Weil jets theory, connections and curvature tensor

R J Alonso-Blanco and J Muñoz-Díaz
Departamento de Matemáticas, Universidad de Salamanca, Plaza de la Merced 1-4, E-37008 Salamanca, Spain.
E-mail: ricardo@usal.es, clint@usal.es

Abstract. We will describe connections in the framework of Weil jets theory which consist of considering jets as ideals and refer all the operations to the ring of functions of the base manifold. With this approach we gain some clarity on certain aspects of the formalism. Moreover, we will obtain, based on it, a slightly new perspective of the curvature tensor. As an application, the Ricci tensor is obtained in a way somewhat more direct than usual, which could be interesting for a better understanding of this important object.

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1. Introduction
The theory of connections is an issue that has been extensively covered because of its important applications in geometry and physics. What we bring here is a small contribution to the understanding of its fundamentals, based on the approach of the theory of jets of Weil (see [7, 19, 1]). In essence, the aforementioned theory of jets (in part by continuing the ideas of [21]) is the development of a perspective in which jets (or contact elements, as they are sometimes called) are understood as ideals of the ring of differentiable functions of the manifold of departure and, as much as possible, all the operations are referred to that ring, a bit in the spirit of algebraic geometry and continuing a tradition originated in Plücker and Lie [11]: are points of a manifold, not only the ordinary ones but also the elements of line, surface, etc. This framework has allowed simplify the presentations (for instance, the prolongation of a tangent vector field to jet spaces is, in a sense, the field itself properly understood) and obtain important applications to the theory of PDE, formal integrability, pseudogroups, differential invariants, etc. (see [2, 3, 8, 9, 12, 13, 14, 15, 16, 17]).

On the other hand, the most primitive concept of connection is that of tangent distribution (of a given dimension) and this is identified exactly with the notion of (first order) jet-field. For a given function, the taking of quotient class by the ideal associated with the jet corresponding at each point, define, already here, a notion of covariant differential of that function. If on the distribution we further require transversality to a given fibred structure or linearity, if we handle a vector bundle, we recover the familiar concepts of connection in a bundle and linear connection in a vector bundle (the same could be done, although we do not develop this possibility here, in the case of principal bundles if we require equivariance under the action of a Lie group, etc.). Definitions of covariant differential or derivative fit in this frame naturally, providing us with a good point of departure.

The description of jets of tangent fields on a manifold \( M \), based on the theory of jets of Weil, allows us to refer all concepts and manipulations to the ring \( C^\infty(M) \) (and its quotient by the
powers of the maximal ideal of a given point). For instance, the 1-jet of a tangent field at a point \( x \in M \) is thus a derivation of the ring \( \mathcal{C}^\infty(M) \) with values in the first order Taylor expansions \( \mathcal{C}^\infty(M)/m_x^2 \). In this way, a connection on the tangent bundle, associate with each tangent vector a derivation of \( \mathcal{C}^\infty(M) \) with values in the ring of first order expansions. As an application, we will see that the torsion tensor when applied on two tangent vectors is given by the commutator of the infinitesimal extensions that the connection determines as above mentioned. Also, we will see that the geodesic field appears quite naturally within this framework.

We will discuss the curvature of a connection within the scheme outlined above by changing the usual point of view. This goes as follows. Each general connection (distribution) defines a certain submanifold of the first order jet space (of a fixed dimension \( m \)): the image \( \nabla(V) \) of the associated jet-field. Two “fields” of tangent \( m \)-planes can be naturally defined along that submanifold. First, the distribution itself, once transported to \( \nabla(V) \). Second, that produced when we take the “value” of sections of the distribution in the corresponding jet. We will see that, in general, both fields of planes no match and we will prove that they are equal only when the connection is integrable. Therefore, the difference between these fields measures the integrability of \( \nabla \); for this purpose we will proceed as follows: given a tangent vector \( D_v \) in the distribution at the point \( v \), we consider the difference between \( \nabla_v D_v \) and the “value” \( D_{\nabla(v)} \) of a section \( D \) of the distribution (extending the given vector); the result is a tangent vector \( R(D_v) \) at the point \( \nabla(v) \), which we will call curvature of the connection at \( D_v \) because the vanishing of \( R \) implies the integrability of \( \nabla \). In the case of a connection on a bundle we consider the operator of two variables: given a tangent vector \( X \) on a point \( x \) of the base manifold and a point \( v \) belonging to the fiber of \( x \), we just will take the value of \( R \) on the horizontal lift of \( X \) to \( v \).

When a linear connection on the tangent bundle \( TM \) is considered, the curvature we have defined above gives an operator depending on two tangent vectors and taking values on the homomorphisms bundle \( T^*M \otimes TM \) (from which the Riemann-Christoffel tensor follows). If, then, we take the trace, we obtain the Ricci tensor without the need to first “twist” the order of the indices as in the usual approach. Perhaps this little advantage could suggest a way for better understanding what the Ricci tensor really means.

The structure of the paper is as follows. In Section 2 we give a short account of the main facts about Weil jets theory in the first order case, the only one that we will need for working with connections; in Section 3 we consider general connections as synonymous of tangent distributions and introduce the covariant differential of a function; then, we specialize the above notions to the case of connections on fiber bundles and linear connections on vector bundles and define the covariant differential of a section; in Section 4 we first describe jets of tangent fields in terms of the ring of functions, and then we take advantage of this description in order to consider connections on the tangent bundle; finally, Section 5 is devoted to our approach to the curvature in all the several cases above considered joint with the Ricci tensor of an affine connection.

2. Weil jets

In this section we will give without proofs the more basic ideas and results about Weil jets theory. In addition, we restrict ourselves to the first order case because this will be sufficient for the purposes of this article. For details we refer to [14, 19, 1, 4].

2.1. Weil bundles

A commutative \( \mathbb{R} \)-algebra is named a Weil algebra if it is finite, local and rational. The most interesting examples for applications are the rings of truncated polynomials

\[
\mathbb{R}_m^k := \mathbb{R}[\epsilon_1, \ldots, \epsilon_m]/(\epsilon_1, \ldots, \epsilon_m)^{k+1},
\]
where \( \epsilon_1, \ldots, \epsilon_m \) are undetermined variables. Algebras \( \mathbb{R}_m^k \) are the models for the rings of Taylor expansions of order \( k \) at a point of an \( m \)-dimensional manifold. For instance, given an smooth manifold \( V \), the class of a parameterized submanifold of dimensi´ on \( m \) at the order \( k \) in the point \( x \in V \) is given by an \( \mathbb{R} \)-algebra morphism

\[
\mathcal{C}^\infty(V) \to \mathbb{R}_m^k.
\]

In general, for a Weil algebra \( A \) and a smooth manifold \( V \), the set

\[
V^A := \text{Hom}_{\mathbb{R} \text{-alg}}(\mathcal{C}^\infty(V), A),
\]

called the manifold of \( A \)-near points (or simply, \( A \)-points) of \( V \), is naturally endowed with the structure of a bundle \( V^A \to V \) called Weil bundle (see [21, 10]).

