Whitney-type extension theorems for jets
generated by Sobolev functions

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Abstract

Let \( L^m_p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), be the homogeneous Sobolev space, and let \( E \subset \mathbb{R}^n \) be a closed set. For each \( p > n \) and each non-negative integer \( m \) we give an intrinsic characterization of the restrictions \( |D^\alpha F|_E : |\alpha| \leq m \) to \( E \) of \( m \)-jets generated by functions \( F \in L^{m+1}_p(\mathbb{R}^n) \). Our trace criterion is expressed in terms of variations of corresponding Taylor remainders of \( m \)-jets evaluated on a certain family of “well separated” two point subsets of \( E \). For \( p = \infty \) this result coincides with the classical Whitney-Glaeser extension theorem for \( m \)-jets.

Our approach is based on a representation of the Sobolev space \( L^{m+1}_p(\mathbb{R}^n) \), \( p > n \), as a union of \( C^m(d)(\mathbb{R}^n) \)-spaces where \( d \) belongs to a family of metrics on \( \mathbb{R}^n \) with certain “nice” properties. Here \( C^m(d)(\mathbb{R}^n) \) is the space of \( C^m \)-functions on \( \mathbb{R}^n \) whose partial derivatives of order \( m \) are Lipschitz functions with respect to \( d \). This enables us to show that, for every non-negative integer \( m \) and every \( p \in (n, \infty) \), the very same classical linear Whitney extension operator as in [33] provides an almost optimal extension of \( m \)-jets generated by \( L^{m+1}_p \)-functions.

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1. Introduction.

1.1. Main definitions and main results.

Given \( m \in \mathbb{N} \) and \( p \in [1, \infty] \), let \( L^m_p(\mathbb{R}^n) \) be the homogeneous Sobolev space of all (equivalence classes of) real valued functions \( f \in L_{1,\text{loc}}(\mathbb{R}^n) \) whose distributional partial derivatives on \( \mathbb{R}^n \) of order \( m \) belong to \( L^p(\mathbb{R}^n) \). \( L^m_p(\mathbb{R}^n) \) is seminormed by

\[
\|f\|_{L^m_p(\mathbb{R}^n)} := \sum_{|\alpha|=m} \|D^\alpha f\|_{L^p(\mathbb{R}^n)}.
\]

As usual, we let \( W^m_p(\mathbb{R}^n) \) denote the corresponding Sobolev space of all functions \( f \in L^p(\mathbb{R}^n) \) whose distributional partial derivatives on \( \mathbb{R}^n \) of all orders up to \( m \) belong to \( L^p(\mathbb{R}^n) \). This space is normed by

\[
\|f\|_{W^m_p(\mathbb{R}^n)} := \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^p(\mathbb{R}^n)}.
\]

We recall that, by the Sobolev imbedding theorem, every function \( f \in L^m_p(\mathbb{R}^n), p > n \), can be redefined, if necessary, on a set of Lebesgue measure zero so that it belongs to the space \( C^{m-1}(\mathbb{R}^n) \). (See e.g., [24], p. 73.) Thus, for \( p > n \), we can identify each element \( f \in L^m_p(\mathbb{R}^n) \) with its unique \( C^{m-1} \)-representative on \( \mathbb{R}^n \). This will allow us to restrict our attention to the case of Sobolev \( C^{m-1} \)-functions.

Given a function \( F \in C^m(\mathbb{R}^n) \) and \( x \in \mathbb{R}^n \), we let

\[
T^m_x[F](y) = \sum_{|\alpha|\leq m} \frac{1}{\alpha!} D^\alpha F(x)(y-x)^\alpha, \quad y \in \mathbb{R}^n,
\]

denote the Taylor polynomial of \( F \) of degree \( m \) at \( x \). By \( P_m(\mathbb{R}^n) \) we denote the space of all polynomials of degree at most \( m \) defined on \( \mathbb{R}^n \).

In this paper we study the following
**Problem 1.1** Let $p \in [1, \infty]$, $m \in \mathbb{N}$, and let $E$ be a closed subset of $\mathbb{R}^n$. Suppose that, for each point $x \in E$, we are given a polynomial $P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n)$. We ask two questions:

1. How can we decide whether there exists a function $F \in L^m_p(\mathbb{R}^n)$ such that $T^{m-1}_x[F] = P_x$ for all $x \in E$?

2. Consider the $L^m_p(\mathbb{R}^n)$-norms of all functions $F \in L^m_p(\mathbb{R}^n)$ such that $T^{m-1}_x[F] = P_x$ on $E$. How small can these norms be?

This problem is a variant of a classical extension problem posed by H. Whitney in 1934 in his pioneering papers [33,34], namely: How can one tell whether a given function $f$ defined on an arbitrary subset $E \subset \mathbb{R}^n$ extends to a $C^m$-function on all of $\mathbb{R}^n$? Over the years since 1934 this problem, often called the Whitney Extension Problem, has attracted a lot of attention, and there is an extensive literature devoted to different aspects of this problem and its analogues for various spaces of smooth functions. Among the multitude of results known so far we mention those in the papers [1,4–6,10–18,22,27–30,35,36]. We refer the reader to all of these papers and references therein, for numerous results and techniques concerning this topic.

In [28] we solved Problem 1.1 for $m = 1$, $n \in \mathbb{N}$ and $p > n$. In the present paper we give a complete solution to Problem 1.1 for arbitrary $m$, $n \in \mathbb{N}$ and $p > n$.

Note that, for the case $p = \infty$, Problem 1.1 was solved by Whitney [33] and Glaeser [18]. In that case the space $L^m_\infty(\mathbb{R}^n)$ can be identified with the space $C^{m-1,1}(\mathbb{R}^n)$ of all $C^m$-functions on $\mathbb{R}^n$ whose partial derivatives of order $m - 1$ all satisfy Lipschitz conditions.

We recall the statement of the classical Whitney [33]-Glaeser [18] extension theorem: Let $E$ be an arbitrary closed subset of $\mathbb{R}^n$. There exists a $C^m$-function $F \in L^m_\infty(\mathbb{R}^n)$ such that $T^{m-1}_x[F] = P_x$ for every $x \in E$ if and only if

$$\sup_{x,y \in E, x \neq y} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_x(x) - D^\alpha P_y(y)|}{||x - y||^{m-|\alpha|}} < \infty. \quad (1.2)$$

Theorem 1.3, our main contribution in this paper, generalizes this result to the case $n < p < \infty$.

Let $P = \{P_x : x \in E\}$ be a family of polynomials of degree at most $m$ indexed by points of a given closed subset $E$ of $\mathbb{R}^n$. (Thus $P_x \in \mathcal{P}_m(\mathbb{R}^n)$ for every $x \in E$.) Following [16] we refer to $P$ as a Whitney $m$-field defined on $E$.

We say that a function $F \in C^m(\mathbb{R}^n)$ agrees with the Whitney $m$-field $P = \{P_x : x \in E\}$ on $E$, if $T^{m-1}_x[F] = P_x$ for each $x \in E$. In that case we also refer to $P$ as the Whitney $m$-field on $E$ generated by $F$ or as the $m$-jet generated by $F$. We define the $L^m_p$-“norm” of the $m$-jet $P = \{P_x : x \in E\}$ by

$$\|P\|_{m,p,E} := \inf \{\|F\|_{L^m_p(\mathbb{R}^n)} : F \in L^m_p(\mathbb{R}^n), T^{m-1}_x[F] = P_x \text{ for every } x \in E\}. \quad (1.3)$$

We also need the following notion:

**Definition 1.2** Let $\gamma \geq 1$ and let $\mathcal{A} = \{\{x_i, y_i\} : i \in I\}$ be a family of two point subsets of $\mathbb{R}^n$. We say that the family $\mathcal{A}$ is $\gamma$-sparse if there exists a collection $\{Q_i : i \in I\}$ of pairwise disjoint cubes in $\mathbb{R}^n$ such that

$$x_i, y_i \in \gamma Q_i \quad \text{and} \quad \text{diam } Q_i \leq \gamma ||x_i - y_i|| \quad \text{for all } i \in I. \quad (1.4)$$
We can now explicitly formulate the above mentioned main result of this paper.

**Theorem 1.3** Let $m \in \mathbb{N}$, $p \in (n, \infty)$, and let $E$ be a closed subset of $\mathbb{R}^n$. There exists an absolute constant $\gamma \geq 1$ for which the following result holds:

Suppose we are given a family $\mathcal{P} = \{P_x : x \in E\}$ of polynomials of degree at most $m-1$ indexed by points of $E$. There exists a $C^{m-1}$-function $F \in L^m_p(\mathbb{R}^n)$ such that

$$T^{m-1}_x[F] = P_x \quad \text{for every} \quad x \in E$$

if and only if the following quantity

$$N_{m,p,E}(\mathcal{P}) := \sup \left\{ \sum_{i=1}^{k} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{\gamma}(x_i) - D^\alpha P_{\gamma}(x_j)|^p}{\|x_i - y_i\|^{(m-|\alpha|)p-n}} \right\}^{1/p}$$

is finite. Here the supremum is taken over all finite $\gamma$-sparse collections $\{(x_i, y_i) : i = 1, \ldots, k\}$ of two point subsets of $E$.

Furthermore,

$$||\mathcal{P}||_{m,p,E} \sim N_{m,p,E}(\mathcal{P}).$$

The constants of equivalence in (1.7) depend only on $m, n$ and $p$.

We refer to this result as a variational criterion for the traces of $L^m_p$-jets.

The variational criterion describes the structure of the linear space

$$\mathcal{J}(L^m_p(\mathbb{R}^n))_{|E} := \{\mathcal{P} = \{T^{m-1}_x[F] : x \in E\} : F \in L^m_p(\mathbb{R}^n)\}$$

of all Whitney $(m - 1)$-fields on $E$ generated by $C^{m-1}$-functions belonging to $L^m_p(\mathbb{R}^n)$. We refer to the space $\mathcal{J}(L^m_p(\mathbb{R}^n))_{|E}$ as the trace jet-space of $L^m_p(\mathbb{R}^n)$ to $E$. The functional $|| \cdot ||_{m,p,E}$ defined above is a seminorm on $\mathcal{J}(L^m_p(\mathbb{R}^n))_{|E}$.

When $E = \mathbb{R}^n$ we simply write $\mathcal{J}(L^m_p(\mathbb{R}^n))$ instead of $\mathcal{J}(L^m_p(\mathbb{R}^n))_{|E}$.

The criterion (1.6) and equivalence (1.7) show which properties of $\mathcal{P}$ on $E$ control its almost optimal extension to a jet generated by a function from $L^m_p(\mathbb{R}^n)$. At first sight, this criterion seems to be extremely difficult to check “in practice” even when $E$ is a finite set. The statement of Theorem 1.3 indicates that this check requires one to verify a very large number of conditions, a number which is very much larger than the cardinality of $E$. (It is the number of all $\gamma$-sparse families of two point subsets of $E$.)

However an examination of our proof for a finite set $E$ shows that, after all, it is only necessary to deal with just one of these families of two point subsets of $E$. This is a certain special $\gamma$-sparse family which is constructed by a particular step in the proof. It is enough to examine the behavior of given field $\mathcal{P}$ only on this particular family. Furthermore, this special family of two point subsets of $E$ generates a certain graph structure on $E$ with rather nice properties.

**Definition 1.4** Let $\gamma \geq 1$ and let $\Gamma$ be a graph whose set of vertices belongs to $\mathbb{R}^n$. The set of edges of $\Gamma$ defines a family $\mathcal{A}$ of two point subsets of $E$ consisting of all pairs of points which are joined by an edge. We say that the graph $\Gamma$ is $\gamma$-sparse if the family $\mathcal{A}$ defined in this way is $\gamma$-sparse, in the sense of Definition 1.2.
In other words, $\Gamma$ is $\gamma$-sparse if there exists a family
\[ \{Q_{xy} : x, y \text{ are vertices of } \Gamma \text{ joined by an edge} \} \]
of pairwise disjoint cubes $Q_{xy}$ such that $x, y \in \gamma Q_{xy}$ and $\text{diam } Q_{xy} \leq \gamma \|x - y\|$ whenever $x$ and $y$ are two arbitrary vertices joined by an edge in the graph $\Gamma$.

The following theorem provides a refinement of the variational criterion given in Theorem 1.3 for finite subsets of $\mathbb{R}^n$.

**Theorem 1.5** Let $E$ be a finite subset of $\mathbb{R}^n$. There exists a constant $\gamma = \gamma(n) \geq 1$ and a connected $\gamma$-sparse graph $\Gamma_E$ whose set of vertices coincides with $E$ which has the following properties:

(i) The degree of each vertex of $\Gamma_E$ is bounded by a constant $C = C(n)$;

(ii) Let $m \in \mathbb{N}$, $n < p < \infty$, and let $P = \{P_x : x \in E\}$ be a Whitney $(m - 1)$-field on $E$. Then

\[
||P||_{m,p,E} \sim \left\{ \sum_{x,y \in E, x \leftrightarrow y} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|^p}{\|x - y\|^{(m-|\alpha|)p-n}} \right\}^{\frac{1}{p}} \tag{1.8}
\]

where the first sum is taken over all points $x, y \in E$ joined by an edge in the graph $\Gamma_E (x \leftrightarrow y)$.

The constants in this equivalence depend only on $m, n$ and $p$.

**Remark 1.6** Using some obvious modifications of the proof of Theorem 1.5 we can also obtain an analogue of $(1.8)$ for $p = \infty$, namely that the equivalence

\[
||P||_{m,\infty,E} \sim \sup_{x,y \in E, x \leftrightarrow y} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|}{\|x - y\|^{m-|\alpha|}} \tag{1.9}
\]

holds, for every finite set $E$, and every Whitney $(m - 1)$-field $P$ on $E$. As the notation indicates, here the supremum is taken over all points $x, y \in E$ joined by an edge in $\Gamma_E$. The constants in this equivalence depend only on $m, n$ and $p$. C.f., (1.2).

Combining the result of Theorem 1.5 with Theorem 1.3 we obtain the following

**Theorem 1.7** Let $m \in \mathbb{N}$, $p \in (n, \infty)$, and let $E$ be a closed subset of $\mathbb{R}^n$. Let $\gamma = \gamma(n) \geq 1$ and $C = C(n) \geq 1$ be the same as in Theorem 1.5. Then the following result holds:

Suppose we are given a family $P = \{P_x : x \in E\}$ of polynomials of degree at most $m - 1$ indexed by points of $E$. There exists a $C^{m-1}$-function $F \in L^m_p(\mathbb{R}^n)$ such that $T^{m-1}_x[F] = P_x$ for every $x \in E$ if and only if the following quantity

\[
\widetilde{\mathcal{N}}_{m,p,E}(P) := \sup_{\Gamma} \left\{ \sum_{x \leftrightarrow y} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|^p}{\|x - y\|^{(m-|\alpha|)p-n}} \right\}^{\frac{1}{p}} \tag{1.10}
\]

is finite. Here the supremum is taken over all finite connected $\gamma$-sparse graphs $\Gamma$ with vertices in $E$ and with the degree of each vertex bounded by $C$. The first sum in (1.10) is taken over all vertices $x, y$ of a graph $\Gamma$ joined by an edge in $\Gamma (x \leftrightarrow y)$.

Furthermore, $||P||_{m,p,E} \sim \widetilde{\mathcal{N}}_{m,p,E}(P)$ with the constants of equivalence depending only on $m, n$ and $p$. 
The next theorem gives another characterization of $L^m_p$-jets on $E$ expressed in terms of $L_p$-norms of certain maximal functions. For each family $P = \{P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\}$ of polynomials we let $P^*_m, E$ denote a certain kind of “sharp maximal function” associated with $P$ which is defined by

$$P^*_m, E(x) := \sup_{y, z \in E, y \neq z} \frac{|P_x(y) - P_z(z)|}{\|x - y\|^m + \|x - z\|^m}, \quad x \in \mathbb{R}^n. \quad (1.11)$$

**Theorem 1.8** Let $m, p, E$ and $P = \{P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\}$ be as in the statement of Theorem 1.3. Then there exists a $C^{m-1}$-function $F \in L^m_p(\mathbb{R}^n)$ such that $T_x^{m-1}[F] = P_x$ for every $x \in E$ if and only if $P^*_m, E \in L^p(\mathbb{R}^n)$.

Furthermore,

$$\|P\|_{m,p,E} \sim \|P^*_m, E\|_{L^p(\mathbb{R}^n)} \quad (1.12)$$

with the constants in this equivalence depending only on $m, n,$ and $p$.

**Remark 1.9** Let us note two interesting results related to the equivalence (1.12).

First, one can show that (1.12) becomes an equality whenever $m = 1$ and $p = \infty$, i.e,

$$\|P\|_{1,\infty,E} = \|P^*_1, E\|_{L^\infty(\mathbb{R}^n)}.$$

This easily follows from the McShane extension theorem [26] which states that every function $f \in \text{Lip}(E)$ extends to a function $F \in \text{Lip}(\mathbb{R}^n)$ such that $\|F\|_{L^\infty(\mathbb{R}^n)} = \|f\|_{\text{Lip}(E)}$. (Recall that $L^1_\infty(\mathbb{R}^n) = \text{Lip}(\mathbb{R}^n)$.)

Secondly, we can express a deep and interesting result proved by Le Gruyer [23] in our notation here by the formula

$$\|P\|_{2,\infty,E} = \frac{1}{2} \|P^*_2, E\|_{L^\infty(\mathbb{R}^n)}.$$

For further results related to the equivalence (1.12) for the case $m = 1, 2$ and $p = \infty$ we refer the reader to papers [20,21,32] and references therein. 

The case $m = 1$ merits particular attention. Theorems 1.3 and 1.8 immediately imply the following characterization of the trace space $L^1_p(\mathbb{R}^n)|_E$.

**Theorem 1.10** Let $p \in (n, \infty)$, and let $E$ be a closed subset of $\mathbb{R}^n$. Let $f$ be a function defined on $E$. Then the following three statements are equivalent:

(i) $f$ extends to a continuous function $F \in L^1_p(\mathbb{R}^n)$;

(ii) The following quantity

$$\Phi_{p,E}(f) := \sup \left\{ \sum_{i=1}^k \frac{|f(x_i) - f(y_i)|^p}{\|x_i - y_i\|^{p-n}} \right\}^{1/p}$$

is finite. Here the supremum is taken over all finite $\gamma$-sparse collections $\{\{x_i, y_i\} : i = 1, ..., k\}$ of two point subsets of $E$, and $\gamma \geq 1$ is a certain absolute constant.
(iii). The following quantity

$$\Psi_{p,E}(f) := \left( \int_{\mathbb{R}^n} \sup_{y,z \in E, y \neq z} \frac{|f(y) - f(z)|^p}{\|y - z\|^p} \, dx \right)^{1/p} < \infty.$$  

Furthermore,

$$\|f\|_{L^1_p(\mathbb{R}^n)|_E} \sim \Phi_{p,E}(f) \sim \Psi_{p,E}(f) \quad (1.13)$$

where

$$\|f\|_{L^1_p(\mathbb{R}^n)|_E} := \inf \left\{ \|F\|_{L^1_p(\mathbb{R}^n)} : F \in L^1_p(\mathbb{R}^n) \cap C(\mathbb{R}^n), F|_E = f \right\}.$$  

The constants of equivalences in (1.13) depend only on $n$ and $p$.

As we have mentioned above, the restrictions of $L^1_p(\mathbb{R}^n)$-functions to subsets of $\mathbb{R}^n$, $p > n$, have been studied in [28]. The trace criteria given in part (ii) and (iii) of Theorem 1.10 are improvements of Theorem 1.1 and Theorem 1.4, part (i), of [28].

Our next result, Theorem 1.11, states that there is a solution to Problem 1.1 which depends linearly on the initial data, i.e., the families of polynomials $\{P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\}$.

**Theorem 1.11** For every closed subset $E \subset \mathbb{R}^n$ and every $p > n$ there exists a continuous linear operator

$$\mathcal{F} : \mathcal{F}(L^m_p(\mathbb{R}^n))|_E \to L^m_p(\mathbb{R}^n)$$

such that for every Whitney $(m - 1)$-field $P = \{P_x : x \in E\} \in \mathcal{F}(L^m_p(\mathbb{R}^n))|_E$ the function $\mathcal{F}(P)$ agrees with $P$ on $E$, i.e.,

$$T_x^{m-1}[\mathcal{F}(P)] = P_x \text{ for every } x \in E.$$  

Furthermore, the operator norm of $\mathcal{F}$ is bounded by a constant depending only on $m, n,$ and $p$.

**Remark 1.12** Actually we show, perhaps surprisingly, that the very same classical linear Whitney extension operator

$$\mathcal{F}_W : \mathcal{F}(C^{m-1}(\mathbb{R}^n))|_E \to C^{m-1}(\mathbb{R}^n),$$

which was introduced in [33] for the space $\mathcal{F}(C^{m-1}(\mathbb{R}^n))|_E$ of Whitney $(m - 1)$-fields on $E$ generated by $C^{m-1}$-functions, has the properties described in Theorem 1.11. See Remark 1.17.

In fact, this “universality” of the Whitney extension operator for the scale of $(m - 1)$-jets generated by the spaces $L^m_p(\mathbb{R}^n)$ for all $p > n$ is the consequence of another result which will be formulated below, namely that it is possible to represent the space $L^m_p(\mathbb{R}^n)$ as a union (see (1.22) below) of $C^{m-1}$-spaces with certain specific Lipschitz properties of higher partial derivatives of order $m - 1$.

Our results mentioned so far deal only with homogeneous Sobolev spaces. But we also wish to treat the non-homogeneous (normed) case. We defer doing this to Section 8. There we present analogues of Theorems 1.3 and 1.11 for spaces of $(m - 1)$-jets generated by functions from the normed Sobolev space $W^m_p(\mathbb{R}^n)$. See Theorems 8.1, 8.2, and 8.12.

**1.2. Our approach: Sobolev-Poincaré inequality and $C^{m,(d)}(\mathbb{R}^n)$-spaces.**
Let us briefly describe the main ideas of our approach. Let $d$ be a metric on $\mathbb{R}^n$ and let $\text{Lip}(\mathbb{R}^n; d)$ be the space of functions on $\mathbb{R}^n$ satisfying the Lipschitz condition with respect to the metric $d$. $\text{Lip}(\mathbb{R}^n; d)$ is equipped with the standard seminorm

$$\|F\|_{\text{Lip}(\mathbb{R}^n; d)} := \sup_{x, y \in \mathbb{R}^n, x \neq y} \frac{|F(x) - F(y)|}{d(x, y)}.$$  

**Definition 1.13** Let $C^{m,(d)}(\mathbb{R}^n)$ be a space of $C^m$-functions on $\mathbb{R}^n$ whose partial derivatives of order $m$ are Lipschitz continuous on $\mathbb{R}^n$ with respect to $d$. This space is seminormed by

$$\|F\|_{C^{m,(d)}(\mathbb{R}^n)} := \sum_{|\alpha| = m} \|D^\alpha F\|_{\text{Lip}(\mathbb{R}^n; d)}.$$  

The main ingredient of our approach is a representation of the Sobolev space $L^m_p(\mathbb{R}^n)$, $p \in (n, \infty)$, as a union of $C^{m-1,(d)}$-spaces where $d$ belongs to a certain family $\mathcal{D}$ of metrics on $\mathbb{R}^n$:

$$L^m_p(\mathbb{R}^n) = \bigcup_{d \in \mathcal{D}} C^{m-1,(d)}(\mathbb{R}^n).$$  

(1.14)

See (1.22).

We obtain this representation using a slight modification of the classical Sobolev-Poincaré inequality for $L^m_p$-functions. More specifically, our aim is to reformulate the Sobolev-Poincaré inequality for $p > n$ in the form of a certain Lipschitz condition for partial derivatives of order $m - 1$ with respect to a certain metric on $\mathbb{R}^n$.

For functions $F \in L^m_p(\mathbb{R}^n)$ it is convenient to use the notation

$$\nabla^m F(x) := \left( \sum_{|\alpha| = m} (D^\alpha F(x))^2 \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n,$$

so that

$$\|F\|_{L^m_p(\mathbb{R}^n)} \sim \|\nabla^m F\|_{L^p(\mathbb{R}^n)}$$

with constants depending only on $n, m$ and $p$. See (1.1).

We recall a variant of the Sobolev-Poincaré inequality for $L^m_p(\mathbb{R}^n)$-functions which holds whenever $p > n$:

Let $q \in (n, p)$ and let $F \in L^m_p(\mathbb{R}^n)$. Then for every cube $Q \subset \mathbb{R}^n$, every $x, y \in Q$ and every multiindex $\beta$, $|\beta| \leq m - 1$, the following inequality

$$|D^\beta F(x) - D^\beta (T^m_y [F])(x)| \leq C (\text{diam } Q)^{m-|\beta|} \left( \frac{1}{|Q|} \int_Q (\nabla^m F(u))^q du \right)^{\frac{1}{q}}$$

(1.15)

holds. Here $C > 0$ is a constant depending only on $n, m$ and $q$. See, e.g. [24], p. 61, or [25], p. 55.

In particular, for every $\alpha$ which satisfies $|\alpha| = m - 1$,

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C \|x - y\| \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} (\nabla^m F(u))^q du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n.$$
where
\[ Q_{xy} := Q(x, ||x - y||). \]

Hence
\[ |D^\alpha F(x) - D^\alpha F(y)| \leq ||x - y|| \sup_{Q \ni x,y} \left( \frac{1}{|Q|} \int_Q h^q(u) \, du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n, \tag{1.16} \]

where \( h = C(n, m, q)||\nabla^m F||. \)

Let \( h \in L_p(\mathbb{R}^n) \) be an arbitrary non-negative function. Inequality (1.16) motivates us to introduce the function
\[ \delta_q(x, y : h) = ||x - y|| \sup_{Q \ni x,y} \left( \frac{1}{|Q|} \int_Q h^q(u) \, du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n. \tag{1.17} \]

By (1.16), for each \( p \in (n, \infty), q \in (n, p), \) and every \( F \in L_m^p(\mathbb{R}^n) \) there exists a non-negative function \( h \in L_p(\mathbb{R}^n) \) such that \( ||h||_{L_p(\mathbb{R}^n)} \leq C(n, p, q)||F||_{L_m^p(\mathbb{R}^n)} \) and
\[ |D^\alpha F(x) - D^\alpha F(y)| \leq \delta_q(x, y : h) \quad \text{for all } \alpha, ||\alpha|| = m - 1, \text{ and } x, y \in \mathbb{R}^n. \tag{1.18} \]

As we prove below, see Theorem 1.15 the converse statement is also true: Let \( n < q < p < \infty \) and let \( F \) be a \( C^{m-1} \)-function on \( \mathbb{R}^n. \) Suppose that there exists a non-negative function \( h \in L_p(\mathbb{R}^n) \) such that (1.18) holds. Then \( F \in L_m^p(\mathbb{R}^n) \) and \( ||F||_{L_m^p(\mathbb{R}^n)} \leq C||h||_{L_p(\mathbb{R}^n)} \) with \( C \) depending only on \( n, m, p, \) and \( q. \)

Thus we have an alternative equivalent definition of the homogeneous Sobolev space \( L_m^p(\mathbb{R}^n) \) in terms of the “Lipschitz-like” conditions (1.18) with respect to the functions \( \delta_q(h) \) whenever \( h \in L_p(\mathbb{R}^n). \)

Of course, these observations lead us to the desired representation (1.14), provided \( \delta_q(h) \) is a metric on \( \mathbb{R}^n. \)

However, in general, \( \delta_q(h) \) is not a metric. Nevertheless, we prove that the geodesic distance \( d_q(h) \) associated with the function \( \delta_q(h) \) is equivalent to \( \delta_q(h). \) Recall that, given \( x, y \in \mathbb{R}^n \) this distance is defined by the formula
\[ d_q(x, y : h) := \inf \sum_{i=0}^{m-1} \delta_q(x_i, x_{i+1} : h), \tag{1.19} \]

where the infimum is taken over all finite sequences of points \( \{x_0, x_1, ..., x_m\} \) in \( \mathbb{R}^n \) such that \( x_0 = x \) and \( x_m = y. \) In Section 2 we prove the following

**Theorem 1.14** Let \( n \leq q < \infty \) and let \( h \in L_{q,loc}(\mathbb{R}^n) \) be a non-negative function. Then for every \( x, y \in \mathbb{R}^n \) the following inequality
\[ d_q(x, y : h) \leq \delta_q(x, y : h) \leq 16 d_q(x, y : h) \tag{1.20} \]
holds.

This leads us to the following result which enables us to explicitly describe the family of metrics \( D \) which provides the representation in (1.14).
Theorem 1.15 Let \( m \in \mathbb{N} \) and let \( n < q < p < \infty \). A \( C^{m-1} \)-function \( F \) belongs to \( L_p^m(\mathbb{R}^n) \) if and only if there exists a non-negative function \( h \in L_p(\mathbb{R}^n) \) such that for every multiindex \( \alpha, |\alpha| = m - 1 \), and every \( x, y \in \mathbb{R}^n \) the following inequality

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq d_q(x, y : h) \tag{1.21}
\]

holds. Furthermore,

\[
\|F\|_{L_p^m(\mathbb{R}^n)} \sim \inf \|h\|_{L_p(\mathbb{R}^n)}.
\]

The constants of this equivalence depend only on \( m, n, q, \) and \( p \).

We prove this theorem in Section 2. By this result, for each \( p \in (n, \infty) \) and each \( q \in (n, p) \), the space \( L_p^m(\mathbb{R}^n) \) can be represented in the form

\[
L_p^m(\mathbb{R}^n) = \bigcup_{d \in \mathcal{D}_{p,q}(\mathbb{R}^n)} C^{m-1,d}(\mathbb{R}^n) \tag{1.22}
\]

where

\[
\mathcal{D}_{p,q}(\mathbb{R}^n) := \{d_q(h) : h \in L_p(\mathbb{R}^n), h \geq 0\}.
\]

In particular, this implies the following representation of the space \( L_p^1(\mathbb{R}^n) \):

\[
L_p^1(\mathbb{R}^n) = \bigcup_{d \in \mathcal{D}_{p,q}(\mathbb{R}^n)} \text{Lip}(\mathbb{R}^n; d).
\]

Representation (1.22) motivates us to study an analog of Problem 1.1 for the spaces \( C^{m,d}(\mathbb{R}^n) \) whenever \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \). The following theorem provides a solution to this problem.

Theorem 1.16 Let \( n < q < p < \infty, m \in \mathbb{N} \), and let \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \). Let

\[
P = \{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\}
\]

be a Whitney \( m \)-field defined on a closed set \( E \subset \mathbb{R}^n \). There exists a function \( F \in C^{m,d}(\mathbb{R}^n) \) which agrees with \( P \) on \( E \) if and only if the following quantity

\[
\mathcal{L}_{m,d}(P) := \sup_{x,y \in E, x \neq y} \sum_{|\alpha| \leq m} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|}{\|x - y\|^{m-|\alpha|}} d(x, y)
\]

is finite. Furthermore,

\[
\mathcal{L}_{m,d}(P) \sim \inf \left\{\|F\|_{C^{m,d}(\mathbb{R}^n)} : F \in C^{m,d}(\mathbb{R}^n), F \text{ agrees with } P \text{ on } E\right\} \tag{1.24}
\]

with constants of equivalence depending only on \( n, p, q, \) and \( m \).

Remark 1.17 When \( d \) is the Euclidean metric this result of course coincides with the Whitney-Glaeser extension theorem. See (1.2).

Theorem 1.16 is also well-known for the space \( C^{m,\omega}(\mathbb{R}^n) := C^{m,d_\omega}(\mathbb{R}^n) \) where

\[
d_\omega(x, y) = \omega(\|x - y\|), \quad x, y \in \mathbb{R}^n,
\]
and \( \omega \) is a concave non-decreasing continuous function on \([0, \infty)\) such that \( \omega(0) = 0 \). See [31], Ch. 6, §2.2.3 and §4.6.

