LATTICE 3-POLYTOPES WITH FIVE LATTICE POINTS

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Abstract. We work out the complete classification of lattice 3-polytopes with exactly 5 lattice points. We first show that for each \( n \) there is only a finite number of (equivalence classes of) 3-polytopes of lattice width larger than one. Since polytopes of width one are easy to classify, we concentrate on an exhaustive classification of those of width larger than one.

For \( n = 4 \), all empty tetrahedra have width one (White). For \( n = 5 \) we show that there are exactly 9 different polytopes of width 2, and none of larger width. Eight of them are the clean tetrahedra previously classified by Kasprzyk and (independently) Reznick.

Keywords: Lattice polytopes, unimodular equivalence, lattice points.

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1. Introduction

A lattice polytope is the convex hull of a finite set of points in \( \mathbb{Z}^d \) (or in a \( d \)-dimensional lattice). A polytope is \( d \)-dimensional if it contains \( d + 1 \) affinely independent points. We call size of \( P \) its number \( P \cap \mathbb{Z}^d \) of lattice points and volume of \( P \) its volume normalized to the lattice:

\[
\text{vol}(\text{conv}\{p_1, \ldots, p_{d+1}\}):=\left|\det\begin{pmatrix} 1 & \cdots & 1 \\ p_1 & \cdots & p_{d+1} \end{pmatrix}\right|
\]

with \( p_i \in \mathbb{Z}^d \). That is, a tetrahedron has volume one if and only if its vertices are an affine lattice basis.

For an integer linear functional \( f : \mathbb{Z}^d \to \mathbb{Z} \), the integer \( \max_{x \in P} f(x) - \min_{x \in P} f(x) \) is called the width of \( P \) with respect to \( f \). The minimum width among all possible (non-constant) choices of \( f \) will be simply called the width of \( P \). Hence, \( P \) has width one if its vertices lie in two consecutive parallel lattice hyperplanes of the lattice.

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Two lattice polytopes $P$ and $Q$ are said $\mathbb{Z}$-equivalent or unimodularly equivalent if there is an affine integer unimodular transformation $f : \mathbb{Z}^d \to \mathbb{Z}^d$ with $f(P) = Q$. We are interested in the complete classification of lattice polytopes in dimension 3, for small size.

The same question in dimension 2 has a relatively simple, and fully algorithmic, answer. Once we fix the size $n$ of $P$, Pick’s formula says that

\begin{equation}
\text{vol}(P) = n + i - 2 \leq 2n - 5,
\end{equation}

where $i \leq n - 3$ is the number of lattice points in the interior of $P$. Also, every lattice polytope contains a unimodular triangle which we may identify, without loss of generality, with the standard unimodular triangle $\{(0,0), (1,0), (0,1)\}$. This implies $A \subset \text{conv}(\{-2n+6, -2n+6, -2n+6, 4n-11, 4n-11, -2n+6\})$. Hence, for each value of $n$ there is a finite number of $\mathbb{Z}$-equivalence classes. Another (more efficient) approach is to observe that every 2-polytope of size $n+1$ can be obtained by adding to a 2-polytope of size $n$ one point at lattice distance one from (at least) one facet. This iterative process should allow the reader to easily check that Figure 1 contains the full list of 2-dimensional polygons with up to five points.

![Figure 1. The 2-dimensional polygons with up to five points](image)

In dimension 3 the situation is completely different. In particular, there are empty tetrahedra with arbitrarily large volume. Here, an empty tetrahedron is the same as a 3-polytope of size four: a tetrahedron with integer vertices and no other integer points. Hence, no analogue of Pick’s Theorem is possible and there is an infinite number of equivalence classes of 3-polytopes of given size.

Still, the following three results are valid in arbitrary fixed dimension $d$. In the first one, a hollow polytope is a lattice polytope with no lattice points in its interior.

**Theorem** (Nill-Ziegler [8, Thm. 1.2]). There is only a finite number of hollow $d$-polytopes that do not admit a lattice projection onto a hollow $(d-1)$-polytope.

**Theorem** (Hensley [5, Thm. 3.4]). For each $k \geq 1$ there is a number $V(k,d)$ such that no lattice $d$-polytope with $k$ interior lattice points has volume above $V(k,d)$.

**Theorem** (Lagarias-Ziegler [7, Thm. 2]). For each $V \in \mathbb{N}$ there is only a finite number of integral equivalence classes of $d$-polytopes with volume $V$ or less.

Combining them with the fact that there is a unique hollow 2-polytope of width larger than one, we get:

**Corollary 1.1.** There are finitely many lattice 3-polytopes of width greater than one for each size $n$. 


Proof. Once we fix $n$, every lattice 3-polytope $P$ with $n$ lattice points falls in one of the following (not mutually exclusive) categories:

- It is not hollow. In this case Hensley’s Theorem gives a bound for its volume. This, in turn, implies finiteness via the Lagarias-Ziegler Theorem.
- It is hollow, but does not project to a hollow 2-polytope. These are a finite family, by the Nill-Ziegler Theorem.
- It is hollow, and it projects to a 2-polytope of width 1. This implies that $P$ itself has width 1.
- It is hollow and it projects to a hollow 2-polytope of width larger than one. The only such 2-polytope is the second dilation of a unimodular triangle. It is easy to check that only finitely many (equivalence classes of) 3-polytopes of size $n$ project onto it: let $P = \text{conv}\{p_1,\ldots, p_n\}$ be a 3-polytope of size $n$ that projects onto $T = \text{conv}\{(0,0), (2,0), (0,2)\}$.

We must have at least one point projecting to each vertex of $T$. That is: there are $p_1 = (0,0,z_1)$, $p_2 = (2,0,z_2)$ and $p_3 = (0,2,z_3)$ in $P$. The unimodular transformation
\[
(x, y, z) \mapsto \left( x, y, z_1 - x \left\lfloor \frac{z_2 - z_1}{2} \right\rfloor - y \left\lfloor \frac{z_3 - z_1}{2} \right\rfloor \right)
\]
allows us to assume that $z_1, z_2, z_3 \in \{0,1\}$. This implies that $P \subset T \times [1-n, n]$, so there is a finite number of possibilities for $P$.

So, it makes sense to classify, for each size $n$, separately the 3-polytopes of width one and those of width larger than one. The latter are a finite list. The former are an infinite set, but easy to describe: they consist of two 2-polytopes of sizes $n_1$ and $n_2$ ($n_1 + n_2 = n$) placed on parallel consecutive lattice planes. For each of the two subconfigurations there is a finite number of possibilities, but infinitely many ways to “rotate” (in the integer sense, that is via an element of $SL(\mathbb{Z}, 2)$) one with respect to the other.

For example, it is a now classical result that all empty tetrahedra have width one. From this, the classification of empty tetrahedra (stated in Theorem 2.4 below) follows easily.

Theorem (White [12, Thm. 1]). Every lattice 3-polytope of size four has width one with respect to (at least) one of its three pairs of opposite edges.

Our main result in this paper is:

Theorem 1.2. There are exactly nine 3-polytopes of size 5 and width two and none of larger width. Eight of them have signature $(4,1)$ and one has signature $(3,1)$.

By the signature of a 3-polytope of size 5 we mean the type of its unique Radon partition. Remember that a set $A$ of $d+2$ points affinely spanning $\mathbb{R}^d$ have a unique (modulo a scalar factor) affine dependence. The Radon partition of $A$ is the set of points with non-zero coefficient in this dependence, separated in two parts according to the sign of their coefficients. Equivalently, it is the unique pair of disjoint subsets $A^+$ and $A^-$ of $A$ such that the relative interiors of $\text{conv}(A^+)$ and $\text{conv}(A^-)$ intersect. The signature $(|A^+|, |A^-|)$ of $d+2$ points in dimension $d$ carries the same information as their order type or oriented matroid. Signatures $(a,b)$ and $(b,a)$ are equivalent, and the five possible signatures of five points in $\mathbb{R}^3$ are $(4,1), (3,2), (2,2), (3,1)$ and $(2,1)$. (See more details on this in Section 2).
More precisely, our classification says:

**Theorem 1.3.** Every lattice 3-polytope of size 5 is unimodularly equivalent to one of the configurations listed in Table 1. The table is irredundant: configurations in different rows, or configurations obtained for different choices of parameters within each row, are inequivalent.

