Algebraic Investigation of the Soft Seven Sphere

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Abstract

We investigate the seven sphere as a soft Lie algebra i.e. an algebra with structure functions instead of structure constants. We calculate its structure functions explicitly and also discuss some relevant points such as the validity of the Jacobi identities. Furthermore, we emphasise some important features such as the pointwise reduction, closure and some other consistency checks.

Ring division algebras play fundamental roles in mathematics from algebra to geometry and topology with many different applications. The applicability of real and complex numbers in physics is not in question. Quaternions which may be represented as Pauli matrices are also important.

The use of octonions in physics is the problem. Non associative algebra appeared for the first time in physics when Jordan, van Neuman and Wigner introduced commutative but non-associative operators – Jordan algebras – for the construction of a new quantum mechanics[1]. More recently, after the proposed eightfold way by Gell-Mann and Ne’eman, there were some octonionnic rivals for $SU(3)$ such as $G_2$, $SO(7)$, $SO(8)$ and others[2]. Another serious step was taken by Günaydin and Gürsey [3] when they present in their

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work a systematic study of the octonionic algebraic structure. The abandon of associativity means the existence of nonobservable states. Günaydin and Gürsey used these nonobservable states to explain the quark confinement phenomena.

Later on octonions entered the Grand Unified Theories era with different applications. In [4], Gürsey suggested that the exceptional group $F_4$ might be used to describe the internal charge space of particles. Even the other exceptional groups $E_6, E_7, E_8$ have been utilized to provide larger GUT models [5]. Starting from the eighties, new applications of ring division algebras in physics were found. The instanton problem [6], supersymmetry [7]. Application of octonions to supergravity spontaneous compactification was a very important and active field of research during the mid eighties. Especially compactification of $d = 11$ supergravity over $S^7$ to 4 dimensions. It is an impossible task to list all the relevant papers, so we direct the interested reader to the physics report [8] written by Duff, Nilsson and Pope where a lot of references are given. We just mention that the first indication of the octonionic nature of this problem appeared in the Englert solution of $d = 11$ supergravity compactification over $S^7$ [9] and a systematic study along this line has been carried out in [10][11]. The relations between superstrings (p-branes) and octonions had been considered from many different points of view, the reader may consult the references given in [12] for details.

Recently, a new line of attack has been proposed. Interpreting octonions as a soft Lie algebra (algebra with structure functions instead of structure constants [13]) proved to be useful for breaking the $N = 8$ superconformal barrier of models composed of spin 2 field, spin 3/2 fields, fermions and spin one current. Englert, Sevrin, Troost, van Proeyen and Spindel [14] constructed two different $N = 8$ superconformal algebras. Latter this soft seven sphere (SSS) algebra was investigated by different people. Berkovits [15] proved that the Green-Schwarz action admits such algebra as an underlying symmetry, Cederwall and Preitschopf [16] made a detailed study of this SSS algebra. Then they with Brink [17] studied the light cone formulation of the Green-Schwarz action using such SSS techniques. Also Samtleben [18] studied further the OPE structure of the $N = 8$ superconformal algebra based on this SSS algebra.

In [13], Sohnius used soft algebras with structure functions that vary over space-time as well as over an internal gauge manifold. In this work we use only soft algebras with structure functions that vary over the internal gauge manifold (the fiber) not the base space-time manifold. In the first section,
we introduce the seven sphere as a soft algebra, we calculate its structure functions explicitly and also discuss some relevant points such as the validity of the Jacobi identity. Furthermore, we emphasis some important features such as the pointwise reduction, closure, and some other consistency checks. In the second section, we give more general construction. We reformulate, in the third section, some standard Lie group results putting them in a form suitable to subsequent applications.

1 Octonions and the Soft Seven Sphere

We start by recalling the non-associative octonion algebra. A generic octonion number is
\[ \varphi = \varphi_0 e_0 + \varphi_i e_i = \varphi_\mu e_\mu, \quad [i = 1..7, \mu = 0..7, \varphi_\mu \in \mathbb{R}] \] (1)
such that \( e_0 = 1 \) and the other seven imaginary units satisfy \( e_i e_j = -\delta_{ij} + f_{ijk} e_k \iff [e_i, e_j] = 2f_{ijk} e_k \) where \( f_{ijk} \) is completely antisymmetric and equals one for any of the following three-cycles \((123), (145), (246), (347), (176), (257), (365)\) and the associator
\[ [e_i, e_j, e_k] = (e_i e_j) e_k - e_i (e_j e_k), \] (2)
is non-zero for any three elements that are not in the same three cycles and is completely antisymmetric The following formula or any of its generalization is thus ambiguous
\[ e_1 e_5 e_7 = \begin{cases} (e_1 e_5) e_7 = -e_3 \\ e_1 (e_5 e_7) = e_3 \end{cases} \] (3)
so the best way is to define the action of the imaginary units in a certain direction. In the spirit of Englert, Sevrin, Troost, Van Proeyen and Spindel [14] (also look at [19] and [20]) we define the left action of octonionic operators \( \delta_i \) by
\[ \delta_i \varphi = (e_i \varphi) \] (4)
implying that the following equation is well defined
\[ \delta_i \delta_j \varphi = \delta_i (\delta_j \varphi) = \delta_i (e_j \varphi) = e_i (e_j \varphi), \] (5)
then eq. (3) reads unambiguously as
\[ \delta_1 \delta_5 e_7 = (e_1 (e_5 e_7)) = e_3. \] (6)
Since octonions are also non-commutative, we must also differentiate between left and right action. Using the barred notation \[21\], we introduce right action as

\[
1\bar{\delta}_i \varphi = (\varphi e_i)
\]

for example

\[
(1\bar{\delta}_i)(1\bar{\delta}_j)\varphi = (1\bar{\delta}_i)(1\bar{\delta}_j\varphi) = 1\bar{\delta}_i(\varphi e_j) = ((\varphi e_j)e_i)
\]

As we shall see shortly, we can express the associator in terms of left and right operators. The imaginary octonionic units generate the seven sphere \(S^7\) which has many properties similar to Lie algebras and/or Lie groups. \(S^7\) and Lie groups are the only non-flat compact parallelizable manifolds \[22\].

