Average case polyhedral complexity of the maximum stable set problem

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Abstract We study the minimum number of constraints needed to formulate random instances of the maximum stable set problem via linear programs (LPs), in two distinct models. In the uniform model, the constraints of the LP are not allowed to depend on the input graph, which should be encoded solely in the objective function. There we prove a $2^{\Omega(n/\log n)}$ lower bound with probability at least $1 - 2^{-n}$ for every LP that is exact for a randomly selected set of instances; each graph on at most $n$ vertices being selected independently with probability $p \geq 2^{-(n/4^2)} + n$. In the non-uniform model, the constraints of the LP may depend on the input graph, but we allow weights on the vertices. The input graph is sampled according to the $G(n, p)$ model. There we obtain upper and lower bounds holding with high probability for various ranges of $p$. We obtain a super-polynomial lower bound all the way from $p = \Omega \left( \frac{\log(\log n)}{n} \right)$ to $p = o \left( \frac{1}{\log n} \right)$. Our upper bound is close to this as there is only an essentially quadratic gap in the exponent, which currently also exists in the worst-case model. Finally, we state a conjecture that would close this gap, both in the average-case and worst-case models.

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1 Introduction

In the last four years, extended formulations have gained considerable interest in various areas, including discrete mathematics, combinatorial optimization, and theoretical computer science. The key idea underlying extended formulations is that with the right choice of variables, various combinatorial optimization problems can be efficiently expressed via linear programs (LPs). This asks for studying the intrinsic difficulty of expressing optimization problems through a single LP, in terms of the minimum number of necessary constraints. In turn, this leads to a complexity measure that we call loosely here ‘polyhedral complexity’ (precise definitions are given later in Sect. 2).

On the one hand, there is an ever-expanding collection of examples of small size extended formulations. For instance, [23] has expressed the minimum spanning tree problem on a planar graph with only a linear number of (variables and) constraints, while in the natural edge variables the LP has an exponential number of constraints. There exist numerous other examples, see e.g., the surveys by [9] and [15].

On the other hand, a recent series of breakthroughs in lower bounds renewed interest for extended formulations [3,5,7,8,13,21,22]. These breakthroughs make it now conceivable to quantify the polyhedral complexity of any given combinatorial optimization problem unconditionally, that is, independently of conjectures such as P ≠ NP, and without extra assumption on the structure of the LP.

Although a polynomial upper bound on the polyhedral complexity yields a polynomial upper bound on the true algorithmic complexity of the problem e.g., through interior point methods—provided that the LP can be efficiently constructed—the converse does not hold in general, as the following recent examples show.

Chen et al. [8] proved that every LP for MAXCUT with approximation factor at most $2 - \varepsilon$ needs at least $n^\Omega(\frac{\log n}{\log \log n})$ constraints, while the approximation factor of the celebrated SDP-based polynomial time algorithm of [14] is close to 1.13. Rothvoß [22] showed a $2^{\Omega(n)}$ lower bound on the size of any LP expressing the perfect matching problem, despite having a polynomial time algorithm by [11]. Braun and Pokutta [4] show that the matching polytope does not admit any fully-polynomial size relaxation scheme (the polyhedral equivalent of an FPTAS).

In this paper, we consider the problem of determining the average case polyhedral complexity of the maximum stable set problem, in two different models: ‘uniform’ and ‘non-uniform’, see Sect. 1.1 below. Roughly, the uniform model asks for a single LP that works for a given set of input graphs. In the non-uniform model the LP can depend on the input graph $G$ but should work for every choice of weights on the vertices of $G$ (in particular, for all induced subgraphs of $G$).

We show that the polyhedral complexity of the maximum stable set problem remains high in each of these models, when the input graph is sampled according to natural distributions. Therefore, we conclude that the (polyhedral) hardness of the maximum stable set problem is not concentrated on a small mass of graphs but is spread out through all graphs.
1.1 Contribution

We present the first strong and unconditional results on the average case size of LP formulations for the maximum stable set problem. In particular, we establish that the maximum stable set problem in two natural average case models does not admit a polynomial size linear programming formulation, even in the unlikely case that \( P = NP \).

**Uniform model** In the *uniform model* the feasible solutions are independent of the instances. The instances will be solely encoded into the objective functions. This ensures that no complexity of the problem is leaked into an instance-specific formulation. A good example of a uniform model is the TSP polytope over \( K_n \) with which we can test for Hamiltonian cycles in any graph with \( n \) vertices by choosing an appropriate objective function.

In the case of the maximum stable set problem, we consider a random collection of input graphs \( G \), where each graph \( G \) with \( V(G) \subseteq [n] \) is included in the collection with probability \( p \geq 2^{-\left(\frac{n}{2}\right) + n} \) and show that with probability at least \( 1 - 2^{-2n} \), every LP for the maximum stable set problem on such collection of input graphs has at least \( 2^{\Omega(n/\log n)} \) inequality constraints.

**Non-uniform model** In the *non-uniform model* we consider the stable set problem for a *specific but random graph*. The polyhedral description may depend heavily on the chosen graph. We sample a graph \( G \) in the Erdős–Rényi \( G(n, p) \) model, i.e., \( G \) has \( n \) vertices, and every pair of vertices is independently connected by an edge with probability \( p \). We then analyze the stable set polytope \( \text{STAB}(G) \) of \( G \). If \( p \) is small enough, so that the obtained graph is sufficiently sparse, it will contain an induced subgraph allowing a polyhedral reduction from the correlation polytope, and of sufficient size. Via this reduction we derive strong lower bounds on the size of any LP expressing \( \text{STAB}(G) \) that hold with high probability. In particular, we obtain superpolynomial lower bounds for \( p \) ranging between \( \Omega \left( \frac{\log^{b+e} n}{n} \right) \) and \( o \left( \frac{1}{\log n} \right) \). For example for \( p = n^{-e} \) and \( e < 1/4 \), any LP has at least \( 2^{\Omega(\sqrt{n^{1+e} \log n})} \) constraints w.h.p. (with high probability), and for \( p = \Omega \left( \frac{\log^{b+e} n}{n} \right) \), any LP has at least \( n^{\Omega(\log^{e/5} n)} \) constraints w.h.p. Figure 1 illustrates our lower bounds. In the figure, \( \text{fc}(G(n, p)) \) denotes the formulation complexity of the stable set problem on \( G \sim G(n, p) \), which is the minimum number of constraints in an LP formulation of the problem, see below.

1.2 Outline

In Sect. 2 we recall basics on extended formulations. We introduce the model of general linear programming formulations in Sect. 2.1. We then establish bounds on the average case complexity for the uniform model of the maximum stable set problem in Sect. 3. In Sect. 4 we consider the non-uniform model and derive lower bounds as well as upper bounds. We conclude with a conjecture in Sect. 5.
2 Preliminaries

We use log to denote the base-2 logarithm. Let $G$ be a graph. We denote by $\alpha(G)$ the maximum size of a stable set in $G$. This is the stability number of $G$. If $S$ is any set, we let $G[S]$ denote the induced subgraph of $G$ on $V(G) \cap S$. For a positive integer $n$ we let $[n] := \{1, 2, \ldots, n\}$.

2.1 Linear programming formulations

The notion of linear programming formulation we use here generalizes both that appearing in the work of [8] on MAX CSPs and the notion of faithful linear encodings, their corresponding pairs of polyhedra, and extended formulations thereof, see [5]. Extended formulation for pairs of polyhedra first appeared in [18, Sect. 4.1] under the name extended relaxation. The motivation for the streamlined model is to give a direct natural approach to understanding polyhedral complexity, in particular in the context of approximate formulations. Our model is in line with [2, 6, 8, 17].

In a nutshell, the presented model focuses on the underlying combinatorial problem, making definitions intuitive, and avoids technicalities arising from the use of polyhedra, in particular, the question of the choice of the right polyhedra. This also eliminates the trap in the polyhedral world of using facets of the polyhedra not having any combinatorial meaning. We would like to stress that there is nothing inherent in the model, which cannot be done with extended formulation (with the minimal polyhedral pair, see Remark 2.4), i.e., the role of new model is just for streamlining the presentation.

As a particularly relevant example for this paper, the stable set polytope (with a suitable outer polyhedron) corresponds to the non-uniform version of the stable set problem. For the uniform version (see below) our model naturally provides a formulation, while there does not seem to be an immediate natural polyhedral formulation.

As a final remark, the model heavily depends on the original combinatorial formulation of the problem, e.g., on the choice of objective functions and feasible solutions,
which is one of the reasons, why the model is not in conflict with any complexity-theoretic assumptions; the other one being that the encoding length of the coefficients are not considered.

We start by defining optimization problems. For the sake of exposition, we restrict ourselves to maximization problems.

