Asymptotics of the spin foam amplitude on simplicial manifold: Euclidean theory

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Abstract

We study the large-\(j\) asymptotics of the Euclidean EPRL/FK spin foam amplitude on a 4D simplicial complex with arbitrary number of simplices. We show that for a critical configuration \(\{j_f, g_{ve}, n_e\}\) in general, there exists a partition of the simplicial complex into three regions: non-degenerate region, type-A degenerate region and type-B degenerate region. On both the non-degenerate and type-A degenerate regions, the critical configuration implies a non-degenerate Euclidean geometry, while on the type-B degenerate region, the critical configuration implies a vector geometry. Furthermore we can split the non-degenerate and type-A regions into sub-complexes according to the sign of Euclidean-oriented 4-simplex volume. On each sub-complex, the spin foam amplitude at the critical configuration gives a Regge action that contains a sign factor \(\text{sgn}(V_4(v))\) of the oriented 4-simplex volume. Therefore the Regge action reproduced here can be viewed as a discretized Palatini action with the on-shell connection. The asymptotic formula of the spin foam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated with different type of geometries.

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1. Introduction

Loop quantum gravity (LQG) is an attempt to make a background independent, non-perturbative quantization of four-dimensional (4D) general relativity (GR)—for reviews, see [1–6]. It is inspired by the classical formulation of GR as a dynamical theory of connections. Starting from this formulation, the kinematics of LQG is well studied and results in a successful kinematical framework (see the corresponding chapters in the books [3, 5]), which is also unique in a certain sense. However, the framework of the dynamics in LQG is still largely open so far. There are two main approaches to the dynamics of LQG; they are (1) the operator...
formalism of LQG, which follows the spirit of Dirac quantization or reduced phase space quantization of constrained dynamical system, and performs a canonical quantization of GR [7–12]; (2) the covariant formulation of LQG, which is currently understood in terms of the spin foam models [1, 13–22]. The relation between these two approaches is well understood in the case of three-dimensional (3D) quantum gravity [23], while for 4D quantum gravity, the situation is much more complicated and there are some attempts [24–28] for relating these two approaches.

This paper is concerning the framework of spin foam models. The current spin foam models for quantum gravity are mostly inspired by the 4D Plebanski formulation of GR [29–31] (or the Plebanski–Holst formulation by including the Barbero–Immirzi parameter $\gamma$), which is a BF theory constrained by the condition that the $B$ field should be ‘simple’, i.e. there is a tetrad field $e^I$ such that $B = \star(e \wedge e)$. Currently, one of the successful spin foam models is the EPRL/FK model defined in [15–19], whose implementation of simplicity constraint is understood in the sense of [32–34]. The EPRL vertex amplitude is shown to reproduce the classical discrete GR in the large-$j$ asymptotics [35, 36]. Recently, the fermion coupling is included in the framework of the EPRL spin foam model [37, 38], and a $q$-deformed EPRL spin foam model is defined and gives discrete GR with a cosmological constant in the large-$j$ asymptotics [39–42].

The semiclassical behavior of the spin foam models is currently understood in terms of the large-$j$ asymptotics of the spin foam amplitude, i.e. if we consider a spin foam model as a state sum

$$Z(\Delta) = \sum_{j_f} \mu(j_f)Z_{j_f}(\Delta),$$

where $\mu(j_f)$ is a measure. We are investigating the asymptotic behavior of the (partial-) amplitude $Z_{j_f}$ as all the spins $j_f$ are taken to be large uniformly. The area spectrum in LQG is given approximately by $A_f = \gamma j_f l_p^2$, so the semiclassical limit of spin foam models is argued to be achieved by taking $l_p^2 \to 0$ while keeping the area $A_f$ comparable to the physical area, which leads to $j_f \to \infty$ uniformly as $\gamma$ is a fixed Barbero–Immirzi parameter. There is another argument relating the large-$j$ asymptotics of the spin foam amplitude to the semiclassical limit, by imposing the semiclassical boundary state to the vertex amplitude [43]. Mathematically, the asymptotic problem is posed by making a uniform scaling for the spins $j_f \mapsto \lambda j_f$, and studying the asymptotic behavior of the amplitude $Z_{\lambda j_f}(\Delta)$ as $\lambda \to \infty$.

There were various investigations for the large-$j$ asymptotics of the spin foam models. The large spin limit is first used in the analysis of the semiclassical limit of the Ponzano–Regge model [59]. They show that for a single 6$j$ symbol, the Ponzano–Regge model amplitude is proportional to a cosine of the Regge action. The asymptotics of the Barrett–Crane vertex amplitude (10$j$ symbol) was studied in [44], which showed that the degenerate configurations in the Barrett–Crane model were non-oscillatory, but dominant. The large-$j$ asymptotics of the FK model was studied in [45], concerning the non-degenerate Riemannian geometry, in the case of a simplicial manifold without boundary. The large-$j$ asymptotics of the EPRL model was initially studied in [35, 36, 46, 47] in both Euclidean and Lorentzian cases, where the analysis was confined into a single 4-simplex amplitude (EPRL vertex amplitude). It was shown that the asymptotics of the vertex amplitude is mainly a cosine of the Regge action in a 4-simplex if the boundary data admit a non-degenerate 4-simplex geometry, and the asymptotics is non-oscillatory if the boundary data do not admit a non-degenerate 4-simplex geometry.

The work presented here generalizes the large-$j$ asymptotic analysis of the Euclidean EPRL spin foam amplitude to the general situation with a 4D simplicial manifold with or
without boundary, with an arbitrary number of simplices. The analysis for the Lorentzian
EPRL model is presented in [48]. The asymptotic behavior of the spin foam amplitude is
determined by the stationary configurations of the ‘spin foam action’, and is given by a sum of
the amplitudes evaluated at the stationary configurations. Therefore, the large-$j$ asymptotics
is clarified as long as we find all the critical configurations and clarify their geometrical
implications. Here for the Euclidean EPRL spin foam amplitude, a critical configuration in
general is given by the data $\{ j_f, g_{\text{ve}} , n_{\text{ef}} \}$ that solve the equations of motion, where $j_f$
is an SU(2) spin assigned to each triangle, $g_{\text{ve}}$ is an SO(4) group variable and $n_{\text{ef}} \in S^2$. In
this work we show that given a general critical configuration, there exists a partition of the
simplicial complex $\Delta$ which contains three types of regions: non-degenerate region, type-
A (BF) degenerate region and type-B (vector geometry) degenerate region. All of the three
regions are simplicial sub-complexes with boundaries, and may be disconnected regions. The
critical configuration implies different types of geometries in different types of regions:

- The critical configuration restricted in the non-degenerate region is non-degenerate in our
  definition of degeneracy. It implies a non-degenerate discrete Euclidean geometry on the
  simplicial sub-complex.
- The critical configuration restricted in the type-A region is degenerate of type-A in our
  definition of degeneracy. But it still implies a non-degenerate discrete Euclidean geometry
  on the simplicial sub-complex.
- The critical configuration restricted in the type-B region is degenerate of type-B in our
  definition of degeneracy. It implies a vector geometry on the simplicial sub-complex.

With the critical configuration, we further make a subdivision of the non-degenerate and
type-A regions into sub-complexes (with boundary) according to their Euclidean-oriented
4-volume $V_4(v)$ of the 4-simplices, such that $\text{sgn} (V_4(v))$ is a constant sign on each sub-
complex. Then in each sub-complex, the spin foam amplitude at the critical configuration
gives an exponential of the Regge action in Euclidean signature. However we emphasize that
the Regge action reproduced here contains a sign factor $\text{sgn} (V_4(v))$ related to the oriented
4-volume of the 4-simplices, i.e.

$$ S = \text{sgn} (V_4) \sum_{\text{Internal } f} A_f \Theta_f + \text{sgn} (V_4) \sum_{\text{Boundary } f} A_f \Theta_f^B, $$

(2)

where $A_f$ is the area of the triangle $f$ and $\Theta_f$ and $\Theta_f^B$ are deficit angle and dihedral angle,
respectively. Recall that the Regge action without $\text{sgn} (V_4)$ is a discretization of Einstein–
Hilbert action of GR. Therefore, the Regge action reproduced here is actually a discrete
Palatini action with the on-shell connection (compatible with the tetrad).

The asymptotic formula of the spin foam amplitude is given by a sum of the amplitudes
evaluated at all possible stationary configurations, which are the products of the amplitudes
associated with different type of geometries.

Additionally, we also show in section 9 that given a spin foam amplitude $Z_{j_f}(\Delta)$ with the
spin configuration $j_f$, any pair of the non-degenerate critical configurations associated with $j_f$
is related to each other by a local parity transformation. The parity transformation is the one
studied in [35, 36] in the case of a single 4-simplex. A similar result holds for any pair of the
degenerate configuration of type-A associated with $j_f$, since it still relates to non-degenerate
Euclidean geometry.

The rest of this paper is organized as follows: in section 2, we give a brief review of the
EPRL/FK spin foam amplitude and write the transition amplitude in a path integral form.
In section 3, we discuss the semiclassical limit we are considering. A detailed discussion of
classical discrete geometry on a simplicial complex is in section 4. The non-degenerate critical
configuration is discussed in detail in sections 5, 6, 7, 8 and 9. The degenerate type-A and type-B configurations are discussed in section 10. In section 11 we give the asymptotics of the spin foam amplitude as a sum over all possible critical configurations.

2. Spin foam amplitude

In this section we briefly review the definition of the Euclidean EPRL spin foam amplitude. We denote $\Delta$ as a simplicial complex and $\Delta^*$ as its dual. The building blocks in $\Delta$ are 4-simplices $\sigma_e$, tetrahedrons $t_e$ and triangles $f$. The corresponding dual building blocks in $\Delta^*$ are vertices $v$, edges $e$ and faces $f$, respectively. We identify the notations of triangle and face, because there is a one-to-one correspondence between the triangles in $\Delta$ and a dual face in $\Delta^*$. The orientation of $\Delta^*$ is determined by the orientation of $e$ and $f$. We call $\Delta^*$ as oriented as long as the orientations of $e$ and $f$ are chosen.

For defining the spin foam model, we introduce more structures to $\Delta^*$. For each internal edge $e$ with $\partial e = (vv')$, we cut it into two half-edges $(ve)$ and $(ev')$ at the middle point of $e$ (we denote the middle point of $e$ also by $e$). The orientation of the half-edge is always from $e$ to $v$. We associate a group element $g_{ve} \in SO(4)$ to each half-edge $(ve)$, and associate an irreducible representation $\text{Irrep}^{j_f \gamma_f \Sigma_f}$ of $\text{SO}(4)$ to each face. At each edge $e$ we associate an SU(2) coherent intertwiner with the resolution of identity [20]

$$\mathbb{I}_H = \int \prod_{f e} d^2 n_{ef} | | j_f, n_{ef} | | j_f, n_{ef} | |,$$

where $\prod_{f e} || j_f, n_{ef} || \equiv \int dh_e \prod_{f e} h_e n_{ef} | j_f, j_f |, h_e \in SU(2)$. As proved in [49], the above integration is essentially over the constraint surface of closure constraint $\sum_{f \in \sigma_e} j_f n_{ef} = 0$. It means that the labels of coherent intertwiner $j_f$ and $n_{ef}$ have geometrical interpretation in the quantum level as a tetrahedron. With the coherent intertwiner, we impose a closure constraint to the spin foam amplitude and associate $e$ with a geometrical tetrahedron $t_e$. In the following discussion we assume all the tetrahedrons $t_e$ are non-degenerate. $\gamma_f j_f$ is the area of the triangle $f$. In the definition of the spin foam amplitude as a state sum, we only sum over the spins with $\sum_{f \in \sigma_e} \epsilon_f j_f \neq 0$, for all $\epsilon_f = \pm 1$ and for all tetrahedrons $t_e$, so that all the geometrical tetrahedrons are non-degenerate. $n_{ef}$ stands for the unit 3-vector normal to the triangle $f$ of the tetrahedron $t_e$. There is a unit 4-vector $u_e = (1, 0, 0, 0)$ orthogonal to all $n_{ef}$. For each edge $e$ connecting to the boundary and connecting to an internal vertex $v$, we regard it as a half-edge and associate it with $g_{ve} \in SO(4)$. We associate the edge $e$ with boundary intertwiners $| j_f, n_{ef} |$ (or $| j_f, n_{ef} |$) with boundary data $j_f, n_{ef}$. Based on the definitions and notations above, we can write down the spin foam model. The definition of the EPRL spin foam model can be found in many articles e.g. [17, 43, 50]. Usually, the spin foam amplitude is written in terms of a product of vertex amplitudes $A_v$ and face amplitudes $A_f$, followed by the sums/integrations over the variables $(j_f, g_{ve}, n_{ef})$

$$Z = \sum_{j_f} \int \prod_{(ve)} dg_{ve} \int \prod_{(ef)} dn_{ef} \prod_v A_v(g_{ve}, j_f, n_{ef}) \prod_f A_f(g_{ve}, j_f, n_{ef}).$$

In the following we are going to write the spin foam amplitude into a ‘path integration’ form as $\int D\mu \, e^\phi$, i.e. we can express the spin foam amplitude as follows:

$$Z(j_f, g_{ve}, n_{ef}, \Sigma_f) = \sum_{j_f} \prod_f \mu(j_f) \int \prod_{(ve)} dg_{ve} \int \prod_{(ef)} dn_{ef} e^{\sum_f S_f},$$
where \( f_b \) and \( f_i \) mean boundary and internal faces, respectively, and

\[
S_f = \sum_{v \in f} \ln(j_f, n_{ef})Y^1 g_{\nu\varepsilon} g_{\nu\epsilon} Y |j_f, n_{ef} \rangle.
\]

(6)

\( S = \sum_f S_f \) is a ‘spin foam action’ for the path integral. It turns out that the critical point of the spin foam action determines the asymptotic behavior of the spin foam amplitude as \( j \to \infty \).

In the above result we have already absorbed the SU(2) integration \( \int dh_e \) in the coherent intertwiner into the integration of \( g_{\nu\epsilon} \). The similar formulas can be found in \([51, 45, 52, 46]\).

Here we use the notation \( g_{\nu\epsilon} \equiv g^{-1}_\nu g_{\epsilon} \) \( Y \) is a projector \( Y : \text{Irrep}^{[SU(2)]} \to \text{Irrep}^{[SO(4)]} \). Using this projector we can totally decompose \( \text{SO}(4) \) group into its self-dual \( g^+ \) and anti-self-dual \( g^- \) parts where \( g^+, g^- \in \text{SU}(2), \forall g \in \text{SO}(4), g = g^-(g^+)^{-1} \) and insert the simplicity condition \( j^+ = (1 \pm \gamma)j/2 \). The above result works for the case with the Barbero–Immirzi parameter \( \gamma < 1 \). The case with \( \gamma > 1 \) will be included in the discussion in section 5.

Moreover, the spin foam action \( S \) can be written in the following form:

\[
S = \sum_f \sum_{v \in f} \ln(j_f, n_{ef})Y^1 g_{\nu\varepsilon} g_{\nu\epsilon} Y |j_f, n_{ef} \rangle
\]

(7)

\[
= \sum_f \sum_{v \in f} \sum_{\pm} 2f_j^n \ln(n_{ef})g_{\nu\varepsilon} g_{\nu\epsilon} |n_{ef} \rangle,
\]

where \( |n \rangle \) is a coherent state in the fundamental representation. It is normalized \( \langle n|n \rangle = 1 \) and can be represented by a spinor \( |n \rangle = \xi_\alpha = (z_0, z_1) \), where \( z_0, z_1 \in \mathbb{C} \). We can identify the spinor with a unit 3-vector \( \mathbf{n} \), where the component of \( \mathbf{n} \) is defined as follows:

\[
|n \rangle \langle n| = \xi_{\alpha} \bar{\xi}_{\alpha} = \frac{1}{2} (\bar{\delta}_{\alpha\alpha} + n \sigma^\alpha_{\alpha} )
\]

(8)

The spin foam action \( S \) is written as a sum of the ‘face action’ \( S_f \) over all the faces. Here in this paper, we are going to compare the spin foam action at the critical point with the Regge action

\[
S_R = \sum_f A_f \Theta_f,
\]

(9)

where \( A_f \) is the area of the triangle \( t \) dual to the face \( f \), and \( \Theta_f \) is the deficit angle in \( f \).

### 3. Semiclassical considerations

In this section we pose the asymptotic problem toward clarifying the semiclassical limit of the EPRL spin foam amplitude.

It is argued that the semiclassical limit in a spin foam formulation is achieved by taking \( \lambda^2 \to 0 \) while keeping the physical area \( A_f = \gamma j_l^2 \) fixed, which implies that \( j \to \infty \) as the limit to obtain the semiclassical approximation. Mathematically, we rescale all the internal and boundary \( js \) with a uniform scaling parameter \( \lambda \). Then the large-\( j \) limit is taken by sending \( \lambda \to \infty \). Here we emphasize that the semiclassical limit is different from the continuum limit, as discussed by Rovelli in \([53]\). The continuum limit of the theory (even within its semiclassical regime) is out of the scope of this paper. With the large-\( j \) limit taken here, we will obtain in some sense the classical GR truncated on the simplicial complex. But to achieve a continuum formulation is out of the scope of this paper.

As discussed in \([45, 35, 36]\), the asymptotic behavior of the spin foam amplitude is determined by the critical points of the spin foam action \( S \), i.e. the stationary phase points of \( S \) satisfying \( \text{Re}(S) = 0 \). The amplitudes at the configurations which do not satisfy these two conditions are all exponentially suppressed in the large-\( j \) limit.
Here we write a spin foam amplitude as
\[ Z(\Delta) = \sum_{j_f} \mu(j_f) Z_{j_f}(\Delta) \]  
(10)
we are studying the asymptotic behavior of the (partial-)amplitude \( Z_{\lambda, j_f}(\Delta) \) as \( \lambda \to \infty \). We do not study the stationary phase with respect to spin \( j_s \), and expect that the sum over spin \( j_s \) should become the sum over all the classical areas once the large-\( j \)-limit is taken. We will clarify the geometric meaning of the face spins in the large-\( j \) regime, i.e. \( \gamma_{j_f} \) is interpreted as the area \( A_f \) of the triangle \( f \). Thus in our calculation, the equation of motion we are considering is given by
\[ \text{Re}(S) = 0 \]  
(11)
\[ \delta_{g_v} S = 0, \]  
(12)
\[ \delta_{n_{te}} S = 0. \]  
(13)
Under the large-\( j \) limit we would like to compare the large-\( j \) regime of the spin foam amplitude with a path integral formulation of area Regge calculus
\[ Z(j_{fe}) \sim \sum_{j_{fi} \text{ large}} \mu(j_f) e^{iS_{\text{critical}}} \sim \int_{j_{fi}} D j_{fe} e^{iS_{\text{Regge}}}. \]  
(14)
Note that the gluing between two 4-simplices is already imposed in the spin foam amplitude because there is only a single set of variables \((j_f, n_{te})\) for each tetrahedron \( t_e \). We will come back to this point later.

