Generalized su(1,1) algebra and the construction of nonlinear coherent states for Pöschl-Teller potential

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ABSTRACT

We introduce a generalization structure of the su(1,1) algebra which depends on a function of one generator of the algebra, \( f(H) \). Following the same ideas developed to the generalized Heisenberg algebra (GHA) and to the generalized su(2), we show that a symmetry is present in the sequence of eigenvalues of one generator of the algebra. Then, we construct the Barut-Girardello coherent states associated with the generalized su(1,1) algebra for a particle in a Pöschl-Teller potential. Furthermore, we compare the time evolution of the uncertainty relation of the constructed coherent states with that of GHA coherent states. The generalized su(1,1) coherent states are very localized.

Keywords: Generalized Heisenberg algebra (GHA), su(1,1) algebra, Coherent state, Pöschl-Teller potential.

I. INTRODUCTION

Coherent states of the harmonic oscillator introduced first by Schrödinger in 1926 [1] are the only quantum states that minimize the Heisenberg uncertainty relation for the position and momentum operators [1, 2]. The dynamic of expectation values of position and momentum operators on these states has the same form as that of their classical counterpart, for this reason, they are called quasi-classical states[3]. More recently, in 1960s Glauber [2], Sudarshan [3] and Klauder [4] studied these states widely in quantum optics showing their important physical applicability. Then, coherent states have been constructed for physical systems other than the harmonic oscillator such as a free particle in a square well potential [5], hydrogen atom [6, 7] and Pöschl-Teller potential [8]. These states were called nonlinear coherent states. Additionally, several generalizations of coherent states have been introduced [5, 9–12] and different approaches for constructing them have been investigated such as Perelomov and Klauder-Gilmore approaches [13, 14]. Subsequently, coherent states and squeezed states can be algebraically constructed by using a unified approach proposed in [15].

In recent years, the generalized Heisenberg algebra (GHA) [16–18] has been constructed and more recently it was considered as an alternative method to construct new solvable models from old ones[19]. By knowing the spectrum of the physical system, the characteristic function of the GHA connecting between two successive dimensionless energy eigenvalues can be derived and the associated algebra can be constructed [16–18]. The GHA has been applied to several physical systems such as Pöschl-Teller potential [8, 19] and Morse potential [20]. Further, the nonlinear coherent states associated with GHA, called the GHA coherent states, have been constructed for these physical systems [5, 8, 21].

In this paper, we will introduce a generalization structure of the su(1,1) algebra and show that a hidden symmetry is present in the sequence of eigenvalues of one algebra generator similarly to the GHA and to the generalized su(2) algebra constructed in [22, 23]. We apply the introduced generalized su(1,1) algebra to the Pöschl-Teller potential and show that it can be applied to practically other physical systems exactly as the GHA. For a particle in a Pöschl-
Teller potential, we construct the coherent states associated with the generalized su(1,1) algebra and compare their behavior with that of GHA coherent states [8]. The generalized su(1,1) coherent states constructed here maintain an important localizability in the time compared with GHA coherent states.

This paper is structured as follows: in section (II) we recall the GHA and introduce the generalized su(1,1) algebra. Then, we construct the associated generalized harmonic oscillator ladder operators. In section (III) we apply the generalized su(1,1) algebra to the Pöschl-Teller potential and find the physical realizations of the algebra generators in terms of position and differential operators. Moreover, in section (IV) we shall construct the generalized su(1,1) Barut-Girardello nonlinear coherent states for the Pöschl-Teller potential and compare the behavior of the time evolution of the uncertainty relation of canonically conjugate operators for both nonlinear coherent states. Finally, our conclusions are given in (V).

II. GENERALIZED SU(1,1) ALGEBRA

A. A review on GHA

The GHA has been constructed and applied to several physical systems [16–18]. This algebra is described by three operators \( H, A \) and \( A^\dagger \) satisfying the following relations

\[
HA^\dagger = A^\dagger f(H),
\]

\[
AH = f(H)A,
\]

and

\[
[A, A^\dagger] = AA^\dagger - A^\dagger A = f(H) - H,
\]

where \( A = (A^\dagger)^\dagger \), \( H \) is the dimensionless Hamiltonian of the physical system under consideration and \( f(H) \) is an analytical function of \( H \), called the characteristic function of the algebra. A large class of Heisenberg algebras can be obtained by particular choices of the analytical function \( f \).

