Pattern formation is ubiquitous in nature from morphogenesis and cloud formation to galaxy filamentation. More often than not, patterns arise in a medium driven far from equilibrium due to the interplay of dynamical instability and nonlinear wave mixing. We report, based on momentum and real space pattern recognition, formation of density patterns with two- (D$_2$), four- (D$_4$) and six-fold (D$_6$) symmetries in Bose-Einstein condensates (BECs) with atomic interactions driven at two frequencies. The symmetry of the pattern is controlled by the ratio of the frequencies. The D$_6$ density waves, in particular, arise from a resonant wave mixing process that coherently correlates and enhances the excitations that respect the symmetry.

How patterns emerge in a homogeneous system is a fundamental question across interdisciplinary research areas including hydrodynamics [1], condensed matter physics [2], nonlinear optics [3], cosmology [4] and bio-chemistry [5, 6]. Two paradigmatic examples are Rayleigh-Bénard convection rolls and Faraday waves [7, 8]. Patterns such as stripes, square lattices and hexagonal lattices form in these systems as a result of wave mixing of different wavelengths [9–11]. A generic model for these phenomena is described by the Swift-Hohenberg equation [12],

$$\frac{\partial u}{\partial t} = \lambda u - (q^2 + \nabla^2) u + f(u),$$  \hspace{1cm} (1)

where $u = u(x, t)$ is the amplitude of a physical field, $t$ is the evolution time, $q$ determines the momentum of the unstable modes and $\lambda$ characterizes the growth rate at small amplitudes. The nonlinear function $f(u)$ describes the mixing of the modes as the amplitude grows. The shape and the symmetry of the resulting pattern crucially depend on the strength and the form of $f(u)$ [12–14].

The onset of pattern formation can be understood from the dynamics and interaction of the excitations in the momentum space, described by the nonlinear amplitude equation [15],

$$\frac{du_i}{dt} = \alpha_i u_i + \sum_{j,k} \beta_{ijk} u_j u_k + O(u^3),$$  \hspace{1cm} (2)

where $u_i$ is the amplitude of the $i$-th excitation mode. Starting from small amplitudes, the modes grow exponentially at the rate $\alpha_i$. The quadratic term becomes important as the mode grows, and the tensor $\beta_{ijk}$ describes the mixing of the modes and determines the resulting pattern. The explicit form of $\beta_{ijk}$ is given by the underlying physics, e.g., Navier-Stokes equation for the hydrodynamic systems [16, 17].

In quantum systems, patterns, often characterized by correlation functions, frequently arise from long-range interactions or dynamics far from equilibrium. In polaritonic quantum fluids, hexagonal patterns emerge due to scattering between polaritons [18]. In cold atoms, Faraday waves induced by modulation of trap frequency [19].
FIG. 2. Formation of density waves with \( D_6 \) symmetry. (A) The scattering length is modulated in two stages. The modulation frequency is 450 Hz in the first ten cycles, which is then superposed with a second modulation of 225 Hz (see text). (B) Examples of \( \text{in situ} \) images at times \( t = 0, 22.6 \) and 45 ms (top row) and the corresponding Fourier transforms (bottom row). At 45 ms, the Fourier transform displays 6 peaks with \( \pi/3 \) angular spacing that break the rotation symmetry. The 6-peak patterns orient randomly in repeated experiments. (C) Pattern recognition based on 185 Fourier transformed images yields six strong peaks (red circles) on the vertices of a hexagon (yellow). Two weaker ones come from patterns with \( D_4 \) symmetry. We remove the contribution from the BECs [30]. (D) Correlations \( g^{(2)} \) of the Fourier modes with angular spacing \( \theta \). The peaks at \( \pi/3, \pi/2 \) and \( \pi \) indicate the strength of the patterns with \( D_6, D_4 \) and \( D_2 \) symmetry respectively.

or interactions [20] occur in one-dimensional (1D) BECs. BECs also develop spin [21] or density wave patterns [22] by quenches of atomic interaction. Droplets in a dipolar BEC form a hexagonal pattern due to Rosensweig instability [23]. Recently, supersolid order, for which a superfluid exhibits spatial correlations, emerges in condensates with spin-orbit coupling [24] or dipolar interactions [25–27].

In this paper, we report formation of various two-dimensional (2D) density wave patterns in a uniform BEC by modulating the atomic interactions at two frequencies (Fig. 1A). The interaction modulation is realized by applying an oscillating magnetic field to the sample [28, 29]. The magnetic field is in the \( z \)-direction, perpendicular to the sample while the pattern forms in the horizontal \( x-y \) plane (Fig. 1B). By changing the ratio of the two modulation frequencies, density patterns with \( D_2, D_4 \) and \( D_6 \) symmetries are observed \( \text{in situ} \) and analyzed. The \( D_6 \) density wave pattern, in particular, results from a novel coherent process that resonantly couples six momentum modes.

To understand the pattern formation process in a driven condensate, we derive the associated quantum nonlinear amplitude equation as [30]

\[
\frac{d\hat{a}_k}{dt} = \gamma_1 \hat{a}_{-k} + \gamma_2 \sum_{k_1} \hat{a}_{k_1} \hat{a}_{k_1-k} - \gamma_2 \sum_{k_2} \hat{a}_{k_2} \hat{a}_{k_2-k}, \tag{3}
\]

where \( \hat{a}_k \) and \( \hat{a}_{-k}^{\dagger} \) are the bosonic annihilation and creation operators with momentum \( \hbar k \), respectively, \( \hbar = h/2\pi \) is the reduced Planck constant, the summations include all resonant scattering processes, and the rate constants \( \gamma_1 \) and \( \gamma_2 \) are given by the modulation strengths. This equation is reminiscent of the classical amplitude equation Eq. 2.

The wave mixing processes leading to \( D_4 \) and \( D_6 \) patterns can be described in two stages (Fig. 1C). In the \( \text{seeding} \) stage, atom pairs with opposite momentum
are generated from the condensate by a single-frequency modulation. Such a process, given by the first term in Eq. (3) seeds and amplifies the primary excitations that spontaneously break the rotational symmetry of the system. In the pattern forming stage, the same or a different frequency component is introduced to the modulation, which stimulates scatterings into a particular pattern with the desired symmetry [32]. This process is described by the nonlinear terms in Eq. (3). Finally, the excitation modes interfere with the BEC to form the density wave \( n(r) \), which we observe. The density wave relates to the excitations \( \hat{a}_k \) as \( \hat{n}(r) = n_0 [1 + N_0^{-1/2} \sum_k (\hat{a}_k + \hat{a}_{-k}^\dagger) e^{i k \cdot r}] \), where \( n_0 \) is the condensate density and \( N_0 \gg 1 \) is the atom number in the condensate.

The experiment starts with a BEC of \( N_0 = 60,000 \) cesium atoms in a dipole trap. Atoms are radially confined in a circular potential well with a radius of 14.5 \( \mu \text{m} \) and a barrier height of \( h \times 140 \text{ Hz} \). In the vertical direction, the sample is confined in a harmonic potential with a \( 1/e^2 \) radius of 0.78 \( \mu \text{m} \). We then apply an oscillating magnetic field near a Feshbach resonance [31] to the BEC, which modulates the atomic s-wave scattering length \( a \). After modulation time \( t \), we perform \textit{in situ} imaging to record the density waves.

