RESURGENCE ANALYSIS OF MEROMORPHIC TRANSFORMS

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Abstract. We consider meromorphic transforms given by meromorphic kernels and study their asymptotic expansions under a certain rescaling. Under decay assumptions we establish the full asymptotic expansion in the rescaling parameter of these transforms and provide global estimates for error terms. We show that the resulting asymptotic series is Borel resummable and we provide formulae for the resulting resurgent function, which allows us to give formulae for the Stokes coefficients. A number of classical functions are obtained by applying such meromorphic transforms to elementary functions, such as Faddeev’s quantum dilogarithm, and our general theory applies to these cases.

1. Introduction

In this paper, we study a certain type of meromorphic transforms which is very common in the study of classical functions. For a meromorphic function $h \in \mathcal{M}(\mathbb{C})$ we let $P_h$ denote its pole divisor. Further for $\gamma \in \mathbb{C}$ we define the rescaled meromorphic function $h_\gamma \in \mathcal{M}(\mathbb{C})$ by

$$h_\gamma(z) = h(\gamma z).$$

Let $W \subset \mathbb{C}$ be an open connected subset and consider a meromorphic kernel $K \in \mathcal{M}(W \times \mathbb{C})$. We use the notation $K_w = K|_{w \times \mathbb{C}}$ for $w \in \mathbb{C}$, with the property that $w \times \mathbb{C}$ is not contained in the pole divisor of $K$, which we will assume for all $w \in W$.

Assume that we have a continuous family $\Gamma_w$, $w \in W$ of curves, possibly non-compact and possibly with a fixed boundary, such that

$$\Gamma_w \subset \mathbb{C} - P_{K_w}$$

parametrized by $w \in W$.

We consider the following subspace $\mathcal{M}_{K,\Gamma}(\mathbb{C}) \subset \mathcal{M}(\mathbb{C})$ consisting of the meromorphic functions $f \in \mathcal{M}(\mathbb{C})$ for which

$$\Gamma_w \subset \mathbb{C} - (P_{K_w} \cup P_{f,\gamma})$$

parametrized by $(w, \gamma) \in W \times U \subset \mathbb{C} \times \mathbb{C}$, where $U \subset \mathbb{C}$ is a non-empty connected open subset such $0 \in \bar{U} - U$ and the integral

$$g_\gamma(w) = \int_{\Gamma_w} K(w, z) f_\gamma(z) \, dz$$

exist for all $(w, \gamma) \in W \times U$. This allows us to consider the following transforms

$$A_{K,\Gamma}^\gamma : \mathcal{M}_{K,\Gamma}(\mathbb{C}) \to \mathcal{O}(W), \quad A_{K,\Gamma}^\gamma(f)(w, \gamma) = g_\gamma(w).$$

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We are in this paper interested in the asymptotic expansion of the transforms \( A_{K,\Gamma}^\gamma \) as \( \gamma \) tends to zero, e.g. the asymptotics of \( g_\gamma \) as \( \gamma \) approached zero.

To this end we will assume that the following integrals
\[
h_m(w) = \int_{\Gamma_w} K(w, z) z^m dz
\]
exist for all \( m \in \mathbb{Z} \) and define \( h_m \in \mathcal{O}(W) \). We will further assume that the pole order of \( K_w \) in zero is independent of \( w \) and denote it \( k_0 \in \mathbb{Z}^+ \cap \{0\} \).

Let now \( n_0 \in \mathbb{Z}^+ \cup \{0\} \) be the pole order of \( f \) in zero and \( a_m \in \mathbb{C} \) the coefficients of the Laurent series of \( f \) at zero
\[
f(z) = \sum_{m=-n_0}^{\infty} a_m z^m
\]
convergent for \( z \in D(0, R_f) \), where \( R_f \) is the minimal distance from \( 0 \in \mathbb{C} \) to \( P_f - \{0\} \). In particular, we can consider the smooth function \( k : (0, R_f) \to \mathbb{R}^+ \) given by
\[
k(r) = \sum_{m=-n_0}^{\infty} |a_m| r^m, \quad r \in (0, R_f),
\]
which we will use below in our analytic estimates. We now define
\[
\hat{A}_{K,\Gamma}^\gamma : \mathcal{M}_{K,\Gamma}(\mathbb{C}) \to \mathcal{O}(W)[\gamma^{-1}, \gamma]
\]
given by
\[
(3) \quad \hat{A}_{K,\Gamma}^\gamma (f) = \sum_{m=-n_0}^{\infty} a_m h_m(w) \gamma^m.
\]
We introduce the following notation for the truncation at \( n \) of the series
\[
\hat{A}_{K,\Gamma}^\gamma_n (f) = \sum_{m=-n_0}^{n} a_m h_m(w) \gamma^m.
\]

In order to get analytic estimated on the reminder term in the asymptotic expansion of \( A_{K,\Gamma}^\gamma \), we will make the following assumptions.

First of all, we will assume that \( \Gamma_w \) can be continuously deformed (relative its boundary if non empty) in side \( \mathbb{C} - (P_{K_w} \cup P_{f_\gamma} - \{0\}) \) to a new contour \( \tilde{\Gamma}_w \) consist of finitely many, say \( d \), smooth arc segment \( \tilde{\Gamma}_{w,j}, j = 1, \ldots, d \). Let \( \Theta(z) \) be the angle between the line through zero and \( z \) and then the tangent line to \( \tilde{\Gamma}_{w,j} \) at \( z \), which is well defined along each of the smooth segments \( \tilde{\Gamma}_{w,j} \). We then require that there exist a positive constant \( b \) such that in each smooth segment
\[
|\cos(\Theta(z))| \geq b, \quad \forall z \in \tilde{\Gamma}_{w,j}, \ j = 1, \ldots, d.
\]

We will also need decay conditions on \( K \) and \( f \). In general we will proceed with the following assumptions.

\footnote{We might want to limit the pole order at 0 of the meromorphic functions \( f \) we allow, and thus only consider the our transform on the subspace \( \mathcal{M}_{K,\Gamma}^{n_0} = \{ f \in \mathcal{M}_{K,\Gamma} | \text{ord}(f) \geq n_0 \} \) and then we only need \( m \geq n_0 \).
\footnote{If a smooth arc segment passes through zero, we split it in two segment at zero, simply to get the right definition of \( d \) in relations to our estimates.}
There exist positive constants $\delta$ and $c$, together with $c_w \in \mathbb{R}_+$ parametrized by $w \in W$ and $\hat{c}_\gamma \in [c, \infty]$ parametrized by $\gamma \in U$, such that
\begin{equation}
|K(w, z)| \leq c_w e^{-\delta |z|^2} \quad \forall (w, z) \in W \times (\hat{\Gamma}_w - \{0\})
\end{equation}
\begin{equation}
|f_\gamma(z)| \leq \hat{c}_\gamma |\gamma z|^{-n_0} \quad \forall (\gamma, z) \in U \times (\hat{\Gamma}_w - \{0\}).
\end{equation}

The specifics of these norm estimated together with the assumptions on $\hat{\Gamma}_w$ are tailored to establish the global control for all $(w, \gamma) \in W \times U$ in (16) in the following theorem.

Let
\begin{align*}
C_{k_0 - 1} &= \frac{d}{bc} c_{k_0 - 1}, \\
C_n &= \frac{d}{bc \sqrt{n - k_0 + 1}} c_n', \\
\text{where} \\
c_n' &= \inf_{0 < r < R_f} \frac{1}{n} \left( \frac{cr^{-n_0}}{n - k_0 + 2} + k(r) \right), \\
\text{if } n \geq k_0 - 1.
\end{align*}

**Theorem 1.** Under the above assumptions, we have an asymptotic expansion in the Poincare sense to all orders
\begin{equation}
A_{K, \Gamma}(f) \sim \hat{A}_{K, \Gamma}(f).
\end{equation}

In fact we have the following estimates for all $n \geq k_0 - 1$
\begin{equation}
|A_{K, \Gamma}(f)(w) - \hat{A}_{K, \Gamma}(f)(f)(w)| \leq C_n c_w \hat{c}_\gamma |\gamma|^{n+1} \delta^{-(n-k_0+2)}(n - k_0 + 2)!
\end{equation}
for all $(w, \gamma) \in W \times U$.

This theorem together with Theorem 5 in Section 2 which covers the cases $-n_0 \leq n < k_0$ are proved in Section 2. The basic principle is to split the integral (11) into the sum of the integral over $\Gamma_w^{0,0}(\hat{r}) = \Gamma_w^{0,0} \cap D(0, \hat{r})$ and over $\Gamma_w^{0,\infty}(\hat{r}) = \Gamma_w^{0,\infty}(\hat{r}) \cap (\mathbb{C} - D(0, \hat{r}))$ for some positive $\hat{r}$. The first of these integral are simply estimated using (5) and (6). For the part of the integral over $\Gamma_w^{0,0}(\hat{r})$, we require that $r = |\gamma| \hat{r} < R_f$ and then we can use the Laurent expansion for $f_\gamma$ to get the needed estimate on that part of the integral to then establish (16). The estimate (14) in Theorem 5 is established along similar lines under slightly different assumptions.

If we have that $R_f = \infty$, then we can make milder conditions dealing with a much simpler case whose discussion we defer to Section 3.

