The Lorentzian Space-Times of the Orientation-Orbifold String Systems

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Abstract

To illustrate our recent discussions of the target space-times in general orbifold-string theories of permutation-type, we return here to a detailed analysis of some simple examples of this type, namely an explicit set of orientation-orbifold string systems. These orientation-orbifold string systems provide twisted, multisector generalizations of ordinary critical open-closed bosonic string systems – each such system exhibiting a unique graviton. Furthermore, each sector $\sigma$ of each of these string systems shows the following properties: a) 26 effective degrees of freedom, b) a Lorentzian space-time with space-time dimension $D(\sigma) \leq 26$, c) an $SO(D(\sigma) - 1,1)$-invariant ordinary string subsystem with quantized intercept less than or equal one, and d) an extra set of $(26 - D(\sigma))$ twisted fields which are $SO(D(\sigma) - 1,1)$ scalars. Subexamples of non-tachyonic strings and four-dimensional strings are noted. Additionally, we discuss certain subsets of physical states of these theories, concluding that these investigations are so far consistent with the no-ghost conjecture for all the Lorentzian orbifold-string theories of permutation-type.

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1 Introduction

The orbifold-string theories of permutation-type [1-8] are multisector candidates for new physical string systems. The expectation [1] that these theories should be physical is based simply on the principles of orbifold theory and the fact that sets of copies of ordinary critical strings are themselves physical. Any “surprising” consistencies encountered in the exploration of these theories – and we shall see a number of these below – should be viewed in this perspective.

Using the techniques of the orbifold program [9-23], we have so far studied only the bosonic prototypes of these theories, including

\[
\frac{U(1)^{26}_K}{H_+}, \quad \left[\frac{U(1)^{26}_K}{H_+}\right]_{\text{open}}, \quad H_+ \subset H(\text{perm})_K \times H_{26}'
\]

\[
\frac{U(1)^{26}}{H_-} = \frac{U(1)^{26} \times U(1)^{26}_{\text{perm}}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26}
\]

where \(U(1)^{26}\) is the critical closed string, \(H(\text{perm})_K\) permutes the copies of the closed string and \(H'_{26}\) is any automorphism group of (the left- and right-movers of) the closed string. The systems in Eq. (1.1b) with \(H(\text{perm})_2 = \mathbb{Z}_2(\text{w.s.})\) are called the orientation-orbifold string theories [20,21,1,3,4,7,8], to which we shall return below.

As large examples of these systems, we have studied the automorphism groups [3,8]

\[
H'_{26} \subset (\pm \mathbb{1})_d \times H(\text{perm})_{26-d}, \quad 1 \leq d \leq 26
\]

for which the divisors \(H_\pm\) in Eq. (1.1) then involve the two permutation groups \(H(\text{perm})_K\) and \(H(\text{perm})'_{26-d}\). Both permutation groups have been systematically analyzed [8] in terms of the respective cycle lengths \(f_j(\sigma)\) (in \(H(\text{perm})_K\)) and \(F_J(\sigma)\) (in \(H(\text{perm})'_{26-d}\)) of their elements

\[
\vec{j} = 0, 1, \ldots, f_j(\sigma) - 1, \quad j = 0, 1, \ldots, N(\sigma) - 1
\]

\[
\vec{J} = 0, 1, \ldots, F_J(\sigma) - 1, \quad J = 0, 1, \ldots, N(\sigma)' - 1
\]

in each twisted sector \(\sigma\) of the orbifold-string theory.

So far we have presented discussions of the following topics:

- the extended actions and new twisted world-sheet permutation gravities [1,6]
- BRST quantization [2,6] of each cycle in the twisted sectors
- the extended physical-state conditions [3-8] at cycle central charge \(\hat{c}_j(\sigma) = 26f_j(\sigma)\)
- the cycle dynamics [3,7,8] of each sector, including the explicit form of the extended Virasoro generators and twisted current algebras of each cycle.
- the equivalent, reduced formulation [3,5,7,8] at reduced cycle central charge \(c_j(\sigma) = 26\)
• interacting examples [4,5]
• target space-times and their symmetries [7,8].

In these investigations we have seen that, although the original formulation of cycle $j$ (at cycle central charge $\hat{c}_j(\sigma) = 26 f_j(\sigma)$) is local [1], the equivalent, reduced formulation (at reduced cycle central charge $c_j(\sigma) = 26$ is often simpler. In any case the two formulations can be straightforwardly reconstructed, each from the other, via a precise 1-1 map which generalizes (the inverse of) the orbifold-induction procedure [9,3,7,8]. In particular, the target space-time dimension of cycle $j$ in sector $\sigma$

$$\hat{D}_j(\sigma) = D_j(\sigma)$$

is invariant under the map. Moreover, these target space-times can be either Lorentzian or Euclidean [8].

Drawing on this general analysis, our task here is to focus on some simple examples in further detail. In particular, we concentrate on the following large example of orientation-orbifold string theories [1,3,4,7,8]

$$U(1)^{26}_{H^-} = \frac{U(1)^{26} \times U(1)^{26}}{H^-},$$

$$H^- \subset \{ \tau_0 \times (\mathbb{1})_{(d)} \times H(\text{perm})'_{26-d}, \tau_- \times (\pm \mathbb{1})_{(d)} \times H(\text{perm})'_{26-d} \}$$

$$1 \leq d \leq 26$$

where $\tau_0 = 1$ and $\tau_-$ is the non-trivial element of the world-sheet orientation-reversing $\mathbb{Z}_2$ called $\mathbb{Z}_2(w.s.)$. These systems include [4] and generalize the ordinary critical open-closed string system, and consist of an equal number of twisted open- and closed-string sectors:

open: $\tau_-, f_j(\sigma) = 2, \hat{c}_j(\sigma) = 52, c_j(\sigma) = 26$ (1.6a)

closed: $\tau_0 = 1, f_j(\sigma) = 1, \hat{c}_j(\sigma) = c_j(\sigma) = 26$. (1.6b)

The open-string sectors appear after the semicolon in Eq. (1.5b), where one may choose the d-dimensional automorphism as either $(\pm \mathbb{1})_{(d)}$ or $(\mathbb{1})_{(d)}$ but not both. The closed-string sectors appear before the semicolon, and together form the ordinary space-time orbifold

$$\frac{U(1)^{26}}{H_{26}^{'d}}, \quad H_{26}^{'d} \subset (\mathbb{1})_{(d)} \times H(\text{perm})'_{26-d}$$

where we have suppressed the division by the trivial element of $\mathbb{Z}_2(w.s.)$. The unique graviton of each orientation-orbifold string system is found in the closed-string sector corresponding to the trivial element of $H(\text{perm})'_{26-d}$. The bulk of this paper focuses on the twisted open-string sectors, but we will return in Secs. 14 and 15 to assemble the full orientation-orbifold string theories.
The systems (1.5) were partially analyzed in Ref. [3], and it is now known [8] that all sectors of all these orientation-orbifold string systems live on sector-dependent Lorentzian target space-times, with space-time symmetry $SO(D_j(\sigma) - 1, 1)$. Here, we will work this out in detail, concluding that each sector of these theories contains an ordinary $D_j(\sigma)$-dimensional string subsector, plus a set of $(26 - D_j(\sigma))$ extra $SO(D_j(\sigma) - 1, 1)$-invariant twisted scalar fields. This gives a total of $c_j(\sigma) = 26$ effective degrees of freedom in each sector, as in the ordinary open-closed string system.

When full computational details are required, we illustrate with the cyclic subset

$$H(\text{perm})'_{26-d} = \mathbb{Z}_{26-d}$$

$$1 \leq d \leq 26.$$  

(1.8a)

(1.8b)

In these cases we compute the number of target space-time dimensions explicitly (see the Tables of Sec. 7), concluding that

$$\hat{D}_j(\sigma) = D_j(\sigma) \leq 26.$$  

(1.9)

Secs. 8 and 9 emphasize in particular the non-tachyonic strings and the four-dimensional strings contained in this subset of theories.

The upper bound (1.9) is in fact a necessary condition for the new string theories to be physical and, indeed, such gratifying consistency – as we have explained above – is expected on general grounds for all $H(\text{perm})'_{26-d}$. Arguments are given in Secs. 4, 7 and 15 that the bound indeed holds for general $H(\text{perm})'_{26-d}$, and slightly stronger conditions on the open-string space-time dimensionalities are obtained in Eq. (7.6).

These results lead us to examine a number of subsets of physical states (see Secs. 10, 11 and 13) in the large example (1.5), concluding that so far these investigations are consistent with the no-ghost conjecture (see Ref. [1] and Sec. 12) for all Lorentzian orbifold-string theories of permutation-type. Sec. 13 notes in particular the ordinary $(D_j(\sigma) \leq 26)$-dimensional open strings which live as subsectors of these theories at quantized intercept less than or equal to one.

2 The Twisted Open-String sectors at $\hat{c} = 52$

The open-string sectors of the orientation-orbifold string systems (1.5)

$$\{\text{open}\} = \{\tau_- \times (\pm 1)_{(d)} \times H(\text{perm})'_{26-d}\}$$

$$1 \leq d \leq 26$$

(2.1a)

(2.1b)

are associated with the non-trivial element $\tau_-$ of $\mathbb{Z}_2(w.s.)$. This element consists of a single $j$-cycle of length 2

$$f_j(\sigma) = 2, \quad \hat{c}_j(\sigma) = 52, \quad j = 0, \quad \bar{j} = 0, 1$$  

(2.2)
but we will for simplicity suppress the single cycle label \( j = 0 \).

Then we may write the extended physical-state conditions and orbifold Virasoro generators of open-string sector \( \sigma \) as follows [3,8]:

\[
(\hat{L}_j((m + \frac{j}{2}) \geq 0) - \frac{17}{8} \delta_{m + \frac{j}{2}, 0})|\chi(\sigma)\rangle = 0, \quad \hat{\chi} \hat{J}_j = 0, 1 \quad (2.3a)
\]

\[
[\hat{L}_j(m + \frac{j}{2}), \hat{L}_k(n + \frac{k}{2})] - (m + \frac{j+k}{2})L_{j+k}(m + n + \frac{j+k}{2}) + \frac{52}{12}(m + \frac{j}{2})(m + \frac{j+k}{2}) - 1)\delta_{m+n+\frac{j+k}{2}, 0} \quad (2.3b)
\]

\[
\hat{L}_j(m + \frac{j}{2}) = \hat{\Delta}_0(\sigma)\delta_{m+\frac{j}{2}, 0} + \frac{1}{4} G^{ab}(d) \sum_{\ell=0}^{1} \sum_{n \in \mathbb{Z}} \sum \hat{J}_{\epsilon a}(p + \frac{\epsilon + \ell}{2}) \hat{J}_{\epsilon, b, j-\ell} (m - p + \frac{\ell - \epsilon}{2}) \circ M + \frac{1}{4} \sum_{L} \frac{1}{F_L(\sigma)} \sum_{\ell=0}^{1} \sum_{p \in \mathbb{Z}} \hat{J}_{\ell, L, \ell}(p + \frac{\ell}{F_L(\sigma)} + \frac{\ell}{2}) \hat{J}_{-L, L, \ell}(m - p - \frac{\ell}{F_L(\sigma)} + \frac{\ell - \epsilon}{2}) \circ M \quad (2.3c)
\]

\[
\hat{\Delta}_0(\sigma) = \frac{13}{8} + \frac{1}{2} \hat{\Delta}_0(\sigma) \quad (2.3d)
\]

\[
\hat{\Delta}_0(\sigma) = \hat{\Delta}_0(\sigma)^{(26-d)} = \frac{1}{4} \sum_{L} \frac{1}{F_L(\sigma)} - 1 \int \left( \frac{2\ell}{F_L(\sigma)} \right) \geq 1 \int \left( \frac{2\ell}{F_L(\sigma)} \right) \geq 0 \quad (2.3e)
\]

\[
a = 0, 1, \ldots, d - 1, \quad G(\sigma) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}_{(d)}, \quad \sum_{L} F_L(\sigma) = 26 - d, \quad F_L(\sigma) \geq 1. \quad (2.3f)
\]

We remind that the choice of either \( \epsilon = 0 \) or \( 1 \) (but not both) corresponds to the choice of \( d \)-dimensional automorphism \( \omega(d) = (1)(d) \) or \( (-1)(d) \) respectively, and that neither choice makes any contribution \( (\hat{\Delta}_0(\sigma))^{(d)} = 0 \) to the conformal-weight shifts \( \hat{\Delta}_0(\sigma) \). The explicit form \( (2.3e) \) of the conformal-weight shift of sector \( \sigma \) is therefore determined entirely by the choice \( \omega(\sigma) \in H(\text{perm})^{26-d} \), which is characterized here by its \( L \)-cycle lengths \( \{F_L(\sigma)\} \). This result was given in Ref. [3] in the notation \( \bar{\chi} = \bar{u} = 0, 1 \) (with small letters for the \( L \)-cycles), and can also be obtained in the present notation as the special case \( f_j(\sigma) = 2 \) in Ref. [8].

In these simple cases, the orbifold Virasoro generators \( \{\hat{L}_j\} \) of sector \( \sigma \) in Eq. (2.3c) satisfy the orbifold Virasoro algebra [9,17, 2-8] of order two in Eq. (2.3b). The orbifold Virasoro algebra has an integral Virasoro subalgebra at \( \hat{c}(\sigma) = 52 \) defined by the generators \( \{\hat{L}_0(m)\} \). The conventional mode normal-ordering indicated here

\[
\hat{\Delta}(\xi) \hat{B}(\eta) \hat{\Delta}(\xi) \equiv \theta(\xi \geq 0) \hat{B}(\eta) \hat{A}(\xi) + \theta(\xi < 0) \hat{A}(\xi) \hat{B}(\eta) \quad (2.4)
\]

will be employed throughout this paper.
In what follows, we often use the nomenclature of Ref. [8], referring to the two kinds of twisted currents in Eq. (2.3c) as those of type $(d)$ $(a = 0, \ldots, d - 1)$ and those of type $(26 - d)$. The rest of the algebra of these currents is easily read from Refs. [3] or [8]

$$\hat{L}_j(m + \hat{j} + \frac{\hat{j}^2}{2}), \hat{J}_{a\hat{l}}(n + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2}) = -(n + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2}) \hat{J}_{a\hat{j} + \hat{l}}(m + n + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2})$$ (2.5a)

$$\hat{L}_j(m + \hat{j} + \frac{\hat{j}^2}{2}), \hat{J}_{LL\hat{L}}(n + \frac{\hat{l}}{2}) = -(n + \frac{\hat{l}}{2}) \hat{J}_{LL\hat{L}}(m + n + \frac{\hat{l}}{2})$$ (2.5b)

$$\hat{J}_{a\hat{l}}(m + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2}), \hat{J}_{\hat{l}L\hat{l}}(n + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2}) = 2(m + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2}) \delta_{\hat{l} + \hat{l}', 0} \mod 2 \delta_{m + n + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2} \mod 0}$$ (2.5c)

$$\hat{J}_{\hat{j}j}(m + \frac{j}{F_L(\sigma)} + \frac{\hat{j}^2}{2}), \hat{J}_{LL\hat{L}}(n + \frac{\hat{l}}{2}) = \delta_{\hat{j} + \hat{j}, 0} \mod 2 \delta_{m + n + \frac{j}{2} \mod 2}$$ (2.5d)

$$1 \leq d \leq 26, a, b = 0, 1, \ldots d - 1, \hat{L} = 0, 1, \ldots F_L(\sigma) - 1, L = 0, 1, \ldots N(\sigma)' - 1$$ (2.5e)

where $N(\sigma)'$ is the number of L-cycles in the element of $H(\text{perm})_{26-d}$. Note that the range of the parameter $d$ cannot be extended to $d = 0$ because $H(\text{perm})_{26}$ is not an automorphism group of the original Minkowski metric $G_{(26)}$ of $U(1)^{26}$.

For computational purposes, we will also need the periodicity conditions and the adjoints of all these operators [8]:

$$\hat{L}_{j\pm 2}(m + \hat{j} + \frac{\hat{j}^2}{2}) = \hat{L}_j(m \pm 1 + \frac{\hat{j}}{2})$$ (2.6a)

$$\hat{J}_{a\hat{j} \pm 2}(m + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2}) = \hat{J}_{a\hat{j}}(m \pm 1 + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2})$$ (2.6b)

$$\hat{J}_{-\epsilon a, \hat{j}}(m + \frac{\hat{j}}{2}) = \hat{J}_{a\hat{j}}(m - \epsilon + \frac{\hat{j} + \frac{\hat{j}^2}{2}}{2})$$ (2.6c)

$$\hat{J}_{L \pm F_L(\sigma), \hat{l}}(m + \frac{\hat{l} + \frac{\hat{l}^2}{2}}{2}) = \hat{J}_{LL\hat{l}}(m + \frac{\hat{l}}{2})$$ (2.6d)

$$\hat{L}_j(m + \frac{\hat{j}}{2})^\dagger = \hat{L}_{-j}(-m - \frac{\hat{j}}{2})$$ (2.6e)

$$\hat{J}_{a\hat{j}}(m + \frac{\hat{j}}{2})^\dagger = \hat{J}_{-\epsilon, a\hat{j}}(-m - \frac{\hat{j}}{2})$$ (2.6f)

$$\hat{J}_{LL\hat{l}}(m + \frac{\hat{l}}{2})^\dagger = \hat{J}_{-L, L - \hat{l}}(-m - \frac{\hat{l}}{2})$$ (2.6g)

It follows in particular that the generators of the Virasoro subalgebra satisfy the conventional generalized hermiticity $\hat{L}_0(m)^\dagger = \hat{L}_0(-m)$.

