We construct the dilaton potential in the gravity dual theory of AdS/QCD for the warp factor of Refs. [1, 2]. Using this AdS$_5$-metric with properties similar to QCD, we find that the gravity dual leads to a meaningful gauge coupling in the region between the charmonium and bottomium mass, but differs slightly from QCD in the extreme UV. When we fix the ultraviolet behavior in accord with the $\beta$-function, we can obtain good agreement with the overall heavy quark-antiquark potential. Although the leading order proportional to $-\alpha_s^4/3r$ differs from perturbative QCD, the full potential agrees quite well with the short distance QCD potential in NNLO.

1 Introduction

Maldacena’s conjecture [3] states that, in the low energy limit, the large-$N_c$ $\mathcal{N} = 4$ super Yang-Mills field theory in four-dimensional space is equivalent to the type $IIB$ string theory in $AdS_5 \times S^5$-space. This may provide a possibility to solve the longstanding problem of strongly coupled QCD in the low energy limit. But a rigorous top-down approach is still far from giving any experimental prediction. Using the idea of holography, bottom-up models can quantitatively reproduce many results of QCD in the low energy limit gained by other existing methods like lattice QCD. The core of these bottom-up models is to find a reasonable non-conformal metric of the $AdS_5$-space, which incorporates relevant physical information.

To obtain such a metric, one can either maintain the basic form of the conformal $AdS_5$-metric and introduce a factor by hand, or assume a very general form of the metric, and calculate its explicit parameters. In this paper, we further investigate the warp factor given in Refs. [1, 2] based on the interpretation of the fifth-dimension $z$ as a coordinate proportional to the inverse energy resolution. The warp factor of Refs. [1, 2] resembles the running coupling in QCD - having a strong inverse logarithmic growth in the infrared. The
Minkowskian form of the metric with the warp factor is

\[ ds^2 = h(z) \frac{1}{(\Lambda z)^2} (-dt^2 + d\vec{x}^2 + dz^2), \]

with

\[ h(z) = \frac{\log (\frac{1}{\epsilon})}{\log \left[ \frac{1}{(\Lambda z)^2 + \epsilon} \right]}, \]

This metric with asymptotically conformal symmetry in the UV and infrared slavery in the IR region yields a good fit to the heavy $Q\bar{Q}$-potential with

\[ \Lambda = 264 \text{ MeV}, \]
\[ \epsilon = \Lambda^2 l_s^2 = 0.48. \]

In Ref. [2], we have used this metric to calculate the expectation value of one circular Wilson loop $\langle W \rangle$ and the correlator of two concentric circular Wilson loops $\langle WW \rangle$ from the Nambu-Goto action of the form:

\[ S_{NG} = \frac{1}{2\pi l_s^2} \int d^2 \xi \sqrt{\det h_{ab}}, \]

where $l_s$ is the string length defined above. The induced world-sheet metric in the Nambu-Goto action is called $h_{ab}$:

\[ h_{ab} = G_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b}. \]

This calculation gives reasonable results for one and two Wilson loops. In addition to the confinement physics, we have derived the gluon condensate at zero temperature. The above metric, Eq. (1), is given in the string frame, and all the calculations mentioned above are done in this frame. Low energy string theory can be approximated further by a gravity action with a background scalar field, the dilaton $\phi$. When the conformal symmetry is broken, the gravity action contains a dilaton potential $V(\phi)$, which is no longer constant and can be calculated from the metric with the help of the Einstein equations. Relating the energy scale of the gauge theory living on the boundary of the $AdS_5$-space and the bulk $z$-coordinate, we can predict the running gauge coupling inside the region between the charmonium and the bottomium mass.

When we check whether this solution of the running coupling is compatible with the behavior of the QCD coupling in the far ultraviolet, we find small deviations. The reason is that our metric in spite of many good phenomenological features has conformal invariance in the ultraviolet. By modifying the dilaton potential $V(\phi)$, we can make the potential
consistent with the QCD $\beta$-function and the heavy $Q\bar{Q}$-potential. We find good agreement with the short distance behavior of the $Q\bar{Q}$-potential. Finally we close the circle of investigation and recalculate the warp factor $\tilde{h}(z)$ of the modified metric in string frame.

The outline of the paper is as follows: After the introduction in Section 1, Section 2 describes the formal construction of the dilaton potential from the metric in the string frame. Section 3 shows how the energy scale of QCD is related to the bulk coordinate $z$ and gives the numerical calculation of the dilaton potential. In Section 4 the resulting QCD running coupling is derived and the ultraviolet behavior of the potential is improved. Section 5 applies the UV-improved dilaton potential to a calculation of the heavy quark potential and the glueball spectrum. We give in Section 6 a discussion about the modified dilaton potential, the running coupling and the new warp factor. Finally, Section 7 gives a final discussion and our conclusions. In Appendix A the infrared and ultraviolet properties of the 5-dim Nambu-Goto theory with the metric of Eqs. (1) and (2) is investigated. In Appendix B we derive the infrared properties of the UV-improved dilaton potential. In Appendix C we show technical details for the analytical computation of the heavy $Q\bar{Q}$-potential in the small distance regime.

2 Construction of the Gravity Dual Theory with Dilaton Potential

We argued in the previous section that the warp factor given by Eq. (1) proposed in Ref. [1] produces reasonable results, as shown in Refs. [1, 2]. But the metric has to be consistent with gravity. In the low energy limit, string theory can be approximated by its gravity dual theory. In a top-down approach one obtains the $IIA/IIB$ effective action for the long-range fields of $D_3$-branes:

$$S_{10D-Gravity} = \frac{1}{2\kappa^2_{10}} \int d^{10}x \sqrt{-G^s} \left[ e^{-2\phi} \left( R + 4(\nabla\phi)^2 \right) - \frac{1}{2 \cdot 5!} F_5^2 \right].$$

Here $R$ is the Ricci-scalar of the gravitational field, $\phi$ is the dilaton field and $F_5$ is the 5-form flux originating from the branes. This string frame action is characterized by the exponential dilaton dependence in front of the curvature scalar. In order to keep the theory simple, we neglect the axion field $a$ and other eventual space filling branes related to quark dynamics. After inserting the equation of motion for $F_5$ back into the action, we transform the action from the string frame to the Einstein frame, in which the Einstein term has the conventional form by a Weyl rescaling of the metric:

$$G^E_{\mu\nu}(X) = e^{-4\phi} G^s_{\mu\nu}(X).$$
Integrating over the $S_5$-space and combining several terms into a dilaton potential $V(\phi)$, we obtain the following five-dimensional action in the Einstein frame [4]:

$$S_{5D-Gravity} = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-G^E} \left( R - \frac{4}{3} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right).$$  \hspace{1cm} (9)

In principle, there can be an additional Gibbons-Hawking term in the action [5, 6]. But since this term does not contribute to the variation with respect to the metric $G^E_{\mu\nu}$, it does not affect the Einstein equations, which determine all the physical quantities at zero temperature. We must emphasize that, at finite temperature, only Einstein equations themselves are not enough to determine all the physical quantities, but the action is also important, hence at finite temperature the Gibbons-Hawking term contributes to the thermodynamic quantities. In this paper, we only focus on the $T = 0$ case, therefore, we can neglect the Gibbons-Hawking term.

We assume that the background dilaton potential denoted by $V(\phi)$ incorporates some information of the 5-form $F_5$ and higher curvature corrections extending the range of applicability. The background field $\phi(z)$ has a $z$-dependence reflecting necessary corrections at higher energies. Let us comment on the dimensionalities of the quantities introduced in the action, Eq. (9). The Ricci scalar $R$ has dimension $[R] = \frac{1}{\text{length}^2}$, $\kappa_5^2$ has dimension $[\kappa_5^2] = \text{length}^3$. Consequently, the bulk field $\phi$ is dimensionless, and the dimension of the dilaton potential $V(\phi)$ is $[V(\phi)] = \frac{1}{\text{length}^2}$. Our space-time metric has Minkowski signature with the sign convention $(-, +, +, +, +)$. We emphasize that all quantities in Eq. (9) have to be in the Einstein frame. In the remaining part of this paper, we simply write $G_{\mu\nu}$, instead of $G^E_{\mu\nu}$. After variation of the Einstein-frame action, Eq. (9), with respect to the metric $G_{\mu\nu}$, one obtains the following equations of motion

$$\left( R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} \right) - \left( \frac{4}{3} \partial_\mu \phi \partial^\nu \phi - \frac{1}{2} G_{\mu\nu} \left( \frac{4}{3} \partial_\sigma \phi \partial^\sigma \phi + V(\phi) \right) \right) = 0. \hspace{1cm} (10)$$

The Einstein equations contain the energy momentum tensor $T_{\mu\nu}$,

$$E_{\mu\nu} = T_{\mu\nu}. \hspace{1cm} (11)$$

With the above chosen signature of the metric and assuming just a flat metric, we obtain $T_{00} = \frac{4}{3} (\phi')^2 + \frac{1}{2} V(\phi)$, where the prime denotes derivative with respect to $z$. Up to a normalization factor of the kinetic term, this agrees with the definition of the energy of a scalar field.  

1 Note in Refs. [4, 7] $V(\phi)$ is defined as the negative of our potential.
The warped metric in the string frame, Eq. (1), becomes the Einstein-frame metric $G_{\mu \nu}$ through Eq. (8):

$$ds_E^2 = e^{-\frac{4}{3} \phi h(z)} \frac{1}{(\Lambda z)^2} (-dt^2 + d\vec{x}^2 + dz^2). \quad (12)$$

The resulting Einstein equations, Eqs. (10), can be solved for the dilaton. Besides the Einstein equations, the action given by Eq. (9) yields another Euler-Lagrange equation, which can be obtained by varying the action with respect to $\phi$:

$$\Box \phi - \frac{3}{8} \frac{dV(\phi)}{d\phi} = 0. \quad (13)$$

This equation contains only redundant information and is not independent of the Einstein equations, Eqs. (10). Hence Eq. (13) needs no further treatment. This is a consequence of the fact that $\phi$ is actually not an independent field, but part of $G_{\mu \nu}$.

Now we turn to the Einstein equations, Eqs. (10). To be consistent with Refs. [4, 7], we use the following form of the space-time metric in the Einstein frame

$$G_{\mu \nu} = e^{2A(z)} \cdot \text{diag}(-1, 1, 1, 1, 1). \quad (14)$$

Comparing this metric with Eq. (12), we have

$$e^{2A(z)} = e^{-\frac{4}{3} \phi h(z)} \frac{1}{(\Lambda z)^2}. \quad (15)$$

We first calculate the tensor $E_{\mu \nu}$ in terms of $A(z)$ and its derivatives. Then we compute the components of the energy-momentum tensor $T_{\mu \nu}$ under the assumption that $\phi = \phi(z)$.

$$-T_{11} = T_{22} = T_{33} = T_{44} = -\frac{2}{3} (\phi')^2 - \frac{1}{2} e^{2A(z)} V(\phi), \quad (16)$$

$$T_{55} = \frac{2}{3} (\phi')^2 - \frac{1}{2} e^{2A(z)} V(\phi). \quad (17)$$

Thus, we end up with only two independent equations of motion, namely

$$3((A'(z))^2 + A''(z)) = -\frac{2}{3} (\phi')^2 - \frac{1}{2} e^{2A(z)} V(\phi), \quad (18)$$

$$6(A'(z))^2 = \frac{2}{3} (\phi')^2 - \frac{1}{2} e^{2A(z)} V(\phi). \quad (19)$$

Adding these equations we obtain a formal expression for the dilaton potential:

$$V(\phi(z)) = -e^{-2A(z)} (9(A'(z))^2 + 3A''(z)). \quad (20)$$
Multiplying Eq. (18) by \((-1)\) and then adding it to Eq. (19) we find an important relation between the dilaton and the metric profile:

\[ (\phi')^2 = \frac{9}{4}(A'(z))^2 - A''(z)). \tag{21} \]

These two equations, Eq. (20) and Eq. (21), are identical with Eq. (2.16) in Ref. [4], up to a different sign of the dilaton potential. The different sign stems from the fact that we have a minus sign in front of the potential in the action, Eq. [9]. We have chosen the potential in this way, because we want the \( T_{00}\)-component of the energy-momentum tensor to be the sum of the kinetic and the potential energy. We see that Eq. (21) depends on the profile \( A(z)\), which is a function of the warp factor \( h(z)\) and the dilaton field \( \phi(z)\) (cf. Eq. (15)). The resulting second order differential equation for \( \phi(z)\) needs two boundary conditions, which we will obtain from the QCD running coupling constant once the bulk coordinate \( z\) is connected with the energy scale \( E(z)\).

