Newton’s law

on an Einstein “Gauss-Bonnet” brane

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Abstract

It is known that Newton’s law of gravity holds asymptotically on a flat “brane” embedded in an anti-de Sitter “bulk”; this was shown not only when gravity in the bulk is described by Einstein’s theory but also in Einstein “Lanczos Lovelock Gauss-Bonnet”’s theory. We give here the expressions for the corrections to Newton’s potential in both theories, in analytic form and valid for all distances. We find that in Einstein’s theory the transition from the $1/r$ behaviour at small $r$ to the $1/r^2$ one at large $r$ is quite slow. In the Einstein Gauss-Bonnet case on the other hand, we find that the correction to Newton’s potential can be small for all $r$. Hence, Einstein Gauss-Bonnet equations in the bulk (rather than simply Einstein’s) induce on the brane a better approximation to Newton’s law.

I. INTRODUCTION

There has been of late an increasing interest for gravity theories formulated within spacetimes with one “large” extra dimension, that is, the idea that our universe may be a four dimensional singular hypersurface, or “brane”, embedded in a five dimensional spacetime, or “bulk”. The Randall-Sundrum scenario [1], where our universe is a four dimensional quasi-Minkowskian edge of a double-sided perturbed anti-de Sitter spacetime which satisfies the five dimensional Einstein equations, was the first explicit model where the linearized four dimensional Einstein equations were found to hold on the brane, at least far from the sources, Newton’s potential being corrected by a small $1/r^2$ term. This claim was thoroughly analyzed and the corrections to Newton’s law exactly calculated, either in the small $r$ limit where it was shown that they diverged in $1/r$ or in the large distance limit [2]. The analysis was then extended to the case when gravity in the bulk is described by the Einstein Gauss-Bonnet equations, and it was found that, in that theory as well, Newton’s law also held on the brane at large distance, [3, 4] with various results on the exact $1/r^2$ corrections, due to more or less careful treatments of “brane bending” and conflicting views on the boundary conditions to be imposed on the brane, see e.g. [5] for a review.

In this paper we describe gravity in the five dimensional bulk by means of the Einstein Gauss-Bonnet equations and use the junction conditions obtained in [6] (which are briefly reviewed in section 2). To obtain the geometry of an almost flat brane imbedded in a perturbed anti-de Sitter bulk and, hence, its gravitational field, we use the approach advocated in [7].

More precisely, we first construct in section 3 the most general linear perturbations of the bulk satisfying the Einstein Gauss-Bonnet equations; it turns out, as already known, [3, 4] that they are the same as in Einstein’s theory (the bulk background being locally anti-de Sitter). Then, in section 4, we use the junction conditions obtained in [6] and write the equations which govern gravity on the brane; we deduce from them a set of equations that can be compared to the usual four dimensional linearized Einstein equations; we see, as already known, [3, 4] that the two sets (after proper identification of Newton’s constant) are identical if only “zero modes” are present in the bulk (however only very special matter sources on the brane are compatible with zero modes only).

We then turn, in section 5, to the case when matter on the brane is a static, spherically symmetric, point source and give the linearized gravitational potentials in terms of well defined integrals which generalize the results obtained in [7] (equations (5.6-5.8) in text). We finally present our main results in sections 6 and 7: by approximating the
integrands by simpler functions we are able to perform the integrations and thus obtain the gravitational potentials in analytic form valid for all distances.

This allows us to see that, when Einstein’s theory governs the bulk, the “correction” to Newton’s potential on the brane goes from its large $1/r$ behaviour at small $r$ to its $1/r^2$ one at large $r$ quite slowly; in the Einstein Gauss-Bonnet case on the other hand, we find that the correction to Newton’s potential can be small for all $r$. We hence conclude that the Einstein “Lanczos Lovelock Gauss-Bonnet” equations in the bulk (rather than simply Einstein’s) induce on the brane a better approximation to Newton’s law and the usual linearized four-dimensional Einstein equations.

Throughout this paper, we denote double partial derivatives $\partial_\mu \partial_\nu$ by $\partial_{\mu\nu}$.

II. THE EINSTEIN “LANCEZ LOVELOCK GAUSS-BONNET” EQUATIONS AND THE ASSOCIATED JUNCTION CONDITIONS

To construct a “$Z_2$-symmetric braneworld” one can proceed as follows: consider a five dimensional Lorentzian manifold $V_+$ with a timelike edge, and superpose $V_+$ and its copy $V_-$ onto each other along the edge; one thus obtains a braneworld, composed of a $Z_2$-symmetric spacetime or “bulk” $V_5$, and a singular surface, or “brane” $\Sigma_4$ whose extrinsic curvature is discontinuous: the extrinsic curvature of $\Sigma_4$ embedded in $V_-$ is the opposite of the extrinsic curvature of $\Sigma_4$ embedded in $V_+$. Some components of the braneworld Riemann tensor therefore exhibit a delta-type singularity in $V_5 \cup \Sigma_4$. In a Gaussian normal coordinate system such that the equation defining the position of the brane is $y = 0$ the braneworld line element can be written as

$$ds^2 |_{y=0} = dy^2 + \bar{g}_{\mu\nu}(x^\rho, |y|)dx^\mu dx^\nu \quad (2.1)$$

where $y > 0$ spans $V_+$ and $y < 0$ spans $V_-$. As for the brane extrinsic curvatures in $V_+$ and $V_-$, they are given by

$$K_{\mu\nu} \equiv \bar{K}_{\mu\nu} = -K_{-\mu\nu} = -\frac{1}{2} \frac{\partial \bar{g}_{\mu\nu}}{\partial y} |_{y=0} \quad (2.2)$$

