PLURICOMPLEX CHARGE AT WEAK SINGULARITIES

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Abstract. Let $u$ be a plurisubharmonic function, defined on a neighbourhood of a point $x$, such that the complex Monge-Ampère operator is well-defined on $u$. Suppose also that $u$ has a weak singularity, in the sense that the Lelong number of $u$ at $x$ vanish. Is it true that the residual mass of the measure $(dd^c u)^n$ vanish at $x$?

To our knowledge there is no known example that falsifies the posed question. In this paper some partial results are obtained. We find that for a significant subset of plurisubharmonic functions with well defined Monge-Ampère mass vanishing Lelong number does implies vanishing residual mass of the Monge-Ampère measure.

1. Introduction and notations

Let us denote the plurisubharmonic functions on a domain $\Omega$ by $\mathcal{PSH}(\Omega)$ and non-positive plurisubharmonic functions by $\mathcal{PSH}^-(\Omega)$. In the same manner, subharmonic functions on $\Omega$ are denoted by $\mathcal{SH}(\Omega)$ and non-positive subharmonic functions by $\mathcal{SH}^-(\Omega)$.

Perhaps the most important parameter to describe the behavior of a plurisubharmonic function near a singularity is the so called Lelong number. The Lelong number of a function $u$ at $x \in \mathbb{C}^n$ can be defined as

$$\nu(u, x) = \lim_{r \to 0} \frac{1}{(2\pi)^n} \int_{\|z-x\| \leq r} dd^c u \wedge (dd^c \log \|z-x\|)^{n-1},$$

where we use the standard, “non-normalized”, differential operators $d := \partial + \bar{\partial}$ and $d^c := i(\bar{\partial} - \partial)$. For any plurisubharmonic function $u$, $dd^c u$ is a positive $(1,1)$-current, thus the integral make sense, and it can be shown that the number is bounded for any plurisubharmonic function, see Lelong’s monograph on the subject [Lel68].

We let $B(r; x)$ denote the ball of radius $r$ with center $x$. Furthermore we use the abbreviated notation $B(r) = B_r = B(r; 0)$. If we define

$$M(u, r, x) = \sup_{z \in B(r; x)} u(z),$$

we have

$$\nu(u, x) = \lim_{r \to 0} \frac{M(u, r, x)}{\log r},$$

confer [Ava61, Kis79]. From this identity it is is immediate that if $F(z)$ is a holomorphic function $\nu(\log \|F(z)\|, x)$ is the weight of the zero at $x$.

For any smooth function $u$ the complex Monge-Ampère operator $(dd^c u)^n$ is well-defined, but for plurisubharmonic functions in particular, smoothness can be considerable relaxed. By Demailly [Dem93] it suffices that $u$ is plurisubharmonic and locally bounded outside a compact subset of the domain of definition. As one would hope, it turns out that $(dd^c \log \|z\|)^n = (2\pi)^n \delta_0$.

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For the general definition of the complex Monge-Ampère operator we refer the reader to the papers [BT76, BT82, Dem93, Ceg98, Ceg04] and the books [Kli91, Dem97]. For some of our results it suffices to have the Monge-Ampère operator defined on plurisubharmonic functions bounded outside a single pole, but for the main part of the paper the full machinery of plurisubharmonic functions with finite pluricomplex energy, from the papers of Cegrell [Ceg98, Ceg04] is needed.

Since we will be using partial integration, estimates, and convergence results in as general setting as possible, we remind you about the definition of some of the energy-classes. Let $\Omega$ be a hyperconvex domain in $\mathbb{C}^n$, i.e. a domain with a continuous plurisubharmonic function $h$ on $\Omega$ such that $\{ z \in \Omega \mid h(z) < c \}$ is relative compact in $\Omega$, for all $c < 0$. The class $\mathcal{E}_0(\Omega)$ is made up of all negative and bounded plurisubharmonic functions $v$ on $\Omega$, such that $\lim_{z \to \zeta} v(z) = 0$, for all $\zeta \in \partial \Omega$, and $\int_{\Omega} (dd^c v)^n < +\infty$. It is well known that $(dd^c v)^n$ is well defined on bounded plurisubharmonic functions, thus the finite mass assumption makes sense.

A function $u \in \mathcal{PSH}(\Omega)$ is said to be in the class $\mathcal{E}(\Omega)$ if there, for every $p \in \Omega$ is a neighbourhood $\omega$ of $p$ and a sequence $\{ u_j \} \subset \mathcal{E}_0(\Omega)$ with $u_j \searrow u$ on $w$, and subject to the total mass condition $\sup_j \int_{\Omega} (dd^c u_j)^n \leq +\infty$. If the neighbourhood $w$ can be chosen as all of $\Omega$ we say that $u \in \mathcal{F}(\Omega)$. Note that for $\Omega$ hyperconvex subset of $\mathbb{C}^2$ it is known that $\mathcal{E}(\Omega) = W^{1,2} \cap \mathcal{PSH}^{-}(\Omega)$ [B/suppress lo04]. The main point of the class $\mathcal{E}$ is that is the largest possible class where the Monge-Ampère operator is well defined.

Often it is clear from the context which domain $\Omega$ we use, or it does not matter much, in that case we often drop the reference to the domain from our notation.

The comparison principle is a very strong tool in pluripotential theory, unfortunately this principle does not hold in $\mathcal{F}$, unless the integrability condition is further strengthen so that $\int_{\Omega} -u(dd^c u)^n < +\infty$ [Ceg98], but still for functions in $\mathcal{F}$ we have for any $\varphi \in \mathcal{PSH}^{-}(\Omega)$, that

\begin{equation}
(3) \quad u \leq v \implies \int_{\Omega} \varphi(dd^c u)^n \leq \int_{\Omega} \varphi(dd^c v)^n.
\end{equation}

(see [Å02]) which, even if it is not as strong as one would like for the purpose of solving the Dirichlet problem for $(dd^c v)^n$, will be enough for our purposes.

Given a point $x \in \mathbb{C}^n$, take an open neighborhood $O_x$ of $x$, and let us denote the residual mass of a measure $\mu$ at $x$, i.e. $\mu(O_x) - \mu(O_x \setminus \{ x \})$, with $\mu(\{ x \})$. Using the Riesz decomposition formula, it is a standard exercise in potential theory to show that for subharmonic functions in $\mathbb{C}^1$:

\begin{equation}
(4) \quad \lim_{r \to 0} \frac{M(u, r, x)}{\log r} = \Delta u(\{ x \}) = 4 \partial \bar{\partial} u(\{ x \}).
\end{equation}

In $\mathbb{C}^n$, $n \geq 2$, it is well-known that the Lelong number is dominated by the Monge-Ampère operator in the following way:

\begin{equation}
(5) \quad (2\pi \nu(u, x))^n \leq (dd^c u)^n(\{ x \}).
\end{equation}

Note that if $u(z_1, z_2) = \max \{ 1/k \log |z_1|, k^2 \log |z_2| \}$, then one can show that $(dd^c u)^2(0) = 4\pi^2 k^2 \delta_0$, and since $\nu(u, 0) = 1/k$, there can be no reverse of the inequality in Equation (5) above.

As has already been pointed out by Cegrell [Ceg04] the inequality in Equation (5) holds whenever $(dd^c u)^n$ is well-defined, and for such functions the set $\{ z \in \mathbb{C}^n \mid \nu(u, z) > 0 \}$ is discrete.

In this paper we try to address the following question:
Main Question Let $u$ be a plurisubharmonic function with well-defined Monge-Ampère mass $u \in \mathcal{E}$, say. Suppose the Lelong number of $u$ at $x$ is 0, is it true that $(dd^c u)^n$ does not charge the point $x$?

Remark First of all we note that this is purely a local problem. There is no difference in asking this question for functions in $\mathcal{E}(\Omega)$ or in $\mathcal{F}(\Omega)$, since if $u \in \mathcal{E}(\Omega)$ and $D$ is an open, relatively compact subset of $\Omega$ there is a function $u_D \in \mathcal{F}(\Omega)$ such that $u_D = u$ in $D$.

