Canonical Forms of Matrices Determining Analytical Manifolds

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Abstract

In this paper many classes of sets of matrices with entries in $F$ ($F \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$) are introduced. Each class with the corresponding topology determines real analytical, complex or symplectic manifold for $F = \mathbb{R}$, $F = \mathbb{C}$ or $F = \mathbb{H}$ respectively. Any such family is called to be a set of canonical forms of matrices. The construction of such canonical forms of matrices is introduced inductively. First basic canonical forms are introduced, and then two operations for obtaining new canonical forms by using the old canonical forms are introduced. All such manifolds have the property that each of them can be decomposed into cells of type $F^i$.

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1 Introduction

In the recent paper [6] are introduced two different classes of canonical forms of matrices over a field $F$. It easily can be generalized for the quaternions $\mathbb{H}$. Note that in [6] and also in this paper the term *matrix in canonical form* does not mean any reduction of a given matrix in a special form, but only means that the corresponding matrix belongs to a given family or set of matrices. The term *canonical form* comes from the example at the end of this section. According to the corresponding topology of the classes of canonical forms of matrices (cfm) in this paper are obtained real analytical (for $F = \mathbb{R}$), complex (for $F = \mathbb{C}$) and symplectic (for $F = \mathbb{H}$) manifolds. In all cases are obtained manifolds such that in special cases are obtained the Grassmann manifolds. In the present paper will be described inductively a large class of cfm yielding to analytical manifolds.

Each set of given cfm consists of $n \times m$ matrices with the following properties.

1°. The first property tells about some restrictions concerning the matrices in the given canonical form (cf), about the first zero coordinates of each vector row. This property depends on the choice of the cf.

2°. The second property is fixed for all cf and it states that any two different vector rows are orthogonal.
3°. The third property is also fixed for all cf and states that any vector row in cf must have norm 1 and the first nonzero coordinate is positive real number.

Thus in order to define a class of cf it is sufficient to specify the property 1°. Note that alternatively the vector rows also can be considered as vectors from $RP^{m-1}$, $CP^{m-1}$ or $HP^{m-1}$ and then the property 3° should be omitted. Note also that $m \geq n$ according to 2°. Indeed, if $m < n$, then that set of cfm is empty set.

We finish the introduction by the basic example concerning the Grassmann manifolds.

Example. Let us consider the set of $n \times m$ matrices ($n \leq m$) such that
1°. If $a_1, \cdots, a_n \in F^m$ are vector rows, then

$$0 \leq t(a_1) < t(a_2) < \cdots < t(a_n) < m,$$

where $t(a_i)$ ($i = 1, \cdots, n$), denotes the number of the first zero coordinates of the vector $a_i$. This set of matrices together with the fixed properties 2° and 3° for cf determines the Grassmann manifold $G_{n,m}(F)$ with the known topology, consisting of all $n$-dimensional subspaces of $F^m$ generated by the vector rows.

2 Construction of different classes of cfm and the corresponding analytical manifolds

First we determine basic canonical forms as unit $n \times n$ matrices for arbitrary $n$.

Further we introduce two operations over the cfm, such that the result is a new canonical form. After introducing the topology on the new cfm we obtain a new analytical manifold (real, complex and symplectic).

i) Inner sum.

Let $C_1, C_2, \cdots, C_p$ be $p$ matrices in cf, not necessary in cf of the same type. Assume that $C_i$ is an $n_i \times m$ matrix ($1 \leq i \leq p$) and let $C$ be the following $n \times m$ block matrix

$$C = \begin{bmatrix}
C_1 \\
C_2 \\
\vdots \\
C_p
\end{bmatrix},$$

where $n = n_1 + \cdots + n_p$. If any two different vector rows of $C$ are orthogonal, we say that $C$ is in cf called inner sum of the cfm $C_1, \cdots, C_p$. Thus the inner sum of given $p$ matrices does not always exist, while the set of matrices of whole cfm is nonempty for sufficiently large $m$. The term "inner" comes from the orthogonality condition and we can say only "sum" of the cfm. The new canonical form will be denoted symbolically by $C_1 + \cdots + C_p$. Note that if
we neglect the orthogonality condition we obtain the Cartesian product of cfm $C_1, \cdots, C_p$.

