Holographic dark energy and $f(R)$ gravity

A Aghamohammadi and Kh Saaidi

1 Department of Physics, Faculty of Science, University of Kurdistan, Sanandaj, Iran
2 Faculty of Science, Islamic Azad University of Sanandaj, Sanandaj, Iran

E-mail: ksaaidi@uok.ac.ir and agha35484@yahoo.com

Received 14 July 2010
Accepted for publication 15 December 2010
Published 17 January 2011
Online at stacks.iop.org/PhysScr/83/025902

Abstract

We investigate the corresponding relation between $f(R)$ gravity and holographic dark energy. We introduce a type of energy density from $f(R)$ that has the same role as holographic dark energy. We obtain the differential equation that specifies the evolution of the introduced energy density parameter based on a varying gravitational constant. We discover the relation for the equation of state parameter for low redshifts that contains varying $G$ correction.

PACS number: 95.36.+x

1. Introduction

Observational data [1–3] indicate that there are several shortcomings of standard gravity [4–6] related to cosmology, large-scale structure and quantum field theory. Several attempts have been made to solve these problems by adding new and unjustified components in order to give an adaptable picture for the present universe [7, 8]. One approach for solving these problems is to add new cosmic fluids to the right-hand side (rhs) of equations, which can cause clustered structures (dark matter) or accelerated dynamics (dark energy) [9, 10]. Although there is no quantum theory of gravity, one can proceed to investigate the nature of dark energy based on some principles of quantum gravity. A significant attempt in this regard is the holographic dark energy (HDE) proposal [11, 12]. As a rule, in quantum field theory, $\rho_\Lambda$ as zero-point energy density is defined based on $L$ (the size of the present universe) as follows:

$$\rho_\Lambda = 3c^2M_\text{P}^2L^{-2},$$

where $c^2$ is a numerical constant of order unity and $M_\text{P}^2 = 8\pi G$ is the reduced Planck mass. This $\rho_\Lambda$ is comparable to the present dark energy [13, 17]. The HDE model may find a solution for the coincidence problem, i.e., why are matter and dark energy densities comparable? However, matter and dark energy have different equations of motion [17]. Lately, the holographic dark energy model has been extended and constrained by various astronomical observations [18]. It has also been extended to involve the spatial curvature contribution [19].

An alternative approach is the extended Einstein’s theory of gravity, for which $f(R)$ gravity is an extended theories of gravity candidate, which was itself advocated to account for this accelerating universe [20–33]. This kind of theory is fruitful and economical compared to other theories. The assumption is that the Ricci scalar of the Einstein–Hilbert action is replaced by an arbitrary function $f(R) = R + h(R)$. The $f(R)$ gravity is assumed to replace dark energy.

In this paper, the evolution of state parameter of holographic dark energy is studied. We investigate the equivalence between $f(R)$ gravity and holographic dark energy. In this regard, we define a dark energy density-like quantity by transmitting some term to the rhs from the generalized Friedmann equation. As a matter of fact, the equivalence between $f(R)$ gravity and dark energy is like the connection between $\Lambda$ terms and dark energy. Note that, at first, Einstein added a $\Lambda$ term to his theory as a geometrical term. As a rule, in most dark energy models investigated, Newton’s gravitation constant is assumed constant. Here, we consider Newton’s gravitation constant to vary with time. There are significant reasons why $G$ can vary with time or be a function of the scale factor [34]. Among them, helio-seismological [35] and astro-seismological data from the pulsating white dwarf star G117-B15A [36] and Hulse–Taylor binary pulsar [37, 38] result in $|G/G| \lesssim 4.10 \times 10^{-11} \text{ yr}^{-1}$ for $z \lesssim 3.5$ [39]. In addition, a varying $G$ has some theoretical benefits such as the discord in the Hubble parameter value [40], the cosmic coincidence problem [41], and alleviating the dark matter problem [42]. The authors of [14] have investigated the generalized
holographic dark energy model, in which Newton’s constant and cosmological constant are scale dependent. They found that the holography energy actually comes back to the quintessence proposal. The authors of [15] have explored the possible cosmological consequences of a varying Newton’s constant. They have obtained a modified equation for the matter density and provided an estimate for the growing post-Newtonian parameter, γ, in the presence of G. Some investigations of a scale-dependent Newton’s constant have also been carried out in [16].

