Maximum independent set (stable set) problem: A mathematical programming model with valid inequalities; Computational testing with binary search and alternate optimal basic solutions (extreme points)

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Abstract

This paper deals with the maximum independent set (M.I.S.) problem, also known as the stable set problem. The basic mathematical programming model that captures this problem is an Integer Program (I.P.) with zero-one variables and only the edge inequalities. We present an enhanced model by adding a polynomial number of linear constraints, known as valid inequalities; this new model is still polynomial in the number of vertices in the graph. We carried out computational testing of the Linear Relaxation of the new Integer Program. We tested about 7000 instances of randomly generated (and connected) graphs with up to 64 vertices (as well as all 64-, 128-, and 256-vertex instances at the “challenge” website OEIS.org). In each of these instances, the Linear Relaxation returned an optimal solution with (i) every variable having an integer value, and (ii) the optimal solution value of the Linear Relaxation was the same as that of the original (basic) Integer Program. Our experience has been that a binary search on the objective function value is a powerful tool which yields a (weakly) polynomial algorithm.

Keywords. Linear programming, Integer programming, Independent set, Stable set, NP completeness, NP hardness, computational complexity, Valid inequalities, Polytope.

AMS classification. 90-05, 90-08, 90C10, 90C27, 90C60, 68Q25, 68Q17.

1 Background

The decision version of the maximum independent set (M.I.S.) problem, also known as the stable set problem, is as follows:

Given a constant $K$ and a graph $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges in $G$, is there a subset $S$ of $V$ such that (i) no two members of $S$ are adjacent to each other in $E$, and (ii) $|S|$ (the cardinality of $S$) is at least $K$?

The number of vertices $|V|$ in a graph is denoted by $N$.

The decision version of M.I.S. is known to be NP-complete [GJ79].
M.I.S. (that is, its optimisation version), is captured by the following basic Integer Program (I.P.).

**Problem 1.**

\[
\text{Maximise } Z_1 = \sum_{j \in V} F[j] \\
\text{Subject to } \\
F[i] + F[j] \leq 1 \quad \forall (i, j) \in E \\
F[j] \in \{0, 1\} \quad \forall j \in V. \tag{1}
\]

Note that for every vertex \(j\), \(F[j]\) is a binary variable which can be assigned exactly one of two values (either zero or one). \(F[j] = 1\) if vertex \(j\) is a member of the Independent set, and zero otherwise.

The constraints of the form \(F[i] + F[j] \leq 1\) are known as **edge inequalities**.

**Number of edge inequalities:** Since there is one such constraint per edge, the number of these constraints is polynomial in \(N\).

Let \(OPT\ (I.P.)\) be the optimal solution value of the Integer Program above.

The **Linear Relaxation** of the above I.P. in (1) is the following Linear Program (L.P.).

**Problem 2.**

\[
\text{Maximise } Z_2 = \sum_{j \in V} F[j] \\
\text{Subject to } \\
F[i] + F[j] \leq 1 \quad \forall (i, j) \in E \\
0 \leq F[j] \leq 1 \quad \forall j \in V. \tag{2}
\]

It is known that Linear Programs are in the computational complexity class \(P\); that is, they can be solved in polynomial time; for example, using algorithms such as the Ellipsoid method or the Interior Point method [FP93]. Hence every instance of Problem 2 can be solved in polynomial time.

Whether we solve Problem 1 or Problem 2, the underlying input instance is the same, which is a graph, say \(G_1 = (V_1, E_1)\). For example, the vertex set can be \(V_1 = \{a, b, c, d\}\), and the edge set can be \(E_1 = \{(a, c), (b, d), (a, d)\}\).

Suppose we solve an instance \(G_k\) of Linear Program [2] in polynomial time (for example, using an Interior Point algorithm); and suppose in the optimal solution, for every vertex \(j\), \(F[j]\) is integer (that is, either zero or one). This means, we have solved the same instance \(G_k\) for the Integer Program (Problem 1) in polynomial time as well. And if we can do this for every instance, then we can conclude that Problem 1 can be solved in polynomial time.

**Maximal Clique:** This is a clique \(Q\) that cannot be enlarged by adding vertices to \(Q\).

## 2 Adding Valid Inequalities to the model

### 2.1 Adding maximal clique constraints

Clique inequalities have been known for a long time. For example, see [GLS12] (the chapter on *Stable Sets in Graphs*).
To the Linear Program in Problem 2, we add *Maximal Clique* inequalities. In any clique, at most one vertex can become a member of the independent set (or stable set) $S$.

For example, if vertices 1, 5, 9, 13 and 15 form a clique in the input graph $G$, at most one of them can become a member of $S$. In other words, this creates the following constraint in our Integer Program (and the corresponding Linear Program):

$$F[1] + F[5] + F[9] + F[13] + F[15] \leq 1.$$  \hspace{1cm} (3)

The essence of the algorithm is as follows:

We use different starting points $(\text{startPt})$, from $(\text{startPt} = 1)$ to $(\text{startPt} = N)$. For every $\text{startPt}$, we vary the second vertex $m$ from $(\text{startPt} + 1)$, go up to $N$, and wrap around to $(\text{startPt} - 2)$. We say that $m$ is where we begin scanning from.

