A minimization approach to hyperbolic Cauchy problems

Enrico Serra, Paolo Tilli

Dipartimento di Scienze Matematiche, Politecnico di Torino
Corso Duca degli Abruzzi, 24, 10129 Torino, Italy
enrico.serra@polito.it, paolo.tilli@polito.it

Abstract

Developing an original idea of De Giorgi, we introduce a new and purely variational approach to the Cauchy Problem for a wide class of defocusing hyperbolic equations. The main novel feature is that the solutions are obtained as limits of functions that minimize suitable functionals in space–time (where the initial data of the Cauchy Problem serve as prescribed boundary conditions). This opens up the way to new connections between the hyperbolic world and that of the Calculus of Variations. Also dissipative equations can be treated. Finally, we discuss several examples of equations that fit in this framework, including nonlocal equations, in particular equations with the fractional Laplacian.

Mathematics subject classification: 35L70, 35L90, 35L15, 49J45.

Keywords: nonlinear hyperbolic equations, minimization, a priori estimates.

1 Introduction

In this paper we introduce a new and purely variational approach to the Cauchy Problem for a wide class of defocusing hyperbolic PDEs having the formal structure

\[ w''(t, x) = -\nabla W(w(t, \cdot))(x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n, \]

with prescribed initial conditions

\[ w(0, x) = w_0(x), \quad w'(0, x) = w_1(x). \]

While a precise setting with all formal details and our main results are given in Section 2, here we confine ourselves to a rather informal description of our approach, focusing on the main ideas that lie behind it and on the possible
new perspectives that it opens up, especially some new connections between the variational world and hyperbolic PDEs of the kind (1).

In (1), $\nabla W$ is the Gâteaux derivative of a functional (e.g. one from the Calculus of Variations) $W : W \to [0, \infty)$, where $W$ is some Banach space of functions in $\mathbb{R}^n$, typically a Sobolev space. If, for instance, $W(u) = 1/2 \int |\nabla u|^2 \, dx$ is the Dirichlet integral and $W = H^1(\mathbb{R}^n)$ then, formally, $-\nabla W(u) = \Delta u$, and (1) reduces to the wave equation $u'' = \Delta u$, much in the same spirit as the heat equation $u' = \Delta u$ is the gradient flow of the Dirichlet integral. Thus, in a sense, (1) can be considered as a “second order gradient flow” for the functional $W$.

Our aim is to initiate and try to develop a rather general program, suggested by De Giorgi in [2] (see also [3]), that offers a new, purely variational approach to equations of the kind (1), possibly with the addition of a dissipative term (see below). We alert the reader that in this paper the term “variational” refers, in the spirit of De Giorgi, to minimization, rather than Critical Point Theory.

The main idea, the abstract counterpart to a specific conjecture stated in [2] and discussed in [9], is to associate with the abstract evolution equation (1) the functional

$$F_\varepsilon(w) = \frac{\varepsilon^2}{2} \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} |w''(t,x)|^2 \, dx \, dt + \int_0^\infty e^{-t/\varepsilon} W(w(t,\cdot)) \, dt.$$  

This functional is to be minimized, for fixed $\varepsilon > 0$, among all functions $w(t,x)$ in spacetime $\mathbb{R}^+ \times \mathbb{R}^n$ subject to the constraints (2), which now play the role of boundary conditions. Assuming the existence of an absolute minimizer $w_\varepsilon$, the Euler–Lagrange equation of (3) formally reads

$$\varepsilon^2 (e^{-t/\varepsilon} w''_{\varepsilon})'' + e^{-t/\varepsilon} \nabla W(w_\varepsilon(t,\cdot))(x) = 0,$$

that is, the fourth–order in time equation

$$\varepsilon^2 w'''_{\varepsilon} - 2\varepsilon w''_{\varepsilon} + w'_{\varepsilon} + \nabla W(w_\varepsilon(t,\cdot))(x) = 0.$$  

The connection with (1) is clear: letting $\varepsilon \downarrow 0$, one formally obtains (1) in the limit. This motivates the following

**Problem 1** (De Giorgi, [2, 3]). Let $w_\varepsilon$ be a minimizer of $F_\varepsilon$ in (3), subject to the boundary conditions (2). Investigate the existence of a limit function

$$w(t,x) = \lim_{\varepsilon \to 0^+} w_\varepsilon(t,x),$$

and see if it solves the Cauchy Problem (1) & (2).

In its generality, as long as the structure of the functional $W$ is unknown, this may sound a little vague. In fact, in [2] De Giorgi raised this general question taking cue from a precise conjecture in a particular case, namely when

$$W(w) = \frac{1}{2} \int_{\mathbb{R}^n} |\nabla w(x)|^2 \, dx + \frac{1}{p} \int_{\mathbb{R}^n} |w(x)|^p \, dx \quad (p \geq 2).$$

2
and (1) becomes the nonlinear wave equation

\[ w'' = \Delta w - w|w|^{p-2} \quad (p \geq 2). \]

In this particular case, Problem 1 has an affirmative answer, [9]. As we will show, however, much can be said on Problem 1 under very mild assumptions on \( W \), and a robust theory can be built that provides several a priori estimates on the minimizers \( w_\varepsilon \). In some cases, basically when \( W(w) \) is quadratic in the highest order derivatives of \( w \), Problem 1 can be completely solved without any other assumption. In all cases, however, up to subsequences the limit (5) always exists and the estimates on \( w_\varepsilon \) entail the fulfillment of (2). When (1) is highly nonlinear, the general estimates still apply, but additional work is needed to get stronger compactness on \( w_\varepsilon \) and possibly obtain (1) in the limit (of course such further estimates, if any, will depend on the particular structure of \( W(w) \), and should be obtained \textit{ad hoc} on a case–by–case basis).

The variational approach suggested by Problem 1 is by genuine minimization, a completely new and unconventional feature, when it comes to hyperbolic equations. The typical case is when \( W \) is a convex (lower semicontinuous, etc.) functional of the Calculus of Variations (possibly depending on \( x, w \) and some of its \textit{spatial} derivatives): in this case \( F_\varepsilon \) in (4) inherits the good properties of \( W \), and the existence of \( w_\varepsilon \) (a minimizer of \( F_\varepsilon \) subject to (2)) is not an issue. Moreover, one may try to exploit several powerful techniques such as the theory of regularity for minimizers to get strong compactness on \( w_\varepsilon \) and pass to the limit in (5).

We believe that these features are a major point of interest of the present work. Indeed on the one hand our results provide a new, general starting point for the investigation of a wide class of hyperbolic problems, and on the other they allow one to use methods (coming from the elliptic theory) that have never been applied before in this context. Thus, our framework might hopefully help in shedding new light on several long–standing open problems in the theory of nonlinear hyperbolic equations.

We also point out that although the fourth order equation (4) has the structure of a singularly perturbed equation, this fact is never used in our results, that are simply based on the properties of minimizers of the functional \( F_\varepsilon \). For instance, no estimates on the third and fourth order derivatives are required.

Our approach also works with an extra (dissipative) term in the right hand side of (1), namely

\[ w''(t, x) = -\nabla W(w(t, \cdot))(x) - \nabla H(w'(t, \cdot))(x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^n \]

where \( H : H \to [0, +\infty) \) is a Gâteaux differentiable functional, defined on a suitable Hilbert space \( H \hookrightarrow L^2(\mathbb{R}^n) \). For the sake of simplicity, contrary to \( W \), we will assume that \( H \) is a \textit{quadratic} form on \( H \). Note that, while \( \nabla W \) is computed at \( w \), \( \nabla H \) is computed at \( w' \): if, for instance, both \( W \) and \( H \) are the Dirichlet integral, then (6) reduces to the strongly damped wave equation \( w'' = \Delta w + \Delta w' \). The reader is suggested to look at Section 7 where we discuss
several examples of hyperbolic problems (with or without dissipative terms) that fit into our scheme.

For equations with dissipative terms the counterpart to Problem 1 is

**Problem 2** (Dissipative case). Let \( w_\varepsilon \) be a minimizer of the functional

\[
\frac{\varepsilon^2}{2} \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} |w''(t, x)|^2 \, dx \, dt + \int_0^\infty e^{-t/\varepsilon} \left\{ W(w(t, \cdot)) + \varepsilon H(w'(t, \cdot)) \right\} \, dt
\]

subject to the boundary conditions (2). Investigate the existence of a limit for \( w_\varepsilon \) as in (5), and see if it solves the Cauchy Problem (6) & (2).

As before, the functional (7) relates to (1) via its Euler–Lagrange equation

\[
\varepsilon^2 (e^{-t/\varepsilon} w''')'' + e^{-t/\varepsilon} \nabla W(w_\varepsilon(t, \cdot))(x) - \varepsilon (e^{-t/\varepsilon} \nabla H(w_\varepsilon'(t, \cdot))(x))' = 0,
\]

namely,

\[
\varepsilon^2 w_\varepsilon''' - 2\varepsilon w_\varepsilon'' + w_\varepsilon'' + \nabla W(w_\varepsilon(t, \cdot))(x) + \nabla H(w_\varepsilon'(t, \cdot))(x) - \varepsilon (\nabla H(w_\varepsilon'(t, \cdot))(x))' = 0
\]

which, formally, reduces to (6) when \( \varepsilon \downarrow 0 \).

Also in the dissipative cases our results provide estimates for the minimizers \( w_\varepsilon \), existence of a limit \( w \), and in general all the properties described above.

A further point of interest is that, as is well known, the energy

\[
\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^n} |w'(t, x)|^2 \, dx + W(w(t, \cdot))
\]

is formally preserved by the solutions of equation (1), while for equation (6) the presence of dissipative terms entails that the preserved quantity is

\[
\mathcal{E}(t) + 2 \int_0^t \mathcal{H}(w'(t, \cdot)) \, dt.
\]

Generally, however, energy conservation is purely formal, since weak solutions are not regular enough to justify the computations needed in its proof. Our solutions are no exception, but in all cases they satisfy the “energy inequalities”

\[
\mathcal{E}(t) \leq \mathcal{E}(0) \quad \text{and} \quad \mathcal{E}(t) + 2 \int_0^t \mathcal{H}(w'(t, \cdot)) \, dt \leq \mathcal{E}(0).
\]

for equations (1) and (6) respectively.

Finally, we point out that our results are stated for functions defined in the whole of \( \mathbb{R}^n \). This choice is motivated as this is a model case of particular interest. However our results hold, without significative changes, also in different contexts, for instance for functions defined on an open subset \( \Omega \) of \( \mathbb{R}^n \) with Dirichlet or Neumann conditions imposed on \( \partial \Omega \).

The paper is organized as follows. The main results are stated in Section 2 and proved in sections 5 and 6. Section 3 contains preliminary results and
Section 4 is devoted to the key argument for the construction of the a priori estimates. Finally, several examples are reported in Section 7.

