It is known that a graded lattice of rank $n$ is supersolvable if and only if it has an EL-labeling where the labels along any maximal chain are exactly the numbers $1, 2, \ldots, n$ without repetition. These labelings are called $S_n$ EL-labelings, and having such a labeling is also equivalent to possessing a maximal chain of left modular elements. In the case of an ungraded lattice, there is a natural extension of $S_n$ EL-labelings, called interpolating labelings. We show that admitting an interpolating labeling is again equivalent to possessing a maximal chain of left modular elements. Furthermore, we work in the setting of an arbitrary bounded poset as all the above results generalize to this case. We conclude by applying our results to show that the lattice of non-straddling partitions, which is not graded in general, has a maximal chain of left modular elements.

1. Introduction

An edge-labeling of a poset $P$ is a map from the edges of the Hasse diagram of $P$ to $\mathbb{Z}$. Our primary goal is to express certain classical properties of $P$ in terms of edge-labelings admitted by $P$. The idea of studying edge-labelings of posets goes back to [12]. An important milestone was [3], where A. Björner defined EL-labelings, and showed that if a poset admits an EL-labeling, then it is shellable and hence Cohen-Macaulay. We will be interested in a subclass of EL-labelings, known as $S_n$ EL-labelings. In [13], R. Stanley introduced supersolvable lattices and showed that they admit $S_n$ EL-labelings. Examples of supersolvable lattices include distributive lattices, the lattice of partitions of $[n]$, the lattice of non-crossing partitions of $[n]$ and the lattice of subgroups of a supersolvable group (hence the terminology). It was shown in [9] that a finite graded lattice of rank $n$ is supersolvable if and only if it admits an $S_n$ EL-labeling. In many ways, this characterization of lattice supersolvability in terms of edge-labelings serves as the starting point for our investigations.

For basic definitions concerning partially ordered sets, see [14]. We will say that a poset $P$ is bounded if it contains a unique minimal element and a unique maximal element, denoted 0 and 1 respectively. All the posets we will consider will be finite and bounded. A chain of a poset $P$ is said to be maximal if it is maximal under inclusion. We say that $P$ is graded if all the maximal chains of $P$ have the same length, and we call this length the rank of $P$. We will write $x \prec y$ if $y$ covers $x$ in $P$ and $x \preceq y$ if $y$ either covers or equals $x$. The edge-labeling $\gamma$ of $P$ is said to be an EL-labeling if for any $y < z$ in $P$,

(i) there is a unique unrefinable chain $y = w_0 \preceq w_1 \preceq \cdots \preceq w_r = z$ such that $\gamma(w_0, w_1) \leq \gamma(w_1, w_2) \leq \cdots \leq \gamma(w_{r-1}, w_r)$, and
Figure 1

(ii) the sequence of labels of this chain (referred to as the increasing chain from $y$ to $z$), when read from bottom to top, lexicographically precedes the labels of any other unrefinable chain from $y$ to $z$.

This concept originates in [3]; for the case where $P$ is not graded, see [4, 5]. If $P$ is graded of rank $n$ with an EL-labelling $\gamma$, then $\gamma$ is said to be an $S_n$ EL-labelling if the labels along any maximal chain of $P$ are all distinct and are elements of $[n]$. In other words, for every maximal chain $0 = w_0 \lessdot w_1 \lessdot \cdots \lessdot w_n = 1$ of $P$, the map sending $i$ to $\gamma(w_{i-1}, w_i)$ is a permutation of $[n]$. Note that the second condition in the definition of an EL-labelling is redundant in this case.

Example 1.1. Any finite distributive lattice has an $S_n$ EL-labelling. Let $L$ be a finite distributive lattice of rank $n$. By the Fundamental Theorem of Finite Distributive Lattices [2, p. 59, Thm. 3], that is equivalent to saying that $L = J(Q)$, the lattice of order ideals of some $n$-element poset $Q$. Let $\omega : Q \to [n]$ be a linear extension of $Q$, i.e., any bijection labelling the vertices of $Q$ that is order-preserving (if $a < b$ in $Q$ then $\omega(a) < \omega(b)$). This labelling of the vertices of $Q$ defines a labelling of the edges of $J(Q)$ as follows. If $y$ covers $x$ in $J(Q)$, then the order ideal corresponding to $y$ is obtained from the order ideal corresponding to $x$ by adding a single element, labeled by $i$, say. Then we set $\gamma(x, y) = i$. This gives us an $S_n$ EL-labelling for $L = J(Q)$. Figure 1 shows a labelled poset and its lattice of order ideals with the appropriate edge-labelling.

A finite lattice $L$ is said to be supersolvable if it contains a maximal chain, called an $M$-chain of $L$, which together with any other chain in $L$ generates a distributive sublattice. We can label each such distributive sublattice by the method described in Example 1.1 in such a way that the $M$-chain is the unique increasing maximal chain. As shown in [13], this will assign a unique label to each edge of $L$ and the resulting global labelling of $L$ is an $S_n$ EL-labelling.

There is also a characterization of lattice supersolvability in terms of left modularity. Given an element $x$ of a finite lattice $L$, and a pair of elements $y \leq z$, it is always true that

$$(x \lor y) \land z \geq (x \land z) \lor y. \quad (1)$$

The element $x$ is said to be left modular if, for all $y \leq z$, equality holds in (1). Following A. Blass and B. Sagan [6], we will say that a lattice itself is left modular if it contains a left modular maximal chain, that is, a maximal chain each of whose
elements is left modular. (One might guess that we should define a lattice to be left modular if all of its elements are left modular, but this is equivalent to the definition of a modular lattice.) As shown in [13], any M-chain of a supersolvable lattice is always a left modular maximal chain, and so supersolvable lattices are left modular. Furthermore, it is shown by L. S.-C. Liu [7] that if \( L \) is a finite graded lattice with a left modular maximal chain \( M \), then \( L \) has an \( S_n \) EL-labeling with increasing maximal chain \( M \). In turn, as shown in [9], this implies that \( L \) is supersolvable, and we conclude the following.

**Theorem 1.** Let \( L \) be a finite graded lattice of rank \( n \). Then the following are equivalent:

1. \( L \) has an \( S_n \) EL-labeling,
2. \( L \) is left modular,
3. \( L \) is supersolvable.

It is shown in [13] that if \( L \) is upper-semimodular, then \( L \) is left modular if and only if \( L \) is supersolvable. Theorem 1 is a considerable strengthening of this. Here we used \( S_n \) EL-labelings to connect left modularity and supersolvability. It is natural to ask for a more direct proof that (2) implies (3); such a proof has recently been provided by the second author in [15].

Our goal is to generalize Theorem 1 to the case when \( L \) is not graded and, moreover, to the case when \( L \) is not necessarily a lattice. We now wish to define natural generalizations of \( S_n \) EL-labelings and of maximal left modular chains.

