources as “phantom” components. The implications of these have since been investigated by several authors (for some examples see [16, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44]. Typically, DEC-violating sources with $w < -1$ are subject to violent instabilities, although these may conceivably be cured in models with higher-derivative kinetic terms [34, 43, 44].

Despite the difficulties of model-building, it is certainly worthwhile to consider the possibility that $w_{\text{eff}} < -1$ when characterizing observational constraints, if only to keep open the possibility of a surprising discovery. In addition, given how little we understand about dark energy, we should keep an open mind about the true explanation for the acceleration of the universe. One alternative that has been explored is a modification of general relativity that would only become important on cosmological scales [45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57]. If the Friedmann equations (2,3) are not valid, the reconstruction (7) of the equation-of-state parameter would similarly be invalid. Therefore, it becomes conceivable that we could measure the effective value $w_{\text{eff}}$ (as defined by (7) to be less than -1, even if the actual dark energy source obeys the DEC, or if there is indeed no dark energy at all. In other words, we could be tricked into thinking that $w_X < -1$ by assuming the validity of general relativity (see also [58] for examples of non-phantom matter behaving as phantom matter).

¿From (7) it is clear that we infer $w_{\text{eff}} < -1$ today if

$$\frac{\dot{H}_0}{H_0^2} > \frac{3}{2} \frac{\alpha}{1 + \alpha},$$ \hspace{1cm} (11)
where the subscript 0 indicates present-day values of quantities.

In this paper we consider a simple class of modifications of general relativity: scalar-tensor theories, featuring a scalar field $\phi$ that interacts non-minimally with gravitation (e.g., through a direct coupling to curvature). A wide variety of alternatives to GR can be cast as scalar-tensor theories, at least in some range of validity. Focusing on Brans-Dicke (BD) theories [57], we examine whether a scalar-tensor theory of gravity could account for the acceleration of the universe and yield $w_{\text{eff}} < -1$, while remaining consistent with other experimental constraints. In particular, the cosmological evolution of the scalar $\phi$ leads to time-dependence of Newton’s gravitational constant $G$, which are constrained by solar-system tests of gravity. We find that only extremely unnatural and contrived models lead to an inference of $w < -1$, even in this wide class of extensions to general relativity.

II. BRANS-DICKE THEORIES

A particular class of theories of gravity beyond General Relativity are Brans-Dicke theories [57] with a potential, which may arise, for example, from the dimensional reduction of a higher dimensional theory. These theories consist of a metric and a Brans-Dicke scalar field $\varphi$, with action

$$S_{\text{BD}} = \int d^4x \sqrt{-g} \left[ \varphi R - \frac{\omega}{\varphi} (\partial_{\mu} \varphi) \partial^{\mu} \varphi - 2V(\varphi) \right] + \int d^4x \sqrt{-g} L_M(\psi_i, g) ,$$

(12)

where $L_M(\psi_i, g)$ is the Lagrangian for matter fields $\psi_i$ and $\omega$ is a constant. In this frame, the Jordan frame, matter is minimally-coupled to gravity and hence test particles fall freely along geodesics of the metric $g_{\mu\nu}$. Scalar-tensor theories such as these have been considered previously as a way to solve the fine-tuning problems of quintessence (see [59, 60, 61]).

The predictions of Brans-Dicke theories differ from those of GR due to the presence of a new scalar component to gravity. Since GR is well-tested in the solar system, these deviations must be smaller than the accuracy of current observations. This can happen in one of two ways. Either the potential is sufficiently confining to render the Brans-Dicke scalar essentially constant in the solar system or, in the presence of a sufficiently weak potential, the parameter $\omega$ must satisfy the bound $\omega > 40000$, obtained using signal timing from the
We will be interested in two possibilities. First, that a smoothly distributed background matter component, described by $\mathcal{L}_M$ and minimally coupled to the Brans-Dicke sector, may fuel faster than exponential expansion. Second, that the Brans-Dicke scalar itself may lead to a similar effect, without appealing to sources outside the gravitational sector of the theory.