4. Discrete geometry on simplicial complex

In this section we discuss the discrete geometry over a non-degenerate simplicial complex \( \Delta \). Here we review the connection formalism of the simplicial geometry (the detailed introduction of the formalism can also be found in e.g. \([54, 60–62]\)). The aim of this section is to give a collection of definitions and variables to describe the discrete Euclidean geometry on \( \Delta \). These geometrical variables will be reconstructed from the critical configurations of the spin foam amplitude in the next sections. In the following we denote the 4D Euclidean vector space by \( \mathbb{E} \).

4.1. The orientation structure of simplicial complex \( \Delta \)

In this subsection, we discuss the orientation structure of simplicial complex \( \Delta \). The simplicial complex \( \Delta \) is a triangulation of the spacetime manifold \( M \). Here we would like to review the definition of the orientation of \( \Delta \), which is necessary to define e.g. oriented 4-volume for each 4-simplex. For convenience, we call the edge of triangle \( \Delta \) ‘Segment’, denoted \( l \), and the vertex of triangle called ‘Point’, denoted \( p \).

4.1.1. The orientation of \( \Delta \). Before defining the orientation of the simplicial complex \( \Delta \), we have to define the orientation of each 4-simplex \( \sigma_e \) dual to \( v \). The orientation of a 4-simplex \( \sigma_e \) can be represented by its ordered five points, i.e. a tuple \([p_1, \ldots, p_5]\). Two orientations are opposite if the two tuples can be related by odd permutation, e.g. \([p_1, p_2, \ldots, p_5] = -[p_2, p_1, \ldots, p_5]\). Because in a 4-simplex \( \sigma_e \), there are five points \( p \) and five tetrahedrons \( t_e \). We can make a duality between \( p \) and \( e \) if \( p \cap t_e = \emptyset \), as shown in figure 1. Thus the orientation of \( \sigma_e \) can also be denoted as \([e_1, \ldots, e_5]\). Since vertex \( v \in \Delta^* \) is dual to \( \sigma_e \), we can also say \([e_1, \ldots, e_5]\) is the orientation of vertex \( v \).
From the orientation of 4-simplex $\sigma_v$ we can induce the orientations of tetrahedrons $t_e$, triangles $f$ and segments $l$ in $\sigma_v$. For example, we define the orientation of $t_{e_1}$ as

$$[e_2, e_3, e_4, e_5] \leftarrow [e_2, e_1, e_3, e_4, e_5],$$

where $\leftarrow$ denotes the induction from the 4-simplex orientation to the orientation of $t_{e_1}$, by deleting the second entry of $[e_2, e_1, e_3, e_4, e_5]$. Similarly, the orientation of $f = t_{e_1} \cap t_{e_2}$ with respect to $t_{e_1}$ can be defined as

$$[e_3, e_4, e_5] \leftarrow [e_3, e_1, e_2, e_4, e_5],$$

where the induction is given by deleting the second and third entries of $[e_3, e_1, e_2, e_4, e_5]$. The orientation of $l$ from point $p_4$ to point $p_5$ with respect to $t_{e_1}$ can be defined as

$$[e_4, e_5] \leftarrow [e_3, e_1, e_2, e_4, e_5].$$

For convenience, we will use a Levi-Civita symbol $\epsilon_{e_1 e_2 e_3 e_4 e_5}^{(v)}$ to denote $[e_1, e_2, e_3, e_4, e_5]$ of $\sigma_v$ in the following discussion. From equation (16), we can find in $\sigma_v$, $\forall f = t_{e_1} \cap t_{e_2}$, the orientations of $f$ with respect to $t_{e_1}$ and $t_{e_2}$ are opposite.

We say that two neighboring 4-simplices $\sigma_v, \sigma_{v'}$ in figure 2 are orientation consistent if the orientations of the tetrahedron $t_{e_1}$ shared by them are opposite with respect to $\sigma_v$ and $\sigma_{v'}$, i.e. $[p_1, p_2, \ldots, p_5] = -[p'_1, p'_2, \ldots, p'_5]$ or $[e_1, e_2, \ldots, e_5] = -[e'_1, e'_2, \ldots, e'_5]$. It means

$$\epsilon_{e_1 e_2 e_3 e_4 e_5}^{(v)} = -\epsilon_{e'_1 e'_2 e'_3 e'_4 e'_5}^{(v')},$$

(18)

From this we can see that the orientations of triangles $f \in t_{e_1}$ and segments $l \in f$ are opposite with respect to $\sigma_v$ and $\sigma_{v'}$ as

$$\epsilon_{e_i e_j e_k}^{(v)} = -\epsilon_{e'_i e'_j e'_k}^{(v')}, \quad \forall i, j, k \neq 1$$

(19)

$$\epsilon_{e_i e_j}^{(v)} = -\epsilon_{e'_i e'_j}^{(v')}, \quad \forall i, j \neq 1.$$  

(20)

We call a given simplicial complex $\Delta$ (or $\Delta^*$) is global oriented if any two neighboring 4-simplices (or vertices) in $\Delta$ are orientation consistent. In the following discussion in this section, we assume the simplicial complex $\Delta$ is global oriented.
4.1.2. Spacetime orientation. We assume the simplicial complex $\Delta$ is a discretization of a manifold $M$ with a global orientation. Therefore we can define an oriented orthonormal frame bundle $e_I^\mu$, where all the orthonormal frames are right handed with respect to the global orientation, or $\text{sgn } \det(e)$ is a constant sign on the manifold $M$. The oriented orthonormal frame bundle has the structure of a principle fiber bundle with the structure group SO(4) in Euclidean signature (or SO(1,3) in Lorentzian signature).

Now we give a discrete analogue of a global spacetime orientation on a simplicial manifold. Given a simplicial complex $\Delta$, we assign a reference frame $\{e(v)\}$ in each 4-simplex $\sigma_v$. We assume for any two frames $\{e(v)\}$ and $\{e(v')\}$ at two different 4-simplices,

$$\text{sgn } \det e(v) = \text{sgn } \det e(v').$$

Then the two reference frames between two neighboring simplices are related by an SO(4) transformation. Then the frames located in different 4-simplices constitute a discrete analogue of oriented orthonormal frame bundle on the simplicial manifold. The SO(4) transformation relating the two frames in different 4-simplices is the discrete spin connection.

Moreover, in the next subsections, we will show that, if there exists a discrete analogue of the oriented orthonormal frame bundle on $\Delta$, i.e. there are frames assigned in the 4-simplices satisfying $\text{sgn } \det e(v) = \text{sgn } \det e(v')$ and being related to each other by SO(4) transformations, then for the oriented volume of 4-simplex, the sign $\text{sgn } V_4(v)$ is a constant, with a consistent orientation on the simplicial complex $\Delta$, i.e.

$$\text{sgn } V_4(v) = \text{sgn } V_4(v'), \quad \forall v, v'.$$

See the next subsection for the definition of $V_4(v)$.

4.2. Discrete geometry in a 4-simplex

Given a simplicial complex $\Delta$, we can define a collection of geometric variables to describe the discrete geometry on a simplicial manifold.

**Definition 4.1 (Segment vector $E_l(v)$).** A segment vector $E_l^I(v)$ is a 4-vector in the tangent space of $v \in \Delta^*$, associated with the oriented segment $l \in \Delta$. The modulus of $E_l^I(v)$ is the length of $l$. $E_l(v)$ should satisfy the following properties:

- When the orientation of $l$ is inversed,
  $$E_{-l}(v) = -E_l(v).$$
\* \* When we sum over the five \( U_e \) identities:

\[
\sigma_v \rightarrow v \in l
\]

(Non-degeneracy) We call \( E^l_1(v) \) non-degenerate if \( \forall p, l \in \sigma, l \cap p \neq \emptyset \), s.t. \( E^l_1(v) \) spans a 4D vector space.

Because a segment \( I \) can be denoted by its end points \( I = [p_1p'_1] \), it can also be denoted by the edges dual to its end points, i.e. \( l = [p_1p'_1] = [ee'] \), where \( [ee'] = [-e'e] \). Thus we can also write \( E^l_1(v) \) as \( E^l_{ee}(v) \). The direction of \( E^l_{ee}(v) \) is from \( e \) to \( e' \). Then the inverse and close properties turn to

\[
E^l_{ee}(v) = -E^l_{e'e}(v), \quad E^l_{ee}(v) + E^l_{e'e}(v) + E^l_{e'e}(v) = 0.
\] (27)

In the following we use both conventions, according to the convenience of the context.

We choose a consistent orientation of all the 4-simplices of \( \Delta \). Then for each 4-simplex \( \sigma_v \in \Delta \), we can define the oriented 4-volume \( V_4(v) \) of \( \sigma_v \)

\[
V_4(v) \equiv \det \left( E^l_{ee_1}(v), E^l_{ee_2}(v), E^l_{ee_3}(v), E^l_{ee_4}(v) \right)
\]

\[
= \frac{1}{4!} \sum_{ijklm} (\epsilon^{ijklm} \epsilon_{ijklm} E_{ee_1} E_{ee_2} E_{ee_3} E_{ee_4}) (v),
\] (28)

which is independent of the index \( i \) by equation (24). Here \( \epsilon^{ijklm} \) and \( \epsilon_{ijklm} \) represent the Lévi-Civita symbol, with \( \epsilon^{ijklm} = \epsilon_{ijklm} \) and \( \epsilon_{ijklm} = \epsilon_{ijklm} \). We define five 4-vectors \( U^e(v) \) orthogonal to \( t_e \) by

\[
U^e_{ee_1}(v) = \frac{1}{3!V_4(v)} \sum_{ijklm} (\epsilon^{ijklm} \epsilon_{ijklm} E_{ee_1} E_{ee_2} E_{ee_3} E_{ee_4}) (v).
\] (29)

We call them frame vectors. Using the above definition and equations (23) (24), we obtain

\[
U^e_{ee_1}(v)E^l_{ee_1}(v) = \delta^l_j - \delta^l_k.
\] (30)

When we sum over the five \( U^e(v) \) in equation (30), we obtain

\[
\sum_i U^e_{ee_i}(v)E^l_{ee_i}(v) = \sum_i \delta^l_j - \sum_i \delta^l_k = 0, \quad \forall \; e_j, e_k,
\] (31)

which implies the closure of \( U^e(v) \) for each 4-simplex \( \sigma_v \)

\[
\sum_i U^e_{ee_i}(v) = 0.
\] (32)

Equations (30) and (32) show that in \( v \), the five vectors are all outpointing to the tetrahedrons from \( \sigma_v \) up to a total reflection \( U^e \rightarrow -U^e \). Also from equation (30) we obtain the following identities:

\[
V_4^{-1}(v) = \det \left( U^{e_1}(v), U^{e_2}(v), U^{e_3}(v), U^{e_4}(v) \right)
\] (33)
The bivector $\Omega_{1v}$ is obtained from $\Omega_{1e} = 3! \sum_{l,m,n} e_{ijklmn} e^{ijkl} U^e_j(v) U^K_k(v) U^e_l(v)$ (34)

$$V_4(v)(U^e_i(v) \wedge U^e_j(v))_{IJ} = \frac{1}{2} \sum_{m,n} e_{ijklmn} e^{ijkl} E^K_{e_m e_n}(v) E^L_{e_m e_n}(v).$$ (35)

Here the last equation gives us a way to construct area bivectors explicitly. In a 4-simplex $\sigma_v$, a triangle can be identified by the two tetrahedrons that share it, or by the three points of the triangle. We define two area bivectors of the triangle shared by $t_{e_1}$ and $t_{e_2}$, which are denoted by $A_{e_1 e_2}$ and $A_{e_2 e_1}$. We define them in the following way:

$$A_{e_1 e_2}(v) = \frac{1}{4} \sum_{m,n} e_{12mn} (E_{e_m e_n}(v) \wedge E_{e_m e_n}(v))$$ (36)

$$A_{e_2 e_1}(v) = \frac{1}{2} (E_{e_1 e_2}(v) \wedge E_{e_1 e_2}(v)).$$ (37)

The bivector $A_{e_1 e_2}(v)$ depends on the orientation of the 4-simplex, while the bivector $A_{e_1 e_2}(v)$ is defined with an orientation of the triangle $f = [p_3, p_4, p_5] \equiv [e_3, e_4, e_5]$, which may or may not be orientation induced from $\sigma_v$. We call $A_{e_1 e_2}(v)$ oriented bivectors and $A_{e_1 e_2}(v)$ non-oriented bivectors. Their relation is $A_{e_1 e_2}(v) = \epsilon_{e_1 e_2 e_3 e_4} A_{e_1 e_2}(v)$ (no sum in $e_l$, $e_m$).

4.3. Gluing condition of many 4-simplices

Given a tetrahedron $t_{e_1} \in \Delta$ which is shared by $\sigma_v$, $\sigma_v'$, as shown in figure 2, we consider the relation between $E^I_1(v)$ and $E^I_2(v')$ for $l \in t_{e_1}$. We define two unit normal vectors $\hat{U}^I_l(v) \equiv U^I_l(v)/|U^I_l(v)|$ and $\hat{U}^I_{l'}(v') \equiv U^I_{l'}(v')/|U^I_{l'}(v')|$, where $U^I_{l'} = \delta^{IJ} U^J_{l'}$. From equation (29), we can find $\forall \ l \in t_{e_1}$

$$\hat{U}^I_{l'}(v)E_{lI}(v) = \hat{U}^I_{l'}(v')E_{lI}(v') = 0.$$ (38)

Thus for $l_1, l_2, l_3 \in t_{e_1}$ but not in the same face, the vectors $E_{l_1}, E_{l_2}, E_{l_3}, \hat{U}_{e_1}$ define two reference frames in both $\sigma_v$ and $\sigma_v'$. To satisfy equation (21), we should have

$$\text{sgn} \ det \left( E_{l_1}(v), E_{l_2}(v), E_{l_3}(v), \hat{U}_{e_1}(v) \right) = \text{sgn} \ det \left( E_{l_1}(v'), E_{l_2}(v'), E_{l_3}(v'), \hat{U}_{e_1}(v') \right),$$ (39)

where $E_{l_1}(v), E_{l_2}(v), E_{l_3}(v)$ and $E_{l_1}(v'), E_{l_2}(v'), E_{l_3}(v')$ span a 3D subspace at $v$ and $v'$, respectively. Because of equations (25), (38) and (39), there exists a unique SO(4) [45, 48] matrix $\Omega_{e_{v'}}$ such that

$$(\Omega_{e_{v'}})E^I_{l_1}(v) = -E^I_{l_1}(v'), \quad (\Omega_{e_{v'}})\hat{U}^I_{l_1}(v) = -\hat{U}^I_{l_1}(v').$$ (40)

The minus in the first equation is because the orientations of any segments $l \in t_{e_1}$ are opposite with respect to two neighboring 4-simplices $\sigma_v$ and $\sigma_v'$. If $l = p_1 p_2$, $E_{l}(v) = E_{p_1 p_2}(v)$, $E_{l}(v') = E_{p_1 p_2}(v')$, we can also rewrite it as

$$(\Omega_{e_{v'}})E^I_{p_1 p_2}(v) = E^I_{p_1 p_2}(v').$$ (41)

The second equation is because

$$\text{det} \left( E_{l_1}(v), E_{l_2}(v), E_{l_3}(v), \hat{U}_{e_1}(v) \right) = -\text{det} \left( E_{l_1}(v'), E_{l_2}(v'), E_{l_3}(v'), \Omega_{ev'}\hat{U}_{e_1}(v') \right) = \text{det} \left( E_{l_1}(v'), E_{l_2}(v'), E_{l_3}(v'), \hat{U}_{e_1}(v') \right).$$ (42)

The first equality is because $\text{det} \Omega_{ev'} = 1$. The second equality implies $\hat{U}_{e_1}(v) = -\Omega_{ev'} \hat{U}_{e_1}(v')$. We call $\Omega_{ev'}$ the spin connection if it satisfies equations (39) and (40).

For an explanation of equation (40), we see a 2D example shown in figure 3 where two triangles share a segment $l = p_2 p_3$. Because of the orientation consistency, the orientation of
Without losing generality, we assume \( E_i(t) = E_{p_ip_i}(t) \) and \( E_i(t') = E_{p_ip_i}(t') \). The first equation in equation (40) implies \( E_i(t) = -\Omega_{tv}E_i(t') \) or \( E_{p_ip_i}(t) = \Omega_{tv}E_{p_ip_i}(t') \). We can also see in figure 3 that the outgoing normals \( U(t) \) and \( U(t') \) should satisfy \( \hat{U}(t) = -\Omega_{tv}\hat{U}(t') \) such that the bases \{\( U(t), E_i(t) \}\) and \{\( U(t'), E_i(t') \)\} are in the same orientation.

Next we will prove the following proposition to show that equation (22) is satisfied.

**Proposition 4.1.** Given two neighboring 4-simplices \( \sigma_v \) and \( \sigma_{v'} \), as shown in figure 2, if the orientation consistency (equation (18)) and the parallel transportation (equation (40)) are satisfied, and \( \Omega_{tv'} \in \text{SO}(4) \), then \( \text{sgn} V_4(v) = \text{sgn} V_4(v') \).

