BERNSTEIN–SATO VARIETIES AND ANNIHILATION OF POWERS

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Abstract. Given a complex germ $f$ near the point $x$ of the complex manifold $X$, equipped with a factorization $f = f_1 \cdots f_r$, we consider the $\mathcal{D}_{X,x}[s_1, \ldots, s_r]$-module generated by $F^S := f_1^{s_1} \cdots f_r^{s_r}$. We show for a large class of germs that the annihilator of $F^S$ is generated by derivations and this property does not depend on the chosen factorization of $f$.

We further study the relationship between the Bernstein–Sato variety attached to $F$ and the cohomology support loci of $f$, via the multivariate version of the classical map $\nabla_A$. We show that for our class of divisors the injectivity of $\nabla_A$ implies its surjectivity. Restricting to reduced, free divisors, we also show the reverse, using the theory of Lie–Rinehart algebras. In particular, we analyze the dual of $\nabla_A$ using techniques pioneered by Narváez–Macarro.

As an application of our results we establish a conjecture of Budur in the tame case: if $V(f)$ is a central, essential, indecomposable, and tame hyperplane arrangement, then the Bernstein–Sato variety associated to $F$ contains a certain hyperplane. By the work of Budur, this verifies the Topological Multi-variable Strong Monodromy Conjecture for tame arrangements. Generalizing results of Budur, Liu, Saumell, and Wang in the reduced and free case, we characterize local systems outside the cohomology support loci of $f$ near $x$ in terms of the simplicity of modules derived from $F^S$.

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1. Introduction

Let $X$ be a smooth analytic space or $\mathbb{C}$-scheme of dimension $n$ with structure sheaf $\mathcal{O}_X$ and with the sheaf of $\mathbb{C}$-linear differential operators $\mathcal{D}_X$. Take a global function $f \in \mathcal{O}_X$. The classical construction of the Bernstein–Sato polynomial of $f$ is as follows:

1. Consider the $\mathcal{O}_X[f^{-1}, s]$-module generated by the symbol $f^s$. This has a $\mathcal{D}_X[s]$-module structure induced by the formal rules of calculus.
2. The Bernstein–Sato ideal $B_f$ of $f$ is
   $$B_f := \mathbb{C}[s] \cap \left( \mathcal{D}_X[s] \cdot f + \text{ann}_{\mathcal{O}_X[s]} f^s \right).$$
3. For $X = \mathbb{C}^n$ and $f$ a polynomial, Bernstein showed in [2] that $B_f$ is not zero. For $f$ local and analytic, Kashiwara [17] proved the same. Since $B_f$, or the local version $B_{f,y}$, is an ideal in $\mathbb{C}[s]$ it has a monic generator, the Bernstein–Sato polynomial of $f$.

The variety $V(B_f)$ contains a lot of information about the divisor of $f$ and its singularities. For example, if $\text{Exp}(\alpha) = e^{2\pi ia}$ and if $M_{f,y}$ is the Milnor Fiber of $f$ at $y \in V(f)$, cf. [23], then Malgrange and Kashiwara showed in [22], [16] that
$$\text{Exp}(V(B_{f,y})) = \bigcup_{y \in V(f) \text{ near } \eta} \{ \text{eigenvalues of the algebraic monodromy on } M_{f,y} \}.$$ Suppose $f$ factors as $f_1 \cdots f_r$. Let $F = (f_1, \ldots, f_r)$. Then there is a generalization of the Bernstein–Sato ideal $B_f$ of $f$ called the multivariate Bernstein–Sato ideal $B_F$ of $F$ obtained in a similar way.

1. Introduce new variables $S := s_1, \ldots, s_r$. Consider the $\mathcal{O}_X[F^{-1}, S]$-module generated by the symbol $F^S = \prod f_k^{s_k}$. Again, this is a $\mathcal{D}_X[S]$-module via formal differentiation.
2. The multivariate Bernstein–Sato ideal $B_F$ is
   $$B_F := \mathbb{C}[S] \cap \left( \mathcal{D}_X[S] \cdot f + \text{ann}_{\mathcal{O}_X[S]} F^S \right).$$
3. For $X = \mathbb{C}^n$ and $f_1, \ldots, f_r$ polynomials, $B_F$ is nonzero, see [18]. Sabbah proved in [24] the corresponding statement for $f_1, \ldots, f_r$ local and analytic. However neither $B_F$ nor $B_{F,y}$ need be principal: cf. Bahloul and Oaku [1].

The significance of $V(B_F)$ or the local version $V(B_{F,y})$ is less developed than the univariate counterparts. Let $f = f_1 \cdots f_r$ be a product of distinct and irreducible germs at $y$ and let $F = (f_1, \ldots, f_r)$. Let $U_{F,y}$ be the intersection of a small ball about $y \in V(f)$ with $X \setminus V(f)$. Denote by $V(U_{F,y})$ the rank one local systems on $U_{F,y}$ with nontrivial cohomology, i.e. the set of rank one local systems $L$ such that $H^k(U_{F,y}, L)$ is nonzero for some $k$. This is the cohomology support locus of $f$ at $y$ in the language of Budur and others. Since local systems can be identified with representations $\pi_1(U_{F,y}) \to \mathbb{C}^*$, regard $V(U_{F,y}) \subseteq (\mathbb{C}^*)^r$. In [7], Budur proposes that the relationship between the roots of the Bernstein–Sato polynomial and the eigenvalues of the algebraic monodromy is generalized by the conjecture
   $$\text{Exp}(V(B_{F,y})) = \bigcup_{y \in V(f) \text{ near } \eta} \text{res}_{\eta}^{-1}(V(U_{F,y})).$$ where $\text{res}_\eta$ restricts a local system on $U_{F,y}$ to a local system on $U_{\eta}$. (This generalization passes through the support of the Sabbah specialization complex in the same
In [7] Budur generalized the to a factorization (not necessarily into linear forms) of a hyperplane arrangement.

**Theorem 1.1.** Let \( f = (f_1, \ldots, f_r) \) be a decomposition of \( f = f_1 \cdots f_r \). If \( f \) is strongly Euler-homogeneous, Saito-holonomic, and tame then

\[
\text{ann}_{\mathcal{D}[S]} F^S = \mathcal{D}[S] \cdot \{ \delta - \sum_{k=1}^{r} s_k \delta \cdot f_k / f_k | \delta \in \text{Der}(- \log f) \}.
\]

The strategy is to take a filtration of \( \mathcal{D}[S] \) and consider the associated graded object of \( \text{ann}_{\mathcal{D}[S]} F^S \). This object can be given a second filtration so its initial ideal is similar to the Liouville ideal of [29]. Section 2 provides the mild generalizations of Gröbner type arguments necessary to transfer properties from this initial ideal to the ideal itself and Section 3 proves nice things about our associated graded objects, culminating in Theorem 1.1. In [21], Maisonobe proves a similar statement in the more restrictive setting of free divisors where many of these methods are not needed. We crucially use one of his techniques.

Not much is known about particular elements of \( V(B_F) \) even when \( F \) corresponds to a factorization (not necessarily into linear forms) of a hyperplane arrangement. In [7], Budur generalized the \(-\frac{n}{d} \) conjecture (see Conjecture 1.3 of [29]) as follows:

**Conjecture 1.2.** (Conjecture 3 in [7]) Let \( f = f_1 \cdots f_r \) be a central, essential, indecomposable hyperplane arrangement in \( \mathbb{C}^n \). Let \( F = (f_1, \ldots, f_r) \) where the \( f_k \) are central hyperplane arrangements, not necessarily reduced, of degree \( d_k \). Then

\[
\{d_1 s_1 + \cdots + d_r s_r + n = 0\} \subseteq V(B_F).
\]

Using Theorem 1.1, we can prove Conjecture 1.2 in the tame case:

**Theorem 1.3.** Let \( f = f_1 \cdots f_r \) be a central, essential, indecomposable, and tame hyperplane arrangement in \( \mathbb{C}^n \). Let \( F = (f_1, \ldots, f_r) \) where the \( f_k \) are central hyperplane arrangements, not necessarily reduced, of degree \( d_k \). Then

\[
\{d_1 s_1 + \cdots + d_r s_r + n = 0\} \subseteq V(B_F).
\]

Conjecture 1.2 was motivated by the formulation of the Topological Multivariable Strong Monodromy Conjecture due to Budur, see Conjecture 5 of [7]. We now state this. First let \( f = f_1 \cdots f_r \) with each \( f_k \in \mathbb{C}[x_1, \ldots, x_n] \) and let \( F = (f_1, \ldots, f_r) \). Given a log resolution \( \mu : Y \rightarrow X \) of \( f \), let \( \{E_i\}_{i \in S} \) be the irreducible components of \( f \circ \mu \), let \( a_{i,j} \) be the order of vanishing of \( f_j \) along \( E_i \), let \( k_i \) be the order of vanishing of the determinant of the Jacobian of \( \mu \) along \( E_i \), and, for \( I \subseteq S \), let \( E_I = \cap_{i \in I} \setminus \cup_{i \in S \setminus I} E_i \). The topological zeta function of \( F \) is

\[
Z_F^{\text{top}}(S) := \sum_{I \subseteq S} \chi(E_I) \cdot \prod_{i \in I} \frac{1}{a_{i,1}s_1 + \cdots + a_{i,r}s_r + k_i + 1}
\]

and this is independent of the resolution. Conjecture 5 of [7] states:
Conjecture 1.4. (Topological Multivariable Strong Monodromy Conjecture) The polar locus of $Z_F^{\text{top}}(S)$ is contained in $V(B_F)$.

By work of Budur in loc. cit., Conjecture 1.2 implies Conjecture 1.4 for hyperplane arrangements. Consequently, we conclude Section 3 with the following:

Corollary 1.5. The Topological Multivariable Strong Monodromy Conjecture is true for (not necessarily reduced) tame hyperplane arrangements.

The paper’s second thread follows the link between $\text{Exp}(V(B_F))$ and the cohomology support loci of $f$ near $r$. The bridge between the two is the $\mathcal{D}_{X,f}$-linear map

$$\nabla_A : \frac{\mathcal{D}_{X,f}[S]F_S}{(S-A)\mathcal{D}_{X,f}[S]F_S} \to \frac{\mathcal{D}_{X,f}[S]F_S}{(S-(A-1))\mathcal{D}_{X,f}[S]F_S},$$

where $(S-A)\mathcal{D}_{X,f}[S]F_S$ is the submodule generated by $s_1 - a_1, \ldots, s_r - a_r$ for $a_k \in \mathbb{C}$ and $\nabla_A$ is induced by $F^S \mapsto F^{S+1}$. In the classical, univariate case the following are equivalent (cf. Björk, 6.3.15 of [3]): (a) $a - 1 \notin V(B_f)$; (b) $\nabla_A$ is injective; (c) $\nabla_A$ is surjective. Showing that (a), (b), and (c) are equivalent in the multivariate case would verify that $\text{Exp}(V(B_{F,r}))$ equals the cohomology support loci of $f$ near $r$. Moreover, under our working hypotheses, it would show that intersecting $V(B_{F,r})$ with appropriate hyperplanes gives $V(B_{F,r})$.

In any case, (a) implies (b) and (c). Under the hypotheses of Section 3, we shall prove that $S - A$ behaves like a regular sequence on $\mathcal{D}_{X,f}[S]F_S$. This allows us to recreate a picture similar to Björk’s and prove, using different methods, the main result of Section 4:

Theorem 1.6. Let $f = f_1 \cdots f_r$ be strongly Euler-homogeneous, Saito-holonomic, and tame and let $F = (f_1, \ldots, f_r)$. If $\nabla_A$ is injective then $\nabla_A$ is surjective.

In Section 5 we strengthen our hypotheses and assume $f$ is reduced and free, that is, we assume $\text{Der}_{X,f}(-\log f)$ is a free $\mathcal{O}_{X,f}$-module. In [24] Narváez–Macarro computed the $\mathcal{D}_{X,f}[s]$-dual of $\mathcal{D}_{X,f}[s]F^S$ for certain free divisors; in [21], Maisonobe shows that this computation easily applies to $\mathcal{D}_{X,f}[s]$-dual of $\mathcal{D}_{X,f}[S]F^S$. For our free divisors we compute the $\mathcal{D}_{X,f}$-dual of $\frac{\mathcal{D}_{X,f}[S]F_S}{(S-A)\mathcal{D}_{X,f}[S]F_S}$ and lift $\nabla_A$ to this dual. Consequently, we prove:

Theorem 1.7. Let $f = f_1 \cdots f_r$ be reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. Then $\nabla_A$ is injective if and only if $\nabla_A$ is injective.

In Section 6 we summarize the relationship between the cohomology support loci of $f$ near $r$, $\text{Exp}(V(B_{F,r}))$, and $\nabla_A$. In [3], the authors characterize membership in the cohomology support loci of $f$ near $r$ in terms of the simplicity of certain perverse sheaves. When $f$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free, we show this characterization can be stated in terms of the simplicity of the $\mathcal{D}_{X,f}$-module $\frac{\mathcal{D}_{X,f}[S]F^S}{(S-A)\mathcal{D}_{X,f}[S]F^S}$.

After this paper’s completion, a preprint [9] by Budur, Veer, Wu, and Zhou was announced showing that if $A - 1 \notin V(B_{F,r})$ then $\nabla_A$ is not surjective. While this makes Section 6 less interesting, it does not affect sections 2-4. Section 5 is still compelling for its complete description of the $\mathcal{D}_{X,f}$-dual of $\nabla_A$.

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2. Initial Ideals

Suppose the commutative Noetherian ring \( R \) is a domain containing a field \( \mathbb{K} \). Consider the polynomial ring over many variables \( R[X] \), graded by the total degree of a non-negative integral vector \( u \). Let \( I \) be an ideal contained in \( (X) \cdot R[X] \). We will establish a relationship between the initial ideal in \( u \)-grading and \( I \) itself. In the standard setting \( R = \mathbb{K} \) and one uses Gröbner basis techniques with impunity (cf. [13] Chapter 15, in particular Section 15). We use the same general strategy, but because \( R \) is not even assumed to be a finitely generated \( \mathbb{K} \)-algebra we have to proceed with care.

The arguments and structure closely follow those of Bruns and Conca [5] but because \( R \neq \mathbb{K} \) we have had to alter many things slightly. First, if \( R = \mathbb{K} \), then for \( I \subseteq R[X] \) an ideal, the monomials not in \( \text{in}_>(I) \) form a basis for \( R/I \) (see [13] Theorem 15.3); this does not hold for \( R \neq \mathbb{K} \). Second, because \( R \) is not a finitely generated \( \mathbb{K} \)-algebra, the maximal ideals of any finitely generated \( R \)-algebra may have different heights. This weakens Theorem 2.8 (b) considerably, (cf. Proposition 3.1 [5]). Finally, because \( u \) may declare non-units to be of degree 0, \( R[X] \) may not be graded local.

Remark 2.1. (a) The monomials of \( R[X] \) are the elements \( x^v = \prod x_i^{v_i} \) for \( v \) a non-negative integral vector. Because the coefficient ring \( R \) is larger than \( \mathbb{K} \) it may be that, for \( r_1, r_2 \in R \), \( r_1 x^v \) and \( r_2 x^v \) generate different ideals.

(b) Just as in the case \( R = \mathbb{K} \) we can declare a monomial ordering \( > \) on \( R[X] \). Because the ordering is inherited from when \( R = \mathbb{K} \) this ordering is Artinian, with the least element being \( 1 \in R \).

c) Every \( f \in R[X] \) has a unique expression in monomials: \( f = \sum r_i m_i \), \( m_i \) a monomial, \( m_i > m_{i+1} \), for some total ordering \( > \) of the monomials.

d) Let the initial term of \( f \) be \( \text{in}_>(f) := r_1 m_1 \), where we appeal to the unique expression of \( f \) above. For \( V \) a \( R \)-submodule of \( R[X] \) let \( \text{in}_>(V) \) be the \( R \)-submodule generated by all the \( \text{in}_>(f) \) elements for \( f \in V \).

e) Given a nonnegative integral vector \( u = (u_1, \ldots, u_n) \) there is a canonical grading on \( R[X] \) given by \( u(x_i) = u_i \). Every monomial \( \sum_{v_i} x_i^{v_i} \) is \( u \)-homogeneous of degree \( \sum v_i u_i \) and every element \( f \in R[X] \) has a unique decomposition into \( u \)-homogeneous pieces. The degree \( u(f) \) is the largest degree of a monomial of \( f \); the initial term \( \text{in}_u(f) \) is the sum of the monomials of \( f \) of largest degree.

Definition 2.2. Let \( f \in R[X] \), \( f = \sum r_i m_i \) its monomial expression, \( u \) a non-negative integral vector defining a grading on \( R[X] \). We introduce a new variable \( t \) by letting \( T = R[X][t] \). Define the homogenization of \( f \) with respect to \( u \) to be

\[
\text{hom}_u(f) := \sum r_i m_i t^{u(f) - u(m_i)} \in T.
\]

For a \( R \)-submodule \( V \) of \( R[X] \) let

\[
\text{hom}_u(V) := \text{ the } R[t]-\text{submodule generated by } \{ \text{hom}_u(f) \mid f \in V \}.
\]

Remark 2.3. (a) If \( I \) is an ideal of \( R[X] \), \( \text{hom}_u(I) \) is an ideal of \( T \).

(b) Let \( u' \) be the non-negative integral vector \( (u, 1) \) and extend the grading on \( R[X] \) to \( T \) by declaring \( t \) to have degree 1. Then \( \text{hom}_u(f) \) is a \( u' \)-homogeneous of degree \( u(f) \).

**Proposition 2.4.** Suppose \( R \) is a Noetherian domain containing the field \( \mathbb{K} \) and let \( I \) be an ideal of \( R[X] \). Then
Let \( \tau \) be a monomial ordering on \( R[X] \), \( u \) a non-negative integral vector. Define a monomial ordering \( \tau u \) on \( R[X] \) by
\[
[m >_{\tau u} n] \iff [u(m) > u(n)] \quad \text{or} \quad [u(m) = u(n), m >_{\tau} n].
\]
and define a monomial ordering \( \tau u' \) on \( T \) by
\[
[mt^i >_{\tau u} nt^j] \iff [u'(mt^i) > u'(nt^j)] \quad \text{or},
\]
\[
[u'(mt^i) = u'(nt^j) \text{ and } i < j] \quad \text{or},
\]
\[
[u'(mt^i) = u'(nt^j) \text{ and } i < j \text{ and } m >_{\tau} n].
\]

**Definition 2.5.** Let \( \tau \) be a monomial ordering on \( R[X] \), \( u \) a non-negative integral vector. Let \( n, m \) be monomials in \( R[X] \). Define a monomial order, \( \tau u \) on \( R[X] \) by
\[
[m >_{\tau u} n] \iff [u(m) > u(n)] \quad \text{or} \quad [u(m) = u(n), m >_{\tau} n].
\]
and define a monomial ordering \( \tau u' \) on \( T \) by
\[
[mt^i >_{\tau u} nt^j] \iff [u'(mt^i) > u'(nt^j)] \quad \text{or},
\]
\[
[u'(mt^i) = u'(nt^j) \text{ and } i < j] \quad \text{or},
\]
\[
[u'(mt^i) = u'(nt^j) \text{ and } i < j \text{ and } m >_{\tau} n].
\]

**Lemma 2.6.** (Compare with 2.3(c) in [5]) Suppose \( R \) is a Noetherian domain containing the field \( K \). For \( V \) a \( R \)-submodule of \( R[X] \),
\[
in_{\tau u}(V)R[t] = in_{\tau u}(hom(V)).
\]
**Proof.** Argue similarly to Proposition 2.3 (c) in [5].

**Proposition 2.7.** (Compare with 2.3(d) in [5]) Suppose \( R \) is a Noetherian domain containing the field \( K \). Let \( I \subseteq X \cdot R[X] \) be an ideal of \( R[X] \) and \( u \) a nonnegative integral vector. Then \( T/\hom_u(I) \) is a torsion-free \( K[t] \) module.

**Proof.** We give a sketch. Suppose \( h \in T, s(t) \in K[t] \) such that \( s(t)h \in \hom_u(I) \). We must show that \( h \in \hom_u(I) \).

Because \( \tau u' \) is a monomial order, \( in_{\tau u'}(s(t)h) = s_k t^k in_{\tau u'}(h) \), for \( s_k \in K \). By hypothesis and Lemma 2.5, \( s_k t^k in_{\tau u'}(h) \in \hom_u(I)R[t] \). By comparing monomials and using the fact we can “divide” an equation by \( t \) if both sides are multiples of \( t \), careful bookkeeping yields the following: there exists \( g \in \hom_u(I) \) such that \( h - g < h \) and \( s(t)(h - g) \in \hom_u(I) \). Repeat the process to continually peel off initial terms and conclude either \( h \in \hom_u(I) \) or there exists \( 0 \neq r \in R \cap \hom_u(I) \). Because \( I \subseteq X \cdot R[X] \), we have \( \hom_u(I) \subseteq X \cdot R[X] \). Therefore no such \( r \) exists and the claim is proved.

The following is the section’s main proposition:

**Proposition 2.8.** (Compare with 3.1 in [5]) Suppose \( R \) is a Noetherian domain containing the field \( K \). Let \( I \subseteq X \cdot R[X] \) be an ideal of \( R[X] \) and \( u \) a non-negative integral vector. Then the following hold:

(a) If \( R[X]/\hom_u(I) \) is Cohen–Macaulay, then \( R[X]/I \) is Cohen–Macaulay;
(b) \( \dim(R[X]/\hom_u(I)) \geq \dim(R[X]/I) \).

**Proof.** (a) We follow the argument of Proposition 3.1 in [5]: first, we show that Cohen–Macaulayness percolates from \( T/\hom_u(I), t \) to \( T/\hom_u(I) \); second, that it descends from \( T/\hom_u(I) \) to \( T/\hom_u(I), t - 1 \).
First we demonstrate the percolation. Since $u'(t) = 1$ and $u(1) = 0$, any maximal $u'$-graded ideal $m^*$ of $T/\text{hom}_u(I)$ contains $t$. (Note that $t \neq 0$ in $T/\text{hom}_u(I)$.) Consider the following commutative diagram:

$$
\begin{array}{ccc}
T/\text{hom}_u(I) & \longrightarrow & T_m/(\text{hom}_u(I))_{m^*} \\
\downarrow & & \downarrow \\
T/(\text{hom}_u(I), t) & \longrightarrow & T_{m^*}/(\text{hom}_u(I), t)_{m^*},
\end{array}
$$

where the horizontal maps are localization at $m^*$ and the vertical maps are quotients by $t$.

It suffices to show that $T/\text{hom}_u(I)$ is Cohen–Macaulay after localizing at a maximal $u'$-graded ideal $m^*$ (cf. Exercise 2.1.27 [6]). Since $t \in m^*$, by assumption $T_{m^*}/(\text{hom}_u(I), t)_{m^*}$ is Cohen–Macaulay. And since $t$ is a non-zero divisor on $T_{m^*}/(\text{hom}_u(I), t)_{m^*}$ by Proposition 2.7, we see $T_{m^*}/(\text{hom}_u(I), t)_{m^*}$ is Cohen–Macaulay (cf. Theorem 2.1.3 in [6]).

