\[ \mathcal{N} = 2 \] Galilean superconformal algebras with a central extension

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Abstract
\[ \mathcal{N} = 2 \] supersymmetric extensions of Galilean conformal algebra (GCA), specified by spin \( \ell \) and dimension of space \( d \), are investigated. Duval and Horváthy showed that the \( \ell = 1/2 \) GCA has two types of supersymmetric extensions, called standard and exotic. Recently, Masterov introduced a centerless super-GCA for arbitrary \( \ell \) which corresponds to the standard extension. We show that Masterov’s super-GCA has two types of central extensions depending on the parity of \( 2\ell \). We then introduced a novel super-GCA for arbitrary \( \ell \) corresponding to the exotic extension. It is shown that the exotic superalgebra also has two types of central extensions depending on the parity of \( 2\ell \). Furthermore, we give a realization of the standard and exotic super-GCAs in terms of their subalgebras. Finally, we present an \( \mathcal{N} = 1 \) supersymmetric extension of GCA with central extensions.

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1. Introduction

Galilean conformal algebra (GCA) is a class of non-semisimple Lie algebras. Its physical relevance is attributed to the fact that it may be regarded as a nonrelativistic counterpart of conformal algebra [1, 2]. Each GCA is specified by two parameters \( d \in \mathbb{N} \) and \( \ell \in \mathbb{N} \), where \( \mathbb{N} \) denotes the set of positive integers. The parameter \( d \) is the dimension of space in which dynamics is considered. This parameter \( d \) also specifies the maximal semisimple subalgebra \( sl(2, \mathbb{R}) \oplus so(d) \) of GCA. On the other hand, the parameter \( \ell \) specifies the \( \ell d \)-dimensional Abelian ideal of GCA which carries the spin \( \ell \) representation of the \( sl(2, \mathbb{R}) \) subalgebra. The Abelian ideal has a central extension for certain values of \( (d, \ell) \) [3–7].

The GCA with \( \ell = 1/2 \), the so-called Schrödinger algebra, is the best studied example of this class of algebras. One may find the Schrödinger algebra appearing in various fields of physics ranging from classical mechanics to high energy physics. Another member with \( \ell = 1 \) is also studied in many physical and mathematical contexts. In particular, the nonrelativistic
version of AdS/CFT correspondence formulated in the context of $\ell = 1/2$ and $\ell = 1$ GCAs [7–10] attracts a renewed interest on GCAs. Readers may refer to the references in [11, 12] for other works on $\ell = 1/2$ and $\ell = 1$ GCAs. For higher values of $\ell$, dynamical systems described by such GCAs have been studied very recently in [13–16]. We thus think that GCA for any $\ell$ is of physical importance.

It is natural to consider supersymmetric extensions of the algebraic structure of physical importance. Several works on the supersymmetrization of GCA have already been performed. A supersymmetric extension of $\ell = 1/2$ GCA has been completed by Duval and Horváthy [17]. In [17] a systematic method to construct the $\mathcal{N}$-supersymmetric extension of Schrödinger algebras in $d$-dimensional space with the central extension is presented. Previous to [17] there were some works on $\ell = 1/2$ super-GCA [18–21]. These works are followed by a recent active study on physical and mathematical aspects of $\ell = 1/2$ superalgebra [22–36].

For GCA with $\ell \geq 1$, a systematic method to obtain the $\mathcal{N}$-supersymmetric extension has not been established yet. However, there are some attempts to extend $\ell \geq 1$ GCA to superalgebra. Such an attempt was initiated by de Azcárraga and Lukierski [37]. They applied the Inönü–Wigner contraction to $su(2; 2; \mathcal{N})$ and obtained $\ell = 1$ and $\mathcal{N} = 2k$ (even) super-GCA in $(3 + 1)$-dimensional spacetime. The Inönü–Wigner contraction was also used by Sakaguchi to obtain $\ell = 1$ super-GCA in $(3 + 1)$- and $(2 + 1)$-dimensional spacetimes [38]. He obtained $\mathcal{N} = 2, 4$ super-GCAs in $(3 + 1)$-dimensional spacetime and $\mathcal{N} = 2, 4, 8$ ones in $(2 + 1)$-dimensional spacetime. Bagchi and Mandal obtained $\ell = \mathcal{N} = 1$ super-GCA in $(3 + 1)$-dimensional spacetime by applying the superspace contraction to $\mathcal{N} = 1$ extended $so(4, 2)$ [39]. A possible general structure of $\ell = 1$ super-GCA was discussed by Fedoruk and Lukierski [40]. Very recently, Masterov introduced the $\mathcal{N} = 2$ extension of GCA for any $d$ and $\ell$ [41]. We remark that central extensions of $\ell \geq 1$ super-GCA are not considered in any works mentioned above.

In this work we investigate $\mathcal{N} = 2$ supersymmetric extensions of GCA with the central extension. The aim of this work is twofold. The first aim is to show that in the case of $\mathcal{N} = 2$, all the supersymmetric extensions known for $\ell = 1/2$ are extended to arbitrary $\ell$. It is shown in [17] that $\ell = 1/2$ GCA has two distinct types of supersymmetric extensions. They are called standard and exotic. Masterov’s $\mathcal{N} = 2$ super-GCA corresponds to the standard supersymmetric extension without central terms. The exotic $\mathcal{N} = 2$ super-GCA is not known yet. We thus start with Masterov’s superalgebra and consider its central extension. Then we introduce a new superalgebra corresponding to the exotic extension and consider its central extension. In this way, we shall complete the generalization of $\ell = 1/2$ result to arbitrary $\ell$. Our second aim is to give a simple realization of the $\mathcal{N} = 2$ standard and exotic super-GCAs. It will be shown that each super-GCA is realized on its subalgebras which is isomorphic to a boson–fermion algebra.

