The ILW hierarchy

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Abstract

In this paper, we present a new hierarchy which includes the intermediate long wave (ILW) equation at the lowest order. This hierarchy is thought to be a novel reduction of the 1st modified KP type hierarchy. The framework of our investigation is Sato theory.

1 Introduction

It is well known that Sato theory was established by M. Sato around 1980 to give a unified viewpoint for integrable soliton equations [1]. A lot of important studies have been based on this magnificent theory since then. In this paper, we focus on one of the basic ideas of the theory, summarized as follows [2].

“Start from an ordinary differential equation and suppose that the solutions satisfy certain dispersion relations. Then, as conditions the coefficients must satisfy, we obtain a set of nonlinear partial differential equations. If we assume a particular set of linear dispersion relations, we obtain the KP hierarchy as the corresponding PDEs.”

It is known that most nonlinear partial differential equations which have $N$-soliton solutions correspond to certain equations of KP hierarchy. But, in the case of the intermediate long wave (ILW) equation, although its $N$-soliton solutions and an inverse scattering transform were well-known [3,4,5], the correspondence to the KP hierarchy has remained unclear.

The ILW equation was proposed by Joseph [6] and Kubota et al. [7] to describe long internal gravity waves in a stratified fluid with finite depth. It is written in the form

$$u_t + \frac{1}{\delta} u_x + 2uu_x + T(u_{xx}) = 0,$$

(1)

where $T(\cdot)$ is the singular integral operator given by

$$T(u) = P \int_{-\infty}^{\infty} \frac{1}{2\delta} \cot \left[ \frac{\pi}{2\delta}(\xi - x) \right] u(\xi) d\xi$$

(2)

($P$ represents the principal value of the integral). Depending on the parameter $\delta$ (which controls the depth of the internal wave layer) (1) reduces to the Korteweg-de Vries (KdV) equation as $\delta \to 0$ or to the Benjamin-Ono (BO) equation as $\delta \to \infty$.

In this paper, we present a new hierarchy which includes the ILW equation at the lowest order. This hierarchy is thought to be a novel reduction of the 1st modified KP type hierarchy. In section 2, we propose a set of differential-difference dispersion relations, and introduce a corresponding 1st modified KP type hierarchy. In section 3, this 2 + 1 dimensional hierarchy will be reduced to a 1 + 1 dimensional hierarchy which contains the ILW equation at its lowest order. In section 4, we discuss more general dispersion relations of differential-difference type.
2 The dispersion relations and their corresponding hierarchy

Let us introduce a pseudo-differential operator,

\[ W = 1 - \frac{U}{2}\partial^{-1} + w_2\partial^{-2} + w_3\partial^{-3} + \cdots + w_m\partial^{-m}, \]

where \( U \) and \( w_j \) \((j = 1, 2, \cdots)\) are functions of continuous variables \( t = (t_1, t_2, \cdots) \) and a discrete variable \( z \). We sometimes use \( x \) instead of \( t_1 \) to follow the convention. \( \partial^{-n} \) denotes

\[ \partial^{-n} = \left( \frac{d}{dt_1} \right)^{-n}. \]

If we employ the Leibniz rule,

\[ \partial^n f(t_1) = \sum_{r=0}^{\infty} \frac{n(n-1)\cdots(n-r+1)}{r!} \left( \frac{\partial^r f}{\partial t_1^r} \right) \partial^{n-r}, \]

then \( \partial^n \) can be a well-defined operator even for negative integer \( n \). Though the theory is developed for the case of \( m \to \infty \) in general, we, in this paper, confine ourselves to (3) for simplicity \([2]\). It is remarked that the essence of the general theory is still kept in this simplification.

Let us consider the ordinary differential equation,

\[ W\partial^m f(t, z) = (\partial^m - \frac{U}{2}\partial^{m-1} + w_2\partial^{m-2} + \cdots + w_m) f(t, z) = 0 \]

which has \( m \) linearly independent solutions, \( f^{(1)}(t, z), f^{(2)}(t, z), \cdots, f^{(m)}(t, z) \). We assume here that \( f^{(1)}(t, z), f^{(2)}(t, z), \cdots, f^{(m)}(t, z) \) satisfy the following dispersion relations,

\[ \begin{cases} i^{1-n}\partial_\tau f^{(j)} + \partial^n f^{(j)} = 0 \\ \Delta_z f^{(j)} = \partial f^{(j)} \end{cases} \quad (j = 1, 2, \cdots, m \text{ and } n = 1, 2, \cdots), \]

where \( \Delta_z \) denotes a difference operator,

\[ \Delta_z g(z) = \frac{e^{2iz\delta \tau} - 1}{g(z)} \frac{g(z + 2i\delta) - g(z)}{2i\delta} \]

(\( \delta \) is an arbitrary constant).

