Abstract. We extend slightly the results of Evens-Mirković, and “compute” the characteristic cycles of Intersection Cohomology sheaves on the transversal slices in the double affine Grassmannian. We propose a conjecture relating the hyperbolic stalks and the microlocalization at a torus-fixed point in a Poisson variety.

1. Introduction

1.1. For a perverse sheaf \( \mathcal{F} \) on a smooth complex variety \( Y \), a basic problem in microlocal geometry is to compute the characteristic cycle \( CC(\mathcal{F}) = \sum_{S \in S_c^Y} c_S T^*S \) a positive integral combination of the closures of the conormal bundles to locally closed strata \( S \subset Y \). This problem was solved by S. Evens and I. Mirković in [3] for Goresky-MacPherson sheaves of \( G[[z]] \)-orbits in the affine Grassmannian \( \text{Gr}_G \) of a semisimple complex group \( G \). Namely, they were able to prove the vanishing of all the Euler obstructions by computing the \( A \)-equivariant Euler characteristic of the complex links (\( A \subset G \) is a Cartan torus). In their approach, it was essential that the strata \( S \) were the orbits of some algebraic group action, and that the conormal fibers could be identified with some algebraic Lie algebras.

The results of [3] suggest the existence of a microlocal fiber functor from the geometric Satake category \( \text{Perv}_{G[[z]]}(\text{Gr}_G) \) to the category of representations of the Langlands dual group \( \tilde{G} \). A conjectural affine analogue \( \text{Gr}_{G}^{\text{aff}} \) of \( \text{Gr}_G \) (the double affine Grassmannian) was introduced in [2]. More precisely, the whole \( \text{Gr}_{G}^{\text{aff}} \) is too infinite-dimensional to handle by the machinery available at the moment, and we are bound to think of it as of a collection of (finite-dimensional) transversal slices. For one thing, the price we have to pay for this lame approach is that the strata are no longer orbits of any group action.

Fortunately, they are still the symplectic leaves of a Poisson structure. This equips the conormal fibers with a Lie algebra structure, and the argument of Evens-Mirković goes through. This is proved in Section 3 after some general preparation in Section 2, answering a question of S. Evens.

An isomorphism of the (would be) microlocal fiber functor with the standard Mirković-Vilonen fiber functor is a relation between the hyperbolic stalks and the microlocalization at the torus-fixed points. It seems to go through in our extended setup, see Conjecture 4.3. If true, it would express the classical \( R \)-matrices of [5, Section 4.8] as the action of the microlocal fundamental group.

There is one more rich class of algebraic Poisson varieties with finitely many symplectic leaves: Nakajima quiver varieties. They satisfy all the technical conditions of [2, Section 4.8] except one (apart from type \( A \) case, finite or affine): the torus fixed point set is not discrete. So already in the simplest examples of type \( D, E \) Kleinian singularities (finite type quiver varieties, corresponding to the zero weight of the adjoint fundamental representation) the Euler obstruction does not vanish. More precisely, it is equal to 1 for all type \( D, E \) Kleinian singularities (there are but 2 strata), so that the multiplicity of the cotangent fiber at the singular point in the characteristic cycle of IC (=constant) sheaf is 2.
1.2. Acknowledgments. This note is a result of patient explanations by V. Ginzburg, A. Kuznetsov, L. Rybnikov, P. Schapira, G. Williamson. M. F. was partially supported by the RFBR grants 12-01-33101, 12-01-00944, the National Research University Higher School of Economics’ Academic Fund award No.12-09-0062 and the AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. This study was carried out within the National Research University Higher School of Economics Academic Fund Program in 2012-2013, research grant No. 11-01-0017. This study comprises research findings from the “Representation Theory in Geometry and in Mathematical Physics” carried out within The National Research University Higher School of Economics’ Academic Fund Program in 2012, grant No 12-05-0014.

