Matrix Recovery from Rank-One Projection Measurements via Nonconvex Minimization

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**Abstract.** In this paper, we consider the matrix recovery from rank-one projection measurements proposed in [Cai and Zhang, Ann. Statist., 43(2015), 102-138], via nonconvex minimization. We establish a sufficient identifiability condition, which can guarantee the exact recovery of low-rank matrix via Schatten-\(p\) minimization \(\min_X \|X\|_{Sp}^p\) for \(0 < p < 1\) under affine constraint, and stable recovery of low-rank matrix under \(\ell_q\) constraint and Dantzig selector constraint. Our condition is also sufficient to guarantee low-rank matrix recovery via least \(q\) minimization \(\min_X \|A(X) - b\|_q^q\) for \(0 < q \leq 1\). And we also extend our result to Gaussian design distribution, and show that any matrix can be stably recovered for rank-one projection from Gaussian distributions via least \(1\) minimization with high probability.

**Key Words and Phrases.** Low-rank matrix recovery, Rank-one projection, \(\ell_q\)-Restricted uniform boundedness, Schatten-\(p\) minimization, Least \(q\) minimization.

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1 Introduction

As is well known to us, a closely related problem to compressed sensing, which was initiated by Candès, Romberg and Tao’s seminal works [16, 17] and Donoho’s groundbreaking work [28], is the low-rank matrix recovery. It aims to recover an unknown low-rank matrix based on its affine transformation

\[
\begin{equation}
\label{eq:11}
b = A(X) + z,
\end{equation}
\]

where \(X \in \mathbb{R}^{m \times n}\) is the decision variable and the linear map \(A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^L\), \(z \in \mathbb{R}^L\) is a measurement error and \(b \in \mathbb{R}^L\) is measurements. The linear map \(A\) can be equivalently specified by \(L\) \(m \times n\) measurement matrices \(A_1, \ldots, A_L\) with

\[
\begin{equation}
\label{eq:12}
[A(X)]_j = \langle A_j, X \rangle,
\end{equation}
\]

where the inner product of two matrices of the same dimensions is defined as \(\langle X, Y \rangle = \sum_{i,j} X_{ij} Y_{ij} = \text{trace}(X^T Y)\). Low-rank matrices arise in an incredibly wide range of settings

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throughout science and applied mathematics. To name just a few examples, we commonly encounter low-rank matrices in contexts as varied as: ensembles of signals [27, 1], system identification [39], adjacency matrices [38], distance matrices [6, 7, 43], machine learning [3, 40, 30], and quantum state tomography [32, 2].

Let \( \|X\|_{S_p} := \|\sigma(X)\|_p = \left( \sum_{j=1}^{\min\{m,n\}} |\sigma_j|^p \right)^{1/p} \) denote the Schatten \( p \)-norm of the matrix \( X \), where \( \sigma(X) = (\sigma_1, \ldots, \sigma_{\min\{m,n\}}) \) is the vector of singular values of the matrix \( X \) and \( \sigma_j \) denote the \( j \)-th largest singular value of \( X \). We should point out that \( \|X\|_{S_1} = \|X\|_* \) the nuclear norm, \( \|X\|_{S_2} = \|X\|_F \) the Frobenius norm and \( \|X\|_{S_{\infty}} = \|X\| \) the operator norm. In 2010, Recht, Fazel and Parrilo [42] generalized the restricted isometry property (RIP) in [19, 28] from vectors to matrices and showed that if certain restricted isometry property holds for the linear transformation \( A \), then the low-rank solution can be recovered by solving the the nuclear norm minimization problem

\[
\min_X \|X\|_* \quad \text{subject to} \quad A(X) = b.
\]

Later, the low-rank matrix recovery problem has been studied by many scholars, readers can refer to [26, 11, 12, 52] under matrix restricted isometry property, and [31, 33] under rank null space property.

In this paper, we consider one matrix recovery model with additional structural assumption—the rank-one projection (ROP), which introduced by Cai and Zhang [12]. Under the ROP model, we observe

\[
b_j = (\beta^j)^T X \gamma^j + z_j, \quad j = 1, \ldots, L,
\]

where \( \beta^j \) and \( \gamma^j \) are random vectors with entries independently drawn from some distribution \( P \), \( z_j \) are random errors. In view of the linear map in \((1.2)\), it can be rewrite as

\[
b_j = [A(X)]_j + z_j = \langle \beta^j(\gamma^j)^T, X \rangle + z_j, \quad j = 1, \ldots, L,
\]

i.e. \( A_j = \beta^j(\gamma^j)^T \) for \( j = 1, \ldots, L \). For this model, Cai and Zhang proposed a constrained nuclear norm minimization method, which can be stated as follows

\[
\min_X \|X\|_{S_1} \quad \text{subject to} \quad A(X) = b.
\]

In the noiseless case, they took \( B = \{0\} \), i.e.,

\[
\min_X \|X\|_{S_1} \quad \text{subject to} \quad A(X) = b.
\]

And in the noisy case, they took \( B = B^{\ell_1}(\eta_1) \cap B^{D_2}(\eta_2) \), i.e.,

\[
\min_X \|X\|_{S_1} \quad \text{subject to} \quad \|b - A(X)\|_1 / L \leq \eta_1, \quad \|A^*(b - A(X))\|_{S_{\infty}} \leq \eta_2.
\]

However, the matrix RIP framework is not well suited for the ROP model and would lead to suboptimal results. Cai and Zhang [12] introduced the restricted uniform boundedness (RUB) condition (see Definition 2.1), which can guarantee the exact recovery of low-rank matrices in the noiseless case and stable recovery in the noisy case through the
constrained nuclear norm minimization. They also showed that the RUB condition are satisfied by sub-Gaussian random linear maps with high probability.

The ROP model can be further simplified by taking $\beta_j = \gamma_j$ if the low-rank matrix $X$ is known to be symmetric. This can be found in many problem, for example, low-dimensional Euclidean embedding \cite{46, 42}, phase retrieval \cite{13, 21, 14, 36} and covariance matrix estimation \cite{9, 10, 25}. In this case, the ROP design can be simplified to symmetric rank-one projections (SROP)

$$b_j = \langle \beta_j (\beta_j)^T, X \rangle + z_j, \quad j = 1, \ldots, L. \quad (1.8)$$

Unfortunately, the original sampling operator $A$ does not satisfy RUB. This occurs primarily because each measurement matrix $A_i$ has non-zero mean, which biases the output measurements. In order to get rid of this undesired bias effect, Cai and Zhang \cite{12} (see also \cite{25}) introduced a set of “debiased” auxiliary measurement matrices as follows

$$\tilde{A}_j = A_{2j-1} - A_{2j}, \quad j = 1, \ldots, \left\lfloor \frac{L}{2} \right\rfloor.$$ (1.9)

By this notation, we can define a linear map $\tilde{A} : S^m \rightarrow \mathbb{R}^{\lfloor L/2 \rfloor}$ by

$$\tilde{b}_j = [\tilde{A}(X)]_j + \tilde{z}_j = \langle \tilde{A}_j, X \rangle + \tilde{z}_j,$$

where $\tilde{b}_j = b_{2j-1} - b_{2j}$, $\tilde{z}_j = z_{2j-1} - z_{2j}$ and $S^m$ denotes the set of all $m \times m$ symmetric matrices. Owing to $A(X) = b$ implying $\tilde{A}(X) = b$, they still considered (1.6) in noiseless case. And note that $\|\tilde{b} - \tilde{A}(X)\|_1 \leq \|b - A(X)\|_1$, therefore they consider

$$\min_X \|X\|_{S_1} \text{ subject to } \|b - A(X)\|_1 / L \leq \eta_1, \quad \|\tilde{A}^*(\tilde{b} - \tilde{A}(X))\|_{S_{\infty}} \leq \eta_2. \quad (1.10)$$

In this paper, we introduce $\ell_p$-RUB condition, which is a natural generalization of RUB condition. And we consider Schatten-$p$ minimization

$$\min_X \|X\|_{S_p}^p \text{ subject to } b - A(X) \in B, \quad (1.11)$$

for ROP model (1.4), where $B$ is a set determined by the noise structure and $0 < p \leq 1$. We consider two types of bounded noises $\cite{8}$. One is $l_q$ bounded noises $\cite{29}$, i.e.,

$$B^{l_q}(\eta_1) = \{z : \|z\|_q / L \leq \eta_1\} \quad (1.12)$$

for some constant $\eta_1$; and the other is motivated by Dantzig Selector procedure $\cite{20}$, where

$$B^{DS}(\eta_2) = \{z : \|A^*(z)\|_{S_{\infty}} \leq \eta_2\} \quad (1.13)$$

for some constant $\eta_2$. In particular, $B = \{0\}$ in noiseless case.

And if $A$ is a symmetric rank-one projection, we use

$$\min_X \|X\|_{S_p}^p \text{ subject to } \tilde{b} - \tilde{A}(X) \in B. \quad (1.14)$$

instead of (1.11).
In the noisy case, we also consider the simpler least $q$-minimization problem
\begin{equation}
\min_{X \in \mathbb{R}^{m \times n}} \| A(X) - b \|_q^q
\end{equation}
which may work equally well or even better than Schatten $p$ minimization problem (1.11) in terms of recovery under certain natural conditions. Apart from simplicity and computational efficiency (see [35, 45] for $q = 2$, [37] for $q = 1$), it has the additional advantage that no estimate $\eta$ of the noise level is required. It was proposed by Candès and Li [14] for $q = 1$, which is used to solve PhaseLift problem [13, 21]. They constructed the dual certificate condition to solve this problem. Later, Kabanava, Kueng, Ravuhut et.al. [33] considered it for $q \geq 1$ for density operators under robust rank null space property. We should point out that when $q = 1$, least $q$-minimization problem in vector case is just the least absolute deviation introduced in [5]. Moreover works about the least absolute deviation, readers can see [41, 48, 49, 47].

