THE CLAIRAUT’S THEOREM ON ROTATIONAL SURFACES IN
PSEUDO EUCLIDEAN 4-SPACE WITH INDEX 2

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Abstract. In this paper, Clairaut’s theorem is expressed on the surfaces of rotation in semi Euclidean 4-space. Moreover, the general equations of time-like geodesic curves are characterized according to the results of Clairaut’s theorem on the hyperbolic surfaces of rotation and the elliptic surface of rotation, respectively.

1. Introduction

The geodesics for rotational surfaces have been studied for a long time and many examples of rotational surfaces have been discovered. To understand the rest of geodesics; we need Clairaut’s Theorem, which is very helpful to understand the geodesics on surfaces of rotation. This gives a well-known characterization of geodesics on surfaces of rotation.

Many studies of surfaces of rotation have received much attention from our researchers. Among them, one can cite our work [1], we described the rotational surfaces using curves and matrices which are the subgroups of rotating a selected axis in Galilean 4-space. We examined the tube surfaces generated by the curve in Galilean 3-space and gave certain results of describing the geodesics on the surfaces [2, 3]. We gave the surfaces of rotational generated by a magnetic curve. Also, we gave the conditions being geodesic on these rotational surfaces in null cone 3-space, with the help of Clairaut’s theorem [3]. In our study [4] we expressed the hyperbolic and the elliptic rotational surfaces using a curve and matrices in 4-dimensional semi-Euclidean space. Goemans constructed a new type of surfaces in Euclidean and Lorentz-Minkowski 4-space and proved the classification theorems of flat double rotational surfaces [5]. Hoffmann and Zhou [6], discussed some issues of displaying 2D surfaces in 4D space, including the behaviour of surface normals under projection, the silhouette points due to the projection, and methods for object orientation and projection center specification.

2. Preliminaries

Let $E^4_2$ denote the 4—dimensional pseudo-Euclidean space with signature $(2, 4)$, that is, the real vector space $\mathbb{R}^4$ endowed with the metric $\langle \cdot, \cdot \rangle_{E^4_2}$ which is defined by

\begin{equation}
\langle \cdot, \cdot \rangle_{E^4_2} = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2
\end{equation}

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or

\[
(2.2) \quad g = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

where \((x_1, x_2, x_3, x_4)\) is a standard rectangular coordinate system in \(E^4_2\).

Recall that an arbitrary vector \(v \in E^4_2 \setminus \{0\}\) can have one of three characters: it can be space-like if \(g(v, v) > 0\) or \(v = 0\), time-like if \(g(v, v) < 0\) and null if \(g(v, v) = 0\) and \(v \neq 0\).

The norm of a vector \(v\) is given by \(\|v\| = \sqrt{g(v, v)}\) and two vectors \(v\) and \(w\) are said to be orthogonal if \(g(v, w) = 0\). A space-like or time-like curve \(x(s)\) has unit speed, if \(g(x', x') = \pm 1\).

Let \((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)\) be any three vectors in \(E^4_2\). The pseudo-Euclidean cross product is given as

\[
x \wedge y \wedge z = \begin{pmatrix}
-i_1 & -i_2 & i_3 & i_4 \\
-x_1 & x_2 & x_3 & x_4 \\
y_1 & y_2 & y_3 & y_4 \\
z_1 & z_2 & z_3 & z_4
\end{pmatrix},
\]

where \(i_1 = (1, 0, 0, 0), i_2 = (0, 1, 0, 0), i_3 = (0, 0, 1, 0), i_4 = (0, 0, 0, 1)\).

The pseudo-Riemannian sphere \(S^3_2(m, r)\) centred at \(m \in E^4_2\) with radius \(r > 0\) of \(E^4_2\) is defined by

\[
S^3_2(m, r) = \{ x \in E^4_2 : \langle x - m, x - m \rangle = r^2 \}.
\]

The pseudo-hyperbolic space \(H^3_1(m, r)\) centred at \(m \in E^4_2\) with radius \(r > 0\) of \(E^4_2\) is defined by

\[
H^3_1(m, r) = \{ x \in E^4_2 : \langle x - m, x - m \rangle = -r^2 \}.
\]

The pseudo-Riemannian sphere \(S^2_3(m, r)\) is diffeomorphic to \(\mathbb{R}^2 \times S\) and the pseudo-hyperbolic space \(H^1_1(m, r)\) is diffeomorphic to \(S^1 \times \mathbb{R}^2\). The hyperbolic space \(H^3_1(m, r)\) is given by

\[
H^3_1(m, r) = \{ x \in E^4_2 : \langle x - m, x - m \rangle = -r^2, x_1 > 0 \}.
\]

### Definition 1. \[10\]

A one-parameter group of diffeomorphisms of a manifold \(M\) is a regular map \(\psi : M \times \mathbb{R} \to M\), such that \(\psi_t(x) = \psi(x, t)\), where

1. \(\psi_t : M \to M\) is a diffeomorphism
2. \(\psi_0 = id\)
3. \(\psi_{s+t} = \psi_s \circ \psi_t\).

This group is attached with a vector field \(W\) given by \(\frac{d}{dt}\psi_t(x) = W(x)\), and the group of diffeomorphism is said to be the flow of \(W\).

### Definition 2. \[10\]

If a one-parameter group of isometries is generated by a vector field \(W\), then this vector field is called as a Killing vector field.

### Definition 3. \[10\]

Let \(W\) be a vector field on a smooth manifold \(M\) and \(\psi_t\) be the local flow generated by \(W\). For each \(t \in \mathbb{R}\), the map \(\psi_t\) is a diffeomorphism of \(M\) and
given a function \( f \) on \( M \), one considers the Pull-back \( \psi_! f \) and one defines the Lie derivative of the function \( f \) as to \( W \) by

\[
L_w f = \lim_{t \to 0} \left( \frac{\psi_! f - f}{t} \right) = \frac{d\psi_! f}{dt} \bigg|_{t=0}.
\]

Let \( g_{\xi} \) be any pseudo-Riemannian metric, then the derivative is given as

\[
L_w g_{\xi} = g_{\xi,\cdot} W^z + g_{\xi} W^z_\xi + g_{\xi} W^z_\xi.
\]

In Cartesian coordinates in Euclidean spaces where \( g_{\xi,\cdot} = 0 \), and the Lie derivative is given by

\[
L_w g_{\xi} = g_{\xi} W^z_\xi + g_{\xi} W^z_\xi.
\]

In \([6, 10, 11, 15]\), the vector \( W \) generates a Killing field if and if only

\[
L_w g = 0.
\]

**Theorem 1.** Let \( \xi \) be a geodesic on a surface of revolution \( \Upsilon \) and let \( \rho \) be the distance function of a point of \( \Upsilon \) from the axis of rotation, and let \( \theta \) be the angle between \( \xi \) and the meridians of \( \Upsilon \). Then, \( \rho \sin \theta \) is constant along \( \xi \). Conversely, if \( \rho \sin \theta \) is constant along some curve \( \xi \) in the surface, and if no part of \( \xi \) is part of some parallel of \( \Upsilon \), then \( \xi \) is a geodesic. \([13]\).