Now we introduce the topology on the new cfm as follows. Let us denote by $\{C_1\}, \ldots, \{C_p\}$ the sets of all matrices in given cf, and assume that the corresponding topologies are known. Then the set of all matrices in the new cf $\{C_1 + \ldots + C_p\}$ is a subset of the Cartesian topology space $\{C_1\} \times \ldots \times \{C_p\}$.

Thus the topology of the new cf $\{C_1 + \ldots + C_p\}$ we define to be the relative topology with respect to the Cartesian topological space $\{C_1\} \times \ldots \times \{C_p\}$.

If $\tau : \{1, \ldots, p\} \to \{1, \ldots, p\}$ is a permutation, then obviously the cfm $C_1 + \cdots + C_p$ and $C_{\tau(1)} + \cdots + C_{\tau(p)}$ determine homeomorphic spaces.

ii) Spreading of a cfm onto a space $\Sigma$.

Let $\{C\}$ be a set of all quadratic $n \times n$ matrices from a given canonical form. Let $k > n$ and $\Sigma$ be a subspace of $F^k$ such that $n \leq \dim \Sigma \leq k$. Specially, $\Sigma$ can be the total space $F^k$.

Spreading $S_{\Sigma}(C)$ over $\Sigma$ is defined to be the set of matrices $CX$ where $X$ is an $n \times k$ matrix in canonical form of the Grassmann manifold $G_{n,k}$ (see example in section 1) such that the row vectors of $X$ belong to $\Sigma$, i.e. $X \in G_{n,\Sigma}$. Applying this spreading for each matrix $C$ from the given cfm we obtain the total set of matrices of the spread canonical form. Notice that $C_1 X_1 = C_2 X_2$ implies $C_1 = C_2$ and $X_1 = X_2$ (see [6]).

The topology on the set of matrices in the spread cf we define inductively. The topology of the set $\{C\}$ of quadratic $n \times n$ matrices must be the topology of inner sum. So assume that it is obtained as a sum $C_1 + \ldots + C_p$. Then the topology of the set of the spread cfm is defined to be the topology of the inner sum of the spread cfm of $C_1, \ldots, C_p$ on $\Sigma$ separately. Hence it is sufficient to determine the topology of each spread cfm $\{C_i\}$ over $\Sigma$. Continuing this process of decreasing the number $n$, we should finally determine the topology of spread cfm of the unit matrix (i.e. basic matrix). But, the spreading of the unit matrix is the Grassmann manifold, whose topology is well known.

Assuming that $M$ is an analytical manifold, then the topology of the spreading $S_{\Sigma}(C)$ is such that it is an analytical manifold which is bundle over $G_{n,\Sigma}$ with projection $\pi : S_{\Sigma}(C) \to G_{n,\Sigma}$ defined by: $\pi(CX)$ is the subspace of $\Sigma$ generated by the vector rows of $CX$, i.e. of $X$, and the fiber is the topology space $M$ induced via the set of matrices $\{C\}$ in initial cf. Note that a spreading of a cf can be done only on a set of square matrices in cf. Thus if $\{C\}$ is a set of $n \times m$ matrices in cf, then we can sum with one or more submatrices in cf (example in section 1) determining the Grassmann manifolds in order to obtain cf on $m \times m$ matrices. Specially we can sum with $m-n$ vector rows or we can sum with a set of $(m-n) \times m$ matrices from the Grassmann canonical form.

