DECOMPOSABLE EDGE POLYTOPES OF FINITE GRAPHS

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ABSTRACT. Edge polytopes is a class of interesting polytope with rich algebraic
and combinatorial properties, which was introduced by Ohsugi and Hibi. In this
paper, we follow a previous study on cutting edge polytopes by Hibi, Li and Zhang.
Instead of focusing on the algebraic properties of the subpolytopes as the previous
study, in this paper, we take a closer look on the graphs whose edge polytopes
are decomposable. In particular, we answer two important questions raised in the
previous study about 1) the relationship between type I and type II decomposable
graphs and 2) characterization of decomposable graphs.

INTRODUCTION

Let \( \mathcal{P} \) be an integral polytope in \( \mathbb{R}^d \). Here we say a polytope is integral if all
vertices of the polytope are integer points. We say that \( \mathcal{P} \) is decomposable if there
exist a hyperplane \( \mathcal{H} \) with \( \mathcal{H} \cap (\mathcal{P} \setminus \partial \mathcal{P}) \neq \emptyset \) such that each of the convex polytope
\( \mathcal{P} \cap \mathcal{H}^{(+)} \) and \( \mathcal{P} \cap \mathcal{H}^{(-)} \) is integral. In this paper, we discuss decomposability
of a special class of integral polytopes, called edge polytopes. Edge polytopes are
introduced by Ohsugi and Hibi in [3]. Edge polytopes are integral polytopes arising
from finite connected graphs. Let \( G \) be a finite connected simple graph with
vertex set \( V = [d] = \{1, \ldots, d\} \) and edge set \( E(G) = \{e_1, \ldots, e_n\} \). Let \( e_i \) be the
\( i \)-th unit coordinate vector of the Euclidean space \( \mathbb{R}^d \). If \( e = (i, j) \) is an edge of \( G \),
then we set \( \rho(e) = e_i + e_j \in \mathbb{R}^d \). The edge polytope \( \mathcal{P}_G \) of \( G \) is the convex hull of
\( \{\rho(e_1), \ldots, \rho(e_n)\} \) in \( \mathbb{R}^d \).

The only integer point belongs to an edge polytope are its vertices. It follows
from this fact that an edge polytope is decomposable if and only if there exist a hyperplane \( \mathcal{H} \) which is not supporting hyperplane of \( \mathcal{P}_G \) and such that for each edge
\( E \) of \( \mathcal{P}_G \) with \( \mathcal{H} \cap E \neq \emptyset \) it follows that \( E \subset \mathcal{H} \) or \( \mathcal{H} \cap E \) is an end point of \( E \).

In [1], type I and II decomposability for edge polytopes are introduced and an
algorithm to decide decomposability is given. In this paper, we carry out the study of
edge polytope decomposability and take a closer look at the graphs with type I and
type II decomposable edge polytopes. In Section 2, we characterize decomposability
of edge polytopes in terms of the underlying graphs.

1. EDGE POLYTOPES AND ITS DECOMPOSABILITY.

Recall that a convex polytope is integral if all of its vertices have integral coor-
dinates: in particular, \( \mathcal{P}_G \) is an integral polytope. Let \( \partial \mathcal{P} \) denote the boundary of

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a polytope $\mathcal{P}$. We say that $\mathcal{P}$ is decomposable if there exists a hyperplane $\mathcal{H}$ of $\mathbb{R}^d$ with $\mathcal{H} \cap (\mathcal{P} \setminus \partial \mathcal{P}) \neq \emptyset$ such that each of the convex polytopes $\mathcal{P} \cap \mathcal{H}^{(+)}$ and $\mathcal{P} \cap \mathcal{H}^{(-)}$ is integral. Here $\mathcal{H}^{(+)}$ and $\mathcal{H}^{(-)}$ are the closed half-space of $\mathbb{R}^d$ with $\mathcal{H}^{(+)} \cap \mathcal{H}^{(-)} = \mathcal{H}$. Such a hyperplane $\mathcal{H}$ is called a separating hyperplane of $\mathcal{P}$. We say a graph $G$ is decomposable if edge polytope $\mathcal{P}_G$ of $G$ is decomposable. A simple graph is a graph with no loops and no multiple edges. Let $G$ be a finite connected simple graph with vertex set $V = [d] = \{1, \ldots, d\}$ and edge set $E(G) = \{e_1, \ldots, e_n\}$. Let $e_i$ be the $i$-th unit coordinate vector of the Euclidean space $\mathbb{R}^d$. If $e = (i, j)$ is an edge of $G$, then we set $\rho(e) = e_i + e_j \in \mathbb{R}^d$. The edge polytope $\mathcal{P}_G$ of $G$ is the convex hull of $\{\rho(e_1), \ldots, \rho(e_n)\}$ in $\mathbb{R}^d$.

