Entanglement between atomic condensates in an optical lattice: effects of interaction range

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We study the area-dependent entropy and two-site entanglement for two state Bose-Einstein condensates in a 2D optical lattice. We consider the case where the array of two component condensates behave like an ensemble of spin-half particles with the interaction to its nearest neighbors and next nearest neighbors. We show how the Hamiltonian of their Bose-Einstein condensate lattice with nearest-neighbor and next-nearest-neighbor interactions can be mapped into a harmonic lattice. We use this to determine the entropy and entanglement content of the lattice.

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I. INTRODUCTION

Since Bose-Einstein condensates (BEC’s) of alkali gases were first observed in a magnetic trap [1], the experimental study of BEC’s has grown rapidly. In particular, multi-component BEC’s have been realized using the different hyperfine states of $^{87}\text{Rb}$ [2] and spinor condensates using different Zeeman states of sodium $F=1$ [3]. In addition, interspecies interactions give rise to interesting phenomena such as phase-separation [4]. Besides this, the quantum phase transition (QPT) [5] from a superfluid phase to a Mott insulator phase has been observed using atoms in optical lattices [6]. BEC’s in optical lattices are also being used to study aspects of quantum entanglement [7] in many-body systems because of their controllability [8]. Indeed, evidence of multi-particle entanglement has been seen in atoms with two internal states prepared in Mott insulator in optical lattices [8].

In this paper, we study the entanglement in the spin waves of a two-component BEC [2] in a 2D optical lattice [6] in which the condensates at each site behave like a “spin magnet” and interact with each other to its nearest neighbors and next nearest neighbors. The low lying excitation of this system are spin waves. Most importantly, a “long-ranged” interaction can in principle be produced by dipole-dipole interactions. With this in mind we note that Bose-Einstein condensation of chromium, with a high magnetic dipole-dipole interaction, has recently been realized [9]. This dipole-dipole interaction strength should also be tunable by magnetic fields, or by engineering the geometry of the trap [11]. Such controllable interspecies, short- and long-ranged interactions are important in producing the multi-particle entangled states.

The study of the entanglement measures leads to a novel perspective on the structure of ground state and its quantum critical behavior. We shall show that this “BEC lattice” is equivalent to a set of harmonic oscillators in the low-excitation regime. This means that the well-known analysis of a quantum harmonic lattice can be used to investigate entanglement in this many-body system. We shall further see that the entropic measure of entanglement is useful to understand our system’s properties of area dependence which used in some parts of quantum field theory [12]. In the case of the Klein-Gordon (KG) field, the entropy of the field is obtained by tracing over the variables outside a region under consideration and the entropy found to be directly proportional to the boundary area of inside region. This was shown by Bombelli et al. and Srednicki by considering a free, massless and scalar KG field, and which is also equivalent to the ground state of a coupled of harmonic oscillators [14].

An analysis of the entropy and entanglement in a 1D harmonic lattice system was from the viewpoint of quantum information theory was given in the reference [14], along with the generalization to the 2D and 3D cases. The area-dependence of the entropy [15] and the quantum correlations [16] in a harmonic lattice including some aspects of the critical behavior were also studied. Nevertheless, the general relationship between area-dependent entropy and the nature of QPT is not yet fully understood [15]. We will examine an aspect of this relationship and provide some new insights.

Two-site entanglement is also a most useful quantity with which to examine the nature of quantum correlations in a lattice system. For the case of an infinite spin-chain, with nearest neighbor interaction, the critical behavior of two-site entanglement was given in references [17]. It is clear from these studies that two-site entanglement shows the non-local nature of a many-body system close to QPT.

This paper is organized as the follows: In Sec. II, we introduce the system of two-component dipolar condensates trapped in a two-dimensional optical lattice. In Sec. III, we make the Gaussian approximation to the system and show that the system can be represented in terms of harmonic oscillators. The quantum phases (QPT) of the system are then determined. In Sec. IV, we study the area-dependent entropy and its behavior closing to a QPT. In Sec. V, we investigate two-site entanglement and then we give out a conclusion.
in which the dipolar interaction potential energy is
\[
U(r) = -\frac{\mu_0}{4\pi} \frac{3(\mathbf{m}_1 \cdot \hat{r})(\mathbf{m}_2 \cdot \hat{r}) - (\mathbf{m}_1 \cdot \mathbf{m}_2)}{r^3},
\]
for \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \) are the two magnetic dipole moments at \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \) respectively, and \( r = |\mathbf{r}_1 - \mathbf{r}_2| \) is the distance between two dipole moments and \( \hat{r} = (\mathbf{r}_1 - \mathbf{r}_2)/r \) is an unit vector.