2. Vanishing of the Euler obstruction

2.1. Review of [3]. We recall the argument of S. Evens and I. Mirković.

Let \( \mathcal{L} \) be a Whitney stratification of a complex manifold \( Y \) and let \( D_{\mathcal{L}}(Y) \) denote the derived category of \( \mathcal{L} \)-constructible sheaves. Let \( \mathcal{F} \) be an object of \( D_{\mathcal{L}}(Y) \). Then \( \mathcal{F} \) defines two important constructible (with respect to \( \mathcal{L} \)) functions on \( Y \). The first one, \( \chi \) is just the Euler characteristic \( \chi(\mathcal{F}_y) \) of the stalk of \( \mathcal{F} \) at a point \( y \). As \( \mathcal{F} \) lies in \( D_{\mathcal{L}}(Y) \), \( \chi \) is \( \mathcal{L} \)-constructible and we denote by \( \chi_{\alpha}(\mathcal{F}) \) its value on a stratum \( \alpha \). The second function, \( c(\mathcal{F}) \) comes from the characteristic cycle

\[
CC(\mathcal{F}) = \sum_{\alpha \in \mathcal{L}} c_{\alpha}(\mathcal{F}) \cdot T^*_{\alpha}(Y),
\]

its value on stratum \( \alpha \) is equal to \( c_{\alpha}(\mathcal{F}) \), which is called the microlocal multiplicity of \( \mathcal{F} \) along \( \alpha \). Here \( T^*_{\alpha}(Y) \subset T^*(Y) \) denotes the conormal bundle to the stratum \( \alpha \).

For any pair of strata \( \alpha \) and \( \beta \) the Euler obstruction \( e_{\alpha,\beta} \) is defined as \( c_{\alpha}(\mathcal{C}_\beta) \), where \( \mathcal{C}_\beta \) is the constant sheaf on \( \beta \), extended by zero on \( Y \).

A covector \( \xi \in T^*_{\alpha}(Y) \) is called \( \mathcal{L} \)-generic if it lies in

\[
T^*_{\alpha}(Y)^r := T^*_{\alpha}(Y) - \bigcup_{\alpha \neq \beta} \overline{T^*_{\beta}(Y)}.
\]

The set of generic elements is open and dense in \( T^*_{\alpha}(Y) \) and its fundamental group is called microlocal fundamental group of the stratum \( \alpha \).

For the following theorem, see e.g. [1] Chapter 9:

**Theorem 2.2.** a) On the Grothendieck group of \( \mathcal{L} \)-constructible complexes,

\[
c_{\alpha} = \sum_{\beta \in \mathcal{L}} e_{\alpha,\beta} \cdot \chi_{\beta}
\]

b) One has \( e_{\alpha,\alpha} = (-1)^{\dim(\alpha)} \) and \( e_{\alpha,\beta} = 0 \) if \( \alpha \nsubseteq \beta \). If \( \alpha \subset \partial \beta \), we choose

- a normal slice \( N \) in \( (Y, \mathcal{L}) \) to \( \alpha \) at a point \( y \in \alpha \),
- a holomorphic function \( \phi \) on \( N \) vanishing at \( y \) and such that \( d_y f \in T^*_{y}(N) \) is \( \mathcal{L} \)-generic,
- a small ball \( B \) around \( y \) in \( Y \),
- a small \( t \in \mathbb{C} \).

Then

\[
e_{\alpha,\beta} = (-1)^{\dim(\alpha)+1} \chi_{\alpha}(\beta \cap \phi^{-1}(t) \cap B)
\]

is, up to sign, the Euler characteristic of a compactly supported cohomology of the intersection of \( \beta \) with a nearby hyperplane \( \phi^{-1}(t) \), near \( \alpha \) and normal to \( \alpha \).

The intersection \( \beta \cap \phi^{-1}(t) \cap B \) is called the complex link of strata \( \alpha \) and \( \beta \).

Now Mirković and Evens argue as follows: having a compact torus \( A^e \) acting on \( Y \) stabilizing \( y \) and \( \beta \), choose \( B \) in Theorem 2.2 \( A^e \)-invariant and try to find a \( A^e \)-invariant function \( \phi \) on
N such that its differential is generic. Then the complex link will be stable under $A^c$-action too and the classical result of Borel can be applied: Let $Z$ be a paracompact space with finite cohomological dimension with the action of a compact torus $A^c$. Then $\chi_e(Z) = \chi_e(Z^{A^c})$, where $Z^{A^c}$ is the fixed-point set. In particular if $Z^{A^c}$ is empty, we get $\chi_e(Z) = 0$. In other words, $e_{a, b} = 0$.

So the main question left is the existence of an $A^c$-invariant function with generic differential. In the next subsection we formulate some technical conditions which guarantee the existence. In the next section we check the technical conditions in our main example.

