Time-dependent backgrounds of 2D string theory

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We study possible backgrounds of 2D string theory using its equivalence with a system of fermions in upside-down harmonic potential. Each background corresponds to a certain profile of the Fermi sea, which can be considered as a deformation of the hyperbolic profile characterizing the linear dilaton background. Such a perturbation is generated by a set of commuting flows, which form a Toda Lattice integrable structure. The flows are associated with all possible left and right moving tachyon states, which in the compactified theory have discrete spectrum. The simplest nontrivial background describes the Sine-Liouville string theory. Our methods can be also applied to the study of 2D droplets of electrons in a strong magnetic field.

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1. Introduction

One of the most important problems of the modern string theory is a search for new nontrivial backgrounds and the study of the underlying string dynamics. In most of the cases the target space metric of such backgrounds is curved and often it incorporates the black hole singularities. In the superstring theories, the supersymmetry allows for some interesting nontrivial solutions which are stable and exact. But the string theory on such backgrounds is usually an extremely complicated sigma-model, very difficult even to formulate it explicitly, not to mention studying quantitatively its dynamics.

The two-dimensional bosonic string theory is a rare case of sigma-model where such a dynamics is integrable, at least for some particular backgrounds, including the dilatonic black hole background. A physically transparent way to study the perturbative (one loop) string theory around such a background is provided by the CFT approach. However, once we want to achieve more ambitious quantitative goals, especially in analyzing higher loops or multipoint correlators, we have to address ourselves to the matrix model approach to the 2D string theory proposed in [1] in the form of the matrix quantum mechanics (MQM) of [2]. The string theory has been constructed as the collective field theory (the Das-Jevicki-Sakita theory), in which the only excitation, the massless ”tachyon”, was related to the eigenvalue density of the matrix field.

In the framework of MQM it is difficult to formulate directly a string theory in a nontrivial background metric since the operators perturbing the metric do not have a simple realization. However, we can perturb the theory by other operators, like tachyon or winding operators. We hope that such a perturbation can also produce a curved metric but in an indirect way. An example of such a duality was given in [3] where the 2D black hole background is induced by a winding mode perturbation.

The 2D string theory in the simplest, translational invariant background (the linear dilaton background) is described by the singlet sector of MQM with inversed quadratic potential. In the singlet sector of the Hilbert space, the eigenvalues have Fermi statistics, and the problem of calculating the $S$-matrix of the string theory tachyon becomes a rather standard problem in a quantum theory of a one-dimensional nonrelativistic free fermionic field. The tree-level $S$-matrix can be extracted by considering the propagation of ”pulses” along the Fermi surface and their reflection off the ”Liouville wall” [4,5].

The very existence of a formulation in terms of free fermions indicates that the 2D string theory should be integrable. For example, the fermionic picture allows to calculate the exact $S$-matrix elements. Each $S$-matrix element can be associated with a single fermionic loop with a number of external lines [6]. One can then expect that the theory is also solvable in a nontrivial, time-dependent background generated by a finite tachyonic source. Dijkgraaf, Moore and Plesser [7] demonstrated that this is indeed the case when the allowed momenta form a lattice as in the case of the compactified Euclidean theory. In [7] it has been shown that the string theory compactified at any radius $R$ possesses the
integrable structure of the Toda lattice hierarchy. However, this important observation had not been, until recently, really exploited. The Toda structure is too general, and it becomes really of use only if accompanied by an initial condition or a constraint (a "string equation"), which eliminates all the solutions but one. Thus the first results for a nontrivial background in case of a general radius were obtained as a perturbative expansion in the tachyon source \[8\].

Recently, the Toda integrable structure of the compactified Euclidean 2D string theory was rediscovered \[9,10\] and used to find the explicit solution of the theory \[10,11\]. Finally, the string equation at an arbitrary compactification radius was found in \[12\]. These papers studied the T-dual formulation of the string theory, where instead of the discrete spectrum of tachyon excitations the winding modes of the string around the compactification circle (Berezinski-Kosterlitz-Thouless vortices) were used. These modes appear in the non-singlet sectors of MQM \[13\] and can be generated by imposing twisted periodic boundary conditions \[14\].

In this paper we return to the study of the Toda integrable structure of tachyon excitations of 2D string theory originated in \[7\]. It describes special perturbations of the ground state within the singlet sector of the MQM. We will construct the Lax operators as operators in the Hilbert space of the singlet sector of MQM. We will be able to find an interpretation of the Toda spectral parameters in terms of the coordinates of the two-dimensional target phase space and interpret the solutions of the Toda hierarchy in terms of the shape of the Fermi sea. In particular, we find the explicit shape of the Fermi sea for the Sine-Liouville string theory.

We can give two different interpretations of our problem. The first one is that of a 2D string theory in Minkowski space in a non-stationary background. The simplest, time-independent ground state of the theory is characterized by a condensation of the constant tachyon mode, which is controlled by the cosmological constant \(\mu\). Here we will study more general, time-dependent backgrounds characterized by a set of coupling constants \(t_{\pm n}\) associated with non-trivial tachyon modes with purely imaginary energies \(E_n = in/R\), where \(R\) is a real number. Since the incoming/outgoing tachyons with imaginary energies have wave functions exponentially decreasing/increasing with time, such a ground state will contain only incoming tachyons in the far past and only outgoing tachyons in the far future. In other words, the right and left vacua are replaced by coherent states depending correspondingly on the couplings \(t_n\) and \(t_{-n}\) \((n = 1, 2, ...)\), which will be identified with the “times” of the Toda hierarchy. The coherent states of bosons modify the asymptotics of the profile of the Fermi sea at far past and future, without changing the number of fermions. The flow of the fermionic liquid is no more stationary, but its time dependence happens to be quite trivial, and the profiles of the Fermi sea at different moments of time are related by Lorentz boosts.
The second interpretation of our analysis is that we consider perturbations of an Euclidean 2D string theory in which the Euclidean time is compactified at some length \( \beta = 2\pi R \). Together with the cosmological term, we allow all possible vertex operators with momenta \( n/R, n \in \mathbb{Z} \). The simplest case of such a perturbation is the Sine-Liouville theory. This case will be considered in detail and the shape of the Fermi sea produced by the Sine-Liouville perturbation will be found explicitly. The solution we have found exhibits interesting thermodynamical properties, which may be relevant to the thermodynamics of the string theory on the dilatonic 2D black hole background.

Let us mention also another possible application of our analysis: the fermionic system that appears in MQM is similar to the problem of two-dimensional fermions in a strong transverse magnetic field [15]. The electrons filling the first Landau level form stationary droplets of Fermi liquid, similar to that of the eigenvalues in the phase space of MQM. The form of such droplets is also described by the Toda hierarchy [16]. Our problem might be related to the situation in which two such droplets are about to touch, which can happen at a saddle point of the effective potential [17].

The paper is organized as follows. In section 2 we will remind the CFT formulation of the 2D string theory. In section 3 the MQM in the inverted harmonic potential in the “light-cone” phase space variables is formulated and the free fermion structure of its singlet sector is revealed. In section 4 the one particle wave functions are studied. In section 5, after the description of the fermionic ground state, tachyonic perturbations are introduced and the equations defining the corresponding time-dependent profile of the Fermi sea are derived. In section 6 the Toda integrable structure of the perturbations restricted to a lattice of Euclidean momenta is derived directly from the Schroedinger picture for the free fermions. In section 7 we recover the solution of the Sine-Liouville model and describe explicitly its Fermi sea profile. In section 8 we reproduce the free energy of the perturbed system from the profile of the Fermi sea. The section 9 is devoted to conclusions and in section 10 we discuss some problems and future perspectives. In particular, we propose a 3-matrix model describing the 2D string theory perturbed by both tachyon and winding modes. In the appendix the one particle wave functions of the type II model, defined on both sides of the quadratic potential, in the “light cone” formalism are presented.

2. Tachyon and winding modes in 2D Euclidean string theory

The 2D string theory is defined by Polyakov Euclidean string action

\[
S(x, g) = \frac{1}{4\pi} \int d^2\sigma \sqrt{\det g} [g^{ab} \partial_a x \partial_b x + \mu],
\]

(2.1)

where the bosonic field \( x(\sigma) \) describes the embedding of the string into the Euclidean time dimension and \( g^{ab} \) is a world sheet metric. In the conformal gauge \( g^{ab} = e^{\phi(\sigma)} \hat{g}^{ab} \), where
$\hat{g}^{ab}$ is a background metric, the dilaton field $\phi$ becomes dynamical due to the conformal anomaly and the world-sheet CFT action takes the familiar Liouville form

$$S_0 = \frac{1}{4\pi} \int_{\text{world sheet}} d^2\sigma \left[ (\partial x)^2 + (\partial \phi)^2 - 4\hat{R}\phi + \mu e^{-2\phi} + \text{ghosts} \right]. \quad (2.2)$$

This action describes the unperturbed linear dilaton string background corresponding to the flat 2D target space $(x, \phi)$.

