Lagrangian dynamics of incompressible thermofluids

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Abstract

A key aspect of fluid dynamics is the correct definition of the phase-space Lagrangian dynamics which characterizes arbitrary fluid elements of an incompressible fluid. Apart being an unsolved theoretical problem of fundamental importance, the issue is relevant to exhibit the connection between fluid dynamics and the classical dynamical systems underlying incompressible and non-isothermal fluid, typically founded either on: a) a configuration-space Lagrangian description of the dynamics of fluid elements; b) a kinetic description of the molecular dynamics, based on a discrete representation of the fluid. The goal of this paper is to show that the exact Lagrangian dynamics can be established based on the inverse kinetic theory (IKT) for incompressible fluids recently pointed out (Ellero et al., 2004-2006, \cite{4}). The result is reached by adopting an IKT approach based on a restricted phase-space representation of the fluid, in which the configuration space coincides with the physical fluid domain. The result appears of potential importance in applied fluid dynamics and CFD.

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I. INTRODUCTION

As it is well known, the description of fluids can be performed choosing either an Eulerian or a Lagrangian point of view. The two approaches are equivalent and, if fluid dynamics were fully understood, one should be able to translate Eulerian properties into Lagrangian ones and vice versa. At least for the treatment of turbulent fluids, we are still quite far from this point \[1\]. The main historical reason of this situation can be understood by looking at the customary statistical approach based on so-called velocity probability density function (pdf)-method for an incompressible fluid (for a review see for example \[2\]). In fact - as it is well known - in this approach the time evolution of the pdf which advances in time the average (in a stochastic sense) fluid velocity is determined by a Fokker-Planck transport equation. As a consequence the corresponding Lagrangian characteristics are necessarily stochastic in nature and therefore difficult to handle. This explains why in the literature, for this purpose, stochastic models of various nature have been adopted, which, however, typically rely on experimental verification rather than (uniquely) on first principles.

Recently, however, an important breakthrough has been achieved by the discovery of the so-called inverse kinetic theory (IKT) approach for incompressible fluids \[3, 4, 5, 6, 7\] which permits a straightforward connection between Eulerian and Lagrangian descriptions. This is achieved by identifying the relevant fluid fields, which are assumed to be defined in a suitable domain \(\Omega \subseteq \mathbb{R}^3\) (fluid domain), with appropriate moments of a suitably-defined kinetic distribution function (KDF) \(f(x,t)\) [with \(x = (r, v) \in \Gamma\), \(x\) and \(\Gamma\) denoting a suitable state-vector and an appropriate phase-space] which is assumed to advance in time by means of Vlasov-type kinetic equation. In such a case, the time-evolution of the KDF is determined by a kinetic equation which, written in the Eulerian form, reads

\[
Lf(x, t) = 0. \tag{1}
\]

Here \(f(x, t)\) denotes the Eulerian representation of the KDF, \(L\) is the streaming operator

\[
Lf \equiv \frac{\partial}{\partial t} f + \frac{\partial}{\partial x} \cdot \{X(x, t)f\}, \quad X(x, t) \equiv \{v, F(x, t)\}
\]

a suitably smooth vector field, while \(v\) and \(F(x, t)\) denote respectively appropriate velocity and acceleration fields. As a main consequence the approach can in principle be used to determine in a rigorous way the Lagrangian formulation for arbitrary complex fluids. Although the choice of the phase space \(\Gamma\) is in principle arbitrary, in the case of incompressible isothermal fluids, it is found \[4\] that the phase-space \(\Gamma\) can always be reduced to the direct-product space \(\Gamma = \Omega \times V\)
(restricted phase-space), where \( \Omega, V \subseteq \mathbb{R}^3 \), \( \Omega \) is an open set denoted as configuration space of the fluid (fluid domain) and \( V \) is the velocity space. This type of approach (based on a restricted phase-space IKT formulation) will be adopted also in the sequel.