We are further interested in the resurgence properties of the Borel resummation of the resulting possibly divergent series $A_{K, \Gamma}(f)$ in $\gamma$. To this end we introduce $g^- \in \mathcal{M}(U)[\gamma]$, given by
\begin{equation}
g^-(w) = \sum_{m=-n_0}^{k_0-1} a_m h_m(w) \gamma^m,
\end{equation}
and $g^+ \in \gamma \mathcal{M}(U)[[\gamma]]$ defined by
\begin{equation}
g^+(w) = \sum_{m=k_0}^{\infty} a_m h_m(w) \gamma^m.
\end{equation}

We will typically expect that the (15) is only an asymptotic expansion, e.g. that the series $g^+(w)$ is divergent, as we will see in various examples in Section 6 and 7. We therefore consider the formal Borel transform
\begin{equation}
B : \gamma \mathcal{O}(U)[[\gamma]] \to \mathcal{O}(U)[[\xi]]
\end{equation}
determined by 

\[ B(\gamma^m) = \frac{\xi^{m-1}}{(m-1)!} \]

and formally extended linearly over \( O(U) \). We recall that \( B \) applied to a divergent Gevrey-1 power series gives a power series with positive radius of convergence.

Let \( \mathbb{R}_+^\theta = e^{i\theta}\mathbb{R}_+ \) for \( \theta \in \mathbb{R} \) and consider the Laplace transform

\[ \mathcal{L}_\theta(\psi)(\gamma) = \int_{\mathbb{R}_+^\theta} e^{-z/\gamma} \psi(z) dz \]

which is certainly well defined for measurable \( \psi \) defined on \( \mathbb{R}_\theta \) provided there exist positive constants \( C, \alpha \) such that

\[ |\psi(z)| \leq Ce^{\alpha|z|}, \forall z \in \mathbb{R}_\theta \]

and

\[ \alpha < \text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta. \]

We observe that \( \mathcal{L}_\theta \circ B(\gamma^m) = \gamma^m \) for all \( m \in \mathbb{Z}_+ \) provided \( \gamma \in \cup_{\alpha > 0} U_{\theta, \alpha} \)

where \( U_{\theta, \alpha} = \{ \gamma \in \mathbb{R} \mid \text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta > |\gamma|^2 \alpha \} \).

We will now state our assumptions which will guarantee that \( B(g_z^+(w)) \) is convergent for all \( w \in W \).

To that end introduce \( \varphi \in \mathcal{M}(\mathbb{C}) \) given by

\[ \varphi(z) = f(z) - \sum_{m=-n_0}^{k_0-1} a_m z^m. \]

and \( \varphi_z \in \mathcal{M}(\mathbb{C}) \) given by

\[ \varphi_z(\gamma) = \varphi(\gamma z). \]

**Theorem 2.** Assume that exists \( \theta, \alpha \in \mathbb{R}, \ 0 < \beta < \delta \) and a constant \( C \), such that \( \mathcal{L}_\theta^{-1}(\varphi_z) \) exist for all \( z \in \Gamma_w, w \in W \) and further that for all \( w \in W \) we have that

\[ |\mathcal{L}_\theta^{-1}(\varphi_z)(\xi)| \leq Ce^{\alpha|\xi|+\beta|z|} |z|^{k_0} \forall (z, \xi) \in \tilde{\Gamma}_w \times V_{w, \theta}, \]

where \( V_{w, \theta} \subset \mathbb{C} \) is an open subset containing the half line \( \mathbb{R}_\theta^+ \).

Then we have that the formal series \( g_z^+ \) is Borel summable and

\[ B_w(\xi) = B(g_z^+(w))(\xi) \]

is well defined in \( V_{w, \theta} \) and \( B_w \in O(V_{w, \theta}) \) is given by

\[ B_w(\xi) = \int_{\Gamma_w} K(w, z) \mathcal{L}_\theta^{-1}(\varphi_z)(\xi) dz. \]

Finally we have that

\[ g^\gamma(w) = g_z^+(w) + \mathcal{L}_\theta(B_w)(\gamma) \forall \gamma \in U_{\theta, \alpha}. \]

The proof of this theorem is provided in Section 4. We first establish the formula \( \text{(10)} \) under the assumptions stated in the theorem and then we can derive the rest by properties of the Laplace transform.

The resurgence properties and Stokes phenomenon of \( B_w \) can now be studied. In general we see that \( \mathcal{L}_\theta(B_w) \) will be constant on sectors where \( B_w \) has no poles and jump when \( e^{i\theta}\mathbb{R}_+ \) hits poles of \( B_w \) along a set of directions \( \theta_{J_w} = \{ \theta_j \mid j \in J_w \} \) index by a set \( J_w, w \in W \). Let the jump at \( \theta_j \) be denote \( \Delta_{\theta_j}(\mathcal{L}_\theta(B_w)) \).
Theorem 3. Under the above assumptions we have that
\[
\Delta_\theta (L_\theta(B_w)) = 2\pi i \sum_{p \in P_B \cap e^{i\theta} \mathbb{R}_+} \text{Res}_{\xi=p} (e^{-\xi/\gamma} B_w(\xi))
\]
provided \(B_w\) decays sufficiently fast in a small sector around \(e^{i\theta} \mathbb{R}_+\), \(j \in J_w, w \in W\).

Let now the principal part of \(\varphi\) at \(p \in P_\varphi\) be denoted
\[
\tilde{\varphi}_{z,p}(\gamma) = \sum_{m=1}^{n_p} \frac{b_{p,m}}{(\gamma z - p)^m},
\]
where we assume that the pole order is universally bounded by some integer \(n_p\) independent of \(z \in \tilde{\Gamma}_w\) and \(w \in W\). We observe that if we let
\[
\tilde{\varphi}'_{z,p}(\gamma) = \sum_{m=1}^{n_p} \frac{b_{p,m}}{p^m} (-1)^m \sum_{l=0}^{m-1} \binom{m}{l} \left( \frac{z}{p} \right)^{m-l} \left( \frac{1}{\gamma} - \frac{z}{p} \right)^{m-l}
\]
then
\[
\tilde{\varphi}_{z,p}(\gamma) = \sum_{m=1}^{n_p} \frac{b_{p,m}}{p^m} (-1)^m + \tilde{\varphi}'_{z,p}(\gamma).
\]

We will assume that there exist an ordering on \(P_\varphi\) such that the series
\[
\tilde{\varphi}'_{z}(\gamma) = \sum_{p \in P_\varphi} \tilde{\varphi}'_{z,p}(\gamma)
\]
with respect to that ordering is uniformly absolutely convergent on
\[
O_\varepsilon = \mathbb{C} - \cup_{p \in P_\varphi} D(p, \varepsilon)
\]
for some \(\varepsilon > 0\) and there exist an entire function \(\tilde{\psi}_z\) parametrized by \(z \in \tilde{\Gamma}_w\) and \(w \in W\) such that
\[
\varphi_z(\gamma) = \tilde{\varphi}'_z(\gamma) + \tilde{\psi}_z(\gamma) \quad \forall \gamma \in O_\varepsilon.
\]
We will see in examples that one actually sometimes get that \(\tilde{\psi}_z = 0\), but in general we will simply assume that \(L_{\theta}^{-1}(\tilde{\psi}_z)\) exist for all \(z \in \tilde{\Gamma}_w\) and \(w \in W\).

Theorem 4. Assume that exists \(\theta, \alpha \in \mathbb{R}, 0 < \beta < \delta\) and a constant \(C_w\), such that for all \(w \in W\) we have that
\[
|L_{\theta}^{-1}(\tilde{\psi}_z)(\xi)| \leq C e^{\alpha |\xi|+\beta |z|} |z|^\delta \quad \forall (z, \xi) \in \tilde{\Gamma}_w \times V_{w,\theta},
\]
where \(V_{w,\theta} \subset \mathbb{C}\) is an open subset containing the half line \(\mathbb{R}_+^\theta\). Assuming further that
\[
\int_{\mathbb{R}_+^\theta} \sum_{p \in P_\varphi} \sum_{m=1}^{n_p} \left| \frac{b_{p,m}}{p^{2m-1}} \right| \int_{\tilde{\Gamma}_w} \left| e^{-\xi/\gamma} K(w,z) e^{\frac{2\pi i}{p}} \left( \frac{|z|}{p} + |\xi| \right)^{m-1} \right| |dz| |d\xi| < \infty
\]
and that
\[
1 \left( \frac{\text{Re}(\gamma)}{|\gamma|^2} \right) \left( \frac{\text{Im}^2(\gamma)}{\sin \theta} \right) > \frac{1}{p} \left( \frac{\text{Im}(z)}{\sin \theta} \right) > 0,
\]
we have for all \((w, \xi) \in W \times V_{w, \theta}\) that

\[
B_w(\xi) = \sum_{p \in P_w} \left( \int_{\Gamma_w} K(w, z) e^{\gamma z} dz \right) \sum_{m=1}^{n_w} \frac{b_{p,m}}{p^m} (-1)^m \sum_{t=0}^{m-1} \binom{m}{l} \left( \frac{z}{p} \right)^{m-t} \frac{\xi^{m-t-1}}{(m-l-1)!} + \int_{\Gamma_w} K(w, z) L_{\theta}^{-1}(\psi_z)(\xi) dz.
\]

The proof of Theorem 3 and 4 is provided in Section 5. The formula (13) is ideally suited to understand the resurgence properties of \(B_w\) and the Stokes coefficients, since it is in examples easy to determined what maximal open subset \(B_w\) extends to from the expression (13), as we will in particular see in Faddeev quantum dilogarithm case in Section 6.

In the final two sections of this paper we provide several examples of our main theorems stated above, which in particular include the case of Faddeev’s quantum dilogarithm.

