ABSTRACT. In this paper we consider the heat equation with strongly singular potentials and prove that it has a "very weak solution". Moreover, we show the uniqueness and consistency results in some appropriate sense. The cases of positive and negative potentials are studied. Numerical simulations are done: one suggests so-called "laser heating and cooling" effects depending on a sign of the potential. The latter is justified by the physical observations.

1. INTRODUCTION

After the pioneering works due to Baras and Goldstein [BG84a], [BG84b], the heat equation with inverse-square potential in bounded and unbounded domains has attracted considerable attention during the last decades, we cite [AFP17], [FM15], [Gul02], [IKM19], [IO19], [Mar03], [MS10] and [VZ00] to name only few.

Our aim is to contribute to the study of the heat equation by incorporating more singular potentials. The major obstacle for considering general coefficients is related to the multiplication problem for distributions [Sch54]. There are several ways to overcome this problem. One way is to use the notion of very weak solutions.

The concept of very weak solutions was introduced in [GR15] for the analysis of second order hyperbolic equations with non-regular time-dependent coefficients, and was applied for the study of several physical models in [MRT19], [RT17a], and in [RT17b]. In these papers the very weak solutions are presented for equations with time-dependent coefficients. In the recent paper [Gar20], the author introduces the concept of the very weak solution for the wave equation with space-depending coefficient. In the present paper we consider the Cauchy problem for the heat equation with a non-negative potential, we allow the potential to be discontinuous or even less regular and we want to apply the concept of very weak solutions to establish a well-posedness result.

In this paper we consider the heat equation with strongly singular potentials, in particular, with a $\delta$-function and with a behaviour like "multiplication" of $\delta$-functions. The existence result of very weak solutions is proved. Also, we show the uniqueness of the very weak solution and the consistency with the classical solution in some appropriate senses. The cases of positive and negative

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potentials are studied. Numerical simulations are given. Finally, one observes so-called "laser heating and cooling" effects depending on a sign of the potential.

2. PART I: NON-NEGATIVE POTENTIAL

In this section we consider the case when the potential $q$ is non-negative. But first let us fix some notations. For our convenience, we will write $f \lesssim g$, which means that there exists a positive constant $C$ such that $f \leq Cg$. Also, let us define

$$
\|u(t, \cdot)\|_k := \|\nabla u(t, \cdot)\|_{L^2} + \sum_{l=0}^{k} \|\partial_t^l u(t, \cdot)\|_{L^2},
$$

for all $k \in \mathbb{Z}_+$. In the case when $k = 0$, we simply use $\|u(t, \cdot)\|$ instead of $\|u(t, \cdot)\|_0$.

Fix $T > 0$. In the domain $\Omega := (0, T) \times \mathbb{R}^d$ we consider the heat equation

$$
(2.1) \quad \partial_t u(t, x) - \Delta u(t, x) + q(x)u(t, x) = 0, \quad (t, x) \in \Omega,
$$

with the Cauchy data $u(0, x) = u_0(x)$, where the potential $q$ is assumed to be non-negative and singular.

In the case when the potential is a regular function, we have the following lemma.

**Lemma 2.1.** Let $u_0 \in H^1(\mathbb{R}^d)$ and suppose that $q \in L^\infty(\mathbb{R}^d)$ is non-negative. Then, there is a unique solution $u \in C^1([0, T]; L^2) \cap C([0, T]; H^1)$ to (2.1) and it satisfies the energy estimate

$$
(2.2) \quad \|u(t, \cdot)\| \lesssim (1 + \|q\|_{L^\infty}) \|u_0\|_{H^1}.
$$

**Proof.** By multiplying the equation (2.1) by $u_t$ and integrating with respect to $x$, we obtain

$$
(2.3) \quad \text{Re} \left( \langle u_t(t, \cdot), u_t(t, \cdot) \rangle_{L^2} + \langle -\Delta u(t, \cdot), u_t(t, \cdot) \rangle_{L^2} + \langle q(\cdot)u(t, \cdot), u_t(t, \cdot) \rangle_{L^2} \right) = 0.
$$

One observes

$$
\text{Re} \langle u_t(t, \cdot), u_t(t, \cdot) \rangle_{L^2} = \|u_t(t, \cdot)\|_{L^2}^2.
$$

Also, we see that

$$
\text{Re} \langle -\Delta u(t, \cdot), u_t(t, \cdot) \rangle_{L^2} = \frac{1}{2} \partial_t \langle \nabla u(t, \cdot), \nabla u(t, \cdot) \rangle_{L^2} = \frac{1}{2} \partial_t \|\nabla u(t, \cdot)\|_{L^2}^2
$$

and

$$
\text{Re} \langle q(\cdot)u(t, \cdot), u_t(t, \cdot) \rangle_{L^2} = \frac{1}{2} \partial_t \langle q^{\frac{1}{2}}(\cdot)u(t, \cdot), q^{\frac{1}{2}}(\cdot)u(t, \cdot) \rangle_{L^2} = \frac{1}{2} \partial_t \|q^{\frac{1}{2}}(\cdot)u(t, \cdot)\|_{L^2}^2.
$$

It follows from (2.3) that

$$
(2.4) \quad \partial_t \left[ \|\nabla u(t, \cdot)\|_{L^2}^2 + \|q^{\frac{1}{2}}(\cdot)u(t, \cdot)\|_{L^2}^2 \right] = -2\|u_t(t, \cdot)\|_{L^2}^2.
$$

Let us denote by

$$
E(t) := \|\nabla u(t, \cdot)\|_{L^2}^2 + \|q^{\frac{1}{2}}(\cdot)u(t, \cdot)\|_{L^2}^2,
$$

the energy functional. It follows from (2.4) that $E'(t) \leq 0$, and thus

$$
E(t) \leq E(0).
$$
By taking into account that \( \| q^{\frac{1}{2}}(\cdot)u_0(\cdot) \|_{L^2}^2 \) can be estimated by
\[
\| q^{\frac{1}{2}}(\cdot)u_0(\cdot) \|_{L^2}^2 \leq \| q(\cdot) \|_{L^\infty} \| u_0(\cdot) \|_{L^2}^2,
\]
we get
\[
\| \nabla u(t, \cdot) \|_{L^2}^2 + \| q^{\frac{1}{2}}(\cdot)u(t, \cdot) \|_{L^2}^2 \leq \| \nabla u_0 \|_{L^2}^2 + \| q(\cdot) \|_{L^\infty} \| u_0 \|_{L^2}^2.
\]
Thus, we have
\[
(2.5) \quad \| q^{\frac{1}{2}}(\cdot)u(t, \cdot) \|_{L^2}^2 \leq \| \nabla u_0 \|_{L^2}^2 + \| q(\cdot) \|_{L^\infty} \| u_0 \|_{L^2}^2
\]
and
\[
\| \nabla u(t, \cdot) \|_{L^2}^2 \leq \| \nabla u_0 \|_{L^2}^2 + \| q(\cdot) \|_{L^\infty} \| u_0 \|_{L^2}^2,
\]
and consequently, one can be seen that
\[
(2.6) \quad \| \nabla u(t, \cdot) \|_{L^2} \leq \left( 1 + \| q \|_{L^\infty} \right)^2 \| u_0 \|_{H^1}.
\]

