The inflation technique solves completely the classical inference problem

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The causal inference problem consists in determining whether a probability distribution over a set of observed variables is compatible with a given causal structure. In (Wolfe et al., 2016), one of us introduced a hierarchy of necessary linear programming constraints which all the observed distributions compatible with the considered causal structure must satisfy. In this work, we prove that the inflation hierarchy is complete, i.e., any distribution of the observed variables which does not admit a realization within the considered causal structure will fail one of the inflation tests. More quantitatively, we show that any distribution of measurable events satisfying the $n^{th}$ inflation test is $O\left(\frac{1}{\sqrt{n}}\right)$-close in Euclidean norm to a distribution realizable within the given causal structure. In addition, we show that the corresponding $n^{th}$-order relaxation of the dual problem consisting in maximizing a $k^{th}$ degree polynomial on the observed variables is $O\left(\frac{k^2}{\sqrt{n}}\right)$-close to the optimal solution.

I. INTRODUCTION

A Bayesian network or causal structure is a directed acyclic graph where vertices represent random variables, each of which is generated by a non-deterministic function depending on the value of its parents. Nowadays, causal structures are commonly used in bioinformatics, medicine, image processing, sports betting, risk analysis and experiments of quantum nonlocality. It is important to remark that, while some of these variables may be directly observable, some others are not. Those are called hidden or latent variables.

Due to the presence of latent variables, determining whether a causal structure may be behind the statistics of the set of observable events, the so-called inference problem, is a very difficult mathematical question. Given a function on the probabilities of the measurable events, the dual of the causal inference problem is the task of computing its maximum value when evaluated over probability distributions compatible with the considered causal structure. While there are a number of more or less effective heuristics to search for probabilistic models compatible with our causality assumptions or optimizing functionals thereof (Koller and Friedman, 2009), the problem of proving the impossibility to accommodate experimental data within a given network or bounding the values of a function of such data is still open. In recent years, though, there have been many advances, see (Fritz, 2012; Fritz and Chaves, 2013; Chaves et al., 2014; Chaves et al., 2014; Chaves, 2016; Bohr Brask and Chaves, 2017).

In (Wolfe et al., 2016), the authors presented the inflation technique, a hierarchy of necessary constraints, verifiable via linear programming, which any distribution realizable within the considered causal structure must satisfy. Notably, the inflation technique allowed the authors to derive polynomial inequalities for the triangle scenario (Fritz and Chaves, 2013), one of the simplest causal structures for which the inference problem is not solved. More generically, the inflation technique’s versatility and practical performance make it a prominent tool to attack the causal inference problem. The inflation technique also leads naturally to a simple sequence of linear programming relaxations of the dual problem when the function to evaluate is a polynomial on the probabilities of the observed variables.

In this paper, we will first study the performance of the inflation method on a particular type of causal structures called causal networks. For these structures, we will show that the hierarchies of relaxations on the inference and the dual problems provided by the inflation technique are complete. In other words: any distribution of observed variables that passes all inflation tests must be realizable within the considered causal network, and the sequence of upper bounds obtained via inflation on the solution $f^*$ of the dual problem converges to $f^*$ asymptotically. Next, we will show that any inference (dual) problem in an arbitrary causal structure can be mapped to an inference (dual) problem in a causal network. Put together, these two results imply that the inflation technique, far from being a relaxation, is an alternative way of understanding general causal structures.

This paper is organized as follows: in Section II, we will formulate the concepts of causal networks and causal structures, and introduce the inference and dual problems. In Section III, we will review the inflation technique to solve the causal inference and dual problems. In Section IV we will prove an extension of the finite de Finetti theorem (Diaconis and Freedman, 1980) for distributions admitting an “inflated” extension. This theorem will allow us, in Section V, to prove that any distribution $P$ passing the $n^{th}$ inflation test is $O\left(\frac{1}{\sqrt{n}}\right)$-close in Euclidean norm to a feasible distribution within the considered causal network. From this theorem it will also follow straightforwardly that the $n^{th}$ inflation relaxation of the dual problem differs from the optimal value on at most $O\left(\frac{k^2}{n}\right)$, where $k$ is
A causal network is a type of causal structure with just two layers: a bottom layer of independently distributed latent random variables \( \{\lambda_1, \lambda_2, ..., \lambda_L\} \) and a top layer of observable random variables \( \{A_1, A_2, ..., A_m\} \), see Figure 1. The observable distribution \( P(A_1, A_2, ..., A_m) \) is generated via non-deterministic functions \( A_x = A_x(\bar{\lambda}_L) \), with \( \bar{\lambda}_L = (\lambda_1, ..., \lambda_L) \) and \( L_x \subset \{1, ..., L\} \). Here (and in the following) the notation \( \bar{v}^S \), where \( \bar{v} \) is a vector with \( N \) entries and \( S \subset \{1, ..., N\} \), will represent the vector with entries \( v_s \), \( s \in S \).

