ALMOST COMPLETE INTERSECTIONS AND STANLEY’S CONJECTURE

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Abstract. Let $K$ be a field and $I$ a monomial ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$. We show that if either: 1) $I$ is almost complete intersection, 2) $I$ can be generated by less than four monomials; or 3) $I$ is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then Stanley’s conjecture holds for $S/I$.

1. Introduction

Throughout this paper, let $K$ be a field and $I$ a monomial ideal of the polynomial ring $S = K[x_1, \ldots, x_n]$.

A decomposition of $S/I$ as direct sum of $K$-vector spaces of the form $D : S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$, where $u_i$ is a monomial in $S$ and $Z_i \subseteq \{x_1, \ldots, x_n\}$, is called a Stanley decomposition of $S/I$. The number $sdepth D := \min\{|Z_i| : i = 1, \ldots, r\}$ is called Stanley depth of $D$. The Stanley depth of $S/I$ is defined to be $sdepth S/I := \max\{sdepth D : D$ is a Stanley decomposition of $S/I\}$.

Stanley conjectured [St] that $\text{depth} S/I \leq sdepth S/I$. This conjecture is known as Stanley’s conjecture. Recently, this conjecture was extensively examined by several authors; see e.g. [A1], [A2], [HP], [HSY], [P], [R], [S2] and [S3]. On the other hand, the present third author [S2] conjectured that there always exists a Stanley decomposition $D$ of $S/I$ such that the degree of each $u_i$ is at most $\text{reg} S/I$. We refer to this conjecture as $h$-regularity conjecture. It is known that for square-free monomial ideals, these two conjectures are equivalent. Our main aim in this paper is to determine some classes of monomial ideals such that these conjectures are true for them.

A basic fact in commutative algebra says that there exists a finite chain $F : I = I_0 \subset I_1 \subset \cdots \subset I_r = S$ of monomial ideals such that $I_i/I_{i-1} \cong S/p_i$ for monomial prime ideals $p_i$ of $S$. Dress [D] called the ring $S/I$ clean if there exists a chain $F$ such that all the $p_i$ are minimal prime ideals of $I$. By [HSY] Proposition 2.2 if $I$ is complete intersection, then the ring $S/I$ is clean. Lemmas 2.4 and 2.8 provide two other classes of clean rings.

Herzog and Popescu [HP] called the ring $S/I$ pretty clean if there exists a chain $F$ such that for all $i < j$ for which $p_i \subseteq p_j$, it follows that $p_i = p_j$. Obviously, cleanness implies pretty cleanness and when $I$ is square-free, it is known that these two concepts coincide; see [HP] Corollary 3.5.

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If $S/I$ is pretty clean, then $S/I$ is sequentially Cohen-Macaulay and depth of $S/I$ is equal to the minimum of the dimension of $S/p$, where $p \in \text{Ass}_S S/I$; see [S1] for an easy proof. If $S/I$ is pretty clean, then [HP Theorem 6.5] asserts that Stanley’s conjecture holds for $S/I$. In fact, if $S/I$ is pretty clean, then [HVZ Proposition 1.3] yields that depth $S/I = \text{depth} S/I$. Also if $S/I$ is pretty clean, then by [S2 Theorem 4.7] $h$-regularity conjecture holds for $S/I$.

We prove that if the monomial ideal $I$ is either almost complete intersection or it can be generated by less than four monomials, then $S/I$ is pretty clean. Thus, for such monomial ideals both Stanley’s and $h$-regularity conjectures hold. Also, we show that if $I$ is the Stanley-Reisner ideal of a locally complete intersection simplicial complex on $[n]$, then $S/I$ satisfies Stanley’s conjecture.