It remains to show that Cohen–Macaulayness descends from $T/\text{hom}_u(I)$ to $T/(\text{hom}_u(I), t - 1)$. By the universal property of localization we have:

$$
\begin{array}{ccc}
T/\text{hom}_u(I) & \longrightarrow & T/(\text{hom}_u(I), t - 1) \\
\downarrow & & \gamma \\
(T/\text{hom}_u(I))[t^{-1}] & &
\end{array}
$$

(2.1)

It is well known (cf. Proposition 1.5.18 in [3]) that

$$(T/\text{hom}_u(I))[t^{-1}] \simeq ((T/\text{hom}_u(I))[t^{-1}])_0[y, y^{-1}].$$

So $\gamma$ of (2.1) induces, where $-0$ denotes the degree 0 elements, the following isomorphisms:

$$T/(\text{hom}_u(I), t - 1) \simeq \frac{(T/\text{hom}_u(I))[t^{-1}]}{(t - 1)(T/\text{hom}_u(I))[t^{-1}]} \simeq ((T/\text{hom}_u(I))[t^{-1}])_0.$$

We have

(2.2)

$$(T/\text{hom}_u(I))[t^{-1}] \simeq (T/(\text{hom}_u(I), t - 1))[y, y^{-1}].$$

Therefore, since Cohen Macaulayness is preserved under localization at a non-zero divisor, all we need to show is that if $B[y, y^{-1}]$ is a Laurent polynomial ring that is Cohen–Macaulay then $B$ is an Cohen–Macaulay. To see this take a $m \in m\text{Spec}(B)$ and look at $(m, y - 1) \in \text{Spec}(B[y])$ and the corresponding prime ideal in $B[y, y^{-1}]$.

Now we move onto (b). The descent part of part (a) gives us the plan:

$$\dim(T/(\text{hom}_u(I), t - 1)) = \dim((T/(\text{hom}_u(I), t - 1))[y, y^{-1}]) - 1$$

$$= \dim((T/\text{hom}_u(I))[t^{-1}]) - 1$$

$$\leq \dim(T/\text{hom}_u(I)) - 1$$

$$= \dim(T/(\text{hom}_u(I), t)).$$

The second equality follows by (2.2). The inequality is not an equality because localization may lower dimension. For the last equality use the fact dimension of a graded ring can be computed by looking only at the height of the graded maximal
ideals (Corollary 13.7 [13]). In $T/\text{hom}_u(I)$, $t$ is of degree 1 and so contained in all graded maximal ideals; since it is a non-zero divisor, its associated primes are not minimal.

Remark 2.9. (a) This proposition generalizes the common geometric picture for $R = \mathbb{K}$. In this setting the map $\mathbb{K}[t] \to T/\text{hom}_u(I)$ gives a flat family whose generic fiber is $R/I$ and whose special fiber is $R/\text{in}_u(I)$. In our generality, it is easy to extend Proposition 2.7 and show that $\mathbb{K}[t] \to T/\text{hom}_u(I)$ is a flat ring map whose generic fiber is $R[X]/\text{in}_u(I)$ and whose special fiber is $R[X]/I$.

(b) We are mostly interested in studying ideals $I \subseteq Y \cdot \mathcal{O}_{X,Y}$ where $\mathcal{O}_X$ is an analytic structure sheaf and the $u$-grading assigns 1 to some $y$’s and 0 to others. Proposition 2.8 applies with $R = \mathcal{O}_{X,Y}$.

3. The $\mathcal{D}_X[S]$-Annihilator of $F^S$

As in the introduction, let $X$ be a smooth analytic space or $\mathbb{C}$-scheme of dimension $n$ and with structure sheaf $\mathcal{O}_X$. Let $f \in \mathcal{O}_X$ be regular with divisor $Y = \text{Div}(f)$ and corresponding ideal sheaf $\mathcal{I}_Y$. Throughout, $Y = \text{Div}(f)$ will not necessarily be reduced. Let $\mathcal{D}_X$ be the sheaf of $\mathbb{C}$-linear differential operators with $\mathcal{O}_X$-coefficients and let $\mathcal{D}_X[s]$ and $\mathcal{D}_X[S] = \mathcal{D}_X[s_1, \ldots, s_r]$ be polynomials rings over $\mathcal{D}_X$.

Recall the order filtration $F_{(0,1)}$ on $\mathcal{D}_X$ induced, in local coordinates, by making every $\partial_{x_k}$ weight one and every element of $\mathcal{O}_X$ weight zero. Denote the differential operators of order at most $k$ as $F^k_{(0,1)}$ and the associated graded object as $\text{gr}_{(0,1)}(\mathcal{D}_X)$. We will be primarily interested in a subset of $\text{Der}_X$ called the logarithmic derivations:

**Definition 3.1.** Let $\text{Der}_X(-\log f) = \text{Der}_X(-\log(Y))$, be the sheaf of logarithmic derivations, i.e. the $\mathcal{O}_X$-module with local generators on $U$ the set

$$\text{Der}_X(-\log f) := \{ \delta \text{ a vector field in } \mathcal{D}_X(U) \mid \delta \bullet \mathcal{I} \subseteq \mathcal{I} \}.$$  

We also put

$$\text{Der}_X(-\log_0 f) := \{ \delta \in \text{Der}_X(-\log f) \mid \delta \bullet f = 0 \}.$$  

Note that $\text{Der}_X(-\log_0 f)$ may depend on the choice of defining equation for $f$, which is why we have fixed a global $f$.

**Definition 3.2.** For $\mathfrak{r} \in X$, we say that $f \in \mathcal{O}_{X,\mathfrak{r}}$ is Euler-homogeneous at $\mathfrak{r}$ if there exists $E_\mathfrak{r} \in \text{Der}_{X,\mathfrak{r}}(-\log f)$ such that $E_\mathfrak{r} \bullet f = f$. If $E_\mathfrak{r}$ vanishes at $\mathfrak{r}$ then $f$ is strongly Euler-homogeneous at $\mathfrak{r}$.

Finally, a divisor $Y$ is (strongly) Euler-homogeneous if there is a defining equation $f$ at each $\mathfrak{r}$ such that $f$ is (strongly) Euler-homogeneous at $\mathfrak{r}$.

**Example 3.3.** Let $f = x(2x^2 + yz)$. Note that $\text{Sing}(f) = \{z - \text{axis}\} \cup \{y - \text{axis}\}$. Along the $z$-axis there is the strong Euler-homogeneity induced by $\frac{1}{2}xf_x + \frac{2}{3}zf_z$; along the $y$-axis there is the strong Euler-homogeneity induced by $\frac{1}{3}xf_x + \frac{2}{3}zf_z$. Thus $f$ is strongly Euler-homogeneous on the singular locus. Since $f$ is automatically strongly Euler-homogeneous on the smooth locus, $f$ is strongly Euler-homogeneous.

**Example 3.4.** Let $f$ be a central hyperplane arrangement, cf. Section 3.6. Then the Euler vector field $\sum x_i \partial_{x_i}$ shows that $f$ is strongly Euler-homogeneous at the origin. For all points in $V(f)$, $f$ is locally given by a hyperplane arrangement; hence $f$ is strongly Euler-homogeneous.
**Definition 3.5.** Define the total order filtration $F_{(0,1,1)}$ as the filtration on $\mathcal{D}_X[S]$ induced by the $(0,1,1)$-weight assignment that, in local coordinates, gives elements of the form $\mathcal{O}_X \partial^u S^v$. \( u, v \) non-negative integral vectors, weight $\sum u_i + \sum v_i$. Let $F^k_{(0,1,1)}$ be the homogeneous operators of weight at most $k$ with respect to the total order filtration. When the context is clear, we will use $F^k_{(0,1,1)}$ to refer to the similarly defined filtration on $\mathcal{D}_X[S]$ (the classical case). Denote the associated graded object associated to $F_{(0,1,1)}$ as $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S])$.

Our principal objective is to study the annihilator of $F^S$—the left $\mathcal{D}_X[S]$-ideal $\text{ann}_{\mathcal{D}_X[S]} F^S$. Take the $\mathcal{D}_X[f^{-1}_1, \ldots, f^{-1}_r, S]$-module generated freely by the symbol $F^S = \prod f_k^{s_k}$.

In most cases $\text{ann}_{\mathcal{D}_X[S]} F^S$ is very hard to compute. In the classical setting, there is a natural identification between the $(0,1,1)$-homogeneous elements of $\text{ann}_{\mathcal{D}_X[S]} f^s$ and $\text{Der}_X(- \log f)$. We will establish a similar correspondence.

**Definition 3.6.** The annihilating derivations of $F^S$ are the elements of the $\mathcal{O}_X$-module

$$\theta_F := \text{ann}_{\mathcal{D}_X[S]} F^S \cap F^1_{(0,1,1)}.$$ 

We say $\text{ann}_{\mathcal{D}_X[S]} F^S$ is generated by derivations when $\text{ann}_{\mathcal{D}_X[S]} F^S = \mathcal{D}_X[S] \cdot \theta_F$.

**Proposition 3.7.** For $f = f_1 \cdots f_r$, let $F = (f_1, \ldots, f_r)$. Then as $\mathcal{O}_X$-modules,

$$
\psi_F : \text{Der}_X(- \log f) \xrightarrow{\sim} \theta_F
$$

where $\psi_F$ is given by

$$\delta \mapsto \delta - \sum_{k=1}^r \frac{s_k}{f_k} \delta \cdot f_k.$$

**Proof.** We first prove the claim locally. By Lemma 3.4 of [14], $\text{Der}_X(- \log f) \cap \text{Der}_X(- \log f_k)$; in particular, $\delta - \sum_{k=1}^r s_k \frac{\delta \cdot f_k}{f_k}$ lies in $\mathcal{D}_{X,+}[S]$.

Fix a coordinate system. Take $P \in \theta_{F,\bar{x}}$, $P = \delta + p(S)$, where $\delta \in \mathcal{D}_{X,+}$ is a derivation and $p(S) = \sum_k b_k s_k \in \mathcal{O}_{X,+}[S]$ is necessarily $S$-homogeneous of $S$-degree 1. Keep the notation $F^S$ and the $f_j$ for the local versions at $x$. By definition, $F^S$ generates a free $\mathcal{O}_{X,+}[f^{-1}, S]$-module, $\sum_k (s_k \frac{\delta \cdot f_k}{f_k} - b_k s_k) = 0$. Fixing a particular $k$, we also have $\delta \cdot f_k \in \mathcal{O}_{X,+} \cdot f_k$; moreover, $\delta \cdot f_k = b_k f_k$. We have shown $\delta \in \cap \text{Der}_{X,+}(- \log f_k)$ and, in fact,

$$\theta_{F,\bar{x}} = \{ \delta - \sum_k b_k s_k \mid \delta \in \text{Der}_{X,+}(- \log f), \delta \cdot f_k = b_k f_k \}.$$

Thus the map $\psi_F : \text{Der}_{X,+}(- \log f) \to \theta_{F,\bar{x}}$ given by $\delta \mapsto \delta - \sum_k \frac{\delta \cdot f_k}{f_k} s_k$ is a well-defined $\mathcal{O}_{X,+}$-linear isomorphism for a fixed coordinate system. Showing that
θ_{F,t} commutes with coordinate change is routine and is effectively shown in Remark 3.17 (b).

Since δ ∈ Der(− log f) precisely when δ ⋅ f = 0 in $\mathcal{O}_X(f)$, membership in Der(− log f) is a local condition. The above shows that $\psi^{-1}_{F,t}$ is an $\mathcal{O}_{X,t}$-isomorphism at all $x$; hence $\psi^{-1}_{F,t}$ is an isomorphism. □

Remark 3.8. (a) We emphasize that the above isomorphism holds at $x \in X$ where some of the $f_k$’s are units. Every derivation is logarithmic for such $f_k$.
(b) $\psi_{F,t}$ identifies $\theta_{F,t}(U) \simeq \mathcal{O}_X(U) \cdot \{\delta - \sum s_k \cdot \delta f_k \mid \delta \in \text{Der}_U(\log f)\}$.

Remark 3.9. (a) Suppose $f$ is Euler-homogeneous at $x \in X$: take $E_{\tau}$ to be a vector field preserving $f$. Then we have a split short exact sequence of $\mathcal{O}_{X,t}$-modules:
$$0 \rightarrow \text{Der}_{X,t}(- \log_0 f) \rightarrow \text{Der}_{X,t}(\log f) \rightarrow \mathcal{O}_{X,t} \cdot E_{\tau} \rightarrow 0.$$ This splitting depends on the choice of $E_{\tau}$ and is not canonical.
(b) For $F$ and $D_{X,t}[S]/F$, applying $\psi_{F,t}$ to the sequence of (a) gives the split short exact sequence
$$0 \rightarrow \psi_{F}(\text{Der}(\log_0 f)) \rightarrow \theta_{F,t} \rightarrow \mathcal{O}_{X,t} \cdot \left(E_{\tau} - \sum s_k \cdot \frac{E_{\tau}}{f_k}\right) \rightarrow 0.$$ Again, this splitting is non-canonical. Just as in [29], $\psi_{F}(\text{Der}(\log_0 f))$ will often have very nice properties and will in general be easier to work with.

3.1. Hypotheses on $Y$ and $F$.

In this subsection we introduce many of the standard hypothesis on $Y$ and $F$ we use throughout the paper.

Definition 3.10. Let $U \subseteq X$ be open and $f \in \mathcal{O}_X(U)$. We will say $F = (f_1, \ldots, f_r)$ is a decomposition of $f$ when $f = f_1 \cdots f_r$.

We will also restrict to divisors $Y$ such that $\text{Der}_X(- \log Y)$ has a light constraint on its dimension.

Definition 3.11. Consider the sheaf of differential forms of degree $k$: $\Omega^k_X = \wedge^k \Omega^1_X$ and their differential $d: \Omega^k_X \rightarrow \Omega^{k+1}_X$. We define the subsheaf of logarithmic differential forms $\Omega_X(\log f)^k$ by
$$\Omega^k_X(\log f) := \{w \in \frac{1}{f} \Omega^k_X \mid d(w) \in \frac{1}{f} \Omega^{k+1}_X\}.$$ See 1.1 and 1.2 in [28] for more details.

We say $f \in \mathcal{O}_X(U), U \subseteq X$ is tame if the projective dimension of $\Omega^k_U(\log f)$ is at most $k$ at each $x \in U$. A divisor $Y$ is tame if it admits tame defining equations locally everywhere. See Definition 3.8 and the surrounding text in [29] for more details on tame divisors. In particular, if $n \leq 3$ then $Y$ is automatically tame.

We will also need a stratification of $X$ that respects the logarithmic structure of $Y$.

Definition 3.12. (Compare with 3.8 in [28]) Define a relation on $X$ by identifying two points $x$ and $y$ if there exists an open $U \subseteq X$, $x, y \in U$ and a derivation $\delta \in \text{Der}_U(- \log(Y \cap U))$ such that (i) $\delta$ is nowhere vanishing on $U$ and (ii) the integral flow of $\delta$ passes through $x$ and $y$. The transitive closure of this relation stratifies $X$ into equivalence classes. The irreducible components of the equivalence
classes are called the *logarithmic strata*; the collection of all strata the *logarithmic stratification*.

We say \( Y \) is *Saito-holonomic* if the logarithmic stratification is locally finite, i.e. at every \( \mathfrak{x} \in X \) there is an open \( U \subseteq X \), \( \mathfrak{x} \in U \), such that \( U \) intersects finitely many logarithmic strata. Equivalently, \( Y \) is Saito-holonomic if the dimension of \( \{ \mathfrak{x} \in X \mid r \ker \{ \text{Der}_X(- \log Y) \otimes \mathcal{O}_{X,\mathfrak{x}}/m_{X,\mathfrak{x}} = i \} \) is at most \( i \).

**Remark 3.13.** (a) Pick \( \mathfrak{x} \in X \) and consider its log stratum \( D \) with respect to \( f = f_1 \cdots f_r \). We can find logarithmic derivations \( \delta_1, \ldots, \delta_m \) at \( \mathfrak{x} \), \( m = \dim D \), that are \( C \)-independent at \( \mathfrak{x} \). Note that each \( \delta_i \) also lies in \( \text{Der}_{X,\mathfrak{x}}(- \log f_i) \). By Proposition 3.6 of [28] there exists a coordinate system so that the generators can be picked to be of the form \( \delta_k = \frac{\partial}{\partial x_{m-k+1}} + \sum_{1 \leq j \leq n-m} g_{j}^{k}(x) \frac{\partial}{\partial y_{j}} \).

(b) By Lemma 3.5 and Proposition 3.6 of [28], the same change of coordinates from \( 3.13 \) fixes the last \( m \) coordinates and satisfies \( \phi_{f}(x_1, \ldots, x_{n-m}, 0) = (x_1, \ldots, x_m, 0) \). Moreover, it simultaneously satisfies \( f_i(\phi_{f}(x_1, \ldots, x_m)) = u_i(x_1, \ldots, x_m) f_i(x_1, \ldots, x_{n-m}, 0) \) where \( u_i(x_1, \ldots, x_m) \) is a unit for \( 1 \leq i \leq m \).

(c) Now assume the logarithmic stratification is locally finite and the log stratum \( D \) of \( \mathfrak{x} \) has dimension 0. So \( D = \{ \mathfrak{x} \} \). Let \( U \supseteq D \) be an open set intersecting finitely many strata different from \( D \). Suppose \( D' \) is another strata intersecting every such \( U \). Then \( D = \{ \mathfrak{x} \} \subseteq \mathcal{D} \). So dim \( D' \geq 1 \). Thus, if \( \mathfrak{x} \) is a point whose logarithmic strata is of dimension 0, then there exists an open \( U \ni \mathfrak{x} \) such that \( U \setminus \mathfrak{x} \) consists only of points whose logarithmic stratum are of positive dimension.

(d) By Lemma 3.4 of [28], for a divisor \( Y \) connected components of \( X \setminus Y \) and \( Y \setminus \text{Sing}(Y) \) are logarithmic strata. In particular, if \( \dim X = n \), any log stratum of dimension \( n \) is comprised solely of points in \( X \setminus \text{Sing}(Y) \).

**Example 3.14.** Let \( f = x(2x^2 + yz) \) and note that \( \text{Sing}(f) = \{ \text{z-axis}\} \cup \{ \text{y-axis}\} \). Since the Euler derivation \( x \partial_x + y \partial_y + z \partial_z \) is a logarithmic derivation, the \( z \)-axis \( \setminus \{ 0 \} \) and the \( y \)-axis \( \setminus \{ 0 \} \) are logarithmic strata. Therefore \( f \) is Saito-holonomic.

**Example 3.15.** By Example 3.14 of [28], hyperplane arrangements are Saito-holonomic.

We will almost always assume our divisors \( Y \) are strongly Euler-homogeneous for reasons that will become clear in Remark 3.17.

### 3.2. Generalized Liouville Ideals.

In Section 3 of [29], Walther defines the *Liouville ideal* \( L_f \) as the ideal in \( \mathfrak{g}_X^{(0,1)}(\mathcal{D}_X) \) generated by the symbols \( \mathfrak{g}_X^{(0,1)}(\text{Der}_X(- \log f)) \). As \( \text{Der}_X(- \log f) \subseteq \text{ann}_{\mathfrak{g}_X} f^s \), \( L_f \) represents the contribution of \( \text{Der}_X(- \log f) \) to \( \mathfrak{g}_X^{(0,1)}(\text{ann}_{\mathfrak{g}_X} f^s) \). When \( f \) is strongly Euler-homogeneous with strong Euler-homogeneity \( E_f \), \( L_f \) is coordinate independent (see Remark 3.2 [29]) and by the splitting of Remark 3.3 (b), \( L_{f,1} \) and \( \mathfrak{g}_X^{(0,1)}(E_f) - s \) generate the obvious elements of \( \mathfrak{g}_X^{(0,1)}(\text{ann}_{\mathfrak{g}_X} f^s) \) of degree one (here \( L_{f,1} \) is extended to \( \mathfrak{g}_X^{(0,1)}(\mathcal{D}_X[s]) \) by an ideal extension).

If we want to generalize this to \( F^S \), there is no obvious inclusion between \( \text{Der}_X(- \log f) \) and \( \text{ann}_{\mathfrak{g}_X} F^S \). In fact, \( \delta \in \text{Der}_X(- \log f) \) is in \( \text{ann}_{\mathfrak{g}_X} F^S \) precisely when \( \delta \in \bigcap \text{Der}_X(- \log f_i) \). Trying to define a generalized Liouville ideal using \( \bigcap \text{Der}_X(- \log f_i) \) would lose too many elements of \( \text{Der}_X(- \log f) \).
Definition 3.16. Recall the isomorphism of $\mathcal{O}_X$-modules from Proposition 3.7

$$\psi_F : \text{Der}_X(- \log f) \cong \theta_F,$$

which is given by

$$\psi_F(\delta) = \delta - \sum_k s_k \frac{\delta \cdot f_k}{f_k}.$$ 

This restricts to a map of sheaves of $\mathcal{O}_X$-modules

$$\psi_F : \text{Der}_X(- \log_0 f) \to \theta_F.$$ 

Let the generalized Liouville ideal $L_F$ by the ideal in $\text{gr}(\mathcal{O}_X[S])$ generated by

the symbols of $\psi_F(\text{Der}_X(- \log_0 f))$ in the associated graded:

$$L_F := \text{gr}(\mathcal{O}_X[S]) \cdot \text{gr}(\text{Der}_X(- \log_0 f)).$$

We also define

$$\widetilde{L}_F := \text{gr}(\mathcal{O}_X[S]) \cdot \text{gr}((\psi_F(\text{Der}_X(- \log f))).$$

Remark 3.17. The splittings of $\text{Der}_{X,r}(\log f)$ into $\text{Der}_{X,r}(\log_0 f)$ and $\mathcal{O}_{X,r} \cdot E_r$, $E_r$ a Euler-homogeneity at $r \in X$, of Remark 3.9 may depend on the choice of equation of $f$ at $r$. Consequently, $L_{F,r}$ may depend on the choice of $f$. Just like in Remark 3.2, if we assume the divisor of $f$ is strongly Euler-homogeneous, then the algebraic properties of $L_{F,r}$ and $\widetilde{L}_{F,r}$ are independent of the choice of equation at $r$.

(a) To this end, let $x$ and $x'$ denote two coordinate systems, $J = (\frac{\partial x_i'}{\partial x_j})$ the Jacobian matrix with rows $i$, columns $j$, and $\partial'$ column vectors of partial differentials in the $x$ and $x'$ coordinates, respectively. Let $\nabla(g)$, $\nabla'(g)$ be the gradient, as a column vector, of $g$ in the two coordinate systems. Finally, express a derivation $\delta$ in terms of the two coordinate systems as $\delta = c_x^j \partial = c_x'^{j'} \partial'$, where $c_x, c_x'$, are column vectors of $\mathcal{O}_X$ functions representing the coefficients of the partials in the $x$ and $x'$ coordinates. Note that in $x'$-coordinates $c_x' = J^T c_x$.

(b) In $x$-coordinates $\psi_F(\delta) = c_x^j \partial - \sum_k s_k \frac{c_x^j \nabla(f_k)}{f_k}$, in $x'$-coordinates $\psi_F(\delta) = c_x'^{j'} \partial' - \sum_k s_k \frac{c_x'^{j'} \nabla'(f_k)}{f_k}$. Thus $\psi_F$ commutes with coordinate change. (Note that strongly Euler-homogeneous is not needed here.)

