Approximation for Fault-Tolerant Virtual Backbone in Wireless Sensor Networks

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Abstract

To save energy and alleviate interferences in a wireless sensor network, the usage of virtual backbone was proposed. Because of accidental damages or energy depletion, it is desirable to construct a fault tolerant virtual backbone, which can be modeled as a $k$-connected $m$-fold dominating set (abbreviated as $(k,m)$-CDS) in a graph. A node set $C \subseteq V(G)$ is a $(k,m)$-CDS of graph $G$ if every node in $V(G) \setminus C$ is adjacent with at least $m$ nodes in $C$ and the subgraph of $G$ induced by $C$ is $k$-connected. In this paper, we present an approximation algorithm for the minimum $(3,m)$-CDS problem with $m \geq 3$. The performance ratio is at most $\gamma$, where $\gamma = \alpha + 8 + 2 \ln(2\alpha - 6)$ for $\alpha \geq 4$ and $\gamma = 3\alpha + 2 \ln 2$ for $\alpha < 4$, and $\alpha$ is the performance ratio for the minimum $(2,m)$-CDS problem. Using currently best known value of $\alpha$, the performance ratio is $\ln \delta + o(\ln \delta)$, where $\delta$ is the maximum degree of the graph, which is asymptotically best possible in view of the non-approximability of the problem. This is the first performance-guaranteed algorithm for the minimum $(3,m)$-CDS problem on a general graph. Furthermore, applying our algorithm on a unit disk graph which models a homogeneous wireless sensor network, the performance ratio is less than 27, improving previous ratio 62.3 by a large amount for the $(3,m)$-CDS problem on a unit disk graph.

Keywords: wireless sensor network; fault-tolerance; connected dominating set; connectivity; approximation algorithm.

1 Introduction

A wireless sensor network (WSN) consists of spatially distributed autonomous sensors to monitor physical or environmental condition, and to cooperatively pass the sensed data through the network. The development of wireless sensor networks was originally motivated by military applications, and today they are widely used in many industrial...
fields and everyday life, such as industrial process monitoring, traffic control, smart home, etc. If all sensors frequently transmit messages in a flooding way, then a lot of energy is wasted and intense interferences are created. To solve these problems, the concept of virtual backbone was proposed by Das and Bhargharan \cite{Das2002} and Ephremides et al. \cite{Ephremides2001} which corresponds to a connected dominating set in a graph.

Given a graph $G = (V, E)$, a subset $C$ of $V$ is said to be a dominating set (DS) of $G$ if any $v \in V \setminus C$ is adjacent with at least one node of $C$. We say that a dominating set $C$ of $G$ is a connected dominating set of $G$ if $G[C]$ is connected, where $G[C]$ is the subgraph of $G$ induced by $C$. Nodes in $C$ are called dominators, while the other nodes are called dominatees.

In WSNs, a sensor may fail due to accidental damage or energy deletion. To make a virtual backbone more robust, it is suggested to use $(k, m)$-CDS.

**Definition 1.1** $(k, m)$-CDS. A node subset $C$ is a $k$-connected $m$-fold dominating set, if every node in $V \setminus C$ has at least $m$ neighbors in $C$ and $G[C]$ is $k$-connected.

In a homogeneous wireless sensor network, all sensors are equipped with omnidirectional antennas with the same transmission radius (say, one unit), and thus the transmission range of every sensor is a disk of radius one. Two sensors can communicate with each other if and only if they fall into the transmission ranges of each other. Such a setting is typically modeled as a unit disk graph (UDG), in which every node of the graph corresponds to a sensor on the plane, and two nodes are adjacent if and only if the Euclidean distance between their corresponding sensors is at most one unit. There are a lot of studies on virtual backbones in UDG (see the book \cite{Zhao2015}), but for general graphs, related studies are rare.

Notice that in a real world, the environment is very complicated, and thus it is rare that the topology can be ideally modeled as a unit disk graph. So, it is meaningful to study virtual backbone in a general graph.

In this paper, we study the minimum $(3, m)$-CDS problem with $m \geq 3$ in a general graph. The strategy used in this paper is greedy. It is well known that if the potential function related with the greedy algorithm is monotone increasing and submodular, then an $O(\ln n)$ approximation ratio can be achieved. An interesting part of this paper is that we constructed a potential function which is NOT submodular, and proposed an analysis to show that the approximation ratio $O(\ln n)$ can still be achieved.

The main result of this paper is that our algorithm works for general graphs with a guaranteed performance ratio $(\alpha + 8 + 2\ln(2\alpha - 6))$ for $\alpha \geq 4$ and a guaranteed performance ratio $(3\alpha + 2\ln 2)$ for $\alpha < 4$, where $\alpha$ is the approximation ratio for the minimum $(2, m)$-CDS problem. In a recent paper, we \cite{Wang2020} proposed a $(\ln(\delta + m - 2) + o(\ln \delta))$-approximation algorithm for the minimum $(2, m)$-CDS problem on a general graph, where $\delta$ is the maximum degree of the graph. Based on it, the algorithm in this paper has performance ratio $\ln(\delta + m - 2) + o(\ln \delta)$. In view of the non-approximability of this problem \cite{Karp1982}, the ratio is asymptotically best possible.

Furthermore, if applying our algorithm on a unit disk graph, then the performance ratio is less than 27. Previous to this work, Wang et al. \cite{Wang2017} obtained a constant approximation algorithm for $(3, m)$-CDS on UDG, and the ratio is further improved in their recent work \cite{Wang2018}, which is $5\alpha$. For example, if the value of $\alpha$ in paper \cite{Wang2020} is used, their
algorithm for (3,3)-CDS on UDG has performance ratio 62.3. Our ratio improves theirs by a large amount.

Our work is based on the brick decomposition of 2-connected graphs, which is commonly known as Tutte’s decomposition. This decomposition is an important tool in graph theory, and was studied extensively by a lot of researchers, including Tutte \cite{26}, Hopcroft and Tarjan \cite{16}, Cunningham and Edmonds \cite{3}, et al. The same decomposition is also used by Wang et al. \cite{31}. However, our method differs a lot from theirs since we are considering general graphs while they only considered unit disk graphs. Furthermore, our method is more refined which can be seen from the improvement on the performance ratio.

The rest of this paper is organized as follows. Section \ref{sec:related} introduces related works. Some preliminary results concerning with the brick decomposition structure of 2-connected graphs are introduced in Section \ref{sec:preliminaries}. In Section \ref{sec:alg}, the algorithm is presented, and the performance ratio is analyzed. Section \ref{sec:conclusion} concludes the paper and discusses some future research directions.

\section{Related work}

The idea of using a CDS as a virtual backbone for WSN was proposed by Das and Bhargharan \cite{5} and Ephremides \textit{et al.} \cite{13}. The minimum CDS problem is NP-hard. In fact, by reducing the minimum set cover problem to the minimum CDS problem, Guha and Khuller \cite{15} proved that a minimum CDS cannot be approximated within $\rho \ln n$ for any $0 < \rho < 1$ unless $NP \subseteq DTIME(n^{O(\log \log n)})$. In the same paper, they proposed two greedy algorithms with performance ratios of $2(H(\delta) + 1)$ and $H(\delta) + 2$, respectively, where $\delta$ is the maximum degree of the graph and $H(\cdot)$ is the harmonic number. This was improved by Ruan \textit{et al.} \cite{22} to $2 + \ln \delta$. Du \textit{et al.} \cite{7} presented a $(1 + \varepsilon)(1 + \ln(\delta - 1))$-approximation algorithm, where $\varepsilon$ is an arbitrary positive real number. In UDGs, a polynomial time approximation scheme (PTAS) for this problem was given by Cheng \textit{et al.} \cite{2}, which was generalized to higher dimensional space by Zhang \textit{et al.} \cite{34}. For distributed algorithms with constant performance ratios, the readers may refer to \cite{12, 19, 20, 27, 28, 32}.

The problem of constructing fault-tolerant virtual backbone was proposed by Dai and Wu \cite{4}. They proposed three heuristic algorithms for the minimum $(k,k)$-CDS problem. However, no theoretical analysis was given. Table \ref{table:heuristics} summarizes results with guaranteed performance ratio for $(k,m)$-CDS. The last two rows are results obtained in this paper. It can be seen that we obtained the first approximation algorithm for $(3,m)$-CDS on a general graph. When the algorithm is applied on UDG, the performance ratio is reduced by a large amount compared with previous ones. For some heuristics on $(k,m)$–MCDS for general $k$ and $m$, the readers may refer to \cite{21, 25, 33}.