It is convenient to perform both a conformal transformation and a field redefinition to obtain an Einstein frame description of the theory.

We define a canonically-normalized version of the Brans-Dicke scalar via

$$e^{\sigma/\sigma^*} = 16\pi G \phi ,$$

where

$$\sigma^* \equiv \sqrt{\frac{2\omega + 3}{16\pi G}} ,$$

with $\omega$ the Brans-Dicke parameter. Denoting the Jordan-frame metric by $g_{\mu\nu}$, the Einstein-frame metric is

$$\bar{g}_{\mu\nu} \equiv e^{\sigma/\sigma^*} g_{\mu\nu}$$

and the resulting action becomes

$$S_E = \int d^4x \sqrt{-\bar{g}} \left[ \frac{1}{16\pi G} \bar{R} - \frac{1}{2} (\partial_\mu \sigma) \partial^\mu \sigma - U(\sigma) \right] + \int d^4x \sqrt{-\bar{g}} \mathcal{L}_M(\psi, \bar{g}, \sigma) ,$$

where

$$U(\sigma) \equiv 2e^{-2\sigma/\sigma^*} V[\varphi(\sigma)] .$$

This description now has the advantage that the gravitational sector is of Einstein form, with a minimally coupled scalar field, but the disadvantage that the matter does not freely-fall along the geodesics of the Einstein-frame metric.

Further, the relationship between the energy-momentum tensors in the two frames is

$$T^{\mu\nu} = \Omega^3 \bar{T}^{\mu\nu} .$$

Therefore, if the energy conditions are not violated in the Einstein frame they will not be violated in the Jordan frame, since $\Omega$ is strictly positive.
Let us now study the Einstein-frame equations of motion. We define the Einstein-frame scale factor and time coordinate by

\[
\bar{a} = e^{\sigma/2\sigma_*} a, \\
\bar{d}t = e^{\sigma/2\sigma_*} dt
\]

and let a prime denote differentiation with respect to \(\bar{t}\), so that the Einstein-frame Hubble parameter is

\[
\bar{H} \equiv \frac{\bar{a}'}{\bar{a}}.
\]

The Friedmann equations are then

\[
\bar{H}^2 = \frac{8\pi G}{3} (\rho + \bar{\rho}) ,
\]

\[
\bar{H}' = -4\pi G \left[(\sigma')^2 + \bar{\rho} + \bar{p}\right] ,
\]

and the equation of motion for \(\sigma\) is

\[
\sigma'' + 3\bar{H} \sigma' + U_{,\sigma} = \frac{1}{2\sigma_*} (\bar{\rho} - 3\bar{p}) .
\]

Here we have also defined \(\rho = e^{2\sigma/\sigma_*} \bar{\rho}\) and \(p = e^{2\sigma/\sigma_*} \bar{p}\), where \(\rho\) and \(p\) are respectively the Jordan frame energy density and pressure of matter. Since \(\sigma\) is a canonically-normalized, minimally-coupled scalar field, its energy density and pressure are given by the usual definitions

\[
\rho_{\sigma} = \frac{1}{2}\sigma'^2 + U(\sigma) ,
\]

\[
p_{\sigma} = \frac{1}{2}\sigma'^2 - U(\sigma) .
\]

Because cosmological observations, and in particular those quantities entering (11), involve Jordan frame quantities, we need to know how to transform between our easily-interpretable Einstein-frame quantities and those in the Jordan frame. This is achieved by the following expressions

\[
H = e^{\sigma/2\sigma_*} \left(\bar{H} - \frac{\sigma'}{2\sigma_*}\right) .
\]
\[ \dot{H} = e^{\sigma/\sigma^*} \left[ \dot{H}' - \frac{\sigma''}{2\sigma_*} + \frac{\sigma'}{2\sigma_*} \left( \dot{H} - \frac{\sigma'}{2\sigma_*} \right) \right]. \] (27)

