Diophantine Equations in Semiprimes

Shuntaro Yamagishi*

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Abstract: A semiprime is a natural number which is the product of two (not necessarily distinct) prime numbers. Let \( F(x_1, \ldots, x_n) \) be a degree \( d \) homogeneous form with integer coefficients. We provide sufficient conditions, similar to those of the seminal work of B. J. Birch [1], for which the equation \( F(x_1, \ldots, x_n) = 0 \) has infinitely many integer solutions with semiprime coordinates. Previously it was known, by a result of Á. Magyar and T. Titichetrakun [12], that under the same hypotheses there exist infinitely many integer solutions to the equation with coordinates that have at most \( 384n^{3/2}d(d+1) \) prime factors.

Key words and phrases: Hardy-Littlewood circle method, Diophantine equations, almost primes

1 Introduction

Solving Diophantine equations in primes or almost primes is a fundamental problem in number theory. For example, the celebrated work of B. Green and T. Tao [5] on arithmetic progressions in primes can be phrased as the statement that given any \( n \in \mathbb{N} \) the system of linear equations \( x_{i+2} - x_{i+1} = x_{i+1} - x_i \) (\( 1 \leq i \leq n \)) has a solution \( (p_1, \ldots, p_{n+2}) \) such that each \( p_i \) is prime and \( p_1 < p_2 < \ldots < p_{n+2} \). A major achievement extending this result in which a more general system of linear equations is considered has been established by B. Green, T. Tao, and T. Ziegler (see [6], [7], [8]) and we refer the reader to [6, Theorem 1.8] for the precise statement. Another important achievement in this area is the well-known theorem of Chen [3] related to the twin prime conjecture. The theorem asserts that the equation \( x_1 - x_2 = 2 \) has infinitely many solutions \( (\ell_1, p_2) \) where \( \ell_1 \) has at most two prime factors and \( p_2 \) is prime.

The main focus of this paper is on equations involving higher degree polynomials. Let \( d > 1 \). Let \( F(x) \) be a degree \( d \) homogeneous form in \( \mathbb{Z}[x_1, \ldots, x_n] \). We are interested in integer solutions \( x \) to the equation

\[
F(x) = 0
\]

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for which all coordinates have small numbers of prime factors. For this to be possible one has to impose appropriate conditions. Let \( \mathbb{Z}_p^\times \) be the units of \( p \)-adic integers. We consider the following conditions.

**Local conditions \((\star)\).** The equation (1) has a non-singular real solution in \((0, 1)^n\), and also has a non-singular solution in \((\mathbb{Z}_p^\times)^n\) for every prime \( p \).

Let \( V^*_F \) be an affine variety in \( \mathbb{A}^d_{\mathbb{C}} \) defined by

\[
V^*_F := \left\{ z \in \mathbb{C}^n : \frac{\partial F}{\partial x_j}(z) = 0 \ (1 \leq j \leq n) \right\}.
\]

(2)

By Euler’s formula it follows that \( V^*_F \) is the singular locus of \( V(F) = \{ z \in \mathbb{C}^n : F(z) = 0 \} \), but we shall consider it as a subvariety of \( \mathbb{A}^d_{\mathbb{C}} \) and let \( \text{codim} \ V^*_F = n - \dim V^*_F \).

For solving general non-linear polynomial equations in primes, the following important result was established by B. Cook and Á. Magyar [4].

**Theorem 1.1.** [4, Theorem 1] Let \( F(x) \in \mathbb{Z}[x_1, \ldots, x_n] \) be a degree \( d \) homogeneous form. Suppose that \( F \) satisfies the local conditions \((\star)\) and \( \text{codim} \ V^*_F \) is sufficiently large with respect to \( d \). Then the equation (1) has an infinite number of solutions \((p_1, \ldots, p_n)\) for which \( p_i \) is prime for each \( 1 \leq i \leq n \).

Here the theorem requires \( \text{codim} \ V^*_F \) to be very large. In fact, the required bound on \( \text{codim} \ V^*_F \) “already exhibit(s) tower type behavior in \( d \)” [4]. We also refer the reader to [17] for the case of quadratic forms. It is expected that a lower bound exponential in \( d \) is sufficient in Theorem 1.1 [4], because this is the case for integer solutions as seen in the work of B. J. Birch [1]. As the requirement on \( \text{codim} \ V^*_F \) in Theorem 1.1 is significantly larger than what is expected, it is natural to consider if one can achieve a result analogous to Theorem 1.1 for almost primes, which are positive integers with a small number of prime factors (counting multiplicity), with smaller \( \text{codim} \ V^*_F \). In this direction, there is a result by Á. Magyar and T. Titichetrakun [12] provided \( \text{codim} \ V^*_F > 2^{d(d-1)} \), which is also the required bound in [1].

**Theorem 1.2.** [12, Theorem 1.1] Let \( F(x) \in \mathbb{Z}[x_1, \ldots, x_n] \) be a degree \( d \) homogeneous form. Suppose that \( F \) satisfies the local conditions \((\star)\) and \( \text{codim} \ V^*_F > 2^{d(d-1)} \). Then the equation (1) has an infinite number of solutions \((\ell_1, \ldots, \ell_n)\) such that \( \ell_i \) has at most \( 384n^{3/2}d(d+1) \) prime factors for each \( 1 \leq i \leq n \).

This result was established by combining sieve methods with the Hardy-Littlewood circle method. In order to keep the amount of notation to a minimum we presented simplified statements of Theorems 1.1 and 1.2 (without quantitative estimates and only the case of one homogeneous form instead of systems of homogeneous forms of equal degree); we refer the reader to the respective papers for the precise statements. We also refer the reader to [10, Section 5.2] and [15, Section 17] for overviews of the progress on a related problem, the Goldbach-Waring problem with almost primes. In a related but different direction, an important method known as the affine linear sieve was introduced and developed by J. Bourgain, A. Gamburd, and P. Sarnak in [2], which established the existence of almost prime solutions to certain quadratic equations in [11]. We refer the reader to [2] and [11], and also a short discussion of this work in [4, Section 1], for more detailed information on this topic.

The main result of this paper improves on the bound on the number of prime factors in Theorem 1.2 with a modest cost on \( \text{codim} \ V^*_F \). In fact we establish a result analogous to Theorem 1.1 for semiprimes,
which are natural numbers with precisely two (not necessarily distinct) prime factors, with an exponential lower bound for \( \text{codim} \ V^*_F \).

**Theorem 1.3.** Let \( F(x) \in \mathbb{Z}[x_1, \ldots, x_n] \) be a degree \( d \) homogeneous form. Suppose that \( F \) satisfies the local conditions \((\ast)\) and \( \text{codim} \ V^*_F > 4^d \cdot 8(2d - 1) \). Then the equation \((1)\) has an infinite number of solutions \((\ell_1, \ldots, \ell_n)\) such that \( \ell_i \) has precisely two (not necessarily distinct) prime factors for each \( 1 \leq i \leq n \).

We note that a more general result, Theorem 5.2, is proved in this paper, where we obtain quantitative estimates on the number of semiprime solutions of a specific shape, from which Theorem 1.3 follows immediately. We present this theorem in Section 5. The proof is based on several key observations. The first observation is that solving the equation \((1)\) in semiprimes is equivalent to solving the equation

\[
F(x_1 y_1, \ldots, x_n y_n) = 0
\]

in primes. This observation appears to be not particularly helpful at first because the only known result for solving general polynomial equations in primes is Theorem 1.1. However, we observe that \( F(x_1 y_1, \ldots, x_n y_n) \) is now a bihomogeneous form (defined in Section 2), and we can in fact exploit this structure to obtain an estimate on the number of prime solutions to \((3)\) efficiently. We employ the work of D. Schindler [13] on bihomogeneous forms to achieve this. Therefore, we do not rely on the sophisticated method of B. Cook and Á. Magyar [4] which would drive up the requirement for \( \text{codim} \ V^*_F \). In particular, our method avoids the use of sieve theory, unlike the work of [12]. Another observation is that the dimensions of the variants (defined in \((7)\)) of the singular locus of \( \{(x, y) \in \mathbb{C}^{2n} : F(x_1 y_1, \ldots, x_n y_n) = 0\} \) are well-controlled by \( \text{dim} \ V^*_F \) (Theorem 5.1), and this plays a crucial role in the proof of Theorem 5.2.

We remark that Theorem 1.2 was improved recently by D. Schindler and E. Sofos in [14]. As a special case of their main result [14, Theorem 1.1], D. Schindler and E. Sofos established [14, Corollary 1.2], which holds when \( F \) is non-singular, \( d \geq 5 \), and \( n > 2^{d-1}(d^2 - 1) \), from which one can obtain a quantitative estimate on the number of solutions to the equation \((1)\) whose coordinates have at most \( O(d \log n / (\log \log n)) \) prime factors. Their approach is based on combining sieve methods and the Hardy-Littlewood circle method. Note we have stated this result by D. Schindler and E. Sofos and Theorem 1.2 in terms of the number of prime factors, but in fact the results were obtained in terms of the smallest prime divisors. Thus they obtained results for a different problem from which the mentioned statements follow immediately.

