Multi-linear Monogamy Relations for Multi-Qubit States

Xian Shi,† Lin Chen,‡∗ and Mengyao Hu†,

1School of Mathematics and Systems Science, Beihang University, Beijing 100191, China
2International Research Institute for Multidisciplinary Science, Beihang University, Beijing 100191, China

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The monogamy of entanglement means that entanglement cannot be freely shared. In 2014, Oliveira et al. [Oliveira et al., Phys. Rev. A. 89, 034303 (2014)] proposed a monogamy relation in the linear version and considered it in terms of entanglement of formation. Here we generalize the above version and consider a multi-linear monogamy relation for a multi-qubit system in terms of entanglement of formation and concurrence. Based on the above results, we present an entanglement criterion for genuine entangled states, also we consider the absolutely maximally entangled states and present what an absolutely maximally entangled state is for a three-qubit system. At last, we apply our results to a three-qubit pure state in terms of quantum discord.

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I. INTRODUCTION

Quantum entanglement is an essential feature of quantum mechanics. It plays an important role in quantum information and quantum computation theory [1], such as superdense coding [2], teleportation [3] and the speedup of quantum algorithms [4].

As a property of multipartite entanglement, monogamy of entanglement presents that entanglement cannot be shared arbitrarily among many parties, which is different from classical correlations [5]. This property has been applied on many areas in quantum information. It can be applied to prove the security of quantum cryptography [6–8] and the bound of the regularization of its Holevo information for arbitrary channels [9]. It can also be applied to distinguish inequivalent classes of pure states in a tripartite system [10, 11]. Recently, the authors showed there exists restrictions of indistinguishability for entangled systems due to monogamy relations [12].

Mathematically, for a tripartite system with parties A, B and C, the general monogamy in terms of an entanglement measure $E$ implies that the entanglement between A and BC satisfies

$$\mathcal{E}_{A|BC} \geq \mathcal{E}_{AB} + \mathcal{E}_{AC},$$

(1)

here $\mathcal{E}_{AB}$ and $\mathcal{E}_{AC}$ means the entanglement between the parties A, B and A, C. This relation was first proved for qubit systems in terms of the 2-tangle [10, 13]. Bai et al. showed that the inequality Eq. (1) is valid in terms of the squared entanglement of formation (EoF) for n-qubit systems [14]. Zhu et al. investigated the monogamy relations related to the concurrence and the entanglement of formation [15]. Recently, the authors in [16, 17] presented generalized monogamy relations, Jin et al. proposed tighter monogamy relations for n-qubit systems [18]. Yu et al. utilized the conversion relation between the coherence and the entanglement to establish the monogamy inequalities for high-dimensional coherence-induced entanglement in terms of the relative entropy of coherence and the negativity [19]. Zhang et al. studied the monogamy relations for multi-qubit quantum systems in product norm [20].

However, it is well known that the EoF ($E$) does not satisfy the inequality Eq. (1). In 2014, Oliveira et al. proposed a linear monogamy relation in terms of EoF and numerically obtained the bound for a three-qubit system. This result indicates that entanglement cannot be freely shared in terms of EoF [21]. In 2015, Liu et al. proved this bound analytically. There they also computed the bound of linear monogamy relation in terms of concurrence for a three-qubit system [22]. Moreover, Cornelio et al. proposed another interesting monogamy relation in terms of the squared concurrence for three-qubit systems [23]. They called the relations multipartite monogamy relations.

One of the motivations of this paper is to better understand the monogamy relations within the theory of multipartite entanglement. Although the authors in [24] mentioned a similar function of a three-qubit pure state in terms of some entanglement measure, they aimed to investigate the robustness of a three-qubit pure state against loss of a qubit. Here we characterize the distribution of the entanglement for an $n$-qubit system in terms of EoF and concurrence. In [24], the authors only showed the function numerically in terms of EoF and the bound of the function in terms of the squared concurrence among three-qubit pure states. Crucially, we present multi-linear monogamy relation in terms of entanglement of formation for a three-qubit pure state analytically. We generalize this bound to a three-qubit mixed state in terms of EoF and concurrence. Also we present only the LU-equivalent class of W state can reach the upper bound among three-qubit mixed states. That is, this can be seen to detect whether a three-qubit pure state is W state. Due to the importance of the W state in

*shixian01@gmail.com
†linchen@buaa.edu.cn (corresponding author)
‡mengyao@buaa.edu.cn
quantum computation and communication [25–28], this result is meaningful.

In this work, we consider a multi-linear monogamy relation in terms of EoF and concurrence for a multi-qubit system. We present that the W state is the unique state that can reach the upper bound of multi-linear monogamy relations in terms of concurrence and EoF up to the local unitary transformations (LU). We also present the condition when the states reach the minimum of the multi-linear monogamy relation in terms of concurrence. At last, we present some applications of our results to build an entanglement criterion and consider the absolutely maximally entangled states for a three-qubit system mainly. We also get a similar bound for the discord of three-qubit pure states.

This article is organized as follows. First we review the preliminary knowledge needed. Then we prove our main results. We present multi-linear monogamy relations in terms of EoF and concurrence. We also present some applications of our results on the entanglement witness. At last, we present some applications of our results on the entanglement witness.

### II. PRELIMINARIES

An $n$-partite pure state $|\psi\rangle_{A_1A_2\cdots A_n}$ is full product if it can be written as

$$|\psi\rangle_{A_1A_2\cdots A_n} = |\phi_1\rangle_{A_1}|\phi_2\rangle_{A_2}\cdots |\phi_n\rangle_{A_n},$$

otherwise, it is entangled. A multipartite pure state is called genuinely entangled if

$$|\psi\rangle_{A_1A_2\cdots A_n} \neq |\phi\rangle_{S}|\varphi\rangle_{S^c},$$

for any bipartition $S|S^c$; here $S$ is a subset of $A = \{A_1, A_2, \cdots, A_n\}$, $S^c = A - S$.

Assume $|\psi\rangle_{AB}$ is a bipartite pure state. Due to the Schmidt decomposition, $|\psi\rangle_{AB}$ can always be written as

$$|\psi\rangle_{AB} = \sum_i \sqrt{\lambda_i} |i\rangle_A |i\rangle_B,$$

here $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, $\{|i\rangle_{A(B)}\}$ is an orthonormal basis of the Hilbert space $A(B)$. First we recall the EoF. The EoF of $|\psi\rangle_{AB}$ is given by

$$E(|\psi\rangle_{AB}) = S(\rho_A) = -\sum_i \lambda_i \log \lambda_i,$$  \hspace{1cm} (4)

here $\lambda_i$ are the eigenvalues of $\rho_A = \text{Tr}_B |\psi\rangle_{AB}\langle\psi|$. For a mixed state $\rho_{AB}$, the EoF is defined by the convex roof extension method,

$$E(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle_{AB}\}} \sum_i p_i E(|\phi_i\rangle_{AB}),$$  \hspace{1cm} (5)

where the minimum is taken over all the decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB}\langle\phi_i|$ with $p_i \geq 0$ and $\sum p_i = 1$.