In the target space this theory possesses only one propagating degree of freedom which corresponds to tachyon field. If the Euclidean time is compactified to a circle of radius $R$, $x(\sigma) \equiv x(\sigma) + 2\pi R$, the spectrum of admissible momenta is discrete: $p_n = n/R, \; n \in \mathbb{Z}$. In this case there is also a discrete spectrum of vortex operators, describing the winding modes (Kosterlitz-Thouless vortices) on the world sheet. A vortex of charge $q_m = mR$ corresponds to a discontinuity $2\pi mR$ of the time variable around a point on the world sheet. The explicit expressions of the vertex operators $V_p$ and the vortex operators $\tilde{V}_q$ in terms of the position field $x = x_R + x_L$ and its dual $\tilde{x} = x_R - x_L$ are

$$V_p = \frac{\Gamma(|p|)}{\Gamma(-|p|)} \int d^2\sigma e^{-ipx} e^{(|p|-2)\phi},$$

$$\tilde{V}_q = \frac{\Gamma(|q|)}{\Gamma(-|q|)} \int d^2\sigma e^{-iq\tilde{x}} e^{(|q|-2)\phi}. \quad (2.3)$$

Any background of the compactified 2D string theory can be obtained (at least in the case of irrational $R$) by a perturbation of the action (2.2) with both vertex and vortex operators

$$S = S_0 + \sum_{n \neq 0} (t_n V_n + \tilde{t}_n \tilde{V}_n). \quad (2.4)$$

Such a theory possesses the T-duality symmetry of the non-perturbed theory $x \leftrightarrow \tilde{x}$, $R \rightarrow 1/R$, $\mu \rightarrow \mu/R$ if one also exchanges the couplings as $t_n \leftrightarrow \tilde{t}_n$. Another general feature of the theory (2.4) is the existence of the physical scaling of various couplings, including the string coupling $g_s$, with respect to the cosmological coupling $\mu$. It can be found from the zero mode shift of the dilaton $\phi \rightarrow \phi + \phi_0$. In this way we obtain

$$t_n \sim \mu^{1-\frac{1}{2}|n|/R}, \quad \tilde{t}_n \sim \mu^{1-\frac{1}{2}R|n|}, \quad g_s \sim \mu^{-1}. \quad (2.5)$$

The scaling (2.5) allows to conclude that at the compactification radius lower than the Berezinski-Kosterlitz-Thouless radius $R_{KT} = 1/2$ all vertex operators are irrelevant. In

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1 At rational values of $R$ there exist additional physical operators (similar to discrete states at the self-dual radius $R = 1$) containing the derivatives of fields $x(\sigma)$ and $\phi(\sigma)$, which we will ignore in this paper.
the interval $1/2 < R < 1$ the only relevant momenta are $p = \pm 1/R$. The theory perturbed by such operators looks as the Sine-Gordon theory coupled to 2d gravity ("Sine-Liouville" theory):

$$S_{SG} = \frac{1}{4\pi} \int d^2\zeta \left[ (\partial x)^2 + (\partial \phi)^2 - 4\hat{R}\phi + \mu e^{-2\phi} + \lambda e^{(\frac{4}{R}-2)\phi} \cos(x/R) \right].$$

(2.6)

It was conjectured by [3] that at $R = 2/3$ and $\mu = 0$ the "Sine-Liouville" is dual to the $\left[ SU(2C)_{k} / SU(2) \times U(1) \right]$ coset model with central charge $c = \frac{3k}{k-2} - 1 = 26$, which describes the 2D string theory in the black hole ("cigar") background.

If we go to the Minkowski space, the perturbation (2.4) is made by tachyons with purely imaginary momenta. In this case there are more general perturbations, generated by tachyons with any real energies. In the next section we will introduce the matrix formulation of the string theory in Minkowski space using light-cone coordinates.

3. Matrix Quantum Mechanics in light cone formulation

The 2D string theory in Minkowski time appears as the collective field theory for the large-$N$ limit of MQM in the inverted oscillator potential [11,13,20,21,22]. The matrix Hamiltonian is

$$H = \frac{1}{2} \text{Tr} (P^2 - M^2),$$

(3.1)

where $P = -i\partial/\partial M$ and $M_{ij}$ is an $N \times N$ hermitian matrix variable. The cosmological constant $\mu$ in (2.2) is introduced as a "chemical potential" coupled to the size of the matrix $N$, which should be considered as a dynamical variable. In this formulation the time coordinate of the string target space coincides with the MQM time (or its Euclidean analogue $x = -it$ in the compactified version of the MQM) and that of the Liouville field is related to the spectral variable of the random matrix.

The tachyon modes of the string theory are represented by the asymptotic states of the collective theory. The scattering operators with real energy $E$ and describing left- and right-moving waves, respectively, are given by [5]

$$T_+^E = e^{-itE} \text{Tr} (M + P)^iE, \quad T_-^E = e^{-itE} \text{Tr} (M - P)^{-iE}. \quad (3.2)$$

These operators can be used to construct the in- and out-states of scattering theory. Namely, for an in-state, one needs a left-moving wave while the out-state is necessarily given by a right-moving one. The vertex operators (2.3) are tachyons with purely imaginary energies and are therefore represented by the following chiral operators in the MQM

$$V_p \rightarrow \begin{cases} 
 e^{-pt} \text{Tr} (M + P)^{|p|}, & p > 0, \\
 e^{-pt} \text{Tr} (M - P)^{|p|}, & p < 0. 
\end{cases}$$

(3.3)
Since we are interested in perturbations with the chiral operators (3.3), it is natural to perform a canonical transformation to light-cone variables

\[ Z_\pm = \frac{M \pm P}{\sqrt{2}} \] (3.4)

and write the matrix Hamiltonian as

\[ H_0 = -\frac{1}{2} \text{Tr} (Z_+ Z_- + Z_- Z_+) \] (3.5)

where the matrix operators \( Z_{\pm} \) obey the canonical commutation relation

\[ [(Z_+)_i^j, (Z_-)_k^l] = -i \delta_i^l \delta_j^k \] (3.6)

Define the right and left Hilbert spaces as the spaces of functions of \( Z_+ \) and \( Z_- \) only, with the scalar product

\[ \langle \Phi_\pm | \Phi'_\pm \rangle = \int d^N z \Phi_\pm (Z_\pm) \Phi'_\pm (Z_\pm). \] (3.7)

The operator of coordinate in the right Hilbert space is the momentum operator in the left one and the wave functions in the \( Z_+ \) and \( Z_- \) representations are related by a Fourier transform. The second-order Schrödinger equation associated with the Hamiltonian (3.1) becomes a first-order one when written in the \( \pm \)-representations

\[ \partial_t \Phi_{\pm}(Z_{\pm},t) = \mp \text{Tr} \left( Z_{\pm} \frac{\partial}{\partial Z_{\pm}} + \frac{N}{2} \right) \Phi_{\pm}(Z_{\pm},t), \] (3.8)

whose general solution is

\[ \Phi_{\pm}(Z_{\pm},t) = e^{\mp \frac{1}{2}N^2 t} \Phi^{(0)}_{\pm} (e^{\mp t} Z_{\pm}). \] (3.9)

The Hilbert space decomposes into a direct sum of eigenspaces labeled by the irreducible representations of \( SU(N) \), which are invariant under the action of the Hamiltonian (3.3). The functions \( \tilde{\Phi}_{\pm}^{(r)} = \{ \Phi_{\pm}^{(r,J)} \}_{J=1}^{\dim(r)} \) belonging to given irreducible representation \( r \) transform as

\[ \tilde{\Phi}_{\pm}^{(r,I)} (\Omega^\dagger Z_{\pm} \Omega) = \sum_J D_{IJ}^{(r)} (\Omega) \tilde{\Phi}_{\pm}^{(r,J)} (Z_{\pm}), \] (3.10)

where \( D_{IJ}^{(r)} \) is the group matrix element in representation \( r \) and \( I, J \) are the representation indices. The wave functions transforming according to given irreducible representation depend only on the \( N \) eigenvalues \( z_{\pm,1}, \ldots, z_{\pm,N} \) and the Hamiltonian (3.3) reduces to its radial part

\[ H_0 = \mp i \sum_k (z_{\pm,k} \partial_{z_{\pm,k}} + \frac{N}{2}), \] (3.11)
A potential advantage of the light-cone approach is that the Hamiltonian (3.11) does not contain any angular part, which is not the case for the standard one (3.1), whose angular term induces a Calogero-like interaction.

In the scalar product (3.7), the angular part can also be integrated out, leaving only the trace over the representation indices:

\[
\langle \Phi^r_\pm | \Phi'^r_\pm \rangle = \sum J \int \prod_k dz_{\pm,k} \Delta^2(z_\pm) \Phi^r_{\pm}(z_\pm) \Phi'^r_{\pm}(z_\pm),
\]

where \(\Delta(z_\pm)\) is the Vandermonde determinant. If we define

\[
\tilde{\Psi}^r_{\pm}(z_\pm) = \Delta(z_\pm) \Phi^r_{\pm}(z_\pm),
\]

these determinants disappear from the scalar product. The Hamiltonian for the new functions \(\tilde{\Psi}^r_{\pm}(z_\pm)\) takes the same form as (3.11) but with a different constant term:

\[
H_0 = \mp i \sum_k (z_{\pm,k} \partial_{z_{\pm,k}} + 1/2).
\]

In the singlet representation, the wave function \(\Psi_{\pm}(z_\pm) \equiv \Psi^{(\text{singlet})}_{\pm}(z_\pm)\) is a completely antisymmetric scalar function. The scalar product in the singlet representation is given by

\[
\langle \Psi_{\pm} | \Psi'_{\pm} \rangle = \int \prod_k dz_{\pm,k} \tilde{\Psi}_{\pm}(z_\pm) \tilde{\Psi}'_{\pm}(z_\pm).
\]

Thus the singlet sector describes a system of \(N\) free fermions. The singlet eigenfunctions of the Hamiltonian (3.14) are represented by Slater determinants of one-particle eigenfunctions discussed in the next section.

