THE SPECTRAL FLOW OF A FAMILY OF TOEPLITZ OPERATORS

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(WITH AN APPENDIX BY KOEN VAN DEN DUNGEN)

Abstract. We show that the (graded) spectral flow of a family of Toeplitz operators on a complete Riemannian manifold is equal to the index of a certain Callias-type operator. When the dimension of the manifold is even this leads to a cohomological formula for the spectral flow. As an application, we compute the spectral flow of a family of Toeplitz operators on a strongly pseudoconvex domain in $\mathbb{C}^n$. This result is similar to the Boutet de Monvel's computation of the index of a single Toeplitz operator on a strongly pseudoconvex domain. Finally, we show that the bulk-boundary correspondence in the tight-binding model of topological insulators is a special case of our result.

In the appendix, Koen van den Dungen reviewed the main result in the context of (unbounded) KK-theory.

1. Introduction

In the study of topological insulators the edge index is often defined as the spectral flow of a certain family of Toeplitz operators on a circle [15,16,18,20,22,25]. The bulk-edge correspondence establishes the equality of the edge index and the bulk index, which can be interpreted as the index of a certain Dirac-type operator. Thus we obtain the equality between the spectral flow of a family of Toeplitz operators and the index of a Dirac-type operator. This is similar but different from the classical “desuspension” result of Baum-Douglas [3] and Booss-Wojciechowski [5] (see also [6, §17]) which establishes an equality between the spectral flow of a family of Dirac-type operators and the index of a Toeplitz operator. The goal of this note is to generalize the bulk-edge correspondence to a formula for the spectral flow of a quite general family of Toeplitz operators.

We note that the result of [3,5] were extended to a family case by Dai and Zhang [13]. It would be interesting to obtain a similar extension of our results.

1.1. A family of Toeplitz operators. Suppose $E = E^+ \oplus E^-$ is a Dirac bundle over a complete Riemannian manifold $M$ and let $D$ be the corresponding Dirac operator on the space $L^2(M, E \otimes \mathbb{C}^k)$. We denote by $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ the kernel of $D$ and by $P : L^2(M, E \otimes \mathbb{C}^k) \to \mathcal{H}$ the orthogonal projection. Let $f = \{f_t\}_{t \in S^1}$ be a periodic family of smooth functions on $M$ with values in the space of Hermitian $k \times k$-matrices. For $t \in S^1$, we denote by $M_{f_t} : L^2(M, E \otimes \mathbb{C}^k) \to L^2(M, E \otimes \mathbb{C}^k)$ the multiplication by $f_t$. The Toeplitz operator is the composition $T_{f_t} := P \circ M_{f_t} : \mathcal{H} \to \mathcal{H} (t \in S^1)$.

Under the assumption that

(i) 0 is an isolated point of the spectrum of $D$;

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we show that $T_{f_t}$ is a family of Fredholm operators. Let $T_{f_t}^±$ denote the restriction of $T_{f_t}$ to $\mathcal{H}^±$ and let $\text{sf}(T_{f_t}^±)$ denote the spectral flow of $T_{f_t}$. In many applications, including the Toeplitz operator on a strongly pseudoconvex domain, $\mathcal{H}^− = \{0\}$ and, hence, $\text{sf}(T_{f_t}^-) = 0$. Thus in those cases we compute $\text{sf}(T_{f_t}^0)$.

1.2. A Callias-type operator. Let $\mathcal{M} = S^1 \times M$ and let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^−$ be the lift of $E$ to $\mathcal{M}$. It is naturally an ungraded Dirac bundle and we denote the corresponding Dirac operator by $\not{D}$. Let $\mathcal{M}_f : L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \to L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$ denote the multiplication by $f$ and consider the Callias-type operator $\mathcal{B}_{cf} := \not{D} + ic\mathcal{M}_f$, where $c > 0$ is a large constant. Our assumptions guarantee that this operator is Fredholm and our main result (Theorem 2.10) states that

$$ \text{sf}(T_{f_t}^+) − \text{sf}(T_{f_t}^-) = \text{ind} \mathcal{B}_{cf}. \tag{1.1} $$

In the appendix Koen van den Dungen presented a KK-theoretical interpretation of this equality.

1.3. The even dimensional case: a cohomological formula. Suppose now that the dimension of $M$ is even. Then the dimension of $\mathcal{M}$ is odd and by the Callias-type index theorem [10] the index of $\mathcal{B}_{cf}$ is equal to the index of a certain Dirac operator on a compact hypersurface $\mathcal{N} \subset \mathcal{M}$. Applying the Atiyah-Singer index theorem we thus obtain a cohomological formula for $\text{ind} \mathcal{B}_{cf}$ and, hence, for $\text{sf}(T_{f_t}^+) − \text{sf}(T_{f_t}^-)$, cf. Corollary 2.14.

1.4. A family of Toeplitz operators on a strongly pseudoconvex domain. Consider a strongly pseudoconvex domain $\overline{M} \subset \mathbb{C}^n$ with smooth boundary. We denote its boundary by $N := \partial \overline{M}$ and consider its interior $M$ as a complete Riemannian manifold endowed with the Bergman metric, cf. [24 §7]. Let $D = \partial + \partial^*$ be the Dolbeault-Dirac operator on the space $\Omega^{n,*}(M, \mathbb{C}^k)$ of $(n, \bullet)$-forms on $M$ with values in the trivial bundle $\mathbb{C}^n$. Let $f = \{f_t\}_{t \in S^1}$ be a periodic family of smooth functions on $\overline{M}$ with values in the space of Hermitian $k \times k$-matrices. We assume that the restriction of $f$ to $N := \partial \overline{M}$ is invertible. Then it follows from [14] §5 that $D$ and $f$ satisfy our assumptions (i)–(iv). Moreover, in this case the space $\mathcal{H}^− = \{0\}$. Thus (1.1) computes $\text{sf}(T_{f_t}^+) = \text{sf}(T_{f_t}^-)$.

Let $\mathcal{N} := S^1 \times N$ denote the boundary of $\overline{M} := S^1 \times M$. Since the restriction of $f$ to $\mathcal{N}$ is invertible, we obtain a direct sum decomposition $\mathcal{N} \times \mathbb{C}^k = \mathcal{F}_{\mathcal{N}^+} \oplus \mathcal{F}_{\mathcal{N}^-}$ of the trivial bundle $\mathcal{N} \times \mathbb{C}^k$ into the positive and negative eigenspaces of $f$. Our Theorem 3.3 states that

$$ \text{sf}(T_{f_t}) = \int_{\mathcal{N}} \text{ch}(\mathcal{F}_{\mathcal{N}^+}), \tag{1.2} $$

where $\text{ch}(\mathcal{F}_{\mathcal{N}^+})$ denotes the Chern character of the bundle $\mathcal{F}_{\mathcal{N}^+}$.

1.5. A tight-binding model of topological insulators and the bulk-boundary correspondence. We consider a tight binding model on the lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$. Note that this model covers not only crystals with square lattice, but many other materials, including the hexagon lattice of graphene, cf. Section 3 of [16].

In the bulk (i.e. far from the boundary) the lattice looks like $\mathbb{Z} \times \mathbb{Z}$. The bulk Hamiltonian is a bounded map $H : l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k)$ which is periodic with period one in both directions of the lattice. By performing the Fourier transform of $H$ (the Bloch decomposition) we obtain a family of self-adjoint $k \times k$-matrices $H(s, t) \ ((s, t) \in S^1 \times S^1)$. 

