A Survey on Compressive Sensing: Classical Results and Recent Advancements

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Abstract

Recovering sparse signals from linear measurements has demonstrated outstanding utility in a vast variety of real-world applications. Compressive sensing is the topic that studies the associated raised questions for the possibility of a successful recovery. This topic is well-nourished and numerous results are available in the literature. However, their dispersity makes it challenging and time-consuming for new readers and practitioners to quickly grasp its main ideas and classical algorithms, and further touch upon the recent advancements in this surging field. Besides, the sparsity notion has already demonstrated its effectiveness in many contemporary fields. Thus, these results are useful and inspiring for further investigation of related questions in these emerging fields from new perspectives. In this survey, we gather and overview vital classical tools and algorithms in compressive sensing and describe significant recent advancements. We conclude this survey by a numerical comparison of the performance of described approaches on an interesting application.

1 Introduction

In traditional sensing, i.e., uniform sampling, we need to densify measurements to obtain a higher-resolution representation of physical systems. But in applications like multiband signals with wide spectral ranges, the required sampling rate may exceed the specifications of the best available analog-to-digital converters (ADCs) [94]. On the other hand, measurements obtained from linear sampling methods approximately carry a close amount of information, which makes them reasonably robust; for example, for packages lost in data streaming applications [89]. Compressive sensing linearly samples sparse signals at a rate lower than the traditional rate provided by the Nyquist-Shannon sampling theorem [130].

Recovering a sparse signal from a linear measurement is of high interest to preserve more storage, to have less computation, energy, and communication time, and to propose effective data compression and transmission methods [9, 24, 47, 54, 65, 94, 95, 139, 143]. There are also specialised motivations behind this interest; for example, the recovery process in compressive sensing requires prior knowledge about original signals, which adds reasonable levels of security and privacy to the compressive sensing based data acquisition framework [133]. Compressive sensing-based face recognition techniques are invariant to rotation, re-scaling and data translation [94]. Besides, compressive sensing has demonstrated outstanding numerical stability in terms of not only noisy measurements but also quantization errors [28]. In machine learning, compressive sensing has demonstrated to boost the performance of pattern recognition methods.

The sparsity assumption as the basis of compressive sensing is not practically restrictive because it is empirically observed that desired signals in an extensive range of applications are sparse, possibly after a

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change of basis though. However, blind compressive sensing techniques are universal in terms of transfer domains, namely, there is no need to know the sparsifying basis while taking linear measurements \cite{57}. Sparsity notion directly leads to NP-hard problems and thus the original problem of compressive sensing is computationally intractable; see Section 2 for the definition of this problem. Two main strategies to tackle this obstacle are convex or nonconvex relaxations and greedy algorithms; see Sections 3 and 4. There are several other effective approaches that have different ideas than these two categories. Hence, this survey attempts to briefly characterize these promising studies as well.

To relax an $\ell_0$-quasi-norm based sparse optimization problem, $\ell_0$-quasi-norm is often replaced by $\ell_p$-(quasi)-norm with $p > 0$. Then, it is investigated that under which conditions on problem parameters and for which $p$’s both original and approximate problems uniquely obtain the same solution. The efficiency of $p = 1$ is well-documented \cite{20, 21, 26, 31, 43, 44, 53}. But since this choice does not lead into strictly convex programs, the solution uniqueness shortcoming needs to be addressed \cite{77, 132}. This case converts to a linear program and since those matrices involved in sparse optimization often inherit large dimensions, well-known methods including simplex and interior-point methods or specified algorithms are applied \cite{18, 99}. The shape of unit balls associated with $0 < p < 1$ encourages examining this case as well. It turns out that the obtained guarantee results of certain values of such $p$ are more robust and stable with much less restrictive conditions compared to the previous case $p = 1$ \cite{31, 32}. Despite these favorable theoretical results, nonconvex relaxations ask for a global minimizer, which is an intractable task. One numerical way to bypass this is by utilizing classical schemes that obtain a local minimizer with an initial point sufficiently close to a global minimizer. For example, the solution of least squared provides an empirical suggestion for this initialization \cite{31}, although there is no guarantee that such a solution is close enough. Nonetheless, a recent theoretical study proves that the main representative problems utilized in the realm of sparse optimization, such as the generalized basis pursuit and LASSO, almost always return a full-support solution for $p > 1$ \cite{104}.

Greedy algorithms in our second class have low computational complexity, especially for relatively small sparsity levels, and yet they are effective. A well-received algorithm in this class which plays a key role is the orthogonal matching pursuit (OMP). To directly solve the original problem of compressive sensing, it exploits the best local direction in each step and adds its corresponding index to the current support set. Then, it estimates the new iteration as the orthogonal projection of the measurement vector onto the subspace generated by columns in the current support set \cite{113, 114}. There is an extensive literature on the capability of this algorithm in identifying the exact support set that vary based on the required number of steps and accounting for noise in the measurement vector. For example, a sufficient condition for recovering an $s$ sparse vector from an exact linear measurement asks for an $(s + 1)$th restricted isometry constant strictly smaller than $(\sqrt{s} + 1)^{-1}$ \cite{74}. This upper bound improves to the necessary and sufficient bound of $1/\sqrt{s + 1}$. A result with the same spirit for the noisy measurement is available that imposes an extra assumption on the magnitude of the smallest nonzero entry of a desired sparse vector \cite{124}. Further, stability is also achievable with the presence of noisy measurements if the number of required steps is increased and $\delta_s + 2\delta_{s+1} \leq 1$ \cite{134}. Nevertheless, there are other algorithms in this class that have the same spirit but attempt to resolve issues of the OMP, such as stagewise OMP \cite{42} and multipath matching pursuit \cite{66}. For example, many algorithms allow more indices to enter the current support set for picking a correct index in each step such as the compressive sampling matching pursuit \cite{81} and subspace pursuit \cite{35}. Other effective greedy algorithms extending the OMP are simultaneous, generalized, and grouped OMP \cite{107, 113, 118, 122}. Nonnegative and more general constrained versions of the OMP obtain similar guarantee results based on variations of the restricted isometry and the orthogonality constants \cite{16, 68, 105}. These algorithms are also generalized for block sparse signals \cite{46, 120}. 

\textit{2}
Compressive sensing (CS), also known as compressed sensing or sparse sampling, is established based on sparsity assumption. Hence, we start with preliminary definitions about sparsity and a closely related concept, compressibility.

