CONTROLLABILITY OF FRACTIONAL DYNAMICAL SYSTEMS: A FUNCTIONAL ANALYTIC APPROACH

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Abstract. In this paper, we investigate controllability of fractional dynamical systems involving monotone nonlinearities of both Lipschitzian and non-Lipschitzian types. We invoke tools of nonlinear analysis like fixed point theorem and monotone operator theory to obtain controllability results for the nonlinear system. Examples are provided to illustrate the results. Controllability results of fractional dynamical systems with monotone nonlinearity is new.

1. Introduction. Fractional differential and integral equations have gained considerable popularity and importance during the past three decades. Many scientific areas are currently paying attention to the fractional calculus concepts and we can refer its adoption in science and engineering, see, for example [2, 24]. The main advantage of the fractional calculus is that the fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Many real world systems are better characterized by using a non-integer order dynamic model based on fractional calculus, for example the Basset problem, the Baglay-Torvik equation and many more models in fluid dynamics. Recently the fractional dynamical systems have attracted lot of attention among control system society even though fractional-order control problems have been investigated as early as 1960s. The generalization to non-integer orders of traditional controllers or control schemes translates into more tuning parameters and more adjustable time of the control system. Thus the fractional-order control system has explicitly concentrated on the practical consequences and inducing an impact on all areas of the control discipline. However, because of the absence of appropriate mathematical methods, fractional-order dynamical systems have been studied only marginally in the theory and practice of control systems. Some successful attempts were undertaken by several researchers (see, for instance [1, 3, 26, 29]) and recently Kaczorek [20] focused on selected problems of fractional control and SISO, MIMO systems. These studies have shown that a fractional-order controller can provide better performance than an integer order one [23]. Also research on fractional order systems continues to be active and extensive throughout the world.

Control system is an interconnection of components forming a system configuration that will provide a desired system response. Controllability is one of the

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structural properties of dynamical systems and it means that it is possible to steer a dynamical system from an arbitrary initial state to arbitrary final state using the set of admissible controls. Study on controllability of fractional dynamical systems provide important issues for many applied problems because the use of fractional order derivatives and integrals in control theory leads to better results than those of integer order ones. To the best of our knowledge, in recent years, a few contributions have been devoted to the controllability results for fractional dynamical systems. Matignon and D’Andréa-Novel [25] discussed the controllability of linear fractional dynamical systems. Vinagre et al. [32] discussed the fundamental ideas of controllability for fractional order systems. Battayab and Djennoune [11] studied the controllability of fractional dynamical systems using rank condition whereas Chen et al. [12] focused on the robust controllability for uncertain fractional-order linear time-invariant systems in state-space form. Guermah et al. [14] presented the results on both controllability and observability of discrete-time fractional order systems. Mozyrska and Torres [27, 28] obtained controllability results and modified energy control for fractional linear control systems in the sense of Riemann-Liouville and Caputo fractional derivatives. More recently, Balachandran et al. [5, 6, 7, 8] investigated the controllability of linear and nonlinear fractional dynamical systems and established sufficient conditions for the controllability of nonlinear fractional dynamical systems of order $0 < \alpha < 1$ and $1 < \alpha < 2$ using fixed point theorems. Based on the above literature, the controllability of nonlinear fractional dynamical systems are studied only by using the technique of fixed point theorems, in particular the nonlinear function satisfies the property of Lipschitz continuity. If in the case of nonlinear function does not satisfies the property of Lipschitz continuity, there is no controllability results for such systems.

Motivated by the above fact, in this paper, we discuss controllability of a nonlinear fractional dynamical system represented by the fractional differential equation in the sense of Caputo fractional derivative of order $\alpha \in (0,1]$ of the form

$$
\frac{C D^\alpha_{t_0+} x(t)}{t} = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in [t_0, t_1],
$$

where, $\frac{C D^\alpha_{t_0+} x(t)}{t} = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} x'(s)ds$. For brevity, the Caputo fractional derivative $\frac{C D^\alpha_{t_0+} x(t)}{t}$ is taken as $\frac{C D^\alpha x(t)}{t}$, the state vector $x(t) \in \mathbb{R}^n$, the control vector $u(t) \in \mathbb{R}^m$, the matrices $A$ and $B$ are real constant matrices of dimensions $n \times n, n \times m$ respectively and $f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function satisfying Caratheodory conditions by employing tools of monotone operator theory and contraction mapping principle.

It should be mentioned that, for $\alpha = 1$, the nonlinear fractional dynamical system (1) reduces to integer-order nonlinear dynamical system

$$
\frac{dx(t)}{dt} = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in [t_0, t_1],
$$

(2)

The controllability results of (2) have been established by using fixed-point theorems by many researchers (see for example [4, 22] and the references therein) and in particular Joshi and George [19] studied the controllability results of (2) by using monotone operator theory and contraction mapping principle.
The controllability results of linear fractional dynamical systems (the absence of nonlinear function $f(t, x(t))$ in (1)) are established by Matignon and D’Andréa-Novel in [25]. However, the motivation of defining controllability Grammian is not mentioned for fractional-order systems is still unsolved among all fractional-order control researchers. Motivated by this fact, we obtain the results by using the concept of linear bounded operators. In this paper, we obtain controllability results of linear and nonlinear fractional dynamical systems by using linear bounded operators, fixed point theory, and monotone operator theory. We consider two types of non-linearities, Lipschitzian and non-Lipschitzian. For Lipschitzian nonlinearities are similar to those of [9] and [10]. Whereas the results concerning monotone nonlinearities are new in the theory of fractional order control systems. There are situations where, in addition to the Lipschitz continuity, the nonlinear function also satisfies monotonicity conditions. In such cases, our result improves the results of the aforementioned authors. The results concerning non-Lipschitzian monotone nonlinearities are new in fractional-order control system. Moreover, together with the controllability results, we also get a stability result for the system with respect to the initial and final states. Also, we give an iterative scheme for the computation of control and state of the fractional dynamical system. The non-Lipschitzian nonlinearities are discussed in a separate section, and we feel that our results in this direction are new for fractional-order control systems.

The rest of the paper is organized as follows: In Section 2, the controllability condition is established for the linear fractional dynamical system with control by using the linear bounded operators and we defined some well-known Definitions and Theorems from the theory of nonlinear functional analysis. In Section 3, we prove that the controllability of the nonlinear system is equivalent to the solvability of a feedback system. This in turn is used in Sections 4 and 5 to obtain the main results regarding controllability of (1) with Lipschitzian and non-Lipschitzian monotone nonlinearities. The results are obtained by using the theory of nonlinear functional analysis and the fractional calculus. Finally, in Section 6, examples are provided to illustrate the theory.

2. Controllability of linear systems and preliminaries. In our investigation, controllability properties of the nonlinear system (1) depend mainly on the properties of its linear part. So we consider the linear fractional dynamical system represented by the fractional differential equation of the form

$$^{C}D^{\alpha}x(t) = Ax(t) + Bu(t), \quad t \in [t_0, t_1],$$

(3)

where the state vector $x$, the control vector $u$ and the matrices $A$ and $B$ are defined as above. Using the method of successive approximation, the solution of (3) can be written as [21, 30]

$$x(t) = E_{\alpha}(A(t - t_0)^{\alpha})x_0 + \int_{t_0}^{t}(t - s)^{\alpha - 1}E_{\alpha,\alpha}(A(t - s)^{\alpha})Bu(s)ds,$$

(4)

where $E_{\alpha,\alpha}(At^{\alpha}) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(ak + \alpha)}$ is the Mittag-Leffler matrix function for a square matrix $A \in \mathbb{R}^{n \times n}$ with $E_{\alpha,1}(At^{\alpha}) = E_\alpha(At^{\alpha})$. The function $E_{\alpha,\alpha}(At^{\alpha})$ is continuous and it satisfies $\|E_{\alpha,\alpha}(At^{\alpha})\| \leq M$ for all $t \in [t_0, t_1]$. 


Definition 2.1. The system (3) is said to be controllable over \([t_0, t_1]\) if for each pair of vectors \(x_0, x_1 \in \mathbb{R}^n\), there exists a control \(u \in L^2([t_0, t_1]; \mathbb{R}^m)\) such that the solution of (3) with \(x(t_0) = x_0\) also satisfies \(x(t_1) = x_1\).

The control \(u(t)\) is said to steer the trajectory from the initial state \(x_0\) to the final state \(x_1\). So the controllability of (3) is equivalent to finding the function \(u(t)\) such that

\[
x_1 = x(t_1) = E_\alpha(A(t_1 - t_0)^\alpha)x_0 + \int_{t_0}^{t_1} (t_1 - s)^{\alpha - 1}E_{\alpha,\alpha}(A(t_1 - s)^\alpha)Bu(s)ds.
\]

Equivalently, the system (3) is controllable if and only if there exists a control \(u\) such that

\[
x_1 - E_\alpha(A(t_1 - t_0)^\alpha)x_0 = \int_{t_0}^{t_1} (t_1 - s)^{\alpha - 1}E_{\alpha,\alpha}(A(t_1 - s)^\alpha)Bu(s)ds. \tag{5}
\]

The above discussion leads to the following controllability condition:

**Theorem 2.2 ([25]).** The system (3) is controllable if and only if the controllability Grammian

\[
W(t_0, t_1) = \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1 - s)^\alpha)ds
\]

is non-singular. A control \(u(t)\) steering the system from the initial state \(x_0\) to the final state \(x_1\) is given by

\[
u(t) = (t_1 - t)^{1-\alpha}B^*E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)W^{-1}(t_0, t_1) [x_1 - E_\alpha(A(t_1 - t_0)^\alpha)x_0].
\]

Without giving any information about how to obtain the controllability Grammian and control function for linear fractional-order control systems used by Matignon and D’Andréa-Novel in [25] is still unsolved among all fractional-order control researchers. Motivated by this fact, using the concept of linear bounded operators, we obtain the results of controllability Grammian and control function for linear fractional-order control systems exactly fulfilled by the above problem.

