A hybridizable discontinuous Galerkin method for the quad-curl problem

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Abstract

The quad-curl problem arises in magnetohydrodynamics, inverse electromagnetic scattering and transform eigenvalue problems. In this paper we investigate a hybridizable discontinuous Galerkin method to solve the quad-curl problem based on a mixed formulation. The divergence-free condition is enforced by introducing a Lagrange multiplier into the system. The analysis is performed for the model problem with low regularity, which is posed on a Lipschitz polyhedron domain.

keywords: quad-curl, HDG method, low regularity

1 Introduction.

Let $\Omega$ be a bounded simply-connected Lipschitz polyhedron in $\mathbb{R}^3$ with connected boundary $\Gamma$. We consider the following mixed quad-curl problem:

Find the vector field $\mathbf{u}$ such that

$$\begin{align*}
\nabla \times \nabla \times \nabla \times \nabla \times \mathbf{u} + \nabla p &= f \quad \text{in } \Omega, \\
\nabla \cdot \mathbf{u} &= g \quad \text{in } \Omega, \\
\mathbf{n}_\Gamma \times \mathbf{u} &= g_1 \quad \text{on } \Gamma, \\
\mathbf{n}_\Gamma \times \nabla \times \mathbf{u} &= g_2 \quad \text{on } \Gamma, \\
p &= 0 \quad \text{on } \Gamma.
\end{align*}$$

(1)
Here \( \mathbf{n}_\Gamma \) is the outward normal unit vector to the domain boundary \( \Gamma \), \( \mathbf{f} \in [L^2(\Omega)]^3 \) is an external source field, \( g \in L^2(\Omega) \) and \( g_1, g_2 \in H^{-\frac{1}{2}}(\text{div}; \Gamma) \cap [H^\delta(\Gamma)]^3 \) are given functions with \( \delta > 0 \), where \( H^{-\frac{1}{2}}(\text{div}; \Gamma) \) is the range space of “tangential trace” of space \( H(\text{curl}; \Omega) \). See [3] for the detailed description of space \( H^{-\frac{1}{2}}(\text{div}; \Gamma) \).

The model problem (1) arises in many areas such as magnetohydrodynamics, inverse electromagnetic scattering and transform eigenvalue problems. The main challenges of designing accurate, robust and efficient numerical methods for (1) are listed as follows.

- The construction of \( H^2 \)-conforming and curl-curl-conforming finite elements for solving the quad-curl problem would be very complicated.

- The quad-curl operator is not positive definite; hence it is difficult to design suitable numerical schemes. Moreover, it makes the error analysis and the design of fast solvers more complicated.

- The regularity of the model problem (1) on general polyhedral domain is still unknown. The existing work on numerical schemes are all based on high regularity assumptions.

- When Lagrange multiplier is introduced to enforce the divergence-free condition, an inf-sup condition is required in order to ensure existence and uniqueness of the approximation of \( p \).

There is only few work on numerical methods for quad-curl problem on three-dimensional domains. In [15], a nonconforming finite element method for the quad-curl model with low order term was studied under the regularity assumption such that

\[
\mathbf{u} \in [H^4(\Omega)]^3. \quad (2)
\]

A discontinuous Galerkin (DG) method using \( H(\text{curl}) \) conforming elements for the quad-curl model problem was investigated in [5], where the following lower regularity requirement was considered:

\[
\mathbf{u} \in [H^2(\Omega)]^3, \quad \nabla \times \mathbf{u} \in [H^2(\Omega)]^3. \quad (3)
\]

A mixed FEM for the quad-curl eigenvalue problem was introduced and analyzed in [13] under the regularity assumption higher than (3), such that

\[
\mathbf{u} \in [H^3(\Omega)]^3, \quad \nabla \times \mathbf{u} \in [H^3(\Omega)]^3. \quad (4)
\]

The quad-curl problem in 2D was studied in [2] based on Hodge decomposition.

Some regularity results of the quad-curl problem on domains with particular geometries also exists in the literature. For example, the following results were proved in [12]: when \( \mathbf{f} \in [L^2(\Omega)]^3 \), \( g_1 = g_2 = 0 \), \( g = 0 \), and the domain has no point and edge singularities, it holds that \( \mathbf{u} \in [H^4(\Omega)]^3 \); when the domain has point and edge singularities, \( \mathbf{u} \) does not belong to \( [H^3(\Omega)]^3 \) in general. In [14], the author proved that on convex polyhedral domains, when \( g_1 = g_2 = 0 \), \( g = 0 \) and \( \nabla \cdot \mathbf{f} = 0 \), there holds

\[
\mathbf{u} \in [H^2(\Omega)]^3, \quad \nabla \times \mathbf{u} \in [H^2(\Omega)]^3, \quad p = 0. \quad (5)
\]
However, there are no regularity results available for general Lipschitz polyhedral domains.

In recent years, the hybridizable discontinuous Galerkin (HDG) method has been successfully applied to solve various types of differential equations. It retains the main advantages of standard DG methods, such as flexible in meshing, easy to design and implement, ideal to be used with hp-adaptive strategy, etc. Moreover, HDG method can significant reduce the number of degrees of freedom, which allows for a substantial reduction in the computational cost. In this paper, we propose and analyze a HDG method for quad-curl model problem (Π), aiming to tackle the difficulties mentioned above. The error analysis is based on the following lower regularity requirement:

$$\mathbf{u} \in [H^s(\Omega)]^3, \nabla \times \mathbf{u} \in [H^{s+1}(\Omega)]^3, \nabla \times \nabla \times \mathbf{u} \in [H^s(\Omega)]^3, p \in H^{1+s}(\Omega),$$

(6)

with $s \in (\frac{1}{2}, 1]$. Actually, such regularity results hold for simply-connected Lipschitz polyhedron.

The rest of this paper is organized as follows. In section 2, we give some preliminaries, including basic notations, the regularity results based a mixed formulation and several projection operators needed for error analysis. In section 3, we propose the HDG method for the quad-curl model problem and show its stability results. In section 4, we derive the convergence analysis of the proposed HDG scheme. In section 5, some numerical experiments are performed to verify our theoretical results.

Throughout this paper, we use $C$ to denote a positive constant independent of mesh size and the frequency $w$, not necessarily the same at its each occurrence. We use $a \lesssim b$ ($a \gtrsim b$) to represent $a \leq Cb$ ($a \geq Cb$), and $a \sim b$ to represent $a \lesssim b \lesssim a$.

2 Preliminaries

2.1 Notations

For any bounded domain $\Lambda \subset \mathbb{R}^s$ ($s = 2, 3$), let $H^m(\Lambda)$ and $H^m_0(\Lambda)$ denote the usual $m^{th}$-order Sobolev spaces on $\Lambda$, and $\| \cdot \|_{m,\Lambda}, | \cdot |_{m,\Lambda}$ denote the norm and semi-norm on these spaces, respectively. We use $(\cdot , \cdot)_{m,\Lambda}$ to denote the inner product of $H^m(\Lambda)$, with $(\cdot , \cdot)_\Lambda := (\cdot , \cdot)_{0,\Lambda}$.

When $\Lambda = \Omega$, we denote $\| \cdot \|_{m} := \| \cdot \|_{m,\Omega}, | \cdot |_{m} := | \cdot |_{m,\Omega}$ and $(\cdot , \cdot) := (\cdot , \cdot)_{\Omega}$. In particular, when $\Lambda \in \mathbb{R}^2$, we use $(\cdot , \cdot)_\Lambda$ to replace $(\cdot , \cdot)_{\Lambda}$; when $\Lambda \in \mathbb{R}^1$, we use $(\cdot , \cdot)_\Lambda$ to replace $(\cdot , \cdot)_{\Lambda}$ to make a distinction. The bold face fonts will be used for vectors (or tensors) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. For integer $k \geq 0$, we denote by $\mathcal{P}_k(\Lambda)$ the set of polynomials defined on $\Lambda$ with degree no greater than $k$.

Let $\mathcal{T}_h = \bigcup \{T\}$ be a shape regular simplex partition of the domain $\Omega$ consists of arbitrary polygons. For any $T \in \mathcal{T}_h$, we let $h_T$ be the infimum of the diameters of spheres containing $T$ and denote the mesh size $h := \max_{T \in \mathcal{T}_h} h_T$. Let $\mathcal{F}_h = \bigcup \{F\}$ be the union of all faces of $T \in \mathcal{T}_h$, and let $\mathcal{F}_h^\circ$ and $\mathcal{F}_h^B$ be the set of interior faces and boundary faces, respectively. We denote by $h_F$ the length of diameter of the smallest circle containing face $F$. For all $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$, we denote by $\mathbf{n}_T$ and $\mathbf{n}_F$ the unit outward normal vectors along $\partial T$ and face $F$, respectively. Broken curl-curl, curl, div and gradient operators with respect to decomposition $\mathcal{T}_h$ are denoted by $\nabla \times \nabla \times, \nabla \times, \nabla \cdot$ and $\nabla$, respectively. For $u, v \in L^2(\partial \mathcal{T}_h)$,
we define the following inner product and the corresponding norm:

\[ \langle u, v \rangle_{\partial T_h} = \sum_{T \in T_h} \langle u, v \rangle_{\partial T}, \quad \|v\|_{T_h}^2 = \sum_{T \in T_h} \|v\|^2_{0,T}, \quad \|v\|_{\partial T_h}^2 = \sum_{T \in T_h} \|v\|^2_{0,\partial T}. \]

Define the following function spaces

\[ H(\text{curl}; \Omega) := \{ v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3 \}, \]
\[ H^s(\text{curl}; \Omega) := \{ v \in [H^s(\Omega)]^3 : \nabla \times v \in [H^s(\Omega)]^3 \} \text{ with } s \geq 0, \]
\[ H_0(\text{curl}; \Omega) := \{ v \in H(\text{curl}; \Omega) : n_T \times v|_{\Gamma} = 0 \}, \]
\[ H(\text{div}; \Omega) := \{ v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega) \}, \]
\[ H_0(\text{div}; \Omega) := \{ v \in H(\text{div}; \Omega) : n \cdot v = 0 \}, \]
\[ H(\text{div}^0; \Omega) := \{ v \in H(\text{div}; \Omega) : \nabla \cdot v = 0 \}, \]
and

\[ X := H(\text{curl}; \Omega) \cap H(\text{div}; \Omega), \]
\[ X_N := H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega), \]
\[ X_T := H(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega). \]

