Dilaton, winding modes and cosmological solutions

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We review some formal aspects of cosmological solutions in closed string theory with duality symmetric “matter” emphasizing the necessity to account for the dilaton dynamics for a proper incorporation of duality. We consider two models: when the matter action is the classical action of the fields corresponding to momentum and winding modes and when the matter action is represented by the quantum vacuum energy of the string compactified on a torus. Assuming that effective vacuum energy is positive one finds that in both cases the scale factor undergoes oscillations from maximal to minimal values with the amplitude of oscillations decreasing to zero or increasing to infinity depending on whether the effective coupling (dilaton field) decreases or increases with time. The contribution of the winding modes to the classical action prevents infinite expansion. Duality is “spontaneously broken” on a solution with generic initial conditions.

Submitted to Classical and Quantum Gravity
10/91

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1. Introduction

Target space duality is known to be one of fundamental properties of string theory [1]. Being in some sense a symmetry between large and small scales it may play an important role in string cosmology [2] if the contributions of the winding modes are included [2-4]. In order for the string theory cosmological equations to be invariant under the “non-static” generalisation of duality (or “σ model duality”) [5] it is necessary to account for the dependence on the dilaton field which transforms under the duality [5-10]. It is the combined metric-dilaton system of equations that is invariant under the duality if the “matter” action is duality symmetric (i.e. contains contributions of both momentum and winding modes) [11].

In this paper we shall discuss some formal aspects of “cosmological” (i.e. time dependent) solutions in the case of duality symmetric “matter” with one of the aims being to illustrate the effect of the winding mode contributions on the evolution of the scale factor. We shall not consider “realistic” cosmological scenarios as this was already done in [12]. The present paper is a review and an extension of certain aspects discussed in [12].

Our choice of the matter actions will be based on the assumption of “adiabaticity” understood in the following sense. Let the space-time be a product of a time line and an N dimensional torus with radii $a_i = e^\lambda_i$ (which we shall usually take to be equal). We shall assume that the matter action can be represented by the classical action of the modes of the string theory compactified on the torus with $a_i$ replaced by functions of time. The resulting action will be invariant under the duality transformation which inverts the radii, shifts the dilaton and interchanges momentum and winding modes. The corresponding equations will describe evolution of time dependent perturbations of the torus vacuum (from a different point of view these equations may be considered as a generalisation of the Kosterlitz–Thouless type renormalisation group equations [13]).

We shall also consider the case when the matter action is represented by the (zero temperature) vacuum energy of the gas of momentum and winding modes (i.e. by the
partition function of the string compactified on the torus) which is automatically invariant under the duality transformation of $a_i$ and the dilaton. Though the vacuum energy vanishes for the superstring a formal analysis of this case is useful since the resulting equations are similar to the system one finds in a more realistic finite temperature case [12].

We shall try not to specify the number $N$ of space dimensions (i.e. keep the value of the central charge $c \sim D - 26 (= N - 25)$ arbitrary) so that some of our results may apply, e.g., to the case of $D = 2$ string theory (cosmological solutions with non-zero $c$ in the matter-free case were discussed in [14-15]).

We shall find that both in the “classical” and “quantum” matter cases the scale factor (the common radius of the torus) oscillates between finite maximal and minimal values. If the effective dilaton coupling is decreasing with time the amplitude of oscillations is also decreasing and the asymptotic value of the radius is the Planck one ($\sqrt{\alpha'}$). Though the duality is spontaneously broken by initial conditions it is thus restored asymptotically. The role of the winding mode contributions is to prevent an infinite expansion in the “classical” matter case [12] (and, modulo the issue of interpretation, to prevent a contraction to zero in the “quantum” matter case).

We shall describe the expansion/contraction in terms of the original unrescaled (“$\sigma$ model”) metric. Though the effective gravitational constant is then time dependent, this is the metric strings directly interact with and hence the one measured by stringy “rods”. Similar point of view seems to be expressed in refs. 16–17 where the question of which metric should be used to measure the expansion in string cosmology was discussed. The use of “$\sigma$ model” metric is particularly natural in the duality symmetric case since it is the unrescaled metric that has a simple transformation law under the duality. We shall assume that the dilaton does not have a non-perturbative potential, i.e. is “massless”.

In Sect.2 we shall present the basic “cosmological” equations for the two choices of the matter sources. In the case of the classical matter we shall consider a “mechanical” interpretation of the resulting system of equations (with the central charge playing the role
of an “energy”) and also note a correspondence with the renormalisation group equations (for a related discussion of the RG equations see [18]).

In Sects.3 and 4 we shall study the behaviour of the solutions in the classical and quantum matter cases respectively. The qualitative analysis shows that the radius is oscillating with time between its extremal values with the amplitude of oscillations increasing to infinity or decreasing to zero depending on initial conditions for the dilaton.

Sect.5 contains some concluding remarks. We shall briefly discuss the “anisotropic” case in which there are “large” and “small” (internal) radii.

In Appendices A and B we shall explicitly solve the basic “classical” and “quantum” systems of equations in the asymptotic region of large radius and demonstrate the existence of a maximal radius of expansion.

2. Basic equations

1. We shall consider only the leading order terms in the low energy expansion of the tree level gravitational effective action of the closed bosonic string theory [19] (we shall absorb the gravitational coupling constant into $\phi$ and often set $\alpha'=1$)

$$S_0 = - \int d^D x \sqrt{-G} \ e^{-2\phi} \left[ c + R + 4(\partial \phi)^2 \right] ,$$

$$c = - \frac{2}{3\alpha'}(D - 26) .$$

The corresponding equations for the gravity plus matter action $S = S_0 + S_m$ are

$$R_{\mu\nu} + 2D_{\mu}D_{\nu} \phi - \frac{1}{2}G_{\mu\nu}L = \frac{e^{2\phi}}{\sqrt{-G}} \frac{\delta S_m}{\delta G_{\mu\nu}} ,$$

$$L \equiv c + R + 4D^2 \phi - 4(\partial \phi)^2 = - \frac{1}{2} \frac{e^{2\phi}}{\sqrt{-G}} \frac{\delta S_m}{\delta \phi} .$$

We shall consider the following cosmological background

$$ds^2 = - dt^2 + \sum_{i=1}^{N} a_i^2(t) dx_i^2 ,$$
\[ a_i = e^{\lambda_i(t)} , \quad \phi = \phi(t) , \quad N = D - 1 . \]

For a proper interpretation of the matter action we shall discuss it is necessary to assume that at least some of coordinates \( x_i \) are periodic.

It is useful to introduce the “shifted” (and rescaled by 2) dilaton field \( \phi \)

\[ \varphi \equiv 2\phi - \sum_{i=1}^{N} \lambda_i , \quad \sqrt{-G} e^{-2\phi} = e^{-\varphi} , \tag{2.5} \]

which is invariant under the duality transformation [6-10]

\[ \tilde{\lambda}_i = -\lambda_i , \quad \tilde{\phi} = \phi - \lambda_i , \quad \tilde{\varphi} = \varphi \tag{2.6} \]

(here \( i = 1, \ldots, N \); more general duality transformations are obtained by combining these basic ones). We are assuming that all the dimensions are compact. In general, the effective coupling is represented by the dilaton (2.5) where only the logarithm of the volume of the compact part of the space is subtracted out. Re-written in terms of \( \lambda_i \) and \( \varphi \) the action and equations (2.3),(2.2) take the form (we drop a constant factor of integral over \( x_i \))

\[ S = - \int dt \ e^{-\varphi} \sqrt{-G_{00}} \left[ c - G_{00} \sum_{i=1}^{N} \dot{\lambda}_i^2 + G_{00} \dot{\varphi}^2 \right] + S_m[G_{00}, \lambda_i, \varphi] , \tag{2.7} \]

\[ S_0 = - \int dt \ e^{-\varphi} \left[ c + \sum_{i=1}^{N} \dot{\lambda}_i^2 - \dot{\varphi}^2 \right] , \tag{2.8} \]

\[ c - \sum_{i=1}^{N} \dot{\lambda}_i^2 + \varphi^2 = -2e^{\varphi} \frac{\delta S_m}{\delta G_{00}} , \tag{2.9} \]

\[ \ddot{\lambda}_i - \dot{\phi} \dot{\lambda}_i = -\frac{1}{2}e^{\varphi} \frac{\delta S_m}{\delta \lambda_i} , \tag{2.10} \]

\[ \ddot{\varphi} - \sum_{i=1}^{N} \dot{\lambda}_i^2 = -e^{\varphi} \left( \frac{\delta S_m}{\delta G_{00}} - \frac{1}{2} \frac{\delta S_m}{\delta \varphi} \right) . \tag{2.11} \]

Similar set of equations was discussed in [10-12]. Eqs.(2.9)-(2.11) follow from the action (2.7) (after the variation \( G_{00} \) is set equal to \(-1\)). If one starts from the action (2.8) (where \( G_{00} = -1 \)) one should add eq.(2.9) as an additional “zero energy” constraint. If the original
matter action is generally covariant the time derivative of (2.9) vanishes identically once (2.10) and (2.11) are satisfied. The examples of matter actions $S_m[G_{00},\lambda_i,\varphi]$ we shall consider will be invariant under reparametrisations of time so that the identity relating the derivative of (2.9) to (2.10),(2.11) will be satisfied. Hence one of eqs.(2.10),(2.11) is redundant and can be dropped (if $\dot{\lambda} \neq 0$, $\dot{\varphi} \neq 0$). The meaning of eq.(2.9) is that the integration constant which appears after integrating once eqs.(2.10),(2.11) is not arbitrary but is proportional to $c$.

