Impact of rheology on the acousto-elastic effect revealed by guided waves in a highly-stretched plate

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We study the propagation of guided elastic waves in a plate made of Ecoflex®, a nearly-incompressible elastomer. The in-plane displacement field in a soft plate is extracted. We measure the phase velocities of two in-plane modes $SH_0$ and $S_0$ (first symmetrical Lamb mode) coexisting in the low frequency limit. While $SH_0$ propagates at the transverse velocity, $S_0$ propagates at the plate velocity. The plate is then subjected to a nearly-uniaxial stress with an elongation reaching 120%. An induced anisotropy is first observed and then characterized by following the phase velocities of both modes in two principal directions. Although these measurements provide an estimate of the initial stress, they are not correctly predicted by the acousto-elastic theory. We thus show that the acousto-elastic theory alone is not sufficient to explain the evolution of velocities in a prestressed elastomer. The origin of this discrepancy is actually explained by the rheological properties of the elastomer, namely the frequency-dependent shear modulus. The implementation of a fractional rheological model in the acousto-elastic theory enables a proper prediction for these velocities up to 80% elongation.

The evaluation of mechanical properties is of paramount importance and the propagation of elastic waves allows to probe those properties deep inside the medium. Such an inverse problem is of fundamental relevance for geophysics imaging [1] or seismology [2]. Similarly, in an industrial context, non-destructive techniques based on ultrasonic waves permit to access the elastic constants of a material [3, 4].

In human body, ultrasonic waves are also used to build medical images of fetus and other organs. Interestingly, in this specific application, the medium is considered as a fluid-like material and the images remain qualitative. Nevertheless, with the emergence of elastography [5, 6] in the last decades, exploiting the propagation of shear elastic waves which are typical of solid-like material has revealed a good tool to probe the stiffness of tissues. Despite some very efficient applications such as the detection of hepatic fibrosis [7, 8], this technique shows some limitations for quantitative measurements [9, 10]. Indeed, the body is mostly made of nearly incompressible media which are highly deformable and the retrieved stiffness seems to depend on applied stresses [11–13]. This is related to a phenomenon known as the acousto-elastic effect [14–17], which is not specific to elastography but refers to the changes in elastic wave velocities with an initial stress. It is for instance at the basis of string instruments, where the tension is finely tuned to adjust the pitch of the musical instrument. The more deformable the medium, the more visible this effect; this is why the case of soft media is of special interest. With the emergence of polymers which can support high deformations, the class of hyperelastic media has emerged, and a wide literature on the subject exists [18–21]. Moreover, this constitutive model is often considered since the velocities of the shear waves directly provide a measurement for the applied stress [3, 4].

In addition, although soft materials are generally viscoelastic, the rheology is often omitted when modelling prestressed soft media. Yet, it was shown that the viscoelasticity must be taken into account to describe hysteresis curves during experimental tensile tests [23], bulk wave velocities changes induced by acousto-elasticity [24, 25], and the dispersion of bulk waves in metamaterials [26, 27], Lamb waves in a plate [28, 29] or in-plane guided waves in a soft strip [30].

In this article, the propagation of guided elastic waves in a pre-stressed soft plate is investigated experimentally and theoretically. First, the stress-induced anisotropy is observed. Notably, a systematic study of the wave speeds along with or transversely to the stress direction demonstrates different behaviours for the two low-frequency modes of interest. Then, the main steps of the acousto-elastic theory are recalled and compared to the experimental measurements. It appears that the theory permits to recover the static stress as in [31], but it fails to fit the measured dispersion curves. This highlights the importance of the polymer rheology in the theory. In the end, a full acousto-viscoelastic theory is presented, and it better catches the physics of the system.
FIG. 1. Experimental setup to measure velocities in a (deformed) plate—(a) A thin plate of Ecoflex® 00-30 with dimensions 60 cm x 60 cm x 3 mm held in a vertical position, clamped to a frame on its top and bottom edges. Sinusoidal vibrations in the \((x_1, x_2)\) plane are generated by a shaker. The experiment is recorded using a CCD camera located 3 m away from the plate. (b) Same experimental configuration, but the frame can be adjusted to impose large deformations, reaching a stretch ratio \(\lambda > 2\) along the vertical axis.

