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Computational experiment on the numerical solution of some inverse problems of mathematical physics

V I Vasil'ev, A M Kardashevsky and PV Sivtsev
North-Eastern Federal University, 58 Belinskogo, 677000 Yakutsk, Russia

Email: vasvasil@mail.ru

Abstract. In this article the computational experiment on the numerical solution of the most popular linear inverse problems for equations of mathematical physics are presented. The discretization of retrospective inverse problem for parabolic equation is performed using difference scheme with non-positive weight multiplier. Similar difference scheme is also used for the numerical solution of Cauchy problem for two-dimensional Laplace equation. The results of computational experiment, performed on model problems with exact solution, including ones with randomly perturbed input data are presented and discussed.

1. Introduction
We have sufficiently wide class of inverse problems for equations with partial derivative, which are recently used for mathematical description of various processes. The important feature of inverse problems arising from the processing of the experimental results is that the initial information is defined approximately, because used measurement devices have a certain level of error. Therefore, methods of solution of these inverse problems must be resistant to small errors of input data.

Inverse problems of mathematical physics, as a rule, belong to the class of ill-posed problems in the classical sense. The ill-posedness is primarily caused by lack of continuous dependence of solutions on the initial data. In this case it is necessary to narrow the class of feasible solutions and to use a special regularization procedure to find sustainable solutions. In addition to the existence and stability of solutions a question of uniqueness is also relevant, which study answers the question of whether available information is sufficient for unique identification of the desired characteristics of the studied object or process.

Construction of mathematical models of various processes leading to inverse problems of mathematical physics, their theoretical study and design of numerical methods are summarized in the monographs [1-3]. Retrospective inverse heat transfer problem is one of the most popular inverse problems of mathematical physics, for which there is no continuous dependence of solutions on the initial data. However, while narrowing the class of solutions we achieve stability - this problem is conditionally correct [1]. In paper [4] a method for solution of retrospective inverse problem for a parabolic equation is proposed. The Cauchy problem for an elliptic equation, as shown by Hadamard, is the first example of incorrect inverse problem, which is unstable in the classical sense. In work [5] as it is proved by M.M. Lavrentiev [6], this problem is conventionally well-posed in the class of sufficiently smooth functions.
2. Retrospective heat transfer problem

In rectangular \((a, b) \times [0, T)\) we search the function \(u(x, t)\), which is a solution of one-dimensional parabolic equation of second order

\[
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x} \left( k(x) \frac{\partial u}{\partial x} \right) = 0, \quad x \in (a, b) = \Omega, \quad 0 \leq t < T. \tag{1}
\]

Suppose that homogeneous Dirichlet boundary conditions are given

\[
u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t < T. \tag{2}
\]

We formulate a retrospective inverse problem, where the required function at the final moment of time is set instead of the initial condition

\[
u(x, T) = \varphi(x), \quad x \in [a, b] = \Omega. \tag{3}
\]

Let the coefficient \(k(x)\) from differential equation be sufficiently smooth, positive and bounded function, i.e. it satisfies the condition \(0 < k_1 \leq k(x) \leq k_2 < \infty\). Then the assigned problem (1)–(3) in the class of bounded functions is conditionally well-posed [1]. It should be noted that for \(k(x) \equiv 1, a = -\infty, b = \infty\) the equation (1) has automodel solution

\[
u(x, t) = \frac{1}{\sqrt{\beta + 4t}} e^{-(x-x_0)^2/\beta + 4t}, \quad \beta > 0, \tag{4}
\]

where \(x_0 \in \Omega\). This function is sufficiently smooth and limited and can be subsequently used to establish the numerical accuracy of the proposed method.

In domain \(\Omega_T = (a, b) \times [0, T)\) let us introduce a uniform rectangular mesh

\[
\omega_h = \omega_x \times \omega_t,
\]

\[
\omega_x = \{ x_i = a + i h, i = 1, \ldots, N; h = (b-a)/N \},
\]

\[
\omega_t = \{ t_j = j \tau, j = 1, \ldots, M; \tau = T / M \}.
\]

On the set of grid functions \(y \in H\), which are \(y(x) = 0, x \notin \omega_h\), we define a grid operator as

\[
Ay = -\left( a(x)y_x \right)_x, x \in \omega_h
\]

as for example, setting \(a(x) = k(x-0.5h)\).

