ENHANCEMENT OF THE ZAKHAROV-GLASSEY’S METHOD
FOR BLOW-UP IN NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper we give a sharper condition for blow-up of the solution to a nonlinear Schrödinger equation with free/Stark/quadratic potential by improving the well known Zakharov-Glassey’s method.

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1. Introduction

We consider, in dimension one, the nonlinear Schrödinger equation
\[ \begin{cases}
  i\hbar \frac{\partial \psi}{\partial t} = H\psi + \nu |\psi|^2 \psi \\
  \psi(t)(x) = \psi_0(x), \quad \|\psi_0\|_{L^2} = 1,
\end{cases} \tag{1} \]
where $H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$ is the linear Schrödinger operator with potential $V(x)$; $\nu \in \mathbb{R}$ represents the strength of the nonlinear perturbation and $\mu > 0$ is the nonlinearity power. Hereafter, for sake of simplicity, we fix the units such that $\hbar = 1$ and $m = 1$, we further assume that $t_0 = 0$. The restriction to dimension one is just to simplify the discussion, but extension to higher dimensions of the ideas in this paper could be possible; however, we do not dwell on this problem here.

The first fundamental question that arises when dealing with a nonlinear Schrödinger equation (1) is the existence of a solution locally in time in some functional space. Thus, for $\psi_0$ in such a space and under some assumptions on the potential $V(x)$, there exists $0 < t^*_+ \leq +\infty$ such that $\psi_t \in C([0,t^*_+))$; furthermore, conservation of the norm
\[ N(\psi_t) = N(\psi_0) \quad \text{where} \quad N(\psi) := \|\psi\|_{L^2}, \tag{2} \]
and of the energy
\[ E(\psi_t) = E(\psi_0) \quad \text{where} \quad E(\psi) := \langle \psi, H\psi \rangle_{L^2} + \frac{\nu}{\mu + 1} \|\psi\|_{L^{2\mu+2}}^{2\mu+2}, \tag{3} \]
are satisfied. Concerning global existence in the future three possibilities may occur:
- $t^*_+ = +\infty$ and $\limsup_{t \to +\infty} \|\psi_t\|_{H^1} < +\infty$, that is the solution is global and bounded;

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- $t^+_* = +\infty$ and \( \limsup_{t \to +\infty} \| \psi_t \|_{H^1} = +\infty \), that is the solution blows up in infinite time;
- \( 0 < t^+_* < +\infty \) and \( \| \psi_t \|_{H^1} \to +\infty \) as \( t \to t^+_* - 0 \), that is the solution blows up in finite time.

A similar analysis can be considered in the past for \( t \leq 0 \).

Our purpose in this paper is to give a blow-up criterion by improving the Zakharov(-Shabat)-Glassey’s method. The method introduced by Zakharov and Shabat [22] and by Glassey [9] (see also the papers by [12, 14, 16]) is quite simple in the case where the virial identity takes a simple form.

Let
\[
I(t) = \langle \psi_t, x^2 \psi_t \rangle_{L^2}
\]
be the moment of inertia. It should be noticed that some textbooks denote (improperly) \( I \) by the name of variance; in fact, we will introduce the variance later. Eventually, in analogy with the usual definition in Classical Mechanics the term \( I \) should be properly called (polar) moment of inertia.

If it can be shown that \( I(T^+_V) = 0 \) (resp. \( I(T^-_V) = 0 \)) for some \( \pm T^+_V > 0 \) then blow-up occurs in the future at some \( t^+_* \in (0, T^+_V] \) (resp. in the past at some \( t^-_* \in [T^-_V, 0) \)). This fact is a consequence of the functional inequality (31) and of the conservation of the norm [2]. In order to prove that \( I(t) \) can take zero value at some instant \( t \) one usually makes use of the virial identity, which in the one-dimensional free model where \( V \equiv 0 \) takes the form
\[
\frac{d^2 I}{dt^2} = C_I + 2\nu \frac{\mu - 2}{\mu + 1} \| \psi_t \|_{L^{2\mu+2}}^2, \quad C_I = 4E(\psi_0).
\] (4)

If, for example, \( \mu = 2 \) and \( \psi_0 \) is such that \( E(\psi_0) < 0 \), then by the virial identity [4] and by the conservation of the energy, the positive quantity \( I(t) \) is an inverted parabola that must then become negative in finite times \( T^+_V \), \( -\infty < T^+_V < 0 < T^+_V < +\infty \), and thus the solution cannot exist for all time and blows up at finite time in the future as well as in the past [15].

This argument is very powerful because of its simplicity, in fact it is based on a pure Hamiltonian information \( E(\psi_0) < 0 \), and it also applies to the super-critical case \( \mu > 2 \). On the other hand, it strongly depends on the virial identity [4] and thus it cannot simply be applied when an external potential \( V(x) \) is present. However, in a sequence of seminal papers by Carles [3, 4, 5, 6] this method has been applied to the case where \( V(x) \) is a quadratic or Stark potential in any dimension.

Our proposal of enhancement of the Zakarov-Glassey’s method is based on a quite simple idea. Let
\[
\langle \hat{x} \rangle^t := \langle \psi_t, x \psi_t \rangle_{L^2}
\] (5)
be the expectation value of the position observable \( x \), where \( \hat{x} \) is the associated operator. Let
\[
\mathcal{V}(t) = \langle \psi_t, (\hat{x} - \langle \hat{x} \rangle^t)^2 \psi_t \rangle = I(t) - [\langle \hat{x} \rangle^t]^2
\]
be the variance. If it can be shown that \( \mathcal{V}(T^+_V) = 0 \) (resp. \( \mathcal{V}(T^-_V) = 0 \)) at some \( \pm T^+_V > 0 \) then blow-up occurs in the future for some \( t^+_* \in (0, T^+_V] \) (resp. in the past for some \( t^-_* \in [T^-_V, 0) \)), by [22]. Since \( \mathcal{V}(t) \leq I(t) \) then we expect to give a sharper condition for the occurrence of the blow-up; the price to pay is to give an expression of the expectation value \( \langle x \rangle^t \), but this problem can be easily overcome.
using the (generalized) Ehrenfest’s Theorem where \( \langle x \rangle^t \) is nothing but the solution of the “classical mechanics equation”. Finally, we must also emphasize the fact that the enhanced Zakharov-Glassey’s method not only gives sharper conditions for the occurrence of blow-up but also allows us to give a better estimate of the instants \( t^\pm \) at which the solution becomes singular because \( |T^V_\pm| \leq |T^I_\pm| \).

