SMALL DATA SCATTERING OF DIRAC EQUATIONS WITH YUKAWA TYPE POTENTIALS IN $L^2_2(\mathbb{R}^2)$

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Abstract. We revisit the Cauchy problem of nonlinear massive Dirac equation with Yukawa type potentials $F^{-1}[(b^2 + |\xi|^2)^{-1}]$ in 2 dimensions. The authors of [10, 4] obtained small data scattering and large data global well-posedness in $H^s$ for $s > 0$, respectively. In this paper we show that the small data scattering occurs in $L^2_2(\mathbb{R}^2)$. This can be done by combining bilinear estimates and modulation estimates of [12, 10].

1. Introduction

We consider the following Cauchy problem for an nonlinear Dirac Hartree-type equation:

\[
\begin{array}{l}
\left\{ \begin{array}{l}
-\mathcal{F}^{-1}[(\mathcal{F}^{-1}[(b^2 + |\xi|^2)^{-1}]) \\ \psi(0) = \psi_0 \in L^2_2(\mathbb{R}^2),
\end{array} \right.
\end{array}
\]

(1.1)

where $D = -i \nabla$, $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{C^2}$, and $\psi : \mathbb{R}^{1+2} \rightarrow \mathbb{C}^2$ is the spinor field represented by a column vector. We define the Dirac matrices $\alpha, \beta$ by dimensions as follows:

$$
\alpha = (\alpha^1, \alpha^2), \quad \alpha^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

The constant $m > 0$ is a physical mass parameter and the symbol $*$ denotes convolution in $\mathbb{R}^2$ and the potential $V$ is defined by $F^{-1}[(b^2 + |\xi|^2)^{-1}]$ for some fixed constant $b > 0$. More explicitly, for a constant $a > 0$,

$$
V(x) = a \int_0^\infty e^{-b^2|x|^2/4r} \frac{dr}{r} \sim \begin{cases} e^{-b|x||\xi|} |\xi|^{-\frac{1}{2}} & |\xi| \gtrsim 1, \\ -\ln |x| & |x| \ll 1. \end{cases}
$$

The equation (1.1) with Yukawa potential was derived by uncoupling the Dirac-Klein-Gordon system in $\mathbb{R}^{1+2}$:

\[
\begin{array}{l}
\left\{ \begin{array}{l}
-\mathcal{F}^{-1}[(b^2 + |\xi|^2)^{-1}]
\end{array} \right.
\end{array}
\]

(1.2)

$$
\begin{array}{l}
\left\{ \begin{array}{l}
-\mathcal{F}^{-1}[(b^2 + |\xi|^2)^{-1}]
\end{array} \right.
\end{array}
\]

(1.2)

Let us assume that the scalar field $\phi$ is a standing wave of the form $\phi(t, x) = e^{\mathcal{F}^{-1}[b^2 + |\xi|^2]}$. Then the Klein-Gordon part of (1.2) becomes

$$
(-\Delta - \lambda^2 + M^2)\phi = \langle \psi, \beta \psi \rangle.
$$

If $b^2 := M^2 - \lambda^2 > 0$, then we get the equation (1.1).

The equation (1.1) obeys mass conservation law. If a solution $\psi$ is sufficiently smooth, then the mass $\|\psi(t)\|^2_{L^2_2}$ is conserved, that is, $\|\psi(t)\|^2_{L^2_2} = \|\psi_0\|^2_{L^2_2}$ for all $t$ within an existence time interval. See [4].

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Now let us consider a scaled function \( \tilde{\psi} \) defined by \( \tilde{\psi}(t,x) = m^{-\frac{3}{2}} \psi \left( \frac{t}{m}, \frac{x}{m} \right) \). Then by a direct calculation \( \tilde{\psi} \) satisfies the equation: 
\[
(-i\partial_t + \alpha \cdot D + \beta)\tilde{\psi} = (\tilde{V} * (\psi, \beta\tilde{\psi}))\beta\tilde{\psi},
\]
where \( \tilde{V} = \mathcal{F}^{-1}[\left( \frac{K}{m^2} + |\xi|^2 \right)^{-1}] \). Since the changed potential is essentially the same type as \( V \) up to constant, for the Cauchy problem \( \text{(1.1)} \) we assume that \( m = 1 \) in this paper.