The first vertex in a maximal clique $Q$ is always the $\text{startPt}$. Next, we check whether vertex $(m)$ can be included (this is possible if there is an edge between $\text{startPt}$ and $m$). Then we check whether $(m + 1 \text{ mod } N)$ can be included in $Q$. We continue this until we reach vertex $(m - 1 \text{ mod } N)$. Once a vertex is included a maximal clique $Q$, it is never removed (which follows the definition of a maximal clique).

This algorithm has loops nested up to 3 levels and each loop is of $N$ iterations, at most. Hence the worst case running time is $O(N^3)$, which is polynomial in $N$.

### 2.1.1 Complexity of the C code that generates maximal clique constraints

**Loops:** The C program contains only for loops, and they are all polynomial in $N$, the number of vertices. The number of such loops is, of course, a constant.

### 2.2 Adding chordless cycle constraints

In addition to the Maximal Clique constraints, we add *Chordless Cycle* constraints to the Linear Program in Problem 2.

We follow the method described in Dial et. al. [DCLJ13]. This method finds each chordless cycle only once.

In polynomial time, we can add only a polynomial number of constraints.

For a cycle $C$ to be chordless, we require that there be no edge in $E$ between any two of the cycle vertices, other than those edges in $C$.

For example, if $C$ consists of the edges $(a, b), (b, c), (c, d), (d, e)$ and $(a, c)$, then the following edges cannot be in $E$: $(a, c), (a, d), (b, d), (b, e)$, and $(c, e)$.

We first list all 3-cycles in the graph and add them as constraints of the form $F[i] + F[j] + F[k] \leq 1$. The number of such cycles is at most $N(N - 1)(N - 2)/6$; hence this step adds a polynomial number of constraints to the Linear Program.

*When the cycle length is more than three:* Suppose the number of vertices in the cycle is $K$. The right hand side (RHS) of this type of constraint is $\lfloor K/2 \rfloor$. For example, if the cycle length is 10, the RHS is $\lfloor 10/2 \rfloor = 5$. However, if the cycle length is 15, the RHS is $\lfloor 15/2 \rfloor = 7$.5 = 7.$
2.2.1 Complexity of the C code that generates chordless cycle constraints

**Loops:** The for loops in the C program are all polynomial in $N$. As for the while loop, it starts with a value of $Tcount$ which is at most $N^3$. In every iteration, the loop counter $Tcount$ is decreased by one. But then, further down the while loop, the $Tcount$ can also increase under certain conditions. Hence this while loop is NOT polynomial.

To overcome this difficulty, we restrict path lengths to a small number, say 5 or 6. Once we implement this restriction, the growth in the number of constraints is polynomial (see Section 2.3).

2.3 Growth in the number of constraints

We used a upper bound of 5 for the path length (see Section 2.2.1).

We observed the following growth in the number of constraints. For each $N$ (number of vertices), we recorded the worst case; that is, the instance with the highest number of constraints. Also for each $N$ in the table below (except the 128-vertex case), we tested at least a 1000 instances.

| Number of Vertices | Number of Constraints |
|--------------------|-----------------------|
| 28                 | 4438                  |
| 30                 | 5395                  |
| 32                 | 6480                  |
| 34                 | 7701                  |
| 36                 | 9066                  |
| 38                 | 10546                 |
| 40                 | 12260                 |
| 42                 | 14105                 |
| 44                 | 16126                 |
| 46                 | 18331                 |
| 48                 | 20681                 |
| 50                 | 23325                 |
| 52                 | 26028                 |
| 54                 | 29098                 |
| 128 (single instance; see Sec. 6.3) | 25405 |

3 The algorithm

After adding maximal clique and chordless cycle constraints, we have an L.P. (Linear Programming) model as follows:
Problem 3.

Maximise \( Z_3 = \sum_{j \in V} F[j] \)

Subject to

- Edge Inequality constraints
- Maximal Clique constraints
- Chordless Cycle constraints

\[ 0 \leq F[j] \leq 1, \quad \forall j \in V. \] (4)

The total number of constraints in the above Linear Program is polynomial in \( N \), the number of vertices.

The algorithm involves the following steps:

1. Solve the original Integer Program in Problem 1 and obtain its optimal solution value, \( OPT \ (I.P.) \). This step is carried out using an Integer Program solver (from either GLPK or Gurobi). For future reference, we say that:

   \( OPT \ (I.P.) = \) Optimal value of the original Integer Program in Problem 1.

   We obtain \( OPT \ (I.P.) \) purely for comparison purposes; to compare \( OPT \ (I.P.) \) with the optimal solution values of the Linear Programs solved in Steps 2, 3 and 4.

2. Solve the Linear Program in Problem 3. If all variables in the optimal solution turn out to be integers (zero or one), and the objective function value \( Z = OPT \ (I.P.) \), we have achieved an optimal Integer solution for Problem 2, hence go to Step 5, and end the algorithm! If not, continue.

3. (The \( F[i] = 0 \) loop) Let \( i = 0 \);

   Do the following until either all variables \( F[j] \) (for every vertex \( j \) in the instance) are integers (that is, either zero or one), or when \( i \) has reached its limit of \( N \):

   (3a) \( i = i + 1 \) (that is, increase \( i \) by one). Set \( F[i] = 0 \). Add this constraint to the model in Problem 3. (If there is any other constraint in the model setting \( F[k] \) (for ANY \( 1 \leq k \leq N \)) to either zero or one, remove them.)