The organization of the rest of the paper is as follows. We devote Sections 2, 3, and 4 to establishing Theorem 2.1, which is of interest on its own, regarding the number of prime solutions to systems of bihomogeneous equations. This is achieved by the Hardy-Littlewood circle method. We cover preliminaries in Section 2, and obtain the minor arcs estimate in Section 3 and the major arcs estimate in Section 4. In Section 5, we establish the main results of this paper by using estimates obtained in the previous sections.

We use the well-known notation \( \ll \) and \( \gg \) of Vinogradov. By an affine variety we mean an algebraic set which is not necessarily irreducible. We use the notation \( e(x) \) to denote \( e^{2 \pi i x} \). We let \( 1_H \) be the characteristic function of the set \( H \). Given \( \omega_1, \ldots, \omega_{h_0} \in \mathbb{C}[x_1, \ldots, x_n] \), we let \( V(\omega_1, \ldots, \omega_{h_0}) = \{ z \in \mathbb{C}^n : \omega_i(z_1, \ldots, z_n) = 0 \ (1 \leq i \leq h_0) \} \).
Furthermore, $\sigma > 0$ provided the system of equations (6) has a non-singular solution in $(\mathbb{Z}_p^\times)^{n_1+n_2}$ for each prime $p$ and the system $G_r(x;y) = 0$ $(1 \leq r \leq R)$ has a non-singular real solution in $(0,1)^{n_1+n_2}$. 

\[ \mathcal{N}_p(g;P_1,P_2) = \sigma_g P_1^{p_1-d_1 R} P_2^{p_2-d_2 R} + O\left(\frac{P_1^{p_1-d_1 R} P_2^{p_2-d_2 R}}{(\log P)^c}\right). \]
We establish Theorem 2.1 by an application of the Hardy-Littlewood circle method. Let \( P = P_1^{d_1} P_2^{d_2} \).

We define the major arcs \( \mathfrak{M}(\vartheta) \) to be the set of points \( \alpha = (\alpha_1, \ldots, \alpha_R) \in [0,1)^R \) satisfying the following: there exist \( 1 \leq q \leq p^{R(d_1+d_2-1)} \) and \( a_1, \ldots, a_R \in \mathbb{Z} \) with

\[
\gcd(q,a_1,\ldots,a_R) = 1 \quad \text{and} \quad \|aq - \alpha_r\| \leq P_1^{-d_1} P_2^{-d_2} p^{R(d_1+d_2-1)} \vartheta \quad (1 \leq r \leq R).
\]

We define the minor arcs to be the complement \( m(\vartheta) = [0,1)^R \setminus \mathfrak{M}(\vartheta) \).

Let us define

\[
S(\alpha) := \sum_{x \in [0,P_1]^{n_1}} \sum_{y \in [0,P_2]^{n_2}} \Lambda^\ast(x) \Lambda^\ast(y) e\left( \sum_{r=1}^R \alpha_r g_r(x;y) \right).
\]

By the orthogonality relation, we have

\[
\mathcal{N}_g(P_1,P_2) = \int_{[0,1)^R} S(\alpha) \, d\alpha = \int_{\mathfrak{M}(\vartheta')} S(\alpha) \, d\alpha + \int_{m(\vartheta')} S(\alpha) \, d\alpha.
\]

For a suitable choice of \( \vartheta' \), we prove estimates for the integral over the minor arcs in Section 3 and over the major arcs in Section 4. In this section, we collect results to set up the proof for these estimates.

We make frequent use of the following basic lemma on the dimensions of affine varieties.

**Lemma 2.2.** Let \( X \) be an irreducible affine variety in \( \mathbb{A}^n_\mathbb{C} \), and let \( \omega \in \mathbb{C}[x_1, \ldots, x_n] \). Suppose \( \emptyset \neq X \cap V(\omega) \) and \( X \not\subseteq V(\omega) \). Then every irreducible component of \( X \cap V(\omega) \) has dimension \( \dim X - 1 \).

Furthermore, if \( Y = \bigcup_{1 \leq i \leq s_0} Y_i \) and \( Z = \bigcup_{1 \leq j \leq t_0} Z_j \) are irreducible affine varieties in \( \mathbb{A}^n_\mathbb{C} \), where \( Y_i \)'s and \( Z_j \)'s are the irreducible components of \( Y \) and \( Z \) respectively, such that \( \emptyset \neq Y_i \cap Z_j \) \( (1 \leq i \leq s_0, 1 \leq j \leq t_0) \), then \( \dim Z - \codim Y \leq \dim(Z \cap Y) \).

**Proof.** The first part of the statement is precisely [9, Exercise I.1.8]. For the second part we recall [9, Proposition I.7.1]: If \( V \) and \( W \) are irreducible affine varieties in \( \mathbb{A}^n_\mathbb{C} \) and \( V \cap W \neq \emptyset \), then \( \dim(V \cap W) \geq \dim V + \dim W - n \). The second part of the statement follows immediately from this result, and we leave the details to the reader.

Let us also recall that given an affine variety \( X \) in \( \mathbb{A}^n_\mathbb{C} \), if \( X \) is defined by homogeneous polynomials then every irreducible component of \( X \) contains \( 0 \). We prove the following lemma regarding \( \dim V_{G,i}^s \), the codimension of \( V_{G,i}^s \) as a subvariety of \( \mathbb{A}^{n_1+n_2}_\mathbb{C} \).

**Lemma 2.3.** Let \( G_1(x;y), \ldots, G_R(x;y) \in \mathbb{Z}[x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}] \) be bihomogeneous of bidegree \( (d_1,d_2) \). Let \( 0 \leq s < n_1 \) and \( 0 \leq t < n_2 \), for each \( 1 \leq r \leq R \), let

\[
\mathfrak{F}_r(x_{r+1}, \ldots, x_{n_1}; y_{r+1}, \ldots, y_{n_2}) = G_r(0, \ldots, 0, x_{s+1}, \ldots, x_{n_1}; 0, \ldots, 0, y_{t+1}, \ldots, y_{n_2})
\]

Then we have

\[
\min\{\dim V_{\mathfrak{F}_1}^s, \dim V_{\mathfrak{F}_2}^s\} \geq \min\{\dim V_{G,1}^s, \dim V_{G,2}^s\} - (s+t)(R+1).
\]
**Proof.** We consider the case $s = 1$ and $t = 0$ as the general case follows by repeating the argument for this case. It is clear from the definition that $\text{Jac}_{\mathbb{R},1}$ is obtained by removing the first column from $\text{Jac}_{\mathbb{R},1}|_{x_1=0}$. Let $W$ be the affine variety in $\mathbb{A}_{\mathbb{C}}^{n_1+n_2}$ defined by the entries of the first column of $\text{Jac}_{\mathbb{R},1}|_{x_1=0}$. In particular, $W$ is defined by $R$ homogeneous polynomials, and hence $\text{codim} W \leq R$. Let $\lambda_1(x,y), \ldots, \lambda_{K_1}(x,y)$ denote the determinants of matrices formed by $R$ columns of $\text{Jac}_{\mathbb{R},1}$. Then we see that $V_{\mathbb{R},1}^*$ is defined by these polynomials. Take a point

$$(0, \tilde{x}_0, y_0) = (0, x_{0, 2}, \ldots, x_{0, n_1}, y_{0, 1}, \ldots, y_{0, n_2}) \in \{ x_1 \in \mathbb{C} : x_1 = 0 \} \times (V_{\mathbb{R},1}^* \cap W).$$

Let $1 \leq k \leq K_1$. Suppose $\lambda_k(x,y)$ corresponds to $R$ columns of $\text{Jac}_{\mathbb{R},1}$ which contains the first column. Then since every entry of the first column of $\text{Jac}_{\mathbb{R},1}$ is 0 at $(0, \tilde{x}_0, y_0)$, we have $\lambda_k(0, \tilde{x}_0, y_0) = 0$. On the other hand, suppose $\tilde{\lambda}_k(x,y)$ corresponds to a collection of $R$ columns which does not contain the first column. In this case $\tilde{\lambda}_k(0, x_2, \ldots, x_{n_1}, y)$ is the determinant of one of the matrices formed by taking $R$ columns of $\text{Jac}_{\mathbb{R},1}$, and hence $(\tilde{x}_0, y_0)$ is a zero of this polynomial. Thus we have $\lambda_k(0, \tilde{x}_0, y_0) = 0$ in this case as well. Therefore, we have shown that

$$\{0\} \subseteq \{ x_1 \in \mathbb{C} : x_1 = 0 \} \times (V_{\mathbb{R},1}^* \cap W) \subseteq V_{\mathbb{R},1}^* \cap V(x_1) \subseteq \mathbb{A}_{\mathbb{C}}^{n_1+n_2}.$$

We know that $\text{dim}(V_{\mathbb{R},1}^* \cap V(x_1))$ is either $(\text{dim} V_{\mathbb{R},1}^* - 1)$ or $\text{dim} V_{\mathbb{R},1}^*$. By Lemma 2.2 we obtain $\text{dim} V_{\mathbb{R},1}^* - R \leq \text{dim} V_{\mathbb{R},1}^*$, and consequently $\text{codim} V_{\mathbb{R},1}^* \geq \text{codim} V_{\mathbb{R},1}^* - (R + 1)$.

