Entire slice regular functions

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Abstract. Entire functions in one complex variable are extremely relevant in several areas ranging from the study of convolution equations to special functions. An analog of entire functions in the quaternionic setting can be defined in the slice regular setting, a framework which includes polynomials and power series of the quaternionic variable. In the first chapters of this work we introduce and discuss the algebra and the analysis of slice regular functions. In addition to offering a self-contained introduction to the theory of slice-regular functions, these chapters also contain a few new results (for example we complete the discussion on lower bounds for slice regular functions initiated with the Ehrenpreis-Malgrange, by adding a brand new Cartan-type theorem). The core of the work is Chapter 5, where we study the growth of entire slice regular functions, and we show how such growth is related to the coefficients of the power series expansions that these functions have. It should be noted that the proofs we offer are not simple reconstructions of the holomorphic case. Indeed, the non-commutative setting creates a series of non-trivial problems. Also the counting of the zeros is not trivial because of the presence of spherical zeros which have infinite cardinality. We prove the analog of Jensen and Carathéodory theorems in this setting.
Chapter 1

Introduction

The theory of holomorphic functions in one complex variable assumes a particular flavor when one considers functions that are holomorphic on the entire plane, namely the entire functions. The reasons for the richness of results that are possible in that context are many, but certainly include the fact that for entire functions we can study growth phenomena in a cleaner way than what would be possible if one were to consider the issues introduced by the boundary of the domain of holomorphy associated to an individual function. While the beautiful issues connected with analytic continuation do not arise in the context of entire functions, the theory acquires much strength from the ability of connecting the growth of the functions to the coefficients that appear in their Taylor series.

The study of entire functions, in addition, has great relevance for the study of convolution equations, where different spaces of entire functions arise naturally from the application of the Paley Wiener theorem (in its various forms), and from the topological vector space approach that was so instrumental in the work of Schwarz, Malgrange, Ehrenpreis, Palamodov, Hörmander.

Further arguments in support of the study of entire functions are supplied by the consideration that not only the most important elementary functions are entire (polynomials, exponentials, trigonometric and hyperbolic functions), but also that many of the great special functions of analysis (the Jacobi theta function, the Weierstrass sigma function, the Weierstrass doubly periodic functions, among others) are entire. Finally, one knows that the solutions of linear differential equations with polynomial coefficients and with constant highest derivative coefficient are entire as well (this includes the elementary functions, as well as the Airy functions, for example).
While many texts exist, that are devoted exclusively to the study of entire functions (see for example [34, 125, 133]), one of the best ways to understand the importance and the beauty of entire functions is the article by de Branges [82].

One might therefore ask whether an analogous analysis can be done for functions defined on the space $\mathbb{H}$ of quaternions. As it is well known, there are several different definitions of holomorphicity (or regularity, as it is often referred to) for functions defined on quaternions. The most well known is probably the one due to Fueter, who expanded on the work of Moisil, to define regular functions those which satisfy a first order system of linear differential equations that generalizes the one of Cauchy-Riemann. These functions are often referred to as Fueter regular, and their theory is very well developed, see e.g. [69, 126]. It is therefore possible to consider those functions which are Fueter regular on the entire quaternionic space $\mathbb{H}$, and see whether there are important properties that can be deduced. However, the mathematician interested in this generalization would encounter immediately the problem that these functions do not admit a natural power series expansion. While this comment would need to be better qualified, it is the basic reason why most of what is known for holomorphic entire functions cannot be extended to the case of Fueter regular entire functions.

In the last ten years, however, many mathematicians have devoted significant attention to a different notion of regularity, known as slice regularity or slice hyperholomorphy. This theory began as a theory of functions from the space of quaternions $\mathbb{H}$ to itself whose restriction to any complex plane contained in $\mathbb{H}$ was in the kernel of the corresponding Cauchy-Riemann operator, see [109, 110]. It immediately appeared that this class of functions, at least on balls centered at the origin, coincides with the class of polynomials and convergent power series of the quaternionic variable, previously introduced in [83]. In particular, there are very natural generalizations of the exponential and trigonometric functions, that happen to be entire slice regular functions. Further studies showed that on suitable open sets called axially symmetric slice domains, this class of function coincides with the class of functions of the form $f(q) = f(x + Iy) = \alpha(x, y) + I\beta(x, y)$ when the quaternion $q$ is written in the form $x + Iy$ ($I$ being a suitable quaternion such that $I^2 = -1$) and the pair $(\alpha, \beta)$ satisfies the Cauchy-Riemann system and the conditions $\alpha(x, -y) = \alpha(x, y)$, $\beta(x, -y) = -\beta(x, y)$. This class of functions when $\alpha$ and $\beta$ are real quaternionic or, more in general, Clifford algebra valued is well known: they are the so-called holomorphic functions of a paravector variable, see [126, 149], which were later studied in the setting of real alternative algebras in [116].
But there are deeper reasons why these functions are a relevant subject of study. Maybe the most important point is to notice that slice regular functions, and their Clifford valued companions, the slice monogenic functions (see [72, 73]), have surprising applications in operator theory. In particular, one can use these functions (collectively referred to as slice hyperholomorphic functions) to define a completely new and very powerful slice hyperholomorphic functional calculus (which extends the Riesz-Dunford functional calculus to quaternionic operators and to \(n\)-tuples of non-commuting operators). These possible applications in operator theory gave great impulse to the theory of functions.

In particular, their Cauchy formulas, with slice hyperholomorphic kernels, are the basic tools to extend the Riesz-Dunford functional calculus to quaternionic operators and to \(n\)-tuples of not necessarily commuting operators. They also lead to the notion of \(S\)-spectrum which turned out to be the correct notion of spectrum for all applications in different areas of quaternionic functional analysis and in particular to the quaternionic spectral theorem.

The function theory of slice regular and slice monogenic functions was developed in a number of papers, see the list of references (note that some of the references are not explicitly quoted in the text) and the comments below. It has also been extended to vector-valued and more in general operator-valued functions, see [23] and the references therein. The monographs [23, 76, 108] are the main sources for slice hyperholomorphic functions and their applications.

It should be pointed out that the theory of slice hyperholomorphic functions is different from the more classical theory of functions in the kernel of the Dirac operator [55, 69, 84], the so-called monogenic functions. While the latter is a refinement of harmonic analysis in several variables, the former has many applications among which one of the most important is in quaternionic quantum mechanics, see [3, 32, 88, 90, 127]. Slice hyperholomorphic functions and monogenic functions can be related using the Fueter theorem, see [51, 56, 64] and the Radon transform, see [148] and the more recent [36].

To give a flavor of the several results available in the literature in the context of slice regular functions, with no claim of completeness, we gather the references in the various areas of research.

**Function theory.** The theory of slice regular functions was developed in the papers [47, 104, 106, 107, 109, 110, 112, 113], in particular, the zeros
were treated in [103, 105, 111] while further properties can be found in [33, 44, 85, 92, 93, 102, 140, 141, 150, 151, 152, 154]. Slice monogenic functions with values in a Clifford algebra and their main properties were studied in [58, 72, 73, 74, 75, 78, 81, 122, 155]. Approximation of slice hyperholomorphic functions are collected in the works [77, 94, 95, 96, 97, 98, 99, 100, 145]. The case of several variables was treated in [2, 79], but a lot of work has still to be done in this direction since the theory is at the beginning. The generalization of slice regularity to real alternative algebra was developed in the papers [26, 116, 117, 118, 119] and, finally, while some results associated with the global operator associated to slice hyperholomorphicity are in the papers [53, 70, 80, 120].

**Function spaces.** Several function spaces have been studied in this framework. In particular, the quaternionic Hardy spaces \( H^2(\Omega) \), where \( \Omega \) is the quaternionic unit ball \( \mathbb{B} \) or the half space \( \mathbb{H}^+ \) of quaternions with positive real part, together with the Blaschke products are in [14, 15, 16] and further properties are in [28, 29, 30]. The Hardy spaces \( H^p(\mathbb{B}) \), \( p > 2 \), are considered in [144]. The Bergman spaces can be found in [50, 52, 54] and the Fock space in [25]. Weighted Bergman spaces, Bloch, Besov, and Dirichlet spaces on the unit ball \( \mathbb{B} \) are considered in [38]. Inner product spaces and Krein spaces in the quaternionic setting, are in [21].

**Groups and semigroups of operators.** The theory of groups and semigroups of quaternionic operators has been developed and studied in the papers [22, 62, 124].

**Functional calculi.** There exists at least five functional calculi associated to slice hyperholomorphicity. For each of it we have the quaternionic version and the version for \( n \)-tuples of operators. The \( S \)-functional calculus, see [9, 45, 48, 49, 59, 60, 61, 71], is the analogue of the Riesz-Dunford functional calculus in the quaternionic setting. Further developments are in [40, 42]. The \( SC \)-functional calculus, see [85], is the commutative version of the \( S \)-functional calculus. For the functional calculus for groups of quaternionic operators based on the Laplace-Stieltjes transform, see [8]. The \( H^\infty \) functional calculus based on the \( S \)-spectrum, see [24], is the analogue, in this setting, of the calculus introduced by A. McIntosh, see [137]. The F-functional calculus, see [15, 41, 63, 67, 68], which is based on the Fueter mapping theorem in integral form, is a monogenic functional calculus in the spirit of the one developed in [128, 129, 130, 138, 134], but it is associated to slice hyperholomorphicity. Finally the W-functional calculus, see [57], is
a monogenic plane wave calculus based on slice hyperholomorphic functions.

*Spectral theory.* The spectral theorem based on the $S$-spectrum for bounded and for unbounded quaternionic normal operators on a Hilbert space was developed in [10, 11, 115]. The case of quaternionic normal matrices was proved in [89] and it is based on the right spectrum, but the right spectrum is equal to the $S$-spectrum in the finite dimensional case. The Continuous slice functional calculus in quaternionic Hilbert spaces is studied in [114].

*Schur Analysis.* This is a very wide field that has been developed in the last five years in the slice hyperholomorphic setting. Schur analysis originates with the works of Schur, Herglotz, and others and can be seen as a collection of topics pertaining to Schur functions and their generalizations; for a quick introduction in the classical case see for example [5]. For the slice hyperholomorphic case see [1, 2, 6, 7, 12, 13, 14, 15, 16, 17, 19, 20] and also the forthcoming monograph [23].

Since the literature in the field is so vast, it became quite natural to ask whether the deep theory of holomorphic entire functions can be reconstructed for slice regular functions. As this book will demonstrate, the answer is positive, and our contribution here consists in showing the way to a complete theory of entire slice regular functions. It is probably safe to assert that this monograph is only the first step, and in fact we only have chosen a fairly limited subset of the general theory of homomorphic entire functions, to demonstrate the feasibility of our project. We expect to return to other important topics in a subsequent volume.

This monographs contains four chapters, beside this introduction. In Chapter 2 and 3 we introduce and discuss the algebra and the analysis of slice regular functions. While most of the results in those chapters are well known, and can be found in the literature, see e.g. [23, 76, 108], we repeated them to make the monograph self-contained. However there are a few new observations (e.g. in section 1.4 where we tackle the composition of slice regular functions and also the Riemann mapping theorem) that do not appear in the aforementioned monographs. There are also a few new results (for example we complete the discussion on lower bounds for slice regular functions initiated with the Ehrenpreis-Malgrange in section 3.4, by adding a brand new Cartan-type theorem in section 3.5).

Chapter 4 deals with infinite products of slice regular functions. The results in this chapter are known, but at least the Weierstrass theorem receives
here a treatment that is different from the one originally given in [113], see also [108]. This treatment leads also to the definition of genus of a a canonical product. The core of the work, however, is Chapter 5, where we study the growth of entire slice regular functions, and we show how such growth is related to the coefficients of the power series expansions that these functions have. This chapter contains new results, the only exception is section 5.5 which is taken from [99]. It should be noted that the proofs we offer are not simple reconstructions (or translations) of the holomorphic case. Indeed, the non-commutative setting creates a series of non-trivial problems, that force us to define composition and multiplication in ways that are not conducive to a simple repetition of the complex case. Also the counting of the zeros is not trivial because of the presence of spherical zeros which have infinite cardinality.

We believe that much work still needs to be done in this direction, and we hope that our monograph will inspire others to turn their attention to this nascent, and already so rich, new field of noncommutative analysis.

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Chapter 2

Slice regular functions: algebra

2.1 Definition and main results

In this chapter we will present some basic material on slice regular functions, a generalization of holomorphic functions to the quaternions. The skew field of quaternions $\mathbb{H}$ is defined as

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k; \ x_0, \ldots, x_3 \in \mathbb{R}\},$$

where the imaginary units $i, j, k$ satisfy

$$i^2 = j^2 = k^2 = -1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$  

It is a noncommutative field and since $\mathbb{C}$ can be identified (in a non unique way) with a subfield of $\mathbb{H}$, it extends the class of complex numbers. On $\mathbb{H}$ we define the Euclidean norm $|q| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

The symbol $\mathbb{S}$ denotes the unit sphere of purely imaginary quaternions, i.e.

$$\mathbb{S} = \{q = ix_1 + jx_2 + kx_3, \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}.$$  

Note that if $I \in \mathbb{S}$, then $I^2 = -1$. For this reason the elements of $\mathbb{S}$ are also called imaginary units. For any fixed $I \in \mathbb{S}$ we define $\mathbb{C}_I := \{x + Iy; \ x, y \in \mathbb{R}\}$. It is easy to verify that $\mathbb{C}_I$ can be identified with a complex plane, moreover $\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I$. The real axis belongs to $\mathbb{C}_I$ for every $I \in \mathbb{S}$ and thus a real quaternion can be associated with any imaginary unit $I$. Any non real quaternion $q$ is uniquely associated to the element $I_q \in \mathbb{S}$ defined by $I_q := (ix_1 + jx_2 + kx_3)/|ix_1 + jx_2 + kx_3|$. It is obvious that $q$ belongs to $\text{(continued on next page)}$
the complex plane $\mathbb{C}_I$.

**Definition 2.1.1.** Let $U$ be an open set in $\mathbb{H}$ and $f : U \to \mathbb{H}$ be real differentiable. The function $f$ is said to be (left) slice regular or (left) slice hyperholomorphic if for every $I \in \mathbb{S}$, its restriction $f_I$ to the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ passing through origin and containing $I$ and $1$ satisfies

$$
\bar{\partial}_I f(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) = 0,
$$
on $U \cap \mathbb{C}_I$. The class of (left) slice regular functions on $U$ will be denoted by $\mathcal{R}(U)$.

Analogously, a function is said to be right slice regular in $U$ if

$$
(f_I \partial_I)(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I \right) = 0,
$$
on $U \cap \mathbb{C}_I$.

It is immediate to verify that:

**Proposition 2.1.2.** Let $U$ be an open set in $\mathbb{H}$. Then $\mathcal{R}(U)$ is a right linear space on $\mathbb{H}$.

Let $f \in \mathcal{R}(U)$. The so called left (slice) $I$-derivative of $f$ at a point $q = x + Iy$ is given by

$$
\partial_I f_I(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) - I \frac{\partial}{\partial y} f_I(x + Iy) \right).
$$

In this case, the right $I$-derivative of $f$ at $q = x + Iy$ is given by

$$
\partial_I f_I(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) - \frac{\partial}{\partial y} f_I(x + Iy)I \right).
$$

Let us now introduce a suitable notion of derivative:

**Definition 2.1.3.** Let $U$ be an open set in $\mathbb{H}$, and let $f : U \to \mathbb{H}$ be a slice regular function. The slice derivative $\partial_s f$ of $f$, is defined by:

$$
\partial_s(f)(q) = \begin{cases} 
\partial_I(f)(q) & \text{if } q = x + Iy, \ y \neq 0, \\
\frac{\partial f}{\partial x}(x) & \text{if } q = x \in \mathbb{R}.
\end{cases}
$$
The definition of slice derivative is well posed because it is applied only to slice regular functions, thus
\[
\frac{\partial}{\partial x} f(x + Iy) = -I \frac{\partial}{\partial y} f(x + Iy) \quad \forall I \in S.
\]
Similarly to what happens in the complex case, we have
\[
\partial_s(f)(x + Iy) = \partial_I(f)(x + Iy) = \partial_x(f)(x + Iy).
\]
We will often write \(f'(q)\) instead of \(\partial_s f(q)\).

It is important to note that if \(f(q)\) is a slice regular function then also \(f'(q)\) is a slice regular function.

Let \(I, J \in S\) be such that \(I\) and \(J\) are orthogonal, so that \(I, J, IJ\) is an orthogonal basis of \(\mathbb{H}\) and write the restriction \(f_I(x + Iy) = f(x + Iy)\) of \(f\) to the complex plane \(C_I\) as \(f = f_0 + If_1 + Jf_2 + Kf_3\). It can also be written as \(f = F + GJ\) where \(f_0 + If_1 = F\), and \(f_2 + If_3 = G\). This observation immediately gives the following result:

**Lemma 2.1.4** (Splitting Lemma). If \(f\) is a slice regular function on \(U\), then for every \(I \in S\), and every \(J \in S\), perpendicular to \(I\), there are two holomorphic functions \(F, G: U \cap C_I \to C_I\) such that for any \(z = x + Iy\), it is
\[
f_I(z) = F(z) + G(z)J.
\]

**Proof.** Since \(f\) is slice regular, we know that
\[
(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y}) f_I(x + Iy) = 0.
\]
Therefore by decomposing the values of \(f_I\) into complex components
\[
f_I(x + Iy) = F(x + Iy) + G(x + Iy)J,
\]
the statement immediately follows.

In this work we will be interested in the case in which slice regular functions are considered on open balls \(B(0; r)\) centered at the origin with radius \(r > 0\). We have:

**Theorem 2.1.5.** A function \(f: B(0; r) \to \mathbb{H}\) slice regular on \(B(0; r)\) has a series representation of the form
\[
f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \cdot \frac{\partial^n f}{\partial x^n}(0) = \sum_{n=0}^{\infty} q^n a_n, \tag{2.1}
\]
uniformly convergent on \(B(0; r)\). Moreover, \(f \in C^\infty(B(0; R))\).
Proof. Let us use the Splitting Lemma to write $f_I(z) = F(z) + G(z)J$, where $z = x + Iy$. Then

$$f_I^{(n)}(z) = \partial_I^{(n)} f(z) = \frac{\partial^n}{\partial z^n} F(z) + \frac{\partial^n}{\partial z^n} G(z)J.$$ 

Now we use the fact that $F$, and $G$ admits converging power series expansions which converge uniformly and absolutely on any compact set in $B(0;r) \cap \mathbb{C}_I$:

$$f_I(z) = \sum_{n \geq 0} z^n \frac{1}{n!} \frac{\partial^n F}{\partial z^n}(0) + \sum_{n \geq 0} z^n \frac{1}{n!} \frac{\partial^n G}{\partial z^n}(0)J \quad \text{=} \quad \sum_{n \geq 0} z^n \frac{1}{n!} \left( \frac{\partial^n f}{\partial z^n}(0) \right)$$

$$= \sum_{n \geq 0} z^n \frac{1}{n!} \left( \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) \right)^n f(0) = \sum_{n \geq 0} z^n \frac{1}{n!} f^{(n)}(0).$$

The function $f$ is infinitely differentiable in $B(0;r)$ by the uniform convergence on the compact subsets. \qed

The proof of the following result is the same as the proof in the complex case.

**Theorem 2.1.6.** Let $\{a_n\}, \ n \in \mathbb{N}$ be a sequence of quaternions and let

$$r = \lim_{n \to \infty} \frac{1}{|a_n|^{1/n}}.$$ 

If $r > 0$ then the power series $\sum_{n=0}^{\infty} q^n a_n$ converges absolutely and uniformly on compact subsets of $B(0;r)$. Its sum defines a slice regular function on $B(0;r)$.

**Definition 2.1.7.** A function slice regular on $\mathbb{H}$ will be called entire slice regular or entire slice hyperholomorphic.

Every entire regular function admits power series expansion of the form (2.1) which converges everywhere in $\mathbb{H}$ and uniformly on the compact subsets of $\mathbb{H}$.

A simple computation shows that since the radius of convergence is infinite we have

$$\lim_{n \to \infty} \frac{1}{|a_n|^{1/n}} = 0,$$

or, equivalently:

$$\lim_{n \to \infty} \frac{\log |a_n|}{n} = -\infty$$

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2.2 The Representation Formula

Slice regular functions possess good properties on specific open sets that we will call axially symmetric slice domains. On these domains, slice regular functions satisfy the so-called Representation Formula which allows to reconstruct the values of the function once that we know its values on a complex plane $\mathbb{C}_I$. As we shall see, this will allow also to define a suitable notion of multiplication between two slice regular functions.

**Definition 2.2.1.** Let $U \subseteq \mathbb{H}$. We say that $U$ is axially symmetric if, for every $x + Iy \in U$, all the elements $x + S y = \{x + Jy \mid J \in S\}$ are contained in $U$. We say that $U$ is a slice domain (or s-domain for short) if it is a connected set whose intersection with every complex plane $\mathbb{C}_I$ is connected.

**Definition 2.2.2.** Given a quaternion $q = x + Iy$, the set of all the elements of the form $x + Jy$ where $J$ varies in the sphere $S$ is a 2-dimensional sphere denoted by $[x + Iy]$.

The Splitting Lemma allows to prove:

**Theorem 2.2.3** (Identity Principle). Let $f : U \rightarrow \mathbb{H}$ be a slice regular function on an s-domain $U$. Denote by $Z_f = \{q \in U : f(q) = 0\}$ the zero set of $f$. If there exists $I \in S$ such that $\mathbb{C}_I \cap Z_f$ has an accumulation point, then $f \equiv 0$ on $U$.

**Proof.** The restriction $f_I = F + GJ$ of $f$ to $U \cap \mathbb{C}_I$ is such that $F, G : U \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ are holomorphic functions. Under the hypotheses, $F$ and $G$ vanish on a set $\mathbb{C}_I \cap Z_f$ which has an accumulation point so $F$ and $G$ are both identically zero. So $f_I$ vanishes on $U \cap \mathbb{R}$ and, in particular, $f$ vanishes on $U \cap \mathbb{R}$. Thus the restriction of $f$ to any other complex plane $\mathbb{C}_L$ vanishes on a set with an accumulation point and so $f_L \equiv 0$. Since

$$U = \bigcup_{I \in \mathbb{C}_I} U \cap \mathbb{C}_I$$

we have that $f$ vanishes on $U$. \qed

The following result shows that the values of a slice regular function defined on an axially symmetric s-domain can be computed if the values of the restriction to a complex plane are known:
Theorem 2.2.4 (Representation Formula). Let $f$ be a slice regular function
defined on an axially symmetric $s$-domain $U \subseteq \mathbb{H}$. Let $J \in \mathcal{S}$ and let $x \pm Jy \in U \cap \mathbb{C}_J$. Then the following equality holds for all $q = x + Iy \in U$:

$$
f(x + Iy) = \frac{1}{2} [f(x + Jy) + f(x - Jy)] + I \frac{1}{2} [J[f(x - Jy) - f(x + Jy)]]
= \frac{1}{2}(1 - IJ)f(x + Jy) + \frac{1}{2}(1 + IJ)f(x - Jy).
$$

(2.2)

Moreover, for all $x + Ky \subseteq U$, $K \in \mathcal{S}$, there exist two functions $\alpha, \beta$, independent of $I$, such for any $K \in \mathcal{S}$ we have

$$
\frac{1}{2} [f(x + yK) + f(x - yK)] = \alpha(x, y), \quad \frac{1}{2} [K[f(x - yK) - f(x + yK)]] = \beta(x, y).
$$

(2.3)

Proof. If $\text{Im}(q) = 0$, then $q$ is real, the proof is immediate. Otherwise let us define the function $\psi : U \to \mathbb{H}$ as follows

$$
\psi(q) = \frac{1}{2} [f(\text{Re}(q) + |\text{Im}(q)|J) + f(\text{Re}(q) - |\text{Im}(q)|J)]
+ \frac{\text{Im}(q)}{|\text{Im}(q)|} J[f(\text{Re}(q) - |\text{Im}(q)|J) - f(\text{Re}(q) + |\text{Im}(q)|J)].
$$

Using the fact that $q = x + yI$, $x, y \in \mathbb{R}$, $y \geq 0$ and $I = \frac{\text{Im}(q)}{|\text{Im}(q)|}$ we obtain

$$
\psi(x + yI) = \frac{1}{2} [f(x + yJ) + f(x - yJ) + IJ[f(x - yJ) - f(x + yJ)]].
$$

Observe that for $I = J$ we have

$$
\psi_J(q) = \psi(x + yJ) = f(x + yJ) = f_J(q).
$$

Thus if we prove that $\psi$ is regular on $U$, the first part of the assertion follows from the Identity Principle. Since $f$ is regular on $U$, for any $I \in \mathcal{S}$ we have, on $U \cap \mathbb{C}_I$

$$
\frac{\partial}{\partial x} 2\psi(x + yI) = \frac{\partial}{\partial x} [f(x + yJ) + f(x - yJ) + IJ[f(x - yJ) - f(x + yJ)]]
= \frac{\partial}{\partial x} f(x + yJ) + \frac{\partial}{\partial x} f(x - yJ) + IJ[\frac{\partial}{\partial x} f(x - yJ) - \frac{\partial}{\partial x} f(x + yJ)]
$$
\[
-J \frac{\partial}{\partial y} f(x + yJ) + J \frac{\partial}{\partial y} f(x - yJ) + IJ \frac{\partial}{\partial y} f(x - yJ) + J \frac{\partial}{\partial y} f(x + yJ)
\]

\[
= -J \frac{\partial}{\partial y} f(x + yJ) + J \frac{\partial}{\partial y} f(x - yJ) - I \frac{\partial}{\partial y} f(x - yJ) + \frac{\partial}{\partial y} f(x + yJ)
\]

\[
= -I \frac{\partial}{\partial y} \left[ f(x+yJ) + f(x-yJ) + IJ [f(x-yJ) - f(x+yJ)] \right] = -I \frac{\partial}{\partial y} 2\psi(x+yI)
\]

i.e.

\[
\frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) \psi(x + yI) = 0.
\] (2.4)

To prove (2.3) we take any \( K \in \mathbb{S} \) and use equation (2.2) to show that

\[
\frac{1}{2} \left[ f(x + yK) + f(x - yK) \right]
\]

\[
= \frac{1}{2} \left\{ \frac{1}{2} \left[ f(x + yJ) + f(x - yJ) \right] + K \frac{1}{2} \left[ J[f(x - yJ) - f(x + yJ)] \right] \right\}
\]

\[
+ \frac{1}{2} \left[ f(x + yJ) + f(x - yJ) \right] - K \frac{1}{2} \left[ J[f(x - yJ) - f(x + yJ)] \right] \right\}
\]

\[
= \frac{1}{2} \left[ f(x + yJ) + f(x - yJ) \right] = \alpha(x, y)
\]

and that

\[
\frac{1}{2} \left[ K [f(x - yK) - f(x + yK)] \right]
\]

\[
= \frac{1}{2} K \left\{ \frac{1}{2} \left[ f(x + yJ) + f(x - yJ) \right] - K \frac{1}{2} \left[ J[f(x - yJ) - f(x + yJ)] \right] \right\}
\]

\[
- \frac{1}{2} \left[ f(x + yJ) + f(x - yJ) \right] - K \frac{1}{2} \left[ J[f(x - yJ) - f(x + yJ)] \right] \right\}
\]

\[
= \frac{1}{2} K \left[ - K [f(x - yJ) - f(x + yJ)] \right]
\]

\[
= \frac{1}{2} \left[ J[f(x - yJ) - f(x + yJ)] \right] = \beta(x, y),
\]

and the proof is complete. \( \square \)

Some immediate consequences are the following corollaries:

**Corollary 2.2.5.** Let \( U \subseteq \mathbb{H} \) be an axially symmetric s-domain, let \( D \subseteq \mathbb{R}^2 \) be such that \( x + Iy \in U \) whenever \((x,y) \in D\) and let \( f : U \to \mathbb{H} \). The function \( f \) is slice regular if and only if there exist two differentiable functions \( \alpha, \beta : D \subseteq \mathbb{R}^2 \to \mathbb{H} \) satisfying \( \alpha(x, y) = \alpha(x, -y) \), \( \beta(x, y) = -\beta(x, -y) \) and the Cauchy-Riemann system

\[
\begin{align*}
\partial_x \alpha - \partial_y \beta &= 0 \\
\partial_x \beta + \partial_y \alpha &= 0
\end{align*}
\] (2.5)
such that
\[ f(x + Iy) = \alpha(x, y) + I\beta(x, y). \] (2.6)

Proof. A function of the form (2.6) where \( \alpha \) and \( \beta \) satisfy the hypothesis in the statement is clearly slice regular. Conversely, a slice regular function on an axially symmetric s-domain satisfies the Representation formula and thus it is of the form (2.6), where \( \alpha \) and \( \beta \) satisfy the Cauchy-Riemann system. The conditions \( \alpha(x, y) = \alpha(x, -y) \), \( \beta(x, y) = -\beta(x, -y) \) can be easily verified from the definition of \( \alpha \) and \( \beta \) given in the Representation Formula.

Remark 2.2.6. Since \( \alpha, \beta \) are quaternion-valued functions for any given \( I \in S \) we select \( J \in S \) orthogonal to \( I \) and we can write (omitting, for simplicity, the arguments of the functions):

\[ f_I = \alpha + I\beta = (a_0 + Ia_1 + Ja_2 + IJa_3) + I(b_0 + Ib_1 + Jb_2 + IJb_3) \]
\[ = (a_0 - b_1) + I(a_1 + b_0) + J(a_2 - b_3) + IJ(a_3 + b_2) \]
\[ = c_0 + Ic_1 + Jc_2 + IJc_3 = (c_0 + Ic_1) + (c_2 + Ic_3)J, \]

where the functions \( a_\ell, b_\ell, c_\ell, \ell = 0, \ldots, 3 \) are real-valued.

Imposing \( (\partial_x + I\partial_y)f_I = 0 \) we obtain the equations

\[ \begin{align*}
\partial_x c_0 - \partial_y c_1 &= 0 \\
\partial_y c_0 + \partial_x c_1 &= 0 \\
\partial_x c_2 - \partial_y c_3 &= 0 \\
\partial_y c_2 + \partial_x c_3 &= 0
\end{align*} \] (2.7)

Requiring that the pair \( \alpha, \beta \) satisfies the Cauchy-Riemann system, we obtain eight real equations:

\[ \partial_x a_i - \partial_y b_i = 0 \quad \partial_x b_i + \partial_y a_i = 0 \quad i = 0, \ldots, 3. \] (2.8)

System (2.7) is equivalent to the request that the functions \( F = c_0 + Ic_1 \) and \( G = c_2 + Ic_3 \) prescribed by the Splitting Lemma are holomorphic. As it can be easily verified, by setting \( c_0 = a_0 - b_1 \), \( c_1 = a_1 + b_0 \), \( c_2 = a_2 - b_3 \), \( c_3 = a_3 + b_2 \) the solutions to system (2.8) give solutions to system (2.7).

The previous remark implies a stronger version of the Splitting lemma for slice regular functions. To prove the result we need to recall that complex functions defined on open sets \( G \subset \mathbb{C} \) symmetric with respect to the real axis and such that \( \overline{f(z)} = f(z) \) are called in the literature intrinsic, see [142].
Proposition 2.2.7 (Refined Splitting Lemma). Let $U$ be an open set in $\mathbb{H}$ and let $f \in \mathcal{R}(U)$. For any $I \in \mathbb{S}$ there exist four holomorphic intrinsic functions $h_\ell : U \cap \mathbb{C}_I \to \mathbb{C}_I$, $\ell = 0, \ldots, 3$ such that

$$f_I(x + Iy) = h_0(x + Iy) + h_1(x + Iy)I + h_2(x + Iy)J + h_3(x + Iy)K.$$ 

Proof. Let us write (omitting the argument $x + Iy$ of the various functions):

$$f_I = a_0 + Ia_1 + Ja_2 + Ka_3 + I(b_0 + Ib_1 + Jb_2 + Kb_3)$$

$$= (a_0 + Ib_0) + (a_1 + Ib_1)I + (a_2 + Ib_2)J + (a_3 + Ib_3)K$$

$$= h_0 + h_1I + h_2J + h_3K.$$ 

It follows from system (2.8) that $h_\ell$, $\ell = 0, \ldots, 3$ satisfy the Cauchy-Riemann system. The fact that the functions $\alpha(x, y)$ and $\beta(x, y)$ defined in the Splitting Lemma are even and odd, respectively, in the variable $y$ implies that $a_\ell(x, y)$ and $b_\ell(x, y)$ are even and odd, respectively, in the variable $y$. Thus

$$h_\ell(x - Iy) = a_\ell(x, -y) + Ib_\ell(x, -y) = a_\ell(x, y) - Ib_\ell(x, y) = \overline{h_\ell(x + Iy)}$$

and so the functions $h_\ell$, $\ell = 0, 1, 2, 3$ are complex intrinsic. \qed

Corollary 2.2.8. A slice regular function $f : U \to \mathbb{H}$ on an axially symmetric $s$-domain is infinitely differentiable on $U$. It is also real analytic on $U$.

Corollary 2.2.9. Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and let $f : U \to \mathbb{H}$ be a slice regular function. For all $x_0, y_0 \in \mathbb{R}$ such that $x_0 + Iy_0 \in U$ there exist $a, b \in \mathbb{H}$ such that

$$f(x_0 + Iy_0) = a + Ib \quad (2.9)$$

for all $I \in \mathbb{S}$. In particular, $f$ is affine in $I \in \mathbb{S}$ on each 2-sphere $[x_0 + Iy_0]$ and the image of the 2–sphere $[x_0 + Iy_0]$ is the set $[a + Ib]$.

Corollary 2.2.10. Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and let $f : U \to \mathbb{H}$ be a slice regular function. If $f(x + Jy) = f(x + Ky)$ for $I \neq K$ in $\mathbb{S}$, then $f$ is constant on $[x + Iy]$. In particular, if $f(x + Jy) = f(x + Ky) = 0$ for $I \neq K$ in $\mathbb{S}$, then $f$ vanishes on the whole 2–sphere $[x + Iy]$. 19
Corollary 2.2.11. Let $U_J$ be a domain in $\mathbb{C}_J$ symmetric with respect to the real axis and such that $U_J \cap \mathbb{R} \neq \emptyset$. Let $U$ be the axially symmetric $s$-domain defined by

$$U = \bigcup_{x+Jy \in U_J, \, I \in S} \{x + Iy\}.$$ 

If $f : U_J \to \mathbb{H}$ satisfies $\overline{\partial}_J f = 0$ then the function

$$\text{ext}(f)(x + Iy) = \frac{1}{2} \left[ f(x + Jy) + f(x - Jy) \right] + I \frac{1}{2} \left[ J[f(x - Jy) - f(x + Jy)] \right]$$

is the unique slice regular extension of $f$ to $U$.

