Non-extremal Localised Branes and Vacuum Solutions in M-Theory

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Abstract

Non-extremal overlapping p-brane supergravity solutions localised in their relative transverse coordinates are constructed. The construction uses an algebraic method of solving the bosonic equations of motion. It is shown that these non-extremal solutions can be obtained from the extremal solutions by means of the superposition of two deformation functions defined by vacuum solutions of M-theory. Vacuum solutions of M-theory including irrational powers of harmonic functions are discussed.
1 Introduction

String theories and M-theory admit various solitonic extended-object solutions [1]-[6]. The description of the p-brane solutions of supergravity has been exploited to elucidate many nonperturbative aspects of string theories as well as of gauge theories, see for example [3]-[11]. These extended solutions include intersecting extremal p-branes as well as non-extremal or “black” p-branes [1]-[7], [12]-[15]. A brane solution whose harmonic function is independent of a number of transverse coordinates (relative transverse or overall transverse) is said to be delocalised over those directions. It was found that many delocalised solutions have non-extremal generalisations [12]-[14], [40]-[45].

In this paper we are interested in more general localised solutions. It will be shown that localised overlapping extremal solutions obtained in [34, 35] admit non-extremal generalisations.

Intersection rules [17, 18] for M-branes and D-branes were found by using the string theory representation of brane, duality and the requirement of supersymmetry [21] or the no-force condition [20]. Another derivation of the intersection rules was obtained in [26, 27, 28, 29] by direct solving the bosonic equations of motion of the low-energy theory. These intersection rules are universal, in the sense that they are not specific for some space-time dimension and therefore do not require the supersymmetry. Starting from these intersection rules one can check [27, 28] the harmonic superposition rule and S,T-dualities. These rules were first obtained for extremal branes and then were generalised [12, 14] to include the intersections of non-extremal branes. Recently it was shown that the intersection rules have a simple geometric meaning as the condition ensuring the symmetric space property of the appropriate σ-model target space [33].

In the last year there was an important progress in describing non-perturbative phenomena in gauge theories using brane configurations. In this approach branes are considered as configurations preserving a part of supersymmetry and one has to deal with intersecting configurations having a brane stretched between other branes. From the point of view of application to non-perturbative study of gauge theories [8]-[11], [46]-[49] via brane consideration one has to find brane configurations as solutions of equations of motion with special localisation properties, in particular solutions with one brane ending on another one. It is a rather complicated problem to write explicitly gravity solutions describing one brane ending on another brane [15, 16, 14, 30]. Brane configurations which in some approximation satisfy the desirable localisation properties were considered in [34, 37, 36, 4] and further were generalised in [37]. Also some examples of partially localised p-branes were constructed in [38, 39].

In this paper we present a construction of non-extremal p-brane solutions which distinguishing characteristic is that branes are localised in the relative transverse directions. We derive an intersecting rule for pair-wise intersections of non-extremal branes. As in the extremal case [37], our non-extremal solutions satisfy a characteristic equation which is different from a standard characteristic equation of intersecting branes with harmonic functions depending only on overall transverse directions [26]-[29].

Applying our general formulae presented in Section 2 to D=11 and D=10 we obtain in Section 3 non-extremal deformations of localised overlapping solutions, found by Khuri [34], Gauntlett, Kastor and Traschen [35], and Tseytlin [36]. It is interesting that these deformations are specified by two different vacuum solutions. Vacuum solutions defining the
deformation functions satisfy to a kind of harmonic superposition. One deformation function corresponds to the Schwarzshild-type vacuum solution and another to a new vacuum solution. There are also more general deformations (see Section 3 for details).

2 Localised Intersection of Two Non-extremal Branes

Our construction uses an algebraic method [26, 27] of solution of the bosonic equations of motion. It is convenient to start with the following expression for the low-energy bosonic action in D-dimensional space-time

\[ S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g}(R - \frac{1}{2}(\nabla \phi)^2 - \frac{e^{-\alpha_1\phi}}{2(d_1 + 1)!} F_{d_1+1}^2 - \frac{e^{-\alpha_2\phi}}{2(d_2 + 1)!} F_{d_2+1}^2), \]

where \( F_{d_a+1} \) \((a = 1,2)\) is a \(d_a\)-form field strength, \( F = dA\), \(\phi\) is a dilaton. The low-energy superstring action contains also the Chern-Simons terms, but we omit them since in the particular interesting examples that we will consider in Section 3 they are irrelevant.

In order to find special solutions which describe localised p-branes we take the line element in the following form

\[ ds^2 = e^{2(A_x + A_y)}(-f_xf_ydt^2 + dz_1^2 + \ldots + dz_{q-1}^2) + e^{2(F_{x1} + F_{y1})}(f_x^{-1}dr_x^2 + r_x^2d\Omega_{r_1-1}) + e^{2(F_{x2} + F_{y2})}(f_y^{-1}dr_y^2 + r_y^2d\Omega_{r_2-1}) + e^{2(B_x + B_y)}(du_1^2 + \ldots + du_s^2). \]

Here \(z_i, i = 1, \ldots q - 1\) belong to the intersection, \(x_\gamma\)'s, \(\gamma = 1, \ldots r_1\), \(r_1 \geq 3\) and \(y_\mu\)'s, \(\mu = 1, \ldots r_2\), \(r_2 \geq 3\) are the relative transverse coordinates and the \(u_k\)'s, \(k = 1, \ldots s\) are the overall transverse coordinates, \(d_1 = q + r_1, d_2 = q + r_2\). We denote \(r_x = \sqrt{x^\gamma x_\gamma}\) and \(r_y = \sqrt{y^\gamma y_\gamma}\). All functions with the subscript \(x\) depend on \(r_x\) only, and the functions with the subscript \(y\) depend on \(r_y\).

We will consider the intersection of two different localised p-branes. They are coupled to \(d_1\)- and \(d_2\)-forms. Consider for example, an electrically charged branes with the following ansätze for the forms

\[ F_{q+r_1-1} = h_1r_x^{r_1-1}\partial_y e^{D_{1x} + D_{1y}} dt \wedge dz_1 \wedge \ldots \wedge dz_{q-1} \wedge dr_x \wedge dr_y \wedge vol_{r_1-1}, \]

\[ F_{q+r_2+1} = h_2r_y^{r_2-1}\partial_x e^{D_{2x} + D_{2y}} dt \wedge dz_1 \wedge \ldots \wedge dz_{q-1} \wedge dr_x \wedge dr_y \wedge vol_{r_2-1}, \]

where \(vol_{r_1-1}\) and \(vol_{r_2-1}\) are volume forms on the \((r_1-1)\)- and \((r_2-1)\)-dimensional spheres, \(h_a\) are constants. \(\partial_x\) and \(\partial_y\) mean derivatives on \(r_x\) and \(r_y\), respectively.

