A GRADIENT-THRESHOLDING ALGORITHM FOR SPARSE
REGULARIZATION

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Abstract. Inverse problems arise in a wide spectrum of applications in fields ranging from engineering to scientific computation. Connected with the rise of interest in inverse problems is the development and analysis of regularization methods, such as Tikhonov-type regularization methods or iterative regularization methods, which are a necessity in most of the inverse problems. In the last few decades, regularization methods motivating sparsity has been the focus of research, due to the high dimensionality of the real-life data, and $L^1$-regularization methods (such as LASSO or FISTA) has been in its center (due to their computational simplicity). In this paper we propose a new (semi-) iterative regularization method which is not only simpler than the mentioned algorithms but also yields better results, in terms of accuracy and sparsity of the recovered solution. Furthermore, we also present a very effective and practical stopping criterion to choose an appropriate regularization parameter (here, it’s iteration index) so as to recover a regularized (sparse) solution. To illustrate the computational efficiency of this algorithm we apply it to numerically solve the image deblurring problem and compare our results with certain standard regularization methods, like total variation, FISTA, LSQR etc.

1. Introduction

1.1. Inverse Problems and Regularization: An inverse problem in general is a problem where the effect (output) is known but the source (input) is not, in contrast to a direct problem where we deduce the effect from the source. Mathematically, an inverse problem is often expressed as the problem of finding/estimating a (source function) $\varphi$, given an (effect data) $g$, which satisfies the following operator equation:

$$T\varphi = g,$$

(1.1)

$T$ being some operator describing the underlying physical process, where the domain and range of the operator $T$ varies depending on the problem (usually, it’s a subset of $L^2(\Omega)$, for $\Omega \subset \mathbb{R}^n$). Typically, the solution of (1.1) is approximated by the solution of the least-square problem, i.e., minimizer of the following functional

$$F(\psi) = \|T\psi - g\|_2^2.$$

(1.2)

However, inverse problems are usually ill-posed, in the sense of violating Hadamard’s third condition: “Continuous dependence of the data”, i.e., even for a slightly perturbed data $g_\delta$, such that $\|g - g_\delta\| \leq \delta$ (usually small), the inverse recovery becomes unstable, $\|\varphi - \varphi_\delta\| >> \delta$ (very large); typically due to the unboundedness of the (pseudo-) inverse operator $T^\dagger$ and the noise present in the data. To counter such
instabilities or the ill-posedness of inverse problems, regularization methods have to be employed. In the last few decades, several regularization methods have been established for linear as well as nonlinear inverse problems. Broadly, there exist two kinds of regularization approaches:

1.2. Tikhonov-type regularization: Tikhonov-type regularization methods are probably the most well known regularization techniques for solving linear as well as nonlinear inverse problems (see [1, 2, 3, 4, 5]), where (instead of minimizing the simple least-square problem (1.2)) one recovers a “regularized solution” by minimizing a (constrained or) penalized functional

$$F(\psi; \lambda, L, \psi_0, p, q) = \|T\psi - g_\delta\|_p^p + \lambda\|L(\psi - \psi_0)\|_q^q,$$  

(1.3)

(for some $p$ and $q$) where $\lambda > 0$ is the called the regularization parameter, $\|g - g_\delta\| \leq \delta$ is the error norm, $\psi_0$ is an initial guess and $L$ is a regularization operator\(^1\), with the null spaces of $T$ and $L$ intersecting trivially. In this approach, the ill-posed problem (1.1) is first converted to a family of well-posed problems (depending on a regularization parameter $\lambda$) and once an appropriate choice of the parameter value (say $\lambda_0$) is estimated, a regularized solution $\varphi^\delta$ of (1.1) is recovered by completely minimizing the associated functional, i.e.,

$$\varphi^\delta_{\lambda_0} = \arg\min_{\psi} F(\psi; \lambda_0, L, \psi_0, p, q).$$  

(1.4)

The choice of an appropriate parameter value $\lambda_0$ is crucial in any regularization method, as it balances between the data fitting term $\|T\psi - g_\delta\|_p^p$ and the regularization term $\|L(\psi - \psi_0)\|_q^q$. If $\lambda$ is too small then minimizing (1.3) over-fits the noisy data (thus, leading to a noisy recovery or, statistically speaking, a high-variance recovery), where as as if $\lambda$ is too big then it under-fits the data (thus, leading to an over-smooth recovery, or statistically, a high-bias recovery). Typically, one recovers a bunch of $\lambda$-dependent regularized solutions $\{\varphi^\delta_{\lambda} : \lambda \in [\lambda_{\min}, \lambda_{\max}]\}$ and then choose an appropriate regularized solution $\varphi^\delta_{\lambda^*}$ depending on some parameter choice criterion, say DP (discrepancy principle), GCV (generalized cross-validation), L-curve, monotone error rule etc., see [6, 7, 8, 9, 10, 11, 12, 13].

1.3. (Semi-) Iterative regularization: Where as, in an iterative regularization method one recovers a regularized solution $\varphi^\delta$ of (1.1) by stopping the minimization process of the simple least-square problem (1.2) (i.e., no additional penalty term) at an appropriate (early) instance, i.e., starting from an initial guess $\psi_0^\delta$, one updates the $m^{th}$ recovered solution ($\varphi^\delta_m$) in a direction ($\xi^\delta_m$) such that the $(m+1)^{th}$ recovered solution ($\varphi^\delta_{m+1}$) is better than the previous, meaning $||\varphi^\delta_{m+1} - \varphi|| < ||\varphi^\delta_m - \varphi||$, and stops the iteration after certain number of steps (i.e., $m \leq m_0(\delta) < \infty$, for some appropriate $m_0(\delta)$, usually depending on the noise level $\delta$). Mathematically, it can be expressed as

$$\varphi^\delta_{m+1} = \varphi^\delta_m + \tau_m \xi^\delta_m, \quad \text{for} \quad m \leq m_0(\delta) < \infty,$$  

(1.5)

where $\tau_m$ is the $m$-th step-length and the descent direction $\xi^\delta_m$ is usually the (negative) gradient direction ($\xi^\delta_m = -T^*(T\varphi^\delta_m - g_\delta)$). When the step-size is fixed ($0 < \tau_m = \tau < \frac{1}{\max\{||\nabla F(\psi; \lambda, L, \psi_0, p, q)\||\}}$) for all $m$, it’s known as Landweber iterations (also known as Richardson iterations) and has been intensively investigated in the literature (see, [14, 15, 1, 2, 16, 17] and references therein). The main drawback of Landweber

