CORRIGENDUM TO “SYNDE蒂CALLY PROXIMAL PAIRS” [J. MATH. ANAL. APPL. 379 (2011) 656–663]

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ABSTRACT. We give a counterexample to Theorem 9 in “Syndetically proximal pairs” [J. Math. Anal. Appl. 379 (2011) 656–663]. We also provide sufficient conditions for the conclusion of Theorem 9 to hold.

1. INTRODUCTION

The reader not familiar with the theory of entropy, in particular this theory in the context of one-dimensional dynamics, is referred to monographs [2, 9, 1].

One of the celebrated results in the one-dimensional dynamics is that if a continuous interval map \( f : [0,1] \to [0,1] \) has positive topological entropy then there is a horseshoe in its iteration [7, 8]. An important consequence of this result is that it has a factor map to the full shift. More specifically, we have the following useful result, see [10, Theorem 8] for this version. Note that the one-sided full shift dynamical system on the alphabet \( \{0, 1, 2, \ldots, m - 1\} \) is denoted by \((\Sigma_m, \sigma)\).

**Theorem 1.1.** Let \( f : [0,1] \to [0,1] \) be a continuous map with positive topological entropy. Then there exist \( n \in \mathbb{N} \), an \( f^n \)-invariant closed set \( X \subset [0,1] \), and a continuous surjection \( \phi : X \to \Sigma_2 \) such that

1. \( \phi \circ f^n(x) = \sigma \circ \phi(x) \) for every \( x \in X \).
2. \( |\phi^{-1}(y)| \leq 2 \) for every \( y \in \Sigma_2 \).
3. The set \( \{y \in \Sigma_2 : |\phi^{-1}(y)| > 1\} \) is at most countable.

It is claimed [10] that the map \( \phi \) in Theorem 1.1 can be chosen to be a homeomorphism. The following result is Theorem 9 in [10].

**Theorem 1.2.** Let \( f : [0,1] \to [0,1] \) be a continuous map with positive topological entropy and let \( m \geq 2 \). Then there exist \( n \in \mathbb{N} \), an \( f^{2n} \)-invariant closed set \( X \subset [0,1] \), and a homeomorphism \( \phi : X \to \Sigma_m \) such that \( \phi \circ f^{2n}(x) = \sigma \circ \phi(x) \) for every \( x \in X \).

After checking the proof of Theorem 1.2 in [10] carefully, we found some gaps in the proof and later realized that we are able to construct a counterexample to the statement of Theorem 1.2. Strictly speaking, we have the following result.

**Theorem 1.3.** There exists a surjective continuous map \( f : [0,1] \to [0,1] \) with positive topological entropy such that for every \( n \in \mathbb{N} \) and every \( f^n \)-invariant closed set \( X \subset [0,1] \) the map \( f^n|_X \) is not topologically conjugate to the full shift dynamical system \((\Sigma_2, \sigma)\).

**Remark 1.4.** Even though Theorem 1.2 turns out to be false, several results using it remain valid, for examples Theorems 10 and 11 in [10], Theorem 6.1 in [4], Theorem 6.7 in [5]. It seems sufficient to use Theorem 1.1 instead of Theorem 1.2 and some standard techniques such as in the proof of Theorem 5.17 in [9].

We also give sufficient conditions for the conclusion of Theorem 1.2 to hold. These conditions cover quite a large class of interval maps.

**Theorem 1.5.** Let \( f : [0,1] \to [0,1] \) be a continuous map. If \( f \) is transitive, then there exist \( n \in \mathbb{N} \) and an \( f^n \)-invariant closed set \( X \subset [0,1] \) such that \( f^n|_X \) is topologically conjugate to the shift dynamical system \((\Sigma_2, \sigma)\).
Recall that a point $x \in [0, 1]$ is equicontinuous if for every $\varepsilon > 0$ there exists an open neighborhood $U$ of $x$ such that $\text{diam}(f^n(U)) < \varepsilon$ for all $n \geq 0$.

