Adaptive Observers for Biophysical Neuronal Circuits

Thiago B. Burghi ©, Member, IEEE, and Rodolphe Sepulchre ©, Fellow, IEEE

Abstract—This article presents adaptive observers for online state and parameter estimation of a class of nonlinear systems motivated by biophysical models of neuronal circuits. We first present a linear-in-the-parameters design that solves a classical RLS problem. Then, building on this simple design, we present an augmented adaptive observer for models with a nonlinearly parameterized internal dynamics, the parameters of which we interpret as structured uncertainty. We present a convergence and robustness analysis based on contraction theory, and illustrate the potential of the approach in neurophysiological applications by means of numerical simulations.

Index Terms—Adaptive observers, conductance-based models, contraction theory, neuroscience, nonlinear systems.

I. INTRODUCTION

With the development and refinement of neural recording technology, controlling the nervous system at the cellular scale may soon become possible. Techniques such as voltage imaging [21] promise to deliver simultaneous subthreshold recordings of large biological neural networks, opening up new possibilities for the design of brain–machine interfaces [32]. But while large-scale technologies are still maturing, closed-loop control of small living neuronal circuits has been a reality since the development of the dynamic clamp [37] electrophysiology technique. Even though the control of such small circuits is not yet done in a systematic fashion, it has enabled important scientific discoveries related to the electrochemical process of neuromodulation [28].

The systematic control of small neural circuits is an open problem [9] that will only become more challenging as the scale of the circuits is increased. The main bottleneck is the ever changing nature of living neurons [38], which implies that any model-based approach to neuronal control must consider online estimation methods. Any such estimation method should deal with the spiking nature of electrophysiological signals, and the consequent nonlinearity of state-space neuronal models [19]. In particular, conductance-based models, introduced in the seminal work [17], have a large number of uncertain parameters and unmeasured states, and dealing with this issue has been an important modeling challenge [16].

The question of estimating conductance-based neuronal models from input–output data has mostly been approached with offline algorithms and output-error [24] model structures, see, e.g., [11], [30], [33]. However, since the neuronal dynamics lack the fading memory property that is essential for performing output-error estimation [23], [24], such methods lead to difficult optimization problems with nonsmooth cost functions [1], [34]; as a consequence, the use of such methods in adaptive schemes is precluded. To deal with these difficulties, some authors have exploited the assumption that the only parameters to be estimated are a neuron’s maximal conductances (including synaptic weights), while other parameters related to ion channel properties can be assumed known. In this case, the neuronal model structure becomes linear-in-the-parameters [5], [18], [31]. An important question related to such approaches is the effect of ion channel model uncertainty.

In this article, we address the problem of online estimation of single-neuron and neuronal network conductance-based models. Our modeling framework acknowledges the linear parametrization of maximal conductances, which are key players in the neuromodulation of neuronal behaviours, from the single-cell to the network scale [9], [10], [28]. At the same time, we highlight the important issue of uncertainty in ion channel models, which define the internal dynamics of a neuron and its synapses. Our first contribution is the design and analysis of a globally convergent adaptive observer based on the classical recursive least squares (RLS) method, which assumes a linear-in-the-parameter neuronal output dynamics and a known nonlinear internal dynamics. Building on that design, we then propose an augmented adaptive observer capable of estimating parametric (structured) uncertainty in a nonlinearly parameterized neuronal internal dynamics.

The observers in this article are aligned with the literature on nonlinear adaptive observers [13], [14], [29]. Our approach is however closer to linear observer design [41] than to nonlinear observer design, since, instead of relying on particular state space observer normal forms [3], [22], we rather rely on contraction theory principles [25]. Contraction analysis provides a framework reminiscent of the linear theory of adaptive control, as well as explicit convergence rates and robustness guarantees grounded in the concept of a virtual system [4], [20].
analysis has been a driving methodology in recent adaptive control research [26], and the present work demonstrates its value for the design of adaptive systems in neuroscience.

The article is organized as follows. The model structure assumptions and their application to conductance-based models are presented in Section II. In Section III, the online estimation problem for a simplified linear-in-the-parameters model structure is studied, and a globally convergent adaptive observer is presented. In Section IV, parametric nonlinear uncertainty in the internal dynamics is introduced, and we present an augmented adaptive observer to solve the estimation problem; we also discuss the effects of measurement noise and unstructured uncertainty. In Section V, we illustrate the performance of the adaptive observers and discuss the potential of the approach in neurophysiology.

A. Notation

For a finite-dimensional vector $x$, we write $n_x := \dim(x)$. For two matrices $X \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{p \times m}$, we write $\text{col}(X, Y) := (X^T, Y^T)^T$. For a matrix $A \in \mathbb{R}^{n \times m}$, $\|A\|$ denotes the spectral norm (the largest singular value of $A$). For a vector $x \in \mathbb{R}^n$ and a symmetric matrix $P \in \mathbb{R}^{n \times n}$, we write $\|x\|_P^2 := x^T P x$, and $\|x\|_I := \|x\|_I^2$ with $I$ the identity matrix.

For a vector-function valued $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$, we write $\partial_x f(x,y) \in \mathbb{R}^{m \times n}$ for the Jacobian of $f(x,y)$ with respect to $x$. We write $A \geq B$ ($A > B$) if $A - B$ is a positive-semidefinite (positive-definite) matrix. This article often uses the formalism of contraction analysis [25], which is briefly recalled in Appendix A1.

II. System Class

This section introduces and motivates the model assumptions of the article. Section II-A defines the basic model structure and states our main assumptions. Section II-B then shows how the model structure and the assumptions are motivated by our main application: The conductance-based model of a neuron. Finally, Section II-C shows that the assumptions extend from single neurons to models of neuronal networks.

A. Problem Statement

This article considers nonlinear state-space systems of the form

$$\dot{v} = \Phi(v, w, u)\theta + a(v, w, u) \tag{1a}$$
$$\dot{w} = A(v, \eta)w + b(v, \eta) \tag{1b}$$

where $v(t) \in \mathbb{R}^{n_v}$ is a measured output, $w(t) \in \mathbb{R}^{n_w}$ are unmeasured internal states, and $\theta \in \mathbb{R}^{n_{\theta}}$ and $\eta \in \mathbb{R}^{n_{\eta}}$ are parameter vectors. We call (1a) the output dynamics, and (1b) the internal dynamics. We assume that $\Phi, A, a$, and $b$ are continuously differentiable functions of the appropriate dimensions.

The model structure (1) is motivated by neuroscience applications discussed in Section II-B. In those applications, the vector $\theta$ is unknown, while $\eta$ is not unknown but uncertain. Thus, we work in the context of structured model uncertainty [35]. Our aim is to design an adaptive observer to estimate $\theta$ and, if necessary, $\eta$. For that purpose, we regard the parameters as part of the state of the system. We will initially consider the constant model

$$\dot{\theta} = 0, \quad \dot{\eta} = 0 \tag{1c}$$

so that $\theta(t) = \theta(0)$ and $\eta(t) = \eta(0)$ for all $t \geq 0$; later, we will study the case where such parameters are time-varying.

We now consider the main assumptions on the properties of (1). These assumptions will also be justified by the applications in Section II-B.

**Assumption 1:** There exists a compact set $U$ such that $u(t) \in U$ for all $t \geq 0$. Furthermore, there exists a compact convex set $V \times W \times \{\theta(0)\} \times \{\eta(0)\}$, which is positively invariant with respect to (1), uniformly in $u$ on $U$.

**Assumption 2:** There exist a symmetric positive definite matrix $M_w > 0$ and a contraction rate $\lambda_w > 0$ such that

$$A(v, \eta)^T M_w + M_w A(v, \eta) \preceq -\lambda_w M_w \tag{2}$$

for all $\{v, \eta\} \in \mathbb{R}^{n_v} \times \mathbb{R}^{n_{\eta}}$. It is assumed that $\|M_w\| = 1$ without loss of generality.

**Remark 1:** When Assumption 1 holds, then without loss of generality we can assume that for all $v \in V$ and $u \in U$, the functions $\Phi(v, w, u)$ and $a(v, w, u)$ are globally Lipschitz and bounded in $w \in \mathbb{R}^{n_w}$. This is because we can replace $w$ by $s_w(w)$ in the arguments of those functions, where $s_w : \mathbb{R}^{n_w} \rightarrow W$ is a smooth saturation function such that $s_w(w) = w$ for all $w \in W$. Doing so does not change the dynamics of (1) within the positively invariant set of Assumption 1.

The reader will note that the system (1) is not in the classical output-feedback canonical form [22], nor in any of the equivalent adaptive observer forms summarized by [3]. The system also does not fit the model structures addressed in the more recent adaptive observer literature, e.g., [13, 40].

B. Conductance-Based Single-Neuron Model

Since the seminal work of Hodgkin and Huxley [17], the nonlinear electrical circuits known as conductance-based models have become the foundation of biophysical modeling in neurophysiology [19]. We now show that any single-neuron conductance-based model can be written in the form (1) in such a way that Assumptions 1 and 2 are satisfied.