It is known [13,14,10] that unlike singlet, which is free of vortices, the adjoint representation contains string states with a vortex-antivortex pair, and higher representations contain higher number of such pairs. In what follows we will concentrate on the fermionic system describing the singlet sector of the matrix model. We will start from the properties of the ground state of the model, representing the unperturbed 2D string background and then go over to the perturbed fermionic states describing the (time-dependent) backgrounds characterized by nonzero expectation values of some vertex operators.
4. Eigenfunctions and fermionic scattering

To study the system of non-interacting fermions we have to start with one-particle eigenfunctions. The one-particle Hamiltonian in the light-cone variables of the previous section can be written as

$$ H_0 = -\frac{1}{2}(\hat{z}_+\hat{z}_- + \hat{z}_-\hat{z}_+), \quad (4.1) $$

where $z_\pm$ turn out to be canonically conjugated variables

$$ [\hat{z}_+, \hat{z}_-] = -i. \quad (4.2) $$

We can work either in $z_+$ or in $z_-$ representation, where the theory is defined in terms of fermionic fields $\psi_\pm(z_\pm)$ respectively. General solutions of the Schrödinger equation with the Hamiltonian (4.1) written in these representations take the form

$$ \psi_\pm(z_\pm, t) = z_\pm^{-1/2} f_\pm(e^{\mp t} z_\pm). \quad (4.3) $$

There are two versions of the theory, referred in [6] as theories of type I and II. In the theory of type I the eigenvalues $\lambda_k$ of the original matrix field $M$ are restricted to the positive half-line. The theory of type II is defined on the whole real axis $\lambda$. Such a string theory splits, at the perturbative level, into two disconnected string theories of type I. Here we will consider the fermion eigenfunctions in the theory of type I. The theory of type II will be considered in Appendix A.

In the light cone formalism it is natural to replace the restriction $\lambda > 0$ with $z_\pm > 0$, which again does not affect the perturbative behavior. In this case the solutions with a given energy are $\psi_\pm^E(z_\pm, t) = e^{-iEt}\psi_\pm^E(z_\pm)$ with

$$ \psi_\pm^E(z_\pm) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z_\pm^2 E E}. \quad (4.4) $$

The functions (4.4) with $E$ real form two complete systems of $\delta$-function normalized orthonormal states

$$ \langle \psi_\pm^E | \psi_\pm^{E'} \rangle \equiv \int_0^\infty dz_\pm \frac{\psi_\pm^E(z_\pm)\psi_\pm^{E'}(z_\pm)}{\sqrt{2\pi}} = \delta(E - E'), \quad (4.5) $$

$$ \int_{-\infty}^\infty dE \frac{\psi_\pm^E(z_\pm)}{\sqrt{2\pi}} \frac{\psi_\pm^E(z_\pm)}{\sqrt{2\pi}} = \delta(z_\pm - z_\pm'). \quad (4.6) $$

As any two representations of a quantum mechanical system, the $z_+$ and $z_-$ representations are related by a unitary operator $\hat{S}$, which in our case is nothing but the Fourier transformation on the half-line. The latter is defined by the integral

$$ [\hat{S}\psi_+](z_-) = \int_0^\infty dz_+ K(z_-, z_+)\psi_+(z_+), \quad (4.7) $$
where there are two choices for the kernel:

\[ K(z_-, z_+) = \sqrt{\frac{2}{\pi}} \cos(z_- z_+) \quad \text{or} \quad K(z_-, z_+) = i \sqrt{\frac{2}{\pi}} \sin(z_- z_+). \tag{4.8} \]

The sine and the cosine kernels describe two possible theories, which differ only on the non-perturbative level. Let us choose the cosine kernel in (4.8) and evaluate the action of the $\hat{S}$-operator on the eigenfunctions. This integral is essentially the defining integral for the $\Gamma$-function:

\[ [\hat{S}^{\pm 1}\psi^E_\pm](z_+) = \frac{1}{\pi} \int_0^\infty dz_\pm \cos(z_+ z_-) z_\pm e^{\mp iE - \frac{1}{2}} = \mathcal{R}(E)\psi^E_+(z_+), \tag{4.9} \]

where

\[ \mathcal{R}(E) = \sqrt{\frac{2}{\pi}} \cosh \left( \frac{\pi}{2} (i/2 - E) \right) \Gamma(iE + 1/2). \tag{4.10} \]

(The sine kernel would give the same result, but with $\cosh$ replaced by $\sinh$). The factor $\mathcal{R}(E)$ is a pure phase

\[ \overline{\mathcal{R}(E)}\mathcal{R}(E) = \mathcal{R}(-E)\mathcal{R}(E) = 1, \tag{4.11} \]

which proves the unitarity of the operator $\hat{S}$. The operator $\hat{S}$ relates the incoming and the outgoing waves and therefore can be interpreted as the fermionic scattering matrix. The factor $\mathcal{R}(E)$ is identical to the the reflection coefficient (the “bounce factor” of [3]), characterizing the scattering off the upside-down oscillator potential. In the standard scattering picture, the reflection coefficient is extracted by comparing the incoming and outgoing waves in the large-$\lambda$ asymptotics of the exact eigenfunction of the inversed oscillator Hamiltonian $H = -\frac{1}{2}(\partial^2_\lambda + \lambda^2)$.

It follows from the above discussion that the scattering amplitude between two arbitrary in and out states is given by the integral with the Fourier kernel (4.8)

\[ \langle \psi^- | \hat{S} \psi^+ \rangle = \langle \hat{S}^{-1} \psi^- | \psi^+ \rangle = \langle \psi^- | K | \psi^+ \rangle \]

\[ \langle \psi^- | K | \psi^+ \rangle \equiv \int_0^\infty dz_+ dz_- \psi^- (z_-) K(z_-, z_+) \psi^+(z_+). \tag{4.12} \]

The integral (4.12) can be interpreted as a scalar product between the in and out states. By (4.5) and (4.9) one finds that the in and out eigenfunctions satisfy the orthogonality relation

\[ \langle \psi^E_- | K | \psi^E_+ \rangle = \mathcal{R}(E)\delta(E - E'). \tag{4.13} \]

The fact that there are two choices for the kernel can be explained as follows. In order to define the theory of type I for the original second-order Hamiltonian $H_0$ [3.1], we should fix the boundary condition at $\lambda = 0$, and there are two linearly independent boundary conditions.
5. String theory backgrounds as profiles of the Fermi sea

The ground state of the MQM is obtained by filling all energy levels up to some fixed Fermi energy which we choose to be \( E_F = -\mu \). Quasiclassically every energy level corresponds to a certain trajectory in the phase space of \( z_+, z_- \) variables. The Fermi sea can be viewed as a stack of all classical trajectories with \( E < E_F \) and the ground state is completely characterized by the curve representing the trajectory of the fermion with highest energy \( E_F \). For the Hamiltonian (1.1) all trajectories are hyperboles \( z_+ z_- = -E \) and the profile of the Fermi sea is given by

\[
    z_+ z_- = \mu. \tag{5.1}
\]

In the quasiclassical limit the phase-space density of fermions is either 0 or 1. Then the low lying collective excitations are represented by deformations of the Fermi surface, \( i.e., \) is the boundary of the region in the phase space filled by fermions. At any moment of time such deformation can be obtained by replacing the constant \( \mu \) on the right hand side of (5.1) with a function of \( z_+ \) and \( z_- \)

\[
    z_+ z_- = M(z_+, z_-). \tag{5.2}
\]

In contrast to the ground state which is stationary, an excited state given by a generic function \( M \) leads to a time dependent profile. However, this dependence is quite trivial: since the Fermi surface consists of free fermions each moving according to its classical trajectory \( z_\pm(t) = e^{\pm t} z_\pm(0) \), where \( z_\pm(0) \) is the initial data, we can always replace (5.2) by the equation for the initial shape. So the evolution of a shape in time is simply its homogeneous extension with the factor \( e^t \) along the \( z_+ \) axis and a homogeneous contraction with the same factor along the \( z_- \) axis.

Below we will find equations that determine the shape of the Fermi sea for a generic perturbation with tachyon operators. Our analysis is in the spirit of the Polchinski’s derivation of the tree-level \( S \)-matrix \([4]\), but we will consider finite, and not only infinitesimal, perturbation.

In terms of the Fermi liquid, the incoming and the outgoing states are defined by the asymptotics of the profile of the Fermi sea at \( z_+ \gg z_- \) and \( z_- \gg z_+ \), correspondingly. If we want to consider such a perturbation as a new fermion ground state, we should change the one-fermion wave functions. The new wave functions are related to the old ones by a phase factor

\[
    \Psi_\pm^E(z_\pm) = e^{\mp i \varphi_\pm(z_\pm;E)} \psi_\pm^E(z_\pm), \tag{5.3}
\]

whose asymptotics at large \( z_\pm \) characterizes the incoming/outgoing tachyon state. We split the phase into three terms

\[
    \varphi_\pm(z_\pm;E) = V_\pm(z_\pm) + \frac{1}{2} \phi(E) + v_\pm(z_\pm;E), \tag{5.4}
\]
where the potentials $V_{\pm}$ are fixed smooth functions vanishing at $z_{\pm} = 0$, while the term $v_{\pm}$ vanishing at infinity and the constant $\phi$ are to be determined. Thus, the potentials $V_{\pm}$ fix unambiguously the perturbation. The time evolution of these states with the original Hamiltonian (1.1) is determined by the eq. (1.3). To see that the state (5.3) contains incoming and outgoing tachyons, it is enough to note that it arises as a coherent state of vertex operators (3.3) acting on the unperturbed wave function (4.4).

