Is the Low CMB Quadrupole a Signature of Spatial Curvature?

G. Efstathiou
Institute of Astronomy, Madingley Road, Cambridge, CB3 OHA.

20 March 2022

ABSTRACT
The temperature anisotropy power spectrum measured with the Wilkinson Microwave Anisotropy Probe (WMAP) at high multipoles is in spectacular agreement with an inflationary Λ-dominated cold dark matter cosmology. However, the low order multipoles (especially the quadrupole) have lower amplitudes than expected from this cosmology, indicating a need for new physics. Here we speculate that the low quadrupole amplitude is associated with spatial curvature. We show that positively curved models are consistent with the WMAP data and that the quadrupole amplitude can be reproduced if the primordial spectrum truncates on scales comparable to the curvature scale.

Key words: cosmic microwave background, cosmology.

1 INTRODUCTION
The recent measurements of the cosmic microwave background (CMB) anisotropies by WMAP (Bennett et al. 2003; Spergel et al. 2003; Peiris et al. 2003) are in striking agreement with a ‘concordance’ cosmology (hereafter referred to as ΛCDM) that has been built up over the last few years using many different data sets (see e.g. Bahcall et al. 1999, for a review). According to this model, the Universe is spatially flat with adiabatic, nearly scale invariant initial fluctuations as predicted in simple inflationary models (see e.g. Liddle and Lyth 2000). In addition, the present day Universe is dominated by a cosmological constant, consistent with the magnitude-redshift relation of Type Ia supernovae (Reiss et al. 1998; Perlmutter et al. 1999) and the large-scale distribution of galaxies (Efstathiou et al. 1990, 2002).

The high precision of the WMAP experiment leads to tight constraints on various cosmological parameters. For example, from the WMAP data alone Spergel et al. (2003) find that the best fit spatially flat ΛCDM models have a Hubble parameter of $h = 0.72 \pm 0.05$, scalar spectral index of $n_s = 0.99 \pm 0.04$ and physical baryonic and cold dark matter densities of $\omega_b = \Omega_b h^2 = 0.024 \pm 0.001$ and $\omega_c = \Omega_c h^2 = 0.12 \pm 0.02$ (all 1σ errors). Furthermore, from the temperature-polarization cross power-spectrum the WMAP data suggest a high optical depth for secondary reionization, $\tau = 0.17 \pm 0.04$, indicating significant star formation at high redshifts (e.g. Wyithe and Loeb 2003; Cen 2003; Haiman and Holder 2003).

There is, however, a potential problem with the simple concordance model. The WMAP results confirm the low amplitude of the CMB quadrupole seen by COBE (Hinshaw et al. 1996). As Bennett et al. (2003) comment, the amplitude of the quadrupole (and to a lesser extent the octopole) is low compared with the predictions of ΛCDM models that otherwise fit the rest of the power spectrum to extraordinarily high precision. The WMAP team present a convincing case that the low CMB multipoles are not significantly affected by foreground Galactic emission (see also Tegmark, de Oliveira-Costa and Hamilton 2003). The discrepancy is evident in a particularly dramatic form in the temperature angular correlation function, which shows an almost complete lack of signal on angular scales $\gtrsim 60$ degrees. According to Spergel et al. (2003), the probability of finding such a result in a spatially flat ΛCDM cosmology is about $1.5 \times 10^{-3}$. This is small enough to be worrying and may indicate the need for new physics. Since this paper was submitted, a number of authors have questioned the interpretation of Spergel et al.’s result (Gaztañaga et al. 2003; Tegmark et al. 2003, Bridle et al. 2003, Efstathiou 2003). The analysis of Efstathiou (2003) suggests that a more realistic probability of finding the observed quadrupole and octopole amplitudes in the concordance ΛCDM cosmology is more like 0.05. The data are suggestive of new physics, but not at a high level of statistical significance. Nevertheless, it is worth investigating models incorporating new physics and to identify any distinctive predictions (apart from low quadrupole and octopole amplitudes) that they might make.