EXPERIMENT

The experiment, presented in figure 1, has been inspired from previous works [28, 30]. A 3 mm thick plate made of a soft elastomer, namely Ecoflex® 00-30, is prepared. This material is assumed to be nearly-incompressible with a Young modulus \(E \sim 75\) kPa. The plate is held vertically and clamped at its bottom and top edges to rigid bars. This configuration allows to apply a static and large stretch in the plate. In figure 1(a), the plate is undeformed and in a natural configuration defined by the \((X_1, X_2, X_3)\) coordinates. After applying an elongation (figure 1b), the coordinates become \((x_1, x_2, x_3) = (\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3)\). In both configurations, a point-like source is made of two magnets fixed on both sides of the plate center. This point source is driven monochromatically with a shaker and generates elastic waves polarized in the \((x_1, x_2)\) plane. The small perturbation defines the new set of coordinates \((x_1', x_2', x_3')\). The shaker can be rotated in order to change the source oscillation direction. The excitation frequency spans a range from 50 Hz to 300 Hz. A 60-frames video is recorded using a Basler full-frame CCD camera (acA-4112-20um) mounted with an 85 mm lens. The limited acquisition frame rate of the camera is overcome using stroboscopic imaging. Each frame is then compared to a reference through a Digital Image Correlation (DIC) algorithm [32] and the in-plane wave field components \((u_1, u_2) = (x_1' - x_1, x_2' - x_2)\) are retrieved.

An example of acquired frame is displayed in figure 2(a) for the undeformed plate. Typical displacement maps obtained when vibrating the source at 200 Hz are also shown as a colour code. Given a source vibrating along \(x_1\) axis (respectively \(x_2\)), the displacement \(\text{Re}[u_1(\omega)]\) (respectively \(\text{Re}[u_2(\omega)]\)) is displayed in figure 2(b) (respectively 2(c)). As a first observation one can notice that rotating the source by 90° involves a rotation of 90° of the displacement map. This demonstrates that the material is isotropic at rest and no privileged axis exists in the undeformed case. By carefully looking at the change of sign along the two main directions one can notice the existence of two distinct wavelengths, one being twice larger than the other. This effect is confirmed by applying spatial Fourier Transform on these wave-fields. After normalization and summation in intensity of the two, the spatial spectrum of the measured waves evidences two concentric circles in figure 2(d), with radii again showing this factor of 2. It corresponds to guided modes with isotropic behaviours.

Given the theoretical framework detailed in [28, 33], only three modes can propagate in this plate at this frequency: the first shear horizontal mode \(SH_0\) and the first two Lamb modes \(S_0\) and \(A_0\). While \(SH_0\) and \(S_0\) are polarized in the \((x_1, x_2)\) plane, the \(A_0\) mode is mainly polarized along \(x_3\) axis at this frequency, and is not observed in this experiment. Furthermore, \(SH_0\) and \(S_0\) are nearly non-dispersive at this frequency, and propagates respectively at the transverse velocity \(V_T\) and at the plate velocity \(V_P\). For an incompressible solid, it is noteworthy that \(V_P = 2V_T\). All these considerations permit to label the two visible circles in figure 2(d).

The same procedure is then repeated after applying a static load to the plate. The stretch ratios in directions...
FIG. 2. Experimental 2D displacement maps in an undeformed and a deformed plate at 200 Hz – (a)-(e) Typical pictures obtained for the initial and the deformed plate. 4 red dots are here to measure \( \lambda_1 = 2.01 \) and \( \lambda_2 = 0.74 \). (b)-(f) 2D map of the displacement \( u_1 \) with a source vibrating along \( x_1 \). (c)-(g) 2D map of the displacement along \( u_2 \) with a source vibrating along \( x_2 \). (d)-(h) Isofrequency contours for the initial and deformed plate. The spatial Fourier transforms of the two previous maps are normalized, squared and summed.

