Paired quantum Hall states on noncommutative two-tori

Vincenzo Marotta\textsuperscript{1}, Adele Naddeo\textsuperscript{2}

Abstract

By exploiting the notion of Morita equivalence for field theories on noncommutative tori and choosing rational values of the noncommutativity parameter $\theta$ (in appropriate units), a one-to-one correspondence between an abelian noncommutative field theory (NCFT) and a non-abelian theory of twisted fields on ordinary space can be established. Starting from this general result, we focus on the conformal field theory (CFT) describing a quantum Hall fluid (QHF) at paired states fillings $\nu = \frac{m}{p_m + 2}$\textsuperscript{[1]}, recently obtained by means of $m$-reduction procedure, and show that it is the Morita equivalent of a NCFT. In this way we extend the construction proposed in \textsuperscript{[2]} for the Jain series $\nu = \frac{m}{2p_m + 1}$. The case $m = 2$ is explicitly discussed and the role of noncommutativity in the physics of quantum Hall bilayers is emphasized. Our results represent a step forward the construction of a new effective low energy description of certain condensed matter phenomena and help to clarify the relationship between noncommutativity and quantum Hall fluids.

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\textsuperscript{1}Dipartimento di Scienze Fisiche, \textit{Universit\`a di Napoli “Federico II”} and INFN, Sezione di Napoli, Compl. universitario M. Sant’Angelo, Via Cinthia, 80126 Napoli, Italy
\textsuperscript{2}CNISM, Unit\`a di Ricerca di Salerno and Dipartimento di Fisica ”E. R. Caianiello”, \textit{Universit\`a degli Studi di Salerno}, Via Salvador Allende, 84081 Baronissi (SA), Italy
1 Introduction

The quantum Hall effect (QHE) is one of the most remarkable many-body phenomena discovered in the last twenty-five years [3][4]. It takes place in a two-dimensional electron gas formed in a quantum well in a semiconductor host material and in the presence of a very high magnetic field [5][6], as a result of the commensuration between the number of electrons $N_e$ and the number of flux quanta $N_\Phi$. The electrons condense into distinct and highly non-trivial ground states ('vacua') formed at each integer (IQHE) [7] or rational fractional value (FQHE) [8] of the filling factor $\nu = \frac{N_e}{N_\Phi}$. In particular at fractional fillings quasi-particles with fractional charge and statistics emerge, and new kinds of order parameters are considered [9]. In such a context strong research interests have been growing in the last years towards a full understanding of the physics of those plateaux which do not fall into the hierarchical scheme [10]. To such an extent a pairing picture, in which pairs of spinless or spin-polarized fermions condense, has been proposed [11] for the non-standard fillings $\nu = \frac{1}{q}$, $q > 0$ and even. As a result the ground state has been described in terms of the Pfaffian (the so called Pfaffian state) and the non-Abelian statistics of the fractional charged excitations evidenced [11][12]. Today quasi-particles with non-Abelian statistics appear very promising in view of the realization of a fault-tolerant quantum computer [13]. Indeed protection against decoherence could be obtained by encoding quantum information in some topological characteristics of the strongly correlated electron system while quantum gates could be implemented by topologically non trivial operations such as braidings of non-Abelian anyons.

Following this line, increasing technological progress in molecular beam epitaxy techniques has led to the ability to produce pairs of closely spaced two-dimensional electron gases. Since then such bilayer quantum Hall systems have been widely investigated theoretically as well as experimentally [4][14, 15]. Strong correlations between the electrons in different layers lead to new physical phenomena involving spontaneous interlayer phase coherence with an associated Goldstone mode. In particular a spontaneously broken $U(1)$ symmetry [16] has been discovered and identified and many interesting properties of such systems have been studied: the Kosterlitz-Thouless transition, the zero resistivity in the counter-current flow, a DC/AC Josephson-like effect in interlayer tunneling as well as the presence of a gapless superfluid mode [17] [18]. Indeed, when tunneling between the layers is weak, the quantum Hall bilayer state can be viewed as arising from the condensation of an excitonic superfluid in which an electron in one layer is paired with a hole in the other layer. The uncertainty principle makes it impossible to tell which layer either component of this composite boson is in. Equivalently the system may be regarded as a ferromagnet in which each electron exists in a coherent superposition of the “pseudospin” eigenstates, which encode the layer degrees of freedom [19][18]. The phase variable of such a superposition fixes the orientation of the pseudospin magnetic moment and its spatial variations govern the low energy excitations in the system. So quantum Hall bilayers are an interesting realization of the pairing picture at non-standard fillings.

On the other hand, noncommutative field theories (NCFT) have attracted much attention in the last years because they provide a non trivial generalization of local quantum field theories, allowing for some degree of non locality while retaining an interesting mathematical structure [20]. One motivation for the relevance of such theories is that the notion of space-time presumably has to be modified at very short distances by introducing a limit to the resolution in which one may probe it. In this way space-time becomes fuzzy at very short distances because the concept of a point is lost.

The simplest framework in which NCFT emerge in a natural way as an effective description of the dynamics is just the Landau problem [21], namely the quantum mechanics of the motion of $N_e$ charged particles in two dimensions subjected to a transverse magnetic field. The strong field limit $B \to \infty$ at fixed mass $m$ projects the system onto the lowest Landau level and, for each particle $I = 1, ..., N_e$, the corresponding coordinates $(\frac{eB}{c} x_I, y_I)$ are a pair of canonical variables which satisfy the commutation relations $[x_I, y_J] = i \delta_{IJ} \theta$, $\theta = \frac{\hbar c}{eB} = l_M^2$ being the noncommutativity parameter. In this picture the electron is not a point-like particle and can be localized at best at the scale of the magnetic length $l_M$. 

Indeed an alternative description is possible, which reminds a system of open strings in a B-field background [24], the one in terms of neutral fields such as the density and the current which carry the quantum numbers of dipoles. As a consequence, the currents and the density of a system of electrons in a strong magnetic field may be described by a noncommutative Chern-Simons theory [24], which captures the granular structure of the fluid and exactly reproduces the quantitative connection between filling fractions and statistics, a crucial feature of quantum Hall fluid (QHF) physics [24][25]. An equivalent formulation is possible as well in terms of a matrix theory similar to that describing D0-branes in string theory [26]. These observations stimulated a new research area, so in the last years deep efforts have been devoted to a whole understanding of the relationship between noncommutative spaces and QHF and its implications. As a result, noncommutativity is related to the finite number $N_e$ of electrons in a realistic sample via the rational parameter $\theta \propto \frac{1}{N_e}$, which sets the elementary area of nonlocality [27][28][29]. In this context the $\theta$-dependence of physical quantities is expected to be analytic near $\theta = 0$ [30] because of the very smooth ultraviolet behavior of noncommutative Chern-Simons theories [31]; as a consequence the effects of the electron’s granularity embodied in $\theta$ can be expanded in powers of $\theta$ via the Seiberg-Witten map [32] and then appear as corrections to the large-$N$ results of field theory.

A significant feature of NCFT is the Morita duality between noncommutative tori [33][34][35], which establishes a relation, via a one-to-one correspondence, between representations of two noncommutative algebras. If we refer to gauge theories on noncommutative tori, Morita duality can be viewed as a low energy analogue of $T$-duality of the underlying string model [36] and, as such, it results a powerful tool in order to establish a correspondence between NCFT and well known standard field theories. Indeed, for rational values of the noncommutativity parameter, $\theta = \frac{1}{n}$, one of the theories obtained by using the Morita equivalence is a commutative field theory of matrix valued fields with twisted boundary conditions and magnetic flux $c$ [37] (which, in a string description, behaves as a $B$-field modulus). Also open Wilson lines of the noncommutative theory are mapped to closed Wilson lines wrapping the torus of the commutative theory [38]. The $U(N)$ commutative theory displays a bit unexpected behavior, for instance in correspondence of particular values of the relevant parameters its renormalized dispersion relation will develop tachyonic modes [39] which appear to be related to the spontaneous $Z_N \times Z_N$ symmetry breaking due to electric flux condensation. In other words, the commutative field theory keeps track of the spontaneous breaking of translational invariance which characterizes its dual (noncommutative) counterpart.

