Positivity Conditions for Cubic, Quartic and Quintic Polynomials

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Abstract

We present a necessary and sufficient condition for a cubic polynomial to be positive for all positive reals. We identify the set where the cubic polynomial is nonnegative but not all positive for all positive reals, and explicitly give the points where the cubic polynomial attains zero. We then reformulate a necessary and sufficient condition for a quartic polynomial to be nonnegative for all positive reals. From this, we derive a necessary and sufficient condition for a quartic polynomial to be nonnegative and positive for all reals. Our condition explicitly exhibits the scope and role of some coefficients, and has strong geometrical meaning. In the interior of the nonnegativity region for all reals, there is an appendix curve. The discriminant is zero at the appendix, and positive in the other part of the interior of the nonnegativity region. By using the Sturm sequences, we present a necessary and sufficient condition for a quintic polynomial to be positive and nonnegative for all positive reals. We show that for polynomials of a fixed even degree higher than or equal to four, if they have no real roots, then their discriminants take the same sign, which depends upon that degree only, except on an appendix set of dimension lower by two, where the discriminants attain zero.

Key words. Cubic polynomials, quartic polynomials, quintic polynomials, the Sturm theorem, discriminant, appendix.

AMS subject classifications. 15A69, 15A83

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1 Introduction

In 1988, Schmidt and Heß [20] presented a necessary and sufficient condition for a cubic polynomial to be nonnegative for all positive reals. In 1994, Ulrich and Watson [23] presented a necessary and sufficient condition for a quartic polynomial to be nonnegative for all positive reals. On the other hand, necessary and sufficient conditions for a quartic polynomial to be positive for all reals have been studied by Gadem and Li [7], Ku [13], Jury and Mansour [10], Wang and Qi [24], and Gao [8].

In this paper, we first present a necessary and sufficient condition for a cubic polynomial to be positive for all positive reals. We identify the set where the cubic polynomial is nonnegative but not all positive for all positive reals, and explicitly give the points where the cubic polynomial attains zero.

We then reformulate the result of Ulrich and Watson [23]. Based upon this, we derive a necessary and sufficient condition for a quartic polynomial to be nonnegative and positive for all reals. Comparing with the existing results in the literature, our condition has three merits. First, it explicitly states a necessary condition. Second, it explicitly states a symmetric relation between two parameters. Third, its geometrical meaning is clear. We also present a theorem on the geometrical properties of the nonnegativity region and the positivity region. In the interior of the nonnegativity region for all reals, there is an appendix curve. The discriminant is zero at the appendix, and positive in the other part of the interior of the nonnegativity region.

By using the Sturm sequences, we present a necessary and sufficient condition for a quintic polynomial to be positive and nonnegative for all positive reals. This is the first for quintic polynomials. It also indicates that such conditions are possible for polynomials of even higher degrees.

Finally, we show that such an appendix exists for all even degree polynomials with their degrees higher than or equal to 4.

These results are useful in automatic control [7, 13, 10] and determining copositivity and strict copositivity of symmetric tensors, and positive semidefiniteness of even order symmetric tensors [14, 15, 16, 21, 22]. These properties are further useful in optimization [18, 19], hypergraphs [2, 3, 17] and physics [5, 9, 11, 12].

Some preliminary knowledge on the Sturm theorem and quadratic polynomials are given in the next section. Then, four sections are about cubic polynomials, quartic polynomials, quintic polynomials and higher even degree polynomials, respectively.

All the polynomials studied in this paper are of real coefficients.
2 Preliminaries

2.1 The Sturm Theorem

Suppose that we have an \( m \)th degree polynomial \( g(t) \). What is the analytically expressed necessary and sufficient condition such that \( g(t) > 0 \) for all \( t \geq 0 \) \((t > 0)\)?

We use the Sturm theorem in classical algebra [1, pp.52-57] to answer this question.

We may construct the Sturm sequence \( \{g_0, g_1, \cdots, g_l\} \). Let \( g_0 = g \), \( g_1 = g' \), and \( g_k = -\text{rem}(g_{k-1}, g_k) \) for \( k \geq 1 \), where \( g' \) is the derivative of \( g \), \( \text{rem}(g_{k-1}, g_k) \) is the remainder of the division of \( g_{k-1} \) by \( g_k \). We have \( g_l \neq 0 \) and \( g_{l+1} = 0 \). Then the length \( l + 1 \) of the Sturm sequence is not greater than \( m + 1 \). The number of variations of the Sturm sequence \( S = \{g_0, g_1, \cdots, g_l\} \) at a real number \( \xi \) is the number of sign changes (ignore zero) of the real number sequence \( \{g_0(\xi), g_1(\xi), \cdots\} \). Denote it as \( V(\xi) \). By the Sturm theorem, the number of real distinct roots of \( g(t) = 0 \), for \( t \geq 0 \), is \( V(0) - V(\infty) \). If \( V(0) - V(\infty) = 0 \), then \( g \) has no positive roots, i.e., \( g(t) > 0 \) for all \( t \geq 0 \). This answers the question.

Note that \( g_l \) is the greatest common divisor (GCD) of \( g \) and \( g' \). If \( g_l \) is not a constant term, then \( g \) has multiple roots. Any root of \( g_l \) with multiple \( m \) is a \( m + 1 \) multiple root of \( g \).

In the following, we simply say that \( g \) is strictly copositive if \( g(t) > 0 \) for all \( t \geq 0 \).

2.2 Quadratic Polynomials

The following result should be known several centuries ago, and is easy to be derived.

Theorem 2.1 Suppose that we have a quadratic polynomial

\[
\phi(t) = t^2 + ut + v,
\]

where \( v \neq 0 \). Then \( \phi(t) > 0 \) for all \( t \geq 0 \) if and only if either (i) \( u \geq 0 \) and \( v > 0 \); or (ii) \( u < 0 \) and \( 4v > u^2 \). If \( u < 0 \) and \( 4v = u^2 \), then \( \phi(t) \geq 0 \) for all \( t \geq 0 \), and this is the only case that \( \phi(t) \geq 0 \) for all \( t \geq 0 \), but not \( \phi(t) > 0 \) for all \( t \geq 0 \).

Furthermore, \( \phi(t) > 0 \) for all \( t \) if and only if \( 4v > u^2 \). If \( 4v = u^2 \), then \( \phi(t) \geq 0 \) for all \( t \), and this is the only case that \( \phi(t) \geq 0 \) for all \( t \), but not \( \phi(t) > 0 \) for all \( t \).
3 Cubic Polynomials

3.1 Nonnegativity Conditions for Non-Degenerate Cubic Polynomials

We call a cubic polynomial non-degenerate if its constant is nonzero. For a non-degenerate cubic polynomial, we may write it as

\[ h(t) = t^3 + pt^2 + qt + r, \]  

(3.1)

where \( r \neq 0 \). If \( r < 0 \), then it cannot be always nonnegative for all positive reals. Thus, we may assume that \( r > 0 \).

By [20], we have the following theorem.

**Theorem 3.1** Suppose that \( h(t) \) is defined by (3.1). Then \( h(t) \geq 0 \) for all \( t \geq 0 \) in and only in the following two cases: (A) \( p \geq 0 \) and \( q \geq 0 \); (B) (D) \( \Delta(h) \leq 0 \). Here,

\[ \Delta(h) = p^2q^2 + 18pqr - 27r^2 - 4p^3r - 4q^3 \]  

(3.2)

is the discriminant of \( h(t) \).

3.2 Positivity Conditions for Non-Degenerate Cubic Polynomials

Suppose that we have a cubic polynomial \( h(t) \) defined by (3.1), with \( r > 0 \).

Then

\[ h(t) = \phi(t)t + r, \]

where \( \phi(t) = t^2 + pt + q \). By Theorem 2.1, if either (I) \( p \geq 0 \) and \( q \geq 0 \), or (II) \( q \geq \frac{p^2}{4} \), then \( \phi(t) \geq 0 \) for all \( t \geq 0 \). This implies that \( h(t) > 0 \) for all \( t \geq 0 \), as \( r > 0 \).

We now study the case which is not included in (I) and (II).

Then \( q < \frac{p^2}{4} \). We now construct the Sturm sequence for \( h(t) \). Let

\[ h_0(t) = h(t) = t^3 + pt^2 + qt + r, \]

\[ h_1(t) = h'(t) = 3t^2 + 2pt + q. \]

Then

\[ h_0(t) - \frac{t}{3}h_1(t) = \frac{p}{3}t^2 + \frac{2q}{3}t + r, \]

\[ h_0(t) - \frac{t}{3}h_1(t) - \frac{p}{9}h_1(t) = \left( \frac{2q}{3} - \frac{2p^2}{9} \right)t + r - \frac{pq}{9}. \]
We have
\[ h_2(t) = q_2 t + r_2, \]
where
\[ q_2 = \frac{2p^2}{9} - \frac{2q}{3}, \quad r_2 = \frac{pq}{9} - r. \] (3.3)

Since \( q < \frac{p^2}{4} \), we have \( q_2 > 0 \).

Then
\[ h_3(t) \equiv r_3 = -h_1 \left( -\frac{r_2}{q_2} \right) = -\frac{3r_2^2}{q_2^2} + \frac{2pr_2}{q_2} - q = \frac{\Delta(h)}{9q_2^2}, \]
where
\[ \Delta(h) \equiv p^2q^2 + 18pqr - 27r^2 - 4p^3r - 4q^3 = -81r_2^2 + 54pq_2r_2 - 27qq_2^2. \]

The expression of \( \Delta(h) \) can be found in [6].

Suppose that \( \Delta(h) = 0 \). Then \( h_2(t) \) is the GCD of \( g \) and \( g' \),
\[ h(t) = (t - \alpha)^2(t + \beta), \]
where
\[ \alpha = -\frac{r_2}{q_2}. \]

We have \( q_2 > 0 \). We now show that \( r_2 < 0 \). Assume that \( r_2 \geq 0 \). This means that \( pq \geq 9r \). Since we have already exclude the case that \( p \geq 0 \) and \( q \geq 0 \), this means \( p < 0, q < 0 \) and \( pq \leq 9r \). These further implies that \( \Delta(h) < 0 \), contradicting the assumption that \( \Delta(h) = 0 \).

We have \( \alpha^2 \beta = r > 0 \). Thus, \( \beta > 0 \) and \( \alpha \neq 0 \). Therefore, \( h(t) \geq 0 \) for all \( t \geq 0 \). Furthermore, in this case, we always have \( t = \alpha > 0 \) such that \( h(t) = 0 \).