Also we assume the following form for the dilaton field

\[ \phi = \phi_x + \phi_y. \]

Examining the Maxwell equations we conclude (see Appendix) that

\[ e^{-D_{2x}} = H_x, \quad e^{-D_{1y}} = H_y, \]

where \(H_x\) and \(H_y\) are harmonic functions

\[ H_x = 1 + \frac{Q_2}{r_x^{r_1-2}}, \quad H_y = 1 + \frac{Q_1}{r_y^{r_1-2}}. \]
The consistency of the Einstein equations gives (see Appendix)

\[ f_x = 1 - \frac{\mu_2}{r_x^{r_1-2}}, \quad f_y = 1 - \frac{\mu_1}{r_y^{r_2-2}}. \] (7)

This is in agreement with the vacuum solutions (see equation (33) below).

The dilaton field is defined in terms of harmonic functions \( H \) and \( f \) via the following equations

\[ \partial_x \phi_x = \frac{\alpha_2 Q_2(r_1 - 2)h_2^2}{2r_x^{r_1-1}f_x} H_x^{-1} + \frac{c_{\phi_1}}{r_x^{r_1-1}f_x}, \quad \partial_y \phi_y = \frac{\alpha_1 Q_1(r_2 - 2)h_1^2}{2r_y^{r_2-1}f_y} H_y^{-1} + \frac{c_{\phi_2}}{r_y^{r_2-1}f_y}, \] (8)

where \( c_{\phi_1} \) and \( c_{\phi_2} \) are some constants.

The metric functions \( A_x \) and \( A_y \) are defined by simple integration of the following equations

\[ \partial_x A_x = \frac{Q_2(r_1 - 2)t_2 h_2^2}{r_x^{r_1-1}f_x} H_x^{-1} + \frac{c_{A_1}}{r_x^{r_1-1}f_x}, \quad \partial_y A_y = \frac{Q_1(r_2 - 2)t_1 h_1^2}{r_y^{r_2-1}f_y} H_y^{-1} + \frac{c_{A_2}}{r_y^{r_2-1}f_y}, \] (9)

where \( c_{A_1} \) and \( c_{A_2} \) are integration constants; \( t_1, t_2, h_1 \) and \( h_2 \) are constants given by

\[ t_1 = \frac{D - d_1 - 2}{2(D - 2)}, \quad t_2 = \frac{D - d_2 - 2}{2(D - 2)}. \] (10)

\[ h_1 = \frac{2}{\sqrt{\Delta_1}} \sqrt{1 + \frac{\mu_1}{Q_1}}, \quad h_2 = \frac{2}{\sqrt{\Delta_2}} \sqrt{1 + \frac{\mu_2}{Q_2}}, \] (11)

where we use the usual notations

\[ \Delta_a = \alpha_a^2 + \frac{2(D - d_a - 2)d_a}{D - 2}. \] (12)

The metric functions \( F \) and \( B \) are defined by integration of the following equations

\[ \partial_x F_{1x} = -\frac{Q_2(r_1 - 2)u_2 h_2^2}{r_x^{r_1-1}f_x} H_x^{-1} + \frac{c_{F_{11}}}{r_x^{r_1-1}f_x}, \quad \partial_y F_{2y} = -\frac{Q_1(r_2 - 2)u_1 h_1^2}{r_y^{r_2-1}f_y} H_y^{-1} + \frac{c_{F_{22}}}{r_y^{r_2-1}f_y}, \] (13)

\[ \partial_x F_{2x} = \frac{Q_2(r_1 - 2)t_2 h_2^2}{r_x^{r_1-1}f_x} H_x^{-1} + \frac{c_{F_{21}}}{r_x^{r_1-1}f_x}, \quad \partial_y F_{1y} = \frac{Q_1(r_2 - 2)t_1 h_1^2}{r_y^{r_2-1}f_y} H_y^{-1} + \frac{c_{F_{12}}}{r_y^{r_2-1}f_y}, \] (14)

\[ \partial_x B_x = -\frac{Q_2(r_1 - 2)u_2 h_2^2}{r_x^{r_1-1}f_x} H_x^{-1} + \frac{c_{B_1}}{r_x^{r_1-1}f_x}, \quad \partial_y B_y = -\frac{Q_1(r_2 - 2)u_1 h_1^2}{r_y^{r_2-1}f_y} H_y^{-1} + \frac{c_{B_2}}{r_y^{r_2-1}f_y}, \] (15)

where \( c \)'s are also integration constants and the constants \( u_1 \) and \( u_2 \) are given by

\[ u_1 = \frac{d_1}{2(D - 2)}, \quad u_2 = \frac{d_2}{2(D - 2)}. \] (16)
The consistency of our ansatz with the equations of motion imply the following restriction on the parameters
\[
\frac{\alpha_1 \alpha_2}{2} + q + 2 - \frac{d_1 d_2}{D-2} = 0,
\] (17)
as well as constraints on the integration constants
\[
qcA_1 + (r_1 - 2)c_{F11} + r_2 c_{F21} + sc_{B1} = 0, \quad qcA_2 + r_1 c_{F12} + (r_2 - 2)c_{F22} + sc_{B2} = 0, \quad (18)
\]
\[
\alpha_2 c_{\phi 1} + 2qcA_1 + 2r_2 c_{F21} + 2(r_1 - 2)\mu_2 = 0, \quad \alpha_1 c_{\phi 2} + 2qcA_2 + 2r_1 c_{F12} + 2(r_2 - 2)\mu_1 = 0, \quad (19)
\]
\[
\alpha_1 c_{\phi 1} + 2qcA_1 + 2(r_1 - 2)c_{F11} + 4c_{F21} = 0, \quad \alpha_2 c_{\phi 2} + 2qcA_2 + 4c_{F12} + 2(r_2 - 2)c_{F22} = 0, \quad (20)
\]
\[
\frac{1}{2} c_{\phi 1} + qc_{A1} + (r_1 - 2)c_{F11} + r_2 c_{F21} + sc_{B1} + \mu_2(r_1 - 2)(c_{A1} - c_{F11}) = 0,
\]
\[
\frac{1}{2} c_{\phi 2} + qc_{A2} + (r_2 - 2)c_{F22} + r_1 c_{F12} + sc_{B2} + \mu_1(r_2 - 2)(c_{A2} - c_{F22}) = 0, \quad (21)
\]
\[
\mu_1 \mu_2(\frac{1}{2} r_1 r_2 - r_1 - r_2 + 2) + \mu_2(c_{A2} - c_{F12})(r_1 - 2) + \mu_1(c_{A1} - c_{F21})(r_2 - 2)
\]
\[
+ c_{\phi 1} c_{\phi 2} + 2qc_{A1} c_{A2} + 2(r_1 - 2)c_{F11} c_{F12} + 2(r_2 - 2)c_{F21} c_{F22} + 4c_{F12} c_{F21} + 2sc_{B1} c_{B2} = 0. \quad (22)
\]

Now the main task is to solve the system of algebraic equations (18), (19), (20), (21), (22) and find the appropriate constants. For simplicity we will consider only particular cases.

