Abstract

We show how to relate the parafermions that occur in the $W_3$ string to the standard construction of parafermions. This result is then used to show that one of the screening charges that occurs in parafermionic theories is precisely the non-trivial part of the $W_3$ string BRST charge. A way of generalizing this pattern for a $W_N$ string is explained. This enables us to construct the full BRST charge for a $W_{2,N}$ string and to prove the relation of such a string to the algebra $W_{N-1}$ for arbitrary $N$. We also show how to calculate part of the BRST charge for a $W_N$ string, and we explain how our method might be extended to obtain the full BRST charge for such a string.
The discovery of extensions of the Virasoro algebra, known as \(W\) algebras, has lead to the study of \(W\) string theories based on these algebras. In the absence of any clear geometrical understanding of \(W\) algebras, a crucial role is played by the BRST charge in such string theories. A review of recent work on \(W\) string theory can be found in [1].

One striking fact that has emerged from a study of the \(W_3\) string is that this theory is closely related to the Ising model. It was recently shown that the Ising model that arises provides a two-scalar realization of the \(c = 1/2\) parafermion theory corresponding to the coset \(SU(2)_{N}/U(1)\) [2]. A consequence of this fact is that the \(W_3\) string has embedded within it a non-linear \(W\) algebra containing an infinite number of generators \(W_s\), one for each spin \(s = 2, \ldots, \infty\). In addition the parafermions could be used to generate the states in the cohomology of the BRST charge.

The two-scalar realization of the parafermion algebra found in the \(W_3\) string differs from the usual realization of this algebra. In this paper, starting from the Wakimoto construction of the \(SU(2)\) level \(k\) currents [3] and using an alternative bosonization for the bosonic spin 0 and spin 1 fields used in this construction, we obtain new expressions for the parafermions. For the case \(k = 2\) these coincide with those found in the \(W_3\) string. For general values of \(k\) we show how to relate the two different realizations of the parafermion algebra, and this enables us to apply results from the theory of parafermions to the \(W_3\) string. In particular, we express the known screening charges of the parafermion theory in terms of the variables of the \(W_3\) string. One of these coincides, for \(k = 2\), with the screening charge \(S\) that was used in [4] to generate physical states of the \(W_3\) string from three basic physical states and that was seen to be needed to calculate scattering amplitudes [4]. Another of the screening charges of the parafermion theory turns out to give the non-trivial part of the \(W_3\) BRST charge.

We argue that this connection between parafermions and \(W\) strings can be generalised, and give evidence that the BRST charge for the \(W_N\) string has a natural decomposition [5,2,6] of the form \(Q(W_N) = Q_0 + Q_1\), where \(Q_1\) is just the second of the two screening charges discussed above, evaluated for \(k = N - 1\). We conjecture further that, using the series of cosets \(SU(N - r + 1)/SU(N - r) \times U(1)\), \(Q(W_N)\) has a decomposition [5,2,6] of the form \(Q_0^{(r)} + Q_1^{(r)}\), where now \(Q_1^{(r)}\) is a
screening charge for the above coset constructed from the $2(N - r)$ bosonic fields that are used to describe this coset in the Wakimoto realization [7].

We can also use this procedure to construct an explicit expression for the BRST charge for the spin-$(2, N)$ string [8] for arbitrary values of $N$. Using this method it follows immediately that such a string theory will contain the algebra $W_{N-1}$ [9].

We begin with the Wakimoto realization of the $SU(2)$ level $k$ affine Lie algebra [3], which is given in terms of a scalar field $\phi$ and a spin-$(1,0)$ pair of bosonic ghosts $\beta, \gamma$ by

$$J^+ = -\beta$$
$$J^3 = \sqrt{2} \beta \gamma - i \sqrt{k + 2} \partial \phi$$
$$J^- = \beta \gamma^2 - i \sqrt{2(k + 2)} \partial \phi \gamma + k \partial \gamma.$$ (1)

The operator product expansions for the fields occurring in these currents are

$$\phi(z)\phi(w) = -\log(z - w)$$ (2)
and

$$\beta(z)\gamma(w) = -\gamma(z)\beta(w) = -\frac{1}{z - w},$$ (3)

and the energy-momentum tensor is

$$T = -\frac{1}{2} (\partial \phi)^2 - \beta \partial \gamma + \frac{i}{\sqrt{2(k + 2)}} \partial^2 \phi.$$ (4)

