The settlement speed of the finite width foundation caused by the extrusion of weak soil layer from compressible base

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Abstract. The article presents the formulation and solution of quantifying the speed of the finite width foundation settlement due to viscoplastic or elastic-plastic extrusion of the weak soil layer from compressible base by analytical and numerical methods. During numerical simulations were used the finite element method with Mohr-Coulomb model (MC) and the model of Soft Soil Creep (SSC). According to the results of analytical solution were obtained the formulas to determine an initial ultimate load and a value and the foundation settlement speed.

1. Introduction

The weak layer existence in the compressible strata with the loaded footing on it leads to high probability of its extrusion from the soil base. It can be a reason of large soil base vertical displacements and also the foundation settlements. Thus, the problem is need to be solved, to quantify this probability. It should be noted that this logical situation take place in soil bases of ground dams, as well as at the base of deep excavations.

This work deals with the quantification of values of settlement and the speed of settling of the foundation of finite width due to the soft soil layer extrusion from the compressible strata (fig.1).

An importance question of the issue is the problem of load distribution, the vertical stress at the top of the weak layer. In different papers accepted the outlines of the plot in the form of a triangle or parabola [2-6]. In the present work adopted a distribution using the normal distribution of Gauss.
Figure 1. The design scheme of interaction of the foundation of finite width \((b=2A)\) with a weak layer of finite thickness \((2h)\) and length \((2l>>2a)\).

2. Initial equations

As a design for the analytical solution of the problem is proposed rheological equation to describe the angular velocity of soil deformation (distortion) of linear-fractional equation of the form [7]:

\[
\dot{\psi} = \frac{\tau}{\eta(t)} \cdot \frac{\tau^*}{\tau - \tau},
\]

where \(\tau\) and \(\tau^*\) - current and limit values of the shear stress \([\text{kH/m}^2]\);
\(\eta(t)\) - time-varying viscosity of the soil \([\text{kH*min/m}^2]\), and

\[
\tau_n^* = \sigma_n' \cdot t g \varphi + c,
\]

where \(\sigma_n'\) - effective stress, \(\varphi\) and \(c\) – parameters of the plain shear.

In the case where the soil hasn’t a pronounced creep properties (sandy loam, sand), it is more convenient to consider the elastic-plastic equation by Timoshenko [8] applied to a soil massif (\(\tau^* = f(c, \varphi)\)):

\[
\dot{\psi} = \frac{\tau}{G} \cdot \frac{\tau^*}{\tau - \tau},
\]

where \(G\) – the shear modulus of the soil, that tends to \(\infty\) when \(\tau \to 0\).

From the analysis of the stress state of the soil half-space by decision of the Flaman [9] implies that the stress state at depth \(z\) changes according to the following formulas [10]:

\[
\sigma_z = \frac{p}{\pi} \left( \arctg \frac{a-x}{z} + \arctg \frac{a+x}{z} \right) \cdot \frac{2ap}{\pi} \cdot \frac{z(x^2 - z^2 - a^2)}{(x^2 + z^2 - a^2)^2 + 4a^2z^2},
\]

\[
\tau_{xz} = \frac{4ap}{\pi} \cdot \frac{xz^2}{(x^2 + z^2 - a^2)^2 + 4a^2z^2}.
\]

However, these equations are cumbersome and inconvenient for the task, due to the necessity of differentiation and integration. Analysis of the stress state showed that for the description of stresses from external pressure (at \(z=\text{const}\)) in the form:

\[
\sigma_z(x) = p_0(z) \cdot e^{-ax^2},
\]
where \( p_0 \) - the maximum value of the \( \sigma_{zp} \) at depth \( z \), that can be defined by the Set of Rules «Soil bases of buildings and structures» [11] or by well-known formulae \( \sigma_{zp} = K_z \cdot p \), where

\[
K_z = f\left(\frac{x}{b \cdot \beta}\right), \ \alpha \ \text{in} \ [1/m^2].
\]