We define the following norm on \( H^s(\text{curl}; \Omega) \) with \( s \geq 0 \):

\[ \|v\|_{s,\text{curl}} = \left( \|v\|^2_s + \|\nabla \times v\|^2_s \right)^{\frac{1}{2}}. \]

### 2.2 Regularity

By introducing \( r = \nabla \times \nabla \times u \) we can rewrite (1) into the following second order system. Find \((r, u, p)\) that satisfies

\[
\begin{aligned}
    r - \nabla \times \nabla \times u &= 0 \quad \text{in } \Omega, \\
    \nabla \times \nabla \times r + \nabla p &= f \quad \text{in } \Omega, \\
    \nabla \cdot u &= g \quad \text{in } \Omega, \\
    n_T \times u &= g_1 \quad \text{on } \Gamma, \\
    n_T \times \nabla \times u &= g_2 \quad \text{on } \Gamma, \\
    p &= 0 \quad \text{on } \Gamma.
\end{aligned}
\]

We assume that the following regularity holds true:

\[ r \in H^s(\Omega), \ u \in H^s(\text{curl}; \Omega), \ \nabla \times u \in H^{1+s}(\text{curl}; \Omega), \text{ and } p \in H^{1+s}(\Omega), \]

with \( s \in \left( \frac{1}{2}, 1 \right] \) The designing of HDG scheme will based on equations (7), and the analysis of HDG scheme will based on the regularity (8).
2.3 Projection operators

2.3.1 $L^2$-projection

For any $T \in \mathcal{T}_h$, $F \in \mathcal{F}_h$ and integer $j \geq 0$, let $\Pi^o_j : L^2(T) \to \mathcal{P}_j(T)$ and $\Pi^\partial_j : L^2(F) \to \mathcal{P}_j(F)$ be the usual $L^2$ projection operators. The following approximation and stability results are standard.

**Lemma 1** For any $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$ and nonnegative integer $j$, it holds

\[
\|v - \Pi^o_j v\|_{0,T} \lesssim h_T^s |v|_{s,T} \quad \forall v \in H^s(T),
\]

\[
\|v - \Pi^\partial_j v\|_{0,\partial T} \lesssim h_T^{s-1/2} |v|_{s,T} \quad \forall v \in H^s(T),
\]

\[
\|v - \Pi^\partial_j v\|_{0,\partial T} \lesssim h_T^{s-1/2} |v|_{s,T} \quad \forall v \in H^s(T),
\]

\[
\|\Pi^o_j v\|_{0,T} \leq \|v\|_{0,T} \quad \forall v \in L^2(T),
\]

\[
\|\Pi^\partial_j v\|_{0,F} \leq \|v\|_{0,F} \quad \forall v \in L^2(F),
\]

where $1/2 < s \leq j + 1$.

2.3.2 $H(\text{div})$-projection

For integer $j \geq 1$, we first define the following local spaces on $T$ and $F$.

\[
\mathcal{D}_j(T) = [\mathcal{P}_{j-1}(T)]^3 \oplus \tilde{\mathcal{P}}_{j-1}(T) \cdot x,
\]

\[
\mathcal{D}_j(F) = [\mathcal{P}_{j-1}(F)]^3 \oplus \tilde{\mathcal{P}}_{j-1}(F) \cdot x,
\]

where $\tilde{\mathcal{P}}_{j-1}$ denotes the subspace of $\mathcal{P}_{j-1}$ consisting of homogeneous polynomials of degree $j - 1$. Then we define the global $H(\text{div})$ space by

\[
X_{h,j} = \{ v_h \in H(\text{div}; \Omega) : v_h|_T \in \mathcal{D}_j(T), \forall T \in \mathcal{T}_h \}.
\]

For any $v \in H(\text{div}; \Omega)$, its $H(\text{div})$-projection $\Pi_{h,j}^\text{div} v \in X_{h,j}$ is defined as follows (see Ref. [10]).

\[
\langle \Pi_{h,j}^\text{div} v \cdot n, w_{j-1} \rangle_T = \langle v \cdot n, w_{j-1} \rangle_T \quad \forall w_{j-1} \in \mathcal{P}_{j-1}(F),
\]

\[
(\Pi_{h,j}^\text{div} v, w_{j-2})_T = (v, w_{j-2})_T \quad \forall w_{j-2} \in [P_{k-2}(T)]^3,
\]

which hold for all $T \in \mathcal{T}_h$, $F \subset \partial T$ and $E \subset \partial F$.

2.3.3 $H(\text{curl})$-projection

For integer $j \geq 1$, we define

\[
Y_{h,j} = \{ v_h \in H(\text{curl}; \Omega) : v_h|_T \in [\mathcal{P}_j(T)]^3, \forall T \in \mathcal{T}_h \},
\]
For any \( v \in H^s(\text{curl}; \Omega) \) (\( s > \frac{1}{2} \)), its \( \text{H(curl)} \)-projection \( \Pi_{h,j}^{\text{curl}} v \in Y_{h,j} \) is defined as follows (see Ref. [11] for details).

\[
\langle \Pi_{h,j}^{\text{curl}} v \cdot \tau, w_j \rangle_E = \langle v \cdot \tau, w_{j-1} \rangle_E \quad \forall w_j \in \mathcal{P}_j(E),
\]

(10a)

\[
\langle \Pi_{h,j}^{\text{curl}} v, w_{j-1} \rangle_F = \langle v, w_{j-1} \rangle_F \quad \forall w_{j-1} \in \mathcal{D}_{j-1}(F),
\]

(10b)

\[
\langle \Pi_{h,j}^{\text{curl}} v, w_{j-2} \rangle_T = \langle v, w_{j-2} \rangle_T \quad \forall w_{j-2} \in \mathcal{D}_{k-2}(T),
\]

(10c)

which hold for all \( T \in \mathcal{T}_h \) and \( F \subset \partial T \).

Note that for each \( T \in \mathcal{T}_h \), the above projection makes sense for \( v \in H^s(\text{curl}; T) \) with \( s > \frac{1}{2} \) (see [1, Lemma 5.1] for details). Moreover, the following approximation properties hold true:

**Lemma 2** [11, 1, 9]. For any \( T \in \mathcal{T}_h \) and \( v \in H^s(\text{curl}; T) \) with \( s > \frac{1}{2} \), if \( v \in [H^1(T)]^3 \) with \( t \in (\frac{1}{2}, k + 1] \), it holds that

\[
\|v - \Pi_{h,k}^{\text{curl}} v\|_{\mathcal{T}_h} \lesssim h^t \|v\|_t.
\]

(11)

Moreover, if \( \nabla \times v \in [H^1(T)]^3 \) with \( t \in (\frac{1}{2}, k) \), then it holds

\[
\|\nabla \times v - \nabla \times \Pi_{h,k}^{\text{curl}} v\|_{\mathcal{T}_h} \lesssim h^t \|\nabla \times v\|_t.
\]

(12)

**Lemma 3** For any integer \( j \geq 1 \), we have the following commuting property

\[
\nabla \times \Pi_{h,j}^{\text{curl}} v = \Pi_{h,j}^{\text{div}} \nabla \times v \quad \forall v \in H^s(\text{curl}; \Omega), \ s > \frac{1}{2}.
\]

(13)

### 2.3.4 H(div)-projection on domain surface

For any \( v \in H^s(\text{curl}; T) \) with \( s > \frac{1}{2} \) and \( T \in \mathcal{T}_h \), we consider \( \Pi_{h,k}^{\text{curl}} v \) restricted to face \( F \subset \partial T \) such that

\[
\langle \Pi_{h,k}^{\text{curl}} v \cdot t_{FE}, w_k \rangle_E = \langle v \cdot t_{FE}, w_k \rangle_E \quad \forall w_k \in \mathcal{P}_k(E), \ E \subset \partial F,
\]

(14)

where \( t_{FE} = n_F \times n_{FE} \). It can be observed that for \( k \geq 2 \),

\[
\langle \Pi_{h,k}^{\text{curl}} v, w_{k-1} \rangle_F = \langle v, w_{k-1} \rangle_F \quad \forall w_{k-1} \in \mathcal{D}_{k-1}(F).
\]

(15)

Note that

\[
v \cdot t_{FE} = v \cdot (n_F \times n_{FE}) = (v \times n_F) \cdot n_E = -(n_F \times v) \cdot n_E,
\]

and

\[
v|_F = (n_F \times v) \times n_F + (v \cdot n_F)n_F, \quad v \cdot n_F = 0.
\]

Hence we can rewrite equations (14) and (15) as

\[
\langle \Pi_{h,k}^{\text{curl}} (n_F \times v) \cdot n_E, w_k \rangle_E = \langle (n_F \times v) \cdot n_E, w_k \rangle_E \quad \forall w_k \in \mathcal{P}_k(E), \ E \subset \partial F,
\]

and

\[
\langle \Pi_{h,k}^{\text{curl}} (n_F \times v), n_F \times w_{k-1} \rangle_F = \langle n_F \times v, n_F \times w_{k-1} \rangle_F \quad \forall w_{k-1} \in \mathcal{D}_{k-1}(F), \ k \geq 2.
\]

The operator \( \Pi_{h,k}^{\text{curl}} (n_F \times \cdot) \) maps from space \( H^s(\text{curl}; T) \) to \( \bigcup_{F \subset \partial Y} [\mathcal{P}_k(F)]^3 \) for each \( T \in \mathcal{T}_h \).

Actually, by denoting \( \Pi_{h,k}^{T,\text{div}} := \Pi_{h,k}^{\text{curl}}|_T \), we observe that \( \Pi_{h,k}^{T,\text{div}} (n_F \times v) \) defines a \( \text{H(div)} \)-projection of \( n_F \times v \) on the domain surface \( \Gamma \).
2.3.5 \( H(\text{curl})\)-projection and \( H^1\)-projection on finite element spaces

In the error analysis, we need the following \( H_0(\text{curl})\)-conforming and \( H^1\)-conforming interpolations.