Substituting the Einstein-frame equations of motion into the expressions (27) and (26), we obtain, after some algebra

\[ \dot{H} = e^{\sigma/\sigma^*} \left\{ -\frac{8\pi G}{(2\omega + 3)} \left[ (\omega + 2) \left( \bar{\rho} + (\sigma')^2 + \omega \bar{p} \right) \right] + \left( \frac{4\pi G}{2\omega + 3} \right)^{1/2} \left[ 4\sigma' \sqrt{\frac{8\pi G}{3} (\bar{\rho} + \rho_\sigma) + U_{,\sigma}} \right] \right\} \] (28)

and

\[ H = e^{\sigma/2\sigma^*} \sqrt{4\pi G} \left[ \sqrt{\frac{2}{3}} \sqrt{\bar{\rho} + \frac{1}{2} (\sigma')^2 + U} - \sqrt{\frac{(\sigma')^2}{2\omega + 3}} \right]. \] (29)

These are the fundamental expressions relating the observable quantities \( H \) and \( \dot{H} \) to sources as measured in the Einstein frame.

III. A PERTURBATIVE APPROACH

A. Acceleration from a Dark Energy Component in Brans-Dicke Theories

Let us now focus on the case in which the source of cosmic acceleration is some dark-energy component in the matter sector (rather than the BD field itself). Since we are interested in the present day, we have \( \sigma = \sigma_0 = 0 \). We assume that the BD field provides a negligible contribution to the evolution of the universe, so that

\[ (\sigma')^2, U(\sigma) \ll \bar{\rho}, \] (30)

and that the theory is consistent with solar system test of gravity, requiring \( \omega \gg 1 \). However, we will keep quantities that are first-order in \( 1/\sqrt{\omega} \), since these are important for our effect. The “matter” sector will consist of ordinary matter plus the dark-energy component \( X \).

Expanding and evaluating (28) and (29) at the present day, after some algebra we obtain

\[ \dot{H}_0 \approx -4\pi G \left( 1 + \frac{w_X}{1 + \alpha} - 4\xi - \eta \right) \bar{\rho}_0, \] (31)
\[ H_0^2 \approx 4\pi G \left( \frac{2}{3} - \xi \right) \tilde{\rho}_0, \]  

where we have introduced the parameters

\[ \xi \equiv \left[ \frac{2{(\sigma'_0)}^2}{3(2\omega + 3)\tilde{\rho}_0} \right]^{1/2} \]

and

\[ \eta \equiv \frac{2U_{,\sigma_0}}{\sqrt{16\pi G(2\omega + 3)\tilde{\rho}_0}}. \]

(Note that, since \( \sigma_0 = 0 \), we have \( \tilde{\rho}_0 = \rho_{X0} + \rho_{m0} \), so we could drop the tildes, but we won’t bother.) We thus have

\[ \frac{\dot{H}_0}{H_0^2} \approx -3 \left( \frac{1 + w_X + 4\xi - \eta}{2 - 3\xi} \right), \]

which implies

\[ w_{\text{eff}} \approx \frac{2w_X - (5\xi + 2\eta)(1 + \alpha)}{2 - 3\xi} \cdot \]

So far we have not assumed that the parameters \( \xi \) and \( \eta \) are small. However, note that

\[ \xi^2 = \frac{2{(\sigma'_0)}^2}{3(2\omega + 3)\tilde{\rho}_0}. \]

Since we have already assumed that \( (\sigma'_0)^2 \ll \tilde{\rho}_0 \), and also that \( \omega \gg 1 \), we see that \( \xi \) is certainly very small. It is therefore legitimate to rewrite (36) as

\[ w_{\text{eff}} \approx w_X - \left( 1 + \frac{5}{2}\alpha \right) \xi - (1 + \alpha)\eta. \]

Note that for \( \alpha = 3/7 \), this is

\[ w_{\text{eff}} \approx w_X - \frac{29}{14}\xi - \frac{10}{7}\eta, \]

and since \( \xi \) is extremely small (less than \( 10^{-3} \)), the second term on the right hand side will have a negligible effect on \( w_{\text{eff}} \).