We use the representation of solution based on the massive Klein-Gordon equation. For this purpose, let us define the energy projection operators \( \Pi_{\pm}(D) \) by
\[
 \Pi_{\pm}(D) := \frac{1}{2} \left( I \pm \frac{1}{(D) \{ \alpha \cdot D + \beta \}} \right),
\]
where \( (D) := \mathcal{F}^{-1}(\xi)\mathcal{F} \) and \( (\xi) := (1 + |\xi|^2)^{\frac{1}{2}} \) for any \( \xi \in \mathbb{R}^2 \). Then we get
\[
\alpha \cdot D + \beta = (D)(\Pi_{+}(D) - \Pi_{-}(D)),
\]
and
\[
(1.3)
\]
\[
(1.4)
\]
We denote \( \Pi_{\pm}(D)\psi \) by \( \psi_{\pm} \). Then the equation \( (1.1) \) becomes the following system of semi-relativistic Hartree equations:
\[
\begin{aligned}
(\Pi_{\pm}(D)\psi_{\pm}) &= \Pi_{\pm}(D)[(V * (\psi, \beta\psi))\beta\psi],
\end{aligned}
\]
with initial data \( \psi_{\pm}(0,\cdot) = \psi_{0,\pm} := \Pi_{\pm}(D)\psi_0 \). The free solutions of \( (1.5) \) are \( e^{\mp it(D)}\psi_{0,\pm} \), respectively, where
\[
e^{\mp it(D)}f(x) = \mathcal{F}^{-1}(e^{\mp it(\xi)}\mathcal{F}f) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{i(x \cdot \xi \mp t(\xi))} \hat{f}(\xi) \, d\xi.
\]
Here \( \mathcal{F}, \mathcal{F}^{-1} \) are Fourier transform, its inverse, respectively. Then by Duhamel’s principle the Cauchy problem \( (1.5) \) is equivalent to solving the integral equations:
\[
\psi_{\pm}(t) = e^{\mp it(D)}\psi_{0,\pm} + i \int_0^t e^{\mp i(t-t')(D)}\Pi_{\pm}(D)[(V * (\psi, \beta\psi))\beta\psi](t') \, dt'.
\]
We call that the solution \( \psi \) scatters forward (or backward) in \( H^s \) if there exist \( \psi^f \in C(\mathbb{R}; H^s) \), linear solutions to \( (-i\partial_t + \alpha \cdot D + \beta)\psi = 0 \), such that
\[
\|\psi(t) - \psi^f(t)\|_{H^s} \to 0 \quad \text{as} \quad t \to +\infty \quad (-\infty, \quad \text{respectively}).
\]
Equivalently, \( \psi \) is said to scatter forward (or backward) in \( H^s \) if there exist \( \psi^\pm_{\pm} := e^{\mp it(D)}\varphi_{\pm} \in H^s \) such that
\[
\|\psi_{\pm}(t) - \psi^\pm_{\pm}(t)\|_{H^s} \overset{t \rightarrow \pm\infty}{\longrightarrow} 0.
\]

Recently, Yang \( \text{(12)} \) and Tesfahun \( \text{(11)} \) showed, independently, small data scattering results on \( H^s(\mathbb{R}^3) \) for \( s > 0 \) in 3 dimensions. They developed the bilinear methods based on the null structure and \( U^p - V^p \) space. At the same time, Tesfahun \( \text{(10)} \) considered 2d problem \( (1.1) \) and obtained the scattering in \( H^s(\mathbb{R}^2)(s > 0) \). In \( \text{(3)} \) the global well-posedness was shown in \( H^s(\mathbb{R}^2) \) for \( s > 0 \) without the smallness of initial data. In \( \text{(5)} \) the authors considered the global well-posedness of 2d Dirac-Klein-Gordon system with data in \( L^2_x \times \dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}} \). There has not been known about the global well-posedness and scattering in \( L^2_x \) of the single equation \( (1.1) \). In this paper, we tackle the scattering problem in \( L^2_x(\mathbb{R}^2) \) and obtain the following theorem.

