Coalescence Model of Rock-Paper-Scissors Particles

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Abstract The rock-paper-scissors game, commonly played in East Asia, gives a simple model to understand physical, biological, psychological and other problems. The interacting rock-paper-scissors particle system is a point of contact between the kinetic theory of gases by Maxwell and Boltzmann (collision model) and the coagulation theory by Smoluchowski (coalescence model). A $2s+1$ types extended rock-paper-scissors collision model naturally introduces a nonlinear integrable system. The time evolution of the $2s+1$ types extended rock-paper-scissors coalescence model is obtained from the logarithmic time change of the nonlinear integrable system. We also discuss the behavior of a discrete rock-paper-scissors coalescence model.

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1 Introduction

The rock-paper-scissors game is commonly played in East Asia. The cyclic dominance systems naturally occurs in biological systems as color morphisms of the side-blotched lizard [23] and strains of Escherichia coli [19]. The types, rock paper and scissors, can also represent social groups, opinions, or survival strategies of organisms in a preferential attachment graph model [9].
Considering simple models [18] for kinetic theory of gases by Maxwell and Boltzmann, we introduced a collision model of rock (type 1), paper (type 2), and scissors (type 3) particles with cyclic dominance Fig. 1, where 2 dominates 1, 3 dominates 2, and 1 dominates 3, and obtained a Lotka-Volterra equation [10] (the Boltzmann equation for the rock-paper-scissors particles). We can extend the argument to the $2s + 1$ types rock-paper-scissors particles [13], which gives a nonlinear integrable system [3, 15, 17, 22] with $s + 1$ conserved quantities like the Toda lattice [28] and the Calogero system [4]. For the case of finite number of particles, the probability of coexistence of types is obtained by using $s + 1$ martingales, which are stochastic version of the $s + 1$ conserved quantities [11, 14]. The rock-paper-scissors lattice model greatly enriches the dynamics as studied in the physics literatures [7, 6, 20, 25, 26, 27].

We introduce a coalescence model of rock-paper-scissors particles as in Fig. 2. For a given initial distribution of the particles of rock (type 1), paper (type 2) and scissors (type 3), what type of the particle will finally survive? As we see in Fig. 3 and Fig. 4 it gives a leader selection problem, which is for another aspect of the previously studied problem[8]. We carried out simulation studies for the coalescence model of rock-paper-scissors particles [12] for finite size fluctuation, where the total number of particles decreases at each step. Cyclic trapping reactions for finite size fluctuations [2] gives insights to the behavior of our model for finite size.

Here we study the time evolution of the coalescence model of rock-paper-scissors particles with sufficiently large number of particles. We apply the study of coalescence of clusters [1, 21, 24] to our problem. Let us follow the argument [21] for the simplest case. Infinite set of master equations that describe how the cluster mass distribution $n_k(t)$ evolves by the rule

$$A_i + A_j \to A_{i+j}$$

, in which clusters of mass $i$ and $j$ irreversibly join to form a cluster of mass $i + j$, is given by

$$\frac{d n_k(t)}{dt} = 1/2 \sum_{i+j=k} n_i n_j - \sum_{j=1}^{\infty} n_j n_k. \quad (2)$$

The first term on the right-hand side is for the creation of a $k$-mers due to the coalescence of two clusters of mass $i$ and $j$, we sum over all such pairs
with \( i + j = k \). The factor \( 1/2 \) in the gain term is needed to avoid double counting.

Instead of Eq. (1), the rock-paper-scissors coalescence in Fig 2 is represented as,

\[
A_i + A_j \rightarrow \begin{cases} 
A_i & \text{if } i - j \equiv 0,1 \pmod{3}, \\
A_j & \text{if } i - j \equiv 2 \pmod{3}.
\end{cases}
\] (3)

There are several solvable cases for the Smoluchowski coalescence equations [1] for the size of clusters. The "Boltzmann equation" for the collision model of the extended \( 2s + 1 \) types rock-paper-scissors particles is a nonlinear integrable system [3, 15, 17, 22]. Consider the time evolution of the relative abundance (concentration ratio), as in [21], for the "Smoluchowski equation" of the coalescence of extended \( 2s + 1 \) types rock-paper-scissors particles. It is obtained by a logarithmic time change of the nonlinear integrable system, as will be shown in Section 4.