The other important entanglement measure is the concurrence ($C$). The concurrence of a pure state $|\psi\rangle_{AB}$ is defined as

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - \text{Tr} \rho_A^2)} = \sqrt{2(1 - \sum_i \lambda_i^2)}.$$  \hspace{1cm} (6)

For a mixed state $\rho_{AB}$, it is defined as

$$C(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle_{AB}\}} \sum_i p_i C(|\phi_i\rangle_{AB}),$$

where the minimum takes over all the decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB}\langle\phi_i|$ with $p_i \geq 0$ and $\sum p_i = 1$.

For a two-qubit mixed state $\rho_{AB}$, Wootters derived an analytical formula [29]:

$$E(\rho_{AB}) = h\left(1 + \frac{1 - C^2_{AB}}{2}\right),$$  \hspace{1cm} (8)

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x),$$

$$C_{AB} = \max \left\{ \sqrt{\mu_1^2 - \mu_2^2 - \mu_3^2 - \mu_4^2}, 0 \right\},$$  \hspace{1cm} (10)

here the $\mu_1, \mu_2, \mu_3, \mu_4$, are the eigenvalues of the matrix $\rho_{AB}(\sigma_y \otimes \sigma_y)\rho_{AB}^\dagger(\sigma_y \otimes \sigma_y)$ with nonincreasing order.

### III. MAIN RESULTS

For a three-qubit pure state $|\psi\rangle_{ABC}$, the pairwise correlations are described by the reduced density operators $\rho_{AB}, \rho_{BC}$ and $\rho_{CA}$. In 2014, Oliveira et al. [21] numerically presented the following inequality is valid for a 3-qubit pure state in terms of EoF and concurrence,

$$E_{A|B} + E_{A|C} \leq \lambda,$$  \hspace{1cm} (11)

here $\lambda$ is a constant, when $E$ is EoF, they conjectured $\lambda = 1.2018$. In 2015, the authors in [22] proved the above inequality for a 3-qubit pure state in terms of EoF analytically, there they denoted the above inequality as the linear monogamy relation.

From the (11), we find that although the EoF does not satisfy (1) for 3-qubit generic states, the entanglement cannot be freely shared in terms of EoF. Here we mainly consider a new linear monogamy relation which we call it multi-linear monogamy relation. The main difference between ours and the linear monogamy relations is that the left hand side takes over all bipartitions within the multipartite entanglement. For the 3-qubit states, it means in terms of some entanglement measure $\mathcal{E}$, the following inequality is valid,

$$M\mathcal{E} = \mathcal{E}_{A|B} + \mathcal{E}_{A|C} + \mathcal{E}_{B|C} \leq \nu.$$  \hspace{1cm} (12)

Here $\nu$ is a constant. We can also generalize the relations to $n$-qubit states $\rho_{A_1A_2\cdots A_n}$, we denote the following inequality in terms of some entanglement measure $\mathcal{E}$ as the multi-linear monogamy relation

$$\sum_{i < j} \mathcal{E}_{i|j} \leq \eta.$$  \hspace{1cm} (13)

Here $\eta$ is a constant.
A. Multipartite linear monogamy relations in terms of EoF

In this subsection, we first present a theorem on the multi-linear monogamy relation in terms of EoF for a three-qubit pure state.

**Theorem 1** For a three-qubit pure state, the W state reaches the upper bound $c_{\text{max}} = 3\beta(\frac{1}{2} + \sqrt{\beta})$ of multi-linear monogamy relation in terms of EoF.

The proof of Theorem 1 is in the APPENDIX VI A.

We can extend this result to the mixed state $\rho_{ABC}$. Assume that $\{s_h, |\phi_h\rangle_{ABC}\}$ is a decomposition of $\rho_{ABC}$, then we have

$$E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC}) = \sum_i p_i E(|\phi_i\rangle_{AB}) + \sum_j q_j E(|\theta_j\rangle_{AC}) + \sum_k r_k E(|\zeta_k\rangle_{BC}) \leq \sum_h s_h (E(\rho^h) + E(\rho^h_{AC}) + E(\rho^h_{BC})) \leq \sum_h s_h \times c_{\text{max}} = c_{\text{max}}.$$  \hspace{1cm} (14)

Here we assume that in the first equality, $\{p_i, |\phi_i\rangle\}$, $\{q_j, |\theta_j\rangle\}$ and $\{r_k, |\zeta_k\rangle\}$ are the optimal decompositions of $\rho_{AB}$, $\rho_{AC}$ and $\rho_{BC}$ in terms of the EoF correspondingly. The first equality is due to the definition of the EoF for the mixed states, the second inequality is due to the equality (48). In the first inequality, we denote $\text{Tr}_C |\phi^h\rangle\langle\phi^h| = \rho^h_{AB}$, $\text{Tr}_B |\phi^h\rangle\langle\phi^h| = \rho^h_{AC}$, $\text{Tr}_A |\phi^h\rangle\langle\phi^h| = \rho^h_{BC}$.

For a three-qubit pure state, Dür et al. [24] showed that there are two inequivalent kinds of genuinely entangled states, i.e. the W-class states and the Greenberger-Horne-Zeilinger (GHZ)-class states. The W-class states $|\psi\rangle$ are all LU equivalent to the following states:

$$|\psi\rangle = r_0|000\rangle + r_1|001\rangle + r_2|010\rangle + r_3|100\rangle,$$  \hspace{1cm} (15)

where $r_1, r_2, r_3 > 0$, and $\sum_{i=0}^{3} |r_i|^2 = 1$. From simple computation, we have $C^2(\rho_{AB}) = 4|r_2r_3|^2, C^2(\rho_{AC}) = 4|r_1r_3|^2, C^2(\rho_{BC}) = 4|r_1r_2|^2$. We see that the function $E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC})$ ranges over $[0, c_{\text{max}}]$ for the W class states. When $|\psi\rangle = |000\rangle + |111\rangle$, $E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC}) = 0$, and as the GHZ class states is dense [30], the function $E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC})$ ranges over $[0, c_{\text{max}}]$.

B. Multipartite linear monogamy relations in terms of concurrence

In this subsection, we present a theorem on the multi-linear monogamy relation in terms of the concurrence for a three-qubit pure state $|\psi\rangle_{ABC}$.

