POLYNOMIAL BEHAVIOR OF SPECIAL VALUES OF PARTIAL ZETA FUNCTION OF REAL QUADRATIC FIELDS AT S=0

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ABSTRACT. We compute the special values of partial zeta function at \( s = 0 \) for family of real quadratic fields \( K_n \) and ray class ideals \( b_n \) such that \( b_n^{-1} = [1, \delta(n)] \) where the continued fraction expansion of \( \delta(n) \) is purely periodic and each terms are polynomial in \( n \) of bounded degree \( d \). With an additional assumptions, we prove that the special values of partial zeta function at \( s = 0 \) behaves as quasi-polynomial. We apply this to obtain that the special values the Hecke’s \( L \)-functions at \( s = 0 \) for a family of for a Dirichlet character \( \chi \) behave as quasi-polynomial as well. We compute out explicitly the coefficients of the quasi-polynomials. Two examples satisfying the condition are presented and for these families the special values of the partial zeta functions at \( s = 0 \).

CONTENTS

1. Introduction
2. Decomposition of \( L \)-function into partial zeta function
   2.1. Partial zeta function of a ray class
   2.2. Decomposition of partial Hecke \( L \)-function
   2.3. Continued fractions
   2.4. Quasi-polynomials
   2.5. Main theorem
3. Shintani-Zagier decomposition and partial zeta values
4. Periodicity of orbit
5. Explicit computation of the coefficients
6. Two examples
   6.1. Case 1: \( f(n) = n^2 + 2 \)
   6.2. Case 2: \( f(n) = 16n^4 + 32n^3 + 24n + 3 \)
References

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1. Introduction

This note is complementary to our previous work “the behavior of Hecke’s L-function at \( s=0 \)” ([10]). In [10], a series of real quadratic fields \( K_n \) with fixed ideals \( b_n^{-1} = [1, \delta(n)] \) and a mod-\( q \) Dirichlet character \( \chi \), which yields a mod-\( q \) ray class character \( \chi_n := \chi \circ N_{K_n/Q} \) for each \( K_n \), are considered. We gave a condition on \( \delta(n) \) to ensure a controlled behavior of the special value at \( s = 0 \) of the Hecke’s partial L-function of \( b_n \) twisted with a ray class character \( \chi_n \) of modulus \( q \) in a certain way. More precisely, we called this property ‘linearity of partial Hecke L-values’ when

\[
L_{K_n}(0, b_n, \chi_n) = A(r)k + B(r)
\]

where \( n = qk + r \) and for some constants \( A(r), B(r) \) associated to \( r = 0, 1, \ldots, r-1 \). The coefficients \( A(r), B(r) \) can be explicitly computed and are shown to be lying inside \( \frac{1}{12q^2} \mathbb{Z}[\chi(1), \chi(2), \ldots, \chi(r-1)] \). Roughly written the criterion is that the terms of the continued fractions of \( \delta(n) \) has to be linear function in \( n \).

The proto-typical result of this linearity appeared first in Biro’s proof of Yokoi’s conjecture (cf. [18] for the conjecture and [2] for the proof). It was the key ingredient with a class number one criterion in solving Yokoi’s conjecture. Later this moral has been extensively applied in solving some class number one problems for some families of real quadratic fields when there is an appropriate class number one criterion. (cf. [2], [3], [7], [8], [9], [11]).

In this paper, we deal with similar phenomenon with almost the same assumptions as in [10], but we have freed the linearity assumption on the terms of continued fraction expansion of \( \delta(n) - 1 \). In this setting, the linearity of the values of \( \zeta \)- or L-values in family is generalized to a ‘polynomial’ expression. Written precisely, the special values as function in \( n \) are taken in a packet of polynomials in a periodic way. Functions of this type are called quasi-polynomials. So in particular the ‘linearity’ in loc. cit. should read as quasi-linear function. A precise definition and a short discussion of quasi-polynomial are presented in Section 2 of this article.

Our main result is as follows:

**Theorem 2.4.** Let \( \{K_n = \mathbb{Q}({\sqrt{f(n)}})\}_{n \in \mathbb{N}} \) be a family of real quadratic fields where \( f(n) \) is a positive square free integer for each \( n \). Suppose \( b_n \) is an integral ideal relatively prime to \( q \) such that \( b_n^{-1} = [1, \delta(n)] \). Assume \( \delta(n) - 1 \) has purely periodic continued fraction expansion

\[
\delta(n) - 1 = [[a_0(n), a_1(n), \cdots, a_{s-1}(n)]]
\]
A part of the result presented here was announced for the Hecke’s $L$-values in a family at the end of [10]. Actually, the consideration of the partial zeta function for ray class ideals instead of the partial Hecke’s $L$-function of an ideal ameliorates the previous in two folds. Firstly, we give here an expression of Hecke’s $L$-function as twisted sum of the ray class partial zeta function:

$$L_{K_n}(\chi, s, b) = \sum_{(C, D) \in \tilde{F}_d} \chi((C + D\delta)b)\zeta_q(s, (C + D\delta)b)$$

(c.f. Proposition [2.2]). If we restrict to the case $d = 1$, using this identification, we can recover directly the linearity of the values of Hecke’s $L$-functions at $s = 0$ (See Corollary [2.5]). For general $d$, we obtain the higher degree generalization of the main result presented in [10]. This is previously announced in loc.cit. Secondly, in principle the partial zeta function of a ray class ideal contains finer information than the partial Hecke’s $L$-function of the associated ideal.

Our main idea is to develop and examine appropriate cone decomposition similar to Shintani and Zagier for partial zeta functions (cf. [16], [19]). Once a simple cone is given, one can evaluate the Riemann sum for the partial zeta value at $s = 0$ over the cone and it is simply expressed using values of Bernoulli polynomial. For a ray class partial zeta values, we need to sum over larger cone than that for ideal class partial zeta or $L$ function. Unfortunately, the decompositions for a family of real quadratic fields with ideals are far from being uniform. But surprisingly again the cone decomposition for the associated ideal class behaves in a uniform way. In particular, if we parameterize the orbit of the ray class ideal $b_n$ acted by the totally positive unit group using a pair of integers $(C, D)$ such that $0 \leq C, D \leq q - 1$, this action has $q$-periodicity in the family parametrized by $n$.

This paper is composed as follows. In Section 2, we recall the notions of ray class partial zeta function and some necessary stuffs. Then we state the main results. Section 3 is devoted to an expression of the partial zeta value at 0, where we use the cone decomposition à la Shintani and Zagier. In Section 4, we show that this domain decomposition is acted
by the totally positive unit group modulo units congruent to 1 modulo \( q \), which has certain invariance property in the family. This concludes the most important part of our main theorem that the special values behave in quasi polynomial for the family satisfying our assumption. In Section 5, we compute the coefficients of the quasi polynomials in the expression of partial zeta values to restrict possible coefficients. Finally, in Section 6, we compute the quasi polynomials for two explicitly chosen families.