Definition 2.2.12. Let $U_J$ be any open set in $\mathbb{C}_J$ and let

$$U = \bigcup_{x+Jy \in U_J, \, I \in S} \{x + Iy\}. \quad (2.11)$$

We say that $U$ is the axially symmetric completion of $U_J$ in $\mathbb{H}$.

Corollary 2.2.5 implies that slice regular functions are a subclass of the following set of functions:

Definition 2.2.13. Let $U \subseteq \mathbb{H}$ be an axially symmetric open set. Functions of the form $f(q) = f(x + Iy) = \alpha(x,y) + I\beta(x,y)$, where $\alpha, \beta$ are continuous $\mathbb{H}$-valued functions such that $\alpha(x,y) = \alpha(x,-y)$, $\beta(x,y) = -\beta(x,-y)$ for all $x + Iy \in U$ are called continuous slice functions.

Let $U \subseteq \mathbb{H}$ be an axially symmetric open set.

Theorem 2.2.14 (General Representation Formula). Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and $f : U \to \mathbb{H}$ be a left slice regular function. The following equality holds for all $q = x + Iy \in U$, $J, K \in S$:

$$f(x + Iy) = (J - K)^{-1} \left[ Jf(x + Jy) - Kf(x + Ky) \right] + I(J - K)^{-1} \left[ f(x + Jy) - f(x + Ky) \right].$$

Proof. If $q$ is real the proof is immediate. Otherwise, for all $q = x + yI$, we define the function

$$\phi(x + yI) = (J - K)^{-1} \left[ Jf(x + yJ) - Kf(x + yK) \right] + I(J - K)^{-1} \left[ f(x + yJ) - f(x + yK) \right]$$

$$= [(J - K)^{-1}J + I(J - K)^{-1}]f(x + yJ) - [(J - K)^{-1}K + I(J - K)^{-1}]f(x + yK).$$
As we said, for all \( q = x \in U \cap \mathbb{R} \) we have
\[
\phi(x) = f(x).
\]

Therefore if we prove that \( \phi \) is slice regular on \( U \), the first part of the assertion will follow from the identity principle for slice regular functions. Indeed, since \( f \) is slice regular on \( U \), for any \( L \in \mathbb{S} \) we have \( \frac{\partial}{\partial y} f(x + Ly) = -L \frac{\partial}{\partial y} f(x) \) on \( U \cap \mathbb{C}_L \); hence
\[
\frac{\partial \phi}{\partial x}(x + yI) = -[\dotProduct{(J - K)^{-1}J} + \dotProduct{I(J - K)^{-1}}] \frac{\partial f}{\partial y}(x + yJ) \\
+ \dotProduct{[(J - K)^{-1}K + \dotProduct{I(J - K)^{-1}}]K \frac{\partial f}{\partial y}(x + yK)}
\]
and also
\[
I \frac{\partial \phi}{\partial y}(x + yI) = I[\dotProduct{(J - K)^{-1}J} + \dotProduct{I(J - K)^{-1}}] \frac{\partial f}{\partial y}(x + yJ) \\
- I[\dotProduct{(J - K)^{-1}K + \dotProduct{I(J - K)^{-1}}}] \frac{\partial f}{\partial y}(x + yK).
\]

It is at this point immediate that
\[
\left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) \phi(x + yI) = 0
\]
and equality (2.12) is proved.

\[ \square \]

### 2.3 Multiplication of slice regular functions

In the case of slice regular functions defined on axially symmetric s-domains we can define a suitable product, called \( \ast \)-product, which preserves slice regularity. This product extends the very well know product for polynomials and series with coefficients in a ring, see e.g. [91] and [132]. The inverse of a function with respect to the \( \ast \)-product requires to introduce the so-called conjugate and symmetrization of a slice regular function. These two notions, as we shall see, will be important also for other purposes.

Let \( U \subseteq \mathbb{H} \) be an axially symmetric s-domain and let \( f, g : U \rightarrow \mathbb{H} \) be slice regular functions. For any \( I, J \in \mathbb{S} \), with \( I \perp J \), the Splitting Lemma guarantees the existence of four holomorphic functions \( F, G, H, K : U \cap \mathbb{C}_I \rightarrow \mathbb{C}_I \) such that for all \( z = x + Iy \in U \cap \mathbb{C}_I \)
\[
f_I(z) = F(z) + G(z)J, \quad g_I(z) = H(z) + K(z)J.
\]
We define the function \( f_I \ast g_I : U \cap \mathbb{C}_I \rightarrow \mathbb{H} \) as

\[
 f_I \ast g_I(z) = [F(z)H(z) - G(z)\overline{K(z)}] + [F(z)K(z) + G(z)\overline{H(z)}]J. \quad (2.13)
\]

Then \( f_I \ast g_I(z) \) is obviously a holomorphic map and hence its unique slice regular extension to \( U \) defined, according to the extension formula (2.10) by

\[
 \text{ext}(f_I \ast g_I)(q),
\]

is slice regular on \( U \).

**Definition 2.3.1.** Let \( U \subseteq \mathbb{H} \) be an axially symmetric s-domain and let \( f, g : U \rightarrow \mathbb{H} \) be slice regular. The function

\[
 (f \ast g)(q) = \text{ext}(f_I \ast g_I)(q)
\]

defined as the extension of (2.13) is called the slice regular product of \( f \) and \( g \). This product is called \( \ast \)-product or slice regular product.

**Remark 2.3.2.** It is immediate to verify that the \( \ast \)-product is associative, distributive but, in general, not commutative.

**Remark 2.3.3.** Let \( H(z) \) be a holomorphic function in the variable \( z \in \mathbb{C}_I \) and let \( J \in S \) be orthogonal to \( I \). Then by the definition of \( \ast \)-product we obtain

\[
 J \ast H(z) = \overline{H(z)}J.
\]

Using the notations above we have:

**Definition 2.3.4.** Let \( f \in \mathcal{R}(U) \) and let \( f_I(z) = F(z)G(z)J \). We define the function \( f_f^c : U \cap \mathbb{C}_I \rightarrow \mathbb{H} \) as

\[
 f_f^c(z) = \overline{F(z)} - G(z)J. \quad (2.14)
\]

Then \( f_f^c(z) \) is a holomorphic map and we define the conjugate function \( f^c \) of \( f \) as

\[
 f^c(q) = \text{ext}(f_f^c)(q).
\]

Note that, by construction, \( f^c \in \mathcal{R}(U) \). Note that if we write the function \( f \) in the form \( f(x + Iy) = \alpha(x, y) + I\beta(x, y) \), then it is possible to show that

\[
 f^c(x + Iy) = \overline{\alpha(x, y)} + I\overline{\beta(x, y)}.
\]
Remark 2.3.5. Some lengthy but easy computations show that if $c$ is a fixed quaternion and $I \in \mathbb{S}$ there exists $J \in \mathbb{S}$ such that $Ic = cJ$. Thus we have

$$|f^c(x + Iy)| = |\alpha(x, y) + I\beta(x, y)| = |\alpha(x, y) + I\beta(x, y)| = |\alpha(x, y) - \beta(x, y)I| = |\alpha(x, y) + J\beta(x, y)|$$

for a suitable $J \in \mathbb{S}$.

Using the notion of $\star$-multiplication of slice regular functions, it is possible to associate to any slice regular function $f$ its ”symmetrization” also called ”normal form”, denoted by $f^s$. We will show that all the zeros of $f^s$ are spheres of type $[x + Iy]$ (real points, in particular) and that, if $x + Iy$ is a zero of $f$ (isolated or not) then $[x + Iy]$ is a zero of $f^s$.

Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and let $f : U \rightarrow \mathbb{H}$ be a slice regular function. Using the notation in (2.15), we consider the function $f_I^s : U \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ defined by

$$f_I^s = f_I \star f_I^c = (F(z) + G(z)J) \star (\overline{F(z)} - G(z)J)$$

Then $f_I^s$ is holomorphic. We give the following definition:

Definition 2.3.6. Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and let $f : U \rightarrow \mathbb{H}$ be slice regular. The function

$$f^s(q) = \text{ext}(f_I^s)(q)$$

defined by the extension of (2.16) is called the symmetrization (or normal form) of $f$.

Remark 2.3.7. Notice that formula (2.16) yields that, for all $I \in \mathbb{S}$, $f^s(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I$.

We now show how the conjugation and the symmetrization of a slice regular function behave with respect to the $\star$-product:

Proposition 2.3.8. Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and let $f, g : U \rightarrow \mathbb{H}$ be slice regular functions. Then

$$(f \star g)^c = g^c \star f^c$$

and

$$(f \star g)^s = f^s g^s = g^s f^s.$$
Proof. It is sufficient to show that \((f \circ g)^c = g^c \circ f^c\). As customary, we can use the Splitting Lemma to write on \(U \cap C_I\) that \(f_I(z) = F(z) + G(z)J\) and \(g_I(z) = H(z) + K(z)J\). We have

\[
f_I \circ g_I(z) = [F(z)H(z) - G(z)K(z)] + [F(z)K(z) + G(z)H(z)]J,
\]
and hence

\[
(f_I \circ g_I)^c(z) = [\overline{F(z)} \overline{H(z)} - \overline{G(z)}K(z)] - [F(z)K(z) + G(z)\overline{H(z)}]J.
\]

We now compute

\[
g_I^c(z) \circ f_I^c(z) = (\overline{H(z)} - K(z)J) \circ (\overline{F(z)} - G(z)J)
\]

\[
= \overline{H(z)} \circ \overline{F(z)} - \overline{H(z)} \circ G(z)J - K(z)J \circ \overline{F(z)} + K(z)J \circ G(z)J,
\]
and conclude by Remark 2.3.3.

**Proposition 2.3.9.** Let \(U \subseteq \mathbb{H}\) be an axially symmetric s-domain and let \(f : U \to \mathbb{H}\) be a slice regular function. The function \((f^s(q))^{-1}\) is slice regular on \(U \setminus \{q \in \mathbb{H} \mid f^s(q) = 0\}\).

**Proof.** The function \(f^s\) is such that \(f^s(U \cap C_I) \subseteq C_I\) for all \(I \in S\) by Remark 2.3.7. Thus, for any given \(I \in S\) the Splitting Lemma implies the existence of a holomorphic function \(F: U \cap C_I \to C_I\) such that \(f_I^s(z) = F(z)\) for all \(z \in U \cap C_I\). The inverse of the function \(F\) is holomorphic on \(U \cap C_I\) outside the zero set of \(F\). The conclusion follows by the equality \((f_I^s)^{-1} = F^{-1}\).

The \(\circ\)-product can be related to the pointwise product as described in the following result:

**Theorem 2.3.10.** Let \(U \subseteq \mathbb{H}\) be an axially symmetric s-domain, \(f, g : U \to \mathbb{H}\) be slice hyperholomorphic functions. Then

\[
(f \circ g)(p) = f(p)g(f(p)^{-1}pf(p)),
\]
(2.18)

for all \(p \in U\), \(f(p) \neq 0\), while \((f \circ g)(p) = 0\) when \(p \in U\), \(f(p) = 0\).

An immediate consequence is the following:

**Corollary 2.3.11.** If \((f \circ g)(p) = 0\) then either \(f(p) = 0\) or \(f(p) \neq 0\) and \(g(f(p)^{-1}pf(p)) = 0\).
2.4 Quaternionic intrinsic functions

An important subclass of the class of slice regular functions on an open set \( U \), denoted by \( \mathcal{N}(U) \), is defined as follows:

\[
\mathcal{N}(U) = \{ f \text{ slice regular in } U : f(U \cap \mathbb{C}_I) \subseteq \mathbb{C}_I, \forall I \in \mathbb{S} \}.
\]

**Remark 2.4.1.** If \( U \) is axially symmetric and if we denote by \( \overline{q} \) the conjugate of a quaternion \( q \), it can be shown that a function \( f \) belongs to \( \mathcal{N}(U) \) if and only if it satisfies \( f(q) = f(\overline{q}) \). In analogy with the complex case, we say that the functions in the class \( \mathcal{N} \) are **quaternionic intrinsic**.

If one considers a ball \( B(0, R) \) with center at the origin, it is immediate that a function slice regular on the ball belongs to \( \mathcal{N}(B(0, R)) \) if and only if its power series expansion has real coefficients. Such functions are also said to be real. More in general, if \( U \) is an axially symmetric \( s \)-domain, then \( f \in \mathcal{N}(U) \) if and only if \( f(q) = f(x + Iy) = \alpha(x, y) + I\beta(x, y) \) with \( \alpha, \beta \) real valued, in fact we have:

**Proposition 2.4.2.** Let \( U \subseteq \mathbb{H} \) be an axially symmetric open set. Then \( f \in \mathcal{N}(U) \) if and only if

\[
f(x) = \alpha(x, y) + I\beta(x, y)
\]

with \( \alpha, \beta \) real valued, \( \alpha(x, -y) = \alpha(x, y) \), \( \beta(x, -y) = -\beta(x, y) \) and satisfy the Cauchy–Riemann system.

**Proof.** If \( \alpha, \beta \) are as above, trivially \( f(q) \) defined by (2.19) belongs to \( \mathcal{N}(U) \). Conversely, assume that \( f(q) \in \mathcal{N}(U) \). Let \( I, J \in \mathbb{S} \), \( I \) and \( J \) orthogonal and let \( f_I(z) = F(z) + G(z)J \) with \( F, G : U \cap \mathbb{C}_I \rightarrow \mathbb{C}_I \) holomorphic. Since \( f \) takes \( \mathbb{C}_I \) to itself, then \( G \equiv 0 \). Then the function \( f_I(z) = F(z) = \alpha(x, y) + I\beta(x, y) \) where \( \alpha, \beta \) are real valued and satisfy the Cauchy–Riemann system. Thus, by the Identity Principle, the function \( f(q) \) can be written as \( f(q) = \alpha(x, y) + Iq\beta(x, y) \). The equalities \( \alpha(x, -y) = \alpha(x, y) \), \( \beta(x, -y) = -\beta(x, y) \) follow from the Representation Formula. \( \Box \)

**Remark 2.4.3.** The class \( \mathcal{N}(\mathbb{H}) \) includes all elementary transcendental functions, in particular

\[
\exp(q) = e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!},
\]

25
\[
\sin(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n+1}}{(2n+1)!},
\]

\[
\cos(q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{2n}}{(2n)!}.
\]

Note that these functions coincide with the analogous complex functions on any complex plane \( \mathbb{C}_I \).

Also the quaternionic logarithm is a quaternionic intrinsic function. Let us recall the definition of quaternionic logarithm, see [108], [126]. It is the function inverse of the exponential function \( \exp(q) \) in \( \mathbb{H} \).

**Definition 2.4.4.** Let \( U \subseteq \mathbb{H} \) be a connected open set. We define a branch of the quaternionic logarithm (or simply a logarithm) on \( U \) a function \( f : U \to \mathbb{H} \) such that for every \( q \in U \)

\[ e^{f(q)} = q. \]

Since \( \exp(q) \) never vanishes, we suppose that \( 0 \notin U \). Recalling that

\[ I_q = \begin{cases} 
\text{Im}(q)/|\text{Im}(q)| & \text{if } q \in \mathbb{H} \setminus \mathbb{R} \\
\text{any element of } \mathbb{S} & \text{otherwise}
\end{cases} \]

we have that for every \( q \in \mathbb{H} \setminus \{0\} \) there exists a unique \( \theta \in [0, \pi] \) such that \( q = |q|e^{\theta I_q} \). Moreover we have \( \theta = \arccos(\text{Re}(q)/|q|) \).

**Definition 2.4.5.** The function \( \arccos(\text{Re}(q)/|q|) \) will be called the principal quaternionic argument of \( q \) and it will be denoted by \( \arg_{\mathbb{H}}(q) \) for every \( q \in \mathbb{H} \setminus \{0\} \).

Below we define the principal quaternionic logarithm.

**Definition 2.4.6.** Let \( \log(x) \) be the natural real logarithm of \( x \in \mathbb{R}^+ \). For every \( q \in \mathbb{H} \setminus (-\infty, 0] \), we define the principal quaternionic logarithm or, in short, principal logarithm) of \( q \) as

\[ \text{Log}(q) = \ln |q| + \arccos \left( \frac{\text{Re}(q)}{|q|} \right) I_q. \]

**Remark 2.4.7.** It is easy to directly verify that the principal quaternionic logarithm coincides with the principal complex logarithm on the complex plane \( \mathbb{C}_I \), for any \( I \in \mathbb{S} \).
We now go back to the properties of intrinsic slice regular functions. The following result has an elementary proof which is left to the reader.

**Proposition 2.4.8** (Algebraic properties). Let $U, U'$ be two open sets in $\mathbb{H}$.

1. Let $f$ and $g \in \mathcal{N}(U)$, then $f + g \in \mathcal{N}(U)$ and $fg \in \mathcal{N}(U)$.
2. Let $f$ and $g \in \mathcal{N}(U)$ such that $g(q) \neq 0$ for all $q \in U$, then $g^{-1}f = fg^{-1} \in \mathcal{N}(U)$.
3. Let $f \in \mathcal{N}(U')$, $g \in \mathcal{N}(U)$ with $g(U) \subseteq U'$. Then $f(g(q))$ is slice regular for $q \in U$.

**Proposition 2.4.9.** Let $U \subseteq \mathbb{H}$ be an axially symmetric $s$-domain and let $\{1, I, J, IJ\}$ be a basis of $\mathbb{H}$. Then

$$\mathcal{R}(U) = \mathcal{N}(U) \oplus \mathcal{N}(U)I \oplus \mathcal{N}(U)J \oplus \mathcal{N}(U)IJ.$$  

**Proof.** By the Refined Splitting Lemma, there exist four functions $h_0, h_1, h_2, h_3$ holomorphic intrinsic on $U \cap \mathbb{C}I$ such that

$$f_1(z) = h_0(z) + h_1(z)I + h_2(z)J + h_3(z)IJ,$$

and $f = \text{ext}(f_1) = \text{ext}(h_0) + \text{ext}(h_1)I + \text{ext}(h_2)J + \text{ext}(h_3)IJ$.

Since the extension of a complex intrinsic function is a quaternionic intrinsic function, we obtain $\mathcal{R}(U) = \mathcal{N}(U) \oplus \mathcal{N}(U)I \oplus \mathcal{N}(U)J \oplus \mathcal{N}(U)IJ$.

To show that the sum is a direct sum, suppose that $f \in \mathcal{N}(U) \cap \mathcal{N}(U)I$. Then there exists $g \in \mathcal{N}(U)$, such that $f = gI$. So there exist $h_1, h_2$ complex intrinsic and holomorphic such that $f = \text{ext}(h_1)$, and $g = \text{ext}(h_2)$. Then $h_1 = h_2I$, and for any $q \in U_I$ one has

$$h_2(q)I = h_1(q) = \overline{h_1(q)} = \overline{h_2(q)I} = -h_2(q)I,$$

then $h_2 = h_1 = 0$, and $\mathcal{N}(U) \cap \mathcal{N}(U)I = \{0\}$.

Similarly one can see that all the other intersections between $\mathcal{N}(U)$, $\mathcal{N}(U)I$, $\mathcal{N}(U)J$, $\mathcal{N}(U)IJ$, are $\{0\}$ and the statement follows. 

2.5 Composition of power series

In this section we first introduce and study a notion of composition of slice regular functions, see also [92]. As it is well know, the composition $f \circ g$ of two slice regular functions is not, in general, slice regular, unless $g$ belongs to
the subclass of quaternionic intrinsic functions. Our choice of the notion of composition is based on the fact that slice regularity is not preserved by the pointwise product, but is preserved by the \( \star \)-product. Thus the power of a function is slice regular only if it is computed with respect to the \( \star \)-product and we will write \( (w(q))^n \) to denote that we are taking the \( n \)-th power with respect to this product. To introduce the notion of composition, we first treat the case of formal power series.

**Definition 2.5.1.** Denoting \( g(q) = \sum_{n=0}^{\infty} q^n a_n \) and \( w(q) = \sum_{n=1}^{\infty} q^n b_n \). We define

\[
(g \star w)(q) = \sum_{n=0}^{\infty} (w(q))^n a_n.
\]

**Remark 2.5.2.** When \( w \in \mathcal{N}(B(0; 1)) \), then \( g \star w = g \circ w \) where \( \circ \) represents the standard composition of two functions. Moreover, when \( w \in \mathcal{N}(B(0; 1)) \) then \( (w(q))^n = (w(q))^n \) so, in particular, \( q^n = q^n \).

**Remark 2.5.3.** The order of a series \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) can be defined as in [37] and we denote it by \( \omega(f) \). It is the lowest integer \( n \) such that \( a_n \neq 0 \) (with the convention that the order of the series identically equal to zero is \( +\infty \)). Assume to have a family \( \{f_i\}_{i \in \mathcal{I}} \) of power series where \( \mathcal{I} \) is a set of indices. This family is said to be summable if for any \( k \in \mathbb{N} \), \( \omega(f_i) \geq k \) for all except a finite number of indices \( i \). By definition, the sum of \( \{f_i\} \) where \( f_i(q) = \sum_{n=0}^{\infty} q^n a_{i,n} \) is

\[
f(q) = \sum_{n=0}^{\infty} q^n a_n,
\]

where \( a_n = \sum_{i \in \mathcal{I}} a_{i,n} \). The definition of \( a_n \) is well posed since our hypothesis guarantees that for any \( n \) just a finite number of \( a_{i,n} \) are nonzero.

**Remark 2.5.4.** In Definition 2.5.1 we require the hypothesis \( b_0 = 0 \). This is necessary in order to guarantee that the minimum power of \( q \) in the term \( (w(q))^n \) is at least \( q^n \) or, in other words, that \( \omega(w(q))^n \geq n \) for all indices. With this hypothesis, the series \( \sum_{n=0}^{\infty} (w(q))^n a_n \) is summable according to Remark 2.5.3 and we can regroup the powers of \( q \).

Let us consider the following example: let \( f(q) = q^2c, g(q) = qa \) and \( w(q) = q^2b \). We have

\[
((f \cdot g) \cdot w)(q) = q^4 b^2 a^2 c
\]

while

\[
(f \cdot (g \cdot w))(q) = q^4 babac
\]
so
\[(f \cdot g) \cdot w \neq f \cdot (g \cdot w).\]

Thus we conclude that:

**Proposition 2.5.5.** The composition \( \cdot \) is, in general, not associative.

However, we will prove that the composition is associative in some cases and to this end we need a preliminary Lemma.

**Lemma 2.5.6.** Let \( f_1(q) = \sum_{n=0}^{\infty} q^n a_n \), \( f_2(q) = \sum_{n=0}^{\infty} q^n b_n \), and \( g(q) = \sum_{n=1}^{\infty} q^n c_n \). Then:

1. \((f_1 + f_2) \cdot g = f_1 \cdot g + f_2 \cdot g;\)
2. if \( g \) has real coefficients \((f_1 \ast f_2) \cdot g = (f_1 \cdot g) \ast (f_2 \cdot g);\)
3. if \( g \) has real coefficients \( f^n \cdot g = (f \cdot g)^n.\)
4. if \( \{f_i\}_{i \in I} \) is a summable family of power series then \( \{f_i \cdot g\}_{i \in I} \) is summable and \( \sum_i (f_i \cdot g) = \sum_i (f_i \cdot g).\)

**Proof.** Point (1) follows from

\[
(f_1 + f_2) \cdot g = \sum_{n=0}^{\infty} g^n (a_n + b_n) \\
= \sum_{n=0}^{\infty} g^n a_n + \sum_{n=0}^{\infty} g^n b_n \\
= f_1 \cdot g + f_2 \cdot g.
\]

To prove (2), recall that \((f_1 \ast f_2)(q) = \sum_{n=0}^{\infty} q^n (\sum_{r=0}^{n} a_r b_{n-r})\) and taking into account that the coefficients of \( g \) are real we have:

\[
((f_1 \ast f_2) \cdot g)(q) = \sum_{n=0}^{\infty} (g(q))^n (\sum_{r=0}^{n} a_r b_{n-r}).
\]

The statement follows since

\[
(f_1 \cdot g)(q) \ast (f_2 \cdot g)(q) = \left( \sum_{n=0}^{\infty} g(q)^n a_n \right) \ast \left( \sum_{m=0}^{\infty} g(q)^m b_m \right) \\
= \sum_{n=0}^{\infty} g(q)^n (\sum_{r=0}^{n} a_r b_{n-r}).
\]
To prove point (3) we use induction. The statement is true for \( n = 2 \) since it follows from (2). Assume that the assertion is true for the \( n \)-th power. Let us show that it holds for \( n + 1 \)-th power. Let us compute

\[
(f^{(n+1)} \cdot g)(q) = ((f^{(n)} \ast f) \cdot g)(q) \overset{(2)}{=} (f \cdot g)^{\ast n} \ast (f \cdot g) = (f \cdot g)^{\ast (n+1)},
\]

and the statement follows.

To show 4 we follow [37, p. 13]. Let \( f_i(q) = \sum_{n=0}^{\infty} q^n a_{i,n} \) so that, by definition

\[
\sum_{i \in I} f_i(q) = \sum_{n=0}^{\infty} q^n \left( \sum_{i \in I} a_{i,n} \right).
\]

Thus we obtain

\[
\left( \sum_{i \in I} f_i(q) \right) \cdot g = \sum_{n=0}^{\infty} g(q)^{\ast n} \left( \sum_{i \in I} a_{i,n} \right) \tag{2.20}
\]

and

\[
\sum_{i \in I} (f_i \cdot g)(q) = \sum_{i \in I} \left( \sum_{n=0}^{\infty} g(q)^{\ast n} a_{i,n} \right). \tag{2.21}
\]

By hypothesis on the summability of \( \{f_i\} \), each power of \( q \) involves just a finite number of the coefficients \( a_{i,n} \) so we can apply the associativity of the addition in \( \mathbb{H} \) and so (2.20) and (2.21) are equal.

**Proposition 2.5.7.** If \( f(q) = \sum_{n=0}^{\infty} q^n a_n, \ g(q) = \sum_{n=1}^{\infty} q^n b_n, \ w(q) = \sum_{n=1}^{\infty} q^n c_n \) and \( w \) has real coefficients, then \((f \cdot g) \cdot w = f \cdot (g \cdot w)\).

**Proof.** We follow the proof of Proposition 4.1 in [37]. We first prove the assertion in the special case in which \( f(q) = q^n a_n \). We have:

\[
((f \cdot g) \cdot w)(q) = \sum_{n=0}^{\infty} g(q)^{\ast n} w(q) = \sum_{n=0}^{\infty} g(q)^{\ast n} \cdot w(q) = (f \cdot (g \cdot w)) (q) = (g \ast w)(q) a_n \tag{2.22}
\]

and

\[
(f \cdot (g \cdot w))(q) = (g \cdot w)(q) a_n \tag{2.23}
\]

Lemma 2.5.6 point (3) shows that \( g^{\ast n} \cdot w = (g \cdot w)^{\ast n} \) and the equality follows. The general case follows by considering \( f \) as the sum of the summable family \( \{q^n a_n\} \) and using the first part of the proof:

\[
(f \cdot g) \cdot w = \sum_{n=0}^{\infty} (g^{\ast n} \cdot w) a_n = \sum_{n=0}^{\infty} (g \cdot w)^{\ast n} a_n = f \cdot (g \cdot w) \tag{2.24}
\]

\( \square \)
Thus we have that \( \sum_{n=0}^{\infty} q^n a_n \) converges in the balls of nonzero radius \( R \) and \( p \), respectively, and let \( h(q) = (f \circ g)(q) \). Then the radius of convergence of \( h \) is nonzero and it is equal to \( \sup \{ r > 0; \sum_{n=1}^{\infty} r^n |b_n| < R \} \).

**Proof.** Let us compute

\[
\left| \left( \sum_{m=1}^{\infty} q^m b_m \right)^* \left( \sum_{m=1}^{\infty} q^m c_m \right) \right| = \left| \sum_{m=1}^{\infty} q^m \left( \sum_{r=0}^{m} b_r c_{m-r} \right) \right| \\
\leq \sum_{m=1}^{\infty} |q|^m \left( \sum_{r=0}^{m} |b_r||c_{m-r}| \right) \\
= \left( \sum_{m=1}^{\infty} |q|^m |b_m| \right) \left( \sum_{m=1}^{\infty} |q|^m |c_m| \right),
\]

and thus the formula

\[
\left| \left( \sum_{m=1}^{\infty} q^m b_m \right)^* \right| \leq \left( \sum_{m=1}^{\infty} |q|^m |b_m| \right)^n
\]

is true for \( n = 2 \). The general formula follows recursively by using (2.22).

So we have

\[
\left| \sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} q^m b_m \right)^* a_n \right| \leq \sum_{n=0}^{\infty} \left| \sum_{m=1}^{\infty} q^m b_m \right|^n |a_n| \\
\leq \sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} |q|^m |b_m| \right)^n |a_n|.
\]

Since the series expressing \( g \) is converging on a ball of finite radius, there exists a positive number \( r \), sufficiently small, such that \( \sum_{n=1}^{\infty} r^n |b_n| \) is finite. Moreover, \( \sum_{n=1}^{\infty} r^n |b_n| = r \sum_{n=1}^{\infty} r^{n-1} |b_n| \to 0 \) for \( r \to 0 \) and so there exists \( r \) such that \( \sum_{n=1}^{\infty} r^n |b_n| < R \). Thus, from (2.23), we deduce

\[
\sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} r^m |b_m| \right)^n |a_n| = \sum_{m=1}^{\infty} r^m \gamma_m < \infty.
\]

Thus we have that \( (g \circ f)(q) = \sum_{m=0}^{\infty} q^n d_m \) and \( |d_m| \leq \gamma_m \) and the radius of convergence of \( g \circ f \) is at least equal to \( r \). 

\[\square\]
Using this notion of composition, it is also possible to define, under suitable conditions, left and right inverses of a power series:

**Proposition 2.5.9.** Let \( g : B(0; R) \to \mathbb{H}, R > 0 \), be a function slice regular of the form \( g(q) = \sum_{n=0}^{\infty} q^n a_n \).

1. There exists a power series \( g_r^{-\bullet}(q) = \sum_{n=0}^{\infty} q^n b_n \) convergent in a disc with positive radius, such that \((g \cdot g_r^{-\bullet})(q) = q\) and \( g_r^{-\bullet}(0) = 0 \) if and only if \( g(0) = 0 \) and \( g'(0) \neq 0 \).

2. There exists a power series \( g_l^{-\bullet}(q) = \sum_{n=0}^{\infty} q^n b_n \) convergent in a disc with positive radius, such that \((g_l^{-\bullet} \cdot g)(q) = q\) and \( g_l^{-\bullet}(0) = 0 \) if and only if \( g(0) = 0 \) and \( g'(0) \neq 0 \).

**Proof.** To prove (1), assume that \( g_r^{-\bullet} \) exists. Then
\[
\sum_{n=0}^{\infty} \left( \sum_{m=1}^{\infty} q^m b_m \right)^n a_n = q.
\]
and so to have equality it is necessary that \( a_0 = 0 \), i.e. \( g(0) = 0 \), and \( b_1 a_1 = 1 \) and so \( a_1 \neq 0 \), i.e. \( g'(0) \neq 0 \). To prove that the condition is sufficient, we observe that for \( n \geq 2 \), the coefficient of \( q^n \) is zero on the right hand side of (2.24) while on the left hand side it is given by
\[
b_n a_1 + P_n(b_1, \ldots, b_{n-1}, a_2, \ldots, a_n),
\]
thus we have
\[
b_n a_1 + P_n(b_1, \ldots, b_{n-1}, a_2, \ldots, a_n) = 0,
\]
where the polynomials \( P_n \) are linear in the \( a_i \)'s and they contain all the possible monomials \( b_{j_1} \ldots b_{j_r} \) with \( j_1 + \ldots + j_r = n \) and thus also \( b_1^n \). In particular we have: \( b_1 a_1 = 1 \) and so \( b_1 = a_1^{-1} \), then
\[
b_2 a_1 + b_1^2 a_2 = 0
\]
and so
\[
b_2 = -a_1^{-2} a_2 a_1^{-1}.
\]
Using induction, we can compute \( b_n \) if we have computed \( b_1, \ldots, b_{n-1} \) by putting (2.25) equal to 0 and using the fact that \( a_1 \) is invertible. This concludes the proof since the function \( g_r^{-\bullet} \) is a right inverse of \( g \) by construction.

We now show that \( g_r^{-\bullet} \) converges in a disc with positive radius following the
proof of [37, Proposition 9.1]. Construct a power series with real coefficients $A_n$ which is a majorant of $g$ as follows: set

$$
\tilde{g}(q) = qA_1 - \sum_{n=2}^{\infty} q^n A_n
$$

with $A_1 = |a_1|$ and $A_n \geq |a_n|$, for all $n \geq 2$. It is possible to compute the inverse of $\tilde{g}$ with respect to the (standard) composition to get the series $\tilde{g}^{-1}(q) = \sum_{n=1}^{\infty} q^n B_n$. The coefficients $B_n$ can be computed with the formula

$$
B_n A_1 + P_n(B_1, \ldots, B_{n-1}, A_2, \ldots, A_n) = 0,
$$

analog of (2.25). Then we have

$$
B_1 = A_1^{-1} = |a_1|^{-1},
$$
$$
B_2 = A_1^{-2}(-A_2)A_1^{-1} \geq |a_1|^{-2}|a_2| \cdot |a_1|^{-1} = |b_1|
$$

and, inductively

$$
B_n = Q_n(A_1, \ldots, A_n) \geq Q_n(|a_1|, \ldots, |a_n|) = |b_n|.
$$

We conclude that the radius of convergence of $g^{-\bullet}$ is greater than or equal to the radius of convergence of $\tilde{g}^{-1}$ which is positive, see [37, p. 27].

Point (2) can be proved with similar computations and the function $g^{-\bullet}$ so obtained is a left inverse of $g$. \qed

**Remark 2.5.10.** As we have seen in Remark 2.4.3 the transcendental functions cosine, sine, exponential are entire slice regular functions. Let $f(q)$ be another entire function, for example a polynomial. It is then possible to define the composed functions

$$
\exp_{\bullet}(f(q)) = e_{\bullet}^{f(q)} = \sum_{n=0}^{\infty} \frac{(f(q))^{*n}}{n!},
$$
$$
\sin_{\bullet}(f(q)) = \sum_{n=0}^{\infty} (-1)^n \frac{(f(q))^{*2n+1}}{(2n+1)!},
$$
$$
\cos_{\bullet}(f(q)) = \sum_{n=0}^{\infty} (-1)^n \frac{(f(q))^{*2n}}{(2n)!}.
$$
Comments to Chapter 2. The material in this chapter comes from several papers. The theory of slice regular functions started in [109] and [110] for functions defined at ball centered at the origin and then evolved in a series of papers, among which we mention [58], [46], [47], into a theory on axially symmetric domains. The theory was developed also for functions with values in a Clifford algebra, see [72], [73], for functions with values in a real alternative algebra, see [117]. The composition of slice regular functions which appears in this chapter is taken from [92].
Chapter 3

Slice regular functions: analysis

3.1 Some integral formulas

In this section we collect the generalizations of the Cauchy and Schwarz integral formulas to the slice regular setting. We begin by stating a result which can be proved with standard techniques, see [35].