\( x_1 \) and \( x_2 \) are measured by manually tracking the displacement of the red diamonds in figure 2(e). In this specific example, the following stretch ratios are measured: \( \lambda_1 = 2.01 \) and \( \lambda_2 = 0.74 \). Comparing the field maps for the two different vibrating directions of figure 2(f) and 2(g) now reveals that the system is no longer invariant by rotation: the initial deformation has induced an anisotropy to the propagation. Again, this effect is nicely caught in the spatial Fourier domain of figure 2(h) where the two concentric circles are now replaced with ellipses. It appears that the elongation does not affect similarly the two types of waves since the two ellipses have different aspect ratios.

To track systematically the anisotropy induced on both \( SH_0 \) and \( S_0 \) modes, a new set of measurements is performed. The point source (figure 1) is replaced by a line source to generate plane waves as shown on the left part of figure 3. And the vibration is reproduced for different static stretch ratio \( \lambda_1 \) and frequencies. Shaking this line source in the \((x_1, x_2)\) plane, with a 45° angle between the displacement and the propagation direction, allows the observation of both \( SH_0 \) and \( S_0 \) in one experiment. The dispersion curves obtained for different stretch ratios \( \lambda_1 \) are plotted in figure 3(a) (respectively 3(b)) for plane waves propagating in the \( x_1 \) direction (respectively the \( x_2 \) direction). The higher the stretch ratio, the darker the curves, as represented on the colorbar. The behaviour depends on the propagation direction: while the slopes are increasing in the parallel direction (in the \( x_1 \) direction), they barely vary in the perpendicular direction (in the \( x_2 \) direction). Note that at the frequency \( f \sim 100 \text{ Hz} \), an accident occurs due to a mechanical resonance of the clamp fixed to the shaker and holding the line source. This small ‘anti-crossing’ has no influence on the measured dispersion relation above 150 Hz and is briefly discussed in SI.

From now on, phase velocities are extracted at an intermediate frequency of 170 Hz and plotted as a function of the stretch ratio \( \lambda_1 \) in figure 3(b) and 3(d) for the parallel and perpendicular directions (numerical values are available in table S1). The velocity of \( SH_0 \) appears to vary linearly with \( \lambda_1 \) in the parallel direction, while this is not the case for...
FIG. 3. Experimental dispersion curves and phase velocities at 170 Hz for a stretched plate — The measurement in the top part (respectively the bottom part) are obtained for plane waves propagating in the $x_1$ direction (respectively in the $x_2$ direction). (a)-(c) Dispersion curves of waves propagating in $x_1$ and $x_2$ directions. The stretch ratio $\lambda_1$ is given by the darkness, as represented on the colorbar. (b)-(d) Phase velocities of $SH_0$ and $S_0$ at 170 Hz extracted and plotted as functions of $\lambda_1$.

the velocity of $S_0$. In addition, velocities of both $SH_0$ and $S_0$ remain almost constant in the perpendicular direction. To understand those observations, one should study the acousto-elastic theory.

ACOUSTO-ELASTIC EFFECT

Acousto-elasticity is the effect that explains the changes in the propagation velocities of elastic waves due to the initial stress. It is a consequence of the initial deformation as a non-linear geometrical effect. Numerous works have been devoted to explaining this phenomenon and we will simply recall its main characteristics (more details can be found in the SI, or in [34, 35]). First, the consequence on the elastic theory for bulk materials is shown: the breaking of the main symmetries of the elastic tensor is responsible for the induced anisotropy. Second, the theory is applied to the guiding geometry of the plate and compared to the experimental data.