Eqs.(2.7)–(2.11) are invariant under the duality transformation (2.6) provided the matter action is duality symmetric. Given a solution of (2.9)–(2.11) we can then generate other solutions by applying duality transformations.

Let us note that in the duality non-invariant case when the matter action is given by the dilaton potential $V(\phi)$, i.e. (we shall assume the isotropic case when all radii are equal)

$$S_m = \int dt \sqrt{-G_{00}} V(\varphi + N\lambda) e^{N\lambda}$$

the system of equations (2.9)–(2.11) reduces to

$$c - N\dot{\lambda}^2 + \dot{\varphi}^2 = e^{\varphi}V,$$

$$\ddot{\lambda} - \dot{\varphi}\dot{\lambda} = -\frac{1}{2}e^{\varphi}(V + V'),$$

$$\ddot{\varphi} - N\dot{\lambda}^2 = \frac{1}{2}e^{\varphi}(V + V').$$

2. As was noted in the Introduction, one of our aims is to understand the effect of the winding modes on cosmological evolution. Let us first consider the case when the matter action is represented by the classical action of the fields corresponding to momentum and winding modes of string theory compactified on a torus. For simplicity we shall consider just two complex scalar fields representing the momentum and winding modes of the tachyon field ($\bar{x}_i$ are the “dual” coordinates; $\alpha' = 1$)

$$T(x,\bar{x}) = \psi \exp(i \sum_{i=1}^{N} m_i x_i) + \bar{\psi} \exp(i \sum_{i=1}^{N} \bar{m}_i \bar{x}_i) + c.c + ... ,$$

(2.12)
\[ S_m = \frac{1}{2} \int dt \, e^\varphi \sqrt{-G_{00}} \left[ G^{00} |\psi|^2 + \left( \sum_{i=1}^{N} m_i^2 e^{-2\varphi_i} - 4 \right) |\psi|^2 \right. \\
+ \left. G^{00} |\varphi|^2 + \left( \sum_{i=1}^{N} \tilde{m}_i^2 e^{2\varphi_i} - 4 \right) |\varphi|^2 + 2U(\psi, \tilde{\psi}) \right], \quad (2.13) \]

where the potential \( U \) starts with quartic terms (cubic interactions involve other modes as well). To describe the case when \( \psi \) and \( \tilde{\psi} \) represent higher level scalar modes the tachyonic mass term \(-4\) should be replaced by \( m_i^2 = 4(n-1) \), \( n = 0, 1, \ldots \). Variation of the total action (2.7), (2.13) over \( G_{00}, \lambda_i, \varphi, \psi, \tilde{\psi} \) gives the following system of equations (cf.(2.9)–(2.11) ; \( G_{00} = -1 \))

\[
\begin{align*}
    c - \sum_{i=1}^{N} \dot{\lambda}_i^2 + \varphi^2 &= \frac{1}{2} \left[ |\dot{\psi}|^2 + \left( \sum_{i=1}^{N} m_i^2 e^{-2\varphi_i} - 4 \right) |\psi|^2 \right. \\
    &\quad \left. + |\dot{\varphi}|^2 + \left( \sum_{i=1}^{N} \tilde{m}_i^2 e^{2\varphi_i} - 4 \right) |\varphi|^2 + 2U \right], \\
    \ddot{\lambda}_i - \dot{\varphi} \dot{\lambda}_i &= \frac{1}{2} \left( m_i^2 e^{-2\varphi_i} |\psi|^2 - \tilde{m}_i^2 e^{2\varphi_i} |\varphi|^2 \right), \quad (2.14) \\
    \ddot{\varphi} - \sum_{i=1}^{N} \dot{\lambda}_i^2 &= \frac{1}{2} \left( |\dot{\psi}|^2 + |\dot{\varphi}|^2 \right), \quad (2.15) \\
    \ddot{\psi} - \dot{\varphi} \dot{\psi} + \left( \sum_{i=1}^{N} m_i^2 e^{-2\varphi_i} - 4 \right) \psi + \frac{\partial U}{\partial \psi} &= 0, \quad (2.16) \\
    \ddot{\tilde{\psi}} - \dot{\varphi} \dot{\tilde{\psi}} + \left( \sum_{i=1}^{N} \tilde{m}_i^2 e^{2\varphi_i} - 4 \right) \tilde{\psi} + \frac{\partial U}{\partial \tilde{\psi}} &= 0. \quad (2.17) \\
\end{align*}
\]

Eqs.(2.14)–(2.18) are invariant under the duality transformation which is the combination of the transformation (2.6) of “moduli” and of the interchanging of the momentum and winding modes,

\[
\lambda_i \to -\lambda_i, \quad \varphi \to \varphi, \quad \psi \leftrightarrow \tilde{\psi}, \quad m_i \leftrightarrow \tilde{m}_i. \quad (2.19)
\]

The system (2.14)–(2.18) is consistent in the sense that the time derivative of the “zero energy” constraint (2.14) is proportional to eqs.(2.15)–(2.18). In fact, eqs.(2.14)–(2.18)
have the following “mechanical” interpretation. Eqs. (2.15), (2.16) (combined with (2.14)) and (2.17), (2.18) follow from the action (2.7), (2.13) (with \( G_{00} = -1 \))

\[
S = -\int dt (K - U) ,
\]

where the kinetic and potential terms are

\[
K = e^{-\varphi} \left( -\dot{\varphi}^2 + \sum_{i=1}^{N} \lambda_i^2 + \frac{1}{2} |\dot{\psi}|^2 + \frac{1}{2} |\ddot{\psi}|^2 \right) ,
\]

\[
U = e^{-\varphi} \left[ -c + \frac{1}{2} \left( \sum_{i=1}^{N} m_i^2 e^{-2\lambda_i} - 4 \right) |\psi|^2 + \frac{1}{2} \left( \sum_{i=1}^{N} m_i^2 e^{2\lambda_i} - 4 \right) |\ddot{\psi}|^2 + U \right] .
\]

The action (2.19)–(2.21) describes a mechanical system with the \( M = 1 + N + 4 \) dimensional configuration space with the Minkowski signature metric \( g_{AB} = e^{\varphi} \eta_{AB} \), \( \eta_{AB} = (-, +, ..., +) \) and the dilaton \( \varphi \) playing the role of a time coordinate. This space is curved. For example, its scalar curvature is \( R = \frac{1}{4}(M - 1)(M - 2)e^{-\varphi} \); it vanishes if \( N = 1 \) (i.e. \( D = 2 \)) and the matter fields are absent. Eq. (2.14) is then recognised as the zero energy condition \( K + U = 0 \) which should be added “by hands”.

