An endpoint estimate for the Kunze-Stein phenomenon and related maximal operators

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Abstract

One of the purposes of this paper is to prove that if $G$ is a noncompact connected semisimple Lie group of real rank one with finite center, then

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G).$$

Let $K$ be a maximal compact subgroup of $G$ and $X = G/K$ a symmetric space of real rank one. We will also prove that the noncentered maximal operator

$$\mathcal{M}_2 f(z) = \sup_{z \in B} \frac{1}{|B|} \int_B |f(z')| dz'$$

is bounded from $L^{2,1}(X)$ to $L^{2,\infty}(X)$ and from $L^p(X)$ to $L^p(X)$ in the sharp range of exponents $p \in (2, \infty]$. The supremum in the definition of $\mathcal{M}_2 f(z)$ is taken over all balls containing the point $z$.

1. Introduction

A central result in the theory of convolution operators on semisimple Lie groups is the Kunze-Stein phenomenon which, in its classical form, states that if $G$ is a connected semisimple Lie group with finite center and $p \in [1, 2)$, then

$$L^2(G) * L^p(G) \subseteq L^2(G).$$

The usual convention, which will be used throughout this paper, is that if $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are Banach spaces of functions on $G$ then the notation $\mathcal{U} * \mathcal{V} \subseteq \mathcal{W}$ indicates both the set inclusion and the associated norm inequality. The inclusion (1.1) was established by Kunze and Stein [10] in the case when the group $G$ is $\text{SL}(2, \mathbb{R})$ (and, later on, for a number of other particular groups) and by Cowling [3] in the general case stated above. For a more complete account of the development of ideas leading to (1.1) we refer the reader to [3] and [4].

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More recently, Cowling, Meda and Setti noticed that if the group \( G \) has real rank one then the inclusion (1.1) can be strengthened. Following earlier work of Lohoué and Rychener [9], the key ingredient in their approach is the use of Lorentz spaces \( L^{p,q}(G) \); they prove in [4] that if \( G \) is a connected semisimple Lie group of real rank one with finite center, \( p \in (1,2) \) and \((u,v,w) \in [1,\infty]^3\) has the property that \( 1 + 1/w \leq 1/u + 1/v \), then

\[
L^{p,u}(G) * L^{p,v}(G) \subseteq L^{p,w}(G).
\]

In particular, \( L^{p,1} \) convolves \( L^p \) into \( L^p \) for any \( p \in [1,2) \). Our first theorem is an endpoint estimate for (1.2) showing what happens when \( p = 2 \).

**Theorem A.** If \( G \) is a noncompact connected semisimple Lie group of real rank one with finite center then

\[
L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G).
\]

Notice that (1.2) follows from Theorem A and a bilinear interpolation theorem ([4, Theorem 1.2]). Unlike the classical proofs of the Kunze-Stein phenomenon, our proof of Theorem A will be based on real-variable techniques only: the inclusion (1.3) is equivalent to an inequality involving a triple integral on \( G \) and we use certain nonincreasing rearrangements to control this triple integral. Easy examples, involving only \( K \)-bi-invariant functions, show that the inclusion (1.3) is sharp in the sense that neither of the \( L^{2,1} \) spaces nor the \( L^{2,\infty} \) space can be replaced with some \( L^{2,u} \) space for any \( u \in (1,\infty) \).

Let \( K \) be a maximal compact subgroup of the group \( G \) and \( X = G/K \) the associated symmetric space. Assume from now on that the group \( G \) satisfies the hypothesis stated in Theorem A and let \( d \) be the distance function on \( X \times X \) induced by the Killing form on the Lie algebra of the group \( G \). Let \( B(x,r) \) denote the ball in \( X \) centered at the point \( x \) of radius \( r \) (with respect to the distance function \( d \)) and let \( |A| \) denote the measure of the set \( A \subset X \). For any locally integrable function \( f \) on \( X \), let

\[
\mathcal{M}_2f(z) = \sup_{z \in B} \frac{1}{|B|} \int_B |f(z')| \, dz',
\]

where the supremum in the definition of \( \mathcal{M}_2f(z) \) is taken over all balls \( B \) containing \( z \). We will prove the following:

**Theorem B.** The operator \( \mathcal{M}_2 \) is bounded from \( L^{2,1}(X) \) to \( L^{2,\infty}(X) \) and from \( L^p(X) \) to \( L^p(X) \) in the sharp range of exponents \( p \in (2,\infty] \).

We recall that the more standard centered maximal operator

\[
\mathcal{M}_1f(z) = \sup_{r > 0} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(z')| \, dz'
\]
is bounded from \( L^1(X) \) to \( L^{1,\infty}(X) \) and from \( L^p(X) \) to \( L^p(X) \) for any \( p > 1 \), as shown in [5] and [12] (without the assumption that \( G \) has real rank one). Notice however that, unlike in the case of Euclidean spaces, balls on symmetric spaces do not have the basic doubling property (i.e. \( |B(z, 2r)| \) is not proportional to \( |B(z, r)| \) if \( r \) is large), thus the maximal operators \( M_1 \) and \( M_2 \) are not comparable. Easy examples (see [7, Section 4]) show that Theorem B is sharp in the sense that the maximal operator \( M_2 \) is not bounded from \( L^{2,u}(X) \) to \( L^{2,v}(X) \) unless \( u = 1 \) and \( v = \infty \).

This paper is organized as follows: in the next section we recall most of the notation related to semisimple Lie groups and symmetric spaces and prove a proposition that explains the role of the Lorentz space \( L^{2,1}(G/K) \) – the subspace of \( K \)-bi-invariant functions in \( L^{2,1}(G) \). In Section 3 we prove Theorem B. As a consequence of Theorem B we obtain in Section 4 a covering lemma on noncompact symmetric spaces of real rank one. In Section 5 we give a complete proof of Theorem A, which is divided into four steps. The main estimate in the proof of Theorem A uses the technique of nonincreasing rearrangements; we return to this technique in the last section and prove a general rearrangement inequality.

We conclude this section with some remarks on semisimple Lie groups of higher real rank. If the group \( G \) has real rank different from 1, then (1.2) fails (the estimate in Lemma 6 and the discussion following Proposition 7 in [1] show that the appropriate spherical function \( \Phi_p \) fails to belong to \( L^{p',\infty}(G) \), where \( p' \) is the conjugate exponent of \( p \)); therefore Theorem A fails to hold. On the other hand, the author has recently proved by a different method in [7] that the \( L^p \) estimate in Theorem B holds on symmetric spaces of arbitrary real rank. In the general case it is not known however whether the maximal operator \( M_2 \) is bounded from \( L^{2,1}(X) \) to \( L^{2,\infty}(X) \).

