DIXMIER TRACES FOR DISCRETE PSEUDO-DIFFERENTIAL OPERATORS

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Dedicated to Professor Lázaro Recht on his 79 birthday

Abstract. In this paper we provide sharp results for the Dixmier traceability of discrete pseudo-differential operators on \( \ell^2(\mathbb{Z}^n) \). In this setting, we introduce a suitable notion of a class of classical symbols which provide a class of Dixmier traceable discrete pseudo-differential operators. We also present a formula for the Dixmier trace of a Dixmier traceable discrete pseudo-differential operator by using the Connes equivalence between the Wodzicki residue and the Dixmier trace. MSC 2010. Primary 47G30; Secondary 58B34, 47L20.

1. Introduction

In this paper we study the membership of global pseudo-differential operators on \( \mathbb{Z}^n \), also known as discrete pseudo-differential operators, associated with the discrete Hörmander symbol classes introduced by Botchway, Kibity and Ruzhansky [2] to the Dixmier ideal of operators on \( \ell^2(\mathbb{Z}^n) \). For this purpose, we use a fundamental tool of non-commutative geometry, namely, the celebrated theorem of Connes on the equivalence of the Wodzicki residue and the Dixmier trace [9]. Recently, due to its applications in number theory, difference equations and many other problems involving the discretization of differential equations, the theory of pseudo-differential operators on \( \mathbb{Z}^n \) has gained an important role in harmonic analysis [31, 46, 33, 25, 30, 44, 45, 46]. Discrete pseudo-differential operators appear early in the classic Fourier analysis (see [17, 21]) as well as discrete counterparts of Calderón-Zygmund singular integral operators.

The question “whether the \( C^* \)-algebra of all bounded linear operators on a Hilbert space \( H \) had a unique non-trivial trace” led to the concept of the Dixmier trace \( \text{Tr}_\omega \). In 1966, Dixmier answered this question in negative in a note published in Comptes Rendus [16]. To obtain the trace he used an invariant mean \( \omega \) on the \( ax + b \) group. In contrast to the usual trace, his trace vanishes on the ideal of trace class operators. Applications of the Dixmier trace in differential geometry as well as in non-commutative geometry were facilitated by the celebrated theorem of Connes [9] which relates the trace to the Wodzicki residue [47] for pseudo-differential operators on a closed manifold. Both the Dixmier trace and the Wodzicki residue play important roles in noncommutative geometry and its

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applications to fractal theory, foliation theory, spaces of non-commuting coordinates, perturbation theory, and quantum field theory [9].

In a recent context, some properties of the pseudo-differential operators on \( \mathbb{Z}^n \), studied by Molahajloo in [29] which were initially introduced by Rabinovich and Rabinovich and Roch [36, 37] but without the symbolic calculus. But still the symbolic calculus for this quantization was missing. More recently, the symbolic classes and symbolic calculus have been introduced and constructed in the work of Botchway, Kibiti, and Ruzhansky [2] by asking the symbolic condition for decay in the space variable rather than frequency variable. They also presented its applications to difference equations, as well as the Gårding and the sharp Gårding inequality on \( \mathbb{Z}^n \). In this setting the pseudo-differential operators on \( \mathbb{Z}^n \) are defined by

\[
t_m f(n') := \int_{\mathbb{T}^n} e^{2\pi i n' \cdot \xi} m(n', \xi)(\mathcal{F} f)(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{Z}^n), \quad n' \in \mathbb{Z}^n.
\]  

(1.1)

Under suitable conditions on the symbol \( m \), we have that \( t_m \) is a continuous linear operator from the discrete Schwartz space \( \mathcal{S}(\mathbb{Z}^n) \) into itself. This is the case of symbols in the discrete Hörmander classes (see e.g. [2]). The references Rabinovich[34, 35], and Rabinovich and Roch[37, 38] are predecessor works on the subject. Analytic and spectral properties as the ellipticity and Fredholmness, nuclearity, the spectral traces, and the \( L^p \) and weak-\( L^p \)-boundedness for these operators have been considered in Botchway, Kibiti, and Ruzhansky [2], Catana [7], Wong[49], Rodriguez [39], Dasgupta and Kumar [10], and the works of the first and third author [5]. In particular, the Calderón-Vaillancourt theorem and the Gohberg Lemma for the discrete Hörmander classes in [2] have been proved in Cardona [6].