**Lemma 1.1.** Suppose that $\mathcal{P}_G$ is an edge polytope arising from finite connected graph $G$. Then, the only integer points in $\mathcal{P}_G$ are its vertices.

**Proof.** Since all the vertices of edge polytope $\mathcal{P}_G$ are of the form $e_i + e_j$, edge polytope lies on the hyperplane $x_1 + \cdots + x_d = 2$. So we may assume that all the integer points in $\mathcal{P}_G$ are of the form $e_i + e_j$. Since every edge $e \in E(G)$ does not belongs to convex hull of $\{\rho(e') : e' \in E(G), e' \neq e\}$, there are no integer points but $\rho(e)$.\qed

The vertices of the edge polytope $\mathcal{P}_G$ of $G$ are $\{\rho(e_1), \ldots, \rho(e_n)\}$, but not all edges of the form $(\rho(e_i), \rho(e_j))$ actually occur. In the recent research, the number of edges of edge polytopes has been discussed (\cite{1}). For $i \neq j$, let $co(e_i, e_j)$ be the convex hull of the pair of $\{\rho(e_i), \rho(e_j)\}$. The edges of $\mathcal{P}_G$ will be a subset of these $co(e_i, e_j)$. For edges $e = (i, j)$ and $f = (k, \ell)$, call the pair of edges $(e, f)$ cycle-compatible with $C$ if there exists a 4-cycle $C$ in the subgraph of $G$ induced by $\{i, j, k, \ell\}$ (in particular, this implies that $e$ and $f$ do not share any vertices). The following result allows us to identify the $co(e_i, e_j)$ that are actually edges of $\mathcal{P}_G$ using the notion of cycle-compatibility.

**Lemma 1.2** (\cite{2}). Let $e$ and $f$ be edges of $G$ with $e \neq f$. Then $co(e, f)$ is an edge of $\mathcal{P}_G$ if and only if $e$ and $f$ are not cycle-compatible.

Since the only integer points of edge polytopes are its vertices, the condition that $\mathcal{P}_G \cap \mathcal{H}^{(+)}$ and $\mathcal{P}_G \cap \mathcal{H}^{(-)}$ are integral is equivalent to the following: There exist a hyperplane $\mathcal{H}$ which is not a supporting hyperplane of $\mathcal{P}_G$ and such that for each edge $E$ of $\mathcal{P}_G$ with $\mathcal{H} \cap E \neq \emptyset$ it follows that $E \subset \mathcal{H}$ or $\mathcal{H} \cap E$ is an end point of $E$. This is by Lemma 1.2 equivalent to the following: for any pair of edges $e, f \in E(G)$ such that $\rho(e) \in \mathcal{H}^{(+)} \cap \partial \mathcal{P}$ and $\rho(f) \in \mathcal{H}^{(-)} \cap \partial \mathcal{P}$, $e$ and $f$ are cycle-compatible.

