Weak $E_2$-Morita equivalences via quantization of the 1-shifted cotangent bundle

Márton Hablicsek

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In this paper we investigate the structure of the convergent quantization of the 1-shifted cotangent bundle $S$ of a smooth scheme $X$ over a perfect field of positive characteristic. The quantization is an $E_2$-algebra over the Frobenius twist $S'$ of the 1-shifted cotangent bundle which restricted to the zero section $X' \rightarrow S'$ is weakly $E_2$-Morita equivalent to the structure sheaf $\mathcal{O}_{X'}$ of the Frobenius twist $X'$ of $X$.

Explicitely, we show that the $(\infty,2)$-category of coherent (left-)modules over $\mathcal{O}_{X'}$ is equivalent to the full subcategory of the $(\infty,2)$-category of coherent category of (left-)modules over the quantization restricted to the zero section are equivalent.

1. Introduction

1.1. Deformation quantizations have interesting features over fields of positive characteristic. For instance, the Poisson-center of a Poisson-variety $(S,\{,\})$ is large: for any local sections $f$ and $g$ of $\mathcal{O}_S$ we have $\{f^p, g\} = 0$. Moreover, any convergent (central) quantization $\mathcal{A}$ of $\mathcal{O}_S$ is an Azumaya algebra (see [4] for more details). In the simplest case, when $S = T^*X$ is the cotangent bundle of a smooth variety $X$ equipped with its natural symplectic structure, the convergent quantization is the sheaf of rings of crystalline differential operators, $\mathcal{D}$. It is an Azumaya algebra over the Frobenius twist $S' = T^*X'$. Moreover it is naturally a trivial Azumaya algebra restricted to the zero section $X' \rightarrow T^*X' = S'$ of the Frobenius twist of the cotangent bundle (see [5]).

*University of Copenhagen, e-mail: mhablicsek@math.ku.dk
Recently, a new approach to deformation quantization was introduced by Calaque, Pantev, Toën, Vaquié and Vezzosi ([13, 6]) and separately by Pridham ([14, 15]) to study quantizations in the context of derived algebraic geometry. The input of the quantization problem is an $n$-shifted symplectic derived stack $S$, the output is a sheaf of $BD_{n+1}$-algebras (deforming the structure sheaf of $S$).

Consider the simplest, non-trivial case, the $-1$-shifted cotangent bundle $T^*[-1]X$ of a smooth variety $X$. The $-1$-shifted cotangent bundle is a derived scheme whose convergent quantization using the Costello-Li framework ([8]) is a Batalin-Vilkovisky algebra structure on the structure sheaf of the $T^*[-1]X$. In our case it is the $\mathcal{O}_X$-linear dual of the de Rham complex of $X$ where the Batalin-Vilkovisky-differential is basically given by the de Rham differential (see [2] and [3] for more details). In positive characteristic, the de Rham complex is naturally a complex of $\mathcal{O}_X'$-module, hence naturally a complex over the Frobenius twist $X'$. The Cartier isomorphisms identify the cohomology sheaves of the de Rham complex: the $d$-th cohomology sheaf is the sheaf of algebraic $d$-forms on $X'$. Moreover, the (dual of) de Rham complex of $X$ can be realized as a line bundle on the Frobenius twist of the $-1$-shifted cotangent bundle, and this line bundle becomes a trivial line bundle once restricted to the zero section $X' \to T^*[-1]X'$ (see [1], [11] for more details).

We summarize the discussion above as follows.

| Shift ($n$) | Structure of the quantization over the Frobenius twist | Structure over the zero section $X' \to T^*[n]X'$ |
|-------------|------------------------------------------------------|-------------------------------------------------|
| $-1$        | Line bundle                                          | Trivial                                         |
| 0           | Azumaya algebra                                      | Trivial                                         |

From the table we can see that the structure of the quantization gets more interesting as $n$ increases. It is natural to ask what kind of structure the convergent quantization of a $1$-shifted symplectic derived stack has. The natural answer to this question would be that the quantization is an $E_{n+1}$-Azumaya algebra over the Frobenius twist, which becomes trivial (an $E_{n+1}$-Morita equivalence) once we restrict it to the zero section $X' \to T^*[n]X'$. The purpose of this paper is to investigate the structure of the quantization of $T^*[1]X$ restricted to the zero section $X' \to T^*[n]X'$.

The notion of $E_n$-Morita equivalence of algebras was recently defined by Haugseng ([12]) (or for the “pointed” version, see [9], [16]). For nice enough $(\infty, 1)$-category $\mathcal{C}$ the $E_n$-algebras over $\mathcal{C}$ form an $(\infty, n+1)$-category. This
category can be described roughly as follows: objects are the $E_n$-algebras, 1-morphisms are $E_{n-1}$ algebras in bimodules, 2-morphisms are $E_{n-2}$-algebras in bimodules over the bimodules, etc. This notion recovers the standard Morita 2-category whose objects are associative algebras, 1-morphisms are bimodules and 2-morphisms are morphisms of bimodules (see \[1\] for Azumaya schemes and \[19\] for derived schemes).

