ON ISOMORPHISMS BETWEEN SIEGEL MODULAR THREEFOLDS

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ABSTRACT. Mukai, in his recent article "Igusa quartic and Steiner surfaces", has constructed an isomorphism between two Siegel modular threefolds which parametrize different moduli spaces and has shown that, as a consequence, the moduli space of principally polarized abelian varieties with level 2 structure, as a modular variety, has a self-morphism of degree 8.

Changing the point of view, we will see that this is in fact a special case of a more general result. In effect, we shall show that for some congruence subgroups $\Gamma$ of the integral symplectic group it is possible to construct an isomorphism between two Siegel modular threefolds. As a consequence the associated modular variety $\text{Proj}(A(\Gamma))$ has a self-morphism of degree 8.

We will also examine the action of the Fricke involution, extending the result to some subgroups contained in the Hecke group of level 4 and computing the ring of modular forms with respect to the latter group. In the last section we shall show that the construction does not generalize directly for higher genera, treating in details the genus 3 case.

1. Introduction

In genus 2 the Satake compactification of the moduli space of principally polarized abelian varieties with level 2 structure is a quartic hypersurface in $\mathbb{P}^4$, called the Igusa quartic. Recently this space has been characterized as a Steiner quartic surface and, as a corollary, it has been shown that the Satake compactification of the moduli space of principally polarized abelian varieties with Göpel triples is isomorphic to the Igusa quartic. Since there is a morphism of degree 8 among these two compactifications of different moduli spaces, the Igusa quartic has a self-morphism of degree 8 $[\text{Mu}]$.

We shall show that this is a special case of a more general result. Indeed we will construct an isomorphism of graded rings of modular forms inducing a self-morphism of degree 8 of the associated modular varieties. Let $\mathbb{H}_2$ be the space of symmetric complex $2 \times 2$ matrices with positive imaginary part, we can consider the action of the integral symplectic group $\text{Sp}(2,\mathbb{Z})$ on this space. There are interesting functions defined on $\mathbb{H}_2$, usually called Siegel modular forms, namely the functions there are in some sense invariant under the action of $\text{Sp}(2,\mathbb{Z})$.

In what follows we’ll first consider the ring of Siegel modular forms with multiplier, with respect to the following two congruence subgroup of level 4 of the integral symplectic group. One is the well known subgroup $\Gamma(2,4) = \{ \gamma = (A\ B) \in \text{Sp}(2,\mathbb{Z}) \mid \gamma \equiv 1_4 \pmod{2}, \text{diag}(B) \equiv \text{diag}(C) \equiv 0 \pmod{4} \}$, the other is a group obtained by $\Gamma(2,4)$ by relaxing the condition for the $B$ block, we’ll denote it by $\Gamma^1(2,4)$.

Considering these rings of modular forms, it will be possible to show that the degree 8 map

$$[x_0, x_1, x_2, x_3] \mapsto [x_0^2, x_1^2, x_2^2, x_3^2]$$
can be regarded as a map between the modular varieties associated to the two groups
\[
\psi: \text{Proj}(A(\Gamma'(2,4))) \to \text{Proj}(A(\Gamma(2,4))).
\]

The principal result of this paper involves two more groups, already known in literature, namely the groups
\[
\Gamma_0(2) = \{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2,\mathbb{Z}) \mid C \equiv 0 \pmod{2} \}, \\
\Gamma_0^0(2) = \{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2,\mathbb{Z}) \mid C \equiv B \equiv 0 \pmod{2} \}.
\]

We shall prove that there is an isomorphism
\[
\Gamma_0(2)/\Gamma_1(2,4) \cong \Gamma_0^0(2)/\Gamma(2,4).
\]
Denoting this group by \(G\) and observing that it acts in the same way on the two copies of \(\mathbb{P}^3\) in (\(\text{I}\)), we'll establish the following

**Theorem.** For any subgroup \(H \subset G\) exist two groups \(\Gamma', \Gamma\) such that
\[
\Gamma(2,4) \subset \Gamma' \subset \Gamma_0^0(2) \\
\Gamma^1(2,4) \subset \Gamma \subset \Gamma_0(2)
\]
and the quotients \(\Gamma'/\Gamma(2,4)\) and \(\Gamma/\Gamma^1(2,4)\) are both isomorphic to \(H\). Hence, it can be constructed an isomorphism of graded ring of modular forms
\[
\Phi_H: A(\Gamma') \to A(\Gamma),
\]
that moreover increases the weights by a factor two.

Let’s observe that the two groups considered in [Mu] are the principal congruence subgroup of level two \(\Gamma_2(2)\) and the group of integral symplectic matrices in block form \(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\) such that \(A \equiv 1_2 \pmod{2}\) and \(C \equiv 0 \pmod{2}\). Hence the statement of the theorem applies to these groups and we recover Mukai’s result from the one above.

As a corollary we have the following:

**Corollary.** For any subgroup \(\Gamma(2,4) \subset \Gamma' \subset \Gamma_0^0(2)\) the modular variety \(\text{Proj}(A(\Gamma'))\) has a map of degree 8 onto itself.

Moreover we will analyze the action of the Fricke involution in genus 2 on the groups introduced so far, we’ll find that only the groups \(\Gamma^1(2,4)\) and \(\Gamma_0(2)\) are fixed while the group \(\Gamma_0^0(2)\) is sent to the Hecke group \(\Gamma_0(4)\). From here, an analog of the above theorem can be shown for a subgroup of a suitable quotient of the Hecke group \(\Gamma_0(4)\), denoting in this way a kind of symmetry for the genus 2 case. With a view toward the situation in higher genera, we will analyze the genus three case that reveals itself to be a more delicate case to treat.

Furthermore we will give some explicit description of the rings of modular forms with respect to some of the group we have just introduced.