There are some vague reasons to believe that, in general, $(dd^c u)^n$ do not charge the point $x$. E.g. it follows directly from a theorem in [Ras01a] that if $u(z_1, \ldots, z_n) = \mathcal{E}(\Omega) \cup L^\infty(D^2 \setminus K)$, for some $0 \in K \subseteq \Omega$. If $u(|z|, w) = u(z, w)$ and $\nu(u, 0) = 0$, then $(dd^c u)^2(\{0\}) = 0$.

In this paper we prove the following two theorems

**Theorem 1.1.** If $u \geq p_\mu$, where $p_\mu$ is the potential of the pluricomplex Green function with a single pole, then vanishing Lelong number implies vanishing residual Monge-Ampère mass. (For the statement in full generality see Theorem 4.10.)

**Theorem 1.2.** Assume that $u \in \mathcal{PSH} \cap L^\infty(D^2 \setminus K)$, for some $0 \in K \subseteq \Omega$. If $u(|z|, w) = u(z, w)$ and $\nu(u, 0) = 0$, then $(dd^c u)^2(\{0\}) = 0$.

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2. Some observations

One of the main tools in analysis, and in particular in pluripotential theory, is partial integration. We will frequently apply partial integration as a technique to estimate the Lelong-number.

Since the following “Hölder like” theorem, a rather non-trivial application of partial integration, proved in Cegrell’s seminal paper [Ceg04], will be one of the main tools we state it here for future reference.

**Theorem 2.1.** Suppose $u, v \in \mathcal{F}$, $h \in \mathcal{E}_0$, and that $p, q$ are positive natural numbers such that $p + q = n$. Then

$$\int_{\Omega} -h \n(dd^c u)^p \wedge (dd^c v)^q \leq \left( \int_{\Omega} -h \n(dd^c u)^n \right)^\frac{p}{n} \left( \int_{\Omega} -h \n(dd^c v)^n \right)^\frac{q}{n}$$

Proof. Cf. [Ceg04].

Let $h_j(z) = \max(1/j \log \|z\|, -1)$, then $\{h_j\}_j$ is an increasing sequence of continuous plurisubharmonic functions on the unit ball of $\mathbb{C}^n$, tending to 0 outside the origin. Assuming that $u \in \mathcal{PSH}(\overline{\Omega})$, we can rewrite the definition of the Lelong number of $u$ at the origin as

$$\lim_{r \to 0} \frac{1}{(2\pi)^n} \int_{B(r)} dd^c u \wedge (dd^c (\log \|z\|))^{n-1} =$$

$$= \lim_{j \to \infty} \frac{1}{(2\pi)^n} \int_{B(1)} h_j dd^c u \wedge (dd^c (\log \|z\|))^{n-1},$$

making the last integral a prime target for partial integration, or for Theorem 2.1.
The pluricomplex Green function with pole at \( w \) was introduced by Klimek [Kli85] and Zahariuta [Zah84]. For any connected open subset \( \Omega \) of \( \mathbb{C}^n \) we have \( g_\Omega(z, w) = \sup\{u(z) \mid u \in \mathcal{P}SH^- (\Omega), \text{ and } \nu(u, w) \geq 1\} \).

Note that a continuity result by Demailly [Dem87], shows that

\[
g_\Omega(z, w) \in C(\overline{\Omega} \times \Omega \setminus \{z = w\}).
\]

Thus with help of the pluricomplex Green function we can construct increasing sequences \( \{h_j\}_j \) of continuous plurisubharmonic functions on any hyperconvex set such that \( h_j \equiv -1 \) in a neighbourhood of any fixed point \( x \) and such that \( h_j \not\equiv 0 \) outside \( x \), just by setting \( h_j = \max\{1/j g_\Omega(z, x), -1\} \). By using this sequence, it follows from Demailly’s comparison theorem of Lelong numbers [Dem93], that we can express the Lelong number as

\[
\nu(u, x) = \lim_{j \to \infty} \frac{1}{(2\pi)^n} \int_{\Omega} -h_j \, dd^c u \wedge (dd^c g_\Omega(z, x))^{n-1}.
\]

Before the main body of the article we will use partial integration to make some observations regarding the question. First of all we note that class of functions without concentrated mass at a point form a convex cone, and the class is closed under taking maximum.

**Proposition 2.2.** Let \( u, v \in \mathcal{E}(\Omega) \). Assume that neither \( (dd^c u)^n \), nor \( (dd^c v)^n \) has an atom at the point \( x \in \Omega \), then the same holds for \( (dd^c (u + v))^n \), and \( (dd^c (\max \{u, \varphi\}))^n \) for any \( \varphi \in \mathcal{E}(\Omega) \).

**Proof.** Choose \( x \) as the origin. Take any \( h \in \mathcal{E}_0 \), then

\[
\int_{\Omega} -h \, (dd^c (u + v))^n = \sum_{j=0}^{n} \binom{n}{j} \int_{\Omega} -h \, (dd^c u)^j \wedge (dd^c v)^{n-j}
\]

and using the “Hölder like” Theorem 2.1

\[
\int_{\Omega} -h \, (dd^c (u + v))^n \leq \sum_{j=0}^{n} \binom{n}{j} \left( \int_{\Omega} -h \, (dd^c u)^n \right)^{\frac{j}{n}} \left( \int_{\Omega} -h \, (dd^c v)^n \right)^{\frac{n-j}{n}}
\]

Let \( \{h_j\} \) be a sequence in \( E_0 \), such that \( h_j \equiv -1 \) in a neighbourhood of the origin and \( h_j \not\equiv 0 \) outside a sequence of shrinking balls \( B(r_j) \). Let \( j \to \infty \) and we have proved the statement.

For the statement about the maximum note that \( \max(u, \varphi) \geq u \). By Equation (3) we have

\[
\int_{\Omega} -h_j \, (dd^c (\max \{u, \varphi\}))^n \leq \int_{\Omega} -h_j \, (dd^c u)^n,
\]

where \( h_j \) is the same sequence as above. Again, letting \( j \to \infty \) proves the statement. \( \square \)

The following proposition is pretty well known, I think. Using partial integration—very much in the same spirit as above—we can prove the following theorem. I should mention that the short and elegant proof is due to Urban Cegrell, and that many of the calculations in this chapter uses the same idea as in this proof.

**Proposition 2.3.** Let \( \Omega \) be hyperconvex and assume \( u \in \mathcal{E}(\Omega) \), \( \nu(u, 0) = 0 \), and \( u(z) \geq \text{const} \cdot \log ||z|| \). Then \( (dd^c u)^n(\{0\}) = 0 \).
Proof. Assume that \( u \in \mathcal{F} \). Note that \( 0 \geq u \geq C \log \|z\| \), for some positive constant \( C \). Take \( h \in \mathcal{E}_0 \), with \( h \equiv -1 \) close to the origin, as above. Then
\[
\int -h(\ddc u)^n = \int -u \ddc h \wedge (\ddc u)^{n-1}
\leq \int -C \log \|z\| \ dd^c h \wedge (\ddc u)^{n-1}
\leq \ldots \leq C^{n-1} \int -h \ddc u \wedge (\ddc (\log \|z\|))^{n-1},
\]
and we get in the manner as above that \((\ddc u)^n(\{0\}) = 0\).

The local statement for \( u \in \mathcal{E} \) follows as in the remark after the main question. \(\square\)

A fundamental idea in this paper is to use estimates and approximation from below. Since, in general, the Monge-Ampère is only well behaved for monotonically decreasing sequences, we need to be sure that we can use approximation from below. To use an approximation from below in conjunction with the Monge-Ampère operator we use a powerful convergence theorem in \( \mathcal{F} \).

