Correspondence between QCD sum rules and constituent quark models

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Abstract. We compare two widely used approaches to the description of hadron properties: QCD sum rules and constituent quark models. Making use of the dispersion formulation of the quark model, we show that both approaches lead to similar spectral representations for hadron observables with an important difference that quark model is based on Feynman diagrams with massive quarks, whereas QCD sum rules are based on the same Feynman diagrams for current quarks with the additional condensate contributions for light quarks and gluons. We give arguments for a similarity of the smearing function in sum rules and the hadron wave function of the quark model. Analyzing the sum rule for the leptonic decay constant of the heavy pseudoscalar meson containing a light $u$ or $s$ quark, we find that the quark condensates at the chiral symmetry-breaking scale $\mu \chi \simeq 1$ GeV, $\langle \bar{u}u \rangle = -(230 \pm 15$ MeV)$^3$ and $\langle \bar{s}s \rangle = - (220 \pm 15$ MeV)$^3$ correspond to constituent quark masses $m_u \simeq 220$ MeV and $m_s \simeq 350$ MeV, respectively. We also obtain the running of the model parameters above the chiral scale $\mu \chi$. The observed correspondence between constituent quark models and QCD sum rules allows a deeper understanding of both methods and their parameters. It also provides a QCD basis for constituent quark models, extending their applicability above the scale of chiral symmetry breaking.

1 Introduction

There are several evidences that static properties of hadrons and their characteristics in processes with momentum transfers not larger than few GeV may be well described treating hadrons as relativistic few-body composite systems of effective particles - constituent quarks. These evidences come from several sources. Among them: (i) hadron spectroscopy where mesons and baryons spectra may be well described in the relativistic constituent quark model \[1\]: (ii) high-energy hadron-hadron and hadron-nucleon scattering at small and intermediate momentum transfers where momentum distributions of secondary particles are well described assuming that mesons and nucleons are bound states of two and three constituent quarks, respectively \[2\]: (iii) photon-hadron scattering at small momentum transfers where the observables speak in favor of the presence of few extended objects inside hadrons \[3\]: (iv) exclusive processes at small and intermediate momentum transfers where the constituent quark picture has been successfully applied to the calculation of elastic and transition form factors. Indeed, various models based on the notion of constituent quarks can be found in the literature, for instance the dispersion approach \[4\],\[5\], the quasipotential approach \[6\], light-front \[7\],[8], instant-form \[9\] and point-form \[10\] quark models. For more details we refer to the review \[11\]. The constituent quark masses are the parameters to be adjusted by describing the data. The typical values of constituent masses are: $m_u = m_d \simeq 220 \div 300$ MeV, $m_s \simeq 350 \div 450$ MeV, $m_c \simeq 1.4 \div 1.6$ GeV and $m_b \simeq 4.8 \div 5.0$ GeV.

The many successes of the constituent quark model to describe the data definitely prompt that this approach provides a relevant description of the nonperturbative QCD physics at low and intermediate momentum transfers. However, on one hand a rigorous derivation of the quark model from the QCD Lagrangian is hard to establish. On the other hand a relationship between QCD Lagrangian and hadron physics is provided by QCD sum rules. Thus it is reasonable to look for a correspondence between quark models and sum rules in order to connect the former to QCD in this way.

In this paper we demonstrate such a correspondence between sum rules and quark models making use of the dispersion formulation for the latter \[12\]. This correspondence not only provides a QCD basis for quark-model calculations but also helps in obtaining a better understanding of the different versions of QCD sum rules and of the parameters which a priori have no clear physical meaning within QCD sum rules. Moreover, it allows to understand the proper running of the constituent quark-model parameters above the chiral symmetry breaking scale $\mu \chi \approx 1$ GeV, opening the possibility to apply constituent quark models above this scale in a controlled way.
1.1 QCD sum rules

The method of QCD sum rules [13] is based on the following theoretical concepts: a complicated structure of the physical QCD vacuum, Operator Product Expansion and quark-hadron duality. The first concept states that the physical QCD vacuum is different from perturbative QCD vacuum; properties of the former may be described in terms of the condensates, i.e. non-vanishing amplitudes of local gauge-invariant operators over the physical vacuum. Perturbative QCD calculations can still be applied far from hadronic thresholds, but require modifications: non-perturbative contributions given by the condensates appear as power corrections to the usual perturbative expressions. These proper modifications of the perturbation theory may be obtained using the Operator Product Expansion.