The main motivation [of this work] is that some of the general understanding recently achieved in simple flows by means of the IKT approach could also give a significant contribution to a wider range of problems. In the sequel, we will concentrate on the issue of a consistent formulation for fluid dynamics based on a phase-space (IKT) description of incompressible fluids, whereby its pressure, velocity (and possibly also thermal) fluctuations are consistently taken into account. In particular we intend to show that the theory admits a well-defined Lagrangian formulation. In particular, the corresponding Lagrangian trajectories can be interpreted in terms of the phase-space dynamics of suitable, classical molecules, i.e., point particles whose dynamics is determined by the phase-space Lagrangian characteristics. The motion of these particles (rather than that of the fluid elements), as they are pushed along erratic trajectories by fluctuations of the fluid-field gradients (in particular, characterizing the fluid pressure and temperature), is fundamental to transport and mixing processes in fluids. In this regard, it is well known that the interaction between (deterministic and/or turbulent) fluctuations of the fluid fields and fluid particles still escapes a consistent theoretical description. Being a subject of major importance for many environmental, geophysical and industrial applications, the issue deserves a careful investigation. A key aspect of fluid dynamics is, therefore, the correct definition of the (phase-space) Lagrangian dynamics which characterizes incompressible fluids. The customary approach to the Lagrangian formulation is based typically on a configuration-space description, i.e., on the introduction of the configuration-space Lagrangian characteristics \( \mathbf{r}(t) \), spanning the fluid domain \( \Omega \). Here \( \mathbf{r}(t) \) denotes the solution of the initial-value problem:

\[
\begin{align*}
\frac{d\mathbf{r}}{dt} &= \mathbf{V}(\mathbf{r}, t), \\
\mathbf{r}(t_o) &= \mathbf{r}_o,
\end{align*}
\]

with \( \mathbf{r}_o \) an arbitrary vector belonging to the closure \( \overline{\Omega} \) of \( \Omega \) and \( \mathbf{V}(\mathbf{r}, t) \) being the velocity fluid field, to be assumed continuous in \( \overline{\Omega} \) and suitably smooth in \( \Omega \). The purpose of this paper is to achieve a phase-space Lagrangian formulation for incompressible thermofluids, extending the formulation previously developed, adopting, instead, an extended-phase-space formulation \[8\]. The present formulation permits to advance in time the relevant fluid fields
by means of the set of Lagrangian equations defined by the vector field $\mathbf{X}(\mathbf{x}, t)$, namely

$$\frac{d\mathbf{x}}{dt} = \mathbf{X}(\mathbf{x}, t),$$

$$\mathbf{x}(t_0) = \mathbf{x}_o,$$  

where $\mathbf{x}_o$ is an arbitrary initial state of $\Gamma$ (closure of the phase-space $\Gamma$). The goal of this paper is to show, in particular, that the exact (phase-space) Lagrangian dynamics can be established based on the inverse kinetic theory (IKT) for incompressible fluids in such a way that the phase space $\Gamma$ coincides with the direct product space $\Gamma = \Omega \times V$, $\Omega$ being the fluid domain and $V$ (velocity space) the set $\mathbb{R}^3$. The result appears relevant in particular for the following reasons: 1) the Lagrangian dynamics here determined permits to advance in time self-consistently the fluid fields, i.e., in such a way that they satisfy identically the requires set of fluid equations. For isothermal fluids, this conclusion is consistent with the results indicated elsewhere by Tessarotto et al. [5]; 2) the Lagrangian dynamics takes into account the specific form of the phase-space distribution function which advances in time the fluid fields; 3) the theory permits an exact description of the motion of those particles immersed in the fluid which follow the Lagrangian dynamics (classical molecules). In detail the plan of the paper is as follows. First, in Section 2, an IKT for incompressible thermofluids, adopting a restricted phase-space description (analogous to that developed in Ref. [5]), is presented. This permits us to determine the appropriate form of the vector field $\mathbf{X}(\mathbf{x}, t)$. For the sake of illustration, we shall consider in particular the case in which the KDF is identified with a local Maxwellian distribution function. Second, in Sec.3, the new set of Lagrangian equations is presented, which are proven to advance uniquely in time the relevant fluid fields of an incompressible thermofluid.