In order to bosonize the fields $\beta, \gamma$ we first introduce a spin-(1,0) fermionic ghost pair $(\xi, \zeta)$ and a bosonic field $\eta$, and write

$$\beta = \partial \xi e^{-\eta}, \quad \gamma = \xi e^\eta,$$ (5)

where

$$\xi(z)\zeta(w) = \frac{1}{z - w}.$$ (6)

Then $\xi$ and $\zeta$ can then be expressed in terms of a bosonic field $\chi$

$$\xi = e^{i\chi}, \quad \zeta = e^{-i\chi},$$ (7)

with the result that the fields $\beta, \gamma$ can be expressed in terms of the two bosonic fields $\chi$ and $\eta$ as follows:

$$\beta = -i \partial \chi e^{-i\chi - \eta}, \quad \gamma = e^{i\chi + \eta}.$$ (8)
In terms of $\chi$ and $\eta$, the energy-momentum tensor for the ghosts is

$$T_{\beta,\gamma} = -\frac{1}{2}(\partial\chi)^2 - \frac{1}{2}(\partial\eta)^2 - \frac{i}{2}\partial^2\chi - \frac{1}{2}\partial^2\eta. \quad (9)$$

In order to obtain the parafermions $\psi_{\pm 1}$ we divide out by the $U(1)$ subalgebra generated by $J^3$. We first observe that, having bosonized the ghost fields, we can write

$$J^3 = \sqrt{2}\partial\eta - i\sqrt{k+2}\partial\phi \quad (10)$$

so that $J^3$ is the derivative of some field. This field can then be used to construct parafermions whose operator product with $J^3$ is regular, namely

$$\psi_1 = -J^+ \exp\left\{ -\frac{\sqrt{2}}{k} \left( \sqrt{2}\eta - i\sqrt{(k+2)}\phi \right) \right\}$$

$$\psi_{-1} = -k^{-1}J^- \exp\left\{ \frac{\sqrt{2}}{k} \left( \sqrt{2}\eta - i\sqrt{(k+2)}\phi \right) \right\} \quad (11)$$

It is convenient to introduce two orthogonal scalar fields $\phi_1$ and $\phi_2$ by

$$\phi_1 = \chi$$

$$\phi_2 = \sqrt{\frac{k+2}{k}}\eta - i\sqrt{\frac{2}{k}}\phi \quad (12)$$

in terms of which the parafermions become

$$\psi_1 = -i\partial\phi_1 \exp\left\{ -i\phi_1 - \sqrt{\frac{k+2}{k}}\phi_2 \right\}$$

$$\psi_{-1} = -\frac{1}{k} \left( i(k+1)\partial\phi_1 + \sqrt{k(k+2)}\partial\phi_2 \right) \exp\left\{ i\phi_1 + \sqrt{\frac{k+2}{k}}\phi_2 \right\}. \quad (13)$$

In deriving these expressions we have made use of the formulae $\beta\gamma = \partial\eta$ and $\beta\gamma^2 = (i\partial\chi + 2\partial\eta)e^{i\chi + \eta}$. The energy-momentum tensor for the fields $\phi_1$ and $\phi_2$ is easily seen to be

$$T = -\frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{i}{2}\partial^2\phi_1 - \frac{1}{2}\partial^2\phi_2. \quad (14)$$

As noted in reference [2], the parafermions that occur in the $W_3$ string are not of this form. In fact there is another way to bosonize the ghost fields $\beta$ and $\gamma$, obtained by taking

$$\beta = \xi e^{-\eta}, \quad \gamma = -\partial\xi e^\eta \quad (15)$$
in place of equation (5). The pair \((\xi, \zeta)\) is bosonized exactly as before, so that in terms of the scalars \(\chi, \eta\) with energy-momentum tensor given by equation (9) we have
\[
\beta = e^{i\chi - \eta}, \quad \gamma = i\partial \chi e^{-i\chi + \eta}.
\] (16)