To obtain the estimate for \( u \), we rewrite the equation (2.1) as follows
\[
(2.7) \quad u_t(t, x) - \Delta u(t, x) = -q(x)u(t, x), \quad (t, x) \in (0, T) \times \mathbb{R}^d.
\]
Here, considering \(-q(x)u(t, x)\) as a source term, we denote it by \( f(t, x) := -q(x)u(t, x) \). By using Duhamel’s principle (see, e.g. [Eva98]), we represent the solution to (2.7) in the form
\[
(2.8) \quad u(t, x) = \phi_t * u_0(x) + \int_0^t \phi_{t-s} * f_s(x) ds,
\]
where \( f_s = f(s, \cdot) \) and \( \phi_t = \phi(t, \cdot) \). Here, \( \phi \) is the fundamental solution (heat kernel) to the heat equation, and it satisfies
\[
\| \phi(t, \cdot) \|_{L^1} = 1.
\]
Now, taking the \( L^2 \)-norm in (2.8) and using Young’s inequality, we arrive at
\[
\| u(t, \cdot) \|_{L^2} \leq \| \phi_t \|_{L^1} \| u_0 \|_{L^2} + \int_0^T \| \phi_{t-s} \|_{L^1} \| f_s \|_{L^2} ds
\]
\[
\leq \| u_0 \|_{L^2} + \int_0^T \| f_s \|_{L^2} ds
\]
\[
\leq \| u_0 \|_{L^2} + \int_0^T \| q(\cdot)u(s, \cdot) \|_{L^2} ds.
\]
We estimate the term \( \| q(\cdot)u(s, \cdot) \|_{L^2} \) as
\[
\| q(\cdot)u(s, \cdot) \|_{L^2} \leq \| q \|_{L^\infty} \| q^{\frac{1}{2}}u(s, \cdot) \|_{L^2},
\]
and using the estimate (2.5), one observes
\[
(2.9) \quad \| u(t, \cdot) \|_{L^2} \leq \left( 1 + \| q \|_{L^\infty} \right)^2 \| u_0 \|_{H^1}.
\]
Summing the estimates proved above, we conclude (2.2).

\[ \square \]

**Remark 2.1.** We can also prove that the estimate
\[ \| \partial_t^k u(t, \cdot) \|_{L^2} \lesssim (1 + \| q \|_{L^\infty}) \| u_0 \|_{H^{2k+1}}, \]
is valid for all \( k \geq 0 \), by requiring higher regularity on \( u_0 \). To do so, we denote by \( v_0 := u \) and its derivatives by \( v_k := \partial_t^k u \), where \( u \) is the solution of the Cauchy problem (2.1). Using (2.9) and the property that if \( v_k \) solves the equation
\[ \partial_t v_k(t, x) - \Delta v_k(t, x) + q(x)v_k(t, x) = 0, \]
with the initial data \( v_k(0, x) \), then \( v_{k+1} = \partial_t v_k \) solves the same equation with the initial data
\[ v_{k+1}(0, x) = \Delta v_k(0, x) - q(x)v_k(0, x), \]
we get our estimate for \( \partial_t^k u \) for all \( k \geq 0 \).

To prove the uniqueness and consistency of the very weak solution, we will also need the following lemma.

**Lemma 2.2.** Let \( u_0 \in H^1(\mathbb{R}^d) \) and assume that \( q \in L^\infty(\mathbb{R}^d) \) is non-negative. Then, the estimate
\[ (2.10) \quad \| u(t, \cdot) \|_{L^2} \lesssim \| u_0 \|_{L^2}, \]
holds for the unique solution \( u \in C^1([0, T] ; L^2) \cap C([0, T] ; H^1) \) of the Cauchy problem (2.1).

**Proof.** Again, by multiplying the equation (2.1) by \( u \) and integrating over \( \mathbb{R}^d \) in \( x \), we derive
\[ \text{Re} \left( \langle u_t(t, \cdot), u(t, \cdot) \rangle_{L^2} + \langle -\Delta u(t, \cdot), u(t, \cdot) \rangle_{L^2} + \langle q(\cdot)u(t, \cdot), u(t, \cdot) \rangle_{L^2} \right) = 0. \]
Using the similar arguments as in Lemma 2.1, we obtain
\[ (2.11) \quad \partial_t \| u(t, \cdot) \|_{L^2}^2 = -\| \nabla u(t, \cdot) \|_{L^2}^2 - \| q^2(\cdot)u(t, \cdot) \|_{L^2}^2 \leq 0. \]
This ends the proof of the lemma. \( \square \)

Now, let us show that the Cauchy problem (2.1) has a very weak solution. We start by regularising the coefficient \( q \) and the initial data \( u_0 \) using a suitable mollifier \( \psi \), generating families of smooth functions \( (q_\varepsilon) \) and \( (u_{0,\varepsilon}) \). Namely,
\[ q_\varepsilon(x) = q * \psi_\varepsilon(x), \quad u_{0,\varepsilon}(x) = u_0 * \psi_\varepsilon(x), \]
where
\[ \psi_\varepsilon(x) = \omega(\varepsilon)^{-1}\psi(x/\omega(\varepsilon)), \quad \varepsilon \in (0, 1), \]
and \( \omega(\varepsilon) \) is a positive function converging to 0 as \( \varepsilon \to 0 \) to be chosen later. The function \( \psi \) is a Friedrichs-mollifier, i.e. \( \psi \in C_0^\infty(\mathbb{R}^d), \psi \geq 0 \) and \( \int \psi = 1. \)
Assumption 2.3. On the regularisation of the coefficient $q$ and the initial data $u_0$ we make the following assumptions: there exist $N, N_0 \in \mathbb{N}_0$ such that
\begin{equation}
\|u_{0, \varepsilon}\|_{H^1} \leq C_0 \omega(\varepsilon)^{-N_0},
\end{equation}
and
\begin{equation}
\|q_{\varepsilon}\|_{L^\infty} \leq C_0 \omega(\varepsilon)^{-N},
\end{equation}
for $\varepsilon \in (0, 1]$. 

Remark 2.2. We note that such assumptions are natural for distributions. Indeed, by the structure theorems for distributions (see, e.g. [FJ98]), we know that every compactly supported distribution can be represented by a finite sum of (distributional) derivatives of continuous functions. Precisely, for $T \in \mathcal{E}'(\mathbb{R}^d)$ we can find $n \in \mathbb{N}$ and functions $f_a \in C(\mathbb{R}^d)$ such that $T = \sum_{|a| \leq n} \partial^a f_a$. The convolution of $T$ with a mollifier yields
\begin{equation}
T * \psi_{\varepsilon} = \sum_{|a| \leq n} \partial^a f_a * \psi_{\varepsilon} = \sum_{|a| \leq n} f_a * \partial^a \psi_{\varepsilon} = \sum_{|a| \leq n} \omega(\varepsilon)^{-|a|} f_a * \left( \omega(\varepsilon)^{-1} \partial^a \psi(x/\omega(\varepsilon)) \right).
\end{equation}
It is clear that $T$ satisfies the above assumptions.

