Classifying CP transformations according to their texture zeros: theory and implications

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We provide a classification of generalized CP symmetries preserved by the neutrino mass matrix according to the number of zero entries in the associated transformation matrix. We determine the corresponding constrained form of the lepton mixing matrix, characterized by correlations between the mixing angles and the CP violating phases. We compare with the corresponding restrictions from current neutrino oscillation global fits and show that, in some cases, the Dirac CP phase characterizing oscillations is highly constrained. Implications for current and upcoming long baseline neutrino oscillation experiments T2K, NOνA and DUNE, as well as neutrinoless double beta decay experiments are discussed. We also study the cosmological implications of such schemes for leptogenesis.

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I. INTRODUCTION

Non-Abelian symmetries provide an attractive framework in terms of which to tackle the long-standing flavor problem in particle physics [1–5]. Assuming light neutrinos to be Majorana particles, as suggested on general grounds [6], we consider the case where the neutrino and charged lepton mass matrices have remnant symmetries, both of flavour and CP types. These would ultimately reflect some unspecified flavour symmetry of the underlying gauge theory, providing also a more general framework to describe mu-tau flavor symmetry in neutrino physics [7, 8]. One can show that flavor symmetries can be generated by performing two successive CP transformations [9, 10]. Conversely the remnant CP transformation of the lepton mass matrices can constrain the lepton flavor mixing in a quite efficient way. In particular, the Majorana and Dirac leptonic CP violating phases can be predicted [7, 8]. If one remnant CP transformation \(X\) is preserved by the neutrino mass matrix the \(X\) matrix should be symmetric and unitary. By performing Takagi factorization we have \(X = \Sigma \Sigma^T\), where \(\Sigma\) is a unitary matrix. Without loss of generality, we shall work in the charged lepton diagonal basis so that the lepton flavor mixing completely arises from the neutrino sector. The invariance of the neutrino mass matrix under the action of \(X\) implies that the lepton mixing matrix \(U\) should
Then the lepton mixing matrix is determined to be of the form:

$$U = \Sigma O_{3 \times 3} \hat{X}^{-1/2},$$

where \( \hat{X}^{-1/2} \) is a diagonal matrix with non-vanishing entries equal to \( \pm 1 \) or \( \pm i \), which is necessary for making neutrino masses positive \[6\]. Without loss of generality, this matrix can be parametrized as

$$\hat{X}^{-1/2} = \text{diag}(1, i^{k_1}, i^{k_2}) ,$$

with \( k_{1,2} = 0, 1, 2, 3 \). The Majorana phases may be changed by \( \pi \) due to \( \hat{X}^{-1/2} \). On the other hand, the \( O_{3 \times 3} \) is a generic three dimensional real orthogonal matrix, and it can be parametrized as

$$O_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where \( \theta_{1,2,3} \) are real parameters, and a possible overall minus sign of \( O_{3 \times 3} \) is omitted since it is irrelevant to flavor mixing parameters. Therefore the lepton mixing matrix is predicted to depend on three free parameters \( \theta_{1,2,3} \) besides the parameters characterizing the residual CP transformation \( X \). In the following, we shall classify all possible forms of the remnant CP transformations according to the number of zero elements in \( X \) and investigate the corresponding predictions for the lepton flavor mixing parameters and their implications for the lepton CP violation in conventional neutrino oscillations, neutrinoless double beta decay as well as leptogenesis.

### A. The angle-phase parametrization

For three light Majorana neutrinos the leptonic mixing matrix can be expressed in terms of three rotation angles and three CP violation phases \[9\]. Here we will take these six independent parameters expressed within the PDG prescription given as \[11\] \(\) 1:

$$U_{\text{PDG}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}^{-1}e^{-i\delta_{\text{CP}}} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{i\delta_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{i\delta_{\text{CP}}} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{i\delta_{\text{CP}}} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{i\delta_{\text{CP}}} & c_{13}c_{23} \end{pmatrix} \mathcal{K},$$

where \( c_{ij} \equiv \cos \theta_{ij} \) and \( s_{ij} \equiv \sin \theta_{ij} \) and \( \mathcal{K} \) is a diagonal matrix of phases chosen as \( \mathcal{K} = \text{diag}(1, e^{i\alpha_{21}/2}, e^{i\alpha_{31}/2}) \). The \( \delta_{\text{CP}} \) is the Dirac CP violation phase and \( \alpha_{21}, \alpha_{31} \) are two Majorana CP violation phases. In this parametrization the mixing angles and magnitudes of the entries of the \( U_{\text{PDG}} \) matrix are related as:

$$\sin^2 \theta_{13} = |(U_{\text{PDG}})_{13}|^2 , \quad \sin^2 \theta_{12} = \frac{|(U_{\text{PDG}})_{12}|^2}{1 - |(U_{\text{PDG}})_{13}|^2} \quad \text{and} \quad \sin^2 \theta_{23} = \frac{|(U_{\text{PDG}})_{23}|^2}{1 - |(U_{\text{PDG}})_{13}|^2} .$$

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1 For a description in the original symmetric form of the lepton mixing matrix see \[7\]. As discussed in \[12\] the symmetric presentation is more transparent in describing \( 0\nu\beta\beta \) while the equivalent PDG prescription is more convenient of describing neutrino oscillations.
The well known Jarlskog-like invariant is defined as
\[ J_{\text{CP}} = \Im \left\{ (U_{\text{PDG}})^*_{11} (U_{\text{PDG}})^*_{23} (U_{\text{PDG}})_{13} (U_{\text{PDG}})_{21} \right\}. \]
It describes CP violation in conventional neutrino oscillations, and takes the form
\[ J_{\text{CP}} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta_{\text{CP}}. \] (7)

This expression gives a very transparent interpretation of the Dirac leptonic CP invariant. There are two additional rephase invariants
\[ I_1 = \Im \left\{ (U_{\text{PDG}})^2_{12} (U_{\text{PDG}})_{11} \right\} \]
and
\[ I_2 = \Im \left\{ (U_{\text{PDG}})^2_{13} (U_{\text{PDG}})_{11} \right\}, \]
associated with the Majorana phases \[ [13–15] \] they take the following form
\[ I_1 = \frac{1}{4} \sin^2 2\theta_{12} \cos^4 \theta_{13} \sin \alpha_{12} \quad \text{and} \quad I_2 = \frac{1}{4} \sin^2 2\theta_{13} \cos^2 \theta_{12} \sin \alpha_{13}', \] (8)
where \( \alpha_{13}' = \alpha_{13} - 2\delta_{\text{CP}} \). These invariants appear in lepton number violating processes such as neutrinoless double beta decay which do not depend, as expected, on the Dirac invariant \( J_{\text{CP}} \).

II. TEXTURE ZEROS OF THE REMNANT CP TRANSFORMATIONS

In order to perform the classification of the remnant CP transformations in terms of texture zeros, it is necessary to establish the way of counting these zero entries in the \( X \) matrix: two zeros off-diagonal counts as one texture zero, while one zero on the diagonal counts as one texture zero \[ [16] \]. The CP transformations are unitary-symmetric matrices and consequently have non-zero determinant.

Hence, the \( X \) matrix cannot be represented by a matrix with six or five zero elements. In other words, the maximum number of zero entries in the CP transformation matrices must be four. The CP transformation matrices with four texture zeros and the corresponding \( \Sigma \) matrices are given in the Table I. For this case the lepton mixing matrix is obtained through Eq. (2), and the mixing parameters are given in the fourth column of Table I.

The type-I \( X \) matrix with four texture zeros corresponds to the so-called \( \mu - \tau \) reflection \[ [8,17–19] \]. It is remarkable that both the atmospheric mixing angle \( \theta_{23} \) and the CP violation phase \( \delta_{\text{CP}} \) are predicted to be maximal for any values of the free parameters \( \theta_i \) while Majorana phases take on CP-conserving values.

For the type-II \( X \) matrix one sees that reactor and atmospheric angles are strongly correlated each other,
\[ \sin^2 \theta_{23} = 1 - \tan^2 \theta_{13}, \quad \sin^2 \theta_{12} = \frac{1}{2} \left( 1 + \tan^2 \theta_{13} \cos 2\theta_3 \right). \] (9)
Given the measured value of reactor angle \( \theta_{13} \), we have \( \sin^2 \theta_{12} \simeq \frac{1}{2} \) and \( \sin^2 \theta_{23} \simeq 1 \). The measured values of the solar and atmospheric mixing angles can not be achieved in this case.

For the residual CP transformation with four texture zeros type-III, the lepton mixing matrix is related to the type-II case through the exchange of the second and third rows,
\[ \Sigma_{III} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \Sigma_{II}. \] (10)

As a consequence, except for the atmospheric angle \( \theta_{23} \) and Jarlskog \( J_{\text{CP}} \), the expressions for the rephasing invariants and mixing angles are the same as those obtained in the previous case. For the measured value of \( \theta_{13} \), the other two angles are \( \sin^2 \theta_{12} \simeq \frac{1}{2} \) and \( \sin^2 \theta_{23} \simeq 0 \). The measured values of the three mixing angles \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) can not be accommodated simultaneously. There is only one CP transformation matrix with three texture zeros that we will...
TABLE I: The CP transformation matrices with four texture zeros and the corresponding $\Sigma$ matrices, where $\alpha$, $\beta$ and $\gamma$ are real. The resulting lepton mixing matrix is obtained through the Eq. (2). Type-I is the conventional $\mu - \tau$ reflection while for type-II and -III $\Sigma$ matrices are related by a permutation of the second and third row. The experimentally measured values of the three mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ can not be reproduced.

denote as type-IV CP transformation matrix and whose explicit form is:

$$\textbf{type-IV} : \quad \mathbf{X} = \text{diag} (e^{i\alpha}, e^{i\beta}, e^{i\gamma}) ,$$

with $\alpha, \beta, \gamma \in \mathbb{R}$. Its Takagi factorization reads as $\Sigma = \text{diag} (e^{i\alpha/2}, e^{i\beta/2}, e^{i\gamma/2})$. The lepton mixing parameters are given by $\sin^2 \theta_{13} = \sin^2 \theta_2$, $\sin^2 \theta_{12} = \sin^2 \theta_3$, $\sin^2 \theta_{23} = \sin^2 \theta_1$, and $\sin \delta_{\text{CP}} = \sin \alpha_{21} = \sin \alpha_{31} = 0$. We see that all the three CP phases are zero or $\pi$ in this case while the lepton mixing angles are unconstrained.

The CP transformation matrices with two texture zeros and the corresponding $\Sigma$ matrices are given in Table II. The explicit form of the lepton matrix can be obtained from Eq. (2) and the mixing parameters are given in the fourth column of Table II. The type-V CP transformation with two texture zeros is exactly the generalized $\mu - \tau$ reflection symmetry [7]. The atmospheric angle and the Dirac CP phase $\delta_{\text{CP}}$ only depend on two parameters $\theta_1$ and $\theta_3$, as they are related by

$$\sin^2 \delta_{\text{CP}} \sin^2 2\theta_{23} = \sin^2 \Theta .$$

Moreover the Majorana phases $\alpha_{21}$ and $\alpha_{31}$ take on CP-conserving values. The phenomenological implications of this interesting mixing pattern for neutrinoless double beta decay as well as conventional neutrino oscillations have been discussed in detail in Ref. [7].