Next we consider the case $i = 2$. In this case $\text{Jac}_{\mathbb{R},2}$ is obtained by setting $x_1 = 0$ in $\text{Jac}_{\mathbb{R},2}$. Thus we have

$$\{0\} \subseteq \{ x_1 \in \mathbb{C} : x_1 = 0 \} \times V_{\mathbb{R},2}^* \subseteq V_{\mathbb{R},2} \cap V(x_1) \subseteq \mathbb{A}_{\mathbb{C}}^{n_1+n_2}.$$

Therefore, it follows that $\text{dim} V_{\mathbb{R},2}^* \leq \text{dim} V_{\mathbb{R},2}$, and consequently we have $\text{codim} V_{\mathbb{R},2}^* \geq \text{codim} V_{\mathbb{R},2} - 1$. Our result is then immediate. \hfill \Box

By applying Cauchy-Schwarz inequality we obtain

$$|S(\alpha)|^2 \ll (\log P_1)^m P_1^R \sum_{y, y' \in [0, P_2]^2} \Lambda^*(y) \Lambda^*(y') \sum_{x \in [0, P_1]^n} e \left( \sum_{r=1}^R \alpha_r (g_r(x, y) - g_r(x, y')) \right).$$

We then apply Cauchy-Schwarz inequality once more and obtain

$$|S(\alpha)|^4 \ll (\log P_1)^{2m} (\log P_2)^{2n} P_1^{2n} P_2^{2n} \sum_{x, x' \in [0, P_1]^n} \sum_{y, y' \in [0, P_2]^2} e \left( \sum_{r=1}^R \alpha_r \partial_r(x, x', y, y') \right),$$

(11)

where

$$\partial_r(x, x', y, y') = g_r(x, y) - g_r(x, y') - g_r(x', y) + g_r(x', y').$$

(12)

In order to simplify our notation we denote $u = (x, x')$ and $v = (y, y')$, and write the sum on the right hand side of (11) as

$$T(\alpha) := \sum_{u \in [0, P_1]^n} \sum_{v \in [0, P_2]^2} e \left( \sum_{r=1}^R \alpha_r \partial_r(u, v) \right).$$

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It is clear from the definition of the polynomial $\mathcal{D}_r(u;v)$ given in (12) that it is a degree $(d_1 + d_2)$ polynomial (in $u$ and $v$) whose homogeneous degree $(d_1 + d_2)$ portion is

$$\mathcal{D}_r(u;v) = \mathcal{D}_r(x,x';y,y') = G_r(x;y) - G_r(x;y') - G_r(x';y) + G_r(x';y').$$

It is then immediate that $\mathcal{D}_r(u;v)$ is a bihomogeneous form of bidegree $(d_1,d_2)$.

Note we have

$$\frac{\partial \mathcal{D}_r}{\partial x_j}(x,x';y,y') = \frac{\partial G_r}{\partial x_j}(x;y) - \frac{\partial G_r}{\partial x_j}(x;y').$$

Let $M_1$ be the matrix obtained by removing $n_1$ columns corresponding to $x'$ (that is $(n_1 + 1)$-th column to $(2n_1)$-th columns) from $\text{Jac}_{\mathbb{D},1}$. It is clear from (13) that $M_1$ is independent of $x'$. Let $V^*_{\mathbb{M}_1} = \{ (x,y,y') \in \mathbb{C}^{n_1 + 2n_2} : \text{rank} \ M_1 < R \}$. Since $M_1|_{y'=0}$ is precisely $\text{Jac}_{\mathbb{G},1}$, we have $(x,y) \in V^*_{\mathbb{G},1}$ if and only if $(x,y,0) \in V^*_{\mathbb{M}_1}$. Therefore, we see that

$$V^*_{\mathbb{G},1} \times \{ y' \in \mathbb{C}^{n_2} : y' = 0 \} \times \{ x' \in \mathbb{C}^{n_1} \} = (V^*_{M_1} \times \{ x' \in \mathbb{C}^{n_1} \} ) \cap V(y_1', \ldots, y_{n_2}') \subseteq \mathbb{A}_2^{2n_1 + 2n_2}.

Let $W = \{ (x,y,y',x') \in \mathbb{C}^{2n_1 + 2n_2} : (x,x',y,y') \in V^*_{\mathbb{D},1} \}$. Then $\text{dim} \ W = \text{dim} \ V^*_{\mathbb{D},1}$. Since $M_1$ is a submatrix of $\text{Jac}_{\mathbb{D},1}$ we have $W \subseteq V^*_{\mathbb{M}_1} \times \{ x' \in \mathbb{C}^{n_1} \}$. Therefore, it follows that

$$\{0\} \subseteq W \cap V(y_1', \ldots, y_{n_2}') \subseteq V^*_{\mathbb{G},1} \times \{ y' \in \mathbb{C}^{n_2} : y' = 0 \} \times \{ x' \in \mathbb{C}^{n_1} \} \subseteq \mathbb{A}_2^{2n_1 + 2n_2}.

Consequently, by Lemma 2.2 we obtain $\text{dim} \ V^*_{\mathbb{D},1} - n_2 \leq n_1 + \text{dim} \ V^*_{\mathbb{G},1}$, which is equivalent to

$$\text{codim} \ V^*_{\mathbb{D},1} = (2n_1 + 2n_2) - \text{dim} \ V^*_{\mathbb{D},1} \geq n_1 + n_2 - \text{dim} \ V^*_{\mathbb{G},1} = \text{codim} \ V^*_{\mathbb{G},1}.

By reversing the roles of $x$ and $x'$ with that of $y$ and $y'$, we also obtain $\text{codim} \ V^*_{\mathbb{D},2} \geq \text{codim} \ V^*_{\mathbb{G},2}$. Therefore, it follows from (8) that

$$\text{codim} \ V^*_{\mathbb{D},i} > 2^{d_1 + d_2} \max \{ 2R(R + 1)(d_1 + d_2 - 1), R(bd_1 + d_2) \} \quad (i = 1,2).

Let $\delta_0 > 0$ be a sufficiently small constant. We now define the following constant

$$K := \frac{\min \{ \text{codim} \ V^*_{\mathbb{D},1}, \text{codim} \ V^*_{\mathbb{D},2} \} - \delta_0}{2^{d_1 + d_2 - 2}}.

In particular, we have

$$K > 4 \max \{ 2R(R + 1)(d_1 + d_2 - 1), R(bd_1 + d_2) \}.

We make use of the following generalization of [13, Lemma 4.3] which gives us an exponential sum estimate on the minor arcs. We remark that owing to a minor oversight in [13, pp. 498], the presence of $\delta_0$ in the statement is necessary. Since the lemma can be obtained by following the argument of [13, Lemma 4.3] in our setting, we omit the details. We shall refer to $\mathcal{B} \subseteq \mathbb{R}^m$ as a box, if $\mathcal{B}$ is of the form $\mathcal{B} = I_1 \times \cdots \times I_m$, where each $I_j$ is a closed or open or half open/closed interval ($1 \leq j \leq m$).
Lemma 2.4. [13, Lemma 4.3] Let \( u = (u_1, \ldots, u_{m_1}) \) and \( v = (v_1, \ldots, v_{m_2}) \). Let \( \mathcal{B}_i \subseteq \mathbb{R}^{m_i} \) be a box with sides \( \leq 1 \) (\( i = 1, 2 \)). Let \( f_1(u; v), \ldots, f_R(u; v) \) be degree \( (d_1 + d_2) \) polynomials with rational coefficients and let their degree \( (d_1 + d_2) \) homogeneous portions be \( \mathcal{F}_1(u; v), \ldots, \mathcal{F}_R(u; v) \) respectively. For each \( 1 \leq r \leq R \), suppose \( \mathcal{F}_r(u; v) \) is a bihomogeneous form of bidegree \( (d_1, d_2) \) with integer coefficients. Let \( \delta_0 > 0 \) be a sufficiently small constant. Let \( P = P_1^{d_1} P_2^{d_2} \), \( 1 \leq b = \frac{\log P_1}{\log P_2} \), \( 0 < \vartheta \leq (bd_1 + d_2)^{-1} \), and

\[
\bar{K} = \frac{\min \{\text{codim}(V^*_{\mathcal{F}_1}), \text{codim}(V^*_{\mathcal{F}_2})\}}{2^{d_1 + d_2 - 2}} - \delta_0.
\]

Consider the exponential sum

\[
\widetilde{T}(\alpha) = \sum_{u \in P_1 \mathcal{B}_1} \sum_{v \in P_2 \mathcal{B}_2} e\left( \sum_{r=1}^R \alpha_r f_r(u; v) \right).
\]

Then we have either

(i) \( \alpha \in \mathcal{M}(\vartheta) \) or (ii) \( |\widetilde{T}(\alpha)| \lesssim P_1^{m_1} P_2^{m_2} P^{-\bar{K}\vartheta} (\log P)^{m_1} \).

