Is the solution to the BCS gap equation continuous in the temperature?

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Abstract

One of long-standing problems in mathematical studies of superconductivity is to show that the solution to the BCS gap equation is continuous in the temperature. In this paper we address this problem. We regard the BCS gap equation as a nonlinear integral equation on a Banach space consisting of continuous functions of both $T$ and $x$. Here, $T(\geq 0)$ stands for the temperature and $x$ the kinetic energy of an electron minus the chemical potential. We show that the unique solution to the BCS gap equation obtained in our recent paper is continuous with respect to both $T$ and $x$ when $T$ is small enough. The proof is carried out based on the Banach fixed-point theorem.

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1 Introduction and preliminaries

We use the unit $k_B = 1$, where $k_B$ stands for the Boltzmann constant. Let $\omega_D > 0$ and $k \in \mathbb{R}^3$ stand for the Debye frequency and the wave vector of an electron, respectively. We denote Planck’s constant by $h > 0$ and set $\hbar = h/(2\pi)$. Let $m > 0$ and $\mu > 0$ stand for the electron mass and the chemical potential, respectively. We denote by $T(\geq 0)$ the temperature, and by $x$ the kinetic energy of an electron minus the chemical potential, i.e., $x = \hbar^2 |k|^2/(2m) - \mu \in [-\mu, \infty)$. Note that $0 < \hbar \omega_D << \mu$.

In the BCS model (see [1,3]) of superconductivity, the solution to the BCS gap equation (1.1) below is called the gap function. We regard the gap function as a function of both $T$ and $x$, and denote it by $u$, i.e., $u: (T, x) \mapsto u(T, x) (\geq 0)$. The BCS gap equation is the following nonlinear integral equation:

$$u(T, x) = \int_\xi^{\hbar \omega D} U(x, \xi) u(T, \xi) \tanh \frac{\sqrt{\xi^2 + u(T, \xi)^2}}{2T} d\xi,$$

where $U(x, \xi) > 0$ is the potential multiplied by the density of states per unit energy at the Fermi surface and is a function of $x$ and $\xi$. In (1.1) we introduce $\varepsilon > 0$, which is
small enough and fixed \((0 < \varepsilon << \hbar \omega_D)\). Note that \(\tanh \frac{\sqrt{\xi^2 + u(T, \xi)^2}}{2T}\) is regarded as 1 when \(T = 0\). It is known that the BCS gap equation (1.1) is based on a superconducting state called the BCS state. In this connection, see [8, (6.1)] for a new gap equation based on a superconducting state having a lower energy than the BCS state.

The integral with respect to \(\xi\) in (1.1) is sometimes replaced by the integral over \(\mathbb{R}^3\) with respect to the wave vector \(k\). Odeh [6], and Billard and Fano [2] established the existence and uniqueness of the positive solution to the BCS gap equation in the case \(T = 0\). In the case \(T \geq 0\), Vansevenant [7] determined the transition temperature (the critical temperature) and showed that there is a unique positive solution to the BCS gap equation. Recently, Hainzl, Hamza, Seiringer and Solovej [4] proved that the existence of a positive solution to the BCS gap equation is equivalent to the existence of a negative eigenvalue of a certain linear operator to show the existence of a transition temperature. Hainzl and Seiringer [5] also derived upper and lower bounds on the transition temperature and the energy gap for the BCS gap equation. However, the solution mentioned above belongs to a function space consisting of functions of the wave vector only, and the temperature is regarded as a parameter. So the temperature dependence of the solution is not discussed; for example, it is not shown that the solution is continuous for \(T \geq 0\). To show that the solution is continuous for \(T \geq 0\) is one of long-standing problems in mathematical studies of superconductivity.

Let

\[(1.2)\]

\[U(x, \xi) = U_1 \quad \text{at all} \quad (x, \xi) \in [\varepsilon, \hbar \omega_D]^2,\]

where \(U_1 > 0\) is a constant. Then the gap function depends on the temperature \(T\) only. We therefore denote the gap function by \(\Delta_1\) in this case, i.e., \(\Delta_1 : T \mapsto \Delta_1(T)\). Then (1.1) leads to the simplified gap equation

\[(1.3)\]

\[1 = U_1 \int_\varepsilon^{\hbar \omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_1(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_1(T)^2}}{2T} d\xi.\]

We now define the temperature \(\tau_1 > 0\), which is the transition temperature originating from the simplified gap equation (1.3).