**Proposition 1.3** (\cite{1}). Let $G$ be a finite connected simple graph on $[d]$ and suppose that $\mathcal{P}_G \subset \mathbb{R}^d$ is decomposable by $\mathcal{H}$. Then we can restrict attention to $\mathcal{H}$ of the following form:

$$\mathcal{H} : \sum_{i} \pm a_i x_i = 0 \ (a_i \in \{0, 1, -1\})$$
Proposition 1.3 allows us to assume that $\mathcal{H}^{(+)}$ contains points $(x_1, \ldots, x_n)$ where $\sum a_i x_i \geq 0$ and $\mathcal{H}^{(-)}$ contains points $(x_1, \ldots, x_n)$ where $\sum a_i x_i \leq 0$. For $(i, j) \in E(G)$, let the sign of $(i, j)$ be the sign of $a_i + a_j$, the signature of $(i, j)$ be $\{a_i, a_j\}$ and the weight of vertex $i$ be $a_i$. These notations enable us to call an edge $(i, j)$ \textit{“positive”}, \textit{“negative”} or \textit{“zero”}, corresponding to whether the associated vertex $\rho((i, j)) \in \mathcal{P}_G$ is in $\mathcal{H}^{(+)} \cap \partial \mathcal{P}$, $\mathcal{H}^{(-)} \cap \partial \mathcal{P}$, or $\mathcal{H}$.

In section 2, we will repeatedly use the following Propositions 1.4 and 1.5:

**Proposition 1.4** ([2]). Let $G$ be a finite simple graph on $[d]$. Then, $\dim \mathcal{P}_G = d - r - 1$, where $r$ is the number of bipartite connected components of $G$.

**Proposition 1.5** ([1]). Suppose $G$ is decomposable. Then we must have at least one positive edge and at least one negative edge, and we can assume one of the following two cases for the vertices of $G$:

(I) There are no vertices with weight 0. All positive edges have signature $\{1, 1\}$ and all negative edges have signature $\{-1, -1\}$.

(II) There is at least one vertex with weight 0. All positive edges have signature $\{1, 0\}$ and all negative edges have signature $\{-1, 0\}$.

We call edge polytope $\mathcal{P}_G$ is \textit{type I} (or \textit{type II}) decomposable if there exist a separating hyperplane $\mathcal{H}$ satisfying condition (I) (or (II)) in Proposition 1.5. We say $G$ is decomposable if edge polytope $\mathcal{P}_G$ of $G$ is decomposable.

**Example 1.6.** Following graph $G$ is type I and type II decomposable.

![Type I and II decompositions](image)

**Figure 1.** Type I and II decompositions

The type I decomposition of $\mathcal{P}_G$ is given with the separating hyperplane $\mathcal{H} : -x_1 - x_2 + x_3 + x_4 - x_5 + x_6 = 0$. The type II decomposition is given with the separating hyperplane $\mathcal{H} : -x_1 + x_4 - x_5 + x_6 = 0$.

By using the fact; for any complete multipartite graph $G$, every vertex disjoint two edges are cycle compatible, we can say the following.

**Lemma 1.7.** Complete multipartite graphs are type I decomposable if there exist vertex disjoint two edges.
Proof. Let $e, f$ be vertex disjoint two edges. Assume that $e$ is a positive edge with weight $\{1, 1\}$, $f$ is a negative edge with weight $\{-1, -1\}$. We can set weights either 1 or $-1$ for all the other vertices. In type I sign arrangement, we do not have a pair of positive and a negative edge that has a common vertex. Thus every pair of positive and negative edges are cycle-compatible, as desired. □

In the following example, we use an easy operation. We consider graph $G$ and path of length 3 $P_3 = (x_0, x_1, x_2, x_3)$. For any edge $\{i, j\} \in E(G)$, we join the path as $x_0 = i, x_3 = j, x_1 \neq k, x_2 \neq k$ $(k \in V(G))$ and get the new graph. We call this operation, attach a 4-cycle to $G$ at edge $\{i, j\}$.

**Example 1.8.** Let $G$ be a graph given in Figure 2 (a). After attaching a 4-cycle to $G$ at edge $\{2, 3\}$, we have the graph Figure 2 (b).

![Graph Diagram](image)

**Figure 2. Attaching 4-cycle**

**Proposition 1.9.** Let $G$ be a finite connected graph. Let $G'$ be a graph obtained by attaching a 4-cycle to any edge of $G$. Then, $P_{G'}$ is type II decomposable regardless of which edge we choose.

