Dynamical supersymmetry for strange quark and $ud$ antidiquark in hadron mass spectrum

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Speculating that the $ud$ diquark with spin 0 has a similar mass to the constituent $s$ quark, we introduce a symmetry between the $s$ quark and the $ud$ diquark. Constructing an algebra for this symmetry, we regard a triplet of the $s$ quarks with spin up and down and the $ud$ diquark with spin 0 as a fundamental representation of this algebra. We further build higher representations constructed by direct products of the fundamental representations. We propose assignments of hadrons to the multiples of this algebra, in which we find in particular that \{$D_s, D_s^*, \Lambda_c$\} and \{$\eta_s, \phi, \Lambda, f_0(1370)$\} form a triplet and a nonet, respectively. We also find a mass relation between them by introducing the symmetry breaking due to the mass difference between the $s$ quark and the $ud$ diquark.

1. Introduction

Symmetries play important roles in hadron physics. Hadrons can be classified into the representations of symmetry groups, and the hadron masses and interactions can be explained by the symmetry properties. In particular, the flavor SU(3) symmetry is one of the most successful examples to understand the hadron spectra. After having discovered strangeness, one collects hadrons having a similar mass and classifies into octets and decuplet of the SU(3) representation [1, 2] according to the celebrated Gell-Mann Nishijima relation [3, 4]. Behind this classification, the up, down and strange quarks are found as the fundamental representation of the symmetry [5]. The flavor SU(3) symmetry is not exact but is broken explicitly with the quark mass difference. The symmetry breaking pattern is also constrained by the symmetry properties. Treating the quark mass difference as a first order perturbation, one obtains the so-called Gell-Mann Okubo mass formulae [2, 6] which relates the masses of the hadrons in the same multiplet. In this way, one finds the substantial objects which carry the fundamental properties of symmetry out of the hadron spectrum. In this paper, regarding the constituent strange quark and the $ud$ diquark as a fundamental object of a symmetry, we find mass relations among the hadrons classified in the same multiplet of the symmetry and discuss the possibility of the existence of the $ud$ diquark as an effective constituent of hadrons.
The diquark is a pair of two quarks and is considered as a strong candidate of the hadron constituent [7–9]. Because the diquark has color charge, it cannot be isolated and should exist inside hadrons. One expects a strong correlation particularly between up and down quarks with spin 0 and isospin 0 due to color magnetic interaction [10] and such strong correlations are also found in Lattice QCD studies [11–14]. The diquark has been investigated in a context of the quark models in Refs. [15–21], and recently it has been found in Refs. [22, 23] that the color electric force between diquark and quark could be weaker than that between quark and antiquark. A QCD sum rule approach [24] has suggested also the \( ud \) diquark as a constituent of the ground states of \( \Lambda, \Lambda_c \) and \( \Lambda_b \) having a constituent diquark mass around 0.4 GeV. The mass of the \( ud \) diquark is not fixed yet. Considering the \( u \) and \( d \) constituent quark mass to be about 0.3 GeV, one expects that the diquark mass be 0.4 to 0.6 GeV depending on the attraction between the \( u \) and \( d \) quark. Such value of the diquark mass is very similar to the constituent strange mass, which may be 0.5 GeV.

In this paper, we introduce a symmetry in which the constituent \( s \) quark and the \( \bar{u}d \) diquark form a fundamental representation thanks to their similar masses and classify hadrons according to the symmetry to discuss the breaking pattern of the symmetry in the mass spectrum of hadrons composed of the \( s \) quark and \( \bar{u}d \) diquark. This is the same approach to find the flavor SU(3) symmetry in the hadron spectrum. While both the \( s \) quark and the \( \bar{u}d \) diquark have the same color charge, they have different spins; the \( s \) quark is a fermion with spin 1/2 and the \( \bar{u}d \) diquark is a boson with spin 0. Thus, the symmetry that we consider here is a supersymmetry which transforms fermions and bosons. This kind of supersymmetry was introduced first in hadron physics in Refs. [25, 26]. There \((p, n, \Lambda)\) and \((\bar{K}^+, K^0, \eta)\) were considered as flavor fundamental representations, and a supersymmetry between these two triplets were investigated using so-called V(3) algebra. A supersymmetry between quark and diquark was discussed also in Refs. [27–29]. Dynamical supersymmetry in nuclear physics was suggested in Ref. [30].

In this paper, in Sec. 2 we define the algebra in which the spin up and down \( s \) quark and the \( \bar{u}d \) diquark with spin 0 form a triplet. In Sec. 3, we discuss the representation of the algebra introduced in Sec. 2, and show examples of the representations for hadrons in Sec. 4. Section 5 presents the symmetry breaking by the mass difference of the \( s \) quark and the \( \bar{u}d \) diquark, and derives a Gell-Mann Okubo type mass formula for \( \phi, \Lambda \) and \( f_0 \). Section 6 is devoted to summary and conclusion.

## 2. Definition of algebra

In this section, we introduce a supersymmetry for a Dirac fermion with spin 1/2 and a charged scalar boson in the flavor space according to Ref. [25].

### 2.1. Field definition

Let us first define the fermion and boson fields. We write the fermion and boson fields as \( \psi \) and \( \varphi \), respectively. The fermion field \( \psi \) has four components, two of them are so-called upper components in the Dirac representation, the others are the lower components, while the scalar field \( \varphi \) is composed of two independent real fields for a charged boson.
The Lagrangians for the free Dirac field and the scalar boson field are written as
\begin{align}
\mathcal{L}_F &= \bar{\psi} i \slashed{\partial} \psi - m \bar{\psi} \psi, \\
\mathcal{L}_B &= \partial_\mu \varphi \partial^{\mu} \varphi - m^2 \varphi \varphi^{\dagger},
\end{align}
respectively. Defining the conjugate momenta as
\begin{align}
\bar{\pi} &= \frac{\partial \mathcal{L}_F}{\partial \dot{\psi}} = i \bar{\psi}^{\dagger}, \\
\pi &= \frac{\partial \mathcal{L}_B}{\partial \dot{\varphi}} = \varphi^{\dagger}, \\
\pi^{\dagger} &= \frac{\partial \mathcal{L}_B}{\partial \dot{\varphi}^{\dagger}} = \dot{\varphi},
\end{align}
we have the corresponding Hamiltonians as
\begin{align}
\mathcal{H}_F &= \bar{\pi} \dot{\psi} - \mathcal{L}_F = \bar{\psi} i \nabla \cdot \gamma \psi + m \bar{\psi} \psi, \\
\mathcal{H}_B &= \pi \dot{\varphi} + \varphi^{\dagger} \pi^{\dagger} - \mathcal{L}_B = \nabla \varphi \cdot \nabla \varphi + m \frac{1}{m} \pi^{\dagger} \pi + m \varphi^{\dagger} \varphi.
\end{align}
Quantization is performed by introducing the equal-time commutation relations for the fermion and boson fields. The field commutation relations are given as
\begin{align}
\{ \psi_\alpha(x), \psi^{\dagger}_\beta(y) \} &= \delta_{\alpha\beta} \delta(x - y), \\
\{ \varphi(x), \pi(y) \} &= \delta(x - y), \quad \{ \varphi^{\dagger}(x), \pi^{\dagger}(y) \} = \delta(x - y),
\end{align}
for fermion, where \( \alpha \) and \( \beta \) stand for the Dirac components, and for boson. These expressions are not symmetric in terms of the fermion and boson fields. In the following we redefine the fields in a symmetric form.

Let us introduce two-component fields, \( \psi^{(+)} \) and \( \psi^{(-)} \), as the eigenfunction of \( \gamma_0 \) with eigenvalue \( \pm 1 \), respectively. In the Dirac representation, \( \psi^{(+)} \) and \( \psi^{(-)} \) are the upper and lower components of the Dirac field, respectively, as
\begin{equation}
\psi = \begin{pmatrix} \psi^{(+)} \\ \psi^{(-)} \end{pmatrix}.
\end{equation}
Their conjugate fields are denoted by
\begin{equation}
\hat{\psi}^{(\pm)} \equiv \bar{\psi}^{(\pm)} \gamma_0 = \psi^{(\pm)\dagger}
\end{equation}
Using the \( \hat{\psi}^{(\pm)} \) field, the anti-commutation relation of the fermion field is written as
\begin{align}
\{ \psi^{(+)}_i(x), \psi^{(+)}_j(y) \} &= \{ \psi^{(-)}_i(x), \psi^{(-)}_j(y) \} = \delta_{ij} \delta(x - y)
\end{align}
for \( i, j = 1, 2 \), and the \( \psi^{(+)} \) and \( \psi^{(-)} \) are anti-commuting.

The pseudoscalar and vector fields, \( \bar{\psi} \gamma_5 \psi \) and \( \bar{\psi} \gamma^i \psi \), are decomposed into \( \hat{\psi}^{(\pm)} \) and \( \psi^{(\pm)} \) as
\begin{align}
\bar{\psi} \gamma_5 \psi &= -\hat{\psi}^{(-)} \psi^{(+)} + \hat{\psi}^{(+)} \psi^{(-)}, \\
\bar{\psi} \gamma^i \psi &= \hat{\psi}^{(-)} \sigma_i \psi^{(+)} + \hat{\psi}^{(+)} \sigma_i \psi^{(-)},
\end{align}
where \( \sigma_i \) is the Pauli matrix in the spin space.
For the boson field, we introduce the following two independent fields
\[
\varphi^+ = \sqrt{\frac{m}{2}} \varphi + i \sqrt{\frac{1}{2m}} \pi^+ \\
\varphi^- = \sqrt{\frac{m}{2}} \varphi - i \sqrt{\frac{1}{2m}} \pi^+ 
\]
and their conjugate fields are denoted by
\[
\varphi^+_\dagger = (\varphi^+)\dagger = \sqrt{\frac{m}{2}} \varphi^+_\dagger - i \sqrt{\frac{1}{2m}} \pi^+_\dagger \\
\varphi^-_\dagger = (\varphi^-)\dagger = \sqrt{\frac{m}{2}} \varphi^-_\dagger + i \sqrt{\frac{1}{2m}} \pi^-_\dagger 
\]
Now, in the similar way to the fermion field, we introduce
\[
\hat{\varphi}^\pm \equiv \bar{\varphi}^\pm \gamma^0 = \pm (\varphi^\pm)\dagger 
\]
where \(\gamma^0\) for the boson field is a 2 by 2 matrix and \(\varphi^\pm\) is the eigenvector of \(\gamma^0\) with eigenvalue \(\pm 1\). It is easy to check that the boson fields \(\varphi^\pm\) and \(\hat{\varphi}^\pm\) satisfy the following commutation relation:
\[
[\varphi^+(x), \hat{\varphi}^+(y)] = [\varphi^-(x), \hat{\varphi}^-(y)] = \delta(x - y) 
\]
The \(\varphi^\pm\) and \(\hat{\varphi}^\pm\) fields are commuting. Now we have the commutation relations (10) and (16) in a symmetric form.