**Definition 1.2.** An EL-labeling \( \gamma \) of a poset \( P \) is said to be interpolating if, for any \( y \preceq u \preceq z \), either

(i) \( \gamma(y, u) < \gamma(u, z) \) or

(ii) the increasing chain from \( y \) to \( z \), say \( y = w_0 \prec w_1 \prec \cdots \prec w_r = z \), has the properties that its labels are strictly increasing and that \( \gamma(w_0, w_1) = \gamma(u, z) \) and \( \gamma(w_{r-1}, w_r) = \gamma(y, u) \).

**Example 1.3.** The reader is invited to check that the labeling of the non-graded poset shown in Figure 2 is an interpolating EL-labeling. In fact, the poset shown is the so-called “Tamari lattice” \( T_4 \). For all positive integers \( n \), there exists a Tamari lattice \( T_n \) with \( C_n = \frac{1}{n+1} \binom{2n}{n} \) elements, where \( C_n \) is the \( n \)th Catalan number. More information on the Tamari lattice can be found in [5, §9], [6, §7] and the references given there, and in [7, §3.2], where this interpolating EL-labeling appears. The Tamari lattice is shown to have an EL-labeling in [5] and is shown to be left modular in [6].

If \( P \) is graded of rank \( n \) and has an interpolating labelling \( \gamma \) in which the labels on the increasing maximal chain reading from bottom to top are \( 1, 2, \ldots, n \), then we can check (cf. Lemma 3.2) that \( \gamma \) is an \( S_n \) EL-labeling.

Our next step is to define left modularity in the non-lattice case. Let \( x \) and \( y \) be elements of \( P \). We know that \( x \) and \( y \) have at least one common upper bound, namely \( 1 \). If the set of common upper bounds of \( x \) and \( y \) has a least element, then we denote it by \( x \vee y \). Similarly, if \( x \) and \( y \) have a greatest common lower bound, then we denote it by \( x \wedge y \).

Now let \( w \) and \( z \) be elements of \( P \) with \( w, z \geq y \). Consider the set of common lower bounds for \( w \) and \( z \) that are also greater than or equal to \( y \). Clearly, \( y \) is in this set. If this set has a greatest element, then we denote it by \( w \wedge_y z \) and we say
that $w \land_y z$ is well-defined (in $[y, \hat{1}]$). We see that $(x \lor y) \land_y z$ is well-defined in the poset shown in Figure 3, even though $(x \lor y) \land z$ is not. Similarly, let $w$ and $y$ be elements of $P$ with $w, y \leq z$. If the set $\{u \in P \mid u \geq w, y \text{ and } u \leq z\}$ has a least element, then we denote it by $w \lor^y z$ and we say that $w \lor^y z$ is well-defined (in $[0, z]$). We will usually be interested in expressions of the form $(x \lor y) \land_y z$ and $(x \lor z) \lor^y y$. The reader that is solely interested in the lattice case can choose to ignore the subscripts and superscripts on the meet and join symbols.

**Definition 1.4.** An element $x$ of a poset $P$ is said to be **viable** if, for all $y \leq z$ in $P$, $(x \lor y) \land_y z$ and $(x \land z) \lor^y y$ are well-defined. A maximal chain of $P$ is said to be viable if each of its elements is viable.

**Example 1.5.** The poset shown in Figure 3 is certainly not a lattice but the reader can check that the increasing maximal chain is viable.

**Definition 1.6.** A viable element $x$ of a poset $P$ is said to be **left modular** if, for all $y \leq z$ in $P$, $(x \lor y) \land_y z = (x \land z) \lor^y y$.

A maximal chain of $P$ is said to be left modular if each of its elements is viable and left modular, and $P$ is said to be left modular if it possesses a left modular maximal chain.

This brings us to the first of our main theorems.

**Theorem 2.** Let $P$ be a bounded poset with a left modular maximal chain $M$. Then $P$ has an interpolating EL-labelling with $M$ as its increasing maximal chain.

The proof of this theorem will be the content of the next section. In Section 3, we will prove the following converse result.

**Theorem 3.** Let $P$ be a bounded poset with an interpolating EL-labelling. The unique increasing chain from $\hat{0}$ to $1$ is a left modular maximal chain.
These two theorems, when compared with Theorem 1, might lead one to ask about possible supersolvability results for bounded posets that aren’t graded lattices. This problem is discussed in Section 4. In the case of graded posets, we obtain a satisfactory result, namely Theorem 4. As a consequence, we have given an answer to the question of when a graded poset $P$ has an $S_n$ EL-labelling. This has ramifications on the existence of a “good 0-Hecke algebra action” on the maximal chains of the poset, as discussed in [9]. However, it remains an open problem to appropriately extend the definition of supersolvability to ungraded posets.

An explicit application of Theorem 3 is the subject of Section 5. As a variation on non-crossing partitions and non-nesting partitions, we define non-straddling partitions. Ordering the set of non-straddling partitions of $[n]$ by refinement gives a poset, denoted $NS_n$, that is generally a non-graded lattice. We define an edge-labelling $\gamma$ for $NS_n$ that is analogous to the usual EL-labelling for the lattice of partitions of $[n]$. In order to show that $NS_n$ is left modular, we then prove that $\gamma$ is an interpolating EL-labelling.

2. Proof of Theorem 2

Throughout this section, we suppose that $P$ is a bounded poset with a left modular maximal chain $M : 0 = x_0 \prec x_1 \prec \cdots \prec x_n = 1$. We want to show that $P$ has an interpolating EL-labelling. Our approach will be as follows: we will begin by specifying an edge-labelling $\gamma$ for $P$ such that $M$ is an increasing chain with respect to $\gamma$. We will then prove a series of lemmas which build on the viability and left modularity properties. These culminate with Proposition 2.6 which, roughly speaking, gives a more local definition for $\gamma$. We will then be ready to show that $\gamma$ is an EL-labelling and is, furthermore, an interpolating EL-labelling.

We choose a label set $l_1 < \cdots < l_n$ of natural numbers. (For most purposes, we can let $l_i = i$.) We define an edge-labelling $\gamma$ on $P$ by setting $\gamma(y, z) = l_i$ for $y \prec z$ if

$$(x_{i-1} \lor y) \land_y z = y \text{ and } (x_i \lor y) \land_y z = z.$$ 

It is easy to see that $\gamma$ is well-defined. We will refer to it as the labelling induced by $M$ and the label set $\{l_i\}$. When $P$ is a lattice, this labelling appears, for example, in [7, 16]. As in [7], we can give an equivalent definition of $\gamma$ as follows.
Lemma 2.1. Suppose \( y < z \) in \( P \). Then \( \gamma(y, z) = l_i \) if and only if

\[
i = \min\{j \mid x_j \lor y \geq z\} = \max\{j + 1 \mid x_j \land z \leq y\}.
\]

Proof. That \( i = \min\{j \mid x_j \lor y \geq z\} \) is immediate from the definition of \( \gamma \). By left modularity, \( \gamma(y, z) = l_i \) if and only if \( (x_{i-1} \land z) \lor z = y \) and \( (x_i \land z) \lor z = z \). In other words, \( x_{i-1} \land z \leq y \) and \( x_i \land z \ngeq y \). It follows that \( i = \max\{j + 1 \mid x_j \land z \leq y\} \).