**Definition 2.1** (Maximization problem) A maximization problem $\Pi = (S, F, *)$ consists of a finite set $S$ of feasible solutions, a finite set $F$ of objective functions where $f: S \to \mathbb{R}$ for each $f \in F$, and an approximation guarantee $f^* \in \mathbb{R}$ so that $f^* \geq \max_{s \in S} f(s)$ for each objective function $f \in F$.

By solving such a maximization problem $\Pi = (S, F, *)$ we mean determining an approximation $\hat{f} \in \mathbb{R}$ of the optimum value $\max_{s \in S} f(s)$ satisfying

$$\max_{s \in S} f(s) \leq \hat{f} \leq f^*,$$

for each $f \in F$. Our next definition specifies what it means to formulate the maximization problem as a linear program. We exemplify the above definition with the maximum matching problem.

**Example 2.2** (The maximum weight matching problem) Let $G$ be a graph. Then the set of feasible solutions $S$ is given by all matchings in $G$ and the set of objective functions $F$ is given by all weight functions on the edges. For a matching $s$ and a weight function $f \in F$, we define $f(s)$ as the total weight of the edges in the matching. Finally, the guarantees $f^*$ are defined as $f^* := \max_{s \in S} f(s)$ for $f \in F$. We obtain the exact maximum weight matching problem, since the guarantees are chosen to be equal to the optimum value.

We will next specify what it means for a linear program to solve a maximization problem.

**Definition 2.3** (LP formulation of a maximization problem) A linear programming formulation of a maximization problem $\Pi = (S, F, *)$ is a linear system $\tilde{A}x \leq \tilde{b}$, $\bar{A}x = \bar{b}$ with $x \in \mathbb{R}^d$ together with realizations of:

(i) feasible solutions as points $x^s \in \mathbb{R}^d$ for each $s \in S$, such that $\tilde{A}x^s \leq \tilde{b}$, $\bar{A}x^s = \bar{b}$; (1)

(ii) objective functions as linear functions on $\mathbb{R}^d$, i.e., for each $f \in F$ there is $w^f \in \mathbb{R}^d$ with

$$(w^f, x^s) = f(s) \quad \text{for all } s \in S.$$ (2)

Conditions (1) and (2) imply that

$$\max_{s \in S} f(s) \leq \max \left\{ (w^f, x) \left| \tilde{A}x \leq \tilde{b}, \bar{A}x = \bar{b} \right. \right\} \quad \text{for all } f \in F.$$
We additionally require that

\[
\max \left\{ \langle w^f, x \rangle \mid \tilde{A}x \leq \tilde{b}, \tilde{A}x = \tilde{b} \right\} \leq f^* \quad \text{for all } f \in \mathcal{F},
\]

so that the optimum value of the LP provides an approximation of the optimum value \( \max_{s \in S} f(s) \). The size of the formulation is number of inequalities in the LP, that is, the number of constraints in \( \tilde{A}x \leq \tilde{b} \). We define the formulation complexity \( fc(\Pi) \) of the problem \( \Pi \) as the minimum size of all its LP formulations.

**Remark 2.4 (Relation to extended formulations)** LP formulations are essentially extended formulations of the minimal polyhedral encoding of the problem, formulated in a language without the overhead of polyhedral concepts. Luckily, in most cases actually the minimal polyhedral encoding is used, so the results in the two kinds of formulation are easily transferable.

First, let us turn extended formulations into LP formulations. If instead of the constraints \( \tilde{A}x \leq \tilde{b}, \tilde{A}x = \tilde{b} \) we used an extended formulation \( \tilde{E}x + \tilde{F}y \leq \tilde{g}, \tilde{E}x + \tilde{F}y = \tilde{g} \) to express the set over which we want to maximize \( \langle w^f, x \rangle \), where \( y \in \mathbb{R}^k \) is an extra (vector) variable, we could redefine the ambient space, the points \( x^s (s \in S) \) and the coefficient vectors \( w^f (f \in \mathcal{F}) \) to eliminate the extra variable \( y \). Indeed, we could work directly in \( \mathbb{R}^{d+k} \) instead of \( \mathbb{R}^d \), consider the vectors \( (w^f, 0) \) instead of \( w^f \) and the points \( (x^s, y^s) \) instead of \( x^s \), where \( y^s \in \mathbb{R}^k \) is chosen so that \( (x^s, y^s) \) satisfies the constraints of the extended formulation. (Such a point \( y^s \) exists because by Condition (1), \( x^s \) is feasible for the LP.)

In Braun et al. [5], we start from a specific encoding of the problem by points \( x^s (s \in S) \) and coefficient vectors \( w^f (f \in \mathcal{F}) \), then infer from this a pair \( (P, Q) \) of nested polyhedra where

\[
P := \text{conv}(x^s \mid s \in S),
\]

\[
Q := \left\{ x \in \mathbb{R}^d \mid \langle w^f, x \rangle \leq f^*, \forall f \in \mathcal{F} \right\},
\]

and finally consider any extended formulation of the pair \( (P, Q) \). There is a minimal linear encoding, over which any LP formulation arises as an extended formulation: let \( \mathbb{R}^\mathcal{F} \) be the ambient space, and \( w^f_{\min} \) be the projection to the \( f \)-coordinate for \( f \in \mathcal{F} \). For every \( s \in S \), let \( x^s_{\min} \in \mathbb{R}^\mathcal{F} \) be the vector with \( f \)-coordinate \( f(x^s_{\min}) \) for every \( f \in \mathcal{F} \). Define the polyhedral pair \( (P_{\min}, Q_{\min}) \) by (4), and this is the minimal encoding, minimizing the dimension of the problem. (One might further restrict to the linear space spanned by \( P_{\min} \), essentially factoring out uninteresting direction in \( \mathbb{R}^\mathcal{F} \), using linear dependence between objective functions.)

For any LP formulation of the problem given by linear system \( \tilde{A}x \leq \tilde{b}, \tilde{A}x = \tilde{b} \) with \( x \in \mathbb{R}^d \), feasible solutions \( x^s \), and objective functions \( w^f \), the linear map \( \pi : \mathbb{R}^d \to \mathbb{R}^\mathcal{F} \) with \( \pi(x)_f := \langle w^f, x \rangle \) and the polyhedron \( K := \{ x \in \mathbb{R}^d : \tilde{A}x \leq \tilde{b}, \tilde{A}x = \tilde{b} \} \) defined by the linear system provides an extended formulation of the same size.
2.2 Size lower bounds from nonnegative rank

The basis of most lower bounds for extended formulations is Yannakakis’s celebrated Factorization Theorem (see [25, 26]), equating the minimal size of an extended formulation with the nonnegative rank of a slack matrix. We will now derive a factorization theorem in a similar spirit that characterizes formulation complexity, however without requiring an initial polyhedral representation but rather directly operating on the slack matrix of a problem.

A rank-\(r\) nonnegative factorization of \(M \in \mathbb{R}^{m \times n}_+\) is a factorization of \(M = TU\) where \(T \in \mathbb{R}^{m \times r}_+\) and \(U \in \mathbb{R}^{r \times n}_+\). This is equivalent to \(M = \sum_{i \in [r]} u_i v_i^T\) for some (column vectors) \(u_i \in \mathbb{R}_+^m\), \(v_i \in \mathbb{R}_+^n\) with \(i \in [r]\). The nonnegative rank of \(M\), denoted by \(\text{rank}_+ M\), is the minimum \(r\) such that there exists a rank-\(r\) nonnegative factorization of \(M\).

We will use the following elementary properties of nonnegative rank.

**Lemma 2.5** Let \(A \in \mathbb{R}^{k \times m}_+, M \in \mathbb{R}^{m \times n}_+\) and \(B \in \mathbb{R}^{n \times l}_+\) be nonnegative matrices. Then \(\text{rank}_+(AMB) \leq \text{rank}_+ M\). In particular, the following operations do not increase the nonnegative rank of a matrix:

(i) deleting, duplicating, permuting rows or columns;
(ii) adding a nonnegative linear combination of rows as a new row;
(iii) adding a nonnegative linear combination of columns as a new column.

By considering the nonnegative factorizations \(M = I_m M\) and \(M = M I_n\), we immediately obtain

\[
\text{rank}_+ M \leq \min\{m, n\}. \quad (5)
\]

In analogy to extended formulations, central for understanding the size of a linear program will be the concept of the slack matrix:

**Definition 2.6** (Slack matrix) Let \(\Pi = (S, F, *)\) be a maximization problem as in Definition 2.1. The slack matrix of \(\Pi\) is the nonnegative \(F \times S\) matrix \(M\), with entries

\[
M(f, s) := f^* - f(s).
\]

We are ready to formulate the factorization theorem for formulation complexity. We provide a proof for the sake of completeness, even though in the equivalent formulation for polyhedral pairs, it already appeared in [18].

