OFF-CENTER OBSERVERS VERSUS SUPERNOVAE IN INHOMOGENEOUS PRESSURE UNIVERSES

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ABSTRACT

Exact luminosity distance and apparent magnitude formulae are applied to the Union2 557 supernovae sample in order to constrain the possible position of an observer outside of the center of symmetry in spherically symmetric inhomogeneous pressure Stephani universes, which are complementary to inhomogeneous density Lemaître–Tolman–Bondi (LTB) void models. Two specific models are investigated. The first allows a barotropic equation of state at the center of symmetry without the need to specify a scale factor function (model IIA). The second has no barotropic equation of state at the center, but has an explicit dust-like scale factor evolution (model IIB). It is shown that even at 3σ CL, an off-center observer cannot be further than about 4.4 Gpc away from the center of symmetry, which is comparable to the reported size of a void in LTB models with the most likely value of the distance from the center at about 341 Mpc for model IIA and 68 Mpc for model IIB. The off-center observer cannot be farther away from the center than about 577 Mpc for model IIB at 3σ CL. It is determined that the best-fit parameters which characterize inhomogeneity are $\Omega_{\text{inh}} = 0.77$ (dimensionless: model IIA) and $\alpha = 7.31 \times 10^{-9} \text{ (s km}^{-1})^2 \text{ Mpc}^{-4/3}$ (model IIB).

Key word: cosmology: observations

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1. INTRODUCTION

Field theoretical discrepancy between observed and calculated values of the cosmological constant lead cosmologists to study non-Friedmannian models of the universe which could explain acceleration by the effect of inhomogeneity (Marra et al. 2007; Uzan et al. 2008; Caldwell & Stebbins 2008; Clarkson et al. 2008). It was claimed that we lived in a spherically symmetric void of density described by the Lemaître–Tolman–Bondi (LTB) inhomogeneous dust spheres model (Lemaître 1933; Tolman 1934; Bondi 1947). However, there is a variety of non-Friedmannian models (Bolejko et al. 2010) which have the advantage of being exact solutions of the Einstein field equations and may also serve to solve the problem. The observational cosmology program of Ellis et al. (1985) suggests that we should perhaps start with model-independent observations of the past light cone, and then eventually draw conclusions related to geometry of the universe. Fundamentally, the homogeneity of the universe needs to be checked, and even if its large-scale structure is subject to the Copernican Principle after some averaging process (Buchert 2000, 2008), then one needs to prove that. In fact, non-Friedmannian models of the universe can fit observations very well and we need observational tools to differentiate between them and standard concordance models.

Assume that the spherical symmetry supported by cosmic microwave background (CMB) data can be the first step toward the task. In this context, the inhomogeneous density $\rho(t, r)$ (dust shells) LTB models are complementary to the inhomogeneous pressure $p(t, r)$ (gradient of pressure shells) Stephani models (Stephani 1967; Krasinski 1983; Dąbrowski 1993) since both of them are spherically symmetric and their only common part is the Friedmann models which can be obtained in the limit of vanishing inhomogeneity. Because most recent interest has concentrated on the former models, we would like to investigate such a complement of LTB models here. Accidentally, the Stephani universes were the first inhomogeneous models ever compared with observational data from supernovae (Dąbrowski 1998) following the derivation of the redshift–magnitude relation for these models (Dąbrowski 1995), which used the series expansion of Kristian & Sachs (1966) both for a centrally placed and an off-center observer. LTB models were first tested observationally by Célérier (2000) and Tomita (2001) and then followed more recently by Biswas et al. (2010), Blomqvist & Mörtsell (2010), Marra & Notari (2011), and Valkenburg et al. (2012). It is worth mentioning that a general (nonspherically symmetric) Stephani model has no spacetime symmetries at all, although its three-dimensional hyperspaces of constant time are maximally symmetric, similar to those of the Friedmann universe, and so it can be a good example of full inhomogeneity for future study. A generalization of an LTB model which is fully spacetime inhomogeneous is the Szekeres model, for which observational aspects have recently been studied by Walters & Hellaby (2012). In this paper, we restrict ourselves to the spherically symmetric Stephani models only.

Our paper is organized as follows. In Section 2, we briefly present some basic properties of inhomogeneous pressure Stephani models, also in comparison to the complementary LTB models. In Section 3, we study an exact luminosity distance formula for an off-center observer in the Stephani universe. In Section 4, we discuss some exact Stephani models which will be useful for further discussion. Section 5 contains the main result of the paper and deals with the constraints on the position of an observer who is away from the center of symmetry by the application of the Union2 supernovae data. In Section 6, we give our conclusions.