In a recent work [2] we followed such an approach focusing on a particular conformal field theory (CFT), the one obtained via $m$-reduction technique [40], which has been successfully applied to the description of a quantum Hall fluid (QHF) at Jain [41][42] as well as paired states fillings [1][43] and in the presence of topological defects [44][45][47]. In particular, we showed by means of the Morita equivalence that a NCFT with $\theta = 2\rho + \frac{1}{m}$ is mapped to a CFT on an ordinary space. We identified such a CFT with the $m$-reduced CFT developed for a QHF at Jain fillings $\nu = \frac{m}{2m+1}$ [41][42], whose neutral fields satisfy twisted boundary conditions. In this way we gave a meaning to the concept of "noncommutative conformal field theory", as the Morita equivalent version of a CFT defined on an ordinary space. The image of Morita duality in the ordinary space is given by the $m$-reduction technique and the corresponding noncommutative torus Lie algebra is naturally realized in terms of Generalized Magnetic Translations (GMT). That introduces a new relationship between noncommutative spaces and QHF and paves the way for further investigations on the role of noncommutativity in the physics of general strongly correlated many body systems [44].

In this work, by making use of Morita duality, we will show that an abelian NCFT with $\theta = \frac{\nu}{2} + \frac{1}{m}$ is mapped to a nonabelian theory of twisted fields on ordinary space, which coincides with the $m$-reduced CFT developed for a QHF at paired states fillings $\nu = \frac{2m+1}{2m+2}$ [1][43]. That extends to non standard fillings the analysis carried out in Ref. [2]. In particular we focus on the $m = 2$ case which is experimentally relevant and describes a system of two parallel layers of 2D electrons gas in a strong perpendicular magnetic field. The consequences of noncommutativity on the physics of the pairing picture in quantum Hall bilayers are analyzed in detail.
The paper is organized as follows.

In Section 2, we review the description of a QHF at paired states fillings $\nu = \frac{m}{pm+2}$ obtained by means of the $m$-reduction procedure \[1, 43\]. We focus mainly on the $m = 2$ case and briefly recall the physics of quantum Hall bilayers on the plane.

In Section 3, we explicitly build up the Morita equivalence between CFTs in correspondence of rational values of the noncommutativity parameter $\theta$ with an explicit reference to the $m$-reduced theory describing a QHF at paired states fillings. Indeed we clearly show that there is a well defined one-to-one correspondence between the fields on a noncommutative torus and those of a non-abelian field theory on an ordinary space.

In Section 4, we show how the noncommutative torus Lie algebra is realized through GMT in the QHF context. The role of the noncommutative tori in the twisted sector of our theory is evidenced and the relevance of non-Abelian statistics of the corresponding ground states is outlined. Finally, the implications on the transport properties of a quantum Hall bilayer with different boundary conditions are briefly discussed.

In Section 5, some comments and outlooks of this work are given.

Finally, in the Appendix we recall the $m$-reduction description of a QHF at paired states fillings on the torus topology, focusing on the special $m = 2$ case \[43\].

2 The $m$-reduction description of a QHF at paired states fillings

In this Section we review how the $m$-reduction procedure on the plane (genus $g = 0$) \[10\] works in describing successfully a QHF at paired states fillings $\nu = \frac{m}{pm+2}$ \[1, 43\]. We focus mainly on the special case $m = 2$ and on the physics of a quantum Hall bilayer.

The idea is to build up an unifying theory for all the plateaux with even denominator starting from the bosonic Laughlin filling $\nu = 1/pm + 2$, which is described by a CFT (mother theory) with $c = 1$, in terms of a scalar chiral field compactified on a circle with radius $R^2 = 1/\nu = pm + 2$ (or the dual $R^2 = 4/pm + 2$). Then the $U(1)$ current is given by $J(z) = i\partial_z Q(z)$, where $Q(z)$ is the compactified Fubini field with the standard mode expansion:

$$Q(z) = q - ip\ln z + \sum_{n \neq 0} \frac{a_n}{n} z^{-n},$$

with $a_n$, $q$ and $p$ satisfying the commutation relations $[a_n, a_{n'}] = n\delta_{n,n'}$ and $[q, p] = i$. Let us notice that the informations about the quantization of momentum and the winding numbers are stored in the lattice geometry induced by the QHE quantization (see Ref. \[1\] for details); in other words the QHE physics fixes the compactification radius. The corresponding primary fields are expressed in terms of the vertex operators $U^\alpha(z) := e^{i\alpha Q(z)}$ with $\alpha^2 = 1, ..., 2 + pm$ and conformal dimension $h = \frac{\alpha^2}{2}$.

Starting with this set of fields and using the $m$-reduction procedure, which consists in considering the subalgebra generated only by the modes in Eq. \[1\], which are a multiple of an integer $m$, we get the image of the twisted sector of a $c = m$ orbifold CFT (daughter theory), the twisted model (TM), which describes the lowest Landau level dynamics. Then the fields in the mother CFT can be factorized into irreducible orbits of the discrete $Z_m$ group which is a symmetry of the daughter theory and can be organized into components which have well defined transformation properties under this group. The general characteristics of the daughter theory is the presence of twisted boundary conditions (TBC) which are induced on the component fields and are the signature of an interaction with a localized topological defect \[44, 45, 47\]. When we generalize the construction to a torus (genus $g = 1$), we find different sectors...
corresponding to different boundary conditions imposed at the ends of the finite two-dimensional (2D) layer, as shown in detail in Refs. [43] [42] and briefly recalled in the Appendix. To compare the orbifold so built with the $c = m$ CFT, we use the mapping $z \rightarrow z^{1/m}$ and the isomorphism defined in Ref. [40] between fields on the $z$ plane and fields on the $z^m$ covering plane given by the following identifications:

$$a_{nm+l} \rightarrow \sqrt{m} a_{n+l/m}, \ q \rightarrow \frac{1}{\sqrt{m}} q.$$

Let us now focus on the special $m = 2$ case, which describes a system consisting of two parallel layers of 2D electrons gas in a strong perpendicular magnetic field. The filling factor $\nu = \nu^{(a)} + \nu^{(b)} = \frac{1}{p+1}$. For $p = 0$ ($p = 1$) it describes the bosonic 220 (fermionic 331) Halperin (H) state [48].

The CFT description for such a system can be given in terms of two compactified chiral bosons $Q^{(a)}$ with central charge $c = 2$. In order to construct the fields $Q^{(a)}$ for the TM, the starting point is the bosonic filling $\nu = 1/2(p+1)$, described by a CFT with $c = 1$ in terms of a scalar chiral field $Q$ compactified on a circle with radius $R^2 = 1/\nu = 2(p+1)$ (or its dual $R^2 = 2/(p+1)$), see Eq. (1). The $m$-reduction procedure generates a daughter theory which is a $c = 2$ orbifold. Its primary fields content can be expressed in terms of a $Z_2$-invariant scalar field $X(z)$, given by

$$X(z) = \frac{1}{2} \left( Q^{(1)}(z) + Q^{(2)}(-z) \right),$$

(2)

describing the electrically charged sector of the new filling, and a twisted field

$$\phi(z) = \frac{1}{2} \left( Q^{(1)}(z) - Q^{(2)}(-z) \right),$$

(3)

which satisfies the twisted boundary conditions $\phi(e^{i\pi z}) = -\phi(z)$ and describes the neutral sector [1]. Such TBC signal the presence of a localized topological defect which couples, in general, the $m$ edges in an $m$-layers system [44] [45] [47]. In the case of our interest, a bilayer system ($m = 2$), we get a crossing between the two edges as sketched in Fig. 1.

Figure 1: The bilayer system, (a) without the topological defect (PBC), (b) with the topological defect (TBC).

The chiral fields $Q^{(a)}$, defined on a single layer $a = 1, 2$, due to the boundary conditions imposed upon them by the orbifold construction, can be thought of as components of a unique “boson” defined on a double covering of the disc (layer) ($z^{(1)}_i = -z^{(2)}_i = z_i$). As a consequence the two layers system becomes equivalent to one-layer QHF (in contrast with the Halperin model in which they appear independent) and the $X$ and $\phi$ fields defined in Eqs. (2) and (3) diagonalize the interlayer interaction. In particular the $X$ field carries the total charge with velocity $v_X$, while $\phi$ carries the charge difference of the two edges with velocity $v_\phi$, i.e. no charge, being the number of electrons the same for each layer (balanced system).