We can now follow the same procedure as before to construct parafermions. In this case the products of \(\beta\) and \(\gamma\) that we need are \(\beta \gamma = \partial \eta\), \(\gamma^2 = (i\partial^2 \chi - (\partial \chi)^2)e^{-2i\chi + 2\eta}\) and \(\beta \gamma^2 = (i\partial^2 \chi + (\partial \chi)^2 + 2i\partial \chi \partial \eta)e^{-i\chi + \eta}\). Using these we obtain the expressions for the parafermions
\[
\psi_1 = \exp \left\{ i\phi_1 - \sqrt{\frac{k+2}{k}} \phi_2 \right\}
\]
\[
\psi_{-1} = -\frac{1}{k} \left\{ i(k+1)\partial^2 \phi_1 + (k+1)(\partial \phi_1)^2 + i\sqrt{k(k+2)}\partial \phi_1 \partial \phi_2 \right\}
\times \exp \left\{ -i\phi_1 + \sqrt{\frac{k+2}{k}} \phi_2 \right\}
\] (17)

Here \(\phi_1\) and \(\phi_2\) have exactly the same forms in terms of \(\chi, \eta\) and \(\phi\) as previously, in equation (12).

In order to relate these parafermions to those found in the \(W_3\) string, we need to transform \(\phi_1, \phi_2\) into the bosonized ghost \(\rho\) and the scalar field \(\varphi\) occurring in the \(W_3\) string. For this purpose it would be sufficient to take \(k = 2\), but there is some advantage to working with general \(k\) at this stage. We thus consider fields \(\rho\) and \(\varphi\) with energy-momentum tensor
\[
T = -\frac{1}{2}(\partial \varphi)^2 - \frac{1}{2}(\partial \rho)^2 - \frac{2k+3}{2}\sqrt{\frac{k}{k+2}}\partial^2 \varphi + \frac{2k+1}{2}i\partial^2 \rho.
\] (18)

The motivation for this expression comes from considering the \(WA_k = W_{k+1}\) string. This can be built, using the Miura transformation, from scalar fields \(\varphi_2, \ldots, \varphi_{k+1}\) and corresponding ghosts \(b_2, c_2, \ldots, b_{k+1}, c_{k+1}\), and the ghosts can then be bosonized in terms of fields \(\rho_2, \ldots, \rho_{k+1}\) in the usual way. The energy-momentum tensor for \(\varphi_j, \rho_j\) is given by
\[
T_j = -\frac{1}{2}(\partial \varphi_j)^2 - \frac{1}{2}(\partial \rho_j)^2 - \frac{2k+3}{2}\sqrt{\frac{(j-1)j}{(k+1)(k+2)}}\partial^2 \varphi_j + i\frac{(j-1)j}{2}\partial^2 \rho_j.
\] (19)

It was observed in [2] that the fields \(\varphi_{k+1}, \rho_{k+1}\) together contribute \(2(k-1)/k + 2\) to the central charge, which is precisely the central charge for the parafermion
system discussed above. The energy-momentum tensor in equation (18) is just that for the fields \( \varphi_{k+1} \) and \( \rho_{k+1} \), except that we have written \( \varphi \) and \( \rho \) instead of \( \varphi_{k+1} \) and \( \rho_{k+1} \).

The linear transformation that takes the energy-momentum tensor of equation (14) into that of equation (18) is

\[
\begin{align*}
\phi_1 &= -(k + 1) \rho - i \sqrt{k(k + 2)} \varphi \\
\phi_2 &= -i \sqrt{k(k + 2)} \rho + (k + 1) \varphi.
\end{align*}
\] (20)

Setting \( k = 2 \) and substituting the above into equation (17), we indeed recover the parafermions found in the \( W_3 \) string.

The transition between the old and new bosonized forms of the ghosts \( \beta, \gamma \) is obtained by making the replacements

\[
-i \partial \chi e^{-i \chi} \rightarrow e^{i \chi}, \quad e^{i \chi} \rightarrow i \partial \chi e^{-i \chi},
\] (21)

and it is not difficult to see that this takes the parafermions of equation (13) into those of equation (17).

