Three new classes of optimal frequency-hopping sequence sets

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Abstract

The study of frequency-hopping sequences (FHSs) has been focused on the establishment of theoretical bounds for the parameters of FHSs as well as on the construction of optimal FHSs with respect to the bounds. Peng and Fan (2004) derived two lower bounds on the maximum nontrivial Hamming correlation of an FHS set, which is an important indicator in measuring the performance of an FHS set employed in practice.

In this paper, we obtain two main results. We study the construction of new optimal frequency-hopping sequence sets by using cyclic codes over finite fields. Let \( C \) be a cyclic code of length \( n \) over a finite field \( \mathbb{F}_q \) such that \( C \) contains the one-dimensional subcode \( C_0 = \{(\alpha, \alpha, \ldots, \alpha) \in \mathbb{F}_q^n | \alpha \in \mathbb{F}_q \} \). Two codewords of \( C \) are said to be equivalent if one can be obtained from the other through applying the cyclic shift a certain number of times. We present a necessary and sufficient condition under which the equivalence class of any codeword in \( C \setminus C_0 \) has size \( n \). This result addresses an open question raised by Ding et al. in [9]. As a consequence, three new classes of optimal FHS sets with respect to the Singleton bound are obtained, some of which are also optimal with respect to the Peng-Fan bound at the same time. We also show that the two Peng-Fan bounds are, in fact, identical.

Keywords: Frequency-hopping sequence set, cyclic code, maximum distance separable (MDS) code, cyclotomic coset.

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1 Introduction

Let \( \ell \) be a positive integer and let \( \mathbb{F} = \{f_0, f_1, \ldots, f_{\ell-1}\} \) be an alphabet of \( \ell \) available frequencies. A sequence \( X = \{x_t\}_{t=0}^{n-1} \) is called a frequency-hopping sequence (FHS) of length \( n \) over \( \mathbb{F} \) if \( x_t \in \mathbb{F} \) for all \( 0 \leq t \leq n-1 \). For two FHSs \( X = \{x_t\}_{t=0}^{n-1} \) and \( Y = \{y_t\}_{t=0}^{n-1} \) of length \( n \) over \( \mathbb{F} \), if \( x_t = y_t \) for all \( 0 \leq t \leq n-1 \), then we say \( X = Y \). Let \( S \) be the set of all FHSs of length \( n \) over \( \mathbb{F} \). Any subset of \( S \) is called an FHS set. For any \( X, Y \in S \), their Hamming correlation is defined by

\[
H_{X,Y}(t) = \sum_{i=0}^{n-1} h[x_i, y_{i+t}], \quad 0 \leq t \leq n-1
\]

where \( h[a,b] = 1 \) if \( a = b \) and 0 otherwise, and the subscript addition is taken modulo \( n \). For any distinct \( X, Y \in S \), we have the following measures,

\[
H(X) = \max_{1 \leq t < n} \{H_{X,X}(t)\}
\]

and

\[
H(X,Y) = \max_{0 \leq t < n} \{H_{X,Y}(t)\}.
\]
Let $\mathcal{F}$ be an FHS set containing $N$ elements. The maximum nontrivial Hamming correlation of the FHS set $\mathcal{F}$ is defined by

$$M(\mathcal{F}) = \max \left\{ \max_{X \in \mathcal{F}} H(X), \max_{X,Y \in \mathcal{F}, X \neq Y} H(X,Y) \right\}.$$ 

As in [9], we use $(n, N, \lambda; \ell)$ to denote the FHS set $\mathcal{F}$ with $N$ elements of length $n$ over an alphabet of size $\ell$, where $\lambda = M(\mathcal{F})$.

Frequency-hopping spread spectrum techniques have been widely used in modern communication systems, such as ultrawideband communications, military communications, Wi-Fi and Bluetooth. In these systems, the receivers may be confronted with the interference caused by undesired signals. It is often desirable to properly select frequency-hopping sequences (FHSs) or FHS sets to mitigate the interference. In order to evaluate the goodness of an FHS set, the maximum nontrivial Hamming correlation is an important indicator. In general, it is desirable to construct an FHS set with a large set size and a low Hamming correlation value, when its length and the number of available frequencies are fixed. However, the parameters of an FHS set are not independent of one another, and they are subjected to certain theoretical limits (e.g., see [9], [15], [18], [22], [25], [29]). Lempel and Greenberger obtained a lower bound on $H(X)$ of any individual FHS $X$ in 1974 [18]. Extending the Lempel-Greenberger bound, Peng and Fan derived the following lower bounds on the maximum nontrivial Hamming correlation of an FHS set ([22, Corollary 1]).

For a real number $a$, let $\lceil a \rceil$ denote the least integer not less than $a$ and let $\lfloor a \rfloor$ denote the integer part of $a$.

Lemma 1.1. (Peng-Fan bounds, [22]) Let $\mathcal{F} \subseteq \mathcal{S}$ be a set of $N$ sequences of length $n$ over an alphabet of size $\ell$. Define $I = \lfloor nN/\ell \rfloor$. Then

$$M(\mathcal{F}) \geq \left\lceil \frac{(nN - \ell)n}{(nN - 1)\ell} \right\rceil = \frac{2InN - (I + 1)I\ell}{(nN - 1)N}. \quad (1.1)$$

and

$$M(\mathcal{F}) \geq 2InN - (I + 1)I\ell. \quad (1.2)$$

Remark 1.2. Yang et al. compared the above two Peng-Fan bounds in [25]; it was shown that the Peng-Fan bound of (1.2) may be tighter than that of (1.1). However, the authors failed to find examples where the bound of (1.2) is strictly tighter than that of (1.1), and finally suggested that the exact relationship between the bounds (1.1) and (1.2) needs to be studied further. In this paper, we show that the two Peng-Fan bounds are, in fact, identical. (See Theorem 1.3 below. It is reasonable to assume that $nN \geq \ell$ and its proof is deferred to the Appendix.)

