On the monodromy problem for the four-punctured sphere

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Abstract
We consider the monodromy problem for the four-punctured sphere in which the character of one composite monodromy is fixed, by looking at the expansion of the accessory parameter in the modulus directly, without taking the limit of the quantum conformal blocks for an infinite central charge. The integrals that appear in the expansion of the Volterra equation involve products of two hypergeometric functions to first order and up to four hypergeometric functions to second order. It is shown that all such integrals can be computed analytically. We give the complete analytical evaluation of the accessory parameter to first and second order in the modulus. The results agree with the evaluation obtained by assuming the exponentiation hypothesis of the quantum conformal blocks in the limit of infinite central charge. Extension to higher orders is discussed.

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1. Introduction

In the papers [1, 2] the following monodromy problem is considered for the Liouville theory on the sphere: Given the singularities in the standard position $0, x, 1, \infty$, and given the class, i.e. the trace of the monodromy for a path encircling both singularities at 0 and $x$, find the value of the accessory parameter realizing such data. Such a problem intervenes in the process of the classical limit of the quantum four-point function; the value of trace of the above described monodromy is then fixed by a saddle-point procedure [3]. In [1, 3] the problem is solved by going over to the quantum formulation for the four-point function and taking the classical limit, i.e. the limit in which the central charge goes to infinity. As the quantum conformal blocks are known as formal power series expansions in $x$, also the classical result so obtained is given as a formal power expansion in $x$. The procedure goes through a process...
of exponentiation of the quantum conformal blocks after which the classical limit \( b \to 0 \) is taken; in such a limit, heavy cancellations take place \[4\]. Several analytic \[1, 5–7\] and numerical \[3, 4\] calculations, also exploiting recursion formulae \[8, 9\] for the conformal blocks, support the validity of such a calculational scheme.

One suspects, on the other hand, that the same results should be obtainable just by exploiting the transformation properties of the ordinary differential equations that underlie the Liouville theory at the classical level. In this note, we shall in fact consider the problem directly at the classical level. In addition, it appears that working without taking the singular limit in which the central charge goes to infinity one might better control the convergence region of the expansion of the accessory parameter as a function of the modulus \[10\].

The approach followed in this paper consists of computing the monodromy along a contour embracing 0 and \( x \) through the usual convergent iteration expansion for the solution of the Heun equation. The monodromy is computed along a contour that avoids the neighborhood of the origin where the kernel is singular, and then we expand the result in \( x \).

In so doing, one is faced to first order, with the computation of integrals containing the product of two hypergeometric functions; if one goes to the second order, the product of four hypergeometric functions in a double integral appears.

In this paper, we show how to analytically compute such integrals, which appear in the expansion of the solution of the Volterra equation. The complete first-order result gives as a byproduct the value of the accessory parameter, which coincides with the one derived in \[3\] and re-derived in \[1, 6\].

For computing the integrals appearing in the first-order result, we exploit the transformation property of the solution of the differential equation under \( SL(2, C) \). In the calculation of the second order, such a technique is not sufficient, and we need a non-invertible transformation that at the infinitesimal level is related to the operator \( L_{-2} = \frac{1}{z} \frac{d}{dz} \). Contrary to the \( SL(2, C) \) transformations, this is not one-to-one in the complex plane.

On the other hand, the procedure we shall describe involves only the solutions along the real \( z \) axis for \( z \geq 1 \), and therefore in that region, for \( |x| < 1 \) the transformation is well defined.

After developing such tools, we give the complete second-order computation for the accessory parameter. In so doing, we employ a formalism apt to be extended to higher-order computations.

The second-order result agrees with the one obtained in \[1, 6\] by considering the classical limit of the quantum conformal blocks combined with the exponentiation hypothesis, and thus, it lends strong support to the exponentiation hypothesis of the conformal blocks in the \( b \to 0 \) limit. We discuss also the extension of the procedure to higher orders.

Obviously, as the determination of the accessory parameter \( C(x) \) is always obtained through the solution of an implicit equation, the fact that the function \( Q(z) \), which represents the energy momentum tensor, has a radius of convergence of 1 in \( x \), for \( z > 1 \), does not assure that the expansion of \( C(x) \) in \( x \) has the same radius of convergence. For achieving rigorous lower bounds on such a radius of convergence, methods similar to those developed in \[10, 11\] for convergence in the coupling strength should be applied. The developed technique can also be applied to the problem of the punctured torus.
2. General setting

The ordinary differential equation associated with the monodromy problem is

\[ y''(z) + Q(z)y(z) = 0 \]  

with

\[ Q(z) = \frac{\delta_0}{z^2} + \frac{\delta}{(z-x)^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_0 - \delta - \delta_1}{z(z-1)} + \frac{C(x)}{z(z-x)(1-z)} \]  

(2)

where \( \delta_j = (1 - \lambda_j^2)/4 \). \( C(x) \) is the accessory parameter to be fixed so that the monodromy along a contour encircling both 0 and \( x \) has trace \( -2 \cos \pi \lambda \), and as such will depend both on \( x \) and \( \delta \). We have

\[
C(0) = \delta_x - \delta_0 - \delta
\]

(3)

and \( C(x) \) is related to one used in [1, 3], which we call \( C_L(x) \), by \( C(x) = x(1-x)C_L(x) \) and thus, \( C(0) = xC_L(x)|_{x=0} \) and \( C(0) = [xC_L(x)]'|_{x=0} - C(0) \).

