Criteria for disintegration of an uncharged conducting liquid jet in a transverse electric field

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Abstract. An uncharged conducting liquid cylindrical column (a jet for applications) placed between a pair of flat electrodes is considered. In the trivial case, when the electric field is absent, the jet with circular cross-section is the only possible equilibrium configuration of the system. In the presence of a potential difference between the electrodes, the jet is deformed by the electrostatic forces: its cross-section stretches along the electric field lines. In the case of the mutual compensation of the electrostatic and capillary forces, a new equilibrium configuration of the jet can appear. In a sufficiently strong field, the balance of the forces becomes impossible, and the jet disintegrates (splits into two separate jets). In the present work, we find the range of the parameters (the applied potential difference and the interelectrode distance), where the problem of finding the equilibrium configurations of the jet has solutions. Also we obtain the conditions under which the solutions do not exist and, consequently, the jet splits. The results are compared with the previously studied limiting case of infinite interelectrode distance.

1. Introduction

Let a cylindrical column of a conducting liquid (a jet for applications) be placed between two planar electrodes situated at a distance \( D \) from each other. If the potential difference \( U \) equals zero, the only possible equilibrium configuration of the jet is a jet of circular cross section. If \( U \neq 0 \), the jet is deformed under the action of electrostatic forces: its cross section is stretched along the lines of force of the field. In the case when the strength of the field (for a given interelectrode distance) exceeds a threshold value, the jet splits into two separate jets (in the simplest case of \( D \to \infty \), the splitting conditions were analyzed in [1–3]). From the applied point of view, interest in this electrostatic problem is associated with the possibility of controlled splitting of jets by an applied transverse electric field [4]. For example, longitudinal splitting of jets under an electric field was observed in the experiments in [4–8].

In the recent paper [9], we have obtained a one-parameter family of exact solutions for the shape of a jet with regard to the finiteness of the interelectrode distance (for infinite interelectrode distance the problem was studied in [1–3]).

It is clear from general considerations that the equilibrium configuration of a jet of given radius \( R_0 \) is defined by two parameters, the interelectrode distance \( D \) and the potential difference \( U \). In the present study, we will construct corresponding approximate two-parametric family of solutions on the basis of deformation of previously obtained in [9] exact particular one-parameter
solution. Using it, we will analyze equilibrium configurations of the jet appearing in the case of the mutual compensation of the electrostatic and capillary pressures. We will determine the conditions under which the solutions exist and also those under which the problem has no solutions (the last case corresponds to the jet splitting).

Also, the solutions for the equilibrium configurations of the jets in a transverse electric field can be useful for the analysis of their instabilities. So, under the action of capillary forces, a cylindrical liquid jet is susceptible to the Plateau–Rayleigh instability [10], which leads to the deformation of its surface in the direction of its axis. In the study of the jet instabilities, it is necessary to know the unperturbed solution of the problem. In the absence of the electric field, the unperturbed solution is trivial: the jet has a circular cross section. In the presence of a weak field, its cross section will be obviously close to elliptical in shape. However, in a strong field, when the cross section of the jet is significantly deformed, the unperturbed solutions have to be found.

2. Original equations
We will assume that the liquid is at rest and that the problem has plane symmetry, i.e., the surface of the liquid column is invariant with respect to the translation along its axis (as a result of the action of electrostatic forces, only the cross section of the column is deformed).

Suppose that the origin of a rectangular system of coordinates \( \{x, y\} \) coincides with the axis of the jet and the jet itself is symmetric with respect to the planes \( y = 0 \) and \( x = 0 \) (figure 1). Define the position of the electrodes by the conditions \( y = \pm D/2 \). The distribution of the potential \( \varphi \) of the electric field is described by the Laplace equation

\[
\varphi_{xx} + \varphi_{yy} = 0,
\]

which should be solved together with the condition of equipotentiality of the surface of the jet \( (\varphi = 0) \), as well as of both electrodes \( (\varphi = \mp U/2) \). As \( |x| \to \infty \), the electric field is uniform: \( \varphi \to -E_\infty y \), where \( E_\infty = U/D \).
The equilibrium shape of the free surface of the conducting liquid is determined from the pressure balance (Laplace–Young) condition:

\[ \varepsilon_0 \left( \frac{\varphi_x^2 + \varphi_y^2}{2} + \kappa \right) + \Delta P = 0, \tag{1} \]

where \( \varepsilon_0 \) is the dielectric constant, \( \kappa \) is the local curvature of the surface, and \( \sigma \) is the surface tension coefficient. The first term in (1) corresponds to the electrostatic pressure, and the second, to the capillary pressure. The constant \( \Delta P \) has the meaning of the pressure difference between the inside and outside of the liquid. Note that \( \Delta P \) is not an independent parameter of the problem. It is determined in the course of the solution of the problem. For instance, in the absence of the electric field (\( E_\infty = 0 \)), when the jet has a circular cross section with radius \( R_0 \), we have \( \Delta P = \sigma / R_0 \). For \( E_\infty \neq 0 \), the condition \( \Delta P < \sigma / R_0 \) is always satisfied.

