A general study of charged particles dynamics near magnetic planets

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Abstract. A general study of the Störmer problem is carried taking into account the combined effect of the dipolar and quadrupolar magnetic terms as well as the gauss coefficients for Earth and Saturn. The resulted trajectories as well as the critical states for various kinds of charged dust grains are studied.

1. Introduction
The problem of the dynamics of a charged dust grain particles has attracted the attention of physicists and astronomers for a lot of time, especially in planetary magnetospheres of magnetic planets, it was first considered by Mendis and Axford (1974) mainly in space to give physical explanation the origin of the accelerated mysterious dark spokes in Saturn’s B ring observed by Voyager 1 and 2. [1]. The starting point of a detailed theoretical study of charged dust grain motion subject to a pure magnetic dipole was modeled by Störmer, this treatment provides interesting results for understanding the radiation belts surrounding magnetic planets (included earth), described as light particles (ions or electrons) with motion affected by magnetic force (Classical störmer problem) [2]. In addition, the treatment of charged dust grains with complicated dynamics need to take into account a small charge to mass ratio (heavier particles), where we cannot consider just the magnetic force in the equations of motion, instead the gravitational field created by the planet, and the corotational electric field generated by the considered magnetized planet. This is the socalled generalized Störmer problem denoted in general by GSP; this interesting treatment is characterized by simplified assumptions of keplerian gravity, specific magnetic field (pure dipole), constant charge without radiation pressure effect. We mention the four distinct Störmer problems: CSP (Classical Störmer Problem) in which a charged particle move in a pure dipole magnetic field, RSP (Rotational Störmer problem) characterized by the electric field due to the planet rotation, GSP (Gravitational Störmer problem) only the keplerian gravity is included and the full interesting system (RGSP) (Rotational Gravitational Störmer Problem) counting both fields action. [3][4]. On the way to understanding different physical behaviors of charged particles located near earth or outer planets or in planetary magnetospheres, an extensive bibliography of works on this topic is given, much of this work is devoted to get various aspects of the problem to answer open questions, such as the dynamics of charged dust grains near non-spherical planets (prolate and oblate) [5][6], the behavior of equatorial orbits affected by strong magnetic fields near a black holes and compact stars [7]. Extensive studies are devoted to treat...
dynamics of grains components near solar corona with the influence of the solar radiation pressure[8]. Some studies [9][10] have treat a special kind of charged components such as sub-microns circumplanetary dust grains in dipolar field case, with the inclusion of non-axisymmetric magnetic field terms which gives more interesting results compared to the aligned axisymmetric one, also the study of stability for all charge to mass ratios mainly affected by multipolar field with interesting applications on Earth, Jupiter, Saturn, Uranus and Neptune. Many analytical and numerical models have been developed to study this problem. Some authors have been interested by the trapped orbits of non equatorial position and also stable non-equatorial (halo) dust grain orbits about Saturn. [11][12], also the mapping of the allowed pulse regions at arbitrary points of the dipole field is treated [13]. Various models have been treated to explain the good results given by dipolar approximation without neglect the interesting features of the simultaneous effect of dipolar and quadrupolar magnetic terms on Störmer problem subject to charged particles near magnetic planet discussed in [14][15], also the martian magnetization is treated for charged particles [17].

In the present paper, we will treat the problem of a charged particle orbiting a magnetic planet especially Earth and Saturn. The purpose is mainly to study the effect of the gauss coefficients on the trajectories created by charged dust grains in equatorial and non equatorial plane and their condition of existence or not, using the general expression of a planetary magnetic field with the simultaneous effect of dipolar and quadrupolar magnetic fields compared to the [3] work. The paper is organized as follows. In section 2 we establish the problem using the dynamical model with quadrupolar effect, we present in section 3 the analytical results with gauss coefficients, we analyze the results and discussion in section 4 and summarize the work in the last section.