Consider, for example, the causal network dubbed “the triangle scenario”, with \( m = L = 3 \), see Figure 2. Denoting \( A^1, A^2, A^3 \) respectively by \( A, B, C \), we have that a probability distribution \( P(A,B,C) \) is realizable in the triangle scenario if \( A, B, C \) are generated via the non-deterministic functions \( A(\lambda_1, \lambda_2), B(\lambda_2, \lambda_3), C(\lambda_3, \lambda_1) \). Alternatively, \( P(A,B,C) \) is realizable in the triangle scenario iff it admits a decomposition of the form:

\[
P(A,B,C) = \sum_{\lambda_1, \lambda_2, \lambda_3} P(\lambda_1)P(\lambda_2)P(\lambda_3)P(A|\lambda_1, \lambda_2)P(B|\lambda_2, \lambda_3)P(C|\lambda_3, \lambda_1).
\]

In a general causal structure, the non-deterministic functions giving rise to the observed variables can also depend on other observed variables. An example is given by the instrumental scenario (Figure 3, left), where \( A \) and \( \lambda \) are, respectively, a free observable and a latent variable, and the observed variables \( B \) and \( C \) are generated via the non-deterministic functions \( B = B(A, \lambda), C = C(B, \lambda) \).
The causal inference problem consists in, given the observable distribution $P(A^1, ..., A^m)$, determine whether it admits a realization within the considered causal structure. More formally:

**Definition 1. The causal inference problem**
Given a causal structure $C$ and a probability distribution over the observed variables $P(A^1, ..., A^m)$, decide if there exists a probability distribution $Q$ for all the random variables in $C$ (observed or latent), compatible with this causal structure and such that the marginal distribution $Q(A^1, ..., A^m)$ coincides with $P(A^1, ..., A^m)$.

The above definition of the causal inference problem is different from the conventional one, where, given a probability distribution over observed variables, one asks which causal models can/cannot accommodate it. However, both problems are oracle-wise equivalent. In the following, we will hence stick to Definition 1.

For illustration, consider this example: if $P(A, B, C)$ is the input of the problem and the causal structure to test is the triangle scenario, then the causal inference problem is solved either by providing probability distributions $P(\lambda^1), P(\lambda^2), P(\lambda^3), P(A|\lambda^1, \lambda^2)P(B|\lambda^2, \lambda^3)P(C|\lambda^3, \lambda^1)$ such that eq. (1) holds, or by proving that no such distributions exist.

In contrast, what we will call the dual inference problem consists in, given a function $F$ on the probabilities of the observed events, to maximize its value among all distributions compatible with a given causal structure $C$. That is:

**Definition 2. The dual inference problem**
Given a causal structure $C$ and a real function $F$ of the distribution $P$ over observed variables $A^1, ..., A^m$, solve the optimization problem

$$f^* = \max F(P),$$

s.t. $P$ admits a realization in $C$, (2)

Note that this definition does not coincide with the standard one in the literature of probabilistic graphical models. There the dual problem consists in, given a causal structure $C$, identifying the set of all restrictions which affect any distribution over observed variables compatible with $C$. Even for the simplest causal structures, the output of such a problem can be too large to store in a normal computer. Thus in this paper we will focus on the above restricted notion of dual.

Coming back to the triangle scenario, an instance of the dual problem (as defined in this paper) would be maximizing $-(P(000)−1/2)−(P(111)−1/2)^2$ over all distributions $P(A, B, C)$ with $A, B, C \in \{0, 1\}$, realizable within the triangle scenario.

There exist a number of variational algorithms to solve problem (2) (Koller and Friedman, 2009). Similarly, there exist many heuristics which scan possible causal realizations of a given distribution of observed variables. However, general practical tools to demonstrate the irrealizability of a probability distribution or to derive upper bounds on the solution of the dual problem are scarce. One of them is the inflation technique, which we describe next.