Theorem 1.3. Let $\mathcal{F} \subseteq \mathcal{S}$ be a set of $N$ sequences of length $n$ over an alphabet of size $\ell$. Define $I = \lfloor nN/\ell \rfloor$. If $nN \geq \ell$, then

$$M(\mathcal{F}) \geq \left\lceil \frac{(nN - \ell)n}{(nN - 1)\ell} \right\rceil = \frac{2InN - (I + 1)I\ell}{(nN - 1)N}. \quad (1.3)$$

Besides the bounds on the Hamming correlation, several bounds on the size of an FHS set were also established. Ding et al. in [9] obtained a number of bounds on the size of an FHS set from certain classical bounds in coding theory.

Lemma 1.4. (Sphere-packing bound on the size of FHS sets, [9]) For any $(n, N, \lambda; \ell)$ FHS set $\mathcal{F}$, where $\lambda < n$ and $\ell > 1$, we have

$$N \leq \frac{\ell^n}{n \left( \sum_{i=0}^{\lfloor (n-\lambda-1)/2 \rfloor} \binom{n}{i} (\ell - 1)^i \right)}. \quad (1.3)$$

Lemma 1.5. (Singleton bound on the size of FHS sets, [9]) For any $(n, N, \lambda; \ell)$ FHS set $\mathcal{F}$, where $\lambda < n$ and $\ell > 1$, we have

$$N \leq \left\lfloor \frac{\ell^{\lambda+1}}{n} \right\rfloor. \quad (1.4)$$
An FHS set is called optimal if one of the bounds (1.1)-(1.4) is met. It is of great interest to construct optimal FHS sets with respect to the bounds. In recent years, numerous constructions of optimal FHS sets have been proposed (e.g., see [1]-[14], [16], [20], [23], [26]-[31], and references therein). Ding et al. generalized the ideas in [24] to construct optimal FHS sets by using some special classes of cyclic codes [9]. This idea was further investigated in [11] to obtain more optimal FHS sets. As shown in [9], there is a natural equivalence relation defined on any cyclic code: two codewords of a cyclic code are said to be equivalent if one can be obtained from the other through applying the cyclic shift a certain number of times. A special class of cyclic codes $C_{(q,m)}$ of length $n = (q^m - 1)/(q - 1)$ over a finite field $F_q$ containing $C_0 = \{(\alpha, \alpha, \cdots, \alpha) \in F_q^n | \alpha \in F_q\}$ was discussed in [9]. It was shown in [9] that if the code length $n$ is a prime number, then the equivalence class of any codeword in $C_{(q,m)} \setminus C_0$ has size $n$; an FHS set is thus obtained by taking exactly one element from every equivalence class of $C_{(q,m)} \setminus C_0$, which turns out to be optimal with respect to the sphere-packing bound (1.3). A natural open question posed in [9, p.3302] is whether the prime-length constraint can be dropped without changing the situation that the equivalence class of any codeword in $C_{(q,m)} \setminus C_0$ has size $n$.

In this paper, we further explore the above idea to construct more optimal FHS sets by using maximum distance separable (MDS) cyclic codes. Let $F_q$ be the finite field with $q$ elements and let $n$ be a positive integer co-prime to $q$. Assume that $C$ is a cyclic code of length $n$ over $F_q$ containing $C_0$, where $C_0 = \{(\alpha, \alpha, \cdots, \alpha) \in F_q^n | \alpha \in F_q\}$. In Section 3, we present a necessary and sufficient condition under which the equivalence class of any codeword in $C \setminus C_0$ has size $n$. This result addresses the aforementioned open question. Actually, it turns out that the prime-length constraint is necessary and cannot be dropped (see Corollary 3.3 in Section 3). In Section 4, using the results in Section 3, we obtain three new classes of optimal FHS sets with respect to the Singleton bound (1.4), some of which are also optimal with respect to the Peng-Fan bound (1.2) at the same time. More precisely, the parameters of the new FHS sets are given as follows:

(i) $$\left(q + 1, \frac{q^{2k+1} - q}{q + 1}, 2k; q\right)$$

where $q = 2^m$ with $m > 1$ being a positive integer, and where $1 \leq k \leq \min\{p - 1, 2^{m-1}\}$ with $p$ being the smallest prime divisor of $q + 1$. This FHS set is optimal with respect to the Singleton bound (1.4).

(ii) $$\left(q + 1, q(q - 1), 2; q\right)$$

where $q$ is an odd prime power. The parameters of this FHS set meet both the Peng-Fan bound (1.2) and the Singleton bound (1.4) at the same time.

(iii) $$\left(n, \frac{n(q^{2k+2} - 1)}{n}, 2k + 1; q\right)$$

where $n > 1$ is an odd divisor of $q + 1$ with $q$ being a prime power, and where $k$ is an integer such that $0 \leq k \leq (n - 3)/2 - M$ with $M \leq (n - 3)/2$ being the largest integer such that $\gcd(M, n) > 1$. This FHS set is optimal with respect to the Singleton bound (1.4). In particular, by taking $k = 0$, we have an FHS set with parameters $$\left(n, \frac{n(q^2 - 1)}{n}, 1; q\right)$$

which meet both the Peng-Fan bound (1.2) and the Singleton bound (1.4).

