Relaxation to a Parity-Time Symmetric Generalized Gibbs Ensemble after a Quantum Quench in a Driven-Dissipative Kitaev Chain

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The construction of the generalized Gibbs ensemble, to which isolated integrable quantum many-body systems relax after a quantum quench, is based upon the principle of maximum entropy. In contrast, there are no universal and model-independent laws that govern the relaxation dynamics and stationary states of open quantum systems, which are subjected to Markovian drive and dissipation. Yet, as we show, relaxation of driven-dissipative systems after a quantum quench can, in fact, be determined by a maximum entropy ensemble, if the Liouvillian that generates the dynamics of the system has parity-time symmetry. Focusing on the specific example of a driven-dissipative Kitaev chain, we show that, similarly to isolated integrable systems, the approach to a parity-time symmetric generalized Gibbs ensemble becomes manifest in the relaxation of local observables and the dynamics of subsystem entropies. In contrast, the directional pumping of fermion parity, which is induced by nontrivial non-Hermitian topology of the Kitaev chain, represents a phenomenon that is unique to relaxation dynamics in driven-dissipative systems. Upon increasing the strength of dissipation, parity-time symmetry is broken at a finite critical value, which thus constitutes a sharp dynamical transition that delimits the applicability of the principle of maximum entropy. We show that these results, which we obtain for the specific example of the Kitaev chain, apply to broad classes of noninteracting fermionic models, and we discuss their generalization to a noninteracting bosonic model and an interacting spin chain.

Introduction. After a quench, generic isolated quantum many-body systems relax locally to a state that is determined, according to the fundamental postulates of statistical mechanics [1], by maximization of entropy, subject to the constraints imposed by integrals of motion [2–4]. For integrable systems, which are characterized by an extensive number of integrals of motion, the resultant equilibrium state is the generalized Gibbs ensemble (GGE) [5–7]. The principle of maximum entropy and, consequently, the structure of the GGE are universal in the sense that model-dependent details affect only the specific form of the integrals of motion and the numerical coefficients that enter the GGE as Lagrange multipliers, and that are determined by the initial state. Likewise, relaxation to the GGE is characterized by a set of universal characteristic traits such as light-cone spreading of correlations [7–9] and linear growth and volume-law saturation of the entropy of a finite subsystem [10–13]. In contrast to this scenario of generalized thermalization of isolated integrable systems, open systems, which are subjected to Markovian drive and dissipation [14], typically evolve toward nonequilibrium steady states that are determined by the interplay of internal Hamiltonian dynamics and the coupling to external reservoirs, and that are, therefore, highly model-dependent [15–22]. In particular, the breaking of conservation laws due to the coupling to external reservoirs entails the eventual loss of any memory of the initial state. That is, the constraints that determine the GGE in isolated systems are lifted, and, consequently, the notion of a maximum entropy ensemble appears to be rendered meaningless. Therefore, the existence of model-independent principles that govern the relaxation dynamics and stationary states of open systems is seemingly ruled out.

How are these drastically different paradigms of relaxation in isolated and driven-dissipative systems connected when \( \gamma \), the strength of the coupling to external reservoirs, is gradually diminished? For any \( \gamma > 0 \), an open system eventually reaches a stationary state that is vitally determined by the coupling to external reservoirs and takes the form of a GGE only in the limit \( \gamma \to 0 \) [23–26]. In this Letter, however, we show that the universal principles that govern generalized thermalization after a quantum quench in isolated systems can retain their validity—in suitably generalized form—even for finite values of \( \gamma \) that are comparable to characteristic energy scales of the system Hamiltonian. This robustness is caused by parity-time (PT) symmetry of the Liouvillian [27–39] that generates the dynamics of the system. Focusing on the specific example of a driven-dissipative generalization [40–42] of the Kitaev chain [43], we find that in the PT-symmetric phase, the quadratic eigenmodes of the adjoint Liouvillian oscillate at different frequencies, but crucially they all decay with the same rate. Consequently, after factoring out exponential decay, dephasing [7, 44] leads to local relaxation to a PT-symmetric GGE (PTGGE). In analogy to the GGE for isolated noninteracting fermionic many-body systems and interacting integrable systems that can be mapped to noninteracting fermions [5–7, 45–47], we specify the PTGGE in terms of eigenmodes of the generator of the postquench dynamics. However, the PTGGE generalizes the GGE to account for noncanonical statistics of these eigenmodes, and the nonconservation of the associated mode occupation numbers renders the PTGGE intrinsically time-dependent. We illustrate relaxation to the PTGGE in terms of the fermion parity and the entropy of a finite subsystem. Thereby, we reveal the directional pumping of fermion parity, which occurs for quenches from the topologically trivial phase of the isolated Kitaev chain to the non-Hermitian topological phase of the driven-dissipative Kitaev chain [42], as a phenomenon that is unique to driven-dissipative systems, and we establish the validity of a dissipative quasiparticle picture [48–50] for values of \( \gamma \) up to the sharply defined boundary of the PT-symmetric phase. Going beyond the example of the Kitaev chain, we show that our re-
sults apply to broad classes of noninteracting fermionic models and, in suitably generalized form, also to models of noninteracting bosons and interacting spins.

Model. We consider a Kitaev chain [43] of length $L$ with hopping matrix element $J$, pairing amplitude $\Delta$, and chemical potential $\mu$, as described by the Hamiltonian

$$H = \sum_{l=1}^{L} \left(-Jc_{l}^{\dagger}c_{l+1} + \Delta c_{l}c_{l+1} + \text{H.c.}\right) - \mu \sum_{l=1}^{L} \left(c_{l}^{\dagger}c_{l} - \frac{1}{2}\right).$$  (1)

The operators $c_{l}$ and $c_{l}^{\dagger}$ annihilate and create, respectively, a fermion on lattice site $l$. Unless stated otherwise, we assume periodic boundary conditions with $c_{L+1} = c_{1}$. The system is prepared in the ground state $|\psi_{0}\rangle$ for $J = \Delta$ and $\mu = 0$. We focus on the limit $\mu_{0} \to -\infty$, such that the initial state is the topologically trivial vacuum state, $|\psi_{0}\rangle = |\Omega\rangle$ with $c_{1}|\Omega\rangle = 0$, but our results are not affected qualitatively by this choice. At $t = 0$, the chemical potential is quenched to a finite value $\mu$, while $J = \Delta$ is kept fixed. At the same time, the system is coupled to Markovian reservoirs. Consequently, the postquench dynamics is described by a quantum master equation for the system density matrix $\rho$ [51, 52],

$$i\frac{d}{dt}\rho = \mathcal{L}\rho + i \sum_{l=1}^{L} \left(2L_{d}L_{t}^{l} - \{L_{t}^{l}, L_{d}\}\right),$$  (2)

where we choose $L_{t} = \sqrt{\gamma_{t}}c_{l} + \sqrt{\gamma_{t}^{*}}c_{l}^{\dagger}$ as a coherent superposition of loss and gain at rates $\gamma_{t}$ and $\gamma_{t}^{*}$, respectively [40–42]. The mean rate $\gamma = (\gamma_{t} + \gamma_{t}^{*})/2$ measures the overall strength of dissipation, whereas the relative rate $\delta = \gamma_{t} - \gamma_{t}^{*}$ is akin to an inverse temperature: For $\delta = 0$, the system evolves for $t \to \infty$ toward a steady state $\rho_{SS}$ with infinite temperature, $\rho_{SS} = \rho_{\infty} = 1/2L$ [42]; In contrast, for $\delta \to \infty$, the steady state is pure, $\rho_{SS} = |\Omega\rangle\langle\Omega|$. Since the initial state $\rho_{0} = |\psi_{0}\rangle\langle\psi_{0}|$ is Gaussian and the Liouvillean $\mathcal{L}$ is quadratic and, therefore, preserves Gaussianity, the time-evolved state $\rho(t) = e^{\mathcal{L}t}\rho_{0}$ is fully determined by the covariance matrix

$$g_{l-r}(t) = \langle\{c_{l}, c_{r}^{\dagger}\}(t)\rangle\langle\{c_{l}, c_{r}^{\dagger}\}(t)\rangle\langle\{c_{l}, c_{r}^{\dagger}\}(t)\rangle\langle\{c_{l}, c_{r}^{\dagger}\}(t)\rangle.$$

where $\langle \cdots (t) \rangle = \text{tr}(\cdots \rho(t))$. The Fourier transform $g_{k} = -i\sum_{l=1}^{L} e^{i\epsilon_{l}^{(k)}} g_{l} / dt = -iz_{k}g_{k} + ig_{k}\epsilon^{(k)} + s_{k}$, where $z_{k}$ and $s_{k}$ can be expressed in terms of Pauli matrices as $z_{k} = -2\gamma_{t}\mathbb{1} - 2\sqrt{\gamma_{t}\gamma_{t}^{*}} + 2\Delta \sin(k)c_{\sigma} - (2J\cos(k) + \mu)c_{\sigma}$ and $s_{k} = -2\Delta \sigma_{\sigma}$. For $\gamma \to 0$, $z_{k}$ reduces to the Bogoliubov-de Gennes Hamiltonian of the isolated Kitaev chain [54], and it has inversion symmetry, $z_{k} = \sigma_{z}z_{k}^{*}\sigma_{z}$, and time-reversal symmetry, $z_{k} = z_{-k}^{*}$. These symmetries are broken when $\gamma > 0$. However, the Liouvillean still has PT symmetry in the sense that the traceless part of $z_{k}$, given by $z_{k}^{\prime} = z_{k} + 2\gamma_{t}\mathbb{1}$, is symmetric under the combined operation of inversion and time-reversal, $z_{k}^{\prime} = \sigma_{z}z_{k}^{*}\sigma_{z}$. PT symmetry implies that there are two types of eigenvectors and associated eigenvalues $\lambda_{k,\pm}$ of $z_{k}$ [53]: PT-symmetric eigenvectors, which come in pairs with eigenvalues $\text{Re}(\lambda_{k,\pm}) = -\text{Re}(\lambda_{-k,\pm})$ and $\text{Im}(\lambda_{k,\pm}) = -2\gamma_{t}$, and PT-breaking eigenvectors, for which $\text{Re}(\lambda_{k,\pm}) = 0$ and $\text{Im}(\lambda_{k,\pm} + i2\gamma_{t}) = -\text{Im}(\lambda_{k,\pm} + i2\gamma_{t})$. The PT-symmetric phase is defined by the eigenvectors of $z_{k}$ being PT-symmetric for all momenta $k$, which is the case for $2\sqrt{\gamma_{t}\gamma_{t}^{*}} < 2|J - \mu|$. Then, the eigenvalues of $z_{k}$ are given by $\lambda_{k,\pm} = -i2\gamma_{t} \pm \omega_{k}^{\pm}$ with $\epsilon_{k}^{\pm} = \epsilon_{k}^{2} - 4\gamma_{t}\gamma_{t}^{*}$ and $\epsilon_{k}^{2} = (2J\cos(k) + \mu)^{2} + 4\Delta^{2}\sigma_{\sigma}^{2}$. For strong dissipation with $2\sqrt{\gamma_{t}\gamma_{t}^{*}} > 2J + |\mu|$, all eigenvectors are PT-breaking. Finally, in the PT-mixed phase at intermediate dissipation, eigenvectors of both types exist.

PT-symmetric GGE. We now focus on relaxation dynamics after a quench to the PT-symmetric phase, which is best described in terms of the eigenmodes of the adjoint Liouvillian [53]. With the matrix $V_{k}$ that diagonalizes $z_{k}$, these modes are given by

$$\frac{dk_{k}}{dt} = V_{k}^{\dagger}c_{k}^{\dagger}c_{k}^{\dagger}V_{k}, \quad V_{k} = \left(\begin{array}{cc}\cos(\theta_{k}) & i\sin(\theta_{k}) \\ i\sin(\theta_{k}) & \cos(\theta_{k})\end{array}\right),$$

where $c_{k} = \frac{1}{\sqrt{2}} \sum_{l=1}^{L} e^{-ikl}c_{l}$, $\tan(\theta_{k}) = -2\Delta \sin(k)/(2J\cos(k) + \mu)$, and $\tan(\phi_{k}) = 2\sqrt{\gamma_{t}\gamma_{t}^{*}}/\omega_{k}$. For $\gamma = 0$, $V_{k}$ reduces to the usual unitary Bogoliubov transformation. When $\gamma > 0$, non-unitarity of $V_{k}$ is reflected in the statistics of the modes $d_{k}$ as expressed through their anticommutation relations:

$$\langle\{d_{k}, d_{k}^{\dagger}\}(t)\rangle = e^{-4\gamma_{t}}\langle\{d_{k}, d_{k}^{\dagger}\}\rangle_{0} + \langle 1 - e^{-4\gamma_{t}}\rangle \langle\{d_{k}, d_{k}^{\dagger}\}\rangle_{SS},$$

where $\langle \cdot \cdot \cdot \rangle_{0} = \text{tr}(\cdots \rho_{0})$ and $\langle \cdot \cdot \cdot \rangle_{SS} = \text{tr}(\cdots \rho_{SS})$ denote expectation values in the initial and steady state, respectively. For anomalous commutors we find

$$\langle\{d_{k}, d_{-k}^{\dagger}\}(t)\rangle = e^{-2\epsilon_{k}\omega_{k}^{\pm} - i2\gamma_{t}^{*}}\langle\{d_{k}, d_{-k}^{\dagger}\}\rangle_{0} + \langle 1 - e^{-2\epsilon_{k}\omega_{k}^{\pm} - i2\gamma_{t}^{*}}\rangle \langle\{d_{k}, d_{-k}^{\dagger}\}\rangle_{SS}. \quad (7)$$

We first consider the case of balanced loss and gain, $\delta = 0$. Then, heating to infinite temperature is reflected in the exponential decay and vanishing in the steady state of the expectation values of both normal and anomalous commutators. Crucially, in the PT-symmetric phase, the decay rate is identical for all momentum modes. Thus, after factoring out exponential decay, the system relaxes locally to a maximum entropy ensemble through dephasing of modes with $\omega_{k} = \omega_{k}^{\mp}$. Since the decay of normal commutators is nonoscillatory, dephasing affects only anomalous commutators. Therefore, we define the PTGGE as the maximum entropy ensemble [55] that is compatible with the statistics given in Eq. (5), and the nondephasing expectation values of normal commutators collected in the diagonal matrix
Figure 1. Subsystem parity after quenches to the trivial (green, \( \mu = -4J \)) and topological (blue, \( \mu = -J \)) PT-symmetric phases for \( \gamma = 0.3J \), \( \delta = 0 \), and \( t = 20 \). The solid lines are obtained from Eqs. (9) and (10), while we set \( \alpha_s = 0.08 \) and \( \alpha_c = 0.11 \) to achieve best agreement with the numerical data shown as dashed lines. Vertical straight and horizontal lines indicate \( t = t_r \) and the PTGGE predictions for the stationary values, respectively. In all figures, \( L \) is chosen large enough to avoid finite-size effects.

\[
\zeta(t) = e^{-4\gamma t} \text{diag}(\langle [d_k, d_k^\dagger] \rangle_0, \langle [d_{-k}, d_{-k}^\dagger] \rangle_0).
\]

We find, in terms of spinors \( D_k = (d_k, d_k^\dagger) \) [53],

\[
\rho_{\text{PTGGE}}(t) = \frac{1}{Z_{\text{PTGGE}(t)}} e^{-2 \sum k \epsilon k \Gamma_k \zeta(t) \zeta(t)^\dagger} D_k \zeta(t) \zeta(t)^\dagger D_k^\dagger,
\]

with normalization \( Z_{\text{PTGGE}}(t) \) such that \( \text{tr}(\rho_{\text{PTGGE}}(t)) = 1 \). The PTGGE reduces to the conventional GGE when \( \gamma = 0 \) such that \( f_k = 1 \) and \( \zeta(t) \) becomes time-independent. Relaxation to the PTGGE in the PT-symmetric phase stands in stark contrast to the appearance of elementary excitations, which are accounted for in Eq. (9) by \( \langle [d_k, d_k^\dagger] \rangle_0 \), before it approaches a stationary value. The value \( t_s \) can be large enough such that relaxation to the PTGGE can be observed if \( \delta \) is sufficiently small. The precise condition on the value of \( \delta \) depends on the observable under consideration. Below, we provide a quantitative discussion for the fermion parity of a finite subsystem.

Relaxation of subsystem parity. To illustrate relaxation to the PTGGE, we consider the fermion parity of a subsystem that consists of \( \ell \) contiguous lattice sites, \( P_\ell = e^{i\gamma \sum \epsilon_k \zeta_k} \). The expectation value \( \langle P_\ell \rangle = \langle \Phi | \rho | \Phi \rangle \) is given by the Pfaffian of the reduced covariance matrix \( \Gamma_\ell = (\Gamma_{k\ell})_{\ell,k=1}^{2\ell} \) [56, 57], where \( \Gamma = iR^T GR \), \( G \) is a block Toeplitz matrix built from the \( 2 \times 2 \) blocks \( g_i \) in Eq. (3), and \( R = \sum_{l=1}^{\ell} \frac{1}{2\ell} \left( 1 - \zeta_i \right) \). For the isolated Kitaev chain, a combined Jordan-Wigner [58] and Kramers-Wannier [59, 60] transformation maps the subsystem parity to order parameter correlations in the transverse field Ising model [53]. Based on the analytical results of Calabrese et al. [45–47] for the relaxation of order parameter correlations in the space-time scaling limit \( t, t \to \infty \) with \( \ell / t \) fixed, in Eqs. (9) and (10) below, we formulate analytical conjectures for the time dependence of the subsystem parity in the driven-dissipative Kitaev chain, which we find to be in excellent agreement with numerical results.

First, we consider quenches to the topologically trivial [42] PT-symmetric phase with \( |\mu| > 2J \). Then, as shown in Fig. 1, for \( \delta = 0 \), the behavior of the subsystem parity in the space-time scaling limit is well described by [53]

\[
\langle P_\ell(t) \rangle \sim P_0 e^{-4\gamma t + \frac{\ell}{2} \min(2\nu_k, |\nu_k|) \text{tr}(\ln(\zeta_k \zeta_k^\dagger))},
\]

where \( \zeta_k = e^{4\gamma t} \zeta_k(t) \) is time-independent. The value \( \gamma = 0.3J \) chosen in Fig. 1 leads to sizeable modifications of statistics and dynamics of Liouvillian as compared to Hamiltonian elementary excitations, which are accounted for in Eq. (9) by the appearance of \( f_k \) and the definition of the velocity \( v_k = \omega_k / \delta_k \) in terms of \( \omega_k \) rather than the Hamiltonian dispersion relation \( \epsilon_k \). Relaxation to the PTGGE is best revealed by considering the rescaled subsystem parity \( e^{4\gamma t} \langle P_\ell(t) \rangle \), which decays up to the Fermi time [46] \( t_F = \ell / (2v_{\max}) \) where \( v_{\max} = \max_k |v_k| \), before it approaches a stationary value. The prefactor \( P_0 \) in Eq. (9) is obtained by fitting the long-time limit of the rescaled subsystem parity to the PTGGE prediction.

For small \( \delta \neq 0 \), we expect \( \langle P_\ell(t) \rangle \) to deviate from Eq. (9) after a crossover time \( t_c \sim (1/\gamma) \text{ln}(c_\ell) \). This expectation is confirmed in Fig. 2, where we also compare numerical results for \( t_s \) with an analytical estimate [53]. The condition to observe relaxation of the subsystem parity to the PTGGE, therefore, reads \( t_F < t_c \).

Directional parity pumping. For quenches to the PT-symmetric phase with \( |\mu| < 2J \), due to nontrivial non-Hermitian topology of the Liouvillian [42], the rescaled subsystem parity repeatedly crosses zero before it relaxes to a stationary value. Physically, these zero crossings can be interpreted as pumping of parity between the subsystem and
its complement. The period of the zero crossings is determined by soft modes of the PTGGE, i.e., momenta $k_{\pm}$ for which the exponent in Eq. (8) vanishes. For the isolated Kitaev chain [7, 46], the soft modes $k_{\pm} = -k_{\mp}$ are locked onto each other by inversion symmetry [53], and the period of zero crossings is given by $t_c = \pi/(2\omega_{k_{\pm}}) = \pi/(2\omega_{k_{\mp}})$. In contrast, for the PTGGE in Eq. (8), we find that due to the breaking of inversion symmetry when $\gamma > 0$, there are two distinct soft modes with $k_{\pm} \neq -k_{\mp}$ [53], and, consequently, two distinct time scales $t_{\pm} = \pi/(2\omega_{k_{\pm}})$. As shown in Fig. 1, for $t < t_F$, the resulting oscillatory decay of the subsystem parity is captured by the following modified space-time scaling limit [53]:

$$\langle P(t) \rangle \sim 2 \cos(\omega_{k_{\pm}} t + \alpha_{\pm}) \cos(\omega_{k_{\mp}} t + \alpha_{\mp}) \langle P(t) \rangle_{\text{nonosc}},$$

where $\alpha_{\pm}$ are undetermined phase shifts and the nonoscillatory part is given by Eq. (9), which also approximately describes the behavior of $\langle P(t) \rangle$ for $t > t_F$.

The two timescales $t_{\pm}$ and $t_{\mp}$ have a clear physical meaning in terms of the exchange of parity through, respectively, the left and right boundaries of the subsystem. This is confirmed numerically in Fig. 3 by considering a chain with open boundary conditions and subsystems $L_t = \{1, \ldots, L\}$ and $R_t = \{L - t + 1, \ldots, L\}$ located at the left and right ends of the chain [53]. Then, zero crossings of $\langle P(t) \rangle$ occur only with period $t_{\pm}$ and $t_{\mp}$, respectively. In contrast, for a chain with periodic boundary conditions, $\langle P(t) \rangle$ exhibits zero crossings at multiples of both $t_{\pm}$ and $t_{\mp}$. As we show in the Supplemental Material [53], the occurrence of different periods of parity pumping for subsystems at the left and right ends of the chain requires both mixedness of the time-evolved state and breaking of inversion symmetry and is, therefore, unique to driven-dissipative systems.

**Evolution of subsystem entropy.** In isolated systems, a key signature of thermalization is provided by the growth and saturation of the von Neumann entropy of a finite subsystem, $S_{vN,\ell} = -\text{tr} (\rho_\ell \ln(\rho_\ell))$. Here, we consider a subsystem that consists of $\ell$ contiguous lattice sites, and whose density matrix $\rho_\ell$ is obtained by taking the trace over the $L-\ell$ remaining sites, $\rho_\ell = \text{tr}_{L-\ell}(\rho)$. Quantitative predictions for the time dependence of $S_{vN,\ell}$ in the space-time scaling limit can be derived from a quasiparticle picture [10–13], according to which the initial state acts as source of pairs of entangled quasiparticles. The ballistic propagation of quasiparticles leads to growth of the subsystem entropy in proportion to the number of pairs of entangled quasiparticles that are shared between the subsystem and its complement.