2.3. Setup. $X$ is an affine complex Poisson variety with finitely many symplectic leaves $X_i$, $i \in L$. The closure of $X_i$ is denoted by $\overline{X}_i$. The Poisson bracket on $R := \mathbb{C}[X]$ is denoted by $\{ , \}$. A reductive group $G$ with Lie algebra $\mathfrak{g}$ acts on $X$ preserving the bracket $\{ , \}$. We assume the existence of a moment map $\mu : X \rightarrow \mathfrak{g}^*$, hence $\mu^* : \text{Sym}(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \rightarrow R$. The fixed point subset $X^A$ for a maximal torus $A \subset G$ consists of a single point $x \in X$, a 0-dimensional symplectic leaf. Additionally, a one-parametric group $\mathbb{C}^\times$ acts on $X$ commuting with $G$. We assume that the corresponding action on $R$ has only positive weights: it induces the grading $R = \mathbb{C} \oplus \bigoplus_{l > 0} R_l$, and $\mathfrak{m}_x = \bigoplus_{l > 0} R_l$. We assume the Poisson bracket has weight $w > 0$ with respect to the one-parametric group $\mathbb{C}^\times$, i.e. $\{R_l, R_{m}\} \subset R_{l+m-w}$. Finally, we assume that $R_0 = 0$ for $0 < l < w$, and $R_w$ coincides with the image of $\mu^*$ on linear functions on $\mathfrak{g}^* : \mathfrak{g}^* \rightarrow R$.

2.4. Cotangent Lie algebra. The Poisson bracket on $R$ induces a Lie algebra structure on the Zariski cotangent space $T^*_x X := \mathfrak{m}_x / \mathfrak{m}_x^2$. In effect, since $x$ is a symplectic leaf, the Poisson bivector vanishes at $x$, and $\{\mathfrak{m}_x, \mathfrak{m}_x\} \subset \mathfrak{m}_x$. By Leibniz rule $\{f, g, h\} = f\{g, h\} + g\{f, h\}$, we also have $\{\mathfrak{m}_x^2, \mathfrak{m}_x\} \subset \mathfrak{m}_x^2$. In other words, the Poisson bracket on $\mathfrak{m}_x$ descends to $\mathfrak{m}_x / \mathfrak{m}_x^2$ and provides it with a structure of Lie algebra, to be denoted $\mathfrak{q}$. The grading on $\mathfrak{m}_x$ gives rise to a grading on $\mathfrak{q} = \bigoplus_{l \geq w} \mathfrak{q}_w$. We define $\mathfrak{q}^m := \mathfrak{q}_{k+m}$. With this new shifted grading, the Lie bracket is homogeneous, so $\mathfrak{q}$ becomes a nonnegatively graded Lie algebra.

Lemma 2.5. $\mathfrak{q}$ is an algebraic Lie algebra, i.e. there exists a complex algebraic Lie group $Q$ with Lie algebra $\mathfrak{q}$.

Proof. By our assumptions in 2.3, $\mathfrak{q}^0$ is a quotient of $\mathfrak{g}$, and hence we can take for $Q^0$ a quotient of $G$. The nilpotent ideal $\mathfrak{q}^{>0}$ integrates to a unipotent algebraic group $Q^{>0}$, and the adjoint action of $\mathfrak{q}^0$ on $\mathfrak{q}^{>0}$ integrates to the action of $G$ on $\mathfrak{q}^{>0}$ (factoring through $G \rightarrow Q^0$). The lemma follows. □

2.6. Calculation of Euler obstructions. There is a Zariski open neighbourhood $U$ of $x$ in $X$, and a closed embedding $U \hookrightarrow Y$ into a smooth algebraic variety which induces an isomorphism on the Zariski tangent spaces $T_x X = T_x U \rightarrow T_x Y$. Dually, $T^*_x Y \rightarrow \mathfrak{q}$. The regular part $\mathfrak{q}^{reg} \subset \mathfrak{q}$ is defined as $\mathfrak{q} \setminus \bigcup_{l \in L, \, \chi_i \neq 0} T^*_X \cap \overline{Y} \cap T^*_X Y$. This is a nonempty open subset of $\mathfrak{q}$. It is clearly independent of the choices of $U$ and $Y$ above. Moreover, if we choose any closed embedding $U \hookrightarrow Y'$ giving rise to a surjection $p : T^*_x Y' \rightarrow \mathfrak{q}$, then the regular part of $T^*_x Y'$ is nothing but $p^{-1}(\mathfrak{q}^{reg})$.

Lemma 2.7. $\mathfrak{q}^{reg} \subset \mathfrak{q}$ is invariant with respect to the adjoint action of $Q$ on $\mathfrak{q}$.