Particularly, if \( f(H) = H + 1 \) and \( H \) is the dimensionless Hamiltonian of the harmonic oscillator, the GHA defined in (1)-(3) recover the ordinary Heisenberg algebra spanned by \( H \) and the creation and annihilation operators of the harmonic oscillator [18]. The Casimir operator of the GHA is given by

\[
\Gamma = A^\dagger A - H = AA^\dagger - f(H).
\]

The irreducible representation of the GHA is given through the eigenvectors \( |n\rangle \) of the Hamiltonian \( H, H |n\rangle = \varepsilon_n |n\rangle \) such that

\[
\varepsilon_{n+1} = f(\varepsilon_n), \quad n = 0, 1, 2, \ldots,
\]

where \( \varepsilon_{n+1}, \varepsilon_n \) are two successive energy eigenvalues. Then, the Fock space representation of the GHA is given through the eigenvalue \( \varepsilon_0 \) corresponding to the ground state \( |0\rangle \) and an eigenvalue \( \varepsilon_n \) is the \( n \)-iterate of \( \varepsilon_0 \) under \( f \), i.e.,

\[
\varepsilon_n = f^n(\varepsilon_0).
\]

Assuming that \( A |0\rangle = 0 \) and by using (1) – (4), we can show that

\[
A^\dagger |n\rangle = N_n |n+1\rangle,
\]

\[
A |n\rangle = N_{n-1} |n-1\rangle,
\]

where

\[
N_n^2 = \varepsilon_{n+1} - \varepsilon_0.
\]

The operators \( A^\dagger, A \) are then the creation and the annihilation operators of GHA, respectively.

B. Generalized su(1,1) algebra

Let us now introduce an algebraic structure that generalizes the su(1,1) algebra and give the associated Fock space representation. Similarly to the GHA, we will show that the generalized su(1,1) algebra can be constructed once the spectrum of the physical system under consideration is known, i.e., once the characteristic function of the algebra is determined. Let \( H, J_+ \) and \( J_- \) be three operators obeying the following relations

\[
HJ_+ = J_+ f(H),
\]

\[
J_-H = f(H)J_-,
\]
and

\[ [J_+, J_-] = (H - f(H))(H + f(H) - 1), \tag{12} \]

where \( J_- = J_+^\dagger \), \( H \) is the dimensionless Hamiltonian of the system under consideration and \( f(H) \) is a given function of \( H \). In fact, \( H \) can be any hermitian operator.

For the particular case, \( f(H) = H + 1 \), the relations (10)-(12) become

\[
[H, J_+] = J_+, \tag{13}
\]

\[
[H, J_-] = -J_-, \tag{14}
\]

and

\[
[J_+, J_-] = -2H. \tag{15}
\]

These relations are the well-known \( su(1,1) \) algebra. Thus, the relations (10)-(12) contains the \( su(1,1) \) algebra by choosing the specific function \( f(H) = H + 1 \), as a particular case. For this reason we call it the generalized \( su(1,1) \) algebra. Let us consider the operator \( C \) given by

\[
C = J_+ J_- - H(H - 1) = J_- J_+ - f(H)(f(H) - 1). \tag{16}
\]

By using (10)-(12), we can show that \( C \) satisfies the following commutation relations

\[
[C, H] = [C, J_\pm] = 0, \tag{17}
\]

showing that \( C \) is the Casimir operator of the algebra (10)-(12). Now, let us provide the Fock space representation of the generalized \( su(1,1) \) algebra introduced in (10)-(12). Let \( |n\rangle \) be an eigenvector of \( H \), \( H |n\rangle = \varepsilon_n |n\rangle \) and let us assume that we have an infinite irreducible representation \( (n = 0, 1, \ldots) \). Applying (10) on \( |n\rangle \), we find that

\[
H(J_+ |n\rangle) = J_+ f(H) |n\rangle = f(\varepsilon_n)(J_+ |n\rangle). \tag{18}
\]

Then, \( J_+ |n\rangle \) is an eigenvector of \( H \) with eigenvalue \( f(\varepsilon_n) \). Let \( \varepsilon_{n+1} = f(\varepsilon_n) \), Thus \( J_+ |n\rangle \) is proportional to \( |n+1\rangle \) showing that \( J_+ \) is a raising operator.

Following the same procedure for \( J_- \), from (11) we have

\[
J_- H |n\rangle = f(H)J_- |n\rangle = \varepsilon_n(J_- |n\rangle), \tag{19}
\]

which implies that \( J_- |n\rangle \) is an eigenvector of \( H \) with eigenvalue \( \varepsilon_{n-1} \), implying that \( J_- |n\rangle \) is proportional to \( |n-1\rangle \).

Thus, \( J_- \) is an annihilation operator. By applying the Casimir (16) on \( |0\rangle \) and assuming that \( J_- |0\rangle = 0 \), we can show that

\[
J_+ |n\rangle = \sqrt{(\varepsilon_{n+1} - \varepsilon_0)(\varepsilon_{n+1} + \varepsilon_0 - 1)} |n + 1\rangle, \tag{20}
\]

\[
J_- |n\rangle = \sqrt{(\varepsilon_n - \varepsilon_0)(\varepsilon_n + \varepsilon_0 - 1)} |n - 1\rangle, \tag{21}
\]

for \( n = 0, 1, \ldots \). The operators \( J_+, J_- \) are then the ladder operators of the generalized \( su(1,1) \) algebra. We note that \( J_- |0\rangle = 0 \), which means that the vacuum state condition is verified. The first two relations defining the generalized \( su(1,1) \) algebra (10)-(11) have the same form as those of the GHA (1)-(2). Furthermore, the GHA and the algebra given in (10)-(12) have the same characteristic function relying two successive dimensionless energy eigenvalues \( \varepsilon_{n+1} \) and \( \varepsilon_n \) of the physical system under consideration. Moreover, since any quantum system having \( \varepsilon_{n+1} - f(\varepsilon_n) \) can be perfectly described by GHA, this system can also be described by the generalized \( su(1,1) \) algebra.