(ii) for each $t \in S^1$ the differential $df_t$ vanishes at infinity on $M$;

(iii) $f$ is invertible at infinity (cf. Definition 2.8);

(iv) $\frac{2\pi}{L} f_t$ is bounded;

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where $\text{ch}(\mathcal{F}_{\mathcal{N}^+})$ denotes the Chern character of the bundle $\mathcal{F}_{\mathcal{N}^+}$.
We assume that the bulk Hamiltonian has a spectral gap at Fermi level $\mu \in \mathbb{R}$. In particular, the operator $H(s, t) - \mu$ is invertible for all $(s, t) \in S^1 \times S^1$. Thus the trivial bundle $(S^1 \times S^1) \times \mathbb{C}^k$ over the torus $S^1 \times S^1$ decomposes into the direct sum $(S^1 \times S^1) \times \mathbb{C}^k = \mathcal{F}_+ \oplus \mathcal{F}_-$ of positive and negative eigenbundles of $H(s, t) - \mu$. The bundle $\mathcal{F}_+$ is referred to as the Bloch bundle. The bulk index is defined by

$$I_{\text{Bulk}} := \int_{S^1 \times S^1} \text{ch}(\mathcal{F}_+).$$

We now take the edge into account, i.e. restrict to the half-lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$. The Fourier transform of the bulk Hamiltonian $H$ in the direction “along the edge” transforms $H$ into a family of self-adjoint translationally invariant operators $H(t) : l^2(\mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z}, \mathbb{C}^k)$. Let $\Pi : l^2(\mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k)$ denote the projection. The edge Hamiltonian is the family of Toeplitz operators $H^\#(t) := \Pi \circ H(t) \circ \Pi : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k)$.

The edge index $I_{\text{Edge}}$ of the Hamiltonian $H$ is the spectral flow of the edge Hamiltonian

$$I_{\text{Edge}} := \text{sf}(H^\#(t)).$$

Both the bulk and the edge Hamiltonians extend to operators on the unit disc $B \subset \mathbb{C}$. The disc is the simplest example of a strongly pseudoconvex domain. Applying (1.2) to this situation we obtain the following bulk-boundary correspondence equality

$$I_{\text{Bulk}} = I_{\text{Edge}}.$$

Thus (1.2) is an extension of (1.3) to general strongly pseudoconvex domains. For this reason we refer to the equality (1.2) as the generalized bulk-boundary correspondence.

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2. The main result

In this section we formulate our main result – the equality between the spectral flow of a family of Toeplitz operators on a complete Riemannian manifold $M$ and the index of a Callias-type operator on $M$.

2.1. The Dirac operator. Let $M$ be a complete Riemannian manifold and let $E = E^+ \oplus E^-$ be a graded Dirac bundle over $M$, cf. [21] [II.5], i.e. a Hermitian vector bundle endowed with the Clifford action

$$c : T^*M \to \text{End}(E), \quad (c(v)) : E^\pm \to E^\mp, \quad c(v)^2 = -|v|^2, \quad c(v)^* = -c(v),$$

and a Hermitian connection $\nabla^E = \nabla^{E^+} \oplus \nabla^{E^-}$, which is compatible with the Clifford action in the sense that

$$[\nabla^E_u, c(v)] = c(\nabla^{LC}_u v), \quad \text{for all} \quad u \in TM,$$

where $\nabla^{LC}$ is the Levi-Civita connection on $T^*M$.

We extend the Clifford action to the product $E \otimes \mathbb{C}^k$ and we denote by $D$ the associated Dirac operator. In local coordinates it can be written as $D = \sum_j c(dx^j) \nabla^E_{\partial_j}$. We view $D$ as an unbounded self-adjoint operator $D : L^2(M, E \otimes \mathbb{C}^k) \to L^2(M, E \otimes \mathbb{C}^k)$.

Throughout the paper we make the following

Assumption 2.2. Zero is an isolated point of the spectrum of $D$. 
Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^- \subset L^2(M, E \otimes \mathbb{C}^k)$ denote the kernel of $D$ and let $P : L^2(M, E \otimes \mathbb{C}^k) \to \mathcal{H}$ be the orthogonal projection. Here $\mathcal{H}^\pm$ is the restriction of $\mathcal{H}$ to $L^2(M, E^\pm \otimes \mathbb{C}^k)$. We denote by $P^\pm : L^2(M, E^\pm \otimes \mathbb{C}^k) \to \mathcal{H}^\pm$ the restriction of $P$ to $L^2(M, E^\pm \otimes \mathbb{C}^k)$.

2.3. The Toeplitz operator. Let $BC(M; k)$ denote the Banach space of bounded continuous functions on $M$ with values in the space Herm($k$) of Hermitian complex-valued $k \times k$-matrices. We denote by $C^\infty_g(M; k) \subset BC(M; k)$ the subspace of bounded $C^\infty$-functions such that $df$ vanishes at infinity of $M$.

For $f \in C^\infty_g(M; k)$ we denote by $M_f : L^2(M, E \otimes \mathbb{C}^k) \to L^2(M, E \otimes \mathbb{C}^k)$ the multiplication by $1 \otimes f$ and by $M^\pm_f$ the restriction of $M_f$ to $L^2(M, E^\pm \otimes \mathbb{C}^k)$.

Definition 2.4. The operator

$$T_f := P \circ M_f : \mathcal{H} \to \mathcal{H}$$

(2.1)
is called the Toeplitz operator defined by $f$. We denote by $T^\pm_f$ the restriction of $T_f$ to $\mathcal{H}^\pm$.

Definition 2.5. We say that a matrix-valued function $f \in C^\infty_g(M; k)$ is invertible at infinity if there exists a compact set $K \subset M$ and $C_1 > 0$ such that $f(x)$ is an invertible matrix for all $x \notin K$ and $\|f(x)^{-1}\| < C_1$ for all $x \notin K$.

The following result is proven in [11, Lemma 2.6]

Proposition 2.6. If $D$ satisfies Assumption 2.2 and $f \in C^\infty_g(M; k)$ is invertible at infinity then the Toeplitz operator $T_f$ is Fredholm.

2.7. A family of self-adjoint Toeplitz operators. Let now $S^1 = \{e^{it} : t \in [0, 2\pi]\}$ be the unit circle. Consider a smooth family $f : S^1 \to C^\infty_g(M; k)$, $t \mapsto f_t$, of matrix valued functions. Assume that $f_t$ is invertible at infinity for all $t \in [0, 2\pi]$. Then $T_{f_t}$ ($t \in S^1$) is a periodic family of self-adjoint Fredholm operators. Our goal is to compute the spectral flow of this family. We make the following

Assumption 2.8. There exists a constant $C_2 > 0$ such that $\|\frac{\partial}{\partial t} f_t(x)\| < C_2$ for all $t \in S^1$, $x \in M$.

If $f_t$ is invertible at infinity for all $t \in S^1$ and satisfies Assumption 2.8, then there exists a compact set $K \subset M$ and a large enough constant $\alpha > 0$ such that

$$\frac{\partial}{\partial t} f_t(x) < \frac{\alpha}{2} f_t(x)^2,$$

for all $x \notin K$. (2.2)

Here the inequality $A < B$ between two self-adjoint matrices means that for any vector $v \neq 0$, we have $\langle Av, v \rangle < \langle Bv, v \rangle$.

