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Vaccination games in prevention of infectious diseases with application to COVID-19

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Abstract
Vaccination coverage is crucial for disease prevention and control. An appropriate combination of compulsory vaccination with voluntary vaccination is necessary to achieve the goal of herd immunity for some epidemic diseases such as measles and COVID-19. A mathematical model is proposed that incorporates both compulsory vaccination and voluntary vaccination, where a decision of voluntary vaccination is made on the basis of game evaluation by comparing the expected returns of different strategies. It is shown that the threshold of disease invasion is determined by the reproduction numbers, and an over-response in magnitude or information interval in the dynamic games could induce periodic oscillations from the Hopf bifurcation. The theoretical results are applied to COVID-19 to find out the strategies for protective immune barrier against virus variants.

Keywords:
Dynamic game
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1. Introduction

The spread of infectious diseases, for example, highly infectious measles, hepatitis B and the recent epidemic of COVID-19, poses a great threat to our lives and to economic development in all regions of the globe [1–3]. Though the prevention measures such as early detection, early quarantine and early treatment are important, vaccination is the most effective to control infectious diseases. In fact, the highly contagious smallpox and polio were eradicated or eliminated through vaccination campaigns [4–6]. The success of a vaccination program is crucially dependent upon the vaccination coverage above which, the spread of infectious disease is controlled. The approach to fulfill that goal is affected by multiple factors, including the attitude of individuals for vaccination, injection risk of disease and side-effect of vaccine. Therefore, it is necessary to find an optimal way to deploy the vaccination program.

An appropriate fraction of compulsory vaccination for individuals with the higher infectious transmissions is important to curb the spread of infectious diseases. For example, employees in public services such as health-care workers, bus drivers and teachers have the much more probability to be infected and to transmit the disease. Therefore, the compulsory vaccination for the leading infectious group is fundamental in a vaccination program. However, the mandatory vaccination is limited by the beliefs of free choices in vaccination or religious faith against vaccination. Thus, the voluntary vaccination applies to the majority of a population. This means that the coverage of voluntary vaccination is critical in order to achieve the goal of immune barrier. However, for voluntary vaccination, the decision is made on the basis of evaluating the cost and benefits of vaccination. For example, the decision is given by comparing the loss from a potential infection with the safety and cost of vaccination [7–9].

Game theory is often applied to vaccination decisions, in which individuals choose the most advantageous strategy by comparing the expected returns of two strategies. The researches focus upon the case where all the individuals in a population are voluntarily vaccinated and they respond linearly to the pay-off information [10–13]. Moreover, since a vaccination decision may be delayed at a certain time, a discrete time delay or a compartment for the delayed vaccination are incorporated into the dynamic game models [14,15]. Notice that the accumulated historical information of disease status plays the key role in a realistic vaccination decision [16,17]. Furthermore, a nonlinear response function for vaccination decision is appropriate in vaccination games [18–20]. In this paper, we propose a mathematical model with the vaccination game, which incorporates the integral information of disease status and introduces a nonlinear response function in decision-making game. Another novelty of the model is that both mandatory vaccination and voluntary vaccination are included, which can be used to analyze the impact of vaccination behaviors on the prevention and control of COVID-19.

The organization of this paper is as follows. In the next section, we formulate the mathematical model of epidemic diseases with behavior...
changes for vaccination. Section 3 presents the mathematical analysis for the stability and Hopf bifurcation of equilibrium points of the model. In Section 4, we use the model to simulate the dynamics of COVID-19 spread in China and analyze the vaccination strategies for immune barrier. The paper ends with some discussions in Section 5.

2. Model

We start with the formulation of mathematical model of epidemic diseases of SIRVS type, which is applicable to describe the transmission dynamics of infectious diseases such as measles, influenza, haemorrhagic fever and COVID-19. Here, the population is simply divided into four groups: susceptible (S), vaccinated (V), acutely infected (I), and recovered (R) individuals. The individuals with vaccination failure become susceptible, susceptible individuals enter the infection compartment after valid infectious contacts with infectors, and infected individuals are admitted to the recovery compartment due to natural recovery, quarantine and treatment. Notice that reinfections in vaccinated individuals or naturally infected individuals are common in the diseases such as COVID-19 and influenza [21–23]. We assume that recovered individuals can return into the susceptible compartment. The flowchart of disease transmission and progression is shown in Fig. 1, and the dynamics of the state variables are described by the following differential equations:

\[
\begin{align*}
\frac{dS}{dt} &= \mu N - \sigma \rho \theta S + (1 - \theta)Sx - \frac{\beta i}{N} + \phi V + \kappa R - \mu S, \\
\frac{dI}{dt} &= \frac{\beta i}{N} - \gamma I - qI - \mu I, \\
\frac{dR}{dt} &= \gamma I + qI - \mu R - \kappa R, \\
\frac{dV}{dt} &= \sigma \rho \theta S + (1 - \theta)Sx - \phi V - \mu V,
\end{align*}
\]