\(^1\)the space for $\psi$ and the norms involved are defined appropriately.
iterations is its slow performance, i.e., it takes a large number of iterations to obtain the optimal convergence rates. To circumvent this drawback many extensions (known as polynomial accelerated Landweber methods) have been proposed and studied in the framework of regularization (for an overview, see [16, 17]). Such iterative methods are typically known as semi-iterative methods, since when dealing with a noisy data \((g_0)\) one encounters a semi-convergent nature of the recovery process, see \([1]\). That is, in the initial stage \(\|\psi^\delta_m - \varphi\| \to 0\) for \(m \leq m_0(\delta) < \infty\), but upon further iterations \(\|\psi^\delta_m - \varphi\| \to \infty\) (or, \(\|\psi^\delta_m\| \to \infty\), as \(m \to \infty\). Hence, to achieve regularization one needs to terminate the descent process at an appropriate early stage, i.e., here the iteration index serves as a regularization parameter. If the iterations are stopped too early (\(m\) too small) then that leads to an over-smooth or high-bias recovery (i.e., under-fitting the data), and if \(m\) is large then it leads to a noisy or high-variance recovery (i.e., over-fitting the noisy data). Typically, the iterations are terminated with the aid of certain stopping rule, such as the discrepancy principle (see [5]). The advantage of these extended semi-iterative methods over the simple Landweber iteration is that, while Landweber iteration (1.5) uses only the last iterate \(\psi^\delta_m\) to construct the new approximate \(\psi^\delta_{m+1}\), in a semi-iterative method one make use of the last few iterates (if not all),

\[
\psi^\delta_{m+1} = \sum_{i=0}^{m} \mu_{m,i} \psi^\delta_i + \tau_m \xi^\delta_m, \quad (1.6)
\]

where \(\xi^\delta_m\) is the descent direction (usually \(\xi^\delta_m = -T^*(T\psi^\delta_m - g_0)\)). Note that, \(\psi^\delta_{m+1} - \psi^\delta_0\) belongs to the Krylov-subspace \(\mathcal{K}_{m+1}(T^*T, \xi^\delta_0)\), which is defined as

\[
\mathcal{K}_{m+1}(T^*T, \xi^\delta_0) := \text{span}\{\xi^\delta_0, (T^*T)\xi^\delta_0, \ldots, (T^*T)^m \xi^\delta_0\}. \quad (1.7)
\]

Hence, these methods are also called as Krylov-subspace methods. One can easily see that choosing \(\mu_{m,i} = \delta_{m,i}\), where \(\delta_{m,m} = 1\) and \(\delta_{m,i} = 0\), for \(m \neq i\), and \(\tau_m = \tau\) in (1.6) would give back Landweber iterations. It can be further shown (see [1, 16]) that there exist sequences \(\mu_m\) and \(\tau_m\) such that \(\psi^\delta_{m+1}\) in (1.6) can be reduced to

\[
\psi^\delta_{m+1} = \psi^\delta_m + \mu_m(\psi^\delta_m - \psi^\delta_{m-1}) + \tau_m \xi^\delta_m, \quad m \geq 1. \quad (1.8)
\]

A specific example of such methods is known as the \(\nu\)-methods (see [18]), which are defined by \(\mu_1 = 0, \tau_1 = \frac{4\nu + 2}{4\nu + 1}\), and for \(m > 1\)

\[
\mu_m = \frac{(m - 1)(2m - 3)(2m + 2\nu - 1)}{(m + 2\nu - 1)(2m + 4\nu - 1)(2m + 2\nu - 3)}, \quad (1.9)
\]

\[
\tau_m = \frac{4(2m + 2\nu - 1)(m + \nu - 1)}{(m + 2\nu - 1)(2m + 2\nu - 1)}. \quad (1.10)
\]

Even further acceleration is possible by adapting a conjugate-gradient type method (see [19]), where \(\mu_{m,i}\) in (1.6) depend on data \(g_0\) (making it a non-linear method).

1.4. Sparsity Regularization: In the last few decades a particular subclass of the regularization methods has drawn a lot of attention, it’s called sparsity regularization or regularization with sparsity, where the support\(^2\) of the solution of (1.1) is significantly smaller than its domain, i.e., \(\mu\{x : \varphi(x) \neq 0\} \subset \Omega_{\varphi}\) \(<\mu(\Omega_{\varphi})\),

\(^2\)the support of a function \(\psi\) is defined as \(\text{supp}_\psi = \{x \in \text{Domain}(\psi) : \psi(x) \neq 0\}\)
where $\Omega_\varphi$ is the domain of $\varphi$ and $\mu$ is the corresponding measure of the space. In practical situations one only has finitely many observations or measured data and hence, the operator equation (1.1) can be represented in its discretized version (when $T$ is linear, or being approximated by a linear operator) as

$$A\hat{x} = b,$$

(1.11)

where $A \in \mathbb{R}^{m \times n}$ (is usually ill-conditioned), $\hat{x} \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. Again, we don’t have the exact measurement of the output ($b$), but rather a noisy one ($b_0$) such that $||b - b_0|| \leq \delta$. From now on (until otherwise stated) we assume $T$ be a linear bounded (injective) operator and, use the symbolic system ($T$, $\varphi$, $g$) to denote the original problem (i.e., $T$ is an operator acting on infinite dimensional spaces, where the functions $\varphi$, $\psi$ and $g$ belongs to) and the system ($A$, $\hat{x}$, $b$) to denote its discretized version (i.e., $A$ is $m \times n$ matrix and, $\hat{x}$, $x$ and $b$ are $n$ and $m$ dimensional vectors, respectively). Here, a vector $x \in \mathbb{R}^n$ is said to be sparse if the number of non-zero elements in $x$ is significantly less than $n$, i.e., $||x||_0 = \sum_i^n \{1 \leq i \leq n : x_i \neq 0\}$, where $x = (x_i)_{i=1}^n << n$, this is analogous to the above definition of the support for a function when the function is replaced by a vector and the continuous measure $\mu$ is replaced by a counting measure. Therefore, ideally one would like to minimize the following functional (for an appropriate $\lambda_0$)

$$F(x, \lambda_0) = ||Ax - b||_2^2 + \lambda_0 ||x||_0,$$

(1.12)

which is known as the best subset selection $[20, 21]$, but due to the non-convexity of $||.||_0$ norm, it’s a computational nightmare (NP-hard problem). One can circumvent this problem by having a convex-relaxation of the $||.||_0$-norm, i.e., considering the $||.||_1$ norm (which is convex) instead of the $||.||_0$ norm, where $||x||_1 = \sum_{i=1}^n |x_i|$ for $x = (x_i)_{i=1}^n$. That is, instead of minimizing (1.12) one minimizes the following functional (for an appropriate $\lambda_0$)

$$F(x, \lambda_0) = ||Ax - b||_2^2 + \lambda_0 ||x||_1,$$

(1.13)