We refer to any function \( \omega \) with these properties as a “modulus of continuity”, and we let \( \mathcal{MC} \) denote the family of all “moduli of continuity”. It should be noted that, for each \( \omega \in \mathcal{MC} \), the metric space \((\mathbb{R}^n, d_\omega)\) is known in the literature as the metric transform of \( \mathbb{R}^n \) by \( \omega \) or the \( \omega \)-metric transform of \( \mathbb{R}^n \).

In [31] it is shown that the classical Whitney extension operator \( \mathcal{F}_W \) constructed in [33] maps the space \( \mathcal{T}(C^{m, \omega}(\mathbb{R}^n))|_E \) into the space \( C^{m, \omega}(\mathbb{R}^n) \) in such a way that, for every Whitney \( m \)-field

\[
P = \{ P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E \}
\]

belonging to \( \mathcal{T}(C^{m, \omega}(\mathbb{R}^n))|_E \), the function \( \mathcal{F}_W(P) \) agrees with \( P \). Furthermore, the operator norm of \( \mathcal{F}_W \) satisfies the inequality

\[
\| \mathcal{F}_W \| \leq C(n, m).
\]

We prove that the same statement is true for an arbitrary metric \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \). Our proof of this property of \( \mathcal{F}_W \) mainly follows the classical scheme used in [31] for the metric \( d_\omega \) with \( \omega \in \mathcal{MC} \). As we show below, see Proposition 2.2, the fact that the proofs of Theorem 1.16 for \( d_\omega \) and for arbitrary \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \) are similar to each other can be explained by the following important property of metrics in \( \mathcal{D}_{p,q}(\mathbb{R}^n) \): for each \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \) there exists a mapping \( \mathbb{R}^n \ni x \mapsto \omega_x \in \mathcal{MC} \) such that

\[
d(x, y) \sim \omega_x(||x - y||), \quad x, y \in \mathbb{R}^n,
\]

with constants of equivalence depending only on \( n, p, \) and \( q \). This equivalence motivates us to refer to the metric space \((\mathbb{R}^n, d)\) as a variable metric transform of \( \mathbb{R}^n \).

Property (1.25) and the representation (1.14) also explain the above-mentioned phenomenon of the universality of the Whitney extension operator for the scale of \( L^m_p(\mathbb{R}^n) \)-spaces when \( p > n \). See Remark 2.3.

The proof of Theorem 1.8 is given in Section 5. It is a consequence of Theorem 1.16 and the representation (1.14).

Theorem 1.3 is proven in Sections 6 and 7. The difficult part of its proof is the sufficiency, which relies on a modification of the Whitney extension method [33] used in the author’s paper [29]. The main feature of that modification is that, instead of treating each Whitney cube separately, as is done in the original extension method in [33], we deal simultaneously with all members of certain families of Whitney cubes. We refer to these families of Whitney cubes as lacunae. See Section 6.

Finally, direct calculations of the \( L^m_p \)-norm of the extension obtained by the lacunary extension method lead us to criterion (1.6) which proves the sufficiency part of Theorem 1.3.

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2. Metrics on $\mathbb{R}^n$ generated by the Sobolev-Poincaré inequality.

Let us fix additional notation. Throughout the paper, $\gamma$, $\gamma_1$, $\gamma_2$, ... and $C, C_1, C_2, ...$ will be generic positive constants which depend only on parameters determining function spaces $(m, n, p, q, ...)$ such that $A/C \leq B \leq CA$.

These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation $C = C(n, p, q)$. We write $A \sim B$ if there is a constant $C \geq 1$ such that $A/C \leq B \leq CA$.

For $x \in \mathbb{R}^n$ we also set $\text{dist}(x, A) := \text{dist}([x], A)$. We put $\text{dist}(A, B) = +\infty$ and $\text{dist}(x, B) = +\infty$ whenever $B = \emptyset$. For each pair of points $z_1$ and $z_2$ in $\mathbb{R}^n$ we let $(z_1, z_2)$ denote the open line segment joining them. Given a cube $Q$ in $\mathbb{R}^n$ by $c_Q$ we denote its center, and by $r_Q$ a half of its side length. (Thus $Q = Q(c_Q, r_Q)$.)

Finally, given a function $g \in L_{1,loc}(\mathbb{R}^n)$ we let $M[g]$ denote its Hardy-Littlewood maximal function:

$$M[g](x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |g| dx, \quad x \in \mathbb{R}^n.$$ 

Here the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$ containing $x$.

2.1. The geodesic distance $d_q(h)$: a proof of Theorem 1.14

Clearly, by definition (1.19), $d_q(x, y : h) \leq \delta_q(x, y : h)$. Prove that

$$\delta_q(x, y : h) \leq 16 d_q(x, y : h). \tag{2.1}$$

Let $w$ be a weight on $\mathbb{R}^n$, i.e., a non-negative locally integrable function. Let $q > 0$ and let $\varphi_q(w) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a function defined by the formula

$$\varphi_q(x, y : w) := \|x - y\| \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q w(u) du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n. \tag{2.2}$$

Proposition 2.1 Let $n \leq q < \infty$. Then for every $x, y \in \mathbb{R}^n$ and every finite family of points $x_0 = x, x_1, ... x_{m-1}, x_m = y$ in $\mathbb{R}^n$ the following inequality

$$\varphi_q(x, y : w) \leq 16 \sum_{i=0}^{m-1} \varphi_q(x_i, x_{i+1} : w) \tag{2.3}$$

holds.

Proof. Let $K$ be an arbitrary cube in $\mathbb{R}^n$ and let $x, y \in K$. Prove that

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u) du \right)^{\frac{1}{q}} \leq 16 \sum_{i=0}^{m-1} \varphi_q(x_i, x_{i+1} : w). \tag{2.4}$$
Let 

$$\widetilde{Q} = Q(x, 2\|x - y\|) = 2Q_{xy}.$$ 

Consider two cases.

The first case. Suppose that there exists \( j \in \{0, \ldots, m - 1\} \) such that \( x_j \in 2Q_{xy} \), but 

\[ x_{j+1} \notin 4Q_{xy}. \]

Hence \( \|x - x_j\| \leq 2\|x - y\| \) and \( \|x - x_{j+1}\| \geq 4\|x - y\| \) so that

$$\|x_i - x_{j+1}\| \geq 2\|x - y\|. \quad (2.5)$$

Since \( x, y \in K \), we have \( \text{diam } K \geq \|x - y\| \) which easily implies the inclusion

$$2Q_{xy} \subset 5K. \quad (2.6)$$

Hence \( x_j \in 2Q_{xy} \subset 5K. \)

Consider the following two subcases. First let us assume that

\[ x_{j+1} \in 8K. \]

Since \( x_j \in 2Q_{xy} \subset 5K \), we have \( x_j, x_{j+1} \in 8K \). Hence, by (2.5) and definition (2.2) of \( \varphi_q \), we obtain

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u)du \right)^{\frac{1}{q}} \leq \|x - x_{j+1}\| \left( \frac{1}{|K|} \int_K w(u)du \right)^{\frac{1}{q}} \leq \|x - x_{j+1}\| \left( \frac{1}{|8K|} \int_{8K} w(u)du \right)^{\frac{1}{q}} \leq 8^\frac{q}{8} \varphi_q(x_j, x_{j+1} : w).$$

Since \( n \leq q \), we have

$$\|x - y\| \left( \frac{1}{|K|} \int_K w(u)du \right)^{\frac{1}{q}} \leq 8\varphi_q(x_j, x_{j+1} : w)$$

proving (2.4) in the case under consideration.

Now consider the second subcase where

\[ x_{j+1} \notin 8K. \]

Since \( x_j \in 5K \), we conclude that \( \|x_j - x_{j+1}\| \geq 3r_K \).

Let \( \overline{Q} := Q(x_j, 2\|x_j - x_{j+1}\|) \). Since \( x_j \in 5K \), for every \( z \in K \) we have

$$\|x_j - z\| \leq \|x_j - c_K\| + \|c_K - z\| \leq 5r_K + r_K = 6r_K \leq 2\|x_j - x_{j+1}\| = r_{\overline{Q}}$$

proving that \( \overline{Q} \supseteq K. \) Clearly, \( r_K \leq r_{\overline{Q}} \) and \( \overline{Q} \ni x_j, x_{j+1} \). 
Hence
\[ \|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} = 2^q \|x - y\| r_K^{\frac{1}{q}} \left( \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^q \|x - y\| r_K^{\frac{1}{q}} \left( \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}}. \]

Since \( x, y \in K \), we have \( \|x - y\| \leq \text{diam} K = 2r_K \). Combining this with inequality \( r_K \leq r_{\overline{Q}} \), we obtain
\[ \|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^q \|x - y\| r_K^{\frac{1}{q}} \left( \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}} \leq 2^q \|x - y\| r_{\overline{Q}} \left( \int_{\overline{Q}} w(u) \, du \right)^{\frac{1}{q}} = 2^q r_{\overline{Q}} \left( \frac{1}{|Q|} \int_Q w(u) \, du \right)^{\frac{1}{q}}. \]

Since \( r_{\overline{Q}} = 2 \|x_j - x_{j+1}\| \) and \( \frac{q}{q} \leq 1 \), we have
\[ \|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^q \|x_j - x_{j+1}\| \left( \frac{1}{|Q|} \int_Q w(u) \, du \right)^{\frac{1}{q}}. \]

But \( x_j, x_{j+1} \in \overline{Q} \) so that, by definition (2.3),
\[ \|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^{\frac{1}{q}} \leq 2^q \varphi_q(x_j, x_{j+1} : w) \]
proving (2.4).

The second case. Suppose that the assumption of the first case is not satisfied, i.e., for each \( i \in \{0, ..., m - 1\} \) such that \( x_i \in 2Q_{xy} \) we have \( x_{i+1} \in 4Q_{xy} \).

Let us define a number \( j \in \{0, 1, ..., m - 1\} \) as follows. If
\[ \{x_0, x_1, ..., x_m\} \subset 2Q_{xy}, \]
we put \( j = m - 1 \). If
\[ \{x_0, x_1, ..., x_m\} \not\subset 2Q_{xy}, \]
then there exists \( j \in 0, 1, ..., m - 1 \), such that
\[ \{x_0, x_1, ..., x_j\} \subset 2Q_{xy} \text{ but } x_{j+1} \not\in 2Q_{xy}. \]

Note that, by the assumption, \( x_{j+1} \in 4Q_{xy} \).
Prove that
\[
\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^\frac{1}{q} \leq 16 \sum_{i=0}^{j} \varphi_q(x_i, x_{i+1} : w).
\]
In fact, since \(x_{j+1} \notin 2Q_{xy} = Q(x, 2\|x - y\|)\) we have
\[
\|x_0 - x_{j+1}\| = \|x - x_{j+1}\| \geq 2\|x - y\|
\]
so that
\[
\sum_{i=0}^{j} \|x_i - x_{i+1}\| \geq \|x_0 - x_{j+1}\| \geq 2\|x - y\|.
\] (2.7)

Recall that, by (2.6), \(2Q_{xy} \subseteq 5K\) so that \(10K \supseteq 4Q_{xy}\). Hence
\[
x_0, x_1, ... , x_j, x_{j+1} \in 10K.
\] (2.8)

By (2.7),
\[
\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^\frac{1}{q} \leq \frac{1}{2} \left( \sum_{i=0}^{j} \|x_i - x_{i+1}\| \right) \left( \frac{1}{|K|} \int_K w(u) \, du \right)^\frac{1}{q} \leq \frac{(10)^\frac{q}{2}}{2} \left( \sum_{i=0}^{j} \|x_i - x_{i+1}\| \right) \left( \frac{1}{10K} \int_{10K} w(u) \, du \right)^\frac{1}{q}.
\]

Since \(10^{\frac{q}{2}} \leq 10\), we obtain
\[
\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^\frac{1}{q} \leq 5 \left( \sum_{i=0}^{j} \|x_i - x_{i+1}\| \right) \left( \frac{1}{10K} \int_{10K} w(u) \, du \right)^\frac{1}{q}.
\]

But, by (2.8), \(x_i, x_{i+1} \in 10K\) for every \(0 \leq i \leq j\), so that
\[
\|x_i - x_{i+1}\| \left( \frac{1}{10K} \int_{10K} w(u) \, du \right)^\frac{1}{q} \leq \varphi_q(x_i, x_{i+1} : w).
\]

Hence
\[
\|x - y\| \left( \frac{1}{|K|} \int_K w(u) \, du \right)^\frac{1}{q} \leq 5 \sum_{i=0}^{j} \varphi_q(x_i, x_{i+1} : w) \leq 5 \sum_{i=0}^{m-1} \varphi_q(x_i, x_{i+1} : w)
\]
proving inequality (2.4).

Thus (2.4) is proven for an arbitrary family of points \(x_0 = x, x_1, ..., x_m = y\) in \(\mathbb{R}^n\). Taking the supremum in this inequality over all cubes \(K \ni x, y\), we finally obtain the statement of the proposition. \(\square\)

Clearly, applying this proposition to \(w := h^q\) we obtain the required inequality (2.1). The proof of Theorem 1.14 is complete. \(\square\)

2.2. Variable metric transforms.

In this subsection we present several important properties of metrics from the family \(\mathcal{D}_{p,q}(\mathbb{R}^n)\). See (1.23).
Proposition 2.2 Let $n \leq q \leq p < \infty$ and let a metric $d \in \mathcal{D}_{p,q}(\mathbb{R}^n)$. Then:

(a). There exists a mapping $\mathbb{R}^n \ni x \to \omega_x \in \mathcal{MC}$ such that

\[
\frac{1}{2} d(x, y) \leq \omega_x(\|x - y\|) \leq 32 d(x, y) \quad \text{for every } x, y \in \mathbb{R}^n;
\]  

(2.9)

(b). Let $x, y, z \in \mathbb{R}^n$ and let $\lambda \geq 1$ be a constant such that $\|y - z\| \leq \lambda \|x - z\|$. Then

\[
d(y, z) \leq 32 \lambda d(x, z)
\]

(2.10)

and

\[
\frac{d(x, z)}{\|x - z\|} \leq 32 \lambda \frac{d(y, z)}{\|y - z\|};
\]

(2.11)

(c). For every $x, y \in \mathbb{R}^n$ and $z \in (x, y)$ the following inequality

\[
d(x, z) + d(y, z) \leq 64 d(x, y)
\]

holds.

Proof. (a). Since $d \in \mathcal{D}_{p,q}(\mathbb{R}^n)$, there exists a non-negative function $h \in L_p(\mathbb{R}^n)$ such that $d = d_q(h)$. Given $x \in \mathbb{R}^n$ we let $v_x$ denote a function on $\mathbb{R}_+$ defined by the formula

\[
v_x(t) := t \sup_{s \geq t} \left( \frac{1}{|Q(x, s)|} \int_{Q(x, s)} h^q(u) \, du \right)^{\frac{1}{q}}, \quad t \geq 0.
\]

(2.12)

Prove that $d(x, y) \sim v_x(\|x - y\|)$. In fact, since $y \in Q(x, s)$ whenever $s \geq \|x - y\|$, by (1.17), $v_x(\|x - y\|) \leq \delta_q(x, y : h)$.

On the other hand, for every cube $Q = Q(a, r) \ni x, y$ we have $\text{diam } Q = 2r \geq \|x - y\|$. Furthermore, $Q(a, r) \subset Q = Q(x, 2r)$ and $|Q| = 2^{n} |Q|$. Hence

\[
\|x - y\| \left( \frac{1}{|Q|} \int_Q h^q(u) \, du \right)^{\frac{1}{q}} \leq 2^{\frac{n}{q}} \|x - y\| \left( \frac{1}{|Q|} \int_Q h^q(u) \, du \right)^{\frac{1}{q}} \leq 2v_x(\|x - y\|)
\]

proving that $\delta_q(x, y : h) \leq 2v_x(\|x - y\|)$.

Combining these inequalities with inequality (1.20) of Theorem 1.14 we obtain:

\[
\frac{1}{2} d(x, y) \leq v_x(\|x - y\|) \leq 16 d(x, y).
\]

(2.13)

Prove that $v_x$ is equivalent to a “modulus of continuity” $\omega_x \in \mathcal{MC}$. Clearly, by (2.12), the function $v_x(t)/t$ is non-increasing. Let us show that $v_x$ is a non-decreasing function.

Let $0 \leq t_1 \leq t_2$. Then $v_x(t_1) = \max(I_1, I_2)$ where

\[
I_1 := t_1 \sup_{s \geq t_2} \left( \frac{1}{|Q(x, s)|} \int_{Q(x, s)} h^q(u) \, du \right)^{\frac{1}{q}} \quad \text{and} \quad I_2 := t_1 \sup_{t_1 \leq s < t_2} \left( \frac{1}{|Q(x, s)|} \int_{Q(x, s)} h^q(u) \, du \right)^{\frac{1}{q}}.
\]

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Since \( t_1 \leq t_2 \), by (2.12), \( I_1 \leq v_\times(t_2) \). On the other hand,

\[
I_2 \leq t_1 \left( \frac{1}{|Q(x, t_1)|} \int_{Q(x, t_2)} h^q(u) \, du \right)^{\frac{1}{q}} = 2^{-\frac{q}{p}} t_1 \left( \int_{Q(x, t_2)} h^q(u) \, du \right)^{\frac{1}{q}}.
\]

Hence

\[
I_2 \leq 2^{-\frac{q}{p}} t_2 \left( \int_{Q(x, t_2)} h^q(u) \, du \right)^{\frac{1}{q}} = t_2 \left( \frac{1}{|Q(x, t_2)|} \int_{Q(x, t_2)} h^q(u) \, du \right)^{\frac{1}{q}} \leq v_\times(t_2)
\]

so that

\[
v_\times(t_1) = \max\{I_1, I_2\} \leq v_\times(t_2).
\]

Let \( \omega_\times \) be the least concave majorant of \( v_\times \). Since \( v_\times \) and \( t/v_\times(t) \) are non-decreasing, for every \( t > 0 \) the following inequality

\[
v_\times(t) \leq \omega_\times(t) \leq 2v_\times(t)
\]

holds. Combining this inequality with (2.13) we obtain the required inequality (2.9).

(b). In part (a) we have proved that for each \( z \in \mathbb{R}^n \) the functions \( v_\times \) is non-decreasing and the function \( v_\times(t)/t \) is non-increasing. Hence \( v_\times(\lambda t) \leq \lambda v_\times(t) \) provided \( \lambda \geq 1 \). Therefore, by (2.13),

\[
d(y, z) \leq 2v_\times(||y - z||) \leq 2v_\times(\lambda ||x - z||) \leq 2\lambda v_\times(||x - z||) \leq 32\lambda d(x, z)
\]

proving (2.10). On the other hand, by the first inequality in (2.13),

\[
\frac{d(x, z)}{||x - z||} \leq \frac{2v_\times(||x - z||)}{||x - z||} \leq 2\lambda \frac{v_\times(||y - z||/\lambda)}{||y - z||} \leq 2\lambda \frac{v_\times(||y - z||)}{||y - z||}
\]

so that, by the second inequality in (2.13),

\[
\frac{d(x, z)}{||x - z||} \leq 32\lambda \frac{d(y, z)}{||y - z||}
\]

proving (2.11).

(c). Since \( z \in (x, y) \), we have \( ||y - z||, ||x - z|| \leq ||x - y|| \) so that by part (b)

\[
d(x, z) \leq 32 d(x, y) \quad \text{and} \quad d(y, z) \leq 32 d(x, y)
\]

proving the statement (c) and the proposition. \( \square \)

**Remark 2.3** Equivalence (2.9) motivates us to refer to the metric space \( (\mathbb{R}^n, d) \) where \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \) as a *variable metric transform* of \( \mathbb{R}^n \); see Remark 1.17. This equivalence shows that given \( x \in \mathbb{R}^n \) the local behavior of the metric \( d \) is similar to the behaviour of a certain regular metric transform \( d_{\omega_\times} := \omega_\times(||\cdot||) \) where \( \omega_\times \in \mathcal{MC} \) is a “modulus of continuity”. The function \( \omega_\times \) varies from point to point, and this is the main difference between a regular metric transform (where \( \omega_\times \) is “constant”, i.e., the same “modulus of continuity” \( \omega \) for all \( x \in \mathbb{R}^n \)) and a variable metric transform.

Nevertheless, in spite of \( \omega_\times \) changes, the metric \( d \in \mathcal{D}_{p,q}(\mathbb{R}^n) \) preserves several important properties of regular metric transforms.

For instance, let \( E \subset \mathbb{R}^n \) be a closed set and let \( x \in \mathbb{R}^n \setminus E \). Let \( \tilde{x} \in E \) be an almost nearest point to \( x \) on \( E \) with respect to the Euclidean distance, i.e., \( ||x - \tilde{x}|| \sim \text{dist}(x, E) \). Then \( \tilde{x} \) is an almost nearest to \( x \) point with respect to the variable majorant \( d \) as well.
Another example is the standard Whitney covering of \( \mathbb{R}^n \setminus E \) by a family of Whitney’s cubes. See, e.g. [31]. It is well known that this covering is universal with respect to the family

\[ \mathcal{M} = \{(\mathbb{R}^n, d_\omega), \ \omega \in \mathcal{MC} \} \]

of all metric transforms, i.e., it provides an almost optimal Whitney type extension construction for the family of Lipschitz spaces with respect to metric transforms. As we shall see below, the same property also holds for variable metric transforms.

Thus there exists a more or less complete analogy between extension methods for regular and variable metric transforms. In the next sections we present several applications of this approach to extensions of jets generated by Sobolev functions.

\[ \triangleright \]

3. Sobolev \( L^m_p \)-space as a union of \( C^{m-1,(d)} \)-spaces.

In this subsection we prove Theorem 1.15.

*(Necessity)*. The necessity part directly follows from inequality (1.18) and inequality (1.20) with \( h = C(n, p, q) \| \nabla^m F \| \).

*(Sufficiency)*. Let \( F \) be a \( C^{m-1} \)-function. Suppose there exists a non-negative function \( h \in L^p(\mathbb{R}^n) \) such that inequality (1.21) holds for every multiindex \( \alpha, |\alpha| = m-1 \). Prove that \( F \in L^m_p(\mathbb{R}^n) \) and \( \| F \|_{L^m_p(\mathbb{R}^n)} \leq C(n, p, q) \| h \|_{L^p(\mathbb{R}^n)} \).

Our proof relies on a result of Calderón [7] which characterizes Sobolev functions in terms of certain sharp maximal functions. See also [8]. Let us recall this characterization of Sobolev spaces.

Given a cube \( Q \subset \mathbb{R}^n \) and a function \( f \in L^q(Q), 0 < q \leq \infty \), we let \( E_m(f; Q) \) denote the normalized local best approximation of \( f \) on \( Q \) in \( L^q \)-norm by polynomials of degree at most \( m-1 \). More explicitly, we define

\[
E_m(f; Q) := \inf_{P \in \mathcal{P}_{m-1}(\mathbb{R}^n)} \left( \frac{1}{|Q|} \int_Q |f - P|_q \ dx \right)^{\frac{1}{q}}.
\]

Given a locally integrable function \( f \) on \( \mathbb{R}^n \), we define its *sharp maximal function* \( f^\#_m \) by letting

\[
f^\#_m(x) := \sup_{r>0} r^{-m} E_m(f; Q(x, r))_{L_q}.
\]

Calderón [7] proved that a locally integrable function \( f \in L^m_p(\mathbb{R}^n) \), \( 1 < p < \infty \), if and only if \( f^\#_m \) is in \( L^p(\mathbb{R}^n) \). Furthermore, the following equivalence

\[
\| f \|_{L^m_p(\mathbb{R}^n)} \sim \| f^\#_m \|_{L^p(\mathbb{R}^n)}
\]

holds with constants depending only on \( n, m \) and \( p \).

Let us show that \( f^\#_m \in L^p(\mathbb{R}^n) \). Our proof of this fact relies on series of auxiliary lemmas. To formulate the first of them we introduce the following notion.

We say that a metric \( d \) on \( \mathbb{R}^n \) is *pseudononconvex* if there exists a constant \( \lambda_d \geq 1 \) such that for every \( x, y, z \in \mathbb{R}^n, z \in (x, y) \), the following inequality

\[
d(x, z) + d(z, y) \leq \lambda_d d(x, y)
\]

holds.
Lemma 3.1 Let $d$ be a pseudoconvex metric on $\mathbb{R}^n$. Then for every $F \in C^{m,(\alpha)}(\mathbb{R}^n)$ and every multiindex $\beta$ with $|\beta| \leq m$ the following inequality
\[
|D^\beta F(x) - D^\beta \left(T^m_y[F]\right)(x)| \leq C\|F\|_{C^{m,(\alpha)}(\mathbb{R}^n)} \|x - y\|^{m-|\beta|} d(x,y), \quad x, y \in \mathbb{R}^n,
\]
holds. Here $C = C(m, \lambda_d)$.

Proof. For $|\beta| = m$ the lemma follows from Definition 1.13 so we can assume that $|\beta| < m$.

Our proof for this case relies on the following well known identity: Let $m > 0$ and let $F$ be a $C^m$-function on $\mathbb{R}^n$. Then for every $x, y \in \mathbb{R}^n$ the following equality
\[
F(x) = T^m_y[F](x) + m \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - y)^\alpha \int_0^1 (1 - t)^{m-1}\left[D^\alpha(D^\beta F)(x + t(x - y)) - D^\alpha(D^\beta F)(y)\right] dt
\]
holds.

Let us apply this identity to $D^\beta F$. We obtain
\[
D^\beta F(x) = T^{m-|\beta|}_y[D^\beta F](x) + (m - |\beta|) \sum_{|\alpha| = m-|\beta|} \frac{1}{\alpha!} (x - y)^\alpha \int_0^1 (1 - t)^{m-|\beta|-1}\left[D^\alpha(D^\beta F)(x + t(x - y)) - D^\alpha(D^\beta F)(y)\right] dt.
\]

Since
\[
T^{m-|\beta|}_y[D^\beta F](x) = D^\beta \left(T^m_y[F]\right)(x)
\]
for every $x, y \in \mathbb{R}^n$, we have
\[
D^\beta F(x) - D^\beta \left(T^m_y[F]\right)(x) = (m - |\beta|) \sum_{|\alpha| = m-|\beta|} \frac{1}{\alpha!} (x - y)^\alpha \int_0^1 (1 - t)^{m-|\beta|-1}\left[D^\alpha(D^\beta F)(x + t(x - y)) - D^\alpha(D^\beta F)(y)\right] dt.
\]
Hence
\[
|D^\beta F(x) - D^\beta \left(T^m_y[F]\right)(x)| \leq m \sum_{|\alpha| + |\beta| = m} ||x - y||^{|\alpha|} \sup_{z \in [x,y]} |D^{\alpha+\beta} F(z) - D^{\alpha+\beta} F(y)|.
\]
Combining this inequality with Definition 1.13, we obtain
\[
|D^\beta F(x) - D^\beta \left(T^m_y[F]\right)(x)| \leq m||x - y||^{m-|\beta|} \|F\|_{C^{m,(\alpha)}(\mathbb{R}^n)} \sup_{z \in [x,y]} d(z,y).
\]
Since $d$ is pseudoconvex, by (3.2),
\[
|D^\beta F(x) - D^\beta \left(T^m_y[F]\right)(x)| \leq \lambda_d m ||x - y||^{m-|\beta|} \|F\|_{C^{m,(\alpha)}(\mathbb{R}^n)} d(x,y)
\]
proving the lemma. □

Let us apply this result to metrics from the family $D_{p,q}(\mathbb{R}^n)$ whenever $q \in (n, p)$, see (1.23). Note that, by part (c) of Proposition 2.2, every metric $d \in D_{p,q}(\mathbb{R}^n)$ is pseudoconvex. This property of $d$ and Lemma 3.1 imply the following
Proposition 3.2 Let \(d \in D_{p,q}(\mathbb{R}^n)\) where \(q \in (n, p)\). Then for every \(F \in C^{m,d}(\mathbb{R}^n)\), every \(x, y \in \mathbb{R}^n\) and every \(\beta, |\beta| \leq m\), the following inequality
\[
|D^\beta F(x) - D^\beta (T^m_y[F])(x)| \leq C||F||_{C^{m,d}(\mathbb{R}^n)}||x - y||^{m - |\beta|} d(x, y)
\]
holds. Here \(C = C(m, n, q, p)\).

We are in a position to finish the proof of the sufficiency part of Theorem 1.15. First recall that, by (1.21), for every \(\alpha, |\alpha| = m - 1\),
\[
|D^\alpha F(x) - D^\alpha F(y)| \leq d(x, y), \quad x, y \in \mathbb{R}^n,
\]
where \(d = d_q(h)\). Thus, by Definition 1.13
\[
F \in C^{m-1,d}(\mathbb{R}^n) \quad \text{and} \quad ||F||_{C^{m-1,d}(\mathbb{R}^n)} \leq C(m, n).
\]
Furthermore, since \(h \geq 0\) and \(h \in L_p(\mathbb{R}^n)\), the metric \(d \in D_{p,q}(\mathbb{R}^n)\), see (1.23), so that, by Proposition 3.2 for every \(x, y \in \mathbb{R}^n\) the following inequality
\[
|F(x) - (T^m_y[F])(x)| \leq C||x - y||^{m-1} d(x, y)
\]
holds. Here \(C = C(m, n, q, p)\).

Hence, by Theorem 1.14 and definition (1.17),
\[
|F(x) - (T^m_y[F])(x)| \leq C||x - y||^m \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q h^\mu(u) \, du \right)^{\frac{1}{\mu}}, \quad x, y \in \mathbb{R}^n,
\]
which leads to the following inequality
\[
|F(x) - (T^m_y[F])(x)| \leq C||x - y||^m (M[h^q])^{\frac{1}{q}}(y), \quad y \in \mathbb{R}^n.
\]

Let \(Q = Q(y, r)\) be a cube in \(\mathbb{R}^n\) centered in \(y\). Then, by the latter inequality, for every \(y \in \mathbb{R}^n\),
\[
\sup_Q |F - T^m_y[F]| \leq C r^m (M[h^q])^{\frac{1}{q}}(y),
\]
so that
\[
r^{-m}E_m(F; Q)_{L_1} = r^{-m} \inf_{P \in \mathcal{F}_{m-1}(\mathbb{R}^n)} \frac{1}{|Q|} \int_Q |F - P| \, du \leq r^{-m} \frac{1}{|Q|} \int_Q |F - T^m_y[F]| \, du
\]
\[
\leq C (M[h^q])^{\frac{1}{q}}(y).
\]
Hence
\[
F^q_m(y) = \sup_{r > 0} r^{-m}E_m(F; Q(y, r))_{L_1} \leq C (M[h^q])^{\frac{1}{q}}(y).
\]
Since \(p > q\), by the Hardy-Littlewood maximal theorem,
\[
\|F^q_m\|_{L_p(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^n} M[h^q]^{\frac{1}{q}} \, du \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} (h^q)^{\frac{1}{q}} \, du \right)^{\frac{1}{q}} = C \|h\|_{L_p(\mathbb{R}^n)},
\]
so that, by (3.1), \(F \in L^m_p(\mathbb{R}^n)\) and \(\|F\|_{L^m_p(\mathbb{R}^n)} \leq C\|h\|_{L_p(\mathbb{R}^n)}\).

The proof of Theorem 1.15 is complete. \(\square\)
4. Extensions of jets generated by $C^{m,d}(\mathbb{R}^n)$-functions.

In this section we prove Theorem 1.16.