Since the functions $\Psi^E_+$ and $\Psi^E_-$ should describe the same one-fermion state, the Fourier transform (4.7) should be diagonal in their basis. We fix the zero mode $\phi$ so that

$$\hat{S} \Psi^E_+ = \Psi^E_-.$$  

This condition can also be expressed as the orthonormality of in and out eigenfunctions (5.3)

$$\langle \Psi^-_+ |K| \Psi^+_+ \rangle = \delta(E_+ - E_-),$$  

with respect to the scalar product (4.12). This requirement fixes the exact form of the wave functions. Let us look at this problem in the quasiclassical limit $E_{\pm} \to \infty$. The scalar product in (5.6) is written as

$$\langle \Psi^-_+ |K| \Psi^+_+ \rangle = \frac{e^{-i\phi_0}}{\pi\sqrt{2\pi}} \int_0^\infty \frac{dz_+ dz_-}{\sqrt{z_+ z_-}} \cos(z_+ z_-) e^{-i\varphi_+(z_+) - i\varphi_-(z_-)} z_+ z_- \delta E_+ - \delta E_-.$$  

At the quasiclassical level it can be evaluated by the saddle point approximation. One obtains two equations for the saddle point

$$z_+ z_- = -E_+ + z_+ \partial \varphi_\pm(z_\pm).$$  

Generically, the two equations (5.8) define two different curves in the $z_+ z_-$ plane, and their compatibility can render at most a discrete set of saddle points. However, if the two solutions define the same curve, we obtain a whole line of saddle points, which implies the existence of a zero mode in the double integral (5.7). The contribution of the zero mode is infinite, amounting to the $\delta$-functional orthogonality relation. Thus, the orthonormality condition is reduced to the compatibility of two equations (5.8) at $E_+ = E_-$. For example, in the absence of perturbations ($\varphi_\pm = 0$), the saddle point equations $z_+ z_- = -E_\pm$ are inconsistent, unless $E_+ = E_-$. For equal energies they coincide leading to a zero mode in the double integral (1.13). The resulting saddle point equation gives the classical hyperbolic trajectory in the phase space of an individual fermion. If $E_+ \neq E_-$, the integrand of (5.7) is a rapidly oscillating function and the integral is zero.

---

3 We assume that only one exponent of $\cos(z_+ z_-)$ gives a contribution. The result will be shortly justified from another point of view.
The equations (5.8) and the requirement of their compatibility can be obtained from another point of view. The perturbed wave function (5.3) can be interpreted as an eigenstate of a new, perturbed Hamiltonian. Indeed, the functions (5.3) are not eigenstates of the original Hamiltonian (4.1) in \( \pm \)-representations
\[
H_0^\pm = \mp i (z_+ \partial z_- + 1/2),
\]
but for each given \( E \) they are evidently eigenstates of the operators
\[
H_\pm^E = H_0^\pm + z_\pm \partial \varphi_\pm (z_\pm; E), \tag{5.9}
\]
where \( \varphi_\pm \) contain so far unknown functions \( v_\pm(z_\pm; E) \). However, the operators \( H_\pm^E \) depend on the energy \( E \) through these functions. Therefore, they can not be considered as Hamiltonians. But one can define the Hamiltonians as solutions of the equations
\[
H_\pm = H_0^\pm + z_\pm \partial \varphi_\pm (z_\pm; H_\pm). \tag{5.10}
\]
Then all functions (5.3) are their eigenstates.

The orthonormality condition (5.6) can be equivalently rewritten as the condition that the Hamiltonians \( H_\pm \) define the action of the same self-adjoint operator \( H \) in the \( \pm \)-representations. In the quasiclassical limit, this is equivalent to the coincidence of the phase space trajectories associated with \( H_+ \) and \( H_- \):
\[
H_+(z_+, z_-) = E \leftrightarrow H_-(z_+, z_-) = E. \tag{5.11}
\]
This condition is equivalent to the compatibility of two equations (5.8).

Note that with respect to the time \( \tau \) defined by the new Hamiltonian, the time dependence of the states characterized by the wave functions (5.3) is given by \( e^{-iE\tau} \). This corresponds to a stationary flow of the Fermi liquid. The profile of the Fermi sea coincides with the classical trajectory of the fermion with the highest energy \( E_F = -\mu \). Its equation can be written in two forms similar to (5.2) which should be consistent
\[
z_+ z_- = M_\pm (z_\pm) \equiv \mu + z_\pm \partial \varphi_\pm (z_\pm; -\mu). \tag{5.12}
\]
These equations are the non-compact version of the equations arising in the conformal map problem which is a semiclassical description of compact Fermi droplet [10]. In that case the potential must be an entire function which is not required in our case. The functional equation (5.12) contains all the information of the tachyon interactions in the tree level string theory. To proceed further with our analysis, we should concretize the form of the perturbing potentials \( V_\pm \). We will show in the next section that the perturbations produced by vertex operators are integrable.
6. Integrable flows associated with vertex operators

6.1. Lax formalism, Toda Lattice structure and string equations

Now we restrict ourselves to the time dependent coherent states made of tachyons with discrete Euclidean momenta $p_n = n/R$. This spectrum of momenta arises when the system compactified on a circle of length $2\pi R$, or heated to the temperature $T = 1/(2\pi R)$. Such perturbations are described by the potentials of the following form

$$V_{\pm}(z_{\pm}) = R \sum_{k \geq 1} t_{\pm k} z_{\pm}^{k/R}. \quad (6.1)$$

In this section we will show that such a deformation is exactly solvable, being generated by a system of commuting flows $H_n$ associated with the coupling constants $t_{\pm n}$. The associated integrable structure is that of a constrained Toda Lattice hierarchy. The method is very similar to the standard Lax formalism of Toda theory, but we will not assume that the reader is familiar with this subject. It is based on rewriting all operators in the energy representation, in which the Fourier transformation $\hat{S}$ is diagonal as we required in the previous section. The energy $E$ will play the role of the coordinate along the lattice formed by the allowed energies $E_n = ip_n$ of the tachyons.

Let us start with the operators $\hat{z}_{\pm} = z_{\pm}$ and $\hat{\partial}_{\pm} = \frac{\partial}{\partial z_{\pm}}$, which are related as

$$-i\hat{\partial}_{-} = \hat{S}\hat{z}_{+}\hat{S}^{-1}, \quad i\hat{\partial}_{+} = \hat{S}^{-1}\hat{z}_{-}\hat{S}. \quad (6.2)$$

The Heisenberg commutation relation $[\hat{\partial}_{\pm}, \hat{z}_{\pm}] = 1$ leads to the operator identity

$$\hat{z}_{-}\hat{S}\hat{z}_{+}\hat{S}^{-1} - \hat{S}\hat{z}_{+}\hat{S}^{-1}\hat{z}_{-} = i. \quad (6.3)$$

Further, the Hamiltonian (4.1) in the $z_{\pm}$-representation $H_0^{\pm} = \mp i(z_{\pm}\hat{\partial}_{\pm} + 1/2)$ is expressed in terms of $\hat{z}_{\pm}$ and $\hat{S}$ as

$$H_0^{-} = -\frac{1}{2} \left( \hat{z}_{-}\hat{S}\hat{z}_{+}\hat{S}^{-1} + \hat{S}\hat{z}_{+}\hat{S}^{-1}\hat{z}_{-} \right) = \hat{S}H_0^{+}\hat{S}^{-1}. \quad (6.4)$$

It follows from the identities

$$H_0^{\pm}\psi_{\pm}^E(z_{\pm}) = E\psi_{\pm}^E(z_{\pm}),
\quad z_{\pm}\psi_{\pm}^E(z_{\pm}) = \psi_{\pm}^{E+1}(z_{\pm}),
\quad \hat{S}\psi_{\pm}^{+1}(z_{\pm}) = R_{\pm}^1(E)\psi_{\pm}^E(z_{\pm}) \quad (6.5)$$

that the operators $H_0^{\pm}$, $\hat{S}$ and $\hat{z}_{\pm}$ are represented in the basis of the non-perturbed wave functions (4.4) by

$$H_0^{\pm} = \hat{E}, \quad \hat{z}_{\pm} = \hat{\omega}^{\pm1}, \quad \hat{S} = R(\hat{E})\hat{l} \equiv \hat{R}\hat{l}, \quad (6.6)$$
where \( \mu_{\psi_{\pm}} = \psi_{\mp} \) and \( \hat{\omega} \) is the shift operator
\[
\hat{\omega} = e^{-i\partial_E}. \tag{6.7}
\]

In order to have a closed operator algebra, we should also define the action of the operators \( \hat{S}^{\pm 1} \) on the functions \( \psi_{\pm,n}^{E\pm}(z_{\pm}) = z_{\pm}^{n}\psi_{\mp}^{E}(z_{\pm}) \). To satisfy the identities (6.2), we should define \( \hat{S}^{\pm 1} \) by the cosine kernel for \( n \) even and by the sine kernel for \( n \) odd. Therefore \( \hat{S}_{\psi_{\pm}}^{E\pm}(z_{\pm}) = \hat{R}'\psi_{\pm}^{E\pm}(z_{\pm}) \) where \( \hat{R}' = R'(E) \) is given by (4.10) with \( \cosh \) replaced by \( \sinh \).

