A MODULAR FRAMEWORK OF FUNCTIONS OF KNOPP AND INDEFINITE BINARY QUADRATIC FORMS

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ABSTRACT. In this paper, we investigate functions introduced by Knopp and complete them to non-holomorphic bimodular forms of positive integral weight related to indefinite binary quadratic forms. We study further properties of our completions, which in turn motivates certain local cusp forms. We then define modular analogues of negative weight of our local cusp forms, which are locally harmonic Maass forms with continuously removable singularities. They admit local splittings in terms of Eichler integrals, and a realization as outputs of a certain theta lift.

1. Introduction and statement of results

Throughout the paper \( D > 0 \) is a non-square discriminant, \( k \in 2\mathbb{N} \), \( Q_D \) denotes the set of integral binary quadratic forms \( Q = [a,b,c] \) of discriminant \( d \in \mathbb{Z} \), and \( \mathbb{H} \) is the complex upper half-plane. In 1975, Zagier [23] introduced the functions

\[
f_{\kappa,D}(\tau) := \sum_{Q \in Q_D} \frac{1}{Q(\tau,1)^{\kappa}}, \quad \tau \in \mathbb{H},
\]

and proved that they are cusp forms if \( \kappa > 1 \) (if \( \kappa = 1 \), one may use Hecke’s trick, see [14, p. 239]). To name a few prominent applications of the \( f_{\kappa,D} \)'s, they are coefficients of the holomorphic kernel function of the Shimura [19] and Shintani [20] lifts due to [14], and they are closely related to central \( L \)-values by [15]. Their even periods are rational according to [16].

Over 30 years ago, Knopp [13, (4.5)] found a term-by-term preimage of each \( f_{\kappa,D} \) under the Bol operator \( \mathbb{D}^{2k-1} \) [8], where \( \mathbb{D} := \frac{1}{2\pi i} \frac{d}{d\tau} \) (compare Proposition 3.1 (2)). To ensure convergence after summing over \( Q \in Q_D \), he changed the sign of \( k \) in his result afterwards, which lead to (here and throughout \( \text{Log} \) denotes the principal branch of the complex logarithm)

\[
\psi_{k+1,D}(\tau) := \sum_{Q \in Q_D} \frac{\text{Log}(\tau - a_+)}{Q(\tau,1)^{k+1}}, \quad a_+ := \frac{-b \pm \sqrt{D}}{2a} \in \mathbb{R}. \tag{1.1}
\]

He furthermore stated that \( \psi_{k+1,D}(\tau+1) = \psi_{k+1,D}(\tau) \), and the behaviour of \( \psi_{k+1,D} \) under modular inversion (see [14, (4.6)]). Correcting a typo there, we find that (see Proposition 3.1 (3))

\[
\tau^{-2k-2} \psi_{k+1,D}\left(-\frac{1}{\tau}\right) - \psi_{k+1,D}(\tau) = \sum_{Q \in Q_D} \frac{\log(\alpha_+)}{Q(\tau,1)^{k+1}} - 2\pi i \sum_{Q=[a,b,c] \in Q_D \atop a<0<c} \frac{1}{Q(\tau,1)^{k+1}}. \tag{1.2}
\]

On the one hand, we observe that \( \psi_{k+1,D} \) is holomorphic and vanishes at \( i\infty \) (this follows by Proposition 3.1 (1) and (3.5)). On the other hand, \( \psi_{k+1,D} \) itself is not modular. Hence, it is natural

\[
2020 \text{ Mathematics Subject Classification.} \ 11F11 \ (\text{primary}); \ 11E16, 11E45, 11F12, 11F27, 11F37 \ (\text{secondary}).
\]

\textbf{Key words and phrases.} Harmonic Maass forms, integral binary quadratic forms, locally harmonic Maass forms, modular completion, theta lifts.

The first author has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No. 101001179).

1We define \( f_{\kappa,D} \) in Zagier’s original normalization, which differs from the normalization used in [4] for instance.

2We alert the reader to the fact that Knopp used the older convention \( T = (\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \).
to “complete” $\psi_{k+1,D}$. Setting $\mathbb{H}^{-} := -\mathbb{H}$ throughout, completions of $\psi_{k+1,D}$ are bimodular forms\footnote{We slightly modify the initial definition by Stienstra and Zagier \cite{21} here to include the domain $\mathbb{H} \times \mathbb{H}^{-}$.} $\Omega_{k+1,D}$ of weight $(2k + 2, 0)$ defined on $\mathbb{H} \times \mathbb{H}^{-}$ such that
\[
\lim_{w \to \infty} \Omega_{k+1,D}(\tau, w) = \psi_{k+1,D}(\tau). \tag{1.3}
\]

Here we construct such completions explicitly. Firstly, we note that the final sum appearing in (1.2) is finite, because $b^2 + 4|ac| = D > 0$ has only finitely many integral solutions. This leads to Knopp’s modular integrals with rational period functions \cite{12}. Roughly speaking, period polynomials describe the obstruction of modularity of Eichler integrals \cite{10} (defined in (1.8)) of such functions are called modular integrals. Parson \cite{15} defined such modular integrals by
\[
\varphi_{k+1,D}(\tau) := \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}} = \sum_{Q = [a,b,c] \in \mathcal{Q}_D} \frac{1}{Q(\tau, 1)^{k+1}}, \quad \text{sgn }([a,b,c]) := \text{sgn}(a) \tag{1.4}
\]
and we recall her result on the $\varphi_{k+1,D}$’s in Lemma 3.3. Secondly, we define
\[
Q_w := \frac{1}{\text{Im}(w)} \left( a |w|^2 + b \text{Re}(w) + c \right), \quad S_Q := \{ \tau \in \mathbb{H} : Q_\tau = 0 \}, \quad E_D := \bigcup_{Q \in \mathcal{Q}_D} S_Q,
\]
for $w \in \mathbb{C} \setminus \mathbb{R}$, $Q \in \mathcal{Q}_D$, as well as the functions
\[
\rho_{k+1,D}(\tau, w) := \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \frac{w-a_Q}{w-Q^*}}{Q(\tau, 1)^{k+1}}, \quad \lambda_{k+1,D}(\tau, w) := 2i \sum_{Q \in \mathcal{Q}_D} \frac{\text{arctan} \frac{Q_\tau}{Q}}{Q(\tau, 1)^{k+1}}. \tag{1.5}
\]
for $w \in \mathbb{H}^{-}$. We refer to Propositions 3.2 and 3.4 for some of their properties. Thirdly, we define
\[
\Omega_{k+1,D}(\tau, w) := \psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, w) + 2\pi i \varphi_{k+1,D}(\tau) + \lambda_{k+1,D}(\tau, w), \tag{1.6}
\]
on $\mathbb{H} \times \mathbb{H}^{-}$ and have the following results.

**Theorem 1.1.** Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^{-}$.

1. The functions $\Omega_{k+1,D}$ are bimodular of weight $(2k + 2, 0)$ that is
\[
\Omega_{k+1,D}(\tau + 1, w + 1) = \Omega_{k+1,D}(\tau, w), \quad \Omega_{k+1,D} \left( \frac{1}{\tau}, -\frac{1}{w} \right) = \tau^{2k+2} \Omega_{k+1,D}(\tau, w).
\]

2. Condition (1.3) holds.

3. We have
\[
\lim_{\tau \to i\infty} \Omega_{k+1,D}(\tau, w) = 0.
\]

4. The functions $\Omega_{k+1,D}$ are holomorphic with respect to $\tau$ and anti-holomorphic with respect to $w$.

5. We have that
\[
\Omega_{k+1,D}(\tau, \tau) = 0.
\]

**Remark.** During the proof of Theorem 1.1 (5), we show that
\[
\text{Log} \left( \frac{\tau-a_Q}{\tau-a_Q^*} \right) - \text{Log} \left( \frac{\tau-a_Q}{\tau-a_Q} \right) + \pi i \text{sgn}(Q) + 2i \text{arctan} \left( \frac{Q_\tau}{Q} \right) = 0.
\]
This implies that the $\Omega_{k+1,D}$’s have representations on $\mathbb{H} \times \mathbb{H}$ as well, and these representations coincide with the functions
\[
\omega_{k+1,D}(\tau, z) := \psi_{k+1,D}(\tau) - \sum_{Q \in \mathcal{Q}_D} \frac{\text{Log} \frac{z-a_Q}{z-a_Q^*}}{Q(\tau, 1)^{k+1}}, \quad (\tau, z) \in \mathbb{H} \times \mathbb{H}.
\]
The $ω_{k+1,D}$’s satisfy
\[
ω_{k+1,D}(τ + 1, z + 1) = ω_{k+1,D}(τ, z), \quad ω_{k+1,D}\left(-\frac{1}{τ}, -\frac{1}{z}\right) = τ^{2k+2}ω_{k+1,D}(τ, z),
\]

\[\lim_{z \to \infty} ω_{k+1,D}(τ, z) = ψ_{k+1,D}(τ).\]

In the course of proving Theorem 1.1 (5), we encounter the functions (see (3.7))
\[
Λ_{k+1,D}(τ) := \sum_{Q \in \mathcal{Q}_D} \text{sgn}(Q_τ) \frac{Q(τ, 1)^{k+1}}{Q(τ, 1)^{k+1}}, \tag{1.7}
\]
to which we refer as local cusp forms. That is, the functions $Λ_{k+1,D}$ behave like cusp forms of weight $2k + 2$ outside $E_D$, however, in addition, have jumping singularities on $E_D$ in the sense of [4]. We refer the reader to Proposition 4.1 for details. By a result of the second author [17, Theorem 1.1], the functions $Λ_{k+1,D}$ can be written in terms of traces of cycle integrals.

Next, we construct negative weight analogues $Ψ_{-k,D}$ of the $Λ_{k+1,D}$’s along the lines of [4]. This is natural, because the $Λ_{k+1,D}$’s are “odd” positive weight analogues of the $f_{k,D}$’s, and the $f_{k,D}$’s recently motivated the introduction of new automorphic objects by the first author, Kane, and Kohnen [3]. To be more precise, we let $β(x; s, w) := f_0^{x}t^{s-1}(1-t)^{-w-1}dt$, $x \in (0, 1)$, $\text{Re}(s), \text{Re}(w) > 0$, be the incomplete $β$-function, $τ = u + iv$ throughout, and we define
\[
Ψ_{-k,D}(τ) := \frac{1}{2} \sum_{Q \in \mathcal{Q}_D} Q(τ, 1)^{k+1} β \left( \frac{Dv^2}{|Q(τ, 1)|^2}; k + \frac{1}{2}, \frac{1}{2} \right), \quad τ \in \mathbb{H} \setminus E_D.
\]

In the spirit of Knopp’s initial preimage of $f_{k,D}$ under the Bol operator (without an additional sign change of $k$), it turns out that $Ψ_{-k,D}$ is a preimage of $Λ_{k+1,D}$ under the Bol operator as well as the shadow operator $ξ_κ := 2ivκ\frac{∂}{∂w}$ due to Bruinier and Funke [7] (up to constants). Such a behaviour is impossible in the situation of a (globally defined) non-trivial harmonic Maass form $\tilde{f}$. If $f$ is a cusp form of weight $2k + 2$, then preimages under $D^{2k+1}$ and $ξ_{-2k}$, respectively, are provided by the holomorphic and non-holomorphic Eichler integrals (see (5.4))
\[
E_f(τ) := -\frac{(2π)^{2k+1}}{(2k)!} \int_{-∞}^{∞} f(w)(τ - w)^{2k}dw, \quad f^*(τ) := (2i)^{-2k-1} \int_{-∞}^{∞} f(-w)(w + τ)^{2k}dw. \tag{1.8}
\]

To be able to insert the local cusp forms $Λ_{k+1,D}$ into each integral in (1.8), we work in a fundamental domain of $\text{SL}_2(\mathbb{Z})$, in which we have just finitely many equivalence classes of geodesics $S_Q$. Integrating piecewise, both Eichler integrals of $Λ_{k+1,D}$ are well-defined on $\mathbb{H} \setminus E_D$. In addition we ensure in Proposition 4.4 that both Eichler integrals of $Λ_{k+1,D}$ exist on $E_D$. This established, we prove the following properties of $Ψ_{-k,D}$. We refer the reader to Subsection 2.2.2 for definitions.