It should be remarked that \( U \) and \( w_j \) are expressible by means of a \( \tau \)-function, which is the Wronskian of \( f^{(1)}(t, z), f^{(2)}(t, z), \cdots, f^{(m)}(t, z) \), i.e.

\[ U = 2 \frac{\tau_2}{\tau}, \quad w_2 = \frac{1}{2} \left( \frac{\tau_2}{\tau} - \frac{\tau_{xx}}{\tau} \right), \cdots. \]

Let \( B_n \) \((n = 1, 2, \cdots)\) and \( C \) be pseudo-differential operators,

\[ B_n = (W\partial^n W^{-1})^+, \quad C = (\bar{W}\partial W^{-1})^+, \]

where \( \bar{W}(z) = W(z + 2i\delta) \) and \((A)^+\) denotes the differential part of the pseudo differential operator \( A \). We use \( \tau \) to denote a shift operator: \( z \to z + 2i\delta \). Then we introduce time evolution equations for \( W \) by

\[ \begin{cases} i^{1-n}\frac{\partial W}{\partial t_1} = B_n W - W \partial^n \\ \bar{W} - \frac{W - \bar{W}}{2i\delta} = CW - \bar{W}\partial \end{cases} \quad (n = 1, 2, \cdots), \]

2
which are Sato type equations. From (12) we get an infinite system of the Zakharov-Shabat type equations

\[
\begin{cases}
  i^{l-l} (C)_t - \frac{B_l - B_t}{2i\delta} + CB_t - B_tC = 0 \\
  i^{l-l} (B_k)_{t_k} - i^{l-k} (B_l)_t + [B_k, B_l] = 0 \\
  \quad (k, l = 1, 2, \ldots).
\end{cases}
\]

Furthermore from (13) we can deduce a system of partial differential-difference equations for \( U \). The first few are explicitly given by

\[
\begin{align*}
  i(\bar{U} - U)_{t_2} + \frac{i}{\delta}(\bar{U} - U)_{x} + (\bar{U} - U)(\bar{U} - U)_{x} + (\bar{U} + U)_{xx} &= 0, \\
  2(\bar{U} - U)_{t_3} - \frac{3i}{2}(\bar{U} - U)_{xt_2} + \frac{1}{2}(\bar{U} - U)_{xxx} &+ \frac{3i}{2\delta}(\bar{U} + U)_{xx} + \frac{6i}{2\delta}(\bar{U} - U)(\bar{U} - U)_x + \frac{3}{2\delta}(\bar{U} - U)_x \\
  &+ \frac{3}{2}(\bar{U} + U)_{xx} (\bar{U} - U) + \frac{3}{2}(\bar{U} + U)_x (\bar{U} - U)_x + \frac{3}{2}(\bar{U} - U)^2(\bar{U} - U)_x = 0, \\
  -i(\bar{U} - U)_{t_4} - \frac{1}{2}(\bar{U} + U)_{t_2 t_2} + \frac{i}{2}(\bar{U} - U)_{xxx} + (\bar{U} + U)_{xxxx} &+ 3(\bar{U} - U)_x (\bar{U} - U)_{xx} - \frac{i}{2}(\bar{U} + U)_{t_2} (\bar{U} - U)_x - i(\bar{U} - U)_{t_2} (\bar{U} + U)_x \\
  &+ \frac{i}{\delta^2}(\bar{U} + U)_{xt_2} + \frac{i}{\delta^2}(\bar{U} - U)_{t_2} + \frac{1}{\delta^2}(\bar{U} + U)_{xt_2} + \frac{i}{\delta^2}(\bar{U} - U)_{xxx} \\
  &+ \frac{1}{\delta^2}(\bar{U} - U)(\bar{U} - U)_{t_2} + (\bar{U} - U)(\bar{U} - U)_{xxxx} - \frac{i}{2}(\bar{U} - U)^2(\bar{U} - U)_{t_2} = 0.
\end{align*}
\]