**Proof.** Without losing generality, we assume \( \epsilon^{e_1e_2e_3e_4e_5}(v) = 1 \). For convenience, we introduce the shorthand notations: \( E_{ij} = E_{e_ie_j}(v), E'_{ij} = E_{e'_ie'_j}(v'), U^1 = U^e_1(v), U'^1 = U^e_1(v') \). The 4-volumes of \( \sigma_v \) and \( \sigma_{v'} \) are given by equation (28)

\[
V_4(v) = -\det(E_{12}, E_{32}, E_{42}, E_{52})
\]

\[
V_4(v') = \det(E'_{12}, E'_{32}, E'_{42}, E'_{52})
\]

where the minus sign for \( V_4(v) \) is because the orientation of \( \sigma_v \) is \([e_1, \ldots, e_5]\) while the orientation of \( \sigma_{v'} \) is \([-e'_1, \ldots, e'_5]\). By using equation (30), we have

\[
\frac{1}{3!} U'_p \epsilon^{ijkl} \det(E_{12}, E_{32}, E_{42}, E_{52}) = E_{12}' \epsilon_{12345} E_{32}' E_{42}' E_{52}'.
\]

Multiply with \( \hat{U}'_i \epsilon^{ijkl} \) and use \( \epsilon^{ijkl} \epsilon_{ijkl} = 3! \delta'_i^j \), then we have

\[
\hat{U}'_i \hat{U}^j_1 \epsilon^{ijkl} \det(E_{12}, E_{32}, E_{42}, E_{52}) = \det(\hat{U}'_1, E_{32}', E_{42}', E_{52}').
\]

Using this result to both \( \sigma_v \) and \( \sigma_{v'} \), we can easily obtain

\[
\text{sgn} V_4(v) = -\text{sgn} \det(\hat{U}'_1, E_{32}', E_{42}', E_{52}')
\]

\[
\text{sgn} V_4(v') = \text{sgn} \det(\hat{U}'_1, E_{32}', E_{42}', E_{52}').
\]

By equation (40) and \( \det \Omega_{tv'} = 1 \), we obtain

\[
\text{sgn} V_4(v) = \text{sgn} V_4(v').
\]

The orientation consistency means if we want to glue two 4-simplices together, the orientation bivectors in \( t_v \) should be opposite with respect to \( t \) and \( t' \). As shown in figure 3, \( E_i(t) = E_{p_ip_i}(t) \) and \( E_i(t') = E_{p_ip_i}(t') \). The first equation in equation (40) implies \( E_i(t) = -\Omega_{tv}E_i(t') \) or \( E_{p_ip_i}(t) = \Omega_{tv}E_{p_ip_i}(t') \). We can also see in figure 3 that the outgoing normals \( U(t) \) and \( U(t') \) should satisfy \( \hat{U}(t) = -\Omega_{tv}\hat{U}(t') \) such that the bases \{\( U(t), E_i(t) \}\) and \{\( U(t'), E_i(t') \)\} are in the same orientation.

Next we will prove the following proposition to show that equation (22) is satisfied.

**Proposition 4.1.** Given two neighboring 4-simplices \( \sigma_v \) and \( \sigma_{v'} \), as shown in figure 2, if the orientation consistency (equation (18)) and the parallel transportation (equation (40)) are satisfied, and \( \Omega_{tv'} \in \text{SO}(4) \), then \( \text{sgn} V_4(v) = \text{sgn} V_4(v') \).

**Proof.** Without losing generality, we assume \( \epsilon^{e_1e_2e_3e_4e_5}(v) = 1 \). For convenience, we introduce the shorthand notations: \( E_{ij} = E_{e_ie_j}(v), E'_{ij} = E_{e'_ie'_j}(v'), U^1 = U^e_1(v), U'^1 = U^e_1(v') \). The 4-volumes of \( \sigma_v \) and \( \sigma_{v'} \) are given by equation (28)

\[
V_4(v) = -\det(E_{12}, E_{32}, E_{42}, E_{52})
\]

\[
V_4(v') = \det(E'_{12}, E'_{32}, E'_{42}, E'_{52}),
\]

where the minus sign for \( V_4(v) \) is because the orientation of \( \sigma_v \) is \([e_1, \ldots, e_5]\) while the orientation of \( \sigma_{v'} \) is \([-e'_1, \ldots, e'_5]\). By using equation (30), we have

\[
\frac{1}{3!} U'_p \epsilon^{ijkl} \det(E_{12}, E_{32}, E_{42}, E_{52}) = E_{12}' \epsilon_{12345} E_{32}' E_{42}' E_{52}'.
\]

Multiply with \( \hat{U}'_i \epsilon^{ijkl} \) and use \( \epsilon^{ijkl} \epsilon_{ijkl} = 3! \delta'_i^j \), then we have

\[
\hat{U}'_i \hat{U}^j_1 \epsilon^{ijkl} \det(E_{12}, E_{32}, E_{42}, E_{52}) = \det(\hat{U}'_1, E_{32}', E_{42}', E_{52}').
\]

Using this result to both \( \sigma_v \) and \( \sigma_{v'} \), we can easily obtain

\[
\text{sgn} V_4(v) = -\text{sgn} \det(\hat{U}'_1, E_{32}', E_{42}', E_{52}')
\]

\[
\text{sgn} V_4(v') = \text{sgn} \det(\hat{U}'_1, E_{32}', E_{42}', E_{52}').
\]

By equation (40) and \( \det \Omega_{tv'} = 1 \), we obtain

\[
\text{sgn} V_4(v) = \text{sgn} V_4(v').
\]
4.4. Discrete geometry of boundary

Now we consider the discrete geometry of the boundary of a given simplicial complex \( \Delta \). We denote the boundary of \( \Delta \) as \( \partial \Delta \). On \( \partial \Delta \), each boundary triangle is exactly shared by two boundary tetrahedrons, as shown in figure 4.

At each boundary node \( e \) (the center of a boundary tetrahedron \( t_e \)), we can also construct the segment vectors as before. For each segment \( l \) of a boundary tetrahedron \( t_e \), we associate it with an oriented vector \( E_l(e) \) at \( e \). The segment vectors \( E_l(e) \) should satisfy the following properties:

Inverse: When the orientation of \( l \) is inverted,
\[
E_{l'}(e) = -E_l(e). \tag{43}
\]

Close: \( \forall f \in \partial \Delta \), if its boundary \( l_1, l_2, l_3 \) orientations are consistent, then
\[
E_{l_1}(e) + E_{l_2}(e) + E_{l_3}(e) = 0. \tag{44}
\]

Gluing: If edge \( e \) touches vertex \( v \), \( \forall f \in \partial t_e, l, l' \in \partial f \)
\[
\delta_{l,l'} E_l(e) E_{l'}(e) = \delta_{l,l'} E_l(v) E_{l'}(v). \tag{45}
\]

Gauge: \( \forall l \in t_e \), the segment vector \( E_l(e) \) is orthogonal to the unit vector \( u = (1, 0, 0, 0) \)
\[
E_l(e) \cdot u = 0. \tag{46}
\]

As before we can also define the induced boundary metric by the boundary segment vectors
\[
g_{l,l'}(e) = \delta_{l,l'} E_l(e) E_{l'}(e). \tag{47}
\]

For each boundary tetrahedron \( t_e \), it lies in the 3D subspace which is orthogonal \( u \). An oriented tetrahedron \( t_e \) can be represented by its ordered four points \( [p_1, p_2, p_3, p_4] \). The orientation of \( t_e \) should be identified with the induced orientation from the 4-simplex \( \sigma_v \) containing \( t_e \), i.e. \( [p_1, p_2, p_3, p_4] \leftarrow [p_1, p, p_2, p_3, p_4], p \in \sigma_v, p \not\in t_e \). We assume all the tetrahedrons are non-degenerate. Then we can define the oriented 3-volume of \( t_e \)
\[
V_3(e) = \frac{1}{3!} \sum_{l,k,l} \epsilon_{lkl} \epsilon_{ijkl} E_{ijkl}(e) E_{p_1p_2p_3p_4}(e), \tag{48}
\]
where \( \epsilon_{ijkl} \equiv \epsilon_{ijkl}(u) \), and \( \epsilon_{ijkl}(v) = \epsilon_{ijkl}(v) \).

Then we can define the 3-vector \( n_{p_j}(e) \) which is normal to the face \( f \in t_e \) and \( f \cap p_j = \emptyset \)
\[
n^p_{l}(e) \equiv \frac{1}{2V_3(e)} \sum_{l,k,l} \epsilon_{ijkl}(ev) \epsilon_{ijkl}(p_j) E_{p_1p_2p_3p_4}(e), \tag{49}
\]
which implies
\[ n_p^b(e)E^I_{p,b}(e) = \delta^I_j - \delta^I_k. \]  
(50)

It is not hard to show the following relations:
\[ \sum_{i=1}^4 n_p(e) = 0 \]  
(51)
\[ \nu_3(e)^{-1} = \frac{1}{3!} \sum_{j,k,l} \epsilon_{jkl} \epsilon_{ijkl} n_p^b(e) n_p^a(e) n_p^c(e) \]  
(52)
\[ E^I_{p,b}(e) = \frac{\nu_3(e)}{2} \sum_{j,k,l} \epsilon_{jkl} \epsilon_{ijkl} n_p^b(e) n_p^a(e) n_p^c(e). \]  
(53)

For the boundary edge \( e \) connecting a vertex \( v \), and for any triple of segments \( l_1, l_2, l_3 \in n_e \), we have the segment vectors at vertex \( v \): \( E_i(v) \) and at \( e: E_i(e) \), where \( i = 1, 2, 3 \). If we consider the unit vector \( \hat{U}_e(v) \) defined before, which is orthogonal to \( E_i(v) \) such that
\[ \text{sgn det} (E_i(v), E_j(v), E_k(v), \hat{U}_e(v)) = \text{sgn det} (E_i(e), E_j(e), E_k(e), \epsilon \hat{U}_e(v)), \]  
(54)
where \( \epsilon = \pm 1 \), then there exists a unique \( \text{SO}(4) \) matrix \( \Omega_{ve} \) such that
\[ \Omega_{ve} \hat{U}_e(v) = \nu_3(e), \quad (\Omega_{ve})^I_J \hat{U}_e^J = \epsilon \hat{U}_e(v). \]  
(55)

Then we find that the 3-volume defined in equation (48) is consistent with the one induced from \( \sigma_e \) up to a sign, i.e. \( \nu V_3(v) = V_3(e) \), while the 3-volume of tetrahedron \( t_e \) induced from \( \sigma_e \) is defined by
\[ V_p^b(v) = \frac{1}{3!} \sum_{j,k,l} \epsilon_{jkl} \epsilon_{ijkl} \hat{U}_e^I \hat{U}_e^J E^I_j E^J_k E^K_l E^K_j (v). \]  
(56)

We can also find an explicit expression between \( V_4(v) \) and \( V_3(v) \) of \( t_e \).
\[ V_3(v) = V_4(v) \hat{U}_e^I(v) \hat{U}_e^J(v), \]  
(57)
where \( \hat{U}_e^I(v) = U^I(v)/|U^I(v)| \). Because of this \( U^I(v) \) can also be written as
\[ U^I(v) = \frac{V_3(v)}{V_4(v)} \hat{U}_e^I(v). \]  
(58)

Because of \( \text{det} \Omega_{ve} = 1 \) and \( \epsilon \text{sgn} V_3(v) = \text{sgn} V_3(e) \), we have \( \text{sgn} (\hat{U}_e^I(v) U^I(v)) = \epsilon \text{sgn} (U^I(e)) \). Furthermore, because of \( \hat{U}_e^I(v) \hat{U}_e^J(v) = 1 \), we obtain \( \text{sgn} V_3(v) = \text{sgn} V_4(v) \).

The above construction for a boundary tetrahedron can be also extended to any internal tetrahedron \( t_e \). As before, we can construct the segment vectors \( E_i(e) \) for any segment \( l \in n_e \). However this time for each edge we have two segment vectors \( E_i(e) \) and \( E_i(e) \) associated with \( \sigma_e \) and \( \sigma_e' \), respectively. Because of the orientation consistency, \( E_i(e) = -E_i(e) \) where the minus sign comes from the opposite orientations of \( l \) induced from different 4-simplices. Moreover because of equation (55), we also obtain
\[ \hat{U}_e(e) = -\hat{U}_e(e), \]  
(59)
where \( \hat{U}_e(e) \equiv \Omega_{e,v} \hat{U}_e(v) \) and \( \hat{U}_e'(e) \equiv \Omega_{e,e'} \hat{U}_e'(v') \). This relation implies that given two neighboring 4-simplices share a tetrahedron \( t_e \), if \( \hat{U}_e(e) \) is future pointing, then \( \hat{U}_e'(e) \) is past pointing, or vice versa. From the relation between \( E_i(e) \) and \( E_i(e) \), \( \hat{U}_e(e) \) and \( \hat{U}_e'(e) \), using equation (56) and \( \text{det} \Omega_{ve} = 1 \), we have
\[ V_3(e) = V_3'(e). \]  
(60)
Come back to the boundary tetrahedrons, because of equations (25) and (45), for two boundary tetrahedrons $t_0$ and $t_1$ that share the triangle $f$, we can get that the induced metric on the triangle $f$ from $t_0$ and $t_1$ is the same

$$\delta_{ij} E_i^0 (e_0) E_j^0 (e_0) = \delta_{ij} E_i^1 (e_1) E_j^1 (e_1)$$

(61)

for any pair of the segments $l_1, l_2$ of the triangle $f$.

For gluing the boundary tetrahedrons $t_0$ and $t_1$, the orientation of the tetrahedrons should be consistent with each other. If the induced orientation of the face $f_1 = t_0 \cap t_1$ is opposite with respect to $t_0$ and $t_1$, i.e. $\epsilon_{pp_1pp_2} (e_0) = - \epsilon_{pp_1pp_2} (e_1)$, the orientations of the two tetrahedrons are consistent.

For defining the dihedral angle of face $f$, we assign a reference frame at the boundary face $f$. In the $f$ frame, we construct the segment vectors $E_i (f)$ for all $l \in f$. Because of the orientation consistency, for each segment $l \in f$, we can define two segment vectors $E_i (f)$ and $E_i (f)$ with respect to two boundary tetrahedrons sharing face $f$. They are opposite since the opposite induced orientations on $f$ from two different tetrahedrons, i.e.

$$E_i (f) = - E_i (f).$$

(62)

Except for satisfying inverse, close, gluing, gauge properties as $E_i (e)$, $E_i (f)$ also satisfies the face gauge, which means there exists a vector $z = (0, 0, 0, 1)$ such that

$$E_i (f) \cdot z = 0, \quad \forall \ l \in f.$$  

(63)

If we consider a normal vector $n_{ef}$ orthogonal to triangle $f$ and $u$ such that

$$\text{sgn } \det (E_i (e), E_i (e), n_{ef}, u) = \text{sgn } \det (E_i (f), E_i (f), z, u),$$

there must be a unique SO(3) matrix $\omega_{ef}$ such that

$$\omega_{ef} \triangleright E_i (f) = E_i (e), \quad \omega_{ef} \triangleright z = n_{ef}.$$  

(65)

Then for the loop holonomy $\Omega_f (f) = \omega_{f0} \omega_{orf} \omega_{ef}$, we always have

$$\Omega_f (f) \triangleright E_{pp_1} (f) = E_{pp_1} (f), \quad \forall \ l = p_1 p_2 \in f,$$

(66)

where $E_{pp_1} (f) = E_i (f) = - E_i (f)$. Then we know that the non-oriented bivector (the triangle $f = (p_3, p_4, p_5)$)

$$A_{\epsilon\epsilon\epsilon\epsilon} (f) = \frac{1}{2} (E_{\epsilon\epsilon\epsilon} (f) \land E_{\epsilon\epsilon\epsilon} (f))$$

(67)

is invariant under the operation of loop spin connection $\Omega_f$, i.e. $A_{\epsilon\epsilon\epsilon\epsilon} (f) = \Omega_f (f) \triangleright A_{\epsilon\epsilon\epsilon\epsilon} (f)$.

### 4.5. The Regge action from a connection formalism

The construction in the previous subsection is essentially a connection formalism for discrete classical geometry both in the bulk and on the boundary. Here we show how to relate the Regge action from this formalism.

In order to writing down the Regge action, we should define the deficit angle $\Theta_f$ for internal faces and dihedral angle $\Theta_f$ for boundary faces. Let us first consider the internal faces $f$. The first step is to write down the explicit expression for the loop spin connections $\Omega_f$ along the boundary of internal faces $f$. For an internal face $f$, the loop spin connection keeps the three segment vectors $E_{pp_1} (v)$ unchanged by equations (36) and (40), where $p_i, p_j$ are the vertices of the triangle $f$,

$$\Omega_f (v) E_{pp_1} (v) = E_{pp_1} (v).$$

(68)
where $\Omega_f(v) \equiv \Omega_{e_1} \ldots \Omega_{e_e}$. The loop spin connection keeps the vectors lying on the plane determined by $E_{p,p}(v)$. It implies that the loop spin connection $\Omega_f(v) \in \text{SO}(4)$ is either a pure boost with a parameter $\theta_f$ or a pure boost connecting $-1 \in \text{SO}(4)$ combined with a $\pi$-rotation on the plane determined by $E_{p,p}(v)$, explicitly,

$$\Omega_f(v) = e^{\omega_{e_e} \pi} \exp(\theta_f \cdot \hat{A}_f(v) + n_f \pi \hat{A}_f(v)), \quad (69)$$

where $A_f(v) \equiv A_{e_1 e_2 \ldots e_e}(v)$ is the non-oriented bivector associated with $f$, and $n_f = 0, 1$. Then we parallel-transport $\Omega_f(v)$ it to a neighboring tetrahedron $t_e$ by using $\Omega_{e_e}$, i.e. $\Omega_f(e) = \Omega_{e_e} \Omega_f(v) \Omega_{e_e}$. We find that the parameter $\theta_f$ is related to the deficit angle. An explicit way to see it is the following: the curvature in the discrete setting is given by the pure boost part of the above spin connection, i.e. the above $\Omega_f(v)$ with $n_f = 0$. Thus to find the relation between the parameter $\theta_f$ with the deficit angle, in the following we only consider $\Omega_f(v)$ with $n_f = 0$ which is the pure boost part of the spin connection [48]. Let $\Omega_f(e) \equiv \Omega_{e_e}$ act on the vector $u$ by using equations (92) and (100), we have

$$(\Omega_{e_e} \triangleright u) \sigma_E^f \equiv \Omega_{e_e}^{-1} \Omega_{e_e}^{-1} = \cos \theta_f \mathbb{I} + \sin \theta_f \mathbf{n}_f \cdot \sigma_E,$$

(70)

where $\mathbf{n}_f$ is the unit vector orthogonal to triangle $f$. It is consistent with the orientation of the non-oriented area bivector $A_f$. Then we can obtain

$$\cos \Theta_f := u \cdot \Omega_{e_e} \triangleright u = \cos \theta_f,$$

(71)

which implies $\theta_f = \pm \Theta_f$. It is not the case that $\theta_f = \Theta_f$ always holds. Suppose we assume the parameter $\theta_f$ would be a Regge deficit angle being a function of segment lengths only, we make a global parity transformation $E_i(v) \mapsto \mathbf{P} E_i(v)$, and correspondingly for the spin connection $\Omega_i \mapsto \mathbf{P} \Omega_i \mathbf{P}$. Then

$$\mathbf{P} \Omega_f \mathbf{P} = \mathbf{P} \exp(\theta_f \bullet A_f) \mathbf{P} = \exp(-\theta_f \bullet \mathbf{P} \triangleright A_f)$$

(72)

implies $\theta_f \mapsto -\theta_f$ under the parity transformation, where the above second equality is because $\mathbf{P} \bullet = - \bullet \mathbf{P}$. However, the parity transformation does not change the segment lengths. Therefore we see that $\theta_f$ does not only depend on the segment lengths. In order to give the relation between $\theta_f$ and deficit angle $\Theta_f$, let us see the discrete version of the Einstein–Hilbert action $S = \int d^4x \sqrt{\mathcal{R}}/2$. For each dual face $f$

$$ S_f = \frac{1}{2} \text{tr} \left( \int_{\Delta_f} \text{sgn} \det(e) \bullet (e \wedge e) \int_f \mathcal{R} \right)$$

$$ \cong \text{sgn} (V_4) \frac{1}{2} \text{tr} (\bullet A_f(e) \ln \Omega_{e_e}) = \text{sgn} (V_4) A_f \theta_f, \quad (73)$$

where $A_f$ is the face area of triangle $f$ and $\text{sgn} (V_4)$ is the sign of the 4-volume of the simplices. Recall that the Regge action $S_f = A_f \Theta_f$ is the discretization of the Einstein–Hilbert action, we find

$$\Theta_f = \text{sgn} (V_4) \theta_f, \quad (74)$$

$\Theta_f$ is the deficit angle of interior face $f$, which measures the curvature located at the triangle $f$.