Writing the mass term of the Hamiltonians in the redefined fields, we obtain
\[
H_{\text{mass}} = m(\hat{\psi}^+_1 \psi^+_1 + \hat{\psi}^+_2 \psi^+_2 + \varphi^+(\varphi^+ - \hat{\psi}^-_1 \psi^+_1 - \hat{\psi}^+_2 \psi^+_2 - \hat{\varphi}^- \varphi^-)) 
\]
if one assumes the same mass \(m\) for the fermion and boson.

2.2. \(V(3)\) algebra

2.2.1. Generators of \(V(3)\). Now let us consider the fermion and boson fields as a triplet for each \((\pm)\) component:
\[
\Psi^\pm = \begin{pmatrix} \psi^\pm_1 \\ \psi^\pm_2 \\ \varphi^\pm \end{pmatrix} 
\]
Hereafter we indicate the \((+)\) and \((-)\) components by the superscript and subscript, respectively:
\[
\Psi^+_i = \Psi^i = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \varphi^3 \end{pmatrix}, \quad \Psi^-_i = \Psi_i = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \varphi^3 \end{pmatrix} 
\]
We introduce transformation among the triplet for each component. We call this algebra by \(V(3)\) accordingly to Ref. [26]. This algebra has \(SU(2)\) as a subalgebra. The fermion field is transformed as a doublet of \(SU(2)\), while the boson field is transformed as a singlet of \(SU(2)\). Regarding the fermion field as quark and the boson field as antidiquark, we also introduce baryon number. The fermion field has baryon number 1/3, while the boson field has baryon number \(-2/3\). We can label each component of the \(V(3)\) representation by the
3rd component of spin and the baryon number \((S_3, B)\) in the similar way of the isospin 3rd component and hypercharge \((I_3, Y)\) for SU(3).

The generators of these transformations can be written as

\[
G^{ij} = \int \hat{\psi}^j(x)\psi^i(x)d^3x, \quad G^{33} = \int \varphi^3(x)\varphi^3(x)d^3x, \quad (20)
\]

\[
G_{ij} = \int \hat{\psi}_i(x)\psi_j(x)d^3x, \quad G_{33} = \int \varphi_3(x)\varphi_3(x)d^3x, \quad (21)
\]

for \(i, j = 1, 2\). With these generators, the triplets transform as

\[
[G^{ij}, \psi^k] = -\delta_{ik}\psi^j, \quad [G^{ij}, \hat{\psi}^k] = \delta_{jk}\hat{\psi}^i, \quad [G^{33}, \varphi^3] = -\varphi^3, \quad [G^{33}, \hat{\varphi}^3] = \hat{\varphi}^3 \quad (22)
\]

\[
[G_{ij}, \psi_k] = -\delta_{ik}\psi_j, \quad [G_{ij}, \hat{\psi}_k] = \delta_{jk}\hat{\psi}_i, \quad [G_{33}, \varphi_3] = -\varphi_3, \quad [G_{33}, \hat{\varphi}_3] = \hat{\varphi}_3 \quad (23)
\]

and

\[
[G^{ij}, \varphi^3] = [G^{ij}, \hat{\varphi}^3] = [G^{33}, \varphi^3] = [G^{33}, \hat{\varphi}^3] = 0, \quad (24)
\]

\[
[G_{ij}, \varphi_3] = [G_{ij}, \hat{\varphi}_3] = [G_{33}, \varphi_3] = [G_{33}, \hat{\varphi}_3] = 0, \quad (25)
\]

for \(i, j = 1, 2\). These transformation rules is easily checked by using the commutation relations for the fields \((10)\) and \((16)\).

We also introduce the generators which transform fermion and boson as

\[
G^{i3} = \int \hat{\psi}^i(x)\varphi^3(x)d^3x, \quad G^{3i} = \int \varphi^3(x)\psi^i(x)d^3x, \quad (26)
\]

\[
G_{i3} = \int \hat{\psi}_i(x)\varphi_3(x)d^3x, \quad G_{3i} = \int \varphi_3(x)\psi_i(x)d^3x, \quad (27)
\]

for \(i = 1, 2\). These generators interchange the fermion and boson fields. For instance, we have

\[
\{G^{i3}, \psi^j\} = \int \{\hat{\psi}^i, \varphi^3, \psi^j\}d^3x = \int \{\hat{\psi}^i, \psi^j\}\varphi^3d^3x - \int \psi^i[\varphi^3, \psi^j]d^3x = \delta_{ij}\varphi^3, \quad (28)
\]

where we have used the commutation relation \((10)\). Here it should be noted that the generator \(G^{i3}\) and field \(\psi^j\) are fermionic and one should use the anticommutation relation for the transformation. The other transformation rules are

\[
[G^{i3}, \varphi^3] = \hat{\psi}^i, \quad \{G^{3i}, \psi^j\} = \delta_{ij}\varphi^3, \quad [G^{3i}, \varphi^3] = -\psi^j, \quad (29)
\]

\[
\{G_{i3}, \psi_j\} = \delta_{ij}\varphi_3, \quad [G_{i3}, \varphi_3] = \hat{\psi}_i, \quad \{G_{3i}, \psi_j\} = \delta_{ij}\varphi_3, \quad [G_{3i}, \varphi_3] = -\psi_i \quad (30)
\]

and

\[
\{G^{i3}, \hat{\psi}^j\} = [G^{i3}, \varphi^3] = \{G^{3i}, \psi^j\} = [G^{3i}, \varphi^3] = 0, \quad (31)
\]

\[
\{G_{i3}, \hat{\psi}_j\} = [G_{i3}, \varphi_3] = \{G_{3i}, \psi_j\} = [G_{3i}, \varphi_3] = 0. \quad (32)
\]

2.2.2. Commutation relations of generators. Now let us show the commutation relations for the V(3) generators. The indices \(i, j, k, l\) stand for 1, 2. The commutation relations for
the 1, 2 components read
\[
[G^{ij}, G^{kl}] = \delta_{jk} G^{il} - \delta_{il} G^{kj}, \quad [G_{ij}, G_{kl}] = \delta_{jk} G_{il} - \delta_{il} G_{kj}.
\] (33)

Some of these commutation relations can be written as
\[
[G^a, G^b] = i\epsilon^{abc} G^c, \quad [G_a, G_b] = i\epsilon^{abc} G_c,
\] (34)

where we have defined
\[
G^a = \int \hat{\psi}(x) \left( \frac{\lambda_a}{2} \right) \psi(x) d^3 x, \quad G_a = \int \hat{\psi}(x) \left( \frac{\lambda_a}{2} \right) x \psi(x) d^3 x
\] (35)

with the Gell-man matrix \( \lambda_a \) for \( a = 1, 2, 3 \). There are the generators of the SU(2) subalgebra. We also introduce the operator for the baryon number as
\[
G^8 = \frac{1}{3} G_{11} + \frac{1}{3} G_{22} - \frac{2}{3} G_{33} = \frac{1}{\sqrt{3}} \int \hat{\psi}(x) (\lambda_8)_{ij} \psi^i(x) d^3 x,
\]
\[
G_8 = \frac{1}{3} G_{11} + \frac{1}{3} G_{22} - \frac{2}{3} G_{33} = \frac{1}{\sqrt{3}} \int \hat{\psi}(x) (\lambda_8)_{ij} x \psi(x) d^3 x,
\] (36)

with \( \lambda_8 \) being the eighth component of the Gell-Man matrix. The other commutation relations for the bosonic generators vanish:
\[
[G^{ij}, G^{33}] = [G^{33}, G^{33}] = 0, \quad [G_{ij}, G_{33}] = [G_{33}, G_{33}] = 0.
\] (37)

This implies that \( G^{ij} \) and \( G_{ij} \) for \( i, j = 1, 2 \) do not change the baryon number.

The commutation relations among the fermionic and bosonic generators are
\[
[G^{ij}, G^{k3}] = \delta_{jk} G^{i3}, \quad [G^{ij}, G^{3k}] = -\delta_{ik} G^{3j},
\]
\[
[G_{ij}, G_{k3}] = \delta_{jk} G_{i3}, \quad [G_{ij}, G_{3k}] = -\delta_{ik} G_{3j},
\] (38)

and
\[
[G^{33}, G^{33}] = -G^{33}, \quad [G^{33}, G^{3i}] = G^{3i},
\]
\[
[G_{33}, G_{3i}] = -G_{3i}, \quad [G_{33}, G_{3i}] = G_{3i}.
\] (39)

We can also show using the spin generators \( G^3 \) and \( G_3 \) that
\[
[G^3, G^{13}] = \frac{1}{2} G^{13}, \quad [G^3, G^{23}] = -\frac{1}{2} G^{23}, \quad [G^3, G^{31}] = -\frac{1}{2} G^{31}, \quad [G^3, G^{32}] = \frac{1}{2} G^{32},
\]
\[
[G_3, G_{13}] = \frac{1}{2} G_{13}, \quad [G_3, G_{23}] = -\frac{1}{2} G_{23}, \quad [G_3, G_{31}] = -\frac{1}{2} G_{31}, \quad [G_3, G_{32}] = \frac{1}{2} G_{32}.
\] (40)

These equations imply that \( G^{13}, G^{32}, G_{13} \) and \( G_{32} \) raise the 3rd component of spin by 1/2, while \( G^{31}, G^{23}, G_{31} \) and \( G_{23} \) lower by 1/2. The commutation relations with the baryon number operators \( G^8 \) and \( G_8 \)
\[
[G^8, G^{13}] = G^{13}, \quad [G^8, G^{23}] = G^{23}, \quad [G^8, G^{31}] = -G^{31}, \quad [G^8, G^{32}] = -G^{32},
\]
\[
[G_8, G_{13}] = G_{13}, \quad [G_8, G_{23}] = G_{23}, \quad [G_8, G_{31}] = -G_{31}, \quad [G_8, G_{32}] = -G_{32},
\] (41)

show that \( G^{13}, G^{23}, G_{13} \) and \( G_{23} \) change the baryon number by 1, while \( G^{31}, G^{32}, G_{31} \) and \( G_{32} \) change the baryon number by \(-1\).
The anticommutation relations for the fermionic generator $s$ are
\[
\{G^{3i}, G^{3j}\} = \delta_{ij} G^{33} + G^{ij}, \quad \{G_{3i}, G_{3j}\} = \delta_{ij} G_{33} + G_{ij},
\] (42)
and the other commutation relations vanish:
\[
\{G^{3i}, G^{3j}\} = \{G^{3i}, G^{3j}\} = 0, \quad \{G_{3i}, G_{3j}\} = \{G_{3i}, G_{3j}\} = 0.
\] (43)