1.6. Explicitly two $E_2$-algebras $A$ and $B$ are $E_2$-Morita equivalent if there exist algebras $N$ and $M$ in $A - B$ and $B - A$ bimodules so that $N \otimes_A^L M$ is $E_1$-Morita equivalent to $B$ and $M \otimes_B^E N$ is $E_1$-Morita equivalent to $A$. It is easy to see that if $A$ is commutative algebra, and $B$ is $E_2$-Morita equivalent to $A$, then $B$ has to be perfect as an $A$-module.

1.7. In our set-up, we consider the 1-shifted cotangent bundle $T^*[1]X$ of a smooth scheme $X$ over a perfect field of characteristic $p$. The quantization of $\mathcal{O}_{T^*[1]X}$ is a variant of the Hochschild cosimplicial complex, which we call the crystalline Hochschild cosimplicial complex. Its restriction to the zero section $X' \to T^*[1]X'$ is the $\mathcal{O}_{X'}$-linear Hochschild cosimplicial complex of $\mathcal{O}_{X}$, which we denote by $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$.

A quick consequence of the above paragraph is that $\mathcal{O}_{X'}$ and $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$ are NOT $E_2$-Morita equivalent as $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$ is not perfect as a $\mathcal{O}_{X'}$-complex.

1.8. On the other hand, we show that the $(\infty, 2)$-category of coherent (left-)modules over $\mathcal{O}_{X'}$ is equivalent to a full subcategory of the $(\infty, 2)$-category of coherent (left-)modules over $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$ generated by $\mathcal{O}_{X}$. Explicitly, we show first that the algebra $\mathcal{O}_{X}$ can be realized as a module over $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$. This construction provides functors $F$ and $G$ between the coherent category of (left-)modules over $\mathcal{O}_{X'}$ and over $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$ as follows:

$$F : \text{Coh}(\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)) \to \text{Coh}(\mathcal{O}_{X'}) \quad F(-) = \mathcal{O}_X \otimes_{\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)} -$$

$$G : \text{Coh}(\mathcal{O}_{X'}) \to \text{Coh}(\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)) \quad G(-) = \mathcal{O}_X \otimes_{\mathcal{O}_{X'}} -$$. 

We prove the following theorem relating the coherent categories of (left-)modules over $\mathcal{O}_{X'}$ and $\mathcal{D}\mathcal{iff}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, \mathcal{O}_X)$.

1.9. **Theorem.** The following two $(\infty, 2)$-categories

- the coherent category of (left-)modules over $\mathcal{O}_{X'}$, where we consider $\mathcal{O}_{X'}$ as an $E_2$-algebra and

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\[1\] Here we regard $\mathcal{O}_{X}$ as a $\mathcal{O}_{X'}$-algebra using the Frobenius morphism $F : X \to X'$. We abuse notation and we will write $\mathcal{O}_{X}$ instead of $F_\ast \mathcal{O}_{X}$.
• the full thick subcategory of the coherent category of (left-)modules over $\text{Diff}_{O_X}(O_{X'}, O_X)$ generated by $O_X$

are equivalent.

Explicitly, we show that for any algebra object $A$ in the coherent categories $\text{Coh}(O_X')$ we have that $\text{FG}(A)$ and $A$ are Morita equivalent.

1.10. Remark: We conjecture that the full thick subcategory of the coherent category of (left-)modules over $\text{Diff}_{O_X}(O_{X'}, O_X)$ generated by $O_X$ is actually equivalent to the $(\infty, 2)$-category of coherent (left-)modules over $\text{Diff}_{O_X}(O_{X'}, O_X)$.

We also expect that our theorem generalizes for higher shifts as well.

1.11. Conjecture. Let $S$ be the $n$-shifted cotangent bundle of a smooth scheme $X$ over a perfect field $k$ of characteristic $p$. Let $\mathcal{A}$ be the convergent quantization of $\mathcal{O}_S$. Consider the Frobenius twist $S'$ of $S$, and the zero section $i : X' \to S'$. Then, the algebra $\mathcal{A}$ can be regarded as an $E_{n+1}$-algebra over $S'$ so that

- Weak Morita equivalence: The $(\infty, n+1)$-category of coherent modules over $i^*\mathcal{A}$ is equivalent to the $(\infty, n+1)$-category of coherent sheaves over $\mathcal{O}_{S'}$ (viewed as an $E_{n+1}$-algebra).

- Weak Azumaya property: Étale locally over $X$, the $(\infty, n+1)$-category of coherent modules over $\mathcal{A}$ is equivalent to the $(\infty, n+1)$-category of coherent sheaves over $\mathcal{O}_{S'}$ (viewed as an $E_{n+1}$-algebra).