**Theorem 2.4.** Assume that \( u_1^0, \ldots, u_n^0 \in \mathcal{F}(\Omega) \), \( \Omega \) hyperconvex. If \( u_p^j \) are monotonically increasing sequences such that \( u_p^j \nearrow u_p \) (a.e), for all \( 1 \leq p \leq n \), then
\[
\ddc u_1^j \wedge \ldots \wedge \ddc u_n^j \to \ddc u_1 \wedge \ldots \wedge \ddc u_n, \text{ as } j \to \infty
\]
in the weak-* topology.

**Proof.** Since monotone convergence implies convergence in capacity, this follows directly from Theorem 1.1 in [Ceg01]. \(\square\)

For decreasing sequences the convergence in weak-* sense is a well-known property. Bedford and Taylor showed in [BT82] that \((\ddc u)^n\) is continuous under decreasing sequences in \( L^\infty \), this is generalized to decreasing sequences in \( \mathcal{E} \) in [Ceg04].

Denote the upper semicontinuous regularization of a function \( f \) by \( f^* \). Functions with one concentrated singularity, and maximal outside this singularity is of special interest.

**Theorem 2.5.** Let \( \Omega \) be a hyperconvex domain containing the origin. If there exist a plurisubharmonic negative function \( u \in \mathcal{F}(\Omega) \), with \( \nu(u,0) = 0 \), such that the Monge-Ampère operator charges the origin, there exist such function with \((\ddc u)^n \equiv 0 \) outside the origin.

**Proof.** Take \( u \in \mathcal{F} \), \( r > 0 \), and define
\[
S_r(z) = \left( \sup \{ v(z) \in \mathcal{PSH}(\Omega) \mid v \leq u \text{ on } B(r); \ v \leq 0 \} \right)^*.
\]

Clearly \( S_r \in \mathcal{F}(\Omega) \) and \( S_r \leq v \) on \( B(r) \). Take \( r' < r \), then we must have that \( S_r \leq S_{r'} \), since \( S_r \leq u \) on \( B(r') \) so \( S_r \) is one of the “competitors” in the class of functions we take supremum of when defining \( S_r^* \). Let \( S = (\lim_{r \to 0} S_r)^* \). Then \( S \) is plurisubharmonic and, by construction, maximal outside the origin.

Since \( S > S_r \), we have that \( \nu(S,0) \leq \nu(S_r,0) \leq \nu(u,0) = 0 \).

Furthermore \((\ddc u)^n(\{0\}) = c, c > 0 \), and we have
\[
\int_{\Omega} (\ddc S_r)^n \geq \int_{B_r} (\ddc u)^n = c.
\]
Since \( S_r \) is an increasing sequence we get that \((\ddc S)^n(\{0\}) \geq c \), according to Theorem 2.4 \(\square\)
Remark. It is important, in the construction of \( S \) above, that \((dd^c u)^n\) charges the origin. Otherwise \((\lim_{r \to 0} S_r)^*\) might be identically zero. For example, take \( u(z) = -\sqrt{-\log \|z\|} \), then a simple calculation yields that \( S_r(z) = \log \|z\|/\log r \), outside \( B(r) \), and therefore \((\lim_{r \to 0} S_r)^* = 0 \).

3. Using estimates from below

A radial subharmonic function whose Laplace mass does not charge the origin can be minorized by any logarithm close to the origin. Let us make this simple observation precise.

Lemma 3.1. Let \( D \) denote the unit disc in \( \mathbb{C}^1 \), and suppose \( u \in \mathcal{SH}(D), u \not\equiv -\infty \) is radial (i.e. \( u(|z|) = u(z) \)). For any \( \epsilon > 0 \), let \( D_\epsilon = \{ z : u(z) > \epsilon \log |z| \} \cup \{0\} \).

Then \( \nu(u,0) = 0 = \partial \bar{\partial} u(\{0\}) \) if and only if \( D_\epsilon \) is a disc of positive radius centered at the origin for all \( \epsilon > 0 \).

Proof. Since \( u \) is radial and \( u \not\equiv -\infty \), it is a convex function in \( \log r \), continuous outside of the (possible) pole in the origin. Assume \( \nu(u,0) = 0 \), Equation (2) gives that there is a sequence \((r_j), r_j \to 0\) such that \( u > \epsilon \log r_j \). If we change variables \( t \to e^t \), we get \( u(t_j) > \epsilon t_j \).

Suppose there is a \( t' < t_k \), such that \( u(t') < \epsilon t' \). Since \( u \) is convex and \( u(t_{k+1}) > \epsilon t_{k+1} \), we have that \( u(t) < \epsilon t \), for \( t < t_{k+1} \), which contradicts the assumption that \( \nu(u,0) = 0 \).

The opposite implication follows from the continuity of the Laplace operator. \( \square \)

Now if we wanted to show that \( M(u, r, 0)/\log r \) tends to zero with \( r \) for radial potentials which does not charge the origin, we could use the Lemma above together with the continuity of the Laplace operator on increasing sequences.

If we want to use this method to deal with more general plurisubharmonic functions we need to replace the comparison with an increasing sequence of logarithms to any sequence of plurisubharmonic functions increasing to zero, since even for non-radial subharmonic functions on the plane it is clearly not the case that vanishing Laplace mass implies that we can estimate the function from below with logarithms.

To deal with more oscillating functions we mimic the sets \( D_k \) in Lemma 3.1 above to make the following convenient definition.

Definition 3.2. Let \( U \) be an open neighbourhood of the origin. We say that \( u \in \mathcal{PSH}(U) \) is of class \( \mathcal{K} \) if there exists an increasing sequence \( \{ f_j \} \subset \mathcal{PSH}^-(U) \) such that \( f_j \not\nearrow 0 \) (a.e.) and \( \forall j, \exists r = r(j) > 0 \) such that

\[
\{ f_j(z) \leq u(z) \} \cup \{0\} \supset B_r,
\]

where \( B_r \) is the ball of radius \( r \), centered at the origin.

It turns out that Definition 3.2 is a handy way to describe functions with no residual Monge-Ampère mass at the origin. The following theorem makes it clear.

Theorem 3.3. Let \( \Omega \) be an hyperconvex domain in \( \mathbb{C}^n \) such that \( 0 \in \Omega \), and let \( u \in \mathcal{F}(\Omega) \), then \( u \in \mathcal{K} \) if, and only if, \((dd^c u)^n(\{0\}) = 0 \).

Proof. Suppose \( u \in \mathcal{K} \), then by definition there is \( f_j \in \mathcal{PSH}^-(\Omega) \) such that \( f_j \not\nearrow 0 \) and \( f_j < u \) on balls \( B(r_j) \).

Define a sequence of functions \( u_j := \max(u, f_j) \). Then \( u_j \in \mathcal{F} \), and \( u_j \not\nearrow 0 \) (a.e.). Clearly \( u_j = u \) on \( B(r_j) \), thus

\[
\int_{B(r_j)} -\varphi(dd^c u_j)^n = \int_{B(r_j)} -\varphi(dd^c u_j)^n \leq \int_{\Omega} -\varphi(dd^c u_j)^n.
\]
Take \( \varphi \equiv -1 \) at a neighbourhood of the origin. Since we have, according to Theorem 2.4, \((dd^c u_j)^n \rightarrow 0\), we get in particular that \( \int_{\Omega} - \varphi (dd^c u_j)^n \rightarrow 0 \). Thus \((dd^c u)^n(\{0\}) = 0\).