A deformation of the \( su(1,1) \) algebra by two analytical functions of an hermitian operator has been constructed in [24]. This deformed algebra contains the \( su(1,1) \) algebra as a particular case and the approach can be applied to particular problems depending on the functions of deformations which can be linear or nonlinear. Our approach differs from the one given in [24] since it can be applied to physical systems whose spectrum is known as the GHA approach. Another particularity of the introduced generalized \( su(1,1) \) algebra (10)-(12) compared with those given in [24] is that here, the corresponding function of deformation is the function relying between two eigenvalues of one generator of the algebra. Another nonlinear deformed \( su(1,1) \) algebras were applied to the Pöschl-Teller potential [25, 26]. However, these approaches are complicated and can not be extended to other physical systems. Here, we have shown that physical systems can be described algebraically in an easier way once the characteristic function is determined. In section (II), we will apply the generalized \( su(1,1) \) algebra introduced in this work to the Pöschl-Teller system and show how this system can be simply described by this algebra. Another motivation of introducing this algebra is the construction of another type of nonlinear coherent states for
physical systems. Let us note that the difference between
our generalized su(1,1) algebra and the deformed su(1,1)
algebras already existed in the literature [25, 26] is exactly
the difference between the GHA and the Generalized de-
formed oscillator algebras introduced in [27]. In the first,
the function of deformation appears in the spectrum such
that \( f(\varepsilon_n) = \varepsilon_{n+1} \). However, both nonlinear algebras
contains the harmonic oscillator algebra as a particular case. In
[28], Berrada et al. introduced a deformed su(1,1) algebras
and constructed the corresponding coherent states. How-
evertheless, the algebras introduced in [28] can not be applied to
physical system and the constructed coherent states are ab-
stract and not related to physical systems. Here, the func-
tion of deformation is related to systems and can be simply
derived once the spectrum is known. Hence, the generalized
su(1,1) constructed here can be applied to physical systems
having the property \( \varepsilon_{n+1} = f(\varepsilon_n) \) and this function is the
characteristic function of the algebra.

C. Generalized su(1,1) algebra and generalized
harmonic oscillator ladder operators

We consider a general system whose generalized su(1,1)
algebra is constructed and \( H \) is its Hamiltonian and let
\( N \) be the corresponding number particle operator \( N \), i.e.,
\( N |n\rangle = n |n\rangle \) for \( n = 0, 1, 2, \ldots \). As already provided for
GHA in [8], we define now an operator \( B \) associated with
the generalized su(1,1) algebra, which with its conjugate
and \( N \) satisfy the algebraic relations satisfied by the cre-
ation and annihilation operators \( a^\dagger \), \( a \) and \( N \) of the
harmonic oscillator. The operator \( B \) is defined by

\[
B = \sqrt{\frac{N + 1}{(f(H) - \varepsilon_0)(f(H) + \varepsilon_0 - 1)}} J^-, \quad (22)
\]

and the operator \( B^\dagger \) is the adjoin of \( B \). The operators
inside the square root in (22) are hermitian. Then, their
square root is well known. The action of \( B \) on a vector \( |n\rangle \)
is

\[
B |n\rangle = \sqrt{n} |n - 1\rangle, \quad n = 0, 1, 2, \ldots \quad (23)
\]

It is then simple to prove that

\[
[B, B^\dagger] = 1, \quad (24)
\]

\[
[N, B] = -B, \quad (25)
\]

\[
[N, B^\dagger] = B^\dagger, \quad (26)
\]

where \( 1 \) is the identity operator. Now, let us introduce
the canonically conjugate position-like and momentum-like
operators \( (X, P) \) associated with the generalized su(1,1)
algebra. They are given by

\[
X = \frac{L}{\sqrt{2}} (B + B^\dagger), \quad (27)
\]

\[
P = \frac{\hbar}{\sqrt{2L}} (B^\dagger - B). \quad (28)
\]

where \( L \) is a constant having the dimension of length. From
(27)-(28), we can show that

\[
[X, P] = i\hbar 1. \quad (29)
\]

III. GENERALIZED SU(1,1) ALGEBRA AND
PÖSCHL-TELLER POTENTIAL

The one-dimensional Pöschl-Teller (PT) is a Model hav-
ing important applications in physics [29, 30]. The associ-
ated Schrödinger equation is

\[
\mathcal{H}\psi_n(x) = \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_{PT}(x) \right) \psi_n(x) = E_n \psi_n(x), \quad (30)
\]

where

\[
V_{PT}(x) = \frac{\hbar^2 \pi^2}{2mL^2} \frac{\nu(\nu + 1)}{\sin^2(\pi x/L)}, \quad (31)
\]

such that \( \nu \geq 0 \), \( 0 < x < L \) and \( m \) is the mass of the
particle in the potential \( V_{PT}(x) \). For \( x \leq 0 \) and \( x \geq L \), the
potential is infinite.