The paper is organized as follows. In Section 2 we recall the Ehrenfest’s generalized Theorem; in Section 3 we review the standard blow-up conditions in the free model where \( V(x) \equiv 0 \) and we show that these conditions can be easily improved by applying the virial equation for the variance \( V(t) \); in Section 4 we consider the case where \( V(x) = \alpha x \), \( \alpha \in \mathbb{R} \), is a Stark potential; in Section 5 we review the blow-up conditions in the case where \( V(x) = \alpha x^2 \), \( \alpha \in \mathbb{R} \), is a quadratic potential and we show that again these conditions can be easily improved by applying the virial equation for the variance \( V(t) \).

Hereafter, for the sake of simplicity, we omit the dependence on the variable \( t \) when this fact does not cause misunderstandings, e.g. \( \psi_x \) instead of \( \psi_x^t \), \( \langle \hat{x} \rangle \) instead of \( \langle \hat{x} \rangle^t \), \( \langle \hat{p} \rangle \) instead of \( \langle \hat{p} \rangle^t \), \( I \) instead of \( I(t) \), \( V \) instead of \( V(t) \), and so on.

### 2. Ehrenfest’s generalized Theorem for NLS

The extension of the Ehrenfest’s Theorem to the nonlinear Schrödinger equation (1) has already been considered by [2, 11]. In fact, by means of a straightforward calculation it follows that

**Proposition 1.** Let \( a = a(x,p) \), \( x, p \in \mathbb{R} \), be a classical observable function with associated operator \( A \), let

\[
\langle A \rangle = \langle \psi_t, A \psi_t \rangle_{L^2}
\]

be its expectation value. Then

\[
\frac{d\langle A \rangle}{dt} = i \langle \psi_t, [H, A] \psi_t \rangle_{L^2} + \imath \nu \langle \psi_t, [\psi_t |^{2\mu}, A] \psi_t \rangle_{L^2},
\]

where \([H, A] = HA - AH \) is the commutator operator between the operators \( H \) and \( A \), and where \([\psi|^{2\mu}, A] \psi = |\psi|^{2\mu} A(\psi) - A(|\psi|^{2\mu} \psi) \). Equation (7) is usually called “Ehrenfest’s generalized Theorem”.

As a consequence it follows that

**Corollary 1.** Let \( x \) be the position observable and let \( \hat{x} = x \) be the associated multiplication operator, then

\[
\frac{d\langle \hat{x} \rangle^t}{dt} = \langle \hat{p} \rangle^t
\]

where \( \hat{p} = -i \frac{\partial}{\partial x} \) is the associated operator to the momentum observable \( p \).
Proof. Corollary 1 immediately follows from (7) since \(|\psi|^2 \hat{x} = 0\); hence
\[
\frac{d\langle \hat{x} \rangle}{dt} = i \langle \psi, [H, \hat{x}] \psi \rangle = i \left\langle \psi, \left[ \frac{\hat{p}^2}{2}, \hat{x} \right] \psi \right\rangle = \langle \hat{p} \rangle.
\]
\[\square\]

Similarly

Corollary 2. Let \( p \) be the momentum observable with associated operator \( \hat{p} = -i \frac{\partial}{\partial x} \), then
\[
\frac{d\langle \hat{p} \rangle}{dt} = -\left\langle \frac{dV}{dx} \right\rangle, \quad \text{where} \quad \left\langle \frac{dV}{dx} \right\rangle_{L^2} = \left\langle \psi_t, \frac{dV}{dx} \psi_t \right\rangle.
\]

Proof. Corollary 2 follows from (7) if we prove that \( \langle \psi, \left[ |\psi|^2 \mu, \hat{p} \right] \psi \rangle = 0 \); indeed
\[
\langle \psi, \left[ |\psi|^2 \mu, \hat{p} \right] \psi \rangle = -i \int_{\mathbb{R}} \bar{\psi} \left[ |\psi|^2 \mu \frac{\partial \psi}{\partial x} - \partial \left( |\psi|^2 \mu \psi \right) \right] dx = -i \int_{\mathbb{R}} |\psi|^2 \mu \left[ \frac{-\partial \psi}{\partial x} + \psi \frac{\partial \bar{\psi}}{\partial x} \right] dx = -i \int_{\mathbb{R}} \rho \frac{\partial \rho}{\partial x} dx = 0
\]
where \( \rho = |\psi|^2 \). Hence
\[
\frac{d\langle \hat{p} \rangle}{dt} = i \langle \psi, [H, \hat{p}] \psi \rangle = i \langle \psi, [V, \hat{p}] \psi \rangle = -\left\langle \frac{dV}{dx} \right\rangle.
\]
\[\square\]

Remark 1. Let \( \psi_t \) be the solution to the NLS (1); then the expectation values \( \langle \hat{x} \rangle \) of the position observable and \( \langle \hat{p} \rangle \) of the momentum observable satisfy to the “classical canonical equation of motion” (8-9). In the case where \( V(x) \) is a free, Stark or quadratic potential then the system (8-9) has an explicit solution that does not depend on the nonlinearity parameter \( \nu \).

Remark 2. Ehrenfest’s generalized Theorem was also proved by [1] for nonlinear Schrödinger equations with a 2 or 3-dimensional confining harmonic potential and under the effect of a rotating force. In such a framework it has also been proved that, under some circumstances (see Proposition 4.3 by [1]), the solution is such that \( \langle \hat{x} \rangle^t \) and \( \langle \hat{p} \rangle^t \) go to \(+\infty\) when \( t \) goes to \( \pm\infty \).

Remark 3. However, we should point out that the Ehrenfest’s generalized Theorem (7) for nonlinear Schrödinger does not give the same result of the usual one for linear Schrödinger equations
\[
\frac{d\langle A \rangle}{dt} = i \langle [H, A] \rangle
\]
if the classical observable is the Hamiltonian function \( h(x, p) = \frac{1}{2}p^2 + V(x) \) with associated operator \( H \); indeed, in such a case
\[
\frac{d\langle H \rangle}{dt} = i\nu \langle \psi, [|\psi|^2 \mu, H] \psi \rangle = -\nu \Im \langle \psi, |\psi|^2 \mu \hat{p}^2 \psi \rangle
\]
is not generically zero. In fact, \( \langle H \rangle \) is an integral of motion only when \( \nu = 0 \); otherwise the integral of motion is the energy \( \mathcal{E}(\psi) \) defined by (3).
3. Blow-up for the free NLS