**Theorem 2.** Let the pseudo Euclidean group be a subgroup of the diffeomorphisms group in \( E^4 \) and let \( W \) be vector field which generate the isometries. Then, the killing vector field associated with the metric \( g \) is given as

\[
W(\xi, \varphi, \theta, \eta) = a(\eta \partial \xi + \xi \partial \eta) + b(\varphi \partial \eta + \eta \partial \varphi) + c(\varphi \partial \xi + \xi \partial \varphi) + d(\varphi \partial \theta + \theta \partial \varphi) + e(\varphi \partial \eta - \eta \partial \varphi) + f(\xi \partial \varphi - \varphi \partial \xi),
\]

where \( a, b, c, d, e, f \in \mathbb{R}^+_0 \), \([5]\).

**Theorem 3.** Let \( W(\xi, \varphi, \theta, \eta) \) be the killing vector field and let \( \gamma = (f_1, f_2, f_3, f_4) \) be a curve in \( E^4 \), then the surfaces of rotation are given as follows

1. For the rotations \( \Omega_1 = \varphi \partial \xi + \xi \partial \varphi \) and \( \Omega_4 = \varphi \partial \eta + \eta \partial \varphi \), the hyperbolic surface of rotation is given as

\[
S_{14}(x, \alpha, s) = \begin{pmatrix}
 f_1 \cosh x + f_3 \sinh x, & f_2 \cosh \alpha + f_4 \sinh \alpha, \\
 f_1 \sinh x + f_3 \cosh x, & f_2 \sinh \alpha + f_4 \cosh \alpha
\end{pmatrix}
\]

and for the planar curve \( \gamma(s) = (f_1(s), 0, 0, f_4(s)) \) the Gaussian curvature \( K \) and the mean curvature vector \( H \) of the rotational surface \( S_{14}(x(t), \alpha(t), s) = (f_1 \cosh x, f_4 \sinh \alpha, f_1 \sinh x, f_4 \cosh \alpha) \) are given as

\[
K = \frac{\left( f_1 f_4 - f_1 f_3 \right)^2 (x \check{\alpha})^2}{f_4^2 \alpha^2 - f_3^2 x^2} + \frac{\left( f_1 f_4 - f_1 f_3 \right)^2 \left( f_1^2 f_4^2 - f_1^2 f_3^2 \right) (f_1 f_4^2 - f_1 f_3^2)}{-f_4^2 + f_3^2},
\]

\[
H = \left\{ \frac{f_1 f_4 (x \check{\alpha} + \check{x} \alpha)}{2 \sqrt{f_4^2 \alpha^2 - f_3^2 x^2}} + \frac{f_1 f_4}{{2} \sqrt{-f_1^2 + f_4^2}} e_3 + \frac{(f_4 f_4^2 - f_1^2 f_1^2)}{{2} \sqrt{-f_1^2 + f_4^2}} e_4 \right\}
\]

where

\[
e_3 = \frac{\left( f_4 \sinh x, f_1 x \cosh \alpha, f_4 \sinh x, f_1 x \sinh \alpha \right)}{\sqrt{f_4^2 \alpha^2 - f_3^2 x^2}},
\]

\[
e_4 = \frac{\left( f_1 \cosh x, f_4 \sinh \alpha, f_4 \sinh x, f_1 \cosh \alpha \right)}{\sqrt{-f_1^2 + f_4^2}}.
\]
(2) For the rotations $\Omega_2 = \eta \partial \xi + \xi \partial \eta$ and $\Omega_3 = \vartheta \partial \varphi + \varphi \partial \vartheta$, the hyperbolic surface of rotation is given as

$$S_{23}(y, z, s) = \left( f_1 \cosh y + f_4 \sinh y, f_2 \cosh z + f_3 \sinh z, f_4 \sinh y + f_1 \cosh y \right),$$

and for the planar curve $\gamma(s) = (f_1(s), f_2(s), 0, 0)$ the Gaussian curvature $K$ and the mean curvature vector $H$ of the rotational surface $S_{23}(y(t), z(t), s) = (f_1 \cosh y, f_2 \cosh z, f_2 \sinh y, f_1 \cosh y)$ are given as

$$K = -\left(\frac{(f_1 f_2' + f_4 f_3')^2 (y')^2}{(f_1 f_2'^2 + f_4 f_3'^2) (f_1' f_2 + f_4' f_3)}\right); \quad H = \begin{pmatrix} f_1 f_2 (y z + y z) \\ 2 \sqrt{f_2^2 + f_4^2} e_3 + f_1 f_2 y^2 f_2' f_3' - f_2 f_4' e_4 \end{pmatrix}.$$

where $e_3 = \frac{(f_2 \cosh y, f_4 \cosh x, f_2 \sinh y, f_4 \sinh x)}{\sqrt{f_2^2 + f_4^2}}$; $e_4 = \frac{(f_2' \cosh y, f_4' \cosh x, f_2' \sinh y, f_4' \sinh x)}{\sqrt{f_2'^2 + f_4'^2}}$.