Now having in mind the topology of any cfm, we are able to prove that each canonical form of $n \times m$ matrices $\{C\}$ yields to an analytical manifold. The coordinates of any such analytical manifold will be constructed in such a way that for any $n \times n$ nonsingular submatrix will be constructed a coordinate neighborhood, like for the standard coordinate neighborhoods for the Grassmann manifolds.
First note that any basic cf of unit \( n \times n \) vectors determines a 0-dimensional manifold, i.e. a point. The first operation must be spreading onto \( F^k \) and hence in the first step we obtain the Grassmann manifolds \( G_{n,k} \) which are analytical (real, complex and symplectic) manifolds.

Further, suppose that a new cfm is obtained via an inner sum of matrices \( C_1, \ldots, C_p \) in the corresponding cf. If these \( p \) cfm determine analytical manifolds, then the set of new cfm is also an analytical manifold. Indeed, let \( G_{n_1, \ldots, n_p, m}(F) \) be the manifold consisting of \((p+1)\)-tuples \((\Pi_1, \Pi_2, \ldots, \Pi_p, \Pi_{p+1})\) of orthogonal subspaces of \( F^m \) where \( \dim \Pi_i = n_i \) for \( i = 1, 2, \ldots, p \) and \( \dim \Pi_{p+1} = m - n_1 - \cdots - n_p \). Note that this is a flag manifold of included subspaces \( V_1 = \Pi_1, V_2 = \Pi_1 + \Pi_2, V_3 = \Pi_1 + \Pi_2 + \Pi_3, \ldots \). Thus \( G_{n_1, \ldots, n_p, m}(F) \) is an analytical manifold. Let us denote by \( M_i \) the analytical manifold which is the fiber of the projection of the matrices \( \{C_i\} \) on the Grassmann manifold \( G_{n_1, m} \). Then the new cf of sum of cfm determines a space which is a bundle with base \( G_{n_1, \ldots, n_p, m}(F) \) and fiber \( M_1 \times \cdots \times M_p \). The projection \( \pi \) is given by

\[
\pi \left( \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix} \right) = (\Pi_1, \Pi_2, \ldots, \Pi_p, \Pi_{p+1}),
\]

where \( \Pi_i \) is generated by the vector rows of \( C_1 \), \( \Pi_2 \) is generated by the vector rows of \( C_2, \ldots \) and \( \Pi_{p+1} \) is the orthogonal complement of \( \Pi_1 + \cdots + \Pi_p \) into \( F^m \). Thus this space of sum of cfm is also an analytical manifold. The coordinate neighborhoods of the new cf can be constructed as follows. Let \( C' \) be any nonsingular \( n \times n \) submatrix of \( C \). Then there exist \( p \) submatrices: \( C'_1 \) submatrix of \( C_1 \) of order \( n_1 \times n_1, \ldots, C'_p \) submatrix of \( C_p \) of order \( n_p \times n_p \), where \( n = n_1 + \cdots + n_p \), such that

1. these matrices are submatrices of \( C' \),
2. the columns of \( C'_1, \ldots, C'_p \) are distinct,
3. the matrices \( C'_1, \ldots, C'_p \) are non-singular, and
4. by deleting the rows and columns of the submatrices \( C'_1, C'_2, \ldots, C'_i \), the rest \((n_{i+1} + \cdots + n_p) \times (n_{i+1} + \cdots + n_p)\) submatrix of \( C' \) is nonsingular \( (i = 1, 2, \ldots, p - 1) \).

Note that such a choice of submatrices \( C'_1, C'_2, \ldots, C'_p \) is possible because by generalization of the Laplace decomposition of the determinants it holds

\[
\det C' = \sum \pm \det D'_1 \cdot \det D'_2 \cdots \det D'_p
\]

where \( D'_j \) is \( n_j \times n_j \) submatrix of \( C' \) and submatrix of \( C_j \) and such that the columns of \( D'_1, \ldots, D'_p \) are distinct.