Proof. Let $C_x \overline{X}_i \subset C_x X \subset T_x X = T_x Y$ be the normal cones. Let $C^{reg}_x \overline{X}_i \subset C_x \overline{X}_i$ be the smooth part. The deformation to the normal cone construction proves that $T^{C^{reg}_x \overline{X}_i}_x Y \cap T^*_x Y = \overline{T^*_X \cap \overline{Y} \cap T^*_X Y}$. Now the LHS is the cone over the closed subvariety of $\mathbb{P} = \mathbb{P}^n T^*_x Y$ projectively dual to $\mathbb{P}C_x \overline{X}_i \subset \mathbb{P}T^*_x Y$. Since $C_x \overline{X}_i \subset \mathfrak{q}^*$ is a Poisson subvariety, it is $Q$-invariant; hence its projective dual is $Q$-invariant as well. □
Theorem 2.8. The Euler obstruction \( e_{x,X} = 0 \) for every \( X_i \neq x \).

Proof. First we consider \( e_{x,X} \). According to \([6, 2.13]\), there is a nonzero \( q \in q^{reg} \) stabilized by a maximal torus \( T \subset Q \). Since all the maximal tori in \( Q \) are \( Q \)-conjugate, while \( q^{reg} \) is \( Q \)-invariant, we can find another nonzero element \( q' \in q^{reg} \) stabilized by the maximal torus \( A \subset G \) (notations of \([3,\, 2]\)). Choose an \( A \)-equivariant closed embedding \( X \hookrightarrow \mathbb{A}^N \), and lift \( q' \) to an \( A \)-invariant linear function \( \phi \) on \( \mathbb{A}^N \). Now the argument of \([3\, proof of Theorem 2.2]\) proves \( e_{x,X} = 0 \).

For an arbitrary symplectic leaf \( x \neq X_i \subset X \) we just replace \( X \) by \( \overline{X} \) in the above argument: all the assumptions of \([3,\, 2]\) are clearly satisfied with the same \( A \subset G \), \( C^\times \). \( \square \)

3. Applications to transversal slices in double affine Grassmannians

3.1. Uhlenbeck spaces. We follow the notations of \([2]\). To begin with, for an almost simple simply connected complex algebraic group \( H \) we denote by \( \mathcal{U}_H^0(\mathbb{A}^2) \) the Uhlenbeck closure of the moduli space \( \text{Bun}_H^0(\mathbb{A}^2) \) of \( H \)-bundles on \( \mathbb{P}^2 \) trivialized along \( \mathbb{P}^1 \) of second Chern class \( a \). It is acted upon by \( H \ltimes \text{Aff}(2) \). Here \( H \) acts by the change of trivialization, while \( \text{Aff}(2) = GL(2) \ltimes \mathbb{G}_m^2 \) (the group of affine motions of \( \mathbb{A}^2 \), i.e. \( \text{Aut}(\mathbb{P}^2, \mathbb{P}^1) \)) acts via the transport of structure. The normal subgroup \( \mathbb{G}_m^2 \) acts on \( \mathcal{U}_H^0(\mathbb{A}^2) \) freely, and the quotient is denoted \( \mathcal{U}_H^{reg}(\mathbb{A}^2) \): the reduced Uhlenbeck space, acted upon by \( H \times GL(2) \). Let \( C^\times \subset GL(2) \) stand for the central subgroup, and let \( G \) stand for \( H \times SL(2) \).

Proposition 3.2. \( X = \mathcal{U}_H^0(\mathbb{A}^2) \) with the action of \( G \times C^\times \) satisfies the assumptions of \([2,\, 2]\).

Proof. In case \( H = SL(r) \) the Uhlenbeck space \( \mathcal{U}_H^0(\mathbb{A}^2) \) was constructed by Donaldson by Hamiltonian reduction from the representation space of the 2-loop quiver; it is usually denoted \( M_0(V,W) \) where \( V = C^a \), \( W = C^b \). This construction provides \( \mathcal{U}_H^0(\mathbb{A}^2) \) with a Poisson structure. For an arbitrary \( H \) and a representation \( \rho : H \rightarrow SL(r) \) we have the corresponding embedding \( \rho_\ast : \mathcal{U}_H^0(\mathbb{A}^2) \rightarrow \mathcal{U}_{\rho H}^0(\mathbb{A}^2) \) for certain \( a' \). The Poisson structure on \( \mathcal{U}_H^0(\mathbb{A}^2) \) restricted to \( \mathcal{U}_H^{reg}(\mathbb{A}^2) \) is independent of \( \rho \) up to a (nonzero) scalar factor. We fix the Poisson structure on \( \mathcal{U}_H^0(\mathbb{A}^2) \), say choosing the adjoint representation \( \rho \).

