Reynolds Number of Transition as a Dynamic Constraint on Statistical Theory of Turbulence.

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Abstract

Iterative coarse-graining procedure based on Wyld’s perturbation expansion is applied to the problem of Navier-Stokes turbulence. It is shown that the low-order calculation gives the fixed-point Reynolds number $Re_{fp}$ (coupling constant) almost identical to the Reynolds number of the recently discovered transition to anomalous scaling of the moments of “velocity derivatives”. Using this result as a dynamic constraint, it is argued that in the vicinity of the fixed point (integral scale) the high-order non-linearities, generated by the procedure, are irrelevant. The infra-red divergencies do not disappear but are contained in the derived equations for the symmetry-breaking large-scale flows (turbulence models or “condensates”), which are source of the small-scale turbulence.

Introduction. For many years patterns, emerging in fluids undergoing transition to turbulence, were a source of fascination not only to scientists but also to artists and philosophers. Further increase of the Reynolds number leads to a complex system of intermittently bursting and dying small-scale structures resembling “worms”, rolls or “pancakes”. Still, even when the Reynolds number is very large, one can discern somewhat blurred, but clearly seen with naked eye silhouettes of large-scale images created at the transition point $Re=Re_{tr}$. An example, which is part of our every-day experience, is “vortex street” behind a fast car moving on a dusty road. These patterns can be made clearer when small-scale fluctuations are filtered out.

Disregarding intermittency, a turbulent flow is characterized by two length scales: integral $L \approx 1/\Lambda_f$ where energy is pumped into the system and dissipation scale $\eta = 1/\Lambda_0$ at which viscous effects balance the non-linearity. According to Kolmogorov’s theory, neither $L$ nor $\eta$ can appear in the expression for the energy spectrum in the “inertial range”. If this is so, the energy spectrum must be $E(k) \propto k^{-5/3}$. Due to infra-red divergencies of renormalized perturbation expansions of turbulence theory, this qualitatively appealing result has never been derived directly from the Navier-Stokes equations. It is well-known that each term in Wyld’s diagrammatic expansion is infra-red (i.r.) divergent, i.e. tends to
infinity together with integral scale $L \approx 1/\Lambda_f$ [1]. Among the earliest attempts to “save” the theory was Kraichnan’s Lagrangian History Direct Interaction Approximation (LHDIA) which was basically a one-loop closure written in Lagrangian coordinates eliminating the effects of a transport of small eddies by the large ones [2]. Still, the theory was unable to deal with subleading contributions appearing in the higher orders. Moreover, according to modern experimental and numerical data, it is quite possible that, due to anomalous scaling, the integral scale does enter the expressions for moments of velocity increments in the inertial range scales $1/\Lambda_0 \ll r \ll L$.

Application of the dynamic renormalization group to statistical theory of fluids was first proposed in Refs. [3]-[4]. These ideas were later generalized to the problem of large-scale features of hydrodynamic turbulence in Refs.[5]-[7]. In the lowest order in powers of the turbulent-viscosity-based Reynolds number (the $\epsilon$-expansion) the renormalization group (RNG) led to an excellent agreement with experimental data on various dimensionless amplitudes characterizing large-scale features of turbulent flows [5]-[7]. Moreover, the method yielded the coarse-grained equations (turbulence models) widely used in modern engineering [8]-[9]. The main drawback of the theory can be illustrated as follows. According to Kolmogorov’s phenomenology the scale-dependent coupling constant (Reynolds number) $Re(r) = u(r)_{rms} r/\nu(r)$, where $\nu(r)$ is the effective turbulent viscosity accounting for the effects of the small-scale fluctuations from the interval $l \sim r$. In the inertial range neither integral nor dissipation scale $\eta$ can enter the result. Therefore $\nu(r) \approx u_{rms}(r) r$ and at the the fixed point of renormalization group $Re(r) \approx O(1) = \text{const}$ is not a small parameter. Technically speaking, this invalidates the low-order truncation of expansion in powers of effective Reynolds number and the reasons for numerical success of the theory are yet to be explored. To correct for this drawback, one has to use non-perturbative methods or resummations of an infinite series in powers of the $O(1)$ parameters.