The important point for evaluating any Lie algebra is the commutator, so let's examine

\[
[\delta_i, \delta_j] \varphi = e_i(e_j\varphi) - e_j(e_i\varphi) = 2f_{ijk}e_k\varphi - 2[e_i, e_j, \varphi]
\]

\[
= 2f_{ijk}e_k\varphi + 2[e_i, \varphi, e_j].
\]

now the last term can be written as

\[
[e_i, \varphi, e_j] = (e_i \varphi)e_j - e_i(\varphi e_j) = -\delta_i(1\bar{\delta}_j)\varphi + (1\bar{\delta}_j)\delta_i\varphi = -[\delta_i, 1\bar{\delta}_j] \varphi,
\]

thus our commutator can be rewritten as

\[
[\delta_i, \delta_j] \varphi = 2f_{ijk}e_k\varphi + 2[e_i, \varphi, e_j] = 2f_{ijk}\delta_k\varphi - 2e_i, 1\bar{\delta}_j] \varphi.
\]

Note that right operators are necessary because the last term the associator can never be written in terms of left operators alone.

After simple calculations, one concludes that the octonionic imaginary units are determined completely by \(1\bar{\delta}_i\) and the following equations

\[
[1\bar{\delta}_i, 1\bar{\delta}_j] \varphi = -2f_{ijk}1\bar{\delta}_k\varphi - 2[\delta_i, 1\bar{\delta}_j] \varphi
\]

\[
\{\delta_i, \delta_j\} \varphi = -2\delta_{ij}\varphi
\]

\[
\{1\bar{\delta}_i, 1\bar{\delta}_j\} \varphi = -2\delta_{ij}\varphi
\]

where the \(\delta_{ij}\) in \(1\bar{\delta}_i\) and \(1\bar{\delta}_j\) are the standard Kronecker delta tensor.

It has been proved in \[13\] \[14\], using three different ways, that the \(\delta_i\) algebra is associative. Thus a representation theory in terms of matrices
should be possible. Indeed, in [20], we have derived an algebra completely isomorphic to (11–14) by exploiting the idea that octonions can be used as a basis for any $8 \times 8$ real matrix. we have two sets of matrices, essentially,

$$
\delta_i \iff (E_i)_{\mu\nu} = \delta_{0\mu} \delta_{i\nu} - \delta_{0\nu} \delta_{i\mu} - f_{i\mu\nu} ,
$$

$$
1|\delta_i \iff (1|E_i)_{\mu\nu} = \delta_{0\mu} \delta_{i\nu} - \delta_{0\nu} \delta_{i\mu} + f_{i\mu\nu} .
$$

The set of matrices $E_i$ and $1|E_i$ have appeared in different octonionic works e.g. [3][10][23]. We suggest that their most appropriate names should be the canonical left and right octonionic structure at the north/south pole of the seven sphere. By explicit calculation, one finds that

$$
[E_i, E_j] \phi = 2f_{ijk} E_k \phi - 2[E_i, 1|E_j] \phi
$$

$$
[1|E_i, 1|E_j] \phi = -2f_{ijk} 1|E_k \phi - 2[E_i, 1|E_j] \phi
$$

$$
\{E_i, E_j\} \phi = -2\delta_{ij} \phi
$$

$$
\{1|E_i, 1|E_j\} \phi = -2\delta_{ij} \phi
$$

where $\phi$ is represented by a column matrix

$$
\phi^t = \begin{pmatrix}
\phi_0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 & \phi_5 & \phi_6 & \phi_7
\end{pmatrix}
$$

The word “isomorphic” above is justified since $\{\delta_i\}$ is associative [14][16] and the same holds obviously for our $\{E_i\}$ as they are written in terms of matrices. Our Jacobian identities are

$$
[\delta_i, [\delta_j, \delta_k]] \phi + [\delta_j, [\delta_k, \delta_i]] \phi + [\delta_k, [\delta_i, \delta_j]] \phi = 0 ,
$$

or

$$
[E_i, [E_j, E_k]] \phi + [E_j, [E_k, E_i]] \phi + [E_k, [E_i, E_j]] \phi = 0 .
$$

We shall return to these identities again at the end of this section.

Fixing the direction of the application for any imaginary octonionic units extracts a part of the algebra that respects associativity, but a certain price has to be paid. The presence of the $\phi$ is essential and either of $\{E_i\}$ or $\{1|E_i\}$ is an open algebra, they don’t close upon the action of the commutator. Here comes the second step, the soft Lie algebra idea. It is clear that the right hand side of (11) or (16) has a complicated $\phi$ dependence. Knowing that the seven sphere has a torsion that varies from one point to another [10][22] and mimicking the Lie group case where the structure constants are proportional
to the fixed group torsion, it is natural to propose that (11) may be redefined as

$$[\delta_i, \delta_j] \varphi = 2f^{(+)}_{ijk}(\varphi) \delta_k \varphi \quad ,$$  \hspace{1cm} (22)

where $f^{(+)}_{ijk}(\varphi)$ are structure functions that vary over the whole $S^7$ manifold. These structure functions $f^{(+)}_{ijk}(\varphi)$ were computed previously using different properties of the associator and some other octonionic identities in [10] [14] [22] [16].

