Exact finite-size corrections in the dimer model on a planar square lattice

Nikolay Sh Izmailian\textsuperscript{1,2}, Vladimir V Papoyan\textsuperscript{2,3,4} and Robert M Ziff\textsuperscript{4}

\textsuperscript{1} Yerevan Physics Institute, Alikhanian Brothers 2, 375036 Yerevan, Armenia
\textsuperscript{2} Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia
\textsuperscript{3} Dubna State University, Dubna, Russia
\textsuperscript{4} Center for the Study of Complex Systems and Department of Chemical Engineering, University of Michigan, Ann Arbor, MI 48109-2800, United States of America

E-mail: izmail@yerphi.am, vpap@theor.jinr.ru and rziff@umich.edu

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Abstract
We consider the dimer model on the rectangular $2M \times 2N$ lattice with free boundary conditions. We derive exact expressions for the coefficients in the asymptotic expansion of the free energy in terms of the elliptic theta functions ($\theta_2, \theta_3, \theta_4$) and the elliptic integral of second kind ($E$), up to 22nd order. Surprisingly we find that ratio of the coefficients $f_p$ in the free energy expansion for strip ($f_p^{\text{strip}}$) and square ($f_p^{\text{sq}}$) geometries $r_p = f_p^{\text{strip}} / f_p^{\text{sq}}$ in the limit of large $p$ tends to 1/2. Furthermore, we predict that the ratio of the coefficients $f_p$ in the free energy expansion for rectangular ($f_p(\rho)$) for aspect ratio $\rho > 1$ to the coefficients of the free energy for square geometries, multiplied by $\rho^{-p-1}$, that is $r_p = \rho^{-p-1} f_p(\rho)/f_p^{\text{sq}}$, is also equal to 1/2 in the limit of $p \to \infty$.

With these results, one can find the asymptotic behavior of the $f_p(\rho)$ from that of the $f_p^{\text{strip}}$, whose asymptotic behavior is derived explicitly here. We find that the corner contribution to the free energy for the dimer model on rectangular $2M \times 2N$ lattices with free boundary conditions is equal to zero and explain that result in the framework of conformal field theory, with two consistent values of the central charge, namely, $c = -2$ for the construction of a conformal field theory using a mapping of spanning trees and $c = 1$ for the height function description. We also derive a simple exact expression for the free energy of open strips of arbitrary width.
Keywords: dimer statistics, exact solutions, conformal field theory, asymptotic expansions, perfect matchings

Supplementary material for this article is available online

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1961, Kasteleyn [1] and independently Temperley and Fisher [2] found an explicit and elegant formula for the number of dimer tilings (or perfect matchings) on an open square lattice of dimensions $2M \times 2N$:

$$Z = M \prod_{j=1}^{M} N \prod_{k=1}^{N} 4 \left[ \cos\left(\frac{j\pi}{2M+1}\right) + \cos\left(\frac{k\pi}{2N+1}\right) \right]. \quad (1)$$

The asymptotic behavior of the logarithm of $N$ was also found as

$$\ln Z \sim (2M+1)(2N+1) \frac{G}{\pi} - (2M+2N+2) \ln(1+\sqrt{2}) + \ln 2 + \ldots$$

as $N \to \infty$, where $G$ is the Catalan constant $G = \sum_{n=0}^{\infty} (-1)^n / (2n+1)^2 = 0.915965594\ldots$

This result represents one of the fundamental exact results of statistical mechanics. The number $Z$ is the partition function for this system, and its logarithm gives the free energy per site, $f = \ln Z / [(2N+1)(2M+1)]$.

The motivation of the present paper was to find higher-order contributions to this asymptotic series. With multiple precision computer algorithms such as exist in Mathematica, it is not difficult to calculate $Z$ explicitly from (1) for $N$ and $M$ in the thousands, and by looking at the behavior it is possible to determine a few higher-order asymptotic coefficients numerically. However, using results from [3, 4] and some new procedures including the automation of calculation, we were able to calculate analytically 22 terms in the asymptotic expansion of $f$. At the same time, we found an intriguing and unexplained simple relation between the coefficients in the asymptotic series for a square, rectangle, and strip, in the limit of high order. Of course, practically it is not really necessary to find so many terms in the asymptotic series, because, first of all, the exact value can be calculated from equation (1), and secondly, for the asymptotic behavior one does not usually need such accuracy. However, we extended it to such a high order to study the behavior of the coefficients of the asymptotic series. In doing so, we discovered the remarkable relation to the asymptotic coefficients for the strip (which has a simple closed-form expression), and to confirm that relation to high precision, we went to 22nd order. This paper is an account of that work. We begin by discussing general finite-size scaling, and then go through calculations for the dimers on rectangle and strip geometries, and give conclusions at the end.

2. Finite-size scaling theory

Finite-size scaling theory, introduced by Fisher and Barber [5] almost five decades ago, has been of interest to scientists working on a variety of critical systems [6], and finds extensive applications in the analysis of experimental, Monte Carlo, and transfer-matrix data, as well as in theoretical developments related to conformal invariance [7–11]. Theories of finite-size
effects have been successful in deriving critical and noncritical properties of infinite systems from their finite counterparts. Exact solutions have played a key role in determining the form of the finite-size scaling. To fully understand the finite-size behaviour, it is valuable to study model systems, especially those which have exact results. Very few models of statistical mechanics have been solved exactly. The dimer model is one of the most prominent examples.

The dimer model is a classical statistical mechanics model, first introduced to represent physical adsorption of diatomic molecules on crystal surfaces [12]. The exact partition functions for the dimer model on a square lattice of dimensions $2M \times 2N$ with both free and toroidal boundary conditions were obtained by Kasteleyn [1], Fisher [13], and Temperley and Fisher [2] in 1961.

Ivashkevich et al [3] proposed a systematic method to compute the finite-size corrections for the partition function of free models on a torus, including the dimer model on a $2M \times 2N$ rectangular lattice, and the Ising and Gaussian models on a $M \times N$ rectangular lattice. In particular they derive all terms of the exact asymptotic expansion of the logarithm of the partition function on a torus for a class of free exactly solvable models of statistical mechanics. Their approach is based on an intimate relation between the terms of the asymptotic expansion and Kronecker’s double series [3]. The work [3] has been further extended by Izmailian, Oganesyan, and Hu [4] to the dimer model on a rectangular $M \times N$ lattice with various boundary conditions and for different parities of lattice sites in vertical $M$ and horizontal $N$ directions. Later Izmailian et al [14] derived the exact asymptotic expansions of the partition functions of the dimer model on the rectangular $(2M-1) \times (2N-1)$ lattice with a single monomer residing on the boundary under free and cylindrical boundary conditions. In particular they show that because of certain non-local features present in the model, the finite-size corrections in a crucial way depend on the parity of the lattice sites in horizontal $N$ and vertical $M$ directions. Furthermore, the change of parity of $M$ or $N$ induces a change of boundary condition [15, 16].

It has been shown [4, 14] that the exact asymptotic expansion of the free energy for dimers on an open rectangular $M \times N$ lattice takes the form

$$f = f_{\text{bulk}} + \frac{2f_1}{M} + \frac{2f_2}{N} + f_{\text{corn}} \frac{\ln S}{S} + f_0 + \sum_{p=1}^{\infty} \frac{f_p}{S^{p+1}}.$$  \hspace{1cm} (3)

Here $S = M \times N$ is the area of the lattice. The bulk free energy is $f_{\text{bulk}}$, the surface free energies are $f_1$ and $f_2$, and the corner free energy is $f_{\text{corn}}$. The leading finite-size correction term is $f_0$ and the subleading correction terms are $f_p$ for $p = 1, 2, 3, ...$. Indeed such a form of the asymptotic expansion for the free energy also holds for the Ising model, the spanning-tree model, the Gaussian model, and resistor networks.