**Theorem 1.2.**

1. The functions $Ψ_{-k,D}$ are locally harmonic Maass forms of weight $-2k$ with continuously (however not differentially) removable singularities on $E_D$.
2. If $τ \in \mathbb{H} \setminus E_D$, then we have, with $c_∞$ defined in equation (5.3),
\[
Ψ_{-k,D}(τ) = c_∞ - \frac{D^{k+\frac{1}{2}}(2k)!}{(4π)^{2k+1}} E_{Λ_{k+1,D}}(τ) + D^{k+\frac{1}{2}}Λ_{k+1,D}(τ).
\]

The functions $Ψ_{-k,D}$ are outputs of theta lifts. To motivate this, we parallel a construction of the first author, Kane, and Viazovska [5]. We employ Borcherds [2] regularization of the Petersson inner product $⟨·, ·⟩_{\text{reg}}$ for $Γ_{0}(4)$ (see Section 2), and define the theta kernel $\theta_{z,k}^{+}(z = x + iy, w \in \mathbb{H}^-)$
\[
θ_{z,k}^{+}(τ, z) := y^{k+1} \sum_{d \in \mathbb{Z}} \sum_{Q \in \mathcal{Q}_D} |Q_z| Q(τ, 1)^{k} e^{-\frac{4σ(|Q_z|^{2})y^{2}}{z^2}} e^{-\frac{2πi dz}.}
\]

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4We explain this terminology in Section 2.

5One may overcome this by weakening the growth condition in Definition 2.6 see [3, Theorem 6.15].

6We use the variable $τ$ for integral weight, and $z$ for half-integral weight. This is opposite to the notation in [5].
The function \( \theta_{-k}^* \) transform like modular forms of weight \( \frac{1}{2} - k \) in \( z \), and of weight \(-2k\) in \( \tau \), see Lemma 2.3. Thus, they give rise to the theta lift \(( F, \text{a harmonic Maass form of weight } \frac{1}{2} - k )\)

\[
\mathcal{L}_{-k}^*(F)(\tau) := (F, \theta_{-k}^*(-\tau, \cdot))_{\text{reg}}.
\]

It suffices to compute \( \mathcal{L}_{-k}^*(F) \) on the Maass–Poincaré series \( \mathcal{P}_{\frac{1}{2} - k, m} \) (defined in (2.4)) as they generate the space of harmonic Maass forms.

**Theorem 1.3.** Let \( \tau \in \mathbb{H} \setminus E_D \). We have, with \( \Gamma \) the usual \( \Gamma \)-function

\[
\mathcal{L}_{-k}^* \left( \mathcal{P}_{\frac{1}{2} - k, D} \right)(\tau) = \frac{D^{\frac{1}{2} - \frac{3}{2}k!}}{4\pi^{\frac{3}{4} + \frac{1}{4}}} \mathcal{L}_{-k, D}(\tau).
\]

The paper is organized as follows. We recall results required for this paper in Section 2. Section 3 is devoted to the proof of Knopp’s initial claims on \( \psi_{k+1, D} \), to some properties of the functions \( \rho_{k+1, D}, \varphi_{k+1, D}, \lambda_{k+1, D} \), and to the proof of Theorem 1.1. In Section 4 we investigate the behaviour of \( \Lambda_{k+1, D}, \tilde{E}_{k+1, D}, \) and \( \Lambda_{k+1, D}^* \) on \( E_D \). Section 5 discusses the properties of \( \Psi_{-k, D} \), and proves Theorem 1.2. In Section 6 we give the proof of Theorem 1.3.

**Acknowledgements**

The authors would like to thank Caner Nazaroglu for very helpful comments and discussions.

2. Preliminaries

### 2.1. Integral binary quadratic forms and Heegner geodesics.

The modular group \( \text{SL}_2(\mathbb{Z}) \) acts on \( \mathcal{Q}_d \) by \(( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \)

\[
Q \circ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) (x, y) := Q(ax + by, cx + dy).
\]

The action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \) is compatible with the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \mathcal{Q}_d \), in the sense that if

\[
(Q \circ \gamma)(\tau, 1) = j(\gamma, \tau)^2 Q(\gamma \tau, 1), \quad j \left( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \tau \right) := cd \tau + d \quad (2.1)
\]

Since \( D > 0 \) is not a square, the two roots \( \frac{1}{2} \sqrt{D} \) of \( Q \in \mathcal{Q}_D \) are real-quadratic and connected by the Heegner geodesic \( S_Q \). We orientate \( S_Q \) counterclockwise (resp. clockwise) if \( \text{sgn}(Q) > 0 \) (resp. \( \text{sgn}(Q) < 0 \)). The orientation of \( S_Q \) in turn determines the sign one catches if \( \tau \) jumps across \( S_Q \). More precisely, one has \( \text{sgn}(Q) \text{sgn}(Q_+) < 0 \) if and only if \( \tau \) lies in the bounded component of \( \mathbb{H} \setminus S_Q \). The unbounded connected component of \( \mathbb{H} \setminus E_D \) is the unique such component containing \( i\infty \) on its boundary. We refer the reader to the beautiful article by Duke, Imamoglu, and Tóth for more on Heegner geodesics.

We next collect some results, which we utilize throughout. The following lemma is straightforward.

**Lemma 2.1.** For \( Q \in \mathcal{Q}_d \), \( d \in \mathbb{Z} \), we have

\[
dv^2 + Q_\tau^2 v^2 = |Q(\tau, 1)|^2.
\]

To determine the weights of our functions, the following lemma is useful.

**Lemma 2.2.** For every \( Q \in \mathcal{Q}_D \) and \( \gamma \in \text{SL}_2(\mathbb{Z}) \), we have

\[
(Q \circ \gamma)_\tau = Q_{\gamma \tau}, \quad \frac{\text{Im}(\gamma \tau)}{|Q(\gamma \tau, 1)|} = \frac{v}{|Q(\gamma \tau, 1)|}.
\]

We also require the following elementary lemma.

\[\text{A good reference is for example Zagier’s book [25] §8}.\]
Lemma 2.3. Let $U \subseteq \mathbb{C}$ be open. Assume that $f : U \to \mathbb{C}$ is real-differentiable and satisfies $f(\tau) = f(\tau)$. Then
\[
\frac{\partial}{\partial \tau} f(\tau) = \frac{\partial}{\partial \tau} f(\tau).
\]

The following differentiation rules are obtained by a direct calculation.

Lemma 2.4. Let $Q \in \mathbb{Q}_D$.
(1) We have
\[
v^2 \frac{\partial}{\partial \tau} Q(\tau) = i \frac{Q(-\tau, 1)}{2}, \quad v^2 \frac{\partial}{\partial \tau} Q(\tau) = i \frac{Q(\tau, 1)}{2}, \quad v^2 \frac{\partial}{\partial \tau} Q(\tau, 1) = i Q_\tau.
\]
(2) We have
\[
\frac{\partial}{\partial \tau} v^2 \frac{\partial}{\partial \tau} Q(\tau, 1) = Q(\tau, 1)^2, \quad 2 v^2 \frac{\partial}{\partial \tau} Q(\tau, 1) = Q(\tau, 1), \quad i v^2 \frac{\partial}{\partial \tau} Q(\tau, 1) = Q_\tau.
\]

Letting $Q'(\tau, 1) := \frac{\partial}{\partial \tau} Q(\tau, 1)$, the following lemma can be verified by direct calculation.

Lemma 2.5. Let $Q \in \mathbb{Q}_D$ and $\tau \in \mathcal{H}$. We have
\[
Q_\tau v + i v Q'(\tau, 1) = Q(\tau, 1), \quad Q(\tau, 1)^2 - 2Q''(\tau, 1)Q(\tau, 1) = D.
\]

2.2. Automorphic forms.

2.2.1. Maass forms and modular forms. We collect the definitions of various automorphic objects appearing in this paper; see [3] for more details on harmonic Maass forms. Let $\kappa \in \frac{1}{2} \mathbb{Z}$, and $N := 1$ if $\kappa \in \mathbb{Z}$ and $N := 4$ if $\kappa \not\in \mathbb{Z}$. The slash operator is defined as $((a \ b) \ c d) \in \Gamma_0(1))$
\[
f(\tau) \big|_{\kappa} \frac{a \ b}{c \ d} := \begin{cases} (c \tau + d)^{-\kappa} f(\gamma \tau) & \text{if } \kappa \in \mathbb{Z}, \\
(\frac{\tau}{d}) \varepsilon_d^\kappa (c \tau + d)^{-\kappa} f(\gamma \tau) & \text{if } \kappa \in \frac{1}{2} + \mathbb{Z}, \end{cases}
\]
where $\varepsilon_d$ is the extended Legendre symbol, and for $d$ odd $\varepsilon_d := 1$ if $d \equiv 1 \pmod{4}$ and $\varepsilon_d := i$ if $d \equiv 3 \pmod{4}$. The weight $\kappa$ hyperbolic Laplace operator is given as
\[
\Delta_\kappa := -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i \kappa v \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).
\]

We require various classes of modular objects.

Definition 2.6. Let $f : \mathbb{H} \to \mathbb{C}$ be a real-analytic function.

(1) We call $f$ a (holomorphic) modular form of weight $\kappa$ for $\Gamma_0(N)$, if $f$ satisfies the following:
(i) We have $f_{|\kappa} \gamma = f$ for all $\gamma \in \Gamma_0(N)$.
(ii) The function $f$ is holomorphic on $\mathbb{H}$.
(iii) The function $f$ is holomorphic at the cusps of $\Gamma_0(N)$.
(2) We call $f$ a cusp form of weight $\kappa$ for $\Gamma_0(N)$, if $f$ is a modular form that vanishes at all cusps of $\Gamma_0(N)$.
(3) We call $f$ a harmonic Maass form of weight $\kappa$ for $\Gamma_0(N)$, if $f$ satisfies the following:
(i) For every $\gamma \in \Gamma_0(N)$ and every $\tau \in \mathbb{H}$ we have that $f_{|\kappa} \gamma = f$.
(ii) The function $f$ has eigenvalue $0$ under $\Delta_\kappa$.
(iii) There exists a polynomial $P_f \in \mathbb{C}[q^{-1}]$ (the principal part of $f$) such that
\[
f(\tau) - P_f(\tau) = O(e^{-\delta v})
\]
as $v \to \infty$ for some $\delta > 0$, and we require analogous conditions at all other cusps of $\Gamma_0(N)$.