For the $\mathbf{X}$ matrix with two texture zeros of type-VI, the mixing angles are given by

$$\sin^2 \theta_{13} = \frac{1}{2} \left( 1 - \cos \Theta \cos 2\theta_1 \right) \cos^2 \theta_2 , \quad \sin^2 \theta_{23} = \frac{2 \sin^2 \theta_2}{2 - (1 - \cos \Theta \cos 2\theta_1) \cos^2 \theta_2} ,$$

$$\sin^2 \theta_{12} = \frac{(1 + \cos \Theta \cos 2\theta_1) \cos^2 \theta_3 + \sin \theta_2 \left( (1 - \cos \Theta \cos 2\theta_1) \sin \theta_2 \sin^2 \theta_3 - \cos \Theta \sin 2\theta_1 \sin 2\theta_3 \right)}{2 - (1 - \cos \Theta \cos 2\theta_1) \cos^2 \theta_2} .$$

(13)
We easily see that the following sum rules are fulfilled,
\[
\sin^2 \theta_{13} \cos^2 \theta_{13} = \sin^2 \theta_2, \\
\cos^2 \theta_{13} \cos^2 \theta_{23} - \sin^2 \theta_{13} = \cos \Theta \cos 2 \theta_1, \\
\sin^2 \theta_{12} (1 - \sin^2 \theta_{23} \cos^2 \theta_{13}) - \cos^2 \theta_{23} \cos^2 \theta_3 - \sin^2 \theta_{13} \sin^2 \theta_{23} \sin^2 \theta_3 \\
= \pm \frac{1}{2} \sqrt{1 - \cos^2 \theta_{23} \cos^2 \theta_3} \left( \cos 2 \theta_{13} - \sin^2 \theta_{23} \right) \tan 2 \theta_1 \sin 2 \theta_3.
\]
(14)

The Jarlskog-like invariant and the invariants associated with the Majorana phases take the following form
\[
J_{CP} = -\frac{1}{4} \sin \Theta \sin \theta_2 \cos^2 \theta_2 \sin 2 \theta_3, \\
I_1 = \frac{1}{4} (-1)^{k_1+1} \sin \Theta \sin \theta_2 \left[ 2 \cos \Theta \sin 2 \theta_1 \sin \theta_2 \cos 2 \theta_3 + \left( \cos^2 \theta_2 + \cos \Theta \cos 2 \theta_1 (1 + \sin^2 \theta_2) \right) \sin 2 \theta_3 \right], \\
I_2 = \frac{1}{2} (-1)^{k_2} \sin \Theta \cos^2 \theta_2 \sin \theta_3 \left[ (1 - \cos \Theta \cos 2 \theta_1) \sin \theta_2 \cos \theta_3 + \cos \Theta \sin 2 \theta_1 \sin \theta_3 \right].
\]
(15)

Notice that the three phases \( \alpha, \beta \) and \( \gamma \) can be absorbed into the charged lepton fields, therefore they are unphysical and hence do not appear in the above mixing parameters. The correlations between the mixing parameters \( \Theta, \delta_{CP} \) and \( \theta_1 \) are displayed in Fig. 1. The blue rings are allowed parameter regions of \( \Theta \) and \( \theta_1 \) predicted by Eq. (14), where \( \theta_{13} \) and \( \theta_{23} \) are compatible with the preferred values from the neutrino oscillations global fit in [20] at the 3\( \sigma \) level. The red points are obtained from a random numerical scan over the parameters \( \Theta \) and \( \theta_{1,2,3} \). We can see the allowed regions of \( \Theta \) from the two different approaches are compatible with each other.

The lepton mixing matrix corresponding to the type-VII \( \mathbf{X} \) matrix with two texture zeros, is related to the one of type-VI by the exchange of the second and the third rows. As a result, the lepton mixing parameters for the type-VII case are the same as those of type-VI except that \( \theta_{23} \) and \( \delta_{CP} \) becomes \( \pi/2 - \theta_{23} \) and \( \pi + \delta_{CP} \) respectively. A detailed analysis of theoretical predictions of the CP transformation matrices with two texture zeros will be given in the next section.

Finally, the CP transformation matrices with one texture zero and the corresponding \( \mathbf{\Sigma} \) matrices are given in Table III. The corresponding expressions for the mixing matrix are obtained by using Eq. (2) and the mixing parameters can be extracted straightforwardly. For the type-VIII CP transformation with one zero element, the mixing angles
FIG. 1: The correlations between the parameters Θ, δ_{CP} (red points) and θ_{1} (blue rings) for the case of type-VI CP transformation, where the three lepton mixing angles are required to lie in the experimentally preferred 3σ regions.

TABLE III: The CP transformation matrices with one texture zeros and the corresponding Σ matrices with c_{θ} ≡ cos Θ and s_{θ} ≡ sin Θ. The lepton mixing matrix is obtained through Eq. (2). The lepton mixing matrix for type-IX and type-X are related by the exchange of the second and third rows.

| Type | X | Σ |
|------|---|---|
| VIII | \[\begin{pmatrix} 0 & e^{iθ}c_{θ} & e^{iβ}s_{θ} \\ e^{iθ}c_{θ} & e^{iγ}s_{θ} & -e^{i(−α+β+γ)}c_{θ}s_{θ} \\ e^{iβ}s_{θ} & -e^{i(−α+β+γ)}c_{θ}s_{θ} & e^{i(−2α+2β+γ)}c_{θ} \end{pmatrix}\] | \[\begin{pmatrix} -ie^{i(α−γ/2)} & e^{i(α−γ/2)} & 0 \\ ie^{iγ/2}c_{θ} & e^{iγ/2}s_{θ} & \sqrt{2}e^{iγ/2}s_{θ} \\ ie^{i(−α+β+γ/2)}c_{θ} & e^{i(−α+β+γ/2)}s_{θ} & −\sqrt{2}e^{i(−α+β+γ/2)}c_{θ} \end{pmatrix}\] |
| IX | \[\begin{pmatrix} e^{iθ}c_{θ} & e^{iθ}c_{θ} & e^{iγ}s_{θ} \\ e^{iθ}c_{θ} & 0 & −e^{i(−α+β+γ)}c_{θ} \\ e^{iθ}s_{θ} & −e^{i(−α+β+γ)}c_{θ} & e^{i(−α+2γ)}c_{θ} \end{pmatrix}\] | \[\begin{pmatrix} -ie^{iα/2}s_{θ} & -ie^{iα/2}s_{θ} & \sqrt{2}ie^{iα/2}s_{θ} \\ iie^{i(−α+2γ)}c_{θ} & iie^{i(−α+2γ)}c_{θ} & 0 \\ iie^{i(−α−2γ)}c_{θ} & iie^{i(−α−2γ)}c_{θ} & \sqrt{2}ie^{i(−α−2γ)}c_{θ} \end{pmatrix}\] |
| X | \[\begin{pmatrix} e^{iθ}c_{θ} & e^{iγ}s_{θ} & e^{iθ}c_{θ} \\ e^{iθ}c_{θ} & −e^{i(−α+β+γ)}c_{θ} & 0 \\ e^{iθ}s_{θ} & e^{i(−α+β+γ)}c_{θ} & 0 \end{pmatrix}\] | \[\begin{pmatrix} -ie^{iα/2}s_{θ} & -ie^{iα/2}s_{θ} & \sqrt{2}ie^{iα/2}s_{θ} \\ iie^{i(−α−2γ)}c_{θ} & iie^{i(−α−2γ)}c_{θ} & 0 \\ iie^{i(−α+2γ)}c_{θ} & iie^{i(−α+2γ)}c_{θ} & \sqrt{2}ie^{i(−α+2γ)}c_{θ} \end{pmatrix}\] |

are given by:

\[
\sin^{2}θ_{13} = \frac{1}{8} \left( 3 - \cos 2θ_{1} - 2 \cos^{2}θ_{1} \cos 2θ_{2} \right),
\]

\[
\sin^{2}θ_{12} = \frac{4 \cos^{2}θ_{1} \cos^{2}θ_{2} - 2 \sin 2θ_{1} \sin 2θ_{2} \sin 2θ_{3} + 4 \left( \sin^{2}θ_{1} \sin^{2}θ_{2} + \cos^{2}θ_{2} \right) \sin^{2}θ_{3}}{5 + \cos 2θ_{1} + 2 \cos^{2}θ_{1} \cos 2θ_{2}},
\]

\[
\sin^{2}θ_{23} = \frac{4 \cos^{2}θ \left( \sin^{2}θ_{2} + \sin^{2}θ_{1} \cos^{2}θ_{2} \right) + 8 \cos^{2}θ_{1} \cos^{2}θ_{2} \sin^{2}θ + 2\sqrt{2} \sin 2θ_{1} \cos^{2}θ_{2} \sin 2θ_{2}}{5 + \cos 2θ_{1} + 2 \cos^{2}θ_{1} \cos 2θ_{2}}.
\]

The smallness of θ_{13} requires θ_{1} ≃ 0, π, and θ_{2} ≃ 0, π. Consequently, the solar mixing angles would be \(\sin^{2}θ_{12} \approx \frac{1}{2}\) which is outside the experimentally allowed ranges [20]. As a result, the measured values of θ_{12} and θ_{13} can not be accommodated simultaneously in this case, and this mixing pattern is not viable. This observation is indeed confirmed in our numerical analysis.
FIG. 2: The correlation between the mixing parameters predicted in the case of type-VI residual CP transformation. All the parameters \( \theta_1, \theta_2, \theta_3 \) and \( \Theta \) are taken to be random numbers in the interval of \([0, 2\pi]\). The three mixing angles \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) are required to be within the experimentally preferred 3\( \sigma \) intervals \([20\)\]. The blue points indicate the numerical results obtained by fixing \( \Theta = \frac{\pi}{7} \) as a benchmark example.

For the type-IX remnant CP transformation with one zero, we find the lepton mixing angles are

\[
\sin^2 \theta_{13} = \frac{1}{4} \left[ 1 + \cos^2 \theta_1 \cos^2 \theta_2 + (3 \cos^2 \theta_1 \cos^2 \theta_2 - 1) \cos 2\Theta - \sqrt{2} \sin 2\theta_1 \cos^2 \theta_2 \sin 2\Theta \right],
\]

\[
\sin^2 \theta_{12} = 2 \left[ \left( \sqrt{2} \cos \Theta (\sin \theta_1 \cos \theta_3 + \cos \theta_1 \sin \theta_2 \sin \theta_3) + \sin \Theta (\cos \theta_1 \cos \theta_3 - \sin \theta_1 \sin \theta_2 \sin \theta_3) \right)^2 + \cos^2 \theta_2 \sin^2 \theta_3 \sin^2 \Theta \right]/ \left[ 3 - \cos^2 \theta_1 \cos^2 \theta_2 + (1 - 3 \cos^2 \theta_1 \cos^2 \theta_2) \cos 2\Theta + \sqrt{2} \sin 2\theta_1 \cos^2 \theta_2 \sin 2\Theta \right],
\]

\[
\sin^2 \theta_{23} = \frac{2 \left( \sin^2 \theta_2 + \sin^2 \theta_1 \cos^2 \theta_2 \right)}{3 - \cos^2 \theta_1 \cos^2 \theta_2 + (1 - 3 \cos^2 \theta_1 \cos^2 \theta_2) \cos 2\Theta + \sqrt{2} \sin 2\theta_1 \cos^2 \theta_2 \sin 2\Theta}.
\] (17)

Also, we see that the mixing angles are correlated as follows,

\[
2 \cos^2 \theta_{13} \cos^2 \theta_{23} = 1 - \cos^2 \theta_1 \cos^2 \theta_2,
\]

\[
\cos 2\theta_{13} = 2 \cos^2 \theta_{13} \cos^2 \theta_{23} \cos^2 \Theta + (1 - 2 \cos^2 \theta_{13} \cos^2 \theta_{23}) (\sqrt{2} \tan \theta_1 \sin 2\Theta - \cos 2\Theta).
\] (18)