II. EULERIAN FORMULATION OF IKT

A first key issue is to prove that the IKT approach developed for isothermal incompressible fluids [3, 4, 5, 6, 7] can also be achieved by adopting the same type of restricted phase-space formulation, obtained identifying the phase space with $\Gamma = \Omega \times V$, where $\Omega$ coincides with the fluid domain and $V = \mathbb{R}^3$ is a 3D velocity space. For definiteness, here we shall assume that the relevant fluids for a thermofluid, namely $\{\rho = \rho_o > 0, \mathbf{V}, p \geq 0, T > 0, S_T\}$, i.e., respectively the constant mass density, the fluid velocity, pressure, temperature and
entropy, in the open set $\Omega$ satisfy the so-called non-isentropic and incompressible Navier-Stokes-Fourier equations (INSFE), i.e.,

$$\nabla \cdot \mathbf{V} = 0,$$

$$\frac{D}{Dt} \mathbf{V} = -\frac{1}{\rho_o} [\nabla p - \mathbf{f}] + \nu \nabla^2 \mathbf{V},$$

$$\frac{D}{Dt} T = \chi \nabla^2 T + \frac{\nu}{2c_p} \left( \frac{\partial V_i}{\partial x_i} + \frac{\partial V_k}{\partial x_k} \right) + \frac{1}{\rho_o c_p} J \equiv K,$$

$$\frac{\partial}{\partial t} S_T \geq 0,$$

where $\frac{D}{Dt} \mathbf{V}$ is the fluid acceleration, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$ the convective derivative and $S_T$ a functional, to be suitably defined, of an appropriate set of fluid fields. These equations are assumed to satisfy a suitable initial-boundary value problem (INSFE problem) so that a smooth (strong) solution exists for the fluid fields $\{\rho = \rho_o > 0, \mathbf{V}, p \geq 0, T > 0\}$. Here the notation is standard. Thus, Eqs. (4), (5) and (6) denote respectively the so-called isochoricity condition, the forced Navier-Stokes equation, written in the Boussinesq approximation, and the Fourier equation. As a consequence, in such a case the force density $\mathbf{f}$ reads

$$\mathbf{f} = \rho_o \mathbf{g} (1 - k_p T) + \mathbf{f}_1,$$

where the first term represents the (temperature-dependent) gravitational force density, while the second one ($\mathbf{f}_1$) the action of a possible non-gravitational externally-produced force. Hence $\mathbf{f}$ can be written also as $\mathbf{f} = -\nabla \phi + \mathbf{f}_R$, where $\phi = \rho_o g z$ and $\mathbf{f}_R = -\rho_o \mathbf{g} k_p T + \mathbf{f}_1$ denote respectively the gravitational potential (hydrostatic pressure) and the non-potential force density. Moreover, in Eq. (6) $J$ is the quantity of heat generated by external sources per unit volume and unit time. Finally, Eq. (7) defines the so-called 2nd principle for the thermodynamic entropy $S_T$. For its validity in the sequel we shall assume that there results everywhere in $\Omega \times \mathcal{T}$

$$\int_\Omega d\mathbf{r} \left( \chi \nabla^2 T + \frac{1}{\rho_o c_p} J \right) \geq 0$$

(externally heated thermo-fluid). In these equations $\mathbf{g}, k_p, \nu, \chi$ and $c_p$ are all real constants which denote respectively the local acceleration of gravity, the density thermal-dilatation coefficient, the kinematic viscosity, the thermometric conductivity and the specific heat at constant pressure. Thus, by taking the divergence of the N-S equation (5), there it follows the Poisson equation for the fluid pressure $p$ which reads

$$\nabla^2 p = -\rho_o \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) + \nabla \cdot \mathbf{f},$$
with \( p \) to be assumed non-negative and bounded in \( \Omega \times I \). In this Section we shall assume that \( f(x,t) \) is a solution of the Eulerian kinetic equation (\( \| \) defined in a suitable extended phase-space \( \Gamma \times I \), where \( I \subseteq \mathbb{R} \) is a suitable time interval. In such a case, we intend to show that \( f(x,t) \) can be defined in such a way that the fluid fields \( \mathbf{V}, p_1 \) can be identified with its velocity moments \( \int d\mathbf{v} G(x,t)f(x,t) \), where respectively \( G(x,t) = 1, \mathbf{v}, u^2/3 \) and \( p_1 \) is the kinetic pressure to be defined as:

\[
p_1 = p_0(t) + p - \phi + \rho_o T/m. \tag{10}
\]