The algebraic relations (6.3) and (6.4) are transformed into algebraic relations among the operators \( \hat{E} \), \( \hat{\omega} \) and \( \hat{R} \) in the \( E \)-space
\[
\hat{R}^{-1}\hat{\omega}\hat{R}'\hat{\omega}^{-1} = -\hat{E} + i/2, \\
\hat{\omega}^{-1}\hat{R}'^{-1}\hat{\omega}\hat{R} = -\hat{E} - i/2. \tag{6.8}
\]

This is equivalent, according to the remark above, to the functional constraint
\[
R(E) = -(i/2 + E) R'(E + i), \tag{6.9}
\]
which is evidently satisfied. To simplify the further discussion, in the rest of this section we will identify the functions \( R(E) \) and \( R'(E) \), thus neglecting the non-perturbative terms \( \sim e^{\pi E} \). Note however that all statements made below can be proved without this identification.

Our aim is to extend the \( E \)-space representation for the basis of wave functions (4.4) perturbed by a phase factor \( e^{\pm i\varphi_{\pm}} \) as in eqs. (5.3)-(5.4), with \( V_{\pm} \) given by (6.1). We assume that the phase \( \varphi_{\pm} \) can be expanded, for sufficiently large \( z_{\pm} \) as a Laurent series
\[
\varphi_{\pm}(z_{\pm}) = R \sum_{k \geq 1} t_{\pm,k} z_{\pm}^{k/R} + \frac{1}{2}\phi - R \sum_{k \geq 1} \frac{1}{k} v_{\pm,k} z_{\pm}^{-k/R}. \tag{6.10}
\]
This assumption will be justified by the subsequent analysis. With the special form (6.10) of the perturbing phase, there exist unitary “dressing operators” \( W_{\pm} \) acting in the \( E \)-space, which transform the “bare” wave functions \( \psi_{\pm}^{E} \) into the “dressed” ones
\[
\Psi_{\pm}^{E} \equiv e^{\mp i\varphi_{\pm}(z_{\pm})} \psi_{\pm}^{E} = W_{\pm} \psi_{\pm}^{E}. \tag{6.11}
\]

It is evident from (6.10) and (3.6) that the dressing operators can be expressed as series in the shift operator \( \hat{\omega} \) with \( E \)-dependent coefficients
\[
W_{\pm} = e^{\mp i\phi/2} \left( 1 + \sum_{k \geq 1} w_{\pm,k} \hat{\omega}^{k/R} \right) e^{\mp iR \sum_{k \geq 1} t_{\pm,k} \hat{\omega}^{k/R}}. \tag{6.12}
\]

\[\text{Note that we should inverse the order of the operators.}\]
In the basis of the perturbed wave functions (6.11), the operators of canonical coordinates \( \hat{z}_\pm \) are represented by two Lax operators

\[
L_{\pm} = W_{\pm} \hat{\omega}^{\pm 1} W_{\pm}^{-1} = e^{\mp i \phi/2} \hat{\omega}^{\pm 1} e^{\pm i \phi/2} \left( 1 + \sum_{k \geq 1} a_{\pm k} \hat{\omega}^{\mp k/R} \right) \tag{6.13}
\]

and the Hamiltonians \( H^0_{\pm} \) are represented (up to a change of sign) by the so-called Orlov-Shulman operators

\[
M_{\pm} = -W_{\pm} \hat{E} W_{\pm}^{-1} = \sum_{k \geq 1} k t_{\pm k} L_{\pm}^{k/R} - \hat{E} + \sum_{k \geq 1} v_{\pm k} L_{\pm}^{-k/R}. \tag{6.14}
\]

In the above formulas all coefficients are functions of \( E \) and \( \vec{t} = \{t_{\pm k}\}_{k=1}^{\infty} \). The Lax and Orlov-Shulman operators act on the wave functions as

\[
L_{\pm} \Psi^E_{\pm}(z_{\pm}) = z_{\pm} \Psi^E_{\pm}(z_{\pm}), \tag{6.15}
\]

\[
M_{\pm} \Psi^E_{\pm}(z_{\pm}) = \left( \sum_{k \geq 1} k t_{\pm k} z_{\pm}^{k/R} - E + \sum_{k \geq 1} v_{\pm k} z_{\pm}^{-k/R} \right) \Psi^E_{\pm}(z_{\pm}) \tag{6.16}
\]

and satisfy the Lax equations

\[
[L_{\pm}, M_{\pm}] = \pm iL_{\pm}, \tag{6.17}
\]

which are the dressed version of the relation \( [\hat{\omega}^{\pm 1}, -\hat{E}] = \pm i \hat{\omega}^{\pm 1} \).

The structure of constrained Toda hierarchy follows from the requirement (5.5), which means that the action of the \( \hat{S} \) operator on the perturbed wave functions is totally compensated by the dressing operators:

\[
W_- = W_+ \hat{R}. \tag{6.18}
\]

The condition (6.18) defines both the Toda structure and a constraint which plays the role of an initial condition for the PDE of the Toda hierarchy.

The Toda structure implies that the tachyon operators generating the perturbation are represented in the \( E \)-space by an infinite set of commuting flows. To show this, we evaluate the variations of the Lax operators with respect to the coupling constants \( t_n \). From the definition (6.13) we have

\[
\partial_{t_n} L_{\pm} = [H_n, L_{\pm}], \tag{6.19}
\]

where the operators \( H_n \) are related to the dressing operators as

\[
H_n = (\partial_{t_n} W_+) W_+^{-1} = (\partial_{t_n} W_-) W_-^{-1}. \tag{6.20}
\]
The two representations of the flows $H_n$ are equivalent by virtue of the relation (6.18). It is important that $\hat{R}$ does not depend on $t_n$’s. A more explicit expression in terms of the Lax operators is derived by the following standard argument. Let us consider the case $n > 0$. From the explicit form of the dressing operators (6.12) it is clear that $H_n = W_+\hat{\omega}^n/R W_+^{-1} + \text{negative powers of } \hat{\omega}^{1/R}$. The variation of $t_n$ will change only the coefficients of the expansions (6.13) of the Lax operators, preserving their general form. But it is clear that if the expansion of $H_n$ contained negative powers of $\hat{\omega}^{1/R}$, its commutator with $L_{-}$ would create extra powers $\hat{\omega}^{-1-k/R}$. Therefore

$$H_{\pm n} = (L_{\pm n}^{n/R})_\geq + \frac{1}{2} (L_{\pm n}^{n/R})_0, \quad n > 0,$$

(6.21)

where the symbol $(\_)_\geq$ means the positive (negative) parts of the series in the shift operator $\hat{\omega}$ and $(\_)_0$ means the constant part. By a similar argument one shows that the Lax equations (6.19) are equivalent to the zero-curvature conditions

$$\partial_{t_m} H_n - \partial_{t_n} H_m - [H_m, H_n] = 0.$$

(6.22)

The equations (6.19) and (6.21) ensure that the perturbations related with the couplings $t_n$ are described by the Toda integrable structure.

The Toda structure leads to an infinite set of PDE’s for the coefficients $w_n$ of the dressing operators, the first of which is the Toda equation for the zero mode of the dressing operators

$$i \frac{\partial}{\partial t_1} \frac{\partial}{\partial t_{-1}} \phi = e^{i\phi(E) - i\phi(E+i/R)} - e^{i\phi(E-i/R) - i\phi(E)}.$$

(6.23)

The uniqueness of the solution is assured by appropriate boundary conditions or additional constraints, known also as string equations. In our case the string equation can be obtained by dressing the equation (6.8) using the formula (6.18), which leads to the following relation between the Lax and Orlov-Shulman operators

$$L_+ L_- = M_+ + i/2, \quad L_- L_+ = M_- - i/2.$$

(6.24)

Similarly, the identity $\hat{R} \hat{E} = \hat{E} \hat{R}$ implies

$$M_- = M_+.$$

(6.25)

Thus, the perturbations of the one-fermion wave functions of the form (6.11) are described by a constrained Toda lattice hierarchy. In the T-dual formulation the same result was proved recently using the standard definition of the Toda Lax operators [12]. The standard Lax and Orlov-Shulman operators are related to $L_{\pm}$ and $M_{\pm}$ by

$$L = L_{+}^{1/R}, \quad \bar{L} = L_{-}^{1/R}, \quad M = M_{+}, \quad \bar{M} = M_{-}.$$

(6.26)
This operators satisfy $R$-dependent constraints

\[ L^R \bar{L}^R = M + i/2, \quad \bar{L}^R L^R = \bar{M} - i/2, \quad M = \bar{M}. \] (6.27)

(The string equations for integer values of $R$ have been discussed previously in \[23,24,25,26\].)

The integrability allows to find the unknown coefficients in the Laurent expansion of the phase (6.10). All the information about them is contained in the so-called $\tau$-function, which is the generating function for the vector fields corresponding to the flows $H_n$. Its existence follows from the zero-curvature condition (6.22) and can be proved in the standard way [27]. The coefficients are related to the $\tau$-function as follows

\[ v_n = \frac{\partial \log \tau}{\partial t_n}, \quad \phi(E) = i \log \frac{\tau(E + i/2R)}{\tau(E - i/2R)}. \] (6.28)

The $\tau$-function can thus be considered as the generating function for the perturbations by vertex operators. As we will show in sec. 8, the logarithm of the $\tau$-function gives the free energy of the Euclidean theory compactified at radius $R$.