### 3 The Energy Scale \( E(z) \) and the Solution for the Dilaton Potential

Before we turn to the numerical solution of Eq. (21), we must do some preparations. Perturbative string theory is based on a topological expansion in the string coupling \( g_s\), which generates factors \( e^{-\chi \phi}\) in the string partition function, where \( \chi\) is the Euler characteristic of the string surface, e.g. \( \chi = 2\) for a sphere. Therefore the string coupling is proportional to \( e^\phi\). We choose the proportionality constant equal to unity, i.e.

\[ g_s = e^\phi. \tag{22} \]

The elastic amplitude of two closed strings (glueballs) of order \( \mathcal{O}(g_s^2)\) corresponds to a contribution \( \mathcal{O}(\alpha^2)\) in the boundary field theory. In general, the \( AdS/CFT\) correspondence relates the string coupling \( g_s\) of perturbative string theory to the Yang-Mills coupling \( \alpha\) in the following manner:

\[ g_s = \frac{g_{YM}^2}{4\pi} = \alpha. \tag{23} \]

In order to simplify the notation, in the following we will use \( \alpha\) to denote both the Yang-Mills coupling related to the string coupling and the QCD running coupling.

The warping of the bulk space relates the bulk coordinate \( z\) to the energy scale \( E(z)\) associated with \( z\) via the gravitational blue-shift. It is pedagogical to use the radial coordinate \( r \propto 1/ z\). In \( SU(N)\), one has \( N\) \( D_3\)-branes as gravitational source located at \( r \equiv \frac{L^2}{z} \rightarrow 0^+ E_{r\rightarrow \infty}\) denotes the value of the energy scale at the holographic boundary

\[ \text{In our notation the radius of the } AdS_5\text{-space is } L = \frac{1}{\Lambda}, \text{ see Eq. (1).} \]
$r \to \infty$, when $E_r$ is the value of the energy at an arbitrary value of $r$. The blue-shift is given by the dimensionless ratio

$$\frac{E_r}{E_{r\to\infty}} = \sqrt{\frac{G_{tt}(r \to \infty)}{G_{tt}(r)}},$$

(24)

where $G_{tt}$ denotes the temporal component of the metric. In the limit $r \to \infty$, we are far away from the branes, where the space-time is asymptotically flat, which yields $G_{tt}(r \to \infty) = -1$. Hence, the blue-shift reads

$$E_{r\to\infty} = E_r \sqrt{-G_{tt}(r)} \quad \text{or equivalently} \quad E_{r\to\infty} = E_z \sqrt{-G_{tt}(z)}.$$  

(25)

In the unmodified $AdS_5$-space, $G_{tt} = -\frac{1}{(\Lambda z)^2}$ and hence $E_{r\to\infty} = E_z \frac{1}{\Lambda z}$. For $z \to 0$, we find the UV regime of the boundary field theory. This is in agreement with the intuition underlying the renormalization group interpretation of the $z$-coordinate, which was instrumental to guess the warp factor $h(z)$. For $E_z$ one can choose an arbitrary value of the energy scale. In order to simplify the expression, we choose the confinement scale given by $\Lambda = 264$ MeV. This leads to the following explicit formulas in the Einstein frame

$$E_{r\to\infty} = e^{-\frac{2}{3} \phi(z)} \frac{\sqrt{h(z)}}{z}$$

(26)

$$= \alpha^{-\frac{2}{3}} \frac{\sqrt{h(z)}}{z}$$

(27)

$$= e^{A(z)} \cdot \Lambda.$$  

(28)

Suppose that we know the value of the coupling constant $\alpha$ at a given energy scale $E = E_{r\to\infty}$, then we can find the corresponding value of $z$ from Eq. (27).

At a given value of $z$, $\phi(z) = \log(\alpha)$ gives just one boundary condition for Eq. (21). In order to obtain the second boundary condition to Eq. (21), we need a second value of the QCD running coupling. It is not easy to choose two appropriate energy values. The reason is the following. In the UV limit it is questionable whether the $AdS/CFT$ correspondence is still valid. On the other hand, in the IR limit, there is no reliable measurement of the running coupling. We think that the region between the charmonium mass and the bottomium mass, i.e. between 3 GeV and 8 GeV, is a reasonable region where the modeling of $AdS/QCD$ with the warp factor of Eq. (1) should work well. Therefore, we propose as input values for $\alpha$ [8]:

$$\alpha(3 \text{ GeV}) = 0.25241, \quad \alpha(8 \text{ GeV}) = 0.18575.$$  

(29)
This also means that we have implicitly set $N_c = 3$ and $N_f = 4$. Although there is no entry in our model for color and flavor, we have fitted our final result to the Cornell potential, therefore we have implicitly made a choice for these parameters.

The fact that one has to fix two integration constants using initial conditions contrasts with the analysis of Refs. [7, 9], where the authors show that just one initial condition is enough. One possibility studied in Refs [7, 9] is fixing one of the parameters by requiring that the bulk singularity is not of the “bad kind”, which means that the singularity should be repulsive to physical fluctuations. In our case the singularity in the infrared is of the “good kind”. As we explain in Appendix A, the reason why we have to handle with two integration constants is that we do not use the perturbative $\beta$-function as starting point.

With the above conditions we solve Eq. (21) numerically, and obtain the dilaton field $\phi$ as a function of $z$ (cf. Fig. 1). Note the strong variations of the dilaton field for small and large $z$. For large $z$, infrared confinement at low energies is felt.

In Appendix A we have investigated the infrared and ultraviolet properties of $\phi(z)$ analytically. Defining

$$\xi = z_{\text{IR}} - z,$$

with

$$z_{\text{IR}} = \sqrt{1 - \epsilon/\Lambda},$$

we find in the infrared

$$\phi(\xi) = \frac{3}{16} \left( \log \frac{\xi}{\omega_{\text{IR}}} \right)^2 + \kappa_{\text{IR}} + \mathcal{O}(\xi \log \xi), \quad \xi \to 0,$$

Fig. 1: The dilaton field profile $\phi$ as a function of the bulk coordinate $z$, computed from the warp factor $h(z)$ of Eq. (2).
with
\[ \omega_{\text{IR}} = 4.55 \text{ GeV}^{-1}, \quad \kappa_{\text{IR}} = -0.758, \quad \omega_{\text{UV}} = 0.1285 \text{ GeV}^{-1}, \quad \kappa_{\text{UV}} = -1.386. \quad (33) \]
and in the ultraviolet
\[ \phi(z) = -\frac{\omega_{\text{UV}}}{z} + \kappa_{\text{UV}} + O(z), \quad z \to 0. \quad (34) \]

The parameters \( \kappa_{\text{UV}} \) and \( \kappa_{\text{IR}} \) are related, in the sense that setting \( \kappa_{\text{UV}} \) in the UV then sets \( \kappa_{\text{IR}} \) in the IR. \( \omega_{\text{UV}} \) and \( \omega_{\text{IR}} \) are related in the same way. From a numerical computation of \( \phi(z) \) in the full regime \( 0 < z < z_{\text{IR}} \) we find that Eq. (33), or equivalently Eq. (29), leads to:
\[ \omega_{\text{UV}} = 0.1285 \text{ GeV}^{-1}, \quad \kappa_{\text{UV}} = -1.386. \quad (35) \]

We show in Fig. 2 the relation between the parameters \( \kappa_{\text{UV}} \) and \( \kappa_{\text{IR}} \) when \( \omega_{\text{IR}} \) and \( \omega_{\text{UV}} \) are fixed to the values quoted in Eqs. (33) and (35). The functional dependence is
\[ \kappa_{\text{UV}} = \kappa_{\text{IR}} - 0.628. \quad (36) \]

Inserting \( \phi(z) \) into Eq. (26), we calculate immediately the associated energy scale \( E_{r \to \infty}(z) \), which is shown in Fig. 3 as a function of the \( z \)-coordinate in the fifth dimension. At the lower edge of the \( z \)-scale, we have a strongly increasing energy \( E_{r \to \infty}(z) \) (cf. Fig. 3).
Consequently, the $A(z)$ in the metric can be calculated from Eq. (28). With $A(z)$, the other Einstein equation, Eq. (20), gives us $V(z)$. Combining $V(z)$ with $\phi(z)$, we obtain the dilaton potential $V(\phi)$, which is shown in Fig. 4. The result shows an approximately constant dilaton potential until $\phi(z) = 1$. Beyond this point the dilaton potential falls rapidly. Recall that in the $AdS$-space, the “cosmological” term is negative and slowly varying due to the asymptotic conformal behavior of the warp factor. In the conformal limit, $\phi' = 0$, and the dilaton potential should have the value $-\frac{12}{L^2}$, which can solve the Einstein equations, Eqs. (20) and (21).

Questions about the stability of the vacuum because of the large and negatively un-
bound dilaton potential have to be analyzed, but it is well known that due to Breitenlohner-Freedman bound negative second order derivatives in the dilaton potential do not cause problems in the presence of gravity [10] [11].

4 Constraining the Dilaton Potential by the QCD $\beta$-Function

From the energy scale $E(z)$ and the dilaton profile $\phi(z)$, we are now able to calculate the value of the strong coupling constant at any energy scale $\alpha = e^{\phi(z)}$. The gravity dual of string theory allows to interpolate the QCD coupling between our boundary values at 3 GeV and 8 GeV in a satisfactory manner, as one sees from the comparison of $\alpha = e^\phi$ in Fig. 5 with the strong coupling from the PDG web tool [8]. The good description of the coupling adds another positive feature to the warp factor proposed in Refs. [1] [2].

![Fig. 5: The running coupling as a function of the energy scale. Full (red) line corresponds to the string theory result that follows from the relation $\alpha = e^\phi$, using the warp factor $h(z)$ of Eq. (2). We show as a dashed (black) line the experimental values of the running coupling from the PDG data [8]. The two points correspond to the input conditions, Eq. (29).](image)

When we investigate the behavior of the running coupling in the deep UV more closely, we expect some contradiction with QCD. As much as conformal behavior is favored in correlation functions, where the leading behavior up to logarithmic corrections is correctly reproduced, we have to deviate from the correct running of the coupling in the deep ultraviolet, since our metric $h(z)$ has scale independence in this limit. We start with the definition of the $\beta$-function:

$$\beta \equiv E \frac{d\alpha}{dE}. \quad (37)$$
Eq. (28) tells us that the energy can be expressed as the product of \( e^A \cdot \Lambda \). Using \( \alpha = e^\phi \), we obtain:
\[
\beta \equiv E \frac{d\alpha}{dE} = e^A \Lambda \cdot \frac{d(e^\phi)}{d(e^A \Lambda)} = \frac{e^\phi d\phi}{dA} = \frac{e^{\phi(z)} \cdot \phi'(z)}{A'(z)}.
\]
(38)

All quantities in the last expression are calculable from the warp factor \( h(z) \). In Fig. 6 we show the \( \beta \)-function from our \( AdS/QCD \) model together with the QCD \( \beta \)-function at two-loop level. In QCD the \( \beta \)-function has the following form [12]:
\[
\beta(\alpha) = -b_0 \alpha^2 - b_1 \alpha^3,
\]
(39)

with
\[
b_0 = \frac{1}{2\pi} \left( \frac{11}{3} N_c - \frac{2}{3} N_f \right), \quad \text{and} \quad b_1 = \frac{1}{8\pi^2} \left( \frac{34}{3} N_c^2 - \left( \frac{13}{3} N_c - \frac{1}{N_c} \right) N_f \right).
\]
(40)

As argued before, we have set \( N_c = 3 \) and \( N_f = 4 \). In this case, \( b_0 = \frac{25}{6\pi} \), and \( b_1 = \frac{77}{12\pi^2} \).

One sees that the agreement is very good near \( \alpha = 0.25 \), i.e. near the charmonium mass region, where the phenomenological adjustment has been done. For smaller values of \( \alpha \) there is a sizable discrepancy. Explicitly in this region, the \( \beta \)-function stemming from the warp factor \( \propto -\alpha \), not \( \propto -\alpha^2 \) as in QCD. This explains the deviation visible in Fig. 6 for small \( \alpha \). We have derived in Appendix A an analytic expression for \( \beta(\alpha) \) in the deep ultraviolet, cf. Eq. (119). The strongly increasing warp factor \( h(z) \) in the infrared leads to a stronger beta function for large \( \alpha \).