**Definition 2.1.** For a vector \( x \in \mathbb{R}^N \), let \( \text{supp}(x) := \{ i \in [N] | x_i \neq 0 \} \) and \( \|x\|_0 := \text{card}(\text{supp}(x)) \). In particular, for \( s \in [N] \), this vector is called \( s \)-sparse if \( \|x\|_0 \leq s \).

**Definition 2.2.** For a vector \( x \in \mathbb{R}^N \), \( s \in [N] \), and \( p > 0 \), let

\[
\sigma_s(x)_p := \min \|x - z\|_p \quad \text{subject to} \quad z \in \mathbb{R}^N \quad \text{and} \quad \|z\|_0 \leq s.
\]

This vector is called \( s \)-compressible or nearly \( s \)-sparse in \( \ell_p \)-norm if \( \sigma_s(x)_p \) is small for some \( p > 0 \).

It turns out that \( \sigma_s(x)_p = \|x - x_S\|_p \), where \( S \subseteq [N] \) contains all the \( s \) largest absolute entries of \( x \) (consequently, \( \sigma_s(x)_p = 0 \) when \( x \) is \( s \)-sparse). Roughly speaking, a vector is called sparse if most of its entries are zero and it is called compressible if it is well-approximated by a sparse vector. Sparsity as a prior assumption for a desired vector is consistent with diverse applications because it is often doable to employ a change of basis technique for finding sparse representations in a transform domain. For instance, wavelet, Radon, discrete cosine, and Fourier transforms are well-known as suitable and efficient.

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The original problem of compressive sensing aims to recover a sparse signal \( x \) from a linear measurement vector \( y = Ax \), where \( A \in \mathbb{R}^{m \times N} \) (with \( m \ll N \)) is the so called the measurement, coding or design matrix. This ultimate goal of compressive sensing is reasonably formulated as follows:

\[
\min_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{subject to} \quad Ax = y.
\]  

(CS)

This problem has demonstrated to revolutionize several real-world applications in both science and engineering disciplines, including but not limited to signal processing, imaging, video processing, remote sensing, communication systems, electronics, machine learning, data fusion, manifold processing, natural language and speech processing, and processing biological signals [70, 90, 92, 96, 140].
In a broad range of real-world applications, we utilize ADCs to map the real value measurements of physical phenomena (over a potentially infinite range) to discrete values (over a finite range). As an illustration, when we use $b$ bits for representing the measurements digitally, the quantization module in an ADC maps measurements to one of the $2^b$ distinct values that introduces error in measurements. Quantization module plays a bottleneck role in restricting the sampling rate of the ADCs because the maximum sampling rate decreases exponentially when the number of bits per measurement increases linearly. Quantization module also is the main source of energy consumption in ADCs. Single-bit (or 1-bit) compressive sensing enables us to reduce the number of bits per measurements to one and introduces a proper model for successful recovery of the original signal. This extreme quantization approach only retains the sign of the measurements, i.e., $y_i \in \{1, -1\}$ for all $i \in [m]$, which results in a significantly efficient, simple, and fast quantization [15, 62].

A recent generalization of (CS) is the so called compressive sensing with matrix uncertainty. This uncertainty finds two formulations that incorporate measurement error as well. The first formulation is the following:

$$\min_{x \in \mathbb{R}^N, E \in \mathbb{R}^{m \times N}} \| (A + E)x - y \|_2^2 + \lambda_E \| E \|_F^2 + \lambda \| x \|_0,$$

where the matrix $E$ is the perturbation matrix. The second one is the following:

$$\min_{x \in \mathbb{R}^N, d \in \mathbb{R}^r} \left\| (A_0 + \sum_{i \in [r]} d_i A_i)x - y \right\|_2^2 + \lambda_d \| d \|_2^2 + \lambda \| x \|_0,$$

where the measurement matrices $A_i$'s for $i = 0, 1, \ldots, r$ are known and the unknown vector $d$ is the uncertainty vector. In quantized compressed sensing, we have either $y = Qx$ or $y = Q(Ax + v)$ where $v$ is the noise such that $v \sim \mathcal{N}(0, \sigma^2 I)$ and $Q: \mathbb{R}^m \mapsto \mathcal{A} \subseteq \mathbb{R}^m$ is the set-valued quantization function. Well-studied examples of such function, map $Ax$ or $Ax + v$ into $\mathcal{A} = \{ x \mid l \leq x \leq u \}$ for some vectors $l$ and $u \in \mathbb{R}^m$ and $\mathcal{A} = \{+1, -1\}^m$ [62, 126, 145]. The former case leads to:

$$\min_{x \in \mathbb{R}^N} \| x \|_0 \quad \text{subject to} \quad l \leq Ax + v \leq u,$$

and the latter one yields the following problem:

$$\min_{x \in \mathbb{R}^N} \| x \|_0 \quad \text{subject to} \quad y = \text{sign}(Ax + v).$$

In some recent applications, we may have nonlinear measurements that encourage the so called cardinality constrained optimization:

$$\min_{x \in \mathbb{R}^N} f(x) \quad \text{subject to} \quad \| x \|_0 \leq s.$$  

Two important examples are $f(x) = \|Ax - y\|^2$ (motivated by linear measurements $y = Ax$) and $f(x) = \sum_{i}^{m}(y_i - x^T A_i x)^2$ (proposed for quadratic measurements of $y_i = x^T A_i x$ for $i = 1, 2, \ldots, m$ and symmetric matrices $A_i$’s) [103, 108]. Many algorithms have been proposed for solving this cardinality constrained problem with specific and general smooth functions [12, 13, 17, 102]. In matrix setting, this problem converts to rank minimization, which has numerous applications [48]. Although we listed some vital variations of compressive sensing above, our focus in this survey is on the original problem to motivate the important tools, algorithms and results in compressive sensing for a linear measurement possibly contaminated with some error or noise.
3 $\ell_p$-Recovery with $0 < p \leq 1$: Main Formulations and Results

Since the problem (CS) is NP-hard [78], it is not tractable and it must be handled indirectly. Because $\ell_p$-norm approximates $\ell_0$-norm as $p$ goes to zero, one approach to tackle the above problem is exploiting the so called Basis Pursuit (BS) problem:

$$\min_{x \in \mathbb{R}^N} \|x\|_1 \text{ subject to } Ax = y.$$ \hspace{1cm} (BP)

Then, one investigates that under which conditions they appoint the same solution. This equivalence property occurs when the solution is highly sparse and the measurement matrix has sufficiently small mutual incoherence [43]. Sparsity is an inevitable assumption in the recovery process. This accompanied by other conditions on a measurement matrix such as the mutual coherence, null space property (NSP) or restricted isometry property (RIP) guarantee a successful recovery through the $\ell_1$-norm problem above.