6.2. The dispersionless (quasiclassical) limit

Let us consider the quasiclassical limit $E \to -\infty$ when all allowed energies are large and negative. In this limit the integrable structure described above reduces to the dispersionless Toda hierarchy [28,29,27], where the operators $\hat{E}$ and $\hat{\omega}$ can be considered as a pair of classical canonical variables with Poisson bracket

\[ \{\omega, E\} = \omega. \] (6.29)

Similarly, all operators become $c$-functions of these variables. The Lax operators can be identified, by eq. (6.13), with the phase space coordinates $z_\pm$. The functions $z_{\pm}(\omega, E)$ are given by eq.

\[ z_\pm(\omega, E) = e^{-\frac{i\pi}{2R} \chi \omega^{\pm 1}} \left( 1 + \sum_{k \geq 1} a_{\pm k}(E) \omega^{\pm k/R} \right), \] (6.30)

where we have introduced the “string susceptibility”

\[ \chi = -R \partial_E \phi = \partial_E^2 \log \tau. \] (6.31)

The equations (6.24) and (6.25) in the quasiclassical limit give

\[ \{z_-, z_+\} = 1, \] (6.32)

\[ z_+ z_- = M_+ = M_. \] (6.33)
It follows from the first equation, (6.32), that eq. (6.30) defines a canonical transformation relating the phase-space coordinates $z_+, z_-$ to $\log \omega$ and $E$. Moreover, the dressing procedure itself can be interpreted as a canonical transformation between $z_+, z_-$ and the “bare” coordinates

$$\omega_+ = \sqrt{-E} \omega, \quad \omega_- = \sqrt{-E}/\omega.$$  

(6.34)

The second equation, (6.33), which is a deformation of the relation

$$\omega_+ \omega_- = -E$$  

(6.35)

satisfied by the “bare” variables, yields the form of the perturbed one-fermion trajectories.

Let us consider the highest trajectory that defines the shape of the Fermi sea. The variable $E$ then should be interpreted as the Fermi energy, which is equal to (minus) the chemical potential, $E_F = -\mu$. Taking into account the expression for $M_\pm$ through $z_\pm$ given by the r.h.s. of (6.16), we can expand the right-hand side either in $z_+$ or in $z_-\nabla+$

$$z_+ z_- = \sum_{k \geq 1} k t_{\pm k} z_\pm^k + \mu + \sum_{k \geq 1} v_{\pm k} z_{\pm}^{-k}.$$  

(6.36)

(These expansions are of course convergent only for sufficiently large values of $z_\pm$.) Eq. (5.36), which is the dispersionless string equation, is a particular case of the relation (5.12), which was derived for an arbitrary potential. In this way we found a direct geometrical interpretation for the dispersionless string equation in terms of the profile of the Fermi sea.

A simple procedure to calculate the coefficients $a_{\pm k}$ has been suggested in [12]. First, note that if all $t_{\pm k}$ with $k > n$ vanish, the sum in (6.30) can be restricted to $k \leq n$. Then it is enough to substitute the expressions (6.30), with $E = -\mu$, in the profile equations (6.36) and compare the coefficients in front of $\omega_\pm^k$. The result will give the generating function of one point correlators, which is contained in the inverse function $\omega(z_\pm)$. In the next section we will apply this method to calculate the profile of the Fermi sea in the important case of the Sine-Liouville string theory.

7. Solution for Sine-Gordon coupled to gravity

The simplest nontrivial string theory with time-dependent background is the Sine-Gordon theory coupled to gravity known also as Sine-Liouville theory. It is obtained by perturbing with the lowest couplings $t_1$ and $t_{-1}$. It can be easily seen that in this case the expansion (6.30) consists of only two terms

$$z_\pm = e^{-\frac{1}{2}\chi \omega^\pm}(1 + a_{\pm} \omega^\mp \frac{\chi}{\pi}).$$  

(7.1)
The procedure described in [12] gives the following result for the susceptibility $\chi$ and the coefficients $a_{\pm}$:

$$\mu e^{\chi} - \left(1 - \frac{1}{R}\right) t_{1} t_{-1} e^{\frac{2R-1}{R^2} \chi} = 1,$$

$$a_{\pm} = t_{\mp 1} e^{\frac{R-1/2}{R^2} \chi}. \tag{7.2}$$

The first equation in (7.2) for the susceptibility $\chi = \partial^2_{\mu} \log \tau$ was found in [10] and it was shown to reproduce the Moore’s expansion of the free energy $F = \log \tau$ [8]. The equation (7.1) with $a_{\pm}$ given in (7.2) was first found in [11] in the form

$$e^{Rh_{\pm}} - x_{\pm} e^{h_{\pm}} = 1, \quad x_{\pm} = e^{-\chi/2R^2} a_{\pm} z_{\pm}^{-1/R}, \tag{7.3}$$

where $h_{\pm}(z_{\pm})$ is the generating function of the one-point correlators

$$h_{\pm}(z_{\pm}) = \sum_{n=1}^{\infty} \frac{1}{n} z_{\pm}^{-n/R} \frac{\partial^2 F}{\partial \mu \partial t_{\pm n}}. \tag{7.4}$$

It is related to $\omega(z_{\pm})$ as follows

$$\omega(z_{\pm}) = e^{\frac{1}{2R} \chi} z_{\pm} e^{-Rh_{\pm}(z_{\pm})}. \tag{7.5}$$

The equation (7.2) for the susceptibility imposes some restrictions on the allowed values of the parameters $t_{\pm 1}$ and $\mu$. Let us consider the interval of the radii $\frac{1}{2} < R < 1$, which is the most interesting from the point of view of the conjectured duality between the Sine-Liouville theory and the 2D string theory in the black hole background at $R = 2/3$ [8]. In this interval, if we choose $t_{1} t_{-1} < 0$, a real solution for $\chi$ exists only for $\mu > 0$. On the other hand, if $t_{1} t_{-1} > 0$, the allowed interval is $\mu > \mu_{c}$, where the critical value is negative:

$$\mu_{c} = - \left(2 - \frac{1}{R}\right) \left(\frac{1}{R} - 1\right)^{\frac{1}{2R-1}} (t_{1} t_{-1})^{\frac{R-1}{2R-1}}. \tag{7.6}$$

Let us note that (7.6) was interpreted in [31] as a critical point of the pure 2D gravity type.

The solution (7.1)-(7.2) allows us to find the explicit shape of the Fermi sea in the phase space. In the model of the type I considered so far there is an additional restriction for the admissible values of parameters since the boundary of the Fermi sea should be entirely in the positive quadrant $z_{\pm} > 0$. This forces us to take $t_{\pm 1} > 0$. The general situation is shown on Fig. 1a. The unperturbed profile corresponds to the hyperbola asymptotically approaching the $z_{\pm}$ axis, whereas the perturbed curves deviate from the

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5 Since the papers [10,11,12,30] considered the case of vortex perturbations, the duality transformation $R \rightarrow 1/R$ should be performed before comparing the formulas.
Fig. 1: Profiles of the Fermi sea \((z_\pm = x \pm p)\) in the theory of the type I at \(R = 2/3\). Fig. 1a contains several profiles corresponding to \(t_1 = t_{-1} = 2\) and values of \(\mu\) starting from \(\mu_c = -1\) with a step 40. For comparison, we also drew the unperturbed profile for \(\mu = 100\). Fig. 1b shows three moments of the time evolution of the critical profile at \(\mu = -1\).

axes by a power law. We see that there is a critical value of \(\mu\), where the contour forms a spike. It coincides with the critical point \(\mu = \mu_c\) given by (7.6). At this point the quasiclassical description breaks down. On the Fig. 1b, the physical time evolution of a profile is demonstrated.

In the same way one can find the solution in the classical limit of the theory of the type II described in Appendix A. In this case we can introduce two pairs of perturbing potentials describing the asymptotics of the wave function at \(z_\pm \to \infty\) and \(z_\pm \to -\infty\). For sufficiently large \(\mu\) the Fermi sea consists of two connected components and the theory decomposes into two theories of type I. However, in contrast to the previous case, there are no restrictions on the signs of the coupling constants. When \(\mu\) decreases, the two Fermi seas merge together at some critical value \(\mu^*\). This leads to interesting (for example, from the point of view of the Hall effect) phenomena, which we intend to discuss elsewhere. Here we will only mention that, depending on the choice of couplings, it can happen that for some interval of \(\mu\) around the point \(\mu^*\), the Toda description is not applicable.

8. Free energy of the perturbed background

As we mentioned, the perturbations considered above appear in a theory at the finite temperature \(T = 1/\beta\), \(\beta = 2\pi R\). The free energy per unit volume \(\mathcal{F}\) is related to the
partition function $Z$ by
\[ Z = e^{-\beta F}. \] (8.1)

The free energy is considered as a function of the chemical potential $\mu$, so that the number of fermions is given by
\[ N = \partial F / \partial \mu. \] (8.2)

We have seen that any perturbation can be characterized by the profile of the Fermi sea. For a generic profile, the number of fermions is given by the volume of the domain in the phase space occupied by the Fermi liquid
\[ N = \frac{1}{2\pi} \int \int_{\text{Fermi sea}} dz_+ dz_. \] (8.3)

It is implied that the integral is regularized by introducing a cut-off at distance $\sqrt{\Lambda}$. For the unperturbed ground state (5.1), one reproduces from (8.2) and (8.3) the well known result for the (universal part of the) free energy
\[ F = \frac{1}{4\pi} \mu^2 \log \mu. \] (8.4)

The authors of [7] identify the partition function (8.1) for the tachyon backgrounds studied in sect. 6 with the $\tau$-function of the Toda hierarchy, defined by (6.28). In this section we will reproduce this statement by direct integration over the Fermi sea, adjusting the above derivation of (8.4) to the case of a general profile of the form (6.36).