It is interesting to note that eqs. (2.15), (2.17), (2.18) generalise the renormalisation group equations for the couplings which correspond to perturbations of the torus vacuum by marginal operators. Let us consider for simplicity the case of one dimensional torus \((N = 1)\). The action which generates the corresponding \( \beta \) - functions is given by [20]

\[
L = \int dx d\tilde{x} \left[ \frac{1}{2} G^{-1} \left( \frac{\partial T}{\partial x} \right)^2 + \frac{1}{2} \tilde{G}^{-1} \left( \frac{\partial T}{\partial \tilde{x}} \right)^2 - 2 T^2 + \frac{1}{4} T^3 + ... \right] ,
\]

\[
\tilde{G} = G^{-1} , \quad G = e^{2\lambda} ,
\]

\[
\beta^T = \dot{T} = \frac{1}{2} \frac{\delta L}{\delta T} , \quad \beta^G = \dot{G} = -\frac{1}{2} \frac{\partial L}{\partial G^{-1}} .
\]

The rotational invariance constraint \( \frac{\partial^2 T}{\partial x \partial \tilde{x}} = 0 \) implies

\[
T = \psi(x) + \tilde{\psi}(\tilde{x}) = \sum_{m=-\infty}^{\infty} \psi_m e^{imx} + \sum_{\tilde{m}=-\infty}^{\infty} \tilde{\psi}_{\tilde{m}} e^{i\tilde{m}\tilde{x}} ,
\]
so that the action and RG equations take the form [13,20]

\[ L = \frac{1}{2} \sum_{m=1}^{\infty} (G^{-1} m^2 - 4) |\psi_m|^2 + \frac{1}{2} \sum_{\tilde{m}=1}^{\infty} (G\tilde{m}^2 - 4) |\tilde{\psi}_{\tilde{m}}|^2 + O(\psi^3, \tilde{\psi}^3), \quad (2.25) \]

\[ \dot{\psi}_m = \frac{1}{2} \frac{\partial L}{\partial \psi_m} = \frac{1}{4} (G^{-1} m^2 - 4) \psi_m + O(\psi^2), \quad (2.26) \]

\[ \dot{\tilde{\psi}}_{\tilde{m}} = \frac{1}{2} \frac{\partial L}{\partial \tilde{\psi}_{\tilde{m}}} = \frac{1}{4} (G\tilde{m}^2 - 4) \tilde{\psi}_{\tilde{m}} + O(\tilde{\psi}^2), \quad (2.27) \]

\[ \dot{G} = -\frac{1}{2} \frac{\partial L}{\partial G^{-1}} = -\frac{1}{4} \left( \sum_{m=1}^{\infty} m^2 |\psi_m|^2 - G^2 \sum_{\tilde{m}=1}^{\infty} \tilde{m}^2 |\tilde{\psi}_{\tilde{m}}|^2 \right). \quad (2.28) \]

Here \( \psi_m, \tilde{\psi}_{\tilde{m}}, \ G = e^{2\lambda} \) are “running” couplings, and the derivatives are taken with respect to the RG parameter \( \tau \). The correspondence between eqs.(2.26)–(2.28) and (2.14)–(2.18) is established as follows. If we drop the second derivative terms in (2.15), (2.17), (2.18), set \( N = 1 \), \( \varphi=4 \) and keep just two modes (\( m \) and \( \tilde{m} \)) then eq. (2.15) becomes identical to (2.28) while eqs. (2.17), (2.18) reduce to (2.26), (2.27). As discussed in [18] this prescription corresponds to taking a semiclassical limit if the cosmological equations (2.14)–(2.18) are interpreted as RG equations in the presence of quantized 2d gravity. To satisfy (2.14) one should have \( c + \varphi^2 \approx 0 \). Given the value of \( c \) in (2.1) (i.e. \( c = 16 \) for \( D = 2, \ \alpha' = 1 \)) this implies that the RG parameter \( \tau \) should be identified with with the euclidean time \( \tau = it \).

We shall return to the analysis of eqs.(2.14)–(2.18) in Sect. 3.

3. Let us now consider the case when the matter action is represented by the quantum vacuum energy of the string modes, i.e. by the partition function of the string compactified on a torus. In general, the partition function has the following duality invariant structure

\[ Z(\lambda, \varphi) = \sum_{n=1}^{\infty} e^{2(n-1)\varphi} \ Z_n(\lambda) = \sum_{n=1}^{\infty} e^{(n-1)\varphi} \ f_n(\lambda), \quad (2.29) \]

\[ Z(\lambda_i, \varphi) = Z(-\lambda_i, \varphi), \ \ f_n(\lambda_i, \varphi) = f_n(-\lambda_i, \varphi). \quad (2.30) \]

In the large radii limit

\[ Z_n(\lambda_i \to \infty) = d_n V, \ \ V \equiv \exp \sum_{i=1}^{N} \lambda_i, \quad (2.31) \]
i.e. \( Z \) is proportional to the volume factor. Comparing with field theory and assuming that string ultraviolet cutoff corresponds to a proper time cutoff one may say that it is the momentum mode contribution that controls the large radius limit. In that sense the contributions of the winding modes (that make \( Z \) symmetric under (2.6) ) control the small radii limit

\[
f_n(\lambda_i \to -\infty) = d_n \exp(-n \sum_{i=1}^{N} \lambda_i), \quad f_1(\lambda_i \to -\infty) = d_1 V^{-1}.
\]

(2.32)

This interpretation is, in fact, ambiguous since from the cutoff field theory point of view both momentum and winding modes contribute similar terms (e.g. with positive and negative powers of radii) to the partition function. The temperature dependence can be included by compactifying the euclidean time [21] (so that the inverse temperature \( \beta \) will couple to \( \sqrt{-G_{00}} \) ) and does not change the duality invariance property of \( Z \). Assuming adiabaticity of evolution we may replace \( \lambda_i, \varphi \) and \( \beta \) by functions of time and take the matter action in the form (see also [22])

\[
S_m = \int dt \sqrt{-G_{00}} F(\lambda, \varphi, \beta \sqrt{-G_{00}}), \quad F = V F, \quad Z = -\beta F.
\]

(2.33)

Let us consider, for example, the one-loop contribution when \( F \) does not depend on \( \varphi \) (this is a good approximation if the effective coupling \( e^{\varphi} \) is small). The energy-momentum tensor can be represented in the form

\[
t_{\mu\nu} = \frac{2}{\sqrt{-G}} \frac{\delta S_m}{G^{\mu\nu}} = \text{diag}(\rho, e^{2\lambda_1} p_1, ..., e^{2\lambda_N} p_N),
\]

(2.34)

\[
\rho = -\frac{2}{V} \delta S_m \frac{\delta G_{00}}{\delta G_{00}} = E, \quad E = F + \beta \frac{\partial F}{\partial \beta},
\]

(2.35)

\[
p_i = -\frac{1}{V} \delta S_m \frac{\delta \lambda_i}{\delta \lambda_i} = P_i, \quad P_i = -\frac{\partial F}{\partial \lambda_i}
\]

(2.36)

The conservation of the energy (following from the invariance under reparametrisations of time ) implies (\( \dot{E} \equiv dE/dt \))

\[
\dot{\rho} + \sum_{i=1}^{N} \dot{\lambda}_i (\rho + p_i) = 0, \quad \dot{E} + \sum_{i=1}^{N} \dot{\lambda}_i P_i = 0.
\]

(2.37)
Since $F = F(\lambda(t), \beta(t))$ eq.(2.37) is equivalent to the conservation of the entropy $S = \beta^2 \partial F/\partial \beta$, $\dot{E} + \sum_{i=1}^{N} \dot{\lambda}_i P_i = \dot{S}/\beta$.

The resulting cosmological equations (2.9) – (2.11) are [12]

$$c - \sum_{i=1}^{N} \ddot{\lambda}_i^2 + \dot{\varphi}^2 = e^{\varphi} E$$ \hspace{1cm} (2.38)

$$\ddot{\lambda}_i - \dot{\varphi} \dot{\lambda}_i = \frac{1}{2} e^{\varphi} P_i$$ \hspace{1cm} (2.39)

$$\ddot{\varphi}_i - \sum_{i=1}^{N} \ddot{\lambda}_i^2 = \frac{1}{2} e^{\varphi} E$$ \hspace{1cm} (2.40)

The time derivative of (2.38) vanishes on eqs.(2.39), (2.40) if eq.(2.37) is satisfied. Eqs.(2.37)–(2.40) are invariant under the duality transformation (2.6) if $F$ is duality symmetric (note that under the duality $E \rightarrow E$, $P_i \rightarrow -P_i$).

Solving the adiabaticity condition one can in principle express the temperature in terms of $\lambda_i$. Then eqs. (2.38)–(2.40) will take the same form as in the zero temperature case with $E$ and $P_i$ replaced by $E(\lambda)$ and $-\partial E/\partial \lambda_i$.