When \( A = \mathbb{R}_m^k \) we will use the notation \( V^A_m \) instead of \( V^{\mathbb{R}_m^k} \). A particular instance is \( A = \mathbb{R}_1^1 \) (dual numbers); in this case, we recover the tangent bundle: \( TV = V^1_1 \).

Weil bundles enjoy of a nice property: if \( A \) and \( B \) are Weil algebras, then \( V^{A \otimes B} = (V^A)^B = (V^B)^A \); for instance, if we take \( B = \mathbb{R}_1^1 \) we get \( T(V^A) = (TV)^A \) or, what is the same, for each \( \phi \in V^A \) we have

\[
T_\phi V^A = \text{Der}_R(\mathcal{C}^\infty(V), A)
\]

where \( A \) is an \( \mathcal{C}^\infty(V) \)-module via \( \phi \).

### 2.2. Jets of submanifolds

Let \( S \subset V \) be an \( m \)-dimensional submanifold of an \( n \)-dimensional smooth manifold \( V \). The class of the submanifolds having a contact of order 1 with \( S \) at a point \( v \in V \) is naturally identified with the ideal

\[
j_1^i S := I_S + m_v^2 \subset \mathcal{C}^\infty(V)
\]

where \( I_S \subset \mathcal{C}^\infty(V) \) (resp. \( m_v \subset \mathcal{C}^\infty(V) \)) is the ideal attached with \( S \) (resp. with \( v \)). For this reason we give the following

**Definition 2.1.** An 1-jet of \( m \)-dimensional submanifold of \( V \) is, by definition, an ideal \( p \subset \mathcal{C}^\infty(V) \) such that

\[
\mathcal{C}^\infty(V)/p \simeq \mathbb{R}_m^1;
\]

equivalently, an 1-jet is the kernel of a surjective (="regular") \( \mathbb{R}_m^1 \)-point. The set of such ideals will be called space of 1-jets of \( m \)-dimensional submanifolds of \( V \) and denoted by \( J^1_m V \).

The space \( J^1_m V \) can be also seen as the quotient of the set of regular \( \mathbb{R}_m^1 \)-points \( V_m^1 \) under the action of the Lie group \( G_m^1 := \text{Aut}_{\mathbb{R} \text{-alg}} \mathbb{R}_m^1 \), so that \( V_m^1 \to J^1_m V \) is a principal fiber bundle.

Later on, because there will be no risk of confusion, we will denote by the same letter, say \( p \), as the ideal \( p \subset \mathcal{C}^\infty(V) \) as the induced quotient map \( p: \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(V)/p \).

For each 1-jet \( p \in J^1_m V \), there is a unique point \( v \in V \), called source of \( p \), such that \( p \subset m_v \); in this way, the space of jets is endowed with a natural projection

\[
\pi_1: J^1_m V \to V, \quad p \mapsto v
\]

**Coordinates.** Let \((U; x_1, \ldots, x_n)\) be a local chart of \( V \). Now, let us consider the open subset \( J^1_m U \) comprised by those jets \( p \in J^1_m V \) such that \( \mathbb{R}[x_1, \ldots, x_m]/p \cap \mathbb{R}[x_1, \ldots, x_m] \simeq \mathbb{R}_m^1 \). In addition, put \( y_j = x_{m+j}, j = 1, \ldots, n - m \). Then, for each \( p \in J^1_m U \), there are numbers \( y_{ji}(p) \) such that \( p \) is generated by functions

\[
y_j - y_j(v) - y_{ji}(p)(x_i - x_i(v)), \quad j = 1, \ldots, n - m,
\]

(2.3)
joint with \( m_2 \). Functions \( x_i, y_j, y_{ji} \) thus defined, give a local chart on \( \mathcal{J}_m^1 U \). It follows easily how the whole of \( \mathcal{J}_m^1 V \) may be covered by local charts.

Finally observe that a first order jet \( p \in \mathcal{J}_m^1 V \), with source \( v \in V \), defines (and is defined by) the \( m \)-plane

\[
L_p := \{ D_v \in T_v V \mid D_v \text{ annihilates } p \} = \ker(d_v p) \subset T_v V
\]

(2.4)

2.3. The tangent space at a jet

The following statement is a basic property and allows us to identify tangent vectors at a jet in \( \mathcal{J}_m^1 V \) as (classes of) derivations of \( C^\infty(V) \) with values in \( C^\infty(V)/p \). This enforces the idea of considering jets as being also “points” of \( V \) besides the ordinary ones.

**Theorem 2.2** ([14]). For each \( p \in \mathcal{J}_m^1 V \) the following isomorphism holds,

\[
T_p \mathcal{J}_m^1 V \simeq \text{Der}_x(C^\infty(V), C^\infty(V)/p)/\sim
\]

where two derivations \( D, D' \in \text{Der}_x(C^\infty(V), C^\infty(V)/p) \) are considered as equivalents if and only if they coincide when applied on the ideal \( p \subset C^\infty(V) \). In particular, \( T_p \mathcal{J}_m^1 V \) is a \( C^\infty(V)/p \)-module.

The above statement follows from (2.1) by taking into account that \( \mathcal{J}_m^1 V \) is the quotient of \( \check{V}_m^1 \) under the action of the Lie group \( \text{Aut}_{\mathcal{R}_{\text{alg}}} \mathcal{R}_m^1 \), by identifying \( C^\infty(V)/p \) with \( \mathcal{R}_m^1 \), and, finally, by computing the tangent space to the orbits. For details, see [1, 14, 19].

In terms of coordinates, the correspondence established in the above theorem is locally given by

\[
\left( \frac{\partial}{\partial x_i} \right)_p = \left[ \frac{\partial}{\partial x_i} \right]_p, \quad \left( \frac{\partial}{\partial y_j} \right)_p = \left[ \frac{\partial}{\partial y_j} \right]_p, \quad \left( \frac{\partial}{\partial y_{ji}} \right)_p = \left[ (x_i - x_i(v)) \frac{\partial}{\partial y_j} \right]_p,
\]

(2.5)

where \( [\ ]_p \) denotes the class modulo equivalence relation \( \sim \).