**Definition 3.1.** For a matrix $A \in \mathbb{R}^{m \times N}$, the mutual coherence is defined as

$$\mu(A) := \max_{i \neq j} \frac{|\langle A_i, A_j \rangle|}{\|A_i\|_2 \|A_j\|_2}.$$ 

This quantity simply seeks the largest correlation between two different columns and below states a sufficient condition for the equivalence of the $\ell_0$ and $\ell_1$-norm problems. The following global 2-coherence is a generalization of the mutual coherence ($\mu = \nu_1$):

**Definition 3.2.** For a matrix $A \in \mathbb{R}^{m \times N}$, its $k$th global 2-coherence is defined as the following:

$$\nu_k(A) := \max_{i \in [N]} \max_{A \subseteq [N] \setminus \{i\}, \|A\| \leq k} \left( \sum_{j \in A} \frac{\langle A_i, A_j \rangle^2}{\|A_i\|_2 \|A_j\|_2} \right)^{1/2}.$$ 

This tool is useful for a successful recovery of a weak version of orthogonal matching pursuit [129], while we focus on the standard mutual coherence to convey the main idea.

**Theorem 3.1.** [45] Assume that the measurement matrix $A$ satisfies $\mu(A) < 0.5(1 + \mu(A)^{-1})$. Then, problems CS and BP uniquely obtain the same solution.

This bound is improved using the RIP.

**Definition 3.3.** Let $A$ be an $m \times N$ matrix. Then, it has the restricted isometry property of order $s$ provided that there exists $\delta_s \in (0, 1)$ such that

$$(1 - \delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_s)\|x\|_2^2 \quad \forall x; \ |x|_0 \leq s.$$ 

This definition demands each column submatrix $A_S$ with $\text{card}(S) \leq s$ to behave like identity matrix, that is, to have eigenvalues in $[1 - \delta_s, 1 + \delta_s]$. Since this definition involves all the $s$-tuples of columns, it is more rigorous than the mutual coherence ($\delta_2 = \mu$). Further, it derives to better upper bounds on the sparsity level of a vector to be recovered, that is, less sparse vectors are properly handled.

**Theorem 3.2.** [73] Problems CS and BP uniquely obtain the same $s$-sparse solution if $\delta_2s < 0.4931$.

The BP recovers all the $s$-sparse signals if $\delta_s < 1/3$ and this bound is sharp [22]. There are numerous similar results for recovering $s$-sparse signals in compressive sensing literature based on the RIP constants which are often in the form of $\delta_s$ or $\delta_{ks} \leq \delta$ for some numbers $k > 0$ and $\delta \in (0, 1)$. Since this condition
is not practically verifiable due to its computational complexity \[\|v_S\|_1 < \|v_{S^c}\|_1; \quad \forall v \in \text{Ker}(A) \setminus \{0\} \text{ and } \forall S \subseteq [N] \text{ with } |S| \leq s.\]

In principle, the measurement pair \((A, y)\) carries all the required information for the recovery process. Since linear systems \(Ax = y\) and \(PAX = PY\) have identical solution sets for a nonsingular matrix \(P\), the measurement pair \((PA, PY)\) has the same information but it is numerically a better choice for a suitable conditioning matrix \(P\). Although the RIP constants of matrices \(A\) and \(PA\) can vastly differ \[137\], the NSP is preserved if either one holds it. In addition, this property is necessary and sufficient for the equivalence between \(\ell_0\) and \(\ell_1\)-norm problems.

**Theorem 3.3.** \[54\] Every \(s\)-sparse vector \(x\) is the unique solution to \(\text{BP}\) with \(y = A\) and only the measurement matrix \(A\) holds the NSP of order \(s\).

The Range Space Property (RSP) is another necessary and sufficient condition given by Zhao \[141\].

**Definition 3.5.** The matrix \(A^T\) has the range space property of order \(s\) if for any disjoint subsets \(S_1\) and \(S_2 \subseteq [N]\) with \(|S_1| + |S_2| \leq s\) there exists a vector \(\eta \in \mathcal{R}(A^T)\) such that

\[\eta_i = 1 \quad \forall i \in S_1; \quad \eta_i = -1 \quad \forall i \in S_2 \text{ and } \|\eta_{(S_1 \cup S_2)^c}\|_\infty < 1.\]

**Theorem 3.4.** \[141\] Every \(s\)-sparse vector \(x\) is exactly recovered by the problem \(\text{BP}\) with \(y = Ax\) if and only if \(A^T\) has the RSP of order \(s\).

The matrix \(A^T\) has the range space property of order \(s\) under several assumptions such as \(s < 0.5(1 + \mu(A)^{-1})\), \(\delta_{2s}(A) < \sqrt{2} - 1\) and the NSP of order \(2s\) \[141\]. Further, an extended version of this property is useful to provide a similar result to the above theorem for the nonnegative sparse recovery.

Taking into account noisy measurements yields in the following Quadratically Constrained Basis Pursuit (QCBP) problem:

\[
\min_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad \|Ax - y\|_2 \leq \eta.
\]

**Theorem 3.5.** \[54\] Given \(x \in \mathbb{R}^N\) and a matrix \(A \in \mathbb{R}^{m \times N}\) such that \(\delta_{2s} < 4/\sqrt{41} \approx 0.6246\), every minimizer \(x^*\) of \(\text{QCBP}\) satisfies \(\|x^* - x\|_2 \leq Cs^{-1/2}\sigma_s(x)_1 + D\eta\) with \(C > 0\) and \(D > 0\).
The following so called Least Absolute Shrinkage and Selection Operator (LASSO) is practically more efficient [50] [116] [275]:

\[
\min_{x \in \mathbb{R}^N} \frac{1}{2} ||y - Ax||^2 + \lambda ||x||_1. \tag{LASSO}
\]

There is a trade-off between feasibility and sparsity in the nature of this problem that is controlled by the regularization parameter \( \lambda \). Not only this problem has been extensively studied [109], but also there are results in terms of its effectiveness in sparse optimization. For instance, we bring the following theorem for deterministic measurement matrices.

