A Mean-Field Model for Extended Stochastic Systems with Distributed Time Delays

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ABSTRACT

A network of noisy bistable elements with global time-delayed couplings is considered. A dichotomous mean field model has recently been developed describing the collective dynamics in such systems with uniform time delays near the bifurcation points. Here the theory is extended and applied to systems with nonuniform time delays. For strong enough couplings the systems exhibit delay-independent stationary states and delay-dependent oscillatory states. We find that the regions of oscillatory states in the parameter space are reduced with increasing width of the time delay distribution function; that is, nonuniformity of the time delays increases the stability of the trivial equilibrium. However, for symmetric distribution functions the properties of the oscillatory states depend only on the mean time delay.

KEY WORDS

Stochastic dynamics, mean-field dynamics, delay differential equation, self-organization

1 Introduction

The understanding of the collective dynamics in extended stochastic systems with long range interactions is relevant for many domains in physics, chemistry, biology and even social sciences. A popular and effective generic model for such systems is the globally coupled bistable-element-network, whose dynamical properties have been studied in the absence (Zanette, 1997) and presence (see Gammaitoni et al., 1998; references therein) of noise and whose relevance for critical phenomena (Dawson, 1983), spin systems (Jung et al., 1992), neural networks (Jung et al., 1992), Camperi and Wand, 1998; Koukalov et al., 2002), genetic regulatory networks (Gardner et al., 2000) and decision making processes in social systems (Zanette, 2000) has been pointed out.

In recent years it has been realized that time delays arising, for example, from the finite propagation speed of signals are ubiquitous in most physical and biological systems. The effects of the delays on the behavior of various dynamical systems have been studied (Zanette, 2000; Jeong et al. 2002), and some significant changes of the dynamical properties have been demonstrated (Nakamura et al. 1994; Bressloff and Coombes 1998; Choi and Huberman, 1985). The effects of uniform time delays in globally coupled networks of phase oscillators has been explored by Yeung and Strogatz (1999). Tsimring and Pikovsky (2001) studied the dynamics of a single noise activated, bistable element with time-delayed feedback. Combining the properties of these two systems Huber and Tsimring (2003) studied the cooperative dynamics of an ensemble of noisy bistable elements with delayed couplings and a mean field theory based on delay-differential master equations was developed. However, the theory assumed identical (i.e. uniform) time delays among all elements. Although for many systems this approximation is justified (Salami et al., 2003; Paulsson and Ehrenberg, 2001), most systems have distributed coupling delays. Thus, we study in this paper the generalized case of distributed time delays in a globally coupled network of bistable elements.

2 The model

Our generalized model for the study of noise-activated, collective dynamical phenomena in extended systems consists of $N$ Langevin equations, each describing the overdamped noise driven motion of a particle in a bistable potential $V = -x^2/2 + x^4/4$, whose symmetry is distorted by the time-delayed couplings to the other network elements,

$$
\dot{x}_i = x_i - x_i^3 + \frac{\epsilon}{N} \sum_{j=1}^{N} x_j(t - \tau_{ij}) + \sqrt{2D}\xi(t),
$$

(1)

where $\tau_{ij}$ are the time delays depending on the two coupled elements $i$ and $j$. The strength of the feedback is $\epsilon$ and $D$ denotes the variance of the Gaussian fluctuations $\xi(t)$, which are $\delta$-correlated and mutually independent $\langle \xi(t)\xi(t') \rangle = \delta(t - t')\delta_{ij}$.

We use an Euler method to explore model (1) numerically and focus our interest on the collective dynamics of the bistable elements, i.e., on the dynamics of the mean field, $X = N^{-1} \sum_{i=1}^{N} x_i$. 


analytical description of networks with distributed time delays.

In this section we derive the dichotomous theory for the description of globally coupled bistable-element-networks with distributed time delays.

For \( \varepsilon = 0 \), the elements are decoupled from each other. They randomly and independently jump from one potential well to the other. Therefore, in this case the mean field \( X = 0 \). For small \( |c| \), the mean field remains zero. However, for a strong enough feedback the system undergoes ordering transitions and demonstrates multistability. That is, for a strong enough positive coupling the systems undergo a pitchfork bifurcation and adopts a non-zero stationary mean field \( X > 0 \), and transitions to a variety of stable oscillatory mean field states via Hopf bifurcations, are observed for strong enough positive and negative feedbacks.

