Abstract

We show the results on the history of the invention of the conjugacy \( h(x) = \frac{2}{\pi} \arcsin \sqrt{x} \) of one-dimensional \([0, 1] \to [0, 1]\) maps \( f(x) = 4x(1-x) \) and \( g(x) = 1 - |1 - 2x| \).

1 Introduction

Not every classical mathematical problem has its final form in the moment of the creation. Moreover, this is true about “classical methods” and “classical tools” of solving mathematical problems. Almost all branches of modern mathematics appeared or obtained the form, which is closed to modern, in 18-19 century. During the reading of mathematical works of that time, it is easy to regard such mistakes of bright mathematicians, which should not be made by scholars and students of first cors of university nowadays. From another hand, it is necessary to remember, that it is due to these (and only these) mistakes the modern mathematics has the level of strictness, that it has today.

We will pay attention to the appearance and the first attempts to study the following construction. Let \( f : I \to I \) be a function, where \( I \) is a set of real numbers. For any \( n \in \mathbb{N} \) denote \( g(x) = f(f(\ldots f(x)\ldots)) \), write \( g = f^n \) for being short and call \( g \) the \( n \)th iteration of \( f \). At the beginning of the 19th century the problems of finding the formula for \( f^n \) by given \( f \) or, conversely, finding \( f \) by given \( g = f^n \) was formulated and studied as deeply as it was possible. Non-succeed attempts to solve these problems, lead to the appearance and active development of Dynamical Systems Theory in the middle and second part of the 20th century.

This article is motivated by the following question. It is known that equality \( g(x) = h(h^{-1}(x)) \) holds for every \( x \in [0, 1] \), where \( f(x) = 4x(1-x) \), \( g(x) = 1 - |1 - 2x| \) and \( h(x) = \frac{2}{\pi} \arcsin \sqrt{x} \) are defined on \([0, 1]\). The mentioned equality appears in all textbooks on one-dimensional dynamics in the chapter about topological conjugation (topological equivalence). A lot of books on the Theory of Dynamical Systems (for instance [4, p. 15], [6, p. 103], [7, p. 316] and [11, p. 341]) refer this formula to [18], but this is just a note on Summer Meeting of the AMS in 1947, which is about one fifth of a page and contains only the remark that \( g \) can be used as a generator a random numbers. During the private talks with Kyiv specialists on one-dimensional dynamics the author failed to get the clearness about the history of the investigation of this formula. All the answers of professors looked like: “This is the mathematical folklore: the fact is known to everybody, but not the authorship. Certainly, one can spend a lot of time and found some history, but nobody can be sure, wether or not this history will appear to be interesting enough”.

During the attempts to understand the history of the formula, which is mentioned above, the author entered the magic world of mathematical texts of 19-20 century, where the solutions of mathematical problems are strongly mixed with ambitions of great mathematicians and appear also at the pages of scientific articles. But, let us look at everything in turn.
2 Functional equations of J. Herschel

John Herschel is a talented scientist of the end of 19th century. For instance, he was presented with the Gold Medal of the Royal Astronomical Society in 1826 and with the Lalande Medal of the French Academy of Sciences in 1825, was a Honorary Member of the St Petersburg Academy of Sciences. Also he was one of the founders of the Royal Astronomical Society in 1820. Nevertheless, John Herschel was an astronomer, but not a “professional mathematician”.

In the same time, J. Herschel stated in [8] series of mathematical problems and make attempts to solve them. His solutions contain some disadvantages, which do not decrease the value of the problems themselves.