The well known stratification \( \mathcal{U}_H^0(\mathbb{A}^2) = \bigsqcup_{0 \leq b \leq a} \text{Bun}_H^b(\mathbb{A}^2) \times \text{Sym}^{a-b} \mathbb{A}^2 \) admits a refinement by the diagonal stratification of \( \text{Sym}^{a-b} \mathbb{A}^2 = \bigsqcup_{\mathfrak{p} \in P(a-b)} \text{Sym}^\mathfrak{p} \mathbb{A}^2 \), see \([1, \text{Section 10}]\) (here \( P(a-b) \) stands for the set of partitions of \( a-b \)). The diagonal strata are nothing but the symplectic leaves of the Poisson structure; in particular, there are finitely many symplectic leaves. We will denote the stratum \( \text{Bun}_H^b(\mathbb{A}^2) \times \text{Sym}^\mathfrak{p} \mathbb{A}^2 \) by \( \mathcal{U}_H^{a,b,\mathfrak{p}} \) for short, and its image in the reduced Uhlenbeck space \( \mathcal{U}_H^{reg}(\mathbb{A}^2) \) will be denoted by \( \mathcal{U}_H^{a,b} \). Thus \( \mathcal{U}_H^a(\mathbb{A}^2) = \bigsqcup_{0 \leq b \leq a} \mathcal{U}_H^{a,b} \) is the decomposition into symplectic leaves. Among those, there is a unique 0-dimensional leaf: the one for \( b = 0 \), \( \mathfrak{p} = (a) \). This point \( x \) is the unique fixed point for any maximal torus \( A \subset G \).

In order to compute the weight \( w \) of the Poisson structure with respect to the \( C^\times \)-action we recall the Hamiltonian reduction construction of \( \mathcal{U}_H^0(\mathbb{A}^2) = M_0(V,W) \). Namely, for \( M = \text{Hom}(W,V) \oplus \text{Hom}(V,W) \oplus \text{End}(V,V) \oplus \text{End}(V,V) \) (with a typical element \( (p,q,A,B) \), and \( M' := \{ (p,q,A,B) : AB - BA + pq = 0 \} \) naturally acted upon by \( GL(V) \), we have \( M_0(V,W) = M'//GL(V) \). The action of \( \mathbb{G}_m^a \) on \( M_0(V,W) \) comes from the action \( (a,b) \cdot (p,q,A,B) = (p,q,A + a Id_V, B + b Id_V) \) on \( M \). Thus \( \mathcal{U}_H^a(\mathbb{A}^2) = M_0(V,W)/\mathbb{G}_m^a = M'//GL(V) \) where \( M' \supset M" := \{ (p,q,A,B) : AB - BA + pq = 0, Tr A = Tr B = 0 \} \). The vector space \( M \) has a natural symplectic structure which gives rise to the Poisson structure on the categorical quotient \( M_0(V,W) \). The action of \( C^\times \) on \( M_0(V,W) \) comes from the dilation action \( t \cdot (p,q,A,B) = (tp,tq,tA,tB) \) on \( M \). Evidently, the symplectic form on \( M \) has weight 2 with respect to this
action, so \( w = 2 \). By the construction of the Poisson structure on \( \mathcal{U}_H^a(\mathbb{A}^2) \), it has weight 2 for arbitrary \( H \) as well.

It remains to find all the functions on \( \mathcal{U}_H^a(\mathbb{A}^2) \) of \( \mathbb{C}^\times \)-weights 1, 2. Again we start with \( H = \text{SL}(r) \). By the classical invariant theory, all the \( \text{GL}(V) \)-invariant functions on \( M'' \) are generated by the following ones: (a) matrix elements of \( qCp \in \text{End}(W) \) where \( C \) is a word in the alphabet \( (A, B) \); (b) traces of \( C \in \text{End}(V) \) where \( C \) is a word in the alphabet \( (A, B) \). Evidently, the \( \mathbb{C}^\times \)-weight of (a) is length(\( C \)) + 2, while weight(\( \text{Tr} C \)) = length(\( C \)). Note that the only words of length 1 are \( A, B \), and their traces vanish by definition of \( M'' \). So there are no functions of weight 1.