Here \( p_0(t) \) (to be denotes as pseudo-pressure [5]) is an arbitrary strictly positive and suitably smooth function defined in \( I \). Moreover, \( m > 0 \) is a constant mass, whose value remains in principle arbitrary. In particular it can be identified with the average mass of the molecules forming the fluid. Finally, the thermodynamic entropy \( S_T \) can be identified with the Shannon statistical entropy functional

\[
S(f(x,t)) = -\int_{\Gamma} dx f(x,t) \ln f(x,t),
\]

provided the function \( p_0(t) \) is a suitably prescribed function and \( f(x,t) \) is strictly positive in the whole set \( \Gamma \times I \).

To reach the proof, let us first show that, by suitable definition of the vector field \( F(x,t) \), a particular solution of the IKE (\( \| \) is delivered by the Maxwellian distribution function:

\[
f_M(x, t) = \rho_o \pi^{\frac{3}{2}} v_{th,p}^3 \exp \left\{ -\frac{u^2}{v_{th,p}^2} \right\} \tag{11}.
\]

Here \( u = \mathbf{v} - \mathbf{V}(r, t) \) and \( v_{th,p} = \sqrt{2p_1(r, t)/\rho} \) are respectively the relative and the thermal velocities.

The following theorem can immediately be proven:

**Theorem - Restricted phase-space INSFE–IKT** Let us assume that: 1) the INSFE problem admits a smooth strong solution in \( \Gamma \times I \), such that the inequality (\( \| \) is fulfilled; 2) the vector field \( F \) is defined as

\[
F(x, t; f) = F_0 + F_1, \tag{12}
\]

where \( F_0, F_1 \) read respectively

\[
F_0(x, t; f) = \frac{1}{\rho_o} \left[ \nabla \cdot \Pi - \nabla p_1 + \mathbf{f}_R \right] + \mathbf{u} \cdot \nabla \mathbf{V} + \nu \nabla^2 \mathbf{V}, \tag{13}
\]

\[
F_1(x, t; f) = \frac{1}{2} \mathbf{u} \left\{ \frac{1}{p_1} A + \frac{1}{p_1} \nabla \cdot \mathbf{Q} - \frac{1}{p_1^2} \left[ \nabla \cdot \Pi \right] \cdot \mathbf{Q} \right\} + \frac{v_{th}}{2p_1} \nabla \cdot \Pi \left\{ \frac{u^2}{v_{th}^2} - \frac{3}{2} \right\}, \tag{14}
\]

where

\[
A \equiv \frac{\partial}{\partial t} (p_0 + p) - \mathbf{V} \cdot \left[ \frac{D}{Dt} \mathbf{V} - \frac{1}{\rho_o} \mathbf{f}_R - \nu \nabla^2 \mathbf{V} \right] + \frac{\rho_o K}{m} \equiv \frac{D}{Dt} p_1. \tag{15}
\]
3) in $\Omega \times I$ the KDF $f(x,t)$ admits the velocity moments $G(x,t) = 1, v, u^2/3$ and $uu$; thus, we denote in particular $Q = \int d^3v u^2 f$ and $\Pi = \int d^3vuuf$; 4) the KDF $f(x,t)$ satisfies identically in in $\Omega \times I$ the constraint equation:

$$ \rho \equiv \int_{\mathbb{R}^3} d^3v f(x,t) = \rho_o > 0, $$

where $\rho_o$ is a positive constant; 5) the entropy integral $S(f(x,t_0)) = -\int_{\Gamma} d^3x f(x,t_0) \ln f(x,t_0)$ exists; 6) in the time interval $I$ the pseudo-pressure $p_0(t)$ is defined so that there results identically:

$$ \int_{\Omega} d^3r \frac{1}{p_1} \left[ \frac{\partial}{\partial t} p_1 + \nabla \cdot Q - \frac{1}{p} \nabla p \cdot Q \right] = 0. $$