(3) For the rotations $\Omega_3 = \xi \partial \varphi - \eta \partial \varphi$ and $\Omega_0 = \vartheta \partial \vartheta - \eta \partial \eta$, the elliptic surface of rotation is given as

$$S_{56}(\beta, \theta, s) = \left( f_1 \cos \beta + f_2 \sin \beta, -f_1 \sin \beta + f_2 \cos \beta, f_3 \sin \theta + f_4 \sin \theta, -f_3 \sin \theta + f_4 \cos \theta \right),$$

and for the planar curve $\gamma(s) = (0, f_2(s), 0, f_4(s))$ the Gaussian curvature $K$ and the mean curvature vector $H$ of the rotational surface $S_{56}(\beta(t), \theta(t), s) = (f_2 \sin \beta, f_2 \cos \beta, f_4 \sin \theta, f_4 \cos \theta)$ are given as

$$K = -\left(\frac{(f_1 f_2' - f_4 f_3')^2 (\beta')^2}{-f_2^2 \beta^2 + f_4^2 \beta^2} + \frac{(-f_2 f_3' + f_4 f_2') (f_1 f_2' - f_4 f_3')^2}{-f_2^2 \beta^2 + f_4^2 \beta^2}\right); \quad H = \frac{f_4 f_2 (\beta' + \theta')}{2 \sqrt{f_2^2 \beta^2 - f_4^2 \beta^2}} e_3 + \frac{f_4 f_2 \beta^2 - f_2 f_4 \theta^2 + f_4 f_2' e_4}{2 \sqrt{f_4^2 \beta^2 - f_2^2 \beta^2}} e_4,$$

where $e_3 = \frac{(-f_2 \sin \beta, f_4 \sin \beta, -f_2 \cos \theta, f_4 \cos \theta)}{\sqrt{-f_2^2 \beta^2 + f_4^2 \beta^2}}$; $e_4 = \frac{(f_2' \sin \beta, f_4' \sin \beta, f_2' \cos \theta, f_4' \cos \theta)}{\sqrt{-f_2^2 \beta^2 + f_4^2 \beta^2}}$.

$\infty < x, y, z, \alpha, \beta, \theta < \infty, s \in I$ and $f_i \in C^\infty$, [5].

3. Clairaut’s theorem on the surfaces of rotation in $E^4_2$

This section will use three different types of surfaces of rotation given the previous section, and will generalize Clairaut’s theorem to these surfaces in $E^4_2$.

3.1. Clairaut’s theorem on the hyperbolic surface of rotation $T^1$. In this section, one will use the hyperbolic surface of rotation parametrized as

$$T^1(x, \alpha, t) = \left( f_1 \cosh x + f_4 \sinh x, f_2 \cosh \alpha + f_4 \sinh \alpha, f_1 \sinh x + f_3 \cos x, f_2 \sinh \alpha + f_3 \cos \alpha \right).$$

Also, one can take the planar curve $\gamma$ for this surface of rotation to be the intersection of $T^1(x, \alpha, t)$ with $\vartheta = 0$ (or $\xi, \eta = 0$) for the coordinate system ($\xi, \eta, \vartheta, \varphi$). Therefore, one can write that the curve $\gamma$ lies on the $\xi \eta$-plane (or $\vartheta \varphi$-plane), and the curve can be written by

$$\gamma(t) = (f_1(t), 0, 0, f_4(t)); f_1, f_4 \in C^\infty,$$
then one has the parametrization
\[ \mathbf{Y}^1(x, \alpha, t) = (f_1 \cosh x, f_4 \sinh \alpha, f_1 \sinh x, f_4 \cosh \alpha); \]
\[ \mathbf{T}_x^1(x, \alpha, t) = (f_1 \sinh x, 0, f_1 \cosh x, 0); \]
\[ \mathbf{T}_t^1(x, \alpha, t) = (f_1' \cosh x, f_4' \sinh \alpha, f_1' \sinh x, f_4' \cosh \alpha). \]

Hence, from the first fundamental form of the surface \( \mathbf{Y}^1 \), one has
\[ \langle \mathbf{T}_x^1, \mathbf{T}_x^1 \rangle = f_1^2; \]
\[ \langle \mathbf{T}_x^1, \mathbf{T}_t^1 \rangle = -f_4^2; \]
\[ \langle \mathbf{T}_t^1, \mathbf{T}_t^1 \rangle = (f_4^2 - f_1^2); \]
\[ (3.1) \quad \langle \mathbf{T}_x^1, \mathbf{T}_x^1 \rangle, \langle \mathbf{T}_x^1, \mathbf{T}_t^1 \rangle, \langle \mathbf{T}_t^1, \mathbf{T}_t^1 \rangle = 0; \]
\[ I_{\mathbf{T}^1} = \begin{pmatrix} f_1^2 & 0 & 0 & 0 \\ 0 & -f_4^2 & 0 & 0 \\ 0 & 0 & f_4^2 - f_1^2 & 0 \end{pmatrix} \]
and one can write Lagrangian equation
\[ L = f_1^2 x^2 - f_4^2 \dot{\alpha}^2 + (f_4^2 - f_1^2) t^2. \]

So, the curve \( \gamma \) is time-like and one writes \( f_4^2 - f_1^2 = -1, f_2^2 > 0, -f_4^2 < 0. \)
Then, one gets
\[ L = f_1^2 x^2 - f_4^2 \dot{\alpha}^2 - t^2. \]

Hence, one can write following equations by using Clairaut’s theorem,
\[ \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial x} \right) = \frac{\partial L}{\partial x} \Rightarrow \frac{\partial}{\partial s} (2f_1^2 \dot{x}) = 0; \]
\[ \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial \alpha} \right) = \frac{\partial L}{\partial \alpha} \Rightarrow \frac{\partial}{\partial s} (-2f_4^2 \dot{\alpha}) = 0, \]
\[ \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial t} \right) = \frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial s} (-2t) = 2f_1' f_1 \dot{x}^2 - f_4' f_4 \dot{\alpha}^2. \]

Assume that \( \gamma(t) \) is a geodesic on the surface \( \mathbf{Y}^1 \). Hence, the curve \( \gamma(t) \) can be written as
\[ \dot{\gamma} = \dot{x} \mathbf{Y}_x^1 + \dot{\alpha} \mathbf{Y}_\alpha^1 + i \mathbf{Y}_t^1, \]

one can note that \( \mathbf{Y}_x^1 = f_1 \mathbf{N}_x \) is a unit space-like vector pointing along \( x \)-axis of the meridians, and \( \mathbf{Y}_\alpha^1 = f_4 \mathbf{N}_\alpha \) is a unit time-like vector pointing along the \( \alpha \)-axis of the parallels. Also \( \mathbf{Y}_t^1 = N_t \) is a unit time-like vector pointing along \( t \)-axis of the parallels. Also, the plane spanned by \( N_t \) and \( N_\alpha \) are time-like and \( \{ \mathbf{Y}_x^1, \mathbf{Y}_\alpha^1, \mathbf{Y}_t^1 \} \) is an orthonormal basis. Also, from (3.1), one gets
\[ \dot{\gamma} = f_1 \dot{x} \mathbf{N}_x + f_4 \dot{\alpha} \mathbf{N}_\alpha + i \mathbf{N}_t. \]