It will be shown below that the Reynolds number $Re(r)$ derived at the fixed point in Refs. [5]-[7] is almost identical to that of smooth” transition to strong turbulence numerically discovered Ref.[10]. Below, using this result as a dynamic constraint on an expansion, we argue that at the fixed point $r = 1/\Lambda_f$ all high-order nonlinear terms (HOT), generated by coarse-graining, sum up to zero, thus justifying the $\epsilon$-expansion introduced in [5]-[7]. It will also become clear that outside the fixed point, in the inertial range, the HOT exponentially grow invalidating some of the theories based on the second-order closures.

Transition to turbulence. For almost a century transition to turbulence has been a major theoretical challenge. There exist a huge literature on this topic which, together with theory of dynamical systems, evolved into a separate field of research. Typically, one searches for instabilities in laminar flows manifested by an exponential growth of some modes $u(k,t)$. We will loosely identify laminar flow as a pattern $u_0$ formed by a small set of excited modes supported in the range of wave-numbers $k \approx \Lambda_f$. All modes with $k > \Lambda_f$ are strongly overdamped, i.e. $u(k) = 0$ for both $k \ll \Lambda_f$ and $k \ll \Lambda_f$.

Landau’s theory. Here we mention just one work which is relevant for considerations pre-
sented below. Assuming that in the vicinity of a transition point imaginary part of complex frequency is much smaller than the real one, Landau considered the Navier-Stokes equations for incompressible fluid. Denoting the velocity field at a transition point \( u_0 \) and introducing an infinitesimal perturbation \( u_1 \) he wrote \( u = u_0 + u_1 \) with \( u_1 = A(t)f(r) \). Based on qualitative considerations, Landau proposed \[ d|A|^2 \over dt = 2\gamma|A|^2 - \alpha|A|^4 \]
where in the vicinity of transition point \( \gamma = c(Re - Re_{tr}) \) and \( \alpha > 0 \). In principle, \( |A|^2 \) must be considered as time-averaged. Landau noted, however, that \( u_1(k) \) is a slow mode and, since the averaging is taken over relatively short time-intervals, the averaging sign in the above equations is not necessary. At small times the solution exponentially grows and then reaches the maximum \( A_{max} \propto \sqrt{Re - Re_{tr}} \). When \( \gamma = Re - Re_{tr} < 0 \), any initial perturbation decays. In this theory, the magnitude of transitional Reynolds number is a free parameter and since the large-scale field \( u_0 \) strongly depends on geometry, external forces, stresses the transition Reynolds number \( Re_{tr} \) is not expected to be a universal constant.

Transition to turbulence: a new angle. A new way of looking at phenomenon of transition to turbulence was introduced in numerical simulations of a flow at a relatively low Reynolds number \( R_\lambda = \sqrt{\nu u_{rms}^2} \geq 4.0 \) [11]. In this work the rms velocity was defined as: \( u_{rms} = \sqrt{u_x^2 + u_y^2 + u_z^2} \) which included not only turbulent fluctuations at \( Re > Re_{tr} \) but also a low-Reynolds number (“non-turbulent”) velocity field \( u_0 \) at \( Re = Re_{tr} \). In this approach transition to turbulence is identified with the first appearance of non-gaussian anomalous fluctuations of velocity derivatives including those of dissipation rate. The flow in a periodic box was generated by a force in the right-side of the Navier-Stokes equation with driven by the force \( F(k,t) = P \sum_{|u(k,t)|^2} \delta_{k,k'} \), where summation is carried over \( k_f = (1,1,2); (1,2,2) \). It is easy to see that the model with this forcing generates flows with constant energy flux \( P = \mathcal{E} = \nu \left( \partial u \over \partial x \right)^2 = \text{const} \) and the variation of the Reynolds number is achieved by variation of viscosity.