The bulk free energy term $f_{\text{bulk}}$ is nonuniversal as are the surface free energies $f_1$ and $f_2$, and the subleading correction terms $f_p$. It is worthwhile to mention that the subleading correction terms $f_p$ are nonuniversal, nevertheless in the $\lim_{p \to \infty}$ and $\lim_{p \to 0}$, they are related to the eigenvalues of the universal conformal operators, the so-called integral of motions (for details see [35, 41]). In contrast, $f_{\text{corn}}$ are believed to be universal [9]. In the $\lim_{p \to \infty}$ and $\lim_{p \to 0}$ the value of $f_0 (p)$ is related to the conformal anomaly $c$ and conformal weights of the underlying conformal theory [10, 11]. Moreover, in a rectangular geometry on a plane with free boundary conditions the leading finite-size correction term $f_0$ contains both universal $f_{\text{univ}}$ and non-universal $f_{\text{nonuniv}}$ contributions [17]. Thus, the term $f_0$ can be written as $f_0 = f_{\text{univ}} + f_{\text{nonuniv}}$. The universal part $f_{\text{univ}}$ of $f_0$, which depends only upon the shape of the system and not upon the underlying lattice, can be calculated by conformal field theory methods, and for rectangular geometry with free boundary conditions is given by [17]
Here \( \rho = M/N \) is the aspect ratio and \( \eta(\rho) \) is the Dedekind eta function. In this formula, \( c \) represents the central charge defining the universality class of the system. This formula also applies for example to percolation theory [18]. However, the nonuniversal part \( f_{\text{nonuniv}} \) of \( f_0 \), which is lattice-dependent, is not calculable via conformal field theory methods. In this paper we will derive the leading finite-size correction term \( f_0 \) for the dimer model on rectangular \( 2M \times 2N \) lattice with free boundary conditions and explain why \( f_0 \) is different from equation (4). We are also especially interested in the universal corner terms \( f_{\text{corn}} \) because they are logarithmic. Cardy and Peschel studied the free energy within CFT [9] and predicted that a corner with an angle \( \pi/2 \) and two edges under free boundary conditions has

\[
f_{\text{corn}}(0, 0) = -\frac{c}{32}.
\]  

(5)

We confirmed this in [19, 20] for the Ising model on the square and triangular lattices with free boundary conditions. Later, Imamura et al [21] and Bondesan et al [22, 23] also used CFT to study the corner terms with different conformally invariant boundary conditions and found that the contribution to the free energy from a corner with two edges under \( a \) and \( b \) boundary conditions is

\[
f_{\text{corn}}(a, b) = \Delta_{a,b} - \frac{c}{32}.
\]  

(6)

This formula, where \( \Delta_{a,b} \) represents the conformal weight of the boundary operator inserted at the corner, was verified in our previous work on the Ising model on the square lattice with different boundary conditions [24]. The contribution to the free energy from all corners \( f_{\text{corn}} \) is given by

\[
f_{\text{corn}} = \sum_i f_{\text{corn}}(a_i, b_i)
\]  

(7)

where summation is taken over all corners of the lattice and \( a_i \) and \( b_i \) denote boundary conditions on two edges of the corner \( i \).

In this paper we will explain why the corner term \( f_{\text{corn}} \) is absent in the asymptotic expansion of the free energy for the dimer model on a \( 2M \times 2N \) rectangular lattice with free boundary conditions.

The subleading correction terms \( f_p \) in the asymptotic expansion of the free energy have been obtained for many models, including the Ising [3, 25–28] and dimer models [3, 4, 14, 29–31] on different lattices under various boundary conditions, the Gaussian model [3], the spanning-tree model [32], and resistor networks [33, 34]. In all these papers the subleading correction terms \( f_p \) (for all \( p \)) are expressed in terms of Kronecker’s double series, which in turn are directly related to elliptic theta functions. While the Kronecker’s double series can be expressed in terms of elliptic theta functions for arbitrary \( p \), in fact in a series of papers [3, 4, 14, 26–34] the exact expressions for the subleading correction terms \( f_p \) for different models have been calculated up to second order (for \( p = 0, 1, 2 \)). In [25], the subleading correction terms \( f_p \) are calculated for the Ising model on a rectangular lattice under Brascamp–Kunz boundary conditions up to fourth order (for \( p = 0, 1, 2, 3, 4 \)).

In recent years, the dimer problem has continued to attract much interest, including the study of the asymptotic behavior. Some references include [31, 35–43].

In this paper we have calculated the exact expressions for the subleading correction terms \( f_p \) in the asymptotic expansion of the free energy for the dimer model on a \( 2M \times 2N \) rectangular
lattice with free boundary conditions explicitly up to 22nd order (for \( p = 0, 1, 2, \ldots, 22 \)), and also for the strip geometry. Surprisingly we find that ratio of coefficients \( f_p \) in the free-energy expansion for the strip \( (f_p^{\text{strip}}) \) and square \( (f_p^{\text{sq}}) \) geometries, multiplied by \( \rho^{-p-1} \), that is \( r_p = \rho^{-p-1}f_p(\rho)/f_p^{\text{sq}} \), in the limit of large \( p \), tends to 1/2. Furthermore, we conjecture that the ratio of coefficients \( f_p \) in the free-energy expansion for rectangular \( (f_p(\rho)) \) for \( \rho > 1 \) and square \( (f_p^{\text{sq}}) \) geometries, multiplied by \( \rho^{-p-1} \), is equal to 1/2 in the limit of \( p \to \infty \).

3. Dimer model on rectangular lattice

Let us consider the dimer model on the rectangular lattice \( L \) of size \( 2M \times 2N \) of \( 4MN \) sites with \( 2M \) rows and \( 2N \) columns, and with free boundary conditions. The partition function of the dimer model is given by

\[
Z_{2M,2N} = \sum {Z^\text{h}_h} {Z^\text{v}_v}
\]

where the summation is taken over all possible dimer covering configurations, \( z_h \) and \( z_v \) are, respectively, the dimer weight in the horizontal and vertical directions, and \( n_h \) and \( n_v \) are, respectively, the number of vertical and horizontal dimers.

In [4] it was shown that the partition function of the dimer model on rectangular \( 2M \times 2N \) lattice under free boundary conditions, equation (1), can be expressed in terms of \( Z_{2M,2N}^\text{free}(z, K, L) \),

\[
Z_{2M,2N}^\text{free}(z) = \rho z^{2MN} \left[ \frac{1 + (z^2)^{1/2} Z_{1/2}(z, 2M + 1, 2N + 1)}{2z^{2N+1} \cosh [(2M + 1) \arcsinh z] \cosh [(2N + 1) \arcsinh z]^{1/2}} \right]^{1/2},
\]

where \( z = z_h/z_v \) and \( Z_{1/2}^2(z, M, N) \) is given by

\[
Z_{1/2}^2(z, M, N) = \prod_{m=0}^{M-1} \prod_{n=0}^{N-1} \frac{1}{4} \left( z^2 \sin^2 \frac{\pi (m + \frac{1}{2})}{M} + \sin^2 \frac{\pi (n + \frac{1}{2})}{N} \right).
\]

For the isotropic case \( (z_v = z_h = z = 1) \), equation (9) reduces to

\[
Z_{2M,2N}^\text{free}(1) = \left[ \frac{Z_{1/2}(2M + 1, 2N + 1)}{\sqrt{2} \cosh [(2M + 1) \arcsinh 1] \cosh [(2N + 1) \arcsinh 1]} \right]^{1/2}
\]

where \( \arcsinh 1 = \ln(1 + \sqrt{2}) \). This is an alternate expression for \( Z \) of equation (1). Based on such results, one can easily write down all the terms of the exact asymptotic expansion of the logarithm of the partition functions for the dimer model using an asymptotic expansion of \( \log Z_{1/2}^2(z, 2M + 1, 2N + 1) \) (see [3, 4])

\[
\ln Z_{1/2}^2(z, 2M + 1, 2N + 1) = \frac{S}{\pi} \int_0^\pi \omega_z(x) dx + \ln \frac{\theta_3(z\rho)}{\eta(z\rho)} - 2\pi \rho \sum_{p=1}^{\infty} \frac{\pi^2 \rho}{S} \frac{\Lambda_{2p} K_{2p+1}(z\rho)}{(2p)!} \frac{1}{2p + 2}
\]

where \( S = (2M + 1)(2N + 1) \) is the area of the lattice with an additional row and column, \( \rho = (2M + 1)/(2N + 1) \) is the aspect ratio, \( \eta(\tau) = e^{-\pi \tau /12} \prod_{n=1}^{\infty} \left[ 1 - e^{-2\pi n \tau} \right] \) is the Dedekind eta function, and \( K_{2p+1}^1(z\rho) \) is a special case of Kronecker’s double series [3], defined in general by
\[ K_{\rho}^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{(m,n)\neq(0,0)} e^{-2\pi i (m\alpha + n\beta)} (n + \tau m)^{-p}. \]  

(13)

The functions \( \theta_2(\tau), \theta_3(\tau), \theta_4(\tau) \) are elliptic theta functions with nome \( q = e^{-\pi \tau}, \tau \) being real here. The Dedekind eta function can be expressed as

\[ \eta(\tau) = [\theta_2(\tau)\theta_3(\tau)\theta_4(\tau)/2]^{1/3}. \]  

(14)

Also in equation (12), we have

\[ \omega_z(k) = \arcsinh(z \sin k). \]  

(15)

The differential operators \( \Lambda_{2p} \) that appear in equation (12) can be expressed via coefficients \( z_{2p} \) of the Taylor expansion of the lattice dispersion relation \( \omega_z(k) \) and derivatives over \( z \) (see appendix A). To get result for the dimer model in the isotropic case we should evaluate equations (12) and (15) at \( z = 1 \). From equations (11) and (12) one can find the exact asymptotic expansion of the free energy \( f = \frac{1}{S} \ln Z_{\text{free}}^{M,N} \) for the dimer model in the isotropic case \( (z = 1) \), which can be written as

\[ f = \frac{1}{S} \ln Z_{\text{free}}^{M,N} = f_{\text{bulk}} + \frac{2}{2N+1} + \frac{2}{2M+1} + \frac{f_0(\rho)}{S} + \sum_{p=1}^{\infty} f_p(\rho) S^{-p-1}. \]  

