On the $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb model

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Abstract

This work concerns the boundary integrability of the $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb model. We constructed the solutions of the graded reflection equations in order to determine the boundary terms of the corresponding spin-1 Hamiltonian. We obtain the eigenvalue expressions as well as its associated Bethe ansatz equations by means of the coordinate Bethe ansatz. These equations provide the complete description of the spectrum of the model with diagonal integrable boundaries.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz).
1 Introduction

Integrability in classical vertex models and quantum spin chains is intimately connected with solutions of the Yang–Baxter equation [1]. This equation provides an unified approach to construct and study physical properties of integrable models [2, 3]. Usually these systems are studied with periodic boundary conditions but more general boundaries can also be considered.

Although the physical properties associated with the bulk of the system are not expected to be influenced by boundary conditions in the thermodynamical limit, there are surface properties such as the interfacial tension where the boundary conditions are of relevance. Moreover, the conformal spectra of lattice models at criticality can be modified by the effect of boundaries [4]. So, there is an increasing interest in the studies of this issue.

Integrable systems with open boundary conditions can also be accommodated within the framework of the Quantum Inverse Scattering Method [5]. In addition to the solution of the Yang–Baxter equation governing the dynamics of the bulk there is another fundamental ingredient, the reflection matrices [6]. These matrices, also referred as $K$ matrices, represent the interactions at the boundaries and compatibility with the bulk integrability requires these matrices to satisfy the so-called reflection equations [5, 6].

The study of general regular solutions of the reflection equations for vertex models based on $q$-deformed Lie algebras [7, 8] has been successfully accomplished. See [9] for instance and references therein. Subsequently, similar success has been obtained with vertex models based on Lie superalgebras [10, 11]. For instance, the diagonal solutions associated with the $U_q[sl(m|n)]$ [12, 13] and $U_q[osp(2|2)]$ symmetries [14] and non-diagonal solutions related to super-Yangians $osp(m|n)$ [15] and $sl(m|n)$ [16, 17]. The most general set of solutions of the reflection equation for the vertex models based on Lie superalgebras are reported in [18].

More recently, the study of general regular solutions of the reflection equations for vertex models based on $q$-deformed Temperley-Lieb algebras [19] has been done with success [20, 21, 22]. The spectra of the corresponding spin chain with integrable open diagonal boundaries were obtained via the coordinate Bethe ansatz [23]. However, this same analysis for vertex models based on graded Temperley-Lieb algebra [24] is still an open problem.

The aim of this paper is to start to bridge this gap by presenting the most general set of solutions of the reflection equation for the $U_q[osp(1|2)]$ Temperley-Lieb vertex model and to supply the most general integrable boundaries for the corresponding spin chain. The description of the spectrum of the model
with diagonal integrable boundaries is obtained with aid of the Bethe ansatz.

This paper is organized as follows. In the next section we present the $\mathcal{R}$-matrix solution of the $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb vertex model. This information allows the way for the analysis of the corresponding reflection equations and in Section 3 we present what we understood to be the most general set of $K$ matrices. In Section 4 we present the Temperley-Lieb Hamiltonian with more general integrable boundary terms, with special interest on diagonal ones. In Section 5 we use the generalization of the coordinate Bethe ansatz as presented in [23], in order to obtain directly the spectrum of the Hamiltonian with diagonal boundary terms. Concluding remarks are discussed in Section 6.

2 $\mathcal{R}$-matrix solution

The Temperley-Lieb is a unital associative algebra generated by $\{I, U_1, U_2, \ldots, U_{L-1}\}$ subject to the relations

\begin{align*}
U_i^2 &= \sqrt{Q} U_i, \\
U_i U_{i \pm 1} U_i &= U_i, \\
U_i U_j &= U_j U_i, \quad |i - j| \geq 2
\end{align*}  \tag{2.1}

where $U_i$ acts non-trivially in the sites $i$ and $i + 1$:

\begin{equation}
U_i = I_1 \otimes \cdots \otimes I_{i-1} \otimes U \otimes I_{i+2} \otimes \cdots \otimes I_L,  \tag{2.2}
\end{equation}

$I$ is the matrix identity and $\sqrt{Q}$ a given number. This tensor algebra governs the dynamics of a completely integrable model in the sense that the global quantity

\begin{equation}
H = \sum_{i=1}^{L-1} U_i  \tag{2.3}
\end{equation}

is an involutive integral of motion.

The Temperey-Lieb algebra provided an algebraic framework for constructing and analyzing different types of integrable lattice models, such as the $Q$-state Potts model, IRF model, $O(n)$ loop model, six vertex model, etc [25]. This equivalence [26] was used in order to obtain the spectra properties of quantum spin chains with periodic boundary conditions and free ends [27].

Given any representation $U_i$, we define the operators

\begin{equation}
\tilde{R}_i(u) = \frac{\sinh(\eta - u)}{\sinh \eta} I_i + \frac{\sinh u}{\sinh \eta} U_i, \quad i = 1, 2, \ldots, L - 1  \tag{2.4}
\end{equation}
where $\eta$ is related to $Q$ through

$$2 \cosh \eta = \sqrt{Q}$$

(2.5)

and it follows from (2.1) that

$$\hat{R}_i(u)\hat{R}_j(v) = \hat{R}_j(v)\hat{R}_i(u), \quad |i-j| \geq 2$$

$$\hat{R}_i(u)\hat{R}_{i+1}(u+v)\hat{R}_i(v) = \hat{R}_{i+1}(v)\hat{R}_i(u+v)\hat{R}_{i+1}(u)$$

(2.6)

where $u$ and $v$ are spectral parameters.

From the $Z_2$-graded vector space, we refer to $U_i$ as a graded vector representation of the Temperley-Lieb algebra and $\hat{R}_i(u)$ as the graded solution of the Yang-Baxter equation (2.6).

The orthosympletic $U_q[osp(M|2n)]$ Temperley-Lieb solutions of the Yang-Baxter equation are well known by Zhang’s paper [24], from which we can write down the $U_q[osp(1|2)]$ solution:

$$\tilde{R}(u) = \frac{\sinh(\eta - u)}{\sinh \eta} I + \frac{\sinh u}{\sinh \eta} U,$$

(2.7)

where $I$ is the 9 by 9 matrix identity and the Temperley-Lieb operator

$$U = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -q^{-1} & 0 & q^{-\frac{1}{2}} & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},$$

(2.8)

which, in a spin chain language, is the projector onto the two-sites spin zero singlet written in the spin-1 basis $\{|+\rangle, |0\rangle, |−\rangle\}$. Here we have used the grading FBF i.e., $|+\rangle = |−\rangle = 1$ and $|0\rangle = 0$.