2.9. A Callias-type operator on $S^1 \times M$. Set $\mathcal{M} = S^1 \times M$. We write points of $\mathcal{M}$ as $(t, x)$, $t \in S^1$, $x \in M$. Denote by $\pi_1 : S^1 \times M \to S^1$ and $\pi_2 : S^1 \times M \to M$ the natural projections. By a slight abuse of notation we denote the pull-backs $\pi_1^* dt$, $\pi_2^* dx \in T^* \mathcal{M}$ by $dt$ and $dx$ respectively. Set

$$\mathcal{E} := \pi_2^* E.$$

Then $\mathcal{E}$ is naturally an ungraded Dirac bundle with Clifford action $c : T^* \mathcal{M} \to \text{End}(\mathcal{E})$ such that $c(dx) = c(dx)$ and $c(dt)$ is given with respect to the decomposition $\mathcal{E} = \pi_2^* E^+ \oplus \pi_2^* E^-$ by the matrix

$$c(dt) = \begin{pmatrix} i \cdot \text{Id} & 0 \\ 0 & -i \cdot \text{Id} \end{pmatrix}.$$
Let $\mathcal{D}$ be the corresponding Dirac operator. With respect to the decomposition
\[ L^2(M, E \otimes \mathbb{C}^k) = L^2(S^1) \otimes L^2(M, E \otimes \mathbb{C}^k), \tag{2.3} \]
it takes the form
\[ \mathcal{D} = c(dt) \frac{\partial}{\partial t} \otimes 1 + 1 \otimes D. \tag{2.4} \]

We remark that, as opposed to $D$, the operator $\mathcal{D}$ is not graded.

Let now $f_t \in C^\infty(M; k)$ ($t \in S^1$) be a smooth periodic family of invertible at infinity matrix-valued functions satisfying Assumption 2.2. We consider the family $f_t$ as a smooth function on $M$ and denote it by $f_t$. Let $\mathcal{M}_f : L^2(M, E \otimes \mathbb{C}^k) \to L^2(M, E \otimes \mathbb{C}^k)$ denote the multiplication by $f_t$. Then the commutator
\[ [\mathcal{D}, \mathcal{M}_f] := \mathcal{D} \circ \mathcal{M}_f - \mathcal{M}_f \circ \mathcal{D} \]
is a zero-order differential operator, i.e., a bundle map $E \otimes \mathbb{C}^k \to E \otimes \mathbb{C}^k$.

From (2.2) and our assumption that $df_t$ vanishes at infinity, we now conclude that there exist constants $c, d > 0$ and a compact set $K \subset M$, called an essential support of $\mathcal{B}_{cf}$, such that
\[ [\mathcal{D}, c \mathcal{M}_f](t, x) < c^2 \mathcal{M}_f(t, x)^2 - d, \quad \text{for all } (t, x) \notin K. \tag{2.5} \]

It follows that
\[ \mathcal{B}_{cf} := \mathcal{D} + ic \mathcal{M}_f \tag{2.6} \]
is a Callias-type operator in the sense of \[1, 10\] (see also \[8, \S 2.5\]). In particular, it is Fredholm.

Our main result is the following

**Theorem 2.10.** Let $E = E^+ \oplus E^-$ be a Dirac bundle over a complete Riemannian manifold $M$ and let $f_t \in C^\infty(M; k)$ ($t \in S^1$) be a smooth periodic family of invertible at infinity matrix-valued functions. Suppose that Assumptions 2.2 and 2.8 are satisfied. Then
\[ sf(T_{f_t}^+) - sf(T_{f_t}^-) = \text{ind} \mathcal{B}_{cf}. \tag{2.7} \]

**Remark 2.11.** Note that, as opposed to $df$, the differential $d f_t$ does not vanish at infinity. Because of this $\mathcal{B}_{cf}$ does not satisfy the conditions of Corollary 2.7 of \[11\] and its index does not vanish in general.

**Remark 2.12.** In our main applications $H^- = \{0\}$. Hence, $sf(T_{f_t}^-) = 0$ and (2.7) computes $sf(T_{f_t}^+)$.  

The proof of Theorem 2.10 is given in Section 5.

### 2.13. The even dimensional case: a cohomological formula.

Suppose now that the dimension of $M$ is even. Then the dimension of $M$ is odd and by the Callias-type index theorem \[1, 10\] the index of $\mathcal{B}_{cf}$ is equal to the index of a certain Dirac operator on a compact hypersurface $N \subset M$. Applying the Atiyah-Singer index theorem we thus obtain a cohomological formula for $\text{ind} \mathcal{B}_{cf}$. We now provide the details of this computation.

Let $N \subset M$ be a hypersurface such that there is an essential support $K \subset M$ of $\mathcal{B}_{cf}$ whose boundary $\partial K = N := S^1 \times N$. In particular, the restriction of $f_t$ to $N$ is invertible and satisfies (2.5). Then there are vector bundles $\mathcal{F}_{N^\pm}$ over $N$ such that
\[ \mathcal{M} \times \mathbb{C}^k = \mathcal{F}_{N^+} \oplus \mathcal{F}_{N^-}, \]
and the restriction of $f_t$ to $\mathcal{F}_{N^+}$ (respectively, $\mathcal{F}_{N^-}$) is positive definite (respectively, negative definite).
Corollary 2.14. Under the conditions of Theorem 2.10 assume that \( \dim M \) is even. Let \( j : N \hookrightarrow M \) be a hypersurface such that there is an essential support \( K \subset M \) of \( \mathcal{B}_c \) whose boundary \( \partial K = N := S^1 \times N \). Then
\[
\text{sf}(T_{f_t}^+) - \text{sf}(T_{f_t}^-) = \int_N \left[ j^* \hat{A}(M) j^* \text{ch}(E/S) \pi_{2*} \text{ch}(\mathcal{F}_{N+}) \right].
\]
(2.8)

Here \( \hat{A}(M) \) is the differential form representing \( \hat{A} \)-class of \( M \), \( \text{ch}(E/N) \) is the differential form representing the graded relative Chern character of \( E \), cf. [4] p. 146, and
\[
\pi_{2*} \text{ch}(\mathcal{F}_{N+}) = \int_{S^1} \text{ch}(\mathcal{F}_{N+}) \in \Omega^*(N)
\]
is the push-forward of \( \text{ch}(\mathcal{F}_{N+}) \) under the map \( \pi_2 : N \to N \).

Proof. Let \( E_N \) denote the restriction of the bundle \( E \) to \( N \). Then \( \mathcal{E}_N := \pi_2^* E_N \) is the restriction of \( \mathcal{E} \) to \( N \). It is naturally a Dirac bundle over \( N \). We denote by \( \mathcal{D}_N \) the induced Dirac-type operator on \( \mathcal{E}_N \otimes \mathcal{F}_{N+} \).

Let \( v \) denote the unit normal vector to \( N \) pointing towards \( K \). Then \( ic(v) : \mathcal{E}_N \to \mathcal{E}_N \) is an involution. We denote by \( \mathcal{E}_N^{\pm 1} \) the eigenspace of \( ic(v) \) with eigenvalue \( \pm 1 \). This defines a grading \( \mathcal{E}_N = \mathcal{E}_N^{+1} \oplus \mathcal{E}_N^{-1} \) on the Dirac bundle \( \mathcal{E}_N \). The operator \( \mathcal{D}_N \) is odd with respect to the induced grading on \( \mathcal{E}_N \otimes \mathcal{F}_{N+} \). (This grading is different from the one induced by the grading on \( E \). Note that the operator \( \mathcal{D}_N \) is not an odd operator with respect to the grading induced by the grading on \( E \)).

