Interplay between density and superconducting quantum critical fluctuations

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Received 27 March 2015, revised 7 July 2015
Accepted for publication 14 July 2015
Published 29 September 2015

Abstract
We consider the case of a density-driven metal-superconductor transition in the proximity of an electronic phase separation. In particular, we investigate the interplay between superconducting fluctuations and density fluctuations, which become quantum critical when the electronic phase separation vanishes at zero temperature into a quantum critical point. In this situation, the critical dynamical density fluctuations strongly affect the dynamics of the Cooper-pair fluctuations, which acquire a more singular character with a $z = 3$ dynamical critical index. This gives rise to a scenario that possibly rules the disappearance of superconductivity when the electron density is reduced by electrostatic gating at the LaAlO$_3$/SrTiO$_3$ interface.

Keywords: inhomogeneous superconductors, quantum critical phenomena, coupled density and superconducting fluctuations

(Some figures may appear in colour only in the online journal)

I. Introduction

There are many electronic systems where some kind of order is established upon changing the electron density. Traditional examples are given by disordered doped semiconductors or MOSFETs, where a metal-insulator transition is found upon reducing the electron density \cite{1} and by heavy fermion systems, where competing phases (like, e.g. antiferromagnetism) are often spoiled by doping, giving rise to metallic states often accompanied by superconductivity \cite{2}. High-temperature superconducting (SC) cuprates and pnictides are other noticeable examples of systems where several electronic phases are realized by changing doping. More recently, it was also found that the two-dimensional electron gas (2DEG) formed at the interface of transition-metal oxides (like, e.g. LaXO$_3$/SrTiO$_3$, with $X = \text{Al,Ti}$, hereafter LXO/STO) become SC when the electron density is varied by electrostatic gating \cite{3, 4}. A zero-temperature SC quantum critical point (SC-QCP) separates the SC phase from the non-SC one \cite{6}. It may happen that the electronic liquid is not homogeneously distributed in the system, as in the presence of an electronic phase separation (PS) for instance. In this case, if the electronic inhomogeneities are static and sizable, i.e. large enough to sustain the ordered phase (e.g. superconductivity) one may expect that an inhomogeneous ordered state can be realized. For instance, one can envisage a state where SC ‘puddles’ are embedded in a non-SC metallic phase with lower electronic density, and link together to establish an overall coherent state, which is destroyed when the electron density is reduced \cite{7, 8}. The phase separation regime itself may be tuned by an external parameter, such as sample preparation and/or electrostatic gate voltage, which controls the average electronic density, and it disappears at zero temperature into a PS quantum critical point (PS-QCP). Around this point the density fluctuations have a quantum nature and keep a dynamical character, giving rise to slow critical density fluctuations at very long
wavelengths. A classical analog of this scenario is realized at finite temperature at the gas-liquid critical point.

Here, we address one single question: what happens if the Quantum Critical Point for the Phase Separation occurs near the one of the density driven ordering transition? For the sake of concreteness, we will explicitly consider the case of density-driven superconductor-to-metal (SC-M) transition, but several of our arguments will apply to other transitions as well. First of all SC pairing is favored (and it may even be induced) by critical density fluctuations. It has been shown in the framework of strongly correlated electrons that near a PS-QCP SC pairing is induced due to the mediation of the nearly critical density fluctuations [11, 24]. On the other hand, it has also been proposed [25] that a density-driven SC transition may favor PS. These remarks indicate that the PS-QCP and the SC-QCP may tend to coalesce. Their interplay then becomes a rather general and intriguing issue.

First of all, in section II we will provide a brief introduction to the PS quantum criticality. Then, in order to provide a first intuitive glimpse of the interplay between quantum-critical density fluctuations and superconductivity, we will consider a toy model. It is based on superconductivity instantaneously taking place as soon as a (dynamical but slow) density fluctuation increases locally the electron density above a critical value (section III). In section IV we will provide general arguments to support the idea that the two QCPs (the one closing the PS dome and the one related to density-driven superconductivity) tend to attract and coalesce. When this occurs, the Cooper-pair propagator is strongly dressed by the density fluctuations and the dynamics of the Cooper-pair fluctuations is strongly affected. Section V contains our concluding remarks.

