The Window Validity Problem in Rule-Based Stream Reasoning

Alessandro Ronca, Mark Kaminski, Bernardo Cuenca Grau and Ian Horrocks
Department of Computer Science, University of Oxford, UK
\{alessandro.ronca, mark.kaminski, bernardo.cuenca.grau, ian.horrocks\}@cs.ox.ac.uk

Abstract

Rule-based temporal query languages provide the expressive power and flexibility required to capture in a natural way complex analysis tasks over streaming data. Stream processing applications, however, typically require near real-time response using limited resources. In particular, it becomes essential that the underpinning query language has favourable computational properties and that stream processing algorithms are able to keep only a small number of previously received facts in memory at any point in time without sacrificing correctness. In this paper, we propose a recursive fragment of temporal Datalog with tractable data complexity and study the properties of a generic stream reasoning algorithm for this fragment. We focus on the window validity problem as a way to minimise the number of time points for which the stream reasoning algorithm needs to keep data in memory at any point in time.

1 Introduction

Query processing over streams is becoming increasingly important for data analysis in domains as diverse as financial trading (Nuti et al. 2011), equipment maintenance (Cosad et al. 2009), or network security (Münz and Carle 2007).

A growing body of research has recently focused on extending traditional stream management systems with reasoning capabilities (Barbieri et al. 2010; Calbimonte, Corcho, and Gray 2010; Amicic et al. 2011; Le-Phuoc et al. 2011; Zaniolo 2012; Özçep, Möller, and Neuenstadt 2014; Beck et al. 2015; Dao-Tran, Beck, and Eiter 2015; Ronca et al. 2018). Languages well-suited for stream reasoning applications are typically rule-based, where prominent examples include temporal Datalog (Chomicki and Imieliński 1988) and DatalogMTL (Brandt et al. 2017). These core languages are powerful enough to capture many other temporal formalisms (Abadi and Manna 1989; Baudinet, Chomicki, and Wolper 1993) and provide the logical underpinning for other expressive languages proposed in the stream reasoning literature (Zaniolo 2012; Beck et al. 2015).

Rules provide the expressive power and flexibility required to naturally capture in a declarative way complex analysis tasks over streaming data. This is illustrated by the following example in network security, where intrusion detection policies (IDPs) are represented in temporal Datalog.

Example 1. Consider a computer network which is being monitored for external threats. Bursts (usually high amounts of data) between any pair of nodes in the network are detected by specialised monitoring devices and streamed to the network’s management centre as timestamped facts. A monitoring task in the centre is to identify nodes that may have been hacked according to a specific IDP, and add them to a blacklist of nodes. In this setting, one may want to know the contents of the blacklist at any given point in time in order to decide on further action. This task is captured by a temporal Datalog query consisting of the rules given next and where Black is the designated output predicate:

\[
\begin{align*}
\text{Brt}(x, y, t) &\land \text{Brt}(z, y, t + 1) \rightarrow \text{Atk}(x, y, t + 1) \quad (1) \\
\text{Atk}(x, y, t) &\land \text{Atk}(x, y, t + 1) \\
&\land \text{Atk}(x, y, t + 2) \rightarrow \text{Black}(x, t + 2) \quad (2) \\
\text{Black}(x, t) &\rightarrow \text{Black}(x, t + 1) \quad (3) \\
\text{Atk}(x, y, t) &\rightarrow \text{Grey}(x, \text{max}, t) \quad (4) \\
\text{Grey}(x, i, t) &\land \text{Succ}(j, i) \rightarrow \text{Grey}(x, j, t + 1) \quad (5) \\
\text{Grey}(x, i, t) &\land \text{Brt}(x, y, t) \rightarrow \text{Black}(x, t) \quad (6)
\end{align*}
\]

Rule (1) identifies two consecutive bursts from nodes \(v\) and \(v'\) to a node \(v''\) in the network as an attack on \(v''\) originated by \(v\). Rule (2) implements an IDP where three consecutive attacks from \(v\) to \(v''\) result in \(v\) being added to the blacklist, where it remains indefinitely (Rule (3)). Rules (4)–(6) implement a second IDP where an attack from \(v\) on any node leads to \(v\) being identified as suspicious and added to a “greylist”. Such list comes with a succession of decreasing warning levels, where the maximum is represented by the constant \(\text{max}\) and where the relationship from each level to the next is captured by a binary, non-temporal, \(\text{Succ}\) predicate. As time goes by, the warning level decreases; however, if at any point during this process node \(v\) generates another burst to any other node in the network, then it gets blacklisted.

Stream processing applications typically require near real-time response using limited resources; this becomes especially challenging in the context of rule-based stream reasoning due to the following reasons:

1. Fact entailment over temporal rule languages is typically intractable in data complexity—\(\text{EXPSPACE}\)-complete in the case of DatalogMTL and \(\text{PSPACE}\)-complete in the case of temporal Datalog. Furthermore, known tractable
fragments are non-recursive \cite{Ronca2018,Brandt2017}, which limits their applicability to certain data analysis applications.

2. In order to adhere to memory limitations and scalability requirements, systems can only keep a limited history of previously received input facts in memory to perform further computations. Rules, however, can propagate derived information both forwards back and future time points and hence query answers can depend on data that has not yet been received as well as on data that arrived in the past. This may force the system to keep in memory a very large (or even unbounded) input history to ensure correctness.

We address the first challenge by introducing in Section 3 the language of forward-propagating queries—a fragment of temporal Datalog that extends plain (non-temporal) Datalog by allowing unrestricted recursive propagation of information into future time points, while at the same time precluding propagation of derived facts towards past time points. Our language is sufficiently expressive to capture interesting analysis tasks over streaming data, such as the one illustrated in our previous example. Moreover, we show that forward-propagating queries can be answered in polynomial time in the size of the input data and hence they are well-suited for data-intensive applications.

To address the second challenge, we take as a starting point a generic algorithm which accepts as input a set of non-temporal background facts and a stream of timestamped facts and outputs as a stream the answers to a forward-propagating query \( Q \). The algorithm is parameterized by a window size \( w \) and a signature \( \Sigma \), which determine the set of facts stored in memory by the algorithm at any point in time. As the algorithm receives the input stream at time point \( \tau \), it computes all implicit \( \Sigma \)-facts and answers to \( Q \) holding at \( \tau \) using only facts held in memory, and subsequently discards all stored facts holding at \( \tau - w \). For the algorithm to be correct, the computed answers for each \( \tau \) over the restricted set of facts in memory must coincide with the answers over the entire stream. Such an assurance, however, can only be given for certain values of \( \Sigma \) and if the window parameter \( w \) is large enough so that facts that may influence answers at later time points are not discarded too early. This motivates the window validity problem, which is to decide whether a given window \( w \) is valid for a given query \( Q \) and signature \( \Sigma \) in the sense that the aforementioned correctness guarantee holds for any input data.

In our prior work \cite{Ronca2018}, we considered an instantiation of the generic stream reasoning algorithm where only explicit facts from the input stream (and hence no entailed facts) are kept in memory. This setting can, however, be problematic in the presence of recursion, in that a recursive query may not admit a valid window. Stream reasoning clearly becomes impractical for such queries since the entire stream received so far must be kept in memory by the algorithm in order to ensure correctness.

To address this limitation, we consider in Section 4 a full materialization variant of the algorithm in which all facts (explicit or implicit) over the entire signature are kept in memory; as a result, when a fact is discarded by the algorithm, its consequences at later time points are not lost. In this setting, we can show that a valid window is guaranteed to exist for any forward-propagating query, and a (possibly larger than needed) window can be obtained syntactically by inspection of the query. From a practical perspective, however, it is important to have a valid window that is as small as possible since the number of facts entailed by the query’s rules at any given time point can be very large. Thus, we investigate in Sections 5 and 6 the computational properties of window validity in this revised setting.

In Section 3 we show that window validity and query containment are interreducible problems, and hence known complexity bounds on temporal query containment transfer directly. In particular, undecidability of window validity for forward-propagating queries follows from the undecidability of query containment for non-temporal Datalog.

To regain decidability, we consider in Section 4 the situation where the set of relevant domain objects can be fixed in advance, in the sense that input facts can refer only to those objects. In Example 1 this assumption amounts to fixing both the nodes in the network and the grey list’s warning levels, and requiring that all input facts mention only these objects. This assumption allows us to ground the non-temporal variables of the query to a set of known objects; such grounding is exponential and results in an object-ground query where all variables are temporal. We show that the window problem is PSPACE-complete for object-ground forward-propagating queries and coNP-complete if the query is also non-recursive. This immediately gives us an EXPSPACE upper bound (coNEXP if queries are additionally assumed to be non-recursive) for the fixed-domain window validity problem; we then prove that these bounds are tight. Our results show that, although window validity is undecidable, we can obtain decidability under reasonable assumptions on the input data. Even under such assumptions, the problem is computationally intractable; however, queries can be assumed to be relatively small in practice and windows can be computed offline, prior to receiving any data.

Finally, for applications where one cannot assume the object domain to be fixed in advance, we propose in Section 5 a sufficient condition for the validity of a window that can be checked in exponential time without additional assumptions.

Complete proofs of all our technical results are deferred to the appendix.

2 Preliminaries

We recapitulate temporal Datalog \cite{Chomicki1988} as a basic language for stream reasoning.

Syntax A signature consists of predicates, constants and variables, where constants are partitioned into objects and non-negative integer time points and variables are partitioned into object variables and time variables. An object term is an object or an object variable. A time term is a time point, a time variable, or an expression of the form \( t + k \) with \( t \) a time variable, \( k \) an integer, and + the integer addition function.

Predicates are partitioned into extensional (EDB) and in-
tensional (IDB) and they come with a non-negative integer
arity $n$, where each position $1 \leq i \leq n$ is of either object or
time sort. A predicate is rigid if all its positions are of object sort
and it is temporal if the last position is of time sort and
all other positions are of object sort. An atom is an expres-
sion $P(s_1, \ldots, s_n)$, where $P$ is an $n$-ary predicate and each
$s_i$ is a term of the required sort; we sometimes use the term
$P$-atom to refer to an atom with predicate $P$. A rigid atom
(respectively, temporal, IDB, EDB) is an atom over a rigid
predicate (respectively, temporal, IDB, EDB).

A rule $r$ is of the form $\bigwedge_i \alpha_i \rightarrow \alpha$, where $\alpha$ and each $\alpha_i$
are rigid or temporal atoms, and $\alpha$ is IDB whenever $\bigwedge_i \alpha_i$
is non-empty. Atom head($r$) = $\alpha$ is the head of $r$, and
body($r$) = $\bigwedge_i \alpha_i$ is the body of $r$. Rules are safe—that is,
all variables occur in the body. A program $\Pi$ is a finite set
of rules. A term, atom, rule, or program is ground if it has
no variables. A predicate $P$ is $\Pi$-dependent on predicate $P'$
if $P$ has a rule with $P$ in the head and $P'$ in the body. A
fact is a ground, function-free rigid or temporal atom, and a
dataset is a (possibly infinite) set of EDB facts. Each fact $\alpha$
corresponds to a rule having empty body and $\alpha$ in the head,
so we use $\alpha$ and its corresponding rule interchangeably.

A query is a pair $Q = (P_Q, \Pi_Q)$ with $P_Q$ a program and
$P_Q$ an IDB output predicate in $\Pi_Q$ not occurring in the body
of any rule in $\Pi_Q$. We also denote with $\Sigma_Q$ the set of all IDB
predicates in $\Pi_Q$. Query $Q$ is
- temporal if $P_Q$ is a temporal predicate;
- Datalog if no temporal predicate occurs in $\Pi_Q$;
- object-ground if $\Pi_Q$ has no object variables; and
- non-recursive if the directed graph induced by the $\Pi_Q$-
dependencies is acyclic.

Semantics Rules are interpreted as universally quantified
first-order sentences. A Herbrand interpretation $H$ is a (pos-
sibly infinite) set of facts. It satisfies a rigid atom $\alpha$ if $\alpha \in H$,
and it satisfies a temporal atom $\beta$ if evaluating the addi-
tion function in $\beta$ yields a fact in $H$. Satisfaction is extended
to conjunctions of ground rules, rules and programs in the
standard way. If $H \models \Pi$, then $H$ is a model of $\Pi$. Program $\Pi$
entails a fact $\alpha$, written $\Pi \models \alpha$, if $H \models \Pi$ implies $H \models \alpha$.
The set of answers to a query $Q$ over a dataset $D$, written
$Q(D)$, consists of each $P_Q$-fact $\alpha$ such that $\Pi_Q \cup D \models \alpha$.

Reasoning We next define two basic reasoning problems,
which we parametrise to specific classes of input queries $Q$
and datasets $D$. Similarly to [Chomicki and Imieliński
1988], we assume from now onwards in all reasoning prob-
lems that numbers in input queries and datasets are coded
in unary; our complexity results may (and almost certainly
will) change if binary encoding is assumed, and we leave
this investigation for future work. Furthermore, we make
the following general assumptions for each $D$: (1) for each
$D' \in D$ and each finite subset $S$ of $D$ there is a finite $D' \subseteq D$
such that $S \subseteq D' \subseteq D$; and (2) for each $D \in D$ and unary
temporal fact $\alpha$, we have $D \cup \{\alpha\} \in D$. The former prop-
erty is a form of compactness closure, whereas the latter is a
closure property under addition of unary temporal facts.

The query evaluation problem $\text{EVAL}_Q^D$, for $Q$ a class of
queries and $D$ a class of finite datasets, is to check whether
$\alpha \in Q(D)$ for $\alpha$ an input fact, $Q \in Q$ and $D \in D$; the data
complexity of $\text{EVAL}_Q^D$ is the complexity for fixed $Q$. Query
evaluation for arbitrary datasets is PSPACE-complete in data
complexity under unary encoding of numbers [Chomicki
and Imieliński 1988], and in $AC^0$ for non-recursive queries.

Let $Q_1$ and $Q_2$ be queries having the same output predi-
cate. Then, $Q_1$ is contained in $Q_2$ with respect to $D$, written
$Q_1 \subseteq_D Q_2$, if $Q_1(D) \subseteq Q_2(D)$ for each $D \in D$. The con-
tainment problem $\text{CONT}_Q^D$ is to check $Q_1 \subseteq_D Q_2$ for given
$Q_1, Q_2 \in Q$. For simplicity, we drop $D$ from $Q_1 \subseteq_D Q_2$
and $\text{CONT}_Q^D$ (respectively, from $\text{EVAL}_Q^D$) whenever $D$ is the
class of all datasets (respectively, of all finite datasets).

Our definition of containment considers infinite datasets,
which is required to capture streams. This does not change
the nature of the problem due to the properties of first-order
logic and our assumptions on $D$; as shown in the appendix,
$Q_1 \subseteq_D Q_2$ if and only if $Q_1 \subseteq_{D'} Q_2$ with $D'$ the class
consisting of all finite datasets in $D$. By standard results in
non-temporal Datalog, it follows that unrestricted containment
is undecidable [Shmueli 1993], and it is coNP-hard for non-
recursive queries [Benedikt and Gottlob 2010].

3 Forward-Propagating Queries

Stream processing applications are data-intensive, requiring
fast response using limited resources. Tractability of query
evaluation in data complexity is thus a key requirement for
logics underpinning stream reasoning systems. Query evalua-
tion in temporal Datalog is, however, PSPACE-complete in
data complexity, which limits its applicability.

In this section we introduce the language of forward-
propagating queries—a fragment of temporal Datalog which
allows unrestricted recursive propagation of derived facts
into the present and future time points, while at the same
time precluding propagation towards past time points.

Definition 2. The offset of a time term $s$ equals zero if $s$ is a
time variable, and it equals $k$ if $s$ is the time point $k$ or
a time term of the form $t + k$. The radius of a rule is zero if its
head is rigid, and it is the maximum difference between the
offset of its head time argument and the offset of a body time
argument otherwise. A rule $r$ is forward-propagating if it is
datalog, or it satisfies all of the following properties:
- it contains no time points;
- it has a single time variable, which occurs in the head;
- its radius is non-negative.

A query $Q$ is forward-propagating, or an fp-query for short,
if so is each rule in $\Pi_Q$. The radius of $Q$ is the maximum
radius amongst the rules in $\Pi_Q$. For $k \geq 0$, we denote as
$Q^k$ the query $(P_Q, \Pi_Q^k)$ with $\Pi_Q^k$ the subset of rules in $\Pi_Q$
with radius at most $k$.

We denote the class of fp-queries as FP, and let OG, NR, 
and OGNR be the subclasses of FP where queries are re-
quired to be object-ground, non-recursive, and both object-
ground and non-recursive, respectively.

Example 3. The query in our running Example 7 is forward-
propagating. Its radius is two, which is justified by Rule 2,
where the offset of the head is two and the offset of the first
body atom is zero.
Algorithm 1: A generic stream reasoning algorithm

Parameters: Temporal fp-query \( Q \), window size \( w \), and a subset \( \Sigma \) of the IDBs in \( Q \) with \( P_Q \in \Sigma \).

Input: Background dataset \( B \), stream \( S \).

1. Assign \( M := B \) and \( \tau := 0 \).
2. loop
   3. Receive \( S|\tau \) and assign \( M := M \cup S|\tau \).
   4. Add to \( M \) all \( \Sigma \)-facts \( \alpha \) holding at \( \tau \) s.t. \( \Pi_Q \cup M \models \alpha \).
   5. Stream out all \( P_Q \)-facts in \( M|\tau \).
   6. If \( \tau \geq w \), remove from \( M \) all facts in \( M|\tau-w \).
   7. \( \tau := \tau + 1 \).

end

The conditions in Definition 2 ensure that the derivation via rule application of a fact \( \alpha \) holding at a time point \( \tau \) can be justified by facts holding at time points no greater than \( \tau \): as a result, one can safely disregard all facts holding after \( \tau \) for the purpose of deriving \( \alpha \).

The restrictions imposed by Definition 2 are sufficient to ensure tractability of query evaluation, while at the same time allowing for temporal recursion. The following theorem shows a stronger result, namely that query evaluation over fp-queries can be reduced to query evaluation over standard non-temporal Datalog.

Theorem 4. Let \( \mathcal{D} \) be a class of finite datasets, let \( Q \in \{ \text{FP}, \text{NR}, \text{OG}, \text{OGNR} \} \), and let \( Q' \) be the Datalog subset of \( Q \). Then, \( \text{EVAL}^\mathcal{Q} \) is \text{LOGSPACE}-reducible to \( \text{EVAL}^{Q'} \).

