A Weak Galerkin Finite Element Method for $p$-Laplacian Problem

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Abstract. In this paper, we introduce a weak Galerkin (WG) finite element method for $p$-Laplacian problem on general polytopal mesh. The quasi-optimal error estimates of the weak Galerkin finite element approximation are obtained. The numerical examples confirm the theory.

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1. Introduction

We consider the following $p$-Laplacian problem,

$$\nabla \cdot \left( |\nabla u|^{p-2} \nabla u \right) = f \quad \text{in} \quad \Omega,$$

$$u = 0 \quad \text{on} \quad \partial \Omega,$$

where $1 < p < \infty$.

The $p$-Laplacian problem has many applications including filtration, power-law materials and quasi-Newtonian flows. Finite element analysis of the $p$-Laplacian has been extensively studied in the literature. The quasi-norm approach introduced in [2] provides sharper error bounds for finite element solutions of the $p$-Laplacian problems. The quasi-norm error estimates have been derived for different finite element approximations in [4, 8, 9].

The weak Galerkin finite element method is an effective and flexible numerical technique for solving partial differential equations. The WG finite element methods. It is a natural extension of the standard Galerkin finite element method where classical derivatives were substituted by weakly defined derivatives on functions with discontinuity. The WG method was first introduced in [15, 16] and then has been applied to solve various PDEs.
such as second order elliptic equations, biharmonic equations, Stokes equations, convection dominant problems, hyperbolic equations, and Maxwell’s equations [1, 3, 5, 6, 11–14, 17–24].

In this paper, we introduce a WG finite element method for solving the \( p \)-Laplacian problem. Error estimates are obtained in different norms. The numerical examples tested on hybrid polygonal meshes confirm the theoretical findings.

2. Finite Element Methods

For any given polygon \( D \subseteq \Omega \), we use the standard definition of Sobolev spaces \( H^s(D) \) with \( s \geq 0 \). The associated inner product, norm, and semi-norm in \( H^s(D) \) are denoted by \( (\cdot, \cdot)_s, \| \cdot \|_s, \) and \( | \cdot |_s, 0 \leq s \), respectively. When \( s = 0 \), \( H^0(D) \) coincides with the space of square integrable functions \( L^2(D) \). In this case, the subscript \( s \) is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, for \( D = \Omega \) the subscript \( D \) is also suppressed.

Let \( \mathcal{T}_h \) be a partition of a domain \( \Omega \) consisting of polygons in two dimension or polyhedra in three dimension satisfying a set of conditions specified in [16]. Denote by \( \mathcal{E}_h \) the set of all edges or flat faces in \( \mathcal{T}_h \), and let \( \mathcal{E}^0_h = \mathcal{E}_h \setminus \partial \Omega \) be the set of all interior edges or flat faces. For every element \( T \in \mathcal{T}_h \), we denote by \( h_T \) its diameter and mesh size \( h = \max_{T \in \mathcal{T}_h} h_T \) for \( \mathcal{T}_h \).

For \( k \geq 1 \), we define the finite element spaces

\[
V_h := \{ v = \{v_0, v_b\} : v|_T \in P_k(T) \times P_k(e), e \in \partial T, \ T \in \mathcal{T}_h \},
\]

\[
V^0_h := \{ v \in V_h : v_b = 0 \text{ on } \partial \Omega \}.
\]

For any \( v = \{v_0, v_b\} \), the weak gradient \( \nabla v \in [P_{k-1}(T)]^d \) is defined on \( T \) by

\[
(\nabla v, \varphi)_T := (v_0, \nabla \varphi)_T - (v_b, \varphi \cdot n)|_{\partial T} \quad \text{for all } \varphi \in [P_{k-1}(T)]^d.
\]  

We introduce also the bilinear forms

\[
s(v, w) := \sum_{T \in \mathcal{T}_h} h_T^{-1} (v_0 - v_b, w_0 - w_b)|_{\partial T},
\]

\[
a(v, w) := \sum_{T \in \mathcal{T}_h} (|\nabla w|^p - 2 \nabla w \cdot \nabla w) + s(v, w).
\]

Let \( Q_0, Q_b \) and \( Q_h \) be the locally defined \( L^2 \) projections onto \( P_k(T), P_k(e) \) and \( [P_{k-1}]^d \) accordingly on each element \( T \in \mathcal{T}_h \) and \( e \in \partial T \). For the true solution \( u \) of (1.1), we define \( Q_h u \) as

\[ Q_h u := \{Q_0 u, Q_b u\} \in V^0_h. \]

Algorithm 2.1. A numerical approximation for (1.1) can be obtained by seeking \( u_h = \{u_0, u_b\} \in V^0_h \) satisfying the following equation:

\[
a(u_h, v) = (f, v_0) \quad \text{for all } \ v = \{v_0, v_b\} \in V^0_h.
\]
Define the functional $J(v)$ by

$$J(v) := \frac{1}{p} \sum_{T \in \mathcal{T}_h} \int_T |\nabla w v|^p + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T} h_T^{-1}(v_0 - v_b)^2 - \int_\Omega f v. \quad (2.5)$$

It is easily seen that for $1 < p < \infty$, the functional $J(v)$ is strictly convex and if $u_h$ is the solution of (2.4), then

$$J(u_h) \leq J(v) \quad \text{for all} \quad v \in V_h.$$ 

Therefore, the weak Galerkin finite element formulation (2.4) is well-posed.