It follows that: A) the local Maxwellian distribution function $f(x,t)$ is a particular solution of the IKE (1) if an only if the fluid fields $\{\rho = \rho_o > 0, V, p, T\}$ satisfy the fluid equations (4)-(6). In such a case there results identically $Q = 0, \Pi = 0$; B) for an arbitrary KDF $f(x,t)$, which satisfies identically assumptions 4)-6), the velocity-moment equations obtained by taking the weighted velocity integrals of Eq.(1) with the weights $G(x,t) = 1, v, u^2/3$ deliver identically the same fluid equations (4)-(6); C) the Shannon entropy $S(f)$ is a monotonic function of time, i.e., there results $(\forall t \in I)$:

$$ \frac{\partial}{\partial t} S(f) \geq 0 $$

(H-theorem), where there results in particular for isothermal fluids (see also [8]) $\frac{\partial}{\partial t} S(f) = 0$ (constant H-theorem). This permits us to identify $S_T = S(f)$ so that the 2nd principle [i.e., the inequality (7)] is satisfied too.

PROOF First, let us assume that a strong solution of the INSFE problem exists which satisfies identically Eqs.(4)-(6) in the set $\Omega \times I$. In such a case it is immediate to prove that $f_M(x,t)$ is a particular solution of the inverse kinetic equation (1). The proof follows upon invoking Eqs.(12)-(15) for the vector field $\mathbf{F}(x,t; f)$ and by direct substitution of the distribution $f_M(x,t)$ in the same equation (Proposition A). Instead, if we assume that in $\Gamma \times I$, $f \equiv f_M(x,t)$ is a particular solution of the inverse kinetic equation, which fulfills identically the constraint equation (16), it follows that the fluid fields $\{\rho = \rho_o > 0, V, p, T\}$ are necessarily solutions of the INSFE equations. This can be proven either: a) by direct substitution of $f \equiv f_M(x,t)$ in Eq.(11) (Proposition A); b) by direct evaluation of the velocity moments of the same equation for $G(x,t) = 1, v, u^2/3$ (Proposition B). In fact, thanks to
assumptions 1-6 the first two moment equation coincide respectively with the isochoricity and Navier-Stokes equations [Eqs. (4) and (5)]. As a consequence the energy equation is also satisfied by the fluid fields $V$ and $p$, namely there results identically in $\Omega \times I$

$$V \cdot \left[ \frac{D}{Dt} V + \frac{1}{\rho_o} [\nabla (p - \phi) - f_R] - \nu \nabla^2 V \right] = 0. \quad (19)$$

Therefore, the third moment equation delivers the Fourier equation [Eq.(6)]. The same proof (for Proposition B) is straightforward if $f \neq f_M(x,t)$. This is reached again imposing the same constraint equation (16) on first velocity-moment of the distribution function $f$. Finally, the proof of Proposition C is achieved by imposing on $\rho_0(t)$ the constraint equation (17) and invoking assumptions 1) and 5). It follows that the thermodynamic entropy can be identified with the Shannon entropy, i.e., letting for all $t \in I$, $S_T(t) = S(f_M(x,t))$. As a final remark, we point out that by properly prescribing the initial and boundary conditions for the KDF, one can show that also the appropriate initial and boundary condition for the fluid fields can be satisfied (see related discussion in Ref. [5]). Furthermore, by construction, due to the constraint (16), $\hat{f}(x,t) = f(x,t)/\rho_o$ is a velocity-space probability density, i.e., there results identically in $\Omega \times I$, $\int \mathbb{R}^3 dV \hat{f}(x,t) = 1$.