One advantage of making contact with the well-known formulation of the parafermions is that one can exploit the knowledge that has been built up in that formulation. In particular it is known [10] that there exist three screening charges \( Q_i = \oint j_i \, dz \), where

\[
\begin{align*}
j_1 &= e^{i(k+1)\phi_1 + \sqrt{k(k+2)}\phi_2} \\
j_2 &= e^{i\phi_1} \\
j_3 &= -i \partial \phi_1 e^{-i \phi_1} - \sqrt{\frac{k}{k+2}} \phi_2.
\end{align*}
\] (22)

Using the replacement rules given above, the latter two currents become

\[
\begin{align*}
j_2 &= i \partial \phi_1 e^{-i \phi_1} \\
j_3 &= e^{i \phi_1} - \sqrt{\frac{k}{k+2}} \phi_2.
\end{align*}
\] (23)

While \( j_2 \) thus becomes a total derivative, so that the corresponding charge is somewhat trivial, the current \( j_3 \) can be rewritten in terms of the fields \( \rho, \phi \) as

\[
j_3 = \exp \left( -i \rho + \sqrt{\frac{k}{k+2}} \phi \right),
\] (24)

6
For $k = 2$ this is precisely the current for the screening charge $S$ that was used to relate the different states in the cohomology of the $W_3$ BRST charge.

In order to find the new form of $j_1$ we need the formula

\[ e^{in\chi} \rightarrow (-1)^{n+1} e^{-i(n+1)\chi} \partial^n e^{i\chi}, \tag{25} \]

which we prove by induction. For $n = 1$ this is precisely equation (21). We now observe that $e^{i(n+1)\chi}$ is the leading order term in the operator product expansion of $e^{i\chi}$ with $e^{in\chi}$, so for the inductive step we need to consider the operator product of $-\partial e^{-i\chi} = i\partial e^{-i\chi}$ with the right hand side of equation (25). One way to do this is to note that

\[ e^{-i(n+1)\chi} \partial^n e^{i\chi}(w) = n! \left[ \oint_w e^{i\chi}, e^{-i(n+1)\chi}(w) \right]. \tag{26} \]

It is easy to show that $\oint e^{i\chi}$ commutes with $-\partial e^{-i\chi}$, so that the operator product we want can be written as

\[ -\partial e^{-i\chi}(z)(-1)^{n+1} e^{-i(n+1)\chi} \partial^n e^{i\chi}(w) \]

\[ = (-1)^{n+1} n! \left[ \oint e^{i\chi}, -\partial e^{-i\chi}(z)e^{-i(n+1)\chi}(w) \right]. \tag{27} \]

The required result follows immediately from evaluating the operator product within the commutator.

Using the above result we find that, in the new formulation of the parafermions,

\[ j_1 = (-1)^{k+1} e^{i\rho} P_{k+1}(\phi_1), \tag{28} \]

where $P_{k+1}$ is a differential polynomial given by

\[ P_{k+1}(\phi_1) = e^{-i\phi_1} \partial^{k+1} e^{i\phi_1}. \tag{29} \]

The corresponding charge is then

\[ Q_1 = \oint dz e^{i\rho} P_{k+1}(\phi_1). \tag{30} \]

We observe that, in the old formulation of the parafermions, the charge obtained by integrating $j_1$ is manifestly nilpotent, and it follows that $Q_1$ in equation (30) must also square to zero. Indeed, for $k = 1$ we can evaluate equation (30) to obtain

\[ Q_1 = -6 \oint e^{i\rho} (T_\phi + T_\rho) \tag{31} \]
which is just the bosonized form of the BRST charge for a one-scalar string with matter energy-momentum tensor $T_\phi$ which has $c = 26$. The fact that the ghost energy-momentum tensor $T_\rho$ does not have the usual factor of $1/2$ relative to $T_\phi$ is a consequence of working with bosonized ghosts. Taking $k = 2$ we find that $Q_1$ is given by

$$Q_1 = 8\sqrt{2} \oint \left\{ 2(\partial \phi)^3 + \frac{21}{\sqrt{2}} \partial^2 \phi \partial \phi + \frac{19}{4} \partial^3 \phi - \frac{9i}{2} \partial^2 \rho \partial \phi 
- \frac{9}{2} (\partial \rho)^2 \partial \phi - \frac{7}{4\sqrt{2}} \partial^3 \rho - \frac{21}{4\sqrt{2}} \partial^2 \rho \partial \rho + \frac{7i}{4\sqrt{2}} (\partial \rho)^3 \right\}. \quad (32)$$

This is just the non-trivial part of the BRST charge for the $W_3$ string written using the variables introduced in reference [11].