In this paper we are interested in providing sharp conditions on \( m \), in order that \( t_m \) in (1.1), belongs to the Dixmier ideal \( L^{1,\infty}(\ell^2(\mathbb{Z}^n)) \). The Dixmier traceability of pseudo-differential operators on smooth manifolds is a classical problem. Indeed, the problem was treated by using the notion of local symbols in the works of Connes, Dixmier, Fedosov, Grubb, Schrohe, [8, 19, 20, 22, 32, 47] and references therein. We refer the reader to [4] in which the Dixmier traceability of pseudo-differential operators on compact manifolds has been studied by using the matrix-valued quantization (see [40]).

Our main result is Theorem 3.3, where we prove that, under the condition \( t_m \in \Psi_{\text{cl}}^{-n}(\mathbb{Z}^n) \) (see Definition 2.4) \( t_m \) being a positive pseudo-differential operator, its Dixmier trace is given, surprisingly, by the formula

\[
\text{Tr}_\omega(t_m) = \frac{1}{n(2\pi)^n} \int_{\mathbb{T}^n} \int_{S^{n-1}} [i^\ast(\tau)(x, \xi)]_{[-n]} d\Sigma(\xi) dV_{\text{vol}} x, \quad \tau(x, k) := m(-k, x),
\]

(1.2)

where \( d\Sigma(\xi) \) is the surface measure associated to the sphere \( S^{n-1} \subset \mathbb{R}^n \), and \( dV_{\text{vol}} x \) is the volume form on \( M = [0, 1)^n \). The integral formula (1.2) shows that we can capture the spectral information in the Dixmier trace of an operator in terms of the geometry of the manifold \( \mathbb{T}^n \times S^{n-1} \), where the density \( d\Sigma(\xi) dV_{\text{vol}} x \) appears, together with the Fourier analysis encoded in the global symbol \( i^\ast \tau = i^\ast(m(-\cdot, \cdot)) \).
In order to prove this formula, we will provide some preliminaries in Section 2 and later we will present the proof in Section 3.

2. PSEUDO-DIFFERENTIAL CALCULUS ON $\mathbb{Z}^n$ AND THE $n$-TORUS $\mathbb{T}^n$

In this paper we will use the usual notation and terminology given in the standard monographs or papers (see e.g. [18, 23, 24, 40, 48, 49, 42]). The discrete Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ is the space of all functions $\phi : \mathbb{Z}^n \to \mathbb{C}$ such that

$$\forall M \in \mathbb{R}, \exists C_M > 0, \text{ such that } |\phi(\xi)| \leq C_M |\xi|^M,$$

(2.1) where $|\xi| = (1 + |\xi|^2)^{\frac{1}{2}}$. The toroidal Fourier transform is defined as follows: for any $f \in C^\infty(\mathbb{T}^n)$,

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} e^{-i2\pi(x,\xi)} f(x) dx, \ \xi \in \mathbb{Z}^n.$$  

The inverse toroidal Fourier transform is given by the following formula: for $f \in C^\infty(\mathbb{T}^n)$

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} e^{i2\pi(x,\xi)} \hat{f}(\xi), \ x \in \mathbb{T}^n.$$  

The fundamental tool in the analysis of periodic operators, that is, operators defined on function spaces on $\mathbb{T}^n$, are the Hörmander class of periodic symbols. For every $m \in \mathbb{R}$, the periodic symbol of Hörmander class of order $m$ and of $(\rho, \delta)$-type, denoted by $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$, $0 \leq \rho, \delta \leq 1$, consists of those functions $a(x, \xi)$ which are smooth in $(x, \xi) \in \mathbb{T}^n \times \mathbb{R}^n$ and which satisfy the following toroidal symbols inequalities

$$|\partial_x^\gamma \partial_{\xi}^\alpha a(x, \xi)| \leq C_{\alpha,\beta} |\xi|^{m - \rho|\alpha | + \delta|\beta |}.$$  

(2.2)

It is well-known fact that Symbols in $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$ can be considered as symbols in $S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ (see [23, 40]) of order $m$ which are 1-periodic in $x$. If $a(x, \xi) \in S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{R}^n)$, the corresponding pseudo-differential operator is defined by

$$a(X, D_x)u(x) = \int_{\mathbb{T}^n} \int_{\mathbb{R}^n} e^{i2\pi(x-y,\xi)} a(x, \xi) u(y) d\xi dy, \ u \in C^\infty(\mathbb{T}^n).$$  