The central object considered within the QCD sum rule method is the correlator of the quark currents over the physical vacuum. One obtains spectral representations for this correlator by two different means: within the modified QCD perturbation theory including condensates, and using hadron saturation. Local quark-hadron duality states that both representations for the spectral density should be equal to each other after a proper smearing (the same for the QCD part and the hadronic part) is applied. The two smeared representations for the spectral density give the two sides of the QCD sum rule.

Sum rules as a technical tool to obtain the resonance properties from QCD is based on attempting to choose the smearing function such that a single resonance dominates the hadronic part of the sum rule, and at the same time only few condensates of the lowest dimension are essential on the QCD side. In many cases it is possible to find smearing functions which satisfy the above competing requirements [14]. Then one obtains the resonance parameters, such as masses, decay constants, or form factors. For details we refer to review papers [15,16] and references therein.

In practice the application of this method leads to the calculation of the spectral densities of the relevant Feynman diagrams with current quarks and gluons, taking the convolution of this spectral density with the smearing function, and adding nonperturbative power corrections described in terms of the condensates. Depending on the choice of the smearing function one obtains various versions of the sum rules (moment, Borel, Gaussian, etc). The resulting sum rules contain physical parameters such as quark masses and condensates, and parameters describing the details of the smearing procedure which may vary from one observable to the other. They are fixed by requiring the stability of the sum rule or from fits to the data.

1.2 Constituent quarks and the dispersion approach

The dispersion approach [12] uses the constituent quarks and is thus conceptually quite different. Nevertheless, technically it is based on calculating the spectral densities of the same diagrams as in the sum rules, but involving the constituent quarks, and taking the convolution of these spectral densities with the wave functions of the participating hadrons. All nonperturbative effects are assumed to be taken into account by introducing the constituent quarks and no other nonperturbative contributions are added. One may treat the wave functions either as some nonperturbative inputs and use simple parameterizations for them, or use the relativistic wave functions obtained from the solutions to an eigenvalue problem [8].

The central observation for comparing sum rules and quark models is that the densities in the spectral representations are the same functions in both methods. The differences come from the following sources: in quark models one uses effective constituent quark masses and the wave functions of hadrons; in sum rules one uses current quark masses, smearing functions which may have no physical meaning, and adds contributions of condensates. The spectral densities determine the main qualitative features of the calculated hadron observables, such as their dependence on the momentum transfer or scaling properties in the heavy-quark mass. On the quantitative side, if both approaches give the correct description of the hadron properties, the constituent masses in quark-model calculations should numerically reproduce the contribution of the condensates in QCD sum rules.

We make this correspondence explicit by analyzing the decay constant of the $B$-meson, a pseudoscalar meson containing heavy $b$ and light $\bar{u}$ quarks. We obtain the constituent mass of the light quark from QCD sum rules in the limit $m_b \to \infty$.

We argue that the above correspondence between sum rules and quark model, and the knowledge of the spectroscopic wave functions in the latter allow to find the optimal smearing function and to favor a specific version of QCD sum rules depending on the hadron considered\(^1\). For instance, for hadrons containing heavy quarks we give arguments in favor of Gaussian sum rules compared to Borel sum rules, while in case of hadrons containing light quarks only Borel sum rules appear to be favored.