It is also notable that the relations
\[
\{G^{31}, G^{13}\} = G^{11} + G^{33} = \left[ \frac{1}{2}(G^{11} + G^{22}) + G^{33} \right] + \frac{1}{2}(G^{11} - G^{22})
\] (44)
\[
\{G^{32}, G^{23}\} = G^{22} + G^{33} = \left[ \frac{1}{2}(G^{11} + G^{22}) + G^{33} \right] - \frac{1}{2}(G^{11} - G^{22})
\] (45)
\[
\{G^{21}, G^{12}\} = G^{11} - G^{22}
\] (46)
implies that the commutation relations can only provide the combinations of the generators,
\[
\sqrt{\frac{1}{3}}(G^{11} + G^{22}) + \sqrt{\frac{2}{3}}G^{33} \quad \text{and} \quad \sqrt{\frac{1}{3}}(G^{11} + G^{22} - G^{33})
\] (47)
which implies that the spin operation for the fermion acts on both $\psi$ and $\bar{\psi}$ in the same direction, we should consider only the subalgebra of $V(3) \otimes V(3)$ which is generated by $G_{AB} = G_{AB}$. As a result, $G_{AB}$ satisfies the same commutation relations of $G_{AB}$ and $G_{AB}$, that is, $G_{AB}$ generates the $V(3)$ algebra.

\[ S = \frac{1}{2} \Sigma = \frac{1}{2} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}, \] (51)
which implies that the spin operation for the fermion acts on both $\psi$ and $\bar{\psi}$ in the same direction, we should consider only the subalgebra of $V(3) \otimes V(3)$ which is generated by $G_{AB} = G_{AB} + G_{AB}$. As a result, $G_{AB}$ generates the same commutation relations of $G_{AB}$ and $G_{AB}$.

\section{Representations of $V(3)$}

\subsection{Fundamental representation: $\Psi^i \oplus \bar{\Psi}_i$}

The triplets, $\Psi^i$ and $\Psi_i$, given in Eq. (19) are the fundamental representations of the $V(3)$ algebra. These fields are transformed by the generators $G_{AB}$ and $G_{AB}$, respectively. Similarly the conjugate fields $\bar{\Psi}^i$ and $\bar{\Psi}_i$ are the complex representation of the fundamental representation.
The commutation relations of these fields for the spin $G^3$ and the baryon number $G^8$ read

$$
[G^3, \psi^1] = -\frac{1}{2} \psi^1, \quad [G^3, \psi^2] = \frac{1}{2} \psi^2, \quad [G^3, \varphi^3] = 0, \quad (52)
$$

$$
[G^8, \psi^1] = -\frac{1}{3} \psi^1, \quad [G^8, \psi^2] = -\frac{1}{3} \psi^2, \quad [G^8, \varphi^3] = \frac{2}{3} \varphi^3. \quad (53)
$$

This implies that $\psi^1, \psi^2$ and $\varphi^3$ fields have the quantum number $(S_3, B)$ as $(1/2, 1/3)$, $(-1/2, 1/3)$ and $(0, -2/3)$, respectively.

The same relations are satisfied for the fields with the subscript. Regarding $\Psi^i \oplus \Psi_i$ as a color triplet “quark” field and $\hat{\Psi}^i \oplus \hat{\Psi}_i$ as a “antiquark” field with a color anti-triplet, we construct the representation of “hadron” as a composites of the quark fields.

With the commutation relations derived in the previous section, we find that $\hat{\Psi}^i \Psi^i$ and $\hat{\Psi}_i \Psi_i$ are invariant under any transformations $G^{AB}$ and $G_{AB}$, respectively, as

$$
[G^{AB}, \hat{\psi}^1 \psi^1 + \hat{\psi}^2 \psi^2 + \varphi^3 \varphi^3] = 0, \quad [G_{AB}, \hat{\psi}_1 \psi_1 + \hat{\psi}_2 \psi_2 + \hat{\varphi}_3 \varphi_3] = 0. \quad (54)
$$

If we take a linear combination of these terms as $\hat{\Psi}^i \Psi^i - \hat{\Psi}_i \Psi_i$, the mass term of Hamiltonian (17) is invariant under the V(3) transformation generated by $G_{AB} = G^{AB} + G^i_j$ as

$$
[G_{AB}, H_{\text{mass}}] = [G_{AB}, m(\hat{\psi}^1 \psi^1 + \hat{\psi}^2 \psi^2 + \varphi^3 \varphi^3) - m(\hat{\psi}_1 \psi_1 + \hat{\psi}_2 \psi_2 + \hat{\varphi}_3 \varphi_3)] = 0. \quad (55)
$$

### 3.2. “Adjoint” Representation: $\hat{\Psi}^i \Psi^j \oplus \hat{\Psi}^i \Psi_j$

Next, we consider composite fields made of two fundamental representations, $\Psi^i \otimes \Psi_i$ and $\hat{\Psi}^i \otimes \hat{\Psi}_i$, which can be regarded as “mesonic” fields. We have two kinds of the combinations:

$$
\hat{\Psi}_i \Psi^j \oplus \hat{\Psi}_j \Psi^i, \quad \hat{\Psi}^i \Psi^j \oplus \hat{\Psi}^i \Psi_j. \quad (56)
$$

The former is favored by the ground state in the nonrelativistic limit, because $\Psi^i$ and $\hat{\Psi}_i$ contain the large components of the Dirac spinor. First, we consider the former combination. Let us introduce

$$
M^i_j = \hat{\Psi}_i \Psi^j, \quad M^j_i = \hat{\Psi}^i \Psi_j. \quad (57)
$$

These fields are algebraically independent and belong to the same representations, as we shall see below. For the Lorentz symmetry we need both fields in an appropriate combination.

Let us see the irreducible representation for $M^i_j$ and $M^j_i$. We start with the $M^i_1$ field, which has $S_3 = -1$ for the 3rd component of spin and $B = 0$ for the baryon number, and therefore $M^i_1$ is a boson field. The generator $G_{21}$ raises $1/2$ for the 3rd component of spin and does not change the baryon number. Having the commutation relations

$$
[G_{21}, M^1_2] = M^2_2 - M^1_1, \quad [G_{21}, \frac{1}{\sqrt{2}} (M^1_2 - M^2_2)] = \sqrt{2} M^1_2, \quad [G_{21}, M^1_2] = 0, \quad (58)
$$

we find that $\{ -M^1_2, \frac{1}{\sqrt{2}} (M^1_1 - M^2_2), M^2_1 \}$ form a spin triplet. Thus, these fields have total spin 1 with $B = 0$. We also find that the $\frac{1}{\sqrt{2}} (M^1_1 + M^2_2)$ field orthogonal to $\frac{1}{\sqrt{2}} (M^1_1 - M^2_2)$ has total spin 0.
Next let us consider the transformation of $M_1^2$ by $G_{23}$ which raises the 3rd component of spin by 1/2 and change the baryon number by −1:

$$[G_{23}, M_1^2] = -M_1^3. \quad (59)$$

This implies that $M_1^3$ has $S_3 = -1/2$ and $B = -1$, and thus, it is a fermion field. We consider the transformation of the field $M_1^3$ by $G_{21}$ changing $S_3$ by +1 as

$$[G_{21}, M_1^3] = M_2^3, \quad (60)$$

which implies that $M_2^3$ has $S_3 = 1/2$ and $B = -1$. Thus, $\{-M_1^3, M_2^3\}$ form a spin doublet with total spin 1/2 and $B = -1$. Considering also the transformation of $M_1^2$ by $G_{31}$ which changes $S_3$ by 1/2 and $B$ by 1 as

$$[G_{31}, M_1^2] = M_3^2 \quad (61)$$

we find that the field $M_3^2$ has $S_3 = -1/2$ and $B = 1$. The commutation relation

$$[G_{21}, M_3^2] = -M_3^1 \quad (62)$$

implies that $\{-M_2^2, M_3^1\}$ form a spin doublet having total spin 1/2 and baryon number $B = 1$. These fields are fermions. Finally the commutation relation

$$\{G_{23}, M_3^2\} = M_2^2 + M_3^3 \quad (63)$$

shows that the field $M_3^3$, which has total spin $S = 0$ and baryon number $B = 0$, is also within this multiplet. This implies that we need all of three components, $\frac{1}{\sqrt{2}}(M_1^1 - M_2^1)$, $\frac{-1}{\sqrt{2}}(M_1^1 + M_2^1)$, $M_3^3$, to express $M_2^2 + M_3^3$.

Similarly for the $M_j^i$ field, we find that $\{-M_1^2, \frac{-1}{\sqrt{2}}(M_1^1 - M_2^1), M_2^1\}$ forms the spin triplet with $B = 0$, $\frac{1}{\sqrt{2}}(M_1^1 + M_2^1)$ has total spin $S = 0$ and baryon number $B = 0$, $\{-M_3^1, M_2^1\}$ and $\{-M_3^1, -M_2^1\}$ are spin doublets with $B = -1$ and $B = 1$, respectively, and $M_3^2$ is the spin singlet with $B = 0$. We also confirm in the same way that $\Psi^i \Psi^j \oplus \bar{\Psi}^i \bar{\Psi}^j$ forms a nonet of $V(3)$.

In this way, we find a nonet representation of $V(3)$ and write $3 \otimes \bar{3} = 9$ instead of $3 \otimes \bar{3} = 1 \oplus 8$ in SU(3).

### 3.3. $\Psi^i \Psi^j \oplus \bar{\Psi}^i \bar{\Psi}^j$ representations

We consider higher dimensional representations composed of two fundamental representations $(\Psi^i \otimes \Psi^j) \oplus (\bar{\Psi}_i \otimes \bar{\Psi}_j)$, which can be regarded as “diquark” fields. Again, $\Psi^i \otimes \Psi^j$ is favored by the ground state in the nonrelativistic limit than $\Psi_i \otimes \Psi^j$. Here we consider the decomposition of $\Psi^i \otimes \Psi^j$ into the irreducible representations of $V(3)$. The other combinations are also decomposed in the same way.

We introduce nine fields, $\Psi_a^i \Psi_b^j$, where $i$ and $j$ are indices of the V(3) fundamental representation running 1 to 3, and $a$ and $b$ are fixed labels representing other quantum numbers such as color. The product $\Psi^i \Psi^j$ has 9 components. Here we consider the decomposition of the 9 components into the irreducible representations of $V(3)$. We will see that $\Psi^i \Psi^j$ is decomposed into a quintet and a quartet representations, that is written as $3 \otimes 3 = 5 \oplus 4$. 