2. **Siegel modular forms with multiplier**

In this paragraph we will introduce the notations and all the tools that we will need in the following. Let
\[
\text{Sp}(g,\mathbb{R}) = \{ \gamma \in M_{2g \times 2g}(\mathbb{R}) \mid ^t \gamma J \gamma = J \}, \text{ where } J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix},
\]
be the symplectic group of degree \(g\). Considering the subset
\[
\text{Sp}(g,\mathbb{Z}) = \text{Sp}(g,\mathbb{R}) \cap M_{2g \times 2g}(\mathbb{Z})
\]
for all $\gamma, \beta$ satisfies the cocycle condition
\[
\gamma(\beta \tau) = \gamma \beta \tau.
\]

A subgroup $\Gamma$ for some $n \in \mathbb{N}$ is called a congruence subgroup of level $n$. Just by definitions we see that $\Gamma(n) \subset \Gamma$ for some $n \in \mathbb{N}$ is called a congruence subgroup of level $n$. Just by definitions we see that $\Gamma(g(n), 2n)$ as
\[
\Gamma(g(n)) = \bigl\{ \gamma \in \text{Sp}(g, \mathbb{Z}) \mid \gamma \equiv 1_2 \pmod{n} \bigr\}.
\]

For even values of the level $n$ it is easily checked that for $\gamma \in \Gamma(n)$
\[
\begin{aligned}
\text{diag}(tAC) &\equiv \text{diag}(C) \pmod{2n}, \\
\text{diag}(tBD) &\equiv \text{diag}(D) \pmod{2n}.
\end{aligned}
\]

Hence we have the characterization
\[
\Gamma(2m, 4m) = \{ \gamma \in \Gamma(2m) \mid \text{diag}(B) \equiv \text{diag}(D) \equiv 0 \pmod{4m} \}.
\]

Furthermore, it can be proved that $\Gamma(2m, 4m)$ is a normal subgroup of the integral symplectic group.

A subgroup $\Gamma \subset \text{Sp}(2, \mathbb{Z})$ such that $\Gamma(n) \subset \Gamma$ for some $n \in \mathbb{N}$ is called a congruence subgroup of level $n$. Just by definitions we see that
\[
\Gamma(g(n)) = \Gamma(n, 2n) \subset \Gamma(n).
\]

Then, for example, the group $\Gamma(g(n), 2n)$ is a congruence subgroup of level $2n$.

There is an action of the integral symplectic group on the space of symmetric complex matrices with positive-definite imaginary part
\[
\mathbb{H}_g = \bigl\{ \tau \in M_{g \times g}(\mathbb{C}) \mid \tau = t \tau, \text{Im} \tau > 0 \bigr\},
\]
called the Siegel upper half-space of degree $g$. The action is defined as a direct generalization of the Möbius transformations on the complex plane. In fact, for $\gamma \in \text{Sp}(g, \mathbb{Z})$ and $\tau \in \mathbb{H}_g$ we define
\[
\gamma \cdot \tau = (A \tau + B)(C \tau + D)^{-1}.
\]

Let $k$ be a positive integer and $\Gamma$ be a subgroup of finite index in $\text{Sp}(g, \mathbb{Z})$. A multiplier system of weight $k$ for $\Gamma$ is a map $v : \Gamma \to \mathbb{C}^*$ such that the map
\[
\varphi : \mathbb{H}_g \times \Gamma \to \mathbb{C} \\
(\tau, \gamma) \mapsto \varphi(\gamma, \tau) := v(\gamma) \text{det}(C \tau + D)^k,
\]
satisfies the cocycle condition
\[
\varphi(\gamma \beta, \tau) = \varphi(\gamma, \beta \cdot \tau) \varphi(\beta, \tau)
\]
for all $\gamma, \beta \in \Gamma$ and $\tau \in \mathbb{H}_g$. 

\[\vphantom{\text{Eq}(\gamma)}\]
With these notations we can define an operator on holomorphic functions \( f : \mathbb{H}_g \to \mathbb{C} \) as
\[
(4) \quad f_{|\gamma,k,v}(\tau) := v(\gamma)^{-1} \det(C\tau + D)^{-k} f(\gamma \cdot \tau).
\]
We will omit the weight and the multiplier when they will be clear by the context.

We say that a holomorphic function \( f \) defined on \( \mathbb{H}_g \) is a modular form of weight \( k \) with respect to \( \Gamma \) and \( v \) if
\[
(5) \quad f_{|\gamma,k,v}(\tau) = f(\tau), \quad \forall \gamma \in \Gamma, \forall \tau \in \mathbb{H}_g,
\]
and if additionally \( f \) is holomorphic at all cusps when \( g = 1 \). We denote by \([\Gamma,k,v]\) the vector space of such functions.

Theta functions with characteristics are classical examples of modular forms with multiplier if we consider their transformation under the action of some of the congruence subgroup defined above.

Given a theta characteristic or \( g \)-characteristic, that is a column vector \([a \ b] \in \mathbb{F}_2^{2g}\), the theta function with characteristic \( \vartheta[a \ b] \) is defined by the series
\[
(5) \quad \vartheta[a \ b](\tau,z) = \sum_{n \in \mathbb{Z}^g} \exp\left((n + a/2)\tau(n + a/2) + 2(n + a/2)(z + b/2)\right),
\]
where \( \exp(\cdot) = e^{2\pi i \cdot} \).

It is a classical result that (5) defines an holomorphic function on \( \mathbb{H}_g \times \mathbb{C}^g \) \(\text{[RF74]}\).

Evaluating this function in \( z = 0 \) we get a holomorphic function on the Siegel upper half space. These functions are usually called theta constants and are denoted by
\[
\vartheta[a \ b](\tau) = \vartheta[a \ b](\tau,0).
\]
Sometimes we will also denote by \( \vartheta_m \) the theta constant with characteristic \( m = [a \ b] \).