(16)

The bulk free energy \( f_{\text{bulk}} \) is given by

\[ f_{\text{bulk}} = \frac{1}{2\pi} \int_0^\pi \omega_1(x) \, dx = \frac{1}{2\pi} \int_0^\pi \arcsinh(\sin x) \, dx = \frac{G}{\pi}, \]  

(17)

where \( G \) is the Catalan constant. The surface free energies \( f_1s \) and \( f_2s \) are given by

\[ f_{1s} = f_{2s} = -\frac{1}{4} \ln(1 + \sqrt{2}). \]  

(18)

We will show that all finite-size correction terms \( f_p(\rho) \) for \( p = 0, 1, 2, \ldots \) are invariant under the transformation

\[ \rho \to 1/\rho \]  

(19)

as one would expect by the symmetry of the rectangular system. The leading coefficient \( f_0 \) is given by

\[ f_0 = \frac{3}{4} \ln 2 + \frac{1}{2} \ln \frac{\theta_3(\rho)}{\eta(\rho)}. \]  

(20)

It is easy to see from equations (B.1) and (B.4) (see appendix B) that \( f_0(\rho) \) is invariant under transformation given by equation (19)

\[ f_0 \left( \frac{1}{\rho} \right) = f_0(\rho). \]  

(21)

The coefficients \( f_p(\rho) \) are given by

\[ f_p(\rho) = -\frac{\pi^{2p+1} \rho^{p+1}}{(2p)! (2p+2)} \left[ \Lambda_{2p} K_{2p+2}(z\rho) \right]_{z=1}. \]  

(22)

Again, it is easy to see from equation (B.5) that \( f_p(\rho) \) is invariant under transformation given by equation (19):
Comparing equations (3) and (16) one can see that there are no corner terms in the asymptotic expansion of the free energy (equation (16)) for the dimer model $2M \times 2N$ lattice with free boundary conditions, while the rectangular lattice with free boundary conditions has four corners. One can explain that paradox if we consider the bijections between dimer coverings, spanning trees and sandpile models.

The bijection between close-packed dimer coverings of lattice $L$ and spanning trees on the odd–odd sublattice $G \subset L$ is well known [15, 16, 44]. The odd–odd sublattice $G \subset L$ contains the sites whose coordinates are both odd. A dimer containing a site of $G$, in blue in figure 1, can be represented as an arrow directed along the dimer from the odd–odd site to the nearest neighbor site of $G$. It is easy to prove that the resulting set of arrows generates a uniquely defined spanning tree (see figure 1). Since the dimers which do not contain a site of $G$ are completely fixed by the others, one has a one-to-one correspondence between dimer coverings on $L$ and spanning trees on $G$. The above construction leads to a set of spanning trees on the odd–odd sublattice $G$, where certain arrows may point out of the lattice from the right vertical side and upper horizontal side as in figure 1. Viewing these vertical and horizontal boundaries of $G$ as roots for the spanning trees, we see that dimer coverings on $L$ map onto spanning trees on $G$ which can grow from any site of the right and upper sides of the lattice $G$. The sites on these boundaries, being connected to roots, are dissipative in the Abelian sandpile model (ASM) language, and form open boundaries. Thus the spanning trees map onto the ASM configurations with one vertical open and one horizontal open dissipative boundary, and the two other closed boundaries.

Thus also the dimer model on rectangular $2M \times 2N$ lattices with free boundary conditions, which maps to spanning trees, also maps onto the ASM configurations with the one vertical and one horizontal open dissipative boundary, and two closed boundaries. Now the contribution to the free energy from all four corners $f_{\text{corn}}$ of the lattice is given by equation (7), where the summation is taken over four corners of the lattice. As one can see from figure 1, we have two corners with boundary conditions on the two sides of the corner that are the same.
(open–open and closed–closed), and two corners with boundary conditions on the two sides of the corner that are different (open–closed and closed–open).

When boundary conditions on both sides of the corner are the same, the boundary operator inserted at the corner is just the identity operator with $\Delta_{aa} = 0$. In the case when the boundary condition on one side of the corner is $a_i = \text{closed}$ and on the other side it is $b_i = \text{open}$, the boundary operator at the corner, which changes the boundary conditions from open to closed or closed to open, has conformal weight $\Delta_{\text{open, closed}} = \Delta_{\text{closed, open}} = -\frac{1}{8}$ [45]. Thus for $f_{\text{corn}}$ we obtain from equation (7) that

$$f_{\text{corn}} = \sum_{i=1}^{4} \left( \Delta_{a_i, b_i} - \frac{c}{32} \right) = -2 \frac{c}{32} + 2 \left( -\frac{1}{8} - \frac{c}{32} \right) = -\frac{1}{4} - \frac{c}{8}. \tag{24}$$

Now plugging $c = -2$ to equation (24) we obtain that the contribution to the free energy from all four corners $f_{\text{corn}}$ is equal to zero

$$f_{\text{corn}} = 0 \tag{25}$$

which is in full agreement with our results for the exact asymptotic expansion of the free energy for the dimer model on a plane, equation (16). Thus our results suggest that the dimer model can be described by a logarithmic conformal field theory with central charge $c = -2$ [15, 16, 36, 46].

Actually there exists another mapping of the dimer model to the height function model known to be described by a logarithmic conformal field theory with central charge $c = 1$, which can also explain the finite-size behavior of the dimer model on the rectangular lattice. Let us consider the dimer model on the rectangular lattice with free boundary conditions. The choice of these boundary conditions imposes very specific boundary conditions for the height variable. Indeed, the value of the height $h$ is (...0 1 0 1 0...) along one boundary (we call it $a$ boundary condition) until the corner and (...0 −1 0 −1 0...) along the next boundary (we call it $b$ boundary condition) [47]. Thus we have four corners, each of them having two different boundary conditions ($a$ and $b$) on either sides of the corner. The boundary operator at the corner, which changes the boundary conditions from $a$ to $b$, has conformal weight $\Delta_{a,b} = 1/32$ [39]. The contribution to the free energy from a corner with two edges under $a$ and $b$ boundary conditions can be obtained from equation (6)

$$f_{\text{corn}}(a, b) = \Delta_{a,b} - \frac{c}{32} = \frac{1}{32} - \frac{1}{32} = 0. \tag{26}$$

Thus one can see that contribution to the free energy from each corner is equal to zero and as result the total contribution to the free energy from all four corners $f_{\text{corn}}$ is also equal to zero, which again is in full agreement with our results for the exact asymptotic expansion of the free energy for the dimer model on a plane, equation (16).

The leading coefficient $f_0$ is given by equation (20). Comparing equations (4) and (20) we can see a clear difference between our results (see equation (20)) and the result for the universal part $f_{\text{univ}}$ of $f_0$ in the rectangular geometry with free boundary conditions [17] (see equation (4)). Again we can explain that paradox by considering the bijections between dimer coverings, spanning trees and the sandpile model. As we show above, the dimer model on the rectangular $2M \times 2N$ lattice with free boundary conditions can be mapped onto ASM configurations with two open and two closed boundaries as shown in figure 1. Thus our result for $f_0$ is related to the rectangular $2M \times 2N$ lattice with two open and two closed boundaries and
is different from the universal part \( f_{\text{uni}} \), which is calculated by conformal field theory methods for rectangular geometry with free boundary conditions on all sides (see equation (4)).

The subleading coefficients \( f_p(\rho) \) for \( p = 1, 2, 3, \ldots \) are given by equation (22). Now using expressions for \( K_{2p+2}(\rho) \) and \( \Lambda_{2p} \), which are given in appendices A and C, and the following relations between the elliptic functions and derivatives of the elliptic functions

\[
\frac{\partial}{\partial z} \ln \theta_3 = \frac{\pi}{4} \theta_4^2 + \frac{\partial}{\partial z} \ln \theta_2
\]

\[
\frac{\partial}{\partial z} \ln \theta_4 = \frac{\pi}{4} \theta_3^2 + \frac{\partial}{\partial z} \ln \theta_2
\]

\[
\frac{\partial}{\partial z} \ln \theta_2 = -\frac{1}{2} \theta_3^2 E
\]

we can express the subleading correction terms \( f_p(\rho) \) in the asymptotic expansion of the free energy for the dimer model on a \( 2M \times 2N \) rectangular lattice with free boundary conditions for any value of \( \rho \) in terms of the elliptic theta functions \( \theta_2, \theta_3, \theta_4 \) and the elliptic integral of the second kind \( E \). In particular in this paper we have calculated the subleading correction terms \( f_p \) in terms of the elliptic functions the elliptic integral of the second kind up to \( p = 22 \). Due to very large expressions for \( f_p(\rho) \) for \( p > 5 \) we have not listed those expressions here, but give the exact numerical values for \( f_p(\rho) \) for particular value of \( \rho \), namely \( \rho = 1 \) and \( \rho = 2 \), up to \( p = 22 \), in table 1.