Lemma 2.2. Suppose that \( y \leq w \leq z \) in \( P \) and let \( x \in M \). Then \( (x \land z) \lor w \) is well-defined and equals \( (x \lor y) \land y \lor w \). Similarly, \( (x \lor y) \land y \) is well-defined and equals \( (x \land z) \lor z \).

Proof. It is routine to check that, in \( [\hat{0}, z] \), \( (x \land z) \lor w \) is the least common upper bound for \( w \) and \( (x \land z) \lor z \), and that, in \( [y, 1] \), \( (x \lor y) \land y \) is the greatest common lower bound lower bound for \( x \lor y \) and \( z \).

Lemma 2.3. Suppose that \( t \leq u \) in \( [y, z] \) and \( x \in M \). Let \( w = (x \lor y) \land y \lor z = (x \land z) \lor z \in [y, z] \). Then \( (w \lor t) \land u \lor (w \land u) \lor t \) are well-defined elements of \([t, u]\) and are equal.

Proof. We see that, by Lemma 2.2,

\[
(x \lor t) \land u = ((x \lor t) \land t) \land u = ((x \lor z) \lor t) \land u = (((x \lor z) \lor z) \lor u) \land u.
\]

Similarly,

\[
(x \land u) \lor t = (w \land t) \lor u
\]

But \( (x \lor t) \land u = (x \land u) \lor u \), yielding the result.

Lemma 2.4. Suppose \( x \) and \( w \) are viable and that \( x \) is left modular in \( P \).

(a) If \( x \leq w \) then for any \( z \) in \( P \) we have \( x \land z \leq w \land z \).

(b) If \( w \leq x \) then for any \( y \) in \( P \) we have \( w \lor y \leq x \lor y \).

Part (b) appears in the lattice case in [7, Lemma 2.5.6] and [8, Lemma 5.3].

Proof. We prove (a); (b) is similar. Assume, seeking a contradiction, that \( x \land z < u \leq w \land z \) for some \( u \in P \). Now \( u \leq z \) and \( u \leq w \). It follows that \( u \leq x \).

Now \( x < x \lor u \leq w \). Therefore, \( w = x \lor u \). So

\[
u = (x \land z) \lor z = (x \lor u) \land u \leq w \land z,
\]

which is a contradiction.

We now prove a slight extension of [7, Lemma 2.5.7] and [8, Lemma 5.4].

Lemma 2.5. The elements of \([y, z]\) of the form \((x_i \lor y) \land y \land z\) form a left modular maximal chain in \([y, z]\).

Proof. Lemma 2.3 gives the viability and left modularity properties. By Lemma 2.4(b), \( x_i \lor y \leq x_i+1 \lor y \). By Lemma 2.3 with \( z = 1 \), we have that \( x_i \lor y \) is left modular in \([y, 1]\). Therefore, \( (x_i \lor y) \land y \land z \leq (x_i+1 \lor y) \land y \land z \) by Lemma 2.4(a).

We are now ready for the last, and most important, of our preliminary results. Let \([y, z]\) be an interval in \( P \). We call the maximal chain of \([y, z]\) from Lemma 2.5 the \textit{induced} left modular maximal chain of \([y, z]\). One way to get a second edge-labelling for \([y, z]\) would be to take the labelling induced in \([y, z]\) by this induced maximal chain. We now prove that, for a suitable choice of label set, this labelling coincides with \( \gamma \).
Proposition 2.6. Let $P$ be a bounded poset, $0 = x_0 \prec x_1 \prec \cdots \prec x_n = 1$ a left modular maximal chain and $\gamma$ the corresponding edge-labelling with label set $\{l_i\}$. Let $y < z$, and define $c_i$ by saying

$$
y = (x_0 \lor y) \land y z = \cdots = (x_{c_i-1} \lor y) \land y z \prec (x_{c_i} \lor y) \land y z = \cdots = (x_{c_{i-1}} \lor y) \land y z \prec \cdots \prec (x_{c_r} \lor y) \land y z.
$$

Let $m_i = l_{c_i}$. Let $\delta$ be the labelling of $[y, z]$ induced by its induced left modular maximal chain and the label set $\{m_i\}$. Then $\delta$ agrees with $\gamma$ restricted to the edges of $[y, z]$.

Proof. Suppose $t \prec u$ in $[y, z]$. Using ideas from the proof of Lemma 2.3,

$$
\delta(t, u) = m_i \iff \left( ((x_{c_i-1} \lor y) \land y z) \lor^z t \land t u = t \land ((x_{c_i} \lor y) \land y z) \lor^z t \land t u = u \iff (x_{c_i-1} \lor t) \land t u = t \land (x_{c_i} \lor t) \land t u = u \iff \gamma(t, u) = l_{c_i}.
$$

Proof of Theorem 2. We now know that the induced left modular chain in $[y, z]$ has (strictly) increasing labels, say $m_1 < m_2 < \cdots < m_r$. Our first step is to show that it is the only maximal chain with (weakly) increasing labels. Suppose that $y = w_0 \prec w_1 \prec \cdots \prec w_r = z$ is the induced chain and that $y = u_0 \prec u_1 \cdots \prec u_s = z$ is another chain with increasing labels.

If $s = 1$ then $y < z$ and the result is clear. Suppose $s \geq 2$. By Proposition 2.6, we may assume that the labelling on $[y, z]$ is induced by the induced left modular chain $\{w_i\}$. In particular, we have that $\gamma(u_i, u_{i+1}) = m_l$ where $l = \min\{j \mid w_j \lor^z u_i \geq u_{i+1}\}$. Let $k$ be the least number such that $u_k \geq w_1$. Then it is clear that $\gamma(u_k, w_1) = m_1$. Note that this is the smallest label that can occur on any edge in $[y, z]$. Since the labels on the chain $\{u_i\}$ are assumed to be increasing, we must have $\gamma(u_0, u_1) = m_1$. It follows that $w_1 \lor^z u_0 \geq u_1$ and since $y \prec u_1$, we must have $u_1 = w_1$. Thus, by induction, the two chains coincide. We conclude that the induced left modular maximal chain is the only chain in $[y, z]$ with increasing labels.

It also has the lexicographically least set of labels. To see this, suppose that $y = u_0 \prec u_1 \prec \cdots \prec u_s = z$ is another chain in $[y, z]$. We assume that $u_1 \neq w_1$ since, otherwise, we can just restrict our attention to $[u_1, z]$. We have $\gamma(u_0, u_1) = m_l$, where $l = \min\{j \mid w_j \geq u_1\} \geq 2$ since $w_1 \neq u_1$. Hence $\gamma(u_0, u_1) \geq m_2 > \gamma(w_0, w_1)$. This gives that $\gamma$ is an EL-labelling. (That $\gamma$ is an EL-labelling was already shown in the lattice case in [7, 16].)