This paper is organized as follows. In the next section we give a brief summary of bosonic GCA and its central extensions. It is pointed out that, combining the central extensions and the two types of supersymmetric extensions, there are four possible super-GCAs with the central extension. In section 3 we study central extensions of the standard super-GCA by Masterov. We introduce the exotic super-GCA with the central extension in section 4. In section 5 we give a realization of the superalgebras introduced in sections 3 and 4. This realization shows an interesting relation of the standard super-GCA and a certain Bose–Fermi oscillator Hamiltonian. Finally we introduced the $\mathcal{N} = 1$ supersymmetric extension of centrally extended GCA in section 6. Another possibility of central extension is discussed in the appendix.
2. Preliminary: bosonic GCA

GCA for a given pair of \((d, \ell)\) has generators

\[ D, \ H, \ C, \ M_{ij}, \ P_i^{(n)} \]

where \(i, j = 1, 2, \ldots, d\) and \(n = 0, 1, \ldots, 2\ell\). The sets of generators \((D, H, C)\) and \((M_{ij})\) span \(sl(2, \mathbb{R})\) and \(so(d)\) subalgebras, respectively. \(P_i^{(n)}\) is a basis of Abelian ideal of the GCA.

The nonvanishing commutation relations are given by \([1, 2]\)

\[
[D, H] = 2H, \quad [D, C] = -2C, \quad [C, H] = D, \\
[M_{ij}, M_{k\ell}] = -\delta_{ik}M_{j\ell} - \delta_{jk}M_{i\ell} + \delta_{il}M_{jk} + \delta_{jk}M_{il}, \\
[H, P_i^{(n)}] = -nP_i^{(n-1)}, \quad [D, P_i^{(n)}] = 2(\ell - n)P_i^{(n)}, \\
[C, P_i^{(n)}] = (2\ell - n)P_i^{(n+1)}, \quad [M_{ij}, P_k^{(n)}] = -\delta_{ik}P_j^{(n)} + \delta_{jk}P_i^{(n)}. \tag{1}
\]

It is known that the algebra has the two types of central extensions \([7]\):

(i) Mass extension existing for any \(d\) and half-integer \(\ell\)

\[
[P_i^{(m)}, P_j^{(n)}] = \delta_{ij} \delta_{m+n, 2\ell} I_m M, \quad I_m = (-1)^{m+\frac{\ell}{2}}(2\ell - m)!m!. \tag{2}
\]

(ii) Exotic extension existing only for \(d = 2\) and integer \(\ell\)

\[
[P_i^{(m)}, P_j^{(n)}] = \epsilon_{ij} \delta_{m+n, 2\ell} \tilde{I}_m \Theta, \quad \tilde{I}_m = (-1)^{m}(2\ell - m)!m!. \tag{3}
\]

where \(\epsilon_{ij}\) is the antisymmetric tensor with \(\epsilon_{12} = 1\). We note that this agrees, up to an overall factor, with the structure constants used in \([13, 42]\).

Next we briefly summarize the supersymmetric extensions of \(\ell = 1/2\) GCA introduced by Duval and Horváth. The supersymmetric extension requires the introduction of supercharges, superconformal charges and fermionic partners of \(P_i^{(n)}\). It is shown in \([17]\) that there exist two possible ways. One is called the standard extension existing for any \(d\), other is the exotic extension existing only for \(d = 2\). One of the main differences between standard and exotic extensions is the number of fermionic partners of \(P_i^{(n)}\). If the standard extension has \(2m\) fermionic partners, then the exotic extension has \(m\). Combination of the central and supersymmetric extensions yields the following four possibilities of super-GCA with the central extension:

- standard super and mass central extension for any \(d\) and half-integer \(\ell\),
- standard super and exotic central extension for \(d = 2\) and integer \(\ell\),
- exotic super and mass central extension for \(d = 2\) and half-integer \(\ell\),
- exotic super and exotic central extension for \(d = 2\) and integer \(\ell\).

We remark that this is not an all possible \(\mathcal{N} = 2\) supersymmetric and central extension, but a generalization of the result in \([17]\) to higher \(\ell\). In the following sections, we show that all four extensions are possible for \(\mathcal{N} = 2\). This will be done based on the definition of Lie superalgebra. Namely we set up a set of (anti)commutation relations, then verify that they satisfy the super Jacobi identity.

Although we do not aim to a full classification of all possible super and central extensions, one may ask the following question. The mass and exotic central extensions make the Abelian ideal non-Abelian. It is known that another type of central extension is possible for an infinite-dimensional \(\ell = 1\) super-GCA in one space dimension \([43]\). Is such a central extension possible for the finite-dimensional super-GCA? This problem is studied in the appendix and we have a negative answer to this question.
3. Standard supersymmetric extension

\( \mathcal{N} = 2 \) standard supersymmetric extension of GCA for any pair of \((d, \ell)\) was introduced by Masterov [41]. With a slight change of notations and conventions, the superalgebra is defined as follows. In addition to the generators of bosonic GCA, we introduce fermionic and the bosonic–fermionic sector reads

\[
\begin{align*}
\{Q, \bar{Q}\} &= 2H, \quad \{S, \bar{S}\} = 2C, \quad \{Q, S\} = D + R, \quad \{\bar{Q}, S\} = D - R, \\
\{Q, \bar{X}^{(n)}_i\} &= -P^{(n)}_i - nJ^{(n-1)}_i, \quad \{\bar{Q}, X^{(n)}_i\} = -P^{(n)}_i + nJ^{(n-1)}_i, \\
\{S, \bar{X}^{(n)}_i\} &= -P^{(n+1)}_i - (n - 2\ell + 1)J^{(n)}_i, \\
\{\bar{S}, X^{(n)}_i\} &= -P^{(n+1)}_i + (n - 2\ell + 1)J^{(n)}_i,
\end{align*}
\]