**Necessity.** Suppose that there exists a function $F \in C^{m,d}(\mathbb{R}^n)$ which agrees with the Whitney $m$-field $P = \{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\}$, i.e., $T^m_x[F] = P_x$ for every $x \in E$. Then, by Proposition 3.2,

$$L_{m,d}(P) := \sum_{|\alpha| \leq m} \sup_{x,y \in E, x \neq y} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|}{|x - y|^{m-|\alpha|} d(x,y)} \leq C \|F\|_{C^{m,d}(\mathbb{R}^n)}$$

where $C = C(m, n, p, q)$. This proves the necessity and the inequality

$$L_{m,d}(P) \leq C \inf \left\{ \|F\|_{C^{m,d}(\mathbb{R}^n)} : F \in C^{m,d}(\mathbb{R}^n), F \text{ agrees with } P \text{ on } E \right\}.$$

**(Sufficiency.)** Let $P = \{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\}$ be a Whitney $m$-field on $E$ and let $\lambda := L_{m,d}(P)$. Suppose that $\lambda < \infty$. Thus for every multiindex $\alpha$, $|\alpha| \leq m$, and every $x, y \in E$ the following inequality

$$|D^\alpha P_x(x) - D^\alpha P_y(x)| \leq |x - y|^{m-|\alpha|} d(x,y) \lambda \quad (4.1)$$

holds.

Prove the existence of a function $F \in C^{m,d}(\mathbb{R}^n)$ such that $T^m_x[F] = P_x$ for every $x \in E$, and $\|F\|_{C^{m,d}(\mathbb{R}^n)} \leq C \lambda$ where $C$ is a positive constant depending only on $m, n, p, q$.

We construct $F$ with the help of a slight modification of the classical Whitney extension method which we present below.

Since $E$ is a closed set, the set $\mathbb{R}^n \setminus E$ is open so that it admits a Whitney covering by a family $W_E$ of non-overlapping cubes. See, e.g., [31], or [19]. These cubes have the following properties:

(i). $\mathbb{R}^n \setminus E = \cup\{Q : Q \in W_E\}$;

(ii). For every cube $Q \in W_E$ we have

$$\text{diam } Q \leq \text{dist}(Q, E) \leq 4 \text{ diam } Q. \quad (4.2)$$

We are also needed certain additional properties of Whitney’s cubes which we present in the next lemma. These properties easily follow from constructions of the Whitney covering given in [31] and [19].

Given a cube $Q \subset \mathbb{R}^n$ let $Q^* := \frac{1}{2} Q$.

**Lemma 4.1** (1). If $Q, K \in W_E$ and $Q^* \cap K^* \neq \emptyset$, then

$$\frac{1}{4} \text{ diam } Q \leq \text{ diam } K \leq 4 \text{ diam } Q;$$

(2). For every $K \in W_E$ there are at most $N = N(n)$ cubes from the family $W_E^* := \{Q^* : Q \in W_E\}$ which intersect $K$;

(3). If $Q, K \in W_E$, then $Q^* \cap K^* \neq \emptyset$ if and only if $Q \cap K \neq \emptyset$.

Let $\Phi_E := \{\varphi_Q : Q \in W_E\}$ be a smooth partition of unity subordinated to the Whitney decomposition $W_E$. Recall the main properties of this partition.

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Lemma 4.2 The family of functions $\Phi_E$ has the following properties:

(a) $\varphi_Q \in C^\infty(\mathbb{R}^n)$ and $0 \leq \varphi_Q \leq 1$ for every $Q \in W_E$;

(b) $\text{supp } \varphi_Q \subset Q^*(:= \frac{9}{8}Q)$, $Q \in W_E$;

(c) $\sum\{\varphi_Q(x) : Q \in W_E\} = 1$ for every $x \in \mathbb{R}^n \setminus S$;

(d) For every cube $Q \in W_E$, every $x \in \mathbb{R}^n$ and every multiindex $\beta$, $|\beta| \leq m$, the following inequality

$$|D^\beta \varphi_Q(x)| \leq C(n,m)(\text{diam } Q)^{-|\beta|}$$

holds.

Let $\theta \geq 1$ be a constant. Let $Q \in W_E$ be a Whitney cube and let $A \in E$ be a point such that

$$\text{dist}(A, Q) \leq \theta \text{dist}(Q, E).$$

(4.3)

We refer to $A$ as a $\theta$-nearest point to the cube $Q$.

Clearly, a point $A \in E$ is a nearest point to $Q$ on $E$ if and only if $A$ is a 1-nearest point to $Q$. Also it can be readily seen that

$$A \in (8\theta + 1) Q$$

(4.4)

provided $A \in E$ is a $\theta$-nearest point to a Whitney cube $Q \in W_E$. Conversely, if $A \in (\gamma Q) \cap E$ where $Q \in W_E$ and $\gamma > 0$ is a constant, then

$$A \text{ is a } \frac{\gamma + 1}{2} \text{- nearest point to } Q.$$  

(4.5)

Suppose that to every cube $Q \in W_E$ we have assigned a $\theta$-nearest point $A_{Q,\theta} \in E$. For the sake of brevity we denote this point by $a_Q$; thus $a_Q = A_{Q,\theta}$. In particular, by (4.3) and (4.4),

$$\text{dist}(a_Q, Q) \leq \theta \text{dist}(Q, E) \text{ and } a_Q \in (8\theta + 1) Q \text{ for every } Q \in W_E.$$  

(4.6)

By $P_Q$ we denote the polynomial $P_{a_Q}$. Finally, we define the extension $F$ by the Whitney extension formula:

$$F(x) := \begin{cases} 
P_s(x), & x \in E, \\
\sum_{Q \in W_E} \varphi_Q(x)P^Q(x), & x \in \mathbb{R}^n \setminus E. 
\end{cases}$$

(4.7)

Let us note that the metric $d$ is continuous with respect to the Euclidean distance, i.e., for every $x \in \mathbb{R}^n$

$$d(x, y) \to 0 \text{ as } \|x - y\| \to 0.$$  

(4.8)

In fact, since $d \in D_{p,q}(\mathbb{R}^n)$, by definition (1.23), $d = d_q(h)$ for some non-negative function $h \in L_p(\mathbb{R}^n)$. Then, by Theorem 1.14

$$d(x, y) \sim \delta_q(x, y : h) = \|x - y\| \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q h^q(u) \, du \right)^{\frac{1}{q}}.$$
Since \( n \leq q, h \in L_{q, loc}(\mathbb{R}^n) \) and \( \text{diam } Q_{xy} = 2 \|x - y\| \to 0 \) as \( \|x - y\| \to 0 \), we have \( d(x, y) \to 0 \) proving (4.8).

Clearly, if a cube \( Q \ni x, y \), then \( |Q| \geq \|x - y\|^n \). (Recall that we measure the distance in the uniform metric.) Since \( n < q < p \), we obtain

\[
d(x, y) \leq C \|x - y\| \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q h^q(u) \, du \right)^{\frac{1}{q}} \leq C \|x - y\| \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q h^p(u) \, du \right)^{\frac{1}{p}}
\]

\[
\leq C \|x - y\| \|x - y\|^{-\frac{m}{n}} \|h\|_{L_p(\mathbb{R}^n)} = C \|x - y\|^{1 - \frac{m}{n}} \|h\|_{L_p(\mathbb{R}^n)} \to 0
\]
as \( \|x - y\| \to 0 \).

Hence, by (4.1), for every multiindex \( \beta, |\beta| \leq m \), we have

\[
D^\beta P_x(x) - D^\beta P_y(x) = o(\|x - y\|^{m - |\beta|}), \quad x, y \in E,
\]

so that the \( m \)-jet \( P = \{P_x \in \mathcal{P}_m(\mathbb{R}^n) : x \in E\} \) satisfies the hypothesis of the Whitney extension theorem \([33]\). In this paper Whitney proved that, whenever \( \theta = 1 \) (i.e., \( a_Q \) is a point nearest to \( Q \) on \( E \)) the extension \( F : \mathbb{R}^n \to \mathbb{R} \) defined by formula (4.7) is a \( C^m \)-function which agrees with the Whitney \( m \)-field \( P \), i.e.,

\[
D^\beta F(x) = D^\beta P_x(x) \quad \text{for every } x \in E \quad \text{and every } \beta, |\beta| \leq m. \tag{4.9}
\]

Stein \([31]\), p. 172, noticed that the approach suggested by Whitney in \([33]\) works for any mapping

\[
W_E \ni Q \mapsto a_Q \in E
\]

provided \( a_Q \) is a \( \theta \)-nearest point to \( Q \in W_E \). In particular, the property \((4.9)\) holds for such a choice of the point \( a_Q \).

Prove that \( \|F\|_{C^{m, (0)(\mathbb{R}^n)}} \leq C \lambda \), i.e., for every multiindex \( \alpha, |\alpha| = m \), and every \( x, y \in \mathbb{R}^n \) the following inequality

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C \lambda \, d(x, y) \tag{4.10}
\]

holds. Here \( C = C(m, n, p, \theta) \).

Consider four cases.

The first case: \( x, y \in E \). Since \( P_y \in \mathcal{P}_m(\mathbb{R}^n) \), for every multiindex \( \alpha \) of order \( m \) the function \( D^\alpha P_y \) is a constant function. In particular, \( D^\alpha P_y(x) = D^\alpha P_y(y) \). Hence, by \((4.9)\),

\[
|D^\alpha F(x) - D^\alpha F(y)| = |D^\alpha P_x(x) - D^\alpha P_y(y)| = |D^\alpha P_x(x) - D^\alpha P_y(x)|
\]

so that, by \((4.1)\),

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq \lambda \, d(x, y)
\]

proving (4.10) in the case under consideration.

The second case: \( x \in E, y \in \mathbb{R}^n \setminus E \). Given a Whitney cube \( K \in W_E \) let

\[
T(K) := \{Q \in W_E : Q \cap K \neq \emptyset\} \tag{4.11}
\]

be a family of all Whitney cubes touching \( K \).
Lemma 4.3 Let $K \in W_E$ be a Whitney cube and let $y \in K^\circ = \frac{9}{8} K$. Then for every multiindex $\alpha$ the following inequality

$$|D^\alpha F(y) - D^\alpha P_{a_k}(y)| \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - |\alpha|} |D^\xi P_{a_Q}(a_K) - D^\xi P_{a_k}(a_K)|$$

holds. Here $C$ is a constant depending only on $n, m, \alpha$ and $\theta$.

Proof. Note that, by part (2) of Lemma 4.1, $\#T(K) \leq N(n)$, and, by part (3) of this lemma,

$$T(K) = \{Q \in W_E : Q^* \cap K^* \neq \emptyset\}.$$

Recall that $P^Q = P_{a_Q}$ and $a_Q \in 9Q$ for every $Q \in W_E$. Let us estimate the quantity

$$I := |D^\alpha F(y) - D^\alpha P_{a_k}(y)|.$$

By formula (4.7) and by part (c) of Lemma 4.1,

$$F(y) - P_{a_k}(y) = \sum_{Q \in W_E} \phi_Q(y)(P^Q(y) - P_{a_k}(y))$$

so that, by part (b) of Lemma 4.1 and by definition (4.11),

$$F(y) - P_{a_k}(y) = \sum_{Q \in T(K)} \phi_Q(y)(P^Q(y) - P_{a_k}(y)).$$

Hence

$$I := |D^\alpha F(y) - D^\alpha P_{a_k}(y)| \leq \sum_{Q \in T(K)} |D^\alpha(\phi_Q(y)(P^Q(y) - P_{a_k}(y)))|$$

so that

$$I \leq \sum_{Q \in T(K)} A_Q(y; \alpha) \tag{4.12}$$

where

$$A_Q(y; \alpha) := |D^\alpha(\phi_Q(y)(P^Q(y) - P_{a_k}(y)))|.$$

Let $Q \in T(K)$. Then

$$A_Q(y; \alpha) \leq C \sum_{|\beta| + |y| = |\alpha|} |D^\beta \phi_Q(y)||D^\gamma(P^Q(y) - P_{a_k}(y))|$$

so that, by part (d) of Lemma 4.2,

$$A_Q(y; \alpha) \leq C \sum_{|\beta| + |y| = |\alpha|} (\text{diam } Q)^{-|\beta|} |D^\gamma(P^Q(y) - P_{a_k}(y))|.$$

Since $Q \cap K \neq \emptyset$, by part (1) of Lemma 4.1 diam $Q \sim$ diam $K$ so that

$$A_Q(y; \alpha) \leq C \sum_{|\beta| + |y| = |\alpha|} (\text{diam } K)^{-|\beta|} |D^\gamma(P^Q(y) - P_{a_k}(y))|. \tag{4.13}$$
Let us estimate the distance between \(a_Q\) and \(y\). By (4.2) and (4.6),

\[
\|a_Q - y\| \leq \text{diam } K^* + \text{diam } Q + \text{dist}(a_Q, Q)
\leq 2 \text{diam } K + \text{diam } Q + \theta \text{dist}(Q, E)
\leq 2 \text{diam } K + (1 + 4\theta) \text{diam } Q.
\]

Since \(Q \cap K \neq \emptyset\), by part (1) of Lemma 4.1,

\[
\|a_Q - y\| \leq 2 \text{diam } K + 4(1 + 4\theta) \text{diam } K \leq 22\theta \text{ diam } K. \tag{4.14}
\]

Let

\[
\tilde{P}_Q := P^z(Q) - P_{ak} = P_{a_Q} - P_{ak}.
\]

Let us estimate the quantity \(|D^y \tilde{P}_Q(y)|\). Since \(\tilde{P}_Q \in \mathcal{P}_m(\mathbb{R}^n)\), we can represent this polynomial in the form

\[
\tilde{P}_Q(z) = \sum_{|\xi| \leq m} \frac{1}{\xi!} D^\xi \tilde{P}_Q(a_K) (z - a_K)^\xi.
\]

Hence

\[
D^y \tilde{P}_Q(z) = \sum_{|\gamma| \leq |\xi| \leq m} \frac{1}{(\xi - \gamma)!} D^\xi \tilde{P}_Q(a_K) (z - a_K)^{\xi-\gamma}
\]

so that

\[
|D^y \tilde{P}_Q(y)| \leq C \sum_{|\gamma| \leq |\xi| \leq m} |D^\xi \tilde{P}_Q(a_K)| \|y - a_K\|^{|\xi| - |\gamma|} \leq C \sum_{|\gamma| \leq |\xi| \leq m} (\text{diam } K)^{|\xi| - |\gamma|} |D^\xi \tilde{P}_Q(a_K)|.
\]

Combining this inequality with (4.13) we obtain

\[
A_Q(y; \alpha) \leq C \sum_{|\beta| + |\gamma| = |\alpha|} (\text{diam } K)^{-|\beta|} |D^\gamma \tilde{P}_Q(y)|
\leq C \sum_{|\beta| + |\gamma| = |\alpha|} (\text{diam } K)^{-|\beta|} \sum_{|\gamma| \leq |\xi| \leq m} (\text{diam } K)^{|\xi| - |\gamma|} |D^\xi \tilde{P}_Q(a_K)|
\leq C \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - |\alpha|} |D^\xi \tilde{P}_Q(a_K)|.
\]

Hence, by (4.12),

\[
I \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - |\alpha|} |D^\xi \tilde{P}_Q(a_K)|
\]

proving the lemma. \(\square\)

Let us apply Lemma 4.3 to an arbitrary multiindex \(\alpha\) of order \(m + 1\). Since \(D^\alpha P = 0\) for every polynomial \(P \in \mathcal{P}_m(\mathbb{R}^n)\), we obtain the following statement.

**Lemma 4.4** Let \(K \in W_E\) be a Whitney cube and let \(y \in K^* = \frac{9}{8}K\). Then for every multiindex \(\alpha\), \(|\alpha| = m + 1\), the following inequality

\[
|D^\alpha F(y)| \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{|\xi| - m - 1} |D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)|
\]

holds. Here \(C\) is a constant depending only on \(m, n,\) and \(\theta\).
Lemma 4.5 Let \( x \in E \) and let \( K \in W_E \) be a Whitney cube. Then for every \( y \in K \) and every \( \alpha, |\alpha| = m \), the following inequality

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C \left( |D^\alpha P_x(x) - D^\alpha P_{a_K}(x)| + \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{\xi-m} |D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)| \right)
\]

holds. Here \( C \) is a constant depending only on \( n, m \) and \( \theta \).

Proof. We have

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq |D^\alpha F(x) - D^\alpha P_{a_K}(y)| + |D^\alpha P_{a_K}(y) - D^\alpha F(y)| = I_1 + I_2.
\]

Since \( P_{a_K} \in P_m(\mathbb{R}^n) \) and \( |\alpha| = m \), the function \( D^\alpha P_y \) is a constant function so that

\[
D^\alpha P_{a_K}(y) = D^\alpha P_{a_K}(x).
\]

Since \( x \in E \), by (4.9), \( D^\alpha F(x) = D^\alpha P_x(x) \) so that, by (4.1),

\[
I_1 := |D^\alpha F(x) - D^\alpha P_{a_K}(y)| = |D^\alpha P_x(x) - D^\alpha P_{a_K}(x)|.
\]

It remains to apply Lemma 4.3 to \( I_2 := |D^\alpha F(y) - D^\alpha P_{a_K}(y)| \), and the lemma follows. \( \square \)

Let us prove inequality (4.10) for arbitrary \( x \in E, y \in \mathbb{R}^n \setminus E \) and \( \alpha, |\alpha| = m \). Let \( y \in K \) for some \( K \in W_E \). By Lemma 4.5

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq C\{J_1 + J_2\}
\]

where

\[
J_1 := |D^\alpha P_x(x) - D^\alpha P_{a_K}(x)|
\]

and

\[
J_2 := \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{\xi-m} |D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)|
\]

First let us estimate \( J_2 \). By (4.1), for every \( \xi, |\xi| \leq m \), and every \( Q \in T(K) \), we have

\[
|D^{\xi} P_{a_Q}(a_K) - D^{\xi} P_{a_K}(a_K)| \leq \lambda \|a_Q - a_K\|^{m-|\xi|} d(a_Q, a_K).
\]

Hence

\[
J_2 \leq \lambda \sum_{Q \in T(K)} \sum_{|\xi| \leq m} (\text{diam } K)^{\xi-m} \|a_Q - a_K\|^{m-|\xi|} d(a_Q, a_K).
\]

But, by (4.14),

\[
\|a_Q - a_K\| \leq \|a_Q - y\| + \|y - a_K\| \leq (22\theta + 1) \text{diam } K
\]

proving that

\[
J_2 \leq C(n, m, \theta) \lambda \sum_{Q \in T(K)} d(a_Q, a_K).
\]
Now prove that for some constant $C = C(n, p, q, \theta)$

$$d(a_Q, y) \leq C \, d(x, y) \quad \text{for every } Q \in T(K).$$  \hfill (4.17)

In fact, by (4.14), $\|a_Q - y\| \leq 22\theta \, \text{diam } K$. Since $x \in E$ and $y \in K$, by (4.2),

$$\text{diam } K \leq 4 \, \text{dist}(K, E) \leq 4\|x - y\|$$

so that $\|a_Q - y\| \leq 88\theta \|x - y\|$. Hence, by part (b) of Proposition 2.2, see inequality (2.10),

$$d(a_Q, y) \leq C \, d(x, y) \text{ proving (4.17)}.$$  

Now we have

$$d(a_Q, a_K) \leq d(a_Q, y) + d(y, a_K) \leq C \, d(x, y)$$

so that, by (4.16),

$$J_2 \leq C \, \lambda \#T(K) \, d(x, y) \leq C \, \lambda \, d(x, y).$$

See part (2) of Lemma 4.1.

On the other hand, by (4.1) and (4.17),

$$J_1 := |D^\alpha P(x) - D^\alpha P_{a_K}(x)| \leq \lambda \, d(x, a_K) \leq \lambda (d(x, y) + d(y, a_K)) \leq C \, \lambda \, d(x, y).$$

Finally,

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C \{J_1 + J_2\} \leq C \lambda \, d(x, y).$$

The third case: $y \in K, K \in W_E$ and $x \in \mathbb{R}^n \setminus K^*$. Since $K^* = \frac{9}{8} K$ and $x \not\in K^*$, we have

$$\|x - y\| \geq \frac{1}{10} \, \text{diam } K.$$

Let $a \in E$ be a point nearest to $x$ on $E$. Then

$$\|a - x\| = \text{dist}(x, E) \leq \text{dist}(y, E) + \|x - y\| \leq \text{dist}(K, E) + \text{diam } K + \|x - y\|$$

so that, by (4.2),

$$\|a - x\| \leq 4 \, \text{diam } K + \text{diam } K + \|x - y\| \leq 81\|x - y\|.$$

Hence, by part (b) of Claim 2.2, see (2.10), $d(a, x) \leq C \, d(x, y)$.

We have

$$\|y - a\| \leq \|x - y\| + \|x - a\| \leq 82\|x - y\|$$

so that again, by (2.10), $d(y, a) \leq C \, d(x, y)$. We obtain

$$|D^\alpha F(x) - D^\alpha F(y)| \leq |D^\alpha F(x) - D^\alpha F(a)| + |D^\alpha F(a) - D^\alpha F(y)|$$

so that, by the result proven in the second case,

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C \lambda (d(x, a) + d(y, a)) \leq C \lambda \, d(x, y).$$

The fourth case: $y \in K, x \in K^* \text{ where } K \in W_E$. The proof of inequality (4.10) in this case is based on the next
Lemma 4.6  Let $K \in W_E$ be a Whitney cube and let $x, y \in K^*$. Then for every multiindex $\alpha, |\alpha| = m$, the following inequality

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C \|x - y\| \sum_{Q \in T(K)} \sum_{|\beta| \leq m} (\text{diam } K)^{|\beta| - m} |D^\beta P_{a_Q}(a_K) - D^\beta P_{a_K}(a_K)|$$

holds. Here $C$ is a constant depending only on $n, m$ and $\theta$.

Proof. Note that the function $F|_{\mathbb{R}^n \setminus E} \in C^\infty(\mathbb{R}^n \setminus E)$, see formula (4.7), so that, by the Lagrange theorem, for every $\alpha, |\alpha| = m$, there exists $z \in [x, y]$ such that

$$|D^\alpha F(x) - D^\alpha F(y)| \leq C \|x - y\| \sum_{|\beta| = m + 1} |D^\beta F(z)|.$$

(4.18)

Since $x, y \in K^*$, the point $z \in K^*$ as well.

Combining this inequality with Lemma 4.4 we obtain the statement of the lemma.

We are in a position to prove inequality (4.10) for arbitrary $y \in K$ and $x \in K^*$. By inequality (4.1), for every cube $Q \in T(K)$ and every $\xi, |\xi| \leq m$,

$$|D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)| \leq \lambda d(a_Q, a_K)(\text{diam } K)^{m-|\xi|}$$

so that, by Lemma 4.18,

$$I := |D^\alpha F(x) - D^\alpha F(y)|$$

$$\leq C \|x - y\| \sum_{Q \in T(K)} \sum_{|\beta| \leq m} (\text{diam } K)^{|\beta| - m} |D^\beta P_{a_Q}(a_K) - D^\beta P_{a_K}(a_K)|$$

$$\leq C \|x - y\| \sum_{Q \in T(K)} \sum_{|\beta| \leq m} (\text{diam } K)^{|\beta| - m} (\lambda d(a_Q, a_K)(\text{diam } K)^{m-|\beta|})$$

$$\leq C \lambda \|x - y\| \sum_{Q \in T(K)} d(a_Q, a_K).$$

Note that, by (4.15) and (4.2),

$$\|a_Q - y\| \leq \|a_Q - a_K\| + \|a_K - y\| \leq 23 \text{diam } K + \text{diam } K + \text{dist}(K, E)$$

$$\leq 23 \cdot 4 \text{dist}(K, E) + 4 \text{dist}(K, E) + \text{dist}(K, E)$$

$$= 97 \text{dist}(K, E) \leq 97 \|y - a_K\|.$$

Hence, by part (b) of Claim 2.2, see (2.10), $d(a_Q, y) \leq C d(y, a_K)$ so that

$$d(a_Q, a_K) \leq d(a_Q, y) + d(y, a_K) \leq C d(y, a_K).$$

This implies the following inequality

$$I \leq C \lambda \left(\#T(K)\right) \frac{\|x - y\|}{\text{diam } K} d(y, a_K).$$

But, by part (2) of Lemma 4.1, $\#T(K) \leq N(n)$ so that

$$I \leq C \lambda \frac{\|x - y\|}{\text{diam } K} d(y, a_K).$$
Since \( x, y \in K^* \), the distance \( ||x - y|| \leq \text{diam } K^* = \frac{9}{5} \text{diam } K \). But

\[
\text{diam } K \leq 4 \text{dist}(K, E) \leq 4||y - a_k||
\]

so that \( ||x - y|| \leq 5||y - a_k|| \). Therefore, by part (b) of Proposition 2.2 see (2.11),

\[
\frac{||x - y||}{||y - a_k||} \leq C d(y, a_k) \leq C d(x, y).
\]

On the other hand,

\[
||y - a_k|| \leq \text{diam } K + \text{dist}(a_k, K) = \text{diam } K + \text{dist}(K, E)
\]

\[
\leq \text{diam } K + 4 \text{ diam } K = 5 \text{ diam } K.
\]

Finally,

\[
I \leq C\lambda \frac{||x - y||}{\text{diam } K} d(y, a_k) \leq 5 C\lambda \frac{||x - y||}{||y - a_k||} d(y, a_k) \leq C\lambda d(x, y).
\]

The proof of Theorem 1.16 is complete. □

**Remark 4.7** For \( m = 0 \) the statement of Theorem 1.16 is true for an arbitrary metric \( d \) on \( \mathbb{R}^n \). In fact, in this case \( C^{0,(d)}(\mathbb{R}^n) = \text{Lip}(\mathbb{R}^n; d) \). By the McShane extension theorem [26], every function \( f \in \text{Lip}(E; d) \) extends to a function \( F \in \text{Lip}(\mathbb{R}^n; d) \) such that \( ||F||_{\text{Lip}(\mathbb{R}^n; d)} = ||f||_{\text{Lip}(E; d)} \). This extension property of Lipschitz functions coincides with the statement of Theorem 1.16 for \( m = 0 \). (Furthermore, in this case the equivalence (1.24) is actually an equality).

We also note that, by the McShane extension formula, the function \( F \) can be chosen in the form

\[
F(x) = \inf_{y \in E} \{ f(y) + d(x, y) \}, \quad x \in \mathbb{R}^n.
\]

This observation enables us to simplify considerably an almost optimal algorithm for extension of functions from the Sobolev space \( L^1_p(\mathbb{R}^n) \). See Remark 5.1 <

5. Extensions of \( L^m_p(\mathbb{R}^n) \)-jets: a proof of Theorem 1.8

(Necessity.) Let \( P = \{ P_x : x \in E \} \) be a Whitney \((m - 1)\)-field so that \( P_x \in \mathcal{P}_{m-1}(\mathbb{R}^n) \) for every \( x \in E \). Let \( F \in L^m_p(\mathbb{R}^n) \) be a \( C^{m-1} \)-function such that \( T^{m-1}_x[F] = P_x \) for all \( x \in E \). Prove that

\[
||P^d_{m,E}||_{L^p(\mathbb{R}^n)} \leq C ||F||_{L^m_p(\mathbb{R}^n)}
\]

with \( C = C(m, n, p) \).

Let \( q = (p + n)/2 \), and let \( Q \) be a cube in \( \mathbb{R}^n \). Let \( y, z \in Q \cap E \) and let \( \tilde{P} := P - P_z \). Let

\[
I_Q := \left( \frac{1}{|Q|} \int_Q (\nabla^m F(u))^q du \right)^{\frac{1}{q}}.
\]

Then, by inequality (1.15), for every multiindex \( \alpha, |\alpha| \leq m - 1 \), the following inequality

\[
|D^\alpha \tilde{P}(y)| (\text{diam } Q)^{q-m} \leq C I_Q
\]

holds. Here \( C = C(m, n, p) \).
Prove that for every \( x \in Q \)

\[
|\tilde{P}(x)| (\text{diam } Q)^{-m} \leq C I_Q. 
\] (5.2)

In fact, since \( \tilde{P} \in \mathcal{P}_{m-1}(\mathbb{R}^n) \),

\[
\tilde{P}(x) = \sum_{|\alpha| \leq m-1} \frac{1}{\alpha!} D^\alpha \tilde{P}(y) (x - y)^\alpha
\]

so that

\[
|\tilde{P}(x)| \leq C \sum_{|\alpha| \leq m-1} |D^\alpha \tilde{P}(y)| |x - y|^{|\alpha|} \leq C \sum_{|\alpha| \leq m-1} |D^\alpha \tilde{P}(y)| (\text{diam } Q)^{|\alpha|}.
\]

Combining this inequality with (5.1), we obtain the required inequality (5.2).

Let us apply this inequality to the cube \( Q = Q(x, R) \) where \( R := ||x - y|| + ||x - z||. \) (Clearly, \( Q \ni y, z \).) We obtain that for every \( y, z \in E \) and every \( x \in \mathbb{R}^n \)

\[
\frac{|P_y(x) - P_z(x)|}{||x - y||^m + ||x - z||^m} \leq C \left( \frac{1}{|Q(x, R)|} \int_{Q(x,R)} (\nabla^m F(u))^p du \right)^{\frac{1}{p}}
\]

\[
\leq C (M[|\nabla^m F|^q])^{\frac{1}{q}}(x).
\]

Hence

\[
P^q_{m,E}(x) \leq C (M[|\nabla^m F|^q])^{\frac{1}{q}}(x),
\]

see (1.11), so that

\[
\|P^q_{m,E}\|_{L_p(\mathbb{R}^n)} \leq C \|M[|\nabla^m F|^q]\|_{L_p(\mathbb{R}^n)}^{\frac{1}{q}} \|F\|_{L_p(\mathbb{R}^n)}^{\frac{1}{p}}
\]

Since \( 1 < q < p \), by the Hardy-Littlewood maximal theorem,

\[
\|P^q_{m,E}\|_{L_p(\mathbb{R}^n)} \leq C \left( \int_{\mathbb{R}^n} \left[ |\nabla^m F|^q \right]^{\frac{1}{q}} du \right)^{\frac{1}{p}} = C \|\nabla^m F\|_{L_p(\mathbb{R}^n)} \sim \|F\|_{L_p(\mathbb{R}^n)}^{\frac{1}{p}}
\]

proving the necessity.

**(Sufficiency.)** Let \( P = \{P_x : x \in E\} \) be a Whitney \((m - 1)\)-field such that \( P^q_{m,E} \in L_p(\mathbb{R}^n) \). Prove the existence of a function \( F \in L_p^m(\mathbb{R}^n) \) such that \( T_x^{m-1} [F] = P_x \) for all \( x \in E \), and

\[
\|F\|_{L_p^m(\mathbb{R}^n)} \leq C(m, n, p) \|P^q_{m,E}\|_{L_p(\mathbb{R}^n)}.
\]

Let \( x, y \in E \), and let \( q := (n + p)/2 \). Let \( h_1 := P^q_{m,E} \) and let \( \bar{P} := P_x - P_y \). Prove that for every multiindex \( \alpha, |\alpha| \leq m - 1 \),

\[
|D^\alpha \bar{P}(x)| \leq C ||x - y||^{m-1-|\alpha|} \delta_q(x, y : h_1)
\] (5.3)

where \( C = C(m, n, p) \). See definition (1.17).