**Corollary 1.6.** Let $f : [0, 1] \to [0, 1]$ be a continuous map. If the set of equicontinuity points of $f$ fails to be dense in $[0, 1]$, then there exist $n \in \mathbb{N}$ and an $f^n$-invariant closed set $X \subset [0, 1]$ such that $f^n|_X$ is topologically conjugate to the shift dynamical system $(\Sigma_2, \sigma)$.

**Proof.** For $k \in \mathbb{N}$, let $S_k$ be the collection of all $x \in [0, 1]$ with the following property: for every open neighborhood $U$ of $x$, there is $n \in \mathbb{N}$ such that $\text{diam}(f^n(U)) \geq 1/k$. Then it may be seen that each $S_k$ is closed (and also $f$-invariant; but this we do not need). Note that $\bigcup_{k=1}^{\infty} S_k$ is the complement of the set of equicontinuity points of $f$ (in other words, $\bigcup_{k=1}^{\infty} S_k$ is the set of sensitivity points of $f$). By the hypothesis of the Corollary, $\bigcup_{k=1}^{\infty} S_k$ contains a nondegenerate interval. Since $S_k$’s are closed, we conclude by Baire category theorem that $\text{int}(S_k) \neq \emptyset$ for some $k \in \mathbb{N}$. Now by Proposition 2.40 of [9], there exists a cycle $L_1, \ldots, L_p$ of closed intervals such that $f$ restricted to $L_1 \cup \cdots \cup L_p$ is transitive. Then $f^n$ restricted to $L_1$ must be transitive. It is enough to apply Theorem 1.5 to the restriction of $f^n$ to $L_1$. \hfill $\square$

**Corollary 1.7.** Let $f : [0, 1] \to [0, 1]$ be a continuous map with positive entropy. If $f$ has a dense set of periodic points, then there exist $n \in \mathbb{N}$ and an $f^n$-invariant closed set $X \subset [0, 1]$ such that $f^n|_X$ is topologically conjugate to the shift dynamical system $(\Sigma_2, \sigma)$.

**Proof.** Since $h(f) > 0$, $f^2$ cannot be identity. Hence by Proposition 3.8 of [9], either $f$ or $f^2$ must be transitive on a nondegenerate closed interval $J \subset [0, 1]$. Apply Theorem 1.5 to $f|_J$ or $f^2|_J$. \hfill $\square$

2. **Proofs of the main results**

**Proof of Theorem 1.3.** In fact we will construct a map $f : [0, 2] \to [0, 2]$ which after normalization to a map $\tilde{f} : [0, 1] \to [0, 1]$ is an example as required.

Start by considering a map $g : [0, 1] \to [0, 1]$ defined by $g(x) = \min\{1, 3/2 - 3|3x - 3/2\}$. In other words, $g$ is a tent map with slope $\pm 3$ and flattened top. Note that there is a Cantor set $C \subset [0, 1]$ such that $(C,g)$ is conjugated (by a homeomorphism $\eta : C \to \Sigma_2$) with $(\Sigma_2, \sigma)$ and if $x \in [0, 1] \setminus C$ then there is $n \geq 0$ such that $g^n(x) = 0$.

Let $\{C_i\}_{i \in \mathbb{A}}$ be the family of all closed subsets of $C$ such that $g^k(C_i) = C_i$ for some $k > 0$ and $(\eta(C_i), \sigma_C)$ is a non-trivial mixing sofic shift in the higher power block representation of the full shift $(\Sigma_2, \sigma)$ which is conjugated with $(\Sigma_2, \sigma_C)$. Since every sofic shift has a labeled graph presentation (see [6, §3.1]), the set $\mathbb{A}$ is countable. Note that it may happen that for some $i \neq j$ we have $C_i \subset C_j$ or even $C_i = C_j$ but $k_i \neq k_j$.