2. Main Results

A simplicial complex $\Delta$ on $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ with the property that if $F \in \Delta$, then all subsets of $F$ are also in $\Delta$. Any singleton element of $\Delta$ is called a vertex. An element of $\Delta$ is called a face of $\Delta$ and the maximal faces of $\Delta$, under inclusion, are called facets. We denote by $F(\Delta)$ the set of all facets of $\Delta$. The dimension of a face $F$ is defined as $\text{dim } F = |F| - 1$, where $|F|$ is the number of elements of $F$. The dimension of the simplicial complex $\Delta$ is the maximal dimension of its facets. A simplicial complex $\Delta$ is called pure if all facets of $\Delta$ have the same dimension. We denote the simplicial complex $\Delta$ with facets $F_1, \ldots, F_t$ by $\Delta = \langle F_1, \ldots, F_t \rangle$. According to Björner and Wachs [BW], a simplicial complex $\Delta$ is said to be (non-pure) shellable if there exists an order $F_1, \ldots, F_t$ of the facets of $\Delta$ such that for each $2 \leq i \leq t$, $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is a pure $(\text{dim } F_i - 1)$-dimensional simplicial complex. If $\Delta$ is a simplicial complex on $[n]$, then the Stanley-Reisner ideal of $\Delta$, $I_{\Delta}$, is the square-free monomial ideal generated by all monomials $x_i x_{i_2} \cdots x_{i_t}$ such that $\{i_1, i_2, \ldots, i_t\} \notin \Delta$. The Stanley-Reisner ring of $\Delta$ over the field $K$ is the $K$-algebra $K[\Delta] := S/I_{\Delta}$. Any square-free monomial ideal $I$ is the Stanley-Reisner ideal of some simplicial complex $\Delta$ on $[n]$. If $F(\Delta) = \{F_1, \ldots, F_t\}$, then $I_{\Delta} = \bigcap_{i=1}^t p_{F_i}$, where $p_{F_i} := (x_j : j \notin F_i)$; see [BH Theorem 5.1.4].

Recall that the Alexander dual $\Delta^\vee$ of a simplicial complex $\Delta$ is the simplicial complex whose faces are $([n]\setminus F)[F \notin \Delta]$. Let $I$ be a square-free monomial ideal of $S$. We denote by $I^\vee$, the square-free monomial ideal which is generated by all monomials $x_{i_1} \cdots x_{i_k}$, where $(x_{i_1}, \ldots, x_{i_k})$ is a minimal prime ideal of $I$. It is easy to see that for any simplicial complex $\Delta$, one has $I_{\Delta^\vee} = (I_{\Delta})^\vee$. A monomial ideal $I$ of $S$ is said to have linear quotients if there exists an order $u_1, \ldots, u_m$ of $G(I)$ such that for any $2 \leq i \leq m$, the ideal $(u_1, \ldots, u_{i-1}) : S u_i$ is generated by a subset of the variables.

Lemma 2.1. Let $I$ be a square-free monomial ideal of $S$. Then $S/I$ is clean if and only if $I^\vee$ has linear quotients.

Proof. Dress [L1] Theorem on page 53] proved that a simplicial complex $\Delta$ is (non-pure) shellable if and only if $K[\Delta]$ is a clean ring. On the other hand, by [HHZ Theorem 1.4], a simplicial complex $\Delta$ is (non-pure) shellable if and only if $I_{\Delta^\vee}$ has linear quotients. Combining these facts, yields our claim. □

Lemma 2.2. Let $I$ and $J$ be two monomial ideals of $S$. Assume that $I = uJ$ for some monomial $u$ in $S$ and $\text{ht } J \geq 2$. If $S/J$ is pretty clean, then $S/I$ is pretty clean too.

Proof. With the proof of [S3 Lemma 1.9], the claim is immediate. □
In what follows for a monomial ideal \( I \) of \( S \), we denote the number of elements of \( G(I) \) by \( \mu(I) \).

**Definition 2.3.** A monomial ideal \( I \) of \( S \) is said to be **almost complete intersection** if \( \mu(I) = \text{ht } I + 1 \).

**Lemma 2.4.** Let \( I \) be an almost complete intersection square-free monomial ideal of \( S \). Then \( S/I \) is clean.