We shall associate to this braneworld the following action

$$S = \int_{V_5} d^5x \sqrt{-g} \bar{L}_{[2]} + 2\kappa \int_{\Sigma_4} d^4x \sqrt{-\bar{g}} L_m - 2 \int_{\Sigma_4} d^4x \sqrt{-\bar{g}} Q \quad (2.3)$$

$g$ is the determinant of the bulk metric coefficients $g_{AB}$, $\bar{g}$ that of the induced brane metric coefficients $\bar{g}_{\mu\nu}$. In the first term:

$$\bar{L}_{[2]} = -2\Lambda + s + \alpha L_{(2)} \quad \text{where} \quad L_{(2)} \equiv s^2 - 4r_{LM}^R R_{LMNP} R_{LMNP} \quad (2.4)$$

is the Einstein Lanczos Lagrangian (also called Einstein Gauss-Bonnet and generalized by Lovelock [8], see e.g. [9] for a review). Here $R^A_{BCD} = \partial_C \Gamma^A_{BD} - ...$ are the components of the Riemann tensor, $\Gamma^A_{BD}$ being the Christoffel symbols and all indices $A$ being moved with $g_{AB}$ and its inverse $g^{AB}$; $r_{BD} \equiv R^A_{BAD}$ are the Ricci tensor components, $s \equiv g^{BD} r_{BD}$ is the scalar curvature; $\Lambda$ is the bulk “cosmological constant” and $\alpha$ a (length)$^2$ parameter (that we shall take to be positive, see e.g. [9]). In the second term, $L_m(\bar{g}_{\mu\nu})$ is the braneworld “tension plus matter” Lagrangian and $\kappa$ a coupling constant which will be ultimately related to Newton’s. The third, boundary, term, which generalizes Gibbons-Hawking’s [10], was first obtained by Myers [11] and its explicit expression is (see [9])

$$Q = 2K + 4\alpha (J - 2\bar{\sigma}_{\mu\nu} K^\nu_{\mu}) \quad (2.5)$$

with $J$ the trace of

$$J^\nu_{\mu} = -\frac{2}{3} K^\mu_{\rho} K^\rho_{\nu} + \frac{2}{3} K K^\mu_{\nu} + \frac{1}{3} K^\nu_{\nu} (K.K - K^2) \quad (2.6)$$

$\bar{\sigma}_{\mu\nu} \equiv \bar{\rho}_{\mu\nu} - \frac{1}{2} \delta_{\mu\nu} \bar{s}$ is the intrinsic Einstein tensor of the brane $\Sigma_4$ and $K^\mu_{\nu}$ its extrinsic curvature [22], all indices $\mu$ being moved with $\bar{g}_{\mu\nu}$ and its inverse $\bar{g}^{\mu\nu}$.

Thanks to this boundary term the variation of $S$ with respect to the metric coefficients is given in terms of their variations only [11], as [3]:

$$\delta S = \int_{V_5} d^5x \sqrt{-g} \sigma_{AB}^{[2]} \delta g^{AB} + \int_{\Sigma_4} d^4x \sqrt{-\bar{g}} (2B_{\mu\nu} - \kappa T_{\mu\nu}) \delta \bar{g}_{\mu\nu} \quad (2.7)$$
The “braneworld equations of motion” are therefore simply taken to be \( \delta S = 0 \), with the metric fixed at the boundaries at infinity only. They are, first, the Einstein Gauss-Bonnet “bulk” equations (which are second order and quasi-linear in the metric coefficients, see e.g. [5] for a review):

\[
\sigma^A_{(2)B} \equiv \Lambda \delta^A_B + \sigma_B^A + \alpha \sigma^A_{(2)B} = 0, \tag{2.8}
\]

where \( \sigma_B^A \equiv r_B^A - \frac{1}{2} \delta_B^A s \) is the bulk Einstein tensor and

\[
\sigma^A_{(2)B} \equiv 2 \left[ R^{ALMN} R_{BLMN} - 2 r^{LM} R^A_{LBM} - 2 r^{AL} r_{BL} + sr^A_B \right] - \frac{1}{2} \delta^A_B L(2) \tag{2.9}
\]

is the Lanczos tensor. As for the brane equations, which, as shown in [6], generalize the Israel junction conditions [12] to Einstein Gauss-Bonnet gravity, they are:

\[
B^\mu_\nu \equiv K^\mu_\nu - K \delta^\mu_\nu + 4\alpha \left( \frac{3}{2} J^\mu_\nu - \frac{1}{2} J \delta^\mu_\nu - \bar{P}_{\mu\nu\sigma} K^{\rho\sigma} \right) = \frac{\kappa}{2} T^\mu_\nu \tag{2.10}
\]

where

\[
\bar{P}_{\mu\nu\sigma} = \bar{R}_{\mu\nu\sigma} + (\bar{r}_{\mu\sigma} \bar{g}_{\nu\rho} - \bar{r}_{\rho\sigma} \bar{g}_{\nu\mu} + \bar{r}_{\rho\nu} \bar{g}_{\mu\sigma} - \bar{r}_{\mu\nu} \bar{g}_{\rho\sigma})
\frac{-1}{2} (\bar{g}_{\rho\sigma} \bar{g}_{\nu\mu} - \bar{g}_{\rho\mu} \bar{g}_{\nu\sigma}) \tag{2.11}
\]

and where \( T_{\mu\nu} \) is defined by \( \delta(\sqrt{-\bar{g}}L_m) = -\frac{1}{2} \sqrt{-\bar{g}} T_{\mu\nu} \delta \bar{g}^{\mu\nu} \) and is interpreted as the stress-energy tensor of “tension plus matter” on the brane. By making a \((4+1)\) decomposition of \( \sigma^A_{(2)B} \) in the Gaussian normal coordinate system (2.1), one can see that

\[
\sigma^\mu_{(2)\nu} = -\nabla_\nu B^{\mu\nu} \tag{2.12}
\]

with \( \nabla_\nu \) the covariant derivative associated with the brane metric \( \bar{g}_{\mu\nu} \), so that the conservation of \( T_{\mu\nu} \),

\[
\nabla_\nu T^{\mu\nu} = 0, \tag{2.13}
\]

is included in the bulk equations (2.8).