2. Decomposition of \( L \)-function into partial zeta function

2.1. Partial zeta function of a ray class. For a number field \( K \) and a fixed positive rational integer \( q \) as a conductor, the partial zeta function of the ray class of an ideal \( \mathfrak{a} \) in \( K \) is defined as

\[
\zeta_q(s, \mathfrak{a}) := \sum_{c \sim \mathfrak{a} \text{ integral}} N(c)^{-s}
\]

where \( c \sim q \mathfrak{a} \) means that \( c = (\alpha) \mathfrak{a} \) for totally positive \( \alpha \equiv 1 \pmod{q} \). For \( c \) to be integral, \( \alpha \) should be an element of \( 1 + q^{-1} \mathfrak{a}^{-1} \) if and only if \( \alpha \in E_q^+ \), where \( E_q^+ \) is the multiplicative group of totally positive units congruent to 1 modulo \( q \). Thus we have

\[
(1) \quad \zeta_q(s, \mathfrak{a}) = \sum_{\alpha \in (1 + qa^{-1})^+/E_q^+} N(\alpha \mathfrak{a})^{-s}
\]

This should not be confused with the partial zeta function for an ordinary ideal class. Note that this definition works not only for an ideal relatively prime to \( q \) but for general ideal of \( K \).

From now on, we assume \( K \) to be a real quadratic field and we consider ray class partial zeta function of an integral ideal \( \mathfrak{b} \) relatively prime to \( q \) such that \( \mathfrak{b}^{-1} = [1, \delta] \). In this case, we have a description of the partial zeta function of other ray class than \( \mathfrak{b} \) but in the same ideal class.

**Lemma 2.1.** Suppose \( C, D \) are integers such that \( 0 \leq C, D \leq q - 1 \) and \( ((C + D\delta)\mathfrak{b}, q) = 1 \). Then we have

\[
\zeta_q(s, (C + D\delta)\mathfrak{b}) = \sum_{\alpha \in \left(\frac{C+D\delta}{q} + \mathfrak{b}^{-1}\right)^+/E_q^+} N(q\mathfrak{b}\alpha)^{-s}.
\]

**Proof.** From (1) we have

\[
(2) \quad \zeta_q(s, (C + D\delta)\mathfrak{b}) = \sum_{\beta \in (1 + q(C + D\delta)^{-1}\mathfrak{b}^{-1})^+/E_q^+} N((C + D\delta)\mathfrak{b}\beta)^{-s}.
\]

put \( \alpha = \beta \frac{C+D\delta}{q} \). This shows the equality. \( \square \)
Let us define
\[ F := \{(C, D) \in \mathbb{Z}^2 | 0 \leq C, D \leq q - 1, (C, D) \neq (0, 0)\} \]
and its subset
\[ F_\delta := \{(C, D) \in \mathbb{Z}^2 | 0 \leq C, D \leq q - 1, ((C + D\delta)b, q) = 1\}. \]

An element \((C, D)\) of \(F\) sends an ideal \(a\) to another ideal \((C + D\delta)a\). If \((C, D) \in F_\delta, (C + D\delta)\) sends \(b\) to another ideal relatively prime to \(q\).

The group \(E^+\) of totally positive units of \(K\) acts on \(F\) by
\[ \epsilon \ast (C + D\delta) = C' + D'\delta \]
where \((C + D\delta)\epsilon + qb^{-1} = C' + D'\delta + qb^{-1}\) for \(\epsilon \in E^+\). Note that this action is inherited to \(F_\delta\).

Let \(\tilde{F}' \subset F\) be a fundamental set for the quotient \(F/E^+\) and \(\tilde{F}_\delta' \subset \tilde{F}'\) be a fundamental set for the quotient \(F_\delta/E^+\).

### 2.2. Decomposition of partial Hecke L-function

Now we consider the partial Hecke L-function for \((b, \chi)\) where \(\chi\) is a ray class character of modulus \(q\). Recall the partial Hecke L-function associated to \((b, \chi)\) is defined as follows:
\[ L_K(s, \chi, b) = \sum_{c \sim b} \frac{\chi(c)}{N(c)^s} \]
where the summation runs over every integral ideal \(c\) principally equivalent to \(b\).

Keeping the notations introduced before, we obtain an expression of partial Hecke L-function of \((b, \chi)\) as sum of ray class partial zeta functions of ray class ideals principally equivalent to \(b\) twisted with values of \(\chi\). Since any ray class associated to an ideal class of \(b\) can be represented in the form \((C + D\delta)b\) and \(E_q^+\) action on \((C, D)\) preserves the ray class, this sum is taken over \(\tilde{F}'\). Moreover, as \(\chi\) values 0 at \((C + D\delta)b\) for \((C, D)\) outside \(\tilde{F}_\delta'\), the summation actually runs over \(\tilde{F}_\delta'\).

Summarizing this discussion, we have

**Proposition 2.2.** Let \(q\) be a positive integer. For an ideal \(b \subset K\) relatively prime to \(q\) and a ray class character \(\chi\) modulo \(q\), we have
\[ L(\chi, s, b) = \sum_{(C, D) \in \tilde{F}_\delta'} \chi((C + D\delta)b) \zeta_q(s, (C + D\delta)b). \]
Proof.

\[ L_K(s, \chi, b) = \sum_{a \sim b \text{ integral}} \chi(a) N(a)^{-s} \]

\[ = \sum_{(C, D) \in F'_3} \chi((C + D \delta)b) \sum_{a \in \left(\frac{C + D \delta}{q} + \mathbb{Z}\right)^+ / \mathbb{Z}} N(qb \alpha)^{-s} \]

Applying Lemma (2.1), we have done the proof. \( \square \)

2.3. Continued fractions. To state our main result properly, we need to recall the notions of continued fractions and quasi polynomials.

Let \([a_0, a_1, \ldots, a_n]\) be the purely periodic continued fraction

\[ [a_0, a_1, a_2, \ldots, a_n, a_0, a_1, \ldots], \]

where

\[ [a_0, a_1, a_2, \ldots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}. \]

2.4. Quasi-polynomials. If \(f(n)\) is a function of \(\mathbb{Z}\) such that

\[ f(n) := c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n) \]

for some periodic functions \(c_k(n)\) then we call \(f(n)\) a quasi-polynomial of degree \(d\). When \(c_k(n)\) has a common period \(q\) for \(k = 0, 1, \ldots, d\), we say \(f(n)\) has (quasi-)period \(q\). We call \(c_k(n)\) the \(k\)-th coefficient (function) of \(f(n)\).