Since the change of variables described by eq. (6.30) is canonical (see eq. (6.32)), one can rewrite the integral (8.3) as
\[ N = \frac{1}{2\pi} \int dE \int \frac{d\omega}{\omega - (E)}. \] (8.5)

The limits of integration over $\omega$ are determined by the cut-off. If we put it at the distance $z_\pm = \sqrt{\Lambda}$, they can be found from the equations
\[ z_\pm (\omega_\pm (E), E) = \sqrt{\Lambda}. \] (8.6)

Taking the derivative with respect to $\mu$, we obtain
\[ \partial_\mu N = -\frac{1}{2\pi} \log \frac{\omega_+(\mu)}{\omega_-(\mu)}. \] (8.7)

The integral over $E$ is also bounded from below, but we do not need to specify the boundary explicitly.
It is enough to keep only the leading order in the cut-off $\Lambda$ for the boundary values $\omega_\pm$. From (8.4) and (8.30) we find to this order

$$\omega_\pm = \left(\Lambda e^{\chi/R}\right)^{\pm 1/2}. \quad (8.8)$$

Combining (8.7) and (8.8), we find

$$\partial_\mu N = -\frac{1}{2\pi} \log \Lambda - \frac{1}{2\pi R} \chi. \quad (8.9)$$

Taking into account the relation (8.2) between the free energy and $N$, we see that

$$\mathcal{F} = -\frac{1}{\beta} \log \tau, \quad \beta = 2\pi R. \quad (8.10)$$

As a result, if we compactify the theory at the time circle of the length $\beta$, our free energy coincides with the logarithm of the Toda $\tau$-function.

The perturbed flow of the Fermi liquid is non-stationary with respect to the physical time $t$ associated with the original Hamiltonian $H_0$. In sec. 5 we constructed a new Hamiltonian, which preserves the shape of the Fermi sea. The latter coincides with the phase-space trajectory (5.11) at $E = -\mu$. Using the fact that $\partial_\mu H(z_+, z_-) = 0$, one can check that

$$\mathcal{F} = \langle H(z_+, z_-) + \mu \rangle. \quad (8.11)$$

Note that the function $H(z_+, z_-)$ is obtained by solving the profile equations (6.36) with respect to $\mu$. One can see, looking at the large $z_\pm$ asymptotics of the profile equations, that it can be written in the following compact form:

$$H(z_+, z_-) = H_0 + \sum_{k \geq 1} k \left( t_k H_k + t_{-k} H_{-k} \right). \quad (8.12)$$

Here the Hamiltonians $H_n(z_+, z_-; \vec{t})$ are given by equations (6.21), with $\omega$ and $E$ expressed as functions of $z_+$ and $z_-$ through eq. (6.30).

9. Conclusions

In this paper we studied the 2D string theory in the presence of arbitrarily strong tachyonic perturbations. In the matrix model language, we dealt with the singlet sector of the MQM described by a system of free fermions in inverted quadratic potential. In the quasiclassical limit the state of the system is described by the shape of the domain in the phase space occupied by the fermionic liquid (the profile of the Fermi sea). In this limit we formulated functional equations, which determine the shape of the Fermi sea, given its
asymptotics at two infinities. There is some analogy between our problem and the problem of conformal maps studied in [12,33,34,35].

This allows to calculate various physical quantities related to the perturbed shape, such as the free energy of the compactified Euclidean theory, in terms of the parameters defining the asymptotic shapes.

For a particular case of perturbations generated by vertex operators (in- and out-going with discrete equally spaced imaginary energies), the system solves the Toda lattice hierarchy where the Toda “times” correspond to the couplings characterizing the asymptotics of the profile of the Fermi sea. Using the “light cone” representation in the phase space, we developed the Lax formalism of the constrained Toda hierarchy, where the constraint coincides with the string equation found recently in [12]. In the dispersionless limit, the Lax formalism reproduces the functional equations for the shape of the perturbed Fermi sea and allows to give a direct physical meaning to the Toda mathematical quantities. In particular, we identified the two spectral parameters of the Toda system with the two chiral coordinates $z_\pm$ in the target phase space.

Let us list the other possible applications of our formalism:

- Calculation of the tachyon scattering matrix for 2D string theory in nontrivial backgrounds.
- Analysis of the systems with more complicated than Sine-Liouville backgrounds. In particular, the quasiclassical analysis of the sections 4 and 7 are not limited to any lattice of tachyon charges and can be used for a mixture of any commensurate or non-commensurate charges. It provides a way of studying the whole space of possible tachyonic backgrounds of the 2D string theory.
- Investigation of the thermodynamics of the 2D string theory in Sine-Liouville type tachyonic backgrounds characterized by a lattice of discrete Matsubara energies [30], [36]. This is supposed to be the thermodynamics of the dilatonic black hole, according to the conjecture of [3].
- The type 2 theory can be used to model the situation where either two quantum Hall droplets approach each other or one droplet is about to split in two.

10. Problems and proposals

We list also some unsolved problems, which could be approached by our formalism:

- To find the metric of the target space for the Sine-Liouville string theory.
- To construct the discrete states of the 2D string theory (at any rational compactification radius) in the framework of our formalism.
- A few important unsolved problems concern the relation between the MQM and CFT formulations of the 2D string theory. We still don’t have a precise mapping of the vertex and vortex operators of the MQM to the corresponding operators of CFT, in spite of the
useful suggestions of the papers [37], [4]. As the result, we did not manage to match the宋-Liouville/Black Hole correlators calculated by [3], [38], and [39] from the CFT approach with the corresponding correlators found in [11] from the Toda approach to the MQM. Another related question is how can the target space integrable structure be seen in the CFT formulation of 2D string theory.

- Description and quantitative analysis of the mixed (vertex&vortex) perturbations in the 2D string theory by means of the MQM. We hope that the “light-cone” formalism might be appropriate to study such perturbations, in spite of the absence of integrability.\(^7\)

Finally, keeping in mind this interesting problem, let us propose here a 3-matrix model whose grand canonical partition function 

\[ Z_N(t_n, \tilde{t}_n) = \sum_{N=0}^{\infty} e^{-2\pi R \mu N} Z_N(t_n, \tilde{t}_n) \]

is the generating function of correlators for both types of perturbations: the parameters \( \tilde{t}_n \) are the couplings of the vortex operators, in the same way as \( t_n \)’s are the couplings of the tachyon (vertex) operators throughout this paper. As usual, the chemical potential \( \mu \) plays the role of the string coupling. The partition function at fixed \( N \) is represented by the integral over two hermitian matrices \( Z_\pm \) and one unitary matrix, the holonomy factor \( \Omega \) around the circle

\[ Z_N(t_n, \tilde{t}_n) = \int [\mathcal{D}Z_+]{t_n}[\mathcal{D}Z_-]{t_n}[d\Omega]{\tilde{t}_n} e^{\text{Tr}(Z_- Z_+ - qZ_\Omega Z_+ \Omega^{-1})}, \quad (10.1) \]

where we denoted \( q = e^{i 2\pi R} \) and introduced the following matrix integration measures

\[ [dZ_\pm]{t_n} = dZ_\pm e^{R \sum_{n>0} t_n \text{Tr} Z_n^R}, \quad (10.2) \]

\[ [d\Omega]{\tilde{t}_n} = [d\Omega]_{SU(N)} e^{\sum_{n \neq 0} \tilde{t}_n \text{Tr} \Omega^n}. \quad (10.3) \]

This 3-matrix integral is nothing but the Euclidean partition function of the upside-down matrix oscillator on a circle of length \( 2\pi R \) with twisted periodic boundary conditions \( Z_\pm(\beta) = \Omega^\dagger Z_\pm(0) \Omega \), needed for the introduction of the winding modes (vortices) in the MQM model of the 2D string theory [14]. It is trivial to show that in particular case of all \( t_n = 0 \) this 3-matrix model reduces to the model with only vortex perturbations of the paper [10]. It can be also shown [36] that in the case of all \( \tilde{t}_n = 0 \) it reduces to the Euclidean version of the model with only tachyonic perturbations studied in this paper. The crucial point here is that the sources for tachyon perturbations are introduced in a single point on the time circle, similarly to the perturbation imposing the initial conditions on the wave function (5.3). The model (10.1) reduces the problem of the study of the backgrounds...
of the compactified 2D string theory with arbitrary tachyon and winding sources, to a three-matrix integral.

We learned from R. Dijkgraaf and C. Vafa that they discovered a similar Toda structure in the $c = 1$ type theory arising in connection to the topological strings.

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Appendix A. Theory of type II

In the theory of type II, the fermions are defined on the whole real line. In this case one should introduce two sets of functions describing, in the quasiclassical limit, fermions at different sides of the potential. They are defined on right (left) semi-axis by

$$
\psi^E_{\pm,>} (z_\pm) = \frac{1}{\sqrt{2\pi}} \frac{z^{\pm iE - \frac{1}{2}}}{\sqrt{1 + e^{2\pi E}}} (z_\pm > 0),
$$

$$
\psi^E_{\pm,<} (z_\pm) = \frac{1}{\sqrt{2\pi}} \frac{(-z_\pm)^{\pm iE - \frac{1}{2}}}{\sqrt{1 + e^{2\pi E}}} (z_\pm < 0)
$$

(A.1)

and the continuation to the other semi-axis is performed according to the rule $z_\pm \to e^{\mp\pi i} z_\pm$. This gives, for $z_\pm > 0$

$$
\psi^E_{\pm,>} (-z_\pm) = \pm ie^{\pi E} \psi^E_{\pm,>} (z_\pm),
$$

$$
\psi^E_{\pm,<} (z_\pm) = \pm ie^{\pi E} \psi^E_{\pm,<} (-z_\pm).
$$

(A.2)

The functions (A.1) satisfy the orthonormality and completeness conditions

$$
\langle \psi^E_{\pm, a} | \psi^{E'}_{\pm, b} \rangle = \delta_{ab} \delta(E - E'),
$$

(A.3)

$$
\int_{-\infty}^{\infty} dE \sum_a \psi^E_{\pm, a} (z_\pm) \psi^{E'}_{\pm, a} (z_\pm') = \delta(z_\pm - z_\pm'),
$$

(A.4)

where the indices $a, b = (> ,< )$ label the right and left wave functions and the scalar product is defined as in (1.5) but with the integral over the whole real axis.