(2.1)

where \( \theta \) is the proportion of susceptible people who are compulsorily vaccinated because they work in high-risk occupations, such as doctors, teachers and staffs in transportation services; other individuals are voluntarily vaccinated with the vaccination proportion \( \sigma \). \( N \) represents the vaccine efficacy (0 ≤ \( \sigma \) ≤ 1); \( \rho \) is the waning rate of immunity in vaccinated individuals; \( \kappa \) is the waning rate of immunity in recovered individuals; \( \beta \) is the effective transmission rate; \( \gamma \) is the natural recovery rate. Furthermore, \( q = \gamma_0 - \gamma_2 \) where \( \gamma_0 \) is the quarantine rate and \( \gamma_2 \) is the treatment rate after quarantine. In addition, we use the coefficient \( \rho \) (0 < \( \rho \) < 1) to describe the limited supply of vaccine.

Note that the birth rate and death rate in the population are identical and the disease-induced death is ignored. It is easy to see that the population size is a constant \( N \). Let \( s = S/N, i = I/N, r = R/N, v = V/N \) so that \( s + i + r + v = 1 \). Then (2.1) is simplified to

\[
\begin{align*}
\frac{ds}{dt} &= \mu - \sigma \rho \theta s + (1 - \theta)Sx - \beta si + \phi v + \kappa (1 - s - i - v) - \mu s, \\
\frac{di}{dt} &= \beta si - \gamma i - qi - \mu i, \\
\frac{dv}{dt} &= \sigma \rho \theta S + (1 - \theta)Sx - \phi v - \mu v,
\end{align*}
\]

(2.2)

where the equation in \( r \) is decoupled.

Notice that the vaccination proportion \( x \) is altered by vaccination behaviors, which is a dynamical decision-making game. To describe such a game, we assume that the cost of vaccination (including the cost of vaccine, the burden of adverse reactions after vaccination, etc.) is \( c_v \), and the cost of infection (including the cost of treatment and the loss) is \( c_i \). According to the dynamics of disease transmission, the rate at which an individual is infected is \( \beta i \). Thus, the infection risk for an unvaccinated individual could be measured by \( \beta i \cdot c_i \). Moreover, \( (1 - \sigma)\beta i \cdot c_i \) is the infection risk of an individual with unsuccessful vaccination, and \( \sigma \beta i \cdot c_i \) is the infection risk of an individual with temporary protection from vaccination. Let \( f_v \) be the pay-off for a vaccination strategy and let \( f_n \) be the pay-off for a non-vaccination strategy. Then we have

\[
\begin{align*}
f_v &= -c_v - (1 - \sigma)\beta i \cdot c_i - \sigma \beta i \cdot c_i, \\
f_n &= -\beta i \cdot c_i.
\end{align*}
\]

(2.3)

Let \( g_1(t) = f_v - f_n \) be the first information function that one defines the vaccination cost. The second one is the speed of new infections, i.e., the number of new cases per day, which is \( g_2(t) = \beta Nsi \). This function characterises the importance in change of disease prevalence. It is assumed that the net information function for decision-making is given by

\[
g(t) = m \cdot g_1(t) + (1 - m) \cdot g_2(t),
\]

(2.4)

where \( m \) is a weight coefficient. If \( m = 1 \), the decision is solely dependent on the vaccination cost. For the case \( m = 0 \), the decision is made by considering only the speed of new infections.

It is assumed that individuals make vaccination decisions by considering both the information of present moment and the information of past time. In the aim to mimic the situation that the accumulated information over a time interval is crucial for vaccination decision, we take the exponential fading memory function \( K(t) = e^{-\alpha t} \) and integrate the information to obtain

\[
M(t) = e \cdot g(t) + \frac{1}{\tau} \int_{t-\tau}^{t} g(\eta) \cdot e^{-\alpha(\tau - \eta)} d\eta,
\]

(2.5)

where \( e > 0 \) reflects an individual’s sensitivity to current information and \( \tau \) is the length of information interval.

