Quark Probability Distribution at Finite Temperature and Density

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Abstract

Quark deconfinement phase transition at finite temperature and density is investigated in the frame of quantum mechanics. By solving the Schrödinger equation for a heavy quark in a thermal mean field, we calculate the quark probability distribution as a function of temperature and density. The confined wave function in the vacuum expands outward rapidly when the temperature and density are high enough. The obtained phase transition line agrees qualitatively with the result of lattice QCD.

It is widely accepted that the collective effect of a multiparton system can change the vacuum structure of Quantum Chromodynamics (QCD)\cite{1}, and the partons confined in a hadron bag can move out when the temperature and density are high enough and form a new state of matter, the so called quark-gluon plasma (QGP)\cite{2}. This kind of new matter may be produced in the early stage of a high-energy heavy-ion collision\cite{3}. The dense partons, produced in the vacuum at high temperature or compressed at high baryon density, make the quark wave functions overlap in a space with dimension of the colliding nuclei. An important question in relativistic heavy ion collisions is how to identify the new state of matter if it is created in the early stage. According to the numerical calculation of lattice QCD\cite{4}, the critical temperature of quark deconfinement is about 150 MeV when the density effect is excluded.

The idea that the overlap of wave functions, or the space extension of wave functions at finite temperature and density is the original reason of quark deconfinement phase transition arises naturally an interesting ques-
tion to study the behavior of quark distribution function in the frame of quantum mechanics. In this letter, we will study a quark moving in a thermal mean field describing multiparton interactions, and obtain the quark probability distribution by solving the Schrödinger equation at finite temperature and density. The temperature and density at which the quark wave function expands outward rapidly can be considered as the critical point of quark deconfinement. Because of the limitation of non-relativistic quantum mechanics, we focus on the motion of a heavy quark at finite temperature and density, corresponding to, for example, the dissociation of $J/\psi$ as a signature of QGP in relativistic heavy ion collisions.

In the string-like models that describe the quark confinement well, one uses a linear potential $V = kr$ to express the interaction between two quarks. To simplify the numerical calculation in the following, we use a three dimensional square well

$$V(r) = \begin{cases} 
0 & 0 < r < a \\
V_0 & a < r < b \\
\infty & r > b 
\end{cases}$$

instead of the linear potential. Here $a$ is the boundary of the region where the motion of the quark is asymptotically free, and can be taken as the radius of the hadron constructed by the considered quarks, $b$ is the space scale of the system, and $V_0$ should be so large that the quark wave function can be confined in the asymptotically free region $r, a$. Since the region of collective interaction in high-energy heavy-ion collisions is of the order of the colliding nuclei, we treat $b$ as the radius of a heavy nucleus. In our numerical calculations, we take the parameters of the confinement potential as $a = 1$ fm, $b = 5$ fm, and $V_0 = 10$ GeV.

At finite temperature and density, the quark will be affected by many-body interactions in the region $r > a$ in addition to the confinement potential. In mean field approximation, the thermodynamic potential of the quasi-particle system is

$$\Omega(T, \mu) = -g \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E_q} (f_q(T, \mu) + f_{\bar{q}}(T, \mu))$$

(2)
with the quark and antiquark densities

\[
\begin{align*}
    f_q(T, \mu) &= \frac{1}{e^{\frac{E_q - \mu}{T}} + 1}, \\
    f_{\bar{q}}(T, \mu) &= \frac{1}{e^{\frac{E_{\bar{q}} + \mu}{T}} + 1},
\end{align*}
\]

(3)

where \( g \) is the quark degree of freedoms of flavors, colors and spins, \( T \) and \( \mu \) are respectively the temperature and baryon chemical potential, \( E_q = \sqrt{m_{eff}^2 + p^2} \) is the quark energy with effective mass \( m_{eff} \) obtained in the mean field. Because the contribution from heavy quarks to the thermodynamic potential is very small compared with light quarks, we will only take \( u \) and \( d \) quarks into account, i.e., \( g = 12 \).

From the rules of thermodynamics, the pressure of the system is

\[
P(T, \mu) = -\Omega(T, \mu) .
\]

(4)

Considering the relation between the potential \( V_{th} \) and the force \( \vec{F}_{th} \) that the heavy quark we study feels in the region \( r > a \),

\[
\vec{F}_{th}(\vec{r}) = -\nabla V_{th}(\vec{r}) ,
\]

(5)

i.e.,

\[
- \frac{dV_{th}(r)}{dr} = -4\pi r^2 P(T, \mu) ,
\]

(6)

and assuming that the potential falls down with increasing \( r \) and vanishes on the boundary of the system, namely \( V_{th}(r = b) = 0 \), the potential \( V_{th} \) can be expressed as

\[
V_{th}(r|T, \mu) = -\int_r^b dr' 4\pi r'^2 P(T, \mu) = \frac{4}{3} \pi (b^3 - r^3) \Omega(T, \mu) .
\]

(7)

With the obtained potential, the motion of the heavy quark at finite temperature and density is described by the Schrödinger equation

\[
i \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left( -\frac{1}{2m} \nabla^2 + V_{eff}(r|T, \mu) \right) \psi(\vec{r}, t) ,
\]

(8)

where \( m \) is the mass of the heavy quark, and \( V_{eff} \) is the effective potential defined as

\[
V_{eff}(r|T, \mu) = \begin{cases} 
0 & 0 < r < a \\
V_0 + V_{th}(r|T, \mu) & a < r < b \\
\infty & r > b 
\end{cases}
\]