Non-linear elasticity

The constitutive law in linear elasticity, known as the Hooke’s law, relates linearly the Cauchy stress $\sigma$ to the strain tensor $\epsilon = \frac{1}{2} \left[ \nabla u + (\nabla u)^T \right]$ where $u = x' - X$:

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl}$$  (1)
The link between the two second-order tensors is the fourth-order elasticity tensor $C_{ijkl}$. Combining this constitutive law with the equation of motion, i.e. $\nabla \cdot \sigma^T = \rho \ddot{u}$, the wave equation is obtained:

$$C_{ijkl} \frac{\partial^2 u_i}{\partial x_j \partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2}$$

(2)

Nevertheless, in non-linear elasticity, depending on the chosen Lagrangian or Eulerian approaches, new tensors have to be considered \[34, 36\]. Here, the Cauchy stress tensor $\sigma$ is kept and the Green-Lagrangian strain tensor $E$ is considered. Like any strain measures, it is based on the use of the deformation gradient $F_s = I + \nabla \chi u_s$, where $u_s = x - X$ is the static displacement, and $I$ is the identity second-order tensor. The corresponding static strain tensor $E_s$ is defined as:

$$E_s = \frac{1}{2} (\nabla \chi u_s + (\nabla \chi u_s)^T + (\nabla \chi u_s)^T \cdot \nabla \chi u_s)$$

(3)

A geometrical non-linearity is therefore evidenced as the last term in the sum. But the mechanical non-linearity must also be thought out. Typically, as introduced, it is common to use an hyperelastic law for soft media where $\sigma$ is also a non-linear function of $E_s$. This constitutive law is detailed in SI and basically relies on a strain energy density function $W$.

**Bulk waves**

To describe waves in a pre-stressed body, an incremental approach is built as described by Ogden and Destrade \[16, 22, 34\]. A wave equation still stands:

$$C_{ijkl} \frac{\partial^2 u'_i}{\partial x_j \partial x_k} = \rho \frac{\partial^2 u'_i}{\partial t^2}$$

(4)

with $u'(x, t) = x' - x$ an incremental displacement and $C_0$ a modified elasticity tensor that depends on $W$ and the stretch ratios $\lambda_i$. All non-linearities are henceforth contained in this modified tensor. The calculation of the tensor coefficients are detailed in SI.

A major consequence of this approach is the loss of some fundamental symmetries in the $C_0$ tensor, as illustrated in figure 4. Indeed, in the linear approach, the elasticity tensor $C_{ijkl}$ is generally represented thanks to the Voigt notation as a 6x6 matrix, but one has to keep in mind that it contains 81 coefficients. A full representation with a 3x3 matrix $(i, j)$ of 3x3 matrices $(k, l)$ is preferred here. In this representation, in figure 4(a), the isotropic elastic tensor $C_{ijkl}$ has only 3 different coefficients: the two Lamé coefficients $\lambda_{Lamé}$ (orange) and $\mu$ (yellow), and a third coefficient which depends on these two constants, $\lambda_{Lamé} + 2\mu$ (black).

Given the fact that a stress is applied along a particular direction in our experiments, one would be tempted to consider a transverse isotropic material where $x_1$ would be the isotropy axis. The elasticity tensor for such an anisotropic medium remains as sparse as the one of the isotropic material but now contains 6 distinct coefficients (figure 4(b)).

Let us now consider a uniaxial tension for an incompressible material $\lambda_2 = \lambda_3 = \lambda_1^{-0.5}$. Using the Neo-Hookean hyperelastic model \[21\], the Lamé constants are also sufficient to describe the modified elasticity tensor $C_{ijkl}$ (see SI). It now contains 7 different coefficients as shown in figure 4(c). And, very interestingly, some fundamental symmetries are broken: $C_{ijkl} \neq C_{ijlk}$ and $C_{ijkl} \neq C_{lkij}$. However, the symmetry $C_{ijkl} = C_{lkji}$ is preserved. Therefore, the Voigt notation is no longer valid and no usual anisotropic model can be used. For the Mooney-Rivlin model, an additional constant $\alpha$ is needed (see SI), and there are 9 different coefficients in the modified elasticity tensor $C_{ijkl}$ (figure 4(d)).