Let \( u \in Q_{xy} := Q(x, ||x - y||) \). Then \( ||u - x||, ||u - y|| \leq 2||x - y|| \), so that, by definition (1.11), the following inequality

\[
|\bar{P}(u)| \leq (||u - x||^m + ||u - y||^m) P^q_{m,E}(u) \leq 4^m ||x - y||^m P^q_{m,E}(u) = 4^m ||x - y||^m h_1(u)
\]

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holds. Hence

\[ |P(u)|^q \leq C \|x - y\|^{mq} h_i^q(u) \quad \text{for every } u \in Q_{xy}. \]

Integrating this inequality over \( Q \) with respect to \( u \) we obtain the following:

\[ \int_{Q_{xy}} |P(u)|^q \, du \leq C \|x - y\|^{mq} \int_{Q_{xy}} h_i^q(u) \, du. \]

Hence

\[
\left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} |P(u)|^q \, du \right)^{\frac{1}{q}} \leq C \|x - y\|^m \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} h_i^q(u) \, du \right)^{\frac{1}{q}}
\]

\[ \leq C \|x - y\|^m \sup_{Q_{xy}} \left( \frac{1}{|Q|} \int_{Q} h_i^q(u) \, du \right)^{\frac{1}{q}}
\]

\[ = C \|x - y\|^{m-1} \delta_q(x, y : h_1). \]

Since \( \tilde{P} \in \mathcal{P}_{m-1}(\mathbb{R}^n) \),

\[
\sup_{u \in Q_{xy}} |\tilde{P}(u)| \leq C(m, n, q) \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} |\tilde{P}(u)|^q \, du \right)^{\frac{1}{q}}.
\]

On the other hand, by Markov’s inequality,

\[
\sup_{u \in Q_{xy}} |D^\alpha \tilde{P}(u)| \leq C(m, n) (\text{diam } Q_{xy})^{-|\alpha|} \sup_{u \in Q_{xy}} |\tilde{P}(u)| \leq C(m, n) \|x - y\|^{-|\alpha|} \sup_{u \in Q_{xy}} |\tilde{P}(u)|.
\]

Hence,

\[
|D^\alpha \tilde{P}(x)| \leq \sup_{u \in Q_{xy}} |D^\alpha \tilde{P}(u)| \leq C(m, n) \|x - y\|^{-|\alpha|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} |\tilde{P}(u)|^q \, du \right)^{\frac{1}{q}}
\]

\[ \leq C \|x - y\|^{m-|\alpha|} \delta_q(x, y : h_1) \]

proving \((5.3)\).

This inequality and Theorem \([1,14]\) imply the following:

\[
|D^\alpha P_x(x) - D^\alpha P_y(x)| \leq C \|x - y\|^{m-|\alpha|} d_q(x, y : h_1).
\]

By definition \((1,23)\), the metric \( d = d_q(h_1) \) belongs to the family \( \mathcal{D}_{p,q}(\mathbb{R}^n) \). Furthermore, by the latter inequality,

\[
\mathcal{L}_{m-1,d}(P) := \sum_{|\alpha| \leq m-1} \sup_{x,y \in E, x \neq y} \frac{|D^\alpha P_x(x) - D^\alpha P_y(x)|}{\|x - y\|^{m-|\alpha|} d(x, y)} \leq C = C(m, n, p),
\]

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so that, by Theorem 1.16, there exists a function \( F \in C^{m-1}(\mathbb{R}^n) \) which agrees with \( P \) on \( E \) (i.e., \( T^{m-1}_x[F] = P_x \) for all \( x \in E \)) and

\[
\|F\|_{C^{m-1}(\mathbb{R}^n)} \leq C(m, n, p) \|L_{m-1, p}(P)\| \leq C(m, n, p).
\]

Thus,

\[
|D^\alpha F(x) - D^\alpha F(y)| \leq d_q(x, y : h_2) \quad \text{for all } \alpha, |\alpha| = m - 1,
\]

where \( h_2 = C(m, n, p) h_1 \), so that, by Theorem 1.15

\[
\|F\|_{L_p^m(\mathbb{R}^n)} \leq C \|h_2\|_{L_p(\mathbb{R}^n)} \leq C \|h_1\|_{L_p(\mathbb{R}^n)} = C \|P_{m,E}\|_{L_p(\mathbb{R}^n)}.
\] (5.5)

Theorem 1.8 is completely proved. □

**Remark 5.1** Note that Theorem 1.16 is the main ingredient of the proof of the sufficiency presented above. As we have noted in Remark 4.7, for the space \( L_p^1(\mathbb{R}^n) \) we can replace Theorem 1.16 with the McShane’s extension formula (4.12) which simplifies essentially the proof of the sufficiency part of Theorem 1.8.

Actually we obtain a new (non-linear) algorithm for extension of \( L_p^1(\mathbb{R}^n) \)-functions whenever \( p > n \). Let us describe the main steps of this algorithm.

Let \( q = (n + p)/2 \) and let \( f \) be a continuous function defined on \( E \).

- **Step 1.** We introduce the “sharp maximal function” associated with \( f \) by

\[
f^\sharp_E(x) := \sup_{y \in E} \frac{|f(x) - f(y)|}{\|x - y\| + \|y - z\|}, \quad x \in \mathbb{R}^n.
\]

- **Step 2.** We introduce a “pre-metric” \( \delta_q(f^\sharp_E) \) associated with the function \( f^\sharp_E \), i.e., a function

\[
\delta_q(x, y : f^\sharp_E) = \|x - y\| \sup_{Q \ni x, y} \left( \frac{1}{|Q|} \int_Q (f^\sharp_E(u))^q \, du \right)^{\frac{1}{q}}, \quad x, y \in \mathbb{R}^n.
\]

See (5.6).

- **Step 3.** Using formula (1.19) we construct the geodesic distance \( d = d_q(x, y : f^\sharp_E) \) associated with the “pre-metric” \( \delta_q(f^\sharp_E) \):

\[
d_q(x, y : f^\sharp_E) := \inf \sum_{i=0}^{m-1} \delta_q(x_i, x_{i+1} : f^\sharp_E)
\]

where the infimum is taken over all finite sequences of points \( \{x_0, x_1, ..., x_m\} \) in \( \mathbb{R}^n \) such that \( x_0 = x \) and \( x_m = y \).

- **Step 4.** Using the McShane’s formula (4.19) we construct a function

\[
F(x) := \inf_{y \in E} \left\{ f(y) + 48 d_q(x, y : f^\sharp_E) \right\}, \quad x \in \mathbb{R}^n.
\] (5.6)
Repeating the proof of the sufficiency part of Theorem 1.8 (for \(m = 1\)), we are able to show that the function \(F\) provides an almost optimal extension of the function \(f\) to a function from the Sobolev space \(L^1_p(\mathbb{R}^n)\) provided \(f \in L^1_p(\mathbb{R}^n)|_E\). In fact, by (5.4),

\[
|f(x) - f(y)| \leq 48 d_q(x, y : f^E_E)
\]

for every \(x, y \in \mathbb{R}^n\), so that, by the McShane’s formula, the function \(F\) is an extension of \(f\) from \(E\) on all of \(\mathbb{R}^n\). Following the proof of the sufficiency part of Theorem 1.8 we conclude that inequality (5.5) holds whenever \(m = 1\) and \(F\) is defined by (5.6). Thus

\[
\|F\|_{L^1_p(\mathbb{R}^n)} \leq C \|f^E\|_{L^p(\mathbb{R}^n)}.
\]

On the other hand, by (1.13),

\[
\|f^E\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^1_p(\mathbb{R}^n)}|_E.
\]

Hence

\[
\|F\|_{L^1_p(\mathbb{R}^n)} \leq C \|f\|_{L^1_p(\mathbb{R}^n)}|_E
\]

proving that \(F\) provides an almost optimal extension of \(f\) to a function from \(L^1_p(\mathbb{R}^n)\). △

6. Lacunae of Whitney’s cubes and a lacunary extension operator.

We prove the sufficiency part of Theorem 1.3 with the help of a modification of the classical Whitney extension method [33] used in the author’s paper [29]. As we have noted in Introduction, the main idea of this approach is that, instead of treating each Whitney cube separately, as is done in [33], we deal simultaneously with all members of certain families of Whitney cubes. We refer to these families of Whitney cubes as lacunae.

In Subsection 6.1 we present main definitions related to this notion, and main properties of the lacunae. For the proof of these results we refer the reader to [29], Sections 4-5.

6.1. Lacunae of Whitney’s cubes.

Let \(E\) be a closed subset of \(\mathbb{R}^n\) and let \(W_E\) be a Whitney covering of its complement \(\mathbb{R}^n \setminus E\) satisfying inequality (4.2). This inequality implies the following property of Whitney’s cubes:

\[
(9Q) \cap E \neq \emptyset \quad \text{for every} \quad Q \in W_E.
\]  (6.1)

By \(LW_E\) we denote a subfamily of Whitney cubes satisfying the following condition:

\[
(10Q) \cap E = (90Q) \cap E.
\]  (6.2)

Then we introduce a binary relation \(\sim\) on \(LW_E\): for every \(Q_1, Q_2 \in LW_E\)

\[
Q_1 \sim Q_2 \iff (10Q_1) \cap E = (10Q_2) \cap E.
\]  (6.3)

It can be easily seen that \(\sim\) satisfies the axioms of equivalence relations, i.e., it is reflexive, symmetric and transitive. Given a cube \(Q \in LW_E\) by

\[
[Q] := \{K \in LW_E : K \sim Q\}
\]

we denote the equivalence class of \(Q\). We refer to this equivalence class as a true lacuna with respect to the set \(E\).

Let

\[
\widetilde{L}_E = LW_E / \sim = \{[Q] : Q \in LW_E\}
\]

be the corresponding quotient set of \(LW_E\) by \(\sim\), i.e., the set of all possible equivalence classes (lacunae) of \(LW_E\) by \(\sim\).
Thus for every pair of Whitney cubes \( Q_1, Q_2 \in W_E \) which belong to a true lacuna \( L \in \tilde{L}_E \) we have

\[
(10Q_1) \cap E = (90Q_1) \cap E = (10Q_2) \cap E = (90Q_2) \cap E.
\] (6.4)

By \( V_L \) we denote the associated set of the lacuna \( L \)

\[
V_L := (90Q) \cap E.
\] (6.5)

Here \( Q \) is an arbitrary cube from \( L \). By (6.4), any choice of a cube \( Q \in L \) provides the same set \( V_L \) so that \( V_L \) is well-defined. We also note that for each cube \( Q \) which belong to a true lacuna \( L \) we have \( V_L = (10Q) \cap E \).

We extend the family \( \tilde{L}_E \) of true lacunae to a family \( L_E \) of all lacunae in the following way. Suppose that \( Q \in W_E \setminus LW_E \), see (6.2), i.e.,

\[
(10Q) \cap E \neq (90Q) \cap E.
\]

In this case to the cube \( Q \) we assign a lacuna \( L := \{Q\} \) consisting of a unique cube - the cube \( Q \) itself. We also put \( V_L := (90Q) \cap E \) as in (6.5).

We refer to such a lacuna \( L := \{Q\} \) as an elementary lacuna with respect to the set \( E \). By \( \hat{L}_E \) we denote the family of all elementary lacunae with respect to \( E \):

\[
\hat{L}_E := \{L = \{Q\} : Q \in W_E \setminus LW_E\}.
\] (6.6)

In [29], Section 4, we prove that for every elementary lacuna \( L = \{Q\} \in \hat{L}_E \) we have

\[
diam Q \leq 2 diam V_L = 2 diam((90Q) \cap E).
\] (6.7)

Finally, by \( L_E \) we denote the family of all lacunae with respect to \( E \):

\[
L_E = \tilde{L}_E \cup \hat{L}_E.
\]

Let us present several important properties of lacunae. Let \( L \in L_E \) and let \( U_L := \cup\{Q : Q \in L\} \). We say that \( L \) is a bounded lacuna if \( U_L \) is a bounded set.

In [29], Section 4, we prove that for every lacuna \( L \in L_E \) there exists a cube \( Q_L \in L \) such that

\[
diam Q_L = \inf \{diam Q : Q \in L\}
\]

provided \( diam V_L > 0 \). We also prove that for every bounded lacuna \( L \in L_E \) there exists a cube \( Q^{(L)} \in L \) such that

\[
diam Q^{(L)} = \sup \{diam K : K \in L\}.
\] (6.8)

For the proof of the following four propositions we refer the reader to [29], Sections 4 and 5.

**Proposition 6.1** (i). If \( E \) is an unbounded set, then every lacuna \( L \in L_E \) is bounded;

(ii). If \( E \) is bounded, then there exists the unique unbounded lacuna \( L_{\text{max}} \in L_E \). The lacuna \( L_{\text{max}} \) is a true lacuna for which \( V_{L_{\text{max}}} = E \).
Proposition 6.2 (i). For every bounded true lacuna \( L \) the following inequalities hold:

\[
40 \, \text{diam} \, Q^{(L)} \leq \text{dist}(V_L, E \setminus V_L) \leq \gamma_1 \, \text{diam} \, Q^{(L)} \tag{6.9}
\]

(ii). For every lacuna \( L \) with \( \text{diam} \, V_L > 0 \)

\[
\text{diam} \, Q_L \leq \gamma_1 \, \text{diam} \, V_L. \tag{6.10}
\]

Here \( \gamma_1 > 0 \) is an absolute constant.

Proposition 6.3 Let \( L \in \mathcal{L}_E \) be a lacuna and let \( Q \in L \). Suppose that there exist a lacuna \( L' \in \mathcal{L}_E \), \( L \neq L' \), and a cube \( Q' \in L' \) such that \( Q \cap Q' \neq \emptyset \). Then:

(i). If \( L \) is a true lacuna, then \( L' \) is an elementary lacuna;

(ii). Either \( \text{diam} \, Q^{(L)} \leq \tau \, \text{diam} \, Q \) or \( \text{diam} \, Q \leq \tau \, \text{diam} \, Q_L \) where \( \tau \) is a positive absolute constant.

This proposition motivates us to introduce the following

Definition 6.4 Let \( L, L' \in \mathcal{L}_E \) be lacunae, \( L \neq L' \). We say that \( L \) and \( L' \) are contacting lacunae if there exist cubes \( Q \in L \) and \( Q' \in L' \) such that \( Q \cap Q' \neq \emptyset \). We refer to the pair of such cubes as contacting cubes.

We write

\[
L \Leftrightarrow L' \quad \text{for contacting lacunae} \quad L, L' \in \mathcal{L}_E.
\]

Proposition 6.5 Every lacuna \( L \in \mathcal{L}_E \) contacts with at most \( M \) lacunae. Furthermore, \( L \) contains at most \( M \) contacting cubes. Here \( M = M(n) \) is a positive integer depending only on \( n \).

6.2. A lacunary projector and centers of lacunae.

One of the main ingredients of the lacunary approach is a mapping \( \mathbb{PR} : \mathcal{L}_E \to E \) whose properties are described in Theorem 6.6 below. We refer to this mapping as a “projector” from \( \mathcal{L}_E \) into the set \( E \). Also given \( L \in \mathcal{L}_E \) we refer to the point \( \mathbb{PR}(L) \in E \) as a center of the lacuna \( L \).

Theorem 6.6 is a refinement of a result proven in [29], Section 5.

Theorem 6.6 There exist an absolute constant \( \gamma \geq 1 \) and a mapping \( \mathbb{PR} : \mathcal{L}_E \to E \) such that:

(i). For every lacuna \( L \in \mathcal{L}_E \) and every cube \( Q \in L \) we have

\[
\mathbb{PR}(L) \in \gamma \, Q \cap E; \tag{6.11}
\]

(ii). Let \( L, L' \in \mathcal{L}_E \) be two distinct lacunae such that \( \mathbb{PR}(L) \neq \mathbb{PR}(L') \). Then for every two cubes \( Q \in L \) and \( Q' \in L' \) such that \( Q \cap Q' \neq \emptyset \) the following inequality

\[
\text{diam} \, Q + \text{diam} \, Q' \leq \gamma \| \mathbb{PR}(L) - \mathbb{PR}(L') \| \tag{6.12}
\]

holds;

(iii). For every point \( A \in E \)

\[
\# \{ L \in \mathcal{L}_E : \mathbb{PR}(L) = A \} \leq C
\]

where \( C = C(n) \) is a constant depending only on \( n \).
Proof. We define the projector $\mathbb{P}_{\mathbb{E}} : \mathcal{L}_E \to E$ in several steps. First consider a true lacuna $L \in \mathcal{L}_E$ such that $\text{diam } V_L = 0$, i.e., $V_L = \{a\}$ for some $a \in E$. In this case we put $\mathbb{P}_{\mathbb{E}}(L) := a$.

Let now $\text{diam } V_L > 0$. Fix points $A_L, B_L \in V_L$ such that

$$\|A_L - B_L\| = \text{diam } V_L.$$  

By $i_L \in \mathbb{Z}$ we denote an integer such that

$$2^{i_L} < \text{diam } V_L = \|A_L - B_L\| \leq 2^{i_L+1}. \quad (6.13)$$

In what follows we will be needed a result proven in [29], Section 4, which states the following: There exists a non-increasing sequence of non-empty closed sets $\{E_i\}_{i \in \mathbb{Z}}, E_{i+1} \subset E_i \subset E, i \in \mathbb{Z}$, such that for every $i \in \mathbb{Z}$ the following conditions are satisfied:

(i). The points of the set $E_i$ are $2^i$-separated, i.e.,

$$\|z - z'\| \geq 2^i \quad \text{for every } z, z' \in E_i; \quad (6.14)$$

(ii). $E_i$ is a $2^{i+1}$-net in $E$, i.e.,

for every $x \in E$ there exists $z \in E_i$ such that $\|x - z\| \leq 2^{i+1}. \quad (6.15)$

Let us apply this statement to the points $A_L$ and $B_L$. Since $E_{i_L-2}$ is a $2^{i_L-1}$-net in $E$, there exist points $A_L, B_L \in E_{i_L-2}$ such that

$$\|A_L - \tilde{A}_L\| \leq 2^{i_L-1} \quad \text{and} \quad \|B_L - \tilde{B}_L\| \leq 2^{i_L-1}. \quad (6.16)$$

Since $\tilde{A}_L, \tilde{B}_L \in E_{i_L-2}$, by (6.15), $\|\tilde{A}_L - \tilde{B}_L\| \geq 2^{i_L-2}$.

Prove that $\tilde{A}_L \neq \tilde{B}_L$ and

$$\{\tilde{A}_L, \tilde{B}_L\} \cap (E_{i_L-2} \setminus E_{i_L+2}) \neq \emptyset. \quad (6.17)$$

In fact, by (6.16),

$$\|A_L - B_L\| \leq \|A_L - \tilde{A}_L\| + \|\tilde{A}_L - \tilde{B}_L\| + \|\tilde{B}_L - B_L\| \leq 2^{i_L-1} + \|\tilde{A}_L - \tilde{B}_L\| + 2^{i_L-1}$$

so that

$$\|\tilde{A}_L - \tilde{B}_L\| \geq \|A_L - B_L\| - 2^{i_L}.$$  

But, by (6.13), $\|A_L - B_L\| > 2^{i_L}$ proving that $\tilde{A}_L \neq \tilde{B}_L$.

Prove that the set $\{\tilde{A}_L, \tilde{B}_L\} \not\subset E_{i_L+2}$. In fact, if $\{\tilde{A}_L, \tilde{B}_L\} \subset E_{i_L+2}$ then $\|\tilde{A}_L - \tilde{B}_L\| \geq 2^{i_L+2}$, see (6.14).

But, by (6.13) and (6.16),

$$\|\tilde{A}_L - \tilde{B}_L\| \leq \|\tilde{A}_L - A_L\| + \|A_L - B_L\| + \|B_L - \tilde{B}_L\| \leq 2^{i_L-1} + 2^{i_L+1} + 2^{i_L-1} = 3 \cdot 2^{i_L} < 2^{i_L+2},$$

a contradiction.

Since $\tilde{A}_L, \tilde{B}_L \in E_{i_L-2}$, the statement (6.17) follows. Thus there exists a point $C_L \in E$ such that

$$C_L \in \{\tilde{A}_L, \tilde{B}_L\} \cap (E_{i_L-2} \setminus E_{i_L+2}).$$
Note that, by (6.16), $C_L \in [V_L]_\delta$ with $\delta = 2^{i_L-1}$. Here given $\delta > 0$ the sign $[\cdot]_\delta$ denotes the open $\delta$-neighborhood of a set. Hence, by (6.13),

$$C_L \in [V_L]_\epsilon \cap (E_{i_L-2} \setminus E_{i_L+2})$$

(6.18)

with $\epsilon = \frac{1}{2} \text{diam } V_L$.

We turn to definition of $\text{PR}_E(L)$ whenever $L$ is a bounded lacuna satisfying the following condition:

$$\text{diam } Q^{(L)} \leq \sigma \text{ diam } V_L$$

(6.19)

where

$$\sigma := 33 \tau.$$  

(6.20)

Recall that $\tau$ is the constant from part (ii) of Proposition 6.3.

In this case we define $\text{PR}_E(L)$ by

$$\text{PR}_E(L) := C_L.$$  

(6.21)

In particular, by (6.7), each elementary lacuna $L \in \hat{L}_E$, see (6.6), satisfies inequality (6.19), so that

$$\text{PR}_E(L) := C_L \quad \text{for every elementary lacuna } L \in \hat{L}_E.$$  

(6.22)

It remains to define $\text{PR}_E(L)$ whenever $L$ is a true lacuna satisfying inequality

$$\text{diam } Q^{(L)} > \sigma \text{ diam } V_L.$$  

(6.23)

First suppose that $L$ is a true bounded lacuna. In this case the cube $Q^{(L)}$ is well defined, see (6.8). Let $j_L \in \mathbb{Z}$ be an integer such that

$$2^{j_L} < \frac{1}{\sigma} \text{ diam } Q^{(L)} \leq 2^{j_L+1}.$$  

(6.24)

Recall that, by (6.9),

$$40 \text{ diam } Q^{(L)} \leq \text{dist}(V_L, E \setminus V_L)$$

so that, by (6.24),

$$\text{dist}(V_L, E \setminus V_L) \geq 40 \sigma (\text{diam } Q^{(L)}/\sigma) \geq 40 \sigma 2^{j_L-1} > 2^{j_L+2}.$$  

(6.25)

On the other hand, by (6.23) and (6.24),

$$\text{diam } V_L < \frac{1}{\sigma} \text{ diam } Q^{(L)} \leq 2^{j_L}.$$  

(6.26)

In particular, by (6.25) and (6.26), $\text{diam } V_L < \text{dist}(V_L, E \setminus V_L)$ which implies that

$$C_L \in V_L.$$  

(6.27)

In fact, by (6.18), $C_L \in [V_L]_\epsilon$ with $\epsilon = \frac{1}{2} \text{diam } V_L$ so that $C_L \notin E \setminus V_L$.

Let us consider now the set $E_{j_L} \subset E$. We know that $\|z - z'\| \geq 2^{j_L}$ for all $z, z' \in E_{j_L}$, and that for each $x \in E$ there exists $z \in E_{j_L}$ such that $\|x - z\| \leq 2^{j_L+1}$.  

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Prove that the set \( V_L \cap E_{jl} \) is a singleton. Let us fix a point \( x_0 \in V_L \). Then there exists a point \( z_0 \in E_{jl} \) such that \( \|x_0 - z_0\| \leq 2^{jL+1} \). Prove that \( z_0 \in V_L \).

In fact, if \( z_0 \in E \setminus V_L \), then, by (6.25),

\[
\|x_0 - z_0\| \geq \text{dist}(V_L, E \setminus V_L) > 2^{jL+2},
\]

a contradiction.

Prove that \( \{z_0\} = V_L \cap E_{jl} \). In fact, if there exists a point \( z_1 \in V_L \cap E_{jl} \), \( z_1 \neq z_0 \), then \( \|z_0 - z_1\| \geq 2^{jL} \) so that

\[
\text{diam } V_L \geq \|z_0 - z_1\| \geq 2^{jL}
\]

which contradicts (6.26).

We denote this unique point of the intersection \( V_L \cap E_{jl} \) by \( D_L \); thus

\[
\{D_L\} = V_L \cap E_{jl} \quad (6.28)
\]

We are now in a position to define \( \mathbb{P} \mathbb{R}_E(L) \) for an arbitrary bounded lacuna satisfying inequality (6.23). In this case we put

\[
\mathbb{P} \mathbb{R}_E(L) := \begin{cases} 
D_L, & \text{if } D_L \in E_{jl} \setminus E_{jl+k}, \\
C_L, & \text{otherwise}.
\end{cases} 
\quad (6.29)
\]

Here

\[
k := \lfloor \log_2(360\sigma) \rfloor + 2. 
\quad (6.30)
\]

It remains to define \( \mathbb{P} \mathbb{R}_E(L) \) for an unbounded lacuna \( L \in \mathcal{L}_E \). By Proposition [6.1], such a lacuna exists if and only if \( E \) is a bounded set. Furthermore, by part (ii) of this proposition, such a lacuna is unique, and \( V_L = E \).

In this case we define \( \mathbb{P} \mathbb{R}_E(L) \) by the formula (6.21). Thus

\[
\mathbb{P} \mathbb{R}_E(L) := C_L \quad \text{provided the lacuna } L \text{ is unbounded.} 
\quad (6.31)
\]

We have defined the projector \( \mathbb{P} \mathbb{R}_E \) on all of the family \( \mathcal{L}_E \) of lacunae of the set \( E \). Prove that this mapping satisfies properties (i)-(iii) of the theorem.

Proof of part (i) of the theorem. Let us prove (i) with any \( \bar{\gamma} \geq 180 \). By formulae (6.21), (6.29) and (6.31), \( \mathbb{P} \mathbb{R}_E(L) \in \{C_L, D_L\} \) for every lacuna \( L \in \mathcal{L}_E \). Note that \( D_L \in V_L \). Since \( V_L = (90Q) \cap E \) for each \( Q \in L \), we conclude that \( V_L \subset 90Q \) proving that \( D_L \subset 90Q \).

By (6.18), \( C_L \) belongs to the \( \varepsilon \)-neighborhood of \( V_L \) with \( \varepsilon = \frac{1}{2} \text{diam } V_L \). But \( V_L \subset 90Q \) so that \( C_L \) belongs to the \( \tilde{\varepsilon} \)-neighborhood of 90Q with \( \tilde{\varepsilon} = 45 \text{ diam } Q \). Hence, \( C_L \subset \tilde{\gamma}Q \) provided \( \tilde{\gamma} \geq 180 \).

Proof of part (ii) of the theorem. Let \( L, L' \in \mathcal{L}_E \) be two distinct lacunae such that their “projections” \( \mathbb{P} \mathbb{R}_E(L) \) and \( \mathbb{P} \mathbb{R}_E(L') \) are distinct as well. Thus \( L \neq L' \) and \( \mathbb{P} \mathbb{R}_E(L) \neq \mathbb{P} \mathbb{R}_E(L') \). Prove that under these conditions inequality (6.12) holds with some absolute constant \( \bar{\gamma} > 0 \).

First we note that, by part (1) of Lemma [4.1]

\[
\frac{1}{4} \leq \text{diam } Q / \text{diam } Q' \leq 4. 
\quad (6.32)
\]

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By part (i) of Proposition 6.3, either $L$ or $L'$ is an elementary lacuna. Thus, without loss of generality, we may assume that $L'$ is an elementary lacuna.

Then, by (6.7),

$$\text{diam } Q' \leq 2 \text{ diam } V_{L'}.$$  \hfill (6.33)

We also recall that, by (6.5),

$$\text{diam } V_L \leq 90 \text{ diam } Q \quad \text{and} \quad \text{diam } V_{L'} \leq 90 \text{ diam } Q'.$$  \hfill (6.34)

We know that in this case $PR_E(L') = C_{L'}$. See (6.22).

Recall that

$$C_{L'} \in E_{i_{L'}-2} \setminus E_{i_{L'}+2}$$  \hfill (6.35)

where $i_{L'}$ is an integer such that

$$2^{i_{L'}} < \text{diam } V_{L'} \leq 2^{i_{L'}+1}.$$  \hfill (6.36)

See (6.18) and (6.13).

The next lemma shows that, under certain restriction on the cube $Q$ the inequality (6.12) holds.

**Lemma 6.7** Suppose that $PR_E(L) = C_L$ and $\text{diam } Q \leq \theta \text{ diam } V_{L}$ where $\theta$ is a positive constant. Then

$$\text{diam } Q + \text{diam } Q' \leq C \theta ||PR_E(L) - PR_E(L')||$$

where $C$ is an absolute constant.

**Proof.** Let $m \in \mathbb{Z}$ be an integer such that $2^{m-1} < \theta \leq 2^m$. Then, by (6.36), (6.34) and (6.32),

$$2^{i_{L'}} \leq \text{diam } V_{L'} \leq 90 \text{ diam } Q' \leq 360 \text{ diam } Q \leq 360 \theta \text{ diam } V_{L}.$$  

Hence, by (6.13),

$$2^{i_{L'}} \leq 360 \theta 2^{i_{L'}+1} \leq 2^{i_{L'}+m+9}$$

proving that $j := i_{L'} - m - 9 \leq i_{L}$.

Since $C_L \in E_{i_{L}}$, $C_{L'} \in E_{i_{L'}},$ we conclude that $C_L, C_{L'} \in E_j$. But $C_L = PR_E(L)$, $C_{L'} = PR_E(L')$, and $PR_E(L) \neq PR_E(L')$ so that $C_L$ and $C_{L'}$ are two distinct points of $E_j$. Therefore, by (6.14), these two points are $2^j$-separated, i.e.,

$$||PR_E(L) - PR_E(L')|| \geq 2^j = 2^{-(m+10)} 2^{i_{L'}+1} \geq 2^{-11} 2^{i_{L'}+1}/\theta.$$  

Hence, by (6.36),

$$\text{diam } V_{L'} \leq 2^{11} \theta ||PR_E(L) - PR_E(L')||$$

so that, by (6.33),

$$\text{diam } Q' \leq 2^{12} \theta ||PR_E(L) - PR_E(L')||.$$  

Finally, by (6.32),

$$\text{diam } Q + \text{diam } Q' \leq 5 \text{ diam } Q' \leq 2^{15} \theta ||PR_E(L) - PR_E(L')||$$

proving the lemma. □
We turn to the proof of the property (ii) of the theorem in general case. We do this in several steps. First we prove (6.12) for the lacuna \( L \) satisfying inequality (6.19). In this case, by (6.21), \( \mathcal{P} \mathcal{R}_\mathbb{E}(L) = C_L \) so that the conditions of Lemma 6.7 are satisfied. This lemma implies inequality (6.12) with a constant \( \bar{y} = C \sigma \) where \( C > 0 \) is an absolute constant.

Let now \( L \) be an unbounded lacuna. By part (i) of Proposition 6.1, in this case the set \( E \) is bounded. Furthermore, \( L \) coincides with the unique unbounded lacuna \( L^{\text{max}} \) which is a true lacuna such that \( V_L = V^{L^{\text{max}}} \).

By (6.32) and (6.33),

\[
\text{diam } Q \leq 4 \text{diam } Q' \leq 8 \text{diam } V_L' \leq 8 \text{diam } E
\]

so that \( \text{diam } Q \leq 8 \text{diam } V_L \). Furthermore, by (6.31), \( \mathcal{P} \mathcal{R}_\mathbb{E}(L) = C_L \). Thus the conditions of Lemma 6.7 are satisfied so that inequality (6.12) for the unbounded lacuna \( L \) holds with some absolute constant \( \bar{y} > 0 \).

It remains to prove (6.12) for a bounded lacuna \( L \) satisfying inequality (6.23). By part (ii) of Proposition 6.3, it suffices to consider two cases.

**The first case: for the cubes \( Q \) and \( Q^{(L)} \) the following inequality**

\[
diam Q^{(L)} \leq \tau \text{diam } Q
\]

holds.

Prove that

\[
j_L + 2 < i_L' < j_L + k - 2.
\]

(Recall that \( j_L, i_L' \) and \( k \) are defined by (6.24), (6.36) and (6.29) respectively.)

We begin with the proof of the inequality \( j_L + 2 < i_L' \). By (6.24), (6.37) and (6.32),

\[
2^{j_L - 1} \leq \text{diam } Q^{(L)} / \sigma \leq \tau \text{diam } Q / \sigma \leq 4 \text{diam } Q' / \sigma
\]

so that, by (6.33) and (6.36),

\[
2^{j_L - 1} \leq 2 \tau \text{diam } V_L' / \sigma \leq 2^{i_L' - 1} / \sigma.
\]

Since \( \sigma > 32 \tau \), see (6.20), we obtain the required inequality \( j_L + 2 < i_L' \).

Prove that \( i_L' < j_L + k - 2 \). By (6.36), (6.34), (6.32) and (6.24),

\[
2^{i_L'} < \text{diam } V_L' \leq 90 \text{diam } Q' \leq 360 \text{diam } Q \leq 360 \text{diam } Q^{(L)} \leq 360 \sigma 2^{j_L}.
\]

Since \( 360 \sigma \leq 2^{k-2} \), see (6.20) and (6.30), the required inequality \( i_L' < j_L + k - 2 \) follows.