The elliptic theta functions and the elliptic integral of the second kind \( E \) at particular values of the aspect ratio \( \rho = 1 \) and \( \rho = 2 \) are given by

\[
\theta_2 = \theta_4 = \frac{(\pi/2)^{1/4}}{\Gamma(3/4)}, \quad \theta_3 = \frac{\pi^{1/4}}{\Gamma(3/4)}, \quad E = \frac{\pi^{3/2}}{4 \Gamma(3/4)^2} + \frac{\Gamma(3/4)^2}{2 \sqrt{\pi}}
\]

(27)

for \( \rho = 1 \) and

\[
\theta_2 = \frac{\pi^{1/4}}{\Gamma(3/4) \sqrt{2(2 + \sqrt{2})}}, \quad \theta_3 = \frac{\pi^{1/4} \sqrt{2 + \sqrt{2}}}{2 \Gamma(3/4)}, \quad \theta_4 = \frac{\pi^{1/4}}{2^{1/8} \Gamma(3/4)},
\]

\[
E = \frac{\pi^{3/2}}{2 \sqrt{2} \Gamma(3/4)^2} + \frac{\Gamma(3/4)^2}{(2 + \sqrt{2}) \sqrt{\pi}}
\]

(28)

for \( \rho = 2 \), where \( \Gamma(z) \) is the gamma function. Now, using the above expressions and the expressions for subleading correction terms \( f_p(\rho) \) in terms of the elliptic theta functions and the elliptic integral, we obtain the exact values for \( f_p(\rho) \) for \( \rho = 1 \) and \( \rho = 2 \) up to \( p = 22 \) (see supplementary materials (stacks.iop.org/JPhysA/52/335001/mmedia)). The difference between exact \( f_{\text{exact}}(N) \) and asymptotic value \( f_{\text{asympt}}(N) \) (for \( p_{\text{max}} = 2 \) and 10) of the free energy for dimer model on the square lattice with side 2\( N \) are given in table 2. Here \( p_{\text{max}} \) is a cut off of the asymptotic series, equation (16).

In figure 2 we plot the behavior of the subleading correction terms \( \rho^{\pm(p+1)} f_p(\rho) \) (+ for \( \rho > 1 \) and − for \( \rho < 1 \)) for (a) \( p = 0 \), (b) \( p = 1 \), (c) \( p = 2 \), and (d) \( p = 7 \) as a function of the aspect ratio \( \rho \).
Table 1. Coefficients $f_p$ in the asymptotic expansion of the free energy, for the square, rectangle of aspect ratio 2, and the infinite strip.

| $p$ | $f_p^{\text{sq}}$ | $f_p(2)$ | $f_p^{\text{strip}}$ |
|-----|-------------------|----------|---------------------|
| 0   | 0.693147180559... | 0.783255480959... | 0.130893693899... |
| 1   | 0.188074112777... | 0.320716568887... | 0.075362478042... |
| 2   | 0.336874969732... | 1.620602115733... | 0.196069159812... |
| 3   | 3.290053550462... | 2.50926174993... | 10^1 | 1.565734461321... |
| 4   | 4.81457307505... | 7.84426067442... | 10^2 | 2.44962573404... | 10^3 |
| 5   | 1.27811955024... | 4.66639855884... | 10^4 | 6.35177661759... | 10^2 |
| 6   | 4.91797597262... | 3.15446270890... | 10^6 | 2.46425620918... | 10^4 |
| 7   | 2.67530735325... | 3.42195018099... | 10^8 | 1.35668213673... | 10^6 |
| 8   | 1.93146021814... | 4.94576151005... | 10^10 | 9.65965464746... | 10^7 |
| 9   | 1.79421498557... | 9.18564336803... | 10^12 | 8.97034559509... | 10^9 |
| 10  | 2.08172298698... | 2.13174241269... | 10^15 | 1.04088960155... | 10^12 |
| 11  | 2.95165319167... | 6.04493217496... | 10^17 | 1.47581343488... | 10^14 |
| 12  | 5.02086224796... | 2.05655123614... | 10^20 | 2.51043848305... | 10^16 |
| 13  | 1.00920433549... | 8.26739400791... | 10^22 | 5.04601683138... | 10^18 |
| 14  | 2.36629755293... | 3.87694313074... | 10^25 | 1.18314914769... | 10^21 |
| 15  | 6.40123719928... | 2.09755719148... | 10^28 | 3.20061827240... | 10^23 |
| 16  | 1.97885518164... | 1.29686257503... | 10^31 | 9.89427623712... | 10^25 |
| 17  | 6.93253270671... | 9.08660917146... | 10^33 | 3.46626631597... | 10^28 |
| 18  | 2.73210457870... | 7.16204825184... | 10^36 | 1.36605229412... | 10^31 |
| 19  | 1.2033420978... | 6.30893685465... | 10^39 | 6.01667104210... | 10^33 |
| 20  | 5.8886356106... | 6.17468197571... | 10^42 | 2.94431780611... | 10^36 |
| 21  | 3.18488175869... | 6.67918114922... | 10^45 | 1.59244087916... | 10^39 |
| 22  | 1.89474701875... | 7.94714500005... | 10^48 | 9.47373509413... | 10^41 |

Table 2. The exact free energy $f^{\text{exact}}(N)$ for a square with side $2N$ calculated from equation (1), and the difference between exact and asymptotic values of the free energy $f^{\text{asympt}}_p(N)$ (for $p_{\text{max}} = 2$ and 10), where $p_{\text{max}}$ is a cut-off of the asymptotic series, equation (16).

| $N$ | $f^{\text{exact}}(N)$ | $f^{\text{exact}}(N) - f^{\text{asympt}}_{2}(N)$ | $f^{\text{exact}}(N) - f^{\text{asympt}}_{10}(N)$ |
|-----|-----------------------|-----------------------------------------------|-----------------------------|
| 2   | 0.143340757538244006 | 6.205110750 × 10^{-6} | −0.00114988029 |
| 4   | 0.20221727435649946996 | 9.710971517 × 10^{-8} | −6.123675796 × 10^{-9} |
| 8   | 0.2421189033545298981 | 4.980854240 × 10^{-10} | 1.777260867 × 10^{-15} |
| 16  | 0.26514827115575958947 | 2.371559964 × 10^{-12} | 1.266298133 × 10^{-22} |
| 32  | 0.278165379921913838 | 1.036113289 × 10^{-14} | 9.508968901 × 10^{-30} |
| 64  | 0.28477020429871780721 | 4.294055324 × 10^{-17} | 6.612762353 × 10^{-37} |
| 128 | 0.2881411930242180726 | 1.729156214 × 10^{-19} | 4.293295929 × 10^{-44} |
| 256 | 0.28984546071399267489 | 6.859432660 × 10^{-22} | 2.676266774 × 10^{-51} |
4. Dimer model on an infinitely long strip

Let us consider the case of an infinitely long strip ($M \to \infty$), with $N$ fixed. The free energy per site ($f_{\text{strip}}(N)$) for that case can be written as

$$
\begin{align*}
\frac{1}{S} \ln Z_{\text{free}}^{2M,2N} &= - \arcsinh \frac{1}{2(2N + 1)} + \lim_{M \to \infty} \frac{1}{2S} \ln Z_{\frac{1}{2}, \frac{1}{2}}^{1, 1, 2M + 1, 2N + 1} \\
\end{align*}
$$

where $S = (2M + 1)(2N + 1)$ is the area of the lattice, with an extra row and column added. In the case of an infinitely long strip ($M, \rho \to \infty$), where $\rho = (2M + 1)/(2N + 1)$, we have

$$
\lim_{M \to \infty} \theta_3(\rho) = 1
$$

and

$$
\lim_{M \to \infty} \eta(\rho) = \lim_{\rho \to \infty} e^{-\pi \rho/12} = 0
$$
\[
\lim_{M \to \infty} \Lambda_2 p K_{2p+2}^1 (\rho) = z_{2p} B_{2p+2}^{1/2}
\]  
(32)

where \( B_{2p+2}^{1/2} \) are Bernoulli polynomials \( B_n(z) \) evaluated at \( z = 1/2 \): \( B_{1/2}^1 = -1/12, B_{1/2}^2 = 7/240, B_{3/2}^1 = -31/1344, B_{3/2}^2 = 127/3840, B_{5/2}^1 = -2555/33792, \ldots \), related to the Bernoulli numbers \( B_n \) by

\[
B_n^{1/2} = \frac{1}{2} B_n \left( \frac{1}{2} \right)
\]

and equation (12) can be rewritten as

\[
f_{\text{strip}} (N) = - \arcsinh \left( \frac{1}{2} \right) + \lim_{M \to \infty} \frac{1}{2} \ln Z_{1, \frac{1}{2}} (2M + 1, 2N + 1)
\]

\[
= \frac{2G}{\pi} - \arcsinh \left( \frac{1}{2} \right) + \frac{\pi}{12} \frac{1}{(2N + 1)^2} - \sum_{p=1}^{\infty} \frac{2\pi^{2p+1}}{(2N + 1)^{2p+2}} \frac{z_{2p} B_{2p+2}^{1/2}}{(2p)!} 2p + 2
\]