and the bosonic–fermionic sector reads

\[
\begin{align*}
[H, S] &= H, \quad [H, \bar{S}] = -\bar{S}, \quad [C, Q] = S, \quad [C, \bar{Q}] = \bar{S}, \\
[D, S] &= -S, \quad [D, \bar{S}] = -\bar{S}, \quad [D, Q] = Q, \quad [D, \bar{Q}] = \bar{Q}, \\
[H, X^{(n)}_i] &= -nX^{(n-1)}_i, \quad [H, \bar{X}^{(n)}_i] = -n\bar{X}^{(n-1)}_i, \\
[D, X^{(n)}_i] &= (2\ell - n - 2)X^{(n-1)}_i, \quad [D, \bar{X}^{(n)}_i] = (2\ell - n - 1)\bar{X}^{(n-1)}_i, \\
[C, X^{(n)}_i] &= (2\ell - n - 1)X^{(n+1)}_i, \quad [C, \bar{X}^{(n)}_i] = (2\ell - n - 1)\bar{X}^{(n+1)}_i, \\
[M_{ij}, X^{(n)}_k] &= -\delta_{ik}X^{(n)}_j + \delta_{jk}X^{(n)}_i, \quad [M_{ij}, \bar{X}^{(n)}_k] = -\delta_{ik}\bar{X}^{(n)}_j + \delta_{jk}\bar{X}^{(n)}_i, \\
[R, Q] &= Q, \quad [R, \bar{Q}] = \bar{Q}, \quad [R, S] = -S, \quad [R, \bar{S}] = \bar{S}, \\
[R, X^{(n)}_i] &= -X^{(n)}_i, \quad [R, \bar{X}^{(n)}_i] = \bar{X}^{(n)}_i, \\
[Q, P^{(n)}_i] &= nX^{(n-1)}_i, \quad [\bar{Q}, P^{(n)}_i] = n\bar{X}^{(n-1)}_i, \\
[Q, J^{(n)}_i] &= -X^{(n)}_i, \quad [\bar{Q}, J^{(n)}_i] = \bar{X}^{(n)}_i, \\
[S, P^{(n)}_i] &= -(2\ell - n)X^{(n)}_i, \quad [S, P^{(n)}_i] = -(2\ell - n)\bar{X}^{(n)}_i, \\
[S, J^{(n)}_i] &= -X^{(n+1)}_i, \quad [\bar{S}, J^{(n)}_i] = \bar{X}^{(n+1)}_i.
\end{align*}
\]

Finally the additional bosonic generators satisfy the relations

\[
\begin{align*}
[H, J^{(n)}_i] &= -nJ^{(n-1)}_i, \quad [D, J^{(n)}_i] = 2(\ell - n - 1)J^{(n)}_i, \\
[C, J^{(n)}_i] &= 2(\ell - n - 2)J^{(n+1)}_i, \quad [M_{ij}, J^{(n)}_k] = -\delta_{ik}J^{(n)}_j + \delta_{jk}J^{(n)}_i.
\end{align*}
\]

As seen from these commutation relations, \(\{Q, \bar{Q}, S, \bar{S}, D, H, C, R\}\) spans sl(2/1) subalgebra and \(\{P^{(n)}_i, X^{(n)}_i, \bar{X}^{(n)}_i, J^{(n)}_i\}\) forms an Abelian subalgebra.

For half-integral values of \(\ell\), one may verify that this standard super-GCA has the mass central extension defined by (2) and

\[
\begin{align*}
X^{(m)}_i, \bar{X}^{(m)}_i &= \delta_{ij}\delta_{m+n,2\ell-1}\alpha_mM, \quad J^{(m)}_i, \bar{J}^{(m)}_i &= \delta_{ij}\delta_{m+n,2\ell-2}\beta_M M,
\end{align*}
\]

where

\[
\alpha_m = (-1)^{m+\ell - \frac{1}{2}(2\ell - 1 - m)} m!, \quad \beta_M = (-1)^{m+\ell + \frac{1}{2}(2\ell - 2 - m)} m!.
\]

For \(d = 2\) and integral values of \(\ell\), one may also verify that it has the exotic central extension. Since the dimension of space is 2, the rotation subalgebra is so(2) generated by only one element \(M_{12}\). The additional nonvanishing commutators are given by (3) and

\[
\begin{align*}
X^{(m)}_i, \bar{X}^{(m)}_i &= \epsilon_{ij}\delta_{m+n,2\ell-1}\tilde{\alpha}_m\Theta, \quad J^{(m)}_i, \bar{J}^{(m)}_i &= \epsilon_{ij}\delta_{m+n,2\ell-2}\tilde{\beta}_M\Theta,
\end{align*}
\]
where
\[
\tilde{a}_m = (-1)^{n+1}(2\ell - 1 - m)! m!, \quad \tilde{\beta}_m = (-1)^m m! (2\ell - 2 - m)!.
\]

(10)

4. Exotic supersymmetric extension

In this section, we define the \( \mathcal{N} = 2 \) exotic supersymmetric extension of GCA. As such an extension for \( \ell = 1/2 \) exists only in two-dimensional space, we set \( d = 2 \). The space rotation is generated by one element \( M_{12} \). We introduce fermionic generators \( Q, Q^*, S, S^*, X^{(n)}_i \)
\( n = 0, 1, \ldots, 2\ell - 1 \) and an additional bosonic generator \( R \). In comparison with the standard super-GCA in the previous section, the number of \( X \)-type generators is half and there is no \( J^{(n)}_i \). We set the following nonvanishing commutators as well as (1). The fermionic sector is as follows:

\[
\begin{align*}
\{Q, Q\} &= \{Q^*, Q^*\} = 2H, & \{S, S\} &= \{S^*, S^*\} = 2C, \\
\{Q, S\} &= \{Q^*, S^*\} = D, \\
\{Q, X^{(n)}_i\} &= -P_i^{(n)}, & \{Q^*, X^{(n)}_i\} &= \sum_k \epsilon_{ik} P_k^{(n)}, \\
\{S, X^{(n)}_i\} &= -P_i^{(n+1)}, & \{S^*, X^{(n)}_i\} &= \sum_k \epsilon_{ik} P_k^{(n+1)}.
\end{align*}
\]

