About construction of orthogonal wavelets with compact support and with scaling coefficient N

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In paper [1] with using of the Cuntz algebra representation some methods of construction of wavelets with scaling coefficient \( N \geq 2 \) are considered. In paper [2] it is shown, a construction of wavelets at the prescribed scaling function \( \varphi(x) \). In this paper a simple method of construction of scaling function \( \varphi(x) \) and orthogonal wavelets with the compact support for any natural coefficient of scaling \( N \geq 2 \) is given. Examples of construction of wavelets for coefficients of scaling \( N = 2 \) and \( N = 3 \) are produced.

1. Scaling functions and wavelets. Let \( N \geq 2 \) is an integer, \( \mathbb{Z} \) is set of all integers and \( L^2(\mathbb{R}) \) is Hilbert space of square integrable functions.

**Definition 1.** Function \( \varphi(x) \in L^2(\mathbb{R}) \) is called \( N \)-scaling, if it can be represented as

\[
\varphi(x) = \sqrt{N} \sum_{n \in \mathbb{Z}} h_n \varphi(Nx - n),
\]

where coefficients \( h_n, n \in \mathbb{Z} \) satisfy to condition \( \sum_n |h_n|^2 < \infty \). The relationship (1) is called the \( N \)-scale equation (refinement equation). The set \( \{h_n\} \) of coefficients of expansion in the equation (1) is called the scaling filter.

**Note 1.** If \( N \)-scaling function \( \varphi(x) \) has the compact support of length \( L \), then the sum in equation (1) is finite, contained at most \( L(N-1) + 1 \) components.

The Fourier transform of \( N \)-scale equation is

\[
\hat{\varphi}(\omega) = H_0 \left( \frac{\omega}{N} \right) \hat{\varphi} \left( \frac{\omega}{N} \right),
\]

where

\[
H_0(\omega) = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} h_n e^{-in\omega}.
\]

The function \( H_0(\omega) \) is called frequency function of scaling function \( \varphi(x) \).

In the orthogonal case translations of scaling function \( \varphi(x-n), n \in \mathbb{Z} \) form orthonormal basis of the subspace \( V_0 \) in \( L^2(\mathbb{R}) \), and translations \( \varphi_{1,n}(x) = \sqrt{N} \varphi(Nx-n), n \in \mathbb{Z} \) on \( 1/N \), form orthonormal basis of the subspace \( V_1 \) in \( L^2(\mathbb{R}) \). Thus \( V_0 \subseteq V_1 \). In the orthogonal case to the scaling function \( \varphi(x) \) corresponds \( N-1 \) wavelets-functions \( \psi^1(x), \ldots, \psi^{N-1}(x) \), for each of which translations \( \psi^k_{0,n}(x) = \psi^k(x-n), n \in \mathbb{Z} \) form orthonormal basis of subspaces \( W^k_0 \) in \( L^2(\mathbb{R}) \), and expansion in the direct sum of orthogonal subspaces \( V_1 = V_0 \oplus W^1_0 \oplus \cdots \oplus W^{N-1}_0 \) be valid.
Wavelets $\psi^1(x), \ldots, \psi^{N-1}(x)$ form orthonormal basis $L^2(\mathbb{R})$:

$$\{\psi^k_{j,n}(x) = \sqrt{N} \psi^k(N j x - n), \ j, n \in \mathbb{Z}, \ k = 1, 2, \ldots, N-1\}.$$  

As wavelets $\psi^1(x), \ldots, \psi^{N-1}(x)$ belong to space $V_1$ they are decomposed on basis of this space,

$$\psi^k(x) = \sqrt{N} \sum_{n \in \mathbb{Z}} g_n^k \varphi(N x - n).$$  \hspace{1cm} (4)

The coefficients $\{g_n^k\}$ is called filters of wavelets $\psi^k(x), \ k = 1, 2, \ldots, N-1$. Let

$$H_k(\omega) = \frac{1}{\sqrt{N}} \sum_{n \in \mathbb{Z}} g_n^k e^{-i\omega n}$$  \hspace{1cm} (5)

- the frequency functions corresponding to wavelets $\psi^1(x), \ldots, \psi^{N-1}(x)$. The Fourier transform of equalities (4) is

$$\tilde{\psi}^k(\omega) = H_k\left(\frac{\omega}{N}\right) \tilde{\varphi}\left(\frac{\omega}{N}\right).$$

For the frequency functions $H_k(\omega)$ the following matrix is unitary $[2], [3]$,

$$H(z) = \begin{pmatrix}
H_0(z) & H_0(\rho z) & \cdots & H_0(\rho^{N-1} z) \\
H_1(z) & H_1(\rho z) & \cdots & H_1(\rho^{N-1} z) \\
\cdots & \cdots & \cdots & \cdots \\
H_{N-1}(z) & H_{N-1}(\rho z) & \cdots & H_{N-1}(\rho^{N-1} z)
\end{pmatrix},$$  \hspace{1cm} (6)

where $z = e^{-i\omega}$ and $\rho = e^{-i2\pi/N}$. The matrix (6) has special view. It is possible to avoid of this special view of the matrix $H(z)$ with Fourier transform on cyclic group $\mathbb{Z}/N\mathbb{Z} = \{1, \rho, \rho^2, \ldots, \rho^{N-1}\}$ $[2]$. We shall define

$$A_{k,j}(w) = \frac{1}{\sqrt{N}} \sum_{z^{N} = w} z^{-j} H_k(z).$$  \hspace{1cm} (7)