Note that if the \( \gamma \) is time-like, since \( \mathbf{N}_x^1 \in Sp\{N_\alpha, N_t\} \) one gets
\[ \dot{\gamma} = f_1 N_x \dot{x} + \left( f_4 N_\alpha \dot{\alpha} + i N_t \right) = N_x \cos \varphi_1 + N_x^1 \sin \varphi_1; \]
\[ = \cos \varphi_1 N_x + \cosh \theta_1 \sin \varphi_1 N_\alpha + \sinh \theta_1 \sin \varphi_1 N_t, \]
where \( \varphi_1 \) and \( \theta_1 \) are the angles between the meridians of the surface and the time-like geodesic \( \gamma \).

Also, from (3.2) and (3.3), one can write
\[ \dot{f}_1 \dot{x} = \cos \varphi_1; \]
\[ f_4 \dot{\alpha} = \cosh \theta_1 \sin \varphi_1; \]
\[ \dot{t} = \sinh \theta_1 \sin \varphi_1. \]
Similarly, for the curve \( \gamma(t) = (0, f_2(t), f_3(t), 0); f_2, f_3 \in C^\infty \). Then, the first fundamental form is written by
\[
I_{S^1} = \begin{pmatrix} -f_3^2 & 0 & 0 \\ 0 & f_2^2 & 0 \\ 0 & 0 & f_3^2 - f_2^2 \end{pmatrix}
\]
and similar calculations are obtained.

Hence, from the Lagrangian equation, one has
\[
2\dot{t} = 2f_1'f_1\dot{x}^2 - f_4'f_4\dot{\alpha}^2.
\]

and
\[
\frac{\partial}{\partial s}(2f_1^2\dot{x}) = 0 \Rightarrow x = \frac{c_1}{2f_1^2} + c_3; \quad \frac{\partial}{\partial s}(-2f_4^2\dot{\alpha}) = 0 \Rightarrow \alpha = -\frac{c_2}{2f_4^2} + c_4, c_i \in \mathbb{R},
\]
which gives that \( 2f_1^2\dot{x} \) and \(-2f_4^2\dot{\alpha} \) are constant along the geodesic curve. It follows that the geodesics are given by
\[
\frac{\partial}{\partial s}(2f_1^2\dot{x}) = 0; \quad \frac{\partial}{\partial s}(-2f_4^2\dot{\alpha}) = 0; \quad \frac{\partial}{\partial s}(-2\dot{t}) = 2f_1'f_1\dot{x}^2 - f_4'f_4\dot{\alpha}^2.
\]

Now, by using the equations (3.4) and from the previous equations, one writes
\[
\begin{align*}
(3.5) & \quad f_1\dot{x} = \cos \varphi_1 \Rightarrow 2f_1^2\dot{x} = 2f_1\cos \varphi_1 = \text{cons}. \\
(3.6) & \quad f_4\dot{\alpha} = \cosh \theta_1 \sin \varphi_1 \Rightarrow -2f_4^2\dot{\alpha} = -2f_4\cosh \theta_1 \sin \varphi_1 = \text{cons}. \\
& \quad -2\dot{t} = -2\sinh \theta_1 \sin \varphi_1 \neq \text{cons}.
\end{align*}
\]
and for the equation \( \frac{\partial}{\partial s} \left( \frac{\partial x}{\partial \varphi_1} \right) = \frac{\partial L}{\partial x} \) which means that
\[
(3.7) \quad x = \int \frac{\cos \varphi_1}{f_1} ds
\]
is constant along the geodesic, conversely, if \( \gamma \) is a curve with \( 2f_1 \cos \varphi_1 = \text{constant} \), the second equation is satisfied, differentiating \( L \) and substituting into the second Euler Lagrangian equation yields the first Lagrangian equation. Furthermore, for the equation \( \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial \varphi_1} \right) = \frac{\partial L}{\partial \alpha} \)
\[
(3.8) \quad \alpha = \int \frac{1}{f_4} \cosh \theta_1 \sin \varphi_1 ds
\]
is constant along the curve \( \gamma \). Therefore, one gives the following theorem as a result of Clairaut’s theorem on the hyperbolic surface of rotation \( \mathbb{Y}^1 \subset E_2^4 \).

**Theorem 4.** Let \( \gamma(t) = (f_1(t), 0, f_2(t), f_3(t)) \) or \( \gamma(t) = (0, f_2(t), f_3(t), 0) \), \( f_1 \in C^\infty \) be a time-like geodesic curve on the hyperbolic surface of rotation \( \mathbb{Y}^1 \) in the \( E_2^4 \), let \( f_1 \) and \( f_2 \) be the distance functions from the axis of rotation to a point on the surface. Therefore, \( 2f_1 \cos \varphi_1 \) and \(-2f_4 \cosh \theta_1 \sin \varphi_1 \) are constant along the curve \( \gamma \) where \( \varphi_1 \) and \( \theta_1 \) are the angles between the meridians of the surface and the time-like geodesic \( \gamma \). Conversely, if \( 2f_1 \cos \varphi_1 \) and \(-2f_4 \cosh \theta_1 \sin \varphi_1 \) are constant along \( \gamma \), if no part of some parallels of the surface of rotation, then \( \gamma \) is time-like geodesic.