If now the bulk is locally an anti-de Sitter (AdS\(_5\)) spacetime, then, because of maximal symmetry:

\[
R_{ABCD} = -\frac{1}{L^2} (g_{AC} g_{BD} - g_{AD} g_{BC}), \tag{2.14}
\]

with

\[
\frac{1}{L^2} = \frac{1}{4\alpha} \left( 1 \pm \sqrt{1 + \frac{4\alpha \Lambda}{3}} \right) \tag{2.15}
\]

in order to satisfy the bulk equations (2.8). (One usually chooses the lower sign so that \( \lim_{\alpha \to 0} L^2 = -\frac{6}{\Lambda} \), that is the Einsteinian value ; however, there is no reason at that level to reject the upper sign which allows, in particular, for an AdS\(_5\) bulk even when \( \Lambda = 0 \); as for the case \( L^2 = 4\alpha \) it has been analyzed in e.g. [13].) If, moreover, one wants the brane to be flat, a convenient coordinate system to describe the anti-de Sitter bulk is

\[
\bar{ds}^2 = dy^2 + e^{-2\frac{|y|}{L}} \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.16}
\]

in which the brane is located at \( y = 0 \) and has an extrinsic curvature given \( K_{\nu}^\mu = \frac{1}{L^2} \delta_{\nu}^\mu \), see [2.12]. The brane equations (2.10) then tell us that

\[
\kappa T_{\nu}^\mu = -\frac{6}{L^2} \delta_{\nu}^\mu \left( 1 - \frac{4\alpha}{3L^2} \right) \tag{2.17}
\]

which is interpreted as the “tension” required for keeping a brane flat in an anti de-Sitter bulk governed by the Einstein Gauss-Bonnet equations.
III. THE BULK PERTURBATIONS

In (quasi) conformally Minkowskian coordinates \( x^A = \{ x^\mu, x^i \}, x^4 = w \), the line element of our perturbed \( Z_2 \)-symmetric anti-de Sitter bulk can be taken to read

\[
ds^2|_5 = \left( \frac{\mathcal{L}}{w} \right)^2 (\eta_{AB} + \gamma_{AB}) dx^A dx^B \quad \text{with} \quad \gamma_{ww} = \gamma_{w\mu} = 0
\]

(3.1)

where \( \mathcal{L}(>0) \) is given by (3.15) and where \( \gamma_{\mu\nu} \) are ten functions of the five coordinates \( (x^\mu, w) \). Since the (unperturbed) brane is located at \( w = 0 \), the coefficients \( \gamma_{\mu\nu} \) will be at first order functions of \( |w| \) only.

If we now impose this metric to satisfy the bulk Einstein Gauss-Bonnet equations (2.8), it is an easy exercise to see, first, that the conditions (all indices being from now on raised with \( \eta^{\mu\nu} \))

\[
\gamma_\rho = 0 \quad , \quad \partial_\rho \gamma_\mu = 0
\]

(3.2)

solve the constraints \( \sigma^w_{[2]w} = 0, \sigma^\mu_{[2]w} = 0 \) at linear order and reduce the ten metric coefficients \( \gamma_{\mu\nu} \) to five, which represent the five degrees of freedom of AdS5 gravitational waves in Einstein Gauss-Bonnet theory (this being so because the Lanczos-Lovelock Lagrangian yields second order field equations, just like Einstein-Hilbert’s).

As for their evolution equation, it then reads (with \( \square \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \))

\[
\left( 1 - \frac{4\alpha}{\mathcal{L}^2} \right) \left( \square \gamma_{\mu\nu} + \partial_{w\mu} \gamma_{w\nu} - \frac{3}{w} \partial_w \gamma_{\mu\nu} \right) = 0 .
\]

(3.3)

The presence of the overall coefficient \( (1 - 4\alpha/\mathcal{L}^2) = \mp \sqrt{1 + \alpha \Lambda/3} \) is a consequence of the quasi-linearity of the Lanczos-Lovelock equations (2.8), see e.g. [5]. The case when the parameters \( \alpha \) and \( \Lambda \) are such that it vanishes (studied in e.g. [13]) will not be considered here. Therefore, the evolution equation for the AdS5 gravitational waves in Einstein and Einstein Gauss-Bonnet theory are the same [3, 4].

As emphasized in [7], the general solution of (3.3) must converge (more precisely be \( L_2 \)) on its domain of definition. Now, the bulk is composed of \( V_+ \) and its copy \( V_- \) joined by the brane \( \Sigma_4 \) and its \( Z_2 \)-symmetric metric (3.1) depends only on \( |w| \). The general solution of (3.3) is therefore the union of two sets. First, the set :

\[
\gamma_{\mu\nu}(x^\mu, w) = \text{Re} \int \frac{d^3k}{(2\pi)^3} \frac{dm}{(2\pi)^2} e^{ik\cdot x^\mu \rho} e_{\mu\nu} w^2 Z_2(m|w|)
\]

with \( k_0 = -\sqrt{k^2 + m^2} \), \( k^\rho e_{\rho\mu} = 0 \), \( \eta^\rho\sigma e_{\rho\sigma} = 0 \)

