Asymmetric voltage noise in superconducting tunnel junctions with the electromagnetic environment

Martin Žonda and Tomáš Novotný

Department of Condensed Matter Physics, Faculty of Mathematics and Physics, Charles University in Prague, Ke Karlovu 5, 121 16 Praha 2, Czech Republic

E-mail: martin.zonda@gmail.com

Received 3 May 2012
Accepted for publication 27 June 2012
Published 30 November 2012
Online at stacks.iop.org/PhysScr/T151/014023

Abstract
We investigate theoretically the $V-I$ characteristics and voltage noise of superconducting tunnel junctions with a small critical current via the matrix-continued-fraction method. Special attention is paid to the large hysteresis in the $V-I$ characteristics and to the voltage-noise anomaly observed in preliminary experiments. The current dependence of the voltage noise shows a strong asymmetry between the forward and the backward current ramping and a discontinuous change in the noise close to its measured maximum occurring at the switching current. We show that both the large hysteresis and the voltage noise anomaly in this current-biased setup are a consequence of the influence of the junction’s electromagnetic environment. Typically, the voltage-noise dependence on the junction current parametrized by the external drive contains a loop that is, however, not observed experimentally because of the implementation of the junction-current ramp. Skipping a part of the loop is responsible for the observed hysteresis as well as the noise anomaly.

PACS numbers: 72.70.+m, 74.50.+r, 85.25.cp

(Some figures may appear in color only in the online journal)

1. Introduction

For small-area Josephson junctions, where the critical current is in the nanoampere range and the capacitance is in femtofarads, the dynamics of the Josephson phase is strongly influenced by the electromagnetic environment [1]. This can have profound effects on the measured $V-I$ characteristics and voltage noise. For example, depending on the realization of the environment the same junction can be in the underdamped regime, with a typical hysteretic behavior of the $V-I$ curves between the forward and the backward source-current ramping, but also in the nonhysteretic overdamped regime.

A recent preliminary experimental study by Pertti Hakonen’s group in Helsinki [2] focusing on the voltage noise of small-area tunnel junctions revealed a strong forward versus backward current-ramping mean-voltage and voltage-noise asymmetry and hysteresis (see figure 1). Furthermore, a discontinuous change in the voltage noise coinciding with the switching current was observed. The purpose of this paper is to show that such surprising effects can be understood as a consequence of the junction’s electromagnetic environment.

The organization of this paper is as follows. First, we introduce in section 2 a simple model circuit close to those used in experiments for obtaining the critical current of real junctions and shortly discuss the matrix continued fraction method used for analyzing the properties of the model circuit. In section 3, we present the basic ideas of our explanation. Finally, we discuss our numerical results and show that they are in qualitative agreement with the experiment in section 4.

2. The circuit and the method

As is well known, the electromagnetic environment can be engineered to place any Josephson junction in the overdamped...
I equivalent to a current source \( R \) and resistance \( I \) by the RCSJ model. The red square contains a junction modeled of Josephson junctions. The black line represents forward current ramping, while the red line corresponds to the backward direction. Unpublished data courtesy of Fay and Hakonen [2].

![Figure 1](image1.png)  
**Figure 1.** Experimental dependence of the mean junction voltage on the bias current (top panel) with the corresponding voltage-noise (in arbitrary units) dependence (bottom panel). The black line represents forward current ramping, while the red line corresponds to the backward direction. Unpublished data courtesy of Fay and Hakonen [2].

![Figure 2](image2.png)  
**Figure 2.** Idealized circuit for the measurement of the supercurrent of Josephson junctions. The red square contains a junction modeled by the RCSJ model.

regime [3]. We investigate in more detail the influence of such environments on the junction properties. The circuit considered is shown in figure 2, where a Josephson junction (red square) with critical current \( I_c \), intrinsic capacitance \( C_j \) and resistance \( R_j \) is biased by a circuit with a capacitor \( C \), a resistor \( R \) and an ideal voltage source \( V_s \). This circuit is equivalent to a current source \( I_s = V_s / R \) in parallel with \( R, C \) and the junction. An identical circuit was used to show that the commonly observed reduction of the maximum supercurrent in an ultrasmall junction is not an intrinsic junction property, but is due to its electromagnetic environment [3]. Its advantage lies in the fact that, in contrast to the usual current-bias method (applied directly to the junction) giving access only to the positive differential resistance part of the \( V-I \) characteristic, all points on the \( V-I \) characteristic are, in principle, achievable. That is because both the current \( I \) through the junction (measured by ideal amperemeter \( I \)) and the junction voltage \( V \) (measured by ideal voltmeter \( V \)) are average quantities adjusted to the global drive.