\section{Preliminaries}

The following lemma is well known in graph theory \cite{1}.
| graph   | $(k, m)$ | ratio                                      | reference |
|---------|---------|--------------------------------------------|-----------|
| general | $(1, m)$| $2H(\delta + m - 1)$                       | [18][35]  |
| general | $(1, m)$| $2 + H(\delta + m - 2)$                    | [38]      |
| general | $(2, m)$| $4 + \ln(\delta + m - 2) + 2\ln(2 + \ln(\delta + m - 2))$ | [24]      |
| UDG     | $(2, 1)$| 72                                         | [29]      |
| UDG     | $(1, m)$| $\begin{cases} 5 + 5/m, & m \leq 5 \\ 7, & m > 5 \end{cases}$ | [23]      |
| UDG     | $(2, m)$| $\begin{cases} 15 + 15/m, & 2 \leq m \leq 5 \\ 21, & m > 5 \end{cases}$ | [23]      |
| UDG     | $(2, m)$| $\begin{cases} 7 + 5/m + 2\ln(5 + 5/m), & 2 \leq m \leq 5 \\ 12.89, & m > 5 \end{cases}$ | [24]      |
| UDG     | $(3, m)$| constant (280 for $m = 3$)                 | [30]      |
| UDG     | $(3, m)$| $5\alpha$, where $\alpha$ is performance ratio for (2, m)-CDS on UDG (62.3 for $m = 3$) | [31]      |
| general | $(3, m)$| $\begin{cases} \alpha + 8 + 2\ln(2\alpha - 6) & \text{for } \alpha \geq 4 \\ 3\alpha + 2\ln 2 & \text{for } \alpha < 4 \end{cases}$ where $\alpha$ is performance ratio for (2, m)-CDS | *         |
| UDG     | $(3, m)$| $\begin{cases} 26.34, & m = 3 \\ 25.68, & m = 4 \\ 26.86, & m \geq 5 \end{cases}$ | *         |

Table 1: Results on $(k, m)$-CDS with guaranteed performance ratio

**Lemma 3.1.** Suppose $H_1$ is a $k$-connected graph and $H_2$ is obtained from $H_1$ by adding a new node $u$ and joining $u$ to at least $k$ nodes of $H_1$. Then $H_2$ is also $k$-connected.

As a consequence, we have the following result.

**Corollary 3.2.** Suppose $G$ is a $k$-connected graph, $k$ and $m$ are two positive integers with $m \geq k$, and $C$ is a $(k, m)$-CDS of $G$. For any $U \subseteq V(G) \setminus C$,

(i) node set $C \cup U$ is also a $(k, m)$-CDS of $G$, and

(ii) no node in $U$ is involved in any $k$-node cut of $G[C \cup U]$.

In the following, we focus on 2-connected graphs.

**Definition 3.3 (2-separator).** Suppose $H$ is a 2-connected graph. A node set $\{u, v\}$ is a 2-separator of $H$ if $H - \{u, v\}$ is not connected. The local connectivity between two nodes $u$ and $v$ in graph $H$ is the maximum number of internally disjoint $(u, v)$-paths in $H$, denoted as $p_H(u, v)$. A 2-separator $\{u, v\}$ is good if $p_H(u, v) \geq 3$, otherwise it is bad.
For example, in Fig.1(a), \( \{u_1, v_1\} \) is a good 2-separator of the first graph and \( \{u_2, v_2\} \) is a bad 2-separator of the graph containing it (which is a 4-cycle). The following lemma characterizes 2-connected graphs without good 2-separators.

**Lemma 3.4** ([36]). Let \( H \) be a 2-connected graph which has no good 2-separator. Then \( H \) is either 3-connected or a cycle.

In view of Lemma 3.4, we say that a 2-connected graph without good 2-separators is a T-brick if it is 3-connected or an R-brick if it is a cycle.

Suppose \( H \) is a \((2, m)\)-CDS of a 3-connected graph \( G \), where \( m \geq 3 \). In view of Corollary 3.2, adding nodes to \( H \) does not incur new 2-separators. So, to augment \( H \) into a \((3, m)\)-CDS, it suffices to eliminate all 2-separators in \( H \). However, the number of 2-separators might be exponential. In order that the algorithm is polynomial, 2-separators have to be eliminated in a neat way. For this purpose, we need a structural characterization of 2-connected graphs, based on the concept of marked components defined as follows.

**Definition 3.5** (S-component and marked S-component). Let \( H \) be a 2-connected graph, \( S = \{u, v\} \) be a 2-separator of \( H \), and \( C \) be a connected component of \( H - S \). The subgraph \( H[C \cup S] \) is called an S-component of \( H \). For an S-component \( H[C \cup S] \), add a virtual edge \( uv \) if \( uv \notin E(H) \) and do nothing if \( uv \in E(H) \), call the resulting graph as a marked S-component.

For example, in the first graph of Fig.1(a), \( S_1 = \{u_1, v_1\} \) is a 2-separator. Splitting off the graph through \( S_1 \) results in three marked \( S_1 \)-components as in the second graph of Fig.1(a). Those dotted edges are virtual edges. The role virtual edges play is to guarantee the 2-connectedness of marked components, as indicated by Lemma 3.6 whose proof can be found in [1].

![Figure 1](image-url)

Figure 1: (a) Decomposition through good separators. Dotted edges are virtual edges which are added to form the marked S-components. (b) Brick structure of graph \( H \). Each ellipse indicates a brick. (c) The brick-tree \( B(H) \).
Lemma 3.6. Let $H$ be a 2-connected graph and $S$ be a 2-separator of $H$. Then the marked $S$-components of $H$ are also 2-connected.

Let $G$ be a 3-connected graph and $H$ be the subgraph of $G$ induced by a $(2,m)$-CDS of $G$. If $H$ has a good 2-separator $S$, then it can be decomposed into several marked $S$-components, which are also 2-connected by Lemma 3.6. If any one of these marked $S$-components has a good 2-separator, it can be further decomposed into smaller marked components. Such a decomposition continues until $H$ is decomposed into marked components without good 2-separators. In other words, $H$ can be iteratively decomposed into $T$-bricks and $R$-bricks through good 2-separators.

Pasting these bricks through those good 2-separators which have been used in the decomposition procedure, we see that the brick structure of $H$ is tree-like in the following sense: Let $B(H)$ be a bipartite graph with bipartition $(\mathcal{B}, \mathcal{S})$, where $\mathcal{B}$ is the set of bricks and $\mathcal{S}$ is the set of good 2-separators used in the above decomposition. A brick $B \in \mathcal{B}$ is adjacent with a 2-separator $S \in \mathcal{S}$ if and only if $S$ is contained in $B$. Notice that there is no sequence of bricks $B_1, \ldots, B_t$ such that $B_i$ shares a 2-separator with $B_{i+1}$ for $i = 1, \ldots, t-1$ and $B_t$ shares a 2-separator with $B_1$ (otherwise $\bigcup_{i=1}^t B_i$ will be 3-connected). So, the graph $B(H)$ is acyclic. Clearly, $B(H)$ is connected. So, $B(H)$ is a tree, which is called the brick-tree of $H$. Such a decomposition is illustrated in Fig.1.

4 Algorithm and Analysis

This section presents our greedy algorithm and analyzes its performance ratio. We first construct a potential function $f$ which will be used in the greedy algorithm, and derive some properties about $f$.

4.1 Potential Function

Definition 4.1 (brick-bridge). Suppose $H$ is a 2-connected graph. A path $P$ is called a brick-bridge of $H$ if it satisfies the following three conditions:

(i) all internal nodes of $P$ are outside of $H$ and the two ends of $P$ are in $H$;

(ii) the two ends of $P$ are nonadjacent in $H$;

(iii) the two ends of $P$ do not belong to a same $T$-brick of $H$.

Denote by $\text{int}(P)$ the set of internal nodes of $P$.

By the above definition, any brick-bridge either “strides over” different bricks or “strides over” non-adjacent nodes of an $R$-brick.