In the next step we choose the coordinate neighborhoods as follows. All \( n(m - n) \) elements which do not belong to \( C' \) may be changed to be close to
the corresponding elements of \( C \). The same choice is for the elements which are simultaneously in the same row as \( C'_i \) and in the same column as \( C'_j \) for \( i > j \), i.e. they may be chosen to be close to the corresponding elements of \( C' \). Also according to the inductive assumption the elements of \( C'_1, C'_2, \ldots, C'_p \) can be chosen in the corresponding coordinate neighborhoods which they induce respectively on \( \{ C_1 \}, \{ C_2 \}, \ldots, \{ C_p \} \). Finally according to the properties 1. - 4. we note that the elements which belong simultaneously in the same row as \( C'_i \) and in the same column as \( C'_j \) for \( i < j \) can uniquely be determined such that the row vectors of \( C \) are orthogonal. Hence we showed that the chosen matrix \( C \) can be covered by a coordinate neighborhood of \( C_1 + \cdots + C_p \). The Jacobi matrices for the described coordinate neighborhoods are analytical functions, because of the inductive assumptions for \( C_1, \ldots, C_p \) and the analytical solutions of linear algebraic systems.

Next we should show how we can associate a coordinate neighborhood for any nonsingular \( n \times n \) submatrix \( C' \) of spread \( n \times k \) matrix \( S_\Sigma(C) \). Without loss of generality we can suppose that \( C \) is a square \( n \times n \) matrix in former cf and by inductive assumption it can be covered with coordinate neighborhoods with analytical elements of the Jacobi matrices. Since \( \{ S_\Sigma(C) \} \) is a bundle with base \( G_{n,k} \) and according to the standard construction for the coordinates induced by any nonsingular \( n \times n \) submatrix and the fact that the fiber is an analytical manifold by the inductive assumption, we obtain the required covering.

Note that inductively it follows that all the manifolds obtained via this method of cf are compact. Finally we can resume the previous results in the following two theorems.

**Theorem 2.1.** The set of all matrices in a chosen canonical form with the introduced topology is a compact analytical (real, complex or symplectic) manifold.

**Theorem 2.2.** Any manifold determined via a cf of \( n \times m \) matrices is a bundle over the base manifold \( G_{n,m}(F) \) or \( G_{n_1,n_2,\ldots,n_p,m}(F) \).

### 3 Some examples

In this section will be considered some examples of cf.

**Example 1.** Assume that \( n_1 = 1, \ldots, n_p = 1 \) (\( n = p \)) and let \( s_1, \ldots, s_n \in \{0, \ldots, m-1\} \) be given numbers. Then each of the cf of \( 1 \times m \) matrices belongs to the space \( FP^{n-1} \). For any \( i \), let \( \Sigma_i \) be the subspace generated by \( e_{s_i+1}, e_{s_i+2}, \ldots, e_m \). Then the inner sum of cf can be described as a set of \( n \) orthogonal projective vectors, such that the first starts with at least \( s_1 \) zeros, the second starts with at least \( s_2 \) zeros and so on. The dimension of this manifold is

\[
mn - \frac{n(n+1)}{2} - s_1 - s_2 - \cdots - s_n.
\]
Let us consider the special case $m = n$. Then there exists a permutation $\tau$ such that $s_{\tau(i)} < i$, because in opposite case the corresponding matrix would not be orthogonal. Hence without loss of generality we assume that $s_i < i$, $(i = 1, 2, \ldots, n)$. Hence there are $n!$ such manifolds. Some of them are homeomorphic. Note that specially, if $s_1 = s_2 = \cdots = s_n = 0$, we obtain the full flag manifold.

**Example 2.** Now let us consider the following example. Suppose that $m_1, \ldots, m_p$ are fixed positive integers such that $m = m_1 + \cdots + m_p$. Let

- $\Sigma_1$ be the subspace generated by $e_1, \ldots, e_{m_1}$,
- $\Sigma_2$ be the subspace generated by $e_{m_1+1}, \ldots, e_{m_1+m_2}$,
- $\Sigma_3$ be the subspace generated by $e_{m_1+m_2+1}, \ldots, e_{m_1+m_2+m_3}$,
- \[ \cdots \]
- $\Sigma_p$ be the subspace generated by $e_{m_1+\cdots+m_{p-1}+1}, \ldots, e_m$.