Let’s do it for the following example which is equivalent to the following eight equations,

$$\begin{align*}
\varphi_3 &= \varphi_1 f^{(+)}_{121}(\varphi) + \varphi_2 f^{(+)}_{122}(\varphi) + \varphi_3 f^{(+)}_{123}(\varphi) + \varphi_4 f^{(+)}_{124}(\varphi) + \varphi_5 f^{(+)}_{125}(\varphi) + \varphi_6 f^{(+)}_{126}(\varphi) + \varphi_7 f^{(+)}_{127}(\varphi) , \\
\varphi_4 &= \varphi_1 f^{(+)}_{212}(\varphi) - \varphi_2 f^{(+)}_{211}(\varphi) - \varphi_3 f^{(+)}_{213}(\varphi) - \varphi_4 f^{(+)}_{214}(\varphi) + \varphi_5 f^{(+)}_{215}(\varphi) + \varphi_6 f^{(+)}_{216}(\varphi) + \varphi_7 f^{(+)}_{217}(\varphi) , \\
\varphi_5 &= \varphi_1 f^{(+)}_{312}(\varphi) - \varphi_2 f^{(+)}_{313}(\varphi) - \varphi_3 f^{(+)}_{311}(\varphi) + \varphi_4 f^{(+)}_{314}(\varphi) + \varphi_5 f^{(+)}_{315}(\varphi) - \varphi_6 f^{(+)}_{316}(\varphi) - \varphi_7 f^{(+)}_{317}(\varphi) , \\
\varphi_6 &= \varphi_1 f^{(+)}_{422}(\varphi) - \varphi_2 f^{(+)}_{423}(\varphi) - \varphi_3 f^{(+)}_{421}(\varphi) + \varphi_4 f^{(+)}_{424}(\varphi) + \varphi_5 f^{(+)}_{425}(\varphi) - \varphi_6 f^{(+)}_{426}(\varphi) - \varphi_7 f^{(+)}_{427}(\varphi) , \\
\varphi_7 &= \varphi_1 f^{(+)}_{532}(\varphi) - \varphi_2 f^{(+)}_{533}(\varphi) - \varphi_3 f^{(+)}_{531}(\varphi) + \varphi_4 f^{(+)}_{534}(\varphi) + \varphi_5 f^{(+)}_{535}(\varphi) - \varphi_6 f^{(+)}_{536}(\varphi) - \varphi_7 f^{(+)}_{537}(\varphi) , \\
\varphi_8 &= \varphi_1 f^{(+)}_{642}(\varphi) - \varphi_2 f^{(+)}_{643}(\varphi) - \varphi_3 f^{(+)}_{641}(\varphi) + \varphi_4 f^{(+)}_{644}(\varphi) + \varphi_5 f^{(+)}_{645}(\varphi) - \varphi_6 f^{(+)}_{646}(\varphi) - \varphi_7 f^{(+)}_{647}(\varphi) .
\end{align*}$$  \hspace{1cm} (25)

We now solve these equations for the seven unknown $f^{(+)}_{ij}(\varphi)$. We find

$$\begin{align*}
f^{(+)}_{121}(\varphi) &= f^{(+)}_{122}(\varphi) = 0 , \\
f^{(+)}_{124}(\varphi) &= +2 \frac{\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4^2 + \varphi_5 + \varphi_6 + \varphi_7}{r^2} , \\
f^{(+)}_{125}(\varphi) &= -2 \frac{\varphi_0 - \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 + \varphi_7}{r^2} , \\
f^{(+)}_{126}(\varphi) &= +2 \frac{\varphi_0 + \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4^2 + \varphi_5 + \varphi_6 + \varphi_7}{r^2} , \\
f^{(+)}_{127}(\varphi) &= -2 \frac{\varphi_0 - \varphi_1 - \varphi_2 - \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6 + \varphi_7}{r^2} , \\
f^{(+)}_{123}(\varphi) &= \frac{\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \varphi_5^2 + \varphi_6^2 + \varphi_7^2}{r^2} ,
\end{align*}$$

where

$$r^2 = (\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \varphi_5^2 + \varphi_6^2 + \varphi_7^2).$$
Along the same lines we can calculate all the structure functions, we give all of them in Appendix A. What we have just calculated is commonly called the (+) torsion \[^{[10]}\], we can find the (–) torsion by replacing the left by right multiplication in (23)

\[
[1|\mathbb{E}_i, 1|\mathbb{E}_j] \varphi = 2f_{ijk}^{(-)}(\varphi)1|\mathbb{E}_k \varphi.
\]  

(26)

we find

\[
\begin{align*}
f_{121}^{(-)}(\varphi) &= f_{122}^{(-)}(\varphi) = 0, \\
f_{124}^{(-)}(\varphi) &= +2\phi_0\phi_5 + \phi_1\phi_2 - \phi_3 \phi_4, \\
f_{125}^{(-)}(\varphi) &= -2\phi_0\phi_6 + \phi_1\phi_7 + \phi_2\phi_4, \\
f_{126}^{(-)}(\varphi) &= +2\phi_0\phi_1 + \phi_2\phi_3 - \phi_4 \phi_5, \\
f_{127}^{(-)}(\varphi) &= -2\phi_0\phi_2 - \phi_1\phi_5 + \phi_3\phi_4, \\
f_{123}^{(-)}(\varphi) &= -2\phi_0\phi_4 - \phi_1\phi_6 + \phi_2\phi_5 + \phi_3\phi_7, \\
\end{align*}
\]

the remaining \(f_{ijk}^{(-)}(\varphi)\) are listed in appendix A.

Let’s pause for a moment and note some of the evident features of these \(f_{ijk}^{(\pm)}(\varphi)\),

- One notices immediately that at \(\varphi'=(1,0,0,0,0,0,0,0) / (-1,0,0,0,0,0,0,0)\), the north / south pole (NP/SP), we recover the octonionic structure constants: \(f_{ijk}^{(+)}(NP/SP) = -f_{ijk}^{(-)}(NP/SP) = f_{ijk}\) and any non-standard cycles vanishes e.g. \(f_{567}^{(\pm)}(NP/SP) = 0\).