We first describe the experimental procedure for the formation of the density waves with D\(_6\) symmetry. In the seeding stage, we apply a single-frequency modulation as \( a(t) = a_{dc} + a_1 \sin \omega t \), where \( \omega / 2\pi = 450 \text{ Hz} \), \( a_1 = 30 \ a_0 \), \( a_{dc} = 2 \ a_0 \) and \( a_0 \) is the Bohr radius. After \( t = 22.2 \text{ ms} \), in the pattern forming stage, we add a second frequency component to the modulation as \( a(t) = a_{dc} + a_1 \sin \omega t + a_2 \sin \omega t / 2 \), where \( a_2 = 25 \ a_0 \) (see Fig. 2A).

We analyze the symmetry of the density wave patterns based on Fourier analysis. In the seeding stage, only stripe patterns appear. In the pattern forming stage, hexagonal lattice patterns with D\(_6\) symmetry emerge, signified by six distinct modes in the Fourier space. The modes are equally spaced by \( \pi / 3 \) in their directions with the same wavenumber \( k_f = \sqrt{m \omega / \hbar} \) [28] (see Fig. 2B), where \( m \) is the atomic mass.

The presence of the D\(_6\) pattern can be further confirmed with a pattern recognition algorithm [32] (see Fig. 2C). To quantify the strength of the patterns, we evaluate the density correlation function \( g^{(2)}(\theta) \equiv \langle |A_x|^2 |A_{x+\theta}|^2 \rangle / \langle |A_x|^4 \rangle^2 \), where \( A_x = \int n(r) e^{-i k_x \cdot r} \) is the Fourier amplitude evaluated at \( k_x \) with magnitude \( |k_x| = k_f \) and angle \( \theta \). The angle brackets denote averaging over both the angles \( \phi \) from 0 to \( 2\pi \) and the images. The evolution of \( g^{(2)} \) confirms the growth of different patterns in the seeding and pattern forming stages (Fig. 2D).

We tailor the modulation waveform to create different patterns. Here three modulation schemes that lead to patterns with D\(_3\), D\(_4\) and D\(_6\) symmetries are reported. Scheme I: we apply the modulation at a single frequency \( \omega \). Scheme II: we modulate at frequency \( \omega \) in the seeding stage and superpose a second frequency \( \omega / 2 \) in the pattern forming stage (Fig. 3A). Scheme III: we modulate at frequency \( \omega / 2 \) and then switch to frequency \( \omega \).

To reveal the density patterns in real space, we employ a 2D pattern recognition algorithm. Since the pattern in each image appears with random orientation and displacement, the algorithm is developed to rotate and align the patterns (Fig. 3A). We determine the orientation of each image as illustrated in Fig. 2C, and align all of them in the same direction. We then translate each of the images independently to maximize the spatial variance of their average. Finally we extract the underlying pattern by averaging all aligned images. To eliminate long wavelength variations that are uncorrelated with the pattern, we filter the density fluctuations at \( |k| \leq 0.75 k_f \) from the images to get the density waves \( \hat{n}(r) \).

The results of the 2D pattern recognition algorithm are shown in Fig. 3B. Single frequency modulation (Scheme I) produces D\(_2\) stripe patterns. Scheme II (\( \omega \rightarrow \omega / 2 \)) results in a hexagonal lattice pattern, consistent with Fig. 2. Scheme III (\( \omega / 2 \rightarrow \omega \)) results in a square lattice pattern. We further determine the strengths of different symmetry components in each image \( P \) based on the fit: \( P = c_2 P_2 + c_4 P_4 + c_6 P_6 \), where \( P_n \) are normalized patterns with \( D_n \) symmetry, and \( c_n \) are the fitting parameters [30]. The results, shown in the bar diagrams of Fig. 3, suggest that different schemes are effective in generating patterns with different symmetries.

Remarkably, all three patterns extend throughout the entire sample. The spatial extent of the patterns can be evaluated from their real space correlation functions \( g^{(2)}(r) \equiv \langle \hat{n}(r_0) \hat{n}(r_0 + r) \rangle / \langle \hat{n}(r_0)^2 \rangle \). Correlations along principle directions, shown in Fig. 3C, extend across the entire sample of diameter 25 \( \mu \text{m} \). Comparing the patterns, we observe that the D\(_6\) pattern is a factor of 5 more pronounced than D\(_4\) even though these two schemes employ similar modulation strengths [30].

The clear difference between the strength of the D\(_4\) and D\(_6\) patterns comes from the coherence of the underlying scattering processes. For D\(_3\) patterns, phase coherence only exists between counter-propagating modes, illustrated in Fig. 4A. We evaluate the two-point phase correlation function of the density waves as \( g^{(4)}(\theta) \equiv \langle A_x A_{x+\theta} \rangle / \langle |A_x|^2 \rangle \), where \( A_x = |A_x| e^{i \phi_x} \) is the Fourier amplitude of the mode with wavenumber \( k_f \) at angle \( \theta \) and \( \phi_x \) is its phase. The result, see Fig. 4B, shows a single peak at \( \theta = \pi \), simply due to the realness of density. The absence of other features, particularly at \( \theta = \pi / 2 \), shows that the density waves in orthogonal directions are incoherent. Close inspection of the phases of orthogonal modes, see inset of Fig. 4B, confirms the absence of correlation.

The D\(_6\) pattern, on the other hand, displays a novel phase coherence in triplets of modes angularly spaced by \( 2\pi / 3 \), see Fig. 4C. Here we evaluate the three-point phase correlation function as

\[
g^{(3/2)}(\theta, \theta') = \frac{\langle A_x A_{x+\theta} A_{x+\theta'} \rangle}{\sqrt{\langle |A_x|^4 \rangle \langle |A_{x+\theta}|^4 \rangle \langle |A_{x+\theta'}|^4 \rangle}}.
\]

The correlation shows two peaks at \( (\theta, \theta') = (2\pi / 3, 4\pi / 3) \)
and \((4\pi/3, 2\pi/3)\) (see Fig. 4D), where \(\theta\) and \(\theta'\) are the relative angles between the three modes. This indicates phase coherence of any three modes angularly separated by \(2\pi/3\). From repeated measurements, we find that the phases of the triplets are statistically constrained to \(\phi_0 + \phi_{2\pi/3} + \phi_{4\pi/3} = 0\) modulo \(2\pi\) with a small standard deviation of \(\delta\phi = 1.1\), see Fig. 4E and F. The phase differences, e.g. \(\phi_0 - \phi_{2\pi/3} - \phi_{4\pi/3}\), as well as other permutations, are uniformly distributed and thus uncorrelated.

The three-point phase correlation is an essential element to understanding the growth and the origin of \(D_6\) patterns in our system. Based on Eq. (3), we show that the strength of the \(D_6\) pattern satisfies the equation of motion \([30]\)

\[
\frac{dA_{rms}}{dt} = \gamma_1 A_{rms} + \gamma_2 g^{(3/2)} A_{rms}^2, \tag{5}
\]

where \(A_{rms}\) is the root-mean-square of the six Fourier amplitudes that constitute the \(D_6\) pattern and \(g^{(3/2)} \equiv g^{(3/2)}(2\pi/3, 4\pi/3)\). A positive \(g^{(3/2)}\) suggests that beyond small amplitudes, the nonlinear wave mixing term dominates and leads to a faster-than-exponential (hyperbolic) growth of the \(D_6\) density waves. The large measured value of \(g^{(3/2)} = 0.58\) explains the strong \(D_6\) pattern that we observe.