In the general case, model (1), in which the time delays depend on both the “transmitting” and the “receiving” element, cannot directly be described in terms of a mean field theory. However, the system becomes mathematically tractable if we assume that the time delay does only depend on the “transmitting” elements \( j \),

\[
\dot{x}_i = x_i - x_i^3 + \frac{\varepsilon}{N} \sum_{j=1}^{N} x_j(t - \tau_j) + \sqrt{2D} \xi(t). \tag{2}
\]

In order to check if such a simplification is justified, numerical simulations of model (1) and (2) are carried out and compared. In these simulations the distribution of the time delays is Gaussian, i.e., it is fully determined by its mean \( \bar{\tau} \) and variance \( \sigma \). Fig. 1 which compares the critical coupling strength of the Hopf bifurcation for different \( \sigma \), suggests that the above simplification is justified in order to study the stability properties of a bistable-element-network with time delays.

This surprising result not only renders possible an analytical description of networks with distributed delays but also implies that the number of operations, which have to be carried out to study such systems numerically, can be reduced from \( \mathcal{O}(N^2) \) to \( \mathcal{O}(N) \).

3 Dichotomous Theory

In this section we derive the dichotomous theory for the description of globally coupled bistable-element-networks with distributed time delays.

The dichotomous theory, valid in the limit of small coupling strength and small noise, neglects intrawell fluctuations of \( x_i \). Thus, each bistable element can be replaced by a discrete two-state system which can only take the values \( s_{1,2} = \pm 1 \). Then the collective dynamics of the entire system is described by the master equations for the occupation probabilities of these states \( n_{1,2} \). This approach has been successfully used in studies of stochastic and coherence resonance (e.g. McNamara and Wiesenfeld, 1989; Gammaitoni et al., 1998; Jung et al., 1992; Tsimring and Pikovsky, 2001; and Huber and Tsimring, 2003) used it to study the special case of a network with uniform time delays. They found that while away from the transition points the system dynamics are well described by a Gaussian approximation, near the bifurcation points a description in terms of a dichotomous theory is more adequate.

In order to apply the dichotomous theory to a network with distributed time delays, we coarse grain system (2). The coarse graining is accomplished as follows: The range of possible time delays is divided up in \( M \) intervals \( I_k \{ k = 1, 2, \ldots, M \} \). The size of the intervals \( \Delta_k \) is chosen, so that the number of bistable oscillators associated with a delay fitting in a particular interval, is for each interval the same \( m = N/M \). In this way oscillator groups are formed whose mean field can be expressed as,

\[
\Omega_k(t) \equiv \frac{1}{m} \sum_{\tau_j \in I_k} x_j(t), \tag{3}
\]

where \( I_k \equiv (\tau_k, \tau_{k+1}] \), \( \tau_k = \sum_{l=1}^{k-1} \Delta_l \) and \( j = 1 \ldots N \).

Assuming that \( \Delta_k \ll \bar{\tau}/\sigma \), where \( \bar{\tau} \) and \( \sigma \) are the mean and the variance of the time delay distribution, Eq. (2) can then be approximated by,

\[
x_i = x_i - x_i^3 + \frac{\varepsilon}{M} \sum_{k=1}^{M} \Omega_k(t - \tau_k) + \sqrt{2D} \xi(t). \tag{4}
\]

The dynamics of a single element \( x_i \) is determined by the hopping rates \( p_{12} \) and \( p_{21} \), i.e., by the probabilities to hop over the potential barrier from \( s_1 \) to \( s_2 \) and from \( s_2 \) to \( s_1 \), respectively. In a globally coupled system, in which the time delays depend only on the transmitting elements, \( p_{12} \) and \( p_{21} \) are identical for all elements and the master equations expressing the dynamics of Eq. (1) in terms of occupation probabilities
\[ \dot{n}_{1,k} = -p_{12}n_{1,k} + p_{21}n_{2,k} \quad (5) \]
\[ \dot{n}_{2,k} = p_{12}n_{1,k} - p_{21}n_{2,k}. \quad (6) \]

The hopping probabilities \( p_{12,21} \) are given by Kramers’ transition rate (Kramers, 1940) for the instantaneous potential well, which for our system in the limit of small noise \( D \) and coupling strength \( \varepsilon \) reads (cf. Tsirring and Pikovsky [2001]),

\[ p_{12,21} = \frac{\sqrt{2 + 3 \alpha}}{2 \pi} \exp \left( \frac{1 + 4 \alpha}{4D} \right), \quad (7) \]

where \( \alpha = (\varepsilon/M) \sum_{k=1}^{M} \Omega_k (t - \tau_k) \).