**Lemma 4** (cf. [6, Proposition 4.5]) For any integer \( k \geq 1 \), let \( \mathbf{v}_h \in [P_k(\mathcal{T}_h)]^3 \), there exists a function \( \Pi_{h,k}^{\text{curl},c} \mathbf{v}_h \in [P_k(\mathcal{T}_h)]^3 \cap H_0(\text{curl}; \Omega) \) such that

\[
\| \Pi_{h,k}^{\text{curl},c} \mathbf{v}_h - \mathbf{v}_h \|_0 \lesssim \| h_1^{1/2} \mathbf{n} \times [\mathbf{v}_h] \|_{0,F_h},
\]

\[
\| \nabla_h \times (\Pi_{h,k}^{\text{curl},c} \mathbf{v}_h - \mathbf{v}_h) \|_0 \lesssim \| h_1^{-1/2} \mathbf{n} \times [\mathbf{v}_h] \|_{0,F_h},
\]

with a constant \( C > 0 \) independent of the mesh size.

**Lemma 5** (cf. [7, Theorem 2.2]) For all \( q_h \in P_h \), there exists an interpolation operator \( I_k : P_h \to P_h \cap H_0^1(\Omega) \) such that

\[
\| \nabla q_h - \nabla I_k q_h \|_{\mathcal{T}_h} \lesssim \| h_1^{1/2} \| q_h \|_{F_h},
\]

where \( \| q_h \| \) stands for the jump of \( q_h \) on \( F_h \).

3 HDG finite element method

3.1 HDG method

For any integers \( k \geq 1 \), we introduce the following finite dimensional spaces:

\[
R_h = [P_{k-1}(\mathcal{T}_h)]^3,
\]

\[
U_h = [P_k(\mathcal{T}_h)]^3,
\]

\[
\hat{U}_h = \{ \hat{\mathbf{v}}_h \in [P_k(\mathcal{F}_h)]^3 : \hat{\mathbf{v}}_h \cdot \mathbf{n}|_{F_h} = 0 \},
\]

\[
\hat{U}_h^g = \{ \hat{\mathbf{v}}_h \in \hat{U}_h : \mathbf{n} \times \hat{\mathbf{v}}_h|_{\Gamma} = \Pi_{h,k}^{\text{div},g} \}, \quad \hat{g} = 0, g_1,
\]

\[
\hat{C}_h = \{ \hat{\mathbf{v}}_h \in [P_{k-1}(\mathcal{F}_h)]^3 : \hat{\mathbf{v}}_h \cdot \mathbf{n}|_{F_h} = 0 \},
\]

\[
\hat{C}_h^g = \{ \hat{\mathbf{v}}_h \in \hat{C}_h : \mathbf{n} \times \hat{\mathbf{v}}_h|_{\Gamma} = \Pi_{k-1}^g \}, \quad \hat{g} = 0, g_2,
\]

\[
P_h = P_k(\mathcal{T}_h),
\]

\[
\hat{P}_h = P_k(\mathcal{F}_h),
\]

\[
\hat{P}_h^0 = \{ \hat{q}_h \in \hat{P}_h : \hat{q}_h|_{\Gamma} = 0 \},
\]

where

\[
P_j(\mathcal{T}_h) = \{ q_h \in L^2(\Omega) : q_h|_T \in P_j(T), \forall T \in \mathcal{T}_h \},
\]

\[
P_j(\mathcal{F}_h) = \{ q_h \in L^2(\mathcal{F}_h) : q_h|_F \in P_j(F), \forall F \in \mathcal{F}_h \}.
\]

The HDG finite element method for (\( \Pi \)) reads:
For all \((s_h, v_h, \tilde{v}_h, d_h, q_h, \tilde{q}_h) \in R_h \times U_h \times \hat{U}_h \times \hat{C}_h \times P_h \times \hat{P}_h\), find \((r_h, u_h, \tilde{u}_h, \hat{c}_h, p_h, \hat{p}_h) \in R_h \times U_h \times \hat{U}_h \times \hat{C}_h \times P_h \times \hat{P}_h\) such that

\[
a_h(r_h, s_h) + b_h(u_h, \tilde{u}_h, \hat{c}_h; s_h) = 0, \tag{19a}
\]

\[
b_h(v_h, \tilde{v}_h, \tilde{d}_h; r_h) + c_h(p_h, \hat{p}_h; v_h) - s_h^n(u_h, \tilde{u}_h, \tilde{v}_h, \tilde{d}_h) = -(f, v_h), \tag{19b}
\]

\[
c_h(q_h, \hat{q}_h; u_h) + s_h^p(p_h, \hat{p}_h; q_h, \hat{q}_h) = (g, q_h), \tag{19c}
\]

where

\[
a_h(r_h, s_h) = (r_h, s_h)_T, \\
 b_h(u_h, \tilde{u}_h, \hat{c}_h; s_h) = -(u_h, \nabla \times \nabla \times s_h)_T - (n \times \tilde{u}_h, \nabla \times s_h)_{\partial T_h} - (n \times \hat{c}_h, s_h)_{\partial T_h}, \\
 c_h(q_h, \hat{q}_h; u_h) = (\nabla \cdot u_h, q_h)_T - (n \cdot u_h, \hat{q}_h)_{\partial T_h}, \\
 s_h^n(u_h, \tilde{u}_h, \hat{c}_h; v_h, \tilde{v}_h, \tilde{d}_h) = (h_F^{-3} n \times (u_h - \tilde{u}_h), n \times (v_h - \tilde{v}_h))_{\partial T_h} \\
 + (h_F^{-1} n \times (\nabla \times u_h - \hat{c}_h), n \times (\nabla \times v_h - \tilde{d}_h))_{\partial T_h}, \\
 s_h^p(p_h, \hat{p}_h; q_h, \hat{q}_h) = (h_F^{-1} (p_h - \tilde{p}_h, q_h - \tilde{q}_h))_{\partial T_h}.
\]

To simplify notation, we define

\[
\sigma := (r, u, \nabla \times u, p, p), \\
\sigma_h := (r_h, u_h, \tilde{u}_h, \hat{c}_h, p_h, \hat{p}_h), \\
\tau_h := (s_h, v_h, \tilde{v}_h, \tilde{d}_h, q_h, \hat{q}_h), \\
\Sigma_h := R_h \times U_h \times \hat{U}_h \times \hat{C}_h \times P_h \times \hat{P}_h, \\
\Sigma_h^g := R_h \times U_h \times \hat{U}_h \times \hat{C}_h \times P_h \times \hat{P}_h, \\
\Sigma_h^0 := R_h \times U_h \times \hat{U}_h \times \hat{C}_h \times P_h \times \hat{P}_h,
\]

and

\[
B_h(\sigma_h, \tau_h) := a_h(r_h, s_h) + b_h(u_h, \tilde{u}_h, \hat{c}_h; s_h) \\
+ b_h(v_h, \tilde{v}_h, \tilde{d}_h; r_h) + c_h(p_h, \hat{p}_h; v_h) - s_h^n(u_h, \tilde{u}_h, \tilde{v}_h, \tilde{d}_h) \\
+ c_h(q_h, \hat{q}_h; u_h) + s_h^p(p_h, \hat{p}_h; q_h, \hat{q}_h), \\
F_h(\tau_h) := -(f, v_h) + (g, q_h).
\]

Then the HDG scheme \((19a) - (19c)\) can be rewritten as:

Find \(\sigma_h \in \Sigma_h^0\) such that

\[
B_h(\sigma_h, \tau_h) = F_h(\tau_h), \quad \forall \tau_h \in \Sigma_h^0. \tag{28}
\]

### 3.2 Stability analysis

We define the following semi-norms on spaces \(U_h \times \hat{U}_h \times \hat{C}_h\) and \(P_h \times \hat{P}_h\):

\[
\|(v, \tilde{v}, \tilde{d})\|_U := \left\| (v, \tilde{v}, \tilde{d}) \right\|_{\text{curl}}^2 + \left\| v \right\|_{\text{div}}^2, \\
\|(q, \tilde{q})\|_P := \left\| h_T \nabla q \right\|_{\partial T_h} + \left\| h_F^{1/2} (q - \tilde{q}) \right\|_{\partial T_h}^2.
\]
where
\[
\|(v, \hat{v}, d)\|_{\text{curl}}^2 := \|\nabla \times \nabla \times v\|_{T_h}^2 + \|h_F^{-3/2}(n \times (v - \hat{v}))\|_{\partial T_h}^2 + \|h_F^{-1/2}(n \times (\nabla \times v - d))\|_{\partial T_h}^2.
\]

Then we define the semi-norms on \(\Sigma_h\) as:
\[
\|\sigma\|_{\Sigma_h} := \left(\|r\|_{T_h}^2 + \|(u, \hat{u}, \hat{c})\|_{U}^2 + \|(p, \hat{p})\|_{P}^2\right)^{1/2}, \tag{31}
\]
\[
\|\tau\|_{\Sigma_h} := \left(\|s\|_{T_h}^2 + \|(v, \hat{v}, \hat{d})\|_{U}^2 + \|(q, \hat{q})\|_{P}^2\right)^{1/2}. \tag{32}
\]

**Lemma 6** The semi-norm \(\|\cdot\|_U\) defines a norm on the space \(U_h \times \hat{U}_h \times \hat{C}_h\).

**Proof 1** Let \((v_h, \hat{v}_h, \hat{d}_h) \in U_h \times \hat{U}_h \times \hat{C}_h\). It suffices to show that \(\|(v_h, \hat{v}_h, \hat{d}_h)\|_U = 0\) leads to \((v_h, \hat{v}_h, \hat{d}_h) = (0, 0, 0)\), which can be checked easily. Actually, note that \(n \times \hat{v}_h = 0\) and \(n \cdot \hat{v}_h = 0\) lead to \(\hat{v}_h = 0\); Similarly, \(n \times \hat{d}_h = 0\) and \(n \cdot \hat{d}_h = 0\) lead to \(\hat{d}_h = 0\).