(c) Suppose $E_r$ is a strong Euler-homogeneity at $r \in X$ for $f$. Recall from Remark 3.2 of [29] that for a unit $u \in \mathcal{O}_{X,r}$, the map $\alpha_u : \text{Der}_{X,r}(\log_0 f) \to \text{Der}_{X,r}(\log_0 uf)$ given by $\alpha_u(\delta) = \delta - \frac{\delta \cdot u}{u + E_r \cdot u}$ is an $\mathcal{O}_{X,r}$-isomorphism that commutes with coordinate change. In particular, let $u = \prod \leq r u_i$ be a product of units and let $uF = (u_1 f_1, \ldots, u_r f_r)$. Then we have an $\mathcal{O}_{X,r}$-isomorphism

$$\psi_F \circ \alpha_u \circ \psi_F^{-1} : \psi_F(\text{Der}_{X,r}(\log_0 f)) \to \psi_F(\text{Der}_{X,r}(\log_0 uf))$$

that commutes with coordinate change.

(d) To be precise,

$$\psi_F \circ \alpha_u \circ \psi_F^{-1}(\delta - \sum_k s_k \frac{\delta \cdot f_k}{f_k}) = \delta - \frac{\delta \cdot u}{u + E_r \cdot u} E_r - \sum_k s_k \frac{\delta \cdot u_k}{u_k}$$

$$= -\sum_k s_k \frac{\delta \cdot f_k}{f_k} + \sum_k s_k \frac{(\delta \cdot u)(E_r \cdot (u_k f_k))}{u_k f_k (u + E_r \cdot u)}.$$
Note that $E_{\xi} \cdot (u_{k} f_{k})$ is a multiple of $f_{k}$ and $\delta \in \text{Der}_{X,\xi}(-\log f_{k})$ so all these fractions make sense.

(e) Inspection reveals that the morphism of graded objectes induced by $\psi_{F} \circ \alpha_{u} \circ \psi_{F}^{-1}$ is an $O_{X,\xi}[S]$-linear endomorphism $\beta_{u}$ on $\text{gr}_{(0,1)} \mathcal{D}_{X,\xi}[S]$, where

$$\beta_{u}(\text{gr}_{(0,1)}(\partial)) = \text{gr}_{(0,1)}(\partial) - \frac{\partial \cdot u}{u + E_{\xi} \cdot u} \text{gr}_{(0,1)}(E_{\xi}) - \sum_{k} s_{k} \frac{\partial \cdot u_{k}}{u_{k}}$$

$$+ \sum_{k} s_{k} (\partial \cdot u)(E_{\xi} \cdot (u_{k} f_{k})) - \frac{\partial \cdot u}{u + E_{\xi} \cdot u}$$

Since the $\mathcal{O}_{X,\xi}$-linear endomorphism of $\text{gr}_{(0,1)}(\mathcal{D}_{X,\xi})$ given by $\text{gr}_{(0,1)}(\partial) \rightarrow \text{gr}_{(0,1)}(\partial) - \frac{\partial \cdot u}{u + E_{\xi} \cdot u} \text{gr}_{(0,1)}(E_{\xi})$ is surjective and injective, $\beta_{u}$ is as well. So $\beta_{u}(L_{F,\xi}) = L_{u}E_{\xi}$. It is clear by (d) that $\beta_{u}$ commutes with coordinate change.

(f) Therefore for strongly-Euler-homogeneous $f$, the local algebraic properties of $\text{gr}_{(0,1)}(\mathcal{D}_{X,\xi}[S]/L_{F})$ are independent of the choice of local equations for the $f_{1}, \ldots, f_{r}$.

(g) It is also clear that $\alpha_{u}$ sends $E_{\xi}$, a strong Euler-homogeneity for $f$, to a strong Euler-homogeneity for $uf$ and so $\beta_{u}(L_{F,\xi}) = \tilde{L}_{u}E_{\xi}$. Hence, if $f$ is strongly Euler-homogeneous then the local properties of $L_{F}$ do not depend on the defining equations of the $f_{k}$.

At the smooth points of $f$, $L_{F}$ and $\tilde{L}_{F}$ are well understood. First, a lemma:

**Lemma 3.18.** Suppose $f = f_{1} \cdots f_{r}$ has the Euler-homogeneity $E_{\xi}$ at $\xi \in X$. Let $F = (f_{1}, \ldots, f_{r})$. Then

$$\text{gr}_{(0,1)}(\psi_{F}(E_{\xi})) \not\in m_{\xi} \mathcal{O}_{X,\xi}[Y][S] \subseteq \mathcal{O}_{X,\xi}[Y][S] \simeq \text{gr}_{(0,1)}(\mathcal{D}_{X,\xi}[S]).$$

**Proof.** Working at $\xi \in X$ and letting $\tilde{f}_{k} = \prod_{j \neq k} f_{j}$:

$$f = E_{\xi} \cdot f = \sum_{k} (E_{\xi} \cdot f_{k}) \tilde{f}_{k} = \left( \sum_{k} E_{\xi} \cdot f_{k} / \tilde{f}_{k} \right) f.$$

So $1 = \sum_{k} E_{\xi} f_{k} \tilde{f}_{k}$ in $\mathcal{O}_{X,\xi}$; thus there exists a $j$ such that $E_{\xi} f_{j} / \tilde{f}_{j} \not\in m_{\xi}$.

As $\psi_{F}(E_{\xi}) = E_{\xi} + \sum_{k} s_{k} E_{\xi} f_{k} / \tilde{f}_{k}$ the claim follows after looking at the symbol $\text{gr}_{(0,1)}(\psi_{F}(E_{\xi})).$ \hfill $\square$

**Proposition 3.19.** Let $f = f_{1} \cdots f_{r}$ be strongly Euler-homogeneous and let $F = (f_{1}, \ldots, f_{r})$. Then locally at smooth points, $L_{F}$ and $\tilde{L}_{F}$ are prime ideals of dimension $n + r + 1$ and $n + r$ respectively. Moreover, for any $\xi \in X$:

$$\dim \text{gr}_{(0,1)}(\mathcal{D}_{X,\xi}[S]) / L_{F,\xi} \geq n + r + 1;$$

$$\dim \text{gr}_{(0,1)}(\mathcal{D}_{X,\xi}[S]) / \tilde{L}_{F,\xi} \geq n + r.$$

**Proof.** Let $\xi \in X$ be a part of the smooth locus of $f$; fix coordinates and choose $\partial_{x_{i}}$ such that $\partial_{x_{i}} \cdot f$ is a unit in $\mathcal{O}_{X,\xi}$. Then $\Gamma = \{ \partial_{x_{k}} - \frac{\partial_{x_{k}} \cdot f}{\partial_{x_{j}}} \partial_{x_{j}} \}_{k=1, k \neq i} \subseteq \text{Der}_{X,\xi}(-\log f)$ is a set of $n - 1$ linearly independent elements. Saito’s Criterion (cf. page 270 of [23]) implies that $\Gamma$ together with $E_{\xi}$, the strong Euler derivation, gives a free basis for $\text{Der}_{X,\xi}(-\log f)$. Hence, $\Gamma$ generates $\text{Der}_{X,\xi}(-\log f)$ freely. As $\mathcal{O}_{X,\xi}[Y][S] / L_{F,\xi} \simeq \mathcal{O}_{X,\xi}[Y][S]$, $L_{F,\xi}$ is a prime ideal of dimension $n + r + 1$. 

By Lemma 3.18 and the choice of $j$ outlined in its proof,

$$\mathcal{O}_{X_f}[Y][S]/\text{gr}_{(0,1,1)}(\psi_F(E_1)) \simeq \mathcal{O}_{X_f}[Y][s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r].$$

Since $E_1$ is a strong Euler-homogeneity, the coefficient of each $y_k$ in $\text{gr}_{(0,1,1)}(\psi_F(E_1))$ lies in $\mathfrak{m}_f$. Thus the coefficient of $y_k$ in

$$\text{gr}_{(0,1,1)}(\psi_F(y_k - \partial_{x_i} \cdot f \cdot y_k)) \in \mathcal{O}_{X_f}[Y][S]/\text{gr}_{(0,1,1)}(\psi_F(E_1))$$

belongs to $\mathcal{O}_{X_f} \setminus \mathfrak{m}_f$. Hence $\mathcal{O}_{X_f}[Y]/\mathcal{L}_{F,1} \simeq \mathcal{O}_{X_f}[y_i][s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_r]$. So $\mathcal{L}_{F,1}$ is a prime ideal of dimension $n + r$.

Since the smooth points are dense, we get the desired inequalities. □

Take a generator $\text{gr}_{(0,1,1)} \left( \delta - \sum s_k \frac{\delta - f}{f} \right)$, $\delta \in \text{Der}_X(-\log f)$, of $\mathcal{L}_{F,1}$. Erasing the $s_k$-terms results in $\text{gr}_{(0,1,1)}(\delta) = \text{gr}_{(0,1,1)}(\delta) \in L_{F,1}$. This process is formalized by filtering $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S])$ in such a way that the $s_k$-terms have degree 0 and then taking the initial ideal of $\mathcal{L}_{F,1}$.

**Definition 3.20.** It is well known that for an open $U \subseteq X$ with a fixed coordinate system $\text{gr}_{(0,1,1)}(\mathcal{D}_X(U)[S]) \simeq \mathcal{O}_{X}(U)[Y][S]$, where $y_i = \text{gr}_{(0,1,1)}(\partial_{x_i})$. Grade this by the integral vector $(0, 1, 0) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}^r$. For example the element $g^Y \cdot S^y$, where $u, v$ are nonnegative integral vectors and $g \in \mathcal{O}_U$, will have $(0, 1, 0)$-degree $\sum_j u_j$. Changing coordinate systems does not affect the number of $y$-terms so this extends to a grading on $\text{gr}_{(0,1,1)}(\mathcal{D}_X(U)[S])$.

Define $\text{in}_{(0,1,0)} L_F$ to be the initial ideal of the generalized Liouville ideal with respect to the $(0, 1, 0)$-grading. See Section 2 for details about initial ideals.

We now have three ideals: $L_F$, $\text{in}_{(0,1,0)} L_F$, and $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_f$, the ideal extension of $L_f$ to $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S])$. Section 2 and Theorem 3.18 shows how some nice properties of $\text{in}_{(0,1,0)} L_F$ transfer to nice properties of $L_F$. The following construction will let us transfer nice properties of $L_f$, and consequently of $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_f$, to nice properties of $\text{in}_{(0,1,0)} L_F$.

**Proposition 3.21.** Assume $f = f_1 \cdots f_r$ is strongly Euler-homogeneous and let $F = (f_1, \ldots, f_r)$, Consider $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_f$, the extension of the Liouville ideal to $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S])$. Then

$$\frac{\text{gr}_{(0,1,1)}(\mathcal{D}_X[S])}{\text{in}_{(0,1,0)} L_F}.$$ 

**Proof.** $L_f$ is generated by the symbols of $\delta \in \text{Der}_X(-\log f)$ in $\text{gr}_{(0,1)}(\mathcal{D}_X)$. Thinking of $\text{gr}_{(0,1)}(\mathcal{D}_X) \subseteq \text{gr}_{(0,1,1)}(\mathcal{D}_X[S])$, $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_f$ will have the generators $\text{gr}_{(0,1,1)}(\delta)$. On the other hand $L_F$ is locally generated by $\text{gr}_{(0,1,1)} \left( \delta - \sum s_k \frac{\delta - f}{f} \right)$ for $\delta \in \text{Der}_X(-\log f)$. Each such generator has $(0, 1, 0)$-initial term $\text{gr}_{(0,1,1)}(\delta)$. So $\text{gr}_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_{F,1} \subseteq \text{in}_{(0,1,0)} L_{F,1}$. □

**Proposition 3.22.** Suppose $f = f_1 \cdots f_r$ is a strongly Euler-homogeneous divisor and let $F = (f_1, \ldots, f_r)$. Then the following data transfer from the Liouville ideal to the initial ideal of the generalized Liouville ideal:

(a) If $\dim \text{gr}_{(0,1)}(\mathcal{D}_X[S])/L_{F,1} = n + 1$, then

$$\dim \text{gr}_{(0,1,1)}(\mathcal{D}_X[S])/L_{F,1} = n + r + 1;$$
(b) If $L_f$ is locally a prime ideal then
\[ \frac{\gr_{(0,1,1)}(\mathcal{D}_X[S])}{\gr_{(0,1,1)}(\mathcal{D}_X[S])} \cdot L_f \simeq \frac{\gr_{(0,1,1)}(\mathcal{D}_X[S])}{\in_{(0,1,0)} L_f}; \]

(c) If $L_f$ is locally Cohen–Macaulay and prime, then $L_F$ is locally Cohen–Macaulay.

Proof. Because $\gr_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_f$ is the extension of $L_f$ into a ring with new variables $S$,
\[ \frac{\gr_{(0,1,1)}(D_X[S])}{\gr_{(0,1,1)}(D_X[S])} \cdot L_f \simeq \frac{\mathcal{O}_X[Y][S]}{\mathcal{O}_X[Y][S]} \cdot L_f \simeq \frac{\mathcal{O}_X[Y][S]}{\mathcal{O}_X[Y][S]}, \]
so if $\dim \gr_{(0,1,1)}(\mathcal{D}_X[S])/L_{f,2} = n + 1$, $\dim \gr_{(0,1,1)}(\mathcal{D}_X[S])/\gr_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_f = n + r + 1$. Similarly if $L_{f,2}$ is prime, then $\gr_{(0,1,1)}(\mathcal{D}_X[S]) \cdot L_{f,2}$ is prime.

The map \( \mathcal{D}_X \) gives $n + r + 1 \geq \dim \in_{(0,1,0)} L_{F,2}$. Using Theorem 2.8 and Remark 2.9 we know $\dim \in_{(0,1,0)} L_{F,2} \geq \dim L_{F,2}$. Then Proposition 3.19 gives $\dim L_{F,2} = n + r + 1$, proving (a).

As for (b), the hypotheses guarantees that the map \( \mathcal{D}_X \) is locally a surjection from a domain to a ring of the same dimension. Hence the map is locally an isomorphism.

To prove (c), recall Theorem 2.8 and Remark 2.9 show that if $\in_{(0,1,0)} L_F$ is locally Cohen–Macaulay, then $L_F$ is locally Cohen–Macaulay. So (b) implies (c). \( \square \)

3.3. Primality of $L_{F,2}$ and $\tilde{L}_{F,2}$.

Now we show that when $f$ is strongly Euler-homogeneous and Saito-holonomic and $F$ a decomposition of $f$, that the conclusions of Proposition 3.22 imply $L_F$ and $L_F$ are locally prime. The method of argument relies on the Saito-holonomic condition: we use the coordinate transformation of Remark 3.13 to reduce the dimension of logarithmic stratum.

Our first proof mirrors the proof of Theorem 3.17 in [29]. Because our situation is a little more technical and because we end up using this argument again in Theorem 3.23 we give full details.

**Theorem 3.23.** Suppose $f = f_1 \cdots f_r$ is strongly Euler-homogeneous and Saito-holonomic and let $F = (f_1, \ldots, f_r)$. If $L_f$ is locally Cohen–Macaulay and prime of dimension $n + 1$, then $L_F$ is locally Cohen–Macaulay and prime of dimension $n + r + 1$. In particular, this happens when $f$ is strongly Euler-homogeneous, Saito-holonomic, and tame.

Proof. If we prove the second sentence, the third will follow by Theorem 3.17 and Remark 3.18 of [29]. By Proposition 3.22 the only thing to prove in the second sentence is that $L_F$ is locally prime. To do this we induce on the dimension of $X$. If $\dim X$ is 1, then $L_{F,2} = 0$ and the claim is trivially true.

So we may assume the claim holds for all $X$ with dimension less than $n$. Suppose $\mathfrak{r}$ belong to a logarithmic stratum $\sigma$ of dimension $k$. If $k = n$, then by Proposition 3.13 and Remark 3.13 $L_{F,2}$ is prime. Now assume $0 < k < n$. By Remark 3.13 we can find a coordinate transformation near $\mathfrak{r}$ such that each $f_i = u_i g_i$, where $u_i$ is a unit near $\mathfrak{r}$ and $g_i(x_1, \ldots, x_n) = f_i(x_1, \ldots, x_{n-k}, 0, \ldots, 0)$, cf. 3.6 of [28]. By Remark 3.17 $L_{F,2}$ is well-behaved under coordinate transformations and multiplication by units, so we may instead prove the claim for $L_{G,2}$, where $g = \prod g_i$ and $G = (g_1, \ldots, g_r)$. Let $X'$ be the space of the first $n - k$ coordinates and $\mathfrak{r}'$ the first $n - k$
coordinates of \( \mathfrak{r} \). When viewing \( g_i' \) as an element of \( \mathcal{O}_{X',\mathfrak{r}} \), call it \( g_i' \). Let \( g' = \prod g_i' \) and \( G' = (g_1', \ldots, g_r') \). Because strongly Euler-homogeneous descends from \( X \) to \( X' \), see Remark 2.8 in \([21]\), local properties \( L_{G'} \) do not depend on the choice of the defining equations for the \( g_i \). Now

\[
\operatorname{Der}_{X,\mathfrak{r}}(-\log g) = \mathcal{O}_{X,\mathfrak{r}} \cdot \operatorname{Der}_{X',\mathfrak{r}}(-\log g') + \sum_{1 \leq j \leq k} \mathcal{O}_{X,\mathfrak{r}} \cdot \partial_{x_{n-k+j}},
\]

where \( \partial_{x_{n-k+j}} \in \operatorname{Der}_{X,\mathfrak{r}}(-\log g_i) \) for each \( 1 \leq j \leq k \) and \( 1 \leq i \leq r \). Therefore \( \mathcal{O}_{X,\mathfrak{r}}[y_1, \ldots, y_n][S]/L_{G,\mathfrak{r}} \simeq \mathcal{O}_{X',\mathfrak{r}}[y_1, \ldots, y_{n-k}][S]/L_{G',\mathfrak{r}} \). Since Saito-holonomicity descends to \( g' \), see 3.5 and 3.6 of \([28]\) and Remark 2.6 of \([29]\), we may instead prove the claim for \( X' \) and \( L_{G',\mathfrak{r}} \). Since \( \dim X' < \dim X \), the induction hypothesis proves the claim.

So we may assume \( \sigma \) has dimension 0. By Remark \([3,13]\) there is some open \( U \ni \mathfrak{r} \), such that \( \mathfrak{r} = U \cap \sigma \) and \( U \setminus \mathfrak{r} \) consists of points whose logarithmic strata are of strictly positive dimension. The discussion above implies \( L_F \) is prime at all points of \( U \setminus \mathfrak{r} \).

Let \( \pi: \text{Spec} \mathcal{O}_X[Y]/[S] \to \text{Spec} \mathcal{O}_X \) be the map induced by \( \mathcal{O}_X \to \mathcal{O}_X[Y]/[S] \). If \( L_F \) is not prime at \( \mathfrak{r} \), it must have more than one irreducible component that intersects \( \pi^{-1}(\mathfrak{r}) \). As \( L_F \) is prime at points of \( U \setminus \mathfrak{r} \), if \( L_{F,\mathfrak{r}} \) is not prime it must have an “extra” irreducible component \( V(\mathfrak{q}) \) lying inside \( \pi^{-1}(\mathfrak{r}) \). By assumption, \( L_{F,\mathfrak{r}} \) is Cohen–Macaulay of dimension \( n + r + 1 \) and so \( V(\mathfrak{q}) \) has dimension \( n + r + 1 \). But \( \pi^{-1}(\mathfrak{r}) \) has dimension \( n + r \). Therefore, \( V(\mathfrak{q}) \) cannot be contained in \( \pi^{-1}(\mathfrak{r}) \) and \( L_{F,\mathfrak{r}} \) does not have this “extra” irreducible component. Thus \( L_{F,\mathfrak{r}} \) is prime. This completes the induction step and proves the claim.

**Proposition 3.24.** Suppose \( f = f_1 \cdots f_r \) is a strongly Euler-homogeneous divisor and let \( F = (f_1, \ldots, f_r) \). If \( L_F \) is locally prime, Cohen–Macaulay, and of dimension \( n + r + 1 \), then \( L_F \) is locally Cohen–Macaulay of dimension \( n + r \). In particular, this happens when \( f \) is strongly Euler-homogeneous, Saito-holonomic, and tame.

**Proof.** Let \( E_f \) be a strong Euler-homogeneity and consider \( \text{gr}_{(0,1,1)}(\psi_F(E_f)) \), which is \((0,1,1)\)-homogeneous of degree 1. The generalized Liouville ideal is generated by the elements \( \psi_F(Der_{X,\mathfrak{r}}(-\log g)) \). If \( \text{gr}_{(0,1,1)}(\psi_F(E_f)) \in L_{F,\mathfrak{r}} \), then \( \psi_F(E_f) \in \psi_F(Der_{X,\mathfrak{r}}(-\log g)) \). This is impossible since \( E_f \notin Der_{X,\mathfrak{r}}(-\log g) \).

Locally, \( \text{gr}_{(0,1,1)}(\mathcal{O}_{X,\mathfrak{r}}[S])/L_{F,\mathfrak{r}} \) is obtained from \( \text{gr}_{(0,1,1)}(\mathcal{O}_{X,\mathfrak{r}}[S])/L_{F,\mathfrak{r}} \) by modding out by a non-zero element, which must be regular. So \( L_{F,\mathfrak{r}} \) is locally Cohen–Macaulay of dimension at most \( n + r \). That locally the dimension \( L_{F,\mathfrak{r}} \) is \( n + r \) follows from the dimension inequality in Proposition \([3,19]\).

The final sentence is true by Theorem \([3,23]\).

This section’s first main result is that \( L_{F,\mathfrak{r}} \) is locally prime when \( f \) is strongly Euler-homogeneous, Saito-holonomic, and tame. The strategy is the same as in Theorem \([3,23]\). Under much stricter hypotheses, and in his language, Maisonobe shows in Proposition 3 of \([21]\), that \( L_{F,\mathfrak{r}} \) is locally prime. Experts will note that we recycle the part of his argument where he reduced dimension in our proof.

**Theorem 3.25.** Assume that \( f = f_1 \cdots f_r \) is strongly Euler-homogeneous and Saito-holonomic and let \( F = (f_1, \ldots, f_r) \). If \( L_F \) is locally Cohen–Macaulay of dimension \( n + r \), then \( L_{F,\mathfrak{r}} \) is locally prime. In particular, \( L_{F,\mathfrak{r}} \) is locally prime,
Cohen–Macaulay, and of dimension \( n + r \) when \( f \) is strongly Euler-homogeneous, Saito-holonomic, and tame.

Proof. By Proposition 3.24 it suffices to prove the first claim. The proof follows the inductive argument of Theorem 3.23 with a slight modification at the end.

If \( \dim X = 1 \), then \( \bar{L}_{F,\mathfrak{r}} \) is generated by \( \psi_F(E_{\mathfrak{r}}) \), \( E_{\mathfrak{r}} \) a strong Euler-homogeneity. By Lemma 3.18 \( \mathcal{O}_{X,\mathfrak{r}}[Y][S]/\bar{L}_{F,\mathfrak{r}} \simeq \mathcal{O}_{X,\mathfrak{r}}[Y][s_1,\ldots,s_{j-1},s_{j+1},\ldots,s_r] \).