III. LAGRANGIAN FORMULATION OF IKT

The results of the previous Section permit us to formulate in a straightforward way also the equivalent Lagrangian form of the inverse kinetic equation [6]. The Lagrangian formulation is achieved in two steps: a) by identifying a suitable dynamical system (here denoted as INSFE dynamical system), which determines uniquely the time-evolution of the kinetic probability density prescribed by IKT. Its flow defines a family of phase-space trajectories, here denoted as phase-space Lagrangian paths (LP’s); b) by proper parametrization in terms of the LP’s the KDF and the inverse kinetic equation, the explicit solution of the initial-value problem defined by the inverse kinetic equation (1) is determined. First, we notice that - in view of the THM - it is obvious that the LP’s must be identified with the phase-space trajectories of a classical dynamical system $x_o \rightarrow x(t) = T_{t,t_o} x_o$ generated by the vector field $X(x,t)$. Hence we shall assume that the initial-value problem (3), which is realized by the
equations
\[
\begin{align*}
\frac{d}{dt} \mathbf{r}(t) &= \mathbf{v}(t), \\
\frac{d}{dt} \mathbf{v}(t) &= \mathbf{F}(\mathbf{r}(t), t; f), \\
\mathbf{r}(t_0) &= \mathbf{r}_o, \\
\mathbf{v}(t_0) &= \mathbf{v}_o,
\end{align*}
\]
(20)
defines a suitably smooth diffeomorphism. Here, by construction:

- denoting by \( \mathbf{x}(t) = \chi(\mathbf{x}_o, t_0, t) \) the solution of the initial-value problem (20), \( \mathbf{x}_o = \chi(\mathbf{x}(t), t, t_0) \) is its inverse. Both are assumed to be suitably smooth functions of the relevant parameters;

- both \( \mathbf{x}(t) = \chi(\mathbf{x}_o, t_0, t) \) and \( \mathbf{x}_o = \chi(\mathbf{x}(t), t, t_0) \) identify admissible LP’s of the dynamical;

- \( \mathbf{r}(t) \) is the Lagrangian trajectory which belongs to the fluid domain \( \Omega \);

- \( \mathbf{v}(t) \) and \( \mathbf{F}(\mathbf{r}(t), t; f) \) are respectively the Lagrangian velocity and acceleration, both spanning the vector space \( \mathbb{R}^3 \). In particular, \( \mathbf{F}(\mathbf{r}(t), t; f) \), which is defined by Eqs. (12)-(15), and depends functionally on the kinetic probability density \( f(\mathbf{x}, t) \), is the Lagrangian acceleration which corresponds to an arbitrary kinetic probability density \( f(\mathbf{x}, t) \);

- \( f(\mathbf{x}, t) \) is a particular solution of the inverse kinetic equation which is subject to the assumptions imposed by the THM.

It follows that the Jacobian \( J(\mathbf{x}(t), t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right| \) of the map \( \mathbf{x}_o \rightarrow \mathbf{x}(t) \), which is generated by Eq. (20) for a generic distribution function \( f(\mathbf{x}, t) \) of this type, reads

\[
J(\mathbf{x}(t), t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right| = \exp \left\{ \int_{t_0}^{t} dt' H(\mathbf{x}(t'), t') \right\},
\]
(21)
where \( H(\mathbf{x}, t) \equiv \frac{3}{2} \left[ \frac{1}{p_1} A + \frac{1}{p_1} \nabla \cdot \mathbf{Q} - \frac{1}{p_1} \left[ \nabla \cdot \Pi \right] \cdot \mathbf{Q} \right] + \frac{1}{p_1} \mathbf{u} \cdot \nabla \cdot \Pi \). Instead, in the case in which there results identically \( f \equiv f_M(\mathbf{x}, t) \), the Jacobian reduces to

\[
J(\mathbf{x}(t), t) = \left| \frac{\partial \mathbf{x}(t)}{\partial \mathbf{x}_o} \right| = \frac{v_{3h,p}^3(t)}{v_{3h,p}^3(t_o)} \exp \left\{ -\frac{u^2(t_o)}{v_{3h,p}^3(t_o)} + \frac{u^2(t)}{v_{3h,p}^3(t)} \right\}.
\]
(22)
Second, it is immediate to prove that the kinetic equation in the Lagrangian representation can be written in the form

\[ J(x(t), t)f(x(t), t) = f(x_o, t_o) \equiv f_o(x_o) \]  