(11)

The bosonic–fermionic sector is as follows:

\[
\begin{align*}
[H, S] &= -Q, & [H, S^*] &= -Q^*, & [C, Q] &= S, & [C, Q^*] &= S^*, \\
[D, S] &= -S, & [D, S^*] &= -S^*, & [D, Q] &= Q, & [D, Q^*] &= Q^*, \\
[H, X^{(n)}_i] &= -nX^{(n-1)}_i, & [C, X^{(n)}_i] &= (2\ell - n - 1)X^{(n+1)}_i, \\
[D, X^{(n)}_i] &= (2\ell - 2n - 1)X^{(n)}_i, & [R, X^{(n)}_i] &= (2\ell + 1) \sum_k \epsilon_{ik} X^{(n)}_k, \\
[M_{12}, S] &= S^*, & [M_{12}, S^*] &= -S, & [M_{12}, Q] &= Q^*, & [M_{12}, Q^*] &= -Q, \\
[R, S] &= 2S^*, & [R, S^*] &= -2S, & [R, Q] &= 2Q^*, & [R, Q^*] &= -2Q, \\
[Q, P_i^{(n)}] &= nX^{(n-1)}_i, & [Q^*, P_i^{(n)}] &= n \sum_k \epsilon_{ik} X^{(n-1)}_k. \\
[S, P_i^{(n)}] &= -(2\ell - n)X^{(n)}_i, & [S^*, P_i^{(n)}] &= -(2\ell - n) \sum_k \epsilon_{ik} X^{(n)}_k.
\end{align*}
\]

(12)

The additional bosonic commutator is as follows:

\[
[R, P_i^{(n)}] = (2\ell - 1) \sum_k \epsilon_{ik} P_k^{(n)}.
\]

(13)

These relations define the \( \mathcal{N} = 2 \) exotic super-GCA without the central extension as they satisfy the super Jacobi identity. This super-GCA has \( osp(2/2) \) subalgebra generated by \( \{Q, Q^*, S, S^*, D, H, C, R\} \). The set of generators \( \{P_i^{(n)}, X^{(n)}_i\} \) forms an Abelian ideal.

Now we consider central extensions of the exotic super-GCA. For half-integer \( \ell \), we introduce the mass central extension defined by (2) and

\[
X^{(n)}_i X^{(n)}_j = \delta_{ij}\delta_{m+n,2\ell-1} \alpha_m M, \quad \alpha_m = (-1)^{m+\ell-\frac{1}{2}} (2\ell - 1 - m)! m!.
\]

(14)

For integer \( \ell \), we introduce the exotic central extension by (3) and

\[
X^{(m)}_i X^{(n)}_j = \epsilon_{ij}\delta_{m+n,2\ell-1} \tilde{\alpha}_m \Theta, \quad \tilde{\alpha}_m = (-1)^{m+1} (2\ell - 1 - m)! m!.
\]

(15)

One may verify by the straightforward computation that these commutation relations satisfy the super Jacobi identity. Thus we have obtained the \( \mathcal{N} = 2 \) exotic super-GCA with mass or exotic central extensions.
5. Realizations

In [14], dynamical systems on which the bosonic GCA with the mass or exotic central extension acts as a symmetry operation were constructed by the orbit method. The construction shows as a byproduct that the bosonic GCA can be realized in terms of $P_l^{(n)}$. As seen in section 2 the central extensions make $P_l^{(n)}$ noncommutative so that we may use them to realize the GCA. In this section we generalize this realization to the super-GCAs introduced in the previous sections. The obtained result is a realization of the abstract Lie superalgebra by its subalgebra. However, this also leads an interesting relation of the super-GCA and a Bose–Fermi Hamiltonian. In the realization we replace the central elements $M, \Theta$ with their eigenvalues $\mu, \theta$, respectively. We use the vector notation such as $P_l^{(n)} = (P_1^{(n)}, P_2^{(n)}, \ldots, P_\mu^{(n)})$.

5.1. Standard super and mass central extension

The super-GCA defined by (1), (2) and (4)–(7) is realized in terms of $P_l^{(n)}, X_j^{(n)}, \tilde{X}_i^{(n)}, J_j^{(n)}$. It is given by the following equations:

\[
H = \frac{1}{2\mu} \left( \sum_{m=1}^{2\ell} \frac{m}{l_m} p^{(2m-1)} p^{(m-1)} + 2 \sum_{m=1}^{2\ell-1} \frac{m}{\alpha_m} x^{(2\ell-1-m)} \tilde{x}^{(m-1)} + \sum_{m=1}^{2\ell-2} \frac{m}{\beta_m} j^{(2\ell-2-m)} j^{(m-1)} \right),
\]

\[
D = \frac{1}{\mu} \left( \sum_{m=0}^{2\ell} \frac{m-\ell}{l_m} p^{(2m-\ell)} p^{(m)} + 2 \sum_{m=0}^{2\ell-1} \frac{m+1-2\ell}{\alpha_m} x^{(2\ell-1-m)} \tilde{x}^{(m)} + \sum_{m=0}^{2\ell-2} \frac{m}{\beta_m} j^{(2\ell-2-m)} j^{(m)} \right),
\]

\[
C = \frac{1}{2\mu} \left( \sum_{m=0}^{2\ell} \frac{m}{l_m} p^{(2m+1)} p^{(m)} + 2 \sum_{m=1}^{2\ell} \frac{m}{\alpha_m} x^{(2\ell+1-m)} \tilde{x}^{(m)} + \sum_{m=1}^{2\ell} \frac{m}{\beta_m} j^{(2\ell+1-m)} j^{(m)} \right),
\]