The table includes, apart from a representative for each class, three invariants of the class: its width, its signature and its volume vector. The volume vector is a vector in $\mathbb{Z}^5$ recording the volumes of the (perhaps degenerate) tetrahedra spanned by each subset of four of the five points. We give volume vectors with a sign convention that makes them have as many positive and negative entries as given by the signature.

| Sign. | Volume vector | W. | Representative |
|-------|---------------|----|---------------|
| (2, 2) | $(-1, 1, 1, -1, 0)$ | 1 | $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1)$ |
| (2, 1) | $(-2q, 0, q, 0, 0)$ | 1 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (-1, 0, 0), (p, q, 1)$ |
| $(-a - b, a, b, 1, -1)$ | $0 < a \leq b$, $\gcd(a, b) = 1$ | 1 | $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (a, b, 1)$ |
| (3, 1)* | $(-3, 1, 1, 0, 0)$ | 1 | $(0, 0, 0), (1, 0, 0), (0, 1, 0), (-1, -1, 0), (0, 0, 1)$ |
| $(-9, 3, 3, 0)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 1, 0), (-1, -1, 0), (1, 2, 3)$ |
| (4, 1)* | $(-4, 1, 1, 1, 1)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 1, 1), (-2, -1, -2)$ |
| $(-5, 1, 1, 1, 2)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 2, 1), (-1, -1, -1)$ |
| $(-7, 1, 1, 2, 3)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (1, 3, 1), (-1, -2, -1)$ |
| $(-11, 1, 3, 2, 5)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (2, 5, 1), (-1, -2, -1)$ |
| $(-13, 3, 4, 1, 5)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (2, 5, 1), (-1, -1, -1)$ |
| $(-17, 3, 5, 2, 7)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (2, 7, 1), (-1, -2, -1)$ |
| $(-19, 5, 4, 3, 7)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (3, 7, 1), (-2, -3, -1)$ |
| $(-20, 5, 5, 5, 5)$ | 2 | $(0, 0, 0), (1, 0, 0), (0, 0, 1), (2, 5, 1), (-3, -5, -2)$ |

**Table 1.** Complete classification of lattice 3-polytopes of size 5.

Those marked with an * are dps

The key to our classification is Theorem 3.3 which says that every 3-polytope of size 5 is in one of the following three classes: (a) it has width one, (b) it has width two with respect to a functional with certain specific properties, or (c) it has signature $(4, 1)$ and a very symmetric volume vector $(-4q, q, q, q)q$ for some $q$. These three types of polytopes are classified, respectively, in Theorem 3.2, Theorems 3.4 and 3.5, and Theorem 3.6.

Let us mention that part of Theorem 1.2 was already known:

- Polytopes of signatures $(2, 2)$ and $(3, 2)$ have width 1 by the following result of Howe [11, Thm. 1.3]: Every lattice 3-polytope with no lattice points other than its vertices has width 1.
- Polytopes of signature $(4, 1)$, which are the same as “clean tetrahedra with a single interior point”, were classified by Kasprzyk [6] and Reznick [9, Thm. 7], who obtained the same list as us.

The classification of polytopes of signatures $(2, 1)$ and $(3, 1)$ is, as far as we know, new.
Our motivation comes partially from the notion of distinct pair-sum lattice polytopes (or dps polytopes, for short), defined as lattice polytopes in which all the pairwise sums \( a + b, a, b \in P \cap \mathbb{Z}^d \) are distinct. Equivalently, they are lattice polytopes containing no three collinear lattice points nor the vertices of a non degenerate parallelogram [3, Lemma 1]. They are also the lattice polytopes of Minkowski length equal to one, in the sense of [1].

Dps \( d \)-polytopes cannot have two lattice points in the same class modulo \((2\mathbb{Z})^d\), hence they have size at most \(2^d\). The nine polytopes of width greater than one stated in Theorem 1.2 are all dps. In fact, for \( d \)-polytopes of size \( d + 2 \), dps is equivalent to the signature not being one of \((2,1)\) or \((2,2)\).

In a subsequent paper [2] we undertake the complete classification of 3-polytopes of size six, obtaining:

**Theorem 1.4 ([2]).** There are exactly 76 3-polytopes of size 6 and width > 1: 74 have width 2 and 2 have width 3. 44 and 1 of those, respectively, are dps.

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2. Preliminaries

2.1. Volume vectors. Since \(\mathbb{Z}\)-equivalence preserves volume, the following is a useful invariant for our classification:

**Definition 2.1.** Let \( A = \{p_1, p_2, \ldots, p_n\} \), with \( n \geq d + 1 \), be a set of lattice points in \( \mathbb{Z}^d \). The volume vector of \( A \) is the vector

\[
w = (w_{i_1 \ldots i_{d+1}})_{1 \leq i_1 < \cdots < i_{d+1} \leq n} \in \mathbb{Z}^{\binom{n}{d+1}}
\]

where

\[
w_{i_1 \ldots i_{d+1}} := \det \begin{pmatrix} 1 & \ldots & 1 \\ p_{i_1} & \ldots & p_{i_{d+1}} \end{pmatrix}.
\]

**Remark 2.2.** The volume vector encodes the unique (modulo a scalar factor) dependence among each set of \( d + 2 \) points \( \{p_{i_1}, \ldots, p_{i_{d+2}}\} \) in \( A \) that affinely span \( \mathbb{R}^d \), which is:

\[
\sum_{k=1,\ldots,d+2} (-1)^{k-1} \cdot w_{I_k} \cdot p_{i_k} = 0
\]

\[
\sum_{k=1,\ldots,d+2} (-1)^{k-1} \cdot w_{I_k} = 0
\]

where \( I_k = \{i_1, \ldots, i_{d+2}\} \setminus \{i_k\} \).

Observe that the definition of volume vector implicitly assumes a specific ordering of the \( n \) points in \( A \). When we say that the volume vector is \(\mathbb{Z}\)-equivalence invariant, this ordering (and the fact that the sign of each volume entry depends on the ordering) has to be taken into account. We now look at the converse question: if two point sets of the same size have the same volume vector, are they necessarily \(\mathbb{Z}\)-equivalent? The answer is *almost* yes.
Theorem 2.3. Let \( A = \{p_1, \ldots, p_n\} \) and \( B = \{q_1, \ldots, q_n\} \) be \( d \)-dimensional sets of lattice points in \( \mathbb{Z}^d \) and suppose they have the same volume vector
\[
 w = (w_I)_{I \in \binom{[d+1]}{d}}
\]
with respect to the given ordering of the points. Then:

1. There is a unique affine map \( f : \mathbb{R}^d \to \mathbb{R}^d \) with \( \det(f) = 1 \) and \( f(p_i) = q_i \) for all \( i = 1, \ldots, n \).

2. If \( \gcd_{I \in \binom{[d+1]}{d}} (w_I) = 1 \) then \( f \) has integer coefficients. Hence, it is a \( \mathbb{Z} \)-equivalence between \( P \) and \( Q \).

Proof. Without loss of generality we may assume that \( w_1, \ldots, w_{d+1} \neq 0 \). This means that \( \text{conv}\{p_1, \ldots, p_{d+1}\} \) and \( \text{conv}\{q_1, \ldots, q_{d+1}\} \), are \( d \)-simplices of the same volume.