We have the following characterization regarding controllability of (3) in terms of linear bounded operators.

Let \(L^2([t_0, t_1]; \mathbb{R}^m) = \left\{ (t_1 - t)^{\alpha - 1}u(t) : u(t) \in L^2([t_0, t_1]; \mathbb{R}^m) \text{ and } \int_{t_0}^{t_1} (t_1 - t)^{\alpha - 1}u(t)dt < \infty \right\}\) such that \(\int_{t_0}^{t_1} (t_1 - t)^{2(\alpha - 1)}|u(t)|^2dt < \infty\). This is a Hilbert space with respect to the inner product \(\langle f_\alpha, g_\alpha \rangle = \int_{t_0}^{t_1} f_\alpha(t)g_\alpha(t)dt\). Let us define a linear operator \(C_\alpha : L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \rightarrow \mathbb{R}^n\) by

\[
C_\alpha v_\alpha = \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)Bv_\alpha(s)ds. \tag{6}
\]

Obviously \(C_\alpha\) is a bounded linear operator, since \(E_{\alpha,\alpha}(A(t_1 - t)^\alpha)\) is bounded for every \(t \in [t_0, t_1]\) and range space of \(C_\alpha\) is a subspace of \(\mathbb{R}^n\). Using the above definition of \(C_\alpha\), equation (5) can be written as \(C_\alpha v_\alpha = w_\alpha\), where \(w_\alpha = x_1 - E_{\alpha,\alpha}(A(t_1 - t_0)^\alpha)x_0\). Thus the controllability problem reduces to the problem of proving that the operator \(C_\alpha\) is surjective.

**Theorem 2.3.** The following statements are equivalent:

(i) The system (3) is controllable.
(ii) The operator \( C_\alpha : L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \rightarrow \mathbb{R}^n \) is onto.

(iii) The operator \( C^*_\alpha : \mathbb{R}^n \rightarrow L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \) is one-to-one.

(iv) The operator \( C_\alpha C^*_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is onto. (Observe that, \( C_\alpha C^*_\alpha \) is a bounded linear operator and hence can be realized by an \( n \times n \) matrix.)

**Proof.** Let \((x_0, x_1)\) be any pair of vectors in \( \mathbb{R}^n \times \mathbb{R}^n \). The controllability problem of (3) reduces to the determination of \( v_\alpha \in L^2_\alpha([t_0, t_1], \mathbb{R}^m) \) such that

\[
\int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)Bv_\alpha(s)ds = w_\alpha = x_1 - E_\alpha(A(t_1 - t_0)^\alpha)x_0.
\]

This is equivalent to solving \( C_\alpha v_\alpha = w_\alpha \) for \( v_\alpha \) in the space \( L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \) for a given \( w_\alpha \), where \( C_\alpha \) is given by (6).

We first note that \( C_\alpha \) is a bounded linear operator. Let \( C^*_\alpha : \mathbb{R}^n \rightarrow L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \) be its adjoint. This is determined as follows: For \( v_\alpha \in L^2_\alpha([t_0, t_1]; \mathbb{R}^m) \) and \( y \in \mathbb{R}^n \), we have

\[
(C_\alpha v_\alpha, y)_{\mathbb{R}^n} = \langle \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)Bv_\alpha(s)ds, y \rangle = \int_{t_0}^{t_1} \langle v_\alpha(s), B^*E_{\alpha,\alpha}(A^*(t_1 - s)^\alpha)y \rangle ds = \langle v_\alpha, C^*_\alpha y \rangle_{L^2_\alpha}.
\]

This gives

\[
(C^*_\alpha y)(t) = B^*E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)y.
\]

To claim that \( C_\alpha \) is onto, it is sufficient to prove that \( C_\alpha C^*_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is onto (equivalently invertibility). That is, given any \( w_\alpha \in \mathbb{R}^n \), find \( y \in \mathbb{R}^n \) such that

\[
C_\alpha C^*_\alpha y = w_\alpha.
\]

If (7) is solvable, then a control \( u(t) \) which satisfies \( C_\alpha v_\alpha = w_\alpha \) is given by \( v_\alpha = C^*_\alpha y = (t_1 - t)^{\alpha-1}u(t) \). This implies that \( u(t) = (t_1 - t)^{1-\alpha}C^*_\alpha y \). Computing \( C_\alpha C^*_\alpha \), we get

\[
C_\alpha C^*_\alpha y = C_\alpha(B^*E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)y) = \int_{t_0}^{t_1} [E_{\alpha,\alpha}(A(t_1 - s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1 - s)^\alpha)ds] y.
\]

The operator \( C_\alpha C^*_\alpha \) is known as controllability Grammian and is denoted by \( W(t_0, t_1) \). Thus we have

\[
W(t_0, t_1) = \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1 - s)^\alpha)ds.
\]

If the controllability Grammian \( W(t_0, t_1) \) is non-singular, we have a control \( u(t) \) given by

\[
u(t) = (t_1 - t)^{1-\alpha}C^*_\alpha(C_\alpha C^*_\alpha)^{-1}w_\alpha = (t_1 - t)^{1-\alpha}B^*E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)W^{-1}(t_0, t_1) [x_1 - E_\alpha(A(t_1 - t_0)^\alpha)x_0]\]

which will steer the state from the initial state \( x_0 \) to the desired final state

\[
x(t_1) = E_\alpha(A(t_1 - t_0)^\alpha)x_0 + \int_{t_0}^{t_1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)BB^*E_{\alpha,\alpha}(A^*(t_1 - s)^\alpha)ds.
\]
Conversely, let the system (3) be controllable. That is, $C_\alpha$ is onto. Equivalently $\mathcal{R}(C_\alpha) = \mathbb{R}^n$. Then, by Theorem 11.2 in [31], $\mathcal{N}(C_\alpha^*) = \mathcal{R}(C_\alpha)^\perp = [\mathbb{R}^n]^\perp = \{0\}$. Hence $C_\alpha^*$ is also one-to-one. This implies $C_\alpha C_\alpha^*$ is also one-to-one, which is proved as under. Assume that $C_\alpha C_\alpha^* y = 0$. This gives

$$0 = \langle C_\alpha C_\alpha^* y, y \rangle = \langle C_\alpha^* y, C_\alpha^* y \rangle = \|C_\alpha^* y\|^2$$

and hence $C_\alpha^* y = 0$, $C_\alpha^*$ is one-to-one implies that $y = 0$. Thus $\mathcal{N}(C_\alpha C_\alpha^*) = \{0\}$. Thus $W(t_0, t_1) = C_\alpha C_\alpha^*: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is one-to-one and onto implying that $C_\alpha C_\alpha^*$ is non-singular. This completes the proof.

**Definition 2.4.** A bounded linear operator $P: \mathbb{R}^n \rightarrow L^2([t_0, t_1]; \mathbb{R}^m)$ is called a steering operator for (3), if for any $y \in \mathbb{R}^n$, $u(t) = (t_1 - t)^{1-\alpha}(Py)(t)$ steers 0 to $y$.

An $m \times n$ matrix function $P(t)$ is called a steering function, if the operator $P$ defined by $(Py)(t) = P(t)y$ is a steering operator. We observe that

1. A bounded linear operator $P: \mathbb{R}^n \rightarrow L^2([t_0, t_1]; \mathbb{R}^m)$ is a steering operator if and only if $C_\alpha P = I$.
2. An $m \times n$ matrix function $P(t)$ is a steering function if and only if

$$\int_{t_0}^{t_1} E_{t_1, t} A(t_1 - t)^\alpha B P(s) ds = I. \quad (8)$$

We have the following characterization regarding controllability of (3) in terms of steering operator and steering function.

**Theorem 2.5.** The following are equivalent

(i) The system (3) is controllable.
(ii) There exists a steering function $P(t)$ for (3).
(iii) There exists a steering operator $P$ for (3).

**Remark 2.1.** If $C_\alpha C_\alpha^*$ is invertible (that is, if the controllability Grammian matrix is non-singular) then

$$(Py)(t) = C_\alpha^* (C_\alpha C_\alpha^*)^{-1} y, y \in \mathbb{R}^n$$

is a steering operator.

