Polytope duality for families of $K3$ surfaces associated to transpose duality

Makiko Mase

Key Words: $K3$ surfaces, toric varieties
AMS MSC2010: 14J28 14M25

Abstract

We consider whether or not transpose-dual pairs, which is a Berglund–Hübsch mirror studied by Ebeling and Ploog [3], extend to a polytope duality that has a potential to be lattice dual.

1 Introduction

Isolated singularities in $\mathbb{C}^3$ are classified by Arnold [1] among which there are classes called bimodal and unimodal. Our notation follows that of Arnold’s. Not only the classification, Arnold also finds that there is a duality among unimodal singularities that is called Arnold’s strange duality. The duality is also related to toric geometry and lattice theory. Ebeling and Ploog [3] find an analogous duality concerning bimodal and other singularities, which is actually a Berglund–Hübsch mirror.

Batyrev’s proposal [2] of polar duality of reflexive polytopes gives a breakthrough in a construction of mirror partner for toric Calabi-Yau hypersurfaces and later complete intersections.

Being origined in physics, there appear a numerical meanings of “mirror” such as cohomological mirror, among which in this article we focus on a relation between Ebeling and Ploog’s transpose duality and Batyrev’s polytope duality associating with bimodal singularities in some manner.

In a series of recent studies, it is concluded that transpose-dual pairs $(Q_{12}, E_{18})$, $(Z_{10}, E_{19})$, $(E_{20}, Z_{20})$, $(Q_{20}, Z_{17})$, $(E_{25}, Z_{19})$, $(Q_{18}, E_{30})$ of singularities can extend to a lattice duality by the author [5] following an extension to polytope duality by the author and Ueda [6]. However, those pairs in the list (∗) below fail to extend to a lattice duality in spite of the fact that they are polytope dual.

(∗) $(Z_{13}, J_{3,0}), (Z_{10}, Z_{1,0}), (Z_{17}, Q_{2,0}), (U_{1,0}, U_{1,0}), (U_{16}, U_{16})$, $(Q_{17}, Z_{2,0}), (W_{1,0}, W_{1,0}), (W_{17}, S_{1,0}), (W_{18}, W_{18}), (S_{17}, X_{2,0})$.

More precisely, for each pair one obtains in [6] reflexive polytopes $\Delta_{[MU]}$ and $\Delta'_{[MU]}$ satisfying that the polar dual of $\Delta_{[MU]}$ is isomorphic to $\Delta'_{[MU]}$ and that $\Delta_{[MU]}$ and $\Delta'_{[MU]}$ respectively contains the Newton polytope of a compactification polynomial of the defining polynomial of singularities. Despite this fact it is concluded in [5] that the corresponding pairs of families $F_{\Delta_{[MU]}}$ and $F_{\Delta'_{[MU]}}$ of $K3$ surfaces are not lattice dual, that is, the Picard lattices $\text{Pic}(\Delta_{[MU]})$
and \( \text{Pic}(\Delta'_{MU}) \) of these families do not satisfy an isometry \( \text{Pic}(\Delta'_{MU}) \cong U \oplus \text{Pic}(\Delta'_{MU}) \). Moreover, for these pairs we can observe that the restriction map \( H^{1,1}(\mathbb{P}_{\Delta_{MU}}, \mathbb{Z}) \to H^{1,1}(\tilde{Z}, \mathbb{Z}) \) for the minimal model of any generic member \( Z \in \mathcal{F}_{\Delta_{MU}} \) is not surjective.

The aim of the study is to consider the following problem arisen by Professor Ashikaga’s question:

**Problem** Let \(((B, f), (B', f'))\) be a transpose-dual pair in the list \((\ast)\) together with their defining polynomials \( f \) and \( f' \). Determine whether or not it is possible to take polynomials \( F \) and \( F' \) that are respectively compactifications of \( f \) and \( f' \), and a reflexive polytope \( \Delta \) such that the following condition \((\ast\ast)\) holds:

\[
\Delta' \subset \Delta, \Delta' \subset \Delta^*, \quad \text{and} \quad L_0(\Delta) = 0.
\]

Here, \( \Delta_F \) and \( \Delta_{F'} \) denote respectively the Newton polytopes of \( F \) and of \( F' \), and \( \Delta^* \) is the polar dual polytope of \( \Delta \).