The Jarlskog-like invariant associated with the Dirac CP phase has the form

\[
J_{CP} = \frac{1}{16} \cos \theta_1 \cos \theta_2 \left[ 4 \sin 2\theta_1 \sin \theta_2 \cos 2\theta_3 - \sin 2\theta_3 \left( 1 - 3 \cos 2\theta_1 + 2 \cos^2 \theta_1 \cos 2\theta_2 \right) \right] \cos 2\Theta
\]
while the invariants associated with the Majorana phases are

\[
I_1 = \left\{ \frac{1}{64\sqrt{2}} \left[ 4 (3 \sin 3 \theta_1 - 7 \cos \theta_1) \sin \theta_2 \cos 2 \theta_3 + (31 \sin \theta_1 - 9 \sin 3 \theta_1 + 6 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \right\} \cos \theta_2 \cos \Theta \sin \Theta + \frac{1}{64\sqrt{2}} \left[ (9 \sin \theta_1 - 15 \sin 3 \theta_1 + 10 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right.
\]
\[
+ 4 (\cos \theta_1 - 5 \cos 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\] \cos \theta_2 \cos 3 \Theta \sin \Theta + \frac{1}{128} \left[ 4 (5 \sin \theta_1 - 3 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3
\]
\[
+ (13 \cos \theta_1 - 9 \cos 3 \theta_1 - 2 (5 - 3 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \cos \theta_2 \sin^2 \Theta
\]
\[
+ \frac{1}{128} \left[ (9 \cos \theta_1 - 21 \cos 3 \theta_1 - 2 (1 - 7 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3
\]
\[
+ 4 (\sin \theta_1 - 7 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\} \cos \theta_2 \sin \Theta \sin 3 \Theta \right\} \cos (k_1 \pi),
\]

\[
I_2 = \left\{ \frac{1}{128\sqrt{2}} \left[ (31 \sin \theta_1 + 15 \sin 3 \theta_1) \cos \theta_2 - 12 \cos^2 \theta_1 \sin \theta_1 \cos 3 \theta_2 \right) \sin 2 \theta_3
\]

while the invariants associated with the Majorana phases are

\[
I_1 = \left\{ \frac{1}{64\sqrt{2}} \left[ 4 (3 \cos 3 \theta_1 - 7 \cos \theta_1) \sin \theta_2 \cos 2 \theta_3 + (31 \sin \theta_1 - 9 \sin 3 \theta_1 + 6 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \right\} \cos \theta_2 \cos \Theta \sin \Theta + \frac{1}{64\sqrt{2}} \left[ (9 \sin \theta_1 - 15 \sin 3 \theta_1 + 10 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right.
\]
\[
+ 4 (\cos \theta_1 - 5 \cos 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\] \cos \theta_2 \cos 3 \Theta \sin \Theta + \frac{1}{128} \left[ 4 (5 \sin \theta_1 - 3 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3
\]
\[
+ (13 \cos \theta_1 - 9 \cos 3 \theta_1 - 2 (5 - 3 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \cos \theta_2 \sin^2 \Theta
\]
\[
+ \frac{1}{128} \left[ (9 \cos \theta_1 - 21 \cos 3 \theta_1 - 2 (1 - 7 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3
\]
\[
+ 4 (\sin \theta_1 - 7 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\} \cos \theta_2 \sin \Theta \sin 3 \Theta \right\} \cos (k_1 \pi),
\]

\[
I_2 = \left\{ \frac{1}{128\sqrt{2}} \left[ (31 \sin \theta_1 + 15 \sin 3 \theta_1) \cos \theta_2 - 12 \cos^2 \theta_1 \sin \theta_1 \cos 3 \theta_2 \right) \sin 2 \theta_3
\]

\[
I_1 = \left\{ \frac{1}{64\sqrt{2}} \left[ 4 (3 \cos 3 \theta_1 - 7 \cos \theta_1) \sin \theta_2 \cos 2 \theta_3 + (31 \sin \theta_1 - 9 \sin 3 \theta_1 + 6 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \right\} \cos \theta_2 \cos \Theta \sin \Theta + \frac{1}{64\sqrt{2}} \left[ (9 \sin \theta_1 - 15 \sin 3 \theta_1 + 10 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right.
\]
\[
+ 4 (\cos \theta_1 - 5 \cos 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\] \cos \theta_2 \cos 3 \Theta \sin \Theta + \frac{1}{128} \left[ 4 (5 \sin \theta_1 - 3 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3
\]
\[
+ (13 \cos \theta_1 - 9 \cos 3 \theta_1 - 2 (5 - 3 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \cos \theta_2 \sin^2 \Theta
\]
\[
+ \frac{1}{128} \left[ (9 \cos \theta_1 - 21 \cos 3 \theta_1 - 2 (1 - 7 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3
\]
\[
+ 4 (\sin \theta_1 - 7 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\} \cos \theta_2 \sin \Theta \sin 3 \Theta \right\} \cos (k_1 \pi),
\]

\[
I_2 = \left\{ \frac{1}{128\sqrt{2}} \left[ (31 \sin \theta_1 + 15 \sin 3 \theta_1) \cos \theta_2 - 12 \cos^2 \theta_1 \sin \theta_1 \cos 3 \theta_2 \right) \sin 2 \theta_3
\]

\[
I_1 = \left\{ \frac{1}{64\sqrt{2}} \left[ 4 (3 \cos 3 \theta_1 - 7 \cos \theta_1) \sin \theta_2 \cos 2 \theta_3 + (31 \sin \theta_1 - 9 \sin 3 \theta_1 + 6 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \right\} \cos \theta_2 \cos \Theta \sin \Theta + \frac{1}{64\sqrt{2}} \left[ (9 \sin \theta_1 - 15 \sin 3 \theta_1 + 10 \cos \theta_1 \sin 2 \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right.
\]
\[
+ 4 (\cos \theta_1 - 5 \cos 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\] \cos \theta_2 \cos 3 \Theta \sin \Theta + \frac{1}{128} \left[ 4 (5 \sin \theta_1 - 3 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3
\]
\[
+ (13 \cos \theta_1 - 9 \cos 3 \theta_1 - 2 (5 - 3 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3 \right] \cos \theta_2 \sin^2 \Theta
\]
\[
+ \frac{1}{128} \left[ (9 \cos \theta_1 - 21 \cos 3 \theta_1 - 2 (1 - 7 \cos 2 \theta_1) \cos \theta_1 \cos 2 \theta_2) \sin 2 \theta_3
\]
\[
+ 4 (\sin \theta_1 - 7 \sin 3 \theta_1) \sin \theta_2 \cos 2 \theta_3 \right\} \cos \theta_2 \sin \Theta \sin 3 \Theta \right\} \cos (k_1 \pi),
\]

\[
I_2 = \left\{ \frac{1}{128\sqrt{2}} \left[ (31 \sin \theta_1 + 15 \sin 3 \theta_1) \cos \theta_2 - 12 \cos^2 \theta_1 \sin \theta_1 \cos 3 \theta_2 \right) \sin 2 \theta_3
\]
\[
-8 \left(6 + (1 + 3 \cos 2\theta_1) \cos 2\theta_3 \cos \theta_1 \sin 2\theta_2 \right) \cos \Theta \sin \Theta \\
+ \frac{1}{128\sqrt{2}} \left[ \left(9 \sin \theta_1 + 25 \sin 3\theta_1 \right) \cos \theta_2 - 20 \cos^2 \theta_1 \sin \theta_1 \cos 3\theta_2 \sin 2\theta_3 \right] \\
-8 \left(2 - (1 - 5 \cos 2\theta_1) \cos 2\theta_3 \cos \theta_1 \sin 2\theta_2 \right) \cos 3\Theta \sin \Theta \\
+ \frac{1}{256} \left[ 2 \left(-1 + 15 \cos 2\theta_1 \right) \cos \theta_2 + (10 - 6 \cos 2\theta_1 \cos 3\theta_2 \right) \cos \theta_1 \sin 2\theta_3 \right] \\
+8 \left(2 + (1 + 3 \cos 2\theta_1) \cos 2\theta_3 \sin \theta_1 \sin 2\theta_2 \right) \sin^2 \Theta \\
+ \frac{1}{128} \left[ \left((-13 + 35 \cos 2\theta_1 \right) \cos \theta_2 + (1 - 7 \cos 2\theta_1 \cos 3\theta_2 \right) \cos \theta_1 \sin \theta_1 \sin 2\theta_3 \right] \\
+4 \left(2 + (5 + 7 \cos 2\theta_1) \cos 2\theta_3 \right) \sin \theta_1 \sin 2\theta_2 \right] \sin \Theta \sin 3\Theta \right) \cos(k_2\pi). 
\]

(21)

For the type-X remnant CP transformation with one zero, we find that the resulting lepton mixing matrix is related

![Diagram](image_url)

**FIG. 4**: The correlation between the mixing parameters predicted in the case of type-X residual CP transformation. All the parameters $\theta_{1,2,3}$ and $\Theta$ are varied randomly in the range $[0, 2\pi]$. The three mixing angles $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$ are required to be within the $3\sigma$ allowed ranges [20]. The blue (green) points are obtained by fixing $\Theta = \frac{2\pi}{3}$, $\frac{2\pi}{5}$ as a benchmark example.

with the previous case through the exchange of the second and third rows. The expressions for the reactor and solar mixing angles, as well as for the $I_1$ and $I_2$ invariants, are the same as those of Eqs. (17, 20, 21), while the atmospheric mixing angle $\sin^2 \theta_{23}$ becomes $1 - \sin^2 \theta_{23}$. The Dirac CP phase $\delta_{CP}$ becomes $\pi + \delta_{CP}$ so that the overall sign of the Jarlskog invariant is reversed.
FIG. 5: The probability distribution of the lepton mixing parameters $\sin^2 \theta_{23}$, $\delta_{\text{CP}}$, $\alpha_{21}$ and $\alpha_{31}$ predicted for the case of type-X residual CP transformation. All the parameters $\theta_{1,2,3}$ and $\Theta$ are taken to be random numbers in the range $[0, 2\pi]$, and the three lepton mixing angles are required to be compatible with experimental data at 3$\sigma$ level [20].

III. DEMOCRATIC CP SYMMETRY AS AN EXAMPLE WITHOUT ZERO ELEMENTS

If no entry of the residual CP transformation $X$ vanishes, the explicit form of $X$ cannot be fixed uniquely. For illustration, in this section we shall study a particular CP symmetry whose elements have the same absolute value. That is to say, the absolute value of each element of $X$ is equal to $1/\sqrt{3}$. In what follows it will be dubbed as democratic CP symmetry. In this case, the most general form of $X$ can be written as

$$
X = \frac{1}{\sqrt{3}} \begin{pmatrix}
    e^{i\alpha} & e^{i\left(\frac{\alpha + \beta}{2} + \beta_3\right)} & e^{i\left(\frac{\alpha + \gamma}{2} + \beta_2\right)} \\
    e^{i\left(\frac{\alpha + \beta}{2} + \beta_3\right)} & e^{i\beta} & e^{i\left(\beta + \gamma + \beta_1\right)} \\
    e^{i\left(\frac{\alpha + \gamma}{2} + \beta_2\right)} & e^{i\left(\beta + \gamma + \beta_1\right)} & e^{i\gamma}
\end{pmatrix}.
$$

(22)

The unitary condition of $X$ implies that $\beta_1$, $\beta_2$ and $\beta_3$ should satisfy the following equalities $e^{i\beta_1} + e^{-i\beta_1} + e^{i(\beta_2 - \beta_3)} = 0$, $e^{i\beta_2} + e^{-i\beta_2} + e^{i(\beta_3 - \beta_1)} = 0$, and $e^{i\beta_3} + e^{-i\beta_3} + e^{i(\beta_1 - \beta_2)} = 0$. It can be easily checked that these equations have four pairs of solutions,

$$
\beta_1 = \beta_2 = \beta_3 = \pm \frac{2\pi}{3}, \quad X_1^\pm = \frac{1}{\sqrt{3}} \begin{pmatrix}
    e^{i\alpha} & e^{i\left(\frac{\alpha + \beta}{2} \pm \frac{2\pi}{3}\right)} & e^{i\left(\frac{\alpha + \gamma}{2} \pm \frac{2\pi}{3}\right)} \\
    e^{i\left(\frac{\alpha + \beta}{2} \pm \frac{2\pi}{3}\right)} & e^{i\beta} & e^{i\left(\beta + \gamma \pm \frac{2\pi}{3}\right)} \\
    e^{i\left(\frac{\alpha + \gamma}{2} \pm \frac{2\pi}{3}\right)} & e^{i\left(\beta + \gamma \pm \frac{2\pi}{3}\right)} & e^{i\gamma}
\end{pmatrix},
$$