In finite-dimensional Hilbert grid space \(H\) we introduce inner product and norm as

\[
(y, v) = \sum_{x \in \omega} yvh, \quad \|y\| = \sqrt{(y, y)}.
\]

It is known that [7] in Hilbert space \(H\) the operator \(A\) is self-adjoint, positive definite \((A = A^* > 0)\) and there is two-sided inequality \(8 k_1 / l^2 \leq \|A\| \leq 4 k_2 / h^2\), which means it is positive definite and limited. In the rectangular space-time mesh \(\omega_{ht}\) we approximate the problem using implicit difference scheme with weight factor \(\sigma\):

\[
\frac{y_{m+1} - y_m}{\tau} - A \left( (1-\sigma) y_{m+1} + \sigma y_m \right) = 0, m = M-1, \ldots, 1, 0,
\]

\[
y_m(x) = \varphi(x), \quad x \in \partial \omega_h. \tag{6}
\]
Figure 1. Results of the proposed difference method (on the left side) and results from paper [8] (on the right side)

Difference scheme (5)–(6) on solution of non-classical problem (1)–(3) has the first order of approximation by \( \tau \) and the second order by \( h \). For its solution at \( \sigma = 0 \) we have following apriori estimate

\[
y_0 \leq e^{2CM \Phi},
\]

where \( C = 2k_2 \tau / h^2 \) is Courant number of explicit scheme for heat transfer equation, \( M \) is amount of time layers. The estimate (7) implies that the satisfactory results on this scheme can be achieved for not too large time intervals (\( T \)) and not too detailed spatial grids.

Figure 2. Results of the proposed difference method (on the left side) and results from paper [8] (on the right side)

The results of numerical calculations are carried out by using implicit difference scheme (5)–(6) for \( \sigma = 0 \) on problem with the exact automodel solution (4) with following initial data:

\[
k(x) = 1, a = -12, b = 12.
\]

It should be noted that since we perform calculations downward by time, computations are carried out by explicit formulas [8]. Figure 1 and Figure 2 demonstrate the results of calculations obtained respectively for \( T = 1, N = 60, M = 500 \) and \( T = 3, N = 24, M = 300 \). On illustration 1 there is a graph of the exact solution and on 2 there is a graph of numerical solution on the selected spatial grid (on the left side) and similar graphs from paper [8] (on the right side). The graphs show that the accuracy of results obtained by implicit scheme on a fairly rough time-space grids do not concede to the accuracy of results presented in paper [8] and received by modified methods of regularization of A.N. Tikhonov.

The typical results of solution of retrospective inverse problem under conditions, where additional
condition $\varphi(x)$ is set with an error $\gamma_x$, are presented. In the experiments the additional condition was perturbed as follows:

$$\varphi_x(x) = \varphi(x) + \gamma \cdot \text{rand}(-1,1), x \in [a,b],$$

where rand($-1,1$) is random variable uniformly distributed on the segment $[a,b]$. To achieve smoother solution as smoothing operator we take $K$-time used three-point formula (filter):

$$\varphi^{k+1}_i = \left( \varphi^{k+1}_{i+1} + 4\varphi^k_i + \varphi^k_{i-1} \right)/6, i=1,\ldots,N-1, k=0,\ldots,K-1;$$

$$\varphi^0_i = \varphi^N_i, i=1,\ldots,N-1.$$

On the Figure 3 the results obtained for following input data: $N=60, m=5, a = -12, b = 12$ are presented. The left graph is obtained for $\gamma=0.01$, $K=12$ and the right for $\gamma=0.01$, $K=24$. The graphs show that growth of $K$ leads to stronger smoothing of the solution.

Let us now present results of computations by difference scheme (5)–(6) with negative weight factor. On the left side on Figure 4 the calculation results for $\sigma = -2, T=1, a = -12, b = 12, N=120, M=85$ are presented. On the right side there are results with presence of perturbation $\delta=0.01$, $K=24, N=120, M=8$. The remaining data is the same. On the left figure: 1 is a graph of the desired exact solution, 2 is a graph of additional condition (function $\varphi(x)$), 3 is a graph of restored initial condition. On the right figure: 1 is desired exact solution smoothed by a filter, 2 is perturbed additional condition, 3 is found initial condition. The graphs illustrate that the choice of negative weight factor allows to calculate on more detailed spatial grid. In case when additional condition is defined with errors the algorithm gives more accurate results.

Note that in [1] has shown that the use of negative multiplier is equivalent to using pseudoparabolic regularization with parameter $\sigma = -\tau\sigma$ so in our case $\delta=1/12$.