The correspondence between constituent quark models and QCD itself (via the sum rules) can be pushed further. Indeed the natural upper limit of applicability of constituent quark models is the scale of chiral symmetry breaking, $\mu_c \simeq 1$ GeV. Below such a scale the constituent mass may be approximated by a constant fixed mainly by spectroscopic data on hadron masses. Above the chiral scale a running of the constituent mass is strongly expected. A similar situation holds as well for the hadron wave functions. The correspondence we found between constituent quark models and QCD sum rules allows us to find out quantitatively the running of both the constituent mass and average momentum inside the hadron just by imposing the appropriate QCD scale dependence of the quark condensates.

\(^1\) Notice that the idea of a similarity of the Borel wave function of QCD sum rules and the light-cone wave function of a hadron in terms of current quarks was discussed in [17].
The paper is organized as follows: in Section 2 the main formulas for the pseudoscalar $B$-meson in the dispersion approach are recalled. In Section 3 we consider a sum rule for the decay constant $f_B$ and its relationship to the quark model, and obtain the constituent quark mass. We also discuss physics motivations for the smearing function. In Section 4 we address and solve quantitatively the relation between the QCD running of the quark condensate and the running of both the constituent mass and average momentum inside the hadron. Finally we summarize our results in the Conclusions.

2 Dispersion approach based on the constituent quark picture

We present here the main formulas of the dispersion approach [12] necessary for our discussion. The relativistic wave function of the $B$-meson consisting of $b$ and $ar{u}$ quarks with masses $m_b$ and $m_u$, is normalized as

$$1 = \int \frac{ds}{(m_b + m_u)^2} |\psi(s)|^2 \rho(s, m_b^2, m_u^2),$$

where $\rho(s)$ is the spectral density of the Feynman loop graph with the $i\gamma^5$ Dirac structures in the vertices (see Fig. 1), given explicitly by

$$\rho(s, m_b^2, m_u^2) = -\frac{N_c}{8\pi^2} \int \frac{dk_u dk_b \delta(m_b^2 - k_b^2)\delta(m_u^2 - k_u^2)\delta(p - k_b - k_u)}{s - (m_b - m_u)^2} \times \text{Sp} \left( (m_b + \hat{k}_b)i\gamma_5(m_u - \hat{k}_u)i\gamma_5 \right)$$

$$= \frac{N_c}{8\pi^2} \frac{x1/2(s, m_b^2, m_u^2)}{x} \times \theta(s - (m_b + m_u)^2).$$

Here $s = p^2$ and $\lambda(s, m_b^2, m_u^2) = (s - m_b^2 - m_u^2)^2 - 4m_b^2m_u^2$. The normalization condition [11] corresponds to the normalization of the charged form factor of the meson at $q^2 = 0$.

The elastic form factor describing the interaction of the $b$ quark with an external vector field is defined as

$$\langle B(p')|\bar{b}\gamma_\mu b|B(p)\rangle = (p + p')_\mu F(q^2),$$

with $p - p' = q$. For $q^2 < 0$ one finds

$$F(q^2) = \int ds ds' \psi(s)\psi(s')\Delta_V(s', s, q^2|m_b^2, m_u^2, m_u^2),$$

where $\Delta_V(s', s, q^2|m_b^2, m_u^2, m_u^2)$ is the double spectral density of the triangle diagram, whose explicit expression can be found in [12]. An important property of the spectral densities is the relation

$$\Delta_V(s', s, q^2|m_b^2, m_u^2, m_u^2) \rightarrow \delta(s - s')\rho(s, m_b^2, m_u^2)$$

valid for $q^2 \rightarrow 0$. For a given wave functions $\psi(s)$ Eq. [11] allows to calculate the form factor for $q^2 \leq 0$. The relation [11], guarantees that $F(q^2 = 0) = 1$.