Let \( f : \mathbb{R}^d \to \mathbb{R}^d \) be the unique affine map with \( f(p_i) = q_i \) for \( i \in \{1, \ldots, d+1\} \). Equality of volumes (with sign) implies that \( \det(f) = 1 \). We claim that \( f(p_i) = q_i \) also for \( i > d + 1 \).

To show this, simply observe that for each point \( p_i \) with \( i > d + 1 \) the affine dependence on \( \{p_1, \ldots, p_{d+1}, p_i\} \) (which is encoded in the volume vector of \( A \)) allows to write \( p_i \) as an affine combination of \( \{p_1, \ldots, p_{d+1}\} \). Since \( f \) preserves affine combinations, \( f(p_i) = q_i \). This finishes the proof of part (1).

For part (2), let \( \Lambda(A), \Lambda(B) \leq \mathbb{Z}^d \) be the affine sublattices spanned respectively by \( A \) and \( B \). Since \( f \) maps \( A \) to \( B \), it maps \( \Lambda(A) \) to \( \Lambda(B) \). The index \([\mathbb{Z}^d : \Lambda(A)]\) is the minimal volume (with respect to \( \mathbb{Z}^d \)) of a basis of \( \Lambda(A) \). Thus the index divides \( w_I \) for all \( I \in \binom{[d+1]}{d} \), and therefore it divides \( \gcd(w_I) \). In particular, if \( \gcd(w_I) = 1 \), then \( \Lambda(A) = \mathbb{Z}^d = \Lambda(B) \). This implies \( f \) maps \( \mathbb{Z}^d \) to itself, so it has integer coefficients. \( \square \)

In the particular case \( n = d + 2 \) (the main interest in this paper), it is more natural to choose as signs for the entries in the volume vector not those coming from the ordering of the points but those coming from the unique dependence among them. We work out this approach at the beginning of Section 3.

2.2. Empty tetrahedra. To review the classification of empty tetrahedra in dimension three one can take as a starting point the fact that every empty tetrahedron has width one with respect to a functional that is constant in two opposite edges. This easily implies:

Theorem 2.4 (Classification of empty tetrahedra, White [12]). Every empty lattice tetrahedron of volume \( q \) is unimodularly equivalent to the following \( T(p, q) \), for some \( p \in \{0, \ldots, q - 1\} \) with \( \gcd(p, q) = 1 \):
\[
 T(p, q) = \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 0, 1), (p, q, 1)\}
\]
Moreover, \( T(p, q) \) is equivalent to \( T(p', q) \) if and only if \( p' = \pm p^{\pm 1} \pmod{q} \).

Definition 2.5. We say that an empty tetrahedron \( T \) in a 3-dimensional lattice \( \Lambda \) is of type \( T(p, q) \) if there is a lattice isomorphism \( \Lambda \cong \mathbb{Z}^3 \) sending \( T \) to \( T(p, q) \).

For some of our results it will be convenient to make a change of coordinates so that instead of having a tetrahedron of volume \( q \) with respect to \( \mathbb{Z}^3 \) we have a tetrahedron whose vertices span \( \mathbb{Z}^3 \) as an affine lattice but we consider it with respect to a finer lattice. The following transformation to achieve this is worked out, for example, in [10].
Let $T = \text{conv}\{p_1, p_2, p_3, p_4\}$ be an empty tetrahedron in $\mathbb{Z}^3$ of a certain volume $q > 1$ and, without loss of generality, assume $T$ has width one with respect to the pair of edges $p_1p_2$ and $p_3p_4$. Consider also the standard tetrahedron

$$T_0 = \text{conv}\{o = (0, 0, 0), e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\},$$

of volume 1. Then there is an affine transformation $\lambda$ sending $T$ to $T_0$ and, more precisely, with

$$\lambda(p_i) = e_i, \ i = 1, 2, 3; \quad \lambda(p_4) = o.$$

Let $\Lambda$ be the image of the lattice $\mathbb{Z}^3$ by this transformation, which is a superlattice of $\mathbb{Z}^3$ of index $q$. Clearly, two empty tetrahedra that produce the same lattice $\Lambda$ are $\mathbb{Z}$-equivalent. We are going to derive the possibilities for this lattice $\Lambda$.

Since $T$ has width one with respect to the edges $p_1p_2$ and $p_3p_4$, we have that

$$\Lambda \subset \{(x, y, z) \in \mathbb{R}^3 : x + y \in \mathbb{Z}\}.$$

Let $H_i := \{(x, y, z) \in \mathbb{R}^3 : x + y = i\}$, for each $i \in \mathbb{Z}$. The different slices $\Lambda \cap H_i$ are obtained from one another by integer translation, so understanding one of them is enough to understand $\Lambda$. We look at the slice

$$\Lambda_0 := \Lambda \cap H_0 = \{(x, y, z) \in \Lambda : x + y = 0\}.$$

$\Lambda_0$ is a superlattice of $\mathbb{Z}^3 \cap H_0$ with index $q$. In particular, the rectangle

$$R := \text{conv}\{(0, 0, 0), (0, 0, 1), (1, -1, 0), (1, -1, 1)\},$$

which is a fundamental rectangle (that is, of lattice area 2) with respect to $\mathbb{Z}^3 \cap H_0$, has area $2q$ with respect to $\Lambda_0$. Moreover, $\Lambda_0$ contains no non-integer points in the integer vertical and horizontal lattice lines of $\mathbb{Z}^3 \cap H_0$, (in particular, on the edges of $R$), since each primitive integer segment along these lines is a lattice translation of $0e_3$ or $e_1e_2$. This implies $\gcd(p, q) = 1$ and also that, by Pick’s formula (1), $R$ contains exactly $q - 1$ lattice points of $\Lambda_0$ in its interior. No two of these points can have one coordinate equal, because otherwise we would have a horizontal or vertical lattice segment in $\Lambda_0$ of length smaller than one, in contradiction to the fact that $o e_3$ and $e_1 e_2$ are primitive in $\Lambda$. As a conclusion:

**Lemma 2.6.** For each $i = 1, \ldots, q - 1$, $R$ contains exactly one point of $\Lambda_0$ with $z = i/q$ and one (which may or may not be the same) with $x = -y = i/q$.

Let $p \in \{1, \ldots, q - 1\}$ for which $(p/q, -p/q, 1/q) \in \Lambda_0$ and $p' \in \{1, \ldots, q - 1\}$ for which $(1/q, -1/q, p'/q) \in \Lambda_0$. Knowing $p$ or $p'$ is enough to recover $\Lambda_0$, and $\Lambda$. More precisely we will have:

$$\Lambda = \langle (p/q, -p/q, 1/q) \rangle + \mathbb{Z}^3 = \langle (1/q, -1/q, p'/q) \rangle + \mathbb{Z}^3.$$

Observe that these two values are not independent. Indeed, if $(a/q, -a/q, b/q)$ lies in $\Lambda_0$ then $pb \equiv a \pmod{q}$ so, in particular, $p' = p^{-1} \pmod{q}$. We leave it to the reader to check that the values of $p$ and $q$ in this description are the same as in Theorem 2.4. That is, if the tetrahedron $T(p, q)$ is sent by an affine map to the standard tetrahedron (with the conventions for the ordering of vertices set up above) then $\mathbb{Z}^3$ is mapped to the following lattice that we call $\Lambda(p, q)$:

$$\Lambda(p, q) := \mathbb{Z}^3 + \langle (p/q, -p/q, 1/q) \rangle = \langle (1, 0, 0) \rangle \oplus \langle (1, -1, 0), (p/q, -p/q, 1/q) \rangle.$$

We call $R(p, q) := \text{conv}\{(0, 0, 0), (0, 0, 1), (1, -1, 0), (1, -1, 1)\}$ the fundamental rectangle of $T(p, q)$. Observe that all lattice points of $\Lambda(p, q)$ lie in an integer translation of $R(p, q)$, as illustrated in Figure 2.
In Section 3 we will need the following result about the fundamental rectangle:

**Lemma 2.7.** Let \( q \geq 2 \) and let \( p \in \{1, \ldots, q - 1\} \) with \( \gcd(p, q) = 1 \). Let \( R(p, q) := \text{conv}\{(0, 0, 0), (0, 0, 1), (1, -1, 0), (1, -1, 1)\} \) be the fundamental rectangle and consider the following two triangles contained in it: \( t_1 = \text{conv}\{(0, 0, 0), (1, -1, 0), (1, -1, 1/2)\} \) and \( t_2 = \text{conv}\{(0, 0, 0), (1, -1, 0), (1/2, -1/2, 1/2)\} \) (see Figure 3). Then, the point \( (p/q, -p/q, 1/q) \in \Lambda(p, q) \) lies in \( t_2 \) for every \( (p, q) \) and it lies in \( t_1 \) if \( p \geq 2 \).