Substituting (9) into (14)-(16), we also have the equations for \( \tau \) by

\[
\begin{align*}
  \left( iD_{t_2} + \frac{i}{\delta} D_x + D_x^2 \right) \bar{\tau} \cdot \tau &= 0, \\
  \left( 4D_{t_3} - 3i D_x D_{t_2} + D_x^3 + \frac{3i}{\delta} D_x^2 - \frac{3}{\delta^2} D_x \right) \bar{\tau} \cdot \tau &= 0, \\
  \left( -2i D_{t_4} - D_x^2 - iD_x D_x^2 + \frac{1}{\delta} D_{t_2} D_x + \frac{i}{\delta^2} D_{t_2} \right) \bar{\tau} \cdot \tau &= 0
\end{align*}
\]

(\( D \) denotes Hirota’s differential operator). It should be remarked that (17)-(19) are essentially the same as the first few of the 1st modified KP hierarchy [8],

\[
\begin{align*}
  (D_{t_1}^2 + D_{t_2}) \tau_n \cdot \tau_{n+1} &= 0, \\
  (D_{t_1}^3 - 4D_{t_3} - 3D_{t_1} D_{t_2}) \tau_n \cdot \tau_{n+1} &= 0, \\
  (-D_{t_1}^2 D_{t_2} + D_{t_2}^2 + 2D_{t_4}) \tau_n \cdot \tau_{n+1} &= 0.
\end{align*}
\]

### 3 Solutions and special reductions

\( N \)-soliton solutions for (17)-(19) will be written in the form

\[
\tau = \begin{pmatrix}
  1 + c_1 e^{\eta(t,p_1) - \eta(t,q_1)} & \ldots & \ldots & 1 + c_N e^{\eta(t,p_N) + \eta(t,q_N)} \\
  l(q_1) + l(p_1) c_1 e^{\eta(t,p_1) - \eta(t,q_1)} & \ldots & \ldots & l(q_N) + l(p_N) c_N e^{\eta(t,p_N) - \eta(t,q_N)} \\
  \vdots & \ddots & \ddots & \vdots \\
  l(q_1)^{N-1} + l(p_1)^{N-1} c_1 e^{\eta(t,p_1) - \eta(t,q_1)} & \ldots & \ldots & l(q_N)^{N-1} + l(p_N)^{N-1} c_N e^{\eta(t,p_N) - \eta(t,q_N)} \\
  \prod_{j' < j} (l(q_{j'}) - l(q_j))
\end{pmatrix}
\]
\[
\sum_{J \subseteq I} \left( \prod_{j \in J} c_j \right) \left( \prod_{j, j' \not\in J} a_{jj'} \right) \exp \left( \sum_{j \in J} \eta(t, p_j) - \eta(t, q_j) \right)
\]  
\tag{23}
\]
where the summation is taken over all subsets \( J \) of \( I = \{1, 2, \cdots, m\} \). \( \eta(t, p), l(p) \) and \( a_{jj'} \) are defined by
\[
\eta(t, p) = pz + \sum_{n=1}^{\infty} i^{n-1} l(p)^n t_n,
\]
\[
l(p) = e^{2i\delta p} - \frac{1}{2i\delta},
\]
\[
a_{jj'} = \frac{l(p_j) - l(p_{j'})}{l(p_j) - l(q_{j'})},
\]
and the \( c_j \) are constants. Notably, the bilinear identity to \( \tau \) is given as follows.

**Lemma 3.1** For arbitrary \( t = (t_1, t_2, \cdots), t' = (t'_1, t'_2, \cdots) \) and \( z, \tau \) satisfies
\[
\oint e^{-i\xi(k, t-t')} (1 + 2i\delta k) \tau(z + 2i\delta, t - i\epsilon(k^{-1}) ) \tau(z, t + i\epsilon(k^{-1})) \frac{dk}{2\pi i} = 0,
\]
where \( \xi(k, t) = \sum_{n=0}^{\infty} t_n k^n \), \( \epsilon(k^{-1}) = (\frac{1}{k}, \frac{1}{2k}, \cdots, \frac{1}{nk}, \cdots) \). The curve is taken around \( \infty \) and excludes the singular points \( il(p_n), il(q_n) \).

**Proof.** Substitute (23) into (27) and we see that \( \text{Res}(il(p_n)) + \text{Res}(il(q_n)) = 0 \) for \( \forall n \).