The main theorem of this paper is stated as follows:

**Main Theorem.** (Theorem 3.1) For each of the following pairs

\[(Z_{1,0}, Z_{1,0}), (U_{1,0}, U_{1,0}), (Q_{17}, Z_{2,0}), (W_{1,0}, W_{1,0}), \]

there exist compactifications \( F, F' \) and reflexive polytopes \( \Delta \) and \( \Delta' \) such that

\[
\ast\ast \quad \Delta^* \simeq \Delta', \Delta_F \subset \Delta, \Delta_{F'} \subset \Delta', \quad \text{and} \quad \text{rank } L_0(\Delta) = 0
\]

hold. Moreover, \( \rho(\Delta) + \rho(\Delta') = 20 \).

It can be conjectured that there do not exist reflexive polytopes for pairs \((Z_{13}, J_{3,0}), (Z_{17}, Q_{2,0}), (U_{16}, U_{16}), (W_{17}, S_{1,0}), (W_{18}, W_{18}), (S_{17}, X_{2,0})\) of singularities satisfying the condition \((\ast\ast)\). We leave the judgement about this conjecture to a further study in the future.

Section 2 is devoted to recall some facts as to a polytope duality associated to singularities. The proof of the main theorem is given in Theorem 3.1 in section 3 where we explicitly give compactifications and reflexive polytopes for these pairs.

**Acknowledgement**

The author would be grateful to Professor T. Ashikaga for his question of the problem that motivated this article after publication of [5], to Professor N. Aoki who was reading through the first draft carefully and making many helpful suggestions, and to Professor M. Kobayashi for his comments and encouragement.

## 2 Preliminary

Recall that a Gorenstein K3 surface is a compact complex connected 2-dimensional algebraic variety \( S \) with at most ADE singularities satisfying \( K_S \sim 0 \) and \( H^1(S, \mathcal{O}_S) = 0 \). If a Gorenstein K3 surface is nonsingular, we simply call it a K3 surface.
Let \( M \simeq \mathbb{Z}^3 \) be a 3-dimensional lattice and \( N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \simeq \mathbb{Z}^3 \) the dual of \( M \) with a natural pairing \((\cdot, \cdot) : N \times M \to \mathbb{Z}\). Let \( \Delta \) be a 3-dimensional polytope, that is, \( \Delta \) is a convex hull of finitely-many points in \( M \otimes \mathbb{Z} \mathbb{R} \). The associated toric 3-fold is denoted by \( \mathbb{P}_\Delta \). The polar dual \( \Delta^* \) of \( \Delta \) is defined by

\[
\Delta^* = \{ y \in N \otimes \mathbb{Z} \mathbb{R} | (y, x) \geq -1 \text{ for all } x \in \Delta \}.
\]

Let us recall a toric description of weighted projective spaces. Let \( a = (a_0, a_1, a_2, a_3) \) be a well-posed quadruple of natural numbers and \( d = a_0 + a_1 + a_2 + a_3 \). Define a 3-dimensional lattice \( \tilde{M} \) by

\[
\tilde{M} := \{ (i, j, k, l) \in \mathbb{Z}^4 | a_0i + a_1j + a_2k + a_3l \equiv 0 \text{ mod } d \} \simeq \mathbb{Z}^3.
\]

Note that the lattice \( \tilde{M} \) is one-to-one corresponding to the set of monomials of weighted degree \( d \): indeed, for each \( (i, j, k, l) \in \tilde{M} \), a monomial \( X_i^0 X_j^1 X_k^2 X_l^3 \) is of weighted degree \( d \). Here, the weight of \( X_i \) is \( a_i \) for \( i = 0, 1, 2, 3 \). Besides, by letting \( \Delta_a \) be a convex hull of all primitive lattice vectors in \( \tilde{M} \), the associated projective toric 3-fold is the weighted projective space of weight \( a \).

The introduction of reflexive polytope in [2] is motivated by mirror symmetry.