2 Some Facts about Cyclic Codes

In this section, we review some basic notation and results about cyclic codes over finite fields. For the details, the reader is referred to [19] or [21].
Let \( \mathbb{F}_q \) be the finite field with \( q \) elements and let \( n \) be a positive integer co-prime to \( q \). A linear code \( \mathcal{C} \) of length \( n \) over \( \mathbb{F}_q \) is called cyclic if it is an ideal of \( \mathbb{F}_q[x]/(x^n-1) \). It follows that any cyclic code \( \mathcal{C} \) of length \( n \) over \( \mathbb{F}_q \) is generated uniquely by a monic divisor \( g(x) \in \mathbb{F}_q[x] \) of \( x^n-1 \), which is referred to as the generator polynomial, and \( h(x) = (x^n-1)/g(x) \) is called its parity-check polynomial. We then know that the irreducible factors of \( x^n-1 \) in \( \mathbb{F}_q[x] \) determine all cyclic codes of length \( n \) over \( \mathbb{F}_q \). Theoretically, the irreducible factors of \( x^n-1 \) in \( \mathbb{F}_q[x] \) can be derived by the \( q \)-cyclotomic cosets modulo \( n \). For any integer \( t \), the \( q \)-cyclotomic coset \( C_t \) of \( t \) modulo \( n \) is defined by

\[
C_t = \left\{ t q^i \pmod{n} \mid t = 0, 1, \ldots \right\}.
\]

Take \( \alpha \) (maybe in some extension field of \( \mathbb{F}_q \)) to be a primitive \( n \)-th root of unity, which means that \( n \) is the smallest positive integer such that \( \alpha^n = 1 \).

Let \( C_0 = \{0\}, C_{i_1}, C_{i_2}, \ldots, C_{i_t} \) be all the distinct \( q \)-cyclotomic cosets modulo \( n \). It is well known that

\[
x^n - 1 = (x-1)M_1(x)M_2(x)\cdots M_t(x)
\]

with

\[
M_j(x) = \prod_{s \in C_{i_j}} (x - \alpha^s), \quad 1 \leq j \leq t,
\]

all being monic irreducible in \( \mathbb{F}_q[x] \). The defining set of \( \mathcal{C} = \langle g(x) \rangle \) is the subset of integers

\[
Z = \left\{ j \mid 0 \leq j \leq n-1, g(\alpha^j) = 0 \right\}.
\]

It is readily seen that the defining set \( Z \) is a union of \( q \)-cyclotomic cosets modulo \( n \).

A linear code of length \( n \) over \( \mathbb{F}_q \) is called an \([n, k, d]\) code if its dimension is \( k \) and minimum (Hamming) distance is \( d \). The following results are well known.

**Lemma 2.1.** (BCH bound for cyclic codes) Let \( \mathcal{C} \) be a cyclic code of length \( n \) over \( \mathbb{F}_q \). Let \( \alpha \) be a primitive \( n \)-th root of unity in some extension field of \( \mathbb{F}_q \). Assume the generator polynomial of \( \mathcal{C} \) has roots that include the set \( \{ \alpha^i \mid i_1 \leq i \leq i_1 + d - 2 \} \). Then the minimum distance of \( \mathcal{C} \) is at least \( d \).

**Proposition 2.2.** (Singleton bound) If \( \mathcal{C} \) is an \([n, k, d]\) linear code over \( \mathbb{F}_q \), then \( d \leq n - k + 1 \).

A linear code achieving this Singleton bound is called a maximum distance separable (MDS) code. A remark is in order at this point. Lemma 2.1 and Proposition 2.2 provide a useful method to construct MDS cyclic codes: If the generator polynomial of a cyclic code \( \mathcal{C} \) has roots precisely equal to the set \( \{ \alpha^i \mid i_1 \leq i \leq i_1 + d - 2 \} \), then the minimum distance of \( \mathcal{C} \) is exactly equal to \( d \). In particular, \( \mathcal{C} \) is an MDS cyclic code with parameters \([n, n-d+1, d]\). Indeed, it follows from Lemma 2.1 that the minimum distance of \( \mathcal{C} \) is at least \( d \). Since the dimension of \( \mathcal{C} \) is equal to \( n - d + 1 \), then the minimum distance of \( \mathcal{C} \) is no more than \( n - (n - d + 1) + 1 = d \), which implies that \( \mathcal{C} \) is an MDS cyclic code with parameters \([n, n-d+1, d]\) (e.g., see [2]). We will construct MDS cyclic codes based on this fact.

### 3 A Necessary and Sufficient Condition

Throughout this paper, \( C_0 \) denotes the cyclic code of length \( n \) over \( \mathbb{F}_q \) with generator polynomial \( 1 + x + \cdots + x^{n-1} \), i.e.,

\[
C_0 = \left\{ \alpha (1 + x + \cdots + x^{n-1}) \mid \alpha \in \mathbb{F}_q \right\}.
\]

We always assume that \( \mathcal{C} \) is a cyclic code of length \( n \) over \( \mathbb{F}_q \) with parity-check polynomial \( h(x) \) such that \( h(1) = 0 \). Note that \( C_0 \) is contained in \( \mathcal{C} \) if and only if \( h(1) = 0 \).

Two codewords \( c_1(x), c_2(x) \) of \( \mathcal{C} \) are said to be equivalent if there exists an integer \( t \) such that \( x^t c_1(x) \equiv c_2(x) \pmod{x^n-1} \). The codewords of \( \mathcal{C} \) then are classified into equivalence classes. We say that a codeword \( c(x) \in \mathcal{C} \) has size \( n \) if the equivalence class containing \( c(x) \) has size \( n \).