First, we introduce some notations,

$$(v,w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (v,w)_T = \sum_{T \in \mathcal{T}_h} \int_T vw,$$

$$(v,w)_{\partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (v,w)_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} vw.$$ 

**Lemma 2.1.** Let $u \in H^1(\Omega)$ be the solution of (1.1), we have

$$\nabla w Q_h u = Q_h \nabla u.$$ 

**Proof:** Using (2.1), the integration by parts and the definition of $Q_h$, we have that for any $\tau \in [P_{k-1}(T)]^d$

$$(\nabla w Q_h u, \tau)_T = - (Q_0 u, \nabla \cdot \tau)_T + (Q_h u, \tau \cdot n)_{\partial T}$$

$$= - (u, \nabla \cdot \tau)_T + (u, \tau \cdot n)_{\partial T}$$

$$= (\nabla u, \tau)_T = (Q_h(\nabla u), \tau)_T,$$

which completes the proof. \qed

Let us recall a trace inequality — viz. if $\phi \in W^{1,p}(T)$, $1 < p < \infty$, then

$$\|\phi\|_{L^p(e)}^{p} \leq C h_T^{-1} \left( \|\phi\|_{L^p(T)}^p + h_T^{1-p}\|\nabla \phi\|_{L^p(T)}^p \right), \quad (2.7)$$

where $e$ is an edge of the polygon $T$, cf. [10].

**Lemma 2.2.** For $v = \{v_0,v_b \} \in V_h$ and $T \in \mathcal{T}_h$, we have

$$\|\nabla v_0\|_{L^p(T)}^p \leq C \left( \|\nabla w v\|_{L^p(T)}^p + h_T^{1-p}\|v_0 - v_b\|_{L^p(\partial T)}^p \right), \quad (2.8)$$

$$\|\nabla w v\|_{L^p(T)}^p \leq C \left( \|\nabla v_0\|_{L^p(T)}^p + h_T^{1-p}\|v_0 - v_b\|_{L^p(\partial T)}^p \right). \quad (2.9)$$

**Proof:** Let $\varphi = Q_h(\|\nabla v_0\|^{p-1} sgn(\nabla v_0))$. Then for $1/q + 1/p = 1$,

$$\|\varphi\|_{L^q(T)}^q = \int_T |Q_h(\|\nabla v_0\|^{p-1} sgn(\nabla v_0))|^q$$
It follows from the trace inequality (2.7), the inverse inequality and (2.10),

\[
\phi = \left( h_T^{1/p} \| \varphi \|_{L^q(\partial T)} \right)^q = h_T^{(p-1)/p} \| \varphi \|_{L^q(\partial T)}^q = h_T \| \varphi \|_{L^q(\partial T)}^q \\
\leq C \left( \| \varphi \|_{L^q(T)}^q + h_T^q \| \nabla \varphi \|_{L^q(T)}^q \right) \leq \| \varphi \|_{L^q(T)}^q \leq C_2 \| \nabla \varphi \|_{L^p(T)}^p.
\]

(2.10)

Using the definitions of the weak gradient and \(Q_b\), and (2.10)-(2.11), we arrive at

\[
\int_T |\nabla v_0|^p = (\nabla v_0, |\nabla v_0|^{p-1} \text{sgn}(\nabla v_0))_T = (\nabla v_0, Q_b (|\nabla v_0|^{p-1} \text{sgn}(\nabla v_0)))_T \\
= (\nabla v_0, \varphi)_T = (\nabla w, \varphi)_T + (v_0 - v_b, \varphi \cdot n)_{\partial T} \\
\leq \| \nabla w \|_{L^p(T)} \| \varphi \|_{L^q(T)} + \| v_0 - v_b \|_{L^p(\partial T)} \| \varphi \|_{L^q(\partial T)} \\
\leq \| \nabla w \|_{L^p(T)} \| \varphi \|_{L^q(T)} + (h_T^{1/p} \| v_0 - v_b \|_{L^p(\partial T)} \left( h_T^{1-1/p} \| \varphi \|_{L^q(\partial T)} \right)^p \\
\leq C \| \nabla w \|_{L^p(T)}^p + \frac{1}{4C_1} \| \varphi \|_{L^q(T)}^q + C \left( h_T^{1/p} \| v_0 - v_b \|_{L^p(\partial T)} \right)^p \\
+ \frac{1}{4C_2} \left( h_T^{1-1/p} \| \varphi \|_{L^q(\partial T)} \right)^q \\
\leq C \left( \| \nabla w \|_{L^p(T)}^p + h_T^{1-p} \| v_0 - v_b \|_{L^p(\partial T)}^p \right) + \frac{1}{2} \| \nabla v_0 \|_{L^p(T)}^p,
\]