Moreover, we assume that these two internal states of atoms can be coupled by applying an external field. We adopt the single-mode approximation [19] to the condensates trapped deeply in each site in which the condensates can be described by the localized mode functions associated with the potential wells. The Hamiltonian of the system is then given by \( (\hbar = 1) \),
\[
H = H_{\text{sys}} + H_{\text{ext}},
\]
where \( a_i \) and \( b_i \) are the annihilation operators of components A and B in the \( i \)-th site. The interaction parameters \( \kappa_i^A (\kappa_i^B) \), \( \kappa_{ij}^A \) and \( \omega_i \) are respectively the intra-component interaction, the inter-component interaction and the coupling rate between the two internal states in the \( i \)-th site. The hopping and the dipole-dipole interaction strengths between the \( i \)-th site and the \( j \)-th site are denoted by \( \Omega_{ij}^A (\Omega_{ij}^B) \) and \( g_{ij}^A (g_{ij}^B) \) for the component A(B) respectively, and \( g_{ij}^{AB} \) is the dipolar strength between the two different component condensates. We consider the atoms in each site to interact with the nearest neighbors and the next-nearest neighbors only as shown in Fig. I. We neglect the interaction between atoms with the larger separations for simplicity and intend to return to their role in the future work. We should emphasize, however, that we do find novel physics associated with the combination of on-site plus neighbor-neighbor interaction.

For the purposes of discussion, we consider the interaction parameters and the number of atoms to be the same for each site, i.e., \( \omega_i = \omega, \kappa = (\kappa_i^A + \kappa_i^B - \kappa_{ij}^A) / 4 \) and \( N_i = N \). Moreover, the atoms between the nearest neighbor with the same dipolar strength is considered, for \( g_{ij}^A = g_a, g_{ij}^B = g_b, g_{ij}^{AB} = g_{ab} \) and \( i, j \) are two indices for two nearest neighbors. In the Mott-insulator limit, \( \Omega \ll \kappa \), the tunneling between the sites is negligible and hence the trapped two-component condensates in each site can be regarded as an ensemble of pseudo spin-half particles. For this case, we can write the Hamiltonian in terms of angular momentum operators in the following form (omit the constant):
\[
H_1 = \sum_i \omega J_i^x + 4\kappa J_i^{z2} + \sum_{<ij>} g_{ij} J_i^x J_j^x, \tag{5}
\]
where \( J_i^x = \frac{(a_i^+ b_i - b_i^+ a_i)}{2}, J_i^y = (a_i^+ b_i^+ a_i b_i - a_i b_i^+ a_i^+ b_i)/2i, J_i^z = (a_i^+ b_i^+ a_i b_i)/2 \), \( N = a_i^+ a_i + b_i^+ b_i \) is the total number of atoms in each site and the dipolar strength \( g_{ij}^{AB} \) of the nearest neighbor and the next-nearest neighbor interaction are \( g \) and \( 2^{-3/2} g \) respectively, for \( g = g_a + g_b - 2g_{ab} \). The factor \( 2^{-3/2} \) appears in the next nearest neighbor comes from the spatial dependence of the square lattice. We also assume that \( |\kappa_i^A - \kappa_i^B| N \) is very small so that we have neglected the linear terms \( J_i^x \). This shows that atomic dipole-dipole interaction systems will be a good place to study the combined effects of nonlinear on-site interaction with interaction between neighbors.
III. QUANTUM PHASES

We assume a sufficiently large external field is applied to the system such that \( \omega \gg \kappa, g \) and in the order of \( \kappa N \). This means that the spin on each site is initially aligned to the negative \( z \)-direction of the angular momentum basis. The small rotation around this negative \( z \)-direction can be described as the motion of a harmonic oscillator in the phase-space plane. To describe this harmonic motion, we can apply the Holstein-Primakoff transformation (HPT) \([20]\) to map the spin operators into the harmonic oscillators. We can see this approximation turns out to be valid in the two-site case even if the number of atoms \( N \) is in the order of several hundred \([21]\). We immediately see from this that the ground state and low-lying eigenstates of our system will behave like a set of coupled harmonic oscillators. We can see this by representing the operators in the position-momentum space: \( q_i = (c_i^+ + c_i)/\sqrt{2\omega} \) and \( p_i = i\sqrt{\omega}(c_i^+ - c_i)/\sqrt{2} \).