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3.3.1. Quintet. Let us start a highest field $\psi_a^1 \psi_b^1$ which has $S_3 = +1$ and baryon number $B = 2/3$. We lower spin quantum number by take the commutation relations

$$\{G^{12}, \psi_a^1 \psi_b^1\} = -\psi_a^1 \psi_b^2 - \psi_b^1 \psi_a^2,$$  
$$\{G^{12}, - (\psi_a^1 \psi_b^2 + \psi_b^2 \psi_a^1)\} = 2\psi_a^2 \psi_b^2,$$  
and then we find that $\{\psi_a^2 \psi_b^2, \frac{1}{\sqrt{2}}(\psi_a^1 \psi_b^2 + \psi_b^2 \psi_a^1), \psi_a^1 \psi_b^1\}$ forms a spin triplet with total spin 1 and baryon number 2/3. Next we consider the transformation of the field $\psi_a^1 \psi_b^1$ by $G^{13}$,

$$\{G^{13}, \psi_a^1 \psi_b^1\} = \varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3,$$  
and consider the transformation of the field $\varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3$ by $G^{12}$ as

$$\{G^{12}, \varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3\} = -\varphi_a^3 \psi_b^2 + \varphi_b^3 \psi_a^2.$$  

Therefore, we find that $\{\varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3, \varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3\}$ form a spin doublet with total spin 1/2 and baryon number $-1/3$.

Taking the transformation of the field $\varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3$ by $G^{13}$ as

$$\{G^{13}, \varphi_a^3 \psi_b^3 - \psi_a^1 \psi_b^3\} = 0,$$

we find that there are no components with baryon number $-4/3$ for the multiplet starting with $\psi_a^1 \psi_b^1$.

Consequently, there are five components,

$$\psi_a^2 \psi_b^2, \quad \frac{1}{\sqrt{2}}(\psi_a^1 \psi_b^2 + \psi_b^2 \psi_a^1), \quad \psi_a^1 \psi_b^1, \quad \frac{1}{\sqrt{2}}(\varphi_a^3 \psi_b^3 - \psi_a^1 \varphi_a^3), \quad \frac{1}{\sqrt{2}}(\varphi_a^3 \psi_b^1 - \psi_a^1 \varphi_a^3)$$

forming a quintet representation $5$. This representation is anti-symmetric under the exchange of indices $a$ and $b$. (Note that $\psi^i$ is a fermion field having $\{\psi_a^i, \psi_b^i\} = 0.$)

3.3.2. Quartet. To find further representation, we start with $\frac{1}{\sqrt{2}}(\psi_a^1 \psi_b^2 - \psi_a^2 \psi_b^1)$ which is orthogonal to $\frac{1}{\sqrt{2}}(\psi_a^1 \psi_b^2 + \psi_a^2 \psi_b^1)$ and has spin 0. Calculating the commutation relations

$$\{G^{23}, \psi_a^1 \psi_b^1 - \psi_a^2 \psi_b^1\} = -\psi_a^1 \varphi_b^3 - \varphi_a^3 \psi_b^1,$$  
$$\{G^{12}, \psi_a^1 \varphi_b^3 + \varphi_a^3 \psi_b^1\} = \psi_a^2 \varphi_b^3 + \varphi_b^3 \psi_a^2,$$  
$$\{G^{13}, \psi_a^1 \varphi_b^3 + \varphi_a^3 \psi_b^1\} = 2\varphi_a^3 \varphi_b^3,$$

we find that four components,

$$\frac{1}{\sqrt{2}}(\psi_a^1 \psi_b^2 - \psi_a^2 \psi_b^1), \quad \frac{1}{\sqrt{2}}(\psi_a^2 \varphi_b^3 + \varphi_a^3 \psi_b^1), \quad \frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^3 + \varphi_a^3 \psi_b^1), \quad \varphi_a^3 \varphi_b^3,$$

form a quartet representation $4$. The first and last terms in Eq. (73) are spin singlets and have baryon number $2/3$ and $-4/3$, respectively, while the middle two terms form a spin doublet with baryon number $-1/3$. This representation is symmetric under the exchange of indices $a$ and $b$.

As a result, we have the decomposition $3 \otimes 3 = 5_A \oplus 4_S$, where $A$ and $S$ stand for antisymmetry and symmetry under the exchange of two fundamental representations.
We decompose the fields composed of three fundamental representations, $\Psi_1 \Psi_2 \Psi_3$, into irreducible representations of $V(3)$, where again $a$, $b$ and $c$ label other quantum numbers. This field configuration is favored by the ground state in the nonrelativistic limit and corresponds to baryons for color singlet. Finally we will find that $\Psi_1 \Psi_2 \Psi_3$ is decomposed into a septet, a quartet and two octet representations, namely we write $3 \otimes 3 \otimes 3 = 7 \oplus 4 \oplus 8 \oplus 8$. Other configurations, $\Psi_1 \Psi_2 \Psi_3$, $\Psi_1 \Psi_2 \Psi_4$ and $\Psi_1 \Psi_4 \Psi_5$ forms the same multiplets.

3.4.1. Septet. We start with a highest component $\psi^1_a \psi^1_b \psi^1_c$ with spin $S_3 = 3/2$ and baryon number $B = 1$. Taking the commutation relations

\begin{align}
[G_{12}, \psi^1_a \psi^1_b \psi^1_c] &= -\psi^2_a \psi^2_b \psi^1_c - \psi^2_a \psi^1_b \psi^2_c^* - \psi^1_a \psi^1_b \psi^2_c^*, \\
[G_{12}, -\psi^2_a \psi^1_b \psi^1_c] &= 2(\psi^2_a \psi^2_b \psi^1_c^* + \psi^2_a \psi^1_b \psi^2_c^* + \psi^1_a \psi^2_b \psi_c^*), \\
[G_{12}, \psi^2_a \psi^1_b \psi^2_c^*] &= -3\psi^2_a \psi^2_b \psi^1_c^*,
\end{align}

we find a spin quartet $\{\psi^1_a \psi^1_b \psi^1_c, 1/\sqrt{3}(\psi^2_a \psi^2_b \psi^1_c^* + \psi^2_a \psi^2_b \psi^1_c^* + \psi^2_a \psi^2_b \psi^1_c^*) \}$ with total spin $3/2$ and baryon number $1$. Next, we consider the transformation of $\psi^1_a \psi^1_b \psi^1_c$ by $G_{13}$ and the spin partners of its product:

\begin{align}
[G_{13}, \psi^1_a \psi^1_b \psi^1_c] &= \varphi^3_a \psi^1_b \psi^1_c - \psi^1_a \varphi^3_b \psi^1_c + \psi^1_a \psi^1_b \varphi^3_c, \\
[G_{12}, \varphi^3_a \psi^1_b \psi^1_c] &= -\varphi^3_a \psi^2_b \psi^1_c + \psi^2_a \psi^1_b \varphi^3_c + \psi^2_a \psi^2_b \varphi^3_c - \psi^1_a \varphi^3_b \psi^1_c, \\
[G_{12}, -\varphi^3_a \psi^2_b \psi^1_c] &= 2(\varphi^3_a \psi^2_b \psi^1_c^* - \psi^2_a \varphi^3_b \psi^1_c^* + \psi^1_a \varphi^3_b \psi^2_c^* - \varphi^3_a \psi^2_b \varphi^3_c). \\
\end{align}

These form spin triplet with baryon number 0. Decreasing the baryon number of these terms further, we find

\begin{align}
[G_{13}, \varphi^3_a \psi^1_b \psi^1_c - \psi^1_a \varphi^3_b \psi^1_c + \psi^1_a \psi^1_b \varphi^3_c] = 0.
\end{align}

This implies that we have a septet representation 7 as

\begin{align}
\psi^1_a \psi^1_b \psi^1_c, \\
1/\sqrt{3}(\psi^2_a \psi^1_b \psi^1_c^* + \psi^2_a \psi^2_b \psi^1_c^* + \psi^1_a \psi^1_b \psi^2_c^*), \\
1/\sqrt{3}(\psi^2_a \psi^2_b \psi^1_c^* + \psi^2_a \psi^1_b \psi^2_c^* + \psi^1_a \psi^2_b \psi^2_c^*), \\
1/\sqrt{3}(\varphi^3_a \psi^1_b \psi^1_c - \psi^1_a \varphi^3_b \psi^1_c + \psi^1_a \psi^1_b \varphi^3_c), \\
1/\sqrt{6}(\varphi^3_a \psi^1_b \psi^2_c^* + \psi^2_a \psi^1_b \psi^1_c^* - \psi^1_a \varphi^3_b \psi^2_c^* + \psi^2_a \varphi^3_b \psi^1_c^* + \psi^1_a \varphi^3_b \psi^2_c^* + \psi^2_a \varphi^3_b \psi^2_c^*), \\
1/\sqrt{3}(\varphi^3_a \psi^2_b \psi^1_c - \varphi^3_b \psi^2_a \psi^1_c - \psi^3_a \psi^3 b \psi^2 c),
\end{align}

where the first four terms have total spin $3/2$ and baryon number 1 and the last three terms have total spin 1 and baryon number 0. These terms are totally anti-symmetric under the exchange of indices $a$, $b$ and $c$.

3.4.2. Quartet. To find further representations, we start with another highest component, $\varphi^3_a \varphi^3_b \varphi^3_c$, which has spin 0 and baryon number $-2$. Increasing its baryon number by $G^{31}$, we
have
\[
[G^{31}, \varphi_a^3 \varphi_b^3 \varphi_c^3] = -\psi_a^1 \varphi_b^3 \varphi_c^3 - \varphi_a^3 \psi_b^1 \varphi_c^3 - \varphi_a^3 \varphi_b^3 \psi_c^1, \tag{82}
\]
which has spin \(S_3 = +1/2\) and baryon number \(B = -1\), and its spin partner can be found by applying \(G^{12}\) on it as
\[
[G^{12}, -\psi_a^1 \varphi_b^3 \varphi_c^3 - \varphi_a^3 \psi_b^1 \varphi_c^3 - \varphi_a^3 \varphi_b^3 \psi_c^1] = -\psi_a^2 \varphi_b^3 \varphi_c^3 - \varphi_a^3 \psi_b^2 \varphi_c^3 - \varphi_a^3 \varphi_b^2 \psi_c^3, \tag{83}
\]
which has spin \(S_3 = -1/2\) and baryon number \(B = -1\). Calculating
\[
\{G^{32}, -\psi_a^1 \varphi_b^3 \varphi_c^3 - \varphi_a^3 \psi_b^1 \varphi_c^3 - \varphi_a^3 \varphi_b^3 \psi_c^1 \} = \varphi_a^3 \psi_b^2 \psi_c^1 + \psi_a^2 \varphi_b^3 \psi_c^1 - \psi_a^1 \varphi_b^3 \psi_c^2 - \varphi_a^3 \psi_b^2 \psi_c^2 - \psi_a^1 \psi_b^2 \varphi_c^3 + \varphi_a^3 \psi_b^2 \varphi_c^3, \tag{84}
\]
we have a further component with spin 0 and baryon number 0 in this multiplet. Thus the second multiplet of three fundamental representations is a quartet representation 4:
\[
\begin{align*}
\frac{1}{\sqrt{6}}(& \varphi_a^3 \varphi_b^1 \psi_c^1 + \psi_a^2 \varphi_b^1 \psi_c^1 - \varphi_a^3 \psi_b^2 \psi_c^1 - \varphi_a^3 \psi_b^1 \psi_c^1 - \psi_a^1 \varphi_b^2 \varphi_c^1 + \psi_a^1 \psi_b^1 \varphi_c^3) \\
\frac{1}{\sqrt{3}}(& \psi_a^1 \varphi_b^3 \varphi_c^3 + \varphi_a^3 \psi_b^3 \varphi_c^3 + \varphi_a^3 \varphi_b^3 \psi_c^3), \quad \frac{1}{\sqrt{3}}(& \varphi_a^3 \psi_b^2 \varphi_c^2 + \varphi_a^3 \varphi_b^2 \varphi_c^2 + \varphi_a^3 \varphi_b^2 \psi_c^2),
\end{align*}
\]
\[
\varphi_a^3 \varphi_b^3 \varphi_c^3. \tag{85}
\]
This representation is symmetric under the exchange of indices \(a, b\) and \(c\).