On the other hand, suppose \( u \in \mathcal{F} \) such that \((dd^c u)^n(\{0\}) = 0\). As in the proof of Theorem 2.5 define a sequence

\[
  f_j(z) = \left( \sup\{ \varphi(z) \in \mathcal{PSH}(D) : \varphi \leq 0 \text{ and } \varphi|_{B(1,j)} \leq u \} \right)^*.
\]

Then \( f_j \) is an increasing sequence of plurisubharmonic functions such that \( f_j \not\nearrow 0 \) (a.e.).

\[\square\]

**Lemma 3.4.** Let \( u \in \mathcal{F}(\Omega) \). If \( v \in \mathcal{K} \) then \( \max(u,v) \in \mathcal{K} \).

**Proof.** Since \( v \) is of class \( \mathcal{K} \) take the increasing sequence \( f_j \) from the definition of the class \( \mathcal{K} \). Now \( \max(u,f_j) \leq \max(u,v) \) on \( B(r_j) \), and \( \max(u,f_j) \not\nearrow 0 \) pointwise (a.e.). Thus \( \max(u,f_j) \) is the required sequence for \( \max(u,v) \). \[\square\]

4. The pluricomplex potential

Lelong [Lel89] has generalized the notion of the pluricomplex Green function and defined the general multipole Green function on a bounded hyperconvex set \( \Omega \subset \mathbb{C}^n \), with weighted poles \( P = \{(a_k,w_k)\}_{k=1}^p \) as

\[
  g_{\Omega}(z,P) = \sup\{u(z) \mid u \in \mathcal{PSH}^{-}(\Omega), \nu(u,w_j) \geq a_j, 1 \leq j \leq p\},
\]

where the weights \( a_k \geq 0 \) and the poles \( w_k \in \Omega \).

If \( P = \{(1,w)\} \) we write \( g_{\Omega}(z,\{(1,w)\}) = g_{\Omega}(z,w) \), for the pluricomplex Green function with a single pole at \( w \) with weight one.

**Definition 4.1.** [Car99] Let \( \mu \) be a finite, positive measure with support in \( \bar{\Omega} \), where \( \Omega \) is a bounded domain in \( \mathbb{C}^n \). We define the pluricomplex potential of \( \mu \) as

\[
  p_\mu(z) = \int_{\Omega} g_{\Omega}(z,w) \, d\mu(w),
\]

and the logarithmic potential of \( \mu \) as

\[
  lp_\mu(z) = \int_{\Omega} \log \|z - w\| \, d\mu(w).
\]

Note that in \( \mathbb{C}^1 \) the pluricomplex potential is just the ordinary Green potential (or minus the ordinary Green potential, depending on taste).

**Lemma 4.2.** Let \( \Omega \) be a hyperconvex domain, and let \( \mu \) be a positive finite Borel measure on \( \Omega \), with support in \( K \subset \Omega \). Then \( p_\mu \) and \( lp_\mu \) is in \( \mathcal{C}(\bar{\Omega} \setminus K) \).

**Proof.** For the logarithmic potential this is clear. For the pluricomplex potential this follows from the aforementioned continuity result of Demailly [Dem87]. \[\square\]

This place us in position to prove a fundamental lemma about pluricomplex potentials.

**Lemma 4.3.** Suppose \( \mu \) is a positive finite Borel measure then \( p_\mu \in \mathcal{F} \).

**Proof.** Let \( K \subset \Omega \). Define \( \mu_K = \chi_K \mu \) where \( \chi_K \) is the characteristic function of \( K \). Let

\[
  p(z) := \int g(z,w) \, d\mu_K(w).
\]

Take \( r \leq \text{dist}(K,\bar{\Omega}) \) and let \( A = \{z \in \Omega : \text{dist}(z,\bar{\Omega}) < r\} \).

Let us define

\[
  \alpha = \inf\{g(z,w) : z \in A, w \in K\}.
\]
Since, according to Lemma 4.2, \( p \) is continuous away from \( K \), there is a point \((z_0, w_0)\), where \( w_0 \in K \) and \( z_0 \in A \) such that \( g(z_0, w_0) = \alpha \).

For \( z \in A \)

\[
(7) \quad p(z) = \int g(z, w) \, d\mu_K(w) \geq \int \alpha \, d\mu_K(w) = \alpha \mu(\Omega).
\]

Since \( g(z, w_0) \in C(\Omega) \) we get that \( \alpha \to 0 \) as \( r \to 0 \), and thus for any \( K \subseteq \Omega \) we have

\[
(8) \quad \lim_{z \to \partial \Omega} p(z) = 0.
\]

Assume for simplicity that \( \mu(\Omega) = 1 \), note that since \( \alpha < 0 \) Equation (7) gives that

\[
\max(p(z), g(z, w_0)) = p(z), \text{ for } z \in A.
\]

Thus if we set \( v(z) := \max(p(z), g(z, w_0)) \) we get, by Stokes’ theorem,

\[
\int (dd^c p)^n = \int (dd^c v)^n \leq \int (dd^c g(z, w_0))^n = (2\pi)^n,
\]

independent of \( K \). This estimate and Equation (8) implies that \( \mu \in \mathcal{F} \).

If \( \mu(\Omega) \neq 1 \) set \( \tilde{p} = (\mu(\Omega))^{-1} p \), and then \( \tilde{p} \in \mathcal{F} \), which implies that \( p \in \mathcal{F} \). \( \square 

**Proposition 4.4.** Let \( u(z) \) be the logarithmic potential or the pluricomplex potential of a finite positive Borel measure with compact support on a hyperconvex domain \( \Omega \). Suppose \( \mu\{\{0\}\} = 0 \), then \((dd^c u)^n\{\{0\}\} = 0. \)

**Proof.** We prove the lemma for the pluricomplex potential. The proof is similar for the logarithmic potential.

Let \( \mu_j = \chi_j \mu \), where \( \chi_j \) is the characteristic function for \( B(1/j) \), and define an increasing sequence in \( \mathcal{F}(\Omega) \) by the formula

\[
f_j(z) = 2 \int_{\Omega} g(z, w) \, d\mu_j + (1/j) \log \|z\|.
\]

Since \( \mu\{\{0\}\} = 0 \), we have \( f_j \nearrow 0 \) pointwise (q.e.). Theorem 4.7 guarantees that \( \{f_j\}_j \subset \mathcal{F} \).

Now, consider the set \( \{u - f_j > 0\} \).

\[
(u - f_j)(z) = \int_{\Omega} g(z, w) \, d\mu(w) - \int_{\Omega} 2g(z, w) \, d\mu_j(w) - \frac{1}{j} \log \|z\|
\]

\[
= \int_{B(1/j)} g(z, w) \, d\mu(w) - \int_{B(1/j)} g(z, w) \, d\mu_j(w) - \frac{1}{j} \log \|z\|
\]

\[
= h(z) - \int_{B(1/j)} g(z, w) \, d\mu_j(w) - \frac{1}{j} \log \|z\|,
\]

where \( h(z) := \int_{B(1/j)} g(z, w) \, d\mu(w) \).

According to Lemma 4.2 \( h(z) \) is continuous on \( B(1/j) \). In particular \( h(z) > -M \) for some \( M > 0 \) on \( B(1/(j + 1)) \), and in addition we have \( -\int_{B(1/j)} g(z, w) \, d\mu_j(w) \geq 0 \), on the same set. Thus

\[
(u - f_j)(z) > -M - \int_{B(1/j)} g(z, w) \, d\mu_j(w) - 1/j \log \|z\| > -M - 1/j \log \|z\| > 0,
\]

as long as \( \|z\| < e^{-jM} \).
Hence, for every \( j \), there is a \( r \), only depending on \( j \), such that if \( z \in B_r \) then 
\( u(z) - f_j(z) > 0 \), thus \( u \in K \) (see Definition 3.2) and according to Theorem 3.3 we have 
\((dd^c u)^n(\{x\}) = 0\). \( \Box \)

**Corollary 4.5.** Let \( \mu \) be a positive finite Borel measure on \( \Omega \). Take \( x \in \Omega \) then 
\( \nu(p_x, x) = \mu(\{x\}) \).

**Proof.** We may decompose Green potential of \( \mu \) as 
\( p_x = \int g(z, w) d\mu(w) = \mu(\{x\})g(z, x) + p_{\mu}(z) \), where \( \mu \) does not have an atom at \( x \). According to Proposition 4.4 the Monge-Ampère measure of \( p_x \) does not charge \( x \), and since the Lelong number is dominated by the Monge-Ampère charge we get 
\( \nu(p_{\mu}, x) = 0 \).

On the other hand, by definition of the Green function, 
\( \nu(\mu(\{x\})g(z, x), x) = \mu(\{x\}) \).