A circuit representation of the model is shown in Fig. 1: A capacitor of capacitance $c > 0$ in parallel with a leak current $I_L$ and a number of intrinsic ionic currents $I_{ion}$. The input current

![Fig. 1. Circuit representation of a neuron with voltage $v$ that is coupled through a synapse to a presynaptic neuron with voltage $v_p$.](image-url)
$u(t) \in \mathbb{R}$ represents the external current, injected with an intracellular electrode. The capacitor voltage $v(t) \in \mathbb{R}$ modeling the neuronal membrane potential evolves according to Kirchhoff’s law,

$$\dot{c}v = -I_L - \sum_{i \in I} I_{ion} + u$$  \hspace{1cm} (3)

where $I = \{i_{on1}, i_{on2}, \ldots, i_{onend(2)}\}$ is the (finite) index set of intrinsic ionic currents. Each current in the circuit is ohmic in nature, but with a conductance that can be nonlinear and voltage dependent. The leak current has a constant conductance and is given by

$$I_L = \mu_L (v - \nu_L)$$  \hspace{1cm} (4)

with $\mu_L > 0$, and the intrinsic currents are modeled by

$$I_{ion} = \mu_i m_{ion}^p h_{ion}^q (v - \nu_{ion})$$  \hspace{1cm} (5a)

$$\tau_{m_{ion}} (v) \dot{m}_{ion} = -m_{ion} + \sigma_{m_{ion}} (v)$$  \hspace{1cm} (5b)

$$\tau_{h_{ion}} (v) \dot{h}_{ion} = -h_{ion} + \sigma_{h_{ion}} (v).$$  \hspace{1cm} (5c)

The constants $\mu_{ion} > 0$ and $\nu_{ion} \in \mathbb{R}$ are called (intrinsic) maximal conductances and reversal potentials, respectively. The exponents $p_{ion}$ and $q_{ion}$ in (5a) are fixed natural numbers (including zero). The static activation functions $\sigma_{m_{ion}} (v)$ and $\sigma_{h_{ion}} (v)$, and time-constant functions $\tau_{m_{ion}} (v)$ and $\tau_{h_{ion}} (v)$, model the nonlinear gating of the ionic conductance. The activation functions are given by sigmoid functions of the form

$$\sigma (v) = \frac{1}{1 + \exp (- (v - \rho)/\kappa)}$$  \hspace{1cm} (6)

where the constants $\rho_{m_{ion}} \in \mathbb{R}$ and $\rho_{h_{ion}} \in \mathbb{R}$ determine the half-activation of those functions, while the constants $\kappa_{m_{ion}} > 0$ and $\kappa_{h_{ion}} < 0$ determine their slopes. Because $\sigma_{m_{ion}} : \mathbb{R} \rightarrow (0, 1)$ and $\sigma_{h_{ion}} : \mathbb{R} \rightarrow (0, 1)$ are monotonically increasing and decreasing, respectively, the states $m_{ion}$ and $h_{ion}$ are called activation and inactivation gating variables, respectively. The time-constant functions are given by bell-shaped functions of the form

$$\tau (v) = \zeta + (\tau - \zeta) \exp (- (v - \zeta^2) / \chi^2)$$  \hspace{1cm} (7)

for all $v \in \mathbb{R}$ and some $\tau, \zeta > 0$ and $\zeta, \chi \in \mathbb{R}$.

**Example 1:** The Hodgkin–Huxley (HH) model [17] includes two intrinsic ionic currents: a transient sodium current $i_{Na}$ and a potassium current $i_K$, so that $I = \{Na, K\}$. The voltage dynamics of the HH model are given by

$$\dot{c}v = -\frac{\mu_Na m_{Na}^3 h_{Na} (v - \nu_{Na})}{i_{Na}} - \frac{\mu_K m_{K}^4 (v - \nu_{K})}{i_{K}}$$

$$- \mu_L (v - \nu_L) + u$$

the dynamics of $m_{Na}$ and $m_{K}$ are given by (5b), and the dynamics of $h_{Na}$ are given by (5c). \triangle

Two basic properties of a single neuron conductance-based model justify the assumptions of Section II-A. The first property is the existence of a positively invariant set.

**Lemma 1:** Consider the neuronal model (3)–(7), and assume $|u| \leq \pi$ for all $t \geq 0$. Let

$$\nu := \max \left\{ \max_{i \in I} \nu_{ion}, \mu_L^{-1} + \nu_L \right\}$$

$$\rho := \min \left\{ \min_{i \in I} \nu_{ion}, -\pi \mu_L^{-1} + \nu_L \right\}$$

whenever $v(0) \in [\rho, \pi]$, $m_{ion}(0) \in [0, 1]$ and $h_{ion}(0) \in [0, 1]$, it follows that

$$v(t) \in [\rho, \pi], \quad m_{ion}(t) \in [0, 1], \quad \text{and} \quad h_{ion}(t) \in [0, 1]$$

for all $i \in I$ and all $t \geq 0$.

**Proof:** See Appendix B.1.

The second basic property of a conductance-based model is the contraction of its internal dynamics.

**Lemma 2:** For all $i \in I$, the dynamics (5b) are globally exponentially contracting, uniformly in $v \in \mathbb{R}$ and in $\{\rho_{m_{ion}}, \kappa_{m_{ion}}, \zeta_{m_{ion}}, \chi_{m_{ion}}, \nu_{m_{ion}}\}$ on $\mathbb{R}^4$. Exponential contraction holds for any (scalar) constant contraction metric and a contraction rate given by $2 \tau_{m_{ion}}^{-1}$. The same holds analogously for the dynamics (5c).

**Proof:** The Jacobian of the vector field of (5b) is $-\tau_{m_{ion}}^{-1}$. From (7), we see that for any $p > 0$, the inequality

$$-\tau_{m_{ion}}^{-1} p - p \tau_{m_{ion}}^{-1} (v) \leq -2 \tau_{m_{ion}}^{-1} \rho$$

holds for any real $v, \rho_{m_{ion}}, \kappa_{m_{ion}}, \zeta_{m_{ion}}, \chi_{m_{ion}}$.

Finally, we formalize the connection between the single neuron model above and the model structure (1).

**Proposition 1:** Consider the neuronal model (3)–(7). Let

$$\omega := \left( m_{ion1}, h_{ion1}, m_{ion2}, h_{ion2}, \ldots \right)^T$$

and let the parameter vector $\eta$ be composed of any number of elements from the set

$$\bigcup_{i=1}^{n} \{\rho_{i_1}, \kappa_{i_2}, \zeta_{i_3}, \chi_{i_4} \}.$$  \hspace{1cm} (8)

Let $\theta$ be defined according to one of the following parametrizations:

$$\theta := \left( \mu_{ion1}, \mu_{ion2}, \ldots \right)^T,$$  \hspace{1cm} or

$$\theta := c^{-1} \left( 1, \mu_{ion1}, \mu_{ion2}, \ldots \right)^T,$$  \hspace{1cm} or

$$\theta := c^{-1} \left( 1, \mu_{ion1}, \mu_{ion2}, \ldots, \mu_{ionv}, \nu_{ion1}, \mu_{ion2}, \ldots \right)^T.$$  \hspace{1cm} (9)

Then the neuronal model is of the form (1), and it satisfies Assumptions 1 and 2.

**Proof:** Given the above parametrization, it can be verified by inspection that (3)–(7) can be written as (1). It then follows immediately from Lemmas 1 and 2 that the neuronal model satisfies Assumptions 1 and 2.

As Proposition 1 points out, a conductance-based model can be parametrized in a number of ways. The choice of parametrization depends on implicit assumptions about which model constants are known, and which need to be estimated. Estimation of the maximal conductances $\mu_{ion}$ is of particular importance in neurophysiological applications, as they can be
regarded as the key parameters for adaptive control of a neuronal network [9, 10]. Maximal conductances vary greatly under the biochemical action of neuromodulators [28]. In contrast, many other constants in a neuron model may be assumed to be known, but with some level of uncertainty.

Example 2: Assume that the capacitance, maximal conductances, and half-activations of the HH model of Example 1 need to be estimated, while other model constants are known. Then we may define \( w := (m_{Na}, h_{Na}, m_k) \) and
\[
\begin{align*}
\theta &:= c^{-1} \left( 1, \mu_{Na}, \mu_K, \mu_L \right)^T \\
\eta &:= \left( \rho_{m_{Na}}, \rho_{h_{Na}}, \rho_{m_k} \right)^T
\end{align*}
\]
so that the dynamics of the HH model are given by (1), with
\[
\Phi(v, w, u) = - \left( -u, w_2^3 w_2 (v - \nu_{Na}), w_3^3 (v - \nu_K), (v - \nu_L) \right)
\]
\[
A(v) = - \text{diag} \left( \tau_{m_{Na}}^{-1} (v), \tau_{h_{Na}}^{-1} (v), \tau_{m_k}^{-1} (v) \right)
\]
\[
b(v, \eta) = - A(v) \text{col} \left( \sigma_{m_{Na}} (v), \sigma_{h_{Na}} (v), \sigma_{m_k} (v) \right)
\]
and \( a = 0 \).

C. Conductance-Based Neural Network Model

The two basic properties of single neuron models discussed in the previous section extend to conductance-based network models. A conductance-based neural network is given by the interconnection of \( n_v \in \mathbb{N} \) single neurons via synapses. For \( i \in \mathcal{N} := \{1, \ldots, n_v\} \), the voltage dynamics of the \( i \)th neuron in the network is described by
\[
\begin{align*}
\dot{c} v_i &= - I_{L,i} - \sum_{ion \in \mathcal{E}} I_{ion,i} - \sum_{p \in \mathcal{S}} \sum_{i \in \mathcal{S}} I_{syn,p,i} + u_i \quad (9)
\end{align*}
\]
where each \( I_{L,i} \) is given by (4) and each \( I_{ion,i} \) is given by (5), as before (in this case a subscript \( i \) is attached to all variables). The additional currents \( I_{syn,p,i} \) above are synaptic currents interconnecting the \( i \)th (postsynaptic) neuron with the \( p \)th (presynaptic) neuron, so that \( \mathcal{P} \subseteq \mathcal{N} \). Since there might exist multiple synapses (based on different neurotransmitters) connecting two neurons, we denote each synaptic type by \( \text{syn} \), and the index set of synaptic types by \( \mathcal{S} \).

Synaptic currents arise from electrochemical connections between neurons [12, Chapter 7]. We consider the model used in [8, 12], which can be written as
\[
I_{syn,p} = \mu_{syn,p} s_{syn,p} (v - \nu_{syn}) \quad (10a)
\]
\[
\tau_{syn} (v_p) s_{syn,p} = - s_{syn,p} + a_{syn} \tau_{syn} (v_p) \sigma_{syn} (v_p) \quad (10b)
\]
with a synaptic time-constant function \( \tau_{syn} \) given by
\[
\tau_{syn} (v_p) = \frac{1}{a_{syn} \sigma_{syn} (v_p) + b_{syn}} \quad (11)
\]
and a synaptic activation function \( \sigma_{syn} \) of the form (6), with \( \rho_{syn} \in \mathbb{R} \) and \( \kappa_{syn} > 0 \). Here, \( s_{syn,p} \) is the synaptic gating variable, \( v_p \) is the membrane voltage of the presynaptic neuron, and \( a_{syn} > 0 \) and \( b_{syn} > 0 \) are constant parameters. The constants \( \mu_{syn,p} > 0 \) and \( \nu_{syn} \in \mathbb{R} \) are (synaptic) maximal conductances and reversal potentials, respectively. Notice that \( 0 < (a_{syn} + b_{syn})^{-1} \leq \tau_{syn} (v_p) \leq b_{syn}^{-1} \) for all \( v_p \in \mathbb{R} \).