If one wants to obtain the general equation of geodesics, one should consider the Euler-Lagrange equations
\[
(3.9) \quad \dot{x} = \frac{dx}{ds} = \frac{1}{f_1} \cos \varphi_1; \dot{\alpha} = \frac{d\alpha}{ds} = \frac{1}{f_4} \cosh \theta_1 \sin \varphi_1
\]
then, adding (3.9) to Lagrangian equation $L$, one has

$$L = f_1^2 \left( \frac{dx}{ds} \right)^2 - f_4^2 \left( \frac{d\alpha}{ds} \right)^2 - \left( \frac{dt}{dx} \right)^2$$

thus

$$\frac{dt}{dx} = f_1 \sqrt{1 - \cosh^2 \theta_1 \tan^2 \varphi_1 - L \sec^2 \varphi_1}$$

or

$$L = f_1^2 \left( \frac{dx}{ds} \right)^2 - f_4^2 \left( \frac{d\alpha}{ds} \right)^2 - \left( \frac{dt}{dx} \right)^2$$

or

$$\frac{dt}{d\alpha} = f_2 \sqrt{\cot^2 \varphi_1 \tan^2 \theta_1 - L \sech^2 \varphi_1 \cosec^2 \varphi_1}.$$

**Theorem 5.** The general equation of geodesics on the hyperbolic surface of rotation $\mathcal{Y}^1(x, \alpha, t) \subset E^4_2$, and for the parameters $\dot{x} = \frac{1}{f_2} \cos \varphi_1$ and $\dot{\alpha} = \frac{1}{f_2} \cosh \theta_1 \sin \varphi_1$, are given by

$$\frac{dt}{dx} = f_1 \sqrt{1 - \cosh^2 \theta_1 \tan^2 \varphi_1 - L \sec^2 \varphi_1}$$

or

$$\frac{dt}{d\alpha} = f_2 \sqrt{\cot^2 \varphi_1 \tan^2 \theta_1 - L \sech^2 \varphi_1 \cosec^2 \varphi_1}.$$

### 3.2. Clairaut’s theorem on the hyperbolic surface of rotation $\mathcal{Y}^2(y, z, t)$

In this section, one will use the hyperbolic surface of rotation parametrized as

$$\mathcal{Y}^2(y, z, t) = \left( f_1 \cosh y + f_4 \sinh y, f_2 \cosh z + f_3 \sinh z, f_2 \sinh z + f_3 \cosh z, f_1 \sinh y + f_4 \cosh y \right).$$

Also, one can take the planar curve $\gamma$ for this surface of rotation to be the intersection of $\mathcal{Y}^2(y, z, t)$ with $\vartheta, \eta = 0$ (or $\xi, \eta = 0$) for the coordinate system $(\xi, \eta, \vartheta, \eta)$, one can write that the curve $\gamma$ lies on the $\eta\xi$-plane (or $\eta\vartheta$-plane), for $\gamma(t) = (f_1(t), f_2(t), 0, 0)$ or $\gamma(t) = (0, 0, f_3(t), f_4(t))$; $f_i \in C^\infty$, then one gets

$$\mathcal{Y}^2(y, z, t) = (f_1 \cosh y, f_2 \cosh z, f_2 \sinh z, f_1 \sinh y)$$

or

$$\mathcal{Y}^2(y, z, t) = (f_4 \sinh y, f_3 \sinh z, f_3 \cosh z, f_4 \cosh y).$$

Here one will use the surface of rotation generated by the curve $\gamma(t) = (f_1(t), f_2(t), 0, 0)$. Furthermore,

$$\mathcal{Y}^2_y = (f_1 \sinh y, 0, 0, f_1 \cosh y); \mathcal{Y}^2_z = (0, f_2 \sinh z, f_2 \cosh z, 0);$$

$$\mathcal{Y}^2_t = (f_1' \cosh y, f_2' \cosh z, f_2' \sinh x, f_1' \sinh y),$$

by resulting in the first fundamental form:

$$\langle \mathcal{Y}^2_y, \mathcal{Y}^2_y \rangle = f_1^2; \langle \mathcal{Y}^2_z, \mathcal{Y}^2_z \rangle = f_2^2; \langle \mathcal{Y}^2_t, \mathcal{Y}^2_t \rangle = -f_2^2 - f_1^2;$$

or

$$(\mathcal{Y}^2_y, \mathcal{Y}^2_y), (\mathcal{Y}^2_z, \mathcal{Y}^2_z), (\mathcal{Y}^2_t, \mathcal{Y}^2_t) = 0;$$

$$I_{\mathcal{Y}^2} = \begin{pmatrix} f_1^2 & 0 & 0 \\ 0 & f_2^2 & 0 \\ 0 & 0 & -f_2^2 - f_1^2 \end{pmatrix}$$

and one can write Lagrangian equation as follows

$$L = f_1^2 y^2 + f_2^2 z^2 + (-f_2^2 - f_1^2) t^2.$$
Here, one is interested the metric in $E^4_3$. So, one takes $\gamma$ to be time-like and one writes $-f_2^2 - f_1^2 = -1, f_1, f_2 > 0$. Then,

$$L = f_1^2 y^2 + f_2^2 z^2 - t^2.$$  

Hence, one can obtain the following equations using Clairaut’s theorem,

$$\frac{\partial}{\partial s} \left( \frac{\partial L}{\partial y} \right) = \frac{\partial L}{\partial y} \Rightarrow \frac{\partial}{\partial s} \left( 2 f_1^2 y \right) = 0; \quad \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial t} \right) = \frac{\partial}{\partial s} \left( 2 f_2^2 z \right) = 0,$$

$$\frac{\partial}{\partial s} \left( \frac{\partial L}{\partial t} \right) = \frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial s} \left( 2t \right) = 2 f_1^2 f_1 y^2 + 2 f_2^2 f_2 z^2.$$

Let $\gamma(t)$ be a curve geodesic on the surface $\Upsilon^2$. Hence, $\gamma(t)$ can be written as follows

$$\dot{\gamma} = \dot{y} \Upsilon_y^2 + \dot{z} \Upsilon_z^2 + i \Upsilon_t^2.$$

So, one can note that $\Upsilon_y^2 = N_t$ is a unit time-like vector pointing along $t$-axis of the meridians, and $\Upsilon_z^2 = f_2 N_z$ is a unit space-like vector pointing along the $z$-axis of the parallels. Also, $\Upsilon_y^2 = f_1 N_y$ is a unit space-like vector pointing along $y$-axis of the parallels. It also follows that the plane spanned by $N_y, N_z$ is space-like and an orthonormal basis. Also, from (3.10), one gets

$$\gamma = t N_t + f_1 y N_y + f_2 z N_z.$$

Note that if $\gamma$ is time-like curve, since $N_t \in Sp\{N_y, N_z\}$, one gets

$$\gamma = f_1 N_y y + f_2 N_z z + t N_t = N_t \cosh \phi_2 + N_t \sinh \phi_2;$$

