Connected sums with $\mathbb{H}P^n$ or $CaP^2$ and the Yamabe invariant

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Abstract

Let $M$ be a $4k$-manifold whose Yamabe invariant $Y(M)$ is non-positive. We show that

$$Y(M^l \ # \ l \mathbb{H}P^k \ # \ m \mathbb{H}P^k) = Y(M),$$

where $l, m$ are nonnegative integers, and $\mathbb{H}P^k$ is a quaternionic projective space. When $k = 4$, we also have

$$Y(M^l \ # \ l \mathbb{CaP}^2 \ # \ m \mathbb{CaP}^2) = Y(M),$$

where $CaP^2$ is a Cayley plane.

1 Introduction

The Yamabe invariant is an invariant of a smooth closed manifold defined using the scalar curvature. Let $M$ be a closed smooth $n$-manifold. By the well-known solution of the Yamabe problem, each conformal class of a smooth Riemannian metric on $M$ contains a so-called Yamabe metric which has constant scalar curvature. Moreover, letting

$$[g] = \{\varphi g \mid \varphi : M \to \mathbb{R}^+ \text{ is smooth}\}$$

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be the conformal class of a Riemannian metric $g$, a Yamabe metric of $[g]$ actually realizes

$$Y(M, [g]) := \inf_{\tilde{g} \in [g]} \frac{\int_M s_{\tilde{g}} \, dV_{\tilde{g}}}{\left( \int_M dV_{\tilde{g}} \right)^{\frac{n}{n-2}},}$$

where $s_{\tilde{g}}$ and $dV_{\tilde{g}}$ respectively denote the scalar curvature and the volume element of $\tilde{g}$. The value $Y(M, [g])$, which is the value of the scalar curvature of a Yamabe metric with the total volume 1 is the Yamabe constant of the conformal class.

In a quest of a “best” Yamabe metric or more ambitiously a “canonical” metric on $M$, one naturally takes the supremum of the Yamabe constants over the set of all conformal classes on $M$. This is possible because by Aubin’s theorem [2], the Yamabe constant of any conformal class on any $n$-manifold is always bounded by that of the unit $n$-sphere $S^n(1) \subset \mathbb{R}^{n+1}$, which is $n(n-1)(\text{Vol}(S^n(1)))^{2/n}$. The Yamabe invariant of $M$, $Y(M)$, is then defined as the supremum of the Yamabe constants over the set of all conformal classes on $M$. This supremum is not always attained, but if it is attained by a metric which is the unique Yamabe metric with total volume 1 in its conformal class, then the metric has to be an Einstein metric. In general, one can hope a singular or degenerate Einstein metric leading to a kind of a “geometrization” from a maximizing sequence of Yamabe metrics. It is also noteworthy that the Yamabe invariant is a topological invariant of a closed manifold depending only on the smooth structure of the manifold.

The Yamabe invariant of a compact orientable surfaces is just $4\pi \chi(M)$ where $\chi(M)$ denotes the Euler characteristic of $M$ by the Gauss-Bonnet theorem. In higher dimensions, it is not an easy task to compute the Yamabe invariant. Nevertheless recently there have been much progresses in dimension 3 and 4. In dimension 3, the geometrization by the Ricci flow gives a lot of answers, and in dimension 4, the Spin$^c$ structure and the Dirac operator are keys for computing the Yamabe invariant. In particular, LeBrun [7, 8] showed that if $M$ is a compact Kähler surface whose Kodaira dimension is not equal to $-\infty$, then

$$Y(M) = -4\sqrt{2\pi} \sqrt{(2\chi + 3\sigma)(\tilde{M})},$$

where $\sigma$ denotes the signature and $\tilde{M}$ is the minimal model of $M$. Now based on this evidence, one can ask if the blowing-up does not change the Yamabe invariant of a closed orientable 4-manifold with nonpositive Yamabe invariant, namely
**Question 1.1** Let $M$ be a closed orientable 4-manifold with $Y(M) \leq 0$. Is there an orientation of $M$ such that $Y(M \# l \mathbb{CP}^2) = Y(M)$ for any integer $l > 0$? What about in higher dimensions?