Now let us turn to \( \eta \). If we Taylor-expand the potential to first order about the present day value \( \sigma = 0 \),

\[ U(\sigma) \approx U_0 \left( 1 + \lambda \frac{\sigma}{\sigma_*} \right), \]

\( 9 \)
where \( U(0) \equiv U_0 \) and \( U,\sigma(0) \equiv \lambda U_0/\sigma_* \), we obtain

\[
\eta \approx \frac{2\lambda U_0}{(2\omega+3)\rho_0}.
\]

Again, since we have assumed \( U_0 \ll \dot{\rho}_0 \) and \( \omega \gg 1 \), this looks very small except for the freedom associated with the dimensionless parameter \( \lambda \). Thus, it is possible to obtain \( w_{\text{eff}} \) detectably below \(-1\) if

\[
\lambda \geq \omega \left( \frac{\dot{\rho}_0}{U_0} \right).
\]

Thus, we certainly can get \( w_{\text{eff}} < -1 \) with matter sources that obey the Dominant Energy Condition. However, we should think about what it means to have \( \eta \) be large while \( \xi \) remains small. From the definitions (33) and (34), this occurs only if the time derivative of the BD field is small while the slope of its potential is very large. This will only happen for very finely-tuned conditions – either a very sudden change in the slope of the potential, or when the initial conditions for the field are chosen such that it has recently been climbing up the potential and is near a local maximum today. Hence, although it is possible to obtain \( w_{\text{eff}} < -1 \) behavior in BD theories, it is by no means natural.

\textbf{B. Acceleration from the Brans-Dicke Scalar with no Dark Energy}

The right hand side of the Einstein-frame Friedmann equations are sourced not only by matter, but also by the BD scalar \( \sigma \). It is natural to ask whether super-exponential acceleration can be obtained purely from this modification of general relativity.

Making the approximation (40), using (26), evaluated today and defining \( x \equiv \sigma_0'/\sqrt{\rho_{\text{cr},0}} \) we obtain

\[
\frac{\dot{H}_0}{H_0^2} = \frac{3}{2} \frac{2w+4}{2w+3} x \left[ \frac{4}{\sqrt{3}} \frac{\sqrt{2w+3}}{2w+4} \sqrt{x^2 + \frac{2U_0}{\rho_{\text{cr},0}} + \frac{2\alpha}{1+\alpha}} - \frac{3\lambda}{\rho_{\text{cr},0}} \frac{w+2}{2w+3} \frac{\alpha}{1+\alpha} \right] - x + \frac{U_0}{\rho_{\text{cr},0}} \frac{3\lambda}{2w+3} - 3 \frac{w+2}{2w+3} \frac{\alpha}{1+\alpha}.
\]

The Friedmann equation can then be rewritten as

\[
\frac{U_0}{\rho_{\text{cr},0}} = \left[ 1 + \sqrt{\frac{3}{4w+6}} x \right]^2 - \frac{x^2}{2} - \frac{\alpha}{1+\alpha},
\]
which allows us to obtain
\[
\frac{\dot{H}_0}{H_0^2} = \frac{1}{2w + 3} \left\{ \frac{3}{2} x \left[ \frac{4}{\sqrt{3}} \sqrt{4w + 6} - 2\sqrt{w}x \right] + 3\lambda \frac{U_0}{\rho_{cr,0}} - 3 \frac{\alpha}{1 + \alpha} (w + 2) \right\}. 
\] (45)

From equation (44) we can see that in the limit \(|x| \ll 1\), which we may term the slow-roll regime, we obtain
\[
\frac{U_0}{\rho_{cr,0}} \approx \frac{1}{1 + \alpha},
\] (46)
which, from (45) yields
\[
w_{\text{eff}} < -1 \quad \text{for} \quad \lambda > \frac{\alpha}{2}
\] (47)
\[
\dot{H}_0 > 0 \quad \text{for} \quad \lambda > \alpha (\omega + 2)
\]

Therefore, given that \(\omega\) is typically required to be much larger than unity, there exists a range in which the universe is not superaccelerating, but in which we infer \(w_{\text{eff}} < -1\), even without an additional dark energy component. Again, though, we require a very large value of the parameter \(\lambda\) characterizing the slope of the potential.

