ON WEAK RIGIDITY AND WEAKLY MIXING ENVELOPING SEMIGROUPS

ETHAN AKIN, ELI GLASNER AND BENJAMIN WEISS

The question we deal with here, which was presented to us by Joe Auslander and Anima Nagar, is whether there is a nontrivial cascade \((X,T)\) whose enveloping semigroup, as a dynamical system, is topologically weakly mixing (WM). After an introductory section recalling some definitions and classic results, we establish some necessary conditions for this to happen, and in the final section we show, using Ratner’s theory, that the enveloping semigroup of the ‘time one map’ of a classical horocycle flow is weakly mixing.

1. Introduction

A cascade is a homeomorphism \(T\) on a compact Hausdorff space \(X\). We call the system \((X,T)\) metric when \(X\) is metrizable. If \(A\) is a closed, invariant subset of \(X\), then the restriction of \(T\) to \(A\) defines the subsystem on \(A\). If \((X_1,T_1)\) is a cascade and \(\pi : X \to X_1\) is a continuous surjection such that \(\pi \circ T = T_1 \circ \pi\) then \(\pi\) is a surjective cascade morphism and \((X_1,T_1)\) is a factor of \((X,T)\).

For \(A,B \subset X\) we let \(N(A,B) = \{i \in \mathbb{Z} : A \cap T^{-i}(B) \neq \emptyset\}\). We write \(N(x,B)\) for \(N(\{x\},B)\) when \(x \in X\). The cascade is transitive when \(N(U,V)\) is nonempty for every pair of nonempty, open subsets \(U,V\) of \(X\). We let \(O(x) = \{T^n : n \in \mathbb{Z}\}\) denote the orbit of \(x\). A point \(x\) is a transitive point when the orbit-closure \(\overline{O(x)} = X\). When \((X,T)\) admits transitive points the system is called point transitive. A point transitive system is transitive and the converse holds when the system is metric. In a metric transitive system the set of transitive points forms a dense \(G_δ\) set by the Baire Category Theorem. The system \((X,T)\) is called totally transitive if for every \(0 \neq n \in \mathbb{Z}\) the system \((X,T^n)\) is transitive. When the homeomorphism is understood, we will refer to the system \(X\).

An ambit \((X,x_0,T)\) is a point transitive cascade \((X,T)\) with a chosen transitive base point \(x_0\). If \(\pi : (X,T) \to (X_1,T_1)\) is a surjective cascade morphism and \((X,x_0,T)\) is an ambit then \((X_1,\pi(x_0),T_1)\) is an ambit factor.

The enveloping semigroup \(E(X,T)\) (or simply \(E(X)\)) is the closure in \(X^X\) of the set \(\{T^i : i \in \mathbb{Z}\}\). If \(T_s\) denotes composition with \(T\) then \((E(X),1_X,T_s)\) is an ambit where \(1_X = T^0\) is the identity map. If \(x \in X\) then the evaluation map is a surjection of ambiits \(ev_x : (E(X),1_X,T_s) \to (\overline{O(x)},x,T)\) taking the enveloping-semigroup onto the orbit closure of \(x\).

For any non-empty index set \(I\) and the product system \((X^I,T^I)\) we can identify \(E(X)\) with \(E(X^I)\) by mapping \(T^i\) to \((T^I)^i\). Hence, for any \(k\)-tuple \((x_1,x_2,\ldots,x_k) \in X^k\) the ambit \(\overline{O(x_1,x_2,\ldots,x_k)}\) is a factor of \(E(X)\) via evaluation at \((x_1,x_2,\ldots,x_k)\). In addition, \(E(X)\) can be expressed as the inverse limit of these factors.

If \(A\) is closed invariant subset of \(X\) then the restriction map \(E(X) \to E(A)\) is a surjective morphism. A surjective morphism \(\pi : (X,T) \to (X_1,T_1)\) induces a surjective morphism \(\pi_* : (X,T) \to (X_1,T_1)\) by mapping \(T^i\) to \(T_1^i\).