Proof. We set the weight 1 for one of the internal point of the attached path and $-1$ for the other one. We set the weight 0 to all the other vertices of $G'$. Then the unique pair of the positive and negative edges are cycle-compatible. □

We call the graph in Figure 3 (a) tri-pan. We consider two tri-pans, and name the vertices as Figure 3 (b) and join two tri-pans as $x = x', y_1 = y'_1$, then we get the following graph in Figure 3 (c).

We call the graph in Figure 3 (c) Similarly, we define $n$-joined tri-pan $T(n)$. For example, Figure 4 is $T(5)$. Let $N_G(k)$ denote that neighbor set of $k$ in $G$.

**Proposition 1.10.** Suppose that $T(n)$ is a $n$-joined tri-pan. Then $P_{T(n)}$ is indecomposable.

Proof. $P_{T(1)}$ is indecomposable because there is no 4-cycle in $T(1)$. Suppose that $n \geq 2$. First, we check type I decomposability. Assume that $P_{T(n)}$ is type I decomposable. Fix an sign arrangement of $G$. Since $P_{T(n)}$ is decomposable, we have at least one
4 cycle with positive and negative edge in $T(n)$ that satisfies cycle-compatibility. Assume that the 4 cycle is $(x, k - 1, y_{k-1}, k)$ in Figure 5. Without losing generality, we may assume weights of vertices $x, k - 1, y_{k-1}, k$ are $1, 1, -1, -1$ or $-1, -1, 1, 1$. In the first case, take a look at the weight of $y_{k-2}$. It is not possible to suppose the weight of $y_{k-2}$ is 1 because then a positive edge $\{y_{k-2}, k - 1\}$ and a negative edge $y_{k-1}, k$ do not satisfy cycle compatibility. On the other hand, we can not set the weight $-1$ for $y_{k-2}$ because again from cycle compatibility of positive and negative edges. Similarly, if we are in the second case, we can not choose the 4 cycle without losing cycle compatibility. Next, we check type II decomposability. Suppose that $T(n)$ is type II decomposable. We have a 4-cycle $(x, k - 1, y_{k-1}, k)$ with weights $\{1, -1, 0, 0\}$. The weight of the edge $\{k - 1, y_{k-1}\}$ must be $\{0, 0\}$. Otherwise, We can not set the weight of the vertex of $N_{T(n)}(k - 1) \cap N_{T(n)}(y_{k-1})$ because one of the following occurs.

(i) A weight of a edge is $\{1, 1\}$ or $\{-1, -1\}$.
(ii) One of the positive edge and one of the negative edge have a common vertex.
Assume that the weight of $\{k - 1, y_{k-1}\}$ is $\{0, 0\}$ then the weight of $k$ is either 1 or $-1$. Let $N(k) \cap N(y_{k-1}) = y_k$. Then, we can not set the weight for $y_k$ because again one of the above occurs.

\[\square\]

**Theorem 1.11.** There exist infinite number of graphs that is both type I and type II decomposable. Similary, one has type I but not type II, type II but not type I and neither type I nor type II.
(i) Let $G$ be a complete multipartite graphs with at least 4 vertices. Let $G'$ be a graph obtained by attaching a 4-cycle at any edge of $G$. Then $P_{G'}$ is type I and II decomposable.

(ii) Let $G$ be a complete graph with at least 4 vertices. Then $P_G$ is type I decomposable and type II indecomposable.

(iii) Let $T(n)$ be a $n$-joined tri-pan with $n \geq 2$ and $G$ be a graph obtained by attaching a 4-cycle at any edge of $T(n)$. Then $P_G$ is type II decomposable and type I indecomposable.

(iv) Let $T(n)$ be a $n$-joined tri-pan. Then $P_{T(n)}$ is indecomposable.

Proof. We know by Proposition 1.9 that the edge polytopes of (i) and (iii) are type II decomposable. Also, (iv) is indecomposable as we shown in Proposition 1.10. We have to show the rest. First we show (i) is type I decomposable. In the complete multipartite graphs every pair of vertex disjoint edges are cycle-compatible. We set the weight 1 for 2 vertices of the chosen edge $e$ which we attached the 4-cycle and set the weight $-1$ for all the other vertices. Then this sign arrangement implies type I since $e$ and all the other edges that has no common vertex with $e$ are cycle-compatible.