Proposition 2: Consider a conductance-based neural network model with voltage output vector \( v = (v_1, \ldots, v_n) \)^T and internal state vector \( w := \text{col}(w^{(1)}, \ldots, w^{(n)}) \) where
\[
w^{(i)} := \begin{pmatrix} m_{ion,i} \delta_{syn,i} (v_i - \nu_{Na}) - \mu_{K,i} m_{K,i}^4 (v_i - \nu_K) \\ - \mu_{Ca,i} m_{Ca,i}^3 (v_i - \nu_{Ca}) - \mu_{G,p,i} s_{G,p,i} (v_i - \nu_G) \\ - \mu_{L,i} (v_i - \nu_L) + u_i \end{pmatrix}
\]
for \( i, p \in \mathcal{N} = \{1, 2\} \) and \( p \neq i \). The gating variables of each neuron, which evolve according to (5b) and (10b), are collected in \( w^{(i)} = (m_{Na,i}, h_{Na,i}, m_{K,i}, m_{Ca,i}, h_{Ca,i}, s_{G,p,i}) \). Let now
\[
\mu^{(i)} = (\mu_{Na,i}, \mu_{Ca,i}, \mu_{K,i}, \mu_{L,i}, \mu_{G,p,i}, \mu_{L,i}, \mu_{Ca,i}) \quad (12)
\]
for \( i, p \in \mathcal{N} = \{1, 2\} \) and \( p \neq i \). Then, we can parameterize the HCO according to
\[
\theta = \text{col} \left( \mu^{(1)}, \mu^{(2)} \right) \quad (13)
\]
with \( \mu^{(1)} \) and \( \mu^{(2)} \) given by (12). Letting \( v = (v_1, v_2)^T \) and \( w = \text{col}(w^{(1)}, w^{(2)}) \), the voltage dynamics of the model can then be written as (1a), where
\[
\Phi(v, w) = \begin{bmatrix} \varphi (u_1, w^{(1)}) & 0 \\ 0 & \varphi (u_2, w^{(2)}) \end{bmatrix}
\]
\[
a(t) = (u_1(t)/c_1, u_2(t)/c_2)^T
\]
with
\[
\varphi (v_i, w^{(i)}) = - \frac{1}{c_i} \begin{pmatrix} m_{Na,i}^3 h_{Na,i} (v_i - \nu_{Na}) \\ m_{K,i}^4 (v_i - \nu_K) \\ m_{Ca,i}^3 h_{Ca,i} (v_i - \nu_{Ca}) \\ s_{G,p,i} (v_i - \nu_G) \\ v_i - \nu_L \end{pmatrix}
\]
for \( i = 1, 2 \) and \( p \neq i \).

\footnote{Gamma-aminobutyric acid (GABA) is a neurotransmitter associated with inhibitory synapses.}
III. ESTIMATION OF THE OUTPUT DYNAMICS

In this section, we simplify the problem statement of Section II-A by considering the case in which there is no uncertain parameter \( \eta \), that is, the internal dynamics are assumed to be perfectly known. Hence, we consider the simplified model

\[
\dot{v} = \Phi(v,w,u)\theta + a(v,w,u) \tag{14a}
\]
\[
\dot{w} = A(v)w + b(v) \tag{14b}
\]
\[
\dot{\theta} = 0 \tag{14c}
\]
satisfying Assumptions 1 and 2. This simplified model already deserves attention. Using the basic properties of Section II, we propose a least squares method in Section III-A, and a RLS-based adaptive observer in Section III-B.

A. Least Squares and Reduced-Order Observer

A simple least squares solution to the problem of estimating the parameters of (14) exploits the following observation.

**Remark 2:** Assumption 2 implies that the system

\[
\dot{\hat{w}} = A(v)\hat{w} + b(v) \tag{15}
\]

is a globally exponentially convergent reduced-order identity observer for the dynamics (14b). More precisely, as \( t \to \infty \) we have \( \hat{w}(t) \to w(t) \) for any piecewise continuous \( w(t) \) and any initial conditions \( \hat{w}(0), w(0) \in \mathbb{R}^n_w \).

We employ the reduced-order observer (15) to obtain estimates \( \hat{w} \) of the internal states \( w \). Contraction of the internal states suggests postulating the predictor model

\[
\dot{\hat{v}} = \Phi(v,\hat{w},u)\hat{\theta} + a(v,\hat{w},u) \tag{16a}
\]
\[
\dot{\hat{w}} = A(v)\hat{w} + b(v) \tag{16b}
\]

which, for \( \hat{w}(0) = w(0) \), reduces to a continuous-time *equation-error* model structure [24], [42]. Classical system identification theory [42, Sec. 2] thus suggests performing parameter estimation by solving the regularized problem

\[
\hat{\theta}(T) = \min_{\theta} V(\hat{\theta}, T) + \hat{\theta}^T R_0(T) \hat{\theta} \tag{17}
\]

with the weighted cost function

\[
V(\hat{\theta}, T) = \frac{1}{T} \int_0^T e^{-\alpha(T-\tau)} \|H\hat{v}(\tau) - H\dot{\hat{v}}(\tau)\|^2 d\tau \tag{18}
\]

where \( \alpha > 0 \) is a *forgetting factor*, introduced to discount the initial error between \( w(0) \) and \( \hat{w}(0) \), \( R_0 \) is a symmetric positive semidefinite matrix, and \( H \) is the operator of a strictly proper LTI filter introduced to avoid differentiating \( v(t) \). Choosing the simple filter

\[
H(s) = \frac{\gamma}{s + \gamma} \tag{19}
\]

leads to

\[
H\dot{v}(t) = \Psi(t)\hat{\theta} + H\hat{a}(t) \tag{20a}
\]
\[
\dot{\Psi}(t) = -\gamma\Psi(t) + \gamma\Phi(v(t),\hat{w}(t),u(t)) \tag{20b}
\]
\[
\dot{\hat{a}}(t) = a(v(t),\hat{w}(t),u(t)) \tag{20c}
\]

which shows that (17)–(18) is now quadratic in \( \hat{\theta} \) (notice \( H\dot{v} \) is obtained by filtering the data \( v \) with \( sH(s) \)). It follows that the batch problem (17) admits a well-known solution based on the normal equation [42, p. 55]. That \( \hat{\theta}(T) \to \theta \) as \( T \to \infty \) will be shown to be a consequence of the convergence properties of the adaptive observer introduced in Section III-B.

B. RLS-Based Adaptive Observer

Consider the system (14). An adaptive observer for this system is given by

\[
\dot{\hat{v}} = \Phi(v,\hat{w},u)\hat{\theta} + a(v,\hat{w},u) + (\gamma I + \Psi P \Psi^T)(v - \hat{v}) \tag{19a}
\]
\[
\dot{\hat{w}} = A(v)\hat{w} + b(v) \tag{19b}
\]
\[
\dot{\hat{\theta}} = \gamma P \Psi^T (v - \hat{v}) \tag{19c}
\]

where \( \gamma > 0 \) is a constant gain, and the matrices \( P \) and \( \Psi \) evolve according to

\[
\hat{\Psi} = -\gamma \Psi + \gamma \Phi(v,\hat{w},u), \quad \Psi(0) = 0 \tag{20a}
\]
\[
\hat{\dot{P}} = \alpha P - P \Psi^T \Psi P, \quad P(0) > 0 \tag{20b}
\]

where \( \alpha > 0 \) is a constant forgetting factor. The assumption that \( \Psi(0) = 0 \) is made without loss of generality.

**Remark 3:** The adaptive observer (21) and (22) relates to a number of designs in the literature. For instance, when we remove the internal dynamics \( (n_w = 0) \) and set \( \alpha = \gamma \), then (21) and (22) are similar to the high-gain design proposed in [13]. Also, if \( w(t) \) is assumed to be known, then by replacing \( \hat{w} \) by \( w(t) \) in (21) we recover a nonlinear variant of the classical linear design of [41]. Finally, setting \( P = I \) and \( \Psi = \Phi \), and removing the adaptive gain and its dynamics (22), it reduces to the design proposed in [3], which can be thought of as being based on the least-mean squares algorithm rather than RLS.

We will show that the convergence of the adaptive observer above does not require a high gain; a discussion on the benefits of tuning \( \alpha \) and \( \gamma \) will also be presented in Section IV-C. Furthermore, we can prove the design (21) and (22) is directly connected to the least squares problem discussed earlier:

**Proposition 3:** The adaptive observer (21) and (22) implements the RLS solution of the least squares problem (17) and (20), with \( R_0(T) = e^{-\alpha T} P^{-1}(0)/T \).

**Proof:** See Appendix B.2.

To show exponential convergence of the adaptive observer, we require a standard persistent excitation condition (see, for instance, [3], [13], [39]).

**Definition 1:** A time-varying matrix \( M(t) \) is said to be persistently exciting (PE) if there exist \( T > 0 \) and \( \delta > 0 \) such that for all \( t \geq 0 \), we have

\[
\int_t^{t+T} M(\tau)M(\tau)^T d\tau \geq \delta I.
\]

**Assumption 3:** The signals \( v(t) \) and \( u(t) \) are such that for any trajectory of (21), the matrix \( \Psi(t)M(t)^T \) is persistently exciting.

It is well-known [41] that Assumption 3 ensures uniform positive-definiteness of \( P(t) \). In our context, we have:

**Lemma 3:** Under Assumptions 1, 2, and 3, the solution \( \Psi(t) \) of (22a) is bounded for all \( t \geq 0 \), and \( P(t) \) is bounded and
uniformly positive definite for all $t \geq 0$. In particular,
\begin{equation}
0 \leq \underline{p}I < P(t) \leq \overline{p}I
\end{equation}
for all $t \geq T$, with
\begin{equation}
\underline{p} = \left( \|P^{-1}(0)\| + \alpha^{-1} \overline{\phi} \right)^{-1}
\end{equation}
and
\begin{equation}
\overline{p} = \delta^{-1} \epsilon \alpha T
\end{equation}
with $\overline{\phi} = \sup_{v \in V, u \in U} \Phi(v, u)$.

\textbf{Proof:} See Appendix B.3.

We can now state a global convergence result\(^3\) for the simple adaptive observer (21) and (22).

\textbf{Theorem 1:} Consider the systems (14) and (21)-(22), and let Assumptions 1, 2, and 3 hold. Let $\gamma > 0$ and $\alpha > 0$. Then, globally, we have
\begin{equation}
\text{col}(\hat{v}(t), \hat{w}(t), \hat{\theta}(t)) \to \text{col}(v(t), w(t), \theta)
\end{equation}
exponentially fast as $t \to \infty$, with a convergence rate given by arbitrary $\lambda < \min\{\alpha, \lambda_w, \gamma\}$.