Re-written in terms of the original dilaton field $\phi$ (see (2.5)) eqs.(2.38)–(2.40) take the form

$$c - \sum_{i=1}^{N} \ddot{\lambda}_i^2 + (2 \dot{\phi} - \sum_{i=1}^{N} \dot{\lambda}_i)^2 = e^{2\phi} \rho$$ \hspace{1cm} (2.41)

$$\ddot{\lambda}_j - (2 \dot{\phi} - \sum_{i=1}^{N} \dot{\lambda}_i) \dot{\lambda}_j = \frac{1}{2} e^{2\phi} p_j$$ \hspace{1cm} (2.42)

$$2 \ddot{\phi} - \sum_{i=1}^{N} \ddot{\lambda}_i - \sum_{i=1}^{N} \ddot{\lambda}_i^2 = \frac{1}{2} e^{2\phi} \rho$$ \hspace{1cm} (2.43)

It is easy to see that if $c = 0$ a solution with $\phi = \text{const}$ may exist only if the energy-momentum tensor (2.34) is traceless, i.e.

$$\sum_{i=1}^{N} p_i = \rho$$ \hspace{1cm} (2.44)

The latter relation (which is not duality symmetric) is not in general satisfied by the free energy which actually appears in string theory. This point was already emphasized in [12] (cf. [3,4]).
A simple way to see that $\phi = \text{const}$ is not in general a solution of the cosmological equations if the one-loop vacuum energy is used as a source is to start with the action rewritten in terms of the “rescaled” metric (cf.\((2.1),(2.3)\))

$$G'_{\mu \nu} = e^{-4\phi/(D-2)} G_{\mu \nu},$$

(2.45)

$$S = -\int d^Dx \sqrt{-G} e^{-2\phi} \left[ c + R + 4(\partial\phi)^2 \right] + \int d^Dx \sqrt{-G} F =$$

$$-\int d^Dx \sqrt{-G'} \left[ R' - \frac{4}{D-2} (\partial\phi)^2 + c e^{4\phi/(D-2)} \right] + \int d^Dx \sqrt{-G'} e^{2D\phi/(D-2)} F.$$  

(2.46)

The equation for $\phi$ which follows from (5.10) is not solved by $\phi = \text{const}$ if $c = 0$ and $F \neq 0$.

4. Before turning to the analysis of eqs. (2.9)-(2.11) with duality symmetric matter let us recall their solution in the absence of matter [15]. If $S_m = 0$ eq.(2.10) implies

$$\lambda_i = k_i e^\varphi, \quad k_i = \text{const}.$$  

Substituting this into eq.(2.9) we finally get (assuming $D > 26$)

$$\varphi = \varphi_0 - \ln \sinh 2bt, \quad \lambda_i = \lambda_{i0} + q_i \ln \tanh bt,$$

(2.47)

$$\sum_{i=1}^{N} q_i^2 = 1, \quad \alpha'b^2 = \frac{1}{6}(D-26), \quad q_i = k_i b^{-1} e^{\varphi_0}.$$  

The solution for $D < 26$ is obtained by the substitution $b \rightarrow ib$; in particular, for $D = 2$

$$\varphi = \varphi_0 - \ln \sin 2bt, \quad \lambda = \lambda_0 \pm \ln \tan bt, \quad \alpha'b^2 = 4.$$  

(2.48)

The solution for the zero central charge ($D = 26$) is given by

$$\varphi = \varphi_0 - \ln t, \quad \lambda_i = \lambda_{i0} + q_i \ln t, \quad \sum_{i=1}^{N} q_i^2 = 1.$$  

(2.49)

This is a Kasner-type solution with some radii infinitely expanding and others infinitely contracting. Such behaviour will be changed by the presence of matter. Note that the duality transformation (2.6) corresponds to $q_i \rightarrow -q_i$, i.e. it relates contracting solutions to expanding ones.
3. Solutions with classical matter

1. In this section we shall study the solutions of the system of equations (2.14)–(2.18). We shall make a number of simplifying assumptions. In what follows we shall consider the isotropic case of all time dependent radii being equal $\lambda_i = \lambda$. This assumption is consistent with eq.(2.15) if the modes $\psi$ and $\tilde{\psi}$ are such that $m_i = m$, $\tilde{m}_i = \tilde{m}$, $i = 1, ..., N$. The corresponding metric (2.4)

$$ds^2 = -dt^2 + e^{2\lambda(t)} \sum_{i=1}^{N} dx_i^2,$$

(3.1)

can be also interpreted as a spacially flat isotropic cosmological metric. Then

$$\phi = 2\phi - N\lambda \ , \ V = e^{N\lambda} \ .$$

(3.2)

To be able to change the value of the central charge we shall assume that there may be a number $K$ of “static” spacial dimensions, i.e.

$$D = 1 + N + K \ , \ c = -\frac{2}{3}(N + K - 25) \ .$$

In general, the duality transformation (2.6)

$$\tilde{\lambda} = -\lambda \ , \ \tilde{\phi} = \phi$$

(3.3)

corresponds to inverting the direction of evolution of the scale factor.

Eqs. (2.14)–(2.18) reduce to

$$c - N\dot{\lambda}^2 + \dot{\phi}^2 = \frac{1}{2} \left[ |\dot{\psi}|^2 + (Nm^2e^{-2\lambda} - 4)|\psi|^2ight]$$

$$+ |\tilde{\dot{\psi}}|^2 + (N\tilde{m}^2e^{2\lambda} - 4)|\tilde{\psi}|^2 + 2U \right],$$

(3.4)

$$\ddot{\lambda} - \dot{\phi}\dot{\lambda} = \frac{1}{2} \left( m^2e^{-2\lambda}|\psi|^2 - \tilde{m}^2e^{2\lambda}|\tilde{\psi}|^2 \right) ,$$

(3.5)

$$\ddot{\phi} - N\dot{\lambda}^2 = \frac{1}{2} \left( |\dot{\psi}|^2 + |\tilde{\dot{\psi}}|^2 \right) ,$$

(3.6)
\[ \ddot{\psi} - \dot{\phi} \dot{\psi} + (Nm^2 e^{-2\lambda} - 4)\psi + \frac{\partial U}{\partial \psi^*} = 0 , \]
\[ \ddot{\tilde{\psi}} - \dot{\phi} \dot{\tilde{\psi}} + (N\tilde{m}^2 e^{2\lambda} - 4)\tilde{\psi} + \frac{\partial U}{\partial \tilde{\psi}^*} = 0 . \] (3.7)

We shall first ignore the dynamics of the matter fields \( \psi \) and \( \tilde{\psi} \) and consider them as "static" sources. Such an assumption may be justified if the full interaction potential (including \( U \) as well as interactions of \( \psi \) and \( \tilde{\psi} \) with other modes which we ignored) has extrema (see, however, the discussion at the end of this section). Then eqs. (3.4)--(3.6) take the form

\[ -N\dot{\lambda}^2 + \dot{\phi}^2 = 2NW(\lambda) , \quad W \equiv \frac{1}{2}\mu e^{-2\lambda} + \frac{1}{2}\tilde{\mu} e^{2\lambda} - C/2N , \] (3.8)

\[ \ddot{\lambda} - \dot{\phi} \dot{\lambda} = \mu e^{-2\lambda} - \tilde{\mu} e^{2\lambda} = -W'(\lambda) , \] (3.9)

\[ \ddot{\phi} - N\dot{\lambda}^2 = 0 , \] (3.10)

where

\[ \mu = \frac{1}{2}m^2|\psi|^2 , \quad \tilde{\mu} = \frac{1}{2}\tilde{m}^2|\tilde{\psi}|^2 , \quad C \equiv c + 2|\psi|^2 + 2|\tilde{\psi}|^2 - U(\psi, \tilde{\psi}) . \] (3.11)

Note that the derivative of eq.(3.8) is still vanishing if (3.10) and (3.11) are satisfied. Let us discuss some general properties of eqs. (3.8)--(3.11) assuming that \( c \), \( m \) and \( \tilde{m} \) are such that \( W \) is strictly positive. Then eqs.(3.8), (3.10) imply that \( \dot{\phi} \neq 0 \) and is always growing. Hence if we start with \( \dot{\phi} < 0 \) (i.e. the effective coupling \( e^{\phi} \) decreasing with time) then \( \dot{\phi} \) will increase towards positive values but will never reach zero. Eq.(3.9) then describes a motion of a particle in a potential \( W > 0 \) (which grows at large positive and negative \( \lambda \)) with a damping term proportional to \( -\dot{\phi} \) [12]. Introducing the energy of the particle \( \mathcal{E} \) we find from (3.8)--(3.10)

\[ \dot{\mathcal{E}} = \dot{\phi} \dot{\lambda}^2 < 0 , \quad \dot{\phi}^2 = 2N\mathcal{E} > 0 , \] (3.12)

\[ \mathcal{E} \equiv \frac{1}{2}\dot{\lambda}^2 + W(\lambda) , \]
i.e. $E$ will decrease with time. The particle coordinate $\lambda(t)$ will thus be oscillating between its maximal positive and negative values with the amplitude of oscillations decreasing to zero (i.e. the particle trajectory be reflecting from the walls of the potential moving down towards its bottom). If at the initial moment the scale factor $a = e^{\lambda}$ is increasing ($\dot{\lambda} > 0$) the expansion will continue until a maximal value of $a$ is reached. After the turning point the contraction will start until a minimal value of $a$ is reached, etc.