Note that each $(C_i, g^k)$ is non-trivial mixing. There is a countable sequence $\{x_i\}_{i \in \mathbb{A}}$ of points (not necessarily pairwise distinct) such that $x_i \in C_i$ is a transitive point of $(C_i, g^k)$ for all $n \geq 1$. Since $C_i$ is perfect, for every nonempty open set $U \subset C_i$ there are points $a, b, c \in U$, $a < b < c$ and $c - a < \varepsilon$. Then $V = (a, c) \cap C_i$ is a nonempty open subset of $C_i$ and for every $y \in V$ we have $(y - \varepsilon, y) \cap C_i \neq \emptyset$ and $(y, y + \varepsilon) \cap C_i \neq \emptyset$. This immediately shows that the set of points $y \in C_i$ such that for every $\varepsilon > 0$ we have $(y - \varepsilon, y) \cap C_i \neq \emptyset$ and $(y, y + \varepsilon) \cap C_i \neq \emptyset$ is residual. Therefore we can require that $x_i$ is such that for any $\varepsilon > 0$ and $n > 0$ there are $s, t > 0$ such that $g^{sn}(x_i) \in (x_i - \varepsilon, x_i)$ and $g^{tn}(x_i) \in (x_i, x_i + \varepsilon)$.

We will perform a construction similar to the standard Denjoy extension of irrational rotation on the unit circle (see e.g. [3, Proposition 4.4.4]). First observe that by the definition $g^{j}(x_i) \neq 0$ for any $j \geq 0$ and $i \in \mathbb{A}$ and hence the set

$$D = \bigcup_{i \in \mathbb{A}} \bigcup_{k \geq 0} g^{-k}(\{x_i^j : j = 0, 1, \ldots\})$$

is countable, because if $z \neq 1$ then $g^{-1}(z)$ has exactly two elements. Furthermore $g(D) = D$, $g^{-1}(D) = D$ and $D \subset (0, 1/3) \cup (2/3, 1)$. Enumerate elements of $D$, say $D = \{z_j\}_{j \in \mathbb{N}}$. Extend $[0, 1]$ to $[0, 2]$ by inserting in place of each $z_j$ an interval $I_j$ of length $2^{-j}$. This
way we have a monotone surjective map $\pi: [0, 2] \to [0, 1]$ which is one to one for each $x \in [0, 2] \setminus \bigcup I_j$ and $\pi(I_j) = z_j$ for $j \in \mathbb{N}$.

We will define a map $f: [0, 2] \to [0, 2]$ in the following way. For $x \not\in \bigcup I_j$ we put $f(x) = \pi \left( \frac{1}{2} g(\pi(x)) \right)$. If $x \in I_j$ then $\pi(x) = z_j \in D$ and hence $g(z_j) = 3$ or $g(z_j) = -3$. There exists $s \in \mathbb{N}$ such that $g(z_j) = z_s$. Then we define $f|_{I_j}: I_j \to I_s$ as a homeomorphic map of constant slope which is increasing when $g(z_j) = 3$ and decreasing in the other case. Observe that the map $f$ defined that way is continuous and $\pi \circ f = f \circ \pi$.