The primary fields are the composite operators $V(z) = U_X(z) \psi(z)$, where $U_X(z) = \frac{1}{\sqrt{\pi}} : e^{i\alpha X(z)} :$ are the vertices of the charged sector with $\alpha^2 = 2(p+1)$. Furthermore the highest weight states of the neutral
sector can be classified in terms of two kinds of chiral operators,
\[ \psi(z) (\bar{\psi}(z)) = \frac{1}{2\sqrt{z}} (e^{i\alpha} \phi(z) \pm ie^{i\alpha} \phi(-z)) \],
which, in a fermionic language, correspond to \( c = 1/2 \) Majorana fermions with periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions \[43\]. As a consequence this theory decomposes into a tensor product of two CFTs, a twisted invariant one with \( c = 3/2 \), realized by the charged boson \( X(z) \) and the Ramond Majorana fermion, which is coupled to the charged sector, while the second one has \( c = 1/2 \) and is realized in terms of the Neveu-Schwarz Majorana fermion. The two Majorana fermions just defined are inequivalent, due to the breaking of the symmetry which exchanges them and that results in a non-Abelian statistics. Such a factorization results much more evident in the construction of the modular invariant partition function, as we briefly recall in the Appendix \[43\]. The bosonized energy-momentum tensor of the \( Z_2 \) twist invariant theory develops a cosine term in its neutral sector which is described by the Ramond fields:
\[ T_{\psi}(z) = -\frac{1}{4} (\partial \phi)^2 - \frac{1}{16z^2} \cos(2\sqrt{2}\phi). \] (4)
It is a clear signature of a tunneling phenomenon which selects out a new stable vacuum, the \( c = 3/2 \) one. If we refer to the bilayer system, we can reduce the spacing between the layers so that the two species of electrons which live on them become indistinguishable: in such a case the tunneling amplitude gets large enough to make the H states flow to the Moore-Read (MR) states \[11\]. In the limit of strong tunneling the velocity of one Majorana becomes zero and the theory reduces to the \( c = 3/2 \) CFT. Let us also point out that \( m \)-ality in the neutral sector is coupled to the charged one exactly, according to the physical request of locality of the electrons with respect to the edge excitations. Indeed our projection, when applied to a local field, automatically couples the discrete \( Z_m \) charge of \( U(1) \) with the neutral sector in order to give rise to a single valued composite field.

Now let us give an interpretation of the existence of these sectors in terms of conformal invariant boundary conditions which are due to the scattering of the particles on localized impurities \[44\][45][47\]. The H sector describes a pure QHF phase in which no impurities are present and the two layers edges are not connected (see Fig. 1(a)). In realistic samples however this is not the case and the deviations from the Halperin state may be regarded as due to the presence of localized impurities. These effects can be accounted for by allowing for more general boundary conditions just as the ones provided by our TM. In fact an impurity located at a given point on the edge induces twisted boundary conditions for the boson \( \phi \) and, as a consequence, a current can flow between the layers. Then a coherent superposition of interlayer interactions could drive the bilayer to a more symmetric phase in which the two layers are indistinguishable due to the presence of a one electron tunneling effect along the edge.

The primary fields content of the theory just introduced on the torus topology will be given in the Appendix.

### 3 \( m \)-reduced CFT for QHF at paired states fillings and Morita equivalence

In this Section we exploit the notion of Morita equivalence on noncommutative tori for rational values of the noncommutativity parameter \( \theta \) and construct a general one-to-one correspondence between NCFTs and CFTs on the ordinary space. We will refer to the \( m \)-reduced theory describing a QHF at paired states fillings, recalled in Section 2. In this way we extend our previous results concerning a QHF at Jain hierarchical fillings \[2\] and further clarify the role of noncommutativity in QHF physics by introducing a new perspective.

The Morita equivalence \[33\][34][35\] is defined in full generality as a one-to-one correspondence between representations of two noncommutative algebras; indeed it conserves all the modules and their associated structures. Let us now consider an \( U(N) \) NCFT defined on the noncommutative torus \( T^2_{\theta} \) and, for simplicity, of radii \( R \). Such a model has \( N \times N \) matrix degrees of freedom and arises from the regularization
of the theory describing a membrane with world-volume topology $T^2 \times R$. In general, this procedure is related to the fact that the symmetry group of area-preserving diffeomorphisms on the membrane can be approximated by $U(N)$ for a surface of any genus $[19][20]$; furthermore it is equivalent to approximating the membrane surface by a finite lattice.

The simplest noncommutative space is $R^2_\theta$ with coordinates satisfying the Heisenberg commutation relation $[x_1, x_2] = i\Theta$. Along with the space $R^2_\theta$ it is natural to consider its compactification $T^2_\theta$ with the Morita duality represented by the following $SL(2, Z)$ action on the parameters:

$$\theta' = \frac{a\theta + b}{c\theta + d}, \quad R' = |c\theta + d| R;$$

(5)

here $\theta = \Theta / (2\pi R^2)$, $a, b, c, d$ are integers and $ad - bc = 1$. An intriguing feature arises for rational values of the noncommutativity parameter, $\theta = -\frac{p}{q}$, when $c\theta + d = \frac{1}{q}$: the Morita transformation (5) sends the NCFT to an ordinary one with $\theta' = 0$ and different radius $R' = \frac{R}{\theta}$, involving in particular a rescaling of the rank of the gauge group $[34][35][36]$. Indeed the dual theory is a twisted $U(N')$ theory with $N' = aN$ and 't Hooft flux $c$. The classes of $\theta = 0$ theories are parametrized by an integer $m$, so that for any $m$ there is a finite number of abelian theories which are related by a subset of the transformations given in Eq. (3). Our main result relies on such consideration and can be expressed as follows.

The $m$-reduction technique applied to the QHF at paired states fillings ($\nu = \frac{m}{pm + 2}$, $p$ even) can be viewed as the image of the Morita map (characterized by $a = \frac{q}{p} (m - 1) + 1$, $b = \frac{q}{p}$, $c = m - 1$, $d = 1$) between the two NCFTs with $\theta = 1$ and $\theta = \frac{p}{q} + \frac{1}{m}$ ($\theta = \nu_0/\nu$, being $\nu_0 = 1/2$ the filling of the starting theory), respectively and corresponds to the Morita map in the ordinary space. The $\theta = 1$ theory is an $U(1)_{\theta=1}$ NCFT while the mother CFT is an ordinary $U(1)$ theory; furthermore, when the $U(1)_{\theta=\frac{p}{q} + \frac{1}{m}}$ NCFT is considered, its Morita dual CFT has $U(m)$ symmetry. As a consequence, the following correspondence Table between the NCFTs and the ordinary CFTs is established:

$$U(1)_{\theta=1} \quad \text{Morita} \quad \rightarrow \quad (a = 1, b = -1, c = 0, d = 1) \quad U(1)_{\theta=0}$$

$$U(1)_{\theta=\frac{p}{q} + \frac{1}{m}} \quad \text{Morita} \quad \rightarrow \quad (a = m, b = -\frac{pm}{q} - 1, c = 1 - m, d = \frac{p}{q} (m - 1) + 1) \quad U(m)_{\theta=0}$$

(6)

Let us notice that theories which differ by an integer in the noncommutativity parameter are not identical because they differ from the point of view of the CFT. In fact, the Morita map acts on more than one parameter of the theory. For instance, the compactification radius of the charged component is renormalized to $R^2_X = p + \frac{d}{m}$, that gives rise to different CFTs by varying $p$ values. Moreover the action of the $m$-reduction procedure on the number $p$ doesn’t change the central charge of the CFT under study but modifies the spectrum of the charged sector $[1][33]$.

In order to show that the $m$-reduction technique applied to the QHF at paired states fillings is the image of the Morita map between the two NCFTs with $\theta = 1$ and $\theta = \frac{p}{q} + \frac{1}{m}$ respectively and corresponds to the Morita map in the ordinary space it is enough to show how the twisted boundary conditions on the neutral fields of the $m$-reduced theory (see Section 2) arise as a consequence of the noncommutative nature of the $U(1)_{\theta=\frac{p}{q} + \frac{1}{m}}$ NCFT.