II. The phase-separation quantum critical point

In most metallic systems the Fermi energy is by far the largest energy scale. This scale is obviously related to the electronic density fluctuations and superconductivity, we will consider a toy model. It is based on superconductivity instantaneously taking place as soon as a (dynamical but slow) density fluctuation increases locally the electron density above a critical value (section III). In section IV we will provide general arguments to support the idea that the two QCPs (the one closing the PS dome and the one related to density-driven superconductivity) tend to attract and coalesce. When this occurs, the Cooper-pair propagator is strongly dressed by the density fluctuations and the dynamics of the Cooper-pair fluctuations is strongly affected. Section V contains our concluding remarks.

\[ ω ≈ − \frac{1}{\epsilon_{qF}^2} \left( \frac{1}{\xi_0^2} \right) \]

where \( N^* \) is the quasiparticle density of states (DOS), \( \Gamma_\alpha \) is the residual effective interaction between quasiparticles, while \( F_0^+ = N^* \Gamma_\alpha \). In the presence of sizable additional interactions \( \Gamma_\alpha \) may become negative and, for large enough quasiparticle DOS, a Pomeranchuk instability [14], with \( F_0^+ \lesssim 1 \) may take place leading to a divergent compressibility, marking the spinodal line of a PS region.

This instability has a generic character and by no means is peculiar of cuprates or strongly correlated systems: it may generically occur whenever a Fermi liquid is characterized by small Fermi energy, sizable quasiparticle DOS, and some attractive force which overcomes the (usually weak) residual repulsion between quasiparticles. The weakly interacting 2DEG at the LNO/STO interface has recently been proposed to undergo such an instability because of a density-dependent Rashba spin–orbit coupling [17, 18] and/or the effective interaction arising from the self-consistent confining potential well that keeps the electrons close to the interface [19].

For our purposes it is convenient to cast the above generic Fermi liquid description on a microscopic (but still rather general) basis within a random-phase approximation. Let us consider an electron gas interacting via some effective interaction \( V(q) \) (henceforth \( q \equiv |\mathbf{q}| \)), which is attractive at small momenta. Then, considering the bare density–density \( \Pi(q, \omega) \) response function (also customarily called the Lindhardt polarization function), one can obtain the resummed density–density response function \( \Pi(q, \omega) \approx (1 + V(q) \Pi(q, \omega)) \). The standard expansion of the Lindhardt function for small momenta and frequency \( \Pi(q, \omega) \approx N*^2 \left( 1 - Aq^2/\xi_0^2 - \gamma_0 \omega / (v \xi_0) \right) \left( k_F \xi_0 \right) \) are the Fermi momentum and Fermi velocity, respectively, while \( A, B \) are dimensionless factors of order 1 depending on space dimensionality, lattice details, and so on) then leads to the density–density propagator

\[ \Pi(q, \omega) = \langle n(q, \omega) n(-q, -\omega) \rangle = -\frac{1}{\epsilon_{qF}^2} \] 

Here \( m_\alpha^0 = [1 + V(0)]N^* \) is the ‘mass’ of the density fluctuations and \( \gamma_0 = |V(0)|/\xi_0^2 \sim \langle \partial^2 \Pi/\partial q^2 \rangle_{q=0} \) and \( \gamma_0 = B |V(0)|/v \xi_0 \) are fermionic scales arising from the small momentum and frequency expansion of the Lindhardt function. When \( V(0) \) has an attractive character and \( N^* \) is not too small, one can find that \( m_\alpha^0 \to 0 \) leading to a second-order quantum instability. One can then cast \( m_\alpha^0 = \epsilon_{qF}^{-2} / \xi_0 \), with \( \epsilon_{qF} \) being the density correlation length ruling the distance from the PS-QCP, where \( m_\alpha^0 \) vanishes. Similar arguments would also allow to cast the PS-QCP in the framework of the Hertz–Millis approach [20, 21]. Near this QCP the Landau damping induces a term \(|\omega| / \xi_0 \). The largest contribution of

4 Very recent quantum Monte Carlo calculations have also shown that even the simple Hubbard model may display PS in the region of intermediate electron–electron repulsion [13].
these fluctuations (i.e. the maximum of their spectral density) occurs when the damping term is of the same order of the $q^2$ term. This leads to an overdamped dynamics with $\omega \sim q^2$ and therefore a dynamical critical index $z = 3$ characterizes the critical density fluctuations [22, 23].