\textbf{Proof:} See Appendix B.4.

One should notice that the persistent excitation Assumption 3 is classical yet difficult to check in practice, since it depends on system trajectories of a nonlinear system. However, the excitable (spiking) behavior of the neuronal circuits considered in this article is an excellent source of excitation that can be reliably tapped through the application of superthreshold applied currents.

\section{IV. Estimation Under Uncertainty}

The design in the previous section assumes no uncertainty in the model, which is unrealistic in a biophysical context. This section addresses different forms of uncertainty. Section IV-A deals with \textit{structured uncertainty} in the internal dynamics, modeled by the uncertain parameter $\eta$ in (1b). Building on the design (21) and (22), we present a locally convergent adaptive observer capable of estimating $\eta$ in addition to the unknown parameters $\theta$ in (1a). Section IV-B then discusses the problem of measurement errors and how the adaptive observer can be modified to mitigate that problem. Finally, Section IV-C discusses the robustness of the adaptive observers with respect to unstructured uncertainty.

\subsection{A. Estimating Uncertain Internal Dynamics Parameters}

To deal with structured uncertainty in the internal dynamics of (1), we augment the simple adaptive observer (21) to estimate the parameter vector $\eta$ as well. To design the observer the following is assumed.

\textbf{Assumption 4:} Assume that compact sets $\Theta \subset \mathbb{R}^{n_\theta}$ and $H \subset \mathbb{R}^{n_\eta}$ are known such that $\theta \in \Theta$ and $\eta \in H$.

\textbf{Remark 4:} Analogously to Remark 1, under Assumptions 1 and 4, we can assume without loss of generality that for all $v \in V$, the functions $A(v, \eta)$ and $b(v, u, \eta)$ are globally Lipschitz bounded in $\eta \in \mathbb{R}^{n_\eta}$.

\(^3\)Theorem 1 can also be proven with classical (nondifferential) Lyapunov arguments, in the fashion of [13], [41]. Our proof relies instead on contraction (differential) analysis, which is useful to the results of the following section.

The augmented adaptive observer is given by
\begin{equation}
\dot{\hat{v}} = \Phi(v, \hat{w}, \hat{u})\hat{\theta} + a(v, \hat{w}, \hat{u}) + (\gamma I + \Psi_v P \Psi_v^T)\hat{v}
\end{equation}
\begin{equation}
\dot{\hat{w}} = A(v, \hat{\eta})\hat{w} + b(v, \hat{\eta}) + \Psi_w P \Psi_v^T\hat{v}
\end{equation}
\begin{equation}
col\left(\hat{\theta}, \hat{\eta}\right) = \gamma P \Psi_v^T\hat{v}
\end{equation}
where $\gamma > 0$ is a constant gain, and the matrices $P$ and $\Psi := \text{col}(\Psi_v, \Psi_w)$ evolve according to
\begin{equation}
\dot{\Psi} = A_\Psi(t)\Psi + \gamma B_\Psi(t)
\end{equation}
\begin{equation}
\dot{P} = \alpha P + \beta I - P \Psi_v^T\Psi_v P
\end{equation}
Here, $\alpha > 0$ and $\beta \geq 0$ are constant hyperparameters, and the matrix functions in (22a) are given by
\begin{equation}
A_{\Psi}(t) = \begin{bmatrix}
-\gamma I & 0_{n_w \times n_v} & A(v, \hat{\eta}) \\
0_{n_w \times n_v} & 0_{n_w \times n_\eta} & \partial_\eta[A(v, \hat{\eta})] \\
0_{n_w \times n_\eta} & 0_{n_w \times n_\eta} & b(v, \hat{\eta})
\end{bmatrix}
\end{equation}
and
\begin{equation}
B_\Psi(t) = \begin{bmatrix}
\Phi(v, \hat{w}, \hat{u}) & 0_{n_w \times n_v} \\
0_{n_w \times n_v} & \partial_\eta[\Phi(v, \hat{w}, \hat{u})] \\
0_{n_w \times n_\eta} & 0_{n_w \times n_\eta}
\end{bmatrix}
\end{equation}
with $\zeta_\Phi$ and $\zeta_\Theta$ smooth saturation functions (see Remark 1). Here, we assume without loss of generality that $\Psi_v(0) = 0$, $\Psi_w(0) = 0$, and $P(0) > 0$.

As before we need a persistent excitation condition:

\textbf{Assumption 5:} The signals $v(t)$ and $u(t)$ are such that for any trajectory of (25), the matrix $\Psi_v(t)^T$ is persistently exciting.

The following result now parallels Lemma 3:

\textbf{Lemma 4:} Under Assumptions 1, 2 and 4 the solution $\Psi(t)$ of (26a) is bounded for all $t \geq 0$. In addition, under Assumption 5, the solution $V(t)$ of (26b) is bounded and uniformly positive definite for all $t \geq 0$. In particular,
\begin{equation}
0 < \underline{p}I < P(t) < \overline{p}I
\end{equation}
for all $t \geq T$, with
\begin{equation}
\underline{p} = \left( \|P^{-1}(0)\| + \alpha^{-1} \overline{\phi} \right)^{-1}
\end{equation}
and
\begin{equation}
\overline{p} = \delta^{-1} \epsilon \alpha T (1 + \beta \delta^{-1} \alpha^{-3} \epsilon^2 \alpha T \epsilon^4)
\end{equation}
with $\epsilon$ a constant independent of $\alpha, \beta, \gamma$.

\textbf{Proof:} See Appendix B.5.

To state our main result, we gather the variables of (1) in
\begin{equation}
x(t) := \text{col}(v(t), w(t), \theta(t), \eta(t))
\end{equation}
and the variables of (25) in
\begin{equation}
x(t) := \text{col}(\hat{v}(t), \hat{w}(t), \hat{\theta}(t), \hat{\eta}(t)).
\end{equation}
We shall prove the result with contraction analysis, using the contraction metric given by
\begin{equation}
M(t) = T(t)^T M(t) T(t)
\end{equation}
with
\begin{equation}
T = \begin{bmatrix}
I & -\Psi \\
0 & \gamma
\end{bmatrix}, \quad M = \begin{bmatrix}
\varepsilon I & 0 \\
0 & M_w
\end{bmatrix}
\end{equation}
and \( \varepsilon > 0 \) (we use lines to delimit block submatrices of the same size). We use this contraction metric to define the set

\[
B(t; \rho) = \left\{ z : \| z - \hat{x}(t) \|_{M(t)} \leq \rho \right\}
\]

for arbitrary \( \rho > 0 \). We now have:

**Theorem 2:** Let Assumption 1, 2, 4, and 5 hold, and let \( \alpha > 0 \), \( \beta \geq 0 \), and \( \gamma > 0 \). Then there exist \( m, \overline{m} > 0 \) such that

\[
0 < mI \preceq M(t) \preceq \overline{m}I
\]

for all \( t \geq 0 \), and there is a constant \( r > 0 \) such that if

\[
x(0) \in B(0; r\overline{m})
\]

then \( x(t) \in B(t; r\overline{m}) \) for all \( t \geq 0 \). Furthermore, if (35) holds, \( \hat{x}(t) \to x(t) \) as \( t \to 0 \), exponentially fast, with rate \( \lambda > \min\{\alpha, \lambda_w, \gamma\} \).

**Proof:** See Appendix B.6.

**Remark 5:** The observer above extends the simpler observer (21) and (22). Indeed, if \( A \) and \( b \) are independent of \( \eta \), then we recover (21) and (22) from (25)–(26): in this case, \( \partial_{\theta}[A(v)\Psi_w(\hat{w}) + b(v)] = 0 \), and \( \Psi_w = A(v)\Psi_w \). Then, if \( \Psi_w(0) = 0 \) we have \( \Psi_w(t) = 0 \) for all \( t \geq 0 \) and (21), (22) is recovered. If \( \Psi_w(0) \neq 0 \), then \( \Psi_w \to 0 \) as \( t \to \infty \) by Assumption 2, and the simpler observer is also recovered.

**Remark 6:** The initial condition assumption (35) implies that \( \| x(0) - \hat{x}(0) \| \leq r \), while the proof of Theorem 2 (in particular, inequality (61) and the argument that follows it) shows that the radius \( r \) is larger for \( \beta > 0 \) than for \( \beta = 0 \). This motivates the inclusion of the term \( \beta I \) in the update rule dynamics (26).

**Remark 7:** Writing \( \Psi_w = \begin{bmatrix} \Psi_{w,1} & \Psi_{w,2} \end{bmatrix} \) with \( \Psi_{w,1} \in \mathbb{R}^{n_w \times n_\theta} \) and \( \Psi_{w,2} \in \mathbb{R}^{n_w \times n_\eta} \), by Assumption 2 we have without loss of generality that \( \Psi_{w,1} = 0 \). Furthermore, Assumption 5 is in a sense also a condition on the excitation of \( \Psi_{w,2} \). To see this, write \( \Psi_v = \begin{bmatrix} \Psi_{v,1} & \Psi_{v,2} \end{bmatrix} \) analogously. Then from (26a), the dynamics of \( \Psi_{v,1} \) are solely driven by \( \Phi(v, \hat{w}, u) \), while the dynamics of \( \Psi_{v,2} \) are solely driven by \( \partial_{\hat{w}}[\Phi(v, \hat{w}, u)\Theta(\hat{\theta}) + a(v, \hat{w}, u)\Psi_{w,2}] \). Thus, part of the persistent excitation of \( \Psi_v \) is directly due to \( \Psi_{w,2} \).

**B. Measurement Errors**

The adaptive observer design of Section IV-A relies on output injection, that is, it assumes that the measurement \( y = v \) has no errors, and injects the measured \( \hat{v} \) in the observer dynamics. This corresponds to an equation error model structure [24]. Assume instead that a measurement error \( \epsilon(t) \) is present, so that

\[
y(t) = v(t) + \epsilon(t).
\]

In this case, (25) and (26) must be redefined by replacing \( v(t) \) with \( y(t) \). It is clear that this introduces measurement errors in the observer dynamics, and it is well known that even if some level of stability is retained, the parameter estimates will be biased. If the bias is too large, one could modify (25) and (26) toward an output error model structure by replacing \( v(t) \) with \( \hat{v}(t) \) in the arguments of \( \Phi, A, a, \) and \( b \). The downside of the output-error approach is that convergence of the adaptive observer may be lost even when the measurements have no errors (the nominal case \( y = v \)). More precisely, a convergence analysis analogous to that of Theorem 2 shows that nominal stability of the output-error-based adaptive observer only holds for sufficiently high values of \( \gamma \). But a high \( \gamma \) is undesirable when measurement errors do occur, as \( \gamma \) contributes to perturbations in the dynamics coming from \( \gamma I e \) and \( \gamma P\Psi_v^T e \).

