Path Integral Approach to Fermionic Vacuum Energy in Non-parallel D1-Branes

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Abstract
The fermionic one loop vacuum energy of the superstring theory in a system of non-parallel D1-branes is derived by applying the path integral formalism.

1 Introduction
The present work is sequel to the previous one in which we applied the path integral technique to derive the one loop vacuum energy (zero point energy) of a bosonic string in a system of non-parallel D1-branes [1]. Here we shall derive the fermionic zero point energy by engaging the path integral formalism for a superstring in the same system of D1-branes.

The path integral approach to the superstring theory is not a new subject. Indeed it stands as an alternative approach to unveil the physics of string theory through calculating the superstring S-matrix elements [2-9]. So, after a quick review of the path integral derivation of the fermionic partition function of the open superstrings, a similar approach is followed to derive the fermionic partition function in a system of angled D1-branes in section 3. We show that the result is in agreement with one derived earlier using the harmonic oscillator representation [10-12].
2 Fermionic Partition Function: Path Integral Approach

From the fermionic part of the superstring action

\[ S_F = \frac{i}{2\pi} \int d^2 \sigma \bar{\Psi}^\mu \rho^\mu \partial_\tau \Psi^\mu + S[\beta, \gamma] \]  

the partition function can be achieved by evaluating the path integral

\[ Z_F = \int D\Psi D\bar{\Psi} e^{iS[\Psi, \bar{\Psi}] + iS[\beta, \gamma]} \]

Here the action \( S[\beta, \gamma] \) stands for the superconformal ghosts action. We skip the explicit derivation of the contribution arising from the integration over the superconformal ghost fields \( \beta \) and \( \gamma \) as its net effect is to decrease the space-time dimensions by 2. After some algebra and upon introducing \( \Psi^\mu = (\psi^\mu, \tilde{\psi}^\mu)^T \) the action (1) (with Euclidean signature for world-sheet and target space manifolds) takes a more simple form

\[ S[\psi, \tilde{\psi}] = \frac{i}{2\pi} \int d^2 \sigma (\psi^\mu \bar{\partial} \psi^\mu + \tilde{\psi}^\mu \partial \tilde{\psi}^\mu) \]

where \( \partial = \partial_\tau + i\partial_\sigma \). Now, we introduce the notation \((\pm, \pm)\) to distinguish the four fermionic spin structures in such a way that the upper (lower) sign denotes the periodic (anti-periodic) boundary conditions along the \( \tau \) and \( \sigma \) directions, respectively [13, 14]. Therefore, the spin structures \((\pm, +)\) arise from the Ramond sector, which for the off-shell fluctuations implies

\[ \psi^\mu(\tau, \sigma) = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}} \frac{\psi^{\mu}_{mn}}{\sqrt{2}} u_{mn}, \quad \tilde{\psi}^\mu(\tau, \sigma) = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}} \frac{\tilde{\psi}^{\mu}_{mn}}{\sqrt{2}} u_{mn} \]

with the eigen-mode \( u_{mn} = e^{i\tau \omega m} e^{-i\sigma} \). For the spin structures \((\pm, -)\), arising from the Neveu-Schwartz sector, we have

\[ \psi^\mu(\tau, \sigma) = \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}} \frac{\psi^{\mu}_{mn}}{\sqrt{2}} u_{mn}, \quad \tilde{\psi}^\mu(\tau, \sigma) = \sum_{m, n \in \mathbb{Z}} \frac{\tilde{\psi}^{\mu}_{mn}}{\sqrt{2}} u_{mn} \]

In a similar way the fourier expansions of \( \tilde{\psi}^\mu \) associated with different spin structures can be achieved via the substitution \( u_{mn}(\tau, \sigma) \rightarrow u_{mn}(\tau, -\sigma) \equiv \tilde{a}_{mn}(\tau, \sigma) \) in equations (4) and (5). The eigen-modes fulfill the orthogonality relation

\[ \langle u_{mn} u_{m'n'} + \tilde{a}_{mn} \tilde{u}_{m'n'} \rangle = 2\pi s \delta_{m+m'} \delta_{n+n'} \]
Here we have defined

\[ \langle Q \rangle = \int_0^s d\tau \int_0^\pi d\sigma \ Q \] (7)

Hence, by taking into account the Grassmannian nature of coefficients, i.e. \( \{\psi_{\mu mn}, \psi_{\nu m'n'}\} = 0 \) one obtains the partition function of the open superstring

\[ Z_\psi = \int D\psi^\mu e^{-S[\psi^\mu, \bar{\psi}^\mu]} = \prod_{mn} \lambda_{mn}^d \] (8)

with \( \lambda_{mn} = -\frac{s}{\pi}(\omega_m + in) \). The above infinite product can be easily calculated with the aid of identities

\[ \prod_{m \in \mathbb{Z}} (mx + y) = 2 \sinh \left( \frac{i\pi y}{x} \right), \quad \prod_{m \in \mathbb{Z} + \frac{1}{2}} (mx + y) = 2 \cosh \left( \frac{i\pi y}{x} \right) \] (9)

and the zeta-function regularizations

\[ \sum_{m \in \mathbb{N}} m = \frac{1}{12}, \quad \sum_{m \in \mathbb{N} - \frac{1}{2}} m = \frac{1}{24} \] (10)