it’s known as $\mathcal{L}^1$-regularization, or more popularly LASSO (least absolute shrinkage and selection operator) which was proposed by Tibshirani in 1996 [22], or Basis Pursuit Denoising by Chen et al. in 1998 [23]. $\mathcal{L}^1$-regularization gained its popularity because of its ability to perform variables/features/model selection and its significance in signal and image processing, references on earlier works promoting the use and application of $\mathcal{L}^1$-regularization (together with its relevance to other research areas, such as compressed sensing etc.) can be found in [24, 25, 26] and references therein. Note that, the functional defined in (1.13) (though convex) is not differentiable, due to the non-differentiability of the $||.||_1$-norm, and hence, simple gradient descent algorithm wouldn’t work for minimizing (1.13), one has to use sub-gradient methods. Also, due to the convexity of the functional (1.13), it can be minimized by many well-established numerical algorithms such as Quadratic programming with linear constraints [27], but the high-dimensionality of the practical problems (such as image deblurring) often precludes the use of such methods. However, because the functional in (1.13) is a combination of convex-differentiable and convex term, one can make use of the proximal operators to find the derivative of the functional defined in (1.13). This leads to one of the most popular methods for minimizing (1.13), which falls in the class of iterative (shrinkage or) soft-thresholding algorithms (ISTA), where in each iteration the gradient of the differentiable term is projected and shrinked by a factor of $\lambda$ (known as
soft-thresholding), see [28, 29, 26, 30, 25, 31, 32]. Specifically, at the \( m \)-th step the iterate \( x^\delta_m \) is improved in the following manner

\[
x^\delta_{m+1} = S_{\lambda \tau}(x^\delta_m + 2\tau A^*(b_\delta - Ax^\delta_m)),
\]

where \( \tau \) is an appropriate stepsize and the operator \( S_{\lambda \tau} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the shrinkage operator defined as, for any \( x \in \mathbb{R}^n \), \( S_{\lambda \tau}(x) = (S_{\lambda \tau}(x)_i)_{i=1}^n \in \mathbb{R}^n \), where

\[
S_{\lambda \tau}(x)_i = \sgn(x_i) * (|x_i| - \lambda \tau)_+,
\]

where \( \sgn(x_i) = \frac{x_i}{|x_i|} \) and \( (|x_i| - \lambda \tau)_+ = \max\{|x_i| - \lambda \tau, 0\} \). So one can see that it’s a forward-backward splitting method, where at each iteration the iterate \( x^\delta_m \) is first moved in the forward direction (i.e., \( x^\delta_{m+1} = x^\delta_m + 2\tau A^*(b_\delta - Ax^\delta_m) \)) and then moved backward by the shrinking operator \( S_{\lambda \tau} \), as defined in (1.15), i.e., truncating out the “smaller values” of \( x^\delta_{m+1} \). One can also observe from (1.15) that (not only the regularization, but also) the sparsity of the recovered regularized solution \( x^\delta \) will be influenced by the regularization parameter \( \lambda \). Hence, for larger \( \lambda \) the recovered solution will not only be overly-smoothed/regularized (or highly-biased) but also be sparser, where as for smaller \( \lambda \) the recovered solution will be noisy (or high-variance) and dense (low sparsity). Even for an appropriate parameter value \( (\lambda_0) \) ISTA solution \( (x_{\lambda_0}) \) still tends to be more biased, due the soft-thresholding, for details see the references mentioned above. ISTA also suffers from a slower convergence rate of the descent process, which can be improved by using FISTA (fast iterative soft-thresholding algorithm) [33], but it still doesn’t compensate the biasness problem. Usually, the biasness problem is handled through a two-step process: (1) either ISTA or FISTA is used for variables/features selection and then, (2) an ordinary least-square (OLS) is performed on the selected variables and the corresponding matrix.

1.5. A new iterative algorithm for sparsity: In this paper, we propose a new (semi-) iterative regularization method which is not only as simple (if not simpler) as ISTA or FISTA from a computational viewpoint, but also improves the sparsity of the recovered solution significantly, even removing the biasness effect shown by the mentioned methods, i.e., improving the accuracy (or relative error) of the recovery. First note that, starting from the simple Landweber iterations (1.5) all the generalizations and extensions (1.6) developed in the (semi-) iterative regularization literature is focused on improving the speed of the convergence of the descent process. However, they do not motivate sparsity during the descent (or minimization) process, even when started with a zero vector \( (x^\delta_0 = 0 \in \mathbb{R}^n) \). Hence, here we provide some different descent directions that advocate sparsity during the recovery process and thus, leading to a sparse regularized solution \( x^\delta \), when started with \( x^\delta_0 = 0 \in \mathbb{R}^n \).

The paper is organized in the following manner. In §2 we provide some descent directions that will not only impart sparsity but also has a regularizing effect in the recovery process. In §3 we prove some results corresponding to the convergence of this method, when dealing with a noisy data, and the sparsity in the regularized solution. In §3 we also compare and contrast our algorithm with ISTA, and present the advantages of our method over ISTA. In §4 we present numerical results to validate the developed theory and, we also compare the numerical results obtained using our method with the results obtained using certain standard regularization methods such as total-variation regularization, FISTA (fast iterative
soft-thresholding algorithm), CGLS method, hybrid LSQR and many others, see Tables 1, 2 and 3. Furthermore, in §5, we also provide a very effective and efficient (practical) stopping criterion to estimate an appropriate value of the regularization parameter (here, the iteration index) so as to obtain a regularized (sparse) solution, even when no noise information is available. We show that this technique is a significant improvement over the simpler Morozov’s discrepancy principle.

2. Sparsity motivating descent directions

In this section, first we work with noise-free data (g) and then extend the results to noisy data (ĝ). Note that if a functional F is (strictly) convex then there is a unique minimizer (say ϕ), which can be attained in an iterative descent process. That is, if there is a sequence of functions {ψm} (or vectors {xm}, in discretized case) such that the corresponding sequence {F(ψm)} is strictly decreasing (F(ψm+1) < F(ψm)), then the sequence {ψm} “converges” (in some sense)⁢ to the minimizer ϕ. The most common descending direction at any step m is the steepest-descent direction, which (for the functional defined in (1.2)) is given by the (negative) gradient of the functional, i.e., ξm = −2T*(Tψm − g), see below. As aforementioned, though it’s the steepest-descent direction, it doesn’t advocate sparsity in the recovery process. Therefore, if sparsity is the subject of interest, we need to find some other (appropriate) descent directions that will support sparsity in the descent process.

