Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers

V. Dotenko

LPTMC, Université Paris VI - 75252 Paris, France, EU and
L.D. Landau Institute for Theoretical Physics - 119334 Moscow, Russia

Abstract – The distribution function of the free-energy fluctuations in one-dimensional directed polymers with δ-correlated random potential is studied by mapping the replicated problem to a many-body quantum boson system with attractive interactions. Performing the summation over the entire spectrum of excited states the problem is reduced to the Fredholm determinant with the Airy kernel which is known to yield the Tracy-Widom distribution.

Copyright © EPLA, 2010

Introduction. – Directed polymers in a quenched random potential have been the subject of intense investigations during the past two decades (see, e.g., [1]). In the most simple one-dimensional case we deal with an elastic string directed along the τ-axis within an interval [0, L]. Randomness enters the problem through a disorder potential $V[\phi(\tau), \tau]$, which competes against the elastic energy. The problem is defined by the Hamiltonian

$$H[\phi(\tau), V] = \int_0^L d\tau \left\{ \frac{1}{2} \frac{\partial}{\partial \tau} \phi(\tau)^2 + V[\phi(\tau), \tau] \right\},$$

(1)

where $V[\phi, \tau]$ is Gaussian distributed with a zero mean $\langle V[\phi, \tau] \rangle = 0$ and the δ-correlations

$$\langle V(\phi, \tau)V(\phi', \tau') \rangle = u\delta(\tau - \tau')\delta(\phi - \phi').$$

(2)

Here the parameter $u$ describes the strength of the disorder. Historically, the problem of central interest was the scaling behavior of the polymer mean squared displacement which in the thermodynamic limit ($L \to \infty$) is believed to have a universal scaling form $\langle \phi^2 \rangle(L) \propto L^{2\zeta}$ (where $\langle \ldots \rangle$ and $\{ \ldots \}$ denote the thermal and the disorder averages), with $\zeta = 2/3$, the so-called wandering exponent. A more general problem for all directed polymer systems of the type, eq. (1), is the statistical properties of their free-energy fluctuations. Besides the usual extensive (linear in $L$) self-averaging part $f_0 L$ (where $f_0$ is the linear free-energy density), the total free energy $F$ of such systems contains disorder-dependent fluctuating contribution $\tilde{F}$, which is characterized by non-trivial scaling in $L$. It is generally believed that in the limit of large $L$ the typical value of the free-energy fluctuations scales with $L$ as $\tilde{F} \propto L^{1/3}$ (see, e.g., [2–5]). In other words, in the limit of large $L$ the total (random) free energy of the system can be represented as

$$F = f_0 L + c L^{1/3} f,$$

(3)

where $c$ is the parameter, which depends on the temperature and the strength of disorder, and $f$ is the random quantity which in the thermodynamic limit $L \to \infty$ is described by a non-trivial universal distribution function $P_\star(f)$. The derivation of this function for the system with δ-correlated random potential, eqs. (1), (2) is the central issue of the present work.

For the string with the zero boundary conditions at $\tau = 0$ and at $\tau = L$ the partition function of a given sample is

$$Z[V] = \int_{\phi(0)=0}^{\phi(L)=0} D[\phi(\tau)] e^{-\beta H[\phi, V]},$$

(4)

where $\beta$ denotes the inverse temperature. On the other hand, the partition function is related to the total free energy $F[V]$ via $Z[V] = \exp(-\beta F[V])$. The free energy $F[V]$ is defined for a specific realization of the random potential $V$ and thus represent a random variable. Taking the $N$-th power of both sides of this relation and performing the averaging over the random potential $V$ we obtain