We recall that, by (6.22), \( \mathcal{P} \mathcal{R}_\mathbb{E}(L') = C_{L'} \) where \( C_{L'} \) is a point satisfying (6.35). Prove that \( C_{L'} \in E \setminus V_L \).

Suppose that this is not true, i.e., \( C_{L'} \in V_L \). By (6.35), \( C_{L'} \in E_{i_L' + 2} \). But, by (6.38), \( i_L' - 2 > j_L \) so that \( C_{L'} \in E_{j_L} \cap V_L \) (because \( E_{i_L' - 2} \subset E_{j_L} \)). On the other hand, by (6.28), the set \( E_{j_L} \cap V_L = \{ D_L \} \) is a singleton so that \( C_{L'} = D_L \).

Note that, by (6.35), \( C_{L'} \notin E_{i_L' + 2} \). Since \( i_L' + 2 < j_L + k \), see (6.38), \( E_{j_L + k} \subset E_{i_L' + 2} \) proving that \( \{ D_L \} = C_{L'} \in E_{j_L} \cap E_{j_L + k} \).

But, by definition (6.29), in this case \( \mathcal{P} \mathcal{R}_\mathbb{E}(L) = D_L \) so that

\[
\mathcal{P} \mathcal{R}_\mathbb{E}(L) = D_L = C_{L'} = \mathcal{P} \mathcal{R}_\mathbb{E}(L').
\]
This contradicts our assumption that $PR_E(L) \neq PR_E(L')$ proving the required imbedding

$$C_{L'} = PR_E(L') \in E \setminus V_L.$$ 

Let us now estimate from below the distance between the points $PR_E(L)$ and $PR_E(L')$. We note that, by (6.27) and (6.28), $C_L, D_L \in V_L$. But the point $PR_E(L) \in \{C_L, D_L\}$ proving that $PR_E(L) \in V_L$ as well. Since $PR_E(L') \in E \setminus V_L$, we have

$$||PR_E(L) - PR_E(L')|| \geq \text{dist}(V_L, E \setminus V_L)$$

so that, by part (i) of Proposition 6.2,

$$40 \text{ diam } Q^{(L)} \leq ||PR_E(L) - PR_E(L')||.$$ 

Since $\text{diam } Q \leq \text{diam } Q^{(L)}$, by (6.32),

$$\text{diam } Q + \text{diam } Q' \leq 5 \text{ diam } Q \leq 5 \text{ diam } Q^{(L)} \leq ||PR_E(L) - PR_E(L')||$$

proving (6.12) with $\bar{\gamma} = 1$.

**The second case:**

$$\text{diam } Q \leq \tau \text{ diam } Q_L.$$ 

(6.39)

Clearly, in this case $\text{diam } V_L > 0$ so that, by part (ii) of Proposition 6.2,

$$\text{diam } Q_L \leq \gamma_1 \text{ diam } V_L \text{ for some absolute constant } \gamma_1 > 0.$$ 

Hence

$$\text{diam } Q \leq \tau \gamma_1 \text{ diam } V_L.$$ 

(6.40)

By definition (6.29), $PR_E(L) = C_L$ provided the point $D_L \not\in E_{j_L} \setminus E_{j_L+1}$. In this case, by (6.40), the conditions of Lemma 6.7 are satisfied. By this lemma inequality (6.12) holds with a constant $\bar{\gamma} = C \tau \gamma_1$ where $C > 0$ is an absolute constant.

Let now $D_L \in E_{j_L} \setminus E_{j_L+1}$ so that, by (6.29), $PR_E(L) = D_L$. Prove that in this case $i_{L'} \leq j_L + m$ where $m := \lceil \log_2 \gamma_2 \rceil$ and $\gamma_2 := 360 \tau \gamma_1$.

By (6.36), (6.34) and (6.32),

$$2^{i_{L'}} \leq \text{diam } V_{L'} \leq 90 \text{ diam } Q' \leq 360 \text{ diam } Q$$

so that, by (6.39) and (6.10),

$$2^{i_{L'}} \leq 360 \tau \text{ diam } Q_L \leq 360 \tau \gamma_1 \text{ diam } V_L = \gamma_2 \text{ diam } V_L \leq 2^{j_L+m}$$

proving the required inequality $i_{L'} \leq j_L + m$.

Recall that $PR_E(L) = D_L \in E_{j_L}$ and $PR_E(L') = C_{L'} \in E_{i_{L'}}$, see (6.35). Hence

$$PR_E(L), PR_E(L') \in E_{i_{L'}-m}$$

so that $PR_E(L)$ and $PR_E(L')$ are $2^{i_{L'}-m}$-separated points, i.e.,

$$||PR_E(L) - PR_E(L')|| \geq 2^{i_{L'}-m}.$$
We also recall that, by (6.36), \( \text{diam } V_L \leq 2^{i+1} \) so that
\[
\text{diam } V_L \leq 2^{m+1} \| PR_{E}(L) - PR_{E}(L') \| \leq 4 \gamma_2 \| PR_{E}(L) - PR_{E}(L') \|.
\]
Combining this inequality with (6.33), we obtain
\[
\text{diam } Q' \leq 8 \gamma_2 \| PR_{E}(L) - PR_{E}(L') \|
\]
so that, by (6.32),
\[
\text{diam } Q + \text{diam } Q' \leq 5 \text{ diam } Q' \leq 40 \gamma_2 \| PR_{E}(L) - PR_{E}(L') \|.
\]
This completes the proof of part (ii) of the theorem.

**Proof of part (iii) of the theorem.** Let \( A \in E \) and let
\[
PR^{-1}_{E}(A) := \{ L \in L_E : PR_{E}(L) = A \}.
\]

By Proposition 6.1, the set \( PR^{-1}_{E}(A) \) contains at most one unbounded lacuna so that, without loss of generality, we may assume that all lacunae from \( PR^{-1}_{E}(A) \) are bounded.

Let us also note that if \( A \) is an isolated point of \( E \), then there exists a unique lacuna \( L_A \in L_E \) such that \( \{ A \} = V_{L_A} \). (Of course, \( L_A \) is a true lacuna.) Conversely, if \( \{ A \} = V_L \) for some \( L \in L_E \), then \( A \) is an isolated point of \( E \) so that \( L = L_A \). This elementary remark enables us to assume that diam \( V_L > 0 \) for each lacuna \( L \in PR^{-1}_{E}(A) \).

Now, by definition (6.21) and (6.29), the point \( PR_{E}(L) \) coincides either with the point \( C_L \), see (6.18), or with the point \( D_L \), see (6.28).

Note that if \( PR_{E}(L) = C_L \), by (6.18),
\[
PR_{E}(L) \in E_{i_L-2} \setminus E_{i_L+2}.
\]
(Recall that \( i_L \in \mathbb{Z} \) is an integer determined by inequalities (6.13).) Hence, by (6.23) and (6.5), diam \( Q_L \sim 2^{i_L} \) with absolute constants in this equivalence.

Let now \( PR_{E}(L) = D_L \) for some lacuna \( L \in PR^{-1}_{E}(A) \). Then, by (6.29),
\[
PR_{E}(L) \in E_{j_L} \setminus E_{j_L+k}.
\]
See (6.30). Furthermore, by (6.24), diam \( Q^{(L)} \sim 2^{i_L} \) with absolute constants.

Summarizing these properties of \( PR_{E}(L) \) we conclude that for each lacuna \( L \in PR^{-1}_{E}(A) \) there exists an integer \( m_L \in \mathbb{Z} \) and a cube \( K_L \in L \) such that
\[
PR_{E}(L) \in E_{m_L} \setminus E_{m_L+k} \tag{6.41}
\]
and
\[
\frac{1}{C_1} 2^{m_L} \leq \text{diam } K_L \leq C_1 2^{m_L} \tag{6.42}
\]
Here \( C_1 > 0 \) is an absolute constant, and \( k \) is defined by (6.30).

Now let \( L, L' \in PR^{-1}_{E}(A) \), i.e., \( PR_{E}(L) = PR_{E}(L') = A \). Then, by (6.41),
\[
A \notin E_{m_L+k} \quad \text{and} \quad A \in E_{m_L}.
\]

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so that \( E_{m_L+k} \neq E_{m_{L'}} \). Hence \( E_{m_L+k} \subset E_{m_{L'}} \) proving that \( m_{L'} \leq m_L + k \). In the same fashion we prove that \( m_L \leq m_{L'} + k \), so that \(|m_L - m_{L'}| \leq k\).

We also note that, by inequality \((6.42)\) (which we apply to \( L' \)), we have

\[
\frac{1}{C_1} 2^{m_{L'}} \leq \text{diam } K_L \leq C_1 2^{m_{L'}}.
\]

Hence,

\[
\text{diam } K_L \leq C_1 2^{m_L} \leq C_1 2^k 2^{m_{L'}} \leq C_1 2^k \text{diam } K_{L'}. 
\]

In the same way we obtain that \( \text{diam } K_{L'} \leq C_1^2 2^k \text{ diam } K_L \). Thus

\[
C_1^2 2^{-k} \leq \text{diam } K_L/ \text{ diam } K_{L'} \leq C_1^2 2^k. \tag{6.43}
\]

We recall that \( K_L \in L \) and \( K_{L'} \in L' \) so that \( K_L \neq K_{L'} \) provided \( L \neq L' \). Thus the family

\[
\mathcal{K}_A := \{ K_L : L \in \mathbb{PR}_E^{-1}(A) \}
\]

consists of pairwise disjoint cubes. We also note that, by part (i) of the theorem (proven earlier),

\[
A \in \gamma K_L \quad \text{for every} \quad L \in \mathbb{PR}_E^{-1}(A)
\]

with \( \gamma = 180 \). Hence,

\[
\bigcup_{K_L \in \mathcal{K}_A} K_L \subset (2\gamma)K^{\max}
\]

where \( K^{\max} \) is a cube from \( \mathcal{K}_A \) of the maximal diameter.

Note that the cubes of the family \( \mathcal{K}_A \) are pairwise disjoint, and the diameters of these cubes are equivalent to the diameter of the cube \( K^{\max} \), see \((6.43)\). Consequently, the number of these cubes, the quantity \( \#\mathcal{K}_A \), is bounded by a constant \( C = C(n) \) depending only on \( n \). But \( \#\mathcal{K}_A = \#\mathbb{PR}_E^{-1}(A) \) which proves part (iii) of the theorem.

The proof of Theorem \((6.6)\) is complete. \( \Box \)

Let us note several simple and useful properties of the projector \( \mathbb{PR}_E \) for a finite set \( E \).

**Proposition 6.8** Let \( E \) be a finite subset of \( \mathbb{R}^n \) and let \( x \in E \). There exists a unique lacuna \( L^{(x)} \) such that \( V_{L^{(x)}} = \{ x \} \). Furthermore, \( L^{(x)} \) is a true lacuna, and

\[
\mathbb{PR}_E \left( L^{(x)} \right) = x. \tag{6.44}
\]

**Proof.** We define \( L^{(x)} \) by

\[
L^{(x)} := \{ Q \in W_E : (90Q) \cap E = \{ x \} \}. \tag{6.45}
\]

Let \( \tilde{\varepsilon} := \text{dist}(x, E \setminus \{ x \})/180 \) and let \( 0 < \varepsilon < \tilde{\varepsilon} \). Then each cube \( Q \in W_E \) such that \( Q \subset Q(x, \varepsilon) \) belongs to \( L^{(x)} \). In fact,

\[
(90Q) \subset Q(x, 90\varepsilon) \subset Q(x, 90\tilde{\varepsilon}) = Q(x, \frac{1}{2} \text{dist}(x, E \setminus \{ x \})).
\]

Hence \( (90Q) \cap E = \{ x \} \) proving that \( Q \in L^{(x)} \).

In particular, \( L^{(x)} \neq \emptyset \). Furthermore, every cube \( Q \in L^{(x)} \) is a lacunary cube, i.e., \((6.2)\) is satisfied. In fact, by \((6.1)\), \( (9Q) \cap E \neq \emptyset \). Since \( (9Q) \cap E = \{ x \} \), we have

\[
(10Q) \cap E = (90Q) \cap E = \{ x \},
\]

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i.e., (6.2) holds. This equality also shows that $L^{(x)}$ is a true lacuna, i.e., an equivalence class with respect to the binary relation $\sim$, see (6.3), on the family $LW_E$ of all lacunary cubes. In addition, $V_{L^{(x)}} = (90Q) \cap E = \{x\}$, see (6.5). Also note that the uniqueness of $L^{(x)}$ directly follows from definitions (6.45) and (6.5).

Prove (6.44). By part (i) of Theorem 6.6 $PR_B(L^{(x)}) \in (\gamma Q) \cap E$ for every $Q \in L^{(x)}$ where $\gamma \geq 1$ is an absolute constant.

Let $\varepsilon_0 := \bar{\varepsilon}/(2\gamma)$, and let $Q \subset Q(x, \varepsilon_0)$. We know that $Q \in L^{(x)}$. But

$\gamma Q \subset Q(x, \bar{\varepsilon} \varepsilon_0) = Q(x, \bar{\varepsilon}/2)$

so that

$(\gamma Q) \cap E \subset Q(x, \bar{\varepsilon}/2) \cap E = \{x\}.$

Hence $PR_B(L^{(x)}) \in (\gamma Q) \cap E = \{x\}$ proving the proposition. □

Theorem 6.6 enables us to compare the number of lacunae with the number of points of a finite set $E \subset \mathbb{R}^n$.

**Proposition 6.9** Let $E$ be a finite subset of $\mathbb{R}^n$. Then

$$\#E \leq \#L_E \leq C(n) \#E.$$ 

**Proof.** By Proposition 6.8 $E \ni x \to L^{(x)} \in L_E$ is a one-to-one mapping, so that $\#E \leq \#L_E$. On the other hand, by part (iii) of Theorem 6.6 the projector $PR_B : L_E \to E$ is “almost” one-to-one, i.e., for each $x \in E$ the number of its sources $\{L \in L_E : PR_B(L) = x\}$ is bounded by a constant $C = C(n)$. Hence $\#L_E \leq C(n) \#E$ proving the proposition. □

**6.3. The graph $\Gamma_E$ and its properties.**

The lacunary projector constructed in Theorem 6.6 generates a certain (undirected) graph with vertices in $E$ which we denote by $\Gamma_E$.

**Definition 6.10** We define the graph $\Gamma_E$ as follows:

- The set of vertices of the graph $\Gamma_E$ coincides with $E$;
- Two distinct vertices $A, A' \in E$, $A \neq A'$, are joined by an edge in $\Gamma_E$ (we write $A \leftrightarrow A'$) if there exist contacting lacunae $L, L' \in L_E$, $L \leftrightarrow L'$, such that

$$A = PR_B(L) \text{ and } A' = PR_B(L').$$

More specifically, notation $A \leftrightarrow A'$ means that $A \neq A'$ and there exist lacunae $L, L' \in L_E$ and cubes $Q \in L$ and $Q' \in L'$ such that $Q \cap Q' \neq \emptyset$, $A = PR_B(L)$ and $A' = PR_B(L')$. See Definition 6.4.

Let us note two important properties of the graph $\Gamma_E$.

**Proposition 6.11** For every closed set $E \subset \mathbb{R}^n$ the graph $\Gamma_E$ has the following properties:

(i) $\Gamma_E$ is a $\gamma$-sparse graph where $\gamma = \gamma(n) \geq 1$ is a constant depending only on $n$;

(ii) $deg_{\Gamma_E}(x) \leq C(n)$ for each vertex $x$ of the graph $\Gamma_E$.

**Proof.** (i) We let $Ed$ denote the family of edges of the graph $\Gamma_E$. Let $u \in Ed$ and let $A_u$ and $A'_u$ be the ends of $u$. (We write $u = (A_u, A'_u)$.) Thus $A_u \neq A'_u$, and $A_u \leftrightarrow A'_u$ in $\Gamma_E$.

By Definition 6.10 there exit distinct contacting lacunae $L_u, L'_u \in L_E$, $L_u \leftrightarrow L'_u$, and cubes $Q_u \in L_u, Q'_u \in L'_u$ such that

$$A_u = PR_B(L_u) \text{ and } A'_u = PR_B(L'_u).$$

(6.46)
since \( A_u = \mathbb{P}_B(L_u) \neq \mathbb{P}_B(L'_u) = A'_u \), by part (ii) of Theorem \[6.6\]

\[
\text{diam } Q_u + \text{diam } Q'_u \leq \tilde{\gamma} \|A_u - A'_u\| \tag{6.47}
\]

where \( \tilde{\gamma} \geq 1 \) is an absolute constant. Note that, by part (i) of this theorem, \( A_u \in \tilde{\gamma}Q \) and \( A'_u \in \tilde{\gamma}Q' \).

Since \( Q_u \cap Q'_u \neq \emptyset \), by part (i) of Lemma \[4.1\] \( Q'_u \subset 5\tilde{\gamma}Q \). Hence,

\[
A_u, A'_u \subset 5\tilde{\gamma}Q. \tag{6.48}
\]

Furthermore, by \( (6.47) \),

\[
\text{diam } Q_u \leq \tilde{\gamma} \|A_u - A'_u\|. \tag{6.49}
\]

Now let us consider a mapping \( T : \text{Ed} \rightarrow W_E \) defined by

\[
T(u) := Q_u, \quad u \in \text{Ed}.
\]

In general, this mapping is not one-to-one. Prove that \( T \) is “almost” one-to-one, i.e., for each \( Q \in W_E \) the set of its sources

\[
T^{-1}(Q) := \{ u \in \text{Ed} : T(u) = Q \}
\]

has the cardinality

\[
\# T^{-1}(Q) \leq M = M(n). \tag{6.50}
\]

Let \( u = (A_u, A'_u) \in T^{-1}(Q) \). In other words, \( u \in \text{Ed} \) is an edge of the graph \( \Gamma_E \) with the ends at points \( A_u \) and \( A'_u \).

Let \( L^{(Q)} \in \mathcal{L}_E \) be a lacuna containing \( Q \). Then, by \( (6.46) \),

\[
A_u = \mathbb{P}_B(L^{(Q)}) \quad \text{for every } u \in T^{-1}(Q)
\]

proving that \( A_u \) depend only on \( Q \) and does not depend on \( u \in T^{-1}(Q) \). We denote this common value of \( A_u \) by \( A^{(Q)} \).

Thus \( u = (A^{(Q)}, A'_u) \) for every \( u \in T^{-1}(Q) \). We know that \( A'_u = \mathbb{P}_B(L'_u) \) where \( L'_u \in \mathcal{L}_E \) is a lacuna contacting to \( L^{(Q)} \) \( (L'_u \leftrightarrow L^{(Q)}), \) see Definition \[6.4\]. But, by Proposition \[6.5\] the number of such lacunae is bounded by a constant \( M = M(n) \) proving the required estimate \( (6.50) \).

Let \( M_Q := \# T^{-1}(Q) \). We know that \( M_Q \leq M \) for every \( Q \in W_E \). Let us enumerate the elements of \( T^{-1}(Q) \):

\[
T^{-1}(Q) = \{ u^{(1)}_{Q}, ..., u^{(M_Q)}_{Q} \}.
\]

Consider a partition of \( Q \) into \( M^p \) equal cubes \( \{H^{(1)}_{Q}, ..., H^{(M^p)}_{Q}\} \) of diameter \( \text{diam } Q/M \). To each edge \( u = u^{(i)}_{Q} \in T^{-1}(Q) \) we assign a cube

\[
K_u := \frac{1}{2} H^{(i)}_{Q}.
\]

Let \( u = (A_u, A'_u) \), i.e., the points \( A_u, A'_u \in E \) are the ends of the edge \( u \). Clearly, \( \text{diam } K_u \leq \text{diam } Q \) so that, by \( (6.49) \),

\[
\text{diam } K_u \leq \tilde{\gamma} \|A_u - A'_u\|.
\]
It is also clear that $Q \subset \lambda K_u$ for some constant $\lambda = \lambda(n) \geq 1$. Hence, by (6.48), $A_u, A_u' \in 5\lambda \gamma K_u$.

Finally, it remains to note that the cubes of the family

$$\{ \frac{1}{2}H_U^{(i)} : Q \in W_E, 1 \leq i \leq M_Q \}$$

are pairwise disjoint proving that the cubes of the family $\{ \mathcal{K}_u : u \in \text{Ed} \}$ are pairwise disjoint as well.

These properties of the family $\{ \mathcal{K}_u : u \in \text{Ed} \}$ show that the graph $\Gamma_E$ satisfies conditions of Definition 1.4 with a constant $\gamma = 5\lambda \gamma$ proving that $\Gamma_E$ is a $\gamma$-sparse graph.

(ii). Let $A \in E$. We have to prove that the set $\text{Ad}(A) := \{ A' \in E : A' \leftrightarrow A \}$ of vertices adjacent to $A$ consists of at most $C = C(n)$ elements.

By Definition 6.10 for every $A' \in \text{Ad}(A)$ there exist contacting lacunae $L, L' \in \mathcal{L}_E (L \lneqq L')$ such that $A = \mathcal{PR}_E(L)$ and $A' = \mathcal{PR}_E(L')$ so that $\# \text{Ad}(A)$ is bounded by the cardinality of the set of lacunae

$$I = \{ L' \in \mathcal{L}_E : A' = \mathcal{PR}_E(L') \text{ and } \exists L \in \mathcal{L}_E \text{ such that } L \lneqq L' \text{ and } \mathcal{PR}_E(L) = A \}.$$

But, by part (iii) of Theorem 6.6 $\# \{ L \in \mathcal{L}_E : \mathcal{PR}_E(L) = A \} \leq C_1(n)$, and, by Proposition 6.5 $\# \{ L' \in \mathcal{L}_E : L' \lneqq L \} \leq C_2(n)$. Hence

$$\text{deg}_{\Gamma_E}(A) = \# \text{Ad}(A) \leq \# I \leq C_1(n)C_2(n)$$

proving part (ii) of the proposition.

The proof of the proposition is complete. \( \square \)

### 7. The lacunary extension operator: a proof of the variational criterion.

#### 7.1. The variational criterion: necessity.

As we have mentioned above, in [28] we have proved a criterion which provides a characterization of the trace space $L_p^1(\mathbb{R}^n)|E$ in terms of certain local oscillations of functions on subsets of the set $E$. Theorem 1.3 which we prove in this section refines and generalizes this criterion to the case of jet-spaces generated by $L_p^m(\mathbb{R}^n)$-functions, $m \geq 1$, $p > n$.

**Proof.** (Necessity.) We prove a slightly more general result which immediately implies the necessity part of Theorem 1.3

**Proposition 7.1** Let $n < p < \infty$. Let $F \in C^{m-1}(\mathbb{R}^n) \cap L_p^m(\mathbb{R}^n)$ and let $P_x = T_x^{m-1}[F]$, $x \in E$.

Then for every constant $\gamma \geq 1$, every family $\{ Q_i : i \in I \}$ of pairwise disjoint cubes in $\mathbb{R}^n$, every collection of points $x_i, y_i \in (\gamma Q_i) \cap \mathcal{E}$, $i \in I$, and every multiindex $\beta, |\beta| \leq m - 1$, the following inequality

$$\sum_{i \in I} \frac{|D^\beta P_{x_i}(x_i) - D^\beta P_{y_i}(x_i)|^p}{(\text{diam } Q_i)^{p-\beta-n}} \leq C \| F \|_{L_p^m(\mathbb{R}^n)}^p$$

holds. Here $C$ is a constant depending only on $m, n, p$ and $\gamma$. 

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Proof. Let \( q = (n + p)/2 \). Let \( Q \) be a cube in \( \mathbb{R}^n \) and let \( K := \gamma Q \). Fix two points \( x, y \in K \). Recall that \( Q_{xy} = Q(x, \|x - y\|) \) so that \( Q_{xy} \subset 2K \). By inequality (1.15), for every multiindex \( \beta, |\beta| \leq m - 1, \)

\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq C \|x - y\|^{m-|\beta|} \left( \frac{1}{|Q_{xy}|} \int_{Q_{xy}} \left( \nabla m F(u) \right)^q du \right)^\frac{1}{q} 
\]

\[
\leq C \|x - y\|^{m-|\beta|} \left( \int_{Q_{xy}} \left( \nabla m F(u) \right)^q du \right)^\frac{1}{q}.
\]

Since \( |\beta| \leq m - 1 \) and \( n < q \), we have \( m - |\beta| - \frac{2}{q} > 0 \). Hence

\[
|D^\beta P_x(x) - D^\beta P_y(x)| \leq C \left( \text{diam } Q \right)^{m-|\beta|} \left( \frac{1}{2K} \int_{2K} \left( \nabla m F(u) \right)^q du \right)^\frac{1}{q}
\]

where \( C \) is a constant depending only on \( m, n, p \) and \( \gamma \). By this inequality,

\[
|D^\beta P_x(x) - D^\beta P_y(x)|^p \leq C \left( \text{diam } Q \right)^{p(m-|\beta|)} \left( \mathcal{M}[\nabla m F^q](z) \right)^\frac{p}{q}
\]

for arbitrary \( z \in Q \). Integrating this inequality over \( Q \) (with respect to \( z \)) we obtain the following:

\[
\frac{|D^\beta P_x(x) - D^\beta P_y(x)|^p}{\left( \text{diam } Q \right)^{(m-|\beta|)p-n}} \leq C \int_Q \left( \mathcal{M}[\nabla m F^q](z) \right)^{\frac{p}{q}} dz.
\]

Hence,

\[
I_\beta := \sum_{i \in I} \frac{|D^\beta P_{x_i}(x_i) - D^\beta P_{y_i}(x_i)|^p}{\left( \text{diam } Q_i \right)^{(m-|\beta|)p-n}} \leq C \sum_{i \in I} \int_{Q_i} \left( \mathcal{M}[\nabla m F^q](z) \right)^{\frac{p}{q}} dz 
\]

\[
\leq C \int_{\mathbb{R}^n} \left( \mathcal{M}[\nabla m F^q](z) \right)^{\frac{p}{q}} dz
\]

so that, by the Hardy-Littlewood maximal theorem,

\[
I_\beta \leq C \int_{\mathbb{R}^n} \left( \nabla m F )^q(z) \right) dz \leq C \|F\|_{L^p_m(\mathbb{R}^n)}^p
\]

proving the proposition. \( \square \)

Let us prove the necessity part of the theorem. Let \( P = \{P_x : x \in E\} \) be a family of polynomials of degree at most \( m - 1 \) indexed by points of \( E \). Suppose there exists a \( C^{m-1} \)-function \( F \in L^m_p(\mathbb{R}^n) \) such that \( T^{m-1}_x[F] = P_x \) for every \( x \in E \).

Let \( \gamma \geq 1 \) and let \( \{x_i, y_i : i = 1, ..., k\} \) be an arbitrary finite \( \gamma \)-sparse collection of two point subsets of \( E \). Then, by Definition 1.2, there exists a family \( \{Q_i, i = 1, ..., k\} \) of pairwise disjoint cubes in \( \mathbb{R}^n \) satisfying condition (1.4), so that, by Proposition 7.1

\[
\sum_{i=1}^k \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{x_i}(x_i) - D^\alpha P_{y_i}(x_i)|^p}{\left( \text{diam } Q_i \right)^{(m-|\alpha|)p-n}} \leq C \|F\|_{L^p_m(\mathbb{R}^n)}^p.
\]
Here $C$ is a constant depending only on $m, n, p,$ and $γ$.

But, by (1.4), diam $Q_i ≤ γ \|x_i − y_i\|$, $i = 1, \ldots, k$, so that

$$\sum_{i=1}^{k} \sum_{|α|≤m-1} \frac{|D^n P_{x_i}(x_i) - D^n P_{y_i}(x_i)|^p}{\|x_i - y_i\|^{(m-|α|)p-n}} \leq γ^{m-p-n} C \|F\|^{p}_{L^n_p(\mathbb{R}^n)}. $$

Hence $N_{m,p,E}(P) ≤ γ^{m-n/p} C^{1/p} \|F\|^{p}_{L^n_p(\mathbb{R}^n)}$, see (1.6), and the proof of the necessity is complete.

### 7.2. The variational criterion: sufficiency.

Let $γ := 10^4 \bar{γ}$ where $\bar{γ}$ is the constant from Theorem 6.6. Let $P = \{P_x ∈ \mathcal{P}_{m-1}(\mathbb{R}^n) : x ∈ E\}$ be a Whitney ($m-1$)-field on $E$.

Let $λ := N_{m,p,E}(P)^p$, see (1.6). Suppose that $λ < ∞$. Then, by (1.6), for every finite $γ$-sparse collection $\{(x_i, y_i) : i = 1, \ldots, k\}$ of two point subsets of $E$ the following inequality

$$\sum_{i=1}^{k} \sum_{|α|≤m-1} \frac{|D^n P_{x_i}(x_i) - D^n P_{y_i}(x_i)|^p}{\|x_i - y_i\|^{(m-|α|)p-n}} \leq λ \quad (7.1)$$

holds.

Let us introduce the lacunary modification of the Whitney extension method. Let $L ∈ L_E$ be a lacuna. For every cube $Q ∈ L$ we put

$$a_Q := \mathbb{P}_{E}(L). \quad (7.2)$$

Here $\mathbb{P}_{E} : L_E → E$ is the “projector” from Theorem 6.6.

Note that, by property (i) of this theorem,

$$a_Q = \mathbb{P}_{E}(L) ∈ \bar{γ}Q \quad \text{for every } Q ∈ L. \quad (7.3)$$

Hence, by (4.5), for each $Q ∈ L$

$$a_Q \quad \text{is a } θ- \text{nearest point to } Q \text{ with } θ = (\bar{γ} + 1)/2. \quad (7.4)$$

Then we construct a function $F$ on $\mathbb{R}^n$ using the Whitney extension formula (4.7).

Let us prove that the function $F$ satisfies the following conditions:

(i). $F$ is a $C^{m-1}$-function such that $T^{m-1}_x[F] = P_x$ for every $x ∈ E$;

(ii). $F ∈ L^n_p(\mathbb{R}^n)$ and

$$\|F\|^{p}_{L^n_p(\mathbb{R}^n)} ≤ C λ^{1/p}$$

where $C$ is a constant depending only on $m, n$ and $p$.

Prove (i). For every multiindex $β, |β| ≤ m − 1$, and every $x, y ∈ E$, by (7.1),

$$\frac{|D^β P_{x}(x) - D^β P_{y}(x)|^p}{\|x - y\|^{(m-|β|)p-n}} ≤ λ.$$ 

Hence

$$|D^β P_{x}(x) - D^β P_{y}(x)| ≤ C λ^{1/p} \|x - y\|^{m-|β|-1} \cdot \|x - y\|^{1-\frac{p}{n}}. $$

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Since \( n < p \),
\[
D^\beta P_\alpha(x) - D^\beta P_\beta(x) = o(||x - y||^{n - |\beta| - 1}) \quad \text{as} \quad y \to x, \ y \in E.
\]

Thus the Whitney \((m - 1)\)-field \( P = \{P_\alpha \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E\} \) satisfies the hypothesis of the Whitney extension theorem \([33]\). Recall that in Section 4 we have constructed the function \( F \) using a modification of the Whitney extension method where for each \( Q \in W_E \) the point \( a_Q \) is \( \theta \)-nearest to \( Q \) with \( \theta = 5.5 \). See \((7.4)\). As we have noted in Section 5, see a remark after \((4.9)\), such a modification provides the statement (i) as well.