Forms in Kohnen’s plus space have the additional property that their Fourier expansion is supported on indices $n$ satisfying $(-1)^{\kappa - \frac{1}{2}} n \equiv 0, 1 \pmod{4}$ with $\kappa \in \mathbb{Z} + \frac{1}{2}$. 

We remark that $\Delta_\kappa$ splits as
\[
\Delta_\kappa = -\xi_{2-\kappa} \circ \xi_{\kappa},
\]
which in turn implies that a harmonic Maass form naturally splits into a holomorphic and a non-holomorphic part. The operator $\xi_\kappa$ annihilates the holomorphic part, while the Bol operator $\mathbb{D}^{1-\kappa}$, $\kappa \in -\mathbb{N}_0$, annihilates the non-holomorphic part (since our growth condition rules out a non-holomorphic constant term in the Fourier expansion). Letting $\ell \in \mathbb{N}$, the Bol operator can be written in terms of the iterated Maass raising operator
\[
(-4\pi)^{\ell-1}\mathbb{D}^\ell = R_{\ell-1}^{\ell-1} := R_{\ell-2} \circ \ldots \circ R_{2-\ell+2} \circ R_{2-\ell}, \quad R_{2-\ell}^0 := \text{id}, \quad R_\kappa := 2i \frac{\partial}{\partial \tau} + \frac{\kappa}{\nu}.
\]
This identity is called Bol’s identity, a proof can for example be found in [3, Lemma 5.3].

2.2.2. Locally harmonic Maass forms. In [1], so-called locally harmonic Maass forms, were introduced (for negative weights). See also [11] for the case of weight 0.

**Definition 2.7** ([1] Section 2). A function $f : \mathbb{H} \to \mathbb{C}$ is called a locally harmonic Maass form of weight $\kappa$ with exceptional set $E_D$, if it obeys the following four conditions:

1. For every $\gamma \in \text{SL}_2(\mathbb{Z})$ we have $f|_\kappa \gamma = f$.
2. For all $\tau \in \mathbb{H} \setminus E_D$, there exists a neighborhood of $\tau$, in which $f$ is real-analytic and in which we have $\Delta_\kappa(f) = 0$.
3. For every $\tau \in E_D$, we have that
\[
f(\tau) = \frac{1}{2} \lim_{\varepsilon \to 0^+} (f(\tau + i\varepsilon) + f(\tau - i\varepsilon)).
\]
4. The function $f$ exhibits at most polynomial growth towards $i\infty$.

We say that a function $f : \mathbb{H} \setminus E_D \to \mathbb{C}$ has jumping singularities on $E_D$ if
\[
\lim_{\varepsilon \to 0^+} (f(\tau + i\varepsilon) - f(\tau - i\varepsilon)) \in \mathbb{C} \setminus \{0\}
\]
for $\tau \in E_D$. Note that this limit depends on the geodesic $S_Q$ on which $\tau$ is located. If
\[
\lim_{\varepsilon \to 0^+} (f(\tau + i\varepsilon) - f(\tau - i\varepsilon)) = 0
\]
for all $\tau \in E_D$, then we say that $f$ has continuously removable singularities on $E_D$.

2.3. A theta lift and Poincaré series. A fundamental domain of $\Gamma_0(4)$ truncated at height $T > 0$ is given by
\[
\mathbb{F}_T(4) := \bigcup_{\gamma \in \Gamma_0(4) \setminus \text{SL}_2(\mathbb{Z})} \gamma \mathbb{F}_T,
\]
where $\mathbb{F}_T := \left\{ z \in \mathbb{H} : |x| \leq \frac{1}{2}, |z| \geq 1, y \leq T \right\}$.

Let $f$ and $g$ satisfy weight $\kappa$ modularity for $\Gamma_0(4)$ and let $d\mu(\tau) := \frac{du dv}{v^2}$. Borcherds regularization of the Petersson inner product of $f$ and $g$ is given by
\[
\langle f, g \rangle_{\text{reg}} := \int_{\Gamma_0(4) \setminus \mathbb{H}} f(w)g(w) \text{Im}(w)^\kappa d\mu(w) := \lim_{T \to \infty} \int_{\Gamma_0(4) \setminus \mathbb{H}} f(w)g(w) \text{Im}(w)^\kappa d\mu(w),
\]
whenever the limit exists. Although the definition of $\mathbb{F}_T(4)$ depends on a set of representatives of $\Gamma_0(4) \setminus \text{SL}_2(\mathbb{Z})$, the inner product is independent of this choice.

We next define the Poincaré series appearing in Theorem [1,3] and follow the exposition from [3, Section 2]. We let $\cdot |_{\text{pr}}$ denote the projection operator into Kohnen’s plus space (see e.g. [3, (6.12)]). We furthermore let $M_{\mu, \nu}$ be the usual $M$-Whittaker function, and define
\[
\mathcal{M}_{\kappa, s}(t) := |t|^{-\frac{\kappa}{2}} M_{s}(t)_{\nu} \mathcal{M}_{s}(\tau)^{-\frac{1}{2}}(|t|).
\]

\[\text{Note that the Poincaré series in [3] are normalized differently.}\]
We then define the Maass–Poincaré series of weights $\kappa \in -\mathbb{N} + \frac{1}{2}$ and indices $m \in \mathbb{N}$ projected into Kohnen’s plus space as (compare [6, (2.12)]), with $\Gamma_\infty := \{ \pm \left( \begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) : n \in \mathbb{Z} \}$,

$$\mathbb{P}_{\kappa,m}(z) = \frac{(4\pi m)^\frac{\kappa}{2}}{\Gamma(2 - \kappa)} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} \left( \mathcal{M}_{\kappa,1 - \frac{\kappa}{2}}(-4\pi my)e^{-2\pi imx} \right) |^\gamma| \pr. \tag{2.4}$$

The functions $\mathbb{P}_{\kappa,m}$ converge absolutely and are harmonic Maass forms, see [3, Theorem 6.11].

We lastly summarize the transformation behaviour of the theta kernel from the introduction.

**Lemma 2.8.**

(1) The function $z \mapsto \theta^*_{-k}(\tau, z)$ is modular of weight $\frac{1}{2} - k$ and is in Kohnen’s plus space.

(2) The function $\tau \mapsto \theta^*_{-k}(\tau, z)$ is modular of weight $-2k$.

**Proof.**

(1) This follows by a result of Vignéras [22]. The application of her result in this case can be found in [6, Section 2]. This part is also contained in [5, Section 2] (up to a local sign factor).

(2) This follows by Lemma 2.5 and equation (2.1). □

3. **Proof of Theorem 1.1**

3.1. **Knopp’s claims on $\psi_{k+1,D}$.** We now discuss the initial claims of Knopp on $\psi_{k+1,D}$.

**Proposition 3.1.**

(1) The functions $\psi_{k+1,D}$ converge absolutely on $\mathbb{H}$ and uniformly towards $i\infty$.

(2) For $n \in \mathbb{N}$, we have

$$\mathbb{D}^{2n-1} \left( \log \left( \frac{\tau - \alpha_+ Q}{\tau - \alpha_- Q} \right) Q(\tau, 1)^{n-1} \right) = -i(2\pi)^{2n-1} (n-1)!^2 D^{n-\frac{1}{2}} \frac{1}{Q(\tau, 1)^n}. \tag{3.1}$$

(3) The functions $\psi_{k+1,D}(\tau + 1) = \psi_{k+1,D}(\tau)$ and (1.2).

**Proof.**

(1) Let $Q = [a,b,c]$ and suppose that $v > 1$. Since $\alpha^\pm_Q \in \mathbb{R}$ are the zeros of $Q$, we have $Q(\tau, 1) = a(\tau - \alpha_+ Q)/(\tau - \alpha_- Q)$ and $v > 1$ implies that $|\tau - \alpha^+_Q| > 1$. Using $|a| \geq 1$ gives

$$\left| \log \left( \frac{\tau - \alpha_+ Q}{\tau - \alpha_- Q} \right) \right| \leq \log \left( \frac{|Q(\tau, 1)|}{|a| \left| \left( \frac{\tau - \alpha_+ Q}{\tau - \alpha_- Q} \right)^2 \right|} \right) + \pi \leq |\log |Q(\tau, 1)|| + \pi,$$

(1) then follows by the properties of $f_{k,D}$ for $k > 1$ (see [23]).

(2) We proceed by induction on $n$. The claims for $n = 1$ and $n = 2$ follow by computing

$$\frac{\partial}{\partial \tau} \log \left( \frac{\tau - \alpha_+ Q}{\tau - \alpha_- Q} \right) = -\sqrt{D} \frac{Q(\tau, 1)}{Q(\tau, 1)^2}, \quad \frac{\partial^3}{\partial \tau^3} \left( \log \left( \frac{\tau - \alpha_+ Q}{\tau - \alpha_- Q} \right) Q(\tau, 1) \right) = \frac{D^\frac{3}{2}}{Q(\tau, 1)^2},$$

utilizing Lemma 2.5 for $n = 2$. To proceed with the induction step, we define for $n \in \mathbb{N}$

$$f_n(\tau) := \log \left( \frac{\tau - \alpha_+ Q}{\tau - \alpha_- Q} \right) Q(\tau, 1)^{n-1}, \quad c_n := (-1)^n (n-1)!^2.$$

Since $Q$ is a polynomial of degree 2, we have, using the Leibniz rule,

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} f_{n+1}(\tau) = \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} (f_n(\tau)Q(\tau, 1)) = f_{n+1}^{(2n+1)}(\tau)Q(\tau, 1) + (2n+1)f_{n+1}^{(2n)}(\tau)Q'(\tau, 1) + (2n+1)n f_{n+1}^{(2n-1)}Q''(\tau, 1).$$
To apply the induction hypothesis, we write $\psi_n^{(2n)}(\tau) = \frac{\partial}{\partial \tau} f_n^{(2n-1)}(\tau)$. Combining with the second identity of Lemma 2.3 then yields

$$\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} f_{n+1}(\tau) = n^2 c_n D^{n+\frac{1}{2}} \frac{1}{Q(\tau, 1)^{n+1}}.$$

Simplifying gives the claim.