It is easy to see, that the sum on the right depends from $w = z^N$. Also transformation (7) accurate within coefficient $\sqrt{N}$ is sample of elements with degrees $z^k N$ in polynomials $H_k(z), z^{-1} H_k(z), \ldots, z^{-N+1} H_k(z)$. Inverse transformation is defined by the formula $[2]$

$$H_k(z) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} z^j A_{k,j}(z^N).$$  \hspace{1cm} (8)

From last relation we shall obtained the following matrix equality:

$$H(z) = \frac{1}{\sqrt{N}} A(z^N) \begin{pmatrix}
1 & 1 & \cdots & 1 \\
z & \rho z & \cdots & \rho^{N-1} z \\
z^{N-1} & \rho^{N-1} z z^{N-1} & \cdots & \rho^{(N-1)^2} z^{N-1}
\end{pmatrix} = A(z^N) R(z).$$  \hspace{1cm} (9)

In this expression the matrix $A(z^N)$ is already arbitrary unitary matrix with polynomial elements. Now specificity of the matrix $H(z)$ go to the matrix

$$R(z) = \frac{1}{\sqrt{N}} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
z & \rho z & \cdots & \rho^{N-1} z \\
z^{N-1} & \rho^{N-1} z z^{N-1} & \cdots & \rho^{(N-1)^2} z^{N-1}
\end{pmatrix}.$$  \hspace{1cm} (10)
Let’s mark, that the matrix $R(z)$ is unitary on the unit circle $z = e^{-i\omega}$.

Specifying the polyphase matrix $A(w)$, we can construct the matrix of frequency functions $H(z)$ by the formula (8) and, together with it, frequency functions of wavelets $H_1(z), \ldots, H_{N-1}(z)$, hence, and wavelets $\psi_1(x), \ldots, \psi_{N-1}(x)$.

In work [2] the scheme of construction of the polyphase matrix $A(z^N)$ is given in the supposition, that polynomial frequency function $H_0(z)$ is prescribed. Then it is possible to consider, that the first row of the matrix $A_{0j}(z^N)$ is known,

$$A_{0j}(w) = \frac{1}{\sqrt{N}} \sum_{z^N = w} z^{-j} H_0(z),$$

and it is necessary to construct remaining row of the matrix $A(w)$.

In the given work we shall give the simple scheme of construction of the unitary matrix $A(w)$ which elements are polynomials with real coefficients. It allows to define both the scaling function $\varphi(x)$ with compact support and with scaling coefficient $N > 2$, and orthogonal wavelets $\psi_1(x), \ldots, \psi_{N-1}(x)$.

2. Scheme of wavelets construction. From above constructions and methods of work [2] follows that orthogonal systems of wavelets can be determine by the unitary matrix $A(w)$ with polynomial elements with using of the formula $H(z) = A(z^N) R(z)$, where $R(z)$ – the special matrix (10). We shall give the simple method of construction enough big set of unitary matrixes $A(w)$ with polynomial elements. It will allow to obtain both the $N$-scaling function with the compact support, and orthogonal wavelets.

Let’s choose any orthogonal matrix $A_0 = \{a_{ij}, i, j = 0, 1, \ldots, N-1\}$ of the order $N \geq 2$. We shall multiply it on the diagonal unitary matrix $D_k(w) = \text{diag}(w^{k_0}, w^{k_1}, \ldots, w^{k_{N-1}})$, where $k = (k_0, k_1, \ldots, k_{N-1})$ is set of integers and $|w| = 1$, and then – on the orthogonal matrix $B_0 = \{b_{ij}, i, j = 0, 1, \ldots, N-1\}$. In outcome we shall obtain unitary matrix

$$A(w) = A_0 D_k(w) B_0,$$

which elements, $A_{ij} = \sum_{s=0}^{N-1} a_{is} b_{sj} w^{ks}$, are polynomials on the variable $w$ with real coefficients.

Now we shall substitute $w = z^N$, where $z = e^{-i\omega}$. We shall obtain the unitary matrix $A(z^N)$ with polynomial elements and real coefficients. We shall multiply it on the unitary matrix $R(z)$. Then we shall obtain the unitary matrix $H(z)$ of frequency polynomial functions $H_0(z), H_1(z), \ldots, H_{N-1}(z)$ of orthogonal system of wavelets $\varphi(x), \psi_1(x), \ldots, \psi_{N-1}(x)$, where the first function $\varphi(x)$ is scaling, and remaining – wavelets. Thus,

$$H(z) = \begin{pmatrix} H_0(z) & H_0(\rho z) & \ldots & H_0(\rho^{N-1} z) \\ H_1(z) & H_1(\rho z) & \ldots & H_1(\rho^{N-1} z) \\ \ldots & \ldots & \ldots & \ldots \\ H_{N-1}(z) & H_{N-1}(\rho z) & \ldots & H_{N-1}(\rho^{N-1} z) \end{pmatrix} = A_0 D_k(z^N) B_0 R(z).$$

From (13) follows the expression for frequency functions:

$$H_k(z) = \frac{1}{\sqrt{N}} \sum_{s,j=0}^{N-1} a_{ks} b_{sj} z^{js} z^{Nks}, \ k = 0, 1, \ldots, N-1.$$  