Using the relation $R = P_g \tilde{R}$ where $P_g$ is the graded permutation operator

$$P_g |\alpha\rangle \otimes |\beta\rangle = (-1)^{|\alpha|} |\beta\rangle \otimes |\alpha\rangle$$

(2.9)
with $\alpha, \beta = +, 0, -, \beta$ the graded $R$-matrix has the form

\[
R = \begin{pmatrix}
-x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & 0 & 0 & 0 & 0 \\
0 & 0 & -x_2 & 0 & \sqrt{q}x_2 & 0 & qx_2 - x_1 & 0 \\
0 & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{q}x_2 & 0 & x_1 + x_2 & 0 & \sqrt{q}x_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 \\
0 & 0 & -x_1 + \frac{1}{q}x_2 & 0 & -\sqrt{q}x_2 & 0 & -x_2 & 0 \\
0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\
\end{pmatrix}
\tag{2.10}
\]

where

\[
x_1 = \frac{\sinh(\eta - u)}{\sinh \eta} \quad \text{and} \quad x_2 = \frac{\sinh(u)}{\sinh \eta},
\tag{2.11}
\]

satisfying a regular condition $R(0) = P_k$.

This $R$ - matrix defines the local structure of Boltzmann weight of a graded 15-vertex model in a two-dimensional lattice. It follows that we can define a row-to-row transfer matrix $t(u)$ as super-trace of a monodromy matrix $T_A(u)$

\[
t(u) = \text{str}_A(T_A(u)),
\]

\[
T_A(u) = R_{L,A}(u)R_{L-1,A}(u) \cdots R_{2,A}(u)R_{1,A}(u),
\tag{2.12}
\]

Here the notation means that the operator $R_{i,A}(u)$ is a matrix in the auxiliary space $A$ corresponding to the horizontal degrees of freedom and its matrix elements are operators on the quantum space $\otimes_{i=1}^L V_i$, where $V_i$ represents the vertical space of states and $i$ the site of one-dimensional lattice of size $L$.

The transfer matrix (2.12) is the functional generator of infinite conserved quantities

\[
t(u) = \exp \left( \sum_{k=1}^{\infty} Q_k u^{k-1} \right)
\tag{2.13}
\]

The commutation relation $[t(u), t(v)] = 0$, $u \neq v$ is provided by (2.6) and it follows that $[Q_k, Q_l] = 0$, $\forall k, l$. Here we note that $Q_2$ can be identified with the global Hamiltonian (2.3), which define the integrable closed spin chain.

The spectra of the orthosympletic Temperley-Lieb Hamiltonian with Martin’s boundary condition [25], periodic boundary condition and free ends are known [28].
3 \( K \)-matrix solution

The notion of quantum integrability was extended to work with open boundary problems [5]. In addition to the graded \( \mathcal{R} \)-matrix describes the bulk dynamics, we have to introduce reflection \( K \) matrices to describe such boundary conditions. These new matrices represent the interactions at the right and left ends of the open spin chain. This is a consequence of the reflection equation, which reads

\[
\mathcal{R}_{12}(u - \mu)K_1^-(u)\mathcal{R}_{21}(u + \mu)K_2^-(\mu) = K_2^-(\mu)\mathcal{R}_{12}(u + \mu)K_1^-(u)\mathcal{R}_{21}(u - \mu). \tag{3.1}
\]

In the case of open boundary conditions, the graded transfer matrix can be written as the super-trace

\[
t(u) = \text{str}_A \left[ K_A^+(u)T_A(u)K_A^-(u) [T_A(-u)]^{-1} \right], \tag{3.2}
\]

where \( K_A^-(u) \) can be chosen as one of the solutions of the reflection equation (3.1). The other boundary matrix \( K_A^+(u) \) is obtained from the previous one by means of the isomorphism [29],

\[
K_A^+(u) = K_A^-(u - \rho)^{st} M, \tag{3.3}
\]

where \( \rho = -\eta \) is the crossing parameter and st means super-transposition.

For the \( \mathcal{U}_q[osp(1|2)] \) Temperley-Lieb model the graded \( M \) matrix is given by

\[
M = \begin{pmatrix}
q^{-1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & q
\end{pmatrix}. \tag{3.4}
\]

The integrable open spin chain is obtained by means of the logarithmic derivative of the transfer matrix (3.2), such that,

\[
H = \sum_{k=1}^{L-1} U_{k,k+1} + \frac{\sinh \eta}{2} \frac{dK_1^-(u)}{du} \bigg|_{u=0} + \frac{\text{str}_A \left[ K_A^+(0)U_LA \right]}{\text{str}_A \left[ K_A^+(0) \right]}. \tag{3.5}
\]

Now, we begin to solve the reflection equation (3.1) for the \( \mathcal{U}_q[osp(1|2)] \) Temperley-Lieb vertex model in order to obtain the boundary terms of (3.5).

Using (2.10), the reflection matrix \((K^-(u))_{ij} = k_{ij}(u), i,j = \{1,2,3\} , \) with \( K_1^-(u) = K^-(u) \otimes I \), \( K_2^-(u) = I \otimes K^-(u) \), \( \mathcal{R}_{12} = \mathcal{R} \) and \( \mathcal{R}_{21} = P_x \mathcal{R} P_y \), the matrix equation (3.1) has 81 functional relations for the \( k_{ij}(u) \) matrix elements, many of them not independent relations. In order to solve these functional equations, we shall proceed as follows. First we consider the \((i,j)\) component of the matrix equation (3.1). By differentiating it with respect to \( v \) and taking \( v = 0 \), we get algebraic equations...
involving the single variable $u$ and nine parameters

$$
\beta_{ij} = \left. \frac{dk_{ij}(u)}{du} \right|_{u=0}, \quad i, j = 1, 2, 3.
$$

(3.6)

Analyzing the reflection equations one can see that they possess a special structure. Several equations exist involving only two non-diagonal elements. They can be solved by the relations

$$
k_{12}(u) = \frac{\beta_{12} k_{13}(u)}{\beta_{13}},
$$

$$
k_{21}(u) = \frac{\beta_{21} k_{13}(u)}{\beta_{13}}, \quad k_{23}(u) = \frac{\beta_{23} k_{13}(u)}{\beta_{13}},
$$

$$
k_{31}(u) = \frac{\beta_{31} k_{13}(u)}{\beta_{13}}, \quad k_{32}(u) = \frac{\beta_{32} k_{13}(u)}{\beta_{13}}.
$$

(3.7)

We are thus left with several equations involving two diagonal elements and $k_{13}(u)$. From the equations (1, 2) and (1, 4) we have

$$
k_{22}(u) = k_{11}(u) + (\beta_{22} - \beta_{11}) \frac{k_{13}(u)}{\beta_{13}},
$$

$$
k_{33}(u) = k_{11}(u) + (\beta_{33} - \beta_{11}) \frac{k_{13}(u)}{\beta_{13}},
$$

(3.8)

respectively.