![Figure 3](image-url)  
**Figure 3.** On the left side there are results for $K=12$, and on the right side there are results for $K=24.
3. Cauchy problem for Laplace equation

Suppose that it is required to find a function \( u(x, y) \) that satisfies two-dimensional Laplace equation on rectangular \((0, L_x) \times (0, L_y)\)

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x, y) \in \Omega
\]  

(8)

and following conditions

\[
u(0, y) = 0, \quad u(L_x, y) = 0, \quad 0 < y \leq L_y, \]

(9)

\[
u(x, 0) = \varphi(x), \quad \frac{\partial u}{\partial y}(x, 0) = \phi(x), \quad 0 \leq x \leq L_x.
\]

(10)

This problem is a particular case of Cauchy problem for elliptic equation. On this problem the function \( u(x, y) \) should be defined in domain \( \Omega \) including upper boundary. But at lower boundary two conditions are set. It is conventionally well-posed on the class of smooth solutions [2].

It should be noted that problem (8)–(10) resembles the initial-boundary value problem for the wave equation. There are two initial conditions for variable \( y \) and a negative definite operator for variable \( x \). Thus, we are dealing with the problem similar to a retrospective inverse problem, but for hyperbolic equation. This fact suggests applying a similar difference scheme.

On rectangular mesh \( \omega = \omega_h \times \omega_\tau \),

\[
\omega_h = \{x_i = ih, \quad i = 1, \ldots, N-1; \quad h = L_x / N\},
\]

\[
\omega_\tau = \{t_j = j\tau, \quad j = 1, \ldots, M-1; \quad \tau = L_x / M\}.
\]

We assign to a problem (8)–(10) the difference scheme with non-positive weight factor

\[
\frac{y_{m+1} - 2y_m + y_{m-1}}{\tau^2} - A(\sigma y_{m+1} + (1 - 2\sigma)y_m + \sigma y_{m-1}) = 0, \quad m = 2, 3, \ldots, M,
\]

(11)

\[
y_{y}(x) = \phi(x), \quad y_{\tau}(x) = y_{y}(x) + \tau \varphi(x) - \frac{\tau^2}{2} A y_{y}, \quad x \in \omega_h.
\]

(12)

Here the grid operator \( A \) is defined as follows

\[
Ay = -y_{\tau}, \quad x \in \omega_h.
\]

The relative error of the numerical method is calculated according to formula
(13)

On Figure 5 we present the results of numerical calculations by the proposed difference scheme (11)–(12) for \( \sigma = 0, L_1 = 1, L_2 = 1 \) on the problem with exact solution

\[
    u(x, y) = \pi^2 \sin(\pi x) e^{\sigma(y-1)}. 
\]

(14)

Figure 5. The results for \( N = 20, M = 28, R = 0.01255 \) (on the left side), \( N = 100, M = 5, R = 0.07022 \) (on the right side)

The figures show results obtained for maximum possible values allowing stability of computational algorithm. The left figure shows the computational results obtained for \( N = 20; M = 28 \) and the right for \( N = 100; M = 5 \). On presented figures: 1 is graph of exact solution, 2 is graph of numerical solution on the selected grid. The graphs demonstrate that the results obtained by the proposed difference scheme provide a good accuracy on the grid coarse by variable \( x \) (the left graph). Increasing the number of partitions by \( x \) leads to increasing of mesh step by \( y \), so the accuracy fails (the right graph).

It should be noted that for large \( N \) calculations will be stable at low values of \( L_2 \).

Now we present the results of calculations carried out by the difference scheme (11)–(12) with a negative weight factor on the example of the exact solution of (12). Figure 6 shows that the use of a negative weighting factor increases the number of partitions of the grid by the variable \( y \). This leads to accuracy increasing.

It should be noted that like in the previous problem, as shown in paper [1] the use of the negative factor is equivalent to using of regularization difference scheme with regularization parameter \( \delta = -\sigma r^2 \) or associated with quasi-inversion method for approximate solution of ill-posed problem.
Figure 6. The results for $N = 20$, $M = 100$, $R = 0.00608$ (on the left side), $N = 100$, $M = 50$, $R = 0.01115$ (on the right side)

4. Conclusion
Direct solution of a retrospective heat transfer problem by using difference scheme with a negative weight factor gives good results even in the presence of random errors in additional conditions at the final moment of time. Numerical solutions of the Cauchy problem by difference scheme with a negative weight factor gives satisfactory results with proper selection of mesh size.

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