What sort of new physics might be required? One possibility is to invoke fluctuations in a quintessence-like scalar field which can introduce features on scales comparable to the present day Hubble radius (Caldwell, Dave and Steinhardt 1998). The difficulty here is to cancel the large inte-
2 Constraints on spatial curvature and the geometrical degeneracy

It is well known that parameters estimated from the CMB anisotropies show strong degeneracies (e.g. Bond et al. 1997; Zaldarriaga, Spergel and Seljak 1997; Efstathiou and Bond 1999). The best known is the geometrical degeneracy between the matter density, vacuum energy and curvature. Models will have nearly identical CMB power spectra if they have identical initial fluctuation spectra and reionization optical depth, and if they have identical values of $\omega_b$, $\omega_c$ and acoustic peak location parameter

$$ R = \frac{\Omega_k^{1/2}}{[\Omega_k]^{1/2}} \sin K \left[ [\Omega_k]^{1/2} y \right], \quad (1a) $$

where

$$ y = \int_{a_r}^1 \frac{da}{[\Omega_m a + \Omega_k a^2 + \Omega_\Lambda a^4]^{1/2}}, \quad (1b) $$

$$ \sin K = \begin{cases} \sinh & \Omega_k > 0 \\ \sin & \Omega_k < 0 \end{cases} \quad (1c) $$

and $a_r$ is the scale factor at recombination normalised to unity at the present day.

The geometrical degeneracy is almost exact and precludes reliable estimates of either $\Omega_\Lambda$ or the Hubble parameter $h$ from measurements of the CMB anisotropies alone. As an example, Figure 1 shows the temperature power spectrum measured by WMAP together with a scale-invariant $\Lambda$ CDM model with $\omega_b$, $\omega_c$, $h$ and $\tau$ fixed to the WMAP best fit values quoted in the Introduction. The other curves in the figure show nearly degenerate models with $\Omega_k = -0.05$, $-0.10$ and $-0.20$ with parameters listed in Table 1. All of these models have been computed using the CMBFAST code of Seljak and Zaldarriaga (1996).

The theoretical power spectra plotted in Figure 1 are, by construction, identical at high multipoles and differ by less than the cosmic variance (indicated by the error bars in Figure 1) at low multipoles. They are therefore statistically indistinguishable using CMB data alone. Closed models with large values of $|\Omega_k|$ require low values of the Hubble constant and conflict with the direct measurement reported by the HST key project team (Freedman et al. 2001). Nevertheless, closed models with $\Omega_k \sim -0.05$ can be adjusted to have values of the cosmological parameters consistent with other data. In fact, Spergel et al. (2003) find that the best fit model to WMAP combined with other data sets is slightly closed with $\Omega_k = -0.02 \pm 0.02$. However, as Figure 1 demonstrates, this value and the error depend critically on the accuracy and interpretation of more complex `astrophysical' (i.e. non-CMB) data. We conclude that closed models are consistent with, and may even be marginally favoured by, the WMAP data.

3 Low CMB multipoles and spatial curvature

The metric of a Friedmann-Robertson-Walker model can be written in terms of a development angle $\chi$ as

$$ ds^2 = dt^2 - a^2(t)[K^{-1} \left| dx^2 + \sin^2 K \left( dr^2 + \sin^2 \theta d\phi^2 \right) \right|]. \quad (2)$$

where $K = -H_0^2 \Omega_k/c^2$ defines a curvature scale $R_c = |K|^{-1/2} = (c/H_0)|\Omega_k|^{-1/2}$. Linear perturbation theory of

### Table 1: Parameters for degenerate models

| $\Omega_k$ | $\Omega_b$ | $\Omega_c$ | $\Omega_\Lambda$ | $h$ |
|-----------|-----------|-----------|----------------|-----|
| 0.00      | 0.0463    | 0.2237    | 0.73           | 0.720 |
| -0.05     | 0.0806    | 0.3894    | 0.58           | 0.546 |
| -0.10     | 0.1114    | 0.5386    | 0.45           | 0.446 |
| -0.20     | 0.1714    | 0.8286    | 0.20           | 0.374 |

Note: These models have identical values of $\omega_b$ and $\omega_c$ fixed to the best fit values from WMAP.