**Definition 2.1** ([2]) Let \( \Delta \) be an integral polytope that contains the origin in its interior. The polytope \( \Delta \) is called reflexive if its polar dual \( \Delta^* \) is also integral.

Not only in a context of mirror, this notion is basically friendly with \( K3 \) surfaces as follows:

**Theorem 2.1** ([2]) Let \( \Delta \) be a 3-dimensional polytope.

(1) The followings are equivalent:

(i) The polytope \( \Delta \) is reflexive.

(ii) The toric 3-fold \( \mathbb{P}_\Delta \) is Fano, in particular, general anticanonical members of \( \mathbb{P}_\Delta \) are Gorenstein \( K3 \).

(2) General anticanonical members of \( \mathbb{P}_\Delta \) are simultaneously resolved by a toric (crepant) desingularization of \( \mathbb{P}_\Delta \) to be \( K3 \) surfaces.

Denote for a reflexive polytope \( \Delta \) by \( \mathcal{F}_\Delta \) a family of (Gorenstein) \( K3 \) surfaces parametrised by the complete anticanonical linear system \( | -K_{\mathbb{P}_\Delta} | \). For a member \( Z \) in \( \mathcal{F}_\Delta \), denote by \( \tilde{Z} \) and \( \mathbb{P}_\Delta \) the minimal models in a cause of the simultaneous resolution.

In the article, we define that a member \( Z \in \mathcal{F}_\Delta \) is generic if the following two conditions are satisfied:

(1) \( Z \) is \( \Delta \)-regular. (See [2] for detail)

(2) The Picard group of \( \tilde{Z} \) is generated by irreducible components of the restrictions of the generator of the Picard group of \( \mathbb{P}_\Delta \).

It is proved in [2] that \( \Delta \)-regularity is a general condition. The condition (2) is also a general condition. Note that all Picard lattices of the minimal models of any generic members are isometric.
Definition 2.2 (1) The Picard lattice $\text{Pic}(\Delta)$ of the family $\mathcal{F}_\Delta$ is the Picard lattice of the minimal model of a generic member. 
(2) $\rho(\Delta) := \text{rank} \text{Pic}(\Delta)$ is called the Picard number of the family $\mathcal{F}_\Delta$. 
(3) Let $r : H^{1,1}(\mathbb{P}_\Delta, \mathbb{Z}) \to H^{1,1}(\mathbb{Z}, \mathbb{Z})$ be the restriction mapping of the cohomology group. The cokernel of $r$ is denoted by $L_0(\Delta)$.

In [6], a notion of transpose duality [3] for singularities is extended to a polytope duality in the sense of the following theorem:

Theorem 2.2 (6) Let $((B, f), (B', f'))$ be a transpose-dual pair together with their defining polynomials $f$ and $f'$ that are respectively compactified to polynomials $F$ and $F'$. Then, there exist reflexive polytopes $\Delta_{[\text{MU}]}$ and $\Delta'_{[\text{MU}]}$ such that 
$$\Delta^*_{[\text{MU}]} \cong \Delta'_{[\text{MU}]}, \quad \Delta_F \subset \Delta_{[\text{MU}]}, \quad \text{and} \quad \Delta_{F'} \subset \Delta'_{[\text{MU}]}.$$ 

However, it depends on the pairs that whether or not $\text{rank} L_0(\Delta_{[\text{MU}]}) = 0$ holds. In section 3, we shall show that some pairs in the list (⋆) do have this property.

We end this section by giving formulas that are needed in the proof of the main theorem. See [4] for details. For a 3-dimensional reflexive polytope $\Delta$, denote by $\Delta[1]$ the set of all edges of $\Delta$, and for an edge $\Gamma \in \Delta[1]$, the dual edge in the polar dual polytope $\Delta^*$ is denoted by $\Gamma^*$. The number of lattice points on an edge $\Gamma$ is denoted by $l(\Gamma)$, whilst $l(\Gamma) - 2$ by $l^*(\Gamma)$. We have

$$\text{rank} L_0(\Delta) = \sum_{\Gamma \in \Delta[1]} l^*(\Gamma)l^*(\Gamma^*).$$  \hfill (1)

$$\rho(\Delta) = \sum_{\Gamma \in \Delta[1]} l(\Gamma^*) - 3.$$ \hfill (2)

Note that $\text{rank} L_0(\Delta) = \text{rank} L_0(\Delta^*)$ by the formula.