Suppose now \( \Delta(h) \neq 0 \). We have
\[ S = \{ h_0, h_1, h_2, h_3 \}, \]
\[ S(\infty) = \{ 1, 1, q_2, \Delta(h) \}, \]
\[ S(0) = \{ 1, q, r_2, \Delta(h) \}. \]

Assume that \( \Delta(h) > 0 \). Then \( V(\infty) = 0 \). Thus, \( h(t) > 0 \) for all \( t \geq 0 \) if and only if \( V(0) = 0 \), i.e., \( q \geq 0 \) and \( r_2 \geq 0 \). However, this implies that \( p > 0 \) and \( q > 0 \), which has already been covered by Case (I).

Assume that \( \Delta(h) < 0 \). Then \( V(\infty) = 1 \). Thus, \( h(t) > 0 \) for all \( t \geq 0 \) if and only if either (i) \( r_2 \leq 0 \), or (ii) \( q \geq 0 \) and \( r_2 \geq 0 \). However, Case (ii) has already been covered by Case (I) as discussed above. Thus, only Case (i) needs to be considered. As discussed above, the case that \( r_2 > 0 \), i.e., \( pq > 9r \), is covered by the condition (I).
\( p \geq 0 \) and \( q \geq 0 \), and (III) \( \Delta(h) < 0 \). Thus, the condition that \( r_2 \leq 0 \) is not necessary here.

Thus, we have one more case such that \( h(t) > 0 \) for all \( t \geq 0 \): (III) \( \Delta(h) < 0 \).

Finally, Case (II) \( q \geq \frac{p^2}{4} \) is covered by the union of Cases (I) and (III).

Thus, we have the following theorem.

**Theorem 3.2** Suppose that \( h(t) \) is defined by (3.1). Then \( h(t) > 0 \) for all \( t \geq 0 \) if and only in the following two cases: (A) \( p \geq 0 \) and \( q \geq 0 \); (B) \( \Delta(h) < 0 \). Here, \( \Delta(h) \) is the discriminant, defined by (3.2).

If \( \Delta(h) = 0 \) and either \( p < 0 \) or \( q < 0 \), then \( h(t) \geq 0 \) for all \( t \geq 0 \), but there is

\[
\alpha = \frac{9r - pq}{2p^2 - 6q} > 0
\]

such that \( h(\alpha) = 0 \).

In all the other cases, there is a \( t > 0 \) such that \( h(t) < 0 \).

4 Quartic Polynomials

4.1 Nonnegativity Conditions

Suppose that we have a quartic polynomial

\[
f(t) = t^4 + \alpha t^3 + \beta t^2 + \gamma t + 1.
\]  

By [23], its discriminant has the form:

\[
\Delta(f) = 4[\beta^2 - 3\alpha \gamma + 12]^3 - [72\beta + 9\alpha \beta \gamma - 2\beta^3 - 27\alpha^2 - 27\gamma^2]^2.
\]  

By [23], we have the following theorem.

**Theorem 4.1** Suppose that \( f \) is defined by (4.4). Let

\[
\Lambda_1 = (\alpha - \gamma)^2 - 16(\alpha + \beta + \gamma + 2), \quad \Lambda_2 = (\alpha - \gamma)^2 - \frac{4(\beta + 2)}{\sqrt{\beta - 2}}(\alpha + \gamma + 4\sqrt{\beta - 2}).
\]

Then \( f(t) \geq 0 \) for all \( t > 0 \) if and only if either

(1) \( \beta < -2 \), \( \Delta(f) \leq 0 \) and \( \alpha + \gamma > 0 \); or

(2) \(-2 \leq \beta \leq 6 \) and either (i) \( \Delta(f) \leq 0 \) and \( \alpha + \gamma > 0 \), or (ii) \( \Delta(f) \geq 0 \) and \( \Lambda_1 \leq 0 \); or

(3) \( \beta > 6 \) and either (i) \( \Delta(f) \leq 0 \) and \( \alpha + \gamma > 0 \), or (ii) \( \alpha > 0 \) and \( \gamma > 0 \); or (iii) \( \Delta(f) \geq 0 \) and \( \Lambda_2 \leq 0 \).
We may combine (1), (2i) and (3i) as
(A) \( \Delta(f) \leq 0 \) and \( \alpha + \gamma > 0 \).

On the other hand, as
\[
f(t) = (t^4 + \beta t^2 + 1) + \alpha t^3 + \gamma t,
\]
if \( \alpha \geq 0, \gamma \geq 0 \) and \( \beta \geq -2 \), then \( f(t) \geq 0 \) for all \( t \geq 0 \). Thus, we have
(B) \( \alpha \geq 0, \gamma \geq 0 \) and \( \beta \geq -2 \).

Then (B) covers (3ii).

We may replace the constraints \( \Lambda_1 \leq 0 \) and \( \Lambda_2 \leq 0 \), by some constraints which are linear with respect to \( \alpha \) and \( \gamma \).

For the condition \( \Lambda_1 \leq 0 \), from the discussion in Sections 2 and 3 of [23], we may replace it by \( |\alpha - \gamma| \leq r \), where \( r \) is the value of \( |\alpha - \gamma| \) at the two intersection points of \( \alpha + \gamma = 0 \) and the curve \( \Gamma_\beta \) in [23]. These are two points where \( \alpha(t) + \gamma(t) = 0 \), for \( \alpha(t) \) and \( \gamma(t) \) given by (7a) and (7b) of [23]. Solving this, we find that \( r = 4\sqrt{\beta+2} \). Thus, we may use \( |\alpha - \gamma| \leq 4\sqrt{\beta+2} \) to replace \( \Lambda_1 \leq 0 \). Note that we may rewrite the constraint \( |\alpha - \gamma| \leq 4\sqrt{\beta+2} \) such that \( \alpha \) and \( \gamma \) are linear there.

For the condition \( \Lambda_2 \leq 0 \), by Figure 1 of [23], we may replace it by \( |\alpha - \gamma| \leq 4\sqrt{\beta+2} \) and \( \alpha + \gamma \geq -4\sqrt{\beta-2} \).

We have the following theorem.

**Theorem 4.2** Suppose that \( g(z) = a z^4 + b z^3 + c z^2 + d z + e \) be a quartic polynomial with real coefficients and \( a > 0 \) and \( e > 0 \). Let
\[
\alpha = ba^{-\frac{3}{4}} e^{-\frac{1}{4}}, \quad \beta = ca^{-\frac{1}{4}} e^{-\frac{1}{4}}, \quad \gamma = da^{-\frac{1}{4}} e^{-\frac{3}{4}},
\]
f is defined by (4.4), and \( \Delta(f) \) is defined by (4.5). Then \( f(t) \geq 0 \) for all \( t > 0 \) if and only if either
(A) \( \Delta(f) \leq 0 \) and \( \alpha + \gamma > 0 \); or
(B) \( \alpha \geq 0, \gamma \geq 0 \) and \( \beta \geq -2 \); or
(C) \( \Delta(f) \geq 0, |\alpha - \gamma| \leq 4\sqrt{\beta+2} \) and either (i) \( -2 \leq \beta \leq 6 \), or (ii) \( \beta > 6 \) and \( \alpha + \gamma \geq -4\sqrt{\beta-2} \).

**Proof** We may convert the general quartic polynomial \( g(z) \) to \( f(t) \), defined by (4.4), as in [23].

We have already discussed to convert Conditions (1), (2) and (3) of Theorem 4.1 to Conditions (A), (B) and (C).

A merit of the format of Theorem 4.2 is that it is somewhat convenient to derive nonnegativity conditions for all \( t \) from conditions (A), (B) and (C).
4.2 Nonnegativity and Positivity Conditions for All Reals

We now consider the conditions that \( f(t) \geq 0 \) \( (f(t) > 0) \) for all \( t \). This is equivalent to the condition that \( f(t) \geq 0 \) \( (f(t) > 0) \) and \( g(t) \geq 0 \) \( (g(t) > 0) \) for all \( t \), where \( g(t) \) is defined by

\[
g(t) = t^4 - \alpha t^3 + \beta t^2 - \gamma t + 1.
\]

Note that \( \Delta(f) = \Delta(g) \). By Theorem 4.2, we have the following theorem.

**Theorem 4.3** Suppose that \( g(z) = az^4 + bz^3 + cz^2 + dz + e \) is a quartic polynomial with real coefficients and \( a > 0 \) and \( e > 0 \). Let

\[
\alpha = ba^{-\frac{3}{4}}e^{\frac{1}{4}}, \quad \beta = ca^{-\frac{1}{2}}e^{\frac{1}{2}}, \quad \gamma = da^{-\frac{1}{4}}e^{-\frac{3}{4}},
\]

\( f \) be defined by (4.4), and \( \Delta(f) \) be defined by (4.5). Then \( f(t) \geq 0 \) for all \( t \) if and only if \( \Delta(f) \geq 0 \), \( |\alpha - \gamma| \leq 4\sqrt{\beta + 2} \) and either (i) \( -2 \leq \beta \leq 6 \), or (ii) \( \beta > 6 \) and \( |\alpha + \gamma| \leq 4\sqrt{\beta - 2} \).

Furthermore, \( f(t) > 0 \) for all \( t \) if and only if either

(A) \( \Delta(f) = 0 \), \( \alpha = \gamma \), \( \alpha^2 + 8 = 4\beta < 24 \); or

(B) \( \Delta(f) > 0 \), \( |\alpha - \gamma| \leq 4\sqrt{\beta + 2} \) and either (i) \( -2 \leq \beta \leq 6 \), or (ii) \( \beta > 6 \) and \( |\alpha + \gamma| \leq 4\sqrt{\beta - 2} \).

**Proof** Let \( \bar{f}(t) = f(-t) \). Then

\[
\bar{f}(t) = t^4 - \alpha t^3 + \beta t^2 - \gamma t + 1.
\]

Then \( f(t) \geq 0 \) for all \( t \) if and only if \( f(t) \geq 0 \) and \( \bar{f}(t) \geq 0 \) for all \( t \geq 0 \). From Theorem 4.2, we have conditions that \( f(t) \geq 0 \) and \( \bar{f}(t) \geq 0 \) for all \( t \geq 0 \). Combining these conditions, i.e., taking the intersection of the conditions that \( f(t) \geq 0 \) for all \( t \geq 0 \), and the conditions that \( \bar{f}(t) \geq 0 \) for all \( t \geq 0 \), we have the conditions for the first part of this theorem.