**Remark on notation.** Throughout the paper, a prime as in $v'$, $v''$ etc. denotes partial differentiation with respect to the time variable $t$. For functions defined in spacetime we will write freely $u(t,x)$ or $u(t)$. So if $u(t,\cdot)$ is an element of a space $X$ and $G$ is a functional on $X$, we will write indifferently $G(u(t,\cdot))$ or $G(u(t))$. Moreover, through the rest of the paper symbols as $\int v \, dx$ will always denote spatial integrals extended to the whole of $\mathbb{R}^n$, and short forms such as $L^2$, $H^1$ etc. will denote $L^2(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ etc. Finally, $\langle \cdot, \cdot \rangle$ will denote the duality pairing between a Banach space $X$ and its dual $X'$, the space $X$ being clear from the context.

### 2 Functional setting and main results

The functional $F_\varepsilon(w)$ to be minimized, subject to the boundary conditions (2), is defined by (3) in the non–dissipative case, and by (7) in the dissipative case. We shall treat the two cases simultaneously, by letting

$$F_\varepsilon(w) = \frac{\varepsilon^2}{2} \int_0^\infty \int_{\mathbb{R}^n} e^{-t/\varepsilon} |w''(t,x)|^2 \, dx \, dt$$

$$+ \int_0^\infty e^{-t/\varepsilon} \left\{ W(w(t,\cdot)) + \kappa \varepsilon \mathcal{H}(w'(t,\cdot)) \right\} \, dt \quad (\kappa \in \{0, 1\}),$$

where the parameter $\kappa \in \{0, 1\}$ plays the role of an on/off variable. Dealing with Problem 2 (dissipative case) one should let $\kappa = 1$, while dealing with Problem 1 (non–dissipative case) one should let $\kappa = 0$ and ignore the functional $\mathcal{H}$.

Concerning the functionals $W$ and $\mathcal{H}$, we make the following assumptions:

**H1** The functional $W : L^2 \to [0, +\infty]$ is lower semicontinuous in the weak topology, i.e.,

$$W(v) \leq \liminf_{k \to \infty} W(v_k) \quad \text{whenever } v_k \rightharpoonup v \text{ in } L^2.$$  

Moreover we assume that $W(v) < \infty \iff v \in W$, a Banach space with

$$C^\infty_0 \hookrightarrow W \hookrightarrow L^2 \quad \text{(dense and continuous inclusions).}$$

We also assume that $W$ is Gâteaux differentiable on $W$, and that its derivative $\nabla W : W \to W'$ satisfies the estimate

$$\|\nabla W(v)\|_{W'} \leq C(1 + W(v)^\theta), \quad C \geq 0, \quad \theta \in (0, 1), \quad \forall v \in W.$$  

**H2** If $\kappa = 1$, we assume that $\mathcal{H} : L^2 \to [0, +\infty]$ is a quadratic functional

$$\mathcal{H}(v) = \begin{cases} \frac{1}{2} B(v, v) & \text{if } v \in H, \\ +\infty & \text{if } v \in L^2 \setminus H \end{cases}$$
where $B : H \times H \to \mathbb{R}$ is a symmetric, bounded, nonnegative bilinear form on a Hilbert space $H$ with the norm $\|v\|_H^2 = \|v\|_{L^2}^2 + 2\mathcal{H}(v)$, and such that

(13) \hspace{1cm} C_0^\infty \hookrightarrow H \hookrightarrow L^2 \quad \text{(dense and continuous inclusions)}.

If $\kappa = 0$, for definiteness we set $\mathcal{H} \equiv 0$ and $H = L^2$.

**Remark 2.1.** If $\nabla^k v$ denotes the tensor of all $k$–th partial derivatives of $v$, a Dirichlet–like functional

$$W(v) = \frac{1}{p} \int_{\mathbb{R}^n} |\nabla^k v(x)|^p \, dx \quad (p > 1)$$

satisfies assumption (H1) with $W$ the Banach space of all $L^2$ functions $v$ such that $\nabla^k v \in L^p$, endowed with its natural norm. Since

(14) \hspace{1cm} \langle \nabla W(v), \eta \rangle = \int_{\mathbb{R}^n} |\nabla^k v(x)|^{p-2} \nabla^k v(x) \cdot \nabla^k \eta(x) \, dx, \quad v, \eta \in W,$

we see that (11) holds with $\theta = 1 - 1/p$. In view of the embeddings (10), the term $\nabla W(w(t, \cdot))$ in equations (4) and (6), as a distribution (note that $W' \hookrightarrow D'$ by (10)), acts as a differential operator (linear when $p = 2$) of order $2k$. Note also that the functional $W$ need not be convex.

**Remark 2.2.** The typical functional $\mathcal{H}$ fulfilling (H2) has the form

(15) \hspace{1cm} \mathcal{H}(v) = \frac{1}{2} \sum_{j \in S} \int_{\mathbb{R}^n} |\partial^j v|^2 \, dx

where $S \subset \mathbb{N}^n$ is any finite set of multi–indices and $\partial^j$ denotes partial differentiation. Here $H$ is the space of those $v \in L^2$ such that $\mathcal{H}(v) < +\infty$, and $\nabla H(v)$, as a distribution (note that $H'$ is a space of distributions by (13)) is the differential operator $\sum_{j \in S} (-1)^{|j|} \partial^{2j}$.

**Remark 2.3.** Assumptions (H1) and (H2) are additively stable. More precisely, if $W_i : L^2 \to [0, \infty]$ ($i = 1, 2$) are two functionals each satisfying (H1) (with Banach spaces $W_i$, constants $\theta_i$ etc.), then the sum $W = W_1 + W_2$ still satisfies (H1), now with $W = W_1 \cap W_2$ normed by $\|\cdot\|_W = \|\cdot\|_{W_1} + \|\cdot\|_{W_2}$ (this makes sense, in view of (10)). In particular, by Young inequality, (11) will hold true with $\theta = \max\{\theta_1, \theta_2\}$.

Finally, a similar argument applies to (H2).

**Theorem 2.4** (non–dissipative case). Given $w_0, w_1 \in W$ and $\varepsilon \in (0, 1)$, under assumption (H1) the functional $F_\varepsilon$ defined in (5) has a minimizer $w_\varepsilon$ in the space $H^2_{\text{loc}}([0, \infty); L^2)$ subject to (2). Moreover:

(a) Estimates. There exists a constant $C$, independent of $\varepsilon$, such that

(16) \hspace{1cm} \int_\tau^{\tau + T} W(w_\varepsilon(t, \cdot)) \, dt \leq CT \quad \forall \tau \geq 0, \quad \forall T \geq \varepsilon,
\[ \int_{\mathbb{R}^n} |w_\varepsilon'(t,x)|^2 \, dx \leq C, \quad \int_{\mathbb{R}^n} |w_\varepsilon(t,x)|^2 \, dx \leq C(1 + t^2) \quad \forall t \geq 0, \]

\[ \|w_\varepsilon''\|_{L^\infty(\mathbb{R}^+; W')} \leq C. \]

(b) Convergence. Every sequence \( w_{\varepsilon_l} \) (with \( \varepsilon_l \downarrow 0 \)) admits a subsequence which is convergent, in the weak topology of \( H^1((0,T);L^2) \) for every \( T > 0 \), to a function \( w \) such that

\[ w \in H^1_{\text{loc}}([0,\infty);L^2), \quad w' \in L^\infty(\mathbb{R}^+;L^2), \quad w'' \in L^\infty(\mathbb{R}^+;W'). \]

Moreover, \( w \) satisfies the initial conditions (2).

(c) Energy inequality. Letting

\[ \mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^n} |w'(t,x)|^2 \, dx + \mathcal{W}(w(t,\cdot)) \]

the function \( w(t,x) \) satisfies the energy inequality

\[ \mathcal{E}(t) \leq \mathcal{E}(0) = \frac{1}{2} \int_{\mathbb{R}^n} |w_1(x)|^2 \, dx + \mathcal{W}(w_0) \quad \text{for a.e. } t > 0. \]

**Theorem 2.5** (dissipative case). Given \( w_0 \in W, \ w_1 \in W \cap H \) and \( \varepsilon \in (0,1) \), under assumptions (H1) and (H2) the functional \( F_\varepsilon \) defined in (7) has a minimizer \( w_\varepsilon \), in the space \( H^2_{\text{loc}}([0,\infty);L^2) \) subject to (2). Moreover, all claims of Theorem 2.4 apply, with the following extensions and modifications:

(a) The additional estimate

\[ \int_0^\infty \mathcal{H}(w'_\varepsilon(t)) \, dt \leq C \]

holds true, while (18) should be replaced with

\[ \|w''_\varepsilon\|_{L^\infty(\mathbb{R}^+;W') + L^2(\mathbb{R}^+;H')} \leq C. \]

(b) The part on \( w'' \) in (19) should be replaced with

\[ w'' \in L^\infty(\mathbb{R}^+;W') + L^2(\mathbb{R}^+;H'). \]

Moreover, the convergence \( w'_\varepsilon \rightarrow w' \) holds in a stronger sense, namely

\[ w'_\varepsilon \rightharpoonup w' \quad \text{weakly in } L^2((0,T);H), \quad \text{for every } T > 0. \]

(c) With the same \( \mathcal{E}(t) \), the inequality (21) is replaced with

\[ \mathcal{E}(t) + 2 \int_0^t \mathcal{H}(w'(t,\cdot)) \, dt \leq \mathcal{E}(0) \quad \text{for a.e. } t > 0. \]
Observe that, under so general assumptions as in Theorem 2.4 (or 2.5), we do not claim that the limit function \( w \) satisfies (1) (or (6)). On the other hand, to our knowledge there are no counterexamples that rule out this possibility. Of course, to perform this step (by which one would completely solve Problem 1 or 2) one should obtain extra estimates exploiting the particular structure of the functional \( \mathcal{W} \), on a case by case basis. In some cases, however, the estimates of Theorem 2.4 (or 2.5 if \( \kappa = 1 \)) are enough to pass to the limit in the main equation, as the following result illustrates.

**Theorem 2.6.** Assume that, for some real number \( m > 0 \),

\[
\mathcal{W}(v) = \frac{1}{2} \|v\|_{H^m}^2 + \sum_{0 \leq k < m} \frac{\lambda_k}{p_k} \int_{\mathbb{R}^n} |\nabla^k v(x)|^{p_k} \, dx \quad (\lambda_k \geq 0, \, p_k > 1).
\]

Then assumption (H1) is fulfilled, if \( \mathcal{W} \) is the space of those \( v \in H^m \) with \( \nabla^k v \in L^{p_k} \) \((0 \leq k < m)\) endowed with its natural norm.

Moreover, the limit function \( w \) obtained via Theorem 2.4 (or 2.5 if \( \kappa = 1 \)) solves, in the sense of distributions, the hyperbolic equation (1) (or (6) if \( \kappa = 1 \)).