As we have explained in Section 3, the assumption $m \geq 3$ guarantees that adding brick-bridges to a $(2,m)$-CDS does not incur new 2-separators. Fig.2 gives us some idea of how the brick-structure is changed after adding internal nodes of some brick-bridge. Roughly speaking, if the brick-bridge $P$ strides over bricks $B$ and $B'$ of $G[C]$, let $Q_{BB'}$ be the unique path on the brick tree of $G[C]$ connecting $B$ and $B'$, and let $Q_{BB'}$ be the set of bricks on $Q_{BB'}$, then all $T$-bricks in $Q_{BB'}$ are merged into a new $T$-brick of $G[C \cup \text{int}(P)]$, and every $R$-brick in $Q_{BB'}$ is divided into smaller $R$-bricks of $G[C \cup \text{int}(P)]$ by this new $T$-brick.
Observation 4.2. Suppose $m \geq 3$, $G$ is a 3-connected graph, and $C$ is a $(2, m)$-CDS of $G$. Let $X$ be a node set of $V(G) \setminus C$ such that $G[X]$ is connected. Denote $B_X = \{(B, B'):$
$B, B'$ are bricks in $G[C]$ and there exists a brick-bridge of $G[C]$ whose internal nodes are in $X$ which strides over $B$ and $B'$. Let $Q_X = \bigcup_{(B, B') \in B_X} Q_{BB'}$.

(i) Those $T$-bricks of $Q_X$ are merged into a bigger new $T$-brick of $G[C \cup X]$, and $X$ is contained in this new $T$-brick.

(ii) Each $R$-brick of $Q_X$ is divided by the new $T$-brick into some smaller $R$-bricks of $G[C \cup X]$.

(iii) If an $R$-brick $R$ of $G[C]$ is divided into $s$ smaller $R$-bricks of $G'[C \cup X]$, say $R_1, \ldots, R_s$, then

$$
\sum_{i=1}^{s} |R_i| \leq \begin{cases} 
|R| + s, & \text{if } s \geq 3, \\
|R| + 1, & \text{if } s = 2, \\
|R| - 1, & \text{if } s = 1,
\end{cases}
$$

where $|R|$ is the number of nodes in $R$.

(iv) For every pair of bricks $(B, B') \in B_X$, all those good 2-separators on the unique path $Q_{BB'}$ in the brick tree of $G[C]$ are contained in the new $T$-brick of $G[C \cup X]$.

For a 2-connected graph $H$, denote by $\mathcal{B}(H)$ the set of bricks of $H$, $\mathcal{R}(H)$ the set of $R$-bricks of $H$, and $\mathcal{T}(H)$ the set of $T$-bricks of $H$. Define

$$f(H) = |\mathcal{T}(H)| + q(H),$$

where $q(H) = \Sigma_{R \in \mathcal{R}(H)} (2|R| - 5)$.

**Lemma 4.3.** Suppose $m \geq 3$, $G$ is a 3-connected graph, and $C$ is a $(2, m)$-CDS of $G$ such that $G[C]$ is not 3-connected. Let $P$ be a brick-bridge of $G[C]$ and let $X = \text{int}(P)$. Then $f(C) \geq f(C \cup X) + 1$. If furthermore, $|Q_X| \geq 2$ and there exists an $R$-brick $R_a \in Q_X$ such that $|R_a| \geq 4$, then $f(C) \geq f(C \cup X) + 2$.

**Proof.** For each $R \in \mathcal{R}(G[C])$, we use $\mathcal{R}_{C,X}^\text{div}(R)$ to denote the set of smaller $R$-bricks arising from the division of $R$ after $X$ is added into $C$, and denote $s(R) = |\mathcal{R}_{C,X}^\text{div}(R)|$. For an integer $j \geq 0$, denote by $\mathcal{R}_j(C)$ (resp. $\mathcal{R}_{\geq j}(C)$) the set of $R$-bricks of $G[C]$ with $s(R) = j$ (resp. $s(R) \geq j$). Notice that every $R \in \mathcal{R}_0(C)$ is completely merged into the new $T$-brick and thus diminished from $\mathcal{R}(G[C \cup X])$. Let $\mathcal{R}_X^\text{rec}(C)$ be the set of $R$-bricks of $G[C]$ which remain the same in $G[C \cup X]$. For simplicity of notation, we use $\mathcal{T}(C)$ to denote $\mathcal{T}(G[C])$ etc. For any $R \in \mathcal{R}(G[C])$, observe that $|R| \geq 3$. Combining this with (iii) of Observation 4.2, we have

$$
\sum_{R' \in \mathcal{R}_{C,X}^\text{div}(R)} (2|R'| - 5) \leq \begin{cases} 
2|R| - 3s(R), & \text{if } s(R) \geq 3, \\
2|R| - 8, & \text{if } s(R) = 2, \\
2|R| - 7, & \text{if } s(R) = 1, \\
2|R| - 6, & \text{if } s(R) = 0.
\end{cases}
$$

(1)
Hence

\[ q(C \cup X) = \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) + \sum_{R \in \mathcal{R}_{G}(C)} \sum_{R' \in \mathcal{R}_{G}(C)} (2|R'| - 5) \]

\[ \leq \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) + \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 7) + \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 8) + \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 3s(R)) \]

\[ = \sum_{R \in \mathcal{R}_{G}(C) \setminus \mathcal{R}_{0}(C)} (2|R| - 5) \]

\[ \leq \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) - \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) - 2|\mathcal{R}_{G}(C)|. \]

Hence

\[ \Delta_X q(C) = q(C \cup X) - q(C) \leq - \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) - 2|\mathcal{R}_{G}(C)|. \]  \hspace{1cm} (2)

By Observation 4.2,

\[ \Delta_X |\mathcal{T}(C)| = |\mathcal{T}(C \cup X)| - |\mathcal{T}(C)| = 1 - |\mathcal{Q}_{X}(C)|, \]  \hspace{1cm} (3)

where \( \mathcal{Q}_{X}^T(C) = \mathcal{Q}_{X} \cap \mathcal{T}(C) \) is the set of \( T \)-bricks of \( G[C] \) which are merged into the new \( T \)-brick of \( G[C \cup X] \). So,

\[ \Delta_X f(C) = \Delta_X |\mathcal{T}(C)| + \Delta_X q(C) \leq 1 - |\mathcal{Q}_{X}(C)| - \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) - 2|\mathcal{R}_{G}(C)|. \]

If the lemma is not true, then \( \Delta_X f(C) = f(C \cup X) - f(C) \geq 0 \), and thus

\[ |\mathcal{Q}_{X}(C)| + \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) + 2|\mathcal{R}_{G}(C)| \leq 1. \]  \hspace{1cm} (4)

It follows that \( \mathcal{R}_{G}(C) = \emptyset, |\mathcal{Q}_{X}(C)| \leq 1, |\mathcal{R}_{G}(C)| \leq 1 \) (since every \( R \)-brick \( R \) has at least three nodes, \( 2|R| - 5 \geq 1 \)). If \( |\mathcal{Q}_{X}(C)| = 1 \), then by the definition of brick-bridge (the two ends of a brick-bridge do not belong to a same \( T \)-brick), we have \( |\mathcal{Q}_{X}| \geq 2 \), and thus \( \mathcal{Q}_{X} \) has at least one \( R \)-brick. Since \( \mathcal{R}_{G}(C) = \emptyset \), this \( R \)-brick belongs to \( \mathcal{R}_{G}(C) \). But then \( \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) \geq 1 \), and the left side of (4) is at least 2. So, all bricks of \( \mathcal{Q}_{X} \) are \( R \)-bricks, and similarly to the above, they belong to \( \mathcal{R}_{0}(C) \). Since \( |\mathcal{R}_{0}(C)| \leq 1 \), this is possible only when the brick-bridge \( P \) strides over non-adjacent nodes of an \( R \)-brick \( R \). It follows that \( |R| \geq 4 \), and thus \( \sum_{R \in \mathcal{R}_{0}(C)} (2|R| - 5) \geq 3 \), again a contradiction. So, \( f(C) \geq f(C \cup X) + 1 \). The first part of the lemma is proved.

