Motion of Curves on Two Dimensional Surfaces and Soliton Equations

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Abstract

A connection is established between the soliton equations and curves moving in a three dimensional space $V_3$. The sign of the self-interacting terms of the soliton equations are related to the signature of $V_3$. It is shown that there corresponds a moving curve to each soliton equations.
Differential geometry and partial differential equations (PDEs) are two different research areas in mathematics. When we study some local properties of surfaces in Euclidean ($E_3$) or Minkowskian ($M_3$) 3-spaces we face with some known PDEs. For instance the Liouville and Sine-Gordon equations describe surfaces of constant Gaussian curvature [1]. Gauss-Codazzi-Mainardi equations describe the surfaces embedded in $E_3$ or in $M_3$. These equations are used for the construction of the soliton connection [2],[3],[4]. Here the differential geometrical tools are utilized to find for example the Backlund transformations and prolongation structures [5] of the soliton equations.

During the last two decades another virtue of the differential geometry arised in soliton theory. The Serret-Frenet equations for the family of curves (the motion of curves) give certain coupled partial differential equations for the curvature ($k$) and torsion ($\tau$) scalars of these curves [6]-[12]. It was shown that some soliton equations like the modified Korteweg de Vries (mKdV), sine-Gordon and nonlinear Schrodinger (NLS) are among the equations which may arise from the motion of space curves. All these considerations were in Euclidean three space $E_3$. This is why only one versions of the nonlinear couplings of the mKdV and NLSEs could have been obtained.

In this work we take a three space $V_3$ with signature $1 + 2\epsilon$, where $\epsilon^2 = 1$. This means that curves in $M_3$ will also be considered. Self interacting terms in the evolution equations of the curvature and the torsion of these curves depend upon signature of the space $V_3$. The sign difference of the self interaction terms is due to signature change of the three space. If for instance a curve $C$ is moving in $E_3$ (or in $M_3$) focusing (or defocusing) versions of mKdV or NLS equations arise.
The motion of the curve \( C \) is described by three functions \( p \), \( q \) and \( w \). The function \( w \) is determined in terms of the others but the functions \( p \) and \( q \) are left arbitrary. Each choice of these functions gives a different class of curves in \( V_3 \). It is in principle possible to convert the differential equations satisfied by the scalars \( k \) and \( \tau \) to any system of coupled two nonlinear PDEs. Here we should remark that not all these equations are integrable. The integrability property of these equations (for each choice of \( p \) and \( q \)) should be examined. The functions \( p \) and \( q \) can be suitably chosen to make the evolution equations satisfied by \( k \) and \( \tau \) integrable. So far, for this purpose \([6]-[12]\) \( p \) and \( q \) were assumed to be local functions of \( k \) and \( \tau \). By this way mKdV, NLS, and complex mKdV equations could be obtained.

On the other hand one may obtain, by a proper choice of \( p \) and \( q \) (since they free), all possible integrable equations. This can be done by relaxing the locality assumptions on the functions \( p \) and \( q \). Sine-Gordon equation is obtained by assuming that \( q = \tau = 0 \) and \( p \) is a nonlocal function of the curvature \( k \) \([8]-[9]\). We show that any integrable system of coupled two nonlinear PDEs can be obtained by assuming nonlocal functional dependence. In this way it is possible to obtain for instance the AKNS \([13]\) hierarchy. Hence in general there exist a curve \( C \) moving in a \( V_3 \) corresponding to any integrable nonlinear differential equation (one or two coupled equations).