(3.12)  

$$= \cosh \phi_2 N_t + \cos \phi_2 \sinh \phi_2 N_y + \sinh \phi_2 \sin \phi_2 N_z,$$

where $\phi_2$ and $\phi_2$ are the angles between the meridians of the surface and the time-like geodesic $\gamma$, and from (3.11) and (3.12), one can write

(3.13)  

$$f_1 y = \cos \phi_2 \sinh \phi_2; f_2 z = \sinh \phi_2 \sin \phi_2; i = \cosh \phi_2.$$

Hence, from the Lagrangian equation one has

$$\dot{t} = f_1^2 f_1 y^2 + f_2^2 f_2 z^2$$

and

$$\frac{\partial}{\partial s} \left( 2 f_1^2 y \right) = 0 \Rightarrow y = \frac{c_1^2}{2 f_1^2} t + c_3; \quad \frac{\partial}{\partial s} \left( 2 f_2^2 z \right) = 0 \Rightarrow z = \frac{c_2^2}{2 f_2^2} t + c_4,$$

which gives that $2 f_1^2 y$ and $2 f_2^2 z$ are constant along the geodesic. It follows that the geodesics can be written as

$$\frac{\partial}{\partial s} \left( 2 f_1^2 y \right) = 0; \quad \frac{\partial}{\partial s} \left( 2 f_2^2 z \right) = 0; \quad \frac{\partial}{\partial s} \left( -\dot{t} \right) = f_1^2 f_1 y^2 + f_2^2 f_2 z^2.$$

Now, from (3.13) one gets

(3.14)  

$$f_1 y = \cos \phi_2 \sinh \phi_2 \Rightarrow 2 f_1^2 y = 2 f_1 \cos \phi_2 \sinh \phi_2 = cons.$$  

(3.15)  

$$f_2 z = \sinh \phi_2 \sin \phi_2 \Rightarrow 2 f_2^2 z = 2 f_2 \sinh \phi_2 \sin \phi_2 = cons.$$  

$$2 t = 2 \cosh \phi_2 \neq cons.$$
for the equation \( \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial y} \right) = \frac{\partial L}{\partial y} \),

(3.16)

\[
y = \int \frac{\cos \theta_2 \sinh \varphi_2}{f_1} ds
\]

is a constant, conversely, for the condition \( 2f_1 \cos \theta_2 \sinh \varphi_2 = \text{constant} \), the second equation is satisfied, differentiating \( L \) and substituting into the second Euler Lagrangian equation yields the first Lagrangian equation. Furthermore, for \( \frac{\partial}{\partial s} \left( \frac{\partial L}{\partial z} \right) = \frac{\partial L}{\partial z} \),

(3.17)

\[
z = \int \frac{\sinh \varphi_2 \sin \theta_2}{f_2} ds
\]

is constant along the curve \( \gamma \). Hence, Clairaut’s theorem is expressed on the hyperbolic surface of rotation given in \( E_4^2 \).

**Theorem 6.** Let \( \gamma(t) = (f_1(t), f_2(t), 0, 0) \) (or \( \gamma(t) = (0, 0, f_3(t), f_4(t)) \), \( f_1 \in C^\infty \) be a time-like geodesic curve on the hyperbolic surface of rotation \( \Upsilon^2 \) in the \( E_4^2 \), and let \( f_1 \) and \( f_2 \) be the distance functions from the axis of rotation to a point on the surface. Then, \( 2f_1 \cos \theta_2 \sinh \varphi_2 \) and \( 2f_2 \sin \theta_2 \sinh \varphi_2 \) are constant along the curve \( \varphi_2 \) and \( \theta_2 \) are the angles between the meridians of the surface and the time-like geodesic curve \( \gamma \). Conversely, if \( 2f_1 \cos \theta_2 \sinh \varphi_2 \) and \( 2f_2 \sin \theta_2 \sinh \varphi_2 \) are constant along the curve \( \gamma \), if no part of some parallels of the surface of rotation, then \( \gamma \) is time-like geodesic.

In order to obtain the general equation of geodesics, one should consider the Euler-Lagrange equations

(3.18)

\[
\hat{y} = \frac{dy}{ds} = \frac{\cos \theta_2 \sinh \varphi_2}{f_1}; \hat{z} = \frac{dz}{ds} = \frac{\sinh \varphi_2 \sin \theta_2}{f_2}.
\]

By adding the equations (3.18) at Lagrangian equation \( L \), one has

\[
L = f_1^2 \left( \frac{dy}{ds} \right)^2 + f_2^2 \left( \frac{dz}{ds} \right)^2 - \left( \frac{dt}{dx} \right)^2 \frac{\cos \theta_2 \sinh^2 \varphi_2}{f_1^2}.
\]

or

\[
L = f_1^2 \left( \frac{dy}{ds} \right)^2 + f_2^4 \left( \frac{dz}{ds} \right)^2 - \left( \frac{dt}{dz} \right)^2 \frac{\sinh \varphi_2 \sin \theta_2}{f_2^2}.
\]

**Theorem 7.** The general equation of geodesics on the hyperbolic surface of rotation \( \Upsilon^2 \subset E_4^2 \), and for the parameters \( \hat{y} = \frac{\cos \theta_2 \sinh \varphi_2}{f_1} \) and \( \hat{z} = \frac{\sinh \varphi_2 \sin \theta_2}{f_2} \), are given by

\[
\frac{dt}{dx} = \frac{f_1 \sqrt{\sinh^2 \varphi_2 - L}}{\cos \theta_2 \sinh \varphi_2} \quad \text{or} \quad \frac{dt}{dz} = \frac{f_2 \sqrt{\sinh^2 \varphi_2 - L}}{\sinh \varphi_2 \sin \theta_2}.
\]
3.3. Clairaut’s theorem on the elliptic surfaces of rotation $T^3(\beta, \theta, s)$. In this section, one will use the elliptic surface rotation parametrized as

$$
\mathbf{Y}^3(\beta, \theta, t) = \left( f_1 \cos \beta + f_2 \sin \beta, -f_1 \sin \beta + f_2 \cos \beta, f_3 \cos \theta + f_4 \sin \theta, -f_3 \sin \theta + f_4 \cos \theta \right),
$$