Problem 1. Find \( f^n \) for the function \( f(x) = 2x^2 - 1 \). The following solution is presented. For every \( x \) denote \( u_0 = x \) and consider the sequence

\[
\begin{align*}
    u_{n+1} &= 2u_n^2 - 1. \\
    \text{(1)}
\end{align*}
\]

To find the formula for \( f^n \) is the same as to find the general formula for \( u_n \), dependent on \( n \) and \( u_0 \). Without any explanations Herschel decides to find the solution in the form

\[
    u_n = \frac{1}{2} \left( C^{2^n} + C^{-2^n} \right),
\]

where \( C \) is a constant, dependent on \( u_0 \). We may assume that the motivation for such form of the solution is the formula \( u_n = \lambda^n \) for the solution of a linear second order difference equation \( u_{n+2} = au_{n+1} + bu_n \). In any way, the direct substitution lets to check that \( \text{(2)} \) satisfies \( \text{(1)} \). After the plug \( u_0 = x \) into \( \text{(2)} \), Herschel has found

\[ C = x + \sqrt{x^2 - 1}, \]

whence rewrites \( \text{(2)} \) as

\[
    f^n(x) = \frac{1}{2} \left( \left( x + \sqrt{x^2 - 1} \right)^{2^n} + \left( x - \sqrt{x^2 - 1} \right)^{2^n} \right). \quad \text{(3)}
\]

He does not explain the possibility of appearance of negative number under the square root.

Problem 2. Find functions \( \varphi \), which satisfy the equality

\[
    \varphi^2(x) = x.
\]

Like above, J. Herschel denotes \( x = u_n \) and \( \varphi(u_n) = u_{n+1} \). Moreover, he considers the sequence \( \varphi(u_n) \), which denotes \( (\varphi(u))_n \). After the “cross multiplication” of equalities

\[
    \begin{cases} 
    (\varphi(u))_n = u_{n+1} \\
    (\varphi(u))_{n+1} = u_n,
    \end{cases}
\]

he obtains \( u_{n+1}(\varphi(u))_{n+1} = u_n(\varphi(u))_n \), i.e. \( u_n(\varphi(u))_n = c \) for all \( n \), where \( c \) is a constant.

Also it follows from equalities \( \text{(5)} \) that \( u_n + (\varphi(u))_n = u_{n+1} + (\varphi(u))_{n+1} \), i.e. \( u_n + (\varphi(u))_n + C = 0 \) for all \( n \) and some constant \( C \). Now Herschel claims that one may state \( f(u_n(\varphi(u))_n) \) instead of \( C \), where \( f \) is an arbitrary function. Thus the equation

\[
    x + \varphi(x) + f(x\varphi(x)) = 0. \quad \text{(6)}
\]

appears. Taking as example the function \( f(x) = a + bx \) Herschel finds \( \varphi \) from equality

\[
    x + \varphi(x) + bx\varphi(x) + a = 0
\]
as
\[ \varphi(x) = -\frac{a + x}{1 + bx} \] (7)
and claims, that it is a solution of (4). He does not notice, that if one plug (7) into (4), then would not get the identity.

**Problem 3.** For given function \( f \) find a function \( \varphi \) such that \( \varphi^n = f \). For this problem Herschel suggests “very simple” solution: find the general formula for \( f^n \) and plug there \( 1/n \) instead of \( n \). For example for the function \( f(x) = 2x^2 - 1 \) Herschel uses (3) to find
\[ \varphi(x) = \frac{1}{2} \left( (x + \sqrt{x^2 - 1})^{\psi x} + (x - \sqrt{x^2 - 1})^{\psi x} \right) \]
and remarks that \( \sqrt{2} \) means the set of \( n \)th complex roots of \( n \)th degree.

**Problem 4.** Let an hyperbola \( AM \) with axis \( CP \) (which coincides with \( x \)-axis) and center \( C \) (which is origin) be given. It is necessary to find the curve \( am \) with the following properties. For any point \( P \) on the \( x \)-axis find the point on \( am \) with the same \( x \)-coordinate and take on the \( x \)-axis the point \( d \) such that \( Cd = Pm \). Again find on \( am \) the point \( l \) with \( x \)-coordinate as of \( d \) and take the third point \( l_3 \) such that \( Cl_3 = dl \). Repeat these actions \( n \) times, where \( n \) is chosen at the very beginning. The question: find \( am \) such that after \( n \) steps we would get the segment \( fk \), whose length equals \( PM \) (see fig. 1a.)?