IV. CONSTRAINTS FROM THE TIME-VARIATION OF NEWTON’S CONSTANT

Thus far we have focused only on the behavior of the effective equation of state parameter. However, in Brans-Dicke theories it is important to remember that, for a weak BD potential, the field \(\phi\) plays the role of a dynamical Newton constant \(G_{\text{eff}}\) via
\[
16\pi G_{\text{eff}} = \frac{1}{\phi} \frac{2w + 4}{2w + 3}.
\] (48)

Experimental constraints on the time variation of Newton’s constant yield
\[
\frac{|\dot{G}_{\text{eff},0}|}{G_{\text{eff},0}} < 6 \times 10^{-12} \text{ yr}^{-1}. \quad (49)
\]

Since (48) gives
\[
\frac{|\dot{G}_{\text{eff}}|}{G_{\text{eff}}} = \Omega^{1/2} \frac{|\phi'|}{\phi} = \Omega^{1/2} \frac{|\sigma'|}{\sigma_*}
\] (50)
this then implies
\[
\frac{|\sigma'|}{\sigma_*} < 6 \times 10^{-12} \text{ yr}^{-1}, \tag{51}
\]
or
\[
|x| = \sqrt{\frac{2w + 3}{6}} \frac{1}{H_0} \frac{|\dot{G}_{\text{eff},0}|}{G_{\text{eff},0}} < \sqrt{\frac{2w + 3}{6}} \frac{5.88 \times 10^{-2}}{h}, \tag{52}
\]
where we have written \( H_0 = 100h \text{ km s}^{-1} \text{Mpc}^{-1} \), with \( h = 0.72 \pm 0.08 \).

We would like to know what range of \( w_{\text{eff}} \) is possible today while remaining consistent with bounds on the time-variation of Newton’s constant. To this end it is convenient to define
\[
y = \frac{1}{H_0} \frac{\dot{G}_{\text{eff},0}}{G_{\text{eff},0}}. \tag{53}
\]
In terms of this variable equation (44) becomes
\[
\frac{U_0}{\rho_{\text{cr},0}} \equiv \gamma(y) = \left(1 + \frac{y}{2}\right)^2 - \frac{2w + 3}{12} y^2 - \frac{\alpha}{1 + \alpha}, \tag{54}
\]
and using equation (53) we obtain
\[
w_{\text{eff}} = -(1 + \alpha) \left[1 + \frac{1}{3} y (4 - wy) + \frac{2\lambda}{2w + 3} \gamma(y) - \frac{2\alpha}{1 + \alpha} \frac{w + 2}{2w + 3}\right]. \tag{55}
\]

Clearly, \( w_{\text{eff}} \) is a function of three parameters: \( \Omega_m/\Omega_X \equiv \alpha = 3/7 \), \( \lambda \), giving the slope of the potential today, and the Brans-Dicke parameter \( \omega \). For fixed \( \omega = 40000 \), we plot this expression as a function of the variables \( y \) and \( \lambda \) in fig. 1. As can be seen, negative values of \( \lambda \) correspond to positive \( w_{\text{eff}} \).

Fixing the parameters \( \lambda \) and \( \omega \) and plotting this function for \( |y| < 5.88 \times 10^{-2} \), we see that, in order to get values of \( w_{\text{eff}} \approx -1.1 \), we need \( \lambda > 0.8 \) for \( \omega = 1 \) (see fig. 2), or \( \lambda > 5000 \) for \( \omega = 40000 \) (see fig. 3). In this last case we must also demand \( |y| < 0.001 \).