Some nonlinear partial differential equations, such as the sine-Gordon and the Liouville equations arise from the surfaces of constant Gaussian curvature. Here we show that such equations and many others may also arise from two dimensional surfaces with vanishing Gaussian curvature, flat surfaces (see also \([14]\)).
Let $V_3$ define a 3-dimensional flat space with the line element
\[ ds^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu \]
where $\mu, \nu = 1, 2, 3$, $x^\mu = (t, y, z)$, and $\eta_{\mu\nu} = \text{diag}(1, \epsilon, \epsilon)$. If $\epsilon = 1$ then $V_3 = E_3$ is an Euclidean three space and if $\epsilon = -1$ then $V_3 = M_3$ is a pseudo-Euclidean (Minkowskian) three space. Hence (1) explicitly takes the form
\[ ds^2 = dt^2 + \epsilon \, dy^2 + \epsilon \, dz^2 \]
Let $S$ be a surface in $V_3$ parametrized by $x^\mu(u, v)$, and let $C$ be a curve on $S$ defined by $\alpha : I \to S$ and parametrized by its arclength $s \in I$. An orthonormal frame $(t^\mu, n^\mu, b^\mu)$ at each point of $C$ is defined by (recalling that $x^\mu_{,s} = t^\mu$)
\[
\eta_{\mu\nu} \, t^\mu \, t^\nu = 1, \quad \eta_{\mu\nu} \, n^\mu \, n^\nu = \epsilon, \quad \eta_{\mu\nu} \, b^\mu \, b^\nu = \epsilon
\]
all the other products vanish. The Serret-Frenet equations are $(x^\mu_{,s} = t^\mu)$
\[
\begin{align*}
t^\mu_{,s} &= k \, n^\mu, \\
n^\mu_{,s} &= -\epsilon \, k \, t^\mu - \tau \, b^\mu, \\
b^\mu_{,s} &= \tau \, n^\mu
\end{align*}
\]
where $k$ and $\tau$ are the curvature and the torsion scalars of the curve $C$ at any point $s$. The vectors $t^\mu$, $n^\mu$ and $b^\mu$ are respectively the tangent, normal and bi-normal vectors to the curve at any point $s$. 

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A curve on $S$ is given by $\alpha^\mu(s) = x^\mu(u(s), v(s))$. This curve may be considered as a member of the family of curves $\beta^\mu_\sigma = x^\mu(u(s, \sigma), v(s, \sigma))$ for a fixed value of $\sigma$. The change (motion) of the curve with respect to the parameter $\sigma$ (on $S$) is given by

$$x^\mu_{,\sigma} = p n^\mu + w t^\mu + q b^\mu$$

(7)

where the $p$, $q$ and $w$ are functions of $s$ and $\sigma$. By using the equation $x^\mu_{,s} = t^\mu$ and (7) we get $w_{,s} = \epsilon k p$ and $t^\mu_{,\sigma}$ (partial derivative of the vector $t^\mu$ with respect to $\sigma$). Using $t^\mu_{,\sigma}$ obtained this way and the first equation (4) of the Serret-Frenet equations one obtains $k_{,\sigma}$ and $n^\mu_{,\sigma}$. Following the similar approach one finds derivatives of the scalars ($k, \tau$) and vectors ($t^\mu, n^\mu, b^\mu$). They are given by

$$t^\mu_{,\sigma} = (p_{,s} + k w + \tau q) n^\mu + (q_{,s} - \tau p) b^\mu$$

(8)

$$n^\mu_{,\sigma} = -\epsilon (p_{,s} + k w + \tau q) t^\mu + \frac{1}{k} [(q_{,s} - \tau p)_{,s} - \tau (p_{,s} + k w + \tau q)] b^\mu$$

(9)

$$b^\mu_{,\sigma} = -\frac{1}{k} [(q_{,s} - \tau p)_{,s} - \tau (p_{,s} + k w + \tau q)] n^\mu - \epsilon (q_{,s} - \tau p) t^\mu$$

(10)

The compatibility conditions give $w_{,s} = \epsilon k p$ and
\[ k,\sigma = (p_s + kw + \tau q),s + \tau (q_s - \tau p) \]  
(11)

\[ \tau,\sigma = -[\frac{1}{k} ((q_s - \tau p)_s - \tau (p_s + kw + \tau q))_s - \epsilon k(q_s - \tau p)] \]  
(12)

These equations may be written in a compact form

\[
\begin{pmatrix}
  k \\
  \tau
\end{pmatrix}_{\sigma} = \mathcal{R}
\begin{pmatrix}
  p \\
  q
\end{pmatrix}
\]  
(13)

where

\[
\mathcal{R} = \begin{pmatrix}
  D^2 + \epsilon k^2 - \tau^2 + \epsilon k_s D^{-1} k & (D \tau + \tau D) \\
  D[\frac{1}{k}(D\tau + \tau D) + \epsilon \tau D^{-1} k] + \epsilon k\tau & -D[\frac{1}{k}D^2 - \frac{\tau^2}{k}] - \epsilon kD
\end{pmatrix}
\]  
(14)

In the special case \( \tau = q = 0 \) which means \( C \) is a plane curve we have

\[ k,\sigma = \mathcal{R} p \]  
(15)

where \( \mathcal{R} \) is the recursion operator of the mKdV equation \( k,\sigma = \mathcal{R} k_s \) given by

\[
\mathcal{R} = D^2 + \epsilon k^2 + \epsilon k_s D^{-1} k
\]  
(16)