Now let us consider the case of a boundary face $f$. The relation $E_{p,p}(f) = \Omega_f(f) \triangleright E_{p,p}(f) (p, p)$ are the vertices of the triangle $f$ implies that $\Omega_f(f)$ can be written in terms of the non-oriented area bivector $A_f(f) \equiv A_{e_1 e_2 \ldots e_e}(f)$ as

$$\Omega_f(f) = e^{\omega_{e_e} \pi} \exp \left( \theta_f^B \bullet \hat{A}_f(f) + n_f \pi \hat{A}_f(f) \right)$$

(75)
with $n_f = 0, 1$. Only the pure boost part of $\Omega_f(f)$ contributes to the extrinsic curvature on the boundary, so we only consider the case with $n_f = 0$ [48]. Then the spin connection becomes

$$\Omega_{\text{out}} = \omega_{n_f} \exp (\theta_f^B \cdot \hat{A}_f(f)) \omega_{\text{in}}.$$  \hspace{1cm} (76)

Acting $\Omega_{\text{out}}$, on the vector $u = (1, 0, 0, 0)$, we obtain the dihedral angle $\Theta_f^B = \pm \theta_f^B$ by

$$\cos \Theta_f^B = u \cdot \Omega_{\text{out}} \cdot u = \cos \theta_f^B.$$  \hspace{1cm} (77)

By a similar discussion as we just did for the deficit angle, $\theta_f^B$ is not exactly the dihedral angle defined in the Regge calculus since it is changed under parity transformation. The relation between $\theta_f^B$ and dihedral angle $\Theta_f^B$ is given by

$$\text{sgn}(V_e) \Theta_f^B = \theta_f^B.$$  \hspace{1cm} (78)

A detailed discussion about this relation can be found in [48] (see also [35, 36]). Here the spin connection is then given by the following dihedral rotation on the plane orthogonal to the triangle $f$

$$\Omega_f(f) = \exp (\text{sgn}(V_e) \Theta_f^B \cdot \hat{A}_f(f)).$$  \hspace{1cm} (79)

Now we give a brief summary of the section. In this section we worked on a global oriented simplicial complex $\Delta$ and defined discrete geometric variable segment vectors $E_l(v)$, $E_l(e)$ and $E_l(f)$ at each vertex $v$, boundary edge $e$ and boundary face $f$, respectively. They are the natural (co-)frames for the discrete geometry. They all satisfy the properties of inverse (equations (23) and (43)), close (equations (24) and (44)) and gluing (equations (25) and (45)). There is a discrete metric $g_{ll}$ defined by $E_l$ and $E_{l'}$ which is the segment length when $l = l'$. We assume the oriented 4-volume $V_4(v)$ has a constant sign $\text{sgn}(V_4(v))$ on the entire complex. From $E_l(v)$ we can define five outpointing vectors $U(v)$ for each $\sigma_v$ which satisfy equations (32) and (30). For two neighboring simplices $\sigma_v$ and $\sigma_{v'}$, their frames are related by $SO(4)$ spin connections $\Omega_{\text{in}}$. The segment vectors $E_l(v)$ and $E_{l'}(v')$, and the unit outpointing vectors $\hat{U}(v)$ and $\hat{U}(v')$ are related by the parallel transportation (equation (40)). The deficit angle and dihedral angle are defined from the spin connection by equations (74) and (78), respectively.

In the following sections we discuss the asymptotic behavior of the Euclidean EPRL spin foam amplitude. We will use the critical configurations $\{j_f, n_{ef}, g_{ve}\}$ to construct (semi-)geometrical variables and to compare them with the ones introduced in this section.

5. Equations of motion

As we discussed in section 3, the asymptotic behaviors of the Euclidean spin foam amplitude are critical configurations that solve equations (11)–(13). The presentation in the following is for the case with the Barbero–Immirzi parameter $\gamma < 1$. However it turns out that the case with $\gamma > 1$ results in the same equations of motion thus the same geometric interpretation.

Firstly, we consider equation (11). Using the definition (equation (7)), we obtain

$$\text{Re}(S) = \sum_f \sum_{vef} \sum_{\pm} 2 j_{ef}^+ \ln \left(1 + R(g_{vv}) n_{ef} \cdot R(g_{vv}) n_{ef} \right) = 0,$$

where $R(g)$ is the vector representation of $g \in SU(2)$. The above equation results in

$$R(g_{ve}) n_{ef} = R(g_{ve}) n_{ef},$$  \hspace{1cm} (80)

which is called gluing condition between tetrahedrons.
Secondly, we consider equation (12). Here we parameterize the group element $g^\pm$ by Euler angles $\theta_i, i = 1, 2, 3$ around the stationary point $\bar{g}^\pm$, i.e. $g^\pm = \exp(\theta_i^F) \bar{g}^\pm$. Evaluate the derivatives over $\theta_i$ on the constraint surface of equation (80), we obtain the following closure condition:

$$\frac{\partial S}{\partial \theta_i} |_{\theta_i = 0} = \sum_{f, e} \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle + \sum_{f, e} \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle$$

$$= 4 \epsilon_{ef}(v) j_f^+ R (\bar{g}_{ve}^\pm) n_{ef} = 0.$$  \hspace{1cm} (81)

where $\epsilon_{ef}(v) = 1$ when the orientations of $f$ and $e$ are agreed, otherwise $\epsilon_{ef}(v) = -1$. As we defined in section 2, the orientation of the half-edges is always from $e$ to $v$. It implies

$$\epsilon_{ef}(v) = -\epsilon_{ef}(v'), \quad \epsilon_{ef}(v) = -\epsilon_{ef}(v).$$  \hspace{1cm} (82)

Finally we consider equation (13). Here we introduce the derivative of the coherent state $|n\rangle$. Since $|n\rangle, |Jn\rangle$ is a basis of the spinor space $\mathbb{C}^2$ and the spinor $|n\rangle$ is normalized, we have

$$\delta |n\rangle = \epsilon |Jn\rangle + i n |n\rangle$$  \hspace{1cm} (83)

$$\delta |n\rangle = (Jn)\bar{e} - i n |n\rangle,$$  \hspace{1cm} (84)

where the parameters $\epsilon \in \mathbb{C}$ and $\bar{e} \in \mathbb{R}$, $J$ is an anti-linear map defined in [36, 55]

$$|Jn\rangle \equiv J \left( \begin{array}{c} z_0 \\ z_1 \end{array} \right) = \left( \begin{array}{c} -\bar{z}_1 \\ z_0 \end{array} \right).$$  \hspace{1cm} (85)

From this definition we can find that $|Jn\rangle$ is orthogonal to $|n\rangle$ because $\langle n|Jn\rangle = 0$. Recall equation (8), the map $J$ sends the 3-vector $\mathbf{n}$ to $-\mathbf{n}$.

Evaluating equation (13) with the derivative equation (83) while taking equation (80) into account, we find that equation (13) is satisfied automatically

$$\delta_{\theta_i} S = j_f^+ \delta_{\theta_i} (\ln(n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle) + j_f^+ \delta_{\theta_i} (\ln(n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)$$

$$= 2 j_f^+ \frac{\delta (n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)}{(n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)} + 2 j_f^+ \frac{\delta (n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)}{(n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)} \frac{(n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)}{(n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle \langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle)} = 0,$$

where the third equality is because

$$\langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle | n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle = 0 \hspace{1cm} (86)$$

$$\langle n_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle | Jn_{ef} | g_{\theta_i}^\pm (\bar{g}_{ve}^\pm, n_{ef}) \rangle = 0 \hspace{1cm} (87)$$

are satisfied on the constraint surface of equation (80).

Thus we summarize the equations of motion at the end of the subsection. Gluing condition:

$$g_{ve}^\pm \triangleright n_{ef} = g_{ve}^\pm \triangleright n_{ef}.$$  \hspace{1cm} (88)

Closure condition:

$$\sum_{f, e} \epsilon_{ef}(v) j_f^+ (g_{ve}^\pm \triangleright n_{ef}) = 0,$$  \hspace{1cm} (89)

with the orientation condition:

$$\epsilon_{ef}(v) = -\epsilon_{ef}(v'), \quad \epsilon_{ef}(v) = -\epsilon_{ef}(v).$$  \hspace{1cm} (90)

The critical configurations $(j_f, g_{ve}, n_{ef})$ are the solutions of the above equations.
6. Semi-geometrical variables

In this section, we construct bivector variables at each vertex $v$ in terms of spin foam variables $(j_f, g_v, n_{ej})$. We call the bivectors constructed in this section the semi-geometrical variables.

We identify any bivectors $X_{ef} \in E$ with $SO(4)$ Lie algebra element $J^{ij} \in so_4$ by using

$$X \equiv X_{ij} J^{ij}. \quad (91)$$

As we know, $so_4$ Lie algebra can be decomposed into two copies $su_2$ Lie algebra, i.e. self-dual and anti-self-dual parts. Give any $J^{ij} \in so_4$ and define $J_i = \frac{1}{2} \epsilon_{ijk} J^{jk}, K^i = J^{i0}$, the generators $J^{\pm i} = \frac{1}{2} (J^i \pm K^i)$ satisfy the following commutation relations:

$$J^{\pm i}, J^{\pm j} = -\epsilon^{ijk} J^{\pm k}$$

$$[J^{\pm i}, J^{\mp j}] = 0. \quad (92)$$

The explicit relation between $SO(4)$ (or Spin(4)) group element and $SU(2) \otimes SU(2)$ is

$$\exp \left( \frac{1}{2} B_{ij} J^{ij} \right) = \exp \left( \sum_{\pm} \left( \frac{1}{2} \epsilon_{ijk} B^{jk} \pm B_{0i} \right) J^{\pm k} \right)$$

$$\exp \left( \frac{1}{2} (\ast B)_{ij} J^{ij} \right) = \exp \left( \sum_{\pm} \left( \frac{1}{2} \epsilon_{ijk} B^{jk} \pm B_{0i} \right) J^{\pm k} \right)$$

$$= \exp \left( \sum_{\pm} B_{1i} J^{\pm k} \right). \quad (92)$$

Based on this decomposition, we define the self-dual and anti-self-dual bivectors in each tetrahedron $t_e$ associated with the faces $f$ of the tetrahedron. The canonical quantization of LQG suggests that the area spectral is given by $\gamma j_f^2$ when $j$ is much larger than 1. So here we define the self-dual bivector $X^+_f$ and anti-self-dual bivector $X^-_f$ for the face $f$ in tetrahedron $t_e$ as

$$iX^\pm_f = 2\gamma j_f (n_{ef}), J^{\pm i} = i(X^\pm_f) \sigma^{\pm i}. \quad (93)$$

Using the above definition we can define the unit bivectors as

$$\hat{X}^\pm_f = X^\pm_f / |X^\pm_f| = (n_{ef}) \sigma^i. \quad (94)$$

where $|X|^2 = X^i X^i, |X^\pm_f| = \gamma j_f$. The parallel transportations of $\hat{X}^\pm_f$ are

$$\hat{X}^\pm_f (v) = \hat{s}_v \hat{X}^{\pm}_f \hat{s}_v^{-1}. \quad (95)$$

Then by using equations (92) and (93) we can write the SO(4) bivector

$$X_{ef} = \sum_{\pm} iX^\pm_f = \gamma j_f \epsilon_{0ijk} n^K_{ef} J^{ij}$$

$$= (X_{ef})_{ij} J^{ij}. \quad (96)$$

where $n^K_{ef} \equiv (0, n^K_{ef})$. Then we can define the unit bivector

$$\hat{X}_{ef} = X_{ef} / |X_{ef}| = \epsilon_{0ijk} n^K_{ef} J^{ij}, \quad (97)$$
where $|X|^2 = \frac{1}{2} X_{IJ} X^{IJ}$, $|X| = \gamma j_f$. Based on equation (94), we can parallel-transport $X_{ef}$ to the nearest vertex $v$ and define a bivector at $v$ by

$$X_{ef}(v) = g_{ve} \triangleright X_{ef}. \quad (98)$$

In $t_e$ frame there is a unit vector $u = (1, 0, 0, 0)$ such that

$$\delta^{IJK} u_I (X_{ef}(v))^J = 0. \quad (99)$$

Any vector $x_I \in E$ can be identified with a $2 \times 2$ matrix $x = x_I \sigma^I_E$. The parallel transformation for this vector is $g^{-1} x_I \sigma^I_E = (g x) I \sigma^I_E$. For $g = (g^+, g^-) \in SO(4)$. (100)

Then the parallel transportation (equation (99)) to vertex $v$ by $g_{ve} \in SO(4)$. By defining $N^e(v) = g_{ve} \triangleright u$, we can have $\delta^{IJK} N^e(v)^J (X_{ef}(v))^K = 0$ which is the simplicity constraint for each face at each vertex.

Now we rewrite equations of motion equations (88) and (89) by using SO(4) bivectors $X_{ef}(v)$ and summarize them as follows:

Gluing condition:

$$X_f(v) \equiv X_{ef}(v) = X_{e'f}(v). \quad (101)$$

Closure condition:

$$\sum_{f \in e} e_{ef}(v) X_{ef}(v) = 0. \quad (102)$$

We can also obtain two more equations from the definitions. In terms of equation (98) we obtain

$$g_{v'v} \triangleright X_{ef}(v) = X_{ef}(v'). \quad (103)$$

We also have the simplicity constraint

$$\delta^{IJK} N^e(v)^J (X_{ef}(v))^K = 0. \quad (104)$$

### 7. Discrete geometry from critical configurations

In this section we use the semi-geometrical variables $X_{ef}(v)$ and $N^e(v)$ to reconstruct the discrete geometrical variables $E_{eef}(v)$ and $U^e(v)$. Here in this section we only discuss the case that all the 4-simplices are non-degenerate (the degenerate case is discussed in section 10).

In our definition and the definition in [45], the non-degeneracy is defined in terms of $Ne(v)$ by

$$\prod_{1 \leq i < j < k < l} \det(N^e(v), N^f(v), N^g(v), N^h(v)) \neq 0. \quad (105)$$

The reconstruction of the non-degenerate geometry in the case of a simplicial manifold without boundary was first introduced in [45].

$N^e(v)$ is determined by the group element $g_{ve}$ for a set of given configuration $\{j_f, g_{ve}, n_{ef}\}$. For Euclidean theory, as discussed in [35], if equation (105) is satisfied, $g_{ve}$ and $g_{ve}'$ should be two different SU(2) group elements, i.e. $g_{ve} \neq g_{ve}'$.

During the following construction, we keep in mind that we are working on an consistently oriented complex $\Delta$, where the orientations of the 4-simplices are defined in section 4.
7.1. Reconstruction of 4-simplex

The following analysis is based on a given non-degenerate critical configuration \{\{f, nf, g_{ve}\}. In the frame of \(v\), we consider two bivectors \(X_{ef}(v)\) and \(X_{ef}(v)\). Because of the simplicity constraint (equation (104)), there are 4D vectors \(V_{ef}\) and \(V_{ef}\) in \(E\) such that \(\mathbf{X}_{ef}(v) = N^e(v) \wedge V_{ef}(v)\) and \(\mathbf{X}_{ef}(v) = N^e(v) \wedge V_{ef}(v)\). Because of the gluing condition (equation (101)), vectors \(N^e(v)\), \(V_{ef}(v)\), \(N^e(v)\) and \(V_{ef}(v)\) are in the same plane. Then this plane is spanned by \(N^e(v)\) and \(N^e(v)\text{, i.e.} V_{ef}(v) = \alpha_{ve}(v)N^e(v) + cN^e(v)\). So the bivector \(X_f(v) = X_{ef}(v) = X_{ef}(v)\) can be written as

\[
X_f(v) = \star \alpha_{ve}(v)(N^e(v) \wedge N^e(v)).
\]  

Using the closure constraint (equation (102)), the above equation turns into

\[
\sum_{j \neq i} \varepsilon_{eef}(v)X_{eef}(v) = \star \left(N^e(v) \wedge \sum_{j \neq i} \varepsilon_{eef}(v)\alpha_{ij}(v)N^j(v)\right)
\]

\[
\equiv \star \left(N^e(v) \wedge \sum_{j \neq i} \beta_{ij}(v)N^j(v)\right)
\]

\[
= 0,
\]

where \(\varepsilon_{eef}(v)\) are the coefficients in the orientation condition (equation (90)) such that \(\varepsilon_{eef}(v) = -\varepsilon_{eef}(v)\equiv \varepsilon_{eef}(v)\equiv -\varepsilon_{ef}(v)\) and \(\beta_{ij}(v) = \varepsilon_{eef}(v)\alpha_{ij}(v)\). Together with the non-degenerate assumption (equation (105)), it implies that there are non-vanishing diagonal elements \(\beta_{i}(v)\) such that

\[
5 \sum_{j=1}^{5} \beta_{ij}(v)N^j(v) = 0.
\]

Otherwise any two of \(N^e(v)\) would be parallel to each other.