   Let \( j = i \).

   (3b) Increase \( j \) by one. Set \( F[j] = 0 \), and add this constraint to the model in Problem 3.

   (3c) Solve this Linear Program. If all variables in the optimal solution turn out to be integers (zero or one), and the objective function value \( Z = OPT \ (I.P.) \), we have achieved an optimal Integer solution for Problem 2, hence go to Step 5 and end the algorithm! If not, continue.

   (3d) Now replace the constraint \( F[j] = 0 \). Remove it and add the constraint \( F[j] = 1 \). Solve this Linear Program. If all variables in the optimal solution turn out to be integers (zero or one), and \( Z = OPT \ (I.P.) \), we have achieved an optimal Integer solution for Problem 2, hence go to Step 5 and end the algorithm! If not, continue.

   (3e) If \( j < N \), go to Step (3b) above, else continue.

   (3f) If \( i < N \), go to Step (3a) above, else go to Step 4.

In this step (Step 3), we solve a quadratic (that is, at most \( N(N - 1) \)) number of Linear Programs.
4. (*The F*[i] = 1 loop*) This loop is similar to the previous (*i*) loop except that here we set \( F[i] = 1 \) rather than \( F[i] = 0 \).

Steps 4(a-f) are defined analogous to Steps 3(a-f).

In this step (Step 4), again, we solve a quadratic (that is, at most \( N(N − 1) \)) Linear Programs.

5. End of the algorithm.. Output the result of the last Linear Program solved.

Odd Anti-Hole constraints may be added.. See the software repository for C code. An odd anti-hole in \( G \) is an odd hole (an odd length chordless cycle) in the complement graph of \( G \).

3.1 d2wNbb

Steps 3 and 4 combined are what we can call \( d2wNbb \), short for “depth 2, width \( n \), branch and bound”. Consider pairs of vertices \((i, j)\). The number of such pairs is \( N(N − 1)/2 \). For each pair, we run Linear Programs with these additional constraints:

1. First, with the two additional constraints \( F[i] = 0 \) and \( F[j] = 0 \);
2. Next, with \( F[i] = 0 \) and \( F[j] = 1 \);
3. Then, with \( F[i] = 1 \) and \( F[j] = 0 \);
4. And finally, with \( F[i] = F[j] = 1 \).

Thus the maximum number of Linear Programs solved is \( 2N(N − 1) \).

**Note:** It is much better to run a Pre-solver (explained below) and obtain a smaller set (or a reduced set) of constraints before entering \( d2wNbb \). Otherwise, we will be executing Pre-solve for every L.P. solved in \( d2wNbb \), which will consume a lot more resources.

*Disadvantage with this method:* Optimal integer solutions to M.I.S. (and very often, even the L.P. relaxation) are usually highly degenerate; i.e., plenty of variables in the optimal basis will be zeroes. Hence different \((F[i], F[j])\) combinations may return the same extreme point of the polytope as optimal solutions.

4 Correctness of the Algorithm

We can be sure that the solution is correct, because:

(a) We include the original Stable Set constraints (the Edge Inequalities) in the larger Linear Program (Problem 3). (We *don’t* include the the 0-1 binary Integer constraints, of course.)

The inclusion of Edge Inequalities ensures feasibility of the original Linear Program (Problem 2).

and

(b) In steps 3(c-d) and 4(c-d), we compare the optimal solution value of the Linear Program with the optimal value \( OPT (I.P.) \) of the given M.I.S. instance. This ensures that the Linear Program achieves the optimal solution value that we seek.

**Note:** Solving Problem 1 needs exponential time in the worst case. Unfortunately, comparison with the optimal solution value of Problem 1 appears to be the *only* method to
confirm optimality of the result that we obtained from Linear programs. In contrast, NP-hard problems such as VERTEX COVER have polynomial time heuristics that guarantee solutions within a factor of 2 of the optimal solution value. There is NO such heuristic for the STABLE SET problem that guarantees a constant factor approximation.

5 Experiments

5.1 Update on 2022-9-29

As for the test problems at the OEIS site (https://oeis.org/A265032/a265032.html), we were able to solve every 128-vertex and 256-vertex instance to optimality in weakly polynomial time using a combination of binary search and $d2wNbb$. The types of constraints used were: Edge inequalities, maximal cliques and 3-cycles. The number of constraints added for each instance is large, but still polynomial in $N$, the number of vertices.

5.2 Solvers

To execute Linear and Integer Programs, we used GLPK Version 5.0 and Gurobi 9.5.1 in a Linux environment. Further information on GLPK is available at: https://www.gnu.org/software/glpk/

Further information on Gurobi can be found at: https://www.gurobi.com

5.3 Testing and Results

In our computational tests, we tested more than 7000 instances, testing connected graphs with up to 64 vertices (as well as all instances up to 256 vertices listed at https://oeis.org/A265032/a265032.html).

In each of the 7000+ instances, the output of the Linear Program in Step 5 of the algorithm provided an optimal solution with

- every variable having an integer value; and
- the optimal solution value was equal to $OPT (I.P.)$ obtained in Step 1.

**Note on tolerance**: All results are within a tolerance limit of $10^{-5}$.

**GLPK vs Gurobi**: We used two solvers, GLPK and Gurobi. The GLPK Linear Program solver always returned an Integer optimal solution at the end of Step 4 of the algorithm (that is, the LP referred to in Step 5).