Since $f$ is an extension of $g$, the topological entropy of $f$ is also positive. Suppose that there exist $m \in \mathbb{N}$ and an $f^m$-invariant closed set $X \subset [0, 2]$ such that $(X, f^m)$ is conjugated to $(\Sigma_2, \sigma)$. Then $(\pi(X), g^m)$ is mixing and infinite. This implies that $\overline{\pi(X)} \cap C \neq \emptyset$ because if $\pi(X) \setminus C \neq \emptyset$ then there exists an open set $U \subset \pi(X)$ and $k > 0$ such that $g^m(U) = \{0\}$ which is impossible. Hence $(\eta(\pi(X)), \sigma^m)$ is a factor of $(X, f^m)$, and therefore is a sofic shift as a factor of a shift of finite type. Therefore there exists an $i \in A$ such that $\pi(X) = C_i$ and $m = k_i$. There exists $r \in \mathbb{N}$ such that $x_i = x_{i-r}$. Let $I_r = [a, b]$. Observe that $f^j(a, b) \cap (a, b) = \emptyset$ for every $j > 0$. In particular $a$ and $b$ are asymptotic, that is $\lim_{j \to \infty} f^j(a) = \lim_{j \to \infty} f^j(b) = 0$. Hence $X$ is mixing and $\pi^{-1}(z_i) = I_i, I_i \subset X \cap \{a, b\}$. Without loss of generality, assume that $a \in X$. Note that the orbit of $x_i$ under $g^k$ intersects both intervals $(x_i - \epsilon, x_i)$ and $(x_i, x_i + \epsilon)$ for every $\epsilon > 0$. Let $\{s_j\}$ be an increasing sequence of positive integers such that $g^{s_j}(x_i) > x_i$ and $\lim_{j \to \infty} g^{s_j}(x_i) = x_i$. Without loss of generality we may also assume that $\lim_{j \to \infty} f^{s_j}(a)$ exists. Since $\pi$ is monotone and $\pi \circ f = g \circ \pi$, we get $f^{s_j}(a) > b$ and so $\lim_{j \to \infty} f^{s_j}(a) = b$. This shows that $a, b \in X$. But for any asymptotic pair $p, q$ in the one sided full shift, there exists $n$ such that $\sigma^n(p) = \sigma^n(q)$. This implies $f^n(a) = f^n(b)$ which contradicts the fact that all intervals $I_j$ are nondegenerate.

Proof of Theorem 1.5. First we recall a well known fact that if $f$ is transitive but not mixing, then there is $0 < \epsilon < 1$ such that $[0, \epsilon]$ is $f^2$-invariant and the restriction of $f^3$ to $[0, \epsilon]$ is mixing (see e.g. [9, Proposition 2.16]). Replacing $f$ by $f^2$ if necessary, we assume that $f$ is mixing. As $f$ has positive topological entropy (see e.g. [9, Proposition 4.70]), there exist $r \in \mathbb{N}$ and disjoint closed intervals $J_0, J_1$ such that $g := f^r$ satisfies $J_0 \cup J_1 \subset g(J_0) \cap g(J_1)$. Without loss of generality assume $J_0$ is to the left of $J_1$, and let $(a, b)$ be the open interval between $J_0$ and $J_1$ (i.e., $a = \max J_0$ and $b = \min J_1$). For later use, keep in mind that $0 < a < b < 1$. By the horseshoe property of $J_0$ and $J_1$, for $k \in \mathbb{N}$ and $w = w_1 \cdots w_k \in \{0, 1\}^k$, we can find closed intervals $J_{w_1 \cdots w_k}$ such that $J_{w_1 \cdots w_k} \subset J_{w_1 \cdots w_{k-1}}$ and $g(J_{w_1 \cdots w_k}) = J_{w_1 \cdots w_{k-1}}$. For $\alpha = w_1 w_2 \cdots \in \Sigma_2$, let $J_\alpha = \bigcap_{w_1 \cdots w_k \in \Sigma_2} J_{w_1 \cdots w_k}$. Then $J_\alpha$ is either a singleton or a nondegenerate closed interval. If possible, let $J_\alpha$ be a nondegenerate closed interval. Since $f$ is mixing and $0 < a < b < 1$, there is $n_0 \in \mathbb{N}$ such that $(a, b) \subset J_n$ for every $n \geq n_0$. On the other hand, $f^n(J_\alpha) = g^n(J_\alpha) \subset J_0 \cup J_1$ for every $n \in \mathbb{N}$. This is a contradiction, and therefore $J_\alpha$ must be a singleton, say $J_\alpha = \{x_\alpha\}$. Letting $X = \{x_\alpha: \alpha \in \Sigma_2\}$, we may check that $X$ is a $g$-invariant closed set and $(X, g)$ is topologically conjugate to $(\Sigma_2, \sigma)$ via the conjugacy $x_\alpha \mapsto \alpha$.

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