Now assume the claim holds for all \( X \) with dimension less than \( n \) and \( \mathfrak{r} \) belongs to a logarithmic stratum \( \sigma \) of dimension \( k \). If \( k = n \), then \( \bar{L}_{F,\mathfrak{r}} \) is prime by Proposition 3.19. If \( 0 < k < n \) we can make the same coordinate transformation as in Theorem 3.23 and instead prove \( \bar{L}_G \) is locally prime where \( g_i(x) = f_i(x_1,\ldots,x_{n-k},0,\ldots,0) \).

Using the notation of Theorem 3.23 \( X' \) is strongly Euler-homogeneous and Saito-holonomic and

\[
\text{Der}_{X,\mathfrak{r}}(-\log g) = \mathcal{O}_{X,\mathfrak{r}} \cdot \text{Der}_{X',\mathfrak{r}}(-\log g') + \sum_{1 \leq j \leq k} \mathcal{O}_{X,\mathfrak{r}} \cdot \partial x_{k+j},
\]

where \( \partial x_{k+j} \in \text{Der}_{X,\mathfrak{r}}(-\log g) \) for each \( 1 \leq j \leq k \) and \( 1 \leq i \leq r \). Moreover, the strong Euler-homogeneity \( E_{\mathfrak{r}} \) for \( g' \) at \( \mathfrak{r} \in X' \) can be viewed as a strong Euler-homogeneity for \( g \) at \( \mathfrak{r} \in X \). Therefore \( \mathcal{O}_{X,\mathfrak{r}}[y_1,\ldots,y_n][S]/\bar{L}_{G,\mathfrak{r}} \simeq \mathcal{O}_{X,\mathfrak{r}}[y_1,\ldots,y_n][S]/\bar{L}_{G',\mathfrak{r}} \). Since \( \dim X' < \dim X \), the induction hypothesis shows that \( \bar{L}_{G',\mathfrak{r}} \) is prime.

So we may assume \( \sigma \) has dimension 0. Let \( \pi : \text{Spec} \mathcal{O}_{X,\mathfrak{r}}[Y][S] \to \text{Spec} \mathcal{O}_X \) be the map induced by \( \mathcal{O}_X \leftarrow \mathcal{O}_{X,\mathfrak{r}}[Y][S] \). Reasoning as in Theorem 3.23 we deduce that if \( \bar{L}_{F,\mathfrak{r}} \) is not prime then there exists an irreducible component \( V(q) \) of \( \bar{L}_{F,\mathfrak{r}} \) contained entirely in \( \pi^{-1}(\mathfrak{r}) \).

By assumption, \( \bar{L}_{F,\mathfrak{r}} \) is Cohen–Macaulay of dimension \( n + r \) and \( V(q) \) has dimension \( n + r \). Let \( E_{\mathfrak{r}} \) be the strong Euler-homogeneity at \( \mathfrak{r} \). Then \( \text{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{r}})) \in \bar{L}_{F,\mathfrak{r}} \subseteq q \) and \( V(q) \subseteq \pi^{-1}(\mathfrak{r}) \cap V(\text{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{r}}))) \). We will show that the intersection of \( V(\text{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{r}}))) \) and \( \pi^{-1}(\mathfrak{r}) \) is proper; since the dimension of \( \pi^{-1}(\mathfrak{r}) \) is \( n + r \) this will show that \( V(q) \), which we know is of dimension \( n + r \), is contained in a closed set of strictly smaller dimension. Therefore no such \( q \) exists and \( \bar{L}_{F,\mathfrak{r}} \) is prime.

Recall \( \text{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{r}})) = \text{gr}_{(0,1,1)}(E_{\mathfrak{r}}) - \sum \frac{E_i F_{j}}{f_{ij}} s_k \). Lemma 3.18 proves that there exists an index \( j \) such that \( \frac{E_i F_{j}}{f_{ij}} \notin m_{\mathfrak{r}} \). So there is a closed point in \( \pi^{-1}(\mathfrak{r}) \) that does not lie in \( V(\text{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{r}}))) \). In particular, the intersection of \( V(\text{gr}_{(0,1,1)}(\psi_F(E_{\mathfrak{r}}))) \) and \( \pi^{-1}(\mathfrak{r}) \) is proper and \( \bar{L}_{F,\mathfrak{r}} \) is prime. This completes the inductive step and finishes the claim. \( \square \)

3.4. The \( \mathcal{D}_X[S] \)-annihilator of \( F^S \).

Let \( \text{Jac}(f) \) be the Jacobian ideal of \( f \). In a given coordinate system, there is a natural \( \mathcal{O}_{X,\mathfrak{r}} \)-linear map

\[
\phi_f : \text{gr}_{(0,1,1)}(\mathcal{D}_X[s]) \to \text{Sym}_{\mathcal{O}_{X,\mathfrak{r}}}(\text{Jac}(f)) \to R(\text{Jac}(f))
\]

given by

\[
s \mapsto ft \text{ and } \text{gr}_{(0,1,1)}(\partial x_k) \mapsto (\partial x_k \cdot f)t.
\]
Its kernel contains $\text{gr}_{(0,1)}(\text{ann}_{\mathcal{O}_{X,S}[s]} f^s)$. (See Section 1.3 in [10] for details.) So we have the containments

$$L_{f,s} + \text{gr}_{(0,1,1)}(\mathcal{O}_{X,S}[s]) \cdot \text{gr}_{(0,1,1)}(E_x - s) \subseteq \text{gr}_{(0,1)}(\text{ann}_{\mathcal{O}_{X,S}[s]} f^s) \subseteq \ker(\phi_f)$$

and equality will hold throughout if $L_{f,s} + \text{gr}_{(0,1,1)}(\mathcal{O}_{X,S}[s]) \cdot \text{gr}_{(0,1,1)}(E_x - s)$ agrees with $\ker(\phi_f)$.

This motivates our analysis of $\text{ann}_{\mathcal{O}_{X,S}[S]} F^S$: we will construct a map $\phi_F$ from $\text{gr}_{(0,1,1)}(\mathcal{O}_X[S])$ into a Rees-algebra like object and squeeze $\text{gr}_{(0,1,1)}(\text{ann}_{\mathcal{O}_{X,S}[S]} F^S)$ between $L_F$ and $\ker(\phi_F)$.

**Definition 3.26.** Let $\text{Jac}(f_i)$ be the Jacobian ideal of $f_i$ and consider the multi-Rees algebra $R(\text{Jac}(f_1), \ldots, \text{Jac}(f_r))$ associated to these $r$ Jacobian ideals. Consider the $\mathcal{O}_{X,S}$-linear map

$$\phi_F : \text{gr}_{(0,1,1)}(\mathcal{O}_X[S]) \to R(\text{Jac}(f_1), \ldots, \text{Jac}(f_r)) \subseteq \mathcal{O}_X[S]$$

given, having fixed local coordinates on $U$, by

$$y_i \mapsto \sum_k \frac{f}{f_k}(\partial_{x_i} \cdot f_k)s_k \text{ and } s_k \mapsto f s_k.$$ 

**Proposition 3.27.** Let $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$. Then

$$\text{gr}_{(0,1,1)}(\text{ann}_{\mathcal{O}_{X,S}[S]} F^S) \subseteq \ker(\phi_F).$$

**Proof.** It is enough to show this locally, so take $P \in \text{ann}_{\mathcal{O}_{X,S}[S]} F^S$ of order $\ell$ under the $(0,1,1)$-filtration. For any $Q$ of order $\ell$ it is always true that $f^\ell Q \cdot F^S \in \mathcal{O}_{X,S}[S]F^S$. Any time a partial is applied to $gF^S$, a $s$-term only comes out of the product rule when the partial is applied to $F^S$. A straightforward calculation shows that the $S$-lead term of $f^\ell P F^S$ is exactly $\phi_F(\text{gr}_{(0,1,1)}(P))F^S$. Since $f^\ell P$ annihilates $F^S$, we conclude $\text{gr}_{(0,1,1)}(P) \in \ker(\phi_F)$. \qed

**Proposition 3.28.** $\ker(\phi_F)$ is a prime ideal of dimension $n + r$.

**Proof.** It is prime. Since Rees rings are domains, to count dimension we squeeze $\phi_F(\text{gr}_{(0,1,1)}(\mathcal{O}_X[S]))$ between two well-behaved multi-Rees algebras: $R((f), \ldots, (f))$ and $R(\text{Jac}(f_1), \ldots, \text{Jac}(f_r))$ (the first multi-Rees algebra is built using $r$ copies of $(f)$). As the latter is the target of $\phi_F$ and $\phi_F(s_i) = f s_i$ this is easy:

$$R((f), \ldots, (f)) \subseteq \phi_F(\text{gr}_{(0,1,1)} \mathcal{O}_X[S]) \subseteq R(\text{Jac}(f_1), \ldots, \text{Jac}(f_r))$$

Moreover, the dimension of a multi-Rees algebra is well known: $R(I_1, \ldots, I_r) = r + \text{dim of the ground ring}$.

So $\phi_F(\text{gr}_{(0,1,1)}(\mathcal{O}_{X,S}[S]))$ is a domain squeezed between subrings of $\mathcal{O}_{X,S}[S]$ of dimension $n + r$. The result then follows by the following lemma:

**Lemma 3.29.** Let $R \subseteq A \subseteq B \subseteq C \subseteq R[X]$ be finitely generated, graded $R$-algebras, whose gradings are inherited from the standard grading on $R[X]$. Assume that $R$ is a universally catenary Noetherian domain. If $\text{dim } A = \text{dim } C$, then $\text{dim } A = \text{dim } B = \text{dim } C$.

**Proof.** Claim: if $m^*$ is a graded maximal ideal of $A$, then $m^*B \neq B$. We prove the contrapositive. So assume $m^* B = B$. Then $m^* R[X] = R[X]$. Write $m^* = (a_1, \ldots, a_n)$ in terms of homogeneous generators $a_i \in A$ and find $r_1, \ldots, r_n$ in $R[X]$ such that $1 = \sum r_i a_i$. Since the degree of 1 is zero, we can assume either $r_i$ and $a_i$
are both degree 0 or \(r_i = 0\). Thus 1 = \(\sum r_i a_i\) occurs in \(m^* \cap R\) and so \(m^* = A\), a contradiction.

Now we argue using a version of Nagata’s Altitude Formula (see [13] Theorem 13.8): \(\dim(B_q) = \dim(A_p) + \dim(Q(A) \otimes_A B)\), for \(q \in \text{Spec } B\) maximal with respect to the property \(q \cap R = p\). Since \(B\) is a finitely generated \(A\)-algebra, and tensors are right exact, \(Q(A) \otimes_A B\) is a finitely generated \(Q(A)\)-algebra. Thus \(\dim(Q(A) \otimes_A B) = \text{trdeg}_{Q(A)}(Q(A) \otimes_A B) = \text{trdeg}_{Q(A)}Q(Q(A) \otimes_A B) = \text{trdeg}_{Q(A)}Q(B) = \text{trdeg}_A B\). Similar statements hold for the other pairs \(A \subseteq C\) and \(B \subseteq C\).

Let \(m \in \text{Spec } A\) such that \(\dim(A_m) = \dim(A)\). By the claim in the first paragraph (so assuming \(m\) is graded if necessary), we can find \(q \in \text{Spec } C\) maximal with respect to the property \(q \cap A = m\). So \(\dim(C_q) = \dim(A_m) + \text{trdeg}_A C\). Therefore \(\dim(C) \geq \dim(A) + \text{trdeg}_A C\) and hence \(\text{trdeg}_A C = 0\). Since we are looking at algebras finitely generated over the appropriate subring, transcendence degree is additive. So \(0 = \text{trdeg}_A B\) and \(0 = \text{trdeg}_B C\).

Let \(m \in \text{Spec } A\) with \(\dim(A_m) = \dim(A)\), as before. Again, using the claim, select \(p \in \text{Spec } B\) maximal with respect to the property \(p \cap A = m\). So \(\dim(B_p) = \dim(A_m) + \text{trdeg}_A B\); hence \(\dim(B) \geq \dim(A)\). Arguing similarly for \(B \subseteq C\) to determine \(\dim(C) \geq \dim(B)\). This ends the proof. \(\square\)

The following is an analogous statement to Corollary 3.23 in [29]:

**Corollary 3.30.** There is the containment

\[\tilde{L}_F \subseteq \text{gr}_{(0,1,1)}(\text{ann}_{\mathcal{D}_X[S]} F^S) \subseteq \ker(\phi_F).\]

If \(f\) is strongly Euler-homogeneous, Saito-holonomic, and tame then all three ideals are equal.

**Proof.** The containments follow from the construction of \(\tilde{L}_F\) and Proposition 3.27. They are equalities when \(f\) is suitably nice because, by Theorem 3.25 and Proposition 3.28 at each \(r \in X\) the outer ideals are prime of the same dimension. \(\square\)

We know \(\mathcal{D}_{X,F}[S] \cdot \theta_F \subseteq \text{ann}_{\mathcal{D}_{X,F}[S]} F^S\) and we have just shown they have the same initial terms under the \((0,1,1)\)-filtration. By the standard strategy of peeling off initial terms of \(\text{ann}_{\mathcal{D}_{X,F}[S]} F^S\) using the appropriate element of \(\mathcal{D}_{X,F}[S] \cdot \theta_F\) we can prove:

**Theorem 3.31.** If \(f = f_1 \cdots f_r\) is strongly Euler-homogeneous, Saito-holonomic, and tame and if \(F = (f_1, \ldots, f_r)\), then the \(\mathcal{D}_{X,S} F\)-annihilator of \(F^S\) is generated by derivations, that is

\[\text{ann}_{\mathcal{D}_{X}[S]} F^S = \mathcal{D}_{X}[S] \cdot \theta_F.\]

**Proof.** Take \(P \in \text{ann}_{\mathcal{D}_{X}[S]} F^S\). By Corollary 3.30 \(\text{gr}_{(0,1,1)}(P) \in \tilde{L}_F\). So there exist \(L_1, \ldots, L_k \in \theta_F, n_1, \ldots, n_k \in \mathcal{D}_X[Y][S]\) such that

\[\text{gr}_{(0,1,1)}(P) = \sum n_i \cdot \text{gr}_{(0,1,1)}(L_i).\]

This element is \((0,1,1)\)-homogeneous of degree \(k\). Note that this means \(k\) is the smallest integer such that \(P \in F^S_{(0,1,1)}\). (Recall \(F_{(0,1,1)}\) is the filtration on \(\mathcal{D}_{X,S}[S]\) making every element of \(\mathcal{D}_{X,S}, wire 0 and giving each \(\partial\) and each \(s_i\) weight 1.) Since each \(\text{gr}_{(0,1,1)}(L_i)\) is \((0,1,1)\)-homogeneous of degree 1, we may assume the \(n_i\) are all \((0,1,1)\)-homogeneous of the same degree. In particular, for each \(i\), there
exists an \( N_i \in \mathcal{D}_X[S] \) such that \( n_i = \text{gr}_{(0,1,1)}(N_i) \). Therefore, \( P - \sum N_i \cdot L_i \in F^{k-1}_{(0,1,1)} \). Since \( P - \sum N_i \cdot L_i \in \text{ann}_{\mathcal{D}_X[S]} F^S \) and \( \text{ann}_{\mathcal{D}_X[S]} F^S \) contains no element of \( F^0_{(0,1,1)} = \partial_X \), an induction argument shows \( P \in \mathcal{D}_X[S] \cdot \theta_F \). 

**Corollary 3.32.** Let \( f = f_1 \cdots f_r \in \mathbb{C}[x_1, \ldots, x_n] \), where each \( f_k \in \mathbb{C}[x_1, \ldots, x_n] \), and let \( F = (f_1, \ldots, f_r) \). If \( f \) is strongly Euler-homogeneous, Saito-holonomic, and tame, then the \( \mathcal{D}_X[S] \)-annihilator of \( F^S \) is generated by derivations, that is

\[
\text{ann}_{\mathcal{D}_X[S]} F^S = \mathcal{D}_X[S] \cdot \theta_F.
\]

More generally, if \( X \) is the analytic space associated to a smooth \( \mathbb{C} \)-scheme and if \( f \) and \( F = (f_1, \ldots, f_r) \) are algebraic, then the conclusion of Theorem 3.31 holds in the algebraic category.

**Proof.** This follows from Theorem 3.31 and the fact algebraic functions have algebraic syzygies. See Theorem 3.26 and Remark 2.11 in [29] for more details. \( \square \)

3.5. **Relations between \( V(B_{F,x}) \) and \( V(B_{f,x}) \).**

**Definition 3.33.** Consider the functional equation

\[
b_{f,x}(s)f^s = Pf^{s+1}
\]

where \( b_{f,x}(s) \in \mathbb{C}[s] \) and \( P \in \mathcal{D}_{X,s}[s] \). Let \( B_{f,x} \) be the ideal in \( \mathbb{C}[s] \) generated by all such \( b_{f,x}(s) \), that is the ideal generated by the Bernstein–Sato polynomial. We may write \( B_{f,x} = (\mathcal{D}_{X,s}[s] \cdot f + \text{ann}_{\mathcal{D}_{X,s}[s]} f^s) \cap \mathbb{C}[s] \). Then the variety \( V(B_{f,x}) \) consists of the roots of the Bernstein–Sato polynomial.

In the multivariate situation we may consider functional equations of the form

\[
b_{F,x}(s)F^s = Pf^{s+1}
\]

where \( b_{F,x}(s) \in \mathbb{C}[s] \) and \( P \in \mathcal{D}_{X,s}[s] \). Just as above, the set of all such \( b_{F,x}(s) \) form an ideal \( B_{F,x} = (\mathcal{D}_{X,s}[s] \cdot f + \text{ann}_{\mathcal{D}_{X,s}[s]} F^s) \cap \mathbb{C}[s] \). The variety \( V(B_{F,x}) \) is called the Bernstein–Sato variety of \( F \).

Under our working hypotheses of strongly Euler-homogeneous, Saito-holonomic, and tame divisors, by Theorem 3.31 the annihilator of \( F^S \) is generated by derivations. Given \( f = f_1 \cdots f_r \), this will let us compare \( V(B_{F,x}) \) and \( V(B_{G,x}) \) where \( F = (f_1, \ldots, f_r) \) and \( G = (f_1, \ldots, f_{r-2}, f_{r-1}f_r) \). Setting \( G = (f) \) lets us compare \( V(B_{F,x}) \) and \( V(B_{f,x}) \).

**Proposition 3.34.** Suppose \( f = f_1 \cdots f_r \) is strongly Euler-homogeneous, Saito-holonomic, and tame. Let \( F = (f_1, \ldots, f_r) \) and \( G = (f_1, \ldots, f_{r-2}, f_{r-1}f_r) \). Then

\[
\frac{\mathcal{D}_X[s_1, \ldots, s_r]}{\text{ann}_{\mathcal{D}_X[s_1, \ldots, s_r]} \mathcal{D}_X F^S + (s_{r-1} - s_r)} \cong \frac{\mathcal{D}_X[s_1, \ldots, s_r-1]}{\text{ann}_{\mathcal{D}_X[s_1, \ldots, s_r-1]} \mathcal{D}_X[s_1, \ldots, s_r-1] G^S}.
\]

**Proof.** By Theorem 3.31 \( \text{ann}_{\mathcal{D}_X[s_1, \ldots, s_r]} F^S \) is generated by \( \delta - \sum_{k=1}^r s_k \frac{s_k f_k}{f_r} \) and \( \text{ann}_{\mathcal{D}_X[s_1, \ldots, s_r-1]} G^S \) is generated by \( \delta - \sum_{k=1}^{r-2} s_k \frac{s_k f_k}{f_r} - s_{r-1} \frac{s_{r-1} f_{r-1} f_r}{f_r} \). Under the quotient map \( \mathcal{D}_X[s_1, \ldots, s_r] \rightarrow \mathcal{D}_X[s_1, \ldots, s_r-1] \rightarrow \mathcal{D}_X[s_1, \ldots, s_r] \).
We can also use Proposition 3.34 to compare the Bernstein–Sato variety of two different, but similar, decompositions $F$ and $G$ of $f$.

**Proposition 3.35.** Suppose $f = f_1 \cdots f_r$ is strongly Euler-homogeneous, Saito-holonomic, and tame. Let $F = (f_1, \ldots, f_r)$ and $G = (f_1, \ldots, f_{r-2}, f_{r-1} f_r)$. Keeping the convention $\mathbb{C}[S] = \mathbb{C}[s_1, \ldots, s_r]$, we have the $\mathbb{C}[S]$-ideal containment

$$B_{F,x} + \mathbb{C}[S] \cdot (s_{r-1} - s_r) \subseteq \mathbb{C}[S] \cdot B_{G,x} + \mathbb{C}[S] \cdot (s_{r-1} - s_r).$$

In particular, let $\Delta : \mathbb{C} \to \mathbb{C}'$ be the diagonal embedding. Then $\Delta(V(B_{f,x})) \subseteq V(B_{F,x})$.

**Proof.** Let

$$I = \mathcal{D}_{X,[S]} \cdot f + \mathcal{D}_{X,[S]} \cdot \text{ann}_{\mathcal{D}_{X,[S]}}[s_1, \ldots, s_{r-1}] \cdot G^S + \mathcal{D}_{X,[S]} \cdot (s_{r-1} - s_r).$$

Suppose $P(S) \in I \cap \mathbb{C}[S]$. Then

$$P(s_1, \ldots, s_{r-1}, s_r) \in \mathcal{D}_{X,[S]} \cdot (s_1, \ldots, s_{r-1}) \cdot f + \text{ann}_{\mathcal{D}_{X,[S]}}[s_1, \ldots, s_{r-1}] \cdot G^S.$$

Therefore $I \cap \mathbb{C}[S] \subseteq \mathcal{D}_{X,[S]} \cdot (s_1, \ldots, s_{r-1}) \cdot f + \text{ann}_{\mathcal{D}_{X,[S]}}[s_1, \ldots, s_{r-1}] \cdot G^S$. The reverse inequality is obvious; therefore $I \cap \mathbb{C}[S] = \mathbb{C}[S] \cdot B_{G,x} + \mathbb{C}[S] \cdot (s_{r-1} - s_r)$.

By Proposition 3.34

$$I \cap \mathbb{C}[S] = \mathbb{C}[S] \cap (\mathcal{D}_{X,[S]} \cdot f + \mathcal{D}_{X,[S]} \cdot \text{ann}_{\mathcal{D}_{X,[S]}}[s] F^S + \mathcal{D}_{X,[S]} \cdot (s_{r-1} - s_r)) \supseteq B_{F,x} + \mathbb{C}[S] \cdot (s_{r-1} - s_r).$$

This proves the claim about $F$ and $G$.

The statement about the diagonal embedding follows by taking varieties and repeating the above process $r - 1$ times.

**Remark 3.36.** Let $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$ the corresponding decomposition of $f$. Let $g_1 = f_1 \cdots f_{i-1}, g_i = f_{i-1} \cdots f_r$ and $G = (g_1, \ldots, g_r)$ the corresponding decomposition of $f$. It is straightforward to argue as in Proposition 3.35 and compare the varieties of $V(B_{F,x})$ and $V(B_{G,x})$.