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2. Preliminaries

Let \( G \) be a noncompact connected semisimple Lie group with finite center, and let \( \mathfrak{g} \) be its Lie algebra. Most of our notation related to semisimple Lie groups and symmetric spaces is standard and can be found for example in [6]. Fix a Cartan involution \( \theta \) of \( \mathfrak{g} \) and let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the associated Cartan decomposition. Let \( \mathfrak{a} \) be a maximal abelian subspace of \( \mathfrak{p} \); we will assume from
now on that the group $G$ has real rank one, i.e., $\dim a = 1$. Let $a^*$ denote the real dual of $a$, let $\Sigma \subset a^*$ be the set of nonzero roots of the pair $(g, a)$ and let $W$ be the Weyl group associated to $\Sigma$. It is well-known that $W = \{ 1, -1 \}$ and $\Sigma$ is either of the form $\{-\alpha, \alpha\}$ or of the form $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Let $m_1 = \dim g_{-\alpha}$, $m_2 = \dim g_{-2\alpha}$, $\rho = \frac{1}{2}(m_1 + 2m_2)\alpha$ and $a_+ = \{ H \in a : \alpha(H) > 0 \}$. Finally let $\pi = g_{-\alpha} + g_{-2\alpha}$, $\bar{\pi} = \exp \pi$, $K = \exp \mathfrak{t}$, $A = \exp \mathfrak{a}$ and $A_+ = \exp \mathfrak{a}_+$ and let $X = G/K$ be a symmetric space of real rank one.

The group $G$ has an Iwasawa decomposition $G = \bar{\pi}AK$ and a Cartan decomposition $G = K\bar{A}_+K$. Our proofs are based on relating these two decompositions, and for real rank one groups one has the explicit formula in [6, Ch.2, Theorem 6.1]. A similar idea was used by Strömberg [12] for groups of arbitrary real rank. Let $H_0 \in a$ be the unique element of $a$ for which $\alpha(H_0) = 1$ and let $a(s) = \exp(sH_0)$ for $s \in \mathbb{R}$ be a parametrization of the subgroup $A$. By [6, Ch.2, Theorem 6.1] one can identify the group $\pi$ with $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ using a diffeomorphism $\pi : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \pi$. This diffeomorphism has the property that if $t \geq 0$ then $\pi(v,w)a(s) \in Ka(t)K$ if and only if

\begin{equation}
(cosh t)^2 = \left[ \cosh s + e^s|v|^2 \right]^2 + e^{2s}|w|^2.
\end{equation}

In addition,

\begin{equation}
a(s)\pi(v,w)a(-s) = \pi(e^{-s}v, e^{-2s}w).
\end{equation}

Let $|\rho| = \rho(H_0) = \frac{1}{2}(m_1 + 2m_2)$ and let $dg$, $d\pi$ and $dk$ denote Haar measures on $G$, $\pi$ and $K$, the last one normalized such that $\int_K 1 \, dk = 1$. Then the following integral formulae hold for any continuous function $f$ with compact support:

\begin{equation}
\int_G f(g) \, dg = C_1 \int_K \int_{\mathbb{R}_+} \int_K \int_{\mathbb{R}} f(k_1a(t)k_2)(\sinh t)^{m_1}(\sinh 2t)^{m_2} \, dk_2 \, dt \, dk_1,
\end{equation}

and

\begin{equation}
\int_G f(g) \, dg = C_2 \int_K \int_{\mathbb{R}} \int_{\pi} f(\pi a(s)k)e^{2|\rho|s} \, d\pi \, ds \, dk
= C_2' \int_K \int_{\mathbb{R}} \int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} f(\pi(v,w)a(s)k)e^{2|\rho|s} \, dv \, dw \, ds \, dk.
\end{equation}

The measures $dv$ and $dw$ are the usual Lebesgue measures on $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, and the constants $C_1$, $C_2$ and $C_2'$ depend on the normalizations of the various Haar measures. We will need a new integration formula, which is the subject of the following lemma.

**Lemma 1.** Suppose that $f : G \to \mathbb{C}$ is a $K$-bi-invariant (i.e., $f(k_1gk_2) = f(g)$ for any $k_1, k_2 \in K$) continuous function with compact support and $F(t) = f(a(t))$ for any $t \in [0, \infty)$. Then for any $s \in \mathbb{R}$

\[ e^{s|\rho|} \int_\pi f(\pi a(s)) \, d\pi = \int_{|s|}^\infty F(t)\psi(t,s) \, dt, \]
where the kernel $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ has the property that $\psi(t, s) = 0$ if $t < |s|$ and
\begin{equation}
\psi(t, s) \approx \sinh t (\cosh t)^{m_2/2} (\cosh t - \cosh s)^{(m_1 + m_2 - 2)/2}
\end{equation}
if $t \geq |s|$.

As usual, the notation $U \approx V$ means that there is a constant $C \geq 1$ depending only on the group $G$ such that $C^{-1} U \leq V \leq C U$. This lemma is essentially proved in [8, Section 5]. For later reference we reproduce its proof.

**Proof of Lemma 1.** For any $t \geq |s|$, let
\begin{equation}
T_{t,s} = \{(v, w) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} : (\cosh t)^2 = \left[ \cosh s + e^s|v|^2 \right]^2 + e^{2s}|w|^2 \}
\end{equation}
be the set of points $P = P(v, w) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ with the property that $\overline{\mathcal{P}(P)} a(s) \in K a(t) K$ (these surfaces will play a key role in the proof of Theorem A). Let $d\omega_{t,s}$ be the induced measure on $T_{t,s}$ such that
\[
\int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} \phi(v, w) \, dv \, dw = \int_{t \geq |s|} \left[ \int_{T_{t,s}} \phi(P) \, d\omega_{t,s}(P) \right] \, dt
\]
for any continuous compactly supported function $\phi$. Then, since the function $f$ is $K$-bi-invariant,
\[
e^{\rho |s|} \int_{\mathbb{R}^N} f(\overline{\mathcal{P}(s)}) \, d\overline{\mathcal{P}} = C e^{\rho |s|} \int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} f(\overline{\mathcal{P}(v, w)} a(s)) \, dv \, dw = C e^{\rho |s|} \int_{t \geq |s|} F(t) \left[ \int_{T_{t,s}} 1 \, d\omega_{t,s} \right] \, dt.
\]
Let $\psi(t, s) = e^{\rho |s|} \int_{T_{t,s}} 1 \, d\omega_{t,s}$ and assume that $m_2 \geq 1$. We make the change of variables $v = [e^{-s}(u \cosh t - \cosh s)]^{1/2} \omega_1$ and $w = e^{-s} \cosh t (1 - u^2)^{1/2} \omega_2$, where $\omega_1 \in S^{m_1 - 1}$ (the $m_1 - 1$ dimensional sphere in $\mathbb{R}^{m_1}$), $\omega_2 \in S^{m_2 - 1}$ and $u \in \left( \frac{\cosh s}{\cosh t} , 1 \right)$. We have
\[
\psi(t, s) = C \sinh t (\cosh t)^{m_2} \int_{\frac{\cosh s}{\cosh t}}^1 (u \cosh t - \cosh s)^{(m_1 - 2)/2} (1 - u^2)^{(m_2 - 2)/2} \, du,
\]
which easily proves (2.5). The computation of the function $\psi$ is slightly easier if $m_2 = 0$ and the result is also given by (2.5).