which yields (2.8). The estimate (2.9) can be proved analogously.

\(\square\)

3. Error Estimates

We first introduce the semi-norms

\[
|v_0|_{L^p(\Omega, \mathcal{G}_h)} := \left( \sum_{t \in \mathcal{G}_h} \int_T |v_0|^p \right)^{1/p}, \quad |v_0|_{W^{1,p}(\Omega, \mathcal{G}_h)} := \left( \sum_{t \in \mathcal{G}_h} \int_T |\nabla v_0|^p \right)^{1/p}
\]

and the quasi-norm

\[
|v|_{(\phi, p)}^2 = \sum_{t \in \mathcal{G}_h} \int_T \left( |\nabla w \phi| + |\nabla w v| \right)^{p-2} |\nabla w v|^2 + s(v, v).
\]

(3.1)

This quasi-norm depends on \(\phi \in W^{1,p}(\Omega, \mathcal{G}_h)\) with \(\nabla w \phi \neq 0\).
Lemma 3.1. For $\nu = \{\nu_0, \nu_b\} \in V_h$ and $\phi \in W^{1,p}(\Omega, T_h)$, the quasi-norm $|\nu|_{(\phi, p)}$ defined in (3.1) has the following properties.

(a) $|\nu|_{(\phi, p)} \geq 0$ and $|\nu|_{(\phi, p)} = 0$ if and only if $\nu = 0$ for any $\nu \in V_h^0$.

(b) If $1 < p \leq 2$, then

$$
\|\nabla_w \nu\|_{L^p(T_h)}^2 \leq C \left( \sum_{T \in T_h} \left( (|\nabla_w \nu| + |\nabla_w \phi|)^p \right) \right)^{(2-p)/p} |\nu|_{(\phi, p)}^2 \quad (3.2)
$$

$$
\|\nabla \nu_0\|_{L^p(\Omega, T_h)}^2 \leq C \left( \sum_{T \in T_h} \left( (|\nabla_w \nu| + |\nabla_w \phi|)^p \right) \right)^{(2-p)/p} |\nu|^2_{(\phi, p)} + s(\nu, \nu). \quad (3.3)
$$

(c) If $2 \leq p < \infty$, then

$$
\|\nabla_w \nu\|_{L^p(T_h)}^p \leq C |\nu|_{(\phi, p)}^2 \quad (3.4)
$$

$$
\|\nabla \nu_0\|_{L^p(\Omega, T_h)}^p \leq C |\nu|^2_{(\phi, p)} + \sum_{T \in T_h} h_T^{1-p} \|v_0 - v_b\|_{L^p(\partial T)}^p. \quad (3.5)
$$

Proof: If $|\nu|_{(\phi, p)} = 0$ for $\nu = \{\nu_0, \nu_b\} \in V_h^0$, the definition of quasi-norm (3.1) implies $\nabla_w \nu = 0$ on $T$ and $\nu_0 = \nu_b$ on $\partial T$. Then by (2.8), we have $\nabla \nu_0 = 0$, i.e. $\nu_0$ is a constant on each element $T$. Combining it with $\nu_0 = \nu_b$ on the interior edges and $v_b = 0$ on the boundary edges, we have $\nu = 0 \in \Omega$, which proves assertion (a).

To prove (3.2), we follow the ideas of [4, 7]. For $1 < p \leq 2$ and $\nu, \phi \in W^{1,p}(\Omega, T_h)$, let $w = (|\nabla_w \nu| + |\nabla_w \phi|)^{p-2}$. Then the Hölder inequality yields