In open systems, the subsystem entropy $S_{\ell} = S_{\ell}^{\text{QP}} + (\ell/L) S_{\ell}^{\text{stat}}$ is the sum of two contributions [48–50, 61]: $S_{\ell}^{\text{QP}}$ measures correlations due to the propagation of quasiparticle pairs, and $S_{\ell}^{\text{stat}} = S_{vN,\ell}$ is the statistical entropy due to the mixedness of the time-evolved state. Based on results of Refs. [49, 50] for weak dissipation $\gamma \sim 1/\ell$, we conjecture that for quenches to the PT-symmetric phase and $\delta = 0$, the quasiparticle-pair contribution $S_{\ell}^{\text{QP}}$ obeys the following space-time scaling limit [53]:

$$S_{\ell}^{\text{QP}} \sim \int_0^\infty \frac{dk}{2\pi} \min(2|\eta_k|, L) \left[ S\left( \xi_k, t, \ell \right) - S\left( g_k(t) \right) \right],$$

where $S(\xi, t, \ell)$ is given by Eq. (10) for weak dissipation $\gamma \sim 1/\ell$, and the subscript “$d$” in last term indicates that due to dephasing, only the nonoscillatory components of the trace are required to capture the space-time scaling limit. At long times $\gamma t \gg 1$, since $\xi_k(t), g_k(t) \sim e^{-\gamma t}$, we can expand $S(\xi) \sim \ln(2) - e^{2}/2$. Then, due to the cancellation of the leading constant term in the difference in Eq. (11), we obtain $S_{\ell}^{\text{QP}} \sim e^{-\gamma \ell}$. Therefore, in analogy to the subsystem parity, relaxation to the PTGGE becomes visible by considering the rescaled quasiparticle-pair entropy $e^{-\gamma \ell} S_{\ell}^{\text{QP}}$. As shown in Fig. 4 the rescaled quasiparticle-pair entropy grows up to the Fermi time $t_F$ before it saturates to a stationary value predicted by the PTGGE.

**Discussion.** An important question concerns the validity of the PTGGE beyond the specific example of the Kitaev chain. As we show in the Supplemental Material [53], our re-
results apply directly to symmetry-preserving deformations of the Kitaev chain, and also to a class of fermionic models with a particle-number conserving Hamiltonian, for which a natural choice of dissipation is provided by incoherent loss and gain. Furthermore, we find that for an interacting spin chain that can be mapped to fermions but with quadratic jump operators, relaxation of a subset of observables is described by the PTGGE. Finally, for a model of noninteracting bosons, we demonstrate relaxation to an ensemble that generalizes the PTGGE for fermions while maintaining the key property of conserving an extensive amount of information about the initial state. It is intriguing to speculate whether PT symmetry can affect also the dynamics of nonintegrable driven-dissipative systems in a similar way so as to induce relaxation to a PT-symmetric Gibbs ensemble.

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In the Supplemental Material, we provide technical details of our analytical and numerical studies of quench dynamics in a driven-dissipative Kitaev chain, and we present additional results for PT-symmetric relaxation in quench dynamics of fermionic and bosonic SSH models with loss and gain and a dimerized XY spin chain with dephasing. We first discuss relevant properties of the isolated Kitaev chain, in particular, its symmetries and the mapping to the transverse field Ising model employed in the main text. Then, we summarize how these properties are affected by Markovian drive and dissipation, and how quench dynamics in the driven-dissipative Kitaev chain can be described analytically and numerically in terms of the time evolution of the covariance matrix. Focusing on the PT-symmetric phase, we present two complementary derivations of the PT-symmetric generalized Gibbs ensemble (PTGGE): The first one is based on the dephased form of the covariance matrix, and the second one, which is outlined in the main text, benefits from the clear picture of quench dynamics that is provided by a description in terms of eigenmodes of the Liouvillian. We then describe how the numerical results for the subsystem parity and entropy shown in the main text are obtained, and we motivate our conjectures for the time-dependence of these quantities in the space-time-scaling limit, which are summarized in Eqs. (9), (10), and (11) in the main text. Finally, we present our results for relaxation to the PTGGE in other models.

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I. ISOLATED KITAEV CHAIN

Here, we summarize aspects of the isolated Kitaev chain that are relevant for our study of quench dynamics in a driven-dissipative Kitaev chain. This includes the diagonalization and ground state of the Kitaev chain for open and periodic boundary conditions, symmetries, and the mapping between the Kitaev chain and the transverse field Ising model.
A. Hamiltonian

The Hamiltonian of the isolated Kitaev chain, as given in the main text, reads
\[
H = \sum_{l=1}^{L} \left( -Jc_l^+c_{l+1} + \Delta c_l^+c_{l+1} + \text{H.c.} \right) - \mu \sum_{l=1}^{L} \left( c_l^+c_l - \frac{1}{2} \right). \tag{1}
\]

All of our results for quench dynamics presented in the main text are obtained for \( J = \Delta \), but we keep \( J \) and \( \Delta \) as distinct parameters in most expressions below. However, we always assume that \( J, \Delta \in \mathbb{R}_{>0} \). Further, we consider both periodic boundary conditions with \( c_{L+1} = c_1 \), and open boundary conditions with \( c_{L+1} = 0 \).

An alternative form of the Hamiltonian, which puts emphasis on translational invariance and, therefore, proves especially convenient when we consider periodic boundary conditions, is given by
\[
H = \frac{1}{2} \sum_{l,l' = 1}^{L} C_l^+ h_{l-l'} C_{l'}, \tag{2}
\]
where we collect fermionic annihilation and creation operators in spinors,
\[
C_l = \begin{pmatrix} c_l \\ c_l^+ \end{pmatrix}, \tag{3}
\]
and where
\[
h_l = h_l \cdot \sigma, \tag{4}
\]
with the vector of Pauli matrices,
\[
\sigma = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}, \tag{5}
\]
and with coefficients given by
\[
h_l = (0, i\Delta (\delta_{l,1} - \delta_{l,-1}), -J (\delta_{l,1} + \delta_{l,-1}) - \mu \delta_{l,0})^T. \tag{6}
\]

Here and in the following, for periodic boundary conditions, each index of a Kronecker delta is to be understood modulo the length of the chain \( L \), e.g., \( \delta_{l-L} = \delta_{l,L} \).

In addition to the above formulation of the Kitaev chain in terms of \( L \) complex or Dirac fermions \( c_l \) with \( l \in \{1, \ldots, L\} \), it is often convenient to work with \( 2L \) Majorana fermions \( w_l \), where \( l \in \{1, \ldots, 2L\} \), which are defined by
\[
w_{2l-1} = c_l + c_l^+, \quad w_{2l} = i(c_l - c_l^+), \tag{7}
\]
and obey the anticommutation relation \( \{w_l, w_{l'}\} = 2\delta_{l,l'} \). Further, it is often convenient to express the transformation to Majorana fermions in terms of spinors of operators as follows:
\[
C_l = \frac{1}{\sqrt{2}} r W_l, \tag{8}
\]
where \( C_l \) is defined in Eq. (3),
\[
W_l = (w_{2l-1}, w_{2l})^T, \tag{9}
\]
and the unitary matrix \( r \) is given by
\[
r = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}. \tag{10}
\]

In terms of Majorana fermions, the Hamiltonian of the isolated Kitaev chain in Eq. (1) can be written as
\[
H = \frac{i}{4} \sum_{l,l' = 1}^{L} W_l^T a_{l-l'} W_{l'}, \tag{11}
\]
where
\[
a_l = -i r^l h_l = a_l \cdot \sigma, \tag{12}
\]
with
\[
a_l = (\Delta (\delta_{l,1} - \delta_{l,-1}), i [J (\delta_{l,1} + \delta_{l,-1}) + \mu \delta_{l,0}], 0)^T. \tag{13}
\]

Here and in the following, the transformations of various quantities between the bases of complex and Majorana fermions, such as the transformation from \( h_l \) to \( a_l \), can be performed efficiently by noting that they correspond merely to permutations of the components in expansions in terms of Pauli matrices, which are here given by \( h_l \) and \( a_l \), respectively. In particular, to calculate
\[
r^l h_{l'} = h_l \cdot (r^l \sigma r), \tag{14}
\]
we use
\[
r^l \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} r = P^T \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}, \tag{15}
\]
where the permutation matrix \( P \) is given by
\[
P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{16}
\]

We thus obtain Eqs. (12) and (13):
\[
r^l h_{l'} = h_l \cdot (P^l \sigma) = (P h_l) \cdot \sigma = i a_l \cdot \sigma = i a_l. \tag{17}
\]

For future reference, we define a real antisymmetric matrix \( A \) as the block Toeplitz matrix built from the \( 2 \times 2 \) blocks \( a_l \),
\[
A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{L-1} \\ a_1 & a_0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{L-1} & \cdots & \cdots & a_0 \end{pmatrix}. \tag{18}
\]

In terms of the matrix \( A \), the Hamiltonian of the Kitaev chain can be written as
\[
H = \frac{i}{4} \sum_{l,l' = 1}^{2L} w_l A_{l-l'} w_{l'}. \tag{19}
\]
B. Covariance matrix

In our studies of quench dynamics, we always deal with Gaussian states $\rho$: The initial state $\rho_0 = |\phi_0\rangle\langle\phi_0|$, where $|\phi_0\rangle$ is the ground state of the isolated Kitaev chain, is Gaussian, and the quadratic Liouvillian we consider below preserves Gaussianity. Gaussian states are fully determined by two-point functions, which are collected in the real and antisymmetric covariance matrix,

$$\Gamma_{l,l'} = \frac{i}{2} \langle [w_l, w_{l'}] \rangle = i \langle w_l w_{l'} \rangle - \delta_{l,l'},$$  \tag{20}$$

where $\langle \ldots \rangle = \text{tr}(\ldots \rho)$. We often find it convenient to work with $2 \times 2$ blocks of $\Gamma$, which can be specified in terms of the Majorana spinors defined in Eq. (9):

$$\gamma_{l,l'} = \begin{pmatrix} \Gamma_{2l-1,2l'-1} & \Gamma_{2l-1,2l'} \\ \Gamma_{2l,2l'-1} & \Gamma_{2l,2l'} \end{pmatrix} = i \begin{pmatrix} \langle W_l W^*_l \rangle - \delta_{l,l'} \mathbb{1} \end{pmatrix}.$$  \tag{21}$$

Switching to the basis of complex fermions, we define a matrix $G$ through the relation

$$G = -i R \Gamma R^\dagger,$$  \tag{22}$$

where the matrix $R$ is given in terms of the $2 \times 2$ matrix $r$ in Eq. (10) by

$$R = \bigoplus_{l=1}^{L} r.$$  \tag{23}$$

We refer to $G$ as covariance matrix in the complex basis, or simply complex covariance matrix. As done in Eq. (21) for the covariance matrix in the Majorana basis, we define blocks of the complex covariance matrix,

$$g_{l,l'} = \begin{pmatrix} G_{2l-1,2l'-1} & G_{2l-1,2l'} \\ G_{2l,2l'-1} & G_{2l,2l'} \end{pmatrix} = -i r \gamma_{l,l'} r^\dagger.$$  \tag{24}$$

In terms of the complex spinors given in Eq. (3), the blocks $g_{l,l'}$ can be written as

$$g_{l,l'} = 2 \langle C_l C^\dagger_{l'} \rangle - \delta_{l,l'} \mathbb{1}.$$  \tag{25}$$

C. Diagonalization and ground state

We proceed to explain how the Hamiltonian of the Kitaev chain in Eq. (1) can be diagonalized and how its ground state can be found. This can be done numerically for finite systems with open or periodic boundary conditions. Analytical progress is possible for a system with periodic boundary conditions. The ground state serves as the initial state in the quench protocol discussed in the main text and further below.

1. Numerical diagonalization

The excitation spectrum and ground state of the Kitaev chain can be found numerically as follows: By employing the algorithm described in Ref. [1] (see also Ref. [2] for a discussion in the context of fermionic Gaussian systems), one obtains a real special orthogonal matrix $O$, which brings $A$ in Eq. (18) to the following canonical block-diagonal form:

$$O A O^\dagger = \bigoplus_{l=1}^{L} \begin{pmatrix} 0 & -e_l \\ e_l & 0 \end{pmatrix},$$  \tag{26}$$

where $e_l \geq 0$ are the energies of elementary excitations. Further, the covariance matrix of the ground state is given by [3]

$$\Gamma_0 = O^\dagger \left( \bigoplus_{l=1}^{L} i \sigma_3 \right) O.$$  \tag{27}$$

Below and in the main text, we focus on quenches from the ground state of the Kitaev chain for $\mu_0 \to -\infty$, i.e., the vacuum state $|\Omega\rangle$, for which $c_l |\Omega\rangle = 0$ for all $l \in \{1, \ldots, L\}$. Then, $O$ is simply the $2L \times 2L$ identity matrix.

2. Analytical diagonalization for periodic boundary conditions

For periodic boundary conditions, analytical progress can be made most conveniently in the complex basis. In particular, we express the fermionic operators $c_l$ in terms of momentum-space operators $\hat{c}_k$ as

$$c_l = \frac{1}{\sqrt{L}} \sum_{k \in \text{BZ}} e^{ikl} \hat{c}_k, \quad \hat{c}_k = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} e^{-ikl} c_l,$$  \tag{28}$$

where the Brillouin zone BZ contains $L$ momenta with spacing $\Delta k = 2\pi/L$, that is, $\text{BZ} = \{ \pi \pm \Delta k, -\pi + \Delta k, \ldots, \pi \}$. The spinors in Eq. (3) transform to

$$C_k = \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} e^{-ikl} C_l,$$  \tag{29}$$

and the Hamiltonian of the Kitaev chain can be written as

$$H = \sum_{k \geq 0} C_k^\dagger \hbar_k C_k,$$  \tag{30}$$

with the Bogoliubov-De Gennes (BdG) Hamiltonian

$$\hbar_k = \sum_{l=1}^{L} e^{-ikl} h_l = \hbar \cdot \sigma,$$  \tag{31}$$

where

$$\hbar = (0, 2\Delta \sin(k), -2J \cos(k) - \mu)^\top.$$  \tag{32}$$

Below, we find it convenient to parameterize the vector $\hbar_k$ in terms of its magnitude,

$$\epsilon_k = |\hbar_k| = \sqrt{(2J \cos(k) + \mu)^2 + 4\Delta^2 \sin(k)^2},$$  \tag{33}$$

and a unit vector

$$\hat{\hbar}_k \equiv \frac{\hbar_k}{\epsilon_k} = (0, \sin(\theta_k), \cos(\theta_k))^\top,$$  \tag{34}$$

$$\theta_k = \arctan\left(\frac{2\Delta \sin(k)}{2J \cos(k) + \mu}\right).$$  \tag{35}$$

$$\epsilon_k = \sqrt{\epsilon_k^2 - \hbar^2},$$  \tag{36}$$

and a unit vector

$$\hat{\hbar}_k = \frac{\hbar_k}{\epsilon_k} = (0, \sin(\theta_k), \cos(\theta_k))^\top,$$  \tag{37}$$

Below, we find it convenient to parameterize the vector $\hbar_k$ in terms of its magnitude,
where the angle $\theta_k$ is defined implicitly by the relation
\[ \varepsilon_k e^{i\theta_k} = -2J \cos(k) - \mu + i2\Delta \sin(k). \]  (35)

The diagonalization of the BdG Hamiltonian corresponds geometrically to a rotation of $h_k$ to the $z$ axis, which can be accomplished by a rotation around the $x$ axis by the angle $\theta_k$:

In particular, with
\[ U_{x,\theta_k} = e^{i\theta_k x}\sigma_x = \begin{pmatrix} \cos(\theta_k/2) & i \sin(\theta_k/2) \\ i \sin(\theta_k/2) & \cos(\theta_k/2) \end{pmatrix}, \]  (36)
we obtain
\[ U^\dagger_{x,\theta_k} h_k U_{x,\theta_k} = e_k \sigma_z. \]  (37)

The corresponding Bogoliubov transformation of fermionic operators reads
\[ D_k = \begin{pmatrix} d_k \\ d_k^\dagger \end{pmatrix} = U^\dagger_{x,\theta_k} C_k. \]  (38)

In terms of the thusly defined fermionic modes $d_k$, the Hamiltonian becomes
\[ H = \sum_{k \geq 0} \varepsilon_k (d_k^\dagger d_k - d_k d_k^\dagger), \]  (39)
which shows that $\varepsilon_k$ determines the dispersion relation of single-particle excitations, and that the ground state $|\psi_0\rangle$ is given by the vacuum state of the operators $d_k$, i.e., $d_k |\psi_0\rangle = 0$ for all $k \in \text{BZ}$. To obtain an explicit expression for the ground state, we have to treat momenta $k \in [0, \pi]$ and $k \in \text{BZ} \setminus [0, \pi]$ separately. For momenta $k \in [0, \pi)$, Eq. (35) implies
\[ \theta_0 = \begin{cases} 0 & \text{for } \mu < -2J, \\ \pi & \text{for } \mu > -2J, \end{cases} \]  (40)
and
\[ \theta_\pi = \begin{cases} 0 & \text{for } \mu < 2J, \\ \pi & \text{for } \mu > 2J. \end{cases} \]  (41)

By inserting these results in the Bogoliubov transformation Eq. (38), we obtain
\[ d_0 = \begin{cases} c_0 & \text{for } \mu < -2J, \\ -ic_0^\dagger & \text{for } \mu > -2J, \end{cases} \]  (42)
and, with $c_\pi = c_{-\pi}$,
\[ d_\pi = \begin{cases} c_\pi & \text{for } \mu < 2J, \\ -ic_\pi^\dagger & \text{for } \mu > 2J. \end{cases} \]  (43)

Therefore, the conditions $d_0 (|\psi_0\rangle = d_\pi |\psi_0\rangle = 0$ are satisfied, if, in the state $|\psi_0\rangle$, the mode with momentum $k = 0$ is occupied when $\mu > -2J$, and the mode with momentum $k = \pi$ is occupied when $\mu > 2J$.

For momenta $k \in \text{BZ} \setminus [0, \pi]$, we employ the following alternative form of the Bogoliubov transformation:
\[ d_k = u_k c_k u_k^\dagger, \]  (44)
where
\[ u_k = e^{i\theta_k/2}(c_{-\gamma_k} - c_{\gamma_k}), \]
\[ = e^{-i\pi/2}c_{-\gamma_k} e^{i\pi/2}c_{\gamma_k} = e^{-i\pi/2}(c_{\gamma_k} + c_{-\gamma_k}) - 1. \]  (45)

From $c_\pi (|\psi_0\rangle = 0$, i.e., $u_k (|\psi_0\rangle$ is the vacuum of the Bogoliubov modes $d_k$. Further, since $u_k$ is unitary, the transformed vacuum $u_k (|\psi_0\rangle$ is normalized.

When we study quench dynamics below, we restrict ourselves to initial states with $\mu < -2J$ such that, according to the above discussion, the modes with $k \in [0, \pi]$ are not occupied. Then, the ground state is given by
\[ |\psi_0\rangle = \prod_{k \in \text{BZ}, [0, \pi]} u_k (|\Omega\rangle \]
\[ = \prod_{k \in \text{BZ}, [0, \pi]} \cos(\theta_k/2) + i \sin(\theta_k/2) c_k^\dagger c_{-k}) |\Omega\rangle. \]  (46)

Finally, we specify the covariance matrix in the ground state. To that end, we first note that due to translational invariance in a system with periodic boundary conditions, the $2 \times 2$ block in Eq. (24) depends only on the difference of its indices, $g_{l,r} = g_{l-r}$. Then, its Fourier transform is given by
\[ g_k = \sum_{l=1}^{L} e^{-ikl} g_l = 2 \langle C_k C_k^\dagger \rangle - 1 = \begin{pmatrix} \langle |c_k, c_k^\dagger\rangle & \langle |c_k, c_{-k}\rangle \\ \langle |c_{-k}, c_k\rangle & \langle |c_{-k}, c_{-k}\rangle \end{pmatrix}. \]  (47)

To obtain an explicit expression for the complex covariance matrix $g_k$ in the ground state, we use the following ground-state expectation values, which follow from the condition $d_k (|\psi_0\rangle = 0$ that defines the ground state,
\[ \langle D_k D_k^\dagger \rangle_0 = \begin{pmatrix} \langle d_k d_k^\dagger \rangle_0 & \langle d_k d_{-k}^\dagger \rangle_0 \\ \langle d_{-k} d_k^\dagger \rangle_0 & \langle d_{-k} d_{-k}^\dagger \rangle_0 \end{pmatrix} = \frac{1 + \sigma_z}{2}, \]  (48)
where $\langle \ldots \rangle_0 = \langle \psi_0 | \ldots | \psi_0 \rangle$. Further, we use Eq. (37), according to which $h_k / \varepsilon_k = \hat{n}_k \cdot \sigma_z = U_{x,\theta_k} \sigma_z U^\dagger_{x,\theta_k}$, and Eq. (38). We thus find
\[ g_{0,k} = 2U_{x,\theta_k} \langle D_k D_k^\dagger \rangle_0 U_{x,\theta_k}^\dagger - 1 = U_{x,\theta_k} \sigma_z U_{x,\theta_k}^\dagger = \hat{n}_k \cdot \sigma_z. \]  (49)

When we rotate this expression to the Majorana basis, we find it convenient to include a prefactor $-i$ relative to Eq. (24), i.e., we define the momentum-space covariance matrix in the Majorana basis as
\[ \gamma_k = r^\dagger g_k r = -i \sum_{l=1}^{L} e^{-ikl} \gamma_l, \quad \gamma_l = i \sum_{k \in \text{BZ}} e^{ikl} \gamma_k. \]  (50)

With this convention, $\gamma_k = \gamma_k^\dagger$ is Hermitian. We note that in terms of spinors of Majorana operators, which we define in analogy to Eq. (29),
\[ W_k = \frac{1}{\sqrt{L}} \sum_{l=1}^{L} e^{-ikl} W_l = \sqrt{2} r^\dagger C_k, \]  (51)
the covariance matrix can be written as
\[ \gamma_k = \langle W_k W_k^\dagger \rangle - I. \] (52)
We obtain the covariance matrix in the ground state from Eq. (49) by using Eqs. (15) and (16),
\[ \gamma_{0,k} = \hat{n}_k \cdot \left( r^\dagger \sigma r \right) = \hat{n}_k \cdot \left( P \hat{n}_k \right) \cdot \sigma = \hat{a}_k \cdot \sigma, \] (53)
where \( \hat{a}_k = a_k/\epsilon_k \) and
\[ a_k = P \hat{b}_k = (2\Delta \sin(k) - 2J \cos(k) - \mu, 0)^\dagger. \] (54)
For future reference, we define the Fourier transform of \( a_l \) in Eq. (12) including the imaginary unity as a prefactor:
\[ a_k = r^\dagger h_k r = i \sum_{l=1}^{L} e^{-ikl} a_l = a_k \cdot \sigma. \] (55)
Again, this convention leads to a Hermitian matrix \( a_k = a_k^\dagger \).

D. Symmetries of the isolated Kitaev chain

We next discuss symmetries of the Kitaev chain, focusing on the BdG Hamiltonian \( h_k \) in Eq. (31). To prepare for a comparison to the driven-dissipative Kitaev chain introduced below, we state two equivalent forms of each symmetry, where the second form follows from Hermiticity of the BdG Hamiltonian, \( h_k = h_k^\dagger \). Particle-hole symmetry (PHS), time-reversal symmetry (TRS), and chiral symmetry (CS), which results from the combination of PHS and TRS, are expressed by

PHS: \[ h_k = -\sigma_y h_k^\dagger \sigma_x = -\sigma_y h_k^\dagger \sigma_x, \] (56)

TRS: \[ h_k = h_k^\dagger = h_k^\dagger, \] (57)

CS: \[ h_k = -\sigma_x h_k \sigma_z = -\sigma_x h_k \sigma_z. \] (58)

We note that TRS is ensured by our choice \( J \in \mathbb{R}_{\geq 0} \) (a non-trivial phase of \( \Delta \) can always be absorbed in a redefinition of the operators \( c_l \)). Then, the time-reversal invariant Kitaev chain belongs to the Altland-Zirnbauer class BDI [4], which, in one spatial dimension, is characterized by an integer-valued winding number [5–7]. The ground state of the Kitaev chain is topologically trivial with a vanishing winding number for \( |\mu| > 2J \), and topologically nontrivial, as indicated by a nonvanishing value of the winding number, for \( |\mu| < 2J \).