We impose the condition
\[
p_j - q_j = l(p_j) - l(q_j) = k_j \tag{28}
\]
on (23). Then, it is reduced to
\[
\tau = \sum_{J \subseteq I} \left( \prod_{j \in J} c_j \right) \left( \prod_{j, j' \not\in J} a_{jj'} \right) \exp \left( \sum_{j \in J} k_j z + \sum_{n=1}^{\infty} \mu_n(k_j) t_n \right)
\]
\tag{29}
for appropriately-defined functions \( \mu_n(k) \). It should be noticed that (29) is the same as the soliton solution of the ILW equation [9]. We here present the 1-soliton solution as an example.

\[
\tau = 1 + c \exp \left[ k z + k t_1 + (k^2 \cot k \delta - \frac{k}{\delta}) t_2 
\right.
\]
\[
+ \frac{1}{4} \left( k^3 - 3k^3 \cot^2 k \delta + \frac{6}{\delta} k^2 \cot k \delta - \frac{3}{\delta^2} k \right) t_3 
\]
\[
+ \frac{1}{2} \left( -k^4 \cot k \delta + k^4 \cot^3 k \delta - \frac{3}{\delta} k^3 \cot^2 k \delta + \frac{1}{\delta} k^3 + \frac{3}{\delta^2} k^2 \cot k \delta - \frac{1}{\delta^3} k \right) t_4 + \cdots 
\]
\tag{30}
\]
We can regard \( z \) and \( t_1 \) as the same variable under this reduction, because the coefficient of \( z \) is equal to that of \( t_1 \) in the exponentiated part of (29). It also should be noticed that

**Lemma 3.2** If \( k_j \)'s are real, \( \mu_n(k_j) \) and \( a_{jj'} \) are also real \((j, n = 1, 2, \cdots)\).

**Proof.**
From (25) and (28), we get
\[
l(p_j) + l(q_j) = -ik_j \cot k_j \delta + \frac{i}{\delta}.
\]
\tag{31}
Because \( l(p_j) + l(q_j) \) is purely imaginary and \( l(p_j) - l(q_j)(= k_j) \) is real, there exist real \( r, \theta \) by which
\[
l(p_j) = re^{i\theta}, \quad l(q_j) = re^{i(\pi-\theta)}.
\]
\tag{32}
By the definition of $\mu_n(k_j)$, we have

$$
\mu_n(k_j) = i^{n-1} (l(p_j) - l(q_j))^n
$$

$$
= i^{n-1} \left( r e^{in\theta} - r e^{in(\pi-\theta)} \right)
$$

$$
= \left\{ \begin{array}{ll}
2i^{n-1}r^n \cos(n\theta) & \text{for } n \text{ odd} \\
2i^{n}r^n \sin(n\theta) & \text{for } n \text{ even.}
\end{array} \right.
$$

(33)

Hence $\mu_n(k_j)$ is real. We also deduce

$$
a_{jj'} = \frac{l(p_j) - l(p_{j'})}{l(q_j) - l(q_{j'})} \cdot \frac{l(q_j) - l(q_{j'})}{l(p_j) - l(p_{j'})}
$$

$$
= \frac{(r e^{i\theta} - r' e^{i\theta}) (r e^{-i\theta} - r' e^{-i\theta})}{(r e^{i\theta} - r' e^{i\theta}) (r e^{-i\theta} - r' e^{-i\theta})}
$$

$$
= \frac{r^2 + r'^2 - rr' \cos(\theta - \theta')}{r^2 + r'^2 - rr' \cos(\theta + \theta')},
$$

(34)

which gives that $a_{jj'}$ is real.

By this reduction, the $x$-shifts can take the place of the $z$-shifts and (14)-(16) are rewritten into the equations for $U_{\pm}(x) := U(x \mp i\delta)$, i.e.

$$
i(U^- - U^+)_t + \frac{i}{6}(U^- - U^+)_x + (U^- - U^+)(U^- - U^+)_x + (U^- + U^+)_xx = 0,
$$