The discrete spectrum of the system is

\[
E_n = \frac{\hbar^2 \pi^2}{2mL^2} (n + \nu + 1)^2, \quad n = 0, 1, 2, \ldots \quad (32)
\]

The associated energy eigenfunctions are given by

\[
\psi_n(x) = c_n(\nu) \sin^{\nu+1}(\pi x/L) C_n^{\nu+1}(\cos(\pi x/L)), \quad (33)
\]

where

\[
c_n(\nu) = \Gamma(\nu + 1) \frac{2^{\nu+1/2}}{\sqrt{L}} \sqrt{n!} \frac{\Gamma(n + 2\nu + \nu)}{\Gamma(n + 2\nu + 2)^2}, \quad (34)
\]
is the normalization constant, $\Gamma$ is the gamma function and $C_n^{\nu+1}$ are the Gegenbauer polynomials \[31\]. From (32), the spectrum of the dimensionless Hamiltonian $H = \frac{2mL^2}{\hbar^2 \pi^2}$ is

$$\varepsilon_n = (n + \nu + 1)^2, \quad n = 0, 1, 2 \ldots . \quad (35)$$

Thus, the characteristic function of the generalized su(1,1) algebra can be easily obtained. We have

$$\varepsilon_{n+1} = (n + \nu + 2)^2 = (\sqrt{\varepsilon_n} + 1)^2. \quad (36)$$

It follows that

$$f(x) = (\sqrt{x} + 1)^2. \quad (37)$$

Consequently, The generalized su(1,1) algebra associated with the physical system is

$$[H, J_+] = 2J_+\sqrt{\varepsilon} + J_+, \quad (38)$$

$$[H, J_-] = -2\sqrt{\varepsilon}J_- - J_-, \quad (39)$$

$$[J_-, J_+] = 2\sqrt{\varepsilon}(2H + 3\sqrt{\varepsilon} + 1). \quad (40)$$

Note that the difference between the GHA [8] and the introduced generalized su(1,1) algebra of the Pöschl-Teller potential is the commutator between the creation and the annihilation operators given in (40).

The Fock space representation of the generalized su(1,1) can be obtained by considering an eigenvector $|n\rangle$. From (20) -(21) and (35), the action of $H$, $J_+$ and $J_-$ on an eigenvector $|n\rangle$ can be given by

$$H |n\rangle = (n + \nu + 1)^2 |n\rangle, \quad n = 0, 1, 2, \ldots, \quad (41)$$

$$J_+ |n\rangle = \sqrt{(n+1)(n+2+\nu)}((n+2+\nu)^2 + (\nu+1)^2 - 1) |n+1\rangle, \quad (42)$$

$$J_- |n\rangle = \sqrt{n(n+2+\nu)}((n+1+\nu)^2 + (\nu+1)^2 - 1) |n-1\rangle. \quad (43)$$

Note that $J_- |0\rangle = 0$. The physical realization of the generalized su(1,1) generators can be easily constructed. By using (41)-(43) and (A2)-(A3), the algebra generators in terms of the position and differential operators can be given by

$$H = -\frac{L^2}{\pi^2} \frac{d^2}{dx^2} + \frac{\nu(\nu+1)}{\sin^2(\pi x/L)}, \quad (44)$$

$$J_+ = \left\{ \frac{L}{\pi} \sin(\frac{\pi x}{L}) \frac{d}{dx} + \cos(\frac{\pi x}{L}) (N + \nu + 1) \right\} k(N), \quad (45)$$

and

$$J_- = k(N) \left\{ -\frac{L}{\pi} \frac{d}{dx} \sin(\frac{\pi x}{L}) + (N + \nu + 1) \cos(\frac{\pi x}{L}) \right\}, \quad (46)$$

where

$$k(N) = \sqrt{\frac{(N+\nu+2)(N+2\nu+3)((N+2+\nu)^2 + (\nu+1)^2 - 1)}{(N+\nu+1)(N+2\nu+2)}}. \quad (47)$$

By using (45)-(46) and (A2)-(A3), we can easily show that $J_+ \psi_n(x) = \sqrt{(\varepsilon_{n+1} - \varepsilon_0)(\varepsilon_{n+1} + \varepsilon_0 - 1)} \psi_{n+1}(x)$ and
\[ J_\pm \psi_n(x) = \sqrt{(\varepsilon_n - \varepsilon_0)(\varepsilon_n + \varepsilon_0 - 1)} \psi_{n \pm 1}(x) \] where \( \varepsilon_n \) is given in (35). This shows how the physical system can be described algebraically by the generalized \( \text{su}(1,1) \) and showing the correspondence between the algebraic approach and the wave function approach. For the Pöschl-Teller potential, the operators \( B \) and \( B^\dagger \) given in (22) and satisfying the harmonic oscillator commutation relations are given by

\[ B = \sqrt{\frac{1}{(N + 2\nu + 3)((N + 2 + \nu)^2 + (\nu + 1)^2 - 1)}} J_- , \tag{48} \]

\[ B^\dagger = J_+ \sqrt{\frac{1}{(N + 2\nu + 3)((N + 2 + \nu)^2 + (\nu + 1)^2 - 1)}} . \tag{49} \]