**Figure 3.** The triangles \( t_1 \) and \( t_2 \) (projected to the plane \( XZ \))

Simplices of type \( T(1, q) \) are somehow special: the fundamental rectangle has all lattice points in the diagonal, and they have width one with respect to two different pairs of opposite edges. They are called tetragonal in [10]. \( T(2, 1) \) is even more special: it has width one with respect to any of the three pairs of opposite edges.

For future reference we include the following statement which can be read as “no vertex of an empty tetrahedron is more special than the others”.

**Lemma 2.8.** Let \( T_0 \) be the standard simplex. Let \( \Lambda(p, q) \) be the lattice of type \((p, q)\), for some \( 1 \leq p < q \). Then, for each \( i \in \{1, 2, 3\} \) there is a \( \mathbb{Z} \)-automorphism \( f_i \) of \( T_0 \) sending \( e_i \) to \( o \) and mapping \( \Lambda(p, q) \) either to itself or to \( \Lambda(p', q) \), where \( p' \equiv p^{-1} \pmod{q} \).

**Proof.** We give the explicit transformation for each case. For \( i = 1 \), the affine map \( f_1(x, y, z) = (1-(x+y+z), z, y) \) is an automorphism of \( \mathbb{Z}^3 \) and maps \((p/q, -p/q, 1/q)\) to \((1-1/q, 1/q, -p/q)\). Thus

\[
    f_1(\Lambda(p, q)) = \mathbb{Z}^3 + (1/q, -1/q, p/q) = \mathbb{Z}^3 + (p'/q, -p'/q, 1/q) = \Lambda(p', q).
\]
Similarly, for \( i = 2 \) the \( \mathbb{Z}^3 \)-automorphism \( f_2(x, y, z) = (z, 1-(x+y+z), x) \) maps \( (p/q, -p/q, 1/q) \) to \( (1/q, 1-1/q, p/q) \), so

\[
f_2(\Lambda(p, q)) = \mathbb{Z}^3 + (1/q, -1/q, p/q) = \mathbb{Z}^3 + (p'/q, -p'/q, 1/q) = \Lambda(p', q).
\]

Finally, for \( i = 3 \) the \( \mathbb{Z}^3 \)-automorphism \( f_3(x, y, z) = (y, x, 1-(x+y+z)) \) maps \( (p/q, -p/q, 1/q) \) to \( (0, 0, 1)-(p/q, -p/q, 1/q) \). Hence \( f_3(\Lambda(p, q)) = \Lambda(p, q) \). \( \square \)

**Remark 2.9.** Let us consider, as in the proof of this lemma, the standard tetrahedron \( T_0 = \{a, e_1, e_2, e_3\} \) and the lattice \( \Lambda(p, q) \). The transformation \( f_3 \) in the proof (exchanging \( o \leftrightarrow e_3 \) and \( e_1 \leftrightarrow e_2 \)) is the only affine map, other than the identity, sending \( T_0 \) to itself and preserving \( \Lambda(p, q) \) for every \( p \) and \( q \). The other 22 symmetries of \( T_0 \) are automorphisms of \( \Lambda(p, q) \) only for particular values of \( (p, q) \).

This means that the sentence “no vertex of an empty tetrahedron is more special than the others” is not true if we fix a particular class \( T(p, q) \subset \mathbb{Z}^3 \) of simplices. If we want to stay within a particular class \( T(p, q) \), and in this class \( p \neq p^{-1} \) (mod \( q \)), then vertices \( (0, 0, 0) \) and \( (1, 0, 0) \) are in one orbit of the unimodular automorphism group of \( T(p, q) \) and \( (0, 0, 1) \) and \( (p, q, 1) \) in another.

Among the methods we use to elaborate the classification, we often have a list of tetrahedra and we need to check that they are empty. Of course, one necessary condition is that all facet triangles are empty, which is equivalent to “unimodular in the lattice planes containing them”. Once we know the three points of a unimodular facet of our tetrahedron \( T \), then there is an affine transformation sending these three points to \( (0, 0, 0), (1, 0, 0) \) and \( (0, 1, 0) \); let \( (a, b, q) \) be the image of the fourth point. (Note: the transformation is not unique, but \( q \) is unique up to a sign, since it equals the volume of \( T \), and \( a \) and \( b \) are determined modulo \( q \)). Then, our problem reduces to knowing when

\[
\text{conv}\{0, 0, 0, 1, 0, 0, 1, 0, (a, b, q)\}
\]

is an empty tetrahedron. The following lemma gives us the answer:

**Lemma 2.10.** The lattice tetrahedron \( T = \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (a, b, q)\} \) is empty (with respect to the integer lattice \( \mathbb{Z}^3 \)) if, and only if, at least one of the following happens:

(i) \( a \equiv 1 \) (mod \( q \)) and \( \gcd(b, q) = 1 \).

(ii) \( b \equiv 1 \) (mod \( q \)) and \( \gcd(a, q) = 1 \).

(iii) \( a + b \equiv 0 \) (mod \( q \)) and \( \gcd(a, q) = 1 \).

**Proof.** Assume without loss of generality that \( q > 0 \). \( T \) is an empty tetrahedron if, and only if, all its edges are primitive and its width equals one with respect to one of the three pairs of edges. In our case, primitivity of edges is equivalent to

\[
\gcd(a, b, q) = \gcd(a-1, b, q) = \gcd(a, b-1, q) = 1
\]

Let us examine the width with respect to the three pairs of edges, depending on the values of \( a \) and \( b \). For this, we compute the linear functional taking value 0 on one edge and value 1 on the other. Width is one with respect to that pair of edges if and only if this functional is integer.

(i) The functional taking value 0 on the segment \( (0, 0, 0)(0, 1, 0) \) and value 1 on \( (1, 0, 0)(a, b, q) \) is \( x + \frac{1-a}{q}z \), which is integer if and only if \( a \equiv 1 \) (mod \( q \)). If this happens, the primitivity conditions (3) become equivalent to \( \gcd(b, q) = 1 \).
(ii) The functional taking value 0 on \((0, 0, 0)(1, 0, 0)\) and value 1 on \((0, 1, 0)(a, b, q)\) is \(y + \frac{b}{q}z\), which is integer if and only if \(b \equiv 1 \pmod{q}\). If this happens, the primitivity conditions become \(\gcd(a, q) = 1\).

(iii) The functional taking value 0 on \((0, 0, 0)(a, b, q)\) and value 1 on \((1, 0, 0)(0, 1, 0)\) is \(x + y - \frac{a}{q}z\) which is integer if and only if \(a \equiv 0 \pmod{q}\). If this happens, the primitivity conditions become \(\gcd(a, q) = 1\).