Finally, we can use the equation (1, 3) to find $k_{11}(u)$:

$$
k_{11}(u) = \frac{k_{13}(u)}{\beta_{13} (x_2(u) \cosh \eta + x_1(u))} \left\{ \frac{x_1(u)x'_2(u) - x'_1(u)x_2(u)}{x_2(u)} - \frac{\beta_{12}\beta_{21}}{2\beta_{13}} x_1(u) - \frac{\beta_{22} - \beta_{11}}{2} x_2(u) - \frac{\beta_{33} - \beta_{11}}{2} (x_1(u) - qx_2(u)) \right\}
$$

(3.9)

where $x'_i(u) = dx_i(u)/du, \, i = 1, 2$.

Now, substituting these expressions into the remaining equations $(i, j)$, we are left with several constraint equations involving the $\beta_{ij}$ parameters.

From the equation (2, 3) we can choose

$$
\beta_{22} - \beta_{11} = \frac{\beta_{12}\beta_{23}}{\beta_{13}} - \frac{\beta_{21}\beta_{13}}{\beta_{23}},
$$

(3.10)

and from the equation (3, 7)

$$
\beta_{33} - \beta_{11} = \frac{\beta_{13}\beta_{32}}{\beta_{12}} - \frac{\beta_{21}\beta_{13}}{\beta_{23}}.
$$

(3.11)
Here we note that $\beta_{11}$ is fixed by the normal condition, $K^{(\cdot)}(0) = I$. Moreover, all the remaining constraint equations are solved by the symmetric relation

$$\beta_{32} \beta_{21} \beta_{13} = \beta_{23} \beta_{12} \beta_{31}, \quad (3.12)$$

from which we can fix $\beta_{32}$ in function of $\beta_{12}, \beta_{13}, \beta_{21}, \beta_{23}$ and $\beta_{31}$. In this way we have obtained a five free parameter solution of (3.1) for the $U_q[osp(1|2)]$ Temperley-Lieb vertex model.

The corresponding $K^{(+)}(u)$ reflection matrix is obtained by the isomorphism (3.3), namely

$$K^{(+)}(u) = \left( \begin{array}{ccc} \frac{1}{q} k_{11}(-u + \eta) & k_{21}(-(u + \eta) & qk_{31}(-u + \eta) \\ -\frac{1}{q} k_{12}(-u + \eta) & k_{22}(-u + \eta) & -qk_{32}(-u + \eta) \\ \frac{1}{q} k_{13}(-u + \eta) & k_{23}(-u + \eta) & qk_{33}(-u + \eta) \end{array} \right) \quad (3.13)$$

where we have used the super-transpostion $(A^t)_{\alpha\beta} = (-1)^{[\alpha][\beta]} A_{\beta\alpha}$ and the graded $M$ matrix (3.4) but, replacing all parameters $\beta_{ij}$ by the new parameters $\alpha_{ij}$.

From the general solution (3.7) to (3.12) one can see $k_{13}(u)$ as an arbitrary function satisfying the normal condition. Therefore the choice

$$k_{13}(u) = \frac{1}{2} \beta_{13} x_2(u) \frac{x_2(u) \cosh \eta + x_1(u)}{x_1(u)x_1'(u) - x_1'(u)x_2(u)} = \frac{1}{2} \beta_{13} \sinh(2u) \quad (3.14)$$

do not imply any restriction as compared to the general case, but simplify our $K^{-}(u)$ - matrix to

$$K^{-}(u) = \left( \begin{array}{ccc} k_{11} & \frac{1}{2} \beta_{21} \sinh(2u) & \frac{1}{2} \beta_{13} \sinh(2u) \\ \frac{1}{2} \beta_{21} \sinh(2u) & k_{11} + \frac{1}{2} (\beta_{22} - \beta_{11}) \sinh(2u) & \frac{1}{2} \beta_{23} \sinh(2u) \\ \frac{1}{2} \beta_{31} \sinh(2u) & \frac{1}{2} \beta_{32} \sinh(2u) & k_{11} + \frac{1}{2} (\beta_{33} - \beta_{11}) \sinh(2u) \end{array} \right) \quad (3.15)$$

and

$$k_{11} = k_{11}(u, \beta) = 1 - \frac{1}{2} \left\{ \frac{\beta_{12} \beta_{23}}{2 \beta_{13}} x_1(u) + \frac{1}{2} (\beta_{22} - \beta_{11}) x_2(u) + \frac{1}{2} (\beta_{33} - \beta_{11}) (x_1(u) - qx_2(u)) \right\} x_2(u) \sinh \eta \quad (3.16)$$

where $\beta_{22} - \beta_{11}, \beta_{33} - \beta_{11}$ and $\beta_{32}$ are given by (3.10), (3.11) and (3.12), respectively.

### 3.1 Reduced $K$- matrix solutions

For particular choice of the free parameters in (3.15) to (3.16), we can derive several reduced solutions. For instance, making $\beta_{21} = \beta_{31} = \beta_{32} = 0$, the $K^{(-)}$ - matrix (3.15) is reduced to a three free parameters
solution, $K^{(-)}_{up}$ - matrix, the up-triangular solution. Similarly, making $\beta_{12} = \beta_{13} = \beta_{23} = 0$ we get a
three free parameters $K^{(-)}_{down}$ - matrix, the down-triangular solution. The corresponding $K^{(+)}$ matrices
are obtained from (3.13). However, in order to obtain all diagonal solutions, it is simpler to solve the
reflection equations again.

Taking into account diagonal $K$ matrices, all reflection equations (3.1) are solved when we find $k_{22}(u)$
and $k_{33}(u)$ as functions of $k_{11}(u)$ provided that the diagonal parameters $\beta_{ii}$ satisfy the constraint equation

\[(\beta_{33} - \beta_{22})(\beta_{33} - \beta_{11})(\beta_{22} - \beta_{11}) = 0 \tag{3.17}\]

From (3.17), of the three matrix elements, two have the same value. Let us normalize one of them to be
equal to 1 such that the other entry is given by

\[k_{pp}(u) = \beta_{pp} \frac{x_2(u)[\Delta_1 x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{pp} x_2(u)[\Delta_2 x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]} \tag{3.18}\]

where $\Delta_1 + \Delta_2 = -q^{-1} + 1 - q$.

