Neutron wave packet tomography

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A tomographic technique is introduced in order to determine the quantum state of the center of mass motion of neutrons. An experiment is proposed and numerically analyzed.

In experimental physics one often faces the following question: “Given the outcomes of a particular set of measurements, which quantum state do they imply?” Such inverse problems may arise for instance when setting up and calibrating laboratory sources of quantum states, or in the analysis of decoherence and other deteriorating effects of the environment, or in some special tasks in quantum information processing such as eavesdropping on a quantum channel in quantum cryptography.

The determination of the quantum state represents a highly nontrivial problem, whose history can be traced back to the early days of quantum mechanics, namely to the Pauli problem [1, 2]: the experimental validation had to wait until quantum optics opened a new era. The theoretical predictions of Vogel and Risken [3] were closely followed by the experimental realization of the suggested algorithm by Smithey et al. [4]. Since then many improvements and new techniques have been proposed: an up-to-date overview can be found in Ref. [4]. Recent progress in instrumentation has made it possible to apply these techniques to a variety of different quantum systems such as fields in optical cavities, polarization and external degrees of freedom of photons, or motional states of atoms.

In this Letter we propose an experiment for determining the quantum states of the center-of-mass motion of neutrons. In accordance with quantum theory, these massive particles can be associated with a wave function describing their motional state. Neutrons are suitable objects for many quantum mechanical experiments due to their interaction with all four basic forces, the ease of detecting them with almost 100% efficiency, and their small coupling to the environment [4]. In marked contrast with light, neutron vacuum field and thermal background can usually be ignored. This makes it possible, for instance, to prepare superpositions of macroscopically separated quantum states—the so-called Schrödinger cat states—that would be extremely difficult to realize with other quantum systems because of their fragility with respect to decoherence. In all experiments performed so far, the existence of the Schrödinger cat states of neutrons has been indirectly demonstrated via interferometric effects, but the full evidence for the nonclassicality of these states, including the presence of the negative values of the reconstructed Wigner function, is still missing.

In the following, we will first briefly review the present neutron interferometric techniques and the means of creating highly nonclassical motional states of neutrons. In the second part of the Letter, an experiment will be proposed for the complete reconstruction of these quantum states.

**Neutron tomography** - The set of measurements that can be done on neutrons to determine their quantum state is severely limited by the very low time resolution of the available detectors. In quantum optics, this obstacle can be overcome by mixing the weak input field with a strong local oscillator. By changing the phase \( \phi \) of the oscillator one can measure the spectral decompositions of all quadratures,

\[
\hat{X}_\phi = \hat{x} \cos \phi + \hat{p} \sin \phi, \quad (1)
\]

\( \hat{x} \) and \( \hat{p} \) being the canonically conjugated operators of position and momentum. Of course, no such local oscillators exist for neutrons. However, notice that massive particles experience a transformation of the type \( \hat{X}_\phi \) in the course of free evolution: \( x(t) = x + (p/m)t \), where \( m \) is the mass. Thus free evolution of the wave packet followed by a position sensitive measurement yields information about a subset of quadratures \( \hat{X}_\phi, \phi \in [0, \pi/2] \). Free evolution was utilized e.g. for the reconstruction of transversal motional states of helium atoms in [7]. Here we are interested in the longitudinal degrees of freedom. Since neutron detectors have very bad time resolution, free evolution alone cannot be used to generate a tomographically complete set of measurements.

Feasible measurements on thermal neutrons consist of measurements of the contrast and phase of interference fringes in an interferometric setup, see Fig. [8] (without momentum kick), and also spectral analysis of the neutron beam using an adjustable Bragg-reflecting crystal plate together with a position sensitive detector. This set of observables is not tomographically complete because the measurable (complex) contrast of the interference pattern \( \Gamma(\Delta x) \) (\( \hbar = 1 \)),

\[
\Gamma(\Delta x) = \langle \psi | e^{i\Delta \hat{p}} | \psi \rangle = \int |a(p)|^2 e^{i\Delta x p} dp, \quad (2)
\]