Figure 1: Rock-paper-scissors collision model [10]

2 Smoluchowski equation for coalescence of rock-paper-scissors particles

The Smoluchowski coagulation model is for the time evolution of distribution of cluster size as Eq. (1), while our coalescence model given by the following
Figure 2: Rock-paper-scissors coalescence model [11].

1), 2), 3) and 4) is for the distribution of particle types of rock, paper and scissors.

1) There are 3 types (rock, paper, scissors) of particles 1, 2, 3 whose numbers of particles at time $t$, are $n_1(t), n_2(t), n_3(t)$ respectively, for which $n_1(t) + n_2(t) + n_3(t) = n(t)$.

2) Each particle coalescence with other particles $dt$ times on the average per time length $dt$.

3) Each particle is in a chaotic bath of particles. Each coalescence pair is equally likely to be chosen.

4) By a coalescence of a particle of type $i$ and a particle of type $j$, the two particles become one particle of type $i$, if $i - j \equiv 0, 1 \pmod{3}$, otherwise become one particle of type $j$ as Eq.(3), as also shown in Fig. 2.

Examples of trees generated by our coalescence model are shown in Fig. 3 and Fig. 4.

We have the master equation

$$\frac{dn_k(t)}{dt} = \frac{1}{2} \sum_{A_i + A_j \rightarrow A_k} n_in_j - \sum_{j=1}^{3} n_jn_k. \quad (4)$$
Figure 3: Possible tree of coalescence model: starting from \( n(0) = 4 \) particle. By the coalescence of a particle of 2 (paper) and the particle of 3 (scissors) the particle of 3 survives. By the coalescence of a particle of 1 (rock) and the particle of 2 the particle of 2 survives. By the coalescence of the two survivors the particle of type 3 (scissors) survives finally.

We have

\[
\begin{align*}
\frac{\partial n_1(t)}{\partial t} &= -\frac{1}{2} n_1(t)(2n_2(t) + n_1(t)) \\
\frac{\partial n_2(t)}{\partial t} &= -\frac{1}{2} n_2(t)(2n_3(t) + n_2(t)) \\
\frac{\partial n_3(t)}{\partial t} &= -\frac{1}{2} n_3(t)(2n_1(t) + n_3(t))
\end{align*}
\]

(5)  (6)  (7)

which will be studied for a general \( 2s + 1 \) hands rock-paper-scissors particles in Section 4.

3 Collision model of \( 2s + 1 \)-rock-paper-scissors particle

Consider the model defined by the following 1), 2), 3), and 4).

1) There are \( 2s + 1 \) types rock-paper-scissors particles 1, 2, ..., \( 2s + 1 \) whose abundances at time \( t \), are \( n_1(t), n_2(t), ..., n_{2s+1}(t) \) respectively, for which \( n_1(t) + n_2(t) + ... + n_{2s+1}(t) = n(t) \).

2) Each particle collides with other particles \( dt \) times on the average per time length \( dt \).

3) Each particle is in a chaotic bath of particles. Each colliding pair is equally likely to be chosen.
4) By a collision a particle of type $i$ and a particle of type $j$ become two particles of type $i$, if $i - j \equiv 0, 1, ..., s (mod \ 2s + 1)$, otherwise become two particles of type $j$ as Eq. (14), as also shown in Fig. 1 for the case $s = 1$.

For the collision model of $2s + 1$ types of rock-paper-scissors particles, we extend the rule of the rock-paper-scissors particles by the cyclic dominance rule,

$$A_i + A_j \rightarrow \begin{cases} 2A_i & \text{if } i - j \equiv 0, ..., s (mod \ 2s + 1) \\ 2A_j & \text{if } i - j \equiv s + 1, ..., 2s (mod \ 2s + 1) \end{cases}, \quad (8)$$

as shown in Fig 1 for the case $s = 1$.