Next we show (ii) is type I decomposable. Let $G = K_d$ with vertex set $[d]$. We can set the weight 1 to vertex 1 and 2 and $-1$ to the others. Since all distinct pair of edges are cycle-compatible, complete graphs are type I decomposable as required. (ii) is not type II decomposable. Let $G = K_d$ with vertex set $[d]$. Suppose that $P_G$ is decomposable. then there exist at least one pair of edges with positive and negative sign, say $e = (i, j)$ and $f = (k, l)$. By symmetry of complete graphs we may assume that weights of $i, j, k$ and $l$ are 1, 0, $-1$ and 0 We can not make this sign arrangement because a positive edge $(i, j)$ and a negative edge $(k, l)$ share a vertex.

(iii) is not type I decomposable. We can not set the weights for the shape of $T(2)$ without breaking cycle-compatability of positive and negative edges, as we see in the proof of Proposition 1.10.

□

2. Characterization of decomposable edge polytopes

In this section, we discuss about decomposable edge polytopes in terms of underlying graphs. Let $V'$ be the subset of $V$, $G[V']$ e the induced subgraph of $G$ with vertex set $V'$ and $N(V')$ be the neighbour set of $V'$. We call a family of vertex set
\{V_1, \ldots, V_n\}, V_i \in V \text{ is a vertex partition of } V \text{ if it satisfies } \bigcup V_i = V \text{ and } V_i \cap V_j = \emptyset \text{ for any } i \neq j. \text{ We call a graph } G \text{ is empty if there exist no edge in } G.\\

**Lemma 2.1.** G is bipartite if and only if there exist a separating hyperplane \( H = \sum a_i x_i \) with \( |a_1| = \ldots = |a_d| = 1 \) such that all signatures of edges of G are 0.

**Proof.** Suppose \( \{V_1, V_2\} \) are a vertex partition of G. We set weight 1 for every vertex in \( V_1 \) and \(-1\) for every vertex in \( V_2 \). It has proved because \( G[V_1] \) and \( G[V_2] \) are not empty.

Conversely suppose we have an odd cycle \( \{v_1, v_2, \ldots, v_{2n+1}, v_1\} \) in G where \( n \) is a positive integer. If weight of \( v_1 \) is 1, then we have to set weight of \( v_2 \) to be \(-1\) and \( v_3 \) to be 1. By continuing this operation, we get that weight of vertex \( v_{2n+1} \) is 1. This contradicts that there is no positive edge. Similarly, a contradiction occur when we start from weight of vertex \( v_1 \) with \(-1\). \( \square \)

**Proposition 2.2.** Suppose that G is a finite connected simple graph on \([d]\).

(i) Edge polytope \( P_G \) is type I decomposable if and only if there exist a vertex partition \( \{V_+, V_-\} \) of G such that \( G_+ = G[V_+] \) and \( G_- = G[V_-] \) are not empty graph and every pair of edges \( (e \in E(G_+), f \in E(G_-)) \) is cycle-compatible.

(ii) Edge polytope \( P_G \) is type II decomposable if and only if there exist a vertex partition \( \{V_1, V_2, V_3, V_4, V_5\} \) of G that satisfies the following conditions. Let \( E_{i,j} \) denote the set of edges between \( V_i \) and \( V_j \).

1. \( G[V_1 \cup V_2] \) is a bipartite graph with a bipartition \( V_1, V_2 \). (2) \( |E_{1,4}| \geq 1, |E_{2,3}| \geq 1 \) and every pair of edges \( e, f \) \( (e \in E_{1,4}, f \in E_{2,3}) \) is cycle-compatible. (3) \( |E_{1,3}| = |E_{1,5}| = |E_{2,4}| = |E_{2,5}| = 0 \).