2.1. Existence of very weak solutions. In this subsection we deal with the existence of very weak solutions. We start by calling the definition of the moderateness.

Definition 1 (Moderateness). Let $X$ be a Banach space with the norm $\| \cdot \|_X$. Then we say that a net of functions $(f_{\varepsilon})_\varepsilon$ from $X$ is $X$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that
\begin{equation}
\|f_{\varepsilon}\|_X \leq c \omega(\varepsilon)^{-N}.
\end{equation}
In what follows, we will use particular cases of $X$. Namely, $H^1$-moderate, $L^\infty$-moderate, and $C([0, T]; H^1)$-moderate families. For the last, we will shortly write $C$-moderate.

Remark 2.3. By assumptions, $(u_{0, \varepsilon})_\varepsilon$ and $(q_{\varepsilon})_\varepsilon$ are moderate.

Now we will fix a notation. By writing $q \geq 0$, we mean that all regularisations $q_{\varepsilon}$ in our calculus are non-negative functions.

Definition 2. Let $q \geq 0$. The net $(u_{\varepsilon})_\varepsilon$ is said to be a very weak solution to the Cauchy problem (2.1), if there exist an $L^\infty$-moderate regularisation $(q_{\varepsilon})_\varepsilon$ of the coefficient $q$ and $H^1$-moderate regularisation $(u_{0, \varepsilon})_\varepsilon$ of the initial function $u_0$, such that $(u_{\varepsilon})_\varepsilon$ solves the regularized equation
\begin{equation}
\partial_t u_{\varepsilon}(t, x) - \Delta u_{\varepsilon}(t, x) + q_{\varepsilon}(x)u_{\varepsilon}(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d,
\end{equation}
with the Cauchy data $u_{\varepsilon}(0, x) = u_{0, \varepsilon}(x)$, for all $\varepsilon \in (0, 1]$, and is $C$-moderate.

With this setup the existence of a very weak solution becomes straightforward. But we will also analyse its properties later on.

Theorem 2.4 (Existence of a very weak solution). Let $q \geq 0$. Assume that the regularisations of the coefficient $q$ and the Cauchy data $u_0$ satisfy the assumptions (2.12) and (2.13). Then the Cauchy problem (2.1) has a very weak solution.
Proof. Using the moderateness assumptions (2.12), (2.13), and the energy estimate (2.2), we arrive at
\[
\|u_\varepsilon(t, \cdot)\| \lesssim \omega(\varepsilon)^{-N} \times \omega(\varepsilon)^{-N_0} \\
\lesssim \omega(\varepsilon)^{-N-N_0},
\]
concluding that \((u_\varepsilon)_x\) is \(C\)-moderate. \(\blacksquare\)

2.2. Uniqueness results. In this subsection we discuss uniqueness of the very weak solution to the Cauchy problem (2.1) for different cases of regularity of the potential \(q\).

2.2.1. The classical case. In the case when \(q \in C^\infty(\mathbb{R}^d)\), we require further conditions on the mollifiers, to ensure the uniqueness.

Definition 3.
- We denote by \(\mathbb{A}_n\), the set of mollifiers defined by
\[
(2.15) \quad \mathbb{A}_n = \left\{ \text{Friedrichs-mollifiers } \psi : \int_{\mathbb{R}^d} x^k \psi(x) dx = 0 \text{ for } 1 \leq k \leq n \right\}.
\]
- We say that \(\psi \in \mathbb{A}_\infty\), if \(\psi \in \mathbb{A}_n\) for all \(n \in \mathbb{N}\).

Remark 2.4. To construct such sets of mollifiers, we consider a Friedrichs-mollifier \(\Phi(x) = a_0\psi(x) + a_1\psi'(x) + ... + a_{n-1}\psi^{n-1}(x)\), where the constants \(a_0, ..., a_{n-1}\) are determined by the conditions in (2.15).

Lemma 2.5. For \(N \in \mathbb{N}\), let \(\psi \in \mathbb{A}_{N-1}\) and assume that \(q \in C^\infty(\mathbb{R}^d)\). Then, the estimate
\[(2.16) \quad |q_\varepsilon(x) - q(x)| \leq C\omega^{N+d-1}(\varepsilon)\]
holds true for all \(x \in \mathbb{R}^d\).

Proof. Let \(x \in \mathbb{R}^d\). We have
\[
|q_\varepsilon(x) - q(x)| \leq \omega^{-1}(\varepsilon) \int_{\mathbb{R}^d} |q(y) - q(x)|\psi \left(\omega^{-1}(\varepsilon)(y-x)\right) dy.
\]
Making the change \(z = \omega^{-1}(\varepsilon)(y-x)\), we get
\[
|q_\varepsilon(x) - q(x)| \leq \omega^{d-1}(\varepsilon) \int_{\mathbb{R}^d} |q(x + \omega(\varepsilon)z) - q(x)|\psi(z)dz.
\]
Expanding \(q\) to order \(N-1\), we get
\[
q(x + \omega(\varepsilon)z) - q(x) = \sum_{k=0}^{N} \frac{1}{(k-1)!} D^{(k-1)}q(x)(\omega(\varepsilon)z)^{k-1} + O(\omega^N(\varepsilon)).
\]
We get our estimate provided that the first \(N-1\) moments of the mollifier \(\psi\) vanish, finishing the proof of the lemma. \(\blacksquare\)
Definition 4. We say that the very weak solution to the Cauchy problem (2.1) is unique, if for all \( \psi, \tilde{\psi} \in \mathbb{A}_\infty \), such that
\[
(2.17) \quad \| u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon} \|_{L^2} \lesssim \omega^k(\varepsilon) \quad \text{and} \quad \| q_{\varepsilon} - \tilde{q}_{\varepsilon} \|_{L^\infty} \lesssim \omega^k(\varepsilon),
\]
for all \( k > 0 \), we have
\[
\| u_{\varepsilon}(t, \cdot) - \tilde{u}_{\varepsilon}(t, \cdot) \|_{L^2} \lesssim \omega^N(\varepsilon),
\]
for all \( N \in \mathbb{N} \), where \((u_{\varepsilon})_\varepsilon\) and \((\tilde{u}_{\varepsilon})_\varepsilon\) solve, respectively, the families of the Cauchy problems
\[
\begin{align*}
\partial_t u_{\varepsilon}(t, x) - \Delta u_{\varepsilon}(t, x) + q_{\varepsilon}(x)u_{\varepsilon}(t, x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
u_{\varepsilon}(0, x) &= u_{0,\varepsilon}(x),
\end{align*}
\]
and
\[
\begin{align*}
\partial_t \tilde{u}_{\varepsilon}(t, x) - \Delta \tilde{u}_{\varepsilon}(t, x) + \tilde{q}_{\varepsilon}(x)\tilde{u}_{\varepsilon}(t, x) &= 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
\tilde{u}_{\varepsilon}(0, x) &= \tilde{u}_{0,\varepsilon}(x).
\end{align*}
\]
Also, the families of functions satisfying the properties (2.17), we call \( \mathbb{A}_\infty \)-negligible initial functions and coefficients, respectively.