Here \( D \) denotes the total derivative with respect to \( s \) and \( D^{-1} \) is its inverse. Choosing , for instance \( p = k_s \) then Eq.(15) reduces to mKdV. The choices of the geometry \( \epsilon = \pm 1 \) we have focusing and defocusing versions of the mKdV equations. Choosing \( p = \mathcal{R}^n k_s \) with \( n = 0, 1, 2, \ldots \) we obtain the infinite
integrable hierarchy of the mKdV equations. As another local choices we need to write Eqs.(11) and (12) in a complexified form

\[ \phi_{,\sigma} = \{ D^2 + i\eta \epsilon \phi D^{-1} \tau \phi^* + |\phi|^2 + \phi_{,s} D^{-1} \phi^* \} (p \rho) + \]
\[ \{-i\eta D^2 - i\eta \epsilon |\phi|^2 - \epsilon \phi D^{-1} \tau \phi^* + i\eta \epsilon \phi D^{-1} \phi_{,s}^* \} (q \rho) \] (17)

where \( \eta^2 = 1 \), \( \rho = e^{i\eta(D^{-1} \tau)} \) and \( \phi = k \rho \) and \( \phi^* \) is the complex conjugate of \( \phi \). When \( p = 0 \) and \( q = k \) then we have the nonlinear Schrodinger (NLS) equation of both versions (\( \epsilon = \pm 1 \)).

\[ i\eta \phi_{,\sigma} = D^2 \phi + \frac{\epsilon}{2} |\phi|^2 \phi \] (18)

Another example is obtained by letting \( p = k_s \) and \( q = -k\tau \). This is the complex mKdV

\[ \phi_{,\sigma} = D^3 \phi + \frac{3}{2} |\phi|^2 \phi_{,s} \] (19)

In all these choices the function \( p \) is choosen as local functions of the \( k \). This means that \( p \) is a function of \( k \) and its partial derivatives with respect to \( s \) and \( \sigma \) to all orders. Other local choices of the function \( p \) in terms of \( k \) may or may not give integrable nonlinear partial differential equations (equations admitting infinitely many generalised symmetries). For each choice of \( p \) one must check whether the resulting equation is integrable \[10, 17, 18\]. The main motivation why integrable equations are trying to be chosen is their position in mathematics and physics.
It is also possible to choose the function $p$ as a nonlocal function of $k$. Choosing for instance $p = R^{-2}k_{,s}$ and letting $k = \theta_{,s}$ we obtain the sine-Gordon equation $\theta_{,s\sigma} = \sin(\theta)$ \cite{3}. Another choice for instance may be $p = R^{-1}(R_{kdv})^n k_{,s}$, where $R_{kdv} = D^2 + 4k + 2k_{,s}D^{-1}$ is the recursion operator of the KdV equation $k_{,\sigma} = k_{,sss} + 6kk_{,s}$. This choice will give the hierarchy of the KdV equation $k_{,\sigma} = (R_{kdv})^n k_{,s}$, for $n = 0, 1, 2, \ldots$. It is clear from these examples that since $p$ is an arbitrary function, Eq. (15) may be reduced to any nonlinear partial differential equation. One can properly choose $p$ so that all scalar integrable nonlinear PDE can be obtained from Eq. (15).

In the general case by choosing $p$ and $q$ properly so that Eq. (14) can be reduced to any system of coupled two nonlinear PDEs. As an example letting

\[
\begin{pmatrix}
p \\
q
\end{pmatrix} = R^{-1}(R_{akns})^n \begin{pmatrix}
k \\
\tau
\end{pmatrix}_{,s}
\]

where $R_{akns}$ is the recursion operator of the AKNS system of equations given by

\[
R_{akns} = \begin{pmatrix}
D + 2kD^{-1}\tau & 2kD^{-1}k \\
-2\tau D^{-1}\tau & -D - 2\tau D^{-1}k
\end{pmatrix}
\]

Eq. (13) reduces to AKNS hierarchy for $n = 0, 1, 2, \ldots$. Hence there corresponds a class of moving curves in $V_3$ to each system of two coupled soliton equations.
The derivatives of the vectors in the frame \( e^\mu_a = (t^\mu, n^\mu, b^\mu) \) may be written in more familiar form \( de^\mu_a = \Omega^h_a e^\mu_h \), where in matrix notation \( \Omega \) is a matrix valued 1-form. Here \( a, b = 1, 2, 3 \) and \( \Omega = \Omega_s ds + \Omega_\sigma d\sigma \), where

\[
\begin{align*}
\Omega_s &= \begin{pmatrix} 0 & k & 0 \\ -\epsilon k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}, & \Omega_\sigma &= \begin{pmatrix} 0 & w_1 & w_0 \\ -\epsilon w_1 & 0 & w_2 \\ -\epsilon w_0 & -w_2 & 0 \end{pmatrix}
\end{align*}
\tag{23}
\]

with

\[
\begin{align*}
w_0 &= q, s - \tau p , & w_1 &= p, s + k w + \tau q \\
w_2 &= \frac{1}{k} [(q, s - \tau p), s - \tau (p, s + k w + \tau q)]
\end{align*}
\tag{24, 25}
\]