2.4. Value of a derivation at a jet. Prolongation of a tangent field

Given a tangent field on \( V \) or, what is the same, if we dispose of a derivation \( D \) of the ring \( C^\infty(V) \), for each jet \( p \in \mathcal{J}_m^1 V \) we can carry out the composition

\[
C^\infty(V) \xrightarrow{D} C^\infty(V) \xrightarrow{p} C^\infty(V)/p
\]

(2.6)

whose class, according Theorem 2.2, defines a tangent vector

\[
D_p := [p \circ D]_p \in T_p \mathcal{J}_m^1 V
\]

(2.7)

We will call \( D_p \) the value of \( D \) at \( p \).

It is not difficult to check that tangent field \( p \mapsto D_p \) thus defined is the usual 1-jet *prolongation*, say \( D^1 \), of the tangent field \( D \) from \( V \) to \( \mathcal{J}_m^1 V \). In this way, prolongation of a field \( D \) is \( D \) itself but considered as “acting” on another points of \( V \).

2.5. The contact system

We associate to each jet \( p \) the set of differentials of functions that belong to the ideal \( p \): The distribution of covectors on \( \mathcal{J}_m^1 V \) defined as

\[
p \mapsto \Omega_p := \pi_1^* d_v p = \pi_1^* \{ d_v f \in T_v^* V \mid f \in p \} \subset T_p^* \mathcal{J}_m^1 V, \quad v = \pi_1(p),
\]

(2.8)

is called the *contact system* (or *contact distribution*). In a sense, we could say that jet \( p \) and contact system \( \Omega_p \) are “the same object”. If we take local coordinates \( x_i, y_j, y_{ji} \), as in (2.3), we obtain that the contact system is generated by 1-forms \( dy_j - y_{ji} dx_i, \ j = 1, \ldots, n - m \).
Dually, the tangent vectors annihilated by the contact system define the so-called Cartan distribution

$$ p \mapsto C_p := \{ X_p \in T_p J^1_m V \mid \pi_1 X_p(f) = 0, \forall f \in p \} \subset T_p J^1_m V $$

(2.9)

(so that $C_p$ is exactly the set of those tangent vectors $X_p$ such that $\pi_1 X_p \in L_p$).

2.6. Jets of sections

When $V$ is endowed with an structure of fiber bundle $\pi: V \to M$, with $\dim M = m$, we consider the open subset

$$ J^1(M, V) := \{ p \in J^1_m V \mid \text{composition } C^\infty(M) \hookrightarrow C^\infty(V) \to C^\infty(V)/p \text{ is surjective} \} $$

called the space of 1-jets of sections of $\pi: V \to M$.

By the very definition, if $p \in J^1(M, V)$ and $x$ is its projection on $M$, it easily follows the isomorphism

$$ C^\infty(M)/m^2_x \simeq C^\infty(V)/p. \quad (2.10) $$

We see that space $J^1(M, V)$, by means of the above identification, equals the set of surjective maps

$$ C^\infty(V) \to C^\infty(M)/m^2_x, $$

such that, when restricted to $C^\infty(M)$ became the factor map $C^\infty(M) \to C^\infty(M)/m^2_x$. Conversely, if $\sigma$ is a local section of $\pi$ defined around $x \in M$, the induced morphism from $C^\infty(V)$ to $C^\infty(M)$, modulo $m^2_x$, gives a jet which is denoted by $j^1_2 \sigma$.

The translation of Theorem 2.2 is

$$ T_p J^1(M, V) = \text{Der}_{\mathbb{R}}(C^\infty(V), C^\infty(M)/m^2_x)/\sim, $$

(recall that $D \sim \overline{D}$ iff $D - \overline{D}$ kills $p$). Further analysis proves that the subspace formed by the tangent vectors in $T_p J^1(M, V)$ which are vertical with respect to projection $\pi_1: J^1(M, V) \to V$ are identified with the module of derivations

$$ \text{Der}_{C^\infty(M)}(C^\infty(V), m_x/m^2_x). $$

2.7. Affine structure

When two jets $p, q \in J^1(M, V)$ project onto the same point $v \in V$, it can be checked that their difference

$$ Z := q - p: C^\infty(V) \to C^\infty(M)/m^2_x, \quad x := \pi(v), $$

(2.11)

is a derivation. Development of this fact leads to the following conclusion: the projection $J^1(M, V) \to V$ is an affine bundle modeled on the vector bundle $Q \to V$ whose fiber at each $v \in V$ is given by $Q_v = \text{Der}_{C^\infty(M)}(C^\infty(V), m_x/m^2_x)$, where $m_x/m^2_x$ is considered as a $C^\infty(V)$-module via an arbitrary jet $p$ with $\pi_1(p) = v$.

When $\pi: V \to M$ is a vector bundle, a section of $V^*$ define a function on $V$; from this it can be derived that $Q_v = V_x \otimes T^*_x M$. Hence, $J^1(M, V)$ is an affine bundle modeled on the vector bundle $\pi^*(V \otimes T^* M) \to V$.

3. Connections

In this section we discuss the concept of connection in the context of Weil jets theory.
3.1. General connections
We give a quite general definition of (first order) connection by considering it as a synonymous of distribution:

**Definition 3.1.** A connection (of dimension $m$) on $V$ is a distribution of $m$-dimensional tangent planes; this is to say, a section

$$V \xrightarrow{\nabla} J^1_m V$$

of the jet bundle projection $J^1_m V \to V$.

For each $v \in V$ we have three equivalent objects:

(i) The 1-jet $\nabla(v)$ which we identify with an ideal $\subset \mathcal{C}^\infty(V)$.

(ii) The quotient map $\mathcal{C}^\infty(V) \xrightarrow{\nabla(v)} \mathcal{C}^\infty(V)/\nabla(v)$ (which we denote by the same symbol).

(iii) The $m$-plane $L_v = \{D \in T_v V \mid D \cdot \nabla(v) = 0\}$ which is the value at $v$ of the distribution $L$ defined by $\nabla$ (this is the $L_{\nabla(v)}$ according (2.4)).