**Definition 1.1** ([1]). The transition temperature is the temperature \(\tau_1 > 0\) satisfying

\[1 = U_1 \int_\varepsilon^{\hbar \omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_1} d\xi.\]

**Remark 1.2.** There is another definition of the transition temperature, which originates from the BCS gap equation (1.1). See [10, Definition 2.4].

The BCS model makes the assumption that there is a unique solution \(\Delta_1 : T \mapsto \Delta_1(T)\) to the simplified gap equation (1.3) and that it is of class \(C^2\) with respect to the temperature \(T\) (see e.g. [1] and [12, (11.45), p.392]). The author [9] has given a mathematical proof of this assumption on the basis of the implicit function theorem. Set

\[(1.4)\]

\[\Delta = \sqrt{(\hbar \omega_D - \varepsilon e^{1/U_1}) (\hbar \omega_D - \varepsilon e^{-1/U_1})} \left/ \sinh \frac{U_1}{U_1} \right.\]
Proposition 1.3 ([3] Proposition 2.2). Let $\Delta$ be as in (1.4). Then there is a unique nonnegative solution $\Delta_1 : [0, \tau_1] \to [0, \infty)$ to the simplified gap equation (1.3) such that the solution $\Delta_1$ is continuous and strictly decreasing on the closed interval $[0, \tau_1]$:

$$\Delta_1(0) = \Delta > \Delta_1(T_1) > \Delta_1(T_2) > \Delta_1(\tau_1) = 0, \quad 0 < T_1 < T_2 < \tau_1.$$  

Moreover, it is of class $C^2$ on the interval $[0, \tau_2)$ and satisfies

$$\Delta_1'(0) = \Delta_1''(0) = 0 \quad \text{and} \quad \lim_{T \uparrow \tau_1} \Delta_1'(T) = -\infty.$$  

Remark 1.4. We set $\Delta_1(T) = 0$ for $T > \tau_1$.

Let $0 < U_1 < U_2$, where $U_2 > 0$ is a constant. We assume the following:

(1.5) \hspace{1cm} U_1 \leq U(x, \xi) \leq U_2 \quad \text{at all} \quad (x, \xi) \in [\varepsilon, \hbar \omega_D]^2, \quad U(\cdot, \cdot) \in C([\varepsilon, \hbar \omega_D]^2).

When $U(x, \xi) = U_2$ at all $(x, \xi) \in [\varepsilon, \hbar \omega_D]^2$, an argument similar to that in Proposition 1.3 gives that there is a unique nonnegative solution $\Delta_2 : [0, \tau_2] \to [0, \infty)$ to the simplified gap equation

(1.6) \hspace{1cm} 1 = U_2 \int_{\varepsilon}^{\hbar \omega_D} \frac{1}{\sqrt{\xi^2 + 2\Delta_2^2(T)^2}} \tanh \frac{\sqrt{\xi^2 + 2\Delta_2^2(T)^2}}{2T} d\xi, \quad 0 \leq T \leq \tau_2.

Here, $\tau_2 > 0$ is defined by

$$1 = U_2 \int_{\varepsilon}^{\hbar \omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_2} d\xi.$$  

We again set $\Delta_2(T) = 0$ for $T > \tau_2$.

Lemma 1.5 ([10] Lemma 1.6). (a) The inequality $\tau_1 < \tau_2$ holds.

(b) If $0 \leq T < \tau_2$, then $\Delta_1(T) < \Delta_2(T)$. If $T \geq \tau_2$, then $\Delta_1(T) = \Delta_2(T) = 0$.

Let $0 \leq T \leq \tau_2$ and fix $T$. The author considered the Banach space $C([\varepsilon, \hbar \omega_D])$ consisting of continuous functions of $x$ only, and dealt with the following subset $V_T$:

(1.7) \hspace{1cm} V_T = \{ u(T, \cdot) \in C([\varepsilon, \hbar \omega_D]) : \Delta_1(T) \leq u(T, x) \leq \Delta_2(T) \text{ at } x \in [\varepsilon, \hbar \omega_D] \}.

Remark 1.6. We denote each element of $V_T$ by $u(T, \cdot)$ since each element of $V_T$ depends on $T$ and we are interested in the temperature dependence of the solution to the BCS gap equation.

On the basis of Proposition 1.3 and the Schauder fixed-point theorem, the author showed the following.