Expanding in \( x \) we have

\[ Q = Q_0 + xQ_1 + x^2Q_2 + O(x^3) \]

(4)

\[
Q_0 = \frac{\delta_x}{z^2} + \frac{\delta_1}{(z-1)^2} + \frac{\delta_\infty - \delta_x - \delta_1}{z(z-1)}
\]

\[
Q_1 = \frac{2\delta - C'(0)}{z^2(z-1)} - \frac{2\delta + C(0)}{z^3(z-1)}
\]

\[
Q_2 = -\frac{C''(0)}{2z^2(z-1)} + \frac{3\delta - C'(0)}{2z^3(z-1)} - \frac{3\delta + C(0)}{z^4(z-1)}
\]

(5)

It is our interest to compute the class of the monodromy along a circuit enclosing both the origin and \( x \). Working near the origin is difficult due to the singular nature of the kernel. Instead, we shall compute the same monodromy along the circuit shown in figure 1. The great advantage in performing such a change in the contour is the fact that the expansion in \( x \) of \( Q(z) \) along the contour is no longer singular and actually is convergent with convergence radius 1. \( Y \) will denote the complex

\[
Y(z) = \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}
\]

(6)

being \( y_k \) two independent solutions of \( y''_k + Q_0y_k = 0 \), canonical at \( z=1 \). They are given by

\[
y_1(z) = (1-z)^{-i\lambda_1} \frac{1 - \lambda_1 - \lambda_\infty - \lambda_x}{2} \left( \frac{1 - \lambda_1 + \lambda_\infty - \lambda_x}{2}, 1 - \lambda_1; 1 - z \right) F \]

\[
y_2(z) = (1-z)^{i\lambda_1} \frac{1 + \lambda_1 + \lambda_\infty + \lambda_x}{2} \left( \frac{1 + \lambda_1 - \lambda_\infty + \lambda_x}{2}, 1 + \lambda_1; 1 - z \right) F
\]

(7)

The constant Wronskian is easily computed at \( z = 1 \)

\[
w_{12} = y_1y_2' - y_1'y_2 = -\lambda_1
\]

(9)

The unperturbed monodromy is computed as follows. We start from the \( Y \) for real \( z \), \( z < 1 \). The continuation to the upper side of the cut \( (1, +\infty) \) is given by
whose asymptotic behavior for large $z$ is

$$
y^+_1(z) \approx -ie^{\frac{\gamma_1}{\pi}}(z - 1)^{\frac{1+\lambda_1}{1+\lambda_2}} z^{\frac{1-\lambda_1}{1+\lambda_2}}
\times F\left(\frac{1 - \lambda_1 + \lambda_{\infty} - \lambda_2}{2}, \frac{1 - \lambda_1 + \lambda_{\infty} + \lambda_2}{2}, 1 - \lambda_1, 1 - z\right)
\equiv -ie^{\frac{\gamma_1}{\pi}}t_1(z) \quad (10)
$$

$$
y^+_2(z) \approx -ie^{\frac{\gamma_1}{\pi}}(z - 1)^{\frac{1+\lambda_1}{1+\lambda_2}} z^{\frac{1-\lambda_1}{1+\lambda_2}}
\times F\left(\frac{1 + \lambda_1 + \lambda_{\infty} + \lambda_2}{2}, \frac{1 + \lambda_1 - \lambda_{\infty} + \lambda_2}{2}, 1 + \lambda_1, 1 - z\right)
\equiv -ie^{\frac{\gamma_1}{\pi}}t_2(z) \quad (11)
$$

whose asymptotic behavior for large $z$ is

$$
Y^+(z) \approx B^+(z) = -i\left(\begin{array}{c}
B_1^{(1)} e^{\frac{\gamma_1}{\pi}z} B_2^{(1)} e^{\frac{\gamma_1}{\pi}z} \\
B_1^{(2)} e^{\frac{\gamma_1}{\pi}z} B_2^{(2)} e^{\frac{\gamma_1}{\pi}z}
\end{array}\right)
\left(z^{\frac{1-\lambda_1}{1+\lambda_2}}
\right) \quad (12)
$$