Proof sketch. To check whether \( \Pi_Q \cup \mathcal{D} \) entails fact \( \alpha \) holding at a time point \( \tau \), it suffices to consider facts (explicitly given or derived) holding at time points in the interval between the minimum time point mentioned in \( \mathcal{D} \) and \( \tau \); such interval contains linearly-many time points due to \( \tau \) being encoded in unary. We can then transform \( \Pi_Q \) in \text{LOGSPACE} into a plain Datalog program \( \Pi' \) by first introducing an object for each time point in the interval, and then grounding the temporal arguments of all rules in \( \Pi_Q \) over these objects. Clearly, it holds that \( \Pi_Q \cup \mathcal{D} \) entails \( \alpha \) if so does \( \Pi' \cup \mathcal{D} \). \( \Box \)

Theorem 4 allows us to immediately transfer known complexity bounds for query evaluation over different classes of Datalog queries to the corresponding class of fp-queries—see, e.g., (Dantsin et al. 2001; Vorobyov and Voronkov 1998). In particular, it follows that evaluation of fp-queries is tractable in data complexity.

Corollary 5. The following complexity bounds hold for the query evaluation problem over classes of fp-queries:
- \( \text{EVAL}^\text{FP} \) is \text{EXP}-complete and \text{P}-complete in data;
- \( \text{EVAL}^\text{NR} \) is \text{PSPACE}-complete and in \( \text{AC}^0 \) in data; and
- \( \text{EVAL}^\text{OG} \) is \text{P}-complete.

4 A Generic Stream Reasoning Algorithm

A stream reasoning algorithm receives as input an unbounded stream \( S \) of timestamped facts and a set \( B \) of rigid background facts, and outputs (also as a stream) the answers to a standing temporal query \( Q \), which is considered fixed.

Algorithm 1 which we describe next, is a generic such algorithm that is applicable to any fp-query. In the algorithm (as well as in the rest of the paper), we denote with \( F[\tau, \tau'] \) the subset of temporal facts in a dataset \( F \) holding in the interval \([\tau, \tau']\), and write \( F|\tau \) for \( F[\tau, \tau] \). Furthermore, from now on we will silently assume all queries to be temporal.

Algorithm 1 is parametrised by an fp-query \( Q \), a non-negative integer window size \( w \) and a signature \( \Sigma \), where the latter two parameters determine the set of facts \( M \) kept in memory by the algorithm at any point in time. The algorithm is initialised in Line 1, where the input set \( B \) of rigid background facts is loaded into memory and the current time \( \tau \) is set to zero. The core of the algorithm is an infinite loop, where each iteration consists of the following four steps and the current time \( \tau \) is incremented at the end of each iteration.
1. The batch of input stream facts holding at \( \tau \) is received and loaded into memory (Line 3).
2. All implicit facts over the relevant signature \( \Sigma \) holding at \( \tau \) are computed and materialised in memory (Line 4).
3. Query answers holding at \( \tau \) are read from memory and streamed out (Line 5).
4. All facts (explicit in \( S \) or implicitly derived) holding at \( \tau - w \) are removed from memory (Line 6).

In order to favour scalability, Algorithm 1 restricts at any point in time the set of facts kept in memory and therefore considered for query evaluation. This, however, carries the obvious risk that valid answers holding over the entire stream may be missed by the algorithm if the facts they depend on are removed from memory too early. Therefore, the window size of the algorithm should be chosen so that the following correctness property is satisfied.

Definition 6. A window size \( w \) is valid for an fp-query \( Q \), a signature \( \Sigma \), and a class \( \mathcal{D} \) of datasets if, when parametrised with \( Q \), \( w \) and \( \Sigma \), and for each input \( (B, S) \) with \( B \cup S \in \mathcal{D} \) and each \( n > 0 \), the set of facts streamed out by Algorithm 1 in the first \( n \) iterations coincides with \( Q(B \cup S)[0, n-1] \).

In prior work (Ronca et al. 2018) we considered an algorithm that does not keep derived facts (other than possibly query answers) in memory and thus only stores EDB facts from the input stream. When applied to an fp-query \( Q \), the algorithm in our previous work can be seen as a variant of Algorithm 1 where \( \Sigma = \{ P_Q \} \). This variant of Algorithm 1 is, however, problematic for recursive queries since no valid window size may exist, in which case the entire stream received so far must be kept in memory to ensure correctness.

Proposition 7. There exists no valid window size for the object-ground fp-query \( Q \) where, for an EDB predicate, \( \Pi_Q = \{ A(t) \rightarrow B(t); B(t) \rightarrow B(t+1); B(t) \rightarrow P_Q(t) \} \), \( \Sigma = \{ P_Q \} \), and the class of all datasets.

To address this limitation, we focus from now onwards on a full materialisation variant of Algorithm 1 in which the signature parameter is fixed to the set \( \Sigma_Q \) of all IDB predicates in \( Q \)—that is, where the algorithm keeps in memory a complete materialisation of the query’s program for the relevant time points. Computing and incrementally maintaining a full materialisation is a common reasoning approach adopted by many rule-based systems (Motik et al. 2015; Motik et al. 2014; Leone et al. 2006; Baget et al. 2015).
this setting, we will be able to ensure existence of a valid window size for any fp-query, and to show that a (maybe larger than needed) valid window size can be obtained syntactically by inspecting the rules in the query one at a time.

Towards this goal, we first analyse the aforementioned stream reasoning algorithm parametrised with query \( Q \), window size \( w \), and signature \( \Sigma_Q \), and show that only the rules in \( Q \) with radius at most \( w \) can contribute to the output.

**Theorem 8.** Consider Algorithm 1 parametrised with \( Q \), \( w \) and \( \Sigma_Q \). On input \( \{B, S\} \), the set of \( P_Q \)-facts streamed out in the first \( n \) iterations coincides with \( Q^w(B \cup S) \mid [0, n-1] \).

**Proof sketch.** We show by induction on \( \tau \) that the set of temporal facts stored in \( M \) right after executing Line 4 of the algorithm’s main loop coincides with the temporal facts entailed by \( \Pi_Q^w \cup B \cup S \) and holding at any \( \tau' \in [\tau - w, \tau] \), which directly implies the statement of the theorem. On the one hand, we show that any derivation from \( \Pi_Q \cup M \) of a fact \( \alpha \) holding at \( \tau \) can involve only rules from \( \Pi_Q^w \); in particular, any derivation involving a rule in \( \Pi_Q \) with radius exceeding \( w \) would require some fact holding at a time point prior to \( \tau - w \); where all such facts were removed from \( M \) in previous iterations of the algorithm. On the other hand, we show that all facts holding at \( \tau \) entailed by \( \Pi_Q^w \cup B \cup S \) admit a derivation involving only facts holding in \( [\tau - w, \tau] \); by the induction hypothesis, all such facts are in \( M \) when Line 4 of the algorithm is executed in the loop’s iteration for \( \tau \).

Theorem 8 immediately yields a characterisation of window size validity in terms of query containment.

**Corollary 9.** A window size \( w \) is valid for an fp-query \( Q \), the signature \( \Sigma_Q \), and a class \( D \) of datasets iff \( Q \subseteq_D Q^w \).

Since \( Q \) and \( Q^w \) coincide unless the radius of \( Q \) exceeds \( w \), we can conclude that the radius of \( Q \) is always a valid window size.

**Corollary 10.** Let \( Q \) be an fp-query. Then, the radius of \( Q \) is a valid window size for \( Q \), \( \Sigma_Q \), and any class \( D \) of datasets.

## 5 The Window Validity Problem

The full materialisation of a query for any given time point may be rather large. Having a valid window size that is as small as possible is thus important for Algorithm 1 to be practically feasible, where even a small improvement on the window size can lead to a significant reduction in the number of facts stored in memory and used for query evaluation.

In particular, the radius of the query yields a valid window size that may be larger than strictly necessary. For instance, our running example query has a radius of two, which would require Algorithm 1 to keep a full materialisation for three consecutive time points; however, the query admits a valid window size of just one since the policy implemented by Rule 2 is subsumed by the other IDP in the example.

We next introduce the **window validity problem**, which is to check whether a given window size is valid for a given query. Due to Corollary 10 computing a valid window of minimal size is clearly feasible using a logarithmic number of calls in the radius of the query to an oracle for this problem. Furthermore, such minimal window can be computed “offline” before Algorithm 1 is applied to any input data.

**Definition 11.** Let \( Q \) and \( D \) be classes of fp-queries and datasets, respectively. Then, \( \text{WINDOW}_{D}^{w} \) is the problem of deciding, given \( Q \in Q \) and \( w \geq 0 \) as input, whether \( w \) is a valid window size for \( Q \), \( \Sigma_Q \), and \( D \).

Corollary 10 provides a straightforward reduction from our problem to query containment. We next show that a reduction in the other direction also exists, which implies that our problem has exactly the same complexity as query containment for all classes of queries we consider.

**Theorem 12.** \( \text{WINDOW}_{D}^{w} \) and \( \text{CONT}_{D}^{w} \) are irreducible in \( \text{LogSpace} \) for each \( Q \in \{ \text{FP}, \text{OG}, \text{NR}, \text{OGNR} \} \) and each class \( D \) of datasets.

**Proof sketch.** Consider queries \( Q_1 \) and \( Q_2 \) in \( Q \), and assume w.l.o.g. that they do not share any IDBs other than the output predicate. In the case \( Q \in \{ \text{OG}, \text{OGNR} \} \) we also assume w.l.o.g. that \( Q_1 \) and \( Q_2 \) are object-free. The key idea in reducing containment to window validity is to merge \( Q_1 \) and \( Q_2 \) into a single query \( Q \) such that

1. both \( Q_1 \) and \( Q_2 \) may contribute to the answers of \( Q \), and
2. only \( Q_2 \) may contribute to the answers of \( Q^w \) if \( w \) is chosen as the maximum radius amongst \( Q_1 \) and \( Q_2 \).

It follows that such \( w \) is a valid window for \( Q \), \( \Sigma_Q \), and \( D \) iff \( Q_1 \subseteq_D Q_2 \). To construct \( \Pi_Q \), we first rename the output predicate in \( \Pi_Q \) and \( \Pi_{Q_2} \) to fresh \( P_Q \) and \( P_{Q_2} \), then union the resulting programs, and finally include the following extra rules (7) and (8), where \( A \) and \( B \) are fresh unary temporal EDB predicates, \( w \) is as before, and \( s = \langle x, t \rangle \) if \( Q_1 \) and \( Q_2 \) are temporal and \( s = x \) otherwise.

\[
A(t - w - 1) \land B(t) \land P_{Q_1}(s) \Rightarrow P_{Q}(x, t) \\
B(t) \land P_{Q_2}(s) \Rightarrow P_{Q}(x, t)
\]

Note that both \( Q_1 \) and \( Q_2 \) contribute to the answers to \( Q \) if the input stream contains facts for \( A \) and \( B \) in all time points. Furthermore, Rule (7) has radius \( w + 1 \); thus, it is not contained in \( Q^w \) and cannot contribute to its answers.

Since the language of fp-queries is an extension of Datalog, it follows from Theorem 12 and standard results on Datalog query containment that window validity is undecidable (Shmueli 1993) in general and coNEXP-hard for non-recursive queries (Benedikt and Gottlob 2010). Furthermore, the results on containment for non-recursive temporal queries in our prior work (Ronca et al. 2018) show that the aforementioned coNEXP lower bound is tight.

**Corollary 13.** Let \( D \) contain all finite datasets. Then,

- \( \text{WINDOW}_{D}^{w} \) is undecidable for any \( Q \) containing all Datalog queries, and
- \( \text{WINDOW}_{D}^{NR} \) is coNEXP-complete.

In the following section we show how to circumvent the undecidability result in Corollary 13 while preserving the full power of forward-propagating queries and, in particular, their ability to express temporal recursion.
6 Window Validity for Fixed Object Domain

We consider the situation where the set of objects relevant to the application domain can be fixed in advance, in the sense that any input set of background facts and any input stream refer only to those objects. This is a reasonable assumption in many applications of stream reasoning. For instance, when analysing temperature readings of wind turbines, one may assume that the set of turbines generating the data remains unchanged; furthermore, for the purpose of analysis we can often also assume that temperature readings themselves can be discretised into relevant levels according to suitable thresholds. In our running example, the set of nodes (pieces of data-generating computer equipment) present in the network is likely to change only rather rarely.

For the remainder of this section, let us fix a finite set $O$ of objects and let us denote with $O$ the class of datasets mentioning objects from $O$ only. Note that $O$ is a valid class of datasets since it trivially satisfies the relevant assumptions in Section 2, thus, problems $\text{WINDOW}_O$ and $\text{CONT}^{\text{QO}}_O$ are well-defined and, by Theorem 12, they are also interreducible for any class of queries $Q$ mentioned in this paper.

In what follows, we show that $\text{WINDOW}_O$ is decidable and establish tight complexity bounds.

6.1 Decidability and Upper Bounds

Fixing $O$ allows us to transform any input $Q$ to $\text{WINDOW}_O$ for $Q \subseteq \text{FP}$ into an object-ground query by grounding the object variables in $Q$ to constants in $O$; this yields an exponential reduction from $\text{WINDOW}_O^O$ to $\text{WINDOW}^{OQ}$. Thus, our first step will be to decide window validity for object-ground queries, and for this we provide a decision procedure for the corresponding query containment problem.

Let us consider fixed, but arbitrary, object-ground (temporal) queries $Q_1$ and $Q_2$ sharing an output predicate $G$. For simplicity, and without loss of generality, we assume that $Q_1$ and $Q_2$ contain no object terms and hence all predicates in the queries are either nullary or unary and temporal.

We first show that there exists a number $b$ of exponential size in $|Q_1| + |Q_2|$ such that $Q_1 \cup Q_2$ holds if and only if $G(\tau) \in Q_1(D)$ and $G(\tau) \notin Q_2(D)$ for some $\tau \in [0,b]$ and some dataset $D$ over time points in $[0,b]$. We do so by constructing deterministic automata $A_1$ and $A_2$ for $Q_1$ and $Q_2$, respectively, and deriving $b$ from well-known bounds for the size of counter-examples to automata containment.

**Lemma 14.** For each $i \in \{1,2\}$, let $\rho_i$ and $p_i$ be the radius and the size of the signature of $Q_i$, respectively. Let $b_i = 1 + 2^{p_i+\rho_i+2}$, and let $b = b_1 \cdot b_2$.

If $Q_1 \nsubseteq Q_2$, then there exists a time point $\tau \in [0,b]$ and a dataset $D$ over time points in $[0,b]$ such that $G(\tau) \in Q_1(D)$ and $G(\tau) \notin Q_2(D)$.

**Proof sketch.** We start with the observation that, given $Q_i$ and a dataset $D$, we can check whether the output predicate is derived at any time point from $\Pi Q_i \cup D$ using our generic stream reasoning algorithm. That is, we can start by loading the rigid facts in $D$ and subsequently reading the temporal facts one time point at a time while maintaining entailments over a window of size $\rho_i$ until the output predicate is derived or $D$ does not contain any further time points.

The correctness of this algorithm relies on the fact that $Q_i$ is forward-propagating and hence $\rho_i$ is a valid window. Based on this, we can construct a deterministic finite automaton $A_i$ that captures $Q_i$ in the following sense: on the one hand, each dataset $D$ corresponds to a word over the alphabet of the automaton, where the first symbol is the set of rigid facts in $D$ and the remaining symbols encode the temporal facts in $D$ one time point at a time on the other hand, each state corresponds to a snapshot of the facts stored in memory by the algorithm, and a state is final if it corresponds to a snapshot in which the output predicate has just been derived. Automaton $A_i$ is defined as follows:

- A state is either the initial state $s_{\text{init}}^i$, or a $(\rho_i + 2)$-tuple where the first component is a subset of the rigid EDB predicates in $Q_i$, and the other components are subsets of the temporal (EDB and IDB) predicates in $Q_i$. A state is final if its last component contains the output predicate $G$.
- Each alphabet symbol is a set $\Sigma$ of EDB predicates occurring in $Q_i$ such that $\Sigma$ does not contain temporal and rigid predicates simultaneously.
- The transition function $\delta_i$ consists of transitions $s_{\text{init}}^i, \Sigma \rightarrow (\Sigma, \emptyset, \ldots, \emptyset)$ such that $\Sigma$ consists of rigid predicates; transitions $(B, M_0, \ldots, M_\rho), \Sigma \rightarrow (B, M'_0, \ldots, M'_\rho)$ such that $\Sigma$ consists of temporal predicates; $M'_j = M_{j+1}$ for each $0 \leq j < \rho_i$; and $M'_\rho$ consists of each predicate $P$ satisfying $\Pi Q_i \cup B \cup H \cup U \models P(\rho_i)$ for $H$ the set of all facts $R(j)$ with $R \in M_j$ and $0 \leq j < \rho_i$, and $U$ the set of all facts $R(\rho_i)$ with $R \in \Sigma$.

The fact that each automaton $A_i$ captures $Q_i$ in the sense described before ensures that the following properties immediately hold:

1. If $Q_1 \nsubseteq Q_2$, then there exists a word that is accepted by $A_1$ and not by $A_2$.
2. For each word of length $n$ accepted by $A_1$ and not by $A_2$, there exists a dataset $D$ over time points in $[0,n-2]$ such that $G(n-2) \notin Q_1(D)$ and $G(n-2) \notin Q_2(D)$.

We finally argue that these properties imply the statement of the lemma. If $Q_1 \nsubseteq Q_2$ then, by Property 1, there is a word accepted by $A_1$ and not by $A_2$. By standard automata results, it follows that there is also a word accepted by $A_1$ and not by $A_2$ having length $n$ bounded by the product of the number of states in $A_1$ and $A_2$, where the number of states in $A_i$ is bounded by $b_i$. By Property 2, there exists a dataset $D$ over time points in $[0,n-2]$ such that $G(n-2) \notin Q_1(D)$ and $G(n-2) \notin Q_2(D)$, where $n$ is bounded by $b$.

Lemma 14 immediately suggests a non-deterministic algorithm for deciding $Q_1 \nsubseteq Q_2$, in which a witness dataset is constructed and checked in each branch. In order to ensure that the space used in each branch stays polynomial, we exploit our observation in the beginning of the proof of Lemma 12. A witness $D$ is guessed one time point at a time until reaching the bound $b$, and $Q_1(D) \nsubseteq Q_2(D)$ is verified incrementally after each guess while keeping in memory just a window of size bounded by the radiuses of $Q_1$ and $Q_2$.

**Lemma 15.** $\text{CONT}^{\text{QO}}$ is in $\text{PSPACE}$.
where a set of propositional symbols containment over non-recursive plain propositional Datalog.

**Theorem 16.** The maximum number of object variables in a rule can be bound in Lemma 15 extends to any class of queries where object variables in a rule from $D_1$. Guess a set $W$ of rigid facts and set $M_1$ and $M_2$ to $D_r$.