In order to the obtained the functions $H_k(z)$ would be frequency functions of orthogonal wavelets, it is necessary, that the sum of coefficients for $H_0(z)$ would be equal to unit, and the sums of coefficients for remaining functions $H_1(z), \ldots, H_{N-1}(z)$ would be equal to zero:

$$\frac{1}{\sqrt{N}} \sum_{s,j=0}^{N-1} a_{0s} b_{sj} = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} a_{0s} \sum_{j=0}^{N-1} b_{sj} = 1,$$
\[
\frac{1}{\sqrt{N}} \sum_{s,j=0}^{N-1} a_{ks} b_{sj} = \frac{1}{\sqrt{N}} \sum_{s=0}^{N-1} a_{ks} \sum_{j=0}^{N-1} b_{sj} = 0, \quad k = 0, 1, \ldots, N - 1.
\]

These equalities can be represented in the matrix view:

\[
\begin{pmatrix}
    a_{00} & a_{01} & \cdots & a_{0,N-1} \\
    a_{10} & a_{11} & \cdots & a_{1,N-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{N-1,0} & a_{N-1,1} & \cdots & a_{N-1,N-1}
\end{pmatrix}
\begin{pmatrix}
    b_{00} + \cdots + b_{0,N-1} \\
    b_{10} + \cdots + b_{1,N-1} \\
    \vdots \\
    b_{N-1,0} + \cdots + b_{N-1,N-1}
\end{pmatrix}
= \begin{pmatrix}
    \sqrt{N} \\
    0 \\
    \vdots \\
    0
\end{pmatrix}.
\tag{15}
\]

Choosing various orthogonal matrices \( A_0 \) and \( B_0 \), which satisfy the equality (15), we obtain various frequency functions of wavelets (14).

For construction enough simple class of orthogonal wavelets with the compact support and scaling coefficient \( N > 2 \), we shall take as an orthogonal matrix \( A_0 \) the following matrix:

\[
A_0 = \begin{pmatrix}
    \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{N}} & \cdots & \frac{1}{\sqrt{N}} \\
    \frac{1}{\sqrt{N}} & -\frac{1}{\sqrt{N}} & 0 & \cdots & 0 \\
    \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \cdots & 0 \\
    \frac{1}{\sqrt{(N-1)}(N-1)} & \frac{1}{\sqrt{(N-1)}(N-1)} & \frac{1}{\sqrt{(N-1)}(N-1)} & \cdots & -(N-1)/\sqrt{(N-1)}
\end{pmatrix}
\]

The matrix \( A_0 \) transform vector of units \( e = (1, 1, \ldots, 1) \) to the vector \( \sqrt{N}e_0 = (\sqrt{N}, 0, \ldots, 0) \), \( A_0 e = \sqrt{N}e_0 \). Then from equality (15) follows, that elements of the orthogonal matrix \( B_0 \) should satisfy to the following system of equations:

\[
\begin{cases}
    b_{00} + b_{01} + \cdots + b_{0,N-1} = 1 \\
    b_{10} + b_{11} + \cdots + b_{1,N-1} = 1 \\
    \vdots \\
    b_{N-1,0} + b_{N-1,1} + \cdots + b_{N-1,N-1} = 1
\end{cases}
\tag{16}
\]

The solution of this system will be any set of orthonormal vectors (rows) which coordinates satisfy to the equation of the plane \( x_0 + x_1 + \cdots + x_{N-1} = 1 \) in \( \mathbb{R}^N \). It is obvious, that coordinates of basis vectors \( e_0 = (1, 0, \ldots, 0) \), \( e_1 = (0, 1, 0, \ldots, 0) \), \( e_{N-1} = (0, 0, \ldots, 1) \) satisfy to this equation. The given solution corresponds to the identity matrix \( B_0 \). Any other solution can be obtained by rotation of the basis solution \( e_0, e_1, \ldots, e_{N-1} \) around of axis \( e = e_0 + e_1 + \cdots + e_{N-1} \), i.e. in the plane \( x_0 + x_1 + \cdots + x_{N-1} = 1 \). We shall find these solutions. We shall take rotation around of axis \( Ox_0 \):

\[
M = \begin{pmatrix}
    1 & 0 & 0 & \cdots & 0 \\
    0 & m_1^1 & m_1^1 & \cdots & m_1^1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & m_{N-1}^1 & m_{N-1}^1 & \cdots & m_{N-1}^1
\end{pmatrix}
\tag{17}
\]

As \( A_0 e = \sqrt{N}e_0 \), then rotation around of axis \( e \) is given by the matrix \( M_e = A_0^{-1} M A_0 \). Then rows of the matrix \( B_0 \) will consist of coordinates of vectors-columns which are obtained from \( e_0, e_1, \ldots, e_{N-1} \) by action on them matrix \( M_e \). Therefore the matrix \( B_0 \) is transposed to \( M_e \). Then

\[
H_M(z) = A_0 D_k(z^N)M_e^T R(z) = A_0 D_k(z^N)A_0^T M^T A_0 R(z),
\tag{18}
\]

where \( M \) - any orthogonal matrix of view (17) and \( D_k(w) = \text{diag}(w^{k_0}, w^{k_1}, \ldots, w^{k_{N-1}}) \).
The formula (18) gives the direct method of construction the big family of frequency functions $H_0(z)$, $H_1(z)$, ..., $H_{N-1}(z)$ and orthogonal wavelets with the compact support $\varphi(x)$, $\psi^1(x)$, ..., $\psi^{N-1}(x)$. Wavelets of the family depend on the orthogonal matrix $M$ of view (17) and of the vector of degrees $k = (k_0, k_1, ..., k_{N-1})$ which it is possible to set arbitrarily.