3.4.3. Octet symmetric. Next, we see another representation by starting with a spin double \(\frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^2 - \psi_a^2 \varphi_b^1)\psi_c^1\), \(\frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^2 - \psi_a^2 \varphi_b^1)\psi_c^1\) with baryon number 1, which are symmetric under the exchange of indices \(a\) and \(b\). We decrease the baryon number of the former term:
\[
\left\{G^{23}, \frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^2 - \psi_a^2 \varphi_b^1)\psi_c^1 \right\} = -\frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^2 \psi_c^1 + \varphi_a^3 \psi_b^1 \psi_c^1). \tag{86}
\]
This term has spin \(S_3 = +1\) and baryon number \(B = 0\). Its spin partners are found by decreasing its spin with \(G^{12}\) sequently as \(\frac{1}{2}(\psi_a^2 \varphi_b^3 \psi_c^1 + \varphi_a^3 \varphi_b^3 \psi_c^1 + \varphi_a^3 \varphi_b^3 \psi_c^2)\) and \(\frac{1}{2}(\psi_a^2 \varphi_b^3 \psi_c^2 + \varphi_a^3 \varphi_b^3 \psi_c^2)\). Thus, these components form a spin triplet with baryon number 0. We further apply \(G^{13}\), and we have
\[
\begin{align*}
G^{13}, \frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^2 \psi_c^1 + \varphi_a^3 \psi_b^1 \psi_c^1) & = -\frac{1}{\sqrt{2}}(2 \varphi_a^3 \varphi_b^3 \psi_c^1 - \varphi_a^3 \varphi_b^3 \varphi_c^3 - \varphi_a^3 \varphi_b^3 \varphi_c^3) \\
\end{align*}
\]
\[
\text{This has spin } S_3 = +1/2 \text{ and baryon number } B = -1. \text{ Its spin partner is found as } \frac{1}{\sqrt{6}}(2 \varphi_a^3 \varphi_b^3 \psi_c^2 - \varphi_a^3 \varphi_b^3 \varphi_c^2 - \varphi_a^3 \varphi_b^3 \psi_c^2). \text{ In addition, we have}
\]
\[
\left\{G^{13}, \frac{1}{\sqrt{2}}(\psi_a^1 \varphi_b^2 - \psi_a^2 \varphi_b^1)\psi_c^1 \right\} = \frac{1}{\sqrt{2}}(\varphi_a^3 \varphi_b^3 \psi_c^1 + \varphi_a^3 \psi_b^2 \psi_c^1 + \varphi_a^3 \varphi_b^3 \psi_c^2), \tag{87}
\]
which has spin 0 and baryon number 0. This can be written as a linear combination of two components as
\[
\begin{align*}
\psi_a^1 \varphi_b^3 \psi_c^1 + \varphi_a^3 \psi_b^1 \varphi_c^3 + \psi_a^2 \varphi_b^3 \varphi_c^3 - \psi_a^1 \varphi_b^3 \varphi_c^3 \\
&= \frac{1}{2}[\psi_a^2 \varphi_b^3 \psi_c^1 + \varphi_a^3 \psi_b^1 \varphi_c^1 + \psi_a^1 \varphi_b^3 \psi_c^1 + \varphi_a^3 \psi_b^1 \psi_c^1] \\
&+ \frac{1}{2}[\psi_a^2 \varphi_b^3 \psi_c^1 + \varphi_a^3 \psi_b^1 \varphi_c^1 - \psi_a^1 \varphi_b^3 \psi_c^1 - \varphi_a^3 \psi_b^1 \psi_c^1 + 2(\psi_a^1 \psi_b^1 \varphi_c^3 - \psi_a^2 \psi_b^1 \varphi_c^3)]. \tag{89}
\end{align*}
\]
Thus \(\frac{1}{\sqrt{12}}(2 \psi_a^2 \varphi_b^3 \psi_c^1 - \psi_a^1 \varphi_b^3 \varphi_c^1 - \varphi_a^3 \varphi_b^3 \psi_c^1 + 2(\psi_a^1 \psi_b^1 \varphi_c^3 - \psi_a^2 \psi_b^1 \varphi_c^3))\) is also a component of the multiplet. Consequently we have eight components in this multiplet forming a
octet 8:

\[
\frac{1}{\sqrt{2}}(\psi^1_a\psi^2_b - \psi^2_a\psi^1_b)\psi^1_c, \quad \frac{1}{\sqrt{2}}(\psi^1_a\psi^2_b - \psi^2_a\psi^1_b)\psi^2_c,
\]

\[
\frac{1}{\sqrt{2}}(\psi^1_a\varphi^3_c + \varphi^3_a\psi^1_b), \quad \frac{1}{\sqrt{2}}(\psi^2_a\varphi^3_c + \varphi^3_a\psi^2_b),
\]

\[
\frac{1}{2}(\psi^2_a\varphi^3_c + \psi^1_a\varphi^3_c) + \varphi^3_a\psi^2_b\psi^2_c + \varphi^3_a\psi^1_b\psi^3_c),
\]

\[
\frac{1}{\sqrt{12}}[\psi^2_a\varphi^3_c + \varphi^3_a\psi^2_b\psi^1_c) - \psi^3_a\varphi^3_c - \varphi^3_a\psi^3_b\psi^3_c) + 2(\psi^1_a\varphi^3_c + \varphi^3_a\psi^1_b\psi^3_c)]
\]

\[
- \frac{1}{\sqrt{2}}(2\varphi^3_a\varphi^3_b\psi^2_c - \psi^3_a\varphi^3_b\varphi^3_c) - \varphi^3_a\psi^3_b\varphi^3_c),
\]

\[
\frac{1}{\sqrt{6}}(2\varphi^3_a\varphi^3_b\psi^2_c - \psi^3_a\varphi^3_b\varphi^3_c) - \varphi^3_a\psi^3_b\varphi^3_c),
\]

3.4.4. Octet asymmetric. Next we start with another spin 1/2 doublet \(\frac{1}{\sqrt{6}}(2\psi^1_a\psi^1_b\psi^2_c - (\psi^1_a\psi^2_c + \psi^2_a\psi^1_c))\), \(\frac{1}{\sqrt{6}}(2\psi^2_a\psi^2_b\psi^1_c - (\psi^1_a\psi^2_b + \psi^2_a\psi^1_b))\) with baryon number 1, which is asymmetric under the exchange of indices a and b. We decrease the baryon number of the former component by considering the transformation of \(G^{23}\):

\[
\left\{G^{23}, \frac{1}{\sqrt{6}}[(2\psi^1_a\psi^1_b\psi^2_c - (\psi^1_a\psi^2_b + \psi^2_a\psi^1_b))\psi^1_c)\right) = \frac{1}{\sqrt{6}}(2\psi^1_a\psi^1_b\varphi^3_c + \psi^1_a\varphi^3_b\psi^1_c) - \varphi^3_a\psi^1_b\varphi^3_c),
\]

which has spin \(S_3 = +1\) and baryon number \(B = 0\). Its spin partners are found by operating \(G^{12}\) as \(\frac{1}{\sqrt{12}}[\psi^1_a\varphi^3_b\psi^1_c - \varphi^3_a\psi^1_b\psi^1_c + \psi^1_a\varphi^3_b\psi^2_c - \varphi^3_a\psi^1_b\psi^2_c) + 2(\psi^1_a\varphi^3_b\varphi^3_c)\), \(\frac{1}{\sqrt{6}}(2\psi^2_a\psi^2_b\varphi^3_c + \psi^2_a\varphi^3_b\varphi^3_c - \varphi^3_a\psi^3_b\varphi^3_c).\) We further decrease the baryon number by using \(G^{13}\):

\[
\left[G^{13}, \frac{1}{\sqrt{6}}(2\psi^1_a\psi^1_b\varphi^3_c + \psi^1_a\varphi^3_b\psi^1_c) - \varphi^3_a\psi^1_b\varphi^3_c)\right) = \frac{1}{\sqrt{6}}(3\varphi^3_a\psi^1_b\varphi^3_c - 3\psi^1_a\varphi^3_b\varphi^3_c),
\]

and its spin partner is found as \(\frac{1}{\sqrt{6}}(\varphi^3_a\psi^2_b\varphi^3_c - \psi^2_a\varphi^3_b\varphi^3_c).\) These components have spin 1/2 and baryon number \(-1\). We also calculate

\[
\left\{G^{13}, \frac{1}{\sqrt{6}}[(2\psi^1_a\psi^1_b\psi^2_c - (\psi^1_a\psi^2_b + \psi^2_a\psi^1_b))\psi^1_c] = \frac{1}{\sqrt{6}}(2\varphi^3_a\psi^1_b\psi^2_c - 2\psi^1_a\varphi^3_b\psi^2_c - \varphi^3_a\psi^1_b\psi^1_c - \psi^1_a\psi^1_b\varphi^3_c) + \psi^2_a\varphi^3_b\psi^1_c) - \varphi^3_a\psi^1_b\psi^1_c).
\]