$$Z^N[V] \equiv Z[N, L] = \exp(-\beta N F[V]),$$

(5)
where the quantity in the lhs of the above equation is called the replica partition function. Substituting here \( F = f_0 L + c L^{1/3} \), and redefining \( Z[N,L] = \tilde{Z}[N,L] \exp\{-\beta N f_0 L\} \) we get
\[
\tilde{Z}[N,L] = \exp(-\lambda N f), \tag{6}
\]
where \( \lambda = \beta c L^{1/3} \). The averaging in the rhs of the above equation can be represented in terms of the distribution function \( P_L(f) \) (which depends on the system size \( L \)). In this way we arrive at the following general relation between the replica partition function \( \tilde{Z}[N,L] \) and the distribution function of the free energy fluctuations \( P_L(f) \):
\[
\tilde{Z}[N,L] = \int_{-\infty}^{+\infty} df \: P_L(f) \: e^{-\lambda N f}. \tag{7}
\]
Of course, the most interesting object is the thermodynamic limit distribution function \( P_\ast(f) = \lim_{L \to \infty} P_L(f) \) which is expected to be the universal quantity. The above equation is the bilateral Laplace transform of the function \( P_L(f) \), and at least formally it allows to restore this function via inverse Laplace transform of the replica partition function \( \tilde{Z}[N,L] \). In order to do so, one has to compute \( \tilde{Z}[N,L] \) for an arbitrary integer \( N \) and then perform analytical continuation of this function from integer to arbitrary complex values of \( N \). In Kardar’s original solution [5], after mapping the replicated problem to interacting quantum bosons, one arrives at the replica partition function for positive integer parameters \( N > 1 \). Assuming a large \( L \to \infty \) limit, one is tempted to approximate the result by the ground-state contribution only, as for any \( N > 1 \) the contributions of excited states are exponentially small for \( L \to \infty \). However, in the analytic continuation for arbitrary complex \( N \) the contributions which are exponentially small at positive integer \( N > 1 \) can become essential in the region \( N \to 0 \), which defines the function \( P(f) \) (in other word, the problem is that the two limits \( L \to \infty \) and \( N \to 0 \) do not commute [6,7]). Thus, it is the neglect of the excited states which is the origin of non-physical nature of the obtained solution.

In my recent paper [8] an attempt has been made to derive the free-energy distribution function via the calculation of the replica partition function \( \tilde{Z}[N,L] \) in terms of the Bethe-Ansatz solution for quantum bosons with attractive \( \delta \)-interactions which involved the summation over the entire spectrum of exited states. Unfortunately, the attempt has failed because on one hand, the calculations contained a kind of a hidden “uncontrolled approximation”, and on the other hand, the analytic continuation of obtained \( Z(N,L) \) was found to be ambiguous.

It turns out that it is possible to bypass the problem of the analytic continuation if instead of the distribution function itself one would study its integral representation
\[
W(x) = \int_x^{+\infty} df \: P_\ast(f), \tag{8}
\]
which gives the probability to find the fluctuation \( f \) bigger than a given value \( x \). Formally the function \( W(x) \) can be defined as follows:
\[
W(x) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp(\lambda N x) \frac{\tilde{Z}[N,L]}{N}. \tag{9}
\]
On the other hand, in terms of the replica approach the function \( W(x) \) is given by the series
\[
W(x) = \lim_{\lambda \to \infty} \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \exp(\lambda N x) \tilde{Z}[N,L]. \tag{10}
\]
In the present paper the replica partition function \( \tilde{Z}[N,L] \) will be calculated (again) by mapping the replicated problem to the \( N \)-particle quantum boson system with attractive interactions. Performing the summation over the entire spectrum of excited states the summation of the series, eq. (10), is reduced to the Fredholm determinant with the so-called Airy kernel which is known to yield the Tracy-Widom distribution. Originally this distribution has been derived in the context of the statistical properties of the Gaussian unitary ensemble [9] while at present it is well established to describe the statistics of fluctuations in various random systems [10–15] which are widely believed to belong to the same universality class as the present model [16–18]. Quite recently, by the other method, the one-point distribution of the solutions of the KPZ-equation (which is equivalent to the present model) has been derived for an arbitrary value of \( L \), and it has been shown that in the limit \( L \to \infty \) this distribution turns into the Tracy-Widom distribution [19,20]. While this manuscript was in course of preparation I have learned that exactly the same result for the system considered in this paper has been independently derived by Calabrese, Le Doussal and Rosso [21].