From the graphs we see that in order to obtain \( w_{\text{eff}} \sim -1.1 \) with \( \omega = 1 \) we require \( \lambda \sim \alpha \). However, this condition implies a severe fine-tuning. The equation for the scalar field requires that the second derivative be negative and much larger in magnitude than the first derivative. Therefore, \( \sigma \) must be close to a maximum today. This is consistent with the toy potential we used as a test.
V. STUDY OF GENERAL SOLUTIONS FOR BD ACCELERATION

To establish the main result of this paper, it was sufficient to use the perturbative approach of the previous two sections. However, it is interesting to consider the equations of motion (21), (22) and (23) in the Einstein frame in generality. Making the following definitions

$$\tau = \sqrt{\rho_{cr,0} \tilde{t}}$$

$$\tilde{\alpha} = \tilde{\alpha}/\tilde{\alpha}_0$$

$$U = \gamma \rho_{cr,0} f \left( \frac{\sigma}{\sigma_*} \right)$$

and redefining a prime to denote a derivative with respect to \( \tau \), the equations become

$$\ddot{H}^2 = \frac{8\pi G}{3} \left[ \frac{\sigma^2}{2} + \gamma f + \frac{\alpha}{1+\alpha} \Omega^{-1/2} \dot{a} a^{-3} \right]$$

FIG. 1: \( w_{eff} \) as a function of \( y \equiv \frac{1}{H_0} \frac{\dot{G}_{\chi}}{G_{\chi}} \) and \( \lambda \).
FIG. 2: $w_{\text{eff}}$ as a function of $y \equiv \frac{1}{H_0} \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}}$ with $\omega = 1$.

\begin{align*}
\sigma'' &= -3\tilde{H}\sigma' - \gamma f,\sigma + \frac{\alpha}{1 + \alpha} \frac{\Omega^{-1/2}}{2\sigma_*} \tilde{a}^{-3} \\
\tilde{H}' &= -4\pi G \left[ \sigma'^2 + \frac{\alpha}{1 + \alpha} \frac{\Omega^{-1/2}}{\tilde{a}^{-3}} \right].
\end{align*}

We focus on the case of $\lambda > \alpha/2$ in order to ensure $w_{\text{eff}} < -1$ today. As we shall see, for some potentials (for example square or exponential ones), $\sigma(\tau)$ has almost zero first derivative ($x \ll 1$) but a much larger and negative second derivative, implying that $\sigma$ achieves a local maximum today. This in turn implies a significant fine tuning of the initial conditions for $\sigma$.

One possibility is to impose that $\sigma'$ is always small, for example $\sigma' = x$ at all times, so that $\sigma'' \approx 0$, and to look for the potential that would satisfy this condition. It is convenient to think of $f(\sigma) = f[\sigma(t)]$, so that $f$ becomes a function of time. For example we could use the Friedmann equation, the scalar field equation and the second Einstein equation. Since

\begin{equation}
\sqrt{\rho_{\text{cr},0}} \frac{1}{a} \frac{d\tilde{a}}{d\tau} \bigg|_0 = \frac{1}{a} \frac{d\tilde{a}}{dt} \bigg|_0 = \tilde{H}_0 = H_0 + \frac{\sigma'_0}{2\sigma_*},
\end{equation}
consistent initial conditions are

\[ \ddot{a}(0) = 1 \tag{63} \]
\[ \ddot{a}'(0) = \sqrt{\frac{8\pi G}{3} + \frac{x}{2\sigma^*}} \tag{64} \]
\[ \sigma(0) = 0 \tag{65} \]
\[ f(0) = 1 \tag{66} \]

We must then solve for the three new unknowns: \( \ddot{a}, \sigma, f \). In this case we find that \( w_{\text{eff}} \) is very close to \(-1\) at all times, as shown in figure [FIG. 3].