How does the three-point phase correlation emerge in a driven condensate? Starting from a condensate seeded by the single-frequency modulation, we see that \(g^{(3/2)}\) increases quickly from zero after the two-frequency

FIG. 3. Density wave patterns in real space. (A) In our pattern recognition algorithm, each of the \textit{in situ} images is first rotated and then translated to overlap the density waves. The translation maximizes the variance of the averaged image \([30]\). (B) Resulting density waves from the algorithm for Scheme I (stripes): single modulation frequency \(\omega\), Scheme II (hexagonal lattice): \(\omega\) followed by \(\omega/2\), and Scheme III (square lattice): \(\omega/2\) followed by \(\omega\). The green lines are guides to the eye to highlight the corresponding pattern. The green arrows show the direction along which the real space correlation is evaluated in panel C. The bar diagrams show the relative weights of \(D_2\), \(D_4\) and \(D_6\) symmetry components from fitting the patterns \([30]\). (C) Real space correlation functions evaluated from the patterns. The oscillations have periods of 2.63(1), 3.05(1) and 2.65(1) \(\mu m\) for schemes I, II and III, respectively. The ratio of the periods is 1.156(2), consistent with theory value \(2/\sqrt{3} \approx 1.155\). The solid lines are guides to the eye.
modulation starts [30]. Theoretically the growth of the correlation is linked to the resonant nonlinear coupling of excitation modes that respect the symmetry and is described by $dg^{(3/2)}/dt = 3\gamma_2 A_{rms}$ for small amplitudes $A_{rms} \ll N_0^{1/2}$. Our measurement is in good agreement with the theory [30]. Given the above, the three-point phase relation $\phi_0 + \phi_{2\pi/3} + \phi_{4\pi/3} = 0$ (see Fig. 4F) can be understood as the phase matching condition that maximizes the correlator $g^{(3/2)}$, which explains the dominance of the D_6 pattern in our experiment.

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I. EXPERIMENTAL PROCEDURE

We start with BECs of 60,000 cesium atoms loaded into a disk-shaped dipole trap with a radius of 14.5 \( \mu m \) in the horizontal direction. The horizontal confinement is provided by a blue-detuned laser at 780 nm. We shape the laser beam profile using a digital micromirror device and project it to the atom plane through a high-resolution objective. The resulting circular potential well has a barrier height of \( h \times 140 \) Hz. Atoms are also tightly confined in the vertical direction with a \( 1/e^2 \) radius of 0.78 \( \mu m \) with a harmonic trapping frequency of 259 Hz.

After preparing the sample, we modulate the magnetic field near a Feshbach resonance, which causes the s-wave scattering length \( a \) of the atoms to oscillate as \( a(t) = a_{dc} + a_1(t) \sin \omega_1 t + a_2(t) \sin (\omega_2 t + \phi) \). We edit the control voltage output from an arbitrary waveform generator to modulate currents in the coils, which leads to the magnetic field being modulated according to a designed waveform. A small positive offset scattering length \( a_{dc} = 2a_0 \) is maintained throughout the experiment to keep the condensate stable. For generating the D\(_2\) density wave pattern, we keep modulating the scattering length at frequency 450 Hz with amplitude 45 \( a_0 \) for 23.8 ms. For D\(_4\) pattern, we first modulate at 225 Hz for 3 cycles with amplitude 45 \( a_0 \) and then switch to 450 Hz with the same amplitude for 24 ms. To generate D\(_6\) pattern, the first 10 cycles of modulation is at 450 Hz with amplitude 30 \( a_0 \), which is then mixed with another frequency component at 225 Hz and amplitude 25 \( a_0 \) for 22.8 ms. The relative phase \( \phi \) between these two frequency components is 0.

We finally perform in situ absorption imaging to observe the resulting density waves in the condensates using the high-resolution objective and a CCD camera. Our imaging system is sensitive to density fluctuations of spatial frequency ranging from 0 up to 3.44 \( \mu m^{-1} \)\(^3\), which covers the density waves we observe at \( k_f = 2.43 \mu m^{-1} \). The individual pixel size of the CCD camera is 0.6 \( \mu m \), which provides a sampling frequency of 4 data points within one wavelength of the density waves.

In order to extract the population of excited modes from their interference with the condensate, we first Fourier transform the images including density waves with 121 \( \times \) 121 pixels. Then in the Fourier space we focus on the ring at \( |k - k_f| \leq 0.1k_f \) and cut it using angular slices of 3° to count the average Fourier magnitude \( A_\theta \) in the direction at angle \( \theta \). In general, the sensitivity of our imaging system varies for signals with different wavenumber. We measure the modulation transfer function \( M(k) \) of thermal atoms and find that the proportional constant of measured strength of density fluctuations at \( k_f \) to its corresponding real strength is \( M(k = k_f) = 0.45 \)\(^3\). The relation between density wave amplitude \( A_\theta \) and population \( |a_k|^2 \) is \( |A_\theta|^2 = 4N_0 \cos^2 (\omega t/2) |a_k|^2 \), where the phase \( \omega t/2 \approx 0.57 \) rad at the time we perform the imaging. Finally the population is evaluated as \( |a_k|^2 = |A_\theta|^2 / [M^2(k = k_f)4N_0\cos^2(\omega t/2)] \). Also, we observe the density waves stroboscopically every 4.4 ms as shown in Fig. S\( ^4 \).

II. QUANTUM DYNAMICS OF PATTERN FORMATION

We start from the general form of Hamiltonian of driven BECs,

\[
H = \int d^3r \Psi^\dagger(r,t) \frac{\hbar^2}{2m} \Psi(r,t) + \int d^3r \Psi^\dagger(r,t) V(r) \Psi(r,t) + \frac{g(t)}{2} \int d^3r \Psi^\dagger(r,t) \Psi^\dagger(r,t) \Psi(r,t) \Psi(r,t),
\]

(S1)

where the interaction strength is modulated as \( g(t) = \frac{4m \hbar^2}{\mu} [a_{dc} + a_1(t) \sin \omega_1 t + a_2(t) \sin (\omega_2 t + \phi)] \). Here \( a_{dc} \) is a small offset scattering length to keep the condensate stable, \( a_{1,2} \) are amplitudes of scattering length modulation and \( \phi \) is the relative phase between the two frequency components \( \omega_1 \) and \( \omega_2 \).

The external potential \( V(r) \) is neglected later because it only serves to determine the initial wavefunction of BECs and doesn’t affect the dynamics. After doing the Fourier transform \( \Psi(r) = \frac{1}{\sqrt{V}} \sum_k \hat{a}_k e^{ikr} \), we obtain the Hamiltonian in momentum space as

\[
H = \sum_k c_k \hat{a}_k^\dagger \hat{a}_k + \frac{g(t)}{2V} \sum_{k_1, k_2, \Delta k} \hat{a}^\dagger_{k_1 + \Delta k} \hat{a}^\dagger_{k_2 - \Delta k} \hat{a}_{k_1} \hat{a}_{k_2},
\]

(S2)

where \( V \) is the volume of condensate and the dispersion is \( c_k = \hbar^2 k^2 / 2m \).