(3.4)

where \( k^2 \equiv k_i k^i \), where the five polarisations \( e_{\mu\nu} \) are a priori arbitrary functions of \( k^i \) and \( m \) and where \( Z_2(mw) = H_2^{(1)}(mw) + a_m H_2^{(2)}(mw) \) is an a priori arbitrary linear combination of second order Hankel functions of first and second kind [13]. The coefficient \( a_m \) is determined by the model at hand and one usually eliminates the mode coming from \( +\infty \), that is sets \( a_m = 0 \). The “zero modes” do not depend on \( w \) and are :

\[
\gamma^{(0)}_{\mu\nu}(x^\mu) = \text{Re} \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x^\mu \rho} e_{\mu\nu}(k^i)
\]

with \( ke_{0\mu} + k^\mu e_{\mu\nu} = 0 \), \( \eta^\rho\sigma e_{\rho\sigma} = 0 \)

(3.5)

The second set is composed of the extra \( L_2 \) modes which converge exponentially as \( |w| \rightarrow +\infty \):

\[
\gamma_{\mu\nu}(x^\mu, w) = \text{Re} \int \frac{d^3k}{(2\pi)^3} \int_0^1 \frac{dA}{(2\pi)^2} e^{ik\cdot x^\mu} w^2 H_2^{(1)}(\sqrt{A}|w|)
\]

with \( k_0 = -\sqrt{k^2 + A} \), \( k^\rho e_{\rho\mu} = 0 \), \( \eta^\rho\sigma e_{\rho\sigma} = 0 \).

(3.6)

The static modes do not depend on time and are :

\[
\gamma^{(s)}_{\mu\nu}(x^\mu, w) = \text{Re} \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot x^\mu} e_{\mu\nu}(k^i) w^2 H_2^{(1)}(ik|w|).
\]

(3.7)
IV. THE LINEARIZED EQUATIONS FOR GRAVITY ON AN EINSTEIN GAUSS-BONNET BRANE

The position of the brane $\Sigma_4$ in the bulk $V_5$ is defined in the coordinate system (3.1) by

$$w = L + \zeta(x^\mu)$$ (4.1)

where the function $\zeta(x^\mu)$ is a priori arbitrary and describes the so-called “brane-bending” effect. The induced metric on $\Sigma_4$ is, at linear order

$$ds^2|_4 = (\eta_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu$$ with $$h_{\mu\nu} = \gamma_{\mu\nu}|_{\Sigma} - 2\frac{\zeta}{L} \eta_{\mu\nu}$$ (4.2)

where $\gamma_{\mu\nu}(x^\mu, w)$ is a solution of (3.2-3.3) and where the index $\Sigma$ means that the quantity is evaluated at $w = L$.

As for its extrinsic curvature (defined by (2.2) in a Gaussian normal coordinate system) it is given, in the system (3.1-3.2), by

$$K^\mu_\nu = \frac{1}{L} \delta^\mu_\nu - \frac{1}{2} \partial_\mu \gamma^\mu_\nu|_{\Sigma} + \partial_\nu \zeta .$$ (4.3)

Now, as we have seen in section 2, the “brane equations” are (2.10). In the case at hand $K^\mu_\nu$ is given by (4.3) and $P^\mu_{\nu\rho\sigma}$ is to be computed by means of the metric (4.2). Splitting $T^\mu_\nu$ into a brane tension and matter stress-energy tensor $S^\mu_\nu$ as

$$\kappa T^\mu_\nu = \frac{1}{L} \delta^\mu_\nu - \frac{1}{2} \partial_\mu \gamma^\mu_\nu|_{\Sigma} + \partial_\nu \zeta$$ (4.4)

where, from now on we shall use the notation

$$\bar{\alpha} \equiv \frac{4\alpha}{L^2} .$$ (4.5)

the brane equations (2.10) read, at linear order

$$\kappa S^\mu_\nu = 2 \left( 1 + \bar{\alpha} \right) (\partial^\mu \zeta - \frac{1}{2} \partial^\nu \gamma^\mu_\nu - \partial_\xi \gamma^\mu_\nu|_{\Sigma} - \bar{\alpha} L \partial_\xi \gamma^\mu_\nu|_{\Sigma} .$$ (4.6)

Equations (4.4) and (4.6) together with (3.4) and (3.7) completely describe gravity on the brane. They have two useful consequences

$$\partial_\nu S^\nu_\rho = 0 \ , \ \kappa S = -6 \left( 1 + \bar{\alpha} \right) \Box \zeta .$$ (4.7)

Let us compare them to the usual four dimensional linearized Einstein equations. In order to do so, we first perform a coordinate transformation $x^\mu \rightarrow x'^\mu = x^\mu + \epsilon^\mu$ such that the new metric coefficients

$$h'^\mu_\nu = \gamma_{\mu\nu}|_{\Sigma} - 2\frac{\zeta}{L} \eta_{\mu\nu} + \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu$$ with $$\Box \epsilon_\mu = -\frac{2}{L} \partial_\mu \zeta$$ (4.8)

satisfy the harmonicity condition

$$\partial_\mu \left( h'^\mu_\rho - \frac{1}{2} \delta^\mu_\rho h'^\rho \right) = 0 .$$ (4.9)

Taking the d’Alembertian of (18), eliminating $\zeta$ by means of (19) and using (22) and (23) we get the following consequence of the brane equations

$$\Box h'^\mu_\nu = -\left( \frac{2}{1 + \bar{\alpha}} \right) \frac{\kappa}{L} \left( S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S \right)$$

$$+ \left( \frac{1 - \bar{\alpha}}{1 + \bar{\alpha}} \right) \left[ -\left( \partial_\xi \gamma_{\mu\nu} \right)|_{\Sigma} + \frac{1}{L} \left( \partial_\xi \gamma_{\mu\nu} \right)|_{\Sigma} \right] .$$ (4.10)

Now, recall that the usual linearized Einstein equations on a four dimensional Minkowski background read, in harmonic coordinates

$$\Box h^\mu_\nu = -16\pi G \left( S_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} S \right)$$ (4.11)
where $G$ is Newton’s constant. One therefore sees that, if we identify, in agreement with [3]
\[8\pi G \equiv \left(\frac{1}{1+\alpha}\right) \frac{\kappa}{L},\] (4.12)
and if only zero modes are present in the bulk, then (4.10) reduces to (4.11).