**Definition 4.6.** Let \( \Omega \) be a hyperconvex domain, let \( g_\Omega(z, w) \) be the pluricomplex Green function with pole at \( w \). We define the class \( \mathcal{P} \) by saying that \( \varphi \in \mathcal{P}(\Omega) \) if \( \varphi \in \mathcal{PSH}^{-}(\Omega) \) and there is a finite positive Borel measure on \( \Omega \) such that 
\[ \varphi(z) \geq \int g_\Omega(z, \zeta) d\mu(\zeta). \]

**Theorem 4.7.** Suppose \( \Omega \) is a hyperconvex domain in \( \mathbb{C}^n \), for dimension \( n \geq 2 \), then \( \mathcal{P}(\Omega) \subseteq \mathcal{F}(\Omega) \).

**Proof.** According to Lemma 4.3 every Green potential is in \( \mathcal{F} \), and since any function \( u \in \mathcal{P} \) vanishes on the boundary and is minorized by a function in \( \mathcal{F} \) the inclusion is clear.

To see that the set \( \mathcal{F}(\Omega) \setminus \mathcal{P}(\Omega) \) is not empty if \( \Omega \subset \mathbb{C}^n \) for \( n \geq 2 \), consider the sequence of functions \( u_N(z) = g_\Omega(z, P_N) \), where the weighted poles for the pluricomplex Green function is \( P_N = \{(k^{-p/n}, w_k)\}_{k=1}^{N} \), for \( 1 < p < 2 \) fixed.

We choose the poles \( \{w_k\} \) such that \( \bigcup\{w_k\} \in \Omega \). By construction of the Green function \( u_N \in \mathcal{F}(\Omega) \) and
\[ \int_{\Omega} (dd^c u_N)^n = (2\pi)^n \sum_{k=1}^{N} (k^{-p/n})^n < +\infty, \]
since \( p > 1 \). Thus if we define \( u := \lim_{N \to \infty} u_N \) we have \( u \in \mathcal{F}(\Omega) \).

However \( u \notin \mathcal{P} \), because if \( u_N \geq 1 \) \( g \text{d}\mu \) we have
\[ k^{-p/n} = \nu(u_N, \omega_k) \leq \nu(p_x, \omega_k) = \mu(\omega_k), \]
where the last equality follows from Corollary 4.5. Then
\[ \mu(\Omega) \geq \sum_{k=1}^{N} k^{-p/n} \to +\infty, \text{ as } N \to \infty, \text{ for } n \geq 2, \]
hence \( u \notin \mathcal{P} \).

Note that since functions in \( \mathcal{F} \) has harmonic majorant 0, it follows from the Riesz representation theorem that \( \mathcal{F} = \mathcal{P} \) in \( \mathbb{C}^1 \).

It is possible to reshape Proposition 4.4 a bit for the purpose of generalizing Proposition 2.3. We begin with a proposition that generalize Proposition 2.2 and is interesting in its own right.

**Proposition 4.8.** Let \( \Omega \) be a hyperconvex domain and take \( x \in \Omega \). If \( f_j \in \mathcal{E}(\Omega) \), 
\( (dd^c f_j)^n(\{x\}) = 0 \), and \( \sum_{j=1}^{\infty} f_j \in \mathcal{E}(\Omega) \), then 
\( (dd^c (\sum_{j=1}^{\infty} f_j))^n(\{x\}) = 0 \).
Proof. Since this is a local statement, we might as well suppose that \( \Omega = B \) and take \( x \) as the origin. Furthermore we might as well assume—after modification outside a neighbourhood of the origin and then translation—that \( f_j \in \mathcal{F} \) and \( \sum_{j=1}^{\infty} f_j \in \mathcal{E}(B) \). Take \( h \in \mathcal{E}_0 \). By partial integration:

\[
\int_B -\left( \sum_{j=1}^{\infty} f_j \right) dd^c h \wedge (dd^c (\sum_{j=1}^{\infty} f_j))^{n-1} = \int_B -h (dd^c (\sum_{j=1}^{\infty} f_j))^{n} < \infty
\]

Now, \( \mu = dd^c h \wedge (dd^c (\sum_{j=1}^{\infty} f_j))^{n-1} \) is a positive finite measure on \( \Omega \), vanishing on pluripolar sets. Hence \( \int \sum_{j=1}^{\infty} f_j > -\infty \) and so

(9) \[
\lim_{N \to \infty} \sum_{j=N}^{\infty} f_j = 0 \text{ a.e. (}\mu\text{).}
\]

Let us define \( h_k(z) = \max (-1, k^{-1} \log \|z\|) \), then \( \{h_k\}_{k=1}^{\infty} \) is an increasing sequence of negative functions, thus \( -h_1 \geq \ldots \geq -h_k \geq \ldots \). By Theorem 2.1 we have

\[
\int -h_k (dd^c (\sum_{j=1}^{\infty} f_j))^{n} = \int -h_k dd^c (\sum_{j=1}^{N-1} f_j) \wedge (dd^c (\sum_{j=1}^{\infty} f_j))^{n-1} + \int -h_k dd^c (\sum_{j=1}^{\infty} f_j) \wedge (dd^c (\sum_{j=1}^{N-1} f_j))^{n-1} \leq \int -h_k dd^c (\sum_{j=1}^{N-1} f_j) \wedge (dd^c (\sum_{j=1}^{\infty} f_j))^{n-1} \leq \left[ \sum_{j=1}^{\infty} \left( \int -h (dd^c f_j)^n \right)^{1/n} \right] \left[ \int -h (dd^c (\sum_{j=1}^{\infty} f_j)) \right] \frac{n-1}{n} + \sum_{j=N}^{\infty} (dd^c f_j) \wedge (dd^c (\sum_{j=1}^{\infty} f_j))^{n-1}.
\]

By Equation (9) the last term in the estimate above can be made arbitrarily small if \( N \) is chosen large enough. By assumption, \( \int_{\{0\}} (dd^c f_j)^n = 0 \), so if we then choose \( k \) big enough \( \int -h_k (dd^c f_j)^n \) can also be made as small as we like, and we have

\[
0 = \lim_{k \to \infty} \int -h_k (dd^c (\sum_{j=1}^{\infty} f_j))^{n} = (dd^c (\sum_{j=1}^{\infty} f_j))^{n}(\{0\}),
\]

which proves the theorem. \(\square\)

Remark Note that the condition \( (dd^c (f_j))^n(\{0\}) = 0, \forall j \) does not suffice to guarantee that \( \sum f_j \in \mathcal{F} \). Take for example \( f_j = g(z, b_j) \), where \( \{b_j\}_{j=1}^{\infty} \) is a sequence such that \( \{b_j\} \subset \Omega \) and \( b_j \neq 0 \). Then \( (dd^c f_j)^n(\{0\}) = 0 \), but because \( \int (dd^c (\sum_{j=1}^{N} f_j))^{n} = (2\pi)^n N \), we have \( \sum f_j \notin \mathcal{F} \).
Proposition 4.9. Suppose \( \mu \) is a finite positive Borel measure on a hyperconvex domain \( \Omega \). Take \( x \in \Omega \), if

\[
u = \int g(z, w) \, d\mu(w),
\]
and \( \nu(u, x) = 0 \) then \((dd^c u)^n(\{x\}) = 0\).

Proof. By Corollary 4.5, \( \nu(u, x) = 0 \) implies that \( \mu(\{x\}) = 0 \), so this is a direct consequence of Proposition 4.4.

Let us demonstrate how it also follows from Proposition 4.8. After scaling we may as well assume that \( B(1) \subset \Omega \), and for convenience we take \( x \) as the origin.

If \( \mu(\{0\}) = 0 \), then

\[
u = \sum_{j=1}^{\infty} \int g(z, w) \, d\mu_j(w),
\]
where \( \mu_j = \chi_j d\mu \), for \( j = 0, 1, \ldots, \chi_0 = \Omega \setminus B_1 \), and \( \chi_j \) is the characteristic function for \( B(1/j) \setminus B(1/(j + 1)) \), for \( j = 1, 2, \ldots \). Then

\[
(dd^c (\int g(z, w) \, d\mu_j(w)))^n(\{0\}) = 0,
\]

and according to Proposition 4.8 \((dd^c u)^n(\{0\}) = 0\).