3 Main result

The chief aim of this section is to prove the following statements.

Theorem 3.1 For pairs $(B, B')$ of singularities, if one takes compactifications $F, F'$ as in Table 1 and polytopes $\Delta, \Delta'$ as in Table 2 then,

(i) $\Delta$ and $\Delta'$ are reflexive,

(ii) $\Delta^*$ is isomorphic to $\Delta'$ up to lattice isometry of $\mathbb{Z}^3$,

(iii) $\Delta_F \subset \Delta$, and $\Delta_{F'} \subset \Delta'$ hold, and

(iv) $\text{rank} L_0(\Delta) = 0$.

Moreover, $\rho(\Delta) + \rho(\Delta') = 20$.

| $B$ | $F$ | $F'$ | $B'$ |
|-----|-----|-----|-----|
| $Z_{1,0}$ | $X^6Y + X^4Z + Z^2 + W^4X^2$ | $X^6Y + X^4Z + Z^2 + W^4$ | $Z_{1,0}$ |
| $U_{1,0}$ | $X^6Y + Y^2Z + Z^2 + W^4X$ | $XZ^2 + X^2Y + Y^2 + W^4$ | $U_{1,0}$ |
| $Z_{2,0}$ | $X^6 + Y^2Z + Z^2 + W^4Z$ | $X^6 + Y^2Z + Z^2 + W^4$ | $Q_{1,0}$ |
| $W_{1,0}$ | $X^6 + Y^2Z + Z^2 + W^4Z$ | $X^6 + Y^2Z + Z^2 + W^4Z$ | $W_{1,0}$ |
Table 1: Compactifications of singularities

| $B$ | vertices of $\Delta$ | vertices of $\Delta'$ | $B'$ |
|-----|----------------------|-----------------------|------|
| $Z_{1,0}$ | $(-1, 0, 1),\ (-1, 0, 0),\ (0, 1, -1),\ (2, 3, -1),\ (2, 2, -1),\ (1, -1, -1),\ (0, -1, -1)$ | $(0, 2, -1),\ (-1, 1, -1),\ (-1, -1, -1),\ (5, -1, -1),\ (4, 0, -1),\ (1, 0, 0),\ (-1, -1, 1)$ | $Z_{1,0}$ |
| $U_{1,0}$ | $(-1, 0, 2),\ (0, 1, 0),\ (1, 2, -1),\ (1, 1, -1),\ (0, -1, 0),\ (0, -1, -1)$ | $(1, 0, -1), (0, -1, -1),\ (-1, -1, -1),\ (-1, 2, -1), (1, 2, -1), (1, 0, 1),\ (0, -1, 2),\ (-1, -1, 2)$ | $U_{1,0}$ |
| $Z_{2,0}$ | $(-1, -1, 2),\ (0, -1, 0),\ (1, -1, 0),\ (1, -1, 1),\ (1, 2, -3),\ (0, 0, -1)$ | $(-1, 2, -1), (-1, -1, 1), (-1, -1, 1), 6, -1, -1),\ (2, 1, -1), (0, -1, 1)$ | $Q_{17}$ |
| $W_{1,0}$ | $(-1, 0, 1),\ (-1, 0, 0),\ (1, 2, -1),\ (2, 3, -1),\ (0, -1, 0)$ | $(-1, -1, -1),\ (5, -1, -1), (1, 3, -1), (1, 3, -1),\ (-1, -1, 1)$ | $W_{1,0}$ |

Table 2: Polytopes that make the pairs polytope dual

**Proof.**

$Z_{1,0}$ case. The defining polynomials of singularities $B = Z_{1,0}$ and $B' = Z_{1,0}$ are the same $f = f' = x^5y + xy^3 + z^2$.

Take a compactification of $f$ as $F = W^{10}X^2 + X^3Y + XY^3 + Z^2$ in the weighted projective space $\mathbb{P}(1, 2, 4, 7)$. Note that $F$ is a different compactification from the one in $[3]$.