We turn to the proof of statement (ii). Let us fix a multiindex \( \beta \) of order \( |\beta| = m - 1 \) and prove that
\[
D^\beta F \in L^1_p(\mathbb{R}^n) \quad \text{and} \quad ||D^\beta F||_{L^1_p(\mathbb{R}^n)} \leq C \lambda^\frac{1}{p}.
\]

The proof of this inequality is based on two auxiliary statements. The first of them is the following combinatorial

**Theorem 7.2** (\([3, 9]\)) Let \( \mathcal{A} = \{Q\} \) be a collection of cubes in \( \mathbb{R}^n \) with covering multiplicity \( M(\mathcal{A}) < \infty \). Then \( \mathcal{A} \) can be partitioned into at most \( N = 2^{n-1}(M(\mathcal{A}) - 1) + 1 \) families of disjoint cubes.

Recall that **covering multiplicity** \( M(\mathcal{A}) \) of a family of cubes \( \mathcal{A} \) is the minimal positive integer \( M \) such that every point \( x \in \mathbb{R}^n \) is covered by at most \( M \) cubes from \( \mathcal{A} \).

The second auxiliary result which we are needed for the proof of the sufficiency is a certain variational description of the space \( L^1_p(\mathbb{R}^n) \) in terms of local oscillations. This result follows from a description of Sobolev spaces obtained in \([2]\); see there §4, subsection 3°.

**Theorem 7.3** Let \( p > n \) and let \( \tau > 0 \). Let \( G \) be a continuous function on \( \mathbb{R}^n \) satisfying the following condition: There exists a constant \( A > 0 \) such that for every finite family \( \{Q_i : i = 1, ..., k\} \) of pairwise disjoint equal cubes in \( \mathbb{R}^n \) of diameter \( \text{diam } Q_i \leq \tau \) and every \( x_i \in Q_i \) the following inequality
\[
\sum_{i=1}^{k} \frac{|G(x_i) - G(c_{Q_i})|^p}{(\text{diam } Q_i)^{p-n}} \leq A
\]
holds. Then \( G \in L^1_p(\mathbb{R}^n) \) and \( ||G||_{L^1_p(\mathbb{R}^n)} \leq C(n, p) A^{\frac{1}{p}} \).

**Proof.** For the case \( \tau = \infty \) this criterion follows from a description of Sobolev spaces obtained in \([2]\); see there §4, subsection 3°.

Prove the result for \( 0 < \tau < \infty \). Using a result from \([2]\) related to an atomic decomposition of the modulus of smoothness in \( L_p \), see there §2, subsection 2°, Theorem 4, and the theorem’s hypothesis, we conclude that the first modulus of continuity of \( G \) in \( L_p(\mathbb{R}^n) \), the function \( \Omega_1(G, t)_{L_p(\mathbb{R}^n)} \), satisfies the following inequality:
\[
\Omega_1(G, t)_{L_p(\mathbb{R}^n)} \leq C(n, p) A^{\frac{1}{p}} t \quad \text{for every} \quad t \in (0, \tau].
\]

But the function \( \Omega_1(G, t)_{L_p(\mathbb{R}^n)} / t \) is a quasi-monotone function on \( \mathbb{R}_+ \), i.e.,
\[
\Omega_1(G, t_2)_{L_p(\mathbb{R}^n)} / t_2 \leq C \Omega_1(G, t_1)_{L_p(\mathbb{R}^n)} / t_1 \quad \text{for every} \quad 0 < t_1 < t_2,
\]
where \( C > 1 \) is an absolute constant.

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Hence,
\[ \Omega_1(G, t)_{L_p(R^n)} / t \leq C \Omega_1(G, \tau)_{L_p(R^n)} / \tau \leq C(n, p) A^{\frac{1}{p}} \]
for every \( t > \tau \)
so that
\[ \Omega_1(G, t)_{L_p(R^n)} \leq C(n, p) A^{\frac{1}{p}} t \]
for every \( t > 0 \).

It is shown in [2], § 4, Subsection 3°, that this property implies that \( G \in L^1_p(R^n) \) and that \( \| G \|_{L^1_p(R^n)} \leq C(n, p) A^{\frac{1}{p}} \). The proof of the lemma is complete. \( \square \)

We turn to the proof of the statement (7.5). We will do this by applying Theorem 7.3 to a function \( G := D^\beta F \). Basing on the following three lemmas we show that this function satisfies the hypothesis of Theorem 7.3.

Let \( Q = \{ Q_1, \ldots, Q_k \} \) be a family of pairwise disjoint equal cubes in \( R^n \). Let \( c_i := c_{Q_i}, i = 1, \ldots, k, \) be the center of the cube \( Q_i \).

**Lemma 7.4** Suppose that
\[ \text{dist}(c_i, E) \leq 40 \text{ diam } Q_i \text{ for all } i = 1, \ldots, k. \] (7.6)

Then for every \( \beta, |\beta| = m - 1 \), and every \( x_i \in Q_i \) the following inequality
\[ \sum_{i=1}^{k} \frac{|D^\beta F(x_i) - D^\beta F(c_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq C \lambda. \]

Here \( C > 0 \) is a constant depending only on \( m, n, \) and \( p \).

**Proof.** First we will prove the lemma for the case \( c_i \in E, i = 1, \ldots, k \). Let
\[ I_1(F) := \sum \left\{ \frac{|D^\beta F(x_i) - D^\beta F(c_i)|^p}{(\text{diam } Q_i)^{p-n}} : i \in \{1, \ldots, k\}, x_i \in E, x_i \neq c_i \right\}. \] (7.7)

Prove that \( I_1(F) \leq \lambda \).

Let us consider a collection of two point sets
\[ \mathcal{A} := \{ (x_i, c_i) : i = 1, \ldots, k, x_i \in E, x_i \neq c_i \}. \]

Prove that \( \mathcal{A} \) is a 1-sparse collection, see Definition 1.2. In fact, since \( x_i, c_i \in Q_i \), there exists a cube \( K_i \) such that \( x_i, c_i \in K_i \subset Q_i \) and \( \text{diam } K_i = |x_i - c_i| \). Since the cubes \( \{ Q_i \} \) are pairwise disjoint, the same is true for the cubes \( \{ K_i \} \) proving that \( \mathcal{A} \) is 1-sparse. By (1.5), \( D^\beta F(x_i) = D^\beta P_{x_i}(x_i) \) and \( D^\beta F(c_i) = D^\beta P_{x_i}(c_i) \). But \( D^\beta P_{x_i} \) is a constant function whenever \( |\beta| = m - 1 \) so that \( D^\beta F(x_i) = D^\beta P_{x_i}(c_i) \). Hence, by the theorem’s hypothesis (with \( |\beta| = m - 1 \)), see (7.1),
\[ I_1(F) \leq \sum \left\{ \frac{|D^\beta P_{x_i}(c_i) - D^\beta P_{x_i}(c_i)|^p}{||x_i - c_i||^{p-n}} : i \in \{1, \ldots, k\}, x_i \in E, x_i \neq c_i \right\} \leq \lambda. \] (7.8)

Let
\[ I_2(F) := \sum \left\{ \frac{|D^\beta F(x_i) - D^\beta F(c_i)|^p}{(\text{diam } Q_i)^{p-n}} : i \in \{1, \ldots, k\}, x_i \in R^n \setminus E \right\}. \] (7.9)
Let
\[ Q := \{ Q_i : x_i \in \mathbb{R}^n \setminus E \}. \]

This family consists of at most \( k \) pairwise disjoint cubes.

We recall that by \( \gamma \) we denote the constant from Theorem 6.6. Let us introduce a family of cubes
\[ \tilde{Q} := \{ 5\gamma Q : Q \in Q \}. \]

Clearly, the covering multiplicity \( MC(\tilde{Q}) \leq C(n) \) so that, by Theorem 7.2, this family can be partitioned into at most \( N(n) \) subfamilies of pairwise disjoint cubes. Therefore in this proof without loss of generality we can assume that the family \( \tilde{Q} \) itself consists of pairwise disjoint cubes.

Fix a cube \( Q \in Q \). Thus \( Q = Q_i \) for some \( i \in \{ 1, ..., k \} \). Let \( x_Q = x_i \). Thus \( x_Q \notin E \) and \( c_Q \in E \) for every \( Q \in Q \). Note also that \( D^\beta F(c_Q) = D^\beta P_{c_Q}(c_Q) \).

Since \( x_Q \notin E \), there exists a Whitney cube
\[ K_Q \in W_E \hspace{1em} \text{which contains} \hspace{1em} x_Q. \quad (7.10) \]

Recall that given a lacuna \( L \in \mathcal{L}_E \) and a cube \( H \in L \) we have \( a_H = \text{PR}_E(L) \), see (7.2). We also recall that by \( T(K_Q) \) we denote the family of Whitney’s cubes intersecting \( K_Q \). See (4.11).

For the sake of brevity let us introduce the following notation:
\[ S(Q) := |D^\beta F(x_Q) - D^\beta F(c_Q)| \]
and
\[ V(Q) := |D^\beta P_{c_Q}(c_Q) - D^\beta P_{a_k}(c_Q)|. \]

Also given \( H \in T(K_Q) \) and a multiindex \( \xi \) with \( |\xi| \leq m - 1 \) let
\[ L(\xi : H, Q) := |D^\xi P_{a_k}(a_k) - D^\xi P_{a_k}(a_k)|. \]

Then, by Lemma 4.5,
\[
S(Q) \leq C \left\{ V(Q) + \sum_{H \in T(K_Q)} \sum_{|\xi| \leq m - 1} \frac{L(\xi : H, Q)}{(\text{diam } K_Q)^{m-1-|\xi|}} \right\}.
\]

Since \#T(K_Q) \leq N(n), see Lemma 4.11, we have
\[
\frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \leq \frac{C V(Q)^p}{(\text{diam } Q)^{p-n}} + C \sum_{H \in T(K_Q)} \sum_{|\xi| \leq m - 1} \left( \frac{\text{diam } K_Q}{\text{diam } Q} \right)^{p-n} \frac{(\text{diam } K_Q)^{m-|\xi| p-n}}{(\text{diam } K_Q)^{(m-1)} p-n}. \]

Prove that \( a_k \in 5\gamma Q \). In fact, since \( x_Q \in K_Q \cap Q \), we have
\[
\text{diam } K_Q \leq 4 \text{dist}(K_Q, E) \leq 4 \text{dist}(x_Q, E) \leq 4\|x_Q - c_Q\| \leq 2 \text{diam } Q \quad (7.11)
\]

Hence, for every \( y \in K_Q \),
\[
\|y - c_Q\| \leq \|y - x_Q\| + \|x_Q - c_Q\| \leq \text{diam } K_Q + r_Q \leq 2 \text{diam } Q + r_Q = 5r_Q
\]
proving that
\[ K_Q \subset 5Q. \quad (7.12) \]
On the other hand, by (7.3), \( a_{K_Q} \in \tilde{\gamma} K_Q \) so that

\[
a_{K_Q} \in 5\tilde{\gamma} Q.
\] (7.13)

In particular, by inequality (7.11),

\[
\frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \leq C \left\{ \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} + \sum_{H \in T(K_Q)} \sum_{|\xi| \leq m-1} \frac{L(\xi : H, Q)^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}} \right\} .
\]

Prove that \( a_H \in \gamma K_Q \) whenever \( H \in T(K_Q) \). Since \( H \cap K_Q \not= \emptyset \), by Lemma 4.1 we have \( \text{diam } H \leq 4 \text{diam } K_Q \) so that \( H \subset 9K_Q \). On the other hand, by Theorem 6.6, \( a_H \in \gamma H \). Hence

\[
a_H \in 9\gamma K_Q \quad \text{for every} \quad H \in T(K_Q)
\] (7.14)

proving that

\[
a_H \in \gamma K_Q.
\] (7.15)

(Recall that \( \gamma := 10^4 \tilde{\gamma} \), see the beginning of this subsection.)

By \( H_Q \) we denote a cube \( H \in T(K_Q) \) for which the quantity

\[
\sum_{|\xi| \leq m-1} \frac{L(\xi : H, Q)^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}}
\]
is maximal on \( T(K_Q) \).

(7.16)

Since \( \#T(K_Q) \leq N(n) \), we obtain the following inequality

\[
\frac{S(Q)^p}{(\text{diam } Q)^{p-n}} \leq C \left\{ \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} + \sum_{|\xi| \leq m-1} \frac{L(\xi : H_Q, Q)^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}} \right\} .
\]

Hence

\[
I_2(F) := \sum_{Q \in Q} \frac{S(Q)^p}{(\text{diam } Q)^{p-n}}
\]

\[
\leq C \left\{ \sum_{Q \in Q} \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} + \sum_{|\xi| \leq m-1} \sum_{Q \in Q} \frac{L(\xi : H_Q, Q)^p}{(\text{diam } K_Q)^{(m-|\xi|)p-n}} \right\}
\]

\[
= C \{ I_2^{(1)}(F) + I_2^{(2)}(F) \}.
\]

Prove that

\[
I_2^{(1)}(F) := \sum_{Q \in Q} \frac{V(Q)^p}{(\text{diam } Q)^{p-n}} \leq \lambda.
\] (7.17)

In fact, we know that \( a_{K_Q} \in \tilde{Q} := 5\tilde{\gamma} Q \) and \( c_Q = c_{\tilde{Q}} \), and the squares of the family \( \tilde{Q} = \{ \tilde{Q} \} \) are pairwise disjoint. This enables us to use the same approach as in the proof of the inequality \( I_1(F) \leq \lambda \). This implies the required estimate \( I_2^{(1)}(F) \leq \lambda \).
Prove that $I_2^{(2)}(F) \leq C \lambda$.

Let $\mathcal{K} := \{K_Q : Q \in Q\}$. We know that $K_Q \subset 5Q \subset \tilde{Q} = 5\gamma Q$ so that the cubes of the family $\mathcal{K}$ are pairwise disjoint. We also recall that, by $\text{(7.15)}$, $a_{H_Q}, a_{K_Q} \in \gamma K_Q$ for every $Q \in Q$. Furthermore, we know that $H_Q \cap K_Q \neq \emptyset$ and that $a_{K_Q} = \mathcal{F}(\mathcal{E}(L), a_{H_Q}) = \mathcal{F}(\mathcal{E}(L'))$ where $L$ and $L'$ are lacunae containing $K_Q$ and $H_Q$ respectively. Therefore, by part (ii) of Theorem 6.6,

$$\text{diam } K_Q \leq \tilde{\gamma} \|a_{K_Q} - a_{H_Q}\| \leq \gamma \|a_{K_Q} - a_{H_Q}\|$$

provided $a_{K_Q} \neq a_{H_Q}$. Thus the family $\{\{a_{K_Q}, a_{H_Q}\} : Q \in Q\}$ of two point subsets of $E$ is $\gamma$-sparse. See Definition 1.2 (We also note that $\text{diam } K_Q \sim \|a_{K_Q} - a_{H_Q}\|$.)

Recall that

$$I_2^{(2)}(F) := \sum_{[\xi] \leq m-1} \sum_{Q \in Q} |D^{\xi} P_{a_{H_Q}}(a_{K_Q}) - D^{\xi} P_{a_{K_Q}}(a_{K_Q})|^p \frac{(\text{diam } K_Q)^{\|\xi\|p-n}}{(\text{diam } K_Q)^{\|\xi\|p-n}}. \quad (7.18)$$

Then, by assumption (7.1),

$$I_2^{(2)}(F) \sim \sum_{[\xi] \leq m-1} \sum_{Q \in Q} \frac{|D^{\xi} P_{a_{H_Q}}(a_{K_Q}) - D^{\xi} P_{a_{K_Q}}(a_{K_Q})|^p}{\|a_{K_Q} - a_{H_Q}\|^{\|\xi\|p-n}} \leq \lambda$$

proving the required inequality $I_2^{(2)}(F) \leq C \lambda$.

We turn to the proof of the lemma in the general case, i.e., for an arbitrary family of equal cubes $\{Q_i : i = 1, \ldots, k\}$ satisfying inequality (7.6). Let $u_i$ be a nearest point on $E$ to the square $Q_i$, $i = 1, \ldots, k$. By (7.6),

$$\|u_i - c_i\| \leq \text{dist}(c_i, E) \leq 40 \text{ diam } Q_i,$$

so that

$$\|u_i - y\| \leq 41 \text{ diam } Q_i \quad \text{for every } y \in Q_i. \quad (7.19)$$

Let

$$\hat{Q}_i := Q(u_i, R_i) \quad \text{where } R_i = 41 \text{ diam } Q_i. \quad (7.20)$$

Then, by (7.19), $\hat{Q}_i \supset Q_i$. Also it can be readily seen that $\hat{Q}_i \subset \gamma' Q_i$ with $\gamma' = 122$. Since the cubes $\{Q_i\}$ are pairwise disjoint, the family $\hat{Q} = \{\hat{Q}_i : i = 1, \ldots, k\}$ has covering multiplicity $M(\hat{Q}) < C(n)$. Therefore, by Theorem 7.2, $\hat{Q}$ can be partitioned into at most $N(n)$ subfamilies each consisting of pairwise disjoint cubes. This enables us to assume that the family $\hat{Q}$ itself consists of pairwise disjoint cubes.

We obtain

$$J := \sum_{i=1}^{k} \frac{|\partial^{\xi} F(x_i) - \partial^{\xi} F(c_i)|^p}{(\text{diam } Q_i)^{\|\xi\|p-n}} \leq C \left\{ \sum_{i=1}^{k} \frac{|\partial^{\xi} F(x_i) - \partial^{\xi} F(u_i)|^p}{(\text{diam } \hat{Q}_i)^{\|\xi\|p-n}} + \sum_{i=1}^{k} \frac{|\partial^{\xi} F(c_i) - \partial^{\xi} F(u_i)|^p}{(\text{diam } \hat{Q}_i)^{\|\xi\|p-n}} \right\} = C \{J_1 + J_2\}.$$
But now \( u_i = c_\tilde{Q}_i \in E \), so that the problem is reduced to the case proven above. Thus \( J_1, J_2 \leq C \lambda \) where \( C = C(m, n, p) \). Hence \( J \leq C \lambda \) proving the lemma. \( \square \)

**Lemma 7.5** Suppose that

\[
\text{dist}(c_i, E) > 40 \text{ diam } Q_i, \quad i = 1, ..., k. \tag{7.21}
\]

Then for every \( \beta, |\beta| = m - 1 \), and every \( x_i \in Q_i \) the following inequality

\[
\sum_{i=1}^{k} \frac{|D^\beta F(x_i) - D^\beta F(c_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq C \lambda
\]

holds. Here \( C = C(m, n, p) \).

**Proof.** For every cube \( Q \in Q \), by (7.21), \( \text{dist}(c_i, E) > 40 \text{ diam } Q > 0 \) so that \( Q \subset \mathbb{R}^n \setminus E \). Let \( K_Q \) be a Whitney cube which contains \( c_i \). Prove that \( Q \subset K_Q^* = \frac{9}{8} K_Q \).

In fact,

\[
\text{diam } Q < \frac{1}{40} \text{ dist}(c_i, E) \leq \frac{1}{40} (\text{diam } K_Q + \text{dist}(K_Q, E))
\]

so that

\[
\text{diam } Q \leq \frac{1}{40} (\text{diam } K_Q + 4 \text{ diam } K_Q) = \frac{1}{8} \text{ diam } K_Q. \tag{7.22}
\]

Hence for every \( z \in Q \) we have

\[
\|z - c_{K_Q}\| \leq \|z - c_Q\| + \|c_Q - c_{K_Q}\| \leq \frac{1}{2} \text{ diam } Q + \frac{1}{2} \text{ diam } K_Q
\]

\[
\leq \frac{1}{2} \cdot \frac{1}{8} \text{ diam } K_Q + \frac{1}{2} \text{ diam } K_Q = \left(\frac{1}{8} + 1\right) r_{K_Q}
\]

so that \( Q \subset \frac{9}{8} K_Q = K_Q^* \). In particular, \( x_Q = x_i \in K_Q^* \) provided \( Q = Q_i \).

Let \( K \in W_E \) and let

\[ Q(K) := \{ Q \in Q : c_Q \in K \}. \]

Let

\[ \mathcal{K} := \{ K \in W_E : Q(K) \neq \emptyset \}. \]

Then for each \( K \in \mathcal{K} \), by Lemma 4.6

\[
|D^\beta F(\bar{Q}) - D^\beta F(c_Q)| \leq C \|x_Q - c_Q\| \sum_{H \in T(K)} \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|}{(\text{diam } K)^{m-|\xi|}}.
\]

By (7.15),

\[
a_K, a_H \in \gamma K \quad \text{for every } H \in T(K) . \tag{7.23}
\]
Now we have

\[ I_K := \sum_{Q \in Q(K)} \frac{|D^\theta F(x_Q) - D^\theta F(c_Q)|^p}{(\text{diam } Q)^{p-n}} \]

\[ \leq C \left\{ \sum_{Q \in Q(K)} \left( \frac{||x_Q - c_Q||}{\text{diam } Q} \right)^p |Q| \right\} \left\{ \sum_{H \in T(K)} \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|}{(\text{diam } K)^{m-|\xi|}} \right\}^p \]

\[ \leq C |K| \left\{ \sum_{H \in T(K)} \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|}{(\text{diam } K)^{m-|\xi|}} \right\}^p. \]

Since \#T(K) \leq N(n), see Lemma 4.1, we obtain

\[ I_K \leq C \sum_{H \in T(K)} \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{m-|\xi|-p-n}}. \]

Let \( \tilde{K} \in T(K) \) be a cube such that the quantity

\[ \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{m-|\xi|-p-n}} \]

takes the maximal value on \( T(K) \).

Then

\[ I_K \leq C \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_H}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{m-|\xi|-p-n}}. \]

This enables us to estimate \( I \) as follows:

\[ I := \sum_{i=1}^k \frac{|D^\theta F(x_i) - D^\theta F(c_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq \sum_{K \in \mathcal{K}} I_K \]

so that

\[ I \leq C \sum_{K \in \mathcal{K}} \sum_{|\xi| \leq m-1} \frac{|D^\xi P_{a_K}(a_K) - D^\xi P_{a_K}(a_K)|^p}{(\text{diam } K)^{m-|\xi|-p-n}}. \] (7.25)

As in the proof of the previous lemma, using Theorem 7.2 we can assume that the cubes of the family \( \mathcal{K} \) are pairwise disjoint. Furthermore, by (7.2) and by part(ii) of Theorem 6.6,

\[ \text{diam } K \leq \gamma \|a_{\tilde{K}} - a_K\| \leq \gamma \|a_K - a_K\| \]
provided \( a_K \neq a_K \). (Recall that \( \gamma = 10^4 \gamma \).) In particular, this inequality and (7.23) imply the following:

\[
\frac{1}{\gamma} \text{diam } K \leq |a_K - a_K| \leq \gamma \text{diam } K.
\]  

(7.26)

Hence

\[
I \leq C \sum_{K \in \mathcal{K}} \sum_{|\xi| \leq m-1} \frac{||D^\xi P_{a_K}(a_K) - D^\xi P_{a_K}(a_K)||^p}{||a_K - a_K||^{(m-|\xi|)p-n}}.
\]  

(7.27)

Note that, by (7.23), \( a_K, a_K \in \gamma K \). Combining this with (7.26) and Definition 1.2 we conclude that the family \( \{a_K, a_K\} : K \in \mathcal{K}\) of two point subsets of \( E \) is \( \gamma \)-sparse. (Recall that \( \mathcal{K}\) consists of pairwise disjoint cubes.) This enables us to apply assumption (7.1) to the right hand side of the above inequality. By this assumption, \( I \leq C \lambda \), and the proof of the lemma is complete. \( \square \)

Combining Lemma 7.4 and Lemma 7.5 with the criterion (7.3), we conclude that \( F \in C^{m-1}(\mathbb{R}^n) \), and for every multiindex \( \beta \) of order \( m - 1 \) the function \( D^\beta F \in L^1_p(\mathbb{R}^n) \) and \( ||D^\beta F||_{L^1_p(\mathbb{R}^n)} \leq C \lambda t^\beta \).

Since weak derivatives commute, the function \( F \in L^m_p(\mathbb{R}^n) \) and \( ||F||_{L^m_p(\mathbb{R}^n)} \leq C \lambda t^\beta \).

The proof of Theorem 1.3 is complete. \( \square \)

### 7.3. A refinement of the variational criterion for finite sets.

In this subsection we prove Theorem 1.3. Our proof is based on the following useful property of the lacunary extension operator.

**Proposition 7.6** Let \( E \) be a closed set and let \( P = \{P_x : x \in E\} \) be a Whitney \((m-1)\)-field on \( E \). Let \( F \) be the function obtained by the lacunary modification (7.2) of the Whitney extension formula (4.7). Then the following inequality

\[
\sum_{|\alpha|=m} ||D^\alpha F||_{L^p(\mathbb{R}^n \setminus E)} \leq C \left\{ \sum_{x,y \in E, x \leftrightarrow y} \sum_{|\beta| \leq m-1} \frac{|D^\beta P_x(x) - D^\beta P_y(y)|^p}{||x - y||^{(m-|\beta|)p-n}} \right\}^{\frac{1}{p}}
\]

holds. Here the first sum is taken over all distinct points \( x \) and \( y \) in \( E \) joined by an edge in the graph \( \Gamma_E \). (In this section we use the notation \( x \leftrightarrow y \).)

The constant \( C \) in this inequality depends only on \( m, n, \) and \( p \).

**Proof.** Let a cube \( K \in W_E \) and let \( u \in K \). Then, by Lemma 4.4, for every multiindex \( \alpha, |\alpha| = m, \)

\[
|D^\alpha F(u)| \leq C \sum_{Q \in T(K), a_Q \neq a_K} \sum_{|\xi| \leq m-1} (\text{diam } K)^{|\xi| - m} |D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)|
\]

where \( C = C(n,m) \). See (7.4).

We recall that \( F|_{\mathbb{R}^n \setminus E} \in C^m(\mathbb{R}^n \setminus E) \), \( T(K) \) is the family of Whitney’s cubes touching \( K \), see (4.11), and \( a_Q \) is defined by (7.2). We also recall that \( \# T(K) \) is bounded by \( C(n) \), see Lemma 4.1.

Integrating the latter inequality on \( K \) (with respect to \( u \)) we obtain

\[
\int_K |D^\alpha F(u)|^p du \leq C \sum_{Q \in T(K), a_Q \neq a_K} \sum_{|\xi| \leq m-1} (\text{diam } K)^{|\xi| - m + p} |D^\xi P_{a_Q}(a_K) - D^\xi P_{a_K}(a_K)|^p
\]

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where $C = C(n, m, p)$.

Let us fix a cube $Q \in T(K)$; thus $Q \cap K \neq \emptyset$. Let $L^{(Q)}$ and $L^{(K)}$ be lacunae containing $Q$ and $K$ respectively. We know that

$$a_Q = \mathcal{PR}_E(L^{(Q)}) \quad \text{and} \quad a_K = \mathcal{PR}_E(L^{(K)}),$$

see (7.2), so that, by part (ii) of Theorem 6.6

$$\text{diam } Q + \text{diam } K \leq \gamma ||a_Q - a_K||$$

provided $a_Q \neq a_K$. Recall that $\gamma$ is an absolute constant. Hence

$$\int_{\mathbb{R} \setminus E} |D^p F(u)|^p \ du \leq C \sum_{Q \in T(K), a_Q \neq a_K} \sum_{|\xi| \leq m-1} \frac{|D^p P_{a_Q}(a_K) - D^p P_{a_K}(a_K)|^p}{||a_Q - a_K||^{(m-|\beta|)p-n}}$$

so that, by Definition 6.10

$$\int_{\mathbb{R} \setminus E} |D^{\beta} F(x)|^p \ du \leq C \sum_{Q \in W_E, a_Q \neq a_K} \sum_{|\xi| \leq m-1} \frac{|D^{\beta} P_{a_Q}(a_K) - D^{\beta} P_{a_K}(a_K)|^p}{||a_Q - a_K||^{(m-|\beta|)p-n}}.$$ 

This estimate implies the following inequality

$$I := \int_{\mathbb{R} \setminus E} |D^p F(u)|^p \ du \leq C \sum_{K \in W_E} \sum_{Q \in W_E, a_Q \neq a_K} \sum_{|\xi| \leq m-1} \frac{|D^p P_{a_Q}(a_K) - D^p P_{a_K}(a_K)|^p}{||a_Q - a_K||^{(m-|\beta|)p-n}}$$

so that

$$I \leq C \sum_{Q \in W_E, a_Q \neq a_K} \sum_{|\xi| \leq m-1} \frac{|D^{\beta} P_{a_Q}(a_K) - D^{\beta} P_{a_K}(a_K)|^p}{||a_Q - a_K||^{(m-|\beta|)p-n}}.$$ 

In turn, this inequality implies the following one:

$$I \leq C \sum_{x, y \in E, x \leftrightarrow y} \# R(x, y) \sum_{|\xi| \leq m-1} \frac{|D^p P(x) - D^p P(y)|^p}{||x - y||^{(m-|\beta|)p-n}}. \quad (7.28)$$

Here given $x, y \in E, x \leftrightarrow y$, by $R(x, y)$ we denote a subset of $W_E \times W_E$ consisting of all pairs $(Q, K)$ of contacting Whitney’s cubes $Q$ and $K$, $Q \cap K \neq \emptyset$, such that there exist lacunae $L^{(Q)}, L^{(K)} \in L_E$, $L^{(Q)} \ni Q$ and $L^{(K)} \ni K$, for which

$$x = \mathcal{PR}_E(L^{(Q)}) \quad \text{and} \quad y = \mathcal{PR}_E(L^{(K)}).$$

But, by Theorem 6.6

$$\# \{L : \mathcal{PR}_E(L) = x \} \leq C_1(n) \quad \text{and} \quad \# \{L : \mathcal{PR}_E(L) = y \} \leq C_1(n).$$

Furthermore, by Proposition 6.5 each lacuna contains at most $M(n)$ contacting cubes. Finally, each Whitney’s cube has common points with at most $N(n)$ Whitney’s cubes, see Lemma 4.1

Hence,

$$\# R(x, y) \leq C_1(n) M(n)^2$$

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so that, by (7.28),
\[
I \leq C_2 \sum_{x,y \in E, x \leftrightarrow y} \sum_{|\beta| \leq m-1} \frac{|D^\beta P_x(x) - D^\beta P_y(y)|^p}{\|x - y\|^{(m-|\beta|)p-n}}
\]
where \(C_2 := C C_1(n) M(n)^2\).

The proof of the proposition is complete. \(\square\)

**Proof of Theorem 1.5.** Let \(E\) be a finite set and let \(P = \{P_x : x \in E\}\) be a Whitney \((m-1)\)-field on \(E\). Let
\[
I := \sum_{x,y \in E, x \leftrightarrow y} \sum_{|\beta| \leq m-1} \frac{|D^\beta P_x(x) - D^\beta P_y(y)|^p}{\|x - y\|^{(m-|\beta|)p-n}}.
\]

Since \(\Gamma_E\) is a \(\gamma\)-sparse graph with \(\gamma = \gamma(n)\), see part (i) of Proposition 6.11 by Definition 1.4 and Proposition 7.1 for every \(C^{m-1}\)-function \(F \in L_p^m(\mathbb{R}^n)\) such that \(T_x^{m-1}\{F\} = P_x\) for all \(x \in E\) the following inequality
\[
I \leq C(m, n, p) \|F\|_{L_p^m(\mathbb{R}^n)}^p
\]
holds. This proves that \(I \leq C \|P\|_{m,p,E}\), see (1.3).

Prove that \(\|P\|_{m,p,E} \leq C I\). In fact, since \(E\) is a finite set, the function \(F\) obtained by the Whitney extension formula (4.7) belongs to \(C^\infty(\mathbb{R}^n)\). Hence
\[
\|F\|_{L_p^m(\mathbb{R}^n)} = \|F\|_{L_p^m(\mathbb{R}^n \setminus E)} = \sum_{|\alpha| = m} \|D^\alpha F\|_{L_p(\mathbb{R}^n \setminus E)}.
\]

Let \(F\) be the function obtained by the lacunary modification of the Whitney extension formula. See (7.2). Then, by Proposition 7.6, \(\|F\|_{L_p^m(\mathbb{R}^n \setminus E)}^p \leq C I\) so that \(\|F\|_{L_p^m(\mathbb{R}^n)}^p \leq C I\). Hence, by (1.3), \(\|P\|_{m,p,E} \leq C I\), and the proof of the theorem is complete. \(\square\)

Let us note the following useful property of the graph \(\Gamma_E\).