In each sector $\sigma$, a useful decomposition of the zero mode of the Virasoro generators is

$$\hat{L}_0(0) = \hat{L}_0(0)^\dagger = \frac{1}{4}(-\hat{P}^2(\sigma) + \hat{R}(\sigma)) + \hat{\Delta}_0(\sigma)$$ (2.7)
where $\hat{R}(\sigma)$ is the generalized number operator $[3,8]$ of sector $\sigma$ and

$$\hat{P}^2(\sigma) = \hat{P}^2(\sigma) = \eta_{(d)}^{\alpha\beta}J_{\alpha\beta}^a(0)J_{\alpha\beta}^a(0)$$

$$- \sum_{L} F_{L(\sigma)}^{-1} (\hat{J}_{0L0}(\sigma)J_{0L0}(\sigma) + \hat{J}_{F_{L(\sigma)}}(0)\hat{J}_{F_{L(\sigma)}}(0) \hat{F}_{L(\sigma)}^{-1}L_{L1}(0))$$

(2.8a)

$$\eta_{(d)} = -G_{(d)} = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{I} \end{pmatrix}_{(d)}$$

(2.8b)

is the momentum-squared operator of the sector. The last term of this expression contributes only when an L-cycle length $F_L(\sigma)$ is even.

We comment further on the set of zero modes $\{\hat{J}(0)_{\sigma}\}$ or momenta of sector $\sigma$, which commute with each other and all other current modes of the sector

$$[\{\hat{J}(0)_{\sigma}\}, \hat{J}(\{\text{all}\})] = 0.$$  

(2.9)

The number of zero modes then defines the target space-time dimension $[7,8]$ of sector $\sigma$, and for the open-string sectors of the orientation-orbifolds we easily count

$$\hat{D}(\sigma) \equiv \dim\{\hat{J}(0)_{\sigma}\} = d + N_O(\sigma)' + 2N_E(\sigma)'$$

(2.10)

$$N(\sigma)' = N_O(\sigma)' + N_E(\sigma)'$$

(2.11)

where $N_{O,E}(\sigma)'$ are respectively the number of L-cycles of odd and even length $F_L(\sigma)$ in $\omega(\sigma) \in \hat{H}(\text{perm})'_26-d$. This result is implicit in Ref. [3] and in agreement with the special case $f_j(\sigma) = 2$ in Ref. [8].

The target space-time dimension (2.10) can also be expressed in the notation

$$\hat{D}(\sigma) = d + \sum_{L} \alpha(L), \quad \alpha(L) \equiv \begin{cases} 1 & \text{for } F_L(\sigma) \text{ odd } \geq 1 \\ 2 & \text{for } F_L(\sigma) \text{ even } \geq 2 \end{cases}$$

(2.12)

where we have introduced the L-cycle function $\alpha(L)$. This function is natural in the orientation-orbifold string systems, and appears as well in the following simple expression for the conformal-weight shifts of sector $\sigma$

$$\hat{\delta}_0(\sigma) = \frac{1}{24}\sum_{L} (\theta(F_L(\sigma) = \text{odd } \geq 1)\frac{F_L^2(\sigma) - 1}{F_L(\sigma)} + \theta(F_L(\sigma) = \text{even } \geq 2)\frac{F_L(\sigma) - 1}{F_L(\sigma)} \{2.13a\})$$

$$= \frac{1}{24}(26 - d - \sum_{L} \frac{\alpha^2(L)}{F_L(\sigma)}), \quad \sum_{L} F_L(\sigma) = 26 - d$$

(2.13b)

after performing the sum over $\hat{L}$ in Eq. (2.3e). Note that the first form of the conformal-weight shift in Eq. (2.13a) shows explicitly that $\hat{\delta}_0(\sigma) \geq 0$, and the double inequalities

$$0 \leq \hat{\delta}_0(\sigma) \leq \frac{1}{24}(26 - d)$$

(2.14a)
\[ 0 \leq \sum_L \frac{\alpha^2(L)}{F_L(\sigma)} \leq 26 - d, \quad \sum_L F_L(\sigma) = 26 - d, \quad 1 \leq d \leq 26 \] (2.14b)

then follow on comparison of the two forms in Eq. (2.13).

The decomposition (2.7) and the extended physical-state condition (2.3a) also lead to the following result for the ground-state momentum-squared \( \hat{P}^2(\sigma)(0) \) of sector \( \sigma \) [3,8]

\[
\hat{\cal J}_{\sigma}(\sigma) \rangle \rangle \hat{J}(0) = 0 \quad (2.15a)
\]

\[
\hat{\cal J}_{L\ell\ell}(\sigma) \rangle \rangle \hat{J}(0) = 0 \quad (2.15b)
\]

\[
\hat{\cal R}(\sigma) \rangle \rangle \hat{J}(0) = 0 \quad (2.15c)
\]

\[
\hat{\cal P}^2(\sigma)(0) \rangle \rangle \hat{J}(0) = \hat{\cal P}^2(\sigma)(0) \rangle \rangle \hat{J}(0) \quad (2.15d)
\]

\[
\hat{\cal P}^2(\sigma)(0) = 2(\hat{\delta}_0(\sigma) - 1) \geq -2 \quad (2.15e)
\]

where \( \hat{\delta}_0(\sigma) \) are the conformal-weight shifts and \( \{0, \hat{J}(0)\rangle \rangle \} \) are the oscillator-free eigenstates of the momenta \( \{\hat{J}(0)\rangle \rangle \} \). At this value of the momentum-squared, the ground-state can also be called the momentum-boosted twist-field state of sector \( \sigma \). The following simple form of the ground-state momentum-squared in terms of the L-cycle function

\[
\hat{\cal P}^2(\sigma)(0) = -\frac{1}{12}(d - 2 + \sum_L \frac{\alpha^2(L)}{F_L(\sigma)}), \quad \sum_L F_L(\sigma) = 26 - d \quad (2.16)
\]

is then obtained from the explicit form (2.13b) of the conformal-weight shifts. Formulae are also known [3,8] for the level-spacing

\[
\Delta(\hat{\cal P}^2(\sigma)) = \Delta(\hat{\cal R}^2(\sigma)) = \begin{cases} 
4|m + \frac{\ell_L}{2}| & \text{for } \hat{\cal J}((m + \frac{\ell_L}{2}) < 0) \\
4|m + \frac{\ell_L}{2}| & \text{for } \hat{\cal J}((m + \frac{\ell_L}{2}) < 0)
\end{cases}
\]

(2.17)

due to the addition of negatively-moded currents of either type to a lower-level state.

Using the adjoint operations in Eq. (2.6) and the current algebras in Eq. (2.5), we finally comment on the norms of the basis-states of sector \( \sigma \). For the one-current states, we easily compute:

\[
||\hat{\cal J}_{\sigma}(\sigma) \rangle \rangle \hat{J}(0)\rangle \rangle ||^2 = 2G(\sigma)^d|m + \frac{\ell_L}{2}| \quad |0, \hat{\cal J}(0)\rangle \rangle \rangle ||^2 \quad (2.18a)
\]

\[
||\hat{\cal J}_{L\ell\ell}(\sigma) \rangle \rangle \hat{J}(0)\rangle \rangle ||^2 = 2F_L(\sigma)|m + \frac{\ell_L}{2}| \quad |0, \hat{\cal J}(0)\rangle \rangle \rangle ||^2 \quad (2.18b)
\]

The norms of higher basis-states are similarly computed, including the strictly positive norms of the general basis-state formed from the “extra” currents of type \((26 - d)\):

\[
|| \prod_{m\ell\ell} \hat{\cal J}_{L\ell\ell}(\sigma) \rangle \rangle \hat{J}(0)\rangle \rangle ||^2 = \prod_{m\ell\ell} (N_{m\ell\ell})! |F_L(\sigma)(2m \ell) + 2\hat{\cal L}| \quad |0, \hat{\cal J}(0)\rangle \rangle \rangle ||^2 \quad (2.19)
\]

\[
= \prod_{m\ell\ell} (N_{m\ell\ell})! |F_L(\sigma)(2m \ell) + 2\hat{\cal L}| \quad |0, \hat{\cal J}(0)\rangle \rangle \rangle ||^2. \quad (2.19)
\]
It is clear from these examples that, as in the ordinary open string, the only negative-norm states in the basis of sector $\sigma$ are those containing an odd number of time-like modes $\{\hat{J}_\epsilon \ell \}$, that is, $a = 0$ of the currents of type $(d)$. This should come as no surprise, since the currents of type $(26 - d)$ are the orbifoldization of copies of purely spatial currents in the untwisted sector $(U(1)^{26}_L \times U(1)^{26}_R)$ of the orbifold.

3 The Equivalent Formulation at $c = 26$

It is now well understood [3-5,7,8] that the description above of the open-string physical states at central charge $\hat{c}(\sigma) = 52$ has an equivalent, reduced formulation at reduced central charge $c(\sigma) = 26$. Indeed, although we limit the discussion here to this case, we are in fact describing only the special case with j-cycle length $f_j(\sigma) = 2$ included in the equivalent descriptions of the general cycle dynamics [7,8] at cycle central charge $\hat{c}_j(\sigma) = 26f_j(\sigma)$ and uniform reduced cycle central charge $c_j(\sigma) = 26$.

In the present case, the reduced formulation of sector $\sigma$ is obtained from the following 1-1 map [3]

$$L(M) \equiv 2\hat{L}_\ell(m + \frac{\ell}{2}) - \frac{13}{4} \delta_{m+\frac{\ell}{2},0} \quad (3.1a)$$

$$J_{\epsilon a}(M + \epsilon) \equiv \hat{J}_{\epsilon a \ell}(m + \frac{\epsilon + \ell}{2}), \quad a = 0, 1, \ldots, d - 1 \quad (3.1b)$$

$$J_{LL}(M + 2\hat{L}_F(\sigma)) \equiv \hat{J}_{LL\ell}(m + \frac{\ell}{2}) \quad (3.1c)$$

$$M \equiv 2m + \ell, \quad \ell = 0, 1 \quad (3.1d)$$

between the hatted operators at $\hat{c}(\sigma) = 52$ and the unhatted operators at $c(\sigma) = 26$. Note that $M \in \mathbb{Z}$ covers the integers once for $\ell$ in its fundamental range $\hat{\ell}$. For the reduced currents of type $(26-d)$ in Eq. (3.1c), we emphasize that the factor 2 in the characteristic moding $\{2\hat{L}/F_L(\sigma)\}$ represents the effect on the element $\omega'(\sigma) \in H(perm)'_{26-d}$ due to the unwinding of the non-trivial element of $\mathbb{Z}_2(w.s.)$. In fact such maps, including the extension to arbitrary j-cycles [7,8], are only modestly-generalized versions of (the inverse of) the so-called orbifold-induction procedure [9] – which was originally used to construct cyclic permutation orbifolds at an early stage of the orbifold program [10-23].

In terms of the unhatted operators, the map gives the following equivalent, reduced formulation of the physical states of sector $\sigma$ at reduced central charge $c(\sigma) = 26$:

$$(L(M \geq 0)) - \delta_{M,0})|\chi(\sigma)) = 0 \quad (3.2a)$$

$$[L(M), L(N)] = (M - N)L(M + N) + \frac{26}{12} M(M^2 - 1)\delta_{M+N,0} \quad (3.2b)$$
\[ L(M) = \tilde{\delta}_0(\sigma)\delta_{M,0} + \frac{1}{2}G^{ab}_{(d)} \sum_{Q \in \mathbb{Z}} \delta J_{La}(Q + \epsilon)J_{-\epsilon,b}(M - Q - \epsilon)_{\sigma}^{\circ} + \]
\[ + \frac{1}{2} \sum_{L} F_L(\sigma)^{-1} \sum_{L=0}^{F_L(\sigma) - 1} \sum_{Q \in \mathbb{Z}} \delta J_{LL}(Q + 2\frac{L}{F_L(\sigma)})J_{-L,L}(M - Q - 2\frac{L}{F_L(\sigma)})_{\sigma}^{\circ} \]  
(3.2c)

\[ F_L(\sigma) \geq 1, \quad \sum_{L} F_L(\sigma) = 26 - d, \quad 1 \geq d \geq 26, \quad a = 0, 1, \ldots, d - 1. \]  
(3.2d)

The Minkowski-space metric \( G_{(d)} \) and the conformal-weight shifts \( \tilde{\delta}_0(\sigma) \) of open-string sector \( \sigma \) are given in Eqs. (2.3f) and (2.3e), (2.13) respectively, and we note in particular that the extended physical-state conditions (2.3a) have been mapped into the ordinary physical-state condition (3.2a) – with unit intercept – for each sector \( \sigma \). It should also be emphasized that the states \( \{|\chi(\sigma)\rangle\} \) described in the reduced formulation are exactly the same physical states described at \( \tilde{c}(\sigma) = 52 \), now rewritten in terms of the reduced currents. Finally, because the map preserves the sign of the mode numbers, the mode normal-ordering in the Virasoro generators (3.2c)

\[ \circ A(\xi)B(\eta)^{\circ}_{\sigma} = \theta(\xi \geq 0)B(\eta)A(\xi) + \theta(\xi < 0)A(\xi)B(\eta) \]  
(3.3)

is isomorphic to the mode normal-ordering defined at \( \tilde{c} = 52 \) in Eq. (2.4).

We remind that the two kinds of reduced currents here are called [8] those of type \( (d) \) \((a = 0, \ldots, d - 1)\) and those of type \( (26 - d) \). The map straightforwardly completes the reduced formulation from the results of the previous section, beginning with the algebra of the reduced currents:

\[ [L(M), J_{ca}(N + \epsilon)] = -(N + \epsilon)J_{ca}(M + N + \epsilon), \quad a = 0, 1, \ldots, d - 1 \]  
(3.4a)

\[ [L(M), J_{LL}(N + \frac{2L}{F_L(\sigma)})] = -(N + \frac{2L}{F_L(\sigma)})J_{LL}(M + N + \frac{2L}{F_L(\sigma)}) \]  
(3.4b)

\[ [J_{La}(M + \epsilon), J_{\epsilon'b}(N + \epsilon')] = G^{(d)}_{ab}(M + \epsilon)\delta_{\epsilon + \epsilon',0} \mod 2\delta_{M + N + \frac{\epsilon + \epsilon'}{2},0} \]  
(3.4c)

\[ [J_{ij}(M + \frac{2L}{F_j(\sigma)}), J_{LL}(N + \frac{2L}{F_L(\sigma)})] = \delta_{jL}F_j(\sigma)(M + \frac{2L}{F_L(\sigma)})\delta_{j + L,0} \mod F_j(\sigma) \delta_{M + N + 2\frac{j + L}{F_j(\sigma)},0} \]  
(3.4d)

\[ \hat{L} = 0, 1, \ldots, F_L(\sigma) - 1, \quad L = 0, 1, \ldots, N(\sigma)' - 1, \quad \sum_{L} F_L(\sigma) = 26 - d. \]  
(3.4e)

We also recall here that the values \( \epsilon = 0 \) or \( 1 \) correspond to the original automorphisms \( (\omega)_d = (1)_d \) or \( (-1)_d \) respectively. As a simple application of the L-cycle data in Eq. (3.4e), let us check explicitly that the number of degrees of freedom in the reduced description of each sector \( \sigma \)

\[ \sum_{a} + \sum_{LL} = d + \sum_{L} F_L(\sigma) = 26 \]  
(3.5)
agrees with the reduced central charge $c(\sigma) = 26$ in the Virasoro algebra (3.2b).

We give next the reduced form of the periodicity relations and the adjoint operations

$$J_{-\epsilon,a}(M - \epsilon) = J_{a}((M - 2\epsilon) + \epsilon)$$  \hspace{1cm} (3.6a)

$$J_{L\pm F_{L}(\sigma)} (M + 2\frac{L\pm F_{L}(\sigma)}{F_{L}(\sigma)}) = J_{L\pm} (M \pm 2 + 2\frac{L}{F_{L}(\sigma)})$$  \hspace{1cm} (3.6b)

$$J_{a}(M + \epsilon) = J_{-\epsilon,a}(-M - \epsilon)$$  \hspace{1cm} (3.6c)

$$J_{LL}(M + 2\frac{L}{F_{L}(\sigma)}) \Rightarrow J_{-L,L}(-M - 2\frac{L}{F_{L}(\sigma)})$$  \hspace{1cm} (3.6d)

$$L(M) \Rightarrow L(-M).$$  \hspace{1cm} (3.6e)

where Eq. (3.6e) is ordinary generalized hermiticity for the reduced Virasoro generators at $c(\sigma) = 26$.