The dynamics of this circuit following from Kirchhoff’s laws and Josephson’s relations is described by the Langevin equations

\[
I(t) = \frac{V_s - V(t)}{R} - C \frac{dV(t)}{dt} + \xi_R(t),
\]

\[
I(t) = \frac{V(t)}{R_j} + C_j \frac{dV(t)}{dt} + L \sin \varphi(t) + \xi_{R_j}(t),
\]

(1)

where two current-noise sources \( \xi_R \) and \( \xi_{R_j} \) associated with (mutually uncorrelated) thermal fluctuations in resistors \( R \) and \( R_j \) are assumed to be simple Gaussian white noises satisfying

\[
\langle \xi_R(t) \rangle = 0, \quad \langle \xi_{R_j}(t_1) \xi_{R_j}(t_2) \rangle = \frac{2k_B T}{R_j} \delta(t_1 - t_2).
\]

(2)

It is useful to introduce the equivalent parallel resistance \( R_p = R R_j / (R + R_j) \) and the equivalent parallel capacitance \( C_p = C + C_j \). Then the quality factor of the circuit reads

\[
Q = \omega_p R_p C_p.
\]

(3)

where \( \omega_p = \sqrt{I_c / \Phi_0 C_p} \) is the plasma frequency with \( \Phi_0 = h / 2e \) the reduced flux quantum. For the purpose of theoretical analysis, it is useful to rewrite equation (1) into the dimensionless form by introducing the following dimensionless quantities [4]—junction voltage \( v = \frac{V}{I_c R j} \), junction current \( i = \frac{i}{I_c} \), time \( \tau = \omega_p \tau \), temperature \( \Theta = \frac{\Theta}{\Theta_p} \) (with \( \Theta_p \) being the Boltzmann constant), the driving force \( i_s = \frac{I_s}{I_c} \), and, finally, the composite Gaussian white noise \( \xi \) with the correlation function

\[
\langle \xi(t_1) \xi(t_2) \rangle = \frac{\omega_p}{I_c} \theta \langle (\xi_R(t_1) - \xi_R(t_1))(\xi_R(t_2) - \xi_R(t_2)) \rangle
\]

\[
= 2 \gamma \Theta \delta(t_1 - t_2),
\]

(4)

where \( \gamma = 1/Q \) is the damping coefficient.

Using these definitions, equation (1) can be reformulated in the dimensionless form \( v(t) = \varphi / \partial \tau \)

\[
\frac{\partial v(\tau)}{\partial \tau} = i_s - \gamma v(\tau) - \sin \varphi(\tau) + \xi(\tau).
\]

(5)

This equation is formally identical to the RCSJ model [5], which is commonly used for the description of Josephson junctions without the environment and, thus, the methods developed for the RCSJ model can be applied. We have used the matrix continued fraction method [6, section 11.5] for the solution of the Fokker–Planck equation associated with the Langevin equation (5) for the probability distribution function \( W(\varphi, v, \tau) \)

\[
\frac{\partial}{\partial \tau} W = -v \frac{\partial W}{\partial \varphi} + \frac{\partial W}{\partial v} \left( \gamma v + \sin \varphi - i_s + \gamma \Theta \frac{\partial}{\partial \varphi} \right) W = -L_{FP} W.
\]

(6)

The probability distribution function is obtained numerically by expanding the \( v \)-part of \( W(\varphi, v, \tau) \) into quantum oscillator basis functions, thus obtaining a tridiagonal coupled system of differential equation, and the \( 2 \pi \)-periodic \( \varphi \)-part into the Fourier series [6, section 11.5]. We restrict our analysis here to
the stationary case when the average value of junction voltage \( \langle v \rangle \) and junction current \( \langle i \rangle \) can be computed as

\[
\langle v \rangle = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv v W(\phi, v, \infty),
\]

\[
\langle i \rangle = i_s - \frac{\gamma}{1 + \rho} \langle v \rangle,
\]

where the environmental parameter \( \rho = R_j/R \), and the voltage autocorrelation function reads [6, section 7.2]

\[
\langle v(\tau)v(0) \rangle = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv v e^{-|\tau|/\sigma} v W(\phi, v, \infty).
\]