Performing simple Gaussian average over the random potential, eq. (2), for the replica partition function, eq. (5), we obtain the standard expression
\[
Z(N,L) = \prod_{a=1}^{N} \int_{\phi_a(0)=0}^{\phi_a(L)=0} D\phi_a(\tau) \: e^{-\beta H_N[\phi]} \tag{11}
\]
where
\[
H_N[\phi] = \frac{1}{2} \int_0^L d\tau \left( \sum_{a=1}^{N} \left[ \partial_\tau \phi_a(\tau) \right]^2 - \beta u \sum_{a \neq b}^{N} \delta \left[ \phi_a(\tau) - \phi_b(\tau) \right] \right) \tag{12}
\]
is the \( N \)-component scalar field replica Hamiltonian and \( \phi \equiv \{\phi_1, \ldots, \phi_N\} \).

According to the above definition this partition function describes the statistics of \( N \) \( \delta \)-interacting (attracting) trajectories \( \phi_a(\tau) \) all starting (at \( \tau = 0 \)) and ending (at \( \tau = L \)
\[ W(x) = \lim_{\lambda \to \infty} \left\{ 1 + \sum_{M=1}^{\infty} \frac{(-1)^M}{M!} \int_{-\infty}^{+\infty} dy_a dy_b \frac{e^{i\lambda n_{a\beta}(y_a + \lambda n_{a\beta} + iy_b)}}{2\pi} \right\} \]

(21)

where \( \delta(k, m) \) is the Kronecker symbol (which allows to extend the summation over \( n_{a\beta}'s \) to infinity). Using the explicit form of the wave functions \( \Psi^{(M)}(x) \) the above expression (after somewhat painful algebra) reduces to (see [8] for details)

\[ Z(N, L) = e^{-\beta N L f_0} \tilde{Z}(N, \lambda) \]

(17)

where \( f_0 = \frac{1}{24} \beta^4 u^2 - \frac{1}{12} \ln(\beta^3 u) \) is the linear (self-averaging) free-energy density, and

\[ \tilde{Z}(N, \lambda) = N! \int_{-\infty}^{+\infty} \frac{dy dp}{4\pi \lambda N} \text{Ai}(y + p^2) e^{\lambda N y} \]

(18)

with \( \text{Ai}(t) \) the Airy function, and instead of the system length \( L \) we have introduced a new parameter \( \lambda = \frac{1}{2} (\beta^5 u^2 L)^{1/3} \). The first term in the above expression is the contribution of the ground state (\( M = 1 \)), and the next terms (\( M \geq 2 \)) are the contributions of the rest of the energy spectrum.

Using the Cauchy double alternant identity

\[ \prod_{\alpha \beta} (a_{\alpha} - a_{\beta})(b_{\alpha} - b_{\beta}) = \prod_{\alpha = 1}^{M} (a_{\alpha} - b_{\alpha}) = (-1)^{N(N-1)/2} \det \left[ \frac{1}{a_{\alpha} - b_{\beta}} \right]_{\alpha,\beta=1,...,M} \]

(19)

the product term in eq. (18) can be represented in the determinant form

\[ \prod_{\alpha\beta} (a_{\alpha} - a_{\beta})(b_{\alpha} - b_{\beta}) = \prod_{\alpha = 1}^{M} (2\lambda n_{\alpha}) \det \left[ \frac{1}{\lambda n_{\alpha} - i\beta_\alpha + \lambda n_{\beta} + i\beta_\beta} \right]_{\alpha,\beta=1,...,M} \]

(20)

Substituting now the expression for the replica partition function \( \tilde{Z}(N, \lambda) \) into the definition of the probability function, eq. (10), we can perform summation over \( N \) (which would lift the constraint \( \sum_{\alpha=1}^{M} n_{a\beta} = N \)) and obtain:

\[ \text{see eq. (21) above} \]