**Remark 2.2.** If there exists a steering operator, then there exists a steering function.

**Proof.** Assume that, the steering operator $(Py)(t) = C_\alpha^* (C_\alpha C_\alpha^*)^{-1} y$ exists. Then by the Definition 2.4, for any $y \in \mathbb{R}^n$, the control function $u(t) = (t_1 - t)^{1-\alpha}(Py)(t)$ steers the system from 0 to $y$ over the time interval $[t_0, t_1]$. This implies that, the control function $u(t)$ satisfies

$$y = \int_{t_0}^{t_1} E_{t_1, t} A(t_1 - t)^\alpha B (Py)(t) dt.$$

By the definition of $C_\alpha, C_\alpha^*$ and $C_\alpha C_\alpha^*$, we have

$$y = \int_{t_0}^{t_1} E_{t_1, t} A(t_1 - t)^\alpha B B^* E_{t_1, t} A^*(t_1 - t)^\alpha W^{-1}(t_0, t_1) y dt.$$
From (8), we get
\[ P_0(t) = B^*E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)W^{-1}(t_0, t_1) \]
is a steering function. \(\square\)

Now we introduce the following necessary concepts from nonlinear functional analysis.

**Definition 2.6.** Let \( X \) be a Banach Space. Let \( \text{Lip} \) be the set of all operators \( F : X \to X \) which satisfy Lipschitz condition, that is, there exists \( \lambda > 0 \) such that
\[ \|Fx_1 - Fx_2\| \leq \lambda\|x_1 - x_2\|, \text{ for all } x_1, x_2 \in X. \]
For \( F \in \text{Lip} \), define
\[ \|F\|_* = \sup_{x_1, x_2 \in X, x_1 \neq x_2} \frac{\|Fx_1 - Fx_2\|}{\|x_1 - x_2\|}. \]

**Definition 2.7.** Let \( H \) be a real Hilbert space. Let \( \mathcal{M} \) be the set of all operators \( G : H \to H \) such that, if \( G \in \mathcal{M} \),
\[ \langle Gx_1 - Gx_2, x_1 - x_2 \rangle \geq \beta\|x_1 - x_2\|^2, \text{ for all } x_1, x_2 \in H \text{ and for some constant } \beta. \]
For \( G \in \mathcal{M} \), define
\[ \mu(G) = \inf_{x_1, x_2 \in H, x_1 \neq x_2} \frac{\langle Gx_1 - Gx_2, x_1 - x_2 \rangle}{\|x_1 - x_2\|^2}. \]
The operator \( G \) is called monotone (strongly monotone) if \( \mu(G) \geq 0 \) \( (\mu(G) > 0) \).

We note that
(i) \( \text{Lip} \subset \mathcal{M} \)
(ii) \( F \in \text{Lip} \) implies \( -\|F\|_* \leq \mu(F) \leq \|F\|_* \)
(iii) \( F, G \in \text{Lip} \) implies \( \|FG\|_* \leq \|F\|_*\|G\|_* \).

**Definition 2.8.** Let \( X \) be a Banach space and \( X^* \) be its dual. Then the operator \( F : X \to X^* \) is called coercive if
\[ \lim_{\|x\| \to \infty} \frac{\langle Fx, x \rangle}{\|x\|} = \infty. \]
Here \( \langle y, x \rangle, y \in X^*, x \in X \), represents the evaluation of \( y \) at \( x \).

**Definition 2.9.** Let \( X \) be a Banach space and \( X^* \) be its dual. Then the operator \( F : X \to X^* \) is said to be of type \( (M) \) if the following conditions hold:
(a) If a sequence \( \{x_n\} \) in \( X \) converges weakly to \( x \) in \( X \) and \( \{Fx_n\} \) converges weakly to \( y \) in \( X^* \) and \( \limsup_{n} \langle Fx_n, x_n \rangle \leq \langle y, x \rangle \), then \( Fx = y \).
(b) \( F \) is continuous from finite dimensional subspace of \( X \) into \( X^* \) endowed with weak* topology.

**Theorem 2.10.** [13] Let \( K \in \mathcal{M} \) be continuous and \( N \in \text{Lip} \), \( \mu(N) > 0 \). If
\[ (\mu(K) + \mu(N))\|N\|^2 > 0, \]
then \( I + KN \) is invertible with \( [I + KN]^{-1} \in \text{Lip} \) and
\[ \|\|I + KN\|^{-1}\|_* \leq \frac{1}{\mu(N)(\mu(K) + \mu(N))\|N\|_*^2}. \]
Theorem 2.11. [16] Let $X$ be a Banach space and let $G : X \to X^*$ be Lipschitz on $X$ with $\|G\| < 1$. Then the operator $N = I + G$ is invertible, $N^{-1}$ is Lipschitz on $X$ and $\|N^{-1}\* \leq \frac{1}{1 - \|G\*}$. 

Theorem 2.12. [17] Let $X$ be a Banach space and $F : X \to X^*$, a mapping of type $(M)$. If $F$ is coercive, then the range of $F$ is all of $X^*$.

Theorem 2.13. [17] Let $T$ be a continuous mapping of a Banach space $X$ into itself such that there exists a positive integer $n \geq 1$ such that $\|T^n x - T^n y\| \leq k\|x - y\|$ for every $x, y \in X$ and for some positive constant $k < 1$. Then $T$ has a unique fixed point.

When $n = 1$, the above Theorem is known as Banach contraction principle.

3. Controllability and feedback formulation. In this section, we discuss the controllability of (1) in terms of the solvability of an equivalent feedback system of the form

$$e_1 = u_1 - G_2 e_2,$$

$$e_2 = u_2 + G_1 e_1,$$

for some appropriate operators $G_1$ and $G_2$. The solvability results for (9) and its equivalent control systems are given in Sections 4 and 5.

A solution of (1) is an absolutely continuous function in $L^2([t_0, t_1]; \mathbb{R}^n)$ which satisfies (1) almost everywhere. A solution $x(t)$ exists for (1) iff $x(t)$ satisfies the integral equation

$$x(t) = E_\alpha(A(t-t_0)\alpha)x_0 + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)ds$$

$$+ \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,x(s))ds.$$

Suppose that the linear system (3) is controllable on $[t_0, t_1]$. There exists a control $u(t)$ which steers the initial state $x_0$ at time $t = t_0$ to the given final state $x_1$ at time $t = t_1$ iff there exists a solution $x(t)$ of (1) satisfying

$$x_1 = x(t_1) = E_\alpha(A(t_1-t_0)^\alpha)x_0 + \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)Bu(s)ds$$

$$+ \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)f(s,x(s))ds.$$

That is

$$x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0 - \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)f(s,x(s))ds$$

$$= \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^\alpha)Bu(s)ds. \quad (10)$$

Since the linear system (3) is controllable, then, by Theorem 2.5, there exists a steering function $P(t)$ for the linear system (3). If there exists $x(t)$ satisfying (10), then the steering control for (1) is given by

$$u(t) = (t_1 - t)^{1-\alpha} P(t) \left[ x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0 ight]$$
ability of the coupled equations

Then the state of the system (1) is given by

\[
x(t) = E_\alpha(A(t-t_0)^\alpha)x_0 + \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, x(s))ds
\]

\[+ \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)ds,
\]

\[
+ \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, x(s))ds,
\]

\[
u(t) = (t_1 - t)^{1-\alpha}P(t)\left[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0
\]

\[+ \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, x(s))ds \right].
\]

Suppose that (12) is solvable, then \(x(t_0) = x_0\) and \(x(t_1) = x_1\). This implies that the system (1) is controllable with the control defined by (11).

Hence the controllability of the nonlinear system (1) is equivalent to the solvability of the coupled equations

\[
x(t) = E_\alpha(A(t-t_0)^\alpha)x_0 + \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)ds
\]

\[+ \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, x(s))ds,
\]

\[
u(t) = (t_1 - t)^{1-\alpha}P(t)\left[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0
\]

\[+ \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s, x(s))ds \right].
\]

Let \(X_1 = L^2([t_0,t_1];\mathbb{R}^m), X_2 = L^2([t_0,t_1];\mathbb{R}^n).\) Define operators \(K, N : X_2 \to X_2, H : X_1 \to X_2\) and \(L : X_2 \to X_1\) as follows

\[
(Kx)(t) = \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)x(s)ds;
\]

\[
(Hu)(t) = \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)Bu(s)ds;
\]

\[
(Lx)(t) = (t_1 - t)^{1-\alpha}P(t)\int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)x(s)ds;
\]

\[
(Nx)(t) = f(t, x(t)).
\]

Clearly \(K, H\) and \(L\) are continuous linear operators and \(N\) is a nonlinear operator, called Nemytskii operator. With these notations, (13) and (14) can be written as a pair of operator equations

\[
x = u_0 + KNx + Hu
\]

\[
u = u_1 - LNx,
\]

where \(u_0(t) = E_\alpha(A(t-t_0)^\alpha)x_0\) and \(u_1(t) = (t_1 - t)^{1-\alpha}P(t)[x_1 - E_\alpha(A(t_1-t_0)^\alpha)x_0].\) Without loss of generality, \(x_0\) can be taken as 0 and hence the nonlinear system (1) is controllable iff the following pair of operator equations

\[
x = KNx + Hu
\]

\[
u = u_1 - LNx
\]
is solvable, where \( u_1(t) = (t_1 - t)^{1-\alpha} P(t)x_1 \). The solvability analysis of the above operator equations depends mainly on the continuous invertibility of the operator \([I-KN]\). Now we make the following assumptions.