**Proposition 2.3.** \(f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n)\) is quasi-polynomial of period \(q\) if and only if for \(n = qk + r\)

\[ f(n) = a_d(r)k^d + a_{d-1}(r)k^{d-1} + \cdots + a_0(r). \]

Moreover, for \(j = 0, 1, \cdots, d\),

\[ c_j(r) = \sum_{i=j}^{d} a_i(r) \binom{i}{j} (-r)^{i-j} q^{-i}. \]

**Proof.** Substitute \(k\) with \(q^{-1}(n - r)\) in the first expression of \(f(n)\). Rearranging this as a form of polynomial in \(n\), we obtain the second expression. \(c_j(r)\) is easily obtain from the rearrangement. \( \square \)
2.5. Main theorem. Our main theorem is as follows:

Theorem 2.4. Let \( \{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n \in \mathbb{N}} \) be a family of real quadratic fields where \( f(n) \) is a positive square free integer for each \( n \). Suppose \( b_n \) is an integral ideal relatively prime to \( q \) such that \( b_n^{-1} = [1, \delta(n)] \). Assume \( \delta(n) - 1 \) has purely periodic continued fraction expansion

\[
\delta(n) - 1 = \left\lfloor \frac{a_0(n), a_1(n), \ldots, a_{s-1}(n)}{1} \right\rfloor
\]

of period \( s \) independent of \( s \) and \( a_i(x) \) are polynomials of degree smaller than \( d \). If \( N_{K_n} (b_n(C + D \delta(n))) \) modulo \( q \) is a function only depending on \( C, D \) and \( r \) for \( n = qk + r \), then

\[
\zeta_q(0, b_n(C + D \delta(n))) = A_0(r) + A_1(r)n + \ldots + A_d(r)n^d
\]

where \( n = qk + r, 0 \leq r < q \) and for some constants \( A_0(r), A_1(r), \ldots, A_d(r) \) depending on \( r \). Furthermore, \( A_i(r) \in \frac{1}{12q^{i+2}} \mathbb{Z} \).

Corollary 2.5. Let \( \chi \) be a Dirichlet character modulo \( q \) for a positive integer \( q \) and \( \chi_n \) be a ray class character modulo \( q \) defined by \( \chi \circ N_{K_n} \). With the same assumption of Theorem 2.4 we have \( L_{K_n}(0, \chi_n, b_n) \) is quasi-polynomial with degree \( d \) and period \( q \) and the values of \( i \)-th coefficient functions are in \( \frac{1}{12q^{i+2}} \mathbb{Z}[\chi(1), \chi(2), \ldots, \chi(q)] \).

Note in [10], the linearity is the quasi-linearity in the second form in Proposition 2.3 written as a polynomial in \( k \), while we take the first form as shape of polynomial in \( n \) in this article.

3. SHINTANI-ZAGIER DECOMPOSITION AND PARTIAL ZETA VALUES

A real quadratic field \( K \) is diagonally embedded into its Minkowski space \( K \cong \mathbb{R}^2 \) by \( \iota = (\tau_1, \tau_2) \), where \( \tau_1, \tau_2 \) are two real embeddings of \( K \). The multiplicative action of \( E_q^+ \) on \( K^+ \) induces an action on \( (\mathbb{R}^2)^+ \) by extending in coordinate-wise way:

\[
\iota \circ (x, y) = (\tau_1(x), \tau_2(y)).
\]

A fundamental domain \( \mathcal{D}_R \) of \( (\mathbb{R}^2)^+/E_q^+ \) is given as

\[
\mathcal{D}_R := \{x \iota(1) + y \iota(e^{-\lambda}) | x > 0, y \geq 0\} \subset (\mathbb{R}^2)^+
\]

where \( e^\lambda \) is the totally positive generator of \( E_q \) and \( \lambda = [E^+ : E_q^+] \).

We fix an integral ideal \( b \) such that \( b^{-1} = [1, \delta] \). Moreover we assume that \( \delta > 1 \) and \( 0 < \delta' < 1 \). If we take the convex hull of \( \iota(b^{-1}) \cap (\mathbb{R}^2)^+ \) in \( (\mathbb{R}^2)^+ \), the lattice points on the boundary are \( \{P_i \}_{i \in \mathbb{Z}} \) for \( P_i \in \iota(b^{-1}) \). \( P_i \) are uniquely determined by imposing that \( P_0 = \iota(1), P_{-1} = \iota(\delta) \) and \( x(P_i) < x(P_{i-1}) \) where \( x(P_k) \) is the first coordinate of \( P_k \). Since

\[
P_{\lambda m} = \iota(e^{-\lambda})
\]
for some positive integer $m$ (See Proposition 2.4 (5) in [10]), $\mathcal{D}_\mathbb{R}$ is further decomposed into $(\lambda \cdot m)$-disjoint union of smaller cones:

$$\mathcal{D}_\mathbb{R} = \bigcup_{i=1}^{\lambda m} \{ xP_{i-1} + yP_i \mid x > 0, \ y \geq 0 \}. $$

Accordingly the fundamental set of the quotient $(\iota((\frac{C+D\delta}{q}) + b^{-1})) \cap (\mathbb{R}^2)^{}/E^+_q$ inside $\mathcal{D}_\mathbb{R}$, which we denote by $\mathcal{Q}$ is given by a disjoint union:

$$ \mathcal{Q} := \bigcup_{i=1}^{\lambda m} \left( \left. \iota \left( \frac{C + D\delta}{q} \right) + b^{-1} \right) \cap \{ xP_{i-1} + yP_i \mid x > 0, \ y \geq 0 \} \right).$$

Since $\{P_{i-1}, P_i\}$ is a $\mathbb{Z}$-basis of $\iota(b^{-1})$, there is a unique $(x^{i}_{C+D\delta}, y^{i}_{C+D\delta}) \in (0, 1] \times [0, 1)$ such that

$$(6) \quad x^{i}_{C+D\delta}P_{i-1} + y^{i}_{C+D\delta}P_i \in \iota \left( \frac{C + D\delta}{q} \right) + b^{-1},$$

for each $i, C, D \in \mathbb{Z}$. Thus

$$(7) \quad \iota \left( \frac{C + D\delta}{q} \right) + b^{-1} \cap \{ xP_{i-1} + yP_i \mid x > 0, \ y \geq 0 \}$$

$$= \{ (x^{i}_{C+D\delta} + n_1)P_{i-1} + (y^{i}_{C+D\delta} + n_2)P_i \mid n_1, n_2 \in \mathbb{Z}_{\geq 0} \}. $$

In [17], Yamamoto found a recursive relation satisfied by $(x^{i}_{C+D\delta}, y^{i}_{C+D\delta})$:

$$(8) \quad x^{i+1}_{C+D\delta} = (b_i x^{i}_{C+D\delta} + y^{i}_{C+D\delta}),$$

$$y^{i+1}_{C+D\delta} = 1 - x^{i}_{C+D\delta}, $$

where $\langle \cdot \rangle$ is as defined as $\langle x \rangle = x - \lfloor x \rfloor$ (resp. 1) for $x \notin \mathbb{Z}$ (resp. for $x \in \mathbb{Z}$) (see (2.1.3) of loc.sit.).