Now we consider

\[
\beta_{lm}(v)5 \sum_{j=1}^{5} \beta_{ij}(v)N^j(v) = \beta_{lm}(v)5 \sum_{j=1}^{5} \beta_{ij}(v)N^j(v)
\]

\[
= \sum_{j \neq m} (\beta_{lm}(v)\beta_{lj}(v) - \beta_{lm}(v)\beta_{lj}(v))N^j(v) = 0.
\]

Because of equation (105), any four \(N^e(v)\) are linearly independent. The above equation turns into

\[
\beta_{lm}(v)\beta_{lj}(v) - \beta_{lm}(v)\beta_{lj}(v) = 0.
\]

We can pick one \(j_0\) for one \(\sigma_c\) and ask \(i = j = j_0\). Then we can obtain

\[
\beta_{lm}(v) = \frac{\beta_{m}(v)\beta_{j_0}(v)}{\beta_{j_0}(v)} = \tilde{e}(v)\beta_{m}(v)\beta_{j_0}(v),
\]

where \(\beta_{j_0}(v) = \beta_{j_0}(v)/\sqrt{\beta_{j_0}(v)}\), \(\tilde{e}(v) = \text{sgn} (\beta_{j_0}(v))\). Then bivector \(\varepsilon_{eef}(v)X_{eef}(v)\) becomes

\[
\varepsilon_{eef}(v)X_{eef}(v) = \tilde{e}(v) \star [(\beta_{j_0}(v)N^j(v)) \wedge (\beta_{j_0}(v)N^j(v))].
\]
Now we can reconstruct the discrete geometrical variable $U^e(v)$ (up to an overall sign in each $\sigma_e$) defined in section 4 by

$$U^e(v) = \pm \frac{\sqrt{2} \beta(v) N^e(v)}{\sqrt{V_4(v)}}$$

(114)

where $V_4(v)$ is defined by

$$V_4(v) := \det(\beta_2(v)N^e_2(v), \beta_3(v)N^e_3(v), \beta_4(v)N^e_4(v), \beta_5(v)N^e_5(v))$$

(115)

from which we obtain

$$V_4^{-1}(v) = \det(U^e_2, U^e_3, U^e_4, U^e_5).$$

(116)

By using equation (34), we can define segment vectors $E_{ee'},(v)$ satisfying the inverse and close properties, such that equation (113) turns into

$$\epsilon_{ee'}(v)X_{ee',}(v) = \tilde{\epsilon}(v) \frac{1}{2} \left( |V_4(v)| \star (U^e(v) \wedge U^{e'}(v)) \right)$$

$$= \epsilon(v) \frac{1}{2} V_4(v) \star (U^e(v) \wedge U^{e'}(v))$$

$$= \epsilon(v) \frac{1}{4} \sum_{m,n} \epsilon_{e'mn} E_{ee',}(v) \wedge E_{ee',}(v)$$

$$= \epsilon(v) \epsilon_{ee'}(v) A_{ee',}(v)$$

(117)

where $\epsilon(v) \equiv \tilde{\epsilon}(v) \text{sgn}(V_4(v))$. In the last equality, we use equation (35). This is the explicit relation between the semi-geometrical bivector $X(v)$ and discrete geometrical bivectors $A(v)$ defined by equation (36).

The above result shows that given a set of non-degenerate critical configurations $\{j_f, n_{ef}, g_{ve}\}$ at a vertex $v$, we can reconstruct a bivector geometry in each 4-simplex $\sigma_v$ [35, 36].

7.2. Gluing the interior 4-simplices

In order to construct a discrete geometry on the entire complex, we discuss the gluing of the geometries of two neighboring vertices $v$ and $v'$ that are linked by edge $e_1$. We still use figure 2 in our discussion.

For convenience, we introduce shorthand notations: $U^i \equiv U^i(v), E_{ij} \equiv E_{ee',}(v), U^{i'} \equiv g_{vv'} U^i(v'), E'_{ij} \equiv g_{vv'} E_{ee',}(v')$, with $e_{i'} \equiv e_1$.

Because $N^e(v) = g_{ve} N^e(v) = g_{ve} N^e(v')$, we have $N^{i'}(v) = g_{ve} N^{i'}(v')$. Together with equation (114), we have

$$\frac{U^i}{|U^i|} = \tilde{\epsilon} \frac{U^{i'}}{|U^{i'}|},$$

(118)

where $\tilde{\epsilon} = \pm 1$. Moreover because of equations (103), (90) and (117), we have

$$\epsilon_{ee',}(v)X_{ee',}(v) = \frac{1}{2} \epsilon V \star (U^i \wedge U^{i'}) = -\frac{1}{2} \epsilon' V' \star (U^i \wedge U^{i'})$$

(119)

where $i \neq 1$ and we use the shorthand notations $\epsilon \equiv \epsilon(v)$, $\epsilon' \equiv \epsilon(v')$. Reminding the orientation consistency that is discussed in section 4, in this notation we should have $\epsilon_{ee',ee',}(v) = -\epsilon_{ee',ee',}(v')$. Then

$$V^{-1} = \tilde{\epsilon} \det(U^2, U^3, U^4, U^5)$$

(120)

$$V'^{-1} = \tilde{\epsilon}' \det(U^2, U^3, U^4, U^5)$$

(121)
where $\hat{e} = -\vec{e} = \pm 1$. Equations (118) and (119) imply that $U^i$ is proportional to $U^1$ and $U'^i$ are the linear combination of $U^1$ and $U^i$. Explicitly,

$$U'^i = -\varepsilon\varepsilon' \hat{e} \frac{|U^1|^2}{|U^1|^2} U^i + a_i U^1,$$  \hspace{1cm} (122)

where $a_i$ are the coefficients such that $\sum_{i=1}^5 U'^i = 0$. Using the above equation, we have

$$\frac{1}{V'} = \hat{e} \varepsilon \varepsilon' \left( -\varepsilon \varepsilon' \hat{e} \right) \frac{|U^1|^2}{|U^1|^2} \varepsilon'^3 \det(U^1, U^2, U^3, U^4)$$

$$= \varepsilon \varepsilon' \left( |U^1|^2 |U^1|^2 \right) \frac{1}{V'}.$$  \hspace{1cm} (123)

This implies $\varepsilon(v) = \varepsilon(v')$. It means the sign factor $\varepsilon(v)$ defined in equation (117) does not depend on $v$.

**Proposition 7.1.** For any semi-geometrical area bivector $X_f(v)$ defined in section 6 from the spin foam model, we can always reconstruct a non-oriented bivector $A_f(v)$ of discrete geometry up to a global sign $\varepsilon$ for the whole simplicial complex $\Delta$. 

**Proof.** We prove the proposition in three steps. We can easily see from equation (117) that a semi-geometrical area bivector $X_f(v)$ corresponds to a non-oriented bivector $A_f(v) \equiv A_{\epsilon_{t_1} \epsilon_{t_2}}(v)$ with a sign factor (recall that $A_{\epsilon_{t_1} \epsilon_{t_2}}(v)$ is defined with the orientation $f = [e_m, e_n, e_k]$)

$$X_{\epsilon_{t1}}(v) = \varepsilon(v) \varepsilon_{\epsilon_{t1}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v).$$  \hspace{1cm} (124)

$\varepsilon_{\epsilon_{t1}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v)$ is a sign factor with a given triangle $f = [p_m, p_n, p_k] = [e_m, e_n, e_k]$. Because $\varepsilon(v) = \varepsilon$ is a global sign, then we focus on proving that $\varepsilon_{\epsilon_{t1}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v)$ is a global sign.

First we prove that it is a constant for each triangle $f$. For this purpose, we only need to prove that the sign is a constant both between two tetrahedrons in one 4-simplex and between two neighboring 4-simplices, sharing the triangle $f$. We consider the situation shown in figure 2. Because $X_{\epsilon_{t1}}(v) = X_{\epsilon_{t2}}(v)$, we can get between two tetrahedrons $t_1$ and $t_2$, in $\sigma_\epsilon$, the sign factor keeps invariant

$$\varepsilon_{\epsilon_{t1}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v) = \varepsilon_{\epsilon_{t2}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v).$$  \hspace{1cm} (125)

Because $X_{\epsilon_{t1}}(v) = g_{v\epsilon} X_{\epsilon_{t1}}(v')$, $A_{\epsilon_{t1} \epsilon_{t2}}(v) = g_{v\epsilon} A_{\epsilon_{t1} \epsilon_{t2}}(v')$, we can get between two neighboring 4-simplices $\sigma_\epsilon$ and $\sigma_{\epsilon'}$, the sign factor keeps invariant

$$\varepsilon_{\epsilon_{t1}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v) = \varepsilon_{\epsilon_{t1}}(v') A_{\epsilon_{t1} \epsilon_{t2}}(v').$$  \hspace{1cm} (126)

Combining equations (125) and (126), we obtain for each face $f$, the sign factor $\varepsilon_{\epsilon_{t1}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v)$ which is a constant $\varepsilon_f$, once we fix an orientation $[e_m, e_n, e_k]$ of $f$.

Secondly we prove that the sign factor is a constant in a 4-simplex between different triangles. For this purpose, we only need to prove that for any two triangles in a tetrahedron of a 4-simplex, the sign factor is a constant. We consider the situation shown in figure 1. Without losing generality, we pick out $t_{e_1} \in \sigma_\epsilon$ and the bivectors $X_{\epsilon_{t1} \epsilon_{t2}}(v), X_{\epsilon_{t1} \epsilon_{t2}}(v)$. For $X_{\epsilon_{t1} \epsilon_{t2}}(v)$ we have

$$X_{\epsilon_{t1} \epsilon_{t2}}(v) = \varepsilon(v) \varepsilon_{\epsilon_{t1}}(v) \varepsilon_{\epsilon_{t1} \epsilon_{t2}}(v) A_{\epsilon_{t1} \epsilon_{t2}}(v).$$  \hspace{1cm} (127)
For $X_{e_1 e_2}(v)$ we have
\[ X_{e_1 e_2}(v) = \varepsilon(v)X_{e_1 e_2}(v)X_{e_2 e_3 e_4}(v)A_{e_4 e_5 e_6}(v). \] 
(128)

If the face orientations of $f_{e_1 e_2}$ and $f_{e_1 e_3}$ agree along $e_1$, $\varepsilon_{e_1 e_2}(v) = \varepsilon_{e_1 e_3}(v)$, we can pick $A_{e_4 e_5 e_6}(v)$ and $A_{e_7 e_8 e_9}(v)$ as the reconstructed non-oriented area bivectors. Then we have
\[ \varepsilon_{e_1 e_2}(v)\varepsilon_{e_1 e_2 e_3 e_4}(v) = \varepsilon_{e_1 e_3}(v)\varepsilon_{e_1 e_2 e_3 e_4}(v). \] 
(129)

If the face orientations of $f_{e_1 e_2}$ and $f_{e_1 e_3}$ are opposite at $e_1$, $\varepsilon_{e_1 e_2}(v) = -\varepsilon_{e_1 e_3}(v)$. We can pick $A_{e_4 e_5 e_6}(v)$ and $A_{e_7 e_8 e_9}(v)$ as the reconstructed non-oriented area bivectors. Then we have
\[ \varepsilon_{e_1 e_2}(v)\varepsilon_{e_1 e_2 e_3 e_4}(v) = \varepsilon_{e_1 e_3}(v)\varepsilon_{e_1 e_2 e_3 e_4}(v). \] 
(130)

The choice of non-oriented area bivectors can always be achieved based on the orientation of $\Delta^*$. Then in one 4-simplex, the sign factor $\varepsilon_{e_1 e_2}(v)\varepsilon_{e_1 e_2 e_3 e_4}(v)$ is also a constant $\hat{\varepsilon}_v = \varepsilon_f$. Then we obtain the following conclusion: for any semi-geometrical bivector $X_f(v)$ constructed from the spin foam critical configuration, we can reconstruct a non-oriented bivector of discrete geometry $A_f(v)$, with a choice of the orientation for each $f$, up to a global sign $\varepsilon$ on the entire simplicial complex
\[ X_f(v) = \varepsilon A_f(v), \] 
(131)

where $\varepsilon \equiv \varepsilon(v)\varepsilon_f$. □

From equation (123) we obtain $|U^j|V = \pm |U'^j|V'$. We define a new type of sign factor
\[ \mu \equiv -\hat{\varepsilon}|U^j|V/|U'^j|V' = -\hat{\varepsilon} \text{ sgn}(VV'). \] 
(132)

Recall equation (34), we obtain
\[ E_{kl} = V'\varepsilon'_{ijklm}n^{ijk}U_{l}^{ijm}U^{k}L_{l} \]
\[ = \mu E_{kl}, \] 
(133)

which implies that the spin foam variables $g_{uv'}$ and SO(4) holonomy $\Omega_{vv'}$ are just different by a sign
\[ g_{uv'} = \mu e^{-\Omega_{uv'}.} \] 
(134)

By the definition of the spin connection in section 4, $\Omega_{vv'}$ is a spin connection as long as sgn $V_i(v) = \text{ sgn} V_i(v')$.

7.3. Reconstruction of boundary

First of all, we can reconstruct the tetrahedron $t_v$ with an edge $e$ connecting to the boundary. Giving a set of non-degenerate boundary data $\{f, n_{ef}\}$ where $f$s are boundary triangles, we have a closure condition for the boundary tetrahedron
\[ \sum_{f \in e} e_{ef}(v)\gamma j n_{ef} = 0, \] 
(135)

where $v$ is the vertex $e$ connecting, and $n_{ef} = (0, n_{ef})$ is lying on the plane orthogonal to $u = (1, 0, 0, 0)$. Then we can reconstruct the discrete geometrical variable $n^p(e)$ defined in section 4 as
\[ n^p(e) = \frac{2\varepsilon_{ef}(v)\gamma j n_{ef}}{|V_3(e)|}. \] 
(136)
where the oriented 3-volume $V_3(v)$ is defined as
\[
\text{sgn} V_3(e) \frac{V_3^2(e)}{8} = \frac{1}{3!} \sum_{jkl} \varepsilon_{ijk} (ev) e^{JKL}(\varepsilon_{efj}(v) \gamma f j_k n_{efj}) (\varepsilon_{efj}(v) \gamma f j_k n_{efj}).
\]

(137)

In this definition $n^0(e)$ satisfies equation (52).

Together with equation (53), we can reconstruct the segment vectors $E_{p,p'}(e)$ defined before. Then we can reconstruct the area bivectors (equation (96)) $X_{ij} = \gamma f j_k (u \wedge n_{efj})$ as
\[
\varepsilon_{efj}(v) X_{ij} = \tilde{\varepsilon}(e) \frac{1}{2} V_3(e) \bullet (u \wedge n_{efj})
\]
\[
= \tilde{\varepsilon}(e) \frac{1}{4} \sum_{jkl} \epsilon^{ijk} (E_{p,p'}(e) \wedge E_{p,p'}(e))
\]
\[
= \tilde{\varepsilon}(e) n_{ij} e_{ijk} (ev)\Lambda_{l,jk}(e),
\]
where $\tilde{\varepsilon}(e) = \text{sgn} V_3(e)$, and here all the boundary segment vectors $E_{p,p'}(e)$ are understood as 4-vectors in $E$ such that $E_{p,p'}(e) = (0, E_{p,p'}^k)$, $k = 1, 2, 3$.

Now we identify the boundary tetrahedron $t_e$ with the tetrahedron in 4-simplex $\sigma_e$ dual to edge $e$. For convenience, we introduce shorthand notations: $E_{ij} \equiv E_{p,p'}(e)$, $E_{ij} \equiv g_{ev} E_{p,p'}(v)$, $\bar{\varepsilon} \equiv \tilde{\varepsilon}(e)$, $n^0 \equiv n^0(e)$. Because the parallel transportation $X_{ef} = g_{ev} \circ X_{ef}(v)$, we have
\[
\bar{\varepsilon} \sum_{jkl} \epsilon^{ijk} (E_{ij} \wedge E_{kl}) = \varepsilon(v) \sum_{jkl} \epsilon^{ijk} (E'_{ij} \wedge E'_{kl}),
\]
\[
(139)
\]
where $p$ is the point belong to $\sigma_e$ but not in $t_e$. It means
\[
\bar{\varepsilon}\bar{V}_3 n^0 = \varepsilon V_3 n^0,
\]
\[
(140)
\]
where $V_3^e$ is defined in the same way as $V_3$ but using $E_{ij}$ instead. Because of $n^0 E_{jk} = \delta_j^i - \delta_k^i = n^0 E'_{jk}$, the above equation turns into
\[
\bar{\varepsilon}\bar{V}_3 n^0 E'_{jk} = \varepsilon(v) V_3 n^0 E'_{jk} = \varepsilon(v) V_3 n^0 E_{jk},
\]
\[
(141)
\]
which implies
\[
E'_{jk} = \bar{\varepsilon}(e) \frac{V_3^e}{V_3} E_{jk}.
\]
\[
(142)
\]
Bring this equation back to equation (139), we obtain
\[
\varepsilon(v) \bar{\varepsilon} \left( \frac{V_3^e}{V_3} \right)^2 = 1,
\]
\[
(143)
\]
which implies
\[
\bar{\varepsilon}(e) = \varepsilon(v) \quad \text{and} \quad |V_3| = |V_3^e|.
\]
\[
(144)
\]
\[
\bar{\varepsilon}(e) = \text{sgn} V_3 \text{ of } V_3^e \text{ is determined by the boundary data. If we choose the orientations of the triangles such that } \varepsilon_{efj}(v) e_{efj} e_{efj} e_{efj} = 1 \text{ identically, the global sign } \varepsilon \text{ relating } X_f(v) \text{ and } A_f(v) \text{ is determined by the boundary data } \varepsilon = \text{sgn } V_3(e) \text{ once we choose sgn } V_3(e) \text{ to be a constant on the boundary.}
\]

Moreover from equation (142), we find that the spin connection equals the on-shell $g_{ve}$ up to a sign $\mu_e$
\[
g_{ve} = \mu_e \Omega_{ve},
\]
\[
(145)
\]
where $\mu_e = \text{sgn } V_3 \text{ sgn } V_3^e \pm 1$. We denote by $V_3^e(v)$ the 3-volume induced from $V_3(v)$ with the normal $\hat{U}_e(v)$. $V_3^e(v)$ is in general different from $V_3^e(v)$ by the discussion in equation (54).

As we show in section 4, sgn $V_3^e = \text{sgn } V_3(v)$, we have
\[
\mu_e = \text{sgn } V_3 \text{ sgn } V_3(v) \text{ sgn } (\hat{U}_e \cdot u).
\]
\[
(146)
\]
Then we can prove the following lemma,
7.4. Reconstruction theorem

In this subsection, we summarize all the discussion in this section as a reconstruction theorem (see also [45] for the case of a simplicial manifold without boundary).