Observe that, for any functional F we have the following Taylor’s expansion

\[ F(\psi + \tau h) = F(\psi) + \tau F'(\psi)[h] + O(\tau^2), \quad (2.1) \]

or equivalently, for sufficiently small \( \tau > 0 \), we can have

\[ F(\psi + \tau h) - F(\psi) \approx \tau F'(\psi)[h], \quad (2.2) \]

where the differential \( F'(\psi) \) for the functional as defined in (1.2) is

\[ F'(\psi)[h] = 2(T*(T\psi - g), h)_{\mathbb{L}^2}, \quad (2.3) \]

and we denote \( \nabla_{\mathbb{L}^2} F := \xi = T^*(T\psi - g) \) as the \( \mathbb{L}^2 \)-gradient of the functional F. Note that, considering \( h = -\xi_m := -T^*(T\psi_m - g) \) (as the descent direction at the \( m^{th} \)-step) we have, from (2.2), \( F(\psi_m + \tau\xi_m) < F(\psi_m), \) i.e., starting from an initial choice \( \psi_0 \) we can have a sequence of functions \( \{\psi_m\} \) such that the sequence \( \{F(\psi_m)\} \) is (strictly) decreasing, and hence, the sequence \( \{\psi_m\} \) converges to the minimizer \( \varphi \), see [1, 14, 2, 3, 4, 5].

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³the nature of convergence, i.e., either strongly or weakly, depends on the functional F.
2.1. Sparsity inducing descent directions:
Splitting the integral in (2.3) into its distinct sub-domains, we have, for any \( \lambda > 0 \)
\[
F'(\psi)[h] = 2 \int_{\Omega} T^*(T\psi - g)h \, d\mu 
= 2 \left[ \int_{|T^*(T\psi - g)| > \lambda} T^*(T\psi - g)h \, d\mu + \int_{|T^*(T\psi - g)| \leq \lambda} T^*(T\psi - g)h \, d\mu \right] 
= 2 \left[ \int_{T^*(T\psi - g) > \lambda} T^*(T\psi - g)h \, d\mu + \int_{0 < T^*(T\psi - g) \leq \lambda} T^*(T\psi - g)h \, d\mu \right. 
+ \left. \int_{T^*(T\psi - g) < -\lambda} T^*(T\psi - g)h \, d\mu + \int_{-\lambda \leq T^*(T\psi - g) < 0} T^*(T\psi - g)h \, d\mu \right].
\]
The above expression (2.4) inspires us to formulate some descent directions that advocates sparsity in the reconstruction during the descent process.

2.1.1. \( \lambda \)-truncated gradient:
From the expression (2.4) if we instead choose
\[
h_{m,\lambda} = -T^*(T\psi_m - g) \cdot \chi_{|T^*(T\psi_m - g)| > \lambda}
\]
(2.5)
(for an appropriate \( \lambda > 0 \)) as the descent direction, then not only will the “smaller” gradient values \(|T^*(T\psi_m - g)| \leq \lambda \) be discarded but also, we have \( F'(\psi_m)[h_{m,\lambda}] < 0 \), i.e., \( F(\psi_{m+1}) < F(\psi_m) \), where \( \psi_{m+1} = \psi_m + \tau m h_{m,\lambda} \) and \( \chi_A \) is the standard characteristic (or indicator) function, defined as,
\[
\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}
\]

Therefore, starting from a zero function \( (\psi_0 \equiv 0) \) and using the \( \lambda \)-truncated gradient, which we denote by \( \xi_{m,\lambda} := h_{m,\lambda} \), as the descent direction one may expect to recover a sparse solution, this is proved below.

First, note that here the value of \( \lambda \) plays an important role in balancing between the sparsity and the accuracy of the recovered solution, where by “accuracy” we mean the relative error of the recovery. If \( \lambda \) is too large (specifically, \( \lambda > \|T^*g\|_\infty \)), then \( \psi_m \equiv 0 \) for all \( m \geq 0 \), i.e., completely sparse, but (obviously) not a very good approximated solution for (1.1). Now if \( \lambda \) is too small, such that there is no truncation of the gradient at any step, then it will be a simple steepest-descent algorithm, which doesn’t guarantee any sparsity, but results in the most accurate estimate for the solution of (1.1). Now we show that one can also obtain a non-trivial sparse (approximate) solution for (1.1) with an appropriate choice of \( \lambda > 0 \).

Note that, for a sequence \( \{\psi_m\} \) if the corresponding sequence \( \{F(\psi_m)\} \) tends to zero \( (F(\psi_m) \to 0) \), then for every \( \epsilon > 0 \), \( \exists M(\epsilon) \in \mathbb{N} \) such that \( |F(\psi_m)| \leq \epsilon \), for all \( m \geq M(\epsilon) \). In other words, for all \( m \geq M(\epsilon) \), \( \|T\psi_m - g\|_2 \leq \epsilon \), and since \( T \) is a bounded linear operator (and so is \( T^* \)), we have \( \|T^*(T\psi_m - g)\|_2^2 \leq \|T^*\|_2 \|T\psi_m - g\|_2 \leq \|T^*\|_2 \|T\psi_m - g\|_2 \leq \|T^*\|_2 \|T\psi_m - g\|_2 \leq \lambda \). That is, for any \( \lambda > 0 \) there is a \( M(\lambda) \in \mathbb{N} \) such that the gradient-norm \( \|\xi_m = T^*(T\psi_m - g)\|_2 \leq \lambda \) (implying \( \|\xi_m \cdot \chi_A\|_2 \leq \lambda \), for any \( A \subset \Omega_{\psi_m} \) with \( \mu(A) > 0 \), for all \( m \geq M(\lambda) \), and hence, if the recovered solution \( (\psi_{M(\lambda)}) \) at \( M(\lambda) \) is sparse, then so are all \( \psi_m \)'s for \( m \geq M(\lambda) \), in fact, \( \psi_m \equiv \psi_{M(\lambda)} \) for all \( m \geq M(\lambda) \). Now the existence of such a \( \lambda \) can be shown as below.
Note that, for the first update, without any truncation of the gradient,

\[(\psi_0 \equiv 0, \xi_0 = T^* g) \quad m=1(\text{no-truncation}) \rightarrow (\psi_1 = \tau_1 T^* g, \xi_1 = T^* (\tau_1 TT^* g - g)) \ (2.7)\]

where \( \tau_1 > 0 \) is the step-size, we have \( ||T^* (\tau_1 TT^* - I) g||_\infty < ||T^* g||_\infty \), since \( T \) is linear and bounded. Now, for any choice of \( \lambda \) satisfying \( ||T^* (\tau_1 TT^* - I) g||_\infty < \lambda < ||T^* g||_\infty \), leads to a non-trivial sparse \( \psi_{1, \lambda} := \tau_1 (T^* g) \cdot (\chi_{|T^* g| > \lambda}) \), since \( \mu(x : |T^* g(x)| \leq \lambda) > 0 \) and \( \mu(x : \lambda < |T^* g(x)| \leq ||T^* g||_\infty) > 0 \), otherwise \( \lambda = ||T^* g||_\infty \). Also, we have \( F(\psi_{1, \lambda}) < F(\psi_0) \), as \( F(\psi_0) = ||T^* g||^2_2 \) and \( F(\psi_{1, \lambda}) = ||T^* (\tau_1 TT^* g \cdot (\chi_{|T^* g| > \lambda}) - g)||^2_2 \), i.e., \( \psi_{1, \lambda} \) is an improvement over \( \psi_0 \) in the approximation of the solution of (1.1). Furthermore, since the gradient at \( m = 1 \), with \( \psi_{1, \lambda} = \tau_1 (T^* g) \cdot (\chi_{|T^* g| > \lambda}) \), is \( \xi_1 = T^*(T \psi_{1, \lambda} - g) \), we have \( ||\xi_1||_\infty \leq ||T^* (\tau_1 TT^* - I) g||_\infty < \lambda \), implying the \( \lambda \)-truncated gradient at \( m = 1 \) will be \( \xi_{1, \lambda} \equiv 0 \), and hence, \( \psi_m = \psi_{1, \lambda} \) for all \( m \geq 2 \). In other words, it says, choosing an appropriate \( \lambda \) would update the initial choice \( \psi_0 \equiv 0 \) to get a sparse \( \psi_{1, \lambda} \neq 0 \), such that, it’s a better approximation for the solution of (1.1), and \( \psi_{m, \lambda} = \psi_{1, \lambda} \), for all \( m \geq 2 \), i.e., the sparsity is not destroyed, though there is no further improvements.