The functions of weight 2 are spanned by the matrix elements of \( qp \), and \( \text{Tr} A^2, \text{Tr} B^2, \text{Tr} AB \). Note that \( \text{Tr} qp = \text{Tr} pq = 0 \). Clearly, the first group of weight 2 functions is lifted from the moment map \( M''/\text{GL}(V) \to \mathfrak{sl}(W)^* = \mathfrak{sl}(r)^* \), while the second one is lifted from the moment map \( M''/\text{GL}(V) \to \mathfrak{sl}(2)^* \). Thus \( \mathfrak{sl}(r) \oplus \mathfrak{sl}(2) \to \mathbb{C}[\mathcal{U}^a_{\text{SL}(r)}]^2 \).

Finally, for arbitrary \( H \xrightarrow{\phi} \text{SL}(r) \), the weight \( k \) functions on \( \mathcal{U}_H^a \) are just the restrictions of weight \( k \) functions on \( \mathcal{U}_{\text{SL}(r)}^a \) under the closed embedding \( \phi_* : \mathcal{U}_H^a \hookrightarrow \mathcal{U}_{\text{SL}(r)}^a \). The commutative diagram of moment maps

\[
\begin{array}{ccc}
\mathfrak{g}^* & \xrightarrow{\phi^*} & \mathfrak{h}^* + \mathfrak{sl}(2)^* \\
\downarrow & & \downarrow \\
\mathfrak{g}^* \oplus \mathfrak{sl}(2)^* & \xleftarrow{\phi^* \oplus \text{Id}} & \mathfrak{h}^* \oplus \mathfrak{sl}(2)^* \\
\end{array}
\]

completes the proof. \( \square \)

3.3. Characteristic cycle of \( \text{IC}(\mathcal{U}_H^a) \). We choose a closed embedding \( \mathcal{U}_H^a \to Y \) into a smooth variety \( Y \), and view \( \text{IC}(\mathcal{U}_H^a) \) as a perverse sheaf on \( Y \).

**Corollary 3.4.** The multiplicity of the conormal bundle \( T^*\mathcal{U}_H^a \cdot Y \) in the characteristic cycle \( \text{CC}(\text{IC}(\mathcal{U}_H^a)) \) equals the total dimension of the stalk of \( \text{IC}(\mathcal{U}_H^a) \) on the stratum \( \mathcal{U}_H^{b, \mathfrak{F}} \) (computed in [1] Theorem 7.10).

**Proof.** The argument of [3] Section 2.5 reduces the proof to the vanishing of the Euler obstructions \( e_{\mathcal{U}_H^{b, \mathfrak{F}}} \), \( \mathcal{U}_H^{b, \mathfrak{F}} \) stands for the closure of the stratum \( \mathcal{U}_H^{b, \mathfrak{F}} \). We first treat the smallest stratum \( x = \mathcal{U}_H^{b, \mathfrak{F}} \) for \( b' = 0 \), \( \mathfrak{F}' = (a) \). This vanishing follows from Theorem 2.8 and Proposition 3.2. In general, the vanishing \( e_{\mathcal{U}_H^{b, \mathfrak{F}}} \), \( \mathcal{U}_H^{b, \mathfrak{F}} \) is 0 is equivalent to the vanishing \( e_{\mathcal{U}_H^{b, \mathfrak{F}}} \), \( \mathcal{U}_H^{b, \mathfrak{F}} \) is 0 where \( \mathcal{U}_H^{b, \mathfrak{F}} \) stands for the closure of the stratum \( \mathcal{U}_H^{b, \mathfrak{F}} \) in the nonreduced Uhlenbeck space \( \mathcal{U}_H^a(\mathbb{A}^2) \). By the factorization principle [1] Proposition 6.5, the desired obstruction is the product \( \prod_{i=1}^m e_{\mathcal{U}_H^{b, \mathfrak{F}}} \), \( \mathcal{U}_H^{b, \mathfrak{F}} \) (where \( m \) is the number of parts of the partition \( \mathfrak{F} \)). The latter factors are already proved to vanish. \( \square \)

3.5. Transversal slices in double affine Grassmannians. Given a cyclic subgroup \( \Gamma_k = \mathbb{Z}/k\mathbb{Z} \subset G = H \times \text{SL}(2) \), a transversal slice \( \mathcal{U}_{H, \mu}^\lambda(\mathbb{A}^2/\Gamma_k) \) is defined in [2] as a certain irreducible component of the fixed point set \( \mathcal{U}_H^a(\mathbb{A}^2)^{\Gamma_k} \). The Poisson structure is restricted from the one on \( \mathcal{U}_H^a(\mathbb{A}^2) \), and the symplectic leaves are the intersections of \( \mathcal{U}_{H, \mu}^\lambda(\mathbb{A}^2/\Gamma_k) \) with the symplectic leaves (the strata of the diagonal stratification) of \( \mathcal{U}_H^a(\mathbb{A}^2) \).