**Theorem 1.1.** If \( \|\psi_0\|_{L^2_x} \) is sufficiently small, then there exists a unique global solution \( \psi \in C(\mathbb{R}; L^2_x) \) to \( (1.1) \), which scatters in \( L^2_x \).
We show Theorem 1.1 by adopting the bilinear method of Yang and Tesfahun. Tesfahun’s method relies on the logarithmic interpolation between $U^p$ spaces, which results in $\varepsilon$-regularity loss for the high-high-low interaction part. To overcome it we use Yang’s bilinear estimates on the bilinear estimate is not strong enough to remove the $\varepsilon$ of 2d Yukawa potential. Unfortunately, our method cannot be applied to 3d problem directly because the bilinear estimate is not strong enough to remove the $\varepsilon$-regularity loss. The 3d scattering problem remains still open in $L^2_x$ and will be treated as a future.

Notations.

1. $\| \cdot \|$ denotes $\| \cdot \|_{L^2_t x}$.

2. (Mixed-normed spaces) For a Banach space $X$ and an interval $I$, $u \in L^q_I X$ iff $u(t) \in X$ for a.e. $t \in I$ and $\| u \|_{L^q(I; X)} := \| u(t) \|_X \|_{L^q(I)} < \infty$. Especially, we denote $L^q_I L^r_x = L^q(I; L^r_x)$, $L^q_t L^r_x = L^q_t L^r_x$, $L^q_t L^r_x = L^q_t L^r_x$.

3. (Littlewood-Paley operators) Let $\rho$ be a Littlewood-Paley function such that $\rho \in C_0^\infty(B(0, 2))$ with $\rho(\xi) = 1$ for $|\xi| \leq 1$ and define $\rho_k(\xi) := \rho \left( \frac{\xi}{2^k} \right) - \rho \left( \frac{\xi}{2^{k+1}} \right)$ for $k \in \mathbb{Z}$. Then we define the frequency projection $P_k$ by $\mathcal{F}(P_k f)(\xi) = \rho_k(\xi) \hat{f}(\xi)$, and also $P_{\leq k} := I - \sum_{k' > k} P_{k'}$. In addition, $P_{k_1 \leq \cdots \leq k_2} := \sum_{k_1 \leq k \leq k_2} P_k$. For $k \in \mathbb{Z}$ we denote $\tilde{P}_k = \rho_{k-1} + \rho_k + \rho_{k+1}$. In particular, $\tilde{P}_k P_k = P_k \tilde{P}_k = P_k$, where $\tilde{P}_k = \mathcal{F}^{-1} \rho_k \mathcal{F}$. Next we define a Fourier localization operators $P_k^1$ as follow:

$$P_k^1 f = \begin{cases} 0 & \text{if } k < 0, \\ P_{\leq 0} f & \text{if } k = 0, \\ P_k f & \text{if } k > 0. \end{cases}$$

Especially, we denote $P_k^1 f$ by $f_k$ for any measurable function $f$.

4. As usual different positive constants depending only on $a, b$ are denoted by the same letter $C$, if not specified. $A \lesssim B$ and $A \gtrsim B$ mean that $A \leq CB$ and $A \geq C^{-1}B$, respectively for some $C > 0$. $A \sim B$ means that $A \lesssim B$ and $A \gtrsim B$.

2. Function spaces

We explain concisely $U^p - V^p$ spaces. For more details, we refer the readers to [6, 7, 8, 9]. Let $1 \leq p < \infty$ and $\mathcal{I}$ be a collection of finite partitions $\{l_0, \ldots, l_N\}$ satisfying $-\infty < t_0 < \cdots < t_N \leq \infty$. If $t_N = \infty$, by convention, $u(t_N) := 0$ for any $u : \mathbb{R} \to L^2_x$ ($\mathbb{R}^2$). Let us define a $U^p$-atom by a step function $a : \mathbb{R} \to L^2_x$ of the form

$$a(t) = \sum_{k=1}^{N} \chi_{[t_{k-1}, t_k)} \phi(t) \text{ with } \sum_{k=1}^{N} \| \phi \|_{L^2_x} = 1.$$ 