In this work, the evolution of state parameter of holographic dark energy is investigated for a type of \( f(R) \) model of gravity. We study and discuss the holographic dark energy-like quantity in the \( f(R) \) model with the time-dependent gravitational constant. Finally, we obtain the equation of state parameter for small redshifts that agrees with time-dependent gravitational constant. Therefore, equation (3.1) reduces to the conclusion.

This paper is organized as follows. In section 2, we will review \( f(R) \) gravity cosmology. In section 3, we find that the holographic dark energy-like quantity with gravitational constant depends on time and we derive the differential equation that specifies the evolution of the dark energy parameter. In section 4, we obtain the parameter of the dark energy equation of state at low redshifts. Section 5 is devoted to the conclusion.

2. Theories of \( f(R) \) gravity

We consider a class of modified gravity that modifies the Einstein–Hilbert action by replacing the Ricci curvature scalar by an arbitrary function of curvature as follows:

\[
S = \int \left( \frac{f(R)}{2} + k L_m \right) \sqrt{-g} d^4 x, \tag{2}
\]

which is introduced in the cosmological context and \( k = 8\pi G \). Variation with respect to a metric yields gravitational equations of motion as

\[
R_{\mu\nu} f' - \frac{1}{2} f g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) f' = k T_{\mu\nu}, \tag{3}
\]

where the prime indicates derivation with respect to the curvature scalar \( R \) and \( \Box \) is the covariant d’Alembert operator \( (\Box \equiv \nabla_\mu \nabla^\mu) \). The equation of motion for a new scalar degree of freedom is given by the trace of equation (3) as \[31\]

\[
\Box f' = \frac{kT}{3} + \frac{2f - R f'}{3}. \tag{4}
\]

We redefine the scalar degree of freedom by

\[
\phi = f' - 1. \tag{5}
\]

Therefore, equation (4) in the form of an equation of motion of the canonical dimensionless scalar field \( \phi \) with a force term \( \mathcal{F} \) and potential \( V \) is as follows:

\[
\Box \phi = V'(\phi) - \mathcal{F}, \tag{6}
\]

where \( V'(\phi) = \frac{1}{2}(2f - R f') \), and the force term that drives the scalar field \( \phi \) is a trace of the stress–energy tensor \( \mathcal{F} = (8\pi G/3)T \). Considering a homogeneous cosmological model in \( f(R) \) gravity with the usual matter field, the length element that describes the expansion of the universe is expressed by a flat Friedmann–Robertson–Walker metric as

\[
ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{7}
\]

and using equations (6) and (7), the scalar degree of freedom \( \phi \) obeys a usual scalar field equation with a force term on the right-hand side as

\[
\phi + 3H \dot{\phi} + V'(\phi) = \mathcal{F}. \tag{8}
\]

Considering the \( tt \) component of gravitational equations from equation (3) for the metric (7), we have

\[
3H (f') - \frac{3}{a} \dot{f'} + \frac{1}{2} f = 8\pi G \rho_f. \tag{9}
\]

Let us write \( \ddot{a} \) with respect to the curvature scalar by using

\[
R = 6 \left( \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 \right). \tag{10}
\]

Finally, equation (9) becomes

\[
H^2 + (\ln f') + \frac{1}{6} \left( \frac{f - R f'}{f'} \right) = \frac{8\pi G}{3} \rho_f. \tag{11}
\]

It is clear that if \( f' \rightarrow 1 \) then equation (11) reduces to the usual Friedmann equation. In the general case, the extra terms are functions of the scalar degree of freedom \( \phi \) and its first time derivative. Hence, we can re-write equation (11) in the following standard form:

\[
H^2 = \frac{8\pi G}{3} \left( \rho_m + \rho_f \right), \tag{12}
\]

where \( \rho_f \) is assumed as the dark energy density \( \rho_f \). We can define \( \rho_f \) as follows:

\[
\rho_f = \frac{3}{8\pi G} \left( (\ln f') + \frac{1}{6} \left( \frac{f - R f'}{f'} \right) \right). \tag{13}
\]

3. The holographic dark energy-like quantity with gravitational constant depending on time

3.1. The density parameter

We will set up the holographic dark energy-like quantity for a Newton’s constant, \( G \), which varies with time. Therefore, we consider the metric given by equation (7). Considering the form of the first Friedmann equation, from equation (12) we have

\[
H^2 = \frac{8\pi G}{3} \left( \frac{\rho_m a^3}{\rho_f} \right). \tag{14}
\]