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2. PROOF OF THE ASYMPTOTIC EXPANSION

To analyse the asymptotics of the transform \(A_{K, \Gamma}^\gamma\), we introduce the function \(\varphi_n \in M(\mathbb{C})\) given by

\[
\varphi_n(z) = f(z) - \sum_{m=-n_0}^{n} a_m z^m.
\]

When we assume that \(n \geq k_0 - 1\), we get that

\[
z \mapsto K(w, z) \varphi_n(\gamma z)\text{ is regular } \forall z \in D(0, \min(R_K(w), \frac{R_f}{|\gamma|}))
\]

for all \(w \in W\), where \(R_K(w)\) is the minimal distance from zero to \(P_{K_w} - \{0\}\). Thus we see that

\[
A_{K, \Gamma}^\gamma(f)(w) - \hat{A}_{K, \Gamma}^\gamma(f)(w) = \int_{\Gamma_w} K(w, z) \varphi_n(\gamma z) dz
\]

and therefore

\[
|A_{K, \Gamma}^\gamma(f) - \hat{A}_{K, \Gamma}^\gamma(f)| \leq \int_{\Gamma_w} |K(w, z)||\varphi_n(\gamma z)||dz|
\]

Let now \(\tilde{r} < \frac{R_f}{|\gamma|}\) and for each \(j = 1, \ldots, d\) consider the decomposition

\[
\tilde{\Gamma}_{w,j} = \hat{\Gamma}_{w,j}(\tilde{r}) \cup \tilde{\Gamma}_{w,j}(\tilde{r}),
\]

where

\[
\hat{\Gamma}_{w,j} = \hat{\Gamma}_{w,j}(\tilde{r}) \cap D(0, \tilde{r}) \quad \text{and} \quad \tilde{\Gamma}_{w,j}(\tilde{r}) = \hat{\Gamma}_{w,j} \cap (\mathbb{C} - D(0, \tilde{r}))
\]
and we let
\[ \tilde{\Gamma}_w^R (\hat{r}) = \bigcup_{j=1}^d \tilde{\Gamma}_{w,j}^R (\hat{r}), \quad \tilde{\Gamma}_w^I (\hat{r}) = \bigcup_{j=1}^d \tilde{\Gamma}_{w,j}^I (\hat{r}). \]

We immediately get the estimate
\[
| A_{K,1}^r (f) (w) - \tilde{A}_{K,1}^r (f) (w) | \leq \int_{\tilde{\Gamma}_w^R (\hat{r})} |K(w, z)| |f(\gamma z)| dz + \sum_{m=-n_0}^n |a_m| \int_{\tilde{\Gamma}_w^R (\hat{r})} |K(w, z)| |\gamma z|^m dz + \int_{\tilde{\Gamma}_w^R (\hat{r})} |K(w, z)| |\varphi_n(\gamma z)| dz.
\]

We now provide estimates for each of these three summands. First of all, it follows directly from our main estimates (5) and (6) that
\[
\int_{\tilde{\Gamma}_w^R (\hat{r})} |K(w, z)| |f(\gamma z)| dz \leq c_w e^{-\delta |z|} |\gamma|^{-n_0} \int_{\tilde{\Gamma}_w^R (\hat{r})} e^{-\delta |z|} |z|^{-k_0-n_0} dz
\]
and
\[
\int_{\tilde{\Gamma}_w^R (\hat{r})} |K(w, z)| |\gamma z|^m dz \leq c_w |\gamma|^m \int_{\tilde{\Gamma}_w^R (\hat{r})} e^{-\delta |z|} |z|^{m-k_0} dz
\]
For fixed \( w \), we now let \( \gamma_j : [t_{j,0}, t_{j,1}] \to \tilde{\Gamma}_{w,j}^\infty (\hat{r}), \quad j = 1, \ldots, d, \) be a smooth parametrization of \( \tilde{\Gamma}_{w,j}^\infty (\hat{r}) \) and for those \( j \) such that \( \tilde{\Gamma}_{w,j}^\infty (\hat{r}) = \emptyset \) we just set \( t_{j,0} = t_{j,1} \). For some \( j \) we might have \( t_{j,1} = \infty \) and we can assume that \( |\gamma_j(t_{j,0})| \leq |\gamma_j(t_{j,1})| \). We see that
\[
\int_{\tilde{\Gamma}_w^R (\hat{r})} e^{-\delta |z|} |z|^l dz \leq \sum_{j=1}^d \int_{t_{j,0}}^{t_{j,1}} e^{-\delta |\gamma_j(t)|} |\gamma_j(t)|^l |\gamma_j'(t)| dt.
\]
But since we have the lower bound \( b \) on \( |\cos(\Theta(z))| \), where \( \Theta(z) \) is the angle discussed above for \( z \in \tilde{\Gamma}_{w,j}^\infty (\hat{r}) \), we see that
\[
\int_{t_{j,0}}^{t_{j,1}} e^{-\delta |\gamma_j(t)|} |\gamma_j(t)|^l |\gamma_j'(t)| dt \leq \frac{1}{b} \int_{|\gamma_j(t)|}^1 e^{-\delta s^l} ds,
\]
since we can make the substitution \( s = |\gamma_j(t)| \) and then use that
\[
|\gamma_j(t)|^l = \frac{\text{Re}(\gamma_j(t))\text{Re}(\gamma_j'(t)) + \text{Im}(\gamma_j(t))\text{Im}(\gamma_j'(t))}{|\gamma_j(t)|} = \frac{|\gamma_j(t)||\gamma_j'(t)| \cos(\Theta(\gamma_j(t)))}{|\gamma_j(t)|},
\]
which in absolute value is bounded below by \( |\gamma_j'(t)| b \).
For \( \hat{r}_1 > \hat{r}_0 \geq \hat{r} \) and \( l \in \mathbb{Z} \cup \{0\} \) we have that
\[
\int_{\hat{r}_0}^{\hat{r}_1} e^{-\delta s^l} ds \leq \frac{1}{\delta} e^{-\delta \hat{r}^l}.
\]
and for $l \in \mathbb{Z}_+$ we have that
\[
\int_{\tilde{r}_0}^{\tilde{r}_1} e^{-\delta s} s^l ds \leq \int_{\tilde{r}}^{\tilde{r}_1} e^{-\delta s} s^l ds
\]
\[
= \sum_{l'=0}^{l} \delta^{-(l'+1)} \frac{l!}{(l-l')!} (e^{-\delta \tilde{r} l-l'} - e^{-\delta \tilde{r} l-l'})
\]
\[
\leq \sum_{l'=0}^{l} \delta^{-(l'+1)} \frac{l!}{(l-l')!} e^{-\delta \tilde{r} l-l'}
\]
Thus we have proved the following two lemmas, which provides the estimates we need in regards to the $\Gamma_{\tilde{r}}^{\gamma, \infty}(\tilde{r})$-part of the integral.

**Lemma 1.** We have that
\[
\int_{\Gamma_{\tilde{r}}} |K(w, z)||f(\gamma z)||dz| \leq \frac{\delta^{-1} e^{-\delta \tilde{r}}}{b} \sum_{m=-n_0}^{k_0-1} |a_m| \gamma^{m_{\tilde{r} m-k_0}}
\]
For the second integral over $\Gamma_{\tilde{r}}^{\gamma, \infty}(\tilde{r})$, we obtain the following estimates.

**Lemma 2.** Provided that $n \geq k_0 - 1$, it follows that
\[
\sum_{m=-n_0}^{n} |a_m| \int_{\Gamma_{\tilde{r}}} |K(w, z)||\gamma z|^m |dz| \leq \frac{\delta^{-1} e^{-\delta \tilde{r}}}{b} \sum_{m=-n_0}^{k_0-1} |a_m| \gamma^{m_{\tilde{r} m-k_0}}
\]
\[
+ \frac{\delta^{-1} e^{-\delta \tilde{r}}}{b} \sum_{m=k_0}^{n} |a_m| \gamma^m \sum_{l=0}^{m-k_0} \delta^{-(l+1)} \frac{(m-k_0)!}{(m-k_0-l)!} \gamma^{m-k_0-l}.
\]
Let us now attend to the $\Gamma_{\tilde{r}}^{\gamma, \partial}(\tilde{r})$-part of the integral. We introduce the notation
\[
k_n(r) = \sum_{m=n+1}^{\infty} |a_m| r^m.
\]

**Lemma 3.** Provided that $r = \tilde{r} |\gamma|$ satisfies $r < R_f$, we have for $n \geq k_0$
\[
\int_{\Gamma_{n}(r)} |K(w, z)||\varphi_n(\gamma z)||dz| \leq \frac{\delta^{-1} e^{-\delta \tilde{r}}}{b} \sum_{m=-n_0}^{k_0-1} |a_m| \gamma^{m_{\tilde{r} m-k_0}} + \frac{\delta^{-1} e^{-\delta \tilde{r}}}{b} \sum_{m=k_0}^{n} |a_m| \gamma^m \sum_{l=0}^{m-k_0} \delta^{-(l+1)} \frac{(m-k_0)!}{(m-k_0-l)!} \gamma^{m-k_0-l}.
\]

**Proof.** We introduce the meromorphic function $\tilde{\varphi}_n \in \mathcal{M}(\mathbb{C})$ given by
\[
\tilde{\varphi}_n(z) = z^{-n-1} \varphi_n(z).
\]
We get that
\[
|\tilde{\varphi}_n(z)| \leq r^{-n-1} \sum_{m=n+1}^{\infty} |a_m| r^m = r^{-n-1} k_{n+1}(r)
\]
for all $z \in D(0, r)$. From this we conclude that
\[
\int_{\Gamma_{n}(r)} |K(w, z)||\varphi_n(\gamma z)||dz| = \int_{\Gamma_{n}(r)} |K(w, z)||\tilde{\varphi}_n(\gamma z)||\gamma z|^n |dz|
\]
\[
\leq c w k_{n+1}(r) r^{-n-1} |\gamma|^n \int_{\Gamma_{n}(r)} e^{-\delta |z|} |z|^{n-k_0+1} |dz|.
\]
We now introduce the notation $\delta$. With reference to Lemma 2 we compute

$$\int_{\tilde{r}(\tilde{r})} K(w, z) |\varphi_n(\gamma z)| dz \leq \frac{d c w k_{n+1}(r) r^{-n-1}}{b} |\gamma|^n + 1 \int_0^r e^{-\delta s n_k^0}ds$$

and

$$\int_0^r e^{-\delta s n_k^0}ds \leq \delta^{k_0-n-2(n-k_0+1)!}$$

We now complete the proof of Theorem 1 based on Lemma 1, 2 and 3.