In this paper, we consider the recovery of the matrix $X \in \mathbb{R}^{m \times n}$ possessing some density, i.e. $\left( \text{tr} \left( (X^T X)^{p/2} \right) \right)^{1/p} = 1$, via nonconvex least $q$ minimization. And our method can be written as
\begin{equation}
\min_{X \in \mathbb{R}^{m \times n}} \| A(X) - b \|_q^q \quad \text{subject to} \quad \left( \text{tr} \left( (X^T X)^{p/2} \right) \right)^{1/p} = 1,
\end{equation}
where $0 < p \leq q \leq 1$. And if $A$ is a symmetric rank-one projection, we use
\begin{equation}
\min_{X \in \mathbb{R}^m} \| \tilde{A}(X) - \tilde{b} \|_q^q \quad \text{subject to} \quad \left( \text{tr} \left( X^p \right) \right)^{1/p} = 1
\end{equation}
instead of (1.16).

The contribution of the present work can be summarized as follows.

1. We introduce the $\ell_q$-RUB for $0 < q \leq 1$, which includes the RUB condition in [12].

2. A uniform and stable $\ell_q$-RUB condition for low-rank matrices' recovery, via Schatten-$p$ minimization ($0 < p < 1$), is given for ROP model. And our condition is also sufficient for SROP model.

3. We obtain that the robust rank null space property of order $r$ can be deduced from the $\ell_q$ RUB of order $(k + 1)r$ for some $k > 1$.

4. A stable $\ell_q$-RUB condition for low-rank matrices’ recovery, via least-$q$ minimization ($0 < q \leq 1$), is also given for ROP model.

5. With high probability, ROP with $L \geq Cr(m + n)$ random projections from Gaussian distribution is sufficient to ensure stable and robust recovery of all rank-$r$ matrices via least absolute deviation estimator.

Throughout the article, we use the following basic notations. We denote $\mathbb{Z}_+$ by positive integer set. For any random variable $x$, we use $\mathbb{E}x$ denote the expectation of $x$. For any vector $x \in \mathbb{R}^L$ and index set $S \subseteq \{1, \ldots, L\}$, let $x_S$ be the vector equal to $x$ on $S$ and to zero on $S^c$. And we denote $I_L$ by $L \times L$ identity matrix. For any matrix $X \in \mathbb{R}^{m \times n}$, we denote $X_{\text{max}(r)}$ as the best rank-$r$ approximation of $X$, and $X_{\text{max}(r)} = X - X_{\text{max}(r)}$ as the error of the best rank-$r$ approximation of $X$. We use the phrase “rank-$r$ matrices” to refer to matrices of rank at most $r$. 
2 Recovery via Schatten-$p$ Minimization

In this section, we introduce $\ell_q$-RUB condition for $0 < q \leq 1$ and consider the recovery of matrices through Schatten-$p$ minimization (1.11). We show that the $\ell_q$-RUB condition of order $(k+1)r$ for any $k > 1$ such that $kr \in \mathbb{Z}_+$, with constants $C_1, C_2$ satisfies $C_2/C_1 < k^{(1/p-1/2)q}$ for $0 < p \leq q \leq 1$ is sufficient to guarantee the exact and and Stable recovery of all rank-$r$ matrices.

2.1 $\ell_q$-RUB and Some Auxiliary Lemmas

In this subsection, we will introduce $\ell_q$-RUB condition for $0 < q \leq 1$ and give some auxiliary lemmas. Firstly, we introduce $\ell_q$-RUB condition, which is a natural generalization of RUB in [12].

Definition 2.1. ($\ell_q$-Restricted Uniform Boundedness) For a linear map $A : \mathbb{R}^{m \times n} \to \mathbb{R}^L$, a positive integer $r$ and $0 < q \leq 1$, if there exist uniform constants $C_1$ and $C_2$ such that for all nonzero rank-$r$ matrices $X \in \mathbb{R}^{m \times n}$

\[
C_1 \|X\|_q S_q \leq \|A(X)\|_q S_q \leq C_2 \|X\|_q S_q,
\]

we say that linear map $A$ satisfies $\ell_q$-Restricted Uniform Boundedness ($\ell_q$-RUB) condition of order $r$ with constants $C_1$ and $C_2$.

Remark 2.2. (RIP-$\ell_2/\ell_q$ for low-rank matrices) If we take $C_1 = 1 - \delta_{rl}^b$ and $C_2 = 1 + \delta_{ru}^r$, then (2.1) becomes

\[
(1 - \delta_{rl}^b) \|X\|_q S_q \leq \frac{1}{L} \|A(X)\|_q S_q \leq (1 + \delta_{ru}^r) \|X\|_q S_q.
\]

And we call that the linear map $A$ satisfies RIP-$\ell_2/\ell_q$ of order $r$ with constants $\delta_{rl}^b$ and $\delta_{ru}^r$. Especially, when $q = 1$, it is the RIP-$\ell_2/\ell_1$ for low-rank matrices introduced in [25]. When $\delta_{rl}^b = \delta_{ru}^r$, it is the restricted $q$-isometry property introduced in [51].

Then we will give some auxiliary lemmas.

The first lemma give out the condition guaranteeing the additivity of Schatten-$p$ norm, which comes from [42, Lemma 2.3] for $q = 1$, and [34, Lemma 2.2] and [51, Lemma 2.1] for $0 < q < 1$.

Lemma 2.3. Let $0 < q \leq 1$. Let $X, Y \in \mathbb{R}^{m \times n}$ be matrices with $X^T Y = O$ and $X Y^T = O$, then the following holds:

1. $\|X + Y\|_q S_q \geq \|X\|_q S_q + \|Y\|_q S_q$;

2. $\|X + Y\|_q S_q \geq \|X\|_q S_q + \|Y\|_q S_q$.

And the second one, which comes from [24, Lemma 2.6], is the cone constraint for matrix’s Schatten-$p$ norm.
Lemma 2.4. Suppose $X, \hat{X} \in \mathbb{R}^{m \times n}$, $R = \hat{X} - X$. If $\|\hat{X}\|_{S_p}^p \leq \|X\|_{S_p}^p$, then we have

$$\|R_{\max(r)}\|_{S_p}^p \leq 2\|X_{\max(r)}\|_{S_p}^p + \|R_{\max(r)}\|_{S_p}^p.$$ 

In order to estimate the Schatten-2 norm of $(\hat{X} - X)_{\max(r)}$ for the stable recovery, we also need the following lemma, which is inspired by [12, Lemma 7.8].

Lemma 2.5. Let

$$f(t) = t^{(2-p)/p}(c - kt),$$

where $c$ is a positive constant independent of $t$. Then we have

$$f(t) \leq \frac{p}{2} \left( \frac{2 - p}{2k} \right)^{(2-p)/p} c^{2/p}, \quad t > 0.$$ 

Proof. By taking derivation of $f(t)$, we have

$$f'(t) = -\frac{1}{p} t^{(2-2p)/p} (2kt - (2 - p)c).$$

Therefore

$$f(t) \leq f \left( \frac{(2 - p)}{2k} c \right) = \frac{p}{2} \left( \frac{2 - p}{2k} \right)^{(2-p)/p} c^{2/p}.$$ 

\[\square\]

2.2 Exact Recovery via via Schatten-$p$ Minimization

In this subsection, we will consider the exact recovery under $\ell_q$-RUB condition.

Theorem 2.6. Let $r$ be any positive integer, and $k > 1$ such that $kr$ be positive integer. Suppose that $A$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_k/C_1 < k^{(1/p - 1/2)q}$ for any $0 < p \leq q \leq 1$, then the Schatten-$p$ minimization (1.11) exact recovers all rank-$r$ matrices. That is, for all rank-$r$ matrices $X$ with $b = A(X)$, we have $\hat{X} = X$, where $\hat{X}$ is the solution of (1.11) with $B = \{0\}$.

Before proving Theorems 2.6, let us first state the well known matrix null space property, which lies in the heart of the proof of this main result.

Lemma 2.7. ([50]) All rank-$r$ matrices $X \in \mathbb{R}^{m \times n}$ with $b = A(X) \in \mathbb{R}^L$ can be exactly recovered by solving problem (1.11) with $B = \{0\}$ if and only if

$$\|R_{\max(r)}\|_{S_p}^p < \|R_{\max(r)}\|_{S_p}^p$$

holds for all $R \in \mathcal{N}(A) \backslash \{O\}$, where $O$ is $m \times n$ zero matrix.
Proof of Theorem 2.6. Our proof follows the idea of [23, Theorem 2.4]. By Lemma 2.7, we only need to show that for all matrices $R \in \mathcal{N}(A) \setminus \{O\}$, one has $\|R_{\max(r)}\|_{S_p}^{p} < \|R_{\max(r)}\|_{S_p}^{p}$. Assume there exists a nonzero matrix $R$ with

\begin{equation}
A(R) = 0
\end{equation}

and

\begin{equation}
\|R_{-\max(r)}\|_{S_p}^{p} \leq \|R_{\max(r)}\|_{S_p}^{p}.
\end{equation}

We denote $l = \min\{m, n\}$ and assume that the singular value decomposition of $R$ is

$$
R = U\Sigma V^T = U\text{diag}(\sigma)V^T,
$$

where $\sigma = (\sigma_1, \ldots, \sigma_l)$ is the singular vector with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0$. Without loss of generality, we assume that $(k + 1)r \leq l$, otherwise we set the undefined entries of $\sigma$ as 0.