The limit point sets \(\alpha(x)\) and \(\omega(x)\) are the sets of limit points of the bi-infinite sequence \(\{T^i(x) : i \in \mathbb{Z}\}\) as \(i\) tends to \(-\infty\) and \(+\infty\) respectively. The orbit-closure of \(x\) consists of

Date: November 05, 2017.
the orbit and $\alpha(x) \cup \omega(x)$. A point $x$ is called recurrent if $x \in \alpha(x) \cup \omega(x)$ and positive recurrent if $x \in \omega(x)$. A point is recurrent if and only if $N(x, U)$ is infinite for every neighborhood $U$ of $x$ and it is positive recurrent if for every such $U$, $N(x, U) \cap N$ is infinite. If $\pi : (X, T) \to (X_1, T_1)$ is a surjective morphism and $x$ is a recurrent point in $X$ then $\pi(x)$ is a recurrent point in $X_1$.

The system is called weakly rigid (see [7]) when $1_X$ is a recurrent point of $E(X)$. This implies that each $k$-tuple $(x_1, x_2, \ldots, x_k)$ is recurrent in $X^k$. Using the inverse limit construction for $E(X)$ one obtains the converse as well. It then follows that $X$ is weakly rigid if and only if there is a filter $\mathcal{F}$ of infinite subsets of $Z$ such that $N(x, U) \in \mathcal{F}$ for every $x \in X$ and neighborhood $U$ of $x$. Weak rigidity is preserved by factors and subsystems.

The system $X$ is weak mixing (hereafter WM) when the product system $X \times X$ is transitive. The transitivity property is preserved by factors and by the operation of inverse limits and so the same is true for WM. Note that a WM system is totally transitive.

A subset of $Z$ is called thick when it contains runs of arbitrary length. The following is a classic result of Furstenberg (see [5]).

1.1. Theorem. For a cascade $(X, T)$ the following are equivalent.

(i) The system $(X, T)$ is WM.
(ii) The product system $(X^I, T^{(I)})$ is WM for every nonempty index set $I$.
(iii) For every pair $U, V$ of nonempty open subsets of $X$, the return time set $N(U, V)$ is thick.
(iv) The system $(X, T)$ is transitive and for every nonempty open subset $U$ of $X$, the return time set $N(U, U)$ is thick.

1.2. Lemma. Let $(X, T)$ be a nontrivial, WM dynamical system. Then every transitive point in $X$ is recurrent.

Proof. Let $x_0 \in X$ be a transitive point. If $x_0$ is not recurrent then it is an isolated point in $X$; i.e. the singleton set \{x_0\} is clopen. Now, in this case, the sets $U = \{(x_0, x_0)\}$ and $V = \{(x_0, Tx_0)\}$ are clopen subsets of $X \times X$ but there is no $k \in Z$ such that $T^kU \cap V \neq \emptyset$. □

2. Some obstructions to WM of $E(X, T)$

Recall that an ambit $(X, x_0, T)$ is an enveloping semigroup if and only if it is point universal; i.e. it satisfies the following condition: For every $x \in X$ there is a (unique) homomorphism of pointed systems $(X, x_0, T) \to (X, x, T)$ (see e.g. [8, Proposition 2.6]).

We will say that $X$ has a WM enveloping semigroup when the system $(E(X), T_s)$ is WM.

Call a subset $F$ of $Z$ diff-thick when the difference set \{i - j : i, j \in F\} is thick.

2.1. Theorem. For a cascade $(X, T)$ the following are equivalent.

(i) The system $(X, T)$ has a WM enveloping semigroup.
(ii) For any $k$-tuple $(x_1, x_2, \ldots, x_k) \in X^k$ the ambit $\mathcal{O}(x_1, x_2, \ldots, x_k)$ is WM.
(iii) There exists a filter $\mathcal{F}$ of diff-thick sets such that for every $x \in X$ and open subset $U$ containing $x$, the return time set $N(x, U)$ is an element of $\mathcal{F}$.

Proof. (i) $\iff$ (ii): Each ambit $\mathcal{O}(x_1, x_2, \ldots, x_k)$ is a factor of $E(X)$ and $E(X)$ is an inverse limit of these factors. WM is preserved by factors and inverse limits.