There is also an action of the integral symplectic group on theta characteristics, it will be useful to understand the action of the integral symplectic group on theta constants.
We define for \( \gamma \in \text{Sp}(g,\mathbb{Z}) \)
\[
(6) \quad \gamma \cdot [a \ b] = \left[ \begin{array}{cc} D & -C \\ -B & A \end{array} \right] \left[ \begin{array}{c} a \\ b \end{array} \right] + \left( \begin{array}{c} \text{diag}(C^tD) \\ \text{diag}(A^tB) \end{array} \right) \pmod{2},
\]
where \( \gamma \cdot [a \ b] \)
Notice that the action is non linear.

Hence, by the classical transformation formula for theta constants (see for example \text{[igu72]}, \text{[RF74]}), we have the following transformation rule for theta constants under the action of an element \( \gamma \in \text{Sp}(g,\mathbb{Z}) \)
\[
(7) \quad \vartheta[a \ b](\gamma \cdot \tau) = \kappa(\gamma) \exp\left(\phi([a \ b](\gamma))\right) \det(C\tau + D)^{1/2} \vartheta[a' \ b'](\tau),
\]
where
\[
\circ \left[ a' \ b' \right] = \gamma^{-1} \cdot \left[ a \ b \right] = \left[ \begin{array}{cc} A & C \\ B & D \end{array} \right] \left[ \begin{array}{c} a \\ b \end{array} \right] - \left( \begin{array}{c} \text{diag}(CA) \\ \text{diag}(DB) \end{array} \right),
\]
\[
\circ \phi([a \ b](\gamma)) = -\frac{1}{8}(4a'BDa + 4b'ACb - 2d'a'BCb) - \frac{1}{4}\text{diag}(A'B)(Da - Cb);
\]
\[
\circ \kappa(\gamma) \text{ is a primitive } 8^{\text{th}} \text{ root of unity depending on } \gamma.
\]

In particular we have that
\[
\vartheta[a \ b](\gamma \cdot \tau) = \kappa(\gamma) \det(C\tau + D)^{1/2} \vartheta[b \ a](\tau).
\]
It can be shown that $\kappa$ is a multiplier system on $\Gamma_g(1,2)$, we will denote it by $v_\vartheta$. Since $\Gamma_g(2) \subset \Gamma_g(1,2)$ we have that

$$\vartheta[\sigma] \in [\Gamma_g(2), 1/2, v_\vartheta].$$

Let us consider now the so called second order theta constants

$$\Theta[a](\tau) = \vartheta[\sigma] (2\tau).$$

By the above formula we can immediately deduce the transformation rule for these functions under the action of an element $\gamma \in \text{Sp}(g, \mathbb{Z})$. We get

$$\Theta[a](\gamma \cdot \tau) = \kappa(\gamma) \exp \left( \phi[\gamma] \right) \det(C\tau + D)^{1/2} \vartheta[\gamma'](\tau),$$

where

- $\gamma = \left( \frac{A}{C/2} \begin{smallmatrix} B \end{smallmatrix} \right)$ if $\gamma = \left( \frac{A}{C} \begin{smallmatrix} B \end{smallmatrix} \right)$
- $\left[ \begin{smallmatrix} a' \\ 0 \end{smallmatrix} \right] = \gamma^{-1} \cdot \left[ \begin{smallmatrix} a \\ 0 \end{smallmatrix} \right] = \left[ \begin{smallmatrix} t_A C/2 \\ t_B C/2 \end{smallmatrix} \right] \left( \begin{smallmatrix} a \\ 0 \end{smallmatrix} \right) - \frac{1}{2} \left( \text{diag}(C,A) \right)$
- $\phi[\sigma] = -\frac{1}{4}(a'B Da) - \frac{1}{2}\text{diag}(A'B) Da$.

Considering the second order theta constant with zero characteristic we get

$$\Theta[0](\gamma \cdot \tau) = \kappa(\gamma) \det(C\tau + D)^{1/2} \Theta[0](\tau).$$

As before it is possible to show that $\kappa$ is a multiplier system, this time with respect to the subgroup $\Gamma_g(2,4)$, we will denote it by $v_\vartheta$. Hence

$$\Theta[0] \in [\Gamma_g(2,4), 1/2, v_\vartheta].$$

It is known that the square of $v_\vartheta$ is non-trivial on $\Gamma_g(2,4)$, in fact we have the following expression for the multiplier system $\kappa$ on $\Gamma_g(2)$

$$\kappa(\gamma)^2 = \kappa(\gamma)^2 = (-1)^{\text{Tr}\left( \frac{A-ar{A}}{2} \right)}.$$

We recall here a formula that we will need in the following paragraphs ([RF74 Appendix II to Chapter II]).

**Proposition 1** (Riemann’s addition formula). Given two $g$-characteristics $[\varphi], [\delta]$, for any $z, w \in \mathbb{C}^g$ and $\tau \in \mathbb{H}_g$ we have that

$$\vartheta[\varphi] (\tau, \frac{z}{\sqrt{2}}) \vartheta[\delta] (\tau, \frac{w}{\sqrt{2}}) = \sum_{\sigma \in \mathbb{F}_2^g} \vartheta[\varphi \cdot \delta + \sigma] (2\tau, z) \vartheta[\varphi \cdot \delta'] (2\tau, w).$$

By the above proposition we get the following identities

$$\Theta[\sigma](\tau) \Theta[\sigma + \varepsilon](\tau) = \frac{1}{2^g} \sum_{\varepsilon' \in \mathbb{F}_2^g} (-1)^{\sigma' \cdot \varepsilon'} \vartheta^2[\varepsilon'](\tau),$$

$$\vartheta^2[\varepsilon'](\tau) = \sum_{\sigma \in \mathbb{F}_2^g} (-1)^{\sigma' \cdot \varepsilon'} \Theta[\sigma](\tau) \Theta[\sigma + \varepsilon](\tau).$$