Write the equation of hyperbola \( AM \) as
\[ y^2 = (1 - e^2)(a^2 - x^2), \]
where \( e \) is the eccentricity of the hyperbola. For the function \( f(x) = \sqrt{(1 - e^2)(a^2 - x^2)} \) we have that
\[ f^n(x) = \sqrt{(e^2 - 1)^n x^2 - \frac{e^2 - 1}{e^2 - 2} ((e^2 - 1)^n - 1) a^2} \] (8)
Moreover, finding a function \( \varphi \) such that \( \varphi^n = f \), Herschel, according to his rule above, plugs \( 1/n \) instead of \( n \) into (8) and obtains the equations of curves, which satisfy the condition of the problem. He does not consider the existence of another curves, which satisfy the former condition. It is interesting, that Herschel does not notice here, that square root has two values, one of which is complex.

![Picture 1](image-url)  
**Pict. 1: Graphical interpretations**

### 3 Conjugacy of C. Babbage and J. Ritt

Charles Babbage pays attention in [2, problem 9] to the study of iterations of functions, precisely to the finding of the function \( \psi \) such that
\[ \psi^n(x) = x \] (9)
for all \( x \). First, he suggests the graphical interpretation of this problem, which is similar to Herschel’s one, which is described above. At the picture [1b], it is necessary to find the cure \( APQT \) (where \( A \) is the origin) with the following properties. For arbitrary point \( B \) find the length of perpendicular \( BP \) and take on the \( x \)-axis the point \( C \) such that \( AC = BP \). Then find the length of the perpendicular \( CQ \) and take its length on the \( x \)-axis. After \( n \) steps we have to get the perpendicular \( FT \), whose length should be equal to the length of the former segment \( AB \). This geometrical construction is exactly the verbal description of the equation [9].

Babbage makes one more important remark. Let \( f \) be some solution of [9] and let \( \varphi \) be an arbitrary invertible function. Then

\[
g(x) = \varphi(f(\varphi^{-1}(x)))
\]

will also be a solution of [9]. In fact, these reasonings about [9] is exactly the invention of the notion of topological conjugacy of maps. Topological conjugacy of function \( f \) and \( g \) can be imagined as the rectangle,

\[
\begin{array}{ccc}
\varphi & f & \varphi \\
\downarrow & & \downarrow \\
g & \rightarrow & g
\end{array}
\]

where we know, that one can come from the left top vertex to the right bottom one either by “top root”, or by “bottom root” with the same result. Coming by the root means here the sequent applying the functions near the arrows to an arbitrary former argument. These diagrams are called “commutative diagrams” and are widely used in different branches of mathematics for the illustration of reasonings. The important corollary of the commutative diagram is the possibility “to continue it to the right” and thus obtain

\[
\begin{array}{ccc}
\varphi & f & \varphi \\
\downarrow & & \downarrow \\
g & \rightarrow & g
\end{array}
\]

which implies that \( g^n(x) = \varphi(f^n(\varphi^{-1}(x))) \). From another hand, the diagram just illustrates the reasonings and the same conclusion could be obtained from the equality [10] without the commutative diagram too.

Joseph Ritt (injustively) says in [16] that C. Babbage earlier claimed that for any fixed solution \( f \) of the equation [9] and any other solution \( g \) there exists an invertible \( \varphi \) such that [10] holds. Suppose (writes Ritt) that \( \varphi(a) = b \) for some \( a \) and \( b \). Then \( \varphi(f^n(a)) = g^n(b) \) for every \( n \in \mathbb{N} \). If for some other \( a' \), \( b' \) the equality \( \varphi(a') = b' \) holds, then for every \( n \) we will have \( \varphi(f^n(a')) = g^n(b') \) (see. pict. 2a). If \( f^n(a) = f^n(a') \) for some \( n \), but \( g^n(b) \neq g^n(b') \), the contradiction with that \( \varphi \) is a function would appear (see pict. 2b).