**Remark 2.7.** In (27), as usual, \( H^m \) is the \( L^2 \) norm of \( |\xi|^m \hat{v}(\xi) \), where \( \hat{v} \) is the Fourier transform of \( v \). The typical case is when \( m \) is integer, so that \( \|v\|_{H^m}^2 \) reduces to \( \|\nabla^m v\|_{L^2}^2 \). In this case (see Remark 2.3) the first term in (27) gives rise to a differential operator of order \( 2m \) in the equations (1) and (6).

On the other hand, in (27) \( m \) may fail to be integer. In this case, however, one can interpret the distribution \( \nabla \mathcal{W}(w) \) in (1) or (6) as a fractional differential operator: this enables us to treat, for instance, equations with the fractional Laplacian (see Example 7).

Several variants are possible in the same spirit. For instance, one may introduce nonconstant coefficients in (27) (and possibly exploit Gårding–type inequalities to make \( \mathcal{W}(v) \) coercive), or consider more general lower–order terms with suitable convexity and growth assumptions (e.g. powers of single partial derivatives as in (14)). Indeed, the central assumption is that \( \mathcal{W} \) be quadratic (and coercive) in the highest order terms, which makes the hyperbolic PDEs (1) and (6) quasilinear.

We end this section by discussing some consequences of assumption (H1) and (H2) which will be used in the sequel. First, (11) implies the linear control

\[
\|\nabla \mathcal{W}(v)\|_{W'} \leq C (1 + \mathcal{W}(v)) \quad C \geq 0, \quad \forall v \in W.
\]

Moreover, (11) entails Lipschitz continuity of \( \mathcal{W} \) along rays, as follows. Given \( a, b \in W \) with \( \|b\|_W = 1 \), the function \( f(\lambda) = \mathcal{W}(a + \lambda b) \) is differentiable and (11) gives \( |f'| \leq C(1 + f^0) \). From well known variants of the Gronwall Lemma, one has \( f(\lambda) \leq C(1 + f(0) + \lambda^{1/(1-\theta)}) \) and so

\[
\sup_{[a,a+b]} \mathcal{W} \leq C \left( 1 + \mathcal{W}(a) + \|b\|_{W}^{\theta} \right), \quad \forall a, b \in W
\]
where \([a, a + b]\) is the segment in \(W\) from \(a\) to \(a + b\). Combining with (28),

\[
\sup_{[a, a+b]} \|\nabla W\|_{W'} \leq C \left( 1 + W(a) + \frac{1}{\delta} \|b\|_W^{\frac{1}{\beta}} \right), \quad a, b \in W.
\]

Then, from Lagrange mean value theorem, for every \(\delta \neq 0\)

\[
\frac{W(a + \delta b) - W(a)}{\delta} \leq \sup_{[a, a + \delta b]} \|\nabla W\|_{W'} \quad \text{and combining with (30),}
\]

\[
\|W(a + \delta b) - W(a)\|_{W'} \leq C \|b\|_W \times \sup_{[a, a+b]} \|\nabla W\|_{W'},
\]

and combining with (30),

\[
\left| \frac{W(a + \delta b) - W(a)}{\delta} - W(a) \right| \leq C \|b\|_W \times \left( 1 + W(a) + \delta \frac{1}{\beta} \|b\|_W^{\frac{1}{\beta}} \right),
\]

a quantitative bound for the Lipschitz constant of \(W\). Thus, in particular,

\[
W(a + \delta b) \leq W(a) + C \delta \|b\|_W \times \left( 1 + W(a) + \delta \frac{1}{\beta} \|b\|_W^{\frac{1}{\beta}} \right),
\]

Finally, assumption (H2) entails that \(\mathcal{H}\) is differentiable in \(H\), with

\[
\langle \nabla \mathcal{H}(v), \eta \rangle = B(v, \eta), \quad \|\nabla \mathcal{H}(v)\|_{H'} \leq \sqrt{2} \mathcal{H}(v), \quad v, \eta \in H.
\]

Moreover, \(\mathcal{H}\) is a fortiori weakly lower semicontinuous in \(L^2\), namely

\[
\mathcal{H}(v) \leq \liminf_{k \to \infty} \mathcal{H}(v_k) \quad \text{whenever } v_k \rightharpoonup v \text{ in } L^2.
\]

3 Existence of minimizers and preliminary estimates

Since the space \(H^2_{\text{loc}}([0, \infty); L^2)\) is invariant under time dilations \(t \mapsto \varepsilon t\), it is convenient to introduce the simpler functional

\[
J_\varepsilon(u) = \int_0^\infty e^{-t} \left( \int \frac{|u''(t, x)|^2}{2e^2} dx + W(u(t)) + \frac{\kappa}{\varepsilon} \mathcal{H}(u'(t)) \right) dt,
\]

equivalent to \(F_\varepsilon\) in (3) in that \(F_\varepsilon(w) = \varepsilon J_\varepsilon(u)\) whenever \(u, w \in H^2_{\text{loc}}([0, \infty); L^2)\) are related by the change of variable \(u(t, x) = \varepsilon w(\varepsilon t, x)\). Of course, the boundary conditions in (2) must be scaled accordingly, namely as in (37).

The existence of minimizers \(w_\varepsilon\) for \(F_\varepsilon\) (as claimed in Theorems 2.4 and 2.5) then follows from the existence of minimizers \(u_\varepsilon\) for \(J_\varepsilon\) and

\[
u_\varepsilon(t, x) = w_\varepsilon(\varepsilon t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n.
\]

Lemma 3.1. Given \(\varepsilon \in (0, 1)\) and \(w_0, w_1 \in W\) (with \(w_1 \in W \cap H\) if \(\kappa = 1\)) the functional \(J_\varepsilon\) has an absolute minimizer \(u_\varepsilon\), in the class of those functions \(u \in H^2_{\text{loc}}([0, \infty); L^2)\) satisfying the boundary conditions

\[
u(0) = w_0, \quad u'(0) = \varepsilon w_1.
\]
Moreover,

\( J_\varepsilon(u_\varepsilon) \leq W(w_0) + C\varepsilon. \)

**Remark 3.2.** Throughout, the symbol \( C \) will always denote (possibly different) constants that are *independent* of \( \varepsilon \) (but may depend on all the other data, including the initial conditions \( w_0, w_1 \)).

**Proof.** The function \( \psi(t, x) = w_0(x) + \varepsilon t w_1(x) \) satisfies the boundary conditions (37). We also have from (32), applied with \( a = w_0, b = w_1 \) and \( \delta = \varepsilon t \), that

\[
W(w_0 + \varepsilon t w_1) \leq W(w_0) + C\varepsilon \left( 1 + W(w_0) + (\varepsilon t)^{1-\theta} \right)
\]

having absorbed \( \|w_1\|_W \) into \( C \). Multiplying by \( e^{-t} \) and integrating, we find that

\[
\int_0^\infty e^{-t}W(\psi(t))
\]

Moreover, if \( \kappa = 1 \), since \( \psi' = \varepsilon w_1 \) and \( w_1 \in H \), from (12) we see that

\[
\frac{\kappa}{\varepsilon} \int_0^\infty e^{-t}H(\psi'(t))
\]

Summing up, \( J_\varepsilon(\psi) \leq W(w_0) + C\varepsilon \): in particular, \( J_\varepsilon \) has a finite infimum and (38) follows as soon as \( J_\varepsilon \) has an absolute minimizer \( u_\varepsilon \). To show this, consider a minimizing sequence \( u_k \) and fix \( T > 0 \). Combining the estimate

\[
\int_0^T \|u_k''(t)\|_L^2 dt \leq e^T \int_0^T e^{-t}\|u_k''(t)\|_L^2 dt \leq 2\varepsilon^2 e^T J_\varepsilon(u_k)
\]

with the initial conditions (37) satisfied by \( u_k \), we see that \( \{u_k\} \) is bounded in \( H^2_{\text{loc}}([0, \infty); L^2) \) whence, up to subsequences, \( u_k(t) \to u(t) \) and \( u_k'(t) \to u'(t) \) in \( L^2 \) for every \( t \geq 0 \), for some \( u \in H^2_{\text{loc}}([0, \infty); L^2) \) that fulfills (37). Now the term involving \( u'' \) in (38) is lower semicontinuous, and the same is true of the other two terms by Fatou’s Lemma and weak convergence in \( L^2 \) of \( u_k(t) \) and \( u_k'(t) \) for fixed \( t \), using (3) and (14). This shows that \( J_\varepsilon(u) \leq \lim inf J_\varepsilon(u_k) \), hence \( u = u_\varepsilon \) is a global minimizer. \( \square \)

In some cases, a weaker version of (38) will be used, namely

\( J_\varepsilon(u_\varepsilon) \leq C. \)

**Remark 3.3.** To simplify notation, given a minimizer \( u_\varepsilon \), we define, for \( t \geq 0 \),

\( W_\varepsilon(t) := W(u_\varepsilon(t, \cdot)) \) and \( H_\varepsilon(t) := H(u_\varepsilon'(t, \cdot)). \)

We also set

\( D_\varepsilon(t) := \frac{1}{2\varepsilon^2} \int |u_\varepsilon''(t, x)|^2 \, dx \) for a.e. \( t > 0 \),

10
so that we write
\begin{equation}
L_\varepsilon(t) := D_\varepsilon(t) + W_\varepsilon(t) + \frac{\kappa}{\varepsilon} H_\varepsilon(t)
\end{equation}
for the locally integrable “Lagrangian”. Finally we introduce the kinetic energy function
\[
K_\varepsilon(t) := \frac{1}{2\varepsilon^2} \int |u'_\varepsilon(t, x)|^2 \, dx \quad \forall t \geq 0.
\]
The notation just introduced will be used systematically in the sequel.

Note that, due to Lemma 3.4 below, \(K_\varepsilon \in W^{1,1}(0, T)\) for all \(T > 0\) and
\begin{equation}
K'_\varepsilon(t) = \frac{1}{\varepsilon^2} \int u'_\varepsilon(t, x) u''_\varepsilon(t, x) \, dx \quad \text{for a.e. } t > 0.
\end{equation}

Lemma 3.4. The minimizers \(u_\varepsilon\) defined by Lemma 3.1 satisfy
\begin{equation}
\int_0^\infty e^{-t} D_\varepsilon(t) \, dt = \int_0^\infty e^{-t} \int \frac{|u''_\varepsilon|^2}{2\varepsilon^2} \, dx \, dt \leq C,
\end{equation}
\begin{equation}
\int_0^\infty e^{-t} K_\varepsilon(t) \, dt = \int_0^\infty e^{-t} \int \frac{|u'_\varepsilon|^2}{2\varepsilon^2} \, dx \, dt \leq C.
\end{equation}

Proof. Estimate (44) follows immediately from (39). The inequality (see [9])
\[
\int_0^\infty \int e^{-t} |v(t, x)|^2 \, dx \, dt \leq 2 \int |v(0, x)|^2 \, dx + 4 \int_0^\infty \int e^{-t} |v'(t, x)|^2 \, dx \, dt,
\]
applied with \(v(t, x) = u'_\varepsilon(t, x)\), shows, using (37) and (44), that
\[
\int_0^\infty \int e^{-t} |u'_\varepsilon|^2 \, dx \, dt \leq 2\varepsilon^2 \int |w_1(x)|^2 \, dx + C \varepsilon^2
\]
and (45) is established since \(w_1 \in L^2\) by (10).

