ON ATYPICAL VALUES AND LOCAL MONODROMIES OF MEROMORPHIC FUNCTIONS

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Abstract. A meromorphic function on a compact complex analytic manifold defines a $C^\infty$ locally trivial fibration over the complement of a finite set in the projective line $\mathbb{CP}^1$. We describe zeta-functions of local monodromies of this fibration around atypical values. Some applications to polynomial functions on $\mathbb{C}^n$ are described.

§1.- Introduction

We want to consider fibrations defined by meromorphic functions. In order to have more general statements we prefer to use the notion of a meromorphic function slightly different from the standard one. Let $M$ be an $n$-dimensional compact complex analytic manifold.

Definition. A meromorphic function $f$ on the manifold $M$ is a ratio $\frac{P}{Q}$ of two non-zero sections of a line bundle $\mathcal{L}$ over $M$. Meromorphic functions $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ (where $P'$ and $Q'$ are sections of a line bundle $\mathcal{L}'$) are equal if $P = UP'$ and $Q = UQ'$ where $U$ is a section of the bundle $\text{Hom}(\mathcal{L}', \mathcal{L}) = \mathcal{L} \otimes \mathcal{L}'^*$ without zeroes (in particular, this implies that the bundles $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic).

A particular important case of meromorphic functions are rational functions $\frac{P(x_0, \ldots, x_n)}{Q(x_0, \ldots, x_n)}$ on the projective space $\mathbb{CP}^n$ ($P$ and $Q$ are homogeneous polynomials of the same degree).

A meromorphic function $f = \frac{P}{Q}$ defines a map $f$ from the complement $M \setminus \{P = Q = 0\}$ of the set of common zeroes of $P$ and $Q$ to the complex line $\mathbb{CP}^1$. The indeterminacy set $\{P = Q = 0\}$ may have components of codimension one. For $c \in \mathbb{CP}^1$, let $F_c = f^{-1}(c)$.

The standard arguments (using a resolution of singularities; see, e.g., [7]) give the following statement.

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Theorem 1. The map $f : M \setminus \{P = Q = 0\} \to \mathbb{CP}^1$ is a $C^\infty$ locally trivial fibration outside a finite subset of the projective line $\mathbb{CP}^1$.

Any fibre $F_{\text{gen}} = f^{-1}(c_{\text{gen}})$ of this fibration is called a generic fibre of the meromorphic function $f$. The smallest subset $B(f) \subset \mathbb{CP}^1$ for which $f$ is a $C^\infty$ locally trivial fibration over $\mathbb{CP}^1 \setminus B(f)$ is called the bifurcation set of the meromorphic function $f$. Its elements are called atypical values.

Remark 1. In addition to some other advantages (see Statement 1 and Remark 4) the described definition of a meromorphic function permits to treat in the same way the following situation. Let $f = \frac{P}{Q}$ be a meromorphic function on $M$ and let $H = \{R = 0\}$ be a hypersurface in $M$ ($R$ is a section of a line bundle). One can be interested to study the map defined by the restriction of the meromorphic function $f$ to $M \setminus (H \cup \{P = Q = 0\})$. Substituting the meromorphic function $f$ by $f' = \frac{PR}{QR}$ one reduces the situation to the discussed one, i.e., the indeterminacy set of the meromorphic function $f'$ coincides with $H \cup \{P = Q = 0\}$ and the meromorphic functions $f$ and $f'$ coincide outside it.

The fundamental group $\pi_1(\mathbb{CP}^1 \setminus B(f))$ of the complement to the bifurcation set acts on homology groups $H_*(F_{\text{gen}}; \mathbb{C})$ of the generic fibre of the meromorphic function $f$. The image of the group $\pi_1(\mathbb{CP}^1 \setminus B(f))$ in the group of automorphisms of $H_*(F_{\text{gen}}; \mathbb{C})$ is called the monodromy group of the meromorphic function $f$. It is generated by local monodromies corresponding to atypical values (see [2]).