In what follows, an important role is played by inversion symmetry (IS) and its breaking due to Markovian drive and dissipation. For the isolated Kitaev chain, inversion is represented by a unitary operator \( I \) that is defined as follows [8]:

open boundary conditions: \[ Ic_l I^\dagger = i c_{L+1-l}, \] (59)
periodic boundary conditions: \[ Ic_l I^\dagger = i c_{-l}. \]

The Hamiltonian in Eq. (1) is symmetric under inversion, \( H = IH^\dagger \). In the transformation of the Hamiltonian, the factor \( i \) on the right-hand side of Eq. (59) cancels in the hopping and chemical potential terms, but is required to leave the pairing term invariant. For periodic boundary conditions, the spinors \( C_k \) in Eq. (29) transform as
\[ IC_k I^\dagger = i c_k C_{-k}. \] (60)

Applying this transformation to the momentum-space representation of the Hamiltonian in Eq. (30) yields, for the BdG Hamiltonian Eq. (31), the following conditions for the correlation and covariance matrices:

PHS: \[ \gamma_k = -\sigma_y \gamma_{-k} \sigma_x = -\sigma_y \gamma_{-k} \sigma_x, \] (52)

PTS: \[ \gamma_k = -\sigma_y h_k^\dagger \sigma_z = -\sigma_y h_k^\dagger \sigma_z. \] (62)

For future reference, we list symmetry properties that can be derived from the symmetries of the BdG Hamiltonian. First, for \( a_k \) defined in Eq. (55), we find

PHS: \[ a_k = -a_{-k}^\dagger = a_{-k}^\dagger, \] (63)

TRS: \[ a_k = \sigma_y a_{-k} \sigma_z = \sigma_y a_{-k} \sigma_z, \] (64)

CS: \[ a_k = -\sigma_y a_k \sigma_z = -\sigma_y a_k \sigma_z, \] (65)

IS: \[ a_k = \sigma_y a_k \sigma_z = \sigma_y a_k \sigma_z, \] (66)

PTS: \[ a_k = \sigma_y a_k^\dagger \sigma_z = \sigma_y a_k^\dagger \sigma_z. \] (67)

Further, PHS yields restrictions on the transformation \( U_{\lambda,k} \) in Eq. (36) that diagonalizes the BdG Hamiltonian. By combining Eqs. (37) and (56) we obtain
\[ \sigma_x U_{\lambda,k}^\dagger \sigma_x h_{\lambda,k} \sigma_x U_{\lambda,k}^\dagger \sigma_x = \epsilon_k \sigma_z. \] (68)

That is, we can identify \( e_k = \epsilon_{-k} \) and
\[ U_{\lambda,k} = \sigma_y U_{\lambda,k}^\dagger \sigma_z, \] (69)

which implies \( \theta_k = -\theta_{-k} \). Alternatively, \( \theta_k \) being an odd function of \( k \) can also be seen as a consequence of IS Eq. (61), which implies
\[ U_{\lambda,k} = \sigma_y U_{\lambda,k}^\dagger \sigma_z. \] (70)

Moreover, in analogy to the PHS of the BdG Hamiltonian \( h_k \) and its counterpart \( a_k \) in the Majorana basis, one can check that also the covariance matrices \( g_k \) and \( \gamma_k \), defined in Eqs. (47) and (50), respectively, obey PHS,
\[ g_k = -\sigma_y \gamma_{-k} \sigma_x, \] (71)

\[ \gamma_k = -\gamma_{-k}^\dagger. \] (72)

In particular, when we expand the covariance matrix in the basis of Pauli matrices as
\[ \gamma_k = \gamma_{1,k} \sigma_z + \gamma_{2,k} \cdot \sigma, \] (73)

PHS implies
\[ \gamma_{1,k} = -\gamma_{1,-k}, \] (74)

and
\[ \gamma_{2,k} = -\gamma_{2,-k}, \] (75)
E. Relation to the transverse field Ising model

Through the Jordan-Wigner transformation, the isolated Kitaev chain can be mapped to the transverse field Ising model, which serves as a paradigmatic model system in studies of quench dynamics and generalized thermalization [9–11]. To establish the connection between our work and the existing literature on quenches in the transverse field Ising model, we provide relevant details of this mapping in the following. In Sec. III C, we exploit this connection to formulate conjectures for the time dependence of the subsystem parity.

Usually, the Jordan-Wigner transformation is applied directly to the Hamiltonian of the transverse field Ising model, which is given by

\[ H_{\text{Ising}} = -J \sum_{i=1}^{L} (\sigma_i^x \sigma_{i+1}^x + h \sigma_i^z). \]  

Spin operators can be expressed in terms of fermionic operators through the Jordan-Wigner transformation [12]:

\[
\sigma_i^+ = e^{i \pi \sum_{j=i+1}^{L} c_j^\dagger c_j^\prime} c_i,
\sigma_i^- = e^{i \pi \sum_{j=i+1}^{L} c_j c_j^\prime} c_i^\dagger,
\sigma_i^z = e^{i \pi c_i c_i^\dagger}.
\]

By means of the Jordan-Wigner transformation, the Hamiltonian of the transverse field Ising model is mapped to two copies of the Kitaev chain Eq. (1) in decoupled sectors with even and odd fermion parity, and \( J = \Delta \) and \( \mu = -2 J h \) [10]. In particular, when we restrict ourselves to positive values of \( h \) (and, therefore, negative values of \( \mu \)), the ferromagnetically ordered phase of the Ising model with \( 0 < h < 1 \) maps to the topological phase of the Kitaev chain with \( \mu < -2 J \), and the paramagnetic phase with \( h > 1 \) maps to the trivial phase with \( -2 J < \mu < 0 \). Long-range order in the ground state in the ferromagnetic phase is revealed as usual by adding a longitudinal field \( J h \), \( \sum_{j=1}^{L} \sigma_j^z \) to the Hamiltonian Eq. (76) and considering the following combination of limits:

\[ \lim_{|t| \to \infty} \lim_{h \to 0} \lim_{L \to \infty} \langle \psi_0 | O_{\ell, t'} | \psi_0 \rangle \neq 0, \]  

where, for \( \ell' > \ell \),

\[ O_{\ell, t'} = \sigma_{\ell}^x \sigma_{\ell'}^x = (c_\ell + c_\ell^\dagger) e^{i \pi \sum_{j=\ell+1}^{L} c_j^\dagger c_j} (c_\ell^\dagger + c_\ell^\prime). \]

That is, longitudinal or order parameter correlations map to the string order parameter \( O_{\ell, t} \) [13] for the Kitaev chain.

We find it convenient to deviate from the route outlined above and employ the Kramers-Wannier duality [14, 15] of the transverse field Ising model before applying the Jordan-Wigner transformation. The Kramers-Wannier self-duality of the Ising model is established by defining new spin operators through the relations

\[ \tau_i^x = \sigma_i^z \sigma_{i+1}^x, \quad \tau_i^z \tau_{i+1}^x = \sigma_i^x. \]

Then, the Hamiltonian maps onto itself with the global energy scale transforming as \( J \to J h \), and the transverse field as \( h \to 1/h \), i.e., the ferromagnetic phase of \( \sigma \) spins maps to the paramagnetic phase of \( \tau \) spins. Consequently, the ferromagnetic and paramagnetic phases of the Ising model for \( \tau \) spins map to the trivial and topological phases of the Kitaev chain, respectively. In particular, the topologically trivial vacuum state, which we choose as the initial state for our studies of quench dynamics, corresponds to the ground state of the Ising model in the strongly ferromagnetic limit. Longitudinal correlations of \( \tau \) spins,

\[ P_{\tau, \ell} = \tau_{\ell}^x \tau_{\ell+1}^x = \prod_{m=1}^{\ell} \tau_{m+1}^x \tau_{m}^x = \prod_{m=1}^{\ell} \sigma_{m}^z, \]  

serve as a “disorder parameter” for \( \sigma \) spins, and the expectation value of \( P_{\tau, \ell} \) is maximized in the paramagnetic phase of \( \sigma \) spins. Through the Jordan-Wigner transformation Eq. (77), \( P_{\tau, \ell} \) maps to the fermion parity of a subsystem that consists of the lattice sites \( \{1, \ldots, \ell\} \),

\[ P_{\ell, \ell'} = e^{i \pi \sum_{m=1}^{\ell} c_m^\dagger c_m} = \prod_{m=1}^{\ell} (i w_{2m-1} w_{2m}). \]

When the expectation value of \( P_{\ell, \ell'} \) is taken in a translationally invariant state, the result depends only on the difference \( \ell' - \ell \). Then, without loss of generality, we set \( \ell = 1 \) and \( \ell' = \ell \), and define the subsystem parity as

\[ P_{\ell} = e^{i \pi \sum_{i=1}^{\ell} c_i^\dagger c_i} = \prod_{i=1}^{\ell} (i w_{2i-1} w_{2i}). \]

As stated in the main text, the expectation value of the subsystem parity is given by the Pfaffian of the reduced covariance matrix [16, 17],

\[ \langle P_{\ell} \rangle = \text{Pf}(Gamma_{\ell}), \]  

where

\[ Gamma_{\ell} = (Gamma_{\ell'})^{2}_{\ell'=-1}. \]

According to the above discussion, the ground-state expectation value \( \langle P_{\ell} \rangle_{0} = \langle \psi_0 | P_{\ell} | \psi_0 \rangle \) can be regarded as a “topological disorder parameter.” The ground-state expectation value \( \langle P_{\ell} \rangle_{0} \) vanishes in the topological phase of the Kitaev chain and is nonzero in the trivial phase. Accordingly, the results of Refs. [9–11] for the time evolution of order parameter correlations for quenches from the ordered to the ordered and disordered phases of the transverse field Ising model, when interpreted as applying to \( \tau \) spins, describe the behavior of \( \langle P_{\ell} \rangle \) for quenches from the trivial to the trivial and topological phases of the Kitaev chain, respectively.

II. DRIVEN-DISSIPATIVE KITAEV CHAIN

We move on to consider quantum quenches in the driven-dissipative Kitaev chain. As specified in the main text, we assume that the system is prepared in the ground state \( |\psi_0 \rangle \) of the Hamiltonian of the isolated Kitaev chain in Eq. (1) with parameters \( J = \Delta \) and \( \mu_0 \). At \( t = 0 \), the chemical potential is quenched to the value \( \mu \), and, concurrently, the system is coupled to Markovian reservoirs.
A. Master equation and Liouvillian

The time evolution of the driven-dissipative Kitaev chain after the quench is described by a quantum master equation in Lindblad form [18, 19],

\[ i \frac{d}{dt} \rho = \mathcal{L} \rho, \quad (86) \]

where we define the Liouvillian \( \mathcal{L} \) as the sum of the Hamiltonian contribution \( \mathcal{H} \) and the dissipator \( \mathcal{D} \),

\[ \mathcal{L} = \mathcal{H} + i \mathcal{D}, \quad (87) \]

with

\[ \mathcal{H} \rho = [H, \rho], \quad (88) \]

and

\[ \mathcal{D} \rho = \sum_{l=1}^{L} (2L_{d}L_{l}^{\dagger} - (L_{l}^{\dagger}L_{l}, \rho)). \quad (89) \]

We consider local jump operators as given in the main text,

\[ L_{l} = \sqrt{\gamma_{l}} c_{l} + \sqrt{\gamma_{l}} c_{l}^{\dagger}. \quad (90) \]

In terms of Majorana operators \( w_{l} \), the jump operators can be represented as

\[ L_{l} = \sum_{j=1}^{2L} B_{l,j} w_{j}, \quad (91) \]

where the coefficients \( B_{l,j} \) form an \( L \times 2L \) matrix. For future reference, we define the bath matrix \( M \) as the product

\[ M = B^{\dagger} B'. \quad (92) \]

According to its definition, the bath matrix is Hermitian; Hermiticity implies the following symmetry conditions for the real and imaginary parts of \( M \), which we denote by \( M_{R} \) and \( M_{I} \), respectively:

\[ M_{R} = M_{R}^{\dagger}, \quad M_{I} = -M_{I}^{\dagger}. \quad (93) \]

In turn, these relations imply

\[ M + M^{\dagger} = 2M_{R}, \quad M - M^{\dagger} = i2M_{I}. \quad (94) \]

Like the matrix \( A \) in Eq. (18), the bath matrix \( M \) is a block Toeplitz matrix, with \( 2 \times 2 \) blocks \( m_{l} \) given by

\[ m_{l} = m_{R,l} + im_{I,l}. \quad (95) \]

The real and imaginary parts, \( m_{R,l} \) and \( m_{I,l} \), respectively, read

\[ m_{R,l} = \frac{\gamma_{l}}{2} \left[ \gamma I + \sqrt{\gamma_{l}} \gamma_{l}^{\dagger} \right], \quad m_{I,l} = \frac{\gamma_{l}}{4} i \delta \gamma_{l}. \quad (96) \]

Here, as in the main text, we introduce the mean rate \( \gamma \), which represents a measure of the overall strength of dissipation, and the difference of loss and gain rates \( \delta \),

\[ \gamma = \frac{\gamma_{l} + \gamma_{l}'}{2}, \quad \delta = \gamma_{l} - \gamma_{l}'. \quad (97) \]

B. Time evolution of the covariance matrix

As stated in Sec. 1B, for the quench protocol specified in the main text, the system is in a Gaussian state at all times, and Gaussian states are fully determined by the covariance matrix. Therefore, in the following, we specify the equation of motion for the covariance matrix and its numerical and analytical solution.

1. Numerical time evolution

The equation of motion for the covariance matrix reads

\[ \frac{d\Gamma}{dt} = -X\Gamma - \Gamma X^{T} - Y, \quad (98) \]

where

\[ X = 4(iH + M_{R}) = -A + 4M_{R}, \quad Y = -8M_{I}, \quad (99) \]

and its formal solution is given by [20]

\[ \Gamma(t) = \Gamma_{1}(t) + \Gamma_{2}(t), \quad (100) \]

with

\[ \Gamma_{1}(t) = e^{-Xt}\Gamma(0)e^{-X'^{T}t}, \]

\[ \Gamma_{2}(t) = -\int_{0}^{t} e^{-X(t-t')}Ye^{-X'^{T}(t-t')}dt'. \quad (101) \]

The initial conditions and the steady state are encoded, respectively, in \( \Gamma_{1}(t) \) and \( \Gamma_{2}(t) \). In particular, \( \Gamma_{1}(0) = \Gamma(0) \) while \( \Gamma_{2}(0) = 0 \), and \( \Gamma_{1}(t) \to 0 \) while \( \Gamma_{2}(t) \to \Gamma_{SS} \) for \( t \to \infty \). For all quenches considered in the main text, the initial state is chosen as the ground state of the Kitaev chain for \( \mu_{0} \to -\infty \), i.e., the vacuum state, and the corresponding covariance matrix \( \Gamma(0) = \Gamma_{0} \) is given in Eq. (27). The integral in the expression for \( \Gamma_{2}(t) \) can be performed explicitly if we express the matrix \( X \) as \( X = V^{\dagger}AV^{-1} \), where \( A \) is a diagonal matrix with the eigenvalues \( \lambda_{l} \) of \( X \) on its diagonal. Then,

\[ \Gamma_{2}(t) = -V \left( [V^{-1}YV^{-T}] \circ K(t) \right) V^{T}, \quad (102) \]

where we use the shorthand notation \( V^{-T} \) to denote the inverse of the transpose, and where \( \circ \) is the Hadamard product (i.e., element-wise) product of \( A \) and \( B \).

\[ (A \circ B)_{l,l'} = A_{l,l'} B_{l,l'}. \quad (103) \]

Further, the matrix \( K(t) \) is defined as

\[ K_{l,l'}(t) = \frac{1 - e^{-\left( \lambda_{l} + \lambda_{l}' \right)t}}{\lambda_{l} + \lambda_{l}'}, \quad (104) \]

Numerically, we find that \( \lambda_{l} + \lambda_{l'} \neq 0 \) for all \( l, l' \) when \( \gamma > 0 \). To obtain the numerical data shown in the plots in the main text, we implement Eqs. (101) and (102).
2. Analytical time evolution for periodic boundary conditions

For periodic boundary conditions, due to translational invariance of the Hamiltonian, the coupling to Markovian baths, and the initial state, all matrices in Eq. (98) are block-circulant Toeplitz matrices. Therefore, Eq. (98) can be rewritten as

\[
\frac{dy_l}{dt} = -\sum_{p=1}^{L} \left( x_{-p} y_p + y_{-p} x_p^T \right) - y_l, \quad \text{(105)}
\]

where the $2 \times 2$ blocks $y_l$ are defined in Eq. (21), and $x_l$ and $y_l$ are defined analogously in terms of the matrices $X$ and $Y$ in Eq. (99), i.e.,

\[
x_l = -a_l + 4m_l, \quad y_l = -8m_l, \quad \text{(106)}
\]

with $a_l$ and the real and imaginary parts of the blocks of the bath matrix given in Eqs. (12) and (96), respectively. In momentum space, the equation of motion for the Fourier transform $\gamma_k$ in Eq. (50) takes the form

\[
\frac{d\gamma_k}{dt} = -i \left( x_k \gamma_k - \gamma_k x_k^T \right) - \gamma_k, \quad \text{(107)}
\]

where we define the Fourier transforms of $x_l$ and $y_l$ with a prefactor $-i$ as

\[
x_k = -i \sum_{l=1}^{L} e^{-ikl} x_l = x_k \mathbb{1} + x_k \cdot \sigma, \quad \text{(108)}
\]

\[
y_k = -i \sum_{l=1}^{L} e^{-ikl} y_l = y \cdot \sigma. \quad \text{(109)}
\]

The coefficients in the expansions of $x_k$ and $y_k$ in terms of Pauli matrices read

\[
x_1 = -iz \gamma, \quad x_k = a_k + i b, \quad b = -2 \sqrt{\gamma} \gamma_0 \epsilon_z, \quad \text{(109)}
\]

where $a_k$ is given in Eq. (54), and

\[
y = -2 \delta \epsilon_z. \quad \text{(110)}
\]

In the above expressions, we denote unit vectors in the $x$, $y$, and $z$ direction by $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$. The solution to Eq. (107) is given by

\[
\gamma_k(t) = \gamma_{1,k}(t) + \gamma_{2,k}(t), \quad \text{(111)}
\]

where

\[
\gamma_{1,k}(t) = e^{-i\omega t} \gamma_k(0) e^{i\omega t},
\]

\[
\gamma_{2,k}(t) = -\int_0^t dt' e^{-i\omega(t-t')} \gamma_k(t') e^{i\omega(t-t')}, \quad \text{(112)}
\]

and the initial conditions are determined by Eq. (53), $\gamma_k(0) = a_{0,k} \cdot \sigma$. In particular, for $\mu_0 \to -\infty$, we find $\gamma_k(0) = \sigma_y$.

3. Spectrum of the covariance matrix

For future reference, we briefly discuss the spectrum of the time-evolved covariance matrix. To that end, it is convenient to define the coefficients $\gamma_{1,k}(t)$ and $\gamma_{2,k}(t)$ in an expansion in terms of Pauli matrices through the relation

\[
\gamma_k(t) = \gamma_{1,k}(t) \mathbb{1} + \gamma_{2,k}(t) \cdot \sigma, \quad \text{(113)}
\]

The eigenvalues of $\gamma_k \cdot \sigma$ are $\pm \epsilon_k$, where we omit the explicit time dependence to lighten the notation. Therefore, the spectrum $\sigma(\gamma_k)$ of the covariance matrix in Eq. (113) is given by

\[
\sigma(\gamma_k) = \{ \pm \epsilon_k \}. \quad \text{(114)}
\]

We note that for an isolated system with unitary dynamics, time evolution corresponds to precession of the vector $\gamma_k(t)$ around $a_k$, with the explicit form of $\gamma_k(t)$ given in Eq. (164) below. Then, $\gamma_{1,k} = 0$, $\epsilon_k = |a_{0,k}| = 1$, and the spectrum is symmetric with respect to zero. This is, in general, not the case for an open system. However, there are restrictions on the spectrum due to PHS, which imply that the union of $\sigma(\gamma_k)$ and $\sigma(\gamma_{-k})$ is again symmetric with respect to zero. This can be seen as follows: Equation (74) implies that $\epsilon_k = \epsilon_{-k}$. Then, with Eq. (73) we obtain

\[
\sigma(\gamma_{-k}) = \{ -\gamma_{1,k} \pm \epsilon_k \}. \quad \text{(116)}
\]

Therefore, when we define

\[
\epsilon_{+k} = \gamma_{1,k} + \epsilon_k, \quad \epsilon_{-k} = -\gamma_{1,k} + \epsilon_k, \quad \text{(117)}
\]

we can write

\[
\sigma(\gamma_k) \cup \sigma(\gamma_{-k}) = \{ \pm \epsilon_{+k}, \pm \epsilon_{-k} \}. \quad \text{(118)}
\]

C. Time evolution of the complex covariance matrix

The equation of motion for the covariance matrix in the basis of complex fermions, which is given in the main text, can be obtained by transforming Eq. (107) according to Eq. (50),

\[
\frac{dg_k}{dt} = -i \left( z_k g_k - g_k z_k^T \right) - s_k, \quad \text{(119)}
\]

where

\[
z_k = r x_k r^\dagger = z_1 \mathbb{1} + z_k \cdot \sigma, \quad s_k = r y_k r^\dagger = s \cdot \sigma. \quad \text{(120)}
\]

By using Eqs. (15) and (16), we obtain

\[
z_1 = -iz \gamma, \quad z_k = b_k - iz \gamma \gamma_0 \epsilon_z, \quad s = -2 \delta \epsilon_z. \quad \text{(121)}
\]

Rather than solving Eq. (119) explicitly, we rotate Eq. (113) to the complex basis to obtain the time-evolved covariance matrix in the form

\[
g_k(t) = r \gamma_k(t) r^\dagger = g_{1,k}(t) \mathbb{1} + g_k(t) \cdot \sigma, \quad \text{(122)}
\]

where

\[
g_{1,k}(t) = \gamma_{1,k}(t), \quad g_k(t) = P^\dagger \gamma_k(t). \quad \text{(123)}
\]
D. Symmetries of the Liouvillian

Before we go into details of quench dynamics which are encoded in the time dependence of the covariance matrix, we discuss relevant symmetries of the Liouvillian $\mathcal{L}$ that generates the dynamics. According to Eqs. (109) and (121), the matrices $x_k$ and $z_k$ can be regarded as generalizations of the matrices $a_0$ and the BdG Hamiltonian $h_k$, respectively, to open systems. Therefore, by considering $x_k$ and $z_k$, we can check directly how the symmetries of the isolated Kitaev chain, which are summarized in Sec. 1D, are affected by dissipation. We find that of the two versions of, e.g., PHS in Eq. (56) for $\gamma = 0$, only one remains valid when $\gamma > 0$. Further, IS and, therefore, PTS, apply only to the traceless parts of $x_k$ and $z_k$, given by $x'_k = x_k + i2y\mathbb{1}$ and $z'_k = z_k + i2y\mathbb{1}$, respectively. This is known as passive PTS [21–23]. In particular, using the terminology of Ref. [24], the symmetries of $x_k$ are given by

\begin{align}
\text{PHS}^1: & \quad x_k = -x_k^*, \\
\text{TRS}^1: & \quad x_k = \sigma_z x_k^* \sigma_z, \\
\text{CS}: & \quad x_k = -\sigma_x x_k^* \sigma_x, \\
\text{IS}^1: & \quad x'_k = \sigma_x x'_k \sigma_y, \\
\text{PTS}: & \quad x'_k = \sigma_x x'_k \sigma_z,
\end{align}

where PTS is the combination of TRS$^1$ and IS$^1$. For the sake of completeness, we also list the symmetries of $z_k$, which result from rotating the above relations to the complex basis:

\begin{align}
\text{PHS}^1: & \quad z_k = -\sigma_z z_k^* \sigma_z, \\
\text{TRS}^1: & \quad z_k = z_k^T, \\
\text{CS}: & \quad z_k = -\sigma_x z_k^* \sigma_x, \\
\text{IS}^1: & \quad z'_k = \sigma_y z'_k \sigma_y, \\
\text{PTS}: & \quad z'_k = \sigma_x z'_k \sigma_z.
\end{align}

We next discuss implications of the above symmetry conditions that are relevant to our studies of quench dynamics. First, we show that IS$^1$ is not sufficient to ensure that the time-evolved covariance matrix obeys IS as specified in Eq. (75). To that end, we apply inversion to the time-evolved covariance matrix Eq. (112), whereby we take into account that the initial state in Eq. (53), as well as $\chi_0$ defined in Eq. (108), obey inversion symmetry in the sense of Eq. (75),

\[
\sigma_y \gamma_{1,-k}(t) \sigma_y = e^{-2iz_i^* t} e^{-i\mathbf{x}_i \cdot \sigma_i t} \gamma_k(0) e^{i\mathbf{x}_i \cdot \sigma_i t},
\]

\[
\sigma_y \gamma_{2,-k}(t) \sigma_y = -\int_0^t dt' e^{-2iz_{i'} t} e^{-i\mathbf{x}_{i'} \cdot \sigma_{i'} t} \gamma_k(0) e^{i\mathbf{x}_{i'} \cdot \sigma_{i'} t}.
\]

By comparing these expressions to Eq. (112), we see that inversion amounts to reversing the sign of the vector $\mathbf{b}$ in Eq. (109), which, in general, does not leave the state invariant. For quenches to the PT-symmetric phase, this can be checked explicitly in Eqs. (157), (158), (161), and (162) below.