### 3.6. Hyperplane Arrangements

Finally let us turn to the algebraic setting and particular to central hyperplane arrangements $A \subseteq \mathbb{C}^n = X$ whose defining equations are given by

$$f_A = \prod L_i,$$

where the $L_i \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous polynomials of degree 1. A central hyperplane arrangement is indecomposable if there is no choice of coordinates $t_1,t_2$, $t_1$ and $t_2$ disjoint, such that $f_A = g_1(t_1) g_2(t_2)$. Many of our results apply to central hyperplane arrangements because they are strongly Euler-homogeneous and Saito-holonomic, see Examples 3.3-3.15.

Write $D_n$ for the $n$th Weyl Algebra $\mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$. Let $F = (f_1, \ldots, f_r)$ be some decomposition of $f_A$ into factors. Construct the $D_n[S]$-module $(D_n[S]$-module) generated by the symbol $f^*$ $(F^S)$ in an entirely similar way as in the
analytic setting. Furthermore, define the roots of the Bernstein–Sato polynomial $B_f$ and the Bernstein–Sato variety $B_F$ just as before. For an algebraic $f$ equipped with an algebraic decomposition $F$, $B_f$ and $B_F$ agree with the analytic versions because algebraic functions have algebraic derivatives and syzygies.

In [29], Budur makes the following conjecture:

**Conjecture 3.37.** (Conjecture 3 in [29]) Let $A$ be a central, essential, indecomposable hyperplane arrangement. Factor $f_A = f_1 \cdots f_r$, where each factor $f_k$ is of degree $d_k$ and the $f_k$ are not necessarily reduced, and let $F = (f_1, \ldots, f_r)$. Then

$$\{d_1s_1 + \cdots + d_rs_r + n = 0\} \subseteq V(B_F).$$

This conjecture is related to the Topological Multivariable Strong Monodromy Conjecture, see Conjecture 1.4 for hyperplane arrangements, which claims that the pole locus of the topological zeta function of $F = (f_1, \ldots, f_r)$ is contained in $V(B_{F,0})$. In Theorem 8 of loc. cit. Budur proves Conjecture 3.37 implies the Topological Multivariable Strong Monodromy Conjecture for hyperplane arrangements. See [4], in particular subsection 1.3 and Theorem 8, for details.

Walther proves in Theorem 5.13 of [29] a weaker version of this conjecture in the classical $f^*$ case: if $f$ is a tame and indecomposable central hyperplane arrangement of degree $d$ then $-n/d \in V(B_f)$. When $f_A$ is tame, $\text{ann}_{D_n[s]} f^*$ and $\text{ann}_{D_n[S]} F^S$ are both generated by $\text{Der}_X(-\log f)$ in similar ways; hence we can argue similarly and prove a weaker version of Conjecture 3.37.

**Theorem 3.38.** Suppose $f_A$ is a central, essential, indecomposable, and tame hyperplane arrangement. Let $F = (f_1, \ldots, f_r)$ be a decomposition of $f_A$ where $f_k$ has degree $d_k$ and the $f_k$ are not necessarily reduced. Then

$$\{d_1s_1 + \cdots + d_rs_r + n = 0\} \subseteq V(B_F).$$

Proof. Since $f_A$ is homogeneous, $\text{Der}_X(-\log f)$ is a graded $\mathbb{C}[X]$-module after giving each $x_i$ degree one and each $\partial_i$ degree $-1$. In the proof of Theorem 5.13 of [29], Walther shows that the indecomposability hypothesis implies there exists a system of coordinates such that $\delta \in \text{Der}_X(-\log f)$ is homogeneous of positive total degree or $\delta = w \sum x_i \partial_i$, $w \in \mathbb{C}$. Fix this system of coordinates and $E = \sum x_i \partial_i$ for the rest of the proof.

By Corollary 3.32 $\text{ann}_{D_n[S]} F^S = D_n[S] \cdot \psi_F(\text{Der}_X(-\log f))$. Recall $\psi_F(\delta) = \delta - \sum \frac{\delta f_k}{f_k} s_k$. If $\delta$ is of positive $(1,-1)$ total degree, then the coefficient of each $s_k$ is either 0 or of positive degree as polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. This shows $\psi_F(\delta) \in D_n[S] \cdot (X)$, where $D_n[S] \cdot (X)$ is the left ideal generated by $x_1, \ldots, x_n$. Because $E + n \in D_n \cdot (X)$,

$$\text{ann}_{D_n[S]} F^S + D_n[S] \cdot f \subseteq D_n[S] \cdot (X) + D_n[S] \cdot \psi_F(E)$$

$$= D_n[S] \cdot (X) + D_n[S] \cdot (E - \sum d_k s_k)$$

$$= D_n[S] \cdot (X) + D_n[S] \cdot (-n - \sum d_k s_k).$$

Suppose $P(S)$ is in the intersection of $D_n[S] \cdot (X) + D_n[S] \cdot (-n - \sum d_k s_k)$ and $\mathbb{C}[S]$. For each root $\alpha$ of $-n - \sum d_k s_k$ there is a natural evaluation map $D_n[S] \rightarrow D_n$ sending $P \mapsto P(\alpha) \in D_n \cdot (X)$. Since $D_n \cdot (X)$ is a proper ideal of $D_n$, $P(\alpha) = 0$.
for all such $\alpha$. Therefore $V(P(S)) \supseteq V(\mathbb{C}[S] \cdot (-n - \sum d_k s_k))$ and we have shown

$$V(B_F) = V((\text{ann}_{D_n[S]} F^S + D_n[S] \cdot f) \cap \mathbb{C}[S]) \supseteq V(-n - \sum d_k s_k).$$

\[ \square \]

**Remark 3.39.** (1) In Theorem 3.38, we only needed tameness to ensure the annihilator of $F^S$ is generated by derivations and we only needed indecomposability to insure $\text{Der}_X(- \log_0 f) \subseteq (X)^2 \cdot \text{Der}_X$. So Theorem 3.38 holds for central, essential, and indecomposable hyperplane arrangements such that $\text{ann}_{D_n[S]} F^S = D_n[S] \cdot \theta_F$.

(2) In an ongoing project, we plan to generalize this argument to find many more hyperplanes that must lie in $V(B_F)$ for central, essential, indecomposable, and tame hyperplane arrangements. When $V(B_F)$ corresponds to the roots of the Bernstein–Sato polynomial, i.e. when $F = (f)$, this will let us compute many, if not all, of the roots of the Bernstein–Sato polynomial lying in $[-1, 0)$.

As outlined in the introduction, Theorem 3.38 is related to the Topological Multivariable Strong Monodromy Conjecture, that is, to Conjecture 1.4.

**Corollary 3.40.** The Topological Multivariable Strong Monodromy Conjecture is true for (not necessarily reduced) tame hyperplane arrangements.

**Proof.** This follows by Theorem 8 of [7] since tameness is a local condition. \[ \square \]

4. The Map $\nabla_A$

In this section we analyze the injectivity of $\mathcal{D}_{X,S}$-map

$$\nabla_A : \frac{\mathcal{D}_{X,S}[S]F^S}{(S - A)\mathcal{D}_{X,S}[S]F^S} \to \frac{\mathcal{D}_{X,S}[S]F^S}{(S - (A - 1))\mathcal{D}_{X,S}[S]F^S}$$

under the nice hypotheses of the previous section. This will, see Section 6, let us better understand the relationship between $V(B_{F,S})$ and the cohomology support loci of $f$ near $r$. The section has two parts: a brief discussion of Koszul complexes associated to central elements over certain non-commutative rings with an application to $\frac{\mathcal{D}_{X,S}[S]F^S}{(S - A)\mathcal{D}_{X,S}[S]F^S}$; a detailed proof that under nice hypotheses, if $\nabla_A$ is injective then it is surjective.

Let’s first give a precise definition of $\nabla_A$.

**Definition 4.1.** (Compare to 5.5 and 5.10, in particular $\rho_{\alpha}$, in [7]) Define

$$\nabla : \mathcal{D}_{X,S}[S]F^S \to \mathcal{D}_{X,S}[S]F^S$$

by sending $s_i \mapsto s_i + 1$ for all $i$. To be precise, in local coordinates declare $\partial^\mu = \prod_i \partial_{x_i}^{\mu_i}$, $S^\nu = \prod_k s_k^{\nu_k}$, and let $S + 1$ be shorthand for replacing each $s_i$ with a $s_i + 1$. Then $\nabla$ is given by the assignment

$$\sum_{u,v} Q_{u,v} \partial^\mu S^\nu \cdot F^S \mapsto \sum_{u,v} Q_{u,v} \partial^\mu (S + 1)^\nu \cdot F^{S+1}.$$

This is a homomorphism of $\mathcal{D}_{X,S}$-modules but not $\mathbb{C}[S]$-linear.

Denote the ideal of $\mathcal{D}_{X,S}[S]$ generated by $s_1 - a_1, \ldots, s_r - a_r$, for $a_1, \ldots, a_r \in \mathbb{C}$ by $(S - A)\mathcal{D}_{X,S}[S]$. Then $\nabla$ is injective and sends $(S - A)\mathcal{D}_{X,S}[S]F^S$ onto...
Using Singular and Macaulay2 we compute \( V(\mathbb{C}) \cap \mathbb{C}^n \) in Proposition 4.4.

\( (a) \) Using Singular and Macaulay2 we compute \( V(\mathbb{C}) \cap \mathbb{C}^n \) in Proposition 4.4.

\( (b) \) Suppose \( B(\mathbb{C}) \) is strongly Euler-homogeneous, Saito-holonomic, and tame. Because \( (S - A) \cdot \mathcal{D}_{X,F}[S] = (s_1 - s_2, \ldots, s_r - a - d) \cdot \mathcal{D}_{X,F}[S] \), by Proposition 4.3(3) there is a commutative square

\[
\begin{array}{ccc}
\mathcal{D}_{X,F}[S]^{P(S-1)FS} & \simeq & \mathcal{D}_{X,F}[S]^{P(S)FS} \\
(S-A)\mathcal{D}_{X,F}[S]^{FS} & \downarrow & (S-A)\mathcal{D}_{X,F}[S]^{FS} \\
\n\end{array}
\]

So \( \nabla_A \) is an injective (surjective) if and only if \( \nabla_a \) is injective (surjective). If we could prove the equivalence of the conditions in Proposition 4.2 then the inclusion, via the diagonal embedding \( V(B_{F,t}) \hookrightarrow V(B_{F,t}) \cap \{s_1 = s_2 = \cdots = s_r\} \) given in Proposition 4.3 would be surjective.

\( \textbf{Example 4.4.} \) Let \( f = x(2x^2 + yz) \) and \( F = (x, 2x^2 + yz) \). This is strongly Euler-homogeneous, Saito-holonomic (cf. Examples 3.3, 3.14), and tame \( (n \leq 3) \).

Using Singular and Macaulay2 we compute \( V(B_{F,0}) = \{s_1 + 1, (s_2 + 1) \prod_{k=3}^n (s_1 + 2s_2 + k) \} \) and \( V(B_{F,0}) = \{s_1 + 1, (s_2 + 1) \prod_{k=3}^n (s_1 + 2s_2 + k) \} \). In this case, the diagonal embedding \( V(B_{F,0}) \hookrightarrow V(B_{F,0}) \cap \{s_1 = s_2\} \) is surjective and, see Remark 4.3, \( \nabla_{-k+1,-k+1} \) is neither surjective nor injective for \( k = 3, 4, 5 \).
The rest of this section is devoted to proving that under the nice hypotheses of the previous section and in the language of Proposition 1.2 that (b) implies (c). Our proof makes use of a Koszul resolution over the central elements $S - A$.

**Conjunction 4.5.** A resolution is a (co)-complex with a unique (co)homology module at its end. An acyclic (co)-complex has no (co)homology. Given a (co)-complex $(C\bullet)\ C_\bullet$ resolving $A$, the augmented (co)-complex $(C\bullet \to A)\ C_\bullet \to A$ is acyclic.

**Definition 4.6.** For a (not necessarily commutative) ring $R$ and a sequence of central $R$-elements $a = a_1, \ldots, a_k$ let $K^\bullet(a)$ be the Koszul co-complex induced by the elements $a$, cf. Section 6 in [15]. For a left $R$-module $M$, let $K^\bullet(a; M) = K^\bullet(a) \otimes M$ be the Koszul co-complex on $M$ induced by $a$. We index $K^\bullet(a)$ so that the right most object is $K_0^0(a)$.

Let $v_1, \ldots, v_k$ be positive integers. If $R$ is commutative and if $K^\bullet(a; M)$ is a resolution, we know $K^\bullet(a_1^{v_1}, \ldots, a_k^{v_k}; M)$ is a resolution, cf. Exercise 6.16 in [15].

We sketch a proof for general $R$ and central $a$.

Proceeding by induction, we must show that $K^\bullet(x^j; M) \otimes K^\bullet(y_2, \ldots, y_k; M)$ is a resolution. Note that $K^\bullet(y_2, \ldots, y_k; M)$ is a resolution. By the spectral sequence of a double co-complex, $K^\bullet(x^j; M) \otimes K^\bullet(y_2, \ldots, y_k; M) = K^\bullet(x^j; M/(y_2, \ldots, y_k)M)$.

Now because $K^\bullet(x, y_2, \ldots, y_k; M)$ is a resolution, so is $K^\bullet(x; M/(y_2, \ldots, y_k)M)$. In other words, multiplication by $x$ on $M/(y_2, \ldots, y_k)M$ is injective. Thus multiplication by $x^j$ is injective and $K^\bullet(x^j; M/(y_2, \ldots, y_k)M)$ is a resolution. This justifies the following:

**Proposition 4.7.** Let $R$ be a, possibly non-commutative, ring, $M$ a $R$-module, $c_1, \ldots, c_r$ central elements of $R$, and $v_1, \ldots, v_r \in \mathbb{Z}$. If $K^\bullet(c_1, \ldots, c_r; M)$ is a resolution, then $K^\bullet(c_1^{v_1}, \ldots, c_r^{v_r}; M)$ is a resolution.

The following lemma is immediate after considering $K^\bullet(c_1, \ldots, c_r; M)$ at the $-1$ slot.

**Lemma 4.8.** Let $R$ be a, possibly noncommutative, ring, $M$ a left $R$-module, $m_i \in M$, and $c_1, \ldots, c_r$ central elements of $R$. Assume $H^{-1}(K(c_1, \ldots, c_r; M)) = 0$. If $c_im_i \in (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_r)M$, then $m_i \in (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_r)M$.

Now return to $\text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S]F^S)$. Under the nice hypothesis of the previous section, it turns out $\text{gr}(S_\pm)(s_1), \ldots, \text{gr}(S_\pm)(s_r)$ act like a regular sequence. More precisely:

**Proposition 4.9.** Let $f = f_1 \cdots f_r$ and let $F = (f_1, \ldots, f_r)$. Suppose that for $\xi \in X$ the following hold:

- $f$ has the strong Euler-homogeneity $E_\xi$ at $\xi$;
- $\text{L}_{\mathcal{F},\xi} \subseteq \text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S])$ is Cohen–Macaulay of dimension $n + r$;
- $\text{L}_{\mathcal{F},\xi} \subseteq \text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S])$ is Cohen–Macaulay of dimension $n$.

Then $K(S; \text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S])/\text{L}_{\mathcal{F},\xi})$ is co-complex of $\text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S])$-modules resolving $\text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S])/(\text{L}_{\mathcal{F},\xi} + \text{gr}(S_\pm)(\mathcal{D}_{X,\xi})) \cong \text{gr}(S_\pm)(\mathcal{D}_{X,\xi})/(\text{L}_{\mathcal{F},\xi} + \text{gr}(S_\pm)(\mathcal{D}_{X,\xi}))$.

**Proof.** The last isomorphism is immediate from the definition of $E_\xi$ and the construction of $L_{\mathcal{F},\xi}$ and $L_{\mathcal{F},\xi}$, see Definition 3.16 and the preceding comments.

Multiplying $\text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S])$ by $s_k$ increases the degree of an element by one. So after doing the appropriate degree shifts, we may view $K^\bullet(S; \text{gr}(S_\pm)(\mathcal{D}_{X,\xi}[S]))$ as
a sequence of graded modules with degree preserving maps. By Proposition 1.5.15 (c) of [6], exactness of such a sequence is a graded local property. A \((0, 1, 1)\)-graded maximal ideal \(m^* \subsetgr(0,1,1)(\mathcal{O}_{X,[S]})\) must contain all the positive degree terms; in particular, \(S \in m^*\). Since \(gr(0,1,1)(\mathcal{O}_{X,[S]})\) modulo the irrelevant ideal is the local ring \(\mathcal{O}_{X,[S]}\) there is only one graded maximal ideal \(m^*\): the ideal generated by the irrelevant ideal and \(m_r \subset \mathcal{O}_{X,[S]}\). Clearly \(\widetilde{L}_{F,r} \subset m^*\) and since the dimension of a graded ring can be computed by only looking at the height at graded maximal ideals, see Corollary 13.7 [13]. \((gr(0,1,1)(\mathcal{O}_{X,[S]})/\widetilde{L}_{F,r})\) is Cohen–Macaulay of dimension \(n+r\). Similar considerations show that \(gr(0,1,1)(\mathcal{O}_{X,[S]})/(\widetilde{L}_{F,r} + gr(0,1,1)(\mathcal{O}_{X,[S]}))\) is Cohen–Macaulay of dimension \(n\).

So localize \(K^*(S;gr(0,1,1)(\mathcal{O}_{X,[S]})/\widetilde{L}_{F,r})\) at \(m^*\). On \((gr(0,1,1)(\mathcal{O}_{X,[S]})/\widetilde{L}_{F,r})\) going modulo \(S\) yields \((gr(0,1,1)(\mathcal{O}_{X,[S]})/(\widetilde{L}_{F,r} + (E))_m^*\). Note that the dimension of \((gr(0,1,1)(\mathcal{O}_{X,[S]})/\widetilde{L}_{F,r})_m^*\) minus the dimension of \((gr(0,1,1)(\mathcal{O}_{X,[S]})/(\widetilde{L}_{F,r} + (E))_m^*\) is the length of the sequence \(S\). By Theorem 2.1.2 of [6], \(S\) is regular on \((gr(0,1,1)(\mathcal{O}_{X,[S]})/\widetilde{L}_{F,r})_m^*\) and so our localized Koszul co-complex is a resolution.

For \(a_1, \ldots, a_r \in \mathbb{C}\), label \(S - A = s_1 - a_1, \ldots, s_r - a_r \in \mathcal{O}_{X,[S]}\). Being central elements, \(S - A\) yields the Koszul co-complex \(K^*(S - A; \mathcal{O}_{X,[S]}F^S)\) of \(\mathcal{O}_{X,[S]}\)-modules. Its terminal cohomology module is \(\mathcal{O}_{X,[S]}F^S/(S - A)\mathcal{O}_{X,[S]}F^S\). We show that under our standard hypotheses on \(f\), i.e. strongly Euler-homogeneous, Saito-holonomic, and tame, that \(s_1 - a_1, \ldots, s_r - a_r\) behaves like a regular sequence for any \(a_i \in \mathbb{C}\).

**Proposition 4.10.** Suppose \(f = f_1 \cdots f_r\) is strongly Euler-homogeneous, Saito-holonomic, and tame and let \(F = (f_1, \ldots, f_r)\). Then \(K^*(S - A; \mathcal{O}_{X,[S]}F^S)\) resolves \(\mathcal{O}_{X,[S]}F^S/(S - A)\mathcal{O}_{X,[S]}F^S\).

**Proof.** Filter \(K^*(S - A; \mathcal{O}_{X,[S]}F^S)\) by

\[ G^pK^*(S - A; \mathcal{O}_{X,[S]}F^S) \simeq K^*(S;gr(0,1,1)(\mathcal{O}_{X,[S]}F^S))^{(1)} \]

where \(F_{(0,1,1)}\) is the total \((0,1,1)\)-weight filtration on \(\mathcal{O}_{X,[S]}\). This induces a filtration on \(\mathcal{O}_{X,[S]}F^S \simeq \mathcal{O}_{X,[S]}/ann_{\mathcal{O}_{X,[S]}F^S}\) since \(ann_{\mathcal{O}_{X,[S]}F^S} = \mathcal{O}_{X,[S]}\). \(\theta_{F,i}\) and \(\theta_{F,r}\) are all \((0,1,1)\)-homogeneous, cf. Theorem 3.31. Moreover, since \(s_k - a_k\) is of weight one, this is a chain co-complex filtration.

It is easy to verify that

\[ gr(G^pK^*(S - A; \mathcal{O}_{X,[S]}F^S)) \simeq K^*(S;gr(0,1,1)(\mathcal{O}_{X,[S]}F^S))/L_{F,r} \]

Replace \(K^*(S - A; \mathcal{O}_{X,[S]}F^S)\) by its augmented co-complex \(K^*(S - A; \mathcal{O}_{X,[S]}F^S) \rightarrow \mathcal{O}_{X,[S]}F^S/(S - A)\mathcal{O}_{X,[S]}F^S\). The associated graded co-complex induced by \(G\) is isomorphic to the augmented co-complex of \(K^*(S;gr(0,1,1)(\mathcal{O}_{X,[S]}F^S))/L_{F,r} \rightarrow gr(0,1,1)(\mathcal{O}_{X,[S]}F^S)/L_{F,r} + (S)\). Theorem 5.25 and Corollary 3.19 of [29] shows that the hypotheses of Proposition 4.13 are met; hence \(K^*(S;gr(0,1,1)(\mathcal{O}_{X,[S]}F^S)/L_{F,r})\) is a resolution. So the first page of the spectral sequence associated to the filtered co-complex of the augmented co-complex \(K^*(S - A; \mathcal{O}_{X,[S]}F^S) \rightarrow \mathcal{O}_{X,[S]}F^S/(S - A)\mathcal{O}_{X,[S]}F^S\) vanishes. Thus \(K^*(S - A; \mathcal{O}_{X,[S]}F^S) \rightarrow \mathcal{O}_{X,[S]}F^S/(S - A)\mathcal{O}_{X,[S]}F^S\) is acyclic; that is, \(K^*(S - A; \mathcal{O}_{X,[S]}F^S)\) resolves \(\mathcal{O}_{X,[S]}F^S/(S - A)\mathcal{O}_{X,[S]}F^S\). □
Finally we can prove the section’s main theorem:

**Theorem 4.11.** Let \( f = f_1 \cdots f_r \) be strongly Euler-homogeneous, Saito-holonomic, and tame and let \( F = (f_1, \ldots, f_r) \). If \( \nabla_A \) is injective, then it is surjective.

**Proof.** For this proof, and this proof alone, write \( \tilde{s}_i = s_i - (a_i - 1) \).

The Plan: If there is some multivariate Bernstein–Sato polynomial \( B(S) \) that does not vanish at \( (a_1 - 1, \ldots, a_r - 1) \), then the claim follows by Proposition 4.12. So pick a multivariate Bernstein–Sato polynomial \( B(S) = \sum A_k \tilde{s}_k, A_k \in \mathbb{C}[S] \).

The idea is to successively “remove” each \( s_k \) factor from each \( A_k \). In doing so, we will produce a finite sequence of polynomials \( B_0, B_1, \ldots \) satisfying the technical condition (4.1) introduced in Step 1, starting with our multivariate Bernstein–Sato polynomial, such that each polynomial uses fewer variables than its predecessor. The terminal polynomial will demonstrate that the cokernel of \( \nabla_A \) vanishes.

