Calculation method of natural characteristics of beam element considering initial state

Jiawei Cui¹, Ailan Che¹*

¹ ChinaShanghai Key Laboratory for Digital Maintenance of Buildings and Infrastructure, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, 800 Dongchuan-Road, Shanghai 200040, China

*Corresponding author’s e-mail: alche@sjtu.edu.cn

Abstract. Natural frequencies and modal shape play an important role in the analysis of the dynamic characteristics of long-span bridges including wind resistance and seismic. The initial static load, which caused by overloading, etc., has a certain impact on the natural characteristics. Therefore, the initial effect of static load on the inherent characteristics of the beam is studied. The vibration control equation of beam with initial state is obtained by using the energy conservation equation and Hamilton principle. Under different boundary conditions, the analytical solution of the natural frequency considering the initial state, and the influence law of the initial static load and the height-span ratio on the natural characteristics are obtained. The results show that the increase in the deflection of the beam caused by the static load will increase the stiffness of the beam, and further lead to the change of the natural characteristics of the beam. Therefore, in the analysis of the natural characteristics of the bridge, the influence of the initial state caused by static load should be fully considered.

1. Introduction

In the rapid development of bridge engineering, long-span bridges with a span of more than 100 m have been used more and more widely due to their many advantages [1]. While undertaking the main transportation mission, the long-span bridge is also under the huge test of the complex construction and operation environment. Once damaged due to earthquake, strong wind, overload, etc., it will cause huge loss of life, property and economic loss [3]. Therefore, the influence of these factors should be fully considered in the analysis of dynamic characteristics related to bridge construction and design. For example, the influence of internal damping force of the structure should be considered in wind resistance analysis [4], the natural vibration period and mode of structure should be considered in vibration control analysis [5], and the vibration modal test should be used in the damage test of the bridge [6]. In the analysis of the dynamic characteristics of long-span bridges, the natural frequencies and natural modes of beam elements are of great significance [7]. In terms of the seismic design of the bridge, the natural frequencies of the beam should be avoided to be the same as the frequency of the seismic wave to reduce the resonance effect. In the bridge damage test, the damage condition of the structure is evaluated by measuring the change of natural frequency [8]. Based on the wide application of natural frequencies and natural modes of beams, how to accurately analyse them has become a very important issue. In the classical mechanic, the natural frequency of the beam is mainly calculated according to the following formula [9]:

\[
\omega_n = \sqrt{\frac{k}{m}}
\]

where \(\omega_n\) is the natural frequency, \(k\) is the stiffness, and \(m\) is the mass.
Where, the two factors are mainly considered: the stiffness of the beam structure \( k \) and the quality of the beam system \( m \), without considering the influence of other factors. At present, the vibration analysis of beams is mainly based on Bernoulli-Euler beam theory [10] and Timoshenko beam theory [11]. The main difference between the two is that Euler beam theory assumes that the bending deformation of the beam unit is the main deformation, and does not consider the effect of shear deformation. It is suitable for solid web beams with very small height-span ratios. Timoshenko beam theory considers shear deformation, and is suitable for beams with large height-span ratio.

With the development of structural mechanics and the research on the natural characteristics of beam elements, many scholars have gradually realized that the natural frequency of beam will be affected by many factors, such as: dynamic load on the beam, damage to the internal structure, initial deflection due to its own gravity, temperature, etc. According to the vibration control equation of Euler beams, Takabatake. analysed the influence of the uniform static load caused by snow load on the simply-supported beam. And the relationship between natural frequency and initial deflection is obtained [12]. Kelly, J.M. obtained the theoretical solution to the relationship between the natural characteristics and the initial rotation of Euler beams [13]. When investigating the mechanical properties of turbine blades, Yin, Y.Q. analysed the changes in the natural characteristics of the cantilever beam that there is a concentrated static load on it based on the Timoshenko beam theory, and the relationship between the natural frequency and the concentrated static load is obtained [14]. It can be seen that these studies focused on the influence of various factors on the natural characteristics of the beam. However, there are few studies in them on the natural characteristics of the beam considering the initial effect.

According to the Hamilton variational principle and Euler beam theory, the influence of initial deflection and initial stress on the change of natural frequency and mode curve when there is concentrated static load in the upper span of a fixed beam is studied, and a theoretical solution of the relationship between them is obtained. The research results can provide a support for applying the natural characteristics considering static load effects to the design of the seismic performance and vibration characteristics of long-span bridges.