Identifying the diagonal indexes by $(1,2,3) \Rightarrow (q^{-1},1,q)$ one can see that $\Delta_1$ is the sum of terms
corresponding to the positions of the entries 1 and $\Delta_2$ is equal to sum of terms corresponding to the
positions of the entries $k_{pp}(u)$. It means that we have six $K^{(-)}(u)$ diagonal solutions, namely

\[D_1^{I} = \text{diag}(k_{11},1,1), \quad D_2^{I} = \text{diag}(1,k_{22},1), \quad D_3^{I} = \text{diag}(1,1,k_{22}) \tag{3.19}\]

with two entries equal to 1 and

\[k_{11}(u) = \frac{\beta_{11} x_2(u)[(1-q)x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{11} x_2(u)[-q^{-1}x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}, \]

\[k_{22}(u) = \frac{\beta_{22} x_2(u)[(-q^{-1} - q)x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{22} x_2(u)[x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}, \]

\[k_{33}(u) = \frac{\beta_{33} x_2(u)[(1-q^{-1})x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{33} x_2(u)[-q x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}. \tag{3.20}\]

The second type has only one entry equal to 1

\[D_1^{II} = \text{diag}(1,k_{22},k_{22}), \quad D_2^{II} = \text{diag}(k_{33},1,k_{33}), \quad D_3^{II} = \text{diag}(k_{11},k_{11},1) \tag{3.21}\]

where

\[k_{11}(u) = \frac{\beta_{11} x_2(u)[-q x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{11} x_2(u)[(1-q^{-1})x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}, \]

\[k_{22}(u) = \frac{\beta_{22} x_2(u)[-q^{-1} - q x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{22} x_2(u)[(1-q)x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}, \]

\[k_{33}(u) = \frac{\beta_{33} x_2(u)[x_2(u) + x_1(u)] + 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}{\beta_{33} x_2(u)[(-q^{-1} - q)x_2(u) + x_1(u)] - 2[x_1(u)x_2'(u) - x_1'(u)x_2(u)]}. \tag{3.22}\]
Again, the prime means derivative in respect to $u$ and $x_i(u)$, $i = 1, 2$ are given by (2.11). Note also that the difference between these one free parameter solutions comes from the partitions of $2 \cosh \eta = -q^{-1} + 1 - q$.
Moreover, we have the symmetry $q \leftrightarrow q^{-1}$ and the corresponding six $K^{(+)}(u)$ diagonal solutions are obtained by the isomorphism (3.3).

4 $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb spin chain

The boundary terms of Hamiltonian (3.5) are directly obtained from the $K^{(\pm)}$ matrices. In particular, the boundary term acting non-trivially in the site 1 has the form

$$ bt_1 = \frac{\sinh \eta}{2} \left. \frac{dK^{(-)}(u)}{du} \right|_{u=0} $$

$$ = \frac{1}{2} \sinh \eta \left( \begin{array}{ccc} t_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & t_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & t_{33} \end{array} \right) $$

where

$$ t_{11} = \frac{1}{2} \left( -\frac{\beta_{12}\beta_{23}}{\beta_{13}} - \frac{\beta_{23}\beta_{13}}{\beta_{21}} + \frac{\beta_{21}\beta_{13}}{\beta_{23}} \right) $$

$$ t_{22} = \frac{1}{2} \left( \beta_{12}\beta_{23} - \beta_{23}\beta_{13} \right) $$

$$ t_{33} = \frac{1}{2} \left( -\frac{\beta_{12}\beta_{23}}{\beta_{13}} + \frac{\beta_{23}\beta_{13}}{\beta_{21}} - \frac{\beta_{21}\beta_{13}}{\beta_{23}} \right) $$

and we remember that $\beta_{32}$ is given by (3.12).

The boundary term acting non-trivially in the site $L$ is given by

$$ bt_L = \frac{\text{str}_A[K_A^{(+)}(0)U_{LA}]}{\text{str}_A[K_A^{(+)}(0)]} $$

$$ = \frac{1}{2} \sinh(\eta) \left( \begin{array}{ccc} \frac{\cos(\gamma)}{\sinh(2\eta)} & q^{\frac{1}{2}}\alpha_{32} & -q\alpha_{31} \\ -q^{-\frac{1}{2}}\alpha_{23} & \frac{\cos(\gamma)}{\sinh(2\eta)} & q^{\frac{1}{2}}\alpha_{21} \\ -q^{-1}\alpha_{13} & -q^{-\frac{1}{2}}\alpha_{12} & \frac{v_3}{\sinh(2\eta)} \end{array} \right) $$

where

$$ v_{11} = 2 + \left( -\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}} + (1 - q^{-1})\frac{\alpha_{23}\alpha_{31}}{\alpha_{21}} + q^{-1}\frac{\alpha_{21}\alpha_{13}}{\alpha_{23}} \right) \sinh(\eta) $$

$$ v_{22} = 2 + \left( (q + q^{-1})\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}} + q\frac{\alpha_{23}\alpha_{31}}{\alpha_{21}} + q^{-1}\frac{\alpha_{21}\alpha_{13}}{\alpha_{23}} \right) \sinh \eta $$

$$ v_{33} = 2 + \left( -\frac{\alpha_{12}\alpha_{23}}{\alpha_{13}} + q\frac{\alpha_{23}\alpha_{31}}{\alpha_{21}} + (1 - q)\frac{\alpha_{21}\alpha_{13}}{\alpha_{23}} \right) \sinh \eta $$

(4.4)
Here we have new five free parameters $\alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{23}$ and $\alpha_{31}$. Note that $\alpha_{32}$ is given by (3.12), replacing $\beta$ by $\alpha$.

With these expressions we have obtained the general integrable boundaries terms for the quantum spin-1 chain associated with the $U_q[osp(1|2)]$ Temperley-Lieb vertex model.

### 4.1 Diagonal boundaries

The choice $\beta_{21} = \beta_{31} = \beta_{32} = 0$ ($\alpha_{21} = \alpha_{31} = \alpha_{32} = 0$) in $t_{b_{1}}$ ($t_{b_{3}}$) gives us the up-triangular right boundary and down-triangular left boundary both with three free parameters and vice-versa for the choice $\beta_{12} = \beta_{13} = \beta_{23} = 0$ ($\alpha_{12} = \alpha_{13} = \alpha_{23} = 0$).

In the next section we will use the coordinate Bethe ansatz in order to diagonalize the Hamiltonian (3.5) with diagonal integrable boundaries. Therefore, let us write explicitly its diagonal entries.

