Analytic derivation of the Alhassid–Whelan arc of regularity

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Abstract. The Alhassid–Whelan arc of regularity is a unique valley of regularity connecting, amidst chaotic regions, the SU(3) and U(5) vertices of the symmetry triangle of the Interacting Boson Model (IBM). Using the contraction of the SU(3) algebra to the algebra of the rigid rotator in the large boson number limit of the IBM, we construct for the first time a symmetry line in the interior of the triangle, along which the SU(3) symmetry is preserved. The line extends from the SU(3) vertex to near the critical line of the first order shape/phase transition separating the spherical and prolate deformed phases. It lies within the Alhassid–Whelan arc of regularity, thus providing an explanation for its existence.

1. Introduction

The study of chaotic properties of the IBM [1], classically and quantum mechanically, led to the discovery [2, 3, 4] of a narrow strip of nearly regular behavior, connecting the U(5) and SU(3) vertices of the symmetry triangle [5] of the IBM, which was called the Alhassid–Whelan arc of regularity, depicted in Fig. 1. The presence of near regularity presupposes the existence of some underlying approximate symmetry. A recent study [6], reveals the existence of a line, which lies very close to the arc and is based on SU(3) symmetry. This line was derived using characteristic signatures of SU(3), like the degeneracies between levels of the $\beta$ band and those (with the same $L$) of the $\gamma$ band.

In this work, we will show the analytical derivation of a line of approximate SU(3) symmetry, by studying Hamiltonians that approximately commute with the SU(3) generators. The derivation takes advantage of the well–known contraction [7] of the SU(3) algebra to the [$R^5$SO(3)] algebra [8, 9] of the rigid rotator [10].

The IBM Hamiltonian used is described is described in Section 2. In Section 3, the equation of the line corresponding to the SU(3) symmetry is derived. The conclusions are presented in Section 4. Necessary values of the commutation relations and matrix elements are given in the Appendices of [11].

2. The IBM Hamiltonian and symmetry triangle

The IBM possesses an overall U(6) symmetry, with three dynamical symmetries, labeled by subgroups of U(6), namely, U(5), SU(3) and O(6), which correspond to vibrational, rotational and $\gamma$–unstable nuclei respectively. The IBM Hamiltonian used by Alhassid and Whelan is
Figure 1. IBM symmetry triangle with the three dynamical symmetries, the Alhassid–Whelan arc of regularity [Eq. (9)], the present line of Eq. (8) (labelled as analytic), the loci of the degeneracies $E(2^+\beta) = E(2^+\gamma)$ (dashed line on the right for $N_B=250$ [6]) and $E(4^+1) = E(0^+_2)$ (dotted line on the left for $N=250$ [6]).

\[ \hat{H}(\eta, \chi) = \hat{H}_1 + \hat{H}_2 = c \left[ \eta \hat{n}_d + \frac{\eta - 1}{N} \hat{Q}_1^{(2)} \cdot \hat{Q}_2^{(2)} \right], \]  

where $N$ is the number of valence bosons, $c$ is a scaling factor, $\hat{H}_1$ and $\hat{H}_2$ denote the first and second terms of the Hamiltonian respectively, $\hat{n}_d = d^\dagger \cdot \hat{d} = \sqrt{5}(d^\dagger d)^{(0)}$ is the d boson number operator, $\hat{Q}_1^{(2)} = (s^\dagger \hat{d} + d^\dagger s)^{(2)}_\xi + \chi (d^\dagger \hat{d})^{(2)}_\xi$ is the quadrupole operator.

The above Hamiltonian contains two parameters, $\eta$ and $\chi$, with the parameter $\eta$ ranging from 0 to 1 and the parameter $\chi$ ranging from 0 to $-\sqrt{7}/2$. The U(5) limit corresponds to $\eta = 1$, the SU(3) limit to $\eta = 0$, $\chi = -\sqrt{7}/2$ and the O(6) limit to $\eta = 0$, $\chi = 0$.

3. The SU(3) symmetry

3.1. Commutation Relations

In order to see whether the Hamiltonian has a certain symmetry, it has to commute with the generators of that symmetry. The generators of the SU(3) algebra [1], are the angular momentum operators

\[ \hat{L}_\xi = \sqrt{10}(d^\dagger \hat{d})^1_\xi, \]  

and the quadrupole operators

\[ \hat{Q}_1^{(2)}_{SU(3),\xi} = (s^\dagger \hat{d} + d^\dagger s)^{(2)}_\xi - \frac{\sqrt{7}}{2} (d^\dagger \hat{d})^{(2)}_\xi. \]  

The Hamiltonian of Eq. (1) commutes with the angular momentum operators, while the first part of the Hamiltonian with the quadrupole operators gives

\[ [\hat{H}_1, \hat{Q}_{SU(3),\nu}^{(2)}] = c\eta [\hat{n}_d, \hat{Q}_{SU(3),\nu}^{(2)}] = c\eta (d^\dagger s - s^\dagger \hat{d})^{(2)}_\nu. \]  