So that, for each $v \in V$, $\nabla(v)$ is given in local coordinates as $x_i(\nabla(v)) = x_i(v)$, $y_j(\nabla(v)) = y_j(v)$, and

$$y_{ji}(\nabla(v)) = \Gamma^j_i(v),$$

for some locally defined functions $\Gamma^j_i \in \mathcal{C}^\infty(V)$. In this way, $\nabla(v) \subset \mathcal{C}^\infty(V)$ is the ideal

$$\left(y_j - y_j(v) - \Gamma^j_i(v)(x_i - x_i(v))\right)_{j=1,\ldots,n-m} + m^2_v; \tag{3.1}$$

the quotient is

$$x_i \mapsto x_i, \quad y_j \mapsto y_j(v) + \Gamma^j_i(v)(x_i - x_i(v)), \mod m^2_v$$

and the associated plane $L_v$ is the kernel of $d_v y_j - \Gamma^j_i(v) d_v x_i$, $j = 1, \ldots, n - m$; this is to say,

$$L_v = (D_{i1}, \ldots, D_{mn}) \quad \text{where} \quad D_{iv} := \left(\frac{\partial}{\partial x_i}\right)_v + \Gamma^j_i(v) \left(\frac{\partial}{\partial y_j}\right)_v. \tag{3.2}$$

On the other hand, as $\nabla(v) \supset m^2_v$, we have the factor map

$$m_v/m^2_v \to m_v/\nabla(v).$$

That allows us to give the following

**Definition 3.2.** The covariant differential of a function $F \in \mathcal{C}^\infty(V)$ at a point $v \in V$ is, by definition, the class of $d_v F \in m_v/m^2_v$ modulo $\nabla(v)/m^2_v$ and we will denote it by

$$d^\nabla_v F \in m_v/\nabla(v) \simeq L^*_v.$$

**Coordinates.** Given $F \in \mathcal{C}^\infty(V)$, we have

$$F \equiv F(v) + \left(\frac{\partial F}{\partial x_i}\right)_v (x_i - x_i(v)) + \left(\frac{\partial F}{\partial y_j}\right)_p (y_j - y_j(v)) \mod m^2_v.$$

Since the jet $\nabla(v) = (x_i(v), y_j(v), y_{ji} = \Gamma^j_i(v))$ represents the ideal (3.1), we get

$$y_j - y_j(v) \equiv \Gamma^j_i(v)(x_i - x_i(v)) \mod \nabla(v),$$

and then, according to definitions,

$$d^\nabla_v F \equiv \left[\left(\frac{\partial F}{\partial x_i}\right)_v + \Gamma^j_i(v) \left(\frac{\partial F}{\partial y_j}\right)_p\right] (x_i - x_i(v)) = (D_{iv} F)(x_i - x_i(v)) \mod \nabla(v),$$

where vectors $D_{iv}$ are given as in (3.2).
3.2. Connections on fiber bundles
When $V$ is endowed with a structure of fiber bundle $\pi: V \to M$, we will consider distributions
\[ \nabla: V \to \mathcal{J}^1(M, V), \]
so that the distribution defined is transversal to the fibers of $\pi$. This kind of object is also known as jet field (see, e.g. [20], p. 145).

In this situation, if $v \in V$ is in the fiber of $x \in M$, we have a natural identification $m_p/\nabla(p) \simeq m_x/m_x^2$ (like in (2.10) for $p = \nabla(v)$), which allow us to interpret the covariant differential of each function $F \in C^\infty(V)$ as an ordinary differential form at $x$:
\[ d_v^\nabla F \in m_v/\nabla(v) \simeq m_x/m_x^2 = T_x^* M. \] (3.3)
In fact, we can consider the covariant differential of a function restricted to each fiber $V_x := \pi^{-1}(x)$, for $x \in M$:
\[ d_v^\nabla F: V_x \to T_x^* M; \quad v \mapsto d_v^\nabla F \] (3.4)

On the other hand, for each $X_x \in T_x M$ and each $v \in V_x$, we will denote by $X_v^\nabla \in T_v V$ the tangent vector defined by composition

\[
\begin{align*}
C^\infty(V) \xrightarrow{\nabla(v)} C^\infty(V)/\nabla(v) & \simeq C^\infty(M)/m_x^2 \xrightarrow{X_x} \mathbb{R}. \\
\end{align*}
\] (3.5)

It is easily checked that, 1) By the very definition, $\pi_* X_v^\nabla = X_x$ and 2) $X_v^\nabla \in L_v$, because map $\nabla(v)$ annihilates ideal $\nabla(v)$. Vector $X_v^\nabla$ is the only one fulfilling these two conditions. In terms of that “lift”, we have
\[ \langle d_v^\nabla F, X_x \rangle = \langle d_v F, X_v^\nabla \rangle = X_x(\nabla(v)F). \] (3.6)

3.3. Connections on vector bundles
When $\pi: V \to M$ is a vector bundle, we will impose on $\nabla: V \to \mathcal{J}^1(M, V)$ the condition of being linear. So, for each linear function $F \in C^\infty(V)$ and each point $x \in M$, the map defined in (3.4) is also linear and then,
\[ d_v^\nabla F \in V_x^* \otimes T_x^* M. \]

On the other hand, each section $\omega$ of the dual bundle $V^* \to M$, defines a linear function, say $F_\omega$, on $V$; that allow us to form its covariant differential
\[ d_\omega^\nabla := d_x^\nabla F_\omega \in V_x^* \otimes T_x^* X, \quad x \in X. \]

For each tangent vector $D_x \in T_x M$, we define the covariant derivative as usual by the coupling
\[ D_x^\nabla \omega := \langle D_x, d_\omega^\nabla \rangle \in V_x^*. \] (3.7)

Coordinates. The linearity of a connection is reflected on the coordinate expressions in the following way. Assume that the $y_j$’s are the coordinates defined by a collection of sections $\alpha_j$ of the dual bundle $V^* \to M$; in this case,
\[ \Gamma_i^j(x, y) = -\Gamma_{ik}^j(x) y_k, \]
for suitable local functions $\Gamma_{ik}^j \in C^\infty(M)$ (the sign minus is traditional).
If $\omega = \sum_{k} f_k \alpha_k$, with $f_k \in C^\infty(M)$, when is considered as a function on $V$ equals $F_\omega = \sum_{k} f_k y_k$ and then,

$$\left(\frac{\partial \nabla \omega}{\partial x_i}\right)_x = \left(\frac{\partial f_k}{\partial x_i}\right)_x - \Gamma^j_{ik}(x)f_j(x) y_k;$$

so that

$$d_x^\nabla \omega = \left(\frac{\partial f_k}{\partial x_i}\right)_x - \Gamma^j_{ik}(x)f_j(x) (\alpha_k)_x \otimes d_x x_i.$$

### 3.4. Covariant differential of a section

Let $\sigma : M \to V$ be a section of $\pi : V \to M$. For each $x \in M$ we have two jets: $j^1_x \sigma$ and $\nabla(\sigma(x))$. The covariant differential of section $\sigma$ at $x$ is, by definition, the difference

$$d_x^\nabla \sigma := j^1_x \sigma - \nabla(\sigma(x))$$

as a map $C^\infty(V) \to C^\infty(M)/m^2$.