(33)

Thus the exact asymptotic expansion of the free energy per site for the dimer model on an infinitely long strip can be written as

\[
f_{\text{strip}} (N) = f_{\text{bulk}} + \frac{2 \pi \, f_s}{2N + 1} + \sum_{p=0}^{\infty} \frac{f_{\text{strip}}^p}{(2N + 1)^{2p+2}}.
\]

(34)

The bulk free energy \( f_{\text{bulk}} \) is given by

\[
f_{\text{bulk}} = \frac{G}{\pi},
\]

where \( G \) is the Catalan constant. The surface free energy \( f_s \) is given by

\[
f_s = f_{1s} = -\frac{1}{4} \ln (1 + \sqrt{2})
\]

(35)

and the coefficients \( f_{\text{strip}}^p \) are given by

\[
f_{\text{strip}}^p = -\pi^{2p+1} \frac{z_{2p} B_{2p+2}^{1/2}}{(2p)! (2p + 2)}.
\]

(36)

The leading \( f_{\text{strip}}^0 \) is given by

\[
f_{\text{strip}}^0 = \frac{\pi}{24} = 0.130 899 693 899 5747\ldots
\]

(37)

For the reader’s convenience we list subleading strip coefficients \( f_{\text{strip}}^p \) up to 22nd order in supplementary materials.

Using known relations for the asymptotic behavior of the \( z_{2p} \) [48] and the \( B_{2p+2}^{1/2} \), we can derive the following formula for the asymptotic behavior of \( f_{\text{strip}}^p \): 

\[
f_{\text{strip}}^p \sim 2^{1/4} \pi^{-1} (p/e)^{2p} [\log (1 + \sqrt{2})]^{-2p-1/2}
\]

(38)

which shows the rapid growth of \( f_{\text{strip}}^p \) with \( p \).

On the basis of conformal invariance, the asymptotic finite-size scaling behavior of the free energy per site of an infinitely long strip of finite width \( \mathcal{N} \) at criticality is expected to have the form [10, 11]

\[
f = f_{\text{bulk}} + \frac{2 f_s}{\mathcal{N}} + \frac{f_0}{\mathcal{N}^2} + \ldots
\]

(39)
where $N$ equal $2N + 1$ in our problem. The value of $f_0$ is related to the conformal charge $c$ of the underlying conformal theory and depends on the boundary conditions in the transversal direction; in the strip geometry it is given by

$$f_0 = \pi \left( \frac{c}{24} - \Delta \right)$$

(40)

where the number $\Delta$ is the conformal weight of the operator with the smallest scaling dimension present in the spectrum of the Hamiltonian with the given boundary conditions.

Again we can explain our result (see equation (37)) in the framework of conformal field theory with two consistent values of the central charge, namely, $c = -2$ for the construction of a conformal field theory using a mapping of spanning trees and $c = 1$ for the height function description. Using a mapping of the dimer model to the spanning trees ($c = -2$), in the case of infinitely long strips we have open–closed boundary conditions in the transversal direction and the conformal weight of the operator with open–closed boundary conditions is equal to $\Delta = -1/8$. Plugging these values of $c$ and $\Delta$ to equation (40) we obtain

$$f_0 = \frac{\pi}{24}$$

(41)

in full agreement with our result (see equation (37)).

Now using a mapping of the dimer model to the height function model ($c = 1$), in the case of infinitely long strip we have $a$–$a$ or $b$–$b$ boundary conditions in the transversal direction and the conformal weight of the operator with $a$–$a$ or $b$–$b$ boundary conditions is equal to $\Delta = 0$. Plugging these values of $c$ and $\Delta$ to equation (40) we again obtain equation (41) in full agreement with our result (see equation (37)).

Once again we confirm that the dimer model on a $2M \times 2N$ rectangular lattice with free boundary conditions can be considered as a model with two open and two closed boundary conditions as shown in figure 1 or as a height function model with alternating $a$ and $b$ boundary conditions.

It is easy to show that

$$f^\text{strip}_p = \lim_{\rho \to \infty} f^\text{strip}_p(\rho) \rho^{-p-1}.$$  

(42)

For let us consider the exact asymptotic expansion of the free energy per site for the dimer model on $2M \times 2N$ rectangular lattice on a plane (see equation (16)). Taking the limit $M \to \infty$ one can obtain the exact asymptotic expansion of the free energy per site for the dimer model on an infinitely long strip

$$f^\text{strip}(N) = \lim_{M \to \infty} \frac{1}{S} \ln Z^\text{free}_{2M,2N} = f^\text{bulk} + \frac{f_{1s}}{2N + 1} + \lim_{M \to \infty} \sum_{p=0}^{\infty} f^\text{strip}_p S^{-p-1}.$$  

(43)

Here $S = (2M + 1)(2N + 1)$ can be rewritten as $S = \rho(2N + 1)^2$, where $\rho = (2M + 1)/(2N + 1)$ is the aspect ratio. Plugging $S = \rho(2N + 1)^2$ back to equation (43) we obtain

$$f^\text{strip}(N) = \lim_{M \to \infty} \frac{1}{S} \ln Z^\text{free}_{2M,2N} = f^\text{bulk} + \frac{2f_{1s}}{2N + 1} + \sum_{p=0}^{\infty} \lim_{\rho \to \infty} f^\text{strip}_p(\rho) \rho^{-p-1}.$$  

(44)

Now comparing equations (34) and (44) we obtain equation (42).
5. Exact results on the strip

For the strip of width 2N, we can also derive a simple exact expression for \( f^{\text{strip}}(N) \) with a single sum. With the help of the identity

\[
\prod_{m=0}^{M-1} \left[ \sinh^2 \omega + \sin^2 \left( \frac{\pi (m + 1/2)}{M} \right) \right] = 4 \cosh^2 (M \omega)
\]  

(45)

the partition function the partition function \( Z_{\frac{1}{2}, \frac{1}{2}}(z, \mathcal{M}, N) \) given by equation (10) can be transformed into simpler form

\[
Z_{\frac{1}{2}, \frac{1}{2}}(z, \mathcal{M}, N) = \prod_{n=0}^{N-1} 2 \cosh \left[ \mathcal{M} \omega_n \left( \frac{\pi (n + 1/2)}{N} \right) \right]
\]  

(46)

where \( \omega_n(k) \) is given by equation (15). In the limit \( \mathcal{M} \to \infty \) from equation (46) we can obtain

\[
\lim_{\mathcal{M} \to \infty} \left[ Z_{\frac{1}{2}, \frac{1}{2}}(z, \mathcal{M}, N) \right] = \prod_{n=0}^{N-1} e^{\arcsinh \left( \frac{\pi (n + 1/2)}{2N+1} \right)}
\]  

(47)

Thus the free energy per site \( (f^{\text{strip}}(N)) \) for infinitely long strip given by equation (29) can be further simplified as

\[
f^{\text{strip}}(N) = \lim_{M \to \infty} \frac{1}{S} \ln \left( \frac{Z_{\text{free}}^{\text{strip}}(2M, 2N)}{Z_{\text{free}}^{\text{strip}}(2M, 2N+1)} \right) = -\frac{\arcsinh 1}{2(2N+1)} + \frac{1}{2(2N+1)} \sum_{n=0}^{2N} \arcsinh \left( \frac{\pi (n + 1/2)}{2N+1} \right)
\]  

(48)

\[
= \frac{1}{2N+1} \sum_{n=0}^{N-1} \arcsinh \left( \frac{\pi (n + 1/2)}{2N+1} \right)
\]  

(49)

\[
= \frac{1}{2N+1} \sum_{n=0}^{N-1} \ln \left( \frac{\pi (n + 1/2)}{2N+1} + \sqrt{1 + \sin^2 \frac{\pi (n + 1/2)}{2N+1}} \right)
\]  

(50)

or equivalently

\[
Z^{\text{strip}}(N) = e^{(2N+1)f^{\text{strip}}(N)} = \prod_{n=0}^{N-1} \left( \frac{\sin \frac{\pi (n + 1/2)}{2N+1} + \sqrt{1 + \sin^2 \frac{\pi (n + 1/2)}{2N+1}}}{2N+1} \right)
\]  

(51)

In transition from equations (48) and (49) we first split summation in equation (48) to two parts

\[
\sum_{n=0}^{2N} \to \sum_{n=0}^{N-1} + \sum_{n=N}^{N-1}
\]

and then in second sum we have changed variable \( n \to 2N - n \).
Note, that with the help of the Euler–Maclaurin summation formula (see, for example, appendix A [3]) one can find from equation (48) the asymptotic expansion of the free energy per site for the dimer model on an infinitely long strip given by equation (33).