2. A dynamical model

We will use the model of the dynamics detailed in [3]. We assume a particle of mass $m$ and charge $q$ orbiting about a rotating magnetic planet (aligned centered planet) [11] with mass $M$ and radius $R$. However, the presence of both gravitational and electromagnetic forces is crucial [12]. The formulation of the general Hamiltonian of this particle in Gaussian units is given as:

$$H = \frac{1}{2}m(P - \frac{q}{c}A(r))^2 + U(r)$$

(1)

Where $c$ is a speed of light in a vacuum and where $r = (x, y, z)$ corresponds to the Cartesian coordinates or the particle position, and $P$ the conjugate momenta of $r$. In addition, the vector potential is represented by $A$, it describe magnetic forces given by:

$$A(r) = \frac{\mu}{r^3} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

(2)

And $U(r)$ is the scalar potential which generate the electric and gravitational interactions. The magnetic field $B$ of the magnetic planet is supposed to be a perfect magnetic dipole of strength $\mu$ aligned to the north - south poles of the planet which reads

$$B = \nabla \times A$$

(3)

Otherwise, the magnetosphere surrounding the planet is taken as rigid conducting plasma that rotates with the same angular velocity as the planet, where the charge $q$ is subject to a corotational electric field, in general it is defined as:

$$E = -\frac{1}{c}(\Omega \times r) \times B$$

(4)
In order to characterize the influence of the charge to mass ratio of the charged particle, we get the gravitational and corotational electric fields influence depending on the two parameters $\sigma_g$ (connected to the gravitational field) and $\sigma_r$ (for the co-rotational electric field) which can take values 1 or 0 [3]. In our treatment to get more interesting results for the dynamics of charged dust grains near magnetic planet, the magnetic field expression is more detailed, similar to the geomagnetic field [16] it can be written in spherical coordinates as:

$$\vec{B}(r, \theta, \varphi, t) = -\nabla V(r, \theta, \varphi, t)$$  \hspace{1cm} (5)

Where the potential is defined by the formulae:

$$V(r, \theta, \varphi, t) = \sum_{l=1}^{n} \sum_{m=0}^{l} \left( \frac{a}{r} \right)^{l+1} \left( g^l_m(t) \cos(m\varphi) + h^l_m(t) \sin(m\varphi) \right) P^l_m \cos \theta$$  \hspace{1cm} (6)

Here, $g^l_m$ and $h^l_m$ represents the Gaussian coefficients depending on the time and the Legendre polynomials $P^l_m \cos \theta$ where $l$ is the quantum number of orbital angular momentum, and $m$ a magnetic number. The general formula of corotational electric field is used in our investigation with development depending on the terms with is:

$$\vec{E} = -\frac{q}{c} \nabla \left( \frac{a^3}{r} (-g^0_1 \sin^2 \theta) + \frac{a^4}{2r^2} (3g^0_2 \sin^2 \theta \cos \theta) \right)$$  \hspace{1cm} (7)

The parametric equation denotes the corotational electric field in spherical harmonics, we can find the treatment [3] as a special case of this expression ($a^3g^0_1 = 1$ and $a^4g^0_2 = 0$), where besides a magnetic dipole another quadrupolar term is considered depending on $g^0_2$ coefficient, the work dealing on the main question posed in the context of the similarity or different possible results of the effect of the quadrupolar magnetic term on the Störmer problem for dynamics of charged dust grains near magnetic planet or outer one. Therefore, the main goal of these studies point on the describe the physical behavior of dust components in arbitrary distances from the center of the considered planets with various physical constraints. For our contribution, the dynamics of charged dust grains is described by the Hamiltonian function in spherical coordinates with the effect of a quadrupolar magnetic term we define:

$$H = \frac{1}{2} (P_r^2 + P_\theta^2) + \frac{1}{2} \omega^2 r^2 \sin^2 \theta - \sigma_g \frac{a^3 g^0_1 \sin^2 \theta}{r} + \sigma_r \frac{a^4 g^0_2 \sin^2 \theta \cos \theta}{2r^2}$$  \hspace{1cm} (8)

Here, to provide a comprehensive view of the behavior or the trajectories of the dust grains charged positively or negatively if we take $a^3g^0_1$ (connected to [3] treatment) and $a^4g^0_2 = 1$ (for quadrupolar term contribution). We investigate the problem by making a comparison of the paper [3] and take into accounting a numerical values of $a$ the planetary radius with the Gauss coefficients $g^0_1$ and $g^0_2$ to get detailed information about their effects and give physical explanations for any different results compared to the dipolar good approximation let say.