\[
\begin{array}{ccc}
\varphi & f & \varphi \\
\downarrow & & \downarrow \\
g & \rightarrow & g
\end{array}
\]

Pict. 2: Reasonings of Ritt

Ritt continues these reasonings. Suppose that \( \varphi(x) = \Psi(x) \) for all \( x \in (a, a') \) and some \( a, a' \), where \( \Psi \) is some function. In other words, suppose that \( \varphi \) is already defined on some interval \( (a, a') \). Then for any \( n \) the equality \( \varphi(f^n(x)) = g^n(\varphi((x))) \) determines \( \varphi \) on \( f^n([a, a']) \) (see pict. 2c). If for some \( n_1, n_2 \) intervals \( f^{n_1}([a, a']) \) and \( f^{n_2}([a, a']) \) intersect, the the contradiction may appear.
Notice, that talking about the contradiction, Ritt does not pay attention to the fact, which gives the contradiction. Indeed, he assumes that two points \((a, b)\) and \((a', b')\) such that \(\varphi(a) = b\) and \(\varphi(a') = b'\) are given and then contradiction appears. But this only means that the graph of \(\varphi\) does not pass through both of these points (but may be pass through one of them). It is not clear, what can generate the conclusion that for some exact solutions \(f\) and \(g\) of (9) there is no any invertible \(\varphi\), which transforms (10) to the identity.

4 Lamerey’s Diagrams

The Cartesian method as a way on plotting the graphs of functions is known from the de Cartes time, i.e. from the first part of the 17th century. In the same time, the problems, which were stated by J. Herschel (and were discussed above), need a bit specifical techniques, and we will pay attention to it now.

![Pict. 3: Lamerey’s Diagrams](image)

When one use the Cartesian method, the arguments of a functions are understood as points of the \(x\)-axis and its values – as points of \(y\)-axis, whence the function appears to be a map from \(x\)-axis to \(y\)-axis. In the same time, we can consider any function \(f\) as a map from the line \(y = x\) to itself, which maps a point \((x, x)\) to \((f(x), f(x))\). The graphical interpretation of this action can be imagined as vertical line from point \((x, x)\) to the graph and then horizontal line to \((f(x), f(x))\), returning to \(y = x\).

For usefulness of such interpretation, let us come back to the problem of finding a continuous function \(\varphi : \mathbb{R} \to \mathbb{R}\) such that \(\varphi^2(x) = \varphi(x)\) for all \(x\) from the domain. Clearly, for every \(x_0\) the equality \(\varphi(\varphi(x_0)) = \varphi(x_0)\) holds. Thus, denote \(y_0 = \varphi(x_0)\) and obtain that \(\varphi(y_0) = y_0\), i.e. the graph of \(\varphi\) passes through the point \((y, y)\) for every \(y\) such that \(y = \varphi(x)\) for some \(x\). This means that there exist \(a, b\) such that \(\varphi(\mathbb{R}) = [a, b]\) and \(\varphi(x) = x\) for every \(x \in [a, b]\) (see pict. 3b).

Also the following reasonings can be illustrated. Notice that the solutions of the equation \(x = \varphi(x)\) are precisely the points of the intersection of the graph of \(\varphi\) and the line \(y = x\). Suppose that for the function \(\varphi\) there exists (unknown) solution \(a\) of the equation \(x = \varphi(x)\) and, moreover, \(x < \varphi(x) < a\) for every \(x < a\) and \(a < \varphi(x) < x\) for each \(x > a\). For arbitrary \(x_0\) consider the sequence \(x_k = \varphi^k(x_0)\). If one plot the graph, then it becomes evident that the sequence \(\{x_k\}\) tends to unknown \(a\) (see pict. 3b). This reasonings were invented in the beginning of the 19th century almost simultaneously by Evarist Galois \cite{Galois} and Adrien Legendre \cite{Legendre} (see also \cite{Legendre}).