(2.3)

The symbol class $S^m_{\rho,\delta}(\mathbb{T}^n \times \mathbb{Z}^n)$, $0 \leq \rho, \delta \leq 1$, consists of those functions $a(x, \xi)$ which are smooth in $x$ for all $\xi \in \mathbb{Z}^n$ and satisfy

$$\forall \alpha, \beta \in \mathbb{N}^n, \exists C_{\alpha,\beta} > 0, \ |\Delta_{\xi}^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha,\beta} |\xi|^{m - \rho|\alpha | + \delta|\beta |}.$$  

(2.4)

The operator $\Delta_{\xi}$ in (2.4) is the difference operator which is defined as follows: if $f : \mathbb{Z}^n \to \mathbb{C}$ is a discrete function and $(e_j)_{1 \leq j \leq n}$ is the canonical basis of $\mathbb{R}^n$, then $\Delta_{\xi}^\alpha$ is defined by

$$(\Delta_{\xi}^\alpha f)(\xi) = f(\xi + e_j) - f(\xi).$$  

(2.5)

If $k \in \mathbb{N}$, denote by $\Delta_{\xi}^k$ the composition of $\Delta_{\xi}$ with itself $k$-times. Finally, if $\alpha \in \mathbb{N}^n$, $\Delta_{\xi}^\alpha = \Delta_{\xi_1}^{\alpha_1} \cdots \Delta_{\xi_n}^{\alpha_n}$. 

The toroidal pseudo-differential operator (or periodic pseudo-differential operator) corresponding to the symbol \( a(x, \xi) \in S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{Z}^n) \) is defined by
\[
a(x, D_x)u(x) = \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i (x, \xi)} a(x, \xi) \hat{u}(\xi), \quad u \in C^\infty(\mathbb{T}^n). \tag{2.6}
\]

Similarly, the Hörmander classes on \( \mathbb{Z}^n \), (see Bothcway, Kibity and Ruzhansky [2]) \( S^m_{\rho, \delta}(\mathbb{Z}^n \times \mathbb{T}^n) \), \( 0 \leq \delta, \rho \leq 1 \), are defined by those functions \( \sigma : \mathbb{Z}^n \times \mathbb{T}^n \to \mathbb{C} \) such that \( \sigma(n', \cdot) \in C^\infty(\mathbb{T}^n) \) for all \( n' \in \mathbb{Z}^n \), and for all multi-indices \( \alpha, \beta \) there exists a constant \( C_{\alpha, \beta} > 0 \) such that
\[
|\partial^\alpha_x \partial^\beta_\xi \sigma(n', x)| \leq C_{\alpha, \beta}(1 + |n'|)^{m-\rho|\alpha|+\delta|\beta|}, \tag{2.7}
\]
for all \( n' \in \mathbb{Z}^n \) and for all \( x \in \mathbb{T}^n \). In this case we denote \( \Psi^m_{\rho, \delta}(\mathbb{Z}^n \times \mathbb{T}^n) = \{ t_\sigma : \sigma \in S^m_{\rho, \delta}(\mathbb{Z}^n \times \mathbb{T}^n) \} \).

The following proposition says that there exists a method to interpolate the second argument of symbols on \( \mathbb{T}^n \times \mathbb{Z}^n \) in a smooth way to get a symbol defined on \( \mathbb{T}^n \times \mathbb{R}^n \). The proof can be found in [40, Theorem 4.5.3].

**Proposition 2.1.** Let \( 0 \leq \delta \leq 1, 0 < \rho \leq 1 \). The symbol \( a \in S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{Z}^n) \) if only if there exists a Euclidean symbol \( a' \in S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{R}^n) \) such that \( a = a'|_{\mathbb{T}^n \times \mathbb{Z}^n} \).

It is an interesting but non trivial fact that the definition of pseudo-differential operator on a torus (2.6) given by Agranovich and by Hörmander (2.3) are equivalent. McLean [26] prove this for all Hörmander classes \( S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{Z}^n) \). A different proof to this fact can be found in [40, Corollary 4.6.13].

**Proposition 2.2.** (Equality of Operators Classes). For \( 0 \leq \delta \leq 1, 0 < \rho \leq 1 \), we have \( \Psi^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{Z}^n) = \Psi^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{R}^n) \).