Recall that Proposition 2.3 stated that for plurisubharmonic functions that can be minorized by a logarithm vanishing Lelong number implies vanishing residual Monge-Ampère mass. We are now in position to generalize that to plurisubharmonic functions minorized by a more general class of functions.

Theorem 4.10. Let \( \Omega \) be a hyperconvex domain and suppose \( u \in \mathcal{P}(\Omega) \). Given a point \( x \in \Omega \) and \( \varphi \in \mathcal{D}(\Omega) \) such that \((dd^c \varphi)^n(\{x\}) = 0\), let \( v \geq u + \varphi \), then \( \nu(v, x) = 0 \) implies that \((dd^c v)^n(\{x\}) = 0\).

Proof. This is entirely a local statement so we might as well suppose that \( x = 0 \) and \( \Omega = B(0, 1) \).

First we assume \( u \in \mathcal{F} \), \( \nu(u, 0) = 0 \) and that \( u \geq p_\mu \) for some positive measure \( \mu \). There is a number \( a \geq 0 \) such that \( p_\mu(z) = a \log \|z\| + s(z) \), where \( s(z) = \int_{B(1, 0)} g(z, w) \, d\mu \), hence for any \( h_k \in \mathcal{P}_0 \):

\[
0 \geq \int hh_k(dd^c u)^n = \int ud\bar{d}h_k \wedge (dd^c u)^{n-1} \\
\geq \int ud\bar{d}h_k \wedge (dd^c f)^{n-1} \geq \int ud\bar{d}h_k \wedge (dd^c (a \log \|z\| + s(z)))^{n-1} \\
= \int ud\bar{d}h_k \wedge (dd^c a \log \|z\|)^{n-1} + \\
+ \sum_{j=0}^{n-2} \binom{n-1}{j} \int ud\bar{d}h_k \wedge (dd^c a \log \|z\|)^j \wedge (dd^c s(z))^{n-1-j}.
\]

Let \( h_k = \max(\log \|z\|/k, -1) \). The first term in Equation (10) tends to zero as \( k \) tends to infinity as in Proposition 2.3. For the other terms we have, according to Theorem 2.1,

\[
\int -ud\bar{d}h_k \wedge (dd^c a \log \|z\|)^j \wedge (dd^c s(z))^{n-1-j} \\
\leq \int -h_kdd^c u \wedge (dd^c a \log \|z\|)^j \wedge (dd^c s(z))^{n-1-j} \\
\leq (2\pi a)^j \left[ \int (dd^c u)^n \right]^j \left[ \int -h_k(dd^c s)^n \right]^j \frac{n-j-1}{2n-j-1},
\]

(11)
which—according to Proposition 4.9—tends to zero as \( k \) tends to infinity.

Now for the general case when \( r \geq \int p_\mu + \varphi \), for some \( \varphi \in \mathcal{E}(\Omega) \) with \( (dd^c \varphi)^n(\{x\}) = 0 \), where we proceed much in the same manner. Again there is a positive number \( a \) such that \( p_\mu = a \log \|z\| + s(z) \), where \( s(z) = p_{\mu'} \) with \( \mu'(\{0\}) = 0 \). Set \( \tilde{\varphi} = s + \varphi \), then \( v(z) \geq a \log \|z\| + \tilde{\varphi}(z) \), where, according to Proposition 2.2, \( \tilde{\varphi} \in \mathcal{E}(\Omega) \) has the property \( (dd^c \tilde{\varphi})^n(\{0\}) = 0 \).

As in the remark following the main question we might as well assume that \( \tilde{\varphi} \in \mathcal{F} \). Take \( h \in \mathcal{E}_0 \) then

\[
\int -h (dd^c v)^n \\
\leq \int -h dd^c v \wedge (dd^c (a \log \|z\| + \tilde{\varphi}))^{n-1} \\
= \int -h dd^c v \wedge (dd^c a \log \|z\|)^{n-1} + \\
+ \sum_{j=1}^{n-1} \binom{n-1}{j} \int -h dd^c v \wedge (dd^c a \log \|z\|)^{n-1-j} \wedge (dd^c \tilde{\varphi})^j.
\]

Choosing \( h = \max(1/k \log \|z\|, -1) \), the first term in the sum above is arbitrarily close to zero as in Proposition 2.3. The remaining terms can be taken cared of in the same way as in Equation (11) above.

\[\square\]

5. EXAMPLES OF FUNCTIONS WITH NO MONGE-AMPERE MASS AT THE POLES

A function \( u : \mathbb{C}^n \to \mathbb{C}^p \) is said to be poly-radial if it is a radial function in every variable separately \( u(|z_1|, \ldots, |z_n|) = u(z_1, \ldots, z_n) \).

If \( u \) is a poly-radial plurisubharmonic function in a neighbourhood of the origin it is a radial subharmonic function along any complex line through the origin. Since radial subharmonic functions are continuous outside the origin, or identically \(-\infty\), it follows that if \( u \) is bounded below away from the origin it has its only possible pole at the origin.

We will need a couple of lemmas about the Lelong number along slices. Given a function \( u : \mathbb{C}^n \cup \Omega \to \mathbb{R} \cup \{-\infty\} \) we define a slice of \( u \) through 0 and \( p \in \mathbb{C}^n \), \( u_p \), as: \( u_p(\zeta) := u(\zeta p), \zeta \in \mathbb{C} \), wherever this expression make sense.

**Lemma 5.1.** \( \nu(u_p, 0) \geq \nu(u, 0) \)

**Proof.** Since \( \log r < 0 \), if \( r < 1 \) we have:

\[
\lim_{r \to 0} \sup_{\|\zeta p\| \leq r} \frac{u(z)}{\log r} \leq \lim_{r \to 0} \sup_{\|k\| \leq r} \frac{u(\zeta p)}{\log r}.
\]

\[\square\]

Using this lemma one can prove that the reverse inequality holds almost everywhere.

**Lemma 5.2.** Assume \( u \in \mathcal{PSH}(B) \), take \( q \in B \) fixed, then \( \nu(u_q, 0) = \nu(u, 0) \) for all \( q \in B \setminus A \), where \( A \) is a pluripolar set.

**Proof.** This is well known, Cf. [CT96], or for a more elegant proof, the survey [Ras01b].

For functions radial in at least one variable Lemma 5.2 above can be considerable strengthen. Namely, if the Lelong number at the origin vanish, it vanish on all slices through the origin, except perhaps along the coordinate axes.
Lemma 5.3. Assume that $u \in PSH(B)$, where $B$ is the unit-ball in $\mathbb{C}^2$, and that $u(|z|, w) = u(z, w)$. Suppose that $\nu(u, 0) = 0$, then for all $y = (y_1, y_2) \in B$, such that $y_1 y_2 \neq 0$, $\nu(u_y, 0) = 0$.

Proof. Take $p = (z, w), q = (z', w') \in \mathbb{C}^2$, with $|z| < |z'|$, and $|w| = |w'| = R$. Since $u(r, w)$ is a increasing function in the radial variable we have that

$$\sup_{|w|=R} u(|z|, w) \leq \sup_{|w'|=R} u(|z'|, w').$$

That is, we have $\nu(u_p, 0) \geq c \nu(u_q, 0)$, for some constant. Take a point $y = (y_1, y_2)$ such that neither $y_1$, nor $y_2$ is zero. According to Lemma 5.2 there is a point $y' = (y'_1, y_2)$ with $|y'_1| < |y_1|$ such that $\nu(u_{y'}, 0) = 0$, but then $0 \geq \nu(u_y, 0)$. □

Now let us demonstrate how we can use the convergence on increasing sequences (Theorem 2.4). Using Lemma 3.1 and Theorem 2.4 we can deal with poly-radial functions. It can be illustrative to see that how this follows directly from the convergence Theorem 2.4.