3 QCD sum rules and the constituent quark mass

We now turn to the calculation of resonance observables in QCD sum rules and start with the decay constant $f_B$. The central object in this case is the correlator of two pseudoscalar currents

$$i \int dx e^{-ix\rho} \langle \text{vac}|\bar{b}(x)\gamma_5 u(x)\bar{u}(0)\gamma_5 b(0)|\text{vac} \rangle.$$
It is clear that as soon as we describe the picture was discussed in [20]. The pseudoscalar interpolating current $\bar{\psi}\gamma_5\psi$ where we have defined the renormalization scheme. The resulting sum rule takes the form

$$\frac{f_B^2 M_B^4}{(m_b + m_u)^2} H(M_B^2) = \int ds \rho(s, m_b^2, m_u^2) H(s) - \bar{m}_u(\bar{q}q) H((m_b + m_u)^2).$$  \hspace{1cm} (7)

Here $m_q \leq 10$ MeV and $\bar{m}_u$ are the short-distance quark masses. $H(s)$ is the function which provides the relevant smearing of the spectral densities on both hadronic and the QCD parts of the sum rule. Let us now rewrite the sum rule (7) in the form

$$1 = \int ds \rho(s, m_b^2, m_u^2) \Phi^2(s) - \bar{m}_b(\bar{q}q)\Phi((m_b + m_u)^2),$$  \hspace{1cm} (8)

where we have defined $\Phi(s)$ as

$$\Phi(s) = \frac{m_b + m_u}{f_B M_B^2} \sqrt{H(s)/H(M_B^2)}.$$  \hspace{1cm} (9)

It is clear that as soon as we describe the $B$ meson by the pseudoscalar interpolating current $b\bar{c}\gamma_5u$, the combination (9) will appear for any observable as the result of applying summing in the corresponding channel. Usually, the smearing functions are not assumed to be universal and are allowed to vary from one observable to another. Imagine however that there exists such a smearing function which turns out to be independent of the quantity described by sum rules and is thus universal. Then up to condensates contributions the expression (9) looks like the normalization condition for this wave function. Comparing the equations (9) and (10) we may expect that the smearing function $\Phi(s)$ and the wave function $\Psi(s)$ are close to each other and that the appearance of the constituent quark effectively accounts for the contribution of the condensates.

To make this point clear let us consider a simple theory without condensates but with confinement. A non-relativistic potential model with a confining potential represents an example of such a theory. In this case the spectrum of states is discrete and one can use sum rules to calculate the bound-state parameters such as their masses and decay constants [14] and to study quark-hadron duality [21]. One can also use sum rules to calculate the transition form factors. The merit of referring to the potential model is that the exact solutions for the above quantities are known. So confronting the exact result with the approximate results obtained by sum rules allows to understand the accuracy of the method and to motivate the choice of the smearing functions. For the resonance masses and the decay constants this was done in [14]. To understand what happens for the form factors, let us study the form factor describing the transition from the state $i$ to the state $j$, $F_{i\rightarrow j}(q^2)$. The corresponding expression is known

$$F_{i\rightarrow j}^{\text{exact}}(q^2) = \int dk\Psi_i(k^2)\Psi_j^*(|(k + q|^2).$$  \hspace{1cm} (10)

Here $\Psi_n(k^2)$ is the wave function of the state $n$ obtained by solving the Schrödinger equation. Eq. (10) can be written in the form of a double dispersion integral

$$F_{i\rightarrow j}^{\text{exact}}(q^2) = \int dz dz' \Psi_i(z) \Psi_j^*(z') \Delta_{\text{NR}}(z, z', q^2|m^2)$$  \hspace{1cm} (11)

where $\Delta_{\text{NR}}(z, z', q^2|m^2)$ is the double spectral density of the triangle diagram of the non-relativistic field theory calculated using the non-relativistic Green functions of free quarks [2]. Since condensates are absent in this theory, the application of sum rules would lead to a similar expression

$$F_{i\rightarrow j}^{\text{SR}}(q^2) = \int dz dz' \Phi_i(z) \Phi_j^*(z') \Delta_{\text{NR}}(z, z', q^2|m^2).$$  \hspace{1cm} (12)

where $\Phi_n(z)$ is the smearing function for the state $n$. Choosing the smearing function $\Phi_n(z)$ equal to the exact function $\Psi_n(z)$ leads to the exact form factor. The same argument applies to other characteristics of the resonances. In this case the existence of the universal smearing function is obvious.