The asymptotic values of $\lambda$ and $\varphi$ correspond to the minimum of the potential $W$

$$
\lambda_* = \frac{1}{4} \ln \left( \frac{\mu}{\tilde{\mu}} \right), \quad \varphi_* = \varphi_0 - Qt ,
$$

$$
Q \equiv [2NW(\lambda_*)]^{1/2} = [2N(\mu\tilde{\mu})^{1/2} - C]^{1/2} , \quad W(\lambda_*) = 0 . \tag{3.13}
$$

If $m = \tilde{m} , \; \psi = \tilde{\psi}$ then $\lambda_0 = 0$ and duality is asymptotically restored at $t \to \infty$. To find the behaviour of the solution at large $t$ we can approximate $W$ near $\lambda = \lambda_*$ by the oscillator potential

$$
W = A + \frac{1}{2} B \xi^2 , \; \lambda = \lambda_* + \xi(t) , \; A = Q^2/2N , \; B = (Q^2 + C)/N = 2(\mu\tilde{\mu})^{1/2} ,
$$

$$
\ddot{\xi} + Q\dot{\xi} + B\xi = 0 , \; \xi \sim e^{\omega t} , \; \omega = -\frac{1}{2} \left( Q \pm \sqrt{Q^2 - 4B} \right)^{1/2} . \tag{3.14}
$$

Depending on the sign of $Q^2 - 4B = 2(N - 4)(\mu\tilde{\mu})^{1/2} - C$ we get either exponential falloff or oscillations with exponentially decreasing amplitude.

The solution in the case $\dot{\varphi} > 0$ can be obtained by time reversal (see (3.8)–(3.10)). We conclude that $e^{\varphi}$ will be increasing and $\lambda$ will oscillate with the amplitude of oscillations exponentially growing with time.

One can also analyse the solutions of (3.8)–(3.10) by studying the behaviour of the trajectories on the $(\lambda , \varphi)$ plane. We have

$$
\dot{f} = N\dot{\lambda}^2 , \quad \dot{\lambda}^2 = \frac{1}{N} ( f^2 - 2NW ) , \quad f \equiv \varphi , \tag{3.15}
$$

$$
f' \equiv \frac{df}{d\lambda} = \pm \sqrt{(f^2 - 2NW)/N} . \tag{3.16}
$$
(near the tuning point $\dot{\lambda} = 0$ eq.(3.10) should be used as well). The trajectories on the $(\lambda, f)$ plane will lie above and below the limiting curves $f = \pm \sqrt{2NW(\lambda)}$. Consider, for example the case when $f > 0, \lambda > 0, \dot{\lambda} > 0$. It is easy to see that the trajectory will move up ($\dot{f} > 0$) towards the right branch of the limiting curve and will hit it in a finite time. We shall present the explicit solution of eq.(3.15) in the region of large $\lambda$ (and for $N = 1$) in Appendix A where we shall demonstrate that the growing $\lambda(t)$ always reaches a maximal value. Since at the turning point $\dot{f} = 0, \dot{\lambda} = 0$ but $\ddot{\lambda} < 0$ the trajectory will be reflected back: $\lambda$ will start decreasing, the trajectory will go up until it will finally hit the left branch of the limiting curve ($a$ will reach its minimal value). If we start with $f < 0$ the trajectory will lie below the lower limiting curve. $f$ will increase along the trajectory so the amplitude of oscillations of $\lambda$ will be decreasing asymptotically to zero. Again, the two cases ($f > 0$ and $f < 0$) are related by time reversal.

2. The above qualitative analysis of the system (3.8)–(3.10) can be repeated for the general system (3.4)–(3.7) with time dependent matter fields. Since according to (3.6) $\ddot{\phi}$ is again non-negative $\dot{\phi}$ will remain negative if we start with decreasing $\phi$. Then the damping terms in the equations for the matter fields (3.7) will cause $\psi, \tilde{\psi}$ to decrease with time towards their “background” (presumably constant) values. The “potential” $W$ in (3.8) will now include the positive kinetic terms for $\psi$ and $\tilde{\psi}$ and hence the energy $\mathcal{E}$ in (3.12) will still satisfy $\dot{\mathcal{E}} < 0$. Since the derivatives of the matter fields will decrease with time the qualitative behaviour of the solution for $\lambda$ (oscillations with decreasing amplitude) will remain the same.

To summarise, if we start with the initial condition $\dot{\phi} < 0$, i.e. the effective coupling $e^{\phi}$ decreasing with time (this is a natural assumption in order to justify the neglect of quantum corrections in the matter action) then the scale $a$ oscillates between the maximal

---

1 If the structure of matter interactions is such that there is no non-vanishing backgrounds for $\psi$ and $\tilde{\psi}$ then the matter field contributions to eqs.(3.4) – (3.7) will be decreasing to zero. Hence the late time solution will be approaching the “free” solution (2.47) and the period of oscillations will become infinite.
and minimal values. If the matter action is “self-dual”, i.e. $\mu = \tilde{\mu}$ then $a_{\text{max}}$ and $a_{\text{min}}$ asymptotically approach the Planck scale $\sqrt{\alpha'} = 1$, i.e. $\lambda = 0$. If one starts with a large value of $a$ it will take, of course, a large number of oscillations before $a_{\text{max}}$ will become of Planck order.

As it is clear from the structure of the potential $W$ the role of the winding mode contribution ($\sim \exp 2\lambda$) to the “classical” system of equations (3.4)–(3.6) is to prevent infinite expansion. The winding mode term in eq. (3.4) is thus opposing the expansion [12]. Without it we would have either the expansion to $\lambda = \infty$ or first contraction to some minimal radius and then infinite expansion. The role of the momentum mode contribution is “dual”: it prevents contraction to $\lambda = -\infty$, i.e. provides the existence of a non-zero minimal radius of contraction.

One may question the validity of the assumption used in deriving (3.8)–(3.10) that the matter fields in (3.4)–(3.7) may be taken to be constant. It may be more appropriate to set constant the rescaled fields $\psi' = \exp(-\varphi/2)\psi$, $\tilde{\psi}' = \exp(-\varphi/2)\tilde{\psi}$ for which there is no $\dot{\varphi}$ term in the analog of eq.(3.7). Then eqs.(3.8)–(3.9) will have the same form but with extra factor of $\exp \varphi$ appearing in front of $W$ (eq.(3.10) will be modified accordingly to preserve the consistency of the system (3.8)–(3.10); being redundant, it may be ignored). Similar system appears in the one-loop “quantum” case discussed in the next section and the behaviour of the solution is similar. Let us note that the presence of the $\exp \varphi$ factor in the r.h.s. of equations corresponding to the classical string modes is suggested also by the form of the combined action (2.7) where $S_m$ is given by the the classical string ($\sigma$ model) action. The absence of the dilaton factor in front of the classical string action implies its presence in the r.h.s. of the corresponding cosmological equations [10]. Hence a condensate of classical string modes is probably described by the rescaled classical fields. The presence of the dilaton factor in the r.h.s. of equations describing the cosmological evolution in the case when the “matter” is represented by classical string modes was also assumed in [12] where the conclusion that the winding modes oppose the expansion was already made.
4. Solutions with quantum matter

1. In this section we shall study the cosmological evolution in the case when the matter action in (2.2)–(2.3) is represented by the (zero temperature) one loop partition function of the string compactified on the torus. The general form of the solution will be the same as in the case of the classical matter discussed in Sect. 3: the scale factor will oscillate between its maximal and minimal values with the amplitude of oscillations decreasing to zero or increasing to infinity depending on whether the dilaton coupling is decreasing or increasing with time.