**Assumptions \( \mathcal{A} \)**

**Ax1** The Mittag-Leffler matrix function \( E_{\alpha,\alpha}(A(t-s)\alpha) \) generated by \( A \) is such that \( \|E_{\alpha,\alpha}(A(t-s)\alpha)\| \leq h(t,s) \), where \( h(\cdot,\cdot) : [t_0,t_1] \times [t_0,t_1] \rightarrow \mathbb{R}^+ \) is a function satisfying

\[
\left( \int_{t_0}^{t_1} \left( \int_{t_0}^{t_1} (t-s)^{2(\alpha-1)} h^2(t,s) \, ds \, dt \right)^{\frac{1}{2}} \right) \equiv k < \infty.
\]

**Ax2** The function \( f(t,x) : [t_0,t_1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) satisfies Caratheodory conditions. That is, \( f \) is measurable with respect to \( t \) for all \( x \in \mathbb{R}^n \) and continuous with respect to \( x \) for almost all \( t \in [t_0,t_1] \). Further \( f \) satisfies a growth condition of the form

\[
\|f(t,x)\| \leq d(t) + \eta \|x\|
\]

for every \( (t,x) \in [t_0,t_1] \times \mathbb{R}^n \) and \( d \in L^2([t_0,t_1]), \eta > 0 \).

**Lemma 3.1.** Under the Assumptions \( \mathcal{A} \) the bounds for \( \|K\|, \|H\| \) and \( \|L\| \) are estimated as \( \|K\| \leq k, \|H\| \leq bk \equiv h \) and \( \|L\| \leq kq_k \equiv \gamma \), where \( b = \|B\|, q = \left( \int_{t_0}^{t_1} (t - \tau)^{2(1-\alpha)} \|P(\tau)\|^2 \, d\tau \right)^{\frac{1}{2}} \) and \( k_1 = \left( \int_{t_0}^{t_1} (t - \tau)^{2(\alpha-1)} h^2(t,\tau) \, d\tau \right)^{\frac{1}{2}} \).

In particular, if \( P(t) = P_0(t) \), then \( \|L\| \leq bk_1k_2c, \) where \( c = \|W^{-1}(t_0,t_1)\| \) and \( k_2 = \left( \int_{t_0}^{t_1} (t - \tau)^{2(1-\alpha)} h^2(t,\tau) \, d\tau \right)^{\frac{1}{2}} \). Moreover the nonlinear operator \( N \) is continuous and bounded from \( X_2 \) into itself.

**Proof.** Using Schwartz inequality and the assumption \( \text{Ax1} \), we have

\[
\|Kx\|^2_{X_2} = \int_{t_0}^{t_1} \| (Kx)(t) \|^2_{\mathbb{R}^n} \, dt
\]

\[
= \int_{t_0}^{t_1} \left\| \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)\alpha)x(s) \, ds \right\|^2 \, dt
\]

\[
\leq \left( \int_{t_0}^{t_1} \left( \int_{t_0}^{t} (t-s)^{2(\alpha-1)} \|E_{\alpha,\alpha}(A(t-s)\alpha)\|^2 \, ds \, dt \right) \right) \|x\|^2_{X_2}
\]

\[
\leq \left( \int_{t_0}^{t_1} \left( \int_{t_0}^{t} (t-s)^{2(\alpha-1)} h^2(t,s) \, ds \, dt \right) \right) \|x\|^2_{X_2};
\]

hence we have \( \|K\| \leq k \). Similarly

\[
\|Hu\|^2_{X_1} = \int_{t_0}^{t_1} \| (Hx)(t) \|^2_{\mathbb{R}^n} \, dt
\]

\[
= \int_{t_0}^{t_1} \left\| \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)\alpha)Bu(s) \, ds \right\|^2 \, dt
\]

\[
\leq \|B\|^2 \left( \int_{t_0}^{t_1} \left( \int_{t_0}^{t} (t-s)^{2(\alpha-1)} h^2(t,s) \, ds \, dt \right) \right) \|u\|^2_{X_1}
\]

\[
\leq b^2k^2\|u\|^2_{X_1}.
\]
hence we have \( \|H\| \leq bk \equiv h \). Also

\[
\|Lx\|_{X^2}^2 = \int_{t_0}^{t_1} \|(Lx)(t)\|_{X^2}^2 dt \\
= \int_{t_0}^{t_1} \left\|(t_1 - t)^{1-\alpha}P(t) \int_{t_0}^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1 - s)^\alpha)x(s)ds \right\|^2 dt \\
\leq \left( \int_{t_0}^{t_1} (t_1 - t)^{2(1-\alpha)} \|P(t)\|^2 \int_{t_0}^{t_1} (t_1 - s)^{2(\alpha-1)} \right. \\
\times \left. \|E_{\alpha,\alpha}(A(t_1 - s)^\alpha)\|^2 ds \right) \|x\|_{X_2}^2 \\
\leq k_1^2 \left( \int_{t_0}^{t_1} (t_1 - t)^{2(1-\alpha)} \|P(t)\|^2 dt \right) \|x\|_{X_2}^2 \\
\leq q^2 k_1^2 \|x\|_{X_2}^2;
\]

hence we have \( \|L\| \leq qk_1 \equiv \gamma \). If \( P(t) = B^*E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)W^{-1}(t_0, t_1) \), we get

\[
q = \left( \int_{t_0}^{t_1} \|(t_1 - t)^{1-\alpha}P(t)\|^2 dt \right)^{\frac{1}{2}} \\
\leq \left( \int_{t_0}^{t_1} (t_1 - t)^{2(1-\alpha)} \|B^*\|^2 \|E_{\alpha,\alpha}(A^*(t_1 - t)^\alpha)\|^2 \|W^{-1}(t_0, t_1)\|^2 dt \right)^{\frac{1}{2}} \\
\leq bk_2 c.
\]

Also, in view of Assumption (\( \mathcal{A}2 \)), the Nemytskii operator \( N \) is continuous and bounded and it satisfies a growth condition of the form

\[
\|Nx\|_{X_2} = \|f(t, x(t))\|_{X_2} \leq \|d(t)\|_{X_2} + \eta \|x\|_{X_2} \leq d + \eta \|x\|_{X_2},
\]

where \( d = \|d(t)\|_{L^2((t_0, t_1))} \).

**Remark 3.1.** The Assumption (\( \mathcal{A}1 \)) made on the matrix \( A \) is not very hard to verify as we see in the following special cases.

**Case (i).** If \( A \) has all its eigenvalues negative, then there exist positive constants \( m \) and \( \sigma \) such that

\[
h(t, s) = mE_{\alpha,\alpha}(-\sigma(t - s)^\alpha)
\]
or suppose that there exists a unitary matrix \( U \) such that \( A = UA_1U^* \), where \( A_1 \) is a diagonal matrix with negative entries. Then the Mittag-Leffler matrix function is given by

\[
E_{\alpha,\alpha}(A(t - s)^\alpha) = U E_{\alpha,\alpha}(A_1(t - s)^\alpha) U^*
\]

and hence

\[
\|E_{\alpha,\alpha}(A(t - s)^\alpha)\| \leq mE_{\alpha,\alpha}(-\sigma(t - s)^\alpha),
\]

where \( m = \|U\|^2, -\sigma \) is the largest eigenvalue of \( A_1 \). In this case, it can be shown easily that

\[
k = \left( \int_{t_0}^{t_1} \int_{t_0}^{t} (t - s)^{2(\alpha-1)} h^2(t, s)dsdt \right)^{\frac{1}{2}}
\]
If \( \gamma = \sigma t \), then

\[
\gamma = qk_1 = qm(t_1 - t_0)^{\alpha - \frac{1}{2}} \left( \sum_{k=0}^{\infty} \frac{(-1)^k C_{k+1} \sigma^k (t_1 - t_0)^{\alpha k}}{(ak + 2\alpha - 1)} \right)^\frac{1}{2},
\]

where \( C_{k+1} = \frac{1}{\Gamma((j+1)\alpha)\Gamma((k-j+1)\alpha)}. \)

For \( P(t) = P_0(t) \), we get

\[
k_2 = \left( \int_{t_0}^{t_1} (t_1 - s)^{2(1-\alpha)\gamma^2(t,s)} ds \right)^\frac{1}{2}
\]

\[
\gamma = bcm^2(t_1 - t_0)^{\alpha - \frac{1}{2}} \left( \sum_{k=0}^{\infty} \frac{(-1)^k C_{k+1} \sigma^k (t_1 - t_0)^{\alpha k}}{(ak + 2\alpha - 1)} \right)^\frac{1}{2},
\]

\[
\times \left( \sum_{k=0}^{\infty} \frac{(-1)^k C_{k+1} \sigma^k (t_1 - t_0)^{\alpha k}}{(ak + 2\alpha - 1)} \right)^\frac{1}{2}.
\]