From right $K^{(-)}$-matrices $D^{[r]}_1, D^{[r]}_2$ and $D^{[r]}_3$, we can compute the type I boundaries

$$
bt^{[1]}_1 = \text{diag}(r(\beta), 0, 0), \quad bt^{[1]}_L = \text{diag}(s_1(\alpha), s_1(\alpha), t_1(\alpha)),
$$

$$
bt^{[2]}_1 = \text{diag}(0, r(\beta), 0), \quad bt^{[2]}_L = \text{diag}(s_2(\alpha), t_2(\alpha), s_2(\alpha)),
$$

$$
bt^{[3]}_1 = \text{diag}(0, 0, r(\beta)), \quad bt^{[3]}_L = \text{diag}(t_3(\alpha), s_3(\alpha), s_3(\alpha))
$$

where $r(\beta) = \frac{1}{2} \beta \sinh \eta$ and $\beta$ is the free parameter. For each $bt^{[i]}_L$ we have $s_i(\alpha)$ and $t_i(\alpha), i = 1, 2, 3$ given by

$$
s_1(\alpha) = \frac{2 + q^{-1} \alpha \sinh \eta}{4 \cosh \eta}, \quad s_2(\alpha) = \frac{2 - \alpha \sinh \eta}{4 \cosh \eta},
$$

$$
s_3(\alpha) = \frac{2 + q \alpha \sinh \eta}{4 \cosh \eta}
$$

and

$$
t_1(\alpha) = \frac{2 + (1 - q) \alpha \sinh \eta}{4 \cosh \eta}, \quad t_2(\alpha) = \frac{2 - (q + q^{-1}) \alpha \sinh \eta}{4 \cosh \eta},
$$

$$
t_3(\alpha) = \frac{2 + (1 - q^{-1}) \alpha \sinh \eta}{4 \cosh \eta}
$$

where $\alpha$ is the corresponding free parameter.

We have more three solutions corresponding to the $K^{(-)}$ matrices (3.21)

$$
bt^{[1]}_1 = \text{diag}(0, r(\beta), r(\beta)), \quad bt^{[1]}_L = \text{diag}(s_1(-\alpha), s_1(-\alpha), t_1(-\alpha)),
$$

$$
bt^{[2]}_1 = \text{diag}(r(\beta), 0, r(\beta)), \quad bt^{[2]}_L = \text{diag}(s_2(-\alpha), t_2(-\alpha), s_2(-\alpha)),
$$

$$
bt^{[3]}_1 = \text{diag}(r(\beta), r(\beta), 0), \quad bt^{[3]}_L = \text{diag}(t_3(-\alpha), s_3(-\alpha), s_3(-\alpha)).
$$

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It follows that we have 6 different integrable boundaries related by the isomorphism (3.3). However, it is worth to note that other combination of the boundaries are allowed $B_{1,L}^{(i,j,[a,b])} = b_1^{(j,[b])} + b_L^{(i,[a])}$ with $i, j = 1, 2, 3, a, b = I, II$ resulting in 36 integrable boundaries for the spin-1 $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb Hamiltonian.

The action of the boundary terms on the Hilbert space is given by

$$B_{1,L}^{(i,j,[a,b])} \left| \frac{1}{\sigma} \ldots \frac{L}{\tau} \right\rangle = E_{(i,j,[a,b])}^{\sigma \tau} \left| \frac{1}{\sigma} \ldots \frac{L}{\tau} \right\rangle$$

where $E_{(i,j,[a,b])}^{\sigma \tau} = l_{\sigma \sigma}^{(i,[a])} + r_{\tau \tau}^{(j,[b])}$ and the sites are indexed by $\sigma, \tau = (1, 2, 3) \equiv (+, 0, -)$. Here we recall that $l_{\sigma \sigma}^{(i,[a])}$ and $r_{\tau \tau}^{(j,[b])}$ are the matrix elements of $b_1^{(i,[a])}$ and $b_L^{(j,[b])}$ respectively.

In the next section, we will restrict ourselves to the case of integrable boundaries related by the isomorphism $B_{1,L}^{(i,[a])} = b_1^{(i,[a])} + b_L^{(i,[a])}$ and we shall use the coordinate Bethe ansatz in order to obtain the eigenvalues of the Hamiltonian (3.5).

## 5 Coordinate Bethe ansatz

We known that the bulk part of the Hamiltonian (3.5) is the projector operator onto the two-site spin zero. This implies that there exist $3 \times 2^{L-1}$ states in a lattice with $L$ sites which are eigenstates of the bulk Hamiltonian with zero eigenvalues. However, these states are also eigenstates of the boundary part of the Hamiltonian $B_{1,L}^{(i,[a])}$ with eigenvalues $E_{(i,[a])}^{\sigma \tau}$. For instance, in a lattice with $L = 4$ sites we have the 24 natural reference states (Grouping according to its $E_{ab}$):

$$
\begin{align*}
|+++\rangle & : \mathcal{E}_{11}, & |0++\rangle & : \mathcal{E}_{21}, & |0+0\rangle & : \mathcal{E}_{31}, \\
|+0+\rangle & : \mathcal{E}_{12}, & |0+0\rangle & : \mathcal{E}_{22}, & |0-0\rangle & : \mathcal{E}_{32}, \\
|+0+\rangle & : \mathcal{E}_{13}, & |0-0\rangle & : \mathcal{E}_{23}, & |0-0\rangle & : \mathcal{E}_{33},
\end{align*}
$$

Moreover, apart from the natural degenerescence of the boundary eigenvalues $E_{(i,[a])}^{\sigma \tau}$, one can see from the structure of the boundary matrix $K^{(\pm)}$ that not all $E_{(i,[a])}^{\sigma \tau}$ are independent. More precisely, with aid
of (4.5) and (4.8) we have

\[
E_{31}^{(1, [a])} = \begin{cases} t_1(\alpha) + r(\beta) \\ t_3(-\alpha) \end{cases}, \\
E_{21}^{(1, [a])} = E_{11}^{(1, [a])} = \begin{cases} s_1(\alpha) + r(\beta) \\ s_1(-\alpha) \end{cases}, \\
E_{32}^{(1, [a])} = E_{33}^{(1, [a])} = \begin{cases} t_1(\alpha) \\ t_1(-\alpha) + r(\beta) \end{cases}, \\
E_{12}^{(1, [a])} = E_{22}^{(1, [a])} = E_{13}^{(1, [a])} = E_{23}^{(1, [a])} = \begin{cases} s_1(\alpha) \\ s_1(-\alpha) + r(\beta) \end{cases},
\]

(5.4)

for the first solutions (1\textsuperscript{st}),

\[
E_{22}^{(2, [a])} = \begin{cases} t_2(\alpha) + r(\beta) \\ t_2(-\alpha) \end{cases}, \\
E_{12}^{(2, [a])} = E_{32}^{(2, [a])} = \begin{cases} s_2(\alpha) + r(\beta) \\ s_2(-\alpha) \end{cases}, \\
E_{21}^{(2, [a])} = E_{23}^{(2, [a])} = \begin{cases} t_2(\alpha) \\ t_2(-\alpha) + r(\beta) \end{cases}, \\
E_{13}^{(2, [a])} = E_{11}^{(2, [a])} = E_{31}^{(2, [a])} = E_{33}^{(2, [a])} = \begin{cases} s_2(\alpha) \\ s_2(-\alpha) + r(\beta) \end{cases},
\]

(5.5)

for the second solutions (2\textsuperscript{nd}) and

\[
E_{13}^{(3, [a])} = \begin{cases} t_3(\alpha) + r(\beta) \\ t_3(-\alpha) \end{cases}, \\
E_{12}^{(3, [a])} = E_{11}^{(3, [a])} = \begin{cases} s_3(\alpha) + r(\beta) \\ s_3(-\alpha) \end{cases}, \\
E_{23}^{(3, [a])} = E_{33}^{(3, [a])} = \begin{cases} t_3(\alpha) \\ t_3(-\alpha) + r(\beta) \end{cases}, \\
E_{21}^{(3, [a])} = E_{22}^{(3, [a])} = E_{31}^{(3, [a])} = E_{32}^{(3, [a])} = \begin{cases} s_3(\alpha) \\ s_3(-\alpha) + r(\beta) \end{cases},
\]

(5.6)

for the third solutions (3\textsuperscript{rd}).