2. INHOMOGENEOUS PRESSURE STEPHANI UNIVERSES

The spherically symmetric inhomogeneous pressure Stephani model is the only spherically symmetric solution of Einstein equations for a perfect-fluid, energy–momentum tensor
\( T^{\mu \nu} = (\rho c^2 + p) u^\mu u^\nu + p g^{\mu \nu} \) (\( p \) is the pressure, \( g^{\mu \nu} \) is the metric tensor, \( c \) is the velocity of light, and \( u^\mu \) is the 4-velocity vector) which is conformally flat (Weyl tensor vanishes) and embeddable in a five-dimensional flat space (Stephani 1967). It is complementary to an LTB spherically symmetric model in the sense that it has inhomogeneous pressure while an LTB model has inhomogeneous density and the only common limit of both models is Friedmann. The metric of the spherically symmetric Stephani model reads as (Dąbrowski 1993)

\[
s^2 = -\frac{a^2}{a^2} \left[ \frac{(V)}{\tau} \right]^2 c^2 dt^2 + \frac{a^2}{V^2} (dr^2 + r^2 d\Omega^2), \tag{2.1}
\]

where

\[
V(t, r) = 1 + \frac{1}{4} k(t)^2 r^2, \tag{2.2}
\]

and \((\cdots) = \partial / \partial t\). The function \(V(t, r)\) with \( k = 0, \pm 1 \) is of the same form for Friedmann models in isotropic coordinates (see Appendix A). \( a(t) \) plays the role of a generalized scale factor, \( k(t) \) represents a time-dependent “curvature index,” and \( r \) is the radial coordinate. Kinematically, Stephani models are characterized by the nonvanishing expansion scalar \( \Theta \) and the acceleration vector \( a_\mu \). LTB models have nonzero expansion \( \Theta \) and the shear tensor \( \sigma_{\mu \nu} \).

Dąbrowski (1993) found two exact spherically symmetric Stephani models: model I which fulfills the condition \((V/\alpha) = 0\), and model II which fulfills the condition \((k/\alpha) = 0\). The metric for model II is simpler since the factor in front of \( dr^2 \) in the metric (2.1) reduces to \((-1/V^2)\). This simplification will further be used in our paper to model the universe. Some models of type I have been investigated in more detail by Barrett & Clarkson (2000).

The metric of model II is given by (Dąbrowski 1995; Balcerzak & Dąbrowski 2013)

\[
ds^2 = -\frac{c^2 dt^2}{V^2} + \frac{a^2(t)}{V^2} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)]. \tag{2.3}
\]

### 3. LUMINOSITY DISTANCE FOR AN OFF-CENTER OBSERVER

Making use of the standard relations for spherical coordinates,

\[
x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta,
\]

we transform the metric (2.3) into the Cartesian coordinate system as follows:

\[
ds^2 = -\frac{1}{V^2} [-c^2 dt^2 + a^2 (dx^2 + dy^2 + dz^2)], \tag{3.5}
\]

where \( V(t, x, y, z) = 1 + (1/4)k(t)(x^2 + y^2 + z^2) \). Due to its fundamental property, the Stephani metric can be transformed into a form expressing its conformal flatness,

\[
ds^2 = \frac{a^2}{V^4} (-d\tau^2 + dx^2 + dy^2 + dz^2), \tag{3.6}
\]

where we have used the conformal time

\[
d\tau = c dt / a(t). \tag{3.7}
\]

Furthermore, we apply another coordinate transformation of the form

\[
x' = x - x_0, \quad y' = y - y_0, \quad z' = z - z_0,
\]

where \((x_0, y_0, z_0)\) is the position of an observer. The position of an observer at the center of symmetry is \((x_0, y_0, z_0) = (0, 0, 0)\). The metric (3.6) in these new coordinates is simply

\[
ds^2 = \frac{a^2}{V^2} [-d\tau^2 + dx'^2 + dy'^2 + dz'^2]. \tag{3.9}
\]

where

\[
V = 1 + \frac{1}{4} k(\tau)(x'^2 + y'^2 + (z' + z_0)^2). \tag{3.10}
\]

Next, we transform the metric back to the spherical coordinate system, but now at the observer’s position \((x_0, y_0, z_0)\) which is outside the center of symmetry, by applying the spherical coordinates at this position,

\[
x' = r' \sin \theta' \cos \varphi', \quad y' = r' \sin \theta' \sin \varphi', \quad z' = r' \cos \theta',
\]

which gives (3.9) in the form

\[
ds^2 = \frac{a^2}{V^2} [-d\tau^2 + dr'^2 + r'^2 (d\theta'^2 + \sin^2 \theta' d\varphi'^2)]. \tag{3.12}
\]

In the new coordinate system \( \{\tau, r', \theta', \varphi'\} \), all null geodesics that reach an observer at \( r' = 0 \) fulfill the following conditions:

\[
d\tau = -dr', \quad \theta' = \text{const.}, \quad \varphi' = \text{const.} \tag{3.13}
\]

Suppose that an object (a supernova) is located at a distance \( r' = \hat{r}' \) and has the coordinates \( \theta' = \hat{\theta}' \) and \( \varphi' = \hat{\varphi}' \) as seen by an observer placed at \((x_0, y_0, z_0)\). Then, the proper area of such an object is given by

\[
dS = \left[ \frac{a^2(\tau)}{V^2(\tau, r')} \right] e^{\hat{\theta}' d\hat{\theta}' d\hat{\varphi}'} \tag{3.14}
\]

where the index “\( e \)” refers to an emitter of light (a supernova). Since the conformal factor \( a^2 / V^2 \) preserves the angles measured in both flat and curved spacetime, the solid angle spanned by an object as seen by an observer is given by

\[
d\Omega = \sin \hat{\theta}' d\hat{\theta}' d\hat{\varphi}' \tag{3.15}
\]