Further, as stated in the main text, and as we explain in detail in the following, PTS has important consequences for the spectrum of $z'_k$ [25]. Since $z'_k = z_k + i2y\mathbb{1} = z_k \cdot \sigma_i$, the eigenvalues of $z'_k$ are given by $\lambda'_{\pm k} = \pm \sqrt{z_k \cdot z_k}$. We denote the corresponding eigenvectors, which are also eigenvectors of $z_k$ with eigenvalues $\lambda_{\pm k} = \lambda'_{\pm k} - i2y$, by $|\psi_{\pm k}\rangle$. Then, the condition of PTS leads to

\[
z'_k (\sigma_i |\psi_{\pm k}\rangle)^* = \lambda'_{\pm k} (\sigma_i |\psi_{\pm k}\rangle)^*.
\]

That is, $\sigma_i |\psi_{\pm k}\rangle$ is an eigenvector of $z'_k$ and the corresponding eigenvalue is given by $\lambda'_{\pm k}$. Now, there are two possibilities: (i) $\lambda'_{\pm k} = \lambda_{\pm k} \in \mathbb{R}$. Then, $\sigma_i |\psi_{\pm k}\rangle = |\psi_{\pm k}\rangle$ (potentially, up to a phase that can be absorbed in a redefinition of $|\psi_{\pm k}\rangle$). Further, also $\lambda'_{\pm k} = -\lambda_{\pm k} \in \mathbb{R}$, and $\sigma_i |\psi_{\pm k}\rangle = |\psi_{\pm k}\rangle$. For the eigenvalues of $z_k$, this implies $\text{Re}(\lambda_{\pm k}) = -\text{Re}(\lambda_{\pm k})$ and $\text{Im}(\lambda_{\pm k}) = -2y$. (ii) $\lambda'_{\pm k} = -\lambda'_{\pm k} = \lambda_{\pm k} \in i\mathbb{R}$, and $\sigma_i |\psi_{\pm k}\rangle = |\psi_{\pm k}\rangle$. For the eigenvalues $z_k$, this implies $\text{Re}(\lambda_{\pm k}) = 0$ and $\text{Im}(\lambda_{\pm k}) = -\text{Im}(\lambda_{\pm k}) - 4y$. We are, therefore, lead to distinguish three phases: In the PT-symmetric phase, (i) applies to all values of $k \in \text{BZ}$, while (ii) applies to all $k$ in the PT-broken phase. Finally, in the PT-mixed phase, both PT-symmetric and PT-breaking momentum modes exist.

Before we study how these phases are realized in the driven-dissipative Kitaev chain, we briefly describe the form of PTS in position space, which is relevant, in particular, for the analysis of systems with open boundary conditions. By taking the inverse Fourier transform of Eqs. (125), (127), and (128) according to Eq. (108), we obtain

\begin{align}
\text{PHS}^1: & \quad x_l = \sigma_z x_{-l}^* \sigma_z, \\
\text{IS}^1: & \quad x'_l = -\sigma_x x'_l \sigma_y, \\
\text{PTS}: & \quad x'_l = -\sigma_x x'_l \sigma_z,
\end{align}

where we define $x'_l = x_l - 2y\mathbb{1}$. For the matrix $X$, which is constructed from the $2 \times 2$ blocks $x_l$ like the matrix $A$ in Eq. (18) is constructed from $a_l$, and the shifted matrix $X' = X - 2y\mathbb{1}$, we find

\begin{align}
\text{PHS}^1: & \quad X = \Sigma_{\ell,l} X^\ell \Sigma_{\ell,l}, \\
\text{IS}^1: & \quad X' = -\Sigma_{\ell,l} X_l^\ell \Sigma_{\ell,l}, \\
\text{PTS}: & \quad X' = -\Sigma_{\ell,l} X'_l \Sigma_{\ell,l},
\end{align}

where, for $\mu \in \{x, y, z\}$,

\[
\Sigma_{\ell,\ell'} = \bigoplus_{l=1}^{\ell} \sigma_{\mu_l}.
\]

and $\Sigma_\ell$ is a $2\ell \times 2\ell$ block anti-diagonal matrix, with $2 \times 2$ identity matrices on the anti-diagonal,

\[
\Xi_\ell = \begin{pmatrix}
\mathbb{1} & \cdots \\
\cdots & \mathbb{1}
\end{pmatrix}.
\]

According to our convention for the Fourier transform in Eq. (108), for a system with periodic boundary conditions, the union of the spectra of the matrices $x_l$ for all $k \in \text{BZ}$ yields the spectrum of $-iX$. Therefore, we focus on the consequences of PTS for the spectrum of the latter. PTS in the form
given in Eq. (141) implies that for each nonzero eigenvalue \( \lambda' \) of \(-iX = -i(X - 2\gamma_1)\), which obeys the eigenvalue equation \(-iX'|\psi\rangle = \lambda'\psi\rangle\), also \(-\lambda'\) is an eigenvalue, and the corresponding eigenvector is given by \( \Sigma_{\xi'\xi}|\psi\rangle \). Further, since \(-iX'\) is purely imaginary, its eigenvalues \( \lambda' \) are either purely imaginary or come in pairs \( \lambda' \) and \(-\lambda'\). Therefore, for each eigenvalue \( \lambda' \), there are three possibilities: (i) There is a pair of real eigenvalues given by \( \pm \lambda' \in \mathbb{R} \). (ii) There is a pair of purely imaginary eigenvalues given by \( \pm \lambda' \in i\mathbb{R} \). (iii) If \( \text{Re}(\lambda) \neq 0 \) and \( \text{Im}(\lambda) \neq 0 \), there are four eigenvalues given by \( \pm \lambda' \) and \( \pm \lambda'' \). Numerically, we find that only cases (i) and (ii) are realized for the driven-dissipative Kitaev chain. That is, the eigenmodes of \(-iX = -i(X + 2\gamma_1)\) are either (i) PT-symmetric, with eigenvalues \( \lambda = \lambda' - i(\gamma) \in \mathbb{R} - i2\gamma \), or (ii) PT-breaking, with \( \lambda \in i\mathbb{R} \).

**E. Spectrum of the Liouvillian**

As can be shown by employing the formalism of third quantization [26, 27], for a system with periodic boundary conditions, the spectrum of the Liouvillian is determined by the spectrum of the matrix \( X_k \) in Eq. (108), or, equivalently, the matrix \( z_k \) defined in Eq. (120), similarly to the spectrum of the Hamiltonian Eq. (30) that can be obtained by populating the single-particle energy levels determined by the BdG Hamiltonian Eq. (31). The eigenvalues of \( z_k \) are given by

\[
\lambda_{\pm,k} = -i2\gamma \pm \omega_k, \tag{144}
\]

where

\[
\omega_k = \sqrt{\epsilon_k^2 - 4\gamma^2g}. \tag{145}
\]

As stated in the main text, the PT-symmetric phase, in which \( \omega_k \in \mathbb{R}_{>0} \), is realized for

\[
2\sqrt{\gamma g} < |2J - |\mu||. \tag{146}
\]

Then, the bands \( \lambda_{\pm,k} \) are separated by a real line gap [24, 28]. We note that for a system with open boundary conditions, we find numerically that the spectrum of \(-iX\) obeys \( \sigma(-iX) \in -i2\gamma + \mathbb{R} \) as suggested by Eq. (144) only for \( |\mu| > 2J \). In contrast, for \( |\mu| < 2J \) there are PT-breaking edge modes with purely imaginary eigenvalues in addition to fully PT-symmetric bulk of the system [28]. For \( \delta = 0 \), the eigenvalues corresponding to the edge modes are 0 and \(-i4\gamma\). Increasing the strength of dissipation leads to the occurrence of PT-breaking modes also in the bulk. In particular, in the PT-broken phase for

\[
2\sqrt{\gamma g} > 2J + |\mu|, \tag{147}
\]

we write \( \omega_k = ik_k \in i\mathbb{R}_{>0} \) with

\[
k_k = \sqrt{4\gamma^2g - \epsilon_k^2}, \tag{148}
\]

such that Eq. (144) becomes

\[
\lambda_{\pm,k} = -i(2\gamma \mp k_k). \tag{149}
\]

In this phase, the bands \( \lambda_{\pm,k} \) are separated by an imaginary line gap. Finally, for intermediate strengths of dissipation, the spectrum is gapless. In the gapless or PT-mixed phase, the bands \( \lambda_{\pm,k} \) cross at exceptional points at \( k = \pm k_o \) at which \( \omega_k \) defined in Eq. (145) vanishes, and which separate PT-symmetric from PT-breaking modes.

For future reference, we note that \( z_k \) can be diagonalized in the PT-symmetric phase by combining an ordinary rotation with a hyperbolic rotation, i.e., a rotation by an imaginary angle. In particular, with

\[
U_k = U_{x,k}U_{y,-ib} = e^{i(b/2)\sigma_z}e^{(b/2)\sigma_y}, \tag{150}
\]

we obtain

\[
z_k = -i2\gamma I + \omega_k U_{x,k}\sigma_z U_{x,k}^{-1}. \tag{152}
\]

By rotating this relation to the real basis, we find that \( x_k \) can be diagonalized as (note the appearance of \( \sigma_y \) instead of \( \sigma_z \))

\[
x_k = -i2\gamma I + \omega_k S_k\sigma_y S_k^{-1}, \tag{153}
\]

where

\[
S_k = r^3 U_k r = U_{z,k}U_{x,-ib} = e^{i(b/2)\sigma_z}e^{(b/2)\sigma_y}. \tag{154}
\]

The symmetries of \( z_k \) impose restrictions on the transformation \( U_k \). Analogously to the derivation of Eq. (70), on can show that inversion symmetry leads to the relation

\[
U_k = \sigma_z U_{-k}\sigma_z, \tag{155}
\]

which implies \( \theta_k = -\theta_{-k} \) and \( \beta_k = \beta_{-k} \).

**III. QUENCH DYNAMICS IN THE PT-SYMMETRIC PHASE AND RELAXATION TO THE PTGGE**

We now focus on quench dynamics in the PT-symmetric phase and, in particular, on relaxation to the PT-symmetric generalized Gibbs ensemble (PTGGE). In the main text, we derive the PTGGE, which describes the late-time dynamics in the PT-symmetric phase, as the maximum-entropy ensemble of eigenmodes of the Liouvillian, and with the modified statistics of these modes as encoded in their anticommutators. Below, we present details of this calculation. But first, we present a derivation of the PTGGE that is based on the explicit form of the time-evolved covariance matrix Eq. (111).

**A. Dephasing and relaxation to the PTGGE**

The most direct way to derive the PTGGE is to determine simplifications of the time-evolved covariance matrix Eq. (111) in the late-time limit due to dephasing. Technical details of the time evolution generated by the matrix \( x_k \), which can be interpreted as a non-Hermitian Hamiltonian, are presented in the Appendix.
1. The covariance matrix in the PT-symmetric phase

First, we provide explicit expressions for the components \( \gamma_{1,k}(t) \) and \( \gamma_{2,k}(t) \) of the time-evolved covariance matrix defined in Eq. (112). With the initial condition \( \gamma_{1,k}(0) = \hat{a}_{0,k} \cdot \sigma \) specified in Eq. (53), we write \( \gamma_{1,k}(t) \) in the form

\[
\gamma_{1,k}(t) = e^{-i\xi_k} \hat{a}_{0,k} \cdot \sigma e^{i\xi_k t} = \gamma_{1,1,k}(t) \mathbb{1} + \gamma_{1,2,k}(t) \cdot \sigma. \tag{167}
\]

Then, Eq. (A.10) leads to

\[
\gamma_{1,1,k}(t) = -\frac{e^{-4\gamma t}}{\omega_k^2} (1 - \cos(2\omega_k t)) \hat{a}_{0,k} \cdot (\mathbf{a}_\epsilon \times \mathbf{b}), \tag{157}
\]

and from Eq. (A.11) we obtain

\[
\gamma_{1,k}(t) = e^{-4\gamma t} \left[ \hat{a}_{0,k} - \frac{\varepsilon_k^2}{\omega_k^2} (1 - \cos(2\omega_k t)) \mathbf{a}_{\perp,k} \right.
\]

\[
+ \left. \frac{\varepsilon_k}{\omega_k} \sin(2\omega_k t) \mathbf{a}_{\parallel,k} \right], \tag{158}
\]

where, with \( \hat{a}_k = \mathbf{a}_k / \varepsilon_k \), the parallel, perpendicular, and out-of-plane components are defined as

\[
\mathbf{a}_{\parallel,k} = (\hat{a}_{0,k} \cdot \hat{a}_k) \hat{a}_k, \quad \mathbf{a}_{\perp,k} = \hat{a}_{0,k} - \mathbf{a}_{\parallel,k}, \quad \mathbf{a}_{\perp,k} = -\hat{a}_{0,k} \times \hat{a}_k. \tag{159}
\]

Analogously, for \( \gamma_{2,k}(t) \)

\[
\gamma_{2,k}(t) = \gamma_{2,1,k}(t) \mathbb{1} + \gamma_{2,2,k}(t) \cdot \sigma, \tag{160}
\]

we find

\[
\gamma_{2,1,k}(t) = \int_0^t dt' \frac{e^{-4\gamma(t-t')}}{\omega_k^2} \times (1 - \cos(2\omega_k (t - t'))) \mathbf{y} \cdot (\mathbf{a}_\epsilon \times \mathbf{b}), \tag{161}
\]

and

\[
\gamma_{2,k}(t) = -\int_0^t dt' e^{-4\gamma(t-t')} \times \left[ \mathbf{y} - \frac{\varepsilon_k^2}{\omega_k^2} (1 - \cos(2\omega_k (t - t'))) \mathbf{y}_{\perp,k} \right.
\]

\[
+ \left. \frac{\varepsilon_k}{\omega_k} \sin(2\omega_k (t - t')) \mathbf{y}_{\parallel,k} \right], \tag{162}
\]

where

\[
\mathbf{y}_{\parallel,k} = (\mathbf{y} \cdot \hat{a}_k) \hat{a}_k, \quad \mathbf{y}_{\perp,k} = \mathbf{y} - \mathbf{y}_{\parallel,k}, \quad \mathbf{y}_{\parallel,k} = -\mathbf{y} \times \hat{a}_k. \tag{163}
\]

The integration in Eqs. (161) and (162) is elementary and we omit the lengthy results.

2. Late-time limit and dephasing

In isolated systems, the defining signature of generalized thermalization after a quantum quench is local observables assuming stationary expectation values that are determined by the GGE. For noninteracting and translationally invariant systems, generalized thermalization occurs through dephasing of momentum modes that oscillate at different frequencies \( \varepsilon_k \neq \varepsilon_{k'} \) for \( k \neq k' \) [29, 30]. In particular, for the isolated Kitaev chain, the component of the time-evolved covariance matrix with momentum \( k \) can be obtained by setting \( \gamma = 0 \) in Eqs. (157), (158), (161), and (162), which yields \( \gamma_{1,1,k}(t) = \gamma_{2,k}(t) = 0 \), such that \( \gamma_{1,k}(t) = \gamma_{2,k}(t) \cdot \sigma \) with

\[
\gamma_{1,k}(t) = a_{\perp,k} + \cos(2\varepsilon_k t) a_{\parallel,k} + \sin(2\varepsilon_k t) a_{\parallel,k}. \tag{164}
\]

The long-time behavior that is established through dephasing is captured by setting the oscillatory components to zero. Thus, with subscript “\( d \)” for dephased,

\[
\gamma_d \equiv a_{\perp,k} = (\hat{a}_{0,k} \cdot \hat{a}_k) \hat{a}_k, \tag{165}
\]

and the generalized Gibbs ensemble (GGE) is the Gaussian state that is uniquely determined by the dephased covariance matrix \( \gamma_{d,k} = \gamma_{d,k} \cdot \sigma \). Since \( \gamma_{d,k} \parallel \hat{a}_k \), the dephased covariance matrix, and, therefore, the GGE, is diagonal in the eigenbasis of the postquench Hamiltonian. The diagonal elements are determined by the conserved mode occupation numbers of the eigenmodes \( d_k \) in Eq. (38) of the postquench Hamiltonian in the initial state \( |\psi_0> \).

\[
\hat{a}_{0,k} \cdot \hat{a}_k = \langle \psi_0 | d_k d_k^\dagger | \psi_0 > = 1 - 2 \langle \psi_0 | d_k^\dagger | d_k | \psi_0 >, \tag{166}
\]

as follows from setting \( \gamma = 0 \) in Eq. (222) below. For quenches in the driven-dissipative Kitaev chain, this picture is modified in two fundamental ways:

(i) The time dependence of the covariance matrix is described by renormalized oscillation frequencies \( \omega_k \) given in Eq. (145) and, crucially, there is additional exponential decay. However, due to PTS, there is only a single relaxation rate, which, for the driven-dissipative Kitaev chain, is determined by the imaginary part \( -\text{Im}(\lambda_{\perp,k}) = 2\gamma \) of the eigenvalues of \( z_k \) in Eq. (144). After factoring out the resulting trivial exponential time dependence, the dispersiveness of the real part \( \text{Re}(\lambda_{\perp,k}) = \pm \omega_k \) induces dephasing and, consequently, relaxation dynamics in analogy to isolated systems. In contrast, the dynamics of open systems without PTS is typically governed by a range of exponential relaxation rates, and the slowest of these rates determines the late-time behavior, while dephasing does not play an important role. For the PT-symmetric driven-dissipative Kitaev chain, as in Eq. (165), dephasing can be accounted for by setting oscillatory contributions to zero. Therefore, the late-time behavior is described by

\[
\gamma_d(t) \sim \gamma_d(t) = \gamma_{d,1,k}(t) + \gamma_{d,2,k}(t) + \gamma_{SS,k}, \tag{167}
\]

where

\[
\gamma_{d,1,k}(t) = e^{-4\gamma t} \gamma'_{d,1,k}, \quad \gamma'_{d,1,k} = \gamma'_{d,1,k} \mathbb{1} + \gamma'_{d,1,k} \cdot \sigma. \tag{168}
\]
where

\[ \gamma_{d,1,k} = -\frac{1}{\omega_k} \hat{a}_{0,k} \cdot (a_k \times b) = \frac{2\varepsilon_k \sqrt{\gamma_l \gamma_g}}{\omega_k} \sin(\Delta \theta_k), \]  

(169)

and

\[ \gamma'_{d,1,k} = \frac{\varepsilon_k^2}{\omega_k} a_{\perp k} \]

\[ = \cos(\Delta \theta_k) \hat{a}_{0,k} + \frac{4\varepsilon_k \gamma_k}{\omega_k} \sin(\Delta \theta_k) \hat{e}_z \times \hat{a}_k. \]

(170)

In these expressions, \( \Delta \theta_k \) is the angle between \( \hat{a}_{0,k} \) and \( \hat{a}_k \).

\[ \cos(\Delta \theta_k) = \hat{a}_{0,k} \cdot \hat{a}_k. \]

(171)

Analogously,

\[ \gamma_{d,2,k}(t) = e^{-4\gamma t} \gamma'_{d,2,k}, \]

(172)

where

\[ \gamma'_{d,2,k} = -\frac{1}{4\gamma} \left[ \frac{1}{\omega_k} y \cdot (a_k \times b) \right] \]  

\[ \left( y - \frac{\varepsilon_k}{\varepsilon_k^2 + \delta^2} (\varepsilon_k y_{\perp k} - 2\gamma y_{\perp k}) \right) \cdot \sigma. \]

(173)

Finally,

\[ \gamma_{SS,k} = \frac{1}{4\gamma} \left[ \frac{1}{\varepsilon_k^2 + \delta^2} y \cdot (a_k \times b) \right] \]  

\[ - \left[ y - \frac{\varepsilon_k}{\varepsilon_k^2 + \delta^2} (\varepsilon_k y_{\perp k} - 2\gamma y_{\perp k}) \right] \cdot \sigma. \]

(174)

(ii) The contribution \( \gamma_{2,k}(t) \) in Eq. (111), which describes the approach to the steady state \( \rho_{SS} \) defined by \( \mathcal{L}(\rho_{SS}) = 0 \) and given explicitly in terms of the covariance matrix in Eq. (174), does not have a counterpart in isolated systems. To understand the physical significance of \( \gamma_{2,k}(t) \), it is instructive to consider once more the isolated Kitaev chain for comparison. There, due to the conservation of mode occupation numbers Eq. (166), a memory of the initial state is kept in Eq. (165), and, therefore, in the GGE. In the case of the driven-dissipative Kitaev chain, this memory of the initial state is contained in \( \gamma_{1,k}(t) \), and fades away due to the exponential damping factor \( e^{-4\gamma t} \) in Eqs. (157) and (158). At the same time, as time progresses, the contribution \( \gamma_{1,k}(t) \) is being overwritten by \( \gamma_{2,k}(t) \), which, for \( t \to \infty \), approaches the stationary form given in Eq. (174). This picture shows that the PTGGE, as a generalization of the GGE, is captured by the dephased form of \( \gamma_{1,k}(t) \) given in Eq. (168),

\[ \gamma_{PTGGE,k}(t) = e^{-4\gamma t} \gamma'_{PTGGE,k}. \]

(175)

where \( \gamma'_{PTGGE,k} = \gamma'_{d,1,k} \), while the growth and eventual dominance of \( \gamma_{2,k}(t) \) restricts the time frame during which relaxation to the PTGGE can be observed. As pointed out in the main text, \( \gamma_{2,k}(t) \) vanishes for \( \delta = \gamma_l - \gamma_g = 0 \). Then, there is no temporal constraint on the validity of the PTGGE. In contrast, when \( \delta \neq 0 \), the PTGGE captures the late-time relaxation of local observables up to a crossover time scale \( t_c \).