By the Callias-type index theorem [1][10] (see also [8] §2.6] where more details are provided)
\[
\text{ind} \mathcal{B}_c = \text{ind} \mathcal{D}_N.
\]
(2.9)

Since \( \mathcal{D}_N \) is an operator on compact manifold \( N \) its index is given by the Atiyah-Singer index theorem. Combining it with (2.4) we obtain
\[
\text{ind} \mathcal{B}_c = \int_N \hat{A}(N) \text{ch}(\mathcal{F}_{N+}) \text{ch}(\mathcal{E}_N/S_N),
\]
(2.10)

where \( \hat{A}(N) \) is the \( \hat{A} \)-genus of \( N \), \( \text{ch}(\mathcal{F}_{N+}) \) is the Chern character of \( \mathcal{F}_{N+} \), and \( \text{ch}(\mathcal{E}_N/S_N) \) is the relative Chern character of the graded bundle \( \mathcal{E}_N \), cf. [4] p. 146.

Since all the structures are trivial along \( S^1 \),
\[
\hat{A}(N) = \pi_2^* \hat{A}(N), \quad \text{ch}(\mathcal{E}_N/S) = \pi_2^* \text{ch}(E_N/S_N)
\]
(2.11)

where \( \hat{A}(N) \) is the \( \hat{A} \)-genus of \( N \) and \( \text{ch}(E_N/S_N) \) is the relative Chern character of the graded bundle \( E_N = E_N^{+1} \oplus E_N^{-1} \).

Since \( \hat{A}(N) \) is a characteristic class it behaves naturally with respect to the pull-backs, i.e.,
\[
\hat{A}(N) = j^* \hat{A}(M).
\]
(2.12)

As for \( \mathcal{E}_N \), the grading \( E_N = E_N^{+1} \oplus E_N^{-1} \) is different from the grading \( E_N = E_N^{+1} \oplus E_N^{-1} \) inherited from the grading on \( E \). However, since \( ic(v) \) is odd with respect to the grading \( E_N = E_N^{+1} \oplus E_N^{-1} \) and \( c(v)^2 = -1 \), we have
\[
E_N^{+1} = \{ e \pm ic(v)e : e \in E_N^+ \}.
\]

It follows that \( E_N^+ \oplus E_N^- \) and \( E_N^{+1} \oplus E_N^{-1} \) are isomorphic as graded Dirac bundles. Hence, we can compute \( \text{ch}(E_N/S_N) \) using the grading \( E_N = E_N^+ \oplus E_N^- \). Even though the relative Chern character is not quite a characteristic class (it depends not only on the connection but also on
the Clifford action and the Riemannian metric) it is well known that it behaves naturally under restrictions to a submanifold
\[
\text{ch}(E_N/S_N) = j^* \text{ch}(E/S),
\]
see, for example, [2] Lemma 7.1. Thus, using (2.11) we now obtain
\[
\text{ch}(E_N/S) = \pi_0^* j^* \text{ch}(E/S).
\]
(2.13)
The equality (2.8) follows now from (2.7), (2.10), (2.12), and (2.13).

\section{A family of Toeplitz operators on a strongly pseudoconvex domain}

In this section we apply Theorem 2.10 to the case when \( M \) is a strongly pseudoconvex domain in \( \mathbb{C}^k \). Our computation of the spectral flow in this case is similar to the computation of index in [17] and [17].

\subsection{A Dirac operator on a strongly pseudoconvex domain}

Let \( M \) be a strongly pseudoconvex domain in \( \mathbb{C}^n \). We denote its boundary by \( N := \partial M \). Let \( g^M \) be the Bergman metric on \( M \), cf. [21, §7]. Then \( (M, g^M) \) is a complete Kähler manifold. We define \( E = \Lambda^{n,*}(T^*M) \) and set \( E^+ = \Lambda^{n,\text{even}}(T^*M), E^- = \Lambda^{n,\text{odd}}(T^*M) \). Then \( E \) is naturally a Dirac bundle over \( M \) whose space of smooth section coincides with the Dolbeault complex \( \Omega^{n,*}(M) \) of \( M \) with coefficients in the canonical bundle \( K = \Lambda^{n,0}(T^*M) \). Moreover, the corresponding Dirac operator is given by
\[
D = \tilde{\partial} + \tilde{\partial}^*,
\]
where \( \tilde{\partial} \) is the Dolbeault differential and \( \tilde{\partial}^* \) its adjoint with respect to the \( L^2 \)-metric induced by the Bergman metric on \( M \).

By [14, §5] zero is an isolated point of the spectrum of \( D \) and \( \mathcal{H} := \ker D \) is a subset of the space \( (n,0) \)
\[
\mathcal{H} := \ker D \subset \Omega^{n,0}(M).
\]
(3.1)
In particular, Assumption 2.2 is satisfied and \( \mathcal{H}^- = \{0\} \).

\subsection{A family of Toeplitz operators}

Let \( C^\infty(M; k) \) denote the space of smooth functions on \( M \) with values in the space of complex-valued self-adjoint \( k \times k \)-matrices. Each \( f \in C^\infty(M; k) \) induces a function on \( (M, g^M) \) which we also denote by \( f \). One readily sees that \( f \in C^\infty_g(M, k) \), cf. [17, §1, Lemma 2]. Then \( f \) is invertible at infinity iff \( f|_{\partial M} \) is invertible.

We now consider the product \( \overline{M} := S^1 \times M \). This is a compact manifold with boundary \( N := S^1 \times N \). We endow the interior \( M := S^1 \times M \) of \( \overline{M} \) with the product of the standard metric on \( S^1 \) and the Bergman metric on \( M \). Let \( C^\infty(M; k) \) denote the space of smooth functions on \( \overline{M} \) with values in the space of self-adjoint \( k \times k \)-matrices. For \( f \in C^\infty(M; k) \), set \( f_t(x) := f(t, x) \) \( (t \in S^1, x \in \overline{M}) \). Then the restriction of \( f_t \) to \( M \) (which is also denoted by \( f_t \)) is a smooth family of functions in \( C^\infty(M, k) \). Let \( T_{f_t} \) be the corresponding family of Toeplitz operators on \( M \). By (3.1), the space \( \mathcal{H}^- = \{0\} \). Hence, \( T_{f_t} = T_{f_t}^+ \), while \( T_{f_t}^- = 0 \).

Assume now that the restriction of \( f \) to \( N = S^1 \times \partial M \) is invertible. Then \( f_t \) are invertible at infinity for all \( t \in S^1 \). Assumption 2.8 is automatically satisfied in this case, since \( \frac{d}{dt} f_t \) is a continuous function on \( \overline{M} \).

As we will see in the next section the following theorem generalizes the bulk-edge correspondence in the theory of topological insulators. More precisely, in the case when \( M \) is the unit
disc in $\mathbb{C}$, the left hand side of (3.2) is equal to the edge index, while its the right hand side is the bulk index.

**Theorem 3.3** (Generalized bulk-edge correspondence). Let $M \subset \mathbb{C}^k$ be a strongly pseudoconvex domain with smooth boundary $N := \partial M$. Set $\overline{M} := S^1 \times \overline{M}$ and let $f \in C^\infty_c(\overline{M}; k)$ be a smooth function with values in the space of self-adjoint $k \times k$-matrices. Assume that $f_t(x) := f(t, x)$ is invertible for all $t \in S^1$, $x \in \partial M$. Then

$$sf(T_{f_t}) = \int_N \text{ch}(\mathcal{F}_{N^+}),$$

where $N := S^1 \times N$ and $\mathcal{F}_{N^+} \subset N \times \mathbb{C}^k$ is a subbundle spanned by eigenvectors of $f|_N$ with positive eigenvalues.