Note that the value of decision-making function varies with disease status, which affects the inclination of individuals for vaccination. Motivated by previous studies [18,24], we define the behavior response by

\[
\text{Response:} M(t) = \frac{1}{1 + e^{-bM(t)}} - \frac{1}{2},
\]

(2.6)

where \( b > 0 \) represents the response intensity. This function is a continuous counterpart of classical Fermi function, which describes behavior switches in discrete imitation dynamics. Then following the classical approach to model imitation dynamics [12,13,16,17,19], we get the dynamical equation for behavior changes:

\[
\frac{dx}{dt} = v(1 - x) \left( \frac{1}{1 + e^{-bM(t)}} - \frac{1}{2} \right),
\]

(2.7)

where \( v > 0 \) is a coefficient for imitation intensity.

Combining the disease transmission dynamics with Eq. (2.6), we get the full mathematical model:

\[
\begin{align*}
\frac{ds}{dt} &= \mu - \sigma \rho \theta s + (1 - \theta)Sx - \beta si + \phi v + \kappa (1 - s - i - v) - \mu s, \\
\frac{di}{dt} &= \beta si - \gamma i + qi - \mu i, \\
\frac{dv}{dt} &= \sigma \rho \theta S + (1 - \theta)Sx - \phi v - \mu v, \\
\frac{dx}{dt} &= v(1 - x) \left( \frac{1}{1 + e^{-bM(t)}} - \frac{1}{2} \right),
\end{align*}
\]

(2.7)
3. Analysis

System (2.7) has a disease-free equilibrium with no voluntary vaccination

\[ E_1 = \left( \frac{\phi + \mu}{\phi + \mu + \alpha \rho \theta}, 0, \frac{\alpha \rho \theta}{\phi + \mu + \alpha \rho \theta}, 0 \right), \]

and a disease-free equilibrium with full voluntary vaccination

\[ E_2 = \left( \frac{\phi + \mu}{\phi + \mu + \alpha \rho \theta}, 0, \frac{\alpha \rho}{\phi + \mu + \alpha \rho}, 1 \right). \]

By the computation methods in papers [25,26], we obtain the basic reproduction number of model (2.7):

\[ R_0 = \frac{\beta}{\gamma + q + \mu}. \] (3.1)

The effective reproduction numbers at disease-free equilibrium points \( E_1 \) and \( E_2 \) are

\[ R_{0v}^0 = R_0 \cdot \frac{\phi + \mu}{\phi + \mu + \alpha \rho \theta}, \]

\[ R_{0v}^1 = R_0 \cdot \frac{\phi + \mu}{\phi + \mu + \alpha \rho}. \]

Set

\[ i_3 = \mu \frac{R_{0v}^0 - 1}{R_{0v}^0} \cdot w, \]

where

\[ w = \frac{\mu + \kappa}{(\mu + \kappa)(\gamma + q + \mu) - \kappa(\gamma + q)}. \]

If \( R_{0v}^0 > 1 \), there is a boundary equilibrium point:

\[ E_3 = \left( \frac{1}{R_0} i_3, \frac{\alpha \rho \theta}{R_0 (\phi + \mu)}, 0 \right). \]

Furthermore, if \( R_{0v}^1 > 1 \), there is an equilibrium

\[ E_4 = \left( \frac{1}{R_0} i_4, \frac{\alpha \rho}{R_0 (\phi + \mu)}, 1 \right). \]

where

\[ i_4 = \mu \frac{R_{0v}^1 - 1}{R_{0v}^1} \cdot w. \]

All of the four equilibrium points are the boundary equilibrium points. Let us look for an internal equilibrium point \((s_5, i_5, v_5, x_5)\) of system (2.7). Setting the right sides of equations in system (2.7) to 0, we obtain

\[ \begin{cases}
\mu - \alpha \rho (s + (1 - \theta)sx) - \beta si + \phi \nu + \kappa(1 - s - i - v) - \mu s = 0, \\
\beta si - (\gamma + q + \mu)i = 0, \\
\alpha \rho (s + (1 - \theta)sx) - \phi \nu - \mu v = 0,
\end{cases} \] (3.3)

and \( M = 0 \). The second equation of (3.3) implies \( s_5 = 1/R_0 \). From \( M = 0 \) we get \( g = 0 \), which implies

\[ i_5 = \frac{mc_0}{m(1 - \phi)/\kappa c_1 + (1 - m)(\gamma + q + \mu)N}. \] (3.4)