1. Guess a set $D_r$ of rigid facts and set $M_1$ and $M_2$ to $D_r$.
2. For each value of $\tau$ from 0 to $b$ as in Lemma 14:
   a. Guess $D_1|_{\tau}$.
   b. Set each $M_i$ to $M_i \cup D_1|_{\tau}$.
   c. Add to each $M_i$ facts $\alpha$ at $\tau$ s.t. $\Pi Q_i \cup M_i \models \alpha$.
   d. If there is a $G$-fact in $M_1|_{\tau}$ and not in $M_2|_{\tau}$, accept.
   e. Remove from each $M_i$ all facts in $M_i|_{\tau}$.
3. Reject.

The algorithm correctly computes the answers over the guessed facts, since it mimics Algorithm 1 and $\rho$ is a valid window for both queries. By Lemma 14, the algorithm finds a witness dataset for non-containment whenever one exists. Furthermore, the algorithm runs in polynomial space since the size of each $M_i$ is polynomial, and a polynomially-sized counter suffices for checking the halting condition. 

Lemma 15 yields a PSPACE upper bound to $\text{WINDOW}_1^{OGNR}$. In turn, it also provides an EXPSPACE upper bound to $\text{WINDOW}_1^{NR}$, which is obtained by first applying to the input query $Q$ a grounding step where object variables from $\Pi Q_i$ are replaced with constants from the object domain. Furthermore, this grounding process is polynomial in the number of domain objects and exponential in the maximum number of object variables in a rule from $\Pi Q_i$; thus, the PSPACE upper bound in Lemma 15 extends to any class of queries where the maximum number of object variables in a rule can be bounded by a constant (which equals zero for $OG$).

**Theorem 16.** The following upper bounds hold:
- $\text{WINDOW}_1^{OGNR}$ is in EXPSPACE; and
- $\text{WINDOW}_1^{NR}$ is in PSPACE for any class $Q$ of fp-queries where the maximum number of object variables in any rule of any $Q \in Q$ is bounded by a constant.

By exploiting results from our prior work (Ronca et al., 2018), we can show that $\text{WINDOW}_1^{OGNR}$ reduces to query containment over non-recursive plain propositional Datalog. The latter can be decided in coNP by universally guessing a set of propositional symbols $D$ and then checking (in polynomial time) that $Q_2(D)$ holds whenever $Q_1(D)$ does, which yields a coNP bound for $\text{WINDOW}_1^{OGNR}$. In turn, this bound yields a coNEXP upper bound for $\text{WINDOW}_1^{NR}$ by means of an exponential grounding step of the object variables. Furthermore, such grounding is polynomial for any class $Q \subseteq NR$ where the maximum number of object variables in any rule is bounded by a constant; hence, the coNP upper bound for $\text{OGNR}$ seamlessly extends to any such class.

**Theorem 17.** The following upper bounds hold:
- $\text{WINDOW}_1^{OGNR}$ is in coNEXP; and
- $\text{WINDOW}_1^{NR}$ is in coNP for any class $Q \subseteq NR$ where the maximum number of object variables in any rule of any $Q \in Q$ is bounded by a constant.

### 6.2 Lower Bounds

We next show that all the upper bounds established in Section 4 are tight. We start by providing a matching PSPACE lower bound to $\text{WINDOW}_1^{OGNR}$.

**Theorem 18.** $\text{WINDOW}_1^{OGNR}$ is PSPACE-hard.

**Proof sketch.** We show hardness for $\text{CONT}_1^{OGNR}$, which implies the theorem’s statement by Theorem 12. The proof is by reduction from the containment problem for regular expressions. Let $R_1$ and $R_2$ be regular expressions over a common finite alphabet $\Sigma$. We construct object-free queries $Q_1$ and $Q_2$ with unary output temporal predicate $G$ such that $R_1 \sqsubseteq R_2$ if and only if $Q_1 \sqsubseteq Q_2$.

Each $Q_1$ is defined such that it captures $R_1$ as described next. We encode words in $\Sigma^*$ using facts over unary temporal EDB predicates $F$ and $A_0$ for each alphabet symbol $\sigma \in \Sigma$. Intuitively, a fact $F(\tau)$ indicates that $\tau$ is the first position of the word, whereas a fact $A_\sigma(\tau')$ with $\tau' \geq \tau$ means that $\sigma$ is the symbol in position $\tau' - \tau$. Queries $Q_1$ are constructed from $R_1$ such that the following property ($\star$) holds for each dataset $D$ over the aforementioned EDB predicates and each time point $\tau$:

$$(*) \quad G(\tau) \in Q_1(D) \iff \text{if only if there exists a word } \sigma_1 \ldots \sigma_n \text{ in the language of } R_1 \text{ such that } D \text{ contains facts } F(\tau-n), A_{\sigma_1}(\tau-n), A_{\sigma_2}(\tau-n+1), \ldots, A_{\sigma_n}(\tau-1).$$

Property ($\star$) implies the statement of the theorem. On the one hand, if $Q_1 \not\sqsubseteq Q_2$, then $G(\tau) \in Q_1(D)$ and $G(\tau) \not\in Q_2(D)$ for some $\tau$ and $D$; by ($\star$), the former implies existence of a word $s \in L(R_1)$ such that $D$ contains the relevant facts, whereas the latter together with the aforementioned property of $D$ implies that $s \not\in L(R_2)$. On the other hand, $R_1 \sqsubseteq R_2$ implies that there exists $s = \sigma_1 \ldots \sigma_n$ with $s \in L(R_1)$ and $s \notin L(R_2)$; let $D_s$ be the dataset consisting of facts

$$F(0), A_{\sigma_1}(0), A_{\sigma_2}(1), \ldots, A_{\sigma_n}(n-1)$$

By ($\star$), we then have $G(n) \in Q_1(D_s)$ and $G(n) \notin Q_2(D_s)$, and hence $Q_1 \not\sqsubseteq Q_2$.

We now define $Q_2 = (G, \Pi_{R_2})$, where $\Pi_{R_2}$ is defined inductively from $R_1$ as described next; note that, for II a program, we denote with $\Pi'$ (resp., $\Pi''$) the program obtained from $II$ by renaming each predicate $P$ not in $\{A_\sigma \mid \sigma \in \Sigma\}$ to a globally fresh predicate $P'$ ($P''$) of the same arity.

1. $R_1 = \emptyset$. Then, $\Pi_{R_2}$ is the empty program.
2. $R_1 = \sigma$ for $\sigma \in \Sigma$. Then, $\Pi_{R_2}$ consists of rule
   $$F(t) \land A_{\sigma}(t) \rightarrow G(t+1).$$
3. $R_1 = \varepsilon$. Then, $\Pi_{R_2}$ consists of rule
   $$F(t) \rightarrow G(t).$$
4. $R_1 = S \cup T$. Then, $\Pi_{R_2}$ extends $\Pi''_S \cup \Pi''_T$ with rules
   $$F(t) \rightarrow F'(t), \quad F(t) \rightarrow F''(t)$$
   $$G'(t) \rightarrow G(t), \quad G''(t) \rightarrow G(t).$$
5. $R_1 = S \circ T$. Then, $\Pi_{R_2}$ extends $\Pi_S \cup \Pi_T$ with rules
   $$F(t) \rightarrow F'(t), \quad G'(t) \rightarrow F''(t), \quad G''(t) \rightarrow G(t).$$
6. $R_1 = S^+$. Then, $\Pi_{R_2}$ extends $\Pi_S$ with rules
   $$F(t) \rightarrow F'(t), \quad G'(t) \rightarrow F''(t), \quad G''(t) \rightarrow G(t).$$

It can be checked using a simple induction that the construction ensures that ($\star$) holds. 

\[ \square \]
Theorem 18 implies PSPACE-hardness of WINDOW$_Q^P$ for any class $Q$ of fp-queries where the maximum number of object variables is bounded by a constant.

We next show a matching EXPSPACE lower bound to the complexity of WINDOW$_Q^P$. To this end, we upgrade the reduction in Theorem 18 to a reduction from the containment problem of succinct regular expressions—regular expression extended with an exponentiation operation $R^k$ where $k$ is coded in binary (Sipser 2006).

**Theorem 19.** WINDOW$_Q^P$ is EXPSPACE-hard.

**Proof sketch.** We show hardness of the corresponding query containment problem, which implies the statement by Theorem 12. Let $R_1$ and $R_2$ be succinct regular expressions over the same vocabulary $\Sigma$. We construct fp-queries $Q_1$ and $Q_2$ over the same unary temporal output predicate $G$ such that $R_1 \subseteq R_2$ if and only if $Q_1 \subseteq Q_2$.

As in the proof of Theorem 18, we construct $Q_1$ such that it captures $R_1$. We encode words as before using unary temporal EDB predicates $F$ and $A_\sigma$ for each $\sigma \in \Sigma$. Also as before, we construct $Q_2$ from $R_2$ such that property (*) holds where $D$ in the formulation of (*) is over objects in $O$.

We now define $Q_1 = \langle G, \Pi_{\text{suc}} \cup \Pi_{R_1} \rangle$, where $\Pi_{R_1}$ will be defined inductively over the structure of $R_1$, and $\Pi_{\text{suc}}$ is a Datalog program that defines in the standard way (Dantsin et al. 2001) rigid IDC successor predicates $\text{suc}^{m}$ of arity $2m$ relating $m$-strings over objects 0 and 1 for each exponent $k$ occurring in $R_i$ with $m = \lceil \log_2 k \rceil$. Now we proceed with the inductive definition of $\Pi_{R_1}$, which is analogous to that in the proof of Theorem 18 with the following additional case, and the minor modification that successor predicates are never renamed apart:

7. $R_i = S^k$ for some succinct regular expression $S$ and $k \geq 2$. Then, $\Pi_{R_i}$ is constructed from $\Pi_S$ as follows. First, we replace each $n$-ary atom $P(p, s)$, for $p$ a vector of object terms and $s$ a temporal term, with $P'(p, x, s)$ for $P'$ a fresh predicate (unique to $P$) of arity $n + m$ with $m = \lceil \log_2 k \rceil$, and $x$ a fixed $m$-vector of fresh object variables. Second, we extend the resulting program with the following rules, where $a$ is the encoding of $k - 1$ as a binary string over 0 and 1:

\[
F(t) \to F'(\overline{0}, t) \\
G'(a, t) \to G(t) \\
G'(x, t) \land \text{suc}^{m}(x, y) \to F'(y, t)
\]

We can show inductively that (*) holds.

To conclude, we turn our attention to the case of non-recursive queries. A matching coNLP lower bound to the complexity of WINDOW$_{QNR}^P$ is obtained by a simple reduction from 3-SAT to the complement of our problem. A matching coNEXP lower bound for WINDOW$_{QNR}^P$ follows by a simple adaptation of the hardness proofs in (Benedikt and Gottlob 2010) for containment in non-recursive Datalog.

**Theorem 20.** WINDOW$_{QNR}^P$ is coNP-hard. Furthermore, WINDOW$_{QNR}^P$ is coNEXP-hard if $O$ has at least two objects.

### 7 A Sufficient Condition for Window Validity

The assumption that the object domain can be fixed in advance may not be reasonable in some applications. For instance, it may be the case that sensor values cannot be naturally discretised into suitable levels according to a threshold, or that new sensors are continuously activated on-the-fly.

As already established, dropping the fixed domain assumption leads to undecidability of window validity for (recursive) fp-queries. In this section, we propose a sufficient condition for the validity of a window that can be checked in exponential time without additional assumptions, and which leads to smaller window sizes compared to the radius of the query. Our condition relies on the notion of uniform containment of two programs $\Pi_1$ and $\Pi_2$ (Sagiv 1988), which is sufficient to ensure containment of any queries $Q_1$ and $Q_2$ on programs in $\Pi_1$ and $\Pi_2$, respectively.

**Definition 21.** An extended dataset $E$ is a (possibly infinite) set of (not necessarily EDB) facts. Program $\Pi_1$ is uniformly contained in program $\Pi_2$, written $\Pi_1 \subseteq^u \Pi_2$, if and only if, for each extended dataset $E$ and each fact $\alpha$, it holds that $\Pi_1 \cup E \models \alpha$ implies $\Pi_2 \cup E \models \alpha$.

A window size $w$ is uniformly valid for an fp-query $Q$ if and only if $\Pi_Q \subseteq^u \Pi_Q^w$.

It is straightforward to check that, given any queries $Q_1$ and $Q_2$, it holds that $\Pi_{Q_1} \subseteq^u \Pi_{Q_2}$ implies $\Pi_{Q_1} \subseteq \Pi_{Q_2}$.

Hence, we can establish that uniform validity is a sufficient condition for window validity, which is more precise than the syntactic condition given by the radius.

**Proposition 22.** Let $Q$ be an fp-query with radius $\rho$, and let $w$ be a non-negative integer. If $w$ is a uniformly valid window size for $Q$, then $w$ is also a valid window size for $\Sigma_Q$ and any class $\mathcal{D}$ of datasets. Furthermore, if $w$ is the smallest uniformly valid window size for $Q$, then $w \leq \rho$.

**Example 23.** Consider the query $Q$ where $\Pi_Q$ consists of the following rules and $A$ is the only EDB predicate:

\[
A(t) \rightarrow P_Q(t) \\
A(t - 1) \land A(t) \rightarrow P_Q(t)
\]

Query $Q$ has radius one. We can see that $w = 0$ is a (uniform) window. Intuitively, this is because the first rule entails the second; thus, $\Pi_Q$ and $\Pi_{Q_{\leq 1}}$ are logically (and hence also uniformly) equivalent.

It is well-known that uniform program containment amounts to checking fact entailment (Sagiv 1988). On the one hand, to check $\Pi_1 \subseteq^u \Pi_2$, it suffices to show that $\Pi_2$ entails each rule $r$ in $\Pi_1$, which can in turn be checked by first “freezing” $r$ into an extended dataset $E$ for the body and a fact $\alpha$ for the head and then verifying whether $\Pi_2 \cup E \models \alpha$.

On the other hand, to check whether $\Pi_1 \cup E \models \alpha$, it suffices to check uniform containment of a single rule $r$ in $\Pi_2$, where $r$ is obtained from $E$ and $\alpha$ by replacing each constant with a fresh variable in the obvious way.

**Theorem 24.** Let $Q \in \{FP, NR, OG, QNR\}$ and let $\mathcal{P}$ be the class of programs that occur in queries from $Q$. Then, uniform window validity over queries in $Q$ and fact entailment over programs in $\mathcal{P}$ are inter-reducible in LOGSPACE.
The following complexity bounds for uniform window validity immediately follow from complexity results for fact entailment.

**Corollary 25.** Uniform window validity over a class \( Q \) of queries is

- EXP-complete if \( Q = \text{FP} \);
- \( \text{PSPACE-complete} \) if \( Q = \text{NR} \);
- \( \text{in P if Q is any subclass of FP where the maximum number of object variables in any rule of any } Q \in \text{Q is bounded by a constant; and} \)
- \( \text{in AC}^0 \) if \( Q \) is any subclass of \( \text{NR} \) where the maximum number of object variables in any rule of any \( Q \in \text{Q} \) is bounded by a constant.

We see uniform validity as a reasonable compromise in practice. On the one hand, it may yield smaller window sizes than the radius of the query, thus reducing the amount of information that a stream reasoning algorithm needs to retain in memory; on the other hand, it can be checked while relying solely on query processing infrastructure, and hence without the need for specialised algorithms.

### 8 Related Work

The formal underpinnings of stream query processing in databases were established in [Babcock et al. 2002][Arasu, Babu, and Widom 2006][Arasu, Babu, and Widom 2006] proposed CQL as an extension of SQL with a window construct, which specifies the input data relevant for query processing at any point in time. CQL has become since then the core of many other stream query languages, including languages for the Semantic Web [Barbieri et al. 2009][Babu, and Widom 2006].

In the context of stream reasoning, [Zaniolo 2012] proposed Streamlog: a language which extends temporal Datalog with non-monotonic negation while at the same time restricting the syntax so that only facts over time points mentioned in the data can be derived. LARS [Beck et al. 2015][Beck, Dao-Tran, and Eiter 2015][Beck, Dao-Tran, and Eiter 2016] is a temporal rule-based streaming language featuring built-in window constructs and negation interpreted according to the stable model semantics. In contrast to temporal Datalog, the semantics of LARS assumes that the number of time points in a model is a part of the input to query evaluation, and hence is restricted to be finite. Stream reasoning has also been considered in ontology-based data access [Calbimonte, Corcho, and Gray 2010][Ozcep, Möller, and Neuenstadt 2014] as well as in the context of complex event processing [Anicic et al. 2011][Dao-Tran and Le-Phuoc 2015].

There are have been several proposals of Datalog extensions for reasoning over static temporal data. The language we consider is a notational variant of DatalogMTL [Chomicki and Imieliński 1988][Chomicki and Imieliński 1989][Chomicki 1990]. Templog is an extension of Datalog with modal temporal operators [Abadi and Manna 1989]; DatalogMTL is an extension with metric temporal logic [Brandt et al. 2017]; and the language proposed by Toman and Chomicki [1998] extends Datalog with integer periodicity constraints.

Our language of fp-queries is related to past temporal logic, where formulae are restricted to refer to past time points only [Manna and Pnueli 1992][Chomicki 1995]. [Chomicki 1995] presents an incremental update algorithm for checking dynamic integrity constraints expressed in past temporal logic; similarly to our stream reasoning algorithm, Chomicki’s update algorithm exploits the idea that the length of the stored history throughout a sequence of updates can be bounded to a value depending only on the query.

The window validity problem was introduced in our prior work [Ronca et al. 2018] based on a generic stream reasoning algorithm that only keeps EDB facts in memory. We established undecidability for unrestricted queries, and provided tight complexity bounds for the non-recursive case. Our current paper extends [Ronca et al. 2018] by generalising window validity to the case where the underpinning stream reasoning algorithm can also keep IDB facts in memory; furthermore, we show decidability and tight complexity bounds for recursive queries under the (rather mild) assumption that the object domain can be fixed in advance. The window validity problem is related to a problem considered in the context of database constraint checking by Chomicki [1995], who obtained positive results for queries in temporal first-order logic. It is also related to the forgetting problem in logic programming [Wang, Sattar, and Su 2005][Eiter and Wang 2008], where the goal is to eliminate predicates while preserving certain logical consequences.

### 9 Conclusion and Future Work

We have studied the window validity problem in stream reasoning and its computational properties for temporal Datalog. We showed that window validity is undecidable; however, decidability can be regained by making mild assumptions on the input data.

We see many avenues for future work. First, it would be interesting to consider window validity for extensions of temporal Datalog (e.g., with comparison atoms or stratified negation) as well as for DatalogMTL. Second, we have assumed throughout the paper that all numbers in input queries and data are coded in unary; it would be interesting to revisit our technical results for the case where binary encoding is assumed instead. Finally, our decidability results do not immediately yield implementable algorithms; we are planning to develop and implement practical window validity checking algorithms under the fixed object domain assumption.

### Acknowledgments

This research was supported by the SIRIUS Centre for Scalable Data Access in the Oil and Gas Domain and the EPSRC projects DBOnto, MaSI, and ED^3.