The total number of particles $n(t)$ is time invariant. The relative abundance for each type of particles [10, 13] is given by the master equation

$$\frac{\partial}{\partial u} P_i(u) = P_i(u)\left(\sum_{j=1}^{s} P_{i-j}(u) - \sum_{j=1}^{s} P_{i+j}(u)\right). \quad (9)$$

Consider $2r + 1$ types out of the $2s + 1$ types $A_1, ..., A_{2s+1}$. If each of the $2r + 1$ types dominates the other $r$ types and is dominated by the other remained $r$ types, then we say the $2r + 1$ types are in a regular tournament.

Take $2r + 1$ particles at random from our system of the $2s + 1$ types $A_1, ..., A_{2s+1}$ of particles Eq. (9). Let $H_r(P(t))$ for $P(t) = (P_1(t), ..., P_{2s+1}(t))$, be the probability that the $2r + 1$ particles whose corresponding $2r + 1$ types...
are in a regular tournament. Then we have the $s+1$ conserved quantities [15]

$$H_r(\vec{P}(t)) = H_r(\vec{P}(0)), \quad \text{for } r = 0, \ldots, s. \quad (10)$$

For example, for the case $2s+1 = 5$, we have the conserved quantities

$$H_0(\vec{P}(t)) = \sum_{i=1}^{5} P_i(t) \quad (11)$$

$$H_1(\vec{P}(t)) = \sum_{i=1}^{5} P_i(t)P_{i+1}P_{i+3} \quad (12)$$

$$H_2(\vec{P}(t)) = P_1(t)P_2(t)P_3(t)P_4(t)P_5(t). \quad (13)$$

Eq. (9) is a nonlinear integrable system [3, 15, 17, 22].

4 Coalescence model of 2s+1-types rock-paper-scissors particles

We consider the coalescence model defined by the following 1), 2), 3) and 4).

1) There are $2s+1$ types rock-paper-scissors particles $1, 2, \ldots, 2s+1$ whose abundances at time $t$, are $n_1(t), n_2(t), \ldots, n_{2s+1}(t)$ respectively, for which $n_1(t) + n_2(t) + \ldots + n_{2s+1}(t) = n(t)$.

2) Each particle coalescence with other particles $dt$ times on the average per time length $dt$.

3) Each particle is in a chaotic bath of particles. Each coalescence pair is equally likely to be chosen.

4) By a coalescence a particle of type $i$ and a particle of type $j$ become one particle of type $i$, if $i - j \equiv 0, \ldots, s (mod \ 2s+1)$, otherwise become one particle of type $j$ as

$$A_i + A_j \rightarrow \begin{cases} A_i & \text{if } i - j \equiv 0, \ldots, s \ (mod \ 2s+1) \\ A_j & \text{if } i - j \equiv s + 1, \ldots, 2s \ (mod \ 2s+1) \end{cases} \quad (14)$$

We have for $i = 1, 2, \ldots, 2s + 1$,

$$\frac{\partial n_i(t)}{\partial t} = -\frac{1}{2}n_i(t)(n_i(t) + 2\sum_{j=1}^{s} n_{i+j}(t)), \quad (15)$$
which shows
\[
\frac{\partial n_i(t)}{\partial t} = \frac{1}{2} n_i(t)(\sum_{j=1}^{s} n_{i-j}(t) - \sum_{j=1}^{s} n_{i+j}(t)) - \frac{1}{2} n_i(t)n(t)
\] (16)
where \( n(t) = n_1(t) + n_2(t) + \cdots + n_{2s+1}(t). \)

Since
\[ \frac{\partial n(t)}{\partial t} = -\frac{1}{2} n(t)^2; \] (17)
we have
\[ n(t) = \frac{2}{t + 2/n(0)}. \] (18)

As in [21]), ”one useful trick that often simplifies master equations is to eliminate the loss terms by considering concentration ratios, rather than the concentrations themselves”. We have from Eq. (16)
\[ \frac{\partial}{\partial t} \frac{n_i(t)}{n(t)} = \frac{1}{2} \frac{n(t) n_i(t)}{n(t)} (\sum_{j=1}^{s} \frac{n_{i-j}(t)}{n(t)} - \sum_{j=1}^{s} \frac{n_{i+j}(t)}{n(t)}). \] (19)