Therefore we find the open superstring partition function as [9] \((q = e^{-\frac{s}{\pi}})\)

\[ Z_\psi = \prod_{mn} \lambda_{mn}^d = \begin{cases} \frac{q^d}{\pi^d} \prod_{n \in \mathbb{N}} (1 + q^{2n})^d, & m \in \mathbb{Z} + \frac{1}{2} \\ q^{-\frac{d}{2}} \prod_{n \in \mathbb{N}} (1 + q^{2n-1})^d, & m \in \mathbb{Z} + \frac{1}{2} \\ q^{-\frac{d}{2}} \prod_{n \in \mathbb{N}} (1 - q^{2n-1})^d, & m \in \mathbb{Z} \end{cases} \] (11)

We shall assign the symbols \( Z^+_{\psi}, Z^-_{\psi}, \) and \( Z^-_{\psi} \) to the terms of equation (11) from above to below, respectively. One must note that \( Z^+_{\psi} = 0 \) because of the well-known property of the Grassmann variables

\[ \int d\psi_{\mu mn} = 0 \] (12)

3 Fermionic Partition Function: The Case of Angled D1-Branes

We specify the position of first D1-brane by

\[ X^i(\tau, 0) = 0, \quad i = 2, ..., d \] (13)

and the second one by

\[ X^2(\tau, \pi) \cos \alpha = X^1(\tau, \pi) \sin \alpha \]

\[ X^r(\tau, \pi) = l_r \] (14)
where $r = 3, \ldots, d$. We denote the deflection angle by $\alpha = \pi a$ and $0 \leq a \leq 1$. Then, the conditions satisfied by the ends of an open string at the boundaries, imposed by the classical equations of motion, read \[1, 10-12\]

\[
\partial_{\sigma}X^1(0, \tau) = X^2(\tau, 0) = 0, \\
\partial_{\sigma}X^1(\tau, \pi) \cos \alpha = -\partial_{\sigma}X^2(\tau, \pi) \sin \alpha
\]

Similarly, for the fermionic degrees of freedom we find

\[
\bar{\epsilon} \rho^0 \psi^1(0, \tau) = \bar{\epsilon} \rho^0 \psi^1(0, \tau) = 0
\]

and

\[
\bar{\epsilon} \rho^0 (\psi^1 + \tan \alpha \psi^2) = \bar{\epsilon} \rho^0 (\psi^2 + \tan \alpha \psi^1) = 0
\]

at the other end $\sigma = \pi$. So, for the classical solutions one finds \[10\]

\[
\left(\begin{array}{c}
\psi^1 \\
\tilde{\psi}^1
\end{array}\right) = \frac{1}{2i} \sum_n \psi_n \left(e^{in(\tau-\sigma)} \pm e^{in(\tau+\sigma)}\right) + \text{complex conjugate}
\]

and

\[
\left(\begin{array}{c}
\psi^2 \\
\tilde{\psi}^2
\end{array}\right) = \frac{1}{2} \sum_n \psi_n \left(e^{in(\tau-\sigma)} \mp e^{in(\tau+\sigma)}\right) + \text{complex conjugate}
\]

The lower sign in expression (18) and (19) corresponds to the NS sector. Now let us consider the fluctuations around the classical solutions in both sectors as

\[
\psi^1 = \frac{1}{2l} \sum_{m,n \in \mathbb{Z}} \left(\psi_{mn} u_m^a - \bar{\psi}_{mn} \bar{u}_m^a \right)
\]

\[
\psi^2 = \frac{1}{2l} \sum_{m,n \in \mathbb{Z}} \left(\psi_{mn} u_m^a + \bar{\psi}_{mn} \bar{u}_m^a \right)
\]

and

\[
\tilde{\psi}^1 = \frac{1}{2l} \sum_{m,n \in \mathbb{Z}} \left(\psi_{mn} \tilde{u}_m^a - \bar{\psi}_{mn} \bar{\tilde{u}}_m^a \right)
\]

\[
\tilde{\psi}^2 = \frac{1}{2l} \sum_{m,n \in \mathbb{Z}} \left(\psi_{mn} \tilde{u}_m^a + \bar{\psi}_{mn} \bar{\tilde{u}}_m^a \right)
\]

where $u_{mn}^a = e^{i\omega_m \tau} e^{-i n a \sigma}$ and $n_a = n + a$. Thus one finds

\[
\sum_{A=1}^{2} \langle \psi^A \tilde{\psi}^A \rangle = \frac{1}{4} \sum_{mn} \sum_{m'n'} \psi_{mn} \left(\begin{array}{cc}
0 & -2l_{m'n'} \langle u_{mn}^a \bar{u}_{m'n'}^a \rangle \\
2l_{m'n'} \langle u_{mn}^a u_{m'n'}^a \rangle & 0
\end{array}\right) \Psi_{m'n'}
\]
where we have introduced $\Psi^t_{mn} = (\psi_{mn}, \bar{\psi}_{mn})$ and $l_a^m = (i\omega_m - n_a)$. This expression when combined with $\sum_{A=1}^2 \langle \bar{\psi}^A \partial \psi^A \rangle$ yields the diagonalized action

$$S = \sum_{mn} \Psi^t_{mn} \begin{pmatrix} 0 & -\lambda^a_{mn} \\ \lambda^a_{mn} & 0 \end{pmatrix} \Psi_{mn}$$