20003-p3
The above expression in nothing else but the expansion of the Fredholm determinant \( \det(1 - K) \) (see, e.g., [26]) with the kernel
\[
K \equiv K[(n, p); (n', p')] = \left[ \int_{-\infty}^{+\infty} dy \, \text{Ai}(y + p^2)(-1)^{n-1}e^{\lambda_n(y + x)} \right] \frac{1}{\lambda n - ip + \lambda n' + ip'}
\]
(22)

Using the exponential representation of this determinant we get
\[
W(x) = \lim_{\lambda \to \infty} \exp \left[ -\sum_{M=1}^{\infty} \frac{1}{M} \text{Tr} K^M \right],
\]
(23)

where
\[
\text{Tr} K^M = \prod_{\alpha=1}^{M} \int_{-\infty}^{+\infty} \frac{dy_\alpha \, dp_\alpha}{2\pi} \, \text{Ai}(y_\alpha + p_\alpha^2) \sum_{n_\alpha = 1}^{\infty} (-1)^{n_\alpha - 1} e^{\lambda_n(y_\alpha + x)}
\]
\[
\times \frac{1}{(\lambda n_1 - ip_1 + \lambda n_2 + ip_2) \ldots (\lambda n_M - ip_M + \lambda n_1 + ip_1)}.
\]
(24)

Substituting here
\[
\frac{1}{\lambda n_\alpha - ip_\alpha + \lambda n_{\alpha+1} + ip_{\alpha+1}} = \int_0^\infty d\omega_\alpha \exp\left[ -(\lambda n_\alpha - ip_\alpha + \lambda n_{\alpha+1} + ip_{\alpha+1})\omega_\alpha \right]
\]
(25)

one can easily perform the summation over \( n_\alpha \)'s. Taking into account that
\[
\lim_{\lambda \to \infty} \sum_{n=1}^{\infty} (-1)^{n-1}e^{\lambda n z} = \lim_{\lambda \to \infty} \frac{e^{\lambda z}}{1 + e^{\lambda z}} = \theta(z)
\]
(26)

and shifting the integration parameters, \( y_\alpha \to y_\alpha - x + \omega_\alpha + \omega_{\alpha-1} \) and \( \omega_\alpha \to \omega_\alpha + x/2 \), we obtain

see eq. (27) above

where by definition it is assumed that \( \omega_0 \equiv \omega_M \). Using the Airy function integral representation, and taking into account that it satisfies the differential equation, \( \text{Ai}(t) = t \text{Ai}(t) \), one can easily perform the following integrals:
\[
\int_0^{\infty} dy \, \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \, \text{Ai}(y + p^2 + \omega + \omega') e^{ip(-\omega - \omega')}
\]
\[
= 2^{-1/3} \int_0^{+\infty} dy \text{Ai}(2^{1/3}\omega + y) \text{Ai}(2^{1/3}\omega' + y)
\]
\[
= \frac{\text{Ai}(2^{1/3}\omega)\text{Ai}'(2^{1/3}\omega') - \text{Ai}'(2^{1/3}\omega)\text{Ai}(2^{1/3}\omega')}{\omega - \omega'},
\]
(28)

Redefining \( \omega_\alpha \to \omega_\alpha 2^{-1/3} \), we find
\[
\lim_{\lambda \to \infty} \text{Tr} K^M = \int \int \ldots \int_{-x/2}^{+\infty} d\omega_1 d\omega_2 \ldots d\omega_M K^*(\omega_1, \omega_2)
\]
\[
\times K^*(\omega_2, \omega_3) \ldots K^*(\omega_M, \omega_1),
\]
(29)

where
\[
K^*(\omega, \omega') = \frac{\text{Ai}(\omega)\text{Ai}'(\omega') - \text{Ai}'(\omega)\text{Ai}(\omega')}{\omega - \omega'}
\]
(30)

is the so-called Airy kernel. This proves that in the thermodynamic limit, \( L \to \infty \), the probability function \( W(x) \), eq. (8), is defined by the Fredholm determinant,
\[
W(x) = \det[1 - K^*] = F_2(-x/2^{2/3}),
\]
(31)