In case \( k = 2 \), the cyclic subgroup \( \Gamma_2 \) is central in \( \text{SL}(2) \), and the centralizer \( G = \mathbb{Z}_{H \times \text{SL}(2)}(\Gamma_2) = \mathbb{Z}_H(\Gamma_2) \times \text{SL}(2) \) acts on \( \mathcal{U}_{H, \mu}^\lambda(\mathbb{A}^2/\Gamma_k) \) preserving the Poisson structure. In case \( k > 2 \), we only have the action of \( G = \mathbb{Z}_{H \times \text{SL}(2)}(\Gamma_k) = \mathbb{Z}_H(\Gamma_k) \times T \) on \( \mathcal{U}_{H, \mu}^\lambda(\mathbb{A}^2/\Gamma_k) \), where \( T \) is the centralizer torus of \( \Gamma_k \) in \( \text{SL}(2) \). The action of central \( \mathbb{C}^\times \subset \text{GL}(2) \) on \( \mathcal{U}_H^a(\mathbb{A}^2) \) preserves \( \mathcal{U}_{H, \mu}^\lambda(\mathbb{A}^2/\Gamma_k) \).
Proposition 3.6. $X = \mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k)$ with the action of $G \times \mathbb{C}^\times$ satisfies the assumptions of [23].

Proof. We essentially repeat the proof of Proposition 3.2 using a representation $\rho : H \to \text{SL}(r)$ we reduce the claim to the case $H = \text{SL}(r)$. In this case the transversal slice $\mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k)$ is nothing but a cyclic quiver variety $M_0(V, W)$ [1 Section 7] (for the cyclic quiver with $k$ vertices). We have $M_0(V, W) = M'$ where $G =$ (GL($V$, W))$^{-}$invariant functions on $X$ is nothing but a cyclic quiver variety $U$. We have $M_0(V, W) = \mathcal{H}(U, V)$ where $M' = \bigoplus_{l \in \mathbb{Z}/k\mathbb{Z}} \text{Hom}(W_l, V_l) \oplus \bigoplus_{l \in \mathbb{Z}/k\mathbb{Z}} \text{End}(V_l, V_{l+1})$. We have $M'$ cut out by the equations $A_i - B_i - B_{i+1} + p_l q_l = 0$, $l \in \mathbb{Z}/k\mathbb{Z}$. The $\mathbb{C}^\times$-action is by dilations, and the $\mathbb{C}^\times$-weight of the Poisson structure is $\omega = 2$. It is not hard to check from the classical invariant theory that all the $GL(V)$-invariant functions on $M'$ are generated by the following ones: (a) matrix elements of $q_m C p_m$ where $C$ is a word in the alphabet $(A_1, B_1)_{l \in \mathbb{Z}/k\mathbb{Z}}$ (not all the words are allowed: only the composable ones); (b) traces of $C$ $A_1 \in \text{End}(V_0)$ where $C$ is a composable word in the alphabet $(A_1, B_1)_{l \in \mathbb{Z}/k\mathbb{Z}}$ starting and ending at the 0-th vertex. Among those, the functions of weight 2 are the matrix elements of $q_m p_m$, and $\text{Tr}(B_1 A_0)$ for $k > 2$ (note that $\text{Tr}(A_1 B_0) = \text{Tr}(B_1 A_0) - \text{Tr}(q_0 p_0)$, and also $\text{Tr}(A_1 A_0, B_1 B_0)$ for $k = 2$. Clearly, these functions are lifted from the moment map $M_0(V, W) \to \bigoplus_{l \in \mathbb{Z}/k\mathbb{Z}} \text{gl}(W_l) \cap \mathfrak{s}l\bigoplus_{l \in \mathbb{Z}/k\mathbb{Z}} \text{End}(V_l)$ $\otimes \mathfrak{t}^*$ in case $k > 2$, and $M_0(V, W) \to \bigoplus_{l \in \mathbb{Z}/k\mathbb{Z}} \text{gl}(W_l) \cap \mathfrak{s}l\bigoplus_{l \in \mathbb{Z}/k\mathbb{Z}} \text{End}(V_l)$ $\otimes \mathfrak{t}(2)^*$ in case $k = 2$.