It is clear that the above procedure enables us to calculate a nilpotent BRST charge for a spin-2 coupled to a spin-(k+1) system for any value of $k$—the non-trivial part of the BRST charge for such a system is simply given by equation (30). Furthermore, it follows from known results of parafermion systems that this spin-(2,k+1) system will contain a $W_k$ algebra with $c = 2(k-1)/(k+2)$. The generating fields of this algebra are obtained as coefficients of powers of $z-w$ arising in the operator product of $\psi_1$ with $\psi_{-1}$ [12]. Since the parafermions commute with the charge $Q_1$, so will the $W$ generators, and thus there will be an action of this $W$ algebra on the physical states of the theory.

It was conjectured in references [2,6] that the BRST charge $Q(W_N)$ for the Miura representation of $W_N$ can be written in the form

$$Q(W_N) = Q_0 + Q_1,$$  \quad (33)

where $Q_1$ is a function of only $\phi_N, b_N,$ and $c_N$, and where $Q_1^2 = 0$. It is natural to suppose that $Q_1$ is precisely the operator given in equation (30). For $W_2$, which is the Virasoro algebra, we have seen above that $Q_1$ is the usual BRST charge. For the $W_3$ string the BRST charge of Thierry-Mieg [13] can be written in the form of equation (33) by making a suitable field redefinition [11], and we have shown that $Q_1$ is indeed given by equation (30) with $k = 2$. More recently the BRST charge for $W_4$ has been found [14], and a decomposition of the form (33) has been found in reference [6]. It has been shown by Duke [15] that in this case also $Q_1$ is given by equation (30), with $k = 3$. 


It is natural to attempt to generalize this construction so as to obtain the full 
BRST charge for the $W_N$ string. As explained above, the $W_N$ string can be 
constructed starting from the Miura realization in terms of fields $\varphi_2, \ldots, \varphi_N$ and 
ghosts $\rho_2, \ldots, \rho_N$. From the energy-momentum tensors of these fields one finds [6] 
that the fields $\varphi_{r+1}, \ldots, \varphi_N$ and $\rho_{r+1}, \ldots, \rho_N$ give a contribution 

$$c^r_N = (r - 1) \left(1 - \frac{r(r + 1)}{n(n + 1)}\right)$$ (34) 

to the central charge. This is the central charge of one of the minimal models 
of the $W_r$ algebra. Whereas the usual Miura construction of the $W_r$ algebra 
involves $r - 1$ scalar fields, the above observation suggests that there should exist 
a realization in terms of $2(N - r)$ bosonic fields for the values of $c$ in equation (34). 
It may be possible to obtain such a realization by generalizing the construction of 
parafermions as the coset $SU(2)_k/U(1)$. In fact, the coset 

$$\frac{SU(N - r + 1)_r}{SU(N - r)_r \times U(1)}$$ (35) 

can be seen to have the central charge $c^r_N$, by using the formula $c = k \dim G/(k + h)$ for the central charge of a level-$k$ affine Lie algebra $G$ with Coxeter number $h$. Furthermore, the Wakimoto construction of $SU(2)_k$ has been generalized to 
$SU(n)_k$ giving a realization of $SU(n)_k$ in terms of $\dim SU(n)$ bosonic fields [7]. 
Consequently, the generalized parafermions corresponding to the coset of equation 
(35) would have a description in terms of $2(N - r)$ bosonic fields. The form 
of these parafermions is unknown, but we note that this coset has the same $W$ 
algorithm corresponding to it as $SU(r)_{N-r} \times SU(r)_1/SU(r)_{N-r+1}$ [16], which has 
a description in terms of $r - 1$ bosonic fields and provides the standard unitary 
representations of $W_r$ [17]. 

We therefore propose that it should be possible to construct some generalized 
parafermions from the coset of equation (35) which can be expressed in terms of 
the fields $\varphi_{r+1}, \ldots, \varphi_N$ and $\rho_{r+1}, \ldots, \rho_N$. These parafermions should generate $W_r$, 
and they should possess a screening charge $Q^{(r)}_1$ such that the BRST charge for 
the $W_N$ string takes the form 

$$Q(W_N) = Q^{(r)}_0 + Q^{(r)}_1$$ (36) 

with $(Q^{(r)}_1)^2 = 0$. Clearly the case $r = N - 1$ reduces to $SU(2)_{N-1}/U(1)$ which 
was discussed above, and for $r = 2$ we should recover the full screening charge of
the Miura realization of the $W_N$ string. In this way we would be able to obtain for the quantum case a series of nested BRST charges, analogous to the classical BRST charges obtained in reference [6]. Further details will be given elsewhere [18].