**Proof.** By (3.1), the space $\mathcal{H}^- = \{0\}$. Hence, $T_{f_t} = T_{f_t}^+$, while $T_{f_t}^- = 0$. In particular,

$$sf(T_{f_t}^-) = 0.$$  

(3.3)

Since $M$ is a domain in a flat space $\mathbb{C}^k$, both $\hat{A}(M) = 1$ and $\text{ch}(E/S) = 1$. Hence, by (2.8) and (3.3) we obtain

$$sf(T_{f_t}) = \int_N \pi_{2*} \text{ch}(\mathcal{F}_{N^+}) = \int_N \text{ch}(\mathcal{F}_{N^+}).$$

□

4. A tight-binding model of topological insulators and the bulk-edge correspondence

In this section we briefly review a standard tight-binding model for two-dimensional topological insulators, following the description in [19] (see also [16]) and show that the bulk-edge correspondence for this model follows immediately from our Theorem 3.3.

4.1. The bulk Hamiltonian. We consider a tight binding model on the lattice $\mathbb{Z}_{\geq 0} \times \mathbb{Z}$. Basically, this means that the electrons can only stay on the lattice sites and the kinetic energy is included by allowing electrons to hop from one site to a neighboring one. Surprisingly, this model covers not only crystals with square lattice, but many other materials, including the hexagon lattice of graphene, cf. Section 3 of [16].

The mathematical formulation of the model is as follows (we present the version suggested in [19]): The “bulk” state space is the space $l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k)$ of square integrable sequences

$$\phi = \{\phi_{ij}\}_{(i,j) \in \mathbb{Z} \times \mathbb{Z}}, \quad \phi_{ij} \in \mathbb{C}^k.$$ 

The bulk Hamiltonian $H : l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k) \to l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k)$ is periodic with period one in both directions of the lattice. By performing a Fourier transform of $H$ (the Bloch decomposition in physics terminology) we obtain a family of self-adjoint $k \times k$-matrices $H(s, t) ((s, t) \in S^1 \times S^1)$. We assume that $H(s, t)$ depend smoothly on $s$ and $t$.

We assume that the bulk Hamiltonian has a spectral gap at Fermi level $\mu \in \mathbb{R}$, i.e. there exists $\epsilon > 0$ such that the spectrum of $H(s, t)$ does not intersect the interval $(\mu - \epsilon, \mu + \epsilon)$ for all $(s, t) \in S^1 \times S^1$. In particular, the operator $H(s, t) - \mu$ is invertible for all $(s, t) \in S^1 \times S^1$. Thus the trivial bundle $(S^1 \times S^1) \times \mathbb{C}^k$ over the torus $S^1 \times S^1$ decomposes into the direct sum of subbundles

$$(S^1 \times S^1) \times \mathbb{C}^k = \mathcal{F}_+ \oplus \mathcal{F}_-.$$
such that the restriction of \( H(s, t) - \mu \) to \( \mathcal{F}_+ \) is positive definite and the restriction of \( H(s, t) - \mu \) to \( \mathcal{F}_- \) is negative definite. The bundle \( \mathcal{F}_+ \) is referred to as the Bloch bundle.

**Definition 4.2.** The bulk index of the Hamiltonian \( H \) is

\[
I_{\text{Bulk}} := \int_{S^1 \times S^1} \text{ch}(\mathcal{F}_+).
\]

(4.1)

**4.3. The edge Hamiltonian.** We now take the boundary into account. The “edge” state space is the space \( l^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z}, \mathbb{C}^k) \) of square integrable sequences of vectors in \( \mathbb{C}^k \) on the half-lattice \( \mathbb{Z}_{\geq 0} \times \mathbb{Z} \).

The Fourier transform of the bulk Hamiltonian \( H : l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k) \rightarrow l^2(\mathbb{Z} \times \mathbb{Z}, \mathbb{C}^k) \) in the direction “along the edge” transforms \( H \) into a family of self-adjoint translationally invariant operators

\[
H(t) : l^2(\mathbb{Z}, \mathbb{C}^k) \rightarrow l^2(\mathbb{Z}, \mathbb{C}^k).
\]

(4.2)

Then \( H(t) \) depends smoothly on \( t \in S^1 \). Let \( \Pi : l^2(\mathbb{Z}, \mathbb{C}^k) \rightarrow l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \) denote the projection.

**Definition 4.4.** The edge Hamiltonian is the family of Toeplitz operators

\[
H^\#(t) := \Pi \circ H(t) \circ \Pi : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \rightarrow l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k), \quad t \in S^1.
\]

(4.3)

**Definition 4.5.** The edge index \( I_{\text{Edge}} \) of the Hamiltonian \( H \) is the spectral flow of the edge Hamiltonian

\[
I_{\text{Edge}} := \text{sf}(H^\#(t)).
\]

(4.4)

**Theorem 4.6** (Bulk-edge correspondence). \( I_{\text{Bulk}} = I_{\text{Edge}} \).

**Proof.** Consider the unit disc \( B := \{ z \in \mathbb{C} : |z| \leq 1 \} \). This is a strongly pseudoconvex domain in \( \mathbb{C} \). We view the bulk Hamiltonian \( H(s, t) \) as a function on \( \partial B \times S^1 \) with values in the set \( \text{Herm}(k) \) of invertible Hermitian \( k \times k \)-matrices.

We endow \( B \) with the Bergman metric and consider the Dolbeault-Dirac operator \( D = \partial + \bar{\partial}^* \) on the space \( L^2\Omega^1\bullet(B, \mathbb{C}^k) = L^2\Omega^{1,0}(B, \mathbb{C}^k) \oplus L^2\Omega^{1,1}(B, \mathbb{C}^k) \) of square-integrable \((1, \bullet)\)-forms on \( B \). Let \( \mathcal{H} := \ker D \). Then \( \mathcal{H} \subset L^2\Omega^{1,0}(B, \mathbb{C}^k) \). Let \( P : L^2\Omega^{1,\bullet}(B, \mathbb{C}^k) \rightarrow \mathcal{H} \) denote the orthogonal projection. As usual, for \( f \in C_b^\infty(B, k) \) we denote by \( T_f := P \circ M_f : \mathcal{H} \rightarrow \mathcal{H} \) the Toeplitz operator.

To each \( u = \{u_j\}_{j \geq 0} \in l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \) we associate a 1-form

\[
\phi(f) := \sum_{j \geq 0} u_j z^j \, dz \in \mathcal{H}.
\]

If \( A : l^2(\mathbb{Z}, \mathbb{C}^k) \rightarrow l^2(\mathbb{Z}, \mathbb{C}^k) \) is a translationally invariant operator, then \( \phi \circ A \circ \phi^{-1} \) is the multiplication operator by the Fourier transform \( A(s) \) of \( A \). Let \( a : \bar{B} \rightarrow \text{Herm}(k) \) be a continuous extension of \( A(s) \) to \( \bar{B} \) (i.e., we assume that \( a|_{\partial B} = A(s) \)) and let

\[
T_a := P \circ M_a(s) : \mathcal{H} \rightarrow \mathcal{H}
\]

be the corresponding Toeplitz operator.