$B^{(j)}_k$ being a well-known matrix

$$
B^{(j)}_k =
\left(\begin{array}{cc}
\Gamma\left(1 - \lambda_1\right)\Gamma\left(\lambda_{\infty}\right)
\frac{1 - \lambda_1 + \lambda_{\infty} + \lambda_2}{2} & \Gamma\left(1 - \lambda_1\right)\Gamma\left(-\lambda_{\infty}\right)
\frac{1 - \lambda_1 - \lambda_{\infty} + \lambda_2}{2} \\
\Gamma\left(1 + \lambda_1\right)\Gamma\left(-\lambda_{\infty}\right)
\frac{1 + \lambda_1 + \lambda_{\infty} + \lambda_2}{2} & \Gamma\left(1 + \lambda_1\right)\Gamma\left(\lambda_{\infty}\right)
\frac{1 + \lambda_1 - \lambda_{\infty} + \lambda_2}{2}
\end{array}\right) \quad (13)
$$

Similarly

$$
y^-_1(z) = ie^{-\frac{\gamma_1}{\pi}}t_1(z), \quad y^-_2(z) = ie^{-\frac{\gamma_1}{\pi}}t_2(z) \quad (14)
$$
We start from \( \varepsilon = + \infty - ie \) equation (15), whose continuation to the upper side of the cut is equation (12). Taking the turn of \( \pi / 2 \) at infinity, we go back to \( \varepsilon = + \infty - ie \) having encircled the origin and \( x \), and we obtain the monodromy matrix to lowest order

\[
M^0 = - (B^+ \begin{pmatrix} e^{-i\lambda_0} & 0 \\ 0 & e^{i\lambda_0} \end{pmatrix} B)^{-1} .
\]

One easily checks that \( \text{tr} M^0 = -2 \cos \pi \lambda_v \). The first order corrections of \( Y \) are provided by

\[
Y(z) + x S_1(z) Y(z)
\]

with

\[
S_1(z) = \frac{1}{w_{12}} \begin{pmatrix}
\int_1^z y_2 Q_1 y_1 dz - \int_1^z y_1 Q_1 y_2 dz \\
\int_1^z y_2 Q_1 y_2 dz - \int_1^z y_1 Q_1 y_1 dz
\end{pmatrix}
\]

\[
w_{12} = -\lambda_1 .
\]

Following the procedure illustrated above using (17) instead of the unperturbed \( Y \), we obtain the new monodromy matrix \( M^0 + \delta M \).

\[
\delta M = x \left( S_1^+ M^0 - M^0 S_1^- \right)
\]

with \( S_1 \equiv S_1(\infty) \), and we have

\[
\text{tr} M = -2 \cos \pi \lambda_v + x \text{tr} \left( S_1^+ - S_1^- \right) M^0
\]

where

\[
(S_1^+ - S_1^-)_12 = \frac{2i \sin \pi \lambda_1}{w_{12}} \int_1^\infty t_1(z) Q_1(z) t_1(z) dz \equiv \frac{2i \sin \pi \lambda_1}{w_{12}} Q_1(1, 1)
\]

\[
(S_1^+ - S_1^-)_21 = \frac{2i \sin \pi \lambda_1}{w_{12}} \int_1^\infty t_2(z) Q_1(z) t_2(z) dz \equiv \frac{2i \sin \pi \lambda_1}{w_{12}} Q_1(2, 2)
\]

We have due to equations (10), (11), (14)

\[
(S_1^+ - S_1^-)_11 = -(S_1^+ - S_1^-)_22 = 0
\]

and thus,

\[
\text{tr} \delta M = x \left[ (S_1^+ - S_1^-)_12 M^0_{21} + (S_1^+ - S_1^-)_21 M^0_{12} \right].
\]

To determine \( C'(0) \), we must impose \( \text{tr} \delta M = 0 \).

The second-order result is obtained by iterating once the result with \( Q_1 \) and adding also the contribution obtained by replacing in (17) \( x Q_1 \) with \( x^2 Q_2 \).
3. First-order calculation

The computation of the first-order result is reasonably simple. We shall adopt here a formalism that is apt to be extended to the second- and higher-order computations.

We are faced to compute the integrals appearing in equations (21) and (22) with $Q_1(z)$ given in equation (5). More generally, we shall compute analytically the indefinite integrals

$$\int_1^\infty \frac{t_j(z) t_k(z)}{z^n(z - 1)} dz$$

for $m \geq 2$ and where $j$ and $k$ take the values of 1 and 2.