### 3 Localised Intersections in D=11 and D=10

#### 3.1 Non-extremal localised M-branes

Equation (17) has the solution \(\alpha_1 = \alpha_2 = 0, q = 2, r_1 = 4, r_2 = 4, s = 1\), which corresponds to intersecting M5-branes. Our system of algebraic equations has the following solutions
\[
c_{\phi 1} = 0, \quad c_{A1} = -\mu_2, \quad c_{F11} = \mu_2, \quad c_{F21} = 0, \quad c_{B1} = 0,
\]
\[
c_{\phi 2} = 0, \quad c_{A2} = -\frac{1}{3}\mu_1, \quad c_{F12} = -\frac{1}{3}\mu_1, \quad c_{F22} = \frac{2}{3}\mu_1, \quad c_{B2} = \frac{2}{3}\mu_1, \quad (23)
\]
or
\[
c_{\phi 1} = 0, \quad c_{A1} = -\frac{1}{3}\mu_2, \quad c_{F11} = \frac{2}{3}\mu_2, \quad c_{F21} = -\frac{1}{3}\mu_2, \quad c_{B1} = \frac{2}{3}\mu_2,
\]
\[
c_{\phi 2} = 0, \quad c_{A2} = -\mu_1, \quad c_{F12} = 0, \quad c_{F22} = \mu_1, \quad c_{B2} = 0. \quad (24)
\]

Thus we get two metrics. The first solution (23) corresponds to the line element
\[
ds^2 = H_x^{-1/3} H_y^{-1/3} f_x^{-2/3}(-f_x f_y dt^2 + dz^2) + H_x^{2/3} H_y^{-1/3} f_x^{1/3} (f_x^{-1} dr_x^2 + r_x^2 d\Omega_3)
\]
whereas the second one \((24)\) leads to the equivalent expression

\[
ds^2 = H_x^{-1/3} H_y^{-1/3} f_y^{-2/3} (-f_x f_y dt^2 + dz^2) + H_x^{2/3} H_y^{2/3} (f_y f_x^{-1} dr_x^2 + r_x^2 d\Omega_3^1) + H_x^{2/3} H_y^{2/3} f_y^{-2/3} du^2.
\]  

Solutions \((25)\) and \((26)\) describe a non-extremal generalisation of the solution \([34]\) describing the M5-branes intersecting on a string. We see that the harmonic functions are independent of the overall transverse direction and depend only on the relative transverse directions. As in the extremal case each M5-brane is localised in the directions tangent to the other M5-brane but is delocalised in the overall transverse direction that separates them.

### 3.2 Non-extremal localised NS-branes

Equation \((17)\) has also the following solution \(\alpha_1 = 1, \alpha_2 = 1, q = 2, r_1 = 4, r_2 = 4, s = 0.\) It describes the intersection of NS5-branes on a string. In this case the integration constants are

\[
c_{\phi_1} = 0, \quad c_{A_1} = -\mu_2, \quad c_{F_{11}} = \mu_2, \quad c_{F_{21}} = 0,
\]

\[
c_{\phi_2} = -\mu_1, \quad c_{A_2} = -\frac{1}{4} \mu_1, \quad c_{F_{12}} = -\frac{1}{4} \mu_1, \quad c_{F_{22}} = \frac{3}{4} \mu_1,
\]  

\[(27)\]

or

\[
c_{\phi_1} = -\mu_2, \quad c_{A_1} = -\frac{1}{4} \mu_1, \quad c_{F_{11}} = \frac{3}{4} \mu_2, \quad c_{F_{21}} = -\frac{1}{4} \mu_2,
\]

\[
c_{\phi_2} = 0, \quad c_{A_2} = -\mu_1, \quad c_{F_{12}} = 0, \quad c_{F_{22}} = \mu_1.
\]  

\[(28)\]

We find the following solutions

\[
ds^2 = H_x^{-1/4} H_y^{-1/4} f_x^{-3/4} (-f_x f_y dt^2 + dz^2) + H_x^{3/4} H_y^{-1/4} f_y^{1/4} (f_y^{-1} dr_y^2 + r_y^2 d\Omega_3^1)
\]

\[
+ H_x^{-1/4} H_y^{3/4} f_y^{1/4} (f_y^{-1} dr_y^2 + r_y^2 d\Omega_3^1).
\]  

\[(29)\]

\[
e^{-\phi_x} = H_x^{1/2} f_y^{-1/2}, \quad e^{-\phi_y} = H_y^{1/2},
\]

\[(30)\]

and

\[
ds^2 = H_x^{-1/4} H_y^{-1/4} f_y^{-3/4} (-f_x f_y dt^2 + dz^2) + H_x^{3/4} H_y^{-1/4} f_y^{1/4} (f_y^{-1} dr_y^2 + r_y^2 d\Omega_3^1)
\]

\[
+ H_x^{-1/4} H_y^{3/4} f_y^{1/4} (f_y^{-1} dr_y^2 + r_y^2 d\Omega_3^1).
\]  

\[(31)\]

\[
e^{-\phi_x} = H_x^{1/2}, \quad e^{-\phi_y} = H_y^{1/2} f_y^{-1/2},
\]

\[(32)\]

Solutions \((29)\) and \((31)\) give a non-extremal generalisation of the solution \([34]\) describing two NS5-branes intersecting on a string.
4 Vacuum Solutions in M-theory and Deformations

A non-extremal deformation of an extremal solution is performed by means of a vacuum solution. Vacuum solution in M-theory is a Ricci flat metric \((R_{\mu\nu} = 0)\) in eleven dimensions, or the solution with the cosmological constant \((R_{\mu\nu} = \Lambda g_{\mu\nu})\).