Putting \( \frac{n_i(t)}{n(t)} = Q_i(t), \ i = 1, \ldots, 2s+1, \) we have
\[ \frac{\partial}{\partial t} Q_i(t) = \frac{1}{t + \frac{2}{n(0)}} Q_i(t)(\sum_{j=1}^{s} Q_{i-j}(t) - \sum_{j=1}^{s} Q_{i+j}(t)). \] (20)

For \( u = \log[t + \frac{2}{n(0)}], \) considering
\[ e^u = t + \frac{2}{n(0)} \] (21)
we have
\[
\frac{\partial}{\partial t} Q_i(e^u - \frac{2}{n(0)}) \]
\[ = \frac{1}{e^u} Q_i(e^u - \frac{2}{n(0)})(\sum_{j=1}^{s} Q_{i-j}(e^u - \frac{2}{n(0)}) - \sum_{j=1}^{s} Q_{i+j}(e^u - \frac{2}{n(0)})). \] (23)

\[
- \sum_{j=1}^{s} Q_{i+j}(e^u - \frac{2}{n(0)}). \] (24)
Since

\[
\frac{\partial u}{\partial t} = \frac{1}{t + \frac{2}{n(0)}} = \frac{1}{e^u} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial u}{\partial t} \frac{\partial}{\partial u} = \frac{1}{e^u} \frac{\partial}{\partial u},
\]

we have

\[
P_i(u) = Q_i(e^u - \frac{2}{n(0)}), \quad \text{for} \quad i = 1, \ldots, 2s + 1.
\]  

(25)

Hence for the solution \( P_i(u) \) to the nonlinear integrable system Eq. (??), we have

\[
Q_i(t) = P_i(\log[t + \frac{2}{n(0)}]), \quad \text{for} \quad i = 1, \ldots, 2s + 1.
\]

(26)

Thus we see the dynamical system of the \( 2s + 1 \) types rock-paper-scissors coalescence model is obtained by a logarithmic time change to Eq. (9) of the \( 2s + 1 \) types rock-paper-scissors collision model.

We can extend our argument to the infinitely many types rock-paper-scissors particles coalescence model. Eq. (9) for the collision model is extended to

\[
\frac{d}{dt}P(x, t) = P(x, t) \left( \int_{x-\pi}^{x} P(y, t)dy - \int_{x}^{x+\pi} P(y, t)dy \right)
\]

(28)

where \( P(x, t) = P(x + 2\pi, t) \) for each \( x \), \( P(x, t) \) being the probability density for a point on the unit circle [5, 16]. The logarithmic time change to Eq. (28) gives the dynamics of infinitely many types rock-paper-scissors particles coalescence model.
5 Surviver for a discrete model

Starting from a given initial state, what type of particle finally survives? Let us consider a discrete time model. We assume one coalescence takes place in a unit time instead of 2) in Section 2. Some possible cases are in Fig. 3 and in Fig. 4 starting from four particles. Here we show the simulation result in Fig. 5 and Fig. 6 As for collision model, we have martingales instead of conserved quantities and can discuss the probability of coexistence of type.

Figure 5: Discrete time coalescence model of rock-paper-scissors particles. At time 0, there are 150 rock particles, 100 paper particle and 50 scissors particles at time 0. A particle of scissors survives finally [12]

Figure 6: Relative abundance for the discrete time model [12]
References

[1] Aldous, D. J. 1999 Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists. Bernoulli, 5(1), 3-48.

[2] Ben-Naim, E., and Krapivsky P. L. 2004 Finite-size fluctuations in interacting particle systems, Physical Review E 69.4: 046113.

[3] Bogoyavlensky O I 1988 Integrable discretizations of the KdV equation, Phys. Lett. A 134 34-38.