where \( K^* \) is the integral operator on \([-x/2^{2/3}, \infty) \) with the Airy kernel, eq. (30). The function \( F_2(s) \) is the Tracy-Widom distribution [9]
\[
F_2(s) = \exp\left( -\int_s^{\infty} dt \, (s - t) q^2(t) \right),
\]
(32)

where the function \( q(t) \) is the solution of the Painlevé II equation, \( q'' = t + 2q^3 \) with the boundary condition, \( q(t \to \pm \infty) \sim \text{Ai}(t) \). This distribution was originally derived as the probability distribution of the largest eigenvalue of a \( n \times n \) random Hermitian matrix in the limit \( n \to \infty \). At present there are exists an appreciable list of statistical systems (which are not always look similar) in which the fluctuations of the quantities which play the role of “energy” are described by the same distribution function \( F_2(s) \). These systems are: the polynuclear growth (PNG) model [10], the longest increasing subsequences (LIS) model [11], the longest common subsequences (LCS) [12], the oriented digital boiling model [13], the ballistic decomposition model [14], the zero-temperature lattice version of the directed polymers with an exponential and geometric site-disorder distribution [15]. Now we can add to this list the model of one-dimensional directed polymers with Gaussian \( \delta \)-correlated random potential.

REFERENCES

[1] HALPIN-HEALY T. and ZHANG Y.-C., Phys. Rep., 254 (1995) 215.
[2] HUSE D. A., HENLEY C. L. and FISHER D. S., Phys. Rev. Lett., 55 (1985) 2924.
[3] HUSE D. A. and HENLEY C. L., Phys. Rev. Lett., 54 (1985) 2708.
[4] KARDAR M. and ZHANG Y.-C., Phys. Rev. Lett., 58 (1987) 2087.
Bethe ansatz derivation of the Tracy-Widom distribution for one-dimensional directed polymers

[5] Kardar M., Nucl. Phys.B, 290 (1987) 582.
[6] Medina E. and Kardar M., J. Stat. Phys., 71 (1993) 967.
[7] Dotsenko V. S., Ioffe L. B., Geshkenbein V. B., Korshunov S. E. and Blatter G., Phys. Rev. Lett., 100 (2008) 050601.
[8] Dotsenko V. and Klumov B., J. Stat. Mech. (2010) P03022.
[9] Tracy C. A. and Widom H., Commun. Math. Phys., 159 (1994) 151.
[10] Pr"ahofer M. and Spohn H., Phys. Rev. Lett., 84 (2000) 4882.
[11] Baik J., Deift P. A. and Johansson K., J. Am. Math. Soc., 12 (1999) 1119.
[12] Majumdar S. N. and Nechaev S., Phys. Rev. E, 72 (2005) 020901(R).
[13] Gravner J., Tracy C. A. and Widom H., J. Stat. Phys., 102 (2001) 1085.
[14] Majumdar S. N. and Nechaev S., Phys. Rev. E, 69 (2004) 011103.
[15] Johansson K., Commun. Math. Phys., 209 (2000) 437.
[16] Brunet E. and Derrida B., Phys. Rev. E, 61 (2000) 6789.
[17] Kolokolov I. V. and Korshunov S. E., Phys. Rev. B, 75 (2007) 140201(R).
[18] Pr"ahofer M. and Spohn H., J. Stat. Phys., 115 (2004) 255.
[19] Sasamoto T. and Spohn H., arXiv:1002.1873; arXiv:1002.1879; arXiv:1002.1883.
[20] Amir G., Corwin I. and Quastel J., arXiv:1003.0443.
[21] Calabrese P., Le Doussal P. and Rosso A., EPL, 90 (2010) 20002, this issue.
[22] McGuire J. B., J. Math. Phys., 5 (1964) 622.
[23] Yang C. N., Phys. Rev., 168 (1968) 1920.
[24] Takahashi M., Thermodynamics of One-dimensional Solvable Models (Cambridge University Press) 1999.
[25] Calabrese P. and Caux J.-S., Phys. Rev. Lett., 98 (2007) 150403.
[26] Mehta M. L., Random Matrices (Elsevier, Amsterdam) 2004.