**Proof.** The claim is obvious when \( \text{ht } I = 0 \). Let \( \text{ht } I = 1 \). Then \( I = (u_1, u_2) \) for some monomials \( u_1 \) and \( u_2 \). We can write \( I \) as \( I = u(u_1', u_2') \), where \( u = \text{gcd}(u_1, u_2) \) and \( u_1', u_2' \) are monomials forming a regular sequence on \( S \). So in this case, the claim is immediate by Lemma 2.2 and [HSY Proposition 2.2]. Now, assume that \( h := \text{ht } I \geq 2 \). By [KTY Theorem 4.4] \( I \) can be written in one of the following forms, where \( A_1, A_2, \ldots, B_1, B_2, \ldots \) are non-trivial square-free monomials which are pairwise relatively prime, and \( p, p' \) are integers with \( 2 \leq p \leq h \) and \( 1 \leq p' \leq h \).

1. \( I_1 = (A_1B_1, A_2B_2, \ldots, A_pB_p, A_{p+1}, \ldots, A_h, B_1B_2 \cdots B_p) \).
2. \( I_2 = (A_1B_1, A_2B_2, \ldots, A_pB'_p, A_{p+1}, \ldots, A_h, B_1B_2 \cdots B_p) \).
3. \( I_3 = (B_1B_2, B_2B_3, B_3B_4, \ldots, A_h) \).
4. \( I_4 = (A_1B_1, A_2B_3, B_2B_4, \ldots, A_h) \).
5. \( I_5 = (A_1B_1, A_2B_2, B_2B_3, A_4, \ldots, A_h) \).
6. \( I_6 = (A_1B_1, A_2B_2, A_2B_3, B_3B_4, A_4, \ldots, A_h) \).

Let \( I = I_1 \). Since \( A_1, A_2, \ldots, A_p, A_{p+1}, \ldots, A_h, B_1, B_2, \ldots, B_p \) are pairwise relatively prime, it turns out that \( A_{p+1}, \ldots, A_h \) is a regular sequence on \( S/(A_1B_1, A_2B_2, \ldots, A_pB_p, B_1B_2 \cdots B_p) \). So, in view of [R Theorem 2.1], we may and do assume that \( I = (A_1B_1, A_2B_2, \ldots, A_pB_p, B_1B_2 \cdots B_p) \). Next, we are going to show that \( I \) is of forest type. Let \( G \) be a subset of \( \{A_1B_1, A_2B_2, \ldots, A_pB_p, B_1B_2 \cdots B_p\} \) with at least two elements. If \( B_1B_2 \cdots B_p \not\in G \), then any \( a \in G \) can be taken as a leaf and any \( b \in G \) different from \( a \) can be taken as a branch for this leaf. If \( B_1B_2 \cdots B_p \in G \), then any \( a \in G \) different from \( B_1B_2 \cdots B_p \) can be taken as a leaf and then \( B_1B_2 \cdots B_p \) is a branch for this leaf. So, \( I \) is of forest type. Thus, since \( I \) is square-free, by [SZ Theorem 1.5], we obtain that \( S/I \) is clean. By the similar argument, one can see that if \( I = I_2 \), then \( S/I \) is clean. Set

\[
J := (C_1B_1B_2, C_2B_1B_3, C_3B_2B_3, A_4, \ldots, A_{h+1}),
\]