The reduced zero modes at $c(\sigma) = 26$ are in 1-1 correspondence with the zero modes of the original formulation at $\hat{c}(\sigma) = 52$

$$J_{0,a} = \hat{J}_{0,a}$$  \hspace{1cm} (3.7a)

$$\dim \{J_{0}\sigma\} = \dim \{\hat{J}_{0}\sigma\}$$  \hspace{1cm} (3.7b)

and we remark that the three types of reduced zero modes \{J_{0,a} \}, J_{0L} \), and $J_{F_{L}(\sigma)/2,L}$ occur respectively at $M = -\epsilon, 0$ and 1. As above, the third type of zero mode exists only when the cycle-length $F_{L}(\sigma)$ is even, and so we verify the invariance of the target space-time dimension

$$D(\sigma) = \hat{D}(\sigma) = d + NO(\sigma)' + 2NE(\sigma)' = d + \sum_{L} \alpha(L)$$  \hspace{1cm} (3.8)

under the reduction procedure (see Eqs. (2.10) and (2.12)).

Similarly, we find the reduced momentum-squared operator

$$L(0) = \frac{1}{2}(-P^{2}(\sigma) + R(\sigma)) + \hat{\delta}_{0}(\sigma)$$  \hspace{1cm} (3.9a)

$$P^{2}(\sigma) = \hat{P}^{2}(\sigma) = \eta_{(d)}^{ab} \sum_{Q \in \mathbb{Z}} J_{a}(0)J_{-\epsilon,b}(0) - \sum_{L} \{\hat{J}_{0L}(0)\hat{J}_{0L}(0) + \hat{J}_{F_{L}(\sigma)/2,L}(0)\hat{J}_{-F_{L}(\sigma)/2,L}(0)\}$$  \hspace{1cm} (3.9b)

$$\hat{J}_{0L}(0) \equiv \frac{1}{\sqrt{F_{L}(\sigma)}} J_{0L}(0), \quad \hat{J}_{F_{L}(\sigma)/2,L}(0) \equiv \frac{1}{\sqrt{F_{L}(\sigma)}} J_{F_{L}(\sigma)/2,L}(0)$$  \hspace{1cm} (3.9c)

where the $d$-dimensional Minkowski metric $\eta_{(d)}^{ab}$ is given in Eq. (2.8b) and we have rescaled the last two terms (of type $(26-d)$ as shown in Eq. (3.9c). All the operators $L(0)$, $P^{2}(\sigma)$ and $R(\sigma) = \hat{R}(\sigma)$ in this decomposition are hermitian, and the explicit form of the generalized number operator $R(\sigma)$ is given in Refs. [3,8].
We turn next to some simple properties of the physical states \{ |\chi(\sigma)\rangle \}, as defined by the reduced physical-state condition (3.20). We remind that the physical states are the same as those defined by the extended physical-state conditions (2.3a). Indeed all states, including the physical states as well as the basis states, are inert under the map (3.1) – which only relabels the operators. For good book-keeping however, we should here imagine having used the map to replace the negatively-moded (hatted) currents of these states in the original formulation at \( \hat{c}(\sigma) = 52 \) by the negatively-moded (unhatted) currents at \( c(\sigma) = 26 \).

More precisely, we have the same ground state in each sector

\[
|0, J(0)\rangle = |0, \hat{J}(0)\rangle
\]

\[
J_{\alpha a}((M + \epsilon) > 0)|0, J(0)\rangle = J_{LL}((n + 2\frac{L}{F_L(\sigma)}) > 0)|0, J(0)\rangle = 0
\]

\[
R(\sigma)|0, J(0)\rangle = 0
\]

\[
P^2(\sigma)_{(0)} = \hat{P}^2(\sigma)_{(0)} = 2(\hat{\delta}_0(\sigma) - 1) = -\frac{\alpha}{16}(d - 2 + \sum_L \frac{\alpha^2(L)}{F_L(\sigma)}), \quad \sum_L F_L(\sigma) = 26 - d
\]

with the same ground-state momentum-squared in Eq. (3.10d). Including the commutators (3.4a,b), the level-spacings of the physical states on the addition of a negatively-moded current

\[
\Delta(P^2(\sigma)) = \Delta(R(\sigma)) = \begin{cases} 
2|M + \epsilon| & \text{for } J_{\alpha a}((M + \epsilon) < 0) \\
2|M + \frac{2L}{F_L(\sigma)}| & \text{for } J_{LL}((M + \frac{2L}{F_L(\sigma)}) < 0)
\end{cases}
\]

are also obtained in the reduced formulation of sector \( \sigma \). Recalling the definition \( M = 2m + \bar{\ell} \) in Eq. (3.1d), it is easily checked that these level-spacings are the same as those given in Eq. (2.17) at \( \hat{c}(\sigma) = 52 \).

The current algebras (3.4b,c) and the adjoint operations (3.6b,c) of the reduced formulation also allow the straightforward computation of the following norms:

\[
||J_{\alpha a}((M + \epsilon) < 0)|0, J(0)\rangle||^2 = G^{(a)}_{\alpha a}|M + \epsilon|||0, J(0)\rangle||^2
\]

\[
||J_{LL}((M + \frac{2L}{F_L(\sigma)}) < 0)|0, J(0)\rangle||^2 = F_L(\sigma)|M + \frac{2L}{F_L(\sigma)}|||0, J(0)\rangle||^2
\]

\[
\prod_{QLL} J_{LL}((Q + \frac{2L}{F_L(\sigma)}) < 0)^{N_{LL}(Q)}|0, J(0)\rangle||^2 = \prod_{QLL} N_{LL}(Q)!|F_L(\sigma)Q + 2\hat{L}|^{N_{LL}(Q)}||0, J(0)\rangle||^2.
\]

Using again the definition \( M = 2m + \bar{\ell} \), we see that these norms are equal to those computed at \( \hat{c}(\sigma) = 52 \) in Eqs. (2.18),(2.19). Indeed, because states are inert under the map, we understand that the inner products \( \langle \alpha | \beta \rangle \) of any two states are invariant under the reduction procedure to \( c(\sigma) = 26 \).

We will return to further analysis of the physical states at various points below, including in particular the discussion of Sections 10-12.
4 The Enhanced Lorentz Symmetry $SO(D(\sigma) - 1, 1)$

In string theory, the target space-time symmetry of the system is determined by the integer-moded sequences [8] — whose zero modes are the momenta of the string. The number of integer-moded sequences in sector $\sigma$

$$D(\sigma) = D(\sigma)^{(d)} + D(\sigma)^{(26-d)} = d + N_O(\sigma)' + 2N_E(\sigma)'$$

(4.1)

is of course the same as the dimension of the target space-time of the sector.

Let us first mention the contribution to this number from the integer-moded sequences of type $(d)$, which are contained entirely in the second term of the Virasoro generators (3.2c). This term can be expressed in the familiar form

$$L(M)_{(d)} \equiv -\frac{1}{2} \eta_{(d)}^{ab} \sum_{Q \in \mathbb{Z}} \circ J_a(Q)J_b(M - Q)\circ_M$$

(4.2a)

$$\eta_{(d)} = -G_{(d)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{(d)}, \quad D(\sigma)^{(d)} = d$$

(4.2b)

after the shift $Q \rightarrow Q - \epsilon$ under the sum and the relabeling

$$J_a(M) \equiv J_{\epsilon a}(M), \quad a = 0, \ldots, d - 1$$

(4.3)

for $\epsilon = 0$ or 1. These generators satisfy a Virasoro algebra by themselves, with central charge $c = d$ and what we may call a “preliminary” Lorentz symmetry $SO(d - 1, 1)$.

But this is not the whole story in the new string theories, because these systems also contain the integer-moded sequences of type $(26 - d)$. We include these contributions as higher values of the index $a$ in the following definition:

$$J_a(M) \equiv \frac{1}{\sqrt{F_L(\sigma)}} \{J_{DL}(M), J_{FL(\sigma)/2}, L(M)\}, \quad a = d, \ldots, D(\sigma) - 1$$

(4.4a)

$$D(\sigma)^{(26-d)} = N_O(\sigma)' + 2N_E(\sigma)' = \sum_L \alpha(L).$$

(4.4b)

The rescaling here is motivated by the form of the momentum-squared operator in Eq. (3.9).

Then we may express the physical-state condition (3.2) of sector $\sigma$ in the following simple form

$$(L(M \geq 0) - \delta_{M,0})|\chi(\sigma)) = 0$$

(4.5a)

$$[L(M), L(N)] = (M - N)L(M + N) + \frac{26}{12}M(M^2 - 1)\delta_{M+N,0}$$

(4.5b)
\[ L(M) = \delta_{M,0} \hat{\delta}_0(\sigma) - \frac{1}{2} \eta_{ab}^{(D(\sigma))} \sum_{Q \in \mathbb{Z}} \frac{1}{F_{L(\sigma)}} \sum_{L \neq 0, F_{L(\sigma)}/2} \sum_{Q \in \mathbb{Z}} \hat{\sigma} J_a(Q) J_b(M - Q) \circ M + \]

\[ + \frac{1}{2} \sum_L \frac{1}{F_{L(\sigma)}} \sum_{L \neq 0, F_{L(\sigma)}/2} \sum_{Q \in \mathbb{Z}} \frac{1}{2 L} J_{LL}(Q + \frac{2iL}{F_{L(\sigma)}}) J_{-L,L}(M - Q - \frac{2iL}{F_{L(\sigma)}}) \circ M \]  \hspace{1cm} (4.5c)

\[ L(0) = \frac{1}{2} (-P^2(\sigma) + R(\sigma)) + \hat{\delta}_0(\sigma) \]  \hspace{1cm} (4.5d)

\[ P^2(\sigma) = \eta_{ab}^{(D(\sigma))} J_a(0) J_b(0) \]  \hspace{1cm} (4.5e)

\[ \eta_{(D(\sigma))} = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)_{(D(\sigma))} \]  \hspace{1cm} (4.5f)

where we have collected all the integer-moded sequences of the sector in the second term of the Virasoro generators. We remind that the explicit form of the conformal-weight shift \( \hat{\delta}_0(\sigma) \) and the ground-state mass-squared \( P^2(\sigma) \circ 0 \) are given respectively in Eqs. (2.13) and (2.16).

Counting the extended Virasoro generators (2.3c) and the reduced Virasoro generators (3.2c), this is the third and – as we shall see – most transparent form of the dynamics of open-string sector \( \sigma \). Before giving the physical interpretation of this result however, we should re-express the remaining structure of the reduced system in our new notation.

After some algebra, we find for the currents in Eq. (4.5c):

\[ [L(M), J_a(N)] = -N J_a(M + N), \quad a = 0, 1, \ldots, D(\sigma) - 1 \]  \hspace{1cm} (4.6a)

\[ [L(M), J_{LL}(N + \frac{2iL}{F_{L(\sigma)}})] = -(N + \frac{2iL}{F_{L(\sigma)}}) J_{LL}(M + N + \frac{2iL}{F_{L(\sigma)}}) \]  \hspace{1cm} (4.6b)

\[ [J_a(M), J_b(N)] = \eta_{ab}^{(D(\sigma))} N \delta_{M+N,0} \]  \hspace{1cm} (4.6c)

\[ [J_{J(J + \frac{2i}{F_{J(\sigma)}}), J_{LL}(N + \frac{2iL}{F_{L(\sigma)}})] = \delta_{JL} F_J(\sigma)(M + \frac{2iL}{F_{L(\sigma)}}) \delta_{J,L,0 \mod F_{L(\sigma)}} \delta_{M+N+\frac{2(J+L)}{F_{L(\sigma)}} 0} \]  \hspace{1cm} (4.6d)

\[ \{J(\text{int})\}, \{J(\text{frac})\} = 0 \]  \hspace{1cm} (4.6e)

\[ J_a(M) \dagger = J_a(-M), \quad J_{LL}(M + \frac{2iL}{F_{L(\sigma)}}) \dagger = J_{-L,L}(-M - \frac{2iL}{F_{L(\sigma)}}). \]  \hspace{1cm} (4.6f)

In this list, the currents shown are divided into the integer-modered currents \( \{J(\text{int})\} \) and the truly fractional-modered currents \( \{J(\text{frac})\} \), with \( \hat{L} \neq 0 \) or \( F_{L(\sigma)}/2 \). Similarly, the periodicity condition for \( \{J(\text{frac})\} \) is the same as given in Eq. (3.6b). We also find the level-spacing

\[ \Delta(P^2(\sigma)) = \Delta(R(\sigma)) = \begin{cases} 2|M| & \text{for } J_a(M < 0), a = 0, 1, \ldots, D(\sigma) - 1 \\
2|M + \frac{2iL}{F_{L(\sigma)}}| & \text{for } J_{LL}((M + \frac{2iL}{F_{L(\sigma)}}) < 0), \hat{L} \neq 0, \frac{F_{L(\sigma)}}{2} \end{cases} \]  \hspace{1cm} (4.7)

in this notation.
We emphasize that the parameter $\epsilon = 0$ or 1 (corresponding to $(\omega_d = (\pm 1)_{(d)})$ no longer appears in this reduced formulation, owing to our redefinition (4.3), so the physical spectrum of each open-string sector is independent of $\epsilon$.

We are now able to state the enhanced or full Lorentz symmetry [8] of open-string sector $\sigma$ in our large example of orientation-orbifold string systems. It is clear from the current algebra (4.6c) and the corresponding adjoint operation (4.6f) that the second term of the Virasoro generators

$$L(M)(D(\sigma)) \equiv -\frac{1}{2} \eta_{ab}^{D(\sigma)} \sum_{Q \in \mathbb{Z}} \circ J_a(Q)J_b(M-Q)\circ M$$

(4.8a)

$$c_{(D(\sigma))}(\sigma) \equiv \sum_{a} D(\sigma) = d + \sum_{L} \alpha(L) = d + NO(\sigma)' + 2NE(\sigma)'$$

(4.8b)

is a set of ordinary $SO(D(\sigma)-1,1)$-invariant open-string Virasoro generators on the $D(\sigma)$-dimensional Lorentzian target space-time. As in the ordinary string, each of the integer-moded currents $\{J_a(M), a = 0, \ldots, D(\sigma)-1\}$ transforms as a $D(\sigma)$-dimensional Lorentz vector under this Lorentz group.

The first and third terms of Eq. (4.5c) form another (commuting) set of Virasoro generators $\{L_{\text{frac}}(M)\}$ for the currents $\{J(\text{frac})\}$, which are extra fractional-moded $SO(D(\sigma)-1,1)$-invariant scalar fields. It is not surprising that the first and third terms of (4.5c) should form a Virasoro subsystem because integer-moded sequences (of either type) can not contribute [8] to the conformal-weight shifts (see Eq. (2.13)). Using the sum rule in Eq. (3.4e), we may compute the central charge of $\{L_{\text{frac}}(M)\}$ by counting as follows:

$$c_{\text{frac}}(\sigma) \equiv \sum_{L} \sum_{L \neq 0} \frac{F_L(\sigma)}{2} = \sum_{L} (F_L(\sigma) - \alpha(L))$$

$$= (26 - d) - (NO(\sigma)'+2NE(\sigma)') = 26 - D(\sigma).$$

(4.9)

This summation tells us that there are exactly $(26-D(\sigma))$ extra fractionally-moded $SO(D(\sigma)-1,1)$-invariant scalar fields $\{J(\text{frac})\}$ in the open-string system, which is consistent with the sum

$$c_{(D(\sigma))}(\sigma) + c_{\text{frac}}(\sigma) = c(\sigma) = 26.$$  

(4.10)

Moreover, since the number of fractionally-moded fields cannot be negative, the computation (4.9) provides a simple argument that the target-space dimensionality satisfies $D(\sigma) \leq 26$ for all $H(\text{perm})_{26-d}$. We shall check this conclusion more directly in Sec. 7, where somewhat stronger conditions are obtained for the open-string dimensionalities $D(\sigma)$. 

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We conclude this section with three simple examples of our development so far:

a) \( d = 26, H(\text{perm})' \) absent with \( \sum_L \frac{\alpha^2(L)}{F_L(\sigma)} \equiv 0 \)

b) \( d = 25, H(\text{perm})' \) trivial with

\[
F_L(\sigma) = N(\sigma)' = N_O(\sigma)' = 1
\]

(4.11)

c) the trivial element \((1\ l)_{26-d} \in H(\text{perm})'_{26-d}, 1 \leq d \leq 26\) with

\[
L = 0, 1, \ldots, 25 - d, \ \hat{L} = 0, \ F_L(\sigma) = 1, \ N(\sigma)' = N_O(\sigma)' = 26 - d.
\]

(4.12)

In all three cases we find the system

\[
L(M) = L(M)_{(26)} = -\frac{1}{2} h_{(26)}^{ab} \sum_Q e \ J_a(Q) J_b(M - Q)\circ_M
\]

(4.13)

\[
D(\sigma) = 26, \ \hat{\delta}_0(\sigma) = 0, \ P^2(\sigma)_{(0)} = -2, \ \Delta(P^2(\sigma)) = 2|M| \text{ for } J_a(M < 0)
\]

(4.14)

so that each of these examples is spectrally-equivalent to an ordinary 26-dimensional \(SO(25, 1)\)-invariant open string.