Finally, the steady-state voltage noise is obtained by time integral of the (connected) voltage autocorrelation function

\[
S = \int_{-\infty}^{\infty} d\tau \langle (v(\tau)v(0)) - \langle v(\tau) \rangle \langle v(0) \rangle \rangle.
\]

3. Basic mechanism

In figure 3(a) we plot a typical curve of the mean junction current \( \langle i \rangle \) dependence on the mean junction voltage \( \langle v \rangle \) for temperature \( \Theta = 0.04 \), quality factor \( Q = 1 \) and different environment parameters \( \rho \). Here, the limit \( \rho \to \infty \) corresponds to the case without the environment. For that case the mean current \( \langle i \rangle \) is a monotonic function of the voltage \( \langle v \rangle \). This changes with the introduction of the environment when a local maximum and a local minimum (decreasing with increasing \( \rho \)) of \( \langle i \rangle \) develop. Figure 3(b) shows the voltage dependence of the voltage noise—it turns out that for a given \( Q \) this quantity does not depend on the environment details and, thus, all curves for various \( \rho \) coincide.

The nonmonotonic dependence of current on the mean voltage has serious consequences for current-biased junctions as illustrated in figure 4(a). The mean voltage is not a function of the junction current since in the range between the current minimum and maximum three different voltage values can be attributed to the same current (green line in figure 4(a)). In practice, when the junction current is ramped up, it will follow the stable branch until it hits the point \( A \) when a step back would be required to keep on the average characteristic (gray curve). Instead, the mean voltage will switch to its other stable value at point \( B \) (red dots). For the reverse ramp, the analogous situation happens between points \( C \) and \( D \) (black dotted curve). Depending on the distance between the current extrema the hysteresis can be large.

The fact that \( \langle v \rangle \) is not a function of \( \langle i \rangle \) also has a profound effect on the voltage noise dependence on \( \langle i \rangle \). Typically, this dependence (understood as a parametric curve of both quantities parametrized by the mean voltage) contains a loop as shown in figure 3(c) and further illustrated in detail in figure 4(b) by the solid gray line. There the points \( A, B, C, D \) correspond to the points in figure 4(a). For the same reasons causing the hysteresis in \( v-I \) characteristics, one will not observe the whole loop in the noise either under the junction-current ramp conditions. The measured noise will leave the loop in the moment when a step back in the current bias (negative differential resistance) is needed to remain on the loop. This coincides with points \( A \) and \( C \), respectively, i.e. with the switching currents for opposite ramp directions. Therefore, we suggest this mechanism as responsible for the discontinuous changes in the noise observed in the experiment close to the switching currents (see the bottom panel of figure 1). Furthermore, the noise loop is not symmetric and, therefore, the height and position of the peaks depend on the direction of the measurement as illustrated again by the black and red dotted lines (the same color code as in figure 1). As we will show later, for some parameters the lower peak becomes negligible and in this case we get qualitatively the same noise behavior as in the experiment. To sum up, we have strong evidence that both the large hysteresis and the asymmetric voltage noise are a consequence of the influence of the electromagnetic environment on the current-biased junction.

4. Results and discussion

In this section, we will show that also other properties of the \( V-I \) curves and noise are qualitatively in agreement with the experiment. The junction used in the experiment has \( R_j = 7.7 k\Omega \), \( I_c = 19.8 nA \) and \( C_j = 5 fF \) and the
The environment was modeled by \( R = 0.1 \, \text{k} \Omega \) and \( C = 0.1 \, \text{pF} \) in all the presented examples.