Theorem 1.7 ([10] Theorem 2.2). Let $T \in [0, \tau_2]$ be fixed. Then there is a unique nonnegative solution $u_0(T, \cdot) \in V_T$ to the BCS gap equation (1.1):

$$u_0(T, x) = \int_{\varepsilon}^{\hbar \omega_D} \frac{U(x, \xi) u_0(T, \xi)}{\sqrt{\xi^2 + u_0(T, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u_0(T, \xi)^2}}{2T} d\xi, \quad x \in [\varepsilon, \hbar \omega_D].$$

Consequently, the gap function $u_0(T, \cdot)$ is continuous with respect to $x$. Moreover, $u_0(T, \cdot)$ satisfies

$$\Delta_1(T) \leq u_0(T, x) \leq \Delta_2(T) \quad \text{at} \quad (T, x) \in [0, \tau_2] \times [\varepsilon, \hbar \omega_D].$$


Remark 1.8. In [10, Theorem 2.2] the author assumed the following:  
\[ U_1 \leq U(x, \xi) \leq U_2 \quad \text{at all} \quad (x, \xi) \in [\varepsilon, \hbar \omega_D]^2, \quad U(\cdot, \cdot) \in C^2([\varepsilon, \hbar \omega_D]^2). \]
But Theorem 1.7 holds true under (1.5).

Remark 1.9. We regard the gap function of Theorem 1.7 as a function of both \( T \) and \( x \), and denote it by \( u_0 \), i.e., \( u_0 : (T, x) \mapsto u_0(T, x) \).

Studying smoothness of the thermodynamical potential with respect to \( T \), the author [10, Theorem 2.10] showed, under an assumption, that the phase transition to a superconducting state is a second-order phase transition without the restriction (1.2) imposed in our recent paper [9]. Moreover, the author obtained the explicit form for the gap in the specific heat at constant volume.

But it has not been shown that the solution \( u_0 \) to the BCS gap equation (1.1) is continuous in \( T \). In this paper we regard the BCS gap equation (1.1) as a nonlinear integral equation on a Banach space consisting of continuous functions of both \( T \) and \( x \).

On the basis of the Banach fixed-point theorem, we show that the solution \( u_0 \) is continuous with respect to both \( T \) and \( x \) when \( T \) is small enough.

The paper proceeds as follows. In section 2 we state our main results without proof. In section 3 we prove our main results.

2 Main results

Let \( U_0 > 0 \) be a constant satisfying \( U_0 < U_1 < U_2 \). An argument similar to that in Proposition 1.3 gives that there is a unique nonnegative solution \( \Delta_0 : [0, \tau_0] \to [0, \infty) \) to the simplified gap equation
\[ 1 = U_0 \int_{\varepsilon}^{\hbar \omega_D} \frac{1}{\sqrt{\xi^2 + \Delta_0(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_0(T)^2}}{2T} d\xi, \quad 0 \leq T \leq \tau_0. \]

Here, \( \tau_0 > 0 \) is defined by
\[ 1 = U_0 \int_{\varepsilon}^{\hbar \omega_D} \frac{1}{\xi} \tanh \frac{\xi}{2\tau_0} d\xi. \]

We set \( \Delta_0(T) = 0 \) for \( T > \tau_0 \). A straightforward calculation gives the following.

**Lemma 2.1.**  
(a) \( \tau_0 < \tau_1 < \tau_2 \).
(b) If \( 0 \leq T < \tau_0 \), then \( 0 < \Delta_0(T) < \Delta_1(T) < \Delta_2(T) \).
(c) If \( \tau_0 \leq T < \tau_1 \), then \( 0 = \Delta_0(T) < \Delta_1(T) < \Delta_2(T) \).
(d) If \( \tau_1 \leq T < \tau_2 \), then \( 0 = \Delta_0(T) = \Delta_1(T) < \Delta_2(T) \).
(e) If \( \tau_2 \leq T \), then \( 0 = \Delta_0(T) = \Delta_1(T) = \Delta_2(T) \).