**III. A QUICK OVERVIEW OF THE INFLATION TECHNIQUE**

Let $P(A, B, C)$ be a distribution realizable in the triangle scenario, and suppose that we generate $n$ independently distributed copies of $\lambda^1, \lambda^2, \lambda^3$, that is, the variables $\{\lambda^1_i, \lambda^2_i, \lambda^3_i : i = 1, ..., n\}$. Then we could define the random variables

$$A_{i,j} \equiv A(\lambda^1_i, \lambda^2_j), B_{i,j} \equiv B(\lambda^2_i, \lambda^3_j), C_{i,j} \equiv C(\lambda^3_i, \lambda^1_j).$$

These variables follow a probability distribution $Q_n(\{A_{i,j}\}, \{B_{i,j}\}, \{C_{i,j}\})$ with the property
A second-order inflation of the triangle scenario.

$$Q_n\left(\{A_{ij} = a_{ij}, B_{kl} = b_{kl}, C_{pq} = c_{pq}\}\right) = Q_n\left(\{A_{ij} = a_{\pi(i)\pi'(j)}, B_{kl} = b_{\pi'(k)\pi''(l)}, C_{pq} = c_{\pi''(p)\pi(q)}\}\right),$$  \hspace{1cm} (4)

for all permutations of \(n\) elements \(\pi, \pi', \pi''\). Moreover, the marginal distribution over the “diagonal variables” \(\{A_{ii}, B_{ii}, C_{ii}\}_i\) also satisfies the identities

$$Q_n\left(\{A_{ii} = a_i, B_{ii} = b_i, C_{ii} = c_i\}_i\right) = \prod_{i=1}^{n} P(a_i, b_i, c_i).$$  \hspace{1cm} (5)

See Figure 4 for the resulting causal network when \(n = 2\).

Given an arbitrary distribution \(P(A, B, C)\), the inflation technique consists in demanding the existence of a distribution \(Q_n\) satisfying (4) and (5). Any such \(Q_n\) will be called an \(n^{th}\) order inflation of \(P\). Clearly, if \(P(A, B, C)\) does not admit a \(n^{th}\) order inflation for some \(n\), then it cannot be realizable in the triangle scenario. Deciding the existence of an \(n^{th}\) order inflation can be cast as a linear program (Alevaris, 2001).

Notice that, for any distribution \(Q_n\) satisfying (4), \(Q_n\left(\{A_{\pi(i)\pi'(i)} = a_i, B_{\pi'(i)\pi''(i)} = b_i, C_{\pi''(i)\pi(i)} = c_i\}_i\right) = Q_n\left(\{A_{ii} = a_i, B_{ii} = b_i, C_{ii} = c_i\}_i\right)\) holds for all permutations \(\pi, \pi', \pi''\). Therefore, any distribution \(Q_n\) subject to the constraints (4), (5) must be such that

$$Q_n\left(\{A_{\pi(i)\pi'(i)} = a_i, B_{\pi'(i)\pi''(i)} = b_i, C_{\pi''(i)\pi(i)} = c_i\}_i\right) = \prod_{i=1}^{n} P(a_i, b_i, c_i).$$  \hspace{1cm} (6)
for all permutations $\pi, \pi', \pi''$ of $n$ elements. Actually, the original description of the inflation technique in (Wolfe et al., 2016) imposes the constraints (6) rather than (4), (5) over the distribution $Q_n$.

Demanding the existence of a distribution $Q_n$ satisfying condition (6) can be shown to enforce over $P(A, B, C)$ exactly the same constraints as demanding the existence of a distribution satisfying (4) and (5). Indeed, as noted in (Wolfe et al., 2016), any distribution $Q_n$ satisfying (6) can be twirled or symmetrized (see below) to a distribution $\hat{Q}_n$ satisfying eqs. (4), (5). For convenience, from now on we will just refer to the formulation of the inflation technique involving the symmetries (4). This formulation has the added advantage that the symmetry constraints can be exploited to reduce the time and memory complexity of the corresponding linear program, see (Gent et al., 2006).

The inflation technique is easy to generalize to relax the property of admitting an explanation in terms of arbitrary causal networks (remember, though, that causal networks are just a particular type of causal structures). Namely, one just adds to each observable variable as many subindices as latent variables it depends on, and makes the total probability distribution invariant under independent permutations of the same type of indices. The symmetry condition to impose is

$$Q_n(\{A^1_{i_1} = a^1(\tilde{i}_1), ..., A^m_{i_m} = a^m(\tilde{i}_m) : \tilde{i}_1, ..., \tilde{i}_m\}) = Q_n(\{A^1_{i_1} = a^1(\tilde{\pi}^L(\tilde{i}_1)), ..., A^m_{i_m} = a^m(\tilde{\pi}^L(\tilde{i}_m)) : \tilde{i}_1, ..., \tilde{i}_m\}),$$

for all vectors $\tilde{\pi} = (\pi^1, ..., \pi^L)$ of $L$ independent permutations (one for each latent variable or index type). Here $\tilde{i}_x$ denotes the tuple of subindices on which variable $A^x$ depends. In addition, one must enforce that $Q_n$ satisfies the compatibility conditions

$$Q_n(\{A^1_{i_1,...,i} = a^1_{i_1}, ..., A^m_{i_m,...,i} = a^m_{i_m}\}_i) = \prod_{i=1}^n P(a^1_i, ..., a^m_i).$$

For further elucidation, consider another causal network. In the bilocality scenario, Fig. 5, we again have three random variables $A, B, C$ which are defined, respectively, via the non-deterministic functions $A(\lambda^1), B(\lambda^1, \lambda^2), C(\lambda^2)$. As before, we assume that the latent variables $\lambda^1, \lambda^2$ are independently distributed.