Therefore, when \( \lambda \) decreases the sparsity of the recovered solution also decreases, but the accuracy of the recovery increases, and vice-versa, i.e., the parameter value \( \lambda \) balances between the sparsity and the accuracy of the recovery. An algorithm to choose an appropriate value of \( \lambda \) such that the constructed sequence \( \{\psi_{m, \lambda}\} \) “converges” to a regularized (sparse) solution \( (\varphi^\delta) \) of (1.1) for a noisy data \( (g_\delta) \) is presented in §3 and, its numerical implementation and results shown in §4.

**Remark 2.1.** Note that, the parameter \( \lambda \) not only serves as a sparsity parameter, but also as a regularization parameter, since (as explained above) for any \( \lambda > 0 \) there is a \( M(\lambda) \in \mathbb{N} \) such that \( \psi_m = \psi_{M(\lambda)} \) for all \( m \geq M(\lambda) \). In other words, when dealing with a noisy data \( (g_\delta) \), where the regularization is achieved by terminating the descent process at an appropriate iteration index, the regularization effect can also be achieved by using the \( \lambda \)-truncated gradient, for an appropriate \( \lambda > 0 \).

**Remark 2.2.** As stated in remark 2.1, the value \( \lambda \) can play the role of the sparsity parameter as well as the regularization parameter, in the \( \lambda \)-truncated gradient descent algorithm, and hence, has double the responsibilities. Now, since in a traditional (semi-) iterative regularization method the regularization effect is handled by the iteration index \( (m) \), we would like to distribute the responsibility of regularization and sparsity to the parameters \( m \) (iteration index) and \( \lambda \), respectively. This will enable us to increase the level of sparsity in the recovered regularized solution without affecting the accuracy of the recovery (i.e. the regularizing effect). One way to achieve that is through the approach stated in 3, and/or having a sequence of iteration-dependent \( \lambda \) values, i.e. \( \lambda_m \)-truncated gradient, which is discussed below.

### 2.1.2. \( \lambda_m \)-truncated gradient:

In order to split the responsibility of regularization and sparsity between the parameters \( m \) (iteration index) and \( \lambda \), respectively, as discussed in remark 2.2, we can use a sequence \( \{\lambda_m > 0\} \) of truncation values, instead of a constant \( \lambda > 0 \), such that in each iteration step \( (m) \) we can truncate the \( m^{th} \) gradient \( (\xi_m) \) based on \( \lambda_m \), i.e., our \( m^{th} \) (sparsity inducing) descent direction \( (h_{m, \lambda_m}) \) will be now

\[ h_{m, \lambda_m} = -T^* (T \psi_m - g) \cdot \chi_{|T^* (T \psi_m - g)| > \lambda_m} , \ (2.8) \]
and we denote \( \xi_m, h_m \) as the \( \lambda_m \)-truncated sparsity. One of the choices for the sequence \( \{\lambda_m\} \) can be based upon the sequence values \( \{||\xi_m = T^* (T \psi_m - g) ||_\infty\} \), such as, \( \lambda_m = \alpha \% * ||\xi_m||_\infty \). In this case, one updates the \( m \)th iterate \( \psi_m, \lambda_m \) to get \( \psi_{m+1}, \lambda_{m+1} \), but only corresponding to top \( (1 - \alpha)\% \) of its gradient \( \xi_m = T^* (T \psi_m, \lambda_m - g) \), that is, \( \psi_{m+1, \lambda_{m+1}} = \psi_m, \lambda_m - \tau_{m+1} \xi_{m, \lambda_{m+1}} \). We implemented this choice of \( \{\lambda_m\} \) in our numerical experiments, for different values of \( \alpha \), and the results are presented in §4, and also a comparison with the \( \lambda \)-truncated gradient approach.

Remark 2.3. Note that, in the \( \lambda \)-truncated gradient method one has to a-priori fix the \( \lambda \)-value for all the iterations and, since it also influence the regularization (or accuracy) of the recovery process, one cannot have a large \( \lambda \)-value, i.e., one has to compromise between the sparsity and regularization, to certain extent. Where as, in the \( \lambda_m \)-truncated gradient method one can have larger \( \lambda_m \) values for the first few iterations (since the \( ||\xi_m||_\infty \) values are large in the initial stage) imposing a lot of sparsity, in addition to not compromising the accuracy of the recovery, as the regularization (or accuracy of the recovery) can be achieved through terminating the iterations independent of the \( \lambda_m \)-values, say discrepancy principle. That is, the stopping criterion for the descent process can be made independent of \( \lambda_m \), say depending on the noise level \( \delta \), leading to greater degrees of freedom for the regularization parameter \( M(\delta) \), instead of \( M(\lambda) \) and the sparsity parameter \( (\lambda_m) \).

In §4 we compare the sparsity and the accuracy of the results obtained by \( \lambda \)-truncated and \( \lambda_m \)-truncated gradient methods.