- Our construction started from a given multiplication table and as there are different choices \[^{[24]}\], we can have different families.

- Over \(S^7\), \(\partial_{S^7} f_{ijk}^{(\pm)}(\varphi) \neq 0\). This is a very important characteristic of the seven sphere.

To manifest the \(\varphi\) dependence, let’s give a non–tivial example. To simplify
the notations, we use here \((i, j, k)^{(\pm)}\) for \(f_{ijk}^{(\pm)}(\varphi)\). At \(\varphi = \frac{\mu + 1}{\sqrt{204}}\), we find

\[
\begin{align*}
(1, 2, 3)^{(+) = -12/17,} & \quad (2, 5, 7)^{(+) = 4/51,} & \quad (1, 5, 6)^{(+) = 1/51,} \\
(1, 4, 5)^{(+) = -6/17,} & \quad (1, 7, 6)^{(+) = 8/51,} & \quad (3, 6, 5)^{(+) = 0,} \\
(4, 3, 7)^{(+) = -2/51,} & \quad (4, 2, 6)^{(+) = 3/17,} \\
(1, 2, 4)^{(+) = 4/17,} & \quad (1, 5, 2)^{(+) = -8/17,} & \quad (3, 5, 4)^{(+) = -44/51,} \\
(5, 6, 7)^{(+) = -10/17,} & \quad (1, 3, 4)^{(+) = -5/17,} & \quad (4, 1, 6)^{(+) = -14/17,} \\
(1, 5, 7)^{(+) = -40/51,} & \quad (3, 5, 7)^{(+) = -2/17,} & \quad (3, 1, 6)^{(+) = -14/51,} \\
(2, 3, 5)^{(+) = -23/51,} & \quad (1, 7, 4)^{(+) = -4/17,} & \quad (4, 5, 6)^{(+) = -16/51,} \\
(2, 6, 5)^{(+) = -38/51,} & \quad (1, 6, 2)^{(+) = -8/17,} & \quad (6, 3, 2)^{(+) = -22/51,} \\
(1, 3, 5)^{(+) = -10/51,} & \quad (2, 4, 3)^{(+) = -2/17,} & \quad (4, 3, 6)^{(+) = -20/51,} \\
(3, 7, 6)^{(+) = -13/17,} & \quad (1, 2, 7)^{(+) = -1/17,} & \quad (4, 2, 7)^{(+) = -16/17,} \\
(2, 7, 3)^{(+) = -16/17,} & \quad (2, 6, 7)^{(+) = -4/51,} & \quad (7, 1, 3)^{(+) = -28/51,} \\
(2, 4, 5)^{(+) = 2/17,} & \quad (4, 7, 5)^{(+) = -7/51,} & \quad (4, 6, 7)^{(+) = -10/51,}
\end{align*}
\]

(27)

We have some kind of dynamical Lie algebra of seven generators with structure “constants” that change their values from one point to another. Let us emphasis the difference between considering \(\mathbb{E}_i\) as an open algebra or as elements of a soft seven sphere, observe that

\[
[\mathbb{E}_1, \mathbb{E}_2] = 2\mathbb{E}_3 - 2 \ [\mathbb{E}_1, 1\mathbb{E}_2],
\]

(28)

but

\[
[\mathbb{E}_1, \mathbb{E}_2] \Phi = 2 f_{123}^{(+)}(\varphi) \mathbb{E}_3 \Phi + 2 f_{124}^{(+)}(\varphi) \mathbb{E}_4 \Phi \\
+ 2 f_{125}^{(+)}(\varphi) \mathbb{E}_5 \Phi + 2 f_{126}^{(+)}(\varphi) \mathbb{E}_6 \Phi + 2 f_{127}^{(+)}(\varphi) \mathbb{E}_7 \Phi.
\]

(29)

At the NP

\[
\Phi_{NP} = (1 0 0 0 0 0 0 0)
\]

(30)

we still have

\[
[\mathbb{E}_1, \mathbb{E}_2] = 2\mathbb{E}_3 - 2 \ [\mathbb{E}_1, 1\mathbb{E}_2]
\]

(31)

whereas

\[
[\mathbb{E}_1, \mathbb{E}_2] \Phi_{NP} = 2 f_{12k}^{(+)}(\varphi_{NP}) \mathbb{E}_k \Phi_{NP}
\]

\[
= 2\mathbb{E}_3 \Phi_{NP}
\]

8
At $\Phi_W \equiv (\varphi_{\mu} = \frac{\mu + 1}{\sqrt{204}})$, we still have

$$[E_1, E_2] = 2E_3 - 2[E_1, 1E_2]$$

we find that

$$[E_1, E_2] \Phi_W = 2f_{12k}^{(\pm)}(\varphi_W)E_k \Phi_W$$

$$= \left( -\frac{24}{17}E_3 + \frac{8}{17}E_4 + \frac{16}{17}E_5 + \frac{16}{17}E_6 - \frac{2}{17}E_7 \right) \Phi_W$$

$$= \begin{pmatrix} \frac{1}{\sqrt{204}} \\ \frac{1}{\sqrt{204}} \\ \frac{1}{\sqrt{204}} \\ \frac{1}{\sqrt{204}} \\ \frac{1}{\sqrt{204}} \\ \frac{1}{\sqrt{204}} \end{pmatrix} = \begin{pmatrix} \frac{-1}{51} \sqrt{51} \\ \frac{-1}{17} \sqrt{51} \\ \frac{-1}{51} \sqrt{51} \\ \frac{-1}{8} \sqrt{51} \\ \frac{-1}{51} \sqrt{51} \\ \frac{-1}{17} \sqrt{51} \end{pmatrix}$$
\[
\begin{pmatrix}
0 & 0 & 0 & 24 & -8 & -16 & 2 \\
0 & 0 & 24 & 17 & -16 & 17 & 17 \\
0 & -24 & 17 & 0 & 0 & 16 & 17 \\
17 & 17 & -8 & 17 & 17 & 17 & 17 \\
16 & 17 & 17 & 2 & 17 & 17 & 17 \\
17 & 17 & -16 & 17 & 17 & 17 & 17 \\
12 & 16 & 17 & 0 & 0 & 0 & 17 \\
17 & 17 & -24 & 17 & 17 & 17 & 17
\end{pmatrix}
\cdot
\begin{pmatrix}
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}} \\
\frac{1}{\sqrt{204}}
\end{pmatrix}
\]