This component can be written as

\[
\psi^2_a\varphi^3_b\psi^1_c - \varphi^3_a\psi^2_b\psi^1_c - 2\psi^1_a\varphi^3_b\psi^2_c - 2\varphi^3_a\psi^1_b\psi^2_c - \varphi^3_a\psi^1_b\varphi^3_c - \psi^2_a\psi^1_b\varphi^3_c)
\]

\[
- \frac{1}{2}[\varphi^3_a\psi^2_b\psi^1_c - \varphi^3_a\psi^2_b\psi^2_c + \varphi^3_a\psi^1_b\psi^2_c - \varphi^3_a\psi^1_b\varphi^3_c + 2(\psi^1_a\varphi^3_b\varphi^3_c) + \psi^1_a\psi^2_b\varphi^3_c]
\]

\[
+ \frac{3}{2}[\psi^3_a\varphi^3_b\psi^1_c - \varphi^3_a\psi^3_b\psi^1_c - \psi^3_a\psi^3_b\psi^2_c + \varphi^3_a\psi^1_b\varphi^3_c).]
\]
Thus, we have the following 8 components in this multiplet:

\[
\begin{align*}
\frac{1}{\sqrt{6}} (2\psi_{a}^{1} \psi_{b}^{2} \psi_{c}^{3} - (\psi_{a}^{1} \psi_{b}^{2} + \psi_{a}^{2} \psi_{b}^{1}) \psi_{c}^{3}), & \quad \frac{1}{\sqrt{6}} (2\psi_{a}^{2} \psi_{b}^{2} \psi_{c}^{1} - (\psi_{a}^{1} \psi_{b}^{2} + \psi_{a}^{2} \psi_{b}^{1}) \psi_{c}^{3}) \\
\frac{1}{\sqrt{12}} [\psi_{a}^{2} \psi_{b}^{3} \psi_{c}^{1} - \varphi_{b}^{3} \psi_{a}^{2} \psi_{c}^{1} + \psi_{a}^{1} \psi_{b}^{3} \psi_{c}^{2} - \varphi_{a}^{3} \psi_{b}^{1} \psi_{c}^{2} + 2(\psi_{a}^{2} \psi_{b}^{3} \varphi_{c}^{2} + \psi_{a}^{1} \psi_{b}^{2} \varphi_{c}^{2})] & \\
\frac{1}{\sqrt{6}} (2\psi_{a}^{1} \psi_{b}^{3} \varphi_{c}^{3} + \psi_{a}^{2} \psi_{b}^{2} \varphi_{c}^{1} - \varphi_{b}^{3} \psi_{a}^{2} \varphi_{c}^{1}), & \quad \frac{1}{\sqrt{6}} (2\psi_{a}^{2} \psi_{b}^{2} \varphi_{c}^{3} + \psi_{a}^{1} \psi_{b}^{1} \varphi_{c}^{2} - \varphi_{a}^{3} \psi_{b}^{1} \varphi_{c}^{2}) \\
\frac{1}{2} (\psi_{a}^{2} \varphi_{b}^{3} \psi_{c}^{1} - \varphi_{a}^{3} \psi_{b}^{2} \psi_{c}^{1} - \psi_{a}^{1} \varphi_{b}^{3} \psi_{c}^{2} + \varphi_{a}^{3} \psi_{b}^{1} \psi_{c}^{2}), & \\
\frac{1}{\sqrt{2}} (3\varphi_{a}^{3} \psi_{b}^{3} \varphi_{c}^{3} - 3\psi_{a}^{3} \varphi_{b}^{3} \varphi_{c}^{3}), & \quad \frac{1}{\sqrt{2}} (\varphi_{a}^{3} \psi_{b}^{3} \varphi_{c}^{3} - \psi_{a}^{3} \varphi_{b}^{3} \varphi_{c}^{3}).
\end{align*}
\]

Then we find that the baryonic representation is

\[
3 \otimes 3 \otimes 3 = 7_A \oplus 4_S \oplus 8_\rho \oplus 8_\lambda.
\]

where subscripts $A$ and $S$ mean totally asymmetry and symmetry under the exchange of indices $a$, $b$ and $c$, respectively, while subscript $\rho$ and $\lambda$ stand for asymmetry and symmetry under the exchange of indices $a$ and $b$, respectively.

4. Representations of hadrons

Here we show examples of the $V(3)$ representations for hadrons. Regarding that the strange constituent quark and the $ud$-diquark have a very similar mass, such as 500 MeV, we assign the fundamental representation of $V(3)$, 3, into a strange quark with spin up, $s^\uparrow$, a strange quark, and the fundamental representation of $V(3)$, 3, into a strange quark with spin down, $s^\downarrow$, a strange quark.

| Table 1 | Possible hadrons in the same multiplet of $V(3)$. The lowest states in each category are considered. The wavefunction of the orbital motion is assumed to be symmetric, while for the multiplets with asterisks, their symmetry property makes the orbital wavefunction asymmetric with orbital or radial excitation. The number in the parenthesis denotes the considerable spin-parity $J^P$ of the state. For the excited state, the possible spin $S$ not total spin $J$ is written. One understands that $c$ is a charm quark as a representative of quarks in the outside of the V(3) multiplets. One can replace the charm quark into another quark such as a bottom quark. |
|---------|---------------------------------------------------------------------------------------------------------------|
| $\Psi_c$ | 3 | $sc$ (1$^-$, 0$^-$), $udc$ (1$^+$) | $D_s^*, D_s$, $\Lambda_c$ |
| $\Psi \Psi$ | 9 | $ss$ (1$, 0^-$), $uds$ (1$^+$), $uds$ (1$^+$), $\bar{udud}$ (0$^+$) | $\phi$, $\eta_s$, $\Lambda$, $f_0$ |
| $c \Psi \Psi$ | 5 | $css$ (2$^-$, 2$^+$), $\bar{udcs}$ (0$^+$, 1$^+$) | $\Omega_c$, $T_{cs}$ |
| $4^*$ | css ($S = \frac{1}{2}$), $\bar{udcs}$ ($S = 0$, 1), $\bar{ududc}$ ($S = \frac{1}{2}$) | $\Omega_c$, $T_{cs}$, $\Theta_c$ |
| $\Psi \Psi \Psi$ | 7 | $sss$ (3$^+$), $\bar{udss}$ (1$^+$) | $\Omega$, $T_{ss}$ |
| $8^*$ | $ss$ (1$, 0^+$), $\bar{udss}$ (0$^+$, 1) | $\bar{udud}$ ($S = \frac{1}{2}$) | $\Omega^*$, $T_{ss}$, $\Theta$ |
| $4^{**}$ | $\bar{udss}$ (0$^-$), $\bar{udud}$ ($S = \frac{1}{2}$), $\bar{ududud}$ (0$^+$) | $T_{ss}$, $\Theta^*$, dibaryon |
quark with spin down, \( s_\downarrow \), and an \( ud \) antidiquark, \( \bar{u}d \), as

\[
\Psi^i = \begin{pmatrix}
\psi^1 \\
\psi^2 \\
\varphi^3
\end{pmatrix} = \begin{pmatrix}
s_\uparrow \\
s_\downarrow \\
ud
\end{pmatrix},
\]

which has color triplet. We compose hadrons out of the \( \Psi \) fields. The possible hadrons are summarized in Table 1. We consider nonrelativistic favor configurations made of \( \Psi_i \) and \( \hat{\Psi}_i \). It is easy to recover the relativistic covariance by making up other components composed of \( \Psi_i \) and \( \hat{\Psi}_i \) appropriately.

### 4.1. Triplet Representation

Since the triplet field \( \Psi \) has color, a single \( \Psi \) cannot form hadrons. We consider color singlet hadrons made of an heavy-quark and the anti-triplet field. First we take the charm quark as one of the heavy quarks and consider hadrons composed of \( c^i \) and \( \hat{\Psi}_j \). The charm quark has spin 1/2 and \( i \) runs from 1 to 2. Here we have six components, \( \hat{\psi}_1 c^1, \hat{\psi}_2 c^1, \varphi_3 c^1, \hat{\psi}_1 c^2, \hat{\psi}_2 c^2, \varphi_3 c^2 \), which are two triplets of \( V(3) \). For the spin eigenstates, we have a spin triplet, a spin singlet and a spin doublet as

\[
D_s^* = \{ \hat{\psi}_2 c^1, \frac{1}{\sqrt{2}}(\hat{\psi}_1 c^1 - \hat{\psi}_2 c^2), \hat{\psi}_1 c^2 \}, \quad D_s = \frac{1}{\sqrt{2}}(\hat{\psi}_1 c^1 + \hat{\psi}_2 c^2), \quad \Lambda_c = \{ \varphi_3 c^1, \varphi_3 c^2 \}.
\]

Here we assign possible hadrons which have the appropriate quantum number. In this way, in the \( V(3) \) symmetry, \( D_s^* \), \( D_s \) and \( \Lambda_c \) are in the same multiplet.

We Introduce the conjugate fields like Eqs. (9) and (15) as,

\[
\hat{D}_s^* = \{ \hat{c}^1 \psi_2, \frac{1}{\sqrt{2}}(\hat{c}^1 \psi_1 - \hat{c}^2 \psi_2), \hat{c}^2 \psi_1 \}, \quad \hat{D}_s = \frac{1}{\sqrt{2}}(\hat{c}^1 \psi_1 + \hat{c}^2 \psi_2), \quad \hat{\Lambda}_c = \{ -\hat{c}^1 \varphi_3, -\hat{c}^2 \varphi_3 \}.
\]

Here we have used \((\hat{\psi}_2 c^1)^\dagger = (c^1)^\dagger(\hat{\psi}_2) = \hat{c}^1 \psi_2 \) and \((\varphi_3 c^1)^\dagger = (c^1)^\dagger(\varphi_3) = -\hat{c}^1 \varphi_3 \) with \( \varphi_3 = -\varphi_3^\dagger \). The mass term for the \( V(3) \) triplet hadron can be written with a common mass \( m_0 \) as

\[
m_0(\hat{D}_s D_s^* + \hat{D}_s D_s + \hat{\Lambda}_c \Lambda_c) = m_0 \left[ \hat{c}^1 (\hat{\psi}_1 \hat{\psi}_1 + \hat{\psi}_2 \hat{\psi}_2 - \varphi_3 \varphi_3) c^1 + \hat{c}^2 (\hat{\psi}_1 \hat{\psi}_1 + \hat{\psi}_2 \hat{\psi}_2 - \varphi_3 \varphi_3) c^2 \right]
\]

\[
= -m_0 \left[ \hat{c}^1 (\hat{\psi}_1 \hat{\psi}_1 + \hat{\psi}_2 \hat{\psi}_2 + \varphi_3 \varphi_3) c^1 + \hat{c}^2 (\hat{\psi}_1 \hat{\psi}_1 + \hat{\psi}_2 \hat{\psi}_2 + \varphi_3 \varphi_3) c^2 \right].
\]

This is invariant under the \( V(3) \) rotation. Thus, if we have the \( V(3) \) symmetry, the masses of \( D_s^* \), \( D_s \) and \( \Lambda_c \) get degenerate.