There is however an important proviso. The usual Einstein equations (4.11) hold for any type of matter (compatible with the harmonicity condition, or, equivalently with the Bianchi identity $\partial_{\rho}S_{\mu\nu}^{\rho} = 0$). By contrast, the linearized equations for gravity on a brane are (4.10), provided the source $S_{\mu\nu}$ satisfies the junction condition (4.12). If only zero modes are present in the bulk, then, since the last two terms in (4.8) are absent because of (4.3), the derivatives $\partial_{\alpha}(S_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}S)$ must be symmetric in $\lambda$ and $\mu$, a property which is satisfied only by very special matter, obeying a very contrived equation of state (see e.g. [7] for details). There is therefore no compelling reason, at that level, to make the identification (4.12), that we shall consider until further notice as a mere notation.

Now, of course, the junction conditions are better seen as boundary conditions on the allowed modes in the bulk. Indeed, if matter on the brane is known, then (4.6-4.7) can be inverted to give $\zeta$ and the last two terms in (4.6) in terms of $S_{\mu\nu}$. With that information one can in principle get, by inversion of (3.4) and (3.7), the polarisations $\epsilon_{\mu\nu}$, and hence the allowed bulk perturbations, in terms of the brane matter source. Then the induced metric on the brane (4.2) is known in terms of the matter variables and can be compared with the usual four dimensional Einstein result. This programme is completed below in the particular case of a point static source.

V. THE GRAVITATIONAL POTENTIALS OF A STATIC POINT SOURCE ON AN EINSTEIN GAUSS-BONNET BRANE

Consider a static, spherically symmetric, point-like source on the brane. Its stress-energy tensor is, (with $r = \{x^i\}$, \(r^2 = x_i x^i\)):
\[S_{00} = M\delta(r), \quad S_{0i} = 0.\] (5.1)

When necessary, we shall go to Fourier space:
\[f \leftrightarrow \hat{f} \quad \text{with} \quad f = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \hat{f}, \]
\[\hat{\delta} = \frac{1}{(2\pi)^{3/2}}, \quad \frac{1}{r} = \sqrt{\frac{2}{\pi} \frac{1}{k^2}} \quad \text{and} \quad \partial_r \frac{1}{r} = -\sqrt{\frac{2}{\pi}} \frac{k_i k_j}{k^2}.\]

The junction conditions (4.6-4.7) then give:
\[\zeta = -\frac{1}{24\pi} \frac{\kappa M}{(1+\alpha) r},\]
\[(1-\alpha) \partial_{\alpha} \gamma^{(s)}_{ij} |_\Sigma + \alpha \mathcal{L} \gamma^{(s)}_{ij} |_\Sigma = -\frac{\kappa M}{3} \delta(\vec{r}) \delta_{ij} - \frac{\kappa M}{12\pi} \partial_r \frac{1}{r},\]
\[(1-\alpha) \partial_{\alpha} \gamma^{(s)}_{00} |_\Sigma + \alpha \mathcal{L} \gamma^{(s)}_{00} |_\Sigma = -\frac{2\kappa M}{3} \delta(\vec{r}),\]
\[(1-\alpha) \partial_{\alpha} \gamma^{(s)}_{0i} |_\Sigma + \alpha \mathcal{L} \gamma^{(s)}_{0i} |_\Sigma = 0.\] (5.2)

Now, the static bulk modes are given by (3.7) : $\gamma^{(s)}_{\mu\nu}(k^i, w) = \epsilon_{\mu\nu}(k^i) w^2 H_2^{(1)} (ik | w |)$ and their $w$-derivatives on $\Sigma$ are (4.7) : $\partial_w \gamma^{(s)}_{\mu\nu} |_\Sigma = \epsilon_{\mu\nu}(k^i) ik \mathcal{L} H_1^{(1)} (ik \mathcal{L})$. Equation (5.2) therefore gives the polarisations in terms of the brane stress-energy tensor as
\[e_{00}(k^i) \left[ (1-\alpha) H_1^{(1)} (ik \mathcal{L}) + i \alpha k \mathcal{L} H_2^{(1)} (ik \mathcal{L}) \right] = -\frac{2}{3} \frac{\kappa M}{(2\pi)^{3/2}} \frac{1}{ik \mathcal{L}^2}, \quad e_{00}(k^i) = 0\]
\[e_{ij}(k^i) \left[ (1-\alpha) H_1^{(1)} (ik \mathcal{L}) + i \alpha k \mathcal{L} H_2^{(1)} (ik \mathcal{L}) \right] = -\frac{1}{3} \frac{\kappa M}{(2\pi)^{3/2}} \frac{1}{ik \mathcal{L}^2} \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right).\] (5.3)

The bulk metric is then known and is:
\[\gamma_{\mu\nu}(x^i, w) = \Re \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \gamma_{\mu\nu}(k^i, w)\]
with \[\gamma_{\mu\nu}(k^i, w) = \frac{\kappa M}{3\mathcal{L}(2\pi)^{3/2}} \frac{w^2 K_2(k | w |)}{k \mathcal{L} [(1-\alpha) K_1(k \mathcal{L}) + \alpha k \mathcal{L} K_2(k \mathcal{L})]} \epsilon_{\mu\nu}.\] (5.4)
where \( \alpha_0 = 2, \alpha_0 = 0 \) and \( \alpha_i = \delta_{ij} - k_i k_j / k^2 \) and where \( K_\nu(z) \) is the modified Bessel function defined as \( K_\nu(z) = \frac{1}{2} e^{\nu \pi i} H^{(1)}_{\nu}(iz) \).