Let us start by recalling that an associative algebra of smooth functions over the noncommutative two-torus $T^2_\theta ([x_1, x_2] = i\Theta)$ can be realized through the Moyal star product:

$$f(x) \ast g(x) = \exp \left( \frac{i\Theta}{2} (\partial_{x_1} \partial_{y_2} - \partial_{x_2} \partial_{y_1}) \right) f(x) g(y) \bigg|_{y=x}.$$ 

(7)
Such functions are identified with general field operators $\Phi$ and, when defined on a torus, can have different boundary conditions associated to any of the compact directions. For the torus we have four different possibilities:

$$
\Phi(x_1 + R, x_2) = e^{2\pi i \alpha_1} \Phi(x_1, x_2), \quad \Phi(x_1, x_2 + R) = e^{2\pi i \alpha_2} \Phi(x_1, x_2),
$$

(8)

where $\alpha_1$ and $\alpha_2$ are the boundary parameters. It is useful to decompose the elements of the algebra in their Fourier components. The Fourier expansion of the general field operator $\Phi_{\vec{\alpha}}$ with boundary conditions $\vec{\alpha} = (\alpha_1, \alpha_2)$ takes the form:

$$
\Phi_{\vec{\alpha}} = \sum_{\vec{n}} \Phi_{\vec{n}} U_{\vec{n} + \vec{\alpha}},
$$

(9)

where we define the generators of the algebra as

$$
U_{\vec{n} + \vec{\alpha}} \equiv \exp \left( 2\pi i \frac{(\vec{n} + \vec{\alpha}) \cdot \vec{J}}{R} \right).
$$

(10)

They give rise to the following commutator:

$$
\left[ U_{\vec{n} + \vec{\alpha}}, U_{\vec{n'} + \vec{\alpha'}} \right] = -2i \sin \left( \frac{2\pi \theta (\vec{n} + \vec{\alpha}) \wedge (\vec{n'} + \vec{\alpha'})}{m} \right) U_{\vec{n} + \vec{\alpha} + \vec{n'} + \vec{\alpha'}},
$$

(11)

where $\vec{\theta} \wedge \vec{q} = \varepsilon_{ij} p_i q_j$.

When the noncommutativity parameter $\theta$ takes a rational value, which we choose as $\theta = \frac{q}{m}$, being $q = \frac{2}{m}m + 1$ and $m$ relatively prime integers, the infinite-dimensional algebra generated by the $U_{\vec{n} + \vec{\alpha}}$ breaks up into equivalence classes of finite dimensional $(m \times m)$ subspaces. Indeed the elements $U_{m \vec{n}}$ generate a center of the algebra and, within Eq. (9), can be treated as ordinary generators defined on a commutative space. That makes possible for the momenta the following decomposition:

$$
\vec{n}' + \vec{\alpha}' = m \vec{n} + \vec{J}, \quad 0 \leq j_1, j_2 \leq m - 1.
$$

(12)

and leads to the splitting of the whole algebra into equivalence classes classified by all the possible values of $m \vec{n}$. Each class is a subalgebra generated by the $m^2$ functions $U_{m \vec{n} + \vec{J}}$ which satisfy the relations

$$
\left[ U_{m \vec{n} + \vec{J}}, U_{m \vec{n'} + \vec{J'}} \right] = -2i \sin \left( \frac{2\pi \theta}{m} \frac{m \vec{n} \wedge \vec{n'}}{m \vec{n} + m \vec{n'} + \vec{J} + \vec{J'}} \right) U_{m \vec{n} + m \vec{n'} + \vec{J} + \vec{J'}}.
$$

(13)

The algebra (13) just introduced is isomorphic to the (complexification of the) $U(m)$ algebra, whose general $m$-dimensional matrix representation can be constructed by means of the following "shift" and "clock" generators [38, 49, 51]:

$$
Q = \begin{pmatrix}
1 & & \\
-\varepsilon & \ddots & \\
& \ddots & 1
\end{pmatrix}, \quad P = \begin{pmatrix}
0 & 1 & & \\
& \ddots & & \\
& & \ddots & 1 \\
& & & \varepsilon^{m-1}
\end{pmatrix},
$$

(14)

being $\varepsilon = \exp(\frac{2\pi i}{m})$. Matrices $P$ and $Q$ are unitary, traceless and satisfy the property:

$$
PQ = \varepsilon QP.
$$

(15)

So the matrices $J_{\vec{J}} = \varepsilon \frac{\hbar}{2\pi} Q^3 P^2$, $j_1, j_2 = 0, \ldots, m - 1$, generate an algebra isomorphic to (13):

$$
\left[ J_{\vec{J}}, J_{\vec{J'}} \right] = -2i \sin \left( \frac{2\pi \theta}{m} \frac{m \vec{J} \wedge \vec{J'}}{m \vec{J} + m \vec{J'}} \right) J_{m \vec{J} + \vec{J'}}.
$$

(16)
Thus the following Morita mapping has been realized between the Fourier modes defined on a noncommutative space and $U(m)$-valued functions defined on a commutative space:

$$
\exp\left(2\pi im \frac{\vec{n} \cdot \vec{x}}{R}\right) \longleftrightarrow \exp\left(2\pi i \frac{\vec{m} \cdot \vec{x}}{R}\right) J^\gamma.
$$

(17)

As a consequence a mapping between the general field operator $\Phi$ on the noncommutative torus $T^2$ and the field $\Phi$ on the dual commutative torus $T^2_{\theta=0}$ is generated as follows:

$$
\Phi = \sum_{n} \exp\left(2\pi im \frac{\vec{n} \cdot \vec{x}}{R}\right) \sum_{j=0}^{m-1} \hat{\Phi} \left(\frac{\vec{n} \cdot \vec{x}}{R} + j\right) U_{\vec{n}+\vec{j}} \longmapsto \Phi = \sum_{j=0}^{m-1} \chi^{(j)} J^\gamma.
$$

(18)

The new field $\Phi$ is defined on the dual torus with radius $R' = \frac{R}{m}$ and satisfies the twist eaters boundary conditions, due to the presence of the 't Hooft magnetic flux $c$:

$$
\Phi (x_1 + R', x_2) = \Omega_1^{+} \cdot \Phi (x_1, x_2) \cdot \Omega_1, \quad \Phi (x_1, x_2 + R') = \Omega_2^{+} \cdot \Phi (x_1, x_2) \cdot \Omega_2,
$$

(19)

with

$$
\Omega_1 = P^c, \quad \Omega_2 = Q,
$$

(20)

where $c$ is an integer satisfying $dq - cm = 1$. Furthermore the field components $\chi^{(j)}$, defined as:

$$
\chi^{(j)} = \exp\left(2\pi i \frac{\vec{j} \cdot \vec{x}}{R}\right) \sum_{n} \hat{\Phi} \left(\frac{\vec{x}}{R} + n\right) \exp\left(2\pi i m \frac{\vec{n} \cdot \vec{x}}{R}\right),
$$

(21)

satisfy the following twisted boundary conditions:

$$
\chi^{(j)} (x_1 + R', x_2) = e^{2\pi ij_2/m} \chi^{(j)} (x_1, x_2),
$$

(22)

$$
\chi^{(j)} (x_1, x_2 + R') = e^{2\pi ij_1/m} \chi^{(j)} (x_1, x_2),
$$

that is

$$
\left(\frac{j_1}{m}, \frac{j_2}{m}\right), \quad j_1 = 0, ..., m - 1, \quad j_2 = 0, ..., m - 1.
$$

(23)

Within a gauge theory context, generally modes with fractional momentum $\vec{j}$ carry the electric flux:

$$
e_i \equiv q_i \epsilon_i j_i \text{mod} m.
$$

(24)

Let us notice that $\chi^{(0,0)}$ is the trace degree of freedom which can be identified with the $U(1)$ component of the matrix valued field or the charged $X$ field within the $m$-reduced theory of the QHF at paired states fillings introduced in Section 2, while the twisted fields $\chi^{(j)}$ with $\vec{j} \neq (0,0)$ should be identified with the neutral ones $\chi^{(j)}$. The commutative torus is smaller by a factor $m \times m$ than the noncommutative one; in fact upon this rescaling also the "density of degrees of freedom" is kept constant as now we are dealing with $m \times m$ matrices instead of scalars. In conclusion, when the parameter $\theta$ is rational we recover the whole structure of the noncommutative torus and recognize the twisted boundary conditions which characterize the neutral fields $\chi^{(j)}$ of the $m$-reduced theory as the consequence of the Morita mapping of the starting NCFT ($U(1)_{\theta=\frac{\pi}{2}+\frac{\pi}{m}}$ in our case) on the ordinary commutative space.