Take a compactification of $f'$ as $F' = W^{14} + X^5Y + XY^3 + Z^2$ in the weighted projective space $\mathbb{P}(1, 2, 4, 7)$. Note that $F'$ is the same compactification as in $[3]$.

The polytope $\Delta$ contains the Newton polytope of $F$: indeed, by taking a basis $e_1 = (-6, 1, 1, 0), e_2 = (2, 1, -1, 0), e_3 = (-7, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials $W^{10}X^2, X^3Y, XY^3, Z^2$ are respectively corresponding to vertices 

$$(0, 1, -1), (2, 2, -1), (1, -1, -1), (-1, 0, 1).$$

The polytope $\Delta'$ contains the Newton polytope of $F'$: indeed, by taking a standard basis $e'_1 = (-2, 1, 0, 0), e'_2 = (-4, 0, 1, 0), e'_3 = (-7, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials $W^{14}, X^5Y, XY^3, Z^2$ are respectively corresponding to vertices

$$(-1, -1, -1), (4, 0, -1), (0, 2, -1), (-1, -1, 1).$$

The dual polytope $\Delta'^*$ of $\Delta'$ is a convex hull of vertices 

$$(0, 0, 1), (-1, -2, -3), (-1, -3, -5), (1, -1, -1), (1, 0, 0), (0, 1, 0), (-1, -1, -3)$$

that is mapped to isomorphically from $\Delta$ by a transformation of $\mathbb{R}^3$ by the matrix

$$M := \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 1 & 2 & 4 \end{pmatrix}$$
that is, \((x, y, z)M = (x', y', z')\) for \((x, y, z) \in \Delta\) and \((x', y', z') \in \Delta'\).

Therefore, \(\Delta\) and \(\Delta'\) are reflexive and the pair is polytope dual.

By the formula (1), one gets \(\text{rank } L_0(\Delta) = \text{rank } L_0(\Delta^*) = 0\) because for all edges in \(\Delta\) satisfy \(I^*(\Gamma)l^*(\Gamma^*) = 0\). In fact, at least either \(\Gamma\) or \(\Gamma^*\) has no lattice points in its interior.

By the formula (2), one can compute that

\[ \rho(\Delta) = 17 - 3 = 14, \quad \rho(\Delta^*) = 9 - 3 = 6 \]

thus one has

\[ \rho(\Delta) + \rho(\Delta^*) = 20. \]

**Case.** The defining polynomials of singularities \(B = U_{1,0}\) and \(B' = U_{1,0}\) are \(f = x^3y + y^2z + z^3, f' = x'z^3 + x^2y' + y^3\), respectively.

Take a compactification of \(f\) as \(F = WX^4 + X^3Y + Y^2Z + Z^3\) in the weighted projective space \(\mathbb{P}(1,2,3,3)\). Note that \(F\) is a different compactification from the one in [3].

Take a compactification of \(f'\) as \(F' = W'^9 + X'Z'^3 + X'^2Y' + Y'^3\) in the weighted projective space \(\mathbb{P}(1,3,3,2)\). Note that \(F'\) is the same compactification as in [3].

The polytope \(\Delta\) contains the Newton polytope of \(F\); indeed, by taking a basis \(e_1 = (-5,1,1,0), e_2 = (1,1,-1,0), e_3 = (-3,0,0,1)\) for \(\mathbb{R}^3\), one can see that monomials \(WX^4, X^3Y, Y^2Z, Z^3\) are respectively corresponding to vertices

\((1,2,-1), (1,1,-1), (0,-1,0), (-1,0,2)\).

The polytope \(\Delta'\) contains the Newton polytope of \(F'\); indeed, by taking a standard basis \(e_1' = (-3,1,0,0), e_2' = (-3,0,1,0), e_3' = (-2,0,0,1)\) for \(\mathbb{R}^3\), one can see that monomials \(W'^9, X'Z'^3, X'^2Y', Y'^3\) are respectively corresponding to vertices

\((-1,-1,-1), (0,-1,2), (1,0,-1), (-1,2,-1)\).