Here the implicit constant is independent of \( \vartheta \), and it is also independent of the coefficients of \((f_r(u; v) - \mathcal{F}_r(u; v))\) for each \( 1 \leq r \leq R \).

We remark that the hypotheses in the statement of Lemma 2.4 are sufficient and the additional assumption [13, lines 1-2, pp.488] is in fact unnecessary; this can be verified by going through the proof of [13, Lemma 4.3] and observing that the expression in [13, line 22, pp.496] is a multilinear form with integer coefficients due to the factor \( d_1!d_2! \) as long as \( F_1, \ldots, F_R \) have integer coefficients. We note the fact that the implicit constant is independent of the lower degree terms of \( f_r(u; v) \) becomes crucial when we apply this lemma in Section 4. We have the following exponential sum estimate as a corollary which we also use in Section 4.

Corollary 2.5. Make all the assumptions of Lemma 2.4. Suppose \( \gcd(q, a_1, \ldots, a_R) = 1 \). Then for any \( \varepsilon > 0 \) we have

\[
\sum_{u \in [0, q-1]^{m_1}} \sum_{v \in [0, q-1]^{m_2}} e\left( \sum_{r=1}^R \frac{f_r(u; v) \cdot a_r}{q} \right) \lesssim q^{m_1 + m_2 - \frac{\bar{K}}{\log d_1^{d_1} d_2^{d_2} - 1} + \varepsilon}.
\]

Proof. See the proof of [13, Lemma 5.5].

3 The minor arcs estimate

From the bound (11) we have \( |S(\alpha)|^4 \lesssim (\log P_1)^{2n_1} (\log P_2)^{2n_2} P_1^{2n_1} P_2^{2n_2} T(\alpha) \). Thus the following is an immediate consequence of applying Lemma 2.4 to \( T(\alpha) \).

Lemma 3.1. Let \( K \) be as in (15) and \( 0 < \vartheta \leq (bd_1 + d_2)^{-1} \). Then we have either

(i) \( \alpha \in \mathcal{M}(\vartheta) \) or (ii) \( |S(\alpha)| \lesssim P_1^{n_1} P_2^{n_2} P^{-K\vartheta/4} (\log P)^{n_1 + n_2} \).

Here the implicit constant is independent of \( \vartheta \).
We define
\[ \sigma := \frac{1}{2} \left( \frac{K}{4} - \max \{ 2R(R+1)(d_1 + d_2 - 1), R(bd_1 + d_2) \} \right), \]
which we know to be positive because of (16). Let us fix \( \delta_0 \) satisfying
\[ 0 < \delta_0 \leq (bd_1 + d_2)^{-1} \quad \text{and} \quad \frac{\delta_0 K}{4} > R + \epsilon_0 \]
for some \( \epsilon_0 > 0 \) sufficiently small, which is possible because of (16).

Let us set
\[ \zeta := \frac{4R(R+1)(d_1 + d_2 - 1) + 4\sigma}{K}, \]
which can be verified to satisfy \( 0 < \zeta < 1 \). Throughout Sections 3 and 4 we let \( C \) to be a sufficiently large positive constant which does not depend on \( P \). Let us define \( \delta_{i+1} = \delta_i + \zeta \delta_i \) \( (0 \leq i \leq M - 1) \), where \( M \) is the smallest positive integer such that \( P^{\delta_M} \leq (\log P)^C \). From the definition of \( M \) it follows that \( (\log P)^C \zeta < P^{\delta_M} = P^{\zeta M} \delta_0 \), for otherwise we have \( P^{\delta_{M-1}} = P^{\delta_M/\zeta} \leq (\log P)^C \) and this is a contradiction. We then obtain \( M \ll \log \log P \). We also remark that from the definition of \( M \) we have
\[ (\log P)^C < P^{\delta_{M-1}}. \]

Let us use the notation \( 0 \leq a \leq q \) to mean \( 0 \leq a_r \leq q \) \( (1 \leq r \leq R) \). The Lebesgue measure of \( \mathcal{M}(\delta_i) \) is bounded by the following quantity
\[ \text{meas}(\mathcal{M}(\delta_i)) \ll \sum_{q \leq P^{R(d_1 + d_2 - 1) \delta_i}} \sum_{0 \leq a \leq q \atop \gcd(q, a_1, \ldots, a_R) = 1} q^{-R} p_1^{-d_1 R} p_2^{-d_2 R} p^{R^2(d_1 + d_2 - 1) \delta_i}, \]
\[ \ll p^{-R + R(R+1)(d_1 + d_2 - 1) \delta_i}. \]
Thus for each \( 0 \leq i \leq M - 1 \), we have by Lemma 3.1 that
\[ \int_{\mathcal{M}(\delta_i) \setminus \mathcal{M}(\delta_{i+1})} |S(\alpha)| \, d\alpha \ll (\log P)^{n_1 + \frac{n_2}{2}} p_1^{n_1} p_2^{n_2} P^{\frac{1}{C} \delta_{M-1}} \text{meas}(\mathcal{M}(\delta_i)) \]
\[ \ll (\log P)^{n_1 + \frac{n_2}{2}} p_1^{n_1} p_2^{n_2} P^{-R - \sigma \delta_i}, \]
where we obtained the final inequality using (19), the relation \( \delta_{i+1} = \zeta \delta_i \), and the definition of \( \zeta \). Since
\[ m(\delta_M) \leq m(\delta_0) \bigcup_{0 \leq i \leq M-1} \mathcal{M}(\delta_i) \setminus \mathcal{M}(\delta_{i+1}), \]
it follows from Lemma 3.1 with \( \delta_0 \) and (20) that
\[ \int_{m(\delta_M)} |S(\alpha)| \, d\alpha \ll \int_{m(\delta_0)} |S(\alpha)| \, d\alpha + M \max_{0 \leq i \leq M-1} \int_{\mathcal{M}(\delta_i) \setminus \mathcal{M}(\delta_{i+1})} |S(\alpha)| \, d\alpha \]
\[ \ll P_1^{n_1} P_2^{n_2} P^{-R - \frac{n_2}{2}} + (\log \log P)(\log P)^{n_1 + \frac{n_2}{2}} P_1^{n_1} P_2^{n_2} P^{-R - \sigma \delta_{M-1}} \]
\[ \ll (\log P)^{n_1 + \frac{n_2}{2} - \sigma C} (\log \log P) P_1^{n_1} P_2^{n_2} P^{-R}, \]
where we obtained the final inequality using (18). Therefore, we have established the following.

**Proposition 3.2.** Given any \( c > 0 \), we have
\[ \int_{m(\delta_M)} S(\alpha) \, d\alpha \ll \frac{P_1^{n_1} P_2^{n_2} P^{-R}}{(\log P)^c}. \]
4 The major arcs estimate

As the material in this section is fairly standard, we keep the details to a minimum and also refer the reader to [4, Sections 6 and 7] or [16, Section 7] where similar work has been carried out. Let us define $C_0$ by $P^{\theta_0} = (\log P)^{C_0}$. It is clear that $C_0$ depends on $P$; however, by the definition of $\theta_0$ we have $C_0 < C_0 \leq C$. By the definition of $\mathcal{M}(\theta_0)$ we can write

\[ \mathcal{M}(\theta_0) = \bigcup_{1 \leq q \leq (\log P)^{C_0(d_1 + d_2 - 1)}} \bigcup_{0 \leq a \leq q} \mathcal{M}_{a,q}(C_0), \]

where

\[ \mathcal{M}_{a,q}(C_0) = \left\{ \alpha \in [0,1)^R : 2|q\alpha - a| < \frac{(\log P)^{C_0(d_1 + d_2 - 1)}}{p} \right\}. \]

It can be verified that the arcs $\mathcal{M}_{a,q}(C_0)$’s are disjoint for $P$ sufficiently large.