Using

\[ \hat{Q}_1^{(2)}_\chi = \hat{Q}_{SU(3),\xi}^{(2)} + \left( \chi + \frac{\sqrt{7}}{2} \right) (d^\dagger \hat{d})^{(2)}_\xi, \]  

\[ \hat{H}_1 = H_1 + \hat{H}_2 \]  

where $H_1$ and $\hat{H}_2$ are the first and second terms of the Hamiltonian respectively, $\hat{n}_d = d^\dagger \cdot \hat{d} = \sqrt{5}(d^\dagger d)^{(0)}$ is the d boson number operator, $\hat{Q}_1^{(2)} = (s^\dagger \hat{d} + d^\dagger s)^{(2)}_\xi + \chi (d^\dagger \hat{d})^{(2)}_\xi$ is the quadrupole operator.
in the second term of the Hamiltonian one gets the intermediate result
\[
[\hat{Q}^{(2)}_\chi, \hat{Q}^{(2)}_\chi, \hat{Q}^{(2)}_{SU(3),\nu}] = \sum_\xi (-1)^\xi \{[\hat{Q}^{(2)}_{SU(3),\xi}, \hat{Q}^{(2)}_{SU(3),\nu}]\hat{Q}^{(2)}_{\chi, -\xi} + \hat{Q}^{(2)}_{\chi, \xi} [\hat{Q}^{(2)}_{SU(3), -\xi}, \hat{Q}^{(2)}_{SU(3),\nu}] + \left(\chi + \frac{\sqrt{7}}{2}\right) \{[(d^\dagger \tilde{d})^{(2)}_{\xi}, \hat{Q}^{(2)}_{SU(3),\nu}]\hat{Q}^{(2)}_{\chi, -\xi} + \hat{Q}^{(2)}_{\chi, \xi} [(d^\dagger \tilde{d})^{(2)}_{-\xi}, \hat{Q}^{(2)}_{SU(3),\nu}]\}\}. \tag{6}
\]

In the large N–limit, Eq. (6) can be simplified. In this limit, the ground state band becomes energetically isolated from all other excitations, so SU(3) effectively reduces to a simple rigid rotator. This situation is known as the contraction of SU(3) to R$^3$[SO(3)] [8, 9]. At the contraction limit, the $Q^{(2)}_{SU(3)}$ operators can be replaced by mutually commuting quantities and terms containing $(d^\dagger \tilde{d})^{(k)}$ can be ignored. As seen from Eq. (5), $\hat{Q}^{(2)}_\chi$ can be replaced by $\hat{Q}^{(2)}_{SU(3)}$ and in its turn, $\hat{Q}^{(2)}_{SU(3)}$ can be replaced by the intrinsic quadrupole moment $q_0$, which is $N\sqrt{2}$ in the present case. So, in the large N–limit, the commutator of the second part of the Hamiltonian becomes
\[
[\hat{H}_2, \hat{Q}^{(2)}_{SU(3),\nu}] = c(\eta - 1)2\sqrt{2} \left(\chi + \frac{\sqrt{7}}{2}\right) (d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu. \tag{7}
\]

In order to get a vanishing commutator, the coefficients of $(d^\dagger s - s^\dagger \tilde{d})^{(2)}_\nu$ in Eqs. (4) and (7) should cancel, leading to the condition
\[
\chi(\eta) = \frac{1}{2\sqrt{2}} \frac{\eta}{(1 - \eta)} - \frac{\sqrt{7}}{2}. \tag{8}
\]

The Alhassid–Whelan arc of regularity [2, 3, 4] is given by the expression [12]
\[
\chi(\eta) = \frac{\sqrt{7} - 1}{2} \eta - \frac{\sqrt{7}}{2}. \tag{9}
\]

Equations (8) and (9) give very similar predictions for values of $\eta$ between 0 and 0.6, i.e. from the SU(3) vertex until quite close to the critical line, as seen in Fig. 2.

3.2. Matrix Elements

We can also consider the matrix elements of $[\hat{H}_1, \hat{Q}^{(2)}_{SU(3),\nu}]$ and $[\hat{H}_2, \hat{Q}^{(2)}_{SU(3),\nu}]$ within the ground state band, in order to justify the replacement of the quadrupole operator $\hat{Q}^{(2)}_{SU(3)}$ by the intrinsic quadrupole moment in the previous subsection.

Using the standard formalism for treating matrix elements, one finds for the first and second part of the Hamiltonian,
\[
\langle [N], (2N, 0), \bar{\chi} = 0, L | [\hat{H}_1, \hat{Q}^{(2)}_{SU(3)}] | [N], (2N, 0), \bar{\chi} = 0, L \rangle = -\frac{2}{3\sqrt{7}} c\eta N R_1 \sqrt{2L + 1}, \tag{10}
\]
\[
\langle [N], (2N, 0), \bar{\chi} = 0, L | [\hat{H}_2, \hat{Q}^{(2)}_{SU(3)}] | [N], (2N, 0), \bar{\chi} = 0, L \rangle = \frac{1}{7\sqrt{2}} c(1 - \eta) \left(\chi + \frac{\sqrt{7}}{2}\right) q_0 R_2 \sqrt{2L + 1}, \tag{11}
\]

where $\bar{\chi}$ is the Vergados quantum number [13] and $R_1, R_2$ are complicated expressions [11] of the number of bosons N and the angular momentum L. In the large N limit and for $L$ not too
small, we obtain $R_1 = 1$ and $R_2 = 16/3$. Replacing the intrinsic quadrupole moment $q_0$ by its value ($N\sqrt{2}$), we see that the two matrix elements vanish when

$$
\chi(\eta) = \frac{\sqrt{7}}{8} \frac{\eta}{(1 - \eta)} - \frac{\sqrt{7}}{2}.
$$

(12)

Expressions (12) and (8) give very similar results as seen in Fig. 2.

**Figure 2.** Location of the arc of regularity, as described by the original Eq. (9), and as predicted by the findings of the present work, Eq. (8) and Eq. (12). The $\eta$ axis has been reversed, in order to correspond directly to Fig. 1.

4. Conclusion

The present work is the first analytical evidence for the existence of an approximate symmetry inside the symmetry triangle of the IBM. The extracted line of SU(3) symmetry follows closely the Alhassid–Whelan arc of regularity, providing thus an explanation for its existence.

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