(9)
By separating the variables of the wave function

\[ \psi(\vec{r},t) = R(r)Y(\theta,\phi)e^{-iEt}, \]

and letting \( \phi(r) = rR(r) \), we obtain the radial equation for a stationary state

\[
- \frac{1}{2m} \frac{d^2\phi(r)}{dr^2} + \left( V_{\text{eff}}(r|T,\mu) + \frac{l(l+1)}{2mr^2} \right) \phi(r) = E\phi(r), \tag{10}
\]

where \( E \) is the eigenvalue of the quark energy. Since we focus in this letter on quark deconfinement at finite temperature and density, we discuss in the following the ground state of the Schrödinger equation only. The corresponding radial equation is then simplified as

\[
- \frac{1}{2m} \frac{d^2\phi_0(r)}{dr^2} + V_{\text{eff}}(r|T,\mu)\phi_0(r) = E_0\phi_0(r). \tag{11}
\]

We first discuss qualitatively the deconfinement phase transition through the behavior of the effective potential as a function of \( T \) and \( \mu \). From the \( T \)- and \( \mu \)-dependence of the thermodynamic potential, we can immediately obtain the conclusion that the confinement potential in the vacuum at \( T = \mu = 0 \) will be suppressed by the thermal mean field \( V_{\text{th}} < 0 \), and the motion of the quark will extend outside when the temperature and density increase. Since the mean field decreases monotonously from the maximum on the boundary \( r = a \) to the minimum on the other boundary \( r = b \), the suppression at \( r = a \) is most strong, and the potential at \( r = b \) does not change at finite temperature and density.

In order to illustrate quantitatively the above estimated quark deconfinement phase transition, we should solve the Schrödinger equation (11) at finite temperature and density to get the quark probability distribution

\[
\rho(r|T,\mu) \propto r^2R^2(r|T,\mu). \tag{12}
\]

We take the considered heavy quark in the Schrödinger equation as a \( c \) quark which forms \( J/\psi \) with \( m = 1.6 \) GeV, and the effective mass of the light \( u \) and \( d \) quarks in \( V_{\text{eff}} \) as their constituent quark mass \( m_{\text{eff}} = 0.35 \) GeV.

In the limit of \( \mu = 0 \), the probability distribution as a function of temperature obtained by solving numerically the Schrödinger equation (11) is shown in Fig.1. In the vacuum with \( T = \mu = 0 \), the chosen parameters for
the square well guarantee that almost the whole quark wave function is re-
stricted in the region \( r < a \), that means quark confinement. As temperature
increases the quark motion extends outside very slowly in the beginning, then
the extension is accelerated rapidly in the neighborhood of \( T = 120 \) MeV,
which indicates that the two \( c \) quarks forming \( J/\psi \) begin to dissociate in the
hot mean field, and finally the quark moves in the whole region of the system
\( 0 < r < b \). The case \( T = 0 \) is similar to that of \( \mu = 0 \), the quark probability
distribution as a function of \( r \) for different chemical potentials is indicated
in Fig. (2). The probability expands outward rapidly in the neighborhood of
\( \mu = 445 \) MeV.

![Figure 1](image)

**Figure 1:** The quark probability distribution at finite temperature for \( \mu = 0 \).

![Figure 2](image)

**Figure 2:** The quark probability distribution at finite chemical potential for
\( T = 0 \).

It is well known that for a particle moving in a central field the most
probable radius describes the bound degree of the stationary state. The most probable radius $r_0(T, \mu)$ of the $c$ quark motion at finite temperature and density defined by

$$\left. \frac{\partial \rho(r|T, \mu)}{\partial r} \right|_{r=r_0} = 0$$

is indicated as a function of $T$ for $\mu = 0$ in Fig. 3. It is clear to see that around $T = 120$ MeV $r_0$ goes up from $a/2$ to $b/2$ very fast. We can determine the critical temperature $T_c$ and chemical potential $\mu_c$ of the quark deconfinement phase transition as the point at which the most probable radius $r_0(T_c, \mu_c)$ reaches $a$, the boundary of the asymptotically free region in the vacuum,

$$r_0(T_c, \mu_c) = a .$$

The critical line which separates the deconfinement phase from the confinement phase is shown in Fig. 4. While the most probable radius $r_0$ jumps from $a/2$ to $b/2$ in a very narrow region around $T_c$, it is still continuous at the critical point, and the phase transition we study is a continuous one. When the temperature and chemical potential exceed the critical values, the region where the quark is asymptotically free will expand, i.e., the position of $a$ will move outward.

![Graph](image.png)

Figure 3: The most probable radius as a function of temperature for $\mu = 0$.

In summary, by describing the collective effect as a thermal mean field and combining it with the confinement potential in the vacuum, we have demonstrated the physical picture of quark deconfinement in the frame of quantum mechanics through solving the Schrödinger equation and calculating the quark probability distribution at finite temperature and density. When
Figure 4: The phase transition line separating the region with quark confinement from that with quark deconfinement.

The temperature and density are high enough, the quark wave function extends outward from inside the hadron. The critical temperature and density around which the wave function extension happens rapidly separates the confinement from deconfinement. The obtained critical temperature $T_c = 120$ MeV at $\mu = 0$ agrees qualitatively with the lattice QCD calculation.

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References

[1] J. I. Kapusta, *Finite Temperature Field Theory*, Cambridge University Press, 1993; M. le Bellac, *Thermal Field Theory*, Cambridge University Press, 2000;

[2] *Quark-Gluon Plasma*, ed. R.C.Hwa (World Scientific, 1990).

[3] C. Y. Wong, *Introduction to High-Energy Heavy-Ion Collisions*, World Scientific, 1995.

[4] F. Karsch, Nucl. Phys. A698(2002)199c.

[5] T. Matsui and H. Satz, Phys. Lett. B178 (1986) 416; C. Gerschel and J. Hübner, Ann. Rev. Nucl. Part. Sci. 49 (1999) 255.