2.1.3. Discretized truncated gradients:

Note that, all of the above developed theories and results are also valid in the discretized settings \( (A, x, b) \), when the function \( \psi_m \) is replaced by the vector \( x_m \) and the continuous measure \( \mu \) is replaced by the counting measure. In fact, it’s much easier when dealing in a discretized system, for example, from the \( m \)th gradient \( d_m = A^* (Ax_m - b) \in \mathbb{R}^n \) we can have

1. the discrete \( \lambda \)-truncated gradient, \( d_{m, \lambda} = (\{d_{m, \lambda}\}_i)_{i=1}^n \), as

\[
(d_{m, \lambda})_i = \begin{cases} 
(d_m)_i & \text{if } |(d_m)_i| > \lambda \\
0 & \text{if } |(d_m)_i| \leq \lambda 
\end{cases} \tag{2.9}
\]

2. the discrete \( \lambda_m \)-truncated gradient, \( d_{m, \lambda_m} = (\{d_{m, \lambda_m}\}_i)_{i=1}^n \), as

\[
(d_{m, \lambda_m})_i = \begin{cases} 
(d_m)_i & \text{if } |(d_m)_i| > \lambda_m \\
0 & \text{if } |(d_m)_i| \leq \lambda_m 
\end{cases} \tag{2.10}
\]

3. the \( k \)-truncated gradient:

In addition, when working in a discretized setting, where the discretized solution space is a finite high-dimensional space, say \( \mathbb{R}^n \), for large \( n \), but the solution \( \hat{\lambda} \) of (1.11) has a sparse structure to it, i.e., \( ||\hat{x}||_0 \leq k << n \), then we also have the luxury to construct a \( k \)-truncated gradient \( d_{m, k} \) by simply projecting the \( (n - k) \)-smaller values of the gradient \( d_m \in \mathbb{R}^n \) to \( 0 \in \mathbb{R}^{n-k} \), i.e., \( d_{m, k} = (d_{m, k}, d_{m, k})_{i=1}^{n-k} \in \mathbb{R}^n \) such that \( ||d_{m, k}||_0 \leq k \), where

\[
(d_{m, k})_i = \begin{cases} 
(d_m)_i & \text{if } (d_m)_i \geq k \text{-th max of } d_m \\
0 & \text{otherwise.} \tag{2.11}
\end{cases}
\]
Remark 2.4. Note that, when using the discrete $\lambda$-truncated gradient ($d_{m,\lambda}$) or $\lambda_m$-truncated gradient ($d_{m,\lambda_m}$), the sparsity of the gradients is dynamically bounded, i.e., $||d_{m,\lambda}||_0 \leq k_m$ or $||d_{m,\lambda_m}||_0 \leq k_m$, where the sparsity bound ($k_m$) of the truncated gradients varies dynamically depending on the $\lambda$ or $\lambda_m$ values and the gradient $d_{m,\lambda}$, i.e., $k_m = k(\lambda, d_m)$ or $k_m = k(\lambda_m, d_m)$, respectively. Where as, the bound is fixed for $k$-truncated gradient ($d_{m,k}$), i.e., $||d_{m,k}||_0 \leq k$, irrespective of the iterations. However, this doesn’t imply that the sparsity of the iterates are $k$-bounded, i.e., $||x_{m,k}||_0 \leq k$, as $x_{m,k}$ are the updates of the previous iterates ($x_{m,k} = x_{m-1,k} - \tau_k d_{m,k}$) and the sparsity of $x_{m-1,k}$ may not be $k$-bounded.

Remark 2.5. Also note that the $k$-truncated gradient can be derived from the $\lambda_m$-truncated gradient, by choosing $\lambda_m = \min\{k, ||d_m||_0\}$-th max of $d_m$, for the positivity of $\lambda_m$. However, we mostly determine the values of $\lambda_m$ as some percentage of $||d_m||_\infty$, i.e., $\lambda_m = \alpha\% \cdot ||d_m||_\infty$, for some $0 \leq \alpha \leq 100$.

2.1.4. $k\lambda_m$-truncated gradient:
Now one can make use of both the gradients $\lambda_m$-truncated and $k$-truncated by either having the

(1) **min-$k\lambda_m$-truncated gradient**: ($d_{m,k\lambda_m}$) which can be defined as

$$d_{m,k\lambda_m} = \begin{cases} d_{m,\lambda_m} & \text{if } ||d_{m,\lambda_m}||_0 \leq ||d_{m,k}||_0 \\ d_{m,k} & \text{if } ||d_{m,\lambda_m}||_0 > ||d_{m,k}||_0, \end{cases}$$

(2.12)

which increases the sparsity of the recovery process, more than the individual ones, or

(2) **max-$k\lambda_m$-truncated gradient**: ($d_{m,k\lambda_m^+}$) which can be defined as

$$d_{m,k\lambda_m^+} = \begin{cases} d_{m,k} & \text{if } ||d_{m,\lambda_m}||_0 \leq ||d_{m,k}||_0 \\ d_{m,\lambda_m} & \text{if } ||d_{m,\lambda_m}||_0 > ||d_{m,k}||_0, \end{cases}$$

(2.13)

which increases the accuracy of the recovery process, more than the individual ones.

The numerical implementations, results and comparison (with the individual gradients: $\lambda_m$-truncated or $k$-truncated gradient) is presented in §4.

3. IMPLEMENTATION ALGORITHMS AND CONVERGENCE RESULTS

3.1. Convergence: In this section we describe how to implement the above truncated gradients, when dealing with a noisy data ($b_\delta$), such that the sequence $\{x_{m,\lambda}^\infty M \mid k \leq k \lambda_m \}$ or $\{\psi_{m,\lambda}^\infty M \}$ converges to a regularized (sparse) solution $x^\delta$ (or $\varphi^\delta$). First note that, as discussed in §1.3, the key to obtain a regularized solution in a (semi-) iterative regularization method (when dealing with a noisy $b_\delta$) is to terminate the iterations at an appropriate index $m_0 < \infty$. A typical choice of the stopping condition is to terminate the minimization process when the residue ($Ax_m^\delta - b_\delta$) has achieved a certain level of accuracy, i.e., when $||b - b_\delta|| \leq \delta$ is known, then terminate the iteration at $M(\delta)$ such that $||Ax_M^\delta - b_\delta|| \leq \eta\delta$, for $\eta > 1$, this is known as Morozov’s discrepancy principle, see [5] for the convergence analysis. In other words, it says, a reconstructed solution ($x^\delta$) is a good approximate of the true solution ($\hat{x}$) if the image of $x^\delta$ under the forward operator $A$, fits the noisy data $b_\delta$ appropriately, i.e., $Ax^\delta$ doesn’t underfit or overfit $b_\delta$. 

For the $\lambda$-truncated gradient, note that $F(x_{m+1}^{\delta,\lambda}) < F(x_{m,\lambda}^{\delta})$, for all $m \leq M(\lambda)$, where $M(\lambda) \in \mathbb{N}$ is the smallest integer such that $||d_{M(\lambda)}^{\delta}||_{\infty} \leq \lambda$, and $F(x_{m,\lambda}^{\delta}) = F(x_{M(\lambda),\lambda}^{\delta})$ for all $m > M(\lambda)$, since the $\lambda$-truncated gradient $d_{M(\lambda),\lambda}^{\delta} \equiv 0$. This implies the iterates $x_{m,\lambda}^{\delta}$ get updated for all $m \leq M(\lambda)$ and remain unchanged after $M(\lambda)$, i.e., $x_{m,\lambda}^{\delta} = x_{M(\lambda),\lambda}^{\delta}$ for all $m > M(\lambda)$. Hence, an appropriate $\lambda$ should be such that $M(\delta) \leq M(\lambda)$, implying the recovered solution $x_{M(\delta),\lambda}^{\delta}$ (being both sparse and regularized) as a good approximate of the true solution ($\hat{x}$), since $||Ax_{M(\delta),\lambda}^{\delta} - b|| \leq \delta$.