**Case (ii).** If \( ||A|| = a \), then \( h(t,s) \) can be taken as \( h(t,s) = E_{\alpha,\alpha}(a(t-s)^\alpha). \) In this case

\[
k = (t_1 - t_0)^{\alpha} \left( \sum_{k=0}^{\infty} \frac{C_{k+1} a^k (t_1 - t_0)^{\alpha k}}{(ak + 2\alpha - 1)(ak + 2\alpha)} \right)^\frac{1}{2},
\]

\[
k_1 = (t_1 - t_0)^{\alpha - \frac{1}{2}} \left( \sum_{k=0}^{\infty} \frac{C_{k+1} a^k (t_1 - t_0)^{\alpha k}}{(ak + 2\alpha - 1)} \right)^\frac{1}{2},
\]
\[ \gamma = q(t_1 - t_0)^{\alpha - \frac{1}{2}} \left( \sum_{k=0}^{\infty} C_{k+1}a^k(t_1 - t_0)^{\alpha k} \right)^{\frac{1}{2}}. \]

For \( P(t) = P_0(t) \), we get

\[ k_2 = (t_1 - t_0)^{3-2\alpha} \left( \sum_{k=0}^{\infty} C_{k+1}a^k(t_1 - t_0)^{\alpha k} \right)^{\frac{1}{2}}, \]

\[ \gamma = bc(t_1 - t_0)^{\frac{1}{2} - \alpha} \left( \sum_{k=0}^{\infty} C_{k+1}a^k(t_1 - t_0)^{\alpha k} \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} C_{k+1}a^k(t_1 - t_0)^{\alpha k} \right)^{\frac{1}{2}}. \]

**Assumptions \([\mathcal{B}]\)**

\( (\mathcal{B}1) \) There exists a constant \( \lambda > 0 \) satisfying

\[ \|f(t, x) - f(t, y)\|_{\mathbb{R}^n} \leq \lambda \|x - y\|_{\mathbb{R}^n}. \]

\( (\mathcal{B}2) \) There exists a constant \( \beta > 0 \) satisfying

\[ \langle f(t, x) - f(t, y), x - y \rangle_{\mathbb{R}^n} \leq -\beta \|x - y\|_{\mathbb{R}^n}^2 \]

for all \( t \in [t_0, t_1] \) and \( x, y \in \mathbb{R}^n \).

Now provide sufficient conditions for invertibility of the operator \([I - KN]\) and its Lipschitz continuity properties.

**Theorem 3.2.** In each of the following cases, the operator \([I - KN]\) is invertible

(i) Assumptions \([\mathcal{A}]\) and \([\mathcal{B}]\) hold with \(-k + \frac{\beta}{\lambda^2} > 0.\)

(ii) Assumptions \([\mathcal{A}]\) and \((\mathcal{B}1)\) hold with \(k\lambda < 1.\)

(iii) Assumption \([\mathcal{A}]\) (with \(h(t, s) = m\)) and \((\mathcal{B}1)\) hold for all \(t, s \in [t_0, t_1].\)

Further, in each case \([I - KN]^{-1}\) is Lipschitz continuous with constants

\[ \frac{\lambda^2}{\beta(\beta - k^2\lambda^2)}, \]

\[ \frac{1}{(1 - \lambda k)} \text{ and } \exp \left( \frac{M\lambda(t_1 - t_0)^{\alpha}}{\alpha} \right) \text{ respectively.} \]

**Proof.** Case (i). When the assumptions \([\mathcal{A}]\) and \([\mathcal{B}]\) are satisfied, it can be shown that \(-N \in \mathcal{M}\) with \(\mu(-N) \geq \beta > 0.\) Also \(-N \in \text{Lip} \) with \(\| - N \|_{\ast} \leq \lambda.\) Since \(K\) is a bounded linear operator, it follows that \(K \in \mathcal{M}\) and

\[ \|Kx_1 - Kx_2\|_{X_2} \leq \int_{t_0}^{t_1} \int_{t_0}^{t} (t - s)^{2(\alpha - 1)}(E_{\alpha, \alpha}(A(t - s)^\alpha))^2 \|x_1 - x_2\|^2_{\mathbb{R}^n} \, ds \, dt \]

\[ \leq k^2 \|x_1 - x_2\|^2_{X_2} \]

and hence \(\|Kx_1 - Kx_2\|_{X_2} \leq k \|x_1 - x_2\|_{X_2};\) which implies \(\|K\|_{\ast} = k.\) Therefore \(\mu(K) \geq -k.\)

The operators \(K\) and \(-N\) satisfy all the conditions of Theorem 2.10 and hence \(I - KN\) is invertible with

\[ \|[(I - KN)^{-1}]_{\ast} \leq \frac{1}{\mu(N)(\mu(K) + \mu(N))^{\frac{1}{2}}} \leq \frac{1}{\beta(-k + \frac{\beta}{\lambda^2})} \]

\[ \leq \frac{\lambda^2}{\beta(\beta - k^2\lambda^2)}. \]
Case (ii). Suppose that assumptions $[\mathscr{A}]$ and $(\mathscr{B}1)$ are satisfied. Then
\[
\|Kn - Kn y\|_{X_2}^2 \leq \int_{t_0}^{t} \int_{t_0}^{t} (t-s)^{2(\alpha-1)}(E_{\alpha,\alpha}(A(t-s)^\alpha))^2 \times \|f(s,x(s)) - f(s,y(s))\|_{X_2}^2 \, ds \, dt \\
\leq k^2 \lambda^2 \|x_n - x_1\|_{X_2}^2;
\]
hence $\|Kn - Kn y\|_{X_2} \leq k\lambda\|x - y\|_{X_2}$. Thus if $k\lambda < 1$ then by the Banach contraction principle, the operator $KN$ has a unique fixed point. This implies that $x = Knx + y$ has a unique solution. Using Theorem 2.11, the operator $I - Kn$ is invertible, $[I - Kn]^{-1}$ is Lipschitz with Lipschitz constant
\[
\|\|I - Kn\|^{-1}\|_* \leq \frac{1}{1 - \|Kn\|_*} \leq \frac{1}{1 - k\lambda}.
\]
Case (iii). Consider the nonlinear Volterra-type integral equation
\[
x(t) = \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,x(s))ds + y(t) \tag{15}
\]
for any arbitrary $y(t) \in X_2$. Define the following iterative scheme
\[
x_{0}(t) = y(t), \\
x_{n+1}(t) = y(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,x_n(s))ds, n = 0, 1, 2, \ldots,
\]
for all $t \in [t_0,t_1]$. Applying Gronwall’s inequality and using the Lipschitz continuity of $f(t,x(t))$ and the boundedness of $E_{\alpha,\alpha}(A(t-s)^\alpha)$ for all $t, s \in [t_0,t_1]$, the existence and uniqueness of the solution (15) can be easily proved. Let us consider two solutions
\[
x_1(t) = y(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,x_1(s))ds, x(t) \in X_2, \tag{16}
\]
and
\[
x_2(t) = y(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)f(s,x_2(s))ds, z(t) \in X_2. \tag{17}
\]
Then
\[
\|x_1(t) - x_2(t)\| \leq \|y(t) - z(t)\| + \int_{t_0}^{t} (t-s)^{\alpha-1}\|E_{\alpha,\alpha}(A(t-s)^\alpha)\| \times \|f(s,x_1(s)) - f(s,x_2(s))\|ds \\
\leq \|y(t) - z(t)\| + M\lambda \int_{t_0}^{t} (t-s)^{\alpha-1}\|x_1(s) - x_2(s)\|ds.
\]
Using Gronwall’s inequality
\[
\|x_1(t) - x_2(t)\| \leq \|y(t) - z(t)\| \exp \left( M\lambda \int_{t_0}^{t} (t-s)^{\alpha-1}ds \right) \\
\leq \|y(t) - z(t)\| \exp \left( \frac{M\lambda(t_1-t_0)^\alpha}{\alpha} \right). \tag{18}
\]
We can write (16) and (17) as $x_1 = [I - Kn]^{-1}y$ and $x_2 = [I - Kn]^{-1}z$. Then
\[
\|x_1 - x_2\| = \|[I - Kn]^{-1}y - [I - Kn]^{-1}z\|. \tag{19}
\]
Using (18) and (19), we get
\[ \| (I - KN)^{-1} y - (I - KN)^{-1} z \| \leq \exp \left( \frac{M \lambda (t_1 - t_0)^\alpha}{\alpha} \right) \| y(t) - z(t) \|. \]

This gives that \( \exp \left( \frac{M \lambda (t_1 - t_0)^\alpha}{\alpha} \right) \) is the Lipschitz constant of \( (I - KN)^{-1} \).

Let us now define the following operators \( G_1 : X_1 \to X_2, G_2 : X_2 \to X_1 \) such that \( G_1 = (I - KN)^{-1} H \) and \( G_2 = LN \). We immediately get the following lemma

**Lemma 3.3.** If \( I - KN \) is invertible, then the controllability of the system (1) is equivalent to the solvability of the feed-back system
\[
\begin{align*}
    u &= u_1 - G_2 x, \\
    x &= G_1 u.
\end{align*}
\]  
(20)

We note that (20) is a particular case of the general feed-back system (9) with \( u_2 = 0 \).

4. **Controllability results with Lipschitzian nonlinearity.** To prove the main controllability results of this paper, we first discuss the solvability of a feed-back system of the type (9).