Let $\text{supp}(\sigma_{\max(r)}) = \{1, \ldots, r\} =: T_0$. We partition $T_0^c = \{r + 1, \ldots, l\}$ as

$$
T_0^c = \bigcup_{j=1}^J T_j,
$$

where $T_1$ is the index set of the $s$ largest entries of $\sigma_{-\max(r)}$, $T_2$ is the index set of the next $s$ largest entries of $\sigma_{-\max(r)}$, and so on. Here, $s \in \mathbb{Z}_+$ is to be determined. The last index set $T_J$ may contain less $s$ elements. Then for $j \geq 2$, we have $|\sigma_i|^p \leq |\sigma_{T_{j-1}}|^p/s$ for any $i \in T_j$. Therefore

$$
\|\sigma_{T_j}\|^2_2 = \sum_{i \in T_j} |\sigma_i|^2 \leq \sum_{i \in T_j} \left(\frac{\|\sigma_{T_{j-1}}\|^p_s}{s}\right)^{2/p} \leq \frac{\|\sigma_{T_{j-1}}\|^2_p}{s^{2/p-1}},
$$

so that

\begin{equation}
\sum_{j \geq 2} \|\sigma_{T_j}\|^q = \sum_{j \geq 2} \left(\frac{\|\sigma_{T_j}\|^2_2}{q/2}\right)^{q/2} \leq \sum_{j \geq 2} \left(\frac{\|\sigma_{T_{j-1}}\|^2_p}{s^{2/p-1}}\right)^{q/2} = \frac{1}{(s^{1/p-1/2})^q} \sum_{j \geq 2} \left(\frac{\|\sigma_{T_{j-1}}\|^p_p}{q/p}\right)^{q/p} \leq \frac{1}{(s^{1/p-1/2})^q} \left(\sum_{j \geq 2} \|\sigma_{T_{j-1}}\|^p_p\right)^{q/p} = \frac{1}{(s^{1/p-1/2})^q} \left(\|\sigma_{-\max(r)}\|^p_p\right)^{q/p},
\end{equation}

where the second line follows from $q/p \geq 1$.

Let $T_{01} = T_0 \cup T_1$, we consider the following identity

\begin{equation}
\|A(R)\|^q_q = \left\|A(R_{T_{01}}) + \sum_{j \geq 2} A(R_{T_j})\right\|^q_q,
\end{equation}

where

$$
R_{T_j} = U\text{diag}(\sigma_{T_j})V^T, \ \forall j = 0, 1, \ldots
$$

First, we give out a lower bound for (2.5). Note that $\text{rank}(R_{T_{01}}) \leq r + s$ and $|T_j| \leq s$. By the $r + s$ order $\ell_q$-RUB condition, we have

$$
\|A(R)\|^q_q \geq \|A(R_{T_{01}})\|^q_q - \sum_{j \geq 2} \|A(R_{T_j})\|^q_q.
$$
\[ \begin{align*}
&\geq C_1 L \| R_{T_01} \|_{S_2}^q - \sum_{j \geq 2} C_2 L \| R_{T_j} \|_{S_2}^q = C_1 L \| R_{T_01} \|_{S_2}^q - C_2 L \sum_{j \geq 2} \| \sigma_{T_j} \|_2^q \\
&\geq C_1 L \| R_{T_01} \|_{S_2}^q - \frac{C_2 L}{(s^{1/p-1/2})^q} (\| \sigma_{\max(r)} \|_p^q)^{q/p},
\end{align*} \]

(2.6)

where the last inequality follows from (2.4). Then by (2.3), we get a lower bound of \( \| A(R) \|_q \) as follows

\[ \| A(R) \|_q \geq C_1 L \| R_{T_01} \|_{S_2}^q - \frac{C_2 L}{(s^{1/p-1/2})^q} (\| \sigma_{\max(r)} \|_p^q)^{q/p} \]

\[ \geq C_1 L \| R_{T_01} \|_{S_2}^q - C_2 L \left( \frac{r}{s} \right)^{(1/p-1/2)q} \| \sigma_{\max(r)} \|_2^q \]

\[ = C_1 L \| R_{T_01} \|_{S_2}^q - C_2 L \left( \frac{r}{s} \right)^{(1/p-1/2)q} \| R_{\max(r)} \|_{S_2}^q \]

(2.7)

We also need an upper bound of \( \| A(R) \|_q \). Using (2.2), we have

(2.8)

\[ \| A(R) \|_q = 0. \]

Combining the lower bound (2.7) with the upper bound (2.8), we get

(2.9)

\[ L \left( C_1 - C_2 \left( \frac{r}{s} \right)^{(1/p-1/2)q} \right) \| R_{T_01} \|_{S_2}^q \leq 0. \]

Taking \( s = kr \) for \( k > 1 \) with \( kr \in \mathbb{Z}_+ \). Note that

\[ \frac{C_2}{C_1} < \left( \frac{s}{r} \right)^{(1/p-1/2)q} = k^{(1/p-1/2)q}. \]

Then (2.9) implies that

\[ \| R_{\max(r)} \|_{S_p} \leq r^{1/p-1/2} \| R_{\max(r)} \|_{S_2} \leq r^{1/p-1/2} \| R_{T_01} \|_{S_2} \leq 0. \]

However, (2.2) implies that

(2.10)

\[ 0 \leq \| R_{\max(r)} \|_{S_p} \leq \| R_{\max(r)} \|_{S_p}, \]

which is a contradiction. \( \square \)

As a direct consequence of Theorem 2.6, we have following two corollaries.

**Corollary 2.8.** Let \( r \) be any positive integer, and \( k > 1 \) such that \( kr \) be positive integer. Suppose that \( \hat{A} \) satisfies \( \ell_q \) RUB of order \((k+1)r\) with \( C_2/C_1 < k^{(1/p-1/2)q} \) for any \( 0 < p \leq q \leq 1 \), then the Schatten-\( p \) minimization (1.14) exact recovers all symmetric rank-\( r \) matrices. That is, for all symmetric rank-\( r \) matrices \( X \) with \( \hat{b} = \hat{A}(X) \), we have \( \hat{X} = X \), where \( \hat{X} \) is the solution of (1.14) with \( B = \{0\} \).
Corollary 2.9. Let $\tau > 1$ and $s = \lceil \tau \frac{2p}{(2-p)q} \rceil$. Suppose that $A$ satisfies RIP-$\ell_2/\ell_q$ of order $s + r$ with

$$\delta_{s+r}^{\text{sup}} + \tau \delta_r^{\text{lb}} < \tau - 1$$

for any $0 < p \leq q \leq 1$, then the Schatten-$p$ minimization (1.11) recovers all rank-$r$ matrices. That is, for all rank-$r$ matrices $X$ with $b = A(X)$, we have $\tilde{X} = X$, where $\tilde{X}$ is the solution of (1.11) with $B = \{0\}$.

However, our results may not be optimal and can be improved further.

Remark 2.10. We emphasize that our $(k+1)r$ order $\ell_q$-RUB condition for $q = 1$ is a litter stronger than the condition $kr$ order RUB condition in [12]. We note that in [12], Cai and Zhang used a sparse representation of a polytope in $\ell_1$ norm (see [11, Lemma 1.1]), which provide a more refined analysis. And Zhang and Li [53, Lemma 2.1] gave out a similar sparse representation in $\ell_p$ norm for $0 < p \leq 1$. However this sparse representation cannot be direct used in our model (1.11). Therefore, we don’t know whether or not it is possible to reduce this $(k+1)r$ order $\ell_q$-RUB condition to $kr$ order for $k > 1$ with $kr \in \mathbb{Z}_+$.

2.3 Stable Recovery via Schatten-$p$ Minimization

In this subsection, we consider the stable recovery. We show that the $\ell_q$-RUB condition of order $(k+1)r$ with constants $C_1, C_2$ satisfies $C_2/C_1 < k^{(1/p-1/2)q}$ is also sufficient to guarantee the stable recovery of all rank-$r$ matrices via the noisy measurements.

Theorem 2.11. Let $r$ be any positive integer and $0 < p \leq q \leq 1$. Let $\tilde{X}^{\ell_q}$ be the solution of the Schatten-$p$ minimization (1.11) with $B = B^{\ell_q}(\eta_1) \cap B^{DS}(\eta_2)$.

1. For any rank-$r$ $X$, let $k > 1$ with $kr$ be positive integer, and $A$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < k^{(1/p-1/2)q}$, then

$$\|\tilde{X} - X\|_S^q \leq \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \frac{1}{\rho_1 L^q} \min \left\{ \frac{2}{L^{1-2q} \eta_1^q}, \frac{2q/p+q}{\rho_1} r^{(1/p-1/2)q} \eta_2^q \right\},$$

where $\rho_1 = C_1 - C_2 (1/k)^{(1/p-1/2)q}$.

2. And for more general matrix $X$, let $k > 2^q/p$ with $kr$ be positive integer, and $A$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < 2^{1-q/p} k^{(1/p-1/2)q}$, then

$$\|\tilde{X} - X\|_S^q \leq \left( \frac{C_2 2^{2q/p-1}}{\rho_2} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \left( 2^{2q/p-1} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \left( \frac{\|X - \max(r)\|_{S_p}}{r^{1/p-1/2}} \right)^q$$

$$+ \left( 2^{q/p-1} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \frac{1}{\rho_2 L^q} \min \left\{ \frac{2}{L^{1-2q} \eta_1^q}, \frac{2q/p+q+1}{\rho_1} r^{(1/p-1/2)q} \eta_2^q \right\},$$

where $\rho_2 = C_1 - C_2 2^{q/p-1} (1/k)^{(1/p-1/2)q}$.
In fact, Theorem 2.11 can be deduced by following Proposition 2.12 for \( B^q(\eta_1) \) and Proposition 2.13 for \( B^{DS}(\eta_2) \). First, we give out the estimate \( \| \hat{X} - X \|_{S_2}^q \) for the \( B = B^q(\eta_1) \).

**Proposition 2.12.** Let \( r \) be any positive integer and \( 0 < p \leq q \leq 1 \). Let \( \hat{X}^q \) be the solution of the Schatten-\( p \) minimization (1.11) with \( B = B^q(\eta_1) \).