\[
M_{ij} = \frac{1}{2\mu} \left( \sum_{m=0}^{2\ell} \frac{1}{l_m} (J_j^{(2m-\ell)} p_i^{(m)} - J_j^{(2m-\ell)} p_i^{(m)}) + 2 \sum_{m=0}^{2\ell-1} \frac{1}{\alpha_m} (X_j^{(2\ell-1-m)} \tilde{x}_i^{(m)} + \tilde{x}_i^{(m)} X_j^{(2\ell-1-m)}) + \sum_{m=0}^{2\ell-2} \frac{1}{\beta_m} (J_j^{(2\ell-2-m)} j_i^{(m)} - j_j^{(2\ell-2-m)} j_i^{(m)}) \right)
\]

\[
R = -\frac{1}{\mu} \sum_{m=0}^{2\ell-1} \frac{1}{\alpha_m} x^{(2\ell-1-m)} \tilde{x}^{(m)} + \ell d,
\]

\[
Q = -\frac{1}{\mu} \sum_{m=1}^{2\ell} \frac{m}{l_m} (p^{(2\ell-m)} + (2\ell-m)j^{(2\ell-m)}) x^{(m-1)},
\]

\[
\tilde{Q} = -\frac{1}{\mu} \sum_{m=1}^{2\ell} \frac{m}{l_m} (p^{(2\ell-m)} - (2\ell-m)j^{(2\ell-m)}) x^{(m-1)},
\]

\[
S = -\frac{1}{\mu} \sum_{m=1}^{2\ell} \frac{m}{l_m} (p^{(2\ell+1-m)} - (m-1)j^{(2\ell-m)}) x^{(m-1)},
\]

\[
\tilde{S} = -\frac{1}{\mu} \sum_{m=1}^{2\ell} \frac{m}{l_m} (p^{(2\ell+1-m)} + (m-1)j^{(2\ell-m)}) x^{(m-1)}.
\]

The mass central extension makes the subalgebra $(P_l^{(n)}, X_j^{(n)}, J_j^{(n)})$ noncommutative. This noncommutative subalgebra is isomorphic to the boson–fermion algebra. Therefore the
realization (16) is nothing but a boson–fermion realization of super-GCA. Let us introduce bosons and fermions by which we may rewrite (16). We define $2\ell$ kinds of bosons in $(d + 1)$-dimensional spacetime:

$$b_i^{(n)} = \begin{cases} \frac{1}{\sqrt{\mu_{n-1}}} p_i^{(n-1)}, & n = 1, 2, \ldots, \ell + \frac{1}{2} \\ \frac{1}{\sqrt{\mu_{n-\ell}}} f_i^{(n-\ell - \frac{1}{2})}, & n = \ell + \frac{3}{2}, \ldots, 2\ell \end{cases}$$ (17)

$$b_i^{(\ell)} = \frac{1}{\sqrt{\mu}} p_i^{(2\ell - n + 1)}, \quad n = 1, 2, \ldots, \ell + \frac{1}{2}$$

$$b_i^{(\ell + \frac{1}{2})} = \frac{1}{\sqrt{\mu}} f_i^{(3\ell - n - \frac{1}{2})}, \quad n = \ell + \frac{3}{2}, \ldots, 2\ell.$$ (18)

We also introduce $2\ell$ kinds of fermion:

$$\alpha_i^{(n)} = \frac{-1}{\sqrt{\mu} \alpha_{n-1}} \bar{X}_i^{(n-1)}, \quad \alpha_i^{(\ell+n)} = \frac{1}{\sqrt{\mu}} X_i^{(\ell-n)},$$

where $n = 1, 2, \ldots, 2\ell$. Then they satisfy the commutation relations:

$$[b_i^{(m)}, b_j^{(n)}] = \delta_{ij} \delta_{mn}, \quad [\alpha_i^{(m)}, \alpha_j^{(n)}] = \delta_{ij} \delta_{mn}.$$ (20)

It is obvious that one can rewrite the realization (16) with these bosons and fermions. One may recognize that the generator $D$, in the realization by the bosons and fermions, is a linear combination of Bose and Fermi oscillators. Then it relates to the following Bose–Fermi oscillator Hamiltonian:

$$\mathcal{H} = \sum_{m=1}^{\ell+\frac{1}{2}} (\ell - m + 1) b_i^{(m)} b_i^{(m)} + \sum_{m=\ell+\frac{1}{2}}^{2\ell} \left( 2\ell - m + \frac{1}{2} \right) b_i^{(m)} b_i^{(m)}$$

$$- \sum_{m=1}^{2\ell} \left( \ell - m + \frac{1}{2} \right) \alpha_i^{(m)} \alpha_i^{(m)}.$$ (21)

The relations are given by

$$D = -2\mathcal{H} - \ell \left( \ell + \frac{1}{2} \right) d,$$

namely one may interpret $D$ as a Hamiltonian. Then each generator of the standard super-GCA is either a raising operator or a lowering operator or a symmetry operator of the eigenvalue of $\mathcal{H}$. This interpretation is different from the ordinary supersymmetric theories where the generator $H$ plays the role of the Hamiltonian. However, we think that this is an interesting observation resulting from the realization (16).