We now prove the second part of this theorem. If \( \Delta(f) > 0 \), it is in the interior of the nonnegativity region, we have \( f(t) > 0 \) for all \( t \). We have condition (B). Now assume that \( \Delta(f) = 0 \). Then \( f(t) \) has a multiple root. To make \( f(t) > 0 \) for all \( t \), this multiple root has to be complex. Thus, \( f(t) = (t^2 + ut + 1)^2 \) with \( u^2 < 4 \). Comparing the coefficients of

\[
f(t) = (t^2 + ut + 1)^2 = t^4 + \alpha t^3 + \beta t^2 + \gamma t + 1,
\]

We have \( \alpha = \gamma = 2u \) and \( u^2 + 2 = \beta \). Then, we have condition (A). \( \square \)
Geometrically, for fixed $\beta$, the nonnegativity region of $(\alpha, \gamma)$ for $g(t)$ is the convex hull of the central convex compact part of the positive sign region in Figure 5b and 5c of [23], while the positivity region is the interior of the nonnegativity region. In Figures 5b of [23], for $2 \leq \beta < 6$, there are two points missing. These two points $(\alpha, \beta, \gamma)$ are defined by $\alpha = \gamma$ and $\alpha^2 + 8 = 4\beta$. At these two points, $\Delta(f) = 0$. These two points are missing in Figure 5b of [23], but do not affect the final results of [23].

Note that $\beta + 2 \geq 0$ is a necessary condition such that $f(t) \geq 0$ for all $t$. This explicitly stated in Theorem 4.3. Actually, if $f(t) \geq 0$ for all $t$, then

$$\beta + 2 = \frac{f(1) + f(-1)}{2} \geq 0.$$ 

The following theorem presents the geometrical features of the nonnegativity region and the positivity region.

**Theorem 4.4** Suppose that $f(t)$ is defined by (4.4). Let

$$S = \{(\alpha, \beta, \gamma) : f(t) > 0 \text{ for all } t\}$$

and

$$\bar{S} = \{(\alpha, \beta, \gamma) : f(t) \geq 0 \text{ for all } t\}.$$ 

Then $\bar{S}$ is a closed convex cone with a recession direction $(0, 1, 0)$ and an apex $(0, -2, 0)$. It is symmetric with respect to $\alpha$ and $\gamma$, i.e., if $(\alpha, \beta, \gamma) \in \bar{S}$, then $(\gamma, \beta, \alpha), (-\alpha, \beta, -\gamma) \in \bar{S}$. For any $(\alpha, \beta, \gamma)$ in the interior of $\bar{S}$, if $2 \leq \beta < 6$, $\alpha = \beta$ and $\alpha^2 + 8 = 4\beta$, we have $\Delta(f) = 0$. Otherwise, we have $\Delta(f) > 0$. For any $(\alpha, \beta, \gamma)$ at the boundary of $\bar{S}$, we have $\Delta(f) = 0$. The set $S$ is the interior of $\bar{S}$.

**Proof** Consider $\bar{S}$. Suppose that $f(t)$ and $g(t)$ are two quartic polynomials with the form of (4.4), $f(t) \geq 0$ and $g(t) \geq 0$ for all $t$. Then $\frac{1}{2}(f(t) + g(t))$ is still a quartic polynomial with the form of (4.4), and is nonnegative for all $t$. This shows that $\bar{S}$ is convex. Similarly, we may show that it is closed. Let $f(t)$ be defined by (4.4), and $\delta > 0$. Let

$$g(t) = t^4 + \alpha t^3 + (\beta + \delta)t^2 + \gamma t + 1.$$ 

Then $g(t) = f(t) + \delta t^2 \geq f(t) \geq 0$ for all $t$. This shows that $(0, 1, 0)$ is a recession direction of $\bar{S}$ and $\bar{S}$ is a cone. From Theorem 4.3, the point $(\alpha, \beta, \gamma) = (0, -2, 0)$ is the unique point of $\bar{S}$ with the smallest value of $\beta$. Hence, it is an apex of $\bar{S}$. The other properties of $\bar{S}$ and $S$ can be derived from Theorem 4.3 accordingly. □
In Figure 1, the $\alpha = \gamma$ plane is exhibited. The set $\bar{S}$ is the area above $\beta = 2|\alpha| - 2$ for $|\alpha| \leq 4$, and the area above the parabola $\beta = \frac{\alpha^2}{4} + 2$ for $|\alpha| \geq 4$. The discriminant $\Delta(f)$ vanishes at the line segments $\beta = 2|\alpha| - 2$ for $|\alpha| \leq 4$ and the parabola $\beta = \frac{\alpha^2}{4} + 2$ for all $\alpha$. The truncated parabola $\beta = \frac{\alpha^2}{4} + 2$ for $|\alpha| < 4$ is in the interior of $\bar{S}$. In Figure 2, the $\alpha = -\gamma$ plane is exhibited. The set $\bar{S}$ is the area above the parabola $\beta = \frac{\alpha^2}{4} - 2$. We may see that $(\alpha, \beta, \gamma) = (0, -2, 0)$ is the apex of the cone $\bar{S}$.

Figure 1. The intersection of $\bar{S}$ and the plane $\alpha = \gamma$

The truncated parabola $\beta = \frac{\alpha^2}{4} + 2$ for $|\alpha| < 4$ in Figure 1 is a special part of the surface $\Delta(f) = 0$. The other part of the surface $\Delta(f) = 0$ is of dimension 2, but the truncated parabola $\beta = \frac{\alpha^2}{4} + 2$ for $|\alpha| < 4$ is of dimension 1. Thus, we call it the appendix of the surface $\Delta(f) = 0$. Such an appendix exists for polynomials with their degrees higher than or equal to 4.
This problem has been studied by Gadem and Li [7], Ku [13], Jury and Mansour [10], Wang and Qi [24], and Gao [8]. We may compare Theorem 4.3 with the result of [24], which is a correction of the result of [13]. The following is the result of [24].

The polynomial considered in [24] has the following form:

\[ f(t) = a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4. \]

Thus,

\[ a_0 = a_4 = 1, \quad a_1 = \frac{\alpha}{4}, \quad a_2 = \frac{\beta}{6}, \quad a_3 = \frac{\gamma}{4}. \]

Let

\[ G = a_0 a_3 - 3a_0 a_1 a_2 + 2a_1^3 = \frac{\gamma}{4} - \frac{\alpha \beta}{8} + \frac{\alpha^3}{32} = \frac{8 \gamma - 4 \alpha \beta + \alpha^3}{32}. \]

\[ H = a_0 a_2 - a_1^2 = \frac{\beta}{6} - \frac{\alpha^2}{16} = \frac{8 \beta - 3 \alpha^2}{48}. \]

\[ I = a_0 a_4 - 4a_1 a_3 + 3a_2^2 = 1 - \frac{\alpha \gamma}{4} + \frac{\beta^2}{12} = \frac{12 - 3 \alpha \gamma + \beta^2}{12}. \]

\[ J = a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 - a_2^3 = \frac{\beta}{6} + \frac{\alpha \beta \gamma}{48} - \frac{\alpha^2 + \gamma^2}{16} - \frac{\beta^3}{216} = \frac{72 \beta + 9 \alpha \beta \gamma - 27 \alpha^2 - 27 \gamma^2 - 2 \beta^3}{432}. \]

\[ \Delta = I^3 - 27J^3 = \left( \frac{12 - 3 \alpha \gamma + \beta^2}{12} \right)^3 - \frac{(72 \beta + 9 \alpha \beta \gamma - 27 \alpha^2 - 27 \gamma^2 - 2 \beta^3)^2}{3 \times 36^2} = \frac{\Delta(f)}{4 \times 12^3}. \]

Hence, \( \Delta \) in [24] has the same sign as \( \Delta(f) \) and plays the same role as \( \Delta(f) \). The result of [24] can be formulated with the language of this paper as follows.
Theorem 4.5  Let $f(t)$ be defined by (4.4). Then $f(t) > 0$ for all $t$ if and only if either
(1) $\Delta(f) = 0$, $G = 0$, $12H^2 - I = 0$ and $H > 0$; or
(2) $\Delta(f) > 0$ and either (i) $H \geq 0$, or (ii) $H < 0$ and $12H^2 - I < 0$.

We have
\[
12H^2 - I = 12 \left( \frac{8\beta - 3\alpha^2}{48} \right)^2 - \frac{12 - 3\alpha\gamma + \beta^2}{12} = \frac{64\beta^2 - 48\alpha^2\beta + 9\alpha^4 - 192 + 48\alpha\gamma - 16\beta^2}{192}
\]
\[
= \frac{16\beta^2 - 16\alpha^2\beta + 3\alpha^4 - 64 + 16\alpha\gamma}{64}.
\]

Corollary 4.6  Let $f(t)$ be defined by (4.4). Then $f(t) \geq 0$ for all $t$ if and only if
$\Delta(f) \geq 0$ and either (i) $H \geq 0$, or (ii) $H < 0$ and $12H^2 - I \leq 0$.

The conclusions of Theorem 4.3 should be the same with the conditions of Theorem
4.5 and Corollary 4.6. However, Theorem 4.3 has three merits. First, it explicitly stated
that a necessary condition is that $\beta \geq -2$. Second, it treats $\alpha$ and $\gamma$ in a symmetric way.
Third, its geometrical meaning is clear, as shown by Theorem 4.4. The set indicated
by Theorem 4.3 (A) and Theorem 4.5 (1) is the appendix of the surface $\Delta(f) = 0$. It
is actually in the interior of the nonnegativity region.

5  Quintic Polynomials

5.1 Quintic Polynomials with Multiple Roots

Suppose that we have a quintic polynomial
\[
g(t) = t^5 + at^4 + bt^3 + ct^2 + dt + e,
\]
where $e > 0$.

As we discussed in Section 2, by construct the Sturm sequence of $g$, we may find
the GCD of $g$ and $g'$ if $g$ has a multiple root.

(A). Suppose that the GCD of $g$ and $g'$ is a linear polynomial. We may assume
that it is $\phi(t) = t - \alpha$. Then $g$ has a double root $\alpha$. Since $e > 0$, $\alpha \neq 0$. We may write that
\[
g(t) = h(t)(t - \alpha)^2,
\]
where $h(t) = t^3 + pt^2 + qt + r$. Then
\[
p = a + 2\alpha, \quad q = b + 2\alpha p + \alpha^2, \quad r = \frac{e}{\alpha^2}.
\]
Since $e > 0$, $r > 0$, i.e., $h$ has the form (3.1). By Theorem 3.2, we have the following
proposition.
Proposition 5.1 Suppose that \( g(t) \) is defined by (5.6), \( g \) and \( g' \) has a GCD \( \phi(t) = t - \alpha. \) Then \( g \) has the form (5.7), where \( h(t) \) has the form (3.1), with \( p, q \) and \( r, \) given by (5.8). Then \( g(t) \geq 0 \) for all \( t \geq 0, \) if and only if \( h(t) \geq 0 \) for all \( t \geq 0. \) If furthermore \( \alpha < 0, \) then \( g(t) > 0 \) for all \( t \geq 0, \) if and only if \( h(t) > 0 \) for all \( t \geq 0. \)

(B). Suppose that the GCD of \( g \) and \( g' \) is a quadratic polynomial. We may assume that it is \( \phi(t) = t^2 + ut + v. \) There are three subcases.