The area distance \( d_A = \sqrt{dS / d\Omega} \) is

\[
d_A = \left[ \frac{a}{V} \right] e^\hat{\varphi}'. \tag{3.16}
\]

The redshift in the Stephani universe (2.3) reads as (Dąbrowski 1995; Balcerzak & Dąbrowski 2013)

\[
1 + z = \frac{a_0 V_e}{a_e V_0}, \tag{3.17}
\]

where index “\( 0 \)” refers to the present. Due to the Etherington (1933) reciprocity theorem, we relate the luminosity distance \( d_L \) with the area distance \( d_A \) as

\[
d_L = (1 + z)^2 d_A, \tag{3.18}
\]

and so finally the luminosity distance is

\[
d_L = \frac{a_0(1 + z)^2}{1 + \frac{4}{3} a_0 r'_0}. \tag{3.19}
\]
Furthermore, we will assume that \( r_0 \) indicates the position of an observer in the coordinate system \( \{ t, r, \theta, \varphi \} \) of the metric (2.3). Since the observational data is given in terms of the apparent magnitude, we apply the standard relation

\[
\mu(z) = 5 \log(d_L) + 25. \tag{3.20}
\]

The same formula (3.19) for the luminosity distance can alternatively be obtained by using the area distance definition of Ellis et al. (1985), which reads as

\[
d_A^2 \sin^2 \gamma = \tilde{g}_{\gamma\gamma} \tilde{g}_{\xi\xi} - \tilde{g}_{\gamma\xi}^2, \tag{3.21}
\]

where \( \tilde{g}_{\mu\nu} \) is the metric expressed in an observer’s frame, i.e., a frame which is centered on the observer in which the angular part of the metric is given by

\[
\Omega^2 = dy^2 + \sin^2 \gamma dx^2. \tag{3.22}
\]

Here, the angles \( \gamma \) and \( \xi \) correspond to the polar and azimuthal angles in this frame. We note that an observer frame which was

\[
\tilde{\beta} \equiv \left. \frac{\partial^2}{\partial \gamma^2} \right|_{\gamma=0} γ \quad \text{and} \quad \tilde{\xi} \equiv \left. \frac{\partial^2}{\partial \xi^2} \right|_{\xi=0} γ
\]

provides that we make the following identifications: \( \gamma \equiv \theta' \) and \( \xi \equiv \phi' \). With such identifications, we immediately obtain

\[
\tilde{g}_{\theta\theta} = g_{\theta'\theta'} = \frac{a^2}{V^2} r^2, \tag{3.23}
\]

\[
\tilde{g}_{\xi\xi} = g_{\xi'\xi'} = \frac{a^2}{V^2} r^2 \sin^2 \theta', \tag{3.24}
\]

\[
\tilde{g}_{\varphi\varphi} = g_{\phi'\phi'} = 0. \tag{3.25}
\]

Finally, the application of (3.21) gives

\[
d_A = \frac{a_c}{V_c} t', \tag{3.26}
\]

which coincides with (3.16).

4. THE MODELS

4.1. Model IIA

A subclass of model II with \( k(t) = \beta a(t) \) (\( \beta = \text{const with units of } [\beta] = \text{Mpc}^{-1} \)) was proposed by Stelmach & Jackačka (2001), and it was assumed that at the center of symmetry the standard barotropic equation of state \( p(t) = w_0 \rho(t) \) was fulfilled. This assumption means that

\[
8 \pi G \frac{3}{c^2} \rho(t) = \frac{A^2}{a^{3(w+1)}(t)} (A = \text{const.}) \tag{4.27}
\]

and allows one to write a generalized Friedmann equation as

\[
\frac{1}{c^2} \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{A^2}{a^{3(w+1)}(t)} - \frac{\beta}{a(t)} \tag{4.28}
\]

with the following equation of state:

\[
p(t) = \left[ w + \frac{\beta}{4}(w+1)a(t)r^2 \right] \dot{\rho}(t)c^2 = w_0 \rho(t)c^2. \tag{4.29}
\]

From (4.28) and (4.29), we can see that the standard dust-filled \( (w = 0) \) Friedmann universe solution \( a(t) \propto t^{2/3} \) is possible outside the center of symmetry only in the limit \( \beta \to 0 \), i.e., when no inhomogeneity is present. In the next subsection, we will present the solution with such a form of the scale factor which admits the inhomogeneity, but no barotropic equation of state at the center.