The proposition is proved.

Corollary 3.7. For a closed embedded $\mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k) \hookrightarrow Y$ into a smooth variety $Y$, the multiplicity of the conormal bundle $T^*_XY$ to a symplectic leaf $S \subset \mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k)$ in the characteristic cycle $CC(\text{IC}(\mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k)))$ equals the total dimension of the stalk of $\text{IC}(\mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k))$ on the stratum $S$.

Proof. The same as the proof of Corollary 3.4.

Remark 3.8. The dimension of the stalk of $\text{IC}(\mathcal{U}_{H,\mu}(\mathbb{A}^2/\Gamma_k))$ on a stratum $S$ is computed by the factorization principle, and [2] Conjecture 4.14.

4. A CONJECTURE

4.1. Chambers. In the setup of [23] we set $a := \text{Lie} A$, and $a_{\mathbb{R}} := X_*(A) \otimes \mathbb{Z}_{\mathbb{R}} \subset a$. We say that a coweight $a \in X_*(A)$ is regular if $X^{\omega(C)} = x$. We assume that the nonregular coweights form a finite union of corank one subgroups in $X_*(A)$. We say that $a \in a_{\mathbb{R}}$ is regular if it lies off the corresponding real hyperplanes. The connected components of $a^{\text{reg}}$ are called chambers.

Given a regular $a \in X_*(A)$ we define the attracting set $\Xi_a \subset X$ as the set of all $z \in X$ such that $\lim_{c \to a} c \cdot z = x$. If $a$ varies in a chamber $\Xi$, then $\Xi_a$ does not change, and so we denote it $\Xi$. The closed embedding $\Xi_a \hookrightarrow X$ is denoted by $\iota.a$. The closed embedding $x \hookrightarrow \Xi$ is denoted by $\iota.x$. The composition $\iota.a \circ \iota.x$ is the hyperbolic restriction. In all the examples of Section 3, $\iota.a \circ \iota.x$ $\text{IC}(X)$ is a vector space in cohomological degree 0.

4.2. Microlocalization. According to the first paragraph of Section 2.6 we have a perverse sheaf $\mu(\text{IC}(X))$ on $T^*_X = q$. According to [1] Proposition 4.4.7, $\mu(\text{IC}(X))$ is well defined, i.e. does not depend on the choice of $U \hookrightarrow Y$. Recall that we have a homomorphism $\eta : a \to q = T^*_X X$. Motivated by [3] and by the examples of Section 3, we propose the following

Conjecture 4.3. (a) $\eta^* \mu(\text{IC}(X))$ is constant in any chamber $\Xi$.

(b) There is a canonical isomorphism $\eta^* \mu(\text{IC}(X))_{\Xi} \overset{\sim}{\longrightarrow} \iota.a \circ \iota.x \text{IC}(X)$. 

References

[1] A. Braverman, M. Finkelberg, D. Gaitsgory, *Uhlenbeck spaces via affine Lie algebras*, Progress in Math. **244** (2006), 17–135. *Erratum*: arXiv:math/0301176v4.

[2] A. Braverman, M. Finkelberg, *Pursuing the double affine Grassmannian. I. Transversal slices via instantons on $A_k$-singularities*, Duke Math. J. **152** (2010), no. 2, 175–206.

[3] S. Evens, I. Mirković, *Characteristic cycles for the loop Grassmannian and nilpotent orbits*, Duke Math. J. **97** (1999), 109–126.

[4] M. Kashiwara, P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften **292** Springer-Verlag, Berlin (1990), x+512pp.

[5] D. Maulik, A. Okounkov, *Quantum groups and quantum cohomology*, arXiv:1211.1287.

[6] R. Steinberg, *Conjugacy classes in Algebraic groups*, Lect. Notes Math. **366**, Springer-Verlag (1974).

Address:

IMU, IITP, and State University Higher School of Economics,
Department of Mathematics,
20 Myasnitskaya st, Moscow 101000, Russia

E-mail address:
fnklberg@gmail.com
dmkubrak@gmail.com