With equation (49) one can easily calculate \( f_{\text{strip}}(N) \) to high precision for \( N \) up to \( 10^6 \).

In comparison, equation (34) represents an asymptotic series, which is non-convergent for a fixed \( N \). Typical of asymptotic series, the difference between successive terms decreases as \( p \) increases, up to a point, and then starts increasing again, eventually exponentially. The best approximation typically occurs when the contribution of the last term in the sum is at a minimum. In this case, the coefficients of the asymptotic series are all positive, so the partial sums in equation (16) are a monotonically increasing function of \( p \). In table 3 we show exact values of \( f \) calculated from equation (49), and compare them with the results of the asymptotic series cut off when the last term is a minimum, to the precision of the last term. As one can see, the exact value and asymptotic value truncated to the precision of the last term are in excellent agreement. Similar behavior will occur for the asymptotic series for the rectangle, equation (16).

The approach from a rectangle to a strip is also interesting to study. From equation (16) one can show that for \( N \) fixed, and large but finite \( M \gg N \), one has

\[
f(N, M) \sim f_{\text{strip}}(N) + \frac{a(N)}{2M + 1} \tag{52}
\]

where \( a(N) = -\frac{1}{2} \ln(1 + \sqrt{2}) + \frac{3}{4} \ln 2/(2N + 1) \), the latter term reflecting a contribution to the asymptotic behavior from \( f_0 \) of equation (20). There are additional corrections that decay exponentially with both \( N \) and \( M \). For example, for \( N = 5 \), \( a(5) = -0.39342675847159342337857 \), but actually \( \lim_{M \to \infty}(2M + 1)(f(5, M) - f_{\text{strip}}(5)) = -0.3934267586439834913176 \), the difference with \( a(5) \) being just \( -1.72390067979787 \times 10^{-10} \). As \( N \) increases, the difference between the observed coefficient to \( 1/(2M + 1) \) and \( a(N) \) decays exponentially. Likewise, the additional corrections to equation (52) with respect to \( M \) also decay exponentially. Plotting \( f(5, M) \) versus \( 1/(2M + 1) \), for example, for \( M = 10000 \), \( 100000 \) and \( 1000000 \), yields a nearly perfect straight line with slope \(-0.393426758638 \), close to value given above, and an intercept of \( 0.2525855505199970 \) that exactly confirms the prediction of \( f_{\text{strip}}(5) \) of equation (49) which gives the identical value to this precision. For large \( N \) and \( M \) with \( M \gg N \), equation (52) exhibits essentially the entire difference between \( f(N, M) \) and \( f_{\text{strip}}(N) \).

6. The ratio of the coefficients \( f_p(\rho) \) in the free energy expansion

Let us consider the ratio of the coefficients \( f_p \) in the free energy expansion for strip \( f_{\text{strip}}^p \) and square \( f_{\text{sq}}^p = f_p(1) \) geometry

\[
r_p = \frac{f_{\text{strip}}^p}{f_{\text{sq}}^p}.
\]

One can clearly see from table 4 that the ratio tends to \( 1/2 \) as \( p \) tends to \( \infty \)

\[
\lim_{p \to \infty} r_p = \frac{1}{2}.
\]

The values of \( r_p \) oscillate above and below \( 1/2 \) with an exponential decay \( \approx e^{-1.12p} \). Let us consider ratio of coefficients \( f_p \) in free energy expansion for rectangular geometry (for arbitrary \( \rho > 1 \)) and square geometry \( (\rho = 1) \) multiplied by \( \rho^{-p-1} \).
\[ r_p(\rho) = \frac{f_p(\rho)\rho^{p-1}}{f_p^{\text{sq}}(\rho)} \text{ for } \rho > 1. \]  

(54) 

Again one can see from table 4 that for \( \rho = 2 \) the ratio \( r_p(2) \) also tends to 1/2 as \( p \) tends to \( \infty \) with an oscillating exponential decay. In fact, for large \( n \) the ratios are almost exactly the same as those for the square system (\( \rho = 1 \)). Furthermore, we find that for arbitrary \( \rho > 1 \), \( r_p(\rho) \) is equal to \( \frac{1}{2} \) in the limit \( p \to \infty \), namely 

\[ \lim_{p \to \infty} r_p(\rho) = \frac{1}{2} \text{ for } \rho > 1. \]  

(55) 

It is easy to see from equations (23) and (54) that \( r_p(\rho) \) transforms under Jacobi transformation \( \rho \to 1/\rho \) as 

\[ r_p \left( \frac{1}{\rho} \right) = \tilde{r}_p(\rho) \]  

(56) 

where the ratio \( \tilde{r}_p(\rho) \) is given by 

\[ \tilde{r}_p(\rho) = \frac{f_p(\rho)\rho^{p+1}}{f_p^{\text{sq}}(\rho)}. \]
Thus one can obtain from equation (56) that for $\rho < 1$ the ratio $\bar{r}_p(\rho)$ tends to $1/2$ as $p$ tends to infinity

$$\lim_{p \to \infty} \bar{r}_p(\rho) = \frac{1}{2} \quad \text{for} \quad \rho < 1. \quad (57)$$

Thus, we find that if the shape $\rho$ of the rectangular $2M \times 2N$ lattice is varied, the ratios $r_p(\rho)$ (for $\rho > 1$) and $\bar{r}_p(\rho)$ (for $\rho < 1$) both tend to $1/2$ as $p$ tends to infinity.

In figure 3 we plot the ratio $\rho^{-p-1}r_p(\rho)/f^\text{sq}$ as a function of $p$ for (a) $\rho = 1$, (b) $\rho = 2$, (c) $\rho = 3$, and (d) $\rho = 7$.

Note that equation (55) cannot apply when $\rho = 1$, because then the rectangle is a square and $r_p(\rho = 1)$ should be 1. Indeed, we find that as $p$ becomes large, the behavior of $r_p(\rho)$ as a function of $\rho$ becomes a step function at $\rho = 1$, as shown in figure 4.

The results given by equations (53), (55) and (57) are very surprising results, and we do not have any physical explanation why those ratios should be finite and equal $1/2$. To get more insight on these effects we plan to consider the ratio $r_p(\rho)$ in other models, such as the spanning tree model under different boundary conditions and the Ising model with Brascamp–Kunz boundary conditions, in the near future.

| $p$   | $r_p = \frac{f^\text{strip}}{f^\text{sq}}$ | $r_p = \frac{2^{-p-1}r_p(2)}{f^\text{sq}}$ |
|-------|------------------------------------------|------------------------------------------|
| 0     | 0.188848 339 243...                     | 0.565 194 162 895...                     |
| 1     | 0.400706 280 144...                     | 0.426 316 738 242...                     |
| 2     | 0.582023 532 257...                     | 0.601 336 607 490...                     |
| 3     | 0.475899 439 722...                     | 0.476 675 704 408...                     |
| 4     | 0.508793 966 746...                     | 0.509 148 251 059...                     |
| 5     | 0.496962 636 744...                     | 0.497 116 857 889...                     |
| 6     | 0.501071 217 692...                     | 0.501 105 333 933...                     |
| 7     | 0.499636 849 243...                     | 0.499 643 260 737...                     |
| 8     | 0.500121 853 753...                     | 0.500 123 707 369...                     |
| 9     | 0.499959 350 873...                     | 0.499 958 667 800...                     |
| 10    | 0.500013 502 309...                     | 0.500 136 213 630...                     |
| 11    | 0.499995 541 158...                     | 0.499 959 569 751...                     |
| 12    | 0.500001 465 697...                     | 0.500 001 473 193...                     |
| 13    | 0.499999 519 812...                     | 0.499 999 521 710...                     |
| 14    | 0.500000 156 880...                     | 0.500 001 573 343...                     |
| 15    | 0.499999 948 878...                     | 0.499 999 948 993...                     |
| 16    | 0.500000 016 621...                     | 0.500 001 665 651...                     |
| 17    | 0.499999 994 606...                     | 0.499 999 994 614...                     |
| 18    | 0.500000 001 747...                     | 0.500 000 174 949...                     |
| 19    | 0.499999 999 435...                     | 0.499 999 999 435...                     |
| 20    | 0.500000 000 183...                     | 0.500 000 183 183...                     |
| 21    | 0.499999 999 941...                     | 0.499 999 999 941...                     |
| 22    | 0.500000 000 019...                     | 0.500 000 019 019...                     |
Figure 3. (a) The ratio $f_{\text{strip}}/f_{\text{sq}}$ as a function of order $p$ for the square ($\rho = 1$). Dots represent our exact results. The solid line is given by $(-1)^p a b^p + 1/2$, with $a = 0.6561$ and $b = 0.3348$ for all four plots, and represents the exponential decay to 1/2. (b) The ratio $\rho^{-p-1} f_{\rho}(\rho)/f_{\rho}$ as function of $\rho$ for $\rho = 2$, (c) for $\rho = 3$, and (d) for $\rho = 7$.