We consider now the case where the external potential is zero: $V(x) \equiv 0$. We assume that

$$
\psi_0 \in \Sigma := H^1(\mathbb{R}) \cap D(\hat{x}),
$$

where $D(\hat{x})$ is the domain of the operator $\hat{x}$. Then the solution $\psi(x,t)$ to (11) locally exists and it belongs to $C((t^{\star}_{\pm}, t^\star_\pm), \Sigma)$ and the conservation of the norm $\|\psi\|_{L^2}$ and of the energy $\mathcal{E}$ hold true (see, e.g., Theorem 3.10 by [19]). If $t^\star_\pm = \pm \infty$ then the solution locally exists; if not, i.e. $t^\star_+ < +\infty$ (resp. $t^-_+ > -\infty$) then

$$
\lim_{t \to t^\star_\pm \mp 0} \|\psi\|_{H^1} = \infty
$$

and thus blow-up occurs in the future (resp. in the past). We observe that blow-up cannot occur when $\nu \geq 0$ because of the conservation of the energy. Furthermore, we can also point out that when blow-up occurs for $\nu < 0$ then we also have that

$$
\lim_{t \to t^\star_\pm \mp 0} \|\psi\|_{L^{2\nu+2}} = \infty
$$

because conservation of the energy.

3.1. Criterion for blow-up by means of the Zakharov-Glassey method.

Estimates of the momentum of inertia can be obtained by means of the one-dimensional virial identity (4) the with initial conditions

$$
\mathcal{I}_0 := \mathcal{I}(0) = \|x\psi_0\|_{L^2}^2
$$

and

$$
\dot{\mathcal{I}}_0 := \frac{d\mathcal{I}(0)}{dt} = 23 \left[ \int_{\mathbb{R}} x\bar{\psi}_0(x) \frac{\partial \psi_0(x)}{\partial x} dx \right] = 2\Re \langle \hat{x}\psi_0, \hat{p}\psi_0 \rangle .
$$

Theorem 5.1 by [19] gives a condition for blow-up in the future (and similarly in the past). Specifically, when $\nu < 0$ and $\mu \geq 2$ then there exists a $t^\star_+ \in (0, +\infty)$ such that

$$
\lim_{t \to t^\star_+ \mp 0} \|\psi\|_{H^1} = \infty
$$

if any of the following conditions is satisfied:

i. $C_\mathcal{I} < 0$;

ii. $C_\mathcal{I} = 0$ and $\mathcal{I}_0 < 0$;

iii. $C_\mathcal{I} > 0$ and $\mathcal{I}_0 \leq -\sqrt{2C_\mathcal{I}\mathcal{I}_0}$;

where $C(\mathcal{I}) = 4\mathcal{E}(\psi_0)$.

The proof of Theorem 5.1 by [19] is quite simple: if $\mu \geq 2$ and $\nu \leq 0$ then (4) implies that

$$
\frac{d^2 \mathcal{I}}{dt^2} \leq C_\mathcal{I}
$$

and thus

$$
\mathcal{I}(t) \leq M(t) := \frac{1}{2} C_\mathcal{I} t^2 + \mathcal{I}_0 t + \mathcal{I}_0
$$

(14)
If any of the three conditions i.-iii. are satisfied then there exists $\tilde{T}_V^+ > 0$ such that $M(\tilde{T}_V^+) = 0$ and thus there exists a $0 < T_V^+ < \tilde{T}_V^+$ such that $J(T_V^+) = 0$. From this fact and from (31) the occurrence of blow-up in the future follows at some $t^*_+ < T_V^+$. 

3.2. Criterion for blow-up by means of the enhanced Zakharov-Glassey method. We improve now the previous criterion by applying the same argument to the analysis of the variance and making use of the Ehrenfest’s generalized Theorem. Indeed, if the potential $V(x)$ is exactly zero then (8-9) imply that

$$\langle \hat{p} \rangle \equiv \hat{p}_0 \quad \text{and} \quad \langle \hat{x} \rangle = \hat{x}_0 t + \hat{x}_0,$$

where $\hat{x}_0 := \langle \hat{x} \rangle|_{t=0}$ and $\hat{p}_0 := \langle \hat{p} \rangle|_{t=0}$. \hfill (15)

Remark 4. We point out that in the free NLS problem the conservation of the momentum $\langle \hat{p} \rangle$ and the fact that the center of mass of the wavepacket $\langle \hat{x} \rangle$ moves at constant speed can be derived by making use of arguments of invariance of space translation (see, e.g. §2.3 by [19]).

Since (15) we have that

$$V(t) = J(t) - \langle \hat{x} \rangle^2 = J(t) - [\hat{p}_0 t + \hat{x}_0]^2 \leq N(t)$$

where

$$N(t) := M(t) - [\hat{p}_0 t + \hat{x}_0]^2 = \left[ \frac{1}{2} C_I \hat{p}_0^2 \right] t^2 + \left[ \hat{I}_0 - 2\hat{p}_0 \hat{x}_0 \right] t + \left[ I_0 - \hat{x}_0^2 \right]$$

Thus we have the following improvement of Theorem 5.1 by [19].

**Theorem 1.** Let $\nu < 0$ and $\mu \geq 2$, let $\psi_0 \in \Sigma$; then we have blow-up in the future if any of the following conditions is satisfied:

i'. $C_I < 2\hat{p}_0^2$;

ii'. $C_I = 2\hat{p}_0^2$ and $\hat{I}_0 < 2\hat{p}_0 \hat{x}_0$;

iii'. $C_I > 2\hat{p}_0^2$ and

$$\left[ \hat{I}_0 - 2\hat{p}_0 \hat{x}_0 \right] \leq -2\sqrt{\left[ \frac{1}{2} C_I \hat{p}_0^2 \right] \left[ I_0 - \hat{x}_0^2 \right]}$$

**Proof.** Indeed, if any of the three conditions i.-iii. are satisfied then there exists $\tilde{T}_V^+ > 0$ such that $N(\tilde{T}_V^+) = 0$ and thus there exists $0 < T_V^+ < \tilde{T}_V^+$ such that $V(T_V^+) = 0$. From this fact and from (32) the occurrence of blow-up follows for some $t^*_+ \leq T_V^+$. \hfill \Box

**Remark 5.** In fact, under condition i'. we have blow-up in the future and in the past, too; under conditions ii'. and iii'. we have blow-up in the future only.

**Remark 6.** We remark that condition i'. for blow-up is not new and it has been already proved under some circumstances, see e.g. Corollary 1.2 by [8] and Theorem 7 by [16].