Theorem 7.3 (Reconstruction theorem). Given a set of data \( \{ j_f, n_{ef}, g_{sv} \} \) be a non-degenerate critical configuration which solves equations (101)–(104) on a simplicial manifold with boundary, there exists a discrete classical Euclidean geometry represented by a set of segment vectors \( E_i(v) \) satisfying equations (23)–(25) in the bulk, and \( E_i(e) \) satisfying equations (43)–(46) on the boundary, such that

1. The semi-geometrical area bivectors \( X_f(v) \) and \( X_{ef} \) from spin foam stationary points can reconstruct non-oriented classical area bivectors \( A_f(v) \) and \( A_{ef} \) up to a global sign \( \varepsilon \):
   \[
   X_f(v) = \varepsilon A_f(v) = \varepsilon \frac{1}{2} (E_l(v) \wedge E_{l'}(v))
   \]
   \[
   X_{ef} = \varepsilon A_{ef} = \varepsilon \frac{1}{2} (E_l(e) \wedge E_{l'}(e)),
   \]
   where \( l, l' \) are the segments of triangle \( f \). Moreover the segment vectors are totally determined up to an inverse sign \( E_l \mapsto -E_l \). With the segment vectors \( E_l(v) \) and \( E_l(e) \) we can reconstruct the discrete metric \( g_{sv} \) from them
   \[
   g_{ll'}(v) = \delta_{ll'} E_l^1(v) E_{l'}^1(v)
   \]
   \[
   g_{ll'}(e) = \delta_{ll'} E_l^1(e) E_{l'}^1(e).
   \]
   The metric is independent of \( v \) and \( e \) because of the gluing conditions equations (25) and (45). The norm of the bivector \( X_f(v) \) is \( \gamma_j f \) which is understood as the area of the triangle \( f \).
2. \( \text{sgn} V_3(e) \) has to be chosen as a constant \( \forall t_e \in \partial \Delta \), then the global sign factor \( \varepsilon \) is fixed to be \( \varepsilon = \text{sgn} V_3(e) \) when we choose the orientations of the triangles such that \( \varepsilon_{v_1v_2}(v)\varepsilon_{v_2v_3}(v) = 1 \) identically.

3. \( \forall \varepsilon = (\varepsilon \varepsilon') \in \Delta^*, l \in t_e \in \Delta, \) the segment vectors in \( E_l(v) \) and \( E_l(v') \) are related by a parallel transformation \( g_{v'v} \in \text{SO}(4) \) associated with edge \( e \) up to a sign \( \mu_e \)

\[
\mu_e E_{e_{12}}(v) = g_{v'v} \triangleright E_{e_{12}}(v'), \quad \forall [e_1e_2] \in t_e. \quad (154)
\]

\( \forall t_e \in \partial \Delta, t_e \in \sigma_v, \) the segment vectors \( E_{e_{12}}(e) \) and \( E_{e_{12}}(v) \) are also related by a parallel transformation \( g_{ee} \in \text{SO}(4) \) associated with half-edge \( (ev) \) up to a sign \( \mu_e \)

\[
\mu_e E_{e_{12}}(v) = g_{ee} \triangleright E_{e_{12}}(e), \quad \forall [e_1e_2] \in t_e. \quad (155)
\]

Thus the critical point of \( g_{v'v} \) and \( g_{ee} \) can be related to \( \text{SO}(4) \) matrices \( \Omega_{v'v} \) and \( \Omega_{ee} \) up to the same sign as the one relate \( E_{e_{12}} \)

\[
g_{v'v} = \mu_e \Omega_{v'v}, \quad g_{ee} = \mu_e \Omega_{ee}. \quad (156)
\]

The simplicial complex \( \Delta \) can be subdivided into sub-complexes \( \Delta_1, \ldots, \Delta_n \) such that (1) each \( \Delta_i \) is a simplicial complex with boundary, (2) within each sub-complex \( \Delta_i, \) \( \text{sgn} V_4 \) is a constant. Then within each sub-complex \( \Delta_i, \) the \( \text{SO}(4) \) matrices \( \Omega_{v'v} \) and \( \Omega_{ee} \) are the discrete spin connection compatible with the segment vectors \( E_{e_{12}} \).

4. Given the boundary triangles \( f \) and boundary tetrahedrons \( t_e \), in order to have non-degenerate solutions to the equations of motion. The spin foam boundary data \( \{j_f, n_f\} \) must satisfy the non-degenerate Regge boundary conditions: (1) For each boundary tetrahedron \( t_e \) and its triangles \( f, \{j_f, n_f\} \) determines four triangle normals \( n_f \) that span a 3D subspace. (2) The boundary data are restricted to be shape matched (equation (149)). (3) The boundary triangulation is consistently oriented such that \( \text{sgn} V_3 \) is a constant on the boundary. If the Regge boundary condition is satisfied, there are non-degenerate solutions of the equations of motion.

8. Spin foam amplitude at non-degenerate critical configurations

The asymptotics of the spin foam amplitude is a sum of the amplitude evaluated at the critical configurations. In this section, we evaluate the spin foam amplitude at the non-degenerate critical configurations. We show that the spin foam action at a non-degenerate critical configuration is almost a Regge action. As we mentioned in the last section, we subdivide the complex \( \Delta \) into sub-complexes \( \Delta_1, \ldots, \Delta_n \) such that (1) each \( \Delta_i \) is a simplicial complex with boundary, (2) within each sub-complex \( \Delta_i, \) \( \text{sgn} V_4 \) is a constant. To study the spin foam (partial-)amplitude \( Z_{j_f}(\Delta) \) at a non-degenerate critical configuration \( \{j_f, n_f, g_{ef}\} \), we only need to study the amplitude \( Z_{j_f}(\Delta_i) \) on the sub-complex \( \Delta_i \). The amplitude \( Z_{j_f}(\Delta) \) can be expressed as

\[
Z_{j_f}(\Delta) = \prod_i Z_{j_f}(\Delta_i). \quad (157)
\]

Therefore the following analysis is in one of \( \Delta_i \).

8.1. Internal faces

We first consider internal faces \( f_i \). For an internal face, the action defined in equation (7) can be rewritten in the following way: by equation (88), the parallel transportation acting on the coherent state \( |n_f\rangle \) gives

\[
g_{v'v}^{\pm} n_f^{\pm} = e^{\pm \phi_{v'v}} |n_f\rangle. \quad (158)
\]
Thus the loop holonomy along the boundary of an oriented face $f$ gives
\[
g^\pm_{e_{v_{1}}e_{v_{2}}e_{v_{3}}e_{v_{4}}e_{v_{5}}|n_{e_{f}}} = \exp \left( i \Phi^\pm_{f} \right)|n_{e_{f}}\rangle,
\]
where $\Phi^\pm_{f} \equiv \sum_{v \in f} \phi^\pm_{v}$. This implies that the loop holonomy can be written as
\[
G^\pm_{f}(e) \equiv g^\pm_{e_{v_{1}}e_{v_{2}}e_{v_{3}}e_{v_{4}}e_{v_{5}}|e_{v_{a}}e_{v_{b}}e_{v_{c}}e_{v_{d}}e_{v_{e}} = \exp \left( i \Phi^\pm_{f} \hat{X}^\pm_{f} \right),
\]
where $\hat{X}^\pm_{f} = i(n_{e_{f}})\sigma^{i}$ as defined in equation (94). Then the action defined in equation (7) turns into
\[
S_f = \sum_{\pm} \sum_{j_{\pm} \in f} 2 j_{\pm} \ln \left[ \text{tr} \left( \frac{1}{2} G^\pm_{f}(v) \left( 1 + \hat{X}^\pm_{f} \right) \right) \right].
\] (161)

By using the following identity
\[
\exp \left( i \Phi^\pm_{f} \right) = \text{tr} \left( \frac{1}{2} \exp \left( i \Phi^\pm_{f} n_{e_{f}} \sigma_{i} \right) \left( 1 + n_{e_{f}} \sigma_{i} \right) \right),
\]
we can rewrite the action in the following form:
\[
S_f = \sum_{\pm} \sum_{j_{\pm} \in f} 2 j_{\pm} \ln \left[ \text{tr} \left( \frac{1}{2} G^\pm_{f}(v) \left( 1 + \hat{X}^\pm_{f} \right) \right) \right].
\] (162)

Then let us use the parallel transformation of $G^\pm_{f}(e)$ and $\hat{X}^\pm_{f}$ to take them to the nearest vertex $v$.
\[
G^\pm_{f}(v) \equiv g^\pm_{e_{v_{1}}e_{v_{2}}e_{v_{3}}e_{v_{4}}e_{v_{5}}|e_{v_{a}}e_{v_{b}}e_{v_{c}}e_{v_{d}}e_{v_{e}}}
\]
\[
\hat{X}^\pm_{f}(v) = g^\pm_{e_{v_{1}}e_{v_{2}}e_{v_{3}}e_{v_{4}}e_{v_{5}}|e_{v_{a}}e_{v_{b}}e_{v_{c}}e_{v_{d}}e_{v_{e}}}
\]
(163)

Because the trace does not change under the parallel transformation, then the action becomes
\[
S_f = \sum_{\pm} \sum_{j_{\pm} \in f} 2 j_{\pm} \ln \left[ \text{tr} \left( \frac{1}{2} G^\pm_{f}(v) \left( 1 + \hat{X}^\pm_{f}(v) \right) \right) \right].
\] (166)

We would like to find the relation between equation (166) and Regge action. Recalling equations (117) and (134) we have the parallel transportation of $X_{f}(v)$ and $E_{pp'}(v)$ under the loop holonomy $G_{f}(v)$
\[
G_{f}(v) \triangleright X_{f}(v) = X_{f}(v)
\] (167)
\[
G_{f}(v) \triangleright E_{pp'}(v) \equiv \exp \left( i \pi \sum_{e_{f} \in f} n_{e_{f}} \right) E_{pp'}(v)
\]
\[
= \cos \left( \pi \sum_{e_{f} \in f} n_{e_{f}} \right) E_{pp'}(v),
\] (168)

where $\exp(i\pi \sum_{e_{f} \in f} n_{e_{f}}) = \prod_{e_{f} \in f} \mu_{e_{f}}$. These equations imply that the loop holonomy $G_{f}(v)$ gives a rotation in the plane orthogonal to the 2-plane determined by $X_{f}(v)$, i.e. in the plane of $\star X_{f}(v)$. Then we can explicitly write the loop holonomy as
\[
G_{f}(v) = \exp \left( \star \hat{A}_{f}(v) \theta_{f} \right) \exp \left( i \pi \sum_{e_{f} \in f} n_{e_{f}} \hat{A}_{f}(v) \right),
\] (169)

where $\hat{A}_{f}(v) = A_{f}(v)/|A_{f}(v)|$. The transformation equation (168) can be shown in the following way. Given a bivector $A_{f}(v) = (E_{1}(v) \wedge E_{2}(v))/2$, we choose two orthogonal bases
\(e_1 = (0, 0, 1, 0), \ e_2 = (0, 0, 1)\) in the plane of \(A_f(v)\), then \(A_f(v) = e_1 \wedge e_2\). We can show that on the 2-plane of \(A_f(v)\)
\[
\exp\left(\pi \sum_{e \in f} n_e A_f(v)\right) = \cos\left(\pi \sum_{e \in f} n_e\right) \mathbb{I}
\]
is a \(\pi\)-rotation.

From the relation between the spin foam variable \(g_{vv'}\) and spin connection \(\Omega_{e'v'}\) (equation (134)), we can obtain the loop spin connection \(\Omega_f(v)\)
\[
\Omega_f(v) = \exp\left(i \pi \sum_{e \in f} n_e \hat{A}_f(v)\right) \exp\left(i \theta_f + \pi \sum_{e \in f} n_e \hat{A}_f(v)\right).
\]

From the discussion in section 4, we have the geometrical interpretation of the parameter \(\theta_f\)
\[
\theta_f = \text{sgn} V_4 / \Theta_f,
\]
where \(\Theta_f\) is the deficit angle in the Regge calculus.

Based on the relation between SO(4) and its self-dual and anti-self-dual decomposition (equations (92) and (100)), the self-dual and anti-self-dual loop holonomy \(G^\pm_f(v)\) are
\[
G^\pm_f(v) = \exp\left[\frac{i}{2} \sum_{e \in f} n_e \hat{A}_f(v) \pm \text{sgn} V_4 / \Theta_f \hat{X}_f^\pm\right]
\]
\[
= \exp\left(i \Phi^f / \hat{X}_f^\pm\right).
\]

We take \(\Phi_f^\pm\) defined above into equation (161), then we can obtain the asymptotic action for interior faces
\[
S_f = -i \varepsilon \sum_{f_i} \gamma j_{f_i} \text{sgn} V_4 / \Theta_f + i \varepsilon \pi \sum_{f_i} j_{f_i} \sum_{e \in f} n_e.
\]

where \(\gamma j_{f_i}\) is the area of triangle \(f_i\) and the first term \(\text{sgn} V_4 / \Theta_f\) is the Regge action for discrete GR when \(\text{sgn} V_4\) is a constant.

### 8.2. Boundary faces

Now we consider the action for boundary face \(f\). Giving a boundary face \(f\) as shown in figure 4, together with the gluing condition (equation (88)), we obtain
\[
g_{e_{v_0}}^\pm \cdots g_{e_{v_f}}^\pm |n_{e_{0f}}\rangle = \exp\left(i \Phi^\pm_{e_{v_0}} \right) |n_{e_{0f}}\rangle.
\]

This implies that the holonomy \(G^\pm_{e_{v_0}} = g_{e_{v_0}}^\pm \cdots g_{e_{v_f}}^\pm\) can be written as
\[
G^\pm_{e_{v_0}} = g(n_{e_{0f}}) \exp\left[i \Phi^\pm_{e_{v_0}} \tau_2\right] g^{-1}(n_{e_{0f}}),
\]
where the SU(2) group element \(g(n)\) is given by
\[
g(n) = |n\rangle \langle z| + |Jn\rangle \langle Jz|
\]
in spin-\(\frac{1}{2}\) representation. In spin-1 representation, it rotates \(z = (0, 0, 1)\) to the 3-vector \(n\). We can also consider \(g(n)\) as a rotation from the reference frame at \(f\) to the reference frame in the tetrahedron \(t_e\).

The action of \(f_i\) can be written as
\[
S_f = \sum_\pm 2 j_f^\pm \Phi^\pm_{e_{v_0}}.
\]
which can be rewritten using equation (162) as

\[
S_f = \sum_{\pm} 2f_0 \ln \left[ \exp \left( \frac{1}{2} g^{-1}(v_0) G_{e_0}^e g(v_0 e) (1 + \hat{X}_e) \right) \right].
\]  

(179)

where \( \hat{X}_e \pm = \sigma_e \).

To find the relation between equation (179) and the Regge action, we redefine the segment vectors \( E_i(e_i) \) as

\[
\tilde{E}_{pp'}(e_i) \equiv g^{-1}(e_{pp'}) \triangleright E_{pp'}(e_i), \quad \forall \, pp' \in f.
\]  

(180)

Since \( E_{pp'}(e_i) \) is orthogonal to \( e_i \), we can get that \( \tilde{E}_{pp'}(e_i) \) is orthogonal to \( z \). For \( p, p' \) vertices of the triangle \( f, \tilde{E}_{pp'}(e_0) \) and \( \tilde{E}_{pp'}(e_1) \) must be related by a rotation in the plane of \( f \). Here we gauge fix this rotation to be identity, i.e. \( \tilde{E}_{pp'}(e_1) = \tilde{E}_{pp'}(e_0) \equiv E_{pp'}(f) \). Then recall the parallel transportation of the bivector \( X_e = G_{e_0}^e \triangleright X_{e_0} \), and equations (134) and (145) we have the following relations \( (p, p' \) are vertices of the triangle \( f) \)

\[
\tilde{G}_{e_{pp'}} \triangleright E_{pp'}(f) = \left( \prod_{e \in f} \mu_e \right) E_{pp'}(f),
\]

\[
\cos \left( \pi \sum_{e \in f} n_e \right) \tilde{E}_{pp'}(f),
\]

(181)

where \( \tilde{G}_{e_{pp'}} \equiv g^{-1}(e_{pp'}) G_{e_0}^e g(e_{pp'}) \). The above equation implies that the parallel transportation \( g^{-1}(e_{pp'}) G_{e_0}^e g(e_{pp'}) \) has the following form (see section 4):

\[
\tilde{G}_{e_{pp'}} = \exp \left( i \hat{\Delta}_f(f) \theta^B_j + \pi \sum_{e \in f} n_e \hat{\Delta}_f(f) \right)
\]

(182)

where \( \theta^B_j \) is the parameter of the dihedral rotation, and \( \hat{\Delta}_f(f) = \epsilon \hat{X}_e \). From equations (134), (145) and (182), we have

\[
\Omega_{e_{pp'}} = \exp \left( i \pi \sum_{e \in f} n_e g(e_{pp'}) \tilde{G}_{e_{pp'}} g^{-1}(e_{pp'}) \right).
\]

(183)

From the discussion in section 4 and [48], the parameter \( \theta^B_j \) relates to the dihedral angle \( \Theta^B_f \) between the two tetrahedrons \( t_{e_0} \) and \( t_{e_1} \) by

\[
\theta^B_j = \Theta^B_f \operatorname{sgn} V_e(v).
\]

(184)

Then taking equation (182) back to equation (179), we can obtain for any boundary faces

\[
S_f = -i \epsilon \sum_{f} \gamma f \sum_{e \in f} \operatorname{sgn} V_e \Theta^B_f + i \pi \sum_{f} j_f \left( \sum_{e \in f} n_e \right).
\]

(185)

8.3. Asymptotic non-degenerate amplitude

As we have shown above, for any set of non-degenerate solutions \( \{f, n_e, g_{ee} \} \) of equations (101) and (102) together with equations (90), (104) and (98), we can always construct a non-degenerate discrete geometry with a global sign ambiguity \( \epsilon \).

We briefly summarize the results we obtain so far. For a given non-degenerate critical configuration \( \{f, n_e, g_{ee} \} \), we can reconstruct the discrete geometric variables \( E_i \) and \( U^e \).

\[ \forall \, \nu \in \Delta^*, \text{we can reconstruct a bivector geometry of 4-simplex.} \]

Given any semi-geometrical bivector \( X_f(v) \) from the critical configuration, there is a non-oriented bivector \( \hat{A}_f(v) = (E_i(v) \wedge E_i(v))/2 \) in discrete geometry such that

\[
X_f(v) = \epsilon \hat{A}_f(v),
\]

(186)

where \( \epsilon \) is a global sign on the entire simplicial complex \( \Delta \).
• ∀ $e \in \Delta^*$, we can associate a spin connection $\Omega_e$ (when $\text{sgn} V_4(v) = \text{sgn} V_4(v')$) by the on-shell $g_{vv'}$ up to a sign $\mu_e$

$$g_{vv'} = \mu_e \Omega_e,$$

where $v$ and $v'$ are the end points of $e$.