Proposition 5.4. Assume $u \in PSH(D^n)$, is poly-radial, and $u^{-1}\{-\infty\} = \{0\}$. Then $\nu(u, 0) = 0$ if, and only if, $(dd^c u)^n(\{0\}) = 0$.

Proof. Let $e_j$ be the unit vector in $\mathbb{C}^n$ having $1 + i \cdot 0$ as its $j$:th coordinate. Note that $u_{e_j}$ is a subharmonic function on the unit disc in $\mathbb{C}$, and since $u^{-1}\{-\infty\} = \{0\}$ it follows that $u_{e_j} \not\equiv -\infty$ and thus $\nu(u_{e_j}, 0) = \nu_j < +\infty$.

Take $\epsilon > 0$ and define

$$v_{\epsilon}(z) := \max((\nu_1 + \epsilon) \log |z_1|, \ldots, (\nu_n + \epsilon) \log |z_n|).$$

Then $v_{\epsilon} < u$ on a polydisc centered at the origin (as in the proof of Theorem 4.2 [Wik04]). Applying Proposition 2.3 we see that $\nu(u, 0) = 0$ implies that $(dd^c u)^n(\{0\}) = 0$. □

Proposition 5.4 also follows more or less directly from Demailly’s comparison principle for generalized Lelong numbers [Dem93], or directly from a general theorem in [Ras01a].

The goal is to generalize the idea of the proof of Proposition 5.4 for a more general class of plurisubharmonic functions. But the main idea is still to compare the function with a smaller function along a line and then to amplify the comparison to a larger domain.

To highlight the methods that are used through this section we start off with a class of functions for which the the answer to the main question is 1 is rather easily seen to be affirmative.

Proposition 5.5. Assume that $u \in \mathcal{F}(D^2)$. Let $e_2 = (0, 1 + 0 \cdot i)$. If $\nu(u_{e_2}, 0) = 0$, i.e. the Lelong number of $u$ along the $z_2$-axis is zero, and for any point $z_2$ $u(0, z_2) \leq u(z_1, z_2)$, for all points $z_1 \in D$, then $(dd^c u)^2$ does not charge the origin.

Proof. Since the Lelong number of $u$ along the $z_2$-axis vanish we have, after changing $u$ near the boundary if necessary, that $u(0, z_2)$ is of class $\mathcal{K}(D)$, thus we can apply Lemma 3.4 to see that $\max(u(z_1, z_2), u(0, z_2)) \in \mathcal{K}$. Since we assumed that $u(0, z_2) \leq u(z_1, z_2)$ we have that $\max(u(z_1, z_2), u(0, z_2)) = u(z_1, z_2)$, thus $u \in \mathcal{K}$. According to Theorem 3.3 $(dd^c u)^n(\{0\}) = 0$. □

Note that according to Lemma 5.1 we have that functions satisfying the conditions in Proposition 5.5 must have Lelong number zero at the origin.

For functions that are radial in at least one of the variables more could be said. We start off with an immediate corollary to Proposition 5.5.
Corollary 5.6. Assume that $u \in \mathcal{F}(D^2)$ Take $e_2 = (0, 1 + 0 \cdot i)$. If $\nu(u_{e_2}, 0) = 0$, and $u(z_1, z_2) = u(z_1, z_2)$ then $(dd^c u)^2$ does not charge the origin.

Proof. Using the maximum principle in the first variable we have $u(0, z_2) \leq u(z_1, z_2)$ and we may apply Proposition 5.5.

Theorem 5.7. Let $\Omega_1$ and $\Omega_2$ be hyperconvex domains in $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ respectively. Suppose $u_1 \in \mathcal{F}(\Omega_1)$ and $u_2 \in \mathcal{F}(\Omega_2)$, then $\max(u_1, u_2) \in \mathcal{F}(\Omega_1 \times \Omega_2)$ and

$$(12) \quad \int_{\Omega_1 \times \Omega_2} (dd^c (\max(u_1, u_2)))^{n_1 + n_2} = \int_{\Omega_1} (dd^c u_1)^{n_1} \int_{\Omega_2} (dd^c u_2)^{n_2}.$$ 

Proof. It is enough to prove a special case of the statement, namely if $u_i \in \mathcal{E}_0(\Omega_i) \cap C(\Omega_i)$ with $\text{supp}\{(dd^c u_i)^{n_i}\} \subset \Omega_i$, for $i = 1, 2$, then $\max(u_1, u_2) \in \mathcal{E}_0(\Omega_1 \times \Omega_2)$ and Equation (12) holds, because according to the main approximation Theorem 2.1 in [Ceg04] there are always sequences $u_i^j \in \mathcal{E}_0(\Omega_i)$ with $\text{supp}\{(dd^c u_i^j)^{n_i}\} \subset \Omega_i$, for $i = 1, 2$, such that $u_i^j \searrow u_i$ and $u_2^j \searrow u_2$, as $j \to \infty$. Hence

$$\lim_{j \to \infty} \max(u_1^j, u_2^j) = \max(u_1, u_2) \in \mathcal{F}(\Omega_1 \times \Omega_2)$$

by definition.

Assume therefore that

$$(13) \quad \text{supp}\{(dd^c u_i)^{n_i}\} \subset \{z \in \Omega_i : u_i(z) < -\delta\}, \quad i = 1, 2,$$

for some $\delta > 0$. Then

$$u_\delta = \max(u_1, u_2, -\delta) = \max \{ \max(u_1, -\delta) + \delta, \max(u_2, -\delta) + \delta \} - \delta$$

so

$$u_\delta + \delta = \max \{ \max(u_1, -\delta) + \delta, \max(u_2, -\delta) + \delta \}.$$ 

Set $\phi_i = \max(u_i, -\delta) + \delta$, for $i = 1, 2$, then

$$\int_{\{\phi_i > 0\}} (dd^c \phi_i)^{n_i} = 0$$

by the assumption of the support of the Monge-Ampère masses in Equation (13).

Thus we can apply a theorem by Blocki (Theorem 7, [Blo00]) to get

$$(dd^c (u_\delta + \delta))^{n_1 + n_2} = (dd^c (\max(u_1, -\delta)))^{n_1} \wedge (dd^c (\max(u_2, -\delta)))^{n_2}.$$ 

To sum up, we have

$$\int_{\Omega_1 \times \Omega_2} (dd^c u_\delta)^{n_1 + n_2} = \int_{\Omega_1} (dd^c (\max(u_1, -\delta)))^{n_1} \int_{\Omega_2} (dd^c (\max(u_2, -\delta)))^{n_2} = \int_{\Omega_1} (dd^c u_1)^{n_1} \int_{\Omega_2} (dd^c u_2)^{n_2}.$$ 

Where the last equality follows from Stokes’ theorem since $\max(u_1, -\delta) = u_1$ close to the boundaries.

Set $u := \max(u_1, u_2)$. Since $u = \max(u_1, u_2) = \max(u_1, u_2, -\delta) = u_\delta$ outside a compact subset of $\Omega_1 \times \Omega_2$ we also get

$$\int_{\Omega_1 \times \Omega_2} (dd^c u_\delta)^{n_1 + n_2} = \int_{\Omega_1 \times \Omega_2} (dd^c u)^{n_1 + n_2} = \int_{\Omega_1} (dd^c u_1)^{n_1} \int_{\Omega_2} (dd^c u_2)^{n_2}.$$ 

Which proves the required special case. \qed
Example 5.8. Fix a number $0 < r < 1$, and let $u(z, w) = \max \{\log |z|, -(\log |w|)^a\} + \log r$, for $0 < a < 1$. A direct computation gives that
\[
\int_{D_{\rho}} (-\log |w|)^a = \frac{a(1 - a)(-\log r)^{a-2}}{r} dr = \frac{a}{(-\log \rho)^{1-a}}.
\]
Let $\rho$ be the solution of $-(-\log \rho)^a = \log r$. By Theorem 5.7 we get that $u \in \mathcal{F}(D_{\rho} \times D_{\rho})$ with total mass
\[
\int_{D_{\rho} \times D_{\rho}} (dd^c u)^2 = \int_{D_{\rho}} dd^c \log |z| \int_{D_{\rho}} dd^c (-\log |z|)^a = \frac{a}{(-\log \rho)^{1-a}}.
\]
Thus $u \in \mathcal{C} \setminus \mathcal{F}(D_1 \times D_1)$, but we may change $u$ outside a neighbourhood of the origin so that $u$ retains all its properties at the origin but still remain in $\mathcal{F}$.