4 The approximate energy

Since integrals with an exponential weight play a major role in our investigation, it is convenient to introduce the following average operator.

Definition 4.1. If \(f : \mathbb{R}^+ \rightarrow [0, +\infty]\) is measurable, we let
\[
\mathcal{A} f (s) := \int_s^\infty e^{-(t-s)} f(t) \, dt, \quad s \geq 0.
\]

Note that \(\mathcal{A} f\) is well defined (possibly \(+\infty\)) as \(f \geq 0\). However, since
\begin{equation}
\mathcal{A} f (0) = \int_0^\infty e^{-t} f(t) \, dt,
\end{equation}
if \( \mathcal{A}f(0) < \infty \) then \( \mathcal{A}f \) is absolutely continuous on intervals \([0, T]\), and

\[
(\mathcal{A}f)' = \mathcal{A}f - f.
\]

In any case, since \( \mathcal{A}f \geq 0 \), starting from \( f \geq 0 \) one can iterate \( \mathcal{A} \), and a simple computation gives

\[
\mathcal{A}^2f(s) = \int_s^\infty e^{-(t-s)(t-s)}f(t)\,dt
\]

and, in particular,

\[
\mathcal{A}^2f(0) = \int_0^\infty e^{-t}f(t)\,dt.
\]

We now introduce a fundamental quantity for our approach.

**Definition 4.2.** Let \( u_\varepsilon \) be a minimizer of \( J_\varepsilon \). The **approximate energy** is the function

\[
E_\varepsilon := K_\varepsilon + \mathcal{A}^2\mathcal{W}_\varepsilon
\]

or, more explicitly,

\[
E_\varepsilon(s) = K_\varepsilon(s) + \int_s^\infty e^{-(t-s)(t-s)}\mathcal{W}(u_\varepsilon(t))\,dt, \quad s \geq 0.
\]

**Remark 4.3.** In (51), the kinetic energy \( K_\varepsilon \) is evaluated pointwise at time \( s \), while the potential energy \( \mathcal{W}_\varepsilon \) is averaged over times \( t \geq s \) via the probability kernel \( e^{-(t-s)(t-s)} \). However, recalling the time scaling \( t \mapsto \varepsilon t \) that links the functionals \( F_\varepsilon \) and \( J_\varepsilon \), in the original time scale the probability kernel in (51) concentrates close to \( s \) as \( \varepsilon \to 0 \). Thus, heuristically, from (36) one expects that \( E_\varepsilon(t/\varepsilon) \approx E(t) \) where \( E(t) \) is the physical energy defined in (20).

Observe that, from (42) and (46), we have

\[
\mathcal{A}\mathcal{W}_\varepsilon(0) \leq \mathcal{A}L_\varepsilon(0) = J_\varepsilon(u_\varepsilon) \leq C
\]

and so \( \mathcal{A}\mathcal{W}_\varepsilon \) is well defined. But since \( \mathcal{A} \) is iterated twice in (50), it is not even clear why \( E_\varepsilon(s) \) should be finite. In fact, as we will show, \( E_\varepsilon(s) \) is finite and decreasing, and this monotonicity will be the key to our estimates.

The monotonicity of \( E_\varepsilon \) will be deduced from the following proposition.

**Proposition 4.4.** Let \( u_\varepsilon \) be a minimizer of \( J_\varepsilon \). For every \( g \in C^2([0, +\infty)) \) such that \( g(0) = 0 \) and \( g(t) \) is constant for large \( t \), there results

\[
\begin{align*}
\int_0^\infty e^{-s}(g'(s) - g(s))L_\varepsilon(s)\,ds \\
- \int_0^\infty e^{-s}(4D_\varepsilon(s)g'(s) + K_\varepsilon(s)g''(s) + 2K_\varepsilon/sH_\varepsilon(s)g'(s))\,ds & = g'(0)\mathcal{R}(u_\varepsilon),
\end{align*}
\]

\[
\mathcal{R}(u_\varepsilon) := \int_0^\infty e^{-s}(4D_\varepsilon(s)g'(s) + K_\varepsilon(s)g''(s))\,ds.
\]
where

\( R(u_\varepsilon) = -\varepsilon \int_0^{\infty} e^{-s} \langle \nabla \mathcal{W}(u_\varepsilon(s)), w_1 \rangle \, ds - \kappa \int_0^{\infty} e^{-s} \langle \nabla \mathcal{H}(u_\varepsilon'(s)), w_1 \rangle \, ds. \)

The quantity \( R(u_\varepsilon) \) is finite, and satisfies the estimate

\( |R(u_\varepsilon)| \leq C(\varepsilon + \kappa \sqrt{\varepsilon}) \leq C\sqrt{\varepsilon}. \)

**Proof.** For every \( \delta \in \mathbb{R} \) with \( |\delta| \) small enough, the function

\( \varphi(t) = \varphi(t, \delta) = t - \delta g(t) \)

is a diffeomorphism of \( \mathbb{R}^+ \) of class \( C^2 \). We denote by \( \psi \) its inverse,

\( \psi(s) = \varphi^{-1}(s), \quad s \geq 0 \)

(the dependence on \( \delta \), which is fixed, is omitted to simplify the notation). For small \( \delta \), we consider the competitor

\[
U(t) = u_\varepsilon(\varphi(t)) + t \delta \varepsilon g'(0)w_1,
\]

which satisfies the boundary conditions \( U(0) = w_0 \) and \( U'(0) = \varepsilon w_1 \), because \( \varphi(0) = 0 \) and \( \varphi'(0) = 1 - \delta g'(0) \). We have

\[
U'(t) = u_\varepsilon'(\varphi(t))\varphi'(t) + \delta \varepsilon g'(0)w_1,
\]

\[
U''(t) = u_\varepsilon''(\varphi(t))|\varphi'(t)|^2 + u_\varepsilon'(\varphi(t))\varphi''(t)
\]

and hence

\[
J_\varepsilon(U) = \int_0^{\infty} e^{-t} \left\{ \frac{1}{2\varepsilon^2} \left\| u_\varepsilon''(\varphi(t))|\varphi'(t)|^2 + u_\varepsilon'(\varphi(t))\varphi''(t) \right\|_{L^2}^2 + \mathcal{W}(u_\varepsilon(\varphi(t)) + t \delta \varepsilon g'(0)w_1) + \frac{\kappa}{\varepsilon} \mathcal{H}(u_\varepsilon'(\varphi(t))\varphi'(t) + \delta \varepsilon g'(0)w_1) \right\} \, dt.
\]

Changing variable in the integral letting \( t = \psi(s) \), that is, \( s = \varphi(t) \), we have

\[
J_\varepsilon(U) = \int_0^{\infty} \psi'(s)e^{-\psi(s)} \left\{ \frac{1}{2\varepsilon^2} \left\| u_\varepsilon''(s)|\varphi'(\psi(s))|^2 + u_\varepsilon'(s)\varphi''(\psi(s)) \right\|_{L^2}^2 + \mathcal{W}(u_\varepsilon(s) + \delta \varepsilon g'(0)w_1\psi(s)) + \frac{\kappa}{\varepsilon} \mathcal{H}(u_\varepsilon'(s)\varphi(\psi(s)) + \delta \varepsilon g'(0)w_1) \right\} \, ds.
\]

Note that, from (55), \( s = \varphi(\psi(s)) = \psi(s) - \delta g(\psi(s)) \), that is,

\[
\psi(s) = s + \delta g(\psi(s)).
\]

In view of the assumptions on \( g \), we have \( \psi(s) \geq s - \delta \|g\|_\infty \) and hence \( e^{-\psi(s)} \leq e^{\delta \|g\|_\infty} e^{-s} \). Furthermore, by (29) and (12),

\[
\mathcal{W}(u_\varepsilon(s) + \delta \varepsilon g'(0)w_1\psi(s)) \leq C\left(1 + \mathcal{W}(u_\varepsilon(s)) + \psi(s)^\frac{1}{2}\right)
\]

13
and
\[
\mathcal{H}(u'_\varepsilon(s)) \leq 2\varepsilon g'(0)u_1 + 2\varepsilon \mathcal{H}(u'_\varepsilon(s)) + CH(w_1).
\]

These inequalities, together with (44), (45) and the finiteness of \(||\varphi'||\infty\) and \(||\varphi''||\infty\), show that \(J_\varepsilon(U)\) is finite and hence \(U\) is an admissible competitor.

Since \(U(t)\) reduces to \(u_\varepsilon(t)\) when \(\delta = 0\), the minimality of \(u_\varepsilon\) implies that

\[
(58) \quad \frac{d}{d\delta} J_\varepsilon(U)\bigg|_{\delta = 0} = 0.
\]

In order to compute this derivative, we differentiate under the integral sign in (56) (reasoning as above for the finiteness of \(J_\varepsilon(U)\), it is easy to prove that this is possible). From (57),

\[
\frac{\partial}{\partial \delta} (\varphi'(s)e^{-\psi(s)})\bigg|_{\delta = 0} = g(s)e^{-s} - g'(s)e^{-s}.
\]

Moreover, elementary computations give

\[
\frac{\partial}{\partial \delta} |\varphi'(\psi(s))|^2\bigg|_{\delta = 0} = -2g'(s), \quad \frac{\partial}{\partial \delta} |\varphi''(\psi(s))|^2\bigg|_{\delta = 0} = -g''(s).
\]

Denoting by \(\Theta(s)\) the function within braces under the integral sign in (56), and recalling (42), there hold

\[
\Theta(s)\bigg|_{\delta = 0} = \frac{1}{2\varepsilon^2} ||u''_\varepsilon(s)||^2_{L^2} + \mathcal{W}_\varepsilon(s) + \frac{\kappa}{\varepsilon} \mathcal{H}_\varepsilon(s) = L_\varepsilon(s)
\]

and, recalling (41) and (43),

\[
\frac{\partial}{\partial \delta} \Theta(s)\bigg|_{\delta = 0} = -\frac{1}{\varepsilon^2} \left( u''_\varepsilon(s), 2u''_\varepsilon(s)g'(s) + u'_\varepsilon(s)g''(s) \right)_{L^2} - \frac{2\kappa}{\varepsilon} g'(s) \mathcal{H}_\varepsilon(s)
\]
\[
+ \varepsilon g'(0)s(\nabla \mathcal{W}(u_\varepsilon(s)), w_1) + \kappa g'(0) \langle \nabla \mathcal{H}(u'_\varepsilon(s)), w_1 \rangle
\]
\[
= -4D_\varepsilon(s)g'(s) - K'_\varepsilon(s)g''(s) - \frac{2\kappa}{\varepsilon} g'(s) \mathcal{H}_\varepsilon(s)
\]
\[
+ \varepsilon g'(0)s(\nabla \mathcal{W}(u_\varepsilon(s)), w_1) + \kappa g'(0) \langle \nabla \mathcal{H}(u'_\varepsilon(s)), w_1 \rangle.
\]