In this work we propose to consider the possibility that such a PS-QCP is present in the phase diagram of LXO/STO interfaces. Then, if varying the density with the gate potential the system explores a region close to this PS-QCP, the physics of the SC-M transition could be influenced by the critical dynamics of the density fluctuations. Of course, these effects would be more pronounced the smaller is the mass term of the density fluctuations $m_n^0 \propto |V_{G}^{PS} - V_{G}^{SC}|$ near the SC-QCP ($V_{G}^{PS}$ is the critical gating at which the PS-QCP occurs, while $V_{G}^{SC}$ is the critical gating at which the SC-M transition occurs). We will see in the following that when this mixed criticality occurs, the most critical (i.e. singular) fluctuations imprint the dynamics of the less critical fluctuations. In the specific case of PS-QCP and SC-QCP proximity we will see that the $z = 2$ SC (Cooper-pair) fluctuations inherit the $z = 3$ dynamics of the underlying density fluctuations.

III. The fluctuating puddle model

We start from a simple ‘toy model’ model, where the critical character of the density-driven SC ordering is neglected, in order to make the above ‘imprinting’ mechanism more explicit. We assume that when a certain critical density $n_c$ is locally reached, a SC order parameter instantaneously arises. In particular, it may happen that a dynamical density fluctuation is created, where locally the density is sufficiently high to sustain superconductivity. In this case, neglecting the transient behavior, one faces a SC puddle, which, however, only survives as long as the density fluctuation survives. It is then natural to infer that the dynamics of the density fluctuation imprints (actually it determines) the dynamics of the SC puddles and leads to an apparent $z = 3$ behavior for SC dynamics.

Owing to the propagator in equation (1), the density fluctuations in momentum space [i.e. the Fourier transform of $\delta n(\mathbf{r}, t) \equiv n(\mathbf{r}, t) - \bar{n}$, $\bar{n}$ being the average density] satisfy the differential equation

$$\frac{1}{q} \partial_q \delta n(q, t) = (\xi_n^{-2} + q^2) \delta n(q, t),$$

where $\xi_n^{-2} = m_n^0/e_n$, and we take the momenta in units of Fermi wavevectors and time in units of inverse Fermi energy. This equation is readily solved giving in, for $q > 0$,

$$\delta n(q, t) = \delta n(q, t = 0) e^{-q^2 t/\xi_n^2}.$$ (2)

The corresponding real-space density can be coupled to the SC order parameter $\Delta(r)$ by a term of the form

$$H_{\Delta} = -g \int \delta n(r) |\Delta(r)|^2 \, d^3r.$$ (3)

This coupling finds a phenomenological justification within a standard Ginzburg–Landau approach, where the quadratic part of the functional has the form

$$\int a(|T - T_c(n)|) |\Delta(r)|^2 \, d^3r.$$  

Inspired by the experimental observation of a SC $T_c$ in LXO/STO heterostructures increasing when the density becomes larger than a critical value $n_c$ [3, 5], and assuming in the Ginzburg–Landau spirit an analytic dependence of the coefficients on the physical parameters, we take $T_c(n) \equiv g(n - n_c)$. In this way SC never occurs for $n < n_c$, while the critical temperature becomes increasingly large for densities increasingly larger than $n_c$. Considering the density variation as a locally varying field, $\delta n(r) \equiv n(r) - n_c$ one obtains a coupling as in equation (3). On the other hand the SC order parameter is described by a Hamiltonian

$$H_{SC} = \int \left[ m_n^0 |\Delta(r)|^2 + \frac{b}{2} |\Delta(r)|^4 \right] \, d^3r,$$

where $m_n^0$ rules the distance from the SC critical point. Here space and time dynamics of the SC order parameter are purposely neglected, within our toy model, in the absence of coupling to the density fluctuations. If the SC-QCP and the PS-QCP coincide at the same critical density $n_c$, and the SC phase is located at higher density, a positive fluctuation of $\delta n = n - n_c > 0$ instantaneously induces a static SC order parameter. On the other hand, if the PS-QCP occurs at slightly lower density $n_p^0 < n_c$, one needs a sufficiently large density fluctuation $\delta n > n_p^0 - n_c$ to create the SC order in the region of the fluctuation. Thus, inside the region with (sufficiently) positive $\delta n(r, t) > m_n^0/b$, one has

$$[\Delta(r, t)]^2 = \frac{g \delta n(r, t) - m_n^0}{b} \theta(g \delta n(r, t) - m_n^0),$$ (4)

where $\theta(x)$ is the Heaviside function. To test the occurrence of a $z = 3$ dynamics for the SC order parameter, for the sake of definiteness, we consider the case when superconductivity takes place in the puddle as soon as $\delta n > 0$ (i.e. $m_n^0 = 0$). We numerically generated at $t = 0$ in one dimension a density fluctuation with a Gaussian space profile and, after Fourier transforming, we let it evolve in time according to equation (2). Transforming back to real space, we insert the new density profile in equation (4) and we identify the time evolution of the frontier of the SC puddle, obtaining the curves of figure 1.