5.2. Standard super and exotic central extension

The super-GCA defined by (1), (3)–(6) and (9) is realized in terms of $p_i^{(n)}, X_i^{(n)}, \bar{X}_i^{(n)}, J_i^{(n)}$. It is given by the following equations:

$$H = -\frac{1}{2g} \sum_{j,k=1}^{2\ell} \epsilon_{jk} \left( \sum_{m=1}^{2\ell} \frac{m}{\ell_m} p_i^{(2\ell - m)} p_k^{(m-1)} + 2 \sum_{m=1}^{2\ell-1} \frac{m}{\alpha_m} X_j^{(2\ell - 1 - m)} \bar{X}_k^{(m-1)} \\
+ \sum_{m=1}^{2\ell-2} \frac{m}{\beta_m} j_i^{(2\ell - 2 - m)} f_k^{(m-1)} \right),$$
\[ D = \frac{1}{\theta} \sum_{j,k=1}^{2^\ell} \epsilon_{jk} \left( \sum_{m=0}^{2^\ell} \frac{\ell - m}{I_m} P_j^{2\ell-m} P_k^{(m)} + \sum_{m=0}^{2\ell-1} \frac{2\ell - 2m - 1}{\alpha_m} X_j^{(2\ell-1-m)} X_k^{(m)} + \sum_{m=1}^{2\ell-2} \frac{\ell - m - 1}{\beta_m} J_j^{(2\ell-2-m)} J_k^{(m)} \right), \]

\[ C = -\frac{1}{2\theta} \sum_{j,k=1}^{2^\ell} \epsilon_{jk} \left( \sum_{m=0}^{2^\ell} \frac{m}{I_m} P_j^{2\ell+1-m} P_k^{(m)} + \sum_{m=0}^{2\ell-1} \frac{m}{\alpha_m} X_j^{(2\ell-m)} X_k^{(m)} + \sum_{m=1}^{2\ell-2} \frac{m}{\beta_m} J_j^{(2\ell-1-m)} J_k^{(m)} \right), \]

\[ M_{12} = \frac{1}{2\theta} \left( \sum_{m=0}^{2\ell} \frac{1}{I_m} P_j^{2\ell-m} P_k^{(m)} + 2 \sum_{m=0}^{2\ell-1} \frac{1}{\alpha_m} X_j^{(2\ell-1-m)} X_k^{(m)} + 2 \sum_{m=0}^{2\ell-2} \frac{1}{\beta_m} J_j^{(2\ell-2-m)} J_k^{(m)} \right), \]

\[ R = \frac{1}{\theta} \sum_{j,k=1}^{2^\ell} \sum_{m=0}^{2\ell-1} \frac{1}{\alpha_m} \epsilon_{jk} X_j^{(2\ell-1-m)} X_k^{(m)} + 2\ell, \]

\[ Q = \frac{1}{\theta} \sum_{j,k=1}^{2^\ell} \sum_{m=0}^{2\ell} \frac{1}{I_m} \epsilon_{jk} (p_j^{2\ell-m} + (2\ell - m) J_j^{2\ell-1-m}) X_k^{(m-1)}, \]

\[ \tilde{Q} = \frac{1}{\theta} \sum_{j,k=1}^{2^\ell} \sum_{m=0}^{2\ell} \frac{1}{I_m} \epsilon_{jk} (p_j^{2\ell-m} - (2\ell - m) J_j^{2\ell-1-m}) X_k^{(m-1)}, \]

\[ S = \frac{1}{\theta} \sum_{j,k=1}^{2^\ell} \sum_{m=0}^{2\ell} \frac{1}{I_m} \epsilon_{jk} (p_j^{2\ell+1-m} - (m - 1) J_j^{2\ell-m}) X_k^{(m-1)}, \]

\[ \tilde{S} = \frac{1}{\theta} \sum_{j,k=1}^{2^\ell} \sum_{m=0}^{2\ell} \frac{1}{I_m} \epsilon_{jk} (p_j^{2\ell+1-m} + (m - 1) J_j^{2\ell-m}) X_k^{(m-1)}. \] (22)

5.3. Exotic super and mass central extension

The super-GCA defined by (1), (2) and (11)–(14) is realized in terms of \( P_j^{(m)}, X_j^{(m)} \). It is given by the following equations:

\[ H = \frac{1}{2\mu} \sum_{m=1}^{2\ell} \frac{m}{I_m} p_j^{2\ell-m} P_j^{(m-1)} + \frac{2\ell - 1}{2\mu} \sum_{m=1}^{2\ell - 1} \frac{m}{\alpha_m} X_j^{(2\ell-1-m)} X_j^{(m-1)}, \]

\[ D = \frac{1}{\mu} \sum_{m=0}^{2\ell} \frac{\ell - m}{I_m} P_j^{2\ell-m} P_j^{(m)} + \frac{1}{2\mu} \sum_{m=0}^{2\ell - 1} \frac{2m + 1 - 2\ell}{\alpha_m} X_j^{(2\ell-1-m)} X_j^{(m)}, \]

\[ C = \frac{1}{2\mu} \left( \sum_{m=1}^{2\ell} \frac{m}{I_m} p_j^{2\ell+1-m} P_j^{(m)} + \sum_{m=1}^{2\ell - 1} \frac{m}{\alpha_m} X_j^{(2\ell-m)} X_j^{(m)} \right), \]

\[ M_{12} = \frac{1}{2\mu} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2^\ell} \epsilon_{jk} p_j^{2\ell-m} P_k^{(m)} , \]

\[ R = -\frac{2\ell + 1}{2\mu} \sum_{m=0}^{2\ell - 1} \sum_{j,k=1}^{2^\ell} \epsilon_{jk} X_j^{(2\ell-1-m)} X_k^{(m)} - (2\ell - 1) M_{12} , \]

\[ Q = -\frac{1}{2\mu} \sum_{m=1}^{2\ell} \frac{m}{I_m} P_j^{2\ell-m} X_j^{(m-1)}, \]
The super-GCA defined by (1), (3), (11)–(13) and (15) is realized in terms of $P_i^{(n)}$, $X_i^{(n)}$. It is given by the following equations:

$$H = - \frac{1}{2\theta} \sum_{j,k=1}^{2\ell} \varepsilon_{jk} \left( \sum_{m=1}^{2\ell} \frac{m}{\alpha_m} P_j^{(2\ell-m)} P_k^{(m)} + \sum_{m=1}^{2\ell-1} \frac{m}{\alpha_m} X_j^{(2\ell-m)} X_k^{(m)} \right),$$