The inductive construction of these polynomials is not hard but technical. Before doing it we prove three claims. The first is that a particular cohomology module of the Koszul co-complex of \( \tilde{s}_1, \ldots, \tilde{s}_r \) on \( \mathcal{D}_{X,S} [S] F^S \) vanishes. We use this to “remove” the \( \tilde{s}_k \) factors. The second and third claims are the technical details comprising the inductive algorithm used to produce these polynomials.

**Claim 1:** For all positive integers \( v_1, \ldots, v_r \),

\[
\begin{equation}
H^{-1} \left( K^\bullet \left( \tilde{s}_1^{v_1}, \ldots, \tilde{s}_r^{v_r}, \frac{\mathcal{D}_{X,S}[S] F^S}{\mathcal{D}_{X,S}[S] F^{S+1}} \right) \right) = 0.
\end{equation}
\]

**Proof of Claim 1:** Consider the commutative diagram of \( \mathcal{D}_{X,S} \)-modules:
\( K^*(S - (A - 1)): \mathcal{D}_{X, f}[S]F^S \to (S - (A - 1))\mathcal{D}_{X, f}[S]F^S + \mathcal{D}_{X, f}[S]F^{S+1} \)

is acyclic. Claim 1 follows by Proposition 4.3.

**Claim 2:** Write \(F^S\) for the image of \(F^S\) in \(\mathcal{D}_{X, f}[S]F^S\). Suppose there exists \(P(S) \in \mathbb{C}[S], 1 \leq j < r, \) positive integers \(n_{j+1}, \ldots, n_r, \) and an integer \(m \geq \max\{n_{j+1}, \ldots, n_r\}\) such that

\[
\left( \prod_{j+1 \leq k \leq r} \tilde{s}_k \right) P(S) \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} - m, \ldots, \tilde{s}_r) \mathcal{D}_{X, f}[S]F^S.
\]

Then for \(m' = \min\{m - n_{j+1}, \ldots, m - n_r\}\) we have

\[
P(S) \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} - m', \ldots, \tilde{s}_r) \mathcal{D}_{X, f}[S]F^S.
\]

**Proof of Claim 2:** The idea is to use Claim 1 and Lemma 4.8 to “remove” each \(\tilde{s}_k n_k\) factor one at a time. We first “remove” the \(\tilde{s}_{j+1} n_{j+1}\) factor.

By hypothesis, there exists \(Q_{j+1} \in \mathcal{D}_{X, f}[S]\) such that

\[
\left( \prod_{j+1 \leq k \leq r} \tilde{s}_k n_k \right) P(S) \cdot F^S - \tilde{s}_{j+1} m Q_{j+1} \cdot F^S
\]

\[
= \tilde{s}_{j+1}^{-n_{j+1}} \left( \left( \prod_{j+2 \leq k \leq r} \tilde{s}_k n_k \right) P(S) \cdot F^S - \tilde{s}_{j+1}^{-m-n_{j+1}} Q_{j+1} \cdot F^S \right)
\]

\[
\in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} - m, \ldots, \tilde{s}_r) \mathcal{D}_{X, f}[S]F^S.
\]

By Claim 1, \(H^{-1}(K(\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} - n_{j+1}, \tilde{s}_{j+2} - m, \ldots, \tilde{s}_r m, \mathcal{D}_{X, f}[S]F^S))\) vanishes. So Lemma 4.3 implies

\[
\left( \prod_{j+2 \leq k \leq r} \tilde{s}_k n_k \right) P(S) \cdot F^S - \tilde{s}_{j+1}^{-m-n_{j+1}} Q_{j+1} \cdot F^S
\]

\[
\in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+2} - m, \ldots, \tilde{s}_r m) \mathcal{D}_{X, f}[S]F^S.
\]

Rearrange to see

\[
\left( \prod_{j+2 \leq k \leq r} \tilde{s}_k n_k \right) P(S) \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} m - n_{j+1}, \tilde{s}_{j+2} m, \ldots, \tilde{s}_r m) \mathcal{D}_{X, f}[S]F^S.
\]

Repeat this process on each remaining factor \(\tilde{s}_k n_k, j + 2 \leq k \leq r\) one at a time to conclude

\[
P(S) \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} m - n_{j+1}, \tilde{s}_{j+2} m - n_{j+2}, \ldots, \tilde{s}_r m - n_r) \mathcal{D}_{X, f}[S]F^S.
\]

**Claim 3:** Suppose \(B_j \in \mathbb{C}[s_{j+1}, \ldots, s_r]\), where \(j < r, \) with \(B_j \in \mathbb{C}[s_{j+1}, \ldots, s_r]\). \((\tilde{s}_{j+1}, \ldots, \tilde{s}_r)\) but \(B_j \notin \mathbb{C}[s_{j+1}, \ldots, \tilde{s}_r] \cdot (\tilde{s}_k)\) for all \(j + 1 \leq k \leq r\). Furthermore, assume that for \(m \geq \max\{n_{j+1}, \ldots, n_r\}\) we have

\[
B_j \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1} m, \ldots, \tilde{s}_r m) \mathcal{D}_{X, f}[S]F^S.
\]
Then, relabeling the $s_k$ if necessary, there exists $B_i \in \mathbb{C}[s_{i+1}, \ldots, s_r]$, where $j < i < r$, $B_i \notin \mathbb{C}[s_{i+1}, \ldots, s_r] \cdot (\tilde{s}_k)$ for $i + 1 \leq k \leq r$, so that for $m' = \min\{m - n_{j+1}, \ldots, m - n_r\}$ we have

$$B_i \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_{i-1}, \tilde{s}_{i+1}^{m'}, \ldots, \tilde{s}_r^{m'}) \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+1}}.$$

**Proof of Claim 3:** Note that the hypotheses imply $j < r - 1$ so the promised choice of $i$ is possible. Since $B_j \notin \mathbb{C}[s_{j+1}, \ldots, s_r] \cdot (\tilde{s}_k)$ for all $j + 1 \leq k \leq r$, there exists a largest $0 \neq I = \{s_{i_1}, \ldots, s_{i_l}\} \subset \{j + 1, \ldots, r\}$ such that $B_j \notin \mathbb{C}[s_{j+1}, \ldots, s_r] \cdot (\tilde{s}_{i_1}, \ldots, \tilde{s}_{i_l})$. Relabel so that $I = \{j + 1, \ldots, i\}$. This means there exist positive integers $n_k$, polynomials $A_i \in \mathbb{C}[S]$, and a polynomial $B_i \in \mathbb{C}[s_{i+1}, \ldots, s_r]$ such that

$$B_j = \left( \prod_{i+1 \leq k \leq r} \tilde{s}_k^{n_k} \right) B_i + \sum_{1 \leq \ell \leq i} \tilde{s}_\ell A_\ell.$$

We may make each $n_k$ large enough so as to assume $B_i \notin \mathbb{C}[s_{i+1}, \ldots, s_r] \cdot (\tilde{s}_k)$ for any $i + 1 \leq k \leq r$. Therefore

$$\left( \prod_{i+1 \leq k \leq r} \tilde{s}_k^{n_k} \right) B_i \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_{i-1}, \tilde{s}_{i+1}^{m'}, \ldots, \tilde{s}_r^{m'}) \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+1}}.$$

Then Claim 3 follows from Claim 2.

**Proof of Theorem.**

**Step 1:** We will inductively construct a sequence of polynomials $B_{i_1}, B_{i_2}, \ldots$, such that (after potentially relabelling the $s_k$) the following hold: 0 $\leq i_t < r$ for each $i_t$; $i_t < i_{t+1}$; $B_{i_t} \in \mathbb{C}[s_{i_t+1}, \ldots, s_r]$; for $m_{i_t}$ arbitrarily large

$$B_{i_t} \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_{i_t}, \tilde{s}_{i_t+1}^{m_{i_t}}, \ldots, \tilde{s}_r^{m_{i_t}}) \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+1}}.$$

We terminate the induction once we produce a $B_i$ such that, in addition to the above properties, $B_i \notin \mathbb{C}[s_{i+1}, \ldots, s_r] \cdot (\tilde{s}_{i+1}, \ldots, \tilde{s}_r)$.

**Base Case:** Take a multivariate Bernstein–Sato polynomial $B(S) \in B_{F,F}$. If $B(S) \notin \mathbb{C}[s_1, \ldots, s_r] \cdot (\tilde{s}_1, \tilde{s}_r)$ then we are done: $B(S) = B_0$ works. (Recall $B(S) \cdot F^S \in \mathcal{D}_{X,F}[S]F^{S+1}$.) Otherwise find the largest $J \subset \{r\}$ such that $B(S) \notin \mathbb{C}[S] \cdot (\tilde{s}_{j_1}, \ldots, \tilde{s}_{j_{|J|}})$. Re-label to assume $J = \{1, \ldots, j\}, j < r$. (We allow $J = \emptyset$, in which case $j = 0$.) This means we can write $B(S)$ as

$$B(S) = \left( \prod_{j+1 \leq k \leq r} \tilde{s}_k^{n_k} \right) B_j + \sum_{1 \leq \ell \leq j} \tilde{s}_\ell A_\ell$$

where $B_j \in \mathbb{C}[s_{j+1}, \ldots, s_r]$ and each $n_k$ a positive integer chosen large enough so that $B_j \notin \mathbb{C}[s_{j+1}, \ldots, s_r] \cdot (\tilde{s}_k)$, for $j + 1 \leq k \leq r$, because $B(S)$ is a multivariate Bernstein–Sato polynomial, $B(S) \cdot F^S \in \mathcal{D}_{X,F}[S]F^{S+1}$. Therefore,

$$\left( \prod_{j+1 \leq k \leq r} \tilde{s}_k^{n_k} \right) B_j \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j) \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+1}}.$$

Now (4.2) trivially implies that for all $m \geq 0$

$$\left( \prod_{j+1 \leq k \leq r} \tilde{s}_k^{n_k} \right) B_j \cdot F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1}^{m}, \ldots, \tilde{s}_r^{m}) \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+1}}.$$
In particular, the above holds for \( m \) arbitrarily large. By Claim 2, there exists \( m_j \) arbitrarily large such that

\[
B_j \bullet F^S \in (\tilde{s}_1, \ldots, \tilde{s}_j, \tilde{s}_{j+1}^{-m_j}, \ldots, \tilde{s}_r) \frac{\mathcal{D}_{X,S}[S]F^S}{\mathcal{D}_{X,S}[S]F^{S+1}}.
\]

Then \( B_j \) is the first element in our sequence of polynomials.

**Inductive Step:** Suppose \( B_j \in \mathbb{C}[s_{j+1}, \ldots, s_r] \) has already been defined. If the algorithm has not terminated, \( j < r \) and \( B_j \notin \mathbb{C}[s_{j+1}, \ldots, s_r] \cdot (\tilde{s}_r) \) for all \( j + 1 \leq k \leq r \). Then use Claim 3 to define \( B_i \), where \( j < i < r \). Note that if \( j = r - 1 \) then \( B_{r-1} \notin \mathbb{C}[s_r] \cdot (\tilde{s}_r) \) and so the algorithm terminates at \( B_{r-1} \).

**Step 2:** Use the terminal polynomial \( B_i \in \mathbb{C}[s_{i+1}, \ldots, s_r] \), \( i < r \), produced by Step 1. This means \( B_i \notin \mathbb{C}[s_{i+1}, \ldots, s_r] \cdot (\tilde{s}_{i+1}, \ldots, \tilde{s}_r) \) and easily implies

\[
B_i \bullet F^S \in (\tilde{s}_1, \ldots, \tilde{s}_r) \frac{\mathcal{D}_{X_S}[S]F^S}{\mathcal{D}_{X,S}[S]F^{S+1}}.
\]

On one hand, since \( B_i \) does not vanish at \( (a_{i+1} - 1, \ldots, a_r - 1) \), \( B_i \bullet F^S \) and \( F^S \) generate the same submodule of \( \mathcal{D}_{X_S}[S]F^S \); on the other hand, \( 0 = B_i \bullet F^S \in (s_{i+1}, \ldots, s_r) \mathcal{D}_{X,S}[S]F^S + \mathcal{D}_{X,S}[S]F^{S+1} \). Since \( \mathcal{D}_{X,S}[S]F^S + \mathcal{D}_{X,S}[S]F^{S+1} \) is generated by \( F^S \), this shows \( 0 = B_i \bullet F^S \in (s_{i+1}, \ldots, s_r) \mathcal{D}_{X,S}[S]F^S + \mathcal{D}_{X,S}[S]F^{S+1} \) for all \( i \), that is, the cokernel of \( \nabla_A \) vanishes.

In light of Proposition 3.10 and Theorem 4.11 to prove, under our working hypotheses, the three conditions of Proposition 3.10 are equivalent, it suffices to show that if \( A - 1 \in \mathbb{V}(B_{F,I}) \) then \( \nabla_A \) is not surjective. We show that this holds for central, tame hyperplane arrangements if we assume \( A - 1 \) lies in a certain hyperplane.

**Proposition 4.12.** Suppose \( f = f_1 \cdots f_r \) is a central, essential, indecomposable, and tame hyperplane arrangement, where each \( f_k \) is of degree \( d_k \) and the \( f_k \) are not necessarily reduced. Let \( F = (f_1, \ldots, f_r) \). If \( A - 1 \notin \{ d_1 s_1 + \cdots + d_r s_r + n = 0 \} \), then \( A - 1 \in \mathbb{V}(B_{F,0}) \) and \( \nabla_A \) is neither surjective nor injective.

**Proof.** An easy extension of the argument in Theorem 3.38 shows that

\[
(3.3) \quad \text{ann}_{\mathcal{D}_{X,0}[S]} F^S + \mathcal{D}_{X,0}[S] \cdot f \subseteq \mathcal{D}_{X,0}[S] \cdot m_0 + \mathcal{D}_{X,0}[S] \cdot (-n - \sum d_k s_k).
\]

Continue arguing as in Theorem 3.38 to show that \( A - 1 \in \mathbb{V}(B_{F,0}) \). Now \( \nabla_A \) is surjective precisely when

\[
\mathcal{D}_{X,0}[S] = \text{ann}_{\mathcal{D}_{X,0}[S]} F^S + \mathcal{D}_{X,0}[S] \cdot f + \sum \mathcal{D}_{X,0}[S] \cdot (s_k - (a_k - 1)).
\]

By (4.3), if \( \nabla_A \) is surjective,

\[
\mathcal{D}_{X,0}[S] \subseteq \mathcal{D}_{X,0}[S] \cdot m_0 + \mathcal{D}_{X,0}[S] \cdot (-n - \sum d_k s_k) + \sum \mathcal{D}_{X,0}[S] \cdot (s_k - (a_k - 1)).
\]

After evaluating each \( s_k \) at \( a_k - 1 \), we deduce \( \mathcal{D}_{X,0} \subseteq \mathcal{D}_{X,0} \cdot m_0 \). This is impossible, therefore \( \nabla_A \) is not surjective.

That \( \nabla_A \) is not injective is a consequence of Theorem 4.11. \( \Box \)
5. Free Divisors, Lie–Rinehart Algebras, and $\nabla_A$

In Definition 3.11 we defined tame divisors. A stronger condition on the divisor is freeness:

**Definition 5.1.** A divisor $Y$ is *free* if it locally everywhere admits a defining equation $f$ such that $\text{Der}_{X,f}(-\log f)$ is a free $\mathcal{O}_{X,f}$-module.

Note that freeness implies tameness because $\Omega_{X,f}(\log f)$ and $\text{Der}_{X,f}(-\log f)$ are dual to each other and if $\Omega_{X,f}(\log f)$ is free, $\Omega^p_{X,f}(\log f) = \bigwedge^p \Omega_{X,f}(\log f)$ (see 1.7 and 1.8 of [28]).

Throughout this section we upgrade our working hypotheses of strongly Euler-homogeneous, Saito-holonomic, and tame to reduced, strongly Euler-homogeneous, Saito-holonomic, and free. The goal is to investigate the surjectivity of the map $\nabla_A$. Let’s give a road map. First we compute Ext modules of $\mathcal{D}_{X,f}[S]/F^S/(S - A)\mathcal{D}_{X,f}[S]F^S$ using [24] and the rich theory of Lie–Rinehart algebras. Lifting a surjective $\nabla_A$ to these Ext-modules will produce an injective map. This injective map acts like $\nabla_{-A}$. By Theorem 4.11 $\nabla_{-A}$ is surjective. Using duality again will show that $\nabla_A$ is injective.

5.1. Lie–Rinehart Algebras and the Spencer Co-Complex $\text{Sp}^*$.

**Definition 5.2.** (Compare with [10], [26] and the appendix of [22]) Fix a homomorphism of commutative rings $k \rightarrow A$. A *Lie–Rinehart algebra* $L$ over $(k, A)$ is a $A$-module $L$ with *anchor map* $\rho : L \rightarrow \text{Der}_k(A)$ that is $A$-linear, a $k$-Lie algebra map, and satisfies, for all $\lambda, \lambda' \in L$, $a \in A$,

$$[\lambda, a\lambda'] = a[\lambda, \lambda'] + \rho(\lambda)(a)\lambda'.$$

We will usually drop $\rho$ and replace $\rho(\lambda)(a)$ with $\lambda(a)$. A morphism $F : L \rightarrow L'$ of Lie–Rinehart algebras over $(k, A)$ is a $A$-linear map that is a morphism of Lie-algebras satisfying $\lambda(a) = F(\lambda)(a)$.

**Example 5.3.** (a) $\text{Der}_k(A)$ is a Lie–Rinehart algebra over $(k, A)$ with the identity as the anchor map.

(b) Any $A$-submodule of $\text{Der}_k(A)$ that is also a $k$-Lie algebra is a Lie–Rinehart algebra over $(k, A)$, with anchor map induced by the inclusion into $\text{Der}_k(A)$.

In particular $\text{Der}_{X,f}(-\log(f))$ is a Lie–Rinehart algebra over $(\mathbb{C}, \mathcal{O}_{X,f})$.

(c) If $L$ is a Lie–Rinehart algebra over $(k, A)$, then $L \oplus A$ is a Lie–Rinehart algebra over $(k, A)$ with anchor map induced by the projection $L \oplus A \rightarrow L, (\lambda, a) \mapsto \lambda$.

So $\text{Der}_{X,f} \oplus \mathcal{O}_{X,f}^\prime$ and $\text{Der}_{X,f}(-\log(f)) \oplus \mathcal{O}_{X,f}^\prime$ are Lie–Rinehart algebras over $(\mathbb{C}, \mathcal{O}_{X,f})$.

Any Lie algebra over $k$ has a universal algebra that is constructed in an entirely similar way to how the symmetric algebra is defined as a quotient of the tensor algebra. Similarly, a Lie–Rinehart algebra $L$ over $(k, A)$ has a universal algebra $U(L)$ that is the symmetric algebra of $A \oplus L$ modulo the appropriate relations. See Section 2 of [20] for details.

**Definition 5.4.** Let $L$ be a Lie–Rinehart algebra over $(k, A)$ with $k \rightarrow A$. Suppose $R$ is a ring (not necessarily a Lie–Rinehart algebra) and $A \rightarrow R$ a ring homomorphism that makes $R$ central over $k$, i.e. images of elements of $k$ are central elements in $R$. Then a $k$-linear map $g : L \rightarrow R$ is *admissible* if:

(a) $g(a\lambda) = ag(\lambda), \text{ for } a \in A, \lambda \in L$ ($g$ is a morphism of $A$-modules);
(b) \(g([\lambda, \lambda']) = [g(\lambda), g(\lambda')]\), for \(\lambda, \lambda' \in L\) (\(g\) is a morphism of Lie-algebras);
(c) \(g(\lambda)a - ag(\lambda) = \lambda(a)1_R\) for \(\lambda \in L\), \(a \in A\).

The following theorem will be our definition of the universal algebra \(U(L)\):

**Theorem 5.5.** (cf. [26]) For any Lie–Rinehart algebra \(L\) over \((k, A)\) there exists a ring \(U(L)\), a ring homomorphism \(A \to U(L)\) making \(U(L)\) central over \(k\), and an admissible map \(\theta : L \to U(L)\) that is universal in the following sense: for any ring \(R\) with a ring homomorphism \(A \to R\) making \(R\) central over \(k\), and any admissible map \(g : L \to R\), there is a unique ring homomorphism \(h : U(L) \to R\) such that \(h \circ \theta = g\). The natural map \(\theta : L \to U(L)\) induces a filtration on \(U(L)\) given by the powers of images of \(\theta\).

We omit the proof of the following proposition. It uses the (not provided) explicit construction of \(U(L)\) and standard universal object arguments.

**Proposition 5.6.** Given a Lie–Rinehart algebra \(L\) over \((k, A)\), consider the direct sum \(L \oplus A\). This is a Lie–Rinehart algebra over \((k, A)\) with anchor map induced by projection: \(L \oplus k \to L \to \text{Der}_k(A)\). Then \(U(L \oplus A) \simeq U(L)[s]\). Moreover, the natural filtration on \(U(L \oplus A)\) corresponds to a “total order filtration” on \(U(L)[S]\), i.e. a filtration where \(s\) has weight one.

**Example 5.7.** (a) The universal Lie–Rinehart algebra of \(\text{Der}_{X,F}\) over \((C, \mathcal{O}_{X,F})\) is \(\mathcal{O}_{X,F}\). The natural filtration is the order filtration.
(b) By repeated application of Proposition 5.6 the universal Lie–Rinehart algebra of \(\text{Der}_{X,F} \oplus \mathcal{O}'_{X,F}\) over \((C, \mathcal{O}_{X,F})\) is \(\mathcal{O}_{X,F}[s_1, \ldots, s_r]\). The natural filtration is the total order filtration \(F_{(0,1,1)}\).
(c) For \(F = (f_1, \ldots, f_r)\) a decomposition of \(f = f_1 \cdots f_r\), the annihilating derivations \(\theta_{F,s}\) constitute a Lie–Rinehart algebra over \((C, \mathcal{O}_{X,F})\). The \(\mathcal{O}_{X,F}\)-map \(\psi_F : \text{Der}_{X,F}(- \log f) \to \theta_{F,s}\) is an isomorphism of Lie–Rinehart algebras over \((C, \mathcal{O}_{X,F})\). So there is a containment of Lie–Rinehart algebras over \((C, \mathcal{O}_{X,F})\):
\[\theta_{F,s} \subseteq \text{Der}_{X,F} \oplus \mathcal{O}'_{X,F}\]
(d) The universal algebra of the Lie–Rinehart algebra \(\text{Der}_{X,F}[s]\) over \((C[S], \mathcal{O}_{X,F}[S])\) is \(\mathcal{O}_{X,F}[S]\). Note that \(s_k\) is contained in the \(0^\text{th}\) filtered part and the filtration is induced by the order filtration.

We care about the formalism of Lie–Rinehart algebras because we want to construct complexes of the universal algebras. In particular, given two Lie–Rinehart algebras \(L \subseteq L'\) the following gives a complex of \(U(L')\)-modules.