Proof. We assume that $n \geq k_0 - 1$ and further that $r = \tilde{r}|\gamma| < R_f$. With reference to Lemma 1, we compute

$$\delta^{-1} |\gamma|^{-n_0} e^{-\delta r} r^{-(k_0+n_0) \leq \delta^{-1} |\gamma|^{-n_0} |\gamma|^{-n_0} \delta^{-(n-k_0+2)(n-k_0+1)!}$$

Thus for $n = k_0 - 1$ we get that

$$\delta^{-1} |\gamma|^{-n_0} e^{-\delta r} r^{-(k_0+n_0) \leq r^{-(n+n_0+1)} |\gamma|^{n_0+n_1+1} |\gamma|^{-n_0} \delta^{-(n-k_0+2)(n-k_0+1)!}$$

and for $n > k_0 - 1$

$$\delta^{-1} |\gamma|^{-n_0} e^{-\delta r} r^{-(k_0+n_0) \leq \frac{r^{-(n+n_0+1)}}{\sqrt{2\pi(n-k_0+1)}} |\gamma|^{n+1} \delta^{-(n-k_0+2)(n-k_0+1)!}$$

since for $m \in \mathbb{Z}_+$ we have by Robbin’s lower bound for $m$ that

$$\sqrt{2\pi m} \sup_{x \in \mathbb{R}_+} x^m e^{-x} \leq \sqrt{2\pi m} m^m e^{-m} \leq m!$$

We now introduce the notation

$$k_{n,m}(r) = \sum_{m=n}^{n'} |a_m| r^m$$

With reference to Lemma 2 we compute

$$\delta^{-1} e^{-\delta r} \sum_{m=-n_0}^{k_0-1} |a_m| |\gamma|^m r^{m-k_0} \leq \delta^{-1} e^{-\frac{r}{|\gamma|}} |\gamma|^{k_0-r} \sum_{m=-n_0}^{k_0-1} |a_m| r^m$$

and

$$\frac{r^{k_0-k_{n_0,k_0-1}(r)} |\gamma|^{n+1} \delta^{-(n-k_0+2)(n-k_0+1)!}}{(n-k_0+1)!}$$

which for $n = k_0 - 1$

$$\delta^{-1} e^{-\delta r} \sum_{m=-n_0}^{k_0-1} |a_m| |\gamma|^m r^{m-k_0} \leq \frac{r^{-n-1} k_{n_0,k_0-1}(r) |\gamma|^{n+1} \delta^{-(n-k_0+2)(n-k_0+1)!}}{\sqrt{2\pi(n-k_0+1)}}$$

and for $n > k_0 - 1$

$$\delta^{-1} e^{-\delta r} \sum_{m=-n_0}^{k_0-1} |a_m| |\gamma|^m r^{m-k_0} \leq \frac{r^{-n-1} k_{n_0,k_0-1}(r) |\gamma|^{n+1} \delta^{-(n-k_0+2)(n-k_0+1)!}}{\sqrt{2\pi(n-k_0+1)}}$$

□
and further if $n \geq k_0$

$$e^{-\delta \tilde{r}} \sum_{m=k_0}^{n} |a_m| |\gamma|^m \sum_{l=0}^{m-k_0} \delta^{-(l+1)} \frac{(m-k_0)!}{(m-k_0-l)!} p^{m-l-k_0}$$

$$\leq \delta^{-1} \sum_{m=k_0}^{n} |a_m| r^{m-k_0} |\gamma|^k_0 \sum_{l=0}^{m-k_0} \frac{(m-k_0)!}{(m-k_0-l)!} \left( \frac{r}{|\gamma|} \right)^{-l}$$

$$\leq |\gamma|^{n+1} \delta^{-(n-k_0+2)} (n-k_0+1)!$$

$$\sum_{m=k_0}^{n} |a_m| r^{m-(n+1)} \sum_{l=0}^{m-k_0} \frac{(m-k_0)!}{(n-k_0+1)!} (n-k_0+1)! \left( \frac{r}{|\gamma|} \right)^{-l} e^{-\delta \tilde{r}}$$

$$\leq |\gamma|^{n+1} \delta^{-(n-k_0+2)} (n-k_0+1)!$$

$$\sum_{m=k_0}^{n} |a_m| r^{m-(n+1)} \sum_{l=0}^{m-k_0} \frac{(m-k_0)!}{(n-k_0+1)!} (n-k_0+1)! \sqrt{2\pi(n-k_0+1-l)}$$

since

$$\frac{(m-k_0)!}{(n-k_0+1)!} (n-k_0+1)! \leq 1$$

and

$$\sum_{l=0}^{m-k_0} \frac{1}{\sqrt{n-k_0+1-l}} \leq \int_{n-m}^{n-k_0+1} \frac{1}{\sqrt{t}} dt \leq 2\sqrt{n-k_0+1}.$$ 

But then we can continue to obtain

$$e^{-\delta \tilde{r}} \sum_{m=k_0}^{n} |a_m| |\gamma|^m \sum_{l=0}^{m-k_0} \delta^{-(l+1)} \frac{(m-k_0)!}{(m-k_0-l)!} p^{m-l-k_0}$$

$$\leq |\gamma|^{n+1} \delta^{-(n-k_0+2)} (n-k_0+1)!$$

$$\sum_{m=k_0}^{n} |a_m| r^{m-k_0} \frac{1}{\sqrt{2\pi}} r^{-n-1} \sum_{l=0}^{m-k_0} |a_m| r^{m-n} \sqrt{n-k_0+1}$$

$$\leq 2\sqrt{n-k_0+1} \frac{1}{\sqrt{2\pi}} r^{-n-1} k_{0,n}(r) |\gamma|^{n+1} \delta^{-(n-k_0+2)} (n-k_0+1)!$$

By appealing to Lemma 3 and collecting terms, we get for $n = k_0 - 1$ that

$$\int_{\Gamma_{\omega}(r)} |K(w,z)||f(\gamma z)||dz| \leq \frac{dc_{w}}{b} |\gamma|^{k_0} \delta^{-1} \inf_{r \in (0,R_f)} \frac{(c_{\gamma} r^{-n_0} + k(r))}{r^{k_0}}$$

and for $n \geq k_0$

$$\int_{\Gamma_{\omega}(r)} |K(w,z)||f(\gamma z)||dz| \leq \frac{dc_{w}}{b} |\gamma|^{n+1} \delta^{-(n-k_0+2)} (n-k_0+1)!$$

$$\inf_{r \in (0,R_f)} \left( \frac{c_{\gamma} r^{-n_0}}{\sqrt{2\pi(n-k_0+1)}} + \frac{k_{0,n-k_0+1}(r)}{\sqrt{2\pi(n-k_0+1)}} + \frac{2\sqrt{n-k_0+1}}{\sqrt{2\pi}} k_{0,n}(r) + k_{n+1}(r) \right)^{n+1}$$

It might however be hard to use this estimate for $n \geq k_0$ in practice, since it requires control of $k_{n-k_0}(r)$, $k_{0,n-r}(r)$ and $k_{n}(r)$ separately and the infimum involves the $\gamma$-dependent $c_{\gamma}$. So to obtain an estimate where the constant only
involves $k(r)$ and is completely $\gamma$ independent. By multiplying and dividing by $n - k_0 + 2$ on the right hand side in the above expression and using

\[ 1 \leq \frac{\tilde{c}_\gamma}{c}, \quad \frac{1}{\sqrt{2\pi}} \leq 1, \quad \frac{2}{\sqrt{2\pi}} \leq 1, \quad \frac{\sqrt{n - k_0 + 1}}{n - k_0 + 2} \leq \frac{1}{\sqrt{n - k_0 + 1}} \]

and

\[ \frac{1}{n - k_0 + 2} \leq \frac{1}{\sqrt{n - k_0 + 1}} \]

together with

\[ k(r) = k_{-n_0,k_0}(r) + k_{k_0+1,n}(r) + k_n(r) \]

we conclude Theorem [I].

We can also establish estimates for $n < k_0 - 1$, but under further assumptions. Set

\[ r_m = \inf\{|z|: z \in \Gamma_{\tilde{w}}(\tilde{r}), \quad (w, \gamma) \in W \times (U \cap D(0, r))\} \]

and

\[ \tilde{C}_n(r, r_m) = \frac{d}{2b} \tilde{c}_n(r, r_m) \]

where

\[ \tilde{c}_n(r, r_m) = \frac{c r^{-n_0} + k(r)(1 + r_n^{n-k_0+1})}{r^{n+1}}. \]

**Theorem 5.** Assume that $r_m > 0$. Then for $n_0 \leq n < k_0 - 1$ and any positive $r < R_f$ we have that

\[ |A_{\tilde{K},r}(f)(w) - \tilde{A}_{\tilde{K},r}(f)(w)| \leq \tilde{C}_n(r, r_m)c_w \tilde{c}_\gamma |\gamma|^{n+1}\delta^{-1} \]

for all $(w, \gamma) \in W \times (U \cap D(0, r))$.