Finally, we show that it is an interpolating EL-labelling. If $y \prec u \prec z$ is not the induced left modular maximal chain in $[y, z]$, then let $y = w_0 \prec w_1 \prec \cdots \prec w_r = z$ be the induced left modular maximal chain. We have that $\gamma(y, u) = m_l$ where

$$
l = \min\{j \mid w_j \lor^z y \geq u\} = \min\{j \mid w_j \geq u\} = r
$$
since $u < z$. Therefore, $\gamma(y, u) = m_r$. Also, $\gamma(u, z) = m_l$ where

$$
l = \max\{j + 1 \mid w_j \land y z \leq u\} = \max\{j + 1 \mid w_j \leq u\} = 1
$$
since $y \prec u$. Therefore, $\gamma(y, u) = m_1$, as required.
3. Proof of Theorem 3

We suppose that \( P \) is a bounded poset with an interpolating EL-labelling \( \gamma \). Let \( \hat{0} = x_0 \prec x_1 \prec \cdots \prec x_n = \hat{1} \) be the increasing chain from \( \hat{0} \) to \( \hat{1} \) and let \( x_i = \gamma(x_{i-1}, x_i) \).

We will begin by establishing some basic facts about interpolating labellings. These results will enable us to show certain meets and joins exist by looking at the labels that appear along particular increasing chains. We will thus show that the \( x_i \) are viable. We will finish by showing that the \( x_i \) are left modular, again by looking at the labels on increasing chains.

Let \( y = w_0 \prec w_1 \prec \cdots \prec w_r = z \). Suppose that, for some \( i \), we have \( \gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1}) \). Then the “basic replacement” at \( i \) takes the given chain and replaces the subchain \( w_{i-1} \prec w_i \prec w_{i+1} \) by the increasing chain from \( w_{i-1} \) to \( w_{i+1} \). The basic tool for dealing with interpolating labellings is the following well-known fact about EL-labellings.

**Lemma 3.1.** Let \( y = w_0 \prec w_1 \prec \cdots \prec w_r = z \). Successively perform basic replacements on this chain, and stop when no more basic replacements can be made. This algorithm terminates, and yields the increasing chain from \( y \) to \( z \).

**Proof.** At each step, the sequence of labels on the new chain lexicographically precedes the sequence on the old chain, so the process must terminate, and it is clear that it terminates in an increasing chain.

We now prove some simple consequences of this lemma.

**Lemma 3.2.** Let \( m \) be the chain \( y = w_0 \prec w_1 \prec \cdots \prec w_r = z \). Then the labels on \( m \) all occur on the increasing chain from \( y \) to \( z \) and are all different. Furthermore, all the labels on the increasing chain from \( y \) to \( z \) are bounded between the lowest and highest labels on \( m \).

**Proof.** That the labels on the given chain all occur on the increasing chain follows immediately from Lemma 3.1 and the fact that after a basic replacement, the labels on the old chain all occur on the new chain. Similar reasoning implies that the labels on the increasing chain are bounded between the lowest and highest labels on \( m \).

That the labels are all different again follows from Lemma 3.1. Suppose otherwise. By repeated basic replacements, one obtains a chain which has two successive equal labels, which is not permitted by the definition of an interpolating labelling.

**Lemma 3.3.** Let \( z \in P \) such that there is some chain from \( \hat{0} \) to \( z \) all of whose labels are in \( \{l_1, \ldots, l_i\} \). Then \( z \preceq x_i \). Conversely, if \( z \preceq x_i \), then all the labels on any chain from \( \hat{0} \) to \( z \) are in \( \{l_1, \ldots, l_i\} \).

**Proof.** We begin by proving the first statement. By Lemma 3.2, the labels on the increasing chain from \( \hat{0} \) to \( z \) are in \( \{l_1, \ldots, l_i\} \). Find the increasing chain from \( x_i \) to \( \hat{0} \). Let \( w \) be the element in that chain such that all the labels below it on the chain are in \( \{l_1, \ldots, l_i\} \), and those above it are in \( \{l_{i+1}, \ldots, l_n\} \). Again, by Lemma 3.2, the increasing chain from \( \hat{0} \) to \( w \) has all its labels in \( \{l_1, \ldots, l_i\} \), and the increasing chain from \( w \) to \( \hat{1} \) has all its labels in \( \{l_{i+1}, \ldots, l_n\} \). Thus \( w \) is on the increasing chain from \( \hat{0} \) to \( \hat{1} \), and so \( w \preceq x_i \). But by construction \( w \preceq z \). So \( x_i \geq z \).

To prove the converse, observe that by Lemma 3.2, no label can occur more than once on any chain. But since every label in \( \{l_{i+1}, \ldots, l_n\} \) occurs on the increasing
chain from \( x_i \) to 1, no label from among that set can occur on any edge below \( x_i \).

The obvious dual of Lemma 3.3 is proved similarly:

**Corollary 3.4.** Let \( z \in P \) such that there is some chain from \( z \) to \( \hat{1} \) all of whose labels are in \( \{l_{i+1}, \ldots, l_n\} \). Then \( z \geq x_i \). Conversely, if \( z \geq x_i \), then all the labels on any chain from \( z \) to \( \hat{1} \) are in \( \{l_{i+1}, \ldots, l_n\} \).

We are now ready to prove the necessary viability properties.

**Lemma 3.5.** \( x_i \lor z \) and \( x_i \land z \) are well-defined for any \( z \in P \) and for \( i = 1, 2, \ldots, n \).

**Proof.** We will prove that \( x_i \land z \) is well-defined. The proof that \( x_i \lor z \) is well-defined is similar. Let \( w \) be the maximum element on the increasing chain from \( \hat{0} \) to \( z \) such that all labels on the increasing chain between \( \hat{0} \) and \( w \) are in \( \{l_1, \ldots, l_i\} \). Clearly \( w \leq z \) and, by Lemma 3.3, \( w \leq x_i \).

Suppose \( y \leq z, x_i \). It follows that all labels from \( \hat{0} \) to \( y \) are in \( \{l_1, \ldots, l_i\} \). Consider the increasing chain from \( y \) to \( z \). There exists an element \( u \) on this chain such that all the labels on the increasing chain from \( \hat{0} \) to \( u \) are in \( \{l_1, \ldots, l_i\} \) and all the labels on the increasing chain from \( u \) to \( z \) are in \( \{l_{i+1}, \ldots, l_n\} \). Therefore, \( u \) is on the increasing chain from \( \hat{0} \) to \( z \) and, in fact, \( u = w \). Also, we have that \( \hat{0} \leq y \leq u = w \leq z \). We conclude that \( w \) is the greatest common lower bound for \( z \) and \( x_i \).

**Lemma 3.6.** \( \hat{0} = x_0 \land z \leq x_1 \land z \leq \cdots \leq x_n \land z = z \), after we delete repeated elements, is the increasing chain in \( [0, z] \). Hence, \((x_i \land z) \lor^z y \) is well-defined for \( y \leq z \). Similarly, \((x_i \lor y) \land_y z \) is well-defined.