Similarly, as in the Friedmann models, one can define the critical density as \( \varrho_c(t) = (3/8\pi G)(\dot{a}(t)/a(t))^2 \), and the density parameter \( \Omega(t) = \varrho(t)/\varrho_c(t) \). After taking \( t = t_0 \), from (4.27) we have

\[
1 = \frac{A^2}{H_0^2 a^{3(1+w)}(t_0)} - \frac{\beta}{H_0^2 a_0} \equiv \Omega(t) + \Omega_{\text{inh}}, \tag{4.30}
\]

and so

\[
\beta = -a_0 H_0^2 \Omega_{\text{inh}} < 0, \tag{4.31}
\]

where \( H = \dot{a}/a \) is the Hubble parameter, the dimensionless parameter \( \Omega_0 \) stands for the barotropic matter content, while \( \Omega_{\text{inh}} \) stands for the inhomogeneity density. A generalized Friedmann equation can be written as

\[
\frac{H^2(t)}{H_0^2} = \Omega_0 a^{-3(1+w)} + \frac{\Omega_{\text{inh}}}{a}, \tag{4.32}
\]

where the form of the function \( a(t) \) is not specified. Using (4.13), (4.32), and the definition of the conformal time, we find that

\[
\dot{\varphi}' = \dot{\varphi}(a) = \frac{1}{H_0} \int_{a_0}^{a} \frac{dx}{\sqrt{(1 - \Omega_{\text{inh}})x^3 - \Omega_{\text{inh}}x^2}}, \tag{4.33}
\]

where \( a_c \) is the value of the scale factor at the moment of an emission of the light ray. For the model (4.32), the redshift (4.37) reads as

\[
1 + z = \frac{a_0(4 - a_c H_0^2 \Omega_{\text{inh}} r_c^2)}{a_c(4 - a_0 H_0^2 \Omega_{\text{inh}} r_0^2)}, \tag{4.34}
\]

with

\[
r_c^2 = (r_0 \sin \theta_0 \cos \varphi_0 + \dot{\varphi}'(a) \sin \theta' \cos \varphi')^2
+ (r_0 \sin \theta_0 \sin \varphi_0 + \dot{\varphi}'(a) \sin \theta' \sin \varphi')^2
+ (r_0 \cos \theta_0 + \dot{\varphi}'(a) \cos \theta')^2, \tag{4.35}
\]

where \( r = r_0, \theta = \theta_0, \) and \( \varphi = \varphi_0 \) indicate the position of an observer in the very first coordinates of the metric (2.3), while \( \theta' \) and \( \varphi' \) are the coordinates of a supernova as seen by an off-center observer in the sky. Solving (4.34) for \( a \), and substituting the outcome back to (4.33), we thus express \( \dot{\varphi}' \) in terms of the redshift \( z \) instead of the scale factor \( a \). The result of this calculation substituted into (3.19) for model IIA gives the luminosity distance expressed in terms the redshift \( z \), the parameters of the model \( \Omega_{\text{inh}}, w, r_0, \theta_0, \varphi_0, \) and \( H_0 \), and the angles \( \theta' \) and \( \varphi' \) at which a supernova is seen by an observer:

\[
d_L = \frac{(1 + z)}{1 - \frac{\Omega_{\text{inh}}}{4} H_0^2 r_0^2} \dot{\varphi}'(\Omega_{\text{inh}}, w, r_0, \theta_0, \varphi_0, H_0, \theta', \varphi', z). \tag{4.36}
\]

We can determine that in the limit \( \Omega_{\text{inh}} \to 0 \), the formula (4.36) reduces to the standard flat Friedmann model filled with a single matter component which satisfies a barotropic equation of state.
Equation (4.40) can also be written as pressure matter to fill in the universe which plays the role of dark energy. The center being at infinity. This, however, is equivalent to having strongly negative and \[ \frac{\sigma}{a} \rightarrow \infty \]. Note that the homogeneous limit \( \Omega_{inh} \rightarrow 0 \) is effectively possible under the shift of the distance to the center being at infinity. This, however, is equivalent to having strongly negative pressure matter to fill in the universe which plays the role of dark energy.

4.2. Model IIB

In this subsection, we consider another type II model (Dąbrowski 1993, 1998) which is basically the same as the Wesson & Ponce de Leon (1989) model. It has a different type of inhomogeneity than model IIA. We start with the same metric (2.3) but instead of assuming that the barotropic equation of state is fulfilled at the center of symmetry, we take exact forms of the scale factor and the curvature function as

\[
\begin{align*}
a(t) &= \sigma t^{2/3}, \\
k(t) &= -\alpha \sigma a(t),
\end{align*}
\]

where the units of the constants are \( [\alpha] = \text{s/km}^{2/3} \text{Mpc}^{-4/3} \) and \( [\sigma] = \text{km s}^{-1} \text{Mpc}^{1/3} \), and time is measured as inverse to Hubble parameter units \( [t] = \text{s Mpc}^{-1} \).

The Einstein equations for such a model are given by (Dąbrowski 1993)

\[
\frac{8\pi G}{c^4} \rho(t) = \frac{4}{3t^2} - \frac{3\alpha}{t^{2/3}},
\]

\[
\frac{8\pi G}{c^4} p(t,r) = \frac{2\alpha}{t^{2/3}} - \frac{4}{3t^{4/3}} r^2 + \alpha^2 \sigma^2 r^2,
\]

from which we can immediately see that at the center of symmetry \( r = 0 \), no barotropic equation of state is fulfilled. An analytic form of the equation of state at the center of symmetry is instead

\[
\rho = p \left( \frac{32\pi^2 G^2}{3\alpha c^6} r^2 - \frac{3}{2} \right).
\]

Equation (4.40) can also be written as

\[
\rho + \frac{3}{2} p = \frac{c^4}{6\pi G t^2}.
\]

The model approaches the dust-filled Friedmann universe if \( \alpha \rightarrow 0 \). The equation of state (4.40) may be fitted to the ideal gas interpretation of the inhomogeneous pressure (Sussman 2000). From (4.39), one can see that there is a finite density singularity of pressure at \( r \rightarrow \infty \).