In the same way, if we consider a bottom quark \( b \), \( B_s^* \), \( B_s \) and \( \Lambda_b \) are in the same multiplet of \( V(3) \) as

\[
B_s^* = \{ \hat{\psi}_2 b^1, \frac{1}{\sqrt{2}}(\hat{\psi}_1 b^1 - \hat{\psi}_2 b^2), \hat{\psi}_1 b^2 \}, \quad B_s = \frac{1}{\sqrt{2}}(\hat{\psi}_1 b^1 + \hat{\psi}_2 b^2), \quad \Lambda_b = \{ \varphi_3 b^1, \varphi_3 b^2 \}.
\]

### 4.2. Mesonic nonet representation

Next let us consider the mesonic nonet representation, \( \hat{\Psi}_i \Psi_j \oplus \hat{\Psi}_i \Psi_j \). We introduce the matrix representation like Eq. (57) as

\[
M_i^j = \hat{\Psi}_i \Psi^j.
\]
The mass term can be written in the matrix form as
\[ M = \begin{pmatrix} M_1^1, & M_1^2, & M_1^3 \end{pmatrix}, \quad \eta_s = \frac{1}{\sqrt{2}}(M_1^1 + M_2^2), \quad f_0 = M_3^3, \]
\[ \Lambda = \{M_3^1, M_3^2\}, \quad \hat{\Lambda} = \{M_2^3, M_1^3\}. \] (114)

Note that the \( \hat{\Psi}_i \hat{\Psi}_j \) combination has a pseudoscalar meson and a vector meson. Thus, \( \phi, \eta_s, f_0 \) and \( \Lambda \) are in the same multiplet of \( V(3) \). Here \( \eta_s \) denotes the pseudoscalar meson composed of the strange quarks.

Introducing the conjugate fields given in Eqs. (9) and (15), we have
\[ \hat{\phi} = \{M_2^1, \frac{1}{\sqrt{2}}(M_1^1 - M_2^2), M_1^2\}, \quad \hat{\eta}_s = \frac{1}{\sqrt{2}}(M_1^1 + M_2^2), \quad \hat{f}_0 = -M_3^3, \]
\[ \hat{\Lambda} = \{-M_3^1, -M_3^2\}, \quad \hat{\Lambda} = \{M_2^3, M_1^3\}. \] (115)

Here we consider the (+) component for these hadrons and we have used
\[ M_{ij} = (M_{i}^j)^\dagger = (\hat{\Psi}_i \hat{\Psi}_j)^\dagger = (\hat{\Psi}_i)^\dagger (\hat{\Psi}_j)^\dagger = (-)^{\delta_{is}} \hat{\Psi}_i \hat{\Psi}_j \equiv (-)^{\delta_{is}} M_{ij}^j. \] (116)

The mass term can be read with a common mass \( m_0 \) as
\[ m(\hat{\phi} \phi + \hat{\eta}_s \eta_s + \hat{f}_0 f_0 + \hat{\Lambda} \Lambda + \hat{\Lambda} \Lambda) \]
\[ = m_0 M_1^1 M_2^1 + M_1^1 M_1^1 + M_2^2 M_2^2 + M_1^2 M_1^2 - M_3^3 M_3^3 \]
\[ - M_3^1 M_3^1 - M_3^2 M_3^2 + M_2^3 M_2^3 + M_1^3 M_1^3 \] (117)
\[ = m_0 \left[ \hat{\psi}^\dagger (\psi_1 \hat{\psi}_1 + \psi_2 \hat{\psi}_2 - \varphi_3 \hat{\varphi}_3) \psi^1 + \hat{\psi}^2 (\psi_1 \hat{\psi}_1 + \psi_2 \hat{\psi}_2 - \varphi_3 \hat{\varphi}_3) \psi^2 \right. \]
\[ \left. + \varphi^3 (\psi_1 \hat{\psi}_1 + \psi_2 \hat{\psi}_2 - \varphi_3 \hat{\varphi}_3) \varphi^3 \right] \] (118)

This is again invariant under the \( V(3) \) transformation. Therefore the masses of \( \phi, \eta_s, f_0 \) and \( \Lambda \) get degenerate if we have the \( V(3) \) symmetry.

For later convenience, we introduce a matrix form of these hadrons as
\[ M = \begin{pmatrix} \frac{1}{\sqrt{2}}(\phi_0 + \eta_s) & \phi_\dagger & \Lambda_\dagger \\ \phi_\dagger & -\frac{1}{\sqrt{2}}(\phi_0 - \eta_s) & \Lambda_\dagger \\ \Lambda_\dagger & -\Lambda_\dagger & f_0 \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} \frac{1}{\sqrt{2}}(\hat{\phi}_0 + \hat{\eta}_s) & \hat{\phi}_\dagger & -\hat{\Lambda}_\dagger \\ \hat{\phi}_\dagger & -\frac{1}{\sqrt{2}}(\hat{\phi}_0 - \hat{\eta}_s) & -\hat{\Lambda}_\dagger \\ -\hat{\Lambda}_\dagger & \hat{\Lambda}_\dagger & -\hat{f}_0 \end{pmatrix} \] (119)

The mass term can be written in the matrix form as
\[ m_0 \text{Tr} [\hat{M} M]. \] (120)

For the configuration \( \hat{\Psi}_i \hat{\Psi}_j \oplus \hat{\Psi}_i \hat{\Psi}_j \), the multiplet is same but the parity is opposite. In this multiplet there are scalar and axial vector mesons instead of pseudoscalar and vector mesons for the configuration \( \hat{\Psi}_i \hat{\Psi}_j \oplus \hat{\Psi}_i \hat{\Psi}_j \). In the terminology of the nonrelativistic quark model, these hadrons are \( p \)-wave hadrons due to the orbital excitation of \( \Psi \).

### 4.3. Other representations

Here we consider the other representations which we do not discuss their mass terms.
4.3.1. $c\Psi\Psi$ hadron. Here we consider hadrons made of $c\Psi\Psi$. For the lowest states, the orbital wavefunction is symmetric. Since $c$ and $\Psi$ are color triplet, for the color single hadron states, the color configuration for the "diquark" $\Psi\Psi$ should be anti-symmetric. Thus, $\Psi\Psi$ forms a quintet of $V(3)$. In the quintet, we have $ss$ with spin $1$ and $uds$ with spin $1/2$. With the charm quark, in this representation we have charmed baryons, $css$, with spin $J = 1/2$ and $3/2$, which are $\Omega_c$, and charmed tetraquarks, $\overline{udcs}$, with spin $J = 0$ and $1$.

Allowing excitation of one $\Psi$, one can take the quartet representation of $V(3)$ for the $\Psi\Psi$ configuration. With a $c$ quark, one has $css$ with spin $S = 1/2$, $\overline{udcs}$ with spin $S = 0$ and $1$, and a pentaquark $\overline{udcud}$ with spin $S = 1/2$. Here we mention only the spin $S$ for these hadron, which can be fixed by the $V(3)$ symmetry, because orbital excitation brings an angular momentum into the system and we cannot fix the total spin $J$ before specifying the angular momentum. To determine the angular momentum, we should fix the details of the orbital wavefunction, which is beyond simple symmetry argument.

4.3.2. Baryonic representation. For the baryonic representation, we consider the $\Psi\Psi\Psi$ configuration. For the color singlet hadrons, the three fields should be totally antisymmetric. For the lowest state in which the orbital wavefunction is symmetric, the septet representation $7$ of $V(3)$ can be assigned. In this representation, there are the $\Omega$ baryon composed of $sss$ with spin $J = 3/2$ and a tetraquark made of $\overline{udss}$ with spin $J = 1$.

If one allows asymmetry in the orbital wavefunction with excitation of one of the $\Psi$ fields, the octet representation $8$ of $V(3)$ is also possible. In this multiplet, we have a exited $\Omega$ baryon, $sss$, with spin $S = 1/2$, tetraquarks, $\overline{udss}$, with spin $S = 0$ and $1$ and a pentaquark $\Theta$, $\overline{ududs}$, with spin $S = 1/2$. Again here we mention spin $S$ for these hadron, because the angular momentum is not fixed.

If one accepts two excitations, one can take the quartet representation $4$ of $V(3)$, which is totally symmetric in the exchange of the $\Psi$ field. In this representation, there are an exited tetraquark $\overline{udss}$ with spin $S = 0$, an excited pentaquark $\overline{ududs}$ with spin $S = 1/2$ and a dibaryon $\overline{ududud}$ with spin $S = 0$.

5. Symmetry Breaking

So far we have discussed the classifications of hadrons into the $V(3)$ multiplets. The symmetry between the $s$ quark and the $\overline{ud}$ diquark is not fundamental and should be broken by their mass difference and interactions. Here we consider the symmetry breaking effect on the degenerate mass of the hadrons in the same multiplet.

For the fundamental representation (107), since one has the spin symmetry between the first and second components, $\psi^1$ and $\psi^2$, the degeneracy of these components is exact. Because there is no constraint by spin symmetry on the third component $\varphi^3$, the mass degeneracy for $\varphi^3$ can be broken. Thus, we may write the mass term for the $\Psi^i$ field as

$$m_0\bar{\Psi}\Psi + \delta m\bar{\Psi}\lambda_8\Psi$$  \hspace{1cm} (121)

with a common mass $m_0$ and a parameter $\delta m$ representing the mass difference. Here $\lambda_8$ is the eighth component of the Gell-Mann matrix in the space of the triplet $\Psi$. In this way, the symmetry breaking of the mass of the fundamental representation can be expressed as $\lambda_8$ in the same way as the symmetry breaking on the quark masses for the flavor $SU(3)$. 

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5.1. **Triplet representation**

As seen in Sec. 4.1, \( \{ D_s^*, D_s, \Lambda_c \} \) form a triplet \( \mathbf{3} \) of \( V(3) \) and get degenerate in the symmetric limit. The symmetry breaking term can be introduced by \( \lambda_8 \) in the \( V(3) \) space. Thus, we may write the mass term for this multiplet \( \Pi = \{ D_s^*, D_s, \Lambda_c \} \) as

\[
m_0 \hat{\Pi} \Pi + \delta m \hat{\Pi} \lambda_8 \Pi
\]  

with a common mass \( m_0 \) and a strength of the symmetry breaking \( \delta m \) that introduces the mass difference in the multiplet. From this mass term, we find the masses of the hadrons in the multiplet as

\[
m_{D_s} = m_{D_s^*} = m_0 + \frac{\delta m}{\sqrt{3}}, \quad m_{\Lambda_c} = m_0 - \frac{2\delta m}{\sqrt{3}}.
\]  

The lack of the spin-spin interaction between the quarks is responsible for the degeneracy of \( D_s \) and \( D_s^* \). We may introduce the mass difference between \( D_s^* \) and \( D_s \) induced by the spin-spin interaction between the \( c \) and \( \bar{s} \) quarks, which breaks the \( V(3) \) symmetry but is not contradict with the spin symmetry. With the spin-spin interaction, we have three parameters which are not fixed by the symmetry argument for the three hadrons. Therefore, the \( V(3) \) symmetry breaking gives us no constraint among these masses.