For $m = 2, 3$ such integrals can be computed by exploiting the transformation properties under $SL(2, C)$ of the solutions. Let us consider the equation

$$R'' + R = 0$$

$$R = \frac{\delta_\nu}{(z - a)^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_{\infty} - \delta_1 - \delta_\nu}{(z - a)(z - 1)}.$$  

The solutions of equation (26) are

$$l_1(z, a) = (z - 1)^{1-j} \left( \frac{z - a}{1 - a} \right)^{1+j}$$

$$F\left(1 - \lambda_1 - \lambda_{\infty} - \lambda_\nu, \frac{1 - \lambda_1 + \lambda_{\infty} - \lambda_\nu}{2}, 1 - \lambda_1; \frac{1 - z}{1 - a}\right)$$

and similarly for $l_2$. Then, writing

$$R = R_0 + aR_1 + O\left(a^2\right)$$

$$R_0 = \frac{\delta_\nu}{z^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_{\infty} - \delta_1 - \delta_\nu}{z(z - 1)} = Q_0, \quad R_1 = \frac{\delta_{\infty} + \delta_\nu - \delta_1}{z^2(z - 1)} - \frac{2\delta_\nu}{z^3(z - 1)}$$

$$\frac{\partial l}{\partial a} = \left. \frac{\partial l}{\partial a} \right|_{a=0}$$

using

$$\ddot{l}_0 + R_0 \dot{l}_0 + R_1 l_0 = 0$$

and $l(z, 0) = t_k(z)$ we have

$$\int_1^\infty t_k R_1 t_j \, dz = \int_1^\infty t_k \left( - \dot{t}_j - R_0 \dot{l}_j \right) \, dz = t_k \dot{l}_j - t_k \ddot{l}_j \mid_1^\infty = t_k \dot{l}_j - t_j \ddot{l}_k \mid_1^\infty.$$

We find

$$R_1(1, 1) \equiv \int_1^\infty t_1 R_1 t_1 \, dz = - \lambda_\nu^2 B_1^{(1)} B_2^{(1)}$$

$$R_1(2, 2) \equiv \int_1^\infty t_2 R_1 t_2 \, dz = - \lambda_\nu^2 B_1^{(2)} B_2^{(2)}.$$  

On the other hand, the change of $\frac{\delta_\nu}{z^2(z - 1)}$ into $\delta_\nu - \epsilon$ induces in $Q_0$ the change

$$Q_0 \to Q_0 + \epsilon Q_0, \quad \epsilon Q_0 = \frac{\epsilon}{z^2(z - 1)}$$

$$\frac{\partial l}{\partial a} = \left. \frac{\partial l}{\partial a} \right|_{a=0}$$

and similarly for $\ddot{l}_2$. Then, writing

$$R_1 = R_0 + aR_1 + O\left(a^2\right)$$

$$R_0 = \frac{\delta_\nu}{z^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_{\infty} - \delta_1 - \delta_\nu}{z(z - 1)} = Q_0, \quad R_1 = \frac{\delta_{\infty} + \delta_\nu - \delta_1}{z^2(z - 1)} - \frac{2\delta_\nu}{z^3(z - 1)}$$

$$\frac{\partial l}{\partial a} = \left. \frac{\partial l}{\partial a} \right|_{a=0}$$

using

$$\ddot{l}_0 + R_0 \dot{l}_0 + R_1 l_0 = 0$$

and $l(z, 0) = t_k(z)$ we have

$$\int_1^\infty t_k R_1 t_j \, dz = \int_1^\infty t_k \left( - \dot{t}_j - R_0 \dot{l}_j \right) \, dz = t_k \dot{l}_j - t_k \ddot{l}_j \mid_1^\infty = t_k \dot{l}_j - t_j \ddot{l}_k \mid_1^\infty.$$
leaving the singularities at \( z = 1 \) and at \( z = \infty \) unchanged. This time, the related change \( \delta t_k \) is simply given by

\[
\delta t_k = \frac{2 \epsilon}{\lambda_v} \delta t_k.
\]

and we can again apply equation (32), replacing \( \dot{t}_k \) with \( \frac{2 \delta t_k}{z_{\infty}} \). Combined with (33), it provides us with the integrals of type (25) with \( m = 2 \) and with \( m = 3 \) appearing in the first-order computation in terms of hypergeometric functions and derivatives thereof.

The integrals appearing in (21), (22) have infinity as the upper extreme of integration for which the derived formulae (32) also hold. As the asymptotic behavior of the hypergeometric functions are given \[12\] by simple powers of \( z \) multiplied by gamma functions, we see that such integrals are expressed in terms of the functions \( \Gamma \) and \( \psi \) where \( \psi(x) = \Gamma'(x)/\Gamma(x) \). We shall denote by \( N_m \) the expression

\[
N_m = \frac{1}{z^m(z - 1)}.
\]

They form a basis for the derivative of \( Q(z) \) with respect to \( x \) to any order, and we shall set

\[
\int_1^{\infty} t_k \frac{1}{z^m(z - 1)} t_j dz = N_m(k, j) .
\]