We now follow the same procedure as for the previous model IIA. Applying the definition of conformal time (3.7),

\[
d\tau = \frac{dt}{a} = \frac{1}{\sigma} t^{-2/3} dt,
\]

and the condition (3.13), we have

\[
\tau = \hat{\tau}(a) = \frac{3}{\sigma} \left( t_o^{1/3} - t'_{e/3} \right) = \frac{3}{\sigma^{2/3}} \left( a_0 - a_{e/2}^{1/2} \right),
\]

where \( t_e \) and \( t_o \) are the times of emission and observation, respectively. The luminosity distance for the model IIB then reads as

\[
d_L = \frac{\sigma t_0^{2/3} (1 + z) \hat{\tau}(a)}{1 - \frac{1}{4\alpha^2 \sigma^2 t_0^{2/3} r_e^2}} + \frac{1}{a_e - 1 - \frac{1}{4\alpha^2 \sigma^2 t_0^{2/3} r_e^2}}
\]

with \( r_e \) given by (4.35).

5. Constraining the Position of an Observer with Supernovae Data

We used a Bayesian framework based on the Metropolis-Hastings Markov Chain Monte Carlo (MCMC) method to constrain the position of an off-center observer in the Stephani models IIA and IIB with the supernovae (SNIa) data. We took the likelihood function to be Gaussian in the form

\[
p(data|\Theta) \propto \exp \left( -\frac{1}{2} z^2 \right),
\]

where

\[
\begin{align*}
\Omega_{inh} \text{ vs. the proper distance of an off-center observer position Dist plane. The contours denote roughly 68%, 95%, and 99% credible regions. One sees that a more distant position of an observer is connected with having more and more negative pressure matter at the center of symmetry and that larger inhomogeneity prevents the observer from being too far from the center.}
\end{align*}
\]
The value of the equation of state of the matter at the center of symmetry is more negative pressure matter at the center of symmetry is accompanied by a higher positive pressure matter being allowed at the center of symmetry, and so the inhomogeneity mimics the acceleration of the universe. On the contrary, more negative pressure matter at the center of symmetry is accompanied by a small inhomogeneity and this matter is the driving force for cosmic acceleration. The most likely inhomogeneity density value is $\Omega_{\text{inh}} = 0.77$ (compare to Godłowski et al. 2004). As for the equation of state of the matter at the center of symmetry, the value $w = 0.093$ is most favorable. The most likely position of an observer away from the center is Dist $= 341$ Mpc ($\chi^2 = 526$).

It also seems that the more distant position of an observer is connected with having increasingly negative pressure matter at the center of symmetry and that larger inhomogeneity prevents the observer from being too far from the center (see Figure 2). Larger inhomogeneity is accompanied by higher positive pressure matter being allowed at the center of symmetry, and so the inhomogeneity mimics the current acceleration of the universe. On the contrary, more negative pressure matter at the center of symmetry is accompanied by small inhomogeneity, and such matter is the driving force for cosmic acceleration (see Figure 3).

In Figure 4, we have plotted the confidence contours for the location of the center of inhomogeneity (center of symmetry) in the sky for the model IIA. On the left is the North Celestial Hemisphere and on the right is the South Celestial Hemisphere. The bold line is for zeroth right ascension (meridian line). The center of inhomogeneity is placed at declination $\delta = -65^\circ 75$ and right ascension $\alpha = 187^\circ 33$ and the distance from the observer is Dist $= 341$ Mpc.

More restrictive results related to the position of an observer from the center are obtained for the model IIB. The off-center observer cannot be farther away from the center than about 215 Mpc at 1$\sigma$ CL, 320 Mpc at 2$\sigma$ CL, and 577 Mpc at 3$\sigma$ CL. From the plot, we see that the inhomogeneity parameter is centered on an inhomogeneity parameter value of $\alpha = 7.31 \times 10^{-9}$ (s/km)$^{2/3}$Mpc$^{-2/3}$ ($\chi^2 = 557$) which corresponds to the distance of 68 Mpc between the center of symmetry and an observer (see Figure 5).

where $\Theta$ denotes the parameters of the considered models and "data" denotes the SNIa data. For the SNIa data, $\chi^2$ takes the form

$$\chi^2_{SN} = \sum_{i,j=1}^{N} (C^{-1})_{ij} \left[ \left( \mu_{\text{obs}}(z_i, \theta_i', \phi_i') - \mu_{\text{pred}}(z_i, \theta_i', \phi_i') \right)^2 \right]$$

$$\times \left[ \left( \mu_{\text{obs}}(z_j, \theta_j', \phi_j') - \mu_{\text{pred}}(z_j, \theta_j', \phi_j') \right)^2 \right].$$

In Figures 1–3, we plot the contours which show the most likely position of an off-center observer in Gigaparsecs in an inhomogeneous pressure Stephani universe IIA as limited by the Union2 sample of $N = 557$ supernovae (Amanullah et al. 2010). The position is measured in terms of a proper distance:

$$\text{Dist} = \int_0^{\alpha_0} \frac{a}{V} \, dr.$$  \hspace{1cm} (5.48)

It is clear that at 1$\sigma$ CL an observer cannot be further than a distance of about 450 Mpc, at 2$\sigma$ CL he cannot be further than about 2.5 Gpc, and at 3$\sigma$ further than about 4.4 Gpc in model IIA. It is apparently an approximate size of a void also reported for LTB models (Garfinkle 2006; February et al. 2010). From the plots, we can also conclude that the inhomogeneity density is nonzero and its most likely value is $\Omega_{\text{inh}} = 0.77$.