The following result establishes a necessary and sufficient condition under which any codeword in \( \mathcal{C} \setminus C_0 \) has size \( n \).
Theorem 3.1. Let $C$ be a cyclic code of length $n$ over $\mathbb{F}_q$ with parity-check polynomial $h(x)$ such that $h(1) = 0$, say $h(x) = (x-1)^{t}h'(x)$. Then the following statements are equivalent:

(i) Any codeword of $C \setminus C_0$ has size $n$.

(ii) Any root of $h'(x)$ is a primitive $n$-th root of unity.

Proof. Assume first that (i) holds. If $n$ is a prime number, then all the roots of $(x^n - 1)/(x - 1)$ are primitive $n$-th roots of unity. In particular, the roots of $h'(x)$ are primitive $n$-th roots of unity, and we are done. Therefore, we can assume that $n$ is a composite number. To get (ii), it is enough to prove that the roots of $(x^n - 1)/(x - 1)$ which have multiplicative order less than $n$ are roots of $g(x) = (x^n - 1)/h(x)$.

Let $\alpha$ be a primitive $n$-th root of unity. Assume to the contrary that there exists $0 < i_0 < n$ with $\gcd(i_0, n) \neq 1$ such that $\alpha^{i_0}$ is not a root of $g(x)$, i.e., $\alpha^{i_0}$ is a root of $(x^n - 1)/(x - 1)$ having order (say $r$) less than $n$ and $g(\alpha^{i_0}) \neq 0$. Let $C_{i_0}$ denote the $q$-cyclotomic coset modulo $n$ containing $i_0$, and assume that $C_0 = \{0\}, C_{i_0}, C_{i_1}, \ldots, C_{i_{t+1}}, \ldots, C_{i_{t+s}}$ are all the distinct $q$-cyclotomic cosets modulo $n$. Without loss of generality, suppose that the defining set of $C$ is $C_{i_1} \cup C_{i_2} \cup \cdots \cup C_{i_t}$, i.e.,

$$g(x) = \prod_{j=1}^{t} M_j(x), \quad \text{where } M_j(x) = \prod_{k \in C_{i_j}} (x - \alpha^k) \text{ for } 1 \leq j \leq t.$$  

Now let

$$m(x) = \prod_{j=1}^{s} M_{t+j}(x), \quad \text{where } M_{t+j}(x) = \prod_{k \in C_{i_{t+j}}} (x - \alpha^k) \text{ for } 1 \leq j \leq s.$$  

It follows that $c(x) = m(x)g(x)$ lies in $C \setminus C_0$. However, from our construction,

$$(x^r - 1)m(x)g(x) \equiv 0 \pmod{x^n - 1}.$$  

This says that $c(x)$ is a codeword of $C \setminus C_0$, and the size of the equivalence class containing $c(x)$ is less than $n$. This is a contradiction.

Assume that (ii) holds. Suppose otherwise that there exists a codeword $c_1(x) \in C \setminus C_0$ such that the equivalence class containing it has size $t < n$. Thus

$$x^t c_1(x) \equiv c_1(x) \pmod{x^n - 1}.$$  

Let $c_1(x) = m_1(x)g(x)$, then we have

$$(x^t - 1)m_1(x)g(x) \equiv 0 \pmod{x^n - 1},$$  

and hence

$$(x^t - 1)m_1(x) \equiv 0 \pmod{h'(x)}.$$  

Recall that all the roots of $h'(x)$ are primitive $n$-th roots of unity. Since the roots of $x^t - 1$ contain no primitive $n$-th root of unity, $h'(x)$ is a divisor of $m_1(x)$. We have arrived at a contradiction since this implies $c_1(x) = m_1(x)g(x) \in C_0$. \hfill $\square$

Example 3.2. Consider cyclic codes of length 9 over $\mathbb{F}_8$. It is easy to verify that all the distinct $8$-cyclotomic cosets modulo 9 are given by $C_0 = \{0\}, C_1 = \{1, 8\}, C_2 = \{2, 7\}, C_3 = \{3, 6\}$ and $C_4 = \{4, 5\}$. Take $\alpha$ to be a primitive ninth root of unity in $\mathbb{F}_{8^4}$. Then

$$x^9 - 1 = (x - 1)M_1(x)M_2(x)M_3(x)M_4(x), \quad \text{with } M_i(x) = \prod_{j \in C_i} (x - \alpha^j), \ 1 \leq i \leq 4$$  

gives the irreducible factorization of $x^9 - 1$ over $\mathbb{F}_8$. It is readily seen that the roots of $M_3(x)$ are primitive third roots of unity. Let $C_1$ and $C_2$ be cyclic codes of length 9 over $\mathbb{F}_8$ with parity-check polynomials $(x - 1)M_1(x)M_2(x)M_4(x)$ and $(x - 1)M_1(x)M_2(x)$, respectively. It follows from Theorem 3.1 that any codeword of $C_i \setminus C_0$ has size 9, for $i = 1, 2$. Note that $C_2$ is an MDS cyclic code with parameters $[9, 5, 5]$. 

As we will show in the next section, Theorem 3.1 is useful in constructing optimal FHS sets. In the rest of this section, Theorem 3.1 is used to answer an open question posed by Ding et al. [9, p.3302].