How can one repair this problem and make the dilaton potential consistent with QCD in the ultraviolet? The basic idea of Refs. [4] and [7] is to use the QCD \( \beta \)-function itself as a starting point, and derive the metric from the \( \beta \)-function. The resulting metric is then unambiguously consistent with the QCD \( \beta \)-function, as expected, and the running coupling calculated from this new metric is necessarily correct. This procedure presents a systematic approach to define the dilaton potential in the ultraviolet. In the infrared, for large positive values of the dilaton field there remains the problem to choose a parametrization of the potential. Calculations of the string tension have been proposed as tests of this parametrization at zero temperature [13] or of the spatial string tension at finite temperature [14]. In our case, we will build on the phenomenological work done in Refs. [1, 2] and will fit the constrained form of the potential to the heavy quark potential.

In the following, we will review the important formulas given in Refs. [4] and [7]. In the so-called “domain wall coordinates” \( du \equiv e^A dz \):
\[
ds^2 = e^{2A}(-dt^2 + d\vec{x}^2) + du^2,
\]
(41)
Fig. 6: The $\beta$-function as a function of the running coupling $\alpha$. We show as a full (red) line the string theory result that follows from the warp factor $h(z)$ of Eq. (2), using Eq. (38). We show for comparison as a full (blue) line the result corresponding to Eq. (51) with the parameters provided in Sec. 5. Dashed (black) line is the QCD $\beta$-function given by Eq. (39).

the Einstein equations become

$$3 \ddot{A} + 12 \dot{A}^2 = V, \quad (42)$$

$$\ddot{A} = -\frac{4}{9} \dot{\phi}^2, \quad (43)$$

where the dot denotes the derivative with respect to $u$. After defining two auxiliary variables

$$W \equiv -\frac{9}{4} \dot{A}, \quad (44)$$

$$X \equiv -\frac{3}{4} \frac{d \log W}{d \phi}, \quad (45)$$

we may rewrite the Einstein equations using $W$:

$$\dot{A} = -\frac{4}{9} W, \quad \dot{\phi} = \frac{dW}{d \phi}, \quad V = \frac{4}{3} \left( \frac{dW}{d \phi} \right)^2 - \frac{64}{27} W^2. \quad (46)$$

$W(\phi)$ plays the role of a superpotential. Several equivalent expressions hold for $X$:

$$X = \frac{\beta(\alpha)}{3\alpha}, \quad (47)$$

$$X = \frac{\dot{\phi}}{3A}, \quad (48)$$

$$X = \frac{1}{3\alpha} \cdot \frac{d\alpha}{dA}. \quad (49)$$
When the $\beta$-function is known, we can calculate $X$ through the first expression, Eq. (47). With this $X$, $W$ is obtained as a solution of Eq. (45). Consequently, $\dot{A}$ and $\dot{\phi}$ can be calculated through Eq. (44) and Eq. (48), respectively. Finally, the general form of the dilaton potential $V$, is determined from the last equation of the three Eqs. (46) as

$$V(\phi) = V_0 \cdot (1 - X^2) \cdot e^{-\frac{8}{3} \int_{-\infty}^{\phi} X(\tilde{\phi}) d\tilde{\phi}}.$$  (50)

Therefore, for a $\beta$-function given over the whole range of $\alpha$, the dilaton potential is fixed. In the IR region, we do not know the correct form of the $\beta$-function. If we want to impose confinement in the IR region, some forms of the $\beta$-function are excluded, but there still can be several possible classes to achieve confinement [7]. It is well known that the definition of the $\beta$-function becomes dependent of the quantity one studies, when the gauge coupling becomes strong. Therefore, the question arises, which parametrization of the $\beta$-function should one choose in the infrared. One possible choice is given by Eq. (5.1) in Ref. [4]. Here we propose another possible choice which combines the correct UV-behavior with some integrable form,

$$\beta(\alpha) = -b_2 \alpha + \left[ b_2 \alpha + \left( \frac{b_2}{\bar{\alpha}} - b_0 \right) \alpha^2 + \left( \frac{b_2}{2\bar{\alpha}^2} - \frac{b_0}{\bar{\alpha}} - b_1 \right) \alpha^3 \right] e^{-\alpha/\bar{\alpha}}.$$  (51)

This parametrization has the required $\beta$-function in the UV region as limit,

$$\alpha \to 0 : \quad \beta(\alpha) \approx -b_0 \alpha^2 - b_1 \alpha^3,$$

and confinement property in the IR region [7].

We can get a constraint on the parameters $b_2$ and $\bar{\alpha}$ by demanding a good behavior for the running coupling. For $\alpha < \bar{\alpha}$ the QCD-coupling is strictly perturbative, whereas for $\alpha > \bar{\alpha}$ the $\beta$-function is characterized by the non-perturbative linear term $-b_2 \alpha$. When we consider the coupling in the region $0.6 \text{ GeV} < E < 15 \text{ GeV}$, we obtain a good fit of the running coupling for values $1.2 < b_2 < 3$ and $\bar{\alpha} > 0.27$ along the line

$$\frac{b_2}{\bar{\alpha}} = 5.09.$$  (52)

The $\chi^2$/d.o.f. is very close to its minimum in the entire region of $b_2$. The perturbative running for energies larger than the charmonium mass is guaranteed by the limit on $\bar{\alpha}$. On the other hand, the $\beta$-function given by Eq. (51) leads to a confining theory if $b_2 \geq 3/2$.

---

3 This interval for $b_2$ will be further reduced after imposing the requirements of confinement and the infrared singularity being repulsive to physical modes, cf. Appendix B.
To see that we need only to study the IR behavior of the corresponding function $X(\alpha)$, cf. Eq. (47). In this limit

$$\lim_{\alpha \to \infty} \left( X(\alpha) + \frac{1}{2} \right) \log \alpha = \lim_{\alpha \to \infty} \left( -\frac{b_2}{3} + \frac{1}{2} \right) \log \alpha \leq 0 \iff b_2 \geq \frac{3}{2},$$

(53)

which constitutes the general criterion for confinement in its version of the $\beta$-function, Ref. [7].

The dilaton potential corresponding to the parametrization of $\beta(\alpha)$ is

$$V(\alpha) = V_0 \left( 1 - \left( \frac{\beta(\alpha)}{3\alpha} \right)^2 \right) \left( \frac{\alpha}{\alpha} \right)^{\frac{b_2}{3}}$$

$$\cdot \text{Exp} \left[ \frac{4}{9} ((2\gamma - 3)b_2 + 4b_0\bar{\alpha} + 2b_1\bar{\alpha}^2) \right]$$

$$\cdot \text{Exp} \left[ \frac{4}{9} e^{\frac{-\alpha}{\alpha}} \left( 3b_2 - 4b_0\bar{\alpha} - 2b_1\bar{\alpha}^2 + \left( \frac{b_2}{\alpha} - 2b_0 - 2b_1\bar{\alpha}\right) \right) \right]$$

$$\cdot \text{Exp} \left( \frac{8b_2}{9} \cdot \text{ExpIntegralE} \left[ 1, \frac{\alpha}{\alpha} \right] \right).$$

(54)

In this expression, $\gamma$ is the Euler’s constant, and $\text{ExpIntegralE}(n, z) \equiv \int_1^\infty \frac{e^{-zt}}{t^n} dt$ is the exponential integral function, $b_0$ and $b_1$ are the coefficients appearing in the QCD $\beta$-function given by Eq. (39), while $V_0$, $\bar{\alpha}$ and $b_2$ are undetermined constants. An interesting exercise is to expand Eq. (54) in the UV. The result is

$$V(\alpha) = V_0 \left\{ 1 + \frac{8}{9} b_0\alpha + \frac{1}{81} \left( 23b_0^2 + 36b_1 \right) \alpha^2$$

$$+ \frac{2}{2187\bar{\alpha}^3} \left[ 54b_2 - 162b_0\bar{\alpha} - 324b_1\bar{\alpha}^2 + 20b_0^3\bar{\alpha}^3 + 189b_0b_1\bar{\alpha}^3 \right] \alpha^3$$

$$+ O(\alpha^4) \right\}. $$

(55)

The leading orders are determined by the UV parameters $b_0$ and $b_1$, and the unknown constants $b_2$ and $\bar{\alpha}$ start contributing at $O(\alpha^3)$. The same feature is shared by the parametrization of Ref. [15], although in this reference the order $\alpha^3$ is replaced by $\alpha^{8/3}$. The behavior of the dilaton potential for larger values of $\alpha$ can be phenomenologically determined by fitting to the heavy quark potential which we will do in the next section.

5 Fit of Parameters to $Q\bar{Q}$ Potential and Running Coupling

With the modified dilaton potential of Eq. (54) we can calculate the heavy quark-antiquark potential. In Ref. [1] the heavy $Q\bar{Q}$-potential was in very good agreement with the Cornell
potential. Now, we recalculate it with the modified dilaton potential $V(\phi)$ to focus on two particular questions: Firstly, how does the potential from string/gravity theory compare with the potential from three loop perturbation theory? This serves as a test whether the improvements on the $\beta$-function pay off in the UV behavior of observables. Secondly, can the long distance string tension help us fix the remaining parameters? In Ref. [1] the calculation was done in the bulk $z$-coordinate, but in this section it is better to work with the variable $\alpha = e^\phi$. A general derivation for the heavy $Q\bar{Q}$-potential using $\alpha = e^\phi$ has been given in Ref. [13]. In the following we will add the correct short distance and long distance analysis for the first time. We refer to Appendix C for further discussions and computation in the small separation limit.

The first step is to derive the explicit form of the metric which is consistent with the $\beta$-function given by Eq. (51). We work in coordinates dependent on the running coupling $\alpha$ as a variable, instead of $z$ or $u$. From the domain wall coordinates relation $e^A dz = du$, one may easily derive

$$e^A dz = \frac{d\alpha}{\alpha \phi}.$$  \hfill (56)

Then using Eqs. (44), (47) and (48) and solving Eq. (45) we get

$$\frac{d\alpha}{dz} = \frac{1}{\bar{\ell}} e^{A-D},$$  \hfill (57)

with $\bar{\ell}$ given by

$$\bar{\ell} \equiv \frac{6}{\sqrt{-3V_0}},$$  \hfill (58)

while $A$ and $e^D$ are functions of $\alpha$ given by

$$A(\alpha) = A_* + \int_{\alpha_*}^{\alpha} \frac{1}{\beta(a)} da,$$  \hfill (59)

and

$$e^D = -\frac{1}{\beta(\alpha)} \exp \left[ \frac{4}{3} \int_0^{\alpha} \frac{\beta(a)}{3a^2} da \right],$$  \hfill (60)

with the fixed constants $\alpha_*$ and $A_*$ defined at the energy $E$:

$$\alpha_* = 0.25241, \quad E = e^{A_*} \Lambda = 3 \text{ GeV}.$$  \hfill (61)

Then the new metric in Euclidean space, which includes the new warp factor $\tilde{h}(z)$, follows from Eq. (1) using Eqs. (15) and (57), and it reads
\[
d s^2 = \tilde{h}(z(\alpha)) \frac{1}{(\Lambda z(\alpha))^2} (-dt^2 + d\vec{x}^2) + e^{\frac{4\phi}{3}} \cdot \ell^2 e^{2D} d\alpha^2,
\]  
with \(\tilde{h}(z)\) given by
\[
\tilde{h}(z) = e^{2A(z)} e^{\frac{4\phi}{3}} (\Lambda z)^2.
\]  

One advantage of the present computation starting from the \(\beta\)-function given by Eq. \((51)\) is that we need \(\alpha_* \) as single input value for \(\alpha\), in contrast to the two values we needed within the formalism based on the warp factor \(h(z)\) of Secs. \([2]\) and \([3]\) cf. Eq. \((29)\).  

Note that in \(A(\alpha)\) the \(\beta\)-function enters the integral in the denominator, and in \(e^D(\alpha)\) it appears once in the denominator and once in the numerator of the exponential function. Taking into account that \(\beta(\alpha) < 0\) one finds that both \(A\) and \(e^D\) become large for small \(\alpha\). For large \(\alpha\), \(e^D\) becomes small, while \(A\) behaves as \(-\frac{1}{b^2} \log \alpha\) in this regime.  

The general procedure to compute \(V_{Q\bar{Q}}\) within the classical approximation is similar to the one used in Refs. \([1, 13]\). The heavy quark potential follows from the Nambu-Goto action for a rectangular Wilson loop with a short spatial side and a much longer time side, i.e.
\[
\langle W \rangle \simeq e^{-T \cdot V} \simeq e^{-S_{NG}}.
\]  

The picture is given by a string stretched between a quark and an antiquark, located at \(x_1 = \frac{\rho}{2}\) and \(x_2 = -\frac{\rho}{2}\) respectively, which dips into the bulk of the background \(AdS_5\)-space. The separation \(\rho\) between two quarks as well as the potential energy can be expressed as functions of \(\alpha_0\), which is the value of \(\alpha\) at the mid-point between the quark and the antiquark.  