**Theorem 2.7** (Factorization theorem for formulation complexity) (C.f., [18, Lemmas 4.1 and 4.2]) Consider a maximization problem \(\Pi = (S, F, *)\) as in Definition 2.1 with slack matrix \(M\). Then

\[
\text{rank}_+ M - 1 \leq \text{fc}(\Pi) \leq \text{rank}_+ M. \quad (6)
\]

**Proof** To prove the first inequality, let \(\tilde{A}x \leq \tilde{b}\), \(\tilde{A}x = \tilde{b}\) be an arbitrary size-\(r\) LP formulation of \(\Pi\), with realizations \(w^f (f \in F)\) of objective functions and \(x^s (s \in S)\).
of feasible solutions. We shall construct a size-\((r + 1)\) nonnegative factorization of \(M\). As \(\max_{x : Ax \leq b} \langle w^f, x \rangle \leq f^*\) by Condition (3), via Farkas’s lemma, we have

\[
f^* - \langle w^f, x \rangle = T(f, 0) + \sum_{j=1}^{r} T(f, j) \left( \bar{b}_j - \bar{A}_j x \right) + \sum_{j=1}^{k} \lambda(f, j) (\bar{b}_j - \bar{A}_j x)
\]

for some nonnegative multipliers \(T(f, j) \in \mathbb{R}_+\) with \(0 \leq j \leq r\), and arbitrary multipliers \(\lambda(f, j)\) for \(0 \leq j \leq k\). By taking \(x = x^s\), and by \(\bar{A} x^s = \bar{b}\), we obtain

\[
M(f, s) = \sum_{j=0}^{r} T(f, j) U(j, s), \quad \text{with} \quad U(j, s) := \begin{cases} 1 & \text{for } j = 0, \\ \bar{b}_j - \bar{A}_j x^s & \text{for } j > 0. \end{cases}
\]

That is, \(M = TU\). By construction, \(T\) is nonnegative. By Condition (1) we also obtain that \(U\) is nonnegative. Therefore \(M = TU\) is a rank-\((r + 1)\) nonnegative factorization of \(M\).

For the second inequality, let \(M = TU\) be a size-\(r\) nonnegative factorization. We shall construct an LP formulation of size \(r\). Let \(T^f\) denote the \(f\)-row of \(T\) for \(f \in F\), and \(U_s\) denote the \(s\)-column of \(U\) for \(s \in S\). Then

\[
T^f U_s = M(f, s) = f^* - f(s).
\]

In the following we represent the vectors \(y \in \mathbb{R}^{r+1}\) via \(y = (x, \alpha)\) with \(x \in \mathbb{R}^r\) and \(\alpha \in \mathbb{R}\). We claim that the linear system

\[
x \leq 0, \; \alpha = 1
\]

with representations

\[
w^f := (T^f, f^*) \; \forall f \in F \; \text{and} \; x^s := (-U_s, 1) \; \forall s \in S
\]

satisfies the requirements of Definition 2.3. Condition (2) clearly follows from (6):

\[
\langle w^f, x^s \rangle = -T^f U_s + f^* = f(s).
\]

Moreover, the \(x^s\) satisfy the linear program (7), because \(U\) is nonnegative, so that Condition (1) is fulfilled. Finally, Condition (3) also follows readily:

\[
\max \left\{ \langle w^f, (x, \alpha) \rangle \mid x \leq 0, \alpha = 1 \right\} = \max \left\{ T^f x + f^* \mid x \leq 0 \right\} = f^*,
\]

as the nonnegativity of \(T\) implies \(T^f x \leq 0\); equality holds e.g., for \(x = 0\). Thus we have constructed an LP formulation with \(r\) inequalities, as claimed. \(\square\)
2.3 Maximum stable set problems

Now we describe two ways in which the maximum stable set problem can be seen as
a maximization problem $\Pi = (S, F, \ast)$ that we use in the rest of the paper.

**Definition 2.8** (The maximum stable set problem–uniform model) We start with a
family of graphs $\mathcal{G}$ with $V(G) \subseteq [n]$ for each $G \in \mathcal{G}$. The goal is to approximate the
stability number of the graphs in $\mathcal{G}$ within a given relative error guarantee $\rho \geq 0$. To
each graph $G \in \mathcal{G}$ we correspond an objective function $f_G$ in the set $F$ of objective
functions. The set $S$ of feasible solutions is taken to be the set of all
subsets of $[n]$. This is natural since, typically, $G$ contains many graphs with many different vertex
sets and many $S \subseteq [n]$ occur as a stable set in some $G \in \mathcal{G}$. We require
(a) $\alpha(G) = \max_{S \in S} f_G(S)$, for all $G \in \mathcal{G}$;
(b) $f_G(S) = |V(G) \cap S|$ whenever $|V(G) \cap S| \leq 1$, for all $G \in \mathcal{G}$ and $S \in S$.

We choose as approximation guarantee $f^*_G := (1 + \rho)\alpha(G)$ for $G \in \mathcal{G}$. This
defines a class of maximization problems that we denote by $\text{STAB}^u(\mathcal{G}, \rho)$: every
set $\mathcal{F}$ of objective functions subject to the above conditions gives a valid approx-
imate computation of the stability number in a graph $G$ chosen from $\mathcal{G}$. The later
derived lower bounds apply to every problem $\Pi$ from this class and we define $fc(\text{STAB}^u(\mathcal{G}, \rho)) := \min \{fc(\Pi) \mid \Pi \in \text{STAB}^u(\mathcal{G}, \rho)\}$. In the exact case, that is, when
$\rho = 0$, we use $\text{STAB}^u(\mathcal{G})$ to mean $\text{STAB}^u(\mathcal{G}, 0)$.

As an example, a concrete maximization problem $\Pi \in \text{STAB}^u(\mathcal{G}, \rho)$ is obtained
by defining the objective functions as

$$f_G(S) := |V(G) \cap S| - |E(G[S])|.$$

Here $f_G(S)$ is a conservative lower bound on the size of a stable set, which one would
naturally obtain from $G[S]$ by deleting one endpoint from every edge in that induced
subgraph. Moreover we have $f_G(S) = |S|$ for stable sets $S$ of $G$, i.e., in this case our
choice is exact. We leave it to the reader to check Condition (a) while Condition (b)
is immediate.

**Remark 2.9** (Correlation polytope as a specific uniform encoding) In the uniform
model, the feasible solutions are all vertex sets to include every possible stable sets.
Therefore, taking into account the form of objective functions, in the polytope world
the uniform model corresponds to a polyhedral pair where the inner polytope is the
 correlation polytope. Recall that the stable set polytope deals with stable sets of a fixed
graph, and therefore corresponds to the non-uniform model defined below.

**Definition 2.10** (The maximum stable set problem–non-uniform model) To each fixed
graph $G$ on $n$ vertices we associate a problem that corresponds to the exact computation
of the stability number of some induced subgraph $H$ of $G$. We will denote this problem
by $\text{STAB}^{nu}(G)$ or (later) simply $\text{STAB}(G)$. More precisely, $\text{STAB}^{nu}(G)$ has feasible
solutions that are sets $S \subseteq V(G)$ which are stable sets of $G$, and objective functions
of the form $f_H(S) := |S \cap V(H)|$ where $H$ is an induced subgraph of $G$. We let
$f^*_H := \alpha(H)$ for the approximation guarantee.
Remark 2.11 (Stable set polytope as a specific non-uniform encoding) Recall that the stable set polytope of $G$ is the convex hull of (a specific encoding of) all stable sets of $G$, i.e., the feasible solutions of the non-uniform problem. The objective functions are linear over the stable set polytope, but need not include all facet-defining linear functions, hence the non-uniform problem corresponds to a polyhedral pair where the inner polytope is the stable set polytope.

In particular, the abuse of notation by using STAB($G$) to denote both the stable set polytope of $G$ and the problem STAB$^{nu}$($G$) is not too severe.

2.4 Unique disjointness

As we will demonstrate, the polyhedral hardness of the maximum stable set problem arises from the unique disjointness (partial) matrix. Recall that the unique disjointness (UDISJ) matrix, which we denote by UDISJ($n$) below, has $2^n$ rows and $2^n$ columns indexed by all size-$n$ 0/1-vectors $a$ and $b$. Its entries are:

$$
	ext{UDISJ}(n)(a, b) = \begin{cases} 
0 & \text{if } a^T b = 1 \\
1 + \rho & \text{if } a^T b = 0.
\end{cases}
$$

(8)

Although UDISJ($n$) is only a partial matrix, i.e., not all of its entries are defined, we will refer to it as a matrix from here on. The fact that it is only partial does not matter for our purpose, as we only care for whether this (partial) matrix occurs as a submatrix of some appropriate slack matrices. The UDISJ matrix has been studied in many disciplines, arguably the most notable being communication complexity.

Theorem 2.12 \(\text{rank}_+ \text{UDISJ}(n) \geq 2^n \cdot \log(3/2)\).