Note that the up expressions in each (\{ \} of (5.4), (5.5) and (5.6) correspond to type I solution while the down expressions correspond to type II solution.

In face of the large number of reference states, the standard construction of the all eigenstates seems to be impracticable. However, in order to obtain the eigenvalues of the Hamiltonian it is enough to work out with a few reference states. In fact, we can take one reference state from each block of these eigenvalues \(E_{\sigma r}^{(i, [a])}\) [23]. From now on, we drop the label for different solutions of the reflection equation from the boundary eigenvalues, such that \(E_{\sigma r}^{(i, [a])} = E_{\sigma r}\).
5.1 Ferromagnetic reference state

We shall start by considering the pseudo particle as a singlet over any of the reference state listed above and extended to \( L \) sites \((|++\cdots+:\mathcal{E}_{11})\). In general, it is convenient to start our ansatz with the following linear combination of the basis states \([30]\),

\[
|\Omega(k)\rangle = \sum_{i=-1}^{1} \epsilon(i) q^{-i+1} |k(i, -i)\rangle
= -\frac{1}{q} |\cdots k+\cdots\rangle + \frac{1}{\sqrt{q}} |\cdots 00\cdots\rangle + |\cdots -\cdots\rangle
\]

(5.7)

where \((\cdots)\) means that the remained sites are defined by the reference state considered. Here \(\epsilon(-1) = \epsilon(0) = 1\), \(\epsilon(1) = -1\) and \(1 \leq k \leq L - 1\). It follows that \(|\Omega(k)\rangle\) is an eigenstate of \(U_{k,k+1}\) such that

\[
U_{k,k+1} |\Omega(k)\rangle = \sqrt{q} |\Omega(k)\rangle, \quad U_{k,k+1} |\Omega(k \pm 1)\rangle = |\Omega(k)\rangle,
\]

\[
U_{k,k+1} |\Omega(j)\rangle = 0, \text{ if } k \neq \{j-1, j, j+1\},
\]

(5.8)

where \(\sqrt{q} = -q^{-1} + 1 - q\).

The action of the Hamiltonian \(H = \sum_{k=1}^{L-1} U_{k,k+1} + B_{1,L}\) over this state results

\[
H |\Omega(k)\rangle = \left(\sqrt{q} + \mathcal{E}_{11}\right) |\Omega(k)\rangle + |\Omega(k - 1)\rangle + |\Omega(k + 1)\rangle,
\]

\[
1 < k < L - 1.
\]

(5.9)

where \(B_{1,L} |\Omega(k)\rangle = \mathcal{E}_{11} |\Omega(k)\rangle\) for \(1 < k < L - 1\), due to ferromagnetic reference state.

In addition, we have for \(k = 1\)

\[
H |\Omega(1)\rangle = \left(\sqrt{q} + \mathcal{E}_{11}\right) |\Omega(1)\rangle + |\Omega(2)\rangle + (B_{1,L} - \mathcal{E}_{11}) |\Omega(1)\rangle,
\]

(5.10)

and for \(k = L - 1\)

\[
H |\Omega(L - 1)\rangle = \left(\sqrt{q} + \mathcal{E}_{11}\right) |\Omega(L - 1)\rangle + |\Omega(L - 2)\rangle + (B_{1,L} - \mathcal{E}_{11}) |\Omega(L - 1)\rangle,
\]

(5.11)

The equations (5.10) and (5.11) can be seen as extensions of (5.9) provided we define two new states

\[
|\Omega(0)\rangle = (B_{1,L} - \mathcal{E}_{11}) |\Omega(1)\rangle
= -q^{-\frac{1}{2}} (\mathcal{E}_{21} - \mathcal{E}_{11}) |00 + \cdots +\rangle + (\mathcal{E}_{31} - \mathcal{E}_{11}) | -+ + \cdots +\rangle
\]

(5.12)

and

\[
|\Omega(L)\rangle = (B_{L,1} - \mathcal{E}_{11}) |\Omega(L - 1)\rangle
= -q^{-1} (\mathcal{E}_{13} - \mathcal{E}_{11}) |+\cdots + -\rangle + q^{-\frac{1}{2}} (\mathcal{E}_{13} - \mathcal{E}_{11}) |+\cdots 00\rangle
\]

(5.13)
From the action of $H$ on these new states we have two closing relations

$$
H \Omega(0) = \Delta_i^{(1)} \Omega(1) + B_{1,L} \Omega(0) \\
\Delta_i^{(1)} = (\mathcal{E}_{21} - \mathcal{E}_{11}) - q(\mathcal{E}_{31} - \mathcal{E}_{11})
$$

(5.14)

and

$$
H \Omega(L) = \Delta_r^{(1)} \Omega(L - 1) + B_{1,L} \Omega(L) \\
\Delta_r^{(1)} = (\mathcal{E}_{12} - \mathcal{E}_{11}) - q^{-1}(\mathcal{E}_{13} - \mathcal{E}_{11})
$$

(5.15)

It follows from (5.12) that the action of $B_{1,L}$ over $\Omega(0)$ depend on the possible choices of the boundary eigenvalues

$$
B_{1,L} \Omega(0) = \begin{cases} 
\mathcal{E}_{31} \Omega(0) & \text{if } \mathcal{E}_{21} = \mathcal{E}_{11} \ (1^{\text{st}}) \\
\mathcal{E}_{21} \Omega(0) & \text{if } \mathcal{E}_{31} = \mathcal{E}_{11} \ (2^{\text{nd}}) \\
\mathcal{E}_{21} \Omega(0) & \text{if } \mathcal{E}_{31} = \mathcal{E}_{21} \ (3^{\text{th}})
\end{cases}
= \mathcal{E}_{(v_1,i)} \Omega(0), \quad i = 1, 2, 3
$$

(5.16)

where we have used the vector notation $v_1 = (3, 2, 2)$ whose components $i \in \{1, 2, 3\}$ represent the different solutions of the reflection equations (5.4, 5.5, 5.6), in this order.