As a consequence, the vector space of modular forms spanned by the functions $\Theta[a]^2_a$, $a \in \mathbb{F}_2^g$, coincides with the vector space of modular forms spanned by the functions $\vartheta^2[\sigma]$ for $b \in \mathbb{F}_2^g$. In what follows we set the notation $\vartheta_b := \vartheta[\sigma]$. 
3. The Genus 2 Case

In this paragraph we will focus our attention on the case \( g = 2 \) and, for the sake of simplicity, we will drop the index 2 in the notations introduced for the congruence subgroups of \( \text{Sp}(2, \mathbb{Z}) \). We will denote the four second order theta constants in genus 2 as

\[
\begin{align*}
f_{00} & := \Theta[0 \ 0], \\ f_{01} & := \Theta[0 \ 1], \\ f_{10} & := \Theta[1 \ 0], \\ f_{11} & := \Theta[1 \ 1].
\end{align*}
\]

Let us consider the ring of modular forms with multiplier with respect to the congruence subgroup \( \Gamma(2, 4) \)

\[
A(\Gamma(2, 4), v_\Theta) = \bigoplus_{k \in \mathbb{Z}} [\Gamma(2, 4), k/2, v_\Theta^k].
\]

It is known \cite{Run93} that this ring is the polynomial ring in the second order theta constants, so

\[
A(\Gamma(2, 4), v_\Theta) = \mathbb{C}[f_{00}, f_{01}, f_{10}, f_{11}].
\]

Hence the Satake compactification \( \mathcal{A}_2^{\text{Sat}}(2, 4) \) of the modular variety associated to this ring is isomorphic to \( \mathbb{P}^3 \).

Let us consider the map

\[
\mathbb{P}^3 \to \mathbb{P}^3
\]

\[
[x_0, x_1, x_2, x_3] \mapsto [x_0^2, x_1^2, x_2^2, x_3^2]
\]

Clearly this is a map of degree 8 of the projective space onto itself without base points. We would like to have a modular interpretation of this map.

3.1. The group. Let us define the group \( \Gamma^1(2, 4) \) that we have presented in the introduction. This is the set of integral symplectic matrices

\[
\gamma = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)
\]

such that

\[
\begin{align*}
A & \equiv D \equiv 1_2 \pmod{2}, \\
C & \equiv 0 \pmod{2}, \quad \text{diag}(C) \equiv 0 \pmod{4} \\
\text{diag}(B) & \equiv 0 \pmod{2}.
\end{align*}
\]

We will discuss the action of this group on the second order theta constants \( f_a \) and the functions \( \Theta_b \) with \( a, b \in \mathbb{F}_2^2 \).

Clearly \( \Gamma(2, 4) \subset \Gamma^1(2, 4) \) and moreover it is a normal in \( \Gamma^1(2, 4) \) since \( \Gamma(2, 4) \) is a normal subgroup of the integral symplectic group. It is easily seen that the quotient \( \Gamma^1(2, 4)/\Gamma(2, 4) \) is isomorphic to the vector space \( \mathbb{F}_2^3 \). So we have that \( \Gamma^1(2, 4)/\Gamma(2, 4) \) is abelian and the index \([\Gamma^1(2, 4) : \Gamma(2, 4)] = 8\).

Consider now the matrices of the form

\[
M_i = \left( \begin{array}{cc} 1 \ 0 \\ 0 \ 1 \end{array} \right), \quad i = 1, 2, 3, \quad \text{where} \quad B_1 = \left( \begin{array}{cc} 2 \ 0 \\ 0 \ 0 \end{array} \right), \quad B_2 = \left( \begin{array}{cc} 0 \ 0 \\ 0 \ 1 \end{array} \right), \quad B_3 = \left( \begin{array}{cc} 1 \ 0 \\ 0 \ 1 \end{array} \right).
\]

We have that \( M_i \in \Gamma^1(2, 4) \ \backslash \ \Gamma(2, 4) \) and \( M_i^2 \in \Gamma(2, 4) \), so we can take the classes of those matrices in \( \Gamma^1(2, 4)/\Gamma(2, 4) \) as a basis for the vector space \( \mathbb{F}_2^3 \).

We now focus on the action of the matrices \( M_i \) on theta constants. By the classical transformation formula \( (7) \) it can be easily proved that if \( \gamma = \left( \begin{array}{cc} 1 \ 0 \\ 0 \ i \end{array} \right) \) then \( S \) is a symmetric matrix then

\[
\Theta[a](\gamma \cdot \tau) = \Theta[a](\tau + S) = e^{-i a(Sa + 2 \text{diag}(S))} \Theta[a + Sa + \text{diag}(S)](\tau),
\]

with \( \varepsilon = \frac{1+i}{\sqrt{2}} \) a primitive 8th root of unity. For the second order theta constants we then get

\[
\Theta[a](\gamma \cdot \tau) = i^a S a \Theta[a](\tau).
\]
Thus, considering the action (4) we have for \( a = (a_1, a_2) \in \mathbb{F}_2^2 \)

\[
\begin{align*}
  f_a(M_1 \cdot \tau) &= (-1)^{a_1} f_a(\tau), \\
  f_a(M_2 \cdot \tau) &= (-1)^{a_2} f_a(\tau), \\
  f_a(M_3 \cdot \tau) &= (-1)^{a_1a_2} f_a(\tau).
\end{align*}
\]

So the group acts by a change of signs.