**Definition 4.1.** Let \( X_1 \) and \( X_2 \) be real Banach spaces. The feed-back system of the type (9) is said to be globally solvable if for every \((u_1, u_2) \in X_1 \times X_2\), there exists a solution \((e_1, e_2) \in X_1 \times X_2\) of (9). If this solution is unique, then it is said to be uniquely globally solvable. We denote this correspondence by \((u_1, u_2) \mapsto (e_1, e_2)\).

**Definition 4.2.** Let \( x \) and \( u \) be the state trajectory and control function of a controllable system (1) with initial and final states \( x_0 \) and \( x_1 \). Denote this correspondence by \((x_0, x_1) \mapsto (x, u)\). The system (1) will be called stable with respect to the initial and final states if there exist constants \( \lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22} \geq 0 \) such that
\[
\| x - y \|_{L^2} \leq \lambda_{11} \| x_0 - y_0 \|_{\mathbb{R}^n} + \lambda_{12} \| x_1 - y_1 \|_{\mathbb{R}^n},
\]
\[
\| u - v \|_{L^2} \leq \lambda_{21} \| x_0 - y_0 \|_{\mathbb{R}^n} + \lambda_{22} \| x_1 - y_1 \|_{\mathbb{R}^n},
\]
for all \((x_0, x_1)\) and \((y_0, y_1)\) such that \((x_0, x_1) \mapsto (x, u)\) and \((y_0, y_1) \mapsto (y, v)\).

From the Definition 4.2, it follows that the system (1) is stable; then small changes in the initial and final states produce small changes in the corresponding trajectory and control.

**Lemma 4.3.** Let \( G_1 : X_1 \to X_2 \) and \( G_2 : X_2 \to X_1 \) belong to Lip. If \( \| G_1 \|_\ast \| G_2 \|_\ast < 1 \), then (9) is uniquely globally solvable. Further the correspondence \((u_1, u_2) \mapsto (e_1, e_2), (v_1, v_2) \mapsto (f_1, f_2)\) satisfy the relation
\[
\| e_1 - f_1 \|_{X_1} \leq \frac{\| u_1 - v_1 \|_{X_1} + \| G_2 \|_\ast \| u_2 - v_2 \|_{X_2}}{1 - \| G_1 \|_\ast \| G_2 \|_\ast},
\]
\[
\| e_2 - f_2 \|_{X_2} \leq \frac{\| u_2 - v_2 \|_{X_2} + \| G_1 \|_\ast \| u_1 - v_1 \|_{X_1}}{1 - \| G_1 \|_\ast \| G_2 \|_\ast}.
\]
Moreover the iterates \( \{e_1^{(n)}\}, \{e_2^{(n)}\} \) defined by
\[
\begin{align*}
e_1^{(n+1)} &= u_1 - G_2 e_2^{(n)}, \\
e_2^{(n)} &= u_2 + G_1 e_1^{(n)}
\end{align*}
\]
converge to the unique solution \((e_1, e_2) \in X_1 \times X_2\) starting from arbitrary \(e_1^{(0)} \in X_1\).

**Proof.** Solvability of (9) for \((e_1, e_2)\) is equivalent to the solvability of
\[
\begin{align*}
e_1 &= u_1 - G_2(u_2 + G_1 e_1), \\
e_2 &= u_2 + G_1 e_1.
\end{align*}
\]
Define \(G : X_1 \to X_2\) by \(Ge_1 = u_2 + G_1 e_1\). Obviously \(G\) belongs to Lip with \(\|G\|_* = \|G_1\|_*\). This gives \(e_1 = u_1 - G_2 Ge_1\). That is
\[
e_1 + G_2 Ge_1 = u_1.
\]
By hypothesis of lemma, we have
\[
\|G_2 G\|_* \leq \|G_2\|_* \|G\|_* = \|G_2\|_* \|G_1\|_* < 1.
\]
Using Theorem 2.13, it follows that, for every \(u_1 \in X_1\), there exists a unique solution \(e_1 \in X_1\) of (23).
Let \((u_1, u_2) \mapsto (e_1, e_2), (v_1, v_2) \mapsto (f_1, f_2)\). Then we get
\[
\begin{align*}
e_1 &= u_1 - G_2(u_2 + G_1 e_1) \\
e_2 &= u_2 + G_1 e_1, \\
f_1 &= u_1 - G_2(v_2 + G_1 f_1) \\
f_2 &= v_2 + G_1 f_1;
\end{align*}
\]
and hence
\[
e_1 - f_1 = u_1 - v_1 - G_2(v_2 - u_2) + G_2 G_1(f_1 - e_1).
\]
This gives
\[
\|e_1 - f_1\| \leq \frac{\|u_1 - v_1\| + \|G_2\|_* \|v_2 - u_2\|}{1 - \|G_1\|_* \|G_2\|_*}.
\]
Similarly
\[
\|e_2 - f_2\| \leq \frac{\|u_2 - v_2\| + \|G_1\|_* \|u_1 - v_1\|}{1 - \|G_1\|_* \|G_2\|_*}.
\]
Moreover the contraction mapping principle implies that the iterates defined by
\[
e_1^{(n+1)} = u_1 - G_2(u_2 + G_1 e_1^{(n)})
\]
converge to the unique solution \(e_1 \in X_1\) of (21) with any \(e_1^{(0)} \in X_1\). This implies that the iterates
\[
\begin{align*}
e_1^{(n+1)} &= u_1 - G_2 e_2^{(n)}, \\
e_2^{(n)} &= u_2 + G_1 e_1^{(n)},
\end{align*}
\]
converge to the unique solution \((e_1, e_2) \in X_1 \times X_2\) of (9).

Now we state the main Theorem.
Theorem 4.4. Suppose that the linear part of the system (1) is controllable and the matrix $A$ and the nonlinear function $f(t,x)$ satisfy the Assumption $[\mathcal{A}]$ and $[\mathcal{B}]$. Let

$$k < \frac{\beta^2}{\lambda^2(\beta + \lambda \gamma)}.$$ 

Then

(i) The system (1) is controllable.
(ii) The system (1) is stable with respect to the initial and final states and
(iii) The control function and state trajectory tuple $(u,x)$ of (1) with initial state $x_0 = 0$ can be approximated by iterates $(u^{(n)}, x^{(n)})$ defined by

$$u^{(n+1)}(t) = u_0(t) - G_2 x^{(n)}(t),$$

$$x^{(n+1)}(t) = G_1 u^{(n+1)}(t),$$

starting from any arbitrary $x^{(0)}(t)$.

Proof. As before, we set $X_1 = L^2([t_0,t_1], \mathbb{R}^m), X_2 = L^2([t_0,t_1], \mathbb{R}^n)$ and $G_1 = [I - KN]^{-1} H : X_1 \to X_2, G_2 = LN : X_2 \to X_1$. Now, using Theorem 3.2 and Lemma 3.1, we have

$$\|G_1\|_* \leq \|[I - KN]^{-1}\|_* \|H\|_* \leq \frac{\lambda^2 h}{\beta(\beta - \lambda^2 k)},$$

$$\|G_2\|_* \leq \|L\|_* \|N\|_* \leq \gamma \lambda.$$ 

By assumption $k < \frac{\beta^2}{\lambda^2(\beta + \lambda \gamma)}$. Hence we have $\|G_1\|_* \|G_2\|_* < 1$. Lemma 3.3 and Lemma 4.3 the nonlinear system (1) is controllable.

Let $(u,x)$ and $(v,y)$ be the control and trajectory pairs corresponding to the initial and final state pairs $(0,x_1)$ and $(0,y_1)$ respectively for the system (1). Then from the Definition 4.2, we get

$$\|x - y\|_{X_2} \leq \|G_1\|_* \|u - v\|_{X_1},$$

$$\|u - v\|_{X_1} \leq \|u_1 - v_1\|_{X_1} - \|G_2\|_* \|x - y\|_{X_2}.$$ 

Then we have

$$\|u - v\|_{X_1} \leq \frac{\|u_1 - v_1\|_{X_1}}{1 - \|G_1\|_* \|G_2\|_*}, \quad (24)$$

$$\|x - y\|_{X_2} \leq \frac{\|G_1\|_* \|u_1 - v_1\|_{X_1}}{1 - \|G_1\|_* \|G_2\|_*}. \quad (25)$$

The value of $\|u_1 - v_1\|_{X_1} \leq q \|x_1 - y_1\|_{\mathbb{R}^n}$. Then the inequalities (24) and (25) become

$$\|u - v\|_{X_1} \leq \lambda_{11} \|x_1 - y_1\|_{\mathbb{R}^n},$$

$$\|x - y\|_{X_2} \leq \lambda_{21} \|x_1 - y_1\|_{\mathbb{R}^n},$$

where $\lambda_{11} = \frac{q}{1 - \|G_1\|_* \|G_2\|_*}$ and $\lambda_{21} = \frac{\|G_1\|_* q}{1 - \|G_1\|_* \|G_2\|_*}$. This gives us the stability of the system (1) with respect to the initial and final states. The uniqueness of $u$ and $x$ and the convergence of the iterates follow directly from Lemma 4.3.
In the above iterative scheme, the computation of $x^{(n)}$ from $u^{(n)}$ at each stage is different, since $[I-KN]^{-1}$ is not known explicitly. However, if $KN$ is a contraction, then $x^{(n)}$ can be computed from $u^{(n)}$ through the iterative procedure given by the contraction mapping principle as we see in the following Theorem.