From the condition of incompressibility of the soil layer, we have \( \varepsilon_x + \varepsilon_z = 0 \) for any \( x \pm l \) and constancy of the horizontal stress \( \sigma_x(z) \) in sections, where \( x = \text{const} \) it follows, that \( \sigma_x = \sigma_x(z^*) \) and depends only on \( x \), and from the equilibrium condition of the layer of the unit length \( dz \), it follows that

\[
\tau_{xz}(x+h) = h \cdot \frac{d\sigma_x}{dx}.
\]

Given that \( \tau_{xz} \) at any cross-section with \( x \leq \pm l \), distributed over the triangle can record what

\[
\tau_{xz}(z) = z \cdot d\sigma_x / dz
\]

Taking into account the expression (6), we obtain:

\[
\tau_{xz} = -2\alpha \cdot p_0 \cdot x \cdot z \cdot e^{\alpha x}
\]

To determine the parameter \( \alpha \), we use equation of equilibrium in the form \( \sum F_z = 0 \), i.e.

\[
p_b = p_0 \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = p_0 \sqrt{\frac{\pi}{\alpha}}.
\]

It follows that:

\[
\sqrt{\frac{\alpha}{\pi}} = \frac{p_0}{p \cdot b} \rightarrow \alpha = \left( \frac{p_0}{p \cdot b} \right)^2 \cdot \pi \rightarrow \left( \frac{2 \arctg \frac{a}{z}}{a} \right)^2.
\]

Expression (6) follows that:

\[
\frac{p_0}{p} = \frac{2}{\pi} \arctg \frac{a}{z} \rightarrow \alpha = \frac{2 \pi}{\pi} \arctg \frac{a}{z} = 2 \arctg \frac{a}{z},
\]

Then we get:

\[
\alpha \approx \left( \frac{\arctg \frac{a}{z}}{a} \right)^2.
\]

From (11) it follows that at a certain distance from \( x = 0 \) and when \( z = h \), there are the maximum shear stress appears (see Fig. 1). To determine the point of extremum, one should take the derivative with respect to \( x \) in (11) assuming that \( z = h \) [12]. Then we get:

\[
\frac{\sqrt{2} \alpha}{2 \pi} \left( \frac{p_0 \cdot b}{p} \right)^2 \cdot \pi \rightarrow \frac{p_0 \cdot b}{p} = \frac{p_0}{p} \cdot \frac{b}{4.44},
\]

where \( p_0 \) still \( \sigma_z(z = d) \) on the axis \( Z \); \( p \) – pressure under the footing.

Substituting this value of \( x \) in (8), we obtain the maximum value of the tangential stresses at the contact of a weak layer with upper and lower layers. Equating it to the ultimate values of shear stresses in accordance with Coulomb's law:

\[
\tau_{xz}^* = \sigma_z \cdot \tan \varphi + c,
\]

\[
\tau_{xz}^* = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2 \pi}} \rightarrow \frac{p_0 \cdot b}{p} = \frac{p_0}{p} \cdot \frac{b}{4.44}.
\]
where \( \sigma_z \) is determined by (5).
Then we have:
\[
\tau_{sz}^{\max} (h) = \sigma_z (z = d) \cdot \tan \phi + c, \tag{14}
\]
where \( \sigma_z = \sigma_z = p_0 \cdot e^{-\alpha z} \).
Taking into account (8), we can obtain the initial critical pressure under the foundation base from the solution of the following equation:
\[
\tau_{sz}^{\max} = -2 \cdot p_0 \cdot \alpha \cdot x \cdot h \cdot e^{-\alpha (x')^2} = p_0 \cdot e^{-\alpha (x')^2} \cdot \tan \phi + c, \tag{16}
\]
where \( \alpha \) is determined by (11), and initial critical pressure \( p_{cr}^{*} \) by the solution to this transcendental equation by using mathematical software (e.g. MathCAD, MathLAB, etc.). It is obvious that with increasing load on the foundation, around point there is a closed region with a length and developed further. Then move on to forecast the speed of settling of the foundation and its cause of the speed of extrusion of the weak layer.