Then the $U^p$ space is defined by

$$U^p = \left\{ u = \sum_{j=1}^{\infty} \lambda_j a_j : a_j \text{ are } U^p \text{-atoms and } \{ \lambda_j \} \in \ell^1, \| u \|_{U^p} < \infty \right\},$$

where the $U^p$-norm is defined by

$$\| u \|_{U^p} := \inf_{\text{representation of } u} \sum_{j=1}^{\infty} |\lambda_j|.$$
We next define $V^p$ as the space of all right-continuous functions $v : \mathbb{R} \to L^2_x$ satisfying that $\lim_{t \to -\infty} v(t) = 0$ and the norm
\[
\|v\|_{V^p} := \sup_{\{t_k\} \in \mathcal{I}} \left( \sum_{k=1}^{N} \|v(t_k) - v(t_{k-1})\|_{L^2_x}^p \right)^{\frac{1}{p}}
\]
is finite.

We introduce several key properties of $U^p$ and $V^p$ spaces.

Lemma 2.1 ([8]). Let $1 \leq p < q < \infty$. Then the following holds.
(i) $U^p$ and $V^p$ are Banach spaces.
(ii) The embeddings $U^p \hookrightarrow V^p \hookrightarrow U^q \hookrightarrow L^\infty(\mathbb{R}; L^2_x)$ are continuous.

These spaces have the useful duality property.

Lemma 2.2 (Corollary of [9]). Let $u \in U^p$ be absolutely continuous with $1 < p < \infty$. Then
\[
\|u\|_{U^p} = \sup \left\{ \int (u', v)_{L^2_t} dt : v \in C_0^\infty, \|v\|_{V^p'} = 1 \right\}.
\]

Now let us define the adapted function spaces $U^p_\pm$, $V^p_\pm$ as follows:
\[
\|u\|_{U^p_\pm} := \|e^{\pm it(D)}u\|_{U^p} \quad \text{and} \quad \|u\|_{V^p_\pm} := \|e^{\pm it(D)}u\|_{V^p}.
\]

Proposition 2.3 (Transfer principle, Proposition 2.19 of [6]). Let
\[
T : L^2_x \times L^2_x \times \cdots \times L^2_x \to L^1_{t,loc}
\]
be a multilinear operator. If
\[
\|T(e^{\pm it(D)}f_1, e^{\pm it(D)}f_2, \ldots, e^{\pm it(D)}f_k)\|_{L^q_t L^r_x} \lesssim \prod_{j=1}^k \|f_j\|_{L^2_x}
\]
for some $1 \leq q, r \leq \infty$ and $\pm_j \in \{\pm\}$, then we have
\[
\|T(u_1, u_2, \cdots, u_k)\|_{L^q_t L^r_x} \lesssim \prod_{j=1}^k \|u_j\|_{U^q_{\pm_j}}.
\]

3. Bilinear estimates

In this section, we list basic bilinear estimates based on the estimates of [10] [11] [12].

Lemma 3.1. Let $k_j \in \mathbb{Z}$, $\psi_j \in V^2_{\pm j}$ ($j = 1, 2$), and $\Pi_{\pm_j}(D)P^1_{k_j}\psi_j = \psi_j$. Then
\[
\|\langle \psi_1, \beta \psi_2 \rangle\| \lesssim 2^{\kappa_k} \|\psi_1\|_{V^2_{\pm_1}} \|\psi_2\|_{V^2_{\pm_2}}
\]
for any $0 < p < 1$.

Proof of Lemma 3.1 For the proof we use the well-known Strichartz estimates (for instance see [2] [3]): Suppose $(q, r)$ satisfies that $2 \leq r < \infty$ and $\frac{1}{q} = \frac{1}{2} - \frac{1}{r}$. Then
\[
\|e^{\pm it(D)}P^1_k f\|_{L^q_t L^r_x} \lesssim \langle 2^k \rangle^\frac{\alpha}{2} \|P^1_k f\|_{L^2_x}.
\]

From (3.1), Proposition 2.3 and Lemma 2.1 we get
\[
\|P^1_k \psi\|_{L^q_t L^r_x} \lesssim 2^{\frac{\alpha}{2}} \|\psi\|_{U^q_{\pm_1}} \lesssim 2^{\frac{\alpha}{2}} \|\psi\|_{V^2_{\pm_1}}
\]
for $2 < q < \infty$. Hence, by \textcolor{red}{[59x-1344]}, we get
\[
\|\langle \psi_1, \beta \psi_2 \rangle \| \lesssim \|\psi_1\|_{L^q_x L^2_t} \|\psi_2\|_{L^q_x L^2_t} \lesssim 2^k k \|\psi_1\|_{L^q_x L^2_t} \|\psi_2\|_{L^q_x L^2_t}
\lesssim 2^k \|\psi_1\|_{L^{q,1}_x L^2_t} \|\psi_2\|_{L^{q,1}_x L^2_t}.
\]
By setting $p = \frac{2}{q}$ the proof finishes. \hfill \qed

The following proposition is key estimate to be used in high-high-low interaction.