(iii) $\implies$ (ii): If $U_1, U_2$ are open sets which meet $\mathcal{O}(z)$ with $z = (x_1, x_2, \ldots, x_k)$ then there exist $i_1, i_2 \in Z$ such that $(T^{(k)})^{i_1}(z) \in U_1, (T^{(k)})^{i_2}(z) \in U_2$. Let $U = (T^{(k)})^{-i_1}(U_1) \cap (T^{(k)})^{-i_2}(U_2)$. Since $\mathcal{F}$ is a filter, $N(z, U) \in \mathcal{F}$. If $(T^{(k)})^{k_1}(z), (T^{(k)})^{k_2}(z) \in U$ then $k_2 - k_1 + (i_2 - i_1) \in N(U_1, U_2)$. Since $N(z, U)$ is diff-thick and the translate of a thick set is thick, $N(U_1, U_2)$ is thick. Hence, $\mathcal{O}(z)$ is WM.
(ii) ⇒ (iii): If $U_\ell$ is a neighborhood of $x_\ell$ for $\ell = 1, \ldots, k$ and $U = U_1 \times \cdots \times U_k$ then $U$ is a neighborhood of $z = (x_1, x_2, \ldots, x_k)$ and so $N(U, U)$ is thick. Furthermore, as above, $N(U, U) = N(z, U) - N(z, U)$ and so $N(z, U)$ is diff-thick. Since $N(z, U) = \bigcap_{\ell=1}^{k} N(x_\ell, U_\ell)$, it follows that $\{N(x, V) : x \in X, V \text{ open in } X \text{ with } x \in V\}$ generates a filter of diff-thick sets.

So the following is a consequence of Lemma 1.2.

2.2. Corollary. If system $(X, T)$ has a WM enveloping semigroup then it is weakly rigid.

2.3. Definition. Given a cascade $(X, T)$ we say that a closed invariant subset $A \subset X$ is isolated if there is an open subset $U \subset X$ such that $A$ is the maximum closed invariant subset of $U$, i.e. if a closed invariant set $B$ is contained in $U$ then $B \subset A$; or, equivalently if $A = \bigcap_{n \in \mathbb{Z}} T^n(U)$.

2.4. Theorem. If $(X, T)$ is a transitive, weakly rigid cascade, then $X$ does not admit a proper isolated, closed, invariant subset. If, in addition, $(X, T)$ is WM, then $X$ is connected.

Proof. We say that $(X, T)$ is pointwise forward recurrent if $x \in \omega(x)$ for every $x \in X$; it is pointwise recurrent if $x \in \omega(x) \cup \alpha(x)$ for every $x \in X$.

Proposition 2.4 of [2], says that if $(X, T)$ is pointwise forward recurrent and $A$ is an isolated closed invariant subset then $A$ is clopen. As the proof is rather technical, we will sketch the idea.

Suppose $A = \bigcap_{n \in \mathbb{Z}} T^n(U)$ for a closed set $U$ whose interior contains $A$. If $X$ is positive forward recurrent then $A = \bigcap_{n \in \mathbb{N}} T^{-n}(U)$, for if $x$ is in the intersection then the forward orbit $\{T^n(x) : n \in \mathbb{N}\}$ is contained in $U$ and so the invariant set $\omega(x)$ is contained in the closed set $U$. Since $U$ is an isolating neighborhood for $A$, we have $\omega(x) \subset A$. Because the system is positive forward invariant, $x \in A$. It follows from Theorem 3.3 (b) of [1] that $A$ is an attractor for $T^{-1}$, i.e. a repeller for $T$. The complementary attractor $B$ is a closed invariant set disjoint from $A$ and if $x \in X \setminus (A \cup B)$ then $\alpha(x) \subset A$ and $\omega(x) \subset B$. Since the system is positive forward invariant, it follows that $X \setminus (A \cup B) = \emptyset$ and so $A$ is clopen.

Now if $(X, T)$ is weakly rigid, then $(X \times X, T \times T)$ is pointwise recurrent, (i.e. $(x, y) \in \omega(x, y) \cup \alpha(x, y)$ for all $(x, y) \in X \times X$). In turn, this implies that either $(X, T)$ or $(X, T^{-1})$ is pointwise forward recurrent. In fact, otherwise there would be points $x \notin \omega(x)$ and $y \notin \alpha(y)$, whence $(x, y) \notin \omega(x, y) \cup \alpha(x, y)$. We conclude that the isolated, closed, invariant proper subset is clopen, contradicting the transitivity of $X$.