5 Ground-State Lemmas

We consider further the ground-state momentum-squared (3.10d) and the double inequality (2.14) for each sector \( \sigma \) of any \( H(\text{perm})'_{26-d} \)

\[
P^2(\sigma)_{(0)} = 2(\hat{\delta}_0(\sigma) - 1) = -\frac{1}{12} (d - 2 + \sum_L \frac{\alpha^2(L)}{F_L(\sigma)})
\]

(5.1a)

\[
F_L(\sigma) \geq 1, \ \sum_L F_L(\sigma) = 26 - d, \ 0 < \sum_L \frac{\alpha^2(L)}{F_L(\sigma)} \leq 26 - d, \ 1 \leq d \leq 25.
\]

(5.1b)

Combining these results we find the double inequality for the ground-state momentum-squared of sector \( \sigma \):

\[
-2 \leq P^2(\sigma)_{(0)} < \frac{1}{12}(2 - d), \ 1 \leq d \leq 25.
\]

(5.2)

The case \( d = 26 \) (and hence \( H(\text{perm})' \) absent) was discussed in the previous section.

Most of the ground states in our large example are therefore tachyonic

\[
-2 \leq P^2(\sigma)_{(0)} < 0, \ 2 \leq d \leq 25
\]

(5.3)

which is not surprising in these bosonic prototypes.

For \( d = 1 \) (and hence \( H(\text{perm})'_{25} \)), the ground-state mass-squared takes the form

\[
P^2(\sigma)_{(0)} = \frac{1}{12}(1 - \sum_L \frac{\alpha^2(L)}{F_L(\sigma)}), \ \sum_L F_L(\sigma) = 25
\]

(5.4a)

\[
-2 \leq P^2(\sigma)_{(0)} < \frac{1}{12}
\]

(5.4b)

so these sectors are the only cases where non-tachyonic ground-states are possible.
6 The Single-Cycle Sectors of $H(\text{perm})'_{26-d} = \mathbb{Z}_{26-d}$

As other simple examples of our results, let us compute explicitly for the single-cycle elements of the cyclic groups

$$\omega(\sigma) \in H(\text{perm})'_{26-d} = \mathbb{Z}_{26-d}, \quad 1 \leq d \leq 25$$

$$\rho(\sigma) = F_L(\sigma) = 26 - d, \quad L = 0, \quad N(\sigma)' = 1$$

where $\rho(\sigma)$ is the order of $\omega(\sigma)$.

Then we easily obtain for $d$ odd the $SO(d,1)$-symmetric sectors

$$L(M) = \hat{\delta}_0(\sigma)\delta_{M,0} - \frac{1}{2}\eta^a_b\sum_{Q \in \mathbb{Z}} \delta_a(Q)J_b(M - Q)\hat{\delta}_M +$$

$$+ \frac{1}{2(26-d)}\sum_{L=1,26-d}^{25-d} \sum_{Q \in \mathbb{Z}} \delta_a(Q + \frac{2i}{26-d})J_{L,0}(M - Q - \frac{2i}{26-d})\delta_M$$

$$P^2(\sigma) = \eta^a_b\delta_a(0)\delta_b(0), \quad D(\sigma) = d + 1$$

$$\hat{\delta}_0(\sigma) = \frac{1}{24}(26 - d - \frac{1}{26-d})$$

$$P^2(\sigma)(0) = -\frac{1}{12}(d - 2 + \frac{1}{26-d}), \quad 1 \leq d \text{ odd} \leq 25$$

and for $d$ even the $SO(d+1,1)$-symmetric sectors

$$L(M) = \hat{\delta}_0(\sigma)\delta_{M,0} - \frac{1}{2}\eta^a_b\sum_{Q \in \mathbb{Z}} \delta_a(Q)J_b(M - Q)\hat{\delta}_M +$$

$$+ \frac{1}{2(26-d)}\sum_{L=0,26-d}^{25-d} \sum_{Q \in \mathbb{Z}} \delta_a(Q + \frac{2i}{26-d})J_{L,0}(M - Q - \frac{2i}{26-d})\delta_M$$

$$P^2(\sigma) = \eta^a_b\delta_a(0)\delta_b(0), \quad D(\sigma) = d + 2$$

$$\hat{\delta}_0(\sigma) = \frac{1}{24}(26 - d - \frac{4}{26-d})$$

$$P^2(\sigma)(0) = -\frac{1}{12}(d - 2 + \frac{4}{26-d}), \quad 2 \leq d \text{ even} \leq 24.$$

The ground-state momentum-squared (6.3d) was given earlier for prime $d$ in Ref. [3].

In agreement with the ground-state lemma (5.3), these single-cycle ground states are indeed tachyonic for all $2 \leq d < 25$. Among these, only the cases $d = 24$ (the non-trivial element of $H(\text{perm})'_2 = \mathbb{Z}_2$ with $F_L(\sigma) = 2$) and $d = 25$ ($H(\text{perm})'_1 = \mathbb{Z}_1$ trivial with
$F_L(\sigma) = 1$) realize the minimum value $P^2(\sigma)(0) = -2$ of the unshifted tachyonic ground state, and it is easily checked that both of these cases

$$d = 24, 25 : \quad L(M) = L(M)_{(26)} = -\frac{1}{2}\eta_{(26)}^{ab} \sum_{Q \in \mathbb{Z}} \circ J_a(Q)J_b(M - Q) \circ,$$

$$\hat{\delta}_0(\sigma) = 0, \quad P^2(\sigma)(0) = -2, \quad D(\sigma) = 26, \quad \Delta(P^2(\sigma)) = 2|M| \quad (6.5)$$

are spectrally equivalent to an ordinary 26-dimensional $SO(25, 1)$-invariant open string.

Indeed, following the discussion of the previous section, the only non-tachyonic ground state is found at $d = 1$, which selects here the non-trivial single-cycle elements of $H(\text{perm})'_{25} = \mathbb{Z}_{25}$:

$$d = 1 : \quad D(\sigma) = 2, \quad P^2(\sigma)(0) = \frac{2}{25} < \frac{1}{12}. \quad (6.6)$$

In this 2-dimensional $SO(1, 1)$-invariant string, the automorphism $(\omega)_{d=1} = (\pm \mathbb{I})_1$ contributes the time dimension while the element of $\mathbb{Z}_{25}$ contributes the single spatial dimension. Further discussion of the non-tachyonic sectors is found in Sec. 8.

The special cases noted in Eqs. (6.5) and (6.6) are in fact the extrema of the single-cycle series, which satisfies more generally the following double inequalities:

$$1 \leq d \leq 25 : \quad 2 \leq D(\sigma) \leq 26, \quad \frac{2}{25} \geq P^2(\sigma)(0) \geq -2. \quad (6.7)$$

Beyond the single cycle series, we shall see below that the minimal (unshifted) ground-state mass-squared $P^2(\sigma)(0) = -2$ is always associated to the maximal (ordinary) target space-time dimension $D(\sigma) = 26$.

### 7 The 24 Cyclic Groups and $D(\sigma) \leq 26$

Following the evaluation of the single-cycle series in the previous section, we expand our inquiry here to include all sectors $\sigma$ of the 24 relevant cyclic groups:

$$\omega(\sigma) = e^{-2\pi i \frac{\sigma}{26-d}} \in H(\text{perm})'_{26-d} = \mathbb{Z}_{26-d} \quad (7.1a)$$

$$1 \leq d \leq 25, \quad \sigma = 0, 1, \ldots, 26 - d. \quad (7.1b)$$

For convenience we have included here the trivial group $\mathbb{Z}_1$ at $d = 25$ and the trivial sector $\sigma = 0$ of each group, both of which have been described above (see Eqs. (4.10) and (4.11)). The cyclic group $\mathbb{Z}_{26-d}$ has $25 - d$ non-trivial elements (sectors) $\sigma = 1, \ldots, 25 - d$. 

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The explicit form (7.1a) of the group elements is useful for counting the number of sectors \( \sigma \) of each cycle-type, and each cycle-type is easily described in our notation as follows

\[
L = 0, 1, \ldots, \frac{26-d}{\rho(\sigma)} - 1, \quad \tilde{L} = 0, 1, \ldots, \rho(\sigma) - 1
\]

(7.2a)

\[
F_L(\sigma) = \rho(\sigma), \quad \forall L
\]

(7.2b)

\[
N(\sigma)' = \sum_L = \frac{26-d}{\rho(\sigma)}, \quad \sum_L F_L(\sigma) = \rho(\sigma) N(\sigma)' = 26 - d
\]

(7.2c)

where \( \rho(\sigma) \) is the order of \( \omega(\sigma) \in \mathbb{Z}_{26-d} \). The results below record the detailed evaluation of the target space-time dimension of sector \( \sigma \) [8]

\[
D(\sigma) = d + N_O(\sigma)' + 2N_E(\sigma)', \quad N(\sigma)' = N_O(\sigma)' + N_E(\sigma)'
\]

(7.3)

and hence the target space-time symmetry \( SO(D(\sigma) - 1, 1) \) of the sector, where \( N_{O,E}(\sigma) \) are respectively the number of cycles of odd and even length \( F_L(\sigma) \) in \( \omega(\sigma) \). We shall also include some further remarks in this section on the ground-state mass-squared \( P^2(\sigma)_{(0)} \) in Eq. (5.1a).

The result \( D(\sigma) = 26 \) and \( P^2(\sigma)_{(0)} = -2 \) have already been recorded for the trivial sector \( \omega(0) = 1 (\sigma = 0 \text{ with } \rho(0) = 1) \) of each group, and this case is always spectrally-equivalent to an ordinary 26-dimensional \( SO(25, 1) \)-invariant open string. We shall therefore list below only the non-trivial sectors \( \sigma = 1, \ldots, 26 - d \) of each group.

We will however encounter the space-time dimension \( D(\sigma) = 26 \) a number of other times in the tables below, and we have checked that each occurrence of this dimension is also equivalent to an ordinary 26-dimensional open string with \( P^2(\sigma)_{(0)} = -2 \).

Let us illustrate our enumeration first with the simplest cases, namely the prime cyclic groups \( \mathbb{Z}_{26-d} \) with \( 26 - d \) prime. Each of these has \( (25-d) \) non-trivial single-cycle sectors \( \sigma \) with \( L = 0 \) and length \( F_0(\sigma) = 26 - d \), shown in Table 1.
Table 1: The prime cyclic groups

| $H(\text{perm})'$ | $d$ | $D(\sigma)$ | symmetry | $D(\sigma)_c$ | symmetry |
|-------------------|-----|-------------|----------|--------------|----------|
| $\mathbb{Z}_{1}$ | 25  | 26 | $SO(25,1)$ | 26 | $SO(25,1)$ |
| $\mathbb{Z}_{2}$ | 24  | 26 | $SO(25,1)$ | 25 | $SO(24,1)$ |
| $\mathbb{Z}_{3}$ | 23  | 24 | $SO(23,1)$ | 24 | $SO(23,1)$ |
| $\mathbb{Z}_{5}$ | 21  | 22 | $SO(21,1)$ | 22 | $SO(21,1)$ |
| $\mathbb{Z}_{7}$ | 19  | 20 | $SO(19,1)$ | 20 | $SO(19,1)$ |
| $\mathbb{Z}_{11}$ | 15  | 16 | $SO(15,1)$ | 16 | $SO(15,1)$ |
| $\mathbb{Z}_{13}$ | 13  | 14 | $SO(13,1)$ | 14 | $SO(13,1)$ |
| $\mathbb{Z}_{17}$ | 9   | 10 | $SO(9,1)$   | 10 | $SO(9,1)$   |
| $\mathbb{Z}_{19}$ | 7   | 8  | $SO(7,1)$   | 8  | $SO(7,1)$   |
| $\mathbb{Z}_{23}$ | 3   | 4  | $SO(3,1)$   | 4  | $SO(3,1)$   |

Only the columns of this table before the double vertical line are relevant for the open-string sectors discussed here. (The last two columns of this and the following tables record the corresponding data for the closed-string sectors, which we will explain in later sections of the paper.) Being single-cycle sectors, these cases are included in the results of the previous section, and we remind here that all these ground states are tachyonic because $d \geq 2$. We notice also that $\mathbb{Z}_{23}$ provides our first 22 examples of $SO(3,1)$-invariant four-dimensional strings, a subject to which we return in Sec. 9.

We continue our survey of the cyclic groups in the following sector notation:

$$n(\sigma)[F_L(\sigma)]^{N(\sigma)'}$$

$$N(\sigma)' = \frac{26-d}{F_L(\sigma)}, \quad L = 0, 1, \ldots, \frac{26-d}{F_L(\sigma)} - 1, \quad \sigma = 1, \ldots, 25 - d.$$  

Here $n(\sigma)$ is the number of sectors $\sigma$ with cycle-type $[F_L]^{N(\sigma)'}$, that is $N(\sigma)'$ cycles of length $F_L(\sigma)$. In the sector notation the non-trivial sectors of the prime cyclic groups above read $(25 - d)[26 - d]$, and Table 2 gives a selection of non-prime cyclic groups in this notation.
Table 2: Some non-prime cyclic groups

We include two remarks about the data in Table 2:
1) The single-cycle sectors of $\mathbb{Z}_{24}(d = 2)$ provide us with 17 more examples of $SO(3,1)$-invariant four-dimensional strings, whose ground-state mass-squared are included in Eq. (6.4d). See also Sec. 9.
2) All the sectors in Table 2 are tachyonic, except for $\mathbb{Z}_{25}$ – whose 24 non-trivial sectors are entirely non-tachyonic:

$$P^2(\sigma)_{(0)} = \begin{cases} \frac{2}{25} & \text{for the 20 single-cycle sectors with } D(\sigma) = 2 \\ 0 & \text{for the 4 sectors with } D(\sigma) = 6. \end{cases}$$ (7.5)

This is in agreement with our ground-state lemmas in Sec. 5 and the single-cycle result in Eq. (6.3d). See also Sec. 8. The cycle-types for the remaining cyclic groups are given in Table 3.
Table 3: The rest of the cyclic groups

Systematics of the permutation groups
We now consider systematic statements for the target space-time dimensionalities \( \{D(\sigma)\} \) associated to the general permutation group \( H(\text{perm})'_{26-d} \).

Surveying the data in the tables above for all the cyclic groups \( H(\text{perm})'_{26-d} = \mathbb{Z}_{26-d} \), one checks explicitly for these groups that

\[
D(\sigma) = d + N_O(\sigma)' + 2N_E(\sigma)', \quad 1 \leq d \leq 25 \tag{7.6a}
\]

\[
2 \leq (D(\sigma) = \text{even}) \leq 26, \quad 1 \leq d \leq 26. \tag{7.6b}
\]
The trivial elements \( (D(\sigma) = 26) \) of each cyclic group and the case of no permutation group \( D(\sigma) = d = 26 \) is included in the double inequality (7.6b).

We remind [8] that the formula for \( D(\sigma) \) in Eq. (7.6a) holds for all permutation groups \( H(\text{perm})_{26-d}' \), and we know from the counting in Eq. (4.9) that \( D(\sigma) \leq 26 \) for the general case. In fact, it is not difficult to see that the full double inequality (7.6b), including \( D(\sigma) \) even, holds as well for all \( H(\text{perm})_{26-d}' \).

We sketch this first for the symmetric groups \( H(\text{perm})_{26-d}' = S_{26-d} \), where the sectors correspond to the ordered partitions of \( (26 - d) \):

\[
S_{26-d} : \quad \sum_L F_L(\sigma) = 26 - d, \quad 1 \leq F_{L+1}(\sigma) \leq F_L(\sigma), \quad 1 \leq d \leq 25. \quad (7.7)
\]

As detailed above for the cyclic groups, this description allows the inspection of the group elements of the 24 relevant non-trivial symmetric groups \( S_2(d = 24) \ldots S_{25}(d = 1) \), leading to the following observations:
a) For each symmetric group, the minimum space-time dimension is achieved for the single-cycle sectors (see Sec. 6) with

\[
D(\sigma) = \begin{cases} 
  d + 2 & \text{for } d \text{ even} \\
  d + 1 & \text{for } d \text{ odd} 
\end{cases} \quad (7.8)
\]

This provides the lower bound \( D(\sigma) \geq 2 \) in Eq. (7.6b) for the symmetric groups, and the equality is achieved among these groups only by the single-cycle elements of \( S_{25}(d = 1) \).
b) In each sector of each symmetric group, the number of cycles of odd length \( F_L(\sigma) \) satisfies

\[
N_0(\sigma)' = \begin{cases} 
  \text{even for } d \text{ even} \\
  \text{odd for } d \text{ odd} 
\end{cases} \quad (7.9)
\]

and therefore

\[
\quad d + N_O(\sigma)' = \text{even for all } d \quad (7.10a)
\]

\[
D(\sigma) = \text{even for all } d. \quad (7.10b)
\]

c) The maximum space-time dimension for each symmetric group is \( D(\sigma) = 26 \), e.g.

\[
D(\sigma) = 26 : \quad [2]^{\frac{26-d}{2}} \text{ for } d \text{ even, } [2]^{\frac{25-d}{2}}[1]^1 \text{ for } d \text{ odd} \quad (7.11)
\]

which provides the last part of the double inequality (7.6b) for the symmetric groups.

