Quantum gauging from classical gauging of nonlinear algebras

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Abstract: We extend our previous theory of the gauging of classical quadratically nonlinear algebras without a central charge but with a coset structure, to the quantum level. Inserting the minimal anomalies into the classical transformation rules of the currents introduces further quantum corrections to the classical transformation rules of the gauge fields and currents which additively renormalize the structure constants. The corresponding Ward identities are the $c \to \infty$ limit of the full quantum Ward identities, and reveal that the $c \to \infty$ limit of the quantum gauge algebra closes on fields and currents. Two examples are given.

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1 Introduction and summary.

The past two years several new gauge theories have been constructed \[1, 2, 3, 4\], which are extensions of two-dimensional gravity \[5\] and which are based on so-called nonlinear Lie algebras \[6, 7\]. The latter are algebras of the kind \[T_A, T_B] = T_C f^{C}_{AB} + T_C T_D V^{DC}_{AB} + \ldots\].

We shall restrict our attention to quadratically nonlinear algebras. To every generator \(T_A\) one associates a gauge field \(h_\mu^A\) and a classical current \(u_{A\mu}\). To construct corresponding quantum currents, one proceeds as follows. One begins by coupling matter to the external gauge fields \(h_\mu^A\) and fixing all local symmetries without anomalies. By integrating out the matter, one finds the induced action \(S^{\text{ind}}\) which depends on the gauge fields and on one (or more) central charge \(c\). This central charge characterizes the matter system and is for example the number of scalar fields coupling to the gauge fields. Defining the quantum currents as \(u_{A\mu} \sim \partial S^{\text{ind}}/\partial h_\mu^A\), one can then derive Ward identities for the remaining anomalous symmetries. These Ward identities provide functional differential equations for the induced action. They are of the generic form \(D_\mu u_{A\mu} = A_{nA}\), where \(D_\mu\) is a covariant derivative and \(A_{nA}\) are the anomalies. The anomalies \(A_{nA}\) are as always local expressions depending on \(h_\mu^A\), but the expression \(D_\mu u_{A\mu}\) is in general nonlocal and very complicated. However, for \(c \to \infty\) it reduces to a local functional of \(h_\mu^A\) and \(u_{A\mu}\). Namely, in addition to terms linear in \(h_\mu^A\), the covariant derivatives \(D_\mu\) also contain terms bilinear in \(h_\mu^A\) and \(u_{A\mu}\), which are due to the nonlinear terms in the algebra.

From this \(c \to \infty\) Ward identity one can extract the \(c \to \infty\) transformation laws of the quantum currents, denoted by \(\delta^\infty u_{A\mu}\), and by requiring invariance of the \(c \to \infty\) Ward identity, one can deduce the corresponding \(c \to \infty\) transformation laws of the gauge fields, denoted by \(\delta^\infty h_\mu^A\). In examples \[3, 4\] it has been found that these \(\delta^\infty h_\mu^A\) and \(\delta^\infty u_{A\mu}\) form a closed gauge algebra. \[\dagger\]

In this article we shall consider nonlinear Lie algebras defined by an operator product expansion (OPE). These algebras are quantum algebras with central charges. To obtain a

\[\dagger\]Since all symmetries without anomalies have been fixed, one does not have gauge fixing terms and corresponding ghosts; consequently, one deals with gauge transformations and not BRST symmetries.
corresponding classical nonlinear Lie algebra without central charges we first take the $c \to \infty$ limit of the full nonlinear quantum algebra (which converts the quantum algebra to a Poisson bracket algebra but with central charge) and then take in the result the $c \to 0$ limit (which removes the central charge). The theory of the gauging of such classical nonlinear Lie algebras was established by us in collaboration with K. Schoutens [8, 9], and it leads to classical $\delta_{cl} h^A_\mu$ and $\delta_{cl} u^A_\mu$ with an open classical nonlinear gauge algebra. Using this formalism as a starting point, we shall construct the $\delta^\infty u^A_\mu$ by adding the minimal anomalies to $\delta_{cl} u^A_\mu$ and requiring closure of the algebra of $\delta^\infty u^A_\mu$. In this way we find that the structure constructs $f^{A_{BC}}$ become replaced by $f^{A_{BC}} + g^{A_{BC}}$ where $g^{A_{BC}}$ can be viewed as additive renormalizations. From these $\delta^\infty u^A_\mu$ one can also (re)construct the $c \to \infty$ Ward identities. In the original classical gauge algebra without central charges, closure of the gauge algebra on $h^A_\mu$ is violated by terms proportional to $D^A_{cl} u^A_\mu$. In the new $c \to \infty$ gauge algebra with central charges, the extra terms in $\delta^\infty u^A_\mu$ due to the anomalies complete $D^A_{cl} u^A_\mu$ to the full $c \to \infty$ Ward identity and as a result, the algebra of $\delta^\infty h^A_\mu$ now also closes. Hence, the gauge algebra becomes uniformly closed due to quantum effects. 