is not sensitive to the phase of \( a(p) = \langle p | \psi \rangle \), and no information about quadratures other than \( p \) is available.
Obviously, the situation would be different if one could shift both the position (phase) and the momentum of the incoming wave packet inside the interferometer. Such a thought experiment is shown in Fig. 1. In that case, the Wigner function describing the ensemble of measured neutrons would be related to the measured contrast of Fig. 1. As a result of the interaction between the neutron and the coil, the $y_+$ ($y_-$) component of the input state will be decelerated (accelerated). Assuming that the region of interaction is short, so that the dispersion of the wave packet of the neutron can be neglected in the coil, in the quasi-monochromatic approximation, the net momentum transfer can be described by the effective unitary operator,

$$U_2 = e^{-i\Delta p \hat{\mathbf{p}}/2} | y_- \rangle \langle y_+ | + e^{i\Delta p \hat{\mathbf{p}}/2} | y_+ \rangle \langle y_- |.$$  (5)

where

$$\Delta p = \frac{2\mu B m}{p_0}.  \tag{6}$$

Prior to detection, the particles are polarized along the +z direction again, so as to erase the which-way information stored in the polarization degree of freedom. The probability of a neutron being detected is given by the norm of the transmitted component,

$$P = \text{Tr}\{\Pi(\Delta p, \Delta x) \rho\},  \tag{7}$$

where $\rho$ refers only to the spatial degrees of freedom and

$$\Pi(\Delta p, \Delta x) = \langle z_+ | U_1^\dagger U_2^\dagger | z_+ \rangle \langle z_+ | U_2 U_1 | z_+ \rangle$$

$$= (1 + e^{i\hat{\mathbf{p}} \cdot \mathbf{L}/p_0})/4 + \text{h.c.} \tag{8}$$

$$= (1 + e^{i\Delta \hat{\mathbf{p}} \cdot \mathbf{L}/p_0})/4 + \text{h.c.} \tag{9}$$

where we denoted

$$\Delta x = \frac{\Delta p L}{p_0} = \frac{2\mu B m L}{p_0^2}. \tag{11}$$
Direct inversion - The detection probability reads
\[ P(\Delta p, \Delta x) = \frac{1}{2} + \frac{1}{2} \text{Re} \{ \Gamma(\Delta p, \Delta x) e^{2i\Delta x \Delta p} \}. \] (12)
Since the beam is quasi-monochromatic one has for \( \delta x = \pi/2p_0 \),
\[ \Gamma(\Delta p, \Delta x + \delta x) \simeq \Gamma(\Delta p, \Delta x) e^{i\pi/2} \] (13)
from which the imaginary part of the complex degree of coherence \( \Gamma(\Delta p, \Delta x) \) can be obtained.

Summarizing, the tomography of a neutron state consists in the following four steps:
(i) A set of pairs of independent variables \( \{B_j, L_j\} \) is chosen covering a certain range \( B \in [0, B_{\text{MAX}}] \) and \( L \in [0, L_{\text{MAX}}] \).
(ii) For each pair \( B_j, L_j \) the shifts \( \Delta p_j \) in (4) and \( \Delta x_j \) in (11) are calculated, and the corresponding intensities \( P(\Delta p_j, \Delta x_j) \) are measured with and without an auxiliary shift \( \delta x = \pi/2 \).
(iii) The complex degree of coherence \( \Gamma(\Delta p_j, \Delta x_j) \) is calculated from the two intensities using Eqs. (12 - 13).
(iv) Finally, the Wigner function of the input neutrons is calculated with the help of inversion formula (11), where the integrals are approximated by sums over \( \Delta x_j \) and \( \Delta p_j \).