Our next proposition explains the role of the Lorentz space $L^{2,1}(G//K)$ which, by definition, is the subspace of $K$-bi-invariant functions in $L^{2,1}(G)$:

**Proposition 2.** The Abel transform
\[
Af(a) = e^{\rho (\log a)} \int_{\mathbb{R}^N} f(\overline{\mathcal{P}(a)}) \, d\overline{\mathcal{P}}
\]
is bounded from $L^{2,1}(G/K)$ to $L^{\infty}(A/W)$. In other words, if $f$ is a locally integrable $K$-bi-invariant function on $G$ and $a \in A$ then:

\begin{equation}
(2.7) \quad e^{\rho(\log a)} \int_{\mathbb{N}} f(\tau a) \, d\tau \leq C \|f\|_{L^{2,1}(G)}.
\end{equation}

**Proof of Proposition 2.** The usual theory of Lorentz spaces (see, for example, [11, Chapter V]) shows that it suffices to prove the inequality (2.7) under the additional assumption that $f$ is the characteristic function of an open $K$-bi-invariant set of finite measure. For any $t \geq 0$, let $F(t) = f(a(t))$, so

\begin{equation}
(2.8) \quad \|f\|_{L^{2,1}(G)} = C \left[ \int_{\mathbb{R}^+} F(t) (\sinh t)^{m_1} (\sinh 2t)^{m_2} \, dt \right]^{1/2}.
\end{equation}

In view of Lemma 1 and (2.8), it suffices to prove that for any $s \in \mathbb{R}$

\begin{equation}
(2.9) \quad \int_{t \geq |s|} F(t) \psi(t, s) \, dt \leq C \left[ \int_{\mathbb{R}^+} F(t) (\sinh t)^{m_1} (\sinh 2t)^{m_2} \, dt \right]^{1/2}
\end{equation}

for any measurable function $F : \mathbb{R}^+ \to \{0, 1\}$. Notice that if $t \geq 1 + |s|$ then

\begin{equation}
\psi(t, s) \approx e^{\rho t}, \quad (\sinh t)^{m_1} (\sinh 2t)^{m_2} \approx e^{2|\rho| t} \quad \text{and it follows from Lemma 3 below that}
\end{equation}

\begin{equation}
(2.10) \quad \int_{t \geq |s| + 1} F(t) \psi(t, s) \, dt \leq C \left[ \int_{t \geq |s| + 1} F(t) (\sinh t)^{m_1} (\sinh 2t)^{m_2} \, dt \right]^{1/2}.
\end{equation}

In order to deal with the integral in $t$ over the interval $[|s|, |s| + 1]$ we consider two cases: $|s| \geq 1$ and $|s| \leq 1$. If $|s| \geq 1$ and $t \in [|s|, |s| + 1]$, then

\begin{equation}
\psi(t, s) \approx e^{\rho |s| (t - |s|)} (\sinh t)^{m_1} (\sinh 2t)^{m_2} \approx e^{2|\rho| |s|} \quad \text{and, since} \quad (m_1 + m_2 - 2)/2 \geq -1/2,
\end{equation}

it follows that

\begin{equation}
\int_{|s|}^{|s| + 1} F(t) \psi(t, s) \, dt \leq C e^{\rho |s|} \int_{|s|}^{|s| + 1} F(t - |s|)^{-1/2} \, dt
\end{equation}

\begin{equation}
= C e^{\rho |s|} \int_{0}^{1} F(|s| + u^2) \, du \leq C \left[ e^{2|\rho| |s|} \int_{0}^{1} F(|s| + u^2) \, du \right]^{1/2}
\end{equation}

\begin{equation}
\leq C \left[ \int_{|s|}^{|s| + 1} F(t) (\sinh t)^{m_1} (\sinh 2t)^{m_2} \, dt \right]^{1/2}.
\end{equation}

One of the inequalities in the sequence above follows from the estimate (2.11) below. This, together with (2.10), completes the proof of the proposition in the case $|s| \geq 1$. The estimation of the integrals over the interval $[|s|, |s| + 1]$ is similar in the case $|s| \leq 1$.

**Lemma 3.** If $\delta \neq 0$ and $d\mu_1(t) = e^{\delta t} \, dt$, $d\mu_2(t) = e^{2\delta t} \, dt$ are two measures on $\mathbb{R}$ then

\begin{equation}
\|f\|_{L^1(\mathbb{R}, d\mu_1)} \leq C_\delta \|f\|_{L^{2,1}(\mathbb{R}, d\mu_2)}.
\end{equation}
Proof of Lemma 3. One can assume that $f$ is the characteristic function of a set. The change of variable $t = (\log s)/\delta$ and the substitution $g(s) = f((\log s)/\delta)$ show that it suffices to prove that

$$
(2.11) \quad \frac{1}{|\delta|} \int_{\mathbb{R}^+} g(s) \, ds \leq C_\delta \left[ \frac{1}{|\delta|} \int_{\mathbb{R}^+} g(s) s \, ds \right]^{1/2}
$$

for any measurable function $g : \mathbb{R}^+ \to \{0, 1\}$, which follows by a rearrangement argument.

3. Proof of the maximal theorem

For any locally integrable function $f : X \to \mathbb{C}$ let

$$
\widetilde{M}_2 f(z) = \sup_{r \geq 1} \frac{1}{|B(z,r)|^{1/2}} \int_{B(z,r)} |f(z')| \, dz'.
$$

Most of this section will be devoted to the proof of the following theorem:

**Theorem 4.** The operator $\widetilde{M}_2$ is bounded from $L^{2,1}(X)$ to $L^{2,\infty}(X)$.

Notice that Theorem B is an easy consequence of Theorem 4: let

$$
\mathcal{M}_2^0 f(z) = \sup_{z \in B, r(B) \leq 1} \frac{1}{|B|} \int_B f(z') \, dz',
$$

$$
\mathcal{M}_2^1 f(z) = \sup_{z \in B, r(B) \geq 1} \frac{1}{|B|} \int_B f(z') \, dz',
$$

where $r(B)$ is the radius of the ball $B$. We can assume that the Killing form on the Lie algebra $\mathfrak{g}$ is normalized such that $|H_0| = 1$. Let $o = \{K\}$ be the origin of the symmetric space $X$. Then the ball $B(o,r)$ is equal to the set of points $\{ka(t) \cdot o : k \in \mathbb{K}, t \in [0,r]\}$ and one clearly has $|B(o,r)| \approx r^{m_1 + m_2 + 1}$ if $r \leq 1$ and $|B(o,r)| \approx e^{2|\rho| r}$ if $r \geq 1$. The operator $\mathcal{M}_2^0$, the local part of $\mathcal{M}_2$, is clearly bounded on $L^p(X)$ for any $p > 1$. On the other hand, if $z$ belongs to a ball $B$ of radius $r \geq 1$, then $B(z,2r)$ contains the ball $B$ and $|B(z,2r)| \approx e^{3|\rho| r} \approx |B|^2$. Therefore

$$
\frac{1}{|B|} \int_B f(z') \, dz' \leq C \int_{B(z,2r)} f(z') \, dz'
$$

which shows that $\mathcal{M}_2^1 f(z) \leq C \mathcal{M}_2 f(z)$, and the conclusion of Theorem B follows by interpolation with the trivial $L^\infty$ estimate.