• ∀ $e \in \partial \Delta^*$, we can construct the segment vectors $E_i(e)$ such that giving any semi-geometrical bivector $X_f(e)$ from the critical configuration, we can find a non-oriented bivector $A_f(e) = (E_i(e) \wedge E_i(e))/2$ in discrete geometry on the boundary that

$$X_f(e) = \varepsilon A_f(e).$$

A non-degenerate critical configuration $(j_f, g_{ve}, n_{ef})$ specifies uniquely a set of variables $(g_{\ell b \ell}, n_e, \varepsilon)$, which include a discrete metric and two types of sign factors.

Given a critical configuration $(j_f, g_{ve}, n_{ef})$ in general, we can divide the triangulation $\Delta$ into sub-triangulations $\Delta_1, \ldots, \Delta_n$, where each of the sub-triangulation is a triangulation with boundary, by a constant $V_4(v(v))$. On each of the sub-triangulation $\Delta_i$, we add the on-shell actions of internal and boundary faces together, we have

$$S_f(g_{\ell b \ell}, n_e, \varepsilon))_{\text{Non-deg}} = \sum_{f_i} S_{f_i} + \sum_{f_e} S_{f_e}$$

$$= -i \varepsilon \text{sgn} V_4 \left( \sum_{f_i} \gamma j_f \Theta_{f_i} + \sum_{f_e} \gamma j_f \Theta_{f_e}^{\varepsilon} \right) + i \varepsilon \pi \left( \sum_e n_e \sum_{f_e} j_f \right).$$

(189)

Here $\Theta_{f_i}$ and $\Theta_{f_e}^{\varepsilon}$ are the deficit angle and the dihedral angle, respectively, which are determined only by the discrete metric $g_{\ell b \ell}$. Moreover ∀ $e \in \Delta^* \sum_{f_e} j_f$ is an integer. It contributes to an overall sign when we exponentiate $S_f(g_{\ell b \ell}, n_e, \varepsilon)$.

We say a spin configuration $j_f$ is Regge-like if there exist the critical configurations solving the equation of motion, which is non-degenerate everywhere. Given a collection of Regge-like spins $j_f$ for each 4-simplex, the discrete metric $g_{\ell b \ell}(v)$ is uniquely determined for the simplex. Furthermore since the areas $\gamma j_f$ are Regge-like, there exists a discrete metric $g_{\ell b \ell}$ in the entire bulk of the triangulation, such that the neighboring 4-simplices are consistently glued together, as we constructed previously. This discrete metric $g_{\ell b \ell}$ is obviously unique by the uniqueness of $g_{\ell b \ell}(v)$ at each vertex. Therefore given the partial amplitude $Z_{j_f}(\Delta)$ with a specified Regge-like $j_f$, all the critical configurations $(j_f, g_{ve}, n_{ef})$ with the same Regge-like $j_f$ correspond to the same discrete metric $g_{\ell b \ell}$, provided a Regge boundary data. The critical configurations from the same Regge-like $j_f$ are classified in the next section.

As a result, for any Regge-like configurations $j_f$ and a Regge boundary data $n_{ef}$, the amplitude $Z_{j_f}(\Delta)_{\text{Non-deg}}$ has the following asymptotic behavior:

$$Z_{j_f}(\Delta)_{\text{Non-deg}} = \sum_x C(x_e) \left[ 1 + \mathcal{O} \left( \frac{1}{\lambda} \right) \right]$$

$$\times \exp \lambda \sum_{\Delta_i} \left[ -i \varepsilon \text{sgn} V_4 \left( \sum_{f_i} \gamma j_f \Theta_{f_i} + \sum_{f_e} \gamma j_f \Theta_{f_e}^{\varepsilon} \right) + i \varepsilon \pi \left( \sum_e n_e \sum_{f_e} j_f \right) \right].$$

(190)

where $x_e$ stands for the non-degenerate critical configurations $(j_f, g_{ve}, n_{ef})$ and $C(x_e)$ is given as follows:

$$C(x_e) = a(x_e) \left( \frac{2\pi}{\lambda} \right)^{\frac{\text{v.e.}}{\lambda}} \frac{e^{\text{vind}(H(x_e))}}{\sqrt{\left| \det H(x_e) \right|}},$$

(191)
where $H(x_c)$ is the Hessian matrix of the action $S_f$ and $H'(x_c)$ is the invertible restriction on \ket{H(x_c)}$; $r(x_c)$ is the rank of Hessian matrix. The on-shell action on exponential gives the Regge action up to the sign factor $\text{sgn} V_{4\Delta}$ of the oriented 4-volume. However if we recall the difference between the Einstein–Hilbert action and Palatini action

$$\mathcal{L}_{EH} = R_g = \text{sgn} \det \left( \epsilon_{\mu}^i \right) \star \left[ e \wedge e \right]_{12} \wedge R^{12}$$

(192)

$$= \text{sgn} \det \left( \epsilon_{\mu}^i \right) \mathcal{L}_{Pl},$$

where $\mathcal{L}_{EH}$ and $\mathcal{L}_{Pl}$ denote the Lagrangian densities of the Einstein–Hilbert action and Palatini action, respectively, and $g$ is a chosen volume form compatible with the metric $g_{\mu\nu} = \eta_{ij} \epsilon^i_{\mu} \epsilon^j_{\nu}$. Since the Regge action is a discretization of the Einstein–Hilbert action, we may consider that the resulting action from the asymptotics is a discretization of the Palatini action with the connection compatible with the tetrad.

9. Parity inversion

Given a tetrahedron $\tau_c$ associated with spins $j_1, \ldots, j_5$, we know that the set of four normals $n_{ej_1}, \ldots, n_{ej_5} \in S^2$, modulo diagonal SO(3) rotation, is equivalent to the shape of $\tau_c$, if the closure condition is satisfied \cite{49, 56}. Given a set of non-degenerate solutions and configurations $\{j_f, \tilde{g}_{ve}, \tilde{n}_{ef}\}$, as discussed above, the Regge-like spin configuration $j_f$ determines a discrete metric $g_{ve}$, which determines the shape of all the tetrahedrons in $\Delta$. The diagonal SO(3) rotation of $n_{ej_1}, \ldots, n_{ej_5}$ is also a gauge transformation of the spin foam action. Thus the gauge equivalence class of the critical configurations $\{j_f, n_{ef}, g_{ve}\}$ with the same Regge-like spins $j_f$ must have the same set of $n_{ef}$. The degrees of freedom of the non-degenerate critical configurations are the freedom of the variables $g_{ve}$ when we fix a Regge-like $j_f$. The degrees of freedom of $g_{ve}$ are encoded in the 4-simplex geometry. Given a set of data $\{j_f, n_{ef}\}$, the non-degenerate critical configurations within each 4-simplex are completely classified \cite{35, 36} and are related by parity transformation.

Given a set of non-degenerate solutions and configurations $\{j_f, \tilde{g}_{ve}, n_{ef}\}$, we can generate many other sets of solutions and configurations $\{j_f, \tilde{g}_{ve}, n_{ef}\}$. As discussed in \cite{35}, the two solutions $g_{ve}$ and $\tilde{g}_{ve}$ are related by local parity in some 4-simplices. In Euclidean theory, within a 4-simplex $\sigma_v$, if $g_{ve} = (g_{ve}^+, g_{ve}^-)$ is a solution of equations of motion, $\tilde{g}_{ve} = (g_{ve}^+, g_{ve}^-)$ is also a solution of the same equations. The semi-geometric variables generated by $g_{ve}$ and $\tilde{g}_{ve}$ are related by local parity transformation, since

$$N^e(v)\sigma^f_L = g_{ve}^- u_v \sigma^f_L (g_{ve}^+)^{-1} = g_{ve}^- (g_{ve}^+)^{-1}$$

(193)

$$\tilde{N}^e(v)\sigma^f_L = g_{ve}^+ u_v \sigma^f_L (g_{ve}^-)^{-1} = g_{ve}^+ (g_{ve}^-)^{-1}.$$  

(194)

Then

$$\tilde{N}^e(v)\sigma^f_L = (N^e(v)\sigma^f_L)^\dagger = N_0^e(v)I - iN^e(v)\sigma^i$$

$$= (P N^e(v))\sigma^f_L,$$

(195)

where $P$ is the parity operator on the Euclidean vector space.

Then let us look at the relation between semi-geometric bivectors $X_f(v)$ and $\tilde{X}_f(v)$, where $\tilde{X}_f(v)$ is the bivector defined by using $\tilde{g}_{ve}$. We have the relations $X^\pm(v) = X^\mp(v)$

$$X^\pm(v) = \frac{1}{2} \epsilon^i_{jk} X_{jkr}(v) \pm X_{0r}(v) X^\pm(v) = \frac{1}{2} \epsilon^i_{jk} \tilde{X}_{jkr}(v) \mp \tilde{X}_{0r}(v).$$

(196)

Then we can easily obtain $X_{0r}(v) = -\tilde{X}_{0r}(v)$ and $X_{jkr}(v) = \tilde{X}_{jkr}(v)$, i.e.

$$\tilde{X}_f(v) = P \triangleright X_f(v).$$

(197)
Reminding equation (106), we can obtain
\[ \tilde{X}_{e'}(v) = \tilde{\alpha}_{e'}(v) \frac{e^e(e')}{(e^n(v) \wedge \tilde{e}^n(v))} = -\frac{\tilde{\alpha}_{e'}(v)}{\alpha_{e'}(v)} P \triangleright X_f(v). \] (198)

Then recall equation (197) we obtain
\[ \tilde{\alpha}_{e'}(v) = -\alpha_{e'}(v). \] (199)

Because \( \tilde{\beta}_{e'}(v) = \alpha_{e'}(v) \varepsilon_{e'}(v) \),
\[ \tilde{\beta}_{e'}(v) = -\beta_{e'}(v). \] (200)

Then \( \tilde{\beta}_{e'}(v) = \tilde{\beta}_{e'}(v) \) since \( \sum_j \beta_{ij} n^j = 0 \). Based on this and the definition of \( \tilde{\beta}_i(v) = \beta_{ij}(v) / \sqrt{|\tilde{\beta}_{ij}(v)|} \) we can obtain
\[ \tilde{\beta}_i(v) = -\beta_i(v). \] (201)

As shown in subsection 7, by using equation (115) we can obtain
\[ \tilde{V}_4(v) = -V_4(v). \] (202)

The minus sign is from the fact that \( \det P = -1 \). Then the global signs \( \varepsilon(v) = \text{sgn}(\beta_{j0}(v)) \text{sgn}(V_4) \) are the same
\[ \tilde{\varepsilon}(v) = \varepsilon(v). \] (203)

This result shows that when we change \( \sigma_\epsilon \) into its parity one \( \tilde{\sigma}_\epsilon \), the global sign stays invariant.

The fact that the local parity changes the sign of the 4-volume of the 4-simplex leads to some interesting consequences. First of all, given any critical configuration \( \{ j_f, g_{ve}, n_{ef} \} \) with a Regge-like spin configuration \( \{ j_f \} \), we can always subdivide the triangulation \( \Delta \) into subcomplexes. Each of the sub-complex has a constant \( \text{sgn} V_4 \). Now we understand that the local parity transforms a configuration \( \{ j_f, g_{ve}, n_{ef} \} \) to a new configuration \( \{ j_f, \tilde{g}_{ve}, \tilde{n}_{ef} \} \), which may have different subdivision according to \( \text{sgn} V_4 \). On the other hand, for each subdivision with a critical configuration \( x_c = \{ j_f, g_{ve}, n_{ef} \} \), there is always another critical configuration \( \tilde{x}_c = \{ j_f, \tilde{g}_{ve}, \tilde{n}_{ef} \} \) obtained from the former one by a global parity, which leaves the subdivision unchanged but changes the 4-volume sign within each sub-complex. Thus the global parity changes the spin foam action at the non-degenerate stationary configuration into its opposite, i.e.
\[ S(x_c) = -S(\tilde{x}_c). \] (204)

Note that the deficit angle, dihedral angle and \( \sum_{e \subset \partial f} n_e \) are unchanged under the global parity, which is shown below.

Then let us obtain the relation between segment vector \( \tilde{E}_i(v) \) and \( E_i(v) \). From equations (114) and (201), we can obtain
\[ \tilde{U}_e(v) = -P U_e(v). \] (205)

Then based on equations (34) and (202), we can have
\[ \tilde{E}_i(v) = -P E_i(v). \] (206)

From the above discussion we can find that the local and global parity inversion \( \tilde{E}_i(v) = -P E_i(v) \) does not change the discrete metric \( g_{ij} = \delta_{ij} E^i_j(v) E^j_i(v) \). Thus the parity configuration \( \tilde{x}_c \) gives the same discrete geometry as \( x_c \), and only makes an O(4) gauge transformation to the segment vectors. The matrix \( \Omega_{e'v'} \) is uniquely determined by \( E_i(v) \) and is a spin connection as long as \( \text{sgn} V_4(v) = \text{sgn} V_4(v') \), as shown in section 8. The global
parity transformation does not change the subdivisions but flips the signs of $\text{sgn} V_4$ in each sub-complex. Given a spin connection $\Omega_{v'}$ in a subdivision, the parity one $\hat{\Omega}_{v'}$ is
\[ \hat{\Omega}_{v'} = \mathbf{P} \Omega_{v'} \mathbf{P}, \] (207)
since $\hat{\Omega}_{v'} \hat{E}_{ee'}(v') = \hat{E}_{ee'}(v)$. One can check that for a 4-vector $V^I$, $\tilde{g}PV = P \tilde{g}V$. Then from
\[ \tilde{g}_{v'v} \hat{E}_{ee'}(v') = -\hat{g}_{v'v} \mathbf{P} E_{ee'}(v') = -\mathbf{P} g_{v'v} E_{ee'}(v') = -\mu \mathbf{P} E_{ee'}(v) = \mu \hat{E}_{ee'}(v), \] (208)
and $\tilde{g}_{v'v} = \tilde{\mu} \hat{\Omega}_{v'}$, we find that $\mu_\epsilon$ is invariant under the global parity transformation
\[ \tilde{\mu}_\epsilon = \mu_\epsilon. \] (209)
Now let us consider a boundary edge. In the case $t_e$ is a boundary tetrahedron, the parity transform the segment vectors $E_i(v)$ as $\tilde{E}_i(v) = -\mathbf{P} E_i(v)$ at vertex $v$, while leaving the boundary segment vectors $E_i(e)$ invariant. Therefore the spin connection $\hat{\Omega}_{v'} \in \text{SO}(4)$ is uniquely determined by
\[ \hat{\Omega}_{v'} \hat{E}_{pp'}(e) = \hat{E}_{pp'}(v), \quad \forall \; pp' \in t_e. \] (210)
Then the relation between the spin connection $\hat{\Omega}_{v'}$ before parity transformation and $\hat{\Omega}_{v'}$ is
\[ \hat{\Omega}_{v'} = -\mathbf{P} \Omega_{v'} \mathbf{T}, \] (211)
where $\mathbf{T} = \text{diag}(-1, 1, 1, 1)$ is the time reversal that keeps the spatial vectors $E_i(e)$ unchanged. Then because of $\tilde{g}E(e) = \mathbf{P} gE(e)$ for the spatial vector $E(e)$, we have
\[ \tilde{g}_{v'v} E_{pp'}(e) = -\mu_\epsilon \mathbf{P} E_{pp'}(v) = \mu_\epsilon \hat{E}_{pp'}. \] (212)
Then the same as before, we have
\[ \tilde{\mu}_e = \mu_\epsilon. \] (213)
Thus for both interior and exterior faces, the product $\prod_{e \in f} \mu_\epsilon$ is invariant under the global parity transformation, i.e.
\[ \prod_{e \in f} \mu_\epsilon = \prod_{e \in f} \tilde{\mu}_e. \] (214)
If we write $\mu_\epsilon = \exp(\text{i} \pi n_e)$ and $\tilde{\mu}_e = \exp(\text{i} \pi \tilde{n}_e)$ as before, then we can set
\[ \sum_{e \in f} n_e = \sum_{e \in f} \tilde{n}_e. \] (215)
Then let us consider the loop spin connection $\hat{\Omega}_f$ of an internal face $f$. Based on the discussion about the relation between $\Omega_{v'}$ and $\hat{\Omega}_{v'}$, we have
\[ \hat{\Omega}_f(v) = \mathbf{P} \hat{\Omega}_f(v) \mathbf{P}. \] (216)
Recall equation (171), we write down the spin connection $\hat{\Omega}_f(v)$ as
\[ \hat{\Omega}_f(v) = \exp \left( \text{i} \pi \sum_{e \in f} \tilde{n}_e \right) \exp \left( \text{sgn}(V_4) \bullet \hat{A}_f(v) \hat{\Theta}_f + \pi \sum_{e \in f} \tilde{n}_e \hat{A}_f(v) \right). \] (217)
From the relations $\text{sgn} V_4 = -\text{sgn} \tilde{V}_4$, $\sum_{e \in f} n_e = \sum_{e \in f} \tilde{n}_e$, $\mathbf{P} \hat{A}_f = -\hat{A}_f$, we obtain
\[ \hat{\Theta}_f = \hat{\Theta}_f, \] (218)
which is consistent with the fact that the deficit angle $\hat{\Theta}_f$ is determined only by the metric $g_{\nu \nu}$ which is invariant under the parity transformation.
For the holonomy \( \tilde{\Omega}_{e_\alpha j} \) of a boundary face \( f \), the relation between \( \tilde{\Omega}_{e_\alpha j} \) and \( \Omega_{e_\alpha j} \) is

\[
\tilde{\Omega}_{e_\alpha j} = T\Omega_{e_\alpha j} T, \tag{219}
\]

As before reminding equation (183), we can obtain

\[
\Theta_f^\beta = \tilde{\Theta}_f^\beta, \tag{220}
\]

which is consistent with the fact that the dihedral angle \( \Theta_f^\beta \) is determined by the metric \( g_{ll} \) which is invariant under the parity transformation.

Among all the critical configurations \( \{j_f, g_{ve}, n_{ef}\} \) with the same Regge-like \( j_f \), there exists only two critical configurations such that the signs of the oriented 4-volumes are the same for all the 4-simplex \( e_\alpha \) in \( \Delta \), while the two configurations are related by a global parity. For any critical configuration, it leads to the subdivision of the triangulation, where each sub-complex has a constant volume sign of the 4-simplices. As we discussed above, we can always make a certain local/global parity transformation to flip the volume sign within some certain sub-complexes, which constructs the configurations such that the volume sign is constant on the entire simplicial complex.