(B1). \( u^2 \neq 4v. \) In this subcase, \( \phi(t) \) has two distinct roots \( \alpha \) and \( \beta. \) Then \( g \) has a double root \( \alpha \) and a double root \( \beta. \) Since \( e > 0, \) \( \alpha \neq 0 \) and \( \beta \neq 0. \) We have

\[
g(t) = \phi(t)^2(t - \gamma) = (t - \alpha)^2(t - \beta)^2(t - \gamma),
\]

where \( \gamma \) is the fifth root of \( g. \) Since \( e > 0, \gamma < 0. \) Then we have the following conclusion.

Proposition 5.2 Suppose that \( g(t) \) is defined by (5.6), \( g \) and \( g' \) has a GCD \( \phi(t) = t^2 + ut + v. \) If \( u^2 \neq 4v, \) then \( g(t) \geq 0 \) for all \( t \geq 0. \) If furthermore \( \phi(t) > 0 \) for all \( t \geq 0, \) then \( g(t) > 0 \) for all \( t \geq 0. \)

We may use Theorem 2.1 to determine if \( \phi(t) > 0 \) for all \( t > 0. \)

(B2). \( u^2 = 4v. \) Then \( \phi \) has a double root \( \alpha = -\frac{v}{2}, \) and \( \alpha \) is a triple root of \( g. \) Since \( e > 0, \alpha \neq 0. \) We have

\[
g(t) = \left( t + \frac{u}{2} \right)^3 \psi(t),
\]

where

\[
\psi(t) = t^2 + \hat{u}t + \hat{v}.
\]

Then

\[
\hat{v} = \frac{8e}{u^3}, \quad \hat{u} = a - \frac{3u}{2}.
\]

We have the following conclusion.

Proposition 5.3 Suppose that \( g(t) \) is defined by (5.6), \( g \) and \( g' \) has a GCD \( \phi(t) = t^2 + ut + v. \) Suppose that \( u^2 = 4v. \) Then \( u \neq 0. \) Let \( \psi(t) \) be calculated by (5.11) and (5.12). Then \( g(t) \geq 0 \) for all \( t \geq 0 \) if and only if \( u > 0 \) and \( \psi(t) \geq 0 \) for all \( t \geq 0, \) and \( g(t) > 0 \) for all \( t \geq 0 \) if and only if \( u > 0 \) and \( \psi(t) > 0 \) for all \( t \geq 0. \)

We may use Theorem 2.1 to determine the situation of \( \psi(t). \)

(C). Suppose that the GCD of \( g \) and \( g' \) is a cubic polynomial. We may assume that it is \( h(t) = t^3 + pt^2 + qt + r = (t - \alpha)^2(t + \beta). \) There are two subcases.

(C1) \( q = \frac{p^2}{4} \) and \( r = \frac{p^3}{27}. \) This implies \( \alpha = -\beta = -\frac{p}{3}, \) and

\[
g(t) = \left( t + \frac{p}{3} \right)^4 (t - \gamma).
\]
Since $e > 0$, $\gamma < 0$, $p \neq 0$. Thus, $g(t) \geq 0$ for all $t \geq 0$. If furthermore $p > 0$, then $g(t) > 0$ for all $t \geq 0$.

(C2) Otherwise, $\alpha \neq -\beta$, $\alpha \neq 0$ and $\beta \neq 0$. We have $\alpha$ from (??) and (3.3). Then

$$\beta = \frac{r}{\alpha^2}.$$ 

We have

$$g(t) = (t - \alpha)^3(t + \beta)^2.$$ 

Then $-\alpha^3\beta^2 = e$. Since $e > 0$, $\alpha < 0$. Thus, we always have $g(t) \geq 0$ for all $t \geq 0$. If furthermore $r > 0$, then $g(t) > 0$ for all $t \geq 0$.

We have the following proposition.

**Proposition 5.4** Suppose that $g(t)$ is defined by (5.6), $g$ and $g'$ has a GCD $h(t) = t^3 + pt^2 + qt + r$. Then $g(t) \geq 0$ for $t \geq 0$. We have $g(t) > 0$ for all $t \geq 0$ if and only if either (i) $q = \frac{p^2}{3}$, $r = \frac{p^3}{27}$ and $p > 0$; or (ii) $r > 0$ and either $q \neq \frac{p^2}{3}$ or $r \neq \frac{p^3}{27}$.

The case that the GCD of $g$ and $g'$ is a quartic polynomial will be analyzed in the next section. In that case, $g(t) > 0$ for all $t \geq 0$.

### 5.2 General Quintic Polynomials

For constructing the Sturm sequence of a quintic polynomial $g(t)$, we keep to use $a, b, c, d$ and $e$ to denote the coefficients of the polynomials $g_i$. We have

$$g_0(t) = g(t) = t^5 + at^4 + bt^3 + ct^2 + dt + e,$$

and

$$g_1(t) = g'(t) = 5t^4 + 4at^3 + 3bt^2 + 2ct + d.$$ 

For $i \geq 2$, we use $b_i, c_i, d_i$ and $e_i$ to denote the coefficients of $g_i(t)$, with $b_i$ for the coefficient of $t^3$, $c_i$ for $t^2$, $d_i$ for $t$ and $e_i$ for the constant term. If there are different coefficients used, we use an additional index to distinguish them. For example, in the following, $e_3$ was used in Case (2), and in Case (3), we use $e_{3,1}$ to denote a new coefficient. We use $g_3$ to denote different polynomials in different parts of this section, while $e_3$ and $e_{3,1}$ are uniquely used in this section, as $g_3$ will not appear in the theorems in the next sections, but $e_3$ and $e_{3,1}$ play a role in establishing those theorems.

However, $e_3$ and $e_{3,1}$ are fractional functions of the coefficients $a, b, c, d$ and $e$. This is not convenient. Then, we use the bar symbol to denote some polynomial functions to replace them. For example, we write $e_3 = \frac{\bar{e}_3}{2ad^2}$, where $\bar{e}_3$ is a polynomial function of the coefficients $a, b, c, d$ and $e$, and we use $\bar{e}_3$ instead of $e_3$ in our theorems. Totally, eleven such fractional functions are replaced.
In the following, \( S(0) \) and \( S(\infty) \) are the Sturm sequence at 0 and \( \infty \). Since only signs are important, we may replace their entries by other numbers as long as the signs are not changed.

Note that when there are no multiple roots, either \( g(t) \) has no positive root, or has at least two positive roots. This means that either \( V(0) - V(\infty) = 0 \) or \( V(0) - V(\infty) \geq 2 \).

We now construct the Sturm sequence for \( g \). We have

\[
g(t) = \frac{t}{5} g_1(t) - \frac{a}{25} g_1(t) = \left( \frac{2b}{5} - \frac{4a^2}{25} \right) t^3 + \left( \frac{3c}{5} - \frac{3ab}{25} \right) t^2 + \left( \frac{4d}{5} - \frac{2ac}{25} \right) t + \left( e - \frac{ad}{25} \right),
\]

\[
g_2(t) = \frac{4a^2 - 10b}{25} t^3 + \frac{3ab - 15c}{25} t^2 + \frac{2ac - 20d}{25} t + \frac{ad - 25e}{25} = b_2 t^3 + c_2 t^2 + d_2 t + e_2.
\]

There are four possibilities.

(1) \( b_2 = c_2 = d_2 = 0 \). We further divide this case to two subcases.

1A) \( e_2 = 0 \). This implies that \( b = \frac{2a^2}{5} \), \( c = \frac{ab}{5} = \frac{2a^3}{25} \), \( d = \frac{ac}{10} = \frac{a^4}{125} \) and \( e = \frac{ad}{25} = \frac{a^5}{3125} \). Thus,

\[
g(t) = \left( t + \frac{a}{5} \right)^5.
\]

Since \( e > 0 \), we have \( a > 0 \). Then, \( g \) is strictly copositive in this subcase.

1B) \( e_2 \neq 0 \). Then \( S = \{g_0, g_1, g_2\} \), and \( b = \frac{2a^2}{5} \), \( c = \frac{ab}{5} \) and \( d = \frac{ac}{10} \), and \( g_2(t) = e_2 \). We have

\[
d = \frac{ac}{10} = \frac{a^2 b}{50} = \frac{a^4}{125} \geq 0.
\]

Then \( V(\infty) = 0 \) if \( e_2 > 0 \). Otherwise \( V(\infty) = 1 \). On the other hand, \( V(0) = 1 \) if \( e_2 < 0 \), \( V(0) = 0 \) if \( e_2 > 0 \). Thus, \( V(0) - V(\infty) \equiv 0 \), and \( g \) is strictly copositive in this subcase.

Thus, \( g \) is strictly copositive in Case (1).

(2) \( b_2 = c_2 = 0 \) and \( d_2 \neq 0 \). Then \( b = \frac{2a^2}{5} \), \( c = \frac{ab}{5} = \frac{2a^3}{25} \),

\[
g_2(t) = d_2 t + e_2,
\]

\[
g_1(t) = 5 t^4 + 4 a t^3 + \frac{6a^2}{5} t^2 + \frac{4a^3}{25} t + d,
\]

\[
\text{rem}\{g_1, g_2\} = g_1 \left( -\frac{e_2}{d_2} \right),
\]

\[
g_3(t) = e_3 = -g_1 \left( -\frac{e_2}{d_2} \right) = \frac{e_3}{25 d_2^2}.
\]
where
\[ e_3 = -125e_2^4 + 100ae_2^3d_2 - 30a^2e_2^2d_2^2 + 4a^3e_2d_2^3 - dd_2^4. \]

We need to divide this case to two subcases.

(2A) \( e_3 = 0 \), i.e., \( \bar{e}_3 = 0 \). In this subcase, \( \alpha = -\frac{c}{d^2} \) is a double root of \( g \). We may apply Proposition 5.1 and Theorem 3.2 to determine the situation of \( g(t) \).