**Lemma 3.1.1.** Let $f, g$ be quaternion valued, continuously (real) differentiable functions on an open set $U_I$ of the plane $\mathbb{C}_I$. Then for every open set $W_I \subset U_I$ whose boundary is a finite union of continuously differentiable Jordan curves, we have

$$\int_{\partial W_I} g ds_I f = 2 \int_{W_I} ((g\overline{\partial}_I)f + g(\overline{\partial}_I f)) d\sigma$$

where $s = x + Iy$ is the variable on $\mathbb{C}_I$, $ds_I = -Ids$ and $d\sigma = dx \wedge dy$.

An immediate consequence of this lemma is the following:

**Corollary 3.1.2.** Let $f$ and $g$ be a left slice regular and a right slice regular function, respectively, on an open set $U \in \mathbb{H}$. For any $I \in \mathbb{S}$ let $U_I = U \cap \mathbb{C}_I$. For every open $W_I \subset U \cap \mathbb{C}_I$ whose boundary is a finite union of continuously differentiable Jordan curves, we have:

$$\int_{\partial W_I} g ds_I f = 0.$$

We are now ready to state the Cauchy formula (see [76] for its proof):
Theorem 3.1.3. Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of continuously differentiable Jordan curves, for every $I \in S$. Let $f$ be a slice regular function on an open set containing $U$ and, for any $I \in S$, set $ds_I = -Ids$. Then for every $q = x + Iy \in U$ we have:

$$f(q) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} -(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}(q - \overline{s})ds_I f(s). \quad (3.1)$$

Moreover the value of the integral depends neither on $U$ nor on the imaginary unit $I \in S$.

The function

$$S_L^{-1}(s, q) = -(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}(q - \overline{s})$$

is called the Cauchy kernel for left slice regular functions. It is slice regular in $q$ and right slice regular in $s$ for $q, s$ such that $q^2 - 2\text{Re}(s)q + |s|^2 \neq 0$.

In the case of right slice regular functions, the Cauchy formula is written in terms of the Cauchy kernel

$$S_R^{-1}(s, q) = -(q - \overline{s})(q^2 - 2\text{Re}(s)q + |s|^2)^{-1}.$$

An immediate consequence of the Cauchy formula is the following result:

Theorem 3.1.4. (Derivatives using the slice regular Cauchy kernel) Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain. Suppose $\partial(U \cap \mathbb{C}_I)$ is a finite union of continuously differentiable Jordan curves for every $I \in S$. Let $f$ be a slice regular function on $U$ and set $ds_I = ds/I$. Let $q, s$. Then

$$f^{(n)}(q) = \frac{n!}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} (q^2 - 2\text{Re}(s)q + |s|^2)^{-n-1}(q - \overline{s})^{(n+1)*}ds_I f(s)$$

$$= \frac{n!}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} [S^{-1}(s, q)(q - \overline{s})^{-1}]^{n+1}(q - \overline{s})^{(n+1)*}ds_I f(s) \quad (3.2)$$

where

$$(q - \overline{s})^{n*} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!}q^{n-k}s^k, \quad (3.3)$$

is the $n$-th power with respect to the $*$-product. Moreover the value of the integral depends neither on $U$ nor on the imaginary unit $I \in S$. 

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Proposition 3.1.5. (Cauchy estimates) Let \( f : U \to \mathbb{H} \) be a slice regular function and let \( q \in U \cap \mathbb{C}_I \). For all discs \( B_I(q, R) = B(q, R) \cap \mathbb{C}_I, R > 0 \) such that \( B_I(q, R) \subset U \cap \mathbb{C}_I \) the following formula holds:

\[
|f^{(n)}(q)| \leq \frac{n!}{R^n} \max_{s \in \partial B_I(q, R)} |f(s)|.
\]

Proof. Let \( \gamma(t) = q + R e^{2\pi i t}, t \in [0, 2\pi) \), then

\[
|f^{(n)}(q)| \leq n! \left| \frac{1}{2\pi} \int_0^{2\pi} ds_I(s - q)^{-(n+1)} \right| \max_{s \in \partial B_I(q, R)} |f(s)| \leq \frac{n!}{R^n} \max_{s \in \partial B_I(q, R)} |f(s)|.
\]

If \( f \) is an entire regular function the Cauchy estimates yield the Liouville Theorem.

Theorem 3.1.6. (Liouville) Let \( f \) be a quaternionic entire function. If \( f \) is bounded then it is constant.

Proof. Let us consider \( q = 0 \) and let \( R > 0 \). We use the Cauchy estimates to show that if \( R \to \infty \) then all the derivatives \( f^{(n)}(0) \) must vanish for \( n > 0 \). We conclude that \( f(q) = f(0) \) for all \( q \in \mathbb{H} \).

In the sequel we will need the Schwarz formula. We first prove the result for slice regular intrinsic functions and then in the general setting, in both cases on a slice, namely on a complex plane \( \mathbb{C}_I \). As customary, given an open set \( U \subseteq \mathbb{H} \), we will denote by \( U_I \) the set \( U \cap \mathbb{C}_I \). To say that a function is harmonic in \( U_I \) means that the functions is harmonic in the two variables \( x, y \) if we denote by \( x + Iy \) the variable in \( \mathbb{C}_I \). Note that in the sequel, we will denote the variable of the complex plane \( \mathbb{C}_I \) in polar coordinates as \( r e^{I\theta} \) and thus we will write \( f(re^{I\theta}) = \alpha(r \cos \theta, r \sin \theta) + I \beta(r \cos \theta, r \sin \theta) \).

Proposition 3.1.7. Let \( f \in \mathcal{N}(U) \), let \( q \in U \), and let \( \alpha, \beta \in C^2(U_{I_q}, \mathbb{R}) \) be harmonic functions on \( U_{I_q} \) such that \( f_{I_q} = \alpha + I_q \beta \). Assume that for a suitable \( \delta > 0 \) the disk \( |z - q| \leq \delta \) is contained in \( U_{I_q} \). Then there exists a real number \( b \) such that, for any \( z \in U_{I_q} \) with \( |z - q| < r < \delta \), the following formula holds

\[
f(z) = I_q b + \frac{1}{2\pi} \int_0^{2\pi} \frac{r e^{Iq\varphi} - (z - q)}{r e^{Iq\varphi} + (z - q)} \alpha(r \cos \varphi, r \sin \varphi) d\varphi,
\]

for any \( z \in U_{I_q} \) such that \( |z - q| < r < \delta \).
Proof. We use the classical Schwarz formula for holomorphic functions on the complex plane \( \mathbb{C} \), assuming that the pair of points \( z, q \in U \) satisfy the hypothesis. We obtain
\[
f(z) = f_I q(z) = I_q b + \frac{1}{2\pi} \int_0^{2\pi} r e^{i\phi} + (z - q) \frac{re^{i\phi} - (z - q)}{re^{i\phi} - (z - q)} \alpha(r, \phi) d\phi.
\]

**Theorem 3.1.8** (The Schwarz formula on a slice). Let \( U \) be an open set in \( \mathbb{H} \), \( f \in \mathcal{R}(U) \) and let \( q \in U \). Assume that for a suitable \( \delta > 0 \) the disk \( |z - q| \leq \delta \) is contained in \( U_I q \). Then there exist \( b \in \mathbb{H} \) and a harmonic \( \mathbb{H} \)-valued function \( \alpha \), such that for any \( z \in U_I q \) with \( |z - q| < r < \delta \)
\[
f(z) = I_q b + \frac{1}{2\pi} \int_0^{2\pi} r e^{i\phi} + (z - q) \frac{re^{i\phi} - (z - q)}{re^{i\phi} - (z - q)} \alpha(r \cos \phi, r \sin \phi) d\phi.
\]

**Proof.** Let \( f \) be a slice regular function on \( U \) and let us set \( I_q = I \) for the sake of simplicity. We use the Refined Splitting Lemma to write \( f(z) = f_I(z) = f_0(z) + f_1(z) I + f_2(z) I J + f_3(z) I J \) where \( f_\ell \) are holomorphic intrinsic functions, i.e. \( f_\ell(\bar{z}) = f_\ell(z) \) for \( z \in U \cap \mathbb{C} \) and \( J \in \mathbb{S} \) is orthogonal to \( I \). Let \( \alpha_\ell, \beta_\ell \in C^2(U, \mathbb{R}) \) be harmonic functions such that \( f_\ell = \alpha_\ell + I \beta_\ell \), \( \ell = 1 \ldots, 3 \). By the complex Schwarz formula, there exists \( b_\ell \in \mathbb{R} \) such that
\[
f_\ell(z) = I b_\ell + \frac{1}{2\pi} \int_0^{2\pi} r e^{i\phi} + (z - q) \frac{re^{i\phi} - (z - q)}{re^{i\phi} - (z - q)} \alpha_\ell(r \cos \phi, r \sin \phi) d\phi.
\]

Then, by setting \( I_0 = 1, I_1 = I, I_2 = J, I_3 = I J \), we have
\[
f(z) = f_I(z) = \sum_{\ell=0}^3 f_\ell(z) I_\ell
\]
\[
= I \left( \sum_{\ell=0}^3 b_\ell I_\ell \right) + \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\phi} + (z - q)}{re^{i\phi} - (z - q)} \left( \sum_{\ell=0}^3 \alpha_\ell(r \cos \phi, r \sin \phi) I_\ell \right) d\phi
\]
\[
= I b + \frac{1}{2\pi} \int_0^{2\pi} \frac{re^{i\phi} + (z - q)}{re^{i\phi} - (z - q)} \alpha(r \cos \phi, r \sin \phi) d\phi,
\]
where
\[
b = \sum_{\ell=0}^3 b_\ell I_\ell, \quad \alpha(r \cos \phi, r \sin \phi) = \sum_{\ell=0}^3 \alpha_\ell(r \cos \phi, r \sin \phi) I_\ell.
\]
In order to prove a Schwarz formula in a more general form, we need two lemmas. To state them we set

\[ D_I := \{ z = u + Iv \in \mathbb{C}_I : |z| < 1 \}, \]

\[ \overline{D}_I := \{ z = u + Iv \in \mathbb{C}_I : |z| \leq 1 \}. \]

**Lemma 3.1.9.** Let \( f : D_I \to \mathbb{C}_I \) be a holomorphic function and let \( \alpha \) be its real part, so that

\[
f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{I\varphi} + z}{e^{I\varphi} - z} \alpha(e^{I\varphi})d\varphi. \tag{3.5}
\]

Then its slice regular extension to the ball \( B \), still denoted by \( f \), is given by

\[
f(q) = \frac{1}{2\pi} \int_0^{2\pi} (e^{I\varphi} - q)^{-*} \star (e^{I\varphi} + q) \alpha(e^{I\varphi})d\varphi. \tag{3.6}
\]

**Proof.** Using the extension operator induced by the Representation Formula we have that

\[
f(q) = \operatorname{ext}(f)(q) = \frac{1}{2} \left[ f(z) + f(z) + I_q I(f(z) - f(z)) \right]
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left[ \frac{e^{I\varphi} + z}{e^{I\varphi} - z} + \frac{e^{I\varphi} + z}{e^{I\varphi} - z} + I_q I \left( \frac{e^{I\varphi} + z}{e^{I\varphi} - z} - \frac{e^{I\varphi} + z}{e^{I\varphi} - z} \right) \right] \alpha(e^{I\varphi})d\varphi
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} (e^{I\varphi} - q)^{-*} \star (e^{I\varphi} + q) \alpha(e^{I\varphi})d\varphi.
\]

\( \Box \)

To prove our next result we need to introduce the Poisson kernel in this framework:

**Definition 3.1.10 (Poisson kernel).** Let \( I \in \mathbb{S} \), \( 0 \leq r < 1 \) and \( \theta \in \mathbb{R} \). We call Poisson kernel of the unit ball the function

\[
P(r, \theta) := \sum_{n \in \mathbb{Z}} r^{|n|} e^{nI\theta}.
\]

Note that it would have seemed more appropriate to write \( P(re^{I\theta}) \) but next result shows that the Poisson kernel does not depend on \( I \in \mathbb{S} \):
Lemma 3.1.11. Let $I \in \mathbb{S}$, $0 \leq r < 1$ and $\theta \in \mathbb{R}$. Then the Poisson kernel belongs to $\mathcal{N}(U)$ and it can be written in the form

$$P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2} = \Re \left[ \frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right] \quad (3.7)$$

Proof. The functions $1 \pm re^{i\theta}$ obviously map $C_I$ into itself. Moreover, note that

$$(1 + re^{i\theta})(1 - re^{i\theta})^{-1} = (1 - re^{i\theta})^{-1}(1 + re^{i\theta})$$

and so the ratio in (3.7) is well defined. Note also that

$$(1 + re^{i\theta})(1 - re^{i\theta})^{-1} = (1 + re^{i\theta}) \sum_{n \geq 0} r^n e^{ni\theta} = 1 + 2 \sum_{n \geq 1} r^n e^{ni\theta},$$

and

$$\Re \left[ \frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right] = 1 + 2 \sum_{n \geq 1} r^n \cos(n\theta) = 1 + \sum_{n \geq 1} r^n (e^{ni\theta} + e^{-ni\theta}) = P(r, \theta).$$

Since

$$\frac{1 + re^{i\theta}}{1 - re^{i\theta}} = \frac{1 + re^{i\theta} - re^{-i\theta} - r^2}{1 - 2r \cos \theta + r^2}$$

we obtain

$$\Re \left[ \frac{1 + re^{i\theta}}{1 - re^{i\theta}} \right] = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.$$ 

\[\square\]

Lemma 3.1.12. Let $\alpha : \mathbb{D}_I \subseteq \mathbb{C}_I \to \mathbb{H}$ be a continuous function that is harmonic on $\mathbb{D}_I$. Then

$$\alpha(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - \varphi) \alpha(e^{i\varphi}) d\varphi \quad (3.8)$$

for all $I \in \mathbb{S}$, $0 \leq r < 1$ and $\theta \in \mathbb{R}$. Moreover, if $\alpha(z) = \frac{1}{2}(f(z) + f(\bar{z}))$ for some holomorphic map $f : \mathbb{D}_I \to \mathbb{H}$, $I \in \mathbb{S}$, such that $f(0) \in \mathbb{R}$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\varphi} + \frac{z}{e^{i\varphi} - \alpha(e^{i\varphi})} d\varphi, \quad (3.9)$$

where $z = re^{i\theta}$. 

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Proof. Let us write \( \alpha = \sum_{\ell=0}^{3} \alpha_\ell I_\ell \) where \( \alpha_\ell \) are real valued and \( I_0 = 1 \), \( I_1 = I \), \( I_2 = J \), \( I_3 = IJ \), where \( J \in S \) is orthogonal to \( I \). The functions \( \alpha_\ell \) are harmonic since the Laplacian is a real operator. Thus (3.8) follows from the classical Poisson formula applied to \( \alpha_\ell \) by linearity. Note also that (3.7) yields:

\[
\alpha(re^{I\theta}) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - \varphi) \alpha(e^{I\varphi}) d\varphi
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \varphi) + r^2} \alpha(e^{I\varphi}) d\varphi
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \text{Re} \left[ \frac{1 + re^{I(\theta-\varphi)}}{1 - re^{I(\theta-\varphi)}} \right] \alpha(e^{I\varphi}) d\varphi,
\]

for all \( I \in S \), \( 0 \leq r < 1 \) and \( \theta \in \mathbb{R} \). Formula (3.8) allows to write the holomorphic functions \( f_\ell \). The function \( f(z) = \sum_{\ell=0}^{3} f_\ell(z)I_\ell \) equals the function in (3.9) and is in the kernel of the Cauchy-Riemann operator \( \partial/\partial \bar{z} \) by construction.

The previous results have been proved on the unit disc of the complex plane \( \mathbb{C} \) but it is immediate to generalize them to a disc centered at the origin with radius \( r > 0 \).

We can now prove:

**Theorem 3.1.13** (Schwarz formula). Let \( U \) be an axially symmetric slice domain in \( \mathbb{H} \), \( f \in \mathcal{R}(U) \) and assume that 0 \( \in U \). Suppose that \( f(0) \in \mathbb{R} \) and that the ball \( B(0; r) \) with center 0 and radius \( r \) is contained in \( U \), for a suitable \( r \). Let \( f(x + I_qy) = \alpha(x, y) + I_q \beta_q(x, y) \) for any \( q \) in the ball \( B(0; r) \subset \mathbb{H} \), then the following formula holds

\[
f(q) = \frac{1}{2\pi} \int_0^{2\pi} (re^{I\ell} - q)^{-x} \ast (re^{I\ell} + q) \alpha(re^{I\ell}) dt.
\]  

(3.10)

**Proof.** Recall that the Representation Formula implies that the function

\[
\alpha(q) = \alpha(x + Iy) = \frac{1}{2} (f(x + Iy) + f(x - Iy))
\]

depends on \( x, y \) only. By Remark ?? it follows that \( \alpha \) is harmonic. Thus the result follows by extension from formula (3.9).

We end this section by proving an analog of the Harnack inequality. We recall that according to the classical Harnack inequality, if \( a \in \mathbb{C} \) and \( R > 0 \)
then any real positive-valued harmonic function \( \alpha \) on the disk \( |z - a| < R \) satisfies
\[
\frac{R-r}{R+r} \alpha(a) \leq \alpha(z) \leq \frac{R+r}{R-r} \alpha(a),
\]
for any \( |z - a| < r < R \). This inequality is a direct consequence of the Poisson Formula.

For slice regular functions we have the following:

**Proposition 3.1.14.** Let \( f \in \mathcal{N}(U) \), let \( q \in U \subseteq \mathbb{H} \), and let \( R > 0 \) such the ball \( |p - q| \) is contained in \( U \). Assume that, for any \( I \in \mathbb{S} \), there exist two harmonic functions \( \alpha, \beta \) on \( U_I = U \cap \mathbb{C}_I \), with real positive values on the disk \( \{ p \in \mathbb{H} \mid |p - q| < R \} \cap U_I \) such that \( f(u + I_p v) = \alpha(u, v) + I_p \beta(u, v) \), for all \( u + I_p v \in U \). Then for any \( p \in U \) with \( |p - q| < r < R \), one has
\[
\frac{R-r}{R+r} |f(q)| \leq |f(p)| \leq \frac{R+r}{R-r} |f(q)|.
\]

**Proof.** Since \( \alpha \) and \( \beta \) are functions of two variables, we can use the classical Harnack inequality and we obtain
\[
\left( \frac{R-r}{R+r} \right)^2 \alpha(a, b)^2 \leq \alpha(u, v)^2 \leq \left( \frac{R+r}{R-r} \right)^2 \alpha(a, b)^2,
\]
\[
\left( \frac{R-r}{R+r} \right)^2 \beta(a, b)^2 \leq \beta(u, v)^2 \leq \left( \frac{R+r}{R-r} \right)^2 \beta(a, b)^2,
\]
where \( q = a + I_q b, p = u + I_p v \), and \( |p - q| < r < R \). The result is obtained by adding, respectively, the terms of the previous inequalities and by taking the square root. \( \square \)

**Corollary 3.1.15.** Let \( f \in \mathcal{R}(U) \), \( q \in U \), and let \( R > 0 \) be such that the ball \( |p - q| \) is contained in \( U \). Assume that the elements \( I_1, I_2, I_3 = I_1 I_2 \in \mathbb{S} \) form a basis of \( \mathbb{H} \) and set \( I_0 = 1 \). If \( f = \sum_{\ell=0}^3 f_\ell I_\ell \) and the functions \( f_\ell \) satisfy the hypothesis of Proposition 3.1.14, then for any \( p \in U \) with \( |p - q| < r < R \), one has
\[
\frac{R-r}{R+r} \sum_{\ell=0}^3 |f_\ell(q)| \leq \frac{R-r}{R+r} \sum_{\ell=0}^3 |f_\ell(p)| \leq \frac{R+r}{R-r} \sum_{\ell=0}^3 |f_\ell(q)|.
\]
3.2 Riemann mapping theorem

The Riemann mapping theorem, formulated by Riemann back in 1851, is a fundamental result in the geometric theory of functions of a complex variable. It has a variety of applications not only in the theory of functions of a complex variable, but also in mathematical physics, in the theory of elasticity, and in other frameworks. This theorem cannot be extended in its full generality to the quaternionic setting, see [100], and in order to understand how it can be generalized, it is useful to recall it in the classical complex case:

**Theorem 3.2.1** (Riemann Mapping Theorem). Let \( \Omega \subset \mathbb{C} \) be a simply connected domain, \( z_0 \in \Omega \) and let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) denote the open unit disk. Then there exists a unique bijective analytic function \( f : \Omega \rightarrow \mathbb{D} \) such that \( f(z_0) = 0 \), \( f'(z_0) > 0 \).

The theorem holds also for simply connected open subsets of the Riemann sphere which both lack at least two points of the sphere.

As we said before, this theorem does not generalize to the case of all simply connected domains in \( \mathbb{H} \). First of all, it is necessary to characterize which open sets can be mapped bijectively onto the unit ball of \( \mathbb{H} \) by a slice regular function \( f \) having prescribed value at one point and with prescribed value of the derivative at the same point. The class of open sets to which the theorem can be extended, as we shall see, is the class of axially symmetric slice domains which are simply connected.

Let us recall that in [87], the so-called *typically real* functions are functions defined on the open unit disc \( \mathbb{D} \), which take real values on the real line and only there. These functions have real coefficients when expanded into power series and so they are complex intrinsic (see [87], p.55). Thus the image of the open unit disc through such mappings is symmetric with respect to the real line.

The following result will be used in the sequel:

**Proposition 3.2.2.** Let \( \Omega \subset \mathbb{C} \) be a simply connected domain such that \( \Omega \cap \mathbb{R} \neq \emptyset \), let \( x_0 \in \Omega \cap \mathbb{R} \) be fixed and let \( f : \Omega \rightarrow \mathbb{D} \) with \( f(x_0) = 0 \), \( f'(x_0) > 0 \) be the bijective analytic function as in Theorem 3.2.1. Then \( f^{-1} \) is typically real if and only if \( \Omega \) is symmetric with respect to the real axis.

*Proof.* Let \( f : \Omega \rightarrow \mathbb{D} \) be the function as in the Riemann mapping theorem. If \( \Omega \) is symmetric with respect to the real axis, then by the uniqueness of \( f \) we have \( \overline{f(z)} = f(\bar{z}) \), see e.g. [4], Exercise 1, p. 232. Thus \( f \) maps bijectively \( \Omega \cap \mathbb{R} \) onto \( \mathbb{D} \cap \mathbb{R} \) and \( f^{-1} \) is typically real. Conversely, assume
that $f^{-1} : \mathbb{D} \to \Omega$ is typically real. Then $\Omega$, being the image of the open unit disc, is symmetric with respect to the real line.

**Corollary 3.2.3.** Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\Omega \cap \mathbb{R} \neq \emptyset$, let $x_0 \in \Omega \cap \mathbb{R}$ be fixed and let $f : \Omega \to \mathbb{D}$ with $f(x_0) = 0$, $f'(x_0) > 0$ be the bijective analytic function as in Theorem 3.2.1. Then $f^{-1}$ is typically real if and only if $f$ is complex intrinsic.

**Proof.** If $f^{-1}$ is typically real then $\Omega$ is symmetric with respect to the real line and $f^{-1}(w) = f^{-1}(\bar{w})$. By setting $w = f(z)$, we have $z = f^{-1}(w)$, and so

$$f(\bar{z}) = f(f^{-1}(w)) = f(f^{-1}(\bar{w})) = \bar{w} = f(z).$$

Thus $f$ is complex intrinsic. Conversely, let $f$ be complex intrinsic: then $f$ is defined on set $\Omega$ symmetric with respect to the real line and $f(\bar{z}) = f(z)$.

So, by Proposition 3.2.2, $f^{-1}$ is complex intrinsic. \qed

To summarize the results, we state the following:

**Corollary 3.2.4.** Let $\Omega \subset \mathbb{C}$ be a simply connected domain such that $\Omega \cap \mathbb{R} \neq \emptyset$, let $x_0 \in \Omega \cap \mathbb{R}$ be fixed and let $f : \Omega \to \mathbb{D}$ with $f(x_0) = 0$, $f'(x_0) > 0$ be the bijective analytic function as in Theorem 3.2.1. Then the following statements are equivalent:

1. $f^{-1}$ is typically real;
2. $f$ is complex intrinsic;
3. $\Omega$ is symmetric with respect to the real line.

**Remark 3.2.5.** Let $\Omega \subset \mathbb{C}$ be symmetric with respect to the real axis and let the function $f$ as in the statement of the Riemann mapping theorem be intrinsic, i.e. $\overline{f(z)} = f(z)$. By identifying $\mathbb{C}$ with $\mathbb{C}_J$ for some $J \in S$ and using the extension formula (2.10), we obtain a function $\text{ext}(f)$ which is quaternionic intrinsic (see the proof of Proposition 2.4.9). Thus $\text{ext}(f) \in \mathcal{N}(U_\Omega)$ where $U_\Omega \subset \mathbb{H}$ denotes the axially symmetric completion of $U$.

**Definition 3.2.6.** We will denote by $\mathcal{R}(\mathbb{H})$ the class of axially symmetric open sets $U$ in $\mathbb{H}$ such that $U \cap \mathbb{C}_I$ is simply connected for every $I \in S$.

The set $U \cap \mathbb{C}_I$ is simply connected for every $I \in S$ and thus it is connected, so $U$ is a slice domain.

In view of Remark 3.2.5 and of Corollary 3.2.4 we have the following:
Corollary 3.2.7 (Quaternionic Riemann mapping theorem). Let $U \in \mathcal{R}(\mathbb{H})$, $\mathbb{B} \subset \mathbb{H}$ be the open unit ball and let $x_0 \in U \cap \mathbb{R}$. Then there exists a unique quaternionic intrinsic slice regular function $f : U \to \mathbb{B}$ which is bijective and such that $f(x_0) = 0$, $f'(x_0) > 0$.

Proof. Let us consider $U_I = U \cap \mathbb{C}_I$ where $I \in \mathbb{S}$. Then $U_I$ is simply connected by hypothesis and symmetric with respect to the real line since $U$ is axially symmetric. Let us set $D_I = \mathbb{B} \cap \mathbb{C}_I$. By Corollary 3.2.4 there exists a bijective, analytic intrinsic map $f_I : U_I \to D_I$ such that $f(x_0) = 0$, $f'(x_0) > 0$. By Remark 3.2.5, $f_I$ extends to $f : U \to \mathbb{B}$ and $f \in \mathcal{N}(U)$. Note that for every $J \in \mathbb{S}$ we have $f|_{\mathbb{C}_J} : U_J \to \mathbb{D}_J$ since $f$ takes each complex plane to itself.

Remark 3.2.8. We now show that the Riemann mapping theorem proved above, holds under the optimal hypotheses, i.e. the class $\mathcal{R}(\mathbb{H})$ cannot be further enlarged. The class of open sets $\mathcal{R}(\mathbb{H})$ contains all the possible simply connected open sets in $\mathbb{H}$ intersecting the real line for which a map $f : U \to \mathbb{B}$ as in the Riemann mapping theorem belongs to the class $\mathcal{N}(U)$, namely to the class of functions for which the composition is allowed. In fact, assume that a simply connected open set $U \subset \mathbb{H}$ is mapped bijectively onto $\mathbb{B}$ by a map $f \in \mathcal{N}(U)$, with $f(x_0) = 0$, $f'(x_0) > 0$, $x_0 \in \mathbb{R}$. Since $f \in \mathcal{N}(U)$, we have that $f|_{\mathbb{C}_I} : U \cap \mathbb{C}_I \to \mathbb{B} \cap \mathbb{C}_I = D_I$ for all $I \in \mathbb{S}$ and so $f$ takes $U \cap \mathbb{R}$ to $\mathbb{B} \cap \mathbb{R}$. By its uniqueness, $f|_{\mathbb{C}_I}$ is the map prescribed by the complex Riemann mapping theorem, moreover $f|_{\mathbb{C}_I}$ takes $U_I \cap \mathbb{R}$ bijectively to $D_I \cap \mathbb{R}$ so $f^{-1}$ is totally real. By Corollary 3.2.4, it follows that $f$ is complex intrinsic, thus $U \cap \mathbb{C}_I$ is symmetric with respect to the real line. Since $I \in \mathbb{S}$ is arbitrary, $U$ must be also axially symmetric, so it belongs to $\mathcal{R}(\mathbb{H})$.

3.3 Zeros of slice regular functions

Zeros of polynomials with coefficients on one side (say, on the right in the present setting) have been studied in several papers over the years. The case of zeros of polynomials, though is a very special case, is perfect to illustrate the general situation that occurs with slice regular functions.

Let us begin by a very well known fact. Consider the equation:

$$(q - a) \star (q - b) = q^2 - q(a + b) + ab = 0.$$ 

Then $q = a$ is a zero while $q = b$, in general, is not. If $b$ does not belong to $[a]$ then, by (2.3.10), the second zero is $b' = (b - \bar{a})^{-1}b - \bar{a}$. If $b \in [a]$ but
b \neq \bar{a} \text{ then } q = a \text{ is the only zero of the equation and it has multiplicity 2.}

If \( b = \bar{a} \), then we obtain

\[
(q - a) \star (q - \bar{a}) = q^2 - 2\text{Re}(a)q + |a|^2 = 0 \quad (3.11)
\]

and the zero set coincides with the sphere \([a]\). Sometimes, we will denote the sphere \([a]\) also as \(a_0 + \mathbb{S}a_1\), if \(a = a_0 + I_a a_1\). When we have a sphere of zeros of \(f\), we shall speak of a spherical zero of \(f\).

It is important to note that the factorization of a quaternionic polynomial is not unique, in fact we have:

**Theorem 3.3.1.** Let \(a, b\) be quaternions belonging to two different spheres. Then

\[
(q - a) \star (q - b) = (q - b') \star (q - a')
\]

if and only if \(a' = c^{-1}ac, b' = c^{-1}bc\) where \(c = b - \bar{a}\).

However, we have cases in which the factorization is unique:

**Theorem 3.3.2.** The polynomial

\[
P(q) = (q - q_1) \star \cdots \star (q - q_r)
\]

where \(q_\ell \in [q_1]\) for all \(\ell = 2, \ldots, r\) and \(q_{\ell+1} \neq \bar{q}_\ell\), for \(\ell = 1, \ldots, r - 1\) is the unique factorization of \(P(q)\).

Any quaternionic polynomial admits a factorization as described below. We do not prove this result here since we will prove a result in the more general framework of slice regular functions. We refer the reader to [111] for more details.

**Theorem 3.3.3.** Let \(P(q)\) be a slice regular polynomial of degree \(m\). Then there exist \(p, m_1, \ldots, m_p \in \mathbb{N}\), and \(w_1, \ldots, w_p \in \mathbb{H}\), generators of the spherical roots of \(P\), so that

\[
P(q) = (q^2 - 2q\text{Re}(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2q\text{Re}(w_p) + |w_p|^2)^{m_p} Q(q), \quad (3.12)
\]

where \(Q\) is a slice regular polynomial with coefficients in \(\mathbb{H}\) having (at most) only non spherical zeroes. Moreover, if \(n = m - 2(m_1 + \cdots + m_p)\) there exist a constant \(c \in \mathbb{H}\), \(t\) distinct 2-spheres \(S_1 = x_1 + y_1\mathbb{S}, \ldots, S_t = x_t + y_t\mathbb{S}\), \(t\) integers \(n_1, \ldots, n_t\) with \(n_1 + \cdots + n_t = n\), and (for any \(i = 1, \ldots, t\)) \(n_i\) quaternions \(\alpha_{ij} \in S_i, j = 1, \ldots, n_i\), such that

\[
Q(q) = \prod_{i=1}^{t} \prod_{j=1}^{n_i} (q - \alpha_{ij}) \cdot c. \quad (3.13)
\]
This factorization result extends to slice regular functions thanks to the following result.

**Theorem 3.3.4.** Let $f$ be a slice regular function on an axially symmetric slice domain $U$, suppose $f \not\equiv 0$ and let $[x + iy] \subset U$. There exist $m \in \mathbb{N}, n \in \mathbb{N}, p_1, \ldots, p_n \in x + yS$ (with $p_i \neq \bar{p}_{i+1}$ for all $i \in \{1, \ldots, n-1\}$) such that

$$f(q) = [(q - x)^2 + y^2]^m (q - p_1) \ast \ldots \ast (q - p_n) \ast g(q)$$  \hspace{1cm} (3.14)

for some slice regular function $g : U \to \mathbb{H}$ which does not have zeros in the sphere $[x + iy]$.

**Proof.** If $f$ is not identically 0 on $U$, and $f$ vanishes at $[x + iy]$, then there exists an $m \in \mathbb{N}$ such that

$$f(q) = [(q - x)^2 + y^2]^m h(q)$$

for some $h$ not identically zero on $[x + iy]$. In fact, suppose that it were possible to find, for all $k \in \mathbb{N}$, a function $h^k(q)$ such that $f(q) = [(q - x)^2 + y^2]^k h^k(q)$. Then, choosing an $I \in S$, the holomorphic map $f_I$ would have the factorization

$$f_I(z) = [(z - x)^2 + y^2]^k h^k_I(z) = [z - (x + Iy)]^k [z - (x - Iy)]^k h^k_I(z)$$

for all $k \in \mathbb{N}$. This would imply $f_I \equiv 0$ and, by the Identity Principle, $f \equiv 0$.

Now let $h$ be a slice regular function on $U$ which does not vanish identically on $[x + iy]$. By Proposition 3.3.10, $h$ has at most one zero $p_1 \in [x + iy]$. If this is the case then

$$g^{[0]} = h(q) = (q - p_1) \ast g^{[1]}(q)$$

for some function $g^{[1]}$ which does not vanish identically on $[x + iy]$. If for all $k \in \mathbb{N}$ there existed a $p_{k+1} \in [x + iy]$ and a $g^{[k+1]}$ such that

$$g^{[k]}(q) = (q - p_{k+1}) \ast g^{[k+1]}$$

then we would have

$$h(q) = (q - p_1) \ast \ldots \ast (q - p_k) \ast g^{[k]}(q)$$

for all $k \in \mathbb{N}$. Thus the symmetrization $h^s$ of $h$ would be such that

$$h^s(q) = [(q - x)^2 + y^2]^k (g^{[k]})^s(q)$$

for all $k \in \mathbb{N}$. By the first part of the proof, this would imply $h^s \equiv 0$. Thus Proposition 3.3.10 implies that $h \equiv 0$, which is a contradiction. Thus there exists an $n \in \mathbb{N}$ such that $g^{[n]}$ does not have zeros in $[x + iy]$ and, setting $g = g^{[n]}$, we have the statement. \qed
We now introduce the notion of multiplicity of a root of a slice regular function in the various cases.