The factor $\log(3/2) \approx 0.585$ in the exponent is the current best one due to [16]; for various approximate case versions see [3,5,7]. The first exponential lower bound was established in [12,13] by combining the seminal work of [20] together with an observation in [24]. This was at the core of the first results establishing high extension complexity for the correlation polytope, cut polytope, stable set polytope, and the TSP polytope in [12,13].

Braun et al. [5] prove that any $2^n \times 2^n$ matrix $M$ with rows and columns indexed by vectors in $\{0, 1\}^n$ satisfying (8) has superpolynomial nonnegative rank, and that this remains true even if we shift the entries of the matrix $M$ by some number $\rho = O(n^{1/2-\epsilon})$. This result was then extended to shifts $\rho = O(n^{1-\epsilon})$ in [7] which then immediately leads to a polyhedral inapproximability of CLIQUE (in the uniform model!) of $O(n^{1-\epsilon})$, matching Håstad’s hardness result for approximating CLIQUE.

The $\rho$-shifted UDISJ matrix is any $2^n \times 2^n$ matrix indexed by pairs $(a, b)$ where $a, b \in \{0, 1\}^n$ such that

$$(\text{UDISJ}(n) + \rho J)(a, b) = \begin{cases} 
\rho & \text{if } a^T b = 1 \\
1 + \rho & \text{if } a^T b = 0.
\end{cases}$$

where $J$ is the all-one matrix of compatible size.
In Braun and Pokutta [3] an information-theoretic approach for studying the nonnegative rank has been developed. This approach allows to lower bound the nonnegative rank of various ‘deformations’ of the UDISJ matrix. The following theorem from [3] will allow us to analyze a specific type of deformation that we will use in the following. Informally speaking, the theorem shows that the UDISJ matrix has high nonnegative rank almost everywhere. Below, we use UDISJ(n, k) to denote the UDISJ matrix UDISJ(n) restricted to subsets of size k.

**Theorem 2.13** Let M be any submatrix of the ρ-shifted UDISJ matrix UDISJ(n, k) + ρJ obtained by deleting at most an α-fraction of rows and at most a β-fraction of columns for some 0 ≤ α, β < 1. Then for 0 < ε < 1:

\[
\text{rank}_+ M \geq 2^{1/8(\rho+1)-(\alpha+\beta)\mathbb{H}[1/4])n-\Theta(n^{1-\varepsilon}) \quad \text{for} \quad k = \Omega(n/\log n).
\]

Here \(\mathbb{H}[\cdot]\) is the binary entropy function, in particular \(\mathbb{H}[1/4] \approx 0.811\).

We finish this section with an easy example of embedding UDISJ(n) into a slack matrix.

**Example 2.14** Let us consider the family \(G\) of all (non-empty) complete graphs, that is, all graphs \(G\) with \(V(G) \subseteq [n]\) and \(\alpha(G) = 1\) (in case \(V(G) = \emptyset\) we have \(\alpha(G) = 0\)). Then the slack matrix \(M\) of any \(\Pi \in \text{STAB}^u(G)\) contains UDISJ(n) as a submatrix (without the row of the empty set \(a = \emptyset\)). To verify this, note that by Condition (b) for every pair of subsets \(a, b \subseteq [n]\) (with \(a \neq \emptyset\)):

\[M(K_n[a], b) = 1 - |a \cap b| = \begin{cases} 0 & \text{if } |a \cap b| = 1 \\ 1 & \text{if } |a \cap b| = 0, \end{cases}\]

where \(K_n\) denotes the complete graph on \([n]\).

### 3 Average case complexity in the uniform model

We will now establish our main result regarding the average case complexity of the uniform model. We obtain that for any random collection of graphs \(\mathcal{G} = \mathcal{G}(n, p)\) where each graph \(G\) with \(V(G) \subseteq [n]\) is picked independently with probability \(p\), the formulation complexity of STAB\(^u\)(\(\mathcal{G}\)) is high. Loosely speaking, the size of any “simultaneous” LP formulation of the maximum stable set problem for all graphs in this random collection \(\mathcal{G}\) is high. In a way, this indicates that the instances of the stable set problem resulting in high polyhedral complexity are not localized in a set of small density.

**Main Theorem 3.1** (Super-polynomial fc of STAB\(^u\)(\(\mathcal{G}\)) w.h.p) Let \(n \geq 40\) and \(p \in [0, 1]\) with \(p \geq 2^{-\left(\frac{n}{2}\right)^2 + n}\). Pick a random family \(\mathcal{G} = \mathcal{G}(n, p)\) of graphs by adding each graph \(G\) with \(V(G) \subseteq [n]\) to the family with probability \(p\), independent of the other \(G\). Then

\[
\mathbb{P}\left[\text{fc(STAB}^u(\mathcal{G})) \geq 2^{\Omega(n/\log n)}\right] \geq 1 - 2^{-2n}.
\]
A crucial point of the proof is a concentration result on \( \alpha(G) \). It is well-known that almost all graphs \( G \) on \( n \) vertices have stability number \( \alpha(G) \approx 2 \log n \). However, the following rough estimate will be sufficient for our purpose, see e.g. [10, Proposition 11.3.4, page 304] for a proof.

**Lemma 3.2** Let \( n \geq 10 \). The probability that a uniformly sampled random graph \( G \) with \( V(G) = [n] \) has \( \alpha(G) \geq 3 \log n \) is at most \( n^{-1} \).

We are ready to prove the main theorem of this section.

**Proof of Main Theorem 3.1** Consider any problem \( \Pi \in \text{STAB}^u(\mathcal{G}) \). Thus, \( \Pi = (\mathcal{S}, \mathcal{F}, *) \) is a maximization problem that corresponds to determining the stability number of the graphs of \( \mathcal{G} \). Recall that \( \mathcal{S} \) is the set of all subsets of \( S \subseteq [n] \) and that \( \mathcal{F} \) contains an objective function \( f_G \) for each graph \( G \in \mathcal{G} \). These functions satisfy \( \max_{S \in \mathcal{S}} f_G(S) = \alpha(G) \) (see Condition (a)). Also, recall that we require \( f_G(S) = |V(G) \cap S| \) whenever \( |V(G) \cap S| \leq 1 \) (see Condition (b)). Finally, since \( \rho = 0 \), the approximation guarantee is \( f_G^* = \alpha(G) \).

Consider the slack matrix \( M \) of \( \Pi \). We want to show that the nonnegative rank of \( M \) is high by embedding a large portion of \( \text{UDISJ}(n) \) into it. By Theorem 2.13, this will imply that \( \text{fc}(\Pi) \) and thus \( \text{fc}(\text{STAB}^u(\mathcal{G})) \) is high, since \( \Pi \in \text{STAB}^u(\mathcal{G}) \) is arbitrary.

The main idea of the proof is that, with extremely large probability, among all sets \( a \subseteq [n] \) of size \( \lfloor n/4 \rfloor \), the collection \( \mathcal{G} \) contains many graphs \( G \) with \( V(G) = a \). For each of these sets \( a \), there will be at least one corresponding graph \( G_a \) with \( \alpha(G_a) \leq 3 \log n \). Restricting to these graphs \( G_a \), the resulting slack matrix contains a large part of the \( O(\log n) \)-shifted UDISJ matrix as a submatrix (in fact, a large fraction of the rows, and all the columns, survive). We apply Theorem 2.13 to conclude.

Now let us turn to the detailed proof. Consider a set \( a \) with \( a \subseteq [n] \) and size \( k := \lfloor n/4 \rfloor \). We say that a graph \( G \) is good for \( a \) if \( V(G) = a \) and \( \alpha(G) \leq 3 \log n \). Set \( a \) is said to be good if some graph \( G \in \mathcal{G} \) is good for \( a \). Otherwise, \( a \) is called bad.