$$
\|\nabla_w \nu\|_{L^p(T_h)}^2 = \left( \sum_{T \in T_h} \int_T w^{-p/2} \left( w^{p/2} |\nabla_w \nu|^p \right) \right)^{2/p} \leq C \left( \sum_{T \in T_h} \int_T w^{-p/2} |\nabla_w \nu|^p \right)^{(2-p)/2} \left( \sum_{T \in T_h} \int_T w |\nabla_w \nu|^2 \right)^{(p/2)(2/p)} \leq C \left( \sum_{T \in T_h} \int_T (|\nabla_w \nu| + |\nabla_w \phi|)^p \right)^{(2-p)/p} \times \left( \sum_{T \in T_h} \int_{\partial T} (|\nabla_w \nu| + |\nabla_w \phi|)^{p-2} |\nabla_w \nu|^2 \right) \leq C \left( \sum_{T \in T_h} \int_T (|\nabla_w \nu| + |\nabla_w \phi|)^p \right)^{(2-p)/p} |\nu|_{(\phi, p)}^2 .
$$
It follows from (2.8) that for $1 < p \leq 2$ and $h < 1$,
\[
\|\nabla v_0\|_{L^p(T)} \leq \|\nabla w v\|_{L^p(T)} + h_T^{(1-p)/p} \|v_0 - v_b\|_{L^2(\partial T)} \leq \|\nabla w v\|_{L^p(T)} + h_T^{-1/2} \|v_0 - v_b\|_{L^2(\partial T)},
\]
which is equivalent to
\[
\|\nabla v_0\|_{L^p(T)} \leq \|\nabla w v\|_{L^p(T)} + h_T^{-1} \|v_0 - v_b\|_{L^2(\partial T)}.
\]
Combining the estimate above and (3.2), we have
\[
\|\nabla v_0\|_{L^p(T)}^2 \leq C \left( \|\nabla w v\|_{L^p(T)}^2 + h_T^{-1} \|v_0 - v_b\|_{L^2(\partial T)}^2 \right).
\]
If $2 \leq p < \infty$, we have
\[
\|\nabla v_0\|_{L^p(T)}^p = \sum_{T \in \mathcal{T}_h} \int_T |\nabla w v|^p \leq \sum_{T \in \mathcal{T}_h} \int_T \left( |\nabla w v| + |\nabla w \phi| \right)^p |v|^2_{(\phi,p)} + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{L^2(\partial T)}^2,
\]
which implies (3.3).

The estimate (3.5) follows immediately from (2.9) and (3.4) which ends the proof of the lemma.

**Lemma 3.2.** If $v, w \in V_h$, then
\[
C|v - w|_{(v,p)}^2 \leq a(v, v - w) - a(w, v - w). \tag{3.6}
\]

**Proof.** According to [4, 7], the inequality
\[
(|\alpha|^{p-2} - |\beta|^{p-2}) \cdot (\alpha - \beta) \geq C|\alpha - \beta|^2(|\alpha| + |\beta|)^{p-2} \tag{3.7}
\]
holds for any $\alpha, \beta \in \mathbb{R}^n$. It follows from (3.7),
\[
a(v, v - w) - a(w, v - w) = \sum_{T \in \mathcal{T}_h} \left( |\nabla w v|^{p-2} \nabla w v, \nabla w (v - w) \right) + s(v, v - w)
\]
Combining the above estimate with the inequality — cf. [4, 7],

\[
\frac{1}{2}(|x| + |y|) \leq |x| + |y - x| \leq 2(|x| + |y|) \quad \text{for all} \quad x, y \in \mathbb{R}, \tag{3.8}
\]

we obtain

\[
a(v, v - w) - a(w, v - w) \geq C|v - w|^2_{(v, p)},
\]

which completes the proof.

\[\square\]

**Lemma 3.3.** Let \(u\) and \(u_h\) be the solutions of (1.1) and (2.4), respectively. Then

\[
a(Q_h u, Q_h u - u_h) - a(u_h, Q_h u - u_h) = \ell_1(u, Q_h u - u_h) + \ell_2(u, Q_h u - u_h) + s(Q_h u, Q_h u - u_h). \tag{3.9}
\]

**Proof.** Let \(a = |\nabla u|^{p-2}\) and \(\tilde{a} = |\nabla_w Q_h u|^{p-2}\). For any \(v = \{v_0, v_b\} \in V_h^0\), testing the Eq. (1.1) by \(\nu_0\) gives

\[
-(\nabla \cdot a \nabla u, v_0) = (f, v_0). \tag{3.10}
\]

Integration by parts and the continuity of the flux give

\[
-(\nabla \cdot a \nabla u, v_0) = (a \nabla u, \nabla v_0)_{\partial T_h} - \langle v_0, a \nabla u \cdot n \rangle_{\partial T_h} = (a \nabla u, \nabla v_0)_{\partial T_h} - \langle v_0 - v_b, a \nabla u \cdot n \rangle_{\partial T_h}.
\]