If we now focus the attention on the \( f_a^2 \), and therefore to the functions \( \vartheta_b^2 \), we have that the action is trivial since

\[
\vartheta_b^2(M_1 \cdot \tau) = \vartheta_{b+\text{diag}(B_1)}^2(\tau) = \vartheta_b^2(\tau).
\]

Then, for \( b \in \mathbb{F}_2^2 \), \( \vartheta_b^2 \) is a modular form with respect to the group \( \Gamma^1(2,4) \). In this way we can also extend the multiplier system \( \nu_\Theta \) defined earlier on \( \Gamma(2,4) \) to the bigger group \( \Gamma^1(2,4) \) letting it be the trivial multiplier on \( \Gamma^1(2,4) \setminus \Gamma(2,4) \).

We now compute the structure of the ring of modular forms \( A(\Gamma^1(2,4), \nu_\Theta^2) \).

**Proposition 2.** Let \( \Gamma^1(2,4) \) as above then

\[
\mathbb{C}[f_a^2] = A(\Gamma^1(2,4), \nu_\Theta^2).
\]

**Proof.** First note that by definition we have

\[
\mathbb{C}[f_a^2] \subset A(\Gamma^1(2,4), \nu_\Theta^2).
\]

Since both rings are integrally closed (in genus 2 there are in fact no relations between second order theta constants), it is enough to prove that the fields of rational functions of the two rings are equal. By what we said above this is trivial, in fact they both have the field of rational functions of \( \mathbb{C}[f_a] \) as an extension of degree 8. \( \square \)

As we have seen, the functions \( \vartheta_b^2 \) result to be modular forms with respect to the group \( \Gamma^1(2,4) \) then from (3) the map

\[
\mathbb{P}^3 \to \mathbb{P}^3
\]

\[
[f_{00}, \ldots, f_{11}] \mapsto [f_{02}^2, \ldots, f_{11}^2]
\]

can be considered as a map between the two modular varieties

\[
\psi : \text{Proj}(A(\Gamma(2,4))) \to \text{Proj}(A(\Gamma^1(2,4))).
\]

Notice that we can omit the multipliers since the modular variety defined by a ring of modular forms is independent from the multiplier.

### 3.2. Two more interesting groups.

Let us now recall the definitions given in the introduction of the groups

\[
\begin{align*}
  \Gamma_0(2) &= \{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2, \mathbb{Z}) \mid C \equiv 0 \pmod{2} \}, \\
  \Gamma_0^0(2) &= \{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0(2) \mid B \equiv 0 \pmod{2} \}.
\end{align*}
\]

It can be easily checked that the group \( \Gamma^1(2,4) \) is normal in \( \Gamma_0(2) \) using the relations involving blocks of an integral symplectic matrix (2).

As we have notice before, the group \( \Gamma(2,4) \) is a normal subgroup of \( \text{Sp}(2, \mathbb{Z}) \) so it is also normal in \( \Gamma_0^0(2) \). We can construct a map

\[
\varphi : \Gamma_0(2)/\Gamma^1(2,4) \to \Gamma_0^0(2)/\Gamma(2,4),
\]

in the following way. We’ll denote the class and the representative element by the same symbol. For a class \( \gamma \in \Gamma_0(2)/\Gamma^1(2,4) \) we can choose a representative of the form

\[
(12) \quad \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & A^{-1} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & B \\ 0 & 1 \end{pmatrix}.
\]
We set \[ \varphi(\gamma) = \begin{pmatrix} 1 & 0 \\ \frac{1}{CA} - 1 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ \frac{A}{A} - 1 & 2B \end{pmatrix}. \]

Roughly speaking, the map \( \varphi \) sends “B” to “2B”.

Let us consider the action of the matrices \( \gamma_1 = \begin{pmatrix} 1 & 2S \\ 0 & 1 \end{pmatrix} \), \( \gamma_2 = \begin{pmatrix} A & 0 \\ \frac{A}{A} - 1 & 2 \end{pmatrix} \), \( \gamma_3 = \begin{pmatrix} 1 & 2S \\ 0 & 1 \end{pmatrix} \) on the \( f_a \)'s and on the \( f^2_a \)'s. By (12) the classes of these matrices are generators for the group \( \Gamma_0(2)/\Gamma(2,4) \) and their images under \( \varphi \) are generators for the group \( \Gamma_0^0(2)/\Gamma(2,4) \).

We have that
\[
\begin{align*}
\varphi_{\gamma_1} f_a &= f_a - \text{diag}(S), \\
\varphi_{\gamma_2} f_a &= f_{2a}, \\
\varphi_{\gamma_3} f_a &= \exp \left( \frac{1}{2} a^t Sa - t \text{ diag}(S)a \right) f_a,
\end{align*}
\]

For simplicity, we will suppress the weight and the multiplier in the notation. For the first two generator it follows that
\[
\begin{align*}
\varphi_{\gamma_1} f_a &= f_a - \text{diag}(S), \\
\varphi_{\gamma_2} f_a &= f_{2a}.
\end{align*}
\]

For the last generator we have to consider the actions
\[
\begin{align*}
\varphi_{\gamma_3} f_a &= \exp \left( \frac{1}{2} a^t Sa - t \text{ diag}(2Sa) \right) f_a, \\
\varphi_{\gamma_3} f_a &= \exp \left( \frac{1}{2} a^t Sa - t \text{ diag}(2Sa) \right) f^2_a.
\end{align*}
\]

Thus, the action of the group \( \Gamma_0(2)/\Gamma(2,4) \) on the polynomial ring \( \mathbb{C}[f^2_a] \) is the same as the action of the group \( \Gamma^0_0(2)/\Gamma(2,4) \) on the polynomial ring \( \mathbb{C}[f_a] \).