Then the cf which is an inner sum has the following form

$$
\begin{bmatrix}
  C_1 & 0 & 0 & \cdots & 0 \\
  0 & C_2 & 0 & \cdots & 0 \\
  0 & 0 & C_3 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & C_p
\end{bmatrix}
$$

as a block matrix of type $(n_1 + n_2 + \cdots + n_p) \times (m_1 + m_2 + \cdots + m_p)$. Obviously the induced analytical manifold is the Cartesian product $M_1 \times M_2 \times \cdots \times M_p$ where $M_i$ is the analytical manifold induced by the $i$-th cf of type $C_i$ on $n_i \times m_i$ matrix ($n_i \leq m_i$).

**Example 3.** Let $C$ be an inner sum of $C_1, \ldots, C_p$ where $C_i$ is an $n_i \times m_i$ matrix, where $n_1 + n_2 + \cdots + n_p = n$ and $m_1 + \cdots + m_p = m = n$. Since $n_i \leq m_i$, it must be $n_i = m_i$ for $i = 1, \ldots, p$, i.e.

$$
C = \begin{bmatrix}
  C_1 & 0 & 0 & \cdots & 0 \\
  0 & C_2 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & C_p
\end{bmatrix}
$$

where $C_i$ is $n_i \times n_i$ matrix in cf. Since $\det C_i \neq 0$ for $i = 1, 2, \ldots, p$, one can verify that the spread $n \times m'$ matrix $C'$ is in cf if and only if $C'$ decomposes into block matrices

$$
C' = \begin{bmatrix}
  C_{11} & C_{12} & \cdots & C_{1p} \\
  C_{21} & C_{22} & \cdots & C_{2p} \\
  \vdots & \vdots & \ddots & \vdots \\
  C_{p1} & C_{p2} & \cdots & C_{pp}
\end{bmatrix}
$$
where $C_{ij}$ is an $n_i \times m'_j$ matrix, where $m'_1, \cdots, m'_p$ are not fixed but $m'_1 + \cdots + m'_p = m'$, such that
i) $C_{ij} = 0$ for $i > j$,
ii) $\text{rank}(C_{ii}) = n_i$,
iii) the row vectors of $C'$ are orthogonal,
iv) the first non-zero coordinate of each vector is a positive real number.

In special case if $\{C_1, \cdots, C_p\}$ are the $\text{cf}$ of full flag manifolds, we obtain the manifold described in [6] sect.3.

Example 4. Let be given $p$ positive integers $n_1, n_2, \cdots, n_p$ and let $n_1 + n_2 + \cdots + n_p = n \leq m$. We consider a set of linearly independent vectors

$$a_{11}, a_{12}, \cdots, a_{1n_1}, a_{21}, a_{22}, \cdots, a_{2n_1}, \cdots, a_{ip}, a_{ip+1}, \cdots, a_{np},$$

of $F^m$ and we denote the matrix with these $n$ row-vectors by $A$. The matrix $A$ is in a $\text{cf}$ if
i) $t_1 < t_2 < \cdots < t_p$, where $t_i = \min\{t(a_{11}), t(a_{12}), \cdots, t(a_{in_i})\}$ and $t(a)$ denotes the number of the first zero coordinates of $a$,
ii) each two different vectors of these $n$ vectors are orthogonal,
iii) each vector row has norm 1 and the first non-zero coordinate is positive real number.

It is not obvious that this $\text{cf}$ belongs to the $\text{cf}$ of matrices introduced inductively in section 2. One can prove that this set of matrices can be considered as a spreading of $n \times n$ matrices with the same property i), ii) and iii). Thus we should consider the case $m = n$. It is clear that the set of $n \times n$ matrices in $\text{cf}$ is an inner sum of $n_1$ vectors in $F^n$ and set of canonical forms of $(n_2 + \cdots + n_p) \times n$ matrices with parameters $n_2, n_3, \cdots, n_p$, projected on the space generated by the vectors $e_2, e_3, \cdots, e_n$. Hence by induction of $p$ we obtain that the considered $\text{cfm}$ is included in the family of manifolds obtained in section 2.