In this scenario, an $n^{th}$-order inflation corresponds to a distribution $Q_n$ over the variables $A_i, B_{jk}, C_l$, where $i, j, k, l$ range from 1 to $n$. $Q_n$ must satisfy the linear constraints:

$$Q_n(\{A_i = a_i, B_{jk} = b_{jk}, C_l = c_l\}) = Q_n(\{A_i = a_{\pi(i)}, B_{jk} = b_{\pi(j)\pi'(k)} C_l = c_{\pi'(l)}\}),$$

for all permutations of $n$ elements $\pi, \pi'$. It is also subject to the identities

$$Q_n(\{A_i = a_i, B_{ii} = b_i, C_i = c_i\}_i) = \prod_{i=1}^n P(a_i, b_i, c_i).$$

Note that the inflation technique also suggests a simple method to solve the dual problem, see Definition 2. Indeed, for any probability distribution $q(X)$, let $q^{\otimes k}(X_1, ..., X_k)$ represent the $k$-variate distribution generated by $k$ independent samples of $q(X)$, and let $Q_n^k(a^1_1, ..., a^m_1, ..., a^1_k, ..., a^m_k)$ denote the marginal probability
admits an n \overline{\text{order inflation}} is a functional such that \hat{\mathcal{F}}(q(X)\otimes k) = F(q). It is immediate that f_n \geq f^* for all n and that the above can be cast as a linear program.

In fact, as shown in (Wolfe et al., 2016), the inflation technique partially solves as well the conventional dual problem, where one asks which constraints any distribution compatible with the considered causal network is subject to. This is achieved by deriving, via combinatorial tools, the facets or linear inequalities which define the set of diagonal marginals of all distributions \mathcal{Q}_n satisfying eq. (9). When applied to a distribution of the form \mathcal{P}^{\otimes n}, each of such linear inequalities translates to a polynomial inequality to be satisfied by any distribution \mathcal{P}(A^1, ..., A^n) over the observed variables admitting an \textit{n}th-order inflation.

In the next two sections, we will prove that, conversely, any distribution \mathcal{P} for the set of measurable events that admits an \textit{n}th-order inflation is \mathcal{O}\left(\frac{1}{\sqrt{n}}\right)-close in Euclidean norm to a distribution realizable in the considered causal network. Similarly, we will show that \mathcal{F}(\mathcal{Q}) = f^* + \mathcal{O}\left(\frac{L^2}{n}\right).

IV. A GENERALIZATION OF THE DE FINETTI THEOREM

The purpose of this section is to prove the following result:

**Theorem 3.** Let \mathcal{Q}_n be any distribution satisfying the symmetry constraints (7), and call \mathcal{Q}_n^k(a_1^{m}, a_1^{m}, ..., a_k^{m}) the marginal probability \mathcal{Q}_n^k(\{A_i^k = a_i^k : x = 1, ..., m, i = 1, ..., k\}). Then, there exist normalized probability distributions \mathcal{Q}_\mu(a_1^{m}, ..., a_m^{m}), achievable in the considered causal network, and probabilities \mu = 0, \sum \mu = 1 such that

\[
D\left(\mathcal{Q}_n^k, \sum \mu \mathcal{Q}_\mu^{\otimes k}\right) \leq \mathcal{O}\left(\frac{Lk^2}{n}\right),
\]

where D(q, r) denotes the statistical distance between the probability distributions q(X), r(X), i.e., D(q, r) = \sum_x |q(x) - r(x)|.