1.12. We also remark that in the case of $-1$-shifted symplectic derived Artin stacks, the convergent quantization is a $p$-torsion element of the Picard group (for the de-Rham complex, see [11]), in the case of symplectic varieties, the convergent quantization is a $p$-torsion element of the Brauer group ([4]).

We wonder whether the quantizations in higher shifts can be realized as $p$-torsion elements of higher Brauer groups ([12]).

1.13. The paper is organized as follows. In Section 2, we collect facts about the Hochschild cosimplicial complex and its variant which we call the crystalline Hochschild cosimplicial complex. We show (Proposition 2.4.7) that the $\mathcal{O}_X$-linear Hochschild cosimplicial complex $\text{Diff}_{O_X}(O_{X'}, O_X)$ is the restriction of the convergent quantization of $T^*[1]X$ to the zero section $X' \to T^*[1]X'$. We also provide a left and right brace module structure of $\mathcal{O}_X$ over the brace algebra $\text{Diff}_{O_X}(O_{X'}, O_X)$ (Proposition 2.4.3). In Section 3, we prove our main theorem, Theorem [1.9].
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2. **The two Hochschild cosimplicial complexes**

In this section, we recollect facts about two cosimplicial complexes: the first one, $\mathcal{D}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$, is a global model for the Hochschild cosimplicial complex equipped with its brace algebra structure, the second one, $\mathcal{D}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$, can be thought of the convergent quantization of the 1-shifted cotangent bundle $T^*[1]X$ equipped with its brace algebra structure. Similarly, for every $\mathcal{O}_X$-bimodule, $P$, we define the complexes $\mathcal{D}(\mathcal{O}_X^\bullet, P)$ and $\mathcal{D}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$, which are cosimplicial complexes equipped with a left brace module structure over the Hochschild complexes (see [7], [6], [10], [20] for more details). Finally, we prove that the $\mathcal{O}_X'$-linear Hochschild cosimplicial complex $\mathcal{D}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$ is the restriction of $\mathcal{D}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$ to the zero section $X' \to T^*[1]X'$. We also provide a left and right brace module structure of $\mathcal{O}_X$ over the brace algebra $\mathcal{D}(\mathcal{O}_X^\bullet, \mathcal{O}_X)$.

2.1. **Grothendieck polydifferential operators**

We begin with the definition of Grothendieck (poly)differential operators.

2.1.1. **Definition.** Let $P$ be an $\mathcal{O}_X$-bimodule, and $\Lambda : \mathcal{O}_X \to P$ a $k$-linear map. Given a sequence of functions $f_0, f_1, \ldots, \in \mathcal{O}_X$, define a sequence of $k$-linear maps $\Lambda_n : \mathcal{O}_X \to P$ given by $\Lambda_{-1} = \Lambda$ and, $\Lambda_n := f_n\Lambda_{n-1} - \Lambda_{n-1}f_n$.

We say that $\Lambda$ is a differential operator of order at most $N$ if for every point $x \in X$ and every section $s \in \mathcal{O}_X$ defined at $x$, there exists a neighborhood $U$ of $x$ and $N \geq 0$ such that for any open subset $V \subset U$ and any choice of functions $f_0, f_1, \ldots, f_N$ on $V$, so that $\Lambda_N(s|_V)$ vanishes.

The Grothendieck differential operators $\mathcal{O}_X \to P$ form a sheaf that we denote by $\mathcal{D}(\mathcal{O}_X, P)$. In the case of $P = \mathcal{O}_X$ we denote the sheaf of differential operators by $\mathcal{D} := \mathcal{D}(\mathcal{O}_X, \mathcal{O}_X)$. The tangent bundle $T_X$ of $X$ has a $(k, \mathcal{O}_X)$ Lie-algebroid structure. The ring $\mathcal{D}$ is the universal PD-enveloping algebra of the $(k, \mathcal{O}_X)$ Lie-algebroid $T_X$. For instance, if $X = \text{Spec} k[x]$, then $\mathcal{D}$ is the PD-polynomial ring of one variable, $k(x)$. Moreover, by definition the ring $\mathcal{D}$ comes with a filtration

$$\mathcal{O}_X = \mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1} \subset \mathcal{D}^{\leq 2} \subset \ldots$$
given by the degree of the differential operators.

**2.1.2. Definition.** A $k$-polylinear map $A : \mathcal{O}_X \times \ldots \times \mathcal{O}_X \to P$ (of $n$ arguments) is a polydifferential operator of (poly)order at most $(N_1, \ldots, N_n)$ if it is a differential operator of order at most $N_j$ in the $j$-th argument whenever the remaining $n - 1$ arguments are fixed.

The polydifferential operators $\mathcal{O}_X \times \ldots \times \mathcal{O}_X \to P$ of $i$ arguments form a sheaf, which we denote by $\mathcal{D}iff(\mathcal{O}_X^i, P)$.