Now suppose that $X$ is not connected and so there exist $U, V$ proper, disjoint clopen sets with union $X$. Then $W = (U \times U) \cup (V \times V)$ is a proper clopen subset of $X \times X$ containing the diagonal $1_X$, which is a non-empty invariant set. Hence, $A = \bigcap_{n \in \mathbb{Z}} (T \times T)^n(W)$ is a non-empty closed invariant set since $W$ is closed. Since $W$ is open, it is an isolating neighborhood for $A$. If $X$ were WM then $X \times X$ would be transitive and weakly rigid and so can contain no such isolated invariant set.

2.5. Corollary. Let $(X, T)$ be a transitive metric cascade.

(1) If $X$ is not connected then its enveloping semigroup $E(X, T)$, as a dynamical system, is not WM.

(2) If $X$ admits a proper isolated closed invariant subset then its enveloping semigroup $E(X, T)$, as a dynamical system, is not WM.

Proof. By Corollary 2.2, under these assumptions, $(X, T)$ must be weakly rigid. Now apply Theorem 2.4. □
2.6. Example. As was shown in [1, page 180] the stopped torus example \((X,T)\) is a connected, topologically mixing, pointwise recurrent system, containing a unique fixed point as its mincenter \(^1\). Moreover, every other point has dense orbit and it follows that the fixed point is a proper isolated closed invariant subset. Thus we conclude that \((X,T)\) is not weakly rigid and that \(E(X,T)\) is not WM.

2.7. Remark. The above proof of Theorem 2.4 relies on Theorem 3.3 (b) of [1] which, in turn, uses a tricky argument of Smale. We will next give a simpler proof of the fact that a weakly rigid WM system is necessarily connected; in fact we prove a slightly stronger result.

2.8. Theorem. A totally transitive, metric, weakly rigid cascade \((X,T)\) is connected.

Proof. We first note that \(\hat{X}\), the canonically defined largest totally disconnected factor of \(X\), is nontrivial, and that \(E(X) \rightarrow E(\hat{X})\). So we now assume that \(X\) is nontrivial and totally disconnected. Such a system always admits a nontrivial symbolic factor \(X \rightarrow Y\) (i.e. \(Y \subset \{0,1\}^\mathbb{Z}\) is a subshift), which by total transitivity is infinite. It then follows that there are four distinct points \(y_i\) \((i = 1, 2, 3, 4)\) such that \(y_1\) and \(y_2\) are right asymptotic while \(y_3\) and \(y_4\) are left asymptotic (see [9, Theorem 10.36], the so called Schwartzman lemma). Now clearly any limit point of the orbit \(\{T^n(y_1,y_2,y_3,y_4) : n \in \mathbb{Z}\}\) has at most three distinct coordinates. In particular the point \((y_1,y_2,y_3,y_4)\) is not recurrent. This implies that \(Y\) is not weakly rigid, contradicting the fact that \(Y\) is a factor of \(X\). (See also Proposition 6.7 in [7].) \(\square\)

2.1. General group actions. In this subsection we deal with actions of more general groups. Given a locally compact topological group \(G\), an action of \(G\) on a compact space \(X\) is given by a continuous homomorphism \(\pi\) from \(G\) into the topological group \(\text{Homeo}(X)\), equipped with the compact open topology. Given such a homomorphism we denote by \((X,G,\pi)\) the corresponding \(G\) dynamical system, or flow. We usually write just \((X,G)\), suppressing the homomorphism \(\pi\) from the notation, and write \(gx\) instead of \(\pi(g)x\), \(x \in X, g \in G\). The notions of transitivity, minimality and weak mixing have clear meaning for such actions. We say that a point \(x \in X\) is recurrent if \(x \in \bigcap\{G \setminus K)x : K\) is a compact neighborhood of \(e\) in \(G\). A point \(x \in X\) is called almost periodic if its orbit closure is minimal.

Now in the proof of Theorem 2.8 we used the Schwartzman lemma. An even shorter proof can be obtained by using Theorem 1.8 from [4]:

2.9. Theorem. Let \(G\) be a countable finitely generated group. Let \((X,G)\) be a flow, with \(X\) a zero dimensional compact metric space, and \(G\) a finitely generated group. Then the following are equivalent.

(i) \((X,G)\) is pointwise recurrent.
(ii) \((X,G)\) is pointwise almost periodic.
(iii) The orbit closure relation is closed.