(2B) \( e_3 \neq 0 \), i.e., \( \bar{e}_3 \neq 0 \). Then
\[
S = \{g_0, g_1, g_2, g_3\},
\]
\[
S(\infty) = \{+\infty, +\infty, d_2, e_3\},
\]
\[
S(0) = \{e, d, e_2, e_3\}.
\]
Clearly, we may replace \( e_3 \) by \( \bar{e}_3 \) here. Then \( V(0) = V(\infty) \) if and only if (i) \( \bar{e}_3 > 0, d_2 \geq 0, d \geq 0, e_2 \geq 0 \); or (ii) \( \bar{e}_3 < 0, d_2 \geq 0 \); or (iii) \( \bar{e}_3 > 0, d_2 < 0 \).

(3) \( b_2 = 0 \) and \( c_2 \neq 0 \). Then \( b = \frac{2a^2}{5} \),
\[
c_2 = \frac{3ab - 15c}{25} = \frac{6a^3 - 75c}{125},
\]
\[
g_2(t) = c_2t^2 + d_2t + e_2,
\]
\[
g_1(t) = 5t^4 + 4at^3 + \frac{6a^2}{5}t^2 + 2ct + d.
\]
\[
g_1(t) - \frac{5t^2}{c_2}g_2(t) = \left(4a - \frac{5d_2}{c_2}\right)t^3 + \left(\frac{6a^2}{5} - \frac{5e_2}{c_2}\right)t^2 + 2ct + d,
\]
\[
g_1(t) - \frac{5t^2}{c_2}g_2(t) - \left(4a - \frac{5d_2}{c_2}\right)\frac{t}{c_2}g_2(t)
\]
\[
= \left[\left(\frac{6a^2}{5} - \frac{5e_2}{c_2}\right) - \left(4a - \frac{5d_2}{c_2}\right)\frac{d_2}{c_2}\right]t^2 + \left[2c - \left(4a - \frac{5d_2}{c_2}\right)\frac{e_2}{c_2}\right]t + d
\]
\[
= \left[\frac{6a^2}{5} - \frac{5e_2}{c_2} + 4ad_2 + \frac{5d_2^2}{c_2^2}\right]t^2 + \left[2c - 4ae_2c_2 + \frac{5d_2e_2}{c_2^2}\right]t + d
\]
\[
= \frac{6a^2c_2^2}{5} - 5(5e_2 + 4ad_2)c_2 + 25d_2^2t^2 + \frac{2cc_2^2 - 4ae_2c_2 + 5de_2}{c_2^2}t + d
\]
\[
g_1(t) - \frac{5t^2}{c_2}g_2(t) - \left(4a - \frac{5d_2}{c_2}\right) \frac{t}{c_2}g_2(t) - \frac{6a^2c_2^2 - 5(5e_2 + 4ad_2)c_2}{5c_2^3}g_2(t) \\
= \left[\frac{2cc_2^2 - 4ae_2c_2 + 5d_2e_2}{c_2^2} - \frac{6a^2c_2^2 - 5(5e_2 + 4ad_2)c_2d_2}{5c_2^3}\right] t \\
+ \frac{d - \frac{6a^2c_2^2e_2 - 5(5e_2 + 4ad_2)c_2e_2 + 25d_2^2e_2}{5c_2^3}}{5c_2^3} \\
= \frac{10cc_2^3 - 20ae_2c_2^2 + 25c_2d_2e_2 - 6a^2c_2^2d_2 + 5(5e_2 + 4ad_2)c_2d_2 - 25d_2^3}{5c_2^3} \\
+ \frac{5c_2^3d - 6a^2c_2^2e_2 + 5(5e_2 + 4ad_2)c_2e_2 - 25d_2^2e_2}{5c_2^3}.
\]

Then

\[
g_3(t) = d_3t + e_{3,1},
\]

where

\[
d_3 = \frac{\bar{d}_3}{5c_2^4}, \quad e_{3,1} = \frac{\bar{e}_{3,1}}{5c_2^4},
\]

\[
\bar{d}_3 = -10cc_2^2 + 20ae_2c_2^2 - 25c_2d_2e_2 + 6a^2c_2^2d_2 - 5(5e_2 + 4ad_2)c_2d_2 + 25c_2d_2^2,
\]

\[
\bar{e}_{3,1} = -5c_2^4d + 6a^2c_2^3e_2 - 5(5e_2 + 4ad_2)c_2^2e_2 + 25c_2d_2^2e_2.
\]

There are three subcases:

(3A) \(d_3 = e_{3,1} = 0\), i.e., \(\bar{d}_3 = \bar{e}_{3,1} = 0\). Then \(g_2\) is the GCD of \(g\) and \(g'\). Let \(u = \frac{d_2}{c_2}\) and \(v = \frac{e_2}{c_2}\). We may use Propositions 5.2, 5.3 and Theorem 3.2 to determine the situation of \(g(t)\).

(3B) \(d_3 = 0\) and \(e_{3,1} \neq 0\), i.e., \(\bar{d}_3 = 0\) and \(\bar{e}_{3,1} \neq 0\). Then \(g_3(t) = e_{3,1}\), \(S = \{g_0, g_1, g_2, g_3\}\),

\[
S(\infty) = \{+\infty, +\infty, c_2, e_{3,1}\},
\]

\[
S(0) = \{e, d, e_2, e_{3,1}\}.
\]

We may replace \(e_{3,1}\) by \(\bar{e}_{3,1}\) here. Then \(V(0) = V(\infty)\) if and only if (i) \(V(0) = V(\infty) = 0\); or (ii) \(V(0) = V(\infty) = 1\); or (iii) \(V(\infty) = 2\), which implies \(V(0) = 2\) by the early discussion.

We may replace \(e_{3,1}\) by \(\bar{e}_{3,1}\) here. Thus, in this case, \(g\) is strictly copositive if and only if either (i) \(\bar{e}_{3,1} > 0\), \(c_2 \geq 0\), \(d \geq 0\), \(e_2 \geq 0\); or (ii) \(\bar{e}_{3,1} < 0\), and \(d \geq 0\) if \(e_2 > 0\); or (iii) \(\bar{e}_{3,1} > 0\), \(c_2 < 0\).

(3C) \(d_3 \neq 0\), i.e., \(\bar{d}_3 \neq 0\). Then

\[
g_2(t) - \frac{c_2}{d_3}tg_3(t) = \left(d_2 - \frac{c_2}{d_3}e_{3,1}\right)t + e_2 = \frac{d_2d_3 - c_2e_{3,1}}{d_3}t + e_2.
\]

\[
g_2(t) - \frac{c_2}{d_3}tg_3(t) - \frac{d_2d_3 - c_2e_{3,1}}{d_3^2}g_3(t) = e_2 - \frac{d_2d_3 - c_2e_{3,1}}{d_3^2}e_{3,1},
\]
\[ g_4(t) = e_4 = \frac{d_2d_3 - c_2e_{3,1}}{d_3^2} e_{3,1} - e_2 = \frac{d_2d_3e_{3,1} - c_2e_{3,1}^2 - e_2d_3^2}{d_3^2} = \frac{\bar{e}_4}{d_3^2}, \]

where
\[ \bar{e}_4 = d_2d_3e_{3,1} - c_2e_{3,1}^2 - e_2d_3^2. \]

There are two further subcases.

(3Ca) \( e_4 = 0 \), i.e., \( \bar{e}_4 = 0 \). Then \( g_3(t) \) is the GCD of \( g \) and \( g' \), and \( \alpha = -\frac{e_{3,1}}{d_3} \) is a double root of \( g \). We may use Proposition 5.1 and Theorem 3.2 to determine the situation of \( g(t) \).

(3Cb) \( e_4 \neq 0 \), i.e., \( \bar{e}_4 \neq 0 \). We have
\[ S = \{g_0, g_1, g_2, g_3, g_4\}, \]
\[ S(\infty) = \{+\infty, +\infty, c_2, d_3, e_4\}, \]
\[ S(0) = \{e, d, e_2, e_{3,1}, e_4\}. \]

We may replace \( d_3, e_{3,1} \) and \( e_4 \) by \( \bar{d}_3, \bar{e}_{3,1} \) and \( \bar{e}_4 \) here. Then \( V(0) = V(\infty) \) if and only if (i) \( V(0) = V(\infty) = 0 \); or (ii) \( V(0) = V(\infty) = 1 \); or (iii) \( V(0) = V(\infty) = 2 \); or (iv) \( V(\infty) = 3 \), which implies \( V(0) = 3 \) by the early note.

Thus, in this case, \( g \) is strictly copositive if and only if either (i) \( \bar{e}_4 > 0, c_2 \geq 0, \bar{d}_3 \geq 0, d \geq 0, \bar{e}_{3,1} \geq 0 \); or (ii) \( \bar{e}_4 < 0, c_2 \geq 0 \) if \( \bar{d}_3 > 0, d \geq 0 \) if \( \max\{e_2, \bar{e}_{3,1}\} > 0, e_2 \geq 0 \) if \( \bar{e}_{3,1} > 0 \); or (iii) \( \bar{e}_4 > 0, \min\{c_2, \bar{d}_3\} < 0, \min\{d, e_2, \bar{e}_{3,1}\} < 0, d \geq 0 \) if \( e_2 > 0 \); or (iv) \( \bar{e}_4 < 0, \bar{d}_3 > 0, c_2 < 0 \).

(4) \( b_2 \neq 0 \). Then
\[ g_2(t) = b_2t^3 + c_2t^2 + d_2t + e_2. \]
\[ g_1(t) - \frac{5}{b_2}g_2(t) = \left(4a - \frac{5c_2}{b_2}\right) t^3 + \left(3b - \frac{5d_2}{b_2}\right) t^2 + \left(2c - \frac{5e_2}{b_2}\right) t + d, \]
\[ g_1(t) - \frac{5}{b_2}g_2(t) - \frac{1}{b_2} \left(4a - \frac{5c_2}{b_2}\right) g(t) \]
\[ = \left[\left(3b - \frac{5d_2}{b_2}\right) - \frac{c_2}{b_2} \left(4a - \frac{5c_2}{b_2}\right)\right] t^2 + \left[\left(2c - \frac{5e_2}{b_2}\right) - \frac{d_2}{b_2} \left(4a - \frac{5c_2}{b_2}\right)\right] t + d - \frac{e_2}{b_2} \left(4a - \frac{5c_2}{b_2}\right) \]
\[ = \frac{3bb_2 - (5d_2 + 4ac_2)b_2 + 5c_2e_2}{b_2^2} t^2 + \frac{2cb_2^2 - (5e_2 + 4ad_2)b_2 + 5c_2d_2}{b_2^2} t + \frac{db_2^2 - 4ae_2b_2 + 5c_2e_2}{b_2^2}. \]
Then
\[ g_3(t) = c_3 t^2 + d_{3,1} t + e_{3,2}, \]
where
\[ c_3 = \bar{c}_3, \quad d_{3,1} = \bar{d}_{3,1}, \quad e_{3,2} = \bar{e}_{3,2}, \]
\[ \bar{c}_3 = -3bb_2 + (5d_2 + 4ac_2)b_2 - 5c_2e_2, \]
\[ \bar{d}_{3,1} = -2cb_2^2 + (5e_2 + 4ad_2)b_2 - 5c_2d_2, \]
\[ \bar{e}_{3,2} = -db_2^2 + 4ae_2b_2 - 5c_2e_2. \]

We have four subcases.