The \( \hat{h}_{\alpha 0} \) Fourier component of the metric on the brane therefore reads

\[
\hat{h}_{\alpha 0}(\vec{k}) = \gamma_{\alpha 0} \Sigma + 2 \frac{\hat{\alpha}}{\mathcal{L}} \frac{\kappa M}{k^2 L (2\pi)^2 (1 + \alpha)} \left[ 1 + \frac{2}{3} (1 - \alpha) kL K_0(kL) + \alpha kL K_0(kL) \right]
\]

(5.5)

and a similar expression for \( \hat{h}_{ij} \).

We then take the Fourier transform and integrate over angles (after elimination of the \( k_i k_j \) term in \( \hat{h}_{ij} \) by a suitable coordinate transformation). We then go to Schwarzschild coordinates by means of another infinitesimal coordinate transformation. Setting \( x = r/L \) and recalling that we introduced the notation \( (\frac{1}{1 + \alpha})^\frac{2}{3} \equiv 8\pi G \), we finally obtain the linearized metric on the brane, as created by a static point-like source, as:

\[
ds^2|_s = -(1 + 2U)dt^2 + (1 - 2V)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]

(5.6)

with

\[
U = -\frac{GM}{r} \left[ 1 + \frac{4}{3\pi} U_\alpha(x) \right], \quad V = -\frac{GM}{r} \left[ 1 + \frac{2}{3\pi} \bar{V}_\alpha(x) \right]
\]

(5.7)

and

\[
U_\alpha(x) = (1 - \alpha) \int_0^{+\infty} du \sin(ux) \frac{K_0(u)}{(1 + \alpha)K_1(u) + \alpha u K_0(u)}, \quad V_\alpha(x) = U_\alpha(x) - x \partial_x U_\alpha(x).
\]

(5.8)

\( U_\alpha(x) \) is a well defined integral that we shall evaluate analytically.

**VI. THE CORRECTIONS TO NEWTON’S LAW ON AN EINSTEIN BRANE FOR ALL DISTANCES**

When the bulk obeys the five dimensional Einstein equations, that is when \( \alpha_0 = 0, U_\alpha(x) \) in (5.8) reduces to

\[
U_0(x) = \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} du \sin(ux) U_0(u) e^{-\epsilon u} \quad \text{with} \quad U_0(u) = \frac{K_0(u)}{K_1(u)}.
\]

(6.1)

By approximating \( U_0(u) \) by its asymptotic expressions, that is \( U_0(u) \sim 1 \) when \( u \rightarrow \infty \) and \( U_0(u) \sim -u \ln u \) when \( u \rightarrow 0 \), it is a straightforward exercise \( \Box \) to see that \( \lim_{x \rightarrow 0} U_0(x) = x^{-1} \) and that \( \lim_{x \rightarrow \infty} U_0(x) = \pi/2x^2 \). Hence we recover the well-known result \( \Box \) (see also \( \Box \)) that, at short distances the “correction” to the gravitational potentials diverges in \( \mathcal{L}/r \)

\[
\lim_{r/\mathcal{L} \rightarrow 0} U = \lim_{r/\mathcal{L} \rightarrow 0} V = -\frac{4GM\mathcal{L}}{3\pi r^2}
\]

(6.2)

whereas at distances large compared with the characteristic scale \( \mathcal{L} \) of the anti-de Sitter bulk the correction is reduced by another \( \mathcal{L}/r \) factor \( \Box \), so that \( \Box \)

\[
\lim_{r/\mathcal{L} \rightarrow \infty} U = -\frac{GM}{r} \left[ 1 + \frac{2}{3} \left( \frac{\mathcal{L}}{r} \right)^2 \right], \quad \lim_{r/\mathcal{L} \rightarrow \infty} V = -\frac{GM}{r} \left[ 1 + \left( \frac{\mathcal{L}}{r} \right)^2 \right].
\]

(6.3)

However one can do better than just obtaining the asymptotic behaviours of the potentials. Indeed it turns out, see Fig. 1, that we can use for all \( u \), to a good approximation

\[
U_0(u) \approx \hat{U}_0(u) \quad \text{with} \quad \hat{U}_0(u) \equiv u \ln \left( 1 + \frac{1}{u} \right).
\]

(6.4)

Using that approximation, \( \Box \) can be integrated for all \( x \) to yield:

\[
U_0(x) \approx \frac{x \cos x - \sin x}{x^2} \text{ci}(x) + \frac{\cos x + x \sin x}{x^2} \text{si}(x) + \frac{\pi}{2x^2}
\]

(6.5)
Fig. 1. The behaviour of $U_0(u)$ and the accuracy of the approximate analytic formula $\bar{U}_0(u)$. The solid curve is $U_0(u)$ while the upper dotted line shows $\bar{U}_0/U_0$. One sees that the relative error is small: $\lesssim 2\%$ at most.

Fig. 2. A logarithmic plot of the function $U_0(x)$. The short-dashed curve is $\pi/(2x^2)$ and the long-dashed curve is $1/x$.

and

$$V_0(x) \approx \frac{3x \cos x + (x^2 - 3) \sin x}{x^2} \text{ci}(x) + \frac{3x \sin x - (x^2 - 3) \cos x}{x^2} \text{si}(x) + \frac{3\pi}{2x^2} - \frac{1}{x}, \quad (6.6)$$

where $\text{si}(x)$ and $\text{ci}(x)$ are the integral sine and cosine functions, see [14].