**IV. COHERENT STATES FOR THE PÖSCHL-TELLER POTENTIAL WITH \( \nu = 0 \)**

**A. Generalized \( \text{su}(1,1) \) nonlinear Coherent states**

In this section, we aim to construct the Barut-Girardello coherent states [32] for the Pöschl-Teller potential associated with the generalized \( \text{su}(1,1) \) algebra. Let us first recall the minimal set of conditions necessary to construct a Klauder’s coherent state. A state \( |z\rangle \) is a Klauder’s coherent state if and only if it satisfies the following conditions

i) normalization \( \langle z | z \rangle = 1 \),

ii) continuity in the label,

\[ ||z\rangle - |z'\rangle| \rightarrow 0 \quad \text{when} \quad |z - z'| \rightarrow 0, \tag{50} \]

iii) completeness or resolution of the identity

\[ \int \int d^2z W(|z|^2) |z\rangle \langle z| = \sum_{n=0}^{\infty} |n\rangle \langle n| = 1, \tag{51} \]

where \( W(|z|^2) \) is a positive function called the weight function. For the sake of simplification, we will take \( \nu = 0 \). Thus, the spectrum (35) now has the same form as a free particle in a square well potential. In the literature, several nonlinear coherent states for Pöschl-Teller have been constructed [8, 26, 33–38]. In [8], the nonlinear coherent state \( |z\rangle_{NL1} \) associated with GHA is defined by

\[ A |z\rangle_{NL1} = z |z\rangle_{NL1} . \tag{52} \]

Remembering that \( N_n = \varepsilon_{n+1} - \varepsilon_0 \). For the Pöschl-Teller potential with \( \nu = 0 \), \( N_n = \sqrt{(n + 1)(n + 3)} \) and \( |z\rangle_{NL1} \) reads

\[ |z\rangle_{NL1} = N_1(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{N_n}!} |n\rangle , \tag{53} \]

where \( N_{n-1} = N_0 N_1 \ldots N_{n-1} \) and by definition \( N_{-1} = 1 \) and \( z \) is a complex number. This vector satisfies the normalization, continuity in the label and resolution of unity [8, 14, 39]. The normalization factor given in (53) is

\[ (N_1(|z|^2))^2 = \frac{|z|^2}{2I_2(2|z|)} , \tag{54} \]

where \( I_2 \) is the modified Bessel function of the second kind [40].

The linear coherent state associated with Pöschl-Teller potential has been constructed in [8]. It is given by

\[ |z\rangle_L = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n}!} |n\rangle , \tag{55} \]

where \( \langle x|n\rangle = \psi_n(x) \) is the wave function (33) of the Pöschl-Teller system. The state (55) is defined as an eigen-
state of the generalized harmonic oscillator annihilation operator [8] and can be constructed, as a particular case, by using the approach provided in [15].

Now, we construct the generalized su(1,1) algebra nonlinear coherent state. Let us consider a vector \( |z\rangle_{NL2} \) satisfying [5]

\[
J_- |z\rangle_{NL2} = z |z\rangle_{NL2}
\]

Let \( \mathcal{N}_n = \sqrt{(\varepsilon_{n+1} - \varepsilon_0)(\varepsilon_{n+1} + \varepsilon_0 - 1)} \), the vector \( |z\rangle_{NL2} \) can be given by

\[
|z\rangle_{NL2} = N_2(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\mathcal{N}_n!} |n\rangle.
\]

where \( N_2(|z|^2) \) is a normalization factor, \( \mathcal{N}_{n-1}! = \mathcal{N}_0 \mathcal{N}_1 \ldots \mathcal{N}_{n-1} \) and \( \mathcal{N}_{-1}! := 0 \).

By using (43) and (57), for the Pöschl-Teller potential (with \( \nu = 0 \)), \( |z\rangle_{NL2} \) can be written as

\[
|z\rangle_{NL2} = N_2(|z|^2) \sum_{n=0}^{\infty} \frac{\sqrt{2z^n}}{(n+1)! \sqrt{n!(n+2)!}} |n\rangle.
\]

We require that \( \langle z | z \rangle_{NL2} = 1 \). Then, the normalization factor \( N_2(|z|^2) \) reads

\[
N_2(|z|^2) = \frac{1}{\sqrt{\mathcal{F}_3(2, 2, 3; |z|^2)}}
\]

where \( \mathcal{F}_3(2, 2, 3; |z|^2) \) is the generalized hypergeometric function. The normalization function \( N_2(|z|^2) \) is defined for all \( |z| \in [0, \infty[ \). Thus, \( z \) in (58) belongs to the whole complex plane. The continuity condition (50) is simply verified.