We claim that, with high probability, the total fraction of bad sets among all \( k \)-sets \( a \) is at most \( \alpha := 1/(24 \log n) \). By Lemma 3.2, the total number of graphs \( G \) with \( V(G) = a \) that are not good for a fixed \( k \)-set \( a \) is at most \( k^{-1/2} \). Thus

\[
\mathbb{P}[a \text{ is bad}] = \mathbb{P}[\mathcal{G} \text{ contains no good graph for } a] \\
\leq (1 - p)^{\left(1 - \frac{1}{2}\right)2} \\
\leq e^{-p\left(1 - \frac{1}{2}\right)2} \\
\leq 2^{-\frac{9}{1}2^{k/4}} \\
\leq 2^{-\alpha 2^{-2^k}}.
\]

where the third inequality follows from \( k \geq n/4 \geq 10 \) and \( p \geq 2^{-\left(\left(\frac{n}{4}\right) + n \right)} \) and the last inequality follows from our choice of \( \alpha \). Let \( X \) denote the random variable with value the number of bad \( k \)-sets \( a \). By Markov’s inequality,

\[
\mathbb{P}[X \geq \alpha \binom{n}{k}] \leq 2^{-2^n}.
\]

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For each good \( k \)-set \( a \), pick a good graph \( G_a \in \mathcal{G} \) arbitrarily, i.e., with \( \alpha(G_a) \leq 3 \log n \). We need two auxiliary nonnegative vectors. Let \( \mathbf{1} \) denote the row vector with all entries 1, and entries indexed by subsets \( S \subseteq [n] \). Let \( \mathbf{u} \) be the vector with entries indexed by the \( G_a \) for all good \( k \)-sets \( a \), and with entries

\[
u_{G_a} := 3 \log n - \alpha(G_a) \geq 0.
\]

Because the slack matrix \( M \) of \( \Pi \) satisfies

\[
M(G_a, S) = \begin{cases} 
\alpha(G_a) - 1 & \text{if } |V(G) \cap S| = 1, \\
\alpha(G_a) & \text{if } |V(G) \cap S| = 0,
\end{cases}
\]

we obtain after applying the rank-1 shift \( \mathbf{u} \mathbf{1} \)

\[
(M + \mathbf{u} \mathbf{1})(G_a, S) = \begin{cases} 
3 \log n - 1 & \text{if } |V(G) \cap S| = 1, \\
3 \log n & \text{if } |V(G) \cap S| = 0.
\end{cases}
\]

By the above, with probability at least \( 1 - 2^{-2n} \), the fraction of bad \( k \)-sets \( a \subseteq [n] \) among all \( k \)-sets is at most \( \alpha \), and hence the matrix \( (M + \mathbf{u} \mathbf{1}) \) contains a \( (3 \log n - 1) \)-shift of UDISHJ(\( n, k \)), with at most an \( \alpha \)-fraction of the rows thrown away. From Theorem 2.13 (with \( \beta = 0 \)), the nonnegative rank of \( M \) is at least

\[
\text{rank}_+ M \geq \text{rank}_+(M + \mathbf{u} \mathbf{1}) - 1 \geq 2^{(1/8(3 \log n + 1) - \alpha \log [1/4] \log n - O(n^{1-\epsilon} + \alpha \log n)) - 1} = 2^{\Omega(n/\log n)}.
\]

Without much additional work, we can obtain a similar lower bound on the average case formulation complexity also in the approximate case, that is, when \( \rho > 0 \).

**Corollary 3.3** (Super-polynomial xc of STAB\( ^u \)(\( \mathcal{G}, \rho \)) w.h.p.) As in Main Theorem 3.1, let \( \mathcal{G} = \mathcal{G}(n, \rho) \) be a random family of graphs such that each graph \( G \) with \( V(G) \subseteq [n] \) is contained in \( \mathcal{G} \) with probability \( p \geq 2^{-n^{1/4} + n} \) independent of the other graphs. Then for all \( 0 < \epsilon < 1/2 \) and \( \rho \leq n^{1-\epsilon} / \log n \), we have that STAB\( ^u \)(\( \mathcal{G}, \rho \)) has formulation complexity \( 2^{\Omega(n^{\epsilon})} \), with probability at least \( 1 - 2^{-2n} \).

**Proof** The proof is identical to Theorem 3.1 subject to minor changes. First, now we use a different factor \( \alpha := 1/[24(1 + \rho) \log n] \). The computation in Eq. (9) still remains valid, as the different choice of \( \alpha \) affects only the last inequality, which remains true, because \( \rho \leq n^{1-\epsilon} / \log n \). Therefore again with probability \( 1 - 2^{-2n} \), with the exception of at most an \( 1/[24(1 + \rho) \log n] \)-fraction, all graphs \( G \in \mathcal{G} \) are good, i.e., have clique number at most \( \alpha(G) \leq 3 \log n \).

The second difference is that due to dilation, the slack entries of \( M \) are a bit different:

\[
M(G, S) = \begin{cases} 
(1 + \rho)\alpha(G) - 1 & \text{if } |V(G) \cap S| = 1, \\
(1 + \rho)\alpha(G) & \text{if } |V(G) \cap S| = 0.
\end{cases}
\]
This provides an embedded copy of a $(3(1 + \rho) \log n - 1)$-shift of \textsc{UDISJ}(n, k) in $M + (1 + \rho)n\mathbf{1}$ with at most an $\alpha$-fraction of rows missing. Hence Theorem 2.13 applies again, but now we replace the $\epsilon$ there with a $\epsilon'$ lying strictly between $\epsilon$ and 1/2, but not depending on $n$. (E.g., $\epsilon' := (\epsilon + 1/2)/2$ is a good choice.) This will make the error term $O(n^{1-\epsilon'}) = o(n^{1-\epsilon})$ in the exponent negligible. We obtain the lower bound: $2^{(1/8(3(1+\rho)\log n+1)-\alpha H[1/4])n-O(n^{1-\epsilon'})} - 1 = 2^{\Omega(n^{1-\epsilon})}$ on formulation complexity, as claimed. \hfill \Box

Observe that the relative approximation guarantee $\rho$ in Corollary 3.3 can be larger than $3 \log n$. The reason why this is possible, contradicting initial intuition, is that the hardness arises from having many different graphs and hence many objective functions to consider simultaneously and the encoding is highly non-monotone. Roughly speaking, graphs with different vertex sets are independent of each other, even if one is an induced subgraph of the other.

4 Average case complexity in the non-uniform model

We now turn our attention to the non-uniform problem \textsc{STAB}$_{nu}$($G$), see Definition 2.10. Thus $G$ is a fixed graph with vertex set $[n]$, the feasible solutions are the stable sets of $G$ and the objective functions correspond to induced subgraphs of $G$. For simplicity of notation, we index the objective functions with the supporting vertex set $a \subseteq [n]$ instead of the induced subgraph $G[a]$. Thus $f_a(S) = |S \cap a|$ for every stable set $S$ of $G$ and $a \subseteq [n]$, and $f^*_n = \alpha(G[a])$.

Notice that an LP formulation for the problem \textsc{STAB}$_{nu}$($G$) is provided by the linear description of the stable set polytope of $G$, or any extended formulation of the stable polytope of $G$. In this sense, \textsc{STAB}$_{nu}$($G$) generalizes the stable set polytope. For the sake of brevity, we denote the problem \textsc{STAB}$_{nu}$($G$) simply by \textsc{STAB}($G$).

We lower bound the formulation complexity of \textsc{STAB}(G(n, p)) for the random Erdős–Rényi graph $G(n, p)$. Our strategy is to embed certain subdivisions of the complete graph $K_t$ as induced subgraphs of $G$, with $t$ as large as possible, using the probabilistic method.

Our construction is parametrized by an even integer $\ell \geq 0$. For a graph $T$, we let $T^\sim$ denote the subdivision of $T$ obtained by replacing each edge $ij$ of $T$ with a path $P_{ij}$ with $2\ell + 3$ edges between $i$ and $j$. We denote $u_{ij}$ and $v_{ij}$ the middle vertices of $P_{ij}$, see Fig. 2. In total, $T^\sim$ has $v := |V(T)| + (2\ell + 2)|E(T)|$ vertices and $e := (2\ell + 3)|E(T)|$ edges.

Our next lemma proves that increasing the parameter $\ell$ decreases the average degree of induced subgraphs of the gadget graph $T^\sim$, which makes it easier to embed $T^\sim$ in $G(n, p)$ for lower values of $p$.

![Fig. 2](image.png) Path $P_{ij}$ replacing edge $ij$ of $T$ in $T^\sim$. There are $\ell + 1$ edges between $i$ and $u_{ij}$, as between $v_{ij}$ and $j$. In the figure, $\ell = 0$
Lemma 4.1 For any graph $T$, the average degree of any induced subgraph of $T^\sim$ is at most $2 + 1/(\ell + 1)$. For $\ell = 0$, the average degree is at most 3.

Proof Consider an induced subgraph of $T^\sim$ with maximum average degree. Clearly, we may assume that $T$ contains at least one edge, so the average degree of the induced subgraph is at least $3/2$. This implies that it does not contain any vertex of degree at most 1, because the deletion of any such vertex would increase the average degree. It follows that the induced subgraph is of the form $H^\sim$ where $H$ is a subgraph of $T$. The average degree of $H^\sim$ can be expressed in terms of that of $H$ as:

$$
\bar{d}(H^\sim) = \frac{2|E(H^\sim)|}{|V(H^\sim)|} = \frac{2(2\ell + 3)|E(H)|}{|V(H)| + (2\ell + 2)|E(H)|} = \frac{(2\ell + 3)\bar{d}(H)}{1 + (\ell + 1)\bar{d}(H)}.
$$

From the last expression we see that the average degree of $H^\sim$ is an increasing function of the average degree of $H$ that tends to $(2\ell + 3)/(\ell + 1) = 2 + 1/(\ell + 1)$ in the limit.