**Remark 2.2.** Let the functions \( \Delta_k \) (\( k = 0, 1, 2 \)) be as above. For each \( \Delta_k \), there is the inverse \( \Delta_k^{-1} : [0, \Delta_k(0)] \to [0, \tau_k] \). Here,
\[ \Delta_k(0) = \sqrt{\left( \hbar \omega_D - \varepsilon e^{1/U_k} \right) \left( \hbar \omega_D - \varepsilon e^{-1/U_k} \right)} \frac{\sinh \frac{1}{U_k}}{U_k} \]
and \( \Delta_0(0) < \Delta_1(0) < \Delta_2(0) \). See [9] for more details.
For \((0 < T_1 < \tau_0)\), set \(T^*_1 = \sup_{T \in [0, T_1]} \Delta^{-1}_0(\Delta_0(T)) > 0\). Let \(T^*_1\) be so small as to satisfy
\[
(2.1) \quad \frac{\Delta_0(0)}{4T^*_1} \tanh \frac{\Delta_0(0)}{4T^*_1} > \frac{1}{2} \left(1 + \frac{4\hbar^2 \omega^2}{\Delta_0(0)^2}\right).
\]
Since \(T < \Delta^{-1}_0(\Delta_0(T))\), \(T_1 > 0\) is also small enough and satisfies \((2.1)\) with \(T^*_1\) replaced by \(T_1\). We now consider the Banach space \(C([0, T_1] \times [\varepsilon, h\omega_D])\) consisting of continuous functions of both \(T\) and \(x\), and deal with the following subset \(V\) of the Banach space \(C([0, T_1] \times [\varepsilon, h\omega_D]):\)
\[
(2.2) \quad V = \{u \in C([0, T_1] \times [\varepsilon, h\omega_D]) : \Delta_1(T) \leq u(T, x) \leq \Delta_2(T) \text{ at } (T, x) \in [0, T_1] \times [\varepsilon, h\omega_D]\}.
\]

**Theorem 2.3.** Assume \((1.5)\). Let \(u_0\) be as in Theorem 1.7 and \(V\) as in \((2.2)\). Then \(u_0 \in V\). Consequently, the gap function \(u_0\) is continuous on \([0, T_1] \times [\varepsilon, h\omega_D]\).

**3 Proof of Theorem 2.3**

Let \(\xi \in [\varepsilon, h\omega_D]\) and \(X \in (\Delta_0(0)/2, \infty)\) be fixed. Then we can regard the following function \(g\) given by
\[
g(T; \xi, X) = \frac{1}{(\xi^2 + X^2)^{3/2}} \left\{\xi^2 \tanh \frac{X^2 Y}{\cosh^2 Y}\right\}, \quad Y = \frac{\sqrt{\xi^2 + X^2}}{2T}
\]
as a function of \(T (\geq 0)\) only. Note that \(g(T; \xi, X) > 0\).

**Remark 3.1.** When \(T = 0\), \(g(T; \xi, X)\) is regarded as \(\frac{\xi^2}{(\xi^2 + X^2)^{3/2}}\), i.e.,
\[
g(0; \xi, X) = \frac{\xi^2}{(\xi^2 + X^2)^{3/2}}.
\]

Let \(T_2 > 0\) be so small as to satisfy
\[
(3.2) \quad \frac{\sqrt{\xi^2 + X^2}}{2T_2} \tanh \frac{\sqrt{\xi^2 + X^2}}{2T_2} > \frac{1}{2} \left(1 + \frac{\xi^2}{X^2}\right).
\]

**Lemma 3.2.** Let \(T_2\) be as in \((3.2)\). Then \(g\) is continuous and strictly increasing on \([0, T_2]\).

**Proof.** At \(T \in (0, T_2)\),
\[
\frac{\partial g}{\partial T}(T; \xi, X) = \frac{2Y^2}{(\xi^2 + X^2)^2 \cosh^2 Y} \left\{2X^2 Y \tanh Y - (\xi^2 + X^2)\right\} > 0.
\]
Define a mapping \( A \) by

\[
Au(T, x) = \int_\varepsilon U(x, \xi) u(T, \xi) \frac{\tan \left( \frac{\xi^2 + u(T, \xi)^2}{2T} \right)}{\sqrt{\xi^2 + u(T, \xi)^2}} d\xi, \quad u \in V.
\]

A straightforward calculation gives the following.

**Lemma 3.3.** Let \( V \) be as in (2.2). Then \( V \) is closed.

**Lemma 3.4.** \( Au \in C([0, T_1] \times [\varepsilon, \hbar \omega D]) \) for \( u \in V \).

**Lemma 3.5.** Let \( u \in V \). Then \( \Delta_1(T) \leq Au(T, x) \leq \Delta_2(T) \) at \( (T, x) \in [0, T_1] \times [\varepsilon, \hbar \omega D] \).

