GEOMETRIC PROPERTIES OF STARLIKE LOG–HARMONIC MAPPINGS OF ORDER ALPHA

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Abstract. In this paper, we estimate the Jacobian of log–harmonic mappings of the form
\[ f(z) = z|z|^{2\beta} h(z) g(z) \quad |z| < 1, \]
where \( h, g \) are analytic in the unit disc, \( g(0) = 1 \) and \( \text{Re}\{\beta\} > -1/2 \). Also, by using of the subordination principle, some geometric properties starlike log–harmonic mappings of order alpha are investigated.

1. Introduction

Let \( \mathcal{H} (\Delta) \) denote the class of analytic functions defined in the unit disk \( \Delta := \{ z : |z| < 1 \} \). Let \( f \) and \( g \) be two members of \( \mathcal{H} (\Delta) \) that satisfy the normalized conditions. We say that \( f \) is subordinate to \( g \), written \( f(z) \prec g(z) \), if there exists a Schwarz function \( w(z) \) such that \( f(z) = g(w(z)) \) for all \( z \in \Delta \). Furthermore, if the function \( g \) is univalent in \( \Delta \), then we have the following equivalence:
\[ f(z) \prec g(z) \iff (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)). \]

Also let \( \mathcal{B}(\Delta) \) denote the set of functions \( w \in \mathcal{H}(\Delta) \) satisfying \( |w(z)| < 1 \) in \( \Delta \). A mapping \( f \) is said to be log–harmonic in \( \Delta \), if there is an analytic function \( w \in \mathcal{B}(\Delta) \) such that \( f \) is a solution of the nonlinear elliptic partial differential equation
\[ \frac{f(z)}{f(z)} = w(z) \frac{f(z)}{f(z)}. \]

It has been shown that if \( f \) is a non–vanishing log–harmonic mapping, then \( f \) can be expressed as \( f(z) = h(z) g(z) \), where both \( h \) and \( g \) are analytic functions and that if \( f \) vanishes at \( z = 0 \) but is not identically zero, then \( f \) admits the following representation
\[ f(z) = z|z|^{2\beta} h(z) g(z), \]
where \( \text{Re}\{\beta\} > -1/2 \), \( h \) and \( g \) are in \( \mathcal{H}(\Delta) \) and \( g(0) = 1 \) (see [3]).

Example 1.1. Let \( f(1) = 1 \) and \( \text{Re}\{\beta\} > -1/2 \). Then the function \( f_\beta(z) = z|z|^{2\beta} \) is a solution of the equation \( (1.1) \) in \( \mathbb{C} \) with \( w(z) = \frac{\beta}{1 + \beta} \). It is a simple exercise that \( f_\beta \) maps the unit disc \( \Delta \) onto itself. The Figure 1 shows the image of the unit disc under the function \( f_\beta \) in two different cases.

Since \( w \in \mathcal{B}(\Delta) \), the Jacobian
\[ J_f(z) = |f_z|^2 \left( 1 - |w(z)|^2 \right) \]
is positive and all non–constant log–harmonic mappings are therefore sense–preserving and open in \( \Delta \). The class of univalent log–harmonic mappings have been studied extensively by many works, see for example [11]–[13].

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Let $f(z) = z|z|^{2\beta} h(z) g(z)$, where $h(0) = g(0) = 1$, be a univalent log–harmonic mapping. We say $f$ is a starlike log–harmonic (SLH) mapping of order $\alpha$, where $0 \leq \alpha < 1$, if
\begin{equation}
\frac{\partial}{\partial \theta} \arg \left( f(re^{i\theta}) \right) = \Re \left\{ \frac{zf_z - \overline{z}f_{\overline{z}}}{f} \right\} > \alpha \quad (z \in \Delta, 0 \leq \alpha < 1).
\end{equation}

We denote by $ST_{LH}(\alpha)$, the set of all starlike log–harmonic mappings of order $\alpha$. If we take $\alpha = 0$, then we get the class of starlike log–harmonic mappings.

The following lemma will be useful.

\textbf{Lemma 1.1.} [6, p. 35] Let $\Xi$ be a set in the complex plane $\mathbb{C}$ and let $b$ be a complex number such that $\Re(b) > 0$. Suppose that a function $\psi : \mathbb{C}^2 \times \Delta \to \mathbb{C}$ satisfies the condition:

$$\psi(i\rho, \sigma; z) \notin \Xi,$$

for all real $\rho, \sigma \leq -|b - i\rho|^2/(2\Re(b))$ and all $z \in \Delta$. If the function $p(z)$ defined by $p(z) = b + b_1 z + b_2 z^2 + \cdots$ is analytic in $\Delta$ and if

$$\psi(p(z), zp'(z); z) \in \Xi,$$

then $\Re(p(z)) > 0$ in $\Delta$.