The key role in the proof of equivalence is played by the map on the field $Q(z)$ of Eq. (1) which, after the Morita action, is defined on the noncommutative space $z \rightarrow z^{1/m} \equiv U_{0,1}$. The algebra defined by the commutation rules in Eq. (13) is realized in terms of the $m^2 - 1$ general operators:

$$
U_{j_1, j_2} = e^{2\pi i \vec{J} \cdot \vec{x}} z^{j_1} e^{j_2}, \quad j_1, j_2 = 0, ..., m - 1, \quad (j_1, j_2) \neq (0,0),
$$

(25)
where $\tilde{\sigma} = iz\partial_z$. By using the decomposition given in Eq. (18), where $\Phi \equiv Q(z)$, we identify the fields $X(z)$ and $\phi^j(z)$ of the CFT defined on the ordinary space. At this point the correspondence becomes explicit due to the equivalence between the correlators evaluated on the noncommutative space (by using the operators $a_{n+m} \rightarrow \sqrt{m}a_{n+m}$, $q \rightarrow \sqrt{\theta}q$) and the corresponding ones obtained for the commutative case. The correlators of the neutral component $\phi^j(z)$ become identical to those given by a set of $m-1$ bosonic fields with twisted boundary conditions (see Refs. [1][43] for details).

From the CFTs point of view, two theories are equivalent if they have the same symmetry (i.e. Virasoro or Kac-Moody algebras, for instance) and the same correlators on the plane and on the torus (i.e. the conformal blocks). Therefore the NCFT built of the field $Q$ on $T^2_\theta$, whose symmetry, correlators and conformal blocks were calculated in accordance in [1][40], is equivalent to the CFT composed of the charged field $X$ and the $m-1$ twisted fields $\phi^j(z)$ with the same symmetry, correlators and conformal blocks (see Refs. [40][43]).

In the following Section we further clarify such a correspondence by making an explicit reference to the QHF physics at paired states fillings. In particular we will recognize the GMT as a realization of the noncommutative torus Lie algebra defined in Eq. (13).

### 4 Generalized magnetic translations and noncommutative torus Lie algebra

In this Section we construct GMT on a torus within the $m$-reduced theory for a QHF at paired states fillings introduced in Section 2 and show that they are a realization of the noncommutative torus Lie algebra. That completes the proof of our main claim (see Section 3): the $m$-reduction map is the counterpart of Morita map in the ordinary space and the $m$-reduced theory keeps track of noncommutativity in its structure. Then we realize that such a structure is shared also by a pure Yang-Mills theory. An emphasis on the particular $m = 2$ case is given, which corresponds to a quantum Hall bilayer with different boundary conditions [14][45][47]. The action of GMT on the characters of our TM is given together with the corresponding physical interpretation. In general, the different possible boundary conditions are associated with different possible impurities (topological defects). In turn, a boundary state can be defined in correspondence to each class of defects [44] and identified with a condensate of 't Hooft electric and magnetic fluxes. In this context the GMT can be identified with the Wilson loop operators and act on the boundary states, that is electric-magnetic condensates [52]. In the language of Kondo effect [53] they behave as boundary condition changing operators. We will report in detail on such issues in a forthcoming publication [52].

It is today well known that QHF are quantum liquids with novel and extremely rich internal structures, the so called topological orders [54]. Topological order is recognized as a kind of order which cannot be obtained by means of a spontaneous symmetry breakdown mechanism and, as such, it is responsible of all the unusual properties of QHF. In this context the role of translational invariance is crucial in determining the relevant topological effects, such as the degeneracy of the ground state wave function on a manifold with non trivial topology, the derivation of the Hall conductance $\sigma_H$ as a topological invariant and the relation between fractional charge and statistics of anyon excitations [55]. All the above phenomenology strongly relies on the invariance properties of the wave functions under a finite subgroup of the magnetic translation group for a $N_e$ electrons system. Indeed their explicit expression as the Verlinde operators [56], which generate the modular transformations in the $c = 1$ CFT, are taken as a realization of topological order of the system under study [54]. In particular the magnetic translations built so far [55] act on the characters associated to the highest weight states which represent the charged statistical particles, the anyons or the electrons.
In our CFT representation of the QHF at paired states fillings \([1, 43]\) we shall see that the noncommutative torus Lie algebra defined in Eq. (16) has a natural and beautiful realization in terms of GMT. We refer to them as generalized ones because the usual magnetic translations act on the charged content of the one point functions \([55]\). Instead, in our TM model for the QHF (see Section 2, Appendix and Refs. \([42, 43]\)) the primary fields (and then the corresponding characters within the torus topology) appear as composite field operators which factorize in a charged as well as a neutral part. Further they are also coupled by the discrete symmetry group \(Z_m\). Then, in order to show that the characters of the theory are closed under magnetic translations, we need to generalize them in such a way that they will appear as operators with two factors, acting on the charged and on the neutral sector respectively. The presence of the transverse magnetic field \(B\) reduces the torus to a noncommutative one and the flux quantization induces rational values of the noncommutativity parameter \(\theta\).

Let us consider a general magnetic translation of step \((n = n_1 + in_1, \overline{n} = n_1 - in_1)\) on a sample with coordinates \((x_1, x_2)\) and denote with \(T^{n, \overline{n}}\) the corresponding generators. Let us denote with \(|\ \rangle\ \rangle\) and \(|\ \rangle\ \rangle\) the layer index because we are dealing with a bilayer system. Within our TM for a QHF at paired states fillings \([1, 43]\) it is possible to show that such generators can be factorized into a group which acts only on the charged sector as well as a group acting only on the neutral sector \([57]\). In this context the classical magnetic translations group considered in the literature corresponds to the TM charged sector. In order to study the action of a GMT on the torus and clarify its interpretation in terms of noncommutative torus Lie algebra, Eq. (18), let us evaluate how the argument of the Theta functions in which the conformal blocks are expressed gets modified. For a bilayer Hall system a translation carried out on the layer \(|\ \rangle\ \rangle\) is expressed gets modified. For a bilayer Hall system a translation carried out on the layer \(|\ \rangle\ \rangle\) produces a shift in the layer Theta argument \(w_i, w_i \rightarrow w_i + \delta_i\), which can be conveniently expressed in terms of the charged and neutral ones \(w_{(n)} = \frac{w + i\overline{w}}{2}\), and in this way we obtain the action on the conformal blocks of the TM. Indeed, from the periodicity of the Theta functions it is easy to show that the steps of the charged and neutral translation can be parametrized by \(\delta_c = \frac{2\pi l + \frac{1}{2} + 2\pi i}{4(p+1)}\) and \(\delta_n = \pm l \pm \frac{1}{2}\) respectively, being \(l = 0, 1; s = 0, ..., p\); and \(i = 0, 1\). The layer exchange is realized by the transformation \(w_n \rightarrow -w_n\) but the TM is built in such a way to correspond to the exactly balanced system in which \(w_n = 0\) (modulo periodicity) so that this operation can be obtained only by exchanging the sign in \(\delta_n\) (independently for \(l\) and \(i\)). Because of the factorization of the effective CFT at paired states fillings into two sub-theories with \(c = 3/2\) and \(c = 1/2\), corresponding to the MR and Ising model respectively (see Section 2, Appendix and Refs. \([1, 43]\)), we find that also GMT exhibit the same factorization \([57]\). As a consequence, conformal blocks of MR and Ising sectors are stable under the transport of electrons and of the neutral Ising fermion.

In order to derive the transformation properties of the TM characters we resort to the following identity:

\[
J_c^{\alpha, \beta} K_a (w|\tau) = K_a \left( w + \frac{\alpha \tau + \beta}{2(p+1)} \right) |\tau\rangle \rangle = e^{-\frac{2\pi i n_{\alpha}}{4(p+1)}} e^{-\frac{2\pi i n_{\beta}}{4(p+1)}} e^{-\frac{\pi i 2 \tau}{4(p+1)^2} w} K_{a+\alpha} (w|\tau) \tag{26}
\]

(the explicit expressions for \(K_a (w|\tau)\) are given in the Appendix). Let us notice that the factor in the above equation must be a phase, so that in order to extend such a formula to any complex \(\tau\) and \(w\) we need to multiply the characters by a non-analytic factor, as introduced in Ref. \([55]\). Upon fixing \(a = 2(p+1) + q\) and \(b = 2(p+1)l' + q'\), where \(q = 2s + i\), and defining \(a \times b = a_1 b_2 - a_2 b_1\) we obtain:

\[
[J_c^a, J_c^b] = -2isin \left( 2\pi \frac{p}{2} I_1 + \frac{1}{2}I_1' + \frac{1}{4}I_1 + i \frac{q}{2} \right) \tag{27}
\]

For \(p\) even, the first term does not give any contribution while different interpretations can be given for the three kinds of excitations. For \(i, i' = 0\) the characters are a \(2(p+1)\)-dimensional representation of the Abelian magnetic translations generated by the transport of electrons \((s, s' = 0)\) and anyons \((s, s' \neq 0)\) respectively. The \(l\) (\(l'\) resp.) index is the \(Z_2\) charge of the parity rule and the coupling between \(l\) and \(i\), for \(i, i' \neq 0\), implies non-Abelian statistics for the quasi-holes.
The $J^{\alpha,\beta}_c$ translation, which acts as $w \rightarrow w + \frac{\beta}{2(p+1)}$, can be viewed as the result of an electric potential $V$ while the $J^\alpha_c$ translation, whose action is realized by $w \rightarrow w + \frac{\alpha}{2(p+1)}$, corresponds physically to increase the flux through the sample, $N_\Phi = \Phi/\Phi_0$, by $\frac{\alpha}{2(p+1)}$ units of the elementary flux quantum $\Phi_0 = hc/e$ and that gives rise to the Laughlin spectral flow.