Finally, since all permutation groups are subgroups of the symmetric groups

\[
H(\text{perm})_{26-d}' \subset S_{26-d} \quad (7.12)
\]

we conclude that the double inequality (7.6b) holds for all \( H(\text{perm})_{26-d}' \).
The double inequality (7.6b) is one of the central results of this paper, holding across all open-string sectors in the large example (1.5) of orientation-orbifold string systems. We shall return to some physical consequences of this result below.

A final remark is essential here. As seen explicitly in the tables above, the dimension \( D(\sigma) \) of these open-string Lorentzian space-times generally vary from sector to sector even in the same orbifold. It follows that the as-yet-unconstructed twist-fields (intertwiners) of these orbifolds will characteristically induce target space-time transitions \( \Delta(D(\sigma)) \neq 0 \).

8 The Non-Tachyonic Strings

We have noted in Sec. 5 and Eq. (7.5) that the only non-tachyonic open-string sectors in \( \{ H(\text{perm})_{26-a} = \mathbb{Z}_{26-a} \} \) are all 24 non-trivial sectors \( \sigma = 1, \ldots, 24 \) of \( \mathbb{Z}_{25} \) (see also Table 2). For future reference, we write out these sectors here in further detail.

20 sectors of type \([25]^1\) with \( SO(1, 1) \) symmetry

\[
L(M) = \frac{26}{25} \delta_{M,0} - \frac{1}{2} \eta_{(2)}^{ab} \sum_{Q \in \mathbb{Z}} \hat{\circ} J_a(Q) J_b(M - Q) \hat{\circ} M + \]

\[
+ \frac{1}{250} \sum_{L=1}^{24} \sum_{Q \in \mathbb{Z}} \hat{\circ} J_{L0}(Q + \frac{2L}{25}) J_{-L,0}(M - Q - \frac{2L}{25}) \hat{\circ} M \quad (8.1a)
\]

\[
P^2(\sigma) = \eta_{(2)}^{ab} J_a(0) J_b(0) \quad (8.1b)
\]

\[
D(\sigma) = 2, \quad P^2(\sigma)(0) = \frac{24}{25}, \quad 1 \leq (\forall \sigma \text{ not a multiple of } 5) \leq 24 \quad (8.1c)
\]

\[
\Delta(P^2(\sigma)) = \begin{cases} 
2|M| & \text{for } J_a(M < 0), \ a = 0, 1 \\
2|M + \frac{2L}{25}| & \text{for } J_{L0}((M + \frac{2L}{25}) < 0), \ \hat{L} = 1, \ldots, 24.
\end{cases} \quad (8.1d)
\]

In this result, the time dimension comes from \( d = 1 \) while the single spatial dimension comes from the old \( \hat{L} = 0 \) current.

4 sectors of type \([5]^5\) with \( SO(5, 1) \) symmetry

\[
L(M) = \delta_{M,0} - \frac{1}{2} \eta_{(6)}^{ab} \sum_{Q \in \mathbb{Z}} \hat{\circ} J_a(Q) J_b(M - Q) \hat{\circ} M + \]

\[
+ \frac{1}{10} \sum_{L=0}^{4} \sum_{\hat{L}=1}^{4} \sum_{Q \in \mathbb{Z}} \hat{\circ} J_{L\hat{L}}(Q + \frac{2\hat{L}}{5}) J_{-L,L}(M - Q - \frac{2\hat{L}}{5}) \hat{\circ} M \quad (8.2a)
\]

\[
P^2(\sigma) = \eta_{(6)}^{ab} J_a(0) J_b(0) \quad (8.2b)
\]
\[ D(\sigma) = 6, \quad P^2(\sigma)_{(0)} = 0, \quad \sigma = 5, 10, 15, 20 \] (8.2c)

\[ \Delta(P^2(\sigma)) = \begin{cases} 2|M| & \text{for } J_a(M < 0), \ a = 0, 1, \ldots, 5 \\ 2|M + \frac{2i}{5}| & \text{for } J_L((M + \frac{2i}{5}) < 0), \ L = 0, 1, \ldots, 4, \ \hat{L} = 1, \ldots, 4. \end{cases} \] (8.2d)

In these sectors the time dimension comes again from \( d = 1 \), while the five spatial dimensions come from the five old \( \hat{L} = 0 \) currents. There remains for \( \mathbb{Z}_{25} \) the trivial open-string sector \( \sigma = 0 \) with space-time dimension 26, which is spectrally-equivalent to the ordinary critical open string.

### 9 The Four-Dimensional Strings

In this section we collect all the \( SO(3, 1) \)-invariant four-dimensional open strings associated with \( \{H(\text{perm})_{26-d} = \mathbb{Z}_{26-d}\} \).

All non-trivial sectors of \( H(\text{perm})_{23} = \mathbb{Z}_{23} \)

Since 23 is prime, \( \mathbb{Z}_{23} \) has 22 non-trivial sectors of single-cycle type \([23]^1\) — all of which are four dimensional strings (see Table 1). In these cases, we include further details for the currents:

\[ L(M) = \frac{22}{23}\delta_{M,0} - \frac{1}{2}\eta_{(4)}^{ab} \sum_{Q \in \mathbb{Z}} \hat{J}_a(Q) J_b(M - Q) \hat{\sigma}_M + \]
\[ + \frac{1}{20} \sum_{\hat{L}=1}^{22} \sum_{Q \in \mathbb{Z}} \hat{J}_{\hat{L}0}(Q + \frac{2\hat{L}}{23}) J_{-\hat{L},0}(M - Q - \frac{2\hat{L}}{23}) \hat{\sigma}_M \] (9.1a)

\[ [L(M), J_a(N)] = -NJ_a(M + N), \quad a = 0, 1, 2, 3 \] (9.1b)

\[ [J_a(M), J_b(M)] = \eta_{(4)}^{ab} N \delta_{M+N,0} \] (9.1c)

\[ [L(M), J_{L0}(N + \frac{2\hat{L}}{23})] = -(N + \frac{2\hat{L}}{23})J_{L0}(M + N + \frac{2\hat{L}}{23}), \quad \hat{L} = 1, \ldots, 22 \] (9.1d)

\[ [J_{\hat{L}0}(M + \frac{2\hat{L}}{23}), J_{L0}(N + \frac{2\hat{L}}{23})] = (23M + 2\hat{L})\delta_{\hat{L}+L,0 \text{ mod } 23} \delta_{M+N+2\hat{L},0} \] (9.1e)

\[ d = 3, \quad D(\sigma) = 4, \quad P^2(\sigma)_{(0)} = -\frac{2}{23}, \quad \sigma = 1, \ldots, 22 \] (9.1f)

\[ P^2(\sigma) = \eta_{(4)}^{ab} J_a(0) J_b(0), \quad \Delta(P^2(\sigma)) = \begin{cases} 2|M| & \text{for } J_a(M < 0), \ a = 0, 1, 2, 3 \\ 2|M + \frac{2i}{23}| & \text{for } J_{L0}((M + \frac{2i}{23}) < 0), \ \hat{L} = 1, \ldots, 22. \end{cases} \] (9.1g)

We remind that the integer-moded currents \( \{J_a(M), a = 0, 1, 2, 3\} \) transform as Lorentz four-vectors under \( SO(3, 1) \), while the extra 22 fractional-moded currents \( \{J_{L0}(M + \frac{2i}{23}), \hat{L} = 1, \ldots, 22\} \) are scalars under the four-dimensional Lorentz group. The remaining (trivial)
open-string sector $\sigma = 0$ of $\mathbb{Z}_{23}$ has, as usual, $D(\sigma) = 26$ and is spectrally-equivalent to the untwisted critical open string.

The 17 single-cycle sectors of $H(\text{perm})'_{24} = \mathbb{Z}_{24}$

The 17 single-cycle sectors [24]$^1$ of $\mathbb{Z}_{24}$ are also four-dimensional strings (see Table 2):

$$L(M) = \frac{143}{144}\delta_{M,0} - \frac{1}{2}\eta^{ab}_{(4)} \sum_{Q \in \mathbb{Z}} J_a(Q)J_b(M - Q)\delta_{M} + \delta_{L_0}$$

$$+ \frac{1}{34} \sum_{L \neq 0,12} \sum_{Q \in \mathbb{Z}} J_{L_0}(Q + \frac{L}{12})J_{-L_0}(M - Q - \frac{L}{12})\delta_{M}$$

(9.2a)

$$[L(M), J_a(N)] = -NJ_a(M + N), \quad a = 0, 1, 2, 3$$

(9.2b)

$$[J_a(M), J_b(N)] = \eta^{ab}_{(4)}N\delta_{M+N,0}$$

(9.2c)

$$[L(M), J_{L_0}(N + \frac{L}{12})] = -(N + \frac{L}{12})J_{L_0}(M + N + \frac{L}{12}), \quad \hat{L} = 1 \ldots 11 \text{ and } 13 \ldots 23$$

(9.2d)

$$[J_{L_0}(M + \frac{L}{12}), J_{L_0}(N + \frac{L}{12})] = 2(12M + 2\hat{L})\delta_{J_{L_0}M+N,0}$$

(9.2e)

$$d = 2, \quad D(\sigma) = 4, \quad P^2(\sigma) = -\frac{1}{72}, \quad \sigma \text{ not a divisor of } 24$$

(9.2f)

Note that here, as in the sectors of $\mathbb{Z}_{23}$ above, there are exactly 22 extra fractional-moded Lorentz scalar fields, i.e. a total of $c(\sigma) = 4 + 22 = 26$ effective degrees of freedom in sector $\sigma$. As seen in Table 2, the multi-cycle sectors of $\mathbb{Z}_{24}$ have higher space-time dimensions $D(\sigma) = 6, 8, 10, 14$ and 26 with correspondingly larger Lorentz symmetries $SO(D(\sigma) - 1, 1)$.

10 Physical States:

The First Four Levels of $H(\text{perm})'_{3} = \mathbb{Z}_{3}$

In this and the following three sections, we present some introductory remarks on the physical states of the open-string sectors in the orientation-orbifold string systems.

We begin here with the low-lying states of the two non-trivial open-string sectors $\sigma = 1, 2$ of the simplest case $H(\text{perm})'_{3} = \mathbb{Z}_{3}$ (see Table 1). These strings are described by the following dynamics in both sectors:

$$\langle L(M \geq 0) - \delta_{M,0} \rangle = 0, \quad \sigma = 1, 2$$

(10.1a)

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Each of these sectors is an SO-moded currents and, owing to the periodicity conditions (10.1k), modes with levels of these sectors:

\[
\text{Eq. (10.1a). The space of physical states can be analyzed as usual at fixed values of the level } \ell = 0, 1, \ldots, 23 (10.1h)
\]

\[
\Delta(P^2(\sigma)) = 2|\text{mode } \#| \text{ for each } J((\text{mode } \#) < 0) (10.1i)
\]

\[
J_a(M)^\dagger = J_a(-M), \quad J_{L_0}(M + \frac{2L}{3})^\dagger = J_{-L_0}(-M - \frac{2L}{3}) (10.1j)
\]

\[
J_{L\pm 3,0}(M + \frac{2(L\pm 3)}{3}) = J_{L_0}(M \pm 2 + \frac{2L}{3}). (10.1k)
\]

Each of these sectors is an SO(23, 1)-invariant 24-dimensional string, where the integer-moded currents \( \{ J_a, a = 0, \ldots, 23 \} \) transform as Lorentz vectors and the extra two fractional-moded currents \( \{ J_{L_0}, \hat{L} = 1, 2 \} \) are Lorentz scalars.

The physical states of each sector \( \sigma \) are the solutions of the physical-state condition in Eq. (10.1a). The space of physical states can be analyzed as usual at fixed values of the level

\[
\text{level } \equiv \text{total mode } \# \equiv \sum |\text{mode } \# \text{ of the currents}| (10.2)
\]

a fact which holds as well in any sector of all the orbifold-string theories of permutation-type.

Among the Lorentz scalars, one finds that there is exactly one negative, fractional mode of each type

\[
J_{10}(-\frac{1}{3}), J_{20}(-\frac{2}{3}), J_{10}(-\frac{4}{3}), J_{20}(-\frac{5}{3}), J_{10}(-\frac{7}{3}), J_{20}(-\frac{8}{3}), \ldots
\]

and, owing to the periodicity conditions (10.1k), modes with \( \hat{L} = -1, -2 \) are equivalent to entries in this list.

After some algebra, we obtain the following list of all physical states for the first four levels of these sectors:

**level 0:**

\[
|P^2(\sigma) = P^2(\sigma)_{(0)} = -\frac{16}{9}) (10.3)
\]
level 1/3:
\[ J_{10}(-1/3)|P^2(\sigma) = -\frac{13}{9}\rangle \] (10.4a)
\[ ||J_{10}(-1/3)|P^2(\sigma) = -\frac{13}{9}\rangle||^2 = || |P^2(\sigma) = \frac{13}{9}\rangle||^2 \] (10.4b)

level 2/3:
\[ J_{20}(-2/3)|P^2(\sigma) = -\frac{4}{9}\rangle \] (10.5a)
\[ ||J_{20}(-2/3)|P^2(\sigma) = -\frac{4}{9}\rangle||^2 = || |P^2(\sigma) = \frac{4}{9}\rangle||^2 \] (10.5b)
\[ J_{10}(-1/3)J_{10}(-1/3)|P^2(\sigma) = -\frac{4}{9}\rangle \] (10.5c)
\[ ||J_{10}(-1/3)J_{10}(-1/3)|P^2(\sigma) = -\frac{4}{9}\rangle||^2 = 2 || |P^2(\sigma) = -\frac{4}{9}\rangle||^2 \] (10.5d)

level 1:
\[ \epsilon(\sigma) \cdot J(-1)|P^2(\sigma) = \frac{2}{9}\rangle, \quad \epsilon(\sigma) \cdot P(\sigma) = 0, \quad \epsilon^2(\sigma) = -1 \] (10.6a)
\[ ||\epsilon(\sigma) \cdot J(-1)|P^2(\sigma) = \frac{2}{9}\rangle||^2 = || |P^2(\sigma) = \frac{2}{9}\rangle||^2 \] (10.6b)
\[ L(-1)|P^2(\sigma) = \frac{2}{9}\rangle = (-P(\sigma) \cdot J(-1) + \frac{1}{3} J_{10}(-\frac{1}{3})J_{20}(-\frac{2}{3}))|P^2(\sigma) = \frac{2}{9}\rangle \] (10.6c)
\[ ||L(-1)|P^2(\sigma) = \frac{2}{9}\rangle||^2 = 0 \] (10.6d)

In these results, we have introduced an alternate notation for the momentum eigenstates
\[ |P^2(\sigma)\rangle \equiv |0, J(0)\rangle \] (10.7)

to conform with standard usage in string theory. The inner products above are defined with
the 24-dimensional Lorentz metric
\[ A(\sigma) \cdot B(\sigma) = g^{ab}_{\left(24\right)} A_a(\sigma) B_b(\sigma) = A_0(\sigma) B_0(\sigma) - \vec{A}(\sigma) \cdot \vec{B}(\sigma) \] (10.8)
where the momenta \{\(P_a(\sigma) = J_a(0)\}\} are the zero modes of the sector. The last result in
Eq. (10.6d) follows either from the Virasoro algebra or the current algebra.

The \(SO(23,1)\) spin of these physical states is easily read as follows

level 0 : tachyonic ground state at \(J = 0\) \hspace{1cm} (10.9a)
level 1/3 : higher tachyon at \(J = 0\) \hspace{1cm} (10.9b)
level 2/3 : two still higher tachyons at \(J = 0\) \hspace{1cm} (10.9c)
level 1 : non-tachyonic states at \(J = 1, 0\) \hspace{1cm} (10.9d)

where \(J = 1\) and \(0\) denote Lorentz vectors and scalars respectively, as is conventional in string
theory. More generally for space-time dimension \(D(\sigma)\) with \(SO(D(\sigma) - 1,1)\) symmetry, the
ground-state lemmas of Sec. 5 and the presence of the \((26 - D(\sigma))\) Lorentz scalars means that
the physical spectra of these bosonic prototypes often begins with a short set of tachyons
(see however Sec. 8).
The oscillator-free momentum eigenstates \(|P^2(\sigma)\rangle\rangle\) have positive norm as usual, so there are no negative-norm states in the first four levels of \(H(\text{perm})'_{3} = \mathbb{Z}_3\).