A similar argument also works for the $\lambda_m$-truncated gradient, for example, if $\lambda_m$ are chosen such that $||\{d_{m}^{\delta} > \lambda_m\}||_0 \geq 1$, then the iterate $x_{m,\lambda_m}^{\delta}$ is updated to $x_{m+1,\lambda_{m+1}}^{\delta} = x_{m,\lambda_m}^{\delta} - \tau_{m+1}d_{m+1,\lambda_{m+1}}^{\delta}$ with $F(x_{m+1,\lambda_{m+1}}^{\delta}) < F(x_{m+1,\lambda_{m+1}}^{\delta})$, i.e., an improvement over the previous iterate. Again, if we are able to satisfy the stopping criterion (i.e., $||Ax_{M(\delta),\lambda M(\delta)}^{\delta} - b|| \leq \delta$, for some $M(\delta) < \infty$), then we are to recover a regularized (sparse) solution $x_{M(\delta),\lambda}^{\delta}$, which is a good approximation to the true solution $\hat{x}$. A choice for such a sequence can be $\{\lambda_m := \alpha \% \ast ||d_{m}^{\delta}||_{\infty}\}$, for $1 \leq \alpha \leq 100$, where one can see that the sparsity increases upon increasing $\alpha$, but may compromise the accuracy for large $\alpha$, typically $\alpha < 50$.

Similarly, one can also prove the convergence to a regularized (sparse) solution, which is a good approximation to the true solution $\hat{x}$, when dealing with the $k$-truncated or $k\lambda_m$-truncated gradients.

3.2. Implementation Algorithm: As one can clearly see, the numerical implementation of the above developed theories is very simple, as one simply truncates the gradient obtained at any stage, based on certain criterion. Typically, first, one recovers a regularized solution without truncating the gradients, i.e., a simple steepest-descent (gradient) minimization. This gives an idea of the number of iterations needed and a base regularized solution to compare with. Then, one performs the truncated-gradients descent algorithms and compares the (sparse) regularized solution with the already obtained regularized solution, just in case. Also, it’s advisable to start with smaller $\lambda$ or $\lambda_m$ values and larger $k$ values, so that it doesn’t compromise the accuracy of the recovery by too much, and then, can either increase ($\lambda$ or $\lambda_m$) or decrease ($k$) the truncation values to increase the sparsity level in the reconstruction process.

3.3. Comparison with ISTA: Comparing with the Iterative Soft/Shrinkage Thresholding Algorithm (ISTA), which is described earlier, one can see that there is some similarity between these two approaches, though they are still inherently different. For example, in ISTA the iterate $x_m^\delta$ is improved in a forward-backward splitting manner to get the new iterate $x_{m+1}^\delta$, i.e.,

$$ x_m^\delta \xrightarrow{\text{forward}} x_{m+1}^\delta = x_m^\delta - 2\tau A^*(Ax_m^\delta - b_\delta) \xrightarrow{\text{backward}} x_{m+1}^\delta = S_\lambda(x_{m+1}^\delta), \quad (3.1) $$

where the shrinkage operator $S_\lambda$, as is defined in (1.15), not only discards the smaller $x_{m+1}^\delta$ values (i.e., $|x_{m+1}^\delta| \leq \tau \lambda \Rightarrow (x_{m+1}^\delta)_l = 0$), but also shrink the larger $x_{m+1}^\delta$ values (i.e., $|x_{m+1}^\delta| > \lambda \Rightarrow \text{sgn}(x_{m+1}^\delta)_l \ast (|x_{m+1}^\delta| - \tau \lambda)$).

Where as, in our method the iterate $x_m^\delta$ is also improved in a forward-backward splitting fashion to get the new iterate $x_{m+1}^\delta$, but the forward-backward splitting
is applied to the gradient ($\xi^\delta_m$) instead, i.e.,
\[
(x_m^0, 0) \xrightarrow{\text{forward}} (x_m^\delta, \xi_m^\delta) \xrightarrow{\text{backward}} (x_m^\delta, \xi_{m,\lambda}^\delta) \mapsto x_{m+1}^\delta = x_m^\delta - 2\tau \xi_{m,\lambda}^\delta, \tag{3.2}
\]
where $\xi_{m,\lambda}^\delta$, as defined in (2.5), truncates the smaller $\xi_m^\delta$ values (i.e., $|\xi_m^\delta| \leq \lambda \Rightarrow (\xi_{m,\lambda}^\delta)_i = 0$, otherwise $(\xi_{m,\lambda}^\delta)_i = (\xi_m^\delta)_i$), that is, thresholding the gradients instead of the iterates.

The most significant advantage of our method over ISTA or FISTA is that of the (overall) computational time. In ISTA or FISTA, to obtain a regularized solution $(x_{\lambda_0})$, one has to completely minimize the functional (1.13) for a certain number of $\lambda$ values ($\lambda \in [\lambda_{\min}, \lambda_{\max}]$), i.e., one has to minimize the functional $F_\lambda$ such that $F_\lambda(x_{M(\epsilon)}) < \epsilon$, for some accuracy threshold parameter $\epsilon > 0$ (usually, leading to a very high $M(\epsilon)$ value for small $\epsilon$), and also for that many $\lambda$ values. In comparison, in our method one has to perform only certain number of iterations such that $F(x_{M(\epsilon)}) \leq (\eta \delta)^2$ (usually, $M(\delta) \ll M(\epsilon)$, much smaller) and also has to perform the minimization only once (or maybe for a few $\lambda$ values, just to be conservative). Furthermore, we also show that (see Tables 1 and 2) the sparsity level in our recovered solution far exceeds than that obtained by any other regularization methods.

Remark 3.1. In our numerical experiments (presented in §4) we simply modified the steepest-descent algorithm by replacing the gradients $\xi_m^\delta$ with the truncated-gradient $\xi_{m,\lambda}^\delta$ the simple truncated gradients as the descent directions, i.e., in (1.5) we used $\xi_{m,\lambda}^\delta = \xi_{m,\lambda}^\delta$ and $\psi_{m,\lambda}^\delta = \psi_{m,\lambda}^\delta$. However, as mentioned in §1.3, one can further boost the rate of convergence by opting the accelerated Landweber iterations (generalized semi-iterative methods or the Krylov-subspace methods, and even conjugate-gradient methods), instead of the simple Landweber iterative approach. Using a boosted algorithm really speeds up the descent process.