The QCD situation is different and more complicated. There are condensates, so one would not expect the universality of the smearing function to be exact. Nevertheless, assuming (and testing) approximate universality may give a hint for choosing the relevant smearing function. To be more precise, let us assume the existence of the function $\Phi(s, m_b^2, m_u^2)$ such that $\Phi(s) = \Phi(s, m_b^2, m_u^2)$ and $\Phi(s) = \Phi(s, m_b^2, m_u^2)$. Then combining (11) and (10) gives the relation

$$\int ds \rho(s, m_b^2, m_u^2) \Phi^2(s, m_b^2, m_u^2) = \int ds \rho(s, m_b^2, m_u^2) \Phi^2(s, m_b^2, m_u^2) - \bar{m}_b(\bar{q}q) \Phi^2((m_b + m_u)^2, m_b^2, m_u^2).$$  \hspace{1cm} (13)

If the condensate contribution is negligible, the constituent quark mass obtained from this equation is equal to the current quark mass. If condensates give a large contribution, a priori it is not guaranteed that the solution for $m_u$ exists at all. We shall see that for realistic values of the quark condensate the solution does exist and gives the value for the constituent quark mass $m_u$ (or $m_s$) in the “expected” range.

### 3.1 The optimal smearing function

In view of the argument from the potential model discussed above, we may expect the existence of the “optimal” smearing function which is close to the wave function
of the constituent quark model. In many applications of quark models a simple Ansatz
\[
\Psi(s) \simeq e^{-k^2/\beta^2}, \quad k = \lambda^{1/2}(s, m_b^2, m_u^2)/2\sqrt{s},
\]
with $\beta \sim A_{QCD}$, was found to give a good approximation to the exact solution of the spectral problem for ground-state mesons and to lead to a good description of data. Thus we choose the trial function $\Phi(s, m_b^2, m_u^2)$ as
\[
\Phi(s, m_b^2, m_u^2) = \frac{m_b + m_u}{f_B M_B^2} e^{-k^2(s, m_b^2, m_u^2)/\beta^2_B}.
\]
Replacing $m \to \overline{m}$ gives the smearing function $\Phi(s)$
\[
\Phi(s) = \frac{\overline{m}_b + m_u}{f_B M_B^2} e^{-k^2(s, \overline{m}_b^2, m_u^2)/\beta^2_B} \approx \frac{\overline{m}_b}{f_B M_B^2} e^{-\delta(s - \overline{m}_b^2)/4m_b^2\beta^2},
\]
where $M_0 \simeq \overline{m}_b$ and in the last equation we used $m_b^2 \gg \beta^2$. This relation prompts to use the Gaussian smearing function for calculating $f_B$ in [23] (i.e. to use the Gaussian sum rule and not the Borel one), and to place the center of the Gaussian near $m_b^2$. Moreover, choosing $M_0 = \overline{m}_b$ improves the suppression of higher-dimension condensates: since $\Phi(s)$ has in this case maximum at $s = \overline{m}_b^2$, the contribution of the dimension-4 condensate proportional to $\delta(s - \overline{m}_b^2)$ vanishes, and thus higher-dimension condensates are suppressed by two powers of the inverse heavy quark mass instead of one power as it occurs in Borel sum rules. So neglecting higher order condensates looks quite safe. In practice one can allow the difference $M_0 - \overline{m}_b$ to take a small nonzero value.

Before closing the subsection notice that in case of mesons consisting of light quarks only one has $k^2 \simeq 4s$, so that for such hadrons Eq. (15) favors the use of Borel sum rules.