It is convenient to introduce dimensionless parameters

\[ \text{Bo}_E = \frac{\varepsilon_0 R_0 E_\infty^2}{2\sigma}, \quad K = \frac{R_0}{D}. \tag{2} \]

The electric Bond number \( \text{Bo}_E \) characterizes the ratio of electrostatic and capillary pressures. The geometric parameter \( K \) defines the ratio of the scales of the problem. The case \( K = 0 \) corresponds to the infinite interelectrode distance. The solution of the problem of finding the equilibrium shape of a jet (if it exists) is defined by these two parameters, \( \text{Bo}_E \) and \( K \).

If a liquid cylindrical conductor is placed into an initially uniform external field, the field distribution becomes distorted (the field does not penetrate into the conductor). As a result, the electrostatic pressure on the jet surface will be minimal (equal to zero) in the plane \( y = 0 \) and maximal in the plane \( x = 0 \). Then, as a consequence of the pressure balance condition (1), the jet cross section will be stretched in the \( y \) direction. This situation is demonstrated in figure 1, where the characteristic shape of the jet cross section in a transverse electric field (it corresponds to \( K = 0.175 \) and \( \text{Bo}_E = 0.16 \)) and the family of equipotential surfaces \( \varphi = \text{const} \)

It should be noted that the considered electrostatic problem is analogous from the mathematical point of view to the problem of the shape of a two-dimensional gas bubble in the potential flow of an ideal liquid in a plane channel. In order to pass to considering this hydrodynamic problem, the electric field potential \( \varphi \) and the dielectric constant \( \varepsilon_0 \) should be replaced by the stream function \( \Psi \) and by the liquid density \( \rho \), respectively (then the Laplace–Young condition (1) becomes the stationary Bernoulli equation [11]). The region occupied by the conducting liquid will correspond to the bubble, and the region with the electric field, to the fluid flow region. Then the equipotential lines in figure 1 will take the meaning of the streamlines \( \Psi = \text{const} \).

3. Exact solutions

At the present time, there are only two known exact particular solutions of the considered electrostatic problem. In 1955, the zero-parameter solution was found by McLeod [12] for the shape of a two-dimensional gas bubble moving in an ideal infinite liquid. In the parametric form the solution is given by

\[ z = \sqrt{\frac{243}{134}} R_0 \left( \exp (i\tau) - \frac{2}{3} \exp (-i\tau) - \frac{1}{27} \exp (-3i\tau) \right), \tag{3} \]

where \( z = x + iy \) and \( \tau \) is the parameter varying in the interval from 0 to \( 2\pi \). This solution also describes a certain equilibrium configuration of a jet placed in an external uniform electric field with infinite interelectrode distance, \( D \gg R_0 \). As it was shown in [13], this configuration is unstable. Note that the solution (3) corresponds to the special case of \( \Delta P = 0 \), when the pressure in the jet coincides with the external pressure. In this situation, the pressure balance condition simplifies significantly that provides the possibility of solving it analytically (see also
In terms of the control parameters (2), the solution (3) corresponds to $\text{Bo}_E \approx 0.247$ and $K = 0$.

Another exact solution was derived by us in [9]. We will use it for constructing a two-parametric approximate solution in section 4. So let us discuss it in details.

It is convenient to introduce the following dimensionless combinations:

$$A = \varepsilon_0 E_{\text{max}}^2/(2\Delta P), \quad p = R_0 \Delta P/\sigma, \quad k = E_{\text{max}}/E_{\infty},$$

where $E_{\text{max}}$ is the maximum value of the electric field that is attained on the surface of the jet at the points $\{x, y\} = \{0, \pm y_{\text{max}}\}$ (minimum value $E = 0$ is attained at $\{x, y\} = \{\pm x_{\text{max}}, 0\}$). Here $x_{\text{max}}$ and $y_{\text{max}}$ are the distances from the axis of the jet to its boundary in the directions $x$ and $y$. In the particular case of $E_{\infty} = 0$, we have $y_{\text{max}} = x_{\text{max}} = R_0$. The electric Bond number can be expressed through the parameters $A$, $p$, and $k$ as $\text{Bo}_E = Ap/k^2$.