(1) For any rank-\( r \) \( X \), let \( k > 1 \) with \( kr \) be positive integer, and \( A \) satisfies \( \ell_q \)-RUB of order \( (k+1)r \) with \( C_2/C_1 < k^{(1/p-2)/q} \), then

\[
\| \hat{X}^q - X \|_{S_2}^q \leq \frac{2}{\rho_1 L^{1-q}} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \eta_1^q,
\]

where \( \rho_1 = C_1 - C_2 (1/k)^{(1/p-1/2)q} \).

(2) And for more general matrix \( X \), let \( k > 2^{2(q-p)} \) with \( kr \) be positive integer, and \( A \) satisfies \( \ell_q \)-RUB of order \( (k+1)r \) with \( C_2/C_1 < 2^{1-q/p} k^{(1/p-2)/q} \), then

\[
\| \hat{X}^q - X \|_{S_2}^q \leq \frac{C_2 2^{2q/p-1}}{\rho_2} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \left( \frac{2^{2q/p-1}}{k} \right)^{(1/p-1/2)q} + 1 \left( \frac{\| X_{\text{max}(r)} \|_{S_p}}{r^{1/p-1/2}} \right)^q
\]

\[+ \frac{2}{\rho_2 L^{1-q}} \left( \frac{2^{q/p-1}}{k} \right)^{(1/p-1/2)q} + 1 \eta_1^q,
\]

where \( \rho_2 = C_1 - C_2 2^{q/p-1} (1/k)^{(1/p-1/2)q} \).

**Proof.** We denote \( l = \min\{m, n\} \). Let \( R = \hat{X}^q - X \). We have the following tube constraint inequality

\[
(2.11) \quad \| A(R) \|_q^2 \leq \| A(\hat{X}^q) - b \|_q^2 + \| b - A(X) \|_q^2 \leq (L\eta_1)^q + (L\eta_1)^q = 2 L^q \eta_1^q.
\]

By Lemma 2.4, we have cone constraint inequality as follows

\[
(2.12) \quad \| R_{\text{max}(r)} \|_{S_p}^p \leq 2 \| X_{\text{max}(r)} \|_{S_p}^p + \| R_{\text{max}(r)} \|_{S_p}^p.
\]

We still assume that the singular value decomposition of \( R \) is

\[
R = U\Sigma V^T = U\text{diag}(\sigma)V^T,
\]

where \( \sigma = (\sigma_1, \ldots, \sigma_l) \) is the singular vector with \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \geq 0 \). And we denote \( T_0 := \text{supp}(\sigma_{\text{max}(r)}) = \{1, \ldots, r\} \). We also partition \( T_0^c = \{r+1, \ldots, l\} \) as

\[
T_0^c = \bigcup_{j=1}^m T_j,
\]

which is the same as the proof of Theorem 2.6. And (2.4) still holds.

Let \( T_{01} = T_0 \cup T_1 \), we also consider identity (2.5).
First, by (2.6) and (2.12), we can estimate a lower bound for (2.5) as follows

\[
\|A(R)\|_q^p \geq C_1 L \|R_{T_01}\|_S_2^q - \frac{C_2 L}{(s^{1/p-1/2})^q} \left( \|\sigma_{\text{max}(r)}\|_p^p + 2\|X_{-\text{max}(r)}\|_{S_p}^p \right)^{q/p}
\]

\[
\geq C_1 L \|R_{T_01}\|_S_2^q - C_2 L \left( \frac{r}{s} \right)^{(1/p-1/2)q} \left( \|\sigma_{\text{max}(r)}\|_2^p + 2\left( \frac{\|X_{-\text{max}(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^p \right)^{q/p}
\]

It follows from \(|a| + |b|)^{q/p} \leq 2^{q/p-1}(|a|^{q/p} + |b|^{q/p})\) for \(q/p \geq 1\) that

\[
\left( \|\sigma_{\text{max}(r)}\|_2^p + 2\left( \frac{\|X_{-\text{max}(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^p \right)^{q/p} \leq 2^{q/p-1} \left( \|\sigma_{\text{max}(r)}\|_2^q + 2^{q/p} \left( \frac{\|X_{-\text{max}(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \right).
\]

But when \(X\) is rank-\(r\), i.e., \(\|X_{-\text{max}(r)}\|_{S_p}^p = 0\), we have a simpler form

\[
\left( \|\sigma_{\text{max}(r)}\|_2^p + 2\left( \frac{\|X_{-\text{max}(r)}\|_{S_p}^p}{r^{1/p-1/2}} \right)^p \right)^{q/p} = \|\sigma_{\text{max}(r)}\|_2^q.
\]

Therefore

\[
\|A(R)\|_q^p \geq C_1 L \|R_{T_01}\|_S_2^q - C_2 L 2^{q/p-1} \left( \frac{r}{s} \right)^{(1/p-1/2)q} \left( \|\sigma_{\text{max}(r)}\|_2^q + 2^{q/p} \left( \frac{\|X_{-\text{max}(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \right)
\]

(2.13)

\[
\geq L \left( C_1 - C_2 2^{q/p-1} \left( \frac{r}{s} \right)^{(1/p-1/2)q} \right) \|R_{T_01}\|_S_2^q - C_2 L 2^{q/p-1} \left( \frac{r}{s} \right)^{(1/p-1/2)q} \left( \frac{\|X_{-\text{max}(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q
\]

or

\[
\|A(R)\|_q^p \geq C_1 L \|R_{T_01}\|_S_2^q - C_2 L \left( \frac{r}{s} \right)^{(1/p-1/2)q} \|\sigma_{\text{max}(r)}\|_2^q
\]

(2.14)

\[
\geq L \left( C_1 - C_2 \left( \frac{r}{s} \right)^{(1/p-1/2)q} \right) \|R_{T_01}\|_S_2^q.
\]

We also need an upper bound of \(\|A(R)\|_q^q\). Using (2.11), we have

(2.15)

\[
\|A(R)\|_q^q \leq 2L^q \eta_1^q.
\]

Combining the lower bound (2.13) (or (2.14)) with the upper bound (2.15), we get

(2.16)

\[
L \left( C_1 - C_2 2^{q/p-1} \left( \frac{r}{s} \right)^{(1/p-1/2)q} \right) \|R_{T_01}\|_S_2^q - C_2 L 2^{q/p-1} \left( \frac{r}{s} \right)^{(1/p-1/2)q} \left( \frac{\|X_{-\text{max}(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \leq 2L^q \eta_1^q
\]

or

(2.17)

\[
L \left( C_1 - C_2 \left( \frac{r}{s} \right)^{(1/p-1/2)q} \right) \|R_{T_01}\|_S_2^q \leq 2L^q \eta_1^q
\]
Therefore

$$\|R_{T_0}\|_{S_2}^q \leq \frac{C_2 \sigma_{s+r}^{2q/p-1}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( \frac{\|X_{\max(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q + \frac{2}{\rho L^{1-q} \eta_1^q}$$

or

$$\|R_{T_0}\|_{S_2}^q \leq \frac{2}{\rho L^{1-q} \eta_1^q}.$$ 

Next, we estimate $\|R_{T_0}\|_{S_2}^q$. Here, we use some idea of the proof of [12, Lemma 7.8]. First, by a simply computation, we have

$$\|R_{T_0}\|_{S_2}^q = \|\sigma_{\max(s+r)}\|_2^q \leq \left( \left( \|\sigma_{\max(s+r)}\|_{P_\infty} \right)^{(2-p)/p} \|\sigma_{\max(s+r)}\|_p \right)^{q/2}$$

$$\leq \left( \left( \sigma_{s+r}^{(2-p)/p} \left( \|\sigma_{\max(r)}\|_p - \sum_{j=r+1}^{s+r} \sigma_j^p \right) \right)^{q/2}$$

$$\leq \left( \left( \sigma_{s+r}^{(2-p)/p} \left( \|\sigma_{\max(r)}\|_p + 2 \|X_{\max(r)}\|_{S_p} - s \sigma_{s+r}^p \right) \right)^{q/2}$$

$$= \left( f(\sigma_{s+r}) \right)^{q/2},$$

where

$$f(t) = t^{(2-p)/p} \left( \|\sigma_{\max(r)}\|_p^2 + 2 \|X_{\max(r)}\|_{S_p}^2 - st \right).$$

We need to estimate the upper bound of $f(\sigma_{s+r}^p)$. By Lemma 2.5, we have

$$f(t) \leq \frac{p}{2} \left( \frac{2 - p}{2s} \right)^{(2-p)/p} \left( \|\sigma_{\max(r)}\|_p^2 + 2 \|X_{\max(r)}\|_{S_p}^2 \right)^{2/p}$$

$$\leq \frac{p}{2} \left( \frac{2 - p}{2} \right)^{(2-p)/p} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( \|\sigma_{\max(r)}\|_2^q + 2 \left( \frac{\|X_{\max(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \right),$$

which implies that

$$\|R_{T_0}\|_{S_2}^q \leq \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( \|\sigma_{\max(r)}\|_2^q + 2 \left( \frac{\|X_{\max(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \right)^{q/p}$$

$$\leq 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( \|\sigma_{\max(r)}\|_2^q + 2 \left( \frac{\|X_{\max(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \right)^{q/p}$$

$$\|R_{T_0}\|_{S_2}^q \leq 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( \|R_{T_0}\|_{S_2}^q + 2^{q/p} \left( \frac{\|X_{\max(r)}\|_{S_p}}{r^{1/p-1/2}} \right)^q \right),$$

(2.21)
or

\[ \|R_{\mathcal{G}}^q\|_{S_2} \leq \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( \|\sigma_{\text{max}(r)}\|_{p/2} \right)^{q/p} \]

(2.22)

\[ \leq \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \|R_{\mathcal{G}}\|_{S_2}^{q} \cdot \]

It follows from (2.21) or (2.22) that

\[ \|R\|_{S_2}^{q} \leq \|R_{\mathcal{G}}\|_{S_2}^{q} + \|R_{\mathcal{G}}\|_{S_2}^{q} \]

(2.23)

\[ \leq \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \|R_{\mathcal{G}}\|_{S_2}^{q} \]

or

\[ \|R\|_{S_2}^{q} \leq \|R_{\mathcal{G}}\|_{S_2}^{q} + \|R_{\mathcal{G}}\|_{S_2}^{q} \]

(2.24)

\[ \leq \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \|R_{\mathcal{G}}\|_{S_2}^{q} \cdot \]

Then substituting (2.18) into (2.23), we have

\[ \|R\|_{S_2}^{q} \leq \left\{ \frac{C_22^{q/p-1}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \right\} \]

+ \left\{ \frac{2}{\rho L^{1-q}} \left( 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( \frac{2 - p}{2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \right\} \eta_1^{q} \]

(2.25)

\[ \leq \left( \frac{C_22^{q/p-1}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \left( 2^{q/p-1} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \left( \|X_{\text{max}(r)}\|_{S_p} \right)^{q} \]

\[ + \frac{2}{\rho L^{1-q}} \left( 2^{q/p-1} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \eta_1^{q} \]

where the last inequality follows from 0 < p/2 ≤ 1 and (1/p - 1/2)q > 0.