If one has a manifold \(M^D\) which is the product of two manifolds, \(M^D = M^n \times M^{D-n}\), then we always have a vacuum solution on \(M^D\) as the sum of vacuum solutions on \(M^n\) and on \(M^{D-n}\). A simple example is the product of the standard Schwarzschild solution and the Euclidean version of the Schwarzschild solution

\[
\begin{align*}
\text{ds}_V^2 &= (-f_x dt^2 + f_y dz_1^2 + \ldots + d z_{q-1}^2) + (f_x^{-1} dr^2 + r_x^2 d\Omega_{r_1-1}) \\
&
+ (f_y^{-1} dr_y^2 + r_y^2 d\Omega_{r_2-1}) + (du_1^2 + \ldots + du_q^2), \quad (33)
\end{align*}
\]

where \(f_x\) and \(f_y\) are \(r_1\)- and \(r_2\)-dimensional harmonic functions.

Note that it is also possible to take the line element defining a vacuum solution of the D-dimensional gravity in another form

\[
\begin{align*}
\text{ds}_V^2 &= f_x^{\nu_0} (-f_x f_y dt^2 + dz_1^2 + \ldots + d z_{q-1}^2) + f_x^{\nu_1} (f_x^{-1} dr_x^2 + r_x^2 d\Omega_{r_1-1}) \\
&
+ f_x^{\nu_2} (f_y^{-1} dr_y^2 + r_y^2 d\Omega_{r_2-1}) + f_x^{\nu_u} (du_1^2 + \ldots + du_q^2), \quad (34)
\end{align*}
\]

where the parameters \(\nu_0, \nu_1, \nu_2\) and \(\nu_u\) are specified by an algebraic system of non-linear equations with the coefficients which depend on \(D\), \(r_1\) and \(r_2\). Generally, these solutions have naked singularities [22].

In particular, for \(D = 11\) one of these solutions has the form

\[
\begin{align*}
\text{ds}_V^2 &= f_x^{-2/3} (-f_x f_y dt^2 + dz_2^2) + f_x^{1/3} (f_x^{-1} dr_x^2 + r_x^2 d\Omega_3) \\
&
+ f_x^{1/3} (f_y^{-1} dr_y^2 + r_y^2 d\Omega'_3) + f_x^{-2/3} du^2. \quad (35)
\end{align*}
\]

This metric is nothing but the vacuum limit of the solution [23].

Notice also the following vacuum solutions of M-theory (i.e. Ricci flat metrics in \(D = 11\) dimensions)

\[
\begin{align*}
\text{ds}^2 &= f_x^{-a} \left[ -f_y dt^2 + f_x^{-1} d\rho_x^2 + \rho_x^2 d\Omega_2 + f_y^{-1} d\rho_y^2 + \rho_y^2 d\Omega'_2 \right] \\
&
+ f_x^{a+b+1} dz_2^2 + f_x^{-b} \left[ du^2 + dw^2 \right] + f_x^{2+b} du^2, \quad (36)
\end{align*}
\]

where

\[
\begin{align*}
f_x &= 1 - \frac{\mu_2}{\rho_x}, & f_y &= 1 - \frac{\mu_1}{\rho_y}, \quad (37)
\end{align*}
\]

and the real parameters \(a\) and \(b\) satisfy the equation

\[
11a^2 + 5a + 2b^2 + b + 5ab = 0. \quad (38)
\]

In the general case the parameters \(a, b\) are irrational. However there are examples when they are rational.
\[ a = -\frac{1}{3}, \quad b = -\frac{1}{3} \quad \text{or} \quad b = \frac{2}{3}, \]  
\[ (39) \]
\[ a = -\frac{1}{2}, \quad b = \frac{1}{2} \quad \text{or} \quad b = \frac{1}{4}. \]  
\[ (40) \]

The pairs \( a = -1/3, b = -1/3 \) and \( a = -1/3, b = 2/3 \) give the same metric.

Other examples are \( a = -1/2, b = 1/2 \) and \( a = -1/2, b = 1/4 \):

\[ ds^2 = f_x^{1/2} \left[ -f_y dt^2 + f_x^{-1} d\rho_x^2 + \rho_x^2 d\Omega_2 + f_y^{-1} d\rho_y^2 + \rho_y^2 d\Omega'_2 \right] \]  
\[ + f_x^{-1/2} \left[ dz^2 + dv^2 + dw^2 \right] + du^2, \]  
\[ (41) \]

\[ ds^2 = f_x^{1/2} \left[ -f_y dt^2 + f_x^{-1} d\rho_x^2 + \rho_x^2 d\Omega_2 + f_y^{-1} d\rho_y^2 + \rho_y^2 d\Omega'_2 \right] \]  
\[ + f_x^{-3/4} dz^2 + f_x^{-1/4} \left[ dv^2 + dw^2 + du^2 \right]. \]  
\[ (42) \]

These vacuum solutions provide us with the following non-extremal solutions of eleven dimensional supergravity

\[ ds^2 = H_x^{-1/3} H_y^{-1/3} \left\{ -f_x^{-a} f_y dt^2 + f_x^{4a+b+1} dz^2 \right\} \]  
\[ + H_x^{2/3} H_y^{-1/3} \left\{ f_x^{-a} \left[ f_x^{-1} d\rho_x^2 + \rho_x^2 d\Omega_2 \right] + f_x^{-b} dv^2 \right\} \]  
\[ + H_x^{-1/3} H_y^{2/3} \left\{ f_x^{-a} \left[ f_y^{-1} d\rho_y^2 + \rho_y^2 d\Omega'_2 \right] + f_x^{-b} dw^2 \right\} + H_x^{2/3} H_y^{2/3} f_x^{4a+b} du^2, \]  
\[ (43) \]

where

\[ H_x = 1 + \frac{Q_2}{\rho_x}, \quad H_y = 1 + \frac{Q_1}{\rho_y}. \]  
\[ (44) \]

5 Conclusion

In this paper we have presented a non-extremal generalisation of localised overlapping brane solutions in M-theory. The harmonic functions specifying this solution are independent of one overall transverse direction and depend on the relative transverse directions. The M5-branes are localised in the directions tangent to the other M5-brane but are delocalised in the overall transverse direction that separates them. In the extremal case there is a solution which contains an extra M2-brane which overlaps each of the M5-branes. This solution still does not describe a localised solution needed to the brane approach to gauge theory, however it would be interesting to find it’s non-extremal deformation.

There exists also a new non-extremal solution which describes two NS5-branes. The harmonic functions specifying this solution depend on the relative transverse directions. These NS5-branes are localised inside the directions tangent to the other NS5-brane.