then one can take the planar curve $\gamma$ for this surface of rotation to be the intersection of $\mathbf{Y}^3(\beta, \theta, s)$ with $\rho, \eta = 0$ (or $\xi, \vartheta = 0$) for the coordinate system $(\xi, \mu, \vartheta, \eta)$. Therefore, the curve can be written by $\gamma(t) = (0, f_2(t), 0, f_4(t))$ (or $\gamma(t) = (f_1(t), 0, f_3(t), 0)$): $f_i \in C^\infty$, then one gets

$$
\mathbf{Y}^3(\beta, \theta, s) = (f_2 \sin \beta, f_2 \cos \beta, f_4 \sin \theta, f_4 \cos \theta)
$$

and resulting in the first fundamental form:

$$
\begin{align*}
\mathbf{Y}^3_{\beta} &= (f_2 \cos \beta, -f_2 \sin \beta, 0, 0); \quad \mathbf{Y}^3_{t} = (0, 0, f_4 \cos \theta, -f_4 \sin \theta); \\
\mathbf{Y}^3_{\theta} &= (f_2 ^2 \sin \beta, f_2 ^2 \cos \beta, f_4 ^2 \sin \theta, f_4 ^2 \cos \theta);
\end{align*}
$$

$$
\langle \mathbf{Y}^3_{\beta}, \mathbf{Y}^3_{\beta} \rangle = -f_2 ^2; \quad \langle \mathbf{Y}^3_{\beta}, \mathbf{Y}^3_{t} \rangle = f_4 ^2; \quad \langle \mathbf{Y}^3_{t}, \mathbf{Y}^3_{t} \rangle = -f_2 ^2 + f_4 ^2; \\
\langle \mathbf{Y}^3_{\beta}, \mathbf{Y}^3_{\theta} \rangle, \langle \mathbf{Y}^3_{\beta}, \mathbf{Y}^3_{t} \rangle, \langle \mathbf{Y}^3_{\theta}, \mathbf{Y}^3_{t} \rangle = 0
$$

and

$$
I_{\mathbf{Y}^3} = \begin{pmatrix} -f_2 ^2 & 0 & 0 \\ 0 & f_4 ^2 & 0 \\ 0 & 0 & -f_2 ^2 + f_4 ^2 \end{pmatrix}.
$$

Hence, one can write Lagrangian equation as follows

$$
L = -f_2 ^2 \beta ^2 + f_4 ^2 \theta ^2 + (-f_2 ^2 + f_4 ^2) t ^2.
$$

If one takes $\gamma$ to be time-like, one can write $-f_2 ^2 + f_4 ^2 = -1, f_4 ^2 > 0, -f_2 ^2 < 0$. Then,

$$
L = -f_2 ^2 \beta ^2 + f_4 ^2 \theta ^2 - t ^2.
$$

Hence, one obtains the following equations using Clairaut’s theorem,

$$
\frac{\partial}{\partial s} \left( \frac{\partial L}{\partial \beta} \right) = \frac{\partial L}{\partial \beta} \Rightarrow \frac{\partial}{\partial s} \left( -2f_2 ^2 \beta \right) = 0; \quad \frac{\partial L}{\partial \theta} \Rightarrow \frac{\partial}{\partial s} \left( 2f_4 ^2 \theta \right) = 0,
$$

$$
\frac{\partial}{\partial s} \left( \frac{\partial L}{\partial t} \right) = \frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial s} \left( 2t \right) = -2f_2 ^2 \beta ^2 + 2f_4 ^2 \theta ^2.
$$

Let $\gamma$ be a geodesic curve on $\mathbf{Y}^3$. Hence, $\gamma(t)$ can be written as follows

$$
\dot{\gamma} = \beta \mathbf{Y}^3_{\beta} + \theta \mathbf{Y}^3_{\theta} + t \mathbf{Y}^3_{t}.
$$

Hence, one can note that $\mathbf{Y}^3_{t} = N_t$ is a unit time-like vector pointing along $t$-axis of the parallels, and $\mathbf{Y}^3_{\beta} = f_2 N_\beta$ is a unit time-like vector pointing along the $\beta$-axis of the parallels. Also, $\mathbf{Y}^3_{\theta} = f_4 N_\theta$ is a unit space-like vector pointing along $\theta$-axis of the meridians and the plane spanned by $N_t$ and $N_\beta$ is time-like and an orthonormal basis. Also, from (4.21), one gets

$$
\dot{\gamma} = i N_t + f_2 \beta N_\beta + f_4 \theta N_\theta.
$$

Note that the $\gamma$ is time-like curve, since $N_\theta ^{\perp} \in Sp\{N_t, N_\beta\}$, one gets

$$
N_\theta ^{\perp} = \cosh \theta_3 N_t + \sinh \theta_3 N_\beta,
$$
(3.21) \( \dot{\gamma} = iN_1 + f_2 \dot{\beta} N_3 + f_4 \dot{\theta} N_\theta = \cos \varphi_3 N_1 + \sin \varphi_3 \cosh \theta_3 N_3 + \sinh \theta_3 \sin \varphi_3 N_\theta \)

where \( \varphi_3 \) and \( \theta_3 \) are the angles between the meridians of the surface and the time-like geodesic curve \( \gamma \).

Furthermore, from (3.20) and (3.21), one writes

(3.22) \( f_2 \dot{\beta} = \sin \varphi_3 \cosh \theta_3; f_4 \dot{\theta} = \sinh \theta_3 \sin \varphi_3; \dot{t} = \cos \varphi_3. \)

Hence, from the Lagrangian equations, one has

\[
-\ddot{t} = f_2 f_2 \dot{\beta}^2 + f_4 f_4 \dot{\theta}^2; \\
\frac{\partial}{\partial s} \left( -2f_2^2 \dot{\beta} \right) = 0 \Rightarrow \beta = -\frac{c^4 t}{2f_2^2} + c^3; \frac{\partial}{\partial s} \left( 2f_4^2 \dot{\theta} \right) = 0 \Rightarrow \theta = \frac{c^4 t}{2f_4^2} + c^3,
\]

which gives that \( 2f_2^2 \dot{\beta} \) and \( 2f_4^2 \dot{\theta} \) are constant along the geodesic, and by using (3.22) one obtains

(3.23) \( f_2 \ddot{\beta} = \sin \varphi_3 \cosh \theta_3 \Rightarrow 2f_2^2 \dot{\beta} = 2f_2 \sin \varphi_3 \cosh \theta_3 = \text{cons.} \)