In order to prove this theorem we first extend out above Lemma. We observe that the proof of the first two Lemma are exactly the same and thus we immediately obtain.

**Lemma 4.** If $n \leq k_0$ then

\[ \sum_{m=-n_0}^{n} |a_m| \int_{\Gamma_{\tilde{w}}(\tilde{r})} |K(w, z)||\gamma z|^m dz \leq \frac{d c_w}{b} \delta^{-1} e^{-\delta \tilde{r}} \sum_{m=-n_0}^{n} |a_m||\gamma|^{n+m-k_0} \]

For the third lemma, we have in the case $n < k_0 - 1$.

**Lemma 5.** Provided that $r = \tilde{r}|\gamma|$ satisfies $r < R_f$ and $r_m > 0$, we have for $n < k_0 - 1$

\[ \int_{\Gamma_{\tilde{w}}(\tilde{r})} |K(w, z)||\varphi_n(\gamma z)||dz| \leq \frac{d c_w \delta^q k(r)^{-n-1}r^{n-k_0+1}}{b} |\gamma|^{n+1}\delta^{-1}. \]

**Proof.** From the first part of the proof of Lemma 3 we conclude that

\[ \int_{\Gamma_{\tilde{w}}(\tilde{r})} |K(w, z)||\varphi_n(\gamma z)||dz| \leq \frac{d c_w \delta^q k(r)^{-n-1}}{b} |\gamma|^{n+1} \int_{r_m}^{\tilde{r}} e^{-\delta s} s^{n-k_0+1} |ds|. \]

Thus for $n < k_0 - 1$ we get that

\[ \int_{\Gamma_{\tilde{w}}(\tilde{r})} |K(w, z)||\varphi_n(\gamma z)||dz| \leq \frac{d c_w \delta^q}{b} k(r)^{-n-1}r^{n-k_0+1} |\gamma|^{n+1}\delta^{-1}. \]

\[ \square \]
We now move on to proving Theorem 5.

**Proof.** We assume that \(|\gamma| \leq r, r_m > 0\) and \(n < k_0 - 1\).

With reference to Lemma 4 we observe that

\[
|\text{Re}(\gamma)|^{n_0} e^{-\delta \tilde{r} (k_0 + n_0)} \leq |\gamma|^{n_0 + n + 1} |\text{Re}(\gamma)|^{-n_0} (|\gamma|^{k_0} - n - n_0 - 1)^{-1}.
\]

Referring to Lemma 4

\[
e^{-\delta \tilde{r}} \sum_{m=-n_0}^{n} |a_m| |\gamma|^m \left( \frac{r}{|\gamma|} \right)^{m-k_0} \leq |\gamma|^{k_0 - k_0} h(r) \leq |\gamma|^{n+1} r^{-n-1} h(r).
\]

The proof is completed by referring to Lemma 5 and collecting terms. □

### 3. The simpler case of a single pole at zero.

In this section we will establish an analog of Theorem 1 in the case where \(R_f = 0\). We will still assume the setup from the introduction with the assumptions (4), (5), however we will not need (6). But we will now make the following extra assumption. Recall that

\[
\tilde{\varphi}_n(z) = \varphi_n(z) = \sum_{m=n+1}^{\infty} a_m z^{m-n-1}.
\]

For \(n \geq k_0 - 1\), we will assume

\[
C_w^{(n)} := \sup_{z \in D_w} |\tilde{\varphi}_n(z)| = C_n(w) < \infty
\]

for all \(w \in W\).

**Theorem 6.** Under the above assumptions, we have an asymptotic expansion in the Poincare sense to all orders

\[
A_{K,\Gamma}(f) \sim \tilde{A}_{K,\Gamma}(f).
\]

In fact we have the following estimates for all \(n \geq k_0 - 1\)

\[
|A_{K,\Gamma}(f)(w) - \tilde{A}_{K,\Gamma}^n(f)(w)| \leq C_w^{(n)} \frac{dc_w \tilde{c}_\gamma}{b} |\gamma|^{n+1} \delta^{n-k_0} (n - k_0 + 1)!
\]

for all \((w, \gamma) \in W \times U\).

**Proof.** The proof starts the same ways as the proof of Theorem 1 so assume that \(n \geq k_0 - 1\), we get that

\[
z \mapsto K(w, z) \varphi_n(\gamma z) \text{ is regular } \forall z \in D(0, R_K(w))
\]

for all \(w \in W\), where we recall that \(R_K(w)\) is the minimal distance from zero to \(P_{Kw} \setminus \{0\}\). Thus we see that

\[
A_{K,\Gamma}(f)(w) - \tilde{A}_{K,\Gamma}^n(f)(w) = \int_{D_w} K(w, z) \varphi_n(\gamma z) dz
\]
and therefore

\[ |A_{K,F}^\gamma(f) - A_{K,F}^{\gamma,n}(f)| \leq \int_{\tilde{\Gamma}_w} |K(w,z)||\tilde{\varphi}_n(\gamma z)||dz| \]
\[ \leq c_w \int_{\tilde{\Gamma}_w} e^{-\delta|z|}|z|^{-k_0}||\tilde{\varphi}_n(\gamma z)||\gamma|^{n+1}|dz| \]
\[ \leq c_w C_{w}^{(n)} |\gamma|^{n+1} \int_{\tilde{\Gamma}_w} e^{-\delta|z|}|z|^{-k_0+n+1}|dz| \]
\[ \leq c_w C_{w}^{(n)} d_b^\gamma |\gamma|^{n+1} \int_0^\infty e^{-\delta s}s^{-k_0+1}ds \]
\[ = c_w C_{w}^{(n)} d_b^\gamma |\gamma|^{n+1} \delta^{-k_0+2}(n - k_0 + 1)! \]

□

We observe that we can under an extra assumption get a formula for the Laurent expansion in $\gamma$ at zero.

**Proposition 1.** Assume that

\[ \lim sup_{m \to \infty} (|a_m|(m - k_0)!)^{1/m} = R^{-1} < \infty \]

then the Laurent series for $g_\gamma(w)$ is

\[ g_\gamma(w) = \sum_{m=-n_0}^{\infty} a_m h_m(w) \gamma^m \]

which converges for $|\gamma| < \delta R$.

**Proof.** We compute for $m \geq k_0$ that

\[ |h_m(w)| \leq \int_{\tilde{\Gamma}_w} |K(w,z)||z|^m|dz| \]
\[ \leq c_w \int_{\tilde{\Gamma}_w} e^{-\delta|z|}|z|^{m-k_0}|dz| \]
\[ = c_w d_b^\gamma \delta^{m-k_0+1}(n - k_0)! \]

The proposition follows immediately from this.

□

4. **Borel resummation**

We will now prove Theorem 2.

**Proof.** By the assumed decay properties of $K$ and $L^{-1}_\theta(\varphi_z)$ we have that

\[ |K(w,z)L^{-1}_\theta(\varphi_z)(\xi)| \leq C e^{\alpha|\xi|e^{-\delta+\beta}|z|}, \quad \forall (w, z, \xi) \in W \times (\tilde{\Gamma}_w - 0) \times V_{w, \theta} \]

where $\delta - \beta > 0$ by assumption. Combining this with the assumption on $\tilde{\Gamma}_w$ we see that the integral

\[ B_w(\xi) = \int_{\tilde{\Gamma}_w} K(w,z)L^{-1}_\theta(\varphi_z)(\xi)dz \]
is absolutely convergent and well defined for \((w, \xi) \in W \times V_{w, \theta}\). Now we further have that 
\[
|e^{-\xi/\gamma} K(w, z) \mathcal{L}_\theta^{-1}(\varphi_z)(\xi)| \leq C_{w,\xi} e^{-\left((\frac{1}{|\gamma|^{3/2}}(\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta + \alpha)|\xi| + (\delta - \alpha)|z|\right)}
\]
for all \((w, z, \xi) \in W \times (\bar{\Gamma}_w - 0) \times V_{w, \theta}\) and \(\gamma \in U_{\theta, \alpha}\). This implies that 
\[
\int_{\mathbb{R}_+^+} \int_{\bar{\Gamma}_w} e^{-\xi/\gamma} K(w, z) \mathcal{L}_\theta^{-1}(\varphi_z)(\xi)|dz||d\xi| < \infty.
\]
Thus by Fubini’s theorem we get that 
\[
\mathcal{L}_\theta(B_w)(\gamma) = \int_{\mathbb{R}_+^+} e^{-\gamma/\xi} \left(\int_{\bar{\Gamma}_w} K(w, z) \mathcal{L}_\theta^{-1}(\varphi_z)(\xi)dz\right) d\xi 
= \int_{\Gamma_w} K(w, z) \left(\int_{\mathbb{R}_+^+} e^{-\gamma/\xi} \mathcal{L}_\theta^{-1}(\varphi_z)(\xi)d\xi\right) dz 
= \int_{\Gamma_w} K(w, z) \varphi_z(\gamma) dz 
= g^+_\gamma(w).
\]
which establishes (11).