Explicitly we find

\[
N_2(1, 1) = \frac{\lambda_\infty B_1^{(1)} B_2^{(1)} \psi(\lambda_1, \lambda_v, \lambda_\infty)}{\lambda_v} \equiv N_2^{(1)} B_1^{(1)} B_2^{(1)} \psi_1
\]

and

\[
N_2(2, 2) = \frac{\lambda_\infty B_1^{(2)} B_2^{(2)} \psi(-\lambda_1, -\lambda_v, -\lambda_\infty)}{\lambda_v} \equiv N_2^{(2)} B_1^{(2)} B_2^{(2)} \psi_2
\]

where we defined

\[
\psi(\lambda_1, \lambda_v, \lambda_\infty) \equiv \psi\left(1 - \frac{\lambda_1 - \lambda_\infty - \lambda_v}{2}\right) - \psi\left(1 + \frac{\lambda_1 + \lambda_\infty - \lambda_v}{2}\right) + \psi\left(1 - \frac{\lambda_1 - \lambda_\infty + \lambda_v}{2}\right) + \psi\left(1 + \frac{\lambda_1 + \lambda_\infty + \lambda_v}{2}\right).
\]

Moreover, in equation (24) we have

\[
M_{12}^0 \det(B^-) = -2 i \sin \pi \lambda_\infty B_1^{(1)} B_2^{(1)}
\]

\[
M_{21}^0 \det(B^-) = 2 i \sin \pi \lambda_\infty B_1^{(2)} B_2^{(2)}
\]

with \( \det B^- = \lambda_1/\lambda_\infty \). For future use, we report below also the values of \( M_{11}^0 \) and \( M_{22}^0 \).

\[
M_{11}^0 \det(B^-) = e^{i\pi \lambda_1 \lambda_\infty} \left(B_1^{(1)} B_2^{(2)} e^{i\epsilon} - B_1^{(2)} B_2^{(1)} e^{-i\epsilon}\right)
\]

\[
M_{22}^0 \det(B^-) = e^{-i\pi \lambda_1 \lambda_\infty} \left(B_1^{(1)} B_2^{(2)} e^{-i\epsilon} - B_1^{(2)} B_2^{(1)} e^{i\epsilon}\right).
\]
The vectors
\[ T(1, 1) = (R_1(1, 1), N_2(1, 1)) = B_1^{(1)} B_2^{(1)} \left(-\frac{\lambda^2_{\infty}}{\lambda_2}, \frac{\lambda_{\infty}}{\lambda_2} \psi_1 \right) \equiv B_1^{(1)} B_2^{(1)} \hat{T}(1, 1) \]  
(44)
\[ T(2, 2) = (R_1(2, 2), N_2(2, 2)) = B_1^{(2)} B_2^{(2)} \left(-\frac{\lambda^2_{\infty}}{\lambda_2}, \frac{\lambda_{\infty}}{\lambda_2} \psi_2 \right) \equiv B_1^{(2)} B_2^{(2)} \hat{T}(2, 2) \]  
(45)
are given by
\[ T(k, k) = A \begin{pmatrix} N_2(k, k) \\ N_1(k, k) \end{pmatrix} \]  
(46)
with
\[ A = \begin{pmatrix} \delta_2 + \delta_{\infty} - \delta_1 - 2\delta_x \\ 1 \\ 0 \end{pmatrix}. \]  
(47)
\[ Q_1 \text{ (see equation (5)) in the basis } N_2, N_3 \text{ is represented by the vector} \]
\[ q_1 = (2\delta - C'(0), -2\delta - C(0)) \]  
(48)
and thus, see equation (24), the equation for \( C'(0) \) becomes
\[ 0 = q_1 \cdot A^{-1} \cdot \left( \hat{T}(1, 1) - \hat{T}(2, 2) \right), \]  
(49)
i.e.
\[ 0 = 2\delta_x (2\delta - C'(0)) - (2\delta + C(0))(\delta_x + \delta_{\infty} - \delta_1) = \]
\[ -2\delta_x (C'(0) + C(0)) - (\delta_x + \delta_{\infty} - \delta_1)(\delta_x - \delta_0 + \delta), \]  
(50)
giving
\[ C'(0) = \left( \frac{\delta_x - \delta_0 + \delta}{2\delta_x} \right)(\delta_x - \delta_{\infty} + \delta_1) - C(0) = \left[ xC_L(x) \right] \bigg|_{x=0} - C(0), \]  
(52)
which is the result of [1, 3, 6] obtained by taking the \( b \to 0 \) limit of the conformal blocks.