Figure 3. Marginalized confidence intervals for inhomogeneous pressure model IIA in the center of symmetry barotropic index $\omega$ vs. the dimensionless inhomogeneity density $\Omega_{\text{inh}}$. The contours denote roughly 68%, 95%, and 99% credible regions. It is obvious that larger inhomogeneity is accompanied by higher positive pressure matter being allowed at the center of symmetry, and so the inhomogeneity mimics the acceleration of the universe. On the contrary, more negative pressure matter at the center of symmetry is accompanied by small inhomogeneity, and this matter is the driving force for cosmic acceleration.

Figure 4. Position of the center of inhomogeneity for model IIA. On the left is the North Celestial Hemisphere and on the right is the South Celestial Hemisphere. The bold line is for zeroth right ascension (meridian line). The most likely value of the declination is $\delta = -65^\circ 75$ and the right ascension is $\alpha = 187^\circ 33$. (A color version of this figure is available in the online journal.)
Figure 5. Marginalized confidence intervals for inhomogeneous pressure model IIB in the inhomogeneity parameter $\alpha$ vs. the proper distance of an off-center observer position $\text{Dist}$ plane. The contours denote roughly 68%, 95%, and 99% credible regions. The best-fit value of the inhomogeneity parameter is $\alpha = 7.31 \times 10^{-9} \text{s} \text{km}^{-3/2} \text{Mpc}^{-4/3}$. Note that this plot excludes the value of $\alpha \rightarrow 0$ since this is the dust limit (Einstein–de-Sitter) of the inhomogeneous model under study which is incompatible with supernovae data. The most likely value of the distance to the center is 68 Mpc ($\chi^2 = 557$).

Figure 6. Position (in radians) of the center of inhomogeneity for model IIB with respect to an Earth observer. The most likely values of the declination is $\delta = 69.35$ and the right ascension is $\alpha = 8.39$.

The confidence contours for the location of the center of inhomogeneity for the model IIB are plotted in Figure 6. With respect to an Earth observer, the center is placed at declination $\delta = 69.35$ and right ascension $\alpha = 8.39$ and the distance to it is 68 Mpc.

6. RESULTS AND CONCLUSIONS

We have presented exact formulae for the luminosity distance and the apparent magnitude of an astronomical object in inhomogeneous pressure Stephani universes for an off-center observer. Two specific Stephani models have been investigated. The first model (IIA) allowed for a barotropic equation of state to be valid at the center of symmetry with no exact function for the scale factor being specified. The second model (IIB) had no barotropic (though still analytic) equation of state at the center, but its scale factor evolution was assumed to be exact and the same as for the dust-filled Friedmann universe. These models then represented different types of inhomogeneity, a fact which made our investigations more general.

Our exact luminosity distance and apparent magnitude formulae have then been applied to a sample of data of Union 2 supernovae (Amanullah et al. 2010) in order to constrain the possible position of an observer outside of the center of symmetry in these inhomogeneous pressure models.

Our results have shown that in model IIA an observer at $1\sigma$ CL cannot be further than about 450 Mpc away from the center, at $2\sigma$ CL he cannot be further than about 2.5 Gpc away, and at $3\sigma$ further than about 4.4 Gpc, which is comparable with the evaluations of very large voids in LTB models (Clarkson & Regis 2011; Grande & Perivolaropoulos 2011). We have also found that the inhomogeneity density has a most likely value of $\Omega_{\text{inh}} = 0.77$ and the equation of state of the matter at the center of symmetry is characterized by a barotropic index value of about $w = 0.093$. The most likely position of an observer away from the center is $\text{Dist} = 341$ Mpc ($\chi^2 = 526$).

More restrictive results related to the position of an observer away from the center have been obtained for model IIB. The off-center observer cannot be farther away from the center than about 215 Mpc at $1\sigma$ CL, 320 Mpc at $3\sigma$ CL, and 577 Mpc at $3\sigma$ CL. We have also shown that the best-fit value of the inhomogeneity parameter is $\alpha = 7.31 \times 10^{-9} \text{s} \text{km}^{-3/2} \text{Mpc}^{-4/3}$, which corresponds to the distance to the center of 68 Mpc ($\chi^2 = 557$).

Model IIA has $\chi^2 = 526$ which is 5 less than for flat $\Lambda$CDM ($\chi^2 = 530.7$). Then, it is consistent with the data according to both Akaike and Bayesian information criteria, but is not preferred over $\Lambda$CDM since to be so $\chi^2$ would need to be lowered by 8 (4 new parameters times 2) in the former case and by 25 (4 new parameters times ln 557) in the latter case. On the other hand, model IIB has a $\chi^2$ 26.3 greater than $\Lambda$CDM and so it is disfavored at more than $4\sigma$.