Figure 4. Plot of the ratio (54) as a function of $\rho$ for various $p$ (dotted line for $p = 5$, dotted-dashed line for $p = 7$, and dashed line for $p = 16$), showing that $r(\rho)$ becomes a step function at $\rho = 1$ as $p \to \infty$. 

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7. Conclusion

We have analyzed the partition function of the dimer model on a rectangular $2M \times 2N$ lattice with free boundary conditions. We have obtained that the corner free energy for the dimer model on a rectangular $2M \times 2N$ lattice with free boundary conditions is equal to zero, due to the strong nonlocality of the dimer model, and explain that result within the framework of conformal field theory with two consistent values of the central charge, namely, $c = -2$ for the construction of a conformal field theory using a mapping of spanning trees and $c = 1$ for the height function description. Using the results of [4] we have calculated the exact expressions for the correction terms $f_p(\rho)$ in the asymptotic expansion of the free energy in the dimer model from $p = 0$ up to $p = 22$. We also found the general asymptotic formula as well as a simple exact expression for the strip. We find that ratio of coefficients $f_p(\rho)$ in the free energy expansion for the strip ($f_p^{\text{strip}}$) and square ($f_p^{\text{sq}}$) geometries $r_p = f_p^{\text{strip}} / f_p^{\text{sq}}$, in the limit of large $p$, tends to 1/2. Furthermore, we predict that the ratio of coefficients $f_p(\rho)$ in free energy expansion for rectangular ($f_p(\rho)$ for $\rho > 1$) and square ($f_p^{\text{sq}}$) geometries, multiplied by $\rho^{p-1}$, that is $r_p = \rho^{p-1}f_p(\rho)/f_p^{\text{sq}}$, is also equal to 1/2 in the limit of $p \to \infty$, and using the Jacobi transformation we find that the ratio $r_p = \rho^{p+1}f_p^{\text{sq}}/f_p(\rho)$ is also equal to 1/2 for $\rho < 1$ in the limit of $p \to \infty$. Knowing these results allows one to directly find the asymptotic behavior of the $f_p(\rho)$ from the asymptotic behavior of $f_p^{\text{amp}}$, which is given in equation (38). These ratios are very surprising simple results, and to gain some insight on them we plan to study the ratio $r_p(\rho)$ in other models, such as the spanning tree model under different boundary conditions and the Ising model with Brascamp–Kunz boundary conditions.

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Appendix A. Expressions for $\Lambda_{2p}$

The differential operators $\Lambda_{2p}$ that appear in equation (12) can be expressed via coefficients $z_{2p}$ of the Taylor series expansion of the lattice dispersion relation $\omega_z(k)$ of equation (15) about $k = 0$, taking $z = 1$ here because we will eventually take $z = 1$ when evaluating (22). We have:

$$\omega_z(k) = k \left( 1 + \sum_{p=1}^{\infty} \frac{z_{2p}}{(2p)!} k^{2p} \right)$$

(A.1)

with $z_2 = -2/3$, $z_4 = 4$, $z_6 = -632/7$, $z_8 = 39440/9$, $z_{10} = -4087712/11$, $z_{12} = 48787520$, $z_{14} = -137605112192/15$, $z_{16} = 2339775197440$, $z_{18} = -14775064298435072/19$, $z_{20} = 6857795892626969600/21$, etc. These $z_{2p}$ can be found simply from equation (15) using Mathematica and are given in the OEIS [48]. Then we have
\[ \Lambda_2 = -\frac{2}{3}, \]
\[ \Lambda_4 = 4 + \frac{4}{3} \frac{\partial}{\partial z}, \]
\[ \Lambda_6 = -\frac{632}{7} - 40 \frac{\partial}{\partial z} - \frac{40}{9} \frac{\partial^2}{\partial z^2}, \]
\[ \vdots \]
\[ \Lambda_p = \sum_{r=1}^{p} \sum \left( \frac{z_{p1}}{p_1!} \right)^{k_1} \cdots \left( \frac{z_{pr}}{p_r!} \right)^{k_r} \frac{p!}{k_1! \cdots k_r!} \frac{\partial^k}{\partial z^k}. \]  
(A.2)

Here summation is over all positive numbers \( \{k_1 \ldots k_r\} \) and different positive numbers \( \{p_1, \ldots, p_r\} \) such that \( p_1 k_1 + \ldots + p_r k_r = p \) and \( k_1 + \ldots + k_r - 1 = k \). In Supplementary Materials we have listed expressions for the \( \Lambda_{2p} \) up to \( p = 22 \).

**Appendix B. Jacobi transformation**

Let us consider the behavior of the theta functions, the Dedekind eta function, and the Kronecker functions \( K_{2p}^{1/2, 1/2} \) under the Jacobi transformation (19). The result for the theta functions and the Dedekind eta function is given by [49]:

\[ \theta_3 \left( \frac{1}{\rho} \right) = \sqrt{\rho} \theta_3(\rho) \]  
(B.1)
\[ \theta_2 \left( \frac{1}{\rho} \right) = \sqrt{\rho} \theta_4(\rho) \]  
(B.2)
\[ \theta_4 \left( \frac{1}{\rho} \right) = \sqrt{\rho} \theta_2(\rho) \]  
(B.3)
\[ \eta \left( \frac{1}{\rho} \right) = \sqrt{\rho} \eta(\rho). \]  
(B.4)

The result for the Kronecker functions \( K_{2p}^{1/2, 1/2} \) can be obtained from the relation between the coefficients in the Laurent expansion of the Weierstrass function and Kronecker functions (see appendix C) and is given by

\[ K_{2p}^{1/2, 1/2} \left( \frac{1}{\rho} \right) = \rho^{2p} K_{2p}^{1/2, 1/2}(\rho). \]  
(B.5)

**Appendix C. Reduction of Kronecker’s double series to theta functions**

The Kronecker functions \( K_{2p}^{0,0}(\tau) \) are related directly to the coefficients \( a_p(\tau) \)

\[ K_{2p}^{0,0}(\tau) = -\frac{(2p)!}{(-4\pi^2)^p (2p-1)} \frac{a_p(\tau)}{(2p-1)!}. \]  
(C.1)
where $a_p(\tau)$ are the coefficients in the Laurent expansion of the Weierstrass function $\wp(z)$ with two periods $\omega_1 = 1$ and $\omega_2 = \tau$

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m)\neq(0,0)} \left[ \frac{1}{(z-nm)^2} - \frac{1}{(z+n\tau)^2} \right]$$

$$= \frac{1}{z^2} + \sum_{p=2}^{\infty} a_p(\tau)z^{2p-2}.$$

The coefficients $a_p(\tau)$ can all be written in terms of the elliptic theta functions with the help of the recursion relation (see [49] page 749)

$$a_p = \frac{3}{(p-3)(2p+1)} \left( a_2a_{p-2} + a_3a_{p-3} + \ldots + a_{p-2}a_2 \right)$$

where first terms of the sequence are

$$a_2 = \frac{\pi^4}{15} \left( \theta_2^8 + \theta_4^2 \theta_4^4 + \theta_4^8 \right)$$

$$a_3 = \frac{2^6}{189} \left[ 2(\theta_2^12 - \theta_4^12) + 3(\theta_4^1 - \theta_2^1)\theta_2^2 \theta_4^2 \right]$$

$$a_4 = \frac{1}{3} a_2^3$$

$$a_5 = \frac{3}{11} a_2 a_3$$

The coefficients $a_2$ and $a_3$ are related to coefficients $g_2$ and $g_3$, (see for example [49] p 749 and 767.)

The Kronecker function $K_{\frac{1}{2p}}^{\frac{1}{2}}(\tau)$ can in turn be related to the function $K_{2p}^{0,0}(\tau)$ by means of simple resummation of Kronecker’s double series

$$K_{\frac{1}{2p}}^{\frac{1}{2}}(\tau) = (1 + 2^{1-2p}) K_{2p}^{0,0}(\tau) - 2^{1-2p} K_{2p}^{0,0}(\tau/2) - 2 K_{2p}^{0,0}(2\tau).$$

Thus, Kronecker functions $K_{\frac{1}{2p}}^{\frac{1}{2}}(\tau)$ can all be expressed in terms of the elliptic theta functions only. For practical calculations the following identities are also helpful

$$2\theta_2^2(2\tau) = \theta_2^3 - \theta_4^3 \quad \theta_2^2(\tau/2) = 2\theta_2^3\theta_3$$

$$2\theta_4^2(2\tau) = \theta_2^3 + \theta_4^3 \quad \theta_2^2(\tau/2) = \theta_2^3 + \theta_3^3$$

$$2\theta_2^3(2\tau) = 2\theta_3\theta_4 \quad \theta_2^2(\tau/2) = \theta_2^2 - \theta_2^2.$$

From the general formulas above we can easily write down the Kronecker functions $K_{\frac{1}{2p+2}}^{\frac{1}{2}}(\tau)$ for all value of $p$. In particular in Supplementary Materials we have listed expressions for the Kronecker functions $K_{\frac{1}{2p+2}}^{\frac{1}{2}}(\tau)$ up to $p = 11$. 