Combining these facts, we obtain that

\[
\frac{\partial}{\partial \delta} (\psi'(s)e^{-\psi(s)} \Theta(s))\bigg|_{\delta = 0} = e^{-s}(g'(s) - g(s)L_\varepsilon(s))\]
\[
- e^{-s} \left( 4D_\varepsilon(s)g'(s) + K'_\varepsilon(s)g''(s) + \frac{2\kappa}{\varepsilon} g'(s) \mathcal{H}_\varepsilon(s) \right)
\]
\[
+ e^{-s} \left( \varepsilon g'(0)s(\nabla \mathcal{W}(u_\varepsilon(s)), w_1) + \kappa g'(0) \langle \nabla \mathcal{H}(u'_\varepsilon(s)), w_1 \rangle \right).
\]

Finally, integrating in \(s\) we see that (58) reduces to (52).
We now prove estimate (54). For the first integral in (53), we have from (11) and Young inequality

\[
\left| \int_{0}^{\infty} e^{-s} \langle \nabla W(u_{\varepsilon}(s)), w_{1} \rangle \, ds \right| \leq \| w_{1} \|_{W} \int_{0}^{\infty} e^{-s} \| \nabla W(u_{\varepsilon}(s)) \|_{W} \, ds
\]

\[
\leq C \int_{0}^{\infty} e^{-s}(1 + W_{\varepsilon}(s)^{\theta}) \, ds = C + C \int_{0}^{\infty} e^{-s} W_{\varepsilon}(s)^{\theta} \, ds
\]

\[
\leq C + C \int_{0}^{\infty} e^{-s} s^{1/(1-\theta)} \, ds + \int_{0}^{\infty} e^{-s} W_{\varepsilon}(s) \, ds \leq C + J_{\varepsilon}(u_{\varepsilon}) \leq C
\]

having used (39), and thus \( |R(u_{\varepsilon})| \leq C \varepsilon \) when \( \kappa = 0 \). If, on the other hand, \( \kappa = 1 \), we also estimate the second integral in (53):

\[
\left| \int_{0}^{\infty} e^{-s} \langle \nabla H(u_{\varepsilon}^{1}(s)), w_{1} \rangle \, ds \right| \leq \| w_{1} \|_{H} \int_{0}^{\infty} e^{-s} \| \nabla H(u_{\varepsilon}^{1}(s)) \|_{H} \, ds
\]

\[
\leq C \int_{0}^{\infty} e^{-s} \sqrt{\mathcal{H}_{\varepsilon}(s)} \, ds \leq C \left( \int_{0}^{\infty} e^{-s} \mathcal{H}_{\varepsilon}(s) \, ds \right)^{1/2} \leq C \varepsilon^{1/2} \sqrt{\varepsilon},
\]

having used (38), Jensen inequality and (39).

\[ \square \]

**Corollary 4.5.** If \( g \geq 0 \) is of class \( C^{1,1} \), satisfies \( g(0) = 0 \) and is affine for large \( t \), then (52) remains true (all integrals being finite). In particular, when \( g(t) = t \), we obtain

\[ (59) \quad \mathcal{A}^{2}L_{\varepsilon}(0) + \frac{2\kappa}{\varepsilon} \mathcal{A} \mathcal{H}_{\varepsilon}(0) + 4AD_{\varepsilon}(0) = AL_{\varepsilon}(0) - R(u_{\varepsilon}). \]

**Remark 4.6.** Since \( L_{\varepsilon}(t) \geq W_{\varepsilon}(t) \), the finiteness of \( \mathcal{A}^{2}L_{\varepsilon}(0) \) in (59) entails that the approximate energy \( \mathcal{E}_{\varepsilon}(s) \) is finite for every \( s \geq 0 \) (in fact, it is absolutely continuous on intervals \([0,T]\), see the discussion after (46)).

**Proof.** By smoothing a truncation of \( g \), one can find an increasing sequence \( g_{k} \) of \( C^{2} \) functions, each eventually constant, such that as \( k \to \infty \)

\[ g_{k}(t) \uparrow g(t), \quad g'_{k}(t) \uparrow g'(t), \quad g''_{k}(t) \to g''(t) \quad \text{pointwise for every } t \geq 0, \]

with \( g'_{k} \) and \( g''_{k} \) uniformly bounded. We now write (52) for \( g_{k} \) and let \( k \to \infty \). Since the functions

\[ e^{-t}L_{\varepsilon}(t), \quad e^{-t}D_{\varepsilon}(t), \quad e^{-t}|K_{\varepsilon}'(t)|, \quad e^{-t}\mathcal{H}_{\varepsilon}(t) \]

are all in \( L^{1}(\mathbb{R}^{+}) \) (either by the finiteness of \( J_{\varepsilon}(u_{\varepsilon}) \) or by Lemma 3.4) and \( g_{k}'(0)R(u_{\varepsilon}) \) does not depend on \( k \), all integrals pass to the limit, except for the integral of \( e^{-t}g_{k}(t)L_{\varepsilon}(t) \) because the \( g_{k} \) are not uniformly bounded. For this term, however, one can use monotone convergence, and the integral of \( e^{-t}g(t)L_{\varepsilon}(t) \) in the limit is finite, by finiteness of all other terms. In particular, one can let \( g(t) = t \) in (52), which (recalling (46) and (49)) yields (59).
Corollary 4.7. For almost every $T > 0$ there results

\begin{equation}
A^2 L_\varepsilon (T) - A L_\varepsilon (T) + K'_\varepsilon (T) = -4 A D_\varepsilon (T) - \frac{2 \kappa}{\varepsilon} A H_\varepsilon (T).
\end{equation}

Proof. Consider the function $g \in C^{1,1}(\mathbb{R})$ defined as

\[
g(t) = \begin{cases} 
0 & \text{if } t \leq 0 \\
t^2/2 & \text{if } t \in (0, 1) \\
t - 1/2 & \text{if } t \geq 1
\end{cases}
\]

and, for $T > 0$ and $\delta > 0$ (we will let $\delta \downarrow 0$), set

\begin{equation}
g_\delta(t) = \delta g((t - T)/\delta).
\end{equation}

Each $g_\delta$ satisfies the assumptions of Corollary 4.5 and $g''_\delta(t) = \frac{1}{\delta} \chi(T, T + \delta)$. Letting $g = g_\delta$ in (52) and rearranging terms, gives

\[
\int_T^\infty e^{-t}(g_\delta(t) - g_\delta(t)) L_\varepsilon (t) \, dt + \frac{1}{\delta} \int_T^{T+\delta} e^{-t} K'_\varepsilon (t) \, dt
\]

\[
= - \int_T^\infty e^{-t}(4 D_\varepsilon(t) g'_\delta(t) + \frac{2 \kappa}{\varepsilon} H_\varepsilon(t) g''_\delta(t)) \, dt.
\]

Note that, as $\delta \to 0$, $g_\delta(t) \to (t - T)^+$ while $g'_\delta(t) \to \chi(T, \infty)$, with bounds $|g_\delta(t)| \leq (t - T)^+$ and $|g'_\delta(t)| \leq 1$. By dominated convergence we can let $\delta \downarrow 0$, thus obtaining for a.e. $T$

\[
\int_T^\infty e^{-t}(t - T) L_\varepsilon (t) \, dt - \int_T^\infty e^{-t} L_\varepsilon (t) \, dt + e^{-T} K'_\varepsilon (T)
\]

\[
= - \int_T^\infty e^{-t}(4 D_\varepsilon(t) + \frac{2 \kappa}{\varepsilon} H_\varepsilon(t)) \, dt,
\]

and multiplying by $e^T$ one obtains (60).

Theorem 4.8. The function $E_\varepsilon$ is finite and decreasing. More precisely,

\begin{equation}
E'_\varepsilon (T) \leq - \frac{\kappa}{\varepsilon} \left(A H_\varepsilon (T) + A^2 H_\varepsilon (T)\right),
\end{equation}

and

\begin{equation}
E_\varepsilon (T) + \frac{2 \kappa}{\varepsilon} \int_0^T H_\varepsilon(t) \, dt \leq \frac{1}{2} \|w_1\|_{L^2}^2 + W(w_0) + C\varepsilon + C\kappa \sqrt{\varepsilon}, \quad \forall T \geq 0.
\end{equation}

Proof. From Remark 4.6 we know that $E_\varepsilon$ is absolutely continuous on intervals $[0, T]$. Hence, differentiating (50) and using (47) written with $f = A W_\varepsilon$ yields

\[
E'_\varepsilon = K'_\varepsilon - A W_\varepsilon + A^2 W_\varepsilon.
\]
But since $W_\varepsilon = L_\varepsilon - D_\varepsilon - \frac{\kappa}{\varepsilon} H_\varepsilon$, using (60) we obtain
\[ E'_\varepsilon = -3AD_\varepsilon - A^2D_\varepsilon - \frac{\kappa}{\varepsilon} A H_\varepsilon - \frac{\kappa}{\varepsilon} A^2 H_\varepsilon, \]
and (62) follows. Choose now $f = \frac{\varepsilon}{2} H_\varepsilon$, so that (62) reads $E'_\varepsilon + Af + A^2f \leq 0$.

Integrating we find
\[ E_\varepsilon(T) + \int_0^T Af \, dt + \int_0^T A^2 f \, dt \leq E_\varepsilon(0). \quad (64) \]

For the former integral, using (47) we have
\[ \int_0^T Af \, dt = \int_0^T f \, dt + Af(T) - Af(0). \]

For the latter, iterating twice the same argument gives
\[ \int_0^T A^2f \, dt = \int_0^T f \, dt + Af(T) + Af(T) - A^2f(0) - Af(0), \]
so that (64), in particular, yields
\[ E_\varepsilon(T) + 2 \int_0^T f(t) \, dt \leq E_\varepsilon(0) + A^2f(0) + 2Af(0) = \]
\[ K_\varepsilon(0) + A^2W_\varepsilon(0) + A^2f(0) + 2Af(0) \leq \frac{1}{2}\|w_1\|_{L^2}^2 + A^2L_\varepsilon(0) + \frac{2\kappa}{\varepsilon} AH(0). \]

Therefore, since $4D_\varepsilon(t) \geq 0$, using (59) we find that
\[ E_\varepsilon(T) + 2 \int_0^T f(t) \, dt \leq \frac{1}{2}\|w_1\|_{L^2}^2 + AL_\varepsilon(0) - R(u_\varepsilon), \]
and since $AL_\varepsilon(0) = J_\varepsilon(u_\varepsilon)$, from (38) we see that (63) follows from (54). \qed

## 5 Proof of the a priori estimates

In this section we prove part (a) of theorems 2.4 and 2.5.