**Definition 3.3.5.** We say that a function \( f \) slice regular on an axially symmetric slice domain \( U \) has a zero at \( [q_0] \subset U \), \( q_0 = x_0 + I y_0 \), \( y_0 \neq 0 \) with spherical multiplicity \( m \), if \( m \) is the largest natural number such that
\[
f(q) = [(q - x_0)^2 + y_0^2]^m \ast g(q)
\]
for some function \( g \) slice regular on \( U \) which does not vanish at \( [q_0] \).

**Definition 3.3.6.** If a function \( f \) slice regular on an axially symmetric slice domain \( U \) has a root \( q_1 \in [q_0] \) in \( U \), \( q_0 = x_0 + I y_0 \), \( y_0 \neq 0 \), then we say that \( f \) has isolated multiplicity \( r \) at \( q_1 \), if \( r \) is the largest natural number such that there exist \( q_2, \ldots, q_r \in [q_0] \), \( q_{\ell+1} \neq q_{\ell} \) for all \( \ell = 1, \ldots, r - 1 \) and a function \( g \) slice regular in \( U \) which does not have zeros in \( [q_0] \) such that
\[
f(q) = (q - q_1) \ast (q - q_2) \ast \ldots \ast (q - q_r) [(q - x_0)^2 + y_0^2]^m \ast g(q),
\]
where \( m \in \mathbb{N} \).

If \( q_0 \in \mathbb{R} \) then we call isolated multiplicity of \( f \) at \( q_0 \) the largest \( r \in \mathbb{N} \) such that
\[
f(q) = (q - q_0)^r g(q)
\]
for some function \( g \) slice regular in \( U \) which does not vanish at \( q_0 \).

**Remark 3.3.7.** If \( f \) vanishes at the points of a sphere \( [q_0] \), all the points of that sphere have the same multiplicity \( m \) as zeros of \( f \), except possibly for one point which may have higher multiplicity. To see an example, it suffices to consider \( f(q) = (q - i) \ast (q - j)(q^2 + 1)^3 \). All points of the unit sphere \( \mathbb{S} \) have multiplicity 3 except for the point \( q = i \) which has multiplicity 5.

In particular, Theorem 2.3.10 implies that for each zero of \( f \ast g \) in \( [q_0] \) there exists a zero of \( f \) or a zero of \( g \) in \( [q_0] \). However, there are examples of products \( f \ast g \) whose zeros are not in one-to-one correspondence with the union of the zero sets of \( f \) and \( g \). It suffices to take \( f(q) = q - i \) and \( g(q) = q + i \) to see that both \( f \) and \( g \) have one isolated zero, while \( f \ast g \) has the sphere \( \mathbb{S} \) as its set of zeros.

We now study the relation between the zeros of \( f \) and those of \( f^c \) and \( f^s \). We need two preliminary steps.

**Lemma 3.3.8.** Let \( U \subseteq \mathbb{H} \) be an axially symmetric slice domain and let \( f \in \mathcal{N}(U) \). If \( f(x_0 + I_0 y_0) = 0 \) for some \( I_0 \in \mathbb{S} \), then \( f(x_0 + I y_0) = 0 \) for all \( I \in \mathbb{S} \).
Proof. Since \( f \in \mathcal{N}(U) \), then \( f(U_I) \subseteq \mathbb{C}_I \) for all \( I \in \mathbb{S} \). Thus \( f(x) \) is real for all \( x \in U \cap \mathbb{R} \). The restriction \( f_{I_0} : U_{I_0} \to \mathbb{C}_{I_0} \) is a holomorphic function mapping \( U \cap \mathbb{R} \) to \( \mathbb{R} \). By the (complex) Schwarz Reflection Principle, \( f(x + yI_0) = \overline{f(x - yI_0)} \) for all \( x + yI_0 \in U_{I_0} \). Since \( f(x_0 + y_0I_0) = 0 \), we conclude that \( f(x_0 - y_0I_0) = 0 \) and the Representation Formula allows to deduce the statement. \( \square \)

**Lemma 3.3.9.** Let \( U \subseteq \mathbb{H} \) be an axially symmetric slice domain, let \( f : U \to \mathbb{H} \) be a regular function and let \( f^* \) be its symmetrization. Then \( f^*(U_I) \subseteq \mathbb{C}_I \) for all \( I \in \mathbb{S} \).

Proof. It follows by direct computation from the definition of \( f^* \). \( \square \)

**Proposition 3.3.10.** Let \( U \subseteq \mathbb{H} \) be an axially symmetric slice domain, let \( f \in \mathcal{R}(U) \) and \( [x_0 + Iy_0] \subset U \).

1. The function \( f^* \) vanishes identically on \([x_0 + Iy_0]\) if and only if \( f^* \) has a zero in \([x_0 + Iy_0]\), if and only if \( f \) has a zero in \([x_0 + Iy_0]\);

2. The zeros of \( f \) in \([x_0 + Iy_0]\) are in one-to-one correspondence with those of \( f^* \);

3. The function \( f \) has a zero in \([x_0 + Iy_0]\) if and only if \( f^c \) has a zero in \([x_0 + Iy_0]\).

Proof. We prove assertion (1). First of all, we note that by Lemma 3.3.8 and Lemma 3.3.9 if \( f^*(x_0 + Iy_0) = 0 \) for some \( I_0 \in \mathbb{S} \) then \( f^*(x_0 + Iy_0) = 0 \) for all \( I \in \mathbb{S} \). The converse is trivial.

If \( q_0 = x_0 + I_0y_0 \) is a zero of \( f \) then, by Theorem 2.3.10 also \( f^* = f \ast f^c \) vanishes at \( q_0 \). Conversely, assume that \( q_0 \) is a zero of \( f^* \). By formula (2.3.10) either \( f(q_0) = 0 \) or \( f^c \) vanishes at the point

\[
f(q_0)^{-1}q_0f(q_0) = x_0 + y_0[f(q_0)^{-1}I_0f(q_0)] \in [q_0].
\]

In the first case, we have concluded, in the second case, we recall that \( f^* \) vanishes not only at \( q_0 \) but on the whole sphere \([q_0]\) and so \( f^*(\bar{q}_0) = 0 \). This fact implies that either \( f(\bar{q}_0) = 0 \) or

\[
f^c(f(\bar{q}_0)^{-1}\bar{q}_0f(\bar{q}_0)) = 0.
\]

In the first case, the Representation Formula yields that \( f \) vanishes identically on \([q_0]\), which implies that, by its definition, also \( f^c \) vanishes on \([q_0]\).

In the second case, \( f^c \) vanishes at the point

\[
f(\bar{q}_0)^{-1}\bar{q}_0f(\bar{q}_0) = x_0 - y_0[f(\bar{q}_0)^{-1}I_0f(\bar{q}_0)] \in [q_0],
\]
thus $f^c$ vanishes at $[q_0]$ and so also $f$ vanishes at $[q_0]$ and the statement follows.
Thus, we have proven that $f$ has a zero in $[q_0]$ if and only if $f^s$ has a zero in $[q_0]$, which leads to the vanishing of $f^s$ on the whole $[q_0]$, which implies the existence of a zero of $f^c$ in $[q_0]$. Since $(f^c)^c = f$, exchanging the roles of $f$ and $f^c$ gives the rest of the statement.

We now study the distribution of the zeros of regular functions on axially symmetric slice domains. In order to obtain a full characterization of the zero set of a regular function, we first deal with a special case that will be crucial in the proof of the main result.

**Lemma 3.3.11.** Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \rightarrow \mathbb{H}$ be a slice regular intrinsic function. If $f \neq 0$, the zero set of $f$ is either empty or it is the union of isolated points (belonging to $\mathbb{R}$) and/or isolated 2-spheres.

*Proof.* We know from Lemma 3.3.8 that the zero set of such an $f$ consists of 2-spheres of the type $x + yS$, possibly reduced to real points. Now choose $I$ in $S$ and notice that the intersection of $C_I$ with the zero set of $f$ consists of all the real zeros of $f$ and of exactly two zeros for each sphere $x + yS$ on which $f$ vanishes (namely, $x + yI$ and $x - yI$). If $f \neq 0$ then, by the Identity Principle, the zeros of $f$ in $C_I$ must be isolated. Hence the zero set of $f$ consists of isolated real points and/or isolated 2-spheres.

We now state and prove the result on the topological structure of the zero set of regular functions.

**Theorem 3.3.12** (Structure of the Zero Set). Let $U \subseteq \mathbb{H}$ be an axially symmetric slice domain and let $f : U \rightarrow \mathbb{H}$ be a regular function. If $f$ does not vanish identically, then the zero set of $f$ consists of isolated points or isolated 2-spheres.

*Proof.* Consider the symmetrization $f^s$ of $f$: by Lemma 3.3.9, $f^s$ fulfills the hypotheses of Lemma 3.3.11. Hence the zero set of $f^s$ consists of isolated real points or isolated 2-spheres. According to Proposition 3.3.10, the real zeros of $f$ and $f^s$ are exactly the same. Furthermore, each 2-sphere in the zero set of $f^s$ corresponds either to a 2-sphere of zeros, or to a single zero of $f$. This concludes the proof.

An immediate consequence is:
Corollary 3.3.13 (Strong Identity Principle). Let $f$ be a function slice regular in an axially symmetric slice domain $U$. If there exists a sphere $[q_0]$ such that the zeros of $f$ in $U \setminus [q_0]$ accumulate to a point of $[q_0]$ then $f$ vanishes on $U$.

3.4 Modulus of a slice regular function and Ehrenpreis-Malgrange lemma

We start by proving two results extending the Maximum and Minimum Modulus Principle to the present setting.

Theorem 3.4.1 (Maximum Modulus Principle). Let $U$ be a slice domain and let $f : U \to \mathbb{H}$ be slice regular. If $|f|$ has a relative maximum at $p \in U$, then $f$ is constant.

Proof. If $p$ is a zero of $f$ then $|f|$ has zero as its maximum value, so $f$ is identically 0. So we assume $f(p) \neq 0$. It is not reductive to assume that $f(p) \in \mathbb{R}, f(p) > 0$, by possibly multiplying $f$ by $\overline{f(p)}$ on the right side. Let $I, J \in S$ be such that $p$ belongs to the complex plane $\mathbb{C}_I$ and $I \perp J$. We use the Splitting Lemma to write $f_I = F + GJ$ on $U_I = U \cap \mathbb{C}_I$. Then, for all $z$ in a neighborhood $V_I = U \cap \mathbb{C}_I$ of $p$ in $U_I$ we have, since $f(p)$ is real:

$$|F(p)|^2 = |f_I(p)|^2 \geq |f_I(z)|^2 = |F(z)|^2 + |G(z)|^2 \geq |F(z)|^2.$$

Hence $|F|$ has a relative maximum at $p$ and the Maximum Modulus Principle for holomorphic functions of one complex variable allows us to conclude that $F$ is constant and so $F \equiv f(p)$. As a consequence, we have

$$|G(z)|^2 = |f_I(z)|^2 - |F(z)|^2 = |f_I(z)|^2 - |f_I(p)|^2 \leq |f_I(p)|^2 - |f_I(p)|^2 = 0$$

for all $z \in V_I$, and so $f_I = F \equiv f(p)$ in $V_I$. From the Identity Principle, we deduce that $f \equiv f(p)$ in $U$.

Theorem 3.4.2 (Minimum Modulus Principle). Let $U$ be an axially symmetric slice domain and let $f : U \to \mathbb{H}$ be a slice regular function. If $|f|$ has a local minimum point $p \in U$ then either $f(p) = 0$ or $f$ is constant.

Proof. Consider a slice regular function $f : U \to \mathbb{H}$ whose modulus has a minimum point $p \in U$ with $f(p) \neq 0$. Such an $f$ does not vanish on the sphere $S$ defined by $p$. Indeed, if $f$ vanished at a point $p' \in S$ then $|f_{|S}|$ would have a global minimum at $p'$, a global maximum and no other
extremal point, which contradicts the hypothesis on \( p \). But if \( f \) does not have zeroes in \( S \), neither \( f^* \) does. Hence the domain \( U' = U \setminus Z_f^* \) of \( f^{-*} \) includes \( S \) (where \( Z_f^* \) denotes the set of zeros of \( f^* \)). Thanks to Theorem 2.3.10

\[
|f^{-*}(q)| = \frac{1}{|f(q)|}, \quad q \in U'
\]

for a suitable \( q \) belonging to the sphere of \( q \). If \( |f| \) has a minimum at \( p \in x + yS \subseteq U' \) then \( |f| \) has a minimum at \( \tilde{p} \in U' \). As a consequence, \( |f^{-*}| \) has a maximum at \( \tilde{p} \). By the Maximum Modulus Principle, \( f^{-*} \) is constant on \( U' \). This implies that \( f \) is constant in \( U' \) and, thanks to the Identity Principle, it is constant in \( U \).

We now prove an analog of the Ehrenpreis-Malgrange lemma, giving lower bounds for the moduli of polynomials away from their zeros, for polynomials with quaternionic coefficients. We begin with a simple result, which deals with the case in which we are interested in finding a lower bound on a sphere centered at the origin. In general, the bounds will be assigned on a 3-dimensional toroidal hypersurface.

**Theorem 3.4.3.** Let \( P(q) \) be a slice regular polynomial of degree \( m \), with leading coefficient \( a_m \). Let \( p \) be the number of distinct spherical zeroes of \( P(q) \), let \( t \) be the number of distinct isolated zeroes and let \( M = p + t \). Given any \( R > 0 \), we can find a sphere \( \Gamma \) centered at the origin and of radius \( r < R \) on which

\[
|P(q)| \geq |a_m| \left( \frac{R}{2(M + 1)} \right)^m.
\]

**Proof.** By using Theorem 3.3.3 we can decompose \( P(q) \) as

\[
P(q) = S(q)Q(q)a_m
\]

where \( S \) is a product of factors of the form

\[
S(q) = (q^2 - 2qRe(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2qRe(w_p) + |w_p|^2)^{m_p}
\]

and \( Q \) as in (3.13). The cardinality of the set \( V = \{ q : q \in \mathbb{H}, P(q) = 0 \} \) is at most \( M \), in fact some isolated zeros may belong to some spheres of zeros. In any case, there exists a subinterval \([a, b]\) of \([0, R]\) of length at least \( \frac{R}{M+1} \) which does not contain any element of \( V \). Let \( \Gamma \) be the 3-sphere centered in the origin and with radius \( \frac{a+b}{2} \).

We now estimate from below the absolute value of \( P(q) \) on a generic point on \( \Gamma \). Since \( P(q) = S(q)Q(q)a_m \), we will estimate the absolute values
of both $S(q)$ and $Q(q)$ on $\Gamma$. To estimate $S(q)$, it is useful to recall that for any pair of quaternions $q$ and $\alpha$, we have

$$q^2 - 2\text{Re}(\alpha)q + |\alpha|^2 = q^2 - \alpha q - \overline{\alpha}q + \overline{\alpha}\alpha = (q - \alpha)(q - (q - \alpha)^{-1}\overline{\alpha}(q - \alpha));$$

and so

$$|q^2 - 2\text{Re}(\alpha)q + |\alpha|^2| = |(q - \alpha)||q - (q - \alpha)^{-1}\overline{\alpha}(q - \alpha)|$$

$$\geq ||q| - |\alpha|| \cdot ||q - (q - \alpha)^{-1}\overline{\alpha}(q - \alpha)||$$

$$= ||q| - |\alpha|| \cdot ||q - |\alpha|||$$

$$= ||q| - |\alpha||^2.$$

Thus we have

$$|S(q)| = |(q^2 - 2q\text{Re}(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2q\text{Re}(w_p) + |w_p|^2)^{m_p}|$$

$$= |q^2 - 2q\text{Re}(w_1) + |w_1|^2|^{m_1} \cdots |q^2 - 2q\text{Re}(w_p) + |w_p|^2|^{m_p}$$

$$\geq ||q| - |w_1||^{2m_1} \cdots ||q - |w_p||^{2m_p}.$$

To estimate $Q(q)$, we first note that, for suitable quaternions $\alpha_1, \ldots, \alpha_N$, we can split $Q(q)$ into linear factors as

$$Q(q) = (q - \alpha_1) \cdots (q - \alpha_N),$$

and the estimate for $Q(q)$ can be obtained recursively as follows. It is immediate that

$$|q - \alpha_N| \geq ||q| - |\alpha_N||.$$

Let $\ell \leq N - 1$ be an integer and set

$$h_{\ell+1}(q) := (q - \alpha_{\ell+1}) \cdots (q - \alpha_N).$$

We now assume that for $\ell \leq N - 1$ we have established

$$|h_{\ell+1}(q)| = |(q - \alpha_{\ell+1}) \cdots (q - \alpha_N)| \geq ||q| - |\alpha_{\ell+1}|| \cdots ||q - |\alpha_N||, \quad (3.15)$$

and we proceed to the estimate for

$$|h_{\ell}(q)| = |(q - \alpha_\ell) \cdots (q - \alpha_N)|.$$
The associativity of the $\star$-product and Theorem 2.3.10 imply that

\[
|(q - \alpha_\ell) \star (q - \alpha_{\ell+1}) \star \cdots \star (q - \alpha_N)| \\
= |(q - \alpha_\ell) \star ((q - \alpha_{\ell+1}) \star \cdots \star (q - \alpha_N))| \\
= |(q - \alpha_\ell) \star (h_{\ell+1}(q))| \\
= |(q - \alpha_\ell) \cdot h_{\ell+1}((q - \alpha_\ell)^{-1}q(q - \alpha_\ell))| \\
= |(q - \alpha_\ell)| \cdot h_{\ell+1}((q - \alpha_\ell)^{-1}q(q - \alpha_\ell))|.
\]

Using

\[
|(q - \alpha_\ell)^{-1}q(q - \alpha_\ell)| = |q|,
\]

and (3.15), we obtain

\[
|(q - \alpha_\ell) \star \cdots \star (q - \alpha_N)| \geq ||q| - |\alpha_\ell|| \cdot \cdots \cdot ||q| - |\alpha_N||.
\]

In conclusion we have:

\[
|Q(q)| \geq ||q| - |\alpha_1|| \cdot \cdots \cdot ||q| - |\alpha_N||.
\]

Since each factor in the decomposition of $P(q)$ is bounded below by $\frac{R}{2(M+1)}$, the statement follows.

Remark 3.4.4. The estimate proved in Theorem 3.4.3 holds also if the sphere $\Gamma$ is centered at any real point $q_0$.

The next result explains what happens if one attempts to estimate $|P(q)|$ from below, on spheres centered on points $q_0$ which are not real.

Theorem 3.4.5. Let $P(q)$ be a slice regular polynomial of degree $m$ with only spherical zeroes, i.e. a polynomial of the form

\[
P(q) = (q^2 - 2q\text{Re}(w_1) + |w_1|^2)^{m_1} \cdots (q^2 - 2q\text{Re}(w_p) + |w_p|^2)^{m_p}a_m,
\]

with $w_1, \ldots, w_p, a_m \in \mathbb{H}$. For any $q_0 = u + vI \in \mathbb{H}$ and for any $R > 0$, there exist $r < R$ and a 3-dimensional compact hypersurface

\[
\Gamma = \Gamma(q_0, r) = \{ x + yI : (x - u)^2 + (y - v)^2 = r^2 \text{ and } I \in \mathbb{S} \},
\]

smooth if $r < v$, such that for every $q \in \Gamma$ it is

\[
|P(q)| \geq |a_m| \left( \frac{R}{2(m+1)} \right)^m.
\]
Proof. Without loss of generality, we can assume $a_m = 1$. The restriction of $P(q)$ to the complex plane $\mathbb{C}_I$ is a complex polynomial with up to $m$ distinct zeroes. Consider the set

$$V = \{|q - q_0| : q \in \mathbb{C}_I, P(q) = 0\};$$

the cardinality of $V$ is at most $m$. So we can find a subinterval $[a, b]$ of $[0, R]$ of length at least $\frac{R}{m+1}$ which does not contain any element of $V$. Let $\Gamma$ be the circle in $\mathbb{C}_I$ centered in $q_0$ and with radius $r = \frac{a+b}{2}$. Then, on $\mathbb{C}_I$, one has

$$|q^2 - 2qRe(w) + |w|^2| = |(q - w)(q - \overline{w})| = |q - w||q - \overline{w}| = |(q - q_0) - (w - q_0)||q - q_0| - (\overline{w} - q_0)| \geq ||q - q_0| - |w - q_0|| \cdot ||q - q_0| - |\overline{w} - q_0||.$$

Since $w$ and $\overline{w}$ are roots of $P(q)$, we have

$$|q^2 - 2qRe(w) + |w|^2| \geq \left(\frac{R}{2(m+1)}\right)^2$$

and so

$$|P(q)| \geq \left(\frac{R}{2(m+1)}\right)^m.$$

Since $P$ has real coefficients, the estimate is independent of $I$, and the bound we have proved holds on $\Gamma$. \qed

We now prove the following simple lemma:

Lemma 3.4.6. Let $q_0 = u + I_0 v$ be a given point in $\mathbb{H}$ and let $w = a + Ib$ be the generic point on the sphere $|w|$. The distance between $w$ and $q_0$ achieves its extremal points at $w^0 = a + I_0 b$ and $\overline{w}^0 = a - I_0 b$.

Proof. Up to a rotation, we may assume that $I_0 = i$ so that $q_0 = u + iv$. An element $I \in \mathbb{S}$ can be written as $I = \alpha i + \beta j + \gamma k$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$ so that

$$w = a + b(\alpha i + \beta j + \gamma k).$$

The square of the distance between $w$ and $q_0$ is therefore given by

$$d^2(\alpha, \beta, \gamma) = a^2 + u^2 - 2au + b^2\alpha^2 + v^2 - 2bav + b^2\beta^2 + b^2\gamma^2.$$

Since $\mathbb{S}$ is a compact set, the function $d^2(\alpha, \beta, \gamma)$ admits at least a maximum and a minimum in $|w|$, which can be computed with standard techniques. A quick computation shows that the maximum and minimum are achieved for $\alpha = \pm 1, \beta = \gamma = 0$. \qed
Theorem 3.4.7. Let \( P(q) \) be a slice regular polynomial of degree \( m \) with only isolated zeroes, i.e. a polynomial of the form

\[
P(q) = (q - \alpha_1) \cdots (q - \alpha_m) a_m,
\]

with \( \alpha_1, \ldots, \alpha_m, a_m \in \mathbb{H} \). For any \( q_0 = u + vI_0 \in \mathbb{H} \) and for any \( R > 0 \), there exist \( r < R \), and a 3-dimensional compact hypersurface \( \Gamma = \Gamma(q_0, r) \), smooth if \( r < v \), such that for every \( q \in \Gamma \) it is

\[
|P(q)| \geq |a_m| \left( \frac{R}{2(2m + 1)} \right)^m.
\]

Proof. As before, it is not reductive to assume that \( a_m = 1 \). For \( t = 1, \ldots, m \), define \( \alpha_0^t = \text{Re}(\alpha_t) + I_0 \text{Im}(\alpha_t) \) and consider the following subset of the real numbers

\[
V = \{|\alpha_0^t - q_0|, |\vec{\alpha}_t^0 - q_0| \text{ for } t = 1, \ldots, m\}.
\]

Given any \( R > 0 \) there is at least one subinterval \([c, d]\) of \([0, R]\) which does not contain any element of \( V \) and whose length is at least \( \frac{R}{2m+1} \). Set \( r = \frac{c+d}{2} \) and define

\[
\Gamma = \{x + Iy : (x-u)^2 + (y-v)^2 = r^2 \text{ and } I \in \mathbb{S}\}.
\]

We now estimate, for \( q \in \Gamma \), the modulus of \( P(p) = (q - \alpha_1) \cdots (q - \alpha_m) \).

By Theorem 2.3.10 there exist

\[
\alpha'_t \in S_{\alpha_t} = \text{Re}(\alpha_t) + \text{Im}(\alpha_t) \mathbb{S},
\]

for \( t = 1, \ldots, m \), such that

\[
|P(p)| = |(q - \alpha_1') \cdots (q - \alpha_m')| = |q - \alpha_1'| \cdots |q - \alpha_m'|.
\]

Let us now estimate each factor:

\[
|q - \alpha_1'| = |(q - q_0) - (\alpha_1' - q_0)| \geq ||q - q_0| - |\alpha_1' - q_0||.
\]

Note that \( ||q - q_0| - |\alpha_1' - q_0|| \) is either \( |q - q_0| - |\alpha_1' - q_0| \) or \( |\alpha_1' - q_0| - |q - q_0| \).

It is evident that these two cases can be treated in the same way, thus we consider

\[
||q - q_0| - |\alpha_1' - q_0|| = |q - q_0| - |\alpha_1' - q_0|.
\]

In order to find a lower bound for this expression, we need a lower bound for \( |q - q_0| \) and an upper bound for \( |\alpha_1' - q_0| \). By Lemma 3.4.6 and since \( \alpha_1' \in S_{\alpha_t} \), we have

\[
|q - q_0| - |\alpha_1' - q_0| \geq |q^0 - q_0| - |\bar{\alpha}_1^0 - q_0|
\]

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where \( q^0 = \text{Re}(q) \pm \text{Im}(q) I_0 \) and \( \tilde{\alpha}_t^0 \) is either \( \alpha_t^0 \) or \( \overline{\alpha}_t^0 \). By the definition of the set \( V \), we obtain that

\[
|q - \alpha'_t| \geq \frac{R}{2(2m + 1)},
\]

and

\[
|P(q)| \geq |a_m| \left( \frac{R}{2(2m + 1)} \right)^m.
\]

As a consequence we can prove a lower bound for general slice regular polynomials:

**Theorem 3.4.8.** Let \( P(q) \) be a slice regular polynomial of degree \( m \), with leading coefficient \( a_m \). Let \( p \) be the number of distinct spherical zeroes of \( P(q) \), and let \( t \) be the number of its distinct isolated zeroes. Given any \( q_0 = u + vI_0 \in \mathbb{H} \) and any \( R > 0 \), we can find \( r < R \) and a compact 3-dimensional hypersurface

\[
\Gamma = \{ x + yI : (x-u)^2 + (y-v)^2 = r^2 \text{ and } I \in S \},
\]

smooth if \( R < v \), on which

\[
|P(q)| \geq |a_m| \left( \frac{R}{2(2m + 1)} \right)^m.
\]

**Proof.** The assertion follows from Theorem 3.4.5 and Theorem 3.4.7.

Theorem 3.4.8 can be reformulated in a way that does not require the knowledge of the nature of the zeros. See the result below. The estimate that one obtains is naturally not as sharp, but it is the exact analog of the corresponding result in the complex case:

**Theorem 3.4.9.** Let \( P(q) \) be a slice regular polynomial of degree \( m \), with leading coefficient \( a_m \). Given any \( q_0 = u + vI_0 \in \mathbb{H} \) and any \( R > 0 \), we can find a compact 3-dimensional hypersurface

\[
\Gamma = \{ x + yI : (x-u)^2 + (y-v)^2 = r^2 \text{ and } I \in S \},
\]

with \( r < R \), smooth if \( R < v \), on which

\[
|P(q)| \geq |a_m| \left( \frac{R}{2(2m + 1)} \right)^m.
\]
The next result is a consequence:

**Theorem 3.4.10.** Let $P(q)$ be a slice regular polynomial of degree $m$, with leading coefficient $a_m$. For any $R > 0$ there exist a natural number $n \leq m$, $n$ quaternions $q_1 = u_1 + v_1I_1, \ldots, q_n = u_n + v_nI_n$, $n$ strictly positive radii $r_1 < R, \ldots, r_n < R$, and $n$ corresponding compact sets

$$D(q_\ell, r_\ell) = \{x + yI : (x - u_\ell)^2 + (y - v_\ell)^2 \leq r_\ell^2 \text{ and } I \in \mathbb{S}\}$$

($\ell = 1, \ldots, n$) bounded, respectively, by the 3-dimensional hypersurfaces $\partial D(q_\ell, r_\ell)$, smooth if $r_\ell < v_\ell$, such that

$$|P(q)| \geq |a_m| \left(\frac{R}{2(2m + 1)}\right)^m$$

for $q$ outside

$$D = \bigcup_{\ell=1}^n D(q_\ell, r_\ell).$$

**Proof.** Given $R > 0$, it is immediate to find $n \leq m$ quaternions $q_\ell = u_\ell + v_\ellI_\ell$, $\ell = 1, \ldots, n$ such that $A = \bigcup_{\ell=1}^n D(q_\ell, R)$ contains all the roots of $P$. We then apply Theorem 3.4.9 and deduce that, for each $q_\ell$ there exists $r_\ell$ such that

$$|P(q)| \geq |a_m| \left(\frac{R}{2(2m + 1)}\right)^m$$

on the 3-hypersurface $\partial D(q_\ell, r_\ell)$. Thus inequality (3.16) holds on

$$B = \bigcup_{\ell=1}^n \partial D(q_\ell, r_\ell).$$

The set $B$ contains the boundary $\partial D$ of $D = \bigcup_{\ell=1}^n D(q_\ell, r_\ell)$, that coincides with the boundary $\partial(\mathbb{H} \setminus D)$ of $\mathbb{H} \setminus D$. The slice regular polynomial $P$ has no zeros in $\mathbb{H} \setminus D$ and since $\lim_{q \to +\infty} |P(q)| = +\infty$, we get that inequality (3.16) holds on each (open) connected component of $\mathbb{H} \setminus D$. In fact, if this were not the case, $|P|$ would have a local minimum at some point $q \in \mathbb{H} \setminus D$ with $P(q) \neq 0$, and by the Minimum Modulus Principle applied to $P$ on $\mathbb{H}$, $P$ would be constant. The statement follows.

### 3.5 Cartan theorem

In this section we prove an analog of Cartan theorem, providing an estimate from below for the modulus of a quaternionic polynomial. Note that in the statement the roots may be repeated.
Theorem 3.5.1. Let $P(q)$ be a polynomial having isolated roots $\alpha_1, \ldots, \alpha_t$ and spherical zeros $[\beta_1], \ldots, [\beta_p]$. Let $n = \deg P$, i.e. $n = t + 2p$ and let $H$ be any positive real number. Then there are balls in $\mathbb{H}$ with the sum of their radii equal to $2H$ such that for each point $q$ lying outside of these balls the following inequality is satisfied:

$$|P(q)| > \left( \frac{H}{e} \right)^n. \quad (3.17)$$

Proof. We divide the proof in steps.

Step 1. If there is a ball of radius $H$ containing all the zeros of the polynomial $P$, we can consider a ball $B$ with the same center and radius $2H$. Then, for any $q \in \mathbb{H} \setminus B$ we have that the distance from $q$ to any isolated zero of $P$ is at least $H$. By Lemma 3.4.5 also the distance from $q$ to any spherical zero is at least $H$. Consider a decomposition of $P$ into factors (see Theorem 3.3.3) and write the factors associated to a spherical zero $[\beta]$ as $(q - \beta) \ast (q - \overline{\beta})$.

From these considerations, it follows that

$$|(q - \alpha_1) \ast \ldots \ast (q - \alpha_t) \ast (q - \beta_1) \ast (q - \overline{\beta}_1) \ldots (q - \beta_p) \ast (q - \overline{\beta}_p)| = |(q - \alpha_1) \ast \ldots \ast (q - \alpha_t)| \cdot |(q - \beta_1) \ast (q - \overline{\beta}_1) \ldots (q - \beta_p) \ast (q - \overline{\beta}_p)|.

Using Theorem 2.3.10 we can rewrite the products $(q - \alpha_1) \ast \ldots \ast (q - \alpha_t)$ and $(q - \beta_1) \ast \ldots \ast (q - \beta_p)$ as pointwise products of the form $(q - \alpha_1) \ast \ldots \ast (q - \alpha_t)$ and $(q - \beta_1) \ast \ldots \ast (q - \beta_p)$ where $\tilde{\alpha}, \ldots, \tilde{\beta} \in [q]$ and so their distance from the zeros of $P$ is at least $H$. So we have

$$|(q - \alpha_1) \ast \ldots \ast (q - \alpha_t)| \cdot |(q - \beta_1) \ast (q - \overline{\beta}_1) \ldots (q - \beta_p) \ast (q - \overline{\beta}_p)|

> H^n > \left( \frac{H}{e} \right)^n,$$

Step 2. If there is no ball as described in Step 1, let us consider the balls with radius $\lambda H/n$ containing exactly $\lambda$ zeros of $P$, where each spherical zero is described in a decomposition of $P$ into linear factors $(q - \beta)$, $(q - \overline{\beta})$ where $\beta, \overline{\beta}$ is any pair of points belonging to the sphere (and thus a sphere counts as two zeros). Let $\lambda_1$ be the largest integer such that a ball $B_1$ of radius $\lambda_1 H/n$ contains exactly $\lambda_1$ zeros of $P$. No ball of radius $\eta H/n$ greater than or equal to $\lambda_1 H/n$ can contain more than $\eta$ zeros. In fact, assume that there is a ball of radius $\eta H/n$, $\eta \geq \lambda_1$ containing $\eta'$ zeros of $P$. Then the concentric ball of radius $\eta' H/n$ contains either $\eta'$ zeros or $\eta'' > \eta'$ zeros. The first case is not possible as $\lambda_1$ was the largest integer. In the second case,
we consider a concentric ball of radius $\eta''H/n$ and we repeat the procedure. Since the number of zeros (isolated or spherical) is finite, we will find a $\lambda$ such that the ball of radius $\lambda H/n > \lambda_1 H/n$ contains $\lambda$ zeros. This is absurd by our choice of $\lambda_1$.

The zeros of $P$ contained in $B_1$ will be said of rank $\lambda_1$.

Step 3. By removing the zeros in $B_1$, we have $n - \lambda_1$ zeros and repeating the above procedure, we construct a ball $B_2$ of radius $\lambda_2 H/n$ containing $\lambda_2$ zeros. We have that $\lambda_2 \leq \lambda_1$. In fact, if $\lambda_2 > \lambda_1$, the ball $B_2$ would have radius larger than $\lambda_1 H/n$ and it would contain $\lambda_2$ points. This contradicts Step 2.

We then remove the $\lambda_2$ points contained in $B_2$ and we iterate the procedure until we obtain a finite sequence of balls $B_1, \ldots, B_r$ of radii $\lambda_1 H/n \geq \ldots \geq \lambda_r H/n$. Each ball $B_i$ contains $\lambda_i$ zeros of $P$ and

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r.$$ 

Note that, by construction,

$$\left( \lambda_1 + \cdots + \lambda_r \right) \frac{H}{n} = H.$$

The zeros of $P$ contained in $B_i$ will be said of rank $\lambda_i$.