Gurobi’s Linear Program solver (using the Barrier method) returned fractional solutions in about 5% of the M.I.S. instances. However, in these instances, once we added yet another constraint of the form $F[j] = 0$ or $F[j] = 1$ to the Linear Program referred to in Step 5, we obtained Integer solutions from Gurobi’s Linear Program solver. Thus in each of these M.I.S. instances, at most an additional $2(N - 1)$ Linear Programs (which is polynomial in $N$) needed to be solved.

**Multiple optimal solutions**: When a Linear Program has multiple optimal solutions, some solvers may hit a solution which is not an extreme point. Furthermore, some of these extreme points may not be integer. In such cases, we were able to overcome the difficulty and obtain an
Integer solution with independent set size equal to \( OPT \ (I.P.) \) by solving at most an additional \( 2(N - 1) \) Linear Programs as described above.

To achieve this, we look for a Linear Programming instance in Steps 3 and 4 whose optimal solution value is equal to \( OPT \ (I.P.) \). This instance is then tested by adding yet another constraint of the form \( F[j] = 0 \) or \( F[j] = 1 \). In other words, we needed to run at most \( 2(N - 1) \) additional Linear Programs, one of which gave us an Integer optimal solution that we desire, with its value equal to \( OPT \ (I.P.) \).

Random graphs where the probability of each edge is 0.5: It was brought to our attention that such graphs are among the hardest for the M.I.S. problem. So far, we have tested about 800 instances of such graphs with vertices ranging from 40 to 45. In each instance, our algorithm (using both GLPK and Gurobi) successfully produced an optimal solution to the M.I.S. instance in polynomial time.

5.4 Test problems at OEIS

As for the test problems at the OEIS site ([https://oeis.org/A265032/a265032.html](https://oeis.org/A265032/a265032.html)), we were able to solve all 64-vertex and 128-vertex instances exactly (obtained optimal solutions) in polynomial time using the algorithm in Section 3.

Testing for the 128-vertex 1ZC instance ([https://oeis.org/A265032/a265032_1zc.128.txt.gz](https://oeis.org/A265032/a265032_1zc.128.txt.gz)) is described below (since we found this to be the hardest among the 128-vertex instances). For this instance, either a Warm Start or Binary Search is required; both these techniques are described below.

5.5 Test problem 128-vertex 1ZC

The linear relaxation (as in Problem 2) of the given M.I.S. instance returned an optimal solution value of 64. After we added maximal clique and chordless cycle constraints (as in Problem 3), the linear relaxation gave us an optimal solution value of 20.66.

Note: For Chordless cycles, we added only 3-cycles for this instance.

- We first added a cardinality constraint \( \sum_{j \in V} F[j] = 20 \) to Problem 3, and applied \( d2wNbb \) (see Sec. 3.1). No integer solution was obtained. This step required solving \( 2N(N - 1) = 2 \times 128 \times 127 = 32512 \) Linear Programs.
- We repeated the above step using the cardinality constraint \( \sum_{j \in V} F[j] = 19 \). Again, no integer solution was obtained. (solving 32512 Linear Programs again)
- Finally, when we used \( \sum_{j \in V} F[j] = 18 \), we obtained an integer solution. This step required solving at most 32512 Linear Programs.

In total, we solved at most \( 6N(N - 1) \) Linear Programs, each with a polynomial number of constraints.

5.6 Binary search

Let us define an integer solution to be one where the value of every \( F[j] \) (\( 1 \leq j \leq N \)) is integer; that is, \( F[j] \in \{0, 1\} \).
In our experience, Binary Search coupled with \( d2wNbb \) has been a powerful combination. It has worked for every instance that we tested so far, i.e., produced an integer optimal solution in weakly polynomial time.

For each value \( Z \) of the objective function checked within the Binary Search framework, we run the \( d2wNbb \) procedure once; thus for each such value of \( Z \) that we check, we run at most \( N(N - 1) \) Linear Programs.

In general, one could perform a binary search on the objective function value \( Z \), once we determine an upper bound (20, in this example), and a lower bound for \( Z \). Thus the number of Linear Programs to be solved is at most \( N(N - 1) \log(UB) \) where UB is the upper bound (hence yielding a weakly polynomial algorithm).

Binary search is carried out between two markers, top and bottom. We fix top to be \( Z \)’s upper bound (= 20, in the above example for the 1ZC-128 problem). If there is a feasible integer solution to the Linear Program where \( (Z = \text{down}) \) is another extra constraint, we try the next instance by increasing the value of \( \text{down} \); otherwise, we try by decreasing the value of \( \text{down} \).

(We want to find the maximum value of \( \text{down} \) where the Linear Program optimal solution is feasible integer.)

If there is a M.I.S. of size \( k \), there is certainly a M.I.S. of size \( k - 1 \); just remove one of the vertices in the solution set. From this, it follows that if there exists a feasible independent set for a particular value of \( \text{down} \), say, \( \text{down}_A \), there is a feasible independent set for every value of \( \text{down} \) that is \( \leq \text{down}_A \). Hence, once we find an integer optimal solution to the linear program with \( (Z = \text{down}_A) \), we continue our binary search above by setting \( Z > \text{down}_A \), not below \( \text{down}_A \).