(23)

where \( f(x(t), t) \) is the Eulerian representation of the KDF and \( f_o(x_o) \) is a suitably smooth initial KDF. Eq. (23) manifestly implies also the time evolution of \( f(x(t), t) \) in terms of the initial distribution function:

\[ f(x(t), t) = \frac{1}{J(x(t), t)} f_o(\chi(x(t), t, t_o)). \]  

(24)

From the mathematical standpoint main consequences of the theory are that: 1) the Lagrangian formulation (of IKT) is uniquely specified by the proper definition of a suitable family of phase-space LP’s; 2) Eq. (24) uniquely specifies the time-evolution of the Eulerian KDF, \( f(x(t), t) \), which is represented in terms of the initial distribution function \( f_o(x_o) \) and the LP’s defined by the INSFE dynamical system; 3) the time-evolution of the fluid fields \( \{\rho = \rho_o > 0, V, p \geq 0, T > 0, S_T\} \) is uniquely specified via the KDF \( f(x(t), t) \); 4) Eq. (24) also provides the connection between Lagrangian and Eulerian viewpoints. In fact the Eulerian KDF, \( f(x, t) \), is simply obtained from Eq. (24) by letting \( x = x(t) \) in the same equation.

As a result, the Eulerian and Lagrangian formulations of IKT, and hence of the underlying moment (i.e., fluid) equations, are manifestly equivalent. From the physical viewpoint, it is worth mentioning the LP’s here defined can be interpreted as phase-space trajectories of the particles of the fluid, to be considered as a set of ”classical molecules”, i.e., point particles with prescribed mass, which interact only via the action of a suitable mean-field force kind. The ensemble motion of these particles has been defined in such a way that it uniquely determines the time evolution both of the kinetic distributions functions and of the relevant fluid fields which characterize the thermofluid.

IV. CONCLUDING REMARKS

In this Note an inverse kinetic theory has been developed for the INSFE problem based on a restricted phase-space representation, i.e., in which the phase space of the kinetic description is identified with the direct product \( \Gamma = \Omega \times V \) defined in such a way that \( \Omega \) coincides with the fluid. We have shown that equivalent IKT approaches can be formulated
both in the Eulerian and Lagrangian viewpoints. The result appears relevant for several reasons, in particular: 1) the Lagrangian dynamics here determined permits to advance in time self-consistently the fluid fields, i.e., in such a way that they satisfy identically the requires set of fluid equations. For isothermal fluids, this conclusion is consistent with the results indicated elsewhere by Tessarotto et al. [5]; 2) the Lagrangian dynamics, defined in the configuration space of the fluid (fluid domain), takes into account the specific form of the phase-space distribution function which advances in time the same fluid fields; 3) the theory permits an exact description of the motion of the "classical molecules" which follow the Lagrangian dynamics. Finally, in our view the formulation here presented is promising in turbulence theory. In fact, the connection between Eulerian and Lagrangian phase-space descriptions based on a IKT approach, here established on rigorous grounds, represents a potentially useful new development.

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Notice

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[1] L. F. Richardson, Proc. R. Soc. London A 110, 709 (1926).
[2] S.B. Pope, *Turbulent flows*, Cambridge University Press, p.463 (2000).
[3] M. Ellero and M. Tessarotto, Bull. Am Phys. Soc. 45 (9), 40 (2000).
[4] M. Tessarotto and M. Ellero, RGD24 (Italy, July 10-16, 2004), AIP Conf. Proc. 762, 108 (2005).
[5] M. Ellero and M. Tessarotto, Physica A 355, 233 (2005).
[6] M. Tessarotto and M. Ellero, Physica A 373, 142 (2007); arXiv: physics/0602140
[7] M. Tessarotto and M. Ellero, Proc. 25th RGD (International Symposium on Rarefied gas Dynamics, St. Petersburg, Russia, July 21-28, 2006), Ed. M.S. Ivanov and A.K. Rebrov (Novosibirsk Publ. House of the Siberian Branch of the Russian Academy of Sciences), p.1001; arXiv:physics/0611113 (2007).
[8] C. Cremaschini and M.Tessarotto, *Inverse kinetic theory for incompressible thermostfuids*, contributed paper at RGD26 (Kyoto, Japan, July 2008); arXiv:0806.4546 (2008).