Similar setup of the problem was employed in [3], but here, we can write the Effective potential in the form taking into account $g^0_2$:

$$U_{eff} = \frac{1}{2} \omega^2 r^2 \sin^2 \theta + \sigma_g \frac{a^3 g^0_1 \sin^2 \theta}{r} + \sigma_r \frac{3a^4 g^0_2 \sin^2 \theta \cos \theta}{2r^2} - \sigma_g$$  \hspace{1cm} (9)

Where the considered potential is depending on the $(r, \theta)$ spherical variables, and other parameters of the system. Specifically, $\delta$ the charge of the dust grain and the orbital frequency $\omega$ which define the direction motion of charged particles in the same or opposite direction to the subject planet rotation.
3. The equilibrium with dipolar and quadrupolar fields

In order to derive the interesting relations between charged particles $\delta$ and orbital frequency $\omega$ to have various comprehensive view of the equilibrium state we the present conditions, it is convenient to define the critical points of the effective potential $U_{\text{eff}}$ which are found as the solutions of the system of equations:

$$\frac{\partial U_{\text{eff}}}{\partial r} = 0 \quad \text{and} \quad \frac{\partial U_{\text{eff}}}{\partial \theta} = 0$$

The Sign of the particle angular velocity $\omega$ defines the nature motion of the charged dust grains, prograde (positive charge sign) or retrograde (negative charge sign) with respect to the planet motion direction. The critical points of the $U_{\text{eff}}$ are found as the solutions of the system equations

$$\frac{\partial U_{\text{eff}}}{\partial r} = -\omega^2 r \sin^2 \theta + \frac{1}{r} a^3 g_1^0 \delta(\sigma_r - \sigma_0) \sin^2 \theta + \sigma_g + \frac{1}{r^3} (3a^4 g_0^0 \delta(\omega - \sigma_r) \sin^2 \theta \cos \theta = 0 \quad (10)$$

$$\frac{\partial U_{\text{eff}}}{\partial \theta} = -\omega^2 r^2 \sin \theta \cos \theta - \frac{2a^3 g_1^0 \delta}{r} \sin \theta \cos \theta + \frac{3a^4 g_0^0 \delta(\sigma_r - \omega) \sin^2 \theta \cos^2 \theta}{r^2} + \frac{3a^4 g_2^0 \delta(\omega - \sigma_r) \sin^3 \theta}{2r^2} = 0 \quad (11)$$

The equivalent non linear system reads:

$$\frac{\partial U_{\text{eff}}}{\partial r} = -\omega^2 r \sin^2 \theta + \frac{1}{r} a^3 g_1^0 \delta(\sigma_r - \sigma_0) \sin^2 \theta + \sigma_g + \frac{1}{r^3} (3a^4 g_0^0 \delta(\omega - \sigma_r) \sin^2 \theta \cos \theta = 0 \quad (12)$$

$$\frac{\partial U_{\text{eff}}}{\partial \theta} = -\frac{\sin \theta \cos \theta}{r}(\omega^2 r^4 + 2a^3 g_1^0 \delta(\omega - \sigma_r) r + 3a^4 g_0^0 \delta(\omega - \sigma_r)(\cos \theta - \frac{\sin \theta \tan \theta}{2}) = 0 \quad (13)$$

The system of non-linear equations gives a part of the dipolar magnetic field influence parameters and our contribution by terms related to the $a^4 g_0^0$, looking at the system it is evident that it depends on a radial and angular parameter $r$ and $\theta$ for the both equations. Besides this to describe the trajectories of charged dust particles in the new conditions we use the methods derived from the model of the generalized Störmer problem [3] Furthermore, if $\theta = \frac{\pi}{2}$ we get equatorial orbits, and when $\theta \neq \frac{\pi}{2}$ we obtain the non-equatorial (Halo) orbits, where their existence is strictly relied on the values of $\omega$ and $\delta$.