As mentioned earlier, we include all Edge Inequalities from Problem 1 (Page 2) in every instance that we solve. Hence during Binary Search, for values of \( Z \) higher than \( OPT(I.P.) \), it is impossible to find an integer feasible solution.

Degeneracy: As mentioned in 3.1 for a given value of \( Z \), the extreme points of the Linear Programs suffer from a high level of degeneracy. Hence for some instances, it may be easier to enumerate all extreme points and pick the integer solution among them, if any.

Which inequalities to include?: While examining extreme points for a particular \( Z \) value, how many constraints should we include, and which ones? It might be beneficial to include fewer constraints, since this means fewer extreme points to examine. For example, we may only include the edge inequalities (and disregard others such as maximal clique constraints and chordless cycle constraints). However, a disadvantage with restricting ourselves to edge inequalities is that the range of \( Z \) values to carry out the binary search will be wider. For example, in one instance, with edge inequalities alone, the optimal value of the LP was 75; but after we added maximal clique and chordless cycle constraints, it reduced to 40.

5.7 Solving OEIS problems with 128 and 256 vertices using Binary Search

As for the test problems at the OEIS site (https://oeis.org/A265032/a265032.html), we were able to solve every 128-vertex and 256-vertex instance to optimality in weakly polynomial time using a combination of binary search and \( d2wNbb \).

For each instance, we used the following constraints: edge inequalities, maximal cliques and 3-cycles. (After generating the constraints, we used Gurobi’s Pre-Solver to remove redundant constraints.)
5.8 Alternate optimal solutions at extreme points

While solving a Linear Program with an added constraint such as \( Z = \sum_{j \in V} F[j] = 19 \), the first optimal solution that we obtain may not be integral. In such a case, we should check feasible vertices (extreme points) in the L.P. polytope with the same \( Z \) (solution value), whether any of them is an integer solution.

Feasible extreme points with the same \( Z \) value form a spanning tree. For example, we could use pivoting in Simplex method tableaux to move from one extreme point to the next. For a detailed description of a method, see for example, Section 5-7 (Page 166) of [GH62], Determination of all optimal solutions.

Two key questions remain about extreme points with the same \( Z \) value: (1) Is the number of such points polynomial (in the number of variables and constraints)? (2) And if so, can we visit all such points in polynomial time?

From the discussion above, if the answer to both these questions is yes, then Problem 1 can be solved in weakly polynomial time using binary search. (The answer to the second question is yes, if we define “polynomial time” to be “polynomial in the size of the input + size of the output”.)

Software such as lrs (http://cgm.cs.mcgill.ca/avis/C/lrs.html) and cdd (https://people.inf.ethz.ch/fukudak/cddhome/) are available to enumerate extreme points for Linear Programs.

5.9 Finding alternate optimal solutions with GLPK and Degeneracy

Typically the BFS (basic feasible solutions) to a Linear Program that models M.I.S. will be degenerate, i.e., some of the variables in the basis will have values of zero.

For all our instances, we explicitly add lower bound constraints of the form “\( x_i \geq 0 \)” for each variable \( x_i \).

About the problem of enumerating all OEP (optimal extreme points) for a Linear Program, we propose the following approach, using GLPK software. This a good way to handle degeneracy.

1. Solve the original Linear Program (call this \( LP_0 \)). Suppose this returns an optimal solution with a value of 50 (for example). Let the objective function be “Max. \( Z = CX \).” Hence \( Z_{max} = 50 \).

2. Now add the constraint “\( CX = 50 \)” to the original LP. This gives us a new LP, which we can call \( LP_1 \). Solve \( LP_1 \). With GLPK, we are able to save the last BFS (basic feasible solution) of \( LP_1 \) to a file, say \( Soln-1.bas \) (using the “-w Soln-1.bas” option in GLPK). \( Soln-1.bas \) is the first OEP (optimal extreme point).

3. In \( Soln-1.bas \), find the lexicographically first non-basic variable.. For example, let us say that this is \( x_5 \). Since we want to avoid degeneracy and go to a new OEP, we modify the lower bound for \( x_5 \). That is, we modify “\( x_5 \geq 0 \)” to “\( x_5 \geq \epsilon \)” where \( \epsilon \) is a very small positive number.. Call this \( LP_2 \).

4. Now run \( LP_2 \) using GLPK, using the previous basis \( Soln-1.bas \) as the starting solution.. In GLPK, you can do this using the “--ini Soln-1.bas” option in the terminal command line... The \( LP_2 \) output should be written to \( Soln-2.bas \).
5. If Step 4 is a failure, that is, if $LP_2$ is infeasible, then check Soln-1.bas, and find the non-basic variable lexicographically after $x_5$. For example, this could be $x_8$. Then the lower bound “$x_5 \geq \epsilon$” should be reset to zero ($x_5 \geq 0$). The lower bound for $x_8$ should be set to $\epsilon$ (that is, $x_8 \geq \epsilon$). Now run this new LP, again using the “--ini Soln-1.bas” option in GLPK.

On the other hand, if Step 4 is a success (that is, if $LP_2$ is feasible), then Solution 2 is the second OEP. Then we do something similar to Step 3... Open the file Soln-2.bas, find the lexicographically smallest non-basic variable (for example, $x_9$), reset the lower bound of $x_5$ to zero ($x_5 \geq 0$), change the lower bound of $x_9$ to $\epsilon$ ($x_9 \geq \epsilon$), and the solve the new LP (call it $LP_3$) in GLPK using the “--ini Soln-2.bas” option.