(3) Translation invariance of $\psi_{k+1,D}$ follows immediately from equation (2.1) and the fact that

$$[a, b, c] \circ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = [a, -2a + b, a - b + c].$$

Again using (2.1) and the fact that

$$[a, b, c] \circ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = [c, -b, a],$$

we obtain that

$$\tau^{-2k-2} \psi_{k+1,D} \left( -\frac{1}{\tau} \right) = \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left( \frac{-\frac{1}{\tau} - \frac{b-\sqrt{\tau}}{a}}{\frac{1}{\tau} - \frac{b+\sqrt{\tau}}{a}} \right) - \log \left( \frac{-\frac{1}{\tau} - \frac{b+\sqrt{\tau}}{a}}{\frac{1}{\tau} - \frac{b-\sqrt{\tau}}{a}} \right)}{Q(\tau, 1)^{k+1}}.$$

Next, we recall that for $z, w \in \mathbb{C} \setminus \mathbb{R}$

$$\log(z) - \log(w) = \log \left( \frac{z}{w} \right) + i \left( \arg(z) - \arg(w) - \arg \left( \frac{z}{w} \right) \right). \quad (3.2)$$

Choosing $z = \frac{1 - b + \sqrt{\tau}}{2a}, w = \frac{1 - b - \sqrt{\tau}}{2a}$ yields

$$\frac{z}{w} = \left( \frac{1 - b - \sqrt{\tau}}{2a} \right) \left( \frac{1 - b + \sqrt{\tau}}{2a} \right) \left( \frac{1 - b + \sqrt{\tau}}{2a} \right) \left( \frac{1 - b - \sqrt{\tau}}{2a} \right) = \alpha_Q^+ 1 \alpha_Q^+ = \operatorname{sgn}(ac) \frac{\alpha_Q^+}{\alpha_Q^+}.$$

Hence $\arg(z) = \arg(\operatorname{sgn}(ac)w)$ and thus $\arg(z) - \arg(w) - \arg \left( \frac{z}{w} \right)$ vanishes whenever $\operatorname{sgn}(ac) = 1$. Therefore the corresponding terms do not contribute to $\arg(z) - \arg(w) - \arg \left( \frac{z}{w} \right)$. If $\operatorname{sgn}(ac) = -1$, we extend $\log$ by its principal value $\log(x) = \log |x| + \pi i$ for $x \in \mathbb{R}^-$. Then we use that

$$\arg(-w) - \arg(w) = -\arg(\operatorname{Im}(w)) \pi, \quad (3.3)$$

and $\arg \left( \frac{z}{w} \right) = \pi$. Hence, $\arg(z) - \arg(w) - \arg \left( \frac{z}{w} \right)$ vanishes if $\operatorname{sgn}(ac) = -1$ and $\operatorname{Im}(w) < 0$. To determine the sign of $\operatorname{Im}(w)$, we calculate that

$$\frac{\tau - \alpha_Q^-}{\tau - \alpha_Q^+} = \frac{\alpha_Q^- \alpha_Q^+ - \left( \alpha_Q^- + \alpha_Q^+ \right) w + w^2 + v^2}{|\tau - \alpha_Q|^2} - i \frac{\left( \alpha_Q^+ - \alpha_Q^- \right) v}{|\tau - \alpha_Q|^2} = \frac{1}{|\tau - \alpha_Q|^2} \left( \frac{\tau Q x}{a} - i \sqrt{D} \frac{v}{a} \right). \quad (3.4)$$

Consequently, we have $\operatorname{Im}(w) > 0$ if and only if $a < 0$. We conclude by (3.2) and (3.3) that

$$\arg(z) - \arg(w) - \arg \left( \frac{z}{w} \right) = \begin{cases} -2\pi & \text{if } a < 0 < c, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\tau^{-2k-2} \psi_{k+1,D} \left( -\frac{1}{\tau} \right) - \psi_{k+1,D}(\tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left( \frac{\alpha_{Q}^+}{\alpha_{Q}^-} \right)}{Q(\tau, 1)^{k+1}} - 2\pi i \sum_{Q \in \mathcal{Q}_D} \frac{1}{Q(\tau, 1)^{k+1}}.$$
By mapping $Q \mapsto -Q$, we arrive at
\[
\sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{\alpha^+ \rho_{\lambda}}{\alpha_Q})}{Q(\tau,1)^{k+1}} = \sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{\alpha^+ \rho_{\lambda}}{\alpha_Q})}{Q(\tau,1)^{k+1}} + \pi i \sum_{Q = [a,b,c] \in \mathbb{Q}_D} \frac{1}{Q(\tau,1)^{k+1}} = \sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{\alpha^+ \rho_{\lambda}}{\alpha_Q})}{Q(\tau,1)^{k+1}}. \tag{3.4} \]

\[ \square \]

Remark. By (3.4) the branch cut of $\log (\frac{w-a}{w-a_Q})$ is the interval $[\alpha_Q, \alpha^+_Q]$ or $[\alpha^+_Q, \alpha_Q]$.

3.2. The functions $\rho_{k+1,D}$, $\varphi_{k+1,D}$, and $\lambda_{k+1,D}$. Adapting the proof of Proposition 3.1 (1), (3) we deduce the following results.

**Proposition 3.2.** Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.
(1) The functions $\rho_{k+1,D}$ converge absolutely on $\mathbb{H} \times \mathbb{H}^-$ and uniformly as $\tau \to i\infty$ resp. $w \to -i\infty$.
(2) We have
\[
\lim_{w \to -i\infty} \rho_{k+1,D}(\tau, w) = 0, \quad \lim_{\tau \to i\infty} \rho_{k+1,D}(\tau, w) = 0.
\]
(3) The functions $\rho_{k+1,D}$ satisfy
\[
\tau^{-2k-2} \rho_{k+1,D}(\tau+1, w+1) = \rho_{k+1,D}(\tau, w),
\]
\[
\tau^{-2k-2} \varphi_{k+1,D}(\tau+1, w) = \varphi_{k+1,D}(\tau, w) = \sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{\alpha^+ \rho_{\lambda}}{\alpha_Q})}{Q(\tau,1)^{k+1}} + 2\pi i \sum_{a \in \mathbb{Q}_D} \frac{1}{Q(\tau,1)^{k+1}}.
\]

We next cite Parson’s [18] result on her modular integral $\varphi_{k+1,D}$.

**Lemma 3.3.** Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.
(1) The functions $\varphi_{k+1,D}$ satisfy $\varphi_{k+1,D}(\tau+1) = \varphi_{k+1,D}(\tau)$, and
\[
\tau^{-2k-2} \varphi_{k+1,D}(\tau+1, w) = \varphi_{k+1,D}(\tau, w) = \sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{\alpha^+ \rho_{\lambda}}{\alpha_Q})}{Q(\tau,1)^{k+1}} + 2\sum_{a \in \mathbb{Q}_D} \frac{1}{Q(\tau,1)^{k+1}}.
\]
Furthermore, we have $\lim_{\tau \to i\infty} \varphi_{k+1,D}(\tau) = 0$.

We continue with some properties of $\lambda_{k+1,D}$.

**Proposition 3.4.** Let $\tau \in \mathbb{H}$, $w \in \mathbb{H}^-$.
(1) The functions $\lambda_{k+1,D}$ converge absolutely on $\mathbb{H} \times \mathbb{H}^-$, and uniformly as $\tau \to i\infty$ resp. $w \to -i\infty$.
(2) The functions $\lambda_{k+1,D}$ are bimodular of weight $(2k+2,0)$, that is
\[
\lambda_{k+1,D}(\tau+1, w+1) = \lambda_{k+1,D}(\tau, w), \quad \lambda_{k+1,D}(\frac{1}{\tau}, \frac{1}{w}) = \tau^{2k-2} \lambda_{k+1,D}(\tau, w).
\]
(3) We have
\[
\lim_{w \to -i\infty} \lambda_{k+1,D}(\tau, w) = -2\pi i \varphi_{k+1,D}(\tau), \quad \lim_{\tau \to i\infty} \lambda_{k+1,D}(\tau, w) = 0.
\]
(4) We have
\[
\lambda_{k+1,D}(\tau, w) = \sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{1+i \alpha_Q}{1-i \alpha_Q})}{Q(\tau,1)^{k+1}} = \sum_{Q \in \mathbb{Q}_D} \frac{\log (\frac{\alpha^+ \rho_{\lambda}}{\alpha_Q})}{Q(\tau,1)^{k+1}}.
\]

**Proof.**
(1) By the definition of $\lambda_{k+1,D}$ in [13], we have
\[
|\lambda_{k+1,D}(\tau)| \leq 2 \sum_{Q \in \mathbb{Q}_D} \frac{1}{|Q(\tau,1)|^{k+1}} \leq \pi \sum_{Q \in \mathbb{Q}_D} \frac{1}{|Q(\tau,1)|^{k+1}}.
\]
The claim follows by the absolute convergence of the \( f_{k,D} \)’s on \( \mathbb{H} \).

(2) Bimodularity is a direct consequence of Lemma 2.2 and equation (2.1).

(3) The assumption that \( D \) is not a square guarantees that the sum defining \( \lambda_{k+1,D} \) runs over quadratic forms \( Q = [a, b, c] \) with \( ac \neq 0 \). To prove the first assertion, we observe that

\[
\frac{Q_w}{\sqrt{D}} \approx a \text{Im}(w)
\]
as \( \text{Im}(w) \to -\infty \), and hence

\[
\lim_{w \to -i\infty} \arctan \left( \frac{Q_w}{\sqrt{D}} \right) = -\frac{\pi}{2} \text{sgn}(Q).
\]
The first claim follows by the definition of \( \varphi_{k+1,D} \) in (1.4). As \( a \neq 0 \), we have \( \frac{1}{Q(\tau, 1)^k} \to 0 \) for \( \tau \to i\infty \). The second claim follows by (1).

(4) The claim follows by rewriting the arctangent in (1.5).

3.3. Proof of Theorem 1.1 We conclude this section with the proof of Theorem 1.1

Proof of Theorem 1.1

(1) This follows by combining Propositions 3.1 (3), 3.2 (3), and 3.4 (2) with Lemma 3.3.

(2) This follows by combining (1.6) with Propositions 3.2 (2) and 3.4 (2).

(3) Proposition 3.1 (1) along with

\[
\lim_{\tau \to i\infty} \log \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right) = 0 \quad (3.5)
\]
implies that \( \psi_{k+1,D} \) is cuspidal. By Propositions 3.2 (2), 3.4 (3), and Lemma 3.3, every function defining \( \Omega_{k+1,D} \) in (1.6) is cuspidal (with respect to \( \tau \)).