**Proof.** We show \( Au(T, x) \leq \Delta_2(T) \). Since

\[
\frac{u(T, \xi)}{\sqrt{\xi^2 + u(T, \xi)^2}} \leq \frac{\Delta_2(T)}{\sqrt{\xi^2 + \Delta_2(T)^2}},
\]

it follows from (1.6) that

\[
Au(T, x) \leq \int_\varepsilon h_{\omega D} \frac{U_2 \Delta_2(T)}{\sqrt{\xi^2 + \Delta_2(T)^2}} \tanh \frac{\sqrt{\xi^2 + u(T, \xi)^2}}{2T} d\xi = \Delta_2(T).
\]

The rest can be shown similarly by (1.3). \( \square \)

Combining Lemma 3.5 with Lemma 3.4 immediately yields the following.

**Lemma 3.6.** Let \( u \in V \). Then \( Au \in V \).

We now show that the mapping \( A : V \to V \) is contractive. We denote by \( \| \cdot \| \) the norm of the Banach space \( C([0, T_1] \times [\varepsilon, \hbar \omega D]) \).

**Lemma 3.7.** There is a constant \( k \) \( (0 < k < 1) \) satisfying

\[
\| Au - Av \| \leq k \| u - v \| \quad \text{for all} \quad u, v \in V.
\]

**Proof.** Let \( u, v \in V \). Then

\[
|Au(T, x) - Av(T, x)| \leq U_2 \int_\varepsilon h_{\omega D} \left| \frac{u(T, \xi)}{\sqrt{\xi^2 + u(T, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + u(T, \xi)^2}}{2T} \right| d\xi - \frac{v(T, \xi)}{\sqrt{\xi^2 + v(T, \xi)^2}} \tanh \frac{\sqrt{\xi^2 + v(T, \xi)^2}}{2T} d\xi.
\]

Note that each of \( \frac{\sqrt{\xi^2 + u(T, \xi)^2}}{2T} \) and \( \frac{\sqrt{\xi^2 + v(T, \xi)^2}}{2T} \) is regarded as 1 when \( T = 0 \). The integrand above becomes

\[
g(T; \xi, c(T, \xi, u, v)) |u(T, \xi) - v(T, \xi)|,
\]
where $g$ is that in (3.1). Here, $c(T, \xi, u, v)$ depends on $T, \xi, u$ and $v$, and satisfies $u(T, \xi) < c(T, \xi, u, v) < v(T, \xi)$ or $v(T, \xi) < c(T, \xi, u, v) < u(T, \xi)$.

As mentioned above, $T_1$ as well as $T^*_1$ satisfies (2.1), and hence satisfies (3.2) with $T_2$ replaced by $T_1$ or $T^*_1$. Therefore, by Lemma 3.2,

$$g(T; \xi, c(T, \xi, u, v)) \leq g(T^*; \xi, c(T, \xi, u, v)),$$

where $T^* = \Delta_2^{-1}(\Delta_0(T))$. Note that $\frac{Z}{\cosh^2 Z} < \tanh Z \ (Z > 0)$. Since the function $Z \mapsto \frac{\tanh Z}{Z}$ is strictly decreasing on $(0, \infty)$, it follows that

$$g(T^*; \xi, c(T, \xi, u, v)) \leq \frac{1}{\sqrt{\xi^2 + \Delta_1(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_1(T)^2}}{2T^*}.$$

Set

$$k = \sup_{T \in [0, T_1]} \int_{\epsilon}^{h_D} \frac{U_2}{\sqrt{\xi^2 + \Delta_1(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_1(T)^2}}{2T^*} d\xi.$$

Then $\|Au - Av\| \leq k \|u - v\|$. Combining $T^* = \Delta_2^{-1}(\Delta_0(T))$ with (1.6) yields

$$k < \sup_{T \in [0, T_1]} \int_{\epsilon}^{h_D} \frac{U_2}{\sqrt{\xi^2 + \Delta_0(T)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_0(T)^2}}{2T^*} d\xi$$

$$= \sup_{T \in [0, T_1]} \int_{\epsilon}^{h_D} \frac{U_2}{\sqrt{\xi^2 + \Delta_2(T^*)^2}} \tanh \frac{\sqrt{\xi^2 + \Delta_2(T^*)^2}}{2T^*} d\xi$$

$$= 1.$$

By the Banach fixed-point theorem (see e.g. Zeidler [11, pp.18-22]), there is a unique $u_1 \in V$ satisfying $Au_1 = u_1$. Let us fix $T \in [0, T_1]$. Then, for each $u \in V$, it follows that $u(T, \cdot) \in V_T$. Here, $V_T$ is that in (1.7). Theorem 1.7 thus imples $u_1 = u_0$. This completes the proof of Theorem 2.3.

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