**Lemma 1** Up to the local unitary transformations, the W state is the unique state that can reach the upper bound in terms of the function $MC(|\psi\rangle) = C_{AB} + C_{BC} + C_{AC}$ for a three-qubit pure state.

We place the proof of the lemma 1 in the APPENDIX VI C.

Next we present an example on the multi-linear monogamy relation in terms of concurrence for a three-qubit mixed state.

**Example 1**

$$\rho = p_1|W\rangle\langle W| + p_2|\overline{W}\rangle\langle \overline{W}|.$$  

Here we denote that $|\overline{W}\rangle = \frac{1}{2}(|110\rangle + |101\rangle + |011\rangle)$.

Through simple computation, $\rho_{AB} = \rho_{AC} = \rho_{BC}$

$$\rho_{AB} = \frac{p_1}{3}|00\rangle\langle 00| + \frac{1}{3}(|01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|),$$

we have $C(\rho_{AB}) = C(\rho_{AC}) = C(\rho_{BC}) = 2 - 2\sqrt{p_1p_2}$.

if $p_1 > 0, i = 1, 2$, we have $MC(\rho_{ABC}) < 2$.

The Lemma 1 can be generalized to the three-qubit mixed states.

**Theorem 2** Up to the local unitary transformations, the W state is the unique state that can reach the upper bound in terms of the function $MC(\cdot)^{\text{max}}$ for a three-qubit mixed state.

The proof of Theorem 2 is placed in the APPENDIX VI D.

Next we present a necessary and sufficient condition when the function $MC(\cdot)$ attains the minimum 0.

**Theorem 3** Assume $|\psi\rangle$ is a three-qubit pure state, then $MC(|\psi\rangle) = 0$ if and only if $|\psi\rangle$ can be represented as $|\psi\rangle = r_0|000\rangle + r_1|111\rangle$ up to local unitary operations when $0 \leq r_0, r_1 \leq 1$.

The proof of Theorem 3 is placed in the APPENDIX VID.

**Theorem 4** Up to the local unitary transformations, the W state is the unique state that can reach the upper bound in terms of the function $E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC})$ for a three-qubit mixed state.
Theorem 4 can be proved in a similar process with the proof of theorem 2.

Next we consider the multi-linear monogamy relations for the $n$-qubit W-class states. These states were first proposed by [31] in order to study the monogamy relations in terms of convex roof extended negativity for higher dimensional systems.

**Example 2**

$$|\phi\rangle_{A_1A_2\ldots A_n} = \sqrt{p}|GW\rangle_n + \sqrt{1 - p}|0\rangle_n.$$  

Here we assume $|GW\rangle = a_1|10\ldots0\rangle + a_2|01\ldots0\rangle + \ldots + a_n|00\ldots1\rangle$, $\sum_i |a_i|^2 = 1$.

Through simple computation, we have $C(\rho_{A_1A_2\ldots A_n}) = 2p|a_1a_2|$.

$$MC(|\phi\rangle) = 2p\sum_{i<j} |a_i a_j| = p\left(\sum_i |a_i|^2 - \sum_i |a_i|^2\right) = p\left(\sum_i |a_i|^2 - 1\right).$$  

(16)

By the method of Lagrange multiplier, we see when $a_i = \frac{1}{\sqrt{n}}, p = 1$, that is, when $|\phi\rangle = |W\rangle$, the value in (16) attains the maximum.

In [32], the authors presented that for an $n$-qubit symmetric pure state $|\phi\rangle$, the maximal value between any pair of qubits in terms of concurrence is $\frac{2}{n}$, and when $|\phi\rangle = |W\rangle$, it attains the maximum. Then we may propose a conjecture.

**Conjecture 1** For an $n$-qubit genuinely entangled pure state $|\phi\rangle$, the maximum $MC(|\phi\rangle)$ is attained when $|\phi\rangle = |W\rangle$.

**Remark 1** Under the Conjecture 1, we can generalize the above results to $n$-qubit mixed states.

First, we prove that when $|\phi\rangle_{A_1A_2\ldots A_n}$ is an $n$-qubit pure state, the maximum of $MC(|\phi\rangle)$ is attained when $|\phi\rangle = |W\rangle$, that is, $\max_{\phi} MC(|\phi\rangle) = n - 1$.

If $|\phi\rangle$ is not genuinely entangled, we can always assume that $|\phi\rangle_{A_1A_2\ldots A_n}$ is biseparable, i.e. $|\phi\rangle_{A_1A_2\ldots A_n} = |\theta_1\rangle_{A_1A_2\ldots A_m} |\theta_2\rangle_{A_{m+1}A_{m+2}\ldots A_n}$, where $|\theta_1\rangle$, $i = 1, 2$ are genuinely entangled. As $|\phi\rangle$ is biseparable,

$$MC(|\phi\rangle) = MC(|\theta_1\rangle) + MC(|\theta_2\rangle) \leq m - 1 + n - m - 1 = n - 2 < n - 1.$$  

(17)

Then by a similar proof of Theorem 2 and the statement above, we can get the results on mixed states: when $\rho_{A_1A_2\ldots A_n}$ is an $n$-qubit mixed state, up to the local unitary transformations, the $W$ state is the unique state that can reach the upper bound in terms of the function $\sum_{i<j} C_{ij}$.

Then we pick $10^5$ four qubit and five qubit pure states randomly and compute their $MC(|\cdot\rangle)$, these results may verify the Conjecture 1 numerically.

In Fig. 1, we present a histogram of the value of $\sum_{i<j} C_{ij}$ for random pure states of four qubits sampled uniformly. Here we find the function $MC(|\cdot\rangle)$ mainly distributes in the section $[0, 1.8]$, and in the section $[1.8, 2]$, there are few states. Fig. 1 supports Conjecture 1 for $n = 4$.

In Fig. 2, we present a histogram of the value of $\sum_{i<j} C_{ij}$ for random pure states of five qubits sampled uniformly. From the figure, we have that the sum of $MC(|\cdot\rangle)$ mainly distributes in the section $[0, 0.6]$. In [33], the authors considered the multipartite correlations in four-qubit pure states. Here through the Fig 2, we have that the quantity of the bipartite correlations of most five-qubit pure states is few, then it seems that compar-
ing with the separable states, the set of the entangled states for five-qubit pure states are bigger.