(23)
We can see that the four admissible CP transformations $X^\pm_1$, $X^\pm_2$, $X^\pm_3$ and $X^\pm_4$ are related to each other as follows

$X^\pm_i = \text{diag}(1,-1)X^\pm_j \text{diag}(1,-1) = \text{diag}(-1,1)X^\pm_k \text{diag}(-1,1) = \text{diag}(-1,-1)X^\pm_3 \text{diag}(-1,-1,1)$. Therefore the Takagi factorization matrix $\Sigma^i$ for $X^\pm_i$ $(i = 1, 2, 3, 4)$ are related with each other as well, $\Sigma^1 = \text{diag}(1,-1,1)\Sigma^2 = \text{diag}(-1,-1,1)\Sigma^3 = \text{diag}(-1,1,1)\Sigma^4$. As a result, we conclude that the four CP transformations $X^+_1$, $X^+_2$, $X^+_3$ and $X^+_4$ give rise to the same lepton mixing matrix up to a phase factor which can be absorbed by redefining the charged lepton fields. Furthermore, it can be easily checked that $\Sigma^+_i$ and $\Sigma^-_i$ can be related by $\Sigma^-_i = \text{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma})\Sigma^+_i$. Therefore the predicted PMNS matrix by $X^+_i$ and $X^-_i$ are complex conjugate of each other up to the phase factor $\text{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma})$ which can also be absorbed by the charged leptons. Hence it is sufficient to only discuss the CP transformation $X_{\chi I} \equiv X^+_1$ which corresponds to $\beta_1 = \beta_2 = \beta_3 = 2\pi/3$ with

$$X_{\chi I} = \frac{1}{\sqrt{3}} \begin{pmatrix}
 e^{i\alpha} & e^{i(\frac{\alpha + \beta}{2} + \frac{\pi}{4})} & e^{i(\frac{\alpha + \gamma}{2} + \frac{\pi}{2})}
 e^{i(\frac{\alpha + \beta}{2} + \frac{\pi}{4})} & e^{i\beta} & e^{i(\frac{\beta + \gamma}{2} + \frac{\pi}{2})}
 e^{i(\frac{\alpha + \gamma}{2} + \frac{\pi}{2})} & e^{i(\frac{\beta + \gamma}{2} + \frac{\pi}{2})} & e^{i\gamma}
\end{pmatrix},$$

(24)

The corresponding Takagi factorization and the prediction for the PMNS matrix can be straightforwardly obtained

$$\Sigma_{\chi I} = \text{diag}(e^{i\alpha}, e^{i\beta}, e^{i\gamma}) e^{-\frac{i\pi}{4}} \begin{pmatrix}
 \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} & 0 \\
 -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\
\end{pmatrix} \text{diag}(1,e^{i\pi/3},1), \quad U = \Sigma_{\chi I} O_{3 \times 3} X_{\nu}^{-1/2}.$$  

(25)

We can read out the lepton mixing angles as

$$\sin^2 \theta_{13} = \frac{1}{6} \left[ 4 \sin^2 \theta_2 + \sqrt{2} \sin 2 \theta_2 \sin \theta_1 + 2 \sin^2 \theta_1 \cos^2 \theta_2 \right],$$

$$\sin^2 \theta_{12} = \left\{ \sin \theta_1 \left( \sqrt{2} \sin 2 \theta_2 - 2 \sin \theta_1 \sin^2 \theta_2 \right) - 4 \cos^2 \theta_2 \sin^2 \theta_3 - 2 \cos^2 \theta_1 \cos^2 \theta_3 + \left[ \sin 2 \theta_1 \sin \theta_2 \right. \\
- \sqrt{2} \cos \theta_1 \cos \theta_2 \sin 2 \theta_3 \right\} / \left(4 \sin^2 \theta_2 + \sqrt{2} \sin 2 \theta_2 \sin \theta_1 + 2 \sin^2 \theta_1 \cos^2 \theta_2 - 6 \right),$$

$$\sin^2 \theta_{23} = 2 \frac{\sin 2 \theta_2 \left( \sqrt{2} \sin \theta_1 + 2 \sqrt{3} \cos \theta_1 \right) - 2 \sin^2 \theta_2 - \cos^2 \theta_2 \left( \sqrt{6} \sin 2 \theta_1 + \cos 2 \theta_1 + 5 \right)}{2 \left(4 \sin^2 \theta_2 + \sqrt{2} \sin 2 \theta_2 \sin \theta_1 + 2 \sin^2 \theta_1 \cos^2 \theta_2 - 6 \right)}.$$  

(26)

For the CP invariants we get

$$J_{CP} = -\frac{1}{48\sqrt{2}} \left\{ 4 \sqrt{2} \sin 2 \theta_2 \sin \theta_1 \cos \theta_2 + 4 \sin 2 \theta_1 \cos 2 \theta_2 \right\} \cos 2 \theta_3 + \left[ 5 \sin \theta_2 \sin^2 \theta_1 \\
+ \sqrt{2} \sin \theta_1 \left( 5 \cos^2 \theta_1 - 1 \right) \cos \theta_2 + (3 \cos^2 \theta_1 + 1) \sin 3 \theta_2 + \sqrt{2} \sin^3 \theta_1 \cos 3 \theta_2 \right] \sin 2 \theta_3 \right\},$$

$$I_1 = \frac{-1 \kappa_1}{12 \sqrt{3}} \cos \theta_1 \cos \theta_2 \left\{ 3 \sqrt{2} \sin^2 \theta_1 - 2 \sin 2 \theta_2 \sin \theta_1 + \sqrt{2} \left( \cos^2 \theta_1 + 1 \right) \cos 2 \theta_2 \right\} \sin 2 \theta_3$$

$$I_3 = \frac{-1 \kappa_3}{12 \sqrt{3}} \cos \theta_1 \cos \theta_2 \left\{ 3 \sqrt{2} \sin^2 \theta_1 - 2 \sin 2 \theta_2 \sin \theta_1 + \sqrt{2} \left( \cos^2 \theta_1 + 1 \right) \cos 2 \theta_2 \right\} \sin 2 \theta_3$$

$$I_2 = \frac{-1 \kappa_2}{12 \sqrt{3}} \cos \theta_1 \cos \theta_2 \left\{ 3 \sqrt{2} \sin^2 \theta_1 - 2 \sin 2 \theta_2 \sin \theta_1 + \sqrt{2} \left( \cos^2 \theta_1 + 1 \right) \cos 2 \theta_2 \right\} \sin 2 \theta_3.$$
\[ +2 \left[ 2 \cos \theta_1 \cos \theta_2 - \sqrt{2} \sin 2 \theta_1 \sin \theta_2 \right] \cos 2 \theta_3 \right), \]

\[ I_2 = \frac{(-1)^k_2}{6\sqrt{3}} (\sin \theta_1 \cos \theta_3 + \sin \theta_2 \sin \theta_3 \cos \theta_1) \left\{ \left[ 2 \sin \theta_1 \cos 2 \theta_2 + \sqrt{2} \sin 2 \theta_2 \left( \cos^2 \theta_1 + 1 \right) \right] \cos \theta_3 \right. \]
\[ \left. - \left[ \sqrt{2} \sin 2 \theta_1 \cos \theta_2 + 2 \sin \theta_2 \cos \theta_1 \right] \sin \theta_3 \right\}. \tag{27} \]

**FIG. 6**: The probability distribution of the lepton mixing parameters \( \sin^2 \theta_{23}, \delta_{CP}, \alpha_{21} \) and \( \alpha_{31} \) predicted for the case of democratic CP symmetry. The parameters \( \theta_1, \theta_2, \theta_3 \) are taken to be random numbers in the range \([0, 2\pi]\), and the three lepton mixing angles are required to be compatible with experimental data at 3\( \sigma \) level [20].

**IV. NUMERICAL ANALYSIS**

Summarizing the above discussion, we see that type-I, -IV, -V, -VI, -VII, -IX, -X and -XI residual CP transformations with zero elements can accommodate the current experimental neutrino oscillation data [20]. In all these cases, the three mixing angles \( \theta_{12}, \theta_{13} \) and \( \theta_{23} \) as well as three CP phases \( \delta_{CP}, \alpha_{21} \) and \( \alpha_{31} \) are found to depend on just four free independent parameters \( \Theta, \theta_1, \theta_2 \) and \( \theta_3 \), where \( \Theta \) characterizes the shape of the residual CP transformations. This characterizes the degree of predictivity of our present framework. For example, the type-I CP transformation corresponds to the widely studied \( \mu - \tau \) reflection, and leads to \( \theta_{23} = 45^\circ, \delta_{CP} = \pm 90^\circ \) and \( \alpha_{21}, \alpha_{31} = 0 \) or \( \pi \) while the solar and reactor mixing angles are not constrained. On the other hand, the type-IV CP transformation with three
texture zeros is diagonal, and corresponds to the conventional CP transformation. As expected, in this case all three CP phases are predicted to vanish. The CP transformation of type-V is the same as the generalized $\mu - \tau$ reflection which has been discussed by us in Ref. \cite{7}. For the case of $\Theta = \pi/2$, our generalized $\mu - \tau$ reflection reduces to the standard $\mu - \tau$ reflection symmetry. This would provide an interesting new starting point for model building if either $\theta_{23}$ or $\delta_{CP}$ were established to be non-maximal by future neutrino oscillation experiments.

As we already mentioned, the lepton mixing matrices for the case of two–zero texture type-VI and type-VII are related by the exchange of the second and the third rows. The mixing angles and CP invariants are given in Eq. (13) and Eq. (15) respectively. In order to visualize the theoretical predictions in a more clear way, we perform a numerical analysis where the free parameters $\theta_{1,2,3}$ and the CP parameter $\Theta$ are scanned over the range of $[0, 2\pi]$, while the mixing parameters are calculated for each point, retaining only points that agree at $3\sigma$ level with experimentally determined mixing angles \cite{20}. The correlations between the mixing parameters and distributions of the mixing parameters are plotted in Fig. 2 and Fig. 3.

One sees that the three CP phases are strongly correlated with each other, the Majorana phase $\alpha_{21}$ around $\pi/5$, $4\pi/5$, $6\pi/5$ and $9\pi/5$ is preferred, and the Majorana phase $\alpha_{31}$ around $3\pi/20$, $17\pi/20$, $23\pi/23$ and $37\pi/20$ is favored. If we set a value to the CP parameter $\Theta$ then the explicit form of the CP transformation $X$ is fixed, so that definite predictions for the CP phases are obtained. As examples, we consider the case that the parameter $\Theta$ takes some specific values $\pi/5$, $2\pi/17$, $\pi/8$, $2\pi/15$ and $\pi/7$, the values of the parameters $\theta_1$, $\theta_2$ and $\theta_3$ are determined by the experimental

| $\Theta$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\delta_{CP}$ | $\alpha_{21}$ (mod $\pi$) | $\alpha_{31}$ (mod $\pi$) |
|----------|------------|------------|------------|-------------|----------------|----------------|
| $\pi/5$  | 48.167° or 131.833° | 49.591° | 304.258° | 147.618° | 159.573° |
| $2\pi/17$ | 48.167° or 131.833° | 118.010° | 136.032° | 32.382° | 23.728° |
| $\pi/8$  | 48.167° or 131.833° | 61.990° | 223.968° | 147.618° | 159.573° |
| $2\pi/15$ | 48.167° or 131.833° | 130.409° | 55.742° | 32.382° | 20.427° |
| $\pi/7$  | 48.167° or 131.833° | 118.541° | 132.071° | 34.250° | 24.956° |
| $\pi/5$  | 48.167° or 131.833° | 61.459° | 227.929° | 147.618° | 159.573° |
| $2\pi/17$ | 48.167° or 131.833° | 129.782° | 60.419° | 34.250° | 21.790° |
| $\pi/8$  | 48.167° or 131.833° | 51.057° | 293.664° | 143.642° | 156.616° |
| $2\pi/15$ | 48.167° or 131.833° | 119.265° | 126.967° | 36.358° | 26.276° |
| $\pi/7$  | 48.167° or 131.833° | 60.735° | 233.033° | 143.642° | 153.724° |
| $\pi/5$  | 48.167° or 131.833° | 128.943° | 66.336° | 36.358° | 23.384° |
| $2\pi/17$ | 48.167° or 131.833° | 52.287° | 285.502° | 141.268° | 154.690° |
| $\pi/8$  | 48.167° or 131.833° | 120.355° | 119.731° | 38.732° | 27.651° |
| $2\pi/15$ | 48.167° or 131.833° | 59.645° | 240.269° | 141.268° | 152.349° |
| $\pi/7$  | 48.167° or 131.833° | 127.713° | 74.498° | 38.732° | 25.310° |
| $\pi/5$  | 48.167° or 131.833° | 54.848° | 269.598° | 138.569° | 152.045° |
| $2\pi/17$ | 48.167° or 131.833° | 122.745° | 104.895° | 41.431° | 28.774° |
| $\pi/8$  | 48.167° or 131.833° | 57.255° | 255.105° | 138.569° | 151.226° |
| $2\pi/15$ | 48.167° or 131.833° | 125.152° | 90.402° | 41.431° | 27.955° |