We also define a different Toeplitz operator

\[
A^\# := \Pi \circ H(t) \circ \Pi : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \rightarrow l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k)
\]

associated with \( A \) (notice that (4.3) is a special case of this construction). Those two Toeplitz operators are closely related. In particular, it is proven in [12] that the difference

\[
T_a - \phi \circ A^\# \circ \phi^{-1}
\]
is compact. We now apply this result to the family of operators $H(t) : l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k) \to l^2(\mathbb{Z}_{\geq 0}, \mathbb{C}^k)$. Let $h : \bar{B} \times S^1 \to \text{Herm}(k)$ be a continuous extension of $H(s,t)$ and set $h_t(s) := h(s,t)$. Then

$$T_{ht} - \phi \circ H^\#(t) \circ \phi^{-1} : \mathcal{H} \to \mathcal{H}, \quad t \in S^1,$$

is a continuous family of compact operators. It follows, [5, Proposition 1.12] (see also [6, Proposition 17.6]), that

$$\text{sf}(T_{ht}) = \text{sf} \left( \phi \circ H^\#(t) \circ \phi^{-1} \right) = \text{sf} \left( H^\#(t) \right).$$

The theorem follows now from definitions of bulk and edge indexes, (4.1), (4.4), and Theorem 3.3.

\[\square\]

5. Proof of Theorem 2.10

The proof of Theorem 2.10 consists of two steps. First (Lemma 5.3) we apply a result of Atiyah, Patodi and Singer [2, Th. 7.4] (see also [23]) to conclude that the spectral flow $\text{sf}(T_{ht})$ is equal to the index of a certain operator on $\mathcal{M}$. Then, using the argument similar to [11] we show that the latter index is equal to the index of $\mathcal{B}_{sf}$.

5.1. Functions with value in $\mathcal{H}$. We view the space $L^2(S^1, \mathcal{H})$ of square integrable functions with values in $\mathcal{H}$ as a subspace of $L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$. Then the family $T_{ht}$ naturally induces an operator $L^2(S^1, \mathcal{H}) \to L^2(S^1, \mathcal{H})$ which we still denote by $T_{ht}$. Let $T_{ht}^\pm$ denote the restriction of $T_{ht}$ to $L^2(S^1, \mathcal{H}^\pm)$.

5.2. The spectral flow as an index. Atiyah, Patodi and Singer, [2, Th. 7.4], proved that the spectral flow of a periodic family of elliptic differential operators $A(t)$ ($t \in S^1$) is equal to the index of the operator $\partial_t - A$ on $S^1$. Robbin and Salomon [23] extended this equality to a much more general family of operators. Applying this result to our situation we immediately obtain

Lemma 5.3. Under the assumptions of Theorem 2.10 we have

$$\text{sf}(T_{ht}^\pm) = \text{ind} \left( \frac{\partial}{\partial t} - T_{ht}^\pm \right) \bigg|_{L^2(S^1, \mathcal{H}^\pm)}.$$  \hspace{1cm} (5.1)

Notice that, since $\text{sf}(T_{ht}^\pm) = -\text{sf}(-T_{ht}^\pm)$, equality (5.1) is equivalent to

$$\text{sf}(T_{ht}^\pm) = -\text{ind} \left( \frac{\partial}{\partial t} + T_{ht}^\pm \right) \bigg|_{L^2(S^1, \mathcal{H}^\pm)}.$$  \hspace{1cm} (5.2)

5.4. Harmonic sections on $\mathcal{M}$. Let $\mathcal{H} \subset L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k)$ denote the kernel of $\mathcal{P}$ and let $\mathcal{P} : L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \to \mathcal{H}$ be the orthogonal projection.

We denote by $P_0 : L^2(S^1) \to L^2(S^1)$ the orthogonal projection onto the subspace of constant functions. Then with respect to decomposition (2.24) we have $\mathcal{P} = P_0 \otimes P$.

To simplify the notation in the computations below we write $P_0$ for $P_0 \otimes 1$, $Q_0$ for $(1 - P_0) \otimes 1$, and $P$ for $1 \otimes P$. Then the space $L^2(S^1, \mathcal{H})$ coincides with the image of the projection

$$P : L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \to L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k).$$

It follows that the projections $\mathcal{P}$ and $Q := 1 - \mathcal{P}$ preserve the space $L^2(S^1, \mathcal{H})$ and their restrictions this space are given by

$$\mathcal{P}|_{L^2(S^1, \mathcal{H})} = P_0, \quad Q|_{L^2(S^1, \mathcal{H})} = Q_0.$$  \hspace{1cm} (5.3)
Lemma 5.5. The operator
\[ [P, \mathcal{M}_f] := P \circ \mathcal{M}_f - \mathcal{M}_f \circ P : L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \to L^2(\mathcal{M}, \mathcal{E} \otimes \mathbb{C}^k) \] (5.4)
is compact.

Proof. The proof of Lemma 2.4 of [11] extends to our situation almost without any changes. We present it here for completeness.

By Assumption 2.2 there exists a small ball \( B \subset \mathbb{C} \) about 0 which does not contain non-zero points of the spectrum of \( 1 \otimes D \). To simplify the notation we write \( D \) for \( 1 \otimes D \).

For \( \lambda \) not in the spectrum of \( D \), let \( R_D(\lambda) := (\lambda - D)^{-1} \) denote the resolvent. By functional calculus we have
\[ P = \frac{1}{2\pi i} \int_{\partial B} R_D(\lambda) d\lambda. \]

Let \( d_x f := df_t(x) \) be the differential of \( f \) along \( M \) (so that \( df = d_x f + \frac{\partial}{\partial t} dt \)). Then we have
\[ [R_D(\lambda), \mathcal{M}_f] = R_D(\lambda) [D, \mathcal{M}_f] R_D(\lambda) = R_D(\lambda) c(d_x f) R_D(\lambda). \]

From Rellich’s Lemma and the fact that \( d_x f \) vanishes at infinity we conclude that \( c(d_x f) R_D(\lambda) \) is compact. Hence \([R_D(\lambda), \mathcal{M}_f]\) is also compact. It follows that
\[ [P, \mathcal{M}_f] = \frac{1}{2\pi i} \int_{\partial B} [R_D(\lambda), \mathcal{M}_f] d\lambda \]
is compact. \( \square \)

Corollary 5.6. The operators \( P \circ \mathcal{M}_f \circ Q \) and \( Q \circ \mathcal{M}_f \circ P \) are compact.

Proof. The operator
\[ P \circ \mathcal{M}_f \circ Q - Q \circ \mathcal{M}_f \circ P = P \circ \mathcal{M}_f - \mathcal{M}_f \circ P \]
is compact by Lemma 5.5. Since the range of \( P \) is orthogonal to the range of \( Q \), it follows that the operators \( P \circ \mathcal{M}_f \circ Q \) and \( Q \circ \mathcal{M}_f \circ P \) are compact. \( \square \)

Lemma 5.7. Under the assumptions of Theorem 2.10 we have
\[ \text{ind} \mathcal{B}_{cf} = \text{ind} \left( \frac{\partial}{\partial t} - T_f^+ \right)_{L^2(S^1, \mathcal{H}^+)} + \text{ind} \left( \frac{\partial}{\partial t} + T_f^- \right)_{L^2(S^1, \mathcal{H}^-)}. \] (5.5)

Proof. Set
\[ A := c(dt) \frac{\partial}{\partial t} + i c.\mathcal{M}_f. \]

Then \( \mathcal{B}_{cf} = 1 \otimes D + A \). Consider a one parameter family of operators
\[ \mathcal{B}_{cf,u} := 1 \otimes D + u A, \quad 0 \leq u \leq 1. \]

It follows from \( 2.8 \) and the vanishing of \( d_x f \) at infinity that for all \( u > 0 \) the operator \( \mathcal{B}_{cf,u} \) satisfies the Callias-condition \( 2.5 \) and, hence, is Fredholm.