**Proof.** We will prove the result for the triangle scenario; the generalization will be obvious. Given an arbitrary distribution of 3n^2 variables \mathcal{Q}_n(\{A_{ij}, B_{kl}, C_{pq}\}), consider its symmetrization \hat{\mathcal{Q}}_n, defined by

\[
\hat{\mathcal{Q}}_n(\{A_{ij} = a_{ij}, B_{kl} = b_{kl}, C_{pq} = c_{pq}\}) = \frac{1}{n^{3n}} \sum_{\pi, \pi', \pi'' \in S_n} \mathcal{Q}_n(\{A_{ij} = a_{\pi(i)\pi'(j)}, B_{kl} = b_{\pi'(k)\pi''(l)}, C_{pq} = c_{\pi''(p)\pi(q)}\}).
\]

Note that \hat{\mathcal{Q}}_n = \hat{\mathcal{Q}}_n. In addition, any distribution \mathcal{Q}_n satisfying the symmetry condition (4) fulfills \hat{\mathcal{Q}}_n = \mathcal{Q}_n and any symmetrized distribution satisfies (4).

Let \delta(\{\hat{a}(i, j), \hat{b}(k, l), \hat{c}(p, q)\}) be the deterministic distribution assigning the values \hat{a}(i, j), \hat{b}(k, l), \hat{c}(p, q) to the random variables \hat{A}_{ij}, \hat{B}_{kl}, \hat{C}_{pq}, for i, j, k, l, p, q \in \{1, ..., n\}. Since any distribution is a convex combination of deterministic points, it follows that any distribution satisfying eq. (4) can be expressed as a convex combination of symmetric distributions of the form \delta(\{\hat{a}(i, j), \hat{b}(k, l), \hat{c}(p, q)\}). For ease of notation, we will assume from now on that the values \{\hat{a}(i, j), \hat{b}(k, l), \hat{c}(p, q)\} are fixed and denote the latter distribution simply by \hat{\mathcal{Q}}.

Call \hat{\mathcal{Q}}^1 the marginal \hat{\mathcal{Q}}(A_{1,1}, B_{1,1}, C_{1,1}). It can be verified, by symmetry, that it is given by the formula:

\[
\hat{\mathcal{Q}}^1(a, b, c) = \frac{1}{n^2} \sum_{i, j, k = 1}^n \delta(\hat{a}(i, j), a)\delta(\hat{b}(j, k), b)\delta(\hat{c}(k, i), c).
\]
Notice that $\tilde{Q}^i(a, b, c)$ can be reproduced in the triangle scenario. Indeed, the latent variables are $i, j, k$, they can take values in $\{1, \ldots, n\}$ and are uniformly distributed.

Consider now the marginal distribution $\tilde{Q}^k \equiv \tilde{Q}(A_{1,1}, B_{1,1}, C_{1,1}, \ldots, A_{k,k}, B_{k,k}, C_{k,k})$. By symmetry, it is expressed as:

$$\tilde{Q}^k(a_1, b_1, c_1, \ldots, a_k, b_k, c_k) = \frac{1}{n^3(n-1)^3 \ldots (n-k+1)^3} \sum_{i,j,k}^{k} \delta(\tilde{a}(i, j), a_x)\delta(\tilde{b}(j, k), b_x)\delta(\tilde{c}(k, i), c_x),$$

(15)

where the sum is taken over all tuples $\tilde{i}, \tilde{j}, \tilde{k} \in \{1, \ldots, n\}^k$ such that $i_x = i_y (j_x = j_y) [k_x = k_y]$ implies $x = y$. That is, there are no repeated indices when we move from one block of variables $(a_x, b_x, c_x)$ to another.

Now, compare $\tilde{Q}^k$ with $(\hat{Q}^1)^{\otimes k}$. It is straightforward that

$$(\hat{Q}^1)^{\otimes k}(a_1, b_1, c_1, \ldots, a_k, b_k, c_k) = \frac{1}{n} \sum_{i,j,k}^{k} \delta(\tilde{a}(i, j), a_x)\delta(\tilde{b}(j, k), b_x)\delta(\tilde{c}(k, i), c_x),$$

(16)

where, this time, the sum contains all possible tuples $\tilde{i}, \tilde{j}, \tilde{k} \in \{1, \ldots, n\}^k$. The statistical distance between the two distributions is bounded by $\frac{1}{n^3(n-1)^3 \ldots (n-k+1)^3}$ times the number of tuples with non-repeated indices (namely, $n^3(n-1)^3 \ldots (n-k+1)^3$), plus $1/n^k$ times the number of tuples with repeated indices (namely, $n^3 - n^3(n-1)^3 \ldots (n-k+1)^3$). The result is

$$D(\tilde{Q}^k, (\hat{Q}^1)^{\otimes k}) \leq 2 \left(1 - \frac{n^3(n-1)^3 \ldots (n-k+1)^3}{n^k} \right).$$

(17)