Continue the above steps until every OEP has been enumerated.

We traverse the OEP’s in a tree-like fashion.

6 Lifting subset constraints to solve problem 128-vertex 1ZC

6.1 Change (and abuse) of terminology

Traditionally, lifting means, in addition to the $F[i]$ of the vertices of the original subset $S$, the $F[i]$ of some other vertices are added to the left side of the constraint.

For example, an original constraint would be of the form $\sum_{i \in S} \alpha_i F[i] \leq \pi$, to which we add the $F[i]$ of some other vertices, say from a set $T$ where $S \cap T = \emptyset$, such that we obtain a new (lifted version of the) constraint of the form:

$$\sum_{i \in S} \alpha_i F[i] + \sum_{i \in T} \beta_i F[i] \leq \pi$$

(5)

where the $\beta$’s are some coefficients.

However, in what follows below, we do not add the $F[i]$ of any new vertex to the original constraint. Instead, we just increase coefficients $\alpha_i$ of existing vertices to $\gamma_i$, so that the new constraint is:

$$\sum_{i \in S} \gamma_i F[i] \leq \pi$$

(6)

where $\gamma_i \geq \alpha_i \ \forall i \in S$.

Let us abuse terminology and also call the procedure below as Lifting. Or perhaps we should call this “Self-Lifting” (since only the subset vertices remain in the lifted version of the constraint).

6.2 Our version of Lifting

Consider a subset $S$ of the vertex set $V$. The subgraph of $G$ induced by $S$ is denoted by $G[S]$. Let $\alpha(S)$ denote the size of the M.I.S. of $G[S]$. The subset constraint for $S$ is:

$$\sum_{i \in S} F[i] \leq \alpha(S).$$

(7)

For example, for the 5-vertex subgraph in Fig. 1(a), $S = \{A, B, C, D, E\}$. Since the size of
the M.I.S. for this subgraph is 2, the subset constraint is:

\[ F[A] + F[B] + F[C] + F[D] + F[E] \leq 2. \] (8)

However, Constraint (8) can be strengthened (or, lifted). Observe that if \( F[C] = 1 \) (which means that \( C \) is a member of the independent set), then all other \( F[i] \)'s must be zero. Thus (8) is a strict inequality. However, to turn this into an equality, we increase the coefficient of \( F[C] \) to 2:

\[ F[A] + F[B] + 2F[C] + F[D] + F[E] \leq 2. \] (9)

Note that the lifting technique cannot be applied if any other vertex is included first in the independent set. For example, suppose \( B \) is chosen first. And suppose we increase its coefficient to 2. Then \( E \) can be chosen next, which gives \( 2F[B] + F[E] = 3 \), which violates the RHS of Constraint (8) which is 2. Hence the coefficient of \( F[B] \) cannot be increased to 2.

Now let’s remove \( E \) in the subgraph of Fig. 1(a) to obtain the 4-vertex subgraph (b) on the right. The constraint

\[ F[A] + F[B] + F[C] + F[D] \leq 2 \] (10)

can be lifted to

\[ F[A] + 2F[B] + 2F[C] + F[D] \leq 2. \] (11)

We can afford to increase the coefficients of \( B \) and \( C \) simultaneously, since these two vertices cannot be in an independent set at the same time. And if either vertex \( B \) or \( C \) is chosen, neither \( A \) nor \( D \) can be chosen.

Now consider the complete bipartite graph \( K_{2,3} \) in Fig. 2. Then \( \alpha(S) \), the size of the M.I.S.,

\[ \alpha(S) \]

is equal to 3 (by choosing \( A, E \) and \( D \)). We can increase the coefficients of \( B \) and \( C \) to 2, but not simultaneously (if we do so, and \( B \) and \( C \) are members of an independent set, then \( 2F[B] + 2F[C] = 4 \), which violates the upper bound of 3). Thus, we can lift either \( B \) or \( C \), but not both at the same time.
This gives rise to two different lifted constraints (both of which can be included in the Linear Program at the same time):

\[ F[A] + 2F[B] + F[C] + F[D] + F[E] \leq 3; \quad \text{and} \]
\[ F[A] + F[B] + 2F[C] + F[D] + F[E] \leq 3. \]

Note that the coefficients of \( B \) and \( C \) cannot be lifted to 3. Suppose we lift the coefficient of \( B \) to 3. Then if \( B \) and \( C \) are in the independent set, \( 3F[B] + F[C] = 3 + 1 = 4 \), which violates the upper bound of 3 on the RHS.

For a list of lifting rules for all 4-vertex and 5-vertex graphs (obtained at http://www.graphclasses.org), see the appendix.

### 6.3 Solving Problem 128-vertex 1ZC using Lifting

For \( N = 128 \), the number of 5-vertex subsets = \( (128)(127)(126)(125)(124)/120 \). Out of these, we chose about 5400 (5435, to be exact) and lifted these constraints, as explained above.

We first ran a Linear Program (call this \( LP_1 \)) with the following constraints: (a) Edge inequalities, (b) Maximal Cliques, (c) Chordless Cycles, (d) Vertex constraints, and (e) the 5435 number of lifted 5-vertex subset constraints. The total number of constraints was about 25400.