5.2. **Mesonic nonet representation**

5.2.1. **Mass formula.** In Sec. 4.2, we have discussed that \( \{ \phi, \eta_s, f_0, \Lambda \} \) form a nonet representation of \( V(3) \) and get degenerate in the symmetric limit. The symmetric mass term of this multiplet is written as Eq. (120) in the matrix form. Now let us introduce the \( V(3) \) breaking effect on the mass term as

\[
m_0 \text{Tr}[\hat{M}M] + \delta m \text{Tr}[\hat{M}\lambda_8 M]
\]  

with a common mass \( m_0 \) and a mass difference parameter \( \delta m \). Thanks to the properties \( \hat{M} = M^\dagger \) and \( \lambda_8^\dagger = \lambda_8 \), we have \( \text{Tr}[\hat{M}\lambda_8 M] = \text{Tr}[M\lambda_8 \hat{M}] \). Thus, the \( V(3) \) breaking term by \( \lambda_8 \) is represented by a single parameter. Here we have not introduced the spin-spin interaction between the strange quarks, which is another source of the \( V(3) \) symmetry breaking.

Calculating the mass term, we obtain

\[
m_0 \text{Tr}[\hat{M}M] + \delta m \text{Tr}[\hat{M}\lambda_8 M] = \left( m_0 + \frac{\delta M}{\sqrt{3}} \right) (\hat{\phi} \phi + \hat{\eta}_s \eta_s) + \left( m_0 - \frac{2\delta M}{2\sqrt{3}} \right) \hat{f}_0 f_0 + \left( m_0 - \frac{\delta M}{2\sqrt{3}} \right) (\hat{\Lambda} \Lambda + \hat{\bar{\Lambda}} \bar{\Lambda})
\]  

This implies that the masses of these hadrons are obtained as

\[
m_\phi = m_{\eta_s} = m_0 + \frac{\delta m}{\sqrt{3}}, \quad m_{f_0} = m_0 - \frac{2\delta m}{\sqrt{3}}, \quad m_{\Lambda} = m_0 - \frac{\delta m}{2\sqrt{3}}.
\]  

The degeneracy between \( \phi \) and \( \eta_s \) can be resolved by introducing the spin-spin interaction between the \( s \) quarks. Eliminating the parameters \( m_0 \) and \( \delta m \) in these mass formulae, we obtain a mass relation

\[
2m_{\Lambda} = m_{f_0} + m_{\bar{s}s},
\]  

where we have introduced \( m_{\bar{s}s} = m_\phi = m_{\eta_s} \). This is a Gell-Mass Okubo mass formula for the \( V(3) \) symmetry.
5.2.2. Discussion. Here let us discuss how the symmetry property of V(3) works in the mass formula (127) for the $\phi, \eta_s, f_0, A$ hadrons. First of all, in the mass formula, we do not take account of the spin-spin interaction between the strange quarks which induces the mass splitting between the spin partners, $\phi$ and $\eta_s$. A simple way to resolve the spin-spin splitting is to take a spin average

$$m_{\bar{s}s} = \frac{1}{4}(3m_{\phi} + m_{\eta_s}).$$  

(128)

This is obtained accordingly to the first order perturbative calculation, in which the mass shifts induced by the spin-spin interaction are given as

$$m_{\phi} = m_{\bar{s}s} + \frac{1}{4}\alpha_{\bar{s}s}, \quad m_{\eta_s} = m_{\bar{s}s} - \frac{3}{4}\alpha_{\bar{s}s},$$  

(129)

for the spin 1 and 0 states, respectively, with a strength parameter $\alpha_{\bar{s}s}$ of the spin splitting for the strange quarks. Nevertheless, we do not use the physical $\eta$ mass to resolve the spin splitting due to the following reason: For the vector meson, it is well-known that the flavor SU(3) breaking induces the mixing between the flavor octet and singlet with isospin $I = 0$ and strangeness $S = 0$ and the quark contents of the $\omega$ and $\phi$ mesons are written with the ideal mixing in which the $\omega$ meson and the $\phi$ meson may be composed of $\frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d)$ and $\bar{s}s$, respectively. Therefore, the quark content of the $\phi$ meson in our picture is consistent with the physical $\phi$ meson. For the pseudoscalar meson, however, the mixing between the flavor octet and singlet is known to be substantially small due to the different origin of the masses for the octet and singlet pseudoscalar mesons. Hence, the flavor content of the physical $\eta$ meson is almost given by the octet representation of the flavor SU(3), that is $\frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$, not purely $\bar{s}s$. This is not consistent with our picture and we cannot directly apply the physical $\eta$ mass to our mass formula.

Here we estimate the magnitude of the spin-spin splitting of $\bar{s}s$ in the following way. The spin-spin interaction is induced by the color magnetic interaction of the one gluon exchange. The strength of the color magnetic interaction is in inverse proportion to the masses of the quarks participating in the spin-spin interaction. Thus, the strength parameter $\alpha_{\bar{s}s}$ for the strange quarks may be written as

$$\alpha_{\bar{s}s} = \frac{\beta}{m_s^2},$$  

(130)

where $\beta$ is a universal parameter of the spin-spin interaction independent of the flavor of the participating quarks. Here we use the spin-spin splitting between $D^*_s$ and $D_s$ composed of the $c$ and $\bar{s}$ quarks, and the observed mass splitting is found to be $0.1438 \pm 0.0004$ GeV [31]. The mass difference can be written as

$$m_{D^*_s} - m_{D_s} = \alpha_{\bar{s}c} = \frac{\beta}{m_cm_c}.$$  

(131)

Taking the observed mass difference as 0.14 GeV and assuming that the masses of the strange and charm quarks be 0.5 GeV and 1.3 GeV, respectively, we obtain $\beta = 0.094$ GeV$^3$. With this value we also reproduce the mass difference of $B^*_s$ and $B_s$ as 0.047 GeV for the $b$ quark mass $m_b = 4$ GeV, while the observed mass difference is found to be $0.0487^{+0.0023}_{-0.0021}$ GeV [31].
Using these values we find the spin averaged mass for $\bar{ss}$ as

$$m_{\bar{ss}} = 0.925 \text{ GeV}$$  \hfill (132)

for $m_\phi = 1.019 \text{ GeV}$. With the $\Lambda$ mass $m_\Lambda = 1.116 \text{ GeV}$, the mass formula (127) suggests that a scalar meson $f_0$ composed of two diquarks $\bar{ud}ud$ has a mass

$$m_{f_0} = 1.320 \text{ GeV}. \hfill (133)$$

The corresponding particle can be found as $f_0(1370)$ in the particle listing of Particle Data Group [31], in which the scalar resonance $f_0(1370)$ is reported as a broad resonance having a pole mass at $(1200-1500) - i(150-250) \text{ MeV}$. The mass of $f_0(1370)$ includes our value as a central value, and $f_0(1370)$ can be a two-diquark state. For further confirmation, we need to investigate whether the property of $f_0(1370)$ shows the nature of a bound state of two diquarks. Such investigation could reveal the existence of the $ud$ diquark inside hadrons as an effective degrees of freedom.

It is also interesting to mention that the mass differences of $m_{f_0} - m_\Lambda$ and $m_\Lambda - m_{\bar{ss}}$ are about 200 MeV, which is the consequence of the $V(3)$ symmetry breaking in the nonet representation. Similarly, the $V(3)$ breaking appearing in the triplet representations shown in Sec. 4.1 can be found in the mass differences of $m_{\Lambda_c} - m_{\bar{s}c}$ and $m_{\Lambda_b} - m_{\bar{s}b}$, where $m_{\bar{s}c}$ and $m_{\bar{s}b}$ stand for the spin averaged masses of $D^*_s-D_s$ and $B^*_s-B_s$, respectively, and these values are also about 200 MeV. If this $V(3)$ breaking found in these hadrons is attributed to the mass difference between the $s$ quark and the $ud$ diquark, the mass of the $ud$ diquark may be 700 MeV if one assumes the constituent strange quark mass to be 500 MeV. Nevertheless, it should be worth mentioning that the $V(3)$ breaking could stem from asymmetry of the interaction of $\bar{s}-Q$ and $(ud)-Q$ as pointed out in Refs. [22, 23]. There they found that the string tension in the color electric confinement potential between quark and diquark is as weak as half of that between quark and anti-quark, even though these two systems have the same color configuration. Further investigation on the symmetry between the quark and diquark should be necessary.

6. Summary

We have introduced a symmetry among the constituent strange quark and the $\bar{ud}$ diquark, supposing that their masses be very similar to each other, say 500 MeV. To investigate the properties of this symmetry, we have constructed an algebra which transforms a fermion with spin 1/2 and a boson with spin 0. Regarding these fermion and boson as a fundamental representation of this algebra, we have built higher representations for mesonic, diquark and baryonic configurations. We have proposed possible assignments of these irreducible representations to existing hadrons, which is summarized in Table 1. Particularly investigating the triplet and nonet representations, we have found that $(\Lambda_c, D_s, D^*_s)$ and $(\eta_s, \phi, \Lambda, f_0(1370))$ could form multiplets, respectively. Introducing a symmetry breaking coming from the mass difference the $s$ quark and the $\bar{ud}$ diquark, we have derived a mass relation among each multiplet. In the nonet representation, we have the mass relation among $\phi, \eta_s, \Lambda, f_0$. In our formulation, both $\phi$ and $\eta_s$ are composed of the $s$ and $\bar{s}$ quarks, while the physical $\eta$ meson is known to be expressed almost as the octet. Thus, estimating the strength of the spin-spin interaction from the mass difference of $D_s$ and $D^*_s$, we have found the spin averaged mass $\phi$ and $\eta_s$ to be 920 MeV. Using the mass relation with this mass and the observed $\Lambda$ mass,
we have found the mass of $f_0$ in the multiplet to be 1320 MeV. This may correspond to the observed $f_0(1370)$ meson. Thus, our model suggests $f_0(1370)$ to be a tetraquark composed of ud and $\bar{u}\bar{d}$ diquarks. In addition, finding the mass differences among the nonet to be 200 MeV, and the difference between the spin averaged mass of $D_s$ and $D_s^*$ ($B_s$ and $B_s^*$) and $\Lambda_c$ ($\Lambda_b$) also to be 200 MeV, we have suggested the mass difference between the constituent $s$ quark and the ud diquark to be 200 MeV. Thus, if we regard the strange quark mass as 500 MeV, the mass of ud diquark may be 700 MeV. Nevertheless, we could have another possibility for the source of the mass difference to be a perspective suggesting that the symmetry breaking comes from the difference of interactions between $\bar{s}-Q$ and ($ud$)-$Q$. It is an open question that the origin of the symmetry breaking between the $s$ quark and the ud diquark, and further investigations on this issue are necessary.

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