Then it is easy to obtain

\[ v_5 = 1 - \frac{1}{R_0} - \frac{i_5}{\mu w}, \]

\[ x_5 = \frac{\beta (\phi + \mu)v_5 - \alpha \rho \theta (\gamma + q + \mu)}{\alpha \rho (1 - \theta) \alpha \rho (\gamma + q + \mu)} . \] (3.5)

Set

\[ R_c = \frac{\mu w}{\mu w - i_5}. \]

Evidently, (3.4) and (3.5) mean

\[ s_5 > 0, \quad i_5 > 0, \quad v_5 < 1. \]

Assume

\[ R_{0v}^1 < R_c < R_0^0. \] (3.6)

It follows from (3.5) that

\[ 0 < x_5 < 1, \quad v_5 > 0. \]

Notice that

\[ 1 < R_c < R_{0v}^1 < R_0. \]
Theorem 3.1. Model (2.7) admits a unique interior equilibrium $E_S = (s_{10}, s_{20}, v_{10}, s_{30})$ if (3.6) is satisfied.

We turn to consider the local stability of the equilibria. Set $E_n = (s_n, x_n, v_n, s_{1n})$ for $n = 1, 2, 3, 4, 5$ and

$$P_0(n) = (m - c_1(1 - \phi)m\lambda_{0} + (1 - m)\beta_0\lambda_{0} - (1 + \epsilon - \alpha\eta)\theta_0^m, \nu_0, \beta_0, \gamma_0, \mu_0, \alpha_0).$$

Linearize the system (2.7) at $E_n$ and rewrite the state variables as $s, i, v$ and $x$ to obtain

$$T(n) = -\alpha\epsilon(1 - \theta)x_n - \beta_0 - \kappa - (1 - \phi)\lambda_{0}\epsilon_x - c_1 + (1 - m)\beta_0\lambda_{0} - (1 + \epsilon - \alpha\eta)\theta_0^m,$$

$$Q_1(n) = Vx_{n}(1 - x_n) + (1 - m)\beta_0\lambda_{0} - \beta_0 - b_0\lambda_{0} = \frac{(1 + \epsilon - \alpha\eta)\theta_0^m}{(1 - \eta)};$$

$$Q_2(n) = Vx_{n}(1 - x_n) + (1 - m)\beta_0\lambda_{0} - \beta_0 - b_0\lambda_{0} = \frac{(1 + \epsilon - \alpha\eta)\theta_0^m}{(1 - \eta)};$$

where

$$\int_{t}^{\tau} \tau_{e}^{\alpha\eta}k_{\eta}^{\alpha\eta}d\tau = \frac{\alpha(1 - \epsilon - \epsilon^{\alpha\eta})}{\lambda + \alpha} \cdot \tau_{e}^{\alpha\eta},$$

we obtain the characteristic equation at $E_0$

$$P_1(n) = Vx_{0}(1 - x_0) + (1 - m)\beta_0\lambda_{0} - \beta_0 - b_0\lambda_{0} = \frac{(1 + \epsilon - \alpha\eta)\theta_0^m}{(1 - \eta)};$$

$$P_2(n) = Vx_{0}(1 - x_0) + (1 - m)\beta_0\lambda_{0} - \beta_0 - b_0\lambda_{0} = \frac{(1 + \epsilon - \alpha\eta)\theta_0^m}{(1 - \eta)};$$

(3.7)

By analyzing the roots of the characteristic equation, we obtain the conditions for the local stability of the boundary equilibria, which are shown in Table 1.

For brevity, we present only the stability analysis for equilibrium $E_5$. First, $\lambda_1 = P_1(3)$ is a characteristic root. The other three characteristic roots solve the equation:

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0.$$

Here,

$$a_1 = \beta_0 + 2\mu + \phi + \alpha\eta\theta_0 + \nu_0,$$

$$a_2 = \beta_0^2\lambda_{0} + (1 - \phi)m\lambda_{0} + (1 - m)\beta_0\lambda_{0} + (1 + \epsilon - \alpha\eta)\theta_0^m,$$

$$a_3 = (1 + \epsilon - \alpha\eta)\theta_0^m.$$

Direct calculations indicate

$$a_1a_2 - a_3 = a_1 \cdot \alpha\eta\theta_0 + \phi(1 - \theta)x_n + \xi(1 - m)\beta_0\lambda_{0} + (1 + \epsilon - \alpha\eta)\theta_0^m > 0.$$