Recall now the global estimates of Theorem 4 in \(U\) on the asymptotic expansion of \(g^+_\gamma(w)\) in \(\gamma\). From this it follows by the above computation that (9) also holds since 
\[
\mathcal{L}_\theta \circ \mathcal{B} = \text{Id}
\]
on \(\gamma \mathcal{M}(U)[[\gamma]]\). 

\[\Box\]

5. Resurgence and Stokes Coefficients

The resurgence properties of \(B_w\) is really nicely understood using our formula 
\[
B_w(\xi) = \int_{\bar{\Gamma}_w} K(w, z) \mathcal{L}_\theta^{-1}(\varphi_z)(\xi)dz.
\]
Further the Stokes lines and the Stokes jumps can also be derived from the these expressions by considering the pole structure of both \(\varphi_z\) and further of \(B_w\) it self as is clear from Theorem 3 and 4 which we no establish starting with Theorem 3.

Proof. We recall the setting of Theorem 3 and the assumptions of this theorem. First of all, we see by Theorem 2 that \(\mathcal{L}_\theta(B_w)\) is at least well defined for small \(\theta\), since \(B_w\) exist on a sector around the real line and by assumption it is bounded there. In general we see that \(\mathcal{L}_\theta(B_w)\) will be constant on sectors where \(b_w\) has no poles and decays at infinity sufficiently fast so as to make \(\mathcal{L}_\theta(B_w)\) well defined. However, if we consider a line \(e^{i\theta_j} \mathbb{R}_+\) which hits the poles of \(B_w\), then \(\mathcal{L}_\theta(B_w)\) is not well-defined along these lines. These are the directions \(\theta_j = \{\theta_j \mid j \in J_w\}\) index by set \(J_w\), \(w \in W\) as introduced in the introduction. Further recall that the jump at \(\theta_j\) was denote \(\Delta_{\theta_j}(\mathcal{L}_\theta(B_w))\). Fix a \(j\). We now assume that there exist \(\theta_0 > \theta\) such that the only no poles of \(B_w\) in between the two lines \(e^{i\theta_j - \theta_0} \mathbb{R}_+\) and \(e^{i\theta_j + \theta_0} \mathbb{R}_+\) are the ones on the line \(e^{i\theta_j} \varepsilon\). Since we are under the assumptions that \(B_w\) decays fast enough in a small sector around \(e^{i\theta_j} \mathbb{R}_+\), we can now consider a contour integral along \(C_{\varepsilon}^r\), where \(C_{\varepsilon}^r\) starts at \(e^{i\theta_j} \varepsilon\), where \(0 < \varepsilon < R_{\varepsilon}\) and follow the circle of radius \(\varepsilon\) to \(e^{i\theta_j - \theta_0}\), then follows the line \(e^{i\theta_j - \theta_0} \mathbb{R}_+\) to say \(e^{i\theta_j - \theta_0} R_n\), now follows the circle of radius \(R_n\) to \(e^{i\theta_j + \theta_0} R_n\) and finally follows \(e^{i\theta_j + \theta_0} \mathbb{R}_+\) from \(e^{i\theta_j + \theta_0} R_n\) to \(e^{i\theta_j + \theta_0} \varepsilon\).
and back to $e^{\theta_j} \xi$ along the circle of radius $\epsilon$. Here we are assuming that $R_n$ are chosen such that $e^{\theta_j}, R_n \notin \mathcal{P}_{B,w}$ and further $R_n \to \infty$ as $n \to \infty$. By the residue theorem we get that

$$
\int_{C_n^2} e^{-\xi/\gamma} B_w(\xi) d\xi = 2\pi i \sum_{p \in \mathcal{P}_{B,w} \cap D(0,R_n)} \text{Res}_{\xi=p}(e^{-\xi/\gamma} B_w(\xi)).
$$

The theorem now follows directly from taking the limit as $n \to \infty$ and $\epsilon \to 0$ and using that the two integrals along the two circle arcs go to zero by the assumptions on $B_w$.

We further prove Theorem 4 as follows.

Proof. Following the same line of arguments as in the proof of Theorem 3 and using the first assumption of Theorem 4 we see that the integral

$$
\Psi^{(2)}_w(\xi) = \int_{\tilde{\Gamma}_w} K(w,z) \mathcal{L}^{-1}_\theta (\psi_z)(\xi) dz
$$

is absolutely convergent and defines $\Psi^{(2)}_w \in \mathcal{O}(V_{w,\theta})$ for all $w \in W$ and further

$$
\mathcal{L}_\theta (\Psi^{(2)}_w)(\gamma) = \int_{\tilde{\Gamma}_w} K(w,z) \psi_z(\gamma) dz
$$

for all $\gamma \in U_{\theta,\alpha}$. Now let

$$
\Psi^{(1)}_w(\xi) = \sum_{p \in \mathcal{P}_w} \left( \int_{\tilde{\Gamma}_w} K(w,z) e^{\frac{\xi}{\gamma}} dz \right) \sum_{m=1}^{n_p} \frac{b_p}{p^m} (-1)^m \sum_{l=0}^{m-1} \left( m \atop l \right) \left( \frac{z}{p} \right)^{m-l} \frac{\xi^{m-l-1}}{(m-l-1)!},
$$

which by the second assumption of Theorem 4 gives a holomorphic function $\Psi^{(1)}_w \in \mathcal{O}(V_{\theta})$ for all $w \in W$. By the third assumption of the theorem we see that

$$
\int_{R_\theta^+} |e^{-\xi/\gamma} e^{\frac{\xi}{\gamma}}| \xi^m |d\xi| < \infty
$$

and in fact we can easily compute that

$$
\int_{R_\theta^+} e^{-\xi/\gamma} e^{\frac{\xi}{\gamma}} \frac{\xi^m}{m!} d\xi = \frac{1}{(\frac{1}{\gamma} - 1)^{m+1}}.
$$

So by the second assumption of the Theorem we can apply Fubini’s theorem, so as to compute

$$
\mathcal{L}_\theta(\Psi^{(1)}_w)(\gamma) = \sum_{p \in \mathcal{P}_w} \sum_{m=1}^{n_p} \frac{b_p}{p^m} (-1)^m \sum_{l=0}^{m-1} \left( m \atop l \right) \int_{\tilde{\Gamma}_w} K(w,z) \left( \frac{z}{p} \right)^{m-l} \int_{R_\theta^+} e^{-\xi/\gamma} e^{\frac{\xi}{\gamma}} \frac{\xi^{m-l-1}}{(m-l-1)!} d\xi dz
$$

$$
= \sum_{p \in \mathcal{P}_w} \sum_{m=1}^{n_p} \frac{b_p}{p^m} (-1)^m \sum_{l=0}^{m-1} \left( m \atop l \right) \int_{\tilde{\Gamma}_w} K(w,z) \left( \frac{z}{p} \right)^{m-l} \left( \frac{1}{\gamma} - \frac{z}{p} \right)^{m-l} dz
$$

$$
= \int_{\tilde{\Gamma}_w} K(w,z) \psi_z(\gamma) dz
$$

But then for all $w, \gamma \in W \times V_{w,\theta}$

$$
\mathcal{L}_\theta(\Psi^{(1)}_w)(\gamma) + \mathcal{L}_\theta(\Psi^{(2)}_w)(\gamma) = \int_{\tilde{\Gamma}_w} K(w,z) (\psi_z + \psi_z)(\gamma) dz = \mathcal{L}_\theta(B_w)(\gamma)
$$
by Theorem 2 from which we conclude the Theorem. □

6. The examples of Faddeev’s quantum dilogarithm $S_\gamma$

We let

$$K_F(w, z) = \frac{e^{wz}}{\sinh(\pi z)} \quad \text{and} \quad f_F(z) = \frac{1}{\sinh(z)}$$

and

$$S_\gamma(w) = \exp\left(\frac{1}{4}g_\gamma^F(w)\right) \quad \forall (w, \gamma) \in \tilde{W}^F \times U^F$$

where

$$g_\gamma^F(w) = \int_{\mathbb{R}+i\epsilon} \frac{e^{wz}}{\sinh(\pi z)z \sinh(\gamma z)} \frac{1}{z} dz$$

and

$$\tilde{W}^F = \{ w \in \mathbb{C} \mid |\text{Re}(w)| < \pi + |\text{Re}(\gamma)| \}, \quad U^F = \{ \gamma \in \mathbb{C} \mid \text{Re}(\gamma) > 0 \}$$

where $S_\gamma \in \mathcal{M}(\mathbb{C})$ is Faddeev’s quantum dilogarithm.

We observe that $k_0 = 2$ and $n_0 = 1$ and further that $R_{K_F} = 1$ and $R_{f_F} = \pi$. We recall that the Laurent series for $f_F$ convergent in $D(0, \pi)$ is

$$f_F(z) = \sum_{m=0}^{\infty} \frac{2(1 - 2^{2m-1})B_{2m}}{(2m)!} z^{2m-1} \quad \forall z \in D(0, \pi),$$

where $B_{2m}$ is the $2m$’th Bernoulli number. In anticipation of the asymptotic expansion we let

$$P^\gamma_n(w) = \frac{1}{4} \sum_{m=0}^{\infty} \frac{2(1 - 2^{2m-1})B_{2m}}{(2m)!} \sum_{m=0}^{\infty} \frac{e^{wz}}{\sinh(\pi z)z} \frac{1}{z} dz.$$
This definition of $U^F_\theta$ guarantees that the poles of $f^F_\gamma$, which are all on the imaginary axis, when $\gamma$ is positive real, never crosses $\hat{\Gamma}^F_w$ as the absolute value of the argument of $\gamma$ grows from zero to $\pi$, not including $\pi$. We observe that

$$\text{Re}(\gamma) \cos \theta(\gamma) \neq \text{Im}(\gamma) \sin \theta(\gamma)$$

for all $\gamma \in U^F_\theta$.