**Theorem 4.5.** Suppose that the linear part of the system (1) is controllable and the matrix $A$ and the nonlinear function $f(t,x)$ satisfy the Assumption $[A]$ and $(B1)$. If

$$k\lambda < 1 \quad \text{and} \quad \frac{\lambda h\gamma}{1-k\lambda} < 1,$$

(i) The system (1) is controllable.  
(ii) The system (1) is stable with respect to the initial and final states and  
(iii) The control vector and state trajectory pair $(u(t), x(t))$ can be approximated by iterate $u^{(n)}(t)$ defined by

$$u^{(n+1)}(t) = (t_1-t)^{1-\alpha} P(t) \left[ x_1 - \int_{t_0}^{t_1} (t_1-s)^{\alpha-1} E_{\alpha,\alpha}(A(t_1-s)^{\alpha}) f(s, x^{(n)}(s)) ds \right].$$

and the state vector approximated by $x^{(n)}(t)$ at $n^{th}$ stage is in turn given by the approximated scheme $\{x_{j}^{(n)}\}$ defined as

$$x_{j+1}^{(n)}(t) = \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^{\alpha}) \left( Bu^{(n)}(s) + f(s, x_{j}^{(n)}(s)) \right) ds.$$

Proof. By using Theorem 4.4, the conclusions (i) and (ii) follow the same way as in the proof of Theorem 4.5. The convergence of the iterative scheme follows directly from contraction mapping principle.

The following Theorem gives another set of sufficient conditions for the controllability of (1) with $f(t,x)$ Lipschitz continuous. We neither require $N$ to be monotone nor $KN$ to be a contraction.

**Theorem 4.6.** Suppose that the linear part of the system (1) is controllable and the matrix $A$ and the nonlinear function $f(t,x)$ satisfy the Assumption $[A]$ (with $h(t,s) = m, m$ being a positive constant) and $(B1)$. Further let $f(t,x)$ be Lipschitz with constant $\lambda$ and $\exp\left(\frac{m\lambda(t_1-t_0)^\alpha}{\alpha}\right) h\lambda\gamma < 1$. Then the conditions of Theorem 4.5 hold true.

Proof. From Theorem 3.2, we have

$$\|G_1\|_* \leq \|I-KN\|_* h \leq \exp\left(\frac{m\lambda(t_1-t_0)^\alpha}{\alpha}\right) h$$

and also we have

$$\|G_2\|_* \leq \lambda\gamma.$$

The remaining part of the proof follows easily.

5. **Controllability results with non-Lipschitzian nonlinearity.** In this section, we discuss controllability results involving non-Lipschitzian $f(t,x)$. We develop the required theory wherein we need to impose only monotonicity type assumption on $-f(t,x)$ (Assumption $(B2)$) but not Lipschitz continuity. This is significant because we have a situation where the nonlinearities have slopes which are bounded below by a non-negative constant but not bounded above.
First we prove the following lemma which gives the solvability of the feed-back system (9).

**Lemma 5.1.** Let $X_1$ and $X_2$ be real Hilbert spaces. Suppose that the operators $G_1 : X_1 \to X_2$ and $G_2 : X_2 \to X_1$ satisfy the following conditions:

(i) $G_1$ is compact, continuous and satisfies growth condition of the type
\[
\|G_1e_1\| \leq h_1 + g_1\|e_1\|, \text{ for all } e_1 \in X_1, h_1, g_1 > 0.
\]

(ii) $G_2$ is continuous and satisfies the growth condition
\[
\|G_2e_2\| \leq h_2 + g_2\|e_2\|, \text{ for all } e_2 \in X_2, h_2, g_2 > 0.
\]

If $(1 - g_1g_2) > 0$, then the feed-back equation (9) is solvable.

**Proof.** Define the operator $T : X_1 \to X_1$ by
\[
Te_1 = G_2(u_2 + G_1e_1).
\]

Then (9) is solvable if and only if
\[
[I + T]e_1 = u_1
\]
is solvable in $X_1$. Since the compact operator $G_1$ and the operator $G_2$ are continuous and satisfy the growth conditions, it can be easily shown that $T$ is compact, continuous and satisfy the growth condition
\[
\|Te_1\| \leq h_2 + g_2\|u_2\| + g_1g_2\|e_1\|.
\]
Since $[I + T]$ is compact and continuous perturbation of the identity map, it is of type (M), (refer, [17]). Further,
\[
([I + T]e_1, e_1) = (e_1, e_1) + (Te_1, e_1)
\geq \|e_1\|^2 - \|Te_1\|\|e_1\|
\geq (\|e_1\| - h_2 - g_2\|u_2\| - g_1g_2\|e_1\|)\|e_1\|
\geq ((1 - g_1g_2)\|e_1\| - (h_2 + g_2\|u_2\| + g_1h_1))\|e_1\|,
\]
since $(1 - g_1g_2) > 0$; it follows that $[I + T]$ is coercive and hence, by Theorem 2.11, the range of $[I + T]$ is all of $X_1$. This, in turn, implies that the feed-back equation (9) is solvable. \qed

Now we make the following assumptions on the system (1).

**Assumptions \([\mathcal{C}]\)**

(\(\mathcal{C}1\)) There exists $\mu > 0$ such that $(-Ax, x) \geq \mu\|x\|^2$ for all $x \in X_2$.

(\(\mathcal{C}2\)) The function $-f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is monotone
\[
(f(t, x) - f(t, y), x - y) \leq 0, \text{ for all } x, y \in \mathbb{R}^n, t \in [t_0, t_1].
\]

**Lemma 5.2.** Under Condition (\(\mathcal{C}1\)), the operator $K : X_2 \to X_2$ satisfies the condition
\[
(Kx, x)_{X_2} \geq \mu \|Kx\|_{X_2}^2, \text{ for all } x \in X_2.
\]

**Proof.** Define
\[
f(t) = \int_{t_0}^{t} (t - s)^{\alpha - 1} E_{\alpha, \alpha}(A(t - s)\alpha)x(s)ds.
\]
(26)
Taking the Caputo fractional derivative of order $\alpha$ on both sides of (26), we get

\[ C^\alpha f(t) = C^\alpha \left( \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)x(s)ds \right) \]

\[ = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} \left( \frac{d}{ds} \int_{t_0}^{s} (s-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(s-\tau)^\alpha)x(\tau)d\tau \right) ds \]

\[ = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} (s-t_0)^{\alpha-1} E_{\alpha,\alpha}(A(s-t_0)^\alpha)x(t_0)ds \]

\[ + \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-s)^{-\alpha} \left( \int_{t_0}^{s} (s-\tau)^{\alpha-1} E_{\alpha,\alpha}(A(s-\tau)^\alpha)x'(\tau)d\tau \right) ds \]

\[ = E_{\alpha,1}(A(t-t_0)^\alpha)x(t_0) + x(t) - E_{\alpha,1}(A(t-t_0)^\alpha)x(t_0) \]

\[ + A \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)x(s)ds \]

\[ = x(t) + A \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)x(s)ds. \]

Then

\[ (Kx, x)_{X_2} = \int_{t_0}^{t_1} (Kx(t), x(t))dt \]

\[ = \int_{t_0}^{t_1} (f(t), C^\alpha f(t)) - A \int_{t_0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha}(A(t-s)^\alpha)x(s)ds)dt \]

\[ = \int_{t_0}^{t_1} (f(t), C^\alpha f(t))dt + \int_{t_0}^{t_1} (f(t), -Af(t))dt \]

\[ \geq \mu \int_{t_0}^{t_1} \|f(t)\|^2 dt = \mu \|Kx\|_{X_2}^2. \]

This gives \((Kx, x)_{X_2} \geq \mu \|Kx\|_{X_2}^2\) for all \(x \in X_2\). \qed

**Theorem 5.3.** Under Assumptions \([\mathcal{A}]\) and \([\mathcal{C}]\), \([I-KN]^{-1}\) exists and is continuous. Further it satisfies the following growth condition

\[ \| [I-KN]^{-1} y \| \leq \frac{d}{\mu} + \left( \frac{n}{\mu} + 1 \right) \|y\|. \]

**Proof.** As in Lemma 5.2, Assumptions \([\mathcal{C}]\) imply that \(-N\) is monotone and \(K\) satisfies

\[ (Kx, x) \geq \mu \|Kx\|_{X_2}^2, \text{ for all } x \in X_2, \]

The invertibility of the operator \([I-KN]\) follows from [15] and boundedness of \([I-KN]^{-1}\) follows from [18].

To prove the growth condition, we proceed as follows. Let \(x\) satisfy the operator equation \(x = KNx + y\). Then we have

\[ (KNx, Nx) = (x - y, Nx) = (x - y, Nx + Ny - Ny) \]

\[ = (x - y, Nx - Ny) + (x - y, Ny). \]
In view of monotonicity of $-N$, we get
\[(KNx, Nx) \leq (x - y, Ny) \leq \|x - y\|\|Ny\|.
\]
This gives
\[\mu\|KNx\|^2 \leq \|x - y\|\|Ny\|.
\]
That is
\[\mu\|x - y\|^2 \leq \|x - y\|\|Ny\|
\]
which is equivalent to
\[\mu\|x - y\| \leq \|Ny\|.
\]
The growth condition on $N$ implies that
\[\|x - y\| \leq d\frac{\eta}{\mu} + \eta\|y\|
\]
which gives
\[\|x\| \leq \frac{d}{\mu} + \left(\frac{\eta}{\mu} + 1\right)\|y\|.
\]
\[\|([I - KN]^{-1}y\| \leq \frac{d}{\mu} + \left(\frac{\eta}{\mu} + 1\right)\|y\|.
\]
(27)

We now prove the following controllability result for the nonlinear system (1).