Let \( m \) be a positive integer such that \( \gcd(m, q - 1) = 1 \). Take \( n = (q^m - 1)/(q - 1) \). Let \( \beta \) be a generator of \( \mathbb{F}_{q^m}^* = \mathbb{F}_{q^m} \setminus \{0\} \) and \( \gamma = \beta^{q-1} \). Define a cyclic code \( C_{(q,m)} \) by

\[
C_{(q,m)} = \left\{ c(x) \mid c(x) \in \mathbb{F}_q[x]_n \text{ and } c(\gamma) = 0 \right\}
\]

where \( \mathbb{F}_q[x]_n \) consists of all polynomials of degree at most \( n - 1 \) over \( \mathbb{F}_q \). It is known that \( C_{(q,m)} \) is an \([n, n-m, 3]\) code with defining set

\[
C_1 = \left\{ q^i \pmod{n} \mid 0 \leq i \leq m-1 \right\}.
\]

Assuming that \( n = (q^m - 1)/(q - 1) \) is a prime number, Ding et al. in [9, Theorem 12] used \( C_{(q,m)} \) to construct an FHS set whose parameters meet the sphere-packing bound (1.3). The following open question was posed in [9, p.3302]: “If the condition that \( n = (q^m - 1)/(q - 1) \) is a prime number is dropped, is it still true that any codeword of \( C_{(q,m)} \setminus C_0 \) has size \( n \)?”

By Theorem 3.1 we have a complete answer to this question.

**Corollary 3.3.** Assume the same notation as previously defined. Then any codeword of \( C_{(q,m)} \setminus C_0 \) has size \( n \) if and only if \( n \) is a prime number.

**Proof.** It is known that \( C_{(q,m)} \) is a cyclic code of length \( n \) over \( \mathbb{F}_q \) with defining set given by (3.2). Thus the generator polynomial of \( C_{(q,m)} \) is

\[
g(x) = \prod_{i \in C_1} (x - \gamma^i).
\]

Let \( h(x) = \frac{x^n - 1}{g(x)} = (x - 1)h'(x) \). Suppose that any codeword of \( C_{(q,m)} \setminus C_0 \) has size \( n \). It follows from Theorem 3.1 that all the roots of \( h'(x) \) are primitive \( n \)-th roots of unity. Suppose otherwise that \( 1 < d < n \) is a divisor of \( n \). Then \( \gamma^d \) must be a root of \( h'(x) \), contradicting Theorem 3.1. \( \square \)

The following result can be proven in a fashion similar to Theorem 3.1.

**Proposition 3.4.** Let \( D \) be a nonzero cyclic code of length \( n \) over \( \mathbb{F}_q \) with parity-check polynomial \( h(x) \), where \( n \) is a positive integer co-prime to \( q \). Then any nonzero codeword of \( D \) has size \( n \) if and only if the roots of \( h(x) \) are primitive \( n \)-th roots of unity.

Note that the result of Proposition 3.4 appeared previously in [17, Theorem 1]. We will apply Theorem 3.1 and Proposition 3.4 to construct optimal FHS sets.

## 4 Optimal FHS Sets from Cyclic Codes

In this section, three families of optimal FHS sets with respect to the Singleton bound (1.4) are obtained. It turns out that some of the FHS sets are also optimal with respect to the Peng-Fan bound (1.2) at the same time. In the light of [9] and [11], we first present two basic facts.

**Fact 1.** If one has found an \([n, k, n-k+1]\) MDS cyclic code \( C \) over \( \mathbb{F}_q \) with \( n > q \) satisfying Theorem 3.1, then an FHS set is obtained by taking exactly one element from every equivalence class of \( C \setminus C_0 \), which results in an \([n, \frac{q^k - 1}{n}, k-1; q]\) FHS set. It is straightforward to verify that this FHS set is optimal with respect to the Singleton bound (1.4).

**Fact 2.** Similarly, if one has found an \([n, k, n-k+1]\) MDS cyclic code \( D \) over \( \mathbb{F}_q \) satisfying Proposition 3.4, then an FHS set is obtained by taking exactly one element from every equivalence class of \( D \setminus \{0\} \), which results in an \([n, \frac{q^k - 1}{n}, k-1; q]\) FHS set. This FHS set is optimal with respect to the Singleton bound (1.4).
We explain the reason for which an \([n, k, n-k+1]\) MDS cyclic code over \(\mathbb{F}_q\) with \(n > q\) satisfying Theorem 8.3 gives rise to an \((n, N, \lambda; \ell) = (n, \frac{2^k-q}{n}, k-1; q)\) FHS set, i.e., Fact 1 holds. It follows from the definition of \(\lambda\) and the Singleton bound (1.4) that \(\lambda \leq n-(n-k+1) = k-1\) and \(N = \frac{2^k-q}{n} \leq \left[ \frac{q+1}{n} \right] \), which forces \(\lambda = k-1\). Fact 2 is also satisfied for a similar reason.

In the following, three families of MDS cyclic codes satisfying Fact 1 or Fact 2 are constructed and their parameters are computed. Consequently, optimal FHS sets with respect to the Singleton bound (1.4) are derived from these MDS cyclic codes. These FHS sets are new in the sense that their parameters have not been covered in the literature.