One can see that, after a short time transient, a $z = 3$ diffusive dynamics (the long-dashed red line is a $y = 1.85 t^{1/3}$ curve for comparison) occurs with the front of the SC puddle increasing like $r_\Delta \sim t^{1/3}$ (black and blue curves corresponding to $\xi_n^{-2} = 10^{-3}$ and $\xi_n^{-2} = 10^{-4}$ respectively). The inset reports the profile of the SC order parameters $\Delta(r)$ at various times in the case $\xi_n^{-2} = 10^{-3}$. In this case, one can also see that, due to the finite extension of the critical density fluctuation $(\xi_n = 10^{-3/2}$ unit lengths), a long-time crossover to a $z = 1$ propagating behavior occurs for $t \gtrsim 10^8$ (short-dashed green line).

A word is now in order to comment on the space and time units for systems like LXO/STO. The small densities involved
Figure 1. Time dependence of the puddle size in the case of SC puddles at any positive density fluctuation ($m^0_{\text{sc}} = 0$). The long dashed (red) straight line identifies the slope $1/3$ characteristic of the $z = 3$ diffusive dynamics. The short dashed (green) straight line identifies the slope $1$ characteristic of the $z = 1$ diffusive dynamics. The thick black curve is for $\xi_f^{-2} = 10^{-3}$, while the thin blue curve is for $\xi_f^{-2} = 10^{-5}$, in inverse of unit lengths square. Inset: spatial profile of the SC order parameter at various times, from $t = 10$ to $t = 10 \times 2^{10}$ (in $E_F$ units). The initial size of the fluctuating puddle is of 3 unit lengths (in $k_F^{-1}$ units).

(n $\approx 0.01$–0.08 el/cell) are such that the Fermi energies are small (a few tens of meV) and the inverse Fermi wavevector is roughly of order of 5–20 lattice spacings. Thus the extension of the puddle before the $z = 1$ crossover is reached can be estimated to be 300–3000 lattice units for $\xi_f^{-2} = 10^{-3}$ or $\xi_f^{-2} = 10^{-5}$, respectively. For instance, in LTO/STO 300 lattice units correspond to about 120 nm.

IV. Perturbative hints

In the above model one finds that a static SC order parameter acquires a critical dynamics ‘riding’ on a $z = 3$ density fluctuation. We now analyze from a simple perturbative point of view the interplay between the PS and the SC quantum criticality recovering the critical character of the SC fluctuations, which was neglected in the above ‘toy model’. We consider the case when the SC order parameter fluctuates both in modulus and phase (like in the original Ginzburg–Landau case) and couples to nearly critical density fluctuations of an otherwise homogeneous metal. This scenario involves the interplay between standard Cooper-pair fluctuations and critical density fluctuations. An obvious prerequisite for this interplay is that the PS-QCP and the SC-QCP are in close proximity. This naturally raises the question whether such proximity requires a fine tuning of the model parameters or it naturally occurs due to some intrinsic ‘attraction’ of the two critical points. On general grounds one expects these two critical points to attract each other. Indeed it was previously shown [11, 12] that a strong effective attraction at small momenta arises near the spinodal line and the QCP of a PS region, which can mediate strong SC pairing. Conversely, it was also shown [25] that the energy gain provided by the condensation energy in the SC phase may induce a PS tendency. This tendency is also supported by a straightforward perturbative analysis showing that the density fluctuations are made softer (and the compressibility correspondingly increases) when the SC-QCP is approached. We calculate the diagram (a) in figure 2, representing the first perturbative corrections to $m^0_{\text{sc}}$ in the $D$ propagator of equation (1), which corresponds to the bare inverse compressibility. The dashed lines represent the customary Cooper-pair fluctuation propagator [26]

$$L(q, \omega) = \langle \Delta(q, \omega) \Delta(-q, -\omega) \rangle = -\frac{1}{\epsilon^0_{\omega} q^2 - i\gamma^0_{\text{sc}}|\omega| + m^0_{\text{sc}}}.$$ (5)

where $\epsilon^0_{\omega}$ and $\gamma^0_{\text{sc}}$ are analogous to $\epsilon_0$ and $\gamma_0$ for the superconducting fluctuations [26] which clearly displays a $z = 2$ ($\omega \sim q^2$) overdamped dynamics.