As stated in the main text and detailed below, the precise value of \( t_c \) depends on the observable under consideration.

3. From the covariance matrix to the density matrix

According to the above discussion, the PTGGE is uniquely and fully determined by the dephased form of the contribution to the covariance matrix \( \gamma_{1,k}(t) \) given in Eq. (168). For the sake of completeness, we describe in the following how the explicit form of the density matrix corresponding to a given covariance matrix \( \gamma_k \) can be found.

The dynamics generated by the Liouvillian \( \mathcal{L} \) in Eq. (86) couples only modes with momenta \( \pm k \), and we can consider the subspaces corresponding to these modes separately. In one such subspace, for \( k \notin [0, \pi] \) such that \( k \neq -k \), a complete set of fermionic operators is given by the doubled spinors of complex \( C_k \) and Majorana \( W_k \) fermions:

\[ C_k = \begin{pmatrix} c_k \\ c_k^\dagger \\ C_{-k} \\ C_{-k}^\dagger \end{pmatrix}, \quad W_k = \begin{pmatrix} W_{1,k} \\ W_{2,k} \\ W_{3,k} \\ W_{4,k} \end{pmatrix}, \]

(176)

which are related by

\[ C_k = \frac{1}{\sqrt{2}} R W_k, \quad R = r \oplus r. \]

(177)

We note that the operators \( W_k \) describe true Majorana fermions, i.e., they are their own antiparticles, in contrast to the Fourier transform of the Majorana modes defined in Eq. (51). In analogy to Eq. (20), we define a covariance matrix of \( W_k \) fermions as

\[ \gamma_k = i \left( \langle W_k W_k^\dagger \rangle - 1 \right) = i R^\dagger \left( 2\langle C_k C_k^\dagger \rangle - 1 \right) R. \]

(178)

To establish the relation between \( \gamma_k \) and the covariance matrix \( \gamma_k \) in Eq. (50) as well as the complex covariance matrix \( g_k \) in Eq. (47), we note that the spinors in Eq. (29) can be expressed in terms of the doubled spinors \( C_k \) as

\[ \begin{pmatrix} C_k \\ C_{-k}^\dagger \end{pmatrix} = P \begin{pmatrix} c_k \\ c_{-k}^\dagger \end{pmatrix} = P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

(179)

Therefore,

\[ \gamma_k = i R^\dagger P \begin{pmatrix} 2 \langle C_k C_k^\dagger \rangle & \langle C_k C_{-k}^\dagger \rangle & \langle C_{-k} C_k^\dagger \rangle & \langle C_{-k} C_{-k}^\dagger \rangle \\ -1 & -1 & -1 & -1 \end{pmatrix} R P. \]

(180)

where

\[ Q = R^\dagger P R. \]

(181)

The corresponding density matrix is given by \[ \rho_k = \frac{1}{Z_k} e^{i W_k \arctanh(\gamma_k) / W_k} = \frac{1}{Z_k} e^{-2 C_k^\dagger \arctanh(\gamma_k) C_k + \text{tr}(\gamma_k)}, \]

(182)
where the normalization $Z_k$ is to be chosen such that $\text{tr}(\rho_k) = 1$, which leads to
\[
\rho_k = \frac{1}{\det(1 + e^{-2 \text{arctanh}(g_k)})} e^{-2g_k \text{arctanh}(g_k)C_k}.
\tag{183}
\]
This form applies also to $k \in [0, \pi]$ when we define, for these momenta, $C_k = c_k$ and $g_k = \langle |c_k, c_k^\dagger| \rangle$. Then, the full density matrix is given by
\[
\rho = \prod_{k > 0} \rho_k.
\tag{184}
\]
According to the discussion in Sec. III A 2, we obtain the PTGGE by inserting $g_k(t) = r_{PTGGE}(t) e^t$, with $r_{PTGGE}(t)$ given in Eq. (175). In the next section, we establish the equivalence between the resulting expression for $\rho_{PTGGE}(t)$ and the form given in Eq. (8) in the main text in terms of eigenmodes of the adjoint Liouvillian.

B. Derivation of the PTGGE from the principle of maximum entropy

As described in the main text, the PTGGE can be obtained as the maximum entropy ensemble that is compatible with the modified statistics and dephased expectation values of commutators of eigenmodes of the Liouvillian. In the following, we present details of this derivation.

1. Eigenmodes of the adjoint Liouvillian

Our goal is to find eigenmodes of the Liouvillian, which generalize the Bogoliubov quasiparticle operators defined in Eq. (38) to the driven-dissipative setting. Specifically, for the isolated Kitaev chain, an operator $O$ evolves in the Heisenberg picture according to $\frac{dO}{dt} = i[H, O]$, i.e., the dynamics is generated by the superoperator $\mathcal{H}O = [H, O]$. The Bogoliubov quasiparticle operators $d_k$, $d_k^\dagger$ are eigenmodes of $\mathcal{H}$ in the sense that they obey the eigenvalue equation $\mathcal{H}d_k = -\epsilon_k d_k$. As we explain below, this definition of eigenmodes of the generator of dynamics cannot directly be generalized to fermionic open systems. Instead of fermionic mode operators $d_k$ themselves, we consider bilinear forms of operators,$\eta_k = [d_k, d_k^\dagger], \quad \chi_k = [d_k, d_k], \tag{185}\]
which in the isolated Kitaev chain obey $\mathcal{H}\eta_k = 0$ and $\mathcal{H}\chi_k = -2\epsilon_k \eta_k$ such that $\langle \eta_k(t) \rangle = \langle \eta_k \rangle_0$ is conserved, while $\langle \chi_k(t) \rangle$ is purely oscillatory and vanishes via dephasing. We generalize this structure to open systems in three steps: (i) We specify the generator of dynamics for operators, and thereby clarify which eigenvalue equation we actually want to solve. (ii) To find a solution to the eigenvalue equation, we make a bilinear ansatz in the form of $\eta_k$, which leads to an explicit expression for the mode operators $d_k$. We find that the expectation value $\langle \eta_k(t) \rangle$ is not conserved but nonoscillatory, i.e., not subject to dephasing. (iii) The form of $\eta_k$ and, in particular, the mode operators $d_k$, completely determines $\chi_k$. As we show, the expectation value $\langle \chi_k(t) \rangle$ is the sum of constant and purely oscillatory contributions. The constant part vanishes when $\delta = 0$, such that, as in the isolated Kitaev chain, $\langle \chi_k(t) \rangle$ does not contribute to the PTGGE.

(i) Adjoint Liouvillian. For any operator $O$, the expectation value $\langle O \rangle = \text{tr}(O\rho)$ obeys the equation of motion
\[
\frac{d}{dt} \langle O \rangle = i\text{tr}(O\mathcal{L}\rho) = i\text{tr}(\rho\mathcal{L}^\dagger O) = i\mathcal{L}^\dagger O,
\tag{186}
\]
with the adjoint Liouvillian,
\[
\mathcal{L}^\dagger = \mathcal{H} - i\mathcal{D},
\tag{187}
\]
where $\mathcal{H}$ is given in Eq. (88), and
\[
\mathcal{D}^\dagger O = \sum_{l=1}^L \left(2L_l^\dagger OL_l - [L_l^\dagger L_l, O]\right)
\tag{188}
\]
As a general remark, we note that the adjoint Liouvillian generates the dynamics of operators only in the sense of expectation values, while the Heisenberg picture for open systems is defined in terms of quantum Langevin equations [34, 35].

Now, if $O$ is an eigenmode of $\mathcal{L}^\dagger$ in the following sense:
\[
\mathcal{L}^\dagger O = \lambda^* (O - O_{SS}),
\tag{189}
\]
where $O_{SS} \in \mathbb{R}$, then,
\[
\frac{d}{dt} \langle O \rangle = i\lambda^* \langle (O) - O_{SS} \rangle,
\tag{190}
\]
which is solved by
\[
\langle O(t) \rangle = O_{SS} + e^{i\lambda^* t} \langle (O(0)) - O_{SS} \rangle.
\tag{191}
\]
For $\text{Im}(\lambda^*) > 0$, we obtain $\langle O(t) \rangle \rightarrow O_{SS} = \langle O_{SS} \rangle = \text{tr}(O_{SS})$ for $t \rightarrow \infty$. As we show in the following, nonoscillatory eigenmodes, for which $\lambda$ is purely imaginary, $\lambda^* = i\kappa$, can be found for operators $O$ in the form of $\eta_k$ in Eq. (185), while an ansatz in the form of $\chi_k$ leads to oscillatory eigenmodes. Here and in the following, we use the term eigenmodes both for the bilinears $\eta_k$ and $\chi_k$ and the fermionic operators $d_k$.

(ii) Nonoscillatory normal commutators. To find nonoscillatory eigenmodes of the adjoint Liouvillian, we make an ansatz in the form of $\eta_k$ in Eq. (185). In fact, for the time being, we do not assume translational invariance, i.e., we do not assume that the eigenmodes can be labelled by the momentum $k$. Then, $\eta = [d, d^\dagger]$ can be written in the form
\[
\eta = \frac{i}{4} \sum_{l,f=1}^{2L} w_l Q_{l,f} w_f,
\tag{192}
\]
where, without loss of generality, $Q$ can be assumed to be antisymmetric, $Q = -Q^\top$. Further, $\eta$ is Hermitian if $Q$ is real, and $\eta$ is traceless by construction. We proceed to insert our ansatz
for \( \eta \) in Eq. (189). The contribution from the Hamiltonian is
given by
\[
H_\eta = -\frac{1}{4} \sum_{l,l'=1}^{2L} w_l [A, Q]_{l,l'} w_{l'},
\]
and applying the adjoint dissipator to \( \eta \) yields
\[
D^\dagger \eta = -\sum_{m,m'=1}^{2L} M_{m,m'} (w_m \eta, w_{m'}) + [w_m, \eta] w_{m'}
= -i \sum_{l,l'=1}^{2L} w_l [M, Q]_{l,l'} w_{l'} + 2 \text{tr}(M_l Q).
\]
where the bath matrix \( M = M_R + iM_I \) is defined in Eq. (92).
We identify the last, constant term in the above expression with the steady-state expectation value \( \eta_{SS} \),
\[
\eta_{SS} = -\frac{2}{\kappa} \text{tr}(M_I Q),
\]
where we also anticipate that \( \lambda^* = i\kappa \) is purely imaginary. Then, Eq. (189) takes the form
\[
\sum_{l,l'=1}^{2L} w_l [A, Q]_{l,l'} w_{l'} = \kappa \sum_{l,l'=1}^{2L} w_l Q_{l,l'} w_{l'},
\]
Next, we multiply this equation by \( w_l w_{l'} \) and use
\[
\frac{1}{2L+1} \text{tr}(w_l w_{l'} w_m w_{m'}) = \frac{1}{2} \left( \delta_{l,l'} \delta_{m,m'} - \delta_{l,m'} \delta_{l',m} + \delta_{l,m} \delta_{l',m'} \right),
\]
to obtain
\[
X^\dagger Q + QX = \kappa Q,
\]
where matrix \( X \) is defined in Eq. (99). Now that we have illustrated the derivation of the eigenvalue equation for the bilinear \( \eta \), let us briefly comment on the possibility of finding eigenmodes of \( \mathcal{L}^I \) that are linear in fermionic operators. In general, such linear eigenmodes cannot be found because there is no closed eigenvalue equation in the space of linear operators. As a simple example, consider a lattice model without Hamiltonian dynamics, \( H = 0 \), and incoherent loss, \( L_I = \sqrt{\gamma} c_I \). Then, by applying the adjoint dissipator Eq. (188) to the fermionic operator \( c_I \) yields
\[
D^\dagger c_I = -\gamma_I \left( 1 + 4 \sum_{l=1}^{L} c^\dagger c_l \right) c_I,
\]
i.e., a sum of linear and cubic terms. In contrast, the second equality in Eq. (188) shows that, if \( O \) is bilinear and \( L_I \) is linear in fermionic operators, also \( D^\dagger O \) is bilinear, and, therefore, a closed eigenvalue equation can always be found for a bilinear ansatz. The situation is very different for bosons. Then, the second equality in Eq. (188) shows that, if both \( O \) and \( L_I \) are linear in bosonic operators, also \( D^\dagger O \) is linear.

To proceed, we exploit translational invariance of the driven-dissipative Kitaev chain, which implies that there is an eigenmode \( \eta_k \), expressed in terms of a matrix \( Q_k \), for each momentum mode \( k \). We refine our ansatz in Eq. (192) as follows:
\[
\eta_k = \sum_{l,l'=1}^{L} W_l q_{k,l-l'} W_{l'},
\]
where the spinors \( W_l \) of Majorana operators are defined in Eq. (9), and where we assume that \( Q_k \) is a block Toeplitz matrix with \( 2 \times 2 \) blocks
\[
q_{k,l-l'} = \begin{pmatrix}
Q_{k,2l-1,2l-1} & Q_{k,2l-1,2l+1} \\
Q_{k,2l+1,2l-1} & Q_{k,2l+1,2l+1}
\end{pmatrix}.
\]
We define the Fourier transform of \( q_{k,l} \) as
\[
q_{k,k'} = \sum_{l=1}^{L} e^{-i\kappa l'} q_{k,l}.
\]
For future reference, we note that, since \( Q \) is real and antisymmetric, the Fourier transform \( q_{k,k'} \) of \( q_{k,l} \) obeys the relations
\[
q_{k,k'} = -q_{-k,-k'}, \quad q_{k,k'} = -q^{\dagger}_{-k,-k'},
\]
which imply \( q_{k,k'} \). By inserting Eqs. (201) and (202) in Eq. (198) we obtain
\[
x_k \cdot q_{k,k'} - q_{-k,k} x_{-k,k'} = i\kappa q_{k,k'}.
\]
In the PT-symmetric phase, according to Eq. (144), the spectrum of \( x_k \) is given by \( \sigma(x_k) = i(2\gamma \pm \omega_k) \), and \( x_k \) can be diagonalized as specified in Eq. (153). This leads to
\[
\omega_k (S_k^\dagger \sigma, S_k \sigma, q_{k,k'} - q_{-k,k} S_k q_{-k,k'}, \sigma, S_k^{-\dagger} \sigma) = i(\kappa - 4\gamma) q_{k,k'},
\]
where we use the shorthand notation \( S_k^{-\dagger} = \left(S_k^\dagger \right)^{-1} \). On the right-hand side, we identify \( \kappa = 4\gamma \), and are thus led to the following commutation relation:
\[
[\sigma, S_k^\dagger q_{k,k'} S_k^{-\dagger} \sigma] = 0.
\]
The matrices that commute with \( \sigma \) are \( \mathbb{1} \) and \( \sigma \), itself. Therefore, the general solution for \( q_{k,k'} \), which obeys the above commutation relation, reads
\[
q_{k,k'} = S_k^\dagger \left( \alpha_{+,k} \sigma + \alpha_{-,k} \mathbb{1} \right) S_k^{-\dagger},
\]
with undetermined coefficients \( \alpha_{+,k} \) and \( \alpha_{-,k} \). This solution can be seen to obey the conditions Eq. (203) if \( \alpha_{+,k} \) and \( \alpha_{-,k} \) are, respectively, symmetric and antisymmetric functions of \( k' \), and by using that PHS \( x_k \) in Eq. (124) implies
\[
S_k = S_{-k}^\dagger,
\]
in analogy to Eq. (69). We proceed to fix the coefficients \( \alpha_{+,k} \) and \( \alpha_{-,k} \) such that \( \eta_k \) can be written in terms of mode operators \( d_k \) as \( \eta_k = [d_k, d_k^\dagger] \). To this end, we first express \( \eta_k \) in terms of complex fermions. By inserting Eq. (51) in Eq. (200) we obtain
\[
\eta_k = \sum_{k' \in \mathbb{Z}} \frac{1}{4} \sum_{k' \in \mathbb{Z}} W_l^\dagger q_{k,k'} W_{k'} = \frac{1}{2} \sum_{k' \in \mathbb{Z}} C_k^\dagger r_{k,k'} r_{k'} C_{k'},
\]
(209)
where, in the last line, we use Eq. (154). For solutions $\eta_k$ with fixed momentum $k$, the sum over $k'$ should collapse to the two terms $k' = \pm k$. Symmetric and antisymmetric combinations of these components are given by
\[
\alpha_{\pm,k,k'} = \frac{\alpha_{0,k}}{2} (\delta_{k',k} \pm \delta_{k',-k}),
\]
with the coefficient $\alpha_{0,k}$ to be determined below. Then, with
\[
P_{z,k} = \frac{1}{2} (\mathbb{1} \pm \sigma_z),
\]
we find
\[
\eta_k = \frac{\alpha_{0,k}}{2} \left( C_k^\dagger U_{z,k}^\dagger P_{z,k} U_{z,k}^\dagger C_k - C_k^\dagger U_{z,k}^\dagger P_{z,k} U_{z,k}^\dagger C_k \right) - \frac{\alpha_{0,k}}{2} \left( C_k^\dagger U_{z,k}^\dagger P_{z,k} U_{z,k}^\dagger C_k^\dagger + C_k^\dagger U_{z,k}^\dagger P_{z,k} U_{z,k}^\dagger C_k \right),
\]
where in the second equality we use that $U_k$, which is defined in Eq. (150), obeys the same PHS as $U_{z,k} R_k$ in Eq. (69). $\eta_k$ takes its final form, $\eta_k = [d_k, d_k^\dagger]$, if we define the eigenmodes of the Liouvillian as
\[
D_k = \begin{pmatrix} d_k \\ d_k^\dagger \end{pmatrix} = V_k^\dagger C_k,
\]
where
\[
V_k = \sqrt{\frac{2}{\text{tr}(U_k^\dagger U_k^{-1})}} U_k^\dagger = \begin{pmatrix} \cos(\frac{\delta_k + \phi_k}{2}) & \sin(\frac{\delta_k + \phi_k}{2}) \\ i \sin(\frac{\delta_k + \phi_k}{2}) & \cos(\frac{\delta_k + \phi_k}{2}) \end{pmatrix}.
\]
The angle $\theta_k$ is defined in Eq. (35), and $\phi_k$ is defined through the equality
\[
e_k e^{i \phi_k} = \omega_k + i 2\sqrt{\gamma_1 \gamma_2},
\]
and is related to $\beta_k$ defined in Eq. (151) via
\[
tan(\beta_k) = -\frac{\delta_k}{\omega_k} \tanh(\beta_k).
\]
When we insert Eq. (213) in Eq. (212), the normalization in Eq. (214) cancels if we set
\[
\alpha_{0,k} = \text{tr}(U_k^\dagger U_k^{-1}) = 2 \cosh(\beta_k).
\]
The normalization in Eq. (214) is chosen such that $[d_k, d_k^\dagger] = 1$. As in the main text, we collect the anticommutators of the modes $d_k$ in a matrix,
\[
\begin{pmatrix} [d_k, d_k^\dagger] & [d_k, d_{-k}] \\ [d_{-k}^\dagger, d_k^\dagger] & [d_{-k}^\dagger, d_{-k}] \end{pmatrix} = f_k \delta_{k,k'},
\]
where
\[
f_k = \begin{pmatrix} [d_k, d_k^\dagger] & [d_k, d_{-k}] \\ [d_{-k}^\dagger, d_k^\dagger] & [d_{-k}^\dagger, d_{-k}] \end{pmatrix} = V_k^\dagger V_k = \mathbb{1} + \sin(\phi_k) \sigma_y.
\]
Next, we evaluate Eq. (195). A short calculation yields
\[
\eta_{SS,k} = \langle [d_k, d_k^\dagger] \rangle_{SS} = -\frac{\delta \omega_k}{4 \epsilon \gamma} \left( \cos(\theta_k) - \frac{2 \sqrt{\gamma_1 \gamma_2}}{\omega_k} \sin(\theta_k) \right).
\]
Finally, to completely specify the time evolution of the expectation value of $\eta_k$, we need to know the expectation value $\langle \eta_k \rangle_0$ in the initial state. Starting from Eq. (212), we use Eqs. (47) to find
\[
\langle [d_k, d_k^\dagger] \rangle = \frac{\delta \omega_k}{\epsilon_k} \text{tr}(U_k^{-1} P_{z,k} U_k^{-1} g_k).
\]
For the ground state, with covariance matrix given in Eq. (49), we obtain
\[
\langle [d_k, d_k^\dagger] \rangle_0 = \cos(\Delta \theta_k - \phi_k),
\]
where $\Delta \theta_k$ is defined in Eq. (171). Putting everything together, we obtain the result stated in the main text:
\[
\langle [d_k, d_k^\dagger] \rangle(t) = e^{-\delta \gamma t} \langle [d_k, d_k^\dagger] \rangle_0 + \left( 1 - e^{-\delta \gamma t} \right) \langle [d_k, d_k^\dagger] \rangle_{SS},
\]
where $\langle [d_k, d_k^\dagger] \rangle_0$ and $\langle [d_k, d_k^\dagger] \rangle_{SS}$ are given in Eqs. (222) and (220), respectively.