From the knowledge of the modified elastic tensor $C_{ijkl}$, and the propagation equation, the bulk wave velocities for any plane waves can be retrieved. Depending on the considered hyperelastic model, the two transverse waves may have degenerated velocities. Those results are summed up in figure 4. The linear isotropic, the transverse isotropic approaches and the two hyperelastic models are considered.

For each model, the velocities of the three bulk waves propagating in the $(x_1, x_2)$ plane are derived and their inverse $\frac{1}{s}$, referred to as the slowness, are plotted for different values of $\lambda_1$ below the corresponding elasticity tensor in figure 4. In the undeformed case, circles, indicating isotropic media, are recovered. The longitudinal velocity
FIG. 4. Elasticity tensors and slowness curves for various models assuming a uniaxial tension – The elasticity tensor $C_{ijkl}$ is represented as a 3x3 matrix $(i,j)$ of 3x3 matrices $(k,l)$, and the slowness curves of bulk waves propagating in the $(x_1, x_2)$ plane are plotted. An equal radial spacing of 0.02 s.m$^{-1}$ is applied. First, a linear isotropic model (a,e) with $\lambda_{\text{Lamé}} = 1$ GPa and $\mu = 46$ kPa and a transversely isotropic solid (b,f) with $C_{44}/C_{66} = 0.82$, $C_{11} = C_{22}$ and $C_{23} = C_{12}$ are considered. Then, a NeoHookean hyperelastic model (c,g) using the same constants predicts degenerated shear waves, while a Mooney-Rivlin hyperelastic model (d,h) uses an additional constant $\alpha = 0.5$. For each hyperelastic model, different stretch ratios $1 \leq \lambda_1 \leq 2$ are considered.

being very large compared to the transverse one for a nearly incompressible medium, the longitudinal slowness curve appears as a single point on the graph. In the transverse isotropic model, the shear wave polarized in the $(x_1, x_2)$ plane (red in figure 4) remains isotropic, while the wave shear polarized along the $x_3$ axis (blue in figure 4) is anisotropic. Taking into account the hyperelasticity, in the NeoHookean model, slowness curves become ellipses and shear waves are degenerated. But with the Mooney-Rivlin model, transverse velocities are now distinct. In particular, the shear wave polarized in the $(x_1, x_2)$ plane propagates slower in the perpendicular direction in a deformed plate, as for the NeoHookean model, while the shear wave polarized along the $x_3$ axis propagates faster.

Hyperelastic predictions for elastic guided waves in a plate

The case of the plate is slightly more complicated since the solutions are no longer plane waves, but guided waves which must satisfy the boundary conditions. However, the changes compared to an undeformed material are all contained in the modified elastic tensor $C_{ijkl}$ as for the bulk case.

For the $SH_0$ mode, everything remains similar to the bulk and its velocity is indeed the same as the one of a shear bulk wave. Assuming a NeoHookean material under a uniaxial tension, it is easy to show that:

$$V_{T,\parallel} = \sqrt{\frac{\mu}{\rho} \lambda_1} \quad \text{and} \quad V_{T,\perp} = \sqrt{\frac{\mu}{\rho} \lambda_2}$$