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A Appendix

In our proofs, we will make use of the following notion of derivations, which is a variant of hyper-resolution derivations restricted to temporal Datalog.

**Definition 26.** Let $\Pi$ be a program, let $F$ be a set of facts, and let $\alpha$ be a fact. A derivation of $\alpha$ from $\Pi \cup F$ is a finite node-labelled tree such that: (i) each node is labelled with a ground instance of a rule in $\Pi \cup F$; (ii) each $\alpha$ is the head of the rule labelling the root; and (iii) for each node $v$, the body of the rule labelling $v$ contains an atom $\alpha$ if and only if $\alpha$ is the head of the rule labelling a child of $v$.

By the completeness of hyper-resolution, it then follows that a temporal Datalog program $\Pi$ entails a fact $\alpha$ from a set of facts $F$ if and only if $\alpha$ has a derivation from $\Pi \cup F$.

**Proposition 27.** Let $\Pi$ be a program, let $F$ be a set of facts, and let $\alpha$ be a fact. Then, $\Pi \cup F \models \alpha$ if and only if there exists a derivation of $\alpha$ from $\Pi \cup F$.

In the rest, whenever a fact $\alpha$ is entailed by a program $\Pi$ and a set of facts $F$, we directly assume the existence of a derivation of $\alpha$ from $\Pi \cup F$, without referring to Proposition 27.

A.1 Proof of a Claim in the Preliminaries

As promised in the preliminaries, we show the following claim.

**Claim 28.** Let $D$ be a class of datasets, and let $D'$ be the class of all finite datasets in $D$. Then, $Q_1 \subseteq_D Q_2$ iff $Q_1 \subseteq_{D'} Q_2$.

**Proof.** Trivially $Q_1 \subseteq_{D'} Q_2$ implies $Q_1 \subseteq_D Q_2$, since $D' \subseteq D$. For the converse, assume $Q_1 \subseteq_{D'} Q_2$. There exists a dataset $D \in D$ and a fact $\alpha$ such that $\alpha \in Q_1(D)$ and $\alpha \notin Q_2(D)$. There exists a finite $D' \subseteq D$ such that $\alpha \in Q_1(D')$, since derivations are finite. By our assumption on the considered classes of datasets, there exists a finite dataset $D''$ with $D' \subseteq D'' \subseteq D$ and $D'' \in D'$; and hence $D'' \in D'$. By monotonicity, it follows that $\alpha \in Q_1(D'')$ and $\alpha \notin Q_2(D'')$. Therefore, $Q_1 \subseteq_{D'} Q_2$. \hfill $\Box$

A.2 Proof of Theorem 4

**Proposition 29.** Let $\Pi$ be a program consisting of forward-propagating rules, let $F$ be a set of facts, and let $\alpha$ be a temporal fact having a derivation $\delta$ from $\Pi \cup F$. Furthermore, let $\tau_{\min}$ be the minimum time point in $F$, and let $\tau$ be the time argument of $\alpha$. Then, each time point occurring in $\delta$ is in $[\tau_{\min}, \tau]$.

**Proof.** Let $\delta$ be a derivation of $\alpha$ from $\Pi \cup F$. We prove the claim by induction on the height $n$ of $\delta$.

In the base case $n = 0$, and hence $\alpha$ is the only atom in $\delta$. Since no time point occurs in $\Pi$ by the properties of forward-propagating rules, it follows that $\tau$ occurs in $F$. Therefore $\tau_{\min} \leq \tau$ by the definition of $\tau_{\min}$, and hence trivially $\tau \in [\tau_{\min}, \tau]$.

In the inductive case $n > 0$, and we assume that each time point occurring in a derivation of a temporal fact $\beta$ from $\Pi \cup F$ of height at most $n - 1$ is in $[\tau_{\min}, \tau']$ where $\tau'$ is the time argument of $\beta$. Let $r$ be the rule labelling the root of $\beta$, let $\beta$ be an atom in the body of $r$, and let $\delta'$ be a derivation of $\beta$ occurring as a subtree in $\delta$. We have two cases. In the first case $\beta$ is rigid, and hence it is clear that each label of a node of $\delta'$ is an instance of a Datalog rule in $\Pi$ by the properties of forward-propagating rules, and hence no time point occurs in $\delta'$. In the other case $\beta$ is temporal. Let $\tau'$ be the time argument of $\beta$. Note that $\tau' \leq \tau$ since $r$ is forward-propagating. It follows that each time point occurring in $\delta'$ is in $[\tau_{\min}, \tau]$ by the inductive hypothesis. \hfill $\Box$

**Theorem 4.** Let $D$ be a class of finite datasets, let $Q \in \{FP, NR, OG, OGNR\}$, and let $Q'$ be the Datalog subset of $Q$. Then, $\text{Eval}^D_Q$ is $\text{LogSpace}$-reducible to $\text{Eval}^D_{Q'}$.

**Proof.** We describe a LogSpace-computable many-one reduction $\varphi$ from $\text{Eval}^D_Q$ to $\text{Eval}^D_{Q'}$. An instance of $\text{Eval}^D_Q$ is $\langle Q, D, \alpha \rangle$ with $Q \subseteq D \subseteq D$, and $\alpha$ a fact. We consider two cases, depending on whether $\alpha$ is rigid or temporal.

Assume that $\alpha$ is rigid. Then, $\varphi$ maps $I$ to $\langle Q_2, D, \alpha \rangle$ where $Q_2 = \{P_Q, \Pi_1\}$ with $\Pi_1$ the Datalog subprogram of $\Pi_Q$. We argue that $\alpha \in Q(D)$ iff $\alpha \in Q_2(D)$. First, we have that $\alpha \in Q_1(D)$ implies $\alpha \in Q(D)$ by monotonicity. Then, for the converse, assume $\alpha \in Q(D)$ and let $\delta$ be a derivation of $\alpha$ from $\Pi_Q \cup D$. Since $\alpha$ is rigid and $Q$ is an fp-query, the rule $r$ labelling the root of $\delta$ is an instance of a Datalog rule in $\Pi_Q$, and hence each atom in $r$ is rigid; inductively the same holds for each label of a node of $\delta$. Therefore $\delta$ is a derivation of $\alpha$ from $\Pi_1 \cup D$, and hence $\alpha \in Q_1(D)$.

Now, assume that $\alpha$ is temporal. We further split into two cases.

In the first case $D$ contains no temporal fact, and we define $\varphi$ as mapping $I$ to $\langle Q_2, D, \alpha \rangle$ with $Q_2 = \langle P_Q, \emptyset \rangle$. We have that $\alpha \in Q(D)$ iff $\alpha \in Q_2(D)$, since $\alpha \notin Q_2(D)$ holds trivially, and $\alpha \notin Q(D)$ holds because $Q$ mentions no time point, by our assumption.

In the other case, we have that $D$ contains a temporal fact. Let $\tau_{\min}$ be the minimum time point in $D$, and let $\tau_\alpha$ be the time argument of $\alpha$. Let $Q_3 = \langle P_Q, \Pi_3 \rangle$ with $\Pi_3$ the program consisting of each rule $r'$ obtained from a rule $r \in \Pi_Q$ by substituting the time variable in $r$—note that there is at most one time variable in $r$ since $Q$ is an fp-query—so that each time argument in $r'$ is in the interval $[\tau_{\min}, \tau_\alpha]$. Since $Q_3$ is time-ground, it is clear that we can build a Datalog query $Q'_3$ equivalent to $Q_3$ by replacing each atom $\beta$ in $Q_3$ with a rigid atom over a fresh predicate that is unique to the predicate and time argument of $\beta$. 


Then, we define $\varphi$ as mapping $I$ to $\langle Q_1, D, \alpha \rangle$. We argue next that the reduction is correct. It suffices to show that $\alpha \in Q(D)$ iff $\alpha \in Q_3(D)$, since $Q_1$ and $Q_2'$ are equivalent. First, we have that $\alpha \in Q_3(D)$ implies $\alpha \in Q(D)$ because each rule in $\Pi_3$ is an instance of a rule in $\Pi_Q$. Then, for the converse, assume $\alpha \in Q(D)$ and let $\delta$ be a derivation of $\alpha$ from $\Pi_Q \cup D$. We have that each time point occurring in $\delta$ is in $[\tau_{\text{min}}, \tau_{\alpha}]$ by Proposition 29 and hence each label of a node of $\delta$ is an instance of a rule of $\Pi_3$. Therefore $\delta$ is a derivation of $\alpha$ from $\Pi_3 \cup D$, and hence $\alpha \in Q_3(D)$.

We finally argue that $\varphi$ can be computed in logarithmic space. It is clear that we can check whether $\alpha$ is rigid or temporal, check whether $D$ contains a temporal fact, compute the minimum time point in $D$ if one exists, compute renamings, etc... in logarithmic space. The critical step is computing $Q_3$. This is doable in logarithmic space because it suffices to consider substitutions mapping time variables to the interval $[\tau_{\text{min}} - \rho, \tau_{\alpha} + \rho]$ with $\rho$ the radius of $Q$, and the former interval has linear size, since we have assumed that numbers in the input $I$ are coded in unary.

A.3 Proof of Theorem 8

Proposition 30. Let $\Pi$ be a program consisting of forward-propagating rules, let $F$ be a set of facts, and let $\alpha$ be a fact. Furthermore, let $\tau$ be the time argument of $\alpha$, and let $B$ be the rigid facts in $F$. If $\Pi \cup F \models \alpha$, then $\Pi \cup B \cup F|_{[0, \tau]} \models \alpha$.

Proof. If $\delta$ is a derivation of $\alpha$ from $\Pi \cup F$, then each time point in $\delta$ is at most $\tau$ by Proposition 29 and hence $\delta$ is a derivation of $\alpha$ from $\Pi \cup B \cup F|_{[0, \tau]}$.

Lemma 31. Consider Algorithm 1 parametrised with $Q$, $w$ and $\Sigma_Q$. On input $\langle B, S \rangle$, the set of temporal facts stored in $M$ right after executing Line 4 in any iteration of the main loop coincides with the set of temporal facts entailed by $\Pi_Q[0] \cup B \cup S$ and holding at any $\tau' \in [\tau - w, \tau]$.

Proof. Let $\langle B, S \rangle$ be an input to Algorithm 1. For each $n \geq 0$, let $M_{\alpha}^4$ and $M_{\alpha}^w$ be the facts stored in $M$ by Algorithm 1 on input $\langle B, S \rangle$ right after Lines 3 and 4, respectively, in the $(n + 1)$-th iteration of the main loop; furthermore, note that $\tau$ has value $n$ in the $(n + 1)$-th iteration of the main loop. Then, consider the following observations.

Observation 1. $M_{\alpha}^w = B \cup S|_0$.

Observation 2. For each $n > 0$, $M_{\alpha}^w = B \cup M_{\alpha}^{n-1}|_{[n-w, \infty)} \cup S|_n$.

Next, we show the two inclusions separately.

$(\subseteq)$ We first show that each temporal fact stored by Algorithm 1 in $M$ in any iteration of the main loop right after executing Line 4 is entailed by $\Pi_Q[0] \cup B \cup S$ and has time argument in $[\tau - w, \tau]$. It suffices to show that each $M_{\alpha}^w$ is a subset of the facts entailed by $\Pi_Q[0] \cup B \cup S$. We prove it by induction on $n \geq 0$.

In the base case $n = 0$. Let $\alpha$ be a fact in $M_{\alpha}^w$. It is clear from the algorithm that (i) $\alpha$ is in $M_{\alpha}^w$ or (ii) $\alpha$ is a temporal fact with time argument zero such that $\Pi_Q \cup M_{\alpha}^w \models \alpha$. In case (i), we have that $\alpha \in B \cup S|_0$ by Observation 1 and hence in $\alpha \in B \cup S$. Therefore the claim holds by monotonicity. In case (ii), we have that $\Pi_Q \cup B \cup S|_0 \models \alpha$ by Observation 1. Let $\delta$ be a derivation of $\alpha$ from $\Pi_Q \cup B \cup S|_0$. By Proposition 29, we have that zero is the only time point in $\delta$. Any instance of a rule with radius bigger than zero contains a time point different from zero, and hence $\delta$ does not contain such an instance. In particular, $\delta$ is a derivation of $\alpha$ from $\Pi_Q \cup B \cup S|_0$. Therefore $\Pi_Q^w \cup B \cup S \models \alpha$ by monotonicity.

In the inductive case $n > 0$, and we assume that $\alpha \in M_{\alpha}^{n-1}$ implies $\Pi_Q \cup B \cup S \models \alpha$. Let $\alpha$ be a fact in $M_{\alpha}^w$. It is clear from the algorithm that (iii) $\alpha$ is in $M_{\alpha}^w$ or (iv) $\alpha$ is a temporal fact with time argument $n$ such that $\Pi_Q \cup M_{\alpha}^w \models \alpha$. We consider the two cases separately.

In case (iii), we have that $\alpha \in B \cup M_{\alpha}^{n-1}|_{[n-w, \infty)} \cup S|_n$ by Observation 2. We have two subcases: if $\alpha \in B \cup S|_n$, then the claim holds by monotonicity; otherwise, we have that $\alpha \in M_{\alpha}^{n-1}|_{[n-w, \infty)}$, and hence the claim holds by the inductive hypothesis.

In case (iv), we have that $\Pi_Q \cup B \cup M_{\alpha}^{n-1}|_{[n-w, \infty)} \cup S|_n \models \alpha$ by Observation 2. Let $F$ be the set of facts entailed by $\Pi_Q \cup B \cup S$. Note that $M_{\alpha}^{n-1} \subseteq F$ by the inductive hypothesis. It follows that $\Pi_Q \cup B \cup F|_{[n-w, \infty)} \cup S|_n \models \alpha$ by monotonicity. Let $\delta$ be a derivation of $\alpha$ from $\Pi_Q \cup B \cup F|_{[n-w, \infty)} \cup S|_n$. Again by Proposition 29, we have that each time point of $\delta$ is in $[n - w, n]$. Any instance of a rule with radius bigger than $w$ contains time points in an interval of size bigger than $w + 1$, and hence $\delta$ does not contain such an instance. In particular, $\delta$ is a derivation of $\alpha$ from $\Pi_Q \cup B \cup F|_{[n-w, \infty)} \cup S|_n$. It follows that $\Pi_Q \cup B \cup F|_{[n-w, \infty)} \cup S|_n \models \alpha$, hence $\Pi_Q \cup B \cup F \cup S \models \alpha$ by monotonicity, and hence $\Pi_Q^w \cup B \cup S \models \alpha$ since $F$ is entailed by $\Pi_Q^w \cup B \cup S$.

$(\supseteq)$ We now show that the set of temporal facts stored by Algorithm 1 in $M$ in any iteration of the main loop right after executing Line 4 contains each fact entailed by $\Pi_Q^w \cup B \cup S$ and having time argument in $[\tau - w, \tau]$. Let $\alpha$ be a temporal fact entailed by $\Pi_Q^w \cup B \cup S$ and having time argument in $[n - w, n]$. It suffices to show that $\alpha \in M_{\alpha}^w$ for every $n \geq 0$. We prove it by induction on $n \geq 0$.

In the base case $n = 0$, and hence $\alpha$ has a time argument in $[-w, 0]$—specifically, such a time argument is zero. By Proposition 30, we have that $\Pi_Q^w \cup B \cup S|_0 \models \alpha$, hence $\Pi_Q^w \cup M_{\alpha}^3 \models \alpha$ by Observation 1, hence $\Pi_Q \cup M_{\alpha}^3 \models \alpha$ by monotonicity, and hence $\alpha \in M_{\alpha}^3$ according to the algorithm.
In the inductive case $n > 0$, and we assume that $M^4_{n-1}$ contains each fact entailed by $\Pi^w_Q \cup B \cup S$ and having a time argument in $[n-1-w, n-1]$. We consider two cases. In the first case we have that $\alpha$ has time argument in $[n-w, n-1]$, hence $\alpha \in M^4_{n-1}$ by the inductive hypothesis, hence $\alpha \in M^4_{n-1}$ by Observation 2 and hence $\alpha \in M^4_{n}$ by the definition of the algorithm. In the second case we have that $\alpha$ has time argument $n$. According to the algorithm, it suffices to show that $\Pi_Q \cup M^3_{n} \models \alpha$. We prove it by induction on the height $m$ of a derivation $\delta$ of $\alpha$ from $\Pi_Q \cup B \cup S$.

In the base case $m = 0$, and hence $\alpha$ is a fact in $B \cup S$. In particular, $\alpha \in S|_n$, and hence $\alpha \in M^3_{n}$ by Observation 2. Therefore $\Pi_Q \cup M^3_{n} \models \alpha$ by monotonicity.

In the inductive case $m > 0$, and we assume that $\Pi_Q \cup M^3_{n} \models \beta$ for each fact $\beta$ with time argument $n$ and having a derivation from $\Pi_Q^{w} \cup B \cup S$ of height at most $m - 1$. Let $r$ be the rule labelling the root of $\delta$, let $\beta$ be an atom in the body of $r$, and let $r'$ be the time argument of $\beta$. First, $r$ is an instance of a rule of $\Pi_Q$, since $\Pi_Q \subseteq \Pi_Q$ by definition. Second, we show that $\Pi_Q \cup M^3_{n} \models \beta$. Note that $r' \in [n-w, n]$, since $r$ is forward-propagating and the radius of $r$ is at most $w$ by the definition of $\Pi_Q$. We have two cases: if $r' = n$, then $\Pi_Q \cup M^3_{n} \models \beta$ by the ‘inner’ inductive hypothesis; otherwise, we have that $r' \in [n-w, n-1]$, hence $\beta \in M^4_{n-1}$ by the ‘outer’ inductive hypothesis, hence $\beta \in M^4_{n}$ by Observation 2 and hence $\Pi_Q \cup M^3_{n} \models \beta$ by monotonicity. The first and the second points imply $\Pi_Q \cup M^3_{n} \models \alpha$.

**Theorem 8.** Consider Algorithm 7 parametrised with $Q$, $w$ and $\Sigma_Q$. On input $(B, S)$, the set of $P_Q$-facts streamed out in the first $n$ iterations coincides with $Q^w(B \cup S)\mid_{[0, n-1]}$.

**Proof.** Consider the $n$-th iteration of Algorithm 1. The output of the algorithm is determined by Line 5, where the algorithm outputs the $P_Q$-facts in the set $M^4_\tau$, which consists of the facts entailed by $\Pi_Q^{w} \cup B \cup S$ having time argument $\tau$, by Lemma 31. The claim then follows from the fact that $\tau = n - 1$.

**A.4 Proof of Theorem 12**

**Theorem 12.** Window$^Q_D$ and Cont$^Q_D$ are interreducible in LOGSPACE for each $Q \in \{FP, OG, NR, OGNR\}$ and each class $D$ of datasets.