It follows from the Routh–Hurwitz criterion that $E_3$ is asymptotically stable if

$$\lambda_1 = P_1(3) = \frac{1}{1 + \epsilon - \alpha\eta} < 0.$$

Let us now analyze the stability of equilibrium $E_5$. For the case of $\tau = 0$, the characteristic equation of the linearized system at $E_5$ is

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0,$$

where

$$b_1 = \phi + 2\mu + \phi + \alpha\eta\theta_0 + \nu_0,$$

$$b_2 = (1 + \epsilon - \alpha\eta)\theta_0^m,$$

$$b_3 = (1 + \epsilon - \alpha\eta)\theta_0^m,$$

$$b_4 = (1 + \epsilon - \alpha\eta)\theta_0^m.$$

In which,

$$L_1(\tau) = Vx_{0}(1 - x_0) + (1 - m)\beta_0\lambda_{0} - \beta_0 - b_0\lambda_{0} = \frac{(1 + \epsilon - \alpha\eta)\theta_0^m}{(1 - \eta)};$$

$$L_2(\tau) = Vx_{0}(1 - x_0) + (1 - m)\beta_0\lambda_{0} - \beta_0 - b_0\lambda_{0} = \frac{(1 + \epsilon - \alpha\eta)\theta_0^m}{(1 - \eta)};$$

By the Routh–Hurwitz criterion, we see that all the characteristic roots have negative real parts if

$$H(\tau) = (b_1b_2(\tau) - b_3(\tau) - b_4(\tau)) > 0,$$

which implies that equilibrium $E_5$ is locally asymptotically stable when $\tau = 0$. Motivated by numerical calculations in Example 3.1, we assume that there is a $\tau = \tau_0$ such that

$$H(\tau_0) = (b_1b_2(\tau_0) - b_3(\tau_0) - b_4(\tau_0)) = 0.$$
Consequently, we can state the following result.

**Theorem 3.2.** If there is a \( v = \nu_0 \) such that Eq. (3.11) holds and \( \frac{d\nu(\nu_0)}{d\nu} \neq 0 \). Then system (2.7) has a Hopf bifurcation at \( v = \nu_0 \) when \( \tau = 0 \).

For the case of \( \tau > 0 \), we consider below the existence of the Hopf bifurcation for the special case \( \epsilon = 0 \). The analysis of the case \( \epsilon > 0 \) is similar and will not be described here. The characteristic equation of the linearized system at \( \epsilon = 0 \) is

\[
\lambda^3 + f_1 \lambda^2 + f_2 \lambda + f_3 + f_4 \lambda + f_5 = \left( f_6 \lambda^2 + f_7 \lambda + f_8 \right) e^{-\lambda \tau} = 0. \tag{3.12}
\]

Here,

\[
\begin{align*}
 f_1 &= \alpha + \beta \nu_0 + 2 \mu + \phi + \omega \theta + \epsilon \phi + \epsilon \omega \theta + \epsilon \phi + \epsilon \omega \theta, \\
 f_2 &= \alpha \beta + \alpha \beta \nu_0 + 2 \mu \theta + \phi \omega + \epsilon \phi \theta + \epsilon \omega \phi + \epsilon \omega \theta, \\
 f_3 &= \alpha \beta \nu_0 + 2 \mu \theta + \phi \omega + \epsilon \phi \theta + \epsilon \omega \phi + \epsilon \omega \theta, \\
 f_4 &= \alpha \beta \nu_0 + 2 \mu \theta + \phi \omega + \epsilon \phi \theta + \epsilon \omega \phi + \epsilon \omega \theta, \\
 f_5 &= \epsilon \phi + \epsilon \omega \theta + \epsilon \phi + \epsilon \omega \theta, \\
 f_6 &= \epsilon \phi + \epsilon \omega \theta + \epsilon \phi + \epsilon \omega \theta, \\
 f_7 &= \epsilon \phi + \epsilon \omega \theta + \epsilon \phi + \epsilon \omega \theta, \\
 f_8 &= \epsilon \phi + \epsilon \omega \theta + \epsilon \phi + \epsilon \omega \theta.
\end{align*}
\]

In the aim to obtain the conditions under which \( E_0 \) keeps the stability for all \( \tau > 0 \), we set \( \epsilon = 0 \) in Eq. (3.12), and then separate the real and imaginary parts of the equation to get two transcendental equations:

\[
\begin{align*}
 f_1 \omega^3 - f_2 \omega^2 + f_3 &= -f_4 \omega \sin \omega \tau + f_5 \omega^2 - f_8 \cos \omega \tau, \\
 f_6 \omega^2 - f_7 \omega + f_8 &= -f_9 \omega \cos \omega \tau - f_6 \omega^2 - f_9 \omega \sin \omega \tau.
\end{align*}
\]