The main aim of the paper is to give an estimate for the Jacobian of log–harmonic mappings of form (1.2) and obtain some geometric properties of starlike log–harmonic function of order $\alpha$.

2. Main Results

In our first result, we give an estimate for the Jacobian of log–harmonic mappings of form (1.2).

\textbf{Theorem 2.1.} If $f(z) = z|z|^{2\beta} h(z) g(z)$ is log–harmonic mapping, then
\begin{equation}
\frac{(1-|\gamma|^2)(1-|z|^2)}{(1+|\gamma||z|^2)^2} |f_z|^2 \leq J_f(z) \leq \begin{cases} 
\frac{(1-|\gamma|^2)(1-|z|^2)}{(1+|\gamma||z|^2)^2} |f_z|^2 & |z| < |\gamma|, \\
|f_z|^2 & |z| \geq |\gamma|,
\end{cases}
\end{equation}

where $\gamma = \beta/(1+\beta)$, $\beta \in \mathbb{C}$, $\Re(\beta) > -1/2$ and $z \in \Delta$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{(a) The boundary curve of $f_i(\Delta)$ (b) The boundary curve of $f_{-1/3+4i}(\Delta)$}
\end{figure}
Proof. Let $f$ be log–harmonic mapping of the form (1.2). Then the dilatation of $f$ can be expressed as (1.1) and the Jacobian of $f$ given by (1.3). A simple calculation of (1.1) and (1.2), gives us

$$w(z) = \frac{f - \beta}{f f_z} = \frac{\beta + z'f(z)}{1 + \beta + z h'(z)}.$$  

Clearly $w(0) = \beta/(1 + \beta) =: \gamma$, where $\beta \in \mathbb{C}$ and $\text{Re}\{\beta\} > -1/2$. Now, by considering the Möbius transformation, we define the function

$$\phi(z) = \frac{w(z) - \gamma}{1 - \overline{\gamma} w(z)} \quad (z \in \Delta).$$

It is not hard to prove that $\phi(z)$ is analytic (because $|1 + \beta| > |\beta|$), $\phi(0) = 0$ and $|\phi(z)| \leq 1$. Moreover, the function $\phi(z)$ satisfies the assumptions of Schwarz lemma which gives

$$|w(z) - \gamma| \leq |z||1 - \overline{\gamma} w(z)| \quad (z \in \Delta).$$

From (2.3), we get $w(z) = (\phi(z) + \gamma)/(1 + \overline{\gamma}\phi(z))$. This shows that the second dilatation $w(z)$ is subordinate to $\phi(z) := z(\phi(z) + \gamma)/(1 + \overline{\gamma}z)$. It is a simple exercise that the linear transformation $\phi(z)$ maps $|z| = r$ onto the disc with the center

$$\left(\frac{x(1 - |z|^2)}{1 - |\gamma|^2 |z|^2}, \frac{y(1 - |z|^2)}{1 - |\gamma|^2 |z|^2}\right)$$

and the radius

$$\frac{(1 - |\gamma|^2)|z|}{1 - |\gamma|^2 |z|^2},$$

where $x = \text{Re}\{\gamma\}$ and $y = \text{Re}\{\gamma\}$. Therefore, by using the above and by the subordination principle, the inequality (2.4) gives that

$$|w(z) - \gamma(1 - |z|^2)| \leq \frac{(1 - |\gamma|^2)|z|}{1 - |\gamma|^2 |z|^2}$$

and concluding the proof. \qed

In the next theorem, by using the subordination principle, we present a necessary and sufficient condition for functions belonging to the class $ST_{LH}(\alpha)$.

**Theorem 2.2.** Let $0 \leq \alpha < 1$ and $\text{Re}\{\beta\} > -1/2$. Then the function $f(z) = z|z|^{2\beta} h(z) g(z)$ belongs to the class $ST_{LH}(\alpha)$ if, and only if,

$$|z h'(z) - z'f(z)| < \frac{2(1 - \alpha)z}{1 - z} \quad (z \in \Delta).$$

**Proof.** Let $f(z) = z|z|^{2\beta} h(z) g(z) \in ST_{LH}(\alpha)$. Then by a simple check we get

$$\frac{zf_z}{f} = 1 + \beta + z h'(z) h(z)$$

and

$$\frac{zf_z}{f} = \beta + z \left(\frac{g'(z)}{h(z)}\right).$$

Thus

$$\frac{zf_z - zf_z}{f} = 1 + z h'(z) h(z) - \overline{z} \left(\frac{g'(z)}{h(z)}\right).$$
Then \( f \in ST_{LH}(\alpha) \) if, and only if,

\[
\alpha < \text{Re} \left\{ \frac{zf' - zf}{f} \right\} = \text{Re} \left\{ 1 + \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right\}
\]

or

\[
\text{Re} \left\{ \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right\} > \alpha - 1.
\]

Equivalently

\[
\text{Re} \left\{ \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right\} > \frac{2(1 - \alpha)z}{1 - z}.
\]

This ends the proof. \( \square \)

We have the following lemma, directly.