**Definition 5.8.** (Compare with 1.1.8 of [10]) Let \(L\) and \(L'\) be Lie–Rinehart algebras over \((k, A)\). The Cartan–Eilenberg–Chevalley–Rinehart–Spencer co-complex associated to \(L \subseteq L'\) and the left \(U(L)\)-module \(E\) is the co-complex \(\text{Sp}^\bullet_{L,L'}(E)\).
Here
\[\text{Sp}^{-r}(L, L') := U(L') \otimes_A \bigwedge^r L \otimes_A E\]
and the \((U(L')\)-linear differential
\[d^{-r} : \text{Sp}^{-r}_{L,L'}(E) \to \text{Sp}^{-(r-1)}_{L,L'}(E)\]
is given by

\begin{equation}
\label{eq:5.1}
\begin{split}
d^{-r}(P \otimes \lambda_1 \wedge \cdots \wedge \lambda_r \otimes e) &= \sum_{i=1}^{r} (-1)^{i-1} P \lambda_i \otimes \hat{\lambda}_i \otimes e - \sum_{i=1}^{r} (-1)^{i-1} P \otimes \hat{\lambda}_i \otimes \lambda_i e \\
&+ \sum_{1 \leq i < j \leq r} (-1)^{i+j} P \otimes [\lambda_i, \lambda_j] \wedge \hat{\lambda}_{i,j} \otimes e.
\end{split}
\end{equation}

(Here \(\hat{\lambda}_{i,j}\) is the wedge of the of all the \(\lambda_i\)’s except \(\lambda_i\) and \(\lambda_j\).) There is a natural augmentation map

\[ U(L') \otimes_A E \to U(L') \otimes_{U(L)} E. \]

When \(E = A\), write \(\text{Sp}^\bullet_{L,L'}(A)\) as \(\text{Sp}^\bullet_{L,L'}\).

In general, the cohomology of \(\text{Sp}^\bullet_{L,L'}(E)\) is mysterious. Using the natural filtration on \(U(L')\), there is a natural filtration on the co-complex. We can compute the co-homology using the spectral sequence associated to this filtration. By a Poincaré-Birkhoff-Whitt theorem for these universal algebras, the associated graded of \(U(L')\) is \(\text{Sym}_A(L')\). So the 0th page of this spectral sequence can be described in terms of a co-complex involving the symmetric algebra. When \(L\) has a basis whose symbols in \(\text{Sym}_A(L')\) constitute a regular sequence and when \(E\) is a free \(A\)-module, the 0th page of this spectral sequence looks like a Koszul co-complex of a regular sequence. Arguing in this fashion gives the following:

**Proposition 5.9.** (Proposition 1.5.3 in [11]) Suppose \(L \subseteq L'\) are Lie–Rinehart algebras over \((k, A)\) and \(E\) a left \(U(L)\)-module free over \(A\). Moreover, suppose \(L, L'\) are free \(A\)-modules of finite rank such that a basis of \(L\) forms a regular sequence in the symmetric algebra \(\text{Sym}_A(L')\). Then \(\text{Sp}^\bullet_{L,L'}(E)\) is a finite free \(U(L')\)-resolution of \(U(L') \otimes_{U(L)} E\).

If we assume the divisor of \(f\) is nice enough, we may use Proposition 5.9 to explicitly resolve \(\mathcal{D}_{X,F}[S]/F^S\).

**Proposition 5.10.** Suppose \(f = f_1 \cdots f_r\) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let \(F = (f_1, \ldots, f_r)\). Then \(\text{Sp}^\bullet_{\theta_{F,F'}, \text{Der}_{X,F} + \mathcal{E}_{X,F}}\) is a free \(\mathcal{D}_{X,F}[S]\)-resolution of \(\mathcal{D}_{X,F}[S]/F^S\).

**Proof.** We argue as in Section 1.6 of [12]. First, note that by Proposition 6.3 of [4] and Corollary 1.9 of [11], that for reduced free divisors being Saito-holonomic is equivalent to being Koszul free, where Koszul free means there is a basis \(\delta_1, \ldots, \delta_n\) of \(\text{Der}_{X,F}(-\log f)\) that \(\text{gr}_{(0,1)}(\delta_1) \cdots \text{gr}_{(0,1)}(\delta_n)\) is a regular sequence in \(\text{gr}_{(0,1)}(\mathcal{D}_{X,F})\). Let \(\delta_1, \ldots, \delta_n\) be such a basis. Then \(s_1, \ldots, s_n, \psi_{F,F}(\delta_1), \ldots, \psi_{F,F}(\delta_n)\) is a regular sequence in \(\text{gr}_{(0,1,1)}(\mathcal{D}_{X,F}[S])\). As these elements are all \((0, 1, 1)\)-homogeneous, we may rearrange them and conclude \(\psi_{F,F}(\delta_1), \ldots, \psi_{F,F}(\delta_n)\) is a regular sequence in \(\text{gr}_{(0,1,1)}(\mathcal{D}_{X,F}[S]) \simeq \text{Sym}_{\mathcal{E}_{X,F}}(\text{Der}_{X,F} + \mathcal{E}_{X,F})\). Now Proposition 5.9 implies that \(\text{Sp}^\bullet_{\theta_{F,F}, \text{Der}_{X,F} + \mathcal{E}_{X,F}}\) is a free \(\mathcal{D}_{X,F}[S]\)-resolution and inspecting the terminal map of this co-complex shows it resolves \(\mathcal{D}_{X,F}[S]/\mathcal{D}_{X,F}[S] \cdot \theta_{F,F}\), which, by Theorem 3.31, is isomorphic to \(\mathcal{D}_{X,F}[S]/F^S\). \(\square\)

When \(f\) is strongly Euler-homogeneous, Saito-holonomic, and tame we showed in Proposition 4.10 that there is a Koszul co-complex resolution of \(\mathcal{D}_{X,F}[S]/(S –
A) $\mathcal{D}_{X,t}[S]F^S$. Using $\text{Sp}^*_{\theta_{F,t};\mathcal{D}_{X,t}} \oplus \varepsilon_{X,t}$, we construct a free $\mathcal{D}_{X,t}[S]$-resolution of $\mathcal{D}_{X,t}[S]F^S/(S-A)\mathcal{D}_{X,t}[S]F^S$.

**Proposition 5.11.** Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. Then there is a finite, free resolution of $\mathcal{D}_{X,t}[S]$-modules

$$\mathcal{D}_{X,t}[S] \to \mathcal{D}_{X,t}[S]F^S/(S-A)\mathcal{D}_{X,t}[S]F^S.$$ 

**Proof.** Since $\text{Sp}^*_{\theta_{F,t};\mathcal{D}_{X,t}} \oplus \varepsilon_{X,t}$ is a finite free resolution of $\mathcal{D}_{X,t}[S]F^S$ by $\mathcal{D}_{X,t}[S]$-modules, this claim amounts to showing $\text{Tor}^k_{\mathcal{D}_{X,t}[S]}((S-A)\mathcal{D}_{X,t}[S], \mathcal{D}_{X,t}[S]F^S)$ vanishes for $k \geq 1$. We can compute this by taking a free $\mathcal{D}_{X,t}[S]$-resolution of the first argument.

Mimicking the first part of the proof of Proposition 5.10, use the $F_{(0,1,1)}$ filtration on $\mathcal{D}_{X,t}[S]$ to define a filtration on the Koszul co-complex $K^*(S-A)$ of $\mathcal{D}_{X,t}[S]$-modules. The only interesting part of spectral sequence associated to this filtered co-complex is the Koszul co-complex $K^*(S)$ on $\mathcal{O}_{X,t}[Y][S]$, which is acyclic. So $K^*(S-A)$ resolves $\mathcal{D}_{X,t}[S]$.

That $K^*(S-A; \mathcal{D}_{X,t}[S]F^S)$ resolves $\mathcal{D}_{X,t}[S]F^S$ is the content of Proposition 5.10. So $\text{Tor}^k_{\mathcal{D}_{X,t}[S]}((S-A)\mathcal{D}_{X,t}[S], \mathcal{D}_{X,t}[S]F^S)$ vanishes for $k \geq 1$ and the claim is proved. 

### 5.2. Dual of $\mathcal{D}_{X,t}[S]F^S/(S-A)\mathcal{D}_{X,t}[S]F^S$

Now that we have resolutions, we can proceed to our first goal: to compute the $\mathcal{D}_{X,t}$-dual of $\mathcal{D}_{X,t}[S]F^S/(S-A)\mathcal{D}_{X,t}[S]F^S$.

**Definition 5.12.** (Compare with Appendix A of [24]) Consider a Lie–Rinehart algebra $L$ over $(k, A)$ that is $A$-projective of constant rank $n$. There is an equivalence of categories from right $U(L)$-modules $Q$ to the left $U(L)$-modules given by $Q^{\left}\left = \text{Hom}_A(w_L, Q)$ where $w_L$ is the dualizing module of $L$, namely, $w_L = \text{Hom}_A(\wedge^n L, A)$. Regard $\mathcal{D}_{X,t}$ as the universal algebra of the Lie–Rinehart algebra $\mathcal{D}_{X,t}$ over $(\mathbb{C}, \mathcal{O}_{X,t})$ and $\mathcal{D}_{X,t}[S]$ as the universal algebra of the Lie–Rinehart algebra $\mathcal{D}_{X,t}[S]$ over $(\mathbb{C}[S], \mathcal{O}_{X,t}[S])$. In the appropriate derived category of left modules, where $N$ is a left $U(L)$-module, let:

$$\mathbb{D}(N) := (\text{RHom}_{\mathcal{D}_{X,t}}(N, \mathcal{D}_{X,t})^{\left};$$

$$\mathbb{D}_{S}(N) := (\text{RHom}_{\mathcal{D}_{X,t}[S]}(N, \mathcal{D}_{X,t}[S])^{\left}.$$

To be clear: to compute $\mathbb{D}(N)$ take an appropriate resolution of left $\mathcal{D}_{X,t}$-modules of $N$, apply $\text{Hom}_{\mathcal{D}_{X,t}}(-, \mathcal{D}_{X,t})$, and then apply the functor $(-)^{\left}$.

The following demystifies how $(-)^{\left}$ works for the above universal algebras. Its proof is entirely similar to the classical case of $(-)^{\left}$ for $\mathcal{D}_{X,t}$-modules.

**Lemma 5.13.** Take a $\ell \times m$ matrix $M$ with entries in $\mathcal{D}_{X,t}[S]$ so that multiplication on the left gives a map of right $\mathcal{D}_{X,t}[S]$-modules $\mathcal{D}_{X,t}[S]^m \to \mathcal{D}_{X,t}[S]^{\ell}$. Here an element $\mathcal{D}_{X,t}[S]^m$ is a column vector. For some fixed coordinate system, define the map $\tau : \mathcal{D}_{X,t}[S] \to \mathcal{D}_{X,t}[S]$, $\tau(x^w r^w s^w) = (-\partial)^{\tau}r^w s^w$. Extend $\tau$ to $\mathcal{D}_{X,t}[S]^m$ in an obvious way and to $M$ by applying $\tau$ to each entry. Then there is a commutative
diagram of left $\mathcal{D}_{X,F}[S]$-modules, where elements in the bottom row are row vectors and $(\cdot)^T$ denotes the transpose:

\[
\begin{array}{c}
\mathcal{D}_{X,F}[S]^m \xleftarrow{M^{\text{left}}}
\mathcal{D}_{X,F}[S]^f
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
\mathcal{D}_{X,F}[S]^m
\xrightarrow{\tau(M)^T}
\mathcal{D}_{X,F}[S]^f.
\end{array}
\]

We have a similar statement for maps $M : \mathcal{D}_{X,F}^m \to \mathcal{D}_{X,F}^f$ of $\mathcal{D}_{X,F}$-modules $(\tau$ has the obvious definition):

\[
\begin{array}{c}
\mathcal{D}_{X,F}^m \xleftarrow{M^{\text{left}}}
\mathcal{D}_{X,F}^f
\end{array}
\xrightarrow{\cong}
\begin{array}{c}
\mathcal{D}_{X,F}^m
\xrightarrow{\tau(N)^T}
\mathcal{D}_{X,F}^f.
\end{array}
\]

\[\square\]

The first step in computing $\mathcal{D}(\frac{\mathcal{D}_{X,F}[S]^m}{S - \mathcal{D}_{X,F}[S]^s})$ is finding a resolution—this is Proposition 5.11. The second is the following technical lemma:

**Lemma 5.14.** Suppose $f = f_1 \cdots f_r$ is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let $F = (f_1, \ldots, f_r)$. As complexes of free $\mathcal{D}_{X,F}$-modules,

\[
\mathcal{D}
\left(\frac{\mathcal{D}_{X,F}[S]}{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} \text{Sp}^\bullet_{\theta_{F,F}, \text{Der}_{X,F} \oplus \sigma_{X,F}^s}\right)
\cong
\frac{\mathcal{D}_{X,F}[S]}{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} \text{Sp}^\bullet_{\theta_{F,F}, \text{Der}_{X,F} \oplus \sigma_{X,F}^s}.
\]

**Proof.** For brevity, abbreviate $\text{Sp}^\bullet_{\theta_{F,F}, \text{Der}_{X,F} \oplus \sigma_{X,F}^s}$ to $\text{Sp}^\bullet$. Write the differential as $d^{-k} : \text{Sp}^{-k} \to \text{Sp}^{-(k-1)}$.

We will first compute the objects and maps of $\mathcal{D}
\left(\frac{\mathcal{D}_{X,F}[S]}{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} \text{Sp}^\bullet\right)$.

Since $\text{Sp}^\bullet$ is a co-complex of finite, free $\mathcal{D}_{X,F}[S]$-modules, $\text{Sp}^{-k} \cong \mathcal{D}_{X,F}[S]^{\left(\begin{array}{c}k \\ -k\end{array}\right)}$. Therefore, as $\mathcal{D}_{X,F}[S]$-modules,

\[
\begin{array}{c}
\mathcal{D}_{X,F}[S]
\end{array}
\xrightarrow{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} \text{Sp}^{-k}
\cong
\begin{array}{c}
\mathcal{D}_{X,F}[S]^{\left(\begin{array}{c}k \\ -k\end{array}\right)}
\end{array}.
\]

On the LHS of (5.2), we have the differential $\frac{\mathcal{D}_{X,F}[S]}{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} d^{-k}$. Think of $d^{-k}$ as a matrix. On the RHS of (5.2) the differential is $\text{eval}_A(d^{-k})$: the matrix $d^{-k}$ except each $s_i$ is replaced with $a_i$. As right $\mathcal{D}_{X,F}$-modules,

\[
\text{Hom}_{\mathcal{D}_{X,F}}\left(\frac{\mathcal{D}_{X,F}[S]}{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} \text{Sp}^{-k}, \mathcal{D}_{X,F}\right) 
\cong \text{Hom}_{\mathcal{D}_{X,F}}\left(\mathcal{D}_{X,F}[S]^{\left(\begin{array}{c}k \\ -k\end{array}\right)}, \mathcal{D}_{X,F}\right)
\cong \mathcal{D}_{X,F}^{\left(\begin{array}{c}k \\ -k\end{array}\right)}.
\]

Making the above identification, $\text{Hom}_{\mathcal{D}_{X,F}}\left(\frac{\mathcal{D}_{X,F}[S]}{(S - A)\mathcal{D}_{X,F}[S]} \otimes_{\mathcal{D}_{X,F}[S]} \text{Sp}^\bullet, \mathcal{D}_{X,F}\right)$ has a differential given by multiplication on the left by $(\text{eval}_A(d^{-k}))^T$: the transpose of $\text{eval}_A(d^{-k})$. To make the Hom complex a complex of left modules we apply the equivalence of categories $(-)^{\text{left}}$. By Lemma 5.13 we get a complex of left $\mathcal{D}_{X,F}$
modules isomorphic to the following, with differential given by matrix multiplication on the right

\[ A_\bullet \colon \ldots \rightarrow \mathcal{O}_{X,F}^{(k-1)} \rightarrow \mathcal{O}_{X,F}^{(k)} \rightarrow \ldots. \]

Now we compute the objects and maps of 
\[ \mathcal{O}_{X,F}[S] \otimes \mathcal{O}_{X,F}[S] \mathbb{D}_S(\mathcal{O}_{X,F}[S]) \]
As right \( \mathcal{O}_{X,F}[S] \)-modules, \( \text{Hom}_{\mathcal{O}_{X,F}[S]} \left( \mathcal{O}_{X,F}[S], \mathcal{O}_{X,F}[S] \right) \simeq \mathcal{O}_{X,F}[S]^{(1)} \). The induced differential is multiplication on the left by \( (d^{-k})^T \). By Lemma \ref{lem:induced-differential}, we can identify the complex obtained by applying \((-)^{left}\) with a complex whose terms are \( \mathcal{O}_{X,F}[S]^{(1)} \) and whose differentials are \( \tau ((d^{-k})^T) \). As left \( \mathcal{O}_{X,F}[S] \)-modules (and so as left \( \mathcal{O}_{X,F}[S] \)-modules),

\begin{equation}
\mathcal{O}_{X,F}[S] \otimes \mathcal{O}_{X,F}[S]^{(1)} \simeq \mathcal{O}_{X,F}[S]^{(1)} \otimes \mathcal{O}_{X,F}[S]^{(1)}.
\end{equation}

The RHS of (5.3) is isomorphic as a left \( \mathcal{O}_{X,F}[S] \)-module to \( \mathcal{O}_{X,F}[S]^{(1)} \). With this identification, the differentials of the complex \( \mathcal{O}_{X,F}[S] \otimes \mathcal{O}_{X,F}[S] \mathbb{D}_S(\mathcal{O}_{X,F}[S]) \) are given by \( \tau ( (d^{-k})^T ) \). Thus the complex of left \( \mathcal{O}_{X,F}[S] \)-modules \( \mathcal{O}_{X,F}[S] \mathbb{D}_S(\mathcal{O}_{X,F}) \) is isomorphic as a complex of left \( \mathcal{O}_{X,F}[S] \)-modules to

\[ B_\bullet : \ldots \rightarrow \mathcal{O}_{X,F}^{(k-1)} \rightarrow \mathcal{O}_{X,F}^{(k)} \rightarrow \ldots. \]

We will be done once we show that \( A_\bullet \) and \( B_\bullet \) are isomorphic complexes of \( \mathcal{O}_{X,F}[S] \)-modules. Because \( \tau ( (\tau A (d^{-k})^T) = \tau (\tau A (d^{-k})) = \tau (\tau (d^{-k})^T). \)

So we have reduced our problem to, in light of Proposition \ref{prop:reduced-strongly-euler-homogeneous}, computing \( \mathbb{D}_S(\mathcal{O}_{X,F}[S]F_S) \). In Corollary 3.6 of \cite{Narvarez-Macarro} Narváez–Macarro does this for \( \mathcal{O}_{X,F}[S]F_S \) with similar working hypotheses as ours and Maisonobe shows in \cite{Maisonobe} that this argument generalizes to \( \mathcal{O}_{X,F}[S]F_S \) as well. (Note that locally quasi-homogeneous immediately implies locally strongly Euler-homogeneous and that for reduced free divisors, being Saito-holonomic is equivalent to being Koszul free, cf. Proposition 6.3 in \cite{D.C.} and Corollary 1.9 in \cite{G.L.}). In our language, Maisonobe’s Proposition 6 of \cite{Maisonobe} becomes the following:

**Proposition 5.15.** (Proposition 6 in \cite{Maisonobe}) Let \( f = f_1 \cdots f_r \) be reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let \( F = (f_1, \ldots, f_r) \). Then, in the category of left derived \( \mathcal{O}_{X,F}[S] \)-modules, there is a canonical isomorphism

\[ \mathbb{D}_S(\mathcal{O}_{X,F}[S]F_S) \simeq \mathcal{O}_{X,F}[S]F^{-S-1}[[n]]. \]

**Remark 5.16.** Narváez–Macarro uses the univariate form of Proposition \ref{prop:reduced-strongly-euler-homogeneous} to show that the Bernstein–Sato polynomial is symmetric: \( b_f(s) = \pm b_f(-s - 2) \), cf. Theorem 4.1 of \cite{Narvarez-Macarro}. A similar argument gives a symmetry of the Bernstein–Sato variety: \( A \in V(B_{F,S}) \) if and only if \( -A - 2 \in V(B_{F,S}) \). Maisonobe proves this and more: he shows, using our language, that reduced, strongly Euler-homogeneous, Saito-holonomic, and free \( f \) has a principal multivariate Bernstein–Sato ideal and the generator \( B(S) \) of \( B_{F,S} \) satisfies the symmetry \( B(S) = \pm B(-S - 2) \), cf. Proposition 7 of \cite{Maisonobe}, Proposition 20 of \cite{G.L.}.
Theorem 5.17. Suppose \( f = f_1 \cdots f_r \) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let \( F = (f_1, \ldots, f_r) \). Then in the derived category of \( \mathcal{D}_{X,f} \)-modules there is a \( \mathcal{D}_{X,f} \)-isomorphism \( \chi_A \) given by

\[
\chi_A : \mathcal{D} \left( \frac{\mathcal{D}_{X,f}[S]F^S}{(S - A)\mathcal{D}_{X,f}[S]F^S} \right) \cong \frac{\mathcal{D}_{X,f}[S]F^{-S-1}}{(S - A)\mathcal{D}_{X,f}[S]F^{-S-1}[n]} \cong \frac{\mathcal{D}_{X,f}[S]F^S}{(S - (A - 1)\mathcal{D}_{X,f}[S]F^S}[n].
\]

Proof. The majority of the work is in proving the first isomorphism of (5.4). By Proposition 5.11 and Lemma 5.14 in the category of derived \( \mathcal{D}_{X,f} \)-modules

\[
\mathcal{D} \left( \frac{\mathcal{D}_{X,f}[S]F^S}{(S - A)\mathcal{D}_{X,f}[S]F^S} \right) \cong \frac{\mathcal{D}_{X,f}[S]}{(S - A)\mathcal{D}_{X,f}[S]} \otimes_{\mathcal{D}_{X,f}[S]} \mathcal{D}_{S} \left( \text{Sp}^*_{\theta_f, \text{Der}X,f} \otimes \mathcal{D}X,f \right).
\]

We argue as in Proposition 5.11 and show that, for \( k \geq 1 \),

\[
\text{Tor}^k_{\mathcal{D}_{X,f}[S]} \left( \frac{\mathcal{D}_{X,f}[S]F^S}{(S - A)\mathcal{D}_{X,f}[S]F^S}, \mathcal{D}_{X,f}[S]F^{-S-1} \right) = 0.
\]

Then by Proposition 5.15

\[
\mathcal{D}_{X,f}[S] \otimes_{\mathcal{D}_{X,f}[S]} \mathcal{D}_{S} \left( \text{Sp}^*_{\theta_f, \text{Der}X,f} \otimes \mathcal{D}X,f \right)
\]

is a free \( \mathcal{D}_{X,f} \)-resolution of

\[
\frac{\mathcal{D}_{X,f}[S]}{(S - A)\mathcal{D}_{X,f}[S]} \otimes_{\mathcal{D}_{X,f}[S]} \mathcal{D}_{X,f}[S]F^{-S-1} \cong \frac{\mathcal{D}_{X,f}[S]F^{-S-1}}{(S - A)\mathcal{D}_{X,f}[S]F^{-S-1}},
\]

proving the first isomorphism of (5.4) (the degree shift follows from Proposition 5.11 as well).