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Appendix D. Expressions for $f_p(\rho)$ from $p = 0$ up to $p = 5$

These give the explicit asymptotic coefficients $f_p(\rho)$ that enter in equation (16) for an arbitrary aspect ratio $\rho$, for order $p = 0$ to $p = 5$.

$$ f_0 = \frac{3}{4} \ln 2 + \frac{1}{2} \ln \frac{\theta_i}{\eta} $$

$$ f_1 = \frac{\pi^3 \rho^4}{12} \left( \frac{11}{120} \theta_1^3 \theta_4^2 + \frac{7}{240} (\theta_2^3 + \theta_4^3) \right) $$

$$ f_2 = \frac{\pi^6 \rho^4}{48384} \theta_4^2 (-5 \theta_1^2 + 5 \theta_2^4 + 4 \theta_4^4 + 2 \theta_4^2 + 3 \theta_4^3) $$

$$ + \frac{\pi^5 \rho^3}{48384} \left( 31 \theta_2^2 + 15 \theta_4^2 + 31 \theta_4^3 \right) \left( 2 \rho E \sqrt{\theta_2^2 + \theta_4^2} - \frac{7}{12} \right) $$

$$ f_3 = \frac{\pi^9 \rho^6}{829440} \left( \theta_5^2 + \theta_4^2 \right)^2 \left( 5 \theta_1^2 + 11 \theta_2^4 + 4 \theta_4^4 + 3 \theta_4^2 + 38 \theta_4^3 - 1 \right) $$

$$ + \frac{\pi^8 \rho^5}{138240} \theta_2^2 \left( 710 \theta_1^2 + 930 \theta_2^4 + 213 \theta_4^2 + 127 \theta_4^3 \right) \left( \theta_2^2 + \theta_4^2 - 2 \rho E \left( \theta_2^2 + \theta_4^2 \right)^{3/2} \right) $$

$$ + \frac{\pi^7 \rho^4}{19353600} \left( 127 \theta_1^4 + 284 \theta_2^4 \theta_2^4 + 186 \theta_4^2 \theta_4^4 + 284 \theta_4^2 \theta_4^4 + 127 \theta_1^4 \theta_4^4 \right) $$

$$ \times \left( 280 \rho E \left( \theta_2^2 + \theta_4^2 \right) - 280 \rho E \sqrt{\theta_2^2 + \theta_4^2} + 79 \right) $$

$$ f_4 = \frac{\pi^{10} \rho^8 \theta_4^2 \left( \theta_2^2 + \theta_4^2 \right)^2}{7962624} \left( \theta_5^0 + 10 \theta_2^4 \theta_2^4 + 14 \theta_4^4 \theta_4^4 + 67 \theta_4^4 \theta_4^4 + 10 \theta_1^4 \theta_2^4 + 5 \theta_4^4 \theta_4^4 \right) $$

$$ + \frac{\pi^{11} \rho^8 \theta_2^4}{1327104} \left( 3 \theta_4^4 + 62 \theta_2^4 \theta_2^4 + 16 \theta_4^4 \theta_4^4 + 3 \theta_4^4 \theta_4^4 + 25 \theta_4^4 \theta_4^4 \right) \left( \theta_2^2 + \theta_4^2 \right)^{3/2} $$

$$ + \frac{\pi^9 \rho^6 \theta_2^4 \theta_4^2}{510935040} \left( 126 \theta_1^4 \theta_2^4 - 4 \theta_2^2 \theta_2^2 + 30 \theta_4^2 \theta_4^4 + 5 \theta_4^2 \theta_4^4 + 5 \theta_4^2 \theta_4^4 + 25 \theta_4^4 \theta_4^4 \right) $$

$$ \times \left( 421 - 1540 \rho E \left( \theta_2^2 + \theta_4^2 + \rho E \left( \theta_2^2 + \theta_4^2 \right) \right) \right) $$

$$ + \frac{\pi^{10} \rho^8 \theta_2^4 \theta_4^2}{306561024} \left( 5 \theta_1^4 \theta_2^4 + 126 \theta_1^4 \theta_2^4 + 100 \theta_4^4 \theta_1^4 + 10 \theta_4^4 \theta_1^4 + 2 \theta_4^4 \theta_4^4 + 5 \theta_4^4 \theta_4^4 \right) $$

$$ + \frac{\pi^9 \rho^8 \theta_2^4 \theta_4^2}{306561024} \left( 5 \theta_2^2 \theta_2^2 + 4 \theta_2^2 \theta_2^2 + 25 \theta_2^2 \theta_2^2 + 3 \theta_2^2 \theta_2^2 \right) \left( \theta_2^2 + \theta_4^2 - 3 \right) $$

$$ \times \left( 280 \rho E \left( \theta_2^2 + \theta_4^2 \right) - 280 \rho E \sqrt{\theta_2^2 + \theta_4^2} + 79 \right) $$

$$ f_5 = \frac{\pi^{15} \rho^9 \theta_2^4 \left( \theta_2^2 + \theta_4^2 \right)^2}{955144800} \left( 26 \theta_1^2 + 33 \theta_2^4 \theta_2^4 + 11 \theta_4^4 \theta_4^4 + 8 \theta_4^4 \theta_4^4 + 2 \theta_4^4 \theta_4^4 + 286 \theta_4^4 \theta_4^4 + 27 \theta_4^4 \theta_4^4 \right) $$

$$ + 4867 \theta_2^2 \theta_2^2 + 4245 \theta_4^2 \theta_4^4 + 1414 \theta_2^2 \theta_2^2 $$

$$ + \frac{\pi^{14} \rho^8 \theta_2^4}{238878720} \left( 15 \theta_1^2 + 18 \theta_2^4 \theta_2^4 + 4 \theta_4^4 \theta_4^4 + 15 \theta_4^4 \theta_4^4 + 3 \theta_4^4 \theta_4^4 + 3 \theta_4^4 \theta_4^4 + 3 \theta_4^4 \theta_4^4 \right) $$

$$ + 4144 \theta_2^2 \theta_2^2 \left( \theta_2^2 + \theta_4^2 \right)^{3/2} \right) $$

$$ \left( \theta_2^2 + \theta_4^2 \right)^2 - 2 \rho E \left( \theta_2^2 + \theta_4^2 \right)^{3/2} \right) $$
\[ \pi^2 \rho \theta (\theta^2 + \theta^4)^2 \left( \frac{9016857600}{3906420^2} \right) + \pi^2 \rho \theta (\theta^2 + \theta^4)^2 \left( \frac{13976900 \theta^2 \theta^4 + 18606420^2 \theta^2 + 28317890 \theta^4 \theta^2}{3906420^2} \right) + 14144770 \theta (\theta^2 + \theta^4 - \rho \theta (\theta^2 + \theta^4)) \\
\frac{263\pi^2 \rho E \theta^4 \sqrt{\theta_1^2 + \theta_2^2}}{9754214400} \left( 707921 \theta^2 + 2214130 \theta^2 \theta^4 + 3899371 \theta_1^2 \theta_2^4 + 5405580 \theta_1^2 \theta_2^4 \right) + \frac{655202360 \theta_1^2 \theta_2^4 + 49540820 \theta_1^2 \theta_2^4 + 14144770 \theta_2^4}{58525886400} \\
\frac{\pi^2 \rho \theta (\theta^2 + \theta^4)^2 \left( 707921 \theta^2 + 15062090 \theta_1^2 \theta_2^4 + 23931620 \theta_1^2 \theta_2^4 + 30124180 \theta_1^2 \theta_2^4 \right) + 3539605 \theta_1^2 \theta_2^4 + 14144770 \theta_2^4 \left( \sqrt{\theta_1^2 + \theta_2^4} \left( 299 + 980 \rho ^2 E^2 (\theta_1^2 + \theta_2^4) \left( 2 \rho E \sqrt{\theta_1^2 + \theta_2^4} - 3 \right) \right) \right) + \frac{83691159552000}{\rho E (\theta_1^2 + \theta_2^4) \left( 789 + 490 \rho E \left( \rho E (\theta_1^2 + \theta_2^4) - 2 \sqrt{\theta_1^2 + \theta_2^4} \right) \right).}

**ORCID IDs**

Nikolay Sh Izmailian  [https://orcid.org/0000-0002-1006-9947](https://orcid.org/0000-0002-1006-9947)

Vladimír V Papoyan  [https://orcid.org/0000-0001-6854-4723](https://orcid.org/0000-0001-6854-4723)

Robert M Ziff  [https://orcid.org/0000-0002-9023-7508](https://orcid.org/0000-0002-9023-7508)

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