**Lemma 7** The semi-norm \(\|(\cdot, \cdot)\|_P\) defines a norm on \(P_h \times \hat{P}_h\). Moreover, for all \((q_h, \hat{q}_h) \in P_h \times \hat{P}_h\), there holds
\[
\|(q_h, \hat{q}_h)\|_P^2 \sim \|h_T \nabla q_h\|_{T_h}^2 + \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2. \tag{33}
\]

**Proof 2** By combining the definition of \(\|(\cdot, \cdot)\|_P\) in (31), the estimate (18) and the triangle inequality, we have
\[
\|(q_h, \hat{q}_h)\|_P^2 = \|h_T \nabla q_h\|_{T_h}^2 + \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2
\leq \|h_T(\nabla q_h - \nabla \hat{q}_h)\|_{T_h}^2 + \|h_T \nabla \hat{q}_h\|_{T_h}^2 + \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2
\leq \|h_T^{-1/2}[q_h]\|_{0, \hat{F}_h}^2 + \|h_T \nabla \hat{q}_h\|_{T_h}^2 + \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2
\leq \|h_T \nabla \hat{q}_h\|_{T_h}^2 + \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2. \tag{34}
\]

On the other hand,
\[
\|h_T \nabla \hat{q}_h\|_{T_h}^2 \leq \|h_T(\nabla \hat{q}_h - \nabla q_h)\|_{T_h}^2 + \|\nabla q_h\|_{T_h}^2
\leq \|h_T^{-1/2}[q_h]\|_{0, \hat{F}_h}^2 + \|h_T \nabla q_h\|_{T_h}^2
\leq \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2 + \|\nabla q_h\|_{T_h}^2
= \|(q_h, \hat{q}_h)\|_P^2. \tag{35}
\]

Therefore, the estimate (33) holds.

Next, we prove \(\|(\cdot, \cdot)\|_P\) is a norm on \(P_h \times \hat{P}_h\). For any \((q_h, \hat{q}_h) \in P_h \times \hat{P}_h\) such that
\[
\|h_T \nabla q_h\|_{T_h}^2 + \|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial T_h}^2 = 0,
\]
we know that \(q_h\) is piecewise constants, and \(q_h = \hat{q}_h\) on every face. Moreover, \(q_h = \hat{q}_h = 0\) on boundary faces. Therefore, \(q_h = \hat{q}_h = 0\). This completes the proof.
Theorem 1 (Discrete inf-sup condition)  The following stability estimates hold true for $B_h$.

\[
\sup_{\tau_h \in \Sigma_h, \tau_h \neq 0} \frac{B_h(\sigma_h, \tau_h)}{\|\tau_h\|_{\Sigma_h}} \geq \|\sigma_h\|_{\Sigma_h}, \tag{36}
\]

\[
\sup_{\sigma_h \in \Sigma_h, \sigma_h \neq 0} \frac{B_h(\sigma_h, \tau_h)}{\|\sigma_h\|_{\Sigma_h}} \geq \|\tau_h\|_{\Sigma_h}. \tag{37}
\]

Proof 3 We use the following five steps to derive (36)--(37).

Step one:
Taking $\tau_h^1 = (r_h, -u_h, -\hat{u}_h, \nabla \hat{c}_h, p_h, \hat{p}_h) \in \Sigma_h^0$, then by the definitions of $\sigma_h$, $B_h$ and the norm $\| \cdot \|_{\Sigma_h}$ (cf. (21), (26) and (31)--(32)), we have

\[
\|\tau_h^1\|_{\Sigma_h} = \|\sigma_h\|_{\Sigma_h}, \tag{38}
\]

and

\[
B_h(\sigma_h, \tau_h^1) = \|r_h\|^2_{T_h} + \|h_F^{-3/2} \nabla \times (u_h - \hat{u}_h)\|_{\partial T_h}^2
+ \|h_F^{-1/2} \nabla \times u_h \times (\nabla - \hat{c}_h)\|_{\partial T_h}^2 + \|h_F^{-1/2} (p_h - \hat{p}_h)\|_{\partial T_h}^2. \tag{39}
\]

Step two:
By taking $\tau_h^2 = (\nabla \times \nabla \times u_h, 0, 0, 0, 0, 0) \in \Sigma_h^0$ we have

\[
\|\tau_h^2\|_{\Sigma_h} = \|\nabla \times \nabla \times u_h\|_{T_h} \leq \|\sigma_h\|_{\Sigma_h}. \tag{40}
\]

By the definition of $B_h$, integration by parts and inverse inequality, we have

\[
B_h(\sigma_h, \tau_h^2) = -(r_h, \nabla \times \nabla \times u_h) + \|\nabla \times \nabla \times u_h\|_{T_h}^2
+ (\nabla \times \nabla \times u_h, \nabla \times (u_h - \hat{u}_h))_{\partial T_h}
+ (\nabla \times \nabla \times u_h, \nabla \times (\nabla \times u_h - \hat{c}_h))_{\partial T_h}
\geq \frac{1}{2} \|\nabla \times \nabla \times u_h\|_{T_h}^2 - C_1 \|r_h\|_{T_h}^2
- C_1 \|h_F^{-3/2} \nabla \times (u_h - \hat{u}_h)\|_{\partial T_h}^2
- C_1 \|h_F^{-1/2} \nabla \times u_h \times (\nabla - \hat{c}_h)\|_{\partial T_h}^2. \tag{41}
\]

Step three:
Let $r_h = h_F^2 \nabla \cdot u_h$, $\tilde{r}_h = h_F [n \cdot u_h]$ on $F_h^0$ and $\hat{r}_h = 0$ on $\Gamma$. Taking $\tau_h^3 = (0, 0, 0, r_h, \tilde{r}_h, \hat{r}_h) \in \Sigma_h^0$, then by the definition of $\| \cdot \|_{\Sigma_h}$ and inverse inequality we have

\[
\|\tau_h^3\|_{\Sigma_h}^2 \leq \|\nabla r_h\|_{\Sigma_h}^2 + \|h_F^{-1/2} (r_h - \tilde{r}_h)\|_{\partial T_h}^2
\lesssim \|h_F \nabla \cdot u_h\|_{T_h}^2 + \|h_F^{-1/2} [n \cdot u_h]\|_{F_h^0}^2 \tag{42}
\]

\[
\leq \|\sigma_h\|_{\Sigma_h}^2.
\]

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Moreover,
\[
B_h(\sigma_h, \tau_h^4) = \|h_T \nabla \cdot u_h\|_{\Omega_h}^2 + \|h_F^{1/2}[n \cdot u_h]\|_{\partial \Omega_h}^2 + \langle h_F^{-1}(p_h - \bar{p}_h), r_h - \bar{r}_h \rangle_{\partial \Omega_h} \geq \frac{1}{2} \|u_h\|_{\text{div}}^2 - C_2 \|h_F^{-1/2}(p_h - \bar{p}_h)\|_{\partial \Omega_h}^2.
\]  

**(Step four):**

We then take \( \tau_h^4 = (0, -\nabla I_k p_h, -n \times \nabla I_k p_h \times n, 0, 0, 0) \in \Sigma_h^0 \). Similar to previous steps, we have
\[
\|\tau_h^4\|_{\Sigma_h^0} = \|h_T \nabla \cdot \nabla I_k p_h\|_{\Omega_h}^2 + \|h_F^{1/2}[\nabla I_k p_h \cdot n]\|_{\partial \Omega_h} \lesssim \|\nabla I_k p_h\|_{\Omega_h}^2 \lesssim \|\sigma_h\|_{\Sigma_h}^2.
\]

Moreover,
\[
B_h(\sigma_h, \tau_h^4) = (\nabla I_k p_h, \nabla \times \nabla \times r_h)_{\Omega_h} + \langle n \times (\nabla \times I_k p_h \times n), \nabla \times r_h \rangle_{\partial \Omega_h} - (\nabla \cdot \nabla I_k p_h, p_h)_{\Omega_h} + \langle n \cdot \nabla I_k p_h, \bar{p}_h \rangle_{\partial \Omega_h} \geq \frac{1}{2} \|\nabla p_h\|_{\Omega_h}^2 - C_3 \|h_F^{-1/2}(p_h - \bar{p}_h)\|_{\partial \Omega_h}^2 - C_3 \|\tau_h\|_{\Omega_h}^2 - C_4 \|\nabla \times (\nu_h - \bar{\nu}_h)\|_{\partial \Omega_h}^2.
\]

**(Step five):**

Take \( C_0 = \max(C_1, C_2) + C_3 + 1 \) and \( \tau_h = C_0 \tau_h^1 + \tau_h^2 + \tau_h^3 + \tau_h^4 \), then it follows from (38)–(45) that
\[
\|\tau_h\|_{\Sigma_h} \lesssim \|\sigma_h\|_{\Sigma_h}, \quad B_h(\sigma_h, \tau_h) \gtrsim \|\tau_h\|_{\Omega_h}^2 + \|\sigma_h\|_{\Sigma_h}^2 + \|\Pi_k \circ \sigma_h\|_{\Sigma_h}^2.
\]

Hence
\[
B_h(\sigma_h, \tau_h) \gtrsim \|\sigma_h\|_{\Sigma_h}^2 \|\tau_h\|_{\Sigma_h},
\]

which implies (36). Since \( B_h \) is symmetric, (37) also holds.

The following corollary is a direct consequence of Theorem 1.  
**Corollary 1** The HDG scheme (28) admits a unique solution \( \sigma_h \in \Sigma_h^0 \).