**Lemma 2.1.** The function \( f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \) (Re\( \beta > -1/2 \)) belongs to the class \( ST_{LH}(\alpha) \) if, and only if,

\[
1 + \text{Re} \left\{ \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right\} > \alpha \quad (0 \leq \alpha < 1).
\]

Following, we present representation theorem for the function \( h(z) \) of mappings of the form \((1.2)\) in the set \( ST_{LH}(\alpha) \).

**Theorem 2.3.** A function \( f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in ST_{LH}(\alpha) \) if, and only if,

\[
h(z) = g(z) \exp \left( \int_{0}^{\alpha} \frac{2(1 - \alpha)\psi(t)}{t(1 - \psi(t))} dt \right),
\]

where \( \psi \) is Schwarz function, \( 0 \leq \alpha < 1 \) and Re\( \beta > -1/2 \).

**Proof.** If \( f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in ST_{LH}(\alpha) \), then by Theorem 2.2 and by definition of subordination, there exits a Schwarz function \( \psi(z) \) such that

\[
\left( \frac{z h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) = \frac{2(1 - \alpha)\psi(z)}{1 - \psi(z)} \quad (z \in \Delta),
\]

or equivalently

\[
\left\{ \log \frac{h(z)}{g(z)} \right\}' = \frac{2(1 - \alpha)\psi(z)}{z(1 - \psi(z))}.
\]

Integrating the last equality \((2.11)\), we get \((2.10)\). On the other hand, it is an easy calculation that a function having the form \((2.10)\) satisfies condition \((2.5)\). \( \square \)

Applying formula \((2.10)\) for \( \psi(z) = z \) gives that the following.

**Example 2.1.** Let \( g(z) \) be an analytic function with \( g(0) = 1 \). Then the function

\[
F_{\alpha,\beta}(z) = z|z|^{2\beta} \frac{|g(z)|^2}{(1 - z)^{2(1 - \alpha)}} \quad (z \in \Delta, 0 \leq \alpha < 1),
\]

is starlike log–harmonic mapping of order \( \alpha \).

**Remark 2.1.** Since \( g(0) = 1 \), thus if we consider the constant function \( g(z) = 1 \) in \((2.12)\) and \( \beta = 0 \), then we obtain

\[
f(z) = \frac{z}{(1 - z)^{2(1 - \alpha)}} \quad (0 \leq \alpha < 1).
\]

Moreover, the Koebe function of order \( \alpha \) (in particular, the Koebe function) is in the class \( ST_{LH}(\alpha) \).
Example 2.2. Let \( \psi(z) = z/(1 + z) \), \( z \in \Delta \). Then it is easy to see that \( \psi \) is Schwarz function. By putting \( \psi(z) = z/(1 + z) \) in the equation (2.10), we get \( h(z) = g(z) e^{2(1-\alpha)z} \). Thus the function \( f_{\alpha,\beta}(z) = z|z|^{2\beta} e^{2(1-\alpha)z}|g(z)|^{-\alpha} \) (\( \text{Re}\{\beta\} > -1/2 \)) belongs to the class \( \text{STLH}(\alpha) \), where \( g \) is analytic function with \( g(0) = 1 \). In particular, the function \( \hat{f}_{0,0}(z) = z e^{2z} \) is SLH mapping. It is a simple exercise that the radius of injectivity of the function \( \hat{f}_{0,0} \) is \( 1/2 \). The Figures 2(a) and 2(b) show that the image of the unit disc and \( |z| < 1/2 \), respectively. However, this example shows that a starlike log–harmonic mapping need not be univalent.