At the last of this section, we consider the $MC(\cdot)$ for a class of pure states in a system with more qubits studied in [34]. They are useful kinds of entanglement states for quantum teleportation and error correction,

$$|\psi\rangle = a|GHZ\rangle_m|W\rangle_n + b|W\rangle_m|GHZ\rangle_n.$$ 

\[ \rho^1 = \frac{|a|^2}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{|b|^2}{m}(|m-2\rangle\langle 00| + (|01\rangle + |10\rangle)(|01\rangle + |10\rangle)), \]
\[ \rho^2 = \frac{|b|^2}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{|a|^2}{n}(|n-2\rangle\langle 00| + (|01\rangle + |10\rangle)(|01\rangle + |10\rangle)), \]
\[ \rho^3 = \frac{a}{\sqrt{2n}}|01\rangle + \frac{b}{\sqrt{2m}}|10\rangle)(\frac{a}{\sqrt{2n}}|01\rangle + \frac{b}{\sqrt{2m}}|10\rangle) + \frac{(n-1)|a|^2}{2n}(|00\rangle\langle 00| + \frac{(n-1)|a|^2}{2m}|10\rangle\langle 10| + \frac{|b|^2}{2m}|11\rangle\langle 11|, \]

then we have

$$\rho^1 = \frac{|a|^2}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{|b|^2}{m}(|m-2\rangle\langle 00| + (|01\rangle + |10\rangle)(|01\rangle + |10\rangle)), \]
\[ \rho^2 = \frac{|b|^2}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{|a|^2}{n}(|n-2\rangle\langle 00| + (|01\rangle + |10\rangle)(|01\rangle + |10\rangle)), \]
\[ \rho^3 = \frac{a}{\sqrt{2n}}|01\rangle + \frac{b}{\sqrt{2m}}|10\rangle)(\frac{a}{\sqrt{2n}}|01\rangle + \frac{b}{\sqrt{2m}}|10\rangle) + \frac{(n-1)|a|^2}{2n}(|00\rangle\langle 00| + \frac{(n-1)|a|^2}{2m}|10\rangle\langle 10| + \frac{|b|^2}{2m}|11\rangle\langle 11|, \]

Here $|a|^2 + |b|^2 = 1$, and $m, n \geq 2$. Due to the shape of $|\psi\rangle$, we have the set of the bipartite reduced density matrices for the pure state $|\psi\rangle$ consist of three kinds:

$$\text{When } m = 2, n = 3, \text{ we have}$$
\[ f(x) = \begin{cases} 1 - 2x & x \in (0, 1/2) \\ 0 & x \in [1/2, 3\sqrt{5} - 6] \\ 2x - \sqrt{3}x^2 - 12x + 9 & x \in (3\sqrt{5} - 6, 1], \end{cases} \]

then we have when $x = 1$, $f(x)$ gets the maximum 2.

When $m = 3, n = 2$, we have
\[ f(x) = \begin{cases} 2(1-x) - \sqrt{3}x^2 + 6x & x \in [0, 7 - 3\sqrt{5}] \\ 0 & x \in [7 - 3\sqrt{5}, 1] \\ 2x - 1 & x \in [1/2, 1], \end{cases} \]

when $x = 0$, $f(x)$ gets the maximum 2.

When $m = n = 3$, we have
\[ f(x) = \begin{cases} 3(1-x) - \sqrt{3}x^2 + 6x & x \in [0, 7 - 3\sqrt{5}] \\ 0 & x \in [7 - 3\sqrt{5}, 3\sqrt{5} - 6] \\ 2x - \sqrt{3}x^2 - 12x + 9 & x \in (3\sqrt{5} - 6, 1], \end{cases} \]

when $x = 0$ or $x = 1$, $f(x)$ gets the maximum, 2.

When $m = 4, n = 2$, we have
\[ f(x) = \begin{cases} 3 - 3x - 6\sqrt{x} & x \in [0, 3 - 2\sqrt{2}] \\ 0 & x \in [3 - 2\sqrt{2}, 1/2] \\ 2x - 1 & x \in [1/2, 1], \end{cases} \]

when $x = 0$, $f(x)$ gets the maximum, 3.
When \( m = 2, n = 4 \), we have

\[
 f(x) = \begin{cases} 
 1 - 2x & x \in [0,1,2] \\
 0 & x \in (1/2, 2\sqrt{2} - 2) \\
 3x - 12\sqrt{\frac{1}{4} - \frac{x}{4}} & x \in [2\sqrt{2} - 2, 1] 
\end{cases},
\]

(27)

then \( \max_{x \in [0,1]} f(x) = 3 \).

Next from simple computation, we have \( \forall m, n \geq 3, h(n) \geq g(m) \), then

\[
 f(x) = \begin{cases} 
 (m - 1)(1 - x) - 2\sqrt{\frac{4 - m}{4m} x^2 + \frac{m - 2}{2m} x} & x \in [0, g(m)] \\
 0 & x \in [g(m), h(n)] \\
 (n - 1)x - n(n - 1)\sqrt{\frac{1}{4} - \frac{x}{n} + \frac{1 - n}{4n} x^2} & x \in (g(n), 1) 
\end{cases}
\]

(28)

when \( x \in [0, g(m)], m \geq 4 \), \( f(x) \) is monotone decreasing, then when \( x = 0 \), \( f(x) = m - 1 \), when \( x \in (g(n), 1) \), \( f(x) \) is monotone increasing, then \( \max_{x \in (g(n), 1)} f(x) = n - 1 \), that is, when \( m \geq 4 \), \( f(x) = \max(m - 1, n - 1) \).

In the next section, we present some applications of our results.

### IV. APPLICATION

The structure of a multipartite entanglement system is complex. In this section, we apply our results above on the genuine entanglement detection. We also make some comments on the absolutely maximally entangled states (AMES), and we apply Theorem 3 to present when a pure state is AMES in a three-qubit system.

#### A. An entanglement criterion for genuine entangled states

On the other hand, an important problem in entanglement theory is to determine whether a multipartite state is genuinely entangled, biseparable or fully separable. An widely accepted method for attacking the problem is to construct entanglement witnesses (EWs) [1]. The EW \( W \) is an Hermitian operator when \( \text{Tr}(W \sigma) \geq 0 \) for every biseparable state \( \sigma \), and \( \text{Tr}(W \rho) < 0 \) for some entangled state \( \rho \). The EW is a theoretical and experimental method compared with mathematical criteria, such as positive partial transpose (PPT) [34] and computable cross norm [35]. For a review we refer readers to the papers [1, 36]. EWs have been constructed to detect the entanglement of many physically realizable states, such as the GHZ diagonal states [37, 38], GHZ-like state [39], noisy Dicke states [40]. In the following we connect EWs to the \( MC(\phi) \).