[4] Calogero, F. 1975 Exactly solvable one dimensional many-body problems, Lett. Nuovo Cimento 13 411-416

[5] Evans, S. N. 1999 Infinitely-Many-Species Lotka-Volterra Equations Arising from Systems of Coalescing Masses, Journal of the London Mathematical Society, 60 171-186.

[6] Feldager, C. W., Mitarai, N., and Ohta, H. 2017 Deterministic extinction by mixing in cyclically competing species, Physical Review E 95.3 032318.

[7] Frachebourg, L. and Krapivsky, P. L. 1998 Fixation in a cyclic Lotka-Volterra model, J. Phys. A: Math. Gen., 31, L287-L293.

[8] Fuchs, M., Hwang, H-K. and Itoh, Y. 2017 From coin tossing to rock-paper-scissors and beyond: a log-exp gap theorem for selecting a leader, Journal of Applied Probability 54 213-235.

[9] Haslegrave, J. and Jordan, J. 2018 Non-convergence of proportions of types in a preferential attachment graph with three co-existing types, Electronic Communications in Probability, 23 (54), 1-12.

[10] Itoh, Y. 1971 Boltzmann equation on some algebraic structure concerning struggle for existence. Proceedings of the Japan Academy, 47 854-858.

[11] Itoh, Y. 1973 On a ruin problem with interaction, Annals of the Institute of Statistical Mathematics, 25 635-641.
[12] Itoh, Y. 1973 Model of struggle for existence, JUSE Symposium on Mathematical Programing 27, Model building and control problem of Ecosystem, Edited by T. Kitagwa, K. Kunisawa, S. Moriguti, 19-40 (in Japanese).

[13] Itoh, Y. 1975 An H-theorem for a system of competing species, Proceedings of the Japan Academy, 51 374-379.

[14] Itoh, Y. 1979 Random collision models in oriented graphs. Journal of Applied Probability, 16 36-44.

[15] Itoh, Y. 1987 Integrals of a Lotka-Volterra system of odd number of variables, Progress of theoretical physics, 78 507-510.

[16] Itoh, Y. 1988 Integrals of a Lotka-Volterra system of infinite species, Progress of Theoretical Physics, 80, 749-751.

[17] Itoh, Y. 2008 A combinatorial method for the vanishing of the Poisson brackets of an integrable Lotka-Volterra system, Journal of Physics A: Mathematical and Theoretical, 42(2), 025201.

[18] Kac, M. (Ed.) 1959 Probability and related topics in physical sciences (Vol. 1), American Mathematical Soc.

[19] Kerr, B., Riley, M. A., Feldman, M. W., and Bohannan, B. J. 2002 Local dispersal promotes biodiversity in a real-life game of rock-paper-scissors, Nature, 418 171-174.

[20] Knebel, J., Kruger, T., Weber, M. F., and Frey, E. 2013 Coexistence and survival in conservative Lotka-Volterra networks, Physical Review Letters, 110 168106.

[21] Krapivsky, P. L., Redner, S., Ben-Naim, E. 2010 A kinetic view of statistical physics, Cambridge University Press.

[22] Narita, K. 1982 Soliton solution to extended Volterra equation, Journal of the Physical Society of Japan, 51 1682-1685.

[23] Sinervo, B. and Lively, C. M. 1996 The rock-paper-scissors game and the evolution of alternative male strategies, Nature, 380 240-243.
[24] Smoluchowski, M. V. 1917 Mathematical theory of the kinetics of the coagulation of colloidal solutions, Z. Phys. Chem. 92 129-168.

[25] Szabo, G., Szolnoki, A., and Izsak, R. 2004 Rock-scissors-paper game on regular small-world networks, Journal of physics A: Mathematical and General, 37(7), 2599.

[26] Tainaka, K. I. 1988 Lattice model for the Lotka-Volterra system, Journal of the Physical Society of Japan, 57 2588-2590.

[27] Tainaka, K., and Itoh, Y. 1991 Topological phase transition in biological ecosystems, EPL (Europhysics Letters), 15 399-404.

[28] Toda M. 1967 Vibration of a chain with nonlinear interaction, Journal of the Physical Society of Japan, 22 431-436.