4. Blow-up for the NLS with Stark potential

Let the potential $V(x) = \alpha x$ be a Stark potential, where $\alpha \in \mathbb{R} \setminus \{0\}$, the occurrence of blow-up in such a case has been considered by [9][13][17]. Again we assume (11).
4.1. Criterion for blow-up by means of the Zakharov-Glassey method. In the case of Stark potentials it has been proved that the solutions to the NLS with a Stark potential can be derived from the ones of the free NLS, see Theorem 2.1 by [5]. Then one can make use of the results obtained in Section 3.1; in particular, Corollary 3.3 by [6] states that blow-up occurs in the past and in future when

\[ \frac{1}{2} \| \psi' \|_{L^2}^2 + \frac{\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2} < 0. \] (17)

4.2. Criterion for blow-up by means of the enhanced Zakharov-Glassey method. If the potential \( V(x) = \alpha x \) is a Stark potential, where \( \alpha \in \mathbb{R} \setminus \{0\} \) then (8) and (9) imply that

\[ \langle \dot{\rho} \rangle t = -\alpha \dot{t} + \dot{p}_0 \quad \text{and} \quad \langle \dot{x} \rangle t = -\frac{1}{2} \alpha t^2 + \dot{p}_0 t + \dot{x}_0 \] (18)

where

\[ \dot{x}_0 = \langle \dot{x} \rangle \bigg|_{t=0} \quad \text{and} \quad \dot{p}_0 = \langle \dot{p} \rangle \bigg|_{t=0}. \]

Estimates of the momentum of inertia can be obtained by means of the one-dimensional virial identity [6] with initial conditions [12][13].

If \( \nu(\mu - 2) \leq 0 \) then (18) and (38) imply that

\[ \frac{d^2 \mathcal{I}}{dt^2} \leq 4 \mathcal{E} - 6\alpha \left( -\frac{1}{2} \alpha t^2 + \dot{p}_0 t + \dot{x}_0 \right) \]

where

\[ \mathcal{E} = \frac{1}{2} \| \psi' \|_{L^2}^2 + \alpha \langle \dot{x} \rangle + \frac{\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2}, \]

and thus

\[ \mathcal{I}(t) \leq \frac{1}{4} \alpha^2 t^4 - \alpha \dot{p}_0 t^3 + [2\mathcal{E} - 3\alpha \dot{x}_0] t^2 + \dot{\mathcal{I}}_0 t + \mathcal{I}_0, \]

\( \mathcal{I}_0 \) and \( \dot{\mathcal{I}}_0 \) are given by [12] and [13]. Therefore,

\[ \mathcal{V}(t) = \mathcal{I}(t) - \left[ \langle \dot{x} \rangle \right]^2 \leq \left[ \| \psi' \|^2 + \frac{2\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2} - \dot{p}_0^2 \right] t^2 + 2 [\Re \langle \dot{x} \psi, \dot{\psi} \rho \rangle - \dot{p}_0 \dot{x}_0] t + \mathcal{V}(0) \]

Thus, we can conclude that

\textbf{Theorem 2.} If

i. \( \| \psi' \|^2 + \frac{2\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2} < \dot{p}_0^2 \) then we have blow-up in the past and in future;

ii. \( \| \psi' \|^2 + \frac{2\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2} = \dot{p}_0^2 \) and \( \Re \langle \dot{x} \psi, \dot{\psi} \rho \rangle - \dot{p}_0 \dot{x}_0 \neq 0 \) we have blow-up in the past or in future;

iii. \( \| \psi' \|^2 + \frac{2\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2} > \dot{p}_0^2 \) and

\[ [\Re \langle \dot{x} \psi, \dot{\psi} \rho \rangle - \dot{p}_0 \dot{x}_0] > \left[ \| \psi' \|^2 + \frac{2\nu}{\mu + 1} \| \psi\|_{L^{2\mu+2}}^{2\mu+2} - \dot{p}_0^2 \right] \mathcal{V}(0) \]

we have blow-up in the past or in the future.

\textbf{Remark 7.} Since \( \| \psi' \|^2 = \| \dot{\psi} \rho \|^2 \) and \( \dot{p}_0 = \langle \psi, \dot{\psi} \rho \rangle \leq \| \dot{\psi} \rho \| \) then conditions i. and ii. holds true only when \( \nu < 0 \).
Remark 8. We remark that the blow-up condition \([17]\) given by \([5]\) agrees with Theorem 2, indeed \([17]\) implies i.

5. Blow-up for NLS with harmonic/inverted oscillator potential

In this section we consider the cases of harmonic oscillator potential \(V(x) = \alpha x^2\), where \(\alpha > 0\), and inverted oscillator potential, where \(\alpha < 0\). The occurrence of blow-up in these cases has been considered by several authors under different assumptions \([3, 4, 6, 7, 10, 18, 20, 21, 23]\).

In this Section we consider the blow-up conditions obtained by means of the enhanced Zakharov-Glassey’s method and then we compare these results with the previous ones obtained by Carles \([3, 4, 6]\).

We require now some preliminary results.

Also in this case assume \((11)\), then local in time existence of the solution to \((1)\) in \(\Sigma\) and conservation of the norm and of the energy \(E\) follows (see, e.g., \([6]\)).

Corollary 2 implies that
\[
\int_0^t \langle \dot{x} \rangle dt = -2\alpha \langle x \rangle; \\

\int_0^t \langle p \rangle dt = -\lambda \langle x \rangle + \langle p \rangle.
\]

In a previous paper \([4]\) devoted to the analysis of the occurrence of blow-up it has been found that, in the case of harmonic/inverted potential, the momentum of inertia \(I\) satisfies to the following equation
\[
\frac{d^2 I}{dt^2} + 8\alpha I = C_I + 2\nu \mu - 2 \frac{\|\psi\|_{L_{2\mu+2}}^2}{\mu + 1} + L_{2\mu+2}^2,
\]
where \(C_I = 4\mathcal{E}(\psi_0)\). (21)

As in the free case we consider now the equation for the variance \(\mathcal{V}\).