### 4.1 Optimal FHS sets of length \(q+1\)

Consider cyclic codes of length \(q+1\) over \(\mathbb{F}_q\). We first study the case \(q = 2^m\), where \(m > 1\) is a positive integer. Let \(s = 2^{m-1}\). By [21, p.324], we know that all the distinct \(q\)-cyclotomic cosets modulo \(q+1\) are given by

\[
C_0 = \{0\} \quad \text{and} \quad C_i = \{i, q+1-i\}, \quad \text{for } 1 \leq i \leq s.
\]

Suppose \(p\) is the smallest prime divisor of \(q+1\). Let \(\alpha \in \mathbb{F}_{q^2}\) be a primitive \((q+1)\)-st root of unity. For any integer \(k\) with \(1 \leq k \leq \min\{p-1, s\}\), let \(C\) be a cyclic code of length \(q+1\) over \(\mathbb{F}_q\) with parity-check polynomial

\[
h(x) = (x-1)M_1(x)M_2(x) \cdots M_k(x), \quad M_j(x) = \prod_{\ell \in C_j} (x-\alpha^i), \quad 1 \leq j \leq k.
\]

Equivalently, \(C\) is the cyclic code of length \(q+1\) over \(\mathbb{F}_q\) with defining set

\[
C_{k+1} \cup C_{k+2} \cup \cdots \cup C_s.
\]

Thus \(C\) is a \([q+1, 2k+1, q-2k+1]\) MDS cyclic code satisfying Theorem 8.3. The above discussion leads to the following result.

**Theorem 4.1.** Let \(q = 2^m\), where \(m > 1\) is a positive integer. Suppose \(p\) is the smallest prime divisor of \(q+1\). For any integer \(k\) with \(1 \leq k \leq \min\{p-1, 2^{m-1}\}\), we have a \((q+1, \frac{2k+1+q}{2}, 2k; q)\) FHS set whose parameters meet the Singleton bound (1.4).

As an additional remark, we observe that if \(m\) is odd in Theorem 4.1 then \(p = 3\). This is simply because 3 is always a divisor of \(2^m + 1\) for any odd \(m\).

**Example 4.2.** Take \(q = 2^3 = 8\) and \(n = q+1 = 9\). Then \(p = 3\) and \(1 \leq k \leq 2\). Thus we have a \((9, (8^2k+1-8)/9, 2k; 8)\) FHS set for \(k = 1, 2\). It is easy to verify that these parameters indeed meet the Singleton bound (1.4).

**Example 4.3.** Take \(q = 2^4 = 16\) and \(n = q+1 = 17\). We have \(p = 17\) and \(\min\{p-1, 2^3\} = 8\), thus \(1 \leq k \leq 8\). Theorem 4.1 applies to give an optimal \((17, \frac{16^{2k+1-1}}{16}, 2k; 16)\) FHS set for any \(1 \leq k \leq 8\) with respect to the Singleton bound (1.4).

We next consider cyclic codes of length \(q+1\) over \(\mathbb{F}_q\) in the case where \(q\) is an odd prime power. It is easy to verify that all the distinct \(q\)-cyclotomic cosets modulo \(q+1\) are given by

\[
C_0 = \{0\}, \quad C_{\frac{q+1}{2}} = \left\{ \frac{q+1}{2} \right\} \quad \text{and} \quad C_i = \{i, q+1-i\}, \quad \text{for } 1 \leq i \leq \frac{q+1}{2} - 1.
\]

Let \(\alpha \in \mathbb{F}_{q^2}\) be a primitive \((q+1)\)-th root of unity. In this case, let \(C\) be a cyclic code of length \(q+1\) over \(\mathbb{F}_q\) with parity-check polynomial \(h(x) = (x-1)M_1(x)\), where \(M_1(x) = (x-\alpha)(x-\alpha^{-1})\). Thus \(C\) is a \([q+1, 3, q-1]\) MDS cyclic code satisfying Theorem 8.3. From Fact 1, there is an optimal \((q+1, q(q-1), 2; q)\) FHS set with respect to the Singleton bound (1.4), where \(q\) is an odd prime power.
We claim that these parameters also meet the Peng-Fan bound (1.2). To see this, note that $M(\mathcal{F}) = 2$ and $I = \lfloor nN/\ell \rfloor = q^2 - 1$. It is clear that

$$\left\lceil \frac{2IN - (1 + 1)\ell}{(nN - 1)N} \right\rceil = \left\lceil \frac{2q(q + 1)^2(q - 1)^2 - q^3(q + 1)(q - 1)}{q(q - 1)(q(q + 1)(q - 1) - 1)} \right\rceil .$$

After simple computations we have

$$\frac{2q(q + 1)^2(q - 1)^2 - q^3(q + 1)(q - 1)}{q(q - 1)(q(q + 1)(q - 1) - 1)} > 1,$$

proving the claim.

Summarizing the previous discussion, we arrive at the following result.

**Theorem 4.4.** Let $q$ be an odd prime power. Then there is a $(q + 1, q(q - 1), 2; q)$ FHS set whose parameters meet both the Peng-Fan bound (1.2) and the Singleton bound (1.4).

**Example 4.5.** Take $q = 5^2 = 25$ and $n = 25 + 1 = 26$. Theorem 4.4 applies to give a $(26, 600, 2; 25)$ FHS set. It is easy to check that both the Peng-Fan bound (1.2) and the Singleton bound (1.4) are met.

### 4.2 Optimal FHS sets of length dividing $q + 1$

Let $n > 1$ be an odd divisor of $q + 1$. Let $\alpha \in \mathbb{F}_q$ be a primitive $n$-th root of unity. It is easy to verify that all the distinct $q$-cyclotomic cosets modulo $n$ are given by

$$C_0 = \{0\} \text{ and } C_i = \{i, n - i\}, \text{ for } 1 \leq i \leq \frac{n - 1}{2}.$$

Let $M \leq (n - 1)/2$ be the largest positive integer satisfying $\gcd(M, n) \neq 1$; if no such integer exists, then $M$ is assumed to be 0. Observe that $\gcd(\frac{n - 1}{2}, n) = 1$, thus $M \leq (n - 3)/2$. Fix a value $k$, $0 \leq k \leq (n - 3)/2 - M$. Let $\mathcal{D}$ be a cyclic code of length $n$ over $\mathbb{F}_q$ with parity-check polynomial

$$h(x) = M_{(n-1)/2}(x)M_{(n-1)/2-1}(x) \cdots M_{(n-1)/2-k}(x)$$

where $M_j(x) = \prod_{i \in C_j}(x - \alpha^i)$. $(n - 1)/2 - k \leq j \leq (n - 1)/2$. It follows that $\mathcal{D}$ is an $[n, 2(k+1), n-2k-1]$ MDS cyclic code satisfying Proposition 3.3.