### 3.2 The constituent quark mass and the quark condensate

Using the Ansatz (13) we can rewrite (13) as
\[
\overline{m}_b(\bar{q}q) = \int ds \rho(s, m_b^2, m_u^2) e^{-k^2(s, m_b^2, m_u^2)/\beta^2_B} - \int ds \rho(s, m_b^2, m_u^2) e^{-k^2(s, m_b^2, m_u^2)/\beta^2_B}.
\]
This relation gives a connection between the quark mass, the constituent mass and the condensate, but involves also $m_b$, $\overline{m}_b$, and the parameter $\beta_B$ of the smearing function. To get rid of complications related to the presence of $m_b$ we go to the limit $m_b \to \infty$. This procedure requires however some care: The spectral density $\rho(s, m_b^2, m_u^2)$ involves radiative corrections which so far have not been included into consideration. As soon as the radiative corrections are included, it becomes crucial which precise definition of the heavy quark mass is used. As known from the literature radiative corrections to the spectral density are big (almost 100%) for the pole heavy quark mass $m_b$, but they are less than 10% if one works with the $\overline{MS}$ running mass $\overline{m}_b$. Having in mind the small size of radiative corrections in terms of the $\overline{m}_b$, we require that the limit $m_b \to \infty$ is taken such that the ratio of the constituent $m_b$ to the $\overline{m}_b$ goes to unity
\[
\overline{m}_b/m_b \to 1.
\]

Then the radiative corrections to Eq. (17) may be safely omitted to the accuracy we are interested in. Changing the integration variable, in the limit $m_b \to \infty$ Eq. (17) takes the form
\[
\overline{m}_b = \frac{N_c}{\pi^2} \int_0^\infty dk k^2 e^{-k^2/\beta_B^2} \times \left\{ \frac{m_u}{\sqrt{m_u^2 + k^2}} - \frac{m_u}{\sqrt{m_u^2 + k^2}} \right\}.
\]

Equation (19) is the first central result of this paper. The quark condensate and the current quark mass $m_u$ depend on the scale, thus requiring scale-dependence of the quark-model parameters $\beta_B$ and $m_u$. We shall discuss this dependence in the next section, and now we analyse the relation (19) at the chiral symmetry breaking scale $\mu_\chi \simeq 1$ GeV.

First of all we note that below $\mu_\chi$ the quark model parameters $m_u(\mu)$ and $\beta_B(\mu)$ can be reasonably taken as constant values fixed by the specific potential model adopted. From the analysis of properties of the $Q\bar{q}$ mesons in [5] one expects the value $\beta_B = 0.6 \pm 0.7$ GeV in the heavy quark limit. Neglecting the current mass of the light quark in numerical estimates, we find that the constituent quark mass $m_u = 220$ MeV (obtained from the description of the meson transition form factors in [22]) and prompted by the analysis of the meson mass spectrum in [25] corresponds to the condensate $\langle \bar{q}q \rangle = -(230 \pm 15)$ MeV$^4$. Interestingly, the condensate value does not vary much for $m_u$ in the range $200 \pm 350$ MeV. For instance for $m_u = 350$ MeV, $\langle \bar{q}q \rangle = -(260 \pm 15)$ MeV$^3$. The error in this estimate results from the variation of $\beta_B$ in the range 0.6$\pm$0.7 GeV.

The obtained condensate is negative in agreement with the Gell-Mann-Oakes-Renner relation [22] and compares favorably with existing estimates $\langle \bar{q}q \rangle(1$ GeV$) = -(242 \pm 15$ MeV$^3$) [26].

A similar analysis can be done for the $s$-quark. The parameter $\beta^*_s$ for the pseudoscalar $QS$ meson was found to be only few percent bigger than $\beta_B$ [5,24]: the strange-quark condensate is $\langle \bar{s}s \rangle/\langle \bar{q}q \rangle = 0.8 \pm 0.3$ [23]. Then allowing the range of values $\langle \bar{s}s \rangle = -(220 \pm 20$ MeV$)^3$, $\beta^*_s = 0.6 \pm 0.7$ GeV and $\overline{m}_s = 110 \pm 10$ MeV leads to a constituent mass of the strange quark in the range $m_s = 350 \pm 30$ MeV.
4 Running of the constituent quark mass and average momentum

Requiring the relation (19) to hold at any scale above $\mu_\chi$ leads to the scale-dependence of the quark-model parameters $\beta_\infty$ and $m_u$.