Example 5.9. More general, let $u(z, w) = \max \{\log |z|, -(\log |w|)^a\}$, for $0 < a < 1$, where $v \in \mathcal{SH}^-(D)$, $v \neq -\infty$. According to Theorem 5.7 $v \in \mathcal{F}$ after we have modified the function close to the boundary. Since $\nu(-(\log |w|)^a, 0) = 0$ we have $\nu(u(z_0), 0) = 0$ and thus, according to Corollary 5.6, $(dd^c u)^2(\{0\}) = 0$.

Example 5.10. Take a sequence of positive numbers $\{a_j\}_{j=0}^{\infty} \subset \mathbb{R}$, such that $\sum_{j=0}^{\infty} a_j < \infty$, and a sequence $\{b_j\}_{j=0}^{\infty} \subset \mathbb{C}$, with $b_j \neq 0$ and $\lim_{j \to \infty} b_j = 0$. Let
\[
u(u, 0) = 0 \text{ Corollary 5.6 implies that } (dd^c u)^2(\{0\}) = 0.
\]

This also follows from Proposition 4.4 since $u \geq l_{\mu}$ on a neighbourhood of the origin, where $\mu = \sum a_j \delta(u_j, 0)$.

For now, we seem to be stuck with a rather annoying condition along the $\mathbb{Z}_2$-axis. When we take away the radially along one of the axis we have to restrict the Lelong number along that axis. However with little effort we can change that condition to a somewhat weaker, or at least more sensible, condition.

Lemma 5.11. Assume that $u \in \mathcal{F}(D^2)$ is radial in the first variable, i.e. $u(|z_1|, z_2) = u(z_1, z_2)$, and let $e_2 = (0, 1)$. If there is a constant $c$, $0 \leq c < +\infty$ such that $\nu(u(z_2), 0) = c$, and $u(|z_1|, z_2) = u(z_1, z_2)$ then
\[
(dd^c u)^2(\{0\}) = (dd^c \max\{c \log |z_2|, u(z_1, z_2)\})^2(\{0\}).
\]

Proof. Set $h_k = \max(-1, 1/k \log |z_1|, 1/k \log |z_2|)$, then $h_k \in \mathcal{E}_0(D^2)$, and as in the second part of the proof of Proposition 2.2 we get
\[
\int_{D^2} -h(dd^c u)^2 \geq \int_{D^2} -h(dd^c (\max\{c \log |z_2|, u(z_1, z_2)\}))^2,
\]
letting $k \to \infty$ we get the inequality
\[
(dd^c u)^2(\{0\}) \geq (dd^c (\max\{c \log |z_2|, u(z_1, z_2)\}))^2(\{0\}).
\]

We will proceed to prove an inequality in the opposite direction.

Let us introduce the auxiliary function $\varphi(\zeta) := u(0, \zeta)$. Note that $\varphi \in \mathcal{SH}^-(D)$. Fix $\zeta \in D$, then we have $\varphi(\zeta) = u(0, \zeta) \leq \sup_{|z|=r} u(z, \zeta) = u(z, \zeta)$, $\forall z \in D$, hence
\[
(14) \quad u = \max\{\varphi, u\}.
\]

Using Riesz decomposition theorem we can write $\varphi = p_{\mu} + h$, where $p_{\mu}$ is the potential of the measure $\Delta \varphi$ and $h$ is harmonic.
Since \( \nu(u_{\mathbf{e}_2}, 0) = (\Delta \varphi)(\{0\}) = c \), we have \( \varphi(\zeta) = c \log |\zeta| + p_{\nu}(\zeta) + h(\zeta) \), where the measure \( \mu' \) has no atom at the origin. By applying Proposition 4.4 to the potential \( p_{\nu} \), we see that there exists an increasing sequence of subharmonic functions \( f_j \), with \( f_j \not
greater 0 \) (except at the origin), and a decreasing sequence of numbers \( r_j \searrow 0 \), such that \( D(r_j) \subset \{ \varphi(\zeta) > c \log |\zeta| + f_j(\zeta) \} \), with \( r_j > 0 \).

Thus on \( D(1) \times D(r_j) \) we have

\[
(15) \quad c \log |z_2| + f_j < u(0, z_2) \leq u(z_1, z_2).
\]

Define a function

\[
v_j(z_1, z_2) := \max(u(z_1, z_2), c \log |z_2| + f_j(z_2)).
\]

Then \( v_j = u \) on \( D(1) \times D(r_j) \), by the Equation (14) and the Inequality (15).

Take the sequence \( \{h_k\} \subset \mathcal{E}_0(D^2) \) as above

\[
\int_{D(1) \times D(r_j)} -h_k(\partial^c u)^2 = \int_{D(1) \times D(r_j)} -h_k(\partial^c v_j)^2 \leq \int_{D^2} -h_k(\partial^c v_j)^2.
\]

Since \( v_j \not\gtrsim \max(u(z_1, z_2), c \log |z_2|) \) pointwise (except at the origin) we get according to Theorem 2.4 that \( (\partial^c v_j)^2 \rightarrow (\partial^c \max(u(z_1, z_2), c \log |z_2|))^2 \). In particular

\[
\int_{D^2} -h_k(\partial^c u)^2 \leq \int_{D^2} -h_k(\partial^c v_j)^2 \rightarrow \int_{D^2} -h_k(\partial^c \max(u(z_1, z_2), c \log |z_2|))^2.
\]

Let \( k \rightarrow \infty \) to get \( (\partial^c u)^2(\{0\}) \leq (\partial^c (\max\{c \log |z_2|, u(z, z_2)\})^2(\{0\}) \), and the desired inequality is proved. \( \square \)

Now we are in position to give the proof of Theorem 1.2, which is a generalization of Proposition 5.4.

**Proof of Theorem 1.2** Since \( u \in \mathcal{PSH} \cap L^\infty(D^2 \setminus K) \) we have that \( u \in \mathcal{F}(D^2) \).

Also \( \nu(u_{\mathbf{e}_2}, 0) = c < +\infty \), since if \( \nu(u_{\mathbf{e}_2}, 0) = +\infty \) then \( u \equiv -\infty \) on a neighbourhood of the origin along the \( z_2 \)-axis, but then \( u \not\in L^\infty(\Omega \setminus K) \). In the same way there is a \( M > 0 \) such that \( u(z, w) > M \log |z| \).

According to Lemma 5.11 we have

\[
(\partial^c u)^2(\{0\}) \leq (\partial^c (\max\{c \log |w|, u(z, w)\}))^2(\{0\}).
\]

Since

\[
\nu(\max\{c \log |w|, u(z, w)\}, 0) \leq \nu(u(z, w), 0) = 0,
\]

and \( \max\{M \log |z|, c \log |w|\} \leq \max\{c \log |w|, u(z, w)\} \), Proposition 2.3 gives

\[
(\partial^c (\max\{c \log |w|, u(z, w)\})(\{0\}))^2 = 0,
\]

thus \( (\partial^c u)^2(\{0\}) = 0 \). \( \square \)

**Remark** Theorem 1.2 seems to be a very curious theorem, indeed. It may be generalized to functions of the type \( u(z_1, \ldots, z_{n-1}, z_n) = u(|z_1|, \ldots, |z_{n-1}|, z_n) \), by a similar reasoning and using the ideas of [Wik04]. However the truly interesting generalization to functions of type \( u(z_1, z_2, \ldots, z_n) = u(z_1, z_2, \ldots, |z_n|) \), which of course would answer the main question, is elusive to the author.
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