Here \( \rho = \rho_m + \rho_f \) is the energy density; \( \rho_m = \rho_{m_0} a^{-3} \) and \( \rho_f \) are dark matter density and the dark energy density-like quantity, respectively; and \( \rho_{m_0} \) indicates the present value of that quantity. Then, from equation (12),
we introduce the effective density parameter $\Omega_{fe} = \Omega_f/(\phi + 1) \equiv (8\pi G/3(\phi + 1))\rho_f$. Substituting equation (1) into $\Omega_{fe}$ we obtain

$$\Omega_{fe} = \frac{c^2}{H^2 L^2}. \quad (15)$$

As one way to obtain an appropriate definition of $L$, we specify it by the future event horizon as [17, 43]

$$L \equiv d_h(a) = a \int_{t_0}^{\infty} \frac{dt'}{a(t')} = a \int_{t_0}^{\infty} \frac{da'}{Ha'^2}. \quad (16)$$

Henceforth, we will use $\ln a$ as an independent variable. Therefore, we define $X = dX/d\eta$, and $X' = dX/d\ln a$, so that $X = X'H$. By differentiation with respect to coordinate time from equation (15) and that into (16), we have $d_h = H_d h - 1$, and one can obtain

$$\frac{\Omega_f'}{\Omega_{fe}} = -\left( \frac{2H}{H^2} + 2 \left( 1 - \frac{\sqrt{\Omega_{fe}}}{c} \right) \right). \quad (17)$$

To clarify the effect of $G$ on $\Omega_{fe}$, we should eliminate $H$ in equation (17) in favor of $\Omega_{fe}$. In this regard, differentiation of the Friedmann equation yields

$$H = \frac{4\pi}{3(\phi + 1)} \left( G' - 3G(1 + \omega) - \frac{G\phi'}{(\phi + 1)} \right) \rho, \quad (18)$$

where we use the fluid equation $\rho = -3H(1 + \omega)\rho$. $\omega$ is

$$\omega = \frac{\omega_1 \rho_f}{\rho} = \frac{\omega_f}{\Omega_m + \Omega_f} = \omega_f \Omega_f, \quad (19)$$

where, according to (12), we use $\Omega_f + \Omega_m = 1$. Now we must obtain the equation of state parameter for $\rho_f$ and corresponding pressure. For this purpose, one can differentiate $\rho_f = 3c^2(1 + \phi)/8\pi GL^2$ and arrive at

$$\dot{\rho}_f = \rho_f \left[ \frac{\chi}{1 + \phi} - 2H \left( 1 - \frac{1}{LH} \right) \right]; \quad (20)$$

then, by making use of the fluid equation $\rho_f = -3H(1 + \omega_f)\rho_f$, we obtain

$$\omega_f = -\frac{1}{3} \left( 1 + \frac{2\sqrt{\Omega_{fe}}}{c} \right) \frac{\phi'}{3H(1 + \phi)}. \quad (21)$$

It is remarkable that for $\phi = 0$, equation (21) reduces to the $\omega_f$ of HDE which was obtained in [43]. Substituting equations (19) and (21) into equation (18) gives

$$\frac{2H}{H^2} = \left[ G' - 3\frac{\phi'}{1 + \phi} \left( 1 - \Omega_{fe} \right) + \Omega_{fe} + 2\Omega_{fe}' \right]. \quad (22)$$

where $G'$ is $G'/G$. Then by substituting (22) into (17), we have

$$\frac{\Omega_{fe}'}{\Omega_{fe}} = -\left[ G - 1 + \Omega_{fe} + (\Omega_{fe} - 1) \left( \frac{2\sqrt{\Omega_{fe}}}{c} + \frac{\phi'}{1 + \phi} \right) \right]. \quad (23)$$