**Pre-Solver:** Then we used Gurobi’s Pre-Solver to remove redundant constraints. This reduced the number of constraints to 9341.

**Note:** For Chordless cycles, we added only 3-cycles for this instance.

\( LP_1 \) returned an optimal solution with a value of 20.66 units. For 47 (out of 128) variables, the value returned was integer (either 0 or 1), and the remaining 81 variables obtained fractional values.

We then added the (integer) values of the 47 variables as constraints to \( LP_1 \), to obtain a new LP (call this \( LP_2 \)). For example, if \( F[6] = 1 \) in the optimal solution to \( LP_1 \), we added the constraint “\( F[6] = 1 \)” to \( LP_2 \). Hence \( LP_2 \) had 47 more constraints than \( LP_1 \).

In discrete optimisation, \( LP_2 \) is referred to as a **warm start**. Rather than beginning the computation from scratch (a **cold** start), we start \( LP_2 \) from a partial solution previously obtained from \( LP_1 \).

Then we applied the \( d2wNbb \) procedure (Sec. 3.1) to \( LP_2 \). This returned a fully integer solution when \( F[i] = F[39] = 1 \) and \( F[j] = F[58] = 1 \). The value of this optimal solution was 18 units, which is the same as that obtained after solving an Integer Program of Problem 1 for the 128-vertex 1ZC instance.

Hence we were able to obtain an optimal integer solution to this instance by solving a polynomial number of Linear Programs, each of polynomial size.

#### 6.3.1 Solving Problem 256-vertex 1ZC

The problem is available at:
https://oeis.org/A265032/a265032_1zc.128.txt.gz

The number of 5-vertex subset constraints added was 14381.
Other constraints added to the Linear Program $LP_1$: (a) Edge inequalities, (b) Maximal Cliques, (c) Chordless Cycles (only 3-cycles), and (d) the 14381 number of lifted 5-vertex subset constraints. The total number of constraints was about 89783.

**Pre-Solver:** Then we used Gurobi’s Pre-Solver to remove redundant constraints. This reduced the number of constraints to 24659.

$LP_1$ returned an optimal solution with a value of 38 units. For 94 (out of 256) variables, the value returned was integer (either 0 or 1). Hence $LP_2$ had 94 more constraints than $LP_1$.

**Warm Start:** Then we applied the $d2wNbb$ procedure to $LP_2$. This returned a fully integer solution when $F[i] = F[8] = 1$ and $F[j] = F[27] = 1$. The value of this optimal solution was 36 units, which is the same as that obtained after solving an Integer Program of Problem 1 for the 256-vertex 1ZC instance.

Warm Start may work sometimes, fail at other times. It cannot be expected to work in every instance. We found the Binary Search approach coupled with $d2wNbb$ to be more reliable.

1DC-256: Independent set: 90345 rows, 257 columns, 467470 nonzeros Pre-solve removed 62504 rows and 1 columns After Pre-solve: 27841 (checked by replacing ":" with something)

### 7 Conclusion

Since we were able to solve several thousand instances exactly (obtained optimal solutions) in theoretical polynomial time, we can conjecture that at least a high proportion of instances of M.I.S. can be solved in polynomial time. More testing is required.

Though the number of constraints is polynomial in $N$, several constraints are generated repeatedly by the C codes for maximal cliques and chordless cycles. A program is needed to remove such duplicate constraints before solving the Linear programs.

#### 7.1 Software Webpage URL

The software programs are available at the following URL:

https://sites.google.com/view/all-optimisation-slides/

We welcome readers to send us their comments and suggestions.

### References

[DCLJ13] Elisângela Silva Dias, Diane Castonguay, Humberto Longo, and Walid Abdala Rfaei Jradi. Efficient enumeration of chordless cycles. *arXiv preprint arXiv:1309.1051*, 2013.

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[GJ79] M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. Freeman (New York), 1979.

[GH62] George Hadley. *Linear Programming*. Addison Wesley, 1962.
A Lifting rules for 5-vertex graphs

There are 34 such graphs listed at https://www.graphclasses.org/smallgraphs.html#nodes5

We sort the vertices in the order of their degrees. The vertex with the lowest degree is Vertex 1 and the one with the highest degree is Vertex 5. Let degree5 = degree of Vertex 5, degree4 = degree of Vertex 4, and so on.

Let numberEdges = number of edges in the graph.

Let coeff[i] = the coefficient of $F[i]$ in the subset constraint.

Let optimalMIS = value of an M.I.S. (maximum independent set).

For the graph types (for example, $P_3 \cup 2K_1$) and graph diagrams, see: https://www.graphclasses.org/smallgraphs.html#nodes5

The type of graph can be uniquely identified by checking the degrees of each of its vertices, the number of edges, and the size of its M.I.S. For instance, $P$ (= 4-Pan = Banner) and its complement $\overline{P}$ have the same number of edges, and the various degrees of their vertices are the same (one vertex of degree 3, three of degree 2, and a leaf vertex). However, the M.I.S. of the former is 3, whereas the M.I.S. of the latter is 2.

For each graph type, the lifting rules are provided below in the form of C code.