Lemma 1. The variance \(\mathcal{V}\) satisfies to the following equation
\[
\frac{d^2 \mathcal{V}}{dt^2} + 8\alpha \mathcal{V} = C_V + 2\nu \mu - 2 \frac{\|\psi\|_{L_{2\mu+2}}^2}{\mu + 1},
\]
where
\[
C_V = -2\hat{p}_0^2 - 4\alpha \hat{x}_0^2 + C_I.
\]

Proof. Indeed, form \([21]\) it turns out that the variance is a solution to the equation
\[
\frac{d^2 \mathcal{V}}{dt^2} + 8\alpha \mathcal{V} = -\frac{d^2 \langle x \rangle}{dt^2} - 8\alpha \langle \dot{x} \rangle^2 + C_I + 2\nu \mu - 2 \frac{\|\psi\|_{L_{2\mu+2}}^2}{\mu + 1},
\]
where \(\langle \dot{x} \rangle\) simply denotes \(\langle \dot{x} \rangle\) t and it is given by \([19]\) (resp. \([20]\)) when \(\alpha > 0\) (resp. \(\alpha < 0\)). We may remark that the term
\[
C = -\frac{d^2 \langle \dot{x} \rangle^2}{dt^2} - 8\alpha \langle \dot{x} \rangle^2
\]
is constant. Indeed,
\[ C = -2 \left( \frac{d(\dot{x})}{dt} \right)^2 - 2 \langle x \rangle \frac{d^2(\dot{x})}{dt^2} - 8 \alpha \langle x \rangle^2 = -2 \left( \frac{d(\dot{x})}{dt} \right)^2 - 4 \alpha \langle x \rangle^2 \]
since \( \frac{d^2(\dot{x})}{dt^2} = -2 \alpha (\dot{x}) \) and thus
\[ \frac{dC}{dt} = -4 \frac{d(\dot{x})}{dt} \frac{d^2(\dot{x})}{dt^2} - 8 \alpha \frac{d(\dot{x})}{dt} = 0. \]
Hence,
\[ C = -2 \left( \frac{d(\dot{x})}{dt} \right)^2 \bigg|_{t=0} - 4 \alpha \dot{x}_0^2 = -2 \ddot{p}_0 - 4 \alpha \dot{x}_0^2 \]
and (22) follows.

We recall that the initial condition associated to (21) and (22) are
\[ V_0 := V(0) = t_0 - \dot{x}_0^2 = \| \dot{x}_0 \|^2 - \dot{x}_0^2 \]
and
\[ \dot{V}_0 := \frac{dV(0)}{dt} = 2 \Re(\dot{x}_0 \dot{p}_0 - \ddot{x}_0 \dot{p}_0) . \]

Let us consider now the differential equation (22) for \( \mu = 2 \) and \( \nu < 0 \). From Lemma 3 in Appendix B we have that \( 0 \leq V(t) \leq \zeta(t) \) where \( \zeta(t) \) is the solution to
\[
\begin{cases}
\frac{d^2 \zeta}{dt^2} + 8 \alpha \zeta = C_V \\
\zeta(0) = V_0 \quad \text{and} \quad \frac{d\zeta(0)}{dt} = \dot{V}_0 .
\end{cases}
\]
If we set \( \Omega = 2 \lambda = \sqrt{8|\alpha|} \) then the solution \( \zeta(t) \) is given by
\[
\zeta(t) = \begin{cases}
\zeta_H(t) := \frac{\dot{V}_0}{\Omega} \sin(\Omega t) + V_0 \cos(\Omega t) + \frac{\Omega}{16} C_V [1 - \cos(\Omega t)] , & \text{if } \alpha > 0 \\
\zeta_H(t) := \frac{\dot{V}_0}{\Omega} \sinh(\Omega t) + V_0 \cosh(\Omega t) - \frac{\Omega}{16} C_V [1 - \cosh(\Omega t)] , & \text{if } \alpha < 0.
\end{cases}
\]

Now, we are ready to apply the enhanced Zakharov-Glassey method.

5.1. Harmonic oscillator - Criterion for blow-up. In the case of the harmonic oscillator potential, where \( \alpha > 0 \), from Lemma 3 in Appendix B it follows that the variance \( V(t) \) is bounded from above by the function
\[ \zeta_H(t) = \sqrt{a^2 + b^2} \sin(\Omega t + \varphi) + c \]
for any \( t \) such that \( |t| \leq \pi \), where \( \varphi \) is a phase term such that
\[ \frac{a}{\sqrt{a^2 + b^2}} = \cos \varphi , \quad \frac{b}{\sqrt{a^2 + b^2}} = \sin \varphi , \quad a := \frac{\dot{V}_0}{\Omega} , \quad b := V_0 - \frac{C_V}{\Omega^2} , \quad c := \frac{C_V}{\Omega^2} . \]
Since \( \zeta_H(\pm \pi/\Omega) = \frac{2}{\Omega} C_V - V_0 \) then we have blow-up in the future and in the past if
\[ 2 \frac{C_V}{\Omega^2} - V_0 \leq 0 . \]
(26)
If not, since by means of a straightforward calculation it follows that
\[
\min_{t \in [-\pi/\Omega, +\pi/\Omega]} V(t) \leq \min_{t \in [-\pi/\Omega, +\pi/\Omega]} \zeta_H(t) = \frac{C_V}{\Omega^2} - \sqrt{\frac{V_0^2}{\Omega^2} + \left( V_0 - \frac{C_V}{\Omega^2} \right)^2}
\]
and then there exists blow-up in the past or in the future if
\[ \dot{V}_0^2 + V_0^2 \Omega^2 - 2V_0 C_V \geq 0. \]  
(27)

Thus, we have proved the following results.

**Theorem 3.** Let \( \psi_0 \in \Sigma \) be the normalized initial wavefunction; let \( \mu \geq 2, \alpha > 0 \) and \( \Omega = \sqrt{8\alpha} \); let \( C_V, V_0 \) and \( \dot{V}_0 \) defined as in (23), (24) and (25). Then, in the focusing nonlinearity case such that \( \nu < 0 \) blow-up occurs in the past and in the future at some instants \( \tilde{T}_- \leq t^* < 0 < t^*_+ \leq \tilde{T}_+ \) if (20) is satisfied; where \( \tilde{T}_\pm \) are the solutions to the equation \( \zeta(t) = 0 \) in the interval \( [-\pi/\Omega, \pi/\Omega] \). If (20) is not satisfied, but (27) holds true then blow-up occurs in the past or in the future in the interval \( [-\pi/\Omega, +\pi/\Omega] \).

**Remark 9.** We compare now the results above with the ones given by Proposition 3.2 [3] where occurrence of blow-up in the past and in the future was proved in the case of harmonic potential where \( \alpha > 0 \), focusing nonlinearity where \( \nu < 0 \), and under the conditions \( \mu \geq 2 \) and
\[ \frac{1}{2} \| \nabla \psi_0 \|^2_{L^2} + \frac{\nu}{\mu + 1} \| \psi_0 \|^2_{L^{2\mu + 2} \Omega} \leq 0. \]  
(28)

In fact, condition (28) implies that (since \( V_0 > 0 \))
\[ \Omega^2 I_0 \geq 8 \epsilon \quad \Leftrightarrow \quad \Omega^2 V_0 - 2C_V \geq 4p_0^2. \]

That is, if (28) occurs then (20) is satisfied (but not vice versa).