**Theorem 3.6.** [116] Let \( y = Ax^* + e \) where \( x^* \) is supported on \( S \subseteq [N] \) with \( |S| \leq s \) and \( e \) is a zero-mean additive observation noise. Assume that the measurement matrix \( A \) satisfies \( ||A^T_s A_s (A^T_s A_s)^{-1}||_{\infty} < 1 \), \( \lambda_{\min}(A^T_s A_s) > m \) and \( \max_{j \in [N]} ||A_j||_2 \leq \sqrt{m} \). Further, assume that \( N = \Theta(\exp(m^n)) \), \( s = \Theta(m^a) \), \( \min_{i \in S} x_i > 1/m^{1-\beta} \) with \( 0 < \alpha + \gamma < \beta < 1 \). For \( \lambda = m^{1-\delta} \) such that \( \delta \in (\gamma, \beta - \alpha) \), the LASSO problem recovers the sparsity pattern \( S \) with probability \( 1 - \exp(-cm^n) \) for a constant \( c > 0 \).

In case of random measurement matrices, this problem needs a sample size \( m > 2s \ln(N - s) \) to achieve exact recovery with a high probability that converges to one for larger problems [116]. An efficient algorithm for solving the BP utilizes a sequence of the LASSO problems [271]. The following so called Basis Pursuit Denoising (BPD) minimizes the feasibility violation while implicitly bounding the sparsity level with a parameter \( \tau \geq 0 \):

\[
\min_{x \in \mathbb{R}^N} ||y - Ax||_2 \quad \text{subject to} \quad ||x||_1 \leq \tau. \tag{BPD}
\]

There are known relations among the optimums of three above problems [54]. The Dantzig Selector (DS) problem that arises in several statistical applications has been also employed in sparse recovery [27] [21]:

\[
\min_{x \in \mathbb{R}^N} ||x||_1 \quad \text{subject to} \quad ||A^T (Ax - y)||_{\infty} \leq \sigma. \tag{DS}
\]

This problem manages noisy measurements and reduces to a linear programming problem. To bring a pertaining result next, we need another quantity.

**Definition 3.6.** Given a matrix \( A \in \mathbb{R}^{m \times N} \), its \( s, s' \)-restricted orthogonality (RO) constant \( \theta_{s,s'} \) is defined as the smallest \( \theta > 0 \) such that

\[
|\langle Ax, Ax' \rangle| \leq \theta ||x||_2 ||x'||_2 \quad \forall x \text{ and } x'; \quad ||x||_0 \leq s \text{ and } ||x'||_0 \leq s'.
\]

**Theorem 3.7.** [27] Let \( x \in \mathbb{R}^N \), and \( y = Ax + e \) such that \( ||A^T (Ax - y)||_{\infty} \leq \sigma \) and \( \delta_{1,5s} + \theta_{s,1,5s} < 1 \). Then, an optimal solution \( x^* \) to DS obeys

\[
||x^* - x||_2 \leq C s^{\frac{1}{2}} \sigma + D s^{\frac{3}{2}} \sigma_s(x)_1,
\]

with

\[
C = \frac{2 \sqrt{3}}{1 - \delta_{1,5s} - \theta_{s,1,5s}} \text{ and } D = \frac{2 \sqrt{2} (1 - \delta_{1,5s})}{1 - \delta_{1,5s} - \theta_{s,1,5s}}.
\]

In particular, if \( x \) is an \( s \)-sparse vector, then \( ||x^* - x||_2 \leq C s^{\frac{1}{2}} \sigma \).

However the choice of \( p = 1 \) is the most interesting as \( \ell_1 \)-norm is the closest convex norm to \( \ell_0 \)-quasinorm [93] and convex optimization is extremely well-nourished, the shape of a unit ball associated with \( \ell_p \)-norm for \( 0 < p < 1 \) motivates researchers to explore the following nonconvex problem:

\[
\min_{x \in \mathbb{R}^N} ||x||_p \quad \text{subject to} \quad Ax = y.
\]
For certain values of such $p$’s, the obtained theoretical guarantees of this scheme are more robust and stable [32, 97, 121]. Further, much less restrictive conditions than $\ell_1$-norm recovery are achievable, this is not always the case though [144]. For instance, the following result implies that a sufficient condition for recovering an $s$-sparse vector in the noiseless case via $\ell_{0.5}$ minimization is $\delta_{ks} + 27\delta_{ks} < 26$, where an analogous result for $\ell_1$-norm recovery requires $\delta_{2s} + 2\delta_3 < 1$. Since it is practically stringent to impose sparsity, it is desired to consider compressible vectors as well.

**Theorem 3.8.** [97] Assume that $A \in \mathbb{R}^{m \times N}$ satisfies

$$
\delta_{ks}(A) + k^{\frac{2}{p} - 1}\delta_{(k+1)s}(A) < k^{\frac{2}{p} - 1} - 1,
$$

for $k > 1$ such that $ks$ is a natural number. Given $x \in \mathbb{R}^N$, let $y = Ax + e$ with $\|e\|_2 \leq \epsilon$. Then, an optimal solution $x^*$ of

$$
\min_{x \in \mathbb{R}^N} \|x\|_p^p \quad \text{subject to} \quad \|Ax - y\|_2 \leq \epsilon,
$$

obeys

$$
\|x^* - x\|_2^p \leq C\epsilon^p + D s^{\frac{p}{2} - 1}[\sigma_s(x)]_p^p,
$$

where

$$
C = 2^p \left[ \frac{1 + k^{\frac{2}{p} - 1}(2/p - 1)^{-\frac{p}{2}}}{(1 - \delta_{(k+1)s})^\frac{2}{p} - (1 + \delta_{ks})^\frac{2}{p} k^{\frac{2}{p} - 1}} \right], \quad D = 2\left(\frac{2 - p}{2 - p}ight)^{p/2} \left[ \frac{1 + (1 + k^{p/2 - 1})(1 + \delta_{ks})^{p/2}}{(1 - \delta_{(k+1)s})^{p/2} - (1 + \delta_{ks})^{p/2} k^{\frac{2}{p} - 1}} \right].
$$

In particular, if $x$ is an $s$-sparse vector, then $\|x^* - x\|_2^p \leq C\epsilon^p$.