Integration by parts, the definitions of the weak derivative and the operator \(Q_h\) imply

\[
(a \nabla u, \nabla v_0)_{\partial T_h} = -(v_0, \nabla \cdot Q_h (a \nabla u))_{\partial T_h} + \langle v_0, Q_h (a \nabla u) \cdot n \rangle_{\partial T_h}
\]

\[
= (a \nabla u, \nabla w v)_{\partial T_h} + \langle v_0 - v_b, Q_h (a \nabla u) \cdot n \rangle_{\partial T_h}
\]

\[
= (a \nabla u, \nabla w v)_{\partial T_h} + (\tilde{a} \nabla_w Q_h u, \nabla w v)_{\partial T_h} + \langle v_0 - v_b, Q_h (a \nabla u) \cdot n \rangle_{\partial T_h}.
\]

Combining the two above equations with (3.10), we arrive at

\[
(|\nabla_w Q_h u|^{p-2} \nabla_w Q_h u, \nabla_w v)_{\partial T_h} = (f, v_0) + \ell_1(u, v) + \ell_2(u, v), \tag{3.11}
\]

where

\[
\ell_1(u, v) = (\nabla_w Q_h u|^{p-2} \nabla_w Q_h u, \nabla_w v)_{\partial T_h} - (|\nabla u|^{p-2} \nabla u, \nabla w v)_{\partial T_h}, \tag{3.12}
\]

\[
\ell_2(u, v) = \langle v_0 - v_0, a \nabla u - Q_h (a \nabla u) \cdot n \rangle_{\partial T_h}. \tag{3.13}
\]
Adding $s(Q_h u, v)$ to the both sides of the Eq. 3.11 yields

$$a(Q_h u, v) = (f, v_0) + \ell_1(u, v) + \ell_2(u, v) + s(Q_h u, v) \quad \text{for all} \quad v \in V_h^0. \quad (3.14)$$

Considering the difference between (3.14) and the Eq. (2.4) with $v = Q_h u - u_h$, we arrive at (3.9).

Lemma 3.4. If $u \in W^{k+1,p}(\Omega)$ and $v \in V_h^0$, then

$$|\ell_1(u, v)| \leq C h^{2k} \int_{\Omega} |D^k u|^p + \epsilon \|v\|_{(u,p)}^2, \quad (3.15)$$

$$|\ell_2(u, v)| \leq C h^{2k} \|v\|_{W^{k,2}(\Omega)}^2 + \epsilon \|v\|_{(u,p)}^2, \quad (3.16)$$

$$|s(Q_h u, v)| \leq C h^{2k} \|u\|_{W^{k+1,2}(\Omega)} + \epsilon \|v\|_{(u,p)}^2, \quad (3.17)$$

where $\ell_1(u, v)$ and $\ell_2(u, v)$ are defined in (3.12) and (3.13), respectively.

Proof: Let us recall two inequalities from [4, 9]. If $\alpha, \beta \in \mathbb{R}^n$, then

$$|\alpha|^{p-2} \alpha - |\beta|^{p-2} \beta \leq C |\alpha - \beta| (|\alpha| + |\beta|)^{p-2}, \quad (3.18)$$

and if $\sigma, \sigma_1, \sigma_2 > 0$, $0 < \epsilon < 1$ and $p > 1$, then

$$(\sigma + \sigma_1)^{p-2} \sigma_1 \sigma_2 \leq \epsilon \gamma (\sigma + \sigma_1)^{p-2} \sigma_1^2 + \epsilon (\sigma + \sigma_2)^{p-2} \sigma_2^2, \quad (3.19)$$

where $\gamma = 1$ if $p > 2$ and $\gamma = 1/(p-1)$ if $1 < p \leq 2$.

Using the inequalities (2.6), (3.18), (3.8) and (3.19) yields

$$|\ell_1(u, v)| = \left| (|D\nabla Q_h u|^p \nabla u, \nabla v)_{\mathcal{G}_h} - (|D\nabla u|^p \nabla u, \nabla v)_{\mathcal{G}_h} \right|$$

$$= \left| (|Q_h \nabla u|^p - Q_h \nabla u, \nabla v)_{\mathcal{G}_h} - (|D\nabla u|^p \nabla u - Q_h \nabla u, \nabla v)_{\mathcal{G}_h} \right|$$

$$\leq \sum_{T \in \mathcal{T}_h} \int_T \left| |Q_h \nabla u|^p - Q_h \nabla u - |D\nabla u|^p \nabla u - Q_h \nabla u | \nabla v | \right|$$

$$\leq C \sum_{T \in \mathcal{T}_h} \int_T \left| \left( |Q_h \nabla u| + |Q_h \nabla u| \right)^{p-2} |D\nabla u - Q_h \nabla u| \right|$$

$$\leq C \sum_{T \in \mathcal{T}_h} \int_T \left| \left( |Q_h \nabla u| + |Q_h \nabla u| \right)^{p-2} |D\nabla u - Q_h \nabla u| \right|^2$$

$$\leq C \sum_{T \in \mathcal{T}_h} \int_T \left| \left( |Q_h \nabla u| + |Q_h \nabla u| \right)^{p-2} |D\nabla u - Q_h \nabla u| \right|^2 + \epsilon \|v\|_{(u,p)}^2$$

$$\leq Ch^{2k} \int_{\Omega} |D\nabla u|^p + \epsilon \|v\|_{(u,p)}^2.$$
Theorem 3.1. Let \( u \) be the finite element solution of the problem (1.1) arising from (2.4). Then:

(a) If \( 1 < p < \infty \), then
\[
|Q_h u - u_h|_{(Q_h u, p)}^2 \leq C(u) h^{2k}. \tag{3.20}
\]

(b) If \( 1 < p \leq 2 \), then
\[
|Q_h u - u_1|_{W^{1,p}(\Omega, h)}^2 \leq C(u) h^{2k}. \tag{3.21}
\]

(c) If \( p \geq 2 \), then
\[
|Q_h u - u_1|_{W^{1,p}(\Omega, h)}^p \leq C(u) h^{2k}. \tag{3.22}
\]
Proof: It follows from (3.6) and (3.9) that
\[ C|Q_h u - u_h|_{(Q_h u, p)}^2 \leq a(Q_h u, Q_h u - u_h) - a(u_h, Q_h u - u_h) \]
\[ = \ell_1(u, Q_h u - u_h) + \ell_2(u, Q_h u - u_h) + s(Q_h u, Q_h u - u_h). \]
Estimating the right-hand side of the above inequality by Lemma 3.4, we obtain
\[ C|Q_h u - u_h|_{(Q_h u, p)}^2 \leq C(u)h^{2k} + 3\epsilon|Q_h u - u_h|_{(Q_h u, p)}^2, \]
where
\[ C(u) = \int_\Omega |\nabla u|^p |D^k u|^2 + |\nabla u|^{p-2} |\nabla u|_H^{p(\Omega)} + |u|_{H^{k+1}(\Omega)}^2. \]
Letting \( \epsilon \) small enough, we obtain (3.20). The definition (3.1) and (3.20) imply
\[ s(Q_h u - u_h, Q_h - u_h) \leq C(u)h^{2k}. \quad (3.23) \]
Using (3.3), (3.23) and (3.20) gives
\[ |Q_0 u - u_0|_{W^1,p(\Omega, T)}^2 \leq C|Q_h u - u_h|_{(Q_h u, p)}^2 + s(Q_h u - u_h, Q_h - u_h) \leq C(u)h^{2k}, \]
which gives (3.21) for \( 1 < p \leq 2 \). Now it follows from (3.1) and (3.20) that
\[ s(e_h, e_h) = \sum_{T \in \mathcal{T}} h^{-1}_T \|e_0 - e_b\|_{L^2(\partial T)}^2 \leq Ch^{2k}, \]
where \( e_h = \{e_0, e_b\} = Q_h u - u_h \). Based on the inequality above, we can assume
\[ \|e_0 - e_b\|_{L^\infty(\partial T)} \leq Ch. \]
Then we have
\[ \sum_{T \in \mathcal{T}} h^{-1}_T \|e_0 - e_b\|_{L^2(\partial T)}^p = \sum_{T \in \mathcal{T}} \int_{\partial T} h^{-p}_T \|e_0 - e_b\|^p ds \]
\[ \leq \sum_{T \in \mathcal{T}} \int_{\partial T} h^{-2p}_T \|e_0 - e_b\|^{p-2} h^{-1}_T \|e_0 - e_b\|^2 ds \]
\[ \leq C \sum_{T \in \mathcal{T}} \int_{\partial T} h^{-1}_T \|e_0 - e_b\|^2 ds \]
\[ \leq Cs(e_h, e_h) \leq Ch^{2k}. \]
Using the above inequality and (3.5), we have
\[ |Q_0 u - u_0|_{W^1,p(\Omega, T)}^p \leq C|Q_h u - u_h|_{(Q_h u, p)}^2 + \sum_{T \in \mathcal{T}} h^{-p}_T \|Q_0 u - u_0 - (Q_b u - u_b)\|_{L^p(\partial T)}^p \]
\[ \leq Ch^{2k}. \]
The proof is complete. \[ \square \]
4. Numerical Results

**Example 4.1.** We solve the $p$-Laplacian equation (1.1) on the unit square $\Omega = (0, 1)^2$. For various $p$ values, we compute the right-hand side function $f$ of the Eq. (1.1) using the substitution

$$u = (x - x^2)(y - y^2).$$

We use the uniform grids shown in Fig. 1. The error and the order of convergence are listed in Table 1. We can see, the optimal order of convergence is achieved in all cases when $p \geq 2$. In particular, a superconvergence appears for the $P_1$ element in the energy norm.