Lemma 4.2 If graph $G$ contains $K_t^\sim$ as an induced subgraph, then

$$
fc(STAB(G)) \geq \text{rank}_+ \text{UDISJ}(t) - 2 \geq 2^t\log(3/2) - 2.
$$

Proof Let $T$ be any graph with $T^\sim$ being an induced subgraph of $G$. Later we will specialize to $T = K_t$. We choose representatives of subsets of $V(T)$ as stable sets and induced subgraphs of $T^\sim$.

For every $b \subseteq V(T)$, we choose an extension to a stable set $S(b)$ of $T^\sim$ by adding as much internal vertices of each path $P_{ij}$ as possible, see Fig. 3. Let $ij \in E(T)$. On the part of $P_{ij}$ from $i$ to $u_{ij}$, as well on the part from $j$ to $v_{ij}$, we alternate between vertices belonging to $T^\sim$ and not belonging to $T^\sim$, with one exception: if both $i, j \in b$ then we drop either $u_{ij}$ or $v_{ij}$ in $S(b)$.

If we start from a maximum stable set $b$ of $T$, we see that $S(b)$ has $|b| + (\ell + 1)|E(T)|$ vertices. Thus $\alpha(T^\sim) \geq \alpha(T) + (\ell + 1)|E(T)|$. This inequality is tight because no stable set $S$ of $T^\sim$ can have more vertices than $S(b)$, where $b := S \cap V(T)$. That is, $|S| \leq |b| + (\ell + 1)|E(T)| - |E(T[b])| \leq \alpha(T) + (\ell + 1)|E(T)|$. For any graph $T$, we get:

$$
\alpha(T^\sim) = \alpha(T) + (\ell + 1)|E(T)|.
$$

For every subset $a \subseteq V(T)$, consider the induced subgraph $T[a]^\sim$ of $T^\sim$. By (10), we have $\alpha(T[a]^\sim) = \alpha(T[a]) + (\ell + 1)|E(T[a])|$. We also consider the induced subgraph $T[a]^\circ$ on all the $u_{ij}$ and $v_{ij}$ in $T[a]^\sim$. This is a matching, so obviously $\alpha(T[a]^\circ) = |E(T[a])|$.

By construction, for all sets $a, b \subseteq V(T)$ we have:

$$
|V(T[a]^\sim) \cap S(b)| = |S \cap a| + (\ell + 1)|E(T[a])| - |E(T[a \cap b])|,
|V(T[a]^\circ) \cap S(b)| = |E(T[a])| - |E(T[a \cap b])|.
$$
Fig. 3 Possible stable sets $S(b)$ extending a given $b \subseteq V(T)$. Black vertices are those which are part of $S(b)$.

From the slack matrix $M$ of $\text{STAB}(G)$, we construct a matrix $N$ with rows and columns indexed by all subsets $a, b$ of $V(T)$ with $a \neq \emptyset$, with entries

$$N(a, b) := M(T[a]^\sim, S(b)) + M(T[a]^\circ, S(b)) = \alpha(T[a]^\sim) - |V(T[a]^\sim) \cap S(b)| + \alpha(T[a]^\circ) - |V(T[a]^\circ) \cap S(b)|$$

$$= \alpha(T[a]) + 2 |E(T[a \cap b])| - |a \cap b|.$$

Specializing to $T = K_t$, we obtain for $a \neq \emptyset$:

$$N(a, b) = 1 + 2 \left( \frac{|a \cap b|}{2} \right) - |a \cap b| = (1 - |a \cap b|)^2 = \begin{cases} 1 & \text{if } |a \cap b| = 0, \\ 0 & \text{if } |a \cap b| = 1. \end{cases}$$

Thus, $N$ contains $\text{UDISJ}(t)$ as a submatrix without the row of the empty set. Now Theorem 2.7 followed by Lemma 2.5 implies

$$\text{fc}(\text{STAB}(G)) \geq \text{rank}_+ M - 1 \geq \text{rank}_+ N - 1 \geq \text{rank}_+ \text{UDISJ}(t) - 2 \geq 2^{t \log(3/2)} - 2.$$ 

\[ \square \]

4.1 Existence of gadgets in random graphs

In this section, we estimate the probability that a random Erdős–Rényi graph $G = G(n, p)$ contains an induced copy of a graph $H$. Recall that in the $G(n, p)$ model, each of the $\binom{n}{2}$ pairs of vertices is connected by an edge with probability $p$, independently from the other edges. The next lemma is key for proving lower bounds on the formulation complexity of $\text{STAB}(G(n, p))$ via embedding $H = T^\sim$ as an induced subgraph. The lemma is formulated in a general for future applications to many types of subgraphs $H$. 

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Lemma 4.3 Let \( H \) be a graph with \( v \) vertices and with all induced subgraphs having average degree at most \( d \). Let \( 0 < p \leq 1/2 \) and

\[
g = g(n, p, v) := \frac{v^2 p^{-d} (1 - p)^{-\frac{v}{2}}}{n - v}.
\]

The probability of \( G(n, p) \) not containing an induced copy of \( H \) satisfies

\[
P\left[ H \text{ ind} \not\subseteq G(n, p) \right] \leq c_0 g^2 \approx 1.23 g^2,
\]

where \( c_0 := \exp(2W(1/\sqrt{2}))/2 \) and \( W \) is the Lambert W-function, the inverse of \( x \rightarrow x \exp x \).

Proof The proof is via the second-moment method.

Let \( S \) be any graph isomorphic to \( H \) with \( V(S) \subseteq V(G) \). Let \( X_S \) be the indicator random variable of \( S \) being an induced subgraph of \( G \). Obviously, the total number \( X \) of induced subgraphs of \( G \) isomorphic to \( H \) satisfies \( X = \sum_S X_S \). We estimate the expectation and variance of \( X \). Let \( e \) denote the number of edges of \( H \), and let \( \text{Aut}(H) \) denote the automorphism group of \( H \). The expectation is clearly

\[
\mathbb{E}[X] = \sum_S \mathbb{E}[X_S] = \binom{n}{v}^\frac{v!}{|\text{Aut}(H)|} p^e (1 - p)^{(\frac{v}{2}) - e}.
\]

The variance needs more preparations. Let now \( S \) and \( T \) be two graphs isomorphic to \( H \) with \( V(S), V(T) \subseteq V(G) \). Using that \( X_S \) and \( X_T \) are independent and thus \( \text{Cov}(X_S, X_T) = 0 \) when \( |V(S) \cap V(T)| \leq 1 \) we get

\[
\text{Var}[X] = \sum_{S, T} \text{Cov}[X_S, X_T] \leq \sum_{|V(S) \cap V(T)| \geq 2} \mathbb{E}[X_S X_T]
= \sum_{|V(S) \cap V(T)| \geq 2} \mathbb{E}[X_S] \mathbb{E}[X_T \mid X_S = 1]
= \mathbb{E}[X] \sum_{T : |V(S) \cap V(T)| \geq 2} \mathbb{E}[X_T \mid X_S = 1].
\]

Note that in the last sum \( S \) is fixed, and by symmetry, the sum is independent of the actual value of \( S \). That is why we could factor it out. We obtain via Chebyshev’s inequality,

\[
P\left[ H \text{ ind} \not\subseteq G(n, p) \right] = P[X = 0] \leq \frac{\text{Var}[X]}{\mathbb{E}[X]^2}
\leq \sum_{T : |V(S) \cap V(T)| \geq 2} \frac{\mathbb{E}[X_T \mid X_S = 1]}{\mathbb{E}[X]}.
\]
We shall estimate $\mathbb{E}[X_T | X_S = 1]$, which is the probability that $H$ is induced in $G$ provided $S$ is induced in $G$, as a function of $k := |V(S) \cap V(T)|$. We assume that $S$ and $T$ coincide on $V(S) \cap V(T)$, and therefore have at most $dk/2$ edges in common, as their intersection is isomorphic to an induced subgraph of $H$, and therefore have average degree at most $d$ by assumption. Hence as $p \leq 1/2$

$$\mathbb{E}[X_T | X_S = 1] = \mathbb{P}\left[ T \subseteq G \mid S \subseteq G \right] \leq p^{e-\frac{d}{2}k}(1-p)^{\frac{d+1}{2}k}.$$  

This is clearly also true if $S$ and $T$ do not coincide on $V(S) \cap V(T)$, as then the probability is 0. Now we can continue our estimation by summing up for all possible $T$ with $k \geq 2$:

$$\sum_T \mathbb{E}[X_T | X_S = 1] \leq \sum_{k=2}^{v} \frac{v!}{(v-k)!} \frac{\binom{n-v}{k} \binom{v}{k}}{\text{Aut} H} p^{e-\frac{d}{2}k}(1-p)^{\frac{d+1}{2}k} \leq \sum_{k=2}^{v} \frac{v^k}{2(k-2)!} \left( \frac{v}{n-v} \right)^k \left( p^{\frac{d}{2}}(1-p)^{\frac{v}{2}} \right)^k \leq \frac{1}{2} g^2 \sum_{k=2}^{v} \frac{1}{(k-2)!} g^{k-2} \leq \frac{1}{2} g^2 \exp(g).$$