**Proof.** (i)(if part) Suppose that G is type I decomposable with vertex partition \( V_1, V_2 \). We set weights of vertices in vertex set \( V_1 \) to be 1 and \( V_2 \) to be \(-1\). Then it follows \( P_G \) is type I decomposable. (only if part) Suppose that G is a decomposable graph with a type I decomposition. Fix a sign arrangement on G that gives type I decomposition. By Proposition 1.5 we may say that every weight is 1 or \(-1\). Let \( V_+ \) (resp. \( V_- \)) denote the set of 1-weighted (resp. \(-1\)-weighted) vertices. Since G is type I decomposable, we must have at least one positive edge in \( G[V_+] \) and at least one negative edge in \( G[V_-] \). We know that every pair of positive and negative edges are cycle compatible. Thus G satisfies (1).

(ii) (if part) Suppose we have such a vertex partition. We set weights of vertices in \( V_1 \) to be 1, \( V_2 \) to be \(-1\), \( V_3, V_4, V_5 \) to be 0. Then it is obvious that G is type II. (only if part) Suppose that G is type II decomposable. Fix a sign arrangement that gives type II decomposition. Let \( V_+, V_-, V_0 \) be the set of, namely \(-1\), 0-weighted vertices. Let \( V_1 = V_+, V_2 = V_- \). Since this sign arrangement gives type II decomposition, we do not have any edge with weight \((1, 1)\) or \((-1, -1)\). Therefore, condition(1) is satisfied. We set \( V_4 = N(V_+) \cap V_0 \) and \( V_3 = V(N_-) \cap V_0 \) and \( V_5 = V \setminus (V_1 \cup V_2 \cup V_4 \cup V_3) \). Note that \( N(V_+) \cap N(V_-) = \emptyset \). Thus, condition (3) is satisfied. Also condition(2) is satisfied because every pair of positive and negative edges are cycle-compatible. \( \square \)
We call subgraph $G'$ of $G$ a spanning subgraph of $G$ if vertex set of $G'$ is $[d]$ with no isolated vertex in $G'$.

**Proposition 2.3.** Suppose that edge polytope $P_G$ of a connected simple graph $G$ with vertex set $[d]$ and edge set $\{e_1, \ldots, e_n\}$ is decomposable with hyperplane $H$. Then, an integral polytope defined by $P_G \cap H$ is again an edge polytope.

*Proof.* We claim that $G_0$ is composed by all zero edges of the decomposition. Since $H$ does not intersect any edge of $P_G$, The convex polytope lying on the separating hyperplane is $P_G \cap H = \text{conv}(\{\rho(e_i) : \rho(e_i) \in H\})$. That coincides with $P_{G_0} = \text{conv}(\{\rho(e_i) : \rho(e_i) \in H\})$. \qed

We claim that, $P_{G_0}$ appearing in Proposition 2.3 is observed by the spanning subgraph $G_0$ of $G$ consists of zero edges of the sign arrangement.

**Proposition 2.4.** Let $G$ be a connected bipartite graph. $P_G$ is type I decomposable if and only if $P_G$ is type II decomposable.

*Proof.* "if" part is given in [1]. Now we have to show "only if" part. Suppose that $P_G$ is type I decomposable. Let $V_1, V_2$ be a bipartition of $G$, put the vertices of $V_1$ on the left and $V_2$ on the right. Since $P_G$ is type I decomposable, let $H$ be a sign arrangement on $G$ that is type I. Then we can see that on both of the left side and the right side, there are both positive and negative signs. Now apply the following change to the sign arrangement $H$: change all the positive vertices on the left to zeros, and change all the negative vertices on the right to zeros. We call this new sign arrangement $H'$. We claim that with this $H'$ makes $G$ type II decomposable. First, it is clear that in $H'$, all the edges have the form $(0,1), (0,-1)$ or $(0,0)$. Then it is left to show that any pair of positive edge and negative edge in $H'$ satisfies the cycle compatibility. This is true because an edge in $H'$ is positive (or negative) if and only if that edge in $H$ is positive (or negative). Since any pair of positive edge and negative edge in $H$ satisfies cycle-compatibility, so does in $H'$. \qed

**Example 2.5.** For the following bipartite graph, we can give a type I sign arrangement as Figure 6 (a). We have two type II sign arrangement obtained by Proposition 2.4. Those are Figure 6 (b) and (c).