where \( C_i \) is either \( A_i \) or 1 for each \( i = 1, 2, 3 \). Since each of \( I_3, I_4, I_5 \) and \( I_6 \) are the particular cases of the ideal \( J \), we can finish the proof by showing that \( S/J \) is clean. Since, by the assumption \( A_4, \ldots, A_{h+1}, B_1, B_2, B_3, C_1, C_2, C_3 \) are pairwise relatively prime, it follows that \( A_4, \ldots, A_{h+1} \) is a regular sequence on \( S/(C_1B_1B_2, C_2B_1B_3, C_3B_2B_3) \). So by [R Theorem 2.1], we can assume that \( J = (C_1B_1B_2, C_2B_1B_3, C_3B_2B_3) \). Set \( T := k[u, v, w, x, y, z] \) and \( L := (uxy, vxz, wyz) \). Since \( B_1, B_2, B_3, C_1, C_2, C_3 \) is a regular sequence on \( S \), by [HSY Proposition 3.3], the cleanness of \( T/L \) implies the cleanness of \( S/J \). So, by Lemma 2.4, it is enough to prove that \( L' \) has linear quotients. As

\[
L' = (x, y) \cap (x, z) \cap (x, u) \cap (y, z) \cap (y, v) \cap (z, u) \cap (u, v, w),
\]

one has \( L' = (xy, xz, xw, yz, yv, zu, uvw) \), which clearly has linear quotients by the given order. 

Let \( u = \prod_{i=1}^ax_i^{a_i} \) be a monomial in \( S = K[x_1, \ldots, x_n] \). Then

\[
w^p := \prod_{i=1}^n \prod_{j=1}^{a_i} x_{i,j} \in K[x_{1,1}, \ldots, x_{1,a_1}, \ldots, x_{n,1}, \ldots, x_{n,a_n}].
\]
Clearly, any page 53 it turns out that $S/I$ is known that $\Delta$ is shellable; see e.g. [TY, Proposition 1.11 and Theorem 1.5]. Hence, by [D, Theorem 2.8.]

Theorem 3.10 implies that $S/I$ is pretty clean if and only if $T/IP$ is clean. Recently, Cimpoeaş [C1] proved that if $S/J$ is an almost complete intersection monomial ideal of $S$, then Stanley’s conjecture holds for $S/J$. The next result shows that in this case $S/J$ is even pretty clean.

**Theorem 2.5.** Let $I$ be an almost complete intersection monomial ideal of $S$. Then $S/I$ is pretty clean.

**Proof.** From [E Proposition 2.3], one has $\text{ht} I = \text{ht} IP$. On the other hand $\mu(I) = \mu(IP)$, and so $IP$ is an almost complete intersection square-free monomial ideal of $T$. Hence, by Lemma 2.4 the ring $T/IP$ is clean. Now, [S3 Theorem 3.10] implies that $S/I$ is pretty clean, as desired. □

In the situation of Theorem 2.5 there is no need that $S/I$ is clean. For instance, although $(x^2, xy)$ is an almost complete intersection monomial ideal, the ring $k[x, y]/(x^2, xy)$ is not clean.

In [C2 Theorem 2.3], it is shown that if $I$ is a monomial ideal of $S$ with $\mu(I) \leq 3$, then Stanley’s conjecture holds for $S/I$. The next result extends this fact.

**Corollary 2.6.** Let $I$ be a monomial ideal of $S$. If $\mu(I) \leq 3$, then $S/I$ is pretty clean.

**Proof.** Clearly, we may assume that $I$ is non zero. Assume that $\mu(I) = 3$ and $\text{ht} I = 1$. Then $I = uJ$, where $u$ is a monomial in $S$ and $J$ is a monomial ideal of $S$ with $\mu(J) = 3$ and $\text{ht} J \geq 2$. By Lemma 2.2 it is enough to prove that $S/J$ is pretty clean. If $\text{ht} J = 2$, then $\mu(J) = \text{ht} J + 1$, and so by Theorem 2.5 $S/J$ is pretty clean. If $\text{ht} J = 3$, then $J$ is complete intersection, and hence by [HSY] Proposition 2.2, $S/J$ is pretty clean.