Let $A_i := x(P_i)$ and $\delta_i = \frac{A_{i-1}}{A_i}$ for all $i \in \mathbb{Z}$. Then from Eq.(7), we obtain the following:

$$\zeta_q(s, (C + D\delta)b) = \sum_{i=1}^{\lambda m} \sum_{n_1, n_2 \geq 0} N((x^{i}_{C+D\delta} + n_1)\delta_i + (y^{i}_{C+D\delta} + n_2))^{-s}A_i^{-s}. $$

From Shintani and Yamamoto’s evaluation of zeta function at $s = 0$ (Lemma 2.5-6 in [10]), we find an expression of the partial zeta value in terms of the values of 1st and 2nd Bernoulli polynomials:

$$(9) \quad \zeta_q(0, (C + D\delta)b) = \sum_{i=1}^{\lambda m} -B_1(x^{i}_{C+D\delta})B_1(x^{i-1}_{C+D\delta}) + \frac{b_i}{2}B_2(x^{i}_{C+D\delta})$$

Moreover we can express $x^{mi+j}_{C+D\delta}, y^{mi+j}_{C+D\delta}$ using epsilon action * defined in (4).
Lemma 3.1. Let $\epsilon$ be the totally positive fundamental unit of $K$. Then for $i > 1$ we have

$$x_{C+D\delta}^{mi+j} = x_{\epsilon^*(C+D\delta)}^j \quad \text{and} \quad y_{C+D\delta}^{mi+j} = y_{\epsilon^*(C+D\delta)}^j,$$

for $j = 0, 1, 2, \cdots, m - 1$.

**Proof.** See Lemma 2.7 in [10]. □

From above lemma and equation (9), we obtain the following Lemma

**Lemma 3.2.**

$$\zeta_q(0, (C + D\delta)b) = \sum_{i=1}^{m} \sum_{j=0}^{\lambda - 1} -B_1(x_{\epsilon^*(C+D\delta)}^i)B_1(y_{\epsilon^*(C+D\delta)}^i) + \frac{b_i}{2} B_2(x_{\epsilon^*(C+D\delta)}^i).$$

Let $F_\delta$ be a set defined in Eq.(3).

**Lemma 3.3.** If $b$ is an integral ideal such that $b^{-1} = [1, \delta]$ with $(b, q) = 1$. Then for $C + D\delta \in F_\delta$, we find that

$$\text{Orb}(C + D\delta) = \{\epsilon^j * (C + D\delta) | j = 0, 1, \cdots, \lambda - 1\},$$

where $\epsilon$ is a totally positive fundamental unit of $E^+$ and $\lambda = [E^+ : E_q^+]$.

**Proof.** By definition of $\epsilon^*$ in (4) we find that $\epsilon^j * (C + D\delta) = C + D\delta$ if and only if

$$(\epsilon^j - 1)(C + D\delta) \in qb^{-1}.$$

Since $(q, b(C + D\delta)) = 1$, (10) is equal to

$$\epsilon^j \in E_q^+.$$

□

Combining Lemma 3.2 and Lemma 3.3, we obtain

**Proposition 3.4.**

$$\zeta_q(0, (C + D\delta)b) = \sum_{(A,B) \in \text{Orb}(C + D\delta)} \sum_{i=1}^{m} -B_1(x_{A+B\delta}^i)B_1(x_{A+B\delta}^{i-1}) + \frac{b_i}{2} B_2(x_{A+B\delta}^i).$$
4. Periodicity of Orbit

In this section, we adapt the discussion of the previous section to a family of real quadratic fields. With the assumptions of this paper, in the considered family, the Shintani-Zagier cone decomposition varies. This variation of decomposition is far from being periodic in \( q \) but the fundamental unit acts on \( F_\delta \) with period \( q \).

We restate the assumption for the family \((K_n, b_n)\). Let \( \{K_n = \mathbb{Q}(\sqrt{f(n)})\}_{n \in \mathbb{N}} \) be a family of real quadratic fields where \( f(n) \) is a positive square free integer for each \( n \). Suppose \( b_n \) is an integral ideal relatively prime to \( q \) such that \( b_{n-1}^{-1} = [1, \delta(n)] \) for a \( \delta(n) \) satisfying \( \delta(n) > 2, 0 < \delta(n) < 1 \). Assume the continued fraction expansion of \( \delta(n) - 1 \) is

\[
\delta(n) - 1 = [[a_0(n), \cdots, a_{s-1}(n)]].
\]

Then it is known that the minus continued fraction expansion of \( \delta(n) \) is

\[
\delta(n) = ((b_0(n), b_1(n), \cdots, b_{m(n)-1}(n)))
\]

where

\[
b_i(n) = \begin{cases} 
  a_{2j}(n) + 2 & \text{for } i = S_j(n) \\
  2 & \text{otherwise}
\end{cases}
\]

and for some index \( S_j(n) \) depending on \( n \). \( S_j(n) \) is defined from \( a_i(n) \) as follows:

\[
S_j(n) = \begin{cases} 
  0 & \text{for } j = 0 \\
  S_{j-1}(n) + a_{2j-1}(n) & \text{for } j \geq 1
\end{cases}
\]

It follows that the period \( m(n) \) of the minus continued fraction of \( \delta(n) \) is

\[
m(n) = \begin{cases} 
  S_{\frac{s}{2}}(n) & \text{for even } s \\
  S_s(n) & \text{for odd } s
\end{cases}
\]

(cf. page 177-178 in [19], Lemma 3.1 in [10]).

Setting \( \mu(s) = 1 \) (resp. 2) for even \( s \) (resp. odd \( s \)), we have

\[
m(n) = S_{\mu(s)s}(n).
\]

Let \( \epsilon_n > 1 \) be the totally positive fundamental unit of \( K_n \) and \( \epsilon_n^{\lambda(n)} > 1 \) be the generator of \( E_\lambda^+(K_n) \). Then \( \lambda(n) = [E_\lambda^+(K_n) : E_\lambda^+(K_n)] \) equals the cardinality of the orbit of \((C, D) \in F_\delta(n)\) as we have discussed in the previous section.
For each \((C, D) \in F_{\delta(n)}\) pair, we have the sequence \(\{x_{C+D\delta(n)}^i, y_{C+D\delta(n)}^i\}_{i \geq -1}\) defined as in Eq. (6) of the previous section:

\[
\begin{align*}
(12) & \quad x_{C+D\delta(n)}^0 = \left\langle \frac{b}{q} \right\rangle, \\
(13) & \quad y_{C+D\delta(n)}^0 = \frac{c}{q}, \\
(14) & \quad x_{C+D\delta(n)}^{i+1} = \left\langle b_i(n)x_{C+D\delta(n)}^i - x_{C+D\delta(n)}^{i-1} \right\rangle, \\
(15) & \quad y_{C+D\delta(n)}^{i+1} = 1 - x_{C+D\delta(n)}^i.
\end{align*}
\]

For \(i \geq 0\), we define \(0 \leq C_i(n), D_i(n) \leq q - 1\) as follows

\[
e_{\delta}^i((C + D)\delta(n)) = C_i(n) + D_i(n),
\]

with \(C_0(n) = C, D_0(n) = D\).