Set \( G := \Gamma_0(2)/\Gamma(2,4) \cong \Gamma^0_0(2)/\Gamma(2,4) \). From the analysis above we get the following

**Theorem 3.** For any subgroup \( H \subset G \) exist two groups \( \Gamma', \Gamma \) such that
\[
\begin{align*}
\Gamma(2,4) &\subset \Gamma' \subset \Gamma^0_0(2) \\
\Gamma^1(2,4) &\subset \Gamma \subset \Gamma^0_0(2)
\end{align*}
\]

and the quotients \( \Gamma'/\Gamma(2,4) \) and \( \Gamma/\Gamma^1(2,4) \) are both isomorphic to \( H \). Hence, it can be constructed an isomorphism of graded ring of modular forms
\[
\Phi_H : A(\Gamma') \to A(\Gamma),
\]

that moreover increases the weights by a factor two.

As an immediate consequence we have a generalization of Mukai’s result:

**Corollary 4.** For every \( \Gamma(2,4) \subset \Gamma' \subset \Gamma^0_0(2) \) the projective variety \( \text{Proj}(A(\Gamma')) \) has a map of degree 8 onto itself.
3.3. **Action of the Fricke involution.** Let us introduce an interesting involution of the Siegel upper half space \( \mathbb{H}_g \) induced by the action of the symplectic group with real coefficients. For any genus \( g \) we define the so-called Fricke involution

\[
J_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1_g \\ -2 & 0 \end{pmatrix}.
\]

The action of the symplectic group \( \text{Sp}(g, \mathbb{R}) \) on \( \mathbb{H}_g \) is defined as in (3). Then for \( \tau \in \mathbb{H}_g \) we have

\[
J_2 \cdot \tau = -\frac{1}{2\tau}.
\]

We are interested in studying the action of the matrix \( J_2 \) on the functions \( f_a \) with \( a \in \mathbb{P}_2^g \). Despite we cannot use (6) to extend the action of \( \text{Sp}(g, \mathbb{R}) \) on theta characteristics to \( \text{Sp}(g, \mathbb{R}) \) it is still possible to use the classical transformation formula for theta functions to compute the action of the matrix \( J_2 \) on theta constants. We will use that the transformation on the Siegel upper-half space \( \tau \mapsto -\frac{1}{\tau} \) is given by the action of the integral symplectic matrix \( J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix} \).

Considering the action on second order theta constants we get

\[
\Theta[a](J_2 \cdot \tau) = \partial \bigg[ \frac{a}{b} \bigg] \bigg( 2 \bigg( -\frac{1}{\tau^2} \bigg) \bigg) = \kappa(J)\det(\tau)^{1/2}\partial \bigg[ \begin{pmatrix} 0 \\ a \end{pmatrix} \bigg] \bigg( \frac{a}{b} \bigg) = \kappa(J)\det(\tau)^{1/2}\Theta[a](\tau),
\]

while considering the action on the functions \( \vartheta_b \) we get

\[
\vartheta \bigg[ \begin{pmatrix} 0 \\ a \end{pmatrix} \bigg] (J_2 \cdot \tau) = \vartheta \bigg[ \begin{pmatrix} \vartheta_b \\ 0 \end{pmatrix} \bigg] \bigg( -\frac{1}{\tau} \bigg) = \kappa(J)\det(2\tau)^{1/2}\vartheta \bigg[ \begin{pmatrix} 0 \\ a \end{pmatrix} \bigg] \bigg( 2\tau \bigg) = 2\kappa(J)\det(\tau)^{1/2}\Theta[b](\tau).
\]

By these equation it is possible to extend the multiplier system \( v_\Theta \) and \( v_\vartheta \) to the Fricke involution \( J_2 \). We set

\[
v_\Theta(J_2) := v_\vartheta(J),
\]

\[
v_\vartheta(J_2) := v_\vartheta(J).
\]

Hence from (4),

\[
f_a|_{J_2, 1/2, v_\vartheta} = \frac{1}{\sqrt{2}}\vartheta_a,
\]

(13)

\[
\vartheta|_{J_2, 1/2, v_\vartheta} = \sqrt{2}f_b.
\]

(14)

Then the Fricke involution switches the polynomial rings

\[
\mathbb{C}[f_a] \leftrightarrow \mathbb{C}[\vartheta_b].
\]

We want to investigate the effect of the conjugation homomorphism defined by \( J_2 \) on an element of the integral symplectic group \( \text{Sp}(2, \mathbb{Z}) \). For \( \gamma \in \text{Sp}(2, \mathbb{Z}) \) we have that

\[
J_2 \cdot \gamma := J_2 \gamma J_2^{-1} = \begin{pmatrix} D & -C/2 \\ -2B & A \end{pmatrix}.
\]

(15)

Then, if \( C \equiv 0 \pmod{2} \) it follows that \( J_2 \cdot \gamma \in \text{M}_{4 \times 4}(\mathbb{Z}) \) and it can be easily verified that the relations (2) are satisfied hence \( J_2 \cdot \gamma \in \text{Sp}(2, \mathbb{Z}) \).

From (13) we can easily write the image of the groups \( \Gamma^1(2, 4), \Gamma_0(2), \Gamma(2, 4), \Gamma_0^0(2) \) under this conjugation homomorphism. For the first two groups we have

\[
J_2 \cdot \Gamma^1(2, 4) = \Gamma^1(2, 4), \quad J_2 \cdot \Gamma_0(2) = \Gamma_0(2),
\]

hence they are fixed.