**Proof.** We show reducibility in the two directions separately.

$(\Leftarrow)$ Consider an instance $I = (Q, w)$ of Window$^Q_D$, and the function $\varphi$ mapping $I$ to the instance $(Q, Q^w)$ of Cont$^Q_D$. Note that (i) $Q^w \in Q$ since removing any number of rules from $Q$ yields a query in $Q$, (ii) $\varphi$ can clearly be computed in logarithmic space, and (iii) $\varphi$ is a many-one reduction since $w$ is a valid window for $Q, \Sigma_Q$ and $D$ if and only if $Q \subseteq_D Q^w$, by Corollary 9.

$(\Rightarrow)$ Now we prove reducibility in the other direction. Let $\psi$ be the reduction given in the proof sketch of Theorem 12. Let $I = (Q_1, Q_2)$ be an instance of Cont$^Q_D$, and let $\psi(I) = (Q, \rho)$. It is clear that $\psi$ can be computed in logarithmic space.

We argue next that $Q \subseteq_D Q_1$. It is clear that $Q \subseteq FP$. If $Q_1, Q_2 \subseteq NR$, then $Q \subseteq NR$ because $Q_1$ and $Q_2$ do not share IDB predicates by our assumption, and rule (7) and rule (8) do not add cycles in the $\Pi_Q$-dependencies, since $P_Q$ does not occur in any body of a rule of $\Pi_Q$. If $Q_1, Q_2 \subseteq OG$, then $Q \subseteq OG$ since rule (7) and rule (8) are object-free because we have assumed that $Q_1$ and $Q_2$ are object-free. The case where $Q_1, Q_2 \subseteq OGNR$ follows from the two previous cases.

We argue next that $\psi$ is a many-one reduction, by showing that $\rho$ is a valid window for $Q, \Sigma_Q$ and $D$ if and only if $Q_1 \subseteq_D Q_2$. In the following, note that $Q^w$ is $Q$ after removing rule 7.

Assume that $\rho$ is a valid window for $Q, \Sigma_Q$ and $D$, and hence $Q \subseteq_D Q^w$. By Corollary 9, we show that $Q_1 \subseteq_D Q_2$. Let $D$ be a dataset in $D$, and let $\alpha$ be a fact in $Q_1(D)$. Let $\tau$ be the time argument of $\alpha$ if $\alpha$ is temporal, and let $\tau = \rho + 1$ otherwise. Let $D'$ be $D$ extended with $A(\tau - \rho - 1)$ and $B(\tau)$. Note that $D' \in D$ by our assumption on the considered classes of datasets. We have that $\alpha \in Q(D')$ by rule (7). Since $Q \subseteq_D Q^w$ as argued above, it follows that $\alpha \in Q^w(D')$, and in particular $\alpha$ is derived by rule (8). Therefore $\alpha \in Q_2(D')$ by the construction of $Q$.

For the converse, assume that $Q_1 \subseteq_D Q_2$. We show that $\rho$ is a valid window for $Q, \Sigma_Q$ and $D$. By Corollary 9, it suffices to show $Q \subseteq_D Q^w$. Let $D$ be a dataset in $D$, and let $\alpha$ be a fact in $Q_1(D)$. We distinguish two cases. In the first case, $\alpha$ is derived by rule (8), and hence $\alpha \in Q^w(D)$. In the other case, $\alpha$ is derived by rule (7), hence $\alpha \in Q_1(D)$ by the construction of $Q$: furthermore we have that $B(\tau) \in D$, where $\tau$ is the time argument of $\alpha$ if $\alpha$ is temporal and just a time point otherwise. It follows that $\alpha \in Q_2(D)$ since $Q_1 \subseteq_D Q_2$ by our assumption, and hence $\alpha \in Q^w(D)$ by rule (8).

**A.5 Proof of Lemma 14**

We begin by restating the automaton construction given in the proof sketch of Lemma 14 and then formally state and prove its correctness. Finally, we use the automata-theoretic characterisation to prove Lemma 14.
Automaton Construction  Let $Q$ be a temporal object-ground fp-query with output predicate $G$. For simplicity, and without loss of generality, we assume that $Q$ contains no object terms and hence all predicates in the query are either nullary or unary and temporal. Furthermore, let $\rho$ be the radius of $Q$. Then, the automaton $A$ capturing $Q$ is as follows.

- A state is either the initial state $s_{init}$, or a $(\rho + 2)$-tuple where the first component is a subset of the rigid EDB predicates in $Q$, and the other components are subsets of the temporal (EDB and IDB) predicates in $Q$. A state is final if its last component contains the output predicate $G$.
- Each alphabet symbol is a set $\Sigma$ of EDB predicates occurring in $Q$ such that $\Sigma$ does not contain temporal and rigid predicates simultaneously.

The transition function $\delta$ consists of

- each transition $s_{init},\Sigma \mapsto \langle \Sigma, \emptyset, \ldots, \emptyset \rangle$ such that $\Sigma$ consists of rigid predicates; and
- each transition $\langle B, M_0, \ldots, M_p \rangle, \Sigma \mapsto \langle B, M'_0, \ldots, M'_p \rangle$ such that: (i) $\Sigma$ consists of temporal predicates; (ii) $M'_i = M_{i+1}$ for each $0 \leq i < \rho$; and (iii) $M'_\rho$ consists of each predicate $P$ satisfying $\Pi Q \cup B \cup H \cup U \models P(\rho)$ for $H$ the set of all facts $R(i)$ with $R \in M_i$ and $0 \leq i < \rho$, and $U$ the set of all facts $R(\rho)$ with $R \in \Sigma$.

Correctness of the Construction  To argue the correctness of the construction, we first show the following auxiliary result.

Proposition 32. Let $\Pi$ be a program consisting of forward-propagating rules, let $F$ be a set of facts, and let $\alpha$ be a temporal fact. Furthermore, let $\tau$ be the time argument of $\alpha$, let $\rho$ be the maximum radius of a rule in $\Pi$, let $B$ be the set of rigid facts in $F$, let $H$ be the set of temporal facts entailed by $\Pi \cup B \cup F\|_{[0,\tau)}$ and having time argument in $[\tau - \rho, \tau)$, and let $U = F\|_{\tau}$. Then, $\Pi \cup F \models \alpha$ if $\Pi \cup B \cup H \cup U \models \alpha$.

Proof. We prove the two implications separately.

$(\Leftarrow)$ Assume $\Pi \cup B \cup H \cup U \models \alpha$. Let $H'$ be the set of facts entailed by $\Pi \cup F$. Note that $B \subseteq F$, $H \subseteq H'$, and $U \subseteq F$. It follows that $\Pi \cup F \cup H' \models \alpha$ by monotonicity, and hence $\Pi \cup F \models \alpha$ since $\Pi \cup F \models H'$ by definition.

$(\Rightarrow)$ Assume $\Pi \cup F \models \alpha$. Let $\delta$ be a derivation of $\alpha$ from $\Pi \cup F$. We prove the claim by induction on the height $n$ of $\delta$. In the base case $n = 0$, and hence $\Pi \cup F \models \alpha$. In particular $\alpha \in U$, and hence the claim holds by monotonicity. In the inductive case $n > 0$, and we assume that $\Pi \cup B \cup H \cup U \models \beta$ holds for each temporal fact $\beta$ with time argument $\tau$ having a derivation from $\Pi \cup F$ of height at most $n - 1$. Let $r$ be the rule labelling the root of $\delta$, and let $\beta$ be an atom in the body of $r$. It suffices to show $\Pi \cup B \cup H \cup U \models \beta$. We distinguish two cases. In the first case $\beta$ is rigid, and hence $\Pi \cup B \models \beta$ since it is clear that any derivation of any rigid fact such as $\beta$ from $\Pi \cup F$ does not involve temporal facts, by the properties of forward-propagating rules; the claim follows by monotonicity. In the other case $\beta$ is temporal. Let $\tau'$ be the time argument of $\beta$. Note that $\tau' \in [\tau - \rho, \tau)$ since $r$ is forward-propagating and its radius is at most $\rho$. We distinguish again two cases. If $\tau' \in [\tau - \rho, \tau)$, then $\beta \in H$ by Proposition 30 and the construction of $H$, and hence the claim holds by monotonicity. Otherwise, we have that $\tau'$ coincides with $\tau$, and hence the claim holds by the inductive hypothesis. \hfill \Box

Automaton $A$ correctly captures $Q$ in the sense of the following Claim 33 and Claim 34.

Claim 33. Let $D$ be a dataset, and let $\rho$ be a non-negative integer. Furthermore, let $\rho$ be the radius of $Q$, let $w$ be the word $\langle \Sigma_0, \Sigma_1, \ldots, \Sigma_n \rangle$ where $\Sigma_0$ is the set of rigid facts in $D$ and each $\Sigma_i$ with $i > 0$ is the set $\{ A | A(i - 1) \in D \}$, let $M_i = \emptyset$ for each $-\rho \leq i \leq 0$, let $M_i = \{ P | \Pi Q \cup \Sigma \models P(i - 1) \}$ for each $1 \leq i \leq n$, and let $s_i = \langle \Sigma_0, M_{i - \rho}, \ldots, M_i \rangle$ for each $0 \leq i \leq n$. Then, $\langle s_{init}, \Sigma_0, s_0, s_1, \ldots, \Sigma_n, s_n \rangle$ is a run of $A$.

Proof. We prove the claim by induction on $n$.

In the base case $n = 0$. We have that $\langle s_{init}, \Sigma_0, s_0 \rangle$ is a run of $A$ by construction—note that $s_0 = \langle \emptyset, \ldots, \emptyset \rangle$.

In the inductive case $n > 0$, and we assume that $\langle s_{init}, \Sigma_0, s_0, \ldots, \Sigma_n, s_n \rangle$ is a run of $A$. We have to show that $\langle s_{init}, \Sigma_0, s_0, \ldots, \Sigma_n, s_n \rangle$ is a run of $A$, for which it suffices to show that $\delta(s_{n - 1}, \Sigma_n) = s_n$.

Let $H$ be the set consisting of each fact $P(i)$ for $P \in M_{n - \rho + 1}$ and $0 \leq i < \rho$, and let $U$ be the set consisting of each fact $P(\rho)$ for $P \in \Sigma_n$. Hence, according to the construction of $\delta$, it suffices to show that $M_n = \{ P | \Pi Q \cup \Sigma_n \models P(\rho) \}$. Now, we have that $M_n = \{ P | \Pi Q \cup \Sigma_n \models P(n - 1) \}$ by construction. Let $H'$ be the set of temporal facts entailed by $\Pi Q \cup \Sigma_n \cup D\|_{[0,n - 1)}$ and having time argument in $[n - 1 - \rho, n - 1)$. It follows that $M_n = \{ P | \Pi Q \cup \Sigma_n \cup H' \cup D\|_{n - 1} \models P(n - 1) \}$ by Proposition 32. Let $H^*$ and $U^*$ be $H'$ and $D\|_{n - 1}$, respectively, after replacing each time point $\tau$ with $\tau - n + 1 + \rho$. Note that the two former datasets are well-formed since each time point occurring in them is at least $n - 1 - \rho$. It follows that $M_n = \{ P | \Pi Q \cup \Sigma_n \cup H^* \cup U^* \models P(\rho) \}$ since $Q$ is an fp-query—indeed, it mentions no time point. Finally, $M_n = \{ P | \Pi Q \cup \Sigma_n \cup H \cup U \models P(\rho) \}$ holds by Claim 33 and Claim 34, which are given next.

Claim 33.1. It holds that $H = H^*$.

We first show $H \subseteq H^*$. Let $\alpha \in H$. By the definition of $H$, we have that $\alpha$ is a temporal fact of the form $P(i)$ with $P \in M_{n - \rho + 1}$, and $0 \leq i < \rho$. It follows that $\Pi Q \cup D \models P(n - \rho + i - 1)$ by the construction of $M_{n - \rho + i}$, hence $\Pi Q \cup D\|_{[0,n - 1)} \models P(n - \rho + i - 1)$ by Proposition 30 and monotonicity, hence $P(n - \rho + i - 1) \in H'$ by the construction of $H'$, hence $P(i) \in H^*$ by the construction of $H^*$, and hence $\alpha \in H^*$. 
We now show $H^s \subseteq H$. Let $\alpha \in H^s$. We have that $\alpha$ is a temporal fact of the form $P(i)$ such that $P(n - 1 - \rho + i) \in H'$ by the construction of $H'$. It follows that $\Pi_0 \cup \Sigma_0 \cup D|_{[0, n-1]} \models P(n - 1 - \rho + i)$ by the construction of $H'$, hence $P \in M_{n-\rho+i}$ by the construction of $M_{n-\rho+i}$, hence $P(i) \in H$ by the construction of $H$, and hence $\alpha \in H$.

This concludes the proof of Claim 33.1.

Claim 33.2. It holds that $U = U^s$.

We first show $U \subseteq U^s$. Let $\alpha \in U$. We have that $\alpha$ is a temporal fact of the form $P(\rho)$ with $P \in \Sigma_n$, hence $P(n - 1) \in D$ by the construction of $\Sigma_n$, hence $P(\rho) \in U^s$ by the construction of $U^s$, and hence $\alpha \in U^s$.

We now show $U^s \subseteq U$. Let $\alpha \in U^s$. We have that $\alpha$ is a temporal fact of the form $P(i)$ such that $P(n - 1) \in D$, hence $P \in \Sigma_n$ by the construction of $\Sigma_n$, hence $P(\rho) \in U$ by the construction of $U$, and hence $\alpha \in U$.

This concludes the proof of Claim 33.2, and hence the overall proof.

Claim 34. Let $(\Sigma_0, \Sigma_1, \ldots, \Sigma_n)$ be a word over the input alphabet with $n \geq 0$, and let $\rho$ be the radius of $Q$. Furthermore, let $D$ be the dataset $\Sigma_0 \cup \{A(i - 1) \mid 1 \leq i \leq n\}$, and $A \in \Sigma_i$, let $M_i = \emptyset$ for each $-\rho \leq i \leq 0$, and let $M_i = \{P \mid \Pi_0 \cup D \models P(i - 1)\}$ for each $1 \leq i \leq n$. If $(s_{init}, \Sigma_0, s_0, \ldots, \Sigma_n, s_n)$ is a run of $A$, then $s_i = \langle \Sigma_0, M_{i-\rho}, \ldots, M_{i-1}, M_i \rangle$ for each $0 \leq i \leq n$.

Proof. Consider a run $(s_{init}, \Sigma_0, s_0, \ldots, \Sigma_n, s_n)$ of $A$. We prove the claim by induction on $n$.

In the base case $n = 0$, hence the considered run is $(s_{init}, \Sigma_0, s_0)$, and hence the claim holds since $\delta(s_{init}, \Sigma_0) = (\Sigma_0, \emptyset, \ldots, \emptyset)$ by construction.

In the inductive case $n > 0$, and we assume that the claim holds if we replace $n$ with $n - 1$. In particular, the inductive hypothesis implies that $s_{n-1} = (\Sigma_0, M_{n-\rho-1}, \ldots, M_{n-2}, M_{n-1})$. By the construction of $\delta$, we have that $s_n$ is of the form $(\Sigma_0, M_0, \ldots, M^n, p, m)$ with $M_0, \ldots, M^n, m$ sets of temporal predicates. In order to prove the claim, it suffices to show that $M_{n+1} = M_{n-1}$ for each $0 \leq i \leq \rho$. By the construction of $\delta$, we have the following.

Claim 34.1. It holds that $M_{n+1} = M_{n-1}$ for each $0 < i < \rho$.

Hence, we are left to prove $M_{n+1} = M_n$. Let $H$ be the set consisting of each fact $P(i)$ for $P \in M^i$ and $0 \leq i < \rho$, and let $U$ be the set consisting of each fact $P(\rho)$ for $P \in \Sigma_n$. Hence, according to the construction of $\delta$, it suffices to show that $M_n = \{P \mid \Pi_0 \cup \Sigma_0 \cup H \cup U \models P(\rho)\}$. Now, we have that $M_n = \{P \mid \Pi_0 \cup D \models P(n - 1)\}$ by construction. Let $H'$ be the set of temporal facts entailed by $\Pi_0 \cup \Sigma_0 \cup D|_{[0, n-1]}$ and having time argument in $[n - 1 - \rho, n - 1]$. It follows that $M_n = \{P \mid \Pi_0 \cup \Sigma_0 \cup H' \cup D|_{[0, n-1]} \models P(n - 1)\}$ by Proposition 32. Let $H^s$ and $U^s$ be $H'$ and $D|_{[0, n-1]}$, respectively, after replacing each time point $\tau$ with $\tau - n + 1 + \rho$. Note that the two former datasets are well-formed since each time point occurring in them is at least $n - 1 - \rho$. It follows that $M_n = \{P \mid \Pi_0 \cup \Sigma_0 \cup H^s \cup U^s \models P(\rho)\}$ since $Q$ is an fp-query— in particular, it mentions no time point. Finally, $M_n = \{P \mid \Pi_0 \cup \Sigma_0 \cup H \cup U \models P(\rho)\}$ holds by Claim 34.2 and Claim 34.3, which are given next.

Claim 34.2. It holds that $H = H^s$.

We first show $H \subseteq H^s$. Let $\alpha \in H$. By the definition of $H$, we have that $\alpha$ is a temporal fact of the form $P(i)$ with $P \in M^i$ and $0 \leq i < \rho$. Let $j = \rho - i$. It follows that $P \in M_{\rho-j}$, hence $P \in M_{n-j}$ by Claim 34.1, hence $P \in M_{n-\rho-j}$, hence $\Pi_0 \cup D \models P(n - \rho + i - 1)$ by the construction of $M_{n-\rho+j}$, hence $\Pi_0 \cup D|_{[0, n-1]} \models P(n - \rho + i - 1)$ by Proposition 30 and monotonicity, hence $P(n - \rho + i - 1) \in H'$, hence $P(i) \in H^s$, and hence $\alpha \in H^s$.

We now show $H^s \subseteq H$. Let $\alpha \in H^s$. We have that $\alpha$ is a temporal fact of the form $P(i)$ such that $P(n - 1 - \rho + i) \in H'$ by the construction of $H'$. It follows that $\Pi_0 \cup D \models P(n - 1 - \rho + i)$ by the construction of $H'$, hence $P \in M_{n-j}$ by the construction of $M_{n-\rho+i}$, let $j = \rho - i$. It follows that $P \in M_{n-j}$, hence $P \in M_{\rho-j}$ by Claim 34.1, hence $P \in M^i$, hence $P(i) \in H$ by the construction of $H$, and hence $\alpha \in H$.

This concludes the proof of Claim 34.2.

Claim 34.3. It holds that $U = U^s$.

We first show $U \subseteq U^s$. Let $\alpha \in U$. We have that $\alpha$ is a temporal fact of the form $P(\rho)$ with $P \in \Sigma_n$. It follows that $P(n - 1) \in D$ by the construction of $D$, hence $P(\rho) \in U^s$ by the construction of $U^s$, and hence $\alpha \in U^s$.