The idea that the anomalies introduce extra terms in $\delta_{cl} u^A_\mu$ which close the gauge algebra of $\delta^\infty h^A_\mu$, was first proposed in [4]. However, as we have found, the extra terms are not only the anomaly itself, but also terms which are of the same form as the term $u^A_\mu f^{C_{AB}} \epsilon^B$. These extra terms can be written as $u^A_\mu g^{C_{AB}} \epsilon^B$ and renormalize the structure constants as mentioned above.

Our results give a very simple algorithm to construct the $c \to \infty$ quantum algebras from the corresponding (usually much simpler) classical algebras. Input is the existence of the $c \to \infty$ algebra, output all (often complicated) quantum corrections in explicit form.

In section 2 we review the gauging of classical nonlinear algebras, and give the example of chiral $W_3$ gravity. In section 3 we extend these results to the quantum level, and work through

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§For those induced gauge theories which can be obtained from WZWN models (with a suitable gauge group) by imposing constraints on the WZWN currents [8], closure of the $\delta^\infty$ algebra is guaranteed, since the local gauge parameters in the induced gauge theories are obtained from those of the WZWN models by requiring the constraints to be preserved by the local symmetries of the induced gauge theories.
the cases of $W_3$ gravity and chiral supergravity. In section 4 we discuss to which OPE our new formalism can be applied, and we comment on the relations to our earlier work.

We should perhaps stress that we have no corresponding results for finite $c$. However, when passing from the induced to the effective theory (i.e., integrating over the gauge fields), one finds that the Ward identities for the effective theory at arbitrary values of $c$ are the same, up to multiplicative renormalizations, as those for the induced theory for $c \to \infty$ [10]. For a discussion of the quantum theory for general $c$ and its renormalizability, see [10, 11].

2 Gauging of classical nonlinear algebras without central charge.

In [8] we developed a theory of classical gauging of nonlinear algebras. The simplest case are the quadratically nonlinear algebras, defined by

$$[T_A, T_B] = m_{AB} + T_C f^{C}_{AB} + T_D T_C V^{CD}_{AB}$$

where $m_{AB}$ are the central charges. When dealing with classical algebras, the brackets may be realized by Poisson brackets, and no ordering ambiguities in the last term arise. However, if the $T_A$ are quantum operators, one must normal-order the last term on the right-hand side. The Jacobi identities restrict the constants $m_{AB}, f^{A}_{BC}$ and $V^{CD}_{AB}$, and those for the quantum algebras contain extra terms due to the nonassociativity of the normal-ordered product [7].

An important subclass are the nonlinear algebras with a coset structure. The generators $T_A$ can then be divided into coset generators $K_\alpha$ and generators $H_i$ which form a subalgebra, such that only $V^{ij}_{\alpha \beta}$ is nonvanishing. In fact, all known quadratically nonlinear Lie algebras are of this kind. For these algebras a nilpotent BRST charge $Q$ has been written down, both at the classical level and at the quantum level [7] . For the classical quadratically nonlinear Lie algebras with coset structure, the Jacobi identities read

$$m_{AD} f^{D}_{BC} + \text{(cyclic in } ABC) = 0$$

*The quantum $Q$ is obtained from the classical $Q$ by a renormalization of the $f^{A}_{BC}$ proportional to

$V^{AB}_{CD} f^{D}_{BE}$.**
\[ f^E_{\ AD} f^D_{\ BC} + 2 m_{\ AD} V^{\ DE}_{\ BC} + \text{(cyclic in } ABC) = 0 \]
\[ V^E_{\ AD} f^D_{\ BC} + f^E_{\ AD} V^{DF}_{\ BC} + f^F_{\ AD} V^{DE}_{\ BC} + \text{(cyclic in } ABC) = 0 \]  

(2.2)

We consider now classical quadratically nonlinear algebras with a coset structure, but without central charges. One associates with every \( T_A \) a gauge field \( h_\mu^A \) and a current \( u_A \). Since in all cases considered the index \( \mu \) of the gauge fields and currents takes only one value (for example, \( h_{++} \) or \( u_{--} \)) we shall suppress the index \( \mu \) of \( u_A^\mu \), but keep it on \( h_\mu^A \) for historical reasons. These gauge fields and currents transform classically as follows

\[
\delta_{cl} h_\mu^A = \partial_\mu \epsilon^A + (f^A_{\ BC} + 2 u_D V^{DA}_{\ BC}) h_\mu^C \epsilon^B \\
\delta_{cl} u_A = u_C (f^C_{\ AB} + u_D V^{DC}_{\ AB}) \epsilon^B 
\]  

(2.3)