To leverage the advantages of both equation error and output error approaches (lower gain \( \gamma \) and lower bias, respectively), we can exploit an additional property of the true system, motivated by the neural systems of Section II-B.

**Assumption 6:** Under Assumption 1, there exist a symmetric positive definite matrix \( M_v > 0 \) and a contraction rate \( \lambda_v > 0 \) such that

\[
\partial_v[\Phi(v) + a(v)]^T M_v + M_v \partial_v[\Phi(v) + a] \leq -\lambda_v M_v
\]

for all \( \{v, w, \theta\} \in V \times W \times \{\theta(0)\} \).

**Remark 8:** For any conductance-based model from Section II-B, Assumption 6 holds with \( M_v = I \) and \( \lambda_v = -2\mu_L/c \).

Assumption 6 motivates the observer structure given by

\[
\dot{\hat{v}} = \Phi(\hat{v}, \hat{w}, u)\hat{\theta} + a(\hat{v}, \hat{w}, u) + (\gamma I + \Psi_v P\Psi_v^T)(y - \hat{v})
\]

\[
\dot{\hat{w}} = A(y, \hat{\eta})\hat{w} + b(y, \hat{\eta}) + \Psi_w P\Psi_v^T(y - \hat{v})
\]

And (26), where \( A_\Phi \) and \( B_\Phi \) are replaced by

\[
A_\Phi(t) = \begin{bmatrix} -\gamma I + \partial_\hat{\theta}[\Phi(\hat{\theta}) + a] & \partial_\hat{w}[\Phi(\hat{\theta}) + a] \\ 0_{n_w \times n_v} & \Phi \end{bmatrix}
\]

and

\[
B_\Phi(t) = \begin{bmatrix} \Phi & 0_{n_v \times n_\eta} \\ 0_{n_w \times n_\eta} & \partial_\eta[\Phi A(y, \eta)\Theta(\hat{\theta}) + b(y, \eta)] \end{bmatrix}
\]

where \( \Phi = \Phi(\hat{v}, \hat{w}, u) \) and \( \hat{a} = a(\hat{v}, \hat{w}, u) \). The following nominal convergence result is immediate.

**Theorem 3:** Under Assumption 6, for \( y = v \), the statement of Theorem 2 also applies to the adaptive observer above.

**Proof:** The proof follows the very same steps as that of Theorem 2, and is hence omitted.

**1) Robustness to Measurement Errors:** The contraction results of Theorems 1, 2, and 3 imply a nominal exponential stability property of the adaptive observer trajectories. Given those results, it is not difficult to show that the convergence properties of the adaptive observers presented above all have some level of robustness with respect to measurement errors of the form (36). A contraction-based robustness analysis along the lines of [4, Sec. III] can be performed to show that for a bounded error \( \epsilon(t) \) and sufficiently small sup\( t > 0 \)\( \| \epsilon(t) \| \), the trajectories of the adaptive observer estimates remain close to the trajectories of the true system states.

**C. Robustness to Unstructured Uncertainty**

It is well known that with exponential contraction comes robustness with respect to small perturbations [4], [25]. In our context, consider a perturbed version of the true system (1), given by

\[
\dot{\hat{v}} = \Phi(v, w, u)\theta + a(v, w, u) + d_w(t, v, w, \theta)
\]

\[
\dot{\hat{w}} = A(v, \eta)w + b(v, \eta) + d_w(t, v, w)
\]

\[
\dot{\hat{\theta}} = d_\theta(t, v, w)
\]

\[
\hat{\eta} = d_\eta(t, v, w)
\]

(37)
where \( d := \text{col}(d_u, d_w, d_\theta, d_\eta) \) models an unstructured uncertainty. The disturbances can be interpreted as model mismatch resulting from unmodeled dynamics, as well as time variation in the true parameters. We assume that the assumptions of Theorem 2 hold for the perturbed system (37), with the set \( \{\theta(t)\} \times \{q(t)\} \) in Assumption 1 replaced by \( \Theta \times H \) from Assumption 4, and that \( \|d\| \leq \bar{d} \) for all \( t \geq 0 \). We gather the state variables of (37) and (25) in \( x = \text{col}(v, w, \theta, \eta) \) and \( \hat{x} = \text{col}(\hat{v}, \hat{w}, \hat{\theta}, \hat{\eta}) \), respectively.

In the proof of Theorem 2, we have shown that there exists an \( r > 0 \) such that the set \( B(t; r/\sqrt{M}) \) given by (33) is contained at all times in a region of contraction with respect to a virtual system containing the trajectories of the nominal true system (1) and of the adaptive observer (25). In this situation, just as in [4, Sec. III], we can show that if the state \( x(0) \) of the perturbed true system (37) belongs to \( B(0; r/\sqrt{M}) \), then

\[
\|x(t) - \hat{x}(t)\| \leq \sqrt{M} \left( e^{-\frac{r}{\lambda}t}\|x(0) - \hat{x}(0)\| + \frac{2\bar{d}}{\lambda} \right) \tag{38}
\]

for as long as the perturbed state \( x(t) \) remains in the contraction region (see also [6]). For small enough \( \bar{d} \), this holds for all \( t \geq 0 \), as the state \( x(t) \) will remain in \( B(t; r/\sqrt{M}) \).

The maximum permissible \( \bar{d} \) can hence be understood as a robustness margin. This (possibly conservative) margin allows for a simple interpretation of the effects that the hyperparameters will have on the robustness of the algorithm. Notice from (31–32), we have

\[
\frac{\bar{M}}{\bar{m}} \leq \frac{\bar{m}}{\mu} \sup_{t \geq 0} \|T(t)\|^2 \leq \frac{\bar{m}}{\mu \inf_{t \geq 0} \sigma_{\text{min}}(T(t))}\n\]

where \( \mu I \leq \tilde{M}(t) \leq \bar{M}I \). Then, from (27) and (32), for \( t \geq T \) we have

\[
\frac{\bar{m}}{\mu} = \frac{\max\{\varepsilon, 1, \varepsilon(\gamma p)^{-1}\}}{\min\{\varepsilon, \lambda_{\text{min}}[M_w, \varepsilon(\gamma p)]^{-1}\}} \tag{39}\]

where \( p \) and \( \tau \) are given by (28) and \( \varepsilon \) is given by (60). The maximum permissible \( \bar{d} \) must decrease when \( \bar{M}/\mu \) increases. Hence, considering a fixed contraction rate \( \lambda < \min\{\alpha, \lambda_w, \gamma\} \), we can extract a few points about the values of the hyperparameters.

First, large values of \( \alpha \) are undesirable, since \( \bar{p}/\mu \) increases exponentially with \( \alpha \). Thus, there is a tradeoff between quickly keeping track of time-varying parameters and robust stability. Second, since the quantities \( \varepsilon \) and \( \varepsilon/\gamma \) are monotonically increasing in \( \gamma, \gamma \) can be increased (up to a point) to enhance robustness. Additionally, notice that the proof of Theorem 2 shows through (61) that increasing \( \gamma \) allows for a larger \( r > 0 \), which also promotes robustness.

Notice these points also apply to the simple adaptive observer (21), with the caveat that the virtual system was in that case proven to be globally contracting, and hence (38) applies for any \( \bar{d} \). Finally, although we have not analyzed the effect of \( \beta \geq 0 \) above, it should be mentioned that the addition of a moderate \( \beta > 0 \) term in (26b) can be regarded as a form of covariance inflation, which has been noted to improve in practice the robustness of Kalman filter-based methods [15]. This effect was observed in the simulations that follow.

V. APPLICATION TO CONDUCTANCE-BASED MODELS

In this section, we illustrate with numerical simulations how the system theoretic adaptive observers discussed in this article perform when applied to problems in electrophysiology.\(^4\)

A. Estimation of Voltage and Ion Channel Dynamics

The primary goal of neuronal system identification is to estimate maximal conductances [5], [11], [18], [30]. An important (and often ignored) point in this approach is the fact that the parameters in ionic channel models are only approximate in nature, and in practice may vary from neuron to neuron. Using the adaptive observer (25) and (26), we now illustrate the real-time estimation of unknown and uncertain parameters of the Hodgkin–Huxley model of Examples 1 and 2.

1) Perfect Measurements: We begin by verifying the behavior of the adaptive observer when no measurement errors are present. We use the biophysical parameters in Appendix C.1. This results in

\[
\theta(t) = \theta(0) = \left(1, 120, 36, 0.3\right)^T
\]

\[
\eta(t) = \eta(0) = \left(-40, -62, -53\right)^T
\]

By contrast, we initialize the observer with

\[
\hat{\theta}(0) = \left(2, 78, 78, 10\right)^T
\]

\[
\hat{\eta}(0) = \left(-20, -20, -20\right)^T
\]

which represents a parsimonious guess over the parameters of an unknown and uncertain spiking conductance-based model. The true HH model and the adaptive observer were simulated subject to the input \( u(t) = 10 + \sin(2\pi t/10) \) for \( t \geq 0 \). The initial conditions of the voltage and gating variables are given by

\[
\text{col}(v(0), w(0)) = (-30, 0.5, 0.5, 0.5)^T \quad \text{and} \quad \text{col}(\hat{v}(0), \hat{w}(0)) = (-30, 0, 0, 0)^T
\]

and the remaining initial conditions of the observer are given by \( \Psi_u(0) = 0, \Psi_w(0) = 0, P(0) = I \). For \( \alpha = 0.1 \) and \( \beta = \gamma = 1 \), the solutions of the true system and of the adaptive observer can be seen in Fig. 2. All the parameter estimates of the adaptive observer converge to the true parameter values. In accordance with Remark 6, changing \( \beta \) to \( \beta = 0 \) while keeping the same initial conditions results in the loss of convergence of the parameter estimates.