### 4 Error estimates

#### 4.1 Primary estimates

**Lemma 8** Let \((r, u, p)\) be the solution of (7), \( \sigma \) and \( J_h \sigma \) be defined by
\[
\sigma := (r, u, u, \nabla \times u, p, p), \quad J_h \sigma := (\Pi_{k-1} r, \Pi_{h,k}^\text{curl} u, n \times \Pi_{h,k}^\text{curl} u \times n, \Pi_{k-1} \nabla \times u, \Pi_k^0 u, \Pi_k^0 p).
\]
Then we have

$$B_h(\mathcal{J}_h \sigma, \tau_h) = F_h(\tau_h) + E_h^{\mathcal{J}}(\sigma; \tau_h) \quad \forall \tau_h \in \Sigma_h^0,$$

where

$$E_h^{\mathcal{J}}(\sigma; \tau_h) = -\langle n \times (\hat{\nu}_h - v_h), \nabla \times (\Pi_{k-1}^0 \nu - \nu) \rangle_{\partial \Omega_h}$$

$$- \langle n \times (\hat{d}_h - \nabla \times v_h), \Pi_{k-1}^0 \nu - \nu \rangle_{\partial \Omega_h}$$

$$- \langle h_F^{-1} n \times (\nabla \times \Pi_{h,k}^{\text{curl}} u - \nabla \times u), n \times (\nabla \times v_h - \hat{d}_h) \rangle_{\partial \Omega_h}$$

$$- \langle \hat{q}_h - q_h, (\Pi_{h,k}^{\text{curl}} u - u) \cdot n \rangle_{\partial \Omega_h}$$

$$+ (h_F^{-1}(\Pi_{k}^0 - p), q_h - \hat{q}_h)_{\partial \Omega_h} - (\nabla \times \Pi_{h,k}^{\text{curl}} u - u).$$

**Proof 4** By the definitions of $a_h$, $b_h$, integration by parts and the fact that $m \leq k - 1$, we arrive at

$$a_h(\Pi_{k-1}^0 \nu, s_h) + b_h(\Pi_{h,k}^{\text{curl}} u, n \times \Pi_{h,k}^{\text{curl}} u \times n; s_h)$$

$$= (\Pi_{k-1}^0 \nu, s_h) - (\Pi_{h,k}^{\text{curl}} u, n \times \nabla \times s_h)$$

$$- \langle n \times n \times \Pi_{h,k}^{\text{curl}} u \times n, \nabla \times s_h \rangle_{\partial \Omega_h} - \langle n \times \Pi_{h,k}^0 \nabla \times u, s_h \rangle_{\partial \Omega_h}$$

$$= (\nu, s_h) - (\nabla \times \Pi_{h,k}^{\text{curl}} u, \nabla \times s_h) - \langle n \times \nabla \times u, s_h \rangle_{\partial \Omega_h}$$

$$= (\nu, s_h) + (\nabla \times (u - \Pi_{h,k}^{\text{curl}} u), \nabla \times s_h)$$

$$- (\nabla \times u, \nabla \times s_h) - \langle n \times \nabla \times u, s_h \rangle_{\partial \Omega_h}.$$

Using integration by parts and the fact that $\nu = \nabla \times u$, and $\nabla \times (u - \Pi_{h,k}^{\text{curl}} u, \nabla \times s_h) = 0$, we get

$$a_h(\Pi_{k-1}^0 \nu, s_h) + b_h(\Pi_{h,k}^{\text{curl}} u, n \times \Pi_{h,k}^{\text{curl}} u \times n; s_h) = 0. \quad (48)$$

By the definitions of $b_h$, $c_h$, $s^u_h$ and integration by parts, one can get

$$b_h(v_h, \hat{v}_h; \Pi_{k-1}^0 \nu) + c_h(\Pi_{k}^0 P, \Pi_{k}^0 P; v_h) - s^u_h(\Pi_{h,k}^{\text{curl}} u, n \times \Pi_{h,k}^{\text{curl}} u \times n; v_h, \hat{v}_h)_{\Omega_h}$$

$$= -\langle v_h, \nabla \times \nabla \times \Pi_{k-1}^0 \nu \rangle_{\Omega_h} - \langle n \times \hat{v}_h, \nabla \times \Pi_{k-1}^0 \nu \rangle_{\partial \Omega_h}$$

$$- \langle n \times d_h, \Pi_{k-1}^0 \nu \rangle_{\partial \Omega_h} + (\nabla \cdot v_h, \Pi_{k}^0 P)_{\Omega_h} - \langle n \cdot v_h, \Pi_{k}^0 P \rangle_{\partial \Omega_h}$$

$$- \langle h_F^{-1} n \times (\nabla \times \Pi_{h,k}^{\text{curl}} u - \Pi_{k}^0 \nabla \times u), n \times (\nabla \times v_h - \hat{d}_h) \rangle_{\partial \Omega_h}$$

$$= -\langle \nabla \times (v_h, \Pi_{k-1}^0 \nu \rangle_{\Omega_h} - \langle v_h, \nabla P \rangle_{\Omega_h}$$

$$- \langle n \times (\hat{v}_h - v_h), \nabla \times \Pi_{k-1}^0 \nu \rangle_{\partial \Omega_h} - \langle n \times (\hat{d}_h - \nabla \times v_h), \Pi_{k-1}^0 \nu \rangle_{\partial \Omega_h}$$

$$- \langle h_F^{-1} n \times (\nabla \times \Pi_{h,k}^{\text{curl}} u - \nabla \times u), n \times (\nabla \times v_h - \hat{d}_h) \rangle_{\partial \Omega_h}.$$

Since $\langle n \times \hat{v}_h, \nabla \times \nu \rangle_{\partial \Omega_h} = 0$ and $\langle n \times \hat{d}_h, \nu \rangle_{\partial \Omega_h} = 0$, it then follows from (??) and integration
by parts that
\[
\begin{align*}
b_h(v_h, \tilde{v}_h; \Pi^0_{h-1} r) + c_h(P_h^p, \Pi^0_p v_h; v_h) - s^u_h(P_{h,k}^0 u, n \times \Pi^0_{h,k} u \times n; v_h, \tilde{v}_h) \\
= -\langle n \times (\tilde{d}_h - \nabla \times v_h), \Pi^0_{k-1} r - r \rangle_{\partial T_h} \\
- \langle n \times (\tilde{d}_h - \nabla \times v_h), \Pi^0_{k-1} r - r \rangle_{\partial T_h} \\
- \langle h^{-1}_F n \times (\nabla \times \Pi^0_{h,k} u - \nabla \times u), \n \times (\nabla \times v_h - \tilde{d}_h) \rangle_{\partial T_h} \\
= -\langle v_h, f \rangle - \langle n \times (\tilde{v}_h - v_h), \nabla \times (\Pi^0_{k-1} r - r) \rangle_{\partial T_h} \\
- \langle n \times (\tilde{d}_h - \nabla \times v_h), \Pi^0_{k-1} r - r \rangle_{\partial T_h} \\
- \langle h^{-1}_F n \times (\nabla \times \Pi^0_{h,k} u - \nabla \times u), \n \times (\nabla \times v_h - \tilde{d}_h) \rangle_{\partial T_h}.
\end{align*}
\]

By the definition of \( c_h \) and integration by parts we have
\[
\begin{align*}
c_h(q_h, \tilde{q}_h; \Pi^0_{h,k} u) &= \langle q_h, \nabla \cdot \Pi^0_{h,k} u \rangle - \langle \tilde{q}_h, \Pi^0_{h,k} u \cdot n \rangle_{\partial T_h} \\
&= -\langle \nabla q_h, \Pi^0_{h,k} u \rangle - \langle \tilde{q}_h - q_h, \Pi^0_{h,k} u \cdot n \rangle_{\partial T_h} \\
&= -\langle \tilde{q}_h - q_h, (\Pi^0_{h,k} u - u) \cdot n \rangle_{\partial T_h} - \langle \nabla q_h, \Pi^0_{h,k} u - u \rangle_{\Omega} + (g, q_h)_{\partial T_h}.
\end{align*}
\]
where we have used the fact \( \langle \tilde{q}_h - q_h, \Pi^0_{h,k} u \cdot n \rangle_{\partial T_h} = 0 \) and \( \nabla \cdot u = g \). By the definition of \( s^p_h \) we get
\[
s^p_h(P^0_p, \Pi^0_p; q_h, \tilde{q}_h) = \langle h^{-1}_F (\Pi^0_p - p), q_h - \tilde{q}_h \rangle_{\partial T_h}.
\]

Finally the desired result (46) follows from the definition (26) and (48) – (51).

We recall the result in [8]. For any \((v, \tilde{v}) \in [L^2(\Omega))^3 \times [L^2(\partial T_h)]^3\), and for any \( T \in T_h \), there exists an interpolation \( I_T(v, \tilde{v}) \in [P_{k+3}(T)]^3 \) such that
\[
\begin{align*}
(I_T(v, \tilde{v}), w_h)_T &= (v, w_h)_T, \\
(I_T(v, \tilde{v}), \tilde{w}_h)_F &= (\tilde{v}, \tilde{w}_h)_F,
\end{align*}
\]
for all \((w_h, \tilde{w}_h) \in [P_F(T))^3 \times [P_F(F)]^3\), and \( F \subset \partial T \). We define \( I_h|_T = I_T \), if \( v_h|_T \in [P_k(T)]^3 \), \( \tilde{v}_h \in [P_F(F)]^3 \) for all \( F \subset \partial T \), it holds
\[
\|v_h - I_h(v_h, \tilde{v}_h)\|_{\partial T_h} \lesssim h^{-1/2}_T(v_h - \tilde{v}_h)_{\partial T_h},
\]
\[
\|\nabla (v_h - I_h(v_h, \tilde{v}_h))\|_{\partial T_h} \lesssim h^{-1/2}_T(v_h - \tilde{v}_h)_{\partial T_h}.
\]
In addition, if \( \tilde{v}_h \in [P_F(F)]^3 \), \( \tilde{v}_h|_{\partial T} = 0 \), then \( I_h|_{\partial T}(v_h, \tilde{v}_h) \in [H^1_0(\Omega)]^3 \).

We define
\[
P_T(v, \tilde{v}) := I_T(v, \tilde{v} + (n \cdot v)n).
\]

**Lemma 9** For any \( T \in T_h \) and \((v, \tilde{v}) \in H^1(T) \times L^2(\partial T)\), we have
\[
\begin{align*}
P_T(v, \tilde{v})(w_h)_T &= (v, w_h)_T, \\
(n \times P_T(v, \tilde{v}) n)_F &= (n \times \tilde{v} n \times \tilde{w}_h)_F.
\end{align*}
\]
for all \((w_h, \tilde{w}_h) \in [P_k(T)]^3 \times [P_k(F)]^3\), and \(F \subset \partial T\). And the following approximation properties hold true for \((v_h, \tilde{v}_h) \in V_h \times \tilde{V}_h\):

\[
\|v_h - \Pi_h(v_h, \tilde{v}_h)\|_{T_h} \lesssim \|h_T^{1/2} n \times (v_h - \tilde{v}_h)\|_{\partial T_h},
\]

\[
\|\nabla \times (v_h - \Pi_h(v_h, \tilde{v}_h))\|_{T_h} \lesssim \|h_T^{1/2} n \times (v_h - \tilde{v}_h)\|_{\partial T_h}.
\]

Moreover, we define \(\Pi_h|_{T} = \Pi_T\), then \(\Pi_h(v_h, v_h) \in H_0(\text{curl}; \Omega)\) for all \((v_h, \tilde{v}_h) \in V_h \times \tilde{V}_h\).