We calculate Tor by using the resolution \( K^*(S - A; \mathcal{D}_{X,f}[S]) \) of \( \frac{\mathcal{D}_{X,f}[S]}{(S - A)\mathcal{D}_{X,f}[S]} \), cf. Proposition 5.11. So we must show that the co-complex

\[
K^*(S - A; \mathcal{D}_{X,f}[S]) \otimes_{\mathcal{D}_{X,f}[S]} \mathcal{D}_{X,f}[S]F^{-S-1} \cong K^*(S - A; \mathcal{D}_{X,f}[S]F^{-S-1})
\]

is a resolution. The \( \mathcal{D}_{X,f} \)-linear map sending each \( s_i \) to \(-s_i - 1\) is a \( \mathcal{D}_{X,f} \)-isomorphism of \( \mathcal{D}_{X,f}[S] \). It induces a \( \mathcal{D}_{X,f} \)-isomorphism \( \mathcal{D}_{X,f}[S]F^{-S-1} \to \mathcal{D}_{X,f}[S]F^S \) and produces an isomorphism of the \( \mathcal{D}_{X,f} \)-co-complexes \( K^*(S - A; \mathcal{D}_{X,f}[S]F^{-S-1}) \) and \( K^*(-S - (A + 1); \mathcal{D}_{X,f}[S]F^S) = K^*(S - (A - 1); \mathcal{D}_{X,f}[S]F^S) \). The latter is a resolution by an argument similar to the proof of Proposition 5.11.

The second isomorphism in the theorem is given by sending \( s_i \) to \(-s_i - 1\) for each \( i \).

Remark 5.18. When \( f \) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free, this result immediately implies \( \mathcal{D}_{X,f}[S]F^S/(S - A)\mathcal{D}_{X,f}[S]F^S \) is a holonomic \( \mathcal{D}_{X,f} \)-module–it has only one nonzero Ext-module sitting in the \(-n^{th}\) position. When \( f \) is tame but not free computing Ext is currently intractable.

5.3. Free Divisors and \( \nabla_A \).

Recall from Definition 4.1 the \( \mathcal{D}_{X,f} \)-linear map

\[
\nabla_A : \frac{\mathcal{D}_{X,f}[S]F^S}{(S - A)\mathcal{D}_{X,f}[S]F^S} \to \frac{\mathcal{D}_{X,f}[S]F^S}{(S - (A - 1))\mathcal{D}_{X,f}[S]F^S}
\]
induced by \( s_i \mapsto s_i + 1 \), for each \( i \). If \( f \) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free, by Proposition \( \text{5.17} \) the complexes \( \mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]}) \) and \( \mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]}) \) can be identified with modules (i.e., Ext vanishes in all but one place). \( \nabla_A \) lifts to a map between the resolutions of \( \mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]} \) and \( \mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]} \) and to the Hom of those resolutions. Therefore \( \nabla_A \) induces a map (thinking of these as modules)

\[
\mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]}) \to \mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]}).
\]

Name this map \( \mathbb{D}((\nabla_A)) \).

**Theorem 5.19.** Suppose \( f = f_1 \cdots f_r \) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let \( F = (f_1, \ldots, f_r) \). Let \( \chi_A \) be the \( \mathcal{D}_{X,S} \)-isomorphism of Theorem \( \text{5.17} \). Then there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]}) & \xrightarrow{\sim} & \mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]}) \\
\mathbb{D}(\nabla_A) \uparrow & & \uparrow \nabla_A \\
\mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]}) & \xrightarrow{\sim} & \mathbb{D}(\mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]})
\end{array}
\]

**Proof.** First consider the \( \mathcal{D}_{X,S} \)-linear map \( \nabla : \mathcal{D}_{X,S}[S]^{[F_S]} \to \mathcal{D}_{X,S}[S]^{[F_S]} \) given by sending \( s_i \mapsto s_i + 1 \) for all \( i \). By Proposition \( \text{5.10} \) the co-complex of free \( \mathcal{D}_{X,S}[S] \)-modules \( \text{Sp}^\bullet_{\mathcal{D}_{X,S}, \text{Der}_{X,S} \otimes \mathcal{O}_{X,S}^\bullet} \) resolves \( \mathcal{D}_{X,S}[S]^{[F_S]} \). For readability, in this proof we will write this co-complex as \( \text{Sp}^\bullet \). Regarding this as a co-complex of \( \mathcal{D}_{X,S} \)-modules, we may lift \( \nabla \) to a chain map. A straightforward computation using \( \text{5.11} \) and the definition of \( \psi_{F,S} \) shows that one such lift is given by

\[
\text{Sp}^{-k} \xrightarrow{\sim} \mathcal{D}_{X,S}[S]^{(\ell)} \xrightarrow{\sigma_{-k}} \text{Sp}^{-k} \xrightarrow{\sim} \mathcal{D}_{X,S}[S]^{(\ell)},
\]

where the dashed line is the lift of \( \nabla \) at the \( -k \) slot and \( \sigma_{-k} \) multiplies each component of the direct sum by \( f \) on the right and sends each \( s_i \) to \( s_i + 1 \) in every component.

We may use the finite, free \( \mathcal{D}_{X,S} \)-resolution of \( \mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]} \) by \( \mathcal{D}_{X,S}[S]^{[F_S]} \otimes \mathcal{D}_{X,S}[S] \text{Sp}^\bullet \) to lift \( \nabla_A \) to a chain map, cf. Proposition \( \text{5.11} \). One such lift is given by

\[
\begin{align*}
\mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]} \text{ Sp}^{-k} & \xrightarrow{\sim} \mathcal{D}_{X,S}[S]^{[F_S]}/(S-A)\mathcal{D}_{X,S}[S]^{[F_S]} \\
\downarrow_{f_{-k}(\nabla_A)} & \downarrow_{\sigma_{-k}} \\
\mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]} \text{ Sp}^{-k} & \xrightarrow{\sim} \mathcal{D}_{X,S}[S]^{[F_S]}/(S-(A-1))\mathcal{D}_{X,S}[S]^{[F_S]}.
\end{align*}
\]
where the \( \ell_{-k}(\nabla_A) \) is the lift of \( \nabla_A \) the \(-k\) slot and \( \sigma^A_{-k} \) is induced by \( \sigma_{-k} \). That is, \( \sigma^A_{-k} \) is given by multiplying each component of the direct sum by \( f \) on the right.

Apply \( \text{Hom}_{\mathcal{D}_{X,f}}(-, \mathcal{D}_{X,f}) \) to the chain map given by the \( l_-k(\nabla_A) \). Then (5.5) implies that at the \(-n\) slot we have

\[
\begin{align*}
\text{Hom}_{\mathcal{D}_{X,f}}(\nabla_A, \mathcal{D}_{X,f}) & \cong \nabla_A[\mathcal{D}_{X,f}]^\ell \bigg|_{\nabla_A[\mathcal{D}_{X,f}]^\ell} \cong \nabla_A[\mathcal{D}_{X,f}]^\ell \bigg|_{\nabla_A[\mathcal{D}_{X,f}]^\ell}, \\
\text{Hom}_{\mathcal{D}_{X,f}}(\nabla_A, \mathcal{D}_{X,f})^\ell & \cong \nabla_A[\mathcal{D}_{X,f}]^\ell \bigg|_{\nabla_A[\mathcal{D}_{X,f}]^\ell} \cong \nabla_A[\mathcal{D}_{X,f}]^\ell \bigg|_{\nabla_A[\mathcal{D}_{X,f}]^\ell},
\end{align*}
\]

where \( \text{Hom}_{\mathcal{D}_{X,f}}(\sigma^A_{-n}, \mathcal{D}_{X,f}) \) is simply multiplication by \( f \) on the right. Since \( \mathcal{D}(\mathcal{D}_{X,f}[S]^F) \) has nonzero homology only at the \(-n\) slot, we may identify this complex with that homology module and the map \( \mathcal{D}(\nabla_A) \) is induced by \( \text{Hom}_{\mathcal{D}_{X,f}}(\sigma^A_{-n}, \mathcal{D}_{X,f}) \), i.e. by multiplication by \( f \) on the right. So (5.6) and Theorem 5.17 give the following commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_{X,f}[S] & \longrightarrow & \mathcal{D}_{X,f}[S]^F \\
\nabla_A \downarrow & & \nabla_A \downarrow \\
\mathcal{D}(\mathcal{D}_{X,f}[S]^F) & \cong & \mathcal{D}(\mathcal{D}_{X,f}[S]^F) \\
\end{array}
\]

A straightforward diagram chase shows that the dashed map is \( \nabla_A \). \( \square \)

**Theorem 5.20.** Suppose \( f = f_1 \cdots f_r \) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free and let \( F = (f_1, \ldots, f_r) \). Then \( \nabla_A \) is injective if and only if it is surjective.

**Proof.** Suppose \( \nabla_A \) is surjective. So there is a short exact sequence of left \( \mathcal{D}_{X,f} \)-modules

\[
0 \rightarrow N \rightarrow \mathcal{D}_{X,f}[S]^F \rightarrow \nabla_A[\mathcal{D}_{X,f}]^\ell \rightarrow 0.
\]

Note that \( N \) is holonomic, see Remark 5.18. Using the long exact sequence of Ext and basic properties of the holonomic double dual of holonomic modules, a standard argument shows that \( \nabla_A \) is surjective if and only if \( \mathcal{D}(\nabla_A) \) is injective. Similar considerations show that \( \nabla_A \) is injective if and only if \( \mathcal{D}(\nabla_A) \) is surjective.

That the injectivity of \( \nabla_A \) implies the surjectivity of \( \nabla_A \) is the content of Theorem 4.11. The reverse implication is true by the the following implications:

\[
\begin{align*}
\nabla_A \text{ is surjective} & \iff \mathcal{D}(\nabla_A) \text{ is injective} \quad \text{[Duality]} \\
& \iff \nabla_{-A} \text{ is injective} \quad \text{[Theorem 5.19]} \\
& \iff \nabla_{-A} \text{ is surjective} \quad \text{[Theorem 4.11]} \\
& \iff \mathcal{D}(\nabla_A) \text{ is surjective} \quad \text{[Theorem 5.19]} \\
& \iff \nabla_A \text{ is injective} \quad \text{[Duality]}.
\end{align*}
\]

\( \square \)
In Section 6 we will use the above theorem to relate cohomology support loci of \( f \) near \( x \) to the Bernstein–Sato variety \( V(B_{F,x}) \) when \( f \) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free.

### 6. Free Divisors and the Cohomology Support Loci

In this section, we will assume \( f_1, \ldots, f_r \) are mutually distinct and irreducible hypersurface germs at \( x \in X \) that vanish at \( x \). Let \( f = f_1 \cdots f_r \) and \( D \) the divisor of \( f \). Note \( f \) is reduced. Take a small open ball \( B_x \) about \( x \) and let \( U_x = B_x \setminus D \).

Define \( U_\eta \) for \( \eta \in D \) and \( \eta \) near \( x \) similarly.

**Definition 6.1.** (Compare with Section 1, [8]) Let \( M(U_\eta) \) denote the rank one local systems on \( U_\eta \). The cohomology support locus \( V(U_\eta) \) is the set of rank one local systems \( L \) on \( U_\eta \) with nonvanishing cohomology. That is

\[
V(U_\eta) := \{ L \in M(U_\eta) \mid \dim H^\bullet(U_\eta, L) \neq 0 \}.
\]

There is a natural map of local systems on \( U_x \) to \( U_\eta \), \( \eta \) near \( x \), given by restriction:

\[
\text{res}_\eta : \{ \text{local systems on } U_x \} \rightarrow \{ \text{local systems on } U_\eta \}.
\]

It is often easier to consider the data about all the cohomology support loci of \( f \) at \( \eta \) for \( \eta \) near \( x \) at once. With this in mind, let the cohomology support loci of \( f \) near \( x \), denoted by \( V(U_x, B_x) \), be defined as

\[
V(U_x, B_x) := \bigcup_{\eta \in D \text{ near } x} \text{res}_\eta^{-1}(V(U_\eta)).
\]

Note that this definition agrees with the notion of "uniform cohomology support locus" as defined in [7], cf. Remark 2.8 [8] and [19].

**Convention 6.2.** For \( A \in \mathbb{C}^r \) and \( k \in \mathbb{Z} \), let \( A - k \) denote \((a_1 - k, \ldots, a_r - k)\).

Let \( j \) be the open embedding of \( U_x \hookrightarrow B_x \). Given a rank one local system \( L \) on \( U_x \), the derived direct image \( Rj_*(L[n]) \) is a perverse sheaf on \( B_x \) and hence of finite length (in the category of perverse sheaves). In Theorem 1.5 of [8], the authors prove that

(6.1) \( V(U_x, B_x) = \{ L \in M(U_x) \mid Rj_*(L[n]) \text{ is not a simple perverse sheaf on } B_x \} \).

Using this Budur proves in Theorem 1.5 of [7], cf. Remark 4.2 of [8], that

(6.2) \( \text{Exp}(V(B_{F,x})) \supseteq V(U_x, B_x) \).

(This inclusion occurs in \( \mathbb{C}^* \) where a local system on \( U_x \) is viewed as a representation \( \{ \pi_1(U_x) \rightarrow \mathbb{C}^* \} \subseteq \mathbb{C}^* \).)

We give a sketch of the proof. For \( A \in \mathbb{C}^r \), there is a cyclic \( \mathcal{D}_{X,x} \)-module generated by \( F^A = \prod f_k^{a_k} \) and with \( \mathcal{D}_{X,x} \)-action given by the rules of calculus. There is a natural \( \mathcal{D}_{X,x} \)-surjection

\[
p_A : \mathcal{D}_{X,x}[S][F^S] \rightarrow \mathcal{D}_{X,x} F^A
\]

where
sitting in the commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}_X[S_1^{\otimes x}] & \xrightarrow{p_A} & \mathcal{D}_X[F^A] \\
\downarrow \nabla_A & & \downarrow \\
\mathcal{D}_X[S_1^{\otimes x}] & \xrightarrow{p_{A-1}} & \mathcal{D}_X[F^{A-1}]
\end{array}
\]

(6.3)

where \(\nabla_A\) is as defined in Section 4 and the right-most inclusion identifies \(F^A\) with \(f F^{A-1}\).

Now assume \(\text{Exp}(A) \notin \text{Exp}(V(B_{F,t}))\). By Proposition 4.3 for every \(k \in \mathbb{Z}\), the map \(\nabla_{A+k}^t\) is an isomorphism. By Proposition 3.2 and 3.3 of [25] there exists an integer \(t \gg 0\) such that \(p_{A-t}\) is an isomorphism. A standard diagram chase using all the commutative squares given by (6.3) then proves \(\mathcal{D}_X[F^A] = \mathcal{D}_X[F^{A-1}]\) and \(p_{A-1+t}\) is an isomorphism. An easy induction argument using all the commutative squares given by (6.3) then proves \(\mathcal{D}_X[F^{A-m}] = \mathcal{D}_X[F^{A+m}]\) for \(m \gg 0\).

Theorem 5.2 in [7] shows that \(\mathcal{D}_X[F^A]\) is a regular, holonomic \(\mathcal{D}_X[t]\)-module and, where \(\text{DR}\) is the DeRham functor, that:

\[
\begin{align*}
\text{DR}(\mathcal{D}_X[F^{A-m}]) &= Rj_*(\text{Exp}(A)[n]), \text{ for } m \in \mathbb{N}, m \gg 0; \\
\text{DR}(\mathcal{D}_X[F^{A+m}]) &= IC(\text{Exp}(A)[n]), \text{ for } m \in \mathbb{N}, m \gg 0.
\end{align*}
\]

(6.4)

Here \(j\) is the open embedding \(U_t \hookrightarrow B_t\), the local system \(\text{Exp}(A)\) is given by the representation \(M(U_t) \to \mathbb{C}^{x^r}\) encoded by \(\text{Exp}(A)\), and \(IC(\text{Exp}(A)[n])\) is the intersection complex on \(B_t\) of the perverse sheaf \(\text{Exp}(A)[n]\). Note that \(IC(\text{Exp}(A)[n])\) and hence \(F^{A+m}\) are simple. So by (6.1), the local system \(\text{Exp}(A)\) is not in \(V(U_t, B_t)\); this proves (6.2).

Trying to prove the converse containment to (6.2) by similar means results in a gap: there is no obvious equivalence between \(\nabla_A^t\) being an isomorphism and elements of \(V(B_{F,t})\). However, we can prove a weaker statement about simplicity of modules:

**Theorem 6.3.** Suppose \(f = f_1 \cdots f_r\) and \(F = (f_1, \ldots, f_r)\), where the \(f_k\) are mutually distinct and irreducible hypersurface germs at \(x\) vanishing at \(x\). Suppose \(f\) is reduced, strongly Euler-homogeneous, Saito-holonomic, and free. If \(A \in \mathbb{C}^r\) such that the rank one local system \(\text{Exp}(A) \notin V(U_t, B_t)\), then, for all \(k \in \mathbb{Z}\), the map \(\nabla_{A+k}^t\) is an isomorphism and \(\mathcal{D}_X[S_1^{\otimes x}]^{(S_{-}(A+k))\mathcal{D}_X[S_1^{\otimes x}]^{(S_{-}(A+k))}}\) is a simple \(\mathcal{D}_X[t]\)-module.

**Proof.** By (6.1) and (6.4), each \(\mathcal{D}_X[F^{A+k}]\) is a simple \(\mathcal{D}_X[t]\)-module. We will show that each \(p_{A+k}\) is an isomorphism. If not, by using Proposition 3.2 and 3.3 of [25], we may assume there exists an integer \(t \in \mathbb{Z}\) such that \(p_{A-t-j}\) is an isomorphism for all \(j \in \mathbb{Z}_{>0}\), but \(p_{A-t-1}\) is not. Then (6.3) yields

\[
\begin{align*}
\mathcal{D}_X[S_1^{\otimes x}]^{(S_{-}(A-t-1))\mathcal{D}_X[S_1^{\otimes x}]^{(S_{-}(A-t-1))}} & \xrightarrow{p_{A-t+1}} \mathcal{D}_X[F^{A-t+1}] \\
\nabla_{A+t+1} & \downarrow \\
\mathcal{D}_X[S_1^{\otimes x}]^{(S_{-}(A-t))\mathcal{D}_X[S_1^{\otimes x}]^{(S_{-}(A-t))}} & \xrightarrow{p_{A-t}} \mathcal{D}_X[F^{A-t}].
\end{align*}
\]

So \(\nabla_{A+t+1}\) is surjective. By Theorem 5.20 \(\nabla_{A+t+1}\) is an isomorphism, from which it follows that \(p_{A-t+1}\) is an isomorphism as well. This contradicts the choice of \(t\);
therefore no such $t$ exists and $p_{A+k}$ is an isomorphism for all $k \in \mathbb{Z}$. It follows from (6.3) that $\nabla_{A+k}$ is an isomorphism for all $k \in \mathbb{Z}$. The statement about simplicity follows from (6.4) and the fact each $p_{A+k}$ is an isomorphism.

As an application, let's compute a subtorus of $V(U_0, B_0)$ where $f = f_1 \cdots f_r$ is a reduced, central, essential, indecomposable, and free hyperplane arrangement and the $f_k$ are mutually distinct linear forms. Since our subtorus originates in $\text{Exp}(V(B_{F,0}))$ this gives partial evidence for the converse of (6.2).

**Corollary 6.4.** Suppose $f = f_1 \cdots f_r$ is a central, essential, indecomposable, and free hyperplane arrangement, where the $f_k$ are mutually distinct linear forms. Let $F = (f_1, \ldots, f_r)$ and $A - 1 \in V(s_1 + \cdots + s_r + n = 0)$. Then $A - 1 \in V(B_{F,0})$ and $\text{Exp}(A - 1) \in V(U_0, B_0)$.

**Proof.** Immediate by Proposition 4.12 and Theorem 6.3.

**Remark 6.5.** (a) The part of Corollary 6.4 concerning $V(U_0, B_0)$ can be obtained by Theorem 1.2 of [8] wherein an explicit formula for $\text{Exp}(U_0, B_0)$ is given.

(b) The same argument of Corollary 6.4 would apply to any reduced, strongly Euler-homogeneous, Saito-holonomic, and free $f = f_1 \cdots f_r$, with the $f_k \in \mathbb{C}[x_1, \ldots, x_n]$ homogeneous of degree $d_k$, mutually distinct, and irreducible, if we further assume that $\text{Der}_{X,V}(-\log f) \subseteq m_{X,0}^2 \cdot \text{Der}_{X,V}$, cf. see Remark 3.39.

So for all such divisors, $V(d_1s_1 + \cdots + s_r + n) \subseteq V(B_{F,0})$ and $\text{Exp}(V(d_1s_1 + \cdots + s_r + n)) \subseteq V(U_0, B_0)$.

(c) In an ongoing project, we intend to show the converse containment to (6.2), and thus prove Budur’s conjecture relating $\text{Exp}(V(B_{F,0}))$ and the cohomology support loci of $f$ near $f$, for reduced, free and central hyperplane arrangements. In this work we plan to show that $\nabla_A$ satisfies the three conditions of Proposition 4.2 from which the conjecture follows by Theorem 6.3.

7. List of Symbols

- $\mathfrak{u}$ is the initial ideal with respect to the $\mathfrak{u}$-grading;
- $F_{(0,1)}$ is the order filtration on $\mathcal{D}_{X,V}$ and $F_{(0,1,1)}$ is the total order filtration on $\mathcal{D}_{X,V}$;
- $\text{gr}_{(0,1)}(\mathcal{D}_{X,V})$ is the associated graded object of $\mathcal{D}_{X,V}$ corresponding to the order filtration and $\text{gr}_{(0,1,1)}(\mathcal{D}_{X,V}[S])$ is the associated graded object corresponding to the total order filtration;
- $\theta_F = \text{ann}_{\mathcal{D}_{X,V}[S]} \cap F_{(0,1,1)}$ are the annihilating derivations of $F^S$;
- $\psi_F : \text{Der}_{X}(-\log f) \to \theta_F$ is the map $\delta \mapsto \delta - \sum \frac{\partial \delta}{\partial f_k} s_k$;
- $L_F = \text{gr}_{(0,1,1)}(\mathcal{D}_{X,V}[S]) \cdot \text{gr}_{(0,1,1)}(\psi_F(\text{Der}_{X}(-\log f)))$ is the generalized Liouville ideal and $L_{F} = \text{gr}_{(0,1,1)}(\mathcal{D}_{X,V}[S]) \cdot \theta_F$;
- $B_{f}$ and $V(B_{f})$ are the classical Bernstein–Sato ideal and variety of $f$ whereas $B_{F}$ and $V(B_{F})$ are the multivariate Bernstein–Sato ideal and variety of $F$;
- $\nabla_A : \mathcal{D}_{X,V}[S][F^S] \to \mathcal{D}_{X,V}[S][F^S]$ is the $\mathcal{D}_{X,V}$-map induced by sending each $s_k$ to $s_k + 1$.
- $\mathbb{S}^p_{\theta_F, \text{Der}_{X,V} \oplus O_{X,V}}$ is a complex associated to the Lie–Rinehart algebras $\text{Der}_{X,V} \oplus O_{X,V}$, cf. Definition 5.8;
- $\mathbb{D}(-) = (\text{RHom}_{\mathcal{D}_{X,V}}(-, \mathcal{D}_{X,V})^\text{left}$, $\mathbb{D}_S(-) = (\text{RHom}_{\mathcal{D}_{X,V}}(-, \mathcal{D}_{X,V}[S])^\text{left}$.
\( \chi_A : D \left( \frac{\mathcal{D}_X[S]}{\mathcal{D}_X[S]} \right) \cong D \left( \frac{\mathcal{D}_X[S]}{\mathcal{D}_X[S]} \right)^{\mathbb{Z}} \), for strongly Euler-homogeneous, Saito-holonomic, and free \( f \), cf. Theorem 5.17.

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