Moreover, the only negative-norm basis states in each sector are (as usual) those with an odd number of time-like modes \(\{J_0(M < 0)\}\). In particular, there are no negative norms in the positive-definite Hilbert space spanned by the Lorentz-scalar fractionally-modeled currents \(\{J_{L0}((M + \frac{2L}{3}) < 0), \hat{L} = 1, 2\}\). These facts are so far consistent with the no-ghost conjecture (see Refs.[1,4,5,7,8] and Sec. 12) for each Lorentzian sector of all the orbifold-string theories of permutation-type.

On the other hand, we have already found in Eqs. (10.6c,d) a zero-norm physical state at level 1, whose decoupling will depend on the gauges [4,5] of the interacting theory. We shall return to this part of the no-ghost conjecture in Sec. 12.

11 The \(\hat{c}(\sigma) = 52\) Description of the Physical States

In the previous section, we solved the reduced \(c(\sigma) = 26\) formulation for the low-lying physical states (10.3)-(10.6) of the sectors \(\sigma = 1, 2\) associated to the non-trivial elements of \(H(\text{perm})'_{3}\). Solving for the physical states was straightforward in the reduced description (4.5c), at least in part because the enhanced Lorentz transformation properties of the currents and states are transparent in this formulation. It is instructive however to reconsider these same physical states as well in the original description of these sectors at \(\hat{c}(\sigma) = 52\).

In the unreduced \(\hat{c}(\sigma) = 52\) description of the same sectors, we remind that the physical states are the solutions to the extended Virasoro conditions (2.3a), using the orbifold Virasoro generators

\[
\hat{L}_j(m + \frac{j}{2}) = \frac{125}{72} \delta_{m+\frac{j}{2},0} \sum_{\ell=0}^{1} \sum_{p \in \mathbb{Z}} \hat{J}_{c\ell}(p + \frac{j+\ell}{2}) \hat{J}_{-c,b}\hat{L}_j\hat{L}_i(m - p + \frac{j-\ell-c}{2}) + \frac{1}{12} \sum_{\ell=0}^{1} \sum_{p \in \mathbb{Z}} \hat{J}_{L0\ell}(p + \frac{j}{3} + \frac{\ell}{2}) \hat{J}_{-L,0}\hat{L}_j\hat{L}_i(m - p - \frac{j}{3} + \frac{j-\ell-c}{2})
\]

and the hatted current algebras (2.5) with \(F_L(\sigma) = 3, L = 0\). In this unreduced form, we cannot see clearly either the “preliminary” Lorentz symmetry \(SO(22,1)\) from \(d = 23\) (masked in the second term), or the last spatial dimension (masked in the last term) of the enhanced Lorentz symmetry \(SO(23,1)\).

Alternately however, the \(\hat{c}(\sigma) = 52\) description of the physical states in question can be straightforwardly obtained from the \(c(\sigma) = 26\) description of the states (10.3)-(10.6) by using the inverse of the map (3.1) and the relabeling definitions (4.3) and (4.4).

This gives the first four levels of \(H(\text{perm})'_{3} = \mathbb{Z}_3\) in the \(\hat{c}(\sigma) = 52\) description:

\[
|P^2(\sigma) = -\frac{16}{9}\rangle = |\hat{P}^2(\sigma) = -\frac{16}{9}\rangle
\]
\[ J_{10}(-\frac{1}{3}) |P^2(\sigma) = -\frac{13}{9}\rangle = \hat{J}_{101}(-\frac{1}{6}) |\hat{P}^2(\sigma) = -\frac{13}{9}\rangle \] (11.2b)
\[ J_{20}(-\frac{2}{3}) |P^2(\sigma) = -\frac{4}{9}\rangle = \hat{J}_{200}(-\frac{1}{3}) |\hat{P}^2(\sigma) = -\frac{4}{9}\rangle \] (11.2c)
\[ \epsilon(\sigma) \cdot J(-1) |P^2(\sigma) = \frac{2}{9}\rangle = (\sum_{a=0}^{22} \epsilon^a(\sigma) \hat{J}_{a,1-\epsilon}(-\frac{1}{2}) + \frac{1}{\sqrt{3}} \epsilon^2(\sigma) \hat{J}_{001}(-\frac{1}{2}))|\hat{P}^2(\sigma) = \frac{2}{9}\rangle \] (11.2d)
\[ 0 = \epsilon(\sigma) \cdot P(\sigma) = \sum_{a=0}^{22} \epsilon^a(\sigma) \hat{J}_{a}(0) + \frac{1}{\sqrt{3}} \epsilon^2(\sigma) \hat{J}_{000}(0), \quad \epsilon^2(\sigma) = -1 \] (11.2e)
\[ L(-1) |P^2(\sigma) = \frac{2}{9}\rangle = 2 \hat{L}_1(-\frac{1}{2}) |\hat{P}^2(\sigma) = \frac{2}{9}\rangle. \] (11.2f)

Note that the momenta of the states are invariant under the map, but the total mode number is multiplied by 1/2, so that the first four levels in the \( \hat{c}(\sigma) = 52 \) description are 0, 1/6, 1/3 and 1/2.

We add three observations at this point:

1) As discussed in Refs. [7,8] and Sec. 4, the target space-time structure of each \( j \)-cycle of the orbifolds of permutation-type is preserved by the map, including in particular the space-time dimensions \( D(\sigma) \) and the enhanced Lorentz symmetry \( SO(D(\sigma) - 1, 1) \) – though the space-time structure is masked in the \( \hat{c}(\sigma) = 52 \) formulation. In the present case for example, the structures

\[ J_a(-1) = \begin{cases} \hat{J}_{a,1-\epsilon}(-\frac{1}{2}) & \text{for } a = 0, 1, \ldots, 22 \\ \frac{1}{\sqrt{3}} \hat{J}_{001}(-\frac{1}{2}) & \text{for } a = 23 \end{cases} \] (11.3a)
\[ J_a(0) = \begin{cases} \hat{J}_{a}(0) & \text{for } a = 0, 1, \ldots, 22 \\ \frac{1}{\sqrt{3}} \hat{J}_{000}(0) & \text{for } a = 23 \end{cases} \] (11.3b)

are Lorentz 24-vectors while the modes \( \hat{J}_{101}(-\frac{1}{3}) \) and \( \hat{J}_{200}(-\frac{2}{3}) \) are still scalars under \( SO(23, 1) \). Although we will not pursue this here, the enhanced space-time symmetry \( SO(23, 1) \) of the generators (11.1) can be made manifest by field-redefinitions closely related to the examples in Eq. (11.3).

2) As noted more generally in Ref. [8] and Sec. 3, the norms and inner products of all states in the \( \hat{c}(\sigma) = 52 \) description are the same as those computed in the reduced description at \( c(\sigma) = 26 \).

3) In particular, the zero-norm state in (Eqs. (10.6c,d) and) Eq. (11.2f) is still associated with the orbifold Virasoro generators. This fact will play a role in the discussion of the following section.
12 The No-Ghost Conjecture

We have conjectured [1,4,5,8] a no-ghost theorem for all the Lorentzian [8] orbifold-string theories of permutation-type, including in particular all the sectors of the large example (1.5) of orientation-orbifold string systems. This conjecture is based on two observations:

1) The untwisted sectors $U(1)^{26K}$ or $(U(1)^{26}_L \times U(1)^{26}_R)$ of the orbifolds of permutation-type are free of negative-norm and zero-norm states.

2) The twisted sectors of the orbifold-string theories of permutation-type are constructed from these untwisted sectors, following known principles of orbifold theory [9-23].

On this basis, one might expect that an orbifoldization in light-cone gauge could establish the no-ghost theorem for these theories – but such an approach has not yet been attempted, and could shroud a number of issues, such as the new world-sheet geometries [1] and the enhanced target space-time symmetries [8] of the new string theories.

We outline here several parts of the conjecture in our covariant construction of the orientation-orbifold string systems:

a) No negative-norm physical states

We expect that the extended physical-state conditions (2.3a) at $\hat{c}(\sigma) = 52$

$$\hat{L}_j((m + \hat{j})_\sigma^2 \geq 0) - \hat{a}_2\delta_{m+\hat{j},0} |\chi(\sigma)\rangle = 0$$

or the equivalent, reduced physical-state condition (3.2a) at $c(\sigma) = 26$

$$(L(M \geq 0) - \delta_{M,0})|\chi(\sigma)\rangle = 0, \quad M = 2m + \hat{j}$$

will eliminate all negative-norm states from the physical spectrum of each sector $\sigma$.

b) Null physical states

We expect that, as in ordinary open-string theory, all null physical states will be linear combinations of mass-shell states of the form

$$2\hat{L}_j(-m - \hat{j})\hat{L}_0(0) = \hat{a}_2 - m - \hat{j} = L(-M)\hat{L}(0) = 1 - M$$

or

$$M = 2m + \hat{j} = 1, 2, \ldots$$

We have encountered a null physical state of this form in Sections 10 and 11. Note that the reduced forms in Eqs. (12.2) and (12.3a) are the same as ordinary open-string theory. Moreover, as in ordinary string theory, all physical states of the form (12.3) are null physical states.
c) Conjectured form of the gauges

We conjecture that all null physical states will decouple by a single family of gauges whose form is expected to be

\[ \hat{W}_j((m + \frac{j}{2}) > 0) \equiv \hat{\beta}(m + \frac{j}{2})\hat{L}_j(m + \frac{j}{2}) - (\hat{L}_0(0) - \hat{a}_2 + m + \frac{j}{2}) \]  

\[ = \frac{1}{2}W(M > 0) \]  

\[ W(M) = \beta(M)L(M) - (L(0) - 1 + M) \]  

where the \( \beta \)'s are model-dependent non-zero constants satisfying

\[ \hat{\beta}_j(m + \frac{j}{2})^* = \beta_{-j}(-m - \frac{j}{2}), \quad \beta(M) \equiv \hat{\beta}_j(m + \frac{j}{2}), \quad \beta(M)^* = \beta(-M). \]  

It is straightforward to see that these gauges satisfy the following relations:

\[ \hat{W}_j((m + \frac{j}{2}) > 0)^\dagger|\hat{L}_0(0) = \hat{a}_2 - m - \frac{j}{2}) \]  

\[ = \hat{\beta}_{-j}(-m - \frac{j}{2})\hat{L}_j(-m - \frac{j}{2})|\hat{L}_0(0) = \hat{a}_2 - m - \frac{j}{2}) \]  

\[ = \frac{1}{2}W(M > 0)^\dagger|L(0) = 1 - M \]  

\[ = \frac{1}{2}\beta(M)L(-M)|L(0) = 1 - M. \]  

In fact such gauges \( \{\hat{W}_j\} \), with non-trivial \( \hat{\beta} \), have been found explicitly [4] in a simple case of the orientation-orbifold string theories (see also Ref. [5]), and the next step would be to check the “activity” of these gauges vis-a-vis the more general twisted vertex operators outlined for the \( \hat{c} = 52 \) formulation in the Appendix of Ref. [4].

It is clear that parts (a) and (b) of the conjecture are true for our examples above in \( H(\text{perm})_2^\prime = \mathbb{Z}_3 \).

Moreover, it is easily checked that parts (a) and (b) are true for all sectors \( \sigma \) thru level one for any choice \( H(\text{perm})_{26-q}^\prime \) in the orientation-orbifold string systems: For all levels less than one, the physical states are formed entirely from the extra \( 26 - D(\sigma) \) Lorentz-scalar fractional-moded currents, with positive-definite norms, while the only physical states at level one are

\[ \epsilon(\sigma)\cdot J(-1)|P^2(\sigma) = P^2(\sigma)_{(0) + 2}, \quad \epsilon(\sigma)\cdot P(\sigma) = 0, \quad \epsilon^2(\sigma) = -1 \]  

\[ L(-1)|P^2(\sigma) = P^2(\sigma)_{(0) + 2}. \]  

The norms of the level-one physical states are respectively positive and zero

\[ ||\epsilon(\sigma)\cdot J(-1)|P^2(\sigma) = P^2(\sigma)_{(0) + 2}||^2 = ||P^2(\sigma) = P^2(\sigma)_{(0) + 2}||^2 \]  

\[ ||L(-1)|P^2(\sigma) = P^2(\sigma)_{(0) + 2}||^2 = \langle P^2(\sigma)|2L(0)|P^2(\sigma) \rangle \]
\[ (P^2(\sigma) + 2\delta_0(\sigma)) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2 |^2 = 0 \]  
(12.8c)

for all open-string sectors \( \sigma \) of all our theories, generalizing our \( \mathbb{Z}_3 \) examples above. Of course, the mass-shell physical state (12.7b) is guaranteed to be null by the physical-state condition, but we have here computed its norm in a way that shows it is independent of the ground-state mass-squared \( P^2(\sigma)_{(0)} = 2(1 + \hat{\delta}_0(\sigma)) \).

We will not present here the physical states at fractional levels between levels one and two, remarking only that there is always the familiar null physical state at level two

\[ (\alpha L(-2) + \beta L^2(-1)) |P^2(\sigma) = P^2(\sigma)_{(0)} + 4 = 2(1 + \hat{\delta}_0(\sigma)) \]  
(12.9a)

\[ 3\alpha = 2\beta \]  
(12.9b)

independent of the conformal-weight shifts. This is the only combination of mass-shell states (12.3) which is physical at level two.

All the null physical states encountered so far in this discussion are linear combinations of the mass-shell states (12.3), and hence removable in principle by the single family of gauges in Eq. (12.4). Moreover the pattern of each of these examples is familiar from the special case

\[ D(\sigma) = 26, \quad \hat{\delta}_0(\sigma) = 0, \quad P^2(\sigma)_{(0)} = -2, \quad L(M) = L(M)_{(26)} \]  
(12.10)

which is ordinary untwisted critical open-string theory. In the following section, we provide further evidence for the no-ghost conjecture in the orientation-orbifold string theories.

### 13 The Ordinary \( D(\sigma) \leq 26 \) String Subsectors

In this last section on the open-string sectors of the orientation-orbifolds, we discuss the ordinary \( D(\sigma) \)-dimensional string subsystem which exists in each open-string sector \( \sigma \).

This subsystem is defined by the subset of physical states \( \{ |\chi[D(\sigma)] \rangle \} \) which contains none of the extra \( (26 - D(\sigma)) \) fractional modes of sector \( \sigma \). On this subset of states, the full physical-state condition (3.2a) of sector \( \sigma \) reduces to the following description:

\[ \langle L(M \geq 0)_{(D(\sigma))} - a(\sigma)\delta_{M,0} |\chi[D(\sigma)] \rangle = 0 \]  
(13.1a)

\[ L(M)_{(D(\sigma))} = -\frac{1}{2} \eta^{ab}_{(D(\sigma))} \sum_{P \in \mathbb{Z}} J_a(P)J_b(M - P)_{\circ M} \]  
(13.1b)

\[ a(\sigma) \equiv 1 - \hat{\delta}_0(\sigma) = -\frac{1}{2}P^2(\sigma)_{(0)} \leq 1 \]  
(13.1c)

\[ 2 \leq (D(\sigma) = \text{even}) \leq 26 \]  
(13.1d)

\[ [L(M)_{(D(\sigma))}, L(N)_{(D(\sigma))}] = (M - N)L(M + N)_{(D(\sigma))} + \frac{D(\sigma)}{12}M(M^2 - 1)\delta_{M+N,0} \]  
(13.1e)

\[ [L(M)_{(D(\sigma))}, J_a(N)] = -NJ_a(M + N), \quad a = 0, 1, \ldots, D(\sigma) - 1 \]  
(13.1f)
\[ [J_a(M), J_b(N)] = N \eta_{ab}^{(D(\sigma))} \delta_{M+N,0} \]  

(13.1g)

This subsector is recognized as an ordinary \( SO(D-1,1) \)-invariant string with \( D = D(\sigma) \leq 26 \) space-time dimensions and a quantized intercept \( a(\sigma) \leq 1 \), given in Eq. (13.1c) in terms of the conformal-weight shifts. Explicit expressions for the conformal-weight shifts and ground-state momentum-squareds are given in Eqs. (2.13b),(2.16) and (3.10d), leading to the explicit form of the quantized intercept of sector \( \sigma \):

\[
a(\sigma) = \frac{1}{24} (d - 2 + \sum L \frac{\alpha^2(L)}{F_L(\sigma)}) \leq 1
\]

(13.2a)

\[
\sum L F_L(\sigma) = 26 - d, \quad 1 \leq d \leq 25.
\]

(13.2b)

The L-cycle function \( \alpha(L) \) is defined in Eq. (2.12). The maximal value of the quantized intercept is obtained only for \( d = 25, 26 \) where

\[
D(\sigma) = 26, \quad \hat{\delta}_0(\sigma) = 0, \quad \hat{P}^2(\sigma)(0) = -2, \quad a(\sigma) = 1
\]

(13.3a)

\[
L(M)_{(26)} = L(M)
\]

(13.3b)

and the full sector \( \sigma \) is in these cases nothing but an ordinary critical open string.