Step 4. Consider the balls $\Sigma_1, \ldots, \Sigma_r$ with the same center as $B_1, \ldots, B_r$ and with the radius of $\Sigma_i$ which is twice the radius of $B_i$, $i = 1, \ldots, r$. Let $q \in \mathbb{H} \setminus (\Sigma_1 \cup \ldots \cup \Sigma_r)$. Consider the closed ball centered at $q$ and with radius $\lambda_0 H/n$ for some $\lambda_0 \in \mathbb{N}$.

By the construction above we have that the ball $B_0$ intersects the balls $B_i$ with radius at least equal to $\lambda_0 H/n$. So this ball can contain only zeros of rank less than $\lambda_0$. If we remove all the zeros of rank greater than or equal to $\lambda_0$, no ball of radius $\lambda H/n$ with $\lambda > \lambda_0$ can contain $\lambda$ of the remaining zeros. We conclude that $B_0$ contains at most $\lambda_0$ zeros of $P$.

Let us label the zeros of $P$ in order of increasing distance from $q$ and let us write them as $\gamma_1, \ldots, \gamma_n$, where the $\gamma_\ell$ are either the isolated zeros $\alpha_i$ or one of the representatives of a spherical zero $[\beta_i]$, i.e. $\beta_i$ or $\overline{\beta}_i$. We have

$$|q - \gamma_\ell| > \ell \frac{H}{n}$$

so that

$$|P(q)| = |(q - \alpha_1) \cdots (q - \alpha_t) [(q - \beta_1) \cdots (q - \beta_p) * (q - \overline{\beta}_p)]|$$

$$> \left( \frac{H}{n} \right)^n n! > \left( \frac{H}{e} \right)^n.$$
Cartan theorem will be used to prove a lower bound for the modulus of a function slice regular in a ball centered at the origin, see Theorem 5.3.7.

Comments to Chapter 3. The integral formulas in this chapter come from [47] and [55], and the Riemann mapping theorem was originally proved in [100]. The results on the zeros come from [105], [108], [111] while the Ehrenpreis-Malgrange lemma is taken from [112]. Finally, Cartan theorem appears here for the first time.
Chapter 4

Slice regular infinite products

4.1 Infinite products of quaternions

In the sequel we will deal with infinite products of quaternions. As in the classical complex case, we will say that an infinite product of quaternions

$$
\prod_{n=1}^{\infty} u_n
$$

converges if the sequence $p_N = \prod_{n=1}^{N} u_n$ converges to a nonzero limit. It is crucial to note that the exclusion of 0 as a limit is due to the fact that, by allowing it, one has that any sequence $\{u_n\}$ containing at least one vanishing term would converge, independently on the sequence. For practical purposes, it is less reductive to assume that the infinite product $\prod_{n=1}^{\infty} u_n$ converges if and only if the sequence $\{u_n\}$ contains a finite number of zero elements and the partial products $p_N$ obtained by multiplying the nonzero factors converges to a nonzero limit. Since in a convergent infinite product the general term $u_n$ tends to 1 as $n \to \infty$, we will write $u_n = 1 + a_n$ where $a_n \to 0$ as $n \to \infty$. In complex analysis, $\prod_{n=1}^{\infty} (1 + a_n)$ converges to a nonzero limit simultaneously with the series $\sum_{n=1}^{\infty} \log(1 + a_n)$. The convergence is not simultaneous if the infinite product tends to zero.

The following proposition extends to the quaternionic setting the corresponding result in the complex case:
Proposition 4.1.1. Let \( a_1, \ldots, a_N \in \mathbb{H} \) and let
\[
p_N = \prod_{n=1}^{N} (1 + a_n), \quad p_N^* = \prod_{n=1}^{N} (1 + |a_n|).
\]
Then
\[
p_N^* \leq \exp(|a_1| + \ldots + |a_N|) \tag{4.1}
\]
and
\[
|p_N - 1| \leq p_N^* - 1. \tag{4.2}
\]

Proof. The proof is based on the properties of the modulus of a quaternion, thus the proof in the complex case applies also here. We repeat it for the sake of completeness. Since for any real number the inequality \( 1 + x \leq e^x \) holds, we immediately have \( p_N^* \leq \exp(|a_1| + \ldots + |a_N|) \). To show (4.2) we use induction. It is clear that (4.2) holds for \( N = 1 \), so we assume that it holds up to \( k \). We have
\[
p_{k+1} - 1 = p_k(1 + a_{k+1}) - 1 = (p_k - 1)(1 + a_{k+1}) + a_{k+1}. \tag{4.3}
\]
From (4.3) we deduce
\[
|p_{k+1} - 1| \leq |p_k - 1| (1 + |a_{k+1}|) + |a_{k+1}| = p_k^* - 1.
\]

\[\square\]

Proposition 4.1.2. Suppose that the sequence \( \{a_n\} \) is such that \( 0 \leq a_n < 1 \). Then
\[
\prod_{n=1}^{\infty} (1 - a_n) > 0 \text{ if and only if } \sum_{n=1}^{\infty} a_n < \infty.
\]

Proof. Let \( p_N = \prod_{n=1}^{N} (1 - a_n) \) then, by construction, \( p_1 \geq p_2 \geq \ldots p_N > 0 \) and so the limit \( p \) of the sequence \( \{p_N\} \) exists. Assume that \( \sum_{n=1}^{\infty} a_n < \infty \), then Theorem 4.2.2 implies \( p > 0 \). To show the converse, note that
\[
p \leq p_N \leq \exp(-a_1 - \ldots - a_N)
\]
and the right hand side expression tends to 0 when \( N \to \infty \) if \( \sum_{n=1}^{\infty} a_n \) diverges to \(+\infty\). \[\square\]

In the complex case it is well known that the argument of a product is equal to the sum of the arguments of the factors (up to an integer multiple of \( 2\pi \)). In the quaternionic case this equality does not hold in general, since the exponents may belong to different complex planes.
Lemma 4.1.3. Let $n \in \mathbb{N}$ and let $\theta_1, \ldots, \theta_n \in [0, \pi)$ be such that $\sum_{i=1}^{n} \theta_i < \pi$. Then for every $\{I_1, \ldots, I_n\} \in \mathcal{S}$ the inequality
\[
\arg_{\mathbb{H}}(e^{\theta_1 I_1} \cdots e^{\theta_n I_n}) \leq \sum_{i=1}^{n} \theta_i
\]
holds.

Proof. For $n = 1$ equality holds and thus the statement is true. Let us proceed by induction and choose $\theta_1, \theta_2, \ldots, \theta_n \in [0, \pi)$ with $\sum_{i=1}^{n} \theta_i < \pi$. Let $\phi \in [0, \pi)$ and $J \in \mathcal{S}$ be such that $e^{\theta_1 I_1} \cdots e^{\theta_{n-1} I_{n-1}} = e^{\phi J}$ and take the product $e^{\phi J} e^{\theta_n I_n}$:
\[
e^{\phi J} e^{\theta_n I_n} = \cos \phi \cos \theta_n + \cos \phi \sin \theta_n I_n + \sin \phi \cos \theta_n J + \sin \phi \sin \theta_n J I_n.
\]
Since $JI_n = -\langle J, I_n \rangle + J \times I_n$ we obtain that
\[
\cos(\arg_{\mathbb{H}}(e^{\phi J} e^{\theta_n I_n})) = \Re(e^{\phi J} e^{\theta_n I_n}) = \cos \phi \cos \theta_n - \sin \phi \sin \theta_n \langle J, I_n \rangle.
\]
However $\langle J, I_n \rangle \leq |J||I_n| = 1$ and $\sin \phi \sin \theta_n \geq 0$ and so
\[
\cos(\arg_{\mathbb{H}}(e^{\phi J} e^{\theta_n I_n})) \geq \cos \phi \cos \theta_n - \sin \phi \sin \theta_n = \cos(\phi + \theta_n).
\]
But the function $\cos(x)$ is decreasing in $[0, \pi]$ and so we deduce
\[
\arg_{\mathbb{H}}(e^{\phi J} e^{\theta_n I_n}) \leq \phi + \theta_n.
\]
Thanks to the induction hypothesis we have the thesis. \qed

Let $\{a_i\}_{i \in \mathbb{N}} \subseteq \mathbb{H}$ be a sequence such that the associated series is absolutely convergent. Then the series itself is convergent, namely $\sum_{i=0}^{\infty} |a_i|$ convergent implies that $\sum_{i=0}^{\infty} a_i$ convergent. This observation is used to prove the following result.

Theorem 4.1.4. Let $\{a_i\}_{i \in \mathbb{N}} \subseteq \mathbb{H}$ be a sequence. If the series $\sum_{i=0}^{\infty} |\log(1 + a_i)|$ converges, then the product $\prod_{i=0}^{\infty} (1 + a_i)$ converges.

Proof. If the series $\sum_{i=0}^{\infty} |\log(1 + a_i)|$ is convergent, then the sequence $\{a_i\}_{i \in \mathbb{N}}$ tends to zero. Thus we can suppose that $1 + a_i \not\in (-\infty, 0]$. Then, for every $i \in \mathbb{N}$, we consider suitable $\theta_i \in [0, \pi)$ and $I_i \in \mathcal{S}$ such that
\[
1 + a_i = |1 + a_i| e^{\theta_i I_i}.
\]
Consequently,
\[ \log(1 + a_i) = \ln |1 + a_i| + \theta_i I_i \]
and so
\[ |\ln |1 + a_i|| \leq |\log(1 + a_i)| \] \hspace{1cm} (4.4)
moreover
\[ |\theta_i| = |\theta_i I_i| \leq |\log(1 + a_i)| \] \hspace{1cm} (4.5)
for every \( i \in \mathbb{N} \).

Our next task is to show that the sequence of the partial products
\[ Q_n = \prod_{i=0}^{n} (1 + a_i) \]
has a finite, nonzero, limit. First of all, note that for every \( n \in \mathbb{N} \)
\[ \prod_{i=0}^{n} (1 + a_i) = \prod_{i=0}^{n} |1 + a_i| \prod_{i=0}^{n} e^{\theta_i I_i}. \]

Our hypothesis and inequality (4.4) show that the series \( \sum_{i=0}^{\infty} \ln |1 + a_i| \) is convergent and hence the infinite product \( \prod_{i=0}^{\infty} |1 + a_i| \) is convergent. Then it is sufficient to prove that the sequence \( R_n = \prod_{i=0}^{n} e^{\theta_i I_i} \subseteq \partial B(0,1) = \{ q \in \mathbb{H} : |q| = 1 \} \) is convergent. To this end, we will show that \( \{ R_n \}_{n \in \mathbb{N}} \) is a Cauchy sequence. For \( n > m \in \mathbb{N} \), we have:
\[ |R_n - R_m| = \left| \prod_{i=0}^{m} e^{\theta_i I_i} - \prod_{i=0}^{m} e^{\theta_i I_i} \right| \]
\[ = \left| \prod_{i=0}^{m} e^{\theta_i I_i} \prod_{i=m+1}^{n} e^{\theta_i I_i} - \prod_{i=0}^{m} e^{\theta_i I_i} \right| \]
\[ = \left| \prod_{i=0}^{m} e^{\theta_i I_i} \prod_{i=m+1}^{n} e^{\theta_i I_i} - 1 \right| \]
\[ = \left| \prod_{i=m+1}^{n} e^{\theta_i I_i} - 1 \right| \leq \arg_{\mathbb{H}} \left( \prod_{i=m+1}^{n} e^{\theta_i I_i} \right), \]
where the last inequality holds since \( |\prod_{i=m+1}^{n} e^{\theta_i I_i}| = 1 \). Inequality (4.5) implies that the series \( \sum_{i=0}^{\infty} \theta_i \) is convergent. Therefore the sequence \( S_n = \sum_{i=0}^{n} \theta_i \) of the partial sums is a Cauchy sequence. As a consequence, for all \( \epsilon > 0 \), there exists \( m_0 \in \mathbb{N} \) such that for all \( n > m > m_0 \),
\[ \sum_{i=m+1}^{n} \theta_i < \epsilon. \]
In particular for $\epsilon < \pi$, by Lemma 4.1.3, we deduce
\[
\arg_{\mathbb{S}} \left( \prod_{i=m+1}^{n} e^{\theta_i t_i} \right) \leq \sum_{i=m+1}^{n} \theta_i < \epsilon,
\]
and this finishes the proof.

The following proposition, which will be used in the sequel, is an expected extension of the analog property in the complex case:

**Proposition 4.1.5.** Let $\log$ be the principal quaternionic logarithm. Then
\[
\lim_{q \to 0} q^{-1}\log(1 + q) = 1.
\]

**Proof.** Consider a sequence $\{q_n\}_{n \in \mathbb{N}} = \{x_n + I_n y_n\}_{n \in \mathbb{N}}$ such that $q_n \to 0$ for $n \to \infty$. It is not reductive to assume that $y_n > 0$. Let us compute
\[
| (x_n + I_n y_n)^{-1}\log(1 + x_n + I_n y_n) - 1 |. \tag{4.7}
\]

It can be easily checked that both the real part and the norm of the imaginary part of $(x_n + I_n y_n)^{-1}\log(1 + x_n + I_n y_n)$ do not depend on $I_n$ and so we can set $I_n = I_0$ for every $n \in \mathbb{N}$ and so we compute
\[
| (x_n + I_0 y_n)^{-1}\log(1 + x_n + I_0 y_n) - 1 |. \tag{4.8}
\]

It is immediate that $(x_n + I_0 y_n)^{-1}\log(1 + x_n + I_0 y_n) \in \mathbb{C}_{I_0}$ for all $n \in \mathbb{N}$ and in the complex plane $\mathbb{C}_{I_0}$ we can use the classical arguments. Thus we conclude that the sequence (4.8) tends to zero and the statement follows.

**Corollary 4.1.6.** The series $\sum_{n=0}^{\infty} |\log(1 + a_n)|$ converges if and only if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

**Proof.** Assume that the series $\sum_{n=0}^{\infty} |\log(1 + a_n)|$ or the series $\sum_{n=0}^{\infty} |a_n|$ are convergent. Then we have $\lim_{n \to \infty} a_n = 0$. Proposition 4.1.5 implies that for any given $\epsilon > 0$
\[
|a_n|(1 - \epsilon) \leq |\log(1 + a_n)| \leq |a_n|(1 + \epsilon)
\]
for all sufficiently large $n$. Thus the two series in the statement are simultaneously convergent.

The following result contains a sufficient condition for the convergence of quaternionic infinite products.

**Theorem 4.1.7.** Let $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{H}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent then the product $\prod_{n=1}^{\infty} (1 + a_n)$ is convergent.

**Proof.** It is a consequence of Theorem 4.1.4 and Corollary 4.1.6.
4.2 Infinite products of functions

In this section we consider infinite products of functions of a quaternionic variable defined in an open set \( U \subseteq \mathbb{H} \). Given a sequence of functions \( \{a_n(q)\} \) which do not vanish on \( U \), we will say that \( \prod_{n=0}^{\infty} a_n(q) \) converges uniformly on the compact subsets of \( U \) if the sequence of partial products \( \{\prod_{n=0}^{N} a_n(q)\} \) converges uniformly on the compact subsets of \( U \) to a non vanishing function. Since there may be factors with zeros, we require that for any fixed compact subset \( K \) at most a finite number of factors vanish in some points of \( K \). We thus give the following definition:

**Definition 4.2.1.** Let \( \{a_n(q)\} \) be a sequence of functions defined on an open set \( U \subseteq \mathbb{H} \). We say that \( \prod_{n=0}^{\infty} a_n(q) \) converges compactly in \( U \) to a function \( f : U \to \mathbb{H} \) if for any compact set \( K \subset U \) the following conditions are fulfilled:

- there exists \( N_K \in \mathbb{N} \) such that \( a_n \neq 0 \) for all \( n \geq N_K \);
- the residual product \( \prod_{n=N_K}^{\infty} a_n(q) \) converges uniformly on \( K \) to a never vanishing function \( f_{N_K} \);
- for all \( q \in K \)
  \[ f(q) = \left( \prod_{n=0}^{N_K-1} a_n(q) \right) f_{N_K}(q). \]

**Theorem 4.2.2.** Suppose \( \{a_n\} \) is a sequence of bounded quaternionic-valued functions defined on a set \( U \subseteq \mathbb{H} \), such that \( \sum_{n=1}^{\infty} |a_n(q)| \) converges uniformly on \( U \). Then the product

\[ f(q) = \prod_{n=1}^{\infty} (1 + a_n(q)) \quad (4.9) \]

converges compactly in \( U \), and \( f(q_0) = 0 \) at some point \( q_0 \in U \) if and only if \( a_n(q_0) = -1 \) for some \( n \).

**Proof.** First of all we observe that the hypothesis on \( \sum_{n=1}^{\infty} |a_n(q)| \) ensures that every \( q \in U \) has a neighborhood in which at most finitely many of the factors \( (1 + a_n(q)) \) vanish. Then, since \( \sum_{n=1}^{\infty} |a_n(q)| \) converges uniformly on \( U \), we have that it is also bounded on \( U \). Let us set

\[ p_N(q) = \prod_{n=1}^{N} (1 + a_n(q)); \]
then Proposition 4.1.1 yields the existence of a (finite) constant such that $|p_N(q)| \leq C$ for all $N \in \mathbb{N}$ and all $q \in U$. For every $\varepsilon$ such that $0 < \varepsilon < \frac{1}{2}$ there exists a $N_0$ such that

$$\sum_{n=N_0}^{\infty} |a_n(q)| < \varepsilon, \quad q \in U. \quad (4.10)$$

Let $M, N \in \mathbb{N}$ and assume that $M > N$. Proposition 4.1.1 and (4.10) show that

$$|p_M(q) - p_N(q)| \leq |p_N(q)| (e^\varepsilon - 1) \leq 2|p_N(q)| \varepsilon \leq 2C \varepsilon, \quad (4.11)$$

thus the sequence $p_N$ converges uniformly to a limit function $f$. Moreover, (4.11) and the triangular inequality give, for $M > N$

$$|p_M(q)| \geq (1 - 2\varepsilon)|p_N(q)|$$
on $U$ and so

$$|f(q)| \geq (1 - 2\varepsilon)|p_N(q)|.$$

Thus $f(q) = 0$ if and only if $p_N(q) = 0$. \qed

We now give the analog of Definition 4.2.1 in the case of the slice regular product:

**Definition 4.2.3.** Let $\{f_n(q)\}$ be a sequence of functions slice regular on an axially symmetric slice domain $U \subseteq \mathbb{H}$. We say that the infinite $\star$-product

$$\prod_{n=0}^{\infty} (1 + f_n(q))$$

converges compactly in $U$ to a function $f : U \to \mathbb{H}$ if for any axially symmetric compact set $K \subset U$ the following conditions are fulfilled:

- there exists $N_K \in \mathbb{N}$ such that $f_n \neq 0$ for all $n \geq N_K$;

- the residual product $\prod_{n=N_K}^{\infty} (1 + f_n(q))$ converges uniformly on $K$ to a never vanishing function $f_{N_K}$;

- for all $q \in K$

$$f(q) = \left( \prod_{n=0}^{N_K-1} (1 + f_n(q)) \right) f_{N_K}(q).$$

In order to relate an infinite product with an infinite slice regular product, we need a technical result:
Lemma 4.2.4. Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of slice regular functions defined on an axially symmetric slice domain \( U \). Let \( K \subseteq U \) be an axially symmetric compact set. Assume that there exists an integer \( N_K \) such that \( 1 + f_n \neq 0 \) on \( K \) for all \( n \geq N_K \). Let

\[
F_{N_K}^m(q) = \prod_{n=N_K}^{*m} (1 + f_n(q)).
\]

Then for all \( m \geq N_K \) and for any \( q \in K \)

\[
F_{N_K}^m(q) = \prod_{i=N_K}^{m} (1 + f_i(T_i(q))) \neq 0
\]

where

\[
T_j(q) = (F_{N_K}^{(j-1)}(q))^{-1}qF_{N_K}^{(j-1)}(q) \quad \text{for} \quad j > N_K
\]

and \( T_j(q) = q \) for \( j = N_K \).

Proof. Let \( q \in K \). The assertion is true for \( m = N_K \) by hypothesis since \( F_{N_K}^{N_K}(q) = 1 + f_{N_K}(q) \neq 0 \). We now proceed by induction. Assume that the assertion is true for \( m = N_K, \ldots, n-1 \). Then

\[
F_{N_K}^{n}(q) = F_{N_K}^{(n-1)}(q) * (1 + f_n(q)). \tag{4.12}
\]

Since \( T_j(q) \) is a rotation of \( q \), it is immediate that

\[
\text{Re}(T_j(q)) = \text{Re}(q) \quad \text{and} \quad |\text{Im}(T_j(q))| = |\text{Im}(q)| \quad \text{for all} \quad j \geq N_K.
\]

Since \( K \) is an axially symmetric set, obviously \( T_j(q) \in K \) if and only if \( q \in K \). By Theorem 2.3.10 and by the induction hypothesis, we have that formula (4.12) rewrites as:

\[
F_{N_K}^{n}(q) = F_{N_K}^{(n-1)}(q)(1 + f_n(T_n(q))).
\]

The factor \( (1 + f_n) \) does not vanish on \( K \) and hence, since \( T_n(q) \in K \), the function \( F_{N_K}^{n}(q) \) does not vanish on \( K \). Using again the induction hypothesis, we have

\[
F_{N_K}^{n-1}(q) = \prod_{i=N_K}^{n-1} (1 + f_i(T_i(q)))
\]
$F_{N_N}^n(q) = \prod_{i=N_N}^{n-1} (1 + f_i(T_i(q))) (1 + f_n(T_n(q)))$

$\prod_{i=N_N}^{n} (1 + f_i(T_i(q))) \neq 0,$

and this concludes the proof. □

**Theorem 4.2.5.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of slice regular functions defined on an axially symmetric slice domain $U$. The infinite $\ast$-product

$$\prod_{n=0}^{\infty} (1 + f_n(q))$$

converges compactly in $U$ if and only if the infinite product

$$\prod_{n=0}^{\infty} (1 + f_n(q))$$

converges compactly in $U$.

**Proof.** It is not reductive to assume that the compact subsets of $U$ are axially symmetric. Let $K$ be a compact, axially symmetric subset of $U$. Let $N_N \in \mathbb{N}$ be such that for every $n \geq N_N$ the factors $1 + f_n(q)$ do not vanish on $K$. By Lemma 4.2.4 the infinite $\ast$-product

$$\prod_{i=N_N}^{\infty} (1 + f_i(q))$$

converges if and only if

$$\prod_{i=N_N}^{\infty} (1 + f_i(T_i(q)))$$

converges. Since $\text{Re}(T_j(q)) = \text{Re}(q)$ and $|\text{Im}(T_j(q))| = |\text{Im}(q)|$ for all $j \geq N_N$, we have that $T_j(q) \in K$ if and only if $q \in K$. This ends the proof. □

The following result will be useful to establish when an infinite product of slice regular functions is slice regular.

**Proposition 4.2.6.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of slice regular functions defined on an axially symmetric slice domain $U \subseteq \mathbb{H}$ and converging uniformly to a function $f$ on the compact sets of $U$. Then $f$ is slice regular on $U$. 71
Proof. We write the restriction of $f_n$ to $\mathbb{C}_I$ as

$$f_n|_{\mathbb{C}_I}(x + Iy) = F_n(x + Iy) + G_n(x + Iy)J$$

using the Splitting Lemma. The functions $F_n, G_n$ are holomorphic for every $n \in \mathbb{N}$. The restriction $f_I(x + Iy) = F(x + Iy) + G(x + Iy)J$ to $\mathbb{C}_I$ of the limit function $f$ is such that $F$ and $G$ are the limit of $F_n$ and $G_n$ respectively. Since $f_n \to f$ uniformly also $F_n \to F$ and $G_n \to G$ uniformly and so both $F$ and $G$ are holomorphic. It follows that $f_I$ is the kernel of $\partial_x + I\partial_y$ and by the arbitrariness of $I \in \mathbb{S}$ the statement follows. \hfill $\square$

**Proposition 4.2.7.** Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of slice regular functions defined on a symmetric slice domain $U$. If the infinite slice regular product

$$\prod_{n=0}^{\infty} (1 + f_n(q))$$

converges compactly in $U$ to a function $f$, then $f$ is slice regular on $U$.

Proof. By the assumption made at the beginning of this section, for any compact set $K \subseteq \mathbb{H}$ there exists an integer $N_K$ such that $1 + f_n \neq 0$ on $K$ if $n \geq N_K$. The $\ast$-product

$$\prod_{n=N_K}^{\infty} (1 + f_n(q))$$

forms a sequence of slice regular functions which converges uniformly on $K$ to a slice regular function $F_{N_K}$ that does not vanish on $K$. Therefore the function $f$ can be written as a finite product of slice regular functions

$$f(q) = \left[ \prod_{n=0}^{N_K-1} (1 + f_n(q)) \right] \ast F_{N_K}(q)$$

and hence it is slice regular on $K$. Moreover, by Lemma 4.2.4, the zero set of $f$ on $K$ coincides with the zero set of the finite product $\prod_{n=0}^{N_K-1} (1 + f_n(q))$. \hfill $\square$

**Proposition 4.2.8.** Let $\{f_n\}$ be a sequence of functions slice regular on an axially symmetric slice domain $U$ and assume that no $f_n$ is identically zero on $U$. Suppose that

$$\sum_{n=1}^{\infty} |1 - f_n(q)|$$
converges uniformly on the compact subsets of $U$. Then $\prod_{n=1}^{\infty} f_n(q)$ converges compactly in $U$ to a function $f \in \mathcal{R}(U)$.

Proof. It is an immediate consequence of Theorem 4.2.2, Theorem 4.2.5 and Proposition 4.2.7.

4.3 Weierstrass theorem

Weierstrass theorem was originally proved in [113]. Here we provide an alternative statement and we will show how to retrieve the result in [113]. Since we need to consider the slice regular composition of the exponential function with a polynomial as introduced in Remark 2.5.10, for the sake of clarity we repeat the definition below.

**Definition 4.3.1.** Let $p(q) = b_0 + q b_1 + \ldots + q^m b_m$, $b_i \in \mathbb{H}$. We define $e^{p(q)}_\star$ as:

$$e^{p(q)}_\star := \sum_{n=0}^{\infty} \frac{1}{n!} (p(q))^\star_n = \sum_{n=0}^{\infty} \frac{1}{n!} (b_0 + q b_1 + \ldots + q^m b_m)^\star_n. \quad (4.14)$$

**Remark 4.3.2.** When the polynomial $p$ has real coefficients, we have $e^{p(q)}_\star = e^{p(q)}$. When $b_i \in \mathbb{C}_I$ for some $I \in \mathbb{C}_I$ we have that, denoting by $z$ the complex variable on $\mathbb{C}_I$, $(e^{p(q)}_\star)|_{\mathbb{C}_I} = e^{p(z)}$, i.e. the restriction of the function $e^{p(q)}_\star$ to $\mathbb{C}_I$ coincides with the complex valued function $e^{p(z)}$. Therefore $e^{p(q)}_\star$ is the slice regular extension to $\mathbb{H}$ of $e^{p(z)}$. Note that sometimes we will write $\exp(p(z))$ or $\exp_\star(p(q))$ instead of $e^{p(z)}$, $e^{p(q)}_\star$.

**Theorem 4.3.3.** Let $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{H} \setminus \{0\}$ be a diverging sequence. Let $\{p_n\}$ be a sequence of nonnegative integers such that

$$\sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty \quad (4.15)$$

for every positive $r$. Set

$$e_{p_n}(q, a_n^{-1}) = e_\star \left( q a_n^{-1} + q^2 a_n^{-2} \ldots + \frac{1}{p_n} q^p a_n^{-p} \right)$$

for all $q \in \mathbb{H}$. Then the infinite $\star$-product

$$\prod_{n=0}^{\infty} (1 - q a_n^{-1}) \star e_{p_n}(q, a_n^{-1}) \quad (4.16)$$

converges compactly in $\mathbb{H}$ to an entire slice regular function. Furthermore, for every $n \in \mathbb{N}$, the function $e_{p_n}(q, a_n^{-1})$ has no zeros in $\mathbb{H}$.
Proof. For all \( n \in \mathbb{N} \) let us set
\[
P_n(q) = (1 - qa_n^{-1}) \ast e_{p_n}(q, a_n^{-1}).
\]
Theorem 4.2.5 yields that the \( \ast \)-product (4.16) converges compactly in \( \mathbb{H} \) if and only if the pointwise product
\[
\prod_{n=0}^{\infty} P_n(q)
\]
converges compactly in \( \mathbb{H} \). Theorem 4.1.7 implies that the product (4.17) converges compactly in \( \mathbb{H} \) if the series
\[
\sum_{n=0}^{\infty} |1 - P_n(q)|
\]
converges compactly in \( \mathbb{H} \). Let \( K \subseteq \mathbb{H} \) be a compact set. Let \( R > 0 \) be such that \( K \subseteq B(0,R) \) and let \( N \in \mathbb{N} \) be the integer such that \( |a_n| > R \) for all \( n \geq N \). For any \( n \in \mathbb{N} \) let us denote by \( I_n \) the element in \( S \) such that \( a_n \in \mathbb{C}I_n \) and let \( p_n(z) \) be the restriction of \( P_n \) to the complex plane \( \mathbb{C}I_n \). Hence, denoting by \( z \) the variable in \( \mathbb{C}I_n \), we have
\[
\wp_n(z) = (1 - za_n^{-1})e^{za_n^{-1} + \frac{1}{2}z^2a_n^{-2} + \frac{1}{6}z^3a_n^{-3} + \frac{1}{24}z^4a_n^{-4} + \cdots}
\]
and we can estimate the coefficients \( c_k \) of the Taylor expansion
\[
\wp_n(z) = 1 - \sum_{k=0}^{\infty} c_k (za_n)^{k+p_n+1}
\]
of \( \wp_n \) at the point 0 as in the complex case, see [143], thus obtaining
\[
0 \leq c_k \leq \frac{1}{p_n + 1}.
\]
(4.18)
Since the slice regular extension \( P_n \) of \( \wp_n \) is unique by the Identity Principle, the coefficients of the power series expansion of \( P_n \) are the same of \( \wp_n \) and so we have:
\[
P_n(q) = 1 - \sum_{k=0}^{\infty} c_k q^{k+p_n+1}a_n^{-(k+p_n+1)}
\]
for every \( q \in \mathbb{H} \). Then
\[
|1 - P_n(q)| = \left| \sum_{k=0}^{\infty} c_k q^{k+n+1}a_n^{-(k+n+1)} \right|
\]
\[
\leq \sum_{k=0}^{\infty} c_k \left( \frac{|q|}{|a_n|} \right)^{(k+p_n+1)}
\]
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and using the estimate (4.18) we obtain

\[
|1 - P_n(q)| \leq \sum_{k=0}^{\infty} \frac{1}{p_n + 1} \left( \frac{|q|}{|a_n|} \right)^{k+p_n+1}
\]

\[
\leq \frac{1}{p_n + 1} \left( \frac{|q|}{|a_n|} \right)^{p_n+1} \left( \sum_{k=0}^{\infty} \left( \frac{|q|}{|a_n|} \right)^k \right).
\]

The series

\[
\sum_{k=0}^{\infty} \left( \frac{|q|}{|a_n|} \right)^k
\]

is a geometric series whose ratio is strictly smaller than 1 when \( n \geq N \), since \( q \in K \subseteq B(0, R) \) and \( |a_n| > R \) for \( n \geq N \). Hence the power series is convergent to the value

\[
\left( 1 - \frac{|q|}{|a_n|} \right)^{-1}
\]

and we obtain

\[
|1 - P_n(q)| \leq \frac{1}{p_n + 1} \left( \frac{|q|}{|a_n|} \right)^{p_n+1} \left( 1 - \frac{|q|}{|a_n|} \right)^{-1}.
\]

Since \( \lim_{n \to \infty} \left( 1 - \frac{|q|}{|a_n|} \right)^{-1} = 1 \) and the series

\[
\sum_{n=0}^{\infty} \frac{1}{p_n + 1} \left( \frac{|q|}{|a_n|} \right)^{p_n+1}
\]

(4.19)

converges on \( K \), also the series

\[
\sum_{n=0}^{\infty} |1 - P_n(q)|
\]

converges on \( K \). The function defined by the \( \ast \)-product in (4.16) is entire by Proposition 4.2.8. The function \( e_{p_n}(q, a_n^{-1}) \) has no zeros since it is the (unique) slice regular extension of a function holomorphic on the complex plane \( \mathbb{C} \) and without zeros on that plane.

**Remark 4.3.4.** Condition (4.15) in the previous result is satisfied when \( p_n = n - 1 \) for all \( n \in \mathbb{N} \). A fortiori, we can choose \( p_n = n \).
Remark 4.3.5. Let \( f \) be an entire slice regular function and let \( \alpha_1, \alpha_2, \ldots \) be its non-spherical zeros and \([\beta_1], [\beta_2], \ldots\) be its spherical zeros. As we discussed in Chapter 2, a spherical zero \([\beta]\) is characterized by the fact that \( f \) contains the factor \( q^2 - 2 \text{Re}(\beta)q + |\beta|^2 \) to a suitable power. Since this factor splits as \((q - \beta) \ast (q - \overline{\beta})\), the spherical zeros can also be listed as pairs of conjugate elements \( \beta, \overline{\beta} \), where \( \beta \) is any element belonging to the given sphere. It is important to remark that, when forming the list of the zeros of a given function, two elements \( \beta, \overline{\beta} \) defining a sphere should appear one after the other. We will say that \( \beta \) (or any other element in \([\beta]\)) is a generator of a spherical zero. If the sphere has multiplicity \( m \) the pair \( \beta, \overline{\beta} \) will be repeated \( m \) times in the list. Also the isolated zeros appear with their multiplicities. With this notation we can list the zeros as \( \gamma_1, \gamma_2, \ldots \) (where \( \gamma_i \) stands for either one of the elements \( \alpha_j \) or \( \beta_j \) or \( \overline{\beta}_j \)) according to increasing values of their moduli. When some elements have the same modulus, we can list them in any order (but keeping together the pairs defining a sphere).