Using Lemma 3.1, we have

\[
x_{C+D\delta(n)}^{m(n)i} = x_{e_{\delta}^i(C+D\delta(n))}^0 = x_{C_i(n)+D_i(n)\delta(n)}^0 = \left\langle \frac{D_i(n)}{q} \right\rangle
\]

and

\[
y_{C+D\delta(n)}^{m(n)i} = 1 - x_{C+D\delta(n)}^{m(n)i-1} = y_{e_{\delta}^i(C+D\delta(n))}^0 = y_{C_i(n)+D_i(n)\delta(n)}^0 = \frac{C_i(n)}{q}.
\]

**Lemma 4.1.** For \(i \geq 1\),

\[
m(n)i = S_{\mu(s)i}(n).
\]

**Proof.** We can rewrite \(S_{\mu(s)i}(n)\) as following:

\[
\sum_{j=1}^{i} \sum_{k=1}^{\delta(s)} a_{2(j-1)\mu(s)k+2k-1}(n) = \sum_{j=1}^{i} \sum_{k=1}^{\delta(s)} a_{2k-1}(n) = iS_{\mu(s)i}(n) = im(n).
\]

Finally we obtain

\[
\left\langle \frac{D_i(n)}{q} \right\rangle = x_{C+D\delta(n)}^{S_{\mu(s)i}(n)}
\]

and

\[
\frac{C_i(n)}{q} = 1 - x_{C+D\delta(n)}^{S_{\mu(s)i}(n)-1}.
\]

Note that for \(j > 0\), \(x_{C+D\delta(n)}^{S_{\mu(s)}(n)}(n)\) and \(x_{C+D\delta(n)}^{S_{\mu(s)}(n)-1}\) are determined only by \(r\) for \(n = qk + r\) (See Property 3 in p.16 and Proposition 3.5 of [10]).

Thus we have \(q\)-invariant property of \(C_i(n), D_i(n)\) in \(n = qk + r\):

**Proposition 4.2.** For \(n, n'\) such that \(n' = n + qk\), assume \((C, D) \in F_{\delta(n)} \cap F_{\delta(n')}\). Then \((C_i(n), D_i(n)) = (C_i(n'), D_i(n')) \in F_{\delta(n)} \cap F_{\delta(n')}.\)
Consequently, we see $\lambda(n) = \lambda(n')$ and $E^+(n)/E^+_q(n) \simeq E^+(n')/E^+_q(n')$ for $n' = n + qk$. Furthermore, from the original assumption that $N((C + D\delta(n)) \pmod{q})$ is determined by $r$ the residue of $n$, we have $F_{\delta(n)} = F_{\delta(n')}$. This identification preserves the $E^+(n)/E^+_q(n)$-action.

**Lemma 4.3.** If $N((C + D\delta(n))b_n) \pmod{q}$ is determined by $r$ the residue of $n \pmod{q}$ for $n = qk + r$ then $F_{\delta(n)}$ is invariant as $k$ varies.

**Proof.** Use the fact $((C + D\delta(n)))b_n, q) = 1$ iff $(N((C + D\delta(n))b_n), q) = 1$. This is again equivalent to $(C, D) \in F_{\delta(n)}$. From the $q$-invariance of $(N((C + D\delta(n))b_n), q) = 1$ in $n$, we obtain that $(C, D) \in F_{\delta(n)}$ iff $(C, D) \in F_{\delta(n')}$ for $n' = n + qk$. \hfill $\square$

In particular, we obtain the $q$-invariance in $n$ of an orbit in $F_{\delta(n)}$ as in the following:

**Proposition 4.4.** If $N((C + D\delta(n))b_n) \pmod{q}$ is a function depending only on $C, D$ and $r$ for $n = qk + r$ then $O\text{r}(C + D\delta(n))$ is invariant as $k$ varies.

5. Explicit Computation of the Coefficients

Recall some calculations from Section 3.2 in [10]. For this we define a sequence $\{v^i_{CD}(r)\}_{i \geq -1}$ for $0 \leq r \leq q - 1$. As we have constrained that $a_i(x) \in \mathbb{Z}[x]$, $(a_i(n))_q$ is independent of $k$ for $n = qk + r$ but determined only by $r$. Thus we can define

$$\gamma_i(r) := (a_i(n))_q = (a_i(r))_q.$$

Let

$$\Gamma_j(r) := \begin{cases} \gamma_{j-1}(r) + \gamma_{2j-1}(r) & \text{for } j \geq 1 \\ 0 & \text{for } j = 0 \end{cases}$$

For $i \geq 1$,

$$c_i(r) = \begin{cases} \gamma_{2j}(r) + 2 & \text{for } i = \Gamma_j(r) \\ 2 & \text{otherwise} \end{cases}$$

Now we can define a sequence $\{v^i_{CD}(r)\}_{i \geq -1}$ satisfying

$$v^{-1}_{CD}(r) = \frac{q - C}{q}, \quad v^0_{CD}(r) = \left\langle \frac{D}{q} \right\rangle$$

and

$$v^{i+1}_{CD}(r) = (c_i(r)v^i_{CD}(r) - v^{i-1}_{CD}(r)).$$

When $C, D$ are fixed and clear from the context, we denote $x^i_{C+D\delta(n)}$ and $v^i_{CD}(r)$ by $x_i(n)$ and $v_i(r)$, respectively.
Proposition 5.1. For $j \geq 0$ and integer $n$ such that $a_{2j+1}(n) \geq q$, $\{x_i(n)\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ has period $q$. Explicitly we have
\[
x_{S_j(n)+q+1}(n) = x_{S_j(n)+1}(n) \text{ for } 0 \leq i \leq a_{2j+1}(n) - q.
\]

Proof. See Proposition 3.4 in [10].

Proposition 5.2. For integers $j \geq 0$, if $n = qk + r$ then
\[
x_{S_j(n)+1}(n) = \nu_{\gamma_j(r)+1}(r) \text{ for } 0 \leq i \leq \gamma_{2j+1}(r).
\]

Proof. See Proposition 3.5 in [10].

For $0 \leq i \leq s - 1$, let $a_i(x) = \sum_{j=0}^d a_{ij}x^j \in \mathbb{Z}[x]$. Then for $0 \leq r \leq q - 1$, there are an unique $\gamma_i(r) \in [1, q] \cap \mathbb{Z}$ such that $a_i(r) = q\tau_i(r) + \gamma_i(r)$ for $\tau_i(r) \in \mathbb{Z}$.