As discussed at the beginning of Section 3, the minimizers $w_\varepsilon$ of $F_\varepsilon$ in (8) (subject to (2)) are related to the minimizers $u_\varepsilon$ of $J_\varepsilon$ in (35) (subject to (37)) by the change of variable (36) and in particular the functions $w_\varepsilon$ satisfy the boundary conditions
\[ w_\varepsilon(0, x) = w_0(x), \quad w'_\varepsilon(0, x) = w_1(x). \quad (65) \]

So the estimates on $w_\varepsilon$ will follow from analogous estimates on $u_\varepsilon$ by scaling.
Proof of (17). Scaling as in (36) and using (63) and (51) yields
\[
\frac{1}{2} \int |w_\varepsilon'(t,x)|^2 \, dx = K_\varepsilon(t/\varepsilon) \leq C,
\]
which proves the first estimate in (17). The second estimate follows immediately from the first and the boundary condition in (65), since \( w_0 \in W \to L^2 \).

Proof of (22). When \( \kappa = 1 \), observe that (63) gives
\[
\int_0^\infty H_\varepsilon(t) \, dt = \int_0^\infty H(u_\varepsilon'(t)) \, dt \leq C \varepsilon,
\]
and (22) follows from (65) and scaling, using (12).

Proof of (16). Since \( L_\varepsilon \geq 0 \), we have from (39)
\[
e^{-2} \int_0^{s+2} W_\varepsilon(t) \, dt \leq \int_0^{s+2} e^{-t} L_\varepsilon(t) \, dt \leq J_\varepsilon(u_\varepsilon) \leq C.
\]
In the same spirit, we have for every \( s \geq 0 \),
\[
e^{-2} \int_{s+1}^{s+2} W_\varepsilon(t) \, dt \leq \int_{s+1}^{s+2} (t-s)e^{-(t-s)} W_\varepsilon(t) \, dt \leq A^2 W_\varepsilon(s) \leq E_\varepsilon(s) \leq C.
\]
which, combined with (67), yields
\[
\int_s^{s+1} W_\varepsilon(t) \, dt \leq C \quad \forall s \geq 0.
\]
Writing \( s = \tau/\varepsilon \) and scaling, recalling (40) we obtain that
\[
\int_\tau^{\tau+\varepsilon} W(w_\varepsilon(z)) \, dz \leq C \varepsilon \quad \forall \tau \geq 0.
\]
Now, if \( \tau \geq 0 \) and \( T \geq \varepsilon \) as in (16), by covering \([\tau, \tau + T]\) with consecutive intervals of length \( \varepsilon \) and using (69) in each interval, one obtains (16).

In the next lemma we are going to use the inequality
\[
\int_t^{t+1} \| \nabla W(u_\varepsilon(t)) \|_{W'} \, dt \leq C \quad \forall t \geq 0,
\]
which follows immediately on combining (11) and (68).

**Lemma 5.1** (Euler–Lagrange equation). Suppose that \( \eta(t,x) = \varphi(t)h(x) \), with \( \varphi \in C^{1,1}([0, +\infty)) \), \( \varphi(0) = \varphi'(0) = 0 \) and \( h \in W \cap H \). Then
\[
\int_0^\infty e^{-t} \left( \frac{1}{\varepsilon^2} \langle u'_\varepsilon, \eta'' \rangle_{L^2} + \langle \nabla W(u_\varepsilon(t)), \eta \rangle + \frac{K}{\varepsilon} \langle \varepsilon \nabla(\varphi' h(t)), \eta \rangle \right) \, dt = 0.
\]
Moreover, the same conclusion holds if \( \eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n) \).
Proof. The Euler–Lagrange equation (71) corresponds to the condition \( f'(0) = 0 \) where \( f(\delta) = J_\varepsilon(u_\varepsilon + \delta \eta) \); it is enough to justify differentiation under the integral sign in (53) in the term involving \( W \) (the term with \( H \) is quadratic due to (12)).

First consider the case where \( \eta = \varphi(t) h(x) \), and set \( v = u_\varepsilon + \delta \eta \) with, say, \( |\delta| \leq 1 \). As \( \varphi \in C^{1,1} \), \( \varphi(t) \) grows at most quadratically as \( t \to \infty \); applying (28) with \( a = u_\varepsilon(t) \) and \( b = \varphi(t) h \), multiplying by \( e^{-t} \) and integrating, one sees that \( J_\varepsilon(v) \) is finite (and \( v \) satisfies the boundary conditions (37)). For a.e. \( t > 0 \), we have

\[
\frac{d}{d\delta} W(u_\varepsilon(t) + \delta \eta(t))|_{\delta=0} = \langle \nabla W(u_\varepsilon(t)), \eta(t) \rangle = \varphi(t) \langle \nabla W(u_\varepsilon(t)), h \rangle
\]

and this function, multiplied by \( e^{-t} \), is integrable on \( \mathbb{R}^+ \) due to (70). Indeed, one can easily check that differentiation in \( \delta \) under the integral sign is justified, now using (51), with \( a \) and \( b \) as before.

Now consider a generic test function \( \eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n) \). Due to (70) and (10), the left hand side of (71) defines a distribution on \( \mathbb{R}^+ \times \mathbb{R}^n \). If \( \eta = \varphi(t) h(x) \) with \( \varphi \in C_0^\infty(\mathbb{R}^+) \) and \( h \in C_0^\infty(\mathbb{R}^n) \), then in particular \( \varphi \in C^{1,1}([0, +\infty)) \) and (71) has just been established. The general case then follows from the fact that test function of the form \( \varphi(t) h(x) \) are dense in \( C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n) \) (see [3], Chap. IV, and in particular Thm. III). □

Proof of (158), (29). These estimates will follow from the following representation formula (proved below) for \( u_\varepsilon'' \), valid for a.e. \( T > 0 \):

\[
\frac{1}{\varepsilon^2} \langle u_\varepsilon''(T), h \rangle_{L^2} = -\mathcal{A}^2 f_1(T) - \frac{\kappa}{\varepsilon} \mathcal{A} f_2(T), \quad h \in \begin{cases} W & \text{if } \kappa = 0, \\ W \cap H & \text{if } \kappa = 1 \end{cases}
\]

where

\[
f_1(t) = \langle \nabla W(u_\varepsilon(t)), h \rangle, \quad f_1(t) = \langle \nabla H(u_\varepsilon(t)), h \rangle.
\]

Note that using (28),

\[
|f_1(t)| \leq \|h\|_W \|\nabla W(u_\varepsilon(t))\|_{W'} \leq C \|h\|_W (1 + \mathcal{W}_\varepsilon(t)).
\]

But since \( \mathcal{A}^2 1 = 1 \) by (158), and \( \mathcal{A}^2 \mathcal{W}_\varepsilon \leq E_\varepsilon \leq C \) by (10) and (13), we have

\[
|\mathcal{A}^2 f_1(T)| \leq \mathcal{A}^2 |f_1|(T) \leq C \|h\|_W \quad \forall T \geq 0.
\]

Thus, if \( \kappa = 0 \), (72) can be seen (via the second inclusion in (10)) as a representation formula for \( u_\varepsilon''(T) \) as an element of \( W' \), and the last estimate gives

\[
\frac{1}{\varepsilon^2} \|u_\varepsilon''(T)\|_{W'} \leq C \quad \text{for a.e. } T \geq 0.
\]

Then, scaling according to (29), one obtains (158).
In addition, if $\kappa = 1$ and $h \in W \cap H$, we have using (63)
\[ |f_2(t)| \leq \|h\|_H \|\nabla H(u'_0(t))\|_{H'} \leq C\|h\|_H \sqrt{\mathcal{H}_t(t)} \]
and thus, using (60), $\|f_2\|_{L^2(\mathbb{R}^+)} \leq C\sqrt{\varepsilon}\|h\|_H$. Therefore, since the operator $A$ maps $L^2(\mathbb{R}^+)$ continuously into itself, we find that
\[ \|Af_2\|_{L^2(\mathbb{R}^+)} \leq C\sqrt{\varepsilon}\|h\|_H. \]

Then, recalling (13), (72) can be written as $u''_\varepsilon/\varepsilon^2 = \Phi_1 + \Phi_2$, with the bounds $\|\Phi_1\|_{L^\infty(\mathbb{R}^+;W')} \leq C$ by (74), and $\|\Phi_2\|_{L^2(\mathbb{R}^+;H')} \leq C/\sqrt{\varepsilon}$ by the previous inequality. Scaling according to (36), this means that $w''_\varepsilon(t) = \Phi_1(t/\varepsilon) + \Phi_2(t/\varepsilon)$, and (23) follows since $\|\Phi_2(t/\varepsilon)\|_{L^2} = \sqrt{\varepsilon}\|\Phi_2\|_{L^2}$.

It remains to prove (T2). For $T, \delta > 0$, we take the $C^{1,1}$ function $g_\delta$ defined in (61). Given $h$ as in (72), we set $\eta(t,x) = g_\delta(t)h(x)$ and we apply Lemma 5.1
As $g''_\delta(t) = \delta^{-1}\chi(T,T+\delta)(t)$, (61) multiplied by $e^{T t}$ reads
\[ \int_T^{T+\delta} e^{T t} e^{-T t} (u''_\varepsilon(t),h)_L dt = -\int_T^\infty e^{-(t-T)} \left( g_\delta(t)f_1(t) + \frac{C}{\varepsilon}g'_\delta(t)f_2(t) \right) dt \]
with $f_1, f_2$ as in (13). Since $|g_\delta(t)| \leq (t - T)^+$ and $|g'_\delta(t)| \leq \chi(T,\infty)$, one can dominate the integrand functions as done above for $f_1$ and $f_2$. Finally, letting $\delta \downarrow 0$, $g_\delta \to (t - T)^+$ and $g'_\delta \to \chi(T,\infty)$, and one obtains (72) for a.e. $T$. 

6 Proof of convergence and energy inequality

In this section we first prove parts (b) and (c) of Theorems 2.4 and 2.5. Then, we prove Theorem 2.6.

In the sequel, we deal with a sequence of minimizers $w_\varepsilon$, as in (b) of Theorem 2.4, and we will tacitly deal with several subsequences. For ease of notation, however, we will denote by $w_\varepsilon$ the original sequence, as well as the subsequences we extract.