Remark 2.5. We note that for any two \( \psi, \tilde{\psi} \in \mathbb{A}_\infty \) the difference of the corresponding regularisations of the coefficient \( q \in C^\infty(\mathbb{R}^d) \) is an \( \mathbb{A}_\infty \)-negligible function, that is,
\[
\| q_{\varepsilon} - \tilde{q}_{\varepsilon} \|_{L^\infty} \lesssim \omega^k(\varepsilon),
\]
for all \( k > 0 \), for all \( \varepsilon \in (0, 1] \). Moreover, \((q_{\varepsilon} - \tilde{q}_{\varepsilon})_{\varepsilon \in (0, 1]}\) is also an \( \mathbb{A}_\infty \)-negligible family of functions.

Theorem 2.6. Let \( T > 0 \). Assume that a non-negative function \( q \in C^\infty(\mathbb{R}^d) \) and \( u_0 \in H^1(\mathbb{R}^d) \) satisfy the conditions (2.12) and (2.13), respectively. Then, the very weak solution of the Cauchy problem (2.1) is unique.

Proof. Let \( \psi, \tilde{\psi} \in \mathbb{A}_\infty \) and consider \((q_{\varepsilon})_\varepsilon\), \((\tilde{q}_{\varepsilon})_\varepsilon\) and \((u_{0,\varepsilon})_\varepsilon\), \((\tilde{u}_{0,\varepsilon})_\varepsilon\) the regularisations of the coefficient \( q \) and the data \( u_0 \) with respect to \( \psi \) and \( \tilde{\psi} \). Assume that
\[
(2.18) \quad \| u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon} \|_{L^2} \leq C_k \omega^k(\varepsilon),
\]
for all \( k > 0 \). Then, \( u_{\varepsilon} \) and \( \tilde{u}_{\varepsilon} \), the solutions to the related Cauchy problems, satisfy the equation
\[
(2.19) \quad \begin{cases}
\partial_t(u_{\varepsilon} - \tilde{u}_{\varepsilon})(t, x) - \Delta(u_{\varepsilon} - \tilde{u}_{\varepsilon})(t, x) + q_{\varepsilon}(x)(u_{\varepsilon} - \tilde{u}_{\varepsilon})(t, x) = f_{\varepsilon}(t, x), \\
(u_{\varepsilon} - \tilde{u}_{\varepsilon})(0, x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x),
\end{cases}
\]
with
\[
f_{\varepsilon}(t, x) = (\tilde{q}_{\varepsilon}(x) - q_{\varepsilon}(x))\tilde{u}_{\varepsilon}(t, x).
\]
Let us denote by \( U_{\varepsilon}(t, x) := u_{\varepsilon}(t, x) - \tilde{u}_{\varepsilon}(t, x) \) the solution to the problem (2.19). Using Duhamel’s principle, \( U_{\varepsilon} \) is given by
\[
U_{\varepsilon}(t, x) = W_{\varepsilon}(t, x) + \int_0^t V_{\varepsilon}(x, t - s; s)ds,
\]
where $W_{\varepsilon}(t, x)$ is the solution to the problem

$$
\begin{cases}
\partial_t W_{\varepsilon}(t, x) - \Delta W_{\varepsilon}(t, x) + q_{\varepsilon}(x)W_{\varepsilon}(t, x) = 0, \\
W_{\varepsilon}(0, x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x),
\end{cases}
$$

and $V_{\varepsilon}(x, t; s)$ solves

$$
\begin{cases}
\partial_t V_{\varepsilon}(x, t; s) - \Delta V_{\varepsilon}(x, t; s) + q_{\varepsilon}(x)V_{\varepsilon}(x, t; s) = 0, \\
V_{\varepsilon}(x, 0; s) = f_{\varepsilon}(s, x).
\end{cases}
$$

Taking $U_{\varepsilon}$ in $L^2$-norm and using (2.10) to estimate $V_{\varepsilon}$ and $W_{\varepsilon}$, we arrive at

$$
\|U_{\varepsilon}(t, \cdot)\|_{L^2} \leq \|W_{\varepsilon}(t, \cdot)\|_{L^2} + \int_0^T \|V_{\varepsilon}(\cdot, t-s)\|_{L^2} ds
\leq \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \int_0^T \|f_{\varepsilon}(s, \cdot)\|_{L^2} ds
\leq \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{q}_{\varepsilon} - q_{\varepsilon}\|_{L^\infty} \int_0^T \|\tilde{u}_{\varepsilon}(s, \cdot)\|_{L^2} ds.
$$

The net $(\tilde{u}_{\varepsilon})_\varepsilon$ is moderate, the uniqueness of the very weak solution follows by the assumption that $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_{\varepsilon \in (0, 1]}$ is an $A_\infty$-negligible family of initial functions, that is,

$$
\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_k \varepsilon^k \quad \text{for all } k > 0,
$$

the application of Lemma 2.5 and Remark (2.5) due to the $A_\infty$-neglibly of the family of coefficients $\tilde{q}_{\varepsilon}$ and $q_{\varepsilon}$. This ends the proof of the theorem. \qed

### 2.2.2. The singular case

In the case when $q$ is singular, we prove uniqueness in the sense of the following definition.

**Definition 5.** We say that the very weak solution to the Cauchy problem (2.1) is unique, if for all families $(q_{\varepsilon})_\varepsilon$, $(\tilde{q}_{\varepsilon})_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, regularisations of the coefficient $q$ and $u_0$, satisfying

$$
\|q_{\varepsilon} - \tilde{q}_{\varepsilon}\|_{L^\infty} \leq C_k \varepsilon^k \quad \text{for all } k > 0
$$

and

$$
\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \quad \text{for all } l > 0,
$$

then

$$
\|u_{\varepsilon}(t, \cdot) - \tilde{u}_{\varepsilon}(t, \cdot)\|_{L^2} \leq C_N \varepsilon^N,
$$

for all $N > 0$, where $(u_{\varepsilon})_\varepsilon$ and $(\tilde{u}_{\varepsilon})_\varepsilon$ solve, respectively, the families of the Cauchy problems

$$
\begin{cases}
\partial_t u_{\varepsilon}(t, x) - \Delta u_{\varepsilon}(t, x) + q_{\varepsilon}(x)u_{\varepsilon}(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
u_{\varepsilon}(0, x) = u_{0,\varepsilon}(x),
\end{cases}
$$

and

$$
\begin{cases}
\partial_t \tilde{u}_{\varepsilon}(t, x) - \Delta \tilde{u}_{\varepsilon}(t, x) + \tilde{q}_{\varepsilon}(x)\tilde{u}_{\varepsilon}(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
\tilde{u}_{\varepsilon}(0, x) = \tilde{u}_{0,\varepsilon}(x).
\end{cases}
$$
Theorem 2.7. Let \( T > 0 \). Assume that \( q \geq 0 \) and \( u_0 \in H^1(\mathbb{R}^d) \) satisfy the moderateness assumptions (2.12) and (2.13), respectively. Then, the very weak solution to the Cauchy problem (2.1) is unique.