2. The vibration control equation considering the initial state
Due to the initial deflection generated under static load, the natural frequency and mode shape of the beam element may change. According to the energy balance equation and the Hamiltonian principle, the vibration governing equation with initial deflection is derived. By applying different forms of static loads and boundary conditions, the natural frequencies and vibration modes of simply supported beams and fixed beams under uniformly distributed static loads and concentrated static loads can be obtained. Based on the above theoretical derivation results, an example is calculated to verify the obtained theory. Figure 1 shows the vibration of a beam with a length of \( l \) under the initial concentrated static load. Among them, the x-axis is the axial direction of the beam, and the z-axis is the vibration direction of the beam. \( p(x,t) \) and \( f(x) \) are the dynamic load and concentrated static load of the beam respectively. \( w(x,t) \) and \( w_0(x) \) are the deflection produced by them.

![Figure 1. The vibration of beam under initial load](image)

The kinetic energy of the beam during vibration is
\begin{equation}
T = \int_0^l \frac{1}{2} \rho A \left( \frac{\partial^2 w}{\partial t^2} \right) dx
\end{equation}

Where, $\rho$ is the density of the beam and $A$ is the cross-sectional area of the beam.

The relationship between strain and deflection during beam vibration is as follows:
\begin{equation}
\varepsilon_x = -z \frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \text{(Non-linear)}
\end{equation}
\begin{equation}
\varepsilon_{ox} = -z \frac{\partial^2 w}{\partial x^2} \text{(Linear)}
\end{equation}

Strain energy produced by $p(x,t)$:
\begin{equation}
U_p = \frac{1}{2} \iiint \sigma_x \varepsilon_x dx dy dz
\end{equation}

Initial strain energy produced by $f(x)$:
\begin{equation}
U_0 = \iiint \sigma_0 \varepsilon_{ox} dx dy dz
\end{equation}

Among them, $\sigma_x$ and $\sigma_{ox}$ are the bending stress produced by dynamic load $p(x,t)$ and static load $f(x)$ respectively.

Thus, the total strain energy can be written as:
\begin{equation}
U = U_p + U_0 = \int_0^l \left[ \frac{1}{2} EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \frac{1}{4} E A \left( \frac{\partial w}{\partial x} \right)^2 + EI \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] dx
\end{equation}

The potential energy generated by external load work is:
\begin{equation}
V = -\int_0^l [f(x) + p(x,t)]w(x,t) dx
\end{equation}

According to the Hamilton variational principle [15], it is determined by the following formula:
\begin{equation}
\delta E = \delta \int_0^l (T - U - V) dt = 0
\end{equation}

Then the boundary conditions are combined to get:
\begin{equation}
\begin{align*}
\delta E &= \int_0^l (T - U - V) dt \\
&= \int_0^l \left[ \int_0^l \left( -\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 w}{\partial x^2}) + \frac{1}{2} \frac{\partial}{\partial x} \left( E A \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \right) \right) \right] \\
&\quad - \frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 w_0}{\partial x^2}) + f(x,t) \right] \delta w dx dt
\end{align*}
\end{equation}

Considering the deflection caused by static load $f(x)$ alone, there is:
\begin{equation}
\frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 w_0}{\partial x^2}) - f(x) = 0
\end{equation}

Then the vibration control equation of the beam can be obtained as:
\begin{equation}
\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} (EI \frac{\partial^2 w}{\partial x^2}) - \frac{1}{2} \frac{\partial}{\partial x} \left( E A \left( \frac{\partial w}{\partial x} \right)^2 \right) = p(x,t)
\end{equation}

The boundary conditions are: $w(x,t)$, $\frac{\partial^2 w}{\partial x^2}$, and true when $x = 0$ or $x = l$.

In order to solve the natural frequency, the governing equation of the beam under free vibration should be solved:
\[
\rho A \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 (EI \frac{\partial^2 w}{\partial x^2})}{\partial x^2} - \frac{1}{2} \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} \left( EA \frac{\partial w}{\partial x} \right) \right] = 0
\]  
(12)

By using the method of separating variables and setting \( w(x,t) = \sum_{i=1}^{\infty} y_i(x) q_i(t) \), the above equation can be reduced to:

\[
D \ddot{q} + Mq = 0
\]  
(13)

Among them,

\[
D_q = \int_0^l \rho A y_i y_i \, dx, \quad M_q = \int_0^l E I y_i y_i \, dx - \frac{1}{2} \int_0^l EA'(w)^2 y_i y_i \, dx - \int_0^l E A w_i w_i y_i y_i \, dx
\]  
(14)

Where, \( w_0 \) is the deflection that considers the initial static load.

The eigenvalues of the above equations are the natural frequencies of each order of the beam.

Substituting the natural frequencies of each order into equation (12), the corresponding mode curve can be obtained.