Proof of part (b): passage to the limit. Regardless of $\kappa \in \{0,1\}$, (17) shows that the $w_\varepsilon$ are equibounded in $H^1_{\text{loc}}([0,\infty);L^2)$. Precisely, for every $T > 0$ there exists a constant $C_T$ such that
\[ \|w_\varepsilon\|_{H^1([0,T];L^2)} = \int_0^T (\|w'_\varepsilon(t)\|_{L^2} + \|w_\varepsilon(t)\|_{L^2}) dt \leq C_T. \]
Thus there exists a function $w \in H^1_{\text{loc}}([0,\infty);L^2)$ such that
\[ w_\varepsilon \to w \quad \text{in } H^1_{\text{loc}}([0,\infty);L^2) \quad \text{and} \quad w_\varepsilon(t) \to w(t) \quad \text{in } L^2 \quad \forall t \geq 0 \]
as $\varepsilon \to 0$. Clearly, the claims on $w'$, $w''$ in (19) and (21) follow from the uniform bounds in (17), (18) and (23). Moreover, when $\kappa = 1$, since $H$ is normed by $\|v\|^2_H = \|v\|^2_{L^2} + 2\mathcal{H}(v)$, (22) combined with (17) provide a uniform bound for $w''_\varepsilon$ in $L^2((0,T);H)$ for every $T > 0$, whence (25).
To prove that \( w \) satisfies (2), we recall that these two conditions are satisfied, by assumption, by each \( w \epsilon \): then the first condition for \( w \) follows easily from the second part of (70), considering \( t = 0 \).

For the second condition, if \( \kappa = 0 \) then (18) and (17) (combined with \( L^2 \rightarrow W' \), that follows from (11) and (13), in particular (25) yields a uniform bound for \( w'' \epsilon \) in \( W_1^{1,\infty}(\mathbb{R}^+; W') \), which guarantees the maintenance, in the limit, of \( w'(0) = w_1 \) (now viewed as an equality in \( W' \)). If \( \kappa = 1 \) then the argument is similar: since \( W \cap H \rightarrow L^2 \) densely by (11) and (13), in particular (25) yields a uniform bound for \( w'' \epsilon \) in \( L^2((0,1); (W \cap H)') \), hence a bound for \( w' \epsilon \) in \( H^1((0,1); (W \cap H)'), \) sufficient to guarantee that \( w'(0) = w_1 \) (now seen as an equality in \( (W \cap H)' \)).

**Proof of part (c): energy inequality.** To obtain (21) and (26) we need the following Lemma, proved in [9] and reformulated here in terms of the operator \( A \).

**Lemma 6.1.** Let \( l(t) \), \( m(t) \) be nonnegative functions in \( L^1_{loc} \), such that
\[
(77) \quad (A^2l)(t) \leq m(t) \quad \text{for a.e. } t > 0.
\]

Then, for every pair of numbers \( a > 0 \) and \( \delta \in (0,1) \),
\[
\left( \int_{\delta a}^{T + a} se^{-s} \, ds \right) \int_{T + \delta a}^{T + a} l(t) \, dt \leq \int_{T}^{T + a} m(t) \, dt \quad \forall T \geq 0.
\]

Recalling (50) and (63), we can apply Lemma 6.1 with \( l(t) = W \epsilon(t) \) and
\[
m(t) = -K \epsilon(t) - \frac{2\kappa}{\epsilon} \int_{0}^{t} H \epsilon(s) \, ds + \frac{1}{2} \|w_1\|_{L^2}^2 + W(w_0) + C \sqrt{\epsilon}
\]
(assumption (77) corresponds to (63) via (50)). This gives, for every \( T \geq 0 \), every \( a > 0 \) and every \( \delta \in (0,1) \),
\[
Y(\delta a) \int_{T + \delta a}^{T + a} W \epsilon(t) \, dt \leq - \int_{T}^{T + a} \left( K \epsilon(t) + \frac{2\kappa}{\epsilon} \int_{0}^{t} H \epsilon(s) \, ds \right) \, dt + aE(0) + aC \sqrt{\epsilon}
\]
where, for simplicity, \( Y(z) = \int_{0}^{z} se^{-s} \, ds \) and \( E(0) \) is defined as in (21). Now, recalling (50), we want to rewrite this estimate in terms of \( w \epsilon \) instead of \( u \), in view of this, it is convenient to first replace \( T \) with \( T/\epsilon \) and \( a \) with \( a/\epsilon \), and then change variable in the integrals according to (36), thus obtaining, rearranging terms,
\[
Y(\delta a/\epsilon) \int_{T + \delta a/\epsilon}^{T + a} W(w \epsilon(t)) \, dt + \int_{T}^{T + a} \left( \frac{1}{2} \|w'(t)\|_{L^2}^2 + 2\kappa \int_{0}^{t} H(w' \epsilon(s)) \, ds \right) \, dt \leq aE(0) + aC \sqrt{\epsilon}, \quad \forall T \geq 0, \quad \forall a > 0, \quad \forall \delta \in (0,1).
Now, for fixed $T,a,\delta$, recalling (76) we let $\varepsilon \to 0$ in the previous estimate. Since $Y(\delta a/\varepsilon) \to 1$, we obtain by semicontinuity

$$
\int_{T+\delta a}^{T+a} \mathcal{W}(w(t)) \, dt + \int_T^{T+a} \left( \frac{1}{2} \|w'(t)\|_{L^2}^2 + 2\kappa \int_0^t \mathcal{H}(w'(s)) \, ds \right) \, dt \leq a\mathcal{E}(0)
$$

(for the integral involving $\mathcal{W}$ one uses Fatou’s Lemma, (76) and (9), while the double integral with $\mathcal{H}$ is a convex and strongly continuous function of $w'_\varepsilon$ in $L^2((0,T+a);H)$, and one may use (25)). Now we let $\delta \to 0^+$ (with $T$ and $a$ fixed), then we divide by $a$ and finally we let $a \to 0^+$, to obtain

$$
\mathcal{W}(w(T)) + \frac{1}{2} \|w'(T)\|_{L^2}^2 + 2\kappa \int_0^T \mathcal{H}(w'(s)) \, ds \leq \mathcal{E}(0) \quad \text{for a.e. } T \geq 0.
$$

When $\kappa = 0$ this reduces to (21), while when $\kappa = 1$ this reduces to (20).

**Lemma 6.2.** For every test function $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, there holds

$$
\int_0^\infty \langle w'_\varepsilon(\tau), \varepsilon^2 \eta'''(\tau) + 2\varepsilon \eta''(\tau) + \eta'(\tau) \rangle_{L^2} \, d\tau
$$

$$
= \int_0^\infty \left( \langle \nabla \mathcal{W}(w_\varepsilon(\tau)), \eta(\tau) \rangle + \kappa \langle \nabla \mathcal{H}(w_\varepsilon(\tau)), \eta(\tau) + \varepsilon \eta'(\tau) \rangle \right) \, d\tau.
$$

*Proof. Let $\psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$. Choosing $\eta = e^t \psi$ in (71) gives

$$
\int_0^\infty \frac{1}{\varepsilon^2} \langle w''_\varepsilon(t), \psi''(t) + 2\psi'(t) + \psi(t) \rangle_{L^2} \, dt
$$

$$
= - \int_0^\infty \left( \langle \nabla \mathcal{W}(w_\varepsilon(t)), \psi(t) \rangle + \kappa \frac{1}{\varepsilon} \langle \nabla \mathcal{H}(w_\varepsilon(t)), \psi(t) + \psi'(t) \rangle \right) \, dt.
$$

Now we replace $u_\varepsilon$ with $w_\varepsilon$ using (39) and, accordingly, we take $\psi$ of the form $\psi(t,x) = \eta(\varepsilon t,x)$, for an arbitrary test function $\eta$. Plugging into the last equation and changing variable $\tau = \varepsilon t$ in each integral, one obtains (78) after integrating by parts the first term.

*Proof of Theorem 2.7.* The functional $v \mapsto 1/2 \|v\|_{H^m}^2$, clearly satisfies assumption (H1), on letting $W = H^m$ and $b = 1/2$: then first part of the claim follows on combining Remarks 2.1 and 2.3. Thus, one may apply Theorem 2.4 (or 2.5 if $\kappa = 1$). We wish to pass to the limit in (78), in particular in the nonlinear term involving $\nabla \mathcal{W}$, namely we wish to prove that (up to subsequences)

$$
\lim_{\varepsilon \to 0} \int_0^\infty \langle \nabla \mathcal{W}(w_\varepsilon(\tau)), \eta(\tau) \rangle \, d\tau = \int_0^\infty \langle \nabla \mathcal{W}(w(\tau)), \eta(\tau) \rangle \, d\tau,
$$

where $w$ is the limit function obtained by Theorem 2.4 (or 2.5). Due to (27) (see also Remark 2.1 and (14)), we have

$$
\int_0^\infty \langle \nabla \mathcal{W}(w_\varepsilon(\tau)), \eta(\tau) \rangle \, d\tau = \int_0^\infty \langle w_\varepsilon(\tau), \eta(\tau) \rangle_{H^m} \, d\tau
$$

$$
+ \sum_{0 \leq k < m} \lambda_k \int_0^\infty \int \left| \nabla^k w_\varepsilon(\tau) \right|^{p_k-2} \nabla^k w_\varepsilon(\tau) \cdot \nabla^k \eta(\tau) \, dx \, d\tau.
$$
Thus, to prove (79), we need strong convergence of $|\nabla^k w_\varepsilon|^{p_k-2} \nabla^k w_\varepsilon$ ($k < m$) in $L^1(Q)$, for every cylinder $Q = (0, T) \times B$ ($B$ being a ball in $\mathbb{R}^n$), and weak convergence of $w_\varepsilon$ in $L^2((0, T); H^m)$.

Now, due to (27), the bounds in (16) (with $\tau = 0$) take the concrete form

$$
\int_0^T \left( \|w_\varepsilon(t)\|_{H^m}^2 + \sum_{0 \leq k < m} \lambda_k \int_{\mathbb{R}^n} |\nabla^k w_\varepsilon(t, x)|^{p_k} \, dx \right) \, dt \leq C_T \quad (T \geq 1),
$$

so that, combining with the second part of (17), $w_\varepsilon$ is weakly compact in $L^2((0, T); H^m)$, while $\nabla^k w_\varepsilon$ is bounded in $L^{p_k}(Q)$ (we focus on those $k$ for which $\lambda_k > 0$). Thus, to conclude, it suffices to check the strong convergence of $\nabla^k w_\varepsilon$ in $L^2(Q)$ (this condition is even stronger than necessary, if $p_k < 3$). Now fix a cylinder $Q = (0, T) \times B$. If $0 \leq k < m$ we have that $H^m(B) \hookrightarrow H^k(B) \hookrightarrow L^2(B)$, and the first injection is compact: thus, combining the bound for $w_\varepsilon$ in $L^2((0, T); H^m)$ with the bound for $w_\varepsilon'$ in $L^2((0, T); L^2(B))$ (from (17), we obtain the strong compactness of $w_\varepsilon$ in $L^2((0, T); H^k(B))$ (see e.g. Thm. 5.1 in [6]), whence $\nabla^k w_\varepsilon$ converges strongly in $L^2(Q)$.