Further one can also ask whether the analogous statement holds true for the “quaternionic blow-up”, i.e. a connected sum with a quaternionic projective space $\mathbb{HP}^n$, or even a connected sum with a Cayley plane $\mathcal{C}P^2$. The purpose of this paper is to prove an affirmative answer to this:

**Theorem 1.2** Let $M$ be a closed $4k$-manifold with $Y(M) \leq 0$. Then

$$Y(M \# l \mathbb{HP}^k \# m \mathbb{HP}^k) = Y(M),$$

where $l, m$ are nonnegative integers. When $k = 4$, we also have

$$Y(M \# l \mathcal{C}P^2 \# m \mathcal{C}P^2) = Y(M).$$

2 **Preliminaries**

A computationally useful formula for the Yamabe constant is

$$|Y(M, [g])| = \inf_{\tilde{g} \in [g]} \left( \int_M |s_{\tilde{g}}|^{\frac{n}{2}} d\mu_{\tilde{g}} \right)^{\frac{2}{n}},$$

where the infimum is attained only by a Yamabe metric. (For a proof, see [8, 13].) So when $Y(M, [g]) \leq 0$, this implies that

$$Y(M, [g]) = -\inf_{\tilde{g} \in [g]} \left( \int_M |s_{\tilde{g}}|^{\frac{n}{2}} d\mu_{\tilde{g}} \right)^{\frac{2}{n}},$$

where $s_{\tilde{g}}$ is defined as $\min\{s_{\tilde{g}}, 0\}$. Therefore when $Y(M) \leq 0$,

$$Y(M) = -\inf_{g} \left( \int_M |s_g|^{\frac{n}{2}} d\mu_g \right)^{\frac{2}{n}} = -\inf_{g} \left( \int_M |s^{-}_{g}|^{\frac{n}{2}} d\mu_g \right)^{\frac{2}{n}}.$$  \hspace{1cm} (1)

Also essential is Kobayashi’s connected sum formula [6, 12].

$$Y(M_1 \# M_2) \geq \begin{cases} -((Y(M_1))^{\frac{n}{2}} + (Y(M_2))^{\frac{n}{2}})^{\frac{2}{n}} & \text{if } Y(M_i) \leq 0 \forall i \\ \min(Y(M_1), Y(M_2)) & \text{otherwise.} \end{cases}$$

We also need to know about the geometry and topology of $\mathbb{HP}^k$ and $\mathcal{C}P^2$. Both has the homogeneous Einstein metric of positive scalar curvature unique
up to constant and can be viewed as the mapping cones of the (generalized) Hopf fibrations \( \pi_1 : S^{4k-1} \to \mathbb{H}P^{k-1} \) with \( S^3 \) fibers and \( \pi_2 : S^{15} \to S^8 \) with \( S^7 \) fibers respectively.

These fibrations have the associated geometries of Riemannian submersion with totally geodesic fibers. In case of \( \pi_1 \), \( S^{4k-1} \) and \( S^3 \) are endowed with the round metric of constant curvature 1, and \( \mathbb{H}P^{k-1} \) is given the homogeneous Einstein metric with curvature ranging between 1 and 4. In case of \( \pi_2 \), the total space and the fibers have the round metric of curvature 1, but the base has the round metric of curvature 4.

We will denote the round \( n \)-sphere with the metric of constant curvature 1 by \( S^n(a) \), i.e. the sphere of radius \( a \) in the Euclidean \( \mathbb{R}^{n+1} \).

### 3 Proof of Theorem

It’s enough to prove for one connected sum. Let \( M' \) be \( M^\sharp \mathbb{H}P^k \) or \( M^\sharp \overline{\mathbb{H}P^k} \), and set \( n = 4k \). First recall that \( \mathbb{H}P^k \) admits a metric of positive scalar curvature meaning that \( Y(\mathbb{H}P^k) > 0 \). Thus by the connected sum formula, \( Y(M') \geq Y(M) \). The idea of the proof is to surger out an \( \mathbb{H}P^{k-1} \) in \( M' \) by performing the Gromov-Lawson surgery \(^4\) to get back \( M \) without decreasing the Yamabe constant much.