Finally, let $Q_n$ be any distribution satisfying eq. (4). Then, $Q_n = \sum_{\mu} p_{\mu} \tilde{Q}_{\mu}$, where $p_{\mu} \geq 0$, $\sum_{\mu} p_{\mu} = 1$ and $\tilde{Q}_{\mu}$ is a distribution of the form $\delta(\{\tilde{a}(i, j), \tilde{b}(j, i), \tilde{c}(p, q)\})$ for every $\mu$. By convexity of the statistical distance, we have that

$$D\left(Q^k_n, \sum_{\mu} p_{\mu} (\hat{Q}^1_{\mu})^{\otimes k}\right) \leq \sum_{\mu} p_{\mu} D(\tilde{Q}^k_{\mu}, (\hat{Q}^1_{\mu})^{\otimes k}) \leq 2 \left(1 - \frac{n^3(n-1)^3 \ldots (n-k+1)^3}{n^k} \right) = O\left(\frac{3k^2}{n}\right).$$

(18)

Extending this result to general causal networks is straightforward, so we will just sketch the proof. First, the action of the corresponding symmetrization over a determinstic distribution equals a distribution $\hat{Q}$ whose 1-marginal $\hat{Q}^1(a^1, \ldots, a^m)$ is a uniform mixture over the tuple of indices $i$ of deterministic distributions of the form $\prod_{x=1}^{m} \delta(a_x, a^x(\tilde{i}_x))$. It thus follows that $\hat{Q}^1$ is realizable within the causal network. The $k$-site marginal $\hat{Q}^k$ is also a uniform mixture of deterministic distributions of a similar type, but where no repeated indices are allowed between the different blocks of variables. The statistical difference between $\hat{Q}^k$ and $(\hat{Q}^1)^{\otimes k}$ is thus bounded by

$$2 \left(1 - \frac{nL(n-1)^L \ldots (n-k+1)^L}{n^k} \right) = O\left(\frac{Lk^2}{n}\right).$$

(19)

V. PROOF OF CONVERGENCE OF THE INFLATION TECHNIQUE

Now we are ready to prove our first main result. Let $P$ be a probability distribution over the observed variables, and suppose that $P$ admits an $n^{th}$-order inflation $Q_n$. Define the second-degree polynomial $N(R) = \sum_{\tilde{a}} (R(\tilde{a}) - P(\tilde{a}))^2$, and let $\tilde{N}$ be a linear functional such that $\tilde{N}(q^{\otimes 2}) = N(q)$ for all distributions $q$. Note that, due to conditions (8), $\tilde{N}(Q_n^2) = N(P) = 0$. Invoking the theorem in the previous section, we have that
Then, by the extended de Finetti theorem, we have that the probabilities of a restricted set \( E \) will show that any causal (dual) inference problem in an arbitrary causal structure can be mapped to an inference and dual problems for these general structures as well? In the following, we given variable can depend, not only on latent variables, but also on the values of other observed variables. Can the just depend on a number of independent latent variables. However, in a general causal structure, the value of a given variable can depend, not only on latent variables, but also on the values of other observed variables. Can the inflation technique solve the inference and dual problems for these general structures as well? In the following, we will show that any causal (dual) inference problem in an arbitrary causal structure can be mapped to an inference

\[ 0 = N(P) = \hat{N}(Q^2_n) = \hat{N} \left( \sum_{\mu} p_\mu Q_{\mu}^{\otimes 2} \right) + O \left( \frac{L}{n} \right) = \sum_{\mu} p_\mu \hat{N}(Q_{\mu}^{\otimes 2}) + O \left( \frac{L}{n} \right) = \sum_{\mu} p_\mu N(Q_{\mu}) + O \left( \frac{L}{n} \right), \]  

(20)

where \( \{Q_\mu\}_\mu \) are distributions realizable within the considered causal structure. The above implies that, for some \( \mu \), \( N(Q_\mu) \leq O \left( \frac{L}{n} \right) \). In other words: for any distribution \( P \) admitting an \( n \)th-order inflation, there exists a realizable distribution \( Q \) that is \( O \left( \sqrt{\frac{L}{n}} \right) \)-close to \( P \) in the Euclidean norm. Since the set of compatible distributions is closed (Rosset et al., 2016), taking the limit \( n \to \infty \) it follows that any distribution \( P \) passing all inflation tests must be realizable.