Ordinary strings with \( D(\sigma) \leq 26 \) and \( a(\sigma) \leq 1 \) are known [24] to have no negative-norm physical states, and these specifications in fact include all of the “ordinary” string subsectors discussed here (see Eq. (13.1d)). Such consistency in the new string theories should not be surprising because the twisted sectors of these theories are constructed from (copies of) critical untwisted theories using only the principles of orbifold theory.

It should be mentioned that each of these “ordinary” open-string subsectors by themselves would give rise to continuous closed-string spectra via non-planar loops. In our theories however, each full open-string sector \( \sigma \) also contains the extra \((26 - D(\sigma))\) fractional-moded Lorentz-scalar degrees of freedom. Since the total number of effective degrees of freedom of each sector is 26, we do not expect any such difficulty in the loops. This intuition is supported by the closed-string spectra of our orbifold construction in the following sections.

Motivated in part by our interest in null-physical states (see Sec. 12), we also give here the explicit solution of Eq. (13.1a) for the low-lying physical states \( \{|\chi[D(\sigma)]\} \) in the ordinary string subsectors.

**Level 0**

Here the only physical state is the familiar positive-norm oscillator-free state

\[
|P^2(\sigma) = P^2(\sigma)(0) = -2a(\sigma)\rangle, \quad (J = 0 \ \text{ground state})
\]

(13.4)

which is also the true ground-state of each full sector \( \sigma \).
Here the only physical state for $D(\sigma) \leq 25$ is the familiar $J = 1$ state
\[
\epsilon(\sigma) \cdot J(-1) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle, \quad \epsilon(\sigma) \cdot P(\sigma) = 0, \quad \epsilon^2(\sigma) = -1
\] (13.5a)
\[
|| \ldots ||^2 = || |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle||^2
\] (13.5b)
which is a massive vector meson with positive norm.

The $J = 0$ state
\[
L(-1)(D(\sigma)) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle = -P(\sigma) \cdot J(-1) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle
\] (13.6a)
\[
||L(-1)(D(\sigma)) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle||^2
\]
\[
= (P^2(\sigma) = P^2(\sigma)_{(0)} + 2 | 2L(0)(D(\sigma)) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle (13.6b)
\]
\[
= -P^2(\sigma)|| |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle||^2
\] (13.6c)
is not physical and not null, except of course at
\[
D(\sigma) = 26, \quad L(M) = L(M)_{(26)}
\] (13.7a)
\[
a(\sigma) = 1, \quad P^2(\sigma)_{(0)} = -2, \quad P^2(\sigma) = 0
\] (13.7b)
where the entire sector $\sigma$ of the orientation-orbifold is an ordinary critical open string.

It is clear that the state (13.6) for $D(\sigma) \leq 25$ is not the null-physical state (10.6c) discussed above
\[
L(-1)|P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle = (L(-1)(D(\sigma)) + \text{extra}) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle
\] (13.8a)
\[
||L(-1) |P^2(\sigma) = P^2(\sigma)_{(0)} + 2\rangle||^2 = 0
\] (13.8b)
which involves the full Virasoro generators $\{L(M)\}$, and which should be removed by the single set of gauges (12.4) associated to the full Virasoro generators. Put another way, we expect that no extra set of gauges based on the subsector generators $\{L(M)_{(D(\sigma))}\}$ will be necessary to remove all the null-physical states.

We find the familiar $J = 2$ physical state
\[
\{\epsilon^a(\sigma) \epsilon^b(\sigma) + \frac{1}{D(\sigma) - 1}(\eta_{D(\sigma)})^{ab} - \frac{P^a(\sigma)P^b(\sigma)}{P^2(\sigma)} J_a(-1) J_b(-1)\} |P^2(\sigma) = P^2(\sigma)_{(0)} + 4\rangle
\] (13.9)
with positive norm for all $D(\sigma) \leq 26$. There is also a $J = 1$ physical state
\[
\{2\epsilon(\sigma) \cdot J(-1) P(\sigma) \cdot J(-1) - P^2(\sigma) \epsilon(\sigma) \cdot J(-2)\} |P^2(\sigma) = P^2(\sigma)_{(0)} + 4\rangle
\] (13.10a)
whose norm is positive except for ground-state mass-squared $P^2(\sigma)_{(0)} = -2$, where the vanishing norm marks again an ordinary $D(\sigma) = 26$ dimensional open string.

To supplement the discussion of the state (13.10), consider also the following $J = 1$ state:

$$L(-1)_{(\sigma)} \epsilon(\sigma) \cdot J(-1) P^2(\sigma) = P^2(\sigma)_{(0)} + 4$$

$$|| \cdots ||^2 = 2(P^2(\sigma)_{(0)} + 4)(P^2(\sigma)_{(0)} + 2) || P^2(\sigma) = P^2(\sigma)_{(0)} + 4 ||^2$$  \hspace{1cm} (13.10b)

This state is neither physical nor null – except at $P^2(\sigma)_{(0)} = -2$ and hence $D(\sigma) = 26$. At such a point the full sector $\sigma$ is again an ordinary critical string, and this state – now proportional to the state (13.10a) – is a familiar null-physical state which is removed by the single set of gauges (12.4) with full Virasoro generators $L(M) = L(M)_{(26)}$. Once again, consistent with our conjecture, no extra family of gauges based on the subsector generators $\{L(M)_{(\sigma)}\}$ is required.

Finally, level two contains a single $J = 0$ physical state:

$$\{P(\sigma) \cdot J(-2) - \frac{(D(\sigma) + 2P^2(\sigma))}{P^2(\sigma)} P(\sigma) \cdot J(-1) \} P^2(\sigma) = P^2(\sigma)_{(0)} + 4$$

$$|| \cdots ||^2 = 2 \{1 - P^2(\sigma) + \frac{2P^2(\sigma)_{(0)} + 4}{D(\sigma)_{(0)}} \} || P^2(\sigma) = P^2(\sigma)_{(0)} + 4 ||^2.$$  \hspace{1cm} (13.12b)

The norm of the $J = 0$ physical state is positive for $P^2(\sigma)$ outside the region between the zeroes of the norm, which are located at the values:

$$P^2_{\pm}(\sigma) = \frac{1}{8}(D(\sigma) - 5 \pm \sqrt{(D(\sigma) - 1)(D(\sigma) - 25)}) = \text{real.}$$  \hspace{1cm} (13.13)

It follows that the norm of the $J = 0$ physical state is strictly positive for

$$|| \cdots ||^2 > 0 : \hspace{1cm} 2 \leq D(\sigma) \leq 24$$  \hspace{1cm} (13.14)

because the zeroes are complex in this range.

Beyond this range, we consider three additional cases.

a) $D(\sigma) = 25$. Here the zeroes of the norm coincide at $P^2(\sigma) = 5/2$, and we find strictly positive norm except zero norm at $P^2(\sigma) = 5/2$, $P^2(\sigma)_{(0)} = -3/2$ where the $J = 0$ state is proportional to

$$-\frac{1}{2}(L(-2)_{(25)} + L^2(-1)_{(25)}) \parallel P^2(\sigma) = \frac{5}{2}.$$  \hspace{1cm} (13.15)

In ordinary $D = 25$ dimensional string theory, this null-physical state would be removed by a set of gauges constructed from the Virasoro generators $\{L(M)_{(25)}\}$, but for us this would be an extra set of gauges, beyond those constructed in (12.4) with the total Virasoro generators.
\{L(M) = L(M)_{(25)} + \text{(extra)}\}$. Fortunately, the space-times of the open-string sectors of the orientation-orbifold string theories satisfy the double-inequality of Sec. 7

\[2 \leq D(\sigma) = \text{even} \leq 26\]  

and therefore contain no such sectors with $D(\sigma) = \text{odd}$. We consider this observation as a non-trivial check in support of our conjecture that the single set of gauges (12.4) is sufficient to remove all null-physical states in the new string theories.

b) $D(\sigma) = 26$. Here we have

\[P_+^2(\sigma) = \frac{13}{4}, \quad P_-^2(\sigma) = 2, \quad P_\sigma^2(\sigma) = -2, \quad P^2(\sigma) = 2.\]  

This is the familiar case of the critical open string, where the $J = 0$ physical state (13.12) is null, proportional to

\[-\frac{1}{5}(2L(-2)_{(26)} + 3L^2(-1)_{(26)})|P^2(\sigma) = 2\]  

\[L(M)_{(26)} = L(M)\]  

and removed by the single set of gauges (12.4).

c) $D(\sigma) \geq 27$. Above $D(\sigma) = 26$, it is well-known [24] that the $J = 0$ physical state (13.22) can have negative norm in the range $P_\sigma^2(\sigma) \leq P^2(\sigma) \leq P_+^2(\sigma)$. Fortunately again, the open-string sectors of the orientation-orbifold string systems contain no such high-dimensional space-times.

Consistent with the low-level examples of this section, we finally emphasize the following. If our single-family gauge conjecture in Sec. 12 is correct then the ordinary-string subsectors (13.1) will contain no null-physical states at all except at $D(\sigma) = 26$ (the critical open string), where our single set of gauges with $\{L(M) = L(M)_{(26)}\}$ is of course sufficient to remove the null-physical states.

## 14 Assembling the Orientation-Orbifolds

We recall here that the general orientation-orbifold string theory [20,21,23,1,3,4,8] has the form

\[\frac{U(1)^{26}}{H_-} = \frac{U(1)^{26}\times U(1)^{26}}{H_-}, \quad H_- \subset \mathbb{Z}_2(\text{w.s.}) \times H'_{26}\]  

where $\mathbb{Z}_2(\text{w.s.}) = (\tau_0 = 1, \tau_-)$ is the group of left-right exchanges on the closed string $U(1)^{26}$, and the space-time symmetry group $H'_{26}$ acts on the left- and right-movers separately. The elements of the group $H_-$ associated to $\tau_-$ correspond to twisted open-string sectors at $\hat{c}(\sigma) = 52$ (and reduced central charge $c(\sigma) = 26$). A large subset of these open-string sectors have been studied at length above, and earlier in Refs. [3,4,8].
The orientation-orbifold string systems always contain an equal number of twisted open- and closed-string sectors, the latter corresponding to those elements of $H_-$ proportional to the trivial element $\tau_0 = 1$ of $\mathbb{Z}_2$ (w.s.). The closed-string sectors therefore comprise the ordinary space-time orbifold

$$\frac{U(1)^{26}}{H'_{26}}$$

all of whose sectors live at central charge $\hat{c}(\sigma) = 26$. Note that we have chosen here to suppress the element $\tau_0 = 1$, whose action is trivial on the closed string $U(1)^{26}$.

More precisely, we focus our attention on the large example (1.5)

$$H_+ \subset \{ H'_{26}; \tau_+ \times (\pm 1)(d) \times H'(\text{perm})'_{26-d} \}$$

(14.3a)

$$H'_{26} \subset (1)(d) \times H'(\text{perm})'_{26-d}$$

(14.3b)

where the closed- and open-string sectors appear respectively before and after the semicolon. Recall from our discussion above that we may choose either $(\pm 1)(d)$ that is $\epsilon = 0$ or $1$ (but not both) for these open-string sectors, and moreover that the physical spectrum of each open-string sector is Lorentzian and independent of $\epsilon$. On the other hand, we are choosing only $\epsilon = 0$ for the closed-string sectors, which then guarantees [8] that all the closed-string sectors in the large example (14.3) are also Lorentzian. Following our practice for the open-string sectors, we will again choose the cyclic groups

$$H'(\text{perm})'_{26-d} = \mathbb{Z}_{26-d}$$

(14.4)

when needed as explicit examples in the closed-string sectors.

The reader should bear in mind that there is in general more than one way to choose the groups $H_+$ and $H'_{26}$ so that $H'_{26} \subset H_+$. As examples, consider the cyclic groups (14.4), where we may choose for any $d$ (see Eq. (7.1)):

$$H_+ = \{ H'_{26}; \tau_+ \times (\pm 1)(d) \times (1, \omega, \ldots, \omega^{25-d})_{26-d} \}$$

(14.5a)

$$H'_{26} = (1)(d) \times (1, \omega, \ldots, \omega^{25-d})_{26-d}$$

(14.5b)

$$\omega \in \mathbb{Z}_{26-d}.$$  

(14.5c)

Then the space-time orbifold $U(1)^{26}/H'_{26}$ in Eq. (14.2) is an ordinary cyclic permutation orbifold [9] on the $(26 - d)$ higher spatial dimensions of $U(1)^{26}$. On the other hand, when $d$ is even we may alternately choose the restricted groups

$$H_+ = \{ H'_{26}; \tau_+ \times (\pm 1)(d) \times (\omega, \omega^3, \ldots, \omega^{25-d})_{26-d} \}$$

(14.6a)

$$H'_{26} = (1)(d) \times (1, \omega^2, \ldots, \omega^{24-d})_{26-d}$$

(14.6b)

where $H'_{26}$ is now proportional to the subgroup $\mathbb{Z}_{(26-d)/2}$ of $\mathbb{Z}_{26-d}$. Other choices of $H'_{26} \subset H_+$ are possible for special values of $d$, but we will not pursue this further here.
15 The Twisted Closed-String Sectors

Using the standard methods of the orbifold program [10-23], we have worked out the physical-state condition and Virasoro generators for each closed-string sector \( \sigma \) corresponding to any \( \omega'(\sigma) \subset H'_{26} \) of the ordinary space-time orbifold \( U(1)^{26}/H'_{26} \). The results for general \( H'_{26} \) are as follows:

\[
\hat{L}(m) = \frac{1}{2} G^{\mu_1 \mu_2 \cdots \mu_s} \sum_{p \in \mathbb{Z}} \hat{J}_{n(r) \mu_1} \hat{J}_{n(r) \mu_2} \cdots \hat{J}_{n(r) \mu_s} (p + \frac{n(r)}{\rho(\sigma)})^s M + \\
\delta_{m,0} \frac{1}{4} \sum \text{dim}[n(r)] (\delta(r) \rho(\sigma) - \delta(r) \rho(\sigma))
\]

\[
\sum_{r \mu} \sum \text{dim}[\bar{n}(r)] = 26
\]

\[
[\hat{L}(m), \hat{L}(n)] = (m - n) \hat{L}(m + n) + \frac{26}{12} m(m^2 - 1) \delta_{m+n,0}
\]

\[
[\hat{L}(m), \hat{J}_{n(r) \mu}(n + \frac{n(r)}{\rho(\sigma)})] = -(n + \frac{n(r)}{\rho(\sigma)}) \hat{J}_{n(r) \mu}(m + n + \frac{n(r)}{\rho(\sigma)})
\]

\[
[\hat{J}_{n(r) \mu}(m + \frac{n(r)}{\rho(\sigma)}), \hat{J}_{n(s) \nu}(n + \frac{n(s)}{\rho(\sigma)})] = (m + \frac{n(r)}{\rho(\sigma)}) \delta \left(m + n + \frac{n(r) + n(s)}{\rho(\sigma)}\right) G_{n(r) \mu; n(s) \nu}(\sigma)
\]

\[
\hat{J}_{n(r) \mu}(m + \frac{n(r) \pm \rho(\sigma)}{\rho(\sigma)}) = \hat{J}_{n(r) \mu}(m \pm 1 + \frac{n(r)}{\rho(\sigma)})
\]

There is also a set of right-mover Virasoro generators \( \{ \tilde{L} \} \) which are copies (with right mover currents \( \{ \tilde{J} \} \)) of the left-mover system given explicitly here.

For these closed-string sectors, the physical-state conditions (15.1a) are quite ordinary, and the twisted metric \( G(\sigma) \) in these results has the usual form [11,13,3,7]:

\[
G_{n(r) \mu; n(s) \nu}(\sigma) = \chi_{n(r) \mu}(\sigma) \chi_{n(s) \nu}(\sigma) U(\sigma)_{n(r) \mu} a U(\sigma)_{n(s) \nu} b G_{ab}
\]

\[
= \delta_{n(r) + n(s), 0 \mod \rho(\sigma)} G_{n(r) \mu; n(s) \nu}(\sigma)
\]

\[
\omega'(\sigma) a U(\sigma) \omega'(\sigma) a = e^{-2\pi \frac{n(r)}{\rho(\sigma)} U(\sigma)_{a \mu}, \omega'(\sigma) \in H'_{26}}
\]

\[
G_{ab} = -\eta_{ab}, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a, b = 0, 1, \ldots, 25.
\]

Here \( \{ n(r) \} \) and \( \rho(\sigma) \) are respectively the spectral indices and the order of \( \omega'(\sigma) \), and the mode normal-ordering follows the standard convention (2.4).