We find that, starting from separated QCPs, the finite bare inverse compressibility $m^0_{n}$ tends to be reduced by the coupling to the SC critical fluctuations and in two dimensions

$$m_n = m^0_{n} - g^2 \log \frac{\Lambda}{m^0_{\text{sc}}}$$

yielding a smaller dressed inverse compressibility $m_n = \kappa^{-1}$. Here $m_n$ and $m^0_{\text{sc}}$ also measure the distances from the PS-QCP and SC-QCP respectively, while $\Lambda$ is a non-universal high-energy cutoff. Therefore, no matter how weak is the coupling between SC and density fluctuations, the inverse compressibility $m_n$ tends to decrease when approaching the SC-QCP (where $m^0_{\text{sc}} \to 0$). A pictorial view of this scenario is reported in figure 3. Although a full renormalization-group calculation is in order to draw definite conclusions, the above perturbative finding indicates that the proximity of the PS-QCP and the SC-QCP is not a fortuitous (and unlikely) coincidence.

Based on the above indication, we now investigate the interplay of the SC and density fluctuations assuming that the SC and PS criticality are sufficiently close to strongly influence each other. We therefore impose a formal coincidence of the two QCPs. Then, for the sake of definiteness, having in mind the 2DEG at the LXO/STO interface, we carry out perturbative calculations in two
dimensions starting from the standard Cooper-pair fluctuation propagator in equation (5), with $z = 2$, and the density fluctuation propagator of equation (1), with $z = 3$. To impose the QCP coincidence maximizing the effect of the critical interplay we set to zero both $m_0^0$ and $m_0^1$ in the calculation of the diagram in figure 2(b).

In appendix A we report the details of the perturbative calculation. We find that the SC fluctuations are strongly renormalized with the self-energy correction dominating the rest of the denominator, so that at first order the SC propagator acquires the form

$$L(q, \omega) \approx - \frac{1}{(c \omega q^2 - 1(\gamma_0 q^2 \omega^{-1}) \omega^2 q^2)}$$

where $q$ and $\omega$ are nonuniversal momentum and frequency cut-offs, and we remind that $q \equiv |\mathbf{q}|$. Again the largest contribution of these fluctuations (i.e. the maximum of their spectral density) occurs when the damping term $\propto \omega^2 q^2$ is of the same order of the $q$ term leading to an overdamped dynamics with $\omega^2 \sim q^2$. And again this corresponds to a dynamical critical index $z = 3$. Notice that the constant terms in the denominator, which would drive the system away from the critical point has been self-consistently set to zero to keep the criticality condition fulfilled. Once more we stress that a full renormalization group treatment is required to firmly establish the behavior near the two QCP, but our perturbative calculations are clearly indicative of the following scenario. The two QCP tend to coalesce and would generically give rise to a first-order transition with a PS between a low-density metallic state and a high-density SC state. However, as it usually occurs, a tuning of some parameter (like gating or magnetic field) may bring the system near a quantum tricritical point, where the SC critical temperature vanishes and, at the same time, the compressibility diverges, $\kappa = m_n^{-1} \rightarrow \infty$ (i.e. the two QCPs merge).