Because both jets, $j^1_x \sigma$ and $\nabla(\sigma(x))$, project onto $\sigma(x)$, we can consider $d_x^\nabla \sigma$ as an element of $V_x \otimes \mathbb{R} T^*_x M$ (see Subsection 2.7). Also, we can define the covariant derivative of $\sigma$ with respect to a vector $D_x \in T_x M$ as

$$D_x^\nabla \sigma = \langle D_x, d_x^\nabla \sigma \rangle \in V_x$$

On the other hand, a section $\sigma$ is a solution of (distribution) $\nabla$ if for each $x \in M$, we have $\sigma_* T_x M = L_\sigma(x)$; by taking incident on both members we get

$$j^1_x \sigma = \nabla(\sigma(x));$$

in words: “in an infinitesimal neighborhood of each point $v = \sigma(x)$, $\sigma$ coincides with the extension $\nabla(\sigma(x))$”. So that, almost by the very definition of covariant differential:

**Proposition 3.3.** Section $\sigma$ is a solution of distribution $\nabla$ if and only if $d_x^\nabla \sigma = 0$.

**Coordinates.** Let us suppose that section $\sigma$ is locally given by $y_j = \sigma_j(x)$; in this case,

$$j^1_x \sigma = (x_i, \sigma_j(x), (\partial \sigma_j/\partial x_i)_x)$$

and

$$\nabla(\sigma(x)) = (x_i, \sigma_j(x), -\sum_k \Gamma^j_{ik}(x)\sigma_k(x)).$$

It follows that

$$d_x^\nabla \sigma = \sum \left(\left(\frac{\partial \sigma_j}{\partial x_i}\right)_x + \Gamma^j_{ik}(x)\sigma_k(x)\right) d_x x_i \otimes \alpha_j.$$
4. Connections on tangent bundles

4.1. Jets of tangent fields

Prior to introduce connections on the tangent bundle, we will describe jets \( p \in J^1(M, TM) \) in a convenient way for our purposes.

We know that such a jet \( p \) is equivalent to give an \( \mathbb{R} \)-algebra morphism

\[
p : C^\infty(TM) \to C^\infty(M) / m_x^2
\]

whose restriction to \( C^\infty(M) \) is the natural quotient map.

On the other hand we have the map “differential”:

\[
\delta : C^\infty(M) \to C^\infty(TM), \quad f \mapsto \delta f
\]

defined by the rule

\[
\delta f(D_x) := D_x f, \quad \text{for each} \quad f \in C^\infty(M), D_x \in TM.
\]

Each jet \( p \in J^1(M, TM) \) is completely determined by its composition with \( \delta \), because its restriction to \( C^\infty(M) \) becomes always the quotient map modulo \( m_x^2 \); in this way, we can identify \( p \) with the derivation

\[
p \circ \delta : C^\infty(M) \to C^\infty(M) / m_x^2.
\]

Conversely, if we have a (local) tangent field \( D \) on \( M \), then its jet \( j^1_x D \) is the only jet (thought as a map) such that the following square commutes,

\[
\begin{array}{ccc}
C^\infty(M) & \xrightarrow{D} & C^\infty(M) \\
\delta \downarrow & & \downarrow \delta \\
C^\infty(TM) & \xrightarrow{j^1_x D} & C^\infty(M) / m_x^2
\end{array}
\]

In other words, we have a natural identification

\[
J^1(M, TM)_x \simeq \text{Der}_{\mathbb{R}} \left( C^\infty(M), C^\infty(M) / m_x^2 \right)
\]

(or also, \( \simeq \text{Der}_{\mathbb{R}} \left( C^\infty(M) / m_x^3, C^\infty(M) / m_x^2 \right) \)).

4.2. Affine connections

We now consider connections on the tangent bundle \( V = TM \to M \); they are the so known affine connections (or sometimes also linear connections).

According with the interpretation of jets in \( J^1(M, TM) \), a linear connection associates with each vector \( D_x \in T_x M \) a derivation \( \tilde{D}_x \) which makes commutative the following diagram:

\[
\begin{array}{ccc}
C^\infty(M) / m_x^3 & \xrightarrow{\tilde{D}_x} & C^\infty(M) / m_x^2 \\
\downarrow & & \downarrow \\
C^\infty(M) / m_x^2 & \xrightarrow{D_x} & C^\infty(M) / m_x
\end{array}
\]

(vertical arrows represent natural quotient maps).
4.3. An example: Levi-Civita connection

Let $g$ be a non-degenerate metric on $M$. Let us fix a point $x \in M$ and let $0_x \in T_x M$ be the null tangent vector at $x$. The value of $g$ at $x$ defines a metric $g_0$ on the vector space $T_x M$ and, then, also on the manifold $T_x M$. It is possible to determine univocally (up to second order) an isometry around $x$ between $(M, g)$ and $(T_x M, g_0)$ which reduces to identity at the level of tangent spaces: if $x_i$ are local coordinates around $x$, with $x_i(x) = 0$ and denote by $t_i$ the linear coordinates that they induce on $T_x M$, a calculation gives

$$ C^\infty(M)/m_3 \rightarrow C^\infty(T_x M)/m_3^3, \quad x_i \mapsto t_i - \frac{1}{2} \Gamma^i_{hk} t_h t_k \quad (4.3) $$

where $\Gamma^i_{hk}$ are the well known Christoffel symbols associated to $g$ (at the point $x$); this is to say: $\partial h_i / \partial x_j - g_{ik} \Gamma^l_{jk} - g_{jl} \Gamma^l_{ik} = 0$. Hence, by means of the above identification, both metric are osculating in the sense of Cartan (see [6], p. 90, or [22], p. 268).