Proof. Let \((q_\varepsilon)_\varepsilon, (\tilde{q}_\varepsilon)_\varepsilon \) and \((u_{0,\varepsilon})_\varepsilon, (\tilde{u}_{0,\varepsilon})_\varepsilon \), regularisations of the coefficient \( q \) and the data \( u_0 \), satisfying
\[
\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k, \quad \text{for all } k > 0,
\]
and
\[
\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l, \quad \text{for all } l > 0.
\]

Then, \((u_\varepsilon)_\varepsilon \) and \((\tilde{u}_\varepsilon)_\varepsilon \), the solutions to the related Cauchy problems, satisfy
\[
(2.20) \quad \left\{ \begin{array}{l}
\partial_t (u_\varepsilon - \tilde{u}_\varepsilon)(t, x) - \Delta (u_\varepsilon - \tilde{u}_\varepsilon)(t, x) + q_\varepsilon(x)(u_\varepsilon - \tilde{u}_\varepsilon)(t, x) = f_\varepsilon(t, x), \\
(u_\varepsilon - \tilde{u}_\varepsilon)(0, x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x),
\end{array} \right.
\]
with
\[
f_\varepsilon(t, x) = (\tilde{q}_\varepsilon(x) - q_\varepsilon(x))\tilde{u}_\varepsilon(t, x).
\]

Let us denote by \( U_\varepsilon(t, x) \) := \( u_\varepsilon(t, x) - \tilde{u}_\varepsilon(t, x) \) the solution to the equation (2.20). Using similar arguments as in Theorem 2.6, we get
\[
\|U_\varepsilon(t, \cdot)\|_{L^2} \lesssim \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|\tilde{q}_\varepsilon - q_\varepsilon\|_{L^\infty} \int_0^T \|\tilde{u}_\varepsilon(s, \cdot)\|_{L^2}.
\]

The family \((\tilde{u}_\varepsilon)_\varepsilon \) is a very weak solution to the Cauchy problem (2.1), it is then moderate, i.e. there exists \( N_0 \in \mathbb{N} \) such that
\[
\|\tilde{u}_\varepsilon(s, \cdot)\|_{L^2} \leq c \varepsilon^{-N_0}(\varepsilon).
\]

On the other hand, we have that \( \|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \), for all \( k > 0 \), and \( \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_l \varepsilon^l \), for all \( l > 0 \). Thus, we obtain that
\[
\|U_\varepsilon(t, \cdot)\|_{L^2} \triangleright \|u_\varepsilon(t, \cdot) - \tilde{u}_\varepsilon(t, \cdot)\|_{L^2} \lesssim \varepsilon^N,
\]
for all \( N > 0 \), showing the uniqueness of the very weak solution. \( \square \)

2.3. Consistency with the classical case. Now we show that if the classical solution of the Cauchy problem (2.1) given by Lemma 2.1 exists then the very weak solution recaptures it.

Theorem 2.8. Let \( u_0 \in H^1(\mathbb{R}^d) \). Assume that \( q \in L^\infty(\mathbb{R}^d) \) is non-negative and consider the Cauchy problem
\[
(2.21) \quad \left\{ \begin{array}{l}
u(t, x) - \Delta u(t, x) + q(x)u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^d, \\
u(0, x) = u_0(x).
\end{array} \right.
\]

Let \( (u_\varepsilon)_\varepsilon \) be a very weak solution of (2.21). Then, for any regularising families \((q_\varepsilon)_\varepsilon \) and \((u_{0,\varepsilon})_\varepsilon \), the net \( (u_\varepsilon)_\varepsilon \) converges in \( L^2 \) as \( \varepsilon \to 0 \) to the classical solution of the Cauchy problem (2.21).
Proof. Consider the classical solution \( u \) to
\[
\begin{cases}
  u_t(t, x) - \Delta u(t, x) + q(x)u(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}^d, \\
  u(0, x) = u_0(x).
\end{cases}
\]
Note that for the very weak solution there is a representation \((u_\varepsilon)_t\) such that
\[
\begin{cases}
  \partial_t u_\varepsilon(t, x) - \Delta u_\varepsilon(t, x) + q_\varepsilon(x)u_\varepsilon(t, x) = 0, & (t, x) \in [0, T] \times \mathbb{R}^d, \\
  u_\varepsilon(0, x) = u_{0,\varepsilon}(x).
\end{cases}
\]
Taking the difference, we get
\[
(2.22)
\begin{align*}
  \partial_t (u - u_\varepsilon)(t, x) - \Delta (u - u_\varepsilon)(t, x) + q_\varepsilon(x)(u - u_\varepsilon)(t, x) = \eta_\varepsilon(t, x), \\
  (u - u_\varepsilon)(0, x) = (u_0 - u_{0,\varepsilon})(x),
\end{align*}
\]
where
\[
\eta_\varepsilon(t, x) = (q_\varepsilon(x) - q(x))u(t, x).
\]
Let us denote \( U_\varepsilon(t, x) := (u - u_\varepsilon)(t, x) \) and let \( W_\varepsilon(t, x) \) be the solution to the auxiliary homogeneous problem
\[
\begin{cases}
  \partial_t W_\varepsilon(t, x) - \Delta W_\varepsilon(t, x) + q_\varepsilon(x)W_\varepsilon(t, x) = 0, \\
  W_\varepsilon(0, x) = (u_0 - u_{0,\varepsilon})(x).
\end{cases}
\]
Then, by Duhamel’s principle, the solution to (2.22) is given by
\[
(2.23)
U_\varepsilon(t, x) = W_\varepsilon(t, x) + \int_0^t V_\varepsilon(x, t - s; s) ds,
\]
where \( V_\varepsilon(x, t; s) \) is the solution to the problem
\[
\begin{cases}
  \partial_t V_\varepsilon(x, t; s) - \Delta V_\varepsilon(x, t; s) + q_\varepsilon(x)V_\varepsilon(x, t; s) = 0, \\
  V_\varepsilon(x, 0; s) = \eta_\varepsilon(t, x).
\end{cases}
\]
As in Theorem 2.7, taking the \( L^2 \)-norm in (2.23) and using (2.10) to estimate \( V_\varepsilon \) and \( W_\varepsilon \), we get
\[
\|U_\varepsilon(t, \cdot)\|_{L^2} \leq \|W_\varepsilon(t, \cdot)\|_{L^2} + \int_0^T \|V_\varepsilon(\cdot, t - s; s)\|_{L^2} ds
\leq \|u_0 - u_{0,\varepsilon}\|_{L^2} + \int_0^\infty \|\eta_\varepsilon(s, \cdot)\|_{L^2} ds
\leq \|u_0 - u_{0,\varepsilon}\|_{L^2} + \|q_\varepsilon - q\|_{L^\infty} \int_0^T \|u(s, \cdot)\|_{L^2} ds,
\]
and taking into account that
\[
\|q_\varepsilon - q\|_{L^\infty} \to 0 \quad \text{as} \quad \varepsilon \to 0
\]
and
\[
\|u_{0,\varepsilon} - u_0\|_{L^2} \to 0 \quad \text{as} \quad \varepsilon \to 0,
\]
consequently, it implies that \( u_\varepsilon \) converges to \( u \) in \( L^2 \) as \( \varepsilon \to 0 \).\qed
3. PART II: NEGATIVE POTENTIAL

In this part we aim to study the case when the potential is negative and to show that the problem is still well-posed. Namely, we consider the Cauchy problem for the heat equation

\[
\begin{aligned}
& \partial_t u(t, x) - \Delta u(t, x) - q(x)u(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
& u(0, x) = u_0(x),
\end{aligned}
\]

where \( q \) is non-negative.