Then, the Hamiltonian can be written explicitly in terms of a set of coupled harmonic oscillators:

\[
H_2 = \sum_i \omega(c_i^+ c_i) + \kappa N(c_i^+ c_i)^2 + \frac{N}{4} \sum_{<i,j>} g_{ij}^2 (c_i^+ + c_i)(c_j^+ + c_j). \tag{6}
\]

This effective Hamiltonian \( H_2 \) is the zero order approximation of this exact Hamiltonian \( H_1 \) and gives a description of spin waves \([9]\). In the thermodynamic limit, this approximation becomes exactly equivalent to the system \( H_1 \). The exact numerical solution of this multi-spin system even for just a few sites is extremely difficult. This analytical although approximate solution is therefore extremely valuable and provides important insight into the physics of this multi-spin system. Indeed, the approximate solution is therefore exact up to 11 decimal place, 1.5, 1.25 are shown, where are denoted by solid line (empty circle), dashed line (empty square) and dotted line (empty up-triangle) respectively.

IV. AREA-DEPENDENT ENTROPY

We can now examine the entanglement properties of the ground state of our system. As the ground state is a Gaussian state, its \( 2M \times 2M \) density matrix can be completely determined from the second order moments \( \langle X_i X_j + X_j X_i \rangle - 2\langle X_i \rangle\langle X_j \rangle \), where \( X_i \) are the quadrature variables \( q_i(p_i) \) and \( i = 1, \ldots, M \). The density matrix of the system \( \rho \) can be expressed purely in terms of the covariances \( \langle X_i X_j \rangle \), since \( \langle X_i \rangle = 0 \) in the ground state.

We now investigate the area-dependent properties of the entropy of the bipartite entanglement of two sets of harmonic oscillators. We consider the \( M \times M \) square harmonic lattice is bisected into a \( L \times L \) square lattice and the remaining \( M^2 - L^2 \) lattice as “inside” and “outside” parts respectively, say 1 and 2. Moreover, we consider a \( L \times L \) “inside” square lattice which is located at the center of the whole lattice whereas the rest of oscillators are called the “outside” part. The bipartite entanglement can be determined by the von-Neumann entropy \( E_{L^2}(\rho) \) of the reduced density matrix which is given by

\[
E_{L^2}(\rho) = -\text{Tr}(\rho_1 \log_2 \rho_1), \tag{8}
\]
where $\rho_1 = \text{Tr}_2(\rho)$ is the reduced density matrix of $\rho$ by tracing out the “outside” subsystem 2. The reduced density matrix $\rho_1$ can be obtained by including the position and momentum correlations in the set of modes 1 only. It is given by

$$\rho_1 = \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix},$$

where $Q$ and $P$ are the position and momentum covariance matrices with the matrix elements $Q_{ij} = \langle q_i q_j \rangle$ and $P_{ij} = \langle p_i p_j \rangle$ of the subsystem 1 respectively. The entropy measure of this bipartite entanglement between two regions $E_{L^2}$ is then found to be [15]

$$E_{L^2} = \sum_i \left( \frac{\nu_i + 1}{2} \log_2 \frac{\nu_i + 1}{2} - \frac{\nu_i - 1}{2} \log_2 \frac{\nu_i - 1}{2} \right),$$

where the symplectic eigenvalues $\nu_i$ are the square root of the eigenvalues of $QP$ or $PQ$. It is important to note that all symplectic eigenvalues must be greater than or equal to 1. We follow the definition of the bipartite entanglement in reference [15] in which they show the entanglement area law being valid in a finite-ranged interaction harmonic system with arbitrary dimension.

We now proceed to examine the relationship of the entropy $E_{L^2}$ and the boundary length $L$. In fact, by using Fannes’ inequality [22], we can argue that the bipartite entanglement entropy of the many-body Hamiltonian $H_1$ and that of harmonic system $H_2$ are rather close for a finite block $L$. Hence, the harmonic system is a good model to study the area law of this exact many-body system. In Fig. 2, the entropy $E_{L^2}$ against the boundary length $L$ with different couplings $g$ are shown. The entropy $E_{L^2}$ is directly proportional to the length $L$ in the non-critical regime. Moreover, in the case of the coupling $g$ closing to the critical coupling $g_c$, is also shown to satisfy the area-law. Therefore, the entropy of the bipartite entanglement is still area-bounded in the vicinity of the QPT. For the finite-size case, we consider the total number of sites are $M \times M = 6400$ in the system as shown in Fig. 2. Our numerical results, for this case are in a good agreement with the infinite lattice case and also satisfy the area-law.