3. Forecast the speed of settlement of the foundation due to extrusion of the weak layer from the base
The speed of extrusion of the weak layer depends on the velocity distribution of the angular strain in any cross-section [13] \( \gamma (x, z) \), i.e.
\[
u(x, z) = \int_{-h}^{h} \gamma (x, z) dz, \tag{17}\]
Depending on the design of the soft soil creep model (1-3), it is possible to obtain different formulas for determining the speed of extrusion of the weak layer [14]. We consider the solution to (17) for different models.
In the case of viscoplastic model we obtain:
\[
u(x, z) = \int_{-h}^{h} \frac{\tau_{sx} - \tau_{sz}^{*}}{\eta (t)} dz, \tag{18}\]
where \( \tau_{sx} \) and \( \tau_{sz}^{*} \) - obtained by (8) and (13) consequently, at that \( \sigma_z = p_0 \cdot e^{-\alpha z} \), then we have:
\[
u(x, z) = \frac{-2 \cdot p_0 \cdot \alpha \cdot e^{-\alpha z}}{\eta (t)} \int \frac{z dz}{1 - \tau_{sx} / \tau_{sz}^{*}} = \frac{-2 \cdot p_0 \cdot \alpha \cdot x \cdot e^{-\alpha z}}{\eta (t)} \int \frac{z dz}{1 - \frac{Az}{B}} = -A \int \frac{z dz}{1 - Cz}, \tag{19}\]
where \( A = 2 p_0 \alpha e^{-\alpha z}, B = p_0 \cdot x \cdot e^{-\alpha z} \cdot \tan \phi + c, C = A / B \);
and finally:
\[
u(x, z) = -\frac{A}{Cz} \left\{ \ln (1 - Cz) \right\} + D \tag{20}\]
when \( z = \pm h \), \( \nu(x, z) = 0 \rightarrow D = -\frac{A}{Cz} \left\{ \ln (1 - Ch) \right\} \), then we have
\[
u(x, z) = \frac{A}{Cz} \left\{ \ln (1 - Cz) \right\} + Cz - \ln (1 - Ch) = \frac{A}{Cz} \left\{ \ln \frac{1 - Cz}{1 - Ch} \right\}. \tag{21}\]
It follows that when \( z = h \), \( \nu(x, z) = 0 \), when \( z = 0 \),
\[
u_{x,z}^{\max} = \frac{A}{Cz} \left\{ \ln (1 - Ch) \right\}. \tag{22}\]
The average speed on the section \( x \) can be determined by the expression:
\[ \bar{u} = \frac{1}{h} \int_0^h \tilde{u}(x, z) dz = \frac{A}{h c^2} \left[ \frac{1 - C_z \ln(1 - C_z) - z \ln(1 - C_h) + (z - h) \ln(1 - C_h) + (z - h)^2}{3} \right]_0^h; \]

\[ \bar{u} = \frac{A}{h c^2} \left[ \left[ (1 - C_h) \ln(1 - C_h) - (1 - 0) \ln 1 \right] - h \ln(1 - C_h) - h^2 \right]. \]  

Substituting here the values of A and C from (18), we obtain the average speed of extruding in any section x of the weak layer.