In the classical case, we have the following energy estimates for the solution of the problem (3.1).

**Lemma 3.1.** Let \( u_0 \in L^2(\mathbb{R}^d) \) and suppose that \( q \in L^\infty(\mathbb{R}^d) \) is non-negative. Then, there is a unique solution \( u \in C([0, T]; L^2) \) to (3.1) and it satisfies the estimate

\[
\|u(t, \cdot)\|_{L^2} \lesssim \exp \left( t\|q\|_{L^\infty} \right)\|u_0\|_{L^2},
\]

for all \( t \in [0, T] \).

**Proof.** Multiplying the equation in (3.1) by \( u \), integrating with respect to \( x \), and taking the real part, we obtain

\[
\text{Re} \left( \langle u_t(t, \cdot), u(t, \cdot) \rangle_{L^2} + \langle -\Delta u(t, \cdot), u(t, \cdot) \rangle_{L^2} - \langle q(\cdot)u(t, \cdot), u(t, \cdot) \rangle_{L^2} \right) = 0,
\]

for all \( t \in [0, T] \). Using similar arguments as in Lemma 2.1 and noting that the term \( \|q(\cdot)u(t, \cdot)\|_{L^2} \) can be estimated by \( \|q\|_{L^\infty}\|u(t, \cdot)\|_{L^2} \), we get

\[
\partial_t\|u(t, \cdot)\|_{L^2} \lesssim \|q\|_{L^\infty}\|u(t, \cdot)\|_{L^2},
\]

for all \( t \in [0, T] \). The desired estimate follows by the application of Gronwall’s lemma. \( \Box \)

Let now assume that the potential \( q \) and the initial data \( u_0 \) are singular. Consider the Cauchy problem for the heat equation

\[
\begin{aligned}
& \partial_t u(t, x) - \Delta u(t, x) - q(x)u(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
& u(0, x) = u_0(x),
\end{aligned}
\]

In order to prove the existence of a very weak solution to (3.3), we proceed as in the case of the positive potential. We start by regularising the equation in (3.3). In other words, using

\[
\psi_\varepsilon(x) = \omega(\varepsilon)^{-1}\psi(x/\omega(\varepsilon)), \quad \varepsilon \in (0, 1],
\]

where \( \psi \) is a Friedrichs mollifier and \( \omega \) is a positive function converging to 0 as \( \varepsilon \to 0 \), to be chosen later, we regularise \( q \) and \( u_0 \) obtaining the nets \((q_\varepsilon)_\varepsilon = (q * \psi_\varepsilon)_\varepsilon \) and \((u_{0,\varepsilon})_\varepsilon = (u_0 * \psi_\varepsilon)_\varepsilon \). For this, we can assume that \( q \) and \( u_0 \) are distributions.

**Assumption 3.2.** We assume that there exist \( N_0, N_1 \in \mathbb{N}_0 \) such that

\[
\|q_\varepsilon\|_{L^\infty} \leq C_0\omega(\varepsilon)^{-N_0},
\]

and

\[
\|u_{0,\varepsilon}\|_{L^2} \leq C_1\omega(\varepsilon)^{-N_1}.
\]
3.1. Existence of very weak solutions. In this subsection we give the definition of a very weak solution adapted to the problem (3.3). For this, we will make use of the same definition of the moderateness as in the non-negative case. Nevertheless, let us recall it here.

**Definition 6 (Moderateness).** Let $X$ be a Banach space with the norm $\| \cdot \|_X$. Then we say that a net of functions $(f_\varepsilon)_\varepsilon$ from $X$ is $X$-moderate, if there exist $N \in \mathbb{N}_0$ and $c > 0$ such that

$$\|f_\varepsilon\|_X \leq c \omega(\varepsilon)^{-N}.$$ 

In what follows, we will use particular cases of $X$. Namely, $L^2$-moderate, $L^\infty$-moderate, and $C([0, T] ; L^2)$-moderate families. For the last, we will shortly write $C$-moderate.

**Definition 7.** Let $q$ be non-negative. Then the net $(u_\varepsilon)_\varepsilon$ is said to be a very weak solution to the problem (3.3), if there exist an $L^\infty$-moderate regularisation $(q_\varepsilon)_\varepsilon$ of the coefficient $q$ and an $L^2$-moderate regularisation $(u_{0,\varepsilon})_\varepsilon$ of $u_0$ such that $(u_\varepsilon)_\varepsilon$ solves the regularized problem

$$(3.6) \quad \begin{cases} \partial_t u_\varepsilon(t, x) - \Delta u_\varepsilon(t, x) - q_\varepsilon(x) u_\varepsilon(t, x) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u_\varepsilon(0, x) = u_{0,\varepsilon}(x), \end{cases}$$

for all $\varepsilon \in (0, 1]$, and is $C$-moderate.

**Theorem 3.3 (Existence of a very weak solution).** Let $q \geq 0$. Assume that the nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ satisfy the assumptions (3.4) and (3.5), respectively. Then the problem (3.3) has a very weak solution.

**Proof.** The nets $(q_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$ are moderate by the assumption. To prove that a very weak solution to the Cauchy problem (3.3) exists, we need to show that the net $(u_\varepsilon)_\varepsilon$, a solution to the regularized problem (3.6), is $C$-moderate. Indeed, using the assumptions (3.4), (3.5) and the estimate (3.2), we get

$$\|u(t, \cdot)\|_{L^2} \lesssim \exp \left( t \omega(\varepsilon)^{-N_0} \right) \omega(\varepsilon)^{-N_1},$$

for all $t \in [0, T]$. Choosing $\omega(\varepsilon) = \left( \log \varepsilon^{-N_0} \right)^{-\frac{1}{N_0}}$, we obtain that

$$\|u(t, \cdot)\|_{L^2} \lesssim \varepsilon^{-tN_0} \times \left( \log \varepsilon^{-N_0} \right)^{\frac{N_1}{N_0}} \lesssim \varepsilon^{-TN_0} \times \varepsilon^{-N_1},$$

where the fact that $t \in [0, T]$ and that $\log \varepsilon^{-N_0}$ can be estimated by $\varepsilon^{-N_0}$ are used. Then the net $(u_\varepsilon)_\varepsilon$ is $C$-moderate, implying the existence of very weak solutions. \hfill \Box

3.2. Uniqueness results. Here, we prove the uniqueness of the very weak solution to the heat equation with a non-positive potential (3.3) in the spirit of Definition 5, adapted to our problem.