□

3. Polytopes with five lattice points

Any set \(A = \{p_1, \ldots, p_{d+2}\} \subset \mathbb{R}^d\) of \(d + 2\) points affinely spanning \(\mathbb{R}^d\) have a unique affine dependence. The order type or oriented matroid of \(A\) is fully characterized by how many coefficients in this dependence have each sign (positive, negative, or zero). The corresponding subsets of \(A\) form the Radon partition of \(A\). (See more details in [4]). We say that \(A\) has signature \((i, j)\) if its unique affine dependence has \(i\) positive and \(j\) negative coefficients.

Since unimodularly equivalent configurations are, in particular, affinely equivalent, they have the same order type. For five points in dimension three the five possibilities for \((i, j)\) are \((2, 1), (2, 2), (3, 2), (3, 1)\) and \((4, 1)\), depicted in Figure 4. (Observe that \((i, j)\) and \((j, i)\) are the same signature).

![Figure 4. The five possible signatures (oriented matroids) of five different points in \(\mathbb{R}^3\)](image)

For five points in dimension three we modify our sign and order conventions for writing the volume vector, in order to make the signature (and its correspondence to subsets of \(A\)) more explicit. More precisely, we take as volume vector for five points \(p_1, \ldots, p_5\) the vector \((v_1, v_2, v_3, v_4, v_5)\) where

\[
\sum v_i p_i = 0, \quad \sum v_i = 0
\]

is the unique affine dependence on \(A\), normalized so that \(|v_i| = \text{vol}(\text{conv}(A \setminus \{i\}))\).

In particular, the signature of \(A\) equals the number of positive and negative entries in the volume vector. Put differently (see Remark 2.2),

\[
(v_1, v_2, v_3, v_4, v_5) = (w_{2345}, -w_{1345}, w_{1245}, -w_{1235}, w_{1234})
\]

where \(w_{ijkl}\) is as in Equation 2.

3.1. Polytopes of signature \((2, \ast)\). It turns out that every lattice 3-polytope \(P\) of size five and signature \((2, \ast)\) has width one:

**Theorem 3.1.** If \(P\) is a lattice 3-polytope of size 5 and with signature \((3, 2)\), \((2, 2)\) or \((2, 1)\), then \(P\) has width one.
Proof. Let $A = \{p_1, p_2, p_3, p_4, p_5\}$ be the lattice point set of a polytope $P = \text{conv} A$ of size five and signature $(2, \ast)$. We assume the points are ordered so that the volume vector $v = (v_1, v_2, v_3, v_4, v_5)$ of $A$ verifies $v_i < v_4 < v_5$, $i = 1, 2, 3$.

Consider the empty tetrahedron $T = \text{conv}\{p_1, p_2, p_3, p_4\}$, of type $T(p, q)$ for $q = v_5$ and $p \in \{1, \ldots, q\}$. By symmetry on $p_1, p_2$ and $p_3$, there is no loss of generality in assuming $T$ to have width one with respect to the pair of edges $p_1p_2$ and $p_3p_4$.

Observe that the barycentric coordinates of $p_5$ with respect to $T$ are $(\frac{-v_1}{v_5}, \frac{-v_2}{v_5}, \frac{-v_3}{v_5}, \frac{-v_4}{v_5})$.

By our hypotheses, the first three are non-negative and the last one is between $-1$ and $0$. This implies that $p_5$ lies in the homothetic dilation of factor two of $T$ from point $p_4$, and it suggests we use the change of coordinates $\lambda$ of Section 2.2. That is, we consider $T$ to be the standard tetrahedron with $p_4 = (0, 0, 0)$, $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$, $p_3 = (0, 0, 1)$, in the lattice $\Lambda(p, q)$.

Then $p_5 \in 2T$, which implies $A$ to have width one with respect to the functional $x + y$ unless $p_5$ turns out to be one of the three lattice points in $2T$ with $x + y = 2$, namely $(2, 0, 0)$, $(1, 1, 0)$ and $(0, 2, 0)$ (see Figure 5). Let us see what happens in these three cases:

- If $p_5 = (2, 0, 0)$ then the intersection of $\text{conv} A$ with the fundamental rectangle contained in $2T$ is the triangle $\text{conv}((1, 0, 0), (0, 1, 0), (1, 0, 1/2))$. By (a translated version of) Lemma 2.7, for this triangle not to contain additional lattice points of $\Lambda(p, q)$ we need $p = 1$. But in this case $T$ has width one with respect also to the functional $y + z$, and so does $P$.
- The case $p_5 = (0, 2, 0)$ is analogous, exchanging the roles of $x$ and $y$.
- If $p_5 = (1, 1, 0)$ then the intersection of $\text{conv} A$ with the fundamental rectangle contained in $2T$ is the triangle $\text{conv}((1, 0, 0), (0, 1, 0), (1/2, 1/2, 1/2))$. Now Lemma 2.7 implies that this triangle has additional lattice points, no matter the value of $p$, unless $q = 1$. But if $q = 1$ then $P$ has width one with respect to $z$.

![Figure 5. The second dilation of an empty tetrahedron](image-url)
The classification of configurations of width one is easy:

**Theorem 3.2.** Let $P$ be a lattice polytope of size five and width one. Then $P$ is unimodularly equivalent to one of the following:

1. $\text{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,0),(0,0,1)\}$, of signature $(2,2)$.
2. $\text{conv}\{(0,0,0),(1,0,0),(0,1,0),(-1,-1,0),(0,0,1)\}$, of signature $(3,1)$.
3. $\text{conv}\{(0,0,0),(1,0,0),(0,0,1),(-1,0,0),(p,q)\}$, for some $0 \leq p \leq \lfloor q/2 \rfloor \in \mathbb{Z}$ with $\gcd(p,q) = 1$. This is of signature $(2,1)$.
4. $\text{conv}\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(a,b,1)\}$, for some $0 < a \leq b$, with $\gcd(a,b) = 1$. This is of signature $(3,2)$.

Moreover, two such polytopes are never equivalent to one another.

**Proof.** Width one means the 5 lattice points of $P$ lie in two consecutive lattice planes. Say $n_0$ points are in $\{z = 0\}$ and $5 - n_0$ in $\{z = 1\}$ with $n_0 \geq 5 - n_0$. That is, $n_0 \in \{3,4\}$.

- If $n_0 = 4$, then the four points at $z = 0$ form one of the three 2-dimensional polytopes of size 4 displayed in the right top row of Figure 1, of signatures $(3,1)$, $(2,2)$ and $(2,1)$ respectively, and the position of the fifth point (within the plane $z = 1$) does not affect the $\mathbb{Z}$-equivalence class of $P$. The cases $(2,2)$ and $(3,1)$ are the configurations of parts (1) and (2). The case $(2,1)$ is the configuration

  \[ \text{conv}\{(0,0,0),(1,0,0),(-1,0,0),(0,1,0),(0,0,1)\} \]

  which, via the map $(x,y,z) \mapsto (x,y,y+z)$, becomes that of part (3) with $(p,q) = (0,1)$.