**Proof of Theorem 4.** Let $\chi_r$ be the characteristic function of the $K$-bi-invariant set $\{g \in G : d(g \cdot o,o) < r\}$. Since the measure of a ball of
radius \( r \) in \( X \) is proportional to \( e^{2|\rho|r} \) if \( r \geq 1 \), one has

\[
\widetilde{M}_2 f(g \cdot o) \approx \sup_{r \geq 1} \left[ e^{-|\rho|r} \int_G f(g' \cdot o) \chi_r(g'^{-1}g) \, dg' \right].
\]

The change of variables \( g = \overline{\pi}a(t)k, \; g' = \overline{\pi}a(t')k' \) and the integral formula (2.4) show that

\[
(3.2) \quad \widetilde{M}_2 f(\overline{\pi}a(t) \cdot o)
\]

\[
\leq C \sup_{r \geq 1} \left[ e^{-|\rho|r} \int_{\mathbb{N}} \left| f(\overline{\pi}a(t') \cdot o) \chi_r(a(-t')\overline{\pi}^{-1}\overline{\pi}a(t)) \, d\overline{\pi} \right| e^{2|\rho|r'} \, dt' \right].
\]

We first deal with the integral over the group \( \mathbb{N} \) and dominate the right-hand side of (3.2) using a standard maximal operator on the nilpotent group \( \mathbb{N} \). For any \( u > 0 \) let \( B_u \) be the ball in \( \mathbb{N} \) defined as the set \( \{ \overline{\pi}(v, w) : |v| \leq u \text{ and } |w| \leq u^2 \} \). Clearly, \( f_{B_u} 1_{B_u} = C u^{2|\rho|} \). The group \( \mathbb{N} \) is equipped with non-isotropic dilations \( \delta_u(\overline{\pi}(v, w)) = \overline{\pi}(uv, u^2w) \), which are group automorphisms, therefore the maximal operator

\[
M_3 f(\overline{\pi}a \cdot o) = \sup_{u > 0} \left[ \frac{1}{u^{2|\rho|}} \int_{B_u} |f(\overline{\pi}^{-1}a \cdot o)| \, d\overline{\pi} \right].
\]

is bounded from \( L^p(\mathbb{N}) \) to \( L^p(\mathbb{N}) \) for any \( p > 1 \) ([13, Lemma 2.2]). For any locally integrable function \( f : X \to \mathbb{R}^+ \) and any \( \overline{\pi} \in \mathbb{N} \) and \( a \in A \) let

\[
M_3 f(\overline{\pi}a \cdot o) = \sup_{u > 0} \left[ \frac{1}{u^{2|\rho|}} \int_{B_u} |f(\overline{\pi}^{-1}a \cdot o)| \, d\overline{\pi} \right].
\]

Since the maximal operator \( \mathcal{N} \) is bounded on \( L^p(\mathbb{N}) \) one has \( ||M_3 f||_{L^p(\mathbb{N})} \leq C_p ||f||_{L^p(X)} \) for any \( p > 1 \). We will now use the function \( \mathcal{M}_3 f \) to control the integral over \( \mathbb{N} \) in (3.2). Notice that (2.1) and (2.2), together with the fact that \( d(ka(t) \cdot o, o) = t \) for any \( t \geq 0 \) and \( k \in K \), show that if \( \chi_r(a(-t')\overline{\pi}a(t)) = 1 \) for some \( \overline{\pi} \in \mathbb{N} \) then \( \overline{\pi} \) has to belong to the ball \( B_{e^{(t'-t)/2}} \); therefore

\[
\int_{\mathbb{N}} f(\overline{\pi}a(t') \cdot o) \chi_r(a(-t')\overline{\pi}^{-1}\overline{\pi}a(t)) \, d\overline{\pi} \leq \int_{B_{e^{(t'-t)/2}}} f(\overline{\pi}^{-1}a(t') \cdot o) \, d\overline{\pi} \\
\leq C e^{\rho(r-t')/2} M_3 f(\overline{\pi}a(t') \cdot o).
\]

If we substitute this inequality into (3.2) we conclude that

\[
(3.3) \quad \widetilde{M}_2 f(\overline{\pi}a(t) \cdot o) \leq C e^{-|\rho|t} \int_{\mathbb{R}} M_3 f(\overline{\pi}a(t') \cdot o) e^{\rho|t'|} \, dt'.
\]

We can now estimate the \( L^{2,\infty} \) norm of \( \widetilde{M}_2 f \): for any \( \lambda > 0 \), the set \( E_{\lambda} = \{ z \in X : M_2 f(z) > \lambda \} \) is included in the set

\[
\{ \overline{\pi}a(t) \cdot o : e^{-|\rho|t} \int_{\mathbb{R}} M_3 f(\overline{\pi}a(t') \cdot o) e^{\rho|t'|} \, dt' > \lambda/C \}.
\]
The measure $dz$ in $X$ is proportional to the measure $e^{2|\rho|t} \, d\overline{m} \, dt$ in $\overline{N} \times \mathbb{R}$ under the identification $z = \overline{m} a(t) \cdot o$. Therefore the measure of this last set is less than or equal to

$$
\frac{C \int_{\overline{N}} \int_{\mathbb{R}} |M_3 f(\overline{m} a(t') \cdot o) e^{\rho |t'|} \, dt'|^2 \, d\overline{m}}{\lambda^2};
$$

hence

$$
(3.4) \quad \| \tilde{M}_2 f \|^2_{L^2,\infty} \leq C \int_{\overline{N}} \left[ \int_{\mathbb{R}} |M_3 f(\overline{m} a(t') \cdot o) e^{\rho |t'|} \, dt'|^2 \right] \, d\overline{m}.
$$

One can now use the following simple lemma to dominate the right-hand side of (3.4):

**Lemma 5.** If $U$ and $V$ are two measure spaces with measures $du$ and $dv$ respectively, and $H : U \times V \to \mathbb{R}_+$ is measurable then

$$
\left[ \int_U \|H(u, .)\|^2_{L^2,1(V, dv)} \, du \right]^{1/2} \leq C \|H\|_{L^2,1(U \times V, du \, dv)}.
$$

The proof of this lemma is straightforward. Combining Lemma 3 (at the end of the previous section) and Lemma 5, one has

$$
(3.5) \quad \int_{\overline{N}} \left[ \int_{\mathbb{R}} |M_3 f(\overline{m} a(t') \cdot o) e^{\rho |t'|} \, dt'|^2 \right] \, d\overline{m} \leq C \|M_3 f\|^2_{L^2,1(X)}.
$$

Finally, since the maximal operator $M_3$ is bounded on $L^p(X)$ for any $p > 1$, it follows by the general version of Marcinkiewicz interpolation theorem that $\|M_3 f\|_{L^2,1(X)} \leq C \|f\|_{L^2,1(X)}$ and Theorem 4 follows from (3.4) and (3.5).