Remark 4.3.6. When the list of zeros contains a spherical zero, namely a pair \( \beta_n, \overline{\beta}_n \), we can choose \( \gamma_n = \beta_n \) and \( \gamma_{n+1} = \overline{\beta}_n \) and, for \( z \) belonging to the complex plane of \( \beta_n, \overline{\beta}_n \), we have

\[
(1 - q\beta_n^{-1}) \ast e \ast (1 - q\overline{\beta}_n^{-1}) \ast \left(1 - q\beta_n^{-1} + \frac{1}{p_n}q^p\beta_n^{pn}\right) \ast \left(1 - q\overline{\beta}_n^{-1} + \frac{1}{p_n}q^p\overline{\beta}_n^{pn}\right) = \text{ext} \left( (1 - z\beta_n^{-1}) \exp \left( z\beta_n^{-1} + \frac{1}{p_n}z^p\beta_n^{pn} \right) \right) \times \left(1 - z\overline{\beta}_n^{-1} \right) \exp \left( z\overline{\beta}_n^{-1} + \frac{1}{p_n}z^p\overline{\beta}_n^{pn} \right) \right) \]

\[
= \text{ext} \left( (1 - z\beta_n^{-1})(1 - z\overline{\beta}_n^{-1}) \exp \left( z\beta_n^{-1} + \frac{1}{p_n}z^p\beta_n^{pn} + z\overline{\beta}_n^{-1} + \ldots \right) \right) \]

\[
= \text{ext} \left( 1 - 2q\frac{\text{Re}(\beta_n)}{|\beta_n|^2} + \frac{1}{|\beta_n|^2} \right) \exp \left( 2q\frac{\text{Re}(\beta_n)}{|\beta_n|^2} + \ldots \right) + \frac{2}{p_n}z^p \frac{\text{Re}(\beta_n)}{|\beta_n|^2} \right) \]

To prove the Weierstrass factorization theorem we need additional information in the case in which a function has isolated zeros only:

Lemma 4.3.7. Let \( f \) be an entire slice regular function whose sequence of zeros \( \{\gamma_n\} \) consists of isolated, nonreal elements, repeated according to their multiplicity. Then there exist \( \delta_n \in [\gamma_n] \) such that

\[
f(q) = h(q) \ast \prod_{n=1}^{\infty} e_{p_n}(q, \delta_n^{-1}) \ast (1 - q\delta_n^{-1},)
\]
where $h$ is a nowhere vanishing entire slice regular function.

**Proof.** By our hypothesis if $f$ vanishes at $\gamma_n$, $f$ cannot have any other zero on $[\gamma_n]$ and, in particular, $f(\bar{\gamma}_n) \neq 0$. Starting from $f$ we will construct a function having zeros $\bar{\gamma}_n$ so that this new function will have spherical zeros. Let us start by considering the function $f_1$ defined by

$$f_1(q) = f(q) \ast (1 - q\delta_1^{-1}) \ast e_{p_1}(q, \delta_1^{-1})$$

where $\delta_1^{-1} = f(\bar{\gamma}_1)^{-1}f(\gamma_1)$ and so $\delta_1 \in [\gamma_1]$. Formula (2.3.10) implies that $f_1(\gamma_1) = 0$ and so $f_1(q)$ has a spherical zero at $[\gamma_1]$ and

$$f_1(q) = \left(1 - 2\frac{\text{Re}(\gamma_1)}{|\gamma_1|^2} + q^2\frac{1}{|\gamma_1|^2}\right)\tilde{f}_1(q)$$

where $\tilde{f}_1(q)$ has zeros belonging to the sequence $\{\gamma_n\}_{n \geq 2}$. Thus $\tilde{f}_1$ is not vanishing at $\bar{\gamma}_2$ and we repeat the procedure to add to $\tilde{f}_1$ the zero $\bar{\gamma}_2$ by constructing the function

$$f_2(q) = \left(1 - 2\frac{\text{Re}(\gamma_1)}{|\gamma_1|^2} + q^2\frac{1}{|\gamma_1|^2}\right)\tilde{f}_1(q) \ast (1 - q\delta_2^{-1}) \ast e_{p_2}(q, \delta_2^{-1})$$

where $\delta_2^{-1} = \tilde{f}(\bar{\gamma}_2)^{-1}\bar{\gamma}_2^{-1}\tilde{f}(\gamma_2)$ and $\delta_2 \in [\gamma_2]$. The function $f_2$ has spherical zeros at $[\gamma_1, [\gamma_2]$ and isolated zeros belonging to the sequence $\{\gamma_n\}_{n \geq 3}$. Note that

$$f_2(q) = f(q) \ast (1 - q\delta_1^{-1}) \ast e_{p_1}(q, \delta_1^{-1}) \ast (1 - q\delta_2^{-1}) \ast e_{p_2}(q, \delta_2^{-1})$$.

Iterating the reasoning, we obtain a function $k(q)$ where

$$k(q) = f(q) \ast \prod_{n=1}^{\infty} (1 - q\delta_n^{-1}) \ast e_{p_n}(q, \delta_n^{-1}). \quad (4.20)$$

By construction, $k(q)$ is an entire slice regular function with zeros at the spheres $[\gamma_n]$ and no other zeros. Since the factors corresponding to the spheres have real coefficients, see Remark 4.3.6, we can pull them on the left and write

$$k(q) = \prod_{n=1}^{\infty} \left(1 - 2q\frac{\text{Re}(\gamma_n)}{|\gamma_n|^2} + q^2\frac{1}{|\gamma_n|^2}\right) \exp \left(2q\frac{\text{Re}(\gamma_n)}{|\gamma_n|^2} + \ldots + 2\frac{1}{p_n}q^{p_n}\frac{\text{Re}(\gamma_n^{p_n})}{|\gamma_n|^{2p_n}}\right) h(q)$$

$$= \left(\prod_{n=1}^{\infty} (1 - q\gamma_n^{-1}) \ast e_{p_n}(q, \gamma_n^{-1}) \ast (1 - q\bar{\gamma}_n^{-1}) \ast e_{p_n}(q, \bar{\gamma}_n^{-1})\right) h(q).$$

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Let us set
\[
S(q) = \prod_{n=1}^{\infty} \left( 1 - 2q \frac{\text{Re}(\gamma_n)}{|\gamma_n|^2} + q^2 \frac{1}{|\gamma_n|^2} \right) \exp \left( 2q \frac{\text{Re}(\gamma_n)}{|\gamma_n|^2} + \ldots + 2 \frac{1}{p_n} q^{p_n} \frac{\text{Re}(\gamma_n^{p_n})}{|\gamma_n|^{2p_n}} \right)
\]
so that we can rewrite
\[
k(q) = S(q) h(q).
\]
We now multiply (4.20) on the right by
\[
\prod_{n=1}^{\infty} e_{p_n}(q, \bar{\delta}_n - 1) \quad (1 - q \bar{\delta}_n^{-1})
\]
and we obtain
\[
k(q) \star \prod_{n=1}^{\infty} e_{p_n}(q, \bar{\delta}_n - 1) \star (1 - q \bar{\delta}_n^{-1}) = f(q) \star S(q),
\]
from which we deduce
\[
S(q) h(q) \star \prod_{n=1}^{\infty} e_{p_n}(q, \bar{\delta}_n - 1) \star (1 - q \bar{\delta}_n^{-1}) = f(q) \star S(q) = S(q) f(q).
\]
By multiplying on the left by \(S(q)^{-1}\) we finally have
\[
h(q) \star \prod_{n=1}^{\infty} e_{p_n}(q, \bar{\delta}_n - 1) \star (1 - q \bar{\delta}_n^{-1}) = f(q)
\]
so the statement follows.

Using the notation in Remark 4.3.5 to denote the list of zeros of a function, we can prove the following:

**Theorem 4.3.8** (Weierstrass Factorization Theorem). Let \(f\) be an entire slice regular function and let \(f(0) \neq 0\). Suppose \(\{\gamma_n\}\) are the zeros of \(f\) repeated according to their multiplicities. Then there exist a sequence \(\{p_n\}\) of nonnegative integers, a sequence \(\{\delta_n\}\) of quaternions, and a never vanishing entire slice regular function \(h\) such that
\[
f(q) = g(q) \star h(q)
\]
where
\[
g(q) = \prod_{n=1}^{\infty} (1 - q \delta_n^{-1}) \star e_{p_n}(q, \delta_n), \quad (4.21)
\]
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\( \delta_n \in [\gamma_n] \). In particular, \( \delta_n = \gamma_n \) if \( \gamma_n \in \mathbb{R} \), \( \delta_n = \beta_n \), and \( \delta_{n+1} = \overline{\beta_n} \) when \([\beta_n]\) is a spherical zero and, in this case, \( p_{n+1} \) may be chosen equal to \( p_n \).

Moreover, for all \( n \in \mathbb{N} \), the function \( e_{p_n}(q, \delta_n^{-1}) \) is given by

\[
e_{p_n}(q, \delta_n^{-1}) = e_*^{q\delta_n^{-1} + \ldots + \frac{1}{p_n}q^n\delta_n^{-p_n}}.
\]

Under the same hypotheses, if \( f \) has a zero of multiplicity \( m \) at 0 then

\[
f(q) = q^m g(q) * h(q),
\]

where \( g \) and \( h \) are as above.

**Proof.** Let us consider first the real and spherical zeros of \( f \). Then, by Theorem 4.3.3,

\[
g(q) = \left( \prod_{n=0}^{*\infty} (1 - q\delta_n^{-1}) * e_{p_n}(q, \delta_n^{-1}) \right)
\]

is a slice regular function which has the listed zeros if the \( \delta_n \) are chosen to be the real zeros of \( f \) or the pairs \( \beta_n, \overline{\beta_n} \). Since all the corresponding factors

\[
(1 - q\delta_n^{-1}) * e_{p_n}(q, \delta_n^{-1})
\]

or

\[
(1 - q\beta_n^{-1}) * e_{p_n}(q, \beta_n^{-1}) * (1 - q\overline{\beta_n}^{-1}) * e_{p_n}(q, \overline{\beta_n}^{-1})
\]

have real coefficients, they commute with each other. Let us set \( f_1(q) = g(q)^-* * f(q) \). The slice regular function \( f_1(q) \) cannot have real of spherical zeros as the zeros of \( f \) cancel with the factors in \( g(q)^-* \). Thus \( f(q) = g(q) * f_1(q) \) where \( f_1 \) can have isolated zeros, if \( f \) possesses isolated zeros.

We now consider the function \( f_c(q) \) and we recall that its zeros are in one-to-one correspondence with the zeros of \( f_1(q) \). By Lemma 4.3.7 we have

\[
f_c(q) = h(q) * \prod_{n=1}^{*\infty} e_{p_n}(q, \delta_n^{-1}) * (1 - q\delta_n^{-1})
\]

where \( h \) is a suitable entire slice regular function. Consequently, we have

\[
f_1(q) = \prod_{n=1}^{*\infty} (1 - q\delta_n^{-1}) * e_{p_n}(q, \delta_n^{-1}) * h^c(q)
\]

and the statement follows. If \( f \) has a zero of multiplicity \( m \) at 0 it suffices to apply the previous reasoning to the function \( q^{-m} f(q) \). \( \square \)
Remark 4.3.9. We note that, in the case of an isolated zero $\alpha_n$, $\delta_n$ must be chosen in a suitable way in order to obtain the desired zero. When the list of zeros contains a spherical zero, namely a pair $\beta_n$, $\bar{\beta}_n$, we can choose $\delta_n = \beta_n$ and $\delta_{n+1} = \bar{\beta}_n$ so that the product contains factors of the form illustrated in Remark 4.3.6.

As a corollary of Theorem 4.3.8, if we factor first the real zeros, then all the spherical zeros using the previous remark, we obtain the theorem as written in [108, 113].

Theorem 4.3.10 (Weierstrass Factorization Theorem). Let $f$ be an entire slice regular function. Suppose that: $m \in \mathbb{N}$ is the multiplicity of $0$ as a zero of $f$, $\{b_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R} \setminus \{0\}$ is the sequence of the other real zeros of $f$, $\{|s_n|\}_{n \in \mathbb{N}}$ is the sequence of the spherical zeros of $f$, and $\{a_n\}_{n \in \mathbb{N}} \subseteq \mathbb{H} \setminus \mathbb{R}$ is the sequence of the non real zeros of $f$ with isolated multiplicity greater than zero. If all the zeros listed above are repeated according to their multiplicities, then there exists a never vanishing, entire slice regular function $h$ and, for all $n \in \mathbb{N}$, there exist $c_n \in |s_n|$ and $\delta_n \in [\text{Re}(a_n) + I|\text{Im}(a_n)|]$ such that

$$f(q) = q^m \mathcal{R}(q) \mathcal{S}(q) \mathcal{A}(q) \star h(q)$$

where

$$\mathcal{R}(q) = \prod_{n=0}^{\infty} (1 - q^{b_n} - 1 + \frac{1}{n} q^n b_n n),$$

$$\mathcal{S}(q) = \prod_{n=0}^{\infty} \left( \frac{q^2}{|c_n|^2} - \frac{2 q \text{Re}(c_n)}{|c_n|^2} + 1 \right) e^{\frac{q \text{Re}(c_n)}{|c_n|^2} + \frac{1}{n} q^n \text{Re}(c_n)} \frac{n}{|c_n|^2},$$

$$\mathcal{A}(q) = \prod_{n=0}^{*\infty} (1 - q^{\delta_n} - 1) \star e_n(g)$$

and where, for all $n \in \mathbb{N}$, $e_n(q, \delta_n) = e^{q \delta_n^{-1} + \frac{1}{n} q^n \delta_n^{-n}}$.

Corollary 4.3.11. Let $f$ be an entire slice regular function. Then $f$ can be written as $f = gh$ where $g$ and $h$ are entire slice regular, moreover $g$ is such that $g = 1$ or $g$ has spherical or real zeros only and $h$ is such that $h = 1$ or $h$ has isolated zeros only.

In the Weierstrass factorization theorem we used the factors

$$G(q^{\delta_n} p_n) = (1 - q^{\delta_n} - 1) \star e_{p_n}(q^{\delta_n} - 1)$$
see (4.21). Note that the decomposition in factors obtained in this way is far from being unique, in fact the sequence \( \{p_n\} \) is not unique. As we have already pointed out, sometimes it is possible to choose all the \( p_n \) equal to a given integer, see Remark 4.3.4. It is then interesting to ask whether there exists the smallest of such numbers. More precisely, assume that the series

\[
\sum_{n=1}^{\infty} |\delta_n^{-1}|^\lambda
\]  

(4.22)

converges for some \( \lambda \in \mathbb{R}^+ \). Let \( p \) be the smallest natural number such that

\[
\sum_{n=1}^{\infty} |\delta_n^{-1}|^{p+1}
\]

converges. Then, for \( |q| \leq R, R > 0 \), also the series

\[
\sum_{n=1}^{\infty} |q\delta_n^{-1}|^p
\]

converges uniformly and so does the product

\[
\prod_{n=0}^{+\infty} G(q\delta_n^{-1}, p).
\]  

(4.23)

This discussion leads to the following definition:

**Definition 4.3.12.** We will say that (4.23) is a canonical product and that \( p \) is the genus of the canonical product.

In the Weierstrass representation of a function it is convenient to choose the canonical product, instead of another representation. Let \( q \) be the sum of the degrees of the exponents in the functions \( e_p \), if it is finite. Then we have:

**Definition 4.3.13.** The genus of an entire function \( f \) is defined as \( \max(p, q) \) where \( p, q \) are as above and are finite. If \( q \) is not finite or the series (4.22) diverges for all \( \lambda \) we say that the genus of \( f \) is infinite.

## 4.4 Blaschke products

Another useful example of infinite product is the one obtained using the so-called Blaschke factors. In the quaternionic setting it is convenient to write a Blaschke factor distinguishing the case of a isolated zero or of a spherical zero. We start by describing the Blaschke factors related to isolated zeros.
Definition 4.4.1. Let \( a \in \mathbb{H}, |a| < 1 \). The function

\[
B_a(q) = (1 - qa)^{-\ast} \ast (a - q) \frac{\bar{a}}{|a|}
\]  

(4.24)
is called a Blaschke factor at \( a \).

Remark 4.4.2. Using Theorem 2.3.10, \( B_a(q) \) can be rewritten in terms of the pointwise multiplication. In fact, by setting \( \lambda(q) = 1 - qa \) we can write \((1 - qa)^{-\ast} = (\lambda^c(q) \ast \lambda(q))^{-1}\lambda^c(q)\).

Applying formula (2.18) to the products \( \lambda^c(q) \ast \lambda(q) \) and \( \lambda^c(q) \ast (a - q) \), the Blaschke factor (4.24) may be written as

\[
B_a(q) = (\lambda^c(q) \ast \lambda(q))^{-1} \lambda^c(q) \ast (a - q) \frac{\bar{a}}{|a|}
\]

\[
= (\lambda^c(q) \lambda(\tilde{q}))^{-1} \lambda^c(q)(a - \tilde{q}) \frac{\bar{a}}{|a|}
\]

\[
= \lambda(\tilde{q})^{-1}(a - \tilde{q}) \frac{\bar{a}}{|a|} = (1 - \tilde{q}a)^{-1}(a - \tilde{q}) \frac{\bar{a}}{|a|},
\]

(4.25)

where \( \tilde{q} = \lambda^c(q)^{-1}q\lambda^c(q) \). Summarizing, we have

\[
B_a(q) = (1 - \tilde{q}a)^{-1}(a - \tilde{q}) \frac{\bar{a}}{|a|}
\]

where \( \tilde{q} = (1 - qa)^{-1}q(1 - qa) \).

The following result immediately follows from the definition:

Proposition 4.4.3. Let \( a \in \mathbb{H}, |a| < 1 \). The Blaschke factor \( B_a \) is a slice hyperholomorphic function in \( \mathbb{B} \).

Theorem 4.4.4. Let \( a \in \mathbb{H}, |a| < 1 \). The Blaschke factor \( B_a \) has the following properties:

1. it takes the unit ball \( \mathbb{B} \) to itself;
2. it takes the boundary of the unit ball to itself;
3. it has a unique zero for \( q = a \).
Proof. Using Remark 4.4.2 we rewrite $B_{a}(q)$ as $B_{a}(q) = (1-\bar{q}a)^{-1}(a-\bar{q})\frac{\bar{a}}{|a|}$. Let us show that $|q| = |\bar{q}| < 1$ implies $|B_{a}(q)|^2 < 1$. The latter inequality is equivalent to

$$|a - \bar{q}|^2 < |1 - \bar{q}a|^2$$

which is also equivalent to

$$|a|^2 + |q|^2 < 1 + |a|^2|q|^2. \quad (4.26)$$

Then (4.26) becomes $(|q|^2 - 1)(1 - |a|^2) < 0$ and it holds when $|q| < 1$. When $|q| = 1$ we set $q = e^{i\theta}$, so that $\bar{q} = e^{i\theta}$ and we have

$$|B_{a}(e^{i\theta})| = |1 - e^{i\theta}\bar{a}|^{-1}|a - e^{i\theta}|\frac{|\bar{a}|}{|a|} = |e^{-i\theta} - \bar{a}|^{-1}|a - e^{i\theta}| = 1.$$

Finally, from (4.25) it follows that $B_{a}(q)$ has only one zero that comes from the factor $a - \bar{q}$. Moreover

$$B_{a}(a) = (1 - \bar{a}a)^{-1}(a - \bar{a})\frac{\bar{a}}{|a|}$$

where

$$\bar{a} = (1 - a^2)^{-1}a(1 - a^2) = a$$

and thus $B_{a}(a) = 0$. \qed

As we have just proved, $B_{a}(q)$ has only one zero at $q = a$ and analogously to what happens in the case of the zeros of a function, the product of two Blaschke factors of the form $B_{a}(q) \ast B_{\bar{a}}(q)$ gives the Blaschke factor with zeros at the sphere $[a]$. Thus we give the following definition:

**Definition 4.4.5.** Let $a \in \mathbb{H}$, $|a| < 1$. The function

$$B_{[a]}(q) = (1 - 2\text{Re}(a)q + q^2|a|^2)^{-1}(|a|^2 - 2\text{Re}(a)q + q^2) \quad (4.27)$$

is called Blaschke factor at the sphere $[a]$.

**Theorem 4.4.6.** Let $\{a_{j}\} \subset \mathbb{B}$, $j = 1, 2, \ldots$ be a sequence of nonzero quaternions and assume that $\sum_{j \geq 1}(1 - |a_{j}|) < \infty$. Then the function

$$B(q) := \prod_{j \geq 1}^{*}(1 - q\bar{a}_{j})^{-1} \ast (a_{j} - q)\frac{\bar{a}_{j}}{|a_{j}|} \quad (4.28)$$

converges uniformly on the compact subsets of $\mathbb{B}$ and defines a slice hyperholomorphic function.
Proof. Let \( b_j(q) := B_{a_j}(q) - 1 \). We rewrite \( b_j(q) \) using Remark 4.4.2 and we have:

\[
b_j(q) = B_{a_j}(q) - 1 = (1 - \tilde{q}a_j)^{-1}(a_j - \tilde{q}\frac{\tilde{a}_j}{|a_j|}) - 1
\]

\[
= (1 - \tilde{q}a_j)^{-1}\left[(a_j - \tilde{q}\frac{\tilde{a}_j}{|a_j|}) - (1 - \tilde{q}a_j)\right]
\]

\[
= (1 - \tilde{q}a_j)^{-1}\left[|a_j| - 1\right] + q\frac{\tilde{a}_j}{|a_j|}
\]

Thus, recalling that \(|\tilde{q}| = |q|\) and \(|q| < 1\), we have

\[
|b_j(q)| \leq 2(1 - |q|)^{-1}(1 - |a_j|)
\]

and since \(\sum_{j=1}^{\infty}(1 - |a_j|) < \infty\) then \(\sum_{j=1}^{\infty}|b_j(q)| = \sum_{j=1}^{\infty}|B_j(q) - 1|\) converges uniformly on the compact subsets of \(B\). The statement follows from Proposition 4.2.8. \(\square\)

To assign a Blaschke product having zeros at a given set of points \(a_j\) with multiplicities \(n_j, j \geq 1\) and at spheres \([c_i]\) with multiplicities \(m_i, i \geq 1\), we may think to take the various factors a number of times corresponding to the multiplicity of the zero. However, we know that the polynomial \((p - a_j)^{\ast n_j}\) is not the unique polynomial having a zero at \(a_j\) with the given multiplicity \(n_j\), thus the Blaschke product \(\prod_{j=1}^{n_j} B_{a_j}(p)\) is not the unique Blaschke product having zero at \(a_j\) with multiplicity \(n_j\).

Thus we have the following result which takes into account the multiplicity of the zeros in the most general way:

**Theorem 4.4.7.** A Blaschke product having zeros at the set

\[Z = \{(a_1, n_1), \ldots, ([c_1], m_1), \ldots\}\]

where \(a_j \in B, a_j\) have respective multiplicities \(n_j \geq 1, a_j \neq 0\) for \(j = 1, 2, \ldots\), \([a_i] \neq [a_j]\) if \(i \neq j\), \(c_i \in B\), the spheres \([c_j]\) have respective multiplicities \(m_j \geq 1, j = 1, 2, \ldots\), \([c_i] \neq [c_j]\) if \(i \neq j\) and

\[
\sum_{i,j \geq 1} \left(n_i(1 - |a_i|) + 2m_j(1 - |c_j|)\right) < \infty \quad (4.29)
\]

is of the form

\[
\prod_{i \geq 1} \left(B_{[c_i]}(q)^{m_i}\right) \prod_{i \geq 1} \prod_{j=1}^{n_i} (B_{a_{ij}}(q)) \quad (4.30)
\]

where \(n_j \geq 1, \alpha_{11} = a_1\) and \(\alpha_{ij}\) are suitable elements in \([a_i]\) for \(j = 2, 3, \ldots\).
Proof. The hypothesis (4.29) and Theorem 4.4.6 guarantee that the infinite product converges. The zeros of the pointwise product \( \prod_{i \geq 1}(B_{[c_i]}(q))^{m_i} \) correspond to the given spheres with their multiplicities. Consider now the product:

\[
\prod_{j=1}^{\ast n_1}(B_{\alpha_{1j}}(q)) = B_{\alpha_{11}}(q) \ast B_{\alpha_{12}}(q) \ast \cdots \ast B_{\alpha_{1n_1}}(q).
\]

From the definition of multiplicity, this product admits a zero at the point \( \alpha_{11} = a_1 \). This zero has multiplicity 1 if \( n_1 = 1 \); if \( n_1 \geq 2 \), the other zeros are \( \tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{1n_1} \) where \( \tilde{\alpha}_{1j} \) belong to the sphere \( [\alpha_{1j}] = [a_1] \). Since there cannot be other zeros on \( [a_1] \) different from \( a_1 \) (otherwise the whole sphere \( [a_1] \) would be a sphere of zeros), we conclude that \( a_1 \) has multiplicity \( n_1 \). This fact can be seen directly using formula (2.18). Let us now consider \( r \geq 2 \) and

\[
\prod_{j=1}^{\ast n_r}(B_{\alpha_{rj}}(q)) = B_{\alpha_{r1}}(q) \ast \cdots \ast B_{\alpha_{rnr}}(q), \tag{4.31}
\]

and set

\[
B_{r-1}(q) := \prod_{i \geq 1} \prod_{j=1}^{\ast (r-1) \ast n_i}(B_{\alpha_{ij}}(q)).
\]

Then

\[
B_{r-1}(q) \ast B_{\alpha_{r1}}(q) = B_{r-1}(q)B_{\alpha_{r1}}(B_{r-1}(q)^{-1}qB_{r-1}(q))
\]

has a zero at \( a_r \) if and only if

\[
B_{\alpha_{r1}}(B_{r-1}(a_r)^{-1}a_rB_{r-1}(a_r)) = 0,
\]

i.e. if and only if

\[
\alpha_{r1} = B_{r-1}(a_r)^{-1}a_rB_{r-1}(a_r).
\]

If \( n_r = 1 \) then \( a_r \) is a zero of multiplicity 1 while if \( n_r \geq 2 \), all the other zeros of the product (4.31) belongs to the sphere \( [a_r] \) thus the zero \( a_r \) has multiplicity \( n_r \). This finishes the proof. \( \square \)

Comments to Chapter 4. Infinite products of quaternions can be treated by adapting the arguments from the complex setting, see for example [143], to the quaternionic setting. Some results on the quaternionic logarithm come from [113]. The Weierstrass theorem was originally proved in [113]. The version of the theorem proved in this chapter is equivalent but obtained
in a slightly different way. The Blaschke products (in the ball case) has been
treated in several articles but appeared for the first time in [16]. Blaschke
products in the half space have been introduced in [14].
Chapter 5

Growth of entire slice regular functions

5.1 Growth scale

The simplest example of entire slice regular functions is given by polynomials (with coefficients on one side). In the classical complex case the growth of a polynomial is related to its degree and thus to the number of its zeros. In the quaternionic case this fact is true up to a suitable notion of "number of zeros". In fact, as we have seen through the book, there can be spheres of zeros, and thus an infinite number of zeros, even when we consider polynomials. However, if we count the number of spheres of zeros and the number of isolated zeros, we still have a relation with the degree of the polynomial. In fact, each sphere is characterized by a degree two polynomial (see (3.11)), and thus each sphere with multiplicity \( m \) counts as a degree \( 2m \) factor, while each isolated zero of multiplicity \( r \) counts as a degree \( r \) factor.

In this section we introduce the notion of order and type of an entire slice regular function, discussing its growth in relation with the coefficients of its power series expansion and with the density of its zeros.

Let \( f \) be an entire slice regular function and let

\[
M_{f_1}(r) = \max_{|z|=r, z \in \mathbb{C}_I} |f(z)|,
\]

and

\[
M_f(r) = \max_{|q|=r} |f(q)|.
\]

As in the complex case, we have
**Proposition 5.1.1.** Let $f$ be an entire slice regular function. Then function $M_f(r)$ is continuous.

Moreover, in the case of intrinsic functions, we have

**Proposition 5.1.2.** Let $f$ be an entire slice regular function which is quaternionic intrinsic. Then

$$M_f(r) = M_{f_I}(r),$$

for all $I \in \mathbb{S}$.

*Proof.* Let us write $f(x + Iy) = \alpha(x, y) + I\beta(x, y)$ where $\alpha, \beta$ are real-valued functions, as $f$ is intrinsic. We have, for $q = x + Iy$,

$$M_f(r) = \sup_{|q|=r} |f(x + Iy)|$$

$$= \sup_{I \in \mathbb{S}} \sup_{x^2 + y^2 = r^2} |f(x + Iy)|$$

$$= \sup_{I \in \mathbb{S}} \sup_{x^2 + y^2 = r^2} (\alpha(x, y)^2 + \beta(x, y)^2)^{1/2}$$

$$= \sup_{x^2 + y^2 = r^2} (\alpha(x, y)^2 + \beta(x, y)^2)^{1/2}$$

$$= \sup_{x^2 + y^2 = r^2} [f_I(x + Iy)] = M_{f_I}(r)$$

and the statement follows. \qed

**Definition 5.1.3.** Let $f$ be a quaternionic entire function. Then $f$ is said to be of finite order if there exists $k > 0$ such that

$$M_f(r) < e^{r^k},$$

for sufficiently large values of $r$ ($r > r_0(k)$). The greatest lower bound $\rho$ of such numbers $k$ is called the order of $f$.

From the definition of order, it immediately follows the inequalities below which will be useful in the sequel:

$$e^{r^{\rho-\epsilon}} < M_f(r) < e^{r^{\rho+\epsilon}}. \quad (5.1)$$

Let us recall the notations

$$\lim_{r \to \infty} \phi(r) = \lim_{r \to \infty} \inf_{t \geq r} \phi(t).$$

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and
\[ \lim_{r \to \infty} \phi(r) = \lim_{r \to \infty} \sup_{t \geq r} \phi(t). \]

Then, the inequalities 5.1 are equivalent to
\[ \rho = \lim_{r \to \infty} \frac{\log(\log M_f(r))}{\log r}. \]

This latter condition can be considered as an equivalent definition of order of an entire function \( f \).

We now show that an entire slice regular function which grows slower than a positive power of \( r \) is a polynomial:

**Proposition 5.1.4.** If there exists \( N \in \mathbb{N} \) such that
\[ \lim_{r \to \infty} \frac{M_f(r)}{r^N} < \infty \]
then \( f(q) \) is a polynomial of degree at most \( N \).

**Proof.** Let \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) and set
\[ p_N(q) = \sum_{n=0}^{N} q^n a_n, \quad g_N(q) = q^{-N-1}(f(q) - p_N(q)). \]

Then, by our hypothesis, \( g_N(q) = \sum_{j=0}^{\infty} q^j a_{N+1+j} \) is an entire regular function which tends to 0 on a sequence of balls \( |q| = r_j \) when \( r_j \to \infty \). Thus \( g_N \) vanishes on a sequence of balls intersected with a complex plane \( \mathbb{C}_I \) and so, by the Identity Principle, \( g_N \equiv 0 \). We conclude that \( f(q) \) coincides with \( p_N(q) \).

From this proposition, it follows that the growth of an entire regular function is larger than any power of the radius \( r \). In the next definition we compare the growth of a function with functions of the form \( e^{Ar^\rho} \):

**Definition 5.1.5.** Let \( f \) be an entire regular function of order \( \rho \) and let \( A > 0 \) be such that for sufficiently large values of \( r \)
\[ M_f(r) < e^{Ar^\rho}. \]

We say that \( f \) of order \( \rho \) is of type \( \sigma \) if \( \sigma \) is the greatest lower bound of such numbers \( A \).

When \( \sigma = 0 \) we say that \( f \) is of minimal type.

When \( \sigma = \infty \) we say that \( f \) is of maximal type.

When \( 0 < \sigma < \infty \) we say that \( f \) is of normal type.
As in the case of the order, one can verify that the type of a function $f$ of order $\rho$ is given by

$$\sigma = \lim_{r \to \infty} \frac{\log(M_f(r))}{r^\rho}.$$ 

**Example 5.1.6.** The function $\exp(q^n\sigma)$, where $n \in \mathbb{N}$, has type $\sigma$ and order $n$.

**Definition 5.1.7.** We will say that the function $f(q)$ is of growth larger than $g(q)$ if the order of $f$ is larger than the order of $g$ or, if $f$ and $g$ have the same order and the type of $f$ is larger than the type of $g$.

**Remark 5.1.8.** From the definition of order and type, it follows that the order of the sum of two functions is not greater than the largest of the order of the summands. If one summand has order larger than the order of the other summand, then the sum has same order and type of the function of larger growth. If two functions have the same order and this is also the order of their sum, then the type of the sum is not greater than the largest of the type of the summands.

We now relate the order and type of a function with the decrease of its Taylor coefficients. Recall that if $f(q) = \sum_{n=0}^{\infty} q^n a_n$ is an entire slice regular function, then

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = 0.$$

**Theorem 5.1.9.** The order and the type of the entire regular function $f(q) = \sum_{n=0}^{\infty} q^n a_n$ can be expressed by

$$\rho = \lim_{n \to \infty} n \log(n) / \log(|a_n|)$$

and

$$(\sigma e^\rho)^{1/\rho} = \lim_{n \to \infty} n^{1/\rho} \sqrt[n]{|a_n|}.$$ 

**Proof.** The proof closely follows the proof of the corresponding results in complex analysis since it depends only on the modulus of the coefficients $a_n$, see e.g. [133]. We insert it for the sake of completeness. First of all we observe that if $f(q)$ is of finite order then, asymptotically (namely for sufficiently large $r$), we have

$$M_f(r) < \exp(Ar^k).$$ 

The Cauchy estimates, see (3.1.5), give

$$|a_n| \leq \frac{M_f(r)}{r^n}.$$ 

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and so

\[ |a_n| \leq \frac{\exp(A r^k)}{r^n}. \]

An immediate computation gives that the maximum of the function \( \frac{\exp(A r^k)}{r^n} \) is given by \( (eAk/n)^{\frac{n}{k}} \) so we conclude that, asymptotically:

\[ |a_n| \leq \left( \frac{eAk}{n} \right)^{\frac{n}{k}}. \]  \( (5.5) \)

Assume now that \( (5.5) \) is valid for \( n \) large enough. Then

\[ |q^n a_n| < r^n \left( \frac{eAk}{n} \right)^{\frac{n}{k}} \]

and if we take \( n > \lfloor 2^k eAk r^k \rfloor = N_r \) (where \( \lfloor x \rfloor \) denotes the integer part of \( x \)), we obtain \( |q^n a_n| < 2^{-n} \) which yields

\[ |f(q)| < \sum_{j=0}^{N_r} r^j |a_j| + 2^{-N_r}. \]

Let \( \mu(r) = \max_j r^j |a_j| \), then

\[ M_f(r) \leq (1 + 2^k eAk r^k) \mu(r) + 2^{-N_r}. \]  \( (5.6) \)

As \( f \) is not a polynomial, then \( M_f(r) \) grows faster than any power of \( r \). From \( (5.5) \) it follows that, asymptotically:

\[ \mu(r) \leq r^n \max_n \left( \frac{eAk}{n} \right)^{\frac{n}{k}} = e^{Ar^k}, \]

since the maximum is attained for \( n = Ar^k \). Using \( (5.6) \) we have

\[ M_f(r) < (2 + 2^k a A k^{r^k}) e^{Ar^k}. \]  \( (5.7) \)

So we have that if \( f \) is of finite order, then \( (5.4) \) holds, but this implies \( (5.5) \), which, in turns, implies \( (5.7) \). Thus the order \( \rho \) equals the greatest lower bound of the numbers \( k \) for which \( (5.5) \) holds, while the type equals the greatest lower bound of numbers \( A \) for which \( (5.5) \) is valid for \( k = \rho \).