**2.1.3.** We can identify the sheaf $\mathcal{D}iff(\mathcal{O}_X^i, P)$ with the tensor product

$$\mathcal{O}_X \otimes \mathcal{O}_X \ldots \otimes \mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X P$$

as follows (here the number of the $\mathcal{O}_X$ terms is $i$). The map

$$\mathcal{O}_X \otimes \mathcal{O}_X \ldots \otimes \mathcal{O}_X \otimes \mathcal{O}_X P \to \mathcal{D}iff(\mathcal{O}_X^i, P)$$

given by

$$A_1 \otimes \ldots \otimes A_i \otimes p \mapsto A_1(-)A_2(-)\ldots A_i(-)p$$

(for local sections $A_i \in \mathcal{O}_X$ and $p \in P$) is clearly an isomorphism. (Here we use the natural $\mathcal{O}_X$-bimodule structure on $\mathcal{O}_X$.)

**2.2. The Grothendieck Hochschild cosimplicial complex**

**2.2.1.** The sheaves of polydifferential operators form a natural cosimplicial complex $\mathcal{D}iff(\mathcal{O}_X^i, P)$ whose $i$-th term is $\mathcal{D}iff(\mathcal{O}_X^i, P)$ and the differentials $d_{i,j}$ are given by

$$d_{i,k}A(g_1, \ldots, g_{i+1}) =
\begin{cases}
  g_1A(g_2, \ldots, g_{i+1}) & k = 0 \\
  A(g_1, \ldots, g_k g_{k+1}, \ldots, g_{i+1}) & 0 < k < i + 1 \\
  A(g_1, \ldots, g_i)g_{i+1} & k = i + 1
\end{cases}$$

where $A : \mathcal{O}_X \times \ldots \times \mathcal{O}_X \to P$ is a polydifferential operator of $i$ arguments and $\{g_1, \ldots, g_{i+1}\}$ is a local section of $\mathcal{O}_X \times \ldots \times \mathcal{O}_X$ (of $i + 1$ arguments).

**2.2.2.** Let $A \in \mathcal{D}iff(\mathcal{O}_X^i, \mathcal{O}_X)$ and $B \in \mathcal{D}iff(\mathcal{O}_X^j, \mathcal{O}_X)$. We define $A \cdot B \in \mathcal{D}iff(\mathcal{O}_X^{i+j}, \mathcal{O}_X)$ as the differential operator mapping $a_1, \ldots, a_{i+j}$ to

$$(-1)^{ij}A(a_1, \ldots, a_i) \cdot B(a_{i+1}, \ldots, a_{i+j}).$$

This product endows the cosimplicial complex $\mathcal{D}iff(\mathcal{O}_X^*, \mathcal{O}_X)$ a cosimplicial algebra structure.
2.3. The crystalline differential operators

2.3.1. Definition. The crystalline ring of differential operators, \( \mathcal{D}_X = \text{Diff}(\mathcal{O}_X, \mathcal{O}_X) \) is defined as the universal enveloping \((\mathfrak{k}, \mathcal{O}_X)\) Lie-algebroid of the tangent bundle \( \mathcal{T}_X \).
2.3.2. Example: in the case of $X = \text{Spec} \ k[x]$, the ring of differential operators $D_X$ is the Weyl-algebra

$$k \langle x, \frac{d}{dx} \rangle / (\frac{d}{dx} x - x \frac{d}{dx} - 1).$$

2.3.3. The algebra $D_X$ is equipped with a filtration

$$\mathcal{O}_X = D_X^{\leq 0} \subset D_X^{\leq 1} \subset D_X^{\leq 2} \subset ...$$

given by the degree of the differential operator. Note that the filtered pieces $D_X^{\leq p-1}$ and $\mathcal{O}_X^{\leq p-1}$ are isomorphic as $\mathcal{O}_X$-bimodules! Given a derivation $t \in T_X$, its $p$-th composite $t^{[p]} := t \circ t \circ ... \circ t$ is also a derivation. This gives rise to a distinguished element $t^{[p]} - t^{[p]}$ in $D_X$ for every derivation $t \in T_X$. The quotient of $D_X$ with the ideal generated by these distinguished elements can be identified with $D_X^{\leq p-1}$. Hence $D_X^{\leq p-1}$ is a split sub-$\mathcal{O}_X$-bimodule of $D_X$. This also provides an algebra structure on $D_X^{\leq p-1}$, moreover,

2.3.4. Proposition. [5] The algebra $D_X^{\leq p-1}$ is an Azumaya algebra over $\mathcal{O}_X$.

2.3.5. Similarly as in Section 2.1, given an $\mathcal{O}_X$-bimodule $P$, we define the crystalline differential operators $\text{Diff}(\mathcal{O}_X, P)$ as the sheaf $D_X \otimes \mathcal{O}_X P$ equipped with its natural bimodule structure. Moreover, we define the polydifferential operators $\text{Diff}(\mathcal{O}_X, P)$ as the tensor product

$$D_X \otimes \mathcal{O}_X \otimes ... \otimes \mathcal{O}_X D_X \otimes \mathcal{O}_X P$$

(here the number of the $D_X$ terms is $i$).