In QCD the current quark mass and the quark condensate are multiplicatively renormalized in such a way that their product is renormalization group invariant (RGI), namely

$$m_u(\mu) = \hat{m}_u Z_{m_u}(\mu)$$  \hspace{1cm} (20)

$$\langle \bar{q}q \rangle(\mu) = \langle \bar{q}q \rangle Z_S(\mu)$$  \hspace{1cm} (21)

where $Z_{m_u}(\mu) = 1/Z_{\hat{m}_u}(\mu)$ at any renormalization scale $\mu$, while $\hat{m}_u$ and $\langle \bar{q}q \rangle$ are RGI quantities. Within the $\overline{MS}$ scheme, at next-to-leading order (NLO) accuracy in the strong coupling constant $\alpha_s(\mu)$ one explicitly has

$$Z_{m_u}(\mu) = \left( \frac{\alpha_s(\mu)}{\pi} \right)^{4/\beta_0} \left[ 1 + \left( \frac{4\gamma_1}{\beta_0} - \frac{\beta_1}{\beta_0^2} \right) \frac{\alpha_s(\mu)}{\pi} \right]$$  \hspace{1cm} (22)

where $\beta_0 = 11 - 2n_f/3$, $\beta_1 = 102 - 38n_f/3$ and $\gamma_1 = (101 - 10n_f/3)/24$.

4.1 Running at high scales

Above the chiral scale the two quark model parameters $\beta_\infty$ and $m_u$ run with the scale $\mu$ and their evolution is expected to be coupled in order to fulfill Eqs. (19) and (21). Qualitatively, the constituent mass $m_u$ decreases when the scale $\mu$ increases, while the parameter $\beta_\infty$, which governs the average momentum of the constituent quark inside the hadron, increases with the scale $\mu$. Thus, above a sufficiently high scale $\mu_0$ ($> \mu_\chi$) the value of the parameter $\beta_\infty(\mu)$ becomes much larger than the value of the mass $m_u(\mu)$, so that in the r.h.s. of Eq. (19) we can neglect both $\hat{m}_u^2$ and $m_u^2$ with respect to $k^2$. This leads to a simple expression for the quark condensate, namely

$$\langle \bar{q}q \rangle(\mu) \sim \frac{N_c}{2\pi^2} \beta_\infty^2(\mu) \left[ m_u(\mu) - \hat{m}_u(\mu) \right]$$  \hspace{1cm} (23)

We have now to impose that the r.h.s. of the above equation runs with the renormalization scale as in Eq. (21), using Eq. (20) for the scale dependence of the current quark mass. In general there is no unique solution. However, if we require that for $\mu \geq \mu_0$ the evolutions of $m_u(\mu)$ and $\beta_\infty(\mu)$ are decoupled, then there is a unique solution provided by

$$m_u(\mu) \underset{\mu \geq \mu_0}{\longrightarrow} \hat{m}_u Z_{m_u}(\mu)$$

$$\beta_\infty(\mu) \underset{\mu \geq \mu_0}{\longrightarrow} \hat{\beta}_\infty(\mu) Z_{m_u}(\mu)$$  \hspace{1cm} (24)

where $\hat{m}_u$ and $\hat{\beta}_\infty$ are RGI quantities satisfying the relation $\langle \bar{q}q \rangle = (N_c/2\pi^2) \hat{\beta}_\infty^2 (\hat{m}_u^2 - \hat{m}_u^2)$. Note that the assumption of the same running for both the constituent and the current quark mass is quite natural and very plausible.