We now show $U^s \subseteq U$. Let $\alpha \in U^s$. We have that $\alpha$ is a temporal fact of the form $P(i)$ such that $P(n - 1) \in D$, hence $P \in \Sigma_n$ by the construction of $\Sigma_n$, hence $P(\rho) \in U$ by the construction of $U$, and hence $\alpha \in U$.

This concludes the proof of Claim 34.3, and hence the overall proof.

Proof of the Main Claim To show Lemma 14, we first observe two properties of our automata. Let $Q_1$ and $Q_2$ be temporal object-ground fp-queries sharing an output predicate $G$. For simplicity, and without loss of generality, we assume that $Q_1$ and $Q_2$ contain no object terms and hence all predicates in the queries are either nullary or unary and temporal. Furthermore, let $A_1$ and $A_2$ be the automata for $Q_1$ and $Q_2$, respectively, built as described in the previous section.

Claim 35. If $Q_1 \not\subseteq Q_2$, then there exists a word of length at least 2 that is accepted by $A_1$ and not by $A_2$. 

Proof. Let $\tau$ be a time point and let $D$ be a dataset such that $G(\tau) \in Q_1(D)$ and $G(\tau) \notin Q_2(D)$. Furthermore, let $\rho_1$ be the radius of $Q_1$, and let $n = \tau + 1$. By Claim 33 there exists a run $(s^1_{\text{init}}, \Sigma_0, s_0, \ldots, \Sigma_n, s_n)$ where $\Sigma_0$ is the set of rigid facts in $D$, each $\Sigma_i$ with $i > 0$ is the set $\{ A \mid A(i - 1) \in D \}$, and $s_n$ is of the form $\varrho = (\Sigma_0, M_n, \ldots, M_0)$. Since $G(\tau) \in Q_1(D)$, we have that $G \in M_n$, hence $s_n$ is final, hence $\varrho$ is an accepting run of $A_2$, and hence $A_2$ accepts the word $w = (\Sigma_0, \ldots, \Sigma_n)$. Note that $w$ has length $n + 1$, and hence at least 2 as required.

It suffices to show that $A_3$ does not accept $w$. We prove it by contradiction, assuming that $A_3$ accepts $w$. There exists an accepting run $\varrho' = (s^2_{\text{init}}, \Sigma_0, s'_0, \ldots, \Sigma_n, s'_n)$. Let $\rho_2$ be the radius of $Q_2$, and let $D' = \{ A(i - 1) \mid 1 \leq i \leq n, A \in \Sigma_i \}$. By Claim 34 we have that $s'_n$ is of the form $(\Sigma_0, M'_{n-\rho_2}, \ldots, M'_0)$ with $M'_n = \{ P \mid D \models P(\tau) \}$. It follows that $G \in M_n$ since $\varrho'$ is accepting, and hence $G(\tau) \in Q_2(D)$ and $G(\tau) \notin Q_2(D)$, which contradicts our initial assumption.

\[ \Box \]

Claim 36. For each word $w$ of length $n \geq 2$ accepted by $A_1$ and not by $A_2$, there exists a dataset $D$ over time points in $[0, n-2]$ such that $G(n-2) \in Q_1(D)$ and $G(n-2) \notin Q_2(D)$.

Proof. Let $w = \langle \Sigma_0, \ldots, \Sigma_{n-1} \rangle$ be a word with $n \geq 2$. Assume that $A_1$ accepts $w$ and $A_2$ does not accept $w$. There exists an accepting run $\varrho = (s^1_{\text{init}}, \Sigma_0, s_0, \ldots, \Sigma_{n-1}, s_{n-1})$ of $A_1$. Let $D$ be the dataset $\Sigma_0 \cup \{ A(i - 1) \mid 1 \leq i \leq n - 1, A \in \Sigma_i \}$. Note that $D$ is over time points in $[0, n-2]$ as required. Let $\rho_1$ be the radius of $Q_1$. By Claim 34 we have that $s_{n-1} = (\Sigma_0, M^n_0, M^n_1, \ldots, M^n_{\rho_1})$ where $M^n_{\rho_1} = \{ P \mid D \models P(n-2) \}$. Since $\varrho$ is accepting, we have that $s_{n-1}$ is final, hence $G \in M^n_{\rho_1}$, and hence $G(n-2) \in Q_1(D)$.

It suffices to show that $G(n-2) \notin Q_2(D)$. We prove it by contradiction, assuming that $G(n-2) \notin Q_2(D)$. Let $\rho_2$ be the radius of $Q_2$, let $M_i = \emptyset$ for each $-\rho_2 \leq i \leq 0$, let $M_i = \{ P \mid D \models P(i-1) \}$ for each $1 \leq i \leq n - 1$, and let $s'_n = (B, M_{-\rho_2}, \ldots, M_0)$ for each $0 \leq i \leq n - 1$. By Claim 33 we have that $\varrho' = (s^2_{\text{init}}, \Sigma_0, s'_0, \ldots, \Sigma_{n-1}, s'_{n-1})$ is a run of $A_2$. Since $G(n-2) \in Q_2(D)$ by our assumption, we have that $s_{n-1}$ is final, and hence $\varrho'$ is an accepting run of $A_2$. Therefore $A_2$ accepts $w$, which contradicts our initial assumption.

\[ \Box \]

Lemma 14. For each $i \in \{1, 2\}$, let $\rho_i$ and $p_i$ be the radius and the size of the signature of $Q_i$, respectively. Let $b_i = 1 + 2(p_i+\rho_i+2)$, and let $b = b_1 \cdot b_2$.

If $Q_1 \not\subseteq Q_2$, then there exists a time point $\tau \in [0, b]$ and a dataset $D$ over time points in $[0, b]$ such that $G(\tau) \in Q_1(D)$ and $G(\tau) \notin Q_2(D)$.

Proof. Let $\tau$ be a time point and let $D$ be a dataset such that $G(\tau) \in Q_1(D)$ and $G(\tau) \notin Q_2(D)$. By Claim 35 there exists a word of length at least 2 accepted by $A_1$ and not by $A_2$. Let $N_1$ be the number of states of $A_1$; also note that $N_1 \leq b_1$, since the set of states of $A_1$ consists of one initial state plus each $(\rho_1+2)$-tuple where each component is a subset of the predicates occurring in $Q_1$. By standard automata results, it follows that there exists a word of length $n$ with $2 \leq n \leq N_1$, $N_2 \leq b_1 \cdot b_2$ that is accepted by $A_1$ and not by $A_2$. By Claim 36 it follows that there exists a dataset $D'$ over time points in $[0, n-2]$ such that $G(n-2) \in Q_1(D')$ and $G(n-2) \notin Q_2(D')$. Therefore $n-2$ and $D'$ are the desired time point and dataset, respectively.

\[ \Box \]

A.6 Proof of Theorem 17

Lemma 37. CONT^{OGNDR} is in coNP.

Proof. First, note that CONT^{OGNDR} is LogSpace-reducible to CONT^Q, with $Q$ the Datalog subset of OGNR, by the results in (Ronca et al. 2018). Therefore it suffices to show that the complement of the latter problem is in NP. We give an algorithm, with input consisting of two object-ground non-recursive Datalog queries $Q_1$ and $Q_2$. The algorithm guesses a subset $D$ of the EDB atoms occurring in $Q_1 \cup Q_2$, and then accepts if $Q_1(D) \not\subseteq Q_2(D)$. It is correct because it clearly suffices to consider datasets which are subsets of the EDB atoms occurring in $Q_1 \cup Q_2$. It runs in polynomial time because each guessed dataset $D$ is of polynomial size and evaluation of propositional queries is in $P$—it amounts to Horn satisfiability.

For the following theorem, note that $O$ is the class of dataset over objects from a given (but arbitrary) set of objects $O$.

Theorem 17. The following upper bounds hold:

- WINDOW^Q is in coNEXP; and
- WINDOW^Q is in coNP for any class $Q \subseteq NR$ where the maximum number of object variables in any rule of any $Q \in Q$ is bounded by a constant.

Proof. We start by noting that WINDOW^{OGNDR} is LogSpace-reducible to CONT^{OGNDR} by Theorem 12 which is in coNP by Lemma 37. Note also that, given any query $Q \in NR$, we can ground its object variables over $O$ in time asymptotically bounded by $|\Pi_Q|^{|O|+O^{|k|}}$, where $O^{|k|}$ is the set of objects occurring in $Q$, and $k$ is the maximum number of object variables in a rule of $Q$; such a grounding yields a query $Q' \in OGNR$ equivalent to $Q$. So, given an instance $I = (Q, w)$ of WINDOW^{NR}, we can first map it to $I' = (Q', w)$ with $Q'$ the object-grounding of $Q$, and then decide whether WINDOW^{OGNDR} holds for $I'$ in coNP, and hence whether WINDOW^{Q} holds for $I$ in coNEXP, since $I'$ is exponential in $I$. For any class $Q \subseteq NR$ where the maximum number of object variables in any rule is bounded by a constant—i.e., $k$ in the expression $|\Pi_Q|^{|O|+O^{|k|}}$ can be considered fixed—we can compute $I'$ in polynomial time, and hence we can decide whether WINDOW^{Q} holds for $I$ in coNP.

\[ \Box \]
A.7 Proof of Theorem 18

Theorem 18. WINDOW\(^{OG}\) is PSpace-hard.

Proof. It suffices to show hardness for CONT\(^{OG}\) which is LOGSPACE-reducible to WINDOW\(^{OG}\) by Theorem 12. Consider the reduction from containment of succinct regular expressions to CONT\(^{OG}\) given in the following Section A.8. If the construction given there is restricted to (ordinary) regular expression, then it is easy to see that we obtain a reduction from containment of regular expressions to CONT\(^{OG}\). The result then follows from the fact that containment of regular expressions is PSPACE-complete—see, e.g., (Sipser 2006).

A.8 Proof of Theorem 19

We first provide the full version of the query construction described in the proof sketch of Theorem 19. We then show the correctness of the construction and, finally, use it for proving Theorem 19.

Query Construction. Consider a succinct regular expressions (SRE) \(R\) over a finite alphabet \(\Sigma\)—see, e.g., (Sipser 2006)—for the definition of SRE. We build a query \(Q_R = \langle G, \Pi_{\text{suc}} \cup \Pi_R \rangle\) where \(\Pi_R\) will be defined inductively over the structure of \(R\), and \(\Pi_{\text{suc}}\) is a Datalog program that defines ‘successor’ predicates \(\text{suc}_{\text{m}}\).

Next, we define the program \(\Pi_{\text{suc}}\). Let 0 and 1 be two fresh objects, intuitively standing for zero and one respectively. We use \(x\), \(0\), and \(\bar{1}\) for denoting tuples of fresh variables, \(0\)'s, and \(\bar{1}\)'s, respectively. We denote the length of a tuple \(t\) as \(|t|\). Let \(B\) be a fresh unary temporal IDB predicate, and let \(\text{suc}_{\text{m}}\) be a rigid IDB predicate of arity \(2m\) for \(m > 0\). Let \(\Pi_{\text{suc}}^{m}\) for \(m > 0\) be the program consisting of rule (9), rule (10), and each rule of the form (11) for \(0 \leq i < m\) where \(|x| = i\) and \(|\bar{1}| = |0| = m - i - 1\).

\[
\begin{align*}
\Pi_{\text{suc}}^{m} & \rightarrow B(0) \quad \text{(9)} \\
\Pi_{\text{suc}}^{m} & \rightarrow B(\bar{1}) \quad \text{(10)} \\
\bigwedge_{j=1}^{m} B(x_j) & \rightarrow \text{suc}_{\text{m}}(x, 0, \bar{1}, x, \bar{1}, 0) \quad \text{(11)}
\end{align*}
\]

Each program \(\Pi_{\text{suc}}^{m}\) and its corresponding predicate \(\text{suc}_{\text{m}}\) describe a finite successor relationship. Formally, \(\Pi_{\text{suc}}^{m} \models \text{suc}_{\text{m}}(i,j)\) holds if and only if (i) \(i, j\) is an \(m\)-tuple over \(\{0, 1\}\), and (ii) \(i + 1 = j\) for \(i, j\) the numbers encoded by \(\bar{1}\) and \(\bar{1}\), respectively. Finally, \(\Pi_{\text{suc}}\) is the union of each \(\Pi_{\text{suc}}^{m}\) for \(m = \lfloor \log_2 k \rfloor\) and \(k\) an exponent occurring in \(R\).

Next we define the program \(\Pi_R\). Let \(G\) be a fresh temporal unary IDB predicate, let \(F\) be a fresh temporal unary EDB predicate, and let \(A_\sigma\) be a fresh temporal unary EDB predicate for \(\sigma \in \Sigma\). For \(\Pi\) a program, we denote with \(\phi(\Pi)\) and \(\psi(\Pi)\) the programs obtained from \(\Pi\) by renaming each predicate \(P\) not in \(\{A_\sigma \mid \sigma \in \Sigma\}\) and different from any \(\text{suc}_{\text{m}}\) to globally fresh predicates \(P^e\) and \(P^o\), respectively, of the same arity as \(P\)—note that this renaming notation is different from the one used in the proof sketch, which we believe to be more succinct but less readable. The program \(\Pi_R\) is defined below, following the inductive definition of SRE. We have three base cases where we define \(\Pi_R\) from scratch, and four inductive cases where we need to assume that we are given the programs for the subexpressions of \(R\).

**Base case 1.** It is the case where \(R = 0\). Then, \(\Pi_R\) is the empty program.

**Base case 2.** It is the case where \(R = \sigma\) for \(\sigma \in \Sigma\). Then, \(\Pi_R\) consists of the following rule.

\[
F(t) \land A_\sigma(t) \rightarrow G(t + 1)
\]

**Base case 3.** It is the case where \(R = \epsilon\). Then, \(\Pi_R\) consists of the following rule.

\[
F(t) \rightarrow G(t)
\]

**Inductive case 1.** It is the case where \(R = S \cup T\) for \(S\) and \(T\) SREs. Then, \(\Pi_R\) extends \(\phi(\Pi_S) \cup \psi(\Pi_T)\) with the following rules.

\[
\begin{align*}
F(t) & \rightarrow F^\phi(t) \quad \text{(14)} \\
F(t) & \rightarrow F^\psi(t) \quad \text{(15)} \\
G^\phi(t) & \rightarrow G(t) \quad \text{(16)} \\
G^\psi(t) & \rightarrow G(t) \quad \text{(17)}
\end{align*}
\]

**Inductive case 2.** It is the case where \(R = S \circ T\) for \(S\) and \(T\) SREs. Then, \(\Pi_R\) extends \(\phi(\Pi_S) \cup \psi(\Pi_T)\) with the following rules.

\[
\begin{align*}
F(t) & \rightarrow F^\phi(t) \quad \text{(18)} \\
G^\phi(t) & \rightarrow G^\psi(t) \quad \text{(19)} \\
G^\psi(t) & \rightarrow G(t) \quad \text{(20)}
\end{align*}
\]
Inductive case 3. It is the case where $R = S^+$ for $S$ an SRE. Then, $\Pi_R$ extends $\phi(\Pi_S)$ with the following rules.

\[
F(t) \rightarrow F^\phi(t) 
\]  
(21)
\[
G^\phi(t) \rightarrow F^\phi(t) 
\]  
(22)
\[
G^\phi(t) \rightarrow G(t) 
\]  
(23)

Inductive case 4. It is the case where $R = S^k$ for $S$ an SRE and $k \geq 2$. Let $m = \lceil \log_2 k \rceil$—i.e., the number of bits to encode numbers in the interval $[0, k-1]$. Let $\mathbf{x}$ and $\mathbf{y}$ be $m$-tuples of fresh object variables, and let $F'$ be a fresh temporal IDB predicate of arity $n + m$ for each temporal (EDB or IDB) predicate $P$ of arity $n$. Then, $\Pi_R$ is constructed from $\Pi_S$ as follows. First, we replace each atom $P(p, s)$ with $F'(p, x, s)$, where $p$ is a vector of object terms and $s$ is a temporal term. Second, we extend the resulting program, with the resulting $a$ as the encoding of $k - 1$ as a binary string over 0 and 1.

\[
F(t) \rightarrow F'(\bar{0}, t) 
\]  
(24)
\[
G'(a, t) \rightarrow G(t) 
\]  
(25)
\[
G'(x, t) \land \text{succ}^m(x, y) \rightarrow F'(y, t) 
\]  
(26)

**Correctness of the Construction** Query $Q_R$, defined as above, correctly captures its corresponding SRE $R$ in the sense of the following Claim 38 and Claim 39. Note that $L(R)$ denotes the language of an SRE $R$.

**Claim 38.** Let $R$ be an SRE, and let $\Pi_R$ be the program for $R$. Furthermore, let $w = \langle \sigma_1, \ldots, \sigma_n \rangle$ be a word in $L(R)$, let $\tau$ be a time point, and let $D$ be a dataset. Assume that $F(\tau) \in D$, and $A_{\sigma_i}(\tau + i - 1) \in D$ for each $1 \leq i \leq n$. Then, $\Pi_R \cup \Pi_{\text{succ}} \cup D \models G(\tau + n)$.

**Proof.** We prove the claim by induction on the structure of $R$.

- **Base case 1.** It is the case where $R = \emptyset$. This case cannot happen, since $L(R) = \emptyset$ contradicts our assumption that $w \in L(R)$.

- **Base case 2.** It is the case where $R = \sigma$ for $\sigma \in \Sigma$. We have that $w = \langle \sigma \rangle$, and hence $A_{\sigma}(\tau) \in D$. We have that $\Pi_R \cup D \models G(\tau + 1)$ by rule (12).

- **Base case 3.** It is the case where $R = \varepsilon$. We have that $w = \varepsilon$, and hence $n = 0$. We have that $\Pi_R \cup D \models G(\tau)$ by rule (13).

In the inductive case, we have to prove that the claim holds in each of the following inductive cases, assuming that the claim holds for the subexpressions of $R$.

- **Inductive case 1.** It is the case where $R = S \cup T$, for $S$ and $T$ SREs. Let $\Pi_S$ and $\Pi_T$ be the programs for $S$ and $T$, respectively. We have that $w \in L(S) \cup L(T)$. We consider two cases separately. In the first case we have that $w \in L(S)$, hence $\Pi_S \cup D \models G(\tau + n)$ by the inductive hypothesis, hence $\phi(\Pi_S) \cup D \cup \{F^\phi(\tau)\} \models G^\phi(\tau + n)$ by the construction of $\phi(\Pi_S)$, and hence $\Pi_R \cup D \models G(\tau + 1)$ by rule (14) and rule (16). In the other case, symmetrically, we have that $w \in L(T)$, hence $\Pi_T \cup D \models G(\tau + n)$ by the inductive hypothesis, hence $\psi(\Pi_T) \cup D \cup \{F^\psi(\tau)\} \models G^\psi(\tau + n)$ by the construction of $\psi(\Pi_T)$, and hence $\Pi_R \cup D \models G(\tau + n)$ by rule (15) and rule (17).