In principle the Nambu-Goto action \(S_{NG}\) can now contain a new string length \(\tilde{l}_s\) compared with the Nambu-Goto action defined in section\([1]\).  

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\[4\] With the \(\beta\)-function of the previous model we cannot define \(e^D\) in the same way because we effectively need a cutoff at small \(\alpha\), cf. Eqs. \((60)\) and \((119)\). In this case the computation of \(d\alpha/dz\) should be done in a different way. Using Eqs. \((44)\), \((48)\) and \((56)\) one has
\[
\frac{d\alpha}{dz} = -\frac{4}{9} \beta(\alpha) W(\alpha) e^{A(\alpha)},
\]  
where \(A(\alpha)\) is given by Eq. \((59)\) and
\[
W(\alpha) = W(\alpha_{ct}) e^{-\frac{4}{3} \int_{\alpha_{ct}}^{\alpha} \frac{d\alpha}{\beta(\alpha)} d\alpha}.
\]  

The cutoff \(\alpha_{ct}\) introduces a new integration constant, which is multiplicative and related to a factor \(e^{\kappa_{UV}}\) in \(\alpha\), cf. Eq. \((34)\). The correct QCD \(\beta\)-function in the UV, however, renders the integration in Eq. \((65)\) finite. No multiplicative constant is needed. Only a single input value \(\alpha_*\) is sufficient.
Fit of Parameters to $Q\bar{Q}$ Potential and Running Coupling

\[ S_{\text{NG}} = \frac{1}{2\pi l_s^2} \int d^2 \xi \sqrt{\det \tilde{h}_{ab}}, \quad (67) \]

where $\tilde{h}_{ab}$ is the new induced worldsheet metric defined by $\tilde{h}_{ab} = \tilde{G}^{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b}$.

To obtain the heavy $Q\bar{Q}$-potential, we need to express the separation $\rho$ between the quark and the antiquark, as well as the value of the potential $V_{Q\bar{Q}}$ as functions of $\alpha_0$, which is the value of $\alpha$ at the mid-point between the quark and the antiquark. Similarly, $A_0 = A(\alpha_0)$ and $\phi_0 = \phi(\alpha_0)$ are the values of $A$ and $\phi$ at the mid-point. The separation between the quark and the antiquark is given by [13]

\[ \rho(\alpha_0) = 2\bar{\ell} e^{-A_0} \int_0^{\alpha_0} e^{D-3\tilde{A}+\tilde{\alpha} - \frac{4}{3} \tilde{\alpha}^2} \frac{d\alpha}{\sqrt{1 - \tilde{\alpha} - \frac{8}{3} e^{-4\tilde{A}}} \cdot e^{-4\tilde{A}}}, \quad (68) \]

with

\[ \tilde{A} \equiv A - A_0, \quad (69) \]
\[ \tilde{\phi} \equiv \phi - \phi_0, \quad (70) \]
\[ \tilde{\alpha} \equiv \frac{\alpha}{\alpha_0}, \quad (71) \]

The bare potential $V_{Q\bar{Q}}$ calculated from the Nambu-Goto action is divergent, so we have to regularize it. The divergence means that the quark-antiquark pair becomes infinitely heavy. An obvious way to remove this divergence is to subtract the rest mass of the two heavy quarks [13] [16]. Then the finite part of the potential is

\[ V_{Q\bar{Q}}(\alpha_0) = \frac{\bar{\ell} \alpha_0^4 e^{-A_0}}{\pi l_s^2} \left[ \int_0^{\alpha_0} d\alpha \cdot \tilde{\alpha} \frac{\alpha_0 e^{D+\tilde{A}} \left(1 - \sqrt{1 - \tilde{\alpha} - \frac{8}{3} e^{-4\tilde{A}}} \right)}{\sqrt{1 - \tilde{\alpha} - \frac{8}{3} e^{-4\tilde{A}}}} - \int_{\alpha_0}^{\infty} d\alpha \cdot \tilde{\alpha}^3 \cdot e^{D+\tilde{A}} \right]. \quad (72) \]

Combining Eq. (68) with Eq. (72), we obtain the heavy quark potential as a function of the separation between the quark and the antiquark. In the numerical computation of the second integral of Eq. (72) we replace the variable $\alpha \rightarrow \frac{\alpha_0}{\alpha}$, so that the integral transforms into an integration between 0 and 1 for the variable $\tilde{\alpha}$, which is much easier to compute. The regularization procedure ensures that the integrals are ultraviolet convergent as can be easily proved.

In order to fix the three parameters $V_0$, $\tilde{\alpha}$ and $b_2$ of the dilaton potential and the string constant $\bar{\ell}_s$, we will study separately the short distance, i.e. the ultraviolet (UV) regime, and the large distance, i.e. the infrared (IR) regime. The parameter $V_0$ is relevant in the UV, while $\tilde{\alpha}$ and $b_2$ become important in the IR, and $\bar{\ell}_s$ naturally scales the potential.
Let us first study the infrared properties of integrals appearing in Eqs. (68) and (72). For this purpose we focus on the large $\alpha$ behavior of the $\beta$-function, where $-b_2\alpha$ is the relevant term, cf. Eq. (51). As we show in Sec. 4, a value $b_2 \geq 3/2$ ensures that the theory is confining. On the other hand, in the limit $\alpha \to \infty$ the argument inside the square root in Eqs. (68) and (72) becomes

$$1 - \tilde{\alpha}^{-\frac{4}{3}} e^{-4\tilde{A}} \simeq 1 - \tilde{\alpha}^{\frac{1}{b_2} - \frac{8}{3}}, \quad 0 \leq \tilde{\alpha} \leq 1,$$

which is negative for $b_2 > 3/2$. So, a value of $b_2 > 3/2$ means that $\alpha_0$ cannot exceed some upper limit $\alpha_0^*$, and in the limit $\alpha_0 \to \alpha_0^*$ then $\rho$ diverges. As an example, $\alpha_0^* = 1.38$ for $b_2 = 1.7$, and $\alpha_0^* = 1.08$ for $b_2 = 2.3$. Note that these upper limits are not too high, taking into account that the PDG data approximately relate $\alpha = 1.45^{+0.94}_{-0.43}$ to $E = 0.6$ GeV (see Eq. (52) and discussion).

The condition that the second integral in Eq. (72) is finite upon integration to infinity

$$\int_0^\infty \frac{1}{b_2 \alpha_0^{\frac{1}{b_2} - \frac{4}{3}} \alpha^{\frac{1}{3} - \frac{1}{b_2} - \frac{8}{3}b_2}} < \infty,$$

necessitates $\frac{1}{3} - \frac{1}{b_2} - \frac{4}{9}b_2 < -1$, which is fulfilled only if $b_2 \neq 3/2$ and positive. The integration (74) is convergent, but this convergence is very slow for the values of $b_2$ we consider here. As an example, the convergence for the value $b_2 = 2.3$ is reached only when $\alpha \approx 10^{29} \alpha_0$. In practice we handle this problem by analytically computing the integral for large values of $\alpha$, where $\beta(\alpha) \approx -2b_2 \alpha$. This approach is excellent for the interval $(5\alpha_0, \infty)$. For values close to $\alpha_0$, i.e. in the interval $(\alpha_0, 5\alpha_0)$, we perform a numerical integration.

The parameters $\tilde{\alpha}$, $b_2$ and $\tilde{l}_s$ must be chosen to reproduce the physical value of the string tension $\sigma = (0.425 \text{ GeV})^2$. From a numerical computation of the heavy $Q\bar{Q}$-potential in the regime $\rho \sim 5 \text{ GeV}^{-1}$, we find that these three parameters are constrained according to the relation

$$\frac{b_2}{\tilde{\alpha}} = 3.51 \text{ GeV} \cdot \tilde{l}_s,$$

as one sees in Fig. 7.

The parameter $\tilde{l}_s$ then follows from Eqs. (52) and (75), and we get:

$$\tilde{l}_s = 1.45 \text{ GeV}^{-1}.$$

This value is different from the string length $l_s = 2.62 \text{ GeV}^{-1}$ used in Section II with the guessed metric. Clearly a readjustment of the form of the metric may also lead to a readjustment of the string length. At this point we argue that the value of $\tilde{l}_s$ is unambiguously fixed, and it is not possible to accommodate a value of $\tilde{l}_s$ equal to $l_s$ within our present analysis.
From our previous analysis and from Appendix B we see that a value $1.5 < b_2 < 2.37$ and $0.29 < \bar{\alpha} < 0.47$ satisfying Eq. (52) ensures that the theory is confining and the running coupling is well reproduced. Even when these intervals are very narrow, one can desire to get concrete values for $b_2$ and $\bar{\alpha}$. To this end we study the lowest $0^{++}$ and $2^{++}$ glueballs. In the presence of both a gravity field and a dilaton field a careful separation of the scalar degrees of freedom has to be made [17, 18]. We take from the second reference [18] the corresponding effective Schrödinger potential, which is given by

$$V_{i}^{\text{Schr.}}(z) = (B_i'(z))^2 + B_i''(z), \quad i = 0, 2,$$

(77)

where the functions $B_0(z)$ and $B_2(z)$ differ for the $0^{++}$ and $2^{++}$ glueballs

$$B_0(z) = \frac{3}{2} A(z) + \frac{1}{2} \log[X^2(z)],$$

(78)

$$B_2(z) = \frac{3}{2} A(z).$$

(79)

$A(z)$ is the Einstein frame scale factor and $X[z] \equiv X[\alpha(z)]$ has been defined in Eq. (57), being the dependence $\alpha(z)$ given by Eq. (57). Then we can solve the Schrödinger equation

$$-\frac{\partial^2}{\partial z^2} + V^{\text{Schr.}}(z) \psi_n(z) = m_n^2 \psi_n(z).$$

(80)

Best values for the glueballs $m_{0^{++}} = 0.921 \text{ GeV}$, $m_{2^{++}} = 1.462 \text{ GeV}$ come out too low for $b_2 = 2.3$ and $\bar{\alpha} = 0.45$, where $b_2$ and $\bar{\alpha}$ are constrained themselves according to Eq. (52).
These values for the parameters, which are close to the limit of good infrared singularity, cf. Appendix B, give optimum results for the glueball spectrum. The same happens in the fit of the running coupling in Sec. 4. This justifies the choice

\[ b_2 = 2.3, \quad (81) \]
\[ \bar{\alpha} = 0.45. \quad (82) \]

It is interesting to note that the glueball spectrum probes large values of \( \alpha \) in the dilaton potential than what the heavy \( Q\bar{Q} \) potential does. For instance, at the energy of the ground state for \( 0^{++} \), the potential is tested up to values of \( \alpha = 11.1 \), and for \( 2^{++} \) up to \( \alpha = 16.4 \).

The resulting running coupling follows from the \( \beta \)-function of Eq. (51) by just using a single input value (61). See the discussion after Eq. (63). The behavior of the running coupling and its comparison to data from PDG is shown in Fig. 8. The PDG values for the strong coupling are obtained via the PDG web tool [8]. To check the numerical consistency, we compare in Tab. 1 the values of the running coupling at several energy scales before and after the modification of the dilaton potential, with the corresponding values from PDG. The \( AdS/QCD \) model can match perturbative QCD-calculations with a very good accuracy of about 1%. This is important if one wants to connect a perturbative Monte Carlo cascade with non-perturbative QCD physics in parton fragmentation.

| Energy Scale [GeV] | \( \alpha_s \) (PDG) | + | - | \( \alpha \) Model 1: \( h(z) \) | \( \alpha \) Model 2: \( \bar{h}(z) \) |
|-------------------|-------------------|---|---|----------------|----------------|
| 12                | 0.16907           | 0.00436 | 0.00426 | 0.16179        | 0.16809        |
| 8.2               | 0.18463           | 0.00525 | 0.00510 | 0.18431        | 0.18492        |
| 5                 | 0.20994           | 0.00688 | 0.00662 | 0.21371        | 0.21279        |
| 2.6               | 0.26708           | 0.01178 | 0.01106 | 0.26809        | 0.26640        |
| 2                 | 0.29942           | 0.01522 | 0.01408 | 0.31023        | 0.29668        |
| 1                 | 0.4996            | 0.05799 | 0.04598 | 0.82393        | 0.42618        |

**Tab. 1:** Values of the running coupling at different energy scales compared with PDG data [8].

The values of the running coupling provided by the old warp factor \( h(z) \) of Eq. (2) are doing quite well in comparison with the ones given by PDG within the errors, but the values obtained from the new dilaton potential (54) are closer to the experimental data.