Let us consider the following special case for $F = R$, $p = 2$, $n_1 = 2$, $n_2 = 1$ and $m = 3$. Then the manifold of canonical vectors consists of the following cells

$$C_1 = \begin{bmatrix} x & * & * \\ y & * & * \\ 0 & * & * \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & * & * \\ 1 & 0 & 0 \\ 0 & * & * \end{bmatrix},$$

where $x, y > 0$. The cell $C_1$ is homeomorphic to $R \times S^1$ because for fixed ratio $\lambda = x/y \in R^+$ it is homeomorphic to $S^1$. The cells $C_2$ and $C_3$ are homeomorphic to $S^1$. Thus the Euler characteristic of the manifold is $\chi = 0$. It can be described such that each point consists of two orthogonal lines $p$ and $q$ through the coordinate origin in $R^3$ such that $q$ lies in the $yz$-plane. The third line which is orthogonal to $p$ and $q$ is uniquely determined by $p$ and $q$. This manifold is homeomorphic to the Klein’s bottle. Note that if we consider the complex and quaternionic cases, then we obtain complex and symplectic manifolds - analogs of the Klein’s bottle.
4 Decomposition into cells

In this section we show the existence of cell decomposition which is analogous to the Schubert’s cell decomposition of the Grassmann manifolds.

By the construction of cf described in section 2 we obtain a large class of compact analytical manifolds, three manifolds for each cfm: real, complex and symplectic. All these manifolds have the following property.

**Theorem 4.1.** All the manifolds obtained via cfm are such that they can be decomposed into disjoint cells of type $F^i$.

**Proof.** Note that the base cf determine 0-dimensional manifolds and each of them is a point, i.e. $F^0$.

First let us prove that if the set of cfm $\{C\}$ satisfies the property of the theorem 4.1, then $\{S_\Sigma(C)\}$ satisfies that property also. Since $\{C\}$ is a set of quadratic matrices, $\{S_\Sigma(C)\}$ consists of all matrices of type $CX$ where $X$ is matrix of the Grassmann manifolds, and the representation is unique. Hence we obtain that the cells of $\{CX\}$ are products of the cells of $\{C\}$ and the cells of $\{X\}$. The cells of $\{C\}$ are of type $F^i$ because of the inductive assumption and the cells of the Grassmann manifolds $G_{n,\Sigma}$ are also of that type and in this case the proof is finished. Indeed, $G_{n,\Sigma}$ can be decomposed into $\binom{r}{n}$ cells of type $F^i$, where $r = \dim \Sigma$.

Suppose that the manifolds determined by the cf $\{C_i\}$ satisfy the property in theorem 4.1. Then we will show that the manifold induced by the sum $C_1 + \cdots + C_p$ also satisfies that property. This reduces to the special case when $C_1, \ldots, C_p$ are spreadings over the corresponding Grassmann manifolds with bases $M_i$ which are quadratic matrices. Indeed this manifold is a bundle over the base $G_{n_1,n_2,\ldots,n_p,m}(F)$ and the fiber $M_1 \times M_2 \times \cdots \times M_p$ and moreover the new manifold is equivalent (but not necessary homeomorphic) to the Cartesian product

$$G_{n_1,n_2,\ldots,n_p,m}(F) \times (M_1 \times M_2 \times \cdots \times M_p).$$

It follows from the fact that the set of matrices for the spreading of quadratic $n_1 \times n_1$ matrices is the product (which is unique) of matrices of $M_i$ and the Grassmann manifold $G_{n_i,m}$. Since $M_i$ satisfies the property in theorem 4.1, and the base manifold $G_{n_1,n_2,\ldots,n_p,m}(F)$ can be decomposed into

$$\frac{m!}{n_1!n_2!\cdots n_p!(m - n_1 - \cdots - n_p)!}$$

cells of type $F^i$, we obtain that the inner sum $C_1 + \cdots + C_p$ also satisfies the property of the theorem 4.1.