**Proposition 3.2 (Proposition 3.6, 3.7 of [12] and Proposition 3.7, 3.9 of [1]).** Let $\Pi_{\pm} \delta(D) P_k \psi_j = \psi_j \in V^2_{\pm, j}$. Assume that $k_1, k_2 \geq 0$, $k \in \mathbb{Z}$ and that $2^k \ll 2^{k_1} \sim 2^{k_2}$. Then we get the following:

(i) If $\pm_1 = \pm_2$, $\|P_k \langle \psi_1, \beta \psi_2 \rangle \| \lesssim 2^{k - \frac{k_1}{2}} \|\psi_1\|_{L^{q,1}_x L^2_t} \|\psi_2\|_{L^{q,1}_x L^2_t}$.

(ii) If $\pm_1 \neq \pm_2$, $\|P_k \langle \psi_1, \beta \psi_2 \rangle \| \lesssim 2^k \|\psi_1\|_{L^{q,1}_x L^2_t} \|\psi_2\|_{L^{q,1}_x L^2_t}$.

4. PROOF OF THEOREM [11]

We prove Theorem [11] by contraction argument. Let us define Banach spaces $X_{\pm}$ and $X_{\pm, p}$ by
\[
X_{\pm} := \left\{ \phi \in C(\mathbb{R}; L^2_x) : \|\phi\|_{X_{\pm}} := \left( \sum_{k \in \mathbb{Z}} \|P_k \phi\|_{L^2_x}^2 \right)^{\frac{1}{2}} \right< \infty \right\}
\]
and $X_{\pm, p} = \{ \psi = \chi_{[0, \infty)}(t)\phi : \phi \in X_{\pm} \}$, respectively. Then by the decomposition $\psi = \psi_+ + \psi_-$, where $\psi_\pm = \Pi_\pm(D) \psi$, we define a complete metric space $X_p(\delta)$ as
\[
X_p(\delta) := \{ \psi \in X_{\pm, p} : \|\psi\|_{X} := \|\psi_+\|_{X_+} + \|\psi_-\|_{X_-} \leq \delta \}
\]
with metric $d(\psi, \phi) := \|\psi - \phi\|_{X}$ and a map $N$ defined by
\[
N(\psi) = \sum_{j} \left[ \chi_{[0, \infty)}(t) e^{\pm i t D} \Pi_\pm(D) \psi_0 + i \sum_{\pm, j = 1, 2, 3} N_\pm(\psi_{\pm, 1}, \psi_{\pm, 3})(t) \right],
\]
where
\[
N_\pm(\psi_1, \psi_2, \psi_3)(t) = \left[ \int_0^t e^{\mp i(t-t') D} \Pi_\pm(D) (V \ast \langle \psi_1, \beta \psi_2 \rangle) \beta \psi_3 dt' \right].
\]
Here $\sum_+ A_\pm$ means that $A_+ + A_-$. The linear part of $N(\psi)$ can be estimated as follows:
\[
(4.1) \quad \left\| \chi_{[0, \infty)} e^{\pm i t D} \Pi_\pm(D) \psi_0 \right\|_{L^2_{x}}^2 = \sum_{k \in \mathbb{Z}} 2^{2sk} \left( \|\chi_{[0, \infty)} P_k \Pi_\pm(D) \psi_0\|_{L^2_x}^2 \right) \sim \|\psi_0\|_{H^s}^2.
\]
For the nonlinear parts for $N_\pm(\psi)(t)$ we prove

**Proposition 4.1.** If $\psi_j \in X_{\pm, j, p}$, then we have
\[
\|N_\pm(\psi_{1, 3, 1}, \psi_{2, 3, 2}, \psi_{3, 3, 3})\|_{X_{\pm}} \lesssim \prod_{j=1}^3 \|\psi_{j, \pm, j}\|_{X_{\pm, j}}.
\]
The proof of Proposition [12,1] is placed in the next section.