For large oscillator groups \( (m \to \infty) \), \( \Omega_k = n_{1,k} s_1 + n_{2,k} s_2 = n_{2,k} - n_{1,k} \) and \( n_{1,k} + n_{2,k} = 1 \) holds. With these terms, we can find the following set of equations:

\[ \dot{\Omega}_k(t) = p_{12} - p_{21} - (p_{21} + p_{12}) \Omega_k(t). \quad (8) \]

The Jacobian matrix of this system is given through,

\[ J = c \begin{pmatrix} a_1 + b & a_2 & \ldots & a_M \\ a_1 & a_2 + b & \ldots & a_M \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_M + b \end{pmatrix}, \quad (9) \]

where \( c = -\sqrt{2} \exp(-1/4D)/(4M \pi D) \), \( a_k = \varepsilon (3D - 4) \exp(-\lambda \tau_k) \) and \( b = 4MD \). With this Jacobian the characteristic equation, determining the stability of the trivial equilibrium \( X = 0 \), becomes

\[ (bc - \lambda)^{M-1} \left( c \left[ b + \sum_{j=1}^{M} a_j \right] - \lambda \right) = 0. \quad (10) \]

The trivial equilibrium loses its stability and undergoes a pitchfork bifurcation, describing the transition to a steady nonzero mean field, when the real solutions of the characteristic equation (i.e. the eigenvalues of the Jacobian) become positive. Thus setting \( \lambda = 0 \) and solving Eq. (10) for \( \varepsilon \) yields the critical coupling for the pitchfork instability,

\[ \varepsilon_p = \frac{4D}{4 - 3D}. \quad (11) \]

A Hopf bifurcation indicating the transition to an oscillatory mean field state occurs when the real part of the complex eigenvalues becomes positive. Therefore, the properties of the corresponding instabilities (i.e. frequencies and coupling strengths at the bifurcation points) can be found by substituting \( \lambda = \mu + i \omega \) into Eq. (10), separating real and imaginary parts and setting \( \mu = 0 \). For the frequencies of the unstable modes we find,

\[ \omega \tau = -\frac{\sqrt{2}}{\pi} \exp(-1/4D) \tau I_s \frac{I_s}{I_c}, \quad (12) \]

where

\[ I_s = \frac{1}{M} \sum_{k=1}^{M} \sin \omega \tau_k, \quad I_c = \frac{1}{M} \sum_{k=1}^{M} \cos \omega \tau_k. \quad (13) \]

For large systems \( N \to \infty \), the number of groups \( M \to \infty \) and thus

\[ I_s = \int_0^{\infty} P(\tau) \sin \omega \tau d\tau, \quad I_c = \int_0^{\infty} P(\tau) \cos \omega \tau d\tau, \quad (14) \]

where \( P(\tau) \) is the time delay distribution function. We can express the time delay distribution function in terms of cumulant moments \( \kappa_n \) (Van Kampen [2003]) and solve the integrals in (14):

\[ I_s = \sin(g_1) \exp(g_2), \quad I_c = \cos(g_1) \exp(g_2), \quad (15) \]

where

\[ g_1 = \sum_{m=0}^{\infty} \frac{(i \omega)^{2m+1}}{(2m+1)!} \kappa_{2m+1}, \quad (16) \]
\[ g_2 = \sum_{m=1}^{\infty} \frac{(i \omega)^{2m}}{(2m)!} \kappa_{2m}. \quad (17) \]

Consequently,

\[ \frac{I_s}{I_c} = \tan(g_1). \quad (18) \]

Since for symmetric distribution functions all odd cumulant moments except the first one \( \kappa_1 = 1 \) are zero, \( I_s/I_c = \tan \omega \tau \) holds. That is, in the case of a symmetric distribution of the time delays, the frequencies of the unstable modes in Eq. (12) depend only on the mean time delay.

Let us now determine the critical coupling of the Hopf bifurcation. For large time delays \( \tau \gg \tau_k \) (\( \tau_k \) is the inverse Kramers escape rate from one well into the other) the low-order solutions of the transcendental equation (12) yield frequencies \( \omega \ll 1 \). Thus the real part of equation (10) can be linearized near \( \omega = 0 \) and the critical coupling of the Hopf bifurcation becomes,

\[ \varepsilon_H = \frac{4D \pi \omega}{(3D - 4) \left( \frac{1}{N} \sqrt{2} \exp(-1/4D) I_s - \frac{1}{N} \pi \omega I_c \right)}. \quad (19) \]

Then, for large systems \( N \to \infty \) the critical coupling is,

\[ \varepsilon_H = \frac{4D}{(4 - 3D)} \frac{I_c}{I_c}. \quad (20) \]

with \( I_c = 3 \sin(\omega \tau) \sin(5 \omega \sigma / 3)/(5 \omega \sigma) \) and \( I_c = \cos(\omega \tau) \exp(-\omega^2 \sigma^2 / 2) \) for a uniform and a Gaussian distribution, respectively.