V. TWO-SITE ENTANGLEMENT

We shall now examine the entanglement between atoms trapped in two different sites. The problem of two-site entanglement reduces to determining the entanglement between any two harmonic oscillators in an ensemble of harmonic oscillators. Indeed, it is analogous to determining two-mode entanglement in quantum optics [24]. The criterion of the inseparability of two-mode system [24] can be applied to the pure and the mixed states and hence it can be used to evaluate two-site entanglement. The two-site reduced density matrix $\rho^{(i,j)}$ can be constructed from the correlation functions of these two sites $i$ and $j$ only. By applying several local transformations, the reduced density matrix of a symmetric Gaussian state can be written as [24]

$$\rho^{(i,j)} = \begin{pmatrix} n & 0 & c & 0 \\ 0 & n & 0 & -c \\ c & 0 & n & 0 \\ 0 & -c & 0 & n \end{pmatrix},$$

where $n = 2(\langle q_i^2 \rangle \langle p_i^2 \rangle)^{1/2} = 2(\langle q_j^2 \rangle \langle p_j^2 \rangle)^{1/2}$ and $c = 2(\langle q_i q_j \rangle \langle p_i p_j \rangle)^{1/2}$. This two-site entanglement parameter can be defined as $\zeta_{i|j} = n - c$ [21]. If $\zeta_{i|j}$ is below one, then it is said to be entangled. Moreover, this parameter can be used to evaluate the amount of entanglement of formation (EOF) [22]. EOF is a function of the parameter $\zeta_{i|j}$ for $0 < \zeta_{i|j} < 1$ [21] [25].

In the infinite lattice limit, we can treat our system as a symmetric Gaussian state due to the preservation of translational symmetry. We are therefore able to evaluate the EOF through the two-mode entanglement parameter in this case. In the finite-size case, we investigate the oscillators locating at the center in the square lattice and the total number of oscillators are odd. This Gaussian state is nearly symmetric if the system size is large enough. We study the two-site entanglement parameter $\zeta_1$ of two adjacent sites. It decreases as the strength $g$ increases as shown in the inset in Fig. 3 but the minimum is not formed at the critical point. This means that the two-site entanglement can be controlled by the strength $g$. In addition, we found that it is only the nearest neighbors that are entangled. The quantum entanglement is thus very short-ranged indeed. In fact, this feature coincides with the intuition of the area-dependent entropy. It is because the oscillators in the “inside” region entangle with the oscillators in the boundary. The entropy is thus at most proportional to the boundary $L$. 

![FIG. 3: The derivative $d\zeta_1/dg$ as a function of $g/\kappa$ is shown with the same parameters to the previous figure. The infinite case (solid line), $M = 41$ (dashed line), 31 (dash-dot line) and 21 (dotted line) are shown respectively.](image-url)
In order to examine how the ground state changes close to a QPT [17], we numerically examine the first derivative of the two-site entanglement $d\zeta / dg$ of the nearest neighbor. In Fig. 3, we see $d\zeta / dg$ diverging as the QPT is approached in the infinite lattice. In the case of a finite lattice there is still an increase, albeit a finite one.

VI. CONCLUSION

We have investigated the entanglement content in the spin waves of the two-component condensates in a 2D square lattice with the finite-ranged interaction close to a QPT. We have shown that the entropy satisfies the area-law and examined the scaling of two-site entanglement in this finite-ranged interaction system. These features of the system should be addressable by recently developed experimental techniques that give access to atom-atom correlation function. Moreover, the physical realization of harmonic chain may lead to applications in quantum information science [27].

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[23] By Fannes’ inequality, the difference of two entropies are given by

$$|E_{L^2}(\rho_1) - E_{L^2}(\rho_2)| \leq 2D(\rho_1, \rho_2) L^2 \log N - 2D(\rho_1, \rho_2) \log 2D(\rho_1, \rho_2),$$

(12)

where $D(\rho_1, \rho_2)$ is the trace distance between the density matrices, $\rho_1$ and $\rho_2$ are the reduced density matrices of the Hamiltonians $H_1$ and $H_2$ respectively. We have truncated the Hilbert space of $H_2$ to $N^2L^2$ dimensions. For the non-increasing property of the trace distance under partial trace and relation of trace distance between the fidelity of two pure states [7], we have

$$D(\rho_1, \rho_2) \leq \sqrt{1 - F^2(|\psi\rangle, |\phi\rangle)},$$

(13)

where $|\psi\rangle$ and $|\phi\rangle$ are the ground state of the Hamiltonians $H_1$ and $H_2$ respectively. By the perturbation theory, the leading approximation of this ground state of $H_1$ can be found as $|\psi\rangle \approx (|\phi\rangle + \epsilon |\psi^{(1)}\rangle) / \sqrt{1 + \epsilon^2}$, $\epsilon$ is small parameter of the $N^{-1}$ correction to the ground state in the order of $O(N^{-1})$ and $|\psi^{(1)}\rangle$ is the first order of this ground state. The fidelity $F(\rho_1, \rho_2)$ is about $1 - \epsilon^2/2$, so it can be shown that $D(\rho_1, \rho_2)$ is less than $\epsilon$ from (14). Therefore, we can see that the difference of two entropies is bounded by a small value [12] as long as $N$ is sufficiently large than $L^2 \log N$. The infinite-size limit can be attained if we first take $N$ to be infinity and then take $L$ to be infinity.

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