Finally, notice that the previous argument can also be used to prove the convergence of the sequence of linear programs (11). In effect, let \( F \) be a polynomial of degree \( k \), with \( f^* = \max_P F(P) \), and let \( f^n \) be defined as in (11). Then, by the extended de Finetti theorem, we have that

\[ f^* \leq f^n = \sum_{\mu} p_\mu F(Q_\mu) + O \left( \frac{Lk^2}{n} \right) \leq f^* + O \left( \frac{Lk^2}{n} \right). \]  

(21)

In certain practical cases, we may not know the full probability distribution of the observable variables, but only the probabilities of a restricted set \( E \) of measurable events. To apply the inflation technique to those cases, rather than fixing the value of all probability products, like in eq. (8), we would impose the constraint

\[ \sum_{i=1}^{n} \sum_{a_i \in e_i} Q_n \{ \{ A_{i_{1}} \cdots = a_{i_{1}}, \ldots, A_{i_{m}} \cdots = a_{i_{m}} \} \} = \prod_{i=1}^{n} P(e_i), \]  

(22)

where \( e_1, \ldots, e_n \in E \). Any distribution \( Q_n \) satisfying both (7) and (22) will also be dubbed an \( n \)th order inflation of the distribution of measurable events.

For example, consider again the triangle scenario (Fig. 2), and assume that our experimental setup just allows us to detect events of the form \( e(a) \equiv \{(A, B, C) : A = B = C = a\} \). Then our set of measurable events is \( E = \cup_a \{ e(a) \} \) and the input of the causal inference problem is the distribution \( \{ P(e), e \in E \} \). An \( n \)th order inflation \( Q_n \) of \( P(e) \) would satisfy eq. (4) and the linear conditions

\[ Q_n \{ \{ A_{i_1} = a_i, B_{i_2} = a_i, C_{i_3} = a_i \} \} = \prod_{i=1}^{n} P(e(a_i)). \]  

(23)

The proofs of convergence presented above easily extend to this scenario. Indeed, choosing the polynomial \( N \) such that \( N(R) = \sum_{e \in E} (P(e) - R(e))^2 \), and following the same derivation as in eq. (20), we conclude that a distribution of measurable events admitting an \( n \)th order inflation is \( O \left( \sqrt{\frac{L}{n}} \right) \)-close in Euclidean norm to a realizable distribution. Similarly, one can bound the speed convergence of the inflation technique when applied to maximize polynomials of a probability distribution of measurable events.

VI. EXTENSION OF OUR RESULTS TO GENERAL CAUSAL STRUCTURES

So far we have been just referring to causal networks, i.e., those causal structures where all observed variables just depend on a number of independent latent variables. However, in a general causal structure, the value of a given variable can depend, not only on latent variables, but also on the values of other observed variables. Can the inflation technique solve the inference and dual problems for these general structures as well? In the following, we will show that any causal (dual) inference problem in an arbitrary causal structure can be mapped to an inference
(dual) problem with a non-trivial set of measurable events in an extended causal network. Hence we can solve it via the inflation technique.

To go from general causal structures to causal networks, we will follow the following three steps.

**FIG. 6 First step: re-wiring.**

The first step is called re-wiring, see Figure 6. Note that, if in the original structure there is an edge from variable $X$ to the latent variable $\lambda$, if we delete the edge and replace it by edges from variable $X$ to the direct successors of $\lambda$ we obtain a causal structure with the same predictive power. Indeed, in the first structure $\lambda$ could carry a copy of $X$ to each of its direct successors: this implies that any probability distribution of observed variables realizable in the re-wired causal structure can also be realized in the original causal structure. Conversely, suppose that in the original causal structure $\lambda$ depends on the variables $\bar{X}$ (we group them all in a vector) as $\lambda = \lambda(\bar{X}, \mu)$, where $\lambda$ is a deterministic function and $\mu$ is an internal random variable. Then one can simulate the same probability distribution of observed variables in the re-wired causal structure by distributing $\mu$ to the direct successors of $\lambda$ and making them compute locally the value of $\lambda$, given $\bar{X}$. Sequentially re-wiring the edges pointing to a latent variable, we end up with an equivalent causal structure where the latent variables just have edges pointing from them and not to them.

**FIG. 7 Second step: unpacking.**

For the second step, unpacking (Figure 7), we will draw inspiration from the notion of interruption, described in (Wolfe and Sainz, 2017). Let $X$ be an observed variable depending on the latent variables $\lambda$ and the observed variables $\tilde{Y}$, and suppose that $\tilde{Y}$ can take $d$ different values, e.g.: $\tilde{Y} \in \{1, \ldots, d\}$. In this step, we will break the edges between $\tilde{Y}$ and $X$ and “unpack” variable $X$ into the observed variables $X(1), \ldots, X(d)$, defined via the expression $X(s) \equiv X(s, \bar{\lambda})$. Unpacking all the variables, we arrive at a causal structure with a few disconnected random variables $\bar{A}$, the observable parents of the causal graph, and a number of observed variables $\bar{B}$ depending on just the latent variables, which by the previous step are independent from each other. The probabilities of the observed variables in the original causal structure can be obtained from the probabilities of a set of measurable events in this new causal structure via the relation