After transferring to the rotating frame with \( \hat{a}_k \rightarrow \hat{a}_k e^{-ik_0 t/\hbar} \) and using the rotating wave approximation to eliminate the fast oscillating terms, the Hamiltonian becomes time-independent:
\[
H_f = \frac{i}{4V} \sum_k g_1 \hat{a}_k^\dagger \hat{a}_{-k}^\dagger \hat{a}_0 \hat{a}_0 + \sum_{k'} g_2 \hat{a}_{k'}^\dagger \hat{a}_{-k'}^\dagger \hat{a}_0 \hat{a}_0 + \sum_{k_1, k_2} e^{-i\phi} \hat{a}_{k_2}^\dagger \hat{a}_{k_1}^\dagger \hat{a}_{k_1} \hat{a}_{k_2} + h.c.,
\]
(S3)

where \(g_1 = 4\pi \hbar^2 a_1/m\) and \(g_2 = 4\pi \hbar^2 a_2/m\) and the summations go over the processes that satisfy the following energy conservation conditions:

\[
\begin{align*}
\epsilon_k + \epsilon_{-k} &= \hbar \omega_1, \\
\epsilon_{k'} + \epsilon_{-k'} &= \hbar \omega_2, \\
\epsilon_{k_2} + \epsilon_{k_1 - k_2} &= \epsilon_{k_1} + \hbar \omega_2.
\end{align*}
\]
(S4)

Here the left/right hand side is the total energy after/before the collision.

Then the equation of motion for \(\hat{a}_k\) is obtained to second order in the Bogoliubov approximation \(\hat{a}_0 \approx \hat{a}_0^\dagger \approx \sqrt{N_0}\) as

\[
\frac{d\hat{a}_k}{dt} = \gamma_1 \hat{a}_k^\dagger + \gamma_2 \sum_{k_1} \hat{a}_{k_1 - k} \hat{a}_k - \gamma_2 \sum_{k_2} \hat{a}_{k_2}^\dagger \hat{a}_{k - k_2},
\]
(S5)

where the growth rates are given by \(\gamma_1 = \frac{N_0 \pi \hbar a_1}{mV}\), \(\gamma_2 = \frac{\sqrt{N_0 \pi \hbar a_2} e^{-i\phi}}{mV}\). Here all the momenta are restricted to the horizontal plane and the magnitude of \(k\) is \(|k| = k_f = \sqrt{m \omega_1 / \hbar}\). We have been using \(\omega_1/2\pi = 450\) Hz and \(\omega_2/2\pi = 225\) Hz.

The formation of density wave patterns originates from the momentum and energy conservation of underlying bosonic stimulated scattering processes (see Fig. 1C). For the \(D_4\) pattern formation under Scheme III, during the modulation of frequency \(\omega_2\), a pair of BEC atoms absorb a quantum of energy \(\hbar \omega_2\) and scatter into a pair of atoms with opposite momenta \(\pm k_1\) at \(|k_1| = k_f / \sqrt{2}\) and energy \(\epsilon_{k_1} = \hbar \omega_2 / 2\). Then one atom with \(k_1\) collides with one BEC atom absorbing another quantum of \(\hbar \omega_2\). One of them scatters into \(k\) with magnitude \(k_f\) and energy \(\epsilon_k = \hbar \omega_2\) at \(45^\circ\) or \(-45^\circ\) relative to \(k_1\). The other one is scattered into \(k_1 - k\) with magnitude \(k_f / \sqrt{2}\) and energy \(\epsilon_{k_1 - k} = \hbar \omega_2 / 2\) at \(-90^\circ\) or \(90^\circ\) relative to \(k_1\). This process is described by the second term on the right hand side (RHS) of Eq. S5. On the other hand, one atom with \(-k_1\) can collide with one BEC atom and one of the scattered atoms has momentum \(k_f\) at \(45^\circ\) or \(-45^\circ\) relative to \(-k_1\). Thus, seeds of 4 momentum modes at \(k_f\) with \(90^\circ\) relative angular spacing are generated. Later, when another modulation of frequency \(\omega_1 = 2 \omega_2\) is applied, those 4 modes get amplified with pairs of BEC atoms scattering into them. This corresponds to the first term on the RHS of Eq. S5. Finally the those 4 momentum modes with \(90^\circ\) angular spacing interfere with the BEC to form the \(D_4\) density wave pattern.

On the other hand, for \(D_6\) pattern formation under Scheme II, a modulation of frequency \(\omega_1\) is first applied to generate pairs of opposite momentum modes \(\pm k\) at \(k_f\) and energy \(\epsilon_k = \hbar \omega_1 / 2\). Then when the second frequency component \(\omega_2 = \omega_1 / 2\) is added, an atom with \(k\) collides with a BEC atom absorbing one energy quantum \(\hbar \omega_2\) and scattering into atoms with \(k_2\) and \(-k_2\) with the same magnitude \(k_f\) and energy \(\epsilon_{k_2} = \epsilon_{-k_2} = \hbar \omega_1 / 2\) at \(\pm 60^\circ\) relative to \(k\). This corresponds to the third term on the RHS of Eq. S5. Also, atoms with momentum \(k_2\) or \(-k_2\) can collide with one BEC atom into atoms with \(k\), corresponding to the second term on the RHS of Eq. S5. Also, one atom with \(-k\) can collide with one BEC atom and scatter into \(-k_2\) or \(-k - k_2\) at \(\pm 60^\circ\) relative to \(-k\). Thus, 6 momentum modes with \(60^\circ\) relative angular spacing are generated and are amplified by the \(\omega_1\) frequency component at the same time. Eventually they interfere with the condensate and form the \(D_6\) density wave pattern.

III. PRINCIPAL COMPONENT ANALYSIS

In order to remove the background of Fourier space in Fig. 2C, we collect 100 images of pure BECs and apply PCA algorithm to construct the bases and subtract the projection onto these bases from the Fourier transform of BECs with density waves.

We first get the Fourier amplitude’s magnitude \(n_i(k)\) of the \(i^{th}\) image of pure BEC atomic density \(n_i(r)\). Each \(p \times p\) square matrix \(n_i(k)\) is rearranged into a \(1 \times p^2\) row vector. Then all the row vectors are arranged to form a rectangular matrix \(M_{ij}\), where \(j\) ranges from 1 to \(p^2 = 121^2\). The mean value of each column is shifted to zero by subtracting the average of experimental realizations, resulting in the data matrix \(X = M - \bar{M}\). Our goal is to diagonalize the covariance matrix \(X^T X\) to find its eigenvectors \(w_j\) and eigenvalues \(\lambda_j\), which corresponds to statistical independent bases (principal components) and variance of X's projection \(X_{ij} w_j\) onto each basis, respectively. We use singular value decomposition (SVD) to perform this diagonalization.
FIG. S1. Principal component analysis for removing the background in Fourier space. (A) The variance of the 
data matrix X’s projection onto each principal components. The y axis is in log scale. (B) The mean of the magnitude 
of Fourier transforms of pure BECs (left) and the first two bases from PCA (middle and right). (C) One example of removing all 
the projection onto PCA bases (middle) from the Fourier space of BEC with density waves (left), only signals from the density 
waves survive (right).