3. The influence of initial state on natural characteristics of beam

3.1 The influence of span on natural frequency of beam

It is assumed that the geometry and material parameters of the fixed beam are the same as those of the Minpu Bridge, which is as shown in Table 1. The initial uniform load \( P_0 \) on the beam varies from \( 0.1 \rho gh \) to \( \rho gh \) and the cross section of the beam is rectangular. According to the actual situation of the Minpu Bridge, the section height is set to 10 m, the section width is set to 40 m, the beam span varies between 100 m and 1000 m, and \( \Delta l = 0.1 \) m.

| Table 1. Beam geometry, load and material parameters |
|-----------------------------------------------|
| Density \( \rho \) | 2600 kg/m³ |
| Elastic Modulus \( E \) | 345 GPa |
| Span \( l \) | 100m-1000m (\( \Delta l = 0.1 \) m) |
| Section height \( h \) | 10 m |
| Section width \( b \) | 40 m |
| Uniform load \( P_0 \) | \( p_0 = 0.1 \rho gh \sim \rho gh \) |

Through the analysis and calculation with Matlab, the influence of the change of beam span on the natural frequency can be obtained as shown in Figure 2. In this figure, the abscissa is the span \( l \) of the beam, and the ordinate is the error \( \Delta \omega \ (\Delta \omega = \omega - \omega_0) \) of the first-order natural frequencies \( \omega \) and \( \omega_0 \) calculated when considering \( P_0 \) and not considering \( P_0 \).

Figure 2. Variation of natural frequency difference of beam structure with span under static load

It can be seen from the curve in the Figure 2 that: (1) As the span increases, the influence of the initial static load on the natural frequency of the beam increases. (2) When the uniform load is constant
and the span of the beam is large, the uniform load has a greater influence on the natural frequency of the beam. Compared with the result obtained without considering the uniform load, the natural frequency of the beam increases significantly. (3) As the static load on the beam increases, the initial deflection has a greater impact on the natural frequency. The uniform load $P_0$ on the beam varies from $0$ to $-pgbh$.

3.2 The effect of static load on natural frequency of beam

When the span of the beam is constant, the influence of the static load on the beam on the natural frequency is considered. The beam geometry, load and material parameters are shown in Table 2. The beam spans are 100 m, 200 m, 300 m, 400 m and 500 m. The cross-section of the beam is rectangular, and the cross-section width and height are 40×10 m. Numerical analysis shows that as $P_0$ increases, the first-order natural frequency of the beam changes as shown in Figure 3. In this figure, the abscissa is the static load $P_0$ on the beam, the ordinate is the error $\Delta \omega (\Delta \omega = \omega - \omega_0)$ of the first-order natural frequencies $\omega$ and $\omega_0$ calculated when considering $P_0$ and not.

Table 2. Characteristics of beam

| Density $\rho$ | 2600 kg/m³ |
|----------------|-------------|
| Elastic Modulus $E$ | 345 GPa |
| Span $l$ | 100 m, 200 m, 300 m, 400 m, 500 m |
| Section height $h$ | 40 m |
| Section width $b$ | 10 m |

Through the above curve, it can be obtained that when the span of the beam is constant, As the uniformly distributed load on the beam increases, the natural frequency of the beam tends to increase. And as the span increases, the influence of the static load on the beam on the natural frequency increases.

Figure 3. The variation of the natural frequency difference with the static load on the beam.

In summary, it can be seen that the span and the size of the static load affect the natural frequency of the beam. Taking a fixed beam as an example, when other factors remain unchanged, as the span of the beam increases, the influence of the initial static load on the natural frequency increases. Similarly, when other factors remain unchanged, as the static load on the beam increases, the influence of the initial static load on the natural frequency increases. Especially for long-span beams, the initial deflection generated by the initial static load will cause significant changes in the natural frequency.

4. Conclusion

This study explores the influence of the initial deflection of the beam on the natural frequency. Through theoretical analysis based on Euler beam theory, the vibration control equation of the beam and the theoretical solutions of natural frequency are obtained when the initial deflection exists. The response curves of each order of the beams are obtained by further analysis. The following conclusions can be drawn:
(1) When the other parameters of the beam remain unchanged, as the static load on the beam increases, the initial deflection of the beam increases and the geometric configuration changes, which cause the stiffness of the beam to increase. So, the natural frequency value of each order of the beam also increases, and the corresponding mode curve of each order of the beam also shifts.

(2) When the other parameters of the beam remain unchanged, as the height-span ratio of the beam becomes smaller, the beam is more likely to produce greater deflection and stiffness, which makes the initial static load's influence on the natural frequency greater.

(3) As the span increases, the influence of the initial static load on the natural frequency of the beam increases. When the uniform load is constant and the span of the beam is large, the uniform load has a greater influence on the natural frequency of the beam. Compared with the result obtained without considering the uniform load, the natural frequency of the beam increases significantly.

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