2) Measurement Errors: Keeping the same input and true system parameters used in the previous section, we now simulate the behavior of the adaptive observer when zero-mean white Gaussian noise of variance \( \sigma^2_{\text{noise}} = 4 \text{ mV}^2 \) is added to the measured \( v \). Using the rms amplitude \( v_{\text{rms}} \approx 27 \text{ mV} \) of the noise-free voltage trace simulated in the previous section, this noise corresponds to a relatively poor signal-to-noise-ratio of \( 10 \log_{10} \frac{v_{\text{rms}}^2}{\sigma^2_{\text{noise}}} \approx 22 \text{ dB} \). We first try the same observer parameters \( \alpha = 0.1 \) and \( \beta = \gamma = 1 \) that led to convergence in the previous section. The result for the worst affected estimate \( \hat{\theta}_2 \) is shown in Fig. 3 (top). While the estimate remains bounded, the

\(^4\)The Julia code used to generate these results can be found on https://github.com/thiagoburghi/online-learning.
Section IV-B to estimate the parameters $\theta$ of the HCO neuronal circuit introduced in Example 3.

Remark 9: When applied to a conductance-based network, the network observers decouple into $n_v$ independent single neuron observers. This is because in a conductance-based network, $\Phi$ is block-diagonal and, by stability of the dynamics of $\Psi$ in (22a) or (26a) and of $R := P^{-1}$ in (42), we can without loss of generality ignore all off-block diagonal terms of the matrices $\Psi(t)$ and $P(t)$.

Following Remark 9, applying the observer of Section IV-B to the HCO of Example 3 yields

$$\dot{v}_i = \varphi_i(\hat{v}_i, \hat{w}_i)\hat{\mu}(i) + e_i^{-1}u_i + (\gamma + \psi_i P_i \hat{\psi}_i^T)(y_i - \hat{v}_i)$$

$$\dot{\hat{\mu}}(i) = \gamma P_i \hat{\psi}_i^T (y_i - \hat{v}_i)$$

$$\hat{\psi}_i = (-\gamma I + \partial_i[\varphi_i(\hat{v}_i, \hat{w}_i)\hat{\mu}(i)])\psi_i + \gamma \varphi_i(\hat{v}_i, \hat{w}_i)$$

$$\dot{P}_i = \alpha P_i + \beta I - P_i \hat{\psi}_i^T \hat{\psi}_i P_i$$

where $P_i(0) > 0$, $i \in \mathcal{N} = \{1, 2\}$, and $y_i = v_i + e_i$. As in the previous section, we define the measurement errors $e_1$ and $e_2$ as white Gaussian noise with $\sigma_{noise} = 4 \text{ mV}^2$.

To illustrate the importance of tracking time-varying parameters, we consider the problem of neuromodulation [28]. Neuromodulators are substances that continuously modulate the opening of ion channels in a neuron’s membrane. This modulatory control can be modeled as a temporal variation of the maximal conductances in a conductance-based model [10]. Here, we consider the case in which the calcium maximal conductances $\mu_{Ca,i}(t)$ and $\mu_{Ca,2}(t)$ of the true HCO model are slowly varied in time, something that is known to change the bursting frequency of the HCO model [8]. A gradual increase in the concentration of calcium ion channels is simulated by

$$\mu_{Ca,1}(t) = \mu_{Ca,2}(t) = 0.11 + \frac{0.07}{1 + \exp\left(-\frac{t - T_f/2}{1250}\right)}$$

where $T_f = 10^4$ milliseconds is the length of the simulation. For $i \in \{1, 2\}$, the remaining maximal conductances of the true HCO model are given by $\mu_{Na,i} = 60$, $\mu_{K,i} = 40$, $\mu_{L,i} = 0.035$, and $\mu_{G,2,i} = 4$.

In the observer above, the reversal potentials, capacitances, activation functions, and time-constant functions are defined according to the nominal parameters of the true model detailed in Appendix A (where initial conditions are also detailed). To simulate an unknown disturbance $d_w$ in the true internal dynamics (see Section IV-C), a random disturbance of at most 1% (following the uniform distribution) is applied to every internal dynamics parameter of the true system.

Fig. 4 illustrates the resulting voltage traces of the true (perturbed) HCO model. The neuromodulatory action on the calcium conductance increases the number of spikes in each burst. For a forgetting rate of $\alpha = 0.0025$, observer gains of $\beta = 0$ and $\gamma = 0.1$, and a constant input $u_1(t) = u_2(t) = -0.65 \, \text{mA/cm}^2$, Fig. 5 shows the trajectories of some of the true and estimated maximal conductances. It can be seen that the estimates converge toward a region close to the true parameters, illustrating the robustness of the convergence property of the observer. The bias

measurement noise considerably affects its convergence properties. However, tuning the observer parameters to $\alpha = 10^{-4}$, $\beta = 10$, $\gamma = 0.1$ and $P(0) = 0.1I$ drastically improves the result, as shown in Fig. 3 (bottom). The oscillation in $\theta_2(t)$ is now less pronounced, and it converges more slowly to a region close to the true $\theta_2$. Similar behaviours hold for the less affected parameters. Comparing the two cases, there is a clear tradeoff between convergence rate and robustness to measurement noise.

B. Estimation of a Neural Circuit Under Neuromodulation

In this section, we illustrate the robustness to noise, model mismatch, and time-varying parameters by using the observer of
adverse control of neuronal maximal conductances [36], opening the way for innovative neurophysiology research. Future work will explore the benefits and limitations of the method in an experimental context.

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**APPENDIX A**

**A. Contraction Analysis**

The system dynamics

\[ \dot{x} = f(x, u) \quad (41) \]

is said to be *exponentially contracting* [25] in \( x \) on \( X \subset \mathbb{R}^n \), uniformly in \( u \) on \( U \subset \mathbb{R}^m \), if there exist a continuously differentiable symmetric matrix \( P(x, t) \), called the contraction metric, and a constant \( \lambda > 0 \), called the contraction rate, such that \( \epsilon_1 I \leq P(x, t) \leq \epsilon_2 I \) for some \( \epsilon_1, \epsilon_2 > 0 \), and \( \partial_x f^T P + P \partial_x f + \dot{P} \leq \lambda P \) for all \( t \geq 0 \), all \( x \in X \), and all \( u \in U \). The set \( X \subset \mathbb{R}^n \) is said to be *positively invariant* with respect to the dynamics (41), uniformly in \( u \) on \( U \subset \mathbb{R}^m \), if \( x(0) \in X \) and \( x(t) \in X \) for all \( t \geq 0 \). It is a well-known fact that if the dynamics (41) are exponentially contracting on a convex positively invariant set \( X \), then all solutions of that system starting in \( X \) converge toward each other exponentially fast, with rate \( \lambda \) (for a proof of this statement, see for instance [20, Lemma 1]).

**B. Proofs**

1) **Proof of Lemma 1**: We begin by noticing that \([0,1] \) is a positively invariant set for (5b) and for (5c), uniformly in \( v \) on \( \mathbb{R} \). This is because the image of the sigmoid (6) is \([0,1] \), which implies none of the gating variables \( m_\text{ion} \) and \( h_\text{ion} \) can leave the set \([0,1] \): for instance, \( m_\text{ion} \geq 0 \) for \( m_\text{ion} = 0 \) and all \( v \in \mathbb{R} \), and \( m_\text{ion} \leq 0 \) for \( m_\text{ion} = 1 \) and all \( v \in \mathbb{R} \). Now, assuming \( m_\text{ion}(0) \in [0,1] \) and \( h_\text{ion}(0) \in [0,1] \), we have \( m_\text{ion} \leq 0 \) for all \( x \in \mathcal{X} \) and all \( t \geq 0 \). This in turn implies \( v \) cannot leave the interval \([\bar{v}, \overline{\bar{v}}] \), which can be verified by inspection of (3)–(5a): if \( v = \bar{v} \), then \( \dot{v} \leq 0 \), whereas if \( v = \overline{\bar{v}} \), then \( \dot{v} \geq 0 \).

2) **Proof of Proposition 3**: The normal equation of the LS problem (17)–(20) with \( R_0(T) = e^{-\alpha T} P^{-1}(0) / T \) is

\[
R(T) \hat{\theta}(T) = \int_0^T e^{-\alpha(T-\tau)} \Psi(\tau)^T (H \dot{\hat{v}}(\tau) - H \hat{u}(\tau)) d\tau,
\]

where

\[
R(t) = e^{-\alpha t} P^{-1}(0) + \int_0^t e^{-\alpha(t-\tau)} \Psi(\tau)^T \Psi(\tau) d\tau. \quad (42)
\]

Differentiating the normal equation by \( T \) and evaluating at \( t \), we obtain the RLS solution

\[
\hat{\theta}(t) = P(t) \Psi(t)^T \left( H \dot{\hat{v}}(t) - \Psi(t) \hat{\theta}(t) - H \hat{u}(t) \right).
\]
Thus, (21)–(22) implements the RLS solution if and only if
\[ H\dot{v}(t) - \Psi(t)\dot{\theta}(t) - H\dot{a}(t) = \gamma(v(t) - \dot{v}(t)). \]  
(43)

To verify the above identity, we first notice that
\[ \frac{d}{dt}(\Psi\dot{\theta}) = -\gamma\Psi\dot{\theta} + \gamma\dot{\theta} + \gamma\Psi P\Psi^T(v - \dot{v}) = -\gamma\Psi\dot{\theta} + \gamma(\dot{v} - (v - \dot{v}) - \dot{a}). \]

Solving the previous equation for \( \Psi\dot{\theta} \), we obtain
\[ \Psi(t)\dot{\theta}(t) = -\gamma Hv(t) - H\dot{a}(t) + \dot{\gamma}(t) \]

We can now recover (43) by adding \( \gamma(v(t) - \dot{v}(t)) \) to both sides of the previous equation and applying the identity
\[ \gamma(v(t) - Hv(t)) = H\dot{v}(t) \]
which can be easily verified from (19).

3) Proof of Lemma 3: Consider the system
\[ \dot{R} = -\alpha R + \Psi^T\Psi \]  
(44)
with \( R(0) = P(0)^{-1} > 0 \), whose solution is given by (42). We claim that \( R(t) \) is uniformly positive definite and bounded for all \( t \geq 0 \), and that \( \bar{\Psi}^{-1}I < R(t) - \bar{\Psi}^{-1}I, \) for \( t > T \) with the bounds given by (24). In this case, (23) follows from setting \( \bar{R}(t) = R(t)^{-1} \) and checking that the identity \( \dot{\bar{R}} = \bar{R}^{-1} = -R^{-1}RR^{-1} \) leads to (22b). To prove the claim, we first notice \( R(t) \geq e^{-\alpha t}R(0) \) for \( 0 \leq t \leq T \). For \( t \geq T \), we can show that \( R(t) \geq \bar{\Psi}^{-1}I \) by following the same steps as in the proof of [41, Lemma 1]. The upper bound \( \bar{\Psi}^{-1}I \) of \( R(t) \) can be obtained as follows:

First, we notice that (22a) yields \( \|\Psi(t)\| \leq \bar{\delta} \) for all \( t \geq 0 \). Then, since (42) is the solution to (44), we have
\[ \|R(t)\| \leq \|R(0)\| + \alpha^{-1}\sup_{\tau \geq 0}\|\Psi(\tau)\|^2 \leq \alpha^{-1}\bar{\delta}^2 \]
proving the claim.