The results of Ref.[11] can be briefly summarized as follows: 1. Extremely well-resolved simulations of the low-Reynolds number flows at \( R_\lambda \geq 9 - 10 \) revealed a clear scaling range \( \left( \partial u \over \partial x \right)^n \propto Re^{\rho_n} \) with the anomalous scaling exponents \( \rho_n \) consistent with the inertial range exponents typically observed only in very high Reynolds number flows \( Re \gg Re_{tr} \). Identical scaling exponents \( \rho_n \) were later obtained in some other flows [12] indicating possibility of a broad universality. 2. For \( R_\lambda < 9 - 10 \) the flow was subgaussian indicating a dynamical system consisting of a small number of modes with the small-scale fluctuations strongly overdamped. This flow can be called “quasilaminar” or coherent. 3. At a transition point
\(R_{\lambda, tr} \approx 9 - 10\) the fluctuating velocity derivatives obey gaussian statistics and at \(R_{\lambda} > 9 - 10\) a strongly anomalous scaling of the moments, typical of high-Reynolds number turbulence, is clearly seen. 4. It has also been noticed that transition is smooth, i.e. velocity field at \(u(R_{\lambda, tr}) - u(R_{\lambda, tr}) \to 0\).

Below, we consider turbulence driven by a force \(F(\Lambda_f)\) supported at the range of scales \(\Lambda_f\). Like in Landau theory, let us denote the velocity field at the very onset of turbulence \(u_0\) and \(Re_{tr} = \frac{\nu u_{rms}}{\nu_{tr} \Lambda_f}\). Keeping forcing fixed, by decreasing viscosity one achieves a large Reynolds number flow. Neglecting for a time being (see below) eddy noise (backscattering) we assume that the integral scale \(\Lambda_f\) is Reynolds number independent and the principle feature of strong turbulence is formation of small-scale fluctuations in the range \(k > \Lambda_f\). In this case the velocity field \(u_0 \approx \text{const}\). In this picture, the large-scale patterns, created in the vicinity of a transition point \(Re_{tr}\), stay unchanged but somewhat blurred by random gaussian corrections which will be considered in detail below. The velocity field in the large-Reynolds number turbulent flow is then \(U = u_0 + u\) so that \(Re = Re_{tr} \frac{\nu_{rms}}{\nu} + \frac{u_{rms}}{\Lambda_f}\nu\) where \(\nu \ll \nu_{tr}\).

An interesting feature of turbulent flows deserves discussion. According to experimental data, when \(r \ll 1/\Lambda_f\), the probability density of velocity increments \(\Delta u_i = u_i(x+r) - u_i(x)\) leads to the moments \(S_n(r) = (\Delta u_i)^n\) characterized by anomalous scaling \(S_n \propto r^{\kappa_n}\). However, in the limit \(r \to \Lambda_f\) the moments obey gaussian statistics disregarding geometric features of the large-scale flow pattern. This quite general observation has not yet been discussed in the literature.

The model. Stirring a very viscous fluid with the force \(F\) supported on a scale \(r \approx 1/\Lambda_f\) leads to formation of a laminar flow field, with velocity \(u(k)\) often proportional to \(F(k)\). Gradually reducing viscosity to the magnitude \(\nu = \nu_{tr}\) one reaches a transition point of marginal stability of a laminar pattern. Upon further decrease of viscosity the flow becomes totally unstable which is accompanied by generation of small-scale modes \(u(k)\) with \(k > \Lambda_f\) where \(F(k) = 0\). In the limit \(\nu \to 0\), the so called inertial range is formed with velocity fluctuations \(u(k)\) filling the interval \(\Lambda_f \leq k \leq \Lambda_0 \to \infty\). According to this picture, turbulent flow is an interplay of the marginally stable , somewhat “noisy”, velocity field \(u_0\) formed at transition with small-scale fluctuations \(u(k)\) with \(k > \Lambda_f\). The integral scale \(\Lambda_f\) corresponding to the top of the inertial range is an essential part of dynamics which is to be dealt with.

Based on these qualitative features, we consider a flow generated by the Navier-Stokes equations with a force \(F(\Lambda_f)\) in the right side:

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu_0 \nabla^2 u + f + F(\Lambda_f);
\]

with \(\nabla \cdot u = \nabla \cdot f = \nabla \cdot u = 0\). If \(f = 0\) and \(\nu = \nu_{tr}\), the model (1) represents a quasilaminar flow \(u_0\) at a transition point. The random force, mimicking small-scale fluctuations is defined by the correlation function:

\[
f_i(k, \omega) f_j(k', \omega') = 2D_0 (2\pi)^{d+1} k^{-y} P_{ij}(k) \delta(k + k') \delta(\omega + \omega'); \quad f_i(k \leq \Lambda_f, t) = 0. \quad (2)
\]
on the interval $\Lambda_f < k \leq \Lambda_0$, so that $\int_i F_j = 0$. The i.r. cut-off $\Lambda_f$, corresponding to the yet unknown top of the inertial range, must be expressed in terms of observables like energy $K = u_{rms}^2/2$ and dissipation rate $\mathcal{E}$, respectively. It is clear that, in a statistically steady state production must be equal to dissipation so that $\mathcal{P} = \mathcal{E} = \nu_0(\partial u_i/\partial x_j)$ and:

$$\mathcal{E} = \mathcal{P} = \mathbf{F} \cdot \mathbf{u} + 2D_0 \frac{S_d}{(2\pi)^d} \Lambda_f^{-y+d} - \Lambda_0^{-y+d} y - d$$

The turbulent field $\mathbf{u}$ is independent on the forcing so that $\mathbf{F} \cdot \mathbf{u} = \mathbf{F} \cdot \mathbf{u}_0$. In the limit $y > d$, the information about the u.v. cut-off $\Lambda_0$ disappears and we can conclude that in the range $y > d$, the model (1)-(3) may lead to universal statistics of velocity fluctuations in the inertial range of scales. As $y \to d$, the energy flux and mean dissipation rate $\mathcal{P} \to \mathcal{F} \cdot \mathbf{u} + 2D_0 \frac{S_d}{(2\pi)^d} \ln \Lambda_0 \Lambda_f = \text{const}$ shows logarithmic dependence on the u.v. cut-off. In the limit $\Lambda_0 \to \Lambda_f$, the turbulent component disappears and the balance between production and dissipation is given by $\mathcal{E} = \mathbf{F} \cdot \mathbf{u}$.

When $Re > Re_{tr}$ one can introduce “turbulent” Reynolds number: $Re_T = u_{rms}/(\nu \Lambda_f)$, and, if the energy spectrum $E(k) \propto k^{-\alpha}$, turbulent kinetic energy is defined as:

$$u_{rms}^2 = 2 \int_{\Lambda_f}^{\Lambda_0} E(k) dk \approx \frac{1}{\alpha - 1} (\Lambda_f^{-\alpha+1} - \Lambda_0^{-\alpha+1})$$

Here, $\Lambda_0$ and $\Lambda_f$ are the u.v. and i.r. cut-offs. In principle, the cut-off $\Lambda_0$ can be chosen as Kolmogorov’s dissipation scale. In this case, $\Lambda_0 \approx (\frac{\nu}{3})^{\frac{3}{4}}$. As $\nu \to 0$, $\Lambda_0 \to \infty$ and the ratio $\Lambda_0/\Lambda_f \to \infty$ simultaneously with $Re_T$. Therefore, dimensionless parameter $Re_{2,0} = \Lambda_0/\Lambda_f$ can also be regarded as a Reynolds number. In the opposite limit $Re_{2,0} \to 1$, the parameter $Re_T \to 0$. The total rms velocity is $(u_{rms}^2) = (u_{0,rms})^2 + u_{rms}^2$, so that as $\Lambda \to \Lambda_f$, $Re \to Re_{tr}$. This property will be important below.

The renormalization group. The renormalization group for fluid flows has been developed in Refs.[3]-[4] and was generalized to enable computations of various dimensionless amplitudes in the low order in the $\epsilon$-expansion in Refs.[5]-[7].

Introducing velocity and length scales $U = \sqrt{D_0/(\nu \Lambda_0^2)}$ and $X = 1/\Lambda_0$, respectively, the equation (2) can be written as (for simplicity we do not change notations for dimensionless variables):

$$\frac{\partial \mathbf{u}}{\partial T} + \lambda_0 \mathbf{u} \cdot \nabla \mathbf{u} = -\lambda_0 \nabla p + \nabla^2 \mathbf{u} + \frac{f + F}{\sqrt{D_0 \nu \Lambda^2}}$$

where the dimensionless coupling constant (“bare” Reynolds number) is: $\lambda_0^2 = \frac{D_0}{\nu \Lambda_0^4}$. We start with the Reynolds number $Re_T \to \infty$ and iteratively eliminating the small-scale fast modes project original equation onto a smaller wave-number space of the large-scale modes.