Given the fading character of the function \( \sigma_z = \sigma_x = p_0 e^{-\alpha z} \), it should be assumed that when \( x \approx \pm 2l, \bar{u} \to 0 \). The maximum amount of extruded out soft soil from weak layer per unit time in the section \( x = x^* \), will be equal:

\[ Q = \overline{u}(x)^* \Delta t \cdot 2h \cdot 1 \]  

If you divide this volume by the area \( 2l \cdot 1 \text{ m}^2 \), where \( l \) – half of the weak layer’s length, when \( \bar{u} \to 0 \), then we get the speed of movement of the top of the layer in the form of:

\[ \bar{V} = \frac{Q}{(2l \cdot 1)} = \overline{u}(x^*) \cdot \Delta t \cdot 2h / (2b \cdot 1) \]  

From the analysis of (25) it follows that when \( z=0 \) \( \overline{u}(x, z) = 0 \), and when \( z=0 \) \( \overline{u}(x, z) \) can be obtained by (23). The isolines are a family of closed curves in the form of elongated ellipses with centre at \( x=x^* \) (fig.1).

The cross section of the surface described by the function \( \overline{u}(x, z) \) when \( x=\text{const} \), is a plot of velocities in a curved shape with a maximum at \( z=h \). To obtain the displacement \( u(x) \) for the case of elastic-plastic extruding enough in the resulting solutions replace the viscosity \( (\eta) \) and shear modulus \( (G) \).

**4. Example of solution**

Consider the simplest case, when the shear rate is determined by Bingham model [15]:

\[ \dot{\gamma} = \frac{\tau_{ec}}{\eta(t)} + \frac{\dot{\sigma}}{K} \]  

Since the external load is constant \( (p = \text{const}) \), and \( \dot{\sigma} = 0 \), then we get:

\[ \dot{u}_{ec} = -\frac{2p_0 \alpha x e^{-\alpha z}}{\eta(t)} \int z dz = -\frac{2p_0 \alpha x e^{-\alpha z}}{\eta(t)} \cdot \frac{z^2}{2} + D, \]

when \( z = \pm h \), \( \dot{u}_{ec} = 0 \rightarrow D = \frac{p_0 \alpha x e^{-\alpha z}}{\eta(t)} \cdot h^2 \).

And finally:

\[ \dot{u}_{ec} = \frac{p_0 \alpha x e^{-\alpha z}}{\eta(t)} \left( h^2 - z^2 \right) \]  

Average settlement by section \( x > \pm 0 \), we can define by the formula:

\[ \bar{u}_{sc} = \frac{A}{l} \int_0^h (h^2 - z^2) dz = \frac{A}{l} \left( h^2 \cdot z - \frac{z^3}{3} \right)_0^h = \frac{A}{l} \left( \frac{2h^3}{3} \right). \]
From the analysis of (28) it follows that the isolines $\dot{u}(x, z) = \text{const}$ are the family of closed curves of elliptic form with the center at the point $x = \pm \frac{p_0}{p} \cdot \frac{b}{4.44}$ (fig.1), with $x=0$, $x=\infty$, $\dot{u}(x, z) = 0$; $z = \pm h$, $\dot{u}(x, z) = 0$.

5. Numerical modeling of the extrusion process.

The solution of the problem shown in figure 1 by the finite element method even in a simple formulation (plane strain) clearly show the process of extruding the weak layer because of the load attached by the foundation plate. For the qualitative calculation have been used the perfectly elastic-plastic model of Mohr–Coulomb.

The results of numerical calculation (displacements and shear stresses) are similar to the graphs produced by the functions in the analytical solution, which indicates the correctness of the chosen direction in the search for exact analytical solution.

Figure 2. Isofields of the horizontal displacements in the soil massif after loading

Figure 3. Curve of the vertical stress at the border level between weak and underlying soil layers
6. Conclusions
   1. The article analytically obtained expression for determining the foundation settlement speed on condition of the weak layer availability at the base. To describe the pressure distribution on the top of this layer have been used the law of normal Gauss distribution, which is quite realistically reflects it.
   2. The solution of determining the initial critical load to the top of a weak layer at the base is given.
   3. For comparison, the received forms of the stress distribution for the given task was made the numerical simulation of geotechnical situation and obtained comparable results, which indicates the correctness of the chosen direction in the analytical solution.

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