This completes the proof of the theorem and moreover it gives a method for finding all of the cells. $\square$

In the paper [10] is given a decomposition of the full flag manifold $G_n(F)$ into $n!$ cells of type $F^i$. Indeed, the following theorem is proved in [10].
Theorem 4.2. The manifold $G_n(F)$ is a disjoint union of $n!$ disjoint cells, such that for each sequence $(i_1, i_2, \cdots, i_{n-1})$, for $0 \leq i_j \leq j$ and $1 \leq j \leq n-1$, there exists a cell $C_{i_1, \cdots, i_{n-1}}$ which is homeomorphic to $F^{i_1} \times F^{i_2} \times \cdots \times F^{i_{n-1}}$.

Note that the theorem 4.1 tells nothing about real manifolds because each real manifold can be decomposed into cells of type $R^i$. But there are complex manifolds which can not be decomposed into disjoint cells of type $C^i$. For example if the torus $T = S^1 \times S^1$ can be decomposed into some cells of type $C^1 = R^2$ and $C^0 = R^0$, then the Euler characteristic is a sum of such cells and it is positive number, which is a contradiction.

The cohomology modules for any manifolds constructed via the canonical forms in section 2 can be found easily. Indeed, we know the cohomology modules for the manifolds $G_{n,m}(F)$ and $G_{n_1, n_2, \cdots, n_m}(F)$. Using the Leray-Hirsch theorem [1] we can find step by step all the cohomology modules for any such manifold. Indeed, according to the theorem 2.2, we have the following theorem.

Theorem 4.3. Let us denote by $P_t(M)$ the polynomial

\[
\dim H^0(M, R) + t \dim H^1(M, R) + t^2 \dim H^2(M, R) + \cdots + t^s \dim H^s(M, R),
\]

for a manifold $M$, where $s = \dim M$,

a) If $M$ is a real analytical manifold obtained via cfm for $F = R$, then $P_t(M)$ is a product of polynomials of types

\[
P_t(G_{p,p+q}) = \frac{(1-t)(1-t^2) \cdots (1-t^{p+q})}{(1-t)(1-t^2) \cdots (1-t^p)(1-t)(1-t^2) \cdots (1-t^q)};
\]

b) If $M$ is a complex manifold obtained via cfm for $F = C$, then $P_t(M)$ is a product of polynomials of types

\[
P_t(G_{p,p+q}) = \frac{(1-t^2)(1-t^4) \cdots (1-t^{2(p+q)})}{(1-t^2)(1-t^4) \cdots (1-t^p)(1-t^2)(1-t^4) \cdots (1-t^q)};
\]

c) If $M$ is a symplectic manifold obtained via cfm for $F = H$, then $P_t(M)$ is a product of polynomials of types

\[
P_t(G_{p,p+q}) = \frac{(1-t^4)(1-t^8) \cdots (1-t^{4(p+q)})}{(1-t^4)(1-t^8) \cdots (1-t^p)(1-t^4)(1-t^8) \cdots (1-t^q)}.
\]

5 About a further generalization

Now will be presented a possible generalization for $F = C$, which can be a subject of further research. We give the definition and the basic results about symmetric products of manifolds.

Let $M$ be an arbitrary set and $m$ be a positive integer. In the Cartesian product $M^m$ we define a relation $\approx$ such that $(x_1, \cdots, x_m) \approx (y_1, \cdots, y_m)$ iff
$y_1, y_2, \ldots, y_m$ is an arbitrary permutation of $x_1, x_2, \ldots, x_m$. This is equivalence relation and the equivalence class represented by $(x_1, \cdots, x_m)$ is denoted by $(x_1, \cdots, x_m) \approx$ and the quotient space $M^m \approx$ is called symmetric product of $M$ and is denoted by $M^{(m)}$.