4. A covering lemma

A simple connection between covering lemmas and boundedness of maximal operators is explained in [2]. In our setting we have:

**Corollary 6.** If a collection of balls $B_i \subset X$, $i \in I$, has the property that $| \bigcup B_i | < \infty$ then one can select a finite subset $J \subset I$ such that

$$
(4.1) \quad (i) \quad \left| \bigcup_{i \in I} B_i \right| \leq C \left| \bigcup_{j \in J} B_j \right|;
$$

$$
(ii) \quad \left\| \sum_{j \in J} \chi_{B_j} \right\|_{L^{2, \infty}(X)} \leq C \left| \bigcup_{i \in I} B_i \right|^{1/2}.
$$
It follows from (4.1) that
\[
\left\| \sum_{j \in J} \chi_{B_j} \right\|_{L^q(X)} \leq C_q \left\| \bigcup_{i \in I} B_i \right\|^{1/q}
\]
for any \( q \in [1, 2) \). Thus, in the terminology of [2], the family of natural balls on symmetric spaces of real rank one has the covering property \( V_q \) if and only if \( q \in [1, 2) \).

5. Proof of the convolution theorem

In this section we will prove Theorem A. In view of the general theory of Lorentz spaces, it suffices to prove that
\[
(5.1) \quad \int \int_{G \times G} f(z)g(z^{-1}z')h(z') \, dz' \, dz \leq C ||f||_{L^{2,1}} ||g||_{L^{2,1}} ||h||_{L^{2,1}}
\]
whenever \( f, g, h : G \to \{0, 1\} \) are characteristic functions of open sets of finite measure. We can also assume that \( g \) is supported away from the origin of the group, for example in the set \( \bigcup_{t > 1} Ka(t)K \). The main part of our argument is devoted to proving that the left-hand side of (5.1) is controlled by an integral involving suitable rearrangements of the functions \( f, g \) and \( h \), as in (5.19). Let \( z = \pi a(t)k \), \( z' = \pi' a(t')k' \) and the left-hand side of (5.1) becomes
\[
(5.2) \quad \int_K \int_K \int_R \int_R I(k, k', t, t') e^{2\rho|t+\bar{t}'|} \, dt \, dt' \, dk \, dk',
\]
where
\[
(5.3) \quad I(k, k', t, t') = \int_{\mathbb{N} \times \mathbb{N}} f(\pi a(t)k)g(k^{-1}a(-t)\pi^{-1}a(t')k')h(\pi' a(t')k') \, d\pi' \, d\pi
\]
We will show how to dominate the expression in (5.2) in four steps.

**Step 1. Integration on the subgroup \( \mathbb{N} \).** As in the proof of the maximal theorems, we start by integrating on \( \mathbb{N} \). Define \( F_1, H_1 : K \times \mathbb{R} \to \mathbb{R}_+ \) by
\[
F_1(k, t) = \int_{\mathbb{N}} f(\pi a(t)k) \, d\pi
\]
and
\[
H_1(k', t') = \int_{\mathbb{N}} h(\pi' a(t')k') \, d\pi'.
\]
Using the simple inequality
\[
\int_{\mathbb{N} \times \mathbb{N}} a(\pi)b(\pi^{-1}\pi')c(\pi') \, d\pi' \, d\pi \leq \left( \int_{\mathbb{N}} b(\pi) \, d\pi \right) \left[ \min \left( \left( \int_{\mathbb{N}} a(\pi) \, d\pi \right), \left( \int_{\mathbb{N}} c(\pi) \, d\pi \right) \right) \right],
\]

which holds for any measurable functions $a, b, c : \mathbb{N} \to [0, 1]$ with compact support, it follows that the integral $I(k, k', t, t')$ in (5.3) is dominated by

$$\min \{ F_1(k, t), H_1(k', t') \} \int_{\mathbb{N}} g(k^{-1}a(-t)\overline{\alpha}(t')k') \, d\overline{\alpha}_1.$$  

By (2.2), the map $\overline{\alpha}_1 \to a(-t)\overline{\alpha}_1 a(t) = \overline{\alpha}_2$ is a dilation of $\overline{\alpha}$ with $d\overline{\alpha}_1 = e^{-2|\rho|t} \, d\overline{\alpha}_2$; therefore

$$\int_{\mathbb{N}} g(k^{-1}a(-t)\overline{\alpha}_1 a(t')k') \, d\overline{\alpha}_1 = e^{-2|\rho|t} \int_{\mathbb{N}} g(k^{-1}(1 \overline{\alpha}_2 a(t')k') \, d\overline{\alpha}_2$$

$$= Ce^{-2|\rho|t} \int_{\mathbb{R}^m \times \mathbb{R}^m} g(k^{-1}(1 \overline{\alpha}_2 a(t')k') \, dv \, dw$$

$$= Ce^{-2|\rho|t} \int_{u \geq |t'-t|} \int_{T_{u, u'-t}} g(k^{-1}(P) a(t')k') \, d\omega_{u, u'-t}(P) \, du.$$  

The surfaces $T_{u, s}$ defined in (2.6) for $\{(u, s) \in \mathbb{R}_+ \times \mathbb{R} : u \geq |s|\}$ and the associated measures $d\omega_{u, s}$ have the same meaning as in the proof of Lemma 1. Let

$$G_1(k, k', u, s) = \left( \int_{T_{u, s}} 1 \, d\omega_{u, s} \right)^{-1} \left[ \int_{T_{u, s}} g(k^{-1}(P) a(u)k') \, d\omega_{u, s}(P) \right]$$

be the average of the function $P \to g(k^{-1}(P) a(s)k')$ on the surface $T_{u, s}$ (the domain of definition of $G_1$ is $\{(k, k', u, s) \in K \times K \times \mathbb{R}_+ \times \mathbb{R} : u \geq |s|\}$, and $G_1(k, k', u, s) \in [0, 1])$. If we substitute this definition in (5.5), we conclude that

$$\int_{\mathbb{N}} g(k^{-1}a(-t)\overline{\alpha}_1 a(t')k') \, d\overline{\alpha}_1$$

$$= Ce^{-|\rho|t|t'+t|} \int_{u \geq |t'-t|} G_1(k, k', u, t' - t) \psi(u, t' - t) \, du.$$  