TABLE IV: The predictions for the Dirac and Majorana CP phases in the case of type-VI residual CP transformation. The parameter $\Theta$ is set to the representative values of $\pi/9$, $2\pi/17$, $\pi/8$, $2\pi/15$ and $\pi/7$. The parameters $\theta_1$, $\theta_2$ and $\theta_3$ are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for NH neutrino mass spectrum \cite{20}.
TABLE V: The predictions for the Dirac and Majorana CP phases in the case of type-VI residual CP transformation. The parameter $\Theta$ is set to the representative values of $\pi/9$, $2\pi/17$, $\pi/8$, $2\pi/15$ and $\pi/7$. The parameters $\theta_1$, $\theta_2$ and $\theta_3$ are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for IH neutrino mass spectrum [20].

The best fit values of the lepton mixing angles from [20]. As a consequence, the lepton mixing matrix is fully fixed up to the factor $X^{-1/2}$, and the values of the CP violating phases can be predicted, as are shown in Table IV for normal hierarchy (NH) and Table V for inverted hierarchy (IH). We can see that different values of the Dirac CP phase $\delta_{CP}$ can be achieved. Note in particular that, for certain values of $\Theta$, $\theta_1$, $\theta_2$ and $\theta_3$, the magnitude of $\delta_{CP}$ can be quite close to 270° which is weakly favored by present data [21].

The lepton mixing matrices for the one zero texture type-IX and -X differ by a permutation of the second and the third rows. The expressions for the mixing angles and CP invariants are given in Eqs. [17, 19, 20, 21]. The numerical results for the correlation among the mixing parameters and probability distributions of the mixing parameters are displayed in Fig. 4 and Fig. 5. The strong correlations between different CP phases emerge once the value of the parameter $\Theta$ is fixed.

We see that $\theta_{23}$ close to maximal mixing is favored for the type-X texture, and both Majorana phases $\alpha_{21}$ and $\alpha_{31}$ tend to close to 0, $\pi$ and $2\pi$. There appears to be no preferred $\delta_{CP}$ phase within the viable parameter space. Furthermore, we study some concrete benchmark cases in which the parameters $\Theta$ take on certain representative values. The value of the parameters $\theta_1$, $\theta_2$ and $\theta_3$ are fixed by the best fit value of the lepton mixing angles. In this way the CP violating phases can be predicted as listed in Table VI and Table VII for NH and IH mass spectrums respectively. Future long baseline facilities DUNE [22], LBNO [23], T2HK [24] can bring us increased precision on the Dirac phase $\delta_{CP}$. If $\delta_{CP}$ was measured to be far from any of the values in Table VI and Table VII, the present
proposal would be disfavored.

| $\alpha$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\delta_{CP}$ | $\alpha_{21}$ (mod $\pi$) | $\alpha_{31}$ (mod $\pi$) |
|----------|-------------|-------------|-------------|---------------|----------------|----------------|
| $\alpha_1$ | 66.743° | 4.290° | 57.595° | 108.232° | 60.800° | 40.840° |
| $\alpha_1$ | 66.743° | 4.290° | 123.826° | 264.695° | 119.200° | 136.503° |
| $\alpha_1$ | 66.743° | 175.710° | 57.595° | 251.768° | 119.200° | 139.160° |
| $\alpha_1$ | 66.743° | 175.710° | 123.826° | 95.305° | 60.800° | 43.497° |
| $\alpha_2$ | 66.215° | 12.491° | 58.980° | 122.397° | 62.920° | 39.600° |
| $\alpha_2$ | 66.215° | 12.491° | 124.976° | 277.881° | 117.080° | 132.164° |
| $\alpha_2$ | 66.215° | 167.509° | 58.980° | 237.603° | 117.080° | 140.400° |
| $\alpha_2$ | 66.215° | 167.509° | 124.976° | 82.119° | 62.920° | 47.836° |
| $\alpha_3$ | 66.060° | 13.984° | 60.881° | 245.654° | 117.249° | 76.422° |
| $\alpha_3$ | 66.060° | 13.984° | 114.766° | 5.228° | 62.751° | 78.747° |
| $\alpha_3$ | 66.060° | 166.016° | 60.881° | 114.346° | 62.751° | 103.578° |
| $\alpha_3$ | 66.060° | 166.016° | 114.766° | 354.772° | 117.249° | 101.253° |
| $\alpha_3$ | 66.499° | 9.096° | 62.706° | 268.830° | 124.491° | 87.338° |
| $\alpha_3$ | 66.499° | 9.096° | 113.502° | 20.806° | 55.509° | 74.823° |
| $\alpha_3$ | 66.499° | 170.904° | 62.706° | 91.170° | 55.509° | 92.662° |
| $\alpha_3$ | 66.499° | 170.904° | 113.502° | 339.194° | 124.491° | 105.177° |

TABLE VI: The predictions for the Dirac and Majorana CP phases in the case of type-X residual CP transformation. The parameter $\Theta$ is set to the representative values of $\pi/9$, $2\pi/17$, $2\pi/9$ and $3\pi/13$. The parameters $\theta_1$, $\theta_2$ and $\theta_3$ are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for NH neutrino mass spectrum [20].

We perform a numerical analysis by treating the parameters $\theta_{1,2,3}$ as random real numbers scanned over the range $[0, 2\pi]$, with the three mixing angles calculated for each point in the parameter space. Subsequently only points which simultaneously are be compatible with experimental data [20] are retained and from these points the CP violating phases are calculated. The predicted distributions of the lepton mixing parameters are shown in Fig. 6.

Since no specific values of $\theta_{12}$ and $\theta_{13}$ are favored within $3\sigma$, and hence they are not shown in the figure. We see that the atmospheric mixing angle $\theta_{23}$ can be either the first octant or the second octant. As regards the CP phases, there appears to be a slight preference for $\delta_{CP} \sim \pi/2$ and $\delta_{CP} \sim 3\pi/2$, and the Majorana phase $\alpha_{21}$ around $\pi/4$, $3\pi/4$, $5\pi/4$ and $7\pi/4$ are favored while the values of $\alpha_{31}$ around 0 and $\pi$ are preferred. For certain values of $\theta_{1,2,3}$, the best fit values of the mixing angles can be reproduced, and the corresponding predictions for CP phases are listed in Table VIII.

V. PHENOMENOLOGICAL IMPLICATIONS

Implications of the generalized $\mu - \tau$ reflection symmetry have already been discussed in Ref. [7]. In this section, we shall consider the phenomenological implications of the residual CP transformations as we have classified above in Tables I, II and III, focussing on the case of “neutrino appearance” oscillation experiments and neutrinoless double beta decay. The cosmological implications for leptogenesis will be studied as well.
| Θ  | θ₁  | θ₂  | θ₃  | δₐₚ | α₂₁ (mod π) | α₃₁ (mod π) |
|----|-----|-----|-----|------|-------------|-------------|
| 2π/17 | 65.802° | 6.854° | 58.218° | 113.294° | 64.070° | 42.230° |
| 2π/17 | 65.802° | 6.854° | 124.081° | 268.239° | 115.930° | 133.106° |
| 2π/17 | 65.802° | 173.146° | 58.218° | 246.706° | 115.930° | 137.770° |
| 2π/17 | 65.802° | 173.146° | 124.081° | 91.761° | 64.070° | 46.894° |
| 2π/17 | 65.244° | 13.626° | 59.379° | 126.150° | 66.506° | 41.401° |
| 2π/17 | 65.244° | 13.626° | 124.950° | 279.935° | 113.494° | 128.680° |
| 2π/17 | 65.244° | 166.374° | 59.379° | 233.850° | 113.494° | 138.599° |
| 2π/17 | 65.244° | 166.374° | 124.950° | 80.065° | 66.506° | 51.320° |
| 3π/17 | 64.729° | 17.577° | 59.985° | 136.166° | 69.451° | 41.655° |
| 3π/17 | 64.729° | 17.577° | 125.180° | 288.512° | 110.549° | 124.550° |
| 3π/17 | 64.729° | 162.423° | 59.985° | 223.834° | 110.549° | 138.345° |
| 3π/17 | 64.729° | 162.423° | 125.180° | 71.488° | 69.451° | 55.450° |
| 4π/17 | 65.352° | 12.624° | 61.950° | 253.073° | 122.689° | 82.112° |
| 4π/17 | 65.352° | 12.624° | 113.570° | 7.050° | 57.320° | 73.409° |
| 4π/17 | 65.352° | 167.376° | 61.950° | 106.927° | 57.320° | 97.888° |
| 4π/17 | 65.352° | 167.376° | 113.570° | 352.950° | 122.689° | 106.591° |
| 4π/17 | 65.587° | 10.048° | 63.026° | 265.635° | 126.593° | 88.250° |
| 4π/17 | 65.587° | 10.048° | 112.818° | 15.195° | 53.407° | 71.217° |
| 4π/17 | 65.587° | 169.952° | 63.026° | 94.365° | 53.407° | 91.750° |
| 4π/17 | 65.587° | 169.952° | 112.818° | 344.805° | 126.593° | 108.783° |

TABLE VII: The predictions for the Dirac and Majorana CP phases in the case of type-X residual CP transformation. The parameter Θ is set to the representative values of $2\pi/17$, $\pi/8$, $2\pi/15$, $3\pi/13$ and $4\pi/17$. The parameters $\theta_1$, $\theta_2$ and $\theta_3$ are fixed by the requirement of reproducing the best fit values of the three lepton mixing angles for III neutrino mass spectrum [20].

| Θ  | θ₁  | θ₂  | θ₃  | δₐₚ | α₂₁ (mod π) | α₃₁ (mod π) |
|----|-----|-----|-----|------|-------------|-------------|
| NH | 13.015° | 2.354° | 127.433° | 65.801° | 59.150° | 172.892° |
| NH | 13.015° | 2.354° | 179.849° | 309.873° | 120.850° | 177.814° |
| NH | 176.556° | 9.122° | 2.170° | 172.589° | 123.407° | 6.067° |
| NH | 176.556° | 9.122° | 53.464° | 285.771° | 56.593° | 175.842° |
| IH | 13.424° | 2.113° | 127.462° | 64.401° | 59.347° | 172.721° |
| IH | 13.424° | 2.113° | 179.962° | 308.266° | 120.653° | 177.239° |
| IH | 176.886° | 9.403° | 2.220° | 173.932° | 123.420° | 6.171° |
| IH | 176.886° | 9.403° | 53.508° | 287.100° | 56.580° | 175.919° |

TABLE VIII: The predictions for the Dirac and Majorana CP phases for the democratic residual CP transformation. The values of $\theta_1$, $\theta_2$ and $\theta_3$ are fixed by the requirement of accommodating the best fit values of the three lepton mixing angles [20].
FIG. 7: In the left panel we show the $\nu_\mu \to \nu_e$ transition probability in matter for a neutrino energy of $E = 1\text{GeV}$. The right panel displays the neutrino-anti-neutrino asymmetry $A_{\mu e}$ in matter. The oscillation parameters are taken within their currently allowed $3\sigma$ regions [20]. The plot corresponds to type-VI residual CP symmetry.