We are not making a projection but a reformulation of the algebra. This fact should always be kept in mind. The same happens in a non-trivial way for the Jacobi identity, i.e.

\[
(f_{ijm}(\varphi)f_{mkt}(\varphi) + f_{jkm}(\varphi)f_{mit}(\varphi) + f_{kim}(\varphi)f_{mj t}(\varphi)) E_t \varphi = 0, \quad (32)
\]

but, in general,

\[
(f_{ijm}(\varphi)f_{mkt}(\varphi) + f_{jkm}(\varphi)f_{mit}(\varphi) + f_{kim}(\varphi)f_{mj t}(\varphi)) \neq 0. \quad (33)
\]

Another important feature is

\[
[\mathbb{E}_i, 1|\mathbb{E}_j] = -2f_{ij k}(\varphi) \mathbb{E}_k + [\mathbb{E}_i, \mathbb{E}_j] = -2f_{ij k}(\varphi) 1|\mathbb{E}_k + [1|\mathbb{E}_i, 1|\mathbb{E}_j] \quad (34)
\]

which is equal to zero iff \(i = j\) but for the soft seven sphere, \([\mathbb{E}_i, 1|\mathbb{E}_j] \varphi = 0\) not only for \(i = j\) but also at the NP/SP for any \(i,j\).

Lastly over any group manifold the left torsion equals minus the right torsion, but for \(S^7\) this is not in general true.

## 2 More General Solutions

In the previous section, we have used brute force to calculate \(f_{ijk}^{(+)}(\varphi)\). There is another way, smarter and easier. We have the following situation

\[
\mathbb{E}_i \mathbb{E}_j \varphi = \left(-\delta_{ij} + f_{ijk}^{(+)}(\varphi) \mathbb{E}_k\right) \varphi \quad (35)
\]

but one can check that our \(\mathbb{E}_i\) defines what Cartan calls pure spinors \[25\],

\[
\varphi^T \mathbb{E}_i \varphi = 0 \quad (36)
\]

10
thus
\[ \varphi^t(\mathbf{E}_i \mathbf{E}_j) \varphi = \varphi^t(-\delta_{ij}) \varphi, \] (37)

using
\[ (\mathbf{E}_k)^{-1} = -\mathbf{E}_k \] (38)

we find
\[ \varphi^t(-\mathbf{E}_k \mathbf{E}_i \mathbf{E}_j) \varphi = \varphi^t\left(f_{ijk}^+ \varphi\right) \] (39)

but
\[ \varphi^t \varphi = r^2 \] (40)

which gives us
\[ f_{ijk}^+ (\varphi) = {\varphi^t(-\mathbf{E}_k \mathbf{E}_i \mathbf{E}_j) \varphi \over r^2}. \] (41)

and
\[ f_{ijk}^- (\varphi) = {\varphi^t(-1|\mathbf{E}_k 1|\mathbf{E}_i 1|\mathbf{E}_j) \varphi \over r^2}. \] (42)

There is another interesting property to note
\[ \varphi^t[\mathbf{E}_i, 1|\mathbf{E}_j] \varphi = 0 \] (43)

which may be the generalization of the standard Lie algebra relation, left and right action commute everywhere over the group manifold.

The left and right torsions that we have constructed are not the only parallelizable torsions of $S^7$. Our $\mathbf{E}_i$ and $1|\mathbf{E}_i$ are given in terms of the octonionic structure constants (15) i.e. the torsion at NP/SP. Considering two new points, we may define new sets of $\mathbf{E}_i$ and $1|\mathbf{E}_i$. As $S^7$ contains an infinity of points, practically, we have an infinity of parallelizable torsion. If our method is self contained and sufficient, we should be able to construct these infinity of pointwise structures. Indeed, $\mathbf{E}_i(\varphi)$ and $1|\mathbf{E}_i(\varphi)$ are in general
\[ \delta_i \iff (\mathbf{E}_i(\varphi))_{\mu\nu} = \delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} - f_{i\mu\nu}^+(\varphi), \] (44)
\[ 1|\delta_i \iff (1|\mathbf{E}_i(\varphi))_{\mu\nu} = \delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} + f_{i\mu\nu}^-(\varphi), \]
in complete analogy with (11,12,13,14). Of course the soft Algebra idea should hold here as well as for the special $(\mathbf{E}_i, 1|\mathbf{E}_i)$ constructed in terms of the north pole torsion. Repeating the calculation in terms of $(\mathbf{E}_i(\varphi), 1|\mathbf{E}_i(\varphi))$

Let us introduce a new vector field $\lambda$,
\[ \lambda^i = (\lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5 \lambda_6 \lambda_7). \] (45)
We define two new generalized structure functions

\[ [E_i(\varphi), E_j(\varphi)] \lambda = 2f^{(++)}_{ijk}(\varphi, \lambda)E_k(\varphi) \lambda \] (46)

\[ [1|E_i(\varphi), 1|E_j(\varphi)] \lambda = 2f^{(--)}_{ijk}(\varphi, \lambda)1|E_k(\varphi) \lambda \] (47)

where \( f^{(\pm \pm)}_{ijk}(\varphi, \lambda) \) have a very complicated structure,

\[ f^{(++)}_{ijk}(\varphi, \lambda) = \frac{\lambda^t (-E_k(\varphi)E_i(\varphi)E_j(\varphi)) \lambda}{r^2} \] (48)

\[ f^{(--)}_{ijk}(\varphi, \lambda) = \frac{\lambda^t (-1|E_k(\varphi)1|E_i(\varphi)1|E_j(\varphi)) \lambda}{r^2} \] (49)