If $\delta$ is small enough that $C\delta^3 \leq \frac{\delta}{8}$ and $\psi_0$ satisfies $C\|\psi_0\|_{L^2_x} \leq \frac{\delta}{2}$, Proposition [11,1] together with linear estimate [4.1] leads us to
\[
\|N(\psi)\|_{X} = \sum_{\pm} \|\Pi_\pm N(\psi)\|_{X_{\pm}} \leq C(\|\psi_0\|_{L^2_x} + \|\psi\|_{X}^3) \leq \delta.
\]
Then by dyadic decomposition we have
\[ d\left(N(\psi), N(\phi)\right) = \|N(\psi) - N(\phi)\|_X \leq C (\|\psi\|_X + \|\phi\|_X)^2 \|\psi - \phi\|_X \leq 4 C \delta^2 \|\psi - \phi\|_X \leq \frac{1}{2} d(\psi, \phi). \]

Hence \( N : X_p(\delta) \rightarrow X_p(\delta) \) is a contraction mapping for sufficiently small \( \delta \) and then we get a unique solution \( \psi_p \in L^\infty([0, \infty); L^2_x) \) to (1.1). The time continuity and continuous dependency on data follow readily from the formula \( \psi_p = \mathcal{N}(\psi_p) \) and Proposition 4.1. By the time symmetry of (1.1) we also obtain a unique solution \( \psi_n \in C((-\infty, 0], L^2_x) \) with the continuous dependency on data. Defining \( \psi = \psi_p + \psi_n \), we get the global well-posedness of (1.1).

Now we move onto the scattering property of (1.5). Since the backward scattering can be treated similarly to the forward one, we omit its proof. For \( k \geq 0 \) let us define
\[ \varphi_\pm := e^{\pm it(D)} P_k^1 N_\pm(\psi), \]
where \( N_\pm(\psi) = \lim_{t \rightarrow \infty} \sum_{\pm, j} N_{ij}(\psi_{\pm, 1}, \psi_{\pm, 2}, \psi_{\pm, 3})(t) \). Then Lemma 2.1 shows that
\[ \varphi_\pm \in V^2_\pm. \]

Since \( \sum_{k \geq 0} \|\varphi_\pm\|_{V^2_\pm} \leq 1 \), we have
\[ \phi_\pm := \lim_{t \rightarrow \infty} \varphi_\pm \in L^2_x \]
and
\[ \|\psi_{\pm}(t) - e^{\mp it(D)} \phi_\pm\|_{L^2_x} \xrightarrow{t \rightarrow \infty} 0. \]

This completes the proof of scattering part.

5. PROOF OF PROPOSITION 4.1

By duality we obtain
\[
\left\| P_k^1 \int_0^t e^{\mp i(t-t')D}(D) \Pi_{\pm}(D)(V \ast \langle \psi_{1, \pm, 1}, \beta \psi_{2, \pm, 2} \rangle) \beta \psi_{3, \pm, 3} \right\|_{U^2_\pm}^2
= \left\| P_k^1 \int_0^t e^{\pm i t(D)} \Pi_{\pm}(D)(V \ast \langle \psi_{1, \pm, 1}, \beta \psi_{2, \pm, 2} \rangle) \beta \psi_{3, \pm, 3} \right\|_{U^2_\pm}^2
= \sup_{\|\phi\|_{V^2} = 1} \left| \int \int (V \ast \langle \psi_{1, \pm, 1}, \beta \psi_{2, \pm, 2} \rangle) \left( \beta \psi_{3, \pm, 3}, \Pi_{\pm}(D) P_k^1 e^{\mp it(D)} \phi \right) dt dx \right|
= \sup_{\|\phi\|_{V^2_\pm} = 1} \left| \int \int (V \ast \langle \psi_{1, \pm, 1}, \beta \psi_{2, \pm, 2} \rangle) \left( \beta \psi_{3, \pm, 3}, P_k^1 \psi_{4, \pm, 4} \right) dt dx \right|.
\]