4. Second-order calculation

The equation that gives \( C''(0) \) is provided by the vanishing of the coefficient of \( x^2 \) in the expansion of
\[ \text{tr} \left( 1 + xS_1^+ + x^2S_2^+ \right) M^0 \left( 1 + xS_1^- + x^2S_2^- \right)^{-1}, \]  
(53)
i.e.
\[ 0 = \text{tr} \left( S_2^+ - S_2^- \right) M^0 - \text{tr} \left( S_1^+ - S_1^- \right) M^0 S_1^-. \]  
(54)
The second-order change in the functions \( \psi_k \) is given by the direct contribution due to \( Q_2 \) and by the second iteration of the contribution of \( Q_1 \).
With regard to the direct contribution we must compute
\[ \int_{1}^{\infty} t_k(z) Q_2(z) t_j(z) \, dz, \]
(55)
with \( Q_2(z) \) given in equation (5), where the new integrals \( N_q(k, j) \) appear. In the basis \( N_2, N_3, N_4, Q_2 \) is represented by the vector
\[ q_2 = ( -C^*(0)/2, 3\delta - C'(0), -3\delta - C(0) ). \]
(56)
The \( N_q(j, k) \) cannot be computed by performing an \( SL(2, C) \) transformation. We shall exploit the new transformation
\[ z = \frac{v - a}{1 - a}, \]
(57)
and use the Schwarzian transformation of \( R_0 \) and the \( -\frac{1}{2} \)-form nature \([13]\) of the solutions \( t_k \).

The above transformation gives rise to
\[ R(v, a) = Q_0 \left( \frac{v^2 - a}{v(1 - a)} \right) \left( \frac{dz}{dv} \right)^2 - [z, v], \]
(58)
where
\[ \{z, v\} = -\frac{3a}{(a + v^2)^2}, \]
(59)
is the Schwarz derivative of the transformation and the new solutions are
\[ t_k(v, a) = \frac{1}{\sqrt{1 - a}} \left( \frac{dz}{dv} \right)^{-\frac{1}{2}} t_k \left( \frac{v^2 - a}{v(1 - a)} \right) = \frac{v}{\sqrt{v^2 + a}} t_k \left( \frac{v^2 - a}{v(1 - a)} \right). \]
(60)

Reverting to the \( z \)-notation for the variable and denoting with the dot the derivative w.r.t. \( a \), we have
\[ R(z, a) = Q_0 \left( \frac{z^2 - a}{z(1 - a)} \right) \left( \frac{z^2 + a}{z^2(1 - a)} \right)^2 + \frac{3a}{(a + z^2)^2}, \]
(61)
with
\[ \dot{R}(z, 0) = \delta_\nu - \delta_1 - \delta_\infty + \frac{3 - 3\delta_1 + 3\delta_\infty + \delta_0}{z^3(z - 1)} - \frac{3 + 4\delta_\nu}{z^4(z - 1)}. \]
(62)
From equation (60), we have
\[ \dot{t}_k(z, 0) = -\frac{1}{2z^2} t_k(z) + \left( z - \frac{1}{z} \right) t_k'(z). \]
(63)
Contrary to the \( SL(2, C) \) transformations, the transformation (57) is not one-to-one in the complex plane. Nevertheless, for \( \|a\| < 1 \) the transformation is well defined, i.e. non-singular along the line \( 1 < z < \infty \), which is our range of application.

We can then exploit again the integration formula (32), with \( \dot{t}_k \) replaced by equation (63) and \( R_1 \) by \( \dot{R} \) of equation (62). One can also verify the correctness of the result by taking explicitly the derivative w.r.t. \( z \) of the obtained formula.

We introduce the three-dimensional vectors \( T^r(k, j) \), with \( r = 1, 2, 3 \) that represent the matrix elements of the variation of \( Q_0 \) under, respectively, the transformation of equation (27)
(r = 1), the transformation of equation (34) (r = 2) and of the transformation of the above equation (57) (r = 3). We have

\[ T(j, k) = A \begin{pmatrix} N_2(j, k) \\ N_3(j, k) \\ N_4(j, k) \end{pmatrix} \]  

(64)

with

\[ A = \begin{pmatrix} \delta_\infty + \delta_v - \delta_1 & -2\delta_v & 0 \\ 1 & 0 & 0 \\ \delta_v - \delta_1 - \delta_\infty & 3 - 3\delta_1 + 3\delta_\infty + \delta_v - 3 - 4\delta_v \end{pmatrix} \]  

(65)

and

\[ T(1, 1) = B_1^{(1)}B_2^{(1)} \begin{pmatrix} -\lambda_\infty^2, \frac{\lambda_\infty}{\lambda_v}, -\lambda_\infty^2 \end{pmatrix} \equiv B_1^{(1)}B_2^{(1)} \hat{T}(1, 1) \]

\[ T(2, 2) = B_1^{(2)}B_2^{(2)} \begin{pmatrix} -\lambda_\infty^2, \frac{\lambda_\infty}{\lambda_v}, -\lambda_\infty^2 \end{pmatrix} \equiv B_1^{(2)}B_2^{(2)} \hat{T}(2, 2). \]  

(66)

The inversion of equation (64) provides the values of the fundamental matrix elements \( N_{m}(j, k), m = 2, 3, 4 \). We notice that the procedure can be extended to all values of \( m \) by considering the variation of the equation \( y'' + R_0 y = 0 \) under the transformation

\[ z = \left( v - \frac{a}{\sqrt{m-1}} \right)/(1 - a). \]  

(67)

Due to the structure of the matrices \( A \), such a procedure is purely iterative, i.e. known \( N_{m}(j, k) \) for \( m = 2, 3, \ldots, n \), the computation of \( T^n(j, k) \) directly provides \( N_{n+1}(j, k) \).