(4A) \( c_3 = d_{3,1} = e_{3,2} = 0 \), i.e., \( \bar{c}_3 = \bar{d}_{3,1} = \bar{e}_{3,2} = 0 \). This implies that \( g \) is the GCD of \( g \) and \( g' \). Let \( p = \frac{e_2}{b_2}, q = \frac{d_2}{b_2} \) and \( r = \frac{e_2}{b_2} \). Then Proposition 5.4 determines the situation of \( g(t) \).

(4B) \( c_3 = d_{3,1} = 0 \) and \( e_{3,2} \neq 0 \), i.e., \( \bar{c}_3 = \bar{d}_{3,1} = 0 \) and \( \bar{e}_{3,2} \neq 0 \).

Then
\[ g_3(t) = e_{3,2}, \]
\[ S = \{g_0, g_1, g_2, g_3\}, \]
\[ S(\infty) = \{+\infty, +\infty, b_2, e_{3,2}\}, \]
\[ S(0) = \{e, d, e_2, e_{3,2}\}. \]

We may replace \( e_{3,2} \) by \( \bar{e}_{3,2} \) here. Then, \( g \) is strictly copositive if and only if either (i) \( \bar{e}_{3,2} > 0, b_2 \geq 0, d \geq 0, e_2 \geq 0 \); or (ii) \( \bar{e}_{3,2} < 0, d \geq 0 \) if \( e_2 > 0 \); or (iii) \( \bar{e}_{3,2} > 0, b_2 < 0 \).

(4C) \( c_3 = 0 \) and \( d_{3,1} \neq 0 \). Then
\[ g_3(t) = d_{3,1} t + e_{3,2}, \]
\[ g_1(t) = e_{4,1} = -g_2 \left( -\frac{e_{3,2}}{d_{3,1}} \right) = \frac{\bar{e}_{4,1}}{d_{3,1}}, \]
where
\[ \bar{e}_{4,1} = b_2 d_{3,1} \bar{e}_{3,2}^3 - 2b_2 d_{3,1} \bar{e}_{3,2} \bar{e}_{3,2}^2 + d_2 \bar{e}_{3,2} \bar{e}_{3,2} - 2d_{3,1} \bar{e}_{3,2} - e_2 d_{3,1}. \]

Again, there are two further subcases.

(4Ca) \( e_{4,1} = 0 \), i.e., \( \bar{e}_{4,1} = 0 \). Then \( g_3(t) \) is the GCD of \( g \) and \( g' \), and \( g \) has a double nonzero root
\[ \alpha = -\frac{\bar{e}_{3,2}}{d_{3,1}}. \]

We may use Proposition 5.1 and Theorem 3.2 to determine the situation of \( g(t) \).

(4Cb) \( e_{4,1} \neq 0 \), i.e., \( \bar{e}_{4,1} \neq 0 \). We have
\[ S = \{g_0, g_1, g_2, g_3, g_4\}. \]
Then $V(0) = V(\infty)$ if and only if (i) $V(0) = V(\infty) = 0$; or (ii) $V(0) = V(\infty) = 1$; or (iii) $V(0) = V(\infty) = 2$; or (iv) $V(\infty) = 3$, which implies that $V(0) = 3$ by the early note.

We may replace $d_{3,1}$, $e_{3,2}$ and $e_{4,1}$ by $\bar{d}_{3,1}$, $\bar{e}_{3,2}$ and $\bar{e}_{4,1}$. Thus, in this subcase, $g$ is strictly copositive if and only if either (i) $\bar{e}_{4,1} > 0$, $b_2 \geq 0$, $d_{3,1} \geq 0$, $d \geq 0$, $e_2 \geq 0$, $\bar{e}_{3,2} \geq 0$; or (ii) $\bar{e}_{4,1} < 0$, $b_2 \geq 0$ if $d_{3,1} > 0$, $d \geq 0$ if $\max\{e_2, \bar{e}_{3,2}\} > 0$, $e_2 \geq 0$ if $\bar{e}_{3,2} > 0$; or (iii) $\bar{e}_{4,1} > 0$, $\min\{b_2, \bar{d}_{3,1}\} < 0$, $\min\{d_2, e_2, \bar{e}_{3,2}\} < 0$, $d \geq 0$ if $e_2 > 0$; or (iv) $\bar{e}_{4,1} < 0$, $\bar{d}_{3,1} > 0$, $b_2 < 0$.

(4D) $c_3 \neq 0$, i.e., $\bar{c}_3 \neq 0$. Then

$$g_3(t) = c_3 t^2 + d_{3,1} t + e_{3,2};$$

$$g_2(t) = b_2 t^3 + c_2 t^2 + d_2 t + e_2,$$

$$g_2(t) - \frac{b_2}{c_3} t g_3(t) = \left(c_2 - \frac{b_2 d_{3,1}}{c_3}\right) t^2 + \left(d_2 - \frac{b_2 e_{3,2}}{c_3}\right) t + e_2,$$

$$g_2(t) - \frac{b_2}{c_3} t g_3(t) - \frac{1}{c_3} \left(c_2 - \frac{b_2 d_{3,1}}{c_3}\right) g_3(t) = \left(d_2 - \frac{b_2 e_{3,2}}{c_3}\right) - \frac{d_{3,1}}{c_3} \left(c_2 - \frac{b_2 d_{3,1}}{c_3}\right) t + e_2 - \frac{e_{3,2}}{c_3} \left(c_2 - \frac{b_2 d_{3,1}}{c_3}\right)$$

$$= \left(d_2 - \frac{b_2 e_{3,2}}{c_3} - \frac{c_2 d_{3,1}}{c_3} + \frac{b_2 d_{3,1}^2}{c_3^2}\right) t + e_2 - \frac{c_2 e_{3,2}}{c_3} + \frac{b_2 d_{3,1} e_{3,2}}{c_3^2}. $$

Then

$$g_4(t) = d_4 t + e_{4,2},$$

where

$$d_4 = \bar{d}_4, \quad e_{4,2} = \bar{e}_{4,2},$$

$$\bar{d}_4 = -\frac{c_2^2 d_2}{c_3} + b_2 \bar{c}_3 \bar{e}_{3,2} + c_2 \bar{c}_3 \bar{d}_{3,1} - b_2 \bar{d}_{3,1}^2,$$

$$\bar{e}_{4,2} = -\frac{c_2^2 e_2}{c_3} + c_2 \bar{c}_3 \bar{e}_{3,2} - b_2 \bar{d}_{3,1} \bar{e}_{3,2}.$$
$S(\infty) = \{+\infty, +\infty, b_2, c_3, e_{4,2}\}$,
$S(0) = \{e, d, e_2, e_{3,2}, e_{4,2}\}$.

Then $V(0) = V(\infty)$ if and only if (i) $V(0) = V(\infty) = 0$; or (ii) $V(0) = V(\infty) = 1$; or
(iii) $V(0) = V(\infty) = 2$; or (iv) $V(\infty) = 3$, which implies $V(0) = 3$ by the early note.

We may replace $c_3$, $e_{3,2}$ and $e_{4,2}$ here by $\bar{c}_3$, $\bar{e}_{3,2}$ and $\bar{e}_{4,2}$. Thus, in this case, $g$ is strictly copositive if and only if either (i) $\bar{e}_{4,2} > 0$, $b_2 \geq 0$, $\bar{c}_3 \geq 0$, $\bar{d}_4 \geq 0$, $\bar{e}_{3,2} \geq 0$; or (ii) $\bar{e}_{4,2} < 0$, $b_2 \geq 0$ if $\bar{c}_3 > 0$, $\bar{d}_4 \geq 0$ if $\max\{\bar{e}_2, \bar{e}_{3,2}\} > 0$, $\bar{e}_2 \geq 0$ if $\bar{e}_{3,2} > 0$; or (iii) $\bar{e}_{4,2} > 0$, $\min\{b_2, \bar{c}_3\} < 0$, $\min\{d, \bar{e}_2, \bar{e}_{3,2}\} < 0$, $\bar{d}_4 \geq 0$ if $\bar{e}_2 > 0$; or (iv) $\bar{e}_{4,2} < 0$, $\bar{c}_3 > 0$, $b_2 < 0$.

(4Dc) $d_4 \neq 0$, i.e., $\bar{d}_4 \neq 0$. Then
\[
g_3(t) = c_3 t^2 + d_{3,1} t + e_{3,2},
g_4(t) = d_4 t + e_{4,2},
g_5(t) \equiv e_5 = -g_3 \left(\frac{e_{4,2}}{d_4}\right) = \frac{\bar{e}_5}{b_2^2 d_4^2},
\]
where
\[
\bar{e}_5 = -\bar{c}_3 \bar{e}_{4,2}^2 + \bar{d}_{3,1} \bar{e}_{4,2} \bar{d}_4 - \bar{e}_{3,2} \bar{d}_4^2.
\]

There are two further subcases.

(4Dc1) $e_5 = 0$, i.e., $\bar{e}_5 = 0$. Then $g_4(t)$ is the GCD of $g$ and $g'$, and $g$ has a double nonzero root
\[
\alpha = -\frac{e_{4,2}}{d_4}.
\]
We may use Proposition 5.5 and Theorem 3.2 to determine the situation of $g(t)$.

(4Dc2) $e_5 \neq 0$, i.e., $\bar{e}_5 \neq 0$. We have
\[
S = \{g_0, g_1, g_2, g_3, g_4, g_5\},
S(\infty) = \{+\infty, +\infty, b_2, c_3, d_4, e_5\},
S(0) = \{e, d, e_2, e_{3,2}, e_{4,2}, e_5\}.
\]
Then $V(0) = V(\infty)$ if and only if (i) $V(0) = V(\infty) = 0$; or (ii) $V(0) = V(\infty) = 1$; or
(iii) $V(0) = V(\infty) = 2$; or (iv) $V(0) = V(\infty) = 3$, or (v) $V(\infty) = 4$, which implies
that $V(0) = 4$ by the early note.