§2.- Zeta-functions of local monodromies

For a map $h : X \to X$ of a topological space $X$ (say, with finite dimensional homologies) into itself, its zeta-function $\zeta_h(t)$ is the rational function defined by

$$\zeta_h(t) = \prod_{q \geq 0} \{\det \left[ id - t h_* | H_q(X; \mathbb{C}) \right] \}^{-1}.$$

Remark 2. This is the definition of the zeta-function from [2]. The zeta-function defined in [1] is the inverse of this one.

Let $\zeta_f^c(t)$ be the zeta-function of the local monodromy corresponding to the value $c \in \mathbb{CP}^1$ (i.e., defined by a simple loop around $c$).

Remark 3. Local monodromy and the corresponding zeta-function are defined for any value $c \in \mathbb{CP}^1$, not only for atypical ones. For a typical value of the meromorphic function $f$, the local monodromy is the identity and its zeta-function is equal to $(1 - t)^{\chi(F_{\text{gen}})}$.

The following statement is a direct consequence of the definitions.

Statement 1. Let $\pi : \widetilde{M} \to M$ be an analytic map of an $n$-dimensional compact complex manifold $\widetilde{M}$ which is an isomorphism outside of the union of the indeterminacy set $\{P = Q = 0\}$ of the meromorphic function $f$ and of a finite number of level sets $f^{-1}(c_i)$. Let $\widetilde{f} = \frac{P \circ \pi}{Q \circ \pi}$ be the lifting of the meromorphic function $f = \frac{P}{Q}$.
to $\tilde{M}$. Then the generic fibre of $\tilde{f}$ coincides with that of $f$ and for each $c \in \mathbb{CP}^1$ one has

$$\zeta^c_{\tilde{f}}(t) = \zeta^c_f(t).$$

**Remark 4.** Even if the indeterminacy set $\{P = Q = 0\}$ of the meromorphic function $f$ has codimension two (i.e., if the hypersurfaces $\{P = 0\}$ and $\{Q = 0\}$ have no common components), in general, this is not the case for the lifting $\tilde{f}$. This is a reason for our definition of a meromorphic function. If one starts from the usual definition, the lifting $\tilde{f}$ of the meromorphic function $f$ can be defined at some points of $\pi^{-1}(\{P = Q = 0\})$. In this case a generic level set of the meromorphic function $\tilde{f}$ differs from that of $f$ and the Statement 1 does not hold. The simplest example is $f = \frac{x}{y}$ in affine coordinates on $\mathbb{CP}^2$, $\pi$ is the blowing-up of the origin in this affine chart.

In order to have somewhat more attractive and unified formulae we would like to use the notion of the integral with respect to the Euler characteristic ([8]). Let $A$ be an abelian group with the group operation $\ast$, let $X$ be a semianalytic subset of a complex manifold. Let $\Psi : X \to A$ be a function on $X$ with values in $A$ for which there exists a finite partitioning of $X$ into semianalytic sets (strata) $\Xi$ such that $\Psi$ is constant on each stratum $\Xi$ (and equal to $\psi_{\Xi}$). Then by definition

$$\int_X \Psi(x) \, d\chi = \sum_{\Xi} \chi(\Xi) \psi_{\Xi},$$

where $\chi(\Xi)$ is the Euler characteristic of the stratum $\Xi$. In the above formula we use the additive notations for the operation $\ast$. In what follows this definition will be used for integer valued functions and also for local zeta-functions $\zeta_x(t)$ which are elements of the abelian group of non-zero rational functions in the variable $t$ with respect to multiplication. In this case in the multiplicative notations the above formula means

$$\int_X \zeta_x(t) \, d\chi = \prod_{\Xi} (\zeta_{\Xi}(t))^{\chi(\Xi)}.$$

In [5], for a germ of a meromorphic function $\varphi = \frac{F}{G}$ on $(\mathbb{C}^n, 0)$, there were defined two Milnor fibres (the zero and the infinite ones), two monodromy transformations and thus two zeta-functions $\zeta^{0}_{\varphi}(t)$ and $\zeta^{\infty}_{\varphi}(t)$. For a value $c$ different from 0 and from $\infty$ one can define the same objects (in particular the zeta-function $\zeta^{c}_{\varphi}(t)$) as zero ones for the germ $\varphi - c = \frac{F - cG}{G}$.