In the current case, we have

\[ Q_2(j, k) = q_2 \cdot A^{-1} \cdot T(j, k). \]  

(68)

We come now to the second iteration of \( Q_1 \). Given the Green function

\[ G(z, z') = \frac{1}{w_{12}} \left( y_1(z)y_2(z') - y_2(z)y_1(z') \right) \Theta(z, z') \]  

(69)

the expression we are confronted with, for the second order change of \( y_k(z) \) is

\[ y_k(z) = \frac{1}{w_{12}} \int_1^z Q_1(z')y_2(z')\delta^{(1)}y_k(z')dz' - y_2(z) \frac{1}{w_{12}} \int_1^z Q_1(z')y_1(z')\delta^{(1)}y_1(z')dz' \]  

(70)

with \( \delta^{(1)}y_k \) given by equation (17). The indefinite integrals appearing in (70) can be obtained from the equation

\[ \hat{y}''(z, a) + \hat{Q}(z, a)\hat{y}(z, a) = 0, \]  

(71)

with

\[ \hat{Q}(z, a) = \frac{\delta_v + sa}{(z - ca)^2} + \frac{\delta_1}{(z - 1)^2} + \frac{\delta_\infty - \delta_1 - \delta_v - sa}{(z - ca)(z - 1)}, \]  

(72)
where we shall impose

\[
\frac{\partial \tilde{Q}(z, a)}{\partial a} \bigg|_{a=0} = \frac{2\delta - C'(0)}{z^2(z - 1)} - \frac{2\delta + C(0)}{z^3(z - 1)} = Q_1(z)
\]

(73)

with \(C(0) = \delta_0 - \delta_0 - \delta\) and \(C'(0)\) given by equation (52). In order to fit the coefficients of \(N_2, N_3\) in \(Q_1\), we have to allow in principle for two parameters, \(c\) and \(s\). We find

\[
c = \frac{\delta + \delta_0 - \delta_0}{2\delta}, \quad s = 0.
\]

(74)

Notice that the transformation leading from \(Q_0(z)\) to \(\tilde{Q}(z, a)\) is of the same type as the one appearing in (26) and (27) of which we know the solutions. It follows that after replacing \(C(0)\) and \(C'(0)\) in (73) with their values, the simplest method to compute the matrix elements \(Q_{jk}(1, k)\) is to use the transformation (28) with \(\alpha\) replaced by \(ca\). One can easily prove that

\[
y_j(z) + x \frac{\partial y_j(z, a)}{\partial a} \bigg|_{a=0} = y_j(z) = x\tilde{y}_j(z)
\]

(75)

has the correct boundary condition \((1 - z)^{(1-k)/2}\) with coefficient 1 at \(z = 1\), as imposed by the solution of the Volterra equation, and thus, \(x\tilde{y}_j(z)\) equals \(\delta^{(1)}y_j\) i.e. it is the first-order correction of \(y_j\). It is expressed in terms of derivatives of the hypergeometric function. The same holds for \(y_2\). Then, the integrals

\[
\int_{1}^{\infty} y_k(z')Q_1(z')\delta^{(1)}y_j(z')dz'
\]

appearing in (70) can be computed as follows. Using

\[
\tilde{Q}(z, a) = Q_0(z) + aQ_1(z) + a^2\tilde{Q}_2(z) + \ldots
\]

(77)

and

\[
y_k^{(0)} + 2\tilde{Q}_2(z)y_k(z) + 2Q_1(z)y_k^{(0)}(z) + Q_0(z)y_k^{(0)}(z) = 0
\]

(78)

we have

\[
\int_{1}^{\infty} y_k(z')Q_1(z')\tilde{y}_j(z')dz' = -\frac{1}{2} \int_{1}^{\infty} y_k(z')\left(y_k^{(0)}(z') + 2\tilde{Q}_2(z')y_j(z') + Q_0(z')\tilde{y}_j(z')\right)dz'
\]

\[
= -\frac{1}{2} \left(y_k(z)\tilde{y}_j(z) - y_k^{(0)}(z)\tilde{y}_j(z)\right) - \int_{1}^{\infty} y_k(z')\tilde{Q}_2(z')y_j(z')dz'.
\]

(79)

Notice that \(\tilde{Q}_2\) is not equal to the \(Q_2\) of equation (5), but it will be expedient for computing the l.h.s. of (79), \(\tilde{Q}_2\), in the base \(N_2, N_3, N_4\), is represented by the vector

\[
\tilde{q}_2 = c^2 \left(0, \delta_0 - \delta_1 + 2\delta_0, -3\delta_0\right).
\]