Lemma 5.3. If $n = qk + r$, then we have
\[
a_i(n) = q \sum_{m=1}^d A_{im}(r)k^m + q\tau_i(r) + \gamma_i(r),
\]
where $A_{im}(r) = \sum_{j=m}^d a_{ij} \binom{j}{m} \gamma_{m-1} r^{j-m}$.

We recall that $\{x_{C+D\delta(n)}^i\}_{S_j(n) \leq i \leq S_{j+1}(n)}$ is and arithmetic progression mod $\mathbb{Z}$ with difference $x_{C+D\delta(n)}^i - x_{C+D\delta(n)}^i$ (See Proposition 3.2 in [10]).

From proposition 5.2, we find that if $n = qk + r$ then
\[
\langle x_{C+D\delta(n)}^{S_j(n)+1} \rangle - \langle x_{C+D\delta(n)}^{S_j(n)} \rangle = \langle \nu_{C_D}^{\gamma_j(r)+1}(r) - \nu_{C_D}^{\gamma_j(r)}(r) \rangle.
\]

Let $d_{C_D}^j(r) := \langle \nu_{C_D}^{\gamma_j(r)+1}(r) - \nu_{C_D}^{\gamma_j(r)}(r) \rangle$.

Now we will express the value $\zeta_q(0, (C+D\delta(n))\nu(n))$ using $qk + r$ where $n = qk + r$. We note that form Proposition 3.4

(17)
\[
\zeta_q(0, (C+D\delta(n))\nu(n)) = \sum_{(A,B) \in \text{Orb}(C+D\delta(n))} \sum_{i=1}^{m(n)} -B_1(x^i_{A+B\delta(n)}) B_1(x^{j-1}_{A+B\delta(n)}) + \frac{b_i(n)}{2} B_2(x^i_{A+B\delta(n)}).
\]

For simplification, we define
\[
F(x, y) := -B_1(x)B_1(y) + B_2(x) = (x - \frac{1}{2})(\frac{1}{2} - y) + x^2 - x + \frac{1}{6}.
\]
Because $b_i(n) = 2$ if $i \neq S_j(n)$ for some $j$, we can find that

$$\sum_{i=1}^{m(n)} - (B_1(x_{C+D\delta(n)}^i)B_1(y_{C+D\delta(n)}^i) + \frac{b_i(n)}{2}B_2(x_{C+D\delta(n)}^i))$$

(18) $$= \sum_{l=1}^{s(n)} \left( - B_1(x_{C+D\delta(n)}^{S_l(n)})B_1(x_{C+D\delta(n)}^{S_l(n)-1}) + \frac{a_{2l}(n) + 2}{2}B_2(x_{C+D\delta(n)}^{S_l(n)}) \right)$$

$$+ \sum_{l=0}^{s(n)-1} \sum_{i=1}^{S_{l+1}(n)-1} F(x_{C+D\delta(n)}^i, x_{C+D\delta(n)}^{i-1})$$

From the fact that $\{x_{C+D\delta(n)}^i\}_{S_l(n) \leq i \leq S_{l+1}(n)}$ is an arithmetic progression mod $\mathbb{Z}$ with difference $d_{CD}^l(r)$ for $n = qk + r$, we obtain the following:

If $1 \leq \gamma \leq q$ and $a_{2l+1}(n) \geq \gamma$ then

$$\sum_{i=S_l(n)+\gamma}^{S_{l+1}(n)+\gamma} F(x_{i}(n), x_{i-1}(n))$$

$$= \frac{1}{12} \left( 6(\gamma d_i(r)^2 + (1 - 2d_i(r))[\nu_{1/r}(r) + d_i(r)] + B_2(x_{S_l(n)+\gamma}(n)) - B_2(x_{S_l(n)}(n)) - \gamma \right)$$

Moreover if $\gamma = q$ then from the periodicity of $x_i(n)$ [See Proposition 5.1], we have the periodicity of the values of the 2nd Bernoulli polynomial:

$$B_2(x_{S_l(n)+q}(n)) = B_2(x_{S_l(n)}(n)).$$

Thus

(19) $$\sum_{i=S_l(n)+\gamma}^{S_{l+1}(n)+\gamma} F(x_{i}(n), x_{i-1}(n)) = \frac{1}{12} \left( 6(qd_i(r)^2 + (1 - 2d_i(r))[\nu_{1/r}(r) + d_i(r)] + \nu_{1/r}(r) + d_i(r)) \right) - q.$$  

We note that $Orb(C + D\delta(n))$ is a set depending only $C, D$ and $r$ for $n = qk + r$ [See Proposition 4.4] under the conditions of the following proposition. Thus for $n = qk + r$ we can define

$$Orb(C + D\delta(n)) =: Orb_{CD}(r).$$

**Proposition 5.4.** If $b_n^{-1} = [1, \delta(n)]$ and

$$\delta(n) - 1 = [[a_0(n), a_1(n), \ldots, a_{s-1}(n)]]$$

for $a_i(x) = \sum_{j=0}^{d} a_{ij}x^j \in \mathbb{Z}[x]$ and if $N((C + D\delta(n))b_n) \pmod{q}$ is a function only depending on $C, D$ and $r$ then we have for $n = qk + r$

$$\zeta_q((C + D\delta(n))b_n) = \sum_{(A,B) \in Orb_{CD}(r)} B_{0}^0(A,B)(r) + B_{1}^1(A,B)(r)k + \cdots B_{d}^d(A,B)(r)k^d,$$
where for \( m \geq 1 \),

\[
B^m_{AB}(r) = \frac{q}{2} \sum_{l=1}^{s\mu(s)} A_{2l,m}(r)B_2(\nu_{AB}^l(r))
\]

\[
+ \frac{1}{12} \sum_{l=0}^{s\mu(s)-1} A_{2l+1,m}(r) \left( 6(qd_{AB}^l(r))^2 + (1 - 2d_{AB}^l(r))[\nu_{AB}^l(r) + d_{AB}^l(r)] - q \right)
\]

and

\[
B^0_{AB}(r) = \sum_{l=1}^{s\mu(s)} -B_1(\nu_{AB}^l(r))B_1(\nu_{AB}^l(r)-1(r)) + \frac{q\tau_{2l}(r) + \gamma_{2l}(r) + 2}{2} B_2(\nu_{AB}^l(r))
\]

\[
+ \frac{1}{12} \sum_{l=0}^{s\mu(s)-1} \left[ \tau_{2l+1}(r) \left( 6(qd_{AB}^l(r))^2 + (1 - 2d_{AB}^l(r))[\nu_{AB}^l(r) + d_{AB}^l(r)] - q \right) + 6B_2(\nu_{AB}^l(r)-1(r)) - 6B_2(\nu_{AB}^l(r)) + \left( (\gamma_{2l+1}(r) - 1) + d_{AB}^l(r) \right) + (1 - 2d_{AB}^l(r))(\nu_{AB}^l(r) + d_{AB}^l(r)(\gamma_{2l+1}(r) - 1)) ] - \gamma_{2l+1}(r) + 1 \right].
\]