**Proof 5** By \(^{52a}\) and \(^{54}\) we have

\[
(\Pi_T(v, \tilde{v}), w_h)_T = (\mathcal{I}_T(v, \tilde{v} + (n \cdot v)n), w_h)_T = (v, w_h)_T.
\]

By \(^{52b}\) and \(^{54}\), it holds

\[
\langle n \times \Pi_T(v, \tilde{v}), n \times \tilde{w}_h \rangle_F = \langle n \times \mathcal{I}_T(v, \tilde{v}), n \times \tilde{w}_h \rangle_F \\
= \langle \mathcal{I}_T(v, \tilde{v}), n \times \tilde{w}_h \times n \rangle_F \\
= \langle \tilde{v}, n \times \tilde{w}_h \times n \rangle_F \\
= \langle n \times \tilde{v}, n \times \tilde{w}_h \rangle_F.
\]

We use \(^{53a}\), \(^{54}\) and the fact \(n \cdot \tilde{v}_h = 0\) to get

\[
\|v_h - \Pi_h(v_h, \tilde{v}_h)\|_{T_h} \lesssim \|h_T^{1/2}(v_h - (\tilde{v}_h + (n \cdot v_h)n))\|_{\partial T_h} \\
= \|h_T^{1/2}(v_h - (n \cdot v_h)n) - (\tilde{v}_h - (n \cdot \tilde{v}_h)n)\|_{\partial T_h} \\
= \|h_T^{1/2} n \times (v_h - \tilde{v}_h) \times n\|_{\partial T_h} \\
\le \|h_T^{1/2} n \times (v_h - \tilde{v}_h)\|_{\partial T_h}.
\]

\(^{57}\) is followed by the proof similar to the above one, \(^{53b}\), and the fact \(\|\nabla \times (v_h - \Pi_h(v_h, \tilde{v}_h))\|_{T_h} \lesssim \|\nabla(v_h - \Pi_h(v_h, \tilde{v}_h))\|_{T_h}\). We use \(^{54}\) to get \(n_F \times \Pi_T(v_h, \tilde{v}_h) = \mathcal{I}_T(n_F \times v, n_F \times \tilde{v})\) on every face \(F \subset \partial T\). Since \(n \times \tilde{v} \in [P_k(F_h)]^3\) and \(n \times \tilde{v}|_{\partial \Omega} = 0\), then \(\mathcal{I}_T(n_F \times v_h, n_F \times \tilde{v}_h)\) is continuous on \(F\) and \(\mathcal{I}_T(n_F \times v_h, n_F \times \tilde{v}_h)|_{\partial \Omega} = 0\), so \(\Pi_h(v_h, \tilde{v}_h) \in H_0(\text{curl}; \Omega)\).

**Lemma 10** We have the following error estimates

\[
E^T_h(\sigma; \tau_h) \lesssim h^2\|\nabla \times \nabla \times r\|_{T_h} \lesssim h^{3/2} \|v_h - \tilde{v}_h\|_{\partial T_h} \\
+ h^s\|r\|_s \left(\|h_T^{-1/2} n \times (\tilde{d}_h - \nabla \times v_h)\|_{0, \partial T_h} + \|h_F^{-3/2} n \times (v_h - \tilde{v}_h)\|_{\partial T_h}\right) \\
+ h^s\|u\|_s \left(\|\nabla q_h\|_{T_h} + \|h_F^{-1/2}(q_h - \tilde{q}_h)\|_{\partial T_h}\right) \\
+ h^{s+1}\|p\|_{s+1} \|h_F^{-1/2}(q_h - \tilde{q}_h)\|_{\partial T_h} \\
+ h^{\min(k-1,s)}\|\nabla \times u\|_{s+1} \|h_F^{-1/2} n \times (\tilde{d}_h - \nabla \times v_h)\|_{0, \partial T_h}.
\]
**Proof 6** For simplicity we define

\[
E_1 = - \langle n \times (\bar{v}_h - v_h), \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h},
\]

\[
E_2 = - \langle n \times (\bar{d}_h - \nabla \times v_h), \Pi_k r - r \rangle |_{\partial \Omega_h},
\]

\[
E_3 = - \langle h_F^{-1} n \times (\nabla \times \Pi_{h,k}^c u - \nabla \times u), n \times (\nabla \times v_h - \bar{d}_h) \rangle |_{\partial \Omega_h},
\]

\[
E_4 = - \langle \bar{q}_h - q_h, (\Pi_{h,k}^c u - u) \cdot n \rangle |_{\partial \Omega_h},
\]

\[
E_5 = \langle h_F^{-1}(\Pi_k^o p - p), q_h - \bar{q}_h \rangle |_{\partial \Omega_h},
\]

\[
E_6 = - (\nabla q_h, \Pi_{h,k}^c u - u) |_{\partial \Omega_h}.
\]

Then by (57) we have

\[
E_h^J(\sigma; \Omega_h) = E_1 + E_2 + E_3 + E_1 + E_4 + E_5 + E_6,
\]

and we will bound each \( E_j \) (1 \( \leq j \leq 6 \)) separately. It follows from the definition of \( E_1 \) and the fact \( \langle n \times \bar{\nabla}_h, \nabla \times r \rangle |_{\partial \Omega_h} = 0 \) that

\[
E_1 = - \langle n \times \bar{\nabla}_h, \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h} + \langle n \times \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h}
\]

\[
= - \langle n \times \bar{\nabla}_h, \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h} + \langle n \times v_h, \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h}.
\]

By (55a), (55b) and integration by parts we have

\[
E_1 = - \langle n \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h}
\]

\[
- (v_h, \nabla \times \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h} + (\nabla \times v_h, \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
= - \langle n \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times (\Pi_k r - r) \rangle |_{\partial \Omega_h} - (v_h, \nabla \times \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
+ (v_h, \nabla \times \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h} + (\Pi_k (v_h, \bar{\nabla}_h), \nabla \times \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
+ (v_h, \nabla \times \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h} + (\Pi_k (v_h, \bar{\nabla}_h), \nabla \times \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
= - (\nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

Due to the fact \( (\nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times r) |_{\partial \Omega_h} - (\Pi_k (v_h, \bar{\nabla}_h), \nabla \times \nabla \times r) |_{\partial \Omega_h} = 0 \), we have

\[
E_1 = - (\nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
+ (v_h - \Pi_k (v_h, \bar{\nabla}_h), \nabla \times \nabla \times r) |_{\partial \Omega_h} + (\nabla \times v_h, \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
= (\nabla \times v_h - \nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
+ (\Pi_k (v_h, \bar{\nabla}_h), \nabla \times \nabla \times r) |_{\partial \Omega_h}.
\]

(58)

Note that we can rewrite the first term on the right-hand side of (58) as

\[
(\nabla \times v_h - \nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times (\Pi_k r - r)) |_{\partial \Omega_h}
\]

\[
= (\nabla \times v_h - \nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times \Pi_k r - \Pi_k^{\text{div}} |_{\partial \Omega_{h,k+2}} \nabla \times r) |_{\partial \Omega_h}
\]

\[
= (\nabla \times v_h - \nabla \times \Pi_k (v_h, \bar{\nabla}_h), \nabla \times \Pi_k r - \nabla \times \Pi_k^{\text{curl}} r) |_{\partial \Omega_h}.
\]

(59)
Therefore by (57), (56), (38), (59) and inverse inequality, we get
\[ |E_1| \lesssim (h^2\|\nabla \times \nabla \times r\|_{\mathcal{T}_h} + h^s\|r\|_s)\|h_F^{-3/2}\n \times (v_h - \hat{v}_h)\|_{\partial \mathcal{T}_h}. \] (60)

We can bound the other \(E_j\) terms as follows.
\[ |E_2| \lesssim \sum_{T \in \mathcal{T}_h} \|h_F^{-1/2}n \times (\hat{d}_h - \nabla \times v_h)\|_{\partial T}^2. \]
\[ |E_3| \lesssim \sum_{T \in \mathcal{T}_h} \|h_F^{-1/2}n \times (\hat{d}_h - \nabla \times v_h)\|_{\partial T} h_T^{\min(k-1,s)}\|\nabla \times u\|_{1+s,T}. \]
\[ |E_4| \lesssim h^s\|u\|_s\|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial \mathcal{T}_h}, \]
\[ |E_5| \lesssim h^s\|p\|_{s+1}\|h_F^{-1/2}(q_h - \hat{q}_h)\|_{\partial \mathcal{T}_h}, \]
\[ |E_6| \lesssim h^s\|u\|_s\|\nabla q_h\|_{\mathcal{T}_h}. \]

This completes the proof.

**Lemma 11** Let \((r, u, p)\) be the solution of (17), then there holds
\[ \|\sigma_h - \mathcal{J}_h \sigma\|_{\Sigma_h} \lesssim h^2\|\nabla \times \nabla \times r\|_{\mathcal{T}_h} + h^{\min(s,k-1)}\|\nabla \times u\|_{s+1} + h^s(\|r\|_s + \|u\|_s + \|p\|_{s+1}). \] (61)

**Proof 7** By (28), Theorem 1 and Lemma 8 we get
\[ \|\sigma_h - \mathcal{J}_h \sigma\|_{\Sigma_h} \lesssim \sup_{\tau_h \in \Sigma_h^0, \tau_h \neq 0} \frac{B_h(\sigma_h - \mathcal{J}_h \sigma, \tau_h)}{\|\tau_h\|_{\Sigma_h}} = \sup_{\tau_h \in \Sigma_h^0, \tau_h \neq 0} \frac{E_h^\mathcal{J}(\sigma_h, \tau_h)}{\|\tau_h\|_{\Sigma_h}}. \] (62)

Then (61) directly follows from (22), (62) and Lemma 10.