The first 99 principal components are kept and the corresponding variances are shown in Fig. S1A. The average of 
n_i(k) is counted as an additional basis w_0. In Fig. S1B, we plot the average of n_i(k) and the two principal components 
that have the largest and second largest variances. Next, we use those constructed bases to remove the background 
in the Fourier space n_d(k) of the atomic densities of BECs with density waves n_d(r). As an example, in Fig. S1C, 
we project one n_d(k) to all the principal components w_j to reconstruct the background. Finally the background is 
subtracted from the original Fourier space and only the signals from density waves are left.

IV. PHASES AND AMPLITUDES OF DENSITY WAVES

In order to precisely determine the spatial phase of the density waves at different directions, we develop the following 
fitting procedure. Since the length scale of the density wave we care about is only around k_f, we first filter out the 
strong low frequency noise below 0.75k_f in the Fourier transform of in situ density profile n(x, y) and inversely 
transform it back to obtain the filtered atomic density ˜n(x, y) as shown in Fig. S2A. ˜n(x, y) is the superposition of 
plane waves at different directions confined in a finite sized BEC, thus the precision of extracting the phase from its 
Fourier transform is limited by the small number of density wave periods. In order to avoid this limitation, we first 
integrate the filtered atomic density along a certain direction θ normalized by the corresponding integrated circular 
BEC area to get the averaged 1D density oscillation n_θ(x) = \int dy n(x, y)/\sqrt{R^2 - x^2}. Then the central part \ |x| ≤ 10 \ μm 
of n_θ(x) is fitted using fit function f(x) = F_θ \cos (k_f x + \phi_θ), where F_θ and \phi_θ are the amplitude and phase of the 
density wave at k_f and angle θ. Here the step size of angle θ is chosen to be 1° for better resolution compared to the 
Fourier transform. The amplitude F_θ and phase \phi_θ are unaffected by density waves in other directions, which only 
contribute noise at spatial frequency smaller than k_f or are completely integrated out.

Fig. S2B shows the angular distribution of density wave amplitudes from Fourier transform compared with that 
from fitting. It can be seen that the results obtained from these two methods are consistent with each other. At the 
angles indicated by the black arrows in Fig. S2B, three examples of the fitting results are shown in Fig. S2C. The 
density oscillation is fit very well when its Fourier amplitude is significant.
FIG. S2. Extraction of the phases and amplitudes of density waves at different directions. (A) The low frequency part at $k < 0.75 k_f$ of the raw in situ atomic density and its Fourier transform (upper row) is filtered and the density fluctuations at $k \geq 0.75 k_f$ and its Fourier transform are obtained (lower row). (B) The angular distribution of the Fourier transform magnitude of original atomic density before the filtering (blue open circles) and the corresponding amplitude from fitting the 1D mean density fluctuation $n_\theta(x)$ of the filtered atomic density (orange solid line). The scale of the left and right y axis differ by a factor of 331, which is one half of the area where density waves exist in unit of $\mu m^2$. (C) Three examples of fitting the 1D mean density fluctuation $n_\theta(x)$ at different directions with various Fourier magnitudes (indicated by arrows in (B)).

V. REAL SPACE PATTERN RECOGNITION ALGORITHM

We consider each individual in situ absorption image as a combination of several common patterns with random orientations and displacements which contribute to the image with different weights. To reveal the common pattern, we align the strongest components from repeated experimental realizations and the weaker ones are averaged to zero. This alignment can be achieved from our real space pattern recognition algorithm.

Here we describe the details of the 2D pattern recognition algorithm (Fig. 3A). We first filter out the low frequency noise at $|k| < 0.75 k_f$ from the in situ absorption images to get a set of $N = 185$ filtered images of atomic density fluctuations, $\tilde{n}_i(x, y), i = 1, \cdots, N$ (see Fig. S2A). Let $T_{\theta_i, r_i}(\tilde{n}_i)$ denote the result of rotating $\tilde{n}_i$ by $\theta_i$ and then translating by $r_i$, where we impose the constraint $|r_i| < 2\pi/k_f$. The objective function $L$ is the spatial variance of the average image $\bar{n}$ after rotating and translating individual images:

$$\bar{n}(\{\theta_i\}, \{r_i\}) = \frac{1}{N} \sum_i T_{\theta_i, r_i}(\tilde{n}_i),$$

(S6)
where $S$ is the total area of the atomic density fluctuations. The optimal rotation angles and translation displacements \( \{ \theta_i \}, \{ r_i \} \) are found by maximizing $L$, and the pattern recognized is $\bar{n}$ with the optimal parameters.

Since the rotation angle $\theta_i$ and displacement $r_i$ are independent degrees of freedom, we perform the optimization of the objective function $L$ in two separate steps. We first find the orientation of each image from the angular distributions of density wave amplitudes $F_p$ obtained from fitting (see Fig. S2B). The rotation angles $\theta_i$ are changed for individual images in order to maximize the variance of the averaged angular distribution \[32\]. Then the angles are fixed to be the ones after the above optimization before we optimize the displacement of each image. Finally, we translate each image $\bar{n}_i$ by $r_i$ to maximize the spatial variance of resulting averaged density fluctuation $\bar{n}$. The recognized common patterns for different modulation schemes are shown in Fig. 3B.

**VI. SYMMETRY DECOMPOSITION OF DENSITY PATTERNS**

We consider each recognized pattern $P$ shown in Fig. 3B as a superposition of normalized two-, four- and six-fold symmetry components $P_{2,4,6}$ with amplitudes $c_{2,4,6}$ and a small offset $c_0$. In order to find the contribution of each symmetry component, we fit the patterns using the following function:

\[
P = c_2 P_2 + c_4 P_4 + c_6 P_6 + c_0,
\]

where

\[
P_2 = \mathcal{R}_{\theta_2} \cos(k_f x + \phi_2),
\]

\[
P_4 = \frac{1}{\sqrt{2}} \mathcal{R}_{\theta_4} [\cos(k_f x + \phi_{4,1}) + \cos(k_f y + \phi_{4,2})],
\]

\[
P_6 = \frac{1}{\sqrt{3}} \mathcal{R}_{\theta_6} \left[ \cos(k_f x + \phi_{6,1}) + \cos(k_f \left( -\frac{1}{2} x + \frac{\sqrt{3}}{2} y \right) + \frac{1}{2} \phi_{6,1} + \frac{\sqrt{3}}{2} \phi_{6,2}) \right] + \cos(k_f \left( -\frac{1}{2} x - \frac{\sqrt{3}}{2} y \right) - \frac{1}{2} \phi_{6,1} - \frac{\sqrt{3}}{2} \phi_{6,2}) \right].
\]

Here $\mathcal{R}_{\theta}[.]$ denotes rotation by angle $\theta$. There are 12 fitting parameters in total: \( \{ c_2, c_4, c_6 \} \) determine the strengths of the symmetry components, $c_0$ determines the overall offset, \( \{ \theta_2, \theta_4, \theta_6 \} \) determine the orientations, and \( \{ \phi_2, \phi_{4,1}, \phi_{4,2}, \phi_{6,1}, \phi_{6,2} \} \) determine the displacements. The optimal fitting parameters are shown in Table S1. One example of the symmetry decomposition results for the D$_6$ density pattern under Scheme II is shown in Fig. S4.