Since the conjugate \( f^c \) of a slice regular functions expanded in power series has coefficients with the same modulus of the coefficients of \( f \), we have:
Corollary 5.1.10. Let $f$ be an entire slice regular function. Its order and type coincide with order and type of its conjugate $f^c$.

Using Theorem 5.1.9 one can construct entire regular functions of arbitrary order and type: a function of order $n$ and type $\sigma$ is $e^{\sigma \rho^n}$ (compare with the classical complex case).

Our next goal is to establish a dependence between the growth of a function and the density of distribution of its zeros and, in particular, we need to define how to count the zeros of a function. As we already observed, if a function has spherical zeros it automatically possess an infinite number of zeros, and so to introduce a notion of counting function, we need to treat in the appropriate way the spherical zeros. To this end, we introduce the following:

**Assumption.** Assume that a function has zeros

$$\alpha_1, \ldots, \alpha_n, \ldots, [\beta_1], \ldots, [\beta_m], \ldots$$

with $\lim_{n \to \infty} |\alpha_n| = \infty$, $\lim_{n \to \infty} |\beta_n| = \infty$.

It is convenient to write a spherical zero as a pair of conjugate numbers, so we have:

$$\alpha_1, \ldots, \alpha_n, \ldots, \beta_1, \bar{\beta}_1, \ldots, \beta_m, \bar{\beta}_m, \ldots$$

We will also assume that we arrange the zeros according to increasing values of their moduli, keeping together the pairs $\beta_\ell, \bar{\beta}_\ell$ giving rise to spherical zeros. Then we rename $\gamma_s, s = 1, 2, \ldots$, the elements in the list so obtained:

$$\gamma_1, \gamma_2, \ldots, \gamma_n, \ldots$$

(5.8)

where $\gamma_s$ denotes one of the elements $\alpha_m$ or $\beta_\ell$ or $\bar{\beta}_\ell$.

As we shall see, the results obtained using this sequence will not depend on the chosen representative $\beta_\ell$ of a given sphere: what will matter is only the modulus $|\beta_\ell|$ which is independent of the representative chosen.

**Definition 5.1.11.** We define the convergence exponent of the sequence $(5.8)$ to be the greatest lower bound of $\lambda$ for which the series

$$\sum_{n=1}^{\infty} \frac{1}{|\gamma_n|^\lambda}$$

(5.9)

converges. If the series (5.9) diverges for every $\lambda > 0$ we will say that the convergence exponent is infinite.
Note that, by its definition, the larger $|\gamma_n|$ become, the smaller becomes $\lambda$.

**Definition 5.1.12.** Consider a sequence of quaternions as in (5.8). Let $n(r)$ be the number of elements in the sequence belonging to the ball $|q| < r$. We say that $n(r)$ is the counting number or counting function of the sequence. The number 
\[ \rho_1 = \lim_{r \to \infty} \frac{\log(n(r))}{\log(r)} \]

is called order of the function $n(r)$. The number 
\[ \Delta = \lim_{r \to \infty} \frac{n(r)}{r^{\rho_1}} \]

is called upper density of the sequence (5.8), and if the limit exists, $\Delta$ is simply called density.

**Proposition 5.1.13.** The convergence exponent of the sequence $\{\gamma_n\}$ equals the order of the corresponding counting function $n(r)$.

**Proof.** The proof of this result is not related to quaternions, see [133]. We repeat it for the reader’s convenience. The series (5.9) is a series of real numbers which can be expressed by a Stieltjes integral in the form
\[ \int_0^\infty \frac{dn(t)}{t^\lambda}, \]
which is equal to
\[ \int_0^r \frac{dn(t)}{t^\lambda} = \frac{n(t)}{t^\lambda} + \lambda \int_0^r \frac{n(t)}{t^{\lambda+1}} dt. \]
If the series (5.9) is convergent, the two positive terms at the right hand side are bounded. Moreover the integral
\[ \int_0^\infty \frac{n(t)}{t^{\lambda+1}} dt \quad (5.10) \]
is convergent since it is increasing and bounded. Consequently, for any $\varepsilon > 0$ and $r > r_0 = r_0(\varepsilon)$ we have
\[ \frac{n(r)}{r^\lambda} = \lambda n(r) \int_r^\infty \frac{dt}{t^{\lambda+1}} \leq \lambda \int_r^\infty \frac{n(t)dt}{t^{\lambda+1}} < \varepsilon. \]
So
\[ \lim_{r \to \infty} \frac{n(r)}{r^{\lambda}} = 0 \]
and the order of \( n(r) \) is not greater than \( \lambda \). Conversely, the preceding reasoning shows that the convergence of the above integral (5.10) implies the convergence of the series (5.9). Let \( \rho_1 \) be the order of \( n(r) \). Then for \( t \) large enough we have
\[ n(t) < t^{\rho_1 + \varepsilon/2}. \]
By setting \( \lambda = \rho_1 + \varepsilon \) we have that (5.10) converges and therefore (5.9) converges. So \( \lambda \) is not greater than the order of \( n(r) \) and the statement follows. \( \square \)

### 5.2 Jensen theorem

In complex analysis, Jensen theorem states that if a function \( f(z) \) is analytic in \( |z| < R \) and such that \( f(0) \neq 0 \), then
\[
\int_0^R \frac{n_f(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})|d\theta - \log |f(0)|,
\]
where \( n_f(t) \) is the number of zeros of \( f \) in the disc \( |z| < t \).

In this section we prove an analog of this theorem in the slice regular case, where \( n_f \) is the counting number of the sequence constructed with the zeros of \( f \) in the ball \( |q| < t \).

To state Jensen theorem we need some preliminary lemmas and the following definition:

**Definition 5.2.1.** Let \( f \) be a function slice regular in a ball centered at the origin and with radius \( R > 0 \). Let \( \gamma = \{ \gamma_n \} \), with \( |\gamma_n| \to \infty \) be the sequence of its zeros written according to the assumption in the previous section. We denote by \( n_{f,I}(t) \) the number of elements in \( \gamma \) belonging to the disc \( \{ q : |q| < t \} \cap \mathbb{C}_I, \) for \( t \leq R \). We denote by \( n_f(t) \) the number of elements in \( \gamma \) belonging to the ball \( \{ q : |q| < t \}, \) for \( t \leq R \).

We note that since the choice of the elements \( \beta_\ell, \overline{\beta_\ell} \) representing the sphere \( [\beta_\ell] \) is arbitrary, on each complex plane \( \mathbb{C}_I \) we can find a pair of representatives of \( [\beta_\ell] \). Thus, \( n_{f,I}(t) \) is the sum of the number of isolated zeros \( \alpha_\ell \) which belong to the disc \( |z| < t \) in the complex plane \( \mathbb{C}_I \) and of twice the number of spheres \( [\beta_\ell] \) in that same disc.

**Lemma 5.2.2.** If \( g \) is quaternionic intrinsic in \( B(0;t) \), \( t > 0 \), then \( n_g(t) = n_{g,I}(t) \).
Proof. If the function \( g \) has a nonreal zero at \( q_0 = x_0 + Iy_0 \), then \( 0 = \overline{g(q_0)} = g(\overline{q_0}) \), thus \( g \) vanishes at \( \overline{q_0} \) and so it has a spherical zero at \( [q_0] \). Assume that the spherical zero \([q_0]\) has multiplicity \(m\), namely

\[
g(q) = ((q - x_0)^2 + y_0^2)^m \tilde{g}(q)
\]

where \( \tilde{g}(x_0 + Iy_0) \neq 0 \) for all \( I \in \mathbb{S} \). Then for every \( I \in \mathbb{S} \), on the complex plane \( \mathbb{C}_I \) the function \( g \) has zeros at \( x_0 \pm Iy_0 \), each of which with multiplicity \( m \).

The real zeros of \( g \) belong to every \( \mathbb{C}_I \). Thus, the number of points \( \alpha_{e_I} \), being real, is constant for all \( I \in \mathbb{S} \). Repeating the reasoning for all the zeros of \( g \), we deduce that \( n_g(t) = n_{g,I}(t) \), for all \( I \in \mathbb{S} \).

**Lemma 5.2.3.** If \( h \) is slice regular in \( B(0; t), t > 0 \), and it has isolated zeros only, then

\[
n_h(t) = \frac{1}{2} n_{h^s}(t).
\]

*Proof.* To prove the statement assume that in the ball \( \{ q : |q| < t \} \) there are \( N(t) \) distinct isolated zeros and that

\[
h(q) = \prod_{r=1}^{N(t)} (q - \alpha_{r1}) \ast \ldots \ast (q - \alpha_{rj_r}) \ast \tilde{h}(q),
\]

where, for all \( r = 1, \ldots, N(t) \), \( j_r \geq 1 \) denotes the multiplicity of the zero \( \alpha_{r1} \in [\alpha_{r1}] \) (note that only the zero \( \alpha_{11} \) can be immediately read from the factorization of \( h \)). Thus \( n_h(t) = \sum_{r=1}^{N(t)} j_r \). Then we have

\[
h^c(q) = \tilde{h}^c(q) \ast \prod_{r=N(t)}^{1} (q - \overline{\alpha_{rj_r}}) \ast \ldots \ast (q - \overline{\alpha_{r1}}),
\]

(\( \prod_{r=N(t)}^{1} \) indicates that we are taking the products starting with the index \( N(t) \) and ending with \( 1 \)) and, by setting \( \alpha_{r1} = x_r + I_r y_r \), \( r = 1, \ldots, N(t) \), we deduce

\[
h^s(q) = \prod_{r=1}^{N(t)} (q - \alpha_{r1}) \ast \ldots \ast (q - \alpha_{rj_r}) \ast \tilde{h}(q) \ast \tilde{h}^c(q) \ast \prod_{r=N(t)}^{1} (q - \overline{\alpha_{rj_r}}) \ast \ldots \ast (q - \overline{\alpha_{r1}})
\]

\[
= \prod_{r=1}^{N(t)} ((q - x_r)^2 + y_r^{2j_r}) \tilde{h}^s(q).
\]

Since \( \tilde{h}^s \) does not vanish on \( |q| < r \) we have that \( n_h(t) = \sum_{r=1}^{N(t)} 2j_r \) and the statement follows. \( \square \)
Lemma 5.2.4. Let \( f \) be a function, not identically zero and slice regular in a domain containing \( B(0; R) \). Then there exist two functions \( g, h \) slice regular in \( B(0; R) \) and continuous in \( B(0; R) \) such that \( f = gh \), where \( g \) has at most spherical only, and \( h \) has at most isolated zeros only.

Proof. We observe that \( f \) can have only a finite number of spherical zeros in \( B(0; R) \). Otherwise, if these zeros were an infinite number, then on a complex plane \( \mathbb{C}_I \) the function \( f \) would admit an infinite number of zeros with an accumulation point in \( B(0; R) \cap \mathbb{C}_I \) and so, by the Identity Principle, the function \( f \) would be identically zero, contradicting the assumption. Thus this finite number of zeros can be pulled out on the left using Theorem 3.3.4. The product of this finite number of factors corresponding to these zeros (and thus all with real coefficients) gives the function \( g \). If \( f \) does not have any spherical zero, we can set \( g = 1 \) on \( B(0; R) \). The factor \( h \) contains at most isolated zeros. Both the functions \( g \) and \( h \) are slice regular and continuous where needed, by construction. \( \square \)

Theorem 5.2.5 (Jensen theorem). Let \( f \) be a function slice regular in a domain containing \( B(0; R) \) and assume that \( f(0) \neq 0 \). Let \( f = gh \) where \( g \) has at most spherical zeros only, and \( h \) has at most isolated zeros only. Then

\[
\int_0^R \frac{n_f(t)}{t} dt = \frac{1}{2\pi} \left[ \int_0^{2\pi} \log |g(Re^{i\theta})| d\theta - \log |g(0)| + \frac{1}{2} \left( \int_0^{2\pi} \log |h^s(Re^{i\theta})| d\theta - \log |h^s(0)| \right) \right].
\]

(5.11)

Proof. First of all we observe that \( n_f(t) = n_g(t) + n_h(t) = n_g(t) + \frac{1}{2} n_{h^s}(t) \)

where the last equality follows from Lemma 5.2.3. Then we have

\[
\int_0^R \frac{n_f(t)}{t} dt = \int_0^R \frac{n_g(t)}{t} dt + \frac{1}{2} \int_0^R \frac{n_{h^s}(t)}{t} dt.
\]

(5.12)

Since both \( g \) and \( h^s \) are quaternionic intrinsic functions we have

\( g_I, h^s_I : B(0, t) \cap \mathbb{C}_I \rightarrow \mathbb{C}_I \)

and so the restrictions \( g_I, h^s_I \) to any complex plane \( \mathbb{C}_I \) are holomorphic functions to which we can apply the complex Jensen theorem, see Theorem
5, p. 14 in [133], namely
\[
\int_0^R \frac{n_g(t)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log |g(Re^{i\theta})| d\theta - \log |g(0)|,
\]
and similarly for \( h_s^f \). Then, the statement follows from Lemma 5.2.2. \( \square \)

We now prove the following consequence of Jensen theorem, characterizing the number of zeros of \( f \) in a ball.

**Proposition 5.2.6 (Jensen inequality).** Let \( f \) be slice regular in the ball centered at the origin and with radius \( er \), where \( r > 0 \) and \( e \) is the Napier number. Let \( f = gh \) where \( g \) has at most spherical zeros only, and \( h \) has at most isolated zeros only and assume that \( |g(0)| = |h(0)| = 1 \). Then
\[
n_f(r) \leq \log(M_g(er)M_h(er)). \tag{5.13}
\]
The bound \( \log(M_g(er)M_h(er)) \) is optimal, as equality may occur.

**Proof.** Since \( |g(0)| = |h(0)| = 1 \) and \( f(0) = g(0)h(0) \) we have \( |f(0)| = 1 \). We can apply Jensen theorem, and the fact that \( n_f(r) \) is evidently a monotone function in \( r \) yields
\[
n_f(r) \leq \int_r^R \frac{n_f(t)}{t} dt \\
\leq \frac{1}{2\pi} \left[ \int_0^{2\pi} \log |g(Re^{i\theta})| d\theta + \frac{1}{2} \int_0^{2\pi} \log(|h^s(Re^{i\theta})|) d\theta \right] \tag{5.14}
\leq \log M_g(R) + \frac{1}{2} \log M_h^s(R),
\]
where we have set \( R = er \). Recalling (2.15), we have
\[
\sup_{|q|=R} |h^c(q)| = \sup_{I \in \mathbb{S}} \sup_{\theta \in [0,2\pi]} |h^c(Re^{i\theta})| \\
= \sup_{I \in \mathbb{S}} \sup_{\theta \in [0,2\pi]} |\alpha(R,\theta) + I\beta(R,\theta)| \\
= \sup_{I \in \mathbb{S}} \sup_{\theta \in [0,2\pi]} |\alpha(R,\theta) + I\beta(R,\theta)| \\
= \sup_{I \in \mathbb{S}} \sup_{\theta \in [0,2\pi]} |h(Re^{i\theta})| = \sup_{|q|=R} |h(q)|
\]

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and so, using \(2.3.10\) and \(2.15\), we deduce
\[
\sup_{|q|=R} |h^s(q)| = \sup_{|q|=R} |h(q) * h^c(q)|
= \sup_{|q|=R} |h(q)||h^c(\bar{q})|
\leq \sup_{|q|=R} |h(q)|^2
\leq \left( \sup_{|q|=R} |h(q)| \right)^2.
\]
(5.15)

From (5.15) we obtain
\[
\log M_g(R) + \frac{1}{2} \log M_{h^s}(R) \leq \log M_g(R) + \log(M_h(R))
= \log(M_g(R)M_h(R)),
\]
and so
\[
n_f(r) \leq \log(M_g(R)M_h(R)),
\]
which proves the first part of the statement. We have equalities, instead of inequalities, in (5.14) if
\[
n_f(r) = \int_r^R \frac{n_f(t)}{t} \, dt
\]
which means that \(f\) cannot have zeros in \(\{q \in \mathbb{H} : r < |q| < R\}\). Moreover, we must have the equality
\[
\int_r^R \frac{n_f(t)}{t} \, dt = \frac{1}{2\pi} \left[ \int_0^{2\pi} \log |g(Re^{i\theta})| \, d\theta + \frac{1}{2} \int_0^{2\pi} \log |h^s(Re^{i\theta})| \, d\theta \right]
\]
which, by Jensen formula, means that \(f\) has no zeros in the ball \(B(0; r)\).

Finally, if we have an equality instead of the last inequality in (5.14) then \(|f(ere^{i\theta})| = M_g(ere)M_h(ere)\) on the sphere \(|q| = er\). Consider now a function of the form
\[
\varphi(q) = e^{n+ia} \prod_{k=1}^n (R^2 - q\bar{a}_k)^{-*} * (q - a_k)er,
\]
where \(|a_k| = r\) and the factor \(e^{n+ia}\) is needed in order to obtain \(|\varphi(0)| = 1\).
Our purpose is to show that such a function \(\varphi\) satisfies equalities in (5.14).

First of all, we note that
\[
(R^2 - q\bar{a}_k)^{-*} * (q - a_k)R = (R^2 - p\bar{a}_k)^{-1}(p - a_k)R
\]

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where $p = s(q)^{-1}qs(q)$, $s(q) = R^2 - qa_k$ and $s(q) \neq 0$ for $q \in B(0; R)$. Let us study when the inequality

$$|R^2 - \bar{a}_kp|^{-1}|p - a_k|R \leq 1$$

is satisfied. The inequality is equivalent to

$$|p - a_k|^2 R^2 \leq |R^2 - \bar{a}_kp|^2$$

but also to

$$(p - a_k)(\bar{p} - \bar{a}_k)R^2 \leq (R^2 - \bar{a}_kp)(R^2 - \bar{pa}_k)$$

that is, after some calculations

$$(R^2 - |a_k|^2)(|p|^2 - R^2) \leq 0.$$ 

Since $(R^2 - |a_k|^2) > 0$, the above inequality is satisfied if and only if $|p|^2 - R^2 \leq 0$ that is if and only if $|p| = |q| \leq er$. We conclude that each factor takes the ball $B(0; R)$ to $B(0; 1)$ and, in particular, it has modulus 1 on the sphere $|q| = R$. As a consequence, $|\varphi(z)| = e^n$ for all $q$ such that $|q| = R$. All the roots $a_k$ of $\varphi$ have modulus $r$ and $\varphi(0) = 1$. If there are pairs $a_k, a_{k+1}$ such that $a_{k+1} = \bar{a}_k$ then the product of the two corresponding factor gives a spherical zero. If there are $r$ such pairs, we will have $r$ spheres of zeros and the product of the corresponding factors, which commute among themselves and with the other factors, form the function $g$ such that $\varphi(q) = g(q)h(q)$. The remaining factors are $n - 2r$ and will give $2(n - 2r)$ spherical zeros of $h^a$. Since the factor $e^{i+\alpha}$ can be suitably split in $e^{2r+i\alpha}$ in front of $g$ and $e^{(n-2r)+i\alpha}$ in front of $h$, it follows by its construction, that the function $\varphi$ satisfies the equalities in (5.11).

5.3 Carathéodory theorem

The real part of a slice regular function does not play the important role that real parts play for holomorphic functions. For example, what plays the role of the real part in the Schwarz formula in the complex case is the function $\alpha(x, y)$ if $f(x + iy) = \alpha(x, y) + I\beta(x, y)$ is a slice regular function in $B(0; R)$, $R > 0$. Thus, also the analog of Carathéodory theorem is stated in terms of the function $\alpha(x, y)$. Let us set

$$A_f(r) = \max_{|x + iy| = r} |\alpha(x, y)|, \quad \text{for} \quad r < R.$$
Lemma 5.3.1. Let \( f \in \mathcal{R}(B(0; R)), \ R > 0 \). Then \( A_f(r) \leq M_f(r) \).

Proof. Since \( \alpha(x, y) = \frac{1}{2}(f(x + Iy) + f(x - Iy)) \) for any \( x + Iy \in B(0; R) \) we have

\[
A_f(r) = \max_{|x + Iy| = r} |\alpha(x, y)|
\]

\[
= \frac{1}{2} \max_{|x + Iy| = r} |f(x + Iy) + f(x - Iy)|
\]

\[
\leq \frac{1}{2} \left( \max_{|x + Iy| = r} |f(x + Iy)| + \max_{|x + Iy| = r} |f(x - Iy)| \right)
\]

\[
= M_f(r)
\]

for \( r < R \). \( \square \)

We can now prove a Carathéodory type inequality, which is a sort of converse of the previous inequality:

Theorem 5.3.2 (Carathéodory inequality). Let \( f \in \mathcal{R}(B(0; R)), \ f(q) = f(x + Iy) = \alpha(x, y) + I\beta(x, y) \) and let \( 0 < r < R \). Suppose, for simplicity, that \( f(0) \in \mathbb{R} \). Then

\[
M_f(r) \leq \frac{2r}{R - r} (A_f(R) - \alpha(0)) + |\alpha(0)|.
\]

Proof. Let us write the function \( f(q) \) using the Schwarz formula in Theorem 3.1.13

\[
f(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (Re^{It} - q)^{-\ast} (Re^{It} + q) \alpha(Re^{It}) dt.
\]  

(5.17)

If we fix two units \( I, J \in \mathbb{S} \) with \( I \) orthogonal to \( J \), we can decompose the function \( \alpha \) as \( \alpha = \alpha_0 + \alpha_1 I + \alpha_2 J + \alpha_3 IJ \) where \( \alpha_\ell \) are harmonic functions, and since

\[
\alpha_\ell(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha_\ell(Re^{It}) dt, \quad \ell = 0, \ldots, 3
\]

we have that

\[
\alpha(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(Re^{It}) dt = 0.
\]
We can add this quantity to the right hand side of (5.17), obtaining
\[
f(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\text{Re}e^{It} - q)^{-*} \ast (\text{Re}e^{It} + q) \alpha(\text{Re}e^{It}) \, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} \alpha(\text{Re}e^{It}) \, dt + \alpha(0)
\]
\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\text{Re}e^{It} - q)^{-*} \ast (\text{Re}e^{It} + q - \text{Re}e^{It} + q) \alpha(\text{Re}e^{It}) \, dt + \alpha(0)
\]
\[
= \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{Re}e^{It} - q)^{-*} \ast q \alpha(\text{Re}e^{It}) \, dt + \alpha(0).
\]
(5.18)

Assume \( f \equiv 1 \). Since \( f(x + Iy) = 1 \) for all \( x + Iy \in B(0; R) \) we have that \( \alpha(x, y) + I\beta(x, y) \equiv 1 \) for all \( x + Iy \in B(0; R) \). By assigning to \( I \), e. g., the values \( i, j, k, (i + j)/\sqrt{2}, (i + k)/\sqrt{2}, (j + k)/\sqrt{2} \) we deduce that \( \beta(x, y) \equiv 0 \) and \( \alpha(x, y) \equiv 1 \). Thus we have
\[
\frac{1}{\pi} \int_{-\pi}^{\pi} (\text{Re}e^{It} - q)^{-*} \ast q \, dt = 0.
\]
(5.19)

From (5.18) and (5.19) we obtain
\[
-f(q) = \frac{1}{\pi} \int_{-\pi}^{\pi} (\text{Re}e^{It} - q)^{-*} \ast q \left( A_f(r) - \alpha(\text{Re}e^{It}) \right) \, dt - \alpha(0)
\]
from which we deduce, since \( A_f(r) - \alpha(\text{Re}e^{It}) \) is nonnegative,
\[
|f(q)| \leq \frac{2r}{R - r} (A_f(R) - \alpha(0)) + |\alpha(0)|
\]
Taking the maximum on \( |q| = r \) of both sides we obtain
\[
M_f(r) \leq \frac{2r}{R - r} (A_f(R) - \alpha(0)) + |\alpha(0)|,
\]
and this concludes the proof.

In the complex case, Carathéodory theorem allows to estimate from below the modulus of a holomorphic function without zeros in a disc centered at the origin. The proof makes use of the composition of the logarithm with the holomorphic function, so this technique cannot be immediately used in the quaternionic case. However, we can still prove a bound from below on the modulus of a slice regular function.

Let us recall the result in the complex case (see [133] Theorem 9, p. 19):
**Theorem 5.3.3.** Let $f$ be a function holomorphic in $|z| \leq R$ which has no zeros in that disk and such that $f(0) = 1$. Then for any $z$ such that $|z| \leq r < R$, the modulus of $f$ satisfies

$$
\log |f(z)| \geq - \frac{2r}{R-r} \log M_f(R).
$$

(5.20)

The result in the quaternionic case is similar, but the value of the constant is different:

**Theorem 5.3.4.** Let $f$ be a function slice regular in $|q| \leq R$ which has no zeros in that ball and such that $f(0) = 1$. Then for any $q$ such that $|q| \leq r < R$, the modulus of $f$ satisfies

$$
\log |f(q)| \geq - \frac{3r + R}{R-r} \log M_f(R).
$$

(5.21)

**Proof.** Let us consider the symmetrization $f_s$ of $f$. This is a slice regular function which is intrinsic so we can use the result in the complex case, see Proposition 5.1.2 and its proof, and for any $I \in \mathbb{S}$ we have

$$
\log |f_s(q)| = \log |f_s(x + Iy)| \geq - \frac{2r}{R-r} \log M_{f_s}(R) = - \frac{2r}{R-r} \log M_{f_s}(R).
$$

(5.22)

From (5.15) we have $M_{f_s}(R) \leq M_f(R)^2$, so we deduce $\log M_{f_s}(R) \leq 2 \log M_f(R)$ or, equivalently

$$
- \log M_{f_s}(R) \geq -2 \log M_f(R).
$$

(5.23)

It follows that

$$
\log |f^s(q)| = \log (|f(q)| |f^c(f(q)^{-1}qf(q))|) = \log |f(q)| + \log |f^c(f(q)^{-1}qf(q))|
$$

$$
\leq \log |f(q)| + \max_{|q| \leq R} \log |f^c(f(q)^{-1}qf(q))|
$$

$$
\leq \log |f(q)| + \log(\max_{|q| \leq R} |f^c(f(q)^{-1}qf(q)))|
$$

$$
\leq \log |f(q)| + \log(M_f(R)).
$$

(5.24)

From (5.22), (5.23), (5.24) we have

$$
\log |f(q)| + \log(M_f(R)) \geq \log |f^s(q)| \geq - \frac{2r}{R-r} \log M_{f_s}(R)
$$

which gives

$$
\log |f(q)| \geq - \frac{4r}{R-r} \log M_f(R) - \log(M_f(R)) = - \frac{3r + R}{R-r} \log M_f(R).
$$

\qed
In order to provide a lower bound for the modulus of a function slice regular in a ball centered at the origin, we need the following result:

**Proposition 5.3.5.** Let \( a_1, \ldots, a_t \in \mathbb{B} \). Given the function

\[
    f(q) = \prod_{k=1}^{t} (1 - q \overline{a_k})^{-*} (q - a_k)
\]

it is possible to write \( f^{-*}(q) \), where it is defined, in the form

\[
f^{-*}(q) = \prod_{k=t}^{*1} (\hat{a}_k - q) \overline{\hat{a}_k} \rightstar \prod_{k=t}^{*1} (q - \hat{a}_k)^{-*}
\]

(5.25)

where \( \hat{a}_k \in [a_k^{-1}] \), \( \hat{a}_k \in [a_k] \), and in particular \( \hat{a}_1 = a_1 \), \( \hat{a}_t = \hat{a}_t^{-1} \).

**Proof.** First of all, note that

\[
    (q - a_k)^{-*} \star (1 - q \overline{a_k}) = (1 - q \overline{a_k}) \star (q - a_k)^{-*}
\]

and so

\[
    f^{-*}(q) = \prod_{k=t}^{*1} (1 - q \overline{a_k}) \star (q - a_k)^{-*}.
\]

We prove (5.25) by induction. Assume that \( t = 2 \) and consider

\[
    f^{-*}(q) = (1 - q \overline{a_2}) \star (q - a_2)^{-*} \star (1 - q \overline{a_1}) \star (q - a_1)^{-*}.
\]

Since the spheres \( [a_2] \) and \( [\overline{a_1}^{-1}] = [a_1^{-1}] \) are different, by Theorem 3.3.1 we have, for suitable \( \hat{a}_1 \in [a_1^{-1}] \), \( \hat{a}_2' \in [a_2] \):

\[
    (q - a_2)^{-*} \star (1 - q \overline{a_1}) = (q - a_2)^{-*} \star (\overline{a_1}^{-1} - q) \overline{a_1}^{-1}
\]

\[
    = (q^2 - 2 \text{Re}(a_2)q + |a_2|^2)^{-1}(q - a_2) \rightstar (\overline{a_1}^{-1} - q) \overline{a_1}^{-1}
\]

\[
    = (q^2 - 2 \text{Re}(a_2)q + |a_2|^2)^{-1}(\hat{a}_1 - q) \rightstar (q - a_2) \overline{\overline{a_1}^{-1} a_2' \overline{a_1}}
\]

\[
    = (q^2 - 2 \text{Re}(a_2)q + |a_2|^2)^{-1}(\hat{a}_1 - q) \rightstar (q - \hat{a}_2)
\]

\[
    = (\hat{a}_1 - q) \overline{a_1} \rightstar (q^2 - 2 \text{Re}(a_2)q + |a_2|^2)^{-1}(q - \hat{a}_2)
\]

\[
    = (\hat{a}_1 - q) \overline{a_1} \rightstar (q - \hat{a}_2)^{-*},
\]

(5.26)

where \( \hat{a}_2 = \overline{a_1}^{-1} a_2' \overline{a_1} \in [a_2] \), so \( f^{-*}(q) \) rewrites as

\[
    f^{-*}(q) = (\overline{a_2}^{-1} - q) \overline{a_2} \rightstar (\hat{a}_1 - q) \overline{a_1} \rightstar (q - \hat{a}_2)^{-*} \star (q - a_1)^{-*}
\]

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and we have the assertion.

Let us assume that the statement is valid for product with \( n \) factors and let us show that it holds for \( n + 1 \) factors.

\[
f^{-\ast}(q) = \prod_{k=n+1}^{*1} (1 - qa_k)^{\ast} (q - a_k)^{\ast}
\]

\[
= \prod_{k=n+1}^{*1} (1 - qa_k)^{\ast} (q - a_k)^{\ast}
\]

\[
= (1 - q\overline{a}_{n+1})^{\ast} (q - a_{n+1})^{\ast} \prod_{k=n}^{*1} (1 - qa_k)^{\ast} (q - a_k)^{\ast}
\]

\[
= (1 - q\overline{a}_{n+1})^{\ast} (q - a_{n+1})^{\ast} \prod_{k=n}^{*1} (\hat{a}_k - qa_k)^{\ast} \prod_{k=n}^{*1} (q - \tilde{a}_k).
\]

(5.27)

Now we use iteratively (5.26) to rewrite first the product \((q - a_{n+1}) \ast (a_n - q)\overline{a}_n\) as \((\hat{a}_n - q)\overline{a}_n \ast (a_n - q)\overline{a}_n\) where \(\hat{a}_n \in [a_n - 1]\) and \(a_n - q)\overline{a}_n \ast (a_n - q)\overline{a}_n\) and so on. Since the leftmost factor is \((1 - q\overline{a}_{n+1}) = (\overline{a}_{n+1} - q)\overline{a}_{n-1}\) and the rightmost factor is \((q - \tilde{a}_1)\), we have the statement for \(n + 1\) factors and the thesis follows.

\[\square\]

**Corollary 5.3.6.** Let \(a_1, \ldots, a_t \in B(0, 2R)\). Given the function

\[
f(q) = (2R)^t \prod_{k=1}^{*1} ((2R)^2 - qa_k)^{\ast} (q - a_k)
\]

it is possible to write \(f^{-\ast}(q)\), where it is defined, in the form

\[
f^{-\ast}(q) = (2R)^{-t} \prod_{k=t}^{*1} ((2R)^2 \hat{a}_k - q)\overline{a}_k \ast \prod_{k=t}^{*1} (q - \tilde{a}_k)^{\ast},
\]

where \(\hat{a}_k \in [a_k - 1]\), \(\tilde{a}_k \in [a_k]\), and in particular \(\hat{a}_1 = a_1\), \(\hat{a}_t = \hat{a}_t^{-1}\) and

\[
f(q) = (2R)^t \prod_{k=t}^{*1} ((2R)^2 \hat{a}_k - q)\overline{a}_k \ast \prod_{k=t}^{*1} (q - \tilde{a}_k)^{\ast}.
\]

**Proof.** The proof follows from the proof of Proposition 5.3.5 with minor modifications. \[\square\]
Theorem 5.3.7. Let $f(q)$ be a function slice regular in the ball with center at the origin and radius $2eR$, where $R > 0$. Let $f$ be such that $f(0) = 1$ and let $f = gh$ where $g$ has at most spherical zeros only and $h$ has at most isolated zeros only. Let $\eta$ be an arbitrary real number belonging to $(0, 3e/2]$. For any $q \in B(0, R)$ outside a family of balls whose radii have sum not exceeding $4\eta R$ we have

$$\log |f(q)| > -\left(5 + \log \left(\frac{3e}{2\eta}\right)\right) \log(M_g(2eR)M_h(2eR)).$$

Proof. Let us assume that $f$ has, in the given ball $B(0, 2eR)$, $t$ isolated zeros $\alpha_1, \ldots, \alpha_t$ and $p$ spherical zeros $[\beta_1], \ldots, [\beta_p]$ (note that some elements may be repeated). Set $n = t + 2p$ and consider the function

$$\phi(q) = (-1)^n (2R)^{2n} \prod_{k=1}^{\star t}((2R)^2 - qa_k) - \star (q - a_k)$$

$$\cdot \prod_{\ell=1}^{p}((2R)^4 - 2(2R)^2Re(\beta_\ell)q + q^2|\beta_\ell|^2)^{-1}(q^2 - 2Re(\beta_\ell)q + |\beta_\ell|^2)$$

$$(a_1 \cdots a_t|\beta_1|^2 \cdots |\beta_p|^2)^{-1}.$$ 

In the sequel, it will be useful to set

$$\phi_1(q) = \prod_{k=1}^{\star t}((2R)^2 - qa_k)^{-1} \star (q - a_k)$$

and

$$\phi_2(q) = \prod_{\ell=1}^{p}((2R)^4 - 2(2R)^2Re(\beta_\ell)q + q^2|\beta_\ell|^2)^{-1}(q^2 - 2Re(\beta_\ell)q + |\beta_\ell|^2).$$

In the $\star$-product above, the elements $a_k \in [\alpha_k]$, $k = 1, \ldots, t$, have to be suitably chosen. More precisely, let us apply Theorem 3.3.4 in order to write $f(q)$ in the form

$$f(q) = \prod_{k=1}^{\star t}(q - a'_k) \prod_{\ell=1}^{p}(q^2 - 2Re(\beta_\ell)q + |\beta_\ell|^2) \star g(q)$$

(5.28)

where $g(q) \neq 0$ and some elements $a'_k$, $\beta_\ell$ may be repeated. Note that $a'_1 = a_1$ while $a'_k \in [\alpha_k]$ are chosen in order to obtain the assigned isolated
zeros. Then in order to construct the function $\phi(q)$ we choose $a_k$ such that
the elements $\tilde{a}_k$ constructed in the proof of Theorem 5.3.5 are such that
$\tilde{a}_k = a_k$.