Unlike renormalization group theories developed for infinite fluids in this work, trying to accommodate information about transition and other finite-size effects, we keep the integral
scale $L = 1/\Lambda_f = O(1)$. This is the major new element of the model. Technical details of all calculations presented below are best described in Ref. 6. Formally introducing modes $u^<(k, t)$ and $u^>(k, t)$ with $k$ from the intervals $k \leq \Lambda_0 e^{-r}$ and $\Lambda_0^{-r} \leq k \leq \Lambda_0$, respectively, and averaging over small-scale fluctuations, leads to equation for the large-scale modes [6]:

$$
\frac{\partial u^<}{\partial t} + u \cdot \nabla u^< = -\nabla p + (\nu_0 + \Delta \nu) \nabla^2 u^< + F + f + \Delta f + \text{HOT}
$$

(5)

where for simplicity correction to “viscosity” is written in the wave-number space:

$$
\Delta \nu = A_d \frac{D_0}{\nu_0^2} \left[ e^{\epsilon r} - 1 \right] \epsilon \Lambda_0^2 + O(\frac{k^2}{\Lambda_0^{d+2}} e^{(r+2)\epsilon} - 1) + O(\hat{\lambda}_0^4)
$$

(6)

$\epsilon = 4 + y - d$ and $A_d = \hat{A}_d (2\pi)^d; \hat{A}_d = \frac{1}{2} \frac{d^2 - d}{d(d+2)}$. Due to Galileo invariance, high-order ($n > 1$) terms (HOT) generated by scale-elimination are of the order:

$$
\text{HOT} = \left[ \sum_{n=2}^{\infty} \lambda_1 2^n \tau_0^{n-1} (\partial_t u^< + u^< \cdot \nabla)^n u^< + O(\hat{\lambda}_0^4 \nabla^2 S_0^2 \frac{1}{\Lambda_0^2} e^{(r+2)\epsilon} - 1) + \cdots \right]
$$

(7)

with $\tau_0 \approx 1/(\nu_0 \Lambda_0^2)$ and $\hat{\lambda}_1 = \hat{\lambda}_0 (e^{\epsilon r} - 1)$. In addition, the expression (6) includes various products of time- and space-derivatives responsible, for example, for the rapid distortion effects (RDE). The high-order nonlinearities generated by the procedure are small if the eliminated shell is very thin but, as will be shown below, they exponentially grow with increase of $r$.

As long as $k \ll \Lambda_0$, the $O(k^2)$ contributions to correction to bare viscosity can be neglected together with the high-order nonlinearities. However, below, by iterating the procedure, we eliminate all modes with $k \geq \Lambda_f$ and therefore the $O(k^2)$ terms in (6) must be treated with care, especially in the interval $k \approx \Lambda_f$. In the mean time, simply neglecting them, gives:

$$
\nu_1(r) = \nu_0 (1 + A_d \hat{\lambda}_0^2 e^{\epsilon r} \frac{1}{\epsilon})
$$

(8)

The induced noise $\Delta f$ will be analyzed below.

Next, starting with the equations (6) defined on the interval $k < \Lambda_0 e^{-r}$, we can eliminate the modes from the next shell of wave-numbers $\Lambda_0 e^{-2r} \leq k \leq \Lambda_0 e^{-r}$ and derive equations of motion with another set of corrected transport coefficients. The procedure can be iterated resulting in the cut-off-dependent viscosity, induced force etc. Setting $r \to 0$ leads to differential recursion equations: denoting $\Lambda(r) = \Lambda_0 e^{-r}$, one obtains:

$$
\frac{d\nu(r)}{dr} = A_d \nu(r) \hat{\lambda}^2(r); \quad \hat{\lambda}^2(r) = \frac{D_0}{\nu^2(r) \Lambda^4(r)}
$$

and:
\[
\frac{d\hat{\lambda}^2(r)}{dr} = \hat{\lambda}^2(r)[\epsilon - 3A_d\hat{\lambda}^2(r)]
\]