In Fig. 2, $U_0(x)$ is plotted as a function of $x = r/\mathcal{L}$. As one can see the transition from the $1/r$ behaviour at short distances to the $1/r^2$ one at large distances is quite slow. This is due to the slow convergence of $U_0(u)$ when $u \to 0$. As seen from the upper-most lines of Fig. 4 below, the Newtonian behaviour is recovered at $r/\mathcal{L} \gtrsim 10$. 
VII. THE CORRECTIONS TO NEWTON’S LAW ON AN EINSTEIN GAUSS-BONNET BRANE

When the bulk obeys the five dimensional Einstein Gauss-Bonnet equations, \( \mathcal{U}_\alpha(x) \) in (5.8) is

\[
\mathcal{U}_\alpha(x) = (1 - \bar{\alpha}) \int_0^{+\infty} du \sin(ux) U_\alpha(u)
\]

with \( U_\alpha(u) = \frac{K_0(u)}{\bar{\alpha}u K_0(u) + (1 + \bar{\alpha})K_1(u)} \). (7.1)

Again, by approximating \( U_\alpha(u) \) by its asymptotic expressions, that is

\[
U_\alpha(u) \sim \frac{1}{\bar{\alpha}u}
\]

when \( u \to \infty \), and \( U_\alpha(u) \sim -\frac{1}{1 + \bar{\alpha}}u \ln u \) when \( u \to 0 \), it is a simple exercise [14] to see that

\[
\lim_{x \to 0} U_\alpha(x) = \frac{\pi}{2} \frac{1 - \bar{\alpha}}{\alpha} \quad \text{and} \quad \lim_{x \to \infty} U_\alpha(x) = \frac{\pi}{2x^2} \frac{1 - \bar{\alpha}}{1 + \bar{\alpha}}.
\]

Hence we recover the known result [3, 4] that at distances large compared with the characteristic scale \( \mathcal{L} \) of the anti-de Sitter bulk the gravitational potentials are corrected by a small \( 1/r^2 \) factor, their precise expressions being

\[
\begin{align*}
\lim_{r/\mathcal{L} \to \infty} U &= -\frac{GM}{r} \left[ 1 + \frac{2}{3} \left( 1 - \frac{\bar{\alpha}}{\alpha} \right) \left( \frac{\mathcal{L}}{r} \right)^2 \right], \\
\lim_{r/\mathcal{L} \to \infty} V &= -\frac{GM}{r} \left[ 1 + \frac{1}{3} \left( 1 - \frac{\bar{\alpha}}{\alpha} \right) \left( \frac{\mathcal{L}}{r} \right)^2 \right].
\end{align*}
\]

(7.2)

When \( \bar{\alpha} = 0 \), (7.2) reduces to its “Einsteinian” expression (6.3). On the other hand the corrections remain finite when \( r/\mathcal{L} \to 0 \) so that the gravitational potentials keep their Newtonian behaviour :

\[
\begin{align*}
\lim_{r/\mathcal{L} \to 0} U &= -\frac{GM}{r} \left[ 1 + \frac{2}{3} \left( 1 - \frac{\bar{\alpha}}{\alpha} \right) \right], \\
\lim_{r/\mathcal{L} \to 0} V &= -\frac{GM}{r} \left[ 1 + \frac{1}{3} \left( 1 - \frac{\bar{\alpha}}{\alpha} \right) \right].
\end{align*}
\]

(7.3)

contrarily to what happens in the Einstein case, see equation (6.2).

Now, here again, one can do better by approximating \( U_\alpha(u) \) for all \( u \). For example, for all \( 0 < \bar{\alpha} < 1 \) (corresponding to the lower sign in [2.15]), a good approximation (to better than 1%) is

\[
U_\alpha(u) \approx \tilde{U}_\alpha(u) \quad \text{with} \quad \tilde{U}_\alpha(u) = \frac{1}{1 + \bar{\alpha}} \left( \frac{u}{1 + u} \right) \ln \left( 1 + \beta \frac{u}{\bar{\alpha} u} \right)
\]

(7.4)

and \( \beta = 1 - \sqrt{\frac{1 + \bar{\alpha}}{1 + \alpha}} \). Unfortunately we were not able to integrate [14] analytically with that approximation. We hence used another approximate form for \( U_\alpha(u) \), valid for all \( \bar{\alpha} \neq 0 \):

\[
U_\alpha(u) \approx \tilde{U}_\alpha(u) \quad \text{with} \quad \tilde{U}_\alpha(u) = \left( \frac{1}{1 + \bar{\alpha}} \right) \left( \frac{u}{1 + u} \right) \ln \left( 1 + \frac{\beta}{u} \right) - \left( \frac{\beta u}{u + \beta} \right)
\]

(7.5)

where \( \beta \) and \( \gamma \) are constants such that

\[
\beta = \frac{(\gamma^2 - 1/3)}{(\gamma - 1/2)^2}, \quad \gamma^2 - 1/3 = \frac{1 + \bar{\alpha}}{\alpha} \left( \gamma - \frac{1}{2} \right)^3 = 0
\]

(7.6)

and

\[
\beta^2 = \frac{1 + \bar{\alpha}}{\alpha} \left( \gamma - \frac{1}{2} \right).
\]

The relative error is found to be very small in both ranges of \( u \gg 1 \) and \( u \ll 1 \) for any \( \bar{\alpha} \). At mildly small values of \( u \), the error becomes as large as 25%, but only for very large \( \bar{\alpha} \) (see Fig. 3 below). Using (7.4), (7.5) becomes

\[
\mathcal{U}_\alpha(x) \approx \left( \frac{1 - \bar{\alpha}}{1 + \bar{\alpha}} \right) \left[ \frac{\beta x \cos(\beta x) - \sin(\beta x)}{x^2} \text{ci}(\beta x) + \frac{\beta x \sin(\beta x) + \cos(\beta x)}{x^2} \text{si}(\beta x) \right]
\]