The main issue with this result is its demand for a global minimizer of a nonconvex function for which there is no known theoretical guarantee. One way to bypass this is by utilizing classical schemes that obtain a local minimizer with an initial point that is sufficiently close to a global minimizer. For example, the solution of least-squares is experimentally suggested for this initialization [31], while there is no guarantee that such solution is in general close enough to a global minimizer. Since solving an unconstrained problem is simpler, another approach for taking advantage of nonconvex $\ell_p$-minimization [1] with $0 < p < 1$ is tackling the following $\ell_p$-regularized least squares problem:

$$
\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - y\|_2^2 + \lambda\|x\|_p^p,
$$

where $\lambda > 0$ is the penalty parameter. Because problems [1] and [2] are equivalent in limit, this parameter must be selected meticulously to obtain an approximate solution for the original problem. Once $\lambda > 0$ is fixed, algorithms such as iteratively-reweighted least squares explained in [101] are beneficial. Furthermore, the choice of $p$ plays a key role in the efficiency of this approach; for example, the best choices in image deconvolution are $p = 1/2$ and $p = 2/3$. The case of $p = 1/2$ is a critical choice because it provides sparser solutions among $p \in [1/2, 1)$ and any $p \in (0, 1/2)$ does not show significantly better performance [127]. As a result, it is essential to have a specialized algorithm for this choice [128].

Despite $p > 1$ leads to strictly convex programming, it is theoretically shown that in this case not only the solution is not sparse, but also each entry is almost always nonzero. The same result holds for all the primary problems mentioned above.

**Theorem 3.9.** [104] Let $p > 1$, $N \geq m$, $\lambda > 0$, and $\tau > 0$. For almost all $(A, y) \in \mathbb{R}^{m \times N}$, a unique optimal solution $x^*_{(A, y)}$ to the any of the problems $\text{BP}$, $\text{QCBP}$, $\text{LASSO}$ and $\text{BPD}$ when $\|\cdot\|_1$ is replaced by $\|\cdot\|_p$, satisfies $|\text{supp}(x^*_{(A, y)})| = N$. 

8
Recent applications in image processing, statistics and data science led to another class of sparse optimization problems subject to the nonnegative constraint, i.e.,

$$\min_{x \in \mathbb{R}^N} \|x\|_0 \quad \text{subject to} \quad Ax = y, \ x \geq 0.$$  \hspace{1cm} (NCS)

This problem is similar to CS but their optimums differ in general. Similar approaches are advantageous to tackle this problem. The main one is the so called Nonnegative Basis Pursuit (NBP):

$$\min_{x \in \mathbb{R}^N} \|x\|_1 \quad \text{subject to} \quad Ax = y, \ x \geq 0.$$  \hspace{1cm} (NBP)

**Theorem 3.10.** \cite{113} Any $x \geq 0$ such that $\|x\|_0 \leq s$ is recovered by NBP with $y = Ax$ if and only if for any index subset $S \subseteq [N]$ such that $|S| \leq s$ there exists $\eta \in \mathcal{R}(A^T)$ such that $\eta_i = 1$ and $\|\eta_{S^c}\|_\infty < 1$.

For a theoretical study on the following nonnegative version of LASSO (NLASSO) formulation, see \cite{61}:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|y - Ax\|_2^2 + \lambda \|x\|_1 \quad \text{subject to} \quad x \geq 0.$$  \hspace{1cm} (NLASSO)

Nonnegativity is an example of possible available prior information in the recovery process, that enables us to employ sparse measurement matrices for recovery of remarkably larger signals \cite{64}. We discuss two other well-received types of prior information. In applications like infrared absorption spectroscopy, the non-zero elements of the original sparse signal are bounded, e.g., $x \in [0,1]$. Imposing this boundary condition reduces the optimal set, and therefore its convex relaxation leads to the bounded/boxed basis pursuit which obtains better recovery properties \cite{11,40,69}.

Majority of studies in compressive sensing focus on continues signals where $x \in \mathbb{R}^N$, even though there are several real-world applications that include discrete signals \cite{69}. Examples are discrete control signal design, black-and-white or gray-scale sparse image reconstruction, machine-type multi-user communications, and blind estimation in digital communications. Here, we aim to reconstruct a discrete signal whose elements take their values from a finite set of alphabet. It is worth highlighting that when $x$ is discrete, even the $ell_1$-norm recovery method leads to an NP-hard problem \cite{67}. This study revealed this additional prior knowledge about the original signal via imposing new constraints even enhances the performance of both $\ell_0$ and $\ell_1$ norm recovery methods. The sum of absolute values is designed to recover discrete sparse signals whose non-zero elements are generated from a finite set of alphabets with a known probability distribution. In binary compressive sensing, we can represent this model as follows:

$$\min_{x \in \mathbb{R}^N} (1-p)\|x\|_1 + p \|x - 1_N\|_1 \quad \text{subject to} \quad Ax = y,$$

where $p$ is the probability of each component being one; analogous to the sparsity rate \cite{63,69,79}. In this approach, when $p$ goes to zero ($x$ is sparse), the SAV behaves like BP. The sum of norms represents the problem of binary compressive sensing as follows:

$$\min_{x \in \mathbb{R}^N} \|x\|_1 + \lambda \|x - \frac{1}{2} 1_N\|_\infty \quad \text{subject to} \quad Ax = y,$$

where the parameter $\lambda > 0$ keeps $\ell_1$ and $\ell_\infty$ balanced \cite{119}. While minimizing $\|x\|_1$ increases the sparsity of $x$, minimizing the term $\|x - \frac{1}{2} 1_N\|_\infty$ forces $x$ to takes its values from $\{0,1\}$. Nonetheless, the smoothed $\ell_0$ gradient descent technique has demonstrated to outperform such approaches in terms of recovery rate and time \cite{69}.

Finally, we emphasize on two important points. First, that most results in sparse recovery are based on asymptotic properties of random matrices but occasionally an application specifies a measurement...
matrix. Thus, it is vital to have tractable schemes for testing such properties in potential situations; see, e.g., [37, 110]. The nature of this requirement, encourages RIPless conditions [25, 138]. Second, all the introduced convex minimization problems can be solved by general purpose interior-point methods, however specified algorithms designed for these problems exist as well; see, e.g., [18, 30, 100, 135].