The work \cite{Lamerey} contains the picture (see pict. 4), where this method is illustrated. The left graph contains a line \(y = x\) (for the technical calculations, mentioned earlier) and the graph \(y = f(x)\), which is used to construct the function \(y = f^2(x)\). The right sketch contains the construction of the graph of the function \(y(x) = \varphi(f(x))\).
by given graphs of $f$ and $\varphi$.

Nowadays the sketches like those on figure 4 are called Lamerey’s Diagrams due to the work [10].

5 Trigonometry of J. Boole

George Boole is known in the history of mathematics as one on the founders of the mathematical logics. In [8] G. Boole suggests the following way of finding the general formula for iterations of the function $f(x) = 2x^2 - 1$.

Keeping in mind the double angle formula $\cos 2x = 2\cos^2 x - 1$, denote $g(x) = 2x$ and $h(x) = \cos x$, whence write the commutative diagram

\[
x \xrightarrow{g} 2x \xrightarrow{h} f(h) = h(g)
\]

Continue the diagram to the right and obtain

\[
x \xrightarrow{g} 2x \xrightarrow{g} \ldots \xrightarrow{g} 2^n x
\]

\[
\cos x \xrightarrow{f} \cos(2x) \xrightarrow{f} \ldots \xrightarrow{f} \cos(2^n x),
\]

whence $f^n(\cos x) = \cos(2^n x)$. The substitution $t = \cos x$ leads to

\[
f^n(t) = \cos(2^n \arccos t)
\]

for all $t \in [-1, 1]$.

6 The J. von Neumann’s and S. Ulam’s generator

John von Neumann studied deeply quantum physics, functional analysis, sets theory and informatics. His name is connected with the architecture of the most modern computers. von Neumann, together with polish mathematician Stanislaw Ulam participated in Manhattan Project.

In the short note [18], which is the thesis of the mating of American Mathematical Society, J. von Neumann and S. Ulam suggested to use iterations of the function

\[
f(x) = 4x(1 - x)
\]
for obtaining “numbers with different distributions”. Nevertheless, they have not explained there, what the “distribution of a number” is.

The complicatedness of calculations, which are dealing with the function (12), is known from the Pier Verhulst’s [19], written in the first part of the 19th century, where he used the formula

\[ p_{k+1} = p_k(m - np_k). \]  

(13)

for the expectation of the number of individuals \( p_k \) in the biological population in the \( k \)th generation, where \( m \) and \( n \) are constants. Verhulst remarked the strong dependence of \( p_k \) on \( p_1, m \) and \( n \). In other words, for the huge \( k \) the small changes of \( p_1, m \) and \( n \) can lead to march more change of \( p_k \). The sequence (13) was called “logistic” due to “logists”, who made calculations in the Ancient Greece. The map (12) is also called logistic map, because it is of the form (13).

In the article [13] von Neumann mentioned the disadvantage of the function (12) as a generator, introduced in [18] and, in the same time, explained, what is meant under the distribution of a number. Suppose that non-decreasing function \( F : [0, 1] \rightarrow [0, 1] \) such that \( F(0) = 0 \) and \( F(1) = 1 \) is given. It is necessary “in some way” to get the sequence of numbers \( \{ x_n \} \subset [0, 1] \) such that the probability of the event \( x_i \in [a, b] \) equals \( F(b) - F(a) \). The result of [18], says von Neumann, is that for almost every \( x_0 \in [0, 1] \) (up to Lebesgue measure), the probability \( x_i \in [a, b] \) equals \( b - a \). He mentions that obtaining the function, which “generates” the sequences with given distribution, is an important problem, for instance, for some calculating methods. In the same time, it is mentioned in [13] that the function, suggested in [18] can not be used with the mentioned goal. The arguments for such impossibility were the following.