3 Some Group Theory

An arbitrary octonion can be associated to \( \mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7 \) where \( \mathbb{R} \) denotes the subspace spanned by the identity \( e_0 = 1 \). Octonions with unit length define the octonionic unit sphere \( S^7 \). The isometries of octonions is described by \( O(8) \) which may be decomposed as

\[ O(8) : \quad H \oplus K \oplus E \] (50)

where \( H \) is the 14 parameters \( G_2 \) algebra of the automorphism group of octonions, \( K \) is the torsionful seven sphere \( SO(7)/G_2 \) and our \( E \) is the round seven sphere \( SO(8)/SO(7) \). In fact the different three non-equivalent representation of \( O(8) \) - the vectorial \( so(8) \) and the two different spinorial \( spin^L(8) \) and \( spin^R(8) \), which are related by triality, can be realized by suitable left and right octonionic multiplication. The reduction of \( O(8) \) to \( O(7) \) induces \( so(8) \rightarrow so(7) \oplus 1, \) \( spin^R(8) \rightarrow spin(7) \) and \( spin^L(8) \rightarrow spin(7) \).

We would like to show how to generate these different Lie algebras entirely from our canonical left/right octonionic structures. We start from the \( 8 \times 8 \) gamma matrices \( \gamma_{i\mu}^j \) in seven dimensions, using \( \delta_{ij} \) as our flat Euclidean metric,

\[ \{ \gamma^i, \gamma^j \} = 2\delta^{ij}1_8 \] ,

(51)

where \( i,j, \ldots = 1,2,\ldots,7 \) and \( \mu,\nu,\ldots = 0,1,2,\ldots,7 \). We can use either of the following choices

\[ \gamma^j_+ = i|E_j \quad \text{or} \quad \gamma^j_- = i1|E_j \] ,

(52)
of course the $i$ in the right hand sides is the imaginary complex unit. This relates our antisymmetric, Hermitian and hence purely imaginary gamma matrices to the canonical octonionic left/right structures. The antisymmetric product of two gamma matrices will be denoted by
\[
\gamma^{ij} = \gamma^{[i} \gamma^{j]} ,
\]
and we have
\[
\gamma^i \gamma^j \gamma^k = \frac{1}{4!} \epsilon^{ijklmnp} \gamma^l \gamma^m \gamma^n \gamma^p .
\]

The matrices $\gamma^{ij}$ span the 21 generators $J^{ij}$ of spin(7) in its eight-dimensional spinor representation. The spinorial representation of spin(7) can be enlarged to the left/right handed spinor representation of spin(8) by different ways. The easiest one is to include either of $\pm E_i$ or $\pm 1|E_i$ defining $J^i = J^{i0}$, so(8) can be written as
\[
\begin{align*}
[J^i, J^i] &= 2J^{ij} \\
[J^i, J^{mn}] &= 2\delta^{im} J^{jn} - 2\delta^{jm} J^{in} \\
[J^{ij}, J^{kl}] &= 2\delta^{jk} J^{il} + 2\delta^{il} J^{jk} - 2\delta^{ik} J^{jl} - 2\delta^{jl} J^{ik} .
\end{align*}
\]

The automorphism group of octonions is $G_2 \subset SO(7) \subset SO(8)$. A suitable basis for $G_2$ is
\[
H_{ij} = f_{ijk} (E_k - 1|E_k) - \frac{3}{2} [E_i, 1|E_j] ,
\]
which implies the linear relations
\[
f_{ijk} H_{jk} = 0 ,
\]
These constraints enforce $H_{ij}$ to generate the 14 dimensional vector space of $G_2$. There are different ways to represent the remaining seven generators, denoted here by K,
\[
\begin{align*}
\frac{so(7)}{G_2} : & \quad K^{\pm i}_v = \pm \frac{1}{2} (E_i - 1|E_i) , \\
\frac{spin(7)}{G_2} : & \quad K^{\pm i}_s = \pm \left( \frac{1}{2} E_i + 1|E_i \right) , \\
\frac{spin(7)}{G_2} : & \quad K^{\pm i}_s = \mp \left( E_i + \frac{1}{2} 1|E_i \right) .
\end{align*}
\]

\footnote{It is interesting to note that this equation may be used as an alternative definition for the octonionic multiplication table.}
Defining the conjugate representation\[\] by

\[\mathbb{E} = -1|\mathbb{E} \quad \text{and} \quad 1|\mathbb{E} = -\mathbb{E},\] (63)

(64) is self-conjugate while (61) is octonionic-conjugate to (62). The vector representation so(7) generated by \(H_{ij} \oplus K_{v}^{\pm i}\) is seven dimensional because \(K_{v}^{\pm i}e_0 = 0\) whereas the spin(7) representation generated by \(H_{ij} \oplus K_{s}^{\pm i}\) is eight dimensional.

To make apparent the role of the automorphism group \(G_2\), the different commutators of \(\mathbb{E}\) and \(1|\mathbb{E}\) may be written as

\[\begin{align*}
[\mathbb{E}_i, \mathbb{E}_j] &= \frac{1}{3} (4H_{ij} + 2f_{ijk}\mathbb{E}_k + 4f_{ijk}1|\mathbb{E}_k) , \\
[1|\mathbb{E}_i, 1|\mathbb{E}_j] &= \frac{1}{3} (4H_{ij} - 4f_{ijk}\mathbb{E}_k - 2f_{ijk}1|\mathbb{E}_k) , \\
[\mathbb{E}_i, 1|\mathbb{E}_j] &= \frac{1}{3} (-2H_{ij} + 2f_{ijk}\mathbb{E}_k - 2f_{ijk}1|\mathbb{E}_k).
\end{align*}\] (64) (65) (66)

or \(G_2\) given by

\[H_{ij} = \frac{1}{2} ([\mathbb{E}_i, \mathbb{E}_j] + [1|\mathbb{E}_i, 1|\mathbb{E}_j] + [\mathbb{E}_i, 1|\mathbb{E}_j]) .\] (67)

Thus as we promised, the \(\mathbb{E}\) and \(1|\mathbb{E}\) are the necessary and the sufficient building blocks for expressing the different Lie algebras and coset representations related to the seven sphere. Note that all the constructions given in this section start from the Clifford algebra relation (52), and the formulation holds equally for \(\mathbb{E}(\varphi)\) and \(1|\mathbb{E}(\varphi)\).