As in Sect. 3 we shall consider the isotropic case,

\[ \lambda_i = \lambda , \quad F(\lambda_1, ..., \lambda_N) = F(\lambda) , \quad P_i = -\frac{\partial F}{\partial \lambda_i} = -\frac{1}{N} \frac{\partial F}{\partial \lambda} \equiv P . \]  

(4.1)

Then eqs.(2.38)–(2.40), (2.37) take the form

\[ c - N \dot{\lambda}^2 + \dot{\phi}^2 = e^\phi E , \]  

(4.2)

\[ \ddot{\lambda} - \dot{\phi} \dot{\lambda} = \frac{1}{2} e^\phi P , \]  

(4.3)

\[ \ddot{\phi} - N \dot{\lambda}^2 = \frac{1}{2} e^\phi E , \]  

(4.4)

\[ \dot{E} + N \dot{\lambda} P = 0 . \]  

(4.5)

2. For completeness let us first discuss the case of a non-zero temperature \( F = F(\lambda(t), \beta(t)) \) and consider (4.2)–(4.5) as a “phenomenological” system of equations which should be supplemented by some equation of state relating \( E \) and \( P \) (for more general analysis of the finite temperature case see [12]). In the case of the usual radiation-type condition

\[ p = \gamma \rho , \quad P = \gamma E , \]  

(4.6)

we find from (4.5)

\[ E = E_0 \exp(-N \gamma \lambda) , \quad P = \gamma E_0 \exp(-N \gamma \lambda) . \]  

(4.7)
Then a particular solution of (4.2)–(4.4) with \( c = 0 \) is given by ( cf. (2.49) ; similar solutions were discussed in [22,11] )

\[
\varphi = \varphi_0 + s \ln t \quad , \quad \lambda = \lambda_0 + q \ln t \, , \quad (4.8)
\]

\[
s = -2/(1 + \gamma^2 N) \, , \quad q = 2\gamma/(1 + \gamma^2 N) \, , \quad (4.9)
\]

\[
E_0 = 4(1 - \gamma^2 N)(1 + \gamma^2 N)^{-2} \exp(N\gamma\lambda_0 - \varphi_0) \, , \quad s - \gamma Nq = -2 \, .
\]

One should assume \( \gamma^2 N < 1 \) to have \( E_0 > 0 \). If \( \gamma = 1/N \) (so that (2.44) is satisfied)

\[
s = -2N/(1 + N) \, , \quad q = 2/(1 + N) \, , \quad \phi = \frac{1}{2}(\varphi + N\lambda) = \text{const} \, . \quad (4.10)
\]

This solution describes a power – law expansion with the dilaton coupling decreasing with time.

The above equation of state (4.6) explicitly breaks the duality invariance of (4.2)–(4.5). As we shall see, the behaviour of the solution is very different in the case when the duality invariance is preserved. The simplest duality invariant equation of state is \( P = 0 \), i.e. \( E = E_0 = \text{const} \). This seems to be a natural choice for an equation of state near the Hagedorn temperature [2]. In general, if the temperature is approximately constant and \( F \) is duality symmetric such a relation will always be true near the Planck scale \( \lambda = 0 \), i.e. \( E(\lambda) = E(-\lambda) \approx \text{const} \, , \quad P = -F'/N \approx 0 \). In that case we find from (4.2)–(4.4)

\[
\dot{\lambda} = Ae^\varphi \, , \quad \dot{\varphi}^2 = N A^2 e^{2\varphi} + E_0 e^\varphi - c \, . \quad (4.11)
\]

For \( c = 0 \) [12]

\[
\varphi = \ln \left| \frac{4E_0}{E_0^2 t^2 - 4NA^2} \right| \, , \quad \lambda = \lambda_0 + \frac{1}{\sqrt{N}} \ln \left| \frac{t - 2\sqrt{NA}/E_0}{t + 2\sqrt{NA}/E_0} \right| \, , \quad (4.12)
\]

i.e. we conclude that \( e^\lambda \) is asymptotically approaching its maximal or minimal values in the case of expansion ( \( A > 0 \) ) or contraction ( \( A < 0 \) ) while the dilaton is decreasing with time. The duality transformation corresponds to \( A \to -A \) or, equivalently, \( t \to -t \). Note that \( \varphi = \text{const} \) is not a solution of (4.2)–(4.4) (and \( \phi \neq 0 \) for \( \gamma = 0 \) , cf.(4.10) ).
3. Let us now turn to the analysis of the solutions of eqs. (4.2)–(4.5) for the duality symmetric zero temperature case, i.e.

\[ E = F(\lambda) = F(-\lambda) , \quad P = -F'/N , \quad F(\lambda \to \pm \infty) \to d_1 e^{\pm N \lambda}. \quad (4.13) \]

Let us first formally assume that \( E \) is positive (we shall consider the case of negative \( E \) later on). Though the Casimir energy in field theory is usually negative, the bosonic string vacuum energy is in any case not well defined (divergent due to tachyon) but the aim of our analysis is only to illustrate some general properties of the system (4.2)–(4.4).

Assuming (4.13) one finds that eq. (4.5) is satisfied automatically and (4.2)–(4.4) take the form

\[ -N \dot{\lambda}^2 + \dot{\varphi}^2 = 2NW , \quad W(\lambda, \varphi) = [e^\varphi F(\lambda) - c]/2N , \quad (4.14) \]

\[ \ddot{\lambda} - \dot{\varphi} \dot{\lambda} = -\partial W/\partial \lambda , \quad (4.15) \]

\[ \ddot{\varphi} - N\dot{\lambda}^2 = NW - \frac{1}{2}c . \quad (4.16) \]

Since \( F(\lambda) \) (and hence \( W(\lambda) \)) has the minimum at \( \lambda = 0 \) there exists the “fixed point” self-dual solution

\[ \lambda = 0 , \quad \dot{\varphi}^2 = F_0 e^\varphi - c , \quad F_0 = F(0) > 0 , \quad (4.17) \]

i.e. if \( c > 0 \) and \( \dot{\varphi} < 0 \)

\[ \varphi = \varphi_0 - 2 \ln |\sin \frac{1}{2} \sqrt{c} t| , \quad e^{\varphi_0} = c/F_0 . \quad (4.18) \]

Solutions with generic initial conditions will “spontaneously break” duality.

A qualitative analysis of a general solution of (4.14)–(4.16) is similar to that of the “classical system” (3.8)–(3.10). Let us consider the case of \( \dot{\varphi} < 0 \). We note that since \( \ddot{\varphi} \) is positive \( \dot{\varphi} \) will grow but will remain negative if it was negative at the initial moment, i.e. \( \varphi \) will be monotonically decreasing. Treating \( \varphi(t) \) as a given function we can then interpret eq. (4.15) as the equation of motion for a particle with the damping term \( \sim -\dot{\varphi} \).
and the time dependent potential \( W \). The “energy” of the particle is decreasing with time (cf. (3.12))

\[
E \equiv \frac{1}{2} \dot{\lambda}^2 + W, \quad \dot{E} = \dot{\varphi}(\dot{\lambda}^2 + e^{\varphi}F/2N) < 0, \quad \varphi^2 = 2N\mathcal{E} > 0.
\]

(4.19)

If we draw the graph of the potential \( W(\lambda, t) \) (which is exponentially rising at \( \lambda \to \pm \infty \) and has a minimum at \( \lambda = 0 \) ) at several consequent moments of time it will be moving down towards the \( \lambda \) axis. We shall assume that \( c \) is such that \( W > 0 \). Since \( \mathcal{E} \) is decreasing the trajectory of the particle will also be moving down reflecting from the walls of the potential. As a result, the radius \( e^\lambda \) will be oscillating from maximal to minimal values with the amplitude of oscillations asymptotically decreasing to zero, i.e. the solution will be approaching the self-dual point \( \lambda(t \to \infty) = 0 \). The solution in the region near \( \lambda = 0 \) where \( F \approx \text{const} \) was already analyzed above (see (4.11), (4.12)) : \( \lambda \) is asymptotically approaching its constant value (zero) while \( e^\varphi \) is decreasing to zero.

If we start with \( \dot{\varphi} > 0 \) the direction of evolution is reversed: the amplitude of oscillations of \( \lambda \) grows with time, i.e. the maximal and minimal values of \( \lambda \) are asymptotically approaching \( \pm \infty \).

It is possible to relate the analysis of the system (4.14)–(4.16) to that of the system (3.15)–(3.16) which appeared in the “classical” case. Introducing the new time coordinate \( d\tau = dt e^{\varphi/2} \) and assuming \( c = 0 \) we find from (4.14)–(4.16)

\[
N\lambda'^2 = f^2 - F(\lambda), \quad 2f' = f^2 - F(\lambda), \quad f \equiv \varphi' = \frac{d\varphi}{d\tau},
\]

(4.20)

\[
\frac{df}{d\lambda} = \frac{1}{2} \sqrt{N[f^2 - F(\lambda)]}^{-1/2}.
\]

(4.21)

Eq.(4.21) is similar to eq.(3.16) and so is the behaviour of the corresponding trajectories on the \( (\lambda, f) \) plane.

In Appendix B we shall present the explicit solution of eqs.(4.14)–(4.16) in the asymptotic region of large \( \lambda \) which demonstrates that a finite extremum value of \( \lambda \) is always
reached in a finite time (i.e. any trajectory on the \((\lambda, f)\) plane always hits a boundary curve).

Let us now consider the case when \(E\) in (4.13) is negative. Then it follows from (4.14) that \(\dot{\lambda}\) can never vanish (we are assuming that \(c\) is non-negative). As a result, there can be no turning points, i.e. the expansion goes to infinity and contraction goes to zero.