2.4. The crystalline Hochschild cosimplicial complex

2.4.1. Similarly as in Section 2.2 we can construct a cosimplical complex $\text{Diff}(\mathcal{O}_X, P)$ from the sheaves $\text{Diff}(\mathcal{O}_X, P)$. We call this cosimplical complex the crystalline Hochschild cosimplicial complex of $P$.

2.4.2. Remark: Even though the crystalline ring of differential operators is a finitely generated algebra over $\mathcal{O}_X$, the homotopy groups of the crystalline Hochschild cosimplicial complex are not locally free sheaves over $\mathcal{O}_X$. 

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2.4.3. The cosimplicial complex $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$ can be thought of as the shifted quantization of the 1-shifted cotangent bundle $T^*[1]X$. We represent the 1-shifted cotangent bundle as the simplicial scheme

$$...T^*X \times_X T^*X \longrightarrow T^*X \longrightarrow X$$

where locally the maps are given by

$$d_{i,k}(x, \omega_1, ..., \omega_i) = \begin{cases} (x, \omega_2, ..., \omega_i) & k = 0 \\ (x, \omega_1, ..., \omega_k + \omega_{k+1}, ..., \omega_i) & 0 < k < i + 1 \\ (x, \omega_1, ..., \omega_{i-1}) & k = i + 1 \end{cases}$$

Hence, the structure sheaf of $T^*[1]X$ is the cosimplicial algebra

$$\mathcal{O}_X \longrightarrow \mathcal{O}_{T^*X} \longrightarrow \mathcal{O}_{T^*X \otimes \mathcal{O}_X, T^*X}$$

which we will denote by $\text{Pol}^*(X)$ where the differentials are provided by the maps representing $T^*[1]X$ as the simplicial scheme above. We denote the Frobenius twist of the cosimplicial algebra $\text{Pol}^*(X)$ by $\text{Pol}^*(X')$, it is the cosimplicial algebra given by the Frobenius twists of algebras and $p$-th powers of differentials

$$\text{Pol}^*(X') = \mathcal{O}_{X'} \longrightarrow \mathcal{O}_{T^*X'} \longrightarrow \mathcal{O}_{T^*X' \otimes \mathcal{O}_X', T^*X'}.$$ 

An important observation is that since the center of $\text{D}_X$ can be identified with $\mathcal{O}_{T^*X'}$, we have that $\text{Pol}^*(X')$ is a sub-complex of $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$. Moreover, the brace structures on $\text{Diff}(\mathcal{O}_X^*, \mathcal{O}_X)$ become trivial on $\text{Pol}^*(X')$.

2.4.4. Clearly, there is a natural morphism, the zero section $X \to T^*[1]X$, and the following statement provides a convergent quantization of the zero section $X \to T^*[1]X$ equipped with its natural Lagrangian structure (see [17] and [10] for more details).

2.4.5. Proposition. We have

$$\text{Diff}(\mathcal{O}_X^*, \text{D}_X) = \mathcal{O}_X$$

(and similarly

$$\text{Diff}(\mathcal{O}_X^*, \text{D}_X^{op})^{op} = \mathcal{O}_X).$$
2.4.6. Note that the bracket on the tangent bundle $T_X$ is $\mathcal{O}_{X'}$-linear as well, hence we can consider the universal enveloping $(\mathcal{O}_{X'}, \mathcal{O}_X)$ Lie-algebroid $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X)$ and the corresponding Hochschild complex $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X, P)$. We compare $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X)$ with the quantization $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X)$.

2.4.7. Proposition. The restriction of the convergent quantization of $T^*[1]X$ to the zero section $X' \to T^*[1]X'$ is the $\mathcal{O}_{X'}$-linear Hochschild cosimplicial complex of $\mathcal{O}_X$:

$$\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X) = \mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X) \otimes_{\text{Pol}(X')} \mathcal{O}_{X'}.$$

Proof. For any non-trivial derivation $t \in T_X$, its $p$-th divided power is not $\mathcal{O}_{X'}$-linear. Moreover for any derivation $t \in T_X$, $t^p$ and $t[p]$ act identically on local functions of $\mathcal{O}_X$. Hence, we identify $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X)$ with $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X)$.

Similarly, $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X)$ can be identified with the tensor product $\mathcal{D}(\mathcal{O}_{X'}, \mathcal{O}_X) \otimes_{\text{Pol}(X')} \mathcal{O}_{X'}$. This implies the statement, both the brace structures and differentials align.