- **Inductive case 2.** It is the case where $R = S \circ T$, for $S$ and $T$ SREs. Let $\Pi_S$ and $\Pi_T$ be the programs for $S$ and $T$, respectively. We have that $w$ is of the form $w_1 w_2$ with $w_1 \in L(S)$ and $w_2 \in L(T)$. Let $w_1 = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $w_2 = \langle \sigma_1', \ldots, \sigma_{n_2} \rangle$. By the inductive hypothesis, we have that $\Pi_S \cup D \models G(\tau + n_1)$, hence $\phi(\Pi_S) \cup D \cup \{F^\phi(\tau)\} \models G^\phi(\tau + n_1)$ by the construction of $\phi(\Pi_S)$, and hence $\Pi_R \cup D \models F^\phi(\tau + n_1)$ by rule (18) and rule (19). Again by the inductive hypothesis, we have that $\Pi_T \cup D \cup \{F^\psi(\tau + n_1)\} \models G(\tau + n_1 + n_2)$, hence $\psi(\Pi_T) \cup D \cup \{F^\psi(\tau + n_1)\} \models G^\psi(\tau + n_1 + n_2)$ by the construction of $\psi(\Pi_T)$, and hence $\Pi_R \cup D \models G(\tau + n_1 + n_2)$ by rule (20). Therefore, $\Pi_R \cup D \models G(\tau + n_1 + n_2)$.

- **Inductive case 3.** It is the case where $R = S^\ast$, for $S$ an SRE. Let $\Pi_S$ be the program for $S$. We have that $w$ is of the form $w_1 w_2 \ldots w_k$ with $k > 0$ and $w_1, w_2, \ldots, w_k \in L(S)$. Let $w_i = \langle \sigma_1, \ldots, \sigma_{n_i} \rangle$ for each $1 \leq i \leq k$. Note that the length of $w$ is $N = \sum_{i=1}^{k} n_i$. The following claim implies that $\Pi_R \cup D \models G^\phi(\tau + N)$, and hence $\Pi_R \cup D \models G(\tau + N)$ by rule (23).

**Claim 38.** For each $1 \leq i \leq k$, it holds that $\Pi_R \cup D \models G^\phi(\tau + \sum_{j=1}^{i-1} n_j)$.

We prove Claim 38 by induction on $i$ from 1 to $k$. In the base case $i = 1$. By the ‘outer’ inductive hypothesis we have that $\Pi_S \cup D \models G(\tau + n_1)$, hence $\phi(\Pi_S) \cup D \cup \{F^\phi(\tau)\} \models G^\phi(\tau + n_1)$ by the construction of $\phi(\Pi_S)$, and hence $\Pi_R \cup D \models G^\phi(\tau + n_1)$ by rule (21). In the inductive case $i > 1$, and we assume that $\Pi_R \cup D \models G^\phi(\tau + \sum_{j=1}^{i-1} n_j)$ for each $1 \leq j \leq i - 1$. By the ‘outer’ inductive hypothesis we have that $\Pi_S \cup D \cup \{F(\tau + \sum_{j=1}^{i-1} n_j)\} \models G(\tau + \sum_{j=1}^{i-1} n_j)$, hence $\phi(\Pi_S) \cup D \cup \{F^\phi(\tau + \sum_{j=1}^{i-1} n_j)\} \models G^\phi(\tau + \sum_{j=1}^{i-1} n_j)$ by the construction of $\phi(\Pi_S)$, and hence $\Pi_R \cup D \models G^\phi(\tau + \sum_{j=1}^{i-1} n_j)$ since $\Pi_R \cup D \models F^\phi(\tau + \sum_{j=1}^{i-1} n_j)$ by the ‘inner’ inductive hypothesis and by rule (22).

This concludes the proof of Claim 38.

- **Inductive case 4.** It is the case where $R = S^k$, for $k \geq 2$ and $S$ an SRE. Let $\Pi_S$ be the program for $S$. We have that $w$ is of the form $w_1 w_2 \ldots w_k$ with $w_1, w_2, \ldots, w_k \in L(S)$. Let $w_i = \langle \sigma_1, \ldots, \sigma_{n_i} \rangle$ for each $1 \leq i \leq k$. The following claim implies that
\( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(b, \tau + \sum_{i=1}^k n_i) \) where \( b \) is the binary encoding of \( k-1 \), and hence \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G(\tau + \sum_{i=1}^k n_i) \) by rule (25).

**Claim 38.** For each \( 1 \leq i \leq k \), it holds that \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(b, \tau + \sum_{j=1}^{i-1} n_j) \) where \( b \) is the binary encoding of \( i-1 \).

We prove Claim 38 by induction on \( i \) from 1 to \( k \). In the base case \( i = 1 \), we have that \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(0, \tau) \) by rule (24). It follows that \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(0, \tau + n_1) \), since the other cases mention no predicate of the form \( \Sigma \).

In particular, \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(b, \tau + \sum_{i=1}^{i-1} n_i) \) where \( b \) is the binary encoding of \( i-1 \), and hence \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(c, \tau + \sum_{i=1}^{i-1} n_i) \) where \( c \) encodes \( i-1 \), by rule (26) and by the construction of \( \Pi_{\text{succ}} \). It follows that \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(c, \tau + \sum_{i=1}^{i-1} n_i) \), and \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G(\tau + \sum_{i=1}^{i-1} n_i) \).

This concludes the proof of Claim 38 and hence the overall proof.

**Claim 39.** Let \( R \) be an SRE, and let \( \Pi_R \) be the program for \( R \). Furthermore, let \( D \) be a dataset. Assume that \( \Pi_R \cup \Pi_{\text{succ}} \cup D \models G(\tau) \). Then, there exists a word \( w = \langle \sigma_1, \ldots, \sigma_n \rangle \in L(R) \) such that \( F(\tau - n) \in D \) and \( A_{\sigma_i}(\tau - n + i - 1) \in D \) for each \( 1 \leq i \leq n \).

**Proof.** We prove the claim by induction on the structure of \( R \). Note that \( \Pi_{\text{succ}} \) needs to be considered in Inductive case 4 only, since the other cases mention no predicate of the form \( \Sigma \).

In the base case, we have to prove that the claim holds for the three base in the inductive definition of \( R \).

**Base case 1.** It is the case where \( R = 0 \). We show that this case cannot happen. We would have that \( \Pi_R = \emptyset \) by construction, and hence \( \Pi_R \cup D \not\models G(\tau) \), which contradicts our initial assumption.

**Base case 2.** It is the case where \( R = \sigma \) for \( \sigma \in \Sigma \). The word \( \langle \sigma \rangle \) is as required since \( F(\tau - 1) \in D \) and \( A_{\sigma}(\tau - 1) \in D \) by the construction of \( \Pi_R \).

**Base case 3.** It is the case where \( R = c \). The empty word is as required since \( F(\tau) \in D \) by the construction of \( \Pi_R \).

In each of the following inductive cases, we have to prove that the claim holds for \( R \) assuming that the claim holds for the subexpressions of \( R \).

**Inductive case 1.** It is the case where \( R = S \cup T \), for \( S \) and \( T \) SREs. Let \( \Pi_S \) and \( \Pi_T \) be the programs for \( S \) and \( T \), respectively. We have that \( \Pi_R \cup D \models G^0(\tau) \) or \( \Pi_R \cup D \models G^0(\tau) \). Let \( D' \) be the dataset consisting of each \( A_{\sigma_i} \)-fact in \( D \) with \( \sigma \in \Sigma \). It is clear from the construction of \( \Pi_R \) that the following claim holds.

**Claim 39.1.** One of the following holds: (i) there exists an integer \( p \geq 0 \) such that \( \Pi_R \cup D \models F^0(\tau - p) \) and \( \psi(\Pi_S) \cup D' \cup \{ F^0(\tau - p) \} \models G^0(\tau) \); (ii) there exists an integer \( q \geq 0 \) such that \( \Pi_R \cup D \models F^0(\tau - q) \) and \( \psi(\Pi_T) \cup D' \cup \{ F^0(\tau - q) \} \models G^0(\tau) \).

By Claim 39.1 it follows that \( \Pi_S \cup D' \cup \{ F(\tau - p) \} \models G(\tau) \) or \( \Pi_T \cup D' \cup \{ F(\tau - q) \} \models G(\tau) \) by the construction of \( \psi(\Pi_S) \) and \( \psi(\Pi_T) \). By the inductive hypothesis, either there is a word \( w = \langle a_1, \ldots, a_n \rangle \) in \( L(S) \) such that \( F(\tau - n) \in D' \cup \{ F(\tau - p) \} \), and \( A_{a_i}(\tau - n + i - 1) \in D' \) for each \( 1 \leq i \leq n \), or there is a word \( w' = \langle b_1, \ldots, b_n \rangle \) in \( L(T) \) such that \( F(\tau - n') \in D' \cup \{ F(\tau - q) \} \), and \( A_{b_i}(\tau - n' + i - 1) \in D' \) for each \( 1 \leq i \leq n' \). Note that in the former case \( n' = n \) and in the latter case \( n' = n \). Finally, note that both \( w \) and \( w' \) are in \( L(R) \), and it is easy to see that in both cases \( D \) satisfies the required properties.

**Inductive case 2.** It is the case where \( R = S \circ T \), for \( S \) and \( T \) SREs. Let \( \Pi_S \) and \( \Pi_T \) be the programs for \( S \) and \( T \), respectively. Let \( D' \) be the dataset consisting of each \( A_{\sigma_i} \)-fact in \( D \) with \( \sigma \in \Sigma \). We have that \( \Pi_R \cup D \models G(\tau) \), and hence the following claim holds by the construction of \( \Pi_R \).

**Claim 39.2.** There exists an integer \( p \geq 0 \) such that \( \psi(\Pi_T) \cup D' \cup \{ F^0(\tau - p) \} \models G^0(\tau) \) and \( \Pi_R \cup D \models F^0(\tau - p) \).

Furthermore, we have that \( \Pi_R \cup D \models F^0(\tau - p) \) implies \( \Pi_R \cup D \models G^0(\tau - p) \)—see rule (19)—and hence the following claim holds by the construction of \( \Pi_R \).

**Claim 39.3.** There exists an integer \( q \geq 0 \) such that \( \psi(\Pi_S) \cup D' \cup \{ F^0(\tau - q) \} \models G^0(\tau - p) \) and \( \Pi_R \cup D \models F^0(\tau - p - q) \).

By Claim 39.2 we have that \( \Pi_T \cup D' \cup \{ F(\tau - p) \} \models G(\tau) \), and hence by the inductive hypothesis there exists a word \( w = \langle a_1, \ldots, a_n \rangle \in L(S) \) such that \( F(\tau - n) \in D' \cup \{ F(\tau - p) \} \) and \( A_{a_i}(\tau - n + i - 1) \in D' \) for each \( 1 \leq i \leq n \). It follows that \( n' = n \), since \( D' \) contains no \( F \)-facts. Furthermore, \( \Pi_R \cup D \models F^0(\tau - p - q) \) by Claim 39.3 again, and hence \( F(\tau - p - q) \in D \). Finally, note that \( w' \in L(R) \), and hence \( w' \) remains in \( D \) and \( D \) are the required word and dataset, respectively.

**Inductive case 3.** It is the case where \( R = S^+ \). Let \( \Pi_S \) be the program for \( S \). It is easy to see from the construction of \( \Pi_R \) that the following claim holds by the inductive hypothesis.
Claim 39.4. Assume that $\Pi_R \cup D \models G^\phi(\tau')$ with $\tau' \leq \tau$. Then, there exists a word $\langle \sigma_1, \ldots, \sigma_n \rangle \in \mathcal{L}(S)$ such that $\Pi_R \cup D \models F^\phi(\tau' - n) \land A_{\sigma_i}(\tau' - n + i - 1) \in D$ for each $1 \leq i \leq n$.

Given the previous claim, we can prove the following one.

Claim 39.5. There exist words $w_1, \ldots, w_k$ for $k > 0$ such that, for each $1 \leq i \leq k$, it holds that:

- $w_i = \langle \sigma_{i_1}^1, \ldots, \sigma_{i_{n_i}}^i \rangle \in \mathcal{L}(S)$,
- $\Pi_R \cup D \models F^\phi(\tau - \sum_{j=i}^k n_j)$,
- $A_{\sigma_j}(\tau + j - 1 - \sum_{\ell=i}^k n_{\ell}) \in D$ for each $1 \leq j \leq n_i$,
- $F(\tau - \sum_{i=1}^k n_i) \in D$.

We omit the proof of Claim 39.5 since it would be essentially similar to the one of Claim 39.8; i.e., it would consist in showing that the claim follows by repeated application of Claim 39.4—which is again similar to Claim 39.7. One major difference with Claim 39.8 is the last point, for which we would need to argue that each derivation of $G(\tau)$ from $\Pi_R \cup D$ has a leaf labelled with $F(\tau - \sum_{i=1}^k n_i)$ because rule (21) is the only rule in $\Pi_R$ having no IDB atom in its body.

Then, the desired word is $w_1 \cdots w_k$, with $k$ each $w_i$ as in Claim 39.5.

Inductive case 4. It is the case where $R = S^k$ for $k \geq 2$ and $S$ an SRE. Let $\Pi_S$ be the program for $S$. Note that, as mentioned at the beginning of this proof, in this case we have to take $\Pi_{\text{succ}}$ into account. Let $D'$ be the dataset consisting of each $A_{\sigma}$-fact in $D$ with $\sigma \in \Sigma$. It is easy to see from the construction of $\Pi_R \cup \Pi_{\text{succ}}$ that the following claim holds.

Claim 39.6. Assume that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(b, \tau')$ with $\tau' \leq \tau$. There exists an integer $p \geq 0$ such that $\Pi_R \cup D' \cup \{F'(b, \tau' - p)\} \models G'(b, \tau')$ and $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(b, \tau' - p)$.

We use the former claim to prove the next one.

Claim 39.7. Assume that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(b, \tau')$ with $\tau' \leq \tau$. Then, there exists a word $\langle \sigma_1, \ldots, \sigma_n \rangle \in \mathcal{L}(S)$ such that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(b, \tau' - n)$ and $A_{\sigma_i}(\tau' - n + i - 1) \in D$ for each $1 \leq i \leq n$.

We prove Claim 39.7. By Claim 39.6, there exists $n \geq 0$ such that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(b, \tau' - n)$ and $\Pi_R \cup D' \cup \{F'(\tau' - n)\} \models G'(\tau')$. It follows that $\Pi_S \cup D' \cup \{F'(\tau' - n)\} \models G(\tau')$ by the construction of $\Pi_R$. By the inductive hypothesis, there exists a word $\langle \sigma_1, \ldots, \sigma_n \rangle \in \mathcal{L}(S)$ such that $F(\tau' - n') \in D' \cup \{F(\tau' - n)\}$ and $A_{\sigma_i}(\tau' - n' + i - 1) \in D'$ for each $1 \leq i \leq n'$. Since $D'$ contains no $F$-fact, we have that $n = n'$, and hence $D$ is as required.

This concludes the proof of Claim 39.7.

Again, we use the former claim for proving the next one.

Claim 39.8. There exist words $w_1, \ldots, w_k$ such that, for each $1 \leq i \leq k$, it holds that:

- $w_i = \langle \sigma_{i_1}^1, \ldots, \sigma_{i_{n_i}}^i \rangle \in \mathcal{L}(S)$,
- $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(b, \tau - \sum_{j=i}^k n_j)$ where $b$ is the binary encoding of $i - 1$,
- $A_{\sigma_j}(\tau + j - 1 - \sum_{\ell=i}^k n_{\ell}) \in D$ for each $1 \leq j \leq n_i$.

We prove Claim 39.8 by induction on $i$ from $k$ to 1.

In the base case $i = k$. As assumed above, we have that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models G(\tau)$. It follows that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(b, \tau)$ where $b$ is the binary encoding of $k - 1$ by the construction of $\Pi_{\text{succ}}$—see rule (23). By Claim 39.7, there exists a word $\langle \sigma_1, \ldots, \sigma_n \rangle \in \mathcal{L}(S)$ such that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(b, \tau - n)$ and $A_{\sigma_i}(\tau - n + i - 1) \in D$ for each $1 \leq i \leq n$. In the inductive case $1 \leq i < k$, and we assume the claim holds if we replace $i$ with $i + 1$. Let $N = \sum_{j=i+1}^k n_j$. By the inductive hypothesis, we have that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(b, \tau - N)$ where $b$ is the binary encoding of $i$. It follows that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models G'(c, \tau - N)$ with $c$ the binary encoding of $i - 1$ by the construction of $\Pi_{\text{succ}}$—see rule (26). By Claim 39.7 there exists a word $\langle \sigma_1, \ldots, \sigma_n \rangle \in \mathcal{L}(S)$ such that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(c, \tau - N - n)$ and $A_{\sigma_i}(\tau - N - n + i - 1) \in D$ for each $1 \leq i \leq n$.

This concludes the proof of Claim 39.8.

By Claim 39.8, there exists a word $w = \langle \sigma_1, \ldots, \sigma_n \rangle \in \mathcal{L}(R)$ such that $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(\bar{0}, \tau - n)$ and $A_{\sigma_i}(\tau - n) \in D$. Furthermore, $\Pi_R \cup \Pi_{\text{succ}} \cup D \models F'(\bar{0}, \tau - n)$ implies $F(\tau - n) \in D$—see rule (24). Therefore $w$ and $D$ are as required. ∎
**Proof of the Main Claim**  We finally show Theorem 19 using the query construction given above. Note that $O$ is the class of datasets over objects from a given (but arbitrary, and possibly empty) set of objects $O$; and also that $\mathcal{L}(R)$ denotes the language of an SRE $R$.

**Theorem 19.** Window$^\mathcal{O}$ is ExpSpace-hard.

**Proof.** It suffices to show that Cont$^\mathcal{O}$ is ExpSpace-hard, since it is LogSpace-reducible to Window$^\mathcal{O}$ by Theorem 12. We show a LogSpace-computable many-one reduction $\varphi$ from the containment problem for succinct regular expressions (SREs) to Cont$^\mathcal{O}$. Then, the claim of the theorem follows from the fact that SRE containment is ExpSpace-hard—see, e.g., (Sipser 2006).

An instance $I$ of the containment problem for SREs is a pair of SREs $R_1$ and $R_2$. Let $Q_{R_1}$ and $Q_{R_2}$ be the queries for $R_1$ and $R_2$ built as described above. Then, $\varphi$ maps $I$ to $(Q_{R_1}, Q_{R_2})$. Such queries can clearly be computed in logarithmic space, and hence the same holds for $\varphi$. We argue next that $\varphi$ is correct, i.e., $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$ if $Q_{R_1} \subseteq \mathcal{O} Q_{R_2}$.