1. $P_3 \cup 2K_1$:
   
   if (degree5==2 && degree4==1 && degree3==1 && degree2==0 && numberEdges==2 && optimalMIS==4) coeff[5] = 2;

2. $P_3 \cup 2K_1$ = Complement of $P_3 \cup 2K_1$:
   
   if (degree5==4 && degree4==4 && degree3==3 && degree2==3 && degree1==2 && numberEdges==8 && optimalMIS==2) { coeff[5] = 2; coeff[4] = 2; }

3. $W_4$:
   
   if (degree5==4 && degree4==3 && degree3==3 && degree2==3 && degree1==3 && numberEdges==8 && optimalMIS==2) coeff[5] = 2;

4. Claw $\cup K_1$:
   
   if (degree5==3 && degree4==2 && degree3==2 && degree2==2 && degree1==0 && numberEdges==3 && optimalMIS==4) coeff[5] = 3;

5. Complement of (Claw $\cup K_1$):
   
   if (degree5==4 && degree4==3 && degree3==3 && degree2==3 && degree1==1 && numberEdges==7 && optimalMIS==2) coeff[5] = 2;

6. $P_2 \cup P_3$:
   
   if (degree5==2 && degree4==1 && degree3==1 && degree2==1 && degree1==1 && numberEdges==3 && optimalMIS==3) coeff[5] = 2;
7. $K_3 \cup 2K_1 =$ Complement of $(K_3 \cup 2K_1)$:
   if (degree5==4 && degree4==4 && degree3==2 && degree2==2 && degree1==2 &&
   numberEdges==7 && optimal_MIS==3) { coeff[5] = 3; coeff[4] = 3; }

8. Bull:
   if (degree5==3 && degree4==3 && degree3==2 && degree2==1 && degree1==1 &&
   numberEdges==5 && optimal_MIS==3) { coeff[5] = coeff[4] = 2; }

9. Cricket = $(K_{1,4} \cup \text{an edge})$:
   if (degree5==4 && degree4==2 && degree3==2 && degree2==1 && degree1==1 &&
   numberEdges==5 && optimal_MIS==3) coeff[5] = 3;

10. Co-Cricket:
    if (degree5==3 && degree4==3 && degree3==2 && degree2==2 && degree1==0 &&
    numberEdges==5 && optimal_MIS==3) coeff[5] = coeff[4] = 2;

11. Big diamond, that is, $K_5$ minus (an edge):
    if (degree5==4 && degree4==4 && degree3==4 && degree2==3 && degree1==3 &&
    numberEdges==9 && optimal_MIS==2) { coeff[5] = 2; coeff[4] = 2; coeff[3] = 2; }

12. Star graph $K_{1,4}$:
    if (degree5==4 && degree4==1 && degree3==1 && degree2==1 && degree1==1 &&
    numberEdges==4 && optimal_MIS==4) coeff[5] = 4;

13. Butterfly (hourglass):
    if (degree5==4 && degree4==2 && degree3==2 && degree2==2 && degree1==2 &&
    numberEdges==6 && optimal_MIS==2) coeff[5] = 2;

14. Fork (Chair):
    if (degree5==3 && degree4==2 && degree3==1 && degree2==1 && degree1==1 &&
    numberEdges==4 && optimal_MIS==3) coeff[5] = 2;

15. Dart:
    if (degree5==4 && degree4==3 && degree3==2 && degree2==2 && degree1==1 &&
    numberEdges==6 && optimal_MIS==3) { coeff[5] = 3; coeff[4] = 2; }

16. Co-Dart:
    if (degree5==3 && degree4==2 && degree3==2 && degree2==1 && degree1==0 &&
    numberEdges==4 && optimal_MIS==3) coeff[5] = 2;

17. $K_{2,3}$:
    This was addressed in the last example in Sec. 5. In the C code, this is done in two steps:
    if (degree5==3 && degree4==3 && degree3==2 && degree2==2 && degree1==2 &&
    numberEdges==6 && optimal_MIS==3) { set coeff[5] = 2 and write the constraint. }
    if (degree5==3 && degree4==3 && degree3==2 && degree2==2 && degree1==2 &&
    numberEdges==6 && optimal_MIS==3) { set coeff[4] = 2 and write the constraint. }

18. P = 4-Pan = Banner:
    if (degree5==3 && degree4==2 && degree3==2 && degree2==2 && degree1==1 &&
    numberEdges==5 && optimal_MIS==3) coeff[5] = 2;
For the graphs not listed above, lifting is not possible.

**B  Lifting rules for 4-vertex graphs**

For the graphs not listed below, lifting is not possible.

1. **$K_{1,3}$** (star graph):
   
   if (degree4==3 && degree3==1 && degree2==1 && degree1==1 && numberEdges==3 && optimal\_MIS==3) coeff[4] = 3;

2. **Paw** (3Pan):
   
   if (degree4==3 && degree3==2 && degree2==2 && degree1==1 && numberEdges==4 && optimal\_MIS==2) coeff[4] = 2;

3. **Diamond** = ($K_4$ minus an edge):
   
   if (degree4==3 && degree3==3 && degree2==2 && degree1==2 && numberEdges==5 && optimal\_MIS==2) { coeff[4] = 2; coeff[3] = 2; }

4. **Co-Paw**:
   
   if (degree1==0 && degree2==1 && degree3==1 && degree4==2 && numberEdges==2 && optimal\_MIS==3) coeff[4] = 2;