- If $n_0 = 3$, then $P$ has three points in the lattice plane $z = 0$, and two in the next plane $z = 1$. There are two possibilities:
  - If the three points at $z = 0$ are collinear, without loss of generality we can assume they are $(-1,0,0)$, $(0,0,0)$ and $(1,0,0)$. One of the points at $z = 1$ can be assumed to be $(0,0,1)$ and the fifth point has coordinates $(p,q,1)$ with $q \neq 0$ (in order to be full-dimensional) and $\gcd(p,q) = 1$ (in order for the edge at $z = 1$ to be primitive). Reflection of the planes $x = 0$ and $y = 0$, if needed, allows us to assume that $q > 0$ and $p \in \{0,1,\ldots,\lfloor q/2 \rfloor\}$ (mod $q$). The map $(x,y,z) \mapsto (x+y,p,q,y,z)$ allows us to, moreover, assume $0 \leq p \leq \lfloor q/2 \rfloor$.
  - If the three points at $z = 0$ are not collinear then they form a unimodular triangle, and without loss of generality we assume they are $(0,0,0)$, $(1,0,0)$ and $(0,1,0)$. One of the points at $z = 1$ can be assumed to be $(0,0,1)$ and the fifth point has coordinates $(a,b,1)$. By the same argument as before, we need $\gcd(a,b) = 1$ and by symmetries with respect to the triangle at $z = 0$ we can assume $0 \leq a \leq b$ (details are left to the reader).

This configuration has volume vector $(-(a+b),a,b,1,-1)$, so it is of type $(3,2)$ unless $a = 0$ (and hence $b = 1$ since $\gcd(0,b) = b$). In the case $(a,b) = (0,1)$ we recover the configuration of part (1), with coordinates $x$ and $z$ exchanged.

This finishes the case study, but we still need to check that different configurations in the list are not equivalent. Configurations of different signature are certainly inequivalent. Within those of signature $(3,2)$, since the volume vector
is primitive. Theorem 2.3 says that different values of \((a, b)\) produce inequivalent configurations. In signature \((2, 1)\), however, the volume vector is \((q, q, 0, 0, -2q)\) so, a priori, configurations with different \(p\) and the same \(q\) could still be equivalent. Let us prove that they are not.

For this, let \(q\) be fixed and let \(p, p' \in \mathbb{Z}\). Let \(P\) and \(P'\) be two of these configurations having \((p, q, 1)\) and \((p', q, 1)\) as their fifth point, respectively. All affine transformations that map \(P\) to \(P'\) must preserve the collinearity of the three points at \(z = 0\), so they fix \((0, 0, 0)\) and either fix or exchange \((1, 0, 0)\) and \((-1, 0, 0)\). Similarly, they either fix \((0, 0, 1)\) and send \((p, q, 1)\) to \((p', q, 1)\), or they send \((p, q, 1)\) to \((0, 0, 1)\) and \((0, 0, 1)\) to \((p', q, 1)\). So we have the following four possibilities:

\[
\begin{align*}
(x, y, z) &\mapsto \left(x + \frac{p'-p}{q}y, y, z\right), & (x, y, z) &\mapsto \left(x + \frac{-p'-p}{q}y + p'z, -y + qz, z\right), \\
(x, y, z) &\mapsto \left(-x + \frac{p'-p}{q}y, y, z\right), & (x, y, z) &\mapsto \left(-x + \frac{-p'-p}{q}y + p'z, -y + qz, z\right).
\end{align*}
\]

For any of them to be integer we need \(p \equiv \pm p' \pmod{q}\).

3.2. Polytopes of signature \((*, 1)\). In this case width can be two, but no more. Our first result proves this except in one special case, to be studied separately:

**Theorem 3.3.** If \(P\) is a lattice 3-polytope of size 5, then one of the following three things happens:

1. \(P\) has width one.
2. \(P\) has width two with respect to a functional that takes the values \(1, 1, 0, 0, -1\) in the five points (in this order), where the first four points form the empty tetrahedron of largest volume.
3. \(P\) has signature \((4, 1)\) and volume vector \((q, q, q, -4q)\) for some \(q \in \mathbb{N}\).

**Proof.** We assume \(P\) to be of signature \((*, 1)\) (since polytopes of other signatures are of width 1 by Theorem 3.1). Let \(A = \{p_1, p_2, p_3, p_4, p_5\}\) be the lattice point set of our polytope \(P = \text{conv} A\) and assume the points are ordered so that the volume vector \(v = (v_1, v_2, v_3, v_4, v_5)\) of \(A\) verifies \(v_4 < 0 \leq v_i \leq v_5\), \(i = 1, 2, 3\). That is, point \(p_4\) lies in the tetrahedron \(\text{conv}\{p_1, p_2, p_3, p_5\} = \text{conv}(A)\) (this corresponds to the “1” in the signature) and the tetrahedron \(T := \text{conv}\{p_1, p_2, p_3, p_4\}\) has the maximum volume among the empty tetrahedra in \(A\). We also assume without loss of generality (by symmetry of the conditions so far on the points \(p_1, p_2\) and \(p_3\)) that \(T\) has width one with respect to the edges \(p_1p_2\) and \(p_3p_4\).

The barycentric coordinates of \(p_5\) with respect to \(T\) are

\[
\left(\frac{-v_1}{v_5}, \frac{-v_2}{v_5}, \frac{-v_3}{v_5}, \frac{-v_4}{v_5}\right).
\]

Our hypotheses translate to the first three being in \([-1, 0]\). Thus, under the change of coordinates \(\lambda\) of Section 2.2, sending \(p_4\) to \((0, 0, 0)\) and \(p_1, p_2\) and \(p_3\) to \((1, 0, 0)\), \((0, 1, 0)\), \((0, 0, 1)\), \(p_5\) goes to a point in the negative unit cube \([-1, 0]^3\) (see Figure 6). Moreover, by the assumption on the width of \(T\), the functional \(x + y\) is integer on \(\Lambda(p, q) = \lambda(\mathbb{Z}^3)\).

In particular, the value of \(x + y\) at \(p_5\) is one of \(\{0, -1, -2\}\). If \(x + y = 0\) or \(-1\) at \(p_5\) we are in cases (1) and (2) of the statement with respect to the functional \(x + y\). So, we assume the value to be \(-2\), which implies \(p_5 = (-1, -1, 0)\) or \(p_5 = (-1, -1, -1)\).
But \( p_5 = (-1, -1, -1) \) is precisely case (3). In the rest of the proof we assume \( p_5 = (-1, -1, 0) \) and show that we are again in case (2) of the statement.

Observe that the volume vector of \( A \) is \((q,q,0,-3q,q)\), so the volume condition in part (2) of the statement is void. Let us change back to coordinates in which our lattice is \( \mathbb{Z}^3 \). Since \( T := \text{conv}\{p_1,p_2,p_3,p_4\} \) is an empty tetrahedron of width one with respect to \( p_1p_2 \) and \( p_3p_4 \) we take without loss of generality:

\[
p_4 = (0,0,0), \quad p_3 = (1,0,0), \quad p_1 = (0,1,0), \quad p_2 = (p,1,q).
\]

Also, since \( p_4 \) is the centroid of \( p_1, p_2 \) and \( p_5 \), we get \( p_5 = (-p,-2,-q) \). Now, Lemma 2.10 says that in order for the tetrahedron \( \text{conv}\{p_4,p_3,p_1,p_5\} \) to be empty we need one of the following conditions:

- \( p = q - 1 \) (and \( \gcd(2,q) = 1 \)). Then \( A \) is as in the statement for the functional \( x - z \).
- \( p = q - 2 \) (and \( \gcd(2,q) = 1 \)). Same, for the functional \( x + y - z \).
- \( -2 \equiv 1 \mod q \) and \( \gcd(p,q) = 1 \). That is, \( q = 3 \) and \( p \in \{1,2\} \). This is a particular case of one of the two above, depending on whether \( p = 2 \) or 1.

\[
\square
\]

So, we need to analyze the two possibilities (2) and (3) of Theorem 3.3. For case (2) we do the following: taking the functional of the theorem to be the \( z \)-coordinate in the lattice \( \mathbb{Z}^3 \), we assume without loss of generality that the first four points are \( p_1 = (0,0,0) \), \( p_2 = (1,0,0) \), \( p_3 = (0,0,1) \) and \( p_4 = (p,q,1) \), with \( \gcd(p,q) = 1 \) and \( 0 \leq p < q \), since they form an empty tetrahedron. We then look at the possibilities for \( p_5 = (a,b,-1) \) that make \( \text{conv}\{p_1,p_2,p_3,p_4,p_5\} \) not to have extra lattice points. Since all such new lattice points would have to have \( z = 0 \), this turns our question into a two-dimensional problem which we solve case by case.
Let us do this first for configurations of signature (3,1).