**Theorem 2** Let \((r, u, p)\) be the solution of (17), then we have
\[ \|\nabla \times (u - u_h)\|_{\mathcal{T}_h} + \|r - r_h\|_{\mathcal{T}_h} + \|\nabla (p - p_h)\|_{\mathcal{T}_h} \lesssim h^2\|\nabla \times \nabla \times r\|_{\mathcal{T}_h} + h^{\min(s,k-1)}\|\nabla \times u\|_{s+1} + h^s(\|r\|_s + \|u\|_s + \|p\|_{s+1}). \]

### 4.2 Error estimates by dual arguments

We assume \(\Theta \in H(\text{div}^0, \Omega)\) and introduce the problem:
\[
\begin{aligned}
&\mathbf{r}^d - \nabla \times \nabla \times \mathbf{u}^d = 0 \quad \text{in } \Omega, \\
&\nabla \times \nabla \times \mathbf{r}^d + \nabla p^d = \Theta \quad \text{in } \Omega, \\
&\nabla \cdot \mathbf{u}^d = \Lambda \quad \text{in } \Omega, \\
&\mathbf{n}_\Gamma \times \mathbf{u}^d = 0 \quad \text{on } \Gamma, \\
&\mathbf{n}_\Gamma \times \nabla \times \mathbf{u}^d = 0 \quad \text{on } \Gamma, \\
&p^d = 0 \quad \text{on } \Gamma.
\end{aligned}
\] (63)
Assume that
\[ \| r^d \|_\alpha + \| u^d \|_{1+\alpha,\text{curl}} \lesssim \| \Theta \|_{T_h} + \| \Lambda \|_{T_h}, \tag{64} \]
where \( \alpha \in (\frac{1}{2}, 1] \) is dependent on \( \Omega \). It is obviously that \( p^d = 0 \). Note that when \( \Omega \) is convex, (64) holds with \( \alpha = 1 \).

**Lemma 12** Let \( \sigma \) and \( \sigma^d \) be the solutions of (7) and (63), respectively. We have
\[
| E_h^J (\sigma; J_h \sigma - \sigma_h) | \lesssim h^{\min(\alpha,k-1)} (\| \Theta \|_{T_h} + \| \Lambda \|_{T_h}) \| J_h \sigma - \sigma_h \|_{\Sigma_h},
\]
\[
| E_h^J (\sigma; J_h \sigma^d) | \lesssim h^{\min(\alpha,k-1)} (h^s \| r \|_s + h^{\min(k-1,s)} \| \nabla \times u \|_{s+1}) (\| \Theta \|_{T_h} + \| \Lambda \|_{T_h}).
\]

**Proof 8** Similar to the proof of **Lemma 10** we get
\[
E_h^J (\sigma; J_h \sigma^d) \lesssim (h^s \| r \|_s + h^{\min(k-1,s)} \| \nabla \times u \|_{s+1}) \| J_h \sigma - \sigma_h \|_{\Sigma_h}
\]
\[
\lesssim h^{\min(\alpha,k-1)} (h^s \| r \|_s + h^{\min(k-1,s)} \| \nabla \times u \|_{s+1}) (\| \Theta \|_{T_h} + \| \Lambda \|_{T_h}).
\]

**Theorem 3** Let \( (r, u, p) \) and \( (r_h, u_h, \hat{u}_h, p_h, \hat{p}_h) \) be the solutions of (7) and (28), respectively, then there holds
\[
\| u - u_h \|_{T_h} + \| p - p_h \|_{T_h} \lesssim h^{\min(\alpha,k-1)} \left( h^2 \| \nabla \times \nabla \times r \|_{T_h} + h^{\min(s,k-1)} \| \nabla \times u \|_{s+1} \right) \tag{65}
\]
\[
+ h^{s+\min(\alpha,k-1)} (\| r \|_s + \| u \|_s + \| p \|_{s+1}) + \| u - \Pi_{\text{curl}}^k u \|_{T_h}.
\]

**Proof 9** We introduce a projection \( \Pi_k^m \). For all \( v \in H^s(\text{curl}; \Omega) \) with \( s > 1/2 \) and \( v_h \in U_h \), such that
\[
\Pi_k^m (v, v_h) = \Pi_{h,k}^{\text{curl}} v + \nabla \sigma_h, \tag{66}
\]
where \( \sigma_h \in \mathcal{P}_k(T_h) \cap H^1_0(\Omega) \) satisfies
\[
(\nabla \sigma_h, \nabla q_h)_{T_h} = (\Pi_{h,k}^{\text{curl}} (v - \Pi_{h,k}^{\text{curl}} v), \nabla q_h)_{T_h} \quad \forall q_h \in \mathcal{P}_k(T_h) \cap H^1_0(\Omega). \tag{67}
\]
From (66) and (67), it holds
\[
(\Pi_{h,k}^{\text{curl}} (v - \Pi_k^m (v, v_h), \nabla q_h) = 0 \quad \forall q_h \in \mathcal{P}_k(T_h) \cap H^1_0(\Omega). \tag{68}
\]
We take $\Lambda = \Pi^0_p - p_h$ in (63) and let $\Theta \in H(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ be the solution of
\[
\begin{align*}
\nabla \times \Theta &= \nabla \times (\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h))) & \text{in } \Omega, \\
\nabla \cdot \Theta &= 0 & \text{in } \Omega, \\
\mathbf{n} \times \Theta &= 0 & \text{on } \Gamma.
\end{align*}
\]

Due to (68) and the result in [4, Lemma 4.5] one has
\[
\|\Theta - (\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h)))\|_{\tau_h} \lesssim h^\alpha \|\nabla \times (\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h)))\|_{\tau_h}.
\]

We obtain the following estimates by (67), [Lemma 4], and an inverse inequality
\[
\|\Pi^{\text{curl},c}_{h,k} u - \Pi^m_k(u, u_h)\|_{\tau_h}
\]
\[
\leq \|\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_{h,k} u)\|_{\tau_h}
\]
\[
\leq \|\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_{h,k} u) - (u_h - \Pi^{\text{curl}}_{h,k} u)\|_{\tau_h} + \|(u_h - \Pi^{\text{curl}}_{h,k} u)\|_{\tau_h}
\]
\[
\lesssim \left(h_F h \|\nabla \times (u_h - \Pi^{\text{curl}}_{h,k} u)\|_{\Omega_h} + \|\Pi^{\text{curl}}_{h,k} u - u_h\|_{\tau_h}\right)
\]
\[
\lesssim \|\Pi^{\text{curl}}_{h,k} u - u_h\|_{\tau_h}.
\]

Similarity, we can get
\[
\|\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h))\|_{\tau_h} \lesssim \|\Pi^{\text{curl}}_{h,k} u - u_h\|_{\tau_h},
\]

and
\[
\|\nabla \times (\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h)))\|_{\tau_h}
\]
\[
\lesssim \left(h_F^{-1/2} \|\nabla \times (u_h - \Pi^{\text{curl}}_{h,k} u)\|_{\Omega_h} + \|\nabla \times (\Pi^{\text{curl}}_{h,k} u - u_h)\|_{\tau_h}\right)
\]
\[
\lesssim \left(\|\sigma - \sigma_h\|_{\Sigma_h} + \|\nabla \times (\Pi^{\text{curl}}_{h,k} u - u_h)\|_{\tau_h}\right)
\]
\[
\lesssim \left(\|\sigma - \sigma_h\|_{\Sigma_h} + h^s\|\mathbf{r}\|_s\right)
\]
\[
\lesssim h^s(\|\mathbf{r}\|_s + \|u\|_s + \|p\|_{s+1}).
\]

It then follows from (70) and (73) that
\[
\|\Theta - (\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h)))\|_{\tau_h} \lesssim h^{s+\alpha}(\|\mathbf{r}\|_s + \|u\|_s + \|p\|_{s+1}).
\]

Follows from the above estimates inequality, it holds that
\[
\|\Theta\|_{\tau_h} \lesssim \|\Theta - (\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h)))\|_{\tau_h} + \|\Pi^{\text{curl},c}_{h,k}(u_h - \Pi^m_k(u, u_h))\|_{\tau_h}
\]
\[
\lesssim h^{s+\alpha}(\|\mathbf{r}\|_s + \|u\|_s + \|p\|_{s+1}) + \|\Pi^{\text{curl}}_{h,k} u - u_h\|_{\tau_h}.
\]

In view of [Lemma 8] we have
\[
B_h(\mathcal{J}_h \mathbf{r}^d, \tau_h) = -(\Theta, \mathbf{v}_h)_{\tau_h} + (\Lambda, q_h)_{\tau_h} + E^\mathcal{J}_h(\sigma^d; \tau_h) \quad \forall \tau_h \in \Sigma^0_h.
\]
We take $\tau_h = \mathcal{J}_h \sigma - \sigma_h \in \Sigma_h^0$, in (76), and use (16), (28) to get

$$- (\Theta, \Pi^{\text{curl}}_{h,k} u - u_h)_{\tau_h} + \|\Pi^p_k p - p_h\|_{\tau_h}^2$$

$$= B_h(\mathcal{J}_h \sigma^d, \mathcal{J}_h \sigma - \sigma_h) - E_h^\sigma(\sigma^d; \mathcal{J}_h \sigma - \sigma_h)$$

$$= E_h^\sigma(\sigma; \mathcal{J}_h \sigma^d) - E_h^\sigma(\sigma^d; \mathcal{J}_h \sigma - \sigma_h).$$

We take $q_h = \tilde{q}_h = \sigma_h$ in (19c) to get

$$-(u_h, \nabla \sigma_h)_{\tau_h} = (g, \sigma_h)_{\tau_h}.$$ 

We use a direct calculation to get

$$(u - \Pi^m_k (u, u_h), u - u_h)_{\tau_h}$$

$$= (u - \Pi^{\text{curl}}_{h,k} u, u - u_h)_{\tau_h} + (-\nabla \sigma_h, u - u_h)_{\tau_h}$$

by the definiton of $\Pi^m_k$

$$= (u - \Pi^{\text{curl}}_{h,k} u, u - u_h)_{\tau_h} + (\sigma_h, \nabla \cdot u)_{\tau_h} + (\nabla \sigma_h, u_h)_{\tau_h}$$

by integration by parts

$$= (u - \Pi^{\text{curl}}_{h,k} u, u - u_h)_{\tau_h}$$

by (11), (78).