According to (11), the contrast \( \Gamma(\Delta p, \Delta x) \) is essentially the Fourier transform of the Wigner function \( W(x, p) \). Therefore the largest values of \( \Delta p \) and \( \Delta x \) are related to the smallest resolved details in \( x \) and \( p \) respectively. Namely (reinserting \( \hbar \)),
\[ \Delta p_{\text{MAX}} = \hbar/\delta p_{\text{min}}, \quad \Delta x_{\text{MAX}} = \hbar/\delta x_{\text{min}}, \] (14)
where \( \delta x_{\text{min}} \) and \( \delta p_{\text{min}} \) denote the \( x \) and \( p \) resolutions. By (11) and (11) one gets
\[ \delta x_{\text{min}} = \frac{\hbar}{2\mu m B_{\text{MAX}}}, \quad \delta p_{\text{min}} = \frac{p_0}{L} \delta x_{\text{min}}. \] (15)
For a neutron of wavelength \( \lambda_0 = 0.37 \text{nm} \), assuming the reasonable values \( L_{\text{MAX}} = 1 \text{m} \) and \( B_{\text{MAX}} = 0.1 \text{T} \) one gets \( \delta x_{\text{min}} = 60 \mu m \) and \( \delta p_{\text{min}} = \hbar 	imes 10^9 \text{m}^{-1} \).

Radon inversion - It is interesting to give an alternative interpretation of the proposed measurement in Fig. 2. Notice, that the POVM elements in Eq. (11) can also be restated in terms of quadrature operators,
\[ \Pi(\Delta p, \Delta x) = (1/4)(1 + e^{i\omega \hat{X}_\theta}) + \text{h.c.}, \] (16)
where (in fixed units)
\[ \hat{X}_\theta = \cos \theta \hat{x} + \sin \theta \hat{p}, \quad \tan \theta = \frac{\Delta x}{\Delta p} = \frac{L}{p_0}, \] (17)
and \( \omega = \sqrt{\Delta x^2 + \Delta p^2} \). Thus, for a fixed \( \theta \), the data contain information about the characteristic function of the quadrature \( \hat{X}_\theta \),
\[ P(\Delta p, \Delta x) = 1/2 + \text{Re} \{ C_{X_\theta}(\omega) \}/2, \] (18)
\[ \langle C_{X_\theta}(\omega) \rangle = \int P_{\text{gauss}}(x) e^{i\omega x} dx. \] (19)

By changing \( L \) one changes the quadrature measured, while \( \omega \), which depends on both \( L \) and \( B \), determines the observed spatial frequency of the probability distribution of this quadrature. The observed quadratures range from \( \hat{x} \) (for \( L = 0 \)) to \( \hat{p} \) (for \( L \rightarrow \infty \)). From the measurement of \( C_{X_\theta}(\omega) \), the “shadows” \( P_{\text{gauss}}(x) \) of the Wigner function can be obtained by the Fourier transformation, which in turn yield the Wigner function by an inverse Radon transformation. This is an alternative way of reconstructing the Wigner function from the measured data in the setup Fig. 2.

Statistical inversion - The procedures outlined above, based on the direct inversion formula (11), have several drawbacks: (i) Realistic data are always noisy. In that case, formula (4) can yield unphysical results, such as the Wigner representation of a non-positive definite operator. (ii) The Wigner function in Eq. (11) depends on the measured data indirectly, through the complex degree of coherence \( \Gamma \), which itself has to be estimated with the help of an auxiliary position shifter. This intermediate step is, certainly, not necessary as all available information about the Wigner function of the incoming neutrons is contained in the raw data measured without any auxiliary position shift. In order to avoid these problems, we propose to use the maximum-likelihood quantum state reconstruction [5, 9, 10]. The main advantages of this method compared to the above direct inversion are: (i) Asymptotically, for large data samples it provides the best performance available. (ii) Any prior information about the measured neutrons and the known statistics of the experiment can be used to increase the accuracy of the reconstruction. (iii) The existing physical constraints can be easily incorporated into the reconstruction. Most notably, this technique guarantees the positivity of the reconstructed density operator. (iv) It can be applied directly to raw counted data.

Assuming that the statistics of the experiment is Poissonian, the maximum-likelihood reconstruction amounts to minimizing the Kullback-Leibler distance (relative entropy) between the measured data \( f(\Delta x, \Delta p) \) and the renormalized theoretical probabilities \( p(\Delta x, \Delta p)/\sum p \) of Eq. (17). As has been shown in [5, 10], the maximum-likelihood density matrix can be obtained as a fixed point of the iterations of a nonlinear operator map.