**Proof.** From the previous proof, we know that \( x_i \land z \) is the maximum element on the increasing chain from \( \hat{0} \) to \( z \) such that all labels on the increasing chain between \( \hat{0} \) and \( x_i \) are in \( \{l_1, \ldots, l_i\} \). The first assertion follows easily from this.

Now apply Lemma 3.5 to the bounded poset \([0, z] \). It has an obvious interpolating labelling induced from the interpolating labelling of \( P \). Recall that our definition of the existence of \((x_i \land z) \lor^z y \) only requires it to be well-defined in \([0, z] \). The result follows.

We conclude that the increasing maximal chain \( \hat{0} = x_0 \preceq x_1 \preceq \cdots \preceq x_n = \hat{1} \) of \( P \) is viable. It remains to show that it is left modular.

**Proof of Theorem 3.** Suppose that \( x_i \) is not left modular for some \( i \). Then there exists some pair \( y \leq z \) such that \((x_i \lor y) \land_y z \geq (x_i \land z) \lor^z y \). Set \( x = x_i \), \( b = (x_i \land z) \lor^z y \) and \( c = (x_i \lor y) \land_y z \). Observe that \( d := x \lor b \geq c \) while \( a := x \land c \leq b \). So the picture is as shown in Figure 4.

By Lemma 3.3, the labels on the increasing chain from \( \hat{0} \) to \( a \) are less than or equal to \( l_i \). Consider the increasing chain from \( a \) to \( c \). Let \( w \) be the first element along the chain. If \( \gamma(a, w) \leq l_i \), then by Lemma 3.3, \( w \leq x_i \), contradicting the fact that \( a = x \land c \). Thus the labels on the increasing chain from \( a \) to \( c \) are all greater than \( l_i \). Dually, the labels on the increasing chain from \( b \) to \( d \) are less than or equal to \( l_i \). But now, by Lemma 3.2, the labels on the increasing chain from \( b \) to \( c \) must be contained in the labels on the increasing chain from \( a \) to \( c \), and also from \( b \) to \( d \). But there are no such labels, implying a contradiction. We conclude that the \( x_i \) are all left modular.
We have shown that if \( P \) is a bounded poset with an interpolating labelling \( \gamma \), then the unique increasing maximal chain \( M \) is a left modular maximal chain. By Theorem 2, \( M \) then induces an interpolating EL-labelling of \( P \). We now show that this labelling agrees with \( \gamma \) for a suitable choice of label set, which is a special case of the following proposition.

**Proposition 3.7.** Let \( \gamma \) and \( \delta \) be two interpolating EL-labellings of a bounded poset \( P \). If \( \gamma \) and \( \delta \) agree on the \( \gamma \)-increasing chain from \( \hat{0} \) to \( \hat{1} \), then \( \gamma \) and \( \delta \) coincide.

**Proof.** Let \( m : \hat{0} = w_0 < w_1 < \cdots < w_r = \hat{1} \) be the maximal chain with the lexicographically first \( \gamma \) labelling among those chains for which \( \gamma \) and \( \delta \) disagree. Since \( m \) is not the \( \gamma \)-increasing chain from \( \hat{0} \) to \( \hat{1} \), we can find an \( i \) such that \( \gamma(w_{i-1}, w_i) > \gamma(w_i, w_{i+1}) \). Let \( m' \) be the result of the basic replacement at \( i \) with respect to the labelling \( \gamma \). Then the \( \gamma \)-label sequence of \( m' \) lexicographically precedes that of \( m \), so \( \gamma \) and \( \delta \) agree on \( m' \). But using the fact that \( \gamma \) and \( \delta \) are interpolating, it follows that they also agree on \( m \). Thus they agree everywhere. \( \square \)

4. Generalizing Supersolvability

Suppose \( P \) is a bounded poset. For now, we consider the case of \( P \) being graded of rank \( n \). We would like to define what it means for \( P \) to be *supersolvable*, thus generalizing Stanley’s definition of lattice supersolvability. A definition of poset supersolvability with a different purpose appears in [16] but we would like a more general definition. In particular, we would like \( P \) to be supersolvable if and only if \( P \) has an \( S_n \) EL-labelling. For example, the poset shown in Figure 3, while it doesn’t satisfy V. Welker’s definition, should satisfy our definition. We need to define, in the poset case, the equivalent of a sublattice generated by two chains.

Suppose \( P \) has a viable maximal chain \( M \). Thus \( (x \lor y) \land z \) and \( (x \land z) \lor \hat{y} \) are well-defined for \( x \in M \) and \( y \leq \hat{z} \) in \( P \). Given any chain \( c \) of \( P \), we define \( R_M(c) \) to be the smallest subposet of \( P \) satisfying the following two conditions:

(i) \( M \) and \( c \) are contained in \( R_M(c) \),

(ii) If \( y \leq \hat{z} \) in \( P \) and \( y \) and \( z \) are in \( R_M(c) \), then so are \( (x \lor y) \land z \) and \( (x \land z) \lor \hat{y} \) for any \( x \) in \( M \).

**Definition 4.1.** We say that a bounded poset \( P \) is *supersolvable* with \( M \)-chain \( M \) if \( M \) is a viable maximal chain and \( R_M(c) \) is a distributive lattice for any chain \( c \) of \( P \).

Since distributive lattices are graded, it is clear that a poset must be graded in order to be supersolvable. We now come to the main result of this section.
**Theorem 4.** Let $P$ be a bounded graded poset of rank $n$. Then the following are equivalent:

1. $P$ has an $S_n$ EL-labelling,
2. $P$ is left modular,
3. $P$ is supersolvable.

**Proof.** Observe that for a graded poset, Lemma 3.2 implies that an interpolating labelling is an $S_n$ EL-labelling, and the converse is obvious. Thus, Theorems 2 and 3 restricted to the graded case give us that $(1) \iff (2)$.

Our next step is to show that $(1)$ and $(2)$ together imply $(3)$. Suppose $P$ is a bounded graded poset of rank $n$ with an $S_n$ EL-labelling. Let $M$ denote the increasing maximal chain $\emptyset = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of $P$. We also know that $M$ is viable and left modular and induces the same $S_n$ EL-labelling. Given any maximal chain $m$ of $P$, we define $Q_M(m)$ to be the closure of $m$ in $P$ under basic replacements. In other words, $Q_M(m)$ is the smallest subposet of $P$ which contains $M$ and $m$ and which has the property that, if $y$ and $z$ are in $Q_M(m)$ with $y \leq z$, then the increasing chain between $y$ and $z$ is also in $Q_M(m)$. It is shown in [9, Proof of Thm. 1] that $Q_M(m)$ is a distributive lattice. There $P$ is a lattice but the proof of distributivity doesn’t use this fact. Now consider $R_M(c)$. We will show that there exists a maximal chain $m$ of $P$ such that $R_M(c) = Q_M(m)$. Let $m$ be the maximal chain of $P$ which contains $c$ and which has increasing labels between successive elements of $c \cup \{0,1\}$. The only idea we need is that, for $y \leq z$ in $P$, the increasing chain from $y$ to $z$ is given by $y = (x_0 \vee y) \wedge y \leq (x_1 \vee y) \wedge y \leq \cdots \leq (x_n \vee y) \wedge y = z$, where we delete repeated elements. This follows from Lemma 2.5 since the induced left modular chain in $[y,z]$ has increasing labels. It now follows that $R_M(c) = Q_M(m)$, and hence $R_M(c)$ is a distributive lattice.