We may replace $c_3, d_4, e_{3,2}, e_{4,2}$ and $e_5$ here by $\bar{c}_3, \bar{d}_4, \bar{e}_{3,2}, \bar{e}_{4,2}$ and $\bar{e}_5$. Thus, in this case, $g$ is strictly copositive if and only if either (i) $\bar{e}_5 > 0$, $b_2 \geq 0$, $\bar{c}_3 \geq 0$, $\bar{d}_4 \geq 0$, $\bar{e}_{4,2} \geq 0$, $\bar{d}_4 \geq 0$, $\bar{e}_2 \geq 0$, $\bar{e}_{3,2} \geq 0$; or (ii) $\bar{e}_5 < 0$, $b_2 \geq 0$ if $\max\{\bar{c}_3, \bar{d}_4\} > 0$, $\bar{e}_3 \geq 0$ if $\bar{d}_4 > 0$, $\bar{d}_4 \geq 0$ if $\max\{\bar{e}_2, \bar{e}_{3,2}, \bar{e}_{4,2}\} > 0$, $\bar{e}_{3,2} \geq 0$ if $\bar{e}_{4,2} > 0$; or (iii) $\bar{e}_5 > 0$, $\min\{b_2, \bar{c}_3, \bar{d}_4\} < 0$, $b_2 \geq 0$ if $\bar{c}_3 > 0$, $\min\{d, \bar{e}_2, \bar{e}_{3,2}, \bar{e}_{4,2}\} < 0$, $\bar{d}_4 \geq 0$ if $\max\{\bar{e}_2, \bar{e}_{3,2}, \bar{e}_{4,2}\} > 0$, $\bar{e}_5 \neq 0$, $\bar{d}_4 \neq 0$, $\bar{c}_3 > 0$, $b_2 < 0$. 

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5.3 Positivity Conditions for Quintic Polynomials

Summarizing the coefficients, we have

\[
\begin{align*}
    b_2 &= \frac{4a^2 - 10b}{25}, & c_2 &= \frac{3ab - 15c}{25}, & d_2 &= \frac{2ac - 20d}{25}, & e_2 &= \frac{ad - 25e}{25}, \\
    \bar{e}_3 &= -125c_4^4 + 100ae_2^3d_2 - 30a^2e_2^2d_2^2 + 4a^3e_2d_2^3 - 3dd_2^4, \\
    \bar{d}_3 &= -10cc_4^3 + 20ae_2^3d_2^3 - 25c_2^3d_2e_2 + 6a^2c_2^3d_2^2 - 5(5e_2 + 4ad_2)c_2^2d_2 + 25c_2d_2^3, \\
    \bar{e}_{3,1} &= -5c_4^4d + 6a^2c_2^3e_2 - 5(5e_2 + 4ad_2)c_2^2e_2 + 25c_2d_2^2e_2, \\
    \bar{e}_4 &= d_2d_3e_{3,1} - c_2\bar{e}_{3,1}^2 - \bar{e}_2d_3^2, \\
    \bar{e}_3 &= -3bb_2 + (5d_2 + 4ac_2)b_2 - 5c_2e_2, \\
    \bar{d}_{3,1} &= -2d_2b_2^2 + (5e_2 + 4ad_2)b_2 - 5c_2d_2, \\
    \bar{e}_{3,2} &= -d_2b_2^2 + 4ae_2b_2 - 5c_2e_2, \\
    \bar{e}_{4,1} &= b_2d_{3,1}\bar{e}_{3,2}^3 - c_2d_2^2\bar{e}_{3,2}^2 + d_2d_{3,1}\bar{e}_{3,2}^3 - e_2d_{3,1}^3, \\
    \bar{e}_4 &= -c_2d_2^2 + b_2c_3\bar{e}_{3,2} + c_2\bar{e}_3d_{3,1} - b_2d_{3,1}^2, \\
    \bar{e}_{4,2} &= -\bar{e}_2e_2 + c_2\bar{e}_3\bar{e}_{3,2} - b_2\bar{d}_{3,1}\bar{e}_{3,2}, \\
    \bar{e}_5 &= -\bar{c}_3\bar{e}_{4,2}^2 + \bar{d}_{3,1}\bar{e}_{4,2}\bar{d}_4 - \bar{e}_{3,2}\bar{d}_4^2.
\end{align*}
\]

We now can state our theorem.

**Theorem 5.5** Let \( g(t) \) be defined by (5.6), with \( e > 0 \). Let the additional coefficients be defined by (5.13-5.24). Then we have 11 cases.

1. \( b_2 = c_2 = d_2 = 0 \). In this case, \( g(t) > 0 \) for all \( t \geq 0 \).

2. \( b_2 = c_2 = 0, \ d_2 \neq 0 \) and \( \bar{e}_3 \neq 0 \). In this case, \( g(t) > 0 \) for all \( t \geq 0 \) if and only if either (i) \( \bar{e}_3 > 0, \ d_2 > 0, \ d \geq 0, \ e_2 \geq 0 \); or (ii) \( \bar{e}_3 < 0, \ d \geq 0 \) if \( e_2 > 0 \); or (iii) \( \bar{e}_3 > 0, \ d_2 < 0 \).

3. \( b_2 = 0, \ c_2 \neq 0, \ d_3 = 0 \) and \( \bar{e}_{3,1} \neq 0 \). In this case, \( g(t) > 0 \) for all \( t \geq 0 \) if and only if either (i) \( \bar{e}_{3,1} > 0, \ c_2 \geq 0, \ d \geq 0, \ e_2 \geq 0 \); or (ii) \( \bar{e}_{3,1} < 0, \ d \geq 0 \) if \( e_2 > 0 \); or (iii) \( \bar{e}_{3,1} > 0, \ c_2 < 0 \).

4. \( b_2 = 0, \ c_2 \neq 0, \ d_3 \neq 0 \) and \( \bar{e}_4 \neq 0 \). In this case, \( g(t) > 0 \) for all \( t \geq 0 \) if and only if either (i) \( \bar{e}_4 > 0, \ c_2 \geq 0, \ d \geq 0, \ e_2 \geq 0 \); or (ii) \( \bar{e}_4 < 0, \ c_2 \geq 0 \) if \( d_3 > 0, \ d \leq 0 \) if \( \max\{e_2,\bar{e}_{3,1}\} > 0, \ e_2 \geq 0 \) if \( \bar{e}_{3,1} > 0 \); or (iii) \( \bar{e}_4 > 0, \ \min\{c_2,\bar{d}_3\} < 0, \ \min\{d,\bar{e}_2,\bar{e}_{3,1}\} < 0, \ d \geq 0 \) if \( e_2 > 0 \); or (iv) \( \bar{e}_4 < 0, \ d_3 > 0, \ c_2 < 0 \).

5. \( b_2 \neq 0, \ c_2 = \bar{e}_{3,1} = 0, \) and \( \bar{e}_{3,2} \neq 0 \). In this case, \( g(t) > 0 \) for all \( t \geq 0 \) if and only if either (i) \( \bar{e}_{3,2} > 0, \ b_2 \geq 0, \ d \geq 0, \ e_2 \geq 0 \); or (ii) \( \bar{e}_{3,2} < 0, \ d \geq 0 \) if \( e_2 > 0 \); or (iii) \( \bar{e}_{3,2} > 0, \ b_2 < 0 \).
(6) $b_2 \neq 0$, $\bar{c}_3 = 0$, $\bar{d}_{3,1} \neq 0$ and $\bar{e}_{4,1} \neq 0$. In this case, $g(t) > 0$ for all $t \geq 0$ if and only if either (i) $\bar{e}_{4,1} > 0$, $b_2 \geq 0$, $\bar{d}_{3,1} \geq 0$, $d \geq 0$, $e_2 \geq 0$, $e_{3,2} \geq 0$; or (ii) $\bar{e}_{4,1} < 0$, $b_2 \geq 0$ if $\bar{d}_{3,1} > 0$, $d \geq 0$ if $\max\{e_2, e_{3,2}\} > 0$, $e_2 \geq 0$ if $e_{3,2} > 0$; or (iii) $\bar{e}_{4,1} > 0$, $\min\{b_2, \bar{d}_{3,1}\} < 0$, $\min\{d, e_2, \bar{e}_{3,2}\} < 0$, $d \geq 0$ if $e_2 > 0$; or (iv) $\bar{e}_{4,1} < 0$, $\bar{d}_{3,1} > 0$, $b_2 < 0$.

(7) $b_2 \neq 0$, $\bar{c}_3 \neq 0$ and $\bar{d}_4 = 0$. In this case, $g(t) > 0$ for all $t \geq 0$ if and only if either (i) $\bar{e}_{4,2} > 0$, $b_2 \geq 0$, $\bar{c}_3 \geq 0$, $d \geq 0$, $e_2 \geq 0$, $e_{3,2} \geq 0$; or (ii) $\bar{e}_{4,2} < 0$, $b_2 \geq 0$ if $\bar{c}_3 > 0$, $d \geq 0$ if $\max\{e_2, \bar{e}_{3,2}\} > 0$, $e_2 \geq 0$ if $\bar{e}_{3,2} > 0$; or (iii) $\bar{e}_{4,2} > 0$, $\min\{b_2, \bar{c}_3\} < 0$, $\min\{d, e_2, \bar{e}_{3,2}\} < 0$, $d \geq 0$ if $e_2 > 0$; or (iv) $\bar{e}_{4,2} < 0$, $\bar{c}_3 > 0$, $b_2 < 0$.

(8) $b_2 \neq 0$, $\bar{c}_3 \neq 0$, $\bar{d}_4 \neq 0$ and $\bar{e}_5 \neq 0$. In this case, $g(t) > 0$ for all $t \geq 0$ if and only if either (i) $\bar{e}_5 > 0$, $b_2 \geq 0$, $\bar{c}_3 \geq 0$, $d \geq 0$, $e_2 \geq 0$, $e_{3,2} \geq 0$; or (ii) $\bar{e}_5 < 0$, $\bar{d}_4 \neq 0$ and $\bar{e}_5 \neq 0$. Let $\alpha = -\frac{\bar{e}_5}{\bar{d}_4}$.

In these eight cases, $g(t) > 0$ for all $t \geq 0$ if and only if $g(t) \geq 0$ for all $t \geq 0$.