The equilibrium shape of the free surface is determined by the expression

$$y = \pm \frac{R_0}{p\sqrt{A}} \arctan \sqrt{A - (A + 1) \tanh^2 (Bpx/R_0)}, \quad |x| \leq x_{\text{max}},$$

where $B = \sqrt{A(A + 1)}$. The interelectrode distance and the sizes of the jet in the $y$ and $x$ directions are given by the formulas

$$D = \frac{\pi kR_0}{pB}, \quad y_{\text{max}} = \frac{R_0}{p\sqrt{A}} \arctan \sqrt{A}, \quad x_{\text{max}} = \frac{R_0}{pB} \arctanh(A/B).$$

The dimensionless parameters $p$ and $k$, characterizing the pressure difference and, respectively, the electric field amplification, are following:

$$p = \sqrt{\ln(A + 1)/(\sqrt{AB})}, \quad k = 1 + \sqrt{A} + 1.$$  

One can see that, for the fixed radius $R_0$, the solution depends on a single control parameter $A$ (all the parameters $p$, $k$, $B$, $D/R_0$ depend only on $A$).

The dimensionless quantity $A \propto E_{\text{max}}^2$ is chosen as the control parameter because the jet deformation is caused not by the electric field at infinity ($E_{\infty}$) but by the maximum field on its boundary ($E_{\text{max}}$). As a consequence, the ratio $y_{\text{max}}/x_{\text{max}}$, which characterizes the degree of deformation of the jet, monotonically increases with $A$:

$$\frac{y_{\text{max}}}{x_{\text{max}}} = \sqrt{A + 1} \arctan \sqrt{A}/((A + 1)).$$

The solution parameter $A$ belongs to the interval $0 \leq A < \infty$. The lower bound corresponds to the absence of external electric field (then the jet has a circular cross section). The upper limit is realized for $E_{\text{max}} \to \infty$ and $\Delta P \to 0$. In this limit, the cross section of the jet is unboundedly stretched in the direction of the applied field.

Thus, the obtained in [9] exact solution is one-parameter. As pointed out above, the general solution of the problem is defined by two dimensionless parameters, the ratio of the scales $K = R_0/D$ and the electric Bond number $\text{Bo}_E$. For the particular solution (4), both of these parameters depend on $A$:

$$\text{Bo}_E(A) = Ap(A)/k^2(A), \quad K = B(A)p(A)/(\pi k(A)).$$

Figure 2 shows the parameters $K$ and $\text{Bo}_E$ corresponding to the zero-parameter solution (3) (the separate point $K = 0$ and $\text{Bo}_E \approx 0.247$) and to the one-parameter family of the exact solutions (4) (the corresponding curve is given by the parametric expressions (5)). One can see that the exact solutions are not known for the whole parametric plane $K$ and $\text{Bo}_E$. In the next section, we will construct approximate two-parametric family of solutions, which describes the equilibrium configurations at arbitrary values of $K$ and $\text{Bo}_E$. 

[(14)].
Figure 2. The values $K$ and $Bo_E$ corresponding to the exact zero-parameter (3) and one-parameter (4) solutions. For the solution (4), parameter $A$ increases counterclockwise along the corresponding curve. The characteristic jet shapes defined by these solutions are also shown.

In figure 2, the characteristic jet configurations for both families of solutions are also displayed. For the one-parameter family of solutions, the degree of the boundary deformation $y_{\text{max}}/x_{\text{max}}$ increases counterclockwise along the corresponding curve in the plane $\{K, Bo_E\}$. For the solution (3) we have $y_{\text{max}}/x_{\text{max}} = 5.5$.

It should be noted that, when constructing the solutions (4), we applied the assumption that the character of relation between the strength of the field and the inclination angle of the free surface of the liquid is invariant under the deformation of the jet [9]. Let us explain the essence of this hypothesis. If a non-deformable cylindrical conductor with a circular cross-section is placed in a uniform external electric field $E_\infty$ ($D$ is considered to be infinite), then the absolute value of the field strength ($E = |\nabla \varphi|$) at the boundary will be defined by the simple expression

$$E = E_{\text{max}} \sin \theta,$$

where $E_{\text{max}} = 2E_\infty$ and $\theta$ is the angle of inclination of the field strength to the $x$ axis (since the vector of strength is normal to the surface, then the angle $\theta$ differs from the inclination angle of the conductor boundary by $\pi/2$). The solution (4) has been found from the assumption that the relation (6) remains valid under deformations of the jet boundary for finite $D$. The auxiliary relation (6) made it possible to reduce the initial problem to successive integration of ordinary differential equations. In [15], the author used a similar assumption to find exact solutions for the equilibrium shape of a periodic system of liquid-metal columns in an external high-frequency magnetic field.