---

† A closed inflationary model that can produce such a truncation is discussed by Contaldi et al. 2003.
FRW models with arbitrary curvature is well developed (see for example, Wilson 1983; Abbott and Schaefer 1986; Lyth and Woszczyna 1995; White and Scott, 1996; Zaldarriaga, Seljak and Bertschinger 1998; Zaldarriaga and Seljak 2000). As is well known, in a closed FRW model the eigenvalues $\beta$ of the the Laplacian are discrete and related to physical comoving wavenumber $k$ by

$$\beta^2 = (1 + k^2 R_c^2). \quad (3)$$

Modes with $\beta = 1$ and 2 are pure gauge modes (Abbott and Schaefer 1986) and so the complete spectrum of modes extends from $\beta = 3$ to infinity.

The mode spectrum in a closed universe therefore contains a characteristic scale – the curvature scale $R_c$. In the absence of a detailed model for the origin of fluctuations in a closed universe it is not obvious how to generalise the concept of ‘scale-invariant’ fluctuations on scales comparable to the curvature scale. For example, if the potential fluctuations are assumed to be constant per logarithmic interval in wavenumber $k$, the initial density fluctuation spectrum in the CMBFAST code must be set to

$$P_\nu(\beta) \propto (\beta^2 - 4)^2 / \beta(\beta^2 - 1), \quad (4a)$$

(see e.g. White and Bunn 1995). This form of the power spectrum was used to compute the theoretical models plotted in Figure 1. Starobinsky (1996) argues for equation (4a) based on slow roll inflation and adopting the conformal vacuum as an initial state. However, the initial conditions of this model are not understood in terms of a fundamental theory.

In the absence of a well-motivated theory, we take the CMB data at face value and investigate a power spectrum that truncates on scales comparable to the curvature scale. For example, the heuristic form

$$P_\nu(\beta) \propto \left( \frac{(\beta^2 - 4)^2}{\beta(\beta^2 - 1)} \right) \left[ 1 - \exp \left( -\frac{(\beta - 3)}{4} \right) \right], \quad (4b)$$

damps the spectrum at low values of $\beta$. The consequences for the CMB anisotropies are shown in Figure 2. The WMAP temperature power spectrum at low multipoles is shown in the upper panel and the TE temperature-polarization cross power spectrum is shown in the lower panel. The closed universe models from Table 1 with the heuristic form of the power spectrum of equation (4b) are also plotted. The important point to note from this Figure is that a truncation of the primordial power spectrum at $\beta \lesssim 5$ can produce a significant change to the low order multipoles even for values of $\Omega_k$ as small as $-0.05$, which are difficult to rule out observationally. The low amplitudes of the CMB quadrupole and octopole may therefore be linked to the curvature scale.

Spergel et al. (2003), Tegmark et al. (2003) and Uzan et al. (2003) comment that the low CMB multipoles may be an indication of a discrete spectrum in a universe of finite size and non-trivial topology. The models of this Section show that a periodic universe is not necessary if the initial fluctuation spectrum truncates at around the curvature scale.

4 CONCLUSIONS

The low amplitude of the quadrupole may be the first tentative indication of discordance with the otherwise remarkably successful $\Lambda$CDM cosmology. It is worth taking seriously as a possible pointer towards new physics.

Although CMB data are consistent with a spatially flat universe, the geometrical degeneracy makes it impossible to constrain $\Omega_k$ accurately from CMB data alone. To pin down $\Omega_k$ to an accuracy of 0.05 or better requires more messy ‘astrophysical’ data, for example, galaxy clustering or direct measurements of the Hubble constant. Unless one has a good understanding of the systematic errors in these more complicated observations, it will not be easy to rule out models with a small positive curvature. Nevertheless, it may be possible with future experiments to constrain $\Omega_k$ to this kind of accuracy.