Then by dyadic decomposition we have
\[
\|N_{\pm, 4}(\psi_{1, \pm, 1}, \psi_{2, \pm, 2}, \psi_{3, \pm, 3})\|^2_{X^4_{\pm, 4}}
= \sum_{k_4 \in Z} \|P_k^1 N_{\pm, 4}(\psi_{1, \pm, 1}, \psi_{2, \pm, 2}, \psi_{3, \pm, 3})\|^2_{U^2_4}
\leq \sum_{k_4 \in Z} \left( \sup_{\|\psi\|_{V^2_\pm} = 1} \sum_{k_1, k_2, k_3 \in Z} \left| \int \int P_k (V \ast \langle \psi_{1, \pm, 1}, \beta \psi_{2, \pm, 2}, k_2 \rangle) \tilde{P}_k (\langle \beta \psi_{3, \pm, 3}, \psi_{4, \pm, 4}, k_4 \rangle) dt dx \right| \right)^2
\leq \sum_{k_4 \in Z} \left( \sup_{\|\psi\|_{V^2_\pm} = 1} (I_1 + I_2 + I_3) \right)^2,
\]

where \( \|\phi\|_X := \|\phi_+\|_{X_+} + \|\phi_-\|_{X_-} \). This yields that \( \mathcal{N} \) is a self-mapping on \( X_p(\delta) \). In particular, we get
where \( \psi_{j,\pm j,k_j} = P_{kj}^\lambda \Pi_{\pm j}(D)\psi_j \) and

\[
I_1 = \sum_{k_1, k_2 \in \mathbb{Z}} | \cdots |, \quad I_2 = \sum_{k_1, k_2 \in \mathbb{Z}} | \cdots |, \quad I_3 = \sum_{k_1, k_2 \in \mathbb{Z}} | \cdots |.
\]

We subdivide \( I_j \) as follows:

\[
I_1 = I_{11} + I_{12} + I_{13} := \sum_{2^k \ll 2^{k_1} \sim 2^{k_2}} | \cdots | + \sum_{2^{k_1} \ll 2^k} | \cdots | + \sum_{2^k \gg 2^{k_1} \gg 2^{k_2}} | \cdots |.
\]

\[
I_2 = I_{21} + I_{22} + I_{23} := \sum_{2^k \ll 2^{k_1} \sim 2^{k_2}} | \cdots | + \sum_{2^{k_2} \ll 2^k} | \cdots | + \sum_{2^k \gg 2^{k_1} \gg 2^{k_2}} | \cdots |.
\]

\[
I_3 = I_{31} + I_{32} + I_{33} := \sum_{2^k \ll 2^{k_1} \sim 2^{k_2}} | \cdots | + \sum_{2^{k_2} \ll 2^k} | \cdots | + \sum_{2^k \gg 2^{k_1} \gg 2^{k_2}} | \cdots |.
\]

It suffices to show that for each \( I_{ij} \) (\( i, j = 1, 2, 3 \))

\[
(5.1) \quad I_{ij} := \sum_{k_4 \in \mathbb{Z}} \left( \sup_{\|\psi_k\|_{L^2} = 1} |I_{ij}| \right)^2 \lesssim \prod_{j=1}^3 \|\psi_{j,\pm j}\|_{K_{\pm j}}^2.
\]

In fact, they can be handled as follows. By Proposition 3.2 we have

\[
I_{11} \lesssim \sum_{k_4 \in \mathbb{Z}} \left( \sum_{2^k \ll 2^{k_1} \sim 2^{k_2}} \left( 2^k \right)^{-2} \|P_k(\chi_{1,k_1,\beta\psi_{2,k_2}})\| \left\| P_k(\beta\psi_{3,k_3}, \Pi_{\pm j}(D)P_k^\lambda \psi_j) \right\| \right)^2
\]

\[
\lesssim \sum_{k_4 \in \mathbb{Z}} \left( \sum_{2^k \ll 2^{k_1} \sim 2^{k_2}} 2^k \left( 2^k \right)^{-2} \|\psi_{1,k_1}\|_{V_{\pm 2}^2} \|\psi_{2,k_2}\|_{V_{\pm 2}^2} \|\psi_{3,k_3}\|_{V_{\pm 2}^2} \right)^2
\]