And substituting (2.19) into (2.24), we have

(2.26)

\[ \|R\|_{S_2}^{q} \leq \frac{2}{\rho L^{1-q}} \left( \left( \frac{1}{k} \left( \frac{1}{(p-1/2)q} + 1 \right) \eta_1^{q} \right) \right. \]

which finishes the proof. \[ \square \]

Next, we consider Dantzig selector constraint \( B^{DS}(\eta_2) \).
Proposition 2.13. Let \( r \) be any positive integer and \( 0 < p \leq q \leq 1 \). Let \( \hat{X}^{DS} \) be the solution of the Schatten-\( p \) minimization (1.11) with \( B = B^{DS}(\eta_2) \).

(1) For any rank-\( r \) \( X \), let \( k > 1 \) with \( kr \) be positive integer and \( A \) satisfies \( \ell_q \)-RUB of order \((k+1)r\) with \( C_2/C_1 < k^{1/(p-2)/q} \), then

\[
\| \hat{X}^{DS} - X \|_{S_2}^q \leq \frac{2^{q/p+1}(1/p-1/2)q}{L^q \rho_1^q} \left( \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \eta_2^q,
\]

where \( \rho_1 = C_1 - C_2 \left( \frac{1}{k} \right)^{(1/p-1/2)q} \).

(2) And for more general matrix \( X \), let \( k > \frac{2^{q/p}(q-p)}{q(p-1)} \) with \( kr \) be positive integer, and \( A \) satisfies \( \ell_q \)-RUB of order \((k+1)r\) with \( C_2/C_1 < 2^{1-q/p} k^{1/(p-2)/q} \), then

\[
\| \hat{X}^{DS} - X \|_{S_2}^q \leq \left\{ \left( \frac{C_2 2^{q/p}}{\rho_2} \right)^{1/(p-1/2)q} \left( \frac{3}{2} \right) 2^{q/p-1} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \left( \frac{\| X_{\text{max}}(r) \|_{S_p}}{\nu} \right)^q \right\}^q
\]

where \( \rho_2 = C_1 - C_2 2^{q/p-1} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \).

Proof. We denote \( l = \min\{m,n\} \). Let \( R = \hat{X}^{DS} - X \). We have the following tube constraint inequality

\[
| \mathcal{A}^*(\mathcal{A}(R))|_{S_{\infty}} \leq | \mathcal{A}^*(\mathcal{A}(\hat{X}^{DS}) - b)|_{S_{\infty}} + | \mathcal{A}^*(b - \mathcal{A}(X))|_{S_{\infty}} \leq \eta_2 = 2\eta_2,
\]

instead of (2.11). By Lemma 2.4, cone constraint inequality (2.12) still holds.

With the same proof of Theorem 2.12, we still have (2.13) (or (2.14)), which gives out a lower bound of \( \| \mathcal{A}(R) \|_q^q \).

But the upper bound of \( \| \mathcal{A}(R) \|_q^q \) needs a completely different proof from that of Theorem 2.12. Using (2.27), we have

\[
\| \mathcal{A}(R) \|_q^q \leq L^{1-q/2} \| \mathcal{A}(R) \|_q^q = L^{1-q/2} \left( \| \mathcal{A}(R) \|_q^q \right)^{q/2}
\]

\[
\leq L^{1-q/2} \left( \| \mathcal{A}^*(\mathcal{A}(R)) \|_{S_{\infty}} \| R \|_{S_1} \right)^{q/2} \leq L^{1-q/2} \left( 2\eta_2 \| R \|_{S_p} \right)^{q/2}
\]

\[
= L^{1-q/2} \left( 2\eta_2 \right)^{q/2} \left( \| R_{\text{max}}(r) \|_{S_p} + \| R_{-\text{max}}(r) \|_{S_p} \right)^{q/(2p)}
\]

\[
\leq L^{1-q/2} \left( 2\eta_2 \right)^{q/2} \left( 2\| \sigma_{\text{max}}(r) \|_{S_p} + 2 \| X_{\text{max}}(r) \|_{S_p} \right)^{q/(2p)}
\]

(2.28)

\[
= \left\{ L^{1-q/2} \left( 2\eta_2 \right)^{q/2} \left( \| R_{T_0} \|_{S_2} \right)^{q/2} \right\}^{1/2},
\]

\[
= \left\{ L^{1-q/2} \left( 2\eta_2 \right)^{q/2} \left( \| R_{T_0} \|_{S_2} + \left( \frac{\| X_{\text{max}}(r) \|_{S_p}}{\nu} \right)^q \right)^{1/2} \right\}^{1/2},
\]

if \( X \) is rank \( r \), otherwise.
Combining the lower bound (2.14) (or 2.13) with the upper bound (2.28), we get

\[
\left\{ L\left(C_1 - C_2\left(1 - \frac{1}{k}\right)\right)\right\}^2 R_{T_{01}}^2 \leq L^2 - q_2 q/p + q_1 q_1 (1/p - 1/2)q_1 q_1 \eta_{12}^q \left(R_{T_{01}}^q \right)^2
\]

or

\[
\left\{ L\left(C_1 - C_2 q/p - 1\right)\left(1/p - 1/2\right)\right\}^2 R_{T_{01}}^q S_2 - C_2 L^2 q/p - 1\left(1/p - 1/2\right)q_1 q_1 \eta_{2}^q \left(R_{T_{01}}^q \right)^2 + \left(\left|X_{-\text{max}}(r)\right|_{S_p}^p\right)^q \right\}
\]

(2.30)

where \( (x)_+ = \max\{x, 0\} \).

First, we consider (2.30). Let \( x = R_{T_{01}}^q S_2 \) and \( y = \left(r^{-1/p + 1/2}\right)\left|X_{-\text{max}}(r)\right|_{S_p}^p \). Note that

\[
\rho = C_1 - C_2 q/p - 1\left(1/p - 1/2\right) > 0.
\]

When

\[
R_{T_{01}}^q S_2 \geq \frac{C_2 q/p - 1\left(1/p - 1/2\right)}{\rho} \left(1/p - 1/2\right)q_1 q_1 \eta_{12}^q \left(\left|X_{-\text{max}}(r)\right|_{S_p}^p\right)^q,
\]

we have

\[
L^2 \rho^2 x^2 - \left(C_2 L^2 \rho^2 q/p - 1\right)\left(1/p - 1/2\right)q_1 q_1 \eta_{12}^q y \leq 0.
\]

Note that for the second order inequality \( ax^2 - bx - c \leq 0 \) for \( a, b, c > 0 \), we have

\[
x \leq \frac{b + \sqrt{b^2 + 4ac}}{2a} \leq \frac{b}{a} + \sqrt{\frac{c}{a}}.
\]

Hence we can get an upper bound of \( x \) as follows

\[
x \leq \frac{C_2 L^2 q/p}{L^2 \rho^2} \left(1/p - 1/2\right)q_1 q_1 \eta_{12}^q \left(\left|X_{-\text{max}}(r)\right|_{S_p}^p\right)^q + \frac{L^2 q_2 q/p + q_1 q_1 (1/p - 1/2)q_1 q_1 \eta_{12}^q}{L^2 \rho^2} \left(R_{T_{01}}^q \right)^2
\]

\[
+ \left[\frac{L^2 q_2 q/p + q_1 q_1 (1/p - 1/2)q_1 q_1 \eta_{12}^q \eta_{2}^q}{L^2 \rho^2} \right] y
\]

\[
\leq \frac{C_2 q/p}{\rho} \left(1/p - 1/2\right)q_1 q_1 \eta_{2}^q \left(\left|X_{-\text{max}}(r)\right|_{S_p}^p\right)^q + \frac{2 q_2 q/p + q_1 q_1 (1/p - 1/2)q_1 q_1 \eta_{12}^q \eta_{2}^q}{L^2 \rho^2} \left(R_{T_{01}}^q \right)^2
\]

\[
+ \frac{1}{2} \left(\frac{q_2 q/p + q_1 q_1 (1/p - 1/2)q_1 q_1 \eta_{12}^q \eta_{2}^q}{L^2 \rho^2} + y\right)
\]

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\[ (2.31) \quad \leq \left( \frac{C_2 2^{4q/p}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + \frac{1}{2} \right) y + \frac{2^{4q/p+1} q_{r(1/p-1/2)q}}{L^q \rho^2} \eta^q. \]

Hence whenever
\[ \| R_{T_{01}} \|^q_{S_2} \geq \frac{C_2 2^{4q/p-1}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( C \max(\|X_\rho\|_{S_p}) \right) \]

or not, we always have
\[ \| R_{T_{01}} \|^q_{S_2} \leq \max \left\{ \frac{C_2 2^{4q/p}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} \left( C \max(\|X_\rho\|_{S_p}) \right)^q, \right. \]
\[ \left. \frac{2^{4q/p+1} q_{r(1/p-1/2)q}}{L^q \rho^2} \eta^q \right\} \]