![Figure 6](image_url)
As the corollary of Proposition 2.2, Proposition 2.3 and Proposition 2.4, we can characterize decompositions of edge polytopes as the followings.

**Theorem 2.6.** Suppose that $G$ is a connected simple graph.

(a) If $G$ has at least one odd cycle and type I decomposable, then $G_0$ is a connected bipartite graph.

(b) If $G$ has at least one odd cycle and type II decomposable, then $G_0$ consists of exactly 2 connected components. One is bipartite and the other is not bipartite.

(c) If $G$ is bipartite and type I decomposable, then $G_0$ is consists of exactly 2 connected components.

**Proof.** In this proof we repeatedly use Proposition 1.4 that claims dimension of edge polytope is reduced by the number of connected bipartite components.

(a) Let $G$ be a connected non-bipartite graph. Then $\dim(\mathcal{P}_G) = d - 1$ because only one connected component is not bipartite. It is known from Lemma 2.1 that $G_0$ is bipartite. If $G_0$ is not connected, then we have at least 2 connected bipartite components. On the other hand, we know that $\dim(\mathcal{P}_G) = d - 1$ and hence $\dim(\mathcal{P}_{G_0}) = \dim(\mathcal{H}) = d - 2$. Thus, $G_0$ is connected.

(b) Since $G$ is type II decomposable, Let $V_1, \ldots, V_5$ be a vertex partition of $G$ given by Proposition 2.2. As we see in the proof of Proposition 2.2 we obtain $G_0 = G[V_1 \cup V_2] \cup G[V_3 \cup V_4 \cup V_5]$. Now we show that $G[V_1 \cup V_2]$ is connected bipartite and $G[V_3 \cup V_4 \cup V_5]$ is connected non-bipartite. $G[V_1 \cup V_2]$ is bipartite because $G[V_1]$ and $G[V_2]$ is empty graphs. $G[V_1 \cup V_2]$ is connected, otherwise $\dim(\mathcal{H}) \leq \dim(\mathcal{P}_G) - 2$, it contradicts $\mathcal{H}$ is a separating hyperplane. Suppose $G[V_3 \cup V_4 \cup V_5]$ is bipartite. Then we have at least 2 bipartite connected components, again a contradiction. That implies $G[V_3 \cup V_4 \cup V_5]$ is not bipartite. We may assume that for any vertex $i \in V_3$ (or $V_4$), there exist a vertex $j \in V_2$ (or $V_1$) such that \{i, j\} is an edge in $G$ and for any vertex. (If we have vertex which does not have neighbors in $V_2$ (or $V_1$), we may move the vertex to $V_5$. Then new vertex partition again satisfies our condition.) Since vertices in $V_3$ and $V_4$ are end points of positive or negative edges, $G[V_3 \cup V_4]$ form a complete bipartite graph. Thus, $G[V_3 \cup V_4]$ is connected. Our edge deleision from $G$ to $G_0$ is exactly the deleision of $E_{1,4}$ and $E_{2,3}$. $G$ is connected and vertices in $V_3$ are connected. Our edge deleision from $G$ to $G_0$ is exactly the deleision of $E_{1,4}$ and $E_{2,3}$. $G$ is connected and vertices in $V_5$ are not connected to $V_1$ and $V_2$. That implies for any vertex $v \in V_5$, there exist a vertex $u \in V_3 \cup V_4$ and a finite path from $v$ to $u$ in $G[V_3 \cup V_4 \cup V_5]$. Therefore, $G[V_3 \cup V_4 \cup V_5]$ is connected.

(c) Since $G$ is connected bipartite, we know $\dim(\mathcal{P}_G) = d - 2$ and hence $\dim(\mathcal{H}) = d - 3$. Since $G_0$ is a subgraph of $G$, all the connected components are bipartite. Then $G_0$ consists of 2 connected bipartite components.

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