\[
\lesssim \|\psi_1\|_{K_{\pm 1}}^2 \|\psi_2\|_{K_{\pm 2}}^2 \sum_{k_4 \in \mathbb{Z}} \|\psi_{3,k_4}\|_{V_{\pm 2}^2}^2 \left( \sum_{k_4 \in \mathbb{Z}} 2^k \left( 2^k \right)^{-2} \right)^2
\]

\[
\lesssim \prod_{j=1}^3 \|\psi_j\|_{K_{\pm j}}^2.
\]
Using Lemma 3.11 and Proposition 3.2

\[ I_{12} \lesssim \sum_{k_4 \in \mathbb{Z}} \left( \sum_{2^{k_2} < 2^{k_1} < 2^k} \langle 2^k \rangle^{-2} 2^{k_1-k_2} \| \psi_{1,k_1} \|_{V^2_{x_1}} \| \psi_{2,k_2} \|_{V^2_{x_2}} 2^{r/2} \| \psi_{3,k_3} \|_{V^2_{x_3}} \right)^2 \]

\[ \lesssim \sum_{k_4 \in \mathbb{Z}} \left( \sum_{2^{k_2} < 2^{k_1} < 2^k} 2^{k_2} \langle 2^{k_1} \rangle^{-2} 2^{r/2(k_2-k_1)} \| \psi_{1,k_1} \|_{V^2_{x_1}} \| \psi_{2,k_2} \|_{V^2_{x_2}} \| \psi_{3,k_3} \|_{V^2_{x_3}} \right)^2 \]

\[ \lesssim \| \psi_1 \|_{X^{2,-3}}^2 \left( \sum_{2^{k_2} < 2^{k_1} < 2^k} 2^{k_2} \langle 2^{k_1} \rangle^{-2} 2^{r/2(k_2-k_1)} \| \psi_{1,k_1} \|_{V^2_{x_1}} \| \psi_{2,k_2} \|_{V^2_{x_2}} \right)^2 \]

\[ \lesssim \prod_{j=1}^3 \| \psi_j \|_{X^{2,-3,j}}^2. \]

\[ I_{13} \text{ and } I_{21} \text{ can be handled by changing the role of } \psi_1, \psi_2, \text{ and } (\psi_1, \psi_2), (\psi_3, \psi_4), \text{ respectively.} \]

As for \( I_{22} \) we apply Lemma 3.1 to both \((\psi_1, \psi_2)\) and \((\psi_3, \psi_4)\) to get

\[ I_{22} \lesssim \sum_{k_4 \in \mathbb{Z}} \left( \sum_{2^{k_2} < 2^{k_1} < 2^k} \langle 2^{k_1} \rangle^{-2} 2^{k_2} \langle 2^{k_1} \rangle^{-1} 2^{r/2(k_2-k_4)} \| \psi_{1,k_1} \|_{V^2_{x_1}} \| \psi_{2,k_2} \|_{V^2_{x_2}} 2^{k_4} \| \psi_{3,k_3} \|_{V^2_{x_3}} \right)^2 \]

\[ \lesssim \sum_{k_4 \in \mathbb{Z}} \| \psi_{1,k_1} \|_{V^2_{x_2}}^2 \left( \sum_{2^{k_2} < 2^{k_4}} 2^{k_2} \langle 2^{k_4} \rangle^{-1} 2^{r/2(k_2-k_4)} \| \psi_{2,k_2} \|_{V^2_{x_1}} \right)^2 \times \left( \sum_{2^{k_3} < 2^{k_4}} 2^{k_3} \langle 2^{k_4} \rangle^{-1} 2^{r/2(k_3-k_4)} \| \psi_{3,k_3} \|_{V^2_{x_3}} \right)^2 \]

\[ \lesssim \prod_{j=1}^3 \| \psi_j \|_{X^{2,-3,j}}^2. \]

\( I_{23} \) is treated similarly by changing the role of \( \psi_1, \psi_2 \). The estimates of \( I_{1j} \) are symmetric to those of \( I_{2j} \).

We have only to change the role of \( \psi_3, \psi_4 \). This completes the proof of Theorem 1.1

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