4 Greedy Algorithms: Main Variations and Results

Greedy algorithms are iterative approaches that take local optimal decisions in each step to eventually obtain a global solution. Hence, their success is due to some conditions on problem parameters. Greedy algorithms are mostly simple and fast that find numerous applications in various contemporary fields, including biology, applied mathematics, engineering, and operations research. This supports the emerging interest in their performance analysis.

Greedy algorithms also are efficient in tackling those problems in compressive sensing. The most popular greedy algorithm to solve this problem is the Orthogonal Matching Pursuit (OMP). Finding an \(s\)-sparse solution for an underdetermined linear system casts as the following sparsity constrained problem:

\[
\min_{x \in \mathbb{R}^N} \|Ax - y\|_2^2 \quad \text{subject to} \quad \|x\|_0 \leq s. \tag{SC}
\]

To tackle the above problem, starting from zero as the initial iteration, the following OMP algorithm picks an appropriate index in each step to add its current support set and it estimates the new iteration as the orthogonal projection of the measurement vector onto the subspace generated by corresponding columns of the current support set [114, 117].

**Orthogonal Matching Pursuit**

| Input: | \(A \in \mathbb{R}^{m \times N}, y \in \mathbb{R}^m\), and initialize with \(x_0 = 0 \in \mathbb{R}^N\), and \(S_0 = \emptyset\). |
|---|---|
| Iteration: | repeat until convergence: |
| | \(S_{n+1} = S_n \cup \{j_{n+1}\}\) such that \(j_{n+1} = \arg \max_{j \in S_n} |\langle y - Ax_n, A_j \rangle|\), |
| | \(x_{n+1} = \arg \min \|Ax - y\|_2^2\) subject to \(\text{supp}(x) \subseteq S_{n+1}\). |
| Output: | \(x^* = x_n\). |

There is an extensive literature on the capability of this algorithm in identifying the exact support set either in at most \(s\) steps or at arbitrary many steps [113, 114, 19].

**Theorem 4.1.** [74, 124] Assume that \(A \in \mathbb{R}^{m \times N}\) satisfies the RIP of order \(s + 1\) such that \(\delta_{s+1} \leq \frac{1}{\sqrt{s+1}}\). Then, the OMP recovers any \(s\)-sparse vector \(x \in \mathbb{R}^N\) using \(y = Ax\) in at most \(s\) iterations.

However, this sufficient bound is not sharp and \(\delta_{s+1} < 1/\sqrt{s+1}\) is the largest sufficient bound possible to have OMP identify the exact support set. In other words, there exists a matrix with \(\delta_{s+1} = 1/\sqrt{s+1}\) that fails to recover any \(s\)-sparse vectors [36, 72]. A result with the same spirit for the noisy measurement requires an extra assumption on the magnitude of the smallest nonzero entry of the desired vector works [123]. Here, we bring the next theorem that assumes \(\delta_s + 2\delta_{31s} \leq 1\). This inequality holds under the condition \(\delta_{31s} \leq 1/3\) which implies the number of required measurements to successfully recover any \(s\)-sparse signal via the OMP is \(O(s \ln N)\), that is, the number of measurements \(m\) must be in the worst case linear in sparsity level \(s\).

**Theorem 4.2.** [132] Assume that \(A \in \mathbb{R}^{m \times N}\) satisfies \(\delta_s + 2\delta_{31s} \leq 1\). Then, given an \(s\)-sparse vector \(x \in \mathbb{R}^N\), the OMP obeys

\[
\|x^{(30s)} - x\|_2 \leq \frac{2\sqrt{6}(1 + \delta_{31s})^2}{1 - \delta_{31s}} \|Ax - y\|_2.
\]
There are also results for the success of this algorithm in case of availability of partial information on the optimal support set under more restrictive conditions [56, 59]. This partial information is of the form of either a subset of the optimal support or an approximate subset with possibly wrong indices. In many practical situations, there is no prior information available about the sparsity of a desired signal but generalized versions of the OMP still are enabled to recover it under a more stringent condition [39].

The OMP is also used in recovering block sparse signals [46, 60, 120]. To elaborate on this, assume that \( x = [x[1]^T, x[2]^T, \ldots, x[L]^T]^T \) with \( x[i] = [x_{d(i-1)+1}, x_{d(i-1)+2}, \ldots, x_{di}]^T, \) \( 1 \leq i \leq L \) (we focus on this case where the number of nonzero elements in different blocks are equal for the sake of simplicity of exposition). Then, \( x \) is called a block \( s \)-sparse signal if there are at most \( s \) blocks (indices) \( i \) such that \( x[i] = 0 \) [46]. The support of this block sparse vector is defined as \( \Omega := \{i|x[i] \neq 0\} \). This definition reduces to the standard sparsity definition for \( d = 1 \). We also represent the measurement matrix \( A \) as \( A = [A[1], A[2], \ldots, A[L]] \), where \( A[i] = [A_{d(i-1)+1}, A_{d(i-1)+2}, \ldots, A_{di}] \), \( 1 \leq i \leq L \). Then, the block orthogonal matching pursuit (BOMP) is as follows.

**Block Orthogonal Matching Pursuit**

| Input: | \( A \in \mathbb{R}^{m \times Ld}, y \in \mathbb{R}^m \), and initialize with \( x_0 = 0 \in \mathbb{R}^{Ld} \), and \( S_0 = \emptyset \). |
|------------------|--------------------------------------------------------------------------------------------------|
| Iteration: | repeat until convergence: \( S_{n+1} = S_n \cup \{j_{n+1}\} \) such that \( j_{n+1} = \arg \max_{j \in [L]} \| A[j]T(y - A[S_n]x[S_n]) \|_2 \), \( x[S_{n+1}] = \arg \min \| A[S_{n+1}]x - y \|_2^2 \). |
| Output: | \( x^* = x_n \). |

To present the exact recovery result of this algorithm, we need two other tools.