We do not present here the solutions which involve an additional M2-brane or NS1-brane. The method that we have used in this paper can be generalised to these two cases and the corresponding calculations will be a subject of a forthcoming paper.

We also have presented here some more involved solutions, which based on a more general class of vacuum solutions. These solutions correspond to smeared extremal brane configurations. Properties of these solutions require a further study.
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Appendix

In Appendix we present the explicit derivation of the system (18), (19), (20), (21), (22) presented in Section 2.

The equations of motion following from the action (1) are

\[ R_{MN} - \frac{1}{2} g_{MN} R = e^{-\alpha_1 \phi} T_{MN}^{(F_{d_1+1})} + e^{-\alpha_2 \phi} T_{MN}^{(F_{d_2+1})} + T_{MN}^{(\phi)}, \]  

(45)

\[ \partial_M \left( e^{-\alpha_1 \phi} \sqrt{-g} F_{M1...M_{d_1}} \right) = 0, \quad \partial_M \left( e^{-\alpha_2 \phi} \sqrt{-g} F_{M1...M_{d_2}} \right) = 0, \]  

(46)

\[ \frac{1}{\sqrt{-g}} \partial_M (\sqrt{-g} g^{MN} \partial_M \phi) + \frac{\alpha_1}{2(d_1 + 1)!} e^{-\alpha_1 \phi} F_{d_1+1}^2 + \frac{\alpha_2}{2(d_2 + 1)!} e^{-\alpha_2 \phi} F_{d_2+1}^2 = 0. \]  

(47)

The energy-momentum tensors for the matter fields have the form

\[ T_{MN}^{(F_{d_a+1})} = \frac{1}{2d_a!} \left( F_{M1...M_{d_a}} F_{N1...M_{d_a}} - \frac{g_{MN}}{2(d_a + 1)!} F_{d_a+1}^2 \right), \]  

(48)

\[ T_{MN}^{(\phi)} = \frac{1}{2} \left( \partial_M \phi \partial_N \phi - \frac{1}{2} g_{MN} (\nabla \phi)^2 \right). \]  

(49)

It is convenient to rewrite the Einstein equations in another form

\[ R_{MN} = e^{-\alpha_1 \phi} G_{MN}^{(F_{d_1+1})} + e^{-\alpha_2 \phi} G_{MN}^{(F_{d_2+1})} + G_{MN}^{(\phi)}, \]  

(50)

where

\[ G_{MN}^{(F_{d_a+1})} = T_{MN}^{(F_{d_a+1})} + \frac{1}{2 - D} g_{MN} T_{d_a+1}, \quad T_{d_a+1} = g^{MN} T_{MN}^{(F_{d_a+1})} \]  

(51)

\[ G_{MN}^{(\phi)} = T_{MN}^{(\phi)} + \frac{1}{2 - D} g_{MN} T^{(\phi)}, \quad T^{(\phi)} = g^{MN} T_{MN}^{(\phi)}. \]  

(52)

For simplification we impose the gauge, that in the extremal case corresponds to the Fock-De-Donder gauge

\[ q A_x + (r_1 - 2) F_{1x} + r_2 F_{2x} + s B_x = 0, \quad q A_y + r_1 F_{1y} + (r_2 - 2) F_{2y} + s B_y = 0. \]  

(53)

For our ansatz (2) the components of the Ricci tensor in the chosen gauge (53) are

\[ R_{tt} = -g_{tt} (e^{-2(F_{1x} + F_{1y})} \frac{1}{r_x^{r_1-1}} \partial_x (r_x^{r_1-1} f_x \partial_x (A_x + \frac{1}{2} \ln f_x))) \]
\[ R_{x_i x_i} = -g_{x_i x_i} e^{-2(F_{1x} + F_{1y})} \frac{1}{r_y^{r_{y-1}}} \partial_y (r_y^{r_{y-1}} f_y \partial_y (A_y + \frac{1}{2} \ln f_y)), \]

\[ R_{u_k u_k} = -g_{u_k u_k} e^{-2(F_{1x} + F_{1y})} \frac{1}{r_x^{r_{x-1}}} \partial_x (r_x^{r_{x-1}} f_x \partial_x B_x) \]

\[ R_{x x x} = -\partial_x^2 (\ln r_x^{r_x-1} f_x) - 2(\partial_x F_{1x} - \frac{1}{2} \partial_x \ln f_x)^2 + \partial_x (2 F_{1x} - \ln f_x) \partial_x (2 F_{1x} + \ln r_x^{r_x-1}) \]

\[ -(\partial_x A_x + \frac{1}{2} \partial_x \ln f_x)^2 - (q - 1)(\partial_x A_x) - (\partial_x F_{1x} - \frac{1}{2} \partial_x \ln f_x)^2 - (r_1 - 1)(\partial_x F_{1x} + \partial_x \ln r_x)^2 \]

\[-r_2 (\partial_x F_{2x})^2 - s(\partial_x B_x)^2 - \partial_x^2 (F_{1x} + \frac{1}{2} \ln f_x) - \partial_x (F_{1x} + \frac{1}{2} \ln f_x) \partial_x (\ln r_x^{r_x-1} f_x), \]

\[ R_{y y y} = -\partial_y^2 (\ln r_y^{r_y-1} f_y) - 2(\partial_y F_{2y} - \frac{1}{2} \partial_y \ln f_y)^2 + \partial_y (2 F_{2y} - \ln f_y) \partial_y (2 F_{2y} + \ln r_y^{r_y-1}) \]

\[ -(\partial_y A_y + \frac{1}{2} \partial_y \ln f_y)^2 - (q - 1)(\partial_y A_y) - (\partial_y F_{2y} - \frac{1}{2} \partial_y \ln f_y)^2 - (r_2 - 1)(\partial_y F_{2y} + \partial_y \ln r_y)^2 \]

\[-r_1 (\partial_y F_{1y})^2 - s(\partial_y B_y)^2 - \partial_y^2 (F_{2y} + \frac{1}{2} \ln f_y) - \partial_y (F_{2y} + \frac{1}{2} \ln f_y) \partial_y (\ln r_y^{r_y-1} f_y), \]

\[ R_{x y y} = -2 \partial_x F_{2x} \partial_y F_{1y} + \partial_x F_{2x} \partial_y (2 F_{2y} + \ln r_y^{r_y-1}) \]

\[ + \partial_y F_{1y} (2 F_{1x} + \ln r_x^{r_x-1}) - \partial_x (A_x + \frac{1}{2} \ln f_x) \partial_y (A_y + \frac{1}{2} \ln f_y) \]

\[-(q - 1) \partial_x A_x \partial_y A_y - \partial_x (F_{1x} + \frac{1}{2} \ln f_x) \partial_y F_{1y} - (r_1 - 1) \partial_x (F_{1x} + \ln r_x) \partial_y F_{1y} \]

\[-\partial_x F_{2x} \partial_y (F_{2y} - \frac{1}{2} \ln f_y) - (r_2 - 1) \partial_x F_{2y} \partial_y (F_{2y} + \ln r_y) - s \partial_x B_x \partial_y B_y, \]

where \( \partial_x \) and \( \partial_y \) as in Section 2 mean the derivatives on \( r_x \) and \( r_y \), respectively.