Let \( \omega = \alpha \), square both sides of the transcendental equation respectively and add the two transcendental equations to obtain:

\[
\begin{align*}
 h(z) \triangleq & z^3 + g_1 z^2 + g_2 z^2 + g_3 z^2 + g_4 z^2 + g_5 = 0, \tag{3.14}
\end{align*}
\]

where

\[
\begin{align*}
 g_1 &= f_1^2 - 2 f_2, \\
 g_2 &= f_1^2 + 2 f_4 - 2 f_6, \\
 g_3 &= f_1^2 - 2 f_4 f_5 - f_6, \\
 g_4 &= f_1^2 - 2 f_4 f_5 - f_6, \\
 g_5 &= f_1^2 - 2 f_4 f_5 - f_6.
\end{align*}
\]

Set

\[
\begin{align*}
 H_1 &= g_1 = \alpha^2 + f_1^2 \beta^2 + (\mu + \nu_0^2) \nu_0^2 + 2 \mu \beta \nu_0 + (\phi + \omega \theta) + \epsilon \phi + \epsilon \omega \theta, \\
 H_2 &= 2 f_8 H_1 > 0, \\
 H_3 &= g_2 H_1 > 0, \\
 H_4 &= g_2 g_3 H_1 > 0, \\
 H_5 &= g_2 g_3 H_1 > 0.
\end{align*}
\]

Suppose that Eq. (3.14) has a positive root \( \tau \). Then \( \pm i \alpha \) with \( \alpha = \sqrt{2} \) are a pair of pure imaginary root of characteristic equation. It follows from the transcendental equations that \( \tau = \tau(0), \; j = 1, 2, \ldots \), are given by

\[
\tau(j) = \frac{1}{\alpha} \arctan \left( \frac{f_6 \omega^2 - f_7 \omega + f_8}{f_6 \omega^2 + f_7 \omega + f_8} \right) + \left( \frac{f_6 \omega^2 - f_7 \omega + f_8}{f_6 \omega^2 + f_7 \omega + f_8} \right) \left( \frac{f_6 \omega^2 - f_7 \omega + f_8}{f_6 \omega^2 + f_7 \omega + f_8} \right) + \left( j - 1 \right) \mu.
\]
shown in Fig. 4. In particular, equilibrium point $E_2$ exists when $15.98 < c_v < 201.365$ and is unstable when $15.98 < c_v < 184$. The system has periodic solutions for $15.98 < c_v < 184$.

### 4. Application to COVID-19

In this section, we use the mathematical model and the published data of COVID-19 spread in China to show how to deploy the vaccination program to establish an immune barrier for disease control. The inactivated vaccine BIBP developed by Sinopharm group is selected, which has 79% protective efficacy [31] and protection duration of six months [32]. According to the medical payment regulations for confirmed and suspected patients formulated by the National Healthcare Security Administration [33], the payoff is set at 5000 Chinese Yuan. In addition, we set $m = 0.4$ to mean that individuals pay more attention to the daily new cases. Moreover, we fix $\rho = 0.01$ according to the supply of vaccine in the initial stage of vaccination in China [34]. The other parameter values in the simulations below are shown in Table 2.

#### 4.1. Vaccination schemes for immune barrier

We focus on the immune barrier which means that all individuals in the population are protected by immunity when the vaccine coverage in the population reaches a threshold proportion. Clearly, the immune barrier is formed when the effective reproduction number $R_v^{0}$ is less than 1. Using the parameter values in Table 2 and fixing $\theta = 0.25$, numerical computations lead to $R_v = 2.2179$, $R_v^{0} = 1.6348 > 1$, $R_v^{1} = 0.9139 < 1$. Thus, the endemic equilibrium point $E_1$ exists, which has no voluntary vaccination. In order to drive $R_v^{0}$ below 1 to suppress the endemic state, we vary $\theta$ to see how the compulsory vaccination proportion affects the reproduction number, which is shown in Fig. 5. The numerical computations indicate $R_v^{0} \leq 1$ for $\theta \geq 0.8535$. Hence, the compulsory vaccination proportion must be above 85.35% to eradicate the disease prevalence.

However, the compulsory vaccination threshold seems too high in practice. Thus, we turn to consider the importance of voluntary vaccination. Since $R_v = 1.0001$, in view of the theoretical analysis, we see that the endemic equilibrium point $E_2$ exists because $0.9139 = R_v^{1} \leq R_v < R_v^{0} = 1.6348$.