(3.24) \( f_4 \ddot{\theta} = \sinh \theta_3 \sin \varphi_3 \Rightarrow 2f_4^2 \dot{\theta} = 2f_4 \sinh \theta_3 \sin \varphi_3 = \text{cons.} \)

2\( t = 2 \cos \varphi_3 \neq \text{cons.} \)

for \( \frac{\partial L}{\partial \dot{\beta}} \left( \frac{\partial L}{\partial \beta} \right) = \frac{\partial L}{\partial \dot{\theta}} \left( \frac{\partial L}{\partial \theta} \right) \), one has \( f_2 \dot{\beta} = \sin \varphi_3 \cosh \theta_3; f_4 \dot{\theta} = \sinh \theta_3 \sin \varphi_3; \dot{t} = \cos \varphi_3, \)

(3.25) \( \beta = \int \frac{\sin \varphi_3 \cosh \theta_3}{f_2} ds \)

is a constant, conversely for the condition \( 2f_2 \sin \varphi_3 \cosh \theta_3 = \text{constant and the equation} \frac{d}{ds} \left( \frac{\dot{L}}{\dot{\beta}} \right) = \frac{\dot{L}}{\dot{\theta}} \)

(3.26) \( v = \int \frac{\sinh \theta_3 \sin \varphi_3}{f_4} ds \)

is constant along the curve \( \gamma \). Hence, the following theorem can be given.

**Theorem 8.** Let \( \gamma(t) = (0, f_2(t), 0, f_4(t)) \) or \( \gamma(t) = (f_1(t), 0, f_3(t), 0) \), \( f_i \in C^\infty \) be a time-like geodesic curve on the elliptic surface of rotation \( \mathcal{Y} \subset E_2^4 \), and let \( f_2 \) and \( f_4 \) be the distance functions from the axis of rotation to a point on the surface. Then, \( 2f_2 \sin \varphi_3 \cosh \theta_3 \) and \( 2f_4 \sinh \theta_3 \sin \varphi_3 \) are constant along the curve \( \gamma \) where \( \varphi_3 \) and \( \theta_3 \) are the angles between the meridians of the surface and the time-like geodesic curve \( \gamma \). Conversely, if \( 2f_2 \sin \varphi_3 \cosh \theta_3 \) and \( 2f_4 \sinh \theta_3 \sin \varphi_3 \) are constant along the curve \( \gamma \), if no part of some parallels of the surface of rotation, then \( \gamma \) is time-like geodesic curve.

For the general equation of geodesics, one should consider the Euler-Lagrange equations

(3.27) \( \ddot{\beta} = \frac{d}{ds} \sin \varphi_3 \cosh \theta_3; \dot{\theta} = \frac{d}{ds} \sinh \theta_3 \sin \varphi_3. \)

By adding the equations in (3.27) at Lagrangian equation \( L \), one has

\[
L = -f_2^2 \left( \frac{d\beta}{ds} \right)^2 + f_4^2 \left( \frac{d\theta}{ds} \right)^2 - \left( \frac{dt}{d\beta} \frac{d\beta}{ds} \right)^2 \frac{\sin^2 \varphi_3 \cosh^2 \theta_3}{f_2^2} = -\sin^2 \varphi_3 - L; \frac{dt}{d\beta} = \frac{i f_2 \sqrt{L + \sin^2 \varphi_3}}{\sin \varphi_3 \cosh \theta_3} \)
or

\[
L = -f_2^2 \left( \frac{d\beta}{ds} \right)^2 + f_4^2 \left( \frac{d\theta}{ds} \right)^2 - \left( \frac{dt}{d\theta} \right)^2 \sinh^3 \theta \sin^3 \varphi^3
\]

\[
\left( \frac{dt}{d\theta} \right)^2 \left( \frac{\sinh \theta_3 \sin \varphi_3}{f_4} \right)^2 = -\sin^2 \varphi_3 - L \to \frac{dt}{d\theta} = i \frac{f_4}{\sinh \theta_3 \sin \varphi_3} \sqrt{\sin^2 \varphi_3 + L}.
\]

**Theorem 9.** The general equation of geodesics on the elliptic surface of rotation \( \Upsilon^3 \subset E^4_2 \), and for the parameters \( \beta = \frac{\sin \varphi_3 \cosh \theta_3}{f_2} \) and \( \Theta = \frac{\sin \theta_3 \sin \varphi_3}{f_4} \), are given by

\[
\frac{dt}{d\beta} = i \frac{f_2 \sqrt{L + \sin^2 \varphi_3}}{\sin \varphi_3 \cosh \theta_3} \quad \text{or} \quad \frac{dt}{d\theta} = i \frac{f_4}{\sinh \theta_3 \sin \varphi_3} \sqrt{\sin^2 \varphi_3 + L}.
\]

4. **Conclusion**

This study generalizes Clairaut’s theorem to pseudo Euclidean 4-space with two index, and reviews Clairaut’s theorem of surfaces of rotation which define a well-known characterization of geodesics on a surface of rotation. Therefore, it is shown that the time-like geodesic curves on the hyperbolic surface of rotation \( \Upsilon^1 \) are completely characterized by \( 2f_1 \cos \varphi_1 \) and \( -2f_4 \cosh \theta_1 \sin \varphi_1 \) being constant, the time-like geodesics on the hyperbolic surface of rotation \( \Upsilon^2 \) are characterized by \( 2f_1 \cos \theta_2 \sinh \varphi_2 \) and \( 2f_2 \sin \varphi_2 \sin \theta_2 \) being constant, and finally the time-like geodesics on the elliptic surface of rotation \( \Upsilon^3 \) are characterized by \( 2f_2 \sin \varphi_3 \cosh \theta_3 \) and \( 2f_4 \sinh \theta_3 \sin \varphi_3 \) being constant, where \( \varphi_i \) and \( \theta_i \) are the angles between the meridians of the surface and the time-like geodesic curve \( \gamma_i, i = 1, 2, 3 \).

The authors are currently working on the properties of these surfaces of rotation with a view to devising suitable metric in \( E^4_2 \) by adapting the type of conservation laws considered in the paper. In our future studies, the physical terms such as specific energy and specific angular momentum will be examined with the help of the conditions obtained by using the Clairaut’s theorem for geodesics on these special surfaces.

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The authors declared that they have no conflict of interest.
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