Now let us analyse the components \( G_{MN}^{(\phi)} \). For our ansatz \( G_{MN}^{(\phi)} \) has three non-trivial components

\[ G_{r x x}^{(\phi)} = \frac{1}{2} (\partial_x \phi_x)^2, \quad G_{r y y}^{(\phi)} = \frac{1}{2} \partial_y \phi_x \partial_y \phi_y, \quad G_{r y y}^{(\phi)} = \frac{1}{2} (\partial_y \phi_y)^2. \]

Also we have

\[ e^{-\alpha_1 \phi} G_{MN}^{(F_{1x} + 1)} + e^{-\alpha_2 \phi} G_{MN}^{(F_{1y} + 1)} = \]

\[-g_{MN}(t_1 h_1^2 e^{-2(F_{2x} + F_{2y})} S_x H_y^2 (\partial_y e^{D_y})^2 + t_2 h_2^2 e^{-2(F_{1x} + F_{1y})} S_y H_x^2 (\partial_x e^{D_x})^2) \]

\[ 10 \]
for $tt'$, $zz'$, $r_x r_x'$ and $r_y r_y'$-components,

$$e^{-\alpha_1 \phi} G^{(F_{d_1 + 1})}_{MN} + e^{-\alpha_2 \phi} G^{(F_{d_2 + 1})}_{MN}$$

$$= g_{MN}(u_1 h_1^2 e^{-2(F_{x} + F_{y})} S_x H_y^2 (\partial_y e^{D_{1y}})^2 + u_2 h_2^2 e^{-2(F_{x} + F_{y})} S_y H_x^2 (\partial_x e^{D_{2x}})^2)$$  \hspace{1cm} (62)$$

for $uu'$-components,

$$e^{-\alpha_1 \phi} G^{(F_{d_1 + 1})}_{MN} + e^{-\alpha_2 \phi} G^{(F_{d_2 + 1})}_{MN}$$

$$= g_{MN}(t_1 h_1^2 e^{-2(F_{x} + F_{y})} S_x H_y^2 (\partial_y e^{D_{1y}})^2 - u_2 h_2^2 e^{-2(F_{x} + F_{y})} S_y H_x^2 (\partial_x e^{D_{2x}})^2)$$  \hspace{1cm} (63)$$

for the components corresponding to $r_1 - 1$-dimensional block and

$$g_{MN}(-u_1 h_1^2 e^{-2(F_{x} + F_{y})} S_x H_y^2 (\partial_y e^{D_{1y}})^2 + t_2 h_2^2 e^{-2(F_{x} + F_{y})} S_y H_x^2 (\partial_x e^{D_{2x}})^2)$$  \hspace{1cm} (64)$$

for the components corresponding to $r_2 - 1$-dimensional block. We use the following notations

$$H_x^2 = \exp(-\alpha_2 \phi_x - 2q A_x - 2r_2 F_{2x}), \quad H_y^2 = \exp(-\alpha_1 \phi_y - 2q A_y - 2r_1 F_{1y}),$$  \hspace{1cm} (65)$$

$$S_x = \exp(-\alpha_1 \phi_x - 2q A_x - 2r_1 F_{1x} + 2D_{1x}), \quad S_y = \exp(-\alpha_2 \phi_y - 2q A_y - 2r_2 F_{2y} + 2D_{2y}).$$  \hspace{1cm} (66)$$

The Maxwell equations for our field configuration read

$$\partial_x (H_x^2 r_x^{r_1 - 1} \partial_x e^{D_{2x}}) = 0, \quad \partial_y (H_y^2 r_y^{r_2 - 1} \partial_y e^{D_{1y}}) = 0,$$  \hspace{1cm} (67)$$

$$\partial_x (S_x e^{2F_{1x} - 2F_{2x} - D_{1x}} \partial_y e^{D_{1y}}) = 0, \quad \partial_y (S_y e^{2F_{2y} - 2F_{1y} - D_{2y}} \partial_x e^{D_{2x}}) = 0.$$  \hspace{1cm} (68)$$

To solve the equations (67) we set

$$e^{D_{2x}} = H_x^{-1}, \quad e^{D_{1y}} = H_y^{-1}.$$  \hspace{1cm} (69)$$

It means that the functions should be harmonic

$$H_x = 1 + \frac{Q_2}{r_x^{r_1 - 2}}, \quad H_y = 1 + \frac{Q_1}{r_y^{r_2 - 2}}.$$  \hspace{1cm} (70)$$

Considering the dilaton equation as well as the Einstein equations it is not difficult to observe that the variables $x$ and $y$ could be easily separated if

$$S_x = 1, \quad S_y = 1.$$  \hspace{1cm} (71)$$

$$2D_{1x} = \alpha_1 \phi_x + 2q A_x + 2r_1 F_{1x}, \quad 2D_{2y} = \alpha_2 \phi_y + 2q A_y + 2r_2 F_{2y}. $$  \hspace{1cm} (72)$$

It leads to further simplifications of the Maxwell equations (68) from which we now get

$$D_{1x} = 2F_{1x} - 2F_{2x}, \quad D_{2y} = 2F_{2y} - 2F_{1y}. $$  \hspace{1cm} (73)$$

We also assume that the parts of the dilaton equation containing differentiation on $r_x$ and $r_y$ are equalised separately

$$\frac{1}{r_x^{r_1 - 1}} \partial_x (r_x^{r_1 - 1} f_x \partial_x \phi_x) = \frac{\alpha_2^2}{2} h_2^2 H_x^2 (\partial_x e^{D_{2x}})^2.$$  \hspace{1cm}
\[
\frac{1}{r_{y}^{r_{2}^{-1}}} \partial_y (r_{y}^{r_{2}^{-1}} f_y \partial_y \phi_y) = \frac{\alpha_1}{2} h_1^2 H_y^2 (\partial_y e^{D_{1y}})^2.
\] (74)