We have also evaluated possible directions in the sky from the Earth to the center of inhomogeneity. For model IIA it is at declination $\delta = -65.75$ and right ascension $\alpha = 187.33$, while for the model IIB it is at declination $\delta = 1.21$ rad = 69.35 and right ascension $\alpha = 0.15$ rad = 8.39.

Though we do not take into account the local motions of an observer (who is comoving) with respect to the CMB in our models, it might be interesting to ask whether or not such directions may coincide with the directions of the Local Group (LG) motion which are claimed to appear at velocity $V_{l,b} = 627 \pm 22 \text{ km s}^{-1}$ toward $(l, b) = (276^\circ \pm 3^\circ, 30^\circ \pm 3^\circ)$ in galactic coordinates (Kogut et al. 1993; Nusser et al. 2014), which has more recently been the subject of investigations of low-redshift local supernovae by Feindt et al. (2013). Our results in galactic coordinates give $(l, b) = (300^\circ : 66^\circ, -2^\circ : 98^\circ)$ for model IIA and $(l, b) = (121\pm36, 6\pm3)$ for model IIB, which does not seem to be very conclusive as far as possible alignment is concerned.

We should emphasize that our tests are based only on supernovae data. We have not discussed any other cosmological tests, such as the CMB shift parameter, baryon acoustic oscillations, and the Sandage–Loeb redshift drift. Usually, supernovae do not impose such strong constraints on the models as the CMB tests, so we think that the restrictions for the position of an off-center observer may be even more severe once they are taken into account.

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APPENDIX A

NONISOTROPIC VERSUS ISOTROPIC RADIAL COORDINATES

The inhomogeneous Stephani metric (2.1) uses the so-called isotropic coordinate \( \tilde{r} \) which is analogous to the isotropic coordinate applied in the homogeneous and isotropic Friedmann–Robertson–Walker metric (and can be obtained from (2.1) in the limit \( k(t) \rightarrow k_0 = 0, \pm 1 \)) as follows:

\[
\begin{align*}
    ds^2 &= -c^2dt^2 + \frac{a^2(t)}{V^2(\tilde{r})}[d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2 \theta d\phi^2)], \\
    &\quad \text{where} \\
    V(\tilde{r}) &= 1 + \frac{1}{4}k_0\tilde{r}^2, \\
    d\tilde{r}^2 &= -c^2dt^2 + a^2(t) \left[ \frac{dr^2}{1 - k_0r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right].
\end{align*}
\]

The relations between these coordinates are (Narlikar 1983) (note that in the above formulae, we have changed the meaning of coordinate \( r \) from the Stephani metric (2.1) to \( \tilde{r} \) in order to adopt standard notation, which commonly uses \( r \) for the nonisotropic coordinate)

\[
r = \frac{\tilde{r}}{1 + \frac{1}{4}k_0\tilde{r}^2} = \frac{\tilde{r}}{V(\tilde{r})},
\]

\[
\tilde{r} = \frac{2r}{1 + \sqrt{1 - k_0r^2}}.
\]

Note that for \( k_0 = +1, \tilde{r} \in (0, 2), r \in (0, 1); \) for \( k_0 = 0, \tilde{r} \in (0, \infty), r \in (0, \infty); \) for \( k_0 = -1, \tilde{r} \in (0, 2), r \in (0, \infty). \) Usually, one defines a radial coordinate \( r \) by introducing a new coordinate \( \chi \) as follows:

\[
r = \frac{\tilde{r}}{V(\tilde{r})} = S(\chi) = \begin{cases} 
    \sin \chi, & k_0 = +1, \\
    \chi, & k_0 = 0, \\
    \sinh \chi, & k_0 = -1,
\end{cases}
\]

and so

\[
\begin{align*}
    dr &= \cos \chi \frac{dS(\chi)}{d\chi} \\
    &= \sqrt{1 - k_0r^2} = \sqrt{1 - k_0S^2(\chi)} \\
    &= \begin{cases} 
    \sqrt{1 - \sin^2 \chi} = \sqrt{1 - r^2}, & k_0 = +1, \\
    \frac{1}{\sqrt{1 + \sin^2 \chi}} = \sqrt{1 + r^2}, & k_0 = -1.
\end{cases}
\end{align*}
\]

On the other hand, we have

\[
\frac{dS(\chi)}{d\tilde{r}} = \frac{dS(\chi)}{d\chi} \frac{d\chi}{d\tilde{r}} = \frac{1 - \frac{1}{4}k_0\tilde{r}^2}{(1 + \frac{1}{4}k_0\tilde{r}^2)^2},
\]

and we can invert it as

\[
\frac{d\tilde{r}}{dr} = \frac{dS(\chi)}{d\chi} \frac{d\chi}{d\tilde{r}} = \frac{1}{(1 - \frac{1}{4}k_0r^2)^2},
\]

where

\[
\frac{dS(\chi)}{d\chi} = \sqrt{1 - k_0S^2(\chi)} = \frac{1 - \frac{1}{4}k_0\tilde{r}^2}{1 + \frac{1}{4}k_0\tilde{r}^2},
\]

and so

\[
\frac{d\chi}{d\tilde{r}} = \left(1 + \frac{1}{4}k_0\tilde{r}^2 \right)^{-1} = \frac{1}{V(\tilde{r})},
\]

which means that

\[
\frac{d\tilde{r}}{V(\tilde{r})} = d\chi.
\]