**Definition 4.1.** [46] Consider a block measurement matrix \( A \) introduced above. The block mutual coherence of \( A \) is defined as

\[
\mu_B(A) := \max_{1 \leq i \neq r \leq L} \frac{1}{d} \| A[i]T A[r] \|_2,
\]

and its subcoherence is defined as

\[
\nu_B(A) := \max_{1 \leq i \leq L} \max_{1 \leq j \leq d} \frac{1}{d} \| A[i]T A[j] \|,
\]

where \( A[i] \) and \( A[j] \) are the \( i \)th and \( j \)th columns of the block \( A[l] \).

**Theorem 4.3.** [120] Let \( x \) be a block sparse vector supported on \( \Omega \) as introduced above and \( y = Ax + e \) where \( \| e \|_2 \leq \eta \) and the measurement matrix satisfies \( (2s - 1)d\mu_B(A) + (d - 1)\nu_B(A) < 1 \). Further, suppose that

\[
\min_{i \in \Omega} \| x[i] \|_2 \geq \frac{\eta \sqrt{2(1 + (2d - 1)\nu_B(A))}}{1 - (2|\Omega| - 1)d\mu_B(A) - (d - 1)\nu_B(A)}.
\]

Then, the BOMP algorithm recovers block sparse vector \( x \) in \( |\Omega| \) iterations.

A similar result for the capability of another extended version of the OMP in recovering joint block sparse matrices is found in [55, 106]. Despite many interesting features of the OMP, its index selection is problematic. Precisely, if a wrong index is chosen, the OMP fails to expel this index so that the exact support cannot be found in this situation. The Compressive Sampling Matching Pursuit (CoSaPM) is an algorithm to resolve this drawback [81]. To find an \( s \)-sparse feasible vector for an underdetermined linear system, this algorithm allows \( 2s \) best potential indices enter the current support set. Then, it keeps \( s \) entries that play the key role in the pertaining projection in the sense that their corresponding entries have most magnitude.
Compressive Sampling Matching Pursuit

Input: \( A \in \mathbb{R}^{m \times N}, y \in \mathbb{R}^m \), and initialize with \( x_0 = 0 \in \mathbb{R}^N \), and \( S_0 = \emptyset \).

Iteration: repeat until convergence:
\[
U_{n+1} = \text{supp}(x_n) \cup L_2 s(A^T(y - Ax_n)),
\]
\[
u_{n+1} = \arg \min \|Ax - y\|_2^2 \quad \text{subject to} \quad \text{supp}(x) \subseteq U_{n+1},
\]
x_{n+1} = H_s(\nu_{n+1}).

Output: \( x^* = x_n \).

In the noiseless measurement case, the following theorem states that any \( s \)-sparse signal, is recovered as the limit point of a sequence generated by the CoSaMP.

**Theorem 4.4.** [54] Assume that \( A \in \mathbb{R}^{m \times N} \) satisfies \( \delta_{8s} < 0.4782 \). Then, for any \( x \in \mathbb{R}^N \) and \( e \), the sequence \( x_n \) generated by CoSaMP, where \( s \) is replaced by \( 2s \), using \( y = Ax + e \) obeys
\[
\|x - x_n\|_2 \leq C s^{-1/2} \sigma_s(x)_1 + D \|e\|_2 + 2 \rho^n \|x\|_2,
\]
where constants \( C, D \) and \( \rho \in (0,1) \) only depend on \( \delta_{8s} \). In particular, if \( \tilde{x} \) denotes a cluster point of this sequence, then
\[
\|x - \tilde{x}\|_2 \leq C s^{-1/2} \sigma_s(x)_1 + D \|e\|_2.
\]

The following generalized orthogonal matching pursuit (gOMP) allows a given number of indices \( t \geq 1 \) enter a current support set. Consequently, a faster exact recovery under the RI constants involved conditions is attained [118, 122]. A recent algorithm combines the BOMP and the gOMP to propose a block generalized orthogonal matching pursuit for recovering block sparse vectors with possibly different number of nonzero elements in each block [91]. A closely related task to problem SC is the following nonnegative sparsity constrained (NSC) problem
\[
\min_{x \in \mathbb{R}^N} \|Ax - y\|_2^2 \quad \text{subject to} \quad \|x\|_0 \leq s \quad \text{and} \quad x \geq 0.
\]

Then, its solution may completely differ from the one to SC. Bruckstein et al. [16] presented an adapted version of the OMP for finding nonnegative sparse vectors of an underdetermined system, namely, the following Nonnegative Orthogonal Matching Pursuit (NOMP). They demonstrated its capability to find sufficiently sparse vectors. In practice, it is probable not to have sparsity level in hand, this challenge is also doable [82].

There is an emerging interest to explore novel greedy algorithms to find a sparsest feasible point of a set [7, 13]. This casts as a minimization problem under sparsity constrained possibly intersecting a desired set. Nevertheless, the main idea of a greedy algorithm is to start with a feasible sparse point (possibly zero) and add several candidate indices to the current support set. Once this set is updated, a new
The following constrained sparse problem:

Iteration is obtained via a projection onto this set. The constrained matching pursuit \[105\] investigates a nonnegative sensing setting as well. For example, thresholding based algorithms use the adjacent matrix for approximating inversion action and exploit the hard thresholding operator to solve the square system \[\ell^2\] outperform algorithms presented in Section 3. The iterative reweighted algorithm is as follows:

where \(P \subseteq \mathbb{R}^N\) is a closed constraint set containing the origin.

This algorithm reduces to the OMP and the NOMP for constraint set \(P\) with \(\mathbb{R}^N\) and \(\mathbb{R}^N_+\), respectively. The performance of the CMP relies on the measurement matrix as well as the involved constraint set. Hence, a class of convex coordinate-admissible (CP) sets is of interest. A nonempty set \(P \subseteq \mathbb{R}^N\) is called CP admissible if for any \(P \subseteq \mathbb{R}^N\) and for any index set \(J \subseteq \text{supp}(x)\), we have \(x \in P\). The conic hull structure of the CP sets is used to extend the RIP and the RO constants over these sets and a sufficient exact recovery condition is then developed based on such constants \[105\]. We emphasize that except the mutual coherence, the remaining introduces tools in this survey are NP-hard to check. This implicitly confirms the significance of random measurement matrices in this context \[112\] because they hold these properties with a high probability.

### Constrained Matching Pursuit

**Input:** \(A \in \mathbb{R}^{m \times N}, y \in \mathbb{R}^m\), \(P \subseteq \mathbb{R}^N\), and initialize with \(x_0 = 0\) and \(S_0 = \emptyset\).