Suppose the conditions for the second part of the lemma are satisfied. If \( f(C) < f(C \cup X) + 2 \), then inequality (4) becomes

\[ |\mathcal{Q}_{X}(C)| + \sum_{R \in \mathcal{R}_{G}(C)} (2|R| - 5) + 2|\mathcal{R}_{G}(C)| \leq 2. \]  \hspace{1cm} (5)

We can not have \( R_{a} \in \mathcal{R}_{0}(C) \), since otherwise the second term is at least 3. Hence inequality (5) is possible only when \( R_{a} \in \mathcal{R}_{G}(C), |\mathcal{R}_{G}(C)| = 1, \mathcal{R}_{0}(C) = \emptyset, \) and \( |\mathcal{Q}_{X}(C)| = 0 \). But then, \( |\mathcal{Q}_{X}| = 1 \), contradicting the assumption that \( |\mathcal{Q}_{X}| \geq 2 \). The second part of the lemma is proved. \( \square \)
Lemma 4.3 says that as long as \( G[C] \) is not 3-connected, the function \( f \) can always be strictly decreased. Furthermore, under the “if” condition of Lemma 4.3, the amount for the decrease can be at least 2.

**Lemma 4.4.** Let \( C \) be a node subset of \( G \) such that \( G[C] \) is 2-connected. Then, \( f(C) = 1 \) if and only if either \( G[C] \) is 3-connected or \( G[C] \) is a triangle.

**Proof.** Notice that \( f \) can also be written as

\[
f(C) = |B(C)| + \sum_{R \in R(C)} (2|R| - 6).
\]

Since every \( R \)-brick has \( |R| \geq 3 \) and \(|B(C)| \geq 1\), we see that \( f(C) = 1 \) if and only if \(|B(C)| = 1 \) and \( \sum_{R \in R(C)} (2|R| - 6) = 0 \). Notice that \( \sum_{R \in R(C)} (2|R| - 6) = 0 \) if and only if either \( R(C) = \emptyset \) or every \( R \in R(C) \) has \(|R| = 3 \). In the first case, the unique brick of \( G[C] \) is a \( T \)-brick, and thus \( G[C] \) is 3-connected. In the second case, the unique brick of \( G[C] \) is a cycle on three nodes, and thus a triangle.

**Lemma 4.5.** Suppose \( m \geq 3 \), graph \( G \) is 3-connected, and \( C \) is a \((2, m)\)-CDS of \( G \) with \( f(C) > 1 \). Then, there exists a brick-bridge \( P \) of \( G[C] \) with at most two internal nodes. Furthermore, if \( G[C] \) is not a cycle, then for any brick \( B \in \mathcal{B}(C) \), there exists such a brick-bridge \( P \) of \( G[C] \) satisfying \( |Q_{\text{int}(P)}| \geq 2 \) and \( B \in Q_{\text{int}(P)} \).

**Proof.** Since \( f(C) > 1 \), by Lemma 4.4 \( G[C] \) is not 3-connected. Let \( S \) be a 2-separator of \( G[C] \), and \( G_1 \) be a connected component of \( G[C] \) \( - \) \( S \), \( G_2 \) be the union of the remaining connected components of \( G[C] \) \( - \) \( S \). Since \( G \) is 3-connected, there is a shortest path \( P = u_0u_1\ldots u_t \) in \( G \) between \( G_1 \) and \( G_2 \). Suppose \( u_0 \in V(G_1) \) and \( u_t \in V(G_2) \). Assume \( t \geq 4 \). Since \( C \) is an \( m \)-fold dominating set with \( m \geq 3 \), we see that \( u_2 \) has at least three neighbors in \( C \), one of which is \( v \notin S \). If \( v \in V(G_1) \), then \( vu_2u_3\ldots u_t \) is a shorter path between \( G_1 \) and \( G_2 \). If \( v \in V(G_2) \), then \( u_0u_1u_2v \) is a shorter path between \( G_1 \) and \( G_2 \). Both cases contradict the shortest assumption on \( P \). So, \( t \leq 3 \) and thus \(|\text{int}(P)| \leq 2 \).

Under the assumption that \( G[C] \) is not a cycle and \( f(C) > 1 \) (which implies that \( G[C] \) is not 3-connected), we see from Lemma 4.3 that any brick \( B \in \mathcal{B}(C) \) contains a good 2-separator. Use this good 2-separator as \( S \) in the above proof. If \( B \) is a \( T \)-brick, then \( B \) \( - \) \( S \) is connected. If \( B \) is an \( R \)-brick, then \( S \) consists of two consecutive nodes on cycle \( B \), and thus \( B \) \( - \) \( S \) is also connected. So, we can take the connected component \( G_1 \) of \( G[C] \) \( - \) \( S \) in the above proof such that \( B \) \( - \) \( S \) \( \subseteq \) \( G_1 \). Then it can be seen that the brick-bridge \( P \) found by the above proof satisfies \(|Q_{\text{int}(P)}| \geq 2 \) and \( B \in Q_{\text{int}(P)} \).

### 4.2 Algorithm

Our greedy algorithm is described in Algorithm 1 with potential function \( f(C) \). Initially, it computes a \((2, m)\)-CDS \( C_0 \) by an existing algorithm, for example the one in [38]. If \( G[C_0] \) is a triangle, then every node in \( V(G) \) \( - \) \( C_0 \) is adjacent with all the three nodes of \( C_0 \) because \( m \geq 3 \). Hence, adding any node into \( C_0 \) results in a \( K_4 \) (complete graph on four nodes) which is a \((3, m)\)-CDS of \( G \). Suppose \( G[C_0] \) is not a triangle. By Lemma 4.3 as long as \( f(C) > 1 \), there exists a brick-bridge \( P \) with at most two internal nodes. By Lemma 4.4, adding \( \text{int}(P) \) strictly decreases the \( f \)-value. The while-loop iterates until \( f(C) \) is decreased to 1, at which time \( G[C] \) is 3-connected by Lemma 4.4.
Algorithm 1 Computation of $(3,m)$-CDS for $m \geq 3$

Input: A 3-connected graph $G = (V,E)$.
Output: A $(3,m)$-CDS $C$ of $G$.

1: Compute a $(2,m)$-CDS $C_0$ by an $\alpha$-approximation algorithm.
2: if $G[C_0]$ is a triangle then
3:   Let $v$ be an arbitrary node in $V(G) \setminus C_0$.
4:   Output $C \leftarrow C_0 \cup \{v\}$.
5: else
6:   $C \leftarrow C_0$.
7:   while $f(C) > 1$ do
8:     Select a brick-bridge $P$ of $G[C]$ with internal node set $\text{int}(P) = X$ such that $|X| \leq 2$ and $\frac{-\Delta x f(C)}{|X|}$ is maximized.
9:     $C \leftarrow C \cup \{X\}$
10: end while
11: Output $C$.
12: end if

4.3 Analysis of Performance Ratio

To analyze the performance ratio of Algorithm 1, we first present a decomposition result on an optimal solution.

Lemma 4.6. Suppose $m \geq 3$, $C$ is a $(2,m)$-CDS of $G$, and $C^*$ is a minimum $(3,m)$-CDS of $G$. Then $C^* \setminus C$ can be decomposed into the union of node sets $C^* \setminus C = Y_1 \cup Y_2 \cup \ldots \cup Y_h$ satisfying the following conditions. For $j = 1,2,\ldots,h$, denote $C_j^* = Y_1 \cup \ldots \cup Y_j$, $C_0^* = \emptyset$. Suppose $l$ is the first index such that $G[C \cup C_l^*]$ is 3-connected.

(i) For $1 \leq j \leq l$, node set $C_j^*$ is completely contained in one T-brick of $G[C \cup C_j^*]$. Denote this brick as $B^{(j)}$, set $B^{(0)} = \emptyset$.

(ii) For $1 \leq j \leq l$, $Y_j = \text{int}(P_j)$, where $P_j$ is a brick-bridge of $G[C]$ and there exists at least one brick of $G[C]$ contained in $B^{(j-1)}$ which also belongs to $Q_{Y_j}(C)$.

(iii) $1 \leq |Y_j| \leq 2$ for $1 \leq j \leq l$.

(iv) $|Y_j \cap C_j^*| \leq 1$ for $1 \leq j \leq l$.

(v) $|Y_j| = 1$, for $j = l+1,\ldots,h$.

Proof. By Corollary 3.2, $G[C^* \cup C]$ is 3-connected. It should be pointed out that all the following paths are taken in $G[C^* \cup C]$. The 3-connectedness of $G[C^* \cup C]$ guarantees the existence of such paths.

Suppose $G[C]$ is not 3-connected. Let $P_1$ be a shortest brick-bridge of $G[C]$ in $G[C^* \cup C]$. By Observation 4.2 and Lemma 4.5, conditions (i) to (iv) are satisfied for $j = 1$.