3. **Construction of orthogonal wavelets with compact support for $N = 2$.**

In the given section we shall show by the example of scale $N = 2$ effectiveness of the wavelets construction scheme explained above. Though the matrix $D_k(w)$ can be anyone, we shall take for example the diagonal matrix $D_1(w) = \text{diag}\{1, w\}$, $|w| = 1$. In case $N = 2$ orthogonal matrixes $A_0$ and $B_0$ can be in the general view:

$$A_0 = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \quad B_0 = \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix}.$$  

Then

$$H(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} \cos u & \sin u \\ -\sin u & \cos u \end{pmatrix} \begin{pmatrix} 1 & 1 \\ z & \rho z \end{pmatrix}.$$  

Frequency functions are

$$H_0(z) = \frac{1}{\sqrt{2}} (\cos t \cos u + (\cos t \sin u)z - (\sin t \sin u)z^2 + (\sin t \cos u)z^3), \quad (19)$$

$$H_1(z) = \frac{1}{\sqrt{2}} (-\sin t \cos u - (\sin t \sin u)z - (\cos t \sin u)z^2 + (\cos t \cos u)z^3), \quad (20)$$

The sum of coefficients of frequency function $H_0(z)$ should be equal to unit, and the sum of coefficients of frequency function $H_1(z)$ should be equal to zero. The system (15) becomes:

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \cos u + \sin u \\ \cos u - \sin u \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix},$$

$$\left\{ \begin{array}{l} \cos u + \sin u = \sqrt{2} \cos t \\ \cos u - \sin u = \sqrt{2} \sin t \end{array} \right.$$  

Solving last system, we obtain, $u = \pi/4 - t$.

Thus, we have constructed the family of frequency functions of the wavelets specified by formulas (19), (20) in which $u = \pi/4 - t$. After elimination of the variable $u$, we obtain::

$$H_0(z) = \frac{1}{4} (1 + \cos 2t + \sin 2t + (1 + \cos 2t - \sin 2t)z +$$

$$+ (1 - \cos 2t - \sin 2t)z^2 + (1 - \cos 2t + \sin 2t)z^3), \quad (21)$$

$$H_1(z) = \frac{1}{4} (-1 + \cos 2t - \sin 2t + (1 - \cos 2t - \sin 2t)z +$$

$$+ (-1 - \cos 2t + \sin 2t)z^2 + (1 + \cos 2t - \sin 2t)z^3). \quad (22)$$

The given frequency functions $H_0(z)$ and $H_1(z)$ coincide with the same, but obtained other methods in work [1]. Various wavelets of Haar, Daubechies wavelets and their analogs include into this family. In the following section some examples are given.

Choosing other matrix $D_k(z^N)$, similarly we can construct other orthogonal wavelets with other support length.

4. **Examples of scaling functions and wavelets for $N = 2$.** We shall calculate values of coefficients of the obtained frequency functions (19), (20) for various parameters
$t$ and $u$ and we shall find corresponding filters and wavelets $\varphi(x)$ and $\psi(x)$. From formulas (21), (22) follows what enough to take parameter values $t$ on interval of length $\pi$. We shall consider the following parameter values $t$: $0, \pm \pi/12, \pm \pi/6, \pm \pi/4, \pm \pi/3, \pm 5\pi/12, \pi/2$.

### 4.1 Parameter values $t = 0$, $u = \pi/4$.

Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(1, 1, 0, 0), \quad g_1 = \frac{1}{\sqrt{2}}(0, 0, -1, 1).$$

We have obtained wavelets of Haar with the support on unit interval. Refinement equations: $\varphi(x) = \varphi(2x) + \varphi(2x - 1)$ and $\psi(x) = -\psi(2x - 2) + \psi(2x - 3)$.

### 4.2 Parameter values $t = \pi/4$, $u = 0$.

Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \quad g_1 = \frac{1}{\sqrt{2}}(-1, 0, 0, 1).$$

We have obtained wavelets of Haar with the support on interval $[0, 3]$. Refinement equations: $\varphi(x) = \varphi(2x) + \varphi(2x - 3)$ and $\psi(x) = -\psi(2x) + \psi(2x - 3)$.

### 4.3 Parameter values $t = \pi/2$, $u = \pi/4$.

Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(0, 0, 1, 1), \quad g_1 = \frac{1}{\sqrt{2}}(-1, 1, 0, 0).$$

This is wavelets of Haar. Scaling function has the support on interval $[2, 3]$. Refinement equations: $\varphi(x) = \varphi(2x - 2) + \varphi(2x - 3)$ and $\psi(x) = -\psi(2x) + \psi(2x - 1)$.

### 4.4 Parameter values $t = \pi/4$, $u = \pi/2$.

Coefficients of wavelets filters:

$$h_0 = \frac{1}{\sqrt{2}}(0, 1, 1, 0), \quad g_1 = \frac{1}{\sqrt{2}}(0, 1, -1, 0).$$

This is wavelets of Haar. Scaling function has the support on interval $[1, 2]$. Refinement equations: $\varphi(x) = \varphi(2x - 1) + \varphi(2x - 2)$ and $\psi(x) = \psi(2x - 1) - \psi(2x - 2)$.