2.4.8. Proposition. We have quasi-isomorphisms of complexes

$$\mathcal{D}(\mathcal{O}_{X'}, \mathcal{D}_X^\leq p-1) = \mathcal{O}_X$$

(and)

$$\mathcal{D}(\mathcal{O}_{X'}, \mathcal{D}_X^\leq p-1)^{op} = \mathcal{O}_X.$$

Proof. The zeroth homotopy sheaf can be computed easily, it is the centralizer of $\mathcal{O}_X$ inside $\mathcal{D}_X^\leq p-1$ which is $\mathcal{O}_X$. In order to show that the higher homotopy sheaves vanish, we consider the map of cosimplicial complexes

$$\varphi : \mathcal{D}(\mathcal{O}_{X'}, \mathcal{D}_X^\leq p-1) \to \mathcal{D}(\mathcal{O}_{X'}, \mathcal{D}_X^\leq p-1)$$

given by the identifications

$$\mathcal{D}(\mathcal{O}_{X'}, \mathcal{D}_X^\leq p-1) = \mathcal{D}_X^\leq p-1 \otimes_{\mathcal{O}_X} \mathcal{D}_X^\leq p-1 \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathcal{D}_X^\leq p-1.$$
(with $i + 1$ terms!) and the quotient map $D_X \to D_X^{\leq p-1}$ given by the ideal generated by the distinguished elements $t^p - t^p$. This map, $\varphi$ splits, the inclusion map $i : D_X^{\leq p-1} \to D_X$ gives an inverse. Moreover, the maps $i$ and $\varphi$ respect the differentials of the Hochschild cosimplicial complexes, hence the complex $\mathcal{D}(\mathcal{O}_X, D_X^{\leq p-1})$ is a direct summand of the complex $\mathcal{D}(\mathcal{O}_X, D_X)$. The latter is quasi-isomorphic to $\mathcal{O}_X$ by Proposition 2.4.5.

Since the higher homotopy sheaves of $\mathcal{D}(\mathcal{O}_X, D_X)$ vanish, the higher homotopy sheaves of $\mathcal{D}(\mathcal{O}_X, D_X^{\leq p-1})$ have to vanish as well.

Finally, we compute the homotopy sheaves of the Hochschild complex of $D_X^{\leq p-1}$ over $\mathcal{O}_X$ using the natural inclusion $\mathcal{O}_X \to D_X^{\leq p-1}$ (which equips $D_X^{\leq p-1}$ with a bimodule structure over $\mathcal{O}_X$).

2.4.9. Proposition.

$$\mathcal{D}(\mathcal{O}_X^{\leq p-1}, D_X^{\leq p-1}) = \mathcal{O}_X'$$

Proof. First, we show that the zeroth homotopy sheaf is $\mathcal{O}_X'$, and then we show that the higher homotopy sheaves vanish. The zeroth term of the complex is $\mathcal{D}(\mathcal{O}_X, D_X^{\leq p-1})$ which can be identified with $\mathcal{O}_X$. Hence, the zeroth homotopy sheaf is the centralizer of $D_X^{\leq p-1}$ inside $\mathcal{O}_X$ which is indeed $\mathcal{O}_X'$.

Next, we show that the higher homotopy sheaves vanish. First, note that we only need to solve the problem locally, so we can assume that $X$ is a spectrum of a polynomial ring. Moreover, using Künneth-formula, we can assume that the polynomial ring is of one variable.

We use the Dold-Kan correspondence, and we resolve $D_X^{\leq p-1} = k\langle x, d \rangle/(dx - xd - 1, d^p)$ with locally free $D^e = D_X^{\leq p-1} \otimes_{\mathcal{O}_X} D_X^{\leq p-1}$-modules. We claim that there is a 2-periodic resolution given by

$$\cdots D^e \xrightarrow{d \otimes 1 - 1 \otimes d} D^e \xrightarrow{d^p - 1 \otimes + d^{p-2} \otimes d + \cdots + 1 \otimes d^{p-1}} D^e \xrightarrow{d \otimes 1 - 1 \otimes d} D^e \xrightarrow{m} D_X^{\leq p-1}.$$  

Here the first map is given by multiplication. The kernel of that map is the (left)ideal generated by $d \otimes 1 - 1 \otimes d$ and by those monomials $d^i \otimes d^j$ for which $i + j \geq p$. However, the latter is also generated by the former.

We turn our attention to the kernel of the map $D^e \xrightarrow{d \otimes 1 - 1 \otimes d} D^e$ given by multiplication from the right by $d \otimes 1 - 1 \otimes d$. Any element of $D^e$ can be written as $\sum_{0 \leq i,j \leq p-1} f_{i,j} d^i \otimes d^j$. We say that the degree of a monomial $f_{i,j} d^i \otimes d^j$ is $i + j$.