The charged characters $K_\alpha(w_c|\tau)$ appearing in Eqs. (40)-(58) are closed under this group but the whole TM conformal blocks are not stable. Nevertheless the translations (27) don’t correspond to physical ones due to the lack of the neutral sector contribution; so we need to add the neutral sector in order to keep track of the layer degree of freedom, i.e. the pseudospin. The magnetic translation of the whole fundamental physical particles, i.e. the electrons with layer index, corresponds to a translation of the charged as well as the neutral component of the TM characters and is given by the shifts $\delta_c, \delta_n = (1, 1)$ in the corresponding Theta function arguments. In this way we find that the characters of the 331 model, Eqs. (51)-(53), (which belong to the untwisted sector of our TM, see Appendix and Ref. [42]) are invariants only under combined translations: $\chi_{(a,s)}^{331} \rightarrow \chi_{(a,s)}^{331}$ for $a = 1, \ldots, 4$. The exchange of the last two characters is the signal of a layer exchange. Indeed such a feature is irrelevant within the 331 model because for a balanced system ($w_n = 0$) the characters $\chi_{(3,s)}^{331}, \chi_{(4,s)}^{331}$ are equivalent. Nevertheless it gives rise to interesting consequences within the Ho model [59], which takes into account the symmetrization of the state with respect to the layer index. In this case there is only one state $\chi_{(3,s)}^{331} + \chi_{(4,s)}^{331}$, which forces the model to lie in the balanced configuration. Within our TM model the balancing implies that the bilayer system develops a defect, which gives rise to a contact point between the two layers so allowing the transport of an electron from a layer to another. That is the signature of the presence of a twist field in the ground state. The factorization of TM into MR model and Ising one implies that the two independent Ising models of the neutral sector are not equivalent because the charged and the neutral sector of the MR model are not completely independent but need to satisfy the constraint $\alpha - p + l = 0 \pmod{2}$ which is the $m$-ality condition (parity rule). It is explicitly realized by constraining the eigenvalues of the fermion parity operator, which is obtained by defining the generalized GSO projector as $P = \frac{1}{2}(1 - e^{i\pi\alpha p_F})$. Here the operator $\gamma_F = (-1)^F$ is defined in such a way to anticommute with the fermion field, $\gamma_F \psi \gamma_F = -\psi$, and to satisfy $(\gamma_F)^2 = 1$ ($F$ being the fermion number). It has eigenvalues $\pm 1$ when acts on states with an even or an odd number of fermion creation operators. In terms of GMT the above constraint implies that charged and neutral translation cannot be independent.

Let us now observe that the emerging noncommutative torus introduces $m^2$ states but only $m$ are the diagonal ones. Due to the traceless condition the neutral massless states are only $m - 1$, so that there is only one state in the bilayer case $m = 2$. The natural interpretation of this scenario is that $m(m-1)$ states are massive and the massless states are selected by the projection operator $P$ above defined. That is also needed in order to fulfill the consistency requirements imposed by modular invariance. The degeneracy of the TM ground state implies the spontaneous breaking of the fermion parity because the selected ground state has not a well defined parity fermion number and non-Abelian statistics. In order to gain a physical insight on the generalized GSO projection just described we point out that there is a correspondence between TM vacua and particular configurations of the bilayer Hall system [44][45]. In this way the twisted ground state corresponds to an exactly balanced system, i.e. the layers are completely equivalent and there is no current flow by a layer to another one. Finally, in order to formally extend the definition of magnetic translations in the neutral sector to the transport of electrons (or anyons) in any vacuum we have to introduce a couple $(a, F)$ of parameters which are defined only modulo 2. Here $a$ is equal to 1 or 0 for twisted and untwisted vacua respectively. Such a definition is consistent with the fusion rules of Ising model. In this way the transport of a particle around another one produces an extra phase $(-1)^{\alpha_1 F_2 - \alpha_2 F_1}$, which completes the Aharonov–Bohm phase arising from the charged sector.

From the above discussion it follows that the classification of excitations in terms of $U(1)$ quantum numbers $l, s, i$ is misleading because the TM characters $\chi_{(l,s)}^{331}$ are invariant under a greater algebra. Therefore the GMT for TM can be decomposed into a subalgebra generating the translations of the $l$-electrons, which is a symmetry of the model, and a spectrum generating algebra which is written in
terms of the $s$ index, the spin $i$ and the $Z_2$ charge $\pm$ quantum numbers. The non-Abelian nature of the MR model is the consequence of the lack of information on the layer localization of the excitations. Indeed the pseudospin fusion rules can be deduced from its interpretation in terms of layer index because, for any $i = 1$, two configurations exist with an excitation on the layer up $\uparrow$ or down $\downarrow$. The MR model depends only on the absolute value of the $\Sigma_\ell$ pseudospin. Therefore the state with $l = 0$ corresponds to the states $|\uparrow\rangle \pm |\downarrow\rangle$ while the states with $l = 1$ are identified with the states $|\uparrow\rangle$ or $|\downarrow\rangle$. The degeneration can be resolved only within the TM by means of the addition of the $T^{sing}$ quantum numbers, which give us information about the localization of the excitation in the terms of layers (i. e. the sign of the pseudospin).

The GMT operators above introduced realize an algebra isomorphic to the noncommutative torus Lie algebra given in Eqs. (13) and (16). Such operators generate the residual symmetry of the $m$-reduced CFT which is Morita equivalent to the NCFT with rational noncommutativity parameter $\theta = \frac{\pi}{2} + \frac{1}{m}$.

Let us now close this Section by making some interesting observations. The GMT operators can be put in correspondence with the $SU(m)$ valued twist matrices associated to the cycles of the two-torus, $J_{a,0}$ and $J_{0,b}$, which satisfy the relation:

$$J_{a,0}J_{0,b} = J_{b,0}e^{-2\pi ib/m}. \quad (28)$$

and may be chosen to be constant. In general, translations of the gauge fields along the two cycles of the torus are equivalent to gauge transformations [51]:

$$A_i (x + R_j) = U_j (x) A_i (x) U^+_j (x) + U_j (x) i\partial_t U^+_j (x), \quad (29)$$

where $R_j = R_j$ are the periodicities of the torus and we put $U_1 = J_{a,0}$ and $U_2 = J_{0,b}$. For adjoint matter the following constraint holds on:

$$U_i (x) U_j (x + R_i) U^+_i (x + R_j) U^+_j (x) = e^{2\pi i c_{ij}/m}, \quad (30)$$

where the twist $c_{ij}$ is an integer and defines non-Abelian t’Hooft magnetic fluxes through the non trivial cycles of the torus. If we consider a Yang-Mills theory on $T^2 \times R^n$ and choose the time-like direction in $R^n$, then in the $A_0 = 0$ gauge the following twisted boundary conditions

$$U_1 = P^c, \quad U_2 = Q, \quad (31)$$

are imposed, which act as:

$$A_i (x + R_j) = U_j (x) A_i (x) U^+_j (x). \quad (32)$$

The choice (31) corresponds to the so called twist eaters matrices which generate the Weyl-t’Hooft algebra [51]. Let us notice that different choices of twisted boundary conditions with the same magnetic flux are related by large gauge transformations, but in the $A_0 = 0$ gauge no paths exist, which connect different twisted boundary conditions at different times. There exist other large gauge transformations which leave the boundary conditions invariant, and their eigenvalues determine the t’Hooft electric fluxes.