We shall presently comment on the factor 2. The gauge commutator closes on \( u_A \)

\[
[\delta_{cl}(\epsilon_1), \delta_{cl}(\epsilon_2)] u_A = \delta_{cl}(\epsilon_3) u_A 
\]  

(2.4)

where the structure constants are source-dependent

\[
\epsilon_3^A = \tilde{f}^A_{\ BC} \epsilon_1^C \epsilon_2^B, \quad \tilde{f}^A_{\ BC} = f^A_{\ BC} + 2 u_D V^{DA}_{\ BC} 
\]  

(2.5)

The covariant derivatives of the sources follow from (3)

\[
D_\mu^{cl} u_A = \partial_\mu u_A - u_C f^C_{\ AB} h_\mu^B - u_D u_C V^{CD}_{\ AB} h_\mu^B 
\]  

(2.6)

and transform in the coadjoint representation

\[
\delta(D_\mu^{cl} u_A) = (D_\mu^{cl} u_C) \tilde{f}^C_{\ AB} \epsilon^B 
\]  

(2.7)

The gauge commutator on \( h_\mu^A \) does not close; it is off by a term proportional to \( D_\mu^{cl} u_D \)

\[
[\delta(\epsilon_1), \delta(\epsilon_2)] h_\mu^A = \delta(\epsilon_3) h_\mu^A - 2 D_\mu^{cl} u_D V^{DA}_{\ BC} \epsilon_1^C \epsilon_2^B 
\]  

(2.8)

Curvatures are defined by

\[
[D_\mu, D_\nu] u_A = - u_C (f^C_{\ AB} + u_D V^{DC}_{\ AB}) R_{\mu \nu}^B 
\]  

(2.9)
and $R_{\mu\nu}^A$ is given by the usual expression, but with $\tilde{f}^A_{BC}$ instead of $f^A_{BC}$. The curvatures transform in the adjoint representation plus a term proportional to the covariant derivative of the sources

$$\delta R_{\mu\nu}^A = \tilde{f}^A_{BC} R_{\mu\nu}^C \epsilon^B + \left\{ 2D_\mu^c u_D V^D A_{BC} h_\nu^C \epsilon^B - \mu \leftrightarrow \nu \right\}$$  \hspace{1cm} (2.10)

The covariant derivatives of the curvature tensor are given by

$$D_\rho^c R_{\mu\nu}^A = \partial_\rho R_{\mu\nu}^A - \tilde{f}^A_{BC} R_{\mu\nu}^C h_\rho^B - (D_\mu^c u_D V^D A_{BC} h_\nu^C h_\rho^B - \mu \leftrightarrow \nu)$$  \hspace{1cm} (2.11)

Note that in (11) the factor 2 which appeared in (10), has been replaced by unity. They satisfy the Bianchi identities

$$D_{[\mu}^c R_{\nu\rho]}^A = 0.$$  \hspace{1cm} (2.12)

The previous results require only the Jacobi identities in (2) with $m_{AB} = 0$. Since the latter remain valid upon scaling $V^{AB}_{CD}$ by a factor, the results in (3) contain a free parameter. As we will see in the next section, inclusion of central charges fixes the scale of $V^{AB}_{CD}$, and the correct results are those given in (3). In our previous work on classical $W$ gravity we had chosen the normalization $\delta_c h_\mu^A = \ldots + u_D V^{DC}_{AB} \ldots$ and $\delta_c u_A = \ldots + \frac{1}{2} u_D V^{DC}_{AB} \ldots$ which differs from the present rules by a factor 2.

The classical nonlinear gauge algebras with $c = 0$ discussed in this section arise as symmetries of the classical action for the coupling of matter to external gauge fields. A typical example is chiral $W_3$ gravity. The classical matter-coupled action reads [1]

$$S_{cl} = \frac{1}{\pi} \int \left[ -\frac{1}{2} \partial_+ \varphi^i \partial_- \varphi^i - h_{++} T_{--} - b_{+++} W_{--} \right] d^2 x$$  \hspace{1cm} (2.13)

with $d_{ijk}$ a totally symmetric constant symbol satisfying $d_{i(jk} d_{\ell)mi} = \delta_{i(jk} \delta_{\ell)mi}$. The symmetries are given by

$$\delta \varphi^i = \epsilon \partial_- \varphi^i + \lambda_{++} \partial_- \varphi^j \partial_- \varphi^k d_{ijk}$$

$$\delta h = \partial_+ \epsilon - h \partial_- \epsilon + \epsilon \partial_- h + (\lambda \partial_- b - b \partial_- \lambda)(-2T_{--})$$

$$\delta b = \epsilon \partial_- b - 2b \partial_- \epsilon + \partial_+ \lambda - h \partial_- \lambda + 2\lambda \partial_- h$$  \hspace{1cm} (2.14)
The local gauge algebra reads [13]