(iii) Oscillatory anomalous commutators. We next show that also $\chi_k = [d_k, d_{-k}]$ is an eigenmode of the Liouvillian in the sense of Eq. (189), however, where $\text{Re}(A^*) \neq 0$, such that the nonconstant term in Eq. (191) is oscillatory. To that end, in analogy to Eq. (209), we write $\chi_k$ as
\[
\chi_k = \frac{1}{2} \sum_{k' \in \mathbb{Z}} C_{k'} r p_{k,k'} r^\dagger C_{k'},
\]
where we define
\[
p_{k,k'} = -2 (\delta_{k,k'} - \delta_{-k,-k'}) r^\dagger V_k^\dagger \sigma_y V_k^\dagger r.
\]
A straightforward calculation shows that $p_{k,k'}$ satisfies an eigenvalue equation similar to Eq. (204),
\[
x_k^\dagger p_{k,k'} - p_{k,k'} x_k = -2 (\omega_k - i 2 \gamma) p_{k,k'},
\]
which leads us to identify $A^* = -2 (\omega_k - i 2 \gamma)$. Consequently, as stated in the main text, we find
\[
\langle [d_k, d_{-k}] \rangle(t) = e^{-\delta \omega_k - i 2 \gamma t} \langle [d_k, d_{-k}] \rangle_0 + \left( 1 - e^{-\delta \omega_k - i 2 \gamma t} \right) \langle [d_k, d_{-k}] \rangle_{SS},
\]
where the steady-state expectation value is given by
\[
\chi_{SS,k} = \langle [d_k, d_{-k}] \rangle_{SS} = \frac{i \delta}{4 (\omega_k - i 2 \gamma)} \sum_{k' \in \mathbb{Z}} \text{tr}(\sigma_y p_{k,k'}) = \frac{\delta \sin(\theta_k)}{\omega_k - i 2 \gamma}.
\]
2. Maximum entropy ensemble

The PTGGE describes the late-time behavior for $\delta = 0$, such that, as can be seen explicitly in Eqs. (220) and (228),
\[
\langle [d_k, d_k^\dagger] \rangle_{SS} = \langle [d_k, d_{-k}] \rangle_{SS} = 0,
\]
and when the contribution from the anomalous commutators becomes negligible due to dephasing,
\[
\langle [d_k, d_{-k}] \rangle(t) \to 0.
\]
To summarize, the PTGGE is characterized by the following expectation values:

\[
\begin{pmatrix}
\langle [d_k, d^\dagger_{-k}](t) \rangle_{\text{PTGGE}} \\
\langle [d^\dagger_{-k}, d_{-k}](t) \rangle_{\text{PTGGE}} \\
\langle [d^3_{-k}, d_{-k}](t) \rangle_{\text{PTGGE}}
\end{pmatrix} = \zeta_k(t),
\]

(231)

where

\[
\zeta_k(t) = e^{-4\gamma t} \gamma_k',
\]

(232)

with

\[
\zeta_k' = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

(233)

Specifically, we define the PTGGE as the maximum entropy ensemble that is compatible with the expectation values of commutators summarized in \(\zeta_k(t)\), and with the statistics of the modes \(d_k\) as determined by their anticommutators in Eq. (219). The same information is stored in the covariance matrix in the complex basis, which, according to Eqs. (47) and (213), is given by

\[
g_{\text{PTGGE},k}(t) = V_k^{-1} \zeta_k(t) V_k^{-1} = e^{-4\gamma t} g'_{\text{PTGGE},k},
\]

(234)

where

\[
g'_{\text{PTGGE},k} = V_k^{-1} \zeta_k V_k^{-1}.
\]

(235)

For given two-point functions, the entropy is maximized for a Gaussian state, which is determined uniquely by Eq. (183). Therefore, with

\[
C'_k \text{arctanh}(g'_{\text{PTGGE},k}(t)) C_k = D_k f_k^{-1} \text{arctanh}(\zeta'_k(t)f_k^{-1}) D_k,
\]

(236)

we obtain the PTGGE as stated in the main text:

\[
\rho_{\text{PTGGE}}(t) = \frac{1}{Z_{\text{PTGGE}}(t)} e^{-2 \sum_{k>0} D_k f_k^{-1} \text{arctanh}(\zeta'_k(t)f_k^{-1}) D_k}.
\]

(237)

Finally, a straightforward calculation shows that \(\gamma_{\text{PTGGE},k}(t) = 4 g'_{\text{PTGGE},k}(t) t\) for \(\gamma_{\text{PTGGE},k}(t)\) given in Eq. (175) and that, therefore, the derivations of the PTGGE in Secs. III A and III B yield the same result. However, the key benefit of the derivation presented in the current section is that it clarifies how the notions of eigenmodes of the generator of the dynamics and of a maximum entropy ensemble generalize to PT-symmetric open systems. In particular, this derivation shows that conservation of occupation numbers of eigenmodes is not necessary for the PTGGE to capture the late-time limit of quench dynamics. Instead, nonoscillatory decay is sufficient.

### C. Subsystem parity

In the following, we first describe how to obtain the data for the subsystem fermion parity in Eq. (84) that is shown in Figs. 1 and 2 in the main text, and the PTGGE prediction that describes the late-time behavior of the subsystem parity and is shown as horizontal lines in the figures. Then, we fill in details of the derivation of the conjectured analytical expressions for the space-time scaling limit of the subsystem parity given in Eqs. (9) and (10) in the main text. Next, we derive the estimate for the crossover time \(t_s\) that is shown in the inset of Fig. 2, and finally, we derive explicit expressions for the soft modes \(k_{s,+}\) and \(k_{s,-}\) and we argue that the existence of two distinct time scales \(t_{s,+}\) and \(t_{s,-}\) is a unique feature of driven-dissipative systems with broken inversion symmetry.

#### 1. Numerical time evolution

In the PT-symmetric phase, it is convenient to define a rescaled covariance matrix as

\[
\Gamma'(t) = e^{4\gamma t} \Gamma(t),
\]

(238)

where \(\Gamma(t)\) is given in Eqs. (101) and (102). Through this rescaling, one can avoid having to perform numerical calculations with extremely small quantities when \(\gamma t \gg 1\). In particular, we obtain the rescaled subsystem parity, which is shown in Figs. 1 and 2 in the main text, directly as

\[
e^{4\gamma t} \langle P(t) \rangle = \Pi(\Gamma'(t)),
\]

(239)

where we use the following property of the Pfaffian: for a \(2\ell \times 2\ell\) matrix \(A\) and \(a \in \mathbb{C}\), \(\Pi(aA) = a^\ell \Pi(A)\).

#### 2. Long-time behavior

As we show in Sec. III B, the PTGGE is equivalent to the dephased covariance matrix \(\gamma_{\text{PTGGE},k}(t)\) in Eq. (175). The corresponding covariance matrix in position space can be found through an inverse Fourier transform according to Eq. (50) and by using Eq. (21). In analogy to Eq. (175), we obtain the late-time behavior in the form

\[
\Gamma(t) \sim \Gamma_{\text{PTGGE}}(t) = e^{-4\gamma t} \Gamma'_{\text{PTGGE}},
\]

(240)

where \(\Gamma'_{\text{PTGGE}}\) is time-independent. The late-time behavior of the subsystem parity, which according to Eq. (84) is just the Pfaffian of the reduced covariance matrix Eq. (85), is thus given by

\[
\langle P(t) \rangle \sim \langle P(t) \rangle_{\text{PTGGE}} = e^{-4\gamma t} \Pi\left(\Gamma'_{\text{PTGGE}}\right).
\]

(241)

We obtain the horizontal lines in Figs. 1 and 2 of the main text by evaluating the right-hand side of the above equation numerically. In particular, we reconstruct \(\Gamma'_{\text{PTGGE}}\) from \(\gamma_{\text{PTGGE},k}\) defined in Eq. (175) with the aid of Eqs. (21) and (50). In the latter, we impose the thermodynamic limit \(L \to \infty\) through

\[
\gamma_l = \frac{i}{L} \sum_{k \in \text{BZ}} e^{i\delta_l} \gamma_k \sim i \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{i\delta_l} \gamma_k.
\]

(242)
3. Space-time scaling limit

As pointed out in Sec. I.E, for the isolated Kitaev chain, the subsystem fermion parity can be mapped to ordered parameter correlations in the transverse field Ising model through the combination of a Jordan-Wigner transformation and the Kramers-Wannier duality. The relaxation of order parameter correlations after a quantum quench is studied in Refs. [9–11]. In particular, for a quench to the topologically trivial phase of the isolated Kitaev chain with |μ| > 2J, which corresponds to a quench to the ferromagnetic phase of the Ising model, Eq. (19) of Ref. [10] yields the following prediction for the behavior of the subsystem parity in the space-time scaling limit ℓ, t → ∞ with ℓ/t fixed:

\[ \langle P_\ell(t) \rangle \sim P_0 e^{\int_0^t \frac{d\gamma}{\mu} \min(2|\nu_{\ell,\ell}| \ln|\cos(\Delta\theta_\ell)|)}, \]

(243)

where \( \nu_{\ell} = d\varepsilon_{\ell}/dk \). The first step to obtain our conjecture in Eq. (9) in the main text for the subsystem parity in the driven-dissipative Kitaev chain is to rewrite the above equation in terms of the covariance matrix in the complex basis:

\[ \ln|\cos(\Delta\theta_\ell)| = \frac{1}{2} \left( \ln|\cos(\Delta\theta_\ell)| + \ln|\cos(\Delta\theta_{\ell,-})| \right) \]

\[ = \frac{1}{2} \left( \ln(\langle |d^-|, d^+_0 \rangle_\ell ||\langle |d^-|, d^+_0 \rangle_\ell \rangle) + \ln(\langle |d^-|, d^+_0 \rangle_\ell ||\langle |d^-|, d^+_0 \rangle_\ell \rangle) \right) \]

\[ = \frac{1}{2} \text{tr} \left( \ln(g_{\text{GGGE,}\ell}) \right), \]

(244)

where by taking the limit \( \gamma \to 0 \) in Eqs. (231), (232), (233), and (234), we find that \( g_{\text{GGGE,}\ell} \) is given by

\[ g_{\text{GGGE,}\ell} = U_{x,0} \begin{pmatrix} \langle |d^-|, d^+_0 \rangle_\ell & 0 \\ 0 & \langle |d^-|, d^+_0 \rangle_\ell \end{pmatrix} U_{x,0}^\dagger. \]

(245)

Then, Eq. (243) can be written as

\[ \langle P_\ell(t) \rangle \sim P_0 e^{\int_0^t \frac{d\gamma}{\mu} \min(2|\nu_{\ell,\ell}| \ln|\cos(\Delta\theta_\ell)|)}, \]

(246)

To generalize this relation to the driven-dissipative Kitaev chain, we note that as in Eq. (241), the exponential decay of \( g_{\text{PTGGE,}\ell}(t) \) in Eq. (234) leads to the appearance of a prefactor \( e^{-4\gamma t} \). Further, we set \( \nu_{\ell} = d\varepsilon_{\ell}/dk \), and we replace \( g_{\text{GGGE,}\ell} \) by \( g'_{\text{PTGGE,}\ell} \). Finally, by using the cyclic invariance of the trace and Eq. (219), we obtain

\[ \text{tr} \left( \ln(g'_{\text{PTGGE,}\ell}) \right) = \text{tr} \left( \ln(g_{\text{GGGE,}\ell}^{\dagger}) \right), \]

(247)

and are thus lead to Eq. (9) of the main text. As stated there, the constant prefactor \( P_0 \) can be determined by fitting the late-time limit,

\[ \lim_{t \to \infty} e^{4\gamma t} \langle P_\ell \rangle \sim P_0 e^{\int_0^t \frac{d\gamma}{\mu} \min(2|\nu_{\ell,\ell}| \ln|\cos(\Delta\theta_\ell)|)}, \]

(248)

to the numerical result in Eq. (241).

Next, we consider quenches to the topological phase of the isolated Kitaev chain for |μ| < 2J, corresponding to quenches to the paramagnetic phase of the transverse field Ising model. According to Eq. (32) of Ref. [10], for \( t < t_F \), the oscillatory decay of the subsystem parity in the space-time scaling limit is given by

\[ \langle P_\ell(t) \rangle \sim P_0 \left( 1 + \cos(2\varepsilon_{\ell, t} + \alpha) \right) e^{2t} \left( \frac{d}{d\mu} \ln|\cos(\Delta\theta_\ell)| \right), \]

(249)

where in the isolated system \( k_\ell = k_{\ell,+} = k_{\ell,-} \). To generalize this prediction to the driven-dissipative Kitaev chain, we proceed as above, and, additionally, we note that the oscillatory prefactor is a complete square due to the identity \( 2\cos(x)^2 = 1 + 2\cos(2x) \). Based on this simple observation, we conjecture that the proper generalization of the prefactor reads as follows:

\[ 2\cos(\varepsilon_{\ell, t} + \alpha/2)^2 \to 2\cos(\omega_{\ell, t} + \alpha_) \cos(\omega_{\ell,-} t + \alpha_+), \]

(250)

where, for \( \gamma \to 0 \), we set \( \alpha_+ = \alpha_- = \alpha/2 \). Thereby we obtain Eq. (10) of the main text.

4. Condition to observe relaxation to the PTGGE

The condition for the existence of a finite time window during which relaxation to the PTGGE can be observed depends on the observable under consideration. Here, we focus on the subsystem parity given in Eq. (84), and on quenches that originate from the trivial phase of the isolated Kitaev chain. For \( \delta = 0 \), when \( \Gamma_2(t) = 0 \) in Eq. (100), the behavior of the subsystem parity is given by \( \langle P_\ell(t) \rangle = \text{Pf}(\Gamma_{1,\ell}(t)) \), and, as shown in Figs. 1 and 2 of the main text, \( \langle P_\ell(t) \rangle \) relaxes on a time scale \( t_F \) to the PTGGE prediction given in Eq. (241) [36]. If \( \delta \neq 0 \), for relaxation of \( \langle P_\ell(t) \rangle \) to the PTGGE prediction to be observable at all, the contribution to \( \langle P_\ell(t) \rangle \) due to \( \Gamma_2(t) \neq 0 \) has to be small at \( t = t_F \). Then, for \( t > t_F \), \( \langle P_\ell(t) \rangle \) obeys the PTGGE prediction up to the crossover time scale \( t_c \). To estimate \( t_c \), we expand the Pfaffian to lowest nontrivial order,

\[ \langle P_\ell \rangle = \text{Pf}(\Gamma_{1,\ell} + \Gamma_{2,\ell}) \sim \text{Pf}(\Gamma_{1,\ell}) \left( 1 + \frac{1}{2} \text{tr} \left( \frac{\Gamma_{2,\ell}}{\Gamma_{1,\ell}} \right) \right), \]

(251)

where we omit the explicit time dependence. Therefore, the condition for relaxation to the PTGGE to be observable can be stated as

\[ \frac{1}{2} \left| \text{tr} \left( \frac{\Gamma_{2,\ell}}{\Gamma_{1,\ell}} \right) \right| \ll 1 \quad \text{for } t = t_F. \]

(252)

At \( t = t_F \), we may estimate \( \Gamma_{1,\ell} \) and \( \Gamma_{2,\ell} \) by their “dephased” expressions, i.e., by keeping only nonoscillatory contributions. According to Eq. (167), and with Eqs. (50) and (21), we can write

\[ \Gamma_1(t) \sim \Gamma_{d,1}(t), \quad \Gamma_2(t) \sim \Gamma_{d,2}(t) + \Gamma_{SS}, \]

(253)

where

\[ \Gamma_{d,1}(t) \sim e^{-4\gamma t} \Gamma_{d,1}, \quad \Gamma_{d,2}(t) \sim e^{-4\gamma t} \Gamma_{d,2}. \]

(254)

Then, Eq. (252) becomes

\[ \frac{1}{2} \left| \text{tr} \left( \frac{\Gamma_{d,2,\ell}}{\Gamma_{d,1,\ell}} \right) + e^{4\gamma t} \text{tr} \left( \frac{\Gamma_{SS,\ell}}{\Gamma_{d,1,\ell}} \right) \right| \ll 1. \]

(255)
A sufficient condition for the validity of this inequality can be obtained by using the triangle inequality. We find

\[ \frac{1}{2} \left( \left| \text{tr} \left( \Gamma_{\Delta,d,\ell} \right) \right| + e^{i \rho L} \left| \text{tr} \left( \Gamma_{\Delta,1,\ell} \right) \right| \right) \ll 1, \]  

(256)

which is satisfied if \( t_F \) is small in comparison to the crossover time scale \( t_x \), for which the left-hand side of the above inequality is equal to one,

\[ t_F \ll t_x = \frac{1}{4\gamma} \ln \left( \frac{2 - \left( \text{tr} \left( \Gamma_{\Delta,d,\ell} / \Gamma_{\Delta,1,\ell} \right) \right)}{\text{tr} \left( \Gamma_{\Delta,\ell} / \Gamma_{\Delta,1,\ell} \right)} \right). \]  

(257)

For \( \delta \to 0 \), by taking into account that \( \Gamma_{\Delta,\ell} \) and \( \Gamma_{\Delta,d,\ell} \) vanish linearly in \( \delta \) according to Eqs. (173), (174), and (110), while \( \Gamma_{\Delta,1,\ell} \) is finite, we find

\[ t_x = \frac{1}{4\gamma} \ln(c_x) \]  

(258)

The limit \( \delta \to 0 \) can be taken explicitly in Eq. (174),

\[ \lim_{\delta \to 0} \frac{\gamma_{\Delta,\ell}}{\delta} = -\frac{1}{2\pi} \left( \frac{1}{e_k} \cdot (a_{ik} \times b) \| I \right. 

\[ \left. - \left\{ \hat{e}_y - \left[ e_k \hat{e}_{y_{1\perp,k}} - 2y e_k e_{y,\perp,k} \right] \right\} \cdot \sigma \right), \]  

(259)

where we set \( b \rightarrow -2y \hat{e}_y \), for \( \delta \to 0 \), and where

\[ e_{y,\perp,k} = (\hat{e}_y \cdot \hat{e}_{\ell,k}) \hat{e}_{\ell,k}, \]  

(260)

\[ e_{y,\perp,k} = \hat{e}_y - e_{y,\perp,k}, \]

\[ e_{\ell,k} = -\hat{e}_y \times \hat{e}_{\ell,k}. \]

This allows us to reconstruct \( \lim_{\delta \to 0} \Gamma_{\Delta,\ell} / \delta \) numerically by employing Eqs. (50) and (21). Similarly, we can obtain \( \Gamma_{\Delta,1,\ell} \) from \( \Gamma_{\Delta,1,\ell} \) and, thereby, calculate \( c_x \). This leads to the semi-analytical estimate in the inset of Fig. 2.

5. Oscillatory dynamics from zero modes

According to Eq. (233), soft modes of the PTGGE are determined by the condition

\[ \cos(\Delta \theta_k - \phi_k) = 0. \]  

(261)

For \( \phi_k = 0 \), due to \( \Delta \theta_k = -\Delta \theta_{-k} \), which according to Eq. (70) is a consequence of IS, the existence of a soft mode \( k_{s,+} > 0 \) implies the existence of another inversion conjugate soft mode \( k_{s,-} = -k_{s,+} \). Due to \( e_k = e_{-k} \), the associated oscillation periods \( t_{x,s} = \pi/(2e_{k_{s,+}}) \) are identical. Interestingly, IS as stated in Eq. (132) implies, via Eqs. (155) and (216), that \( \phi_k = \phi_{-k} \). However, the sum \( \Delta \theta_k - \phi_k \) that appears in Eq. (261) does not have definite transformation behavior under inversion, just as the sums and differences of \( \theta_k \) and \( \phi_k \) that appear in the Liouvillean eigenmode transformation matrix \( \hat{V}_k \) in Eq. (214). Consequently, the solutions \( k_{s,\pm} \) of Eq. (261) are not related by inversion. In particular, for quenches with pre- and post-quench parameters given by, respectively, \( J_0 = \Delta_0 \) and \( \mu_0 \), and \( J = \Delta \) and \( \mu \), we find

\[ k_{s,\pm} = \mp \text{sgn}(\mu) \arccos \left( \frac{-1}{2(J_0 \mu + J_0 \mu^2)} \right) \times \left\{ \gamma \gamma_{\xi} \left[ 4J_0^2 \mu^2 - J_0^2 \left( 4J_0^2 + 2 \gamma \gamma_{\xi} \right) \mu^2_0 + J_0^2 \mu_0^2 \right] \right\}. \]  

(262)

In the limit \( \mu_0 \to -\infty \) as in the examples considered in the main text, the above general form reduces to

\[ k_{s,\pm} = \mp \text{sgn}(\mu) \arccos \left( \frac{-1/2}{\mu \mp \text{sgn}(\mu) \gamma \gamma_{\xi}} \right). \]  

(263)

6. Necessary conditions for \( t_{s,+} \neq t_{s,-} \)

As claimed in the main text, different periods of parity pumping for subsystems at the left and right ends of the chain require both mixedness of the time-evolved state and breaking of inversion symmetry and are, therefore, unique to driven-dissipative systems. To see that this is the case, consider, for simplicity, an equal bipartition into subsystems \( L_{L/L_2} = \{1, \ldots, L/2 \} \) and \( R_{L/L_2} = \{L/2 + 1, \ldots, L \} \), corresponding to the left and right halves of the chain. Then, for a pure state, the eigenvalues of \( \Gamma_L \) and \( \Gamma_R \), which determine the entanglement spectra for subsystems \( L_{L/L_2} \) and \( R_{L/L_2} \), are identical [37]. Consequently, due to \( \text{det}({\Gamma}_{\ell}) = P{\Gamma}_{\ell}^2 = (P_{\Gamma})^2 \) zero crossings of \( \langle P_L \rangle \) and \( \langle P_R \rangle \) occur at the same time. In contrast, for a mixed state, the spectra of \( \Gamma_L \) and \( \Gamma_R \) are, in general, different. However, if the state is symmetric under inversion, with inversion center in the middle of the chain, the matrices \( \Gamma_L \) and \( \Gamma_R \) are, as we show below, unitarily equivalent. Consequently, even for mixed states, the entanglement spectra and the subsystem parities, given by the Pfaffians, are the same. Therefore, as stated above, both mixedness of the state and inversion symmetry breaking are necessary for \( t_{s,+} \neq t_{s,-} \).

To show that \( \Gamma_L \) and \( \Gamma_R \) are unitarily equivalent for an inversion-symmetric state, we first note that Eq. (59) for open boundary conditions implies the following transformation behavior of the spinors of Majorana operators defined in Eq. (9):

\[ IW_{L}^\dagger = i\sigma_y W_{L+1-\dagger}. \]  

(264)

Therefore, with Eq. (21), we find that the \( 2 \times 2 \) blocks of the covariance matrix of an inversion-symmetric state \( \rho = I \rho^\dagger \) obey the condition

\[ \gamma = \sigma_y \gamma_{L+1-\dagger}. \]  

(265)

which implies that the full covariance matrix behaves as

\[ \Gamma = \Sigma_{x,t} \Xi_{L} \Sigma_{x,t} \Sigma_{x,\ell}, \]  

(266)

where \( \Sigma_{x,\ell} \) and \( \Xi_{\ell} \) are defined in Eqs. (142) and (143), respectively. The product \( \Sigma_{x,\ell} \Xi_{\ell} \) can be seen to be unitary by noting
that both $\Sigma_y,\ell$ and $\Xi_\ell$ are Hermitian and involutory. Therefore, the restriction of Eq. (266) to the left half of the chain,

$$
\Gamma_L = \Sigma_{y,1/2} \Xi_{1/2} \Gamma_R \Xi_{1/2} \Sigma_{y,1/2},
$$

(267)

confirms the claimed unitary equivalence between $\Gamma_L$ and $\Gamma_R$.

D. Subsystem entropy

Figure 4 in the main text shows the quasiparticle-pair contribution to the subsystem entropy, which is defined as

$$
S_{\text{QP}}^{\Sigma_{\ell}} = S_{\Sigma_{\ell}} - \frac{1}{L} S_{\text{stat}}.
$$

(268)

In the following, we describe how to calculate the subsystem entropy $S_{\Sigma_{\ell}}$ and the statistical entropy $S_{\text{stat}} = S_{\Sigma_{\ell,L}}$, as well as the PTGGE prediction that captures the late-time behavior of $S_{\Sigma_{\ell}}$. Further, we present a short derivation that motivates our conjecture for the space-time scaling limit of $S_{\text{QP}}^{\Sigma_{\ell}}$ given in Eq. (11) in the main text. In this section, we restrict ourselves to the case $\delta = 0$.