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\footnote{n.b. this definition is not matrix conjugation.}
A The structure function of the soft seven sphere

The seven standard cycles are given by

\[
\begin{align*}
f_{123}^{(+)}(\varphi) &= -f_{123}^{(-)}(\varphi) = \varphi_0^2 - \varphi_6^2 - \varphi_5^2 + \varphi_2^2 - \varphi_3^2 + \varphi_1^2 - \varphi_7^2, \\
f_{145}^{(+)}(\varphi) &= -f_{145}^{(-)}(\varphi) = \varphi_0^2 - \varphi_6^2 + \varphi_1^2 - \varphi_2^2 - \varphi_3^2 + \varphi_7^2 - \varphi_5^2, \\
f_{176}^{(+)}(\varphi) &= -f_{176}^{(-)}(\varphi) = \varphi_0^2 + \varphi_6^2 - \varphi_1^2 - \varphi_2^2 + \varphi_3^2 + \varphi_7^2 + \varphi_5^2, \\
f_{246}^{(+)}(\varphi) &= -f_{246}^{(-)}(\varphi) = \varphi_0^2 + \varphi_6^2 + \varphi_1^2 + \varphi_2^2 - \varphi_3^2 - \varphi_7^2 - \varphi_5^2, \\
f_{257}^{(+)}(\varphi) &= -f_{257}^{(-)}(\varphi) = \varphi_0^2 - \varphi_6^2 - \varphi_1^2 - \varphi_2^2 + \varphi_3^2 + \varphi_7^2 + \varphi_5^2, \\
f_{347}^{(+)}(\varphi) &= -f_{347}^{(-)}(\varphi) = \varphi_0^2 - \varphi_6^2 + \varphi_1^2 + \varphi_2^2 - \varphi_3^2 - \varphi_7^2 - \varphi_5^2, \\
f_{365}^{(+)}(\varphi) &= -f_{365}^{(-)}(\varphi) = \varphi_0^2 + \varphi_6^2 - \varphi_1^2 - \varphi_2^2 - \varphi_3^2 + \varphi_7^2 + \varphi_5^2.
\end{align*}
\]

and the non-standard subset

\[
\begin{align*}
f_{124}^{(+)}(\varphi) &= +2 \varphi_0 \varphi_6 - \varphi_0 \varphi_2 - \varphi_1 \varphi_6 + \varphi_1 \varphi_2, \\
f_{125}^{(+)}(\varphi) &= -2 \varphi_0 \varphi_6 - \varphi_0 \varphi_2 - \varphi_1 \varphi_6 - \varphi_1 \varphi_2, \\
f_{126}^{(+)}(\varphi) &= +2 \varphi_0 \varphi_6 - \varphi_0 \varphi_2 + \varphi_1 \varphi_6 + \varphi_1 \varphi_2, \\
f_{127}^{(+)}(\varphi) &= -2 \varphi_0 \varphi_6 - \varphi_0 \varphi_2 + \varphi_1 \varphi_6 - \varphi_1 \varphi_2, \\
f_{143}^{(+)}(\varphi) &= +2 \varphi_1 \varphi_6 + \varphi_2 \varphi_6 + \varphi_3 \varphi_6 - \varphi_4 \varphi_6, \\
f_{146}^{(+)}(\varphi) &= -2 \varphi_1 \varphi_6 + \varphi_2 \varphi_6 - \varphi_3 \varphi_6 + \varphi_4 \varphi_6, \\
f_{175}^{(+)}(\varphi) &= +2 \varphi_3 \varphi_6 + \varphi_4 \varphi_6 - \varphi_5 \varphi_6 + \varphi_6 \varphi_6, \\
f_{245}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 + \varphi_5 \varphi_6 - \varphi_6 \varphi_6, \\
f_{253}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 - \varphi_3 \varphi_6 - \varphi_4 \varphi_6 + \varphi_5 \varphi_6, \\
f_{256}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 - \varphi_3 \varphi_6 + \varphi_4 \varphi_6 + \varphi_5 \varphi_6, \\
f_{345}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 - \varphi_3 \varphi_6 + \varphi_4 \varphi_6 - \varphi_5 \varphi_6, \\
f_{346}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 - \varphi_4 \varphi_6 + \varphi_5 \varphi_6, \\
f_{347}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 + \varphi_4 \varphi_6 + \varphi_5 \varphi_6, \\
f_{361}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 + \varphi_5 \varphi_6 - \varphi_6 \varphi_6, \\
f_{362}^{(+)}(\varphi) &= -2 \varphi_2 \varphi_6 - \varphi_3 \varphi_6 + \varphi_5 \varphi_6 + \varphi_6 \varphi_6, \\
f_{365}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 + \varphi_4 \varphi_6 - \varphi_5 \varphi_6, \\
f_{367}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 + \varphi_4 \varphi_6 + \varphi_5 \varphi_6, \\
f_{367}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 - \varphi_3 \varphi_6 + \varphi_4 \varphi_6 - \varphi_5 \varphi_6, \\
f_{367}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 - \varphi_5 \varphi_6 - \varphi_6 \varphi_6, \\
f_{367}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 - \varphi_5 \varphi_6 + \varphi_6 \varphi_6, \\
f_{456}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 - \varphi_5 \varphi_6 - \varphi_6 \varphi_6, \\
f_{467}^{(+)}(\varphi) &= +2 \varphi_2 \varphi_6 + \varphi_3 \varphi_6 + \varphi_5 \varphi_6 + \varphi_6 \varphi_6.
\]
For right actions, the non-standard cocycles are