In this way the structure of GMTs within the $m$-reduced CFT describing the physics of a QHF at paired states fillings coincides with that of a general $U(m)$ Yang-Mills gauge theory on a twisted two-torus. Indeed noncommutativity makes the $U(1)$ and $SU(m)$ sectors of the decomposition:

$$U(m) = U(1) \times \frac{SU(m)}{Z_m} \quad (33)$$

not decoupled because the $U(1)$ photon interacts with the $SU(m)$ gluons [60]. That reflects the nontrivial coupling between the different topological sectors in the $m$-reduced theory, or the interplay between charged and neutral degrees of freedom in the QHF picture. In this sense the $m$-reduced theory retains
some degree of noncommutativity while the $m$-reduction map is the image of the Morita map in the ordinary space.

The Wilson loop operators of Yang-Mills theory could be identified as GMT in QHF context while the t’Hooft electric and magnetic flux condensates could be identified with boundary states. In this way some features of noncommutative gauge theories are recovered in condensed matter systems and that has been recognized to be relevant, for instance, in the context of topological quantum computation [13]. Furthermore the interaction in string and $D$-brane theory is introduced by means of gauge theories; one could think that interactions between topological defects and QHF are described by the same gauge theories. In order to clarify this analogy, let us refer for simplicity to the $m = 2$ case, corresponding to a quantum Hall bilayer. Recently such a system has been shown to be well described within a boundary CFT formalism and the possible defects (impurities) which are compatible with conformal invariance as well as their stability (boundary entropy [61]) have been studied in detail [44][45][47]. To each class of defects corresponds a different boundary state. For the bilayer system two possible boundary conditions are identified, as shown in Fig. 1, the periodic (PBC) and twisted (TBC) boundary conditions respectively, which give rise to different topological sectors (vacua) on the torus. The defects break the GMT symmetry so that different backgrounds can be connected by special GMT. The breaking of the residual symmetry of the CFT can be recognized in the condensation of electric-magnetic fluxes represented by the defects. The quarks in this picture appear as domain walls (or defects) interpolating between different vacua and may carry fractional quantum numbers which differ from those of the electron. The coupling of two QHF layers at the boundary is a topological one and it is well described [45] by a boundary magnetic term of the kind [62]:

$$S_{\text{mag}} = \frac{i \beta}{4\pi} \int_0^T dt \left( X \partial_t Y - Y \partial_t X \right)_{s=0},$$

(34)

where $X$ and $Y$ are two massless scalar fields in $1+1$ dimensions and $\beta = 2\pi B$ is related to the magnetic field orthogonal to the $X - Y$ plane. If we resort to a string analogy, such a term allows for exchange of momentum of the open string moving in an external magnetic field. Indeed within the worldsheet field theory for open strings attached to $D$-branes, it can be written as [63]:

$$S_{\partial \Sigma} = -\frac{i}{2} \oint_{\partial \Sigma} dt B_{ij} \dot{y}^i(t) \dot{y}^j(t),$$

(35)

where $t$ is the coordinate of the boundary $\partial \Sigma$ of the string worldsheet residing on the $D$-brane worldvolume, $B_{ij}$ is a magnetic field on the $D$-branes and $y^i(t)$ are the open string endpoint coordinates. Notice that Eq. (35) formally coincides with that of the Landau problem in the strong field limit and then canonical quantization of the coordinates $y^i(t)$ will induce a noncommutative geometry on the $D$-brane worldvolume. In this way noncommutative field theories emerge as effective descriptions of the string dynamics in a background $B$-field. The deep relation between the boundary CFT given by Eq. (34) and NCFT, as derived by its string counterpart [35], will be fully exploited within the system of two QHF layers analyzed here in a forthcoming publication [52].

5 Conclusions and outlooks

This work relies strongly on the power of Morita equivalence, which allows one to establish a one-to-one correspondence between representations of two noncommutative algebras, and on a peculiar feature of such a correspondence: if the starting field theory is defined on a noncommutative torus and the noncommutativity parameter $\theta$ is rational, the dual theory is a commutative field theory of matrix valued fields with twisted boundary conditions and magnetic flux $c_{\theta}$ [34][35]. In this paper, by extending a recent proposal [2], we have shown by means of the Morita equivalence that a NCFT with $\theta = \frac{p}{2} + \frac{1}{m}$ is mapped to a CFT on an ordinary space. We identified such a CFT with the $m$-reduced CFT developed in [1][13] for a QHF at paired states fillings, whose neutral fields satisfy twisted boundary conditions. The
m-reduction technique appears as the image in the ordinary space of the Morita duality and, as such, retains some noncommutative features: the non trivial interplay between charged and neutral degrees of freedom as well as the structure of GMT, which realize the noncommutative torus Lie algebra in the QHF context. In this way a picture emerges on the interplay between noncommutativity and QHF physics, which is very different from the ones developed in the literature in the last years [23, 27, 28, 29, 30]. The identification of objects of pure Yang-Mills theory such as Wilson loop operators and t’Hooft electric and magnetic flux condensates with GMT and boundary states respectively in the context of a QHF bilayer system will be carried out in detail in a forthcoming publication together with a careful analysis of the relation between the corresponding boundary CFT and NCFT provided by string theory [32].

Recently, the twisted CFT approach provided by m-reduction and developed for a QHF at paired states fillings [14, 43] has been successfully applied to Josephson junction ladders and arrays of non trivial geometry in order to investigate the existence of topological order and magnetic flux fractionalization in view of the implementation of a possible solid state qubit protected from decoherence [64, 65, 66] as well as to the study of the phase diagrams of the fully frustrated XY model (FFXY) on a square lattice [67] and of a general spin-1/2 antiferromagnetic two leg ladder with Mobius boundary conditions in the presence of various perturbations [68]. So, it could be interesting to generalize our approach in order to further elucidate the topological properties of Josephson systems with non trivial geometry as well as the deep nature of the quantum critical points and of the deconfinement of excitations with fractional quantum numbers in antiferromagnetic spin ladders and the consequences imposed upon them by the request of space-time noncommutativity. That could open new perspectives on the general relation between noncommutativity and the physics of strongly correlated electron systems.

On the other hand, the m-reduction technique has been recently employed in order to shed new light on the analogy between string theory and QHF physics [69]. By generalizing the quantum Hall soliton introduced in Ref. [70] the tachyon condensation in a non-BPS system of D-branes [71] has been found similar to the tunneling between two layers of QHF at paired states fillings. In that context the role of the Z_2 (Z_m) symmetry has been pointed out: indeed such a discrete symmetry couples the charged and the neutral vertices hinting to a more general description in terms of Matrix Theory with U(2) (U(m)) symmetry [26]. Following the line introduced in this work, Morita duality could help us to shed new light on the connections between a system of interacting D-branes and the physics of quantum Hall fluids through the unifying framework of Matrix String Theory and CFT [72].

**Appendix: m-reduction procedure for a QHF at paired states fillings on the torus**

In this Appendix we give the whole primary fields content on the torus topology of the theory introduced in Section 2. We focus on the particular case m = 2, which describes the physics of a quantum Hall bilayer.

On the torus, the primary fields are described in terms of the conformal blocks of the MR and the Ising model [13]. The MR characters \( \chi^{MR}_{(\lambda,s)} \) with \( \lambda = 0, \ldots, 2 \) and \( s = 0, \ldots, p \), are explicitly given by:

\[
\chi^{MR}_{(0,s)}(w|\tau) = \chi_{\lambda,s}(\tau)K_{2s}(w|\tau) + \chi_{\lambda,s}(\tau)K_{2s(p+s)+2}(w|\tau) \quad (36)
\]

\[
\chi^{MR}_{(1,s)}(w|\tau) = \chi_{\lambda,s}(\tau)(K_{2s+1}(w|\tau) + K_{2s+1(p+s)+3}(w|\tau)) \quad (37)
\]

\[
\chi^{MR}_{(2,s)}(w|\tau) = \chi_{\lambda,s}(\tau)K_{2s}(w|\tau) + \chi_{\lambda,s}(\tau)K_{2s(p+s)+2}(w|\tau). \quad (38)
\]