We use (16) to get

$$\|\Pi^{\text{curl},c}_{h,k} (u_h - \Pi^m_k (u, u_h)) - (u_h - \Pi^m_k (u, u_h))\|_{\tau_h}$$

$$\lesssim h\|h^{-1/2} n \times [u_h - \Pi^m_k (u, u_h)]\|_{0, \mathcal{F}_h}$$

$$= h\|h^{-1/2} n \times [u_h]\|_{0, \mathcal{F}_h} + C h\|h^{-1/2} n \times [u_h - \Pi^{\text{curl}}_{h,k} u]\|_{0, \mathcal{F}_h}$$

$$= h\|h^{-1/2} n \times [u_h - \tilde{u}_h]\|_{0, \mathcal{F}_h} + h\|h^{-1/2} n \times [u_h - \tilde{u}_h]\|_{0, \mathcal{F}_h}$$

$$\lesssim h\|\sigma_h - \mathcal{J}_h \sigma\|_{\Sigma_h}$$

$$\lesssim h^{s+1}(\|r\|_s + \|u\|_s + \|p\|_{s+1}).$$

We use a direct calculation to get

$$(\Pi^{\text{curl}}_{h,k} u - \Pi^m_k (u, u_h), \Pi^{\text{curl}}_{h,k} u - u_h)$$

$$= (\Pi^{\text{curl}}_{h,k} u - \Pi^m_k (u, u_h), u - u_h)$$

$$+ (\Pi^{\text{curl}}_{h,k} u - \Pi^m_k (u, u_h), \Pi^{\text{curl}}_{h,k} u - u)$$

$$= (-\nabla \sigma_h, u - u_h) + (\Pi^{\text{curl}}_{h,k} u - \Pi^m_k (u, u_h), \Pi^{\text{curl}}_{h,k} u - u)$$

by the definiton of $\Pi^m_k$

$$= (\sigma_h, \nabla \cdot u) + (\nabla \sigma_h, u_h)$$

$$+ (\Pi^{\text{curl}}_{h,k} u - \Pi^m_k (u, u_h), \Pi^{\text{curl}}_{h,k} u - u)$$

by integration by parts

$$= (\Pi^{\text{curl}}_{h,k} u - \Pi^m_k (u, u_h), \Pi^{\text{curl}}_{h,k} u - u)$$

by (11), (78)

$$\leq C \|\Pi^{\text{curl}}_{h,k} u - u_h\|_{\tau_h} \|\Pi^{\text{curl}}_{h,k} u - u\|_{\tau_h}$$

by (71).
By using (77), one can obtain
\[
\|\Pi_{h,k}^{\text{curl}} u - u_h\|_{\mathcal{T}_h}^2 = (\Pi_{h,k}^{\text{curl}} u - u_h, \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]
\[
= -(\Theta, \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h} + (\Theta - (\Pi_{h,k}^{\text{curl},c}(u_h - \Pi_k^m(u, u_h))), \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]
\[
+ (\Pi_{h,k}^{\text{curl}} u - \Pi_h^m(u, u_h), \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]
\[
+ (\Pi_{h,k}^{\text{curl},c}(u_h - \Pi_k^m(u, u_h)) - (u_h - \Pi_k^m(u, u_h)), \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]
\[
= L_{\mathcal{T}_h}(\sigma; J_h\sigma^d) - L_{\mathcal{T}_h}(\sigma^d; J_h\sigma - \sigma_h) - \|\Pi_h^o p - p_h\|_{\mathcal{T}_h}^2
\]
\[
+ (\Theta - (\Pi_{h,k}^{\text{curl},c}(u_h - \Pi_k^m(u, u_h))), \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]
\[
+ (\Pi_{h,k}^{\text{curl}} u - \Pi_h^m(u, u_h), \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]
\[
+ (\Pi_{h,k}^{\text{curl},c}(u_h - \Pi_k^m(u, u_h)) - (u_h - \Pi_k^m(u, u_h)), \Pi_{h,k}^{\text{curl}} u - u_h)_{\mathcal{T}_h}
\]

which together with Lemma 12, (74), (80) and (81) implies (65).

5 Numerical experiments

All numerical tests in this section are programmed in C++. When implementing the HDG method (19a)–(19c), all the interior unknowns \(r_h, u_h\) and \(p_h\) are eliminated. The only global unknowns of the resulting system are \(\hat{u}_h, \hat{c}_h\) and \(\hat{p}_h\); and then \(r_h, u_h\) and \(p_h\) can be recovered locally. This is the unique feature of HDG method. The solver for the linear system is chosen as GMRES, which uses AMG as preconditioner. We take \(\mathcal{T}_h\) to be a uniform simplex decomposition of \(\Omega\) in all examples.

5.1 Smooth case

We take \(\Omega = (0, 1)^3\). The functions \(r, f, g\) and \(g_T\) are determined according to the following true solutions
\[
u_1 = \sin(y) \sin(z), \quad u_2 = \sin(z) \sin(x), \quad u_3 = \sin(x) \sin(y), \quad p = 0.
\]
The \(L_2\) errors are reported in Table 1 and Table 2 for \(k = 1\) and \(k = 2\), respectively. According to Theorem 2 and Theorem 3, we would have
\[
\|u - u_h\|_{\mathcal{T}_h} + \|r - r_h\|_{\mathcal{T}_h} + \|p - p_h\|_{\mathcal{T}_h} \leq C \quad k = 1,
\]
\[
\|u - u_h\|_{\mathcal{T}_h} + h\|r - r_h\|_{\mathcal{T}_h} + \|p - p_h\|_{\mathcal{T}_h} \leq Ch^2 \quad k = 2.
\]
It can be observed that the orders of convergence are better than predicted. This may due to the fact that the exact solution has high smoothness. Actually, when the true solution is smooth enough, one may derive error analysis of HDG method for the quad-curl problem similarly to the biharmonic problem and obtain better convergence rates (probably optimal with respect to \(k\) for different stabilization parameters). This will be our future work.
### Table 1: Results for $k = 1$

| $h^{-1}$ | $\|r - r_h\|_{\tau_h}/\|r\|_{\tau_h}$ | $\|u - u_h\|_{\tau_h}/\|u\|_{\tau_h}$ | $\|p - p_h\|_{\tau_h}$ | DOF |
|----------|-----------------------------------|-----------------------------------|---------------------------|-----|
| 2        | 3.57E-01                          | 1.34E-01                          | 1.05E-02                  | 1320|
| 4        | 1.92E-01                          | 3.42E-02                          | 1.64E-03                  | 9508|
| 8        | 1.02E-01                          | 8.67E-03                          | 2.13E-04                  | 71808|
| 16       | 5.72E-02                          | 2.20E-03                          | 2.70E-05                  | 557568|

### Table 2: Results for $k = 2$

| $h^{-1}$ | $\|r - r_h\|_{\tau_h}/\|r\|_{\tau_h}$ | $\|u - u_h\|_{\tau_h}/\|u\|_{\tau_h}$ | $\|p - p_h\|_{\tau_h}$ | DOF |
|----------|-----------------------------------|-----------------------------------|---------------------------|-----|
| 2        | 3.85E-02                          | 2.73E-02                          | 1.60E-03                  | 2880|
| 4        | 1.26E-02                          | 2.11E-03                          | 6.41E-05                  | 20736|
| 8        | 4.90E-03                          | 1.56E-04                          | 3.40E-06                  | 156672|

### 5.2 Singular solution on L-shaped domain

We take $\Omega = (-1, 1)^3/(-1, 0) \times (-1, 0) \times (-1, 1)$. The functions $f, g$ and $g_T$ are determined according to the following true solutions

$$u_1 = tr^{t-1}\sin[(t - 1)\theta], \quad u_2 = tr^{t-1}\cos[(t - 1)\theta], \quad u_3 = 0, \quad r = 0, \quad p = 0.$$  

By taking $t = 0.9$ and $t = 1.4$, we have $u \in [H^{0.9-\epsilon}(\Omega)]^3$ and $u \in [H^{1.4-\epsilon}(\Omega)]^3$, respectively, for arbitrarily small $\epsilon > 0$. The results for $k = 1$ are reported in Table 3 and Table 4. In this case, we have $\nabla \times u = 0$, therefore, by Theorem 2 and Theorem 3 we have

$$\|u - u_h\|_{\tau_h} + \|r - r_h\|_{\tau_h} + \|p - p_h\|_{\tau_h} \leq C h^{t-\epsilon} \|u\|_{t-\epsilon} \quad t = 0.9, 1.4.$$  

We observe that optimal convergence rate with respect the regularity for $\|u - u_h\|_{\tau_h}$ is obtained, which verifies the theoretical results. Moreover, the convergence rates for $\|r - r_h\|_{\tau_h}$ and $\|p - p_h\|_{\tau_h}$ are better than predicted.

### Table 3: Results for $k = 1, t = 0.9$

| $h^{-1}$ | $\|r - r_h\|_{\tau_h}$ | $\|u - u_h\|_{\tau_h}$ | $\|p - p_h\|_{\tau_h}$ | DOF |
|----------|------------------------|------------------------|------------------------|-----|
| 2        | 2.73E-03               | 7.35E-02               | 3.50E-02               | 1034|
| 4        | 2.65E-03               | 4.20E-02               | 2.29E-02               | 7304|
| 8        | 1.07E-03               | 2.36E-02               | 8.00E-03               | 54560|
| 16       | 3.89E-04               | 1.27E-02               | 2.46E-03               | 420992|
Table 4: Results for \( k = 1, t = 1.4 \)

| \( h^{-1} \) | \( \| r - r_h \|_{\mathcal{V}_h} \) Error | Rate | \( \| u - u_h \|_{\mathcal{V}_h} \) Error | Rate | \( \| p - p_h \|_{\mathcal{V}_h} \) Error | Rate | DOF |
|----------|-------------------------|------|-------------------------|------|-------------------------|------|------|
| 2        | 5.38E-03                |      | 1.49E-01                |      | 8.29E-02                |      | 1034 |
| 4        | 2.92E-03                | 0.88 | 6.34E-02                | 1.23 | 3.64E-02                | 1.19 | 7304 |
| 8        | 7.93E-04                | 1.88 | 2.59E-02                | 1.29 | 8.61E-03                | 2.08 | 54560|
| 16       | 1.81E-04                | 2.13 | 9.97E-03                | 1.38 | 1.80E-03                | 2.26 | 420992|

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