5.2. Inverted oscillator - Criterion for blow-up. In the case of the inverted oscillator potential where \( \alpha < 0 \) a similar argument proves that the variance \( V(t) \) is bounded from above by the function \( \zeta(t) \) for any \( t \in \mathbb{R} \). Then, it follows that

**Theorem 4.** Let \( \psi_0 \in \Sigma \) be the normalized initial wavefunction; let \( \mu \geq 2, \alpha < 0 \) and \( \Omega = \sqrt{8|\alpha|} \); let \( C_V, V_0 \) and \( \dot{V}_0 \) defined as in (23), (24) and (25). Let it now
\[ a := \frac{\dot{V}_0}{\Omega}, \quad b := V_0 + \frac{C_V}{\Omega^2} \quad \text{and} \quad c := -\frac{C_V}{\Omega^2}. \]

Then, in the focusing nonlinearity case such that \( \nu < 0 \) blow-up occurs if
i. \( b < -|a| \); in that case blow-up occurs in both the past and in the future.
ii. \( |a| < b \) and \( \sqrt{b^2 - a^2} + c \leq 0 \); in that case blow-up occurs only in the future (if \( a < 0 \)) or only in the past (if \( a > 0 \)).
iii. \( |a| > |b| \); in that case we have blow-up in the past if \( a > 0 \) or in the future if \( a < 0 \).
iv. \( |a| = |b| \); in that case we have blow-up if \( bc < 0 \), in particular we have blow-up in the past if \( a > 0 \) or in the future if \( a < 0 \).

**Proof.** Let us introduce the function \( \zeta(t) = \zeta_I(t) \) where \( \tau = \Omega t \), then
\[ \zeta(\tau) := a \sinh(\tau) + b \cosh(\tau) + c, \quad \zeta(0) = V(0) > 0, \]
and where \( a, b \) and \( c \) are defined above. If
1) \( |a| < |b| \) then \( \zeta'(|\tau_1|) = 0 \) where \( \tau_1 = \text{arctanh} \left( \frac{-b}{a} \right) \). In particular, if:
1a) \( b < 0 \) then \( \lim_{\tau \to \pm \infty} \zeta(\tau) = -\infty \) and thus there exists \( \mathcal{T}_- < 0 < \mathcal{T}_+ \) such that \( \zeta(\mathcal{T}_\pm) = 0 \). In such a case we have blow-up in the past and in the future.
1b) $0 < b$ then $\lim_{\tau \to +\infty} \zeta(\tau) = +\infty$. We compute now

$$\zeta(\tau_k) = \sqrt{b^2 - a^2} + c.$$ 

Thus, if

$$\sqrt{b^2 - a^2} + c \leq 0$$

then we have blow-up in the future if $a < 0$ or in the past if $a > 0$.

2) $|a| > |b|$ then $\zeta(\tau)$ is a monotone increasing (resp. decreasing) function if $a > 0$ (resp. $a < 0$) such that $\lim_{\tau \to +\infty} \zeta(\tau) = \pm \infty$ (resp. $\mp \infty$); therefore there exists $T_0 < 0$ (resp. $0 < T_+)$ such that $\zeta(T_0) = 0$ (resp. $\zeta(T_+) = 0$), and thus we have blow-up in the past (resp. in the future).

Lemma 2.

The following inequality holds true: let $a > 0$ such that $\lim_{\tau \to -\infty} \zeta(\tau) = c$ and $\lim_{\tau \to +\infty} \zeta(\tau) = 0$. Hence, if:

3a) $a > 0$ then $d\zeta(0) > 0$ and then $d\zeta(\tau) > 0$ for any $\tau$; furthermore, $\lim_{\tau \to -\infty} \zeta(\tau) = c$ and $\lim_{\tau \to +\infty} \zeta(\tau) = +\infty$. Thus, if $c < 0$ then there exists $T_0 < 0$ such that $\zeta(T_0) = 0$ and so we have blow-up in the past.

3b) $a < 0$ then $d\zeta(0) < 0$ and then $d\zeta(\tau) < 0$ for any $\tau$; furthermore, $\lim_{\tau \to -\infty} \zeta(\tau) = c$ and $\lim_{\tau \to +\infty} \zeta(\tau) = -\infty$. Thus, if $c > 0$ then there exists $0 < T_+ < 0$ such that $\zeta(T_+) = 0$ and so we have blow-up in the future.

4) $a = -b$ then $d\zeta(0) \neq 0$ for any $\tau$. Hence, if:

4a) $a > 0$ then $d\zeta(0) > 0$ and then $d\zeta(\tau) > 0$ for any $\tau$; furthermore, $\lim_{\tau \to -\infty} \zeta(\tau) = -\infty$ and $\lim_{\tau \to +\infty} \zeta(\tau) = c$. Thus, if $c > 0$ then there exists $T_0 < 0$ such that $\zeta(T_0) = 0$ and so we have blow-up in the past.

4b) $a < 0$ then $d\zeta(0) < 0$ and then $d\zeta(\tau) < 0$ for any $\tau$; furthermore, $\lim_{\tau \to -\infty} \zeta(\tau) = -\infty$ and $\lim_{\tau \to +\infty} \zeta(\tau) = c$. Thus, if $c < 0$ then there exists $T_0 < 0$ such that $\zeta(T_0) = 0$ and so we have blow-up in the past.

Collecting all these results then Theorem 4 follows.

Theorem 1.1.

For example, [6] proved that in the case of inverted potential, where $\alpha < 0$, and focusing nonlinearity, where $\nu < 0$, under the condition $\mu \geq 2$ and

$$\frac{1}{2} \| \nabla \psi_0 \|_{L^2}^2 + \mu + 1 \| \psi_0 \|_{L^{2\mu+2}}^2 \leq -|\alpha| \| x \psi_0 \|_{L^2}^2 - \sqrt{2|\alpha|} \| R \langle \hat{x} \psi_0, \hat{x} \psi_0 \rangle \|$$

(29)

then blow-up occurs in the future and in the past at some instant. By means of a straightforward calculation it can be proved that if condition (29) is satisfied, then condition i. of Theorem 4 holds true, but not vice versa.