The dual polytope \(\Delta^*\) of \(\Delta'\) is a convex hull of vertices

\((0,0,1), (-1,0,0), (-1,1,0), (0,1,0), (1,0,0), (0,-1,-1)\)

that is mapped to isomorphically from \(\Delta\) by a transformation of \(\mathbb{R}^3\) by the matrix

\[
M = \begin{pmatrix}
2 & 2 & 1 \\
-1 & 0 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

that is, \((x, y, z)M = (x', y', z')\) for \((x, y, z) \in \Delta\) and \((x', y', z') \in \Delta'\).

Therefore, \(\Delta\) and \(\Delta'\) are reflexive and the pair is polytope dual.

By the formula (1), one gets \(\text{rank } L_0(\Delta) = \text{rank } L_0(\Delta^*) = 0\) because for all edges in \(\Delta\) satisfy \(I^*(\Gamma)l^*(\Gamma^*) = 0\). In fact, at least either \(\Gamma\) or \(\Gamma^*\) has no lattice points in its interior.

By the formula (2), one can compute that

\[ \rho(\Delta) = 20 - 3 = 17, \quad \rho(\Delta^*) = 6 - 3 = 3 \]
thus one has
\[ \rho(\Delta) + \rho(\Delta^*) = 20. \]

\[ Z_{2,0} \text{ and } Q_{17} \text{ case} \] The defining polynomials of singularities \( B = Z_{2,0} \) and \( B' = Q_{17} \) are \( f = x^3y + xy^3 + z^2 \), \( f' = x^3y + y^3 + xz^2 \), respectively.

Take a compactification of \( f \) as \( F = W^7Y + X^5Z + XY^3 + Z^2 \) in the weighted projective space \( \mathbb{P}(1, 1, 3, 5) \). Note that \( F \) is the same compactification as in [3].

Take a compactification of \( f' \) as \( F' = W^7 + X^5Y + WY^3 + XZ^2 \) in the weighted projective space \( \mathbb{P}(1, 1, 2, 3) \). Note that \( F' \) is the same compactification as in [3].

The polytope \( \Delta \) contains the Newton polytope of \( F \): indeed, by taking a basis \( e_1 = (-3, 3, 0, 0), \ e_2 = (-8, 0, 1, 1), \ e_3 = (-6, 1, 0, 1) \) for \( \mathbb{R}^3 \), one can see that monomials \( W^7Y, X^5Z, XY^3, Z^2 \) are respectively corresponding to vertices
\[ (0, 0, -1), (1, -1, 1), (1, 2, -3), (-1, -1, 2). \]

The polytope \( \Delta' \) contains the Newton polytope of \( F' \): indeed, by taking a standard basis \( e_1' = (-1, 1, 0, 0), \ e_2' = (-2, 0, 1, 0), \ e_3' = (-3, 0, 0, 1) \) for \( \mathbb{R}^3 \), one can see that monomials \( W^7, X^5Y, WY^3, XZ^2 \) are respectively corresponding to vertices
\[ (-1, -1, -1), (4, 0, -1), (-1, 2, -1), (0, -1, 1). \]

The dual polytope \( \Delta'^* \) of \( \Delta' \) is a convex hull of vertices
\[ (-1, -3, -4), (0, -2, -3), (0, 1, 0), (1, 0, 0), (0, 0, 1), (-1, -2, -3) \]
that is mapped to isomorphically from \( \Delta \) by a transformation of \( \mathbb{R}^3 \) by the matrix
\[
M := \begin{pmatrix}
1 & 1 & 1 \\
1 & 3 & 4 \\
1 & 2 & 3
\end{pmatrix}
\]
that is, \( M(x, y, z) = (x', y', z') \) for \( (x, y, z) \in \Delta \) and \( (x', y', z') \in \Delta' \).

Therefore, \( \Delta \) and \( \Delta' \) are reflexive and the pair is polytope dual.

By the formula (1), one gets \( \operatorname{rank} L_0(\Delta) = \operatorname{rank} L_0(\Delta^*) = 0 \) because for all edges in \( \Delta \) satisfy \( l^*(\Gamma)^* l^*(\Gamma^*) = 0 \). In fact, at least either \( \Gamma \) or \( \Gamma^* \) has no lattice points in its interior.