We give now the appropriate weight function satisfying the completeness relation (51). Let \( z = re^{i\theta} \) where \( 0 \leq r < \infty \) and \( 0 \leq \theta < 2\pi \). Substituting (58) in (51), it follows that

\[
\sum_{n,n'=0}^{\infty} \left\{ \frac{1}{2(n+1)!(n'+1)!} \right\} \int_0^{2\pi} W(r^2) r N_2^2(r^2) r^{n+n'} d\theta \int_0^{2\pi} e^{i\theta-n'} d\theta \sum_{n=0}^{\infty} \left\{ \frac{2\pi}{((n+1)!)^2 n!(n+2)!} \right\} \int_0^{\infty} x^n W(x) N_2^2(x) dx \}|n\rangle \langle n| = 1, \quad (x = r^2),
\]

|n\rangle \langle n| = 1, \quad (x = r^2),

where we have used the fact that \( \int_0^{2\pi} e^{i(n-n')\theta} d\theta = 2\pi \delta_{n,n'} \).

Remembering that \( \sum_{n=0}^{\infty} |n\rangle \langle n| = 1 \), the completeness condition (60) reduces to

\[
\int_0^{\infty} x^n g(x) = \frac{(n+1)!^2 n!(n+2)!}{2},
\]

where \( g(x) = \pi N_2^2(x) W(x) \). Now, we apply the methods recalled in (B) to solve the moment problem (61). We have

\[
2 \int_0^{\infty} dx x^n K_0(2\sqrt{x}) = ((n+1)!)^2,
\]

and

\[
2 \int_0^{\infty} dx x^n K_2(2\sqrt{x}) = n!(n+2)!,
\]

where \( K_n(x) \) is the modified Bessel function of the second kind. Thus, By using (B5)-(B6), the moment problem (61) can be solved by

\[
g(x) = \int_0^{\infty} 2x K_0(2\sqrt{x/t}) K_2(2\sqrt{t}) \frac{dt}{t},
\]

Finally, the weight function satisfying the completeness condition (51), in this case, is

\[
W(x) = \frac{1}{\pi N_2^2(x)} \int_0^{\infty} 2x K_0(2\sqrt{x/t}) K_2(2\sqrt{t}) \frac{dt}{t}.
\]
The behavior of the \( W(x) \) is shown in figure (1). We can see that \( W(x) \) is a positive function. Consequently, the vector \( |z\rangle_{NL2} \) is a Klauder coherent state. Hence, two kinds of nonlinear coherent states can be constructed for the Pöschl-Teller potential, one is associated with GHA and the second is associated with the generalized \( su(1,1) \) algebra. Note that the coherent states constructed here differs from the ones constructed in [36–38] where the factorization of the Hamiltonian of the Pöschl-Teller potential in terms of the ladder operators is needed. Here, we have constructed the generalized \( su(1,1) \) nonlinear coherent states for the Pöschl-Teller potential in an algebraic manner by using the methods followed in [5, 8].

![Graph](image)

**FIG. 1.** The behavior of the weight function for the Pöschl-Teller potential in terms of \( x = |z|^2 \).

### B. Time evolution of nonlinear coherent states of the Pöschl-Teller potential

We compare now the behaviors of the time evolution of the uncertainty relation for both nonlinear coherent states.

For simplification of the calculations, from now on, we will consider that the Fock space starts from \( n = 1 \) instead of \( n = 0 \). Then, the GHA nonlinear coherent state (53) can be now written as

\[
|z\rangle_{NL1} = N_1(|z|^2) \sum_{n=1}^{\infty} \frac{z^{n-1}}{\prod_{i=2}^{n}(i^2 - 1)} |n\rangle, \tag{66}
\]

while the generalized \( su(1,1) \) nonlinear coherent state introduced in (58) becomes

\[
|z\rangle_{NL2} = N_2(|z|^2) \sum_{n=1}^{\infty} \frac{\sqrt{2}z^{n-1}}{n!\sqrt{(n-1)!(n+1)!}} |n\rangle. \tag{67}
\]

The time evolution of a given state can be obtained by the action of the evolution operator

\[
U(t) = e^{-iHt/\hbar}. \tag{68}
\]

Let \( \frac{\hbar^2\pi^2}{2m\ell^2} = 1 \). This means that we choose a particular Pöschl-Teller system. Then, at \( t \) we have

\[
|z,t\rangle_{NL1} = N_1(|z|^2) \sum_{n=1}^{\infty} \frac{z^{n-1}e^{-in^2t/\hbar}}{\prod_{i=2}^{n}(i^2 - 1)} |n\rangle, \tag{69}
\]

and

\[
|z,t\rangle_{NL2} = N_2(|z|^2) \sum_{n=1}^{\infty} \frac{\sqrt{2}z^{n-1}e^{-in^2t/\hbar}}{n!\sqrt{(n-1)!(n+1)!}} |n\rangle. \tag{70}
\]