In practice, we need analyze the change of \( MC(\phi) \) of \( n \)-qubit pure states \( |\phi\rangle \) under the white noise. Let

\[
 \rho(p) = p|\phi\rangle\langle\phi| + (1 - p)\frac{I}{2^n}, \quad p \in (0, 1).
\]

By definition and simple computation, one can show that the \( MC(\rho) \) monotonically increases with \( p \). As an example, we assume that \( |\phi\rangle \) is the \( n \)-qubit W state. Hence

\[
 \rho(p) = p|W_n\rangle\langle W_n| + (1 - p)\frac{I}{2^n}, \quad p \in (0, 1)
\]

(29)

here we can find that the state (29) is symmetric. From computation, we have

\[
 C(\rho_{i\neq j}) = \max(0, \frac{2p}{n} - 2\sqrt{\frac{(1-p)(n+3np-8p)}{16n}}),
\]

(30)

that is, when

\[
 16p^2 - n^2(1-p)^2 + 4np(n-2)(p-1) > 0,
\]

(31)

\[
 MC(\rho) = (n-1)p - (n-1)\sqrt{n(1-p)(n+3np-8p)}.
\]

(32)

When \( n = 3 \), from the analysis above, we have when \( p \geq \frac{\sqrt{35} - 5}{8} \), the state \( \rho(p) \) is a genuinely entangled state.

In [40], the authors showed when \( p \leq \frac{\sqrt{7} - 3}{8} \), the state \( \rho(p) \) is fully separable, there the authors present an optimal entanglement witness \( \hat{W} \) for the \( \rho(p) \) when \( n = 3 \), the
witness $\hat{W}$ is written as
\[
\hat{W} = \frac{1}{d} |\langle 000 \rangle\langle 000 | - |\langle 001 \rangle\langle 001 | + (|\langle 100 \rangle\langle 100 | + |\langle 010 \rangle\langle 010 | + |\langle 001 \rangle\langle 010 | + |\langle 011 \rangle\langle 011 | + |\langle 101 \rangle\langle 110 |)|\langle 011 |\rangle\langle 011 |
\]
Ref. [41] has shown that the state $\rho(p)$ is fully separable when $p \leq \frac{\sqrt{3}}{8+\sqrt{3}}$, they also showed when $p \in (\frac{\sqrt{3}}{8+\sqrt{3}}, 0.2095]$, the state $\rho(p)$ is biseparable but not fully separable. Here if Conjecture 1 is true, then $MC(\rho) > n-2$, implies that $\rho(p)$ is a genuinely entangled state. In particular, this is true when $n = 3$ by Theorem 4. Thus proving Conjecture 1 is meaningful to the investigation of entanglement detection.

B. absolutely maximally entangled state

Next we investigate an absolutely maximally entangled states (AMES) in $n$-qubit systems. A pure multipartite entangled state is called AMES if all reduced density operators obtained by tracing out at least half of the particles are maximally mixed [42]. So the function $MC(\cdot)$ of every AMES is zero, though the converse fails because the $n$-qubit non-AMES state may have separable reduced density operators. An example is the GHZ state. Hence, we have constructed a necessary condition by which a multipartite state is an AMES. Next we consider the AMES in a 3-qubit system.

Corollary 1 The sole class of AMES $|\psi\rangle$ in an 3-qubit system are the states that are $LU$ equivalent to $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000 \rangle + |111 \rangle)$.

We place the proof of the corollary in the APPENDIX VIE.

C. An upper bound of the sum of all bipartite quantum discord

Here we present an upper bound of the sum of all bipartite quantum discord for a three-qubit pure state. The quantum discord was first presented by Henderson and Vedral [43], Ollivier and Zurek [44] independently. Quantum discord is a measure of nonclassical correlation. It is defined as
\[
\delta_{AB}^- = I_{AB} - J_{AB}^-
= I_{AB} - \max_{\{\Pi_{A}^B\}}(S(\rho_A) - \sum_x p_x S(\rho_A^x)),
\]
where the maximum takes over all the positive-operator-valued-measurements $\{\Pi_{A}^B\}$ performed on the subsystem $B$, $p_x = \text{Tr} \Pi_{A}^B \rho_{AB} \Pi_{A}^B$, and $\rho_A^x = \text{Tr}_B (\Pi_{A}^B \rho_{AB} \Pi_{A}^B) / p_x$. From its definition, it quantifies at least how much a bipartite state of one system is changed on average by the measurement of the other system. In the last decade, there are some results suggesting that quantum discord plays an important role in quantum information and computation tasks [45–49]. Recently, Guo et al. considered the complete monogamy relation for multi-party quantum discord [50].

Next we recall a conservation law for distributed EoF and quantum discord of a three-qubit pure state [51],
\[
E_{AB} + E_{AC} = \delta_{AB}^- + \delta_{AC}^-, \quad (32)
\]
The law depends on the Koashi-Winter (KW) relation $E_{AB} + J_{AC}^m = S_A$ [52].

Here we present an upper bound of the sum of all the bipartite discord for a three-qubit pure state, from equation (32), we have
\[
\delta_{AB}^- + \delta_{BC}^- + \delta_{CA}^- + \delta_{BA}^- + \delta_{AC}^- + \delta_{CB}^-
= E_{AB} + E_{AC} + E_{BC} + E_{BA} + E_{CA} + E_{CB}
\leq 2 \times c_{max} = 2c_{max}.
\]
Thus we have quantum discord owns a multi-linear monogamy relation for a three-qubit pure state.

V. CONCLUSIONS

Here we have mainly considered the shareability of the entanglement for a multi-qubit state in terms of the EoF. We have presented that up to the local unitary transformations, the $W$ state is the unique that can reach the upper bound of $MC(\cdot)$ for a three-qubit state, these results may tell us that the entanglement cannot be shared freely for a three-qubit system. We have also picked $10^5$ four-qubit and five-qubit pure states randomly and computed their $MC(\cdot)$, which have verified the Conjecture 1 numerically. Finally, we also have presented some applications of our results. We think the methods here we used can be generalized to consider the upper bound of the multi-linear monogamy relation in terms of other bipartite entanglement measures such as, Rényi entanglement for an $n$-qubit pure state. We believe that our results are helpful on the study of monogamy relations for multipartite entanglement systems.

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VI. APPENDIX

A. The proof of Theorem 1

For a three-qubit pure state, the W state reaches the upper bound of multi-linear monogamy relation in terms of EoF.