Assume that the vaccination cost $c_v$ is so small that one neglects it to take $c_v = 0$. Then we get from (3.4) and (3.5) that the infection proportion $i_0 = 0$ and the effective vaccination ratio is given by $v_0 = 1 - 1/R_v$, which means that the voluntary vaccination ratio $x_0$ is crucial for the herd immunity threshold. Next, we consider how the vaccination ratio $x_0$ for the herd immunity is influenced by the proportion of compulsory vaccination in Fig. 6, which demonstrates that $x_0$ decreases with the increase of $\theta$. More specifically, when $\theta = 0.6$ we have $x_0 = 0.6339$, that is, when the proportion of compulsory vaccination is 60%, the proportion 63.39% of voluntary vaccination is enough to form an immune barrier. To gain intuitions how the combinations of compulsory vaccination and voluntary vaccination form the immune barrier, we present Table 3:

For the case where the perceived vaccination cost $c_v$ is significant or the infection cost is small, one would expect that the individual’s willingness for vaccination declines and therefore, more people don’t confer enough immunity and get infected. Indeed, this is supported by the numerical simulations in panel (a) of Fig. 7 that indicates the infection proportion at equilibrium point $E_0$ is an increasing function of vaccination cost $c_v$. Similarly, since the weight coefficient $m$ is inclined to the vaccination cost, the infection proportion at equilibrium point $E_0$ is an increasing function of $m$, as shown in panel (b) of Fig. 7.

#### 4.2. Vaccination strategies for variants

Virus variants emerge during the pandemic progression of COVID-19. These variants weaken the protective immunity of vaccines and are more contagious than the original SARS-CoV-2 strain. Table 4 shows the resultant ratios by the main variants of coronavirus in disease transmissions and damage to immunity. Fig. 8 demonstrates how these variants affects the dynamics of vaccination and infectious transmissions.

In this paper, we propose the mathematical model with both compulsory vaccination and voluntary vaccination, and use the game theory to describe the behavior dynamics of individuals who consider the cost and benefit of vaccination. To mimic the realistic decision-making process, we take the decision function that is distributed on an interval for the past information. Furthermore, we adopt the nonlinear response function to the pay-off of vaccination, which is an extension of the classical Fermi function in discrete imitation dynamics. We compute the basic reproduction number $R_0$ and the effective reproduction number $R_v$, and analyze the existence and stability of the equilibrium points of the model. We find that the model admits the Hopf bifurcation when the imitation intensity is larger or the length of information interval is larger.

We use the model (2.7) to simulate the dynamics of COVID-19 spread in China. The influences of the key parameters on disease prevalence are given in Fig. 7. With the aid of the published data, we show...
how to deploy the vaccination program to establish an immune barrier for disease control. First, we compute the threshold value of vaccine coverage for immune barrier where there are only mandatory vaccinations, which looks too high to be implemented in practice. For this reason, we calculate the combinations of compulsory vaccination with voluntary vaccination to form the immune barrier in Table 3. This is of importance in designing the vaccination scheme. Indeed, once an upper percentage of the individuals with voluntary vaccination is estimated, one can select from the combination the proportion of mandatory vaccination to establish the immune barrier in practice.

Big challenges in fighting pandemic COVID-19 are the emergence of virus variants, which weaken the protective immunity of vaccines and are more contagious than the original SARS-CoV-2 strain. We show that the effective reproduction number $R_0$ for Omicron strain is larger than unity even though all individuals are vaccinated with two doses of Pfizer vaccine. Then we consider the effect of booster shots in Table 5, which shows that the effective reproduction numbers are always larger than unity for different vaccine efficacy and full mandatory vaccination. As a result, we strengthen the vaccination effect by superposing the quarantine of infectious individuals. Table 6 presents the combinations of voluntary vaccination proportion and quarantine rate to form the immune barrier and their corresponding vaccination coverage where the proportion of mandatory vaccination is fixed as $\theta = 0.25$.

This is helpful to select an optimal quarantine intensity on the status of population immunity in practice. In all, our mathematical modelling and analysis give the theoretical basis to design an optimal strategy of vaccination, which is aided with quarantine, to control a high contagious variant of infectious disease.

The mathematical model (2.7) captures the main features of infectious diseases with vaccination games. Notice that the latent period and asymptomatic phase of some diseases such as COVID-19 and measles contribute more or less to the disease outbreaks. It will be interesting to introduce latent compartment and asymptomatic compartment into the model to describe the disease transmissions more accurately. We leave these as future researches.