**Iteration:** repeat until convergence:

\[
S_{n+1} = S_n \cup \{j_{n+1}\} \quad \text{such that} \quad j_{n+1} = \arg \max_{j \in S_n} 2\langle y - Ax_n, A_j \rangle_{+},
\]

\[
x_{n+1} = \arg \min \|Ax - y\|_2^2 \quad \text{subject to} \quad \text{supp}(x) \subseteq S_{n+1} \quad \text{and} \quad x \geq 0.
\]

**Output:** \(x^* = x_n\).

There are many other effective algorithms in sparse optimization that are useful for the compressive sensing setting as well. For example, thresholding based algorithms use the adjacent matrix \(A^*\) for approximating inversion action and exploit the hard thresholding operator to solve the square system \(A^*Ax = A^*y\) via a fixed-point method \[14\] \[29\] \[38\] \[49\] \[52\]. Iterative reweighted algorithms often outperform algorithms presented in Section 3. The iterative reweighted \(\ell_p\)-algorithm is as follows:

\[
x_{n+1} \in \arg \min_{x \in \mathbb{R}^N} \|W_n x\|_p \quad \text{subject to} \quad \|Ax - y\|_2 \leq \epsilon,
\]

where \(W_n\) is a diagonal weight matrix defined based on the current iteration \(x_n\). For example, \(W_k = \text{diag}(w_n^1, w_n^2, \ldots, w_n^N)\) where \(w_n^i = 1/(|x_n^i| + \gamma)^t\) with \(t\) and \(\gamma > 0\), which encourages small elements to approach zero quickly \[1\] \[28\] \[33\]. The so-called smoothed \(\ell_0\)-norm, uses smooth approximations to tackle the \(\ell_0\) quasi-norm \[75\] \[76\].

In traditional compressive sensing approaches, we recover sparse signals from \(m = \mathcal{O}(s \log N/s)\) linear measurements. Model-based compressive sensing enables us to substantially reduce the number of measurements to \(m = \mathcal{O}(s)\) without losing the robustness of the recovery process \[10\]. In fact, model
based compressive sensing includes the structural dependencies between the values and locations of the signal coefficients. For example, modeling the problem of binary compressive sensing with a bi-partite graph and representing the recovery process as an edge recovery scheme recovers binary signals more accurately, compared to binary $\ell_1$-norm and tree-based binary compressive sensing methods [80].

On the other hand, quantum annealers are type of adiabatic quantum computers that tackle computationally intensive problems, which are intractable in the realm of classical computing. From a problem solving point of view, quantum annealers receive coefficients of an Ising Hamiltonian as input and return a solution that minimizes the given energy function in a fraction of a second [84]. Recent studies have revealed that well-posed binary compressive sensing and binary compressive sensing with matrix uncertainty problems are tractable in the realm of quantum computing [5, 6].

5 Numerical Performance Analysis

In this section, we numerically examine the performance of the most prevalent sparse recovery methods introduced in this survey in recovering unigram text representation from their linear embedding (measurement). To describe the sparse recovery task of interest, we start by an introductory explanation on document (text) unigram representations and embeddings. Suppose that a vocabulary $\mathcal{V}$ is available, namely, a set of all the considered words in a context. Then, a unigram representation of a document is simply a vector in $\mathbb{R}^{|\mathcal{V}|}$, where its $i$th entry counts the occurrences of the $i$th word of $\mathcal{V}$ in that document [85, 87]. Unigram representations are highly sparse as the size of a vocabulary is often too large. These representations are expensive to store and further may not be able to effectively capture the semantic relations among the words of a vocabulary, so text embeddings are naturally favorable.

We start by defining word embeddings and discuss text embeddings afterward. The goal in word embedding is to encode each word of a vocabulary into a vector representation in a much lower dimensional space $m \ll |\mathcal{V}|$ such that the semantic relations among words are preserved. Word embeddings have lately gained much attentions in broad applications of the natural language processing such as classification, question answering and part of speech tagging. There are different linear and nonlinear effective approaches for word embedding, for example, pretrained models Global Vectors For Word Representation (GloVe) [88], word2vec [71] and Rademacher embeddings. The GloVe is a neural (network) based word embedding, where the word2vec is mainly based on matrix decomposition. Assuming that word embeddings are available for all the words of a vocabulary, a text embedding is simply a linear combination of them with the coefficients coming from its unigram representation. In other words, we have $Ax^\text{unigram}_T = y^\text{embedding}_T$, where $A \in \mathbb{R}^{m \times |\mathcal{V}|}$ is an embedding (or measurement matrix) generated from a specific methodology, $x^\text{unigram}_T \in \mathbb{R}^{|\mathcal{V}|}$ is the unigram representation of a text, and $y^\text{embedding}_T \in \mathbb{R}^m$ is its (unigram) embedding. The measurement matrices in our experiments are generated via GloVe embedding method and the Radamacher distribution with the following probability mass function:

$$f(k) = \begin{cases} 1/2 & \text{if } k = -1, \\ 1/2 & \text{if } k = +1. \end{cases}$$

These matrices are: $m \times 17000$ for MR movie reviews [86] and $m \times 20000$ for SUBJ subjectivity dataset [85], respectively. Embedding sizes for our experiments are $m = 50, 100, 200, 300$ and $1600$.

These figures confirm the efficiency of $\ell_1$-recovery and greedy algorithms in recovering unigram representations of 50 documents, where success is achieved if the relative error is smaller than $10^{-7}$. We avoid a detailed explanation on the implemented algorithms for this specific application rather finish this survey with a more general conclusion as follows. The $\ell_p$-recovery with $0 < p \leq 1$, where $p < 1$ promises better theoretical results than $p = 1$, and greedy algorithms are both effective in recovering
sparse vectors in various applications if the measurement matrix inherits several properties explained in this survey. In the constrained case, the constraint set also plays a crucial role in the sparse recovery conditions as well [105]. Nevertheless, these properties often provide sufficient conditions and for some applications specified measurement matrices can work properly as well and even better than random matrices [4]. Further, except when the sparsity level is relatively small, the $\ell_p$-recovery with $0 < p \leq 1$ is superior to the greedy approach. For a more comprehensive study on the computational complexities and efficiency of these algorithms or related studies, see, e.g., [2, 51, 90, 136].

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