(4) As each function defining \( \Omega_{k+1,D} \) in (1.6) is holomorphic as a function of \( \tau \), we obtain the assertion with respect to \( \tau \) directly. To verify that \( \Omega_{k+1,D} \) is anti-holomorphic as a function of \( w \), we compute by Lemmas 2.1 and 2.4 (1) that

\[
2i \frac{\partial}{\partial w} \arctan \left( \frac{Q_w}{\sqrt{D}} \right) = -\frac{\sqrt{D}}{Q(w, 1)}.
\]

By (3.1), we deduce that

\[
2i \frac{\partial}{\partial w} \arctan \left( \frac{Q_w}{\sqrt{D}} \right) = \frac{\partial}{\partial w} \log \left( \frac{w - \alpha_Q}{w - \alpha_{-Q}} \right).
\]

By (1.5) and (1.6), we conclude that

\[
\frac{\partial}{\partial w} \Omega_{k+1,D}(\tau, w) = 0.
\]

(5) We first inspect the functions \( \psi_{k+1,D} - \rho_{k+1,D} \). By definitions (1.1) and (1.3) we have

\[
\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \tau) = \sum_{Q \in \mathcal{Q}_D} \frac{\log \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right) - \log \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right)}{Q(\tau, 1)^k + 1}.
\]

We note that

\[
\log \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right) - \log \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right) \equiv \log \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right) \left( \frac{\tau - \alpha_{-Q}}{\tau - \alpha_Q} \right) (\text{mod } 2\pi i),
\]
and we determine the multiple of \(2\pi i\) now. From (3.4), we deduce that
\[
\frac{(\tau - \alpha_q^-)(\tau - \alpha_q^+)}{(\tau - \alpha_q^-)(\tau - \alpha_q^+)} = \frac{Q - i}{Q + i}.
\]

We use (3.2) and hence need to compute
\[
\log \left( \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+} \right) - \log \left( \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+} \right) - \log \left( \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+} \right) = i \left( \arg \left( \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+} \right) - \arg \left( \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+} \right) - \arg \left( \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+} \right) \right).
\]

(3.6)

Note that for \(z \in \mathbb{C} \setminus \mathbb{R}\)
\[
\arg(z) - \arg(\tau) - \arg \left( \frac{z}{\tau} \right) = \begin{cases} 0 & \text{if } \text{Re}(z) > 0, \\ 2\pi & \text{if } \text{Re}(z) < 0 \text{ and } \text{Im}(z) > 0, \\ -2\pi & \text{if } \text{Re}(z) < 0 \text{ and } \text{Im}(z) < 0. \end{cases}
\]

We use this for \(z = \frac{\tau - \alpha_q^-}{\tau - \alpha_q^+}\). By (3.4), (3.6) thus becomes
\[
\begin{align*}
0 & \quad \text{if } aQ \tau > 0, \\
2\pi i & \quad \text{if } a < 0, \text{ and } Q \tau > 0, \\
-2\pi i & \quad \text{if } a > 0, \text{ and } Q \tau < 0.
\end{align*}
\]

Consequently, we catch some multiple of \(2\pi i\) if and only if \(\text{sgn}(Q \tau) \text{ sgn}(Q) < 0\). We infer that
\[
\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \tau^*) = \sum_{Q \in \mathcal{Q}_D} \log \left( \frac{Q + i}{Q - i} \right) + \frac{2\pi i}{Q(\tau, 1)^{k+1}} + \pi i \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}.
\]

Combining with (1.4) gives
\[
\psi_{k+1,D}(\tau) - \rho_{k+1,D}(\tau, \tau^*) + \varphi_{k+1,D}(\tau, \tau^*) = \sum_{Q \in \mathcal{Q}_D} \log \left( \frac{Q + i}{Q - i} \right) + \frac{2\pi i}{Q(\tau, 1)^{k+1}} + \pi i \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q \tau)}{Q(\tau, 1)^{k+1}},
\]

(3.7)

which is modular of weight \(2k + 2\) by (2.1) and Lemma 2.2. To finish the proof, we inspect \(\lambda_{k+1,D}(\tau, \tau^*)\). Combining \(Q \tau = -Q \tau^*\) with Proposition 3.3 (4) yields
\[
\lambda_{k+1,D}(\tau, \tau^*) = -\sum_{Q \in \mathcal{Q}_D} \log \left( \frac{Q + i}{Q - i} \right).
\]

By (3.3), we obtain
\[
\log \left( \frac{Q + i}{Q - i} \right) - \log \left( \frac{Q + i}{Q - i} \right) = -\pi i \text{sgn}(Q \tau),
\]

from which we conclude the claim using (1.6).
4. The function $\Lambda_{k+1,D}$

4.1. Local cusp forms. Recall the definition of $\Lambda_{k+1,D}$ in (1.7).

Remark. Let $d(z,w)$ denote the hyperbolic distance between $z, w \in \mathbb{C}$ with $y \text{Im}(w) > 0$. Since $D > 0$, we have (with $\tau_{a,b,c} := -b + \frac{i}{2|a|} \sqrt{D}$) $Q = \cosh(d(\tau, \tau_Q))$. This yields an alternative representation of $\Lambda_{k+1,D}$ as well as of $\lambda_{k+1,D}$.

We next prove our claim for $\Lambda_{k+1,D}$ from the introduction.

Proposition 4.1. The functions $\Lambda_{k+1,D}$ are local cusp forms.

Proof. We observe that the $\Lambda_{k+1,D}$’s converge absolutely on $\mathbb{H}$ utilizing absolute convergence of the $f_k$’s. We directly deduce that the $\Lambda_{k+1,D}$’s are holomorphic. Using Lemma 2.2 and (2.1) shows that the $\Lambda_{k+1,D}$’s are modular of weight $2k + 2$. If $v > \sqrt{D}$, then $\text{sgn}(Q) = 1$. Thus, cuspidality of the $\Lambda_{k+1,D}$’s follows by cuspidality of the $f_k$’s for $k > 1$. The local behaviour and the jumping singularities are dictated by $\text{sgn}(Q)$. □

4.2. The local behaviour of $\Lambda_{k+1,D}$. We next provide the behaviour of $\Lambda_{k+1,D}$ on $E_D$.

Proposition 4.2. If $\tau \in E_D$, then we have that

$$\lim_{\varepsilon \to 0^+} (\Lambda_{k+1,D}(\tau + i\varepsilon) - \Lambda_{k+1,D}(\tau - i\varepsilon)) = 2 \sum_{Q \in \mathbb{Q}_D, \varepsilon > 0} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}.$$

Remark. The sum on the right-hand side is finite by [4] Lemma 5.1 (1)].

Proof of Proposition 4.2. We adapt the proof of [4] Proposition 5.2. We write

$$\Lambda_{k+1,D}(\tau \pm i\varepsilon) = \left( \sum_{Q \in \mathbb{Q}_D, \varepsilon > 0} \frac{\text{sgn}(Q)}{Q(\tau \pm i\varepsilon, 1)^{k+1}} \right) \frac{\text{sgn}(Q)}{Q(\tau \pm i\varepsilon, 1)^{k+1}}.$$

The properties of $f_k$ imply that $\Lambda_{k+1,D}$ converges absolutely on $\mathbb{H}$ and uniformly towards $i\infty$, which permits us to interchange the sums with the limit, and argue termwise in the following.

If $Q \neq 0$, then $\tau \pm i\varepsilon$ are in the same connected component of $\mathbb{H} \setminus E_D$ for $\varepsilon > 0$ sufficiently small. Combining with [4] (5.4), we deduce that for $\varepsilon > 0$ sufficiently small

$$\text{sgn}([a, b, c]_{\tau \pm i\varepsilon}) = \text{sgn}([a, b, c]_{\tau - i\varepsilon}) = \delta \text{sgn}(a),$$

where

$$\delta := \text{sgn} \left( \frac{\tau + i\varepsilon + b}{2a} \pm \frac{\sqrt{D}}{2|a|} \right) = \text{sgn} \left( \frac{\tau - i\varepsilon + b}{2a} \pm \frac{\sqrt{D}}{2|a|} \right) = \pm 1.$$

Thus

$$\lim_{\varepsilon \to 0^+} \left( \frac{\text{sgn}(Q_{\tau \pm i\varepsilon})}{Q(\tau \pm i\varepsilon, 1)^{k+1}} - \frac{\text{sgn}(Q_{\tau - i\varepsilon})}{Q(\tau - i\varepsilon, 1)^{k+1}} \right) = \lim_{\varepsilon \to 0^+} \delta \left( \frac{\text{sgn}(Q)}{Q(\tau + i\varepsilon, 1)^{k+1}} - \frac{\text{sgn}(Q)}{Q(\tau - i\varepsilon, 1)^{k+1}} \right) = 0.$$

If $Q = 0$, then $\tau \pm i\varepsilon$ are in different connected components of $\mathbb{H} \setminus E_D$ for all $\varepsilon > 0$. This is justified by [4] (5.6), namely

$$\left| \tau - i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} < \left| \tau + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|} = 0 < \left| \tau + i\varepsilon + \frac{b}{2a} \right| - \frac{\sqrt{D}}{2|a|}$$

for every $\varepsilon > 0$. Combining with [4] (5.4) implies that $\text{sgn}(Q_{\tau \pm i\varepsilon}) = \pm \text{sgn}(Q)$, and consequently

$$\lim_{\varepsilon \to 0^+} \left( \frac{\text{sgn}(Q_{\tau \pm i\varepsilon})}{Q(\tau \pm i\varepsilon, 1)^{k+1}} - \frac{\text{sgn}(Q_{\tau - i\varepsilon})}{Q(\tau - i\varepsilon, 1)^{k+1}} \right) = 2 \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}.$$

□

We next inspect the sum appearing in Proposition 4.2.
Lemma 4.3. The sum

$$\sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}$$

does not vanish identically on $E_D$.

Proof. Let $\tau \in E_D$. Then we have $\tau \in S_\mathcal{Q}$ for some $\mathcal{Q} \in \mathcal{Q}_D$. On the one hand, the sum in the lemma has a pole of order $k + 1 > 0$ at $\alpha^\mathcal{Q}_\mathcal{Q}$, and hence both limits

$$\lim_{\tau \to \alpha^\mathcal{Q}_\mathcal{Q}} \left| \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}} \right|$$

tend towards $\infty$. On the other hand, the sum is continuous on $S_\mathcal{Q}$, and the contribution from the terms corresponding to $Q \neq \mathcal{Q}$ is finite at $\alpha^\mathcal{Q}_\mathcal{Q}$.

4.3. The local behaviour of $\mathcal{E}_{\Lambda_{k+1,D}}$ and $\Lambda^*_{k+1,D}$. We next prove that the Eichler integrals of $\Lambda_{k+1,D}$ exist on $E_D$.

Proposition 4.4. Let $\tau \in E_D$. Then we have

$$\lim_{\varepsilon \to 0^+} \left( \mathcal{E}_{\Lambda_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{\Lambda_{k+1,D}}(\tau - i\varepsilon) \right) = -\frac{2(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau + w, 1)^{k+1}} w^{2k} dw,$$

$$\lim_{\varepsilon \to 0^+} \left( \Lambda^*_{k+1,D}(\tau + i\varepsilon) - \Lambda^*_{k+1,D}(\tau - i\varepsilon) \right) = -\frac{2}{(2\pi i)^{2k+1}} \int_{i\varepsilon}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau - w, 1)^{k+1}} w^{2k} dw.$$

Remark. As remarked after Proposition 4.2, the sums inside the integrals on the right-hand sides of Proposition 4.4 are finite. They can be written as integrals over a bounded domain, because the integrands vanish as soon as $\tau - w$ moves out of $\mathbb{H}$. If $\tau \pm w \notin E_D$, then the sums are in fact empty. Hence, the integrals on the right-hand side of Proposition 4.4 exist.