Finally, we will show that $(3) \Rightarrow (2)$. We suppose that $P$ is a bounded supersolvable poset with M-chain $M$. Suppose $y \leq z$ in $P$ and let $c$ be the chain $y \leq z$. For any $x$ in $M$, $x \vee y$ is well-defined in $P$ (because $M$ is assumed to be viable) and equals the usual join of $x$ and $y$ in the lattice $R_M(c)$. The same idea applies to $x \wedge z$, $(x \vee y) \wedge y$ and $(x \wedge z) \vee y$. Since $R_M(c)$ is distributive, we have that

$$(x \vee y) \wedge y \leq z = (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z) = (x \wedge z) \vee y = (x \wedge z) \vee^2 y$$

in $R_M(c)$ and so $M$ is left modular in $P$. \qed

**Remark 4.2.** We know from Theorem 1 that a graded lattice of rank $n$ is supersolvable if and only if it has an $S_n$ EL-labelling. Therefore, it follows from Theorem 4 that the definition of a supersolvable poset when restricted to graded lattices yields the usual definition of a supersolvable lattice. (Note that this is not *a priori* obvious from our definition of a supersolvable poset.)

**Remark 4.3.** The argument above for the equality of $R_M(c)$ and $Q_M(m)$ holds even if $P$ is not graded. However, in the ungraded case, it is certainly not true that $Q_M(m)$ is distributive. The search for a full generalization of Theorem 1 thus leads us to ask what can be said about $Q_M(m)$ in the ungraded case. Is it a lattice? Can we say anything even in the case that $P$ is a lattice?

5. Non-straddling partitions

Let $\Pi_n$ denote the lattice of partitions of the set $[n]$ into blocks, where we order partitions by refinement: if $y$ and $z$ are partitions of $[n]$ we say that $y \leq z$ if every
block of $y$ is contained in some block of $z$. Equivalently, $z$ covers $y$ in $\Pi_n$ if $z$

 is obtained from $y$ by merging two blocks of $y$. Therefore, $\Pi_n$ is graded of rank $n - 1$. $\Pi_n$

 is shown to be supersolvable in [13] and hence has an $S_{n-1}$ EL-labelling, which we denote be $\delta$. In fact, it will simplify our discussion if we use the label set $\{2, \ldots, n\}$ for $\delta$, rather than the label set $[n - 1]$. We choose the $M$-chain, and hence the increasing maximal chain for $\delta$, to be the maximal chain consisting of the bottom element and those partitions of $[n]$ whose only non-singleton block is $[i]$, where $2 \leq i \leq n$. In the literature, $\delta$ is often defined in the following form, which can be shown to be equivalent. If $z$ is obtained from $y$ by merging the blocks $B$ 

 and $B'$, then we set

$$\delta(y, z) = \max\{\min B, \min B'\}.$$ 

For any $x \in \Pi_n$, we will say that $j \in \{2, \ldots, n\}$ is a block minimum in $x$ if $j = \min B$ 

for some block $B$ of $x$. In particular, we see that $\delta(y, z)$ is the unique block minimum in $y$ that is not a block minimum in $z$.

Recall that a non-crossing partition of $[n]$ is a partition with the property that if some block $B$ contains $a$ and $c$ and some block $B'$ contains $b$ and $d$ with $a < b < c < d$, then $B = B'$. Again, we can order the set of non-crossing partitions of $[n]$ by refinement and we denote the resulting poset by $NC_n$. This poset, which can be shown to be a lattice, has many nice properties and has been studied extensively. More information can be found in R. Simion’s survey article [11] and the references given there. Since $NC_n$ is a subposet of $\Pi_n$, we can consider $\delta$ restricted to the edges of $NC_n$. It was observed by Björner and P. Edelman in [3] that this gives an EL-labelling for $NC_n$ and we can easily see that this EL-labelling is, in fact, an $S_{n-1}$ EL-labelling (once we subtract 1 from every label).

We are now ready to state our main definition for this section, which should be compared with the definition above of non-crossing partitions.

**Definition 5.1.** A partition of $[n]$ is said to be non-straddling if whenever some block $B$ contains $a$ and $d$ and some block $B'$ contains $b$ and $c$ with $a < b < c < d$, then $B = B'$.

This definition is also very similar to that of non-nesting partitions, as defined by A. Postnikov and discussed in [10, Remark 2] and [1]. The only difference in the definition of non-nesting partitions is that we do not require $B = B'$ if there is also an element of $B$ between $b$ and $c$. So, for example, $\{1, 3, 5\}\{2, 4\}$ is a non-nesting partition in $\Pi_5$ but is not a non-straddling partition. We say that $\{1, 3, 5\}\{2, 4\}$ is a straddling partition, that $1 < 2 < 4 < 5$ is a straddle, and that the blocks $\{1, 3, 5\}$ and $\{2, 4\}$ form a straddle.

Let $NS_n$ be the subposet of $\Pi_n$ consisting of those partitions that are non-straddling. To distinguish the interval $[x, y]$ in $\Pi_n$ from the interval $[x, y]$ in $NS_n$, we will use the notation $[x, y]_{\Pi_n}$ and $[x, y]_{NS_n}$, respectively. We note that the meet in $\Pi_n$ of two non-straddling partitions is again non-straddling, implying that $NS_n$ is a meet-semilattice. Since $\{1, 2, \ldots, n\}$ is a top element for $NS_n$, we conclude that $NS_n$ is a lattice. On the other hand, $NS_n$ is not graded. For example, consider those elements of $\Pi_n$ that cover $\{1, 4\}\{2, 5\}\{3, 6\}$, as represented in Figure 5(a). $\{1, 2, 4, 5\}\{3, 6\}$, $\{1, 3, 4, 6\}\{2, 5\}$ and $\{1, 4\}\{2, 3, 5, 6\}$ are all straddling partitions, so $\{1, 4\}\{2, 5\}\{3, 6\}$ is covered in $NS_n$ by $\{1, 2, 3, 4, 5, 6\}$. Figure 5(b) shows $[\{1, 4\}\{2, 5\}\{3\}\{6\}, \hat{1}]_{NS_n}$.