**Remark 5.** One can easily see that these notions are invariant with respect to projective transformations of the projective line $\mathbb{CP}^1$. Thus $\zeta^{c}_{\varphi}(t)$ has to be defined only for one $c$, say, for $c = 0$.

For the aim of convenience, in [5] we considered only germs of meromorphic functions $\varphi = \frac{F}{G}$ with $F(0) = G(0) = 0$. At a point which does not belong to the indeterminacy set of the germ $\varphi = \frac{F}{G}$ (i.e., if $F(0)$ or $G(0)$ is different from 0) one
can use the following version of that definition. The Milnor fibre of the germ of the meromorphic function $\varphi$ corresponding to a value $c \in \mathbb{CP}^1$ is empty unless $\varphi(0) = c$ (and thus the corresponding zeta-function $\zeta^c_\varphi(t)$ is equal to 1). The Milnor fibre of the germ $\varphi$ corresponding to the value $\varphi(0)$ coincides with the usual Milnor fibre of the holomorphic germ obtained from $\varphi$ by a projective transformation of $\mathbb{CP}^1$ which sends $\varphi(0)$ to $0 \in \mathbb{C}^1 \subset \mathbb{CP}^1$ (and thus the zeta-function $\zeta^{\varphi(0)}_\varphi(t)$ coincides with the usual zeta-function of this holomorphic germ).

Let $c$ be a point of the projective line $\mathbb{CP}^1$. For a point $x \in M$, let $\zeta^c_{f,x}(t)$ be the corresponding zeta-function of the germ of the meromorphic function $f$ at the point $x$, let $\chi^c_{f,x}$ be its degree $\deg \zeta^c_{f,x}(t)$.

**Theorem 2.**

$$\zeta^c_f(t) = \int_{\{P=Q=0\} \cup F_c} \zeta^c_{f,x}(t) d\chi,$$

$$\chi(F_{\text{gen}}) - \chi(F_c) = \int_{F_c} (\chi^c_{f,x} - 1) d\chi + \int_{\{P=Q=0\}} \chi^c_{f,x} d\chi. \quad (2)$$

**Proof.** The proof follows the lines of the proof of Theorem 1 in [4]. Without any loss of generality one can suppose that $c = 0$. There exists a modification $\pi : X \rightarrow M$ of the manifold $M$ which is an isomorphism outside the set $\{P = Q = 0\} \cup \{f = 0\} \cup \{f = \infty\} = \{P = 0\} \cup \{Q = 0\}$ such that $D = \pi^{-1}(\{P = 0\} \cup \{Q = 0\})$ is a normal crossing divisor in the manifold $X$. Then at each point of the exceptional divisor $D$ in a local system of coordinates one has $P \circ \pi = u \cdot y_1^{k_1} \cdots y_n^{k_n}$, $Q \circ \pi = v \cdot y_1^{\ell_1} \cdots y_n^{\ell_n}$ with $u(0) \neq 0$, $v(0) \neq 0$, $k_i \geq 0$ and $\ell_i \geq 0$. There exist Whitney stratifications $S'$ and $S^*$ of $M$ and $X$ respectively such that:

1. the map $\pi$ is a stratified morphism with respect to these stratifications;
2. for each stratum $\Xi^* \in S^*$ the germs of the liftings $P \circ \pi$ and $Q \circ \pi$ at points of $\Xi^*$ have normal forms $u \cdot y_1^{k_1} \cdots y_n^{k_n}$ and $v \cdot y_1^{\ell_1} \cdots y_n^{\ell_n}$ where $(k_1, \ldots, k_n)$ and $(\ell_1, \ldots, \ell_n)$ do not depend on a point of $\Xi^*$.