### 4.5 Parameter values $t = \pi/12$, $u = \pi/6$.

Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3 + \sqrt{3}, 1 + \sqrt{3}, 1 - \sqrt{3}, 3 - \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8}(-3 + \sqrt{3}, 1 - \sqrt{3}, -1 - \sqrt{3}, 3 + \sqrt{3}).$$

The result will be wavelets with coefficients which are obtained by permutation of coefficients of the classical Daubechies wavelets with the support of length 3. Refinement equations:

$$
\varphi(x) = \frac{3 + \sqrt{3}}{4} \varphi(2x) + \frac{1 + \sqrt{3}}{4} \varphi(2x - 1) + \frac{1 - \sqrt{3}}{4} \varphi(2x - 2) + \frac{3 - \sqrt{3}}{4} \varphi(2x - 3).
$$

In figure 1 graphs of wavelets are shown.

### 4.6 Parameter values $t = 5\pi/12$, $u = -\pi/6$.

Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3 - \sqrt{3}, 1 - \sqrt{3}, 1 + \sqrt{3}, 3 + \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8}(-3 - \sqrt{3}, 1 + \sqrt{3}, -1 + \sqrt{3}, 3 - \sqrt{3}).$$

This example differs from previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function can be obtained from scaling function of example 4.5 with the using of argument replacement: $\varphi(3 - x)$. It follows from the fact: if $\varphi(x)$ — scaling function with the compact support $[0, L]$ and the filter $\{h_n\}$ then function $\varphi(L - x)$ also is scaling with the filter $\{h_{L-n}\}$. The corresponding wavelet also can be obtained from previous as:
Figure 1: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t = \pi/12, u = \pi/6$

Figure 2: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t = 5\pi/12, u = -\pi/6$

$-\psi(3-x)$. The graph of scaling function $\varphi(x)$ can be obtained from the graph of the Fig.1 by mirroring about the line $x = 3/2$. For the graph of the wavelet $\psi(x)$ it is necessary to add still mirroring about axis $Ox$ (Fig. 2).

4.7. **Parameter values** $t = -\pi/12, u = \pi/3$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8} (1 + \sqrt{3}, 3 + \sqrt{3}, 3 - \sqrt{3}, 1 - \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8} (-1 + \sqrt{3}, 3 - \sqrt{3}, -3 - \sqrt{3}, 1 + \sqrt{3}).$$

The result will be Daubechies wavelets with the support of length 3. Refinement equation:

$$\varphi(x) = \frac{1 + \sqrt{3}}{4} \varphi(2x) + \frac{3 + \sqrt{3}}{4} \varphi(2x - 1) + \frac{3 - \sqrt{3}}{4} \varphi(2x - 2) + \frac{1 - \sqrt{3}}{4} \varphi(2x - 3).$$

In figure 3 graphs of wavelets are shown.

4.8. **Parameter values** $t = -5\pi/12, u = 2\pi/3$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8} (1 - \sqrt{3}, 3 - \sqrt{3}, 3 + \sqrt{3}, 1 + \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8} (-1 - \sqrt{3}, 3 + \sqrt{3}, -3 + \sqrt{3}, 1 - \sqrt{3}).$$

This example differs from the previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function can be obtained from Daubechies scaling function with the help of argument replacement: $\varphi(3-x)$, and wavelet is $-\psi(3-x)$.

4.9. **Parameter values** $t = \pi/6, u = \pi/12$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8} (3 + \sqrt{3}, 3 - \sqrt{3}, 1 - \sqrt{3}, 1 + \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8} (-1 - \sqrt{3}, 1 - \sqrt{3}, -3 + \sqrt{3}, 3 + \sqrt{3}).$$
The result will be wavelets with coefficients which are obtained by permutation of Daubechies wavelets coefficients. The refinement equation:

$$\varphi(x) = \frac{3 + \sqrt{3}}{4} \varphi(2x) + \frac{3 - \sqrt{3}}{4} \varphi(2x - 1) + \frac{1 - \sqrt{3}}{4} \varphi(2x - 2) + \frac{1 + \sqrt{3}}{4} \varphi(2x - 3).$$

In figure 4 graphs of wavelets are shown.

**4.10. Parameter values** $t = \pi/3$, $u = -\pi/12$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8} (1 + \sqrt{3}, 1 - \sqrt{3}, 3 - \sqrt{3}, 3 + \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8} (-3 - \sqrt{3}, 3 - \sqrt{3}, -1 + \sqrt{3}, 1 + \sqrt{3}).$$

This example differs from the previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function can be obtained from the previous scaling function by replacement of argument: $\varphi(3 - x)$, and wavelet is $-\psi(3 - x)$.