Using (73) one can integrate both sides of these equations to get

\[
\partial_x \phi_x = \frac{\alpha_2 Q_2 (r_1 - 2) h_2^2}{2 r_{x}^{r_{1}^{-1}} f_x} H_x^{-1} + \frac{c_{\phi_1}}{r_{x}^{r_{1}^{-1}} f_x}, \quad \partial_y \phi_y = \frac{\alpha_1 Q_1 (r_2 - 2) h_1^2}{2 r_{y}^{r_{2}^{-1}} f_y} H_y^{-1} + \frac{c_{\phi_2}}{r_{y}^{r_{2}^{-1}} f_y},
\] (75)

where \(c_{\phi_1}\) and \(c_{\phi_2}\) are some constants. Equations (72) and (73) can be used to express \(F_{1x}\) and \(F_{2y}\) through the other variables

\[
2(r_1 - 2)F_{1x} = -\alpha_1 \phi_x - 2qA_x - 4F_{2x}, \quad 2(r_2 - 2)F_{2y} = -\alpha_2 \phi_y - 2qA_y - 4F_{1y}.
\] (76)

Now let us consider the Einstein equations. The equation on the \(tt\)-component under assumption (71) and after a separate equalising the terms depending on \(r_x\) and \(r_y\) gives

\[
\frac{1}{r_{x}^{r_{1}^{-1}}} \partial_x (r_{x}^{r_{1}^{-1}} f_x \partial_x (A_x + \frac{1}{2} \ln f_x)) = t_2 h_2^2 H_x^2 (\partial_x e^{D_{2x}})^2,
\]

\[
\frac{1}{r_{y}^{r_{2}^{-1}}} \partial_y (r_{y}^{r_{2}^{-1}} f_y \partial_y (A_y + \frac{1}{2} \ln f_y)) = t_1 h_1^2 H_y^2 (\partial_y e^{D_{1y}})^2,
\] (77)

where \(t_1\) and \(t_2\) are constants

\[
t_1 = \frac{D - d_1 - 2}{2(D - 2)}, \quad t_2 = \frac{D - d_2 - 2}{2(D - 2)}.
\] (78)

The equations on the \(zz\)-components under the same assumptions are

\[
\frac{1}{r_{x}^{r_{1}^{-1}}} \partial_x (r_{x}^{r_{1}^{-1}} f_x \partial_x A_x) = t_2 h_2^2 H_x^2 (\partial_x e^{D_{2x}})^2, \quad \frac{1}{r_{y}^{r_{2}^{-1}}} \partial_y (r_{y}^{r_{2}^{-1}} f_y \partial_y A_y) = t_1 h_1^2 H_y^2 (\partial_y e^{D_{1y}})^2.
\] (79)

The consistency condition of the equations (77) and (73) gives the equations on the functions \(f_x\) and \(f_y\)

\[
\frac{1}{r_{x}^{r_{1}^{-1}}} \partial_x (r_{x}^{r_{1}^{-1}} f_x \partial_x (\ln f_x)) = 0, \quad \frac{1}{r_{y}^{r_{2}^{-1}}} \partial_y (r_{y}^{r_{2}^{-1}} f_y \partial_y (\ln f_y)) = 0.
\] (80)

Thus we have

\[
f_x = 1 - \frac{\mu_2}{r_{x}^{r_{1}^{-2}}}, \quad f_y = 1 - \frac{\mu_1}{r_{y}^{r_{2}^{-2}}}.
\] (81)

The substitution of (79) in (78) gives

\[
\partial_x A_x = \frac{Q_2 (r_1 - 2) t_2 h_2^2}{r_{x}^{r_{1}^{-1}} f_x} H_x^{-1} + \frac{c_{A1}}{r_{x}^{r_{1}^{-1}} f_x}, \quad \partial_y A_y = \frac{Q_1 (r_2 - 2) t_1 h_1^2}{r_{y}^{r_{2}^{-1}} f_y} H_y^{-1} + \frac{c_{A2}}{r_{y}^{r_{2}^{-1}} f_y},
\] (82)

where \(c_{A1}\) and \(c_{A2}\) are integration constants.

The Einstein equations on the \(uu\)-components yield

\[
\frac{1}{r_{x}^{r_{1}^{-1}}} \partial_x (r_{x}^{r_{1}^{-1}} f_x \partial_x B_x) = -u_2 h_2^2 H_x^2 (\partial_x e^{D_{2x}})^2,
\]
\[
\frac{1}{r_y^{r_2-1}} \partial_y(r_y^{r_2-1} f_y \partial_y B_y) = -u_1 h_1^2 H_y^2 (\partial_y e^{D_1 y})^2, \tag{83}
\]

or
\[
\partial_y B_x = -\frac{Q_2(r_1 - 2)u_2 h_2^2}{r_x^{r_1-1} f_x} H_x^{-1} + \frac{c_{B1}}{r_y^{r_2-1} f_y}, \quad \partial_y B_y = -\frac{Q_1(r_2 - 2)u_1 h_1^2}{r_y^{r_2-1} f_y} H_y^{-1} + \frac{c_{B2}}{r_y^{r_2-1} f_y}, \tag{84}
\]

where \(c_{B1}\) and \(c_{B2}\) are some arbitrary constants and the constants \(u_1\) and \(u_2\) are given by
\[
u_1 = \frac{d_1}{2(D - 2)}, \quad u_2 = \frac{d_2}{2(D - 2)}. \tag{85}\]

The consideration of the \(\varphi\varphi\)-components of the Einstein equations where \(\varphi\) is one of the angles on the \((r_1 - 1)\)- or \((r_2 - 1)\)-sphere except the relations considered above gives
\[
\frac{1}{r_x^{r_1-1}} \partial_x(r_x^{r_1-1} f_x \partial_x F_{2x}) = t_2 h_2^2 H_x^2 (\partial_x e^{D_2 x})^2, \quad \frac{1}{r_y^{r_2-1}} \partial_y(r_y^{r_2-1} f_y \partial_y F_{1y}) = t_1 h_1^2 H_y^2 (\partial_y e^{D_1 y})^2, \tag{86}
\]

or
\[
\partial_x F_{2x} = \frac{Q_2(r_1 - 2)t_2 h_2^2}{r_x^{r_1-1} f_x} H_x^{-1} + \frac{c_{F21}}{r_y^{r_2-1} f_y}, \quad \partial_y F_{1y} = \frac{Q_1(r_2 - 2)t_1 h_1^2}{r_y^{r_2-1} f_y} H_y^{-1} + \frac{c_{F12}}{r_y^{r_2-1} f_y}, \tag{87}
\]