(35)

$$
2(U^- - U^+)_t - \frac{3i}{2}(U^- - U^+)_{xt} + \frac{1}{2}(U^- - U^+)_{xxx} + \frac{3i}{2\delta}(U^- + U^+)_{xx}
$$

$$
+ \frac{3i}{\delta}(U^- - U^+)(U^- - U^+)_{xx} - \frac{3}{2\delta^2}(U^- - U^+)_{xx} + \frac{3}{2}(U^- + U^+)_{xx}(U^- - U^+)
$$

$$
+ \frac{3}{2}(U^- + U^+)_{xx}(U^- - U^+) + \frac{3}{2}(U^- - U^+)^2(U^- - U^+)_x = 0,
$$

(36)

$$
i(U^- - U^+)_t - \frac{1}{2}(U^- + U^+)_tx + \frac{i}{2}(U^- - U^+)_{xx} + (U^- + U^+)_xxx
$$

$$
+ 3(U^- - U^+)(U^- - U^+)_{xx} - \frac{i}{2}(U^- + U^+)_{tt} + (U^- - U^+)_{tx} - i(U^- - U^+)_{tx} + i(U^- - U^+)_t (U^- + U^+)_x
$$

$$
- \frac{i}{2}(U^- + U^+)_{tt}(U^- - U^+) + \frac{1}{2\delta^2}(U^- - U^+)_{tt} + \frac{1}{2\delta}(U^- + U^+)_{tx} + \frac{i}{\delta}(U^- - U^+)_{xxx}
$$

$$
- \frac{1}{\delta}(U^- - U^+)(U^- - U^+)_{tx} + (U^- - U^+)(U^- - U^+)_{xxx} - \frac{i}{2}(U^- - U^+)^2(U^- - U^+)_{tx} = 0.
$$

(37)

If we consider lemma 3.2 and suppose that $U(x)$ is analytic in the horizontal strip between $\text{Im } x = -i\delta$ and $\text{Im } x = i\delta$, we can introduce a dependent variable $u$ which satisfies [9]

$$
u = \frac{i}{2}(U^- - U^+),
$$

(38)

$$
T(u) = \frac{1}{2}(U^- + U^+).
$$

(39)

Substituting this $u$ into (35)-(37), we obtain

$$
u_{tx} + \frac{1}{\delta}u_x + 2u u_x + T(u_{xx}) = 0,
$$

(40)

$$
-4u_{tx} - 3T(u_{tx}) - u_{xxx} - 6u T(u_{xx}) - 6u_x T(u_x) + 12u^2 u_x
$$

$$
- \frac{12}{\delta}u u_x + \frac{3}{\delta} T(u_{xx}) + \frac{3}{\delta^2} u_x = 0,
$$

(41)

$$
-2u_{tx} - T(u_{tx}) + u_{xtx} + 4u^2 u_x - 2u_x T(u_x) - 4u_{tx} T(u_x)
$$

$$
-2u T(u_{tx}) + 2T(u_{xxx}) - 12u_{xxx} u_x - 4u u_{xxx} + \frac{1}{\delta} T(u_{xx}) + \frac{2}{\delta} u_{xxx} - \frac{4}{\delta} u u_x + \frac{1}{\delta^2} u_x = 0.
$$

(42)
Because the lowest order is nothing but the intermediate long wave equation, this hierarchy should be called the ILW hierarchy.

4 More general dispersion relations

Other possible differential-difference dispersion relations than (7) can be

\[
\begin{cases}
    i^{1-n} \partial_{t_n} f^{(j)} = \partial^n f^{(j)} \\
    i^{s-1} \Delta_z f^{(j)} = \partial^s f^{(j)},
\end{cases}
\]

where \( s \) is some fixed positive integer, \( j = 1, 2, \cdots, m \) and \( n = 1, 2, \cdots \). Notably, (43) corresponds to (7) when \( s = 1 \).

For each \( s \), we can deduce the corresponding hierarchy in the same way as in the preceding sections. Hence we just present the bilinear identity which generates the hierarchy for \( \tau \) [8],

\[
\oint e^{-i\xi(k, t-L)}(1 - 2\delta(-k)^s)\tau(z + 2i\delta, t - i\epsilon(k^{-1}))\tau(z, t + i\epsilon(k^{-1}))\frac{dk}{2\pi i} = 0. \tag{44}
\]