**Proposition 7.7** Let \(E\) be a finite subset of \(\mathbb{R}^n\). Then \(\Gamma_E\) is a connected graph.

Furthermore, for every \(\bar{x}, \bar{y} \in E\) there is a path \(\{z_0, ..., z_m\}\) joining \(\bar{x}\) to \(\bar{y}\) in \(\Gamma_E\) (i.e., \(z_0 = \bar{x}, z_m = \bar{y}\), and \(z_i \leftrightarrow z_{i+1}, i = 0, ..., m - 1\)) such that
\[
\sum_{i=0}^{m-1} |z_i - z_{i+1}| \leq C(n) \|ar{x} - \bar{y}\|.
\]

**Proof.** Let \(f\) be a function on \(E\). We know that \(L_{1,0}^1(\mathbb{R}^n)\) is isometrically isomorphic to \(\text{Lip}(\mathbb{R}^n)\).

On the other hand, by McShane’s theorem [26], \(\|f\|_{\text{Lip}(E)} = \|f\|_{\text{Lip}(\mathbb{R}^n)_{E}}\) so that \(\|f\|_{\text{Lip}(E)} = \|f\|_{L_1^1(\mathbb{R}^n)_{E}}\).

Hence, by (1.9),
\[
\|f\|_{\text{Lip}(E)} \sim \sup_{x,y \in E, x \leftrightarrow y} \frac{|f(x) - f(y)|}{\|x - y\|} \quad (7.29)
\]
with the constants in this equivalence depending only on \(n\).

Using this equivalence prove that \(\Gamma_E\) is a connected graph. Otherwise there exists a non-empty subset \(E' \subset E\), \(E' \neq E\), such that \(x \leftrightarrow y\) for every \(x \in E'\) and \(y \in E \setminus E'\). Let \(f := \chi_{E'}\). Then \(\|f\|_{\text{Lip}(E)} > 0\), while the right hand side of (7.29) equals 0, a contradiction.
Prove the second statement of the proposition, i.e., the equivalence of the Euclidean and the geodesic metrics in the graph $\Gamma_E$. As usual the geodesic metric $d_{\Gamma_E}$ is defined by the formula

$$d_{\Gamma_E}(x, y) := \inf \sum_{i=0}^{m-1} \|z_i - z_{i+1}\|$$

where the infimum is taken over all paths $\{z_0, ..., z_m\}$ joining $x$ to $y$ in $\Gamma_E$. (Thus $z_0 = x$, $z_m = y$, and $z_i \leftrightarrow z_{i+1}$ for all $i = 0, ..., m - 1$.)

Let $f(z) := d_{\Gamma_E}(z, \bar{x})$, $z \in E$. Then for every $x, y \in E$, $x \leftrightarrow y$,

$$|f(x) - f(y)| = |d_{\Gamma_E}(x, \bar{x}) - d_{\Gamma_E}(y, \bar{x})| \leq d_{\Gamma_E}(x, y) \leq \|x - y\|$$

so that, by (7.29), $\|f\|_{\text{Lip}(E)} \leq C(n)$. Hence,

$$d_{\Gamma_E}(\bar{x}, \bar{y}) = |f(\bar{x}) - f(\bar{y})| \leq C(n)\|\bar{x} - \bar{y}\|$$

proving the proposition. $\square$

**Remark 7.8** Note that Theorem 1.7 easily follows from Theorems 1.3 and 1.5. In fact, the necessity part of Theorem 1.7 and the inequality $\bar{N}_{m,p,E}(P) \leq C \|P\|_{m,p,E}$ directly follow from Definition 1.4 and the necessity part of Theorem 1.3. Let us prove the sufficiency.

Let $P = \{P_x : x \in E\}$ be a Whitney $(m - 1)$-field on $E$ such that $\bar{N}_{m,p,E}(P) < \infty$. See (1.10). Let $\gamma \geq 1$ be the same as in Theorem 1.3, and let $A = \{[x_i, y_i] : i = 1, ..., k\}$ be an arbitrary finite $\gamma$-sparse collections of two point subsets of $E$. By $S$ we denote a subset of $E$ defined by

$$S_A := \bigcup_{i=1}^{k} \{x_i, y_i\}.$$  

Since $S_A$ is finite, by Theorem 1.5 there exists a connected tree $\Gamma_{S_A}$ whose set of vertices coincide with $S_A$ such that the conditions (i) and (ii) of this theorem are satisfied. Hence, by the assumption, (1.10) and the equivalence (1.8), there exists a function $\bar{F} \in C^{m-1}(\mathbb{R}^\ell)$ which agrees with $P$ on $S_A$ such that $\|\bar{F}\|_{W^m_p(\mathbb{R}^\ell)} \leq C\bar{N}_{m,p,E}(P)$.

We again apply the necessity part of Theorem 1.3 to the set $S_A$ and the function $\bar{F}$, and obtain the following inequality:

$$\left\{ \sum_{i=1}^{k} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{x_i}(x_i) - D^\alpha P_{y_i}(x_i)|^p}{\|x_i - y_i\|^{(m-|\alpha|)p-n}} \right\}^{1/p} \leq C \|\bar{F}\|_{W^m_p(\mathbb{R}^\ell)} \leq C\bar{N}_{m,p,E}(P).$$

Since $A$ is arbitrary, by (1.6), $N_{m,p,E}(P) \leq C\bar{N}_{m,p,E}(P)$ so that, by (1.7), $\|P\|_{m,p,E} \leq C\bar{N}_{m,p,E}(P)$. This completes the proof of Theorem 1.7.

8. $W^m_p(\mathbb{R}^\ell)$-jets on closed subsets of $\mathbb{R}^\ell$.

In this section we prove an analogue of Theorem 1.3 for the normed Sobolev space, i.e., a variational criterion for jets generated by $W^m_p(\mathbb{R}^\ell)$-functions.

Let $E$ be a closed subset of $\mathbb{R}^\ell$, and let $P = \{P_x : x \in E\}$ be a family of polynomials of degree at most $m - 1$ indexed by points of $E$. We define the $W^m_p$-"norm" of $P$ by

$$\|P\|_{m,p,E} := \inf \left\{ \|F\|_{W^m_p(\mathbb{R}^\ell)} : F \in W^m_p(\mathbb{R}^\ell), T_x^{m-1}[F] = P_x \text{ for every } x \in E \right\}. \tag{8.1}$$

In this section given $\varepsilon > 0$ we let $E_\varepsilon$ denote the $\varepsilon$-neighborhood of $E$. 59
Theorem 8.1  Let \( m \in \mathbb{N}, \ p \in (n, \infty), \) and let \( \hat{\varepsilon} > 0. \) Fix a number \( \theta \geq 1 \) and let \( V : E_{\hat{\varepsilon}} \to E \) be a measurable mapping such that

\[
\|V(x) - x\| \leq \theta \text{dist}(x, E) \quad \text{for every} \quad x \in E_{\hat{\varepsilon}}. \tag{8.2}
\]

There exists a constant \( \hat{\gamma} \geq 1 \) depending only on \( \theta \) for which the following result holds:

Suppose we are given a family of polynomials \( P = \{P_\alpha \in \mathcal{P}_{m-1}(\mathbb{R}^n) : x \in E \}. \) There exists a \( C^{m-1} \)-function \( F \in W_p^m(\mathbb{R}^n) \) such that

\[
T_{x}^{m-1}[F] = P_\alpha(x) \quad \text{for every} \quad x \in E \tag{8.3}
\]

if and only if the function

\[
P^{(V)} := P_{V(x)}(x), \quad x \in E_{\hat{\varepsilon}}, \tag{8.4}
\]

belongs to \( L_p(E_{\hat{\varepsilon}}) \), and the quantity \( \mathcal{N}^*(P) = \mathcal{N}^*(P : m, n, p, E, \hat{\varepsilon}) \) defined by

\[
\mathcal{N}^*(P) := \sup \left\{ \sum_{j=1}^{k} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{x_{j}}(x_i) - D^\alpha P_{x_j}(y_i)|^p}{(\text{diam } Q_j)^{(m-|\alpha|)p-n}} \right\}^{1/p} \tag{8.5}
\]

is finite. Here the supremum is taken over all finite families \( \{Q_i : i = 1, \ldots, k\} \) of pairwise disjoint cubes contained in \( E_{\hat{\varepsilon}} \), and all choices of points \( x_i, y_i \in (\hat{\gamma}Q_i) \cap E. \)

Furthermore,

\[
\|P\|_{m,p,E}^* \sim \|P^{(V)}\|_{L_p(E_{\hat{\varepsilon}})} + \mathcal{N}^*(P). \tag{8.6}
\]

The constants of this equivalence depend only on \( m, n, p, \hat{\varepsilon} \) and \( \theta. \)

See also Remark 8.10. Note that for the case \( m = 1, \) i.e., for the space \( W_1^1(\mathbb{R}^n) \), this result has been proven in \([28]).\n
Now given a closed \( E \subset \mathbb{R}^n \) we define a special mapping \( V_E : \mathbb{R}^n \to E \) which will enable us to refine the result of Theorem 8.1. This mapping is generated by the lacunary projector \( \mathbb{P} \mathbb{R}_{E} \) constructed in Subsection 6.2, see Theorem 6.6.

We define \( V_E \) as follows. We put

\[
V_E(x) := x \quad \text{for every} \quad x \in E. \tag{8.7}
\]

Let now \( x \in \mathbb{R}^n \setminus E \) and let \( Q \in W_E \) be a Whitney cube containing \( x. \) By \( L(Q) \) we denote the (unique) lacuna which contains \( Q. \) See Subsection 6.1. Then we define \( V_E(x) \) by letting

\[
V_E(x) := \mathbb{P} \mathbb{R}_{E}(L(Q)). \tag{8.8}
\]

Clearly, \( V_E \) is a measurable mapping which is well defined on the set

\[
S := E \cup \left( \bigcup_{Q \in W_E} \text{int}(Q) \right).
\]

It is also clear that the Lebesgue measure of the set \( \mathbb{R}^n \setminus S \) is zero, so that \( V_E \) is well defined a.e. on \( \mathbb{R}^n. \) Note that, by property (4.2) of Whitney’s cubes and by property (i) of Theorem 6.6, the mapping \( V = V_E \) satisfies inequality (8.2) with some absolute constant \( \theta. \)
Theorem 8.2 Let \( m \in \mathbb{N}, p \in (n, \infty), \varepsilon > 0 \). There exists an absolute constant \( \gamma \geq 1 \) for which the following result holds:

Suppose we are given a family \( P = \{ P_x : x \in E \} \) of polynomials of degree at most \( m - 1 \) indexed by points of \( E \). There exists a \( C^{-1} \)-function \( F \in W^m_p(\mathbb{R}^n) \) such that (8.3) is satisfied if and only if the function

\[
P^{(V, \varepsilon)} := P_{V(x)}(x), \quad x \in E, \tag{8.9}
\]

belongs to \( L^p(E, \varepsilon) \) and the quantity \( N^p(P) = N^p(m, n, p, \varepsilon, E) \) defined by

\[
N^p(P) := \sup \left\{ \sum_{i=1}^{k} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{\gamma_i}(x_i) - D^\alpha P_{\gamma_j}(x_j)|^p}{\|x_i - x_j\|^{(m-|\alpha|)p-n}} \right\}^{1/p}
\]

is finite. Here the supremum is taken over all finite \( \gamma \)-sparse collections \( \{[x_i, y_i] : i = 1, \ldots, k\} \) of two point subsets of \( E \) with \( \|x_i - y_i\| \leq \varepsilon, i = 1, \ldots, k \).

Furthermore,

\[
\|P\|^*_{m, p, E} \sim \|P^{(V, \varepsilon)}\|_{L^p(E, \varepsilon)} + N^p(P). \tag{8.11}
\]

The constants of this equivalence depend only on \( m, n, p, \) and \( \varepsilon \).

See also Remark 8.11.

We turn to the proofs of Theorems 8.1 and 8.2.

(Necessity.) The proof of the necessity relies on the following auxiliary lemma.

Lemma 8.3 Let \( F \in W^m_p(\mathbb{R}^n) \cap C^{m-1}(\mathbb{R}^n) \). Then the function \( \tilde{P} := T^{m-1}_{V(x)}[F](x), \ x \in E, \) belongs to \( L^p(E, \varepsilon) \) and

\[
\|\tilde{P}\|_{L^p(E, \varepsilon)} \leq C\|F\|_{W^m_p(\mathbb{R}^n)}. \tag{8.12}
\]

Here \( C = C(m, n, p, \theta, \varepsilon) \).

Proof. By (8.2), \( V(x) = x \) on \( E \) so that \( \tilde{P}(x) = T^{m-1}_{V(x)}[F](x) = F(x) \) for every \( x \in E \). Hence

\[
\|\tilde{P}\|_{L^p(E, \varepsilon)} \leq \|F\|_{L^p(E, \varepsilon)} + \|F - \tilde{P}\|_{L^p(E, \varepsilon)} = \|F\|_{L^p(E, \varepsilon)} + \|F - \tilde{P}\|_{L^p(E \setminus E)}. \tag{8.13}
\]

Let \( q := (p + n)/2 \). Prove that for every \( x \in E, \) the following inequality

\[
|F(x) - \tilde{P}(x)| \leq C_1 \left( M[|\nabla^m F|^q](x) \right)^{\frac{1}{q}} \tag{8.14}
\]

holds. In fact, let \( Q = Q(x, ||x - V(x)||) \). Since \( x \in E, \) we have \( \text{dist}(x, E) \leq \varepsilon \) so that, by (8.2), \( \text{diam} \ Q \leq 2\theta \varepsilon \).

Let us apply to \( x \) and \( y = V(x) \) the Sobolev-Poincaré inequality (1.15) with \( \beta = 0 \). By this inequality,

\[
|F(x) - \tilde{P}(x)| = |F(x) - T^{m-1}_{V(x)}[F](x)| \leq C \left( \text{diam} \ Q \right)^m M[|\nabla^m F|^q](x) \left( \int_Q |\nabla^m F(u)|^q du \right)^{\frac{1}{q}} \leq C_1 \left( M[|\nabla^m F|^q](x) \right)^{\frac{1}{q}} \tag{8.15}
\]

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proving (8.14). Here \( C_1 \) is a constant depending only on \( m, n, p, \theta, \) and \( \varepsilon. \)

Hence,
\[
\|F - \tilde{P}\|_{L^p(E_\varepsilon; \mathbb{E})} \leq C_1 \|(M[(\nabla^m F)^\theta])^{1/2}\|_{L^p(E_\varepsilon; \mathbb{E})} \leq C_1 \|(M[(\nabla^m F)^\theta])^{1/2}\|_{L^p(\mathbb{R}^n)}.
\]

Since \( q < p, \) by the Hardy-Littlewood maximal theorem,
\[
\|F - \tilde{P}\|_{L^p(E_\varepsilon; \mathbb{E})} \leq C_2 \|\nabla^m F\|_{L^p(\mathbb{R}^n)} \leq C_2 \|F\|_{W^1_p(\mathbb{R}^n)}.
\]

Combining this inequality with (8.13) we obtain the required inequality (8.12). \( \square \)

Now the necessity part of Theorem 8.1 and Theorem 8.2 and the inequalities
\[
\|P(V)\|_{L^p(E_\varepsilon)} + \mathcal{N}^*(\mathcal{P}) \leq C \|\mathcal{P}\|_{m,p,E}^* \quad \text{and} \quad \|P^{(V)}\|_{L^p(E_\varepsilon)} + \mathcal{N}^*(\mathcal{P}) \leq C \|\mathcal{P}\|_{m,p,E}^*
\]
directly follow from Lemma 8.3, Proposition 7.1 and definition (8.1).

**Proof of the sufficiency part of Theorem 8.2.** The proof of the sufficiency is based on a slight modification of the lacunary extension method suggested in Subsection 7.2.

Let \( L \in \mathcal{L}_E \) be a lacuna and let a cube \( Q \in L. \) We recall that
\[
a_Q := \mathbb{P} \mathbb{R}_\varepsilon(L) \quad \text{and} \quad P^{(Q)} := P_{a_Q}.
\]

See (7.2).

Let \( \varepsilon > 0 \) and let \( \delta := 10^{-5} \varepsilon. \) Let \( \mathcal{P} = \{ P_x : x \in E \} \) be a Whitney \((m-1)\)-field on \( E. \) Given a cube \( Q \in W_E \) we let \( P^{(Q, \varepsilon)} \) denote a polynomial of degree at most \( m-1 \) defined by the following formula:
\[
P^{(Q, \varepsilon)} := \begin{cases} P^{(Q)}, & \text{if } \text{diam } Q < \delta, \\ 0, & \text{if } \text{diam } Q \geq \delta. \end{cases}
\]

Finally, we define the extension \( F_\varepsilon = F(x; \varepsilon, \mathcal{P}), x \in \mathbb{R}^n, \) by
\[
F_\varepsilon(x) := \begin{cases} P_x(x), & x \in E, \\ \sum_{Q \in W_E} \varphi_Q(x)P^{(Q, \varepsilon)}(x), & x \in \mathbb{R}^n \setminus E. \end{cases}
\]

Let \( F \) be the extension operator constructed in Subsection 7.2, i.e., the operator given by the Whitney extension formula
\[
F(x) := \begin{cases} P_x(x), & x \in E, \\ \sum_{Q \in W_E} \varphi_Q(x)P^{(Q)}(x), & x \in \mathbb{R}^n \setminus E. \end{cases}
\]

Let us note two properties of the function \( F_\varepsilon. \)

**Lemma 8.4** (i). If \( \text{dist}(x, E) < \delta/4, \) then \( F_\varepsilon(x) = F(x); \)
(ii). \( \text{supp } F_\varepsilon \subset E_\tau \) where \( \tau = 20\delta. \)
Hence, by (8.15), (3) of Lemma 4.1, statement of part (i) of the lemma. 

Recall that \( Q^* := \frac{2}{3} Q \). Then, by property (c) of Lemma 4.2 and by (8.17) and (8.16),

\[
F(x) = \sum_{Q \in A_x} \varphi_Q(x) p^{(Q)}(x) \quad \text{and} \quad F_\varepsilon(x) = \sum_{Q \in A_x} \varphi_Q(x) p^{(Q, \varepsilon)}(x). \tag{8.18}
\]

Let \( K \in W_E \) and let \( x \in K \). Then, by (4.2),

\[
\text{diam} \ K \leq \text{dist}(K, E) \leq \text{dist}(x, E) < \delta/4.
\]

Since \( Q^* \cap K \neq \emptyset \) for each \( Q \in A_x \), by property (1) of Lemma 4.1 \( \text{diam} \ Q \leq 4 \text{ diam} \ K \), so that

\[
\text{diam} \ Q < \delta \quad \text{for every} \quad Q \in A_x.
\]

Hence, by (8.15), \( p^{(Q, \varepsilon)} = p^{(Q)} \) for all \( Q \in A_x \). Combining this property with (8.18) we obtain the statement of part (i) of the lemma.

(ii). Let \( \text{dist}(x, E) \geq 20\delta \). Prove that \( F_\varepsilon(x) = 0 \).

By (8.15) and (8.18), it suffices to show that \( \text{diam} \ Q \geq \delta \) for every \( Q \in A_x \).

Let \( K \in W_E \) and let \( x \in K \). (Clearly, \( K \in A_x \).) Then, \( Q^* \cap K \neq \emptyset \) for each \( Q \in A_x \) so that, by part (3) of Lemma 4.1 \( Q \cap K \neq \emptyset \). Hence, by part (1) of this lemma, \( \text{diam} \ Q \geq \frac{1}{4} \text{ diam} \ K \). But \( K \in W_E \) so that

\[
20\delta \leq \text{dist}(x, E) \leq \text{dist}(K, E) + \text{diam} \ K \leq 5 \text{ diam} \ K.
\]

Hence \( \text{diam} \ K \geq 4\delta \) proving the required inequality \( \text{diam} \ Q \geq \delta \). \( \square \)

Let \( \gamma \geq 1 \) be the same absolute constant as in Theorem 1.3. Suppose that the Whitney \((m-1)\)-field \( P = \{P_x : x \in E\} \) satisfies the conditions of the sufficiency part of Theorem 8.2. Thus:

(i) The function \( P^{(\gamma, \varepsilon)} \) belongs to \( L_p(E_\varepsilon) \). See (8.9) and (8.8);

(ii) Let \( \widetilde{\lambda} := N^\gamma(P)^p \), see (8.10). Then \( \widetilde{\lambda} < \infty \) so that for every finite \( \gamma \)-sparse collection \( \{(x_i, y_i) : i = 1, \ldots, k\} \) of two point subsets of \( E \) with

\[
\|x_i - y_i\| \leq \varepsilon, \quad i = 1, \ldots, k, \tag{8.19}
\]

the following inequality

\[
\sum_{i=1}^k \sum_{|a| \leq m-1} |D^a P_{x_i}(x_i) - D^a P_{y_i}(x_i)|^p \leq \widetilde{\lambda} \tag{8.20}
\]

holds.

In Section 7 we have proved that the function \( F \) defined by the formula (8.17) belongs to \( C^{m-1}(\mathbb{R}^n) \) and agrees with \( P \) on \( E \), i.e., (8.3) holds. Since, by Lemma 8.4, \( F_\varepsilon \) coincides with \( F \) on an open neighborhood of \( E \), and \( F_\varepsilon \in C^\infty(\mathbb{R}^n \setminus E) \), the function \( F_\varepsilon \) has similar properties, i.e., \( F_\varepsilon \in C^{m-1}(\mathbb{R}^n) \) and \( F_\varepsilon \) agrees with \( P \) on \( E \).
Prove that

$$\|P\|_{m,p,E}^* \leq C \left( \bar{\lambda}^{\frac{1}{p}} + \|P^{(V_k)}\|_{L_p(E_\lambda)} \right) \quad (8.21)$$

where $C$ is a constant depending only on $m, n, p,$ and $\varepsilon$.

We will estimate the $L^m_p$-seminorm of $F_\varepsilon$ using the scheme of the proof suggested in Subsection 7. More specifically, in Lemmas 8.5 and 8.7 we will prove analogues of Lemmas 7.4 and 7.5 for the function $F_\varepsilon$. Finally, in Lemma 8.9 we will estimate the $L_p$-norm of $F_\varepsilon$.

Let $\eta := \delta/\bar{\gamma}$ where $\bar{\gamma}$ is the constant from Theorem 6.6. Let $Q = \{Q_1, \ldots, Q_k\}$ be a family of pairwise disjoint equal cubes in $\mathbb{R}^n$ such that

$$\text{diam } Q_i \leq \eta \quad \text{for every } i = 1, \ldots, k. \quad (8.22)$$

Let $c_i := c_{Q_i}$, $i = 1, \ldots, k$, be the centers of these cubes.

**Lemma 8.5** Suppose that the family $Q$ satisfies the condition (7.6).

Then for every $\beta, |\beta| = m - 1,$ and every $x_i \in Q_i$ the following inequality

$$\sum_{i=1}^{k} \frac{|D^\beta F_\varepsilon(x_i) - D^\beta F_\varepsilon(c_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq C \bar{\lambda}$$

holds. Here $C > 0$ is a constant depending only on $m, n, p$ and $\varepsilon$.

**Proof.** By Lemma 8.4, $F_\varepsilon(x) = F(x)$ for $x \in E_{\delta'}$ where $\delta' = \delta/4$. Hence

$$D^\beta F_\varepsilon(x) = D^\beta F(x) \quad \text{for every } \beta, |\beta| = m - 1, \text{ and every } x \in E_{\delta'}.$$

Following the proof of Lemma 7.4 we first consider the case where $c_i \in E$ for each $i = 1, \ldots, k$. Let $I_1(F_\varepsilon)$ be the quantity defined by (7.7). Prove that $I_1(F_\varepsilon) \leq \bar{\lambda}$.

In fact, since $x_i \in Q_i$ and diam $Q_i \leq \eta$, the point $x_i \in E_\eta$. Hence, by part (i) of Lemma 8.4 $D^\beta F(x_i) = D^\beta F_\varepsilon(x_i)$ and $D^\beta F(c_i) = D^\beta F_\varepsilon(c_i)$ for every $\beta, |\beta| = m - 1$, and every $i = 1, \ldots, k$. Hence $I_1(F_\varepsilon) = I_1(F)$.

Since $x_i, c_i \in Q_i$ and

$$\text{diam } Q_i \leq \eta < \delta/4 < \varepsilon,$$

we conclude that $\|x_i - c_i\| < \varepsilon$ as it is required in (8.19). Now, using the same argument as in the proof of Lemma 7.4, we obtain the desired inequality $I_1(F_\varepsilon) \leq \bar{\lambda}$. C.f. (7.8).

Let $I_2(F_\varepsilon)$ be the quantity defined by (7.9). Prove that $I_2(F_\varepsilon) \leq C \bar{\lambda}$.

Again we will follow the proof given in Lemma 7.4 for a similar estimate of $I_2(F)$.

Using the same argument as for the case of $I_1(F)$, we conclude that $I_2(F_\varepsilon) = I_2(F)$. Then we literally follow the proof of Lemma 7.4 after the definition (7.9) of $I_2(F)$. This leads us to the corresponding estimate

$$I_2(F_\varepsilon) \leq C \{ I_2^{(1)}(F_\varepsilon) + I_2^{(2)}(F_\varepsilon) \}$$

where $I_2^{(j)}(F_\varepsilon)$, $j = 1, 2$, are defined in the same way as $I_2^{(j)}(F)$. See (7.17) and (7.18).

By repeating the argument of Lemma 7.4 we show that $I_2^{(1)}(F_\varepsilon) \leq \bar{\lambda}$. The unique additional requirement which we have to check is the inequality $\|a_{Q_0} - c_{Q_0}\| \leq \varepsilon$. See (8.19).

But we know that $a_{Q_0} \in 5\gamma Q$, see (7.13), and diam $Q \leq \eta$, see (8.22). Hence

$$\|a_{Q_0} - c_{Q_0}\| \leq 5\gamma \text{ diam } Q \leq 5\gamma \eta = 5 \delta. \quad (8.23)$$
Recall that $\delta = 10^{-5}\varepsilon$ so that $\|a_{K_0} - c_Q\| \leq \varepsilon$.

We turn to estimates of $I_2^{(2)}(F_{\varepsilon})$. Following the proof of the inequality $I_2^{(2)}(F) \leq C\lambda$ in Lemma 7.4 we obtain the required inequality $I_2^{(2)}(F_{\varepsilon}) \leq C\lambda$ provided

$$\|a_{H_0} - a_{K_0}\| \leq \varepsilon. \tag{8.24}$$

See (8.19). We recall that the cubes $K_0$ and $H_0$ are determined by (7.10) and (7.16) respectively.

But in the proof of this lemma we show that $K_0 \subset 5Q, a_{H_0} \in 9\sqrt{\gamma}K_0,$ and $a_{K_0} \in 5\sqrt{\gamma} Q$. See (7.12), (7.13) and (7.14). Hence $a_{H_0}$, $a_{K_0} \in 45\sqrt{\gamma} Q$ so that

$$\|a_{H_0} - a_{K_0}\| \leq 45\sqrt{\gamma} \text{ diam } Q \leq 45\sqrt{\gamma} \eta = 45\delta \leq \varepsilon. \tag{8.25}$$

This proves the required inequality (8.24).

Finally we turn to the general case, i.e., to an arbitrary family of $\{Q_i : i = 1, ..., k\}$ of equal cubes of diameter at most $\eta$ satisfying inequality (7.6). We again follow the proof of this part of Lemma 7.4. This enables us to reduce the proof to the case of the family $\tilde{Q} = \{\tilde{Q}_i : i = 1, ..., k\}$ of cubes defined by (7.20). These cubes are centered at $E$ so the proof is reduced to the previous cases proven above.

Also we know that

$$\text{diam } \tilde{Q}_i \leq \gamma' \text{ diam } Q_i \leq \gamma' \eta$$

where $\gamma' := 122$. Thus the proof is actually reduced to the known case but with $\eta' = \gamma' \eta$ instead of $\eta$.

We recall that $\eta = \delta/\sqrt{\gamma}$. We use equality in inequalities (8.23) and (8.25) to prove that the left hand side of these inequalities are bounded by $\varepsilon$. Clearly, the same is true whenever the constant $\eta$ in these inequalities is replaced by $\eta'$ satisfying the equality $\eta' = \gamma' \delta/\sqrt{\gamma}$. In fact, in this case the right hand sides of (8.23) and (8.25) are bounded by $5\gamma'\delta$ and $45\gamma'\delta$ respectively. Since $\delta = 10^{-5}\varepsilon$, in the both cases the right hand side does not exceed $\varepsilon$ which proves the lemma. $\square$

The following properties of polynomials are well known.

**Lemma 8.6** Let $P \in \mathcal{P}_m(\mathbb{R}^n)$ and let $1 \leq p < \infty$. Let $Q, \tilde{Q}$ be two cubes in $\mathbb{R}^n$ such that $Q \subset \tilde{Q}$. Then

$$\sup_{\tilde{Q}} |P| \leq C \left( \text{diam } \tilde{Q}/ \text{ diam } Q \right)^m \sup_Q |P|$$

and

$$\sup_{\tilde{Q}} |P| \leq C \left\{ \frac{1}{|Q|} \int_Q |P(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

Furthermore, for every $\xi, |\xi| \leq m$, the following inequality

$$\sup_Q |D^\xi P| \leq C \left( \text{diam } Q \right)^{-|\xi|} \sup_Q |P|$$

holds. Here $C$ is a constant depending only on $m$ and $n$.

**Lemma 8.7** Suppose that the family $Q$ satisfies the condition (7.27).

Then for every $\beta, |\beta| = m - 1$, and every $x_i \in Q_i$ the following inequality

$$\sum_{i=1}^k \frac{|D^\beta F_{\varepsilon}(x_i) - D^\beta F_{\varepsilon}(c_i)|^p}{(\text{diam } Q_i)^{p-n}} \leq C \left\{ \lambda + \|P^{(v_\varepsilon)}\|_{L_p(E_\varepsilon)}^p \right\}

holds. Here $C > 0$ is a constant depending only on $m, n, p$ and $\varepsilon$. 65
Proof. We literally follow the proof of Lemma 7.5 until the inequality (7.27). But before to continue the proof from this point let us make the following remark: Let \( \hat{Q} \in Q \) and, as in the proof of this lemma, let \( K_{\hat{Q}} \) be a cube such that \( c_{\hat{Q}} \in K_{\hat{Q}} \). Note that if \( Q \subset \mathbb{R}^n \setminus E, \) then, by part (ii) of Lemma 8.4, \( F_{\hat{Q}} |_{Q} \equiv 0. \) This enables us to assume that \( \text{dist}(Q, E) \leq \tau = 20\delta. \)

Now, by (4.2) and (7.22),
\[
\text{diam} K_{\hat{Q}} \leq \text{dist}(K_{\hat{Q}}, E) \leq \text{dist}(c_{\hat{Q}}, E) \leq \text{dist}(Q, E) + \text{diam} Q \leq \tau + \frac{1}{8} \text{diam} K_{\hat{Q}}
\]
proving that \( \text{diam} K_{\hat{Q}} \leq 8\tau/7 \leq 24 \delta. \) Then, by part (1) of Lemma 4.1,
\[
\text{diam} H \leq 4 \cdot 24 \delta = 96 \delta \quad \text{for every cube } H \in W_E \text{ such that } H \cap K_{\hat{Q}} \neq \emptyset.
\]
(8.26)

We will use this estimate at the end of the proof.