To illustrate what happens when \(W\) is negative let us consider the special case of \(N = 1\). If there are no extra static space dimensions, i.e. \(D = 2\), \(c = 16\) then we may use the expression for the vacuum energy (2.29), (2.30) of a \(D = 2\) string compactified on a circle which is known to all loop orders [23]

\[
f_1 = d_1(e^\lambda + e^{-\lambda}) , \quad f_2 = d_2(e^{2\lambda} + e^{-2\lambda} + 10/7) , \quad ... ,
\]

where \(d_1\) is negative (see also [12]). Assuming again that the effective coupling \(e^\varphi\) is sufficiently small so that we can consider only the one-loop term \(F = f_1\) let us describe another way of solving (4.14)–(4.16) for \(N = 1\) in the large radius limit. If \(\lambda\) is large eqs.(4.14)–(4.16) can be represented in the form

\[
\ddot{\rho} - \dot{\lambda}\dot{\rho} = 0 , \quad \dot{\rho}(\dot{\rho} - 2\dot{\lambda}) = d_1 e^\rho - c , \quad \rho \equiv \varphi + \lambda , \quad \rho \in (4.23)
\]

i.e.

\[
\dot{\rho} = Be^\lambda , \quad -2\ddot{\rho} + \dot{\rho}^2 = d_1 e^\rho - c . \quad \rho \in (4.24)
\]

Integrating once the equation for \(\rho\) (introducing \(y = \exp(-\frac{1}{2}\rho)\)) we find

\[
\dot{\rho}^2 = q e^\rho - d_1\rho e^\rho - c , \quad q = \text{const} . \quad \rho \in (4.25)
\]

As a result, if the radius (which according to (4.24) is proportional to \(\dot{\rho}\)) is large and growing, it grows indefinitely.
5. Concluding remarks

1. It useful to clarify the meaning of the duality invariance (2.19), (3.3) of our systems of equations (3.4)–(3.7) and (4.14)–(4.16). Let us consider, for example, (3.4)–(3.7) and assume for simplicity that \( m = \tilde{m} \). Then the duality invariance implies that for each solution \((\lambda, \varphi, \psi, \tilde{\psi})\) there exists another solution \((-\lambda, \varphi, \tilde{\psi}, \psi)\). If we consider the simplified system (3.8)–(3.10) and further assume that \( \mu = \tilde{\mu} \) then the potential in (3.8) \( W = 2\mu \cosh 2\lambda - C/2N \) will (like the potential in (4.14)) have the exact symmetry under \( \lambda \rightarrow -\lambda \). Though there will exist the symmetric (“self-dual”) solution (cf.(3.13), (4.17), (4.18)) \( \lambda = 0, \varphi = \varphi_0 - Qt \) in general duality will be “spontaneously broken” (by the initial condition \( \lambda(0) \neq 0 \)) on a time–dependent solution. As we have seen, however, the duality is asymptotically restored at late times (for a generic choice of initial conditions for \( \lambda, \dot{\lambda} \) and \( \varphi < 0 \)) : the oscillating solution \( \lambda(t) \rightarrow 0 \) at \( t \rightarrow \infty \). For all the oscillating solutions the duality is true in the following “average” sense : the average value of \( \lambda(t) \) is zero. Also, the behaviour of a given solution at large positive and large negative \( \lambda \) is similar. This is in agreement with one’s expectation that duality should be a symmetry between processes at large and small distances.

2. Let us now discuss a model in which some of time dependent space dimensions are “uncompactified”, i.e. the vacuum energy depends on their scale factor in a trivial way. Let us consider the zero temperature case and set \( c = 0 \). Denoting the common scale of compact (noncompact) dimensions by \( \lambda (\Lambda) \) and their number by \( N (n) \) we find from eqs.(2.38)–(2.40)

\[
- n\ddot{\Lambda}^2 - N\dot{\Lambda}^2 + \varphi^2 = e^{\varphi + n\Lambda} F , \quad F = F(\lambda) , \tag{5.1}
\]

\[
\ddot{\Lambda} - \dot{\varphi} \dot{\Lambda} = -\frac{1}{2} e^{\varphi + n\Lambda} F , \tag{5.2}
\]

\[
\ddot{\lambda} - \dot{\varphi} \dot{\lambda} = -\frac{1}{2} N^{-1} e^{\varphi + n\Lambda} F' , \tag{5.3}
\]

\[
\ddot{\varphi} - n\ddot{\Lambda}^2 - N\dot{\Lambda}^2 = \frac{1}{2} e^{\varphi + n\Lambda} F . \tag{5.4}
\]
Assuming \( F \) is positive a qualitative analysis of this system is similar to that of eqs.(4.14)–(4.16). Since \( \ddot{\phi} > 0 \) the dilaton continues to decrease monotonically, providing the damping terms in the equations for \( \Lambda \) and \( \lambda \). Combining eqs.(5.1), (5.2) and (5.3) one finds (cf. (B.1),(B.2))

\[
\ddot{\Lambda} - \ddot{\phi} = \dot{\phi}(\dot{\Lambda} - \dot{\phi}) \; , \; \dot{\Lambda} = \dot{\phi} + Ae^{\phi} .
\] (5.5)

As a result, one concludes that \( \Lambda \) first expands to its maximal value and then contracts to zero (note that there is no winding mode contribution which may protect \( \Lambda \) from contraction to zero) while \( \lambda \) oscillates between maximal and minimal values approaching the self-dual point \( \lambda = 0 \).

Since \( \Lambda \) in (5.1)–(5.4) corresponds to the non-compact dimensions it is more natural to introduce another “shifted” dilaton field as an effective coupling (cf. (2.5))

\[
\overline{\phi} \equiv 2\phi - N\lambda \; , \; \phi = \overline{\phi} - n\Lambda .
\] (5.6)

\( \overline{\phi} \) “absorbes” only the volume factor corresponding to compact dimensions and like \( \phi \) is invariant under the duality \( \lambda \to -\lambda \). The weak coupling regime corresponds to \( \overline{\phi} \) decreasing with time. \( \overline{\phi} \) in general will not be monotonic but will be decreasing at late times (if \( \dot{\phi} < 0 \)). Let us consider the special solution with constant “internal” scale \( \lambda = 0 \), i.e. \( \lambda \) sitting at the minimum of \( F \). Then (5.1),(5.2) and (5.4) become identical to the system (B.1) (with \( N,\lambda,d_1 \) replaced by \( n,\Lambda,F(0) \)) the solution of which is given by (B.8),(B.9),(B.12). The behaviour of \( \overline{\phi} \)

\[
\overline{\phi} = \phi + n\Lambda = \overline{\phi}_0 + (\sqrt{n}+1)/(\sqrt{n}-1) \ln z - 2(n+1)/(n-1) \ln(1+z)
\] (5.7)

is similar to that of \( \Lambda \) : it first grows with \( t \), reaches its maximum (at \( z = (n+1)^2/(n-1)^2 \)) and then starts decreasing. Since \( \Lambda \) reaches its maximum at a later time \( t \) (\( z = (n+1)/(n-1) \)) there exists the interval of \( t \) in which the non-compact dimensions are still expanding while the dilaton coupling is already decreasing.
3. Cosmological solutions for the space being a product of a number of large flat dimensions and an “internal” torus (and a finite temperature one–loop vacuum energy as a source) were previously discussed in refs.[3,4] with a conclusion that the presence of the winding modes prevents internal dimensions from expanding, stabilizing them at late times near the Planck scale. The dilaton dynamics was, however, ignored, i.e. the original dilaton $\phi$ (see (2.5)) was implicitly assumed to be constant. We would like to emphasize that this assumption is inconsistent in general: as was already noted in Sect.2 the system (5.1)–(5.4) does not have solutions with

$$\phi \equiv \frac{1}{2}(\varphi + n\Lambda + N\lambda) = \text{const}$$

unless (2.44) is true. The latter condition can be satisfied only in the limiting cases when the contributions of the winding modes can be ignored.  

In this paper we considered only models at zero temperature. As we have already mentioned the more “realistic” finite temperature case was discussed in [12].

I would like to acknowledge J. Bagger and M. Tsypin for help and C. Vafa for collaboration on ref.[12] on which most of the present paper is based. I am also grateful to Trinity College, Cambridge for a financial support.

Appendix A . Asymptotic solution of the system (3.8)–(3.10)

Here we shall present the solution of the “classical” system (3.8)–(3.10) in the case of $N = 1$. Our aim is to demonstrate the correctness of the qualitative analysis of the behaviour of the solution given in Sect. 3 which predicts the existence of a finite maximal radius of expansion. In order to solve eq.(3.16) explicitly we shall consider the region of

$$\dot{\varphi} = -N\dot{\Lambda}$$

into (5.1)–(5.4) gives $F = 0$.  