In fact, if \(X\) is metric, zero dimensional, weakly rigid, and WM, then applying this theorem to \(X \times X\), we conclude that \(X\) is trivial. Since this latter proof applies as well in the case where the acting group \(G\) is finitely generated, we obtain the following generalization.

2.10. Theorem. For any finitely generated group \(G\) and metric system \((X,G)\)

(1) If \(X\) is weakly rigid and WM then \(X\) is connected.
(2) If \(E(X,G)\) is WM then \(X\) is connected.

\(^1\)The mincenter of a dynamical system is the closure of the union of its minimal subsystems.
2.11. Remark. A word of warning is perhaps needed here. In general, for non-commutative groups, Furstenberg’s result, Theorem 1.1, does not necessarily hold (see e.g. [6, Section 4]). However, it does hold in the following two important cases (i) when $G$ is abelian, and (ii) when $(X, G)$ admits a fully supported $G$-invariant probability measure $\mu$ such that the system $(X, \mu, G)$ is measure theoretically weakly mixing (see [3, Theorem 6.12]).

3. The horocycle flow

3.1. Theorem. The enveloping semigroup of a classical horocycle flow $(X, \{U_t\}_{t \in \mathbb{R}})$, where $G = \text{PSL}(2, \mathbb{R})$, $\Gamma < G$ is a uniform lattice, $X = G/\Gamma$ and $U_t(g\Gamma) = u_tg\Gamma = (\frac{1}{\lambda} g)\Gamma$, $t \in \mathbb{R}$, $g \in G$, is WM. The same holds for the discrete flow $(X, U_1)$.

Proof. Recall that $E(X, \{U_t\}_{t \in \mathbb{R}})$ is isomorphic, as a flow, to the infinite pointed product of the family of pointed systems $\{(X, x) : x \in X\}$, i.e. $E(X, \{U_t\}_{t \in \mathbb{R}}) = \bigvee_{x \in X} (X, x) = \{U_t x : t \in \mathbb{R}\} \subset X^X$, where $x \in X^X$ is the identity map $x(x) = x$, $\forall x \in X$. It follows that $E(X, \{U_t\}_{t \in \mathbb{R}})$ is the inverse limit of the family of finite pointed systems

$$\{X(\{x_1, x_2, \ldots, x_n\}) = (X, x_1) \vee (X, x_2) \vee \cdots \vee (X, x_n)\},$$

where we range over the directed collection of, unordered, $k$-tuples $\{x_1, x_2, \ldots, x_n\} \subset X$. It therefore suffices to show that each $X(\{x_1, x_2, \ldots, x_n\})$ is WM. The fact that this is indeed the case is a direct corollary of Ratner’s theory, as follows:

By Ratner’s orbit closure theorem we have

$$X(\{x_1, x_2, \ldots, x_n\}) = \{U_t X(x_1, x_2, \ldots, x_n) : t \in \mathbb{R}\} = H(x_1, x_2, \ldots, x_n),$$

where $H < G \times G \times \cdots \times G$ (n times) is a closed connected subgroup of $G^n$ containing the subgroup $\{u_t \times u_t \times \cdots \times u_t : t \in \mathbb{R}\}$ and there is a discrete uniform lattice $\Lambda < H$ so that $X(\{x_1, x_2, \ldots, x_n\}) = H(x_1, x_2, \ldots, x_n) \cong H/\Lambda$. By unique ergodicity of $(G/\Gamma, \{U_t\}_{t \in \mathbb{R}})$, the Haar measure $\lambda$ on $H/\Lambda$ is an n-fold self-joining of $\mu$, the unique invariant probability measure on $G/\Gamma$. Now apply Ratner’s joining theorem (see [11, Theorem 7, page 283]) to conclude that $\lambda$ has the following form: There is, for some $q$, $1 \leq q \leq n$, a partition

$$I_j \subset I = \{1, 2, \ldots, n\}, \quad I_j = \{i_1^{(j)}, i_2^{(j)}, \ldots, i_{n_j}^{(j)}\}, \quad j = 1, \ldots, q,$$

and there are elements $a_{ik}^{(j)} \in G$, $k = 1, \ldots, n_j$, $j = 1, \ldots, q$, with