Figure 2. [(a)] The boundary curve of \( \hat{f}_{0,0}(\Delta) \) [(b)] The boundary curve of \( \hat{f}_{0,0}(|z| < 1/2) \)

Theorem 2.4. Let \( f(z) = z|z|^{2\beta} h(z) g(z) \in \text{STLH}(\alpha) \), where

\[
\begin{align*}
h(z) &= 1 + \sum_{n=1}^{\infty} a_n z^n \\
g(z) &= 1 + \sum_{n=1}^{\infty} b_n z^n,
\end{align*}
\]

be a log–harmonic mapping, and let

\[
\frac{zh'(z)}{h(z)} = \sum_{n=1}^{\infty} h_n z^n \quad \text{and} \quad \frac{zg'(z)}{g(z)} = \sum_{n=1}^{\infty} g_n z^n.
\]

If

(2.13) \[ \sum_{n=1}^{\infty} |h_n| < 1 - |\beta| \]

and

(2.14) \[ \sum_{n=1}^{\infty} (|h_n| + |g_n|) \leq 1 - 2|\beta|, \]

then \( f \) is sense preserving.
Proof. To show that $f$ is sense preserving, we need to prove that $|w(z)| < 1$ where $w(z)$ be given by (2.2). We have

$$
|w(z)| = \left| \frac{\bar{\beta} + z \frac{g'(z)}{g(z)}}{1 + \beta + z \frac{h'(z)}{h(z)}} \right|
= \left| \frac{\bar{\beta} + \sum_{n=1}^{\infty} g_n z^n}{1 + \beta + \sum_{n=1}^{\infty} h_n z^n} \right|
\leq \left| \frac{\beta + \sum_{n=1}^{\infty} |g_n| |z|^n}{1 - |\beta| - \sum_{n=1}^{\infty} |h_n| |z|^n} \right|
< \left| \frac{\beta}{1 - |\beta| - \sum_{n=1}^{\infty} |h_n|} \right| \leq 1.
$$

This proves the theorem.

Theorem 2.5. Let $1/2 \leq \alpha < 1$. If the function $f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$ belongs to the class $ST_{LH}(\alpha)$, then

(2.15) $\Re \left\{ \frac{h(z)}{g(z)} \right\} > \mu(\alpha) := \frac{1}{3 - 2\alpha} \quad (z \in \Delta)$.

Proof. For convenience, we put $\mu(\alpha) := \mu$ and note that $1/2 \leq \mu < 1$ when $1/2 \leq \alpha < 1$. Define

(2.16) $p(z) = \frac{1}{1 - \mu} \left( \frac{h(z)}{g(z)} - \mu \right)$.

Then $p(z)$ is analytic function in $\Delta$ and $p(0) = 1$. A simple check gives

$$
1 + z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} = 1 + \frac{(1 - \mu)zp'(z)}{\mu + (1 - \mu)p(z)} := \phi(p(z), zp'(z)),
$$

where

(2.17) $\phi(x, y) = 1 + \frac{(1 - \mu)y}{\mu + (1 - \mu)x}$.

By Lemma 2.1, since $f(z) = z|z|^{2\beta}h(z)\overline{g(z)} \in ST_{LH}(\alpha)$, we can consider

$$
\{ \phi(p(z), zp'(z)) : z \in \Delta \} \subset \{ w \in \mathbb{C} : \Re\{w\} > \alpha \} =: \Omega_\alpha.
$$

Now, for $\rho, \sigma \leq -\frac{1}{2}(1 + \rho^2)$, we get

$$
\Re\{\phi(i\rho, \sigma)\} = \Re \left\{ 1 + \frac{(1 - \mu)\sigma}{\mu + (1 - \mu)i\rho} \right\}
= 1 + \frac{\mu(1 - \mu)\sigma}{\mu^2 + (1 - \mu)^2\rho^2}
\leq 1 - \frac{1}{2}\mu(1 - \mu)Q(\rho),
$$

where

(2.18) $Q(\rho) := \frac{1 + \rho^2}{\mu^2 + (1 - \mu)^2\rho^2}$.

It is easy to see that

$$
Q'(\rho) = \frac{2(2\mu - 1)\rho}{(\mu^2 + (1 - \mu)^2\rho^2)^2}
$$

and $Q'(0) = 0$ occurs at only $\rho = 0$ and satisfies $Q(0) = 1/\mu^2$. Also

$$
\lim_{\rho \to \pm\infty} Q(\rho) = \frac{1}{(1 - \mu)^2}.
$$
Thus we have

\[
\frac{1}{\mu^2} \leq Q(\rho) < \frac{1}{(1 - \mu)^2} \quad (1/2 \leq \mu < 1).
\]

Hence

\[
\Re\{\phi(i\rho, \sigma)\} \leq 1 - \frac{1}{2} \mu(1 - \mu) \frac{1}{\mu^2} = \frac{3\mu - 1}{2\mu} = \alpha,
\]

and this shows that \(\Re\{\phi(i\rho, \sigma)\} \not\in \Omega_\alpha\). Moreover, by Lemma 1.1 we get \(\Re\{p(z)\} > 0\) in \(\Delta\), and concluding the proof.

\[\square\]

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