Then, we consider \( (2.29) \). Note that
\[ \rho = C_1 - C_2 \left( \frac{1}{k} \right)^{(1/p-1/2)q} > 0. \]

Therefore
\[ (2.33) \quad \| R_{T_{01}} \|^q_{S_2} \leq \frac{2^{4q/p+1} q_{r(1/p-1/2)q}}{L^q \rho^2} \eta^q. \]

Note that \( (2.23) \) and \( (2.24) \) still holds.
Then substituting \( (2.32) \) into \( (2.23) \), we have

\[ \| R \|^q_{S_2} \leq \left\{ \frac{C_2 2^{4q/p}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + \frac{1}{2} \right\} \left( 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( 2 - \frac{p}{2} \right) \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \]
\[ + \frac{2^{4q/p+1} q_{r(1/p-1/2)q}}{L^q \rho^2} \left( 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( 2 - \frac{p}{2} \right) \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \]
\[ \leq \left\{ \frac{C_2 2^{4q/p}}{\rho} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + \frac{3}{2} \right\} \left( 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( 2 - \frac{p}{2} \right) \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \]
\[ \times \left( C \max(\|X_\rho\|_{S_p}) \right)^q \]
\[ + \frac{2^{4q/p+1} q_{r(1/p-1/2)q}}{L^q \rho^2} \left( 2^{q/p-1} \left( \frac{p}{2} \right)^{q/2} \left( 2 - \frac{p}{2} \right) \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \]
where the last inequality follows from $0 < p/2 \leq 1$ and $(1/p - 1/2)q > 0$.

And substituting (2.33) into (2.24), we have

\begin{equation}
||R||^q_{S_2} \leq \frac{2^{q/p+q+1}(1/p-1/2)q}{L^2 \rho^2} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \eta_2^q,
\end{equation}

which finishes the proof. \hfill \Box

**Remark 2.14.** What should be noticed is that Zhang, Huang and Zhang [51] also considered matrix recovery via Schatten-$p$ minimization with $B = B^{\ell_2(\eta)}_q - \ell_2$ norm constraint, under the $\ell_p$-RUB condition. The problem (1.11) considered in Proposition 2.12 and Theorem 2.11 are different from that of they considered.

If $A : \mathbb{S}^n \rightarrow \mathbb{R}^L$ is a SROP, we consider Schatten-$p$ minimization (1.14) instead of (1.11) and have the following results.

**Corollary 2.15.** Let $r$ be any positive integer, $0 < p \leq q \leq 1$ and $L = \lfloor L/2 \rfloor$. Let $X^\ell_q$ be the solution of the Schatten-$p$ minimization (1.14) with $B = B^{\ell_q(\eta_1)}_q \cap B^{\ell_p(\eta_2)}_p$.

(1) For any symmetric rank-$r$ $X$, let $k > 1$ with $kr$ be positive integer, and $\hat{A}$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < k^{(1/p-2)q}$, then

\begin{equation}
||\hat{X} - X||^q_{S_2} \leq \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \frac{1}{\rho_1 L^q} \min \left\{ \frac{2}{L^{1-2q}} \eta_1^q, \frac{2^{q/p+q}}{\rho_1} r^{(1/p-1/2)q} \eta_2^q \right\},
\end{equation}

where $\rho_1 = C_1 - C_2 (1/k)^{(1/p-1/2)q}$.

(2) And for more general symmetric matrix $X$, let $k > 2^{2(q-p)}$ with $kr$ be positive integer, and $\hat{A}$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < 2^{1-2q/p}k^{(1/p-2)q}$, then

\begin{equation}
||\hat{X} - X||^q_{S_2} \leq \left( \frac{C_2 2^{q-1}}{\rho_2} \right)^{(1/p-1/2)q} \left( \frac{1}{k} \right)^{(1/p-1/2)q} + 1 \right) \frac{1}{\rho_2 L^q} \min \left\{ \frac{2}{L^{1-2q}} \eta_1^q, \frac{2^{q/p+q+1}}{\rho_1} r^{(1/p-1/2)q} \eta_2^q \right\},
\end{equation}

where $\rho_2 = C_1 - C_2 2^{q-1}(1/k)^{(1/p-1/2)q}$.
3 Robust Null Space Property and $\ell_q$-RUB

In this section, we will investigate the relationship between the RUB and robust rank null space property.

The robust null space property with $\ell_2$ bound $\|Ax\|_2$ was first introduced by Sun in [44], which is called sparse approximation property. And this name was first used by Foucart and Rauhut in [31]. And they also introduced the robust null space property with Dantzig selector bound $\|A^*Ax\|_\infty$. In [31], Foucart and Rauhut also introduced the robust rank null space property.

Inspired by [44], we introduce $(\ell_t, \ell_p)$-robust rank null space property as follows.

**Definition 3.1.** Let $0 < p, t \leq \infty$ and $0 < q < \infty$. A linear map $A$ satisfies $(\ell_t, \ell_p)$-robust rank null space property of order $r$ for $\ell_q$ bound with constants $D$ and $\beta$ if

\[
\|X_{\text{max}(r)}\|_{S_t}^p \leq D\|A(X)\|_{S_p}^q + \beta \frac{\|X_{\text{max}(r)}\|_{S_t}^p}{r(1/p-1/t)p}
\]

holds for all $X \in \mathbb{R}^{m \times n}$.

And a linear map $A$ satisfies the $(\ell_t, \ell_p)$-robust rank null space property of order $r$ for Dantzig selector bound with constants $D$ and $\beta$ if

\[
\|X_{\text{max}(r)}\|_{S_t}^p \leq D\|A^*A(X)\|_{S_p}^r + \beta \frac{\|X_{\text{max}(r)}\|_{S_t}^p}{r(1/p-1/t)p}
\]

holds for all $X \in \mathbb{R}^{m \times n}$.

By similar proofs as that of Theorems 2.12 and 2.13, we have the following theorem, which implies that $(\ell_2, \ell_p)$-robust rank null space property of order $r$ can be induced from $\ell_q$-RUB of order $(k+1)r$ for any fixed $k > 1$ with $kr \in \mathbb{Z}_+$.

**Theorem 3.2.** Let $r \in \mathbb{Z}_+$, and $k > 1$ with $kr \in \mathbb{Z}_+$.

(1) Suppose that $A$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < k^{(1/p-1/2)q}$ for any $0 < p \leq q \leq 1$, then

\[
\|X_{\text{max}(r)}\|_{S_2}^p \leq \frac{1}{(C_1L)^p/q} \|A(X)\|_{S_q}^p + \left(\frac{C_2}{C_1k^{(1/p-1/2)q}}\right)^{p/q} \frac{\|X_{\text{max}(r)}\|_{S_p}^p}{r(1/p-1/t)p}
\]

\[=: D_1\|A(X)\|_{S_q}^p + \beta_1 \frac{\|X_{\text{max}(r)}\|_{S_p}^p}{r(1/p-1/t)p},\]

i.e., $A$ satisfies $(\ell_2, \ell_p)$ robust rank null space property of order $r$ for $\ell_q$ bound with constant pair $(D_1, \beta_1)$.

(2) Suppose that $A$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < k^{(1/p-1/2)q}/4$ for any $0 < p \leq q \leq 1$, then

\[
\|X_{\text{max}(r)}\|_{S_2}^p \leq \left(\frac{2^{p+1}p}{C_1^{2p/q}L^p}\right)^{p/q} \frac{1}{r(1/p-1/t)p} \|A^*A(X)\|_{S_p}^r + \left(\frac{2C_2}{C_1k^{(1/p-1/2)q}} + \frac{1}{2}\right)^{p/q} \frac{\|X_{\text{max}(r)}\|_{S_p}^p}{r(1/p-1/t)p}
\]
3.2 also holds for vector case. We only need to let $X$ be any diagonal matrix, and let each element $A_j$ of linear map $A$ be diagonal matrix.

4 Recovery via Least $q$ Minimization

In this section, we will consider the matrix’s recovery via least $q$ minimization (1.16). We show that the $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < k^{(1/p-1/2)q}$ for $0 < p \leq q \leq 1$ is also sufficient to recover any matrix $X \in \mathbb{R}^{m \times n}$. We also extend our result to Gaussian design distribution and assert that $L \geq Cr(m+n)$ measurements is sufficient to guarantee all rank-$r$ matrices’ stable recovery with high probability.

4.1 Stable Recovery via Least $q$ Minimization

**Theorem 4.1.** Let $r \in \mathbb{Z}^+$, and $k > 1$ with $kr \in \mathbb{Z}^+$. Suppose that $A$ satisfies $\ell_q$-RUB of order $(k+1)r$ with $C_2/C_1 < k^{(1/p-1/2)q}$ for $0 < p \leq q \leq 1$. Let $\hat{X}$ be the solution of

$$\min_{Y \in \mathbb{R}^{m \times n}} \|A(Y) - b\|_q^p \text{ subject to } \left(\text{tr}(Y^T Y)^{p/2}\right)^{1/p} = 1$$

where $b = A(X) + z$, then

$$\|\hat{X} - X\|_{S_p}^p \leq \frac{2(1 + \beta_1)^2}{1 - \beta_1} \|X_{-\text{max}(r)}\|_{S_p}^p + \frac{2p/q(3 + \beta_1)D_1}{1 - \beta_1} \|z\|_q^p,$$

where $D_1 = (C_1L)^{-p/q} > 0$ and $0 < \beta_1 = (C_2k^{-(1/p-1/2)q}/C_1)^{p/q} < 1$.

Theorem 4.1 can be deduced from the following stronger result and Theorem 3.2.