4.2 Running at intermediate scales $\mu_\chi \leq \mu \leq \mu_0$

For $\mu_\chi \leq \mu \leq \mu_0$ the scale dependencies of $m_u(\mu)$ and $\beta_\infty(\mu)$ are coupled in order to fulfill Eqs. (19) and (21). Requiring that the constituent mass has the same scale-dependence as the current mass for $\mu \geq \mu_\chi$

$$m_u(\mu) = m_u(\mu_\chi) Z_{m_u}(\mu)/Z_{m_u}(\mu_\chi)$$  \hspace{1cm} (25)

we obtain

$$\beta_\infty(\mu) = \beta_\infty(\mu_\chi) Z_{\beta}(\mu)/Z_{\beta}(\mu_\chi)$$  \hspace{1cm} (26)

where $m_u(\mu_\chi)$ and $\beta_\infty(\mu_\chi)$ are respectively the values of $m_u$ and $\beta_\infty$ up to the chiral scale $\mu_\chi$. From Eq. (26) one has that $Z_{\beta}(\mu) \simeq Z_{m_u}(\mu)$ for $\mu \geq \mu_0$, while for $\mu_\chi \leq \mu \leq \mu_0$ the value of $Z_{\beta}(\mu)$ can be obtained numerically from Eq. (19) making use of (21). In Fig. 2 we report $Z_{m_u}(\mu) = Z_{\hat{m}_u}(\mu)$ given by Eq. (22) and $Z_{\beta}(\mu)$ obtained from (19) with $\hat{m}_u = 0$, $m_u(1\, \text{GeV}) = 250\, \text{MeV}$ and $\langle \bar{q}q \rangle(1\, \text{GeV}) = -(242\, \text{MeV})^4$ for $u$- and $d$-quark, and $m_s(1\, \text{GeV}) = 110\, \text{MeV}$, $m_s(1\, \text{GeV}) = 350\, \text{MeV}$ and $\langle ss \rangle(1\, \text{GeV}) = -(220\, \text{MeV})^4$ for $s$-quark. Clearly, the scale dependence of $\beta_\infty(\mu)$ is quite close to the one of the mass in case of the light constituent $u$- and $d$-quark, while larger differences appear only around the chiral scale $\mu_\chi$ in case of the constituent $s$-quark.

5 Conclusions and outlook

We have considered the relationship between QCD sum rules and the constituent quark model formulated in the form of spectral representations. Our main results are:

1. We compared the normalization condition for the wave function of a heavy-light pseudoscalar meson in constituent quark model with the QCD sum rule for the decay constant of the same pseudoscalar meson in QCD. We noticed that if one uses a specific version of QCD sum rules, in which the duality smearing function is close to the hadron wave function of the quark model, then effects related to condensates in QCD may be described in terms of the appearance of an effective constituent quark masses.

2. We gave arguments in favor of choosing the smearing functions of QCD sum rules close to the hadron wave functions of the constituent quark model. Applying sum rules for bound-state transition form factors in a confining potential model (a theory with confinement but without condensates), we have seen that the choice of the smearing functions equal to the bound-state wave functions leads to the exact result for the form factors. Although the condensates in QCD violate this exact relation, the approximate similarity of the smearing function with the wave function seems to remain a useful concept.

The knowledge of the hadron wave functions of the constituent quark model may then suggest the “optimal” choice of the smearing wave function of QCD sum rules, and thus the specific version of sum rules (Borel or Gaussian) to be used. For instance, the quark model wave functions speak in favor of using the Gaussian sum rules for...
The observed correspondence between QCD sum rules and constituent quark models may have two important applications. First, it allows us to understand the parameters of the constituent quark model on the QCD basis and it also opens the possibility to apply such a model beyond the scale of chiral symmetry breaking. Second, it provides a physical motivation and control over the smearing functions in QCD sum rules. In spite of the obvious successes of the constituent quark model mentioned in the beginning of this paper, it is not easy to provide a reliable error estimate for its predictions. The context of QCD sum rules can give a firm theoretical basis for the quark model picture of hadrons.

The most interesting problem where the formulated ideas may be applied and tested is the physics of form factors. This work is in progress.

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