An alternative possibility is to look for a potential \( f \) that leads to a constant \( w_{\text{eff}} \) less than \(-1\). This requires

\[ \frac{\sigma''}{2\sigma^*} = \ddot{H} + \frac{\sigma'}{2\sigma^*} \left( \ddot{H} - \frac{\sigma'}{2\sigma^*} \right) + \frac{3}{2} \left( 1 + w_{\text{eff}} \right) \left( \ddot{H} - \frac{\sigma'}{2\sigma^*} \right)^2 - 4\pi G w_{\text{eff}} \frac{\alpha}{1 + \alpha} \Omega^{-3/2} \ddot{a}^{-3}. \tag{67} \]

Using this relation, the second Einstein equation and the scalar field equations, with...
FIG. 4: $w_{\text{eff}}$ as a function of both $y \equiv \frac{1}{H_0}$ and $\lambda$ with $\omega = 40000$.

We find that the potential is not even a function any more ($f$ becomes double-valued). The potential is a curve and $\sigma = 0$ is a point at which the curve is continuous but not differentiable. This tells us something about how unnatural a potential like this is. So in this case a constant $w_{\text{eff}}$ does not seem likely at all.

The general issue here is the following. Since $\sigma' = x \ll 1$, equation (44) implies that $\gamma \approx 1/(1 + \alpha)$. Thus, in order to achieve $w_{\text{eff}} < -1$ (e.g. $w_{\text{eff}} = -1.2$) with $\omega = 1$, we find that the potential is not even a function any more ($f$ becomes double-valued).
equation (47) yields $\lambda \sim \alpha$. Further, equation (62) implies $\bar{H}_0 \approx H_0/\sqrt{\rho_{cr,0}}$ and therefore, since $\sigma_* \bar{H}_0 \sim 1$, the scalar field equation (60) evaluated at the present day is

$$\sigma_* \sigma'' \approx -\frac{\alpha}{2(1 + \alpha)}, \quad (73)$$

where we have used $f_{,\sigma}(0) = \lambda/\sigma_*$. This requires

$$\left| \frac{\sigma''}{\sigma_0} \right| \sigma_* \gg 1 \quad \text{or} \quad \left| \frac{\sigma''}{\sigma_0} \right| M_p \gg 1. \quad (74)$$

In other words, the function $\sigma(\tau)$ must either be close to a maximum, having a small first derivative and a negative second derivative or, as in section IIIA, close to a sudden change in the potential. Similar behavior also occurs for $\omega = 40000$ with $\lambda \sim 5000$.

VI. CONCLUSIONS

The observed acceleration of the universe presents a fascinating yet daunting challenge to particle cosmology. The development of a theoretical framework in which the correct magnitude of the dark energy density and its relatively recent dominance are explained in a manner consistent with both particle physics and general relativity has so far eluded researchers.

Another problem is posed by the equation of state parameter of the dark energy or, equivalently, the time evolution of its energy density. Even for the most economical possibility, a cosmological constant, we are faced with a problem that has confounded theorists for decades. The cosmological constant problem remains even when we turn to other dark energy models. If $-1/3 > w > -1$ then, assuming a solution to the cosmological constant problem, there are many proposals to obtain the requisite acceleration. To date, however, all of these face fine tuning problems or worse, when considered as serious particle physics or gravitational models.

A more severe problem exists in the observationally allowed range $w < -1$. Such a source for the Einstein equations [16, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 38, 39, 40, 41] must violate the dominant energy condition and hence may lead to instabilities of the vacuum or the propagation of energy outside the light cone.
One way around this problem is to consider theories in which the Friedmann equations are not valid so that it is possible to measure the effective value $w_{\text{eff}}$ to be less than $-1$, even if the actual dark energy source obeys the DEC, or if there is no dark energy at all.

In this paper we have considered Brans-Dicke (BD) theories, to see if such an effect can occur naturally in this framework, while remaining consistent with other experimental constraints such as those on the time-dependence of Newton’s gravitational constant $G$. We have demonstrated that only extremely unnatural and contrived models lead to an inference of $w < -1$, even in this wide class of extensions to general relativity. In particular, it is necessary to fine tune the behavior of the Brans-Dicke scalar so that it is approaching a maximum today, having small first derivative with respect to time and yet large second derivative.

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