**Theorem 4.2.** Let $0 < p \leq t \leq \infty$.

1. If $A : \mathbb{R}^{m \times n} \to \mathbb{R}^L$ satisfies the $(\ell_t, \ell_p)$-robust rank null space property of order $r$ for $\ell_q$ bound with constants $D_1 > 0$ and $0 < \beta_1 < 1$, then

$$\|Y - X\|_{S_t}^p \leq \frac{(1 + \beta_1)^2}{1 - \beta_1} \frac{1}{r^{(1/p-1/t)p}} (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{-\text{max}(r)}\|_{S_p}^p) + \frac{(3 + \beta_1)D_1}{1 - \beta_1} \|A(Y - X)\|_q^p,$$

2. If $A : \mathbb{R}^{m \times n} \to \mathbb{R}^L$ satisfies the $(\ell_t, \ell_p)$-robust rank null space property of order $r$ for Dantzig selector bound with constants $D_2 > 0$ and $0 < \beta_2 < 1$, then

$$\|Y - X\|_{S_t}^p \leq \frac{(1 + \beta_2)^2}{1 - \beta_2} \frac{1}{r^{(1/p-1/t)p}} (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{-\text{max}(r)}\|_{S_p}^p) + \frac{(3 + \beta_2)D_2}{1 - \beta_2} \|A^*A(Y - X)\|_{S_{\infty}}^p.$$
The proof requires some auxiliary lemmas. We start with a matrix version of Stechkin's bound. It follows immediately from [31, Proposition 2.3 of Chapter 2].

Lemma 4.3. Let \( X \in \mathbb{R}^{m \times n} \) and \( 0 < r \leq \min\{m, n\} \). Then for any \( 0 < p \leq t \leq \infty \),

\[
\|X_{-\max(r)}\|_{S_t} \leq \frac{\|X\|_{S_p}}{r^{1/p - 1/t}}.
\]

In order to get a similar cone constraint for matrix’s Schatten-\( p \) norm, we also need the following lemma which was given by Yue and So in [50] and Audenaert [4].

Lemma 4.4. Let \( X, Y \in \mathbb{R}^{m \times n} \) be given matrices. Suppose that \( f : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a concave function satisfying \( f(0) = 0 \). Then, for any \( k \in \{1, \ldots, \min\{m, n\}\} \), we have

\[
\sum_{j=1}^{k} |f(\sigma_j(X)) - f(\sigma_j(Y))| \leq \sum_{j=1}^{k} f(\sigma_j(X - Y)).
\]

Lemma 4.5. Let \( 0 < p \leq t \leq \infty \).

1. If \( A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{L} \) satisfies the \((\ell_t, \ell_p)\)-robust rank null space property of order \( r \) for \( \ell_q \) bound with constants \( D_1 > 0 \) and \( 0 < \beta_1 < 1 \), then

\[
\|Y - X\|_{S_p}^p \leq \frac{2D_1}{1 - \beta_1} r^{(1/p - 1/t)p} \|A(Y - X)\|_{\ell_q}^p + \frac{1 + \beta_1}{1 - \beta_1} \left( \|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{-\max(r)}\|_{S_p}^p \right).
\]

2. If \( A : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{L} \) satisfies the \((\ell_t, \ell_p)\)-robust rank null space property of order \( r \) for Dantzig selector bound with constants \( D_2 > 0 \) and \( 0 < \beta_2 < 1 \), then

\[
\|Y - X\|_{S_p}^p \leq \frac{2D_2}{1 - \beta_2} r^{(1/p - 1/t)p} \|A^*A(Y - X)\|_{\ell_\infty}^p + \frac{1 + \beta_2}{1 - \beta_2} \left( \|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{-\max(r)}\|_{S_p}^p \right).
\]

Proof. Our proof follows the idea of [31, Section 4.3]. Let \( R = Y - X \). Lemma 2.3 states that

\[
\|R\|_{S_p}^p = \|R_{\max(r)}\|_{S_p}^p + \|R_{-\max(r)}\|_{S_p}^p \leq r^{(1/p - 1/t)p} \|R_{\max(r)}\|_{S_t}^p + \|R_{-\max(r)}\|_{S_p}^p.
\]

Applying the \((\ell_t, \ell_p)\)-robust rank null space property of \( A \), we obtain that

\[
\|R_{\max(r)}\|_{S_t}^p \leq D_1 \|A(R)\|_{\ell_q}^p + \frac{\|R_{-\max(r)}\|_{S_p}^p}{r^{(1/p - 1/t)p}}.
\]

Therefore, we have

\[
\|R\|_{S_p}^p \leq D_1 r^{(1/p - 1/t)p} \|A(R)\|_{\ell_q}^p + (1 + \beta_1) \|R_{-\max(r)}\|_{S_p}^p.
\]

Next, we estimate the upper bound of \( \|R_{-\max(r)}\|_{S_p}^p \). Since \( x \rightarrow |x|^p \) is concave on \( \mathbb{R}_+ \) for any \( p \in (0, 1] \), by taking \( f(\cdot) = (\cdot)^p \) in Lemma 4.4, we immediately obtain

\[
\|Y\|_{S_p}^p = \|\sigma(Y)\|_{S_p}^p + \|\sigma(Y)\|_{S_p}^p \geq \|\sigma(X)\|_{S_p}^p + \|\sigma(R)\|_{S_p}^p + \|\sigma(R)\|_{S_p}^p - \|\sigma(X)\|_{S_p}^p.
\]
where \( T = \text{supp}(\sigma(X)_{\max(r)}) \). By rearranging the terms in the above inequality, we get
\[
\|R_{- \max(r)}\|_{S_p}^p \leq \|\sigma(R)_{T^c}\|_{S_p}^p \leq \|Y\|_{S_p}^p - \|\sigma(X)_{T}\|_{S_p}^p + \|\sigma(X)_{T^c}\|_{S_p}^p + \|\sigma(R)_{T}\|_{S_p}^p
\]
\[
\leq (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p) + \|R_{\max(r)}\|_{S_p}^p
\]
\[
\leq (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p) + r^{(1/p-1/2)}\|R_{\max(r)}\|_{S_t}^p
\]
\[
\leq (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p) + D_1 r^{(1/p-1/2)}\|A(R)\|_{q}^p + \beta_1 \|R_{- \max(r)}\|_{S_p}^p,
\]
where the last line follows from (4.1). Note that \( 0 < \beta_1 < 1 \). Therefore,
\[
(4.3) \quad \|R_{- \max(r)}\|_{S_p}^p \leq \frac{1}{1 - \beta_1} (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p) + \frac{D_1}{1 - \beta_1} r^{(1/p-1/2)}\|A(R)\|_{q}^p.
\]

And substituting (4.3) into (4.2), we obtain
\[
\|R\|_{S_p}^p \leq (1 + \beta_1) \left( \frac{1}{1 - \beta_1} (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p) + \frac{D_1}{1 - \beta_1} r^{(1/p-1/2)}\|A(R)\|_{q}^p \right)
\]
\[
+ D_1 r^{(1/p-1/2)}\|A(R)\|_{q}^p + \frac{1 + \beta_1}{1 - \beta_1} (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p),
\]
which finishes the proof of item (1).

If \( \mathcal{A} \) satisfies the \((\ell_t, \ell_p)\)-robust rank null space property for Dantzig selector bound, we replace (4.1) with
\[
(4.4) \quad \|R_{\max(r)}\|_{S_t}^p \leq D_2 \|A^* A(R)\|_{S_{\infty}}^p + \beta_2 \frac{\|R_{- \max(r)}\|_{S_p}^p}{r^{(1/p-1/2)/p}}.
\]

Then by a similar proof as item (1), we get
\[
\|R\|_{S_p}^p = \frac{2D_2}{1 - \beta_2} r^{(1/p-1/2)/p}\|A^* A(R)\|_{S_{\infty}}^p + \frac{1 + \beta_2}{1 - \beta_2} (\|Y\|_{S_p}^p - \|X\|_{S_p}^p + 2\|X_{- \max(r)}\|_{S_p}^p),
\]
which finishes the proof of item (2).

Proof of Theorem 4.2. Let \( R = Y - X \), then we have
\[
(4.5) \quad \|Y - X\|_{S_t}^p = \left( \|R_{\max(r)}\|_{S_t}^p + \|R_{- \max(r)}\|_{S_t}^p \right)^{p/t} \leq \|R_{\max(r)}\|_{S_t}^p + \|R_{- \max(r)}\|_{S_t}^p,
\]
where the inequality follows from \( 0 < p \leq t \). By the \((\ell_t, \ell_p)\) robust rank null space property of \( \mathcal{A} \) with respect to \( \ell_p \),
\[
(4.6) \quad \|R_{\max(r)}\|_{S_t}^p \leq D_1 \|A(R)\|_{q}^p + \beta_1 \frac{\|R_{- \max(r)}\|_{S_p}^p}{r^{(1/p-1/2)/p}} \leq D_1 \|A(R)\|_{q}^p + \beta_1 \frac{\|R\|_{S_p}^p}{r^{(1/p-1/2)/p}}.
\]

Next we estimate \( \|R_{- \max(r)}\|_{S_t}^p \). Lemma 4.3 implies that
\[
(4.7) \quad \|R_{- \max(r)}\|_{S_t}^p \leq \left( \frac{\|R\|_{S_p}^p}{r^{(1/p-1/2)/p}} \right)^p.
\]
Then substituting (4.6) and (4.7) into (4.5) yields

\[(4.8) \quad \|Y - X\|^p_{S_t} \leq D_1 \|A(R)\|^p_q + (1 + \beta_1) \frac{\|R\|^p_{S_k}}{r(1/p-1/q)}.
\]