The function $\psi(u, s)$ was computed in the proof of Lemma 1 and is given by (2.5). Finally, if we substitute this last formula in (5.4), we find that the integral $I(k, k', t, t')$ is dominated by

$$Ce^{-|\rho|t|t'+t|} \min \{ F_1(k, t), H_1(k', t') \} \int_{u \geq |t'-t|} G_1(k, k', u, t' - t) \psi(u, t' - t) \, du,$$

which shows that the left-hand side of (5.1) is dominated by

$$C \int_K \int_K \int_{\mathbb{R}} \int_{|t'-t|} \min \{ F_1(k, t), H_1(k', t') \}$$

$$G_1(k, k', u, t' - t) \psi(u, t' - t) e^{\rho|t|t'} \, du \, dt' \, dt' \, dk.$$  

For later use, we record the following properties of the functions $F_1$ and $H_1$:

$$||f||_{L^2,1(G)} = \left[ C_2 \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} F_1(k, t) e^{2|\rho|t} \, dt \, dk \right]^{1/2},$$

$$||h||_{L^2,1(G)} = \left[ C_2 \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} H_1(k', t') e^{2|\rho|t'} \, dt' \, dk' \right]^{1/2}.$$
Step 2. Integration on the subgroup $A$. Let $\chi_1$ and $\chi_2$, be the characteristic functions of the sets \( \{ k, k', t, t' \} : F_1(k, t) \leq H_1(k', t') \} \) and \( \{ (k, k', t, t') : H_1(k', t') \leq F_1(k, t) \} \) respectively. For any $k, k', t, t'$ one has

\[
\left\{ \begin{array}{l}
F_1(k, t)\chi_1(k, k', t, t') \leq H_1(k', t'), \\
H_1(k', t')\chi_2(k, k', t, t') \leq F_1(k, t),
\end{array} \right.
\tag{5.9}
\]

Since $\chi_1 + \chi_2 \geq 1$, the expression (5.7) is less than or equal to the sum of two similar expressions of the form

\[
C \int_K \int_K \int_{\mathbb{R}^2} \int_{u \geq |t' - t|} F_1(k, t)\chi_1(k, k', t, t') G_1(k, k', u, t' - t) e^{\rho(t + t')} du dt' dt dk' dk.
\]

The change of variable $t' = t + s$ in the expression above shows that it is equal to

\[
C \int_K \int_K \int_{\mathbb{R}^2} \int_{u \geq |s|} F_1(k, t)\chi_1(k, k', t + s) G_1(k, k', u, s) e^{2\rho t} e^{\rho s} du dt ds dk' dk,
\tag{5.10}
\]

and the first of the inequalities in (5.9) becomes

\[
F_1(k, t)\chi_1(k, k', t + s) e^{2\rho t} \leq H_1(k', t + s) e^{2\rho t}.
\tag{5.11}
\]

Let $F(k) = \left[ \int_{\mathbb{R}} F_1(k, t) e^{2\rho t} dt \right]^{1/2}$, $H(k') = \left[ \int_{\mathbb{R}} H_1(k', t') e^{2\rho t'} dt' \right]^{1/2}$ and

\[
A(k, k', s) = \int_{\mathbb{R}} F_1(k, t)\chi_1(k, k', t + s) e^{2\rho t} dt.
\]

The expression (5.10) becomes

\[
C \int_K \int_K \int_{\mathbb{R}} \int_{u \geq |s|} A(k, k', s) G_1(k, k', u, s) e^{\rho s} du ds dk' dk.
\tag{5.12}
\]

Clearly, $A(k, k', s) \leq F(k)^2$ (since $\chi_1 \leq 1$) and $A(k, k', s) \leq e^{-2\rho s} H(k')^2$ by (5.11); therefore

\[
e^{\rho s} A(k, k', s) \leq \left\{ \begin{array}{ll}
e^{\rho s} F(k)^2 & \text{if } e^{\rho s} \leq H(k') / F(k), \\
e^{-\rho s} H(k')^2 & \text{if } e^{\rho s} \geq H(k') / F(k).
\end{array} \right.
\]

If we substitute this inequality in (5.12) we find that the left-hand side of (5.1) is dominated by

\[
C \int_K \int_K \int_{e^{\rho s} \leq H(k') / F(k)} \int_{u \geq |s|} F(k)^2 G_1(k, k', u, s) e^{\rho s} du ds dk' dk
\]

\[
+ C \int_K \int_K \int_{e^{\rho s} \geq H(k') / F(k)} \int_{u \geq |s|} H(k')^2 G_1(k, k', u, s) e^{-\rho s} du ds dk' dk.
\tag{5.13}
\]
We pause for a moment to note that our estimates so far, together with the proof of Lemma 1 in the second section, suffice to prove that $L^{2,1}(G) \ast L^{2,1}(G//K) \subseteq L^{2,\infty}(G)$: if $g$ is a $K$-bi-invariant function, then $G_1(k,k',u,s)$ depends only on $u$, and (2.9) shows that

$$\int_{u \geq |s|} G_1(k,k',u,s) \psi(u,s) \, du \leq C \|g\|_{L^{2,1}}.$$

As a consequence, both terms in (5.13) are dominated by

$$C \|g\|_{L^{2,1}} \int_K \int_K F(k) H(k') \, dk' \, dk;$$

therefore

$$\int_{G \times G} f(z) g(z^{-1}z') h(z') \, dz' \, dz \leq C \|g\|_{L^{2,1}} \int_K \int_K F(k) H(k') \, dk' \, dk \leq C \|f\|_{L^{2,1}} \|g\|_{L^{2,1}} \|h\|_{L^{2,1}}.$$

Here we used the fact that, as a consequence of (5.8),

\begin{equation}
\|f\|_{L^{2,1}(G)} = \left[ C_2 \int_K F(k)^2 \, dk \right]^{1/2},
\end{equation}

\begin{equation}
\|h\|_{L^{2,1}(G)} = \left[ C_2 \int_K H(k')^2 \, dk' \right]^{1/2}.
\end{equation}

**Step 3. A rearrangement inequality.** In the general case (if $g$ is not assumed to be $K$-bi-invariant) we will show that both terms in (5.13) are dominated by some expression of the form

$$C \int_0^1 \int_0^1 \int_{\mathbb{R}^+} F^*(x) H^*(y) G^{**}(x,y,u) e^{\rho |u|} \, du \, dy \, dx$$

where $F^*, H^* : (0,1] \to \mathbb{R}^+$ are the usual nonincreasing rearrangements of the functions $F$ and $H$ (recall that the measure of $K$ is equal to 1) and $G^{**} : (0,1] \times (0,1] \times \mathbb{R}^+ \to \{0,1\}$ is a suitable "double" rearrangement of $g$. The precise definitions are the following: if $a : K \to \mathbb{R}^+$ is a measurable function then the nonincreasing rearrangement $a^* : (0,1] \to \mathbb{R}^+$ is the right semicontinuous nonincreasing function with the property that

$$|\{k \in K : a(k) > \lambda\}| = |\{x \in (0,1] : a^*(x) > \lambda\}| \text{ for any } \lambda \in [0, \infty).$$