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The map \( \mathcal{D}^e \xrightarrow{d \otimes 1 \otimes d} \mathcal{D}^e \) sends the element \( \sum_{0 \leq i,j \leq p-1} f_{ij} d^i \otimes d^j \) to the element

\[
\sum_{0 \leq i,j \leq p-1} (f_{ij} d^{i+1} \otimes d^j - f_{ij} d^i \otimes d^{j+1}).
\]

Notice that the degree of each monomial is increased by 1. As a consequence, it is clear that if the element \( \sum_{0 \leq i,j \leq p-1} f_{ij} d^i \otimes d^j \) is in the kernel of \( \mathcal{D}^e \xrightarrow{d \otimes 1 \otimes d} \mathcal{D}^e \), then degree-wise it is in the kernel, i.e.

\[
\sum_{0 \leq i,j \leq p-1} f_{ij} d^i \otimes d^j
\]

is in the kernel as well for every \( k \). Then, a simple calculation implies that \( f_{ij} = 0 \) if \( i + j < p - 1 \), and that the kernel of \( \mathcal{D}^e \xrightarrow{d \otimes 1 \otimes d} \mathcal{D}^e \) is the (left)ideal generated by \( d^p - 1 \otimes 1 + d^{p-2} \otimes d + ... + 1 \otimes d^{p-1} \). A similar calculation can be done for the kernel of \( \mathcal{D}^e \xrightarrow{d \otimes 1 \otimes d} \mathcal{D}^e \).

We turn our attention now to compute \( \mathcal{D}iff_{\mathcal{O}_X}((\mathcal{D}^e_X)^{\leq p-1}, \mathcal{D}^e_X) \).

We note that \( \mathcal{D}iff_{\mathcal{D}^e}((\mathcal{D}^e_X, \mathcal{D}^e_X^{\leq p-1}) = k[x] \), because the image \( t \) of 1 in \( \mathcal{D}^e \) has to satisfy \( xt - tx = 0 \). As a consequence using our two-periodic resolution the complex \( \mathcal{D}iff_{\mathcal{O}_X}((\mathcal{D}^e_X^{\leq p-1}), \mathcal{D}^e_X^{\leq p-1}) \) becomes

\[
k[x] \xrightarrow{d/dx} k[x] \xrightarrow{d^{p-1}/dx^{p-1}} k[x] \xrightarrow{d/dx} ...
\]

where the first map is derivation, and the second one is the \( p-1 \)-st composite of the derivation. This complex is exact except at degree 0 concluding our proof.

We conclude this section by showing that the \( \mathcal{D}iff_{\mathcal{O}_X}((\mathcal{O}^*_X, \mathcal{O}_X)) \)-linear Hochschild cosimplicial complex of \( \mathcal{O}_X \) can be identified with \( \mathcal{O}_X' \) with its trivial brace algebra structure.

2.4.10. Theorem. We have a quasi-isomorphism of \( E_2 \)-algebras

\[
\mathcal{D}iff_{\mathcal{D}iff_{\mathcal{O}_X'}}(\mathcal{O}^*_X, \mathcal{O}_X)(\mathcal{O}^*_X, \mathcal{O}_X) = \mathcal{O}_X'.
\]

Proof. First, we use Proposition 2.4.8 to equip \( \mathcal{O}_X \) with a \( \mathcal{D}iff_{\mathcal{O}_X}((\mathcal{O}^*_X, \mathcal{O}_X)) \)-structure. Then, the left hand-side of the Theorem becomes

\[
\mathcal{D}iff_{\mathcal{D}iff_{\mathcal{O}_X'}}(\mathcal{O}^*_X, \mathcal{O}_X)(\mathcal{D}iff_{\mathcal{O}_X'}(\mathcal{O}_X, \mathcal{D}^{\leq p-1}_X), \mathcal{D}iff_{\mathcal{O}_X'}(\mathcal{O}_X, \mathcal{D}^{\leq p-1}_X)).
\]
We can see that the $E_2$-module $\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{D}_X^{\leq p-1})$ is generated by $\mathcal{D}_X^{\leq p-1}$ over $\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{O}_X)$. Therefore the complex

$$\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{D}_X^{\leq p-1})$$

is determined by the complex

$$\text{Diff}_{\mathcal{X}}((\mathcal{D}_X^{\leq p-1})^i, \text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{D}_X^{\leq p-1})).$$

However, it is not a quasi-isomorphism, elements of the above complex do not give rise to elements of the complex

$$\text{Diff}_{\mathcal{X}}((\mathcal{D}_X^{\leq p-1})^i, \text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{D}_X^{\leq p-1})).$$

The requirement that the map has to be $\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{O}_X)$-linear shows that only elements of

$$\text{Diff}_{\mathcal{X}}((\mathcal{D}_X^{\leq p-1})^i, \mathcal{D}_X^{\leq p-1}))$$

will give rise elements of the original complex, and as a consequence, it is quasi-isomorphic to the original complex. Finally, Proposition 2.4.9 implies the statement of the Theorem.