As

$$\frac{v!}{(v-k)!} \leq \left( \frac{v}{k} \right) \frac{(n-v)^{v-k}}{(v-k)!} \leq \left( \frac{v}{k} \right)^2 \frac{k!}{(n-v)^k} \leq \frac{1}{k!} \left( \frac{v}{n-v} \right)^k.$$}

The lemma follows: the probability of $H$ not being an induced subgraph is at most $e^g g^2/2$. This upper bound is 1 exactly if $g = 2W(1/\sqrt{2})$. For $g \leq 2W(1/\sqrt{2})$, we obtain the upper bound in the lemma. For $g \geq 2W(1/\sqrt{2})$, the upper bound in the lemma is at least 1, so the statement is obvious. \qed

**4.2 High formulation complexity with high probability**

In order to obtain lower bounds on the formulation complexity of the maximum stable set problem of $G = G(n, p)$, via Lemmas 4.3 and 4.2, taking $H$ to be $K^\sim_i$. We obtain the following result:

**Main Theorem 4.4** (Super-polynomial xc of STAB($G(n, p)$) w.h.p) With high probability, the maximum stable set problem of the random graph $G(n, p)$ has at least the following formulation complexity, depending on the size of $p$: 

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(i) For $p = \omega(1/\sqrt[4]{n})$ and fixed $0 < c < 2/\sqrt{3} \approx 1.1547$, we have

$$ P \left[ f_c(\text{STAB}(G(n, p))) \geq 2^{\sqrt{\frac{\ln n}{p}} \log(3/2)} \right] = 1 - o(1). \quad (11) $$

(ii) For $c > 0$ and $c/\sqrt{n} \leq p = o(1)$ we have

$$ P \left[ f_c(\text{STAB}(G(n, p))) \geq 2^{\frac{\ln(3/2)}{p \ln(1/p)}} \right] = 1 - O\left(\frac{1}{c^6}\right). \quad (12) $$

(iii) Moreover, for any fixed $c > 0$ for all $1/n < p \leq c/\sqrt{n}$ and $0 < \delta < 1$

$$ P \left[ f_c(\text{STAB}(G(n, p))) \geq 2^{\delta \sqrt{\frac{\ln n}{\ln(1/p)}} \log(3/2)} \right] \geq 1 - O(\delta^8). \quad (13) $$

As an illustration of Main theorem 4.4, we include concrete lower bounds in special cases of interest.

**Corollary 4.5** For every fixed $0 < \varepsilon < 1$, we have

$$ P \left[ f_c(\text{STAB}(G(n, n^{-\varepsilon}))) \geq 2^{\sqrt{(1-4\varepsilon)n^2 \ln n \log(3/2)}} \right] = 1 - o(1) \quad \text{for } \varepsilon < 1/4, \quad (14) $$

$$ P \left[ f_c(\text{STAB}(G(n, n^{-\varepsilon}))) \geq 2^{\sqrt{n^2 \ln n} \log(3/2)} \right] = 1 - o(1) \quad \text{for } \varepsilon < 1/3, \quad (15) $$

$$ P \left[ f_c(\text{STAB}(G(n, n^{-\varepsilon}))) \geq 2^{\frac{n(1-\varepsilon)^2}{\ln n} \log(3/2)} \right] = 1 - o(1) \quad \text{for } \varepsilon \geq 1/3. \quad (16) $$

Below the $p = n^{-\varepsilon}$ range, we obtain

$$ P \left[ f_c(\text{STAB}(G(n, (\ln^{6+\varepsilon} n)/n))) \geq 2^{\ln^{1+\varepsilon/5} n \log(3/2)} \right] = 1 - o(1), \quad (17) $$

and (at the other end of the range) for fixed $\delta > 0$,

$$ P \left[ f_c(\text{STAB}(G(n, \delta \ln^{-1} n))) \geq n^{\delta^{-1/2} \log(3/2)} \right] = 1 - o(1). \quad (18) $$

**Proof** Equations (14) and (18) are special cases of (11). For Eq. (14), we choose $p = n^{-\varepsilon}$ and $c = 1$. For Eq. (11), we choose $p = \delta \ln^{-1} n$ and $c = 1.1$, a bit larger than 1, then the square root in (11) becomes

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\[
\sqrt{c \ln np^4 \over p} = \sqrt{c \ln n + 4 \ln(\delta) + 4 \ln^{-1} n \over \delta} \ln n
\]

\[= c \delta^{-1/2} (1 + o(1)) \ln n > \delta^{-1/2} \ln n,
\]

proving the equation.

Equation (15) follows from Eq. (12) via \( p = n^{-\varepsilon} \). Equations (16) and (17) are special cases of Eq. (13). Equation (16) is the case \( p = n^{-1+\varepsilon} \) and \( \delta = \sqrt{\varepsilon} \ln^{-1} n \). For Eq. (17), we choose \( p = (\ln^{6+\varepsilon} n) / n \) and \( \delta = \ln^{-\varepsilon / 20} n \), then the interesting part of the exponent is

\[
\delta \sqrt{\ln(1/p)} = \ln^{-\varepsilon / 20} n \sqrt{\ln^{6+\varepsilon} n \over \ln n - \ln^{6+\varepsilon} n} > \ln^{-\varepsilon / 20} n \sqrt{\ln^{6+\varepsilon} n \over \ln n} = \ln^{1+\varepsilon / 5} n
\]

proving the claim. \( \square \)

Now we are going to prove the main theorem of Sect. 4.2.

**Proof of Main Theorem 4.4** We apply Lemma 4.3 to the graph \( H := K_t \) together with Lemma 4.2 to obtain:

\[
\mathbb{P} \left[ \text{fc}(\text{STAB}(G(n, p))) \geq 2^{t \log(3/2)} \right] \geq \mathbb{P} \left[ K_t \subseteq G(n, p) \right] \\
\geq 1 - c_0 \frac{v^4 p^{-d} (1-p)^{-v}}{(n-v)^2} \\
\geq 1 - c_0 (1 + o(1)) \frac{v^4 p^{-d} e^{pv}}{n^2} \quad \text{if } v = o(n).
\]

Here \( v \) is the number of vertices of \( H \), and every induced subgraph of \( H \) should have average degree at most \( d \). We shall estimate the last fraction \( v^4 p^{-d} e^{pv} / n^2 \), using the \( d \) provided by Lemma 4.1. Below we will tacitly assume \( t = \omega(1) \), which is w.l.o.g because \( \text{fc}(\text{STAB}(G(n, p))) \geq n \) always.

Now we shall substitute various values for \( p, t, d, \ell \) to obtain the equations of the theorem. We will verify \( v = o(n) \) and \( v^4 p^{-d} e^{pv} / n^2 = o(1) \) to obtain an \( 1 - o(1) \) lower bound from the last inequality.

For establishing (11), we choose

\[
\ell := 0 \quad \quad t := \left[ c \sqrt{\ln(np^4) \over p} \right] \quad \quad d := 4.
\]

Note that for \( p \geq 1/\sqrt{n} \),

\[
v = t + 3 \left( \frac{t}{2} \right) = \left( \frac{3}{2} + o(1) \right) t^2 = \left( \frac{3}{2} + o(1) \right) c^2 \ln (np^4) \over p
\]

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\[ \leq \left( \frac{3}{2} + o(1) \right) c^2 \sqrt[4]{n} \ln n = o(n), \]

and hence

\[ \frac{v^4 p^{-d} e^{p^v}}{n^2} = \left( \frac{3}{2} + o(1) \right)^4 \left( pt^2 \right)^4 e^{(3/2+o(1))pt^2-2\ln(np^4)} \]

\[ \leq \left( \frac{3}{2} + o(1) \right)^4 c^8 \left( \ln np^4 \right)^4 \]

\[ \times \exp \left\{ \left[ \left( \frac{3}{2} + o(1) \right) c^2 - 2 \right] \ln(np^4) \right\} = o(1), \]

as \( np^4 = \omega(1) \) by assumption. This finishes the proof of (11).