Assume now that $a : K \times K \to \mathbb{R}^+$ is a measurable function. For almost every $k \in K$ let $a^*(k,y), y \in (0,1]$, be the nonincreasing rearrangement of the function $k' \to a(k,k')$ and let $a^{**}(x,y)$ be the nonincreasing rearrangement of the function $k \to a(k,y)$ (clearly $a^{**} : (0,1] \times (0,1] \to \mathbb{R}^+$). The following lemma summarizes some of the well-known properties of nonincreasing rearrangements (see for example [11, Chapter V]):
LEMMA 7. (a) If \( a : K \to \mathbb{R}_+ \) is a measurable function then
\[
\left[ \int_K a(k)^2 \, dk \right]^{1/2} = \left[ \int_{(0,1]} a^*(x)^2 \, dx \right]^{1/2}.
\]

(b) If \( a : K \times K \to \mathbb{R}_+ \) is a measurable function then
\[
\begin{align*}
(i) & \quad \int_K \int_K a(k,k') \, dk \, dk' = \int_0^1 \int_0^1 a^*(x,y) \, dy \, dx. \\
(ii) & \quad \text{The function } a^* \text{ is nonincreasing: } a^*(x,y) \leq a^*(x',y') \text{ whenever } x \geq x' \text{ and } y \geq y'. \\
(iii) & \quad \text{For any measurable sets } D, E \subset K \text{ with measures } |D| \text{ and } |E| \\
& \quad \int_D \int_E a(k,k') \, dk \, dk' \leq \int_0^{|D|} \int_0^{|E|} a^*(x,y) \, dy \, dx.
\end{align*}
\]

Returning to our setting, let \( F^* \) and \( H^* \) be the nonincreasing rearrangements of \( F \) and \( H \), let \( \tilde{g} : K \times K \times \mathbb{R}_+ \to (0,1) \) be given by \( \tilde{g}(k,k',u) = g^{-1}(a(u)k') \) and let \( G^* : (0,1] \times (0,1] \times \mathbb{R}_+ \to (0,1] \) be the double rearrangement of the function \( \tilde{g} \) (i.e., \( G^* (.,.,u) \) is the double rearrangement of \( \tilde{g}(.,.,u) \) for all \( u \geq 0 \)). Recall that we assumed that the function \( g \) is the characteristic function of a set included in \( \bigcup_{a>1} Ka(u)K \); therefore
\[
(5.15) \quad \|g\|_{L^2,1(G)} \approx \left[ \int_{\mathbb{R}_+} \int_0^1 \int_0^1 G^*(x,y,u) e^{2|\rho|u} \, dy \, dx \, du \right]^{1/2}.
\]

We will now show how to use these rearrangements to dominate the two expressions in (5.13). For any integers \( m, n \) let \( D_m = \{ k \in K : F(k) \in [e^{\rho|m|}, e^{\rho(m+1)}] \} \), \( E_n = \{ k' \in K : H(k') \in [e^{\rho|n|}, e^{\rho(n+1)}] \} \) and let \( D_{-\infty} = \{ k \in K : F(k) = 0 \} \), \( E_{-\infty} = \{ k' \in K : H(k') = 0 \} \) such that \( K = \bigcup_m D_m = \bigcup_n E_n \). Let \( \delta_m \), respectively \( \epsilon_n \), be the measures of the sets \( D_m \), respectively \( E_n \), as subsets of \( K \). The first of the two expressions in (5.13) is dominated by
\[
(5.16) \quad C \sum_{m,n} \int_{D_m} \int_{E_n} \int_{s \leq (n-m+1)} \int_{u \geq |s|} e^{2|\rho|(m+1)} G_1(k,k',u,s) \psi(u,s) e^{|\rho|s} \, du \, ds \, dk' \, dk.
\]

Combining the definition (5.6) of the function \( G_1 \) (recall that the surfaces \( T_{u,s} \) are defined as the set of points \( P \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \) with the property that \( \pi(P) u(s) \in Ka(u)K \)), the fact that \( dk \) is a Haar measure on \( K \) and the last statement of Lemma 7, we conclude that
\[
\int_{D_m} \int_{E_n} G_1(k,k',u,s) \, dk' \, dk \leq \int_0^{\delta_m} \int_0^{\epsilon_n} G^*(x,y,u) \, dy \, dx.
\]
for any \( s \) with the property that \( |s| \leq u \). Substituting this inequality in (5.16), we find that the expression in (5.16) is dominated by

\[
C \sum_{m,n} \int_{\mathbb{R}^+} e^{2|\rho|m} \left[ \int_0^{\delta_m} \int_0^{\varepsilon_n} G^{**}(x, y, u) \, dy \, dx \right]
\]

\[
\left[ \int_{s \leq (n-m+1), |s| \leq u} \psi(u, s) e^{|\rho|s} \, ds \right] \, du.
\]

The formula (2.5) shows that the last of the integrals in the expression above is dominated by \( Ce^{|\rho|u} e^{|\rho|(n-m)} \); therefore the first of the two expressions in (5.13) is dominated by

\[
C \int_{\mathbb{R}^+} \sum_{m,n} \left[ e^{|\rho|(m+n)} \int_0^{\delta_m} \int_0^{\varepsilon_n} G^{**}(x, y, u) \, dy \, dx \right] e^{|\rho|u} \, du.
\]

Let

\[
S(x, y) = \sum_{m,n} \left[ e^{|\rho|(m+n)} \chi_{\delta_m}(x) \chi_{\varepsilon_n}(y) \right],
\]

where \( \chi_{\delta_m}, \chi_{\varepsilon_n} \) are the characteristic functions of sets \((0, \delta_m), (0, \varepsilon_n)\). If \( m_x = \max\{m : \delta_m > x\} \) and \( n_y = \max\{n : \varepsilon_n > y\} \) then \( S(x, y) \leq Ce^{|\rho|(m_x+n_y)} \). Clearly \( F^*(x) \geq e^{|\rho|m_x}, H^*(y) \geq e^{|\rho|n_y} \); therefore the expression (5.18) is dominated by

\[
C \int_{\mathbb{R}^+} \int_0^1 \int_0^1 F^*(x)H^*(y)G^{**}(x, y, u)e^{|\rho|u} \, dy \, dx \, du.
\]

One can deal with the second of the two expressions in (5.13) in a similar way; therefore

\[
\int \int_{G \times G} f(z)g(z^{-1}z')h(z') \, dz' \, dz
\]

\[
\leq C \int_{\mathbb{R}^1} \int_0^1 \int_0^1 F^*(x)H^*(y)G^{**}(x, y, u)e^{|\rho|u} \, dy \, dx \, du.
\]

**Step 4. Final estimates.** Let \( K \) be a suitable constant (to be chosen later) and let \( \mathcal{U} = \{(x, y, u) : F^*(x)H^*(y) \leq Ke^{|\rho|u}\} \) and \( \mathcal{V} = \{(x, y, u) : F^*(x)H^*(y) \geq Ke^{|\rho|u}\} \). By (5.15),

\[
\int_{\mathcal{U}} F^*(x)H^*(y)G^{**}(x, y, u)e^{|\rho|u} \, dy \, dx \, du
\]

\[
\leq \int_{\mathbb{R}^+} \int_0^1 \int_0^1 K G^{**}(x, y, u)e^{2|\rho|u} \, dy \, dx \, du \leq CK\|g\|_{L^2,1}^2.
\]

Using Lemma 7(a), (5.14) and the fact that \( G^{**}(x, y, u) \leq 1 \) one has
\[ \int_V F^*(x)H^*(y)G^{**}(x, y, u)e^{\rho|u|} \, dy \, dx \, du \leq C \int_0^1 \int_0^1 \left[ \frac{F^*(x)H^*(y)}{K} \right]^2 \, dy \, dx \leq C \|f\|_{L^2,1}^2 \|h\|_{L^2,1}^2. \]

Finally one lets \( K = (\|g\|_{L^2,1})^{-1} (\|f\|_{L^2,1} \|h\|_{L^2,1}) \) and the theorem follows.