V. Possible realization: LXO/STO interface

V.A. A sketchy phase diagram for LXO/STO interfaces

The above analysis could find a possible realization in the SC-M transition occurring in the 2DEG at the interface of LXO/STO heterostructures. Figure 4 schematically depicts a possible scenario for this system as it emerges from experimental evidences [37] and theoretical modelling of the LXO/STO interfaces [19]. A word is in order to explain the meaning of the axes in the phase diagram of figure 4. The vertical axis $V_G$ obviously represents the gating potential which is customarily used to tune the electron density at the interface. In this schematic representation it is immaterial to specify whether the gating electrode is in the back of the STO substrate or is deposited on top of the LXO layer. The horizontal axis needs a more detailed explanation. In this system, the electronic density $n$ is controlled by two terms: the as-grown part $n(V_G = 0) \equiv n_0$, which is strongly sample dependent, and the electrostatic one $\delta n = CV_G$, where $C$ is the capacitance of the gate. The contribution to the 2DEG carrier density introduced by gating cannot phase-separate, because it is tightly bound to the uniformly distributed countercharges on the metallic electrodes. On the contrary, $n_0$ may depend on many more parameters.
hardly determinable factors: the thickness of the LXO over-layer and the related electronic reconstruction according to the polarity-catastrophe mechanism [38], the number of oxygen vacancies, the amount and strength of disorder scattering or trapping the 2DEG carriers... Owing to this highly complicated physics, we decided to phenomenologically ‘hide’ these factors in the simple parameter \( n_0 \), which simply represents the final electron density at the interface once all these effects have acted in the specific real sample. Thus in this simplified scheme differently prepared heterostructures have different starting densities \( n_0 \) to which the gate potential adds or subtracts an amount \( \delta n \) of electrons. Of course the physical system does not distinguishes between the intrinsic \( n_0 \) electrons and the gate-added \( \delta n \) ones. This is why we choose to represent on the horizontal axis the total \( n = n_0 + \delta n \). It should however be borne in mind that the horizontal axis is just a phenomenological representation of the complicated ‘chemistry’ of the samples: for each \( V_G \), \( \delta n \) is kept constant while the physics is varied along the axis by the parameter \( n_0 \). In this sense \( n = n_0 + \delta n \) only partially parallels the gating parameter \( V_G \) (via \( \delta n \)), while it contains far richer physical effects via the \( n_0 \) part. This renders the two axes, \( V_G \) and \( n = n_0 + \delta n \), non equivalent. Then, depending on the position of \( n_0 \) within the coexistence dome, different interfaces are characterized by different weights of the two separated phases, according to Maxwell’s rule.

The non-linear electric field dependence of the STO dielectric constant (for the back-gating case) and the self-consistent character of the potential felt by the confined electrons at the interface [19] produce a ‘trajectory’ in the \( n - V_G \) plane which is specific for each sample starting from different \( n_0 \)’s and evolve differently when their density is varied by \( V_G \) (the variation of density \( \delta n \) being a function of \( V_G \)). In figure 4 we represent typical evolutions by the dotted (black, blue or green) lines.

We then consider the vertical critical line (red long-dashed line in figure 4) at a critical (total) density \( n_c \) above which the system becomes SC. This density driven transition is assumed on a purely phenomenological basis without any assumption on the underlying microscopic mechanisms giving rise to SC when \( n > n_c \) (filling of bands which are more delocalized and/or with a larger DOS, or with states having a larger coupling to phonons, ...). Our only assumption is that only the (total) electron density of the 2DEG matters to trigger SC.

From a detailed phenomenological analysis of transport [28, 29], diamagnetic [30], and tunnelling [31] data it was found that the 2DEG at LXO/STO interfaces is markedly inhomogeneous, strongly suggesting the occurrence of an electronic PS represented in figure 4 by the blue shaded region. This means that the as-grown system should be already in the PS region and it is a natural born inhomogeneous system with denser electronic puddles (eventually becoming SC at sufficiently low temperature) embedded in a less dense metallic matrix. While the system stays in the phase-separated region when \( V_G \) is increased, depleting the 2DEG with a negative gating eventually leads to a (weakly localized) homogeneous metallic state (i.e. with absent or only a minor amount of SC regions). This means that the system goes down along the dotted lines changing its average density and exiting the PS region.

Regarding superconductivity, one has to distinguish two situations. When PS is well established, the puddles are static regions with larger electron density and, when the temperature is lowered, they become individually SC and their mutual coherence establishes a global SC state. This state may be closer to the description of [7] or may give rise to a more traditional percolative transition [28–30]. In this regime, when the SC-M transition is driven by a magnetic field [8], a quantum critical crossover occurs, which even allows to infer a characteristic size of the puddles of order of 100 nm. At short distances (intra-puddle) SC fluctuations at intermediate temperature are present in static puddles. The physics inside each puddle is characteristic of a non-disordered quantum XY model \((z = 1, \nu = 2/3)\). When, decreasing the temperature, the SC coherence length increases, the system explores the physical regime of inter-puddle couplings (see also [7]) and a large critical index \( \nu \) is necessary to comply with the Harris criterion (the inter-puddle static landscape is intrinsically disordered, see below). Our scenario of coupled PS-SC criticality is instead intrinsically different because the puddle support is itself fluctuating and the system is essentially homogeneous. In this case we expect one single criticality and no crossover should be present. In this novel and markedly different scenario, when the average density is lowered (the negative gating region), the puddles are smaller, less dense, more distant, and have a dynamical fluctuating character. Specifically, if the system exits the PS region near the PS-QCP (see, e.g. the grey shaded area and the blue and black dotted trajectories in figure 4), these puddles are better represented by dynamical (albeit slow) quantum critical density fluctuations. Our work precisely focuses on this occurrence and it analyzes the critical properties of the system when it undergoes a SC-M quantum phase transition not too far from the PS-QCP. It is exactly in this region that a \( z \nu = 3/2 \) scaling is observed [37]. Within the scheme proposed here, this experimental observation of \( z \nu = 3/2 \) could arise according, to the scenario of section 1, from a \( z = 3 \) dynamical critical behavior of the SC fluctuations together with a mean-field-like exponent \( \nu = 1/2 \). We find it appealing to consider and worth exploring the possibility of this alternative scenario, in which the SC fluctuations are coupled to (and imprinted by) nearly critical density fluctuations.