For a given sequence \( \{ x_i \} \subset [0, 1] \) such that \( x_{i+1} = 4x_i(1 - x_i) \) define \( \alpha_i \) by \( x_i = \sin^2 \pi \alpha_i \). Now notice that \( \alpha_{i+1} = 2\alpha_i \mod 1 \), i.e., \( \alpha_{i+1} \) is the fractional part of \( 2\alpha_i \). Denote the binary decomposition of \( \alpha_1 \) as \( \alpha_1 = \beta_1 \beta_2 \beta_3 \ldots \), whence the binary decomposition of \( \alpha_k \) would be \( \alpha_k = \beta_k \beta_{k+1} \beta_{k+2} \ldots \). Now von Neumann makes conclusion that in real computer we can not take a number with infinitely many random binary digits, whence the sequence \( \{ \alpha_i \} \) (and, correspondingly, \( \{ x_i \} \)), being considered “on the real computer” will become 0 after finitely many iterations.

Moreover, von Neumann made in [13] a very non-ethical think: he wrote that is was S. Ulam, who suggested (12) as a random generator and, thus, “forget” that Ulam was co-author of [18] too. The continuation of this history a much more interesting. Is it easy no see that the formula \( x_i = \sin^2 \pi \alpha_i \) does not define the sequence \( \alpha_i \) with properties, which are mentioned in [18]. If one would find \( \alpha \in [0, 1] \) then there is no one-to-one correspondence, because \( \alpha_i = k + (\frac{(-1)^k}{\pi}) \arcsin \sqrt{x_i}, k \in \{0, 1\} \) and it is not clear, it is necessary to take \( k \) being 0 or 1. If suppose that \( \alpha_1 \in [0, 0.5] \), then the formula \( \alpha_{i+1} = 2\alpha_i \mod 1 \) would give a number from the interval \( (0.5, 1) \) after finitely many steps for every \( \alpha_1 \neq 0 \).

Remark, that the function \( \alpha(x) = \frac{1}{\pi} \arcsin \sqrt{x} \) is the bijection between the intervals \([0, 1]\) and \([0, 0.5]\). Thus, commutative diagram

\[
\begin{array}{ccc}
[0, 1] & \xrightarrow{f} & [0, 1] \\
\downarrow{\alpha} & & \downarrow{\alpha} \\
[0, 0.5] & \xrightarrow{g} & [0, 0.5]
\end{array}
\]

defines a function \( g : [0, 0.5] \rightarrow [0, 0.5] \) by \( g(x) = \alpha(f(\alpha^{-1}(x))) \). Remind that \( \alpha^{-1}(x) = \sin^2 \pi x \). The evident technical calculations give the following:

\[
g(x) = \frac{1}{\pi} \arcsin \sqrt{4 \sin^2(\pi x)(1 - \sin^2(\pi x))} = \frac{\arcsin(|\sin(2\pi x)|)}{\pi}.
\]
Simplification of the absolute value function in the argument of arcsin leats to the dichotomy either \( x \leq 0.25 \) or \( x > 0.25 \) (this is naturally, since we have already seen some “problems” with the middle of the interval for \( x \) in the von Neumann’s work). In fact,

\[
g(x) = \begin{cases} 
2x & \text{for } x \in [0, 0.25] \\
1 - 2x & \text{for } x \in (0.25, 0.5] 
\end{cases} \tag{14}
\]

We have mentioned already, that commutative diagrams are connected with the conjugacy. Let us give the formal definition. Functions \( f : A \to A \) and \( g : B \to B \), where \( A \) and \( B \) are sets of real numbers, are called topologically conjugated, if there exists an invertible function \( h : A \to B \) such that \( h(f(x)) = g(h(x)) \) for all \( x \in A \).