We define

\[ \psi_h(t) = \sum_{0 \leq v \leq t} \Lambda^*(v). \]

We use the notation $x \equiv h_1 \pmod{q}$ to mean $x_j \equiv h_{1,j} \pmod{q}$ for each $1 \leq j \leq n_1$, and similarly for $y \equiv h_2 \pmod{q}$. We also denote $h = (h_1, h_2)$. Recall the definition of $S(\alpha)$ given in (9). Let $\alpha = a/q + \beta \in [0,1)^R$. In a similar manner as in [4, (6.1)], we can express $S(\alpha)$ as

\[ \sum_{h \in (\mathbb{Z}/q\mathbb{Z})^{n_1+n_2}} e\left( \sum_{r=1}^R a_r g_r(h_1;h_2)/q \right) \int_{(t_1,t_2) \in [0,P_1]^{n_1} \times [0,P_2]^{n_2}} e\left( \sum_{r=1}^R \beta_r g_r(t_1; t_2) \right) d\psi_h(t), \]

where $d\psi_h(t)$ denotes the product measure

\[ d\psi_{h_1,1}(t_1) \times \ldots \times d\psi_{h_{n_1,1}}(t_{1,n_1}) \times d\psi_{h_2,1}(t_{2,1}) \times \ldots \times d\psi_{h_{n_2,2}}(t_{2,n_2}). \]

Let $\phi$ be Euler’s totient function. For a positive integer $q$, let $\mathbb{U}_q$ be the group of units in $\mathbb{Z}/q\mathbb{Z}$. Let $\mathcal{B}_0 = [0,1]^{n_1+n_2}$ and

\[ J(\mathcal{B}_0, \tau) = \int_{(v_1,v_2) \in \mathcal{B}_0} e\left( \sum_{r=1}^R \tau_r \cdot G_r(v_1;v_2) \right) dv. \]

We denote $P_1^{d_1} P_2^{d_2} \beta = (P_1^{d_1} P_2^{d_2} \beta_1, \ldots, P_1^{d_1} P_2^{d_2} \beta_R)$. With these notations we have the following lemma.

**Lemma 4.1.** Let $c' > 0$, $q \leq (\log P)^{C_0}$, and $h \in (\mathbb{Z}/q\mathbb{Z})^{n_1+n_2}$. Suppose $\alpha = a/q + \beta \in \mathcal{M}_{a,q}(C_0)$. Then we have

\[ \int_{(t_1,t_2) \in [0,P_1]^{n_1} \times [0,P_2]^{n_2}} e\left( \sum_{r=1}^R \beta_r g_r(t_1; t_2) \right) d\psi_h(t) \]

\[ = \mathbb{1}_{\mathcal{U}_q^{n_1+n_2}}(h) \frac{P_1^{d_1} P_2^{d_2}}{\phi(q)^{n_1+n_2}} J(\mathcal{B}_0, P_1^{d_1} P_2^{d_2} \beta) + O(P_1^{d_1} P_2^{d_2}/(\log P)^c). \]
We omit the proof of Lemma 4.1 because it can be established by following the argument of [4, Lemma 6] in our setting and the changes required are minimal.

Let us define
\[
J(L) = \int_{\tau \in [-L,L]} J(\mathbb{B}_0, \tau) \, d\tau.
\]

It then follows by [13, Lemma 5.6] that under our assumptions on \(G\), namely (8), we have
\[
\mu(\infty) = \int_{\tau \in \mathbb{R}} J(\mathbb{B}_0, \tau) \, d\tau,
\]
which is called the singular integral, exists, and that
\[
\left| \mu(\infty) - J(L) \right| \ll L^{-1}.
\]

We note that \(\mu(\infty)\) is the same as what is defined in [13, (5.3)], and we have
\[
\mu(\infty) > 0
\]
provided the system of equations \(G_r(\mathbf{x}; \mathbf{y}) = 0\) \((1 \leq r \leq R)\) has a non-singular real solution in \((0, 1)^{n_1 + n_2}.

Let us define the following sums:
\[
S_{a,q} = \sum_{k \in U_{q}^{n_1+n_2}} e \left( \sum_{r=1}^{R} g_r(k_1,k_2) \cdot a_r/q \right),
\]
\[
A(q) = \sum_{0 \leq a < q \ \gcd(q,a)=1} \frac{1}{\phi(q)^{n_1+n_2}} S_{a,q}, \quad \text{and} \quad \mathfrak{S}(P) = \sum_{q \leq (\log P)^{C_0R(d_1+d_2-1)}} A(q).
\]

Then by combining Lemma 4.1, (22), and the definition of major arcs, we obtain the following.

**Lemma 4.2.** Given any \(c > 0\), we have
\[
\int_{\mathfrak{M}(\mathfrak{d}(q))} S(\alpha) \, d\alpha = \mathfrak{S}(P)\mu(\infty)P_1^{n_1-d_1}P_2^{n_2-d_2}P_3^{d_3-R}\]
\[
+ O \left( \frac{P_1^{n_1-d_1}P_2^{n_2-d_2}P_3^{d_3-R}}{(\log P)^{C_0R(d_1+d_2-1)}} \sum_{q \leq (\log P)^{C_0R(d_1+d_2-1)}} q |A(q)| + \frac{P_1^{n_1-d_1}P_2^{n_2-d_2}P_3^{d_3-R}}{(\log P)^c} \right),
\]
where the summation in the \(O\)-term is over \(1 \leq q \leq (\log P)^{C_0R(d_1+d_2-1)}\).

We still have to deal with the term \(\mathfrak{S}(P)\), and this is done in the following section.
4.1 Singular Series

We now bound $S_{a,q}$ when $q$ is a prime power. In order to simplify the exposition let us define

$$B := \min\{\operatorname{codim} V_{Q,1}, \operatorname{codim} V_{Q,2}\} \quad \text{and} \quad Q := \frac{1}{2} \cdot \frac{B}{2d_1 + 2d_2 - (R + 1)(d_1 + d_2)}.$$

Since $d_1 + d_2 \geq 3$ and (8) implies $Q > 4R(d_1 + d_2 - 1)/(d_1 + d_2)$, we can verify that

$$Q > \frac{1 + R(2d_1 + 2d_2 + 1)}{2d_1 + 2d_2} \quad \text{and} \quad Q > \frac{R + 1}{1 - \frac{R + 1}{2d_1 + 2d_2}} > R + 1. \quad (27)$$

**Lemma 4.3.** Let $p$ be a prime and let $q = p^t$, $t \in \mathbb{N}$. Let $0 \leq a < q$ with $\gcd(q, a) = 1$. Let $\varepsilon > 0$ be sufficiently small. Then we have the following bounds

$$S_{a,q} \ll \begin{cases} p^{-\varepsilon} q^{n_1 + n_2 - \frac{Q}{R}} & \text{if } t \leq 2(d_1 + d_2), \\ p^{2\varepsilon} q^{n_1 + n_2 - \frac{Q}{R}} & \text{if } t > 2(d_1 + d_2), \end{cases}$$

where the implicit constants are independent of $p$ and $t$.

**Proof.** We consider the two cases $t \leq 2(d_1 + d_2)$ and $t > 2(d_1 + d_2)$ separately. We begin with the case $t \leq 2(d_1 + d_2)$. In this case we apply the inclusion-exclusion principle (see [4, (7.3)]) and express $S_{a,q}$ as

$$\sum_{I_1 \subseteq \{1, 2, \ldots, n_1\}} \sum_{I_2 \subseteq \{1, 2, \ldots, n_2\}} (-1)^{|I_1| + |I_2|} \sum_{v \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{I_1}} \sum_{k \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{I_1}} \mathcal{S}_{I_1, I_2}(k; r) e\left(\sum_{r=1}^{R} g_r(k_1; k_2) \cdot a_r/q\right), \quad (28)$$

where $\mathcal{S}_{I_1, I_2}(k; v)$ is the characteristic function of the set $\{k \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{n_1 + n_2}: k_i = pv_i, j \in I_i, i = 1, 2\}$. Here we are using the notations $k = (k_1, k_2)$ where $k_i \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{n_i}$, and $\mathbf{v} = (v_1, v_2)$ where $v_i = (v_{i,j}, \ldots, v_{i,j_{I_i}}) \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{|I_i|}$ and $I_i = \{j_1, \ldots, j_{|I_i|}\}$. We now bound the summand in the expression (28) by further considering two cases, $|I_1| + |I_2| > \frac{3B}{2^{n_1 + n_2} - (R + 1)}$ and $|I_1| + |I_2| \leq \frac{3B}{2^{n_1 + n_2} - (R + 1)}$. In the first case $|I_1| + |I_2| > \frac{3B}{2^{n_1 + n_2} - (R + 1)}$, we use the following trivial estimate

$$\left| \sum_{\mathbf{v} \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{I_1}} \sum_{k \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{I_1}} \mathcal{S}_{I_1, I_2}(k; \mathbf{v}) e\left(\sum_{r=1}^{R} g_r(k_1; k_2) \cdot a_r/q\right) \right| \leq p^{(t-1)(|I_1| + |I_2|)} q^{n_1 + n_2} |I_1| - |I_2| \leq q^{n_1 + n_2} - Q^{-\varepsilon}. \quad (29)$$