An application of Lemma 4.5, we obtain

\[
\|Y - X\|^p_{S_t} \leq \frac{(1 + \beta_1)}{r(1/p-1/q)} \left( \frac{2D_1}{1 - \beta_1} \|A(R)\|^p_q + \frac{1 + \beta_1}{1 - \beta_1} (\|Y\|^p_{S_p} - \|X\|^p_{S_p} + 2\|X_{\max(r)}\|^p_p) \right)
\]

\[
+ D_1 \|A(R)\|^p_q
\]

\[
= \frac{(1 + \beta_1)^2}{1 - \beta_1} \frac{1}{r(1/p-1/q)} (\|Y\|^p_{S_p} - \|X\|^p_{S_p} + 2\|X_{\max(r)}\|^p_p) + \frac{(3 + \beta_1)D_1}{1 - \beta_1} \|A(R)\|^p_q.
\]

And if \(A\) satisfies the \((\ell_t, \ell_p)\)-robust rank null space property for Dantzig selector bound, then by a similar proof, we have

\[
\|Y - X\|^p_{S_t} \leq \frac{(1 + \beta_2)^2}{1 - \beta_2} \frac{1}{r(1/p-1/q)} (\|Y\|^p_{S_p} - \|X\|^p_{S_p} + 2\|X_{\max(r)}\|^p_p) + \frac{(3 + \beta_2)D_2}{1 - \beta_2} \|A^*(R)\|^p_{S_{\infty}}.
\]

\[\square\]

**Proof of Theorem 4.1.** Note that \(A\) satisfies \(\ell_q\)-RUB of order \((k + 1)r\) with \(C_2/C_1 < k^{(1/p-1/2)q}\) for any \(0 < p \leq q \leq 1\). Therefore Theorem 3.2 implies the validity of the \((\ell_2, \ell_q)\) robust rank null space property for \(\ell_p\) bound with parameters \(D_1 = (C_1L)^{-p/q} > 0\) and \(0 < \beta_1 = (C_2k^{-(1/p-1/2)q}/C_1)^{p/q} < 1\). Then Theorem 4.2 leads to us that

\[(4.9) \quad \|\hat{X} - X\|^p_{S_2} \leq \frac{(1 + \beta_1)^2}{1 - \beta_1} \frac{1}{r(1/p-1/2)p} (\|\hat{X}\|^p_{S_p} - \|X\|^p_{S_p} + 2\|X_{\max(r)}\|^p_p)
\]

\[
+ \frac{(3 + \beta_1)D_1}{1 - \beta_1} \|A(\hat{X} - X)\|^p_q.
\]

We consider recovering density matrix. This property assures

\[(4.10) \quad \|\hat{X}\|^p_{S_p} - \|X\|^p_{S_p} = \text{tr}((\hat{X}^T\hat{X})^{p/2}) - \text{tr}((X^TX)^{p/2}) = 0.
\]

And

\[(4.11) \quad \|A(\hat{X} - X)\|^p_q \leq (\|A(\hat{X}) - b\|_q + \|b - A(X)\|_q^{p/q})^{p/q} \leq (\|z\|_q + \|z\|_q^{p/q}) = 2^{p/q}\|z\|_q^p.
\]

Combining (4.10), (4.11) with (4.9), we get

\[
\|\hat{X} - X\|^p_{S_2} \leq \frac{2(1 + \beta_1)^2}{1 - \beta_1} \frac{\|X_{\max(r)}\|^p_p}{r(1/p-1/2)p} + \frac{2^{p/q}(3 + \beta_1)D_1}{1 - \beta_1} \|z\|_q^p.
\]

\[\square\]

**Remark 4.6.** Kabanava, Kueng, Rauhut et al. [33] considered the least-\(q\) minimization for \(q \geq 1\). Here, we extend it to nonconvex case, i.e., least-\(q\) minimization for \(0 < q < 1\).
4.2 Matrix Recovery for ROP from Gauss Distribution via LAD

This subsection aims to show that recovering low-rank matrices through ROP $\mathcal{A}$ from Gaussian distribution $\mathcal{P}$ is still possible. First, we recall Gaussian random variable $\mathcal{A}$.

A linear map $\mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^L$ is called ROP from distribution $\mathcal{P}$ if $\mathcal{A}$ is defined as in (1.4) with all the entries of $\beta^j$ and $\gamma^j$ independently drawn from the distribution $\mathcal{P}$. The following Theorem 4.7 shows that recovering low-rank matrices through ROP $\mathcal{A}$ from the standard normal distribution $\mathcal{P}$ via least 1 minimization (1.16) is possible.

**Theorem 4.7.** Let $r \in \mathbb{Z}_+$. Suppose that $\mathcal{A}$ is a ROP from the standard normal distribution. Let $\hat{X}$ be the solution of

$$ \min_{Y \in \mathbb{R}^{m \times n}} \|\mathcal{A}(Y) - b\|_1 \quad \text{subject to} \quad \left( \text{tr}((Y^T Y)^{p/2}) \right)^{1/p} = 1 $$

where $b = \mathcal{A}(X) + z$. Then there exist uniform constants $c_1$, $c_2$, $C_3$ and $C_4$ such that, whenever $L \geq c_1 r (m + n)$, $\hat{X}$ obeys

$$ \|\hat{X} - X\|_{S_2}^p \leq C_3 \left[ \frac{\|X - \mathcal{A}(X)\|_{\text{tr}}^p}{r^{(1/p - 1/2)p}} \right] + C_4 \frac{\|z\|_1^p}{L^p} $$

with probability at least $1 - e^{-c_2 L}$.

**Proof.** Suppose $k = 10$, by [12, Theorem 2.2], we can find a uniform constant $c_0$ and $c_2$ such that if $L \geq c_0 (k + 1) r (m + n)$, $\mathcal{A}$ satisfies RUB of order $11r$ and constants $C_1 = 0.32$, $C_2 = 1.01$ with probability at least $1 - e^{-c_2 L}$. Hence, we have $c_1 = 11c_0$ and $c_2$ such that if $L \geq c_1 r (m + n)$, $\mathcal{A}$ satisfies RUB of order $11r$ and constants satisfying $C_2/C_1 < 10^{1/p - 1/2}$ with probability at least $1 - e^{-c_2 L}$. Then it follows from Theorem 4.1 that

$$ \|\hat{X} - X\|_{S_2}^p \leq 2 \left( 1 + \beta_1 \right) \frac{\|X - \mathcal{A}(X)\|_{\text{tr}}^p}{r^{(1/p - 1/2)p}} + \frac{2p/q (3 + \beta_1)}{1 - \beta_1} \|z\|_q^p $$

holds with probability at least $1 - e^{-c_2 L}$, where $C_3, C_4$ are constants depending only on $p$. \qed

**Corollary 4.8.** For the PhaseLift introduced in [13, 21], we consider

$$ (4.12) \min_{X \in \mathbb{R}^n} \|\tilde{A}(X) - \tilde{b}\|_1 \quad \text{subject to} \quad X \succeq O \text{ and } \text{tr}(X) = 1. $$

Then Theorem 4.7 implies that ROP with $L \geq Cm$ random projections from Gaussian distribution is sufficient to ensure the stable recovery of all symmetric rank-1 matrix $X = xx^T$ with high probability.

**Remark 4.9.** Note that the condition of Theorem 4.1 is $\ell_q$-RUB for $0 < p \leq q \leq 1$. Therefore, if we can show that ROP $\mathcal{A}$ from some distribution satisfies the $\ell_q$-RUB condition for $0 < q < 1$ with high probability, then we can get a better conclusion.
5 Conclusions and Discussion

In this paper, we consider the matrix recovery from rank-one projection measurements via nonconvex minimization.

First in Section 2, after introducing the \( \ell_q \)-RUB (Definition 2.1), we consider exact and stable recovery of rank-\( r \) matrices from rank-one projection measurements via Schatten-\( p \) minimization (1.11). And we show that \( \ell_q \)-RUB condition of order \((k + 1)r \) with \( C_2/C_1 < k^{(1/p - 1/2)q} \) for some \( k > 1 \) with \( kr \in \mathbb{Z}_+ \) and \( 0 < p \leq q \leq 1 \) is sufficient for exact recovery of all rank-\( r \) matrices (Theorem 2.6) in Subsection 2.2. Subsection 2.3 considers extensions to the noisy case. We get stable recovery via Schatten-\( p \) model (1.11) with \( B = B^{\ell_q}(\eta_1) \cap B^{DS}(\eta_2) \) (Theorem 2.11), by combining \( B = B^{\ell_q}(\eta_1) \) (Proposition 2.12) with \( B = B^{DS}(\eta_2) \) (Proposition 2.13). And our condition is also sufficient for symmetric rank-one projections (Corollary 2.8 and Corollary 2.15).

By the proofs of Theorems 2.12 and 2.13, we also obtain that the robust rank null sapce property of order \( r \) can be deduced from the \( \ell_q \)-RUB of order \((k + 1)r \) for some \( k > 1 \) (Theorem 3.2) in Section 3.

And in Section 4, we consider the stable recovery via the least \( q \) minimization (1.16) for \( 0 < q \leq 1 \) under \( \ell_q \)-RUB condition. We show that our condition in Theorem 2.6 is still sufficient (Theorem 4.1). And we also consider recovering matrix for ROP \( A \) from Gaussian distribution via least 1 minimization, and we show that with high probability, ROP with \( L \geq Cr(m + n) \) random projections from Gaussian distribution is sufficient to ensure stable recovery of all rank-\( r \) matrices (Theorem 4.7).

However, our \((k + 1)r \) \((k > 1)\) order RUB condition for \( q = 1 \) (Theorem 2.6) is a litter stronger than the \( kr \) order RUB condition in [12]. Therefore, this condition may be improved further (Remark 2.10). And note that Theorem 4.1 show that \( \ell_q \)-RUB condition can guarantee the stable recovery via least \( q \) minimization (1.16). Therefore, finding a ROP \( A \) from some distribution satisfying the \( \ell_q \)-RUB condition for \( 0 < q < 1 \) is one direction of our future research (Remark 4.9).

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