Since $0 < \text{ht} I \leq \mu(I)$, in all other cases, it follows that $I$ is either complete intersection or almost complete intersection. Thus, the proof is completed by [HSY Proposition 2.2] and Theorem 2.5 □

**Definition 2.7.** ([TY Definition 1.1 and Lemma 1.2]) A simplicial complex $\Delta$ on $[n]$ is said to be *locally complete intersection* if $\{\{1\}, \{2\}, \ldots, \{n\}\} \subseteq \Delta$ and $(I_\Delta)_p$ is a complete intersection ideal of $S_p$ for all $p \in \text{Proj} S/I$.

A simplicial complex $\Delta$ is said to be *connected* if for any two facets $F$ and $G$ of $\Delta$, there exists a sequence of facets $F = F_0, F_1, \ldots, F_{q-1}, F_q = G$ such that $F_i \cap F_{i+1} \neq \emptyset$ for all $0 \leq i < q$. Also, a simplicial complex $\Delta$ on $[n]$ is said to be *$n$-pointed path* (resp. *$n$-gon*) if $n \geq 2$ (resp. $n \geq 3$) and, after a suitable change of variables,

$$\mathcal{F}(\Delta) = \{\{i, i+1\}| 1 \leq i < n\}$$

(resp.

$$\mathcal{F}(\Delta) = \{\{i, i+1\}| 1 \leq i < n\} \cup \{\{n, 1\}\}.$$

Clearly, any $n$-pointed path (resp. $n$-gon) is one-dimensional and pure.

Let $\Delta$ be a connected simplicial complex on $[n]$ which is locally complete intersection. Then, it is known that $\Delta$ is shellable; see e.g. [TY Proposition 1.11 and Theorem 1.5]. Hence, by [D] Theorem on page 53 it turns out that $S/I_\Delta$ is clean. So, we record the following:

**Lemma 2.8.** Let $\Delta$ be a connected simplicial complex on $[n]$ which is locally complete intersection. Then $S/I_\Delta$ is clean.
Let $\Delta$ be as in Lemma 2.8. Then $S/I_\Delta$ is clean, and so \cite{HP} Theorem 6.5 implies that $S/I_\Delta$ satisfies Stanley’s conjecture. In Theorem 2.11 we prove that the later assertion holds without assuming that $\Delta$ is connected.

**Proposition 2.9.** Let $I \subset S_1 = K[x_1, \ldots, x_m]$, $J \subset S_2 = K[x_{m+1}, \ldots, x_n]$ be two monomial ideals and $S = K[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]$. Assume that $\text{depth}(S) > 0$ and $\text{depth}(S_1/J) > 0$. Then Stanley’s conjecture holds for $S/(I, J, \{x_i x_j\}_{1 \leq i \leq m, m+1 \leq j \leq n})$.

**Proof.** For convenience, we set $Q_1 := (x_1, \ldots, x_m)$, $Q_2 := (x_{m+1}, \ldots, x_n)$ and $Q := (x_i x_j)_{1 \leq i \leq m, m+1 \leq j \leq n}$. So, $Q = Q_1 \cap Q_2$.

By the assumption, we have $Q_1 \notin \text{Ass}_{S_1} S/I$ and $Q_2 \notin \text{Ass}_{S_2} S/J$. Hence

$$(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \notin \text{Ass}_{S} S/(I, Q_2)$$

and

$$(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) \notin \text{Ass}_{S} S/(J, Q_1),$$

and so

$$\text{depth}(\frac{S}{(J, Q_1) \cap (I, Q_2)} = \text{depth}(\frac{S}{Q_1 + Q_2}) > 0 \text{ depth}(\frac{S}{Q_1 + Q_2}).$$

Now, in view of the exact sequence

$$0 \rightarrow \frac{S}{(J, Q_1) \cap (I, Q_2)} \rightarrow \frac{S}{(J, Q_1)} \oplus \frac{S}{(I, Q_2)} \rightarrow \frac{S}{Q_1 + Q_2} \rightarrow 0,$$

\cite{V} Lemma 1.3.9] implies that

$$\text{depth}(\frac{S}{(J, Q_1)}) = \text{depth}(\frac{S}{(J, Q_1) \cap (I, Q_2)}) = 1.$$

Now the proof is complete, because \cite{C2} Theorem 2.1 yields that for any monomial ideals $L$ of $S$ if $\text{depth}(S/L) \leq 1$, then Stanley’s conjecture holds for $S/L$. 