(80)

We know \(\tilde{y}_k(z)\) and, \(\tilde{y}_j(z)\) and, as a result, we know the l.h.s. of equation (79), thus providing the second iteration of the Volterra equation in terms of hypergeometric functions and derivatives thereof. Explicitly, we find in equation (54)

\[
\text{tr} \left(S_2^+ - S_2^\mp M^0 = 4 \frac{\sin \pi \lambda_4}{\lambda_4} \sin \pi \lambda_2 R_1^{(1)} B_1^{(1)} B_2^{(2)} \right)
\]

\[
\times \left\{ \left(q_2 - \tilde{q}_2\right) \cdot A^{-1} \cdot \left(\tilde{F}(1, 1) - \tilde{F}(2, 2)\right) - \lambda^2 \lambda_4 c^2 \right\}.
\]

(81)
The computation of the term $-\text{tr} \left( S_1^\ell - S_1^\rho \right) M^\rho S_1^\rho$ in equation (54) requires simply the knowledge of $S_1^\ell$, which we have already computed, and it cancels the term $-\lambda \lambda c^2$ in the curly brackets in the above equation. To summarize, the equation for $\ddot{C}(0)$ is given by

$$0 = \left( g_2 - \tilde{q}_2 \right) \cdot A^{-1} \cdot \left( \tilde{T}(1, 1) - \tilde{T}(2, 2) \right). \quad (82)$$

Due to the structure of $\tilde{T}(j, k)$—see equation (66)—the vector $\tilde{T}(1, 1) - \tilde{T}(2, 2)$ has a single entry different from zero. We have for $C''(0) \equiv \left[ x C_2(x) \right]''_{x=0} = 2C'(0) - 2C(0)$

$$C''(0) = \frac{C(0) - 3 \delta + c^2 (2 \delta + \delta_\infty - \delta_1)}{\delta_1} \frac{C(0) + 3 \delta - 3 \delta_\infty}{\delta_1} \left[ 3 \delta_1^2 + 3 \delta_\infty^2 + 3 \delta_{\infty} (1 + \delta_\infty) + \delta_1 (3 + 2 \delta_{\infty}) - 3 \delta_{\infty} (1 + 2 \delta_1 + 2 \delta_{\infty}) \right] \delta_1 (3 + 4 \delta_1)$$

where $C(0)$, $C'(0)$ and $c$ are given, respectively, by equations (3), (52), (74). The value of $C''(0)$ agrees with the one obtained in [6] and [1] by taking the $b \to 0$ limit of the conformal blocks, thus providing strong support to the exponentiation hypothesis.

The procedure can also be pushed to higher orders even if we need a systematic organization of the mixed contributions.

Integrals similar to those discussed here appear in the accessory parameter problem for the torus, which is dealt with in [14]. There, the logarithmic part was derived to all orders and successfully compared to the saddle-point prediction on the quantum theory, while the presence of integrals of the above-mentioned type hampered the analytical evaluation of the $q$ term of the expansion, $q$ being the nome of the torus. Now we have the possibility of computing analytically not only the $q$ term, but also the $q^2$ term.

5. Conclusions

In this paper, we developed, for the monodromy problem considered in papers [1, 2] an analytical technique to compute the expansion of the accessory parameter in term of the invariant cross ratio directly, without taking the limit of the quantum conformal blocks for infinite central charge. In the first-order computation, we have shown how the integrals containing products of two hypergeometric functions, which appear in the first iteration of the Volterra equation, can be computed analytically in terms of $\Gamma$-functions and derivatives thereof, i.e. $\psi$-functions. This method consists in exploiting the transformation properties of the kernel under $\text{SL}(2C)$ and the $-1/2$-form nature of the solutions.

The computation to second order involves double integrals of the products of four hypergeometric functions, and we show how these can also be computed analytically. In such second-order computations, we need a transformation that at the infinitesimal level is akin to the $L_{-2}$ generator of Virasoro algebra. The transformation is not one-to-one on the complex plane, but is well defined in the region needed for our computations.

Both our first- and second-order results agree with the ones obtained from the classical limit of the quantum conformal blocks under the exponentiation hypothesis [1–6], and thus, they lend strong support to such an exponentiation hypothesis in the classical $b \to 0$ limit.

Regarding the extension to higher orders, we have shown how the fundamental matrix elements $N_{\mu}(j, k)$ can all be computed by a simple iterative procedure. As it happens already to the second order in the $n$-th order computation one has both direct contribution, from the $n$-order term in the expansion of the kernel and mixed contributions due to lower-order
expansion. One would need a systematic organization of such mixed contribution to go to arbitrary order.

In the full treatment of the torus with one source [14], integrals of the product of hypergeometric functions appeared, which are similar to those discussed above. The possibility of computing them analytically will extend the results reported in [14].

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