![Figure 1: Example 4.1. The first three grids.](image)

Table 1: Example 4.1. Error profile and convergence order.

| grid | $|Q_hu - u_h|_0$ | $\sigma(h^n)$ | $|Q_hu - u_h|_{(\omega,p)}$ | $\sigma(h^n)$ |
|------|----------------|----------------|----------------------------|----------------|
| The $P_1$ element, $p = 3.5$ | | | | |
| 5    | 0.2662E-03     | 1.94           | 0.2777E-03                 | 1.89           |
| 6    | 0.6856E-04     | 1.98           | 0.7301E-04                 | 1.95           |
| 7    | 0.1733E-04     | 2.00           | 0.1865E-04                 | 1.98           |
| 8    | 0.4350E-05     | 2.00           | 0.4700E-05                 | 1.99           |
| The $P_1$ element, $p = 1.5$ | | | | |
| 5    | 0.1733E-02     | 2.01           | 0.3389E-02                 | 1.15           |
| 6    | 0.4402E-03     | 2.00           | 0.1356E-02                 | 1.34           |
| 7    | 0.1111E-03     | 2.00           | 0.5198E-03                 | 1.39           |
| 8    | 0.1111E-03     | 2.00           | 0.5198E-03                 | 1.39           |
| The $P_2$ element, $p = 3.5$ | | | | |
| 3    | 0.2729E-03     | 2.59           | 0.7354E-03                 | 1.80           |
| 4    | 0.3465E-04     | 3.11           | 0.1404E-03                 | 2.49           |
| 5    | 0.3895E-05     | 3.22           | 0.2178E-04                 | 2.75           |
| 6    | 0.4579E-06     | 3.12           | 0.3507E-05                 | 2.66           |
| The $P_2$ element, $p = 1.75$ | | | | |
| 3    | 0.1203E-02     | 3.10           | 0.1016E-01                 | 1.54           |
| 4    | 0.1760E-03     | 2.89           | 0.2307E-02                 | 2.23           |
| 5    | 0.2731E-04     | 2.75           | 0.5412E-03                 | 2.14           |
| 6    | 0.4342E-05     | 2.68           | 0.1397E-03                 | 1.98           |
### Table 1: Example 4.1. Error profile and convergence order (cont’d).

| grid | $||Q_0u - u_0||_0$ | $\sigma(h^n)$ | $||Q_hu - u_h||_{(0,p)}$ | $\sigma(h^n)$ |
|------|-------------------|----------------|--------------------------|----------------|
|      |                   |                |                          |                |
| The $P_1$ element, $p = 3.5$ | | | | |
| 3    | 0.1233E-03        | 4.04           | 0.3614E-03               | 2.38           |
| 4    | 0.8943E-05        | 3.93           | 0.4202E-04               | 3.22           |
| 5    | 0.7836E-06        | 3.58           | 0.4818E-05               | 3.19           |
| 6    | 0.8408E-07        | 3.25           | 0.6076E-06               | 3.02           |
| The $P_1$ element, $p = 1.9$ | | | | |
| 3    | 0.2383E-03        | 3.28           | 0.2827E-02               | 2.09           |
| 4    | 0.3342E-04        | 2.94           | 0.8671E-03               | 1.77           |
| 5    | 0.4677E-05        | 2.89           | 0.2499E-03               | 1.83           |
| 6    | 0.6728E-06        | 2.83           | 0.6793E-04               | 1.90           |
| The $P_4$ element, $p = 3.5$ | | | | |
| 3    | 0.1930E-04        | 5.70           | 0.4612E-04               | 4.54           |
| 4    | 0.6088E-06        | 5.16           | 0.2541E-05               | 4.33           |
| 5    | 0.1956E-07        | 5.05           | 0.1350E-06               | 4.31           |
| The $P_4$ element, $p = 1.9$ | | | | |
| 3    | 0.1899E-03        | 2.70           | 0.4957E-02               | 1.08           |
| 4    | 0.3582E-04        | 2.49           | 0.1702E-02               | 1.60           |
| 5    | 0.5286E-05        | 2.81           | 0.5015E-03               | 1.79           |

**Example 4.2.** We solve again the $p$-Laplacian equation (1.1) on the unit square $\Omega = (0, 1)^2$. For various $p$ values, we compute the right-hand side function $f$ of (1.1) using the substitution

$$u = \sin(2\pi x) \sin(2\pi y).$$

We use some kind of pentagon grids shown in Fig. 2. The error and the order of convergence are listed in Table 2. The order of convergence matches that of our theory.