Appendix A. Functional inequalities

Lemma 2. The following inequality holds true: let $y \in \mathbb{R}$ and let

$$\Gamma := \langle (x - y)^2 f, f \rangle_{L^2}$$

for any test function $f \in L^2(\mathbb{R}, dx)$ such that $xf \in L^2(\mathbb{R}, dx)$. Then, for any $q \geq 0$:

$$\| f \|_{L^{2q+2}} \leq C \sqrt{\Gamma} \| f \|_{L^2} \| f' \|_{L^2}^{q+1},$$

(30)

for some positive constant $C$, where $f' = \frac{df}{dx}$.
Proof. Indeed:
\[
\|f\|_{L^{q+2}}^{2q+2} = \int_{\mathbb{R}} \frac{\partial(x - y)}{\partial x} f^{q+1} \bar{f}^{q+1} dx
\]
\[
= -(q + 1) \int_{\mathbb{R}} (x - y) |f|^{2q} \left[ f' \bar{f} + f \bar{f}' \right] dx.
\]
Hence
\[
\|f\|_{L^{q+2}}^{2q+2} \leq 2(q + 1) \|(x - y)f\|_{L^2} \|f\|_{L^\infty} \|f'\|_{L^2}.
\]
Now, recalling that from the Gagliardo-Nirenberg inequality one has that
\[
\|f\|_{L^\infty} \leq \sqrt{2} \|f'\|^{1/2}_{L^2} \|f\|^{1/2}_{L^2},
\]
then it follows that
\[
\|f\|_{L^{q+2}}^{2q+2} \leq C \sqrt{\Gamma} \|f\|_{L^2}^{q} \|f'\|_{L^2}^{q+1},
\]
for some positive constant \(C\).

\[\square\]

Corollary 3. In particular, if \(y = 0\) and \(q = 0\) then we have that for some positive constant \(C\):
\[
\|f\|_{L^2}^2 \leq C \sqrt{\Gamma} \|f'\|_{L^2},
\]
where \(\Gamma(0) = \|xf\|_{L^2} = I\) is the moment of inertia; if \(y = \langle \hat{x} \rangle = \langle f, xf \rangle_{L^2}\) and \(q = 0\) then we have that:
\[
\|f\|_{L^2}^2 \leq C \sqrt{\Gamma} \|f'\|_{L^2},
\]
where \(\Gamma(\langle \hat{x} \rangle) = \|(x - \langle \hat{x} \rangle)f\|_{L^2}^2 = V\) is the variance.

Appendix B. Comparison between solutions of the harmonic/inverted oscillator

Let \(V_\pm(t)\) be the solution to the differential equation
\[
\left\{ \begin{array}{l}
\frac{d^2 V_\pm}{dt^2} \pm \Omega^2 V_\pm = C + f(t) \\
V_\pm(0) = V_{\pm,0} \quad \text{and} \quad \frac{dV_\pm(0)}{dt} = \dot{V}_{\pm,0}
\end{array} \right.,
\]
where \(C\) is a constant factor and \(f(t) \leq 0\) for any \(t\); and let \(\zeta_\pm(t)\) be the solution to the differential equation
\[
\left\{ \begin{array}{l}
\frac{d^2 \zeta_\pm}{dt^2} \pm \Omega^2 \zeta_\pm = C \\
\zeta_\pm(0) = V_{\pm,0} \quad \text{and} \quad \frac{d\zeta_\pm(0)}{dt} = \dot{V}_{\pm,0}
\end{array} \right..
\]
Then, the difference \(Z_\pm(t) = V_\pm(t) - \zeta_\pm(t)\) solves the differential equation
\[
\left\{ \begin{array}{l}
\frac{d^2 Z_\pm}{dt^2} \pm \Omega^2 Z_\pm = f(t) \\
Z_\pm(0) = 0 \quad \text{and} \quad \frac{dZ_\pm(0)}{dt} = 0
\end{array} \right..
\]
Hence, we have that
\[
Z_+(t) = \frac{1}{\Omega} \int_0^t \sin [\Omega(t - s)] f(s) ds \leq 0 \quad \text{if} \quad \Omega |t| \leq \pi
\]
and
\[
Z_-(t) = \frac{1}{\Omega} \int_0^t \sinh [\Omega(t - s)] f(s) ds \leq 0, \forall t \in \mathbb{R}.
\]
In conclusion,
Lemma 3. Let $V_\pm$ be the solution to (33), and let

$$
\zeta_+(t) = \frac{\dot{V}_{+,0}}{\Omega} \sin(\Omega t) + V_{+,0} \cos(\Omega t) + \frac{1}{\Omega^2} C_\zeta [1 - \cos(\Omega t)]
$$

and

$$
\zeta_-(t) = \frac{\dot{V}_{-,0}}{\Omega} \sinh(\Omega t) + V_{-,0} \cosh(\Omega t) - \frac{1}{\Omega^2} C_\zeta [1 - \cosh(\Omega t)]
$$

be the solution to (34). Then

$$
V_+(t) \leq \zeta_+(t), \forall t \in \left[\frac{-\pi}{\Omega}, \frac{\pi}{\Omega}\right]
$$

and

$$
V_-(t) \leq \zeta_-(t), \forall t \in \mathbb{R}.
$$

APPENDIX C. A formal touch - the virial identity

Here we formally derive the virial identity for any real-valued potential $V(x)$. Hereafter, we denote $\psi_t$ by $\psi$ and $\psi' = \frac{\partial \psi}{\partial x}$, $\psi'' = \frac{\partial^2 \psi}{\partial x^2}$, $\dot{\psi} = \frac{\partial \psi}{\partial t}$, $\dot{I} = \frac{d}{dt} I$, $\ddot{I} = \frac{d^2}{dt^2} I$, and so on.