\[ f_{124}^{(-)}(\varphi) = +2 \varphi_0 \varphi_7 + \varphi_5 \varphi_2 - \varphi_0 \varphi_1 - \varphi_1 \varphi_2 , \]
\[ f_{125}^{(-)}(\varphi) = -2 \varphi_0 \varphi_6 + \varphi_3 \varphi_5 + \varphi_1 \varphi_7 + \varphi_2 \varphi_4 , \]
\[ f_{126}^{(-)}(\varphi) = +2 \varphi_0 \varphi_5 + \varphi_1 \varphi_4 - \varphi_2 \varphi_3 - \varphi_1 \varphi_6 , \]
\[ f_{127}^{(-)}(\varphi) = -2 \varphi_0 \varphi_4 - \varphi_6 \varphi_2 - \varphi_1 \varphi_5 + \varphi_3 \varphi_7 , \]
\[ f_{143}^{(-)}(\varphi) = +2 \varphi_2 \varphi_6 - \varphi_3 \varphi_5 - \varphi_2 \varphi_4 + \varphi_1 \varphi_7 , \]
\[ f_{146}^{(-)}(\varphi) = -2 \varphi_3 \varphi_0 + \varphi_4 \varphi_7 + \varphi_1 \varphi_5 + \varphi_5 \varphi_6 , \]
\[ f_{175}^{(-)}(\varphi) = +2 \varphi_3 \varphi_0 + \varphi_1 \varphi_2 - \varphi_3 \varphi_6 - \varphi_4 \varphi_7 , \]
\[ f_{247}^{(-)}(\varphi) = -2 \varphi_0 \varphi_1 - \varphi_7 \varphi_6 - \varphi_4 \varphi_5 - \varphi_3 \varphi_2 , \]
\[ f_{147}^{(-)}(\varphi) = +2 \varphi_2 \varphi_0 + \varphi_4 \varphi_6 - \varphi_5 \varphi_7 - \varphi_1 \varphi_3 , \]
\[ f_{243}^{(-)}(\varphi) = -2 \varphi_0 \varphi_5 - \varphi_1 \varphi_4 - \varphi_7 \varphi_2 + \varphi_3 \varphi_6 , \]
\[ f_{203}^{(-)}(\varphi) = +2 \varphi_0 \varphi_4 + \varphi_1 \varphi_5 - \varphi_6 \varphi_2 - \varphi_2 \varphi_3 , \]
\[ f_{245}^{(-)}(\varphi) = -2 \varphi_2 \varphi_0 - \varphi_4 \varphi_6 - \varphi_3 \varphi_7 - \varphi_1 \varphi_2 , \]
\[ f_{256}^{(-)}(\varphi) = +2 \varphi_0 \varphi_1 + \varphi_3 \varphi_2 - \varphi_7 \varphi_6 - \varphi_4 \varphi_5 , \]
\[ f_{301}^{(-)}(\varphi) = +2 \varphi_0 \varphi_4 - \varphi_1 \varphi_2 - \varphi_3 \varphi_6 + \varphi_5 \varphi_7 , \]
\[ f_{362}^{(-)}(\varphi) = -2 \varphi_0 \varphi_7 + \varphi_2 \varphi_3 + \varphi_6 \varphi_6 + \varphi_5 \varphi_5 , \]
\[ f_{345}^{(-)}(\varphi) = -2 \varphi_2 \varphi_0 + \varphi_5 \varphi_7 + \varphi_1 \varphi_3 - \varphi_6 \varphi_6 , \]
\[ f_{346}^{(-)}(\varphi) = +2 \varphi_0 \varphi_1 - \varphi_3 \varphi_2 - \varphi_7 \varphi_6 - \varphi_4 \varphi_5 , \]
\[ f_{367}^{(-)}(\varphi) = +2 \varphi_2 \varphi_0 - \varphi_4 \varphi_6 - \varphi_3 \varphi_7 + \varphi_1 \varphi_3 , \]
\[ f_{135}^{(-)}(\varphi) = -2 \varphi_0 \varphi_7 + \varphi_3 \varphi_5 - \varphi_1 \varphi_2 , \]
\[ f_{136}^{(-)}(\varphi) = -2 \varphi_2 \varphi_0 - \varphi_1 \varphi_3 - \varphi_4 \varphi_6 + \varphi_5 \varphi_7 , \]
\[ f_{237}^{(-)}(\varphi) = +2 \varphi_0 \varphi_1 + \varphi_2 \varphi_4 - \varphi_6 \varphi_2 - \varphi_3 \varphi_7 , \]
\[ f_{267}^{(-)}(\varphi) = -2 \varphi_0 \varphi_7 + \varphi_2 \varphi_5 - \varphi_4 \varphi_7 - \varphi_1 \varphi_2 , \]
\[ f_{156}^{(-)}(\varphi) = -2 \varphi_2 \varphi_6 + \varphi_3 \varphi_5 + \varphi_4 \varphi_7 + \varphi_1 \varphi_2 , \]
\[ f_{157}^{(-)}(\varphi) = +2 \varphi_0 \varphi_1 - \varphi_4 \varphi_5 + \varphi_7 \varphi_2 - \varphi_3 \varphi_6 , \]
\[ f_{567}^{(-)}(\varphi) = +2 \varphi_0 \varphi_4 + \varphi_6 \varphi_2 + \varphi_1 \varphi_5 + \varphi_3 \varphi_7 . \]
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