To prove by contradiction, let’s assume \( Y(M') > Y(M) + 2c > Y(M) \) for a constant \( c > 0 \) such that \( c \) satisfies \( c < \frac{|Y(M)|}{2} \) if \( Y(M) < 0 \). Let \( g \) be an unit-volume Yamabe metric on \( M' \) with scalar curvature \( s_g = Y(M', [g]) = Y(M) + 2c \). Let \( W \) be an \( \mathbb{H}P^{k-1} \) embedded in \( M' \). Take a \( \delta \)-tubular neighborhood \( N(\delta) = \{ x \in M' | dist_g(x, W) < \delta \} \) of \( W \) for \( \delta > 0 \). We will take \( \delta \) small enough so that \( N(\delta) \) is diffeomorphic to \( \mathbb{H}P^k \) and the boundary of \( N(\delta) \) is diffeomorphic to \( S^{4k-1} \).

We perform a Gromov-Lawson surgery described in \(^{11}\)\(^{12}\) on \( N(\delta) \) along \( W \) keeping the scalar curvature bigger than \( s_g - c \) to get a cylindrical end isometric to \( (S^{4k-1} \times [0, 1], \hat{g} + dt^2) \), where \( (S^{4k-1}, \hat{g}) \) is a Riemannian submersion onto \( (W, g_W = g|_W) \) with totally geodesic fibers isometric to \( S^3(\varepsilon) \), the round 3-sphere of radius \( \varepsilon \ll 1 \). Here, the horizontal distribution is given by the connections on the normal bundle. By arranging \( \varepsilon \) sufficiently small, \( \hat{g} \) has positive scalar curvature. Moreover the volume of the deformed metric can be made arbitrarily small, say \( \nu \ll 1 \). (For a proof, one may refer to \(^{12}\). Also a different method bypassing this is given in the remark below.)

Now let’s take a homotopy \( H_\delta(t) = \lambda(t)g_W + (1 - \lambda(t))g_{std} \) of smooth
metrics on $W$ from $g_W$ to the homogeneous Einstein metric $g_{std}$ of $\mathbb{H}P^{k-1}$ with curvature ranging from 1 to 4, where $\lambda : [0, 1] \to [0, 1]$ is a smooth decreasing function with the property that it is 1 for $t$ near 0 and 0 near 1. This induces a homotopy $H_1(t)$ of smooth metrics on $S^{4k-1}$ through a Riemannian submersion with totally geodesic fibers $S^3(\varepsilon)$. And then we homotope the horizontal distribution to that of the Hopf fibration through a Riemannian submersion with totally geodesic fibers $S^3(\varepsilon)$. Let’s denote this homotopy on $S^{4k-1}$ be $H_2(t)$ for $t \in [1, 2]$. When $\varepsilon$ is sufficiently small, $H_1(t) + dt^2$ and $H_2(t) + dt^2$ will give a metric of positive scalar curvature on $S^{4k-1} \times [0, 2]$, because it is a Riemannian submersion with totally geodesic fibers onto $\mathbb{H}P^{k-1} \times [0, 2]$. We concatenate this part to the above one obtained from the Gromov-Lawson surgery to get a smooth metric with the boundary isometric to the squashed sphere $S^{4k-1} \times [0, 2]$. We concatenate this part to the above one obtained from the Gromov-Lawson surgery to get a smooth metric with the boundary isometric to the squashed sphere $S^{4k-1}$ coming from the Hopf fibration. Let’s denote this metric on the boundary by $h_\varepsilon$ for a later purpose.