Proof of Proposition 4.4. As $\tau \pm i\varepsilon, w \notin E_D$ for every $\varepsilon > 0$, we utilize (1.7). Changing variables gives

$$\lim_{\varepsilon \to 0^+} \left( \mathcal{E}_{\Lambda_{k+1,D}}(\tau + i\varepsilon) - \mathcal{E}_{\Lambda_{k+1,D}}(\tau - i\varepsilon) \right) = -\frac{(2\pi i)^{2k+1}}{(2k)!} \int_{i\varepsilon}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \left( \frac{\text{sgn}(Q(\tau + w, 1))}{Q(\tau + w, 1)^{k+1}} - \frac{\text{sgn}(Q(\tau - w, 1))}{Q(\tau - w, 1)^{k+1}} \right) w^{2k} dw,$$

$$\lim_{\varepsilon \to 0^+} \left( \Lambda^*_{k+1,D}(\tau + i\varepsilon) - \Lambda^*_{k+1,D}(\tau - i\varepsilon) \right) = \frac{1}{(2\pi i)^{2k+1}} \left( -\int_{2i(\tau + \varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q(\tau - w, 1))}{Q(\tau - w, 1)^{k+1}} w^{2k} dw + \int_{2i(\tau - \varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q(\tau - w, 1))}{Q(\tau - w, 1)^{k+1}} w^{2k} dw \right),$$

where we use that $Q_{\tau} = -Q_{\bar{\tau}}$ in the case of $\Lambda^*_{k+1,D}$. We next justify interchanging the limits $\varepsilon \to 0^+$ with the holomorphic Eichler integral. By (1.7), $\Lambda_{k+1,D}$ vanishes at $i\infty$, and converges uniformly to $i\infty$ as the sign-function is bounded (using that $f_{\kappa,D}$ converges uniformly towards $i\infty$ for $\kappa > 1$). By modularity of $\Lambda_{k+1,D}$, both assertions hold towards $0$ as well. In other words, the integral converges uniformly, and this permits the exchange of the limit $\varepsilon \to 0^+$ with the integral.\footnote{If $\text{Im}(\tau + w) > \frac{\pi}{2\tau}$, then $\tau + w$ lies in the unbounded component of $\mathbb{H} \setminus E_D$.}
We consider the holomorphic Eichler integral first. If \( \tau + w \notin E_D \), then the limit inside the integral vanishes, because \( \tau + w + i\varepsilon \) and \( \tau + w - i\varepsilon \) are in the same connected component for \( \varepsilon \) sufficiently small. If \( \tau + w \in E_D \), then we apply Proposition 4.2 to obtain

\[
\lim_{\varepsilon \to 0^+} \left( \mathcal{E}_{k+1,D}(\tau + i\varepsilon) - \mathcal{E}_{k+1,D}(\tau - i\varepsilon) \right) = -\frac{(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \lim_{\varepsilon \to 0^+} (\Lambda_{k+1,D}(\tau + w + i\varepsilon) - \Lambda_{k+1,D}(\tau + w - i\varepsilon)) w^{2k} dw
\]

\[
= -\frac{2(2\pi i)^{2k+1}}{(2k)!} \int_0^{i\infty} \sum_{Q \in \mathcal{Q}_D, \tau + w = 0} \frac{\text{sgn}(Q)}{Q(\tau + w, 1)^{k+1}} w^{2k} dw.
\]

Now, we treat the non-holomorphic Eichler integrals, and first split one of them as

\[
\int_{2i(v-\varepsilon)}^{2i(v+\varepsilon)} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} w^{2k} dw = \left( \int_{2i(v-\varepsilon)}^{2i(v+\varepsilon)} \int_{2i(v+\varepsilon)}^{2i(v+\varepsilon)} \sum_{Q \in \mathcal{Q}_D} \frac{\text{sgn}(Q)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} w^{2k} dw. \right)
\]

We note that

\[
\lim_{\varepsilon \to 0^+} \int_{2i(v+\varepsilon)}^{2i(v+\varepsilon)} \frac{\text{sgn}(Q)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} w^{2k} dw = 0,
\]

because the integrand is bounded in the domain of integration, which has measure 0 as \( \varepsilon \to 0^+ \). Hence, it remains to consider the integral from \( 2i(v + \varepsilon) \) to \( i\infty \). If \( \tau - w \notin E_D \), then we have

\[
\lim_{\varepsilon \to 0^+} \int_{2i(v+\varepsilon)}^{i\infty} \left( -\frac{\text{sgn}(Q)}{Q(\tau - w + i\varepsilon, 1)^{k+1}} + \frac{\text{sgn}(Q)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} \right) w^{2k} dw = 0,
\]

as in the previous case, because \( \tau - w \pm i\varepsilon \) are in the same connected component for \( \varepsilon \) sufficiently small. If \( \tau - w \in E_D \), then we obtain

\[
\lim_{\varepsilon \to 0^+} \int_{2i(v+\varepsilon)}^{i\infty} \sum_{Q \in \mathcal{Q}_D} \left( -\frac{\text{sgn}(Q)}{Q(\tau - w + i\varepsilon, 1)^{k+1}} + \frac{\text{sgn}(Q)}{Q(\tau - w - i\varepsilon, 1)^{k+1}} \right) w^{2k} dw = -2 \int_{2i\varepsilon}^{i\infty} \sum_{Q \in \mathcal{Q}_D, Q_{\tau - w} = 0} \frac{\text{sgn}(Q)}{Q(\tau - w, 1)^{k+1}} w^{2k} dw
\]

by Proposition 4.2 exactly as in the previous case. \( \square \)

5. The function \( \Psi_{-k,D} \) and the proof of Theorem 1.2

5.1. Convergence of \( \Psi_{-k,D} \). We first establish convergence of \( \Psi_{-k,D} \).

**Proposition 5.1.** The sum defining \( \Psi_{-k,D} \) converges compactly for every \( \tau \in \mathbb{H} \setminus E_D \), and does not converge on \( E_D \).

**Proof.** If \( \tau \in \mathbb{H} \setminus E_D \), then \( \text{sgn}(Q\tau) = \pm 1 \) and thus the claim follows directly by [4 Proposition 4.1] after summing over all narrow equivalence classes there. (The class number of positive discriminants is finite.) If \( \tau \in E_D \), then the incomplete \( \beta \)-function reduces to a constant depending only on \( k \) according to Lemma 2.1. Hence, the sum defining \( \Psi_{-k,D} \) does not converge on \( E_D \) as the sum is infinite and \( \beta(1; k + \frac{1}{2}, \frac{1}{2}) \neq 0 \). \( \square \)

5.2. Behaviour of \( \Psi_{-k,D} \) under differentiation. We inspect the behaviour of \( \Psi_{-k,D} \) under differential operators.

**Proposition 5.2.** Let \( \tau \in \mathbb{H} \setminus E_D \).

1. We have

\[
\xi_{-2k}(\Psi_{-k,D}) = D^{k+\frac{1}{2}} \Lambda_{k+1,D}.
\]
(2) We have
\[ \mathbb{D}^{2k+1}(\Psi_{-k,D}) = -\frac{D^{k+\frac{1}{2}}(2k)!}{(4\pi)^{2k+1}} \Lambda_{k+1,D}. \]

(3) We have
\[ \Delta_{-2k}(\Psi_{-k,D}) = 0. \]

Define
\[ g_n^{[1]}(\tau) := Q(\tau,1)^n \beta \left( \frac{Dv^2}{|Q(\tau,1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right), \quad n \in \mathbb{N}_0. \]

The proof of Proposition 5.2 is based on the following three technical lemmas.

**Lemma 5.3.** We have for \( n \in \mathbb{N}_0 \)
\[ g_n^{[1]}(\tau) = \frac{n + \frac{1}{2}}{n + 1} Q(\tau,1)g_n^{[1]}(\tau) - \frac{D^{n+\frac{1}{2}}}{n + 1} v^{2n+2} |Q_\tau|. \]

**Proof.** By [11, (8.17.20)], we have that
\[ \beta(x; a, b) \beta(1; a, b) = \frac{x^a(1-x)^b}{a\beta(1; a, b)}. \]

This gives that
\[ \beta \left( \frac{Dv^2}{|Q(\tau,1)|^2}; n + \frac{3}{2}, \frac{1}{2} \right) = \frac{\beta(1; n + \frac{3}{2}, \frac{1}{2})}{\beta(1; n + 1, \frac{1}{2})} \left( \beta \left( \frac{Dv^2}{|Q(\tau,1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) - \left( \frac{Dv^2}{|Q(\tau,1)|^2} \right)^{n+\frac{1}{2}} \left( 1 - \frac{Dv^2}{|Q(\tau,1)|^2} \right)^{\frac{1}{2}} \right) . \]

Using Lemma 2.1 we compute
\[ \left( \frac{Dv^2}{|Q(\tau,1)|^2} \right)^{n+\frac{1}{2}} \left( 1 - \frac{Dv^2}{|Q(\tau,1)|^2} \right)^{\frac{1}{2}} = \frac{D^{n+\frac{1}{2}}v^{2n+2} |Q_\tau|}{|Q(\tau,1)|^{n+\frac{3}{2}}}, \]

and since \( \frac{\beta(1; n + \frac{3}{2}, \frac{1}{2})}{\beta(1; n + 1, \frac{1}{2})} = \frac{n + \frac{1}{2}}{n + 1}, \) we obtain the claim. \( \square \)

**Lemma 5.3** motivates to define the auxiliary function
\[ g_n^{[2]}(\tau) := \frac{D^{n-\frac{1}{2}}v^{2n} |Q_\tau|}{Q(\overline{\tau},1)^n}. \]

The second technical lemma treats the image of \( g_n^{[2]} \) under differentiation.

**Lemma 5.4.** We have for \( n \in \mathbb{N} \)
\[ \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[2]}(\tau) = 0. \]

**Proof.** We prove the claim by induction. If \( n = 1 \), then the claim follows by applying Lemma 2.4 (1) three times.

For the induction step, Lemma 2.3 (1) yields that
\[ \frac{\partial}{\partial \tau} \left( v^\ell Q_\tau \right) = -\frac{i}{2} v^{\ell+1} Q_\tau + \frac{i}{2} v^\ell Q(\overline{\tau},1) \]
for every \( \ell \in \mathbb{N}_0 \). Noting that \( \frac{\partial^2}{\partial \tau^2} (v^\ell Q(\overline{\tau},1)) = 0 \), we obtain
\[ \frac{\partial^{\ell+2}}{\partial \tau^{\ell+2}} (v^{\ell+1} Q_\tau) = -\frac{i}{2} (\ell + 1) \frac{\partial^{\ell+1}}{\partial \tau^{\ell+1}} (v^\ell Q_\tau). \]
Consequently, we find that
\[
\frac{\partial^{2n+3}}{\partial \tau^{2n+3}} g_n^{[2]}(\tau) = - \frac{D(2n+2)(2n+1)}{4Q(\tau,1)} \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[2]}(\tau).
\]
The right-hand side vanishes by the induction hypothesis, as desired. □

The third lemma contains the main technical claim.

**Lemma 5.5.** We have for \( n \in \mathbb{N}_0 \)
\[
\frac{\partial^{2n+1}}{\partial \tau^{2n+1}} g_n^{[1]}(\tau) = \frac{i(-1)^{n+1} D^{n+\frac{1}{2}}(2n)! \text{sgn}(Q_\tau)}{2^{2n} Q(\tau,1)^{n+1}}.
\]

**Proof.** We prove the lemma by induction.

**Step 1:** The case \( n = 0 \)
We apply the Fundamental Theorem of Calculus, Lemma 2.4 and Lemma 2.4, yielding
\[
\frac{\partial}{\partial \tau} \beta \left( \frac{Dv^2}{Q(\tau,1)^2}; n + \frac{1}{2}, \frac{1}{2} \right) = \frac{-iD^{n+\frac{1}{2}}}{2^{2n}} v^{2n} \text{sgn}(Q_\tau) Q(\tau,1)
\]
for every \( n \in \mathbb{N}_0 \). In particular, this proves the desired identity for \( n = 0 \).