Thus:

\[
\nu(r) = \nu_o[1 + \frac{3}{\epsilon}A_d\hat{\lambda}_0^2(e^{\epsilon r} - 1)]^{\frac{1}{2}} = \nu_0[1 + \frac{3A_dD_0S_d}{\epsilon
\nu^3_0(2\pi)^d}(\frac{1}{\Lambda^\epsilon(r)} - \frac{1}{\Lambda^0_0})]^{\frac{1}{2}}
\]

\[
\hat{\lambda}(r) \equiv \left[\frac{D_0S_d}{(2\pi)^d\nu^3(\Lambda^\epsilon(r))^d}\right]^{\frac{1}{2}} = \hat{\lambda}_0e^{\epsilon r/2}[1 + \frac{3}{\epsilon}A_d\hat{\lambda}_0^2(e^{\epsilon r} - 1)]^{-\frac{1}{2}}
= \hat{\lambda}_0e^{\epsilon r}(1 + \frac{3A_dD_0S_d}{\epsilon\nu^3_0(2\pi)^d}(\frac{1}{\Lambda^\epsilon(r)} - \frac{1}{\Lambda^0_0}))^{-\frac{1}{2}}
\]

and the solution for the “induced” coupling constant \(\hat{\lambda}_1\) we have:

\[
\hat{\lambda}_1(r) = \frac{\sqrt{\epsilon}\nu^2}{\sqrt{\lambda^2(0) + 3A_d(e^{\epsilon r} - 1)}}
\]

The fixed-point dimensionless coupling constant:

\[
\hat{\lambda}^* = (\frac{\epsilon}{3A_d})^{\frac{1}{2}} \approx 1.29\sqrt{\epsilon} \approx 2.58.
\]

for \(\epsilon = 4\). We also see that, even when \(\hat{\lambda}_1(0) = 0\), at the fixed point \(\hat{\lambda}^* \approx \hat{\lambda}_1^*\).

**Parameters.** All calculations presented below, made in the lowest order of expansion in powers of \(\hat{\lambda}^* (\epsilon\text{-expansion})\), are described in great detail in Ref.[6]. Eliminating all modes from the interval \(k \geq \Lambda_0e^{-r}\) and setting \(\epsilon = 4\) gives:

\[
\nu(k) = (\frac{3}{8}A_d2D_0)^{\frac{1}{4}}k^{-\frac{1}{4}} \approx 0.42(\frac{2D_0S_d}{(2\pi)^d})^{\frac{1}{2}}k^{-\frac{1}{4}}
\]

and from the linearized equation at the fixed point:

\[
u(k) \approx \mathcal{G}(k,\omega)f = \frac{f + \mathcal{F}(\Lambda_f)}{-i\omega + \nu(k)k^2}
\]

we derive Kolmogorov’s spectrum valid in the range \(k > \Lambda_f\):

\[
E(k) = \frac{1}{2(2\pi)^{d+1}}\int_{-\infty}^{\infty} TrV_{ij}(k\omega)d\omega = \frac{1}{2(\frac{3}{8}A_d)^{\frac{1}{4}}}(2D_0S_d)^{\frac{1}{2}}k^{-\frac{1}{4}} = 1.186(2D_0S_d)^{\frac{1}{2}}k^{-\frac{1}{4}}
\]

where \((2\pi)^{d+1}V_{ij}(k,\omega) = \frac{u^\omega_<(k,\omega)u^\omega_<(k',\omega')}{\delta(k+k')\delta(\omega+\omega')}\). In the so called EDQNM approximation, which is exact at the Gaussian fixed point, the force amplitude \(D_0\) can be related to the mean dissipation rate [6], [13], [14]:

\[7\]
\[ 2D_0 S_d / (2\pi)^d \approx 1.59 \mathcal{E}; \quad E(k) = C_K \mathcal{E}^{\frac{2}{3}} k^{-\frac{5}{3}}; \quad C_K = 1.61 \] (14)

Let us identify the infra-red cut-off \( \Lambda_f = \Lambda(r) \approx 1/L \) with the wave-number corresponding to the top of the inertial range. In the large \( \text{Re-limit} \) \( \Lambda_0 / \Lambda_f \gg 1 \), the total energy of the inertial range turbulent fluctuations is evaluated readily:

\[ \mathcal{K} = \int_{\Lambda_f}^{\infty} E(k) dk = \frac{3}{2} C_K \left( \frac{\mathcal{E}}{\Lambda_f} \right)^{\frac{2}{3}} = \frac{3}{2} \cdot 1.61 \left( \frac{3}{8} \hat{A}_d 1.59 \right)^{\frac{1}{3}} \frac{\mathcal{E}}{\nu(\Lambda_f) \Lambda_f^2} \approx 1.19 \frac{\mathcal{E}}{\nu(\Lambda_f) \Lambda_f^2} \] (15)

and, setting \( k = \Lambda_f \) gives the expression for effective viscosity in equation for the large-scale dynamics in the interval of scales \( k < \Lambda_f \):

\[ \nu_T \equiv \nu(\Lambda_f) \approx 0.084 \frac{\mathcal{K}^2}{\mathcal{E}}; \quad 10.0 \times \nu(\Lambda_f)^2 \Lambda_f^2 = \mathcal{K} \] (16)