The 1-form \( \Omega \) defines a connection with zero curvature. This is due to the flatness of the space \( V_3 \). Vanishing of the curvature of \( \Omega \), i.e., \( d\Omega - \Omega \Omega = 0 \), is due to the evolution equations given in \((13)\). In order to compare this connection 1-form with the soliton connection 1-form we write it in more suitable form \([2, 3]\)

\[
\Omega = \begin{pmatrix} 0 & \pi_0 & \pi_1 \\ -\epsilon \pi_0 & 0 & \pi_2 \\ -\epsilon \pi_1 & -\pi_2 & 0 \end{pmatrix}
\tag{26}
\]

where the 1-forms \( \pi_0 \), \( \pi_1 \) and \( \pi_2 \) are given by
These 1-forms satisfy (from the zero curvature condition)

\[ d\pi_0 + \pi_1 \pi_2 = 0, \quad d\pi_1 - \pi_0 \pi_2 = 0, \quad d\pi_2 + \epsilon \pi_0 \pi_1 = 0 \]  

(28)

An $SL(2, R)$ valued soliton connection 1-form $\Gamma$ may be given in terms of the 1-forms $\pi_0$, $\pi_1$ and $\pi_2$

\[ \Gamma = \begin{pmatrix} \theta_0 & \theta_1 \\ \theta_2 & -\theta_0 \end{pmatrix} \]  

(29)

where

\[ \theta_0 = \alpha \pi_0, \quad \theta_1 = \alpha_1 (\pi_1 + \frac{1}{2\alpha} \pi_2), \quad \theta_2 = \alpha_2 (\pi_1 - \frac{1}{2\alpha} \pi_2). \]  

(30)

Here we have $4 \alpha^2 + \epsilon = 0$, $\alpha_1 \alpha_2 = \alpha^2$. Let $\Psi$ be a $2 \times 2$ matrix valued (0-form) function of $s$ and $\tau$. Then $d\Psi = \Gamma \Psi$ defines the Lax equation without a spectral parameter. A constant may be introduced by performing a gauge transformation $\Gamma' = S \Gamma S^{-1} + dS S^{-1}$. Here $S$ is $2 \times 2$ matrix valued function of $s$, $\sigma$ and the spectral parameter. In this way we set up a correspondence between a curve $C$ moving in a space $V_3$ with a soliton connection.

The line element (1) on the surface $S$, using the parameters $(s, \sigma)$ of the moving $C$, reduces to
\[ ds^2 = (ds + w \, d\sigma)^2 + \epsilon \left( p^2 + q^2 \right) d\sigma^2 \]  \hspace{0.5cm} (31)

The Gaussian curvature \( K \) of \( S \) with the first fundamental form given in (31) is different from zero in general. On the other hand by the choice \( \tau = q = 0 \), the line element becomes

\[ ds^2 = (ds + w \, dt)^2 + \epsilon p^2 \, dt^2 \]  \hspace{0.5cm} (32)

The Gaussian curvature \( K \) becomes

\[ K = \frac{1}{4p} (k_\sigma - R \, p) \]  \hspace{0.5cm} (33)

which vanishes by virtue of the equation \((13)\). Hence all the curves related to the eqn\((13)\) trace flat 2-surfaces. It was usually believed that integrable equations arise from the curved surfaces. For instance the sine-Gordon equation arise from the surface with the line element \( ds^2 = \cos^2(\theta) \, d\sigma^2 + \sin^2(\theta) \, ds^2 \), which describe surfaces of constant negative Gaussian curvature \([1]\). Here we show that all integrable equations including the sine-Gordon equation may also arise from flat 2-surfaces (for mKdV see \([14]\)).