2 Let us note also that no solution of (5.1)–(5.4) exists with $\varphi = \text{const}$ (the substitution of $\dot{\varphi} = -N\dot{\Lambda}$ into (5.1)–(5.4) gives $F = 0$).
large positive $\lambda$ in which the potential $W$ can be approximated by $W = \frac{1}{2} \bar{\mu} e^{2\lambda}$. Assuming that $\dot{\lambda} > 0$ and absorbing the positive constant $\bar{\mu}$ into $\lambda$ we find that eq. (3.16) takes the form

$$f' = \sqrt{f^2 - e^{2\lambda}}, \quad (A.1)$$

or, equivalently,

$$k' = -h(k), \quad h \equiv k - \sqrt{k^2 - 1}, \quad k(\lambda) \equiv f(\lambda) e^{-\lambda}, \quad k = \frac{1}{2} (h + h^{-1}). \quad (A.2)$$

Integrating over $h$ we get

$$-\frac{1}{2} \ln |h| - \frac{1}{4h^2} = \lambda - \lambda_0 \equiv \lambda. \quad (A.3)$$

Here the integration constant $\lambda_0$ should be large positive. Suppose that $f > 0$, i.e. $f > 1, k > 1$ so that $h$ changes from 1 at $k = 1$ to 0 at $k = \infty$. Then $\lambda$ increases from some large negative value at initial large $k$ to its maximal value $-1/4$ at $k = 1$. The maximal value of $\lambda$ (or the value of $\lambda_0$) is determined by initial conditions for $\lambda$ and $f$. We have thus proved that for the given exponentially growing potential the rising trajectory on the $(\lambda, f)$ plane will always hit the boundary curve so that the maximal value of $\lambda$ will be reached in a finite time.

The case of $f < 0$, $\dot{\lambda} > 0$ is analysed in a similar way. Then $h$ is strictly negative and (A.3) is still valid. Both $k$ and $\lambda$ increase from their initial negative values to their maximal values $k = -1$ and $\lambda = -1/4$.

If $\dot{\lambda} < 0$ at $t = 0$ $\lambda$ will be decreasing towards negative values and will finally reach the region where the constant and $e^{-2\lambda}$ terms in the potential $W$ (3.8) cannot be ignored. The solution in asymptotic region of large negative $\lambda$ can be found by the duality transformation: $\lambda \rightarrow -\lambda$. The potential will be dominated by the term $e^{-2\lambda}$ which will imply the existence of a minimal radius of contraction. The full solution can be found by "sewing" the solutions in the two asymptotic regions.
There is another way of representing the solution of (3.8)–(3.10) in the asymptotic regions of large $\pm \lambda$. Consider, for example, $\lambda < 0$ and $\dot{\lambda} > 0$, $\dot{\varphi} < 0$, i.e. the case of expansion starting from small radius. Introducing the “conformal time” $\tau$

$$d\tau = e^{-\lambda}dt, \quad F' \equiv \frac{dF}{d\tau} = e^\lambda F,$$

we find from (3.8)–(3.10) ($N = 1$)

$$\rho'' - \rho'^2 = 0, \quad -\rho'^2 + 2\rho' \varphi' = \mu, \quad \rho \equiv \lambda + \varphi,$$

$$\rho = \rho_0 - \ln |\tau|, \quad \varphi = \varphi_0 - \frac{1}{2} \ln |\tau| - \mu \tau^2/4, \quad \tau \equiv \tau - \tau_0,$$

$$\lambda = \lambda_0 - \frac{1}{2} \ln |\tau| + \mu \tau^2/4, \quad dt = d\tau \tau^{-1/2} \exp(\lambda_0 + \mu \tau^2/4).$$

**Appendix B. Asymptotic solution of the system (4.14)–(4.16)**

Below we shall solve eqs.(4.14)–(4.16) in the region where $\lambda$ is large and positive (the solution for large negative $\lambda$ can then be found by the duality transformation). Then the potential $W$ in (4.14) is given by (see (2.31),(4.13); we shall assume that $d_1$ is positive)

$$W = \overline{W}/2N, \quad \overline{W} = d_1 e^{\varphi + N\lambda},$$

and our system takes the form (the terms proportional to $c$ can be neglected in the large $\lambda$ limit)

$$-N\ddot{\lambda}^2 + \dot{\varphi}^2 = \overline{W}, \quad \ddot{\lambda} - \dot{\varphi} \dot{\lambda} = -\frac{1}{2} \overline{W}, \quad \ddot{\varphi} - N\ddot{\lambda}^2 = \frac{1}{2} \overline{W}.$$  \hspace{1cm} (B.1)

Combining these equations we get

$$\ddot{\lambda} - \ddot{\varphi} = \ddot{\varphi}(\dot{\lambda} - \dot{\varphi}), \quad \dot{\lambda} = \dot{\varphi} + Ae^\varphi,$$

$$\ddot{\varphi} = \frac{1}{2}(N\ddot{\lambda}^2 + \dot{\varphi}^2).$$  \hspace{1cm} (B.2)

$$\ddot{\varphi} = \frac{1}{2}(N\ddot{\lambda}^2 + \dot{\varphi}^2).$$  \hspace{1cm} (B.3)
To have an expansion with decreasing dilaton, i.e., \( \dot{\lambda} > 0 \), \( \dot{\phi} < 0 \) we should take \( A < 0 \). Substituting the expression for \( \dot{\lambda} \) in (B.2) into (B.3) we get the equation for \( \phi \) which is easy to solve by introducing the new time variable \( \tau \)

\[
d\tau = dt \ e^{\phi} , \quad \varphi' = \frac{d\varphi}{d\tau} = e^{-\varphi} \dot{\varphi} , \quad \lambda' = \varphi' + A ,
\]

so that

\[
\varphi'' = \frac{1}{2} [N(\varphi' + A)^2 - \varphi'^2] .
\]

Let us first consider the case of \( N > 1 \). Integrating (B.5) we find

\[
\left| \frac{\varphi' + u}{\varphi' + v} \right| = e^{-\sqrt{NA}\tau} ,
\]

\[
u = \frac{\sqrt{NA}}{\sqrt{N} - 1} , \quad v = \frac{\sqrt{NA}}{\sqrt{N} + 1} .
\]

As a result (we assume that \( -u < \varphi' < -v \))

\[
\varphi' = -\frac{u + vz}{1 + z} , \quad z \equiv e^{-\sqrt{NA}\tau} ,
\]

\[
\varphi = \varphi_0 + \frac{1}{\sqrt{N} - 1} \ln z - \frac{2}{N - 1} \ln (1 + z) ,
\]

\[
\lambda = \lambda_0 + \frac{1}{\sqrt{N} \sqrt{N - 1}} \ln z - \frac{2}{N - 1} \ln (1 + z) .
\]

The first equation in (B.2)

\[
\varphi'^2 - N\lambda'^2 = d_1 e^{-\varphi + N\lambda} ,
\]

is then satisfied if

\[
A^2 = \frac{1}{4N} (N - 1)d_1 .
\]

The relation between the original time \( t \) and \( z \) is

\[
dt = d\tau e^{-\varphi} = -(NA^2)^{-1/2} dz z^a(1 + z)^b , \quad a = -\frac{\sqrt{N}}{\sqrt{N} - 1} , \quad b = \frac{2}{N - 1} ,
\]

which can be approximated by

\[
t \sim z^{-h} = \exp(\sqrt{N} Ah\tau) , \quad 0 < h < 1 , \text{i.e. } z \text{ decreasing to zero corresponds to } t \text{ growing to infinity.}
\]
As it is clear from (B.9), \( \lambda(z) \) first grows to its maximal value at \( z_* = \frac{\sqrt{N}+1}{\sqrt{N} - 1} \) and then decreases. The function \( \varphi(z) \) is monotonically decreasing with \( z \to 0 \). We conclude that a maximal radius of expansion is reached in a finite time (\( z_* \) corresponds to a turning point of the trajectory). Once the contraction starts we eventually reach the region of small \( \lambda \) where other terms in \( W(4.14) \) (which are present because of the contributions the winding modes) cannot be ignored.

If \( N = 1 \) the integration of (B.5) gives

\[
\varphi = \varphi_0 - \frac{1}{2} A \tau + A^{-1} e^{A \tau}, \quad \lambda = \lambda_0 + \frac{1}{2} A \tau + A^{-1} e^{A \tau}.
\]

(B.13)

A different way of solving (B.1) in the \( N = 1 \) case was discussed in Sect.4 (see (4.23)–(4.25)).
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