In figure 5 we plot the \( V-I \) characteristics for different temperatures and the current dependence of the voltage noise for the same parameters is plotted in figure 6. The voltage noise is expressed in units of the thermal Johnson–Nyquist noise \( S_T = 2 k_B T R_p \) of the equivalent parallel resistor \( R_p \). In all figures the solid gray line represents the full static solution of the circuit figure 2 and the dotted red and black lines show the supposed measured curves when the junction-current ramps are considered. From the \( V-I \) characteristics, one can see that the measured hysteresis can be rather large and increases with decreasing temperature. For the chosen parameters the upper drop in the mean junction voltage is approximately 0.1 mV. It is of the same order as in the experiment (figure 1). There is one difference, however. While the asymptotic \( V-I \) lines are almost horizontal in the experiment (figure 1) they follow the \( V = R_I I \) line in our theory. We believe that this discrepancy would be fixed by using a nonlinear model for the junction. The nonlinear model gives a better description of real junctions for the voltages close to the gap voltage \( V_G \) [7, section 2.3.2]. A simple piecewise-linear approximation \( (R_J(V) = R_{(a)} \text{ at } |V| < V_G, \quad R_J(V) = R_{(b)} \text{ at } |V| > V_G \) with \( R_{(a)} \gg R_{(b)} \) should be enough to show that these measured horizontal top and bottom \( V-I \) lines have their origin in the almost infinite slope change of \( I \) close to the \( V = V_G \) known from typical \( V-I \) characteristics of a Josephson junction [7, section 2.3.2] (note that a vertical line in the \( I-V \) curve is horizontal in the \( V-I \) characteristic).

Our calculated voltage noise properties are also in good agreement with the experiment [2] (experimental data are unpublished as yet). With decreasing the temperature, the location of the higher asymmetric noise peak shifts farther away from the zero current and the peak becomes steeper. The lower peak is very small even for low temperatures. The very same trends are observed also in the experiment [2]. A closer look at the experimental data in figure 1 reveals also a small drop in the noise close to \( I = 0 \) (marked with the green circle). The realistic environment present in the experiment was apparently more complex than the one assumed here and, thus, quantitative differences are expected, yet our qualitative explanation should be valid also for more realistic environments.

An important question is the origin of the voltage noise. As demonstrated in figure 7 we are not dealing exclusively with the thermal noise associated with the combination of the junction differential resistance \( R_D = dV/dI \) and resistance \( R \). Figure 7 shows the voltage noise \( S \) divided by the equivalent noise \( S_0 = 2 k_B T R_D \) of the junction’s differential resistance versus the mean junction current \( I \) for a wide range of temperatures (for other parameter values see the main text). The normalized-noise unitary value at zero junction current reflects the fluctuation–dissipation theorem.

In conclusion, there are several possible methods for the verification of our hypothesis. Depending on the details of the measurement the true nature of the \( V-I \) characteristic could be revealed by using the voltage ramp instead of the

---

**Figure 5.** Junction voltage \( V \) versus current junction \( I \) for different temperatures (for other parameter values see the main text).

**Figure 6.** Analogue of figure 5 for the voltage noise \( S \) normalized by \( S_T = 2 k_B T R_p \).

**Figure 7.** Voltage noise \( S \) normalized by the equivalent noise \( S_0 = 2 k_B T R_D \) of the junction’s differential resistance versus the mean junction current \( I \) for a wide range of temperatures (for other parameter values see the main text).
junction-current ramp. The other possibility is to use a larger number of measurements and slower current ramping.

Acknowledgments

We thank Aurelien Fay and Pertti Hakonen for sharing their experimental data with us before publication and for useful discussions. We also thank Gabriel Niebler and Tero Heikkilä for valuable input at early stages of this work. We acknowledge support from the Czech Science Foundation via grant no. 204/11/J042 and the Charles University Research Center ‘Physics of Condensed Matter and Functional Materials’.

References

[1] Joyez P, Vion D, Goët M, Devoret M H and Esteve D 1999 J. Supercond. 12 757–66
[2] Fay A and Hakonen P 2012 in preparation
[3] Steinbach A, Joyez P, Cottet A, Esteve D, Devoret M H, Huber M E and Martinis J M 2001 Phys. Rev. Lett. 87 137003
[4] Kautz R L 1996 Rep. Prog. Phys. 59 935–92
[5] Stewart W C 1968 Appl. Phys. Lett. 12 277–80
  McCumber D E 1968 J. Appl. Phys. 39 3113–8
[6] Risken H 1989 The Fokker–Planck Equation 2nd edn (Berlin: Springer)
[7] Likharev K K 1996 Dynamics of Josephson Junctions and Circuits 3rd edn (Amsterdam: Gordon and Breach)
[8] Voss R 1981 J. Low Temp. Phys. 42 151–63