The twisted closed-string currents above exhibit the conventional orbifold fractions \( n(\rho) \) at \( \hat{c}(\sigma) = 26 \), and there is no distinct reduced formulation for these cases. These results can
be understood as cases of trivial cycle length $f_j(\sigma) = 1$ in the more general reduced orbifold fraction $f_j(\sigma)n/\rho$ [7,8] at $c(\sigma) = 26$.

Next, we give the explicit form of the closed-string Virasoro generators in the case of the large example (14.3), where the space-time orbifold has the form

\[
\frac{U(1)^{26}}{H(\text{perm})_{26-d}}. \tag{15.3}
\]

Including now the adjoint operations [13,8], these results are as follows:

\[
\hat{L}(m) = \delta_{m,0}\frac{1}{2\pi} \sum_L (F_L(\sigma) - \frac{1}{F_L(\sigma)}) - \frac{1}{2} \eta^{ab}_{(D(\sigma),c)} \sum_{p \in \mathbb{Z}} \overset{\circ}{J}_a(p)\overset{\circ}{J}_b(m - p)\overset{\circ}{M} +
\]

\[
+ \frac{1}{2} \sum_L \frac{1}{F_L(\sigma)} \sum_{L' = 1}^{F_L(\sigma)-1} \sum_{p \in \mathbb{Z}} \overset{\circ}{J}_{LL}(p + \frac{L}{F_L(\sigma)})\overset{\circ}{J}_{-LL}(m - p - \frac{L}{F_L(\sigma)})\overset{\circ}{M} \tag{15.4a}
\]

\[
[\hat{L}(m), \hat{L}(n)] = (m - n)\hat{L}(m + n) + \frac{26}{12} m(m^2 - 1)\delta_{m+n,0} \tag{15.4b}
\]

\[
[\hat{L}(m), J_a(n)] = -nJ_a(m + n), \quad a = 0, 1, \ldots, D(\sigma) - 1 \tag{15.4c}
\]

\[
[\hat{L}(m), J_{LL}(n + \frac{L}{F_L(\sigma)})] = -(n + \frac{L}{F_L(\sigma)})\hat{J}_{LL}(m + n + \frac{L}{F_L(\sigma)}) \tag{15.4d}
\]

\[
\hat{L} = 1, \ldots, F_L(\sigma) - 1 \tag{15.4e}
\]

\[
[\hat{J}_a(m), \hat{J}_b(n)] = m\eta^{ab}_{(D(\sigma),c)} \delta_{m+n,0} \tag{15.4f}
\]

\[
[\hat{J}_{LL}(m + \frac{L}{F_L(\sigma)}), \hat{J}_{MM}(n + \frac{M}{F_M(\sigma)})] = \delta_{LM}(m + \frac{L}{F_L(\sigma)} + \frac{M}{F_M(\sigma)})F_L(\sigma)\delta_{m+n+\frac{L+M}{F_L(\sigma)},0} \tag{15.4g}
\]

\[
\hat{J}_{L+\pm F_L(\sigma),LL}(m + \frac{L\pm F_L(\sigma)}{F_L(\sigma)}) = \hat{J}_{LL}(m \pm 1 + \frac{L}{F_L(\sigma)}) \tag{15.4h}
\]

\[
J_a(m)^\dagger = J_a(-m) \tag{15.4i}
\]

\[
\hat{J}_{LL}(m + \frac{L}{F_L(\sigma)})^\dagger = \hat{J}_{-LL}(-m - \frac{L}{F_L(\sigma)}) \tag{15.4j}
\]

\[
\hat{L}(m)^\dagger = \hat{L}(-m) \tag{15.4k}
\]

\[
\sum_L F_L(\sigma) = 26 - d, \quad \sum_a + \sum_{L \neq 0} = 26, \quad \sum_L = N(\sigma)' \tag{15.4l}
\]

Here, as above for the twisted open-string sectors, $F_L(\sigma)$ is the length of cycle $L$ in $\omega'(\sigma) \in H(\text{perm})_{26-d}$ and $N(\sigma)'$ is the total number of cycles in $\omega'(\sigma)$. The closed-string target space-time dimensionality $D(\sigma),c$ is evaluated in the next paragraph.

Following our development of the open-string sectors above, we have here directly collected all the integer-moded sequences in the second term of the Virasoro generators (15.4a). This includes in particular the $\hat{L} = 0$ terms for each cycle $L$ in $\omega'(\sigma)$. (See the rescaling for
\{J_{0L}(M)\} in Eq. (4.4a); there is no analogue of the integer-modeled sequence \{J_{F_L(\sigma)/2,L}(M)\} in the closed-string sectors.) Thus we see that closed-string sector \(\sigma\) exhibits the enhanced Lorentz symmetry \(SO(D(\sigma)c - 1,1)\), where the number of closed-string target space-time dimensions \(D(\sigma)c\) is

\[
2 \leq D(\sigma)c = d + N(\sigma)' = d + N_O(\sigma)' + N_E(\sigma)' \leq 26 \tag{15.5a}
\]

\[
2 \leq D(\sigma) = d + N_O(\sigma)' + 2 N_E(\sigma)' \text{ even} \leq 26. \tag{15.5b}
\]

The quantities \(N_{O,E}(\sigma)'\) are again the number of cycles of odd and even length \(F_L(\sigma)\) in \(\omega'(\sigma)\). In Eq. (15.5b) we have also provided for comparison the number (7.6) of target space-time dimensions in the corresponding open-string sector \(\sigma\). (We remind that both the open- and closed-string space-time dimensions are associated to the same element \(\omega'(\sigma) \in H(\text{perm})'_{26-d}\) as \(\tau_0 \times \omega'(\sigma)\) and \(\tau_- \times \omega'(\sigma)\) respectively.)

Notice that the following triple inequality

\[
2 \leq D(\sigma)c \leq D(\sigma) \leq 26 \tag{15.6}
\]

follows for all \(H(\text{perm})'_{26-d}\) by comparison of the explicit formula (15.5a) with our previous result (15.5b). Moreover, the equality \(D(\sigma)c = D(\sigma)\) is achieved only when all cycles of \(\omega'(\sigma)\) have odd length, and we remark in particular that the space-time dimensionalities of the closed strings are not necessarily even.

The last two columns of Tables 1, 2 and 3 in Sec. 7 give the explicit values of the closed-string target-space dimensionalities for the non-trivial elements of the cyclic groups \(H(\text{perm})'_{26-d} = \mathbb{Z}_{26-d}\). We shall return below to some special non-trivial cases in these tables, mentioning here only that the trivial closed-string sector \(\sigma = 0\) with \(\omega'(0) = 1\) is an ordinary untwisted \(D(0)c = 26\) dimensional closed string for all \(H(\text{perm})'_{26-d}\).

For completeness, we have used the usual closed-string conditions \(\hat{J}(0)c = \hat{J}(0)R \equiv \hat{J}(0)/\sqrt{2}\) to find the ground-state mass-squared of each closed-string sector \(\sigma\) of the full space-time orbifold in Eq. (15.3):

\[
P^2(\sigma)c_{(0)} = -\frac{1}{6} \{(d - 2) + \sum_L \frac{1}{F_L(\sigma)}\} \geq -4 \tag{15.7a}
\]

\[
F_L(\sigma) \geq 1, \quad \sum_L F_L(\sigma) = 26 - d, \quad 1 \leq d \leq 25. \tag{15.7b}
\]

This result for closed-string sector \(\sigma\) should be compared with the corresponding ground-state mass-squared of open-string sector \(\sigma\) in Eq. (3.10d). As for the open-string sectors, this result can be extended to \(d = 26\) by ignoring the summation over \(L\), and we conclude that the extrema of Eqs. (15.5a) and (15.7a) are obtained only in the two cases with trivial \(H(\text{perm})'\)

\[
d = 25, 26: \quad D(0)c = 26, \quad P^2(0)c_{(0)} = -4 \tag{15.8}
\]

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where the only sector $\sigma = 0$ is again an ordinary untwisted critical closed string.

It is not difficult to check from Eq. (15.7a) that all the sectors of the space-time orbifold (15.3) are tachyonic, except possibly for the case $H(\text{perm})'_25$ where

$$d = 1 : \quad P^2(\sigma)_{(0)}^c = \frac{1}{6}(1 - \sum_L \frac{1}{F_L(\sigma)}), \quad \sum_L F_L(\sigma) = 25. \quad (15.9)$$

Indeed one sees explicitly from the cycle-data in Table 2 that all non-trivial sectors of $\mathbb{Z}_{25}$ are non-tachyonic

$$20[25]^1 : \quad D(\sigma)_c = 2, \quad P^2(\sigma)_{(0)}^c = \frac{4}{25} \quad (15.10a)$$

$$4[5]^5 : \quad D(\sigma)_c = 6, \quad P^2(\sigma)_{(0)}^c = 0 \quad (15.10b)$$

with enhanced Lorentz symmetry $SO(1,1)$ and $SO(5,1)$ respectively.

As a simple subclass of examples, consider the closed- and open-string sectors corresponding to the non-trivial elements of the prime cyclic groups. Here we find $(25 - d)$ non-trivial single-cycle closed- and open-string sectors with

$$H(\text{perm})'_{20-d} = \mathbb{Z}_{20-d}, \quad d = \text{prime}, \quad \sigma = 1, \ldots, 25 - d \quad (15.11a)$$

$$D(\sigma)_c = D(\sigma) = d + 1 \quad (15.11b)$$

$$P^2(\sigma)_{(0)}^c = 2P^2(\sigma)_{(0)} = -\frac{1}{6}(d - 2 + \frac{1}{26-d}) \quad (15.11c)$$

where the open-string sectors were given above in Eq. (6.3).

Among these closed-string sectors, we mention in particular the 22 non-trivial closed-string sectors of $H(\text{perm})'_{23} = \mathbb{Z}_{23}$, which are four-dimensional closed strings with enhanced Lorentz symmetry $SO(3,1)$:

$$\hat{L}(m) = \frac{22}{23} \delta_{m,0} - \frac{1}{2} \eta_{(4)}^{ab} \sum_{p \in \mathbb{Z}} \hat{J}_a(p)\hat{J}_b(m - p) M +$$

$$+ \frac{1}{23} \sum_{\hat{L} = 1}^{22} \sum_{p \in \mathbb{Z}} \hat{J}_{\hat{L}0}(p + \frac{\hat{L}}{23})\hat{J}_{-\hat{L},0}(m - p - \frac{\hat{L}}{23}) M \quad (15.12a)$$

$$d = 3, \quad D(\sigma)_c = 4, \quad P^2(\sigma)_{(0)} = 2P^2(\sigma)_{(0)} = -\frac{4}{23}, \quad \sigma = 1, \ldots, 22. \quad (15.12b)$$

In fact, the left-mover Virasoro generators of these 22 closed four-dimensional strings are isomorphic to the 22 four-dimensional open-string sectors given for $\mathbb{Z}_{23}$ in Eq. (9.1). This conclusion follows by mode-relabeling under the mode sums, so that $\{2\hat{L}/F_L(\sigma)\} \sim \{\hat{L}/F_L(\sigma)\}$ when $F_L(\sigma)$ is odd (see the conclusions of Ref. [3]).

Surveying the Tables of the cyclic groups in Sec. 7, we find that the only other four-dimensional $SO(3,1)$-invariant closed string is the single sector $1[12]^2$ of $H(\text{perm})'_{24} = \mathbb{Z}_{24}$. 43
We conclude this section with some general remarks about the full orientation-orbifold string systems (14.3), including both the open- and closed-string sectors, first for the cyclic groups $H(\text{perm})_{26-d} = \mathbb{Z}_{26-d}$ as described by the general sector schematics in Eq. (7.2). One universal feature of all these orbifolds is the presence of the ordinary critical $c = 26$ open-closed string systems (see Ref. [4]) as the following two sectors

$$\tau_\pm \times (\pm \mathbb{1})_{(d)} \times (\mathbb{1})_{(26-d)}, \quad \tau_0 \times (\mathbb{1})_{(d)} \times (\mathbb{1})_{(26-d)}$$

(15.13a)

$$D(\sigma) = D(\sigma)_c = 26, \quad P^2(\sigma)_{(0)} = 2P^2(\sigma)_{(0)} = -4$$

(15.13b)

$$\mathbb{1} = \tau_0 \times (\mathbb{1})_{(d)} \times (\mathbb{1})_{(26-d)}$$

(15.14)

Both of which correspond to the trivial element of $\mathbb{Z}_{26-d}$. We remind that the physical spectrum of the open-string sector $\tau_\pm$ is independent of the choice $(\pm \mathbb{1})_{(d)}$, and for the cases $d = 25, 26$ these two sectors form the entire orientation-orbifold. Similarly, the ordinary critical open-closed string subsystem (15.13) persists for the trivial elements $(\mathbb{1})_{(26-d)}$ of all $H(\text{perm})_{26-d}$.

Moreover, for all $H(\text{perm})_{26-d}$, the closed-string sector

$$\mathbb{1} = \tau_0 \times (\mathbb{1})_{(d)} \times (\mathbb{1})_{(26-d)}$$

(15.14)

is the *only* ordinary closed-string sector, so that each orientation-orbifold string system (14.3) contains exactly *one graviton per orbifold* – in keeping with our conventional understanding of gravity in string theory.

It should be emphasized however that the orientation-orbifold string systems are uniquely simple in this regard. In contrast, the generalized permutation orbifolds [8]

$$\frac{U(1)^{26}}{H_+}, \quad H_+ \subset H(\text{perm})_K \times (\pm \mathbb{1})_{(d)} \times H(\text{perm})_{26-d}$$

(15.15)

are composed entirely of closed-string sectors, and can exhibit *multiple, presumably non-interacting gravitons* (see Ref. 5 and Sec. 10 of Ref. 7).

## 16 Conclusions and Directions

We have studied the following large example of orientation-orbifold string theories [1-4.6-8]

$$\frac{U(1)^{26}}{H_-} = \frac{U(1)^{26}}{H_+} \times \frac{U(1)^{26}}{H_-}$$

(16.1a)

$$H_- \subset \{ H'_{26}; \quad \tau_\pm \times (\pm \mathbb{1})_{(d)} \times H(\text{perm})_{26-d} \}$$

(16.1b)

$$H'_{26} \subset (\mathbb{1})_{(d)} \times H(\text{perm})_{26-d}, \quad 1 \leq d \leq 26$$

(16.1c)
in some detail, emphasizing in particular the cyclic permutation groups $H(\text{perm})^'_{26-d} = \mathbb{Z}_{26-d}$. These new string theories provide multi-sector generalizations of ordinary critical open-closed string theory, each system containing an equal number of twisted open- and closed-string sectors. Moreover, each system includes the ordinary critical open-closed string theory as the special subsystem with the unique graviton of the theory. The open-string sectors (see Secs. 2-7) have a local description [1,6] at $\hat{c} = 52$ and an equivalent, reduced description [3-5,7,8] of the physical states at $c = 26$, while the closed strings (see Secs. 14 and 15) form an ordinary space-time orbifold with all sectors at $\hat{c} = c = 26$.

We have found that the open- and closed-string sectors of these theories have sector-dependent Lorentzian [8] target space-time dimensions

$$D(\sigma), D(\sigma)_c \leq 26$$

(16.2)

for all sectors $\sigma$ associated to each $H(\text{perm})^'_{26-d}$, including the corresponding Lorentz symmetries $SO(D(\sigma) - 1, 1)$ and $SO(D(\sigma)_c - 1, 1)$. The enhancement mechanism [8] which underlies these symmetries is detailed in Sec. 4. See also the discussion of Sec. 7, and in particular the Tables given there for the cyclic groups – as well as the general results in Eqs. (15.5-6). We emphasize with Refs. [3,8] that these constructions are generically new, and certainly not ordinary compactifications. Additionally, we have here pointed out those special cases of non-tachyonic or four-dimensional strings (see Secs. 8,9 and 15) associated to the cyclic groups.

We have also included a number of introductory, successful tests of the no-ghost conjecture [1-5] for these theories (see Secs. 10-13).

It seems that the orientation-orbifold string systems studied here are the simplest among the orbifold-string theories of permutation-type [1-8], not least because they exhibit only a single graviton per orbifold and contain ordinary ghost-free $D(\sigma) \leq 26$ - dimensional string subsystems with quantized intercept $a(\sigma) \leq 1$ (see Sec. 13). A next step in this program should be the study of the twisted $\hat{c} = 52$ open-string vertex operators of the orientation-orbifolds and the construction of the open-string sectors at tree level, following the text and Appendix of Ref. 4. Given our historical understanding [24], one expects that these sectors will be the simplest in which to elevate the no-ghost conjecture to a theorem for the new string theories.

We conclude this paper with a final remark on another important direction in the program. On the basis of the enhanced target-space Lorentz symmetries [8] studied here for the bosonic prototypes, we expect correspondingly-enhanced target-space supersymmetries – and non-tachyonic spectra – in the superstring generalizations [1] of the orientation-orbifold string systems.
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