4. Approximate solutions

Let us pass to dimensionless variables by the substitutions

$$x \to x R_0, \quad y \to y R_0, \quad \varphi \to \varphi E_{\text{max}} R_0.$$

Below, instead of the interelectrode distance $D$, we will use its dimensionless analog $d = D/R_0 = K^{-1}$.

We introduce the complex potential $w = \varphi - i\psi$ of the electric field, which is an analytic function of the complex variable $z$. The function $\psi$ is the harmonic conjugate of the potential
ϕ; the condition ψ = const defines the lines of force of the electric field (see also [2,16]). Also we introduce a complex strength of the field $F = dw/dz$. Represent it as

$$F = -E \exp(-i\theta),$$

where $E$ is the absolute value of the field strength and $\theta$ is the angle of inclination of the field strength to the $x$ axis.

It is convenient to take the complex electric field strength $F$ as the unknown function and the complex potential $w$ as a dependent variable. The latter corresponds to the conformal mapping of the domain bounded by the electrodes and the free surface into the strip

$$-d/(2k) \leq \varphi \leq d/(2k), \quad -\infty < \psi < \infty.$$ 

Then the surface of the jet is mapped onto the segment $|\psi| \leq \psi_0$ and $\varphi = 0$ (the constant $\psi_0$ will be determined later). As a result, the initial problem with unknown boundary is reduced to a much simpler problem on a strip. In this strip, the complex strength of the field $F$ should be an analytic function of the variable $w$. The force balance condition (1) takes the compact form

$$ApE^2 - E\theta\psi + p = 0,$$

(7)

where we have taken into account that the curvature of the surface can be written as $\kappa = -E\theta\psi$ [17, 18].

The exact particular solution for the distribution of electric field potential in space (in the implicit form) is following [9]:

$$z(w) = -iw + \frac{i}{p\sqrt{A}} \arctan \sqrt{A + (A + 1)\tan^2(Bpw)}.$$ 

(8)

The shape of the free surface (4) can be found from (8) by the substitution $w = -i\psi$. In dimensionless notations, it is given by the expression

$$y = \pm \frac{1}{p\sqrt{A}} \arctan \sqrt{A - (A + 1)\tanh^2(Bpx)}.$$ 

(9)

Let us construct now a two-parameter family of solutions with use of the following approximation for the boundary shape:

$$y = \frac{\sqrt{c}}{p_n\sqrt{A_n}} \arctan \left( \sqrt{A_n - (A_n + 1)\tanh^2(B_np_n\sqrt{c}x)} \right),$$ 

(10)

where $A_n$ and $c$ are some constants, and

$$B_n = \sqrt{A_n} (A_n + 1), \quad p_n = \sqrt{\ln(A_n + 1)/(\sqrt{A_n}B_n)}.$$ 

This expression is obtained from the exact solution (9) for the jet shape by its stretching along the $y$ axis and compressing along the $x$ axis in $\sqrt{c}$ times (the cross-section area of the jet is invariant under this transformation). It is convenient to use this approximation for the analysis of jet configurations, since for it the field distribution in the interelectrode space can be found analytically. This distribution is given by the expression similar to (8):

$$z(w) = -iw + \frac{i\sqrt{c}}{p_n\sqrt{A_n}} \arctan \left( \sqrt{A_n + (A_n + 1)\tan^2(B_np_n\sqrt{c}w)} \right).$$ 

(11)

For this representation of the problem solution, we have

$$x_{\max} = \psi_0 = \frac{\artanh(A_n/B_n)}{p_nB_n\sqrt{c}}, \quad y_{\max} = \frac{\sqrt{c} \arctan \sqrt{A_n}}{p_n\sqrt{A_n}}.$$ 

For $c = 1$, the formula (10) for the jet shape gives the exact solution for the pressure balance condition. In this case $A_n = A$, $B_n = B$, and $p_n = p$. For $c \neq 1$, this expression can give only an
approximate solution of the problem. Using the so-called two-point method (see, for example, [3]), we require that the force balance condition be satisfied only at two points, \{0, \pm y_{\text{max}}\} and \{\pm x_{\text{max}}, 0\}. This leads to the equations

\[
Ap - c^{3/2}p_n(A_n + 1) + p = 0, \quad -c^{-3/2}p_n + p = 0,
\]

where we have used the formulas

\[
E = \sin \theta, \quad \theta = -\arctan \left\{ c \cot \left[ \arcsin \left( \frac{B_n \tanh(B_n p_n \sqrt{c} \psi)}{A_n} \right) \right] \right\},
\]

following from (11).