(7.7)
(1 + \bar{\alpha}) U_{\bar{\alpha}}(u)

\begin{align*}
(1 + \bar{\alpha}) U_{\bar{\alpha}}(u)
\end{align*}

\begin{align*}
\text{Fig. 3. The behaviour of } U_{\bar{\alpha}}(u) \text{ for several values of } \bar{\alpha} \text{ and the accuracy of the approximate } \\
\text{formula } \tilde{U}_{\bar{\alpha}}(u). \text{ The solid lines are } (1 + \bar{\alpha}) U_{\bar{\alpha}}(u) \text{ for the cases } \bar{\alpha} = 0.01, 0.3, 0.6, 0.99 \text{ and } \\
\infty, \text{ from top to bottom. The dotted lines are the ratio } \tilde{U}_{\bar{\alpha}}/U_{\bar{\alpha}} \text{ for the same values of } \bar{\alpha}, \\
\text{also from top to bottom. The upper two dotted lines are almost degenerate in the figure.}
\end{align*}

\begin{align*}
&+ \frac{\pi}{2x^2} - \frac{\beta}{x} + \beta^2 \gamma \left[ \text{ci}(\beta \gamma x) \sin(\beta \gamma x) - \cos(\beta \gamma x) \sin(\beta \gamma x) \right], \\
&V_{\bar{\alpha}}(x) \approx \frac{1 - \bar{\alpha}}{1 + \alpha} \left\{ 3 \beta x \cos(\beta x) + \frac{(x^2 \beta^2 - 3)}{x^2} \sin(\beta x) \right. \\
&\left. \quad + \frac{3 \beta x \sin(\beta x)}{x^2} - \left( \frac{\beta^2 x^2 - 3}{x} \right) \cos(\beta x) \right\} \sin(\beta x) + \frac{3 \pi}{2x^2} - \frac{3 \beta}{x} \\
&+ \beta^2 \gamma \left[ \text{ci}(\beta \gamma x) \left( \sin(\beta \gamma x) - \beta \gamma x \cos(\beta \gamma x) \right) \right. \\
&\left. - \text{si}(\beta \gamma x) \left( \cos(\beta \gamma x) + \beta \gamma x \sin(\beta \gamma x) \right) \right].
\end{align*}

On the graphs below, see Fig. 4, we plotted $U(r)$ and $V(r)$ (as a function of $x = r/L$, and divided by the Newtonian potential $U_N \equiv -GM/r$). As one can see the gravitational potentials become closer and closer to their Newtonian values, for all distances, when $\bar{\alpha} \to 1$.

We therefore conclude that the Einstein “Lanczos Lovelock Gauss-Bonnet” equations in the bulk (rather than simply Einstein’s) induce on the brane a better approximation to Newton’s law and the usual linearized four-dimensional Einstein equations. This is all the more so as the coupling parameter $\bar{\alpha}$ tends to 1.

Let us be more precise: at short distances the potential $U(r)$ remains in $1/r$ when $\bar{\alpha} \neq 0$ and hence $L$ needs no longer be smaller than, say 200 $\mu$m, the upper limit that present experiments impose in the $\bar{\alpha} = 0$ case for which $U(r) \propto 1/r^2$.

As for the coupling constant $G_e = \frac{2 + \bar{\alpha}}{3\alpha} G$ which appears in (7.3) we can identify it to Newton’s constant, as measured in terrestrial laboratories (and known at present with a 0.15% accuracy, see e.g. [17]).

If now, $L$ is taken to be of geophysical size (say $\sim 1$ km $-100$ kms) then the transition from the short to large distance regimes does not contradict present experiments, whatever the value of $\bar{\alpha}$ (not too close to zero), as it occurs “safely” in a region when gravity is poorly known.

At large distances ($r/L \gg 1$) the potentials $U(r) \sim V(r)$ are also in $1/r$, with however a coupling constant $G$ which differs from $G_e$ (identified with Newton’s constant as measured in the lab). Now, there is no astro-nomical evidence that $G_e$ should be equal to $G$, as only the product $GM$ comes into play. There is on the other hand some astro-physical evidence that they should not be too different, otherwise solar models would go astray. If we tolerate that $G$ and $G_e$ differ by, say, 10% then $0.85 < \bar{\alpha} < 1.15$.

To summarize: if $L$ is of geophysical size, and $0.85 < \bar{\alpha} < 1.15$ (but $\neq 1$) then linearized gravity on an Einstein Gauss-Bonnet brane is compatible with Newtonian gravity and the light deflection experiments (which test the equality
Figs. 4. The relative corrections to $U$ and $V$ as functions of $x$, in the upper and lower panels, respectively. In both figures, the curves from top to bottom correspond to the cases $\bar{\alpha} = 0, 0.3, 0.6, 0.9$ and $\infty$.

We thus see that bulk Einstein Gauss-Bonnet gravity can relax the constraint on the value of $\mathcal{L}$ (that is, the characteristic size of the bulk) at a fairly reasonable price: $\bar{\alpha}$ must be fine-tuned to be not too far from its critical value $(1 - \bar{\alpha}) \equiv (1 - 4\alpha/\mathcal{L}^2) = \sqrt{1 + \alpha\Lambda/3} = 0$.

Now, to see if such parameter ranges can indeed be allowed, it will be necessary to analyze carefully their implications in the cosmological context as well [18]. Investigations in this direction, however, are beyond the scope of the present paper.

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