If $M$ is a topological space, then the quotient space $M^{(m)}$ is also a topological space. The space $M^{(m)}$ is introduced quite early [4], but mainly it was studied in [12]. The space $(R^n)^{(m)}$ is a manifold only for $n = 2$ [12]. If $n = 2$, then $(R^2)^{(m)} = C^{(m)}$ is homeomorphic to $C^m$. Indeed, using that $C$ is algebraically closed field, it is obvious that the mapping $\varphi : C^{(m)} \to C^m$ defined by

$$\varphi((z_1, \cdots, z_m) \approx) = (\sigma_1, \sigma_2, \cdots, \sigma_m)$$

is a bijection, where $\sigma_i (1 \leq i \leq m)$ is the $i$-th symmetric function of $z_1, \cdots, z_m$, i.e.

$$\sigma_i(z_1, \cdots, z_m) = \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq m} z_{j_1} \cdot z_{j_2} \cdots z_{j_i}.$$ 

The mapping $\varphi$ is also a homeomorphism. Moreover, $M^{(m)}$ is a complex manifold if $M$ is 1-dimensional complex manifold [7]. For example, if $M$ is a sphere, i.e. the complex manifold $CP^1$, then $M^{(m)}$ is the projective complex space $CP^m$. Using the permutation products it is easy to see how $M^{(m)} = CP^m$ decomposes into disjoint cells $C^0, C^1, \cdots, C^m$. Let $\xi \in M$. Then we define $((x_1, \cdots, x_m) \approx) \in M_i$ if exactly $i$ of the elements $x_1, \cdots, x_m$ are equal to $\xi$. Thus

$$M^{(m)} = M_0 \cup M_1 \cup \cdots \cup M_m = (M \setminus \{\xi\})^{(m)} \cup (M \setminus \{\xi\})^{(m-1)} \cup \cdots \cup (M \setminus \{\xi\})^{(0)} = C^{(m)} \cup C^{(m-1)} \cup \cdots \cup C^{(0)}.$$ 

Some recent results about symmetric products of manifolds are obtained in [11, 2]. This theory about symmetric products has an important role in the theory of the topological commutative vector valued groups [8, 9, 5].

We mentioned in the section 1 that the property $3^0$ can be omitted by assuming that the row vectors of the cfm are projective vectors, i.e. they are elements of $CP^{m-1}$. Indeed, for any such vector $v = (v_1, \cdots, v_m) \in CP^m$ we join a polynomial

$$P(z) = v_m z^{m-1} + v_{m-1} z^{m-2} + \cdots + v_1$$

and hence its complex roots $(z_1, \cdots, z_{m-1}) \approx$ up to a permutation. Here $z_1, \cdots, z_{m-1} \in C^* = C \cup \{\infty\}$ such that if $v_m = v_{m-1} = \cdots = v_{m-s+1} = 0$ and $v_{m-s} \neq 0$, then exactly $s$ of the roots are equal to $\infty$.

Now instead of the complex manifold $S^2 = C \cup \{\infty\}$ we should consider an arbitrary 1-dimensional complex manifold $M$. Then any $1 \times m$ canonical form of matrices induces the complex manifold $M^{(m-1)}$ and it can be considered as a projective space over $M$. The idea for generalization is the following. For any
cfm the vector rows should be considered as elements of $M^{(m-1)}$. The zero initial values of the vector rows correspond to the multiplicity of a chosen point $\xi$ on the chosen 2-dimensional surface. If one manage to determine the corresponding orthogonality conditions, then he/she will obtain a complex manifold which corresponds to the considered cfm and the basic 1-dimensional complex manifold $M$. At this moment we know only the projective space over given 2-dimensional real surface, which is the symmetric product of the surface, but do not know the Grassmann manifold over given 2-dimensional real surface.

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