Then only one parameter \( V_0 \) or \( \bar{\ell} = \frac{6}{\sqrt{-3V_0}} \) remains to be fixed. Focusing on the UV regime, we can perform an analytical study of the short distance regime by expanding...
Eqs. (68) and (72) in powers of $\alpha_0$. The details of the computation are provided in Appendix C. The result in NNLO is

$$V_{Q\bar{Q}}(\rho) = -\frac{2\bar{\ell}^2}{\pi l_s^2} \frac{\alpha_0^{4/3}(\rho)}{\rho} \left\{ 0.359 + 0.533b_0\alpha_0(\rho) + (1.347b_0^2 + 0.692b_1)\alpha_0^2(\rho) + O(\alpha_0^3) \right\}. \tag{83}$$

By reversing Eq. (68), cf. Eq. (152) in Appendix C, we get the following functional form for $\alpha_0$ as a function of the separation $\rho$

$$\alpha_0(\rho) = \frac{1}{b_0 \log \left( \frac{1.32\rho}{\rho} \right) + \frac{b_1}{b_0} \log \left( b_0 \log \left( \frac{1.32\rho}{\rho} \right) \right)} - \frac{\left( 0.079b_0^2 + \frac{b_1^2}{b_0} \right)}{\left( b_0 \log \left( \frac{1.32\rho}{\rho} \right) + \frac{b_1}{b_0} \log \left( b_0 \log \left( \frac{1.32\rho}{\rho} \right) \right) \right)^3} + O \left( \log^{-4} \left( \frac{1.32\rho}{\rho} \right) \right). \tag{84}$$

The heavy quark potential can be directly compared with perturbation theory (PT). $V_{Q\bar{Q}}$ is computed in PT as an expansion in powers of the QCD running coupling $\alpha_0(\rho)$. It has the form

$$V_{PT}(\rho) = -\frac{N_c^2 - 1}{2N_c} \cdot \frac{\alpha_0(\rho)}{\rho}, \tag{85}$$

where up to the third order

$$\alpha_0 = \alpha_{PT} \left\{ 1 + (a_1 + 4\pi\gamma_E b_0) \frac{\alpha_{PT}}{4\pi} \right\}$$

$$+ \left[ 4\pi\gamma_E (2a_1 b_0 + 4\pi b_1) + 4\pi^2 \left( \frac{\pi}{3} - 4\gamma_E \right) b_0^2 + a_2 \right] \frac{\alpha_{PT}^2}{16\pi^2} \right\}. \tag{86}$$
In this expression $b_0$ and $b_1$ were defined in Eq. (39), $\alpha_{\text{PT}}$ is the perturbative QCD running coupling, and the coefficients $a_1$ and $a_2$ were calculated by Fischler [20] and by Peter [21] and Schröder [22], respectively. We use the convention of Ref. [19] and Ref. [22]:

$$a_1 = \frac{31}{9} N_c - \frac{10}{9} N_f,$$

$$a_2 = \left( \frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right) N_c^2 - \left( \frac{5081}{324} + \frac{16}{3} \zeta(3) \right) N_c N_f + \left( \frac{55}{12} - 4\zeta(3) \right) \frac{N_f}{N_c} + \frac{100}{81} N_f^2. \tag{88}$$

Practically, we use the two-loop running coupling constant, which has the following form

$$\alpha_{\text{PT}}(\rho, d) = \frac{1}{b_0 \cdot \log \left( \frac{\rho}{\mu} \right) + b_1 \cdot \log \left( 2\log \left( \frac{\rho}{\mu} \right) \right)}, \tag{89}$$

where $d$ is an undetermined parameter which relates the scale $\mu$ of the running coupling with the distance $\rho$, i.e. $\mu = d/\rho$.

The expansion (83) is similar to that of PT, Eqs. (85)-(86), except that there is an extra power $\alpha_0^{1/3}$ at every order. This is a common prediction of all the renormalization group revised models constructed by the general procedure of Kiritsis et al., cf. Refs. [4, 7]. At first sight this difference is a matter of concern. In order to fix the unknown parameter $V_0$ of the dilaton potential, we proceed in the following way: We find that Eq. (83) depends on the factor $\bar{\ell}_s^2 = -12/(V_0\bar{l}_s^2)$, and a numerical comparison between the leading orders in Eqs. (83) and (85) in the regime $0.06 \text{ GeV}^{-1} < \rho < 0.20 \text{ GeV}^{-1}$ gives us the value $-V_0\bar{l}_s^2 = 1.31$, which is confirmed in a big range of $\bar{l}_s$. Using the determined value of $\bar{l}_s$ we get:

$$V_0 = -0.623 \text{ GeV}^2. \tag{90}$$

A convenient choice for the parameter $d$ follows from a direct comparison between the argument inside the logarithms in Eqs. (81) and (89),

$$d = 1.32\bar{l}_s \Lambda = 1.53. \tag{91}$$

We choose the value of $\Lambda$ given in Eq. (3). Another possibility would be to use a $\Lambda$ which follows from QCD studies for the running coupling. In either case the value of $d$ is chosen in such a way that it compensates a change in $\Lambda$.

We compare in Fig. 9 the numerical result for the heavy quark-antiquark potential from Eq. (72), with the perturbative result of QCD given by Eqs. (85) and (86), in the small
distance regime $0.06 \text{ GeV}^{-1} < \rho < 0.20 \text{ GeV}^{-1}$. The upper bound in $\rho$ is motivated by the fact that our result for the running coupling fits well the experimental data in the regime $E > 4 - 5 \text{ GeV}$. We also show in dashed lines the short distance expansion of Eq. (83) up to leading order $\mathcal{O}(\alpha_0^{4/3})$, next-to-leading order $\mathcal{O}(\alpha_0^{7/3})$ and next-to-next-to-leading order $\mathcal{O}(\alpha_0^{10/3})$. We see that the comparison with the numerical computation of $V_{\bar{Q}Q}$ and perturbation theory is quite accurate, although it seems to be that the series (83) is slowly convergent. It is important to note that the full string potential is rather close to the perturbative potential in spite of the expansion containing different powers in the Yang-Mills coupling.

![Figure 9](image)

**Fig. 9:** The heavy quark-antiquark potential as a function of the distance $\rho$ for small $\rho$. The result stemming from the dilaton potential of Eq. (54) is shown as a full (blue) line. It follows from a numerical computation of Eqs. (68) and (72). The perturbative computation, Eqs. (85) and (86), is displayed as a dashed (red) line. The short distance expansion, Eq. (83), is displayed up to leading order (LO), next-to-LO and next-to-next-to-LO, as dashed (green) lines from bottom to top, respectively. We consider in this plot $\bar{t}_s = 1.45 \text{ GeV}^{-1}$ and $V_0 = -0.623 \text{ GeV}^2$.

A full numerical computation gives the heavy $Q\bar{Q}$-potential from Eqs. (68) and (72). The result is shown in Fig. 10. One can try to fit our result with the Cornell form of the potential:

$$V_{Q\bar{Q}}^{\text{Cornell}}(\rho) = -\frac{a}{\rho} + \sigma \cdot \rho + C,$$

and obtains the values

$$a = 0.42, \quad \sigma = (0.415 \text{ GeV})^2, \quad C = -0.14 \text{ GeV},$$

which are rather close to the accepted values [23, 24], and also to the values obtained in Ref. [1]. In the numerical computation of Eq. (72) we have conveniently normalized the result by adding a constant $C$ in order that the $Q\bar{Q}$-potential vanishes close to $\rho = 2 \text{ GeV}^{-1}$. 
Fig. 10: The heavy quark-antiquark potential as a function of $\rho$. Our result stemming from the dilaton potential of Eq. (54) is shown as a full (blue) line. It follows from a numerical computation of Eqs. (68) and (72). The dashed (red) line corresponds to the Cornell potential, Eq. (92).

6 Dilaton Potential and New Warp Factor

With the three parameters $V_0, \bar{\alpha}$ and $b_2$ determined in the previous section we can obtain the dilaton potential Eq. (54), the running coupling and the modified warp factor $\bar{h}(z)$ of Eq. (63).

The parameters governing the UV-asymptotic behavior of the dilaton potential are $b_0 = \frac{25}{6\pi}$ and $b_1 = \frac{77}{12\pi^2}$, the known coefficients of the perturbative $\beta$-function, in addition to $V_0$, as can be seen in Eq. (55). So, the potential is consistent with the QCD $\beta$-function at two-loop level.

The dilaton potential is plotted in Fig. 11 and compared with the one obtained from the warp factor $h(z)$ of Eq. (2). The comparison shows that the modified dilaton potential becomes slightly flatter. This flattening is in line with the result given by Ref. [15], although the parameters provided in this reference together with our value of $\bar{l}_s$ produce a potential of the order of $10^4$ GeV$^2$ in the regime of physical interest, $\alpha \approx 0.3$, in contrast to the value $\sim 1$ GeV$^2$ given by our potential.

It is interesting to compute the warp factor $\bar{h}(z)$ that follows from the new dilaton potential of Eq. (54). This way we can close the circle of investigation in the present paper. The mathematical procedure is as follows: From Eqs. (57) we get a first order differential equation

$$\frac{d\alpha}{dz} = \frac{\sqrt{-3V_0}}{6} e^{A(\alpha) - D(\alpha)}, \quad (93)$$

where $A(\alpha)$ and $D(\alpha)$ are given by Eqs. (59) and (60) respectively. This equation can be
solved by imposing the boundary condition $\alpha(z \to 0) \to 0$, due to the asymptotic freedom. The solution is delicate, because there are problems to find a numerical solution of Eq. (93) near the boundary $z = 0$. At this point the l.h.s. of Eq. (93) is divergent. To overcome this difficulty, we proceed in three steps. First, we consider the lowest order perturbative expansion of the $\beta$-function and rewrite Eq. (93) in the deep UV,

$$\frac{d\alpha_{\text{UV}}}{dz} = b_0 \frac{\sqrt{-3V_0}}{6} \alpha_{\text{UV}}^2 e^{\frac{1}{\alpha_{\text{UV}}}}.$$  \hspace{1cm} (94)

The solution of this equation is:

$$\alpha_{\text{UV}}(z) = -\frac{1}{b_0 \log (\bar{\Lambda} z)}; \hspace{1cm} (95)$$

with

$$\bar{\Lambda} = \frac{\sqrt{-3V_0}}{6} = 237 \text{ MeV}.$$  \hspace{1cm} (96)

Note that the value of $\bar{\Lambda}$ is rather close to the one determined using $h$, cf. Eq. (4). This small discrepancy could be improved when higher orders in the perturbative expansion are considered, as it has been observed in the analytical computation of the $Q\bar{Q}$-potential, see Appendix B, cf. Eqs. (163) and (164).

Choosing $z_0 = 1.2 \cdot 10^{-4} \text{ GeV}^{-1}$ we obtain $\alpha(z_0) = 0.0718$ as initial value in the deep UV to find the numerical solution of Eq. (93). In the second step we consider three
orders in the UV expansion of the $\beta$-function, Eq. (51), and $A(\alpha)$, and solve Eq. (93) from $z_0 = 1.2 \cdot 10^{-4}$ GeV$^{-1}$ to $z_1 = 0.125$ GeV$^{-1}$ with

$$A(\alpha) = C_A + \frac{1}{b_0 \alpha} + \frac{b_1}{b_0} \log \alpha + \frac{1}{6b_0^2 \alpha^3} \left( b_0 b_2 - 3b_0^2 \bar{\alpha} - 6b_0 b_1 \bar{\alpha}^2 - 6b_1^2 \bar{\alpha}^3 \right) \alpha + \mathcal{O}(\alpha^2), \quad (97)$$

where $C_A = -0.1057$ is a constant to make $A(\alpha)$ consistent with the input condition (61). Finally from $z_1$ to infinity we use the input functions to solve the equation fully numerically. We have checked the stability of the solution changing $z_0$ and $z_1$, and considering higher orders in the expansion, Eq. (97). Once we know $\alpha(z)$, the corresponding warp factor is easily computed, and it reads

$$\bar{h}(z) = e^{2A(\alpha(z))} (\alpha(z))^\frac{4}{3} (\Lambda z)^2. \quad (98)$$

The strength in the Nambu-Goto action, Eq. (5), is determined by the factor $h(z)/l_s^2$, and so it is more relevant to consider this quantity when comparing different models. The result $\bar{h}(z)/l_s^2$ is shown in Fig. 12 and compared to the warp factor of Eq. (2) that was first proposed in Ref. [1]. $l_s^2$ is approximately a factor 3 larger than $\bar{l}_s^2$, and this is reflected also in the values of $h(z)$ compared to $\bar{h}(z)$. The numerical agreement is rather good up to $z \simeq 2$ GeV$^{-1}$, in spite of the fact that $\bar{h}(z)$ vanishes in the UV. $\bar{h}(z)$ has a singularity at $z_{IR} = 3.65$ GeV$^{-1}$, which occurs at slightly larger values than in $h(z)$, for which $z_{IR} = \sqrt{1 - \epsilon/\Lambda} = 2.73$ GeV$^{-1}$.