It follows that, since the \( f_a^2 \) are linear combination of the \( \vartheta_a^2 \) and viceversa, by (8) and (9), the polynomial ring \( \mathbb{C}[f_a^2] = \mathbb{C}[\vartheta_b^2] \) is invariant under the conjugation by the Fricke involution.
For the other two groups we have that

\[ J_2 \Gamma_0(2) = \Gamma_0(4) := \{ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2, \mathbb{Z}) \mid C \equiv 0 \pmod{4} \}, \]

while \( J_2 \Gamma(2,4) \) is the group of integral symplectic matrices

\[ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ such that } \begin{cases} A \equiv D \equiv 1_2 \pmod{2}, \\ C \equiv 0 \pmod{4}, \text{ diag}(C) \equiv 0 \pmod{8}, \\ \text{diag}(B) \equiv 0 \pmod{2}. \end{cases} \]

We can exploit this action to compute the ring of modular forms with respect to the groups \( J_2 \Gamma(2,4) \) and \( \Gamma_0(4) \). From \([10]\) and \([13]\) it follows that

\[ A(J_2 \Gamma(2,4), \vartheta) = \mathbb{C}[\vartheta_b]. \]

In the same way, we can compute the ring of modular forms \( A(\Gamma_0(4)) \). By \([15]\) it is known that the ring \( A(\Gamma_0(2)) \) is the polynomial ring

\[ A(\Gamma_0(2)) = \mathbb{C}[x, y, z, k], \]

where

\[ x = \frac{1}{4} (\vartheta_{00}^4 + \vartheta_{01}^4 + \vartheta_{10}^4 + \vartheta_{11}^4), \]
\[ y = (\vartheta_{00}\vartheta_{01}\vartheta_{10}\vartheta_{11})^2, \]
\[ z = \frac{1}{16384} \left( \vartheta_{01}^4 \vartheta_{10}^4 - \vartheta_{11}^4 \right)^2, \]
\[ k = \frac{1}{10995046576} \left( \vartheta_{00}^4 \vartheta_{01}^4 \vartheta_{10}^4 \vartheta_{11}^4 \right)^2. \]

Applying \([8]\) and \([9]\) it is possible to write also the other two forms \( z, k \) in terms of the theta constants \( \vartheta_b \). The computation has been done with the help of the software \textit{Mathematica}. We get

\[ z = \frac{1}{16384} \left( \vartheta_{00}^8 + \vartheta_{01}^8 + \vartheta_{10}^8 + \vartheta_{11}^8 + 8\vartheta_{00}\vartheta_{01}^2\vartheta_{10}\vartheta_{11}^2 - 2\vartheta_{00}\vartheta_{01}^4 + -2\vartheta_{00}^2\vartheta_{10}^4 - 2\vartheta_{01}^4\vartheta_{10}^4 - 2\vartheta_{00}\vartheta_{01}^4\vartheta_{11}^4 - 2\vartheta_{01}^4\vartheta_{10}^4\vartheta_{11}^4 \right), \]

\[ k = \frac{1}{10995046576} \left( \vartheta_{00}^6 \vartheta_{01}^2 \vartheta_{10}^2 \vartheta_{11}^2 + \vartheta_{00}^2 \vartheta_{01}^6 \vartheta_{10}^2 \vartheta_{11}^2 + \vartheta_{00}^2 \vartheta_{01}^2 \vartheta_{10}^6 \vartheta_{11}^2 + \vartheta_{00}^2 \vartheta_{01}^2 \vartheta_{10}^2 \vartheta_{11}^6 + -\vartheta_{00}^4 \vartheta_{01}^4 \vartheta_{10}^4 \vartheta_{11}^4 - \vartheta_{00}^4 \vartheta_{01}^4 \vartheta_{10}^4 \vartheta_{11}^4 - \vartheta_{01}^4 \vartheta_{00}^4 \vartheta_{10}^4 \vartheta_{11}^4 - \vartheta_{01}^4 \vartheta_{00}^4 \vartheta_{10}^4 \vartheta_{11}^4 \right). \]

Since the map in Theorem \([8]\) increases the weight by a factor two to find the ring of modular forms with respect to the group \( \Gamma_0(4) \) we only have to halve the weight of the four modular forms \( x, y, z, k \) introduced above. Then we have to consider the
polynomial ring $\mathbb{C}[X,Y,Z,K]$, where
\[ X = (\vartheta^2_{00} + \vartheta^2_{01} + \vartheta^2_{10} + \vartheta^2_{11}), \]
\[ Y = \vartheta_{00}\vartheta_{01}\vartheta_{10}\vartheta_{11}, \]
\[ Z = (\vartheta^4_{00} + \vartheta^4_{01} + \vartheta^4_{10} + \vartheta^4_{11} + 8\vartheta_{00}\vartheta_{01}\vartheta_{10}\vartheta_{11} - 2\vartheta^2_{00}\vartheta^2_{01} + \\
- 2\vartheta^2_{00}\vartheta^2_{10} - 2\vartheta^2_{01}\vartheta^2_{11} - 2\vartheta^2_{01}\vartheta^2_{10} - 2\vartheta^2_{01}\vartheta^2_{11} - 2\vartheta^2_{10}\vartheta^2_{11}) = \\
= (X^2 + 8Y - 4\vartheta^2_{00}\vartheta^2_{10} - 4\vartheta^2_{00}\vartheta^2_{11} - 4\vartheta^2_{01}\vartheta^2_{10} - 4\vartheta^2_{01}\vartheta^2_{11} - 4\vartheta^2_{10}\vartheta^2_{11}) \]
\[ K = (\vartheta^3_{00}\vartheta_{01}\vartheta_{10}\vartheta_{11} + \vartheta_{00}\vartheta^3_{01}\vartheta_{10}\vartheta_{11} + \vartheta_{00}\vartheta_{01}\vartheta^3_{10}\vartheta_{11} + \vartheta_{00}\vartheta_{01}\vartheta_{10}\vartheta^3_{11} + \\
- \vartheta^2_{00}\vartheta^2_{01}\vartheta^2_{10} - \vartheta^2_{00}\vartheta^2_{01}\vartheta^2_{11} - \vartheta^2_{00}\vartheta^2_{10}\vartheta^2_{11} - \vartheta^2_{01}\vartheta^2_{10}\vartheta^2_{11}) = \\
= (XY - \vartheta^2_{00}\vartheta^2_{01}\vartheta^2_{10} - \vartheta^2_{00}\vartheta^2_{01}\vartheta^2_{11} - \vartheta^2_{00}\vartheta^2_{10}\vartheta^2_{11} - \vartheta^2_{01}\vartheta^2_{10}\vartheta^2_{11}). \]

We can conclude in this way that the ring of modular forms $A(\Gamma_0(4))$ is the polynomial ring in four symmetric polynomials in the $\vartheta_b$, $b \in \mathbb{F}_2^3$.