4. Some cosmology applications

Here, we should follow up an expression related to the equation of state parameter-like quantity at the present time. We have derived the representation for $\Omega_{fe}$; hence we can find the $\omega(z)$ form, at small redshifts. According to [44], one can measure $\omega$ by using $\rho_f$ as $\rho_f \sim a^{-3(1+\omega)}$. Expanding $\rho_f$ we have

$$\ln \rho_f = \ln \rho_f^0 + \frac{d \ln \rho_f}{d \ln a} \ln a + \frac{1}{2} \left( \frac{d^2 \ln \rho_f}{d (\ln a)^2} \right) (\ln a)^2 + \cdots. \quad (24)$$

where the derivatives are taken at the present time $a_0 = 1$. Then, $w(z)$ is given, at small redshifts $\ln a = -\ln(1+z) = -z$ up to second order, as

$$\omega(z) = -1 - \frac{1}{3} \left( \frac{d \ln \rho_f}{d \ln a} - \frac{1}{2} \left( \frac{d^2 \ln \rho_f}{d (\ln a)^2} \right) (z) \right). \quad (25)$$

Succinctly, we can rewrite (25) as

$$\omega = \omega_0 + \omega_1 z. \quad (26)$$

We replace

$$\rho_f = \frac{3(\phi + 1)}{8\pi G} \Omega_{fe} = \frac{\rho_{m0} \Omega_{fe} a^{-3}}{1 - \Omega_{fe}}. \quad (27)$$

Finally, calculating the derivatives and making some simplification we obtain $\omega_0, \omega_1$ as follows:

$$\omega_0 = -\frac{1}{3} \frac{\Omega_{fe}}{\Omega_{fe} (1 - \Omega_{fe})}. \quad (28)$$

$$\omega_1 = \frac{1}{6 \Omega_{fe} (1 - \Omega_{fe})} \left[ \Omega_{fe}'' - \frac{2 \Omega_{fe} (2\Omega_{fe} - 1)}{\Omega_{fe} (1 - \Omega_{fe})} \right]. \quad (29)$$

Now, substituting equation (23) into (28) and (29), we obtain

$$\omega_0 = \frac{1}{3(1 - \Omega_{fe})} \left[ G - 1 + \Omega_{fe} \right]$$

$$+ (\Omega_{fe} - 1) \left( \frac{2\sqrt{\Omega_{fe}}}{c} + \frac{\phi'}{1 + \phi} \right), \quad (30)$$

$$\omega_1 = \frac{1}{6(1 - \Omega_{fe})} \left[ \chi (2\Omega_{fe} - 1) + \xi \right], \quad (31)$$

where $\chi, \eta, \xi$ are defined by

$$\chi = G - 1 + \Omega_{fe} + (\Omega_{fe} - 1) \left( \frac{2\sqrt{\Omega_{fe}}}{c} + \frac{\phi'}{1 + \phi} \right),$$

$$\eta = \chi + \Omega_{fe} \Omega_{fe} \left( \frac{3\sqrt{\Omega_{fe}}}{c} + \frac{\phi'}{1 + \phi} \right) - \frac{\sqrt{\Omega_{fe}}}{c}, \quad (32)$$

$$\xi = \chi \eta - G' - \left( \frac{\phi''}{1 + \phi} - \frac{\phi'}{(1 + \phi)^2} \right) (\Omega_{fe} - 1).$$

In [45] we obtained $\phi = dh(R)/dR = \alpha \ln t$ and $H = \gamma/t$, in which $t$ is the cosmic time. As a result, equations (30) and
(32) in terms of time are obtained as
\[
\omega_0 = \frac{1}{3(1 - \Omega_{fe})} \left[ \mathcal{G} - 1 + \Omega_{fe} + (\Omega_{fe} - 1) \times \left( \frac{2\Omega_{fe}}{\alpha} \right) \right], \tag{33}
\]
where
\[
\chi = \mathcal{G} - 1 + \Omega_{fe} + (\Omega_{fe} - 1) \left( \frac{2\Omega_{fe}}{\alpha} \right) \left( \frac{\alpha}{\gamma (1 + \alpha \ln t)} \right),
\eta = \chi + \Omega_{fe} + (\Omega_{fe} - 1) \left( \frac{3\Omega_{fe}}{\alpha} \right) \left( \frac{\alpha}{\gamma (1 + \alpha \ln t)} \right) - \sqrt{\Omega_{fe}},
\xi = \chi \eta - \mathcal{G} + \frac{\alpha^2}{\gamma^2 (1 + \alpha \ln t)^2} (\Omega_{fe} - 1). \tag{34}
\]