In our further analysis it will be useful record the following fact: there exists a bijection
\[
i^* : S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{Z}^n) \to S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{R}^n) \tag{2.8}
\]
which associates every symbol \( \sigma \in S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{Z}^n) \) to an Euclidean symbol \( i^* \sigma \in S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{R}^n) \) such that
\[
\sigma(x, D_x) f(x) = (i^* \sigma)(X, D_x) f(x), \quad f \in C^\infty(\mathbb{T}^n). \tag{2.9}
\]
The existence of the mapping mapping \( i^* \) in (2.10) is a natural consequence of Proposition 2.2. This can be constructed explicitly in terms of the periodisation technique developed by Ruzhansky and Turunen in [40, Section 4.2].

**Definition 2.3** (Classical symbols on \( \mathbb{T}^n \)). A classical pseudo-differential operator \( A = a(X, D_x) \) on \( \mathbb{T}^n \) is an operator with symbol \( a \in S^m_{\rho, \delta}(\mathbb{T}^n \times \mathbb{R}^n) \), admitting an asymptotic expansion
\[
a(x, \xi) \sim \sum_{j=0}^{\infty} a_{m-j}(x, \xi) \tag{2.10}
\]
where \( a_{m-j}(x, \xi) \) are homogeneous functions with respect to \( \xi \) of degree \( m - j \) for \( \xi \) far from to zero.
The set of classical symbols of order $m$ on $\mathbb{T}^n$ will be denoted by $S^m_{cl}(\mathbb{T}^n \times \mathbb{R}^n)$ and the corresponding class of periodic pseudo-differential operators will be denoted by $\Psi^m_{cl}(\mathbb{T}^n)$.

Now, with the help of the classical symbols on $\mathbb{T}^n$, we introduce classical pseudo-differential operators on $\mathbb{Z}^n$. We start with the following definition of the classical symbols of $\mathbb{Z}^n$.

**Definition 2.4 (Classical symbols on $\mathbb{Z}^n$).** A symbol $\sigma \in S^m_{1,0}(\mathbb{Z}^n \times \mathbb{T}^n)$ is called classical, if the symbol $\tau(x, k) := \sigma(-k, x)$ satisfies that $i^* \tau \in S^m_{cl}(\mathbb{T}^n \times \mathbb{R}^n)$.

The set of classical symbols of order $m$ on $\mathbb{Z}^n$ will be denoted by $S^m_{cl}(\mathbb{Z}^n \times \mathbb{T}^n)$ and the corresponding class of periodic pseudo-differential operators will be denoted by $\Psi^m_{cl}(\mathbb{Z}^n)$.

Periodic and also discrete Hörmander classes of pseudo-differential operators are closed under compositions, adjoints and parametrices. We refer the reader to [2, 40, 41] for details.

### 3. Dixmier Traces for Pseudo-Differential Operators on $\mathbb{Z}^n$

In this section, we present our main result. We begin this section by recalling some basic facts on Dixmier trace and related results.

Let $H$ be a Hilbert space and let $\mathcal{L}(H)$ be the algebra of all bounded linear operators on $H$. We say that a linear operator $A \in \mathcal{L}(H)$ is in the Dixmier class $\mathcal{L}^{(1,\infty)}(H)$ if

$$\sum_{1 \leq n \leq N} s_n(A) = O(\log(N)), \quad N \to \infty,$$

where $s(A) := \{s_n(A)\}_{n}$ denotes the sequence of singular values of $A$, i.e. the square roots of the eigenvalues of the non-negative self-adjoint operator $A^*A$, and $s_n(A)$ are arranged in increasing order. So, $\mathcal{L}^{(1,\infty)}(H)$ is endowed with the norm

$$\|A\|_{\mathcal{L}^{(1,\infty)}(H)} = \sup_{N \geq 2} \frac{1}{\log(N)} \sum_{1 \leq n \leq N} s_n(A).$$

The Dixmier trace $\text{Tr}_\omega$ of a positive operator $A$ in $\mathcal{L}^{(1,\infty)}(H)$ can be formulated by using an average function $\omega$ of the following form

$$\text{Tr}_\omega(A) = \lim_{\omega} \frac{1}{\log(N)} \sum_{1 \leq n \leq N} s_n(A),$$

where $\lim_{\omega}$ is a positive linear functional on $l^\infty(\mathbb{N})$, the set of bounded sequences, satisfying:

- $\lim_{\omega}(\alpha_n) \geq 0$ for all $\alpha_n \geq 0$;
- $\lim_{\omega}(\alpha_n) = \lim(\alpha_n)$ whenever the ordinary limit exists;
- $\lim_{\omega}(\alpha_1, \alpha_2, \alpha_3, \ldots) = \lim_{\omega}(\alpha_n)$.