(25)

where we have gained

$$\langle \bar{u}^a_{mn} u^a_{m'n'} + \bar{\psi}^a_{mn} \tilde{\psi}^a_{m'n'} \rangle = 2\pi s \delta_{n,n'} \delta_{m,m'}$$

(26)

Thus integration over the fields $\psi^1$ and $\psi^2$, or equivalently over $\Psi$, yields

$$Z_{\psi^1,\psi^2} = \int D\Psi e^{-S[\Psi]} = \prod_{mn} \det \begin{pmatrix} 0 & -\lambda^a_{mn} \\ \lambda^a_{mn} & 0 \end{pmatrix}^{1/2} = \prod_{mn} \lambda^a_{mn}$$

(27)

Therefore, on invoking the well-known formula

$$\sum_{n=1}^\infty (n + a) = \frac{1}{24} - \frac{1}{2} \left(a + \frac{1}{2}\right)^2$$

(28)

we obtain

$$Z_{\psi^1,\psi^2} = \begin{cases} q^{\frac{1}{2} + a(a-1)}(1 + q^{2a}) \prod_{n \in \mathbb{N}}(1 + q^{2n+2a})(1 + q^{2n-2a}), & m \in \mathbb{Z} + \frac{1}{2} \\ q^{-\frac{1}{2} + a^2} \prod_{n \in \mathbb{N}}(1 + q^{2n+2a-1})(1 + q^{2n-2a-1}), & m \in \mathbb{Z} + \frac{1}{2} \\ q^{-\frac{1}{2} + a^2} \prod_{n \in \mathbb{N}}(1 - q^{2n+2a-1})(1 - q^{2n-2a-1}), & m \in \mathbb{Z} \end{cases}$$

(29)

The remaining bosonic degrees of freedom either satisfy the Neumann or the Dirichlet boundary condition. So, the corresponding fermionic degrees of freedom are characterized by the condition

$$\psi(\tau, \pi) = \pm \eta \tilde{\psi}(\tau, \pi)$$

(30)

with $\eta = 1$ ($\eta = -1$) for the Neumann (Dirichlet) boundary condition. The minus sign refers to the Dirichlet boundary condition [15]. However, for a typical fermionic degree of freedom the partition function regardless to the boundary condition satisfied by its bosonic counterpart is given by equation (11) (with $d = 1$). So, by taking into account the invariance under the modular transformation, we find the fermionic partition function $Z_F = Z_\psi Z_\beta$, in $d = 10$ dimensions as

$$Z_F = \frac{1}{2} \left[- Z_{\psi^1,\psi^2}^+ (Z^+_\psi)^6 + Z_{\psi^1,\psi^2}^- (Z^-_\psi)^6 + Z_{\psi^1,\psi^2}^\pm (Z^\pm_\psi)^6 \right]$$

(31)
where the factor $\frac{1}{2}$ comes from the GSO projection. A similar result for fermionic partition function is derived earlier by applying the harmonic oscillator representation [10-12]. The one loop vacuum energy of the system becomes
\[
\mathcal{A} = \ln Z = \int_{0}^{\infty} \frac{ds}{s} Z(s)
\]
where the superstring partition function in non-parallel D1-brane setup will be
\[
Z = Z_F Z_B
\]
\[
= T q^{\frac{2}{\pi s}} \frac{(1 - q^{2a})^{-1}}{\prod_{n} (1 - q^{2n})^{-6} (1 - q^{2n+2a})^{-1} (1 - q^{2n-2a})^{-1}} \times \frac{1}{2} \left[ -8 (1 + q^{2a}) \prod_{n=1} (1 + q^n)^6 (1 + q^{2n+2a})(1 + q^{2n-2a}) \right. \\
\left. + q^{-1+a} \prod_{n\in\mathbb{N}} (1 + q^{2n-1})^6 (1 + q^{2n+2a-1})(1 + q^{2n-2a-1}) \right] \\
+ q^{-1+a} \prod_{n\in\mathbb{N}} (1 - q^{2n-1})^6 (1 - q^{2n+2a-1})(1 - q^{2n-2a-1})
\]

Here the partition function of bosonic part (in 26 dimensions) is [1, 10-12]
\[
Z_B = \frac{T q^{\frac{2}{\pi s}}}{\sqrt{2\pi s}} \frac{(1 - q^{2a})^{-a(a-1)-2}}{\prod_{n\in\mathbb{N}} (1 - q^{2n})^{-22} (1 - q^{2n+2a})^{-1} (1 - q^{2n-2a})^{-1}}
\]
where $Y^2$ stands for the distance between D-branes. The total interval of interaction time $T$ arises from integration over the zero mode of $X^0$.

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