4) Proof of Theorem 1: We prove this result using the virtual system idea of contraction analysis [20, 25]: We construct a so-called virtual system whose solutions contain the solutions of both (14) and (21); then we show that the virtual system is globally exponentially contracting; this will imply that any solutions of (14) and (21) converge exponentially fast toward each other. We consider the virtual state vector
\[ \bar{x} = \text{col}(\bar{v}, \bar{w}, \bar{\theta}) \]
and the virtual system given by
\[ \begin{align*}
\dot{\bar{v}} &= \bar{f}(t, \bar{w}, \bar{\theta}) + (\gamma I + \Psi P\Psi^T)(v - \bar{v}) \\
\dot{\bar{w}} &= A(v)\bar{w} + b(v) \\
\dot{\bar{\theta}} &= \gamma P\Psi^T(v - \bar{v})
\end{align*} \]  
(45)

By construction of \( \bar{f}(t, \bar{w}, \bar{\theta}) \), any solutions \( x = \text{col}(v, w, \theta) \) of (14) and \( \bar{x} = \text{col}(\bar{v}, \bar{w}, \bar{\theta}) \) of (21) are particular solutions of the virtual system (45); notice that \( v(t), \bar{w}(t), \) and \( u(t) \) are not states of the virtual system.

To show that the virtual system is globally exponentially contracting, we use the differential Lyapunov function
\[ \delta V(t, \delta \bar{x}) = \delta \bar{x}^T T(t)^T \bar{M}(t) T(t) \delta \bar{x} \]  
(46)
where
\[ T = \begin{bmatrix} I & 0 & -\bar{x} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \bar{M} = \begin{bmatrix} \varepsilon I & 0 & 0 \\ 0 & M_w & 0 \\ 0 & 0 & \varepsilon(\gamma P)^{-1} \end{bmatrix} \]  
(47)
and \( \delta \bar{x} \) is the state vector of the differential system \( \delta \bar{x} = J\delta \bar{x} \), with
\[ J = \begin{bmatrix} -\gamma I + \Psi P\Psi^T & \partial_w \bar{f}(t, \bar{w}, \bar{\theta}) \Phi(v, \bar{w}, \bar{u}) \\ 0 & A(v) \\ -\gamma P\Psi^T & 0 \end{bmatrix} \]  
(48)
the Jacovian of the vector field of (45). It can easily be verified that
\[ \begin{align*}
\delta V(t, \delta \bar{x}, \bar{x}) &= \delta \bar{x}^T T(t)^T \left( \bar{J}^T \bar{M} + \bar{M} \bar{J} + \dot{\bar{M}} \right) T(t) \delta \bar{x} \\
&= \begin{bmatrix} -2\varepsilon \gamma I & \varepsilon \partial_{\bar{w}} \bar{f}(t, \bar{w}) & -\varepsilon \Psi \\ -\varepsilon \Psi & A(v)^T M_w + M_w A(v) & 0 \\ -\gamma^{-1}\varepsilon(\Psi^T \Psi + \alpha P^{-1}) & 0 & 0 \end{bmatrix} \leq Q
\end{align*} \]  
(49)
where the upper bound matrix \( Q \) is given by
\[ Q = \begin{bmatrix} -\varepsilon \gamma I & \varepsilon \partial_{\bar{w}} \bar{f}(t, \bar{w}) & 0 \\ -\varepsilon \Psi & -\lambda_w M_w & 0 \\ -\varepsilon(\gamma P)^{-1} & 0 & -\varepsilon(\gamma P)^{-1} \end{bmatrix} \]  
(50)
Finally, notice that
\[ \partial_{\bar{w}} \bar{f}(t, \bar{w}) = \partial_{\bar{w}} (\Phi(v, \bar{w}, u)\theta + a(v, \bar{w}, u)) \]
is bounded (Remark 1). Thus, for any \( \lambda < \min\{\alpha, \lambda_w, \gamma\} \) choosing
\[ \varepsilon = (\lambda_w - \lambda)(\gamma - \lambda)\lambda_{\min}[M_w] \sup \|\partial_{\bar{w}} \bar{f}(t, \bar{w})\|^2 \]
ensures \( Q \leq -\lambda \bar{M} \) and hence (52) holds globally, and the virtual system is globally exponentially contracting.
5) Proof of Lemma 4: From Assumptions 1 and 4 and Remarks 4 and 1, we have that the off-diagonal term in $A_g(t)$ and the nonzero terms in $B_g(t)$ are bounded for all $t \geq 0$. Let us define $\bar{\sigma} := \sup \|\Phi(v, w, u)\|$, $\bar{a} := \sup \|\partial_w \Phi(v, w, u)\| + a(v, w, u)$ and $b = \max_\eta \sup \|\partial_\eta \Phi(v, w, u)\|$, where the sups extend over $v \in V$, $w \in U$, $\theta \in \Theta$ and $\hat{\eta} \in H$. To show $\Psi(t)$ is bounded, first denote each column of $\Psi_w(t)$ by $\psi_j^w(t)$, $j = 1, \ldots, n_\theta + n_u$. Recalling that $\psi_j^w(0) = 0$, it follows from (26a) that $\psi_j^w(t)$ is 0 for $j = 1, \ldots, n_\theta$ and all $t \geq 0$. Furthermore, Assumption 2 implies that the dynamics $\dot{\psi}_j^w = A(v, \hat{\eta})\psi_j^w$ is contracting, and hence it follows from (26a) that

$$\|\psi_j^w(t)\| \leq \gamma \frac{2}{\kappa} \sqrt{\max(M_w)} \|\bar{\sigma}\|$$

for $j = n_\theta + 1, \ldots, n_\theta + n_u$ and all $t \geq 0$. Since $\|\psi_j^w(t)\| \leq n_\theta \max_j \|\psi_j^w(t)\|$ for each $t$, from (26a) we have

$$\|\psi_j^w(t)\| \leq \bar{c} \frac{2}{\kappa} \sqrt{\max(M_w)} \|\bar{\sigma}\|$$

for all $t \geq 0$, where $\bar{c}$ is independent of the hyperparameters $\alpha$, $\beta$, and $\gamma$. Hence, $\Psi(t)$ is bounded.

To show that (27) holds, we first define the two systems

$$\begin{align*}
\dot{P} &= \alpha P - P \Psi^T \Psi P - \beta I \\
\dot{\Psi} &= \alpha \Psi + \beta I
\end{align*}$$

(54)

with $\dot{T}(0) = P(0) = 0$. Then, by the Comparison Theorem for the differential Riccati equations [2, Th. 4.1.4], it follows that

$$\dot{P}(t) \leq P(t) \leq \dot{T}(t)$$

for all $t \geq 0$. Just as in Lemma 3, we have $\dot{P}(t) \geq \rho I$ for all $t \geq 0$, with $\rho$ now given by (28). On the other hand, we have

$$\dot{T}(t) = e^{\alpha t} \dot{T}(0) + \beta \frac{1}{\alpha}(e^{\alpha t} - 1)I$$

for all $t \geq 0$ and hence $\dot{P}(t)$ is upper bounded for $t \in [0, T]$. To find an upper bound for $\dot{P}(t)$ for all $t \geq T$, we can use [7, Lemma 2]. Comparing (26b) to the $P$ dynamics in [7, Lemma 2], we see that [7, Eqs. (1-2)] in our case are given by

$$\dot{O} = -\alpha O + \Psi^T \Psi$$

$O(0) = 0$

$\dot{D} = -\alpha D + \beta O^2$, $D(0) = 0$.

(55)

It follows from [7, Lemma 2] that the inequality

$$P(t) \leq O^{-1}(t) + O^{-1}(t)D(t)O^{-1}(t)$$

holds as soon as $O^{-1}(t)$ exists. This is guaranteed from $t \geq T$, since, using Assumption 5 and following the same steps in the proof of [41, Lemma 1], we have

$$O(t) \geq \delta e^{-2\alpha T}I$$

(57)

for all $t \geq T$. Furthermore, from (55) we have

$$\|D(t)\| \leq \frac{\beta}{\alpha} \sup_{t \geq 0} \|O(t)\|^2 \leq \frac{\beta}{\alpha} \sup_{t \geq 0} \|\Psi(t)\|^4 \leq \frac{\beta}{\alpha} \bar{c}^4$$

for all $t \geq 0$. Hence, it follows from (56), (57) and the previous equation that $P(t) \leq \overline{p}I$ for all $t \geq T$, with $\overline{p}$ given by (28).

6) Proof of Theorem 2: To begin, notice by Lemma 4, the metric $M(t) = T(t)^T \dot{M}(t)T(t)$ given by (32) is bounded, and $\dot{M}(t)$ is uniformly positive definite. Furthermore, since $T(t)$ is uniformly full column rank, $M(t)$ is also uniformly positive definite. This proves (34).

In the rest of the proof, in similar fashion to the proof of Theorem 1, we consider the virtual state vector

$$\hat{x} := \col(\hat{v}, \hat{w}, \hat{\theta}, \hat{\eta})$$

and the virtual system

$$\begin{align*}
\dot{\hat{v}} &= \hat{f}(t, \hat{w}, \hat{\theta}) + (\gamma I + \Psi_u \Psi^T)(v - \hat{v}) \\
\dot{\hat{w}} &= \hat{g}(t, \hat{w}, \hat{\theta}) + \Psi_w \Psi^T(v - \hat{v})
\end{align*}$$

(58)

and where

$$\begin{align*}
\hat{f}(t, \hat{w}, \hat{\theta}) &= \Phi(v, \hat{w}, u)\theta + \Phi(v, \hat{w}, u)(\hat{\theta} - \theta) + a(v, \hat{w}, u) \\
\hat{g}(t, \hat{w}, \hat{\eta}) &= A(v, \hat{\eta})w + A(v, \hat{\eta})(\hat{w} - w) + b(v, \hat{\eta}).
\end{align*}$$

By construction of $\hat{f}$ and $\hat{g}$, any solutions $x = \col(v, w, \theta, \eta)$ of (1) and $\hat{x} = \col(\hat{v}, \hat{w}, \hat{\theta}, \hat{\eta})$ of (25) are particular solutions of the virtual system (58).