**Fixed-point Reynolds number and irrelevant variables.** The expression (16) gives effective viscosity accounting for all turbulent fluctuations from the interval \( 1/\Lambda_0 \leq r < 1/\Lambda_f \) acting on the almost-coherent-large scale flow on the scales \( r \approx L = 1/\Lambda_f \). Using (15) -(16):

\[ R_{\lambda,fp} = 2 \mathcal{K} \sqrt{5/(3\mathcal{E} \nu(\Lambda_f))} = \sqrt{20/(3 \times 0.084)} = 9.0 \] The same parameter can be expressed in terms of the fixed-point coupling constant:

\[ \hat{\lambda}^* = \sqrt{\frac{D_0 S_d / (2\pi)^d}{\nu_T^3 \Lambda_f^4}} = \sqrt{\frac{0.8 \mathcal{E}}{\nu_T^2 \Lambda_f^4}} = \sqrt{\frac{0.8 \times 400 \mathcal{E} \nu_T}{u_{rms}^2}} = \sqrt{\frac{0.8 \times 400 \times \frac{5}{3}}{R_{\lambda,fp}^{fp}}} = \sqrt{\frac{4}{3 \hat{A}_d}} = 2.58 \]

and \( R_{\lambda,fp} \approx 9.0 \) very close to Reynolds number of transition \( R_\lambda \approx 9 - 10 \), obtained from direct numerical simulations of Ref.[11]. This outcome correlates with an observation that in the flows past various bluff bodies, the Reynolds number based on the measured “turbulent viscosity”, integral length-scale and large-scale rms velocity is \( R_{\lambda,T} = O(10) \), independent on the “bare” (classic) Reynolds number calculated with molecular viscosity [15].

In derivation of parameters (9)-(11) all \( O(k^4) \) contributions to the propagator and nonlinearities (HOT) remained undetermined and were neglected. As follows from relation (10), each term of the expansion is not small: indeed, the initially negligible small coupling constant \( \hat{\lambda}_1 \) exponentially grows to the \( O(1) \) value at the fixed point. Still, the procedure led to various dimensionless amplitudes in a surprisingly good agreement with experimental data. In addition, it enabled derivation of turbulence models widely used in engineering [3]-[9]. The reasons for this “surprise” can be understood as follows. 1. *The above calculations gave for viscosity at the integral scale \( \nu_T = \nu(\Lambda_f) = \nu_r \).* 2. As was shown numerically in Ref.[11], transition to turbulence is “smooth” meaning that neither velocity nor it’s spatial derivative are discontinuous at transition point. Therefore, at the integral scale:
\[
\frac{Du_0}{Dt} = -\nabla p + \nu_{tr} \nabla^2 u_0 + F(\Lambda_f);
\]
\[
\frac{Du^{fp}}{Dt} = -\nabla p + \nu^{fp} \nabla^2 u^{fp} + F(\Lambda_f) + \psi + \text{HOT} \tag{17}
\]

where \( \frac{D}{Dt} \equiv \partial_t + \mathbf{u} \cdot \nabla \) and \( \psi \) is induced gaussian “eddy noise” (backscattering). The field \( u_0 \) describes a coherent, slowly- varying or stationary transitional flow and the time -ensemble - averaged noise \( \bar{\psi} = 0 \) (see below). Since \( \nu_{tr} = \nu^{fp} \) and smoothness of transition means \( u^{fp} = u_0 \), it follows from (17): at the integral scale (fixed point) all high-order non-
linearities, generated by the coarse-graining disappear, i.e. \( \text{HOT} = 0 \).