**V.B. What about long-range Coulomb forces?**

The above scenario is based on the presence of long-ranged nearly critical density fluctuations. Obviously, such fluctuations are only possible if the long-range Coulomb electron–electron interaction is strongly screened, otherwise these fluctuations...
would become high-energy barely damped plasmonic excitations, yielding a dynamical critical index \( z = 1 \) [32, 33]. It has also been shown [34] that a moderate screening, like the one due to a metallic gate close to a Josephson-Junction array, may still lead to \( z = 1 \) and a rather large value of the correlation length critical index \( \nu > 2 \). This seem to be the case inside the PS dome with static puddles, when the magnetic field drives the SC-M transition [8]. The question therefore becomes: to what extent our strong-screening assumption is valid, leading to the strong Landau-damping effects responsible for the large, \( z = 3 \), value of the dynamical critical index? The answer naturally depends on the physical system at hand. If the SC transition occurs in a very low-density system, long-range Coulomb interaction likely plays a relevant role and the scenario of the previous sections would hardly apply. On the other hand, when the SC transition takes place in a metallic state with high mobility and relatively high electron densities, all interactions become short ranged and the SC fluctuations may be strongly damped. The text-book case of the metal-to-superconductor transition driven by Cooper-pair fluctuations naturally realizes this case with \( z = 3 \) (see equation (5)). The same critical index characterizes the SC-QCP of [35]. These cases correspond to SC transitions in metallic states where the compressibility is finite and the density fluctuations are non critical (very) massive fluctuations. On the other hand, if the metal becomes SC close to a point where the compressibility tends to diverge (i.e. the density fluctuations are low-energy critical fluctuations), as it occurs near the PS-QCP, then our claim is that a scenario with \( z = 3 \) becomes possible. LXO/STO interfaces, where \( n \approx 10^{13} - 10^{14} \) electrons cm\(^{-2}\) likely belong to this class of systems where a PS-QCP may occur and the screening plays a relevant role, thereby leaving the possibility of large critical density fluctuations open. Moreover, one should also remember that SrTiO\(_3\), where the interface charges reside, has a huge dielectric constant ranging from 300 to 25 000 [36], strongly enhancing the screening.

**V.C. What about disorder?**

Disorder is another relevant issue to be discussed. We consider systems where the microscopic disorder is weak and rather homogeneously distributed. In this case the metallic character of the systems is poorly affected by the impurity scattering, at most resulting in weak localization of no relevance to our purposes (like in LXO/STO). On the other hand, the presence of a PS naturally entails the occurrence of a strongly inhomogeneous state emerging from this otherwise weakly disordered metal. This inhomogeneity would poorly affect transport properties because it mostly enhances forward scattering, leaving the mobility high. However, such static inhomogeneity would greatly affect the critical properties raising the question of the Harris criterion to be satisfied in order to have a true phase transition. On the contrary, the metallic state near the PS-QCP is not disordered because the density fluctuations are fully dynamical excitations of the electron gas and they are not a static (quenched) source of disorder. In this case the Harris criterion simply does not apply (at least, in its standard form). In other words one can argue that gating brings the system close to a PS-QCP, where no static puddles are present and the density critically fluctuates in a clean metallic state, so that the Harris criterion does not apply.

We finally remark that our scenario may also apply to those cases where SC fluctuations couple to longitudinal nematic modes (also characterized by \( z = 3 \), see, e.g. [43]) possibly explaining the occurrence of a SC-QCP with \( \nu = 3/2 \), reported in a gate doped La\(_{2-}\)Sr\(_2\)CuO\(_4\) sample [44].