\[ [\delta(\epsilon_1), \delta(\epsilon_2)] = \delta(\epsilon_3 = -\epsilon_1 \partial_\epsilon \epsilon_2 + \epsilon_2 \partial_\epsilon \epsilon_1) \]

\[ [\delta(\epsilon), \delta(\lambda)] = \delta(\hat{\lambda} = -\epsilon \partial_\epsilon \lambda + 2\lambda \partial_\epsilon \epsilon) \]

\[ [\delta(\lambda_1), \delta(\lambda_2)] \begin{pmatrix} \varphi^i \\ h \\ b \end{pmatrix} = \delta(\hat{\epsilon}) \begin{pmatrix} \varphi^i \\ h \\ b \end{pmatrix} + \left( \begin{array}{c} 2\pi k \partial_\varphi \varphi^i \partial S/\partial h \\ -2\pi k \partial_\varphi \varphi^i \partial S/\partial \varphi^i \\ 0 \end{array} \right) \]

\[ \hat{\epsilon} = -2kT_{\epsilon}, \ k = -\lambda_1 \partial_\epsilon \lambda_2 + \lambda_2 \partial_\epsilon \lambda_1 \] (2.15)

Hence on \( \varphi^i \) it is closed up to the \( h \) field equation, and on \( h \) up to the \( \varphi \) field equation. The terms with field equations form separately a trivial symmetry, which explains the relative minus sign and the common factor \((2\pi k \partial_\varphi \varphi^i)\).

Defining \( u = -\frac{1}{2}(\partial \varphi)^2 \) and \( v = -\frac{1}{3}d_{ijk} \partial \varphi^i \partial \varphi^j \partial \varphi^k \) with \( \partial = \partial_- \) one finds

\[ \delta_{cl} u = 2\partial \epsilon u + \epsilon \partial u + 2\lambda \partial v + 3\partial \lambda v \]

\[ \delta_{cl} v = 3\partial \epsilon v + \epsilon \partial v - 4(\partial \lambda uu + \lambda u \partial u) \] (2.16)

In order that this corresponds to (3), the antisymmetry of \( f^{A\alpha i} \) in \( \alpha, i \) requires that \( \int [\lambda \delta(\epsilon)v + \epsilon \delta(\lambda)u] d^2x \) vanishes. This fixes the scale (and the form) of the \( \delta(\lambda)u \) terms with respect to the \( \delta(\epsilon)v \) terms (which accounts for the absence of a factor \( \frac{1}{15} \) which is present in our earlier work [3]). One easily verifies that these currents satisfy the same classical closed gauge algebra as the gauge field \( b \), except that \( \hat{\epsilon} \) is now twice as large

\[ [\delta_{cl}(\lambda_1), \delta_{cl}(\lambda_2)] \begin{pmatrix} u \\ v \end{pmatrix} = \delta_{cl}(\hat{\epsilon} = -4(\lambda_2 \partial \lambda_1 - \lambda_1 \partial \lambda_2)u) \begin{pmatrix} u \\ v \end{pmatrix} \] (2.17)

In section 4 we shall explain this difference by a factor 2, but now we continue with (17).

In our earlier work we stated that the classical gauge algebra closes on the gauge fields up to field equations. Although this is correct, from our present perspective we prefer to interpret the term \(-2\pi k \partial_\varphi \varphi^i \partial S/\partial \varphi^i\) in (15) as a covariant derivative of the currents, namely as \(-D_\mu \epsilon u_D V^{DA}_{BC} \lambda_1^C \lambda_2^B \) with \( u_D = u, V^{DA}_{BC} \lambda_1^C \lambda_2^B = -2(\lambda_2 \lambda_1^B - \lambda_1 \lambda_2^B) \) and \( D_\mu \epsilon u_D \) given by

\[ D_\mu \epsilon u = (\partial_+ - 2\partial_- h - h \partial_-)u - (3\partial_- b + 2b \partial_-)v \] (2.18)
The transformation rules $\delta_{cl} h_{\mu}^A$ in (3), which correspond to the gauging of the algebra, are obtained from the transformation rules in (14), which leave the classical action invariant, by rescaling the $T$ term in (14) by a factor 2, for reasons to be explained in section 4. The gauge algebra for $h, b, u$ and $v$ (with $\delta_{cl}(\lambda)h = (\lambda \partial_- b \partial_- \lambda)(-4u)$) is then an example of a classical nonlinear gauge algebra with $c = 0$.