Eq. (12) and (20) have a multiplicity of solutions indicating that multistability occurs in the system beyond a certain coupling strength.
function. The mean time delay is \( \bar{\tau} = 100 \).

4 Results

Eq. 11, 12 and 20 are used to determine the phase diagram and the frequencies of the unstable oscillatory modes \( f = \omega/(2\pi) \) of a bistable-element-network with uniformly distributed time delays. The theoretical predictions are verified with numerical simulations of the Langevin model (1). The results are shown in Fig. 2 and Fig. 3.

The figures show that near the bifurcation points the predictions by the dichotomous theory are reasonably good for small noise \( (D \leq 0.3) \) and consequently in this regime the Langevin models 11 and 12 are dynamically equivalent.

Fig. 2 also shows that the regions in the phase space where mean field oscillations occur are reduced with increasing width \( \sigma \) of the time delay distribution function, meaning that nonuniformity of the time delays inhibits the occurrence of Hopf bifurcations, i.e., increases the stability of the trivial equilibrium.

In Fig. 3 the bifurcation diagrams including higher order solutions of Eq. 12 and 20 are presented. This Figure shows that for a strong enough positive \( (\varepsilon = \varepsilon_p) \) coupling the trivial equilibrium loses its stability via a pitchfork bifurcation while for a strong enough negative coupling \( (\varepsilon = \varepsilon_{H-}) \) a Hopf bifurcation determined by the primary solution of Eq. 12 and 20 occurs. The higher order solutions of these equations provide the multistability of the system. For a positive feedback several oscillatory states with frequencies \( f_k \approx k/\bar{\tau} \) are observed for \( \varepsilon > \varepsilon_{H+}^k \) \( \{k = 1, 2, \ldots\} \). The transition points are ordered as follows, \( 0 < \varepsilon_{st} < \varepsilon_{H+}^1 < \varepsilon_{H+}^2 \ldots \)

If the feedback is negative, the system has oscillatory solutions with periods \( f_l \approx (2l + 1)/(2\bar{\tau}) \) for \( \varepsilon < \varepsilon_{H-}^l \{l = 0, 1, \ldots\} \), where \( 0 > \varepsilon_{H-}^0 > \varepsilon_{H-}^1 \ldots \).

In the above annotation a (+/-) index means that the corresponding value is associated with a negative and positive feedback, respectively.

5 Summary and conclusions

We generalized a dichotomous theory based on delay-differential master-equations to account for the dynamics of globally coupled networks of bistable elements with nonuniform time-delays. As in the case of uniform time delays these systems possess a nonzero stationary mean field for a strong enough positive feedback whose properties are time delay independent and a multiplicity of time delay dependent stable oscillatory states for both positive and negative feedback.

For symmetric time delay distributions the frequencies of the oscillations depend only on the mean time delay (see Eq. 14 and 15).

\[ \bar{\tau} \]

\[ f_{1(\bar{\tau})} \]

\[ \text{theory} \]

\[ \text{Langevin, } \sigma=20 \]

\[ \text{Langevin, } \sigma=30 \]

\[ \text{Langevin, } \sigma=40 \]

This should not be confused with uniform time delays, which means that the delay for each coupling is the same.
Figure 4. Upper panels and lower left panel: Phase diagrams of systems with uniformly distributed time delays, where $\bar{\tau} = 100$ and $\sigma = 0$, 10, 40. The dotted line depicts the critical coupling of the pitchfork bifurcation and the other lines depict those of the primary Hopf bifurcation as well as some higher order solutions (i.e. solution 1-16) of Eq. (12) and (20). Lower right panel: The frequencies of the corresponding unstable modes which do not depend on $\sigma$, but slightly vary with the noise strength $D$. 
of small noise and coupling strength. Far away from the transition points a theoretical description of the mean field dynamics can be found using a Gaussian approximation [Desai and Zwanzig, 1978; Huber and Tsimring, 2003]. However, a theoretical approach for the description of the dynamics in the regime of strong noise near the bifurcation points is still lacking.

This paper discusses the dynamics of globally coupled systems with time delays. It is assumed that all elements are coupled with uniform (i.e., identical) strength $\varepsilon$. However, many networks have sparse couplings and nonuniform coupling strength, which may endow the system with a complexer dynamics. This issue will be discussed in an upcoming paper.

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