Let (3) be the energy integral of motion (here we make no assumptions about the values of the mass $m$ and of the Planck constant $\hbar$):

$$
E(\psi) := \frac{\hbar^2}{2m} \langle \psi', \psi' \rangle + \langle \psi, V \psi \rangle + \frac{\nu}{\mu + 1} \|\psi\|^{2\mu + 2}_{L^{2\mu + 2}},
$$

Let $I(t) = \langle \dot{x}^2 \rangle_t = \langle \psi_t, x^2 \psi_t \rangle_{L^2}$ be the momentum of inertia. It satisfies to the following virial identity:

$$
\frac{d^2 I}{dt^2} = \frac{4}{m} E - \frac{2}{m} \left[ \langle \psi, xV' \psi \rangle + 2 \langle \psi, V \psi \rangle \right] + \frac{2\nu(\mu - 2)}{m(\mu + 1)} \|\psi\|^{2\mu + 2}_{L^{2\mu + 2}}.
$$

(35)

In order to compute the derivatives of $I(t)$ from (35) it follows that

$$
\dot{I} = i \frac{\hbar}{m} \langle \psi, [H, \dot{x}^2] \psi \rangle
$$

since $\|\psi\|^{2\mu, \dot{x}^2} = 0$. From this fact and since

$$
[H, \dot{x}^2] \psi = -\frac{\hbar^2}{2m} (2\psi + 4x\psi')
$$

then

$$
\dot{I} = -i \frac{\hbar}{m} \|\psi\|^2 - 2i \frac{\hbar}{m} \langle x\psi, \psi' \rangle.
$$

(36)

From equation (36) and since the norm $\|\psi\|$ is a constant function with respect to the time then

$$
\ddot{I} = -2i \frac{\hbar}{m} \langle x\dot{\psi}, \psi' \rangle - 2i \frac{\hbar}{m} \langle x\psi, \dot{\psi}' \rangle = 2i \frac{\hbar}{m} \langle \psi, \dot{\psi} \rangle + 4i \frac{\hbar}{m} \langle x\dot{\psi}, \psi' \rangle
$$

where

$$
\langle \psi, \dot{\psi} \rangle = i \frac{\hbar}{m} \langle \psi, H \psi + \nu\psi'^{2\mu} \psi \rangle = -i \frac{\hbar}{m} E - \frac{i \nu}{\hbar \mu + 1} \|\psi\|^{2\mu + 2}_{L^{2\mu + 2}}.
$$
because $\dot{\psi} = -\frac{i}{\hbar} H \psi - i\frac{\nu}{\hbar} |\psi|^{2\mu} \psi$, and

$$\langle \dot{x} \psi, \psi' \rangle = -\frac{i}{\hbar} \langle H \psi + \nu |\psi|^{2\mu} \psi, x \psi' \rangle = \frac{i}{\hbar} B + \frac{i\nu}{\hbar} A,$$

where

$$B = \langle H \psi, x \psi' \rangle \quad \text{and} \quad A = \langle |\psi|^{2\mu} \psi, x \psi' \rangle.$$

By integrating by parts then

$$A = \int_{\mathbb{R}} x \tilde{\psi}^{\mu+1} \psi' dx$$

$$= -\int_{\mathbb{R}} \psi^{\mu+1} \tilde{\psi'} dx - (\mu + 1) \int_{\mathbb{R}} x \tilde{\psi}^{\mu+1} \psi' dx - \mu \int_{\mathbb{R}} x \tilde{\psi}^{\mu+1} \psi^\mu \psi' dx$$

from which it follows that

$$(A + \bar{A}) = -\frac{1}{\mu + 1} \|\psi\|^{2\mu+2}_{L^{2\mu+2}}.$$

Now, let

$$B = B_1 + B_2 \quad \text{where} \quad B_1 = -\frac{\hbar^2}{2m} \langle \psi'', x \psi' \rangle \quad \text{and} \quad B_2 = \langle V \psi, x \psi' \rangle.$$

A straightforward calculation yields to

$$B_2 = -\langle V \psi', x \psi \rangle - \langle (xV)' \psi, \psi \rangle = -\bar{B}_2 - \langle (xV)' \psi, \psi \rangle,$$

hence

$$(B_2 + \bar{B}_2) = -\langle (xV)' \psi, \psi \rangle.$$

Similarly

$$B_1 = \frac{\hbar^2}{2m} \langle \psi'', x \psi' \rangle = \frac{\hbar^2}{2m} \langle \psi', \dot{x} \psi'' \rangle + \frac{\hbar^2}{2m} \langle \psi', \psi' \rangle$$

$$= -\bar{B}_1 + \mathcal{E} - \langle \psi, V \psi \rangle - \frac{\nu}{\mu + 1} \|\psi\|^{2\mu+2}_{L^{2\mu+2}}$$

from which follows that

$$(B_1 + \bar{B}_1) = \mathcal{E} - \langle \psi, V \psi \rangle - \frac{\nu}{\mu + 1} \|\psi\|^{2\mu+2}_{L^{2\mu+2}}.$$

In conclusion:

$$\ddot{I} = \frac{2i}{m} \left[ \frac{i}{\hbar} \mathcal{E} - \frac{i\nu}{\hbar} \|\psi\|^{2\mu+2}_{L^{2\mu+2}} \right] + \frac{4}{m} \left[ \frac{i}{\hbar} A + \frac{i\nu}{\hbar} B \right]$$

$$= \frac{2}{m} \mathcal{E} + \frac{2\nu}{m(\mu + 1)} \|\psi\|^{2\mu+2}_{L^{2\mu+2}} + \frac{4}{m} \Re [B + \nu A]$$

$$= \frac{4}{m} \mathcal{E} - \frac{2}{m} \langle \psi, x V' \psi \rangle + \frac{2\nu}{m(\mu + 1)} \|\psi\|^{2\mu+2}_{L^{2\mu+2}}$$

Thus (35) follows.

**Remark 11.** We remark that the virial identity (35) in the particular cases $V(x) \equiv 0, \ V(x) = \alpha x$ and $V(x) = \alpha x^2$, for $\alpha \in \mathbb{R}$, respectively becomes

$$\frac{d^2 I}{dt^2} = \frac{4}{m} \mathcal{E} + \frac{2\nu(\mu - 2)}{m(\mu + 1)} \|\psi\|^{2\mu+2}_{L^{2\mu+2}}, \quad \text{if} \ V(x) \equiv 0, \quad (37)$$
\[
\frac{d^2 I}{dt^2} = \frac{4}{m} E - \frac{6\alpha}{m} (\dot{x})^2 - \frac{2\nu(\mu - 2)}{m(\mu + 1)} \|\psi\|_{L^{2\mu+2}}^{2\mu+2}, \text{ if } V(x) = \alpha x, \tag{38}
\]

and

\[
\frac{d^2 I}{dt^2} = \frac{4}{m} E - \frac{8\alpha}{m} (\dot{x})^2 - \frac{2\nu(\mu - 2)}{m(\mu + 1)} \|\psi\|_{L^{2\mu+2}}^{2\mu+2} \nonumber
\]

\[
= \frac{4}{m} E - \frac{8\alpha}{m} I + \frac{2\nu(\mu - 2)}{m(\mu + 1)} \|\psi\|_{L^{2\mu+2}}^{2\mu+2}, \text{ if } V(x) = \alpha x^2. \tag{39}
\]

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