3 Gauging at the quantum level.

To extend the gauging of the classical theory to the quantum level, we must integrate out the matter. This will introduce an anomaly which shows up as a violation of current conservation, described by a Ward identity of the form $\partial_+ u_A = (An)_A + \ldots$. The minimal anomaly is due to the variation of the two-point function with two external gauge fields, under the leading variation $\delta h_{\mu}^A = \partial_\mu \epsilon^A$. By varying the Ward identity, we find the minimal anomaly in the transformation rules of the currents

$$\delta_{\text{min}} u_A = m_{AB} \epsilon^B$$

For example, in ordinary gravity, the anomaly appears as $\partial_+ u = \partial_-^3 h + \ldots$, and in $W_3$ gravity this result is extended to the spin 3 current $v$ as $\partial_+ v = \partial_-^5 b + \ldots$. Then (19) corresponds to $\delta u = \partial_-^3 \epsilon + \ldots$ and $\delta v = \partial_-^5 \lambda + \ldots$.

The classical transformation rules of the currents plus the terms due to the minimal anomaly are only part of the $c \to \infty$ transformation rules

$$\delta^\infty u_A = u_C (f^C_{AB} + u_D V^{DC}_{AB}) \epsilon^B + m_{AB} \epsilon^B + \ldots$$

We shall now make the following assumptions (to be discussed later) concerning the terms denoted by dots in (20)

(i) the $u_i$ transform under $\epsilon^A$ as in the classical theory (except for minimal anomalies)

$$\delta^\infty (\epsilon^A) u_i = m_{iA} \epsilon^A + \delta_{cl} (\epsilon^A) u_i$$
(ii) the $u_i$ transform under $\epsilon^i$ as in the classical theory. This follows actually from the antisymmetry of the structure constants and (i), if the general result in (23) is to hold.

(iii) the minimal anomalies respect the coset structure

$$m_{\alpha i} = 0$$

In fact, in all applications $m_{AB}\epsilon^B$ will always be proportional to $\partial^m\epsilon^A$ with $m$ a positive integer.

(iv) the $\delta^\infty$ gauge algebra closes.

In section 4 we shall discuss which quantum algebras fulfill these assumptions. We can then compute the composite parameters $\epsilon_3^A$ of the $\delta^\infty$ gauge algebra by evaluating the gauge commutator on $u_i$, keeping only the field independent terms (which are due to the minimal anomaly) and rewriting them as the minimal anomaly in terms of $\epsilon_3^A$. In formula

$$\left[\delta(\epsilon_1^A), \delta(\epsilon_2^A)\right] u_i = m_{CD}\epsilon_1^D f^C_{\ i B} \epsilon_2^B - 1 \leftrightarrow 2$$

$$= m_{iB}\epsilon_3^B$$

(3.21)

The composite parameters $\epsilon_3$ is parametrized as follows:

$$\epsilon_3^A = f^{A}_{BC} \epsilon_1^C \epsilon_2^B + g^{A}_{BC} \epsilon_1^C \epsilon_2^B$$

$$+ 2u_D V^{DA}_{\ BC} \epsilon_1^C \epsilon_2^B$$

(3.22)

The $f^{A}_{BC}$ are the structure constants of the classical theory but the $g^{A}_{BC}$ are determined by this expression and are the corrections induced by the minimal anomaly. Given the minimal anomaly $m_{AB}\epsilon^B$ with a given overall scale, the Jacobi identity in (2) fixes the scale of $V^{AB}_{\ CD}$.

This results in a factor unity in front of the $V$ term in $\delta u_A$

$$\delta^\infty u_A = m_{AB}\epsilon^B + u_C(f^C_{\ AB} + g^C_{\ AB} + u_D V^{DC}_{\ AB})\epsilon^B$$

(3.23)

The claim is that with these transformation rules, the gauge algebra closes on all currents $u_A$. To prove this, one needs all three Jacobi identities in (2).
It may be helpful to give an example at this point. The $c \to \infty$ limit of the full quantum transformation rules of the currents of $W_3$ gravity is given by

$$
\delta^\infty u = -4 \partial^3 \epsilon + \epsilon \partial u + 2 u \partial \epsilon + 2 \lambda \partial v + 3 \partial \lambda v
$$

$$
\delta^\infty v = -4 \partial^5 \lambda + \epsilon \partial v + 3 \partial \epsilon v - 4(\partial \lambda uu + \lambda u \partial u)
$$

$$
+ \left[ 2 \partial \lambda - 3 + 9 \lambda' \partial - 2 + 15 \lambda'' \partial + 10 \lambda''' \right] u
$$

(3.24)

We shall reobtain them from the corresponding classical transformation laws by our method. These classical transformation laws are obtained by dropping the minimal anomalies ($\partial^3 \epsilon$ and $\partial^5 \lambda$) as well as the terms in square brackets. The latter correspond to the terms due to $g^{A_{BC}}$, but we pretend at this point that we do not know them. (In section 4 we shall show that in general the $g^{A_{BC}}$ terms in $\delta u_\alpha$ correspond to all terms linear in currents and proportional to $\epsilon^\beta$).