3. Proof of Theorem 1.9

In this section we prove our main theorem, Theorem 1.9. We begin with the following statement.

3.1. Theorem. We have

$$\mathcal{O}_X \otimes_{\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{O}_X)} \mathcal{O}_X = \mathcal{D}_X^{\leq p-1}.$$

Proof. This statement is very similar to Theorem 4.3 in [10]. We highlight the key steps.

By Proposition 2.4.8, we can equip $\mathcal{O}_X$ with a left and right brace module structure over $\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{O}_X)$. Hence the complex

$$\mathcal{O}_X \otimes_{\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{O}_X)} \mathcal{O}_X$$

can be represented as

$$\mathcal{D}^\ast(X) := \text{Diff}_{\mathcal{X}}(\mathcal{O}_X, (\mathcal{D}_X^{\leq p-1})^{\text{op}})^{\text{op}} \otimes_{\text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{O}_X)} \text{Diff}_{\mathcal{X}}(\mathcal{O}_X, \mathcal{D}_X^{\leq p-1}).$$

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The morphism
\[ D^{≤p-1}_X \otimes_{\mathcal{O}_X} D^{≤p-1}_X \to \mathcal{D}(X) \]

Consider the map
\[ \chi : D^{≤p-1}_X \to D^{≤p-1}_X \otimes_{\mathcal{O}_X} D^{≤p-1}_X \]
given by
\[ \partial \mapsto d(\partial) + i(\partial) \]
where \(d\) is the Hochschild differential
\[ D^{≤p-1}_X \to \mathcal{D}(\mathcal{O}_X, D^{≤p-1}_X) = D^{≤p-1}_X \otimes_{\mathcal{O}_X} D^{≤p-1}_X \]
and \(i\) is the map
\[ D^{≤p-1}_X \to D^{≤p-1}_X \otimes_{\mathcal{O}_X} D^{≤p-1}_X \]
sending \(\partial\) to \(1 \otimes \partial\). It is easy to show that the map \(\chi\) is an algebra homomorphism and the composite map
\[ D^{≤p-1}_X \to D^{≤p-1}_X \otimes_{\mathcal{O}_X} D^{≤p-1}_X \to \mathcal{D}(X) \]
is a map of dg-algebras.

Finally, we show that the above map is a quasi-isomorphism. By Proposition 2.4.8 we know that \(\mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op})^{op}\) is quasi-isomorphic to \(\mathcal{O}_X\), moreover the action of \(\mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op})^{op}\) on \(\mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op})^{op}\) becomes the natural action of \(\mathcal{D}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X\) on \(\mathcal{O}_X\) and \(\mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op})^{op}\) just acts trivially if \(i > 0\). Therefore, we have
\[ \mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op})^{op} \otimes \mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op})^{op} \cong \mathcal{O}_X \otimes_{\mathcal{O}_X} D^{≤p-1}_X = D^{≤p-1}_X. \]

**3.2.** Consider the functors \(F\) and \(G\) between the coherent category of (left)-modules over \(\mathcal{O}_X\) and over \(\mathcal{D}(\mathcal{O}_X, \mathcal{O}_X)\) given by:
\[ F : \text{Coh}(\mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op}), \mathcal{O}_X) \to \text{Coh}(\mathcal{O}_X, \mathcal{O}_X) \quad F(-) = \mathcal{O}_X \otimes \mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op}) (-) \]
\[ G : \text{Coh}(\mathcal{O}_X) \to \text{Coh}(\mathcal{D}(\mathcal{O}_X, (D^{≤p-1}_X)^{op}), \mathcal{O}_X) \quad G(-) = \mathcal{O}_X \otimes \mathcal{O}_X, \mathcal{O}_X \]
It is clear that the composite functor \(FG(-)\) is given by \(D^{≤p-1}_X \otimes \mathcal{O}_X, \mathcal{O}_X\), hence for any algebra object \(A\) in the coherent category \(\text{Coh}(\mathcal{O}_X)\) we have that \(FG(A)\) and \(A\) are Morita equivalent. As a consequence for any object
B of the coherent category of left-modules over $\mathcal{D}iff_{\mathcal{O}X'}(\mathcal{O}^\bullet_X, \mathcal{O}_X)$ which is of the form $G(A)$ we have $GF(B)$ is Morita-equivalent to $B$. This implies that $F$ and $G$ are essentially surjective functors between the $(\infty,2)$-categories of the coherent category of (left)-modules over $\mathcal{O}_{X'}$ (where we consider $\mathcal{O}_{X'}$ as an $E_2$-algebra) and of the full thick subcategory of the coherent category of (left)-modules over $\mathcal{D}iff_{\mathcal{O}X'}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$. This concludes the proof the Theorem 1.9.

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