The dispersion of Lamb waves in a stretched plate is more difficult to calculate \cite{37,38}. Nonetheless, the velocity of the $S_0$ mode in the low-frequency limit has been derived by Rogerson and Fu in the equation (3.22) of their work \cite{39} for incompressible hyperelastic models. Using the NeoHookean model, the plate velocities write:

$$\rho V_{P,\parallel}^2 = \mu \left( \lambda_1^2 + 3\lambda_2^2 \right)$$

$$\rho V_{P,\perp}^2 = \mu \left( \lambda_2^2 + 3\lambda_2^2 \right)$$

At this point, it is also important to mention that the deformation does not perfectly match the one expected for an uniaxial tension. In fact, the top and bottom clamps induce a slightly different configuration where $\lambda_2 = \lambda_1^{-0.41} \neq \lambda_1^{-0.5}$. This is checked by a calibration procedure described earlier using the four red diamonds in figures 2(a) and 2(e).
FIG. 5. Hyperelastic model predictions for velocities of $SH_0$ and $S_0$. In dashed grey lines are presented the NeoHookean model predictions, and in dotted blue lines the Mooney-Rivlin ones, in the parallel (a) and perpendicular (b) directions. No fitting is performed for the NeoHookean model since we directly have $\mu = \rho V_T^2 = 46 \text{ kPa}$ in the undeformed plate, but one fitting is performed for the Mooney-Rivlin model for $1 \leq \lambda_1 \leq 1.8$, and it provides $\alpha = 0.32$. (c) Slowness curves are plotted for the Mooney-Rivlin model using previous fitting parameters.

FAILURE OF THE ACOUSTO-ELASTIC THEORY

Now that the hyperelastic predictions for the velocities of $SH_0$ and $S_0$ have been established, it is straightforward to compare them to the experimental measurements. This comparison is presented in figure 5(a) and 5(b). The grey lines correspond to the predictions using the NeoHookean model with $\mu = 46 \text{ kPa}$ and the blue lines correspond to the Mooney-Rivlin model with $\mu = 46 \text{ kPa}$ and $\alpha = 0.32$. The $\alpha$ coefficient has been obtained using a least-square procedure involving the four velocities for $1 \leq \lambda_1 \leq 1.8$. For figure 5(c), the tensor of a compressible Mooney-Rivlin model was first derived using Mathematica [40] and the same previous constants. Then, the slowness curves were computed using an adaptation from an open-source code [41] [42, 43]. Those slowness curves should be compared to the isofrequency contours displayed in figure 2(d) and 2(h).

Although the parallel velocity of $SH_0$ is rightfully modelled as a linear function of the stretch ratio $\lambda_1$, it appears that none of those models can predict the slope. Furthermore, it is expected from both models that the perpendicular velocity of $SH_0$ decreases with $\lambda_1$ while the experimental measurements show it remains almost unchanged. Regarding these observations, we have tested various existing hyperelastic models [17, 21], adding unknown mechanical constants to be determined during the least-square procedure, but none of them is able to capture both the slope of the parallel velocity and the almost unchanged perpendicular velocity of $SH_0$.

A second limit appears when comparing those results to an independent static experiment. In figure 6 the static stress in the plate is plotted as a function of the measured stretch ratio $\lambda_1$ (magenta stars) during a tensile test. A static shear modulus $\mu_0 = 27 \text{ kPa}$ can be deduced using the initial slope given by the Young modulus $E = 3\mu_0$. Assuming a NeoHookean model and a uniaxial tension, the stress exactly writes: $\sigma_1 = \mu_0 (\lambda_1^2 - 1/\lambda_1)$ and is plotted in a black dashed line in figure 6.

This value is markedly different from the one used in figure 5 ($\mu = 46 \text{ kPa}$) to match the transverse velocity in the undeformed plate. Surprisingly, this static stress can also be recovered from dynamic experiments thanks to a model-independent observable, as used in [31] to map stresses in soft thin films:

$$\rho V_{T,\parallel}^2 - \rho V_{T,\perp}^2 = \sigma_1 - \sigma_2 \quad (7)$$

In figure 6 this quantity is plotted as a function of $\lambda_1$ in orange symbols for different frequencies. It matches well with all of the data points and implies a very interesting feature in terms of applications: one can measure a static and local stress in a plate through dynamic perturbations [31]. All the more remarkable, it remains true for all the measured frequencies.