**VII. HYPERBOLIC GROWTH OF D$_6$ PATTERN**

As we have shown in Fig. 4, for D$_6$ pattern, only the Fourier modes separated by $2\pi/3$ are coupled together. Since each Fourier mode \( A_\theta = \sqrt{N_0}(a_{\theta}e^{-i\omega t} + \hat{a}_{-\theta}^\dagger e^{i\omega t}) \) consists of two opposite momentum modes, six momentum modes separated by $\pi/3$ are coupled together. Let’s first consider a simple model where there are only six modes $\hat{a}_{i}, \ i = 1, 2, \cdots, 6$, separated by $\pi/3$ with momentum $|k_i| = k_f$. Under driving Scheme II, the equation of motion reads,

\[
\frac{d\hat{a}_i}{dt} = \gamma_1 \hat{a}_{i+3}^\dagger + \gamma_2 (\hat{a}_{i+2}^\dagger \hat{a}_{i+1} + \hat{a}_{i-2}^\dagger \hat{a}_{i-1}) - \gamma_2 \hat{a}_{i+1}^\dagger \hat{a}_{i-1},
\]

where the addition of indices is modulo 6, e.g. $4 + 3 = 1$.

Here we consider the case where the relative phase $\phi = 0$ between the two frequency components and thus $\gamma_2$ becomes real. After the first 10 cycles of single frequency modulation, the population of each mode is amplified to be
larger than the quantum fluctuation. Thus we approximate the operators $\hat{a}_i$ by complex numbers $\tilde{a}_i$. The equation of motion of amplitude for each mode becomes,

$$\frac{d\tilde{a}_i}{dt} = \gamma_1\tilde{a}_{i+3} + \gamma_2(\tilde{a}_{i+2}\tilde{a}_{i+1} + \tilde{a}_{i-2}\tilde{a}_{i-1} - \tilde{a}_{i+1}\tilde{a}_{i-1}).$$

(S13)

At the beginning of the two-frequency modulation, we set the population of each mode $n_i(0) = |\tilde{a}_i(0)|^2$ to satisfy a thermal distribution $p(n) = e^{-n/\bar{n}}/\bar{n}$ with the mean population $\bar{n}$ and $\tilde{a}_i(0) = \tilde{a}_{i+3}(0)$ with its phase randomly distributed from 0 to $2\pi \mathbb{R}$. Because the growth rates $\gamma_1$ and $\gamma_2$ are real, at any later time $t$, we always have

$$\tilde{a}_i = \tilde{a}_{i+3}.$$  

(S14)

Then the Fourier amplitude $A_{\theta_i} = \sqrt{N_0}(\tilde{a}_i e^{-i\omega t} + \tilde{a}_{i+3} e^{i\omega t}) = 2\sqrt{N_0}\tilde{a}_i \cos \omega t$ and Eq. S13 reduces to

$$\frac{d\tilde{a}_i}{dt} = \gamma_1\tilde{a}_i + \gamma_2\tilde{a}_{i+1}\tilde{a}_{i-1}.$$  

(S15)

Multiplying $\tilde{a}_i^*$ on both sides of Eq. S15 and summing their complex conjugates, we get

$$\frac{d|\tilde{a}_i|^2}{dt} = 2\gamma_1|\tilde{a}_i|^2 + 2\gamma_2\Re[\tilde{a}_i^*\tilde{a}_{i+1}\tilde{a}_{i-1}],$$  

(S16)

where $\Re[\cdot]$ means taking the real part. Similarly,

$$\frac{d|\tilde{a}_{i-1}|^2}{dt} = 2\gamma_1|\tilde{a}_{i-1}|^2 + 2\gamma_2\Re[\tilde{a}_{i-1}^*\tilde{a}_{i+1}\tilde{a}_{i-2}],$$  

(S17)

$$\frac{d|\tilde{a}_{i+1}|^2}{dt} = 2\gamma_1|\tilde{a}_{i+1}|^2 + 2\gamma_2\Re[\tilde{a}_{i+1}^*\tilde{a}_{i+2}\tilde{a}_i].$$  

(S18)

| Parameters | Scheme I | Scheme II | Scheme III |
|------------|----------|-----------|------------|
| $c_2$ | $\mu m^{-2}$ | 0.302(8) | -1.49(4) | -0.32(1) |
| $c_4$ | $\mu m^{-2}$ | -0.080(8) | -0.41(3) | -0.26(1) |
| $c_6$ | $\mu m^{-2}$ | 0.070(6) | 1.55(4) | 0.072(8) |
| $c_0$ | $\mu m^{-2}$ | -0.009(3) | 0.02(2) | 0.005(4) |
| $\theta_2$ | rad | 1.594(2) | -0.497(2) | 0.274(2) |
| $\theta_4$ | rad | 1.537(6) | -0.608(6) | 0.206(2) |
| $\theta_6$ | rad | 1.455(6) | -0.547(1) | 0.008(7) |
| $\phi_2$ | rad | 0.96(3) | 5.96(3) | 4.93(3) |
| $\phi_{4,1}$ | rad | 4.4(2) | 6.2(1) | -1.72(5) |
| $\phi_{4,2}$ | rad | 5.6(1) | 2.5(1) | 3.84(4) |
| $\phi_{6,1}$ | rad | 1.2(1) | 3.43(4) | 2.6(2) |
| $\phi_{6,2}$ | rad | 2.7(1) | 5.64(3) | 3.8(2) |

Table S1. Optimal fitting parameters for symmetry decomposition.
Using Eq. S14, it can be seen,
\[
\Re[\hat{a}_i^{\dagger}\hat{a}_{i+1}\hat{a}_{i-1}] = \Re[\hat{a}_{i+3}\hat{a}_{i+1}\hat{a}_{i-1}]
\]
\[
= \Re[\hat{a}_{i-1}^{\dagger}\hat{a}_i\hat{a}_{i-2}] = \Re[\hat{a}_{i+1}^{\dagger}\hat{a}_{i+2}\hat{a}_i].
\]
(S19)

Thus by subtracting two of the equations out of Eqs. (S16) to (S18) and taking the average value on both sides of the equations, we have
\[
\frac{dn_{i,i-1}}{dt} = 2\gamma_1 n_{i,i-1},
\]
(S20)

where the population difference \(n_{i,i-1} = (\langle |\hat{a}_i|^2 \rangle - \langle |\hat{a}_{i-1}|^2 \rangle, n_{i,i+1} \) and \(n_{i-1,i+1} \) also satisfy Eq. S20. Since at the beginning \(|\hat{a}_i|^2, |\hat{a}_{i-1}|^2, \) and \(|\hat{a}_{i+1}|^2 \) satisfy the same distribution \(p(n) = \langle |\hat{a}_i(0)|^2 \rangle = \langle |\hat{a}_{i+1}(0)|^2 \rangle \), which means the population differences \(n_{i,i-1}(0) = n_{i,i+1}(0) = n_{i-1,i+1}(0) = 0 \). Thus according to Eq. S20 at any later time \(t \), the population differences \(n_{i,i-1} = n_{i,i+1} = n_{i-1,i+1} = 0 \), i.e.
\[
\langle |\hat{a}_i|^2 \rangle = \langle |\hat{a}_{i-1}|^2 \rangle = \langle |\hat{a}_{i+1}|^2 \rangle.
\]
(S21)