Once identification (4.3) is obtained, we can transfer the trivial connection defined on the vector space $T_x M$ to the manifold $M$: for each $D_x = a_i(\partial / \partial x_i)_x$, $a_i \in \mathbb{R}$, we take its image (under (4.3)) $\widetilde{D}_{0_x} = a_i(\partial / \partial t_i)_0_x \in T_{0_x}(T_x M)$; the trivial connection consist of associating to vector $\widetilde{D}_{0_x}$ the derivation $\overline{\nabla}_{\widetilde{D}_{0_x}} = a_i(\partial / \partial t_i)$ (this is to say, maintaining constant the coefficients). Finally, the derivation that we associate with $D_x$ is the composition

$$ C^\infty(M)/m_3 \xrightarrow{(4.3)} C^\infty(T_x M)/m_3^3 \xrightarrow{\widetilde{D}_{0_x}} C^\infty(T_x M)/m_3^2 \xrightarrow{(4.3)^{-1}} C^\infty(M)/m_2 $$

so that $\widetilde{D}_x(x_i) = \overline{\nabla}_{\widetilde{D}_{0_x}}(t_i - \frac{1}{2} \Gamma^i_{hk} t_h t_k) = a_i - \Gamma^i_{hk} a_h x_k$.

4.4. Torsion tensor and covariant derivatives

Let $D_1, x$ a tangent vector at $x$ and $D_2$ a vector field (both on $M$). According the precedent definitions, we have the covariant derivative

$$ D^\nabla_{(1, x)} D_2 = D_{1, x} \circ (j^1_x D_2 - \nabla(D_2, x)) \quad (4.4) $$

If we follow the definitions and denote by $\widetilde{D}_{2, x}$ the derivation representing the jet $\nabla(D_2, x)$, then

$$ D^\nabla_{(1, x)} D_2 = D_{1, x} \circ D_2 - D_{1, x} \circ \widetilde{D}_{2, x} \quad (4.5) $$

On the other hand, following the usual definition, the torsion of $\nabla$ can be computed as

$$ \text{Tor}_\nabla(D_{1, x}, D_{2, x}) = D^\nabla_{(1, x)} D_2 - D^\nabla_{(2, x)} D_1 - [D_1, D_2]_x = D_{1, x} \circ D_2 - D_{1, x} \circ \widetilde{D}_{2, x} - (D_{2, x} \circ D_1 - D_{2, x} \circ \widetilde{D}_{1, x}) - [D_1, D_2]_x $$

As a consequence, we get a nice interpretation of torsion tensor.

**Proposition 4.1.** The value of the torsion when applied on two tangent vectors $D_{1, x}$, $D_{2, x}$, is the commutator of the corresponding jet extensions defined by the connection. This is to say, it
is the obstruction to the square
\[
\begin{array}{c}
\frac{C^\infty(M)}{m_x^3} \xrightarrow{\partial_{1,x}} \frac{C^\infty(M)}{m_x^2} \\
\frac{\tilde{D}_{2,x}}{D_{2,x}} \xrightarrow{D_{1,x}} \frac{C^\infty(M)}{m_x^2} \simeq \mathbb{R}
\end{array}
\]
(4.6)
to be commutative.

Besides (4.5) there is another covariant derivative, also associated with \(\nabla\), which seems to be quite ‘natural’,
\[
D_{1,x}^{\nabla}D_2 := [\tilde{D}_{1,x}, D_2]
\]
(4.7)

Covariant derivative \(D_{1,x}^{\nabla}D_2\) is the Lie derivative of \(D_2\) along the jet extension of \(D_{1,x}\) defined by the connection.

Covariant derivatives (4.5) and (4.7) differ by the torsion tensor so that they define the conjugate connection one of each other (see, for instance, [5], p. 232 or [18], p. 78). From this point of view, we can say that “Covariant derivatives are also Lie derivatives”.

4.5. Geodesic field

As another example, we will show here how easily the second order equation of geodesic curves of a connection on \(TM\) appears.

With each tangent vector \(D_x \in T_x M \subset TM\), connection \(\nabla\) associates a jet \(\nabla(D_x) \in J^1(M, TM)\); hence, we can perform the composition
\[
\frac{C^\infty(TM)}{\nabla(D_x)} \xrightarrow{\nabla(D_x)} \frac{C^\infty(M)}{m_x^2} \xrightarrow{D_x} \mathbb{R}^1
\]
(4.8)
(here we consider \(D_x\) as a point of Weil bundle \(M^1 = TM\) which, according to 2.1 is a tangent vector, say \(G(D_x)\), in \(T_{D_x} TM\)). In this way, a tangent field
\[
G: TM \rightarrow TTM, \quad D_x \mapsto G(D_x),
\]
known as the geodesic field associated with connection \(\nabla\), is defined (although it depends only on the symmetrized connection).

It is not too difficult to formally check that integral curves of \(G\) are the geodesic curves of \(\nabla\) (curves with null acceleration). Nevertheless, a calculation in coordinates is an alternative way to show that the assertion in this paragraph is correct.

5. Curvature

We come back to consider general \((m\text{-dimensional})\) connections \(\nabla: V \rightarrow J^1_m V\). Later, we will need the following result.

**Lemma 5.1.** Let \(D \in \text{Der}_R(C^\infty(V), C^\infty(V))\) be a section of the distribution \(L\) (induced by \(\nabla\)); then the value \(D_{\nabla(v)}\) of \(D\) at a jet \(\nabla(v) \in J^1_m(V)\) (see Equation (2.7)), depends uniquely on the vector \(D_v\).
Proof. Let suppose $D_v = 0$ and let, locally, $L = \{D_1, \ldots, D_m\}$, for some vector fields $D_i$. Each section of $L$, by construction, kills the ideal $\nabla(v)$ at the point $v$, so that
\[D_i(\nabla(v)) \subset m_v, \quad i = 1, \ldots, m.\]

On the other hand, $D = \sum_i f_i D_i$, for appropriate functions $f_i \in m_v$, because $D$ vanishes at $v$ by assumption.

As a consequence, $D(\nabla(v)) \subset m_v^2 \subset \nabla(v)$, and, therefore,
\[D\nabla(v) := [\nabla(v) \circ D] = 0 \in T_{\nabla(v)} J^1_m V,\]
(see Theorem 2.2).

We have accordingly two canonical fields of tangent $m$-planes along $\nabla(V)$: the distribution $L$ itself transported to $\nabla(V)$ and the distribution $H$ obtained by taking values as in the lemma above. Concretely, to each point $\nabla(v) \in \nabla(V) \subset J^1_m V$ we can attach two well defined $m$-planes contained within $T_{\nabla(v)} J^1_m V$:

(i) $\nabla_i L_v \subseteq T_{\nabla(v)} \nabla(V)$, where $L_v$ is the distribution $L$ at $v$.

(ii) $H_{\nabla(v)} := \{\text{Set of values at } \nabla(v) \text{ of the sections of } L\}$.