(9) (i) $b_2 = c_2 = 0$, $d_2 \neq 0$ and $\bar{e}_3 = 0$. Let $\alpha = -\frac{\bar{e}_3}{\bar{d}_2}$.

(ii) $b_2 = 0$, $c_2 \neq 0$, $\bar{d}_3 \neq 0$ and $\bar{e}_4 = 0$. Let $\alpha = -\frac{\bar{e}_{3,1}}{\bar{d}_3}$.

(iii) $b_2 \neq 0$, $\bar{c}_3 = 0$, $\bar{d}_{3,1} \neq 0$ and $\bar{e}_{4,1} = 0$. Let $\alpha = -\frac{\bar{e}_{3,2}}{\bar{d}_{3,1}}$.

(iv) $b_2 \neq 0$, $\bar{c}_3 \neq 0$, $\bar{d}_4 \neq 0$ and $\bar{e}_5 = 0$. Let $\alpha = -\frac{\bar{e}_{4,2}}{\bar{d}_4}$.

Then we may apply Proposition 5.1 and Theorem 3.2 to determine the situation of $g(t)$.

(10) (i) $b_2 = 0$, $c_2 \neq 0$ and $\bar{d}_3 = \bar{e}_{3,1} = 0$. Let $u = \frac{\bar{d}_2}{\bar{c}_2}$ and $v = \frac{\bar{e}_2}{\bar{c}_2}$.

(ii) $b_2 \neq 0$, $\bar{c}_3 \neq 0$ and $\bar{d}_4 = \bar{e}_{4,2} = 0$. Let $u = \frac{\bar{d}_{3,1}}{\bar{c}_3}$ and $v = \frac{\bar{e}_{3,2}}{\bar{c}_3}$.

Then we may use Propositions 5.2, 5.3 and Theorem 3.2 to determine the situation of $g(t)$.

(11) $b_2 \neq 0$ and $\bar{c}_3 = \bar{d}_{3,1} = \bar{e}_{3,2} = 0$. Let $p = \frac{\bar{c}_2}{b_2}$, $q = \frac{\bar{d}_2}{b_2}$ and $r = \frac{\bar{e}_2}{b_2}$. Then Proposition 5.4 determines the situation of $g(t)$.

Proof The 11 cases are summarized from the discussion in the last subsection. Cases (1-8) are corresponding to Cases (1), (2B), (3B), (3Cb), (4B), (4Cb), (4Db) and (4Dc2) of the last subsection, respectively. Case (9) is corresponding Cases (2A), (3Ca), (4Ca) and (4Dc1) of the last subsection. Case (10) is corresponding Cases (3A) and (4Da) of the last subsection. Case (11) is corresponding to Case (4A) of the last subsection. □
Higher Even Degree Polynomials and Appendices

We consider non-degenerate even degree polynomials with their degrees higher than or equal to 4. Suppose that
\[ g(t) = \sum_{i=1}^{m} a_i t^{m-i}, \]
where \( m \geq 4 \) is even, \( a_0 = a_m = 1, a_1, \ldots, a_{m-1} \) are real numbers. Denote
\[ S = \{(a_1, \ldots, a_{m-1}) \in \mathbb{R}^{m-1} : g(t) > 0 \text{ for all } t\}, \]
\[ \bar{S} = \{(a_1, \ldots, a_{m-1}) \in \mathbb{R}^{m-1} : g(t) \geq 0 \text{ for all } t\} \]
and
\[ A = \{(a_1, \ldots, a_{m-1}) \in \mathbb{R}^{m-1} : g(t) = (t^2 + ut + v)^2 \phi(t), u^2 < 4v, \phi(t) \geq 0 \text{ for all } t\}. \]
We call \( A \) the appendix of \( \bar{S} \). We have the following theorem.

**Theorem 6.1** In the above setting, \( \bar{S} \) is a closed convex set, \( S \) is its interior, and \( A \subset S \). The dimension of \( S \) and \( S \) is \( m-1 \), while the dimension of \( A \) is \( m-3 \). The discriminant \( \Delta(g) \) is equal to zero on \( A \) and the boundary of \( \bar{S} \), and has the same sign at the other part of \( \bar{S} \).

**Proof** Suppose that
\[ g(t) = \sum_{i=1}^{m} a_i t^{m-i}, \quad \hat{g}(t) = \sum_{i=1}^{m} \hat{a}_i t^{m-i}, \]
g, \( \hat{g} \geq 0 \) for all \( t \), \( a_0 = a_m = \hat{a}_0 = \hat{a}_m = 1 \). Then \( \frac{1}{2}(g(t) + \hat{g}(t)) \geq 0 \) for all \( t \). This shows that \( \bar{S} \) is convex. Taking limiting points, we see that \( \bar{S} \) is closed. Similarly, \( S \) is also convex. Consider \( \min\{g(t)\} \). By the continuity property, we see that \( S \) is an open set, and \( \bar{S} \) is the closure of \( S \). Then \( S \) is the interior of \( \bar{S} \). For any \( (a_1, \ldots, a_{m-1}) \in A \), \( g(t) > 0 \) for all \( t \). Thus, \( A \subset S \). We also see that there is an \( \epsilon > 0 \) such that \( (a_1, \ldots, a_{m-1}) \in S \) for \( |a_i| \leq \epsilon, i = 1, \ldots, m-1 \). This shows that the dimension of \( \bar{S} \) and \( S \) is \( m-1 \). Consider the number of independent parameters of \( g(t) \) in \( A \), we conclude that the dimension of \( A \) is \( m-3 \). For \( g(t) \) in \( A \), \( g(t) \) has multiple roots. Thus \( \Delta(g) = 0 \). On a boundary point of \( \bar{S} \), as it neighbors some parts not in \( \bar{S} \), we also have \( \Delta(g) = 0 \). On the other part of \( S \), as it is in the interior of \( \bar{S} \), and \( g(t) \) has no multiple roots there, we have \( \Delta(g) \neq 0 \). As \( A \) is of dimension \( m-3 \), the other parts is connected. Thus, \( \Delta(g) \) has the same sign there. \( \Box \)
This theorem shows that, in the above setting, for all \( g(t) \) without real roots, \( \Delta(g) \) takes the same sign, which depends upon \( m \) only, except at an appendix set of dimension lower by two, where \( \Delta(g) = 0 \).

Note that such an appendix also exists for odd degree polynomials with their degrees higher than 4. In Subsection 5.1, for the subcase (B1), if \( u^2 < 4v \), then there exists also an appendix similarly. However, as the property that \( g(t) \geq 0 \) for all \( t \) only works for even degree polynomials, only for even degree polynomials, the properties of the appendix are distinguished.

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References

[1] S. Basu, R. Pollack and R. Marie-François, Algorithm in Real Algebraic Geometry, 2nd ed., Springer, Berlin, 2006.

[2] H. Chen, Z. Huang and L. Qi, “Copositivity detection of tensors: Theory and algorithm”, J. Optim. Theory Appl. 174 (2017) 746-761.

[3] H. Chen, Z. Huang and L. Qi, “Copositive tensor detection and its application in physics and hypergraphs”, Comput. Optim. Appl. 69 (2018) 133-158.

[4] H. Chen and Y. Wang, “Higher order copositive tensors and its application”, J. Appl. Anal. Comput. 8 (2018) 1863-1885.

[5] F.S. Faro and I.P. Ivanov, “Boundedness from below in the \( U(1) \times U(1) \) three-Higgs-doublet model”, Phys. Rev. D 100 (2019) 035038.

[6] I.M. Gelfand, M.M. Kapranov and A.V. Zelevinsky, Discriminants, Resultants and Multidimensional Determinants, Birkhäuser, Boston, 1994.

[7] R.N. Gadem and C.C. Li, “On positive definiteness of quartic forms of two variables”, IEEE Trans. Automat. Control AC-9 (1964) 187-188.

[8] Y. Gao, “A necessary and sufficient condition for the positive definite problem of a binary quartic form”, arXiv:2009.01033, 2020.

[9] I.P. Ivanov, M. Köpke and M. Mühlleitner, “Boundedness from below in the \( U(1) \times U(1) \) three-Higgs-doublet model”, Phys. Rev. D 100 (2018) 035038.
[10] E.I. Jury and M. Mansour, “Positivity and nonnegativity of a quartic equation and related problems”, IEEE Trans. Automat. Control 26 (1981) 444-451.

[11] K. Kannike, “Vacuum stability of a general scalar potential of a few fields”, Eur. Phys. J. C 76 (2016) 324.

[12] K. Kannike, “Erratum to: Vacuum stability of a general scalar potential of a few fields”, Eur. Phys. J. C 78 (2018) 355.

[13] W.H. Ku, “Explicit criterion for the positive definiteness of a general quartic form”, IEEE Trans. Automat. Control AC-10 (1965) 372-373.

[14] L. Li, X. Zhang, Z. Huang and L. Qi, “Test of copositive tensors”, J. Indust. Manag. Optim. 15 (2019) 881-891.

[15] J. Liu and Y. Song, “Analytical expressions of copositivity for 3rd-order symmetric tensors and applications”, arXiv:1911.10284, (2019).

[16] J. Nie, Z. Yang, and X. Zhang, “A complete semi-definite algorithm for detecting copositive matrices and tensors”, SIAM J. Optim. 28 (2018) 2902-2921.

[17] L. Qi, “Symmetric nonnegative tensors and copositive tensors”, Linear Algebra Appl. 439 (2013) 228-238.

[18] L. Qi, H. Chen and Y. Chen, Tensor Eigenvalues and Their Applications, Springer, New York, 2018.

[19] L. Qi and Z. Luo, Tensor Analysis: Spectral Theory and Special Tensors, SIAM, Philadelphia, 2017.

[20] J.W. Schmidt and W. Heß, “Positivity of cubic polynomials on intervals and positive spline interpolation”, BIT Numer. Math. 28 (1988) 340-352.

[21] Y. Song and L. Qi, “Necessary and sufficient conditions of copositive tensors”, Linear Multilinear Algebra 63 (2015) 120-131.

[22] Y. Song and L. Qi, “Analytical expressions of copositivity for fourth-order symmetric tensors”, Anal. Appl. DOI:10.1142/S0219530520500049 (2020).

[23] G. Ulrich and L.T. Watson, “Positivity conditions for quartic polynomials”, SIAM J. Sci. Comput. 15 (1994) 528-544.

[24] F. Wang and L. Qi, “Comments on ‘Explicit criterion for the positive definiteness of a general quartic form’”, IEEE Trans. Automat. Control 50 (2005) 416-418.