Similarly, the eigenvalue problem $B_{1,L} \Omega(L)$ depend on the possible choices of the boundary eigenvalues in (5.13)

$$
B_{1,L} \Omega(L) = \begin{cases} 
\mathcal{E}_{13} \Omega(L) & \text{if } \mathcal{E}_{12} = \mathcal{E}_{11} \ (3^{\text{th}}) \\
\mathcal{E}_{12} \Omega(L) & \text{if } \mathcal{E}_{13} = \mathcal{E}_{11} \ (2^{\text{nd}}) \\
\mathcal{E}_{12} \Omega(L) & \text{if } \mathcal{E}_{12} = \mathcal{E}_{13} \ (1^{\text{st}})
\end{cases}
= \mathcal{E}_{(u_1,j)} \Omega(L), \quad j = 1, 2, 3
$$

(5.17)

where $u_1 = (2, 2, 3)$.

Taking into account these relations, valid for the ferromagnetic reference state, we can reconstruct all steps of the coordinate Bethe ansatz presented in the reference [23].

### 5.2 One-particle state

In the first non-trivial sector, we assume the following ansatz for the eigenstates

$$
\Psi_1 = \sum_{k=1}^{L-1} A(k) \Omega(k).
$$

(5.18)
Imposing the eigenvalue equation \( H \Psi_1 = E_1 \Psi_1 \) is fulfilled, we obtain a set of equations for the function \( A(k) \)

\[
\left( E_1 - \sqrt{Q} - E_{11} \right) A(k) = A(k-1) + A(k+1), \quad 1 < k < L - 1
\]

and its extensions to include \( k = 1 \) and \( k = L - 1 \)

\[
(E_1 - E_{(v_1,1)} A(0) = \Delta_l^{(1)} A(1) \tag{5.20}
\]

and

\[
(E_1 - E_{(u_1,1)} A(L) = \Delta_r^{(1)} A(L - 1) \tag{5.21}
\]

Taking the ansatz for the plane wave amplitude

\[
A(k) = a(\theta) \xi^k - a(-\theta) \xi^{-k}, \tag{5.22}
\]

we have the following expression for the energy eigenvalues

\[
E_1(\xi) = E_{11} + \sqrt{Q} + \xi + \xi^{-1}. \tag{5.23}
\]

After we fix the parameter \( \xi = e^{i\theta} \) and the ratio of the amplitudes \( a(\theta)/a(-\theta) \) we are left with the Bethe ansatz equation

\[
\xi^{2L} = \left( \frac{E_1(\theta) - E_{(v_1,1)}}{E_1(\theta) - E_{(v_1,1)} - \xi \Delta_l^{(1)}} \right) \left( \frac{E_1(\theta) - E_{(u_1,1)} - \xi \Delta_r^{(1)}}{E_1(\theta) - E_{(u_1,1)} - \xi^{-1} \Delta_r^{(1)}} \right) \equiv F_l(\theta) F_r(\theta) \tag{5.24}
\]

where \( E_1(\theta) = \sqrt{Q} + \varepsilon_{11} + 2 \cos \theta \).

5.3 Two-particle state

In the next particle sector, we have two interacting pseudo-particles, which can be represented as a product of two pseudo-particles eigenstates, as given by

\[
\Psi_2 = \sum_{k_1+1 < k_2} A(k_1, k_2) |\Omega(k_1, k_2)\rangle \tag{5.25}
\]

It follows from the notation for one-particle state that

\[
|\Omega(k_1, k_2)\rangle = \sum_{i, j = -1}^1 \epsilon(i) \epsilon(j) q^{-1/2} |k_1(i, -i); k_2(j, -j)\rangle. \tag{5.26}
\]
where $\epsilon(1) = -1$ and $\epsilon(0) = \epsilon(-1) = 1$.

We can split the action of the Hamiltonian on the state $|\Omega(k_1, k_2)\rangle$ in four cases: (i) The case where two pseudo-particles are separated in the bulk, (ii) The case where the pseudo-particles are separated but one of them or both are at the boundaries (iii) The case where the particles are neighbours in the bulk (iv) The case where the particles are neighbours at the boundaries. Similarly to one-particle state we need to introduce new states and see the action of the Hamiltonian on them. These equation are not reported here, but we can follow the construction of two-particle states of [23] in order to verify the results reported below.

Take into account the eigenvalue equation $(H \Psi_2 = E_2 \Psi_2)$, one can obtain the two-particle eigenvalue

$$E_2 = 2\sqrt{Q} + \mathcal{E}_{11} + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1},$$  \hspace{1cm} (5.27)

provided that the following parametrization for the plane wave amplitudes is assumed

$$A(k_1, k_2) = \sum_{\varepsilon} \varepsilon_P a(\theta_1, \theta_2) \xi_1^{k_1} \xi_2^{k_2}, \hspace{1cm} (5.28)$$

where the sum extends over all permutations and negations of momenta $(\theta_i)$, such that $\xi_i = e^{i\theta_i}$, and $\varepsilon_P$ is the signature of permutations and negations. This structure already reflects the existence of the boundary reflections.

From the phase shift relations we obtain the corresponding Bethe ansatz equations

$$\xi_1^{2L} = F_l(\theta_1) F_r(\theta_1) \left( \frac{s(\theta_1, \theta_2)}{s(\theta_2, \theta_1)} \right) \left( \frac{s(\theta_2, -\theta_1)}{s(-\theta_1, \theta_2)} \right)$$  \hspace{1cm} (5.29)

and

$$\xi_2^{2L} = F_l(\theta_2) F_r(\theta_2) \left( \frac{s(\theta_2, \theta_1)}{s(\theta_1, \theta_2)} \right) \left( \frac{s(\theta_1, -\theta_2)}{s(-\theta_2, \theta_1)} \right). \hspace{1cm} (5.30)$$

The defining relations for the boundary factors are

$$F_l(\theta_a) = \left( \frac{E_1(\theta_a) - \mathcal{E}_{(v_1)_a} - \xi_a \Delta^{(1)}_1}{E_1(\theta_a) - \mathcal{E}_{(u_1)_a} - \xi_a^{-1} \Delta^{(1)}_1} \right) \hspace{1cm} (5.31)$$

and

$$F_r(\theta_a) = \left( \frac{E_1(\theta_a) - \mathcal{E}_{(u_1)_a} - \xi_a \Delta^{(1)}_1}{E_1(\theta_a) - \mathcal{E}_{(v_1)_a} - \xi_a^{-1} \Delta^{(1)}_1} \right) \hspace{1cm} (5.32)$$

where $a = 1, 2$. Moreover

$$E_1(\theta_a) = \sqrt{Q} + \mathcal{E}_{11} + 2 \cos \theta_a \hspace{1cm} (5.33)$$
and

\[ s(\theta_a, \theta_b) = 1 + \xi_a \xi_b + \xi_a \sqrt{Q}, \quad a \neq b \]  

\[ (5.34) \]

\section*{5.4 \( m \)-particle state}

The generalization to any number \( m \) of pseudo-particles goes along the same lines as before. Therefore, we just present the final results.