**Theorem 3.4.** Every polytope of signature (3,1) and size 5 has volume vector $(-3q, q, q, q, 0)$ with $q \in \{1, 3\}$ and is unimodularly equivalent to one of

1. $\text{conv}\{(0,0,0),(1,0,0),(0,0,1),(-1,0,-1),(0,1,0)\}$ (of width one) or
2. $\text{conv}\{(0,0,0),(1,0,0),(0,0,1),(-1,0,-1),(2,3,1)\}$ (of width two).

**Proof.** By Theorem 3.3 there is no loss of generality in taking the following coordinates for our points:

$$p_1 = (0,0,0), \quad p_2 = (1,0,0), \quad p_3 = (0,0,1), \quad p_4 = (p,q,1), \quad p_5 = (a,b,-1)$$

for some $0 \leq p < q$, $\gcd(p,q) = 1$. Four of the points form a (3,1) circuit. One of them must be $p_5$ and, by Lemma 2.8 there is no loss of generality in assuming that $p_1$ is the centroid of $p_2$, $p_3$ and $p_5$, so $p_5 = (-1,0,-1)$.

The intersection of $P$ with $z = 0$ is then the triangle with vertices

$$p_2 = (1,0,0), \quad p_3 + p_5 = \left(\frac{-1}{2},0,0\right), \quad v = \frac{p_4 + p_5}{2} = \left(\frac{p - 1}{2}, \frac{q}{2}, 0\right).$$

The question is what possibilities for the third vertex $v$ make this triangle not contain any lattice points other than $(0,0,0)$ and $(1,0,0)$. One necessary condition is $q$ to be odd, because if $q$ is even then $\left(\frac{p - 1}{2}, \frac{q}{2}, 0\right)$ itself is a lattice point. (Remember that $\gcd(p,q) = 1$).

![Figure 7](image)

**Figure 7.** The case analysis in the proof of Theorem 3.4. Red squares represent the points $p_1$ and $p_2$ of $P$ in the displayed plane $z = 0$. The red crossed square is the intersection of $p_3p_5$ with that same plane. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge $p_4p_5$ with the plane $z = 0$.

But other conditions are easy to write. For example, in order for $(0,1,0)$ not to be in the triangle, $v$ must be outside the wedge with apex at $(0,1,0)$ and rays in the directions of $(1,2,0)$ and $(-1,1,0)$. (This is the central dark wedge in Figure 7). The same consideration for the other lattice points of the form $(k,1,0)$ defines analogous wedges so that at the end the only half-integer points not excluded by the wedges are those with $q = 1$ or with $q = 3$ and $p = 2 \pmod{3}$. If $q = 1$ then we get width one, and the configuration is equivalent to the first one in the statement. If $q = 3$ then all possibilities for $p$ are equivalent to one another. Taking $p = 2$ we get the second configuration in the statement. □
With similar arguments, but a more involved case study, we get:

**Theorem 3.5.** Apart of those with volume vector of the form \((-4q, q, q, q, q)\), every polytope of size five and signature \((4, 1)\) is equivalent to the one having the points \((0, 0, 0), (1, 0, 0), (0, 0, 1)\) together with one of the following six pairs:

- \((1, 2, 1)\) and \((-1, -1, -1)\), volume vector \((-5, 1, 1, 1, 2)\).
- \((1, 3, 1)\) and \((-1, -2, -1)\), volume vector \((-7, 1, 1, 2, 3)\).
- \((2, 5, 1)\) and \((-1, -2, -1)\), volume vector \((-11, 1, 3, 2, 5)\).
- \((2, 5, 1)\) and \((-1, -1, -1)\), volume vector \((-13, 3, 4, 1, 5)\).
- \((2, 7, 1)\) and \((-1, -2, -1)\), volume vector \((-17, 3, 5, 2, 7)\).
- \((3, 7, 1)\) and \((-2, -3, -1)\), volume vector \((-19, 5, 4, 3, 7)\).

**Proof.** As before, Theorem 3.3 allows us to take the following coordinates for our points:

\[
p_1 = (0, 0, 0), \quad p_2 = (1, 0, 0), \quad p_3 = (0, 0, 1), \quad p_4 = (p, q, 1), \quad p_5 = (a, b, -1),
\]

for some \(0 \leq p < q\), \(\gcd(p, q) = 1\). Without loss of generality (by Lemma 2.8) let \(p_1\) be the interior point.

Then the volume vector of our configuration \(P = \text{conv}\{p_1, p_2, p_3, p_4, p_5\}\) is

\[
((a - 2)q - bp, pb - qa, q + b, -b, q)
\]

with sign vector \((-+, +, +, +, +)\). Since we assumed the normalized volume of \(T = \text{conv}\{p_1, p_2, p_3, p_4\}\) to be the biggest (see part (2) of Theorem 3.3) we have

\[
(4) \quad 0 < -b < q, \quad 0 < pb - qa \leq q.
\]

We want to find out what values of \(a, b, p, q\) make the intersection of \(P\) with \(\{z = 0\}\) not have other lattice points than \(p_1\) and \(p_2\). This intersection must now contain \(p_1\) in its interior (see Figure 8) and it equals the triangle \(t\) with vertices

\[
p_2 = (1, 0, 0), \quad \frac{p_3 + p_5}{2} = \left(\frac{a}{2}, \frac{b}{2}, 0\right), \quad \frac{p_4 + p_5}{2} = \left(\frac{p + a}{2}, \frac{q + b}{2}, 0\right).
\]

![Figure 8. The setting for the proof of Theorem 3.5](image)

To get more symmetric parameters we set \(c = a + p\) and \(d = b + q > 0\). This turns equations (4) into

\[
(5) \quad b < 0 < d, \quad 0 < cb - da \leq d - b.
\]
Since the rest of the proof happens in the plane $z = 0$, we drop the last coordinate for every point. We want to study what values of $a, b, c, d \in \mathbb{Z}$ satisfying equations (5) have $(0, 0)$ in and $(1, 0)$ as the only lattice points in the triangle $t = \text{conv}\{(1, 0), (a/2, b/2), (c/2, d/2)\}$.

By symmetry of pairs $(a, b)$ and $(c, d)$ let us further assume that $|d| \leq |b|$ and observe that, via the transformations $(x, y) \mapsto (x \pm y, y)$, we are only interested in $c$ modulo $d$. Let us look at the possible values of $(c, d)$. For $d = 1$ we take $c = 0$. For $d > 1$, taking into account that $(c/2, d/2)$ must be outside the wedge symmetric to the triangle $(0, 1)p_1p_2$ at point $(0, 1)$ we conclude that $c/2 \not\in [1 - d/2, 0]$, which is equivalent to $c \not\in [2 - d, 0]$. Thus, the only remaining value for $c$ (mod $d$) is $c = 1$.

![Diagram](image)

Figure 9. The case analysis in the proof of Theorem 3.5 for the three possibilities of $(c, d)$. Red squares represent the points $p_1$ and $p_2$ of $P$ in the displayed plane $z = 0$. The red crossed square is the intersection of $p_4p_5$ with that same plane. Black dots are the lattice points in the plane and black crosses represent the possible intersection points of the edge $p_3p_5$ and the plane $z = 0$.