Assume $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$. Then we show $Q_{R_1} \subseteq \mathcal{O} Q_{R_2}$. Let $\tau$ be a time point and let $D$ be a dataset in $\mathcal{O}$ such that $G(\tau) \in Q_{R_1}(D)$. Hence, we have to show that $G(\tau) \in Q_{R_2}(D)$. By Claim 39, there exists a word $w = (\sigma_1, \ldots, \sigma_n) \in \mathcal{L}(R_1)$ such that $F(\tau - n) \in D$, and each $A_{\sigma_i}(\tau - n + i - 1) \in D$ for $1 \leq i \leq n$. It follows that $w \in \mathcal{L}(R_1)$ since $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$ by our assumption, and hence $G(\tau) \in Q_{R_2}(D)$ by Claim 38.

For the converse, assume $Q_{R_1} \subseteq \mathcal{O} Q_{R_2}$. Then we show $\mathcal{L}(R_1) \subseteq \mathcal{L}(R_2)$. Let $w = (a_1, \ldots, a_n)$ be a word in $\mathcal{L}(R_1)$. Let $D$ be the dataset consisting of $F(0)$ and $A_{a_i}(i - 1)$ for each $1 \leq i \leq n$. Note that $D \in \mathcal{O}$, since $D$ mentions no objects. It follows that $G(n) \in Q_{R_2}(D)$ by Claim 38 and hence $G(n) \in Q_{R_2}(D)$ since we have assumed that $Q_{R_1} \subseteq \mathcal{O} Q_{R_2}$. By Claim 39, there exists a word $w' = (b_1, \ldots, b_n)$ in $\mathcal{L}(R_2)$ such that $F(n - n') \in D$ and $A_{b_i}(n - n' + i - 1)$ for each $1 \leq i \leq n'$. We have that $n' = n$, since $F(0)$ is the only $F$-fact in $D$ by construction. Furthermore, we have that $b_i = a_i$ for each $1 \leq i \leq n$, since $A_{a_i}(i - 1)$ is the only fact in $D$ of the form $A_{a}(i - 1)$ for any $a$. Therefore $w' = w$, and hence $w \in \mathcal{L}(R_2)$ as required. $\square$

### A.9 Proof of Theorem 20

**Lemma 40.** Cont$^\text{OGNR}$ is coNP-hard.

**Proof.** We prove the claim by giving a LogSpace-computable many-one reduction $\varphi$ from 3-SAT to the complement of Cont$^\text{Q}$ with $Q$ the propositional Datalog subclass of ognr.

Now we describe the reduction $\varphi$. Let $\alpha$ be a 3-CNF formula. Let $g$ be a fresh IDB nullary predicate—i.e., a propositional variable. Let $c_1, \ldots, c_n$ be fresh IDB nullary predicates corresponding to the clauses of $\alpha$. For each $1 \leq i \leq n$, let $l_{i,1}, l_{i,2}$ and $l_{i,3}$ be fresh IDB nullary predicates corresponding to the literals of the clause corresponding to $c_i$. Let $Q_1$ be the query $\langle g, \Pi_1 \rangle$ where $\Pi_1$ is the program consisting of rule (27), and each rule of the form (28) for $1 \leq i \leq n$ and $1 \leq j \leq 3$. Let $Q_2$ be the query $\langle g, \Pi_2 \rangle$ where $\Pi_2$ is the program consisting of each rule of the form (29) for $l_{i,j}$ and $l_{p,q}$ corresponding to complementary literals over the same propositional variable—e.g., literals $a$ and $\neg a$, where $a$ is a propositional variable.

\[
\begin{align*}
\text{c_1 \wedge \cdots \wedge c_n} & \rightarrow g \\
\text{l_{i,j}} & \rightarrow c_i \\
\text{l_{i,j} \wedge l_{p,q}} & \rightarrow g
\end{align*}
\] (27) (28) (29)

Note that $Q_1$ and $Q_2$ are clearly non-recursive propositional Datalog queries. Then, $\varphi$ maps $\alpha$ to $\langle Q_1, Q_2 \rangle$. We argue next that the reduction is correct, i.e., $\alpha$ is satisfiable if $Q_1 \not\subseteq Q_2$. In the following, in a slight abuse of notation, we identify any $c_i$ with its corresponding clause, and any $l_{i,j}$ with its corresponding literal.

Assume that $\alpha$ is satisfiable, i.e., there exists a satisfying assignment $f$ for $\alpha$. We show that $Q_1 \not\subseteq Q_2$. Let $D$ be the dataset consisting of each positive literal $l_{i,j}$ whose propositional variable is made true by $f$, and each negative literal $l_{i,j}$ whose propositional variable is made false by $f$. Clearly, for each pair of complementary literals $l_{i,j}$ and $l_{p,q}$ sharing the same propositional variable, we have that $l_{i,j} \not\in D$ or $l_{p,q} \not\in D$. By the construction of $Q_2$, it follows that $\Pi_2 \cup D \not\models g$. Now, for each clause $c_i$ of $\alpha$, there exists a literal $l_{i,j}$ made true by $f$, hence $l_{i,j} \in D$, and hence $\Pi_1 \cup D \models c_i$ by one of the rules of the form (28). It follows that $\Pi_1 \cup D \models g$ by rule (27). Therefore $Q_1 \not\subseteq Q_2$.

For the converse, assume $Q_1 \not\subseteq Q_2$. We show that $\alpha$ is satisfiable. There is a dataset $D$ such that $\Pi_1 \cup D \models g$ and $\Pi_2 \cup D \not\models g$. Let $f$ be the assignment for $\alpha$ such that $l_{i,j}$ is made true by $f$ if $l_{i,j} \in D$. Since $\Pi_2 \cup D \not\models g$ and by the rules of the form (29), there is no pair of complementary literals $l_{i,j}$ and $l_{p,q}$ sharing the same propositional variable and being both in $D$, and hence $f$ is a well-formed assignment for $\alpha$. Let $i \in [1, n]$. Since $\Pi_1 \cup D \models g$ and by rule (27), we have that $\Pi_1 \cup D \models c_i$. Hence there exists $j$ such that $\Pi_1 \cup D \models l_{i,j}$, hence $l_{i,j} \in D$, and hence $l_{i,j}$ is made true by $f$. According to the construction of $f$. Therefore $f$ is a satisfying assignment for $\alpha$. $\square$

**Lemma 41.** Cont$^\text{Q}$ is coNEXP-hard if $O$ contains at least two objects.

**Proof.** Assume that $O$ contains at least two objects. We show that Cont$^\text{Q}$ is already coNEXP-hard when $Q$ is the class DNR of non-recursive Datalog queries. In particular, we give a LogSpace-computable many-one reduction $\varphi$ from the exponential
tiling problem to the complement of \textsc{Cont}^\textsc{dnr}. The claim then follows from the fact that the exponential tiling problem is \textsc{NExp}-complete—see, e.g., Section 3.2 of \cite{Johnson1990}.

Our reduction $\varphi$ is a straightforward adaptation of a reduction from the exponential tiling problem to the complement of \textsc{Cont}^\textsc{dnr} given in \cite{BenediktGottlob2010}. Such a reduction does not directly apply to \textsc{Cont}^\textsc{dnr} because it makes use of an unbounded number of objects. Specifically, it requires datasets containing a number of objects which is exponential in the size of the input to the exponential tiling problem, since each object represents one coordinate in the corridor that we are given to tile. In our reduction, instead, coordinates are encoded using just two objects. The ‘downside’ of such an encoding is that it makes use of predicates whose arity is linear in the size of the input to the exponential tiling problem—but that is clearly irrelevant to our purposes.

The exponential tiling problem. An instance $I$ of \textsc{ExpTiling} is a 5-tuple $\langle n, r, H, V, T_0 \rangle$ where $n$ and $r$ are non-negative integers (coded in unary), $H$ and $V$ are subsets of $\{1, r\} \times \{1, r\}$, and $T_0$ is a total function from $\{0, n-1\}$ to $\{1, r\}$. A tiling for $I$ is a total function $T$ from $[0, 2^n - 1] \times [0, 2^n - 1]$ to $\{1, r\}$ such that $T(0, j) = T_0(j)$ for each $j \in [0, n-1]$, $\langle T(i, j), T(i+1, j) \rangle \in H$ for each $i \in [0, 2^n - 2]$ and each $j \in [0, 2^n - 1]$, and $\langle T(i, j), T(i, j+1) \rangle \in V$ for each $i \in [0, 2^n - 1]$ and each $j \in [0, 2^n - 2]$. Finally, \textsc{ExpTiling} holds for $I$ iff there is a tiling for $I$.

Given an instance $I$ of \textsc{ExpTiling} as described above, we define $\varphi(I)$ as the pair $\langle Q_1, Q_2 \rangle$ with $Q_1 = \langle \text{goal}, \Pi_1 \rangle$ and $Q_2 = \langle \text{goal}, \Pi_2 \rangle$, where \text{goal} is a fresh nullary predicate, and $\Pi_1$ and $\Pi_2$ are described next. In the following construction, note that all the predicates considered are rigid, and all the variables are of object sort.

Construction of the left-hand program. Next we build $\Pi_1$. Consider the following predicates. Let \text{zero} and \text{one} be fresh EDB unary predicates. Intuitively, \text{zero}(a) means that $a$ represents zero, and \text{one}(b) means that $b$ represents one. Furthermore, let $eq$ be a fresh IDB binary predicate. Intuitively, $eq(a, b)$ means that $a$ and $b$ represent the same bit. Then, $\Pi_1$ contains rule (30) and rule (31).

\begin{align*}
\text{zero}(x) \land \text{zero}(y) & \rightarrow eq(x, y) \\
\text{one}(x) \land \text{one}(y) & \rightarrow eq(x, y)
\end{align*}

Consider the following predicates. Let \text{tiledBy}_i be a fresh predicate of arity $2 \cdot n$ for each $i \in [1, r]$. Intuitively, \text{tiledBy}_i(a, b) means that the cell with coordinates $\langle p, q \rangle$, where $p$ and $q$ are the integers represented by $a$ and $b$, is tiled with the $i$-th tile. Furthermore, let $V_i$ be a fresh predicate of arity $i + n$ for each $i \in [0, n]$. Intuitively, $V_i(a, b)$ means that all the cells with coordinates $\langle p, q \rangle$ are tiled for every integer $p$ whose least significant $i$ bits are represented by $a$, and $q$ is the integer represented by $b$. Then, $\Pi_1$ contains each rule of the form (32) for $i \in [1, r]$, and each rule of the form (33) for $i \in [1, n]$.

\begin{align*}
\text{tiledBy}_i(x_1, \ldots, x_n, y) & \rightarrow V_n(x_1, \ldots, x_n, y) \\
V_i(x_1, \ldots, x_i, y) \land V_i(z_1, \ldots, z_i, y) & \land \bigwedge_{j=1}^{i-1} eq(x_j, z_j) \land \text{zero}(x_i) \land \text{one}(z_i) \rightarrow V_{i-1}(x_1, \ldots, x_{i-1}, y)
\end{align*}

Consider the following predicates. Let $H_i$ be a fresh predicate of arity $i$, for each $i \in [0, n]$. Intuitively, $H_i(a)$ means that all the cells with coordinates $\langle p, q \rangle$ are tiled for every $p \in [0, 2^n - 1]$ and every integer $q$ whose least significant $i$ bits are represented by $a$. Then, $\Pi_1$ contains rule (34), and each rule of the form (35) for $i \in [1, n]$.

\begin{align*}
H_0(x_1, \ldots, x_n) & \rightarrow \text{goal} \\
H_i(x_1, \ldots, x_i) \land H_i(y_1, \ldots, y_i) & \land \bigwedge_{j=1}^{i-1} eq(x_j, y_j) \land \text{zero}(x_i) \land \text{one}(y_i) \rightarrow H_{i-1}(x_1, \ldots, x_{i-1})
\end{align*}

Finally, $\Pi_1$ contains rule (36).

Construction of the right-hand program. Next we build $\Pi_2$. First, $\Pi_2$ contains rule (30) and rule (31). Then, $\Pi_2$ contains rule (37).

\begin{align*}
\text{zero}(x) \land \text{one}(x) & \rightarrow \text{goal}
\end{align*}

Consider the following predicates. Let $\text{succ}$ be a fresh predicate of arity $2 \cdot n$. Intuitively, $\text{succ}(a, b)$ means that $a$ and $b$ represent two integers $p$ and $q$, respectively, such that $p + 1 = q$. Let $u_1, \ldots, u_n, y, z$ be fresh variables. Then, $\Pi_2$ contains rule (38), rule (39), and each rule of the form (40) for $0 \leq i < n$, where $u$ is the $i$-tuple $\langle u_1, \ldots, u_i \rangle$, $v$ is the $(n-i-1)$-tuple $\langle v, \ldots, v \rangle$, and $w$ is the $(n-i-1)$-tuple $\langle w, \ldots, w \rangle$.

\begin{align*}
\text{zero}(x) & \rightarrow B(x) \\
\text{one}(x) & \rightarrow B(x) \\
\text{zero}(w) \land \text{one}(v) \land \bigwedge_{j=1}^{i-1} B(u_j) & \rightarrow \text{succ}(u, w, v, u, v, w)
\end{align*}

In the rest of the construction, consider the following fresh variables. Let $x = \langle x_1, \ldots, x_n \rangle$, let $x' = \langle x_1', \ldots, x_n' \rangle$, let $y = \langle y_1, \ldots, y_n \rangle$, and let $y' = \langle y_1', \ldots, y_n' \rangle$. Then, $\Pi_2$ contains each rule of the form (41) for $i, j \in [1, r]$ with $i \neq j$.

\begin{align*}
\bigwedge_{k=1}^{n} eq(x_k, x'_k) \land \bigwedge_{k=1}^{n} eq(y_k, y'_k) \land \text{tiledBy}_i(x, y) \land \text{tiledBy}_j(x', y') & \rightarrow \text{goal}
\end{align*}
Then, \( \Pi_2 \) contains each rule of the form \( (42) \) for \( j, k \in [1, r] \) with \( \langle j, k \rangle \notin V \), and each rule of the form \( (43) \) for \( j, k \in [1, r] \) with \( \langle j, k \rangle \notin H \).

\[
\begin{align*}
\bigwedge_{i=1}^{n} eq(x_i, x'_i) \land \text{succ}(y, y') \land \text{tiledBy}_j(x, y) \land \text{tiledBy}_k(x', y') & \rightarrow \text{goal} \\
\bigwedge_{i=1}^{n} eq(y_i, y'_i) \land \text{succ}(x, x') \land \text{tiledBy}_j(x, y) \land \text{tiledBy}_k(x', y') & \rightarrow \text{goal}
\end{align*}
\]  

(42)  

(43)

Then, \( \Pi_2 \) contains each rule of the form \( (44) \) for \( j \in [0, n - 1] \) and each \( k \in [1, r] \) with \( k \neq T_0(j) \), where \( A_{i} \) is zero if the least significant \( i \)-th bit of the binary encoding of \( j \) is 0 and one otherwise.

\[ \bigwedge_{i=1}^{n} A_{i}(x_i) \land \bigwedge_{i=1}^{n} \text{zero}(y_i) \land \text{tiledBy}_k(x, y) \rightarrow \text{goal} \]  

(44)

**Correctness of the reduction.** We now argue that the reduction \( \varphi \) is correct, i.e., there exists a tiling for \( I \) if \( Q_1 \nsubseteq Q_2 \). We show the two implications separately.

(\( \Rightarrow \)) Assume that \( T \) is a tiling for \( I \). Let \( \bar{0} \) and \( \bar{1} \) be two distinct objects in \( O \)—intuitively standing for 0 and 1. Let \( D \) be the dataset consisting of \( \text{zero}(\bar{0}) \), \( \text{one}(\bar{1}) \), and each fact \( \text{tiledBy}_i(a, b) \) for \( T(p, q) = i \), \( a \) the \( n \)-tuple of \( \bar{0} \)'s and \( \bar{1} \)'s encoding \( p \), and \( b \) the \( n \)-tuple of \( \bar{0} \)'s and \( \bar{1} \)'s encoding \( q \). It is easy to verify that \( \text{goal} \in Q_1(D) \) and \( \text{goal} \notin Q_2(D) \).

(\( \Leftarrow \)) Assume that there is a dataset \( D \) such that \( \text{goal} \in Q_1(D) \) and \( \text{goal} \notin Q_2(D) \).

For each tuple \( \langle a_1, \ldots, a_n \rangle \) of objects such that \( \text{zero}(a_i) \in D \) or \( \text{one}(a_i) \in D \) for each \( i \in [1, n] \), let \( \text{bin}(a) \) be the number in \( [0, 2^n - 1] \) whose \( i \)-th bit is 0 if \( \text{zero}(a_i) \in D \) and 1 otherwise. Note that \( \text{bin} \) is a well-defined function, since for each object \( a_i \), \( \text{zero}(a_i) \notin D \) or \( \text{one}(a_i) \notin D \), by rule (\( 37 \)) since \( \text{goal} \notin Q_2(D) \).

Let \( T \) be the relation consisting of each tuple \( (p, q, i) \) for \( \text{tiledBy}_i(a, b) \in D \), \( \text{bin}(a) = p \), and \( \text{bin}(b) = q \). In order to show that \( T \) is a tiling for \( I \), we have to show that (i) \( T \) is a total function over the domain \( [0, 2^n - 1] \times [0, 2^n - 1] \), (ii) \( (T(i, j), T(i + 1, j)) \in H \) for each \( i \in [0, 2^n - 2] \) and each \( j \in [0, 2^n - 1] \), (iii) \( (T(i, j), T(i, j + 1)) \in V \) for each \( i \in [0, 2^n - 1] \) and each \( j \in [0, 2^n - 2] \), and (iv) \( T(0, j) = T_0(j) \) for each \( j \in [0, n - 1] \). We have that (i) holds since \( T \) is total by rules (\( 32 \))–(\( 36 \)) and \( T \) is functional by rule (\( 41 \)). We have that (ii) holds by rule (\( 42 \)). We have that (iii) holds by rule (\( 43 \)). We have that (iv) holds by rule (\( 44 \)).

**Theorem 20.** \( \text{WINDOW}^\text{OGNR} \) is coNP-hard. Furthermore, \( \text{WINDOW}^\text{OR} \) is coNEXP-hard if \( O \) has at least two objects.

**Proof.** \( \text{WINDOW}^\text{OGNR} \) is coNP-hard because it is at least as hard as \( \text{CONT}^\text{OGNR} \) by Theorem \( 12 \) and the former problem is coNP-hard by Lemma \( 40 \). \( \text{WINDOW}^\text{OR} \) is coNEXP-hard if \( O \) contains at least two objects because it is at least as hard as \( \text{CONT}^\text{OR} \) by Theorem \( 12 \) and the former problem is coNEXP-hard by Lemma \( 41 \).