**4.11. Parameter values** $t = -\pi/3$, $u = 7\pi/12$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8} (1 - \sqrt{3}, 1 + \sqrt{3}, 3 + \sqrt{3}, 3 - \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8} (-3 + \sqrt{3}, 3 + \sqrt{3}, -1 - \sqrt{3}, 1 - \sqrt{3}).$$

The result will be wavelets with coefficients which are obtained by coefficients permutation of Daubechies wavelets with the support of length 3. Refinement equations:

$$\varphi(x) = \frac{1 - \sqrt{3}}{4} \varphi(2x) + \frac{1 + \sqrt{3}}{4} \varphi(2x - 1) + \frac{3 + \sqrt{3}}{4} \varphi(2x - 2) + \frac{3 - \sqrt{3}}{4} \varphi(2x - 3).$$
Figure 5: Graphs of functions $\varphi(x)$ and $\psi(x)$ for $t = -\pi/3$, $u = 7\pi/12$

In figure 5 graphs of wavelets are shown.

4.12. Parameter values $t = -\pi/6$, $u = 5\pi/12$. Coefficients of wavelets filters:

$$h_0 = \frac{\sqrt{2}}{8}(3 - \sqrt{3}, 3 + \sqrt{3}, 1 + \sqrt{3}, 1 - \sqrt{3}), \quad g_1 = \frac{\sqrt{2}}{8}(-1 + \sqrt{3}, 1 + \sqrt{3}, -3 - \sqrt{3}, 3 - \sqrt{3}).$$

This example differs from the previous only that coefficients of the filter $\{h_n\}$ go upside-down. In this case scaling function and wavelet can be obtained from the previous by replacement of argument: $\varphi(3 - x), -\psi(3 - x)$.

5. Construction of wavelets in case $N = 3$. In this section we shall show the scheme of scaling function and wavelets construction for $N = 3$. Though the diagonal matrix $D_k(w)$ can be anyone, we shall take for example the diagonal matrix $D_1(w) = \text{diag}(1, w, 1)$, $|w| = 1$. The matrix $A_0$ is:

$$A_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix}.$$ 

Elements of the second orthogonal matrix $B_0$ should satisfy to conditions:

$$b_{00} + b_{01} + b_{02} = 1, \quad b_{10} + b_{11} + b_{12} = 1, \quad b_{20} + b_{21} + b_{22} = 1.$$

The solution of this system will be any set of orthonormal vectors which coordinates satisfy to the equation of the plane $x_0 + x_1 + x_2 = 1$. It is obvious, that coordinates of basis vectors $e_1, e_2, e_3$ satisfy to this equation of plane. For this solution the matrix $B_0$ it is identity. And we obtain the wavelets of Haar,

$$H_0(z) = 1/3(1 + z^2 + z^4), \quad H_1(z) = \frac{1}{\sqrt{6}}(1 - z^4), \quad H_2(z) = \frac{1}{3\sqrt{2}}(1 - 2z^2 + z^4).$$

The maximum degree of frequency function $H_0(z)$ is equal to four, the support length $L$ is equal to two, as it is find from the formula $L(N-1) + 1 = \text{deg}(H_0(z)) + 1$. 

9
It is easy to see, that scaling function $\varphi(x)$ is characteristic function of interval $[0,2)$, $\varphi(x) = \chi_{[0,2)}(x)$. The refinement equation and wavelets (Fig. 4):

$$\varphi(x) = \varphi(3x) + \varphi(3x - 2) + \varphi(3x - 4),$$

$$\psi^1(x) = \frac{\sqrt{3}}{\sqrt{2}}(\varphi(3x) - \varphi(3x - 4)), $$

$$\psi^2(x) = \frac{1}{\sqrt{2}}(\varphi(3x) - 2\varphi(3x - 2) + \varphi(3x - 4)).$$

Any other solution can be obtained by rotation of basis vectors $e_0, e_1, e_2$ in the plane $x_0 + x_1 + x_2 = 1$. We shall find these solutions. Let

$$M(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

- the matrix of rotations around of the axis $e_0$. Then

$$M(t) = A_0^{-1}M(t)A_0 = \frac{1}{3} \begin{pmatrix} 1 + 2 \cos t & 1 - \cos t + \sqrt{3} \sin t & 1 - \cos t - \sqrt{3} \sin t \\ 1 - \cos t - \sqrt{3} \sin t & 1 + 2 \cos t & 1 - \cos t + \sqrt{3} \sin t \\ 1 - \cos t + \sqrt{3} \sin t & 1 - \cos t - \sqrt{3} \sin t & 1 + 2 \cos t \end{pmatrix}. $$

Let’s make rotation $M(t)e_k$ of column vectors $e_0 = (1,0,0), e_1 = (0,1,0), e_2 = (0,0,1)$, and we obtain rows of the required matrix $B_0(t)$:

$$B_0(t) = \frac{1}{3} \begin{pmatrix} 1 + 2 \cos t & 1 - \cos t - \sqrt{3} \sin t & 1 - \cos t + \sqrt{3} \sin t \\ 1 - \cos t + \sqrt{3} \sin t & 1 + 2 \cos t & 1 - \cos t - \sqrt{3} \sin t \\ 1 - \cos t - \sqrt{3} \sin t & 1 - \cos t + \sqrt{3} \sin t & 1 + 2 \cos t \end{pmatrix}. $$

Then $H(t, w) = A_0D_1(w)B_0(t)R(z)$ where the matrix $A_0D_1(z^N)$ is represented by the formula (23), $B_0(t)$ – by the formula (24) and the matrix $R(z)$ is

$$R(z) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ z & \rho z & \rho^2 z \\ z^2 & \rho z^2 & \rho^2 z^2 \end{pmatrix}.$$