Suppose that we have found subsets $Y_1, \ldots,Y_j$ satisfying conditions (i) to (iv) and $[C \cup C_j^*]$ is not 3-connected. Let $S$ be a 2-separator of $G[C \cup C_j^*]$ which is contained in $B^{(j)}$. As we have noticed by Corollary 3.2, any 2-separator of $G[C \cup C_j^*]$ is also a 2-separator of $G[C]$. Hence, if we denote by $B_1$ the set of bricks of $G[C]$ contained in $B^{(j)}$ which also contain $S$, then $|B_1| \geq 1$. Let $G_1$ be the union of those connected components of $G[C] - S$ containing $(\bigcup_{B \in B_1} B) - S$, and let $G_2$ be the union of remaining connected components of $G[C^* - S]$. If $G_2$ is not 3-connected, then by Lemma 4.5, we can find new paths for $G_2$ and continue the process. If $G_2$ is 3-connected, we can take $G_2$ as a new $G[C]$ and continue the process.
components of $G[C] - S$. Similarly to the proof of Lemma 4.3, a shortest path $P$ in $G[C^* \cup C]$ between $G_1$ and $G_2$ has at most two internal nodes. Notice that $B^{(i)} \cap C - S$ is contained in $G_1$. So $P$ contains a brick-bridge $P'$ of $G[C \cup C_j^*]$, and $B^{(i)} \in Q_{int}(P')$. It follows that $B^{(i)}$ is contained in the new $T$-brick of $G[C \cup C_j^* \cup int(P)]$. Taking $P_{j+1} = P$, by Observation 4.2, conditions $(i)$ to $(iv)$ are satisfied for $j + 1$.

For $j \geq l$, it suffices to take $Y_{j+1}$ to be an arbitrary node in $C^* \setminus (C \cup C_l^*)$.

In the following proofs, condition $(i)$ of Lemma 4.6 is very important for a guaranteed performance ratio. The idea of condition $(i)$ is that when $Y_1, \ldots, Y_l$ are added sequentially, we are expanding ONE $T$-brick (instead of merging bricks here and there in a messy way), any brick of $G[C]$ which has empty intersection with this $T$-brick remains the same.

**Lemma 4.7.** Suppose $m \geq 3$, $C$ is a $(2, m)$-CDS of $G$, and $C^*$ is a minimum $(3, m)$-CDS of $G$. Let $C^* \setminus C = Y_1 \cup Y_2 \cup \ldots \cup Y_l$ be the decomposition as in Lemma 4.6, and let $l$ be the first index such that $G[C \cup C_l^*]$ is $3$-connected. Then for any $j = 1, \ldots, l$,

$$- \Delta_{Y_j} f(C \cup C_{j-1}^*) \leq - \Delta_{Y_j} f(C) + 6. \quad (6)$$

Furthermore, if every $R$-brick of $G[C]$ has length three, then for any $j = 1, \ldots, l$,

$$- \Delta_{Y_j} f(C \cup C_{j-1}^*) \leq - \Delta_{Y_j} f(C). \quad (7)$$

**Proof.** The first part of the lemma is the result of the following two claims and the definition of $f$.

**Claim 1.** $\Delta_{Y_j} |T(C)| - \Delta_{Y_j} |T(C \cup C_{j-1}^*)| \leq 1$.

In fact, by equation (3),

$$\Delta_{Y_j} |T(C)| - \Delta_{Y_j} |T(C \cup C_{j-1}^*)| = |Q^T_{Y_j}(C \cup C_{j-1}^*)| - |Q^T_{Y_j}(C)|. \quad (8)$$

Since $C_{j-1}^*$ is completely contained in one $T$-brick of $G[C \cup C_{j-1}^*]$ (see Lemma 4.6 $(i)$), we have $|Q^T_{Y_j}(C \cup C_{j-1}^*)| - |Q^T_{Y_j}(C)| \leq 1$. Claim 1 is proved.

**Claim 2.** $\Delta_{Y_j} q(C) - \Delta_{Y_j} q(C \cup C_{j-1}^*) \leq 5$.

The validity of Claim 2 is achieved by a series of sub-claims. The readers may refer to Fig 4 to help understanding the following proofs.

By the definition of $q$,

$$\Delta_{Y_j} q(C) - \Delta_{Y_j} q(C \cup C_{j-1}^*) = \Delta_{Y_j} \sum_{R \in R(C)} (2 |R| - 5) - \Delta_{Y_j} \sum_{R \in R(C \cup C_{j-1}^*)} (2 |R| - 5). \quad (9)$$

Let $R$ be an $R$-brick of $G[C]$. If $R$ contributes to the first term of (9), then by Observation 4.2, $R$ is divided by the new $T$-brick $B$ of $G[C \cup Y_j]$ containing $Y_j$. By Lemma 4.6 $(i)$, it can be seen that

**SubClaim 2.1.** $V(B^{(j)}) \cap V(R) = (V(B^{(j-1)}) \cup V(B)) \cap V(R)$.

As in the proof of Lemma 4.3, denote by $R^*_{div}(R)$ the set of smaller $R$-bricks of $G[C \cup X]$ arising from the division of $R$ after $X$ is added into $C$. 

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Figure 4: (a) The solid lines indicate $G[C]$. Each rectangle represents a $T$-brick. Each rounded rectangle represents an $R$-brick. Together with the dashed lines, we have $G[C \cup C_{j-1}^*]$. (b) is the brick decomposition of $G[C \cup C_{j-1}^*]$. The blackened lines indicate $B^{(j-1)}$. (c) depicts $G[C \cup Y_j]$, the dashed lines are the edges added together with the addition of $Y_j$. (d) is the brick decomposition of $G[C \cup Y_j]$. The blackened lines indicate $B$. (e) is the brick decomposition of $G[C \cup C_{j-1}^* \cup Y_j] = G[C \cup C_{j}^*]$. The blackened lines indicate $B^{(j)}$. In (f), an $R$-brick is divided into smaller $R$-bricks of $G[C \cup Y_j]$ by the new $T$-brick $B$. The double circled nodes are in $V(B) \cap V(R)$. The center part belongs to the new $T$-brick $B$. The top arc belongs to $E(B) \cap E(R)$. The dashed lines are virtual edges.
SubClaim 2.2. \[ \sum_{R' \in \mathcal{R}^{\text{div}}_{C,C,Y_j}(R)} (2|R'| - 5) - (2|R| - 5) = |E(B) \cap E(R)| - 3|V(B) \cap V(R)| + 5. \]

For simplicity of statement, suppose \( s = |\mathcal{R}^{\text{div}}_{C,Y_j}(R)| \) and \( \mathcal{R}^{\text{div}}_{C,Y_j}(R) = \{ R_1, \ldots, R_s \} \) (see Fig.4(f) for an illustration). For \( i = 1, \ldots, s \), denote by \( S_i = V(R_i) \cap V(B) = \{ v_{i1}, v_{i2} \} \). By Observation 4.2, the subgraph of \( B \) induced by \( (E(B) \cap E(R)) \cup \{ v_{i1}v_{i2}, v_{i2}v_{i3}, \ldots, v_{is}v_{is+1} \} \) is a cycle. So,

\[ |V(R)| = \sum_{i=1}^{s} |R_i| + |V(B) \cap V(R)| - 2s \] (10)

and

\[ s + |E(B) \cap E(R)| = |V(B) \cap V(R)|. \] (11)

It follows that

\[ \sum_{i=1}^{s} (2|R_i| - 5) = 2(|V(R)| - |V(B) \cap V(R)| + 2s) - 5s = 2(|V(R)| - |V(B) \cap V(R)|) - (|V(B) \cap V(R)| - |E(B) \cap E(R)|) = 2|R| - 3|V(B) \cap V(R)| + |E(B) \cap E(R)|. \] (12)

Then, SubClaim 2.2 follows.