![Figure 2: Example 4.2. The first two pentagon grids.](image-url)
Table 2: Example 4.2. Error profile and convergence order.

| grid | $||Q_0u-u_0||_0$ | $O(h^n)$ | $||Q_hu-u_h||_{(0,p)}$ | $O(h^n)$ |
|------|------------------|---------|----------------------|---------|
| The $P_1$ element, $p = 3.5$ | | | | |
| 5    | 0.1941E+00       | 2.37    | 0.3297E+01           | 2.05    |
| 6    | 0.4351E-01       | 2.16    | 0.8440E+00           | 1.97    |
| 7    | 0.1044E-01       | 2.06    | 0.2528E+00           | 1.74    |
| The $P_1$ element, $p = 1.75$ | | | | |
| 5    | 0.1941E+00       | 2.37    | 0.3297E+01           | 2.05    |
| 6    | 0.4351E-01       | 2.16    | 0.8440E+00           | 1.97    |
| 7    | 0.1044E-01       | 2.06    | 0.2528E+00           | 1.74    |
| The $P_2$ element, $p = 3.5$ | | | | |
| 4    | 0.1021E+00       | 2.76    | 0.3449E+01           | 1.75    |
| 5    | 0.1330E-01       | 2.94    | 0.9168E+00           | 1.91    |
| 6    | 0.1652E-02       | 3.01    | 0.2354E+00           | 1.96    |
| The $P_2$ element, $p = 1.9$ | | | | |
| 4    | 0.1021E+00       | 2.76    | 0.3449E+01           | 1.75    |
| 5    | 0.1330E-01       | 2.94    | 0.9168E+00           | 1.91    |
| 6    | 0.1652E-02       | 3.01    | 0.2354E+00           | 1.96    |

Example 4.3. We solve the $p$-Laplace equation (1.1) on the domain $\Omega = (-1,1)^2$. This time, the solution has a singularity. For various $p$ values, we compute the right-hand side function $f$ of (1.1) using the substitution exact solution

$$u = \begin{cases} 
0, & \text{if } r < 0.4, \\
(r-0.4)^4, & \text{if } r \geq 0.4,
\end{cases}$$  

(4.3)

where $r = \sqrt{x^2+y^2}$. We use the square grids shown in Fig. 1. The numerical error and the order of convergence are listed in Table 3. We plot the numerical solution and its error in Figs. 3 and 4. It seems the error is caused more by the approximation of boundary value in nonlinear differential equations than by the singularity (the positive preserving computation.)

Figure 3: Example 4.3. The $P_2$ solution and its error on grid 6, $p = 3.5$. 
Figure 4: The $P_2$ solution and its error for Example 4.3 (4.3) on grid 6, $p = 1.9$.

Table 3: Example 4.3. Error profile and convergence order.

| grid | ||$Q_h u - u_0$|| | $O(h^n)$ | ||$Q_h u - u_h$|| | $O(h^n)$ |
|------|-----------------|--------|-----------------|--------|
| 6    | 0.1172E+00      | 1.96   | 0.1653E+00      | 1.02   |
| 7    | 0.2996E-01      | 1.98   | 0.6478E-01      | 1.36   |
| 8    | 0.7584E-02      | 1.99   | 0.2085E-01      | 1.64   |

The $P_1$ element, $p = 1.75$

| grid | ||$Q_h u - u_0$|| | $O(h^n)$ | ||$Q_h u - u_h$|| | $O(h^n)$ |
|------|-----------------|--------|-----------------|--------|
| 6    | 0.6791E-02      | 2.01   | 0.4515E-02      | 1.97   |
| 7    | 0.1701E-02      | 2.01   | 0.1146E-02      | 1.99   |
| 8    | 0.4256E-03      | 2.01   | 0.2888E-03      | 1.99   |

The $P_2$ element, $p = 3.5$

| grid | ||$Q_h u - u_0$|| | $O(h^n)$ | ||$Q_h u - u_h$|| | $O(h^n)$ |
|------|-----------------|--------|-----------------|--------|
| 4    | 0.2165E+00      | 2.71   | 0.3119E+00      | 1.66   |
| 5    | 0.2879E-01      | 2.97   | 0.1067E+00      | 1.58   |
| 6    | 0.3536E-02      | 3.06   | 0.3055E-01      | 1.82   |

The $P_2$ element, $p = 1.9$

| grid | ||$Q_h u - u_0$|| | $O(h^n)$ | ||$Q_h u - u_h$|| | $O(h^n)$ |
|------|-----------------|--------|-----------------|--------|
| 4    | 0.1084E-01      | 3.09   | 0.1109E-01      | 2.82   |
| 5    | 0.1361E-02      | 3.06   | 0.1870E-02      | 2.62   |
| 6    | 0.1690E-03      | 3.04   | 0.2048E-03      | 3.23   |

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