Multiplying all these matrixes and choosing elements of the first column, we obtain,

$$H_0(t, z) = \frac{1}{9} \left(2 + \cos t - \sqrt{3} \sin t + (2 - 2 \cos t)z + (2 + \cos t + \sqrt{3} \sin t)z^2 + \\
(1 - \cos t + \sqrt{3} \sin t)z^3 + (1 + 2 \cos t)z^4 + (1 - \cos t - \sqrt{3} \sin t)z^5 \right),$$

$$H_1(t, z) = \frac{1}{3\sqrt{6}} \left(1 + 2 \cos t + (1 - \cos t - \sqrt{3} \sin t)z + (1 - \cos t + \sqrt{3} \sin t)z^2 - \\
- (1 - \cos t + \sqrt{3} \sin t)z^3 - (1 + 2 \cos t)z^4 + (1 + \cos t + \sqrt{3} \sin t)z^5 \right),$$

$$H_2(t, z) = \frac{1}{9\sqrt{2}} \left(-1 + 4 \cos t + 2\sqrt{3} \sin t - (1 - \cos t + 3\sqrt{3} \sin t)z - \\
2 \cos t - \sqrt{3} \sin t + (2 - 2 \cos t)z + (2 + \cos t + \sqrt{3} \sin t)z^2 + \\
(1 - \cos t + \sqrt{3} \sin t)z^3 + (1 + 2 \cos t)z^4 + (1 - \cos t - \sqrt{3} \sin t)z^5 \right).$$
Figure 6: Graphs of functions $\varphi(x)$, $\psi_1(x)$ and $\psi_2(x)$ for $t = 0$

\[-(1 + 5 \cos t + \sqrt{3} \sin t)z^2 +
+(1 - \cos t + \sqrt{3} \sin t)z^3 + (1 + 2 \cos t)z^4 + (1 - \cos t - \sqrt{3} \sin t)z^5\] . (27)

6. Examples of scaling functions and wavelets for $N = 3$. We shall calculate coefficients of the obtained frequency functions (25), (26) and (27) for various parameter values $t$. The obtained filters allow to find corresponding wavelets $\varphi(x)$, $\psi_1(x)$ and $\psi_2(x)$ by usual methods [5], [3]. It is enough to find scaling function $\varphi(x)$. Wavelets - functions $\psi_1(x)$ and $\psi_2(x)$ are defined by formulas

$$\psi_1(x) = \sqrt{N} \sum_{n \in \mathbb{Z}} g_1^n \varphi(Nx - n), \quad \psi_2(x) = \sqrt{N} \sum_{n \in \mathbb{Z}} g_2^n \varphi(Nx - n)$$

with known filters $\{g_1^n\}$ and $\{g_2^n\}$ and function $\varphi(x)$.

Let’s consider the following parameter values $t$: 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $\pi$, $4\pi/3$. For each case graphs of wavelets-functions are shown.

6.1. Value of parameter $t = 0$. This case has already been considered above. It is wavelets of Haar with the support $[0, 2]$ (Fig. 6).

6.2. Value of parameter $t = \pi/6$. Filters of scaling function $\varphi(x)$ and wavelets $\psi_1(x)$ and $\psi_2(x)$:

$$h_0 = \frac{\sqrt{3}}{9} (2, 2 - \sqrt{3}, 2 + \sqrt{3}, 1, 1 + \sqrt{3}, 1 - \sqrt{3})$$,

$$g_1 = \frac{\sqrt{6}}{18} (3 + \sqrt{3}, -3 + \sqrt{3}, \sqrt{3}, -\sqrt{3}, -3 - \sqrt{3}, 3 - \sqrt{3})$$,

$$g_2 = \frac{\sqrt{6}}{18} (-1 + 3\sqrt{3}, -1 - \sqrt{3}, -1 - 2\sqrt{3}, 1, 1 + \sqrt{3}, 1 - \sqrt{3})$$.

The refinement equation:

$$\varphi(x) = \frac{1}{3} (2 \varphi(3x) + (2 - \sqrt{3}) \varphi(3x - 1) + (2 + \sqrt{3}) \varphi(3x - 2) + \varphi(3x - 3) +
+(1 + \sqrt{3}) \varphi(3x - 4) + (1 - \sqrt{3}) \varphi(3x - 5)).$$

Graphs of wavelets are shown in figure 7

6.3. Value of parameter $t = \pi/4$. Filters of scaling function $\varphi(x)$ and wavelets $\psi_1(x)$ and $\psi_2(x)$:

$$h_0 = \frac{\sqrt{3}}{18} (4 + \sqrt{2} - \sqrt{6}, 4 - 2\sqrt{2}, 4 + \sqrt{2} + \sqrt{6}, 2 - \sqrt{2} + \sqrt{6}, 2 + 2\sqrt{2}, 2 - \sqrt{2} - \sqrt{6}).$$
Figure 7: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/6$

Figure 8: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/4$

$$g_1 = \frac{\sqrt{6}}{36}(2\sqrt{3} + 2\sqrt{6}, -3\sqrt{2} + 2\sqrt{3} - \sqrt{6}, 3\sqrt{2} + 2\sqrt{3} - \sqrt{6}, -3\sqrt{2} - 2\sqrt{3} + \sqrt{6},$$

$$-2\sqrt{3} - 2\sqrt{6}, 3\sqrt{2} - 2\sqrt{3} + \sqrt{6}),$$

$$g_2 = \frac{\sqrt{6}}{36}(-2 + 4\sqrt{2} + 2\sqrt{6}, -2 + \sqrt{2} - 3\sqrt{6}, -2 - 5\sqrt{2} + \sqrt{6}, 2 - \sqrt{2} + \sqrt{6},$$

$$2 + 2\sqrt{2}, 2 - \sqrt{2} - \sqrt{6}).$$

Graphs of wavelets are shown in figure 8.