Figure 9 shows the possibilities for point $(a, b)$ for the first three cases of $(c, d)$, namely $(c, d) = \{(0, 1), (1, 2), (2, 3)\}$. The figures must be read in the same way as Figure 7. Each lattice point creates an excluded wedge for the positions of $(a, b)$. The only novelty is that now we have also an excluded (open) half-plane, the one defined by $|b| > d$, so that the allowed region (the white region in the picture) gets smaller and smaller and it becomes lattice-point-free (and eventually empty) for $d \geq 4$ (picture left to the reader).

The crosses in the three pictures (9, 5 and 2, respectively) give a priori 16 possibilities for the points $(a, b)$ and $(c, d)$, that is, for the points $p_4$ and $p_5$ of our original problem. Namely:

| $(c, d)$ | $(a, b)$ | $(p, q)$ |
|---------|---------|---------|
| (0, 1)  | (-1, -1)| (1, 2)  |
| (0, 1)  | (-2, -1)| (2, 2)  |
| (0, 1)  | (-3, -1)| (3, 2)  |
| (0, 1)  | (-1, -2)| (1, 3)  |
| (0, 1)  | (-3, -2)| (3, 3)  |
| (0, 1)  | (-5, -2)| (5, 3)  |
| (0, 1)  | (-2, -3)| (2, 4)  |
| (0, 1)  | (-5, -3)| (5, 4)  |
| (0, 1)  | (-3, -4)| (3, 5)  |
| (1, 2)  | (-3, -2)| (4, 4)  |
| (1, 2)  | (-5, -3)| (6, 5)  |
| (1, 2)  | (-3, -4)| (4, 6)  |
| (1, 2)  | (-4, -5)| (5, 7)  |
| (1, 3)  | (-2, -3)| (3, 6)  |
| (1, 3)  | (-3, -4)| (4, 7)  |
Now, these 16 points reduce to only six possibilities by excluding those with \( \gcd(p, q) \neq 1 \) (which produce extra lattice points at \( z = 1 \)) or \( p \geq q \) (which are repeated from others). This six are distinguished by having \((p, q)\) in boldface in the table above. To check that they are all inequivalent and that they coincide with those in the statement we simply need to compute their volume vectors.

| \((a, b)\) | \((p, q)\) | \(((a - 2)q - bp, pb - qa, q + b, -b, q)\) |
|----------|----------|----------------------------------|
| \((-1, -1)\) | \((1, 2)\) | \((-5, 1, 1, 1, 2)\) |
| \((-1, -2)\) | \((1, 3)\) | \((-7, 1, 1, 2, 3)\) |
| \((-3, -4)\) | \((3, 5)\) | \((-13, 3, 1, 4, 5)\) |
| \((-2, -3)\) | \((3, 5)\) | \((-11, 1, 2, 3, 5)\) |
| \((-4, -5)\) | \((5, 7)\) | \((-17, 3, 2, 5, 7)\) |
| \((-3, -4)\) | \((4, 7)\) | \((-19, 5, 3, 4, 7)\) |

Since the volume vector are all primitive, they completely characterize the configuration (Theorem 2.3). The representatives in the statement have been chosen to have smaller coordinates.

We finally need to deal with configurations having volume vector \((-4q, q, q, q, q)\).

**Theorem 3.6.** Every polytope of size five and signature \((4, 1)\) with a symmetric volume vector \((q, q, q, q, -4q)\), is equivalent to the one having the points \((0, 0, 0), (1, 0, 0), (0, 0, 1)\) together with one of the following two pairs:

- \((1, 1, 1)\) and \((-2, -1, -2)\), volume vector \((-4, 1, 1, 1, 1)\).
- \((2, 5, 1)\) and \((-3, -5, -2)\), volume vector \((-20, 5, 5, 5, 5)\).

Both configurations have width two with respect to the functional \(x - z\).

**Proof.** As usual, we take the first four points to form a tetrahedron of type \(T(p, q)\) with \(0 \leq p < q\) and \(\gcd(p, q) = 1\), that is:

\[ p_4 \rightarrow (0, 0, 0), \quad p_3 \rightarrow (1, 0, 0), \quad p_1 \rightarrow (0, 0, 1), \quad p_2 \rightarrow (p, q, 1). \]

Without loss of generality (Lemma 2.8) we assume \(p_4\) to be the barycenter of the other four points, so that

\[ p_5 = (-p - 1, -q, -2). \]

The convex hull of \(P\) consists of four tetrahedra glued together, all of normalized volume \(q\); the one we started with and the following three:

- \(T_1 = \{(0, 0, 0), (1, 0, 0), (0, 0, 1), (-p - 1, -q, -2)\}\),
- \(T_2 = \{(0, 0, 0), (1, 0, 0), (p, q, 1), (-p - 1, -q, -2)\}\), and
- \(T_3 = \{(0, 0, 0), (0, 0, 1), (p, q, 1), (-p - 1, -q, -2)\}\).

What we need to check is which values of \(p\) and \(q\) make these three tetrahedra empty. If \(q = 1\) then we take \(p = 1\) and get the first possibility in the statement. All tetrahedra are unimodular and therefore empty.

We consider \(q > 1\) for the rest of the proof and use Lemma 2.10 to evaluate for which values of \(p\) and \(q\) these tetrahedra are empty.

- For \(T_1\), via the exchange of coordinates \(y \leftrightarrow z\), the lemma says that \(T_1\) is empty if and only if at least one of the following happens:
  
  (i) \(p \equiv -2 \pmod{q}\) and \(\gcd(q, 2) = 1\), i.e. \(q\) has to be odd and \(p = q - 2\).
  
  (ii) \(3 \equiv 0 \pmod{q}\) and \(\gcd(p + 1, q) = 1\), i.e. \(q = 3\) and \(p = 1\) (particular case of (i)).
(iii) \( p \equiv -3 \pmod{q} \) and \( \gcd(p+1,q) = 1 \), i.e. \( q \) has to be odd (if it were even, then either \( p \) or \( p+1 \) would have a common factor 2 with \( q \)) and \( p = q - 3 \).

So tetrahedron \( T_1 \) is empty if and only if \( q \) is odd and \( p \in \{ q-2, q-3 \} \).

- For \( T_2 \) consider the unimodular transformation

\[
(x, y, z) \mapsto (x - pz, z, y - qz)
\]

that sends \( T_2 \) to

\[
T_2' = \operatorname{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (p-1, -2, q)\}.
\]

So, the conditions of the lemma are

(i') \( p \equiv 2 \pmod{q} \) and \( \gcd(q, 2) = 1 \), i.e. \( q \) has to be odd and \( p = 2 \).

(ii') \( 3 \equiv 0 \pmod{q} \) and \( \gcd(p-1,q) = 1 \), i.e. \( q = 3 \) and \( p = 2 \) (particular case of (i)).

(iii') \( p \equiv 3 \pmod{q} \) and \( \gcd(p-1,q) = 1 \), i.e. \( q \) has to be odd and \( p = 3 \).

That is, tetrahedron \( T_2 \) is empty if and only if \( q \) is odd and \( p \in \{2, 3\} \).

Putting both together, for \( T_1 \) and \( T_2 \) to be empty we need that \( q = 5 \) and \( p \in \{2, 3\} \). Both possibilities make \( T_3 \) have width 1 as well: \( \{(0,0,0), (0,0,1), (2,5,1), (-3,-5,-2)\} \) has width one with respect to \( y - 2x \), and \( \{(0,0,0), (0,0,1), (3,5,1), (-4,-5,-2)\} \) has width one with respect to \( y + z - 2x \).

A priori, this could lead to two different configurations with \( q = 5 \). Not surprisingly, the following matrix represents a \( \mathbb{Z} \)-equivalence mapping the configuration with \( p = 2 \) to the one with \( p = 3 \):

\[
\begin{pmatrix}
1 & -1 & 3 \\
0 & -1 & 5 \\
0 & 0 & 1
\end{pmatrix}
\]

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