We have seen how the non-trivial part of the BRST charge for the $W_3$ string and for the $W_2,s$ string can be constructed using parafermions. It is of interest also to consider the action of the parafermions on the physical states of the $W_3$ string, contained in the cohomology of its BRST charge. These states can be found by acting with screening charges $S, S_1$ and $S_2$ and with the picture-changing operator $P$ on three basic states [4]. In the notation of [4], the three basic states are $V(a,0) = \phi(h,0)V_X(a)$, where $a = 1, 15/16, 1/2$, $h = 1 - a$, $\phi(h,0)$ is an Ising primary field of weight $h$, and $V_X(a)$ is a vertex operator constructed from the fields of the matter sector. We can then construct the following series of states:

$$V(15/16,m) = (S^3P)^mV(15/16,0), \quad V(a,m) = (S^3P)\bar{V}(a,m-1), \quad \bar{V}(a,m) = SPV(a,m)$$

for intercepts $a = 1$ and $a = 1/2$ [4]. We can also find series of conjugate states. Thus $V(1,n), \bar{V}(1,n), V(1/2,n)$ and $\bar{V}(1/2,n)$ have conjugates $P(S_1P)^n\bar{V}(1,0), \quad P(S_1P)^nV(1,0), \quad P(S_1P)^n\bar{V}(1/2,0)$ and $P(S_1P)^nV(1/2,0)$ respectively, while $V(15/16,n)$ has conjugate vertices $P(S_1P)^{n/2}V(15/16,0)$ for $n$ even and $P(S_1P)^{(n+1)/2}V(15/16,1)$ for $n$ odd. It is straightforward to verify that these conjugates indeed have the correct $\phi$ and $\rho$ momenta to give a non-zero scalar product. The low level conjugates were found in collaboration with B. Nilsson [19].

The $W_3$ string also includes the discrete physical state [20]

$$D(0) = (ce + \frac{5}{87}\sqrt{58i}\partial e e)e^{i(8iQ/7)}. \quad (37)$$

These vertices have the same $\phi$ momenta as $V(1,n)$, and we can create the following infinite number of discrete physical vertices

$$S^3P\bar{D}(n-1) = D(n), \quad \bar{D}(n) = (SP)D(n) \quad (38)$$

and similarly form their conjugates.

As outlined in reference [2], physical states can also be created using the parafermions $\psi_{\pm 1}$. The parafermion fields $\psi_{\pm 1}$ are given by $\psi_1 = \phi(1/2,1)$ and
\[ \psi_{-1} = \phi(1/2, 2), \] and, using the notation of reference [21], the parafermionic primary fields \( \phi_0, \phi_1 \) and \( \phi_{-1} \) are given by \( \phi(1, 1), \phi(1/16, 1) \) and \( \phi(1/16, 2) \) respectively. In fact, one can act on the basic state \( \phi(a, 0) \) with \( \psi_{-1} \) to obtain the states above, and with \( \psi_1 \) to obtain their conjugates. Let us first consider Ising weight \( 1/16 \) states. \( \psi_{-1}(z)\phi(1/16, n) \) has an operator product expansion in powers of \( (z-w)^{p+1/2} \), \( p \in \mathbb{Z} \), all the coefficients of which are annihilated by \( Q_1 \). For small enough \( n \) the most singular term in the OPE is \( (z-w)^{-1/2} \), and the coefficient of this term is a primary field of weight \( 1/16 \) which is \( \phi(1/16, n + 1) \). For larger \( n \) it is no longer the case that \( (z-w)^{-1/2} \) is the most singular term, but one can construct \( \phi(1/16, n + 1) \) out of the coefficient of \( (z-w)^{-1/2} \) by subtracting Virasoro descendents of the coefficients of \( (z-w)^{-r} \) for \( r \geq 1/2 \). A similar construction works for \( \psi_1 \) acting on \( P\phi(1/16, 0) \) to create the conjugates.

In a similar way we can obtain \( \phi(1/2, n) \) from the operator product expansion \( \psi_{-1}(z)\phi(0, n)(w) \), and \( \phi(0, n+1) \) from \( \psi_{-1}(z)\phi(1/2, n)(w) \). We can also get \( \phi(a, n) \) for \( a = 1/2, 0 \) in this way, and acting with \( \psi_1 \) gives the conjugate states. Further details of this construction will be given elsewhere.

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