In this work we considered the motion of a curve in a three space \( V_3 \). This condition may be relaxed, but for an arbitrary \( V_n \) where \( n > 3 \) the evolution equations corresponding to the geometrical scalars \((k, \tau, ...)\) of the curves become quite complicated. It is perhaps more physical and significant to consider the case \( n = 4 \). This corresponds to classical strings moving in four dimensional Minkowskian space. Hence it will be quite interesting to see the correspondance between strings and the soliton equations with four...
dependent variables. Let \( x^\mu(s, \sigma) \) denote the strings in \( M_4 \). In a similar manner we define curves \( x^\mu(s) \) parametrised with arclength \( s \) and its variations \( x^\mu(s, \sigma) \). We have the orthonormal tetrad \( (t^\mu, n^\mu, b_1^\mu, b_2^\mu) \) with

\[
\eta_{\mu\nu} t^\mu t^\nu = 1, \quad \eta_{\mu\nu} n^\mu n^\nu = \epsilon \quad (34) \\
\eta_{\mu\nu} b_1^\mu b_1^\nu = \epsilon, \quad \eta_{\mu\nu} b_2^\mu b_2^\nu = \epsilon \quad (35)
\]

where \( \eta_{\mu\nu} = diag(1, \epsilon, \epsilon, \epsilon) \), the Greek letters run from 1 to 4. Here \( \epsilon = -1 \) but we keep it to compare the results obtained here with previous sections. The Serret-Frenet equations governing the motion of the tetrad are as follows

\[
t^\mu_s = k n^\mu, \quad (36) \\
(37) \\
b_1^\mu_s = \tau_2 n^\mu + \tau_3 b_2^\mu, \quad (38) \\
b_2^\mu_s = \tau_1 n^\mu - \tau_3 b_1^\mu \quad (39)
\]

where \( k, \tau_1, \tau_2, \tau_3 \) are the geometrical scalars describing the curvature, and torsions in each three space directions respectively. Here we have \( t^\mu = x^\mu_s \).

Letting

\[
x^\mu_{,\sigma} = p n^\mu + w t^\mu + q_1 b_1^\mu + q_2 b_2^\mu \quad (40)
\]

where \( p, w, q_1, q_2 \) are functions of \( s \) and \( \sigma \). Here it is clear that when \( \tau_2, \tau_3 \) and \( q_2 \) vanish we get the same equations obtained in the previous sections for three dimensional spaces. Hence the strings in four dimensions has a
very direct correspondance with the integrable evolution equations. This connection and further progress on the motion of curves in a four dimensional space will be communicated elsewhere.

In this work we established a connection between the curves moving in a three space with arbitrary signature (-1 or 3) and soliton equations. We showed that to each soliton (integrable) equation there exists a class of curve moving either in an Euclidean ($E_3$) or pseudo-Euclidean ($M_3$) three spaces. The signature of $V_3$ and the sign of the self interacting terms in the soliton equations are directly related. We also showed that many integrable nonlinear PDEs may also arise from flat surfaces contrary to the common belief so far [4].

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References

[1] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces (Ginn, Boston, 1909), (reprinted Dover, New York, 1960).

[2] M. Crampin, F. A. E. Pirani Abd D. C. Robinson, Lett. Math. Phys. 2, 15 (1977).

[3] M. Gürses and Y. Nutku, J. Math. Phys 22, 1353 (1981).

[4] B. G. Konopelchenko, SIAM 96, 9 (1996)
[5] H. Wahlquist and F. B. Estabrook, *Phys. Rev. Lett.* **31**, 1386 (1973).

[6] H. Hasimoto, *J. Fluid. Mech.* **51**, 477 (1972).

[7] G. L. Lamb, *Phys. Rev. Lett* **37**, 235 (1976); *J. Math. Phys.* **18**, 1654 (1977).

[8] K. Nakayama, H. Segur, and M. Wadati, *Phys. Rev. Lett* **69**, 2603 (1992).

[9] M. Lakshmanan, *J. Math. Phys.* **20**, 1667 (1978).

[10] J. Ciesliński, P.K.H. Gragert and A. Sym, *Phys. Rev. Lett* **57**, 1507 (1986).

[11] R.E. Goldstein and D.M. Petrich, *Phys. Rev. Lett* **67**, 3203 (1991).

[12] J. Langer and R. Perline, *J. Nonlinear Sci* **1**, 71 (1991).

[13] M. Ablowitz and H. Segur, *Solitons and Inverse Scattering Transform*, SIAM, Philadelphia, 1981.

[14] A. Sym, *Lett. Nuov. Cimento* **39**, 193 (1984).

[15] M. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey.

[16] A. Fokas, *SIAM* **77**, 253 (1987).

[17] A.V. Mikhailov, V.V. Sokolov, and A.B. Shabat, in *What is Integrability?* Ed V. E. Zakharov (Berlin: Springer) p 115, 1991.
[18] M. Gürses and A. Karasu, *J. Math. Phys* 36, 3485 (1995).