We begin our program by adding the minimal anomalies, which we parametrize as $\alpha \partial^3 \epsilon$ and $\beta \partial^5 \lambda$ with $\alpha$ and $\beta$ constants to be determined. Next we consider second variations of $\delta^\infty u$. Under two variations with $\epsilon^i \equiv \epsilon$ we find, collecting all field-independent terms,

$$
[\delta^\infty(\epsilon_1), \delta^\infty(\epsilon_2)] u = \alpha \epsilon_2 \partial (\partial^3 \epsilon_1) + 2 (\partial^3 \epsilon_1) \partial \epsilon_2 - 1 \leftrightarrow 2 + \ldots
$$

$$
= \alpha \partial^3(-\epsilon_1 \partial \epsilon_2 + \epsilon_2 \partial \epsilon_1) + \ldots
$$

(3.25)

This is the same result as for $\delta_{cl}(\epsilon)$ in (15), and shows that $g^{A_{ij}} = 0$. Next we evaluate the commutator of $\delta(\epsilon_1)$ and $\delta(\epsilon_2)$, keeping only the terms linear in currents

$$
[\delta^\infty(\epsilon), \delta^\infty(\lambda)] u = (2 \lambda \partial + 3 \lambda')(\epsilon v' + 3 \epsilon' v)
$$

$$
-(\epsilon \partial + 2 \epsilon')(2 \lambda v' + 3 \epsilon' v) = (2 \lambda \partial + 3 \lambda') v = \delta^\infty(\lambda) v
$$

(3.26)

with

$$
\tilde{\lambda} = -\epsilon \partial \lambda + 2 \lambda \partial \epsilon
$$

(3.27)

Again this is the same result as for $\delta_{cl}$ in (15), and shows that also $g^{A_{\alpha j}} = 0$. However, for the commutator of two coset transformations $\delta(\epsilon_{1\alpha})$ and $\delta(\epsilon_{2\alpha})$ on $u_i$ we find a nontrivial result.
Retaining again only the field-independent terms, we find

\[
\begin{align*}
[\delta(\lambda_1), \delta(\lambda_2)]u & = \beta(2\lambda_2\partial + 3\lambda'_2)(\partial^2\lambda_1) - 1 \leftrightarrow 2 + \ldots \\
& = \beta\partial^3(-2\lambda_1\lambda''_2 + 3\lambda'_1\lambda'_2 - 3\lambda''_1\lambda'_2 + 2\lambda''_1\lambda_2) + \ldots \\
& = \beta\partial^3(\tilde{\epsilon}) + \ldots
\end{align*}
\]

Since classically there is no field-independent contribution to this commutator, \(f^i_{\alpha\beta} = 0\) and hence the present \(\tilde{\epsilon}\) is completely due to a new structure constant \(g^i_{\alpha\beta}\). This \(g^i_{\alpha\beta}\) contributes a term \(\delta v_{\alpha} = u_i g^i_{\alpha\beta} \epsilon^\beta\), given by

\[
\delta v = (10\lambda''' + 15\lambda''\partial_- + 9\lambda'\partial_-^2 + 2\lambda\partial_-^3)u
\]

(3.29)

(3.28)

It follows most easily by extracting \(\lambda_2\) from \(\int u g^i_{\alpha\beta} \lambda_1^\beta \lambda_2^\alpha d^2x\). This is precisely the term in square brackets in \(\delta^\infty v\) in (24). To complete the transformation rules of the \(W_3\) currents, we evaluate \([\delta(\epsilon), \delta(\lambda)]v\) which yields \(\alpha = \beta = -4\). These results agree with (24).

As a second example, consider induced (3,0) supergravity \([4]\). The OPE has one central charge, \(\sigma\). The transformation rules for the currents in the limit \(\sigma \to \infty\) are given by

\[
\begin{align*}
\delta^\infty u & = \epsilon u' + 2\epsilon' u + \lambda'_a v^a + \frac{3}{2} \eta'_q q^i + \frac{1}{2} \eta'^i q'_i [\partial_-^3 \epsilon] \\
\delta^\infty v_a & = \epsilon v'_a + \epsilon' v^a - \frac{i}{2} \epsilon_{abc} \lambda_b v_c - \frac{i}{2} \epsilon^{aij} \eta q_j [\partial_- \lambda^a] \\
\delta^\infty q_i & = \epsilon q'_i + \frac{3}{2} \epsilon' q_i - \frac{i}{2} \epsilon_{aij} \lambda^a q^j - \frac{1}{4} \eta^i v j v_i \\
& \quad [\partial_-^2 \eta_i u - \frac{i}{2} \epsilon_{ij} (2\eta'_j + \eta_j \partial) v^a + \partial_- \eta_i]
\end{align*}
\]

(3.30)