Proof. Here we denote that

\[
 f(x) = h\left(\frac{1 + \sqrt{1 - x}}{2}\right)
\]

\[
 x = C^2_{AB}
\]

\[
 y = C^2_{AC} + C^2_{AB},
\]

\[
 c = C^2_{AC} + C^2_{AB} + C^2_{BC},
\]

\[
 g(x, y) = f(x) + f(y - x) + f(c - y),
\]

\[
 \frac{\partial g}{\partial x} = f'(x) - f'(y - x) = 0,
\]

\[
 \frac{\partial g}{\partial y} = f'(y - x) - f'(c - y) = 0.
\]

As \( f''(x) < 0 \), \( f'(x) \) is monotonously decreasing [53], \(-f'(y - x)\) is also monotonously decreasing in terms of \( x \), and by the equality (35), we have

\[
 C^2_{AB} = C^2_{AC} = C^2_{BC}
\]

is the only case when the (35) is valid. Furthermore, if \( f(x) \) is a monotonic function [53], we have when (36) is valid, \( E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC}) \) achieves the upper bound for a three-qubit pure state.

From [54], we have that a three-qubit pure state \( |\psi\rangle_{ABC} \) can be written in the generalized Schmidt decomposition:

\[
 |\psi\rangle = l_0|000\rangle + l_1 e^{i\theta}|100\rangle + l_2|101\rangle + l_3|110\rangle + l_4|111\rangle,
\]

here \( \theta \in [0, \pi), l_i \geq 0, i = 0, 1, 2, 3, 4, \sum_{i=0}^{4} l_i^2 = 1 \). From simple computation, we have

\[
 C^2_{AB} = 4l_0^2 l_1^2, C^2_{AC} = 4l_0^2 l_2^2,
\]

\[
 C^2_{BC} = 4l_2^2 l_3^2 + 4l_1^2 l_4^2 - 8l_1 l_2 l_3 l_4 \cos \theta.
\]
As $f(x)$ is monotone [53], then we only need to obtain the maximum of $4l_0^2l_2^2$ by using the Lagrange multiplier,  
\[ \max \left( E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC}) \right) =: c_{\max} = 3h\left(\frac{1}{2} + \frac{\sqrt{5}}{6}\right). \]  
(48)

When computing (41) - (47) equal to 0, we have $l_0 = l_2 = l_3 = \frac{1}{\sqrt{3}}$, $l_1 = l_4 = 0$ is the only case when $C^2(\rho_{AB})$ attains the maximum, that is,  
\[ \max_{|\psi\rangle_{ABC}} \left( E(\rho_{AB}) + E(\rho_{AC}) + E(\rho_{BC}) \right) = c_{\max} = 3h\left(\frac{1}{2} + \frac{\sqrt{5}}{6}\right). \]  
(48)

When computing (41) - (47) equal to 0, according to (45), we have that at least one of the equalities in the set \( \{\mu = 0, l_1 = 0, l_2 = 0, l_4 = 0, \sin \theta = 0\} \) is valid. Then by using the method of exclusion, we could get the result. By the way, in the method of exclusion, we mainly use that when $|\psi\rangle$ is separable, the function $ME(\cdot)$ cannot get the maximum.

When we take the operation $\sigma_x$ on the first system, we get the W state $\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. \qed

**B. The proof of Lemma 1**

**Up to the local unitary transformations, the W state is the unique state that can reach the upper bound in terms of the function $MC(\cdot)$ for a three-qubit pure state.**

**Proof.** We'll compute the maximum of the six classes of a three-qubit state respectively according to [55].

**Case i:** When $|\psi\rangle_{ABC}$ is $A - B - C$, $MC(\psi) = 0$.

**Case ii:** When $|\psi\rangle_{ABC}$ is biseparable, if $|\psi\rangle_{ABC} = |\phi_1\rangle_A \otimes |\phi_2\rangle_B$, $\rho_{AC} = |\phi_2\rangle_B$, $MC(\psi) = C_{AB} = 0$, $MC(\psi) = C_{AB} \leq 1$, the other cases are similar.

**Case iii:** When $|\psi\rangle_{ABC}$ belongs to the W class, according to the formula (15), $MC(\psi) = 2r_2r_3 + 2r_1r_3 + 2r_1r_2$. Trivially, when $r_1 = r_2 = r_3$, that is, $|\psi\rangle_{ABC} = |W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$, $MC(\psi)$ gets the maximum.

**Case iv:** When $|\psi\rangle_{ABC}$ belongs to the GHZ class, according to [55], assume $|\psi\rangle = M_1 \otimes M_2 \otimes M_3(\text{GHZ})$, $M_1 = (\bar{u}_i, \bar{\varepsilon}_i) = (u_i \cos \theta_i, u_i \sin \theta_i)T$, $\bar{\varepsilon}_i = (v_i \cos(\phi_i + \theta_i), v_i \sin(\phi_i + \theta_i))T$,
\[ MC(\psi) = \frac{|c_1s_1s_2| + |c_2s_1s_3| + |c_3s_1s_2|}{r + c_1c_2c_3}, \]  
(49)

Here we denote $c_i = \cos \phi_i, s_i = \sin \phi_i, i = 1, 2, 3$, $2r = \frac{\sqrt{14}}{2} + \frac{\sqrt{41}}{2}$, $r \geq 1$. In order to let $MC$ be the maximum, assume $r = 1$. When $c_i < 0$, $MC$ will get maximum. Then let $\phi_i \in [0, \frac{\pi}{2}]$, $MC(\psi) = \frac{c_1s_2s_3 + c_2s_1s_3 + c_3s_1s_2}{c_1c_2c_3}$. Next we will prove $MC(\psi) \leq 2$, first we define $l(c_1, c_2, c_3)$ as follows,
\[ l(c_1, c_2, c_3) = \begin{cases} 0, & r = 1, \end{cases} \]
\[ = c_1 \cos(\phi_2 - \phi_3) + c_2 \cos(\phi_1 - \phi_3) + c_3 \cos(\phi_1 - \phi_2) - c_1c_2c_3, \]

Assume $c_1 > c_2 \geq c_3$, then we obtain
\[ l(c_1, c_2, c_3) \leq l(c_1, c_1, c_1). \]

When $c_1 = c_2 = c_3$, the function $l(c_1, c_1, c_1)$ is a monotonic function of $c_1$. When $c_1 \rightarrow 1$, the function $l(c_1, c_1, c_1)$ gets the maximum, that is, $MC(\psi) \rightarrow 2$. Then we prove that if $|\psi\rangle_{ABC}$ is a GHZ class state, then $MC(\psi|_{ABC}) \leq 2$. However, from the above analysis, when $MC(|\psi\rangle_{ABC}) = 2$, we have $c_1 = c_2 = c_3 \rightarrow 1$, that is, the matrix $M_i$ is singular, this is impossible. \qed

**C. The proof of Theorem 2**

**Up to the local unitary transformations, the W state is the unique state that can reach the upper bound in terms of the function $MC(\cdot)$ for a three-qubit state.**

**Proof.** Combining with Lemma 1, we only need to present that the mixed states cannot reach the upper bound of the multi-linear monogamy relations in terms of concurrence.