The time evolution of \( X(t) \), \( P(t) \), \( X^2(t) \) and \( P^2(t) \) on the state (69) can be easily obtained. They were calculated in [8] and can be written as

\[
\langle X(t) \rangle_{NL1} = \frac{L}{\sqrt{2}} \langle z,t|B + B^\dagger|z,t\rangle_{NL1} = \frac{\sqrt{2}\ell r^2}{I_2(2r)} \sum_{n=1}^{\infty} \frac{z^{2n-1}\cos((2n+1)t/\hbar - \theta)}{\sqrt{n + 2\Gamma(\eta)\Gamma(n + 2)}}, \tag{71}
\]
\begin{equation}
\langle P(t) \rangle_{NL1} = \frac{i\hbar}{\sqrt{2L}} \langle z,t|B^\dagger - B|z,t \rangle_{NL1} \\
\quad = -\sqrt{2\hbar^2 I_2(2r)} \sum_{n=1}^{\infty} \frac{r^{2n-1}\sin((2n+1)t/\hbar - \theta)}{\sqrt{n+2}\Gamma(n+2)}.
\end{equation}

\begin{equation}
\langle X^2(t) \rangle_{NL1} = \frac{L^2}{2} \langle z,t|B^2 + (B^\dagger)^2 + B^\dagger B + BB^\dagger|z,t \rangle_{NL1} \\
\quad = \frac{L^2}{\sqrt{2L}} \sum_{n=1}^{\infty} \frac{r^{2n}\sqrt{n+1}\cos((4n+4)t/\hbar - 2\theta)}{\sqrt{n+2}(n^2+4n+3)\Gamma(n+1)\Gamma(n+2)} + \frac{L^2}{2},
\end{equation}

\begin{equation}
\langle P^2(t) \rangle_{NL1} = \frac{\hbar^2}{2L^2} \langle z,t|B^2 + (B^\dagger)^2 - B^\dagger B - BB^\dagger|z,t \rangle_{NL1} \\
\quad = \frac{\hbar^2}{L^2} \sum_{n=1}^{\infty} \frac{r^{2n}\sqrt{n+1}\cos((4n+4)t/\hbar - 2\theta)}{\sqrt{n+2}(n^2+4n+3)\Gamma(n+1)\Gamma(n+2)} - \frac{\hbar^2}{2L^2}.
\end{equation}

We repeat the same calculations for the nonlinear coherent state (70), we find that
\begin{equation}
\langle X(t) \rangle_{NL2} = 2\sqrt{2L}N_z^2(r^2) \sum_{n=1}^{\infty} \frac{r^{2n-1}\sqrt{n}\cos((2n+1)t/\hbar - \theta)}{n!(n+1)!\sqrt{(n-1)!n!(n+1)!(n+2)!}},
\end{equation}
\begin{equation}
\langle P(t) \rangle_{NL2} = \frac{2\sqrt{2L}hN_z^2(r^2)}{L} \sum_{n=1}^{\infty} \frac{r^{2n-1}\sqrt{n}\sin((2n+1)t/\hbar - \theta)}{n!(n+1)!\sqrt{(n-1)!n!(n+1)!(n+2)!}},
\end{equation}
\begin{equation}
\langle X^2(t) \rangle_{NL2} = 2L^2N_z^2(r^2) \left( \sum_{n=1}^{\infty} \frac{r^{2n}\sqrt{n(n+1)}\cos((4n+4)t/\hbar - 2\theta)}{n!(n+2)!\sqrt{(n-1)!(n+3)!}} + \sum_{n=1}^{\infty} \frac{(n-1)r^{2n-2}}{n!(n-1)!(n+1)!} \right) + \frac{L^2}{2},
\end{equation}
\begin{equation}
\langle P^2(t) \rangle_{NL2} = \frac{-2L^2hN_z^2(r^2)}{L^2} \left( \sum_{n=1}^{\infty} \frac{r^{2n}\sqrt{n(n+1)}\cos((4n+4)t/\hbar - 2\theta)}{n!(n+2)!\sqrt{(n-1)!(n+3)!}} - \sum_{n=1}^{\infty} \frac{(n-1)r^{2n-2}}{n!(n-1)!(n+1)!} \right) + \frac{h^2}{2L^2}.
\end{equation}

The time evolution of the uncertainty relation is given by
\begin{equation}
\Delta X(t) \Delta P(t) = \sqrt{\langle X^2(t) \rangle - \langle X(t) \rangle^2} \times \sqrt{\langle P^2(t) \rangle - \langle P(t) \rangle^2}.
\end{equation}

It can be easily obtained for both nonlinear coherent states from (71)-(78).

In [8], the time evolution (79) of linear coherent states and GHA nonlinear coherent states were calculated and it was shown that the GHA nonlinear ones are the more localized. The time evolution of coherent states has been studied in several works [8, 41]. Particularly, in [41] the authors studied the time evolution and the phase-space dynamics of su(2) Morse potential coherent states by using the Wigner function.

Let us now compare the behavior of the time evolution of the uncertainty relation of the generalized su(1,1) coherent states and that of GHA ones. In Figures (2)-(4) the time evolution (79) of both nonlinear coherent states are shown for \( r = 0.1, r = 5 \) and \( r = 10; \theta = 0 \) and \( \hbar = 1 \).

Analyzing these figures, we can see that for both nonlinear coherent states, the uncertainty relations oscillates between 0.5\( \hbar \) and maximum values. We see also that the uncertainties of both coherent states are equal in few points and
that the uncertainty of generalized su(1,1) coherent states is always smaller than that of GHA coherent states. We can conclude that our nonlinear coherent states are more localized than GHA ones. We can see also in figures (2)-(4) that the maximum uncertainty for both nonlinear coherent states increases with increasing \( r \) which indicates that the uncertainty approaches to 0.5\( \hbar \) for small values of \( r \). Since the generalized su(1,1) nonlinear coherent states for the Pöschl-Teller potential are more localized. Then, they take to wave packet more closer to the classical trajectory than that of GHA nonlinear coheren states.