They represent the field content of the Z_2 invariant \( c = 3/2 \) CFT [11] with a charged component \( \langle K_{\alpha}(w|\tau) \rangle = \frac{\epsilon^{(p+1)}(\pi w)^2}{\eta(\tau)} \Theta \left[ \frac{\alpha}{4(p+1)} \right] (2(p+1) w) \langle 4(p+1) \rangle \rangle \), where we introduce a non-analytic factor as in Ref. [68] and a neutral component (\( \chi_{\lambda,s} \), the conformal blocks of the Ising Model).
The characters of the twisted sector are given by:

\[ \chi^+_{(0, s)}(w|\tau) = \bar{\chi}^+ \left( \chi^{MR}_{(0, s)}(w|\tau) + \chi^{MR}_{(2, s)}(w|\tau) \right) \]  

(39)

\[ \chi^+_{(1, s)}(w|\tau) = \left( \bar{\chi}_0 + \bar{\chi}_+ \right) \chi^{MR}_{(1, s)}(w|\tau) \]  

(40)

which do not depend on the parity of \( p \);

\[ \chi^-_{(0, s)}(w|\tau) = \bar{\chi}^- \left( \chi^{MR}_{(0, s)}(w|\tau) - \chi^{MR}_{(2, s)}(w|\tau) \right) \]  

(41)

\[ \chi^-_{(1, s)}(w|\tau) = \left( \bar{\chi}_0 - \bar{\chi}_+ \right) \chi^{MR}_{(1, s)}(w|\tau) \]  

(42)

for \( p \) even, and

\[ \chi^-_{(0, s)}(w|\tau) = \bar{\chi}^- \left( \chi_{0} - \chi_{\pm} \right) \left( K_{2s} (w|\tau) + K_{2(p+s)+2} (w|\tau) \right) \]  

(43)

\[ \chi^-_{(1, s)}(w|\tau) = \bar{\chi}^- \left( \chi_{0} - \chi_{\pm} \right) \left( K_{2s+1} (w|\tau) - K_{2(p+s)+3} (w|\tau) \right) \]  

(44)

for \( p \) odd. Notice that the last two characters are not present in the TM partition function and that only the symmetric combinations \( \chi^+_{(1, s)} \) can be factorized in terms of the \( c = \frac{1}{2} \) and \( c = \frac{1}{2} \) theory. That is a consequence of the parity selection rule (parity), which gives a gluing condition for the charged and neutral excitations. Furthermore the characters of the untwisted sector are given by:

\[ \bar{\chi}^+_{(0, s)}(w|\tau) = \bar{\chi}^0_0 \chi_{(0, s)}(w|\tau) + \bar{\chi}^+_0 \chi^{MR}_{(2, s)}(w|\tau) = \chi^{331}_{(0, s)}(w|\tau) \]  

(45)

\[ \bar{\chi}^+_{(1, s)}(w|\tau) = \bar{\chi}^0_0 \chi_{(2, s)}(w|\tau) + \bar{\chi}^+_0 \chi^{MR}_{(0, s)}(w|\tau) = \chi^{331}_{(1, s)}(w|\tau) \]  

(46)

\[ \bar{\chi}^-_{(0, s)}(w|\tau) = \bar{\chi}^- \chi_{(0, s)}(w|\tau) - \bar{\chi}^- \chi^{MR}_{(2, s)}(w|\tau) \]  

(47)

\[ \bar{\chi}^-_{(1, s)}(w|\tau) = \bar{\chi}^- \chi^{MR}_{(0, s)}(w|\tau) - \bar{\chi}^- \chi^{MR}_{(0, s)}(w|\tau) \]  

(48)

\[ \bar{\chi}^-_{(s)}(w|\tau) = \bar{\chi}^- \chi^{MR}_{(1, s)}(w|\tau) = \chi^{331}_{(3, s)}(w|\tau) + \chi^{331}_{(4, s)}(w|\tau) \]  

(49)

where \( \chi^{331}_{(l, s)}(w|\tau) \) are the characters of 331 model [48]:

\[ \chi^{331}_{(0, s)}(w|\tau) = K^0 (0|\tau) K_{2s} (w|\tau) + K^2 (0|\tau) K_{2(p+s)+2} (w|\tau) \]  

(50)

\[ \chi^{331}_{(2, s)}(w|\tau) = K^2 (0|\tau) K_{2s} (w|\tau) + K^0 (0|\tau) K_{2(p+s)+2} (w|\tau) \]  

(51)

\[ \chi^{331}_{(3, s)}(w|\tau) = K^1 (0|\tau) K_{2s+1} (w|\tau) + K^3 (0|\tau) K_{2(p+s)+3} (w|\tau) \]  

(52)

\[ \chi^{331}_{(4, s)}(w|\tau) = K^3 (0|\tau) K_{2s+1} (w|\tau) + K^1 (0|\tau) K_{2(p+s)+3} (w|\tau) \]  

(53)

\( K^i (0|\tau) \) being the characters of the \( c = 1 \) Dirac theory. Notice that \( K^3 (-w|\tau) = K^1 (w|\tau) \), so that only for a balanced system the two characters can be identified while \( K^{0(2)} (-w|\tau) = K^{0(2)} (w|\tau) \). Let us also point out that, as evidenced from Eq. [50], one character of the TM is identified with two characters of the 331 model. In this way the degeneracy of the ground state on the torus is reduced from \( 4 (p + 1) \) to \( 3 (p + 1) \) when switching from 331 to TM, a clear signature of a transition from an Abelian statistics to a non-Abelian one. Such a transition is due to the presence of two inequivalent Majorana fermions together with the breaking of the symmetry which exchanges them. In conclusion, while in the Halperin model the fundamental particles are Dirac fermions with a well defined layer index, in the TM they are given in terms of symmetric \( \psi \) and antisymmetric \( \bar{\psi} \) fields, that is as a superposition of states belonging to different layers. As such, they behave in a different way under twisted boundary conditions.

We point out that the partition function on the torus can be written as:

\[ Z(\tau) = \frac{1}{2} \left( \sum_{s=0}^{P} 2 \left| \bar{\chi}(s)(0|\tau) \right|^2 + Z^+_{\text{untwist}}(0|\tau) + Z^-_{\text{untwist}}(\tau) + Z^+_{\text{twist}}(\tau) + Z^-_{\text{twist}}(\tau) \right) \]  

(54)
for $p$ even, where:

$$Z^+_{\text{untwist}}(\tau) = \sum_{s=0}^{p} \left( |\chi_{(0,s)}(0|\tau)|^2 + |\chi_{(1,s)}(0|\tau)|^2 \right)$$

(55)

$$Z^-_{\text{untwist}}(\tau) = \sum_{s=0}^{p} \left( |\chi_{(0,s)}(0|\tau)|^2 + |\chi_{(1,s)}(0|\tau)|^2 \right)$$

(56)

$$Z^+_{\text{twist}}(\tau) = \sum_{s=0}^{p} \left( |\chi_{(0,s)}(0|\tau)|^2 + |\chi_{(1,s)}(0|\tau)|^2 \right)$$

(57)

$$Z^-_{\text{twist}}(\tau) = \sum_{s=0}^{p} \left( |\chi_{(0,s)}(0|\tau)|^2 + |\chi_{(1,s)}(0|\tau)|^2 \right)$$

(58)

while for $p$ odd we get simply:

$$Z(\tau) = \frac{1}{2} \left( \sum_{s=0}^{p} 2 |\bar{\chi}_{(s)}(0|\tau)|^2 + Z^+_{\text{untwist}}(0|\tau) + Z^-_{\text{untwist}}(\tau) + Z^+_{\text{twist}}(\tau) \right).$$

(59)

As recalled above, the two Majorana fermions are not completely equivalent and that reflects in the factorization of the partition function in the MR and Ising (non-invariant) one:

$$Z(\tau) = Z_{MR}(\tau)Z_{\text{Ising}}(\tau)$$

(60)

where $Z_{MR}$ is the modular invariant partition function of the MR $c = 3/2$ theory:

$$Z_{MR}(\tau) = \sum_{s=0}^{p} \left( |\chi^{MR}_{(0,s)}(0|\tau)|^2 + |\chi^{MR}_{(1,s)}(0|\tau)|^2 + |\chi^{MR}_{(2,s)}(0|\tau)|^2 \right)$$

(61)

and $Z_{\text{Ising}}$ is the partition function of the Ising $c = 1/2$ theory:

$$Z_{\text{Ising}}(\tau) = |\bar{\chi}_0(\tau)|^2 + |\bar{\chi}_1(\tau)|^2 + |\bar{\chi}_{16}(\tau)|^2.$$ 

(62)

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