3.1. Equatorial orbits

Assuming that the motion is in the equatorial plane, such kind of orbits appear when $\theta = \frac{\pi}{2}$, and the second equation of the system vanishes, which satisfy the parametric equation:

$$\omega^2 r^4 + 2a^3 g_1^0 \delta(\omega - \sigma_r) r + 3a^4 g_0^0 \delta(\omega - \sigma_r)(\cos \theta - \frac{\sin \theta \tan \theta}{2}) = 0 \quad (14)$$

We mention that, the above parametric equation depends on the radial and angular parts taking a positive radius in all this treatment, in this context the gauss coefficients are considered for dipolar $g_1^0$ and $g_0^0$ for a quadrupolar contribution negative sign for Earth and positive for Saturn planet. These numerical values are not depending in $\omega$ and $\delta$ parameters which not the case in ref [3]. The earth magnetic field is modeled by the general expression in equation 6 with radius $a = 6378$ km, $g_1^0 = -0.29615$G and $g_0^0 = -0.24996$G derived from IGRF-13 model not taken into account in ref [3], here the effect of quadrupolar magnetic term is not presented because of the angular part which is the case of vanishing this interesting part for more details see [15]. Using this parameters give resulting trajectories created near earth in the equatorial plane, we remark
that from Fig(1) et Fig(2) for the most interesting and general case of generalized Störmer the RGSP see [3]. It is obvious that, the existence more restricted than the yellow one which permit to get circular orbits with the effect of the corotational and gravitational fields on one hand, in the other hand the numerical values of Saturn are given by the radius \( a = 60330 \text{ km} \), \( g_1^0 = +0.21535 \text{G} \) and \( g_2^0 = +0.01642 \text{G} \), the fig(3) exhibit the possible existence which are modified compared to [3]. From fig(4) and fig(5) it can be conclude that in a very small range of charges \( \delta \) the orbital frequency \( \omega \) is significant for earth the orbits in the green regions are not considerable and for Saturn the orbits exist with negligible modification compared to work cited in [3].

![Figure 1](image1.png)

**Figure 1.** Equatorial orbits existence regions in the plane(\( \delta, \omega \)) for Earth green color in Gaussian units and ref [3] yellow color

![Figure 2](image2.png)

**Figure 2.** Equatorial orbits existence regions in the plane(\( \delta, \omega \)) for ref [3] with constant variables equal to the unity \( a = g_1^0 = g_2^0 = 1 \)

![Figure 3](image3.png)

**Figure 3.** Equatorial orbits existence regions in the plane(\( \delta, \omega \)) for Saturn orange color in Gaussian units and ref [3] yellow color.

### 3.2. Non equatorial orbits

Non equatorial or Halos orbits (defined as orbits which do not cross the equatorial plane) appear when \( \theta \neq \frac{\pi}{2} \) and their description is connected to the solution of the parametric equations
Figure 4. Curves of constant radius for equatorial orbits with small charges bounded by green existence regions for Earth.

Figure 5. Curves of constant radius for equatorial orbits with small charges bounded by orange existence regions for Saturn.

depending respectively to the partial radial \( r \) and angular \( \theta \) parameters. They are described by solution from a system of equations (12) and (13) given by:

\[
Q(r, \omega) = 0
\]

(15)

And

\[
A(\theta, \omega) = 0
\]

(16)

To calculate Halos orbits around magnetic planet, the constraint of the quadrupolar term is also considered, in this investigation we use purely analytical solutions to check the existence/persistence of these trajectories \((d_1 = a^3 g_1^3 \delta \text{ and } d_2 = a^3 g_2^3 \delta)\). Which gives:

\[
Q(r, \omega) = 324 d_2^4 (\sigma_r - \omega)^4 + 4r^4 (-\sigma_g + d_1 (\sigma_r - \omega) + r^3 \omega^2) \\
(5r^3 \omega^2 + d_1 \delta (\omega - \sigma_r) (r^3 \omega^2 + 2d_1 \omega - \sigma_r^2) - (27d_2^3 r^2 (\sigma_r - \omega)^2) \\
((3\sigma_g - 2d_1 \sigma_r)^2 + 4d_1 (3\sigma_g - 2d_1 \sigma_r) \omega + 4(d_1^2 - 3\sigma_g r^3) \omega^2 + 8r^6 \omega^4)) = 0
\]