**Step 2:** The case \( n = 1 \)
Using (5.1) and the first identity of Lemma 2.5, we compute that
\[
R_{-2n} \left( g_n^{[1]}(\tau) \right) = -2n Q_\tau g_n^{[1]}(\tau) + \frac{2 D^{n+\frac{1}{2}} v^{2n} \text{sgn}(Q_\tau)}{Q(\tau,1)^{n+1}}.
\]
Lemma 5.3 with \( n \mapsto n - 1 \) gives
\[
\frac{g_n^{[1]}(\tau)}{Q(\tau,1)} = \frac{n - \frac{1}{2}}{n} g_{n-1}^{[1]}(\tau) - \frac{D^{n-\frac{1}{2}} v^{2n} \text{sgn}(Q_\tau)}{n Q(\tau,1)^n} Q(\tau,1).
\]
Plugging into the previous equation and applying Lemma 2.5 yields
\[
R_{-2n} \left( g_n^{[1]}(\tau) \right) = -2(n - 1) Q_\tau g_n^{[1]}(\tau) + \frac{2 D^{n-\frac{1}{2}} v^{2n-2} \text{sgn}(Q_\tau)}{Q(\tau,1)^{n}}.
\]
We compute
\[
R_{2-2n} \left( \frac{2 D^{n-\frac{1}{2}} v^{2n-2} \text{sgn}(Q_\tau)}{Q(\tau,1)^{n}} \right) = 0,
\]
by Lemma 2.2.1. We infer that
\[
R_{2-2n} \circ R_{-2n} \left( g_n^{[1]}(\tau) \right) = -2(n - 1) \left( Q_\tau R_{2-2n} \left( g_n^{[1]}(\tau) \right) - g_n^{[1]}(\tau) \frac{Q(\tau,1)}{v^2} \right).
\]
Now, we suppose that \( n = 1 \). Then the previous equation gives
\[
R_0 \circ R_{-2} \left( g_1^{[1]}(\tau) \right) = -Q_\tau R_0 \left( g_0^{[1]}(\tau) \right) + g_0^{[1]}(\tau) \frac{Q(\tau,1)}{v^2}.
\]
We then compute, using (5.1)
\[
R_0 \left( g_0^{[1]}(\tau) \right) = 2r \frac{\partial}{\partial \tau} \beta \left( \frac{Dv^2}{Q(\tau,1)^2}; 1, \frac{1}{2}, \frac{1}{2} \right) = 2D^{\frac{1}{2}} \frac{\text{sgn}(Q_\tau)}{Q(\tau,1)}.
\]
Combining this with the previous equation we obtain
\[
R_2 \circ R_0 \circ R_{-2} \left( g_1^{[1]}(\tau) \right) = R_2 \left( -2Q_\tau D^{\frac{1}{2}} \frac{\text{sgn}(Q_\tau)}{Q(\tau,1)} + \beta \left( \frac{Dv^2}{Q(\tau,1)^2}; 1, \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\tau,1)}{v^2} \right).
\]
By Lemma 2.4 (1) and 5.1, we calculate that

\[
\frac{\partial}{\partial \tau} \left( -2Q_n \frac{D_{\tau} \text{sgn}(Q_{\tau})}{Q(\tau, 1)} + \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\tau, 1)}{v^2} \right)
\]

\[
= -iQ(\tau, 1) \frac{D_{\tau} \text{sgn}(Q_{\tau})}{Q(\tau, 1)} + 2Q_n \frac{D_{\tau} \text{sgn}(Q_{\tau})}{Q(\tau, 1)^2} Q'(\tau, 1) - \frac{iD_{\tau} \text{sgn}(Q_{\tau})}{Q(\tau, 1)} v^2
\]

\[
+ i v^2 \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; \frac{1}{2}, \frac{1}{2} \right) \frac{Q(\tau, 1)}{v^2}.
\]

Hence, by Lemma 2.1 and the first identity of Lemma 2.5

\[
R_2 \circ R_0 \circ R_{-2} \left( g_{n}^{[1]}(\tau) \right) = \frac{4D_{\tau} \text{sgn}(Q_{\tau})}{Q(\tau, 1)^2}.
\]

We can directly conclude the claim using Bol’s identity (2.3).

**Step 3: Application of Lemmas 5.3 and 5.4 and reducing to 2n + 2 derivatives**

Employing Lemma 5.3 and Lemma 5.4 with \(n \mapsto n + 1\) yields

\[
\frac{\partial^{2n+3}}{\partial \tau^{2n+3}} g_{n+1}^{[1]}(\tau) = \frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left( Q(\tau, 1)g_{n}^{[1]}(\tau) \right) \quad (5.2)
\]

By equation (5.1), we compute that

\[
\frac{\partial}{\partial \tau} \left( Q(\tau, 1)g_{n}^{[1]}(\tau) \right) = (n + 1)Q(\tau, 1)^n Q'(\tau, 1) \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) - iD_{\tau}^{n+\frac{1}{2}} v^{2n} \text{sgn}(Q_{\tau}).
\]

We observe that the final term gets annihilated by differentiating \(2n + 1\) times and thus

\[
\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left( Q(\tau, 1)g_{n}^{[1]}(\tau) \right) = \left( n + \frac{1}{2} \right) \frac{\partial^{2n+2}}{\partial \tau^{2n+2}} \left( Q(\tau, 1)^n Q'(\tau, 1) \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) - iQ(\tau, 1)^n Q'(\tau, 1) \right).
\]

**Step 4: Reducing to 2n + 1 derivatives**

By equation (5.1), we furthermore calculate that

\[
\frac{\partial}{\partial \tau} \left( Q(\tau, 1)^n Q'(\tau, 1) \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) \right) = nQ(\tau, 1)^{n-1} Q'(\tau, 1)^2 \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right)
\]

\[
+ Q(\tau, 1)^n Q''(\tau, 1) \beta \left( \frac{Dv^2}{|Q(\tau, 1)|^2}; n + \frac{1}{2}, \frac{1}{2} \right) - iQ(\tau, 1)^n Q'(\tau, 1) \frac{D_{\tau}^{n+\frac{3}{2}} v^{2n} \text{sgn}(Q_{\tau})}{|Q(\tau, 1)|^2 Q(\tau, 1)}.
\]

By the first identity of Lemma 5.5, the final term may be rewritten as

\[
-iQ(\tau, 1)^n Q'(\tau, 1) \frac{D_{\tau}^{n+\frac{3}{2}} v^{2n} \text{sgn}(Q_{\tau})}{|Q(\tau, 1)|^2 Q(\tau, 1)} = -\frac{D_{\tau}^{n+\frac{3}{2}} v^{2n-1} \text{sgn}(Q_{\tau})}{Q(\tau, 1)^n} + \frac{D_{\tau}^{n+\frac{3}{2}} v^{2n} |Q_{\tau}|}{Q(\tau, 1)^n Q(\tau, 1)}.
\]

Again the final term gets annihilated upon differentiating \(2n + 1\) times. Consequently, we obtain, by the second identity of Lemma 5.5

\[
\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left( Q(\tau, 1)g_{n}^{[1]}(\tau) \right) = \left( n + \frac{1}{2} \right) \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \left( Dn \frac{g_{n}^{[1]}(\tau)}{Q(\tau, 1)} + (2n + 1)Q''(\tau, 1)g_{n}^{[1]}(\tau) + Dg_{n}^{[2]}(\tau) \frac{Q_{\tau}}{Q(\tau, 1)} \right).
\]
Step 5: Application of the induction hypothesis
We use Lemma 5.3 with \( n \mapsto n - 1 \), to obtain
\[
\frac{g_n^{[1]}(\tau)}{Q(\tau, 1)} = \frac{n - \frac{1}{2}}{n} g_{n-1}^{[1]}(\tau) - \frac{g_n^{[2]}(\tau)}{n Q(\tau, 1)},
\]
and hence, using step 4,
\[
\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left( Q(\tau, 1)g_n^{[1]}(\tau) \right) = \left( n + \frac{1}{2} \right) \frac{\partial^{2n+1}}{\partial \tau^{2n+1}} \left( D \left( n - \frac{1}{2} \right) g_{n-1}^{[1]}(\tau) + (2n + 1)Q''(\tau, 1)g_n^{[1]}(\tau) \right).
\]
The induction hypothesis for \( n \) and \( n - 1 \), and the fact that \( Q''(\tau, 1) \) is independent of \( \tau \), then gives
\[
\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left( Q(\tau, 1)g_n^{[1]}(\tau) \right) = \frac{(-1)^n i D^{n+\frac{1}{2}} (n + \frac{1}{2}) (2n)! \text{sgn}(Q_{\tau})}{4^n} \left( \frac{1}{n} \frac{\partial^2}{\partial \tau^2} \frac{1}{Q(\tau, 1)^n} - (2n + 1) \frac{Q''(\tau, 1)}{Q(\tau, 1)^{n+1}} \right).
\]

Step 6: Simplifying the expressions
Using the second identity of Lemma 2.3, we compute
\[
\frac{1}{n} \frac{\partial^2}{\partial \tau^2} \frac{1}{Q(\tau, 1)^n} - (2n + 1) \frac{Q''(\tau, 1)}{Q(\tau, 1)^{n+1}} = \frac{D(n + 1)}{Q(\tau, 1)^{n+2}}.
\]
Inserting this into the result from step 5 yields
\[
\frac{n + \frac{1}{2}}{n + 1} \frac{\partial^{2n+3}}{\partial \tau^{2n+3}} \left( Q(\tau, 1)g_n^{[1]}(\tau) \right) = \frac{(-1)^n i \left( n + \frac{1}{2} \right) (n + 1) (2n)! D^{n+\frac{1}{2}} \text{sgn}(Q_{\tau})}{4^n Q(\tau, 1)^{n+2}}.
\]
By equation (5.2), we ultimately arrive at the claim of the lemma (with \( n \mapsto n + 1 \)). \( \square \)

We are now ready to prove Proposition 5.2.

Proof of Proposition 5.2
(1) By Lemma 2.3 and equation (5.1), we obtain
\[
\frac{\partial}{\partial \tau} \left( \frac{D^{v^2}}{|Q(\tau, 1)|^2} : k + \frac{1}{2} \right) = i D^{k+\frac{1}{2}} v^{2k} \text{sgn}(Q_{\tau}) |Q(\tau, 1)|^{2k} Q(\tau, 1).
\]
This implies the claim.
(2) Lemma 5.5 implies that
\[
\frac{1}{2} \frac{D^{2k+1}}{2^{2k+1}} (g_k^{[1]}(\tau)) = -\frac{D^{k+\frac{1}{2}} (2k)! \text{sgn}(Q_{\tau})}{(4\pi)^{2k+1}} Q(\tau, 1)^{k+1},
\]
from which we deduce the claim by (1.7).
(3) The claim follows directly from (2.2) along with part (1) and (1.7). \( \square \)

5.3. Further properties of \( \Psi_{-k,D} \) and the proof of Theorem 1.2. We prove the local behaviour of \( \Psi_{-k,D} \) first. Similar as in the proof of Proposition 4.2, we obtain.