We want to close it up by a 4$^k$-ball $B^{4^k}$ equipped with a metric of positive scalar curvature. To construct such a metric we resort to the Gromov-Lawson surgery again. Take a sphere $S^{4k}$ with any metric of positive scalar curvature and let $p$ be any point on it. As before, we perform a Gromov-Lawson surgery in a sufficiently small neighborhood of $p$ to get a 4$^k$-ball with the positive scalar curvature and the cylindrical end isometric to $S^{4k-1}(\varepsilon') \times [0, 1]$ for a $\varepsilon' > 0$. And then we take a homothety of the whole thing by $\frac{1}{\varepsilon'}$, so that the boundary is isometric to the round sphere $(S^{4k-1}(1), h_1)$. In order to glue this to the above obtained part, we have to homotope the metric on the boundary. We take a homotopy $H_3(t) = \lambda(t)h_1 + (1 - \lambda(t))h_\varepsilon$ for $t \in [0, 1]$.

**Lemma 3.1** The metric $H_3(t)$ on $S^{4k-1}$ has positive scalar curvature for every $t \in [0, 1]$.

**Proof.** Note that $h_1$ and $h_\varepsilon$ differ only by the size of the Hopf fiber. So for each $t$, $H_3(t)$ also has the same Riemannian submersion structure with the fiber isometric to the round 3-sphere of radius $r(t) := \lambda(t) + (1 - \lambda(t))\varepsilon$. By the O’Neill’s formula [3],

$$s_{H_3(t)} = \frac{1}{r^2(t)} s_f + s_b \circ \pi - r^2(t)|A|^2,$$

where $s_f$, $s_b$, and $A$ denote the scalar curvature of the fiber and the base, and the integrability tensor for $t = 0$ respectively. Thus $s_{H_3(t)}$
is constant for each $t$ and increases as $t$ increases. From the fact that $s_{H_3(0)} \equiv (4k - 1)(4k - 2) > 0$, the result follows.

Nevertheless the metric $H_3(t) + dt^2$ on $S^{4k-1} \times [0, 1]$ may not have positive scalar curvature in general. But due to Gromov and Lawson’s lemma in [4], for a sufficiently large constant $L > 0$, $H_3(t) + dt^2$ on $S^{4k-1} \times [0, L]$ has positive scalar curvature. Now we have a desired $4k$-ball to be glued to the part made previously out of $M'$.

After the gluing, what we get is just $M$ with a specially devised smooth metric which we denote by $\bar{g}$. Remember that the scalar curvature of $\bar{g}$ is bigger than $s_g - c$.

Now we will derive a contradiction. In case that $Y(M) = 0$,

$$s_{\bar{g}} > s_g - c = Y(M) + c > Y(M) = 0,$$

which is a contradiction. In case of $Y(M) < 0$, we do the surgery so that $\nu^2 < \frac{2c}{Y(M) + c}$. Then noting that $s_g < 0$,

$$-(\int_M |s_{\bar{g}}|^{\frac{n}{2}}d\mu_{\bar{g}})^{\frac{2}{n}} > -(\int_{M' - N(\delta)} |s_g|^{\frac{n}{2}}d\mu_g + |s_g - c|^{\frac{n}{2}}\nu)^{\frac{2}{n}}$$

$$> -(\int_{M'} |s_g|^{\frac{n}{2}}d\mu_g)^{\frac{2}{n}} + (s_g - c)^{\frac{2}{n}}$$

$$= Y(M', [g]) + (Y(M) + c)^{\frac{2}{n}}$$

$$> (Y(M) + 2c) - 2c$$

$$= Y(M).$$

This gives a contradiction to the formula (1), and completes a proof for the $\mathbb{H}P^k$ case.

The case of $CaP^2$ can be proved in the same way using the fact that $CaP^2$ also admits a metric of positive scalar curvature, and is the mapping cone of the (generalized) Hopf fibration $\pi : S^{15} \to S^8$ with $S^7$ fibers as explained in the previous section.