$$D = \sum_{j,k=1}^{2\ell} \varepsilon_{jk} \left( \frac{1}{\theta} \sum_{m=1}^{2\ell} \frac{m}{\alpha_m} P_j^{(2\ell-m)} P_k^{(m)} + \frac{1}{2\theta} \sum_{m=0}^{2\ell-1} \frac{m}{\alpha_m} X_j^{(2\ell-m)} X_k^{(m)} \right),$$

$$C = - \frac{1}{2\theta} \sum_{j,k=1}^{2\ell} \varepsilon_{jk} \left( \sum_{m=1}^{2\ell} \frac{m}{\alpha_m} P_j^{(2\ell-m)} P_k^{(m)} + \sum_{m=1}^{2\ell-1} \frac{m}{\alpha_m} X_j^{(2\ell-m)} X_k^{(m)} \right),$$

$$M_{12} = \frac{1}{2\theta} \sum_{m=0}^{2\ell} \frac{m}{\alpha_m} P_j^{(2\ell-m)} P_k^{(m)},$$

$$R = - \frac{2\ell + 1}{2\theta} \sum_{m=0}^{2\ell} \frac{1}{\alpha_m} X_j^{(2\ell-m)} X_k^{(m)} - (2\ell - 1)M_{12},$$

$$Q = \frac{1}{\theta} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2} \frac{m}{\alpha_m} \varepsilon_{jk} P_j^{(2\ell-m)} X_k^{(m-1)},$$

$$Q^* = - \frac{1}{\theta} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2} \frac{m}{\alpha_m} P_j^{(2\ell-m)} X_k^{(m-1)},$$

$$S = \frac{1}{\theta} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2} \frac{m}{\alpha_m} \varepsilon_{jk} P_j^{(2\ell+1-m)} X_k^{(m-1)},$$

$$S^* = - \frac{1}{\theta} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2} \frac{m}{\alpha_m} P_j^{(2\ell+1-m)} X_k^{(m-1)}.$$ (23)

5.4. Exotic super and exotic central extension

The super-GCA defined by (1), (3), (11)–(13) and (15) is realized in terms of $P_i^{(n)}$, $X_i^{(n)}$. It is given by the following equations:

$$Q^* = - \frac{1}{\mu} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2} \frac{m}{I_m} \varepsilon_{jk} P_j^{(2\ell-m)} X_k^{(m-1)},$$

$$S = - \frac{1}{\mu} \sum_{m=1}^{2\ell} \frac{m}{I_m} P_j^{(2\ell+1-m)} X_k^{(m-1)},$$

$$S^* = - \frac{1}{\mu} \sum_{m=1}^{2\ell} \sum_{j,k=1}^{2} \frac{m}{I_m} \varepsilon_{jk} P_j^{(2\ell+1-m)} X_k^{(m-1)}.$$ (24)

6. $N = 1$ supersymmetric extension

In this section we give an $N = 1$ supersymmetric extension of GCA for any $(d, \ell)$ and their central extensions. We remark that the $N = 1$ extension is not obtained by Inönü–Wigner contraction [37, 38]. $N = 1$ super-GCA for $\ell = 1$ is discussed in [39, 40]; however, the central extension is not considered.
We introduce fermionic generators $Q, S$ and $X^{(n)}_i$, $n = 0, 1, \ldots, 2\ell - 1$ but no additional bosonic ones. Then we set up the following anti-commutation relations for fermionic generators:

$$\{Q, Q\} = 2H, \quad \{S, S\} = 2C, \quad \{Q, S\} = D,$$

$$\{Q, X^{(n)}_i\} = -P^{(n)}_i, \quad \{S, X^{(n)}_i\} = -P^{(n+1)}_i,$$  \hspace{1cm} (25)

and the commutation relations for pairs of bosonic and fermionic generators:

$$[H, S] = -Q, \quad [C, Q] = S, \quad [D, S] = -S, \quad [D, Q] = Q,$$

$$[H, X^{(n)}_i] = -nX^{(n-1)}_i, \quad [D, X^{(n)}_i] = (2\ell - 2n - 1)X^{(n)}_i,$$

$$[C, X^{(n)}_i] = (2\ell - n - 1)X^{(n+1)}_i, \quad [M_{ij}, X^{(n)}_k] = -\delta_{jk}X^{(n)}_i + \delta_{jk}X^{(n)}_j,$$

$$[Q, P^{(n)}_i] = nX^{(n-1)}_i, \quad \{S, P^{(n)}_i\} = -(2\ell - n)X^{(n)}_i.$$  \hspace{1cm} (26)

We assume that other (anti-)commutators vanish. Then one can verify that these relations are compatible with the super Jacobi identity so that define an $\mathcal{N} = 1$ super-GCA without central extensions.

Now we turn to central extensions. For half-integer $\ell$ and any $d$, we may introduce the mass central extension given by (14). While for integer $\ell$ and $d = 2$, we introduce the exotic central extension given by (15). It is not difficult to verify that these central extensions do not contradict with the super Jacobi identity. Therefore we have obtained the $\mathcal{N} = 1$ super-GCA with the mass or exotic central extension.

7. Concluding remarks

We discussed $\mathcal{N} = 2$ supersymmetric extensions of GCA and their central extensions. We have shown that the supersymmetric and the central extensions of $\ell = 1/2$ GCA presented in [17] are able to generalize to higher $\ell$. We do not claim that we give an exhaustive list of $\mathcal{N} = 2$ super-GCA. For instance, it is shown in [17] that for a given $\mathcal{N}$, the supersymmetric extension of $\ell = 1/2$ GCA is determined by a pair of positive integers $(N_+, N_-)$ satisfying $N_+ + N_- = \mathcal{N}$. The $\mathcal{N} = 2$ extensions presented in this paper correspond to $N_+ = 2, N_- = 0$. Thus super-GCA that corresponds to other pairs of $(N_+, N_-)$ may exist.