The parameters \(\epsilon\) and \(\lambda^a(a = 1, 3)\) correspond to the stress tensor \(u\) and the \(SO(3)\) current \(v_a\), while \(\eta^i(i = 1, 3)\) correspond to the supersymmetry currents \(q_i\). The currents \(u\) and \(v_a\) constitute the subalgebra currents \(u_i\), and \(q_i\) corresponds to the coset currents \(u_{\alpha}\), but in \(\delta q_i\) there is only a nonlinear term with \(vv\) but no terms with \(vu\) or \(uu\). (Actually, we can also consider \(u\) as a coset current. The coset currents \(u_{\alpha}\) are then \(q_i\) and \(u\), while \(v_a\) are then the only subalgebra currents \(u_i\). It will be shown in section 4 that this division does not lead to a corresponding classical algebra. In general, only the \(u_A\) for which \(\delta u_A\) contains nonlinear terms are to be considered as coset currents).
We obtain the classical algebra by deleting the central charges and the terms in $\delta q_i$ which are linear in currents and linear in $\eta_i$. These terms have been put in square brackets. The reader may now reconstruct the complete $\delta^\infty$ algebra by evaluating $[\delta(\epsilon^i), \delta(\epsilon^j)], [\delta(\epsilon^a), \delta(\epsilon^\beta)]$ and $[\delta(\epsilon^i), \delta(\epsilon^a)]$ on $u_i$ where $\epsilon^i = \{\epsilon, \lambda^a\}$ and $\epsilon^a = \eta^i$. We have rescaled the results of [4] such that the $\epsilon$ terms in $\delta_{cl}(u)$ and $\delta_{cl}(v_a)$ match the $\epsilon$ and $\lambda^c$ terms in $\delta_{cl}q_i$

$$\int (\epsilon^i u_A f^A_{\alpha} \epsilon^\alpha + \epsilon^\alpha u_A f^A_{\alpha} \epsilon^i) = 0$$  

(3.31)

At this point we have determined all structure constants of the $\delta^\infty$ algebra, as well as the transformation rules $\delta^\infty u_A$. To obtain the transformation rules $\delta^\infty h^A\mu$, we proceed as in the classical case, but with $f^A_{BC}$ replaced by $f^A_{BC} + g^A_{BC}$, and define

$$\delta h^A\mu = \delta \mu \epsilon^A + (f^A_{BC} + g^A_{BC} + 2u_D V^{DA}_{\;BC}) h^C\mu \lambda^B$$  

(3.32)

The origin of the factor 2 we explained before. For $W_3$ gravity this yields

$$\delta h = \partial_+ \epsilon - h^\prime \epsilon + (\lambda b^\prime - b^\prime \lambda)(-4u) + (2\lambda b'' - 3\lambda' b' + 3\lambda' b'' - 2\lambda'' b)$$

$$\delta b = \epsilon b^\prime - 2b^\prime \epsilon + \partial_+ \lambda - h^\prime \lambda + 2h^\prime \lambda$$  

(3.33)

Direct evaluation of the commutator of two gauge transformations of $\delta h^A\mu$ yields

$$[\delta(\epsilon_1), \delta(\epsilon_2)] h^A\mu = \delta(\epsilon_3) h^A\mu - 2(D^\infty_{\mu} u_D - m_{DE} h^E\mu) V^{DA}_{\;BC} \epsilon_1^C \epsilon_2^B$$  

(3.34)

where $D^\infty_{\mu}$ is obtained from $D^\infty_{\mu}$ by replacing $f^A_{BC}$ by $f^A_{BC} + g^A_{BC}$. This result is obtained by using the second and third Jacobi identity in (2). If the expression within parentheses vanishes

$$D^\infty_{\mu} u_D = m_{DE} h^E\mu$$  

(3.35)

the gauge algebra closes uniformly on $u_A$ and $h^A\mu$. We now shall prove that this identity is the Ward identity for the $c \to \infty$ induced action. We shall do so by showing that variation of this Ward identity produces the correct $\delta^\infty u_A$ transformation rules. This argument was first given in [H], but we repeat it here for completeness.
Varying all gauge fields $h_{\mu}^A$ in (34) into $\partial_+ \epsilon^A$, and the leading term $\partial_+ u$ into $\partial_+ \delta u$, one can make all terms total $\partial_+$ derivatives because the difference is terms proportional to $\partial_+ u$ which can be converted into terms with $\partial_-$ derivatives by using repeatedly the Ward identity. Extracting the overall $\partial_+$, the remainder yields indeed the correct $\delta u_A$.

4 Comments.

We have considered classical nonlinear gauge algebras without central terms, and obtained the $c \to \infty$ limit of the corresponding quantum algebras, by completing the former. This is thus a kind of Noether procedure applied to gauge algebras. We shall now address the question in which cases this method works.