FIG. 8: The transition probability $P(\nu_\mu \to \nu_e)$ at a baseline of 295km which corresponds to the T2K experiment. The neutrino oscillation parameters are taken within the currently allowed $3\sigma$ regions [20]. The plot corresponds to the case of type-VI residual CP symmetry.

A. CP violation in conventional neutrino oscillations

The existence of leptonic CP violation would manifest itself as the differences in the oscillation probabilities involving neutrinos and anti-neutrinos in vacuum [25]:

$$\Delta P_{\alpha \beta} \equiv P(\nu_\alpha \to \nu_\beta) - P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = -16 J_{\alpha \beta} \sin \Delta_{21} \sin \Delta_{23} \sin \Delta_{31},$$

where we have adopted standard definitions $\Delta_{kj} \equiv \Delta m^2_{kj} L/(4E)$ and $\Delta m^2_{kj} = m^2_k - m^2_j$, $L$ is the baseline and $E$ stands for the energy of neutrino beam. The Jarlskog invariant is identified as

$$J_{\alpha \beta} = \Im \left( U_{\alpha 1} U_{\beta 2}^* U_{\alpha 2} U_{\beta 1}^* \right) = \pm J_{CP},$$  \hspace{1cm} (28)
where the positive (negative) sign holds for (anti-)cyclic permutation of the flavour indices \( e, \mu \) and \( \tau \). For the oscillation between electron and muon neutrinos, the transition probability of \( \nu_\mu \to \nu_e \) in vacuum is given by \(^{[25]}\)

\[
P(\nu_\mu \to \nu_e) \simeq P_{\text{atm}} + 2\sqrt{P_{\text{atm}}P_{\text{sol}} \cos(\Delta_{32} + \delta_{\text{CP}})} + P_{\text{sol}},
\]

where \( \sqrt{P_{\text{atm}}} = \sin \theta_{23} \sin 2\theta_{13} \sin \Delta_{31} \) and \( \sqrt{P_{\text{sol}}} = \cos \theta_{23} \cos \theta_{13} \sin 2\theta_{12} \sin \Delta_{21} \). As a result, the oscillation probability asymmetry between neutrinos and anti-neutrinos in vacuum is of the form:

\[
A_{\mu e} = \frac{P(\nu_\mu \to \nu_e) - P(\bar{\nu}_\mu \to \bar{\nu}_e)}{P(\nu_\mu \to \nu_e) + P(\bar{\nu}_\mu \to \bar{\nu}_e)} = \frac{2\sqrt{P_{\text{atm}}P_{\text{sol}} \sin \Delta_{32} \sin \delta_{\text{CP}}}}{P_{\text{atm}} + 2\sqrt{P_{\text{atm}}P_{\text{sol}} \cos \Delta_{32} \cos \delta_{\text{CP}}} + P_{\text{sol}}}. \tag{30}
\]

In order to accurately describe realistic long baseline neutrino oscillation experiments such as T2K, NO\( \nu \)A or the DUNE proposal, it is important to include the matter effect associated with neutrino propagation inside the Earth. Indeed the latter could induce a fake CP violation effect. In this case the expressions for \( \sqrt{P_{\text{atm}}} \) and \( \sqrt{P_{\text{sol}}} \) in matter take the form:

\[
\sqrt{P_{\text{atm}}} = \sin \theta_{23} \sin 2\theta_{13} \sin (\Delta_{31} - aL) \Delta_{31}, \quad \sqrt{P_{\text{sol}}} = \cos \theta_{23} \sin 2\theta_{12} \frac{\sin(aL)}{aL} \Delta_{21}, \tag{31}
\]

where \( a = G_F N_e / \sqrt{2} \), \( G_F \) is the Fermi constant and \( N_e \) is the density of electrons. The parameter \( a \) is approximately equal to \((3500 \text{ km})^{-1}\) for \( \rho Y_e = 3.0 \text{ g cm}^{-3} \), where \( Y_e \) is the electron fraction \(^{[25]}\).

In Fig. 7 we show the \( \nu_\mu \to \nu_e \) transition probability as well as the neutrino-anti-neutrino asymmetry in matter, when the residual CP transformation matrix \( X \) is assumed to be type-VI. In this figure we require the oscillation mixing angles lie within their currently allowed 3\( \sigma \) region \(^{[20]}\).

![Diagram](image.png)

FIG. 9: The transition probability \( P(\nu_\mu \to \nu_e) \) for type-VI residual CP symmetry case at a baseline of 810km which corresponds to the NO\( \nu \)A experiment. The neutrino oscillation parameters are taken within their currently allowed 3\( \sigma \) regions \(^{[20]}\).

In Figs. 8, 9 we show the corresponding behavior of the transition probability \( P(\nu_\mu \to \nu_e) \) in terms of neutrino energy \( E \), as well as of the CP parameter \( \Theta \) describing our approach, when the CP symmetry matrix \( X \) to be type-VI, for baseline values 295 and 810 km, which correspond to the current T2K and NO\( \nu \)A experiments, respectively. One sees that the allowed values of the CP parameter \( \Theta \) describing our approach are quite restricted.
B. Neutrinoless double decay

The rare decay \( (A, Z) \to (A, Z + 2) + e^- + e^- \) is the lepton number violating process “par excellence”. Its observation would establish the Majorana nature of neutrinos irrespective of their underlying mass generation mechanism \[26, 27\]. Within the simplest “long-range” light neutrino exchange mechanism its amplitude is sensitive to the Majorana phases. As discussed in \[12\] the most convenient parametrization of the lepton mixing matrix for the description of neutrinoless double decay is the fully symmetric one \[6\]. However, instead of using the “symmetrical” description as in \[7\], here we stick to the PDG form \[11\].

![FIG. 10: The effective mass \( |m_{ee}| \) describing neutrinoless double beta decay for type-I CP symmetry. Both reactor and solar mixing angles are required to be within the experimental 3σ interval \[20\], while the atmospheric mixing angle is predicted to be maximal with \( \theta_{23} = \pi/4 \). For the inverted neutrino mass ordering, the cyan region corresponds to \( (k_1, k_2) = (0, 0), (0, 1) \), and the purple area corresponds to \( (k_1, k_2) = (1, 0), (1, 1) \). For the normal ordering, the brown, magenta, blue and dark green regions correspond to \( (k_1, k_2) = (0, 0), (0, 1), (1, 0) \) and \( (1, 1) \) respectively. The red and blue dashed lines indicate the 3σ boundaries allowed by current neutrino oscillation data \[20\] for inverted and normal neutrino mass ordering, respectively. For comparison we show also the most stringent upper bound from \( 0\nu\beta\beta \) searches, as well as current Planck sensitivity.

Notice that both Majorana phases \( \alpha_{21} \) and \( \alpha_{31} \) can be shifted by \( \pi \) by the matrix \( \hat{X}^{-1/2} \) in Eq. \[3\]. Under the transformation \( k_1 \to k_1 + 1 \) (\( k_2 \to k_2 + 1 \)), we have \( \alpha_{21} \to \alpha_{21} + \pi \) (\( \alpha_{31} \to \alpha_{31} + \pi \)). Hence without loss of generality, we can focus on four different cases \( (k_1, k_2) = (0, 0), (0, 1), (1, 0) \) and \( (1, 1) \).

We illustrate our results for the effective \( 0\nu\beta\beta \) mass parameter \( |m_{ee}| \) by considering the type-I CP symmetric scheme, given in Fig. \[10\] the type-VI case, given in Fig. \[11\] as well as the results for type-IX given in Fig. \[12\] The residual CP transformations of type-V, type-VII and type-X don’t lead to new results for the effective mass \( |m_{ee}| \).
FIG. 11: The effective mass $|m_{ee}|$ describing the neutrinoless double beta decay amplitude. Using the current neutrino oscillation parameters at $3\sigma$ \cite{20} one obtains the regions delimited by the red and blue dashed lines for inverted and normal neutrino mass ordering, respectively. In contrast to such generic case, the blue and orange regions correspond to letting $\Theta$ and $\theta_{1,2,3}$ as free parameters in the type-VI case, while the green and magenta regions correspond to $\Theta = \frac{\pi}{7}$ with $\theta_{1,2,3}$ free.

since $\theta_{23}$ is not involved in Eq. (32). The experimental errors on the mass-squared splittings are not considered, and the best fit values from \cite{20} are used with $\Delta m^2_{21} = 7.60 \times 10^{-5} \text{eV}^2$ and $|\Delta m^2_{31}| = 2.48 \times 10^{-3} \text{eV}^2$ for normal ordering and $|\Delta m^2_{31}| = 2.38 \times 10^{-3} \text{eV}^2$ for inverted ordering. Notice that the red and blue dashed lines (for inverted and normal neutrino mass ordering, respectively) denote the regions allowed at $3\sigma$ level by current neutrino oscillation data \cite{20} for a generic model, without any special residual CP symmetry. For comparison we display the results for various CP symmetric cases. We also indicate the disfavored band associated to the most stringent upper bound

$$|m_{ee}| < 0.120 \text{ eV}$$

which follows from the EXO-200 experiment \cite{31,32} in combination with results from the first phase of the KamLAND-ZEN experiment \cite{33}. On the other hand the cosmological upper limit on the mass of the lightest neutrino corresponding to the latest Planck result is

$$\sum_i m_i < 0.230 \text{ eV}$$

at the 95% confidence level \cite{34}.

The results of this section are summarized in Figs. 10, 11 and 12 corresponding to the schemes based on type-I, type-VI and type-IX remnant CP symmetries respectively. They clearly show that the attainable values for the effective
FIG. 12: The effective mass $|m_{ee}|$ describing the neutrinoless double beta decay amplitude. Using the current neutrino oscillation parameters at $3\sigma$ one obtains the regions delimited by the red and blue dashed lines for inverted and normal neutrino mass ordering, respectively. In contrast to such generic case, the blue and orange regions correspond to the type-IX case, with $\Theta$ and $\theta_{1,2,3}$ taken as free parameters, while the green and magenta regions correspond to $\Theta = \frac{\pi}{7}$ with $\theta_{1,2,3}$ free. Mass parameter $|m_{ee}|$ cover more restrictive ranges than those expected in generic, non-CP symmetric schemes.

In particular, as illustrated by the green regions in the lower panels of Figs. 11 and 12, the generalized CP symmetry assumption may prevent the destructive interference amongst individual neutrino contributions. This leads to lower bounds for the $0\nu\beta\beta$ decay rates even for normal hierarchical neutrino mass spectra. This behavior is a reminiscent of situations already encountered in the framework of specific flavour symmetry based models [35–38].

In the case of $\Sigma$ matrix type-XI, the predictions for the effective mass of the neutrinoless double beta decay are shown in Fig. 13. As one can read off from this figure, the effective mass $|m_{ee}|$ is around 0.026 eV or 0.040 eV for IH neutrino mass spectrum, which are within the sensitivity of planned $0\nu\beta\beta$ decay experiments. In the case of NH spectrum, the value of $|m_{ee}|$ is bounded from below: $|m_{ee}| \geq 0.00065$ eV for $(k_1, k_2) = (0, 0)$, $|m_{ee}| \geq 0.00056$ eV for $(k_1, k_2) = (0, 1)$, and $|m_{ee}| \geq 0.0011$ eV for $(k_1, k_2) = (1, 0), (1, 1)$.

C. Leptogenesis

The origin of matter-antimatter asymmetry in the Universe is a puzzling and unexplained phenomenon. Although Sakharov discovered that CP violation is a necessary condition for explaining the matter-antimatter asymmetry of the
FIG. 13: The effective mass $|m_{ee}|$ describing neutrinoless double beta decay for the democratic CP symmetry. The red and blue dashed lines indicate the $3\sigma$ regions allowed by current neutrino oscillation data [20] for inverted and normal neutrino mass ordering, respectively. The blue and orange areas denote the possible values of $|m_{ee}|$ where $\theta_{1,2,3}$ are treated as free parameters and the three mixing angles are in the experimentally preferred $3\sigma$ range.