Therefore, unlike $\Pi_n$ and $NC_n$, $NS_n$ cannot have an $S_{n-1}$ EL-labelling. However, we can ask if it has an interpolating EL-labelling. We see that the following
three ways of defining an edge-labelling $\gamma$ for $NS_n$ are equivalent. Observe that if $y < z$ in $NS_n$, then $z$ is obtained from $y$ by merging the blocks $B_1, B_2, \ldots, B_r$ of $y$ into a single block $B$ in $z$. We set

\[
\gamma(y, z) = \begin{cases} 
\text{second smallest element of } \{\min B_1, \ldots, \min B_r\} & \text{if } y \text{ is a block,} \\
\text{smallest block minimum in } y \text{ that is not a block minimum in } z & \text{else.}
\end{cases}
\]

See Figure 5(b) for examples. Note that the label set for $\gamma$ is $\{2, 3, \ldots, n\}$ and that if $r = 2$, then $\gamma(y, z) = \delta(y, z)$. We see that the chain

\[
\hat{0} < \{1, 2\}\{3\} \cdots \{n\} < \{1, 2, 3\}\{4\} \cdots \{n\} < \cdots < \{1, 2, \ldots, n - 1\}\{n\} < \hat{1}
\]

is an increasing maximal chain in $NS_n$ under $\gamma$.

**Theorem 5.** The edge-labelling $\gamma$ is an interpolating EL-labelling for $NS_n$.

Applying Theorem 3, we get the following result:

**Corollary 5.2.** $NS_n$ is left modular.

In preparation for proving Theorem 5, we wish to get a firmer grasp on $NS_n$. Suppose $x, y \in NS_n$. While the meet of $x$ and $y$ in $NS_n$ is just the meet of $x$ and $y$ in $\Pi_n$, the situation for joins is more complicated. The next lemma, crucial to the proof that $\gamma$ is an EL-labelling, helps us to understand important types of joins. From now on, unless otherwise specified, $x \lor y$ with $x, y \in NS_n$ will denote the join of $x$ and $y$ in $NS_n$. Furthermore, if $l_0 < l_1 < \cdots < l_r$ are block minima in $y$, then $\langle l_i \rangle$ will denote the block of $y$ with minimum element $l_i$, and $\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle$ will denote the minimum element $z \in NS_n$ for which the elements of $\langle l_0 \rangle, \langle l_1 \rangle, \ldots, \langle l_r \rangle$ are all in a single block. Note that $z$ is well-defined, since it is the meet of all those elements of $NS_n$ that have the required elements in a single block.

**Lemma 5.3.** Suppose $l_0 < l_1 < \cdots < l_r$ are block minima in $y$ and that

\[
y \lor (\langle l_0 \rangle \cup \langle l_i \rangle) = y \lor (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle).
\]

Then

\[
y \lor (\langle l_i \rangle \cup \langle l_j \rangle) = y \lor (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle).
\]

for any $0 \leq i < j \leq r$. 

---

**Figure 5**

(a) 

(b)
Applying Lemma 5.3, we get that 

\[ u \text{ that } y \text{ obtained from } z \text{ containing } l \]

Therefore, these two blocks are merged in 

\((i)\) We first prove the existence of 

Suppose \( y, z \) has to appear on every maximal chain of 

\[ (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle), \]

as required. Therefore, it suffices to let \( j = r \).

Since \( y \cup (\langle l_0 \rangle \cup \langle l_1 \rangle) = y \cup (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_{r-1} \rangle) = y \cup (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle) \),

we know that \( (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_{r-1} \rangle) \) forms a straddle with \( l_r \). There are two ways in which this might happen.

Suppose we have \( a < b < c < d \) with \( a, d \in (\langle l_0 \rangle \cup \cdots \cup \langle l_{r-1} \rangle) \) and \( b, c \in (\langle l_r \rangle) \).

Suppose \( d \in \langle l_s \rangle \) in \( y \). Then, since \( l_s < l_r \leq b < c \), we have that \( l_s < b < c < d \) is a straddle in \( y \), which contradicts \( y \in NS_n \).

Secondly, suppose we have \( a < b < c < d \) with \( a, d \in (\langle l_r \rangle) \) and \( b, c \in (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_{r-1} \rangle) \). Suppose \( b \in \langle l_s \rangle \) and \( c \in (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_{r-1} \rangle) \). Now \( c > b > a \geq l_r > l_s, l_t \).

If \( s = t \) then \( y \) has a straddle, so we can assume that \( l_s = l_t \) and that \( l_t \neq l_s \), with the argument being similar if \( l_t \neq l_s \). If \( l_t < l_s \), then \( l_t < l_i < c < d \) is a straddle when we merge blocks \( \langle l_i \rangle \) and \( \langle l_r \rangle \) in \( y \). Therefore,

\[ y \cup (\langle l_i \rangle \cup \langle l_r \rangle) = y \cup (\langle l_0 \rangle \cup \langle l_1 \rangle \cup \cdots \cup \langle l_r \rangle) \]

by the induction assumption. If \( l_t > l_i \), then \( l_t < l_i < l_r < c \) is a straddle when we merge blocks \( \langle l_i \rangle \) and \( \langle l_r \rangle \) in \( y \), also implying (3).

Lemma 5.4. Suppose \( y < z \) in \( NS_n \) and that \([y, z]_\Pi_n \) has edge labels \( l_1 < l_2 < \cdots < l_s \) under the edge-labelling \( \delta \).

(i) There is exactly one edge of the form \( y \cong w \) with \( \gamma(y, w) = l_1 \) in \([y, z]_\Pi_n \).

(ii) On any unrefinable chain \( y \cong u_0 \cong u_1 \cong \cdots \cong u_k = z \) in \( NS_n \), the label \( l_1 \) has to appear.

Proof. (i) We first prove the existence of \( w \). Let \( l_0 \) be the minimum of the block of \( z \) containing \( l_1 \) and set \( w = y \cup (\langle l_0 \rangle \cup \langle l_1 \rangle) \). Suppose \( y < u \leq w \). We know \( w \) is obtained from \( y \) by merging the blocks \( \langle l_0 \rangle, \langle l_1 \rangle, \langle l_1 \rangle, \langle l_i \rangle, \langle l_{r-1} \rangle \), for some \( 0 \leq r < s \). Applying Lemma 5.3, we get that \( u = w \) and so \( y \cong u \). By definition of \( \gamma \), we have that \( \gamma(y, w) = l_1 \).

It remains to prove uniqueness. Suppose \( w' \in NS_n \) with \( y < w' \in [y, z] \). If \( \gamma(y, w') = l_1 \), then we see that the blocks \( \langle l_0 \rangle \) and \( \langle l_1 \rangle \) must be merged in \( w' \). Therefore, these two blocks are merged in \( w \cup w' \), which is thus greater than \( y \).

Since \( y < w, w \), we conclude that \( w = w' \).