The chapter of mathematics “topology” studies geometrical “figures” up to some transformations, which roughly can be imagined like if figures be made from a material, which admits stretches and compressing. The word combination “topological conjugation” can be explained as the change of the scale while the construction of the graph of a function. We shall explain more carefully what “the change of the scale” is. Let us construct the graph of the function \( f(x) = 4x(1-x) \) for \( x \in [0, 1] \) (it is the parabola inside the square \([0, 1] \times [0, 1] \), with branches going down). Let \( h : [0, 1] \to [a, b] \) be a continuous invertible function, which (we will say defines the change of coordinates). Let us take each point on the segment \([0, 1] \) of \( x \)-axis and \( y \)-axis and write \( h(x) \) near it. After this, we may say, that each point \((x, y)\) of the square \([0, 1] \times [0, 1] \) obtains the new coordinates \((\tilde{x}, \tilde{y}) \in [a, b] \times [a, b]\), but the line \( y = x \) remains to be \( y = x \). Nevertheless, the parabola, which was (!) the graph of the function \( f(x) = 4x(1-x) \) appears to the the graph of some another function \( \tilde{f} : [a, b] \to [a, b] \). If \( \alpha : [0, 1] \to [0, 0.5] \) would be instead of \( h \) above, then it follows from our calculation, that the obtained “new function” is \( g \) of the form (14).

After \( g \) has been found, it is necessary to “want” to obtain one more function, which will be “similar” to \( g \), but will be \([0, 1] \to [0, 1] \) instead of \([0, 0.5] \to [0, 0.5] \). If apply the topological conjugacy \( \beta(x) = 2x \) to \( g \), then obtain \( g_1(x) = \beta(g(\beta^{-1}(x))) \), which is

\[
g_1(x) = \begin{cases} 
2x & \text{for } x \in [0, 0.5), \\
2 - 2x & \text{for } x \in (0.5, 1]. 
\end{cases} \tag{15}
\]

Notice that graphs of functions (14) and (15) looks the same if the numbers are marked uniformly on coordinate axis. The conjugation of \( f \) and \( g_1 \) can be easily illustrated by the following commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[0, 1] \\
h \\
[0, 0.5]
\end{array}
\end{array}
\end{array}
\xrightarrow{f}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[0, 1] \\
\alpha \\
[0, 0.5] \\
\beta \\
[0, 1]
\end{array}
\end{array}
\end{array}
\xrightarrow{g_1}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[0, 1] \\
\alpha \\
[0, 0.5] \\
\beta \\
[0, 1]
\end{array}
\end{array}
\end{array}
\xrightarrow{h}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[0, 1] \\
\alpha \\
[0, 0.5] \\
\beta \\
[0, 1]
\end{array}
\end{array}
\end{array}
\xrightarrow{h}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
[0, 1] \\
\alpha \\
[0, 0.5] \\
\beta \\
[0, 1]
\end{array}
\end{array}
\end{array}
\]

and the conjugacy (the function, which defines the conjugation) is

\[
h(x) = \beta(\alpha(x)) = \frac{2}{\pi} \arcsin \sqrt{x}. \tag{16}
\]