5. Conclusion

Astrophysical observations imply that the state parameter of dark energy must be changeable on the cosmic timescale. One obvious contender is the holographic dark energy. In the HDE, the parameter \(c\) can take various values, but we set \(c = 1\). In this work, the evolution of state parameter of holographic dark energy has been investigated for a type of \(f(R)\) model of gravity. We have studied the holographic dark energy-like quantity in the \(f(R)\) model with time-dependent gravitational constant. We have shown that the equation of state parameter for small redshifts for two values \(t = 0, \infty\) is
\[
\omega_f(z) = -0.87 + 0.074z. \tag{35}
\]

This result gives a suitable estimate for the state parameter that agrees with other studies [17, 43]. To evaluate \(\omega_f\) for all times, we need the observational data for calculating the constants \(\alpha\) and \(\gamma\). Hence, we think that the modified gravity yields a suitable estimate from \(\omega\) at different times that is consistent with the present local gravity experiments. However, there are some viable \(f(R)\) models that satisfy the solar system constraint; therefore one can discover other models of \(f(R)\) gravity that likely yield better result for those studies. Hence, this approach of holographic cosmology that is based on modified gravity is more general than the approaches studied in [17, 43].

References

[1] Riess A G et al 1998 116 1009
Perlmutter S et al 1999 Astrophys. J. 517 565
[2] Spergel D N et al 2003 Astrophys. J. Suppl. 148 175
[3] Tegmark M et al 2004 Phys. Rev. D 69 103501
[4] Wald R M 1984 General Relativity (Chicago: The University of Chicago Press)
[5] Weinberg S 1984 Gravitation and Cosmology (New York: Wiley)
[6] Misner C W, Thorne K S and Wheeler J A 1970 Gravitation (San Francisco, CA: W. H. Freeman)
[7] Halverson N W et al 2002 Astrophys. J. 568 38
[8] Deffayet C et al 2002 Phys. Rev. D 65 044023
[9] Capozziello S et al 2002 Int. J. Mod. Phys. D 11 483
[10] Copeland E J, Sami M and Tsujikawa S 2010 Int. J. Mod. Phys. D 19 1753
[11] Thomas S D 2002 Phys. Rev. Lett. 89 081301
[12] Zhao W 2007 Phys. Lett. B 655 97
[13] Li H, Guo Z K and Zhang Y Z 2006 Int. J. Mod. Phys. D 15 869
[14] Wang B, Gong Y and Abdalla E 2005 Phys. Lett. B 624 141
[15] Hořava P and Milic D 2000 Phys. Rev. Lett. 85 1610
[16] Guberina B, Horvat R and Nikolic H 2005 J. Cosmol. Astropart. Phys. JCAP05(2005)001
[17] Freese K and Lewis M 2002 Phys. Lett. B 540 1
[18] Carroll S M 2005 Phys. Rev. D 71 044023
[19] Åslund M and Brandbyge M 2003 Int. J. Mod. Phys. D 12 1753
[20] Capozziello S, Carloni S and Troisi A 2003 arXiv:astro-ph/0303150
[21] Nojiri S and Odintsov S D 2004 Gen. Relativ. Gravit. 36 1765
[22] Arkani-Hamed N, Cheng H-C, Luty M A and Mukohyama S 2004 J. High Energy Phys. JHEP05(2004)043528
[23] Capozziello S, Carloni S and Troisi A 2003 arXiv:astro-ph/0303041
[24] Nojiri S and Odintsov S D 2003 Phys. Rev. D 68 123512
[25] Carroll S M 2005 Phys. Rev. D 71 063513
[26] Abdalla M C B and Odintsov S D 2005 Class. Quantum Gravity 22 L135
[27] Capozziello S, Nojiri S, Odintsov S D and Troisi A 2006 Phys. Lett. B 639 135
[28] Appleby S A and Battye R A 2007 Phys. Lett. B 654 7
[29] Easson D A 2004 Int. J. Mod. Phys. A 19 5343
[30] Aghmohammadi A, Saadí K and Abolhassani M R 2010 Int. J. Theor. Phys. 49 709
[31] Aghmohammadi A, Saadí K and Abolhassani M R 2010 Int. J. Theor. Phys. 49 709
[32] Setare M R and Jamil M 2010 J. Cosmol. Astropart. Phys. JCAP01(2010)010
[33] Riess A G et al 2004 Astrophys. J. 607 665
[34] Aghmohammadi A et al 2009 Phys. Scr. 80 065008