(ii) Consider the chain \( y \cong u_0 \cong u_1 \cong \cdots \cong u_k = z \) as a chain in \( \Pi_n \). Since \( \delta \) is an \( S_{n-1} \) EL-labelling for \( \Pi_n \) (once we subtract 1 from every label), the label \( l_1 \) has to appear on every maximal chain of \([y, z]_\Pi_n \). It particular, it has to appear in one of the intervals \([u_i, u_{i+1}]_\Pi_n \) for \( 0 \leq i < k \). Therefore, by (2), we get that \( \gamma(u_i, u_{i+1}) = l_1 \) for some \( 0 \leq i < k \). \( \square \)

Proposition 5.5. The edge-labelling \( \gamma \) is an EL-labelling for \( NS_n \).
Proof. Consider \( y, z \in NS_n \) with \( y < z \). Suppose \([y, z]_{Π_n}\) has edge labels \( l_1 < l_2 < \cdots < l_s \). By (2), these are the only edge labels that can appear in \([y, z]_{NS_n}\). We now describe a recursive construction of an unrefinable chain \( \lambda : y = w_0 < w_1 < \cdots < w_k = z \) in \( NS_n \). We let \( w_1 \) be the \( w \) of Lemma 5.4, i.e. \( w_1 \) is that unique element of the interval \([y, z]\) in \( NS_n \) that covers \( y \) and satisfies \( γ(y, w_1) = l_1 \). Obviously, the labels in the interval \([w_1, z]\) are all greater than the \( l_1 \). Now we apply the same argument in the interval \([w_1, z]\) to define \( w_2 \) and repeat until we have constructed all of \( λ \). Clearly, \( λ \) is then an increasing chain. By Lemma 5.4(ii), it has the lexicographically least set of labels. By Lemma 5.4(i), it is the only increasing chain from \( y \) to \( z \). □

Proof of Theorem 5. Suppose we have \( y < u < z \) in \( NS_n \) with \( γ(y, u) > γ(u, z) \). Let \( y = w_0 < w_1 < \cdots < w_k = z \) be the unique increasing chain of \([y, z]\) in \( NS_n \). By Lemma 5.4, we know that \( γ(w_0, w_1) = γ(u, z) = l_1 \), the smallest edge label of \([y, z]_{Π_n}\).

To show that \( γ(y, u) = γ(w_{k−1}, w_k) \), we have to work considerably harder. We will continue to write \( \langle m \rangle \) to denote the block of \( u \) whose minimum is \( m \) and we suppose that \( u \) is obtained from \( y \) by merging blocks \( \langle m_0 \rangle, \langle m_1 \rangle, \ldots, \langle m_s \rangle \) of \( y \), with \( m_0 < m_1 < \cdots < m_s \). We will write \( \langle l \rangle_u \) to denote the block of \( u \) whose minimum is \( l \), and we suppose that \( z \) is obtained from \( u \) by merging blocks \( \langle l_0 \rangle_u, \langle l_1 \rangle_u, \ldots, \langle l_r \rangle_u \), with \( l_0 < l_1 < \cdots < l_r \). With the structure of the chain \( y < u < z \) thus fixed, we now can deduce information about the structure of the increasing chain.

If \( l_0 \) and \( m_0 \) are distinct and are both block minima in \( z \), then all the \( l_i \)'s and \( m_j \)'s are distinct. It follows that \( z \) is obtained from \( y \) by merging blocks \( \langle l_0 \rangle, \langle l_1 \rangle, \ldots, \langle l_r \rangle \) and separately merging blocks \( \langle m_0 \rangle, \langle m_1 \rangle, \ldots, \langle m_s \rangle \). Since \( y < u < z \), we get that \( k = 2 \) and \( γ(y, u) = γ(w_1, z) = m_1 \). We assume, therefore, that \( m_0 = l_i \) for some \( 0 ≤ i ≤ r \).

As usual, we let \( w_1 = y ∨ (\langle l_0 \rangle ∪ \langle l_1 \rangle) \). Now consider

\[
w = y ∨ (\bigcup_{i : l_i < m_1} \langle l_i \rangle).
\]

Since \( l_1 < m_1 \), we know that \( w ≥ w_1 \). Let \( B \) denote the the block of \( w \) containing all \( \langle l_i \rangle \) satisfying \( l_i < m_1 \). Since \( m_0 < m_1 \), we know that \( m_0 ∈ B \). In fact, if we can show that \( m_1 ∉ B \), then we can now complete the proof. Indeed, assume \( m_1 ∉ B \) and let \( w' = w ∨ (B ∪ \langle m_1 \rangle) \). Now \( w' \) has \( m_0 \) and \( m_1 \) in the same block and so satisfies \( w' ≥ u \), since \( u = y ∨ (\langle m_0 \rangle ∪ \langle m_1 \rangle) \). Also, \( w' \) has \( l_0 \) and \( l_1 \) in the same block and so satisfies \( w' ≥ z \), since \( z = u ∨ (\langle l_0 \rangle ∪ \langle l_1 \rangle) \). Hence, \( w' = z \).

By Lemma 5.3 (substitute \( w \) for \( y \), and \( m_1 < \cdots < m_s \) for \( l_1 < \cdots < l_r \), we see that \( w ≤ w' \). Now \( γ(w, w') = m_1 \), while the edge labels of \([y, w]_{Π_n}\) all come from the set \( \{l_i \mid l_i < m_1\} \), implying that \( w \) is on the increasing chain between \( y \) and \( z \). Therefore, \( w = w_{k−1} \) and so \( γ(w_{k−1}, w_k) = γ(y, u) \).

It remains to show that \( m_1 ∉ B \). In fact, we will show that \( m_j ∉ B \) for any \( j ≥ 1 \). Consider the set:

\[
\hat{B} = \bigcup_{i : l_i < m_1} \langle l_i \rangle
\]

We will show that \( \hat{B} \) does not form a straddle with any \( \langle m_j \rangle \) for \( j ≥ 1 \). From that, it follows immediately that \( B = \hat{B} \), and therefore that \( m_1 ∉ B \), as desired.

For \( j ≥ 1 \), if \( \langle m_j \rangle \) is a singleton, then \( \hat{B} \) does not form a straddle with \( \langle m_j \rangle \). So suppose that \( |\langle m_j \rangle| ≥ 2 \). Let \( m'_j \) denote the second smallest element of \( \langle m_j \rangle \). Observe the following:
If \( \langle m_0 \rangle \) contains an element greater than \( m_j' \), then \( \langle m_0 \rangle \) and \( \langle m_j \rangle \) form a straddle in \( y \), which is impossible.

If \( \langle m_0 \rangle \) has more than one element between \( m_j \) and \( m_j' \), then we can draw the same conclusion.

Consider those \( l_i < m_1 \) with \( l_i \neq m_0 \). If \( \langle l_i \rangle \) contains an element greater than \( m_j \), then \( \langle l_i \rangle \cup \langle m_0 \rangle \cup \langle m_1 \rangle \cup \cdots \cup \langle m_s \rangle \) forms a straddle in \( u \) with \( \langle m_0 \rangle \cup \langle m_1 \rangle \cup \cdots \cup \langle m_s \rangle \), which is impossible.

Combining these three observations, we see that \( \hat{B} \) contains no elements greater than \( m_j' \), and at most one element between \( m_j \) and \( m_j' \). In particular, it does not form a straddle with \( \langle m_j \rangle \), as desired. \( \square \)

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