Notice that SubClaim 2.2 provides an expression for each \( R \in \mathcal{R}(C) \) in the first term of the right-hand side of (9). Estimation on the second term of the right-hand side of (9) can make use of SubClaim 2.2. In fact, consider those \( R \)-bricks in \( \mathcal{R}^{\text{div}}_{C,C,Y_j}(R) \) (where \( R \) is the \( R \)-brick in SubClaim 2.2), they are further divided into smaller \( R \)-bricks when \( Y_j \) is added into \( C \cup C_{j-1}^s \) (see Fig.4(b) and (e)). Making use of SubClaim 2.2 (replacing \( Y_j \) by \( C_{j}^s \) and \( C_{j-1}^s \), and replacing \( B \) by \( B^{(j)} \) and \( B^{(j-1)} \), correspondingly), it can be estimated that

\[ \Delta_{Y_j} = \sum_{R' \in \mathcal{R}^{\text{div}}_{C,C,Y_j}(R)} (2|R'| - 5) \]

\[ = \left( \sum_{R' \in \mathcal{R}^{\text{div}}_{C,C,Y_j}(R)} (2|R'| - 5) - (2|R| - 5) \right) - \left( \sum_{R' \in \mathcal{R}^{\text{div}}_{C,C,Y_j-1}(R)} (2|R'| - 5) - (2|R| - 5) \right) \]

\[ = |(E(B^{(j)}) \setminus E(B^{(j-1)})) \cap E(R)| - 3|(V(B^{(j)}) \setminus V(B^{(j-1)})) \cap V(R)|. \] (13)

So, for each \( R \in \mathcal{R}(C) \), if we denote by \( g(R) \) the total value of those terms in the right-hand side of (9) which are related with \( R \), then by SubClaim 2.2 and (13), it can be seen that \( g(R) \) has the following expression:

\[ g(R) = 5 + 3|(V(B^{(j)}) \setminus V(B^{(j-1)})) \cap V(R)| - |V(B) \cap V(R)| \]

\[ + |E(B) \cap E(R)| - |(E(B^{(j)}) \setminus E(B^{(j-1)})) \cap E(R)|. \] (14)
Notice that (9) can be rewritten as the following:

**SubClaim 2.3.** \( \Delta_{Y_j} q(C) - \Delta_{Y_j} q(C \cup C_{j-1}^*) = \sum_{R \in \mathcal{A}} g(R) \), where \( \mathcal{A} = \{ R \in \mathcal{R}(C) \setminus \mathcal{R}(C \cup C_{j-1}^*) : R \text{ is divided by } B \} \).

The reason why only those \( R \)-bricks in \( \mathcal{R}(C) \setminus \mathcal{R}(C \cup C_{j-1}^*) \) are considered is as follows: If \( R \in \mathcal{R}(C) \cap \mathcal{R}(C \cup C_{j-1}^*) \), then the changes on \( R \) are the same in the two terms of (9), which will cancel. The reason why only those \( R \)-bricks divided by \( B \) are considered is the following: for any \( R \)-brick \( R \) which is not divided by \( B \), adding \( Y_j \) does not change \( R \), neither does it change any smaller \( R \)-bricks in \( \mathcal{R}_{C,C_{j-1}^*}^\Delta(R) \).

The next subclaim estimates the upper bound for \( g(R) \). Suppose \( |V(B) \cap V(R)| - |(V(B^{(j)}) \setminus V(B^{(j-1)})) \cap V(R)| = t(R) \).

**SubClaim 2.4.**

\[
g(R) \leq \begin{cases} 
5, & t(R) = 0, \\
-1, & t(R) \geq 1 \text{ and } (15) \text{ occurs}, \\
5 - 2t(R) - 1, & t(R) \geq 1 \text{ and } (15) \text{ does not occur.}
\end{cases}
\]

By SubClaim 2.1, it can be seen that \( t(R) \) can be rewritten as \( t(R) = |V(B) \cap V(B^{(j-1)}) \cap V(R)| \). So, \( t(R) \geq 0 \). If \( t(R) = 0 \), then \( V(B) \cap V(B^{(j-1)}) \cap V(R) = \emptyset \), and thus \( (E(B) \cap E(R)) \cap (E(B^{(j-1)}) \cap E(R)) = \emptyset \). By noticing that \( V(B) \cap V(R) \subseteq V(B^{(j)}) \cap V(R) \) by SubClaim 2.1, and thus \( E(B) \cap E(R) \subseteq E(B^{(j)}) \cap E(R) \), we have \( E(B) \cap E(R) \subseteq (E(B^{(j)}) \setminus E(B^{(j-1)})) \cap E(R) \), and thus \( g(R) \leq 5 \) by (14). When \( t(R) \geq 1 \), by recalling that \( R \) is a cycle, we see that \( |E(B) \cap E(B^{(j-1)}) \cap E(R)| \leq t(R) - 1 \) unless

\[
V(B) \cap V(R) = V(B^{(j)}) \cap V(R) = V(R). \quad (15)
\]

If (15) occurs, then we see from (14) that \( g(R) = 5 - 2|R| \leq -1 \). Otherwise, \( |E(B) \cap E(R)| - |E(B^{(j)}) \setminus E(B^{(j-1)}) \cap E(R)| = |E(B) \cap E(B^{(j-1)}) \cap E(R)| \leq t(R) - 1 \) and thus \( g(R) \leq 5 - 2t(R) - 1 \) by (14). SubClaim 2.4 is proved.

**SubClaim 2.5.** If \( |\mathcal{A}| \geq 2 \), then for every \( R \in \mathcal{A} \), \( t(R) \geq 2 \).

In fact, since such an \( R \)-brick does not belong to \( \mathcal{R}(C \cup C_{j-1}^*) \), it is divided by \( B^{(j-1)} \). For \( R, R' \in \mathcal{A} \), consider the unique path \( Q_{RR'} \) in the brick tree of \( G[C] \) connecting \( R \) and \( R' \), the first 2-separator incident with \( R \), say \( S \), must belong to \( V(B^{(j-1)}) \) (by Observation 4.2 (iv)). Since both \( R \) and \( R' \) are divided by \( B \), for the same reason, \( S \subseteq V(B) \). So, \( t(R) = |V(B) \cap V(B^{(j-1)}) \cap V(R)| \geq |S| = 2 \). SubClaim 2.5 is proved.

Combining SubClaim 2.4 and SubClaim 2.5, if \( |\mathcal{A}| \geq 2 \), then \( g(R) \leq 0 \) for any \( R \in \mathcal{A} \). Otherwise, \( g(R) = 0 \) if \( \mathcal{A} = \emptyset \) and \( g(R) \leq 5 \) if \( |\mathcal{A}| = 1 \). Then Claim 2 follows from SubClaim 2.3.

Combining Claim 1 and Claim 2, the first part of this lemma is proved.

In the case that every \( R \)-brick of \( G[C] \) has length three, we see from Observation 4.2 (ii) that after adding a node set, any \( R \)-brick either diminishes or remains the same. Denote by \( \mathcal{R}_{C,X}^{\Delta} \) the set of \( R \)-bricks diminished after adding \( X \) into \( C \). By Lemma 4.6 (i)
and (ii), we see that $\mathcal{R}_{C \cup C_j}^{\dim} \subseteq \mathcal{R}_{C,Y_j}$. Hence
\[
\Delta_{Y_j} q(C) - \Delta_{Y_j} q(C \cup C^*_{j-1}) = - \sum_{R \in \mathcal{R}_{C,Y_j} \setminus \mathcal{R}_{C \cup C^*_{j-1}}} (2|R| - 5) \leq 0. \tag{16}
\]
Furthermore, if $\Delta_{Y_j}|T(C)| - \Delta_{Y_j}|T(C \cup C^*_{j-1})| = 1$, then $|Q_{Y_j}^T(C \cup C^*_{j-1})| - |Q_{Y_j}^T(C)| = 1$ by (8), which is possible only when adding $C^*_{j-1}$ into $C$ creates a new $T$-brick, which occurs only when every brick in $Q_{C^*_{j-1}}(C)$ is an $R$-brick. Combining this with Lemma 4.6 (ii), we see that $\mathcal{R}_{C,Y_j} \setminus \mathcal{R}_{C \cup C^*_{j-1}}$ contains at least one $R$-brick, and thus inequality (16) becomes
\[
\Delta_{Y_j} q(C) - \Delta_{Y_j} q(C \cup C^*_{j-1}) = - \sum_{R \in \mathcal{R}_{C,Y_j} \setminus \mathcal{R}_{C \cup C^*_{j-1}}} (2|R| - 5) \leq -1.
\]
Then the second part of this lemma follows from the definition of $f$. \hfill \Box

In the following, we use $X_1, X_2, \ldots, X_g$ to denote the sets chosen by Algorithm 1 in the order of their selection into set $C$. For $1 \leq i \leq g$, denote $C_i = C \cup X_1 \cup X_2 \cdots \cup X_i$.  