In this section, we shall study the phenomenological consequence for leptogenesis if there is only one residual CP transformation in the neutrino sector. We shall consider the classical scenario of leptogenesis from the lightest right-handed (RH) neutrino $N_1$ decay in the type-I seesaw model. In the RH neutrino and charged lepton mass basis, the type-I seesaw lagrangian can be written as

$$-\mathcal{L} = y_\alpha \bar{L}_\alpha H l_{\alpha R} + \bar{N}_R \lambda_{\alpha} \tilde{H} L_\alpha + \frac{1}{2} M_i \bar{N}_i R N_j R + h.c.$$

(34)

where $L_\alpha$ and $l_{\alpha R}$ denote the standard model left-handed (LH) lepton doublet and RH lepton singlet fields with $\alpha = e, \mu, \tau$ and $H$ is the Higgs doublet field with the vacuum expectation value $v = \langle H^0 \rangle = 174$ GeV. The light
neutrino mass matrix is given by the well-known seesaw formula
\[ m_\nu = v^2 \lambda^T M^{-1} \lambda = U^* m U^\dagger, \] (35)
where we denote \( M = \text{diag}(M_1, M_2, M_3) \) and \( m = \text{diag}(m_1, m_2, m_3) \), and \( m_i \) are the light neutrino mass eigenvalues. The most general neutrino Yukawa coupling matrix compatible with the low energy data is given by [45]:
\[ \lambda = \sqrt{M} R \sqrt{m} U^\dagger / v, \] (36)
where \( R \) is generally a complex orthogonal matrix fulfilling \( RR^T = R^T R = 1 \).

The temperature of the universe at the very early time was extremely high and the lightest RH neutrino \( N_1 \) is in thermal equilibrium. As the temperature drops down to \( M_1 \), the \( N_1 \) decay process \( N_1 \rightarrow Hl_\alpha \) and its inverse process start to go out of equilibrium and an asymmetry between leptons and antileptons is induced accordingly. As the temperature of universe goes down to the critical temperature of the electroweak phase transition, the sphaleron interactions convert lepton asymmetry to baryon asymmetry. One can define the CP asymmetry generated by \( N_1 \) decays as [46-50]
\[ \epsilon_\alpha \equiv \frac{\Gamma(N_1 \rightarrow Hl_\alpha) - \Gamma(N_1 \rightarrow \bar{Hl}_\alpha)}{\sum_{\alpha} \Gamma(N_1 \rightarrow Hl_\alpha) + \Gamma(N_1 \rightarrow \bar{Hl}_\alpha)} \]
(37)
\[ = \frac{1}{8\pi(\lambda\lambda_1^i)_{ij}} \sum_{j \neq 1} \left\{ \text{Im}[(\lambda\lambda^\dagger)_1^i \lambda_1^j \lambda_j^* \alpha^*_\alpha] g(x_j) + \text{Im}[(\lambda\lambda^\dagger)_1^j \lambda_1^i \lambda_i^* \alpha^*_\alpha] \frac{1}{1 - x_j} \right\}, \] (38)
where \( x_j = M_j^2 / M_1^2 \) and \( g(x) \) is the loop function with
\[ g(x) = \sqrt{x} \left[ \frac{1}{1 - x} + 1 - (1 + x) \ln \left( \frac{1 + x}{x} \right) \right], \quad x \gg 1 \rightarrow -\frac{3}{2\sqrt{x}}. \] (39)
As usual, we assume a hierarchical RH neutrinos mass spectrum \( M_1 \ll M_2 \ll M_3 \) which implies \( x_j \gg 1 \). As a consequence, the flavored CP asymmetries are approximately given by [43, 50, 55]:
\[ \epsilon_\alpha = -\frac{3M_1}{16\pi v^2} \Im \left( \sum_{ij} \sqrt{m_i m_j} m_j R_{ij} R_{ij}^* U_{\alpha i}^* U_{\alpha j} \right) / \sum_j m_j |R_{ij}|^2, \] (40)
Besides the CP parameter \( \epsilon_\alpha \), the final baryon asymmetry depends on the flavour-dependent washout mass parameters,
\[ \bar{m}_\alpha = \left| \lambda_1^\alpha \right|^2 v^2 / M_1 = \left| \sum_j m_j^{1/2} R_{ij} U_{\alpha j}^* \right|^2. \] (41)
At temperatures \( T \sim M_1 > 10^{12} \) GeV where all lepton flavors are out of equilibrium, the total lepton asymmetry \( \epsilon_1 \) is the sum of the \( \epsilon_\alpha \),
\[ \epsilon_1 = \sum_\alpha \epsilon_\alpha = -\frac{3M_1}{16\pi v^2} \sum_i m_i^2 \Im \left( R_{1i}^2 \right) / \sum_j m_j |R_{ij}|^2, \] (42)
which exactly the standard one-flavor result [42, 43]. In the present work we shall be concerned with temperatures \( 10^9 \leq T \sim M_1 \leq 10^{12} \) GeV. In this mass window only the interactions mediated by the \( \tau \) Yukawa coupling are in equilibrium and the final baryon asymmetry is well approximated by [48, 52]
\[ Y_B \simeq -\frac{12}{37} g^* \left[ \epsilon_2 \eta \left( \frac{417}{589} \bar{m}_2 \right) + \epsilon_\tau \eta \left( \frac{390}{589} \bar{m}_\tau \right) \right], \] (43)
where the number of relativistic degrees of freedom $g^*$ is taken to be $g^* = 106.75$ as in the standard model. The combined asymmetry $\epsilon_2 = \epsilon_e + \epsilon_\mu$ comes from the indistinguishable $e$ and $\mu$ flavored leptons and $\tilde{m}_2 = \tilde{m}_e + \tilde{m}_\mu$. The efficiency factor $\eta(\tilde{m}_\alpha)$ accounts for the washing out of the total lepton asymmetry due to inverse decays,

$$\eta(\tilde{m}_\alpha) \simeq \left[ \left( \frac{\tilde{m}_\alpha}{8.25 \times 10^{-3}\text{eV}} \right)^{-1} + \left( \frac{0.2 \times 10^{-3}\text{eV}}{\tilde{m}_\alpha} \right)^{-1.16} \right]^{-1}. \quad (44)$$

For the mass range of $M_1 < 10^9\text{GeV}$, all the three flavours are distinguishable. and the final value of the baryon asymmetry can be approximated by [48].

$$Y_B \simeq -\frac{12}{37} g^* \left[ \epsilon_e \eta \left( \frac{151}{179} \tilde{m}_e \right) + \epsilon_\mu \eta \left( \frac{344}{537} \tilde{m}_\mu \right) + \epsilon_\tau \eta \left( \frac{344}{537} \tilde{m}_\tau \right) \right], \quad (45)$$

The baryon asymmetry will be typically too small to account for the observed value in this case.

In the same fashion as studying lepton flavor mixing in previous sections, we assume that the seesaw Lagrangian of Eq. (34) is invariant under a CP transformation. We suppose that the lagrangian in Eq. (34) is invariant under the following CP transformation

$$\text{CP} : \nu_L \mapsto iX\gamma_0\nu_L^\dagger, \quad N_R \mapsto i\tilde{X}_N^\dagger N_R^c. \quad (46)$$

For the symmetry to hold, the neutrino Yukawa coupling matrix $\lambda$ and the RH neutrino mass matrix $M$ have to fulfill

$$\tilde{X}_N^\dagger \lambda X = \lambda^*, \quad \tilde{X}_N^\dagger M \tilde{X}_N = M^*. \quad (47)$$

One can immediately see that $\tilde{X}_N$ should be a diagonal matrix with elements equal to $\pm 1$, i.e. $\tilde{X}_N = \text{diag}(\pm 1, \pm 1, \pm 1)$, where the $\pm$ signs can be chosen independently. From Eq. (47) we can derive that the postulated residual CP symmetry leads to the following constraints on the $R$–matrix and lepton mixing matrix $U$ [44]:

$$R^* = \tilde{X}_N R \tilde{X}, \quad U^\dagger U = \tilde{X}, \quad (48)$$

where $\tilde{X} = \text{diag}(\pm 1, \pm 1, \pm 1)$. Note that the same constraint on the $R$–matrix from the CP invariance was found in Refs. [52, 56] As a consequence, the lepton mixing matrix is determined up to an orthogonal matrix $U = \Sigma O_{3 \times 3} \bar{X}^{-1/2}$ as shown in Eq. (2). On the other hand, depending on the values of $\tilde{X}_N$ and $\tilde{X}$, each element of the $R$–matrix satisfies $R_{ij}^* = \pm R_{ij}$ so that $R_{ij}$ is either real or pure imaginary while $R_{ij}^2$ must be real. Hence the total lepton asymmetry $\epsilon_1$ is always predicted to be vanishing $\epsilon_1 = 0$, no matter what form the residual CP $X$ takes. Thus, at temperatures where all lepton flavors are out of equilibrium and the one–flavor approximation is valid, no baryon asymmetry can be generated in the present framework. In the rest of the paper, we shall focus on the flavor dependent leptogenesis, with $M_1$ having a value in the interval of interest $10^9 \text{GeV} \leq M_1 \leq 10^{12} \text{GeV}$.

Notice that both diagonal matrices $\tilde{X}$ and $\tilde{X}_N$ are not constrained by the symmetry. In order to classify different possible cases in a concise and systematical way, we shall separate out $\tilde{X}$ and $\tilde{X}_N$ explicitly and introduce the following notations:

$$U' = U \tilde{X}^{1/2}, \quad R' = \tilde{X}_N^{1/2} R \tilde{X}_N^{1/2}, \quad K_j = (\tilde{X}_N)_{11} (\tilde{X})_{jj}, \quad \text{with } j = 1, 2, 3. \quad (49)$$

Then $U'$ would be a real matrix, and $K_j$ is either $+1$ or $-1$. Furthermore, the CP asymmetry $\epsilon_\alpha$ and the washout mass parameter $\tilde{m}_\alpha$ can be written as

$$\epsilon_\alpha = -\frac{3M_1}{16\pi v^2} \left[ \sum_{ij} \sqrt{m_i m_j} m_j R_{ij}' U_{\alpha i} U_{\alpha j}^* K_j \right], \quad \tilde{m}_\alpha = \sum_j m_j^{1/2} R_{ij}' U_{\alpha j}^* \tilde{m}_\alpha \frac{1}{2}. \quad (50)$$
Obviously only the elements $R'_{1i}$ of the first row of $R'$ are relevant to $\epsilon_\alpha$ and $\tilde{m}_\alpha$. The orthogonal condition $RR^T = 1$ gives rise to

$$R'_{11}^2 K_1 + R'_{12}^2 K_2 + R'_{13}^2 K_3 = 1.$$ (51)

The most general parametrization of the elements $R'_{11}, R'_{12}$ and $R'_{13}$ for different possible values of $K_1, K_2, K_3$ are listed in Table IX. Note that the values $(K_1, K_2, K_3) = (-, -, -)$ is not admissible since the constraint in Eq. (51) can not be fulfilled in that case.

| $(K_1, K_2, K_3)$ | $(R'_{11}, R'_{12}, R'_{13})$ |
|-------------------|---------------------------------|
| $(+, +, +)$       | $(\cos \rho \cos \varphi, \cos \rho \sin \varphi, \sin \rho)$ |
| $(-, +, +)$       | $(\sinh \rho, \cosh \rho \cos \varphi, \cosh \rho \sin \varphi)$ |
| $(+, - , +)$      | $(\cosh \rho \sin \varphi, \sin \rho, \cosh \rho \cos \varphi)$ |
| $(+, +, -)$       | $(\cosh \rho \cos \varphi, \sin \rho \sin \varphi, \sin \rho \cos \varphi)$ |
| $(+, - , -)$      | $(\cosh \rho \sin \varphi, \cosh \rho \cos \varphi, \sin \rho \sin \varphi)$ |
| $(-, +, -)$       | $(\sin \rho \cos \varphi, \sin \rho \sin \varphi, \cosh \rho \cos \varphi)$ |

TABLE IX: The parametrization of the first row of the $R'$ matrix for the possible values of $K_1, K_2$ and $K_3$, where both $\varphi$ and $\rho$ are real parameters.