The \( N \)-soliton solution to the hierarchy is written in the form

\[
\tau = \left| \begin{array}{ccccc}
1 + c_1 e^{H(t,p_1)-H(t,q_1)} & \cdots & 1 + c_N e^{H(t,p_N)-H(t,q_N)} \\
L(q_1) + L(p_1) c_1 e^{H(t,p_1)-H(t,q_1)} & \cdots & L(q_N) + L(p_N) c_N e^{H(t,p_N)-H(t,q_N)} \\
\vdots & \cdots & \vdots \\
L(q_1)^{N-1} + L(p_1)^{N-1} c_1 e^{H(t,p_1)-H(t,q_1)} & \cdots & L(q_N)^{N-1} + L(p_N)^{N-1} c_N e^{H(t,p_N)-H(t,q_N)} \\
\prod_{j,j' > j} (L(q_{j'}) - L(q_j))
\end{array} \right| \prod_{j \in J} C_j \prod_{j,j' \in J} A_{jj'} \exp \left( \sum_{j \in J} H(t,p_j) - H(t,q_j) \right), \tag{45}
\]

where the summation is taken over all subsets \( J \) of \( I = \{1, 2, \cdots, m\} \). \( H(t,p), L(p) \) and \( A_{jj'} \) are defined by

\[
H(t,p) = pz + \sum_{n=1}^{\infty} i^{n-1} L(p) t_n, \tag{46}
\]

\[
i^{1-s} L(p)^s = \frac{e^{2i\delta p} - 1}{2i\delta}, \tag{47}
\]

\[
A_{jj'} = \frac{L(p_j) - L(p_{j'})}{L(p_j) - L(q_{j'})} \frac{L(q_j) - L(q_{j'})}{L(q_j) - L(p_{j'})}. \tag{48}
\]

It should be noticed that \( L(p) \) is multi-valued. It is easy to check that this soliton solution satisfies the bilinear identity (44).

We impose the reduction condition,

\[
p_j - q_j = i^{s-1} (L(p_j)^s - L(q_j)^s) = (-1)^{s-1} \frac{c^{2i\delta p_j} - c^{2i\delta q_j}}{2i\delta} = k_j \quad (j = 1, 2, \cdots). \tag{49}
\]

Substituting (49) into (45), we have

\[
\tau = \sum_{J \in I} \left( \prod_{j \in J} C_j \right) \left( \prod_{j,j' \in J, j' > j} A_{jj'} \right) \exp \left( \sum_{j \in J} k_j z + \sum_{n=1}^{\infty} M_n(k_j) t_n \right) \tag{50}
\]

for appropriately-defined functions \( M_n(k) \). We can regard \( z \) and \( t_n \) as the same variable under this reduction. Furthermore, the following lemma holds.
LEMMA 4.1 If $k_j$'s are real, $M_n(k_j)$ and $A_{ij'}$ are also real ($i = 1, 2, \ldots$).

Proof. From (48) and (50), we see that

$$L(p_j)^s - L(q_j)^s = (i)^{1-s}k_j,$$

$$L(p_j)^s + L(q_j)^s = (i)^s \left( (-1)^sk_j \cot k_j \delta + \frac{2}{\delta} \right).$$

If $s$ is odd, $L(p_j)^s - L(q_j)^s$ is real and $L(p_j)^s + L(q_j)^s$ is purely imaginary. Hence there exists real $r, 0 \leq \theta \leq 2\pi$ by which

$$L(p_j)^s = re^{i\theta}, \quad L(q_j)^s = re^{i(\pi-\theta)}.$$  

If $s$ is even, $L(p_j)^s - L(q_j)^s$ is purely imaginary and $L(p_j)^s + L(q_j)^s$ is real. Then we have

$$L(p_j)^s = re^{i\theta}, \quad L(q_j)^s = re^{-i\theta}.$$  

For both cases, we choose $L(p_j), L(q_j)$ as

$$L(p_j) = \sqrt{r}e^{i\frac{\theta}{2}}, \quad L(q_j) = \sqrt{r}e^{i(\pi-\frac{\theta}{2})}.$$  

Because $L(p_j) - L(q_j)$ is defined to be real and $L(p_j) + L(q_j)$ to be purely imaginary, we see that $M_j, A_{ij'}$ are real by means of lemma 3.2. This, at the same time, takes care of the multi-valuedness of the function $L(p)$.