As is defined in Eq. 4, the three point correlation function at \((\theta, \theta') = (2\pi/3, 4\pi/3)\) is,
\[
g^{(3/2)} = g^{(3/2)}(2\pi/3, 4\pi/3) = \frac{A_\varphi A_{\varphi+2\pi/3} A_{\varphi+4\pi/3}}{\sqrt{|\langle \hat{a}_i|^2 \rangle| |\langle \hat{a}_{i+1}|^2 \rangle| |\langle \hat{a}_{i+2}|^2 \rangle|}}
\]
\[
= \frac{\Re[\langle \hat{a}_{i+3}\hat{a}_{i+1}\hat{a}_{i-1} \rangle]}{\sqrt{|\langle \hat{a}_{i+3}|^2 \rangle| |\langle \hat{a}_{i+1}|^2 \rangle| |\langle \hat{a}_{i-1}|^2 \rangle|}}.
\]
(S22)

Since the average of the product \(A_\varphi A_{\varphi+2\theta} A_{\varphi+2\theta'}\) is performed over all the angles with \(0 \leq \varphi \leq 2\pi \), it always comes in with its complex conjugate, which guarantees that the three point phase correlation function is real. Also, the other possible definitions with one or more of the Fourier amplitudes in \(A_\varphi A_{\varphi+2\theta} A_{\varphi+2\theta'}\) are equivalent to Eq. 4 with angular shifts in \(\theta \) and \(\theta'\), which doesn’t show more information. Then we take the average value on both sides of Eq. S16 and plug in Eqs. S14, S21 and S22 to get
\[
\frac{d\langle |\hat{a}_i|^2 \rangle}{dt} = 2\gamma_1 \langle |\hat{a}_i|^2 \rangle + 2\gamma_2 g^{(3/2)} |\langle |\hat{a}_i|^2 \rangle|^{3/2}.
\]
(S23)

Let’s define the root mean square (RMS) of \(\hat{a}_i \) as \(A_{rms} = \sqrt{\langle |\hat{a}_i|^2 \rangle} \) and plug it into Eq. S23 we finally arrive at the equation of motion,
\[
\frac{dA_{rms}}{dt} = \gamma_1 A_{rms} + \gamma_2 g^{(3/2)} A_{rms}^2.
\]
(S24)

Insert the initial value \(A_{rms}(0) \), we obtain the solution of Eq. S24
\[
A_{rms}(t) = \frac{e^{\gamma_1 t}}{1/A_{rms}(0) - \gamma_2 \int_0^t g^{(3/2)}(t') e^{\gamma_1 t'} dt'}.
\]
(S25)

This solution exhibits hyperbolic growth that hits a finite time singularity at \(t_c \) which satisfies,
\[
\int_0^{t_c} g^{(3/2)}(t') e^{\gamma_1 t'} dt' = \frac{1}{\gamma_2 A_{rms}(0)}.
\]
(S26)

As long as \(g^{(3/2)}(t) \) decays slower than \(e^{-\gamma_1 t} \), a finite time singularity exists.

However, in our experiment, if we look at the mean population \(n_m \) of the modes at all directions and images, it doesn’t show clear deviation from simple exponential growth. Thus we choose the observable as the mean population \(n_s = (n_{\pi/3} + n_{-\pi/3})/2 \) at \(\pm \pi/3 \) relative to the strongest mode in each image. Because the nonlinear coupling between these three adjacent modes, \(n_s \) grows faster than \(n_m \) and can deviate from exponential growth. If only modulation of single frequency \(\omega_1 \) is applied, \(n_s = n_m \), because they are independent and share the same statistics.

Let’s say we always choose \(\hat{a}_i \) as the strongest mode among all sets of coupled six modes, which have larger fluctuation to begin with. The other two modes \(\hat{a}_{i-1} \), \(\hat{a}_{i+1} \) at \(\pm \pi/3 \) relative to it begin with the mean population as all the other modes with \(\langle |\hat{a}_{i+1}|^2 \rangle = \langle |\hat{a}_{i-1}|^2 \rangle \). We model the effect of the strongest sets of modes as an enhancement of \(\gamma_2 \) by a factor of \(\alpha \). Thus the solution of \(n_s \) is,
FIG. S4. The evolution of three point correlation $g^{(3/2)}$ and the mean population at $\pm \pi/3$ relative to the strongest modes during the D$_6$ pattern formation process. (A) The growth of three point correlation $g^{(3/2)}$ as a function of mean population $n_m$ of modes at all directions in all images. Solid line is the theory curve from fitting using Eq. S31. (B) The growth of mean population $n_s$ of modes at $\pm \pi/3$ relative to the strongest modes versus the mean population per mode $n_m$ under scattering length modulation Scheme II (red squares) compared with that under Scheme I (blue circles). Both the x and y axis are in log scale. The red and blue solid lines are theory curves from fitting using Eq. S32 and $y = ax$, respectively. The vertical dashed line is the theory prediction when the growth of population $n_s$ diverges during D$_6$ pattern formation.

\[
\frac{n_s^{1/2}}{1/A_{\text{rms}}(0) - \alpha \gamma_2 \int_0^T g^{(3/2)}(t') e^{\gamma_1 t'} dt'} = e^{\gamma_1 t}.
\]  

(S27)

Because the nonlinear term is relatively weak for mean population $n_m$ of all modes, it grows approximately exponentially as

\[
n_m = A_{\text{rms}}^2(t) \approx A_{\text{rms}}^2(0) e^{2 \gamma_1 t}.
\]  

(S28)

Then $n_s$ as a function of the mean population $n_m$ of all modes is

\[
n_s = \frac{n_m}{\left(1 - \frac{1}{2} \alpha \epsilon \int_{n_m(0)}^{n_m(t)} g^{(3/2)}(n_m')/\sqrt{n_m'n_m'} dn_m' \right)^2},
\]  

(S29)

where $\epsilon = \gamma_2/\gamma_1$ is the ratio of the two rate constants.
In order to know how \( n_s \) grows as a function of \( n_m \), we need to determine the evolution of the three point correlation \( g^{(3/2)} \) as a function of \( n_m \). Combining Eqs. (S15), (S16), (S21) and (S22), we have

\[
\frac{dg^{(3/2)}}{dt} = 3\gamma_2 A_{rms}[g^{(2)} - (g^{(3/2)})^2],
\]