As follows from the parameter estimates given after Eq. (15), the proposed tomography scheme using thermal neutrons will likely have sufficient resolution in momentum. On the other hand, even for well monochromatized thermal beams, the resolution in position is expected to be worse (possibly even by several orders of
exceeding the corresponding coherence lengths of the in-
spatially separated Gaussian states (Schrödinger cats), 

The imaging of non-classical states is a much more
difficult task than typical coherence lengths. The sim-
ulations in Figure 3 illustrate the effect of the restricted
range of $\Delta p$ on the reconstruction. Consider first the re-
construction of a minimum uncertainty Gaussian wave
packet in its moving frame, parameterized by its co-
erherence length $l_{coh}$, $|\Psi_G\rangle \propto \int \exp(-k^2 l_{coh}^2)|k\rangle dk$. (The
choice of a minimum uncertainty state is only for illus-
trative purposes.) Provided the apparatus has a suf-
cient spatial resolution, $\delta x_{\min} < l_{coh}$, a faithful recon-
struction is readily obtained, see the upper left panel. More realistic measurement with $\delta x_{\min} > l_{coh}$ would ob-
viously yield a Wigner function smoothed out along the $x$
axis. However, the states measured in a real experiment
are not going to be minimum uncertainty states. The ex-
perimenter will rather deal with time evolved states
$|\Psi(T)\rangle \propto \int \exp(ik^2T/2m - k^2 l_{coh}^2)|k\rangle dk$ that are strongly
affected by dispersion. As a consequence, the wave packet
spread very soon becomes larger than the resolution
limit, $\delta x_T \sim T/(ml_{coh}) \gg \delta x_{\min}$, and a good recon-
struction can be achieved with a realistic apparatus. Compare
the upper middle and right panels, showing reconstruc-
tions with a sufficient resolution $\delta x_{\min} = l_{coh}/2$ and a
reduced (but more realistic) resolution $\delta x_{\min} = 10 l_{coh}$.

The imaging of non-classical states is a much more
delicate task. Let us consider the superpositions of
spatially separated Gaussian states (Schrödinger cats),
$|\Psi_{cat}\rangle \propto [1 + \exp(i p \Delta)]|\Psi_G\rangle$. Such states can be pre-
pared e.g. by means of a double loop perfect crystal in-
terferometer \cite{11}. As has been shown \cite{8}, the preparation
of thermal neutron cat states is possible, with separations
exceeding the corresponding coherence lengths of the in-
dividual components $\Delta \gg l_{coh}$. Provided the apparatus
has sufficient resolution, the nonclassicality of this state
is manifested by the negative regions of the reconstructed
Wigner function, see the lower left panel of Fig. 3. Tak-
ing dispersion into account, the ordering of the relevant
parameters is $l_{coh} < \Delta \ll \delta x_T$, and one can easily obtain
$\delta x_{\min} < \delta x_T$. As the simulations show, a realistic mea-
surement whose position resolution is much worse than
the coherence lengths of the individual cat state compo-
ents tends to wipe out the negative regions of the re-
constructed Wigner function; compare the lower middle
and right panels of Fig. 3. On the other hand, the main
features of such exotic states, such as their non-Gaussian
character and also the global spatial properties of which
little is known today, should still be accessible to a re-
alistic wave packet tomography. To resolve more subtle
quantum interference effects of the order of the coher-
ence length, more refined experimental techniques may
however be needed. An idea could be to replace thermal
neutrons by ultracold neutrons, for which much larger
momentum shifts $\Delta p$ (and thus much smaller $\delta x_{\min}$, pos-
ibly even smaller than $\Delta$) can be obtained.

In conclusion, we have proposed and analyzed an ex-
perimental scheme for determining the motional states
of neutrons. With the help of a magnetic field and free
propagation, this apparatus realizes quadrature measure-
ments on neutrons by measuring overlaps of the two
transformed components of the initial state. This is an
analog of the quantum optical homodyne detection in
neutron optics, achieved without the use of a strong co-
erherent source of neutrons.

This work was supported by the Bilateral Research
Program between Italy and the Czech Republic PH1 on
“Decoherence and Quantum Measurements” and by the
Research Project MSM 6198959213 of the Czech Ministry
of Education.

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