Since \( D \) and \( \frac{\partial}{\partial t} \) commute with \( P \) and \( Q \)
\[ P \circ \mathcal{B}_{cf,u} \circ Q = u c P \circ \mathcal{M}_f \circ Q, \quad Q \circ \mathcal{B}_{cf,u} \circ P = u c Q \circ \mathcal{M}_f \circ P. \]

Thus these operators are compact by Corollary 5.6. It follows that
\[ \text{ind} \mathcal{B}_{cf,u} = \text{ind} P \circ \mathcal{B}_{cf,u} \circ P|_{\text{im} P} + \text{ind} Q \circ \mathcal{B}_{cf,u} \circ Q|_{\text{im} Q}. \] (5.6)
The operator $Q \circ B_{c,f} \circ Q|_{\text{Im}Q} = Q \circ D \circ Q|_{\text{Im}Q}$ is invertible. Hence, it is Fredholm and its index is equal to 0. We conclude that $Q \circ B_{c,f,u} \circ Q|_{\text{Im}Q}$ $(0 \leq u \leq 1)$ is a continuous family of Fredholm operators with

$$\text{ind} Q \circ B_{c,f,u} \circ Q|_{\text{Im}Q} = 0.$$ 

From (5.6) we now obtain

$$\text{ind} B_{c,f} = \text{ind} B_{c,f,1} = \text{ind} P \circ B_{c,f,1} \circ P|_{\text{Im}P}$$

$$= \text{ind} P^+ \circ \left( \frac{\partial}{\partial t} - icM_f \right) \circ P^+|_{\text{Im}P^+} + \text{ind} P^- \circ \left( - \frac{\partial}{\partial t} - icM_f \right) \circ P^-|_{\text{Im}P^-}$$

$$= \text{ind} \left( \frac{\partial}{\partial t} - T_{f_1}^+ \right)|_{L^2(S^1,\mathcal{H}^+)} + \text{ind} \left( \frac{\partial}{\partial t} + T_{f_1}^- \right)|_{L^2(S^1,\mathcal{H}^-)}.$$ 

5.8. **Proof of Theorem 2.10.** Theorem 2.10 follows now from (5.1), (5.2), and (5.5). □

**REFERENCES**

[1] N. Anghel, *On the index of Callias-type operators*, Geom. Funct. Anal. 3 (1993), no. 5, 431–438.
[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. III*, Math. Proc. Cambridge Philos. Soc. 79 (1976), no. 1, 71–99.
[3] P. Baum and R. G. Douglas, *K homology and index theory*, Operator algebras and applications, Part I (Kingston, Ont., 1980), 1982, pp. 117–173. MR679098
[4] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Springer-Verlag, 1992.
[5] B. Booss and K. Wojciechowski, *Desuspension of splitting elliptic symbols. I*, Ann. Global Anal. Geom. 3 (1985), no. 3, 337–383. MR813137
[6] B. Boos-Bavnbek and K. P. Wojciechowski, *Elliptic boundary problems for Dirac operators*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1993.
[7] L. Boutet de Monvel, *On the index of Toeplitz operators of several complex variables*, Invent. Math. 50 (1978/79), no. 3, 249–272. MR520928
[8] M. Braverman and S. Cecchini, *Callias-type operators in von Neumann algebras*, The Journal of Geometric Analysis 28 (2018), no. 1, 546–586.
[9] M. Braverman and G. Maschler, *Equivariant APS index for dirac operators of non-product type near the boundary*, arXiv preprint [arXiv:1702.08105], to appear in Indiana University Mathematics Journal (201702).
[10] U. Bunke, *A K-theoretic relative index theorem and Callias-type Dirac operators*, Math. Ann. 303 (1995), no. 2, 241–279. MR1348799 (96e:58148)
[11] U. Bunke, *On the index of equivariant Toeplitz operators*, Lie theory and its applications in physics, III (Clausthal, 1999), 2000, pp. 176–184. MR1888382
[12] L. A. Coburn, *Singular integral operators and Toeplitz operators on odd spheres*, Indiana Univ. Math. J. 23 (1973/74), 433–439. MR0322595
[13] X. Dai and W. Zhang, *Higher spectral flow*, J. Funct. Anal. 157 (1998), no. 2, 432–469. MR1638328
[14] H. Donnelly and C. Fefferman, *L²-cohomology and index theorem for the Bergman metric*, Ann. of Math. (2) 118 (1983), no. 3, 593–618.
[15] P. Elbau and G. M. Graf, *Equivalence of bulk and edge Hall conductance revisited*, Comm. Math. Phys. 229 (2002), no. 3, 415–432. MR1924362
[16] G. M. Graf and M. Porta, *Bulk-edge correspondence for two-dimensional topological insulators*, Comm. Math. Phys. 324 (2013), no. 3, 851–895. MR3132359
[17] E. Guentner and N. Higson, *A note on Toeplitz operators*, Internat. J. Math. 7 (1996), no. 4, 501–513. MR1408836
[18] Yasuhiro Hatsugai, *Chern number and edge states in the integer quantum Hall effect*, Phys. Rev. Lett. 71 (1993), no. 22, 3697–3700. MR1246070
[19] S. Hayashi, *Bulk-edge correspondence and the cobordism invariance of the index* (201611), available at 1511.08073.
THE SPECTRAL FLOW OF A FAMILY OF TOEPLITZ OPERATORS

[20] J. Kellendonk, T. Richter, and H. Schulz-Baldes, Edge current channels and Chern numbers in the integer quantum Hall effect, Rev. Math. Phys. 14 (2002), no. 1, 87–119. MR1877916

[21] H. B. Lawson and M.-L. Michelsohn, Spin geometry, Princeton University Press, Princeton, New Jersey, 1989.

[22] E. Prodan and H. Schulz-Baldes, Bulk and boundary invariants for complex topological insulators, Mathematical Physics Studies, Springer, [Cham], 2016. From K-theory to physics.

[23] J. Robbin and D. Salamon, The spectral flow and the Maslov index, Bull. London Math. Soc. 27 (1995), no. 1, 1–33.

[24] E. M. Stein, Boundary behavior of holomorphic functions of several complex variables, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972. Mathematical Notes, No. 11. MR0473215

[25] Y. Yu, Y.-S. Wu, and X. Xie, Bulk-edge correspondence, spectral flow and Atiyah-Patodi-Singer theorem for the $Z_2$-invariant in topological insulators, Nuclear Phys. B 916 (2017), 550–566. MR3611419

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Appendix:
A perspective from (unbounded) KK-theory
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We consider the assumptions and notation of Section 2. The aim of this short appendix is to review Theorem 2.10 from the perspective of (unbounded) KK-theory. For simplicity, we will assume that $f = \{f_t\}_{t \in S^1}$, viewed as an $M_k(\mathbb{C})$-valued function on $S^1 \times M$, is chosen such that $(1 + f^2)^{-1}$ vanishes at infinity. This assumption ensures that the operator $M_f$ (multiplication by $f$), acting on the Hilbert $C_0(S^1 \times M)$-module $\Gamma_0(S^1 \times M, E \otimes \mathbb{C}^k)$, has compact resolvents, so that $(C, \Gamma_0(S^1 \times M, E^+ \oplus E^-), M_f)$ is an unbounded Kasparov $C^*$-module. It also means we do not need the (sufficiently large) constant $c > 0$, and we simply set $c = 1$.