Many works on the supersymmetric extension of $\ell \geq 1$ GCA focus on the study of algebraic structure. Although a relation between $\ell = 1$ super-GCA and superconformal mechanics is discussed in [40], we know very few on physical implication or representation theory of $\ell \geq 1$ super-GCA. The Bose–Fermi oscillator Hamiltonian given in section 5 provides another physical model relating to super-GCA. However, the physical application of $\ell \geq 1$ super-GCA is still an open problem. Representation theory of super-GCA is also a problem to be studied. We give a realization of the super-GCA in section 5. However, more mathematical works such as classification of irreducible representations (see [33, 34] for $\ell = 1/2$) should be performed for further understanding and physical applications of super-GCA.

We close this paper by mentioning that super-GCA is easily extended to infinite-dimensional algebras [23, 39, 41, 43]. They have (super) Virasoro algebra as a subalgebra. Because of this, we think that the infinite-dimensional extension of the superalgebras introduced in this paper may be an interesting object from a mathematical and physical point of view.

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Appendix.

The central extensions considered in sections 3 and 4 make the Abelian ideal of super-GCA non-Abelian. In this appendix we study another possibility of central extension inspired from the infinite-dimensional super-GCA. In [43], an infinite-dimensional super-GCA corresponding to \( d = \ell = 1 \) is considered and the superalgebra has the central extension different from those in sections 3 and 4. We study the possibility of such a central extension for the finite-dimensional super-GCA considered in this work. To keep contact with the infinite-dimensional algebra in [43], \( d \) is set to be 1 but \( \ell \) remains arbitrary. Thus we start with the \( \mathcal{N} = 2 \) standard supersymmetric extension of GCA without central extension. Since it has the subalgebra \( \mathfrak{sl}(2/1) \) generated by \( \{Q, \bar{Q}, S, \bar{S}, D, H, C, R\} \), we introduce the following central terms:

\[
[D, P^{(n)}] = 2(\ell - n)P^{(n)} + \alpha^{(n)}, \quad [H, P^{(n)}] = -nP^{(n-1)} + \beta^{(n)}, \\
[C, P^{(n)}] = (2\ell - n)P^{(n+1)} + \gamma^{(n)}, \\
\{Q, \bar{X}^{(n)}\} = -P^{(n)} - nJ^{(n-1)} + c_1^{(n)}, \quad \{\bar{Q}, X^{(n)}\} = -P^{(n)} + nJ^{(n-1)} + c_1^{(n)}, \\
\{S, \bar{X}^{(n)}\} = -P^{(n+1)} - (n - 2\ell + 1)J^{(n)} + c_2^{(n)}, \\
\{\bar{S}, X^{(n)}\} = -P^{(n+1)} + (n - 2\ell + 1)J^{(n)} + c_2^{(n)},
\tag{A.1}
\]

The Jacobi identity for \( \{D, H, P^{(n)}\} \) and \( \{D, C, P^{(n)}\} \) gives the relations

\[
na^{(n-1)} + 2(\ell - n + 1)\beta^{(n)} = 0, \quad (2\ell - n)\alpha^{(n+1)} - 2(\ell - n - 1)\gamma^{(n)} = 0,
\tag{A.2}
\]

respectively. We also have another relation from the Jacobi identity for \( \{C, H, P^{(n)}\} \):

\[
\alpha^{(n)} + (2\ell - n)\beta^{(n+1)} + ny^{(n-1)} = 0.
\tag{A.3}
\]

The super Jacobi identity for \( \{H, Q, \bar{X}^{(n)}\}, \{H, \bar{Q}, X^{(n)}\}, \{C, S, \bar{X}^{(n)}\} \) and \( \{C, \bar{S}, X^{(n)}\} \) provides us further relations on the central elements:

\[
\beta^{(n)} = nc_1^{(n-1)} = nc_1^{(n-1)}, \quad \gamma^{(n)} = -(2\ell - n)c_2^{(n)} = -(2\ell - n)c_2^{(n)}.
\tag{A.4}
\]

We see from the relations in (A.2) that \( \alpha^{(\ell)} = 0 \). Applying this to (A.3) we have the relation \( \beta^{(\ell+1)} + \gamma^{(\ell-1)} = 0 \). We see from (A.2) again that \( \beta^{(n)}(n \neq \ell + 1) \) and \( \gamma^{(n)}(n \neq \ell - 1) \) are functions of \( \alpha^{(n)} \). Thus by setting \( \alpha^{(n)} = 2(\ell - n)\alpha^{(n)} \) one may write as follows:

\[
\beta^{(n)} = \begin{cases} 
-nc_1^{(n-1)} & n \neq \ell + 1 \\
-\gamma^{(\ell-1)} & n = \ell + 1,
\end{cases} \quad \gamma^{(n)} = \begin{cases} 
(2\ell - n)\alpha^{(n+1)} & n \neq \ell - 1 \\
\gamma^{(\ell-1)} & n = \ell - 1.
\end{cases}
\]

It follows that

\[
c_1^{(n)} = c_1^{(n)} = \begin{cases} 
-\alpha^{(n)} & n \neq \ell \\
-\gamma^{(\ell-1)} & n = \ell + 1,
\end{cases}
\]

and

\[
c_2^{(n)} = c_2^{(n)} = \begin{cases} 
-\alpha^{(n+1)} & n \neq \ell - 1 \\
-\gamma^{(\ell-1)} & n = \ell + 1.
\end{cases}
\]

This means that the central terms in (A.1) are absorbed by the following redefinitions of \( P^{(n)} \):

\[
\begin{cases} 
P^{(n)} + \alpha^{(n)} \rightarrow P^{(n)} & n \neq \ell \\
P^{(\ell)} + \gamma^{(\ell-1)} \rightarrow P^{(\ell)} & n = \ell.
\end{cases}
\]

Therefore the central extensions in (A.1) are all trivial.
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