The discussion above establishes the following theorem.

**Theorem 4.6.** Let $n > 1$ be an odd divisor of $q + 1$. Let $M \leq (n - 3)/2$ be the largest integer such that $\gcd(M, n) > 1$. For any integer $k$ with $0 \leq k \leq (n - 3)/2 - M$, we have an optimal $(n, (q^{2k+1}-1)/n, 2k+1; q)$ FHS set with respect to the Singleton bound (1.4).

In particular, by taking $k = 0$, we have an $(n, (q^2 - 1)/n, 1; q)$ FHS set, whose parameters meet both the Peng-Fan bound (1.2) and the Singleton bound (1.4).

Note that the parameters $(n, (q^2 - 1)/n, 1; q)$ appeared previously in [11, Corollary 12]. In comparison, our method is quite neat and clear for understanding.

**Remark 4.7.** Theorems 4.3, 4.4 and 4.6 generate three new families of optimal FHS sets with respect to the Singleton bound (1.4), some of which are also optimal with respect to the Peng-Fan bound (1.1). Most previously known optimal FHS sets in the literature are constructed with respect to the Peng-Fan bound (e.g., see [23, TABLE I] or [27, TABLE II]). Optimal FHS sets with respect to the Singleton bound have been mainly studied in [9, 11] and [25]. [11, Corollary 12] gives the first class of FHS sets simultaneously meeting the Peng-Fan bound and the Singleton bound. Theorem 4.4 gives one more example of such FHS sets.

**Example 4.8.** Take $q = 2^8 = 512$, then $q + 1 = 513 = 3^3 \times 19$. Choose $n = 27$, thus $M = 12$ and $k = 0$. It follows from Theorem 4.6 that we have a $(27, 9709, 1; 512)$ FHS set, whose parameters meet both the Peng-Fan bound (1.2) and the Singleton bound (1.4).
Example 4.9. Take $q = 2^5 = 32$, then $q + 1 = 33 = 3 \times 11$. Choose $n = 11$, thus $M = 0$ and $0 \leq k \leq 4$. It follows from Theorem 4.6 that we have an $(11, (32k^2 + 1)/11, 2k + 1; 32)$ FHS set for any $0 \leq k \leq 4$, whose parameters meet the Singleton bound 142.

Appendix

We give a detailed proof of Theorem 1.3.

Proof of Theorem 1.3. Let

\[ PF1 = \frac{(nN - \ell)n}{(nN - 1)\ell} \quad \text{and} \quad PF2 = \frac{2InN - (I + 1)\ell}{(nN - 1)N}. \]

We want to prove that $[PF1] = [PF2]$. If $nN = \ell$, then $I = 1$, thus $[PF1] = 0$ and $[PF2] = 0$. Hereafter, we assume that $nN > \ell$.

Suppose $nN = \ell I + J$ with $0 \leq J < \ell$. [25] Proposition 13 says that $[PF2] \geq [PF1]$.

\[
PF2 - PF1 = \frac{2InN - (I + 1)\ell}{(nN - 1)N} - \frac{(nN - \ell)n}{(nN - 1)\ell} \\
= \frac{2InN\ell - (I + 1)\ell^2}{(nN - 1)N\ell} - \frac{(nN - \ell)nN}{(nN - 1)\ell N} \\
= \frac{2(nN - J)nN - (nN - J + \ell)(nN - J) - (nN - \ell)nN}{(nN - 1)\ell N} \quad (4.1) \\
= \frac{(\ell - J)J}{(nN - 1)\ell N} \\
\geq 0.
\]

If $J = 0$, i.e., $\ell$ is a divisor of $nN$, then $PF1 = PF2$, and we are done. We can assume, therefore, that $\ell$ does not divide $nN$. In particular, $\ell > 1$. Note that $I = \lfloor nN/\ell \rfloor \geq 1$. Since $nN = \ell I + J$ with $J \neq 0$, we have $nN > \ell I$. Clearly, $2I \geq I + 1$, which implies that $2InN - (I + 1)\ell > 0$ and $[PF2] \geq 1$.

We first consider the case $n \leq \ell$. In this case, $[PF1] = 1$. Our task is thus to show that $[PF2] = 1$. To this end, it is enough to show that

\[ nN^2 - (2In + 1)N + I^2\ell + I\ell > 0. \quad (4.2) \]

This is obvious because, regarding the expression on the left hand side of (4.2) as a quadratic in the variable $N$, its discriminant $\Delta$ is less than 0, i.e.,

\[ \Delta = 4(I^2n^2 - I^2\ell n) + 4(In - In\ell) + 1 < 0. \]

Therefore, we have obtained that $[PF1] = [PF2] = 1$ in the case where $nN > \ell$ and $n \leq \ell$.