On the other hand, suppose $|I_1| + |I_2| \leq \frac{3B}{2^{n_1 + n_2} - (R + 1)}$. Let us label $s = (s_1, \ldots, s_{n_1 - |I_1|})$ and $w = (w_1, \ldots, w_{n_2 - |I_2|})$ to be the remaining variables of $x$ and $y$ after setting $x_j = 0$ for each $j \in I_1$ and $y_{j'} = 0$ for each $j' \in I_2$ respectively. For each $1 \leq r \leq R$, let $f_r(s; w)$ be the polynomial obtained by substituting $x_j = pv_{1,j}$ ($j \in I_1$) and $y_{j'} = pv_{2,j'}$ ($j' \in I_2$) to the polynomial $g_r(x; y)$. Thus $f_r(s; w)$ is a polynomial in $s$ and $w$ whose coefficients may depend on $p$ and $w$. With these notations we have

$$\sum_{k \in (\mathbb{Z}/q^{I_2\mathbb{Z}})^{n_1 + n_2}} \mathcal{S}_{I_1, I_2}(k; \mathbf{v}) e\left(\sum_{r=1}^{R} g_r(k_1; k_2) \cdot a_r/q\right) = \sum_{s \in [0,q-1]^{n_1 - |I_1|}} \sum_{w \in [0,q-1]^{n_2 - |I_2|}} e\left(\sum_{r=1}^{R} f_r(s; w) \cdot a_r/q\right). \quad (30)$$
We can also deduce easily that the homogeneous degree $(d_1 + d_2)$ portion of the polynomial $f_r(s; w)$, which we denote $\mathfrak{F}_r(s; w)$, is obtained by substituting $x_j = 0 \ (j \in I_1)$ and $y_{j'} = 0 \ (j' \in I_2)$ to $G_r(x; y)$. In particular, it is independent of $p$ and $v$. It then follows from Lemma 2.3 that

$$\min\{\text{codim}(V_{f_1}^*), \text{codim}(V_{f_2}^*)\} \geq B - (R + 1)(|I_1| + |I_2|) \geq \left(1 - \frac{1}{2d_1 + d_2 - 1}\right)B.$$

Let $\varepsilon' > 0$ be sufficiently small. Thus by Corollary 2.5 we obtain

$$\sum_{s \in [0, q^{-1}]} \sum_{r = 1}^R e\left(\sum_{r = 1}^R g_r(k_1; k_2) \cdot \frac{a_r}{q}\right) \ll q^{n_1 + n_2 - |I_1| - |I_2|} \left(\sum_{r = 1}^R g_r(k_1; k_2) \cdot \frac{a_r}{q}\right).$$

Consequently, we have from (30) that

$$\left|\sum_{v \in (Z/p^{r-1}Z)^{|I_1| + |I_2|}} \sum_{k \in (Z/qZ)^{n_1 + n_2}} \mathfrak{F}_{I_1, I_2}(v; k) \cdot e\left(\sum_{r = 1}^R g_r(k_1; k_2) \cdot \frac{a_r}{q}\right)\right| \leq q^{n_1 + n_2 - |I_1| - |I_2|} - \varepsilon.$$ 

in this case as well. By applying the estimates (29) and (31) in (28), we obtain the desired estimate for the case $t \leq 2(d_1 + d_2)$.

We now consider the case $t > 2(d_1 + d_2)$. By the definition of $\mathfrak{F}_{a, q}$ we have

$$\mathfrak{F}_{a, q} = \sum_{k \in \mathbb{U}^{n_1 + n_2}_p} \sum_{b_1 \in [0, p^{r-1} - 1]^{n_1}} \sum_{b_2 \in [0, p^{r-1} - 1]^{n_2}} e\left(\sum_{r = 1}^R g_r(k_1 + pb_1; k_2 + pb_2) \cdot \frac{a_r}{q}\right).$$

For each fixed $k \in \mathbb{U}^{n_1 + n_2}_p$, we have

$$g_r(k_1 + pb_1; k_2 + pb_2) = p^{d_1 + d_2}G_r(b_1; b_2) + \sigma_{r, p, k}(b) \quad (1 \leq r \leq R),$$

where $\sigma_{r, p, k}(b)$ is a polynomial in $b = (b_1, b_2)$ of degree at most $d_1 + d_2 - 1$. Clearly every monomial of $\sigma_{r, p, k}(b)$ has degree in $b_i$ strictly less than $d_i$ for one of $i = 1$ or 2, and its coefficients are integers which may depend on $p$ and $k$. We let

$$c_r(b_1; b_2) = G_r(b_1; b_2) + \frac{1}{p^{d_1 + d_2}} \sigma_{r, p, k}(b) \quad (1 \leq r \leq R).$$

We can then express the inner sum on the right hand side of (32) as

$$\sum_{b \in [0, p^{r-1} - 1]^{n_1 + n_2}} e\left(\sum_{r = 1}^R c_r(b_1; b_2) \cdot \frac{a_r}{q/p^{d_1 + d_2}}\right).$$
We have that each $c_r$ has coefficients in $\mathbb{Q}$, and its degree $(d_1 + d_2)$ homogeneous portion $G_r$ has coefficients in $\mathbb{Z}$. We apply Lemma 2.4 with $\mathcal{B}_1 = [0, 1]^{n_1}$, $\mathcal{B}_2 = [0, 1]^{n_2}$, $\alpha_r = \alpha_r/p^{t-1}$ $(1 \leq r \leq R)$, $P_1 = P_2 = p^{t-1}$, and $P = p^{(t-1)(d_1 + d_2)}$. Let $\theta = \frac{1}{2(d_1 + d_2)(d_1 + d_2 - 1)(R + 1)} < \frac{1}{d_1 + d_2}$. Suppose that there exist $\alpha_1, \ldots, \alpha_R$, and $1 \leq q \leq P^{R(d_1 + d_2 - 1)/2}$ such that $\gcd(q, \alpha_1, \ldots, \alpha_R) = 1$ and

$$2|q\alpha_r - \alpha_r| \leq P^{d_1} P^{2d_2} p^{R(d_1 + d_2 - 1)/2}$$

$(1 \leq r \leq R)$. Note from $t + 1 > 2(d_1 + d_2)$ it follows that $(t - d_1 - d_2) > \frac{t - 1}{2}$. Then it is not possible that $p^{t-1} - d_2$ divide $q$, because

$$1 \leq q \leq p^{R(d_1 + d_2 - 1)/2} < p^{\frac{t - 1}{2}} < p^{t-1} - d_2.$$

Since $\gcd(q, \alpha_1, \ldots, \alpha_R) = 1$ and $q = p^t$, without loss of generality we assume $\gcd(a_1, p) = 1$. Then $q\alpha_1$ is not an integer. Thus we have

$$\frac{1}{p^{t-1} - d_2} \leq |q\alpha_1 - \alpha_1| < \frac{1}{2} P^{d_1} P^{2d_2} p^{t-1} \leq \frac{1}{p(t-1)(d_1 + d_2 - 1/2)}$$

which is a contradiction, because $t - d_1 - d_2 < (t - 1)(d_1 + d_2 - 1/2)$. Therefore, we are in the alternative (ii) of Lemma 2.4, and the expression (33) is bounded by

$$\ll P^{n_1} P^{n_2} P^{-\theta} \frac{\beta - \delta_0}{2^{n_1 + n_2 - 2}} (\log P)^{n_1} \ll (p^{t-1})^{n_1 + n_2 - \frac{\beta - \delta_0}{2(d_1 + d_2 - 1)(R + 1)2^{n_1 + n_2 - 2}} + \epsilon') \leq (p^{t-1})^{n_1 + n_2 - Q - \epsilon}. \quad (34)$$

Thus we can bound (32) by (33) and (34) as follows

$$|S_{a,q}| \ll p^{n_1 + n_2} (p^{t-1})^{n_1 + n_2 - Q - \epsilon} \leq p^{Q - \epsilon} q^{n_1 + n_2 - Q}.$$

By a similar argument as in [9, Chapter VIII, §2, Lemma 8.1], one can show that $A(q)$ is a multiplicative function of $q$. We omit the proof of the following lemma as it is a basic exercise involving the Chinese remainder theorem and manipulating summations.

**Lemma 4.4.** Suppose $q, q' \in \mathbb{N}$ and $\gcd(q, q') = 1$. Then we have $A(qq') = A(q)A(q')$.

Recall we defined the term $\mathcal{S}(P)$ in (25). For each prime $p$, we define

$$\mu(p) = 1 + \sum_{l=1}^{\infty} A(p^l), \quad (35)$$

which converges absolutely under our assumptions on $g$. Furthermore, the following limit exists

$$\mathcal{S}(\infty) := \lim_{L \to \infty} \sum_{q \leq L} A(q) = \prod_{p \text{ prime}} \mu(p), \quad (36)$$

which is called the singular series. We prove these statements in the following Lemma 4.5.
Lemma 4.5. There exists $\delta_1 > 0$ such that for each prime $p$, we have $\mu(p) = 1 + O(p^{-1-\delta_1})$ where the implicit constant is independent of $p$. Furthermore, we have

$$\left| \mathcal{S}(P) - \mathcal{S}(\infty) \right| \ll (\log P)^{-C_0(d_1+d_2-1)\delta_2}$$

for some $\delta_2 > 0$.

Therefore, the limit in (36) exists, and the product in (36) converges. We leave the details that these two quantities are equal to the reader.