It is useful to have the derivatives of one coordinate with respect to the other as follows:

\[
\frac{d\tilde{r}}{dr} = \frac{d\chi}{d\tilde{r}} = \frac{dS(\chi)}{d\chi} \frac{d\chi}{d\tilde{r}} = \frac{1}{(1 - \frac{1}{4}k_0r^2)^2}.
\]

The application of the coordinate transformation given by Equation (A14) allows us to transform the Stephani metric (2.1) to a form analogous to that of the nonisotropic coordinate Friedmann metric (A3) (compare Sussman 2000), i.e.,

\[
ds^2 = -\frac{a^2}{\dot{a}^2} \left[ \frac{V_r}{V} \chi \right]^2 c^2dt^2 + \frac{a^2}{V^2} \left[ \frac{dr^2}{1 - k_0r^2} + r^2d\Omega^2 \right],
\]

where

\[
V_r = 1 + k(t)S^2(\chi/2),
\]

and

\[
S(\chi/2) = \begin{cases} 
    \sin \frac{\chi}{2}, & k_0 = +1, \\
    \chi, & k_0 = 0, \\
    \sinh \frac{\chi}{2}, & k_0 = -1.
\end{cases}
\]

Using the nonisotropic coordinate (A6), one may express Equation (A18) as

\[
S(\chi/2) = \frac{1}{\sqrt{2}}(1 - \sqrt{1 - k_0r^2})^{1/2},
\]

and so the metric (2.1) can be expressed in nonisotropic coordinates as follows:

\[
ds^2 = -\frac{a^2}{\dot{a}^2} \left[ \frac{V_r}{V} \chi \right]^2 c^2dt^2 + a^2 \left[ \frac{dr^2}{1 - k_0r^2} + r^2d\Omega^2 \right],
\]

where

\[
V_r = 1 + \frac{k(t)}{2}(1 - \sqrt{1 - k_0r^2}).
\]

Note that for model II \(((k/a)\dot{r} = 0)\), we obtain a simpler metric which is an analog of the metric (2.3).
It is possible to transform the nonisotropic Friedmann–Robertson–Walker (FRW) coordinate metric \((A3)\) into the flat Minkowski metric by using a conformal transformation of the form
\[
ds_2^2 = \Phi^2 ds_M^2 \tag{B1}
\]
with \(\Phi\) being the conformal factor and where
\[
ds_M^2 = -c^2 dt^2 + d R^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{B2}
\]
In terms of the \(\chi\) coordinate, the Friedmann metric \((A3)\) reads as
\[
ds_\chi^2 = -c^2 dt^2 + a^2(t)[d\chi^2 + S^2(\chi)(d\theta^2 + \sin^2 \theta d\phi^2)], \tag{B3}
\]
which by a simple coordinate transformation of the form
\[
ct = c dT = a(\tau) d\tau \tag{B4}
\]
can be presented as
\[
ds_\chi^2 = a^2(\tau) d\tau^2 = a^2(\tau)[-d\tau^2 + d\chi^2 + S^2(\chi) \times (d\theta^2 + \sin^2 \theta d\phi^2)], \tag{B5}
\]
and so \(\Phi(\tau) = a(\tau)\) is the conformal factor.

The tangent vectors to a null geodesic component in Minkowski space are solved easily as
\[
\begin{align*}
k_M^\tau &= \frac{dT}{ds} = 1, \\
k_M^R &= \frac{dR}{ds} = \pm \sqrt{1 - \frac{h^2}{R^2}}, \\
k_M^\theta &= \frac{d\theta}{ds} = 0, \\
k_M^\phi &= \frac{d\phi}{ds} = \frac{h}{R^2}. \\
\end{align*} \tag{B6}
\]

The tangent vectors for the metric \(d\bar{z}^2\) in Equation \((B5)\) are given by
\[
\begin{align*}
k^\tau &= \frac{dT}{d\bar{z}} = 1, \\
k^\chi &= \frac{d\chi}{d\bar{z}} = \pm \sqrt{1 - \frac{h^2}{S^2(\chi)}}, \\
k^\theta &= \frac{d\theta}{d\bar{z}} = 0, \\
k^\phi &= \frac{d\phi}{d\bar{z}} = \frac{h}{S^2(\chi)}. \\
\end{align*} \tag{B7}
\]

The transformation rule for the tangent vectors reads as (Hawking & Ellis 1999)
\[
k_{FRW}^{\mu} = \Phi^{-2} \frac{\partial x^\mu}{\partial \bar{z}^\nu} k_M^{\nu}. \tag{B8}
\]
where \(\partial x^\mu / \partial \bar{z}^\nu\) includes a coordinate transformation from the coordinates \(x^\mu\) to \(\bar{z}^\nu\) necessary to bring the metric into a flat form so that we have
\[
k_{FRW}^\tau = \Phi^{-2} \frac{dT}{d\tau} = \frac{1}{a}. \tag{B9}
\]