If we now come back to the modular varieties associated to the groups we have obtained by acting by conjugation under the Fricke involution we clearly get, by theorem 3, the following theorem for the group $G' := \Gamma_0(4)/J_2.\Gamma(2,4)$.

**Theorem 5.** For any subgroup $H' \subset G'$ exist two groups $\Delta', \Delta$ such that
\[ J_2.\Gamma(2,4) \subset \Delta' \subset \Gamma_0(4), \]
\[ \Gamma^1(2,4) \subset \Delta \subset \Gamma_0(2), \]
and the quotients $\Delta'/J_2.\Gamma(2,4)$ and $\Delta/\Gamma^1(2,4)$ are both isomorphic to $H'$. Hence can be constructed an isomorphism of graded ring of modular forms
\[ \Psi_{H'} : A(\Delta') \rightarrow A(\Delta), \]
that moreover increases the weights by a factor two.

Note that since the groups $\Gamma^1(2,4)$ and $\Gamma_0(2)$ are fixed by the Fricke involution, also the chain of groups between them is fixed, but not the single groups. Then, if $\Gamma''$ corresponds to $\Gamma'$ in theorem 3 then $J_2.\Gamma''$ corresponds to $J_2.\Gamma'$ in theorem 5 but the two groups are in general different from the starting ones.

As before, we have the following corollary

**Corollary 6.** For every $J_2.\Gamma(2,4) \subset \Delta' \subset \Gamma_0(4)$ the projective variety $\text{Proj}(A(\Delta'))$ has a map of degree 8 onto itself.

4. Higher genera, the genus 3 case

In genus three there is a non trivial algebraic relation between second order theta constants that must be taken in consideration in doing the construction made in the previous section. From [Run95] the ring of modular forms with respect to the group $\Gamma_3(2,4)$ is no more the polynomial ring $\mathbb{C}[f_a]$ but
\[ A(\Gamma_3(2,4)) = \mathbb{C}[f_a]/(R_{16}), \]
where

\[ R_{16} = 2^3 \sum_{m \text{ even}} \vartheta_{16}^{m}(\tau) - \left( \sum_{m \text{ even}} \vartheta_{8}^{m}(\tau) \right)^2. \]

By (9) it is possible to write this relation as

\[ R_{16} = P_8(f_{000}, \ldots, f_{111}) + q \cdot Q_4(f_{000}, \ldots, f_{111}), \]

with \( P_8 \) and \( Q_4 \) polynomials in the \( f_a^2 \)'s of degree 8 and 4 respectively and \( q = \prod_{a \in \mathbb{F}_2^3} f_a \).

We define as in (11) the subgroup \( \Gamma_1(2, 4) \) of \( \text{Sp}(2, \mathbb{Z}) \). If the ring of modular forms in genus 3 with respect to this group is \( \mathbb{C}[f_a^2] \), maybe with some relation coming from the relation \( R_{16} \), we could made the same constructions as above to get similar results.

Instead, by the expression (16) of the relation it is easily checked that \( q \) is also an element of the ring of invariants with respect to the action of the group \( \Gamma(2, 4) \), thus in genus 3 we have to consider no more the polynomial ring in the \( f_a^2 \)'s but at least the ring

\[ R := \mathbb{C}[f_a^2, q]/(P_8 + qQ_4, q^2 = \prod_{a \in \mathbb{F}_2^3} f_a^2). \]

We can certainly say that \( R \subset A(\Gamma^1(2, 4), \nu_0^2) \). We want to show that \( A(\Gamma^1(2, 4), \nu_0^2) = R \).

To do this we need to examine the normality of the quotient ring

\[ S := \mathbb{C}[x_0, \ldots, x_8]/(P, Q), \]

where

\[ P = P_8(x_0, \ldots, x_7) + x_8Q_4(x_0, \ldots, x_7), \]
\[ Q = x_8^2 - x_0 \cdots x_7. \]

For a noetherian ring \( A \), Serre’s criterion for normality states that \( A \) is normal if and only if \( A \) satisfies conditions

\[ \begin{align*}
R1 : & \text{ regularity in codimension one,} \\
S2 : & \text{ every prime } P \text{ of codimension at least 2 satisfies depth } A_P \geq 2,
\end{align*} \]

where \( A_P \) is the localization of the ring at the prime ideal \( P \).

The ring \( S \) is a complete intersection ring, hence it is a Cohen-Macaulay ring and so the condition \( S2 \) is satisfied. With the help of the software Macaulay2 [M2] it is possible to verify that also the condition \( R1 \) for the ring \( S \) is satisfied. The normality of the ring \( R \) follows at once from the normality of the ring \( S \). Hence we can conclude that

\[ A(\Gamma^1(2, 4), \nu_0^2) = \mathbb{C}[f_a^2, q]/(P_8 + qQ_4, q^2 = \prod_{a \in \mathbb{F}_2^3} f_a^2). \]

Then for the genus three case it is not possible to give a modular interpretation of the map of degree 27

\[ [p^7] \rightarrow \mathbb{P}^7, \]
\[ [y_0, \ldots, y_7] \mapsto [y_0^2, \ldots, y_7^2]. \]

In genera higher than 3 many algebraic relations appear between second order theta constants, hence a possible interpretation of the map that squares the coordinates of a projective space of suitable dimension would need a deeper analysis. Then we see how the genus two case is peculiar from this point of view since the principal results of this work cannot be generalized directly to higher genera.
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