By the formula (2), one can compute that
\[ \rho(\Delta) = 18 - 3 = 15, \quad \rho(\Delta^*) = 8 - 3 = 5 \]
thus one has
\[ \rho(\Delta) + \rho(\Delta^*) = 20. \]

W_{1,0} \text{ case} The defining polynomials of singularities \( B = B' = W_{1,0} \) are the same \( f = f' = x^6 + y^2z + z^2 \).

Take a compactification of \( f \) as \( F = X^6 + Y^2Z + Z^2 + W^6Z \) in the weighted projective space \( \mathbb{P}(1, 2, 3, 6) \). Note that \( F \) is a different compactification from the one in [3].
Take a compactification of $f'$ as $F' = X'^6 + Y'^2 Z' + Z'^2 + W'^{12}$ in the weighted projective space $\mathbb{P}(1, 2, 3, 6)$. Note that $F'$ is the same compactification as in [3].

The polytope $\Delta$ contains the Newton polytope of $F$: indeed, by taking a basis $e_1 = (-5, 1, 1, 0)$, $e_2 = (1, 1, -1, 0)$, $e_3 = (-6, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials $X^6$, $Y^2 Z$, $Z^2$, $W^6 Z$ are respectively corresponding to vertices $(2, 3, -1), (0, -1, 0), (-1, 0, 1), (-1, 0, 0)$.

The polytope $\Delta'$ contains the Newton polytope of $F'$: indeed, by taking a standard basis $e'_1 = (-2, 1, 0, 0)$, $e'_2 = (-3, 0, 1, 0)$, $e'_3 = (-6, 0, 0, 1)$ for $\mathbb{R}^3$, one can see that monomials $X'^6$, $Y'^2 Z'$, $Z'^2$, $W'^{12}$ are respectively corresponding to vertices $(5, -1, -1), (-1, 1, 0), (-1, -1, 1), (-1, -1, -1)$.

The dual polytope $\Delta'^*$ of $\Delta'$ is a convex hull of vertices $(0, 1, 0), (-1, -1, -3), (0, -1, -2), (1, 0, 0), (0, 0, 1)$ that is mapped to isomorphically from $\Delta$ by a transformation of $\mathbb{R}^3$ by the matrix

$$M := \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

that is, $M(x, y, z) = (x', y', z')$ for $(x, y, z) \in \Delta$ and $(x', y', z') \in \Delta'$.

Therefore, $\Delta$ and $\Delta'$ are reflexive polytopes and the pair is polytope dual.

By the formula (1), one gets $\rho_0(\Delta) = \rho_0(\Delta'^*) = 0$ because for all edges in $\Delta$ satisfy $l^*(\Gamma) l^*(\Gamma^*) = 0$. In fact, at least either $\Gamma$ or $\Gamma^*$ has no lattice points in its interior.

By the formula (2), one can compute that

$$\rho(\Delta) = 21 - 3 = 18, \quad \rho(\Delta'^*) = 5 - 3 = 2$$

thus one has

$$\rho(\Delta) + \rho(\Delta'^*) = 20. \quad \square$$

References

[1] Arnol’d, V. I., Critical points of smooth functions and their normal forms, Russian Math. Surveys 30, 1–75 (1975).

[2] Batyrev, V.V., Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, J. Alg. Geom. 3, 493–535 (1994).

[3] Ebeling, W. and Ploog, D., A geometric construction of Coxeter-Dynkin diagrams of bimodal singularities, Manuscripta Math. 140, 195–212 (2013).

[4] Kobayashi, M., Duality of weights, mirror symmetry and Arnold’s strange duality, Tokyo J. Math., 31, 225–251 (2008).

[5] Mase, M., A mirror duality for families of $K3$ surfaces associated to bimodal singularities, Manuscripta Math.(online 26 September 2015) 149, 389–404 (2016).
[6] Mase, M. and Ueda, K, A note on bimodal singularities and mirror sym-
metry, Manuscripta Math.(online 31 August 2014) 146, 153–177 (2015).

Makiko Mase

e-mail: mtmase@arion.ocn.ne.jp
Department of Mathematics, Rikkyo University
171-8501 3-34-1 Nishilkebukuro, Toshimaku, Tokyo, Japan.