6.4. Value of parameter $t = \pi/3$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$ are

$$h_0 = \frac{\sqrt{3}}{9}(1, 1, 4, 2, 2, -1),$$

$$g_1 = \frac{\sqrt{2}}{6}(2, -1, 2, -2, -2, 1), \quad g_2 = \frac{\sqrt{6}}{18}(4, -5, -2, 2, 2, -1).$$

The refinement equation:

$$\varphi(x) = \frac{1}{3}(\varphi(3x) + \varphi(3x - 1) + 4\varphi(3x - 2) + 2\varphi(3x - 3) + 2\varphi(3x - 4) - \varphi(3x - 5)).$$

Graphs of wavelets are shown in figure 9.

6.5. Value of parameter $t = \pi/2$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:
Figure 9: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/3$

Figure 10: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi/2$

The refinement equation:

$$h_0 = \frac{\sqrt{3}}{9}(2 - \sqrt{3}, 2, 2 + \sqrt{3}, 1 + \sqrt{3}, 1, 1 - \sqrt{3}),$$

$$g_1 = \frac{\sqrt{6}}{18}(\sqrt{3}, -3 + \sqrt{3}, 3 + \sqrt{3}, -3 - \sqrt{3}, -\sqrt{3}, 3 - \sqrt{3}),$$

$$g_2 = \frac{\sqrt{6}}{18}(-1 + 2\sqrt{3}, -1 - 3\sqrt{3}, -1 + \sqrt{3}, 1 + \sqrt{3}, 1, 1 - \sqrt{3}).$$

The refinement equation:

$$\varphi(x) = \frac{1}{3}((2 - \sqrt{3})\varphi(3x) + 2\varphi(3x-1) + (2 + \sqrt{3})\varphi(3x-2) + (1 + \sqrt{3})\varphi(3x-3) +$$

$$+ \varphi(3x-4) + (1 - \sqrt{3})\varphi(3x-5)).$$

6.6. Value of parameter $t = 2\pi/3$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$ is

$$h_0 = \frac{1}{\sqrt{3}}(0, 1, 1, 0, 0),$$

$$g_1 = \frac{1}{\sqrt{2}}(0, 0, 1, -1, 0, 0),
\quad g_2 = \frac{1}{\sqrt{6}}(0, -2, 1, 1, 0, 0).$$
Figure 11: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = 2\pi/3$

Figure 12: Graphs of functions $\varphi(x)$, $\psi^1(x)$ and $\psi^2(x)$ for $t = \pi$

It is wavelets of Haar. The scaling function $\varphi(x)$ is characteristic function of interval $[1/2, 3/2)$, $\varphi(x) = \chi_{[1/2, 3/2)}(x)$. The refinement equation and wavelets:

$$\varphi(x) = \varphi(3x - 1) + \varphi(3x - 2) + \varphi(3x - 3),$$

$$\psi^1(x) = \frac{\sqrt{3}}{\sqrt{2}}(\varphi(3x - 2) - \varphi(3x - 3)), $$

$$\psi^2(x) = \frac{1}{\sqrt{2}}(-2\varphi(3x - 1) + \varphi(3x - 2) + \varphi(3x - 3)).$$

In figure 11 graphs of wavelets are shown.

6.7. Value of parameter $t = \pi$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

$$h_0 = \frac{\sqrt{3}}{9}(1, 4, 1, 2, -1, 2),$$

$$g_1 = \frac{\sqrt{2}}{6}(-1, 2, 2, -2, 1, -2), \quad g_2 = \frac{\sqrt{6}}{18}(-5, -2, 4, 2, -1, 2).$$

The refinement equation:

$$\varphi(x) = \frac{1}{3}(\varphi(3x) + 4\varphi(3x - 1) + \varphi(3x - 2) + 2\varphi(3x - 3) - \varphi(3x - 4) + 2\varphi(3x - 5)).$$

Graphs of wavelets are shown in figure 12.
6.8. **Value of parameter** $t = 4\pi/3$. Filters of scaling function $\varphi(x)$ and wavelets $\psi^1(x)$ and $\psi^2(x)$:

$$h_0 = \frac{1}{\sqrt{3}}(1, 1, 0, 0, 0, 1),$$
$$g_1 = \frac{1}{\sqrt{2}}(0, 1, 0, 0, 0, -1), \quad g_2 = \frac{1}{\sqrt{6}}(-2, 1, 0, 0, 0, 1).$$

The refinement equation and frequency functions:

$$\varphi(x) = \varphi(3x) + \varphi(3x - 1) + \varphi(3x - 5),$$

$$H_0(z) = \frac{1}{3}(1 + z + z^5), \quad H_1(z) = \frac{1}{\sqrt{6}}(z - z^5), \quad H_2(z) = \frac{1}{3\sqrt{2}}(-2 + z + z^5).$$

Let’s mark, that scaling function $\varphi(x)$ has a complicated structure. Its support has fractal properties.

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