$$\Gamma_0^{(j)} = \bigcap_{k=1}^{n_j} a_{ik}^{(j)} \Gamma (a_{ik}^{(j)})^{-1}$$

uniform lattices, such that, each $(X^{I_j}, \lambda_j, \{U_t\}_{t \in \mathbb{R}})$ is isomorphic to the horocycle flow on $(G/\Gamma_0, \rho_0, \{U_t\}_{t \in \mathbb{R}})$, and $\lambda = \prod_{j=1}^{q} \lambda_j$. In particular then the measure preserving dynamical system $(H/\Lambda, \lambda, \{U_t\}_{t \in \mathbb{R}})$ is measure theoretically weakly mixing, hence also topologically WM.

Finally, the same arguments will work for the discrete flow $(X, U_1)$. \hfill \Box

3.2. Example. The case where $\Gamma$ is a nonarithmetic maximal uniform lattice is special. We see that $E(X, \{U_t\}_{t \in \mathbb{R}}) = E(G/\Gamma, \{U_t\}_{t \in \mathbb{R}}) \cong X^X$, as dynamical systems, where $[X]$ denotes the collection of $\{U_t\}_{t \in \mathbb{R}}$ orbits in $G/\Gamma$. Note that both the $\mathbb{R}$-flow $E(X, \{U_t\}_{t \in \mathbb{R}})$ and the discrete system $E(X, U_1)$, have the property that any point $x \in X^X$ whose coordinates have the property that no two of them belong to the same $\{U_t\}_{t \in \mathbb{R}}$ orbit, has a
dense orbit in $X^{[X]}$. Of course card $(X^{[X]}) = \mathfrak{c}$, the cardinality of the continuum. Projecting on any set of four coordinates we obtain a real flow with 4-fold minimal self joinings. The corresponding $U_1$ cascade has the same property when we allow for off-diagonals resulting from the $\{U_t\}_{t \in \mathbb{R}}$ flow. This does not contradict the result of J. King [10, page 756], which says that there is no infinite minimal cascade $(X, T)$ with 4-fold minimal self-joinings. The reason is that here, as it turns out, the ‘future $\varepsilon$-bounded pair’ produced in Theorem 21 of [10], must lie on the same $\{U_t\}_{t \in \mathbb{R}}$ orbit.

3.3. Remark. It is easy to see that if $(X, T)$ is a dynamical system whose enveloping semigroup is WM then so is the associated system $(\hat{X}, T)$ obtained from $X$ by collapsing its mincenter to a point (see e.g. the remark at the beginning of Section 2). This leads us to the following:

3.4. Problem. Is there a nontrivial metric connected cascade $(X, T)$ with trivial mincenter whose enveloping semigroup is WM? Note that such a system is necessarily both proximal and weakly rigid.

References

[1] E. Akin, The general topology of dynamical systems, Graduate Studies in Mathematics, 1, American Mathematical Society, Providence, RI, 1993.
[2] E. Akin, On chain continuity, Discrete Contin. Dynam. Systems 2, (1996), 111–120.
[3] E. Akin, J. Auslander and E. Glasner, The topological dynamics of Ellis actions, Mem. Amer. Math. Soc., 195, (2008), no. 913.
[4] J. Auslander, E. Glasner and B. Weiss, On recurrence in zero dimensional flows, Forum Math. 19, (2007), 107–114.
[5] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory, Princeton university press, Princeton, N.J., 1981.
[6] E. Glasner, Topological weak mixing and quasi-Bohr systems, Isr. J. Math. 148, (2005), 277–304.
[7] E. Glasner and D. Maon, Rigidity in topological dynamics, Ergod. Th. Dynam. Sys. 9, (1989), 309–320.
[8] E. Glasner and M. Megrelishvili, Hereditarily non-sensitive dynamical systems and linear representations, Colloq. Math., 104 (2006), 223–283.
[9] W. H. Gottschalk and G. A. Hedlund, Topological Dynamics, AMS Colloquium Publications, Vol. 36, 1955.
[10] J. King, A map with topological minimal self-joinings in the sense of del Junco, Ergod. Th. & Dynam. Sys. (1990), 10, 745–761
[11] M. Ratner, Horocycle flows, joinings and rigidity of products, Ann. of Math. (2) 118, (1983), no. 2, 277–313.