**Definition 8.** Let the regularisations $(q_\varepsilon)_\varepsilon$ and $(\tilde{q}_\varepsilon)_\varepsilon$ of $q$ and the regularisations $(u_{0,\varepsilon})_\varepsilon$ and $(\tilde{u}_{0,\varepsilon})_\varepsilon$ of $u_0$ satisfy Assumption 3.2. Then we say that the very weak solution to the heat equation (3.3) is unique, if for all families $(q_\varepsilon)_\varepsilon$, $(\tilde{q}_\varepsilon)_\varepsilon$ and $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$, satisfying

$$\|q_\varepsilon - \tilde{q}_\varepsilon\|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0,$$

then the very weak solution $u_\varepsilon$ is unique. 

\end{document}
and
\[ \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_I \varepsilon^l \text{ for all } l > 0, \]
we have
\[ \|u_{\varepsilon}(t, \cdot) - \tilde{u}_{\varepsilon}(t, \cdot)\|_{L^2} \leq C_N \varepsilon^N \]
for all \( N > 0 \), where \((u_{\varepsilon})_\varepsilon\) and \((\tilde{u}_{\varepsilon})_\varepsilon\) solve, respectively, the families of the Cauchy problems
\[
\begin{cases}
\partial_t u_{\varepsilon}(t, x) - \Delta u_{\varepsilon}(t, x) - q_{\varepsilon}(x)u_{\varepsilon}(t, x) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
u_{\varepsilon}(0, x) = u_{0,\varepsilon}(x),
\end{cases}
\]
and
\[
\begin{cases}
\partial_t \tilde{u}_{\varepsilon}(t, x) - \Delta \tilde{u}_{\varepsilon}(t, x) - \tilde{q}_{\varepsilon}(x)\tilde{u}_{\varepsilon}(t, x) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
\tilde{u}_{\varepsilon}(0, x) = \tilde{u}_{0,\varepsilon}(x).
\end{cases}
\]

**Theorem 3.4.** Let \( T > 0 \). Assume that the nets \((q_{\varepsilon})_\varepsilon\) and \((u_{0,\varepsilon})_\varepsilon\) satisfy the assumptions (3.4) and (3.5), respectively. Then, the very weak solution to the Cauchy problem (3.3) is unique.

**Proof.** Let us consider \((q_{\varepsilon})_\varepsilon\), \((\tilde{q}_{\varepsilon})_\varepsilon\) and \((u_{0,\varepsilon})_\varepsilon\), \((\tilde{u}_{0,\varepsilon})_\varepsilon\), regularisations of the \(q\) and \(u_0\), satisfying
\[ \|q_{\varepsilon} - \tilde{q}_{\varepsilon}\|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0 \]
and
\[ \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_I \varepsilon^l \text{ for all } l > 0. \]
Then, \((u_{\varepsilon})_\varepsilon\) and \((\tilde{u}_{\varepsilon})_\varepsilon\), the solutions to the related Cauchy problems, satisfy
\begin{equation}
\begin{cases}
\partial_t (u_{\varepsilon} - \tilde{u}_{\varepsilon})(t, x) - \Delta (u_{\varepsilon} - \tilde{u}_{\varepsilon})(t, x) - q_{\varepsilon}(x)(u_{\varepsilon} - \tilde{u}_{\varepsilon})(t, x) = f_{\varepsilon}(t, x), \\
(u_{\varepsilon} - \tilde{u}_{\varepsilon})(0, x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x),
\end{cases}
\end{equation}
with
\[ f_{\varepsilon}(t, x) = (q_{\varepsilon}(x) - \tilde{q}_{\varepsilon}(x))\tilde{u}_{\varepsilon}(t, x). \]
Let us denote by \(U_{\varepsilon}(t, x) := u_{\varepsilon}(t, x) - \tilde{u}_{\varepsilon}(t, x)\) the solution to the equation (3.7). Arguing as in Theorem 2.6 and using the estimate (3.2), we arrive at
\[
\|U_{\varepsilon}(t, \cdot)\|_{L^2} \lesssim \exp \left( t\|q_{\varepsilon}\|_{L^\infty} \right)\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} + \|q_{\varepsilon} - \tilde{q}_{\varepsilon}\|_{L^\infty} \int_0^T \exp \left( s\|q_{\varepsilon}\|_{L^\infty} \right)\|\tilde{u}_{\varepsilon}(s, \cdot)\|_{L^2} ds.
\]
On the one hand, the net \((q_{\varepsilon})_\varepsilon\) is moderate by the assumption and \((\tilde{u}_{\varepsilon})_\varepsilon\) is moderate as a very weak solution. From the other hand, we have that
\[ \|q_{\varepsilon} - \tilde{q}_{\varepsilon}\|_{L^\infty} \leq C_k \varepsilon^k \text{ for all } k > 0, \]
and
\[ \|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{L^2} \leq C_I \varepsilon^l \text{ for all } l > 0. \]
By choosing \(\omega(\varepsilon) = (\log \varepsilon^{-N_0})^{-\frac{1}{N_0}}\) for \(q_{\varepsilon}\) in (3.4), it follows that
\[ \|U_{\varepsilon}(t, \cdot)\|_{L^2} = \|u_{\varepsilon}(t, \cdot) - \tilde{u}_{\varepsilon}(t, \cdot)\|_{L^2} \lesssim \varepsilon^N, \]
for all \( N > 0 \), ending the proof. \( \square \)
3.3. **Consistency with the classical case.** We conclude this section by showing that if the coefficient and the Cauchy data are regular then the very weak solution coincides with the classical one, given by Lemma 3.1.

**Theorem 3.5.** Let $u_0 \in L^2(\mathbb{R}^d)$. Assume that $q \in L^\infty(\mathbb{R}^d)$ is non-negative and consider the Cauchy problem for the heat equation

$$
\begin{cases}
  u_t(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\
  u(0,x) = u_0(x).
\end{cases}
$$

(3.8)

Let $(u_\epsilon)_\epsilon$ be a very weak solution of the heat equation (3.8). Then, for any regularising families $(q_\epsilon)_\epsilon$ and $(u_0,\epsilon)_\epsilon$, the net $(u_\epsilon)_\epsilon$ converges in $L^2$ as $\epsilon \to 0$ to the classical solution of the Cauchy problem (3.8).