Thus, the initial pressure balance condition (7) is replaced by the algebraic expressions (12). From these expressions we obtain relations between the parameters \(A\), \(p\) of the key equation (7) and the solution parameters \(A_n\), \(c\):

\[
A(A_n, c) = A_n c^3 - 1 + c^3, \quad p(A_n, c) = c^{-3/2} p_n(A_n).
\]

The approximate solution (11) satisfies the necessary conditions at infinity and on the electrodes for

\[
k(A_n, c) = 1 + c \sqrt{A_n + 1}, \quad d(A_n, c) = \pi k(A_n, c) / (B_n p_n \sqrt{c})
\]

Finally, the problem parameters \(B_{OE}\) and \(K\) can be expressed through the parameters \(c\) and \(A_n\):

\[
B_{OE}(A_n, c) = A(A_n, c) p(A_n, c) k^{-2}(A_n, c),
\]

\[
K(A_n, c) = d^{-1}(A_n, c).
\]

Thus we deal with the two-parameter solution.

Varying the parameters \(c\) and \(A_n\), we can find the domain of existence of the solutions in the parametric plane \(\{B_{OE}, K\}\). This domain is shown in figure 3. One can see that, under a sufficiently strong electric field (or the electric Bond number), equilibrium configurations do

\[\text{Figure 3.}\] The domain of existence of the solutions is shown in the plane \(B_{OE}\) and \(K\). The characteristic jet shapes corresponding to the boundary of this domain are presented. Also, three curves in the parametric plane for the solutions with \(c = 0.5, 1, 2\) are shown.
not exist and hence the considered problem has no solutions. It is possible to conclude that the capillary forces cannot compensate the destabilizing effect of the electrostatic forces, and the jet disintegrates. In addition, in this figure the characteristic jet shapes corresponding to the boundary of the domain of existence of the solutions are presented.

Also, there are three curves in figure 3 that correspond to the solutions with \( c = 0.5, 1, 2 \). One can see that the curves with different values of \( c \) intersect. This means that, for given values of \( \text{Bo}_E \) and \( K \), two different solutions with different degrees of deformation of the jet surface can coexist.

5. Concluding remarks

In the present study, we obtain a two-parameter family of approximate solutions for the shape of an uncharged cylindrical jet placed between a pair of flat electrodes with regard to the finiteness of the interelectrode distance. We find the range of the parameters (the electric Bond number \( \text{Bo}_E \), which is responsible for the electrostatic pressure influence, and the geometrical parameter \( K \)), for which the solutions exist (see figure 3). Also we obtain the conditions under which the problem has no solutions and, as a consequence, the jet splits.

It can be seen from figure 3 that as the interelectrode distance decreases (the parameter \( K \) increases), the threshold value of the electric field (i.e., of the electric Bond number \( \text{Bo}_E \)) for the jet disintegration also decreases. It should be noted that the accuracy of finding the boundary of the solutions existence domain is maximal in the vicinity of the exact solution, which corresponds to \( c = 1 \). The accuracy decreases with distance from the exact solution and becomes minimal close to the axes \( K = 0 \) and \( \text{Bo}_E = 0 \). It is clear from the geometric considerations that the condition \( 2R_0 < D \) is always valid (otherwise, the liquid touches the electrodes). Then, in the absence of the electric field (\( \text{Bo}_E = 0 \)), the boundary is given by \( K = 0.5 \), whereas our approach gives \( K = 0.37 \). For the infinite interelectrode distance (\( K = 0 \)), our approximate expressions (13) and (14) yield \( \text{Bo}_E = 0.32 \), whereas the analysis of [2, 3] gives smaller value \( \text{Bo}_E = 0.28 \).

Acknowledgments

This work was performed in the framework of state program 0389-2014-0006 and supported in part by the Russian Foundation for Basic Research (projects No. 16-08-00228 and 17-08-00430) and by the Presidium of the Russian Academy of Sciences (program “Electrophysics and electronics of powerful pulse systems”).

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