Let us come bach to the George Boole’s work [3]. Commutative diagram (11) does not define a topological conjugation, because the map \( g \) there is not \( B \to B \) such that \( h(x) = \cos x \) is the bijection between \( B \) and \( h(B) \).
where $v$ for all $g$ conjugated to $g$ that
$h$ as follows:
the graph of $x$ symmetrical in $(or, which is the same, to $y$ chaos.
article [12] is written in the second part of the 20th century and is dedicated to the Li and Yorke's notion of
importance or the reasonings, which were suggested by Babbage during the attempts of the solution. The
which Babbage treated, has, in some sense, the only trivial solution. Nevertheless, his does not decrease the
dedicated to Joseph Ritt's work. Let $g : [0, 1] \to [0, 1]$ be of the form (15) and $g_2 : [0, 1] \to [0, 1]$ be defined as follows:
$$ g_2(x) = \begin{cases} l(x) & \text{for } x \in [0, v], \\ r(x) & \text{for } x \in (v, 1], \end{cases} $$
where $v \in (0, 1)$ is a parameter, function $l$ increase, $r$ decrease such that $l(0) = r(1) = 0$. In other words, the graph of $g_2$ is “similar” to the graph of $g_1$, but $g_2$ is not necessary piecewise linear and is not necessary symmetrical in $x = 0.5$. Ulam asks: which conditions should satisfy $g_2$ for being topologically equivalent to $g_1$
(or, which is the same, to $y = 4x(1 - x)$)? Suppose that $\tilde{h} : [0, 1] \to [0, 1]$ is such that $g_2(x) = \tilde{h}(g_1(\tilde{h}^{-1}(x)))$
for all $x \in [0, 1]$. Since $\tilde{h}$ is invertible, then $\tilde{h}(0) \in \{0, 1\}$. Now $g_2(0) = 0$ and $\tilde{h}(g_1(0)) = g_2(\tilde{h}(0))$ imply $\tilde{h}(0) = 0$, whence $\tilde{h}$ increase. Moreover, since $g_1^n(0) = 0$ for all $n \geq 1$, then it follows from $\tilde{h}(g_1^n(x)) = g_2^n(\tilde{h}(x))$
that $g_2^n(\tilde{h}(x)) = 0$, whenever $g_1^n(x) = 0$. Ulam makes a conclusion from these reasonings that $g_2$ is topologically conjugated to $g_1$ is and only if that set $M = \{x \in [0, 1] : g_2^n(x) = 0 \text{ for some } k\}$ is dense in $[0, 1]$.

As about the map $\psi$, which satisfies the functional equation (9), Melvyn Nathanson has proved the following theorem in [12]: if $\psi : [0, 1] \to [0, 1]$ is a continuous function such that $\psi^p(x) = x$ for all $x \in [0, 1]$, then $\psi^2(x) = x$ for all $x \in [0, 1]$. Moreover, if $p$ is even, then $\psi(x) = x$ for all $x \in [0, 1]$. In other words, the problem, which Babbage treated, has, in some sense, the only trivial solution. Nevertheless, his does not decrease the
importance or the reasonings, which were suggested by Babbage during the attempts of the solution. The
article [12] is written in the second part of the 20th century and is dedicated to the Li and Yorke’s notion of
chaos.

The formula (10) or the topological conjugacy of maps (12) and (15) was published at first by Ottis Richard
in [15]. Nevertheless, Richard thanks there S. Ulam “for many helpful and stimulating conversations on the
subject of this paper” and writes especially about the formula (16) (which is used for the calculating of the invariant measure) that it is S. Ulam, which noticed this formula at first. Stanislaw Ulam published (16) the first time 8 years later in [17]. The article [17] does not contain any references to John von Neumann as the author of idea of getting (16). Some modern books on Dynamical Systems theory refer (16) as Ulam’s map.

7 Final remarks

Sometimes mathematicians make mistakes. Mathematicians of the worldwide level sometimes make mistakes too. Sometimes mathematicians quarrel one with each other and tracks of this can be seen in their articles. Nevertheless, the mistake of a scholar and the mistake of a mathematician of the worldwide level are “different sings”.

John Herschel’s problems with solutions, which were published in year 1814, contain mistakes. But they also contain the foundations of the theory of one-dimensional dynamical systems - a theory, which was developed in the second part of 20th century. The idea, which lead to the notion of topological equivalency, is clearly formulated in year 1815 during the attempts of the description of a class of functions, which, in some thence, is empty (but it is the result of 1970th).

Lamerey’s Diagrams, which were described in 1897, were used, in fact, by Adrien Legendre in 1808 and Evariste Galois in 1830. John von Neumann refuse his participation in the preparation of the work, which contained a mistake, but this lead to that he loosed the authority for the more important result, which is mentioned now without the name of von Neumann.

Notice, that pictures 1 (about Herschel’s and Babbage’s works) and 4 (about Pincherle’s work) are prepared by the graphical environment of \LaTeX{}, i.e. they are not copies of the original articles, but saves the meaning and the notations of the original ones.

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