In the deep UV, i.e. at leading order in the perturbative expansion of the $\beta$-function, the analytical solution of Eq. (93) has a simple form:

$$\bar{h}_{UV}(\alpha) = \left( \frac{\Lambda}{\bar{\Lambda}} \right)^2 \alpha_{UV} \left( \frac{1}{-b_0 \log (\Lambda z)} \right)^\frac{4}{3}. \quad (99)$$

This functional form of the calculated warp factor is very similar to the guessed warped factor besides the power $4/3$.

7 Discussion and final remarks

The analogy of the bulk coordinate $z$ with the inverse energy resolution has triggered the guessed warp factor in Refs. [1, 2], which was based on a naive equivalence with the running
coupling of QCD. A careful analysis of the resulting dilaton potential gives the evolution of the dilaton field in the bulk and consequently the running of the QCD gauge coupling. The infrared physics of the model of Refs. [1, 2] was satisfactory to fit the string tension, but it fails to give a good UV-behavior for the $\beta$-function. Because of the second order Einstein equations and the correlated behavior of the dilaton in the infrared and ultraviolet, which is not constrained by the correct QCD $\beta$-function, one needs two boundary conditions to interpolate the gauge coupling between the charmonium and bottomonium masses. This feature weakens the idea of holography for the old ansatz, which determines the field theoretic behavior of our 4-dimensional world from the physics of gravity in 5-dimensional anti-de Sitter space.

In order to have the Yang-Mills theory as a ‘hologram’ of the physics happening in five dimensions we assumed a new ansatz that improves the UV-behavior by using the QCD $\beta$-function as a constraint. Thereby we found a dilaton potential which is consistent with QCD in the UV region. The resulting short distance heavy quark potential $r \cdot V_{Q\bar{Q}}(r)$ has a similar shape as the 3-loop expression derived by Brambilla et al. [19]. The numerical comparison of the QCD and string heavy $Q\bar{Q}$-potential is rather good in spite of the fact that the leading term in string theory proportional to $\alpha^{4/3}$ deviates from the QCD-potential proportional to $\alpha$. By calculating the NNLO-expansion, we show that the expansion in $\alpha$ is slowly converging. With a new string length $\bar{l}_s$ in the Nambu-Goto action we can also match the long range $Q\bar{Q}$-interaction. We have closed the circle of investigation by

Fig. 12: The warp factor divided by $l_s^2$ as a function of $z$. The factor of Eq. (2) proposed in Ref. [1] is shown as a dashed (red) line. The full line corresponds to the warp factor that follows from the dilaton potential of Eq. (54) using the parameters determined in Sec. 5.
computing the warp factor corresponding to the new ansatz, and found that the scaled warp
factor is similar to that of Refs. [1, 2] in the region of interest \(0.5 \text{ GeV}^{-1} \leq z \leq 2 \text{ GeV}^{-1}\).

In the procedure we have proposed, we fix three parameters \((b_2, \bar{\alpha}, V_0)\) corresponding
to the dilaton potential and one parameter \(\bar{l}_s\) equal to the string length by using three
constraints, namely a good behavior of the \(Q\bar{Q}\)-potential in the IR and in the UV, and
a good behavior of the running coupling in the regime \(0.6 \text{ GeV} \leq E \leq 15 \text{ GeV}\). New
parameters may have to be included in order to describe the glueball spectrum or further
observables with precision. One important point of this analysis is the general criterion for
confinement of Ref. [7] and the requirement that the infrared singularity being repulsive to
physical modes, which help us to set a narrow window in our parameter set, in particular
for the parameters controlling the infrared behavior of the theory \((b_2, \bar{\alpha})\).

Recently the question “How Well Does AdS/QCD Describe QCD?” [25] has been asked,
and depending on the feature the answer varied - the accuracy was estimated between 10%
and 25%. In our case of pure gluon dynamics, we have shown that the accuracy is much
better, and therefore we look optimistically towards further tests of the action in finite
temperature calculations [26].

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Appendix A: Infrared and Ultraviolet Properties of the Gravity Dual Theory

In this appendix we study the infrared (IR) and the ultraviolet (UV) properties of the 5-dim Nambu-Goto theory with the “guessed” metric \( h(z)/(\Lambda z)^2 \), cf. Eq. (2), which we develop in Secs. 2 and 3. We apply technical details of Refs. [4, 7], which are shortly reviewed in Sec. 4. The warp factor \( h(z) \) has a singularity at \( z_{IR} = \sqrt{1 - \epsilon}/\Lambda \), so the bulk coordinate \( z \) is restricted to \( z < z_{IR} \). The IR expansion of \( h(z) \) is

\[
h(\xi) = \frac{\log(\xi)}{2\sqrt{1 - \epsilon}} + \frac{(1 - 2\epsilon)\log(\xi)}{4(1 - \epsilon)} + \mathcal{O}(\xi), \quad \xi \to 0,
\]

with

\[
\xi = z_{IR} - z.
\]

Using Eqs. (15) and (21) we get a second order differential equation for \( \phi(\xi) \), which writes

\[
\phi''(\xi) + \left( \frac{1}{\xi} - \frac{3 + 2\epsilon}{2\sqrt{1 - \epsilon}} + \mathcal{O}(\xi) \right) \phi'(\xi) - \frac{3}{8} \left( \frac{1}{\xi^2} + \frac{3 + 2\epsilon}{\sqrt{1 - \epsilon}} + \mathcal{O}(\xi^0) \right) = 0.
\]

This equation can be solved for several orders in the IR expansion of \( h(\xi) \). The theory of differential equations gives the general solution by adding to the special solution of the inhomogeneous equation the full set of homogeneous solutions. The result is

\[
\phi(\xi) = \frac{3}{16} (\log \xi)^2 + c_1 \log \xi + c_2 + \mathcal{O}(\xi \log \xi).
\]

The parameters \( c_1 \) and \( c_2 \) are two unknown constants corresponding to the homogeneous solutions which have to be fixed by two conditions. Obviously the three terms that we show explicitly in Eq. (104) correspond to the lowest orders of the homogeneous and inhomogeneous solutions. At this point it is preferable to write the solution in this way

\[
\phi(\xi) = \frac{3}{16} \left( \log \frac{\xi}{\omega_{IR}} \right)^2 + \kappa_{IR} + \mathcal{O}(\xi \log \xi),
\]

where the constants \( \omega_{IR} \) and \( \kappa_{IR} \) are related to \( c_1 \) and \( c_2 \) as

\[
\omega_{IR} = e^{-\frac{2}{3}c_1}, \quad \kappa_{IR} = -\frac{4}{3}c_1^2 + c_2.
\]

Setting \( \omega_{IR} \) corresponds to setting the scale. We can solve Eq. (21) numerically for the full range of \( z < z_{IR} \) using the IR behavior of Eq. (105) as a boundary condition. In the concrete calculation we choose \( \phi'(\xi_1) \) and \( \phi''(\xi_1) \) with \( \xi_1 \) very small. Doing that, we have checked that the result of Fig. 1 is exactly reproduced for

\[
\omega_{IR} = 4.55 \text{ GeV}^{-1}, \quad \kappa_{IR} = -0.758.
\]
These numbers are stable in the deep infrared near the singularity, \(\xi_1 \sim 10^{-7} - 10^{-4} \text{ GeV}^{-1}\). So, this choice of the constants \(\omega_{\text{IR}}\) and \(\kappa_{\text{IR}}\) is equivalent to the boundary conditions of Eq. (29).

The IR behavior of \(A(\xi)\) can be obtained from Eqs. (15), (101) and (105), and it reads
\[
A(\xi) = -\frac{1}{8} \left( \log \frac{\xi}{\omega_{\text{IR}}} \right)^2 - \frac{1}{2} \left( \log \frac{\xi}{\omega_{\text{IR}}} \right) - \frac{2}{3} \kappa_{\text{IR}} + \frac{1}{2} \log \left( \frac{\log \xi}{2(1 - \epsilon)^{2/3} \omega_{\text{IR}} \Lambda} \right) + \mathcal{O}(\log \xi), \quad \xi \to 0 .
\]

Once that we know \(A(\xi)\), the IR asymptotics of the superpotential \(W(\alpha)\) can be computed using Eq. (44) and taking into account the domain wall coordinates relation \(e^A dz = du\). It reads,
\[
W(\alpha) = W_\infty \alpha^{2/3} (\log \alpha)^{4/3} e^{-4 \sqrt{3} \sqrt{\log \alpha}} \left( 1 - \frac{3 + 2 \kappa_{\text{IR}}}{2 \sqrt{3}} \left( \log \alpha \right)^{1/2} + \mathcal{O}\left( (\log \alpha)^{-1} \right) \right), \quad \alpha \to \infty ,
\]
where the constant \(W_\infty\) is
\[
W_\infty = \frac{3}{4} \sqrt{\frac{6(1 - \epsilon)^{2/3}}{\log \left( \frac{1}{\epsilon} \right)} \frac{\Lambda}{\omega_{\text{IR}}}} .
\]

The dependence in \(\kappa_{\text{IR}}\) appears at \(\mathcal{O}(\left( \log \alpha \right)^{-1/2})\) in the bracket of Eq. (109). The fact that the superpotential \(W(\alpha)\) grows faster than \(\alpha^{2/3}\) in the infrared ensures that the theory is confining and also that there is a mass gap in the spectrum \([7, 9]\). Note that this is true independently of the values of \(\omega_{\text{IR}}\) and \(\kappa_{\text{IR}}\).

The dilaton potential can be obtained from \(W\) using the third relation of Eq. (46), or equivalently from \(A(z)\) and Eq. (20). In this regime it behaves as
\[
V(\alpha) = V_\infty \alpha^{2/3} (\log \alpha)^{4/3} e^{-4 \sqrt{3} \sqrt{\log \alpha}} \left( 1 - \frac{2(2 + \kappa_{\text{IR}})}{\sqrt{3}} \left( \log \alpha \right)^{1/2} + \mathcal{O}\left( (\log \alpha)^{-1} \right) \right), \quad \alpha \to \infty ,
\]
where the constant \(V_\infty\) is
\[
V_\infty = -\frac{16}{9} W_\infty^2 .
\]

The general form of the potential in the infrared that has been studied in Refs. \([4, 7]\) is
\[
V(\alpha) \sim \alpha^{2Q} (\log \alpha)^P , \quad \alpha \to \infty .
\]

The solution of Eq. (111) corresponds to \(Q = 2/3\) and \(P = 1\) in this notation, in addition to an extra factor \(e^{-4 \sqrt{3} \sqrt{\log \alpha}}\) which is dominant with respect to \(\log \alpha\), but subdominant with respect to \(\alpha^{2/3}\). This subdominance, in addition to the fact that \(Q < 2\sqrt{2}/3\) ensures that
the IR singularity is of the good kind according to the criterion of Gürsöy et al. [7], which means that the singularity should be repulsive to physical fluctuations. To avoid any doubt about the extra factor, it is possible to prove that it can be removed in Eqs. (109) and (111) by shifting the dilaton field
\[ \bar{\phi} = \phi + \frac{\sqrt{3}}{2}, \]
which doesn’t affect the IR asymptotics. The resulting expression for the asymptotics of the dilaton potential is:
\[ V(\bar{\alpha}) \sim \bar{\alpha}^{\frac{4}{3}} \log \bar{\alpha}, \quad \bar{\alpha} \to \infty. \]

Note that the asymptotics of Eq. (108) has not been studied by the authors of Ref. [4, 7], and so this case is not listed in Table 1 of Ref. [7]. One can add our result to this table. The IR behavior of \( X(\alpha) \) can be computed from Eqs. (45) and (109):
\[ X(\alpha) = -\frac{1}{2} - \frac{\sqrt{3}}{4} \frac{1}{(\log \alpha)^{\frac{1}{2}}} - \frac{3}{8} \frac{1}{\log \alpha} - \frac{\sqrt{3}(3 + 2\kappa_{\text{IR}})}{16} \frac{1}{(\log \alpha)^{\frac{3}{2}}} + \ldots, \quad \alpha \to \infty. \]

The term \( \propto 1/(\log \alpha)^{\frac{1}{2}} \) comes from the extra factor \( e^{\frac{2}{\sqrt{3}} \sqrt{\log \alpha}} \) in \( W(\alpha) \). In our case, the limit
\[ \lim_{\alpha \to \infty} \left( X(\alpha) + \frac{1}{2} \right) \log \alpha = K, \]
leads to \( K = -\infty \).