C. Phase-space trajectories for \( X \) and \( P \)

We now analyze the phase space trajectories of \((X, P/\hbar)\) of generalized su(1,1) coherent states. In the figures (5)-(6), we show \((X, P/\hbar)\) for \( r = 0.01 \) and \( r = 0.5 \), respectively. In these two graphs, we have taken \( L = \hbar = 1 \). We can see that the form of the phase space trajectory is an ellipse for \( r = 0.01 \) and \( r = 0.5 \). The phase space trajectory of GHA nonlinear coherent states were obtained in [8]. It is an ellipse only for very small values of \( r \).

V. FINAL COMMENT

In this paper, we have introduced a generalization structure of the su(1,1) algebra which depends on a given function of one hermitian operator, called the characteristic
examined the Pöschl-Teller potential and have expressed the generators of the constructed generalized su(1,1) algebra in terms of position and differential operators. Then, we have constructed the pair of canonically conjugate operators satisfying the bosonic relations of the harmonic oscillator algebra. Furthermore, we have constructed the Barut-Girardello coherent states for Pöschl-Teller potential and compare the behavior of the time evolution of the uncertainty relation of the constructed nonlinear coherent states and that of GHA coherent states. The generalized su(1,1) coherent states are more localized. Thence, we have seen that the uncertainty almost equal $\hbar/2$ for small values of radius of convergence of the constructed nonlinear coherent states. An interesting result is that uncertainty oscillates between a minimum value (which is $\hbar/2$) and maximum values close to $\hbar/2$ compared with GHA nonlinear coherent states whose uncertainty oscillates between $\hbar/2$ and maximum values which are so large than $\hbar/2$.

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**Appendix A: Gegenbauer recurrence formulae**

The energy eigenfunctions associated with Schrödinger equation (30) are given by

$$\psi_n(x) = c_n(\nu) \sin^{\nu+1}(\pi x/L)C_n^{\nu+1}(\cos(\pi x/L)), \quad \text{where} \quad c_n(\nu) = \Gamma(\nu + 1) \frac{2^{\nu + 1/2}}{\sqrt{L}} \frac{n! (n + \nu + 1)}{\Gamma(n + 2\nu + 2)}, \quad (A1)$$

and $C_n^{\nu+1}(\cos(\pi x/L))$ are the Gegenbauer polynomials.

By using the recurrence formulae satisfied by Gegenbauer polynomials [31], one can show that

$$\cos(\pi x/L)\psi_n(x) = \frac{1}{2\sqrt{n + \nu + 1}} \left( \sqrt{\frac{n(n + 2\nu + 1)}{n + \nu}} \psi_{n-1}(x) + \sqrt{\frac{(n + 1)(n + 2\nu + 2)}{n + \nu + 2}} \psi_{n+1}(x) \right) \quad (A2)$$

and that

$$-\frac{L}{\pi} \sin(\pi x/L) \frac{d\psi_n(x)}{dx} = \frac{\sqrt{n + \nu + 1}}{2} \left( \sqrt{\frac{n(n + 2\nu + 1)}{n + \nu}} \psi_{n-1}(x) - \sqrt{\frac{(n + 1)(n + 2\nu + 2)}{n + \nu + 2}} \psi_{n+1}(x) \right) \quad (A3)$$
Appendix B: Mellin transform

Let \( l(x) \) be an analytical function. The Mellin transform \([31, 42, 43]\), \( l^*(s) \) of the function \( l(x) \) is defined by

\[
l^*(s) := \int_0^\infty l(x)x^{s-1}dx, \tag{B1}
\]

where \( s \) is a complex variable. The function \( l(x) \) is the inverse Mellin transform of the function \( l(s) \) and it reads as

\[
l(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} l^*(s)x^{-s}ds, \tag{B2}
\]

where \( c \) is a complex number and \( i^2 = -1 \). Let \( l(x) \) and \( h(x) \) be two functions, the Mellin convolution reads

\[
x^a \int_0^\infty t^bh\left(\frac{x}{t}\right)h(t)dt = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} l^*(s+a)h^*(s+a+b+1)x^{-s}ds, \tag{B3}
\]

where \( a, b \) are arbitrary complexes. From (B3), it follows that if

\[
l_n! = \int_0^\infty x^nl(x)dx, \quad \text{and} \quad h_n! = \int_0^\infty x^nh(x)dx, \tag{B4}
\]

where \( l_n! = l_n!l_{n-1} \ldots l_0 \) and \( h_n! = h_n!h_{n-1} \ldots h_0 \). Then, the weight function defined by

\[
a(x) = \int_0^\infty l(x/t)h(t)\frac{dt}{t}, \tag{B5}
\]

is the solution of the moment problem

\[
l_n!h_n! = \int_0^\infty x^n a(x)dx. \tag{B6}
\]

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