**Acknowledgments**

The authors gratefully thank L Benfatto, V Brosco, C Castellani, and C Di Castro, for stimulating discussions. This work has been supported by the Region Ile-de-France in the framework of CNano IdF, OXYMORE and Sesame programs, by DGA and CNRS through a PICS program (S2S) and ANR JCJC (Nano-SO2DEG). SC and MG acknowledge financial support form Sapienza Universit\`a di Roma with the Project no. C26H13KZS9.

**Appendix A. Perturbative calculation of the SC-fluctuation self-energy**

In this appendix we provide some details about the perturbative calculation of the self-energy diagrams in figures 2(a) and (b), representing the dressing of density and SC fluctuations, respectively, in the presence of a coupling between the SC order parameter and the density fluctuation of the form (3), within a Ginzburg--Landau approach.

Given the expression of the density and Cooper-pair fluctuation propagators (1) and (5), the diagrams in figure 2(a) and (b), in \( d \) spatial dimensions, read

\[
\Pi_{\alpha}(\mathbf{q}, \omega) = i g^2 \int L(\mathbf{q} + \mathbf{k}, \omega + \nu) L(\mathbf{k}, \nu) \, d\nu \, d^d\mathbf{k}
\]

and

\[
\Pi_{\beta}(\mathbf{q}, \omega) = i g^2 \int L(\mathbf{q} + \mathbf{k}, \omega + \nu) D(\mathbf{k}, \nu) \, d\nu \, d^d\mathbf{k},
\]

respectively. To extract the wavevector and frequency dependence in the hydrodynamic regime [small \( q \equiv |\mathbf{q}| \) and \( \omega \), and, as dictated by the singular propagator (1), small \( \omega/\nu \)], we write the identity

\[
\Pi_{\alpha}(\mathbf{q}, \omega) = [\Pi_{\alpha}(\mathbf{q}, \omega) - \Pi_{\alpha}(\mathbf{q}, 0)] + [\Pi_{\alpha}(\mathbf{q}, 0) - \Pi_{\alpha}(0, 0)] + \Pi_{\alpha}(0, 0),
\]

for \( \alpha = \alpha, \beta \). The last term is a constant correction that shifts the critical point and since we enforce the condition of reaching a common QCP, we shall neglect it, and assume that the parameters are such that criticality is reached by both density and Cooper-pair fluctuation at a common QCP with the shifted propagators.

The first term [\( \Pi_{\alpha}(\mathbf{q}, \omega) - \Pi_{\alpha}(\mathbf{q}, 0) \)] yields at small \( \omega \) the hydrodynamic frequency dependence of the self-energy,
therefore the second term $[\Pi_+(q,0) - \Pi_-(0,0)]$ yields at small $q \equiv |q|$ the hydrodynamic wavevector dependence of the self-energy.

The numerical calculation shows that the diagram in figure 2(a) does not alter the leading dependencies of the propagator (1), given that Cooper-pair fluctuations, starting with $z = 2$ bare propagator, are less singular than density fluctuations.

On the contrary, the diagram in figure 2(b) yields corrections to the propagator (5) that dominate with respect to its bare wavevector and frequency dependence, changing its scaling form from $z = 2$ to $z \approx 3$. For instance, in two spatial dimensions, which is the case for the 2DEG at the LXO/STO interface, $[\Pi_+(q,0) - \Pi_-(0,0)] \propto q$ overshadows the $q^2$ in the denominator of equation (5), and $[\Pi_+(q) - \Pi_-(0)] \propto q^\omega \omega^q \omega^2$ dominates its $\omega$ dependence and yields the propagator (6).

For instance, in figure A1 we plot the imaginary part of $[\Pi_+(q) - \Pi_-(0)]/q [\text{in units of } g^2/(\gamma^2 \kappa_0)]$ as a function of $\omega q^2$ for various $q = 0.01, 0.1, 1$ [wavevectors are measured here in units of $(\kappa_0^2)^{1/2}$ and frequencies in units of $(\gamma^2)$. The parameters where taken such that $c_\omega^0 = \gamma_0/(\kappa_0^2)^{3/2}$, for the sake of definiteness. The various curves clearly display a low-frequency quadratic dependence (with a very weak residual $q$ dependence, that is logarithmic, and does not change the main scaling), that translates into the $\omega^2 q^2$ dependence mentioned above.

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