In the present formalism, we show that in general the lepton mixing angles and CP phases depend on three parameters $\theta_{1,2,3}$, and two more parameters $\rho$ and $\varphi$ are involved in prediction for the baryon asymmetry. In what follows, we shall apply the above general results to the cases of type-V and type-VI residual CP symmetries with $\Theta = \frac{2\pi}{5}$ and $\frac{\pi}{7}$ respectively. The values of $\theta_{1,2,3}$ are determined to reproduce the best fit values of the three lepton mixing angles [20]. As a typical example, we choose the RH neutrino mass $M_1 = 5 \times 10^{11}$ GeV, the lightest neutrino mass is taken to be $m_1 (or m_3) = 0.01$ eV, and the mass-squared splittings $\Delta m^2_{21}$ and $|\Delta m^2_{31}|$ are fixed at their best fit values [20].

- Type-V CP symmetry with $\Theta = \frac{2\pi}{5}$

It is the so-called generalized $\mu - \tau$ reflection symmetry [7]. The explicit form of the $X$ matrix, its Takagi factorization matrix $\Sigma$ and the corresponding predictions for mixing parameters are collected in Table XI. From Eq. (48) we know that the $R$–matrix and the mixing matrix $U$ have the following properties

$$R_{1j} = R^*_{1j} K_j, \quad U_{1j} = e^{i\alpha} U^*_{1j} (\tilde{X})_{jj}.$$ (52)

It follows that the CP asymmetry $\epsilon_\alpha$ is vanishing $\epsilon_\alpha = 0$ independent of the value of $\Theta$ in this case. The remaining two CP asymmetries $\epsilon_\mu$ and $\epsilon_\tau$ are related as $\epsilon_\mu = - \epsilon_\tau$, which is inferred from general prediction $\epsilon_1 = \sum_\alpha \epsilon_\alpha = 0$. In order to accommodate the best fit values of the mixing angles [20], we take $\theta_1 = 58.026^\circ [59.106^\circ]$, $\theta_2 = 8.60^\circ [8.70^\circ]$ and $\theta_3 = 145.4^\circ [145.4^\circ]$ for NH and in square brackets for IH of the neutrino masses, respectively. Then one can predict the CP violating phases $\delta_{CP} = 253.727^\circ [254.022^\circ]$, $\alpha_{21} (mod \pi) = 0^\circ [0^\circ]$ and $\alpha_{31} (mod \pi) = 147.454^\circ [148.045^\circ]$. Note that the Dirac phase $\delta_{CP}$ is rather close to its present best fit value [20], although the
statistical significance is quite low. Since the baryon asymmetry $Y_B$ depends on $\rho$ and $\varphi$, we display the contour regions of $Y_B/Y_B^{\text{obs}}$ in the plane $\varphi$ versus $\rho$ in Fig. 14. We see that successful leptogenesis can happen except for NH neutrino mass spectrum with $(K_1, K_2, K_3) = (-, -, +)$.

- Type-VI CP symmetry with $\Theta = \frac{\pi}{7}$

In this case, we take $\theta_1 = 178.345^\circ [176.995^\circ]$, $\theta_2 = 48.167^\circ [48.442^\circ]$ and $\theta_3 = 57.255^\circ [58.255^\circ]$ so that the best fitting values of the lepton mixing angles are reproduced exactly. Accordingly, the CP phases are determined to be $\delta_{CP} = 255.105^\circ [249.292^\circ]$, $\alpha_{21}$ (mod $\pi$) = $138.569^\circ [138.360^\circ]$ and $\alpha_{31}$ (mod $\pi$) = $151.226^\circ [150.738^\circ]$. We plot the contour regions for $Y_B/Y_B^{\text{obs}}$ in the $\rho - \varphi$ plane in Fig. 15. As can be seen, we can have successful leptogenesis except for the cases of NH neutrino masses with $(K_1, K_2, K_3) = (-, -, +)$ and IH with $(K_1, K_2, K_3) = (+, -, -)$.

We also study the predictions for leptogenesis in the type-XI case. As an example, we choose $\theta_1 = 176.556^\circ [176.886^\circ]$, $\theta_2 = 9.122^\circ [9.403^\circ]$ and $\theta_3 = 53.464^\circ [53.508^\circ]$ to reproduce the best fit values of the neutrino mixing angles. Then the CP violating phases can be predicted as $\delta_{CP} = 285.771^\circ [287.100^\circ]$, $\alpha_{21}$ (mod $\pi$) = $56.593^\circ [56.580^\circ]$ and $\alpha_{31}$ (mod $\pi$) = $175.842^\circ [175.919^\circ]$. The contour region for $Y_B/Y_B^{\text{obs}}$ in the plane $\varphi$ versus $\rho$ is shown in Fig. 16. The existing matter–antimatter asymmetry can be reproduced for appropriate values of $\rho$ and $\varphi$ except the case of NH with $(K_1, K_2, K_3) = (-, -, +)$.

VI. CONCLUSIONS

In this paper we have given a full classification of generalized CP symmetries preserved by the neutrino mass matrix, taking as basis the number of zero entries in the transformation matrix. We have determined the corresponding constrained form of the lepton mixing matrix. We have shown how this results in correlations between the lepton mixing angles and the Majorana and Dirac CP violating phases. We have also mapped out the corresponding restrictions that follow from current neutrino oscillation global fits and found that, in some cases, the Dirac CP violating phase characterizing neutrino oscillations is highly constrained. Focussing on the expected CP asymmetries for the “golden” oscillation channel we have derived implications for current long baseline neutrino oscillation experiments T2K, NOνA, forecasting also the corresponding results for the upcoming long baseline DUNE experiment. We have also discussed the predicted ranges for the effective neutrino mass parameter characterizing the neutrinoless double beta decay rates. Finally we have also studied the cosmological implications of such schemes for leptogenesis.

The results of this paper are quite general in the sense that they are independent of how the assumed residual CP symmetry is dynamically achieved. If the residual CP symmetry $X$ originates from the breaking of the generalized CP symmetry compatible with a finite flavor symmetry group $G_f$, the admissible form of the residual CP transformation would be strongly constrained to satisfy the consistency condition. If $X$ has at least one zero entry, it would belong to one of the cases studied in the present work. The corresponding prediction for lepton mixing matrix could be straightforwardly obtained by exploiting the master formula Eq. (2). In this paper we have discussed the possible mixing patterns which can be achieved in this method, and the resulting phenomenological predictions for neutrino oscillation, neutrinoless double beta decay and leptogenesis. By comparing with the extensively studied scenarios with two residual CP transformations preserved in the neutrino sector [85, 75], one expects to obtain new phenomenologically viable mixing patterns and new predictions for the CP violation phases.
FIG. 14: Predictions for $Y_B/Y_{B}^{obs}$ as a function of $\rho$ and $\varphi$ in the case of type-V residual CP transformation with $\Theta = 2\pi/5$. We have chosen $M_1 = 5 \times 10^{11}$ GeV, $m_1$ (or $m_3$) = 0.01 eV. The mass-squared differences $\Delta m^2_{21}$ and $|\Delta m^2_{31}|$ are taken to be the best fit values [20]. We set $\theta_1 = 58.026^\circ [59.106^\circ]$, $\theta_2 = 8.60^\circ [8.70^\circ]$ and $\theta_3 = 145.4^\circ [145.4^\circ]$ to reproduce the best fitting values of the mixing angles [20]. The dashed lines denote the precisely measured value of the baryon asymmetry $Y_{B}^{obs} = 8.66 \times 10^{-11}$ [57]. Note that successful leptogenesis is not possible for NH neutrino masses with $(K_1, K_2, K_3) = (-, -, +)$. 
FIG. 15: Predictions for $Y_B/Y_B^{obs}$ as a function of $\rho$ and $\varphi$ in the case of type-VI residual CP transformation with $\Theta = \pi/7$. We have chosen $M_1 = 5 \times 10^{11}$ GeV, $m_1$ (or $m_3$) = 0.01 eV. The mass-squared differences $\Delta m^2_{21}$ and $|\Delta m^2_{31}|$ are taken to be the best fit values \[20\]. We set $\theta_1 = 178.345^\circ$ [$176.995^\circ$], $\theta_2 = 48.167^\circ$ [48.442$^\circ$] and $\theta_3 = 57.255^\circ$ [58.255$^\circ$] to reproduce the best fitting values of the mixing angles \[20\]. The dashed lines denote the precisely measured value of the baryon asymmetry $Y_B^{obs} = 8.66 \times 10^{-11}$ \[57\]. Note that successful leptogenesis is not possible for NH neutrino masses with $(K_1, K_2, K_3) = (-, -, +)$ and IH case with $(K_1, K_2, K_3) = (+, -, -)$. 

FIG. 16: Predictions for $Y_B/Y_B^{obs}$ as a function of $\rho$ and $\varphi$ for the case of democratic CP transformation. We have chosen $M_1 = 5 \times 10^{11}$ GeV, $m_1$ (or $m_3$) = 0.01 eV. The mass-squared differences $\Delta m^2_{21}$ and $|\Delta m^2_{31}|$ are taken to be the best fit values \[20\]. We set $\theta_1 = 176.556^\circ [176.886^\circ]$, $\theta_2 = 9.122^\circ [9.403^\circ]$ and $\theta_3 = 53.464^\circ [53.508^\circ]$ so as to reproduce the best fit values of the neutrinos mixing angles \[20\]. The dashed lines denote the precisely measured value of the baryon asymmetry $Y_B^{obs} = 8.66 \times 10^{-11}$ \[37\]. Note that successful leptogenesis is not possible for NH neutrino masses with $(K_1, K_2, K_3) = (-, -, +)$. 
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Appendix A: Definition domain of $O_{3 \times 3}$

In this appendix, we would like to discuss the domain of the parameters $\theta_1$, $\theta_2$ and $\theta_3$ in the $O_{3 \times 3}$ matrix. In Eq. (4), $O_{3 \times 3}$ is parameterized as

$$O_{3 \times 3}(\theta_1, \theta_2, \theta_3) \equiv \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_1 & \sin \theta_1 \\
0 & -\sin \theta_1 & \cos \theta_1
\end{pmatrix}
\begin{pmatrix}
\cos \theta_2 & 0 & \sin \theta_2 \\
0 & 1 & 0 \\
-\sin \theta_2 & 0 & \cos \theta_2
\end{pmatrix}
\begin{pmatrix}
\cos \theta_3 & \sin \theta_3 & 0 \\
-\sin \theta_3 & \cos \theta_3 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (A1)$$

where $\theta_1$, $\theta_2$ and $\theta_3$ can freely vary in the range of $[0, 2\pi)$. Notice that $O_{3 \times 3}$ has the following properties

$$O_{3 \times 3}(\theta_1, \theta_2, \theta_3 + \pi) = O_{3 \times 3}(\theta_1, \theta_2, \theta_3) \text{diag}(-1, -1, 1),$$

$$O_{3 \times 3}(\theta_1, \theta_2 + \pi, \theta_3) = O_{3 \times 3}(\theta_1, \theta_2, \pi - \theta_3) \text{diag}(1, -1, -1),$$

$$O_{3 \times 3}(\theta_1 + \pi, \theta_2, \theta_3) = O_{3 \times 3}(\theta_1, \pi - \theta_2, \theta_3) \text{diag}(-1, -1, 1), \quad (A2)$$

where the diagonal matrices can be absorbed into the matrix $\hat{X}_\nu^{-1/2}$. As a result, the fundamental interval of the parameters $\theta_{1,2,3}$ can be taken to be $[0, \pi)$.

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