**Proof.** Thus from equation (18) and lemma 5.3 we obtain the following:

\[
\sum_{i=1}^{m(n)} \left( B_1(x_{A+B\delta(n)}^i)B_1(y_{A+B\delta(n)}^i) + \frac{b_i(n)}{2} B_2(x_{A+B\delta(n)}^i) \right) =
\]

\[
\sum_{l=1}^{s\mu(s)} \left( -B_1(x_{A+B\delta(n)}^{S_l(n)})B_1(x_{A+B\delta(n)}^{S_l(n)-1}) \right)
\]

\[
+ \frac{q \sum_{m=1}^{k^m} A_{2l,m}(r)k^m + q\tau_{2l}(r) + \gamma_{2l}(r) + 2}{2} B_2(x_{A+B\delta(n)}^{S_l(n)}) \right]
\]

\[
\sum_{l=0}^{s\mu(s)-1} \sum_{i=S_l(n)+1}^{S_l(n)+k^m} F(x_{A+B\delta(n)}^i, x_{A+B\delta(n)}^{i-1}) \right)
\]

Since \( \{F(x_{A+B\delta(n)}^i, x_{A+B\delta(n)}^{i-1})\}_{S_l(n)+1 \leq i \leq S_{l+1}(m) - 1} \) has period \( q \), we have the following:
From Proposition 2.3, we complete the proof of theorem 2.4. We note that

\[ S_i(n) + q \sum_{m=1}^{d} A_{2l+1,m} r^m + q \tau_{2l+1}(r) + \gamma_{2l+1}(r) - 1 \]

\[ \sum_{i=S_i(n)+1}^{S_i(n)+q} F(x^{i}_{A+B\delta(n)}, x^{i-1}_{A+B\delta(n)}) \]

\[ = \left( \sum_{m=1}^{d} A_{2l+1,m}(r) k^m + \tau_{2l+1}(r) \right) \sum_{i=S_i(n)+1}^{S_i(n)+q} F(x^{i}_{A+B\delta(n)}, x^{i-1}_{A+B\delta(n)}) \]

\[ + \sum_{i=S_i(n)+1}^{S_i(n)+\gamma_{2l+1}(r)-1} F(x^{i}_{A+B\delta(n)}, x^{i-1}_{A+B\delta(n)}) \]

We note that

\[ x^{S_i(n)}_{A+B\delta(n)} = v^{\Gamma_i(r)}_{AB}(r) \]

\[ x^{S_i(n)-1}_{A+B\delta(n)} = v^{\Gamma_i(r)-1}_{AB}(r) \]

\[ x^{S_i(n)+\gamma_{2l+1}(r)-1}_{A+B\delta(n)} = v^{\Gamma_i+1(r)-1}_{AB}(r) \]

Form (19), (20) and (23) - (25), we find that (22) is equal to the following:

\[ \left[ \sum_{m=1}^{d} A_{2l+1,m}(r) k^m + \tau_{2l+1}(r) \right] \cdot \]

\[ \left[ 6(qd^l_{AB}(r))^2 + (1 - 2d^l_{AB}(r))[v^{\Gamma_i(r)}_{AB}(r) + d^l_{AB}(r)q]_1 - q \right] \]

\[ + 6B_2(v^{\Gamma_i(r)-1}_{AB}(r)) - 6B_2(v^{\Gamma_i(r)}_{AB}(r)) + 6(\gamma_{2l+1}(r) - 1)d^l_{AB}(r)^2 \]

\[ + (1 - 2d^l_{AB}(r))[v^{\Gamma_i(r)}_{AB}(r) + d^l_{AB}(r)(\gamma_{2l+1}(r) - 1)]_1 - \gamma_{2l+1}(r) + 1 \]

\[ \square \]

Proof of Theorem 2.4

We note that $v^{\Gamma_i(r)}_{AB}(r)$, $v^{\Gamma_i(r)-1}_{AB}(r)$ and $d^l_{AB}(r) \in \frac{1}{q} \mathbb{Z}$. Thus for $i = 0, 1, \cdots, d,$

\[ B^i_{AB}(r) \in \frac{1}{12q^2} \mathbb{Z}. \]

From Proposition 2.3, we complete the proof of theorem 2.4.
6. TWO EXAMPLES

We present here two examples of \((K_n, b_n)\) showing polynomial behaviour of \(\zeta_q(0, b_n)\). For both examples, we take \(b_n = (C + D\delta(n))O_{K_n}\), for \(C + D\delta(n) \in F_{\delta(n)}\). The first one is a family of real quadratic fields already appeared in a literature dealing the associated class number one problem. The second family is given by a quartic polynomial.

6.1. Case 1: \(f(n) = n^2 + 2\). The following example is one of so-called Richard-Degert type. The quasi-linearity was studied in [11] to solve the class number one problem for the family. From the computation of partial zeta values for ray class in this article, one can recover exactly the result in loc.cit. for partial Hecke’s L-values associated to a mod-q Dirichlet character \(\chi\).

For square free \(f(n) = n^2 + 2\), let \(K_n = \mathbb{Q}((\sqrt{f(n)}). We fix \(b_n = O_{K_n}\) the ring of integers in \(K\). \(O_{K_n} = [1, \delta(n)]\) and

- \(\delta(n) = \sqrt{f(n)} + n + 1\).
- The continued fraction of \(\delta(n) - 1\) is \([2n, n]\).
- The totally positive fundamental unit \(\epsilon_n\) is \(n^2 + 1 + n\sqrt{f(n)}\).

One sees that \(Orb(C + D\delta(n))\) depends only on \(C, D\) and \(r\). For the sake of simplicity and from the periodicity of the orbit, we may well denote \(Orb(C + D\delta(n))\) by \(Orb_{CD}(r)\).

Now we can express the partial zeta values at \(s = 0\):

\[
\zeta_q(0, (C + D\delta(n))O_{K_n}) = A_0(r) + A_1(r)k + \ldots A_d(r)k^d,
\]

where

\[
A_i(r) = \sum_{(A,B) \in Orb_{CD}(r)} B_{AB}^i(r),
\]
for

\[ B_{AB}^0(r) = \left( \frac{1}{2} - \nu_{AB}^\Gamma_{(r)}(r) \right) \left( \nu_{AB}^\Gamma_{(r)-1}(r) - \frac{1}{2} \right) + \frac{r + 1}{6} (6\nu_{AB}^\Gamma_{(r)}(r)^2 - 6\nu_{AB}^\Gamma_{(r)}(r) + 1) \]

\[ + \frac{\tau_1(r)}{12} (6q_d_{AB}^0(r)^2 - 6q_d_{AB}^0(r) + 12q_d_{AB}^0(r)^2 - q) \]