**Lemma 4.8.** For $1 \leq i \leq g$, we have $-\Delta_{X_i} f(C_{i-1}) \geq 1$ by Lemma 4.3. Furthermore, if $\mathcal{R}(C_{i-1})$ contains at least one $R$-brick of length at least 4, then $\frac{-\Delta_{X_i} f(C_{i-1})}{|X_i|} \geq 1$.

**Proof.** For $1 \leq i \leq g$, we have $-\Delta_{X_i} f(C_{i-1}) \geq 1$ by Lemma 4.3. Then $\frac{-\Delta_{X_i} f(C_{i-1})}{|X_i|} \geq \frac{1}{2}$ follows from $|X_i| \leq 2$.

If $G[C_{i-1}]$ is not a cycle and $\mathcal{R}(C_{i-1})$ contains at least one $R$-brick of length at least 4, then by Lemma 4.5 there exists a brick-bridge $P$ with $X = \text{int}(P)$ such that $|Q_X| \geq 2$, $|X| \leq 2$ and $R \in Q_X$. By Lemma 4.3 and the greedy rule of Algorithm 1, we have $-\Delta_X f(C_{i-1}) \geq 2$ and $\frac{-\Delta_{X_i} f(C_{i-1})}{|X_i|} \geq \frac{-\Delta_X f(C_{i-1})}{|X|} \geq 1$.

Notice that $G[C_i]$ cannot be a cycle for $i > 0$. Recall that the case that $G[C_0]$ is a triangle is dealt with separately in Algorithm 1. In the case that $G[C_0]$ is a cycle of length at least 4, consider an arbitrary node $v \in V(G) \setminus C_0$. Since $C_0$ is a $(2, m)$-CDS and $m \geq 3$, node $v$ must have two neighbors $u_1, u_2$ in $C_0$ which are not consecutive on cycle $G[C_0]$. Let $P = u_1vu_2$. Then $P$ is a brick-bridge of $C_0$ and $\frac{-\Delta_{X_i} f(C_0)}{|X_1|} \geq \frac{-\Delta_{X_i} f(C_0)}{|v|} \geq 1$. The lemma is proved. \hfill \Box

Now, we are ready to prove the performance ratio.

**Theorem 4.9.** Algorithm 1 is a polynomial-time $\gamma$-approximation for the minimum $(3, m)$-CDS problem, where $\gamma = (3\alpha + 2\ln 2)$ for $\alpha < 4$ and $\gamma = (\alpha + 8 + 2\ln(2\alpha - 6))$ for $\alpha \geq 4$, $\alpha$ is the performance ratio for the minimum $(2, m)$-CDS problem.

**Proof.** By Corollary 3.2, every $C_i$ is a $(2, m)$-CDS for $0 \leq i \leq g$. Suppose $q$ is the first index such that $\mathcal{R}(C_q)$ contains no $R$-bridge of length at least four. Let $C^*$ be a minimum $(3, m)$-CDS of $G$. Denote $|C^*| = t$.

**Claim 1.** $|C_0| \leq \alpha t$.
Since $C_0$ is an $\alpha$-approximation for the minimum $(2, m)$-CDS problem, and because the size of a minimum $(2, m)$-CDS is no greater than the size of a minimum $(3, m)$-CDS, the claim follows.

For $0 \leq i \leq g$, denote $a_i = f(C_i) - 6t - 1$ and $b_i = f(C_i) - 1$.

**Claim 2.**

$$|X_{i+1}| \leq \begin{cases} \min\{a_i - a_{i+1}, 2t \frac{a_i - a_{i+1}}{a_i}\}, & \text{for } 0 \leq i \leq q - 1, \\ \min\{2(b_i - b_{i+1}), 2 \frac{b_i - b_{i+1}}{b_i}\}, & \text{for } q \leq i \leq g - 1. \end{cases}$$

For any fixed $i$ with $0 \leq i \leq g - 1$, decompose $C^* \setminus C_i$ into $Y_{j}^{(i)}$, $Y_{2}^{(i)}$, ..., $Y_{h_i}^{(i)}$ satisfying those conditions of Lemma 4.6. For $1 \leq j \leq h_i$, denote $C_{j}^{*} = Y_{1}^{(i)} \cup Y_{2}^{(i)} \cup \ldots \cup Y_{j}^{(i)}$. Set $C_{0}^{*} = \emptyset$. Suppose $l_i$ is the first index such that $G[C_i \cup C_{l_i}^*]$ is 3-connected.

First, consider $C_i$ with $i = 0, 1, \ldots, q - 1$. By Lemma 4.7, for $1 \leq j \leq l_i$,

$$-\Delta_{Y_{j}} f(C \cup C_{j-1}^{*}) \leq -\Delta_{Y_{j}} f(C) + 6. \quad (17)$$

By the greedy rule of Algorithm 1, we have

$$\frac{-\Delta_{X_{i+1}} f(C_i)}{|X_{i+1}|} \geq \frac{-\sum_{j=1}^{l_i} \Delta_{Y_{j}^{(i)}} f(C_i)}{|Y_{j}^{(i)}|}, \quad \text{for } j = 1, 2, \ldots, l_i. \quad (18)$$

By Lemma 4.6,

$$\sum_{j=1}^{l_i} |Y_{j}^{(i)}| \leq |C^* \setminus C_i| + l_i \leq 2t. \quad (19)$$

Combining inequalities (17), (18), (19) with the assumption that $G[C_i \cup C_{l_i}^*]$ is 3-connected (and thus $f(C_i \cup C_{l_i}^*) = 1$ by Lemma 4.4), we have

$$\frac{-\Delta_{X_{i+1}} f(C_i)}{|X_{i+1}|} \geq \frac{-\sum_{j=1}^{l_i} \Delta_{Y_{j}^{(i)}} f(C_i)}{|Y_{j}^{(i)}|} \geq \frac{\sum_{j=1}^{l_i} (-\Delta_{Y_{j}^{(i)}} f(C_i \cup C_{j-1}^{*}) - 6)}{2t} = \frac{-(f(C_i \cup C_{l_i}^*) - f(C_i)) - 6l_i}{2t} \geq \frac{-(f(C_i \cup C_{l_i}^*) - f(C_i)) - 6t}{2t} = \frac{f(C_i) - 6t - 1}{2t}. \quad (20)$$

The above inequality can be rewritten as

$$\frac{a_i - a_{i+1}}{|X_{i+1}|} \geq \frac{a_i}{2t}, \quad \text{for } 0 \leq i \leq q - 1. \quad (21)$$

and thus

$$|X_{i+1}| \leq 2t \frac{a_i - a_{i+1}}{a_i}, \quad \text{for } 0 \leq i \leq q - 1. \quad (22)$$
Next, consider \( C_i \) with \( q \leq i \leq g - 1 \). By the second part of Lemma 4.7, we have

\[-\triangle_{Y_j} f(C \cup C'_{j-1}) \leq -\triangle_{Y_j} f(C).\] (23)

Similar to the derivation of inequalities (21) and (22), we have

\[\frac{b_i - b_{i+1}}{|X_i+1|} \geq \frac{b_i}{2t} \quad \text{for} \quad q \leq i \leq g - 1\] (24)

and

\[|X_{i+1}| \leq 2t \frac{b_i - b_{i+1}}{b_i} \quad \text{for} \quad q \leq i \leq g - 1.\] (25)

By Lemma 4.3,

\[\frac{f(C_i) - f(C_{i+1})}{|X_{i+1}|} \geq \begin{cases} 
1, & \text{for} \ 0 \leq i \leq q - 1, \\
1/2, & \text{for} \ q \leq i \leq g - 1.
\end{cases}\] (26)

By the definition of \( a_i \) and \( b_i \), \( f(C_i) - f(C_{i+1}) = a_i - a_{i+1} = b_i - b_{i+1} \). So \( |X_{i+1}| \leq a_i - a_{i+1} \) for \( 0 \leq i \leq q - 1 \) and \( |X_{i+1}| \leq 2(b_i - b_{i+1}) \) for \( q \leq i \leq g - 1 \). Claim 2 is proved.

Claim 3. If \( a_0 \geq 2t \), then \( \sum_{i=0}^{q-1} |X_{i+1}| \leq 8t + 2t \ln(a_0/t) \).