These two $m$-plans are different. In fact, as we will see, $H_{\nabla(v)}$ can be not tangent to the submanifold $\nabla(V) \subset J^1_m V$.

Computations. Let $x_i, y_j$ local coordinates such that the tangent fields
\[D_i := \frac{\partial}{\partial x_i} + \Gamma^j_i \frac{\partial}{\partial y_j}, \quad i = 1, \ldots, m,\]
span distribution $L$. A calculation gives
\[\nabla_*(D_i)_v = \left(\frac{\partial}{\partial x_i}\right)_{\nabla(v)} + \Gamma^j_i(v) \left(\frac{\partial}{\partial y_j}\right)_{\nabla(v)} + (D_i)_v(\Gamma^k_i) \left(\frac{\partial}{\partial y_k}\right)_{\nabla(v)}\]
\[(D_i)_{\nabla(v)} = \left(\frac{\partial}{\partial x_i}\right)_{\nabla(v)} + \Gamma^j_i(v) \left(\frac{\partial}{\partial y_j}\right)_{\nabla(v)} + (D_r)_v(\Gamma^k_i) \left(\frac{\partial}{\partial y_k}\right)_{\nabla(v)},\]
where we can see that all is identical except by a permutation on indexes $r$ and $i$ in the last coefficients of both expressions.

In the sequel we will proceed to analyze what this difference intrinsically is.

Lemma 5.2. Let $L$ be the distribution induced by $\nabla$ and let $p$ be a jet in $J^1_m V$; then
\[p \in \nabla(V) \iff \langle \omega_p, D_p \rangle = 0, \quad \text{for all } D \in L, \omega_p \in \Omega_p,\]
where $\Omega_p$ is the contact system at $p$ (see Subsection 2.5).

Proof. Let $v \in V$ be the source point of the jet $p$ and $D \in L$. Since $\Omega_p = \pi^*_v d_v f$, we have $\omega_p = \pi^*_v d_v f$ for some function $f \in p$:
\[\langle \omega_p, D_p \rangle = \langle d_v f, \pi_1^* D_p \rangle = \langle d_v f, D_v \rangle = D_v f.\]
From which we derive the equivalence
\[\langle \omega_p, D_p \rangle = 0, \quad \forall D \in L, \omega_p \in \Omega_p \iff d_v p = d_v \nabla(v) \iff p = \nabla(v)\]
As a consequence we get intrinsic and symbolically the equations of $\nabla(V) \subset J^1_mV$:

$$\langle \Omega, L^1 \rangle = 0 \quad (5.1)$$

where we denote by $L^1$ the set of prolongations of vector fields in $L$ to $J^1_mV$, so that $L^1 = \{ D^1_p := D^1_p \mid D^1 \in L \}$ (here, $D^1_p$ denotes the tangent vector that vector field $D^1$ on $J^1_mV$ gives at point $p$; instead, $D_p$ is the value at $p$ of a vector field $D$ on the base manifold $V$ according Subsection 2.4).

The previous lemma going to be used to obtain a geometric characterization of the symmetries of the distribution $L$. The statement is, moreover, that one would expect.

**Proposition 5.3.** Let $\mathcal{D}$ be a tangent vector field on $V$. Then,

$$\mathcal{D} \text{ is a symmetry of } \nabla \iff \mathcal{D}_p \in T_p\nabla(V), \text{ for all } p \in \nabla(V).$$

*(this is to say, if the prolongation of $\mathcal{D}$ to $J^1_mV$ is tangent to submanifold $\nabla(V))$.

**Proof.** Let $D \in L$, then:

$$\mathcal{D} (\Omega, D^1) = \langle L_{\mathcal{D}}\Omega, D^1 \rangle + \langle \Omega, [\mathcal{D}^1, D^1] \rangle$$

But, it is well known that prolongation of vector fields to jet spaces holds always the following properties: $L_{\mathcal{D}}\Omega \subset \Omega$ (preserves the contact system) and $[\mathcal{D}^1, D^1] = [\mathcal{D}, D]^1$ (defines a Lie algebra morphism); in this way, taking into account Equations (5.1), we have that

$$\mathcal{D}^1 (\Omega, D^1) = \langle \Omega, [\mathcal{D}, D]^1 \rangle$$

holds on $\nabla(V)$. Hence, for every point $p = \nabla(v)$ we get

$$\mathcal{D}^1_p (\Omega, D^1) = \langle \pi^*_v d_v p, [\mathcal{D}, D]^1_p \rangle = \langle d_v \nabla(v), [\mathcal{D}, D]_v \rangle.$$

\qed

**Corollary 5.4.** The distribution $L$ induced by $\nabla$ on $V$ is involutive (or completely integrable) if and only if for all $D \in L$ and $p \in \nabla(V)$ we have $D_p \in T_p\nabla(V)$; that is to say, the jet prolongation of each section of $L$ is tangent to the submanifold $\nabla(V)$.

Consequently, we can measure the integrability of $L$ (or $\nabla$) by comparing vectors in $\nabla_L$ and $H$ in the following way.

**Definition 5.5.** For each $D_v \in L_v$ let us define the curvature of $\nabla$ at $D_v$ by

$$R(D_v) := \nabla_s D_v - D_{\nabla(v)} \in Q\nabla(v),$$

where $D \in L$ is an arbitrary section extending $D_v$ and $Q \subset T J^1_mV$ denotes the vertical tangent sub-bundle w.r.t. projection $\pi^1: J^1_mV \rightarrow V$.

According to the above corollary, the map of vector bundles $R: L \rightarrow Q|\nabla(V)$ (on $V \simeq \nabla(V)$) thus defined holds, as the name of curvature given to it suggests, that

$$R = 0 \text{ if and only if } \nabla \text{ is completely integrable.}$$

**Local computation.** Let $x_i, y_j$, local coordinates on $V$ such that $D_i := \frac{\partial}{\partial x_i} + \Gamma^h_i \frac{\partial}{\partial y_h}$, $i = 1, \ldots, m$, span $L$. Then

$$R(D_{iv}) = \left( D_{iv} \Gamma^h_r - D_{rv} \Gamma^h_i \right) \left( \frac{\partial}{\partial y_h} \right)_{\nabla(v)} = \left( \frac{\partial \Gamma^h_i}{\partial x_i} + \Gamma^j_i \frac{\partial}{\partial y_j} - \Gamma^j_i \frac{\partial \Gamma^j_r}{\partial y_j} \right)_v \left( \frac{\partial}{\partial y_h} \right)_{\nabla(v)}$$
5.1. Case of a fiber bundle
Let $\nabla$ be a connection on the fiber bundle $\pi: V \to M$. Now we can take a tangent vector $X \in T_xM$, lift it horizontally up to a point $v \in V_x$ (see (3.5) and then take the curvature; so, we define

$$R(X; v) := R(X^\nabla)_v$$

If, in addition, $\pi$ is a vector bundle and $\nabla$ is linear, the same is applied but, now, $R(X; v)$ depends linearly on both arguments, $X \in T_xM$ and $v \in V_x$.