The measured static shear modulus $\mu_0$ differs from $\rho V_T^2$ at all frequencies. This observation highlights the frequency dependence of the shear modulus $\mu$. It is consistent with previous works [28, 30] where the rheology of the used polymer...
FIG. 6. **Applied static stress is measured with different methods** – The observable $\rho V_{T,||}^2 - \rho V_{T,\perp}^2$ is plotted in orange symbols for different frequencies. It is compared to an independent static measurement (magenta stars) similar to a tensile test. Two theoretical predictions assuming a NeoHookean model and a uniaxial tension are plotted in black ($\mu_0 = 27$ kPa) and grey ($\mu = 46$ kPa) dashed line.

is well described by a fractional Kelvin-Voigt model:

$$\mu(\omega) = \mu_0 [1 + (i\omega\tau)^n]$$

with $\tau = 290 \mu s$ and $n = 0.29$. The Kelvin-Voigt model is a commonly used viscoelastic model, and its fractional derivative counterpart $(i\omega)^n$ with $0 < n < 1$ originates from so-called memory effects, where the relaxation function is given by a power-law decay as detailed in [44–46].

### VISCO-HYPERELASTICITY

From the last observations, it appears that the material rheological properties are essential in the problem and must be taken into account. Following the work of Destrade, Saccomandi and Ogden [47], the Cauchy stress tensor is rewritten as the sum of a static and a dynamic part, where the static part is still given by the hyperelastic theory, but the dynamic part writes:

$$\sigma_{\text{dynamic}} = 2\nu D + \beta (BD + DB)$$

where $B = FF^T$ is the left Cauchy-Green tensor, $D = \frac{1}{2} [L + L^T]$ and $L = \partial_t F \cdot F^{-1}$. Here, the fractional derivative of the viscoelastic model in equation [8] needs to be introduced in the $L$ tensor. Like in a recent theoretical study [29], it becomes:

$$L = \partial_n F \cdot F^{-1}$$

From this expression, it is now possible to rewrite the elasticity tensor at a given frequency, using an additional fractional viscous term:

$$C_{\omegaijkl} = C_{ijkl} + \left(\nu + \beta \frac{\lambda_\|^2 + \lambda_\perp^2}{2}\right)(i\omega)^n$$

Considering the limit $(\nu, \beta) \to 0$, we recover the hyperelastic predictions $C_{ijkl}$. In an undeformed plate, $(\lambda_\|) \to 1$ and the rheology described in equation [8] must be recovered, leading to $\nu + \beta = \mu_0 \tau^n$. This condition is satisfied.
FIG. 7. Viscoelastic Mooney-Rivlin model predictions for velocities of $SH_0$ and $S_0$.—The fitting of velocities in the parallel (a) and perpendicular (b) directions for $1 \leq \lambda_1 \leq 1.8$ provides $\alpha = 0.54$ and $\beta' = 0.6$. (c) Slowness curves are plotted using previous fitting parameters.

rewriting $\beta = \beta' \mu_0 \tau^n$ and $\nu = (1 - \beta') \mu_0 \tau^n$ so there is only one remaining unknown constant in this fractional viscous part $\beta'$. From this, it is easy to derive the previous velocities using $C_{\omegaijkl}$ instead of $C_{ijkl}$.

Doing so leads to new predictions as illustrated in figure 7. Compared to the hyperelastic predictions presented in figure 5 both the parallel (figure 7(a)) and perpendicular (figure 7(b)) velocities of $SH_0$ are now well captured. The same remark for the $S_0$ velocities are valid. The fitting of the four velocities for $1 \leq \lambda_1 \leq 1.8$ provides $\alpha = 0.54$ and $\beta' = 0.6$. Slowness curves plotted in figure 7(c) are from now on similar to the experimental spatial Fourier transform presented in figure 2(d) and 2(h). Those corrections were possible thanks to the viscoelastic part in the Cauchy stress tensor, and in particular thanks to the frequency-dependence of the new elasticity tensor $C_{ijkl}$, resulting from the fractional derivative model.