If we consider lemma 4.1 and suppose that $U(t_s)$ is analytic in the horizontal strip between $\text{Im } t_s = -i\delta$ and $\text{Im } t_s = i\delta$, we can introduce a dependent variable $u$ which satisfies

$$u = \frac{i}{2}(U(t_s + i\delta) - U(t_s - i\delta)), \quad T_s(u) = \frac{1}{2}(U(t_s + i\delta) + U(t_s - i\delta)),$$

where

$$T_s(u(t_s)) = P \int_{-\infty}^{\infty} \frac{1}{2\delta} \cot \left[ \frac{\pi}{2\delta}(\xi - t_s) \right] u(\xi)d\xi.$$  

As another concrete example, different from the ILW hierarchy, we apply the preceding argument to the case $s = 2$. From (44), we get

$$\int e^{-i\xi(k, k' \cdot t)} (1 - 2\delta k^2) \tau(z + 2i\delta, t - i\xi(k^{-1})) \tau(z, t + i\xi(k^{-1})) \frac{dk}{2\pi i} = 0,$$

which generates the hierarchy for $\tau$. The first few hierarchy equations are

$$\begin{cases}
-2D_{t_3} + D_z^4 + 3iD_zD_{t_2} + \frac{3}{\delta}D_z \cdot \tau = 0, \\
6D_{t_4} - 4iD_{t_3}D_z - 3iD_z^2 - iD_z^4 - \frac{12j}{\delta}D_z^2 - \frac{6}{\delta}D_{t_2} \cdot \tau = 0.
\end{cases}$$

This hierarchy is essentially the same as the 2nd modified KP hierarchy [8]

$$\begin{align*}
(2D_{t_3} + D_z^4 + 3D_zD_{t_2}) \tau_n \cdot \tau_{n+2} &= 0, \\
(D_z^4 - 4D_zD_{t_3} - 3D_z^2 - 6D_{t_2}) \tau_n \cdot \tau_{n+2} &= 0, \\
\cdots
\end{align*}$$

The 1-soliton solution for this hierarchy is written in the form

$$\tau = 1 + C \exp[H(t, p) - H(t, q)],$$

$$H(t, p) = (p - q)z + \sum_{n=1}^{\infty} i^{n-1}L(p)^n t_n,$$

$$L(p)^2 = \frac{-e^{2isp} + 1}{2\delta}.$$
By the reduction condition

\[ p - q = i (L(p)^2 - L(q)^2) = e^{2i\delta p} - e^{2i\delta q} \]
\[ = \frac{k}{2i\delta} \]

(67)

(64) is reduced to

\[
\tau = 1 + \exp \left[ k z + \sqrt{k \cot k\delta - \delta^{-1} + \sqrt{(k \cot k\delta - \delta^{-1})^2 + k^2}} t_1 + k t_2 \right. \\
\left. + \sqrt{k \cot k\delta - \delta^{-1} + \sqrt{(k \cot k\delta - \delta^{-1})^2 + k^2}} \times \left( -k \cot k\delta + \delta^{-1} + \sqrt{(k \cot k\delta - \delta^{-1})^2 + k^2/\sqrt{2}} \right) t_3 + \cdots \right] .
\]

(68)

Thus the \( t_2 \)-shifts take the place of the \( z \)-shifts. If we impose the analytic condition as before on \( U \), there exists \( u \) which satisfies

\[ u = \frac{i}{2} (U(t_2 + i\delta) - U(t_2 - i\delta)) , \]

(69)

\[ T_2(u) = \frac{1}{2} (U(t_2 + i\delta) + U(t_2 - i\delta)) . \]

(70)

Now, (60) and (61) are expressible by means of \( u \) as

\[
2u_{t_3} - u_{xxx} + 3T_2(u_{xt_2}) - 3(uT_2(u_x))_x + 3u^2 u_x \\
- 3 \left( u \int^{x}_{-\infty} u_x \, dx \right)_x + \frac{3}{\delta} u_x = 0 ,
\]

(71)

\[
6u_{t_4} + T_2(u_{xxxx}) - 4(uu_x)_x + 12T_2(u)T_2(u_x) + 4u^3 u_x \\
- 6(6u^2T_2(u_x))_x + 4T_2(u_{xt_2}) - 4u_x \int^{x}_{-\infty} u_t_3 d\xi - 4uu_{t_3} + 3T_2(u_{xt_2}) \\
- 6u_{t_2} \int^{x}_{-\infty} u_{tx} \, d\xi + \frac{12}{\delta} T_2(u_{xx}) - \frac{24}{\delta} uu_x - \frac{6}{\delta} u_{t_2} = 0 .
\]

(72)

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