To prove this Claim, we first prove the following inequality:

\[\sum_{i=0}^{g-1} |X_{i+1}| \leq \begin{cases} 
2t \ln \frac{a_0}{a_q} + 2t + 2t \ln \frac{a_0 + 6t}{t}, & \text{if} \ a_q \geq 2t, \\
4t - a_q + 2t \ln \frac{a_0}{2t} + 2t \ln \frac{a_0 + 6t}{t}, & \text{if} \ -5t \leq a_q < 2t, \\
14t + a_q + 2t \ln \frac{a_0}{2t}, & \text{if} \ a_q < -5t.
\end{cases}\] (27)

The sequence \( a_1, a_2, \ldots, a_q \) is monotone decreasing with respect to \( i \) and the function \( \min \{1, \frac{2t}{x} \} \) is monotone decreasing with respect to \( x \). Therefore, if \( a_0 \geq 2t \), then by Claim 2, we can estimate \( \sum_{i=0}^{q-1} |X_{i+1}| \) by an integral as follows:

\[\sum_{i=0}^{q-1} |X_{i+1}| \leq \int_{a_q}^{a_0} \min \{1, \frac{2t}{x} \} \, dx \]

\[= \begin{cases} 
2t \int_{a_q}^{a_0} \frac{1}{x} \, dx, & \text{if} \ a_q \geq 2t, \\
\int_{a_q}^{a_0} 1 \, dx + 2t \int_{2t}^{a_q} \frac{1}{x} \, dx, & \text{if} \ a_q < 2t,
\end{cases}\]

\[= \begin{cases} 
2t \ln(a_0/a_q), & \text{if} \ a_q \geq 2t, \\
2t - a_q + 2t \ln(a_0/2t), & \text{if} \ a_q < 2t.
\end{cases}\] (28)

Similar argument yields,

\[\sum_{i=q}^{g-1} |X_{i+1}| \leq \begin{cases} 
2(t - b_q) + 2t \ln(b_q/t), & \text{if} \ b_q \geq t, \\
2(b_q - b_q), & \text{if} \ b_q < t.
\end{cases}\]
Notice that \( b_q = 0 \) and \( b_q = a_q + 6t \). So

\[
\sum_{i=q}^{g-1} |X_{i+1}| \leq \begin{cases} 
2t + 2t \ln((a_q + 6t)/t), & \text{if } a_q \geq -5t, \\
2(a_q + 6t), & \text{if } a_q < -5t.
\end{cases}
\] (29)

Combining (28) and (29), inequality (27) follows.

Next, we estimate the right hand side of (27). If \( a_q \geq 2t \), then

\[
\ln \left( \frac{a_0}{a_q} \right) + \ln \left( \frac{a_q + 6t}{t} \right) = \ln \left( \frac{a_0(a_q + 6t)}{a_q t} \right) = \ln \left( \frac{a_0}{t} \right) + \ln \left( \frac{a_q + 6t}{a_q} \right)
\]

\[
= \ln \left( \frac{a_0}{t} \right) + \ln \left( 1 + \frac{6t}{a_q} \right) \leq \ln \left( \frac{a_0}{t} \right) + 4.
\] (30)

So in this case, \( \sum_{i=0}^{g-1} |X_{i+1}| \leq 2t + 2t \ln 4 + 2t \ln(a_0/t) < 4.78t + 2t \ln(a_0/t) \). It is easy to see that when \( z = -4t \), function \( -z + 2t \ln((z + 6t)/t) \)
achieves its maximum value \( 4t + 2t \ln 2 \).

So in the case \(-5t \leq a_q < 2t \), we have \( \sum_{i=0}^{g-1} \leq 8t + 2t \ln(a_0/t) \). If \( a_q < -5t \), then

\( \frac{a+b}{t} \leq 8t + 2t \ln(a_0/2t) < 7.62t + 2t \ln(a_0/t) \). In any case, Claim 3 is true.

Claim 4. If \( a_0 < 2t \), then \( \sum_{i=0}^{g-1} |X_{i+1}| \leq a_0 + 6t + 2t \ln 2 \).

If \( a_0 < 2t \), then \( a_i < 2t \) for any \( i = 0, 1, \ldots, g-1 \). In this case, by Claim 2, \( \sum_{i=0}^{g-1} |X_{i+1}| \) can be estimated as

\[
\sum_{i=0}^{g-1} |X_{i+1}| = \sum_{i=0}^{g-1} |X_{i+1}| + \sum_{i=q}^{g-1} |X_{i+1}|
\]

\[
\leq \sum_{i=0}^{g-1} (a_i - a_{i+1}) + \sum_{i=q}^{g-1} \min\{2, \frac{2t}{b_i}\} \cdot (b_i - b_{i+1})
\]

\[
\leq \begin{cases} 
0 - a_q + 2t + 2t \ln \left( \frac{a_q + 6t}{t} \right), & \text{if } a_q \geq -5t, \\
a_0 - a_q + 2a_q + 12t, & \text{if } a_q < -5t,
\end{cases}
\]

If \( a_q \geq -5t \), then by using (31), we have \( \sum_{i=0}^{g-1} |X_{i+1}| \leq a_0 + 6t + 2t \ln 2 \). If \( a_q < -5t \), then \( \sum_{i=0}^{g-1} |X_{i+1}| \leq a_0 + a_q + 12t \leq a_0 + 7t \). In any case, Claim 4 is true.

Claim 5. For any 2-connected graph \( H \), \( f(H) \leq 2|V(H)| - 5 \).

We prove the Claim by induction on the number of nodes of \( H \). If \( |V(H)| = 3 \), then \( H \) is a triangle and \( f(H) = 2|V(H)| - 5 \). Suppose the Claim is true when \( V(H) = n - 1 \). Consider the case \( |V(H)| = n \). If \( H \) is a cycle or 3-connected, then by the definition of potential function \( f \), we have \( f(C_0) = 2n-5 \) or \( f(C_0) = 1 \leq 2n-5 \). Otherwise, let \( S \) be a good 2-separator and \( C_1^S, C_2^S, \ldots, C_l^S \) be the marked \( S \)-components of \( H \). By Lemma 3.6
2-connected and thus \( f(C_i^S) \leq 2|C_i^S| - 5 \) for \( 1 \leq i \leq l \) by induction hypothesis. Since \( \sum_{i=1}^{l} |C_i^S| = |V(H)| + 2l - 2 \) and any brick of \( H \) is completely contained in some \( C_i^S \), so \( f(H) = \sum_{i=1}^{l} f(C_i^S) \leq \sum_{i=1}^{l} (2|C_i^S| - 5) = 2|V(H)| - l - 4 \leq 2|V(H)| - 5 \). Claim 5 is proved.

Combining Claim 1 and Claim 5,

\[
a_0 = f(C_0) - 1 - 6t < (2\alpha - 6)t. \tag{32}
\]

So, if \( \alpha < 4 \), then \( a_0 < 2t \). By Claim 4 and inequality \( \sum_{i=1}^{g} |X_i| \leq (2\alpha + 2 \ln 2)t \). If \( \alpha \geq 4 \), then \( 2\alpha - 6 \geq 2 \). We see from Claim 3, Claim 4, and inequality \( \sum_{i=1}^{g} |X_i| \leq (8 + 2 \ln(2\alpha - 6))t \) holds no matter whether \( a_0 \geq 2t \) or \( a_0 < 2t \).

Combining the above analysis with Claim 1 and the fact

\[
|C_g| = |C_0| + |X_1| + \ldots + |X_g|,
\]

we see that \( C_g \) is a \( \gamma \)-approximation.

\[\square\]

5 Conclusion

In this paper, we have presented a polynomial-time \( \gamma \)-approximation algorithm for the minimum \((3, m)\)-CDS problem, \( \gamma = \alpha + 8 + 2\ln(2\alpha - 6) \) for \( \alpha \geq 4 \) and \( \gamma = 3\alpha + 2\ln 2 \) for \( \alpha < 4 \), where \( \alpha \) is the approximation ratio for the minimum \((2, m)\)-CDS problem. This is the first performance guaranteed approximation algorithm for minimum \((3, m)\)-CDS on a general graph and also gives a big improvement on performance ratio of previously known approximation algorithms on unit disk graphs.

For future studies, a natural question is whether the general \((k, m)\)-CDS problem also admits an approximation within factor \( \ln \delta + o(\ln \delta) \). Recently, CDS considering routing-cost has been studies extensively \[6, 10, 11, 37\]. However, nothing has been done on fault-tolerant issue. This is also a direction for our further research.

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