\[
\frac{1}{r_x^{r_1-1}} \partial_x(r_x^{r_1-1} f_x \partial_x F_{1x}) = -u_2 h_2^2 H_x^2 (\partial_x e^{D_2 x})^2, \quad \frac{1}{r_y^{r_2-1}} \partial_y(r_y^{r_2-1} f_y \partial_y F_{2y}) = -u_1 h_1^2 H_y^2 (\partial_y e^{D_1 y})^2, \tag{88}
\]

or
\[
\partial_x F_{1x} = -\frac{Q_2(r_1 - 2)u_2 h_2^2}{r_x^{r_1-1} f_x} H_x^{-1} + \frac{c_{F11}}{r_y^{r_2-1} f_y}, \quad \partial_y F_{2y} = -\frac{Q_1(r_2 - 2)u_1 h_1^2}{r_y^{r_2-1} f_y} H_y^{-1} + \frac{c_{F22}}{r_y^{r_2-1} f_y}. \tag{89}
\]

Now let us examine the consistency conditions of equations (83), (84), (85), (86), (87), (88) on one hand and algebraic restrictions (53), (63), (64), (88), (89) on the other. The gauge condition (53) gives the relations between the integration constants
\[
qc_{A1} + (r_1 - 2)c_{F11} + r_2 c_{F21} + sc_{B1} = 0, \quad qc_{A2} + r_1 c_{F12} + (r_2 - 2)c_{F22} + sc_{B2} = 0. \tag{90}
\]

From the equation (63) we have
\[
\alpha_2 c_{\phi 1} + 2qc_{A1} + 2r_2 c_{F21} + 2(r_1 - 2)\mu_2 = 0, \quad \alpha_1 c_{\phi 2} + 2qc_{A2} + 2r_1 c_{F12} + 2(r_2 - 2)\mu_1 = 0. \tag{91}
\]

\[
h_1 = \frac{2}{\sqrt{\Delta_1}} \sqrt{1 + \frac{\mu_1}{Q_1}}, \quad h_2 = \frac{2}{\sqrt{\Delta_2}} \sqrt{1 + \frac{\mu_2}{Q_2}}, \tag{92}
\]

where we use the usual notations
\[
\Delta_a = \alpha_a^2 + \frac{2(D - d_a - 2)d_a}{D - 2}. \tag{93}
\]
The third type of relations (76) gives us the restriction on the parameters

\[
\frac{\alpha_1 \alpha_2}{2} + q + 2 - \frac{d_1 d_2}{D - 2} = 0, \tag{94}
\]

as well as new constraints on the integration constants

\[
\alpha_1 c_{\phi 1} + 2qcA_1 + 2(r_1 - 2)c_{F11} + 4c_{F21} = 0, \quad \alpha_2 c_{\phi 2} + 2qcA_2 + 4c_{F12} + 2(r_2 - 2)c_{F22} = 0. \tag{95}
\]

Let us consider the equations on the \( r_x r_x^- \) and \( r_y r_y^- \)-components. After separation of the variables we have

\[
\frac{1}{2}\left( \partial_x^2 \phi_x^2 \right) + q(\partial_x^2 A_x)^2 + (r_1 - 2)(\partial_x^2 F_{1x})^2 + r_2(\partial_x^2 F_{2x})^2 + s(\partial_x B_x)^2
\]

\[
- \partial_x \ln f_x(\partial_x F_{1x} - \partial_x A_x) = \frac{1}{2} h_2^2 \frac{f_x^{-1} H_x^2 (\partial_x e^{D_2x})^2}{}, \tag{96}
\]

\[
\frac{1}{2}(\partial_y^2 \phi_y)^2 + q(\partial_y^2 A_y)^2 + (r_2 - 2)(\partial_y^2 F_{2y})^2 + r_1(\partial_y^2 F_{1y})^2 + s(\partial_y B_y)^2
\]

\[
- \partial_y \ln f_y(\partial_y F_{1y} - \partial_y A_y) = \frac{1}{2} h_1^2 \frac{f_y^{-1} H_y^2 (\partial_y e^{D_1y})^2}{}, \tag{97}
\]

These equations give us another restrictions

\[
\frac{1}{2} c_{\phi 1}^2 + qc_{A1}^2 + (r_1 - 2)c_{F11}^2 + r_2 c_{F21}^2 + sc_{B1}^2 + \mu_2 (r_1 - 2)(c_{A1} - c_{F11}) = 0,
\]

\[
\frac{1}{2} c_{\phi 2}^2 + qc_{A2}^2 + (r_2 - 2)c_{F22}^2 + r_1 c_{F12}^2 + sc_{B2}^2 + \mu_1 (r_2 - 2)(c_{A2} - c_{F22}) = 0. \tag{98}
\]

Besides, we obtain the restriction on the constants \( c_{\phi 1} \) and \( c_{\phi 2} \), particularly, if \( \alpha_1 \neq \alpha_2 \) we have to put \( c_{\phi 1} = c_{\phi 2} = 0 \).

Finally we have to consider the \( r_x r_y \)-component of the Einstein equations

\[
\frac{1}{2} \partial_x \phi_x \partial_y \phi_y + q \partial_x A_x \partial_y A_y + 2 \partial_x F_{2x} \partial_y F_{1y} + (r_1 - 2) \partial_x F_{1x} \partial_y F_{1y} + (r_2 - 2) \partial_x F_{2x} \partial_y F_{2y} + s \partial_x B_x \partial_y B_y
\]

\[
- \frac{1}{2} \partial_x \ln f_x(\partial_y F_{1y} - \partial_y A_y) - \frac{1}{2} \partial_y \ln f_y(\partial_x F_{2x} - \partial_x A_x) + \frac{1}{4} \partial_x \ln f_x \partial_y \ln f_y = 0. \tag{99}
\]

Taking into account the equations (73), (82), (84), (89) and (87) after straightforward calculations we obtain that the equation (99) is satisfied if the constants are subjects of the additional relation

\[
\mu_1 \mu_2 \left( \frac{1}{2} r_1 r_2 - r_1 - r_2 + 2 \right) + \mu_2 (c_{A2} - c_{F12})(r_1 - 2) + \mu_1 (c_{A1} - c_{F21})(r_2 - 2)
\]

\[
+ c_{\phi 2} + 2qcA_1cA_2 + 2(r_1 - 2)c_{F11}c_{F12} + 2(r_2 - 2)c_{F21}c_{F22} + 4c_{F12}c_{F21} + 2sc_{B1}c_{B2} = 0. \tag{100}
\]

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