One important aspect of this analysis is that one has to fix two integration constants, \( \omega_{\text{IR}} \) and \( \kappa_{\text{IR}} \), using initial conditions. This contrasts with the analysis of Refs. [7, 9], where they show that just one initial condition is enough. One possibility studied in Refs. [7, 9] is fixing one of the parameters by requiring that the bulk singularity is not of the “bad kind”, which means that the singularity should be repulsive to physical fluctuations, cf. Eq. (E.28) of Ref. [9]. In our case the singularity of our solution at \( z = z_{\text{IR}} \) is of the good kind, independently of the values of \( \omega_{\text{IR}} \) and \( \kappa_{\text{IR}} \), cf. Eqs. (109), (111), (112) and compare with Eqs. (E.27) and (E.29) of Ref. [9]. To give a new perspective to this issue, we can study the UV behavior of the dilaton using \( h(z) \) of Eq. (2). Following the same procedure that we explained for the IR but considering an expansion at small \( z \), one gets
\[ \phi(z) = -\frac{\omega_{\text{UV}}}{z} + \kappa_{\text{UV}} + \frac{(\omega_{\text{UV}} \Lambda)^{2}}{\epsilon \log \left( \frac{1}{\epsilon} \right) \omega_{\text{UV}}} \frac{z}{z^{2}} + \mathcal{O}(z^{2}), \quad z \to 0, \]
where \( \omega_{\text{UV}} \) is a parameter setting the scale and \( \kappa_{\text{UV}} \) is another parameter. They play the role of \( \omega_{\text{IR}} \) and \( \kappa_{\text{IR}} \) respectively in the UV. In fact these parameters are related, in the sense
that setting $\kappa_{\text{UV}}$ in the UV then sets $\kappa_{\text{IR}}$ in the IR (the same for $\omega_{\text{IR}}$ and $\omega_{\text{UV}}$), cf. Sec. 3. From Eq. (118) and using the same procedure that we explain above, one gets

$$\beta(\alpha) = -\frac{3}{2} \alpha - \frac{9}{4} \alpha \log \alpha - \frac{9(3 + 2\kappa_{\text{UV}})}{8} \alpha + \mathcal{O}\left(\frac{\alpha}{(\log \alpha)^2}\right),$$

(119)

which has the drawback not to be consistent with asymptotic freedom as found in QCD and has motivated our improvement in Sec. 4. Note that the parameter $\kappa_{\text{UV}}$ enters in the expansion of $\beta(\alpha)$, Eq. (119). A well defined function $\beta(\alpha)$ from perturbation theory doesn’t have this parameter, and this is the starting point of the program followed in Refs. [4, 7, 9]. As we explain in Sec. 3 we fix $\kappa_{\text{UV}}$ and the scale parameter $\omega_{\text{UV}}$ (equivalently $\kappa_{\text{IR}}$ and $\omega_{\text{IR}}$) by using two input values for $\alpha$, cf. Eq. (29).

Appendix B: Infrared Properties of the Improved Gravity Dual Theory

In this appendix we study the infrared properties of the 5-dim Nambu-Goto theory with the improved metric $\tilde{h}(z)/\Lambda z^2$, cf. Eq. (62), which is proposed in Sec. 4 and further developed in Secs. 5 and 6. The ultraviolet properties are fully dictated by the UV behavior of the $\beta$-function, so we will just focus on the IR asymptotics. As we will see later, the warp factor $\tilde{h}(z)$ has a singularity at some finite value $\tilde{z}_{\text{IR}}$, so that the coordinate $z$ is restricted to $z < \tilde{z}_{\text{IR}}$, like in the guessed metric $h(z)/\Lambda z^2$ studied in Appendix A.

The IR expansion of $A(z)$ can be computed from the function $A(\alpha)$ and the functional dependence $\alpha(z)$. In the IR, i.e. $\alpha \to \infty$, Eq. (59) can be integrated out using the expression of the $\beta$-function given in Eq. (51). One gets

$$A(\alpha) = A(\alpha_c) - \frac{1}{b_2} \log \left(\frac{\alpha}{\alpha_c}\right) + \ldots \quad \alpha \to \infty.$$  

(120)

The computation of $e^D$ involves an integration from 0 to $\alpha$, and so it must be performed more carefully in order to retain the UV convergence, cf. Eq. (60). Inserting the full expression of $\beta(\alpha)$ into the integrand of Eq. (60), and considering an expansion at large $\alpha$, one gets

$$\frac{4}{3} \int_0^\alpha \frac{\beta(\alpha)}{3a^2} da = -\frac{4}{9} b_2 \log \left(\frac{\alpha}{\alpha_c}\right) + C_0 + \mathcal{O}(\alpha^{-4})$$

$$-\frac{2}{9} e^{-\frac{\alpha}{\alpha_c}} \left( b_2 - 2\bar{\alpha}(b_0 + b_1 \bar{\alpha}) \right) \frac{\alpha}{\bar{\alpha}} + \mathcal{O}(\alpha^0), \quad \alpha \to \infty,$$

(121)

where for simplicity we have defined the constant

$$C_0 = \frac{2}{9} [b_2 (3 - 2\gamma_E) - 2\bar{\alpha}(2b_0 + b_1 \bar{\alpha})].$$

(122)
The integration is convergent in the UV, and the dominant contribution in the IR comes from the logarithmic term in the r.h.s. of Eq. (121). From Eq. (60), and using Eqs. (51) and (121) one can easily compute the IR asymptotics of $e^D$. It reads

$$e^D = \frac{e^{C_0}}{b_2\alpha} \left( \frac{\alpha}{\bar{\alpha}} \right)^{-\frac{4}{9}b_2} \left[ 1 + O(e^{-\frac{\alpha}{\bar{\alpha}}}) \right], \quad \alpha \to \infty. \tag{123}$$

The functional dependence $\alpha(z)$ can be computed from Eq. (57), which one can write in the following way

$$dz = \bar{\ell} e^{D-A} d\alpha. \tag{124}$$

Then the function $z(\alpha)$ follows from an integration of Eq. (124),

$$\int_{z_1}^z dz = \bar{\ell} \int_{\alpha_1}^{\alpha(z)} e^{D(\alpha)-A(\alpha)} da, \quad \alpha_1 = \alpha(z_1). \tag{125}$$

Inserting Eqs. (120) and (123) into the r.h.s. of Eq. (125), we can perform the integration analytically. After inverting the solution, one gets

$$\alpha(z) = \left[ C_*(z_1 - z) + \frac{1}{b_2 - \frac{4}{9}b_2} \right]^{-\frac{b_2}{b_2 - \frac{4}{9}b_2}} + \ldots = \frac{1}{\left( C_* \cdot (\bar{z}_{IR} - z) \right)^{\frac{1}{b_2 - \frac{4}{9}b_2}}} + \ldots, \quad z \to \bar{z}_{IR}, \tag{126}$$

where we have defined

$$\delta = \frac{1}{b_2 - \frac{4}{9}b_2} - 1, \quad \bar{z}_{IR} = z_1 + \frac{\alpha_1 - \frac{4}{9}b_2}{C_*}. \tag{127}$$

and the constant $C_*$ is

$$C_* = \frac{1}{\ell \cdot \delta} e^{A(\alpha_*) - C_0} \cdot \frac{\alpha_1}{\alpha_1^{\frac{1}{b_2 - \frac{4}{9}b_2}}}. \tag{128}$$

Inserting Eq. (126) into Eq. (120) then one finally gets the IR asymptotics of $A(z)$, which reads

$$A(z) = \delta \cdot \log(\bar{z}_{IR} - z) + \ldots, \quad z \to \bar{z}_{IR}. \tag{129}$$

The IR behavior of $\phi(z)$ can be obtained from Eqs. (21) and (129), and it reads

$$\phi(z) = -\frac{3}{2} \sqrt{\delta(1+\delta)} \log(\bar{z}_{IR} - z) + \ldots, \quad z \to \bar{z}_{IR}, \tag{130}$$

which corresponds to the asymptotics of $\log \alpha(z)$, cf. Eq. (126), after taking into account Eq. (127).
The procedure to compute the IR asymptotics of the superpotential $W$ and the dilaton potential $V$ is explained in Appendix A. The superpotential reads

$$W(\alpha) = \frac{9}{4} \delta \cdot C_{*}^{1+\delta} \cdot \alpha^{\frac{4}{3}b_{2}} + \ldots, \quad \alpha \to \infty,$$  \hspace{1cm} (131)$$

and the dilaton potential

$$V(\alpha) = \frac{4}{3} C_{*}^{2+2\delta} \cdot \delta^{2} \cdot (b_{2}^{2} - 9) \alpha^{\frac{8}{3}b_{2}} + \ldots, \quad \alpha \to \infty.$$  \hspace{1cm} (132)$$

Within the notation of Refs. [4, 7], cf. Eq. (113), the solution of Eq. (132) corresponds to $Q = \frac{4}{9}b_{2}$. As it has been discussed in Ref. [7], the asymptotics of Eq. (120) leads to a confining theory whenever $Q > 2/3$, and this is fulfilled in our case for $b_{2} > 3/2$. The IR behavior of $X(\alpha)$ reads

$$X(\alpha) = -\frac{1}{3}b_{2} + \ldots, \quad \alpha \to \infty$$  \hspace{1cm} (133)$$

so that the limit

$$\lim_{\alpha \to \infty} \left( X(\alpha) + \frac{1}{2} \right) \log \alpha = K,$$  \hspace{1cm} (134)$$

leads to $K = -\infty$ in the confining case, i.e. $b_{2} > 3/2$, as it was explained in Sec. 4, cf. Eq. (53).

Giving the IR asymptotics of Eq. (113), a good singularity according to the criterion of Ref. [7] is obtained when $Q < 2\sqrt{2}/3$. This means that our theory is confining and it presents a good singularity for values of $b_{2}$ in the range

$$\frac{3}{2} < b_{2} < \frac{3\sqrt{2}}{2}. \hspace{1cm} (135)$$

Nevertheless, as we will see below the upper bound is too conservative, and a slightly larger value is obtained when computing the IR asymptotics of the effective Schrödinger potential for the glueball spectrum. To this end, first we will analyze the $2^{++}$ sector, whose potential is

$$V_{2}^{\text{Schr.}}(z) = (B_{2}'(z))^{2} + B_{2}''(z),$$  \hspace{1cm} (136)$$

where

$$B_{2}(z) = \frac{3}{2} A(z). \hspace{1cm} (137)$$

After inserting the IR asymptotics of $A(z)$, Eq. (120), into Eq. (137) and computing the derivatives, one gets the following asymptotics for the effective Schrödinger potential.
\[ V_2(z) = \frac{3}{2} \delta \left( \frac{3}{2} \delta - 1 \right) \frac{1}{(z_{\text{IR}} - z)^2} + \ldots, \quad z \to z_{\text{IR}}. \]  

(138)

The potential diverges to \(+\infty\) whenever \(\delta > 2/3\), or equivalently

\[ b_2 < \frac{3}{2} \sqrt{\frac{5}{2}} \approx 2.37. \]  

(139)

Otherwise the Schrödinger potential would diverge to \(-\infty\) at \(z \to z_{\text{IR}}\), so that the singularity would be attractive to physical fluctuations. So, the condition Eq. (139) ensures that the singularity is of the good king according to the criterion of Gürsoy et al. [7].

The computation in the \(0^{++}\) sector is similar. In this case the Schrödinger potential reads

\[ V_0(z) = (B'_0(z))^2 + B''_0(z), \]  

(140)

where

\[ B_0(z) = \frac{3}{2} A(z) + \frac{1}{2} \log[X^2(z)], \]  

(141)

and

\[ X(z) = \frac{\beta(\alpha(z))}{3\alpha(z)}. \]  

(142)

Using the asymptotics of \(A(z)\), Eq. (129), in Eq. (141) and the asymptotics of \(\alpha(z)\), Eq. (126), in Eq. (142), one gets for \(V_0(z)\) the same IR asymptotics as for \(V_2(z)\), cf. Eq. (138).

We plot in Fig. 13 the effective Schrödinger potentials for the computation of the \(0^{++}\) and \(2^{++}\) glueball spectrum using the guessed value \(b_2 = 2.3\).

Appendix C: Heavy \(Q\bar{Q}\)-Potential at Short Distance

In this appendix we give technical details for the analytical computation of the heavy quark-antiquark potential in the small separation limit as an expansion in powers of \(\alpha_0\). The relevant formulas are Eqs. (68) and (72). The result up to NLO has been derived in Ref. [13].