**Corollary 2.10.** Let $\Delta_1$ and $\Delta_2$ be two non-empty disjoint simplicial complexes and $\Delta := \Delta_1 \cup \Delta_2$. Then Stanley’s conjecture holds for $S/I_\Delta$.

**Proof.** For two natural integers $m < n$, we may assume that $\Delta_1$ and $\Delta_2$ are simplicial complexes on $[m]$ and $\{m+1, \ldots, n\}$, respectively. Then $K[\Delta_1] = K[x_1, \ldots, x_m]/I_{\Delta_1}$ and $K[\Delta_2] = K[x_{m+1}, \ldots, x_n]/I_{\Delta_2}$, and so

$$K[\Delta] = K[x_1, \ldots, x_m, x_{m+1}, \ldots, x_n]/(I_{\Delta_1}, I_{\Delta_2}, \{x_i x_j\}_{1 \leq i \leq m, m+1 \leq j \leq n}).$$

We claim that $\text{depth}(K[x_1, \ldots, x_m]/I_{\Delta_1}) > 0$ and $\text{depth}(K[x_{m+1}, \ldots, x_n]/I_{\Delta_2}) > 0$. Because if for example $\text{depth}(K[x_1, \ldots, x_m]/I_{\Delta_1}) = 0$, then $I_{\Delta_1} = (x_1, \ldots, x_m)$. But, this implies that $\Delta_1 = \emptyset$ which contradicts our assumption on $\Delta_1$. Now, the claim is immediate by Proposition 2.8. 

**Theorem 2.11.** Let $\Delta$ be a locally complete intersection simplicial complex on $[n]$. Then Stanley’s conjecture holds for $S/I_\Delta$. 

Proof. If $\Delta$ is connected, then Lemma 2.8 yields the claim. Otherwise, by [TY, Theorem 1.15], $\Delta$ is the disjoint union of finitely many non-empty simplicial complexes. So, in this case the assertion follows by Corollary 2.10. □

In [HP, Corollary 4.3] it is shown that if $S/I$ is pretty clean, then it is sequentially Cohen-Macaulay. In [S1] this fact is reproved by a different argument and, in addition, it is shown that depth of $S/I$ is equal to the minimum of the dimension of $S/p$, where $p \in \text{Ass} S/I$. Also if $S/I$ is pretty clean, then by [S2, Theorem 4.7] $h$-regularity conjecture holds for $S/I$. This implies part a) of the following remark.

**Remark 2.12.** Let $I$ be a monomial ideal of $S$.

a) Assume that either:
   i) $I$ is almost complete intersection,
   ii) $\mu(I) \leq 3$; or
   iii) $I$ is the Stanley-Reisner ideal of a connected simplicial complex on $[n]$ which is locally complete intersection.

Then both Stanley’s and $h$-regularity conjectures hold for $S/I$. Also, in each of these cases $S/I$ is sequentially Cohen-Macaulay and $\text{depth } S/I = \min \{ \dim S/p | p \in \text{Ass } S/I \}$.

b) We know that if $S/I$ is pretty clean, then Stanley’s conjecture holds for $S/I$. By using Corollary 2.10, we can provide an example of a monomial ideal $I$ of $S$ such that Stanley’s conjecture holds for $S/I$, while it is not pretty clean. To this end, let $\Delta_1$, $\Delta_2$ and $\Delta$ be as in Corollary 2.10 and $\dim \Delta_i > 0$, $i = 1, 2$. Evidently, $\Delta$ is not shellable, and so [D, Theorem on page 53] implies that $S/I_\Delta$ is not pretty clean. On the other hand, Stanley’s conjecture holds for $S/I_\Delta$ by Corollary 2.10.

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