The terms in (78) other than $\nabla W(w_\varepsilon)$ are linear in $w_\varepsilon$, and using (b) of Theorem 2.4 one can pass to the limit in (78) when $\kappa = 0$. Finally, if $\kappa = 1$, recalling (12) and (33), also the term involving $\nabla H(w_\varepsilon)$ passes to the limit in (78), using (25) and (33). In either case, taking the limit in (78) one obtains

$$
\int_0^\infty \langle w'(\tau), \eta'(\tau) \rangle_{L^2} \, d\tau = \int_0^\infty \left( \langle \nabla W(w(\tau)), \eta(\tau) \rangle + \kappa \langle \nabla H(w'(\tau)), \eta(\tau) \rangle \right) \, d\tau
$$

that is, $w$ is a weak solution of (11) (or (6), if $\kappa = 1$).

7 Examples

In this section we show how several concrete problems fit into the general scheme described above. Let us begin with equations without dissipative terms.

1. Linear equations. These are obtained when $W$ is a quadratic functional, e.g.

$$
W(v) = \frac{1}{2} \sum_{j \in \mathcal{R}} |\partial^j v|^2 \, dx,
$$

where $\mathcal{R} \subset \mathbb{N}^n$ is a finite set of multiindices and $\partial^j$ denotes partial differentiation. In this case the natural choice for the domain of $W$ is $W = \{ v \in L^2 \mid \partial^j v \in L^2, \forall j \in \mathcal{R} \}$, and assumption (H1) is fulfilled (in particular, (11) is satisfied with $\theta = 1/2$).

Reasoning as in Remark 2.1, the hyperbolic equation corresponding to (11) is

$$
w'' = -\sum_{j \in \mathcal{R}} (-1)^{|j|} \partial^j w.
$$
In this case Theorem 2.6 applies ($\kappa = 0$), and Problem 1 can be completely solved. Concrete instances are the linear wave equation $w'' = \Delta w$, the Klein–Gordon equation $w'' = \Delta w - w$, or the bi–harmonic wave equation $w'' = -\Delta^2 w$.

2. Defocusing NLW. This matches De Giorgi’s original conjecture in [2], and has been dealt with in [9]. It corresponds to the choice

$$
W(v) = \int \left( \frac{1}{2} |\nabla v|^2 + \frac{1}{p} |v|^p \right) \, dx
$$

in [9], for some $p > 2$. Here, by Remark 2.1, (11) takes the concrete form

$$w'' = \Delta w - |w|^{p-2} w = 0.$$

Letting $W = H^1 \cap L^p$, assumption (H1) is satisfied (with $\theta = 1 - 1/p$ in (11)), and all the results in [9] are recovered as an application of Theorem 2.6.

3. Sine–Gordon equation. If we let

$$W(v) = \int \left( \frac{1}{2} |\nabla v|^2 + 1 - \cos v \right) \, dx
$$

with domain $W = H^1$, then (11) becomes the sine–Gordon equation

$$w'' = \Delta w - \sin w.$$

Then (H1) is fulfilled with $\theta = 1/2$, and Theorem 2.6 applies. Note that the functional $W$ associated with this problem is not convex.

4. Quasilinear wave equations. Powers other than 2 on the gradient term in $W$ give rise to quasilinear wave equations. For example

$$W(v) = \frac{1}{p} \int |\nabla v|^p \, dx, \quad \text{or} \quad W(v) = \int \left( \frac{1}{p} |\nabla v|^p + \frac{1}{q} |v|^q \right) \, dx \quad (p, q > 1)$$

correspond, respectively, to the quasilinear wave equations

$$w'' = \Delta_p w \quad \text{and} \quad w'' = \Delta_p w - |w|^{q-2} w,$$

where $\Delta_p$ is the $p$–laplacian. Assumption (H1) is satisfied, for the former equation, letting $W = \{ v \in L^2 \mid \nabla v \in L^p \}$ and $\theta = 1 - 1/p$, while for the latter one may set $W = \{ v \in L^2 \mid \nabla v \in L^p, \ v \in L^q \}$ and $\theta = 1 - 1/\max\{p, q\}$. In both cases Theorem 2.4 applies, while Theorem 2.6 cannot be applied (unless $p = 2$).

It is an open problem, however, to establish if the last claim of Theorem 2.6 (passage to the limit in the equation) still applies when $p \neq 2$. A positive answer would settle the long–standing open question of global existence of weak solutions for this kind of equations (see [11]).
5. Higher order nonlinear equations. Just to give an example (see for instance [7]), consider
\[ W(v) = \int \left( \frac{1}{2} |\Delta v|^2 + \frac{1}{p} |\nabla v|^p + \frac{1}{q} |v|^q \right) \, dx \quad (p, q > 1). \]
Then (1) becomes the nonlinear vibrating–beam equation
\[ w'' = -\Delta^2 w + \Delta_p w - |w|^{q-2} w, \]
where $\Delta^2$ is the biharmonic operator. Here $W = \{ v \in H^2 \mid \nabla v \in L^p, \, v \in L^q \}$, while $\theta = 1 - 1/\max\{2, p, q\}$. Here Theorem 2.6 applies, and provides global existence.

6. Kirchhoff equations. The general scheme presented in this paper allows one to treat also nonlocal problems. A typical example is the Kirchhoff equation
\[ w'' = \left( \int |\nabla w|^2 \, dx \right) \Delta w. \]
Here one chooses
\[ W(v) = \frac{1}{4} \left( \int |\nabla v|^2 \, dx \right)^2, \quad W = H^1 \]
(note that (11) holds with $\theta = 3/4$), and Theorem 2.4 applies (while it is an open problem to see if the last claim of Theorem 2.6 is true in this case). More generally, if $W(v) = \frac{1}{2} \Phi \left( \int |\nabla v|^2 \, dx \right)$ for some appropriate function $\Phi$, one formally obtains the equation
\[ w'' = \Phi' \left( \int |\nabla w|^2 \, dx \right) \Delta w \]
(the appropriate constant $\theta$ in (11) will depend on $\Phi$).

7. Wave equations with the fractional Laplacian. Given $s \in (0, 1)$, we may consider the nonlocal energy
\[ W(v) = c \int \int \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dxdy + \frac{\lambda}{p} \int |v|^p \, dx \quad (c > 0, \, \lambda \geq 0, \, p > 1), \]
with domain $W = H^s \cap L^p$ (or simply $H^s$, if $\lambda = 0$). It is well known (see e.g. [3]) that the first integral is the natural energy associated with the fractional Laplacian $(-\Delta)^s$, so that (for a proper choice of $c$, see [3]) (11) becomes
\[ w'' = (-\Delta)^s w - \lambda |w|^{p-2} w, \]
a (nonlinear, if $\lambda > 0$) wave equation with the fractional Laplacian. One may check that assumption (H1) is satisfied, with $\theta = 1/2$ (or $\theta = 1 - 1/\max\{2, p\}$ if $\lambda > 0$) in (11). Here one may apply Theorem 2.6 thus obtaining global existence.
The next examples concern dissipative equations with a structure as in (6); these are related to the functional in (7), as stated in Problem 2. We will mainly focus on the choice of the functional $H$, thus obtaining dissipative variants of the previous examples.

8. Telegraph type equations. These are obtained letting

$$H(v) = \frac{1}{2} \int |v|^2 \, dx \quad \text{(with domain } H = L^2),$$

thus fulfilling assumption (H2). Since $\langle \nabla H(v), \cdot \rangle = \langle v, \cdot \rangle_{L^2}$, by Remark 2.2 this choice of $H$ generates the term $-w'$ in the right-hand-side of (6).

If, for instance, $W$ is as in Example 2, then we can obtain the nonlinear telegraph equation

$$w'' = \Delta w - |w|^{p-2} w - w'.$$

In this case one can solve Problem 2 completely, since Theorem 2.6 can now be applied with $\kappa = 1$.

If, on the other hand, $W$ is as in Example 4, then one obtains

$$w'' = \Delta p w - |w|^{q-2} w - w'$$

and so on. In fact, in practice, the term $-w'$ can be inserted in any of the above examples (Theorem 2.6 can always be applied, while Theorem 2.6 should now be applied with $\kappa = 1$, when possible).

9. Strongly damped wave equations. The term “strongly damped” usually denotes the presence of $\Delta w'$ in the equation (see e.g. [5]). We can treat this case by letting

$$H(v) = \frac{1}{2} \int |\nabla v|^2 \, dx \quad \text{(with domain } H = H^1)$$

so that $\nabla H(v)$ corresponds to $-\Delta v$ by Remark 2.2. Then, building on Example 2, we may consider

$$w'' = \Delta w - |w|^{p-2} w + \Delta w'$$

(for which Theorem 2.6 applies with $\kappa = 1$), or quasilinear versions such as

$$w'' = \Delta_p w - |w|^{q-2} w + \Delta w'.$$

The last equation does not satisfy the assumptions of Theorem 2.6 (unless $p = 2$). In a forthcoming paper, however, we will show that the claim of Theorem 2.6 is in fact true, for every $p, q > 1$.

10. Other damped equations. In each of Examples 1–3 one can add several dissipative terms. For example, by Remark 2.2 the choice

$$H(v) = \frac{1}{2} \int (|\Delta v|^2 + |\nabla v|^2 + |v|^2) \, dx$$

would introduce, in any given equation, the term $-\Delta^2 w' + \Delta w' - w'$. 

26
References

[1] P. Cherrier, A. Milani, *Linear and quasi-linear evolution equations in Hilbert spaces*. Graduate Studies in Mathematics, 135. AMS, Providence, RI, 2012.

[2] E. De Giorgi, Conjectures concerning some evolution problems. A celebration of John F. Nash, Jr. *Duke Math. J.* **81** (1996), no. 2, 255–268.

[3] E. De Giorgi, *Selected papers*. Edited by L. Ambrosio, G. Dal Maso, M. Forti, M. Miranda and S. Spagnolo. Springer–Verlag, Berlin, 2006.

[4] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

[5] V. Kalantarov, S. Zelik, Finite-dimensional attractors for the quasi-linear strongly-damped wave equation. *J. Diff. Eq.* **247** (2009), no. 4, 1120-1155.

[6] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier–Villars, Paris 1969.

[7] L.A. Peletier, W.C. Troy, *Spatial patterns. Higher order models in physics and mechanics*. Progress in Nonlinear Differential Equations and their Applications, 45. Birkhäuser Boston, 2001.

[8] L. Schwartz, *Théorie des distributions*. Publications de l’Institut de Mathématique de l’Université de Strasbourg, No. IX-X. Hermann, Paris 1966.

[9] E. Serra, P. Tilli, Nonlinear wave equations as limits of convex minimization problems: proof of a conjecture by De Giorgi. *Ann. of Math. (2)***175*** (2012), 1551–1574.