In order to compute the series up to NNLO, we need to consider the QCD \(\beta\)-function up to order \(\alpha_0^3\), so we have to take into account the parameters \(b_0\) and \(b_1\) (see Eq. (39)). We have explicitly checked that the order \(\alpha_0^4\) in the \(\beta\)-function contributes at higher orders in the potential. Here and in the following we use the notation \(\alpha = \tilde{\alpha} \cdot \alpha_0\). The variable \(\tilde{\alpha}\) is in the interval \(0 \leq \tilde{\alpha} \leq 1\) in the integral of Eq. (68) and the first integral of Eq. (72).
Fig. 13: The effective Schrödinger potentials for the $0^{++}$ (full red line) and $2^{++}$ (dashed blue line) glueball spectrum using $b_2 = 2.3$. We also show in full green line the running coupling.

First we will consider the computation of $\rho(\alpha_0)$ which is given by Eq. (68). The function $A(\alpha)$ can be computed using Eq. (59), and the result is

$$A(\alpha) = \frac{1}{b_0 \alpha} + \frac{b_1}{b_0^2} \log \left( \frac{\alpha}{b_0 + b_1 \alpha} \right) + \mathcal{O}(\alpha).$$

(143)

The function $e^D$ follows from Eq. (60), and after an expansion in powers of $\alpha_0$ it reads

$$e^D = \frac{1}{b_0 \alpha^2} \left( 1 + d_1 \alpha_0 + d_2 \alpha_0^2 + \mathcal{O}(\alpha_0^3) \right),$$

(144)

where

$$d_1 = - \left( \frac{4}{9} + \frac{b_1}{b_0^2} \right) b_0 \tilde{\alpha},$$

(145)

$$d_2 = \frac{1}{81 b_0^2} \left( 8 b_0^4 + 18 b_0^2 b_1 + 81 b_1^2 \right) \tilde{\alpha}^2.$$

(146)

The integration in $\tilde{\alpha}$ cannot be computed analytically even after this expansion. In order to avoid this problem, we consider the following series

$$\frac{e^{-3 \tilde{A}}}{\sqrt{1 - \tilde{\alpha}^{-3} e^{-4 \tilde{A}}}} = \sum_{n=1}^{\infty} \frac{(2n)(2n-1)!!}{(2n-1)(2n)!!} \tilde{\alpha}^{8(1-n)} e^{(1-4n) \tilde{A}}.$$

(147)

Note that $\tilde{A} = A(\alpha) - A(\alpha_0)$ is also a function of the coupling constant $\alpha_0$, so it makes sense to consider an expansion in $\alpha_0$ for the above expression. The result is

$$e^{(1-4n) \tilde{A}} = e^{(1-4n) \frac{\alpha_0 - \alpha}{\alpha_0^{5/2} \alpha_0^2}} \left[ 1 + t_1 \alpha_0 + t_2 \alpha_0^2 + \mathcal{O}(\alpha_0^3) \right],$$

(148)
where

\begin{align*}
t_1 &= (1 - 4n)(1 - \tilde{\alpha}) \frac{b_2^2}{b_0^2}, \quad (149) \\
t_2 &= -(1 - 4n)(1 - \tilde{\alpha}) \frac{b_3^2}{2b_0^2} \left[ b_0^2 (1 + \tilde{\alpha}) - b_1 (1 - 4n)(1 - \tilde{\alpha}) \right]. \quad (150)
\end{align*}

In Eq. (148) there is a dependence in \( \alpha_0 \) that cannot be expanded due to an essential singularity. Combining Eq. (144) and Eq. (148), we get

\[ e^D e^{(1-4n)A} = \frac{1}{b_0 \alpha_0^2} e^{(1-4n)\frac{A}{\alpha_0}} \alpha^{(1-4n)\frac{b_3}{b_0}} \left[ 1 + (d_1 + t_1) \alpha_0 + (d_2 + d_1 t_1 + t_2) \alpha_0^2 + O(\alpha_0^3) \right]. \quad (151)\]

Using this expression and taking into account Eq. (68) and Eq. (147), we can compute the integration in \( \tilde{\alpha} \) analytically for every order in \( \alpha_0 \) and \( n \), and the result is a summation of terms involving the incomplete Gamma functions. Making a further expansion in \( \alpha_0 \) one gets

\[ \rho(\alpha_0) = 2 \overline{\ell} e^{-A_0} \left[ \rho_0 + \rho_1 \alpha_0 + \rho_2 \alpha_0^2 + O(\alpha_0^3) \right], \quad (152)\]

where

\[ \rho_0 = \sum_{n=1}^{\infty} \frac{(2n)(2n-1)!!}{(2n-1)(4n-1)(2n)!!} \approx 0.596, \]

\[ \rho_1 = \frac{16}{9} b_0 \sum_{n=1}^{\infty} \frac{n(n-1)(2n-1)!!}{(2n-1)(4n-1)^2(2n)!!} \approx 0.0471 b_0, \quad (153)\]

\[ \rho_2 = \sum_{n=1}^{\infty} \frac{(4n)(2n-1)!!}{81(2n-1)(4n-1)^3(2n)!!} \left[ 4(22 + n(40n - 53))b_0^2 + 9(4n-1)(8n-5)b_1 \right] \approx 0.0860b_0^2 + 0.180b_1. \]

We truncate the summation of the infinite series to \( \sum_{1}^{20000} \cdots \), by which the residue contribution can be neglected to \( 10^{-3} \) relative precision.

Next, we will focus on the computation of \( V_{QQ} \), which is given by Eq. (72). There are two contributions which we call \( V \) and \( V_s \), corresponding to the first and second integrals respectively, so we write \( V_{QQ} = V - V_s \). Note that for the first one, the expression is similar to that of \( \rho(\alpha_0) \), and so we can apply the same procedure that we explained above. In order to make the integration analytical we consider the series

\[ e^{\tilde{A}} \frac{1 - \sqrt{1 - \tilde{\alpha}^{-\frac{5}{3} e^{-4\tilde{A}}}}}{\sqrt{1 - \tilde{\alpha}^{-\frac{5}{3} e^{-4\tilde{A}}}}} = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \tilde{\alpha}^{-\frac{5}{3} n} e^{(1-4n)\tilde{A}}. \quad (154)\]
Using Eqs. (72), (144), (148) and (154) we can compute analytically the integration in $\tilde{\alpha}$ in the same way as we did for $\rho(\alpha_0)$. Finally we arrive at this result

$$V(\alpha_0) = \frac{\bar{\ell} \alpha_0^4 e^{A_0}}{\pi l_s^2} \left[ v_0 + v_1 \alpha_0 + v_2 \alpha_0^2 + \mathcal{O}(\alpha_0^3) \right], \quad (155)$$

where

$$v_0 = \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(4n-1)(2n)!!} \approx 0.398,$$

$$v_1 = \frac{8}{9} b_0 \sum_{n=1}^{\infty} \frac{(n-1)(2n-1)!!}{(4n-1)^2(2n)!!} \approx 0.0423 b_0, \quad (156)$$

$$v_2 = \frac{2}{81} \sum_{n=1}^{\infty} \frac{(2n-1)!!}{(4n-1)^3(2n)!!} \left[ 4(22 + n(40n - 53))b_0^2 + 9(4n - 1)(8n - 5)b_1 \right]$$

$$\approx 0.0640b_0^2 + 0.131b_1.$$

The computation of $V_s(\alpha_0)$ is a little bit tricky because it does not allow similar expansion techniques. First of all it is convenient to change the integration limits from $(\alpha_0, \infty)$ to $(0, 1)$, and so we consider the variable replacement $\alpha \rightarrow \frac{\tilde{\alpha}}{\alpha}$. Then we get

$$V_s = \frac{\bar{\ell} \alpha_0^4 e^{A_0}}{\pi l_s^2} \int_{\alpha_0}^{\infty} \tilde{\alpha}^4 \cdot e^{D+\tilde{A}} d\tilde{\alpha} = \frac{\bar{\ell} \alpha_0^4 e^{A_0}}{\pi l_s^2} \int_0^1 \tilde{\alpha}^{-\frac{15}{4}} \cdot e^{D+\tilde{A}} d\tilde{\alpha} =: \frac{\bar{\ell} e^{A_0}}{\pi l_s^2} \int_0^1 \tilde{\alpha}^{-\frac{15}{4}} f(\tilde{\alpha}, \alpha_0) d\tilde{\alpha}, \quad (157)$$

where we have defined the function

$$f(\tilde{\alpha}, \alpha_0) = \alpha_0^5 e^{D+\tilde{A}}, \quad (158)$$

which involves all the interesting dependence in $\alpha_0$. We can expand this function in powers of $\alpha_0$ by considering its derivatives

$$f(\tilde{\alpha}, \alpha_0) = f(\tilde{\alpha}, 0) + f'(\tilde{\alpha}, 0)\alpha_0 + \frac{1}{2} f''(\tilde{\alpha}, 0)\alpha_0^2 + \mathcal{O}(\alpha_0^3), \quad (159)$$

where $f'$ stands for derivative with respect to the second argument. This expansion can be easily done analytically, but the integration in $\tilde{\alpha}$ must be computed numerically. By inserting Eq. (159) into Eq. (157), and after making the integration, finally we get

$$V_s(\alpha_0) = \frac{\bar{\ell} \alpha_0^4 e^{A_0}}{\pi l_s^2} \left[ 1 + 0.889 b_0 \alpha_0 + (2.17b_0^2 + 1.11b_1)\alpha_0^2 + \mathcal{O}(\alpha_0^3) \right]. \quad (160)$$

Note that the contribution of the substraction term is larger than Eq. (155) by approximately a factor 3 in the regime which we are interested in. Eq. (160) reproduces well the
full numerical computation of Eq. (157) up to \( \alpha_0 \approx 0.19 \), with an error of 5%. This \( \alpha_0 \) corresponds to distance \( \rho \approx 0.20 \text{ GeV}^{-1} \).

Combining Eqs. (155), (156) and (160), we obtain for the \( Q\bar{Q} \)-potential

\[
V_{Q\bar{Q}}(\rho) = V - V_s = -\frac{\tilde{\ell}\alpha_0^4 e^{\tilde{A}_0}}{\pi l_s^2} \left[ 0.602 + 0.847 b_0 \alpha_0 + (2.106 b_0^2 + 0.979 b_1) \alpha_0^2 + \mathcal{O}(\alpha_0^3) \right].
\]  

From Eq. (152) and Eq. (161) we can compute the dimensionless quantity \( \rho V_{Q\bar{Q}}(\rho) \) up to order \( \alpha_0^2 \), and we get the final result

\[
V_{Q\bar{Q}}(\rho) = -\frac{2\tilde{\ell}^2}{\pi l_s^2} \frac{\alpha_0^{4/3}(\rho) \log \left( \frac{c}{\rho} \right)}{\rho} \left[ 0.359 + 0.533 b_0 \alpha_0(\rho) + (1.347 b_0^2 + 0.692 b_1) \alpha_0^2(\rho) + \mathcal{O}(\alpha_0^3) \right].
\]  

The function \( \alpha_0(\rho) \) can be obtained by reversing Eq. (152). To do that we make first an expansion of \( \tilde{A}_0 \) in \( \alpha_0 \). The resulting expression is

\[
\alpha_0(\rho) = \frac{1}{b_0 \log \left( \frac{\rho}{b_0} \right) - \frac{b_1}{b_0} \log \alpha_0(\rho)} - \frac{\left( b_0 \frac{\rho^2}{b_0^2} \right) \alpha_0}{\left( b_0 \log \left( \frac{\rho}{b_0} \right) - \frac{b_1}{b_0} \log \alpha_0(\rho) \right)^2} + \mathcal{O} \left( \frac{\alpha_0^2}{\log^2 \left( \frac{\rho}{b_0} \right)} \right),
\]  

where

\[
c \equiv 2 \rho_0 b_0^{1/3} \tilde{\ell} \approx 1.32 \tilde{\ell},
\]  

with the parameters \( \rho_0 \) and \( \rho_1 \) defined in Eq. (153). Then we can use an iteration method to obtain the solution order by order. For example, the first order is given by \( \left( b_0 \log \left( \frac{\rho}{b_0} \right) \right)^{-1} \). If one substitutes this expression in the r.h.s. of Eq. (163), then we get the NLO approximation.

Combining Eq. (162) and Eq. (163) we have the \( Q\bar{Q} \)-potential as a function of the separation \( \rho \) between the quark and the antiquark in the limit of small separations, i.e., \( \rho \to 0 \). This result agrees with Ref. [13] at leading order. We corrected a mistake in this reference at next to leading order for \( V_s \). Fully NNLO expressions are provided here.

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