The functional $\text{Tr}_\omega$ can be defined for positive operators, and later extended to the whole ideal $\mathcal{L}^{(1,\infty)}(H)$ by linearity (in this case $\text{Tr}_\omega$ is not necessarily a positive functional).

Let us assume that $M$ is a closed manifold. In the framework of noncommutative geometry, a remarkable result due to Connes shows that any classical
pseudo-differential operator, of order of the minus of the dimension of the manifold underlying, acting on $L^2(M)$ lies to the Dixmier measurable class, and moreover the Dixmier trace coincides with the noncommutative residue, which is the unique trace (up to by factors) in the algebra of pseudo-differential operators with classical symbols on a closed manifold.

Now, we recall that for a compact manifold $M$ of dimension $κ$ without boundary, a classical pseudo-differential operator $A$ on $M$ of order $m$, is defined by means of local symbols, this means that for any local chart $U$, the operator $A$ reads

$$Au(x) = \int_{T^*_x U} e^{ix\xi} \sigma^A(x, \xi) \hat{u}(\xi) \, d\xi$$

where $\sigma^A(x, \xi)$ is a smooth function on $T^*_x U \cong U \times \mathbb{R}^n$, $T^*_x U = \mathbb{R}^n$, and admits an asymptotic expansion

$$\sigma^A(x, \xi) \sim \sum_{j=0}^{\infty} \sigma^A_{m-j}(x, \xi)$$

(3.3)

where $\sigma^A_{m-j}(x, \xi)$ are homogeneous functions with respect to $\xi$ of degree $m-j$ for $\xi$ far away from the section zero. The set of classical pseudo-differential operators of order $m$ is denoted by $Ψ^m_{cl}(M)$. For $A \in Ψ_{cl}(M)$, for $x \in M$, $\int_{|\xi|=1} \sigma_{-κ}(x, \xi) \, dΣ(\xi)$, defines a local density which can be glued over $M$. So, a linear functional called the noncommutative residue is defined by

$$\text{res}(A) = \frac{1}{\pi(2\pi)^{\frac{κ}{2}}} \int_M \int_{|\xi|=1} \sigma_{-κ}^A(x, \xi) \, dΣ(\xi) \, dV_{Vol}x.$$  

(3.4)

which vanishes on non-integer order classical operators.

In order to provide a trace formula for pseudo-differential operators on $\mathbb{Z}^n$ we will use the following identity in [2]

$$t_\sigma = \mathcal{F}_{\mathbb{Z}^n}(\tau(x, D_x)^*) \mathcal{F}_{\mathbb{Z}^n}^{-1},$$

(3.5)

and the following fundamental result due to Connes [8].

**Theorem 3.1** (A. Connes). Let $M$ be a closed manifold of dimension $κ$. Then every classical and positive pseudo-differential operator $A$ of order $-κ$ lies in $L^{1,\infty}(L^2(M))$ and

$$\text{Tr}_{\text{w}}(A) = \text{res}(A).$$

(3.6)

**Corollary 3.2.** Let $τ(x, D_x)$ be a classical positive pseudo-differential operator with symbol $τ \in S^{-n}_{1,0}(\mathbb{T}^n \times \mathbb{R}^n)$. Then $τ(x, D_x)$ belongs to $L^{1,\infty}(L^2(\mathbb{T}^n))$ and

$$\text{Tr}_{\text{w}}(τ(x, D_x)) = \text{res}(τ(x, D_x)).$$

(3.7)

The following theorem is our main result on the Dixmier traceability of pseudo-differential operators on $\mathbb{Z}^n$. 
Theorem 3.3. Let us assume that $t_σ ∈ Ψ^{−n}_cl(ℤ^n)$ is a positive discrete pseudo-differential operator. Then the operator $t_σ$ is Dixmier traceable on $ℓ^2(ℤ^n)$ and its Dixmier trace is given by

$$\text{Tr}_ω(t_σ) = \frac{1}{n(2π)^n} \int_{T^n} \int_{S^{n−1}} \left[ i^*(τ)(x, ξ) \right]_{(−n)} dσ(ξ) d_{\text{Vol}}x, \quad τ(x, k) := σ(−k, x), \quad (3.8)$$

where $[i^*(τ)(x, ξ)]_{(−n)}$ in the homogeneous term of order $−n$ is the asymptotic expansion of the symbol $i^*(τ)(x, ξ)$ in homogeneous components as in (2.10).