Proof. For any $t \in \mathbb{N}$, we know that $\phi(p^t) = p^t(1 - 1/p) \geq 1/7p^t$. Therefore, by considering the two cases as in the statement of Lemma 4.3 we obtain

$$|\mu(p) - 1| \ll \sum_{1 \leq t \leq 2(d_1 + d_2)} p^{Rd} p^{-(n_1 + n_2)t} p^{n_1 + n_2} + \sum_{t > 2(d_1 + d_2)} p^{Rd} p^{-(n_1 + n_2)t} p^{Q(n_1 + n_2)t - tQ}$$

$$\ll p^{RQ} + p^Q p^{-(2d_1 + 2d_2 + 1)(Q - R)}$$

$$\ll p^{-1-\delta_1}$$

for some $\delta_1 > 0$, where the last inequality follows from (27). We note that the implicit constants in $\ll$ are independent of $p$ here.

Let $q = p_1^{t_1} \cdots p_v^{t_v}$ be the prime factorization of $q \in \mathbb{N}$. Without loss of generality, suppose we have $t_j \leq 2(d_1 + d_2)$ ($1 \leq j \leq v_0$) and $t_j > 2(d_1 + d_2)$ ($v_0 < j \leq v$). Note we can assume the implicit constant in Lemma 4.3 is 1 for $p$ sufficiently large with the cost of $p^{-e}$. By a similar calculation as above and the multiplicativity of $A(\cdot)$, it follows that

$$A(q) \ll q^{R - Q} \left( \prod_{j=v_0+1}^v p_j^Q \right) \leq q^{R - Q} \cdot q^{-\delta_2} \leq q^{-1-\delta_2}$$

(37)

for some $\delta_2 > 0$, where we obtained the last inequality from (27). We note that the implicit constant in $\ll$ is independent of $q$ here. Therefore, we obtain

$$\left| \mathcal{S}(P) - \mathcal{S}(\infty) \right| \leq \sum_{q > (\log P)^{C_0(d_1+d_2-1)R}} |A(q)| \ll (\log P)^{-C_0(d_1+d_2-1)R\delta_2}.

Using the bound (37), we obtain that the first term in the $O$-term of (26) is bounded by

$$\frac{P_1^{p_1} P_2^{p_2}}{P^R (\log P)^{C_0R(d_1+d_2-1)}} \sum_{1 \leq q \leq (\log P)^{C_0R(d_1+d_2-1)}} q|A(q)|$$

$$\ll \frac{P_1^{p_1} P_2^{p_2}}{P^R (\log P)^{C_0R(d_1+d_2-1)}} \sum_{1 \leq q \leq (\log P)^{C_0R(d_1+d_2-1)}} q^{-\delta_2}$$

$$\ll \frac{P_1^{p_1} P_2^{p_2}}{P^R (\log P)^{C_0R(d_1+d_2-1)}} (\log P)^{C_0R(d_1+d_2-1)(1-\delta_2)}$$

$$\ll \frac{P_1^{p_1} P_2^{p_2}}{P^R (\log P)^{C_0R(d_1+d_2-1)}} (\log P)^{-C_0R(d_1+d_2-1)\delta_2}.$$
Let \( v_t(p) \) denote the number of solutions \((x, y) \in (\mathbb{U}_p)^{n_1+n_2}\) to the congruence relations \( g_r(x; y) \equiv 0 \pmod{p^t} \) \((1 \leq r \leq R)\). It is then a basic exercise (see [16, pp. 58]) to deduce

\[
1 + \sum_{j=1}^{t} A(p^j) = \frac{p^R}{\phi(p^t)^{n_1+n_2}} v_t(p).
\]

Therefore, under our assumptions on \( g \) we obtain

\[
\mu(p) = \lim_{t \to \infty} \frac{p^R}{\phi(p^t)^{n_1+n_2}} v_t(p).
\]

We can then deduce by an application of Hensel’s lemma that \( \mu(p) > 0 \), if the system (6) has a non-singular solution in \((\mathbb{Z}_p)^{n_1+n_2}\). From this it follows in combination with (36) and Lemma 4.5 that if the system (6) has a non-singular solution in \((\mathbb{Z}_p)^{n_1+n_2}\) for every prime \( p \), then

\[
\mathcal{G}(\infty) = \prod_{p \text{ prime}} \mu(p) > 0. \tag{39}
\]

By combining (38) and Lemmas 4.2 and 4.5, we obtain the following.

**Proposition 4.6.** Given any \( c > 0 \), under our assumptions on \( g \) the following holds

\[
\int_{\mathcal{M}(\theta)} S(\alpha) \ d\alpha = \mathcal{G}(\infty) \mu(\infty) P^{n_1-Rd_1} P^{n_2-Rd_2} + O \left( \frac{P^{n_1-Rd_1} P^{n_2-Rd_2}}{(\log P)^c} \right),
\]

where \( P^{\theta} = (\log P)^{C_0} \).

Finally, it is clear that Theorem 2.1 follows from (10) and Propositions 3.2 and 4.6. The fact that under suitable local conditions, \( \sigma_g = \mathcal{G}(\infty) \mu(\infty) > 0 \) follows from (23) and (39).

## 5 Proof of Theorem 1.3

We begin this section by proving the following theorem.

**Theorem 5.1.** Let \( d > 1 \). Let \( F(x) \in \mathbb{Z}[x_1, \ldots, x_n] \) be a degree \( d \) homogeneous form. Let us define a bihomogeneous form

\[
G(x; y) = F(x_1 y_1, \ldots, x_n y_n).
\]

Then we have

\[
\min \{ \operatorname{codim} V_{G,1}^*, \operatorname{codim} V_{G,2}^* \} \geq \frac{\operatorname{codim} V_{E}^*}{2}.
\]

**Proof.** Let \( X \) be an irreducible component of \( V_{G,1}^* \) such that \( \dim X = \dim V_{G,1}^* \). By relabeling the variables if necessary, let us suppose we have

\[
X \not\subseteq V(y_j) \quad (1 \leq j \leq m) \quad \text{and} \quad X \subseteq V(y_j) \quad (m+1 \leq j \leq n)
\]

for some \( 0 \leq m \leq n \).

Claim 1: There exists \((z_1, \ldots, z_m) \in (\mathbb{C} \setminus \{0\})^m\) such that

\[
\dim X \cap (\bigcap_{1 \leq j \leq m} V(y_j - z_j)) \geq \dim X - m.
\]
Proof of Claim 1. First we show that there exists \((z_1, \ldots, z_m) \in (\mathbb{C} \setminus \{0\})^m\) such that \(X \cap (\cap_{1 \leq j \leq m} V(y_j - z_j)) \neq \emptyset\). Suppose such \((z_1, \ldots, z_m)\) does not exist. Then we have \(X = \cup_{1 \leq j \leq m} X \cap V(y_j)\). Since \(X\) is irreducible, this implies \(X = X \cap V(y_{j_0})\) for some \(1 \leq j_0 \leq m\); we have a contradiction because \(X \nsubseteq V(y_{j_0})\).

Let \(P = (x_0, z_1, \ldots, z_m, 0) \in X\) with \((z_1, \ldots, z_m) \in (\mathbb{C} \setminus \{0\})^m\). Let us consider

\[
\emptyset \neq X \cap V(y_1 - z_1) = \cup_{1 \leq j \leq \ell_1} W_{1,j},
\]

where \(W_{1,j}\)'s are the irreducible components of \(X \cap V(y_1 - z_1)\). Recall if \(Z\) is an irreducible affine variety and \(H\) is a hypersurface, then we have one of: \(Z \cap H = Z\), \(Z \cap H = \emptyset\) and every irreducible component of \(Z \cap H\) has dimension \(\dim Z - 1\). Therefore, it follows that the \(\dim W_{1,j} \geq \dim X - 1\) for each \(1 \leq j \leq \ell_1\).

Next without loss of generality suppose \(P \in W_{1,1}\). Let us consider

\[
\emptyset \neq W_{1,1} \cap V(y_2 - z_2) = \cup_{1 \leq j \leq \ell_2} W_{2,j},
\]

where \(W_{2,j}\)'s are the irreducible components of \(W_{1,1} \cap V(y_2 - z_2)\). By the same argument as above, we obtain

\[
\dim W_{2,j} \geq \dim W_{1,1} - 1 \geq \dim X - 2 \quad (1 \leq j \leq \ell_2).
\]

By continuing in this manner, we obtain the result. 