Consider a quantum algebra with central term $c$, which is quadratically nonlinear. Let it be of the generic form

$$[K, K] = c + K + H + \frac{1}{c} : H H : + \ldots$$
$$[K, H] = K + H + \ldots$$
$$[H, H] = c + H + \ldots$$

(4.36)

where the terms indicated by dots correspond to similar terms but down by at least one factor of $c$. For example, in the central term one may have $(c + 1 + \ldots)$ and in front of the $: H H :$ term one may have a factor which starts as $\frac{1}{c} + \frac{1}{c^2} + \ldots$. The $W_3$ algebra [6] and the $N$ extended Bershadsky-Knizhnik superconformal algebras [12] are examples of this structure. Rescale now $K = cK'$ and $H = cH'$, and divide by $c^2$. Then all term are of the form

$$[T_A', T_B'] = \frac{1}{c} [1 + K' + H' + : H'H' :] + O \left( \frac{1}{c^2} \right)$$

(4.37)

Moreover, the normal-ordered term $: H'H' :$ differs from just the product of two $H'$ by some reorderings which are of order $\frac{1}{c}$. Hence, to leading order in $c$, one can replace $: H'H' :$ by a product, and use Poisson brackets (simple contractions) instead of quantum commutators since further contractions are down again by factors $1/c$. The leading terms in $1/c$ in the Jacobi identities form themselves an identity, and this is just the $c \to \infty$ algebra.
Next make a second rescaling in this $c \to \infty$ algebra, which scales $H'$ back to $H$, and $K'$ to $K$. This leads to an algebra of the form

\[
[K, K] = c + K + H + \frac{1}{c} HH \\
[K, H] = K + H \\
[H, H] = c + H
\]  

which satisfies the classical (Poisson brackets) Jacobi identities with central charges in [7]. By further rescaling $K = \tilde{K}/\sqrt{c}$ and subsequently taking the limit $c \to 0$, we find

\[
[\tilde{K}, \tilde{K}] = HH \\
[\tilde{K}, H] = \tilde{K} , \quad [H, H] = H
\]  

(4.39)

This consistent classical nonlinear algebra corresponds to the $\delta_{cl}$ gauge algebra. Hence our procedure works if the $c \to \infty$ algebra is reductive ($f^{i}_{\alpha j} = 0$), and we obtain $\delta_{cl}u_{\alpha}$ by deleting the central term and all terms bilinear in $u_{A}$ and $\lambda_{\beta}$.

It is crucial that all $[K, K]$ commutators have nonlinear $HH$ terms. Suppose, for example, that one could subdivide the $K_{\alpha}$ generators into a set $k_{\alpha}$ which all have nonlinear terms in the commutators among themselves, and a set $k_{a}$ which do not form with $H_{i}$ a subalgebra. Hence generically

\[
[k_{\alpha}, k_{\beta}] = c + K + H + \frac{1}{c} HH \\
[k_{a}, H] = H + k_{a} + k_{\alpha}
\]  

(4.40)

Rescaling only $k_{\alpha}$ as $k_{\alpha} = \tilde{k}_{\alpha}/\sqrt{c}$ now produces a singularity in the last term with $k_{\alpha}$, preventing the $c \to \infty$ limit.

Finally we explain why the gauge algebra of the transformation rules of the fields $\varphi^{i}, h, b$ of $W_{3}$ gravity which leave the classical action invariant, differs from the classical gauge algebra on the currents in (17). The reason is that invariance of the interactions $\int (hT + bW)d^{2}x$ requires that $\delta h$ and $\delta b$ transform contragrediently to $T$ and $W$, thus with a factor unity in front of the
$V$ term. (The classical action is then invariant because the $\partial_\mu \epsilon^A$ variation of $h_\mu^A$ contribute $-u_A \partial_\mu \epsilon^A$, whereas the $\varphi$-kinetic term varies into $-(\partial_\mu u_A)\epsilon^A$). On the other hand, uniform closure on $h_\mu^A$ and $u_A$ requires a factor 2 in front of the $V$ term in $\delta h$, and this explains why the $\lambda$ commutator on $b$ in (15) is a factor 2 smaller than the $\lambda$ commutator on $u$ and $v$ in (17).

In [9] we considered a different approach to the gauging of nonlinear algebras, in which we introduced in addition to $h_\mu^A$ further scalars $t_A$ instead of currents $u_A^\mu$. These scalars played the role of Higgs scalars in a $d = 4$ Yang-mills model. Whether scalars or currents, one can also view them as auxiliary fields which close a gauge algebra.

As was shown in this paper many properties of Lie algebras carry over to the case of non-linearly generated Lie algebras. However, two main features which are well understood in the case of linear Lie algebras are not at all established in the case of non-linear algebras. The first is how to tensor representations such as to obtain new representations. This problem is obviously directly related to the presence of non-linearities in the commutation relations. The second and probably most important problem is the lack of understanding of the geometry behind non-linearly generated Lie algebras.

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