We now evaluate $\phi(0)$: by Theorem 2.3.10, since 0 is real, it suffices to take
the product of the factors evaluated at the origin and so

$$
\phi(0) = (-1)^n (2R)^2 \prod_{k=1}^{t} (2R)^{-2} (-a_k) \prod_{\ell=1}^{p} (2R)^{-4} |\beta_\ell|^2
(a_1 \cdots a_t |\beta_1|^2 \cdots |\beta_p|^2)^{-1}
= (a_1 \cdots a_t |\beta_1|^2 \cdots |\beta_p|^2)(a_1 \cdots a_t |\beta_1|^2 \cdots |\beta_p|^2)^{-1}
= 1.
$$

Let us now evaluate $|\phi(2Re^{I\theta})|$ for $I \in S$. By Theorem 2.3.10, the evaluation
of the $\star$-product

$$
\prod_{k=1}^{t} ((2R)^2 - q\overline{a_k})^{-*} (q - a_k)
$$

at the point $2Re^{I\theta}$ gives

$$
\prod_{k=1}^{t} ((2R)^2 - 2Re^{I_k\theta}\overline{a_k})^{-1}(2Re^{I_k\theta} - a_k)
$$

where $I_1 = I$ while $I_2, \ldots, I_t$ have to be chosen according to Theorem 2.3.10.

Then we have

$$
|\prod_{k=1}^{t} ((2R)^2 - 2Re^{I_k\theta}\overline{a_k})^{-1}(2Re^{I_k\theta} - a_k)|
= (2R)^{-t} \prod_{k=1}^{t} |(2R) - e^{I_k\theta}\overline{a_k})^{-1}||2Re^{I_k\theta} - a_k|
= (2R)^{-t} \prod_{k=1}^{t} |(2R - e^{I_k\theta}\overline{a_k})^{-1}||e^{I_k\theta} - a_k|
= (2R)^{-t} \prod_{k=1}^{t} |(2Re^{-I_k\theta} - \overline{a_k})^{-1}||2Re^{I_k\theta} - a_k)|
= (2R)^{-t}.
$$
Similarly,
\[
\left| \prod_{\ell=1}^{p} ((2R)^4 - 2(2R)^3\text{Re}(\beta_\ell)e^{I_\ell \theta} + (2\text{Re}^{I_\ell \theta})^2|\beta_\ell|^2)^{-1} \right. \\
\left. \times ((2\text{Re}^{I_\ell \theta})^2 - 2\text{Re}(\beta_\ell)(2\text{Re}^{I_\ell \theta}) + |\beta_\ell|^2) \right| \\
\times (2R)^{-2p} \prod_{\ell=1}^{p} |(2R)^2 - 2(2R)\text{Re}(\beta_\ell)e^{I_\ell \theta} + e^{2I_\ell \theta}|\beta_\ell|^2|^{-1} \\
\times \left| (2R)^2e^{2I_\ell \theta} - 2(2R)\text{Re}(\beta_\ell)e^{I_\ell \theta} + |\beta_\ell|^2 \right| \\
= (2R)^{-2p}.
\]

Thus we have
\[
|\phi(2\text{Re}^{I_\ell \theta})| = (2R)^n (|a_1| \cdots |a_t| |\beta_1|^2 \cdots |\beta_p|^2)^{-1}.
\]

We now consider the function
\[
\psi(q) = \phi^{-*}(q) \ast f(q)
\]
which, by definition, is slice regular. Note also that, by Corollary \[5.3.6\] and formula \[5.28\] we have
\[
\psi(q) = \phi^{-*}(q) \ast f(q)
\]
\[
= (2R)^{-t} \prod_{k=t}^{*1} ((2R)^2\hat{a}_k - q) \overline{a}_k \ast \prod_{k=t}^{1} (q - \hat{a}_k)^{-*} \\
\ast \prod_{k=1}^{*t} (q - a'_k) \prod_{\ell=1}^{p} (q^2 - 2\text{Re}(\beta_\ell)q + |\beta_\ell|^2) \ast g(q)
\]
\[
= (2R)^{-t} \prod_{k=t}^{*1} ((2R)^2\hat{a}_k - q) \overline{a}_k \prod_{\ell=1}^{p} (q^2 - 2\text{Re}(\beta_\ell)q + |\beta_\ell|^2) \ast g(q)
\]
and so \(\phi\) does not have zeros in the ball \(|q| \leq 2R\), in fact the factors
\[
(2R)^2 - qa_t, \quad (2R)^4 - 2(2R)^2\text{Re}(\beta_\ell)q + q^2|\beta_\ell|^2,
\]
have roots which are clearly outside that ball and \(g(q)\) does not vanish. Applying Theorem \[5.3.4\] we have that for any \(q\) such that \(|q| \leq R < 2R\),
the modulus of \( f \) satisfies
\[
\log |\psi(q)| \geq -5 \log M_f(2R) \\
= -5 \log M_f(2R) + 5 \log M_\phi(2R) \\
= -5 \log M_f(2R) + 5 \log |\phi(2Re^{I\theta})| \\
= -5 \log M_f(2R) \\
\geq -5 \log M_f(2eR) \\
\geq -5 \log (M_g(2eR)M_h(2eR)) \\
\]
where we used the fact that \(|\phi(2Re^{I\theta})| > 1\). We now need a lower bound on
\[
\phi(q) = (-1)^n(2R)^{2n}\phi_1(q)\phi_2(q)(a_1 \cdots a_t|\beta_1|^2 \cdot |\beta_p|^2)^{-1}.
\]
To this end, it is useful to write \( \phi_1(q) \), using Corollary 5.3.6 in the form
\[
\phi(q) = (2R)^{-t} \prod_{k=1}^{*t}(q - \tilde{a}_k) \prod_{k=t}^{*1}((2R)^2\tilde{a}_k - q)(a_k)^{-*}. 
\]
Then we have
\[
|\prod_{k=t}^{*1}((2R)^2\tilde{a}_k - q)(a_k)|^{-*} = \prod_{k=t}^{1}((2R)^2\tilde{a}_k - q_k)(a_k)^{-1},
\]
where \( q_k \) are suitable elements in \([q]\) (computed applying Theorem 2.3.10), and it is immediate that
\[
\prod_{k=t}^{1}((2R)^2\tilde{a}_k - q_k)(a_k) \leq (6R^2)^t,
\]
so we deduce
\[
|\prod_{k=t}^{*1}((2R)^2\tilde{a}_k - q_k)(a_k)|^{-*} \geq \frac{1}{(6R^2)^t}.
\]
Applying Cartan Theorem 3.5.1 we have that outside some exceptional balls
\[
\prod_{k=1}^{*t}(q - \tilde{a}_k) \geq \left(\frac{2nR}{n}\right)^t.
\]
A lower bound on \( \phi_2(q) \) can be provided by observing that \( \phi_2(q) \) can be written as
\[
\phi_2(q) = \prod_{\ell=1}^{p}(2R)^4 - 2(2R)^2Re(\beta_\ell)q + q^2|\beta_\ell|^2)^{-1} \prod_{\ell=1}^{p}(q^2 - 2Re(\beta_\ell)q + |\beta_\ell|^2). 
\]
Computations similar to those done in the case of $\phi_1(q)$ show that
\[
\left| \prod_{\ell=1}^{p} ((2R)^4 - 2(2R)^2 \text{Re}(\beta_\ell)q + q^2|\beta_\ell|^2)^{-1} \right| \geq (6R^2)^{2p}
\]
and
\[
\left| \prod_{\ell=1}^{p} (q^2 - 2\text{Re}(\beta_\ell)q + |\beta_\ell|^2) = \prod_{\ell=1}^{p} (q - \beta_\ell) \star (q - \overline{\beta_\ell}) \right| \geq \left( \frac{2\eta R}{n} \right)^{2p}.
\]
Therefore outside the exceptional balls we have
\[
|\phi(q)| \geq \frac{(2R)^n}{|a_1 \cdots a_t| |\beta_1|^2 \cdots |\beta_p|^2} \left( \frac{2\eta R}{n} \right)^{n} \frac{1}{(6R^2)^n} \geq \left( \frac{2\eta}{3e} \right)^n.
\]
Moreover, Proposition 5.2.6 gives
\[
n = n_f(2R) \leq \log(M_g(2eR)M_h(2eR)),
\]
and so, outside the exceptional balls
\[
\log |\phi(q)| \geq \log \left( \frac{2\eta}{3e} \right) \log(M_g(2eR)M_h(2eR)).
\]
Since
\[
\log |f(q)| = \log |\psi(q)| + \log |\phi(q)| \geq - \left( 5 + \log \left( \frac{3e}{2\eta} \right) \right) \log M_f(2eR)
\]
the statement follows.

\[
\square
\]

### 5.4 Growth of the $\star$-product of entire slice regular functions

Given two entire slice regular functions, it is a natural question to ask if we can characterize order and type of their $\star$-product if we know order and type of the factors. The answer is positive and is contained in the next result:
**Theorem 5.4.1.** Let $f$ and $g$ be two entire functions of order and type $\rho_f$, $\sigma_f$ and $\rho_g$, $\sigma_g$, respectively, and let $\rho_{f\ast g}$, $\sigma_{f\ast g}$ be order and type of the product $f \ast g$. Then:

1. If $\rho_f \neq \rho_g$ then $\rho_{f\ast g} = \max(\rho_f, \rho_g)$ and $\sigma_{f\ast g}$ equals the type of the function with larger order.

2. If $\rho_f = \rho_g$, one function has normal type $\sigma$ and the other has minimal type, then $\rho_{f\ast g} = \rho_f = \rho_g$ and $\sigma_{f\ast g} = \sigma$.

3. If $\rho_f = \rho_g$, one function has maximal type and the other has at most normal type, then $\rho_{f\ast g} = \rho_f = \rho_g$ and $\sigma_{f\ast g}$ has maximal type.

**Proof.** We show that (2) holds, the other two statements follow with similar arguments.

Recalling (5.1), denoting by $\rho$ the order of $f$ and $g$, and assuming that $f$ has normal type $\sigma_f$, we have

$$M_f(r) < e^{(\sigma_f + \varepsilon/2)r^\rho},$$

and

$$M_g(r) < e^{(\varepsilon/2)r^\rho},$$

and so

$$M_{f\ast g}(r) = \max_{|q|=r} |(f \ast g)(q)|$$

$$\leq \max_{|q|=r} |f(q)| \max_{|q|=r} |g(q)|$$

$$= M_f(r)M_g(r)$$

$$< e^{(\sigma_f + \varepsilon)r^\rho}.$$

We now need a lower bound for $M_{f\ast g}(r)$. First of all, we can find a positive number $R_1$, large enough, such that for two given positive numbers $\varepsilon$ and $\delta$ the inequality

$$M_f(R_1) > e^{(\sigma_f - \varepsilon/2)R_1^\rho}$$

holds, and for all $R \geq R_1$

$$M_g(R) < e^{\delta R^\rho}.$$

We are now in need to use Theorem 5.3.7. To this end we assume, with no loss of generality, that $g(0) = 1$. In fact, if this is not the case, we can multiply $g(q)$ on the left by the factor $q^{-m}c$ for suitable $c \in \mathbb{H}$ and $m \in \mathbb{N}$, without changing its order and type. We now assume $0 < \delta < 1,$
\( R = R_1(1 - \delta)^{-1} \) and take \( \eta = \delta/8 \). Inside the ball centered at the origin and with radius \( R \), we exclude the balls described in Theorem 5.3.7. Note that the sum of their diameters is less than \( \delta R \). So there exists \( r_1 \in (R_1, R) \) such that the ball centered at the origin and with radius \( r_1 \) does not meet any of these excluded balls. By Theorem 5.3.7 on this ball of radius \( r_1 \) we have

\[
\log |g(q)| > -\left( 5 + \log \left( \frac{12e}{\delta} \right) \right) \log M_g(2eR), \tag{5.31}
\]

moreover, since \( R_1 < r_1 < R_1(1 - \delta)^{-1} \) and (5.30) is in force, we have

\[
M_f(r_1) > M_f(R_1) > e^{(\sigma - \varepsilon/2)R_1^\rho} > e^{(\sigma - \varepsilon/2)(1 - \delta)^\rho r_1^\rho}.
\]

This latter inequality and (5.31) yield

\[
\log M_{f*f}(r_1) > (\sigma - \varepsilon/2)(1 - \delta)^\rho r_1^\rho - \left( 5 + \log \left( \frac{12e}{\delta} \right) \right) \log M_g(2eR).
\]

Since

\[
\log M_g(2eR) < \delta(2eR)^\rho < \delta(1 - \delta)^{-\rho}(2e)^\rho r_1^\rho,
\]

we finally have

\[
\log M_{f*f}(r_1) > \left[ (\sigma - \varepsilon/2)(1 - \delta)^\rho - \left( 5 + \log \left( \frac{12e}{\delta} \right) \right) \delta(1 - \delta)^{-\rho}(2e)^\rho \right] r_1^\rho.
\]

For any given \( \varepsilon \) we can choose \( \delta \) such that

\[
\left[ (\sigma - \varepsilon/2)(1 - \delta)^\rho - \left( 5 + \log \left( \frac{12e}{\delta} \right) \right) \delta(1 - \delta)^{-\rho}(2e)^\rho \right] \geq \sigma - \varepsilon
\]

and so

\[
M_{f*f}(r_1) > e^{(\sigma - \varepsilon)r_1^\rho}.
\]

This ends the proof of point (2). \( \square \)

Recalling that \( f^s = f \ast f^c \) and Corollary 5.1.10 we immediately have:

**Corollary 5.4.2.** Given an entire slice regular function \( f \), the order and type of \( f^s \) coincide with the order and type of \( f \).

As a consequence of Theorem 5.4.1 we can now provide a relation between the convergence exponent of the sequence of zeros of an entire regular function \( f \) and its order. The bound differs from the one in the complex case, unless we consider intrinsic functions. In fact we have:
**Theorem 5.4.3.** The convergence exponent of the sequence of zeros of an entire regular function does not exceed twice its order.

**Proof.** Suppose that \( f(0) = 1 \) and let us use Proposition 5.2.6 and the notation therein. Then

\[
\rho_1 = \lim_{r \to \infty} \frac{\log n_f(r)}{\log r} \leq \lim_{r \to \infty} \frac{\log(l \log M_g(\text{er}))}{\log \text{er}} \leq \frac{\log(l \log M_h(\text{er}))}{\log \text{er}} = \rho_g + \rho_h,
\]

where \( \rho_g \) and \( \rho_f \) denote the order of \( g \) and \( h \), respectively. By Theorem 5.4.1 the order \( \rho \) of \( f \) equals \( \max(\rho_g, \rho_h) \) so

\[
\rho_1 \leq 2 \rho.
\]

If \( f(0) \neq 1 \), it is sufficient to normalize it or, if \( f \) has a zero of order \( k \) at 0, it is sufficient to consider the function \( \tilde{f}(q) = k!q^{-k}f(q)(\partial_x f(0))^{-1} \). This function \( \tilde{f}(q) \) has the same order as \( f \) and the assertion is true for this function.

In two particular cases, one of which is the case of intrinsic functions, we have the bound which holds in the complex case:

**Theorem 5.4.4.**

1. The convergence exponent of the sequence of zeros of an entire, intrinsic, slice regular function does not exceed its order.

2. The convergence exponent of the sequence of zeros of an entire slice regular function having only isolated zeros does not exceed its order.

**Proof.** If \( f \) is intrinsic then it does not have nonreal isolated zeros (see Lemma 3.3.11) thus, using the notation of the previous theorem, \( f = g \) and \( h \) can be set equal 1, so that

\[
\rho_1 = \lim_{r \to \infty} \frac{\log n_f(r)}{\log r} \leq \frac{\log(l \log M_g(\text{er}))}{\log \text{er}} = \rho_g = \rho.
\]

If \( f \) has only isolated zeros, then \( f = h \) and \( g \) can be set equal 1, so the proof above shows the statement, changing \( g \) with \( h \). \( \Box \)
5.5 Almost universal entire functions

In this section we consider a kind of universal property for quaternionic entire functions. This property, in the quaternionic case, has been originally treated in [99]. The result in the complex plane was proved by Birkhoff [31]. Later, Seidel and Walsh generalized the result to simply connected sets as in the following theorem, see [146]:

**Theorem 5.5.1.** There exists an entire function $F(z)$ such that given an arbitrary function $f(z)$ analytic in a simply connected region $R \subseteq \mathbb{C}$, for suitably chosen $a_1, a_2, \ldots \in \mathbb{C}$ the relation

$$\lim_{n \to \infty} F(z + a_n) = f(z)$$

holds for $z \in R$ uniformly on any compact set in $R$.

Recall that if $g$ is a polynomial with quaternionic coefficients (written on the right), then the standard composition $f \circ g$ is not, in general, slice regular. However, if $g$ is quaternionic intrinsic, and so when $g$ is a polynomial with real coefficients, then $f \circ g$ is slice regular. Thus, for our purposes we will select real numbers $a_n$.

We also need the following

**Definition 5.5.2.** Let $\Omega \subseteq \mathbb{H}$ be an open set and let $\mathcal{K}(\Omega)$ be the set of all axially symmetric compact sets $K \subset \Omega$, such that $\mathbb{C}_I \setminus (K \cap \mathbb{C}_I)$ is connected for some (and hence for all) $I \in S$.

To prove our next result, we need the Runge-type approximation result in [77] Theorem 4.11] stated below:

**Theorem 5.5.3.** Let $K \in \mathcal{K}(\mathbb{H})$ and let $U$ be an axially symmetric open subset of $\mathbb{H}$ containing $K$. For any $f \in \mathcal{R}(U)$, there exists a sequence of polynomials $P_n(q)$, $n \in \mathbb{N}$, such that $P_n \to f$ uniformly on $K$, as $n \to \infty$.

We can now prove the following:

**Theorem 5.5.4.** There exists an entire slice regular function $F(q)$ such that given an arbitrary function $f(q)$ slice regular in a region $U \in \mathcal{R}(\mathbb{H})$, for suitably chosen $a_1, a_2, \ldots \in \mathbb{R}$ the relation

$$\lim_{n \to \infty} F(q + a_n) = f(q)$$

holds for $q \in U$ uniformly on any compact set in $\mathcal{K}(\mathbb{H})$ contained in $U$. 

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Proof. We follow the proof in [146], and we define the spheres

\[ S(4^n, 2^n) = \{ q \in \mathbb{H} ; |q - 4^n| = 2^n \} \quad \text{for} \quad n = 1, 2, \ldots \]

and the spheres

\[ \Sigma(0, 4^n + 2^n + 1) = \{ q \in \mathbb{H} ; |q| = 4^n + 2^n + 1 \} \quad \text{for} \quad n = 1, 2, \ldots . \]

It turns out that \( S(4^n, 2^n) \) and \( S(4^m, 2^m) \) are mutually exterior if \( n \neq m \), moreover \( \Sigma(0, 4^n + 2^n + 1) \) contains in its interior \( S(4^j, 2^j) \) for \( j = 1, 2, \ldots, n \) but it does not contain \( S(4^j, 2^j) \) nor its interior points if \( j > n \). The function \( F \) can be constructed as limit of polynomials with rational quaternionic coefficients, i.e. with coefficients of the form \( a_0 + ia_1 + ja_2 + ka_3 \) and \( a_i \) rational. Note that this assumption on the coefficients is not restrictive since it does not change properties like (uniform) convergence to a given function.

Thus we consider all the polynomials with rational quaternionic coefficients and we arrange them in a sequence \( \{ P_n \} \).

Let \( \pi_1(q) \) be a polynomial such that

\[ |P_1(q - 4) - \pi_1(q)| < \frac{1}{2} \]

for \( q \) on \( S(4, 2) \) or inside \( S(4, 2) \). The polynomial exists since, for example, one may choose \( \pi_1(q) = P_1(q - 4) \). Let us define the function

\[ \tilde{P}_2(q) = \begin{cases} P_2(q - 4^2) & \text{for} \ q \ \text{on or inside} \ S(4^2, 2^2) \\ \pi_1(q) & \text{for} \ q \ \text{on or inside} \ \Sigma(0, 4 + 2 + 1). \end{cases} \]

Theorem 5.5.3 applied to \( \tilde{P}_2 \) shows that there exists a polynomial \( \pi_2(q) \) such that

\[ |P_2(q - 4^2) - \pi_2(q)| < 1/2^2 \quad \text{on or inside} \ S(4^2, 2^2), \]

\[ |\pi_1(q) - \pi_2(q)| < 1/2^2, \quad \text{on or inside} \ \Sigma(0, 4 + 2 + 1). \]

Thus we can define inductively

\[ \tilde{P}_n(q) = \begin{cases} P_n(q - 4^n) & \text{for} \ q \ \text{on or inside} \ S(4^n, 2^n) \\ \pi_{n-1}(q) & \text{for} \ q \ \text{on or inside} \ \Sigma(0, 4^{n-1} + 2^{n-1} + 1) \end{cases} \]

and using Theorem 5.5.3 we can find a polynomial \( \pi_n(q) \) such that

\[ |P_n(q - 4^n) - \pi_n(q)| < 1/2^n, \quad \text{on or inside} \ S(4^n, 2^n), \]

\[ |\pi_{n-1}(q) - \pi_n(q)| < 1/2^n \quad \text{for} \ q \ \text{on or inside} \ \Sigma(0, 4^{n-1} + 2^{n-1} + 1). \]
The sequence \( \{\pi_n(q)\} \) converges uniformly inside the balls \( \Sigma \), and hence uniformly on every bounded set. So the function \( F(q) = \lim_{n \to \infty} \pi_n(q) \) converges at any point in \( \mathbb{H} \) and it is entire. We now show that it satisfies (5.32), see Proposition 4.1 in [74]. Let \( f \) be a slice regular function in a region \( U \in \mathcal{R}(\mathbb{H}) \). By Theorem 5.5.3 there exist a subsequence \( \{P_{n_k}(q)\} \) of the polynomials considered above, such that

\[
\lim_{k \to \infty} P_{n_k}(q) = f(q) \tag{5.33}
\]

in \( U \), uniformly on every \( K \in \mathfrak{F}(\mathbb{H}) \) contained in \( U \). We apply the above construction and consider \( q \in \mathbb{S}(4^n, 2^n) \). We have

\[
F(q) = \pi_n(q) + (\pi_{n+1}(q) - \pi_n(q)) + (\pi_{n+2}(q) - \pi_{n+1}(q)) + \ldots
\]

and

\[
|F(q) - P_n(q - 4^n)| \leq |P_n(q - 4^n) - \pi_n(q)| + |\pi_{n+1}(q) - \pi_n(q)| + |\pi_{n+2}(q) - \pi_{n+1}(q)| + \ldots
\]

\[
< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots = \frac{1}{2^n - 1}
\]

and so

\[
\lim_{n \to \infty} [F(q + 4^n) - P_n(q)] = 0
\]

for \( q \) in any bounded set. Thus, by using (5.33), we deduce

\[
\lim_{k \to \infty} |F(q + 4^{n_k}) - f(q)| \leq \lim_{k \to \infty} |F(q + 4^{n_k}) - P_{n_k}(q)| + \lim_{k \to \infty} |P_{n_k}(q) - f(q)| = 0
\]

which concludes the proof. \( \square \)

**Remark 5.5.5.** The proof shows that the sequence \((a_n)_{n \in \mathbb{N}}\) depends on the approximated function \( f \).

We now present another interesting result, proved in the complex case by MacLane [136], showing that there exists a slice regular function whose set of derivatives is dense in \( \mathcal{R}(\mathbb{H}) \). The proof below follows the proof given in [27].

**Theorem 5.5.6.** There exists an entire slice regular function \( F \) such that the set \( \{F^{(n)}\}_{n \in \mathbb{N}} \) is dense in \( \mathcal{R}(\mathbb{H}) \).
Proof. We begin by defining a linear operator $I$ acting on the monomials $q^n$ as

$$I(q^n) = \frac{q^{n+1}}{n+1}$$

and then we extend by linearity to the set of all polynomials. We note that

$$I^k(q^n) = \frac{q^{n+k}}{(n+k)\ldots(n+1)},$$

and that, for $|q| \leq r$, we have the inequality

$$|I^k(q^n)| \leq \frac{r^{n+k}}{(n+k)\ldots(n+1)} \leq \frac{r^{n+k}}{k!}.$$ 

As a consequence, we have

$$\max_{|q| \leq r} |I^k(q^n)| \to 0 \quad \text{for } k \to \infty.$$ 

For any $\delta > 0$ and any $r > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$\max_{|q| \leq r} |I^k(q^n)| < \delta, \quad \text{for } k \geq k_0.$$

Now observe that, given $f \in \mathcal{R}(\mathbb{H})$, $\varepsilon > 0$ and $m \in \mathbb{N}$, if $|f(q)| \leq \delta$ for $|q| \leq r$, then for $|q| \leq r/2$ the Cauchy estimates give

$$|f^{(j)}(q)| \leq \frac{j! \max_{|q| \leq r} |f(q)|}{(r/2)^j} \leq \frac{j! 2^j}{r^j} \delta < \varepsilon,$$

for any $j = 0, \ldots, m$, if $\delta$ is sufficiently small. Thus, if we consider a polynomial $P(q)$ and any $r > 0$, $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$ we have

$$\max_{|q| \leq r} |(I^k(P))^{(j)}| < \varepsilon, \quad (5.34)$$

for $j = 0, \ldots, m$.

Let now $\{P_n\}_{n \in \mathbb{N}}$ be a sequence of polynomials dense in $\mathcal{R}(\mathbb{H})$ and let

$$\sum_{j=1}^{\infty} I^{k_j}(P_j)(q),$$

where the integers $k_j$ are constructed below. Note also that we denote $I^{k_j}(P_j)(q)$ by $Q_j(q)$. We set $k_1 = 0$, so $Q_1 = P_1$; then we choose $k_2 >$
\[ k_1 + \deg(P_1) \] and, since we set \( Q_2 = I^{k_2}(P_2) \), by (5.34) we can select \( k_2 \) to be such that
\[ |Q_2(q)| < 1/2^2, \quad \text{for } |q| \leq 2. \]
Then, inductively, for \( n \geq 3 \), we choose \( k_n > k_{n-1} + \deg(P_{n-1}) \), we set \( Q_n = I^{k_n}(P_n) \) and we select \( k_n \) to be such that
\[ |Q_n(q)| \leq \frac{1}{2^n}, \]
\[ |Q'_n(q)| \leq \frac{1}{2^n}, \]
\[ \ldots \]
\[ |Q^{(k_{n-1})}_n(q)| \leq \frac{1}{2^n} \]
for \( |q| \leq n \). The series
\[ \sum_{j=1}^{\infty} I^{k_j}(P_j)(q) = \sum_{j=1}^{\infty} Q_j(q) \]
converges uniformly on bounded subsets in \( \mathbb{H} \) to a slice regular function \( F(q) \), in fact (5.35) implies that:
\[ \max_{|q| \leq j} |Q_j(q)| \leq \frac{1}{2^j}. \]

We now show that the function \( F \) is such that \( \{F^{(n)}\}_{n \in \mathbb{N}} \) is dense in \( \mathcal{R}(\mathbb{H}) \). To this end, let \( f \in \mathcal{R}(\mathbb{H}) \) and let \( r > 0 \), \( \varepsilon > 0 \) be arbitrary. Let \( n_0 \in \mathbb{N} \) be such that \( n_0 > r \) and \( 1/2^{n_0-1} < \varepsilon \). Recalling that the sequence \( \{P_n\} \) is dense in \( \mathcal{R}(\mathbb{H}) \), there exists \( n \in \mathbb{N}, n > n_0 \) such that \( \max_{|q| \leq n_0} |f(q) - P_n(q)| < \varepsilon \). The assumptions on the number \( k_n \) imply that \( Q^{(k_n)}_j(q) = 0 \) for \( j = 1, \ldots, n-1 \) and \( Q^{(k_n)}_n(q) = P_n(q) \), thus
\[ |f(q) - F^{(k_n)}(q)| = |f(q) - P_n(q) - \sum_{j=n+1}^{\infty} Q^{(k_n)}_j(q)| \leq |f(q) - P_n(q)| + \sum_{j=n+1}^{\infty} |Q^{(k_n)}_j(q)| < 2\varepsilon, \]
and so
\[ \max_{|q| \leq n_0} |f(q) - F^{(k_n)}(q)| < 2\varepsilon \]
and the statement follows. \( \square \)
5.6 Entire slice regular functions of exponential type

As in the complex setting, it makes sense to introduce also in the quaternionic setting the notion of entire functions of exponential type. These are functions at most of first order and normal type and their exponential type is defined as

$$\sigma = \lim_{r \to \infty} \frac{\log M_f(r)}{r}.$$ 

In other words, we have the following:

**Definition 5.6.1.** An entire slice regular function $f$ is said to be of exponential type if there exist constants $A, B$ such that

$$|f(q)| \leq Be^{A|q|}$$

for all $q \in \mathbb{H}$.

Functions of order less than 1 or of order 1 and minimal type, are said to be of exponential type zero. As a nice application of our definition of the composition of the exponential function with a slice regular function, in particular a polynomial, we show how to associate to each entire slice regular function $f$ of exponential type its Borel transform.

**Definition 5.6.2.** Let $f$ be an entire slice regular function of exponential type, and let $f(q) = \sum_{k=0}^{\infty} q^{k} a_k / k!$, $a_k \in \mathbb{H}$. The function

$$\phi(q) = \mathcal{F}(f)(q) = \sum_{k=0}^{\infty} q^{-(k+1)} a_k$$

is called the Borel transform of $f(q)$.

**Remark 5.6.3.** If $\sigma$ is the exponential type of a function $f$, then

$$\sigma = \lim_{k \to \infty} |a_k|^{1/k}.$$ 

Thus the Borel transform $\phi$ of $f$ is slice regular for $|q| > \sigma$.

Let $q, w \in \mathbb{H}$ and consider the function

$$e_q^{qw} = \sum_{n=0}^{\infty} \frac{1}{n!} q^n w^n.$$
This function can be obtained through the Representation Formula: take
\[ z = x + Iw y \] on the same complex plane as \( w \). Then \( e^{z w} = e^{z w} \) and we can
extend this function for any \( q = x + Iy \in \mathbb{H} \)
\[ e^{q w} = \text{ext}(e^{z w}) = \frac{1}{2}(e^{z w} + e^{\bar{z} w}) + \frac{1}{2} Iw(e^{z w} - e^{\bar{z} w}). \]

This is useful to obtain an integral formula for an entire function \( f \) in terms
of its Fourier-Borel transform.

**Theorem 5.6.4.** An entire slice regular function \( f \) can be written in terms
of its Fourier-Borel transform \( \phi \) as
\[
f(q) = \frac{1}{2\pi} \int_{\Sigma \cap \mathbb{C}_I} e^{q w} dw_I \phi(w), \quad dw_I = -Idw,
\]
where \( \Sigma \) surrounds the singularities of \( \phi \).

**Proof.** First of all, we select a basis \( 1, I, J, IJ \) of \( \mathbb{H} \), so that we can write
\[
f(q) = \sum_{k=0}^{\infty} \frac{q^k}{k!} (a_{k0} + Ia_{k1} + Ja_{k2} + IJa_{k3})
= \sum_{k=0}^{\infty} \frac{q^k}{k!} (a_{k0} + Ia_{k1}) + \sum_{k=0}^{\infty} \frac{q^k}{k!} (a_{k2} + Ia_{k3})J
= F(q) + G(q)J.
\]
The Fourier-Borel transform of \( F \) and \( G \) are
\[
\mathcal{F}F(q) = \sum_{k=0}^{\infty} q^{-(k+1)}(a_{k0} + Ia_{k1}), \quad \mathcal{F}G(q) = \sum_{k=0}^{\infty} q^{-(k+1)}(a_{k2} + Ia_{k3}),
\]
and it is immediate that
\[
\phi(q) = \mathcal{F}f(q) = \mathcal{F}F(q) + \mathcal{F}G(q)J.
\]
When \( z = x + Iy \) then \( F \) and \( G \) are entire holomorphic functions and so we
have, for \( w \in \mathbb{C}_I \)
\[
F(z) = \frac{1}{2\pi} \int_{\gamma} e^{z w} \mathcal{F}(F)(w) dw_I^{-1} = \frac{1}{2\pi} \int_{\gamma} e^{z w} dw_I \mathcal{F}(F)(w)
\]
\[
G(z) = \frac{1}{2\pi} \int_{\gamma} e^{z w} \mathcal{F}(G)(w) dw_I^{-1} = \frac{1}{2\pi} \int_{\gamma} e^{z w} dw_I \mathcal{F}(G)(w)
\]
where $\gamma$ can be chosen to surround both the singularities of $F(F)$ and $F(G)$. By the Splitting Lemma we have

$$f(z) = F(z) + G(z)J$$

$$= \frac{1}{2\pi} \int_{\gamma} e^{zw} dw_1(F(F)(w) + G(F)(w))J$$

$$= \frac{1}{2\pi} \int_{\gamma} e^{zw} dw_1 \phi(w).$$

Now we can reconstruct $f(q)$ using the Representation Formula and the previous formula (5.36). This leads to

$$f(q) = f(x + I_q y)$$

$$= \frac{1}{2\pi} \int_{\gamma} \left[ \frac{1}{2} (e^{zw} + e^{\bar{z}w}) + \frac{1}{2} I_q I(e^{z \bar{w}} - e^{\bar{z}w}) \right] dw_1 \phi(w)$$

$$= \frac{1}{2\pi} \int_{\gamma} e^{qw} \star dw_1 \phi(w),$$

and since $w$ is arbitrary and $\gamma$ can be chosen to be $\Sigma \cap \mathbb{C}_I$, the statement follows.

As we said at the beginning of the section, we introduced the Borel transform as an application of the slice regular composition of the exponential function. This is obviously the beginning of a theory that we plan to further develop.

**Comments to Chapter 5.** The material in this Chapter is new, except for the section on the universal property of entire functions which is taken from [99]. The complex version of the results in this chapter may be found in the book [133].
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