Let us fix \((z_1, \ldots, z_m) \in (\mathbb{C} \setminus \{0\})^m\) as in Claim 1. Let \(z_{m+1} = \cdots = z_n = 0\). Then we have

\[
\dim X \cap (\cap_{1 \leq j \leq n} V(y_j - z_j)) = \dim X \cap (\cap_{1 \leq j \leq m} V(y_j - z_j)) \\
\geq \dim X - m \\
= \dim V_{G,1} - m.
\]

We also have

\[
X \cap (\cap_{1 \leq j \leq n} V(y_j - z_j)) \\
\subseteq V_{G,1} \cap (\cap_{1 \leq j \leq n} V(y_j - z_j)) \\
= \left\{ x \in \mathbb{C}^n : \frac{\partial F}{\partial x_1}(x_1 z_1, \ldots, x_m z_m, 0) = \cdots = \frac{\partial F}{\partial x_m}(x_1 z_1, \ldots, x_m z_m, 0) = 0 \right\} \\
\times \{ y \in \mathbb{C}^n : y_j = z_j \quad (1 \leq j \leq n) \}.
\]

For each \(1 \leq k \leq n\), let us define

\[
T_k = \left\{ x \in \mathbb{C}^n : \frac{\partial F}{\partial x_1}(x) = \cdots = \frac{\partial F}{\partial x_k}(x) = x_{k+1} = \cdots = x_n = 0 \right\}.
\]

Then it follows from (41) that

\[
\dim X \cap (\cap_{1 \leq j \leq n} V(y_j - z_j)) \leq (n - m) + \dim T_n.
\]

Claim 2: We have

\[
\max_{1 \leq k \leq n} \dim T_k \leq \frac{n + \dim V_E^F}{2}.
\]
Proof of Claim 2. First we have
\[
\dim T_{k+1} - 1 \leq \dim T_k \leq \dim T_{k+1} + 1.
\]
This is because the dimension of
\[
\{ x \in \mathbb{C}^n : \frac{\partial F}{\partial x_1}(x) = \cdots = \frac{\partial F}{\partial x_k}(x) = x_{k+2} = \cdots = x_n = 0 \}
\]
is either \( \dim T_{k+1} \) or \( \dim T_{k+1} + 1 \). Furthermore, intersecting this set with \( V(x_{k+1}) \), which is \( T_k \), either reduces the dimension by 1 or the dimension stays the same. Therefore, we have \( \dim T_{k+1} - 1 \leq \dim T_k \leq \dim T_{k+1} + 1 \). Here it is important that we are only dealing with homogeneous forms, because every irreducible component of an affine variety \( Z \) defined by homogeneous forms contains \( 0 \); therefore, any hypersurface \( H \) defined by a homogeneous form intersects every irreducible component of \( Z \), and thus we always have \( \dim Z \cap H \geq \dim Z - 1 \) in this case.

Let \( L_1, \ldots, L_n \) be a set of integers satisfying \( L_n = \dim V^*_F, 0 \leq L_k \leq k \) (1 \( \leq k \leq n \)) and
\[
L_{k+1} - 1 \leq L_k \leq L_{k+1} + 1 \quad (1 \leq k \leq n - 1).
\]
Then it is a basic exercise to show that the largest possible value of \( \max_{1 \leq k \leq n} L_k \) for any such set of integers is \( k_0 \), where
\[
k_0 = \begin{cases} 
\frac{n + \dim V^*_F}{2} & \text{if } n \equiv \dim V^*_F \pmod{2}, \\
\frac{n + \dim V^*_F}{2} - \frac{1}{2} & \text{if } n \not\equiv \dim V^*_F \pmod{2}. 
\end{cases}
\]
Since we can choose \( L_k = \dim T_k \) (1 \( \leq k \leq n \)), the result follows. \( \square \)

Therefore, by combining (40), (42) and (43), we obtain
\[
\text{codim } V^*_G, 1 \geq 2n - \dim V^*_G, 1 \geq \frac{n - \dim V^*_F}{2} = \frac{\text{codim } V^*_F}{2}.
\]
By symmetry we obtain the same bound for \( \text{codim } V^*_G, 2 \) as well. \( \square \)

Let \( d > 1 \). Throughout this section we let \( f(x) \) be a degree \( d \) polynomial in \( \mathbb{Z}[x_1, \ldots, x_n] \), and denote its degree \( d \) homogeneous portion by \( F(x) \). We now solve the equation
\[
f(x) = 0
\]
in semiprimes.

Let \( N = N_1N_2 \) where \( N_1 \geq N_2 \). Let us define
\[
\mathcal{N}_2(f;N;N_1,N_2) = \sum_{z_1 \in [0,N]} \cdots \sum_{z_n \in [0,N]} \prod_{j=1}^n (\log p_j)(\log q_j) \cdot \mathbb{1}_V(f)(z).
\]

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It is clear that \( \mathbb{N}_2(f; N; N_1, N_2) \) is the number of semiprime solutions \((p_1 q_1, \ldots, p_n q_n) \in [0, N]^n\) to the equation (44), where \( p_j \geq q_j, p_j \in [0, N_1] \cap \not\mathcal{P}, \) and \( q_j \in [0, N_2] \cap \not\mathcal{P}, \) counted with weight \( \prod_{1 \leq j \leq n} (\log p_j)(\log q_j). \) We also consider the following modification of the local conditions (\(*\)) given in Section 1.

**Local conditions (\(*\)).** The equation 

\[
F(x) = 0
\]

has a non-singular real solution in \((0,1)^n\), and the equation (44) has a non-singular solution in \((\mathbb{Z}_p^\times)^n\) for every prime \( p. \)

It is clear that these conditions are identical to the local conditions (\(*\)) when the polynomial in consideration is homogeneous. We prove the following theorem.

**Theorem 5.2.** Let \( \delta \leq 1/2. \) Suppose that \( f \) satisfies the local conditions (\(*\)) and

\[
\text{codim } V_F^* > 2 \cdot 4^d \max \left\{ 4(2d - 1), \frac{d}{\delta} \right\}.
\]

Then we have

\[
\mathbb{N}_2(f; N; N^{1-\delta}, N^{\delta}) \gg N^{n-d}.
\]

By taking \( \delta = 1/2 \) in the above theorem, the following is an immediate corollary which also implies Theorem 1.3.

**Corollary 5.3.** Suppose that \( f \) satisfies the local conditions (\(*\)) and \( \text{codim } V_F^* > 4^d \cdot 8(2d - 1). \) Then we have \( \mathbb{N}_2(f; N; \sqrt{N}, \sqrt{N}) \gg N^{n-d}. \)

**Proof of Theorem 5.2.** We define \( g(x; y) = f(x_1 y_1, \ldots, x_n y_n), \) and denote its degree 2d homogeneous portion by \( G(x; y) = F(x_1 y_1, \ldots, x_n y_n), \) which is bihomogeneous in \( x \) and \( y \) of bidegree \((d, d). \) It is clear that if \((x, y) = (p_1, \ldots, p_n, q_1, \ldots, q_n) \in ([0, N^{1-\delta}]^n \times [0, N^{\delta}]^n) \cap \not\mathcal{P}^n \) is a prime solution to the equation \( g(x; y) = 0, \) then \((p_1 q_1, \ldots, p_n q_n) \in [0, N]^n\) is a semiprime solution to the equation (44). Therefore, by taking into account possible repetitions we have

\[
\mathbb{N}_2(f; N; N^{1-\delta}, N^{\delta}) \geq \frac{1}{2^n} \mathbb{N}_\not\mathcal{P}(g; N^{1-\delta}, N^{\delta}).
\]

(47)

By Theorem 5.1, we have

\[
\min\{\text{codim } V_{G,1}^*, \text{codim } V_{G,2}^*\} \geq \frac{\text{codim } V_F^*}{2} > 4^d \max \left\{ 4(2d - 1), \frac{d}{\delta} \right\}.
\]

(48)

It follows that the bihomogeneous form \( G \) satisfies (8) with \( d_1 = d_2 = d, \ P_1 = N^{1-\delta}, \ P_2 = N^{\delta}, \ R = 1, \) and \( b = \frac{1}{2^n}. \) Therefore, Theorem 2.1 gives us

\[
\mathbb{N}_\not\mathcal{P}(g; N^{1-\delta}, N^{\delta}) = \sigma_g N^{n-d} + O \left( \frac{N^{n-d}}{\log N} \right)
\]

(49)

for some \( c > 0. \)
We now prove that $\sigma_g$ in (49) is in fact positive. Suppose the equation (46) has a non-singular real solution $(\xi_1, \ldots, \xi_n) \in (0, 1)^n$. Then it can be verified that $(\xi_1, \ldots, \xi_n, 1/2, \ldots, 1/2) \in (0, 1)^{2n}$ is a non-singular real solution to the equation $G(x; y) = 0$. Similarly if the equation (44) has a non-singular solution $(\xi_1, \ldots, \xi_n) \in (\mathbb{Z}_p^\times)^n$, then $(\xi_1, \ldots, \xi_n, 1, \ldots, 1) \in (\mathbb{Z}_p^\times)^{2n}$ is a non-singular solution in $(\mathbb{Z}_p^\times)^{2n}$ to the equation $g(x; y) = 0$. Thus it follows from Theorem 2.1 that $\sigma_g > 0$. Therefore, we obtain from (47) and (49) that $N_2(f;N;N^{1-\delta},N^\delta) \gg N^{n-d}$.

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AUTHOR

Shuntaro Yamagishi
Mathematisch Instituut
Universiteit Utrecht
Utrecht, Nederland
s.yamagishi@uu.nl