One applies the following version of the formula of A’Campo ([1]) and also its local variant. Let $S_{k,\ell}$ be the set of points of $X$ in a neighbourhood of which the functions $P \circ \pi$ and $Q \circ \pi$ in some local coordinates have the forms $u \cdot y_1^k$ and $v \cdot y_1^\ell$ respectively ($u(0) \neq 0$, $v(0) \neq 0$).

**Statement 2.**

$$\zeta^0_f(t) = \prod_{k>\ell \geq 0} (1 - t^{k-\ell}) \chi(S_{k,\ell}).$$

After that, the arguments from Theorem 1 in [4] work literally. The difference between $(\chi^c_{f,x} - 1)$ and $\chi^c_{f,x}$ in the two integrals in (2) reflects the fact that the Euler characteristic of the local level set $F_\varepsilon \cap B_\varepsilon(x)$ ($B_\varepsilon(x)$ is the ball of small radius $\varepsilon$ centred at the point $x$) of the germ $f$ is equal to 1 at a point $x$ of the level set $F_\varepsilon$ and is equal to 0 at a point $x$ of the indeterminacy set $\{P = Q = 0\}$. In the first case this local level set is contractible and in the second one it is the difference between two contractible sets.
Let us denote $(-1)^{n-1}$ times the first integral in (2) by $\mu_f(c)$ and $(-1)^{n-1}$ times the second one by $\lambda_f(c)$. Let $\mu_f = \sum_{c \in \mathbb{CP}^1} \mu_f(c)$ and $\lambda_f = \sum_{c \in \mathbb{CP}^1} \lambda_f(c)$ (in each sum only finite number of summands are different from zero).

**Theorem 3.**

$$\mu_f + \lambda_f = (-1)^{n-1} (2 \cdot \chi(F_{\text{gen}}) - \chi(M) + \chi(\{P = Q = 0\})) .$$

**Proof.** One has

$$\int_{\mathbb{CP}^1} \chi(F_c) \, d\chi = \chi(M \setminus \{P = Q = 0\}) = \chi(M) - \chi(\{P = Q = 0\}).$$

Therefore

$$\chi(M) - \chi(\{P = Q = 0\}) = \int_{\mathbb{CP}^1} \chi(F_{\text{gen}}) \, d\chi + \int_{\mathbb{CP}^1} (\chi(F_c) - \chi(F_{\text{gen}})) \, d\chi =$$

$$= 2 \cdot \chi(F_{\text{gen}}) - (-1)^{n-1} \sum_{c \in \mathbb{CP}^1} (\mu_f(c) + \lambda_f(c)) = 2 \cdot \chi(F_{\text{gen}}) + (1)^n(\mu_f + \lambda_f). \quad \Box$$

Let $\tilde{f}$ be the restriction of $f$ to $M \setminus \{Q = 0\}$, $\tilde{f} : M \setminus \{Q = 0\} \to \mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}$. Notice that the fibres of both maps $f$ and $\tilde{f}$ over values $c \in \mathbb{C}$ coincide.

**Corollary 1.**

$$\chi(F_{\text{gen}}) = \chi(M) - \chi(\{Q = 0\}) + (-1)^{n-1}(\lambda_f - \lambda_f(\infty) + \mu_f - \mu_f(\infty)).$$

Let $f$ be the meromorphic function on the projective space $\mathbb{CP}^n$ defined by a polynomial $P$ in $n$ variables (see below). If $P$ has only isolated critical points in $\mathbb{C}^n$, then $\mu_f(c)$ is the sum of the Milnor numbers of the critical points of the polynomial $P$ with critical value $c$, $\lambda_f(c)$ is equal to the invariant $\lambda_P(c)$ studied in [3]. Therefore $\mu_f(c)$ and $\lambda_f(c)$ can be considered as generalizations of those invariants. One has $\mu_f = \mu_P + \mu_f(\infty)$, $\lambda_f = \lambda_P + \lambda_f(\infty)$, where $\mu_P = \sum_{c \in \mathbb{C}} \mu_P(c)$, $\lambda_P = \sum_{c \in \mathbb{C}} \lambda_P(c)$. Notice that in this case Corollary 1 turns into the well known formula $\chi(F_{\text{gen}}) = 1 + (-1)^{n-1}(\lambda_P + \mu_P)$.