**Remark**

Since the smallness of $\nu$ was used only in the case of $Y(M) < 0$, we will show a way of proof without using it when $Y(M) < 0$. As done in LeBrun [9], instead of doing surgery on $(N(\delta), g)$, we first take a conformal change $\varphi g$ of $(M', g)$ such that $\varphi \equiv 1$ outside $N(\delta)$ and the scalar curvature of $\varphi g$
is positive on a much smaller neighborhood $N(\delta')$ of $W$. Moreover one can arrange that it satisfies

$$-(\int_{M'} |s_{\varphi g}|^{\frac{2}{n}} d\mu_{\varphi g})^\frac{2}{n} > -(\int_{M'} |s_g|^{\frac{2}{n}} d\mu_g)^\frac{2}{n} - \epsilon$$

for any $\epsilon > 0$. (This is possible because the codimension of $W$ is $\geq 3$.) Let’s just say $\epsilon < c$. Then we perform a Gromov-Lawson surgery on $(N(\delta'), \varphi g)$ keeping the scalar curvature positive. The rest is the same and finally we get

$$-(\int_{M} |s_{\bar{g}}|^{\frac{2}{n}} d\mu_{\bar{g}})^\frac{2}{n} = -(\int_{M'} |s_{\varphi g}|^{\frac{2}{n}} d\mu_{\varphi g})^\frac{2}{n}$$

$$> (Y(M) + 2c) - c$$

$$> Y(M).$$

\[ \square \]

4 Example and Final remark

Obviously the theorem is vacuous for the case of $\mathbb{H}P^1$ which is diffeomorphic to $S^4$.

Example

Let $H$ be a closed Hadamard-Cartan manifold, i.e. one with a metric of nonpositive sectional curvature. By the well-known theorem of Gromov and Lawson [5] on the enlargeable manifolds, $H$ cannot carry a metric with positive scalar curvature. Therefore $Y(H) \leq 0$. Applying our theorem to $H$, one has

$$Y(H_\# \mathbb{H}P^k \# m \bar{\mathbb{H}P^k}) = Y(H).$$

For a specific example, take $M = T^n \times H$, where $T^n$ is an $n$-dimensional torus and $H$ is as above, e.g. a product of closed real hyperbolic manifolds. Now since $M$ has an obvious $F$-structure, its Yamabe invariant is actually 0 by collapsing the $T^n$-part. (Refer to Paternain and Petean [10].) Thus

$$Y(M_\# \mathbb{H}P^k \# m \bar{\mathbb{H}P^k}) = 0.$$  

Similar examples can also be constructed for $CaP^2$. \[ \diamond \]
Going back to the question 1.1 addressed in the introduction, our argument does not apply to the case of complex projective space $\mathbb{C}P^k$. We still have the fact that $\mathbb{C}P^k$ is the mapping cone of the Hopf fibration $\pi : S^{2k-1} \to \mathbb{C}P^{k-1}$ with $S^1$ fibers. So the $\mathbb{C}P^{k-1}$ is embedded as a submanifold of codimension 2 which is one less for the Gromov-Lawson surgery to work. Moreover the statement corresponding to the theorem 1.2 can not be true at least in dimension 4. This is because of Wall’s stabilization theorem [14]. Let $M$ be a simply-connected closed smooth 4-manifold. Then there exists integers $l, m$ such that

$$M \# l \mathbb{C}P^2 \# m \mathbb{C}P^2 = a \mathbb{C}P^2 \# b \mathbb{C}P^2,$$

where $a = l + \frac{1}{2}(b_2(M) + \sigma(M))$ and $b = m + \frac{1}{2}(b_2(M) - \sigma(M))$. But we know that $Y(a \mathbb{C}P^2 \# b \mathbb{C}P^2) > 0$. Thus the Yamabe invariant changes drastically by taking connected sums with both $\mathbb{C}P^2$ and $\mathbb{C}P^2$. We do not know whether the stabilization phenomenon of the Yamabe invariant is prevalent also in higher dimensions. But at least the question 1.1 is worth investigating in dimension both 4 and higher.

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