The unitary operator relating $z_\pm$ representations is given by the Fourier kernel on the whole line $K(z_-, z_+)$

$$
[\hat{S} \psi_+](z_-) = \int_{-\infty}^{\infty} dz_+ K(z_-, z_+) \psi_+(z_+).
$$

(A.5)
It acts on the eigenfunctions (A.1) by the following matrix

\[
\hat{S}^{\pm 1} \psi^{\pm a}(z_{\mp}) = \sum_b [S^{\pm 1}(E)]^{ab} \psi^{\pm b}(z_{\mp}), \quad S(E) = R(E) \begin{pmatrix} 1 & -ie^{\pi E} \\ -ie^{\pi E} & 1 \end{pmatrix}, \quad (A.6)
\]

where

\[
R(E) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} (E-i/2) \Gamma(iE+1/2)}. \quad (A.7)
\]

In this case the reflection coefficient \( R(E) \) is not a pure phase because of the tunneling through the potential described by the off-diagonal elements of the \( S \)-matrix. Nevertheless, the whole \( S \)-matrix is unitary

\[ S^\dagger(E)S(E) = 1. \quad (A.8) \]

The scalar product between left and right states is given by

\[
\langle \psi^- | K | \psi^+ \rangle = \int_{-\infty}^{\infty} dz_+ dz_- \overline{\psi_- (z_-)} K(z_-, z_+) \psi_+ (z_+). \quad (A.9)
\]

It is clear that the matrix of scalar products of states (A.1) coincides with \( S(E)\delta(E-E') \). The corresponding completeness condition is

\[
\int_{-\infty}^{\infty} dE \sum_{a,b} \overline{\psi_{-a} (z_-)} (S^{-1})^{ab} \psi_{+b} (z_+) = \frac{1}{\sqrt{2\pi}} e^{-iz_+ z_-}. \quad (A.10)
\]
References

[1] V. Kazakov and A. A. Migdal, *Nucl. Phys.* **B311** (1988) 171.
[2] E. Brezin, C. Itzykson, G. Parisi, and J.-B. Zuber, *Comm. Math. Phys.* **59** (1978) 35.
[3] V. Fateev, A. Zamolodchikov, and Al. Zamolodchikov, *unpublished*.
[4] J. Polchinski, “What is string theory”, *Lectures presented at the 1994 Les Houches Summer School “Fluctuating Geometries in Statistical Mechanics and Field Theory*”, hep-th/9411028.
[5] A. Jevicki, Developments in 2D string theory, hep-th/9309115.
[6] G. Moore, M. Plesser, and S. Ramgoolam, “Exact S-matrix for 2D string theory”, *Nucl. Phys.* **B377** (1992) 143, hep-th/9111035.
[7] R. Dijkgraaf, G. Moore, and M.R. Plesser, “The partition function of 2d string theory”, *Nucl. Phys.* **B394** (1993) 356, hep-th/9208031.
[8] G. Moore, “Gravitational phase transitions and the sine-Gordon model”, hep-th/9203061.
[9] J. Hoppe, V. Kazakov, and I. Kostov, “Dimensionally reduced SYM$_4$ as solvable matrix quantum mechanics”, *Nucl. Phys.* **B571** (2000) 479, hep-th/9907058.
[10] V. Kazakov, I. Kostov, and D. Kutasov, “A Matrix Model for the Two Dimensional Black Hole”, *Nucl. Phys.* **B622** (2002) 141, hep-th/0101011.
[11] S. Alexandrov and V. Kazakov, “Correlators in 2D string theory with vortex condensation”, *Nucl. Phys.* **B610** (2001) 77, hep-th/0104094.
[12] I. Kostov, “String Equation for String Theory on a Circle”, *Nucl. Phys.* **B624** (2002) 146, hep-th/0107247.
[13] D. Gross and I. Klebanov, *Nucl. Phys.* **B344** (1990) 475; *Nucl. Phys.* **B354** (1990) 459.
[14] D. Boulatov and V. Kazakov, “One-Dimensional String Theory with Vortices as Upside-Down Matrix Oscillator”, *Int. J. Mod. Phys.* **8** (1993) 809, hep-th/9012228.
[15] H.A. Fertig, *Phys. Rev.* **B36** (1987) 7969.
[16] O. Agam, E. Bettelheim, P. Wiegmann, and A. Zabrodin, “Viscous fingering and a shape of an electronic droplet in the Quantum Hall regime”, cond-mat/0111333.
[17] H.A. Fertig, *Phys. Rev.* **B387** (1988) 996.
[18] I. Klebanov, *Lectures delivered at the ICTP Spring School on String Theory and Quantum Gravity*, Trieste, April 1991, hep-th/9108019.
[19] E. Brezin, V. Kazakov and Al. Zamolodchikov, *Nucl. Phys.* **B338** (1990) 673.
[20] G. Parisi, *Phys. Lett.* **B238** (1990) 209, 213.
[21] D. Gross and N. Miljkovic, *Phys. Lett.* **B238** (1990) 217.
[22] P. Ginsparg and J. Zinn-Justin, *Phys. Lett.* **B240** (1990) 333.
[23] T. Eguchi and H. Kanno, “Toda lattice hierarchy and the topological description of the c = 1 string theory”, *Phys. Lett.* **B331** (1994) 330, hep-th/9404050.
[24] T. Nakatsu, “On the string equation at $c = 1$”, Mod. Phys. Lett. **A9** (1994) 3313, hep-th/9407096.

[25] K. Takasaki, “Toda lattice hierarchy and generalized string equations”, Comm. Math. Phys. **181** (1996) 131, hep-th/9506089.

[26] A. Zabrodin, “Dispersionless limit of Hirota equations in some problems of complex analysis”, Theor. Math. Phys. **129** (2001) 1511; Theor. Math. Phys. **129** (2001) 239, math.CV/0104169.

[27] K. Takasaki and T. Takebe, “Integrable Hierarchies and Dispersionless Limit”, Rev. Math. Phys. **7** (1995) 743, hep-th/9405096.

[28] I. Krichever, Func. Anal. i ego pril., **22:3** (1988) 37 (English translation: Funct. Anal. Appl. **22** (1989) 200); “The $\tau$-function of the universal Witham hierarchy, matrix models and topological field theories”, Comm. Pure Appl. Math. **47** (1992), hep-th/9205110.

[29] K. Takasaki and T. Takebe, “Quasi-classical limit of Toda hierarchy and $W$-infinity symmetries”, Lett. Math. Phys. **28** (93) 165, hep-th/9301070.

[30] V. A. Kazakov and A. Zeytlin, “On free energy of 2-d black hole in bosonic string theory”, JHEP **0106** (2001) 021, hep-th/0104138.

[31] E. Hsu and D. Kutasov, “The Gravitational Sine-Gordon Model”, Nucl. Phys. B**396** (1993) 693, hep-th/9212023.

[32] P. Wiegmann and A. Zabrodin, “Conformal maps and dispersionless integrable hierarchies”, Comm. Math. Phys. **213** (2000) 523, hep-th/9909147.

[33] I. Kostov, I. Krichever, M. Mineev-Weinstein, P. Wiegmann, and A. Zabrodin, “$\tau$-function for analytic curves”, hep-th/0005259.

[34] A. Boyarsky, A. Marshakov, O. Ruchhayskiy, P. Wiegmann, and A. Zabrodin, “On associativity equations in dispersionless integrable hierarchies”, Phys. Lett. B**515** (2001) 483, hep-th/0105260.

[35] M. Mineev-Weinstein, P. B. Wiegmann, A. Zabrodin, “Integrable Structure of Interface Dynamics”, Phys. Rev. Lett. **84** (2000) 5106, nlin.SI/0001007.

[36] S. Yu. Alexandrov, V. A. Kazakov, I. K. Kostov, work in progress.

[37] G. Moore, N. Seiberg, and M. Staudacher, “From loops to states in 2D quantum gravity”, Nucl. Phys. B**362** (1991) 665.

[38] J. Teschner, “The deformed two-dimensional black hole”, Phys. Lett. B**458** (1999) 257, hep-th/9902189.

[39] T. Fukuda and K. Hosomichi, “Three-point Functions in Sine-Liouville Theory”, JHEP **0109** (2001) 003, hep-th/0105217.

[40] E. Witten, “Ground Ring of two dimensional string theory”, Nucl. Phys. B**373** (1992) 187, hep-th/9108004.

[41] S. Mukhi and C. Vafa, “Two dimensional black-hole as a topological coset model of $c=1$ string theory”, Nucl. Phys. B**407** (1993) 667, hep-th/9301083.

[42] D. Ghoshal and C. Vafa, “$c=1$ String as the Topological Theory of the Conifold”, Nucl. Phys. B**453** (1995) 121, hep-th/9506122.