Local computations. Case of a fiber bundle. In a local chart as above,

$$R(\partial/\partial x^i; v) = \left(\left(\frac{\partial}{\partial x^i}\right)_v^\nabla \Gamma^h_i - \left(\frac{\partial}{\partial x^i}\right)_v^\nabla \Gamma^h_r\right) \left(\frac{\partial}{\partial y^h}\right)_v\nabla (v)$$

If the connection $\nabla$ depends linearly on the fiber coordinates $y_j$, then

$$\Gamma^j_{ik}(x, y) = -\Gamma^j_{ik}(x) y_k$$

for some functions $\Gamma^j_{ik} \in C^\infty(M)$ (we continue the tradition with a minus sign); we get

$$R((\partial/\partial x^i)_x; v) = \left(\frac{\partial \Gamma^h_i}{\partial x^i} - \frac{\partial \Gamma^h_r}{\partial x^r} + \Gamma^h_{ij} \frac{\partial \Gamma^j_l}{\partial y^r} - \Gamma^h_{rl} \frac{\partial \Gamma^j_i}{\partial y^r}\right) \left(\frac{\partial}{\partial y^h}\right)_v\nabla (v)$$

whose coefficients contain the familiar expressions of the curvature tensor.

5.2. Case of affine connections. The Ricci tensor
When working with connections on the tangent bundle, the vertical bundle $Q_p \subset T_p J^1(M, TM)$ is identified with

$$V_x \otimes T^* x M = T_x M \otimes T^*_x M.$$

So,

$$R(X_x; Y_x) = R(X^\nabla_y)_v \in T_x M \otimes T^*_x M,$$

from which we easily could recover the usual Riemann-Christoffel (3,1)-tensor (homomorphism $R(X_x; Y_x)$ must not be confused with the usual commutator of covariant derivatives).

Now, we can consider the natural coupling:

$$c : TM \otimes T^* M \to \mathbb{R}, \quad Z_x \otimes \omega_x \to (Z_x, \omega_x).$$

In this way, Ricci tensor is directly obtained by taking trace,

$$\text{Ric} := c \circ R, \quad \text{or} \quad \text{Ric} (X_x, Y_x) = c (R(X_x; Y_x)),$$

as a little calculation would show.

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References
[1] Alonso-Blanco R J 2000 Jet manifold associated to a Weil bundle Arch. Math. (Brno) 36 195–9
[2] Alonso-Blanco R J 2003 A transformation of PDE systems related to Drach theorem J. Math. Anal. Appl. 288 530–9
[3] Alonso-Blanco R J, Jiménez S and Rodríguez J 2009 Some canonical structures of Cartan planes in jet spaces and applications Differential Geometry, Symmetries and Integrability, 5th Abel Symp., Tromsø, Norway 2008 ed B Kruglikov et al (Berlin: Springer) 1–20
[4] Alonso-Blanco R J and Muñoz J 2004 The contact system for the A-jet manifolds Arch. Math. (Brno) 40 (3) 233–48
[5] Bishop R L and Goldberg S I 1968 Tensor Analysis on Manifolds (New York: The Macmillan Co.)
[6] Cartan E 1928 Leçons sur la Géométrie des Espaces de Riemann (Paris: Gauthier-Villars)
[7] Jiménez S, Muñoz J and Rodríguez J 2002 On the reduction of some systems of partial differential equations to first order systems with only one unknown function Proc. of the 8th Int. Conf. Differential Geometry and its Applications, Opava, Czech Republic 2001 ed O Kowalski et al (Opava: Silesian University) pp 187–95
[8] Jiménez S 2011 Symmetries of PDE systems and correspondences between jet spaces J. Math. Anal. Appl. 378 23–41
[9] Jiménez S, Muñoz J and Rodríguez J 2005 Correspondences between jet spaces and PDE systems J. Lie Theory. 15 197–218
[10] Kolář I, Michor P W and Slovák J 1993 Natural Operations in Differential Geometry (Berlin: Springer-Verlag)
[11] Lie S 1888 Theorie der Transformationsgruppen (Leipzig: Teubner)
[12] Muñoz J, Muriel J and Rodríguez J 1999 The canonical isomorphism between the prolongation of the symbols of a nonlinear Lie equation and its attached linear Lie equation Proc. of the 7th Int. Conf. Differential Geometry and its Applications, Brno, Czech Republic, 1998 ed I Kolář et al (Brno: Masaryk Univ.) pp 255–61
[13] Muñoz J, Muriel J and Rodríguez J 2000 Integrability of Lie equations and pseudogroups J. Math. Anal. Appl. 252 32–49
[14] Muñoz J, Muriel J and Rodríguez J 2000 Weil bundles and jet spaces Czechoslovak Math. J. 50 (125) 721–48
[15] Muñoz J, Muriel J and Rodríguez J 2001 A remark on Goldschmidt’s theorem on formal integrability J. Math. Anal. Appl. 254 275–90
[16] Muñoz J, Muriel J and Rodríguez J 2001 The contact system on the (m, l)-jet spaces Arch. Math.(Brno) 37 291–300
[17] Muñoz J, Muriel J and Rodríguez J 2003 On the finiteness of differential invariants J. Math. Anal. Appl. 284 266–82
[18] Nomizu K 1956 Lie Groups and Differential Geometry (Tokyo: Math. Soc. of Japan)
[19] Rodríguez J 1990 Sobre los espacios de jets y los fundamentos de la teoría de los sistemas de ecuaciones en derivadas parciales Ph. D. Thesis Universidad de Salamanca
[20] Saunders D J 1989 The Geometry of Jet Bundles London Math. Soc. Lecture Note Ser. 142 (Cambridge: Cambridge Univ. Press)
[21] Weil A 1953 Théorie des points proches sur les variétés différentiables Colloque de Géométrie Différentielle, C.N.R.S. 111–7
[22] Willmore T J 1959 An Introduction to Differential Geometry (New York: Oxford Univ. Press)