We continue the proof of the lemma as follows. Let
\[
\mathcal{K}^{(1)} := \{K \in \mathcal{K} : \text{diam} K \leq \eta\} \quad \text{and let } \mathcal{K}^{(2)} := \{K \in \mathcal{K} : \text{diam} K > \eta\}.
\]
be a partition of the family \( \mathcal{K}. \) Then
\[
I := \sum_{i=1}^{k} \frac{|D^{\beta}F_{\hat{Q}}(x_{i}) - D^{\beta}F_{\hat{Q}}(c_{i})|^p}{(\text{diam } Q_i)^{p-n}} \leq C \{I^{(1)} + I^{(2)} \} \quad (8.27)
\]
where
\[
I^{(j)} := \sum_{K \in \mathcal{K}^{(j)}} \sum_{|\xi| \leq m-1} \frac{|D^{\xi}P_{a_{k}}(a_{K}) - D^{\xi}P_{a_{k}}(a_{K})|^p}{|a_{K} - a_{K}||^{(m-|\xi|)p-n}}, \quad j = 1, 2.
\]

Recall that \( \hat{K} \) is a cube touching \( K \) and satisfying (7.24).

Let us prove that
\[
I^{(1)} \leq C \lambda \quad \text{with } C = C(m, n, p, \epsilon). \quad (8.28)
\]
In fact, literally following the proof of Lemma 7.5 after the inequality (7.25), we obtain (8.28) provided \( |a_{K} - a_{K}| \leq \epsilon. \) But, by (7.14), the points \( a_{K}, a_{K} \in 9\gamma K \) so that
\[
|a_{K} - a_{K}| \leq 9\gamma \text{ diam } K \leq 9\gamma \eta = 9\delta \leq \epsilon.
\]

Let us estimate \( I^{(2)}. \) Let \( K \in \mathcal{K}^{(2)}. \) Since \( |a_{K} - a_{K}| \sim \text{diam } K, \) see (7.26),
\[
I^{(2)} \leq C \sum_{K \in \mathcal{K}^{(2)}} \sum_{|\xi| \leq m-1} \frac{|D^{\xi}P_{a_{k}}(a_{K}) - D^{\xi}P_{a_{k}}(a_{K})|^p}{(\text{diam } K^{(m-|\xi|)p-n})}.
\]

We know that \( a_{K} \in \bar{K} := \gamma K, \) see part (i) of Theorem 6.6. Since \( \text{diam } \hat{K} \leq 4 \text{ diam } K \) and \( \hat{K} \cap K \neq \emptyset, \) the cube \( \hat{K} \subset 9K \) so that \( \hat{K} \subset \bar{K}. \)

Now, by Lemma 8.6 for every \( \xi, |\xi| \leq m-1, \) the following inequality
\[
|D^{\xi}P_{a_{k}}(a_{K})| \leq \sup_{\hat{K}} |D^{\xi}P_{a_{k}}| \leq C \sup_{\hat{K}} |D^{\xi}P_{a_{k}}| \leq C (\text{diam } \hat{K})^{-|\xi|} \sup_{\hat{K}} |P_{a_{k}}| \leq C (\text{diam } K)^{-|\xi|} \left( \frac{1}{|K|} \int_{\hat{K}} |P_{a_{k}}(x)|^p \, dx \right)^{\frac{1}{p}}
\]

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holds. Hence,
\[
\frac{|D^k P_{a_k}^{(K)}|}{(\text{diam } K)^{(m-|\xi|)p-n}} \leq C^p (\text{diam } K)^{-mp} \int |P_{a_k}^{(K)}(x)|^p \, dx.
\]

Since \( K \in \mathcal{K}_2 \), its diameter is at least \( \eta \) so that
\[
\frac{|D^k P_{a_k}^{(K)}|}{(\text{diam } K)^{(m-|\xi|)p-n}} \leq C^p \eta^{-mp} \int |P_{a_k}^{(K)}(x)|^p \, dx \leq C_1 \int |P_{a_k}^{(K)}(x)|^p \, dx
\]
where \( C_1 \) is a constant depending only on \( m, n, p, \) and \( \varepsilon \). (Recall that \( \eta := \sqrt{\gamma} \) depends on \( \varepsilon \).)

By (8.8) and (8.9), \( P_{a_k}^{(K)}(x) = P_{V_k}(x) = P^{(V)}(x) \) for every \( x \in \mathcal{K} \), so that
\[
\frac{|D^k P_{a_k}^{(K)}|}{(\text{diam } K)^{(m-|\xi|)p-n}} \leq C_1 \int |P^{(V)}(x)|^p \, dx.
\]

In the same way we prove that
\[
\frac{|D^k P_{a_k}^{(K)}|}{(\text{diam } K)^{(m-|\xi|)p-n}} \leq C_1 \int |P^{(V)}(x)|^p \, dx.
\]

Let us now introduce a family \( \mathcal{A} \) of cubes by letting
\[
\mathcal{A} := \{ K, \mathcal{K} : K \in \mathcal{K}^{(2)} \}.
\]

Clearly, by Lemma [4.1] the covering multiplicity of \( \mathcal{A} \) is bounded by a constant \( C = C(n) \). We have
\[
I^{(2)} \leq C \sum_{K \in \mathcal{K}^{(2)}} \sum_{|\xi| \leq m-1} \frac{|D^k P_{a_k}^{(K)}| + |D^k P_{a_k}^{(K)}|}{(\text{diam } K)^{(m-|\xi|)p-n}} \leq C_2 \sum_{K \in \mathcal{A}} \int |P_{a_k}^{(K)}(x)|^p \, dx = C_2 \sum_{K \in \mathcal{A}} \int |P^{(V)}(x)|^p \, dx
\]
with \( C_2 = C_2(m, n, p, \varepsilon) \). Since the covering multiplicity of \( \mathcal{A} \) is bounded by \( C(n) \), we obtain
\[
I^{(2)} \leq C_2 \int_{U_{\mathcal{A}}} |P^{(V)}(x)|^p \, dx
\]
where \( U_{\mathcal{A}} := \cup \{ K : K \in \mathcal{A} \} \).

Prove that \( U_{\mathcal{A}} \subset E_\varepsilon \). In fact, by (8.26), \( \text{diam } K \leq 96 \delta \) for every \( K \in \mathcal{A} \). Since each cube \( K \in \mathcal{A} \) is a Whitney cube, for every \( y \in K \) we have
\[
\text{dist}(y, E) \leq \text{dist}(K, E) + \text{diam } K \leq 5 \text{ diam } K \leq 480 \delta = 480 \cdot 10^{-5} \varepsilon < \varepsilon \quad (8.29)
\]
proving that \( K \subset E_\varepsilon \). Hence \( U_{\mathcal{A}} \subset E_\varepsilon \) so that
\[
I^{(2)} \leq C \int_{U_{\mathcal{A}}} |P^{(V)}(x)|^p \, dx \leq C \int_{E_\varepsilon} |P^{(V)}(x)|^p \, dx = \|P^{(V)}\|_{L^p(E_\varepsilon)}^p.
\]

Combining this inequality with (8.28) and (8.27) we obtain the statement of the lemma. \( \square \)
Corollary 8.8 The function \( F_e \) belongs to the space \( L^m_p(\mathbb{R}^n) \) and its seminorm in this space satisfies the following inequality:

\[
\|F_e\|_{L^m_p(\mathbb{R}^n)} \leq C \left( \frac{1}{\varepsilon} + \|P^{(V_e)}\|_{L^p(E_e)} \right).
\] (8.30)

Here \( C \) is a constant depending only on \( m, n, p, \) and \( \varepsilon \).

Proof. Theorem 7.3, Lemma 8.5 and Lemma 8.7 imply the following property of \( F_e \): the function \( F_e \) is a \( C^{m-1} \)-function such that for every multiindex \( \beta \) of order \( m - 1 \) the function \( \partial^{\beta} F_e \in L^1_p(\mathbb{R}^n) \) and

\[
\|\partial^{\beta} F_e\|_{L^1_p(\mathbb{R}^n)} \leq C \left( \frac{1}{\varepsilon} + \|P^{(V_e)}\|_{L^p(E_e)} \right).
\]

Since weak derivatives commute, the function \( F_e \in L^m_p(\mathbb{R}^n) \) and inequality (8.30) holds. \( \square \)

Lemma 8.9 The following inequality

\[
\|F_e\|_{L^p(\mathbb{R}^n)} \leq C \|P^{(V_e)}\|_{L^p(E_e)}
\] (8.31)

holds. Here \( C = C(m, n, p, \varepsilon) \).

Proof. We recall that, by (8.7), \( V_E(x) = x \) on \( E \) so that, by (8.16), \( P^{(V_e)}(x) = F_e(x) \), \( x \in E \). Also, by Lemma 8.4, \( \text{supp} F_e \subset E_{\tau} \) with \( \tau = 20 \delta \). Hence

\[
\|F_e\|_{L^p(\mathbb{R}^n)}^p = \|P^{(V_e)}\|_{L^p(E)}^p + \|F_e\|_{L^p(\mathbb{R}^n \setminus E)}^p = \|P^{(V_e)}\|_{L^p(E)}^p + \|F_e\|_{L^p(E_e \setminus E)}^p
\] (8.32)

Fix a point \( y \in E_{\tau} \setminus E \). Let \( y \in K \) where \( K \in W_e \) is a Whitney cube. Let us apply Lemma 4.3 to \( y \), the cube \( K \), the function \( F_e \) and \( \alpha = 0 \). By this lemma,

\[
|F(y) - P_{a_k}(y)| \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m - 1} (\text{diam } K)^{m-1} |\partial^\xi P_{a_0}(a_K) - \partial^\xi P_{a_K}(a_K)|
\]

so that

\[
|F(y) - P_{a_k}(y)| \leq C \sum_{Q \in T(K)} \sum_{|\xi| \leq m - 1} (\text{diam } K)^{m-1} |\partial^\xi P_{a_0}(a_K)|.
\]

We know that \( a_K \in \bar{K} := \bar{\gamma}K \), see Theorem 6.6, so that, by Lemma 8.6, for every \( Q \in T(K) \)

\[
(\text{diam } K)^{m-1} |\partial^\xi P_{a_0}(a_K)| \leq (\text{diam } K)^{m-1} \sup_K |\partial^\xi P_{a_0}| \leq C \sup_K |P_{a_0}|.
\]

Since \( Q \subset 9K \) (recall that \( Q \) and \( K \) are touching Whitney cubes) and \( \bar{\gamma} > 9 \), we have \( Q \subset \bar{K} \). Again, applying Lemma 8.6 we get

\[
(\text{diam } K)^{m-1} |\partial^\xi P_{a_0}(a_K)| \leq C \sup_K |P_{a_0}| \leq C \sup_Q |P_{a_0}| \leq C \left( \frac{1}{|Q|} \int_Q |P_{a_0}(x)|^p \, dx \right)^{\frac{1}{p}}
\]

provided \( Q \in T(K) \) and \( |\xi| \leq m - 1 \). The same lemma implies the following inequality

\[
|P_{a_0}(y)| \leq C \left( \frac{1}{|K|} \int_K |P_{a_0}(x)|^p \, dx \right)^{\frac{1}{p}}.
\]
Hence, for every \( y \in K \)
\[
|F_\varepsilon(y)|^p \leq 2^p |F_\varepsilon(y) - P_\alpha(y)|^p + |P_\alpha(y)|^p \leq C \sum_{Q \in T(K)} \left\{ \frac{1}{|Q|} \int |P_{a_0}(x)|^p \, dx \right\}^{\frac{1}{p}}
\]
so that
\[
\int_{K} |F_\varepsilon(y)|^p \, dy \leq C \sum_{Q \in T(K)} \frac{|K|}{|Q|} \int |P_{a_0}|^p \, dx 
\leq C \sum_{Q \in T(K)} \int |P_{a_0}|^p \, dx = C \sum_{Q \in T(K)} \int |P^{(V_\varepsilon)}|^p \, dx.
\]

Let us introduce a family \( \mathcal{A} \) of cubes by letting
\[
\mathcal{A} := \{ Q \in T(K) : K \cap E_\varepsilon \neq \emptyset \}.
\]

Prove that \( Q \subset E_\varepsilon \) for every \( Q \in \mathcal{A} \). In fact, let \( K \cap E_\varepsilon \neq \emptyset \), and let \( Q \in T(K) \). Then \( \text{diam}(K, E) \leq \tau \), so that \( \text{diam} K \leq \text{dist}(K, E) \leq \tau \). Therefore for each \( Q \in T(K) \)
\[
\text{diam} Q \leq 4 \text{diam} K \leq 4\tau = 80\delta.
\]

Using the same idea as in the proof of (8.29), we obtain the required inclusion \( Q \subset E_\varepsilon \).

We also note that the covering multiplicity of the family \( \mathcal{A} \) is bounded by a constant \( C = C(n) \).

This easily follows from Lemma 4.1.

Finally, we obtain
\[
\int_{E_\varepsilon \setminus E} |F_\varepsilon(y)|^p \, dy \leq C \sum_{Q \in \mathcal{A}} \int_{Q} |P^{(V_\varepsilon)}(x)|^p \, dx \leq C \int_{U_\mathcal{A}} |P^{(V_\varepsilon)}(x)|^p \, dx
\]
where \( U_\mathcal{A} := \bigcup \{ Q : Q \in \mathcal{A} \} \).

Since \( Q \subset E_\varepsilon \) for each \( Q \in \mathcal{A} \), we conclude that \( U_\mathcal{A} \subset E_\varepsilon \) so that
\[
\int_{E_\varepsilon \setminus E} |F_\varepsilon(y)|^p \, dy \leq C \int_{E_\varepsilon} |P^{(V_\varepsilon)}(x)|^p \, dx.
\]

Combining this inequality with inequality (8.32), we obtain (8.31) proving the lemma. \( \square \)

We are in a position to finish the proof of the sufficiency part of Theorem 8.2. We know that the function \( F_\varepsilon \) agrees with the Whitney \((m - 1)\)-field \( P = \{ P_\alpha : x \in E \} \) so that, by (8.1),
\[
\|P\|_{m,p,E}^* \leq \|F_\varepsilon\|_{W^{m,p}(\mathbb{R}^n)}.
\]

It is well known, see, e.g. [24], p. 21, that for every \( F \in W^{m,p}(\mathbb{R}^n) \) the following equivalence
\[
\|F\|_{W^{m,p}(\mathbb{R}^n)} \sim \|F\|_{L^p(\mathbb{R}^n)} + \|F\|_{L^p(\mathbb{R}^n)}
\]
holds with constants depending only on \( m, n, \) and \( p \). Hence,
\[
\|P\|_{m,p,E}^* \leq C \{ \|F_\varepsilon\|_{L^p(\mathbb{R}^n)} + \|F_\varepsilon\|_{L^p(\mathbb{R}^n)} \}.
\]
Combining this inequality with Corollary 8.8 and Lemma 8.9, we obtain (8.21).

The proof of Theorem 8.2 is complete. \qed

**Proof of the sufficiency part of Theorem 8.1** Let \( \hat{\gamma} := 2\gamma + 12\theta \) where \( \overline{\gamma} \) is the constant from inequality (6.11). Let \( \gamma \) be the same constant as in Theorem 8.2. (Recall that \( \gamma \) coincides with the constant from Theorem 1.3)

Suppose that a Whitney \((m - 1)\)-field \( P = \{ P_x : x \in E \} \) satisfies the conditions of the sufficiency part of Theorem 8.1 Thus:

(a) The function \( P^{(V)} \) belongs to \( L_{p}(E_{\hat{\varepsilon}}) \). See (8.4);

(ii) Let \( \lambda := \mathcal{N}^{*}(P)^{p}, \) see (8.5). Then \( \lambda < \infty \) so that for every finite family \( \{ Q_{i} : i = 1, \ldots, k \} \) of pairwise disjoint cubes contained in \( E_{\hat{\varepsilon}} \), and every choice of points \( x_{i}, y_{i} \in (\hat{\gamma} Q_{i}) \cap E \) the following inequality

\[
\sum_{i=1}^{k} \sum_{|a| \leq m - 1} \frac{|D^a P_{x_{i}}(x_{i}) - D^a P_{y_{i}}(x_{i})|}{(\text{diam } Q_{i})^{(m-|a|)p-n}} \leq \lambda \tag{8.33}
\]

holds.

Prove that \( P = \{ P_x : x \in E \} \) satisfies the hypothesis of the sufficiency part of Theorem 8.2 with

\[
\varepsilon := \hat{\varepsilon} / (2\gamma^2). \tag{8.34}
\]

More specifically, we will prove that the function \( P^{(V_{\varepsilon})} \) defined by (8.9) belongs to \( L_{p}(E_{\varepsilon}) \), and for every finite \( \gamma \)-sparse collection \( \{ x_{i}, y_{i} \} : i = 1, \ldots, k \) of two point subsets of \( E \) with \( \| x_{i} - y_{i} \| \leq \varepsilon, i = 1, \ldots, k \), the inequality (8.20) holds with

\[
\overline{\lambda} := \gamma^{mp-n} \lambda. \tag{8.35}
\]

Furthermore, we will show that

\[
\| P^{(V_{\varepsilon})} \|_{L_{p}(E_{\varepsilon})} \leq C \{ \| P^{(V)} \|_{L_{p}(E_{\hat{\varepsilon}})} + \lambda^{\frac{1}{2}} \}. \tag{8.36}
\]

Here \( C > 0 \) is a constant depending only on \( m, n, p, \hat{\varepsilon} \) and \( \theta \).

Let \( \{ x_{i}, y_{i} \} : i = 1, \ldots, k \) be a finite \( \gamma \)-sparse collection of two point subsets of \( E \) with \( \| x_{i} - y_{i} \| \leq \varepsilon, i = 1, \ldots, k \). Then, by Definition 1.2 there exists a collection \( \{ Q_{i} : i = 1, \ldots, k \} \) of pairwise disjoint cubes in \( \mathbb{R}^{n} \) such that \( x_{i}, y_{i} \in \gamma Q_{i} \) and

\[
\text{diam } Q_{i} \leq \gamma \| x_{i} - y_{i} \|, \quad i = 1, \ldots, k. \tag{8.37}
\]

Hence, \( \text{diam } Q_{i} \leq \gamma \varepsilon \). Since \( x_{i} \in \gamma Q_{i} \), we have \( \| x_{i} - c_{Q_{i}} \| \leq \gamma \text{diam } Q_{i} / 2 \), so that for every \( y \in Q_{i} \) the following inequality

\[
\text{dist}(y, E) \leq \| y - c_{Q_{i}} \| + \text{dist}(c_{Q_{i}}, E) \leq \text{diam } Q_{i} / 2 + \| x_{i} - c_{Q_{i}} \| \leq \gamma \text{diam } Q_{i} \leq \gamma^{2} \varepsilon.
\]

Since \( \varepsilon < \hat{\varepsilon} / \gamma^2 \), see (8.34), we have \( \text{dist}(y, E) < \hat{\varepsilon} \) proving that \( Q_{i} \subset E_{\hat{\varepsilon}} \) for every \( i = 1, \ldots, k \). Therefore, by the assumption, the inequality (8.33) holds for the collection \( \{ x_{i}, y_{i} : i = 1, \ldots, k \} \).
Hence, by (8.37),

$$I := \sum_{i=1}^{k} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{x_i}(x_i) - D^\alpha P_{y_i}(x_i)|^p}{\|x_i - y_i\|^{(m-|\alpha|)p-n}} \leq \sum_{i=1}^{k} \gamma^{(m-|\alpha|)p-n} \sum_{|\alpha| \leq m-1} \frac{|D^\alpha P_{x_i}(x_i) - D^\alpha P_{y_i}(x_i)|^p}{(\text{diam } Q_i)^{(m-|\alpha|)p-n}},$$

so that, by (8.33), $I \leq \gamma^{m-p-n} \lambda = \tilde{\lambda}$.

This proves inequality (8.20) with $\lambda$ and $\varepsilon$ defined by (8.35) and (8.34) respectively.

Prove inequality (8.36). We know that $P^{(V)}|_E = P^{(V)} = f_\varepsilon$ where

$$f_\varepsilon(x) := P_\varepsilon(x), \quad x \in E.$$

Hence,

$$\|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p = \|f_\varepsilon\|_{L_p(E)}^p + \|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p \leq \|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p + \|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p.$$

Let

$$\mathcal{A}_\varepsilon := \{Q \in W_E : Q \cap E \neq \emptyset\}.$$

Note that for every $Q \in \mathcal{A}_\varepsilon$ there exists a point $y \in Q$ such that $\text{dist}(y, E) < \varepsilon$. Hence we have $\text{dist}(Q, E) < \varepsilon$ so that, by (4.2),

$$\text{diam } Q \leq \text{dist}(Q, E) < \varepsilon \quad \text{for every } Q \in \mathcal{A}_\varepsilon. \quad (8.38)$$

Now let $x \in Q$. Then

$$\text{dist}(x, E) \leq \text{dist}(y, E) + \|x - y\| \leq \text{dist}(y, E) + \text{diam } Q.$$

Since $Q \in W_E$,

$$\text{dist}(x, E) \leq \text{dist}(y, E) + \text{dist}(Q, E) \leq 2 \text{dist}(y, E) < 2\varepsilon$$

proving that $Q \subset E_{2\varepsilon}$. In particular,

$$Q \subset E_{\tilde{\varepsilon}} \quad \text{for every } Q \in \mathcal{A}_\varepsilon. \quad (8.39)$$

See (8.34).

We obtain

$$\|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p \leq \sum_{Q \in \mathcal{A}_\varepsilon} \|P^{(V)}\|_{L_p(Q)}^p \leq 2^p \left\{ \sum_{Q \in \mathcal{A}_\varepsilon} \|P^{(V)}\|_{L_p(Q)}^p + I \right\}$$

where

$$I := \sum_{Q \in \mathcal{A}_\varepsilon} \|P^{(V)} - P^{(V)}\|_{L_p(Q)}^p.$$

Hence,

$$\|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p \leq 2^p \left\{ \|P^{(V)}\|_{L_p(\mathcal{U}_\varepsilon)}^p + I \right\}$$

where $\mathcal{U}_\varepsilon := \cup\{Q : Q \in \mathcal{A}_\varepsilon\}$. By (8.39), $\mathcal{U}_\varepsilon \subset E_{\tilde{\varepsilon}}$ so that

$$\|P^{(V)}\|_{L_p(E \setminus \varepsilon)}^p \leq 2^p \left\{ \|P^{(V)}\|_{L_p(\mathcal{U}_\varepsilon)}^p + I \right\}. \quad (8.40)$$
Prove that \( I \leq C(\varepsilon) \lambda \).
Without loss of generality we may assume that the family \( \mathcal{A}_k \) is finite, i.e.,
\[
\mathcal{A}_k = \{Q_i : i = 1, ..., k\}
\]
for some positive integer \( k \). Furthermore, since the covering multiplicity of \( \mathcal{A}_k \) is bounded by a constant \( C = C(n) \) (recall that \( \mathcal{A}_k \subseteq W_e \)), we may assume that the cubes of the family \( \mathcal{A}_k \) are pairwise disjoint. See Theorem 7.2.

We recall that, by definition of \( P^{(V)} \), see (8.8) and (8.9), for every cube \( Q \in \mathcal{A}_k \)
\[
P^{(V)}(x) = P_{a_Q} \quad \text{for every } x \in Q.
\]
(8.41)
Here \( a_Q := \mathbb{P}_{\mathbb{R}_e}(L^{(Q)}) \) and \( L^{(Q)} \) is the (unique) lacuna containing \( Q \). We know that \( a_Q \in \gamma Q \), see (6.11).
Let \( Q = Q_i \in \mathcal{A}_k \) and let
\[
\tau_Q = \tau_{Q_i} := 2^{-\frac{i}{2}} \lambda^\frac{1}{2} \|Q\|^{-\frac{1}{2}} \quad i = 1, ..., k.
\]
By \( x_Q = x_{Q_i} \), we denote a point in \( Q \) such that
\[
\text{ess sup}_Q |P^{(V)} - P^{(V)}| \leq |P^{(V)}(x_Q) - P^{(V)}(x_Q)| + \tau_Q.
\]
Then
\[
\|P^{(V)}(x) - P^{(V)}\|_{L_p(Q)} \leq (|P^{(V)}(x_Q) - P^{(V)}(x_Q)| + \tau_Q)^p |Q| \leq 2^p |P^{(V)}(x_Q) - P^{(V)}(x_Q)|^p |Q| + 2^p \tau_Q^p |Q|.
\]
Combining this inequality with (8.41) and (8.4), we obtain
\[
\|P^{(V)}(x) - P^{(V)}\|_{L_p(Q)}^p \leq 2^p |P_{a_Q}(x_Q) - P_{b_Q}(x_Q)|^p |Q| + 2^{p-i} \lambda
\]
(8.42)
promised \( Q = Q_i \). Here \( b_Q := V(x_Q) \).
By (8.2), \( \|b_Q - x_Q\| \leq \theta \text{dist}(x_Q, E) \) so that
\[
\|b_Q - c_Q\| \leq \|b_Q - x_Q\| + \|x_Q - c_Q\| \leq \theta \text{dist}(x_Q, E) + \text{diam } Q \leq \theta \text{dist}(Q, E) + 2 \text{ diam } Q.
\]
But \( Q \in W_k \) so that \( \text{dist}(Q, E) \leq 4 \text{ diam } Q \). Hence
\[
\|b_Q - c_Q\| \leq 4 \theta \text{ diam } Q + 2 \text{ diam } Q \leq 6 \theta \text{ diam } Q
\]
proving that \( b_Q \in 12 \theta Q \). (Recall that \( \theta \geq 1 \).
Let \( H_Q := P_{a_Q} - P_{b_Q} \). Since \( H_Q \in \mathcal{P}_{m-1}(\mathbb{R}^n) \),
\[
H_Q(x) = \sum_{|a| \leq m-1} \frac{1}{\sigma^a} D^a H_Q(a_Q) (x - a_Q)^a, \quad x \in \mathbb{R}^n.
\]
Since \( x_Q \in Q \) and \( a_Q \in \gamma Q \), we have \( \|x_Q - a_Q\| \leq \gamma \text{ diam } Q \) so that
\[
|H_Q(x_Q)| \leq \sum_{|a| \leq m-1} |D^a H_Q(a_Q)| \|x_Q - a_Q\|^{|a|} \leq \gamma^m \sum_{|a| \leq m-1} |D^a H_Q(a_Q)| (\text{diam } Q)^{|a|}
\]
Now, by (8.42),
\[ \|P^{(V_x)} - P^{(V)}\|_{L_p(Q)}^p \leq C \sum_{|\alpha| \leq m-1} |D^\alpha H_Q(a_Q)|^p (\text{diam } Q)^{|\alpha|p+n} + 2^{-i} \lambda \]

where \( Q = Q_x \). Hence,
\[
I = \sum_{Q \in A_k} \|P^{(V_x)} - P^{(V)}\|_{L_p(Q)}^p
\leq C \sum_{Q \in A_k} \sum_{|\alpha| \leq m-1} |D^\alpha H_Q(a_Q)|^p (\text{diam } Q)^{|\alpha|p+n} + 2^{|k|} \sum_{i=1}^k 2^{-i} \lambda .
\]

By (8.38), \( \text{diam } Q < \varepsilon \) so that
\[ I \leq C \varepsilon^m \sum_{Q \in A_k} \sum_{|\alpha| \leq m-1} |D^\alpha H_Q(a_Q)|^p (\text{diam } Q)^{|\alpha|p+n} + 2^{p+1} \lambda . \]

Recall that \( a_Q \in \bar{\gamma}Q \) and \( b_Q \in (12\theta)Q \) so that \( a_Q, b_Q \in (\bar{\gamma} + 12\theta)Q = \hat{\gamma}Q \). Hence, by (8.33),
\[ I \leq C \varepsilon^m \lambda + 2^{p+1} \lambda . \]
proving the required inequality \( I \leq C(\varepsilon) \lambda \).

Combining this inequality with inequality (8.40), we conclude that inequality (8.36) holds. Thus all conditions of the hypothesis of Theorem 8.2 are satisfied (with \( \lambda \) defined by (8.35)). By this theorem there exists a \( C^{m-1} \)-function \( F \in W^m_p(\mathbb{R}^n) \) which agrees with \( P \) on \( E \) (i.e., (8.3) holds). Furthermore, by (8.11),
\[ \|P\|_{m,p,E}^* \leq C \{ \|P^{(V)}\|_{L_p(E)}^\dagger + \lambda^{\dagger/2} \} \]
so that, by (8.36),
\[ \|P\|_{m,p,E}^* \leq C \{ \|P^{(V)}\|_{L_p(E)}^\dagger + \lambda^{\dagger/2} + (\gamma^{m-p-n})^{\dagger/2} \} \]
proving the required inequality
\[ \|P\|_{m,p,E}^* \leq C \{ \|P^{(V)}\|_{L_p(E)}^\dagger + \lambda^{\dagger/2} \} . \]

The proof of Theorem 8.1 is complete. \( \square \)

**Remark 8.10** The equivalence (8.6) and Lemma 8.6 imply the following “discrete” version of the result of Theorem 8.1. Let \( Q \in W_E \) be a Whitney cube with \( \text{diam } Q \leq \hat{\varepsilon} \). Let \( z_Q \in Q \) be an arbitrary point in \( Q \) and let \( t_Q := V(z_Q) \). Finally, let \( f_p(x) := P_{z}(x), x \in E \).

Then for every Whitney \((m-1)\)-field \( P = \{P_x : x \in E\} \) on \( E \) the following equivalence
\[ \|P\|_{m,p,E}^* \sim \|f_p\|_{L_p(E)}^\dagger + \mathcal{N}^\dagger(P) + \left( \sum_{Q \in W_E, \text{diam } Q \leq \hat{\varepsilon}} \sum_{|\alpha| \leq m-1} (\text{diam } Q)^{|\alpha|p+n} |D^\alpha P_{t_Q}(Q)|^p \right)^\dagger \]
holds. The constants of this equivalence depend only on \( m, n, p, \hat{\varepsilon}, \) and \( \theta \). <
Remark 8.11 Using the result of Theorem 8.2, Lemma 8.6 and properties of lacunae described in Subsections 6.1 and 6.2, we also obtain a corresponding “discrete” version of the criterion (8.11):

Let $L \in L_{E}$ be a lacuna, and let $\text{diam} \ L := \sup\{\text{diam} \ Q : Q \in L\}$. Let $s_{L} := \mathbb{P}\mathbb{R}_{E}(L)$, then for every Whitney $(m - 1)$-field $P = \{P_{x} : x \in E\}$

$$\|P\|_{m,p,E} \sim \|f_{P}\|_{L_{p}(E)} + K^{L}(P) + \left\{ \sum_{L \in \mathcal{L}_{E}} \sum_{|\alpha| \leq m - 1} \min\{\varepsilon, \text{diam} \ L\}^{|\alpha|+n} |D^{\alpha}P_{x}(s_{L})|^{p} \right\}^{\frac{1}{p}}.$$ 

The constants in this equivalence depend only on $m, n, p$ and $\varepsilon$. 

Our last result is an analogue of Theorem 1.11 for the normed Sobolev space $W^{m}_{p}(\mathbb{R}^{n})$, $p > n$. Let $\varepsilon = 1$. We know that the operator $J(W^{m}_{p}(\mathbb{R}^{n}))|_{E} \ni P \rightarrow F_{\varepsilon}$ defined by formula (8.16) provides an almost optimal “extensions” of $(m - 1)$-jets generated by Sobolev $W^{m}_{p}(\mathbb{R}^{n})$-functions. Since this operator is linear, we obtain the following

**Theorem 8.12** For every closed subset $E \subset \mathbb{R}^{n}$ and every $p > n$ there exists a continuous linear operator $\tilde{F} : \mathcal{J}(W^{m}_{p}(\mathbb{R}^{n}))|_{E} \rightarrow W^{m}_{p}(\mathbb{R}^{n})$ such that for every Whitney $(m - 1)$-field $P = \{P_{x} : x \in E\} \in \mathcal{J}(W^{m}_{p}(\mathbb{R}^{n}))|_{E}$

the function $\tilde{F}(P)$ agrees with $P$ on $E$.

Furthermore, the operator norm of $\tilde{F}$ is bounded by a constant depending only on $m, n$, and $p$.

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