We turn to (12) and (13). We will choose a positive \( \ell \) to approximately minimize the fraction in terms of the other parameters. To ease computation, let

\[ \gamma := \frac{2\ell + 3}{2} > 1. \]

Then the parameters \( v \) and \( d \) look like

\[ d = 2 + \frac{4}{2\ell + 3} = 2 + \frac{2}{\gamma}, \]

\[ v = t + (2\ell + 3) \left( \frac{1}{2} \right) = \gamma t^2 + (1 - \gamma)t < \gamma t^2. \]

Hence

\[ \frac{v^4 p^{-d} e^{p^v}}{n^2} < \frac{\gamma^4 t^8 e^{p \gamma t^2 + 2\ln(1/p))}/\gamma}{p^2 n^2}. \]

The \( \gamma \) minimizing the expression is

\[ \frac{\sqrt{4 + 2pt^2 \ln(1/p) - 2}}{pt^2} = \frac{2 \ln(1/p)}{\sqrt{4 + 2pt^2 \ln(1/p)} + 2}, \]

but we use an approximation as \( \ell \) needs to be an even integer. Therefore we choose

\[ \ell = 2 \left[ \frac{\ln(1/p)}{\sqrt{4 + 2pt^2 \ln(1/p)} + 2} - \frac{3}{4} \right]. \]

We will verify later that actually \( \ell = \omega(1) \). Hence

\[ \gamma = (1 + o(1)) \frac{2 \ln(1/p)}{\sqrt{4 + 2pt^2 \ln(1/p)} + 2} = (1 + o(1)) \frac{\sqrt{4 + 2pt^2 \ln(1/p)} - 2}{pt^2}, \]
\[
\frac{v^4 p^{-d} e^{p v}}{n^2} < (1 + o(1)) \left( \frac{2pt^2 \ln(1/p)}{\sqrt{np^3}} \right)^4 \frac{e^{(2+o(1))\sqrt{4+2pt^2 \ln(1/p)}}}{\left( \sqrt{4 + 2pt^2 \ln(1/p) + 2} \right)^4} \\
= (1 + o(1)) \frac{e^{(2+o(1))\sqrt{4+2pt^2 \ln(1/p)}}}{\left( \sqrt{4 + 2pt^2 \ln(1/p) - 2} \right)^4} \\
\tag{19}
\]

Now we shall substitute various values for \( p \) and \( t \) to obtain the equations of the theorem.

We will need to verify \( \ell = \omega(1) \) and \( v = o(n) \) for every choice.

For Eq. (13), i.e., in the case \( 1/n < p \leq c/\sqrt[n]{n} \), we neglect the exponential term in (19) for the choice of \( t \):

\[
t = \left\lceil \delta \sqrt{\frac{\ln(\sqrt[n]{n}/c)}{\ln(1/p)}} \right\rceil.
\]

Here \( 0 < \delta < 1 \) is an additional parameter. Rearranging gives us

\[
2pt^2 \ln(1/p) = (1 + o(1))\delta^2 \sqrt{np^3} \leq (1 + o(1))\delta^2 c^{3/2} \leq (1 + o(1))c^{3/2},
\]

so in particular,

\[
\ell \geq 2 \left\lceil \frac{\ln(\sqrt[n]{n}/c)}{\sqrt{4 + (1 + o(1))c^{3/2}} + 2} - \frac{3}{4} \right\rceil = \omega(1)
\]

\[
v < \gamma t^2 = O(1/p) = O(\sqrt[n]{n}) = o(n).
\]

Finally,

\[
\frac{v^4 p^{-d} e^{p v}}{n^2} \leq (1 + o(1)) \frac{e^{(2+o(1))\sqrt{4+(1+o(1))c^{3/2}}}}{\left( \sqrt{4 + (1 + o(1))\delta^2 \sqrt{np^3} - 2} \right)^4} \\
\leq (1 + o(1))e^{(2+o(1))\sqrt{4+(1+o(1))c^{3/2}}} \left( \frac{1}{1/4 + (1 + o(1))\delta^2} \right)^4 = O(\delta^8),
\]

as claimed.

For Eq. (12), i.e., when \( c/\sqrt[n]{n} \leq p = o(1) \), we choose

\[
t = \left\lceil \frac{1}{\sqrt{p \ln(1/p)}} \right\rceil.
\]

This provides the estimate

\[
2pt^2 \ln(1/p) = 2 + o(1),
\]
hence $\ell = \Theta(\ln(1/p)) = o(1)$, and $v < \gamma r^2 = O(1/p) = O(\sqrt{n}) = o(n)$. Finally,

$$v^4 p^{-d} e^{pv} n^2 = \frac{(1 + o(1)) e^{(2 + o(1)) \sqrt{4 + (2 + o(1))} - 2^4}}{(np^3)^2} = O\left(\frac{1}{(np^3)^2}\right) = O\left(\frac{1}{c}^6\right).$$

as $np^3 \geq c^3$. 

Main Theorem 4.4 gives super-polynomial lower bounds all the way from $p = \Omega\left(\frac{\log n + \epsilon}{n}\right)$ to $p = O\left(\frac{1}{\log n}\right)$. The key for being able to cover the whole regime is to have the gadgets depend on the parameter choice. Notice that for $p < 1/n$ a random graph almost surely will have all its components of size $O(\log n)$, making the stable set problem easy to solve, so that we essentially leave only a small polylog gap.

4.3 Upper bound on formulation complexity with high probability

We now complement Main Theorem 4.4 with an upper bound, which is close to the lower bound, up to an essentially quadratic gap in the exponent.

**Theorem 4.6** (Upper bound on the xc of $\text{STAB}(G(n, p))$ w.h.p) For $0 < p \leq 1/2$,

$$\mathbb{P}\left[\text{fc}(\text{STAB}(G)) \geq 2^{\Omega\left(\frac{\ln n}{p}\right)}\right] \leq n^{-\Omega\left(\frac{\ln n}{p}\right)}.$$ 

In particular, for $p = n^{-\epsilon}$, we obtain $\mathbb{P}\left[\text{fc}(\text{STAB}(G)) \geq 2^{\Omega(p^{\epsilon} \ln^2 n)}\right] = o(1)$ and similarly for $p = \delta \ln^{-1} n$, we get $\mathbb{P}\left[\text{fc}(\text{STAB}(G)) \geq n^{\Omega\left(\frac{\ln^3 n}{\delta}\right)}\right] = o(1)$.

The upper bound stated in Theorem 4.6 is actually an upper bound on the number of stable sets in $G$, i.e., follows from (5).

**Proof of Theorem 4.6** By standard arguments (see, e.g., [10, Chapter 11, page 300]), for $G = G(n, p)$ we have

$$\mathbb{P}[\alpha(G) \geq r] \leq \left(n e^{-p(r-1)/2}\right)^r$$

and thus for $r = 4\frac{\ln n}{p}$ we get

$$\mathbb{P}\left[\alpha(G) \geq 4 \frac{\ln n}{p}\right] \leq \left(n \frac{1}{\sqrt{e}}\right)^{-4 \frac{\ln n}{p}}.$$ 

Therefore, with very high probability, we have $\alpha(G) \leq 4\frac{\ln n}{p}$. Using the inequality $\sum_{i=0}^{-k} \binom{n}{i} \leq (n + 1)^k$, we get
The result then follows directly from (5).

5 Concluding remarks

We conclude with the following conjecture whose validity, we believe, is necessary to strengthen the result, close the remaining gap, as well as establishing truly exponential lower bounds on the extension complexity of further combinatorial problems.

**Conjecture 5.1 (Sparse Graph Conjecture)** There exists an infinite family \((T_k)_{k \in \mathbb{N}}\) of template graphs such that, denoting by \(t_k\) the number of vertices of \(T_k\): (i) \(\text{fc}(\text{STAB}(T_k^\sim)) = 2^{\Omega(t_k)}\); (ii) \(T_k\) has bounded average degree; (iii) \(t_k \leq t_{k+1}\) but at the same time \(t_{k+1} = O(t_k)\).

The existence of such a family would have various consequences.

**Exact case.** Assuming the Sparse Graph Conjecture we would obtain that the extension complexity of polytopes (see, e.g., [12, 13] for definitions) for important combinatorial problems considered in [1, 12, 13, 19] including (among others) the stable set polytope, knapsack polytope, and the 3SAT polytope would have truly exponential extension complexity, that is \(2^{\Omega(n)}\) extension complexity, where \(n\) is the dimension of the polytope.

The recent groundbreaking result of [22] gives \(2^{\Omega(n)}\) bounds for the extension complexity of the matching polytope and TSP polytope. These bounds are also tight up to constants, but this time the upper bound does not come from the number of vertices but rather from the number of facets and dynamic programming algorithms, respectively. Notice that the dimension of both polytopes is \(d = \Theta(n^2)\), thus the bounds are in fact \(2^{\Omega(\sqrt{d})}\).

**Average case.** As observed above, there is a quadratic gap in the best current lower and upper bounds on the worst-case extension complexity of the stable set polytope: \(2^{\Omega(\sqrt{n})}\) versus \(2^n\) respectively. This is reflected in the results we obtain here. Assuming the Sparse Graph Conjecture we could reduce the gap between upper and lower bounds to a logarithmic factor. Moreover, our results could be strengthened to establish super-polynomial lower bounds on the average-case formulation complexity of \(\text{STAB}(G(n, p))\) up to constant probability \(p\).

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