Proposition 5.6. Let \( \tau \in E_D \).
(1) We have
\[
\lim_{\varepsilon \to 0^+} (\Psi_{-k,D}(\tau + i\varepsilon) - \Psi_{-k,D}(\tau - i\varepsilon)) = 0.
\]
(2) We have
\[
\frac{1}{2} \lim_{\varepsilon \to 0^+} (\Psi_{-k,D}(\tau + i\varepsilon) + \Psi_{-k,D}(\tau - i\varepsilon)) = \Psi_{-k,D}(\tau).
\]
We have
\[
\lim_{\varepsilon \to 0^+} \left( \frac{\partial}{\partial \tau} \Psi_{k,D}(\tau + i\varepsilon) - \frac{\partial}{\partial \tau} \Psi_{k,D}(\tau - i\varepsilon) \right) = iD^{k+\frac{1}{2}} e^{2k} \sum_{Q \in \mathbb{Q}, Q_\tau = 0} \frac{\text{sgn}(Q)}{Q(\tau, 1)^{k+1}}.
\]

Secondly, we require the constant from [4] (4.2), (7.3)] (using a different normalization)
\[
c_\infty := \frac{\pi D^{k+\frac{1}{2}}}{2^{2k}(2k+1)} \sum_{a \geq 1} \sum_{0 \leq b < 2a, b^2 \equiv D \pmod{4a}} \frac{1}{a^{k+1}},
\]
which can be evaluated using a result of Zagier [24, Proposition 3].

As a third ingredient, we have, for every \( \tau \in \mathbb{H} \setminus E_D \),
\[
\xi_{-2k} \left( \Lambda_{k+1,D}^* (\tau) \right) = \Lambda_{k+1,D}(\tau), \quad \mathbb{M}^{2k+1} \left( \Lambda_{k+1,D}^* (\tau) \right) = 0,
\]
\[
\xi_{-2k} \left( \mathcal{E}_{\Lambda_{k+1,D}} (\tau) \right) = 0, \quad \mathbb{M}^{2k+1} \left( \mathcal{E}_{\Lambda_{k+1,D}} (\tau) \right) = \Lambda_{k+1,D}(\tau).
\]
The third claim follows by holomorphicity of \( \mathcal{E}_{\Lambda_{k+1,D}} \), while the second claim holds as \( \Lambda_{k+1,D}^* \) (as a function of \( \tau \)) is a polynomial of degree at most \( 2k \) by [1,8]. The first and fourth claim follow by a standard calculation using the integral representations from [1,8] directly.

Now, we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.**

(2) We define
\[
f(\tau) := \Psi_{k,D}(\tau) + \frac{D^{k+\frac{1}{2}} (2k)!}{(4\pi)^{2k+1}} \mathcal{E}_{\Lambda_{k+1,D}} (\tau) - D^{k+\frac{1}{2}} \Lambda_{k+1,D}^* (\tau).
\]
Combining Proposition 5.2 with (5.4), we deduce that
\[
\xi_{-2k}(f) = \mathbb{M}^{2k+1}(f) = 0.
\]
Hence, \( f \) is a polynomial in \( \tau \) of degree at most \( 2k \). By Proposition 5.6 (1), \( \Psi_{k,D} \) has no jumps on \( E_D \). Thus, we may freely select an arbitrary connected component of \( \mathbb{H} \setminus E_D \) to compute \( f \).

Choosing the connected component of \( \mathbb{H} \setminus E_D \) containing \( i\infty \), we are in the same situation as in the induction start during the proof of [4, Theorem 7.1]. In other words, the function \( f \) is in fact constant, and this constant was computed in [4, Lemma 7.3]. We infer that \( f \) coincides with \( c_\infty \).

(1) We verify the four conditions in Definition 2.7:

(i) Modularity of weight \(-2k\) follows by Lemma 2.2 and equation 2.4.

(ii) Local harmonicity with respect to \( \Delta_{-2k} \) outside \( E_D \) is shown in Proposition 5.2 (3).

(iii) The required behaviour on \( E_D \) is given in Proposition 5.6 (2).

(iv) The function \( \Psi_{-k,D} \) is of at most polynomial growth towards \( i\infty \) by virtue of its splitting in Theorem 1.2 (2). Being more precise, \( \Lambda_{k+1,D} \) admits a Fourier expansion of the shape \( \sum_{n \geq 1} c(n) e^{2\pi i n \tau} \), where the Fourier coefficients \( c(n) \) depend on the connected component of \( \mathbb{H} \setminus E_D \) in which \( \tau \) is located. Letting \( \Gamma(s,y) \) denote the incomplete \( \Gamma \)-function, we obtain for \( v \gg 1 \)
\[
\mathcal{E}_{\Lambda_{k+1,D}} (\tau) = \sum_{n \geq 1} \frac{c(n)}{n^{2k+1}} e^{2\pi i n \tau}, \quad \Lambda_{k+1,D}(\tau) = \sum_{n \geq 1} \frac{c(n)}{(4\pi n)^{2k+1}} \Gamma(2k + 1, 4\pi n v) e^{-2\pi i n \tau}.
\]
We observe that the holomorphic Eichler integral vanishes as \( \tau \to i\infty \), and the same holds for the non-holomorphic Eichler integral due to [4, §8.11 (i)]. This proves that
\[
\lim_{\tau \to i\infty} \Psi_{-k,D}(\tau) = c_\infty.
\]

Proposition 5.6 (1) yields that the singularities of \( \Psi_{-k,D} \) on \( E_D \) are continuously removable. Combining Proposition 5.6 (3) with Lemmas 2.3 and 4.3 shows that \( \Psi_{-k,D} \) has no differentiable continuation to \( E_D \). This completes the proof. \( \square \)
6. Proof of Theorem 1.3

We finish this paper with the proof of Theorem 1.3.

**Proof of Theorem 1.3.** We follow [5, Sections 4,5] and shift $k \mapsto k + 1$ in the calculations there. The treatment of Borcherds regularization can be adapted from [5, Section 4] to our case straightforwardly. This implies that the integral over the unbounded region of $\mathcal{F}_T(4)$ vanishes as $T \to \infty$, while the truncated integral over the bounded region converges to the usual Petersson inner product as $T \to \infty$. We may use the usual unfolding argument and the computation of the integral over the real part both exactly as in [5, Section 4]. Combining, this yields

$$\sum_{-k} \left( \mathcal{F}_{\frac{1}{2}-k,D} \right) (\tau) = \frac{1}{6\Gamma \left( k + \frac{3}{2} \right) \left( 4\pi D \right)^{\frac{1}{2} + \frac{k}{2}}} \sum_{Q \in \mathbb{Q}_D} |Q_1| Q(\tau, 1)^k I \left( \frac{Dv^2}{|Q(\tau, 1)|^2} \right),$$

where (compare [5, (5.2)])

$$I(t) := \int_0^\infty \mathcal{M}_{\frac{1}{2} - k, \frac{1}{2} + \frac{k}{4}} (-x) e^{\frac{x}{2} - t} x^{-\frac{1}{2}} dx.$$

The evaluation is permitted, since $\tau \notin E_D$ gives $\frac{Dv^2}{|Q(\tau, 1)|^2} < 1$ by Lemma 2.1, so the series on the right-hand side converges. This can be seen directly after rewriting $I$ in the upcoming sentence, and comparing with Proposition 5.1. The integral $I$ can be evaluated mutans mutandis as in [5], giving

$$I(t) = k! \left( k + \frac{1}{2} \right) (1 - t)^{-\frac{1}{2} - \frac{k}{2}} \beta \left( t; k + \frac{1}{2}, \frac{1}{2} \right).$$

We conclude the theorem by Lemma 2.1. $\square$

**References**

[1] R. Boisvert, C. Clark, D. Lozier, and F. Olver, *NIST Handbook of Mathematical Functions*, Cambridge University Press (2010).

[2] R. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. 132 (1998), 491–562.

[3] K. Bringmann, A. Folsom, K. Ono, and L. Rolen, *Harmonic Maass forms and mock modular forms: theory and applications*, American Mathematical Society Colloquium Publications 64, American Mathematical Society, Providence, RI (2017).

[4] K. Bringmann, B. Kane, and W. Kohnen, *Locally harmonic Maass forms and the kernel of the Shintani lift*, Int. Math. Res. Not. (2015), 3185–3224.

[5] K. Bringmann, B. Kane, and M. Viazovska, *Theta lifts and local Maass forms*, Math. Res. Lett. 20 (2013), 213–234.

[6] K. Bringmann, B. Kane, and S. Zwegers, *On a completed generating function of locally harmonic Maass forms*, Compos. Math. 150 (2014), 749–762.

[7] J. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. 125 (2004), 45–90.

[8] J. Bruinier, K. Ono, and R. Rhoades, *Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues*, Math. Ann. 342 (2008), 673–693.

[9] W. Duke, O. Imamoğlu, and A. Tóth, *Cycle integrals of the j-function and mock modular forms*, Ann. of Math. (2) 173 (2011), 947–981.

[10] M. Eichler, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. 67 (1957), 267–298.

[11] M. Hövel, *Automorphe Formen mit Singularitäten auf dem hyperbolischen Raum*, Ph.D. Thesis, TU Darmstadt, (2012).

[12] M. Knopp, *Rational period functions of the modular group*, Duke Math. J. 45 (1978), 47–62.

[13] M. Knopp, *Modular integrals and their Mellin transforms*, (1900), 327–342, *Analytic number theory*, Allerton Park, IL, (1985), Progr. Math. 85, Birkhäuser Boston, Boston, MA.

[14] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. 271 (1985), 237–268.

[15] W. Kohnen and D. Zagier, *Values of L-series of modular forms at the center of the critical strip*, Invent. Math. 64 (1981), 175–198.

[16] W. Kohnen and D. Zagier, *Modular forms with rational periods*, *Modular forms*, Durham, (1983) Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, (1984), 197–249.

[17] A. Mono, *Locally harmonic Maass forms of positive even weight*, https://arxiv.org/abs/2104.03127.

[18] L. Parson, *Modular integrals and indefinite binary quadratic forms*, *A tribute to Emil Grosswald: number theory and related analysis*, Contemp. Math. 143, Amer. Math. Soc., Providence, RI (1993), 513–523.
[19] G. Shimura, *On modular forms of half integral weight*, Ann. of Math. (2) **97** (1973), 440–481.

[20] T. Shintani, *On construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.

[21] J. Stienstra, D. Zagier, *Bimodular forms and holomorphic anomaly equation*, In: Workshop on Modular Forms and String Duality, Banff International Research Station, 2006, https://www.birs.ca/workshops/2006/06w5041/report06w5041.pdf

[22] M.-F. Vignéras, *Séries thêta des formes quadratiques indéfinies*, Séminaire Delange-Pisot-Poitou, 17e année (1975/76), Théorie des nombres: Fasc. 1, Exp. No. 20, Secrétariat Math., Paris, (1977).

[23] D. Zagier, *Modular forms associated to real quadratic fields*, Invent. Math. **30** (1975), 1–46.

[24] D. Zagier, *Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields*, Modular functions of one variable, VI, Proc. Second Internat. Conf., Univ. Bonn, Bonn (1976), Springer, Berlin, (1977) 105–169. Lecture Notes in Math., Vol. 627.

[25] D. Zagier, *Zetafunktionen und quadratische Körper*, Hochschultext, Springer-Verlag, Berlin, 1981.

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