A connection of the present work to Landau’s theory of transition to turbulence [10] is possible. Let us consider the linearized equation of motion in the vicinity of the fixed point where \( \mathbf{u} = u_0 + \mathbf{u}_1 \):
\[
\frac{\partial \mathbf{u}_1}{\partial t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_1 + \mathbf{u} \cdot \nabla \mathbf{u}_0 = -\nabla p_1 + \nu \nabla^2 \mathbf{u}_1 + \text{HOT} \tag{18}
\]

If the first unstable mode \( \mathbf{u}_1 \propto Ae^{i\omega} \), then according to Landau’s theory: \( u_1 \propto A_{max} \propto \sqrt{Re - Re_{tr}} \)
and all nonlinearities are calculated from the balance:

\[
\text{HOT} \approx u_0 \cdot \nabla \mathbf{u}_1 \approx u_0 \Lambda_f \sqrt{Re - Re_{tr}}
\]

Large-scale dynamics. Now we would like to discuss the large-scale flow in the interval \( k \approx \Lambda_f \), where the bare force \( \mathbf{f}(k) = 0 \) and therefore the equation of motion is:
\[
\frac{\partial \mathbf{u}^<}{\partial t} + \mathbf{u}^< \cdot \nabla \mathbf{u}^< = -\nabla p + \nu(\Lambda_f) \nabla^2 \mathbf{u}^< + F + \psi \tag{19}
\]

with induced noise evaluated in Ref.[6]:
\[
\psi_i(k,\omega)\psi_j(k',\omega') = (2\pi)^{d+1}2D_Lk^2P_{ij}(k)\delta(k+k')\delta(\omega+\omega') \tag{20}
\]

where
\[
D_L = D_0 \frac{d^2 - 2}{20d(d+2)} \frac{\hat{\lambda}^* \hat{\lambda}^*}{\Lambda_f^5} = D_0 \frac{0.155}{\Lambda_f^5} \tag{21}
\]

The induced force \( \psi \) is the result of small-scale turbulent fluctuations on the large-scale dynamics which is often called “backscattering”. In the most important limit \( k \to \Lambda_f \)
\( D_0/D_L \approx 1/0.155 \approx 7.0 \). Therefore, the induced force, while being numerically not too large,
is responsible for both blurring of the large-scale transitional patterns and for the observed
gaussian statistics of the large-scale velocity fluctuations in the high-Reynolds number flows.
Summary and conclusions. 1. The coarse-graining procedure based on Wyld’s diagrammatic expansion leads to the Navier-Stokes-like equations with an infinite number of additional higher-order nonlinearities. In the lowest-order of expansion, the calculated fixed-point Reynolds number \( \text{Re}_{fp} \approx \text{Re}_{tr} \) where \( \text{Re}_{tr} \) is a numerically computed Reynolds number of transition to turbulence. Since the numerically discovered transition is “smooth”, we assumed that at the transition point both the velocity fields \( u_0 = u_{fp} \) and their spatial derivative are equal \( (\nabla_i u_{0,j} = \nabla_i u_{fp,j}) \) and, as a result, at the fixed point all additional to the NS equations high-order nonlinearities are irrelevant. 2. Comparison with Landau’s theory of transition to turbulence shows that the nonlinear terms are \( O(\sqrt{\text{Re} - \text{Re}_{tr}}) \rightarrow 0. \) 3. The infra-red divergencies appearing in the each term of the expansion do not disappear but are summed up into equations of motion for the large-scale features of the flow.

Previous theories of turbulence used all sorts of field-theoretical approaches to entirely remove infra-red divergencies. In this work we show that any large-Reynolds-number flow includes the symmetry-breaking large-scale field responsible for the small-scale turbulence production. The i.r. divergencies of turbulence theory do not disappear but are contained in the equations of motion for the large-scale quasi-laminar (coherent) component reflecting geometry, physical mechanisms of production etc. The symmetries of turbulent flow like a universal behavior of the moments of derivatives and structure functions are recovered at somewhat smaller scales. This result probably has a wide range of applicability. For example, Polyakov, in his conformal field theory of small-scale dynamics of two-dimensional turbulence attributed the infra-red problems appearing in the vicinity of the forcing scale to ( “non-conformist”) symmetry-breaking ”condensates”, i.e. large-scale flows generating “direct” enstrophy cascade [16].

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