(S30)

where \( g^{(2)} \) is the two point correlation function at \( \theta = \pi/3 \), i.e., \( g^{(2)} = \langle |\psi_0^0|^2 |\psi_{x+2\pi/3}^0|^2 \rangle / \langle |\psi_0^0|^2 \rangle^2 = (\langle |\tilde{a}_{i+1}^0|^2 |\tilde{a}_i^0|^2 \rangle + \langle |\tilde{a}_i^0|^2 |\tilde{a}_{i-1}^0|^2 \rangle + \langle |\tilde{a}_{i+1}^0|^2 |\tilde{a}_{i-1}^0|^2 \rangle)/3(\langle |\tilde{a}_i^0|^2 \rangle)^2 \). In the perturbation regime where the population of modes in directions separated by \( \pi/3 \) are almost uncorrelated, i.e. \( g^{(2)} \approx 1 \), the three point phase correlation is given by

\[
g^{(3/2)} = 1 - \frac{2}{1 + \exp[6\epsilon(\sqrt{n_m} - A_{rms}(0))]}.
\]

(S31)

Inserting the above result into Eq. (S29) we arrive at

\[
n_s = n_m[1 - \alpha\epsilon(A_{rms}(0) - \sqrt{n_m}) + \frac{\alpha}{3} \ln(1 - g^{(3/2)})]^{-2}.
\]

(S32)

Since our model only considered 6 excited modes, here the mean population \( n_m \) of a single mode is 1/6 of the total mean population. In our experiment, the total number of excited modes at \( |k| = k_f \) is \( N_{mod} = 1.62/Rk_f \approx 136 \) [28]. In order to generalize Eq. (S31) and Eq. (S32) for multiple sets of 6 modes with \( \pi/3 \) angular spacing, we need to do the replacements: \( n_m \rightarrow N_{mod} n_m, A_{rms}(0) \rightarrow \sqrt{N_{mod}/6} A_{rms}(0) \) and \( n_s \rightarrow N_{mod} n_s \). This is equivalent to replace \( \epsilon \) by \( \sqrt{N_{mod}/6} \epsilon = \sqrt{N_{mod}/6} \gamma_2/\gamma_1 \).

Using Eq. (S31) to fit the data with \( \epsilon \) and \( A_{rms}(0) \) as fitting parameters as shown in Fig. S4A, we get \( \epsilon = 0.08 \) and \( A_{rms}(0) = 0.98 \). Thus \( \gamma_2/\gamma_1 = 0.01 \), which is consistent with the experimental value 0.003. The discrepancy is attributed to the exclusion of other collision processes that are also involved in the experiment, such as the pair generation from BEC at \( |k| = k_f \sqrt{2} \) and secondary collision processes that lead to \( D_4 \) pattern. Then we use the value of \( \epsilon \) and \( A_{rms}(0) \) to fit the three point correlation function \( g^{(3/2)} \) and set \( \alpha \) as another fitting parameter to fit \( n_s \) versus \( n_m \) as shown by the red solid line in Fig. S4B, which gives \( \alpha = 2.78 \). On the other hand, for single frequency modulation under Scheme I, we use the fit function \( y = ax \) and the best fit is obtained with \( a = 0.98 \) as shown by the blue solid line in Fig. S3B.

**VIII. EVOLUTION OF THE PHASE RELATION OF MODES FORMING D_6 DENSITY WAVE PATTERN**

In order to study how the phase relation of modes that form hexagonal lattices evolve from completely uncorrelated to concentrated around the plane \( \phi_0 + \phi_{2\pi/3} + \phi_{4\pi/3} = 0 \), we perform numerical calculation based on Eq. (S13). Here we consider BEC with depletion, which couple to multiple sets of 6 modes with \( \pi/3 \) angular spacing at the same time. The corresponding equations of motion are:

\[
\frac{d\tilde{a}_i}{dt} = [\gamma_1 N_0(t) - \gamma_c]\tilde{a}_{i+3}^\dagger + \gamma_2\sqrt{N_0(t)}(\tilde{a}_{i+2}^\dagger \tilde{a}_{i+1} + \tilde{a}_{i-2}^\dagger \tilde{a}_{i-1} - \tilde{a}_{i+1} \tilde{a}_{i-1})
\]

(S33)

\[
N_0(t) = N_0 - \sum_{i=1}^{N_{mod}} \langle |\tilde{a}_i|^2 \rangle,
\]

(S34)

where the growth rates \( \gamma_1' = \gamma_1/N_0 \) and \( \gamma_2' = \gamma_2/\sqrt{N_0} \). The decay rate due to modes flying out of the condensate is \( \gamma_c \sim v/R \), where the velocity of the modes \( v = \hbar k_f/m \) and \( R \) is the radius of the condensate.

The simulation starts from the beginning of the second pattern forming stage. At the end of the first seeding stage, the population \( n \) in each mode \( \tilde{a}_i \) is therally distributed according to the probability distribution \( p(n) = e^{-n/\bar{n}}/\bar{n} \) with the mean population \( \bar{n} = 2 \). The phase of each mode \( \tilde{a}_i \) is uniformly distributed from 0 to \( 2\pi \) and the modes in opposite directions are correlated as \( \tilde{a}_i = \tilde{a}_{i+3}^\dagger \). The simulation is repeated for 5000 times and each time the initial conditions of the phase and amplitude are independently sampled from their distributions. We finally take the phase of \( \tilde{a}_i e^{-i\omega t} + \tilde{a}_{i+3}^\dagger e^{i\omega t} \) as the phase of Fourier modes in the lab frame. The amplitude of scattering length modulation \( a_1 \) and \( a_2 \) and the escape rate \( \gamma_c \) are chosen as fitting parameters while all the other parameters are the same as our experiment for Scheme II. The green line in Fig. 4F is the result after 22.4 ms evolution time, using the initial condition of mean population at 22.6 ms in our experiment. The corresponding amplitudes of modulation are \( a_1 = 24 \ a_0, a_2 = 68.5 \ a_0 \) and \( \gamma_c = 39 \) Hz. The evolution of the phase distribution of \( \phi_0 + \phi_{2\pi/3} + \phi_{4\pi/3} \) within
FIG. S5. **Evolution of the phase distribution of $\phi_0 + \phi_{2\pi/3} + \phi_{4\pi/3}$ during D$_6$ density wave pattern formation process.** The upper panel is from the experiment under Scheme II where we perform *in situ* imaging of the condensate at different times. The peak position of the phase distribution oscillates between 0 and $\pi$ and gets more concentrated as time evolves. The lower panel is from the numerical calculation with the modulation amplitudes $a_1 = 22.5 \, a_0$ and $a_2 = 63.5 \, a_0$ for frequency components of 450 Hz and 225 Hz, respectively. The escaping rate of momentum modes is 39 Hz. Other parameters are the same as the experiment.

Individual Floquet periods is also calculated as shown in the lower panel of Fig. S5, which is consistent with the experimental result in the upper panel. The peak position $\phi_{\text{peak}}$ of the phase distribution oscillates between 0 and $\pi$, due to the standing wave nature of the density waves. This also means the real space pattern changes back and forth between hexagonal lattice ($\phi_{\text{peak}} = 0$) and honeycomb lattice ($\phi_{\text{peak}} = \pi$). However, in the rotating frame, the phase distribution is always centered at 0, thus the three point correlation $g^{(3)/2}$ is always positive. This ensures the hyperbolic growth since the second term in Eq. 5 is positive.