A closed Universe possesses a characteristic curvature scale $R_c = (c/H_0)|\Omega_k|^{-1/2}$. The proposal explored in this note is that the low CMB quadrupole amplitude may be related to a truncation of the primordial fluctuation spectrum on the curvature scale. If this, admittedly speculative, pro-
posal were right, we would need to add the small positive curvature to the list of ‘cosmic coincidences’ (most notably the small value of Λ) that plague modern cosmology. Furthermore, a positive cosmological curvature would require a radical shift from the simple inflationary paradigm, which many cosmologists might feel is too high a price to pay to explain two or three of the thousand or more CMB multipoles. However, discrepancies between theory and observations must be taken seriously, wherever and whenever they occur.

Acknowledgments: I thank Sarah Bridle, Anthony Challinor, Antony Lewis, Martin Rees and Jochen Weller for helpful comments.

REFERENCES
Abbott L.F., Schaefer R.K., 1986, ApJ, 308, 456.
Bahcall N.A., Ostriker J.P., Perlmutter S., Steinhardt P.J., 1999, Science, 284, 1481.
Bennett C.L. et al, 2003, ApJ in press, astro-ph/0302207.
Bond J.R., Efstathiou G., Tegmark M., 1997, MNRAS, 291, L33.
Bridle S.L., Lewis A.M., Weller J., Efstathiou G., 2003, MNRAS, 374, L72.
Caldwell R.R., Dave R., Steinhardt P.J., 1998, PRL, 80, 1582.
Cen R., 2003, ApJ, 591, L5.
Contaldi C. Peloso M., Kofman L., Linde A., 2003, astro-ph/0303636.
DeDeo S., Caldwell R.R., Steinhardt P.J. 2003, PRD, D67, 103509.
Efstathiou G., 2003, MNRAS, submitted. astro-ph/0306431.
Efstathiou G., Sutherland W. J., Maddox S. J., 1990, Nature 348, 705.
Efstathiou G., Bond J.R., 1999, MNRAS, 304, 75.
Efstathiou G. et al., 2002, MNRAS, 330, L39.
Ellis G., Stoeger W., McEwan P., Dunsby P., 2002, Gen. Rel. Grav., 34, 1445.
Freedman W.L. et al., 2001, ApJ., 553, 47.
Gaztañaga E., Wagg J., Multamaki T., Montana A., Hughes D.H., 2003, submitted to MNRAS. astro-ph/0304178.
Gratton S., Lewis A., Turok N., 2002, Phys.Rev. D65, 043513.
Haiman Z., Holder G.P., 2003, astro-ph/0302403.
Hinshaw G., Banday A.J., Bennett C.L., Gorski K.M., Kogut A., Smoot G.F., Wright E.L., 1996, ApJ, 464, L17.
Liddle A.R., Lyth D.H., 2000, Cosmological Inflation and Large-Scale Structure, Cambridge University Press, Cambridge.
Linde A.D., 1995, Phys. Lett. B, 351, 99.
Linde A.D., 2003, astro-ph/0303245.
Lyth D.H., Woszczy A., 1995, Phys. Rev. D, 52, 3338.
Peiris H.V. et al., 2003, submitted to ApJ, astr-ph/0302225.
Perlmutter S. et al., 1999, ApJ, 517, 565.
Reiss A.G. et al., 1998, AJ, 116, 1009.
Seljak U., Zaldarriaga M., 1996, ApJ, 469, 437.
Spergel D.N. et al., 2003, submitted to ApJ, astr-ph/0302209.
Starobinsky A.A., 1996, in Cosmoparticle Physics, eds M. Yu. Khlopov, M.E. Prokhorov, A.A. Starobinsky, J. Tran Thanh Van, Edition Frontiers, p 43.
Tegmark M., de Oliveira-Costa A., Hamilton A., 2003, astro-ph/0302496.
Uzan J-P., Kirchner U., Ellis G.F.R., 2003, astro-ph/0302597.
Wilson M., 1983, ApJ., 273, 2.
White M., Bunn E., 1995, ApJ, 450, 477.
White M., Scott D., 1996, ApJ, 459, 415.
Wyithe S., Loeb A., 2003, ApJ, 586, 693.
Zaldarriaga M., Spergel D.N., Seljak U., 1997 ApJ, 488, 1.