Now let $n = s\ell + r$ with $s > 0$ and $0 < r \leq \ell - 1$ (recall that $\ell$ is not a divisor of $n$). Let $rN = t\ell + J$ with $0 \leq T \leq \ell - 1$. We then see that

\[ PF1 = \frac{(nN - 1 + 1 - \ell)n}{(nN - 1)\ell} = s + \frac{r}{\ell} - \frac{(\ell - 1)n}{(nN - 1)\ell}. \quad (4.3) \]

Combining Equations (4.1) and (4.3),

\[ PF2 = PF1 + \frac{(\ell - J)J}{(nN - 1)\ell N} = s + \frac{r}{\ell} + \frac{(\ell - J)J}{(nN - 1)\ell N} - \frac{(\ell - 1)n}{(nN - 1)\ell}. \quad (4.4) \]

We now consider two subcases separately.

Subcase 1: $\ell \geq \frac{(\ell - 1)n}{(nN - 1)\ell}$, i.e., $r(nN - 1) \leq (\ell - 1)n$. In this subcase,

\[ rnN - r = r(nN - 1) \leq (\ell - 1)n < \ell n - r, \]

\[ r(nN - 1) \leq (\ell - 1)n \leq \ell n - r, \]

\[ r(nN - 1) \leq (\ell - 1)n < \ell n - r, \]

\[ r(nN - 1) \leq (\ell - 1)n \leq \ell n - r, \]
which implies \( rN < \ell \). This means that \( t = 0 \) and \( J = rN \). Clearly,
\[
\frac{(\ell - 1)n}{(nN - 1)\ell} - \frac{r}{\ell} = \frac{(\ell - 1)n - r(nN - 1)}{(nN - 1)\ell}.
\]
It is readily seen that \((\ell - 1)n - r(nN - 1) < (nN - 1)\ell\). Thus
\[
-1 < \frac{r}{\ell} - \frac{(\ell - 1)n}{(nN - 1)\ell} \leq 0.
\]
This leads to \([PF1] = s\). In order to show that \([PF2] = [PF1] = s\), by Equation \([4.4]\), it suffices to prove that
\[
\frac{r}{\ell} + \frac{(\ell - J)j}{(nN - 1)\ell N} - \frac{(\ell - 1)n}{(nN - 1)\ell} = \frac{rN(nN - 1) + (\ell - rN)rN - (\ell - 1)nN}{(nN - 1)\ell N} \leq 0,
\]
or equivalently,
\[
(r(nN - 1) + (\ell - rN)r - (\ell - 1)n = (n - r)(rN - \ell + 1) \leq 0.
\]
We are done because \( rN < \ell \) and \( n > r \).

Subcase 2: \( \frac{r}{\ell} > \frac{(\ell - 1)n}{(nN - 1)\ell} \). In this subcase, it is clear that \([PF1] = s + 1\). We claim that
\[
\frac{r}{\ell} + \frac{(\ell - J)j}{(nN - 1)\ell N} - \frac{\ell - 1}{(nN - 1)\ell} \leq 1,
\]
then \([PF2] = s + 1 = [PF1]\) and this will complete the proof.

Since
\[
\frac{(\ell - J)j}{(nN - 1)\ell N} = \frac{(\ell + \ell - rN)(rN - t\ell)}{(nN - 1)\ell N} = \frac{rN\ell + 2rNt\ell - r^2N^2 - t\ell^2 - t^2\ell^2}{(nN - 1)\ell N},
\]
it follows that
\[
\frac{r}{\ell} + \frac{(\ell - J)j}{(nN - 1)\ell N} - \frac{(\ell - 1)n}{(nN - 1)\ell} = \frac{rnN^2 - rN + rN\ell + 2rNt\ell - r^2N^2 - t\ell^2 - t^2\ell^2 - \ell nN + nN}{(nN - 1)\ell N}.
\]
We are left to prove that
\[
\ell nN^2 \geq rnN^2 - rN + rN\ell + 2rNt\ell - r^2N^2 - t\ell^2 - t^2\ell^2 - \ell nN + nN + \ell N. \quad (4.5)
\]
We prove Inequality \([4.5]\) by induction on \( t \geq 0 \). For the base step \( t = 0 \),
\[
\ell nN^2 - (rnN^2 - rN + rN\ell - r^2N^2 - \ell nN + nN + \ell N)
= n(\ell N^2 - r^2N^2) + rN - rN\ell + r^2N^2 + N(\ell n - n - \ell)
\geq (\ell + 1)(\ell N^2 - r^2N^2) - \ell N + r^2N^2
\]
where the last inequality holds because \( n > \ell > r \). To obtain the desired result, it is enough to show that \((\ell + 1)(\ell N - rN) - r\ell + r^2N \geq 0\). By a simple computation:
\[
(\ell + 1)(\ell N - rN) - r\ell + r^2N \geq \ell^2 - 2r\ell + \ell - r + r^2 = (\ell - r)^2 + \ell - r > 0.
\]
Now suppose Inequality \([4.5]\) holds true for any \( t > 0 \). For the inductive step,
\[
rnN^2 - rN + rN\ell + 2(t + 1)rN\ell - r^2N^2 - (t + 1)\ell^2 - (t + 1)^2\ell^2 - \ell nN + nN + \ell N
\]
is equal to
\[
2rN\ell - 2\ell^2 - 2t\ell^2 = 2\ell(rN - \ell t - \ell) = 2\ell(J - \ell) < 0.
\]
By the inductive hypothesis,

\[ rnN^2 - rN + rN\ell + 2(t + 1)rN\ell - r^2N^2 - (t + 1)^2\ell^2 - \ell nN + nN + \ell N \leq \ell nN^2. \]

This completes the proof.

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