### 6. A general rearrangement inequality

We will now extend the rearrangement inequality (5.19) to the case when \( f, g, h \) are arbitrary measurable functions (not just characteristic functions of sets). For any measurable function \( f : G \to \mathbb{R}_+ \) we define the function \( F^* : [0, 1] \to \mathbb{R}_+ \) by the following procedure: first, let \( \tilde{f} : K \times (0, \infty) \to \mathbb{R}_+ \) be defined, for almost every \( k \in K \), as the usual nonincreasing rearrangement of the function \( f_k : N \times A \to \mathbb{R}_+, f_k(\pi a) = f(\pi ak) \) with respect to the measure \( e^{2\rho(\log a)} \, da \). Using the function \( \tilde{f} \) we define the function \( \tilde{F} : (0, 1] \times (0, \infty) \to \mathbb{R}_+ \): for each \( r > 0 \) fixed, the function \( \tilde{F}(., r) \) is the usual the nonincreasing rearrangement of the function \( k \to \tilde{f}(k, r) \). Finally let

\[ (6.1) \quad F^*(x) = \frac{1}{2} \int_0^\infty \tilde{F}(x, r) r^{-1/2} \, dr \]

be the \( L^2,1 \) norm of the function \( r \to \tilde{F}(x, r) \). Notice that this definition of the function \( F^* \) agrees with our earlier definition if \( f \) is a characteristic function.

**Theorem 8.** If \( f, g, h : G \to \mathbb{R}_+ \) are measurable functions then

\[ (6.2) \quad \int \int_{G \times G} f(z)g(z^{-1}z')h(z') \, dz' \, dz \leq C \int_{\mathbb{R}_+} \int_0^1 \int_0^1 F^*(x)H^*(y)G^{**}(x, y, u)\phi(u) \, dy \, dx \, du, \]

where \( G^{**} : (0, 1] \times (0, 1] \times \mathbb{R}_+ \to \mathbb{R}_+ \) is the double rearrangement of the function \((k, k', u) \to g(k^{-1}a(u)k')\) (the same definition as before), \( F^* \) and \( H^* \) are defined in the previous paragraph and \( \phi(u) = u^{m_1 + m_2} \) if \( u \leq 1 \) and \( \phi(u) = e^{\rho|u|} \) if \( u \geq 1 \).

**Proof of Theorem 8.** Notice that

\[ \phi(u) \approx \sup_{r \in [-u, u]} e^{-|\rho|r} \int_{s \leq r, |s| \leq u} \psi(u, s)e^{\rho|s|} \, ds. \]

Notice also that if \( f \) and \( h \) are characteristic functions of sets then (6.2) is equivalent to (5.19). If \( f, h \) are simple positive functions, one can write (uniquely up to sets of measure zero) \( f = \sum c_i f_i, h = \sum d_j h_j \), where \( c_i, d_j > 0 \) and \( f_i \)
and \(h_j\), are characteristic functions of sets \(U_i\) and \(V_j\) with the property that for all \(i\) and \(j\) one has \(U_{i+1} \subset U_i\) and \(V_{j+1} \subset V_j\). Simple manipulations involving rearrangements show that \(F^* = \sum_{i=1}^{M_1} c_i F_i^*\) and \(H^* = \sum_{j=1}^{M_2} d_j H_j^*\) (this explains the reason why we chose the apparently complicated definition of the function \(F^*\) in (6.1)), and (6.2) follows by summation. Finally, a standard argument shows that (6.2) holds for arbitrary measurable functions \(f\), \(g\) and \(h\) for which the right-hand side integral in (6.2) converges.

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**References**

[1] J.-Ph. Anker, \(L_p\) Fourier multipliers on Riemannian symmetric spaces of the noncompact type, *Ann. of Math.* **132** (1990), 597–628.

[2] A. Cordoba and R. Fefferman, A geometric proof of the strong maximal theorem, *Ann. of Math.* **102** (1975), 95–100.

[3] M. Cowling, The Kunze-Stein phenomenon, *Ann. of Math.* **107** (1978), 209–234.

[4] M. Cowling, Herz’s “principe de majoration” and the Kunze-Stein phenomenon, in *Harmonic Analysis and Number Theory*(Montreal, PQ, 1996), 73–88, *CMS Conf. Proc.* **21**, A.M.S., Providence, RI, 1997.

[5] J.-L. Clerc and E. M. Stein, \(L^p\)-multipliers for noncompact symmetric spaces, *Proc. Natl. Acad. Sci. USA* **71** (1974), 3911–3912.

[6] S. Helgason, *Geometric Analysis on Symmetric Spaces*, A.M.S., Providence, RI, 1994.

[7] A. D. Ionescu, A maximal operator and a covering lemma on non-compact symmetric spaces, *Math. Res. Lett.* **7** (2000), 83–93.

[8] T. H. Koornwinder, Jacobi functions and analysis on noncompact semisimple Lie groups, in *Special Functions: Group Theoretical Aspects and Applications*, 1–85, *Math. Appl.* **8**, Marcel Dekker, New York, 1984.

[9] N. Lohoué and T. Rychener, Some function spaces on symmetric spaces related to convolution operators, *J. Funct. Anal.* **55** (1984), 200–219.

[10] R. A. Kunze and E. M. Stein, Uniformly bounded representations and harmonic analysis of the \(2 \times 2\) unimodular group, *Amer. J. Math.* **82** (1960), 1–62.

[11] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Math. Series, No. 32, Princeton Univ. Press, Princeton, NJ, 1971.

[12] J.-O. Strömberg, Weak type \(L^1\) estimates for maximal functions on noncompact symmetric spaces, *Ann. of Math.* **114** (1981), 115–126.

[13] N. J. Weiss, Fatou’s theorem for symmetric spaces, in *Symmetric Spaces* (Short Courses, Washington Univ., St. Louis, MO, 1969–1970), 413–441, *Pure and Appl. Math.* **8**, Marcel Dekker, New York, 1972.

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