We use the differential Lyapunov equation $\dot{\Psi} = \partial_t \Psi^T T \dot{M}T \partial_t \Psi := \partial_t \Psi^T \dot{M} \partial_t \Psi$, with $T$ and $\dot{M}$ given by (32) and $\partial_t \Psi$ the state of the differential system $\partial_t \hat{x} = J \partial_t \hat{x}$. The Jacobian $J$ of the vector field of (58) is given by

$$J = \begin{bmatrix}
-\Psi \Psi^T & 0 & 0 & \hat{A}_\Psi(t) \\
0 & -\Psi \Psi^T & 0 & \hat{B}_\Psi(t) \\
\hat{A}_\Psi(t) & 0 & -\Psi \Psi^T & 0 \\
\hat{B}_\Psi(t) & \hat{A}_\Psi(t) & 0 & -\Psi \Psi^T
\end{bmatrix}$$

(59)

where

$$\hat{A}_\Psi(t) = \begin{bmatrix}
-\gamma I & \partial_\theta \Phi(v, \hat{w}, \hat{u}) - \partial_\theta a(v, \hat{w}, u) \\
0_{n_\theta \times n_\theta} & A(v, \hat{\eta})
\end{bmatrix}$$

$$\hat{B}_\Psi(t) = \begin{bmatrix}
\Phi(v, \hat{w}, u) & 0_{n_\theta \times n_\theta} \\
0_{n_\theta \times n_\theta} & \partial_\eta A(v, \hat{\eta})w + b(v, \hat{\eta})
\end{bmatrix}$$

(recall the lines delimit block submatrices of the same size).

As in the previous proof, $\partial_t \hat{\Psi}(t, \partial_t \hat{x}, \hat{x})$ satisfies (49), with $\dot{\hat{f}}$ given by (50) but $J$ now given by (59) and $T$ now given by (32). Computing $\dot{J}$ while replacing $\Psi$ by (26a), we obtain

$$\dot{J} = \begin{bmatrix}
\dot{\hat{A}}_\Psi(t) & \dot{\hat{B}}_\Psi(t) - \hat{B}_\Psi(t) \\
0 & -\hat{B}_\Psi(t)
\end{bmatrix}$$

Since the metric $M(t)$ is bounded and uniformly positive definite and since $\dot{\Psi}^T(t, \partial_t \hat{x}, \hat{x})$ satisfies (49), to prove our result, we will find a contraction region in the state space where (52) holds for all $t \geq 0$. Computing the left-hand side of (32) from (32) and $\dot{J}$ above, we obtain

$$\dot{J}^T \dot{M} + \dot{M}J + \dot{\hat{M}} \leq Q$$

where the upper bound matrix $Q = Q^T(t)$ is given by

$$Q = \begin{bmatrix}
gamma I & \epsilon \partial_\theta \hat{f}(t, \hat{w}) & \epsilon \gamma^{-1} \Delta_1 \Psi_w \Psi^T & 0 \\
\epsilon \partial_\theta \hat{f}(t, \hat{w}) & -\epsilon \gamma^{-1} \Delta_1 \Psi_w \Psi^T & 0 & \epsilon \gamma^{-1} \Delta_2 \Psi_w \Psi^T \\
\epsilon \gamma^{-1} \Delta_1 \Psi_w \Psi^T & 0 & \epsilon \gamma^{-1} \Delta_2 \Psi_w \Psi^T & 0 \\
\epsilon \gamma^{-1} \Delta_2 \Psi_w \Psi^T & 0 & 0 & \epsilon \gamma^{-1} \Delta_2 \Psi_w \Psi^T
\end{bmatrix}$$
where we used $\Psi_w = \begin{bmatrix} 0 & \Psi_{w,2} \end{bmatrix}$ (see Remark 7). Here,
\[
\Delta_1 = \partial_{\tilde{w}}[\Phi(v, \tilde{w}, u)\theta + a(v, \tilde{w}, u)] - \partial_w \left[ \Phi(v, \tilde{w}, u)\theta + a(v, \tilde{w}, u) \right] + \partial_{\tilde{w}}[\Phi(v, \tilde{w}, u)(\theta - \varsigma_0(\hat{\theta}))]
\]
and
\[
\Delta_2 = \partial_{\tilde{w}}[A(v, \tilde{w})w + b(v, \tilde{w})] - \partial_{w}[A(v, \tilde{w})w + b(v, \tilde{w})] + \partial_{\tilde{w}}[A(v, \tilde{w})(w - \varsigma_w(\tilde{w}))].
\]

We now wish to find a region of the state space, where $Q \subseteq -\lambda \tilde{M}$ for all $t \geq 0$. For that purpose, let $\lambda < \min\{\alpha, \lambda_w, \gamma\}$, and consider an arbitrary number $\zeta \in (0, 1)$. Then using Schur’s complement, we can show that the choice
\[
\varepsilon = (1 - \zeta)^2(\lambda_w - \gamma - \lambda)\lambda_{\text{min}}[M_w \sup_{t \geq 0} \| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2]^{-2} \quad (60)
\]
leads to
\[
- Q - \lambda \tilde{M} > \begin{bmatrix} \varepsilon(\alpha - \lambda) \lambda_{\text{min}}[M_w \sup_{t \geq 0} \| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2]^{-2} \end{bmatrix},
\]
Hence, it follows from Schur’s complement that the right-hand side of the inequality above is positive semidefinite if and only if
\[
\frac{\varepsilon}{\gamma} \left( (\alpha - \lambda) \lambda_{\text{min}}[M_w \sup_{t \geq 0} \| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2]^{-2} \right) \geq \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \frac{1}{\gamma} \frac{\| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2}{\| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2} \end{bmatrix},
\]
By Lemma 4, the above will hold if
\[
(\alpha - \lambda) \lambda_{\text{min}}[M_w \sup_{t \geq 0} \| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2]^{-2} \geq \frac{1}{\gamma} \frac{\| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2}{\| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2} + \frac{\| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2}{\| \partial_{\tilde{w}}f(t, \tilde{w}) \|^2},
\]
where we recall $\zeta \in (0, 1)$ is arbitrary.

By the continuity and global Lipschitz properties of $\Delta_1$ and $\Delta_2$, as well as by boundedness of $\Psi_{w,2}$, there exists a sufficiently small $r > 0$ such that for each $t \geq 0$, whenever $\| x(t) - \tilde{x}(t) \| \leq r$, the inequality (61) holds for all $\tilde{x} \in \mathbb{R}^{n_a + n_b + n_a + n_b}$ such that $\| \tilde{x} - \tilde{x}(t) \| \leq r$. Thus, as long as $\| x(t) - \tilde{x}(t) \| \leq r$ for all $t \geq 0$, the ball $B(t; r\sqrt{\tilde{M}})$ given by (33) is contained in a region of attraction at all times, which implies that any trajectory of (58) starting in $B(0; r\sqrt{\tilde{M}})$ remains in $B(t; r\sqrt{\tilde{M}})$ for $t \geq 0$ and converges exponentially fast to $\tilde{x}(t)$ with rate $\lambda$ (see [25, Th. 2]). But since $x(t)$ is a valid trajectory of (58), it follows that if $x(0) \in B(0; r\sqrt{\tilde{M}})$, then $x(t)$ remains in $B(t; r\sqrt{\tilde{M}})$, and exponential convergence of $x(t)$ to $\tilde{x}(t)$ is guaranteed.

C. Model Parameters

1) Hodgkin–Huxley Model: HH model parameters were adapted from [19, pp. 46–47]. Voltage dynamics parameters are given as follows:

| $\mu_{Na}$ | $\mu_K$ | $\mu_L$ | $\nu_{Na}$ | $\nu_K$ | $\nu_L$ | $c$ |
|---------|---------|---------|-----------|---------|---------|------|
| 120     | 36      | 0.3     | 55        | -77     | -54.4   | 1    |

All activation functions are of the form (6), and all time-constant functions are of the form (7), with parameters given in the table below.

2) Half-Center Oscillator: Both neurons in the HCO of Section V-B have identical nominal capacitances, reversal potentials, and internal dynamics. The reversal potentials are given by $\nu_{Na} = 50$, $\nu_K = \nu_G = -80$, $\nu_{Ca} = 120$, and $\nu_L = -49$ mV; the capacitances are given by $c_1 = c_2 = 1$. The internal dynamics were adapted from [8, p. 2474]. All activation functions are of the form (6), and all intrinsic time-constant functions are of the form (7), with parameters given in the table below. The synaptic time-constant (11) has $a_G = 2$ and $b_G = 0.1$.

| $\mu_{Na}$ | $\mu_K$ | $\mu_L$ | $\nu_{Na}$ | $\nu_K$ | $\nu_L$ | $c$ |
|---------|---------|---------|-----------|---------|---------|------|
| $-35.5$ | 5.29    | 0.06    | 42.37     | $-387.92$ | 133.78 |
| $h_{Na}$ | $-48.9$ | -5.18   | 1.50      | 2.50    | -62.90    | 10.00 |
| $h_K$   | $-12.3$ | 11.8    | 0.80      | 6.65    | -76.62    | 61.42 |
| $h_{Ca}$ | $-67.1$ | 7.20    | 1.01      | 40.03   | -117.58   | 62.87 |
| $h_{Ca}$ | $-82.1$ | -5.5    | 40.49     | 126.51  | -92.48    | -50.24 |

We have chosen the HCO initial conditions $v(0)$ and $w(0)$ from the trajectory observed at steady-state oscillations with $\mu_{Ca,1} = \mu_{Ca,2} = 0.11$. The adaptive observer initial conditions were arbitrarily set to $v(0) = (-50, -50)^T$, $\psi(1)(0) = 0$, $\mu_{Na,1}(0) = \mu_{Na,2}(0) = 80$, $\mu_{K,1}(0) = \mu_{K,2}(0) = 80$, $\mu_{Ca,1}(0) = \mu_{Ca,2}(0) = 1$, $\mu_{L,1}(0) = \mu_{L,2}(0) = 1$, $\mu_{Ca,1}(0) = \mu_{Ca,2}(0) = 1$, $\psi(1)(0) = \psi(2)(0) = 0$, and $P(1)(0) = P(2)(0) = 0.11$.

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