10. Degenerate critical configurations

In this section, we discuss the degenerate critical configurations. The degeneracy means that there exists four different edges \( e_1, e_2, e_3, e_4 \) connecting vertex \( v \), with a degenerate critical configuration \( \{j_f, g_{ve}, n_{ef}\}; N^v(\nu) = g_{ve} \triangleright \nu \) satisfy

\[
det(N^v(\nu), N^{\nu'}(\nu), N^{\nu_0}(\nu), N^{\nu_1}(\nu)) = 0. \tag{221}
\]

10.1. Classification

Lemma 3 in [35] shows that within each 4-simplex, all five normals \( N_e(v) \) from a degenerate critical configuration \( \{j_f, g_{ve}, n_{ef}\} \) are parallel and more precisely \( N_e(v) = u = (1, 0, 0, 0) \) once we fix the gauge. This result implies that the self-dual and anti-self-dual parts of SO(4) group element \( g_{ve} \) are the same, since

\[
(g_{ve} u) I_s^v = g_{ve} u I_s^v (g_{ve} u)^{-1} = u I_s^v. \tag{222}
\]

In the following discussion of this subsection, if \( g_{ve}^- \) and \( g_{ve}^+ \) are the same, we denote them as \( h_{ve} \equiv g_{ve}^- = h_{ve}^+ \).

There are two types of degenerate solutions of the equations of motion. We call them type-A and type-B, respectively.

Type-A: Here the type-A degenerate configurations are from the configurations that are used to reconstruct the non-degenerate geometry. For a Regge-like \( \{j_f\} \), as discussed before, we can always solve the equations of motion to obtain a non-degenerate critical configuration \( \{j_f, n_{ef}, g_{ve}\} \) such that we can reconstruct a non-degenerate classical discrete geometry. While as discussed in [35], the equations of motion (equations (88) and (89)) in fact coincide with the equations from SU(2) BF theory. For Regge-like \( \{j_f\} \), within a 4-simplex, the equations of motion have two different groups of SU(2) solutions \( g_{ve}^- \) and \( g_{ve}^+ \). In addition to the non-degenerate solutions \( (g_{ve}^-, g_{ve}^+) \) and \( (g_{ve}^+, g_{ve}^-) \), they imply two degenerate SO(4) solutions \( (g_{ve}^+, g_{ve}^+) \) and \( (g_{ve}^-, g_{ve}^-) \). We call the degenerate configurations \( \{j_f, n_{ef}, g_{ve}\} \) defined in this way within all simplices as type-A configurations.
Type-B: The equations of motion (equations (88) and (89)) may only have one group of SU(2) solutions within a 4-simplex, i.e. we can only find a single SO(4) solution $g_{ve} = (h_{ve}, h_{ve})$ [35]. We call the configurations and solutions $\{j_f, n_{ef}, g_{ve}\}$ defined in this way within all simplices as type-B configurations.

10.2. Type-A asymptotics

The type-A degenerate configurations are constructed from the non-degenerate critical configurations, which have $g_{ve}^+ \neq g_{ve}^-$ in all 4-simplices, as shown in section 8.

Given two type-A degenerate solutions $\{g_{ve}^+, g_{ve}^-\}$ and $\{g_{ve}^+, g_{ve}^-\}$, we canonically associate the solution $\{g_{ve}^+, g_{ve}^+\}$ to the non-degenerate solution $\{g_{ve}^-, g_{ve}^+\}$ for the geometric interpretation, while we associate canonically the other solution $\{g_{ve}^+, g_{ve}^+\}$ to $\{g_{ve}^+, g_{ve}^+\}$. Therefore $\{g_{ve}^+, g_{ve}^+\}$ and $\{g_{ve}^+, g_{ve}^+\}$ have two different geometric interpretations as non-degenerate geometries, which are related by a parity transformation. In particular, $\{g_{ve}^+, g_{ve}^+\}$ and $\{g_{ve}^+, g_{ve}^+\}$ imply different $s_i V_4$ for the 4-simplex oriented volume.

For an internal face $f$, the loop holonomy $G_f(v)$ around it is given by equation (169). The self-dual $G_f^+(v)$ and anti-self-dual $G_f^-(v)$ parts are

$$G_f^\pm(v) = \exp \left[ \frac{i}{2} \varepsilon \left( \pi \sum_{e \in f} n_e \mp \text{sgn} \, V_4 \Theta_f \right) \hat{X}_f^\pm \right],$$

where we have made a subdivision of the complex $\Delta$ into sub-complexes $\Delta_1 \ldots \Delta_n$, such that $\text{sgn} \, V_4$ is a constant on each $\Delta_i$. For the type-A configuration in the bulk, the solution of the loop holonomy would be either $G_f^+(v) = (G_f^+(v), G_f^+(v))$ or $G_f^-(v) = (G_f^-(v), G_f^-(v))$. At these solutions the action $S_f$ becomes

$$S_f = 2i j_f \Phi^+, \quad \text{or} \quad S_f = 2i j_f \Phi^-,$$

where $\Phi^\pm = \varepsilon \left( \pi \sum_{e \in f} n_e - \text{sgn} \, V_4 \Theta_f \right) / 2$. Explicitly, they can write down the action as

$$S_f = i j_f \varepsilon \left( \pi \sum_{e \in f} n_e - \text{sgn} \, V_4 \Theta_f \right).$$

Note that $\text{sgn} \, V_4$ flips sign between the two solutions $\{g_{ve}^+, g_{ve}^+\}$ and $\{g_{ve}^-, g_{ve}^+\}$.

For a boundary triangle $f$ shared by two boundary tetrahedrons $t_0, t_1$, the holonomy is defined by equation (182). The self-dual $G_{t_0}^\pm$ and anti-self-dual $G_{t_0}^\mp$ parts are

$$G_{t_0}^\pm = g(n_{t_0 f}) \exp \left[ \frac{i}{2} \varepsilon \left( \pi \sum_{e \in t_0} n_e - \text{sgn} \, V_4 \Theta_f \right) \sigma_e \right] g^{-1}(n_{t_0 f}).$$

For the type-A configuration, the solution of the loop holonomy would be either $G_{t_0}^+ = (G_{t_0}^+, G_{t_0}^+)$ or $G_{t_0}^- = (G_{t_0}^-, G_{t_0}^-)$. In this solution the action $S_{t_0}$ becomes

$$S_{t_0} = i j_{t_0} \varepsilon \left( \pi \sum_{e \in t_0} n_e - \text{sgn} \, V_4 \Theta_f \right).$$

Adding the asymptotic actions of internal and boundary faces together, we can obtain the total action

$$S_f(g_{ij}, n_{ef}, \varepsilon)_{\text{Type-A}} = \sum_{f_i} S_{f_i} + \sum_{f_e} S_{f_e},$$

$$= -i \varepsilon \, \text{sgn} \, V_4 \left( \sum_{f_i} j_{f_i} \Theta_f + \sum_{f_e} j_{f_e} \Theta_f^e \right) + i \varepsilon \pi \left( \sum_{e} n_e \sum_{f \in e} j_f \right).$$
The action $S_f(g_{ij}, n_\nu, \varepsilon)$ is almost the same as the non-degenerate one $S_f(g_{ij}, n_\nu, \varepsilon)|_{\text{Non-deg}}$ in equation (189). The only difference is that in the type-A case, the action is $\gamma$ independent.

As a result, for any Regge-like configurations $j_f$ and Regge boundary data $n_{e_f}$, we can have a type-A asymptotics by summing over all type-A degenerate critical configurations $x_c$

$$Z_{fj}(\Delta)|_{\text{Type-A}} \sim \sum_{x_c} C(x_c)|_{\text{Type-A}} \left[ 1 + \mathcal{O} \left( \frac{1}{\lambda} \right) \right] \times \exp \lambda \sum_{\Delta_c} \left[ -i\epsilon \sgn V_f \left( \sum_{j_f} j_f, \Theta_f + \sum_{f_e} \Theta^e_{f_e} \right) + i\epsilon \pi \left( \sum_e n_e \sum_{f \in e} j_f \right) \right].$$

(229)

10.3. Type-B asymptotics

A type-B degenerate configuration $\{j_f, g_{ve}, n_{e_f}\}$ gives a so-called vector geometry on the complex $\Delta$. The vector geometry is determined by the discrete geometric variables $V_f(v)$ and $V_f(\epsilon)$ which are 3-vectors. They are interpreted as the normal to the triangle $f$. Given type-B degenerate configurations $\{j_f, n_{e_f}, g_{ve}\}$, we can reconstruct them by using semi-geometrical variables $X_{e_f} \equiv X^e_{f} = \gamma j_f n_{e_f}$

$$V_f(v) \equiv 2X_{e_f}$$

(230)

$$V_f(\epsilon) \equiv h_{ve} > 2X_{f_e}. \quad (231)$$

The same as the discussion in the non-degenerate case, because of the parallel transportation of the vector $n_{e_f}$, the loop holonomy of an internal face $f_i$ and the holonomy of a boundary face $f_e$ can be written, respectively, in the following way:

$$G_{f_i}(\epsilon) = \exp(i\theta_{f_i} n_{e_f} \cdot \sigma) \quad (232)$$

$$G_{f_i}(e_1 e_0) = g(n_{e_{f_i}}) \exp(i\phi_{f_i} \sigma) g^{-1}(n_{e_{f_i}}). \quad (233)$$

Thus the action becomes

$$S_f = 2i j_f \phi_f. \quad (234)$$

For given vector geometry variables $V_f(e)$ and $V_f(\epsilon)$, we can uniquely determine the solutions $h_{v\epsilon}$ as $\exp(i\Phi_{v\epsilon}) \in SO(3)$. However in the spin foam model what we are using is the spinor representation of the $SU(2)$ group. Because $SU(2)$ is the double cover of $SO(3)$, $\forall h_{v\epsilon} \in SO(3)$, there are two $SU(2)$ elements $h^1_{v\epsilon}$ and $h^2_{v\epsilon}$ with $h^1_{v\epsilon} = -h^2_{v\epsilon}$ corresponding to the same vector geometry $V_f(e)$ and $V_f(\epsilon)$. Thus $\phi_f$ is given by

$$\phi_{f_i} = \frac{1}{2} \Phi_{f_i} + \pi \sum_{e \in f_i} n_e \quad (235)$$

$$\phi_{f_e} = \frac{1}{2} \Phi_{f_e} + \pi \sum_{e \in f_e} n_e. \quad (236)$$

where $\Phi$ is an $SO(3)$ angle determined by the vector geometry only and $n_e = 0, 1$ correspond to solutions $h^1_{v\epsilon}$ and $h^2_{v\epsilon}$, respectively.

Then inserting the angles $\phi_{f_i}$ and $\phi_{f_e}$ back to the action $S_f$, we obtain

$$S_f|_{\text{Type-B}} = \sum_{f_i} j_f \Phi_{f_i} + \sum_{f_e} j_f \Phi_{f_e} + i2\pi \left( \sum_e n_e \sum_{f \in e} j_f \right). \quad (237)$$
\[ \sum_{f \in e} j_f \] is an integer. So \( 2 \sum_{f \in e} j_f \) is an even number. Thus when we exponentiate \( S_f \big|_{\text{Type-B}} \) to obtain the amplitude, the phase factor \( \exp \left( i \frac{\pi}{\lambda} \sum_{f \in e} j_f \right) = 1 \). Thus \( \exp S_f \big|_{\text{Type-B}} \) is a function of vector geometry only. We can give a type-B asymptotics by summing over all type-B degenerate configurations \( x_c \).

\[ Z_{j_f}(\Delta) \big|_{\text{Type-B}} \sim \sum_{x_c} C(x_c) \big|_{\text{Type-B}} \left[ 1 + O \left( \frac{1}{\lambda^2} \right) \right] \exp \left( i \sum_{f_i} j_f \Phi_{f_i} + i \sum_{f_e} j_f \Phi_{f_e} \right). \]  

(238)

Note that if we make a suitable gauge fixing for the boundary data, we can always set \( \Phi_{f_e} = 0 \); see e.g. [35].

11. General critical configurations and asymptotics

For a given critical configuration \( \{j_f, g_{ve}, n_{ef}\} \) in the most general circumstance, we can always divide the complex \( \Delta \) into the non-degenerate region, type-A degenerate region and type-B degenerate region, according to the properties of critical configuration restricted in the regions. In the non-degenerate region and type-A degenerate region, we make further subdivision into the regions with \( V_4 > 0 \) or \( V_4 < 0 \). See figure 5 for an illustration.

Therefore for a generic spin configuration \( j_f \), the asymptotics of the spin foam amplitude \( Z_{j_f}(\Delta) \) is given by a sum over all possible critical configurations \( x_c \), which in general gives different subdivisions of \( \Delta \) into the five types of regions

\[ Z_{j_f}(\Delta) \sim \sum_{x_c} C(x_c) \left[ 1 + O \left( \frac{1}{\lambda^2} \right) \right] A_{j_f}(\Delta_{\text{Nondeg}}) A_{j_f}(\Delta_{\text{Deg-A}}) A_{j_f}(\Delta_{\text{Deg-B}}), \]  

(239)

where \( x_c \) labels a general critical configuration \( \{j_f, g_{ve}, n_{ef}\} \) admitted by the spin configuration \( j_f \) and boundary data. The amplitudes \( A_{j_f}(\Delta_{\text{Nondeg}}), A_{j_f}(\Delta_{\text{Deg-A}}), A_{j_f}(\Delta_{\text{Deg-B}}) \) are given, respectively, by

\[ A_{j_f}(\Delta_{\text{Nondeg}}) = \prod_{i=1}^{n(x_c)} \exp -i \left( \varepsilon \text{sgn} (V_4) \sum_{\text{internal } f} \gamma j_f \Theta f \right. \]

\[ + \left. \varepsilon \text{sgn} (V_4) \sum_{\text{boundary } f} \gamma j_f \Theta f^B + \pi \sum_{e} n_e \sum_{f \subset e} \right] \Delta_{\text{Nondeg}, \Delta}(x_c). \]
\[ A_{j_f}(\Delta_{\text{Deg-A}}) = \prod_{j=1}^{n'(x_c)} \exp \left(-i\lambda \sum_{\text{internal } f} j_f \Theta_f \right) + \varepsilon \sum_{\text{boundary } f} j_f \Phi_f \] 
\[ A_{j_f}(\Delta_{\text{Deg-B}}) = \exp \left(-i\lambda \sum_{\text{internal } f} j_f \Theta_f \right) \] 

If we defined the physical area as \( A_f = \gamma j_f \), then the type-A action turns into 
\[ S_f|_{\text{Type-A}} = \frac{i\varepsilon \text{sgn } V_4}{\gamma} \left[ \sum_f A_f \Theta_f + \sum_{\text{boundary } f} j_f \Phi_f \right] \] 
and the type-B action turns into 
\[ S_f|_{\text{Type-B}} = \frac{i}{\gamma} \left[ \sum_f A_f \Phi_f + \sum_{\text{boundary } f} j_f \Phi_f \right] \] 

Here we consider the case when the Barbero–Immirzi parameter \( \gamma \ll 1 \) mentioned in [57, 58, 46]. Then the type-A degenerate parts (equation (241)) and type-B degenerate parts (equation (242)) oscillate much more violently than the non-degenerate amplitude \( A_{j_f}(\Delta_{\text{Nondeg}}) \). When we sum over all spins \( j_f \) to obtain the total spin foam amplitude, we expect that the non-degenerate critical configurations are dominating the large-\( j \) asymptotics in the case of \( \gamma \ll 1 \). Our conjecture is suggested by the Riemann–Lebesgue lemma, which states that 
\[ \int_{-\infty}^{\infty} f(x) e^{i\alpha x} \, dx = 0, \quad \text{as } \alpha \to \pm \infty. \] 

12. Conclusion and discussion

In this work we study the large-\( j \) asymptotic behavior of the Euclidean EPRL spin foam amplitude on a 4D simplicial complex with an arbitrary number of simplices. The asymptotics of the spin foam amplitude is determined by the critical configurations of the spin foam action. Here we show that, given a stationary configuration \( \{j_f, g_{ve}, n_{ef}\} \) in general, there exists a partition of the simplicial complex \( \Delta \) into three types of regions: non-degenerate region, type-A (BF) region and type-B (vector geometry) region. All of the three regions are simplicial sub-complexes with boundaries. The stationary configuration implies different types of geometries in different types of regions, i.e. (1) the critical configuration restricted in the non-degenerate region implies a non-degenerate discrete Euclidean geometry; (2) the critical configuration restricted in the type-A region is degenerate of type-A in our definition of degeneracy, but it still implies a non-degenerate discrete Euclidean geometry; (3) the critical configuration restricted in the type-B region is degenerate, and implies a vector geometry.

With the critical configuration \( \{j_f, g_{ve}, n_{ef}\} \), we can further make a subdivision of the non-degenerate region and type-A region into sub-complexes (with boundary) according to their Euclidean-oriented 4-volume \( V_4(v) \) of the 4-simplices, such that \( \text{sgn } (V_4(v)) \) is a constant sign on each sub-complex. Then in each sub-complex the spin foam amplitude at the critical configuration gives an exponential of the Regge action in Euclidean signature. However we should note that the Regge action reproduced here contains a sign factor \( \text{sgn } (V_4(v)) \) related
to the oriented 4-volume of the 4-simplices. Therefore the Regge action reproduced here is actually a discretized Palatini action with on-shell connection.

Finally the asymptotic formula of the spin foam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the product of the amplitudes associated with different type of geometries.

We give the critical configurations of the spin foam amplitude and their geometrical interpretations explicitly. However we did not answer the question such as whether the non-degenerate critical configurations are dominating the large-$j$ asymptotic behavior in general or not, although we expect that the non-degenerate configurations are dominating when the Barbero–Immirzi parameter $\gamma$ is small. To answer this question in general requires a detailed investigation about the rank of the Hessian matrix in general circumstances. We leave the detailed study about its rank to the future research.

In this work we show that given a Regge-like spin configuration $j_f$ on the simplicial complex, the stationary configurations $\{j_f, g_{ve}, n_{ef}\}$ are non-degenerate, and there is a unique stationary configurations $\{j_f, g_{ve}, n_{ef}\}$ with the oriented 4-volume $V_4(v) > 0$ (or $V_4(v) < 0$) everywhere. We can regard the critical configuration $\{j_f, g_{ve}, n_{ef}\}$ with $V_4(v) > 0$ as a classical background geometry, and define the perturbation theory with the background field method. Thus with the background field method, the n-point functions in the spin foam formulation can be investigated as a generalization of \cite{57, 58} to the context of a simplicial manifold.

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