1. Subsystem entropy from the covariance matrix

The von Neumann entropy of a subsystem that consists of $\ell$ contiguous lattice sites is given by [38]

$$
S_{\Sigma_{\ell}} = -\text{tr}(\rho_{\Sigma_{\ell}} \ln(\rho_{\Sigma_{\ell}})) = \sum_{i=1}^{\ell} S_{\ell}(\xi_i),
$$

(269)

where

$$
S_{\ell}(\xi) = -\frac{1 + \xi}{2} \ln\left(\frac{1 + \xi}{2}\right) - \frac{1 - \xi}{2} \ln\left(\frac{1 - \xi}{2}\right),
$$

(270)

and where $\pm i\xi_i$ with $0 \leq \xi_i \leq 1$ and $l \in \{1, \ldots, \ell\}$ are the eigenvalues of the reduced covariance matrix $\Gamma_{\ell}$ defined in Eq. (85). When $\delta = 0$, the time-evolved covariance matrix in Eq. (100) reduces to $\Gamma(t) = \Gamma_{\ell}(t)$, which, according to Eq. (156), is exponentially small for $\gamma t \gg 1$. Then, to calculate the subsystem entropy Eq. (269), we can restrict ourselves to the lowest nontrivial order in the expansion of $S_{\ell}(\xi)$ in powers of $\xi$,

$$
S_{\ell}(\xi) = \ln(2) - \xi^2/2 + O(\xi^4).
$$

(271)

In particular, from the rescaled covariance matrix defined in Eq. (238), we obtain the spectrum of the reduced rescaled covariance matrix:

$$
\sigma(\Gamma_{\ell}^r(t)) = \{\pm i\xi_i^r(t)\} = \{\pm e^{i\gamma t} \xi_i(t)\}.
$$

(272)

For $\gamma t \gg 1$, the subsystem entropy can be written in terms of the eigenvalues $\xi_i(t)$ as

$$
S_{\Sigma_{\ell}}(t) = \ell \ln(2) - \frac{\ell}{2} \sum_{i=1}^{\ell} \xi_i^2(t) = \ell \ln(2) - \frac{e^{-\gamma t}}{2} \sum_{i=1}^{\ell} \xi_i^2(t).
$$

(273)

2. Statistical entropy

Rather than calculating the statistical entropy $S_{\text{stat}}^{\Sigma_{\ell}} = S_{\Sigma_{\ell,L}}$ numerically for a finite system, we perform an analytical calculation in the thermodynamic limit as detailed in the following: The covariance matrix is block-diagonal in momentum space, with $4 \times 4$ blocks $y_k$ given in Eq. (178). According to Eqs. (180) and (118), the spectrum of $y_k$ is given by $\sigma(y_k) = \sigma(y) \cup \sigma(y_{-k}) = \{\pm \xi_{-k}, \pm \xi_{-k}\}$. Therefore, in the thermodynamic limit,

$$
S_{\text{stat}}^{\Sigma_{\ell}}(t) = \sum_{k \in \mathbb{Z}} \sum_{\sigma = \pm} \mathcal{S}(\xi_{\sigma,k}(t)) \rightarrow L \int_0^\infty \frac{dk}{2\pi} \sum_{\sigma = \pm} \mathcal{S}(\xi_{\sigma,k}(t)).
$$

(274)

The eigenvalues $\xi_{\pm,k}$ are given in Eq. (117), where, for $\delta = 0$, $\gamma_{1,\pm,k}(t) = \gamma_{1,1,k}(t)$ is given in Eq. (157), and $\xi_{\ell}(t) = |\gamma_{1,\ell}(t)|$ can be obtained by using Eq. (A.16), which yields

$$
\xi_{\ell}(t) = e^{-\gamma t} \left(1 - \frac{\xi_k^2}{\omega_k^2} \sin(\Delta \theta_k t)^2 \left[2(1 - \cos(2\omega_k t)) \right] \right)^{1/2},
$$

(275)

where $\Delta \theta_k$ is defined in Eq. (171). At late times, when $\gamma t \gg 1$, the eigenvalues $\xi_{\pm,k}(t) \sim e^{-\gamma t}$ are exponentially small, and we can use the expansion in Eq. (271) to simplify Eq. (274). To make the exponential time dependence explicit, we define the rescaled quantities

$$
\gamma_{1,\pm,k}^r(t) = e^{i\gamma t} \gamma_{1,\pm,k}(t), \quad \xi_{\ell}^r(t) = e^{i\gamma t} \xi_{\ell}(t),
$$

(276)

in terms of which we obtain the statistical entropy in the form

$$
S_{\text{stat}}^{\Sigma_{\ell}}(t) \sim L \ln(2) - e^{-\gamma t} \int_0^\infty \frac{dk}{2\pi} \left(\gamma_{1,\ell}^r(t)^2 + \xi_{\ell}^2(t)^2\right).
$$

(277)

3. Numerical time evolution

At short times $\gamma t \ll 1$, we calculate the covariance matrix for a finite system numerically according to Eq. (100), and we use Eqs. (269) and (274) to obtain the data shown in Fig. 4 in the main text. For longer times $\gamma t \gg 1$, a numerically stable procedure that avoids calculations with extremely small quantities is to take the difference between Eqs. (273) and (277), which yields

$$
S_{\text{QP}}^{\Sigma_{\ell}}(t) \sim -e^{-\gamma t} \left[\frac{1}{2} \sum_{\ell=1}^{\ell} \xi_i^2(t) - \ell \int_0^\infty \frac{dk}{2\pi} \left(\gamma_{1,\ell}^r(t)^2 + \xi_{\ell}^2(t)^2\right)\right],
$$

(278)

where the exponential time dependence is contained in the prefactor that is determined analytically.

4. Late-time behavior

The late-time limit that is described by the PTGGE can be obtained from Eq. (278) by properly taking into account dephasing of oscillatory contributions. Namely, $\xi_i(t)$ should be
replaced by $\xi_{d,l}$, where $\pm i \xi_{d,l}$ are the eigenvalues of $\Gamma_{PTGGE}$ defined in Eq. (240). In contrast, in the terms that describe the statistical entropy, oscillatory terms have to be dropped only after taking the squares of $\gamma_{1,k}^2(t)$ and $\xi_k^2(t)$,

$$S_{\text{vN},t}^{\text{OP}}(t) \sim -e^{-8\pi t} \left[ \frac{1}{2} \sum_{j=1}^\ell \left( \xi_{d,l}^2 - \ell \int_0^\infty \frac{dk}{2\pi} \left( \gamma_{1,k}^2 + \xi_k^2 \right) \right) \right]_d,$$  

(279)

where, according to Eq. (157),

$$\left( \gamma_{1,k}^2 \right)_d = \frac{3}{2\omega_k} \left[ \hat{a}_{1,k} \cdot (\mathbf{a}_k \times \mathbf{b}) \right]^2,$$  

(280)

and, with Eq. (275),

$$\left( \xi_k^2 \right)_d = 1 + \frac{6\xi_k^2}{\omega_k^2} \sin(\Delta \theta_k)^2.$$  

(281)

5. Space-time scaling limit

The quasiparticle picture [38–40] suggests the following functional form for the time dependence of the subsystem entropy after a quench in an isolated integrable system:

$$S_{\text{vN},t}(t) \sim \int_0^\infty \frac{dk}{2\pi} \min(2|v_k|\ell, \ell) s_k,$$  

(282)

where $v_k$ is the quasiparticle velocity, and the weight $s_k$ determines the contribution of quasiparticles with momentum $k$ to the entropy. The above form holds in the space-time scaling limit, $\ell, t \to \infty$ with $\ell/t$ fixed, and does not capture $O(1)$ contributions such as area-law entanglement of the initial state. To provide a quantitative description of the evolution of the subsystem entropy for a specific model, concrete expressions for $v_k$ and $s_k$ are required. According to Ref. [39], the weight $s_k$ can be fixed to reproduce the stationary value of the entropy that is reached for $t \to \infty$. For free fermionic models, this leads to

$$s_k = -\langle n_k \rangle_0 \ln(\langle n_k \rangle_0) - (1 - \langle n_k \rangle_0) \ln(1 - \langle n_k \rangle_0),$$  

(283)

where the occupation numbers $\langle n_k \rangle_0 = \langle \phi_0 \vert d_k^\dagger d_k \vert \phi_0 \rangle$ of eigenmodes of the postquench Hamiltonian are determined by the initial state. Further, $v_k$ is given by the group velocity $v_k = d\epsilon_k/dk$, where $\epsilon_k$ is the dispersion relation of quasiparticles.

To derive Eq. (11) of the main text, which we conjecture to generalize Eq. (282) to PT-symmetric and potentially strongly dissipative open systems, we first rewrite Eq. (282) in terms of the covariance matrix in the complex basis. Under the assumption that $|v_k| = 1$, which is satisfied for the Kitaev chain, the range of integration Eq. (282) can be restricted as

$$S_{\text{vN},t}(t) \sim \int_0^\infty \frac{dk}{2\pi} \min(2|v_k|\ell, \ell) (s_k + s_{-k}).$$  

(284)

Next, by noting that, for $S(\xi)$ given in Eq. (270),

$$s_k = S(2\langle n_k \rangle_0 - 1),$$  

(285)

and with $2n_k - 1 = -[d_k, d_k^\dagger]$ and $S(\xi) = S(-\xi)$, we obtain

$$s_k + s_{-k} = \text{tr}(S(g_{\text{GGE},k})), \quad (286)$$

where $g_{\text{GGE},k}$ is defined in Eq. (245), and we use the cyclic invariance of the trace. We are thus led to the desired expression for the subsystem entropy in the space-time scaling limit in terms of the complex covariance matrix:

$$S_{\text{vN},t} \sim \int_0^\infty \frac{dk}{2\pi} \min(2|v_k|\ell, \ell) \text{tr}(S(g_{\text{GGE},k})).$$  

(287)

Further, in analogy to our approach for the subsystem parity in Sec. III C 3, to account for modifications of quasiparticle dynamics and statistics due to strong dissipation, we set $v_k = do_k/dk$ and replace $g_{\text{GGE},k}$ by $g_{\text{PTGGE},k}(t)$ given in Eq. (234). In particular, to show explicitly the effect of modified quasiparticle statistics, which are described by the anti-commutation relations in Eq. (219), we use the cyclic invariance of the trace to write

$$\text{tr}(S(g_{\text{PTGGE},k}(t))) = \text{tr}(S(\xi_k(t) f_k^{-1})).$$  

(288)

Finally, we subtract the statistical entropy to single out the contribution due the propagation of pairs of entangled quasiparticles [41–43]. To that end, we rewrite the statistical entropy in Eq. (274) by using $S(\xi) = S(-\xi)$ as

$$\sum_{\sigma=\pm} S(\xi_{\sigma,k}(t)) = \text{tr}(S(\gamma_k(t))) = \text{tr}(S(g_k(t))).$$  

(289)

In the space-time scaling limit, oscillatory contributions dephase in the integration over momenta, and, therefore, we have to keep only the nonoscillatory part, which we denote by $\text{tr}(S(g_k(t))_d)$. Then, following Ref. [42], we obtain the quasiparticle-pair contribution to the subsystem entropy in the form given in Eq. (11) in the main text. For $\gamma t \gg 1$, we can simplify Eq. (11) by using the expansion of $S(\xi)$ given in Eq. (271). First, for the subsystem entropy in Eq. (287), and with Eq. (232), we obtain

$$\text{tr}(S(\xi_k(t) f_k^{-1})) \sim \ln(2) - \frac{e^{-8\pi t}}{2} \text{tr}\left(\left(\xi_k^{-1} f_k^{-1}\right)^2\right).$$  

(290)

Further, for the statistical entropy, as in Eq. (279), we find

$$\text{tr}(S(g_k(t))_d) \sim \ln(2) - e^{-8\pi t} \left(\gamma_{1,k}^2 + \xi_k^2\right)_d.$$  

(291)

By combining these results, we lead to

$$S_{\text{vN},t}^{\text{OP}}(t) \sim -e^{-8\pi t} \int_0^\infty \frac{dk}{2\pi} \min(2|v_k|\ell, \ell) \times \left\{ \frac{1}{2} \text{tr}\left(\left(\xi_k^{-1} f_k^{-1}\right)^2\right) - \left(\gamma_{1,k}^2 + \xi_k^2\right)_d \right\}. \quad (292)$$

This is the form used to obtain the solid lines in Fig. 4 of the main text for $\ell/t_F \geq 0.5$. 


IV. UNIVERSALITY OF PT-SYMMETRIC RELAXATION

Generalized thermalization, i.e., relaxation to the GGE, is expected to occur universally in the quench dynamics of isolated integrable systems. Is relaxation to the PTGGE, which we have discussed so far for the specific example of a driven-dissipative Kitaev chain, an equally universal property for PT-symmetric and integrable driven-dissipative systems? In the following, we address this question by studying the quench dynamics of a selection of PT-symmetric models. We demonstrate relaxation to the PTGGE for two broad classes of noninteracting fermionic many-body systems, and—for a subset of observables—also for an interacting spin chain that can be mapped to fermions with quadratic jump operators. For noninteracting bosons, we show that the relaxation dynamics is captured by a dephased ensemble that generalizes the PTGGE for fermions while maintaining the key property of conserving an extensive amount of information about the initial state. Crucially, the same mechanism—dephasing within a subspace that is spanned by eigenmodes of the Liouvillian with equal decay rates—underlies relaxation to both the PTGGE for fermions and the dephased ensemble for bosons. Therefore, while the PTGGE in the form presented in the main text is not as general for driven-dissipative integrable systems as the GGE for isolated integrable systems, our results indicate that the underlying mechanism is indeed very general.

Before we enter the detailed discussion of other models, let us point out that relaxation to the PTGGE as described in the main text is not tied to the particular form of the Kitaev chain and the jump operators discussed there. Instead, our specific choice of Hamiltonian and jump operators is to be understood as one particular representative of a broad class of models. To see why this is the case, we recall the necessary conditions for relaxation to the PTGGE of the driven-dissipative Kitaev chain: (i) PT symmetry of the Liouvillian; (ii) the existence of a fully PT-symmetric phase; (iii) for relaxation to the PTGGE to persist to arbitrary times (for simplicity, we restrict our discussion of other models to this case), the steady state must be an infinite-temperature state. But these requirements leave considerable freedom to modify the model. On the one hand, this concerns the Hamiltonian: For example, relaxation to the PTGGE persists for long-range hopping and pairing [44]. On the other hand, also the jump operators can be modified: An infinite-temperature steady state is reached whenever the jump operators are Hermitian, i.e., when they take the form given in Eq. (91) with real coefficients $B_{l,l'}$. However, PT symmetry poses restrictions on the coefficients $B_{l,l'}$. For example, for purely local dissipation, the choices $L_l = \sqrt{\gamma_{2l-1}}$ and $L_l = \sqrt{\gamma_{2l+1}}$ are compatible with PT symmetry. Moreover, for both choices, a fully PT-symmetric phase exists up to a finite critical value of $\gamma$. These considerations extend straightforwardly to systems in higher dimensions.

A. Fermionic SSH model with loss and gain

Noninteracting fermionic systems can broadly be classified into models with and without pairing terms in the Hamiltonian, and the Kitaev chain, on which our discussion has focused so far, is a particular representative of the former class. In the following, we consider the SSH model [45, 46] as a paradigmatic example of a one-dimensional tight-binding chain without pairing. We present a short summary of results that demonstrate relaxation to the PTGGE. A more detailed discussion will be provided elsewhere [44].

The SSH model describes hopping along a one-dimensional bipartite lattice with sublattices $A$ and $B$. For a system that consists of $L$ unit cells, the Hamiltonian reads

$$H_{SSH} = \sum_{l=1}^{L} \left( J_1 c_{A,l}^\dagger c_{B,l} + J_2 c_{A,l+1}^\dagger c_{B,l} + \text{H.c.} \right),$$

(293)

where $J_1, J_2 \in \mathbb{R}$. We find it convenient to parameterize $J_1$ and $J_2$ in terms of the total and relative hopping amplitudes,

$$J = \frac{1}{2} (J_1 + J_2), \quad \Delta J = \frac{1}{2} (J_1 - J_2).$$

(294)

The topologically nontrivial phase of the SSH model is realized for $\Delta J < 0$ [5–7].

Isolated quadratic fermionic systems without pairing terms in the Hamiltonian do not exhibit anomalous correlations, and, therefore, their dynamics is captured by a closed equation of motion for the covariance matrix which is defined as

$$G_{l,l'}^{s,s'} = \langle [c_{s,l}, c_{s',l'}]^\dagger \rangle.$$  

(295)

For such systems, a natural choice of dissipation which results again in a closed equation of motion for the covariance matrix and can lead to an infinite-temperature steady state, is loss and gain with jump operators

$$L_{l,s,l} = \sqrt{\gamma_{s,l}} c_{s,l}, \quad L_{g,s,l} = \sqrt{\gamma_{s,l}} c_{s,l}^\dagger,$$

(296)

where $\gamma_{s,l}$ and $\gamma_{g,s,l}$ are the loss and gain rates on sublattice $s \in \{A, B\}$, respectively. In particular, the steady state is at infinite temperature if the loss and gain rates are equal, $\gamma_{s,l} = \gamma_{g,s,l}$ [44]. Focusing on this case, we set $\gamma_{A} = \gamma_{g,A} = \gamma + \Delta \gamma$, and $\gamma_{B} = \gamma_{g,B} = \gamma - \Delta \gamma$. The time evolution of the density matrix is then given by a master equation in Lindblad form,

$$\frac{d\rho}{dt} = [H_{SSH}, \rho] + \frac{i}{\hbar} (D_l \rho + D_g \rho),$$

(297)

with dissipators $D_l$ and $D_g$ that describe loss and gain, respectively. To state the resulting equation for the covariance matrix, it is convenient to introduce 2$L$ fermionic operators $c_l$ such that $c_{A,l} = c_{2l-1}$ and $c_{B,l} = c_{2l}$, and to write the Hamiltonian in the form

$$H_{SSH} = \sum_{l,f=1}^{2L} c_{l,f}^\dagger H_{l,f} c_{l,f},$$

(298)

and the jump operators as

$$L_{d,f} = \sum_{l=1}^{2L} B_{l,f} c_{l,f}, \quad L_{g,f} = \sum_{l=1}^{2L} B_{g,f} c_{l,f}^\dagger.$$  

(299)
In terms of the coefficient matrices \( B_I \) and \( B_g \), we define the bath matrices

\[
M_I = B_I^*B_I, \quad M_g = B_g^*B_g^\dagger.
\]  

(300)

Finally, we write the covariance matrix as

\[
G_{l,f} = \langle [c_l, c_{f}^\dagger] \rangle.
\]  

(301)

Then, the equation of motion for the covariance matrix reads

\[
\frac{dG}{dt} = -i \left( ZG - GZ^\dagger \right) + 2 \left( M_I - M_g \right).
\]  

(302)

where \( Z = H - i \left( M_I + M_g \right) \). The matrix \( Z \) has PT symmetry, and a fully PT-symmetric phase is realized for sufficiently small values of \( \gamma \) and \( \Delta \gamma \) [44]. For equal loss and gain rates, also the bath matrices are identical, \( M_I = M_g \), and the inhomogeneity in Eq. (302) vanishes. Consequently, the covariance matrix decays to zero for \( t \to \infty \), which reflects the fact that the steady state has infinite temperature.

In Fig. 1, we illustrate relaxation to the PTGGE with the time evolution of the quasiparticle-pair contribution to the subsystem entropy following a quench from the ground state with \( \Delta J_0 = J \). We consider a subsystem that consist of \( \ell \) unit cells. As shown in the inset of the figure, the numerical data for the quasiparticle-pair entropy converges with subsystem size as \( 1/\ell \) to our analytical prediction. Details of the underlying analytical and numerical computations will be presented in a forthcoming publication [44].

Finally, let us emphasize that, as we have already discussed for the Kitaev chain, relaxation to the PTGGE does not require the specific forms of the Hamiltonian and the jump operators given in Eqs. (293) and (296), respectively. Instead, both can be modified as long as the requirements (i)-(iii) stated above are satisfied. Therefore, the behavior shown in Fig. 1 is representative for a whole class of models.

**B. Bosonic SSH model with loss and gain**

Next, we consider a bosonic version of the SSH model with loss and gain. As anticipated above, the PT-symmetric relaxation dynamics of noninteracting bosons is described by a dephased ensemble that differs from the PTGGE for fermions. The nature and origin of the modifications for bosons can be understood as follows: The Gaussian states which describe the noninteracting systems we consider are fully determined by two-point correlations. However, the decomposition of a two-point function such as \( \langle c_I c_f^\dagger \rangle \) into correlation- and statistical contributions differs for fermions and bosons. In both cases we can write \( c_I c_f^\dagger = \delta_{l,f} + \text{correlation} \), and correlations in the expectation value of the commutator which, therefore, features in the covariance matrix Eq. (301).

For the fermionic SSH model, we have chosen equal loss and gain rates such that the steady state is the infinite-temperature state, \( \rho_{SS} = \rho_{\infty} = 1/2L \). Therefore, taking the expectation value of any operator in the steady state reduces to taking the trace of that operator. But the trace of a commutator vanishes, and, consequently, the covariance matrix vanishes in the steady state, \( \langle [c_I, c_f^\dagger] \rangle_{SS} = \text{tr} \left( [c_I, c_f^\dagger] \right) = 0 \). In the PT-symmetric phase, the decay of the covariance matrix is described by a single exponential factor, \( G(t) = e^{-\gamma t}G'(t) \), and dephasing among PT-symmetric modes induces relaxation of \( G'(t) \) according to \( G'(t) \to G'_{\text{PTGGE}} \).

In contrast, for bosons, statistics are contained in the commutator, \( \{a_l, a_f^\dagger\} = \delta_{l,f} \), and correlations are described by the covariance matrix defined as the expectation value of the anticommutator,

\[
G_{l,f} = \langle [a_l, a_f^\dagger] \rangle.
\]  

(303)

However, the anticommutator \( \{a_l, a_f^\dagger\} = 1 + 2a_l^\dagger a_l \) is positive-definite, and its expectation value is generically nonzero. This leads to a covariance matrix of the general form

\[
G(t) = e^{-2\gamma t}G'(t) + G_{SS}.
\]  

(304)

with a nonvanishing steady-state contribution \( G_{SS} \). While the fact that this constant term is nonzero for any choice of dissipation represents a fundamental difference to fermionic systems, the structure of the time-dependent term is the same for both fermions and bosons: PT symmetry leads to decay with a single exponential factor, and, also for bosons, relaxation of \( G'(t) \to G'_{\text{PTGGE}} \) is induced by dephasing. Therefore, the late-time relaxation dynamics of a PT-symmetric bosonic system is described by a dephased ensemble,

\[
G(t) \sim G_d(t) = e^{-2\gamma t}G_d' + G_{SS}.
\]  

(305)

In analogy to the fermionic PTGGE, information about the initial state is contained in the bosonic dephased ensemble in the term \( G_d' \). Additionally, information about the final state enters through both \( G_d' \) and \( G_{SS} \).