The eigenstates are obtained as a product of \( m \) pseudo-particle eigenstates (5.7)

\[ \Psi_m = \sum_{\{k_1, \ldots, k_m\} \in \{k_{i+1} < k_{i+1}\}} A(k_1, \ldots, k_m) |\Omega(k_1, \ldots, k_m)\rangle \]  

\[ (5.35) \]

where

\[ |\Omega(k_1, \ldots, k_m)\rangle \]

\[ = \sum_{\{i_1, \ldots, i_m\} = -1} \epsilon(i_1) \cdots \epsilon(i_m) q^{\frac{i_1 + \cdots + i_m + m}{2}} |k_1(i_1, -i_1), \ldots, k_m(i_m, -i_m)\rangle \]  

\[ (5.36) \]

with the signs \( \epsilon(-1) = \epsilon(0) = 1 \) and \( \epsilon(1) = -1 \). The energy eigenvalues are given by the sum of single pseudo-particle energies

\[ E_m = E_{11} + \sum_{a=1}^{m} \left( \sqrt{Q} + \xi_a + \xi_a^{-1} \right), \]

\[ (5.37) \]

where \( m \) ranges from 0 to \( L/2 \), and the corresponding Bethe ansatz equations depend on the phase shift of two pseudo-particles and on the boundary factors:

\[ \xi_{sa}^{2L} = F_1(\theta_a) F_r(\theta_a) \prod_{\substack{b=1 \\text{ to } m \\text{ to } 5.1 \\text{ in } L-\text{sites}}} \frac{s(\theta_a, \theta_b)}{s(\theta_b, \theta_a)} \left( \frac{s(\theta_a, -\theta_a)}{s(-\theta_a, \theta_b)} \right) \]

\[ a = 1, 2, \ldots, m. \]  

\[ (5.38) \]

\section*{5.5 Other reference states}

We already known from [23] that to obtain the whole spectrum of the Hamiltonian we have to consider additional reference states. This has to be done for each different boundary eigenvalues \( \mathcal{E}_{\sigma \tau} \). As a result of that, we must have as many as reference states and consequently Bethe ansatz equations as boundary eigenvalues.

In principle, we have nine boundary eigenvalues \( \mathcal{E}_{\sigma \tau} \). If one choose one reference state for each boundary eigenvalues (e.g the first state of each block of 5.1 extended to \( L \)-sites) and proceed along the
same lines as the previous subsection, we obtain nine eigenvalues expressions

\[ E_{m}^{(\sigma,\tau)} = \mathcal{E}_{\sigma\tau} + \sum_{a=1}^{m} \sqrt{Q} + \xi_a + \xi_a^{-1}, \quad (5.39) \]

as well as its associated Bethe ansatz equations

\[ \xi_{sa}^{2L} = F_{l}^{(\sigma,\tau)}(\theta_a)F_{r}^{(\sigma,\tau)}(\theta_a) \prod_{b=1}^{m} \left( \frac{s(\theta_a,\theta_b)}{s(\theta_b,\theta_a)} \right) \left( \frac{s(\theta_b,-\theta_a)}{s(-\theta_a,\theta_b)} \right), \quad (5.40) \]

with

\[ F_{l}^{(\sigma,\tau)}(\theta_a) = \left( \frac{E_{1}^{(\sigma,\tau)}(\theta_a) - \mathcal{E}_{(v_\sigma)i,\tau} - \xi_a \Delta_{l}^{(\sigma)}}{E_{1}^{(\sigma,\tau)}(\theta_a) - \mathcal{E}_{(v_\sigma)i,\tau} - \xi_a^{-1} \Delta_{l}^{(\sigma)}} \right), \quad (5.41) \]

and

\[ F_{r}^{(\sigma,\tau)}(\theta_a) = \left( \frac{E_{1}^{(\sigma,\tau)}(\theta_a) - \mathcal{E}_{(u_\tau)i} - \xi_a \Delta_{r}^{(\sigma)}}{E_{1}^{(\sigma,\tau)}(\theta_a) - \mathcal{E}_{(u_\tau)i} - \xi_a^{-1} \Delta_{r}^{(\sigma)}} \right), \quad (5.42) \]

where

\[ E_{1}^{(\sigma,\tau)}(\theta_a) = \sqrt{Q} + \mathcal{E}_{\sigma\tau} + 2 \cos \theta_a \quad (5.43) \]

The \( \Delta_{[l,r]}^{(\sigma)} \) means that we are considering the reference states listed in (5.1), extended to \( L \)-sites and identified by \( \mathcal{E}_{\sigma1} \). Explicitly, the reference state \(|++\cdots++\rangle\): \( \mathcal{E}_{21} \) (ferromagnetic), the reference state \(|0\cdots++\rangle\): \( \mathcal{E}_{31} \)

\[ \Delta_{l}^{(2)} = -(\mathcal{E}_{31} - \mathcal{E}_{21})q - (\mathcal{E}_{11} - \mathcal{E}_{21})q^{-1}, \quad \Delta_{r}^{(2)} = -(\mathcal{E}_{11} - \mathcal{E}_{12})q - (\mathcal{E}_{13} - \mathcal{E}_{12})q^{-1} \quad (5.44) \]

and the reference state \(|-+\cdots++\rangle\): \( \mathcal{E}_{31} \)

\[ \Delta_{l}^{(3)} = (\mathcal{E}_{21} - \mathcal{E}_{31}) - (\mathcal{E}_{11} - \mathcal{E}_{31})q^{-1}, \quad \Delta_{r}^{(3)} = (\mathcal{E}_{12} - \mathcal{E}_{13}) - (\mathcal{E}_{11} - \mathcal{E}_{13})q. \quad (5.45) \]

The remaining index \((v_\sigma)i\) are defined by \( v_2 = (3,1,1), v_3 = (1,2,1) \) and the \((u_\tau)i\) are given by \( u_2 = (1,1,3), u_3 = (1,2,1) \).

It is not all because we also have to consider the remained reference states (5.2) and (5.3). However, we can see from (5.4) to (5.6) that most of these equations degenerate into each other, resulting in four equations for each integrable boundary.
6 Conclusion

In this paper the reflection $K$ - matrix solution of the spin-1 $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb vertex model is presented. This step paves the way for the analysis of the corresponding open Hamiltonian for which we present what we hope to be the most general set of integrable boundary terms. We obtained the spectrum of the spin-1 $\mathcal{U}_q[osp(1|2)]$ Temperley-Lieb spin chain with diagonal open boundary conditions. We have identified that this model has large number of possible reference states. By selecting a small subset of these states, we manage to obtain four eigenvalue expressions and its associated Bethe ansatz equations by means of a generalization of the coordinate Bethe ansatz, already used in [23] and we also leave the problem of counting of the spectral multiplicities as an open question.

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