Let now $0 < \delta < \pi \cos \theta$.

**Proposition 2.** We have for all $t \in \mathbb{R} - \{0\}$ and $w \in W^F_\theta$ that

$$|K_F(w, e^{i\theta}t)| \leq \frac{\sqrt{2}}{\sqrt{\pi}(1 - e^{-2\pi \cos \theta})} \frac{1}{\delta(1 - \alpha)} e^{-\alpha |t|} t^{-2}, \quad \forall \alpha \in (0, 1)$$

and for all $t \in \mathbb{R} - \{0\}$ and $\gamma \in U^F_\theta$ that

$$|f_{F, \gamma}(e^{i\theta}t)| \leq \frac{\sqrt{2}}{\sqrt{\pi}(1 - e^{-2\pi \cos \theta})} \frac{|\gamma|}{\text{Re}(\gamma) \cos \theta - \text{Im}(\gamma) \sin \theta} |\gamma| t^{-1}.$$

From this proposition we see that

$$\delta^F = \alpha \delta, \quad c^F_w = \frac{\sqrt{2}}{\sqrt{\pi}(1 - e^{-2\pi \cos \theta})} \frac{1}{(1 - \alpha) \delta}$$

and

$$c^F_{\gamma} = \frac{\sqrt{2}}{\sqrt{\pi}(1 - e^{-2\pi \cos \theta})} \frac{|\gamma|}{\text{Re}(\gamma) \cos \theta - \text{Im}(\gamma) \sin \theta} \geq \frac{\sqrt{2}}{\sqrt{\pi}(1 - e^{-2\pi \cos \theta})} = c^F.$$  

**Proof.** For $t \in \mathbb{R}_+$, $\alpha \in (0, 1)$ and $w \in W^F_\theta$, using that

$$(\pi - \text{Re}(w)) \cos \theta + \text{Im}(w) \sin \theta > \delta > 0,$$

and recalling that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we compute

$$|K_F(w, z)| \leq \frac{2 e^{-t((\pi - \text{Re}(w)) \cos \theta + \text{Im}(w) \sin \theta)}}{|1 - e^{-2\pi t (\cos \theta + i \sin \theta)}| |t|} \leq \frac{2 t e^{-(1-\alpha)\delta} e^{-\alpha \delta t}}{(1 - e^{-2\pi t \cos \theta})^2} \leq \frac{2 e^{-\alpha \delta t}}{\delta(1 - \alpha) \sqrt{2\pi}(1 - e^{-2\pi \cos \theta})^2}.$$  

Where we have used that

$$\frac{te^{-(1-\alpha)\delta t}}{(1 - e^{-2\pi t \cos \theta})^2} \leq \frac{1}{\delta(1 - \alpha) \sqrt{2\pi}(1 - e^{-2\pi \cos \theta})},$$

which completes the proof for $t \in \mathbb{R}_+$. The proof for $t \in \mathbb{R}_-$ is completely similar.

For $t \in \mathbb{R}_+$ and $\gamma \in U^F_\theta$ we have that

$$\frac{1}{|\sinh(\gamma e^{i\theta}t)|} = \frac{2}{1 - e^{-2t(\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta)}} e^{-t(\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta)} \leq \frac{2 t (\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta)}{|1 - e^{-2t(\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta)}|} \frac{|\gamma|}{\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta} \frac{1}{|\gamma t|} \leq \frac{2}{\sqrt{2\pi}(1 - e^{-2\pi \cos \theta})} \frac{|\gamma|}{\text{Re}(\gamma) \cos \theta + \text{Im}(\gamma) \sin \theta} \frac{1}{|\gamma t|}$$

and the proof for $t \in \mathbb{R}_-$ is completely similar. □
By Proposition 2 we conclude for \((w, \gamma) \in W \times U^F\) that
\[
\varrho^F_{\gamma}(w) = \int_{e^{i(\mathbb{R}+i\epsilon)}} e^{u z} \frac{1}{\sinh(\pi z) \sinh(\gamma z)} dz
\]
and
\[
h^F_{\gamma}(w) = \int_{e^{i(\mathbb{R}+i\epsilon)}} e^{u z} \frac{1}{\sinh(\pi z) z^m} dz
\]
are absolutely convergent and thus they are well defined holomorphic functions on
\(W^F\). Thus we have for \((w, \gamma) \in W^F \times U^F\) that
\[
P^\gamma_n(w) = \frac{1}{\alpha} \sum_{m=0}^{n} 2(1 - 2^{2m-1})B_{2m} \gamma^{2m-1} h^F_{2m-1}(w).
\]
So we can conclude that for \((w, \gamma) \in W^F \times U^F\) the required estimates (5) and (6) are satisfied and thus our Theorem 1 therefore applies. We see that in this example
\[
k^F(r) = \frac{1}{\sin(r)}, \quad r \in (0, \pi)
\]
and that
\[
c'_n = \inf_{r \in (0, \pi)} b_n(r),
\]
where
\[
b_n(r) = \frac{1}{r^{n+1}} \left( \frac{c^F_n + 1}{\sin(r)} \right), \quad r \in (0, \pi).
\]
We observe that \(b_n(r) \to \infty\) for \(r \to 0\) and \(r \to \pi\), thus there exist \(r_n \in (0, \pi)\) such that
\[
c'_n = b_n(r_n).
\]
Since
\[
b'_n(r) = -\frac{1}{r^{n+3}} \left( c^F_n + \frac{1}{n} \frac{r(n+1)}{\sin(r)} + \frac{r^2 \cos(r)}{\sin^2(r)} \right)
\]
which is negative for \(r \leq \frac{\pi}{2}\), hence \(r_n \in \left( \frac{\pi}{2}, \pi \right)\). In particular
\[
c'_1 \leq \left( \frac{2c^F_1 + 1}{\pi} \right)^{n+1} \cdot
\]
\[
C^2_{2n} = \frac{2}{\sqrt{2n+1} - \sqrt{\pi(1 - e^{-2\pi\cos(\theta)})}} c'_2, \quad C^2_{2n} = \inf_{\alpha \in (0,1)} \frac{1}{1 - \alpha} \frac{1}{\alpha^{2n+1}} c'_2.
\]

**Theorem 7.** We have the following estimates for all positive integers \(n\)
\[
| \log(S_n(w)) - P^\gamma_n(w) | \leq \tilde{C}^F_{2n} |\gamma|^{2n\delta-2n-1} (2n)!
\]
for all \((w, \gamma) \in W^F \times U^F\) and all \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\) where
\[
\tilde{C}^F_{2n} = \frac{8 (1 + \frac{1}{\alpha})^{2n+1} (\frac{c^F_n + 1}{\pi^n} + 1) (\frac{2}{\pi})^{2n+1} |\gamma|}{\pi \sqrt{2n + 1} (1 - e^{-2\pi\cos(\theta)})(1 - e^{-2})(\text{Re}(\gamma) \cos \theta - \text{Im}(\gamma) \sin \theta)}.
\]

This is similar, but not identical, to the estimates obtained, using advanced techniques from the theory of resurgence, in [GK], in the case where \(\text{Re}(\gamma) > 0\), e.g. in the case our \(\theta = 0\). Ours is more general, since it does not require that \(\text{Re}(\gamma) > 0\), in fact, we see that as we vary \(\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\) the above theorem applies to all \(\gamma \in \mathbb{C} - \mathbb{R}_-\). We remark that our general Theorem 5 of course also applies and gives a corresponding estimate for the case \(n = 0\).
Let us now move on to the applications of Theorem 8 and 11. We will now establish the needed estimates in order to apply these two theorem. We recall that for all $z \in \mathbb{C} - i\pi \mathbb{Z}$

$$\frac{1}{\sinh(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{(-1)^n z}{z - i\pi n} + \frac{(-1)^n z}{z + i\pi n} \right) = \frac{1}{z} + 2 \sum_{n=1}^{\infty} \frac{(-1)^n z}{z^2 + \pi^2 n^2},$$

where the last series converges uniformly on

$$O_z = \mathbb{C} - \bigcup_{n \in \mathbb{Z} - \{0\}} D(i\pi n, \varepsilon)$$

for all $\varepsilon > 0$. Thus

$$\varphi_z^F(\gamma) = \frac{1}{\sinh(\gamma z)} - \frac{1}{\gamma z} = \frac{2z}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \left( \frac{1}{\gamma} \right)^2 + \left( \frac{z}{\pi n} \right)^2}$$

converges uniformly in

$$\gamma \in O_{\varepsilon,z} = \mathbb{C} - \bigcup_{n \in \mathbb{Z} - \{0\}} D\left(\frac{i\pi n}{z}, \varepsilon\right)$$

for $\varepsilon > 0$ and $z \neq 0$. Let us now check that when

$$\frac{1}{|\gamma|^2} \left( \frac{\text{Re}(\gamma)}{\text{Im}(\gamma)} \right) \cdot \left( \cos \theta \cdot \sin \theta \right) > \frac{1}{\pi} \left( \frac{\text{Im}(z)}{\text{Re}(z)} \right) \cdot \left( \cos \theta \cdot \sin \theta \right)$$

then

$$\mathcal{L}_\theta^{-1}\left( \frac{1}{(\frac{z}{\pi})^2 + (\frac{\xi}{\pi})^2} \right)(\xi) = \cos(\frac{\xi z}{\pi n}).$$