§3.- **Applications to polynomials**

A polynomial $P : \mathbb{C}^n \to \mathbb{C}$ defines a meromorphic function $f = \frac{P}{x_0^n}$ on the projective space $\mathbb{CP}^n$ ($d = \deg P$). For any $c \in \mathbb{CP}^1$ the local monodromy of the polynomial and its zeta-function $\zeta_f^c(t)$ are defined (in fact they coincide with those of the meromorphic function $f$). The described technique gives the following statements for polynomials. Let us remind that for $x \in \{P = c\} \subset \mathbb{C}^n$ the zeta-function $\zeta_{f,x}^c(t)$ is the usual zeta-function of the germ of the polynomial $P$ at $x$. 


Theorem 4. For $c \in \mathbb{C} \subset \mathbb{C}P^1$,

$$\zeta^c_P(t) = \left( \int_{\{P=0\} \cap \mathbb{C}P_{\infty}^{n-1}} \zeta^c_{f,x}(t) \, d\chi \right) \cdot \left( \int_{\{P=c\}} \zeta^c_{P,x}(t) \, d\chi \right).$$

(3)

For the infinity value,

$$\zeta^\infty_P(t) = \int_{\mathbb{C}P_{\infty}^{n-1}} \zeta^\infty_{f,x}(t) \, d\chi.$$ 

For a generic $c' \in \mathbb{C}$,

$$\chi(\{P = c'\}) - \chi(\{P = c\}) = \int_{\{P=0\} \cap \mathbb{C}P_{\infty}^{n-1}} \chi^c_{f,x} \, d\chi + \int_{\{P=c\}} (\chi^c_{P,x} - 1) \, d\chi.$$ 

Remark 6. In [6] we consider the zeta-function of the local monodromy (corresponding to a finite value $c$) of the polynomial $P$ near infinity which is just the first factor in the formula (3). If that zeta-function is different from 1 then the value $c$ is atypical at infinity.

Let $H = \{R = 0\}$ be a hypersurface in $\mathbb{C}^n$ ($R : \mathbb{C}^n \to \mathbb{C}$ is a polynomial). The polynomial $P$ restricted to the complement of the hypersurface $H$ defines a $C^\infty$ locally trivial fibration outside a finite set in $\mathbb{C}$. For each $c \in \mathbb{C}$ as well as for $c = \infty$, the local monodromy of this fibration and its zeta-function $\zeta^c_{P/H}(t)$ are defined. The described fibration is nothing else but the fibration for the meromorphic function $\tilde{f} = \frac{P \cdot R}{R}$. It implies the following result.

Theorem 5. For $c \in \mathbb{C} \subset \mathbb{C}P^1$,

$$\zeta^c_{P/H}(t) = \int_{\{P=c\} \cup \{P=0\} \cap \mathbb{C}P_{\infty}^{n-1}} \zeta^c_{f,x}(t) \, d\chi.$$ 

For a generic $c' \in \mathbb{C}$,

$$\chi(F_{c'} \setminus H) - \chi(F_c \setminus H) = \int_{\{P=0\} \cap \mathbb{C}P_{\infty}^{n-1}} \chi^c_{f,x} \, d\chi + \int_{\{P=c\} \cap H} \chi^c_{f,x} \, d\chi + \int_{F_{c'} \setminus H} (\chi^c_{P,x} - 1) \, d\chi.$$ 

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