**Theorem 5.4.** Suppose that the linear part of the nonlinear system (1) is controllable and the matrix $A$ and the nonlinear function $f(t, x)$ satisfy the Assumptions $A$ and $C$. If $1 - \left(\frac{\eta}{\mu} + 1\right)\gamma\eta h > 0$, then the system (1) is controllable.

**Proof.** Let $X_1 = L^2([t_0, t_1], \mathbb{R}^m)$ and $X_2 = L^2([t_0, t_1], \mathbb{R}^n)$. By Lemma 3.3, the controllability of the system (1) is equivalent to the solvability of the feed-back system
\[u = u_1 - G_2x,
\]
\[x = G_1u,
\]
(28)

where, $G_2 = LN : X_2 \rightarrow X_1, G_1 = [I - KN]^{-1}H : X_1 \rightarrow X_2$. By Theorem 5.3, $[I - KN]^{-1}$ is continuous and satisfies the growth condition (27). Since the operator $H$ is compact and continuous, it follows that the operator $G_1$ is continuous and compact with growth condition
\[\|G_1u\| \leq \|[I - KN]^{-1}\|\|Hu\| \leq \frac{d}{\mu} + \left(\frac{\eta}{\mu} + 1\right)h\|u\|.
\]
Similarly
\[\|G_2x\| \leq \|L\|\|Nx\| \leq \gamma d + \gamma\|x\|.
\]
The operators $G_1$ and $G_2$ satisfy all the conditions of Lemma 5.1 with $1 - \left(\frac{\eta}{\mu} + 1\right)\gamma\eta h > 0$ and hence (28) is solvable, which in turn implies the controllability of the nonlinear system (1).
6. Examples. In this section, we apply the results established in the previous sections to the following fractional dynamical systems.

Example 6.1. Consider the nonlinear fractional dynamical system represented by the fractional differential equation of order 3/4 as

$$CD^{3/4}x_1(t) = x_2(t) \cdot \frac{1}{4200} (\sin(x_1(t)) + 32x_1(t)), \quad CD^{3/4}x_2(t) = -x_1(t) + u(t), \quad t \in [0, 1],$$

with initial conditions $x_1(0) = 0$ and $x_2(0) = 0$.

The problem (29) can be expressed as

$$CD^{3/4}x(t) = Ax(t) + Bu(t) + f(t, x(t)), \quad t \in [0, 1],$$

$$x(0) = x_0,$$

where $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $f(t, x(t)) = \begin{bmatrix} \frac{1}{4200} (\sin(x_1(t)) + 32x_1(t)) \\ 0 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Let us consider the final point $x(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

The controllability of this system is to determine whether there exists a control function $u(t)$ to steer the states of the system from the initial point $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the final point $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ in time $[0, 1]$. The Mittag-Leffler matrix function for given matrix $A$ is

$$E_{3/4}(A^{3/4}) = \begin{bmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{bmatrix},$$

where $P_1 = \frac{1}{2} \left[ E_{3/4}(it^{3/4}) + E_{3/4}(-it^{3/4}) \right]$ and $P_2 = \frac{i}{2} \left[ E_{3/4}(-it^{3/4}) - E_{3/4}(it^{3/4}) \right]$. The controllability Grammian matrix for this system is

$$W(0, 1) = \int_0^1 E_{3/4,3/4}(A(1 - \tau)^{3/4})BB^*E_{3/4,3/4}(A^*(1 - \tau)^{3/4})d\tau$$

$$= \begin{bmatrix} 0.2948 & 0.2242 \\ 0.2242 & 0.3089 \end{bmatrix} > 0,$$

and the given nonlinear function satisfies the following slope restrictions

$$0 < \beta < \frac{f(t, x) - f(t, y)}{x - y} < \lambda, \quad \text{for all } x, y \in \mathbb{R}.$$

Then it follows that $f(t, x(t))$ is Lipschitz continuous with constant $\lambda = 33/4200$ and strongly monotone with constant $\beta = 31/4200$. Also Lemma 3.1 gives us the following estimates $k = 1.5901, k_1 = 2.5499, k_2 = 1.9934, h = 1.5901, \gamma = 66.4302$ and $a = 1, b = 1, c = 13.0693$. By Theorem 2.3, the linear system is controllable and the condition $k = 1.5901 < \frac{\beta^2}{t^2} \frac{\lambda^2(\beta + \lambda\gamma)}{\lambda^2} = 1.6671$. Hence, it follows, from Theorem 4.4, that the nonlinear system (29) is controllable on $[0, 1]$. 
According to Theorem 4.5, the values $k\lambda = 0.0125 < 1$ and $\frac{\lambda h\gamma}{1-k\lambda} = 0.8404 < 1$; then the nonlinear system (29) is controllable on $[0, 1]$ and the control vector and state trajectory pair $(x(t), u(t))$ can be approximated by iterate $u^{(n)}(t)$ defined by

$$u^{(n+1)}(t) = (t_1 - t)^{\frac{3}{4}} P(t) \left[ x_1 - \int_{t}^{t_1} (t_1 - s)^{-\frac{1}{4}} E_{3/4,3/4} \left( A(t_1 - s)^{3/4} \right) f(s, x^{(n)}(s)) ds \right].$$

and the state vector approximated $x^{(n)}(t)$ at $n^{th}$ stage is in turn given by the approximated scheme $\{x_j^{(n)}\}$ defined as

$$x_{j+1}^{(n)}(t) = \int_{0}^{t} (t - s)^{-\frac{1}{4}} E_{3/4,3/4} \left( A(t - s)^{3/4} \right) \left( Bu^{(n)}(s) + f(s, x_j^{(n)}(s)) \right) ds.$$

The controlled trajectories $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ and steering control $u(t)$ are computed and are depicted in Figure 1 and Figure 2.

**Figure 1.** The trajectory of the system (29) steers from the initial state $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to the final state $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ during the interval $[0, 1]$.

**Figure 2.** The steering control $u(t)$ of the system (29) during the interval $[0, 1]$. 
Example 6.2. Consider the nonlinear fractional dynamical system represented by the fractional differential equation of order 8/9 as

\[ CD^{8/9}x_1(t) = \frac{1}{2} x_2(t), \]
\[ CD^{8/9}x_2(t) = \frac{1}{2} x_1(t) + 2u(t) + \sqrt{x_2(t)} + \frac{x_2(t)}{120}, \quad t \in [0, 1], \tag{30} \]

with initial conditions \( x_1(0) = 1 \) and \( x_2(0) = 2 \).

Comparing with (1), \( A = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, f(t, x(t)) = \begin{bmatrix} 0 \\ \sqrt{x_2(t) + \frac{x_2(t)}{120}} \end{bmatrix}, x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \) and \( x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \). The Mittag-Leffler matrix function for given matrix \( A \) is

\[ E_{8/9}(At^{8/9}) = \begin{bmatrix} Q_1 & Q_2 \\ Q_2 & Q_1 \end{bmatrix}, \]

where

\[ Q_1 = \frac{1}{2} \left[ E_{8/9}(0.5t^{8/9}) + E_{8/9}(-0.5t^{8/9}) \right] \quad \text{and} \]
\[ Q_2 = \frac{1}{2} \left[ E_{8/9}(0.5t^{8/9}) - E_{8/9}(-0.5t^{8/9}) \right]. \]

The controllability Grammian matrix for this system is

\[ W(0, 1) = \int_0^1 E_{8/9, 8/9}(A(1 - \tau)^{8/9})BB^*E_{8/9, 8/9}(A^*(1 - \tau)^{8/9})d\tau \]
\[ = \begin{bmatrix} 0.4550 & 1.2024 \\ 1.2024 & 3.9227 \end{bmatrix} > 0 \]

and the given nonlinear function satisfies the Assumptions \([\mathcal{A}]\) and \([\mathcal{U}]\) with \( \eta = 1/120 = 0.0083 \). The eigenvalues of \( A \) are \( \pm \frac{1}{\sqrt{2}} \) and hence \((-Ax, x) \geq \mu \|x\|^2\), where \( \mu = \frac{1}{\sqrt{2}} \). Also Lemma 3.1 gives us the following estimates \( k = 0.9666, k_1 = 1.4271, k_2 = 1.2093, h = 1.9331, \gamma = 52.1508 \) and \( a = 0.5, b = 2, c = 15.1094 \). By Theorem 2.3, the linear system is controllable and the condition as \( 1 - \left( \frac{\eta}{\mu} + 1 \right)h \gamma \eta = 0.1500 > 0 \). It follows, from Theorem 5.4, that the nonlinear system (30) is controllable on \([0, 1]\), where \( f \) is not Lipschitz continuous.

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