Due to Lemma 1, for a three-qubit pure state $|\psi\rangle$, $MC(|\psi\rangle)$ gets the maximum, only when $|\psi\rangle$ is LU equivalent to $|W\rangle$. Assume $\rho$ is a three-qubit mixed system, \( \{(p_i, |\phi_i\rangle)| i = 1, 2, \ldots, k\} \) is an optimal decomposition of $\rho$, we can always assume $k = 2$. For the cases when $k > 2$, we can prove similarly. As $|\phi_i\rangle$ is LU equivalent to $|W\rangle$, we can always assume \( \{(p_1, |W\rangle), (p_2, U_1 \otimes U_2 \otimes U_3 |W\rangle)\} \) is a decomposition of $\rho$, then
\[ MC(\rho) = C(\rho_{AB}) + C(\rho_{AC}) + C(\rho_{BC}) = C(\sigma_1) + C(\sigma_2) + C(\sigma_3) \]  
(50)
here we assume
\[\sigma_1 = \frac{p_1}{3} |00\rangle \langle 00| + 2|\phi^+\rangle \langle \phi^+| + \frac{p_2}{3} \tau_1,\]
\[\sigma_2 = \frac{p_1}{3} |00\rangle \langle 00| + 2|\phi^+\rangle \langle \phi^+| + \frac{p_2}{3} \tau_2,\]
\[\sigma_3 = \frac{p_1}{3} |00\rangle \langle 00| + 2|\phi^+\rangle \langle \phi^+| + \frac{p_2}{3} \tau_3,\]
\[|\phi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),\]

Here we denote \(\tau_1 = [(U_1 \otimes U_2)|00\rangle \langle 00|(U_1 \otimes U_2)\dagger] + 2(U_1 \otimes U_2)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_2)\dagger\), \(\tau_2 = [(U_1 \otimes U_2)|00\rangle \langle 00|(U_1 \otimes U_3)\dagger] + 2(U_1 \otimes U_3)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_3)\dagger\), \(\tau_3 = [(U_2 \otimes U_3)|00\rangle \langle 00|(U_2 \otimes U_3)\dagger] + 2(U_2 \otimes U_3)|\phi^+\rangle \langle \phi^+|(U_2 \otimes U_3)\dagger\), then we have
\[C(\sigma_1) \leq \frac{2}{3} C(p_1|\phi^+\rangle \langle \phi^+| + p_2(U_1 \otimes U_2)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_2)\dagger)\]

By the Lemma 2, 3 and 4, we have \((U_1 \otimes U_2)|\phi^+\rangle = e^{ix}|\phi^+\rangle\) is a sufficient and necessary condition for \(C(p_1|\phi^+\rangle \langle \phi^+| + p_2(U_1 \otimes U_2)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_2)\dagger) = 1\), where \(e^{ix}\) is a global phase factor. Then we can get the similar result for \(\sigma_2\) and \(\sigma_3\). As \(Sr(\{01\}) = Sr(\{10\}) = 1\), here we denote that \(Sr(\cdot)\) is the Schmidt rank, then we have \(U_i = \begin{pmatrix} e^{i\theta_i} & 0 \\ 0 & 1 \end{pmatrix}, i = 1, 2, 3\), that is, \(\rho = |W\rangle \langle W|\). Then we finish the proof.

Lemma 2 Assume \(A, B \in Pos(\mathbb{H})\), here we denote that \(Pos(\mathbb{H})\) is a linear space consisting of all the semidefinite positive operators of a bounded Hilbert space \(\mathbb{H}\). Here we denote that \(\text{Eig}(A)\) and \(\text{Eig}(B)\) are two sets consisting of all the eigenvalues of the matrix \(A\) and \(B\) respectively. If the biggest elements in the set \(\text{Eig}(A)\) and \(\text{Eig}(B)\) are \(1\) or less, then the biggest element in the set \(\text{Eig}(AB)\) is \(1\) or less.

Proof. Assume that the eigenvalues of \(A\) are \(\lambda_i\) with its eigenvector \(|\alpha_i\rangle\), the eigenvalues of \(B\) are \(\mu_j\) with its eigenvector \(|\beta_j\rangle\), and the eigenvalues of \(AB\) are \(\chi_k\) with its eigenvectors \(|\gamma_k\rangle\). Here we always assume that the range of \(A\) and \(B\) are nonsingular, then we have
\[AB|\gamma_k\rangle = AB \sum_j x_{jk} |\beta_j\rangle = A \sum_j \mu_j x_{jk} |\beta_j\rangle = \sum_{ij} \lambda_i \mu_j x_{jk} y_{ij} |\alpha_i\rangle,\]

in the formula (52), we denote that \(|\gamma_k\rangle = \sum_j x_{jk} |\beta_j\rangle\) and \(|\beta_j\rangle = \sum_i y_{ij} |\alpha_i\rangle\). From the equality (52) and (53), we have
\[\chi_k = \frac{\sum_{ij} \lambda_i \mu_j x_{jk} y_{ij}}{\sum_{ij} x_{jk} y_{ij}} \leq 1.\]

Theorem 3 Rank \(|p_1|\phi^+\rangle \langle \phi^+| + p_2(U_1 \otimes U_2)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_2)\dagger) = 1\) if and only if \((U_1 \otimes U_2)|\phi^+\rangle = e^{ix}|\phi^+\rangle\), here \(e^{ix}\) is a global phase factor.

Proof. Here we denote \(\sigma = p_1|\phi^+\rangle \langle \phi^+| + p_2(U_1 \otimes U_2)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_2)\dagger\). If \((U_1 \otimes U_2)|\phi^+\rangle \neq e^{ix}|\phi^+\rangle\), then \((U_1 \otimes U_2)|\phi^+\rangle\) and \(|\phi^+\rangle\) are linear independent, \(\dim(span\{|\phi^+\rangle, (U_1 \otimes U_2)|\phi^+\rangle\}) = 2\), \(\dim(span\{|\phi^+\rangle, (U_1 \otimes U_2)|\phi^+\rangle\})^\perp = n - 2\). As \(|\alpha\rangle \in \dim(span\{|\phi^+\rangle, (U_1 \otimes U_2)|\phi^+\rangle\}^\perp\), \(\sigma |\alpha\rangle = 0\), that is, \(\text{Rank}(\sigma) \leq 2\). As we cannot find a nontrivial vector \(|\beta\rangle\) in the subspace \(span\{|\phi^+\rangle, (U_1 \otimes U_2)|\phi^+\rangle\}\) such that \(\sigma |\beta\rangle = 0\), then we finish the proof.