Assuming (18) we see that

$$\int_{\mathbb{R}_+} \left| e^{-\xi/\gamma} \cos\left( \frac{\xi z}{\pi n} \right) \right| d\xi \leq |\theta| \left( \int_{\mathbb{R}_+} e^{-t\left( \frac{\text{Re}(\gamma)}{|\gamma|^2} + \frac{\text{Im}(z)}{\pi n} \right) \cos \theta + \left( \frac{\text{Re}(\gamma)}{|\gamma|^2} + \frac{\text{Re}(z)}{\pi n} \right) \sin \theta} \right) dt$$

$$+ \int_{\mathbb{R}_+} e^{-t\left( \frac{\text{Re}(\gamma)}{|\gamma|^2} - \frac{\text{Im}(z)}{\pi n} \right) \cos \theta + \left( \frac{\text{Re}(\gamma)}{|\gamma|^2} - \frac{\text{Re}(z)}{\pi n} \right) \sin \theta} \right) dt$$

$$= \frac{|\theta|}{\left( \frac{\text{Re}(\gamma)}{|\gamma|^2} + \frac{\text{Im}(z)}{\pi n} \right) \cos \theta + \left( \frac{\text{Re}(\gamma)}{|\gamma|^2} + \frac{\text{Re}(z)}{\pi n} \right) \sin \theta}$$

$$+ \frac{|\theta|}{\left( \frac{\text{Re}(\gamma)}{|\gamma|^2} - \frac{\text{Im}(z)}{\pi n} \right) \cos \theta + \left( \frac{\text{Re}(\gamma)}{|\gamma|^2} - \frac{\text{Re}(z)}{\pi n} \right) \sin \theta}.$$
By Theorem 2 equation (10), we can compute

\[
B_w(\xi) = \int_{\Gamma^F_w} K_F(w, z) L^{-1}(\varphi^F_r)(\xi) dz = \int_{\Gamma^F_w, \gamma} \frac{e^{wz}}{\sinh(\pi z)} \left( \frac{2z}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \left( \frac{\xi z}{\pi n} \right) \right) dz
\]

\[
= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_{\Gamma^F_w, \gamma} \frac{e^{wz}}{\sinh(\pi z)} \cos \left( \frac{\xi z}{\pi n} \right) dz
\]

\[
= \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left( \int_{\Gamma^F_w, \gamma} \frac{e^{(w+i\xi)z}}{\sinh(\pi z)} dz + \int_{\Gamma^F_w, \gamma} \frac{e^{(w-i\xi)z}}{\sinh(\pi z)} dz \right),
\]

which is indeed the expression in Theorem 4 in this particular case. However, we must justify this interchange of the integral and the sum by Fubini’s theorem, thus we must establish that

\[
(20) \quad \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{e^{i\theta}(R+ir)} \left| \frac{e^{wz}}{\sinh(\pi z)} \cos \left( \frac{\xi z}{\pi n} \right) \right| dz < \infty.
\]

To this end we observe for \( z \in e^{i\theta}R_\pm \) that

\[
\left| \frac{e^{wz}}{\sinh(\pi z)} \cos \left( \frac{\xi z}{\pi n} \right) \right| \leq e^{-|z|} \left| (\pi - \text{Re}(w)) + \frac{i}{\pi} \text{Im}(\xi) \cos \theta + (\text{Im}(w) + \frac{i}{\pi} \text{Im}(\xi)) \sin \theta \right|
\]

\[
+ e^{-|z|} \left| (\pi - \text{Re}(w)) - \frac{i}{\pi} \text{Im}(\xi) \cos \theta + (\text{Im}(w) - \frac{i}{\pi} \text{Im}(\xi)) \sin \theta \right|
\]

(21)

Thus we now see that

\[
V_{w, \theta} = \left\{ \xi \in \mathbb{C} \mid \left( \pi - \text{Re}(w) \right), \left( \cos \theta \sin \theta > \frac{1}{\pi} \left( \text{Im}(\xi) \cdot \text{Re}(\xi) \right) \cdot \left( \cos \theta \sin \theta \right) \right) \right\}.
\]

To conclude (20), we first deform the contour such that outside the unit disc the contour is along \( e^{i\theta}R \) and inside the unit disc \( z \) along contour satisfies

\[
\frac{1}{|z|^2} \left( \frac{\text{Re}(\gamma)}{\text{Im}(\gamma)} \right) \cdot \left( \cos \theta \sin \theta > \frac{1}{\pi} \left( \text{Im}(z) \right) \cdot \left( \cos \theta \sin \theta \right) > 0
\]

Now, for the part of the integral inside the unit disc we have uniform bound on \| \cos(\frac{\xi z}{\pi n}) \|, so when combined with the sum over \( n \), it is clearly finite. The integrals over \( e^{i\theta}(-\infty, 1] \) and \( e^{i\theta}[1, \infty) \), are by the estimate (21) universally bounded in \( n \).

As we observed above, differentiating

\[
\frac{1}{2i} \text{Li}_2(-e^{iw}) = \frac{1}{4} \int_{\Gamma^F_w} \frac{e^{wz}}{\sinh(\pi z) z^2} dz
\]

twice in \( w \) we obtain

\[
\frac{1}{1 + e^{-iw}} = \frac{i}{2} \int_{\Gamma^F_w} \frac{e^{wz}}{\sinh(\pi z)} dz.
\]

Thus we proved that

**Theorem 8.** For all \( 0 < \delta < \pi \cos \theta \), \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), \( w \in W^F_\theta \) and \( \xi \in V_{w, \theta} \) we have that

\[
B_w(\xi) = \frac{2}{i\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \left( \frac{1}{1 + e^{-(w+i\xi)\pi/n}} + \frac{1}{1 + e^{-(w-i\xi)\pi/n}} \right)
\]
We observe that this formula matches the formula obtained in \cite{GK} in the special case where $\theta = 0$, e.g. in the case where $\text{Re}(\gamma) > 0$.

From this we can of course now apply Theorem 3 to obtain formula for the Stokes coefficients, since they following immediately from that theorem and the above expression for $B_w$. We observe that $B_w$ actually extends to a meromorphic function on all of $\mathbb{C}$, e.g. $B_w \in \mathcal{M}(\mathbb{C})$ for all $w \in \mathbb{C} - (\pi + 2\pi \mathbb{Z})$.

Furthermore its poles are

$$P_{B_w} = \pm i\pi \mathbb{Z}_+(\pi - w + 2\pi \mathbb{Z}).$$

7. Further examples

As further examples of interesting meromorphic transformations we recall a few classical examples, to which our main Theorems, possible with small modifications, applies. Let $d(z, X)$ be the distance from $z$ to $X$.

7.1. The Gamma function $\Gamma$. We now let

$$K_\Gamma(w, z) = \frac{i e^{(w-1)\log(-z)}}{2\sin(\pi z)} \quad \text{and} \quad f_\Gamma(z) = e^{-z}$$

and

$$\Gamma^\Gamma_w = \{ z \in \mathbb{C} \mid d(z, \mathbb{R}_+) = \varepsilon \}$$

oriented from $\infty + i\varepsilon$ to $\infty - i\varepsilon$. We then have that

$$\Gamma_\gamma(w) = g^\Gamma_\gamma(w)$$

where $\Gamma_1$ is Euler’s Gamma function.

7.2. One over the Gamma function $\frac{1}{\Gamma}$. We now let

$$K_\frac{1}{\Gamma}(w, z) = \frac{1}{2\pi i} e^{-w\log(z)} \quad \text{and} \quad f_\frac{1}{\Gamma}(z) = e^{z}$$

and

$$\Gamma^\frac{1}{\Gamma}_w = \{ z \in \mathbb{C} \mid d(z, \mathbb{R}_-) = \varepsilon \}$$

oriented from $-\infty - i\varepsilon$ to $-\infty + i\varepsilon$. We then have that

$$\frac{1}{\Gamma_\gamma(w)} = g^\frac{1}{\Gamma}_\gamma(w).$$

7.3. Riemann zeta function $\zeta$. We set

$$K_\zeta(w, z) = \frac{\Gamma(1 - w)e^{(w-1)\log(z)}}{2\pi i} \quad \text{and} \quad f_\zeta(z) = \frac{1}{e^z - 1}$$

and

$$\Gamma^\zeta_w = \{ z \in \mathbb{C} \mid d(z, \mathbb{R}_+) = \varepsilon \}$$

oriented from $-\infty - i\varepsilon$ to $-\infty + i\varepsilon$. We then have that

$$\zeta_\gamma(w) = g^\zeta_\gamma(w)$$

where $\zeta = \zeta_1$. 
7.4. Gauss hypergeometric function $\,_{2}F_{1}$. We set
\[
K_{2F_{1}}(w, z) = e^{z \log(-w)} \quad \text{and} \quad f_{\,_{2}F_{1}}(z) = \frac{\Gamma(-z)\Gamma(z + a)\Gamma(z + b)}{\Gamma(z + c)}
\]
and
\[
\Gamma_{w}^{\,_{2}F_{1}} = i\mathbb{R}
\]
with the orientation induced from the usual one on $\mathbb{R}$. Then
\[
\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}\,_{2}F_{1}^\gamma(a, b; c; w) = \frac{1}{2\pi i}g_{\,_{2}F_{1}}(w)
\]
with $\,_{2}F_{1}(a, b; c; w) = \,_{2}F_{1}(a, b; c; w)$ Gauss hypergeometric function.

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