CRediT authorship contribution statement

Jingwen Ge: Investigation, Writing – original draft. Wendi Wang: Supervision, Investigation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

![Graphs of Hopf bifurcations for ODE case and DDE case in model (2.7).](image)

Fig. 2. Graphs of Hopf bifurcations for ODE case and DDE case in model (2.7).
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Fig. 3. Bifurcation diagram of equilibria with the change of $\alpha$. The red, blue, pink and green curves correspond to the equilibrium points $E_1$, $E_3$, $E_4$ and $E_5$, respectively. In addition, the solid (dotted) lines indicate that the corresponding equilibrium points are stable (unstable), and $H$ represents the Hopf bifurcation point.

Fig. 4. Bifurcation diagram of equilibria with the change of $c_v$. The red, blue, pink and green curves correspond to the equilibrium points $E_1$, $E_3$, $E_4$ and $E_5$, respectively. The solid (dotted) lines indicate that the corresponding equilibrium points are stable (unstable), and $H$ represents the Hopf bifurcation point.

Fig. 5. Graph of effective reproduction number $R_0$ versus the proportion of mandatory vaccination $\theta$. 
Immunity threshold calculated above. Vaccination coverage in each group corresponds to the herd immunity. The proportions of compulsory vaccination and voluntary vaccination required to form an immune barrier is a decreasing function of the compulsory vaccination fraction $\theta$.

**Fig. 6.** The proportion of voluntary vaccination $x_v$ required to form an immune barrier is a decreasing function of the compulsory vaccination fraction $\theta$.

**Table 3**
The proportions of compulsory vaccination $\theta$ and voluntary vaccination $x_v$ required to achieve herd immunity. Vaccination coverage in each group corresponds to the herd immunity threshold calculated above.

| $\theta$ | 0.6 | 0.5 | 0.4 | 0.3 | 0.2 |
|----------|-----|-----|-----|-----|-----|
| $x_v$    | 0.6339 | 0.7071 | 0.7559 | 0.7908 | 0.8169 |
| Vaccination coverage | 0.85356 | 0.85355 | 0.85354 | 0.85356 | 0.85352 |

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![Fig. 6](image-url)

**Table 4**
Transmission risks and reduced efficacy of strains of coronavirus.

| Strain | Found site | Incidence | Vaccine efficacy (name of vaccine) |
|--------|------------|-----------|-----------------------------------|
| Alpha  | UK         | +43 % [41] | 70.4 % (ChAdOx1 sCoV-19 vaccine) [46] |
| Beta   | South Africa | +50 % [42] | 57 % (NVX-CoV2373 vaccine) [47] |
| Gamma  | Brazil      | +70 % [43] | 50.38 % (Corona Vac vaccine) [47] |
| Delta  | India       | +97 % [44] | 59 % (SARS-CoV-2 inactivated vaccine) [48] |
| Omicron| South Africa | +170.87 % | 65.5 % (BNT162b2 vaccine) [49] |

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Table 5

The effective reproduction number $R_0$ is always larger than unity and is increased with the decrease of vaccine efficacy $\sigma$ where $\theta = 1$.

| $\sigma$ | 0.75 | 0.6 | 0.65 | 0.6 | 0.55 | 0.5 |
|---------|------|-----|------|-----|------|-----|
| $R_0$   | 1.1864 | 1.2534 | 1.3285 | 1.4132 | 1.5094 | 1.6196 |

Table 6

The combinations of voluntary vaccination proportion with quarantine rate to form the immune barrier and their corresponding vaccination coverage where $\theta = 0.25$.

| Quarantine rate $q_0$ | 1 | 1.5 | 2 | 2.5 | 3 |
|-----------------------|----|-----|---|-----|---|
| Voluntary vaccination $x_i$ | 0.985 | 0.704 | 0.5049 | 0.3564 | 0.2415 |
| Vaccine coverage | 0.9888 | 0.778 | 0.6286 | 0.5173 | 0.4311 |

Fig. 8. The proportions of infection for the five mutant strains as time evolves.

Fig. 9. Panel (a) shows that the effective reproduction number $R_0$ is decreasing with quarantine rate $q_0$ and $R_0 \leq 1$ for $q_0 > 0.978$. Panel (b) indicates that a reasonable compulsory vaccine coverage $\theta$ is applicable for an adequate quarantine rate $q_0$ in order to form an immune barrier.