\[ + \frac{1}{12} (6\nu_{AB}^\Gamma_{(r)-1}(r)^2 - 6\nu_{AB}^\Gamma_{(r)-1}(r) + 1) - \frac{1}{12} (6\nu_{AB}^0(r)^2 - 6\nu_{AB}^0(r) + 1) \]

\[ + \frac{1}{2} \gamma_1(r) - 1) d_{AB}^0(r)^2 + \frac{1 - 2d_{AB}^0(r)}{2} (d_{AB}^0(r) \gamma_1(r) - d_{AB}^0(r)) \]

\[ + \nu_{AB}^0(r) - \nu_{AB}^\Gamma_{(r)}(r) - \frac{\gamma_1(r) - 1}{12} , \]

\[ B_{AB}^1(r) = \frac{q^2}{6} (6\nu_{AB}^\Gamma_{(r)}(r)^2 - 6\nu_{AB}^\Gamma_{(r)}(r) + 1) \]

\[ + \frac{q}{12} (6q_d_{AB}^0(r)^2 + 6q_d_{AB}^0(r) - 12q_d_{AB}^0(r)^2 - q) \]

and

\[ d_{AB}^0(r) = \frac{\langle (2r + 1)B + A \rangle_q}{q} \]

\[ \nu_{AB}^0(r) = \frac{\langle B \rangle_q}{q} \]

\[ \nu_{AB}^\Gamma_{(r)}(r) = \frac{\langle (2r^2 - r)B + (r - 1)A \rangle_q}{q} \]

\[ \nu_{AB}^\Gamma_{(r)-1}(r) = \frac{\langle (2r^2 - 3r - 1)B + (r - 2)A \rangle_q}{q} \]

\[ \gamma_1(r) = \frac{\langle 2r + 1 \rangle_q}{q} \]

\[ \tau_1(r) = \frac{2r + 1 - (2r + 1)q}{q} . \]

6.2. Case 2: \( f(n) = 16n^4 + 32n^3 + 24n + 3 \). For square free \( f(n) = 16n^4 + 32n^3 + 24n^2 + 12n + 3 \), let \( K_n := \mathbb{Q}(\sqrt{f(n)}) \). Let us fix \( b_n = O_{K_n} \).

If \( O_{K_n} \) is \([1, \delta(n)]\) for \( \delta(n) \) described as before. For this family, we have:

- \( \delta(n) = \sqrt{f(n)} + \lfloor \sqrt{f(n)} \rfloor + 1 \),
- \( \delta(n) - 1 = \lfloor 8n^2 + 8n + 2, 2n + 1 \rfloor \).
- The totally positive fundamental unit \( \epsilon_n \) is \((2n + 1)^3 + 1 + (2n + 1)\sqrt{f(n)} \).
We can again easily check that \( N(C + D\delta(n)) \) is invariant modulo \( q \) for \( n = qk + r \).

For \( C + D\delta(n) \in F_{\delta(n)} \), we have

\[
\varepsilon_n * (C_i(n) + D_i(n)\delta(n)) = C_{i+1}(n) + D_{i+1}(n)\delta(n),
\]

where

\[
C_q(n) = C, \quad D_q(n) = D
\]

\[
C_{i+1}(n) = ((64n^5 + 160n^4 + 168n^3 + 104n^2 + 38n + 7)D_1(n)
+ (8n^3 + 12n^2 + 6n + 2)C_i(n))_q
\]

\[
D_{i+1}(n) = \left((16n^3 + 24n^2 + 14n + 4)D_1(n) + (2n + 1)C_i(n)\right)_q
\]

Let \( n = qk + r \) for \( 0 \leq r < q \). Denoting \( \text{Orb}(C + D\delta(n)) \) by \( \text{Orb}_{CD}(r) \), the partial zeta value at 0 is

\[
\zeta_q(0, (C + D\delta(n))\Omega_{K_n}) = A_0(r) + A_1(r)k + \cdots + A_d(r)k^d,
\]

where \( A_i(r) = \sum_{(A,B) \in \text{Orb}_{CD}(r)} B_{AB}^i(r) \), for

\[
B_{AB}^0(r) = \left( \frac{1}{2} - v_{AB}^{\Gamma_1(r)}(r) \right) \left( v_{AB}^{\Gamma_1(r)-1}(r) - \frac{1}{2} \right)
+ \frac{8r^2 + 8r + 6}{12} \left( 6v_{AB}^{\Gamma_1(r)}(r)^2 - 6v_{AB}^{\Gamma_1(r)}(r) + 1 \right)
+ \frac{\tau_1(r)}{12} \left( 6qd_{AB}^0(r)^2 - 6qd_{AB}^0(r) + 12qd_{AB}^0(r)^2 - q \right)
+ \frac{1}{12} \left( 6v_{AB}^{\Gamma_1(r)-1}(r)^2 - 6v_{AB}^{\Gamma_1(r)-1}(r) + 1 \right)
- \frac{1}{12} \left( 6v_{AB}^0(r)^2 - 6v_{AB}^0(r) + 1 \right)
+ \frac{1 - 2d_{AB}^0(r)}{2} \left( d_{AB}^0(r)\gamma_1(r) - d_{AB}^0(r) + v_{AB}^0(r) - v_{AB}^{\Gamma_1(r)}(r) \right)
- \frac{\gamma_1(r) - 1}{12},
\]

\[
B_{AB}^1(r) = \frac{2q^2 + 4q^2r}{3} \left( 6v_{AB}^{\Gamma_1(r)}(r)^2 - 6v_{AB}^{\Gamma_1(r)}(r) + 1 \right)
+ \frac{q}{6} \left( 6qd_{AB}^0(r)^2 + 6qd_{AB}^0(r) - 12qd_{AB}^0(r)^2 - q \right),
\]

\[
B_{AB}^2(r) = \frac{2q^3}{3} \left( 6v_{AB}^{\Gamma_1(r)}(r)^2 - 6v_{AB}^{\Gamma_1(r)}(r) + 1 \right),
\]
and

\[ d_{AB}^0(r) = \frac{\langle (8r^2 + 8r + 3)B + A \rangle_q}{q} \]
\[ v_{AB}^0(r) = \frac{\langle B \rangle_q}{q} \]
\[ v_{AB}^{\Gamma_1(r)}(r) = \frac{\langle (4r^2 + 2r + 1)B + 2rA \rangle_q}{q} \]
\[ v_{AB}^{\Gamma_1(r)-1}(r) = \frac{\langle 4r^2B + (2r - 1)A \rangle_q}{q} \]
\[ \gamma_1(r) = \langle 2r + 1 \rangle_q \]
\[ \tau_1(r) = \frac{2r + 1 - \langle 2r + 1 \rangle_q}{q} \]

**Remark 6.1.** The family of [6.2] has not been touched in literature in the context of class number problem or other particular problems in arithmetic. Especially, most known families of real quadratic fields, where class number problems are solved, are Richaud-Decherd type which is generated by some quadratic polynomials. It would be highly interesting if one could answer those questions for other types of families than R-D types.

**References**

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