One should also note that the curves are no longer well captured for a stretch ratio $\lambda \geq 1.8$. In fact, a Mooney-Rivlin model remains a weakly non-linear hyperelastic model. When performing static measurements such as a tensile test, it is common to use other hyperelastic models such as Gent, Fung-Demiray or Arruda-Boyce [21]. Here, it is our choice to avoid adding new material constants in the Cauchy stress tensor expression especially since the dynamic part in equation 20 can also become more complex and rewrites as in [47]:

$$\sigma_{\text{dynamic}} = 2\nu D + \beta (BD + DB) + \gamma (B^2D + DB^2) + \ldots$$

Moreover, by considering the complex-evaluated elasticity tensor $C_{ijkl}$, equation 20 is no longer true and the quantity $\rho V_{T,||}^2 - \rho V_{T,\perp}^2$ does not exactly match the applied static stress. However, in view of figure 6 this quantity still remains a good estimation of this stress.

CONCLUSION

In this paper, an experimental setup is introduced to measure the phase velocities of two in-plane guided modes in a plate made of a soft elastomer: the $SH_0$ and $S_0$ modes. In the first part, a high deformation is applied to the soft plate and an induced anisotropy is observed for those two modes. Given the initial deformation, the non-linear governing equations can in fact be reduced to a linear equation for elastic waves provided that the elasticity tensor is modified. Notably, the underlying symmetries of the Voigt notation are broken and none of the classical anisotropic elasticity tensors are able to explain the observed anisotropy. Then, the phase velocities and their evolutions with the stretch ratio are systematically studied. These measurements confirm that the initial stress can rightfully be estimated using a material-independent quantity $\rho V_{T,||}^2 - \rho V_{T,\perp}^2$. In the same time, they evidence the failure of the usual acoustic-elastic effect for soft elastomers. In fact, it is crucial to account for the rheology. Here, we introduce a fractional derivative model to properly fit the measured phase velocities. A generalization of this approach to other
rheological models is possible by adjusting the dynamic stress tensor. In the same way, they are of practical interest for rheological characterization of soft materials since this method captures the influence of the applied stress on the measured viscoelastic properties. Finally, this theoretical framework bridges the gap between elastic wave physics and rheology, but also paves the way for robust quantitative elastography.

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MATERIAL AND METHODS

Sample preparation: A thin plate of Ecoflex®-0030 (Smooth-On) with dimensions 60 cm x 60 cm x 3 mm is prepared by equally mixing parts A and B and a first layer is poured in the sample mould. After ten minutes, black carbon grains are deposited for displacement tracking. After 2 hours, the second layer is poured and the sample is cured for 6 hours. The rheological properties are measured with a conventional rheometer (MCR501, Anton-Paar), which operates in the plate-plate configuration.

Shaking: The excitation is performed by a shaker (TIRAvib 51120, TIRA), driven by an external arbitrary wave generator (AWG 33 220, Keysight), which is itself connected to a power amplifier (analog amplifier BAA 500, TIRA).

Image acquisition: The motion is captured by a charge coupled device (CCD) camera (acA4112-20um, Basler) with a 4112 x 3008-pixel sensor, mounted with a 85-mm zoom lens (Nikon). Stroboscopic imaging is used to overcome the camera frame rate limitations and a 60 frames movie is recorded at a given frequency.

Post-processing: The in-plane displacement field is extracted using a Digital Image Correlation (DIC) algorithm and the complex monochromatic displacement is computed at the excitation frequency. With a line source, a 1D spatial Fourier Transform is computed for each of the component of the displacement field and the maxima provide the phase velocities of the two modes $SH_0$ and $S_0$.

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