Lemma 4 Assume \(\theta = p_1|\phi^+\rangle \langle \phi^+| + p_2U_1 \otimes U_2)|\phi^+\rangle \langle \phi^+|(U_1 \otimes U_2)\dagger\) with \(\text{Rank}(\theta) = 2\), then \(\text{Rank}(\theta |\sigma_y \otimes \sigma_y\rangle |\theta\rangle) = 2\).

Proof. As \(\sigma_y \otimes \sigma_y\) is invertible, we only need to prove \(\text{Rank}(\theta |\sigma_y \otimes \sigma_y\rangle |\theta\rangle) = 2\). As \(\text{Rank}(\theta |\sigma_y \otimes \sigma_y\rangle |\theta\rangle) = \text{Rank}(\theta |\sigma_y \otimes \sigma_y\rangle |\theta\rangle)\), then if we can prove \(\text{Rank}(\theta |\theta\rangle) = 2\), we finish the proof. As \(\theta = |\theta\rangle\), then \(\text{Rank}(\theta |\theta\rangle) = \text{Rank}(|\theta\rangle) = 2\).
First we provide two methods to prove that the second state is LU equivalent to the state $|\psi\rangle = (l_0|000\rangle + l_1e^{i\theta}|111\rangle)|00\rangle$.

When $MC(|\psi\rangle) = 0$, we have that
\[
\begin{align*}
l_0l_2 = l_0l_3 = 0, \\
l_2^2l_3^2 + l_2^2l_4^2 = 2l_1l_2l_3l_4\cos\theta.
\end{align*}
\]
(56)

When $l_0 \neq 0$, we have $l_2 = l_3 = 0$, that is, $l_1l_4 = 0$, then $|\psi\rangle = l_0|000\rangle + l_1|111\rangle$, or $|\psi\rangle = (l_0|0\rangle + l_1e^{i\theta}|1\rangle)|00\rangle$.

When $|\psi\rangle$ is the second state, let $U_1$ be a unitary on the first system such that $U_1(|l_0|0\rangle + l_1e^{i\theta}|1\rangle) = |0\rangle$, then the second state is LU equivalent to the state $|00\rangle$.

Below we denote $U_i$ be a unitary on the $i$-th system, $i = 1, 2, 3$.

When $l_0 = 0$, then from the third formula in (55),
\[
(l_2l_3 \cos\theta - l_1l_4)^2 + l_2^2l_3^2\sin^2\theta = 0,
\]
(57)

that is, $l_2l_3 \cos\theta = l_1l_4$, $l_2l_3 \sin\theta = 0$. When $\sin\theta = 0$ and $l_i \geq 0$, $i = 1, 2, 3, 4$, $\cos\theta = 1$. That is, $(l_1, l_2) = (l_3, l_4)$, then $|\psi\rangle = |1\rangle(|0\rangle + a|1\rangle)(l_1|0\rangle + l_2|1\rangle)$. When $U_2(|0\rangle + a|1\rangle) = \sqrt{1 + a^2}|0\rangle$, $U_3(l_1|0\rangle + l_2|1\rangle) = \sqrt{l_1^2 + l_2^2}|0\rangle$, then we obtain that the above state is LU equivalent to $|00\rangle$.

When $l_2 = 0$, then $l_1l_3 = 0$. If $l_1 = 0$, then $|\psi\rangle$ can be represented as $|\psi_1\rangle = l_3|110\rangle + l_4|111\rangle$. Let $U_3(l_3|0\rangle + l_4|1\rangle) = |1\rangle$, then it is LU equivalent to the state $|11\rangle$. If $l_4 = 0$, then $|\psi\rangle$ can be represented as $|\psi_2\rangle = e^{i\theta}l_1|100\rangle + l_3|110\rangle$. Let $U_2(e^{i\theta}l_1|0\rangle + l_3|1\rangle) = |1\rangle$ and $U_3 = \sigma_x$, then it is LU equivalent to the state $|11\rangle$. The case when $l_3 = 0$ is similar to the case when $l_2 = 0$. Then we finish the proof. \hfill \Box

E. The proof of Corollary 1

The sole class of AMES $|\psi\rangle$ in an 3-qubit system are the states that are LU equivalent to $|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$.

**Proof.** First we provide two methods to prove that $\rho_{AB}$, $\rho_{AC}$ and $\rho_{BC}$ are separable. Assume $|\psi\rangle_{ABC}$ is an AMES state in an 3-qubit state, then
\[
\rho_A = \rho_B = \rho_C = \frac{I}{2}.
\]
Next as $\rho_A = \frac{I}{2}$, then any purification state $|\phi\rangle_{AB'}$ of $\rho_A$ can be written as
\[
|\phi\rangle_{ABC} = (I_A \otimes U_{BC})(|00\rangle + |11\rangle)|0\rangle \sqrt{2},
\]
here $U_{BC}$ is a unitary operator, then we have
\[
|\phi\rangle_{ABC} = |\phi_1\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_2\rangle,
\]
that is, $r(|\phi_{BC}\rangle) = 2$, then due to Theorem 1 in [56], we have $\rho_{BC}$ is separable. Similarly, we have $\rho_{AB}$ and $\rho_{AC}$ is separable.

Here we provide the other method to prove that $\rho_{AB}$, $\rho_{AC}$ and $\rho_{BC}$ are separable. As $\rho_A = \rho_B = \frac{I}{2}$, from [57], we have
\[
\begin{align*}
\rho_{AB} &= \lambda_1\Psi^+ + \lambda_2\Psi^- + \lambda_3\Phi^+ + \lambda_4\Phi^-,
\end{align*}
\]
(59)

As $\rho_{AC}$ and $\rho_{AB}$ are with the same spectrum, then we have only two of $\lambda_i$, $i = 1, 2, 3, 4$ are $\frac{1}{2}$. Then $\rho_{AB}$ is separable. Similarly, we have $\rho_{AC}$ and $\rho_{BC}$ are separable.

As all of $\rho_{AB}$, $\rho_{AC}$ and $\rho_{BC}$ are separable, then we have
\[
MC(|\psi\rangle) = 0,
\]
then from Theorem 3, we have $|\psi\rangle = r_0|000\rangle + r_1|111\rangle$ up to unitary operations when $r_0, r_1 \in [0, 1]$. As $\rho_A = \rho_B = \rho_C = \frac{I}{2}$, then $|\psi\rangle = \frac{1}{2}(|000\rangle + |111\rangle)$ up to local unitary operations. \hfill \Box