Theorem 2.10 states that we have the equality

$$ sf(T^+_{f_t}) - sf(T^-_{f_t}) = \text{ind} \mathcal{B}_f \in \mathbb{Z}. \quad (A.1) $$

In the context of KK-theory, the right-hand-side of this equality should be viewed as an element in $KK^0(C, \mathbb{C})$. The left-hand-side naturally defines an element in $KK^1(C, C(S^1))$ (cf. §2.3), given as the (odd!) class of the regular self-adjoint Fredholm operator

$$ T_f := \begin{pmatrix} T^+_{f_t} & 0 \\ 0 & -T^-_{f_t} \end{pmatrix} $$

on the Hilbert $C(S^1)$-module $C(S^1, (\mathcal{H}^+ \oplus \mathcal{H}^-) \otimes \mathbb{C}^k)$, where $T^\pm_{f_t} = \{T^\pm_{f_t}(t)\}_{t \in S^1}$ is given by $T^\pm_{f_t}(t) := T^\pm_{f_t}$, and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ denotes the kernel of $D$. Of course, these KK-groups are both isomorphic to $\mathbb{Z}$, and we have a natural isomorphism $\cdot \otimes_{C(S^1)} [-i\partial_t] : KK^1(C, C(S^1)) \to KK^0(C, \mathbb{C})$ (which sends the spectral flow of a family $A(t)$ to the index of $\partial_t - A$, as described
in \cite{2}. Thus we rewrite Eq. (A.1) as (cf. Eq. (5.7))

\[ \mathcal{T}_t \otimes_{C(S^1)} [-i\partial_t] = \text{ind} \mathcal{B}_t \in KK^0(\mathbb{C}, \mathbb{C}). \]

Now let us consider the right-hand-side of this equality. It is well understood that the index class of the Callias-type operator \( \mathcal{B}_t = \mathcal{D} + i\mathcal{M}_t \) is given by the Kasparov product \( \mathcal{B}_t = [\mathcal{M}_t] \otimes_{C_0(S^1 \times M)} [\mathcal{D}] \), cf. \cite{3}. The class of \( \mathcal{D} \) is simply given as the exterior Kasparov product \( \mathcal{D} = [D] \otimes [-i\partial_t] \) of the Dirac operator \( D \) on \( M \) with \(-i\partial_t\) on \( S^1 \). Using the properties of the Kasparov product, we then obtain

\[ \text{ind} \mathcal{B}_t = [\mathcal{B}_t] = [\mathcal{M}_t] \otimes_{C_0(S^1 \times M)} ([D] \otimes [-i\partial_t]) = ([\mathcal{M}_t] \otimes_{C_0(M)} [D]) \otimes_{C(S^1)} [-i\partial_t]. \]

Since the Kasparov product with \([-i\partial_t]\) gives an isomorphism, Eq. (A.1) can be rewritten as

\[ [\mathcal{T}_t] = [\mathcal{M}_t] \otimes_{C_0(M)} [D] \in KK^1(\mathbb{C}, C(S^1)). \]

The Kasparov product on the right-hand-side can be computed \cite{2} Example 2.38, and is represented by the regular self-adjoint operator (with compact resolvents)

\[ a_t := \begin{pmatrix} \mathcal{M}_t^+ & D^- \\ D^+ & -\mathcal{M}_t^- \end{pmatrix} \]

on the Hilbert \( C(S^1) \)-module \( C(S^1, L^2(M, E^+ \oplus E^-)) \). Theorem \ref{2.10} can then be reproven by showing the equality \([\mathcal{T}_t] = [a_t] \) in \( KK^1(\mathbb{C}, C(S^1)) \).

**Proposition A.1.** We have the equality

\[ [\mathcal{T}_t] = [a_t] \in KK^1(\mathbb{C}, C(S^1)). \]

**Proof.** The proof is closely analogous to the proof of Lemma \ref{5.4}. Let \( P = P^+ \oplus P^- \) denote the projection onto the kernel of \( D \), and write \( Q = 1 - P \). Since \( PDP = 0 \), we have the equality \( \mathcal{T}_t = P a_t P \) (where we used the definition of the Toeplitz operators \( T_{f_t} := PM_{f_t}P \)). Hence we need to show that \( P a_t P \) and \( a_t \) define the same class in \( KK^1(\mathbb{C}, C(S^1)) \). By Corollary \ref{5.6} we know that

\[ P a_t Q = \begin{pmatrix} P^+ & \mathcal{M}_t^+ Q^+ \\ 0 & -P^- \mathcal{M}_t^- Q^- \end{pmatrix} \]

is compact, and similarly for \( Q a_t \). This implies that \( P a_t P \) and \( Q a_t Q \) are both Fredholm, and that \([a_t] = [P a_t P] + [Q a_t Q]\). Rescaling the function \( f \) by \( c > 0 \), we see that the operator \( Q a_t Q \) is Fredholm for any \( c > 0 \). Furthermore, since \( D \) is invertible on \( \text{Ran} Q \), we find for \( c = 0 \) that \( Q a_0 Q = QDQ \) is invertible, and therefore its class in \( KK^1(\mathbb{C}, C(S^1)) \) is trivial. Since we have a continuous path of Fredholm operators for \( 0 \leq c \leq 1 \), we conclude that the class of \( Q a_t Q \) is also trivial. Thus we obtain

\[ [a_t] = [P a_t P] + [Q a_t Q] = [P a_t P]. \]

The statement and proof of Proposition A.1 do not rely on the notion of spectral flow, but merely consider the Fredholm operator \( \mathcal{T}_t \) and its odd \( KK \)-class. Hence Proposition A.1 can straightforwardly be generalised to the case where we replace \( S^1 \) by an arbitrary compact space. We thus obtain the following:

**Theorem A.2.** Let \( E = E^+ \oplus E^- \) be a graded Dirac bundle over a complete Riemannian manifold \( M \), and let \( D \) be the associated Dirac operator. Assume that zero is an isolated point of the spectrum of \( D \), and let \( P \) denote the projection onto the kernel of \( D \). Let \( S \) be a compact
topological space, and let $f = \{f_t\}_{t \in S} \in C(S \times M, M_k(\mathbb{C}))$ be given by a continuous family of smooth $M_k(\mathbb{C})$-valued functions $f_t$ on $M$ such that $(1 + f^2)^{-1}$ vanishes at infinity. We consider the Toeplitz operator $T_f = (P \otimes 1) \, M_f (P \otimes 1)$ on the Hilbert $C(S)$-module $C(S, H \otimes \mathbb{C}^k)$. Then we have the equality

$$[T_f] = [\mathcal{M}_f] \otimes_{C_0(M)} [D] \in KK^1(C, C(S)).$$

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References

[1] S. Baaj and P. Julg, Théorie bivariante de Kasparov et opérateurs non bornés dans les $C^*$-modules hilbertiens, C. R. Acad. Sci. Paris Sér. I Math. 296 (1983), 875–878.
[2] Simon Brain, Bram Mesland, and Walter D. van Suijlekom, Gauge theory for spectral triples and the unbounded Kasparov product, J. Noncommut. Geom. 10 (2016), 135–206.
[3] Ulrich Bunke, A K-theoretic relative index theorem and Callias-type Dirac operators, Math. Ann. 303 (1995), no. 1, 241–279.
[4] G. G. Kasparov, The operator K-functor and extensions of $C^*$-algebras, Izv. Akad. Nauk SSSR 44 (1980), 571–636.
[5] K. van den Dungen, The index of generalised Dirac-Schrödinger operators, 2017. arXiv:1710.09206.