**Proof.** Let us denote the classical solution and the very weak one by $u$ and $(u_\epsilon)_\epsilon$, respectively. It is clear, that they satisfy

$$
\begin{cases}
  u_t(t,x) - \Delta u(t,x) - q(x)u(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\
  u(0,x) = u_0(x),
\end{cases}
$$

and

$$
\begin{cases}
  \partial_t u_\epsilon(t,x) - \Delta u_\epsilon(t,x) - q_\epsilon(x)u_\epsilon(t,x) = 0, & (t,x) \in (0,T) \times \mathbb{R}^d, \\
  u_\epsilon(0,x) = u_0,\epsilon(x),
\end{cases}
$$

respectively. Let us denote by $V_\epsilon(t,x) := (u_\epsilon - u)(t,x)$. Using the estimate (3.2) and the same arguments as in the positive potential case, we show that

$$
\|V_\epsilon(t,\cdot)\|_{L^2} \lesssim \exp \left( t\|q_\epsilon\|_{L^\infty} \right) \|u_0,\epsilon - u_0\|_{L^2} + \|q_\epsilon - q\|_{L^\infty} \int_0^T \exp \left( s\|q_\epsilon\|_{L^\infty} \right) \|u(s,\cdot)\|_{L^2} ds.
$$

By taking into account that

$$
\|q_\epsilon - q\|_{L^\infty} \to 0 \quad \text{as} \quad \epsilon \to 0
$$

and

$$
\|u_0,\epsilon - u_0\|_{L^2} \to 0 \quad \text{as} \quad \epsilon \to 0,
$$

from the other hand, due to the facts $q_\epsilon$ is bounded as a regularisation of an essentially bounded function and $\|u(s,\cdot)\|_{L^2}$ is bounded as well as $u$ is a classical solution, we conclude that $(u_\epsilon)_\epsilon$ converges to $u$ in $L^2$ as $\epsilon \to 0$. $\square$

4. **Numerical experiments**

In this Section, we do some numerical experiments. Let us analyse our problem by regularising a distributional potential $q(x)$ by a parameter $\epsilon$. We define $q_\epsilon(x) := (q \ast \varphi_\epsilon)(x)$, as the convolution with the mollifier $\varphi_\epsilon(x) = \frac{1}{\epsilon} \varphi(x/\epsilon)$, where

$$
\varphi(x) = \begin{cases}
  c \exp \left( \frac{1}{x^2 - 1} \right), & |x| < 1, \\
  0, & |x| \geq 1,
\end{cases}
$$
FIGURE 1. In these plots, we analyse behaviour of the temperature in three different cases. In the top left plot, the graphic of the initial function is given. In the further plots, we compare the temperature function $u$ which is the solution of (4.1) at $t = 2, 6, 10$ for $\epsilon = 0.2$ in three cases. Case 1 is corresponding to the potential $q$ equal to zero. Case 2 is corresponding to the case when the potential $q$ is a $\delta$-function with the support at point 40. Case 3 is corresponding to a $\delta^2$-like function potential with the support at point 40.

with $c \simeq 2.2523$ to have $\int_{-\infty}^{\infty} \varphi(x)dx = 1$. Then, instead of (2.1) we consider the regularised problem

$$
\begin{align*}
(4.1) \quad & \partial_t u_\epsilon(t, x) - \partial_x^2 u_\epsilon(t, x) + q_\epsilon(x)u_\epsilon(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},
\end{align*}
$$


with the initial data $u_t(0, x) = u_0(x)$, for all $x \in \mathbb{R}$. Here, we put

$$u_0(x) = \begin{cases} 
\exp\left(\frac{1}{(x-50)^2-0.25}\right), & |x - 50| < 0.5, \\
0, & |x - 50| \geq 0.5.
\end{cases}$$

Note that $\text{supp } u_0 \subset [49.5, 50.5]$.

In the non-negative potential case, for $q$ we consider the following cases, with $\delta$ denoting the standard Dirac’s delta-distribution:

Case 1: $q(x) = 0$ with $q_\varepsilon(x) = 0$;

Case 2: $q(x) = \delta(x - 40)$ with $q_\varepsilon(x) = \varphi_\varepsilon(x - 40)$;

Case 3: $q(x) = \delta(x - 40) \times \delta(x - 40)$. Here, we understand $q_\varepsilon(x)$ as follows $q_\varepsilon(x) = \left(\varphi_\varepsilon(x - 40)\right)^2$;

In Figure 1, we study behaviour of the temperature function $u$ which is the solution of (4.1) at $t = 2, 6, 10$ for $\varepsilon = 0.2$ in three cases: the first case is corresponding to the potential $q$ equal to zero; the second case is corresponding to the case when the potential $q$ is a $\delta$-function with the support at point 40; the third case is corresponding to a $\delta^2$-like function potential with the support at point 40. By comparing these cases, we observe that in the second and in the third cases a place of the support of the $\delta$-function is cooling down faster rather that zero-potential case. This phenomena can be described as a “point cooling” or "laser cooling" effect.

In Figure 2, we compare the temperature function $u$ at $t = 0.01, 1.0, 10.0$ for $\varepsilon = 0.2$ in the second and third cases: when the potential is $\delta$-like and $\delta^2$-like functions with the supports at
In these plots, we analyse behaviour of the solution of the heat equation (4.3) with the negative potential. In the top left plot, the graphic of the temperature distribution at the initial time. In the further plots, we compare the temperature function \( u \) at \( t = 1, 2, 4, 6, 10 \) for \( \varepsilon = 0.8, 0.5, 0.2 \). Here, the case of the potential with a \( \delta \)-like function behaviour with the support at point 30 is considered.

In Figures 1 and 2, we analyse the equation (4.1) with positive potentials. Now, in Figure 3, we study the following equation with negative potentials:

\[
\partial_t u_\varepsilon(t, x) - \partial_x^2 u_\varepsilon(t, x) - q_\varepsilon(x) u_\varepsilon(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R},
\]

with the same initial data \( u_0 \) as in (4.2). In these plots, we compare the temperature function \( u \) at \( t = 1, 2, 4, 6, 10 \) for \( \varepsilon = 0.8, 0.5, 0.2 \) corresponding to the potential with a \( \delta \)-like function with the support at point 30. Numerical simulations justify the theory developed in Section 3. Moreover, we observe that the negative \( \delta \)-potential case a place of the support of the \( \delta \)-function is heating up. This phenomena can be described as a "point heating" or "laser heating" effect. Also, one observes that our numerical calculations prove the behaviour of the solution related to the parameter \( \varepsilon \).

All numerical computations are made in C++ by using the sweep method. In above numerical simulations, we use the Matlab R2018b. For all simulations we take \( \Delta t = 0.2, \Delta x = 0.01 \).
4.1. **Conclusion.** The analysis conducted in this article showed that numerical methods work well in situations where a rigorous mathematical formulation of the problem is difficult in the framework of the classical theory of distributions. The concept of very weak solutions eliminates this difficulty in the case of the terms with multiplication of distributions. In particular, in the potential heat equation case, we see that a delta-function potential helps to loose/increase energy in a less time, the latter causing a so-called "laser cooling/heating" effect in the positive/negative potential cases.

Numerical experiments have shown that the concept of very weak solutions is very suitable for numerical modelling. In addition, using the theory of very weak solutions, we can talk about the uniqueness of numerical solutions of differential equations with strongly singular coefficients in an appropriate sense.

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