(17)

\[
A(\theta, \omega) = \frac{27}{128} d_2 (\sigma_r - \omega)^4 (7923d_2^3 \omega^2 + 1408d_1^4 (\sigma_r - \omega) \cos \theta + 13320d_2^2 \omega^2 \cos 2\theta \\
+ 576d_1^4 (\sigma_r - \omega) \cos 3\theta + (7900d_2^2 \omega^2 \cos (4\theta) + (64d_1^4 (\sigma_r - \omega) \cos 5\theta)) \\
- 8d_2 (\sigma_g (\sigma_r - \omega)) (2\sigma_g \cos \theta (\sigma_g^2 + 18d_1^2 (\sigma_r - \omega)^2 \sin^4 \theta + \sin \theta (6d_1 \sigma_g (\omega - \sigma_r) \sin (2\theta)) \\
- (\sigma_g^3 + 2d_1 (\sigma_r - \omega) \sin^2 \theta (-5\sigma_g^2 - 9d_1 (\sigma_r - \omega) \sin^2 \theta) (2\sigma_g + 3d_1 (-\sigma_r + \omega) \sin^2 \theta)))) \tan \theta = 0
\]

(18)

We mention that, in the ref [3] periodic circular orbits exists in the a yellow bounded regions in fig(7). For Earth and Saturn existence halos orbits from fig(3) and fig (8) respectively, it is shown that the used numerical parameters affect the creation of the trajectories in a plane above or below the equatorial plane. It is important to said that in very small charges the situation is different for earth fig (9) the effect is considerable than the Saturn existence regions in fig(10) for the RGSP. It’s important to mention that a family of curves for the four known Generalized
Figure 6. Halos orbits existence regions in the plane($\delta, \omega$) for Earth green color in Gaussian units and ref [3] yellow color.

Figure 7. Halos orbits existence regions in the plane($\delta, \omega$) for ref [3] with constant variables equal to the unity $a=g_1^0=g_2^0=1$.

Figure 8. Halos orbits existence regions in the plane($\delta, \omega$) for Saturn orange color in Gaussian units and ref [3] orange color.

Störmer cases is considered in detail in ref [3], with the inclusion of the gravitational and co-rotational fields, with the individual effect too which is not the case in our treatment where the interesting results are presented only. The plane($\delta, \omega$) is divided into parts for Earth where the existence region(green color) not include the Halos orbits with the effect of the gravitational forces only see fig(11), contrarily to the Saturn planet which gives halos orbits with a small number than the work of paper [3] shown in fig(12) with constant radial and angular parts of equations which describe non equatorial orbit.
4. Results and discussion

The aim of this study is to analyze the dynamics of charged particles located in magnetosphere of magnetic planets like earth, Saturn, Jupiter, Neptune and Uranus, compared to the generalization of Störmer problem interesting work in ref [3]. From different plots based on the equation of motion and a Hamiltonian of the system with the parametric equations depending on dipolar and quadrupolar magnetic term using analytical solutions we summarize: The trajectories exist in the equatorial and non equatorial (Halos) plane for Earth and Saturn planets with modified behavior of created trajectories.

From region figures fitted for the two planets using gauss coefficients numerical values for each planet considered experimental measured values in the equator besides in the plane parallel above or below the equatorial one, these trajectories exist but in different regions which mean that geomagnetic parameters has significant effects on the created regions for positive radius $r$. With the effect of a quadrupolar magnetic term and the same conditions of parameters the
situation is different see fig(13) and fig (14) where the Halo orbits exist although the small charge and significant orbital frequency with more halos orbits than the results of [15].

5. Conclusion
The study of the behavior of a charged dust grains particles is crucial and can be done in different ways using multiple analytical and computational methods, base on the results teated in [3] as a reference to compare, the present paper, reproduce a general dynamics of charged particles used other term not considered in the literature work which represent a special case of the general contribution of our work. Comparing our new results with known one, we mention that the effect of the gauss coefficient is remarkable for earth case the orbits are created in regions not allowed with ref[3], contrarily to Saturn were the orbits are located in the same regions with small modifications. From the graphs with the effect of $q_0^2$ is significant on the charged particles affected by the simultaneous effect of the corotational and gravitational forces. Finally we conclude that the quadrupolar term give a good approximation in study of dynamics of dust...
grains with and without Gauss coefficients.

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