To illustrate relaxation to this dephased ensemble, we study quench dynamics of a bosonic SSH model that is described by
the Hamiltonian
\[
H_{\text{BSSH}} = \sum_{l=1}^{L} \left( J_{1} a_{A,l}^\dagger a_{B,l} + J_{2} a_{A,l+1}^\dagger a_{B,l} + \text{H.c.} \right),
\]
(306)
where \( a_{s,l} \) and \( a_{s,l}^\dagger \) with \( s \in \{ A, B \} \) are bosonic annihilation and creation operators, respectively, and obey the canonical commutation relations \( [a_{s,l}, a_{s',l'}^\dagger] = \delta_{s,s'} \delta_{l,l'} \) and \( [a_{s,l}, a_{s',l'}] = 0 \). As in the fermionic case, we consider local loss and gain, described by the jump operators
\[
L_{s,l} = \sqrt{\gamma_{s,l}} a_{s,l}, \quad L_{s,l} = \sqrt{\gamma_{s,l}^*} a_{s,l}^\dagger.
\]
(307)
For fermions, we have focused exclusively on the case of equal loss and gain rates, \( \gamma_{l,s} = \gamma_{g,s} \). Here we consider the more general case of unequal loss and gain rates on both sublattices, \( \gamma_{l,s,A} = \gamma_{l,s} + \Delta \gamma_{l,s} \) and \( \gamma_{l,s,B} = \gamma_{l,s} - \Delta \gamma_{l,s} \). We find it convenient to express the loss and gain rates in terms of the four parameters \( \gamma, \delta, \Delta \gamma, \) and \( \Delta \), which are defined as follows:
\[
\gamma = \frac{\gamma_{l,s} + \gamma_{g,s}}{2}, \quad \delta = \gamma_{l,s} - \gamma_{g,s}
\]
(308)
\[
\Delta \gamma = \frac{\Delta \gamma_{l,s} + \Delta \gamma_{g,s}}{2}, \quad \Delta = \Delta \gamma_{l,s} - \Delta \gamma_{g,s}.
\]
(309)
The equation of motion for the bosonic covariance matrix defined in Eq. (303) reads [44]
\[
d\rho = -i \left( ZG - GZ^\dagger \right) + 2 \left( M_{1} + M_{2} \right),
\]
with \( Z = H - i \left( M_{1} - M_{2} \right) \) and where the Hamiltonian matrix \( H \) and the bath matrices \( M_{1} \) and \( M_{2} \) are defined in Eqs. (298) and (300), respectively. The only but important difference from the fermionic version in Eq. (302) is the sign of the gain matrix \( M_{2} \). Due to this sign change, the inhomogeneity in Eq. (309), which is the sum of two positive-semidefinite matrices, is always nonzero for nonvanishing dissipation. In particular, it is nonzero for equal loss and gain rates, \( \delta = \Delta = 0 \), when the inhomogeneity in the fermionic equation of motion (302) vanishes. This reflects the fact that, as explained above, the covariance matrix is generically nonzero in the steady state. Nevertheless, in the fully PT-symmetric phase, which is realized for sufficiently small values of loss and gain rates, all eigenvalues \( \lambda_{i} \) of \( Z \) have constant imaginary part, which leads to the general form of the covariance matrix given in Eq. (304). Dephasing among modes with different real parts \( \text{Re}(\lambda_{i}) \) induces relaxation to an ensemble that is determined by \( G_{\ell}(t) \) defined in Eq. (305). As we proceed to show, relaxation of \( G'(t) \rightarrow G'_{\ell} \) is clearly visible in quasiparticle-pair contribution to the subsystem entropy.

We thus consider quench dynamics and choose a mixed initial state \( \rho_{0} \) as follows: For \( J_{2} = 0 \), the Hamiltonian in Eq. (306) is diagonal when it is expressed in terms of bosonic modes \( b_{s,l} \) that are defined as
\[
\begin{pmatrix}
[b_{A,l}^\dagger] \\
[b_{B,l}^\dagger]
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_{A,l}^\dagger \\ a_{B,l}^\dagger \end{pmatrix},
\]
(310)
where the mixed initial state \( \rho_{0} \) is determined by two parameters \( \mu_{A} \) and \( \mu_{B} \),
\[
\rho_{0} = \frac{1}{Z_{0}} e^{-\sum_{\ell,A} \mu_{A} b_{A,\ell}^\dagger b_{A,\ell}} e^{-\sum_{\ell,B} \mu_{B} b_{B,\ell}^\dagger b_{B,\ell}}.
\]
(311)
where the normalization \( Z_{0} \) ensures that \( \text{tr}(\rho_{0}) = 1 \).

Further, the loss and gain rates are chosen so as to lead to a particularly simple steady state. The steady-state occupation of a single bosonic lattice site is determined by the ratio between loss and gain rates. Therefore, to achieve a spatially uniform distribution of particles, we choose equal ratios on both sublattices, \( \gamma_{l,s} / \gamma_{g,s} = \gamma_{l,s} / \gamma_{g,s} \) or, equivalently, \( \gamma / \Delta \gamma = \delta / \Delta \). The resulting uniform distribution is also stationary under unitary evolution generated by the SSH Hamiltonian in Eq. (306). The covariance in the steady state is thus given by \( G_{SS} = (\delta / \gamma) I \).

As for the fermionic case, the entanglement entropy can be calculated from the covariance matrix [47, 48]. In particular, for a system without anomalous correlations, the von Neumann entropy of a subsystem that consists of \( \ell \) unit cells is given by
\[
S_{\times N,\ell} = \sum_{i=1}^{2\ell} \xi_{i} S(\xi_{i}),
\]
(312)
where \( \xi_{i} \geq 1 \) are the \( 2\ell \) eigenvalues of the covariance matrix \( G_{\ell} \) restricted to a subsystem that consists of \( \ell \) unit cells, and
\[
S(\xi) = \frac{1 + \xi}{2} \ln \left( \frac{1 + \xi}{2} \right) - \frac{1 - \xi}{2} \ln \left( \frac{1 - \xi}{2} \right).
\]
(313)
The function \( S(\xi) \) for bosons differs from the fermionic version given in Eq. (270) only by the sign of the first term. A procedure similar to that described in Sec. III D for fermions leads to the following result for the late-time asymptotics of the quasiparticle-pair contribution to the subsystem entropy:
\[
S_{\times N,\ell}^{\text{QP}}(t) \sim \frac{S_{2}}{2} e^{-\xi_{0} t} \left[ \sum_{i=1}^{2\ell} \xi_{i}^{2} - 2 \ell \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left( \delta_{i,1}^{2} + \delta_{i,\ell}^{2} \right) \right],
\]
(314)
where $S_2 = d^2 S/d\xi^2$ is the second derivative of Eq. (313), evaluated at $\xi = \delta/\gamma$, and $\xi_{<\gamma}$ are the eigenvalues of $G_\gamma$ in Eq. (305). Further, $G'_{\ell}(t)$ in Eq. (304) is a block Toeplitz matrix with $2\times 2$ blocks $g'_\ell(t)$, which we decompose in momentum space as $g'_\ell(t) = g'_{1,\ell}(t) 1 + g'_{2,\ell}(t) \cdot \sigma$. This decomposition defines both $g'_{1,\ell}(t)$ and $g'_{2,\ell}(t) = |\mathbf{g}'_{\ell}(t)|$. As in Eq. (279), the subscript “$d$” in the last term in Eq. (314) indicates that oscillating terms should be dropped after taking the square.

In Fig. 2, the quasiparticle-pair contribution to the entanglement entropy is shown for quenches to the trivial and topological PT-symmetric phases. The characteristic signatures of the quasiparticle picture in the fermionic model transfer well to the bosonic case, consequently reinforcing particular universal aspects of a PT-symmetric ensemble description. Accordingly, after rescaling with $\sqrt{\gamma}$, the bosonic entropy shows initial ballistic growth, followed by saturation roughly at the characteristic Fermi time $\tau_F$, which depends on the modified quasiparticle velocity. The late-time asymptotic saturation of the entropy is well predicted by the dephased ensemble through Eq. (314).

### C. Dimerized XY model with dephasing

In the study of quench dynamics of isolated integrable systems, an important role is played by one-dimensional spin chains, or, equivalently, lattice models of hard-core bosons. Paradigmatic examples are the XX model [9–11, 49, 50] and the transverse field Ising model [9–11, 49, 50]. For these models of interacting spins or bosons, analytical progress is facilitated by mapping them to noninteracting fermions through the Jordan-Wigner transformation. For example, pertinent GGEs can be expressed in terms of occupation numbers of free fermionic modes. In contrast, for interacting systems that cannot be mapped to noninteracting systems, the construction of the GGE is much more complicated [29]. Therefore, to address the question whether our results for PT-symmetric relaxation of driven-dissipative integrable systems also apply to interacting systems, attempting to generalize these results to interacting systems that can be mapped to noninteracting fermions [50] presents itself as a natural first step. However, applying the Jordan-Wigner transformation to driven-dissipative spin chains with jump operators that are linear in spin operators does not lead to noninteracting fermionic models with jump operators that are linear in fermionic operators. Instead, the Jordan-Wigner transformation in Eq. (77) maps spin flips corresponding to the jump operators $\sigma^x_{\ell}$ to nonlocal products of fermionic operators. Dephasing, which is described by local Hermitian jump operators,$$
abla_i = \frac{\sqrt{\gamma}}{2}\sigma^x_i, \quad (315)
$$
takes a much simpler but still quadratic form in fermionic operators. Thus, in the following, we will study quench dynamics of a dimerized XY spin chain [51] with dephasing as a canonical and experimentally relevant form of dissipation, which occurs naturally and can also be induced deliberately, e.g., in quantum simulations of the XY model with trapped ions [52]. We emphasize that there is a fundamental difference between master equations with linear and quadratic Hermitian jump operators such as the driven-dissipative Kitaev chain and the XY model with dephasing, respectively: In the former case, the Gaussianity of an initially Gaussian state is preserved; in the latter case, an initially Gaussian state evolves into a mixture of Gaussian states [53], and, therefore, nontrivial higher-order connected correlations are generated. Nevertheless, a master equation with quadratic Hamiltonians and quadratic Hermitian jump operators still leads to a closed equation of motion for the covariance matrix. As we will show, the relaxation of the covariance matrix—and, therefore, of all spin observables that can be expressed in terms of the covariance matrix—is described by a PTGGE even though the state of the system is not Gaussian.

The dimerized XY spin chain [51] is described by the following Hamiltonian:

$$H_{XY} = \frac{1}{2} \sum_{l=1}^{2L} \left( J - (-1)^l \Delta J \right) \left( \sigma^x_l \sigma^x_{l+1} + \sigma^y_l \sigma^y_{l+1} \right), \quad (316)$$

where $\sigma^x_{l} \sigma^y_l$ are Pauli matrices that act on the spin at site $l$ with $l \in \{1, \ldots, 2L\}$. We choose the spatially uniform component of the XY couplings as antiferromagnetic, $J > 0$. The spin-Peierls term with dimerization strength $\Delta J$ describes a modulation of the XY couplings across successive bonds between the values $J \pm \Delta J$ [51]. This modulation results in dimer order in the ground state, which is detected by a nonvanishing expectation value of the operator [54]

$$d_i = \sigma^x_i \sigma^x_{i+1} - \sigma^x_{i-1} \sigma^x_i + \text{H.c.} \quad (317)$$

In particular, for $\Delta J = J$, the spin chain is fully dimerized and the ground state is given by a product of singlets across the odd bonds, which leads to the dimerization order parameter taking the value $\langle d_{2l-1} \rangle_0 = -1 = -\langle d_{2l} \rangle_0$.

With dephasing as given in Eq. (315), the master equation for the XY chain reads

$$\frac{d\rho}{dt} = L_{XY} \rho = [H_{XY}, \rho] - i \sum_{l=1}^{2L} [L_i, [L_i, \rho]]. \quad (318)$$

Since the jump operators are Hermitian, the steady state is again at infinite temperature. Further, the Liouvillian $L_{XY}$ of the XY model with dephasing has PT symmetry [55].

To numerically study quench dynamics, we switch to a representation of the XY model in terms of fermions. This representation can be obtained through the Jordan-Wigner transformation in Eq. (77), which maps the XY Hamiltonian in Eq. (316) to

$$H_{XY} = -\sum_{l=1}^{2L} \left( J - (-1)^l \Delta J \right) \left( c^\dagger_l c_{l+1} + \text{H.c.} \right). \quad (319)$$

That is, the Hamiltonian takes the form of the SSH model in Eq. (293), with hopping amplitudes $J_{1,2} = -(J \pm \Delta J)$. Further, the jump operators for dephasing in Eq. (315) are quadratic in fermionic operators,

$$L_i = \sqrt{\gamma} c_i^\dagger c_i, \quad (320)$$
where we omit a constant contribution that cancels due to the commutator structure in Eq. (318).

In analogy to Eq. (301) for the SSH model with linear dissipation, we define the covariance matrix as

$$G_{l,l'} = \langle [c_l, c_{l'}^\dagger] \rangle,$$

(321)

where $l, l' \in \{1, \ldots, 2L\}$. To formulate the equation of motion for the covariance matrix, we write the Hamiltonian and the jump operators in the form

$$H_{XY} = \sum_{l,l'=1}^{2L} c_l^\dagger H_{l,l'} c_{l'}, \quad L_d = \sum_{m,m'=1}^{2L} c_m^\dagger B_{l,m,m'} c_{m'},$$

(322)

where, for dephasing as described in Eq. (320), the elements of the matrix $B_l$ are given by $B_{l,m,m'} = \sqrt{\gamma} \delta_{lm} \delta_{m'}$. The covariance matrix evolves in time according to

$$\frac{dG}{dt} = -i[H, G] - \sum_{l=1}^{2L} [B_l, [B_l, G]].$$

(323)

To illustrate the effect of dephasing most transparently, we decompose the covariance matrix into diagonal and offdiagonal elements as $G = G_{\text{OD}} + G_{\text{DD}}$. Then, Eq. (323) reduces to

$$\frac{dG}{dt} = -i[H, G] - 2\gamma G_{\text{OD}},$$

(324)

which shows that dephasing affects only offdiagonal elements. This equation can be simplified further by taking advantage of the inversion symmetry of the SSH model: One can show that for translationally invariant and inversion-symmetric initial states, the diagonal component $G_D$ vanishes at all times [44]. This applies, in particular, to the ground state of the SSH Hamiltonian. Then, with $G = G_{\text{OD}}$, we obtain

$$\frac{dG}{dt} = -i(ZG - GZ^\dagger),$$

(325)

where $Z = H - i\gamma \mathbb{1}$; that is, time evolution with dephasing differs from the dynamics of isolated system only by an overall decay of $G$ at the rate $2\gamma$. Consequently, the PTGGE which describes relaxation of the covariance matrix and, therefore, of all expectation values of spin observables that can be expressed in terms of the covariance matrix, is obtained simply by multiplying the Lagrange multipliers in the GGE by $e^{-2\gamma t}$.

In contrast to the combination of loss and gain discussed in Sec. IV A, the dynamics and statistics of elementary excitations are not affected by dephasing.

To illustrate the resulting dynamics, we consider quenches from the ground state for $\Delta J_0 = J$ with covariance matrix

$$G_0 = \bigoplus_{i=1}^{2L} \sigma_z,$$

(326)

where $Z = H - i\gamma \mathbb{1}$; that is, time evolution with dephasing differs from the dynamics of isolated system only by an overall decay of $G$ at the rate $2\gamma$. Consequently, the PTGGE which describes relaxation of the covariance matrix and, therefore, of all expectation values of spin observables that can be expressed in terms of the covariance matrix, is obtained simply by multiplying the Lagrange multipliers in the GGE by $e^{-2\gamma t}$.

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To illustrate the resulting dynamics, we consider quenches from the ground state for $\Delta J_0 = J$ with covariance matrix

$$G_0 = \bigoplus_{i=1}^{2L} \sigma_z,$$

(326)

to postquench values $\Delta J \neq \Delta J_0$. Relaxation of the covariance matrix to the PTGGE is shown in Fig. 3 with the time evolution of the dimerization order parameter Eq. (317), which can be expressed as $\langle d_l \rangle = \text{Re}(G_{l-1,l} - G_{l,l+1})$. As stated below Eq. (317), the dimerization order parameter takes the value $\langle d_{2l-1} \rangle = -1$ in the initial ground state. Due to dephasing, it decays exponentially, but after rescaling with $e^{2\gamma t}$, it approaches a stationary value as predicted by the PTGGE.

Since the state of the system after the quench is not Gaussian, the entanglement entropy is not fully determined by the covariance matrix. However, it is still informative to neglect deviations from Gaussianity and to study the time evolution of the quasiparticle-pair entropy computed from the covariance matrix alone [41]. In Fig. 4, we show the time evolution of

Figure 3. Dimerization order parameter $\langle d_{2l-1} \rangle$ across odd bonds. The initial value for a fully dimerized chain is $\langle d_{2l-1} \rangle_0 = -1$. After rescaling with $e^{2\gamma t}$, the dimerization order parameter approaches a stationary value that is determined by the PTGGE and indicated by horizontal lines in the figure.

Figure 4. Quasiparticle-pair contribution to the subsystem entropy, calculated from the covariance matrix and thus neglecting deviations from Gaussianity, after quenches of the dimerized XY model with dephasing for $\Delta J = 0.3J$ and at $t = 20$. For $\gamma = 0$, the quasiparticle-pair entropy reduces to the entanglement entropy, and its stationary value is determined by the conventional GGE; for any $\gamma > 0$, the rescaled quasiparticle-pair entropy approaches a stationary value that is independent of $\gamma$ and determined by the PTGGE.

The PTGGE prediction is reached at $\gamma t \gg 1$ for very small values of $\gamma$ with $\gamma < 1/\tau_F \approx 0.07J$. In contrast, for $\gamma > 1/\tau_F$, the curves collapse, and converge towards the PTGGE value at $t \approx \tau_F$. The value of $\tau_F$ does not depend on $\gamma$, which reflects the fact that dephasing does not modify the dynamics of quasiparticle excitations.
the rescaled quasiparticle-pair entropy, which is defined here as $e^{i\gamma t}S_{QP}^{\ell}(t)$ and can be calculated from the covariance matrix as outlined in Sec. IV A. The fact that dephasing does not affect the dynamics and statistics of quasiparticle excitations has two notable consequences that can be seen in the figure: First, the PTGGE prediction for the rescaled quasiparticle-pair entropy is independent of the value of $\gamma$; and second, also the Fermi time, which marks the transition from the initial linear increase of the rescaled entropy to the much slower approach to the stationary value, does not depend on $\gamma$.

**Appendix: Time evolution induced by a non-Hermitian $2 \times 2$ Hamiltonian**

In this short technical appendix, we summarize some relations that allow us to determine the time evolution of the components of the covariance matrix in Eq. (112). We consider a non-Hermitian $2 \times 2$ Hamiltonian $x$ that is given by

$$x = x_1 \mathbb{I} + x \cdot \sigma, \quad x = a + b,$$

(A.1)

where $a, b \in \mathbb{R}^3$. Further, we define

$$\omega = \sqrt{x \cdot x} = \sqrt{e^2 + i2a \cdot b - b^2},$$

(A.2)

where $e = |a|$, and $b = |b|$. In general, $\omega \in \mathbb{C}$. Finally, we define a vector $\hat{x} = x / \omega$. In terms of these quantities, the exponential of $x$ can be written as

$$e^{-is\tau} = e^{-isx_1} (\cos(\omega t) \mathbb{I} - i \sin(\omega t) \hat{x} \cdot \sigma).$$

(A.3)

Using this relation, and with

$$\sigma \cdot (x \cdot \sigma) = x \mathbb{I} + i \hat{x} \times \sigma,$$

(A.4)

it is straightforward to show that

$$\tau = e^{-is\tau} e^{isx_1} = e^{i2Im(x_1)t} \left\{ [\cos(\omega t)]^2 \sigma \right.$$

$$- i \sin(\omega t) \cos(\omega t) (\hat{x} - i \hat{x} \times \sigma)$$

$$+ i \sin(\omega t) \cos(\omega t) (\hat{x}^* + i \hat{x}^* \times \sigma)$$

$$+ [\sin(\omega t)]^2 \left[ (x \cdot \sigma) \hat{x}^* + \hat{x} \cdot \sigma \right) - i \hat{x} \times \hat{x}^* - (\hat{x} \cdot \hat{x}^*) \sigma \left. \right\}.$$  

(A.5)

Next, we consider the scalar product of $\tau$ and $a_0 \in \mathbb{R}^3$, which we denote by $\gamma$:

$$\gamma = a_0 \cdot \tau = \gamma_1 \mathbb{I} + \gamma \cdot \sigma.$$  

(A.6)

We obtain

$$\gamma_1 = 2 e^{2Im(x_1)t} \left\{ \frac{\sin(\omega t) \cos(\omega t)}{\omega^2} a_0 \cdot (a \times b) \right\}$$

(A.7)

and

$$\gamma = e^{2Im(x_1)t} \left\{ \left[ (\cos(\omega t))^2 - |\sin(\omega t)|^2 \right] \hat{x} \cdot \hat{x}^* a_0 
- 2Re(x_1) \cos(\omega t) (a_0 \times \hat{x}) 
+ 2|\sin(\omega t)|^2 Re((a_0 \cdot \hat{x}) \hat{x}^*) \right\}.$$  

(A.8)

In the main text, we are primarily concerned with the case $a \cdot b = 0$ and $e > b$, such that

$$\omega = \sqrt{e^2 - b^2} \in \mathbb{R}>0,$$

(A.9)

and $a_0 \cdot b = 0$. These conditions lead to the following simplifications of the above expressions for $\gamma_1$ and $\gamma$:

$$\gamma_1 = -\frac{e^{2Im(x_1)t}}{\omega^2} \left( 1 - \cos(2\omega t) \right) a_0 \cdot (a \times b),$$

(A.10)

and

$$\gamma = e^{2Im(x_1)t} \left\{ a_0 - \frac{e^2}{\omega^2} \frac{\omega}{\cos(\omega t)} (1 - \cos(2\omega t)) \right\} a_0,$$

(A.11)

where, with $\hat{a} = a / e$, the parallel, perpendicular, and out-of-plane components are given by

$$a_\parallel = (a_0 \cdot \hat{a}) \hat{a}, \quad a_\perp = a_0 - a_\parallel, \quad a_0 = -a_0 \times \hat{a}.$$  

(A.12)

Finally, we calculate the norm of the vector $\gamma$, which we denote by $\xi = |\gamma|$. To that end, we use that $a_\parallel, a_\perp,$ and $a_0$ are mutually orthogonal, and that

$$a_0 \cdot a_\perp = |a_\perp|^2 = |a_0|^2 = e^2 - (a_0 \cdot \hat{a})^2.$$  

(A.13)

We further assume that $a_0 = \hat{a}_0$ is a unit vector, such that $a_0 = |a_0| = 1$. Then,

$$\hat{a}_0 \cdot a_\perp = |a_\perp|^2 = |a_0|^2 = 1 - (\hat{a}_0 \cdot \hat{a}) = \sin(\Delta \theta)^2,$$

(A.14)

where $\Delta \theta$ is the angle between $\hat{a}_0$ and $\hat{a}$,

$$\cos(\Delta \theta) = \hat{a}_0 \cdot \hat{a},$$

(A.15)

and we obtain

$$\xi^2 = e^{4Im(x_1)t} \left\{ 1 - \frac{e^2}{\omega^2} \sin(\Delta \theta)^2 \left[ 2(1 - \cos(2\omega t)) \right. 
- \frac{e^2}{\omega^2} (1 - \cos(2\omega t))^2 - \sin(2\omega t)^2 \left. \right] \right\}.$$  

(A.16)
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