$$P(\bar{A} = \bar{a}, \bar{B} = \bar{b}) = P(\{\bar{A} = \bar{a}\} \cap \{B^i(\bar{a}^A_i, \bar{b}^B_i) = \bar{b}^i\}_i),$$  \hspace{1cm} (24)$$

where $I^A_i, I^B_i$ are, respectively, the superindices of the variables $\bar{A}, \bar{B}$ who are direct predecessors of $B^i$ in the original causal structure.
In the last step, re-parenting (Figure 8), we will introduce a new latent variable \( \lambda_A \) for each observable parent \( A \) of the original graph, and an edge from \( \lambda_A \) to \( A \). At the end of this step, we have a layer of observed variables depending only on the another layer of independent latent variables. This is none other but a causal network! The original causal inference (dual) problem is hence mapped to an inference (dual) problem in a causal network with a set \( E \) of measurable events given by

\[
E = \left\{ \{ \bar{A} = \bar{a} \} \cap \{ B^i (\bar{a}^{(\lambda)}, \bar{b}^{(\nu)}) = \bar{b}^i \}_{\bar{a}, \bar{b}} \right\}. \tag{25}
\]

When dealing with an inference problem with observable parents \( \bar{A} \), an alternative to the re-parenting step is simply to erase the observed parents from the structure and consider the resulting causal network with constraints

\[
P(\{ B^i (\bar{a}^{(\lambda)}, \bar{b}^{(\nu)}) = \bar{b}^i \}_{\bar{a}}) = P(B = b | \bar{A} = \bar{a}). \tag{26}
\]

Applying the inflation technique to this second network is computationally cheaper, since it has less observed variables. Note though that this mapping does not allow us to optimize polynomials of \( P(\bar{A}, B) \), but of \( P(B|\bar{A}) \), so one cannot use it to solve the causal dual problem.

For illustration, consider again the instrumental scenario, see Fig. 3. Suppose that the random variables \( A, B \) can just take two possible values, 0 or 1. Then we can define the vector variable \( (B(0), B(1)) \) via the non-deterministic function \( (B(0, \lambda), B(1, \lambda)) \). Similarly, we can unpack \( C \) into the vector \( (C(0), C(1)) \equiv (C(0, \lambda), C(1, \lambda)) \). The resulting causal network after re-wiring, unpacking and re-parenting is depicted on the right-hand side of Fig. 7. Let \( \hat{P}(A, B(0), B(1), C(0), C(1)) \) be the distribution of the observed variables of the new network. Then, our set of measurable events is \( \{ A = a, B(a) = b, C(b) = c \}_{a,b,c} \), and the correspondence with the original distribution is \( P(A = a, B = b, C = c) = \hat{P}(A = a, B(a) = b, C(b) = c) \).

Alternatively, we can erase variable \( A \) from the causal network and take the distribution \( \hat{P}(B(0), B(1), C(0), C(1)) \) of observed variables to be our fundamental object. Then, the set of measurable events is \( \{ B(a) = b, C(b) = c \}_{a,b,c} \), and the corresponding probabilities are \( P(B = b, C = c | A = a) = \hat{P}(B(a) = b, C(b) = c) \).

VII. CONCLUSION

In this paper we have proven that the hierarchy of tests proposed in (Wolfe et al., 2016) to bound the set of distributions admitting a representation in a given causal network is complete, in the sense that any distribution that does not admit such a representation will fail one of the tests. More quantitatively, we showed that any distribution \( P \) passing the \( n \)th test is \( O\left( \frac{1}{\sqrt{n}} \right) \)-close in Euclidean norm to a distribution \( Q \) realizable within the considered causal network. We also proved that the \( n \)th linear programming relaxation provided by the inflation technique to the solution of the dual problem is \( O\left( \frac{1}{n} \right) \) away from optimality.

To top it up, we showed that any causal inference (dual) problem in a general causal structure can be mapped to a causal inference (dual) problem in an extended causal network with a non-trivial set of measurable events. This is a regime for which we also proved the convergence of the inflation technique. Put together, these two results thus show that the inflation technique is much more than a useful machinery to derive statistical limits. It is an alternative way to define causal structures!

For future work, it would be interesting to adapt our results to the quantum case, for which the inflation technique can also be applied (Wolfe et al., 2016). That would require an extension of the finite quantum de Finetti
theorem (Koenig and Renner, 2005) that held for certain positive-semidefinite structures subject to certain symmetry constraints. An arduous task!

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