Proof. Let us consider the following identity due to Botchway, Kibiti and Ruzhansky [2]

$$t_σ = Ψ_{Z^n}(τ(x, D_x)) F_{Z^n}^{-1}.$$  \hspace{1cm} (3.9)

First we will prove that the set of the singular values of $t_σ$ agrees with the set of singular values of $τ(x, D_x)^*$. We show this by using the fact that the spectrum is stable under unitary transformations. Since the Fourier transform on $Z^n$ is a unitary operator, we have that

$$s(τ(x, D_x)^*) = \{ √λ : λ ∈ \text{Spectrum}(τ(x, D_x) τ(x, D_x)^* ) \}$$

$$= \{ √λ : λ ∈ \text{Spectrum}(Ψ_{Z^n}(τ(x, D_x)) τ(x, D_x)^* F_{Z^n}^{-1} ) \}$$

$$= \{ √λ : λ ∈ \text{Spectrum}(Ψ_{Z^n}(τ(x, D_x)) F_{Z^n}^{-1} τ(x, D_x)^* F_{Z^n}^{-1} ) \}$$

$$= \{ √λ : λ ∈ \text{Spectrum}(Ψ_{Z^n}(τ(x, D_x)) F_{Z^n}^{-1} \circ t_σ ) \}$$

$$= \{ √λ : λ ∈ \text{Spectrum}(t_σ^* t_σ ) \} = s(t_σ),$$

where in the last line we have used $t_σ^* = Ψ_{Z^n}(τ(x, D_x)) F_{Z^n}^{-1}$. Now, it is clear that $t_σ$ is Dixmier traceable if and only if $τ(x, D_x)^* is Dixmier traceable and in a such case $\text{Tr}_ω(t_σ) = \text{Tr}_ω(τ(x, D_x)^*)$. The positivity of $t_σ$ implies that $τ(x, D_x)$ is also positive. Because $t_σ ∈ Ψ^{−n}_cl(ℤ^n)$ we have that $τ(x, D_x) ∈ Ψ^{−n}_cl(ℤ^n)$ and consequently $τ(x, D_x)$ is a compact operator on $L^2(ℤ^n)$, (see e.g. [2], also [27] for $n = 1$). In particular $τ(x, D_x)$ is bounded and taking into account its positivity we conclude that $τ(x, D_x)$ is self-adjoint. So, if we assume the Dixmier traceability of $τ(x, D_x) = τ(x, D_x)^*$, we deduce $\text{Tr}_ω(τ(x, D_x)^*) = \text{Tr}_ω(τ(x, D_x))$. Since $τ(x, D_x) ∈ Ψ^{−n}_cl(ℤ^n)$ by the Connes Theorem 3.1 in the form of Corollary 3.2, we conclude that $τ(x, D_x)$ is Dixmier traceable and its Dixmier trace can be computed from its noncommutative residue. So, we have

$$\text{Tr}_ω(τ(x, D_x)) = \text{Tr}_ω((i^*τ)(X, D_x)) = \text{res}(i^*τ)(X, D_x)$$

$$= \frac{1}{n(2π)^n} \int_{T^n} \int_{S^{n−1}} [i^*(τ)(x, ξ)]_{(−n)} dσ(ξ) d_{\text{Vol}}x.$$  \hspace{1cm} (3.10)

Consequently we obtain

$$\text{Tr}_ω(t_σ) = \frac{1}{n(2π)^n} \int_{T^n} \int_{S^{n−1}} [i^*(τ)(x, ξ)]_{(−n)} dσ(ξ) d_{\text{Vol}}x.$$  \hspace{1cm} (3.10)

Thus, we finish the proof of the theorem.  \hspace{1cm} □
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