4. Numerical Results

To test the numerical viability and efficiency of the developed algorithm a MATLAB code was written to solve the image deblurring inverse problem. The mechanism behind blurring an image results in a Fredholm-type integral equation of first kind, with the kernel as a Gaussian spread function, i.e., the operator describing the underlying physical process can be written as, for any $s \in \Omega_1 \subset \mathbb{R}^2$,
\[
g(s) = (T \varphi)(s) := \int_{\Omega} K(s, t) \varphi(t) dt, \quad \Omega \subset \mathbb{R}^2, \tag{4.1}
\]
where the kernel $K(s, t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(s-t)^2}{2\sigma^2}}$, $\sigma$ determining the spread and the ill-posedness of the problem (ill-posedness increases with the increase in the $\sigma$ value, here we considered $\sigma = 10$, highly ill-posed). Such integral operator equations (4.1) are classical inverse problems and can be quite ill-posed. The operator equation (4.1) is then discretized using either Galerkin or Nyström methods to yield the following linear discrete ill-posed problem
\[
Ax = b, \tag{4.2}
\]
where the matrix $A \in \mathbb{R}^{m \times n}$ (blurring matrix) is the discretized representation of the operator $T$ and the vectors $x \in \mathbb{R}^n$ (true image, Figure 1), $b \in \mathbb{R}^m$ (blurry image) are the discretized source and effect functions. MATLAB functions in [34,
35, 36] determine the discretizations $A \in \mathbb{R}^{n \times n}$, the scaled discrete approximations of $x \in \mathbb{R}^n$ and $b := Ax \in \mathbb{R}^m$, where we considered $n = m = 65536$, i.e., a $256 \times 256$ image. Specifically, we used the routine $\text{blur}()$, to generate the deblurring matrix $A$, and the routine $\text{PRblur}()$, to generate the true image, from [34, 36, 35]. To test the stability of the method, we add a normally distributed (with mean zero) error vector $\epsilon_b \in \mathbb{R}^m$ to the original $b$ to get a perturbed vector $b_\delta \in \mathbb{R}^m$ (blurred noisy image, Figure 1), such that the relative error $\left(\frac{||\epsilon_b||_2}{||b||_2}\right)$ in the noisy data is around 10\%, unless otherwise stated. In particular, when using the discrepancy principle to terminate the descent process we assume $\delta = ||\epsilon_b||_2$ to be known.

We also compare the results obtained using our method (with different truncated gradients) with the results obtained using certain standard iterative regularization methods, such as (i) Least squares minimization with $L^1$-norm (IRell1), (ii) (heuristic) total variation penalization (IRhtv), (iii) Conjugate Gradient (CGLS), (iv) FISTA for constrained LS (IRfista), (v) modified residual norm steepest descent (heuristic) total variation penalization (IRhtv), (vi) Conjugate Gradients (CGLS), (vii) $L^1$-norm penalization term (IRirn), (viii) hybrid FGMRES for enforcing $L^1$-norm penalization (IRhybrid_fgmres), (ix) hybrid GMRES for square systems (IRhybrid_gmres), (x) hybrid LSQR (IRhybrid_lsqr), (xi) restarted Krylov subspace method (IRrestart), where the routines for the above mentioned methods are taken from [35], with the stopping rule as the discrepancy principle (i.e., in the ‘options’ settings for terminating the iterations the ‘stoprule’ is set to be ‘DP’). The comparisons can be seen in Tables 1 and 2.

**Example 4.1.** In the first case, we perform a constraint minimization of the functional $F$ defined in (1.2), i.e., $\min_{x \geq 0} F(x)$, since we consider the pixel values to be non-negative. This is achieved by simply projecting the negative values of the iterates to zero during the descent process, i.e., $x^m = ((x^m_i)_{i=1}^n)$ such that $(x^m_i) = \max\{(x^d_i), 0\}$. We do the same for the above mentioned other iterative methods by turning the ‘nonnegativity’, ‘on’ and having ‘xMin’, ‘0’ in the ‘options’ settings. The recovered deblurred image is shown in Figure 1e and, the relative errors and sparsity comparisons are presented in Table 1. Note that, $\hat{x} \in \mathbb{R}^n$, for $n = 65536$, and the sparsity of the true image ($\hat{x}$) is 58858. We can see from Table 1, that using truncated gradients, we are able to attain sparsity level much better than most of the other methods. Also, note that the truncated gradient $d^\delta_{m,k,\lambda\epsilon}$ always yields better sparsity level than using the corresponding individual ones ($d^\delta_{m,\lambda\epsilon}$ or $d^\delta_{m,k}$). Hence, if some a-priori knowledge about the sparsity of the true solution ($\hat{x}$) is known, then one can combine it with the $\lambda_m$-truncated gradients to significantly improve the sparsity of the recovery.

**Example 4.2.** Note that, in Example 4.1, the sparsity in the recovery is also heavily influenced by the non-negativity constraint ($x \geq 0$), since the negative co-efficients of any iterate $x^d_m$ is projected to zero, i.e., $(x^d_m)_i = \max\{(x^d_m)_i, 0\}$. Hence, to observe the real sparsity effect of these methods we instead have the following constraint minimization: $\min_{x \geq -10^{-100}} F(x)$, i.e., the co-efficients of the iterate $x^d_m = ((x^d_m)_i)_{i=1}^n$ are modified as $(x^d_m)_i = \max\{(x^d_m)_i, -10^{-100}\}$. We do the same for the other mentioned iterative methods by turning the ‘nonnegativity’, ‘off’ and having ‘xMin’, ‘$-10^{-100}$’ in the ‘options’ settings. From a computational viewpoint it does not effect the accuracy (or relative error) of the recovery (since $-10^{-100} \approx 0$), as can be seen in Table 1, but we do see a significant lapse in the sparsity level, implying that the sparsity in those methods arose from the constraint.
However, we notice that using our method, though the sparsity level decreases, the recovered solution is still quite sparse, especially for high truncation values.

**Example 4.3.** In Example 4.1 the true image $\hat{x} \in \mathbb{R}^{65536}$ is too sparse (sparsity of $\hat{x}$ is 58858), and hence, are the recovered regularized sparse solutions. In this example, we repeat the same procedure, constraint minimization: $\min_{x \geq 0} F(x)$ as performed in example 4.1, but on a non-sparse true image ($\hat{x} \in \mathbb{R}^{65536}$, with \# of zeroes in $\hat{x}$ as 1) with lot of smaller values, i.e., we consider the Hubble space telescope, see Figure 1b. The recovered deblurred image is shown in Figure 1f and, the relative errors and sparsity comparisons are presented in Table 2. Note that, unlike the truncated gradient $d^m_{m,k} \lambda_m$ which supports sparsity, the truncated gradient $d^m_{m,k} \lambda_m^+$ supports the accuracy of the recovery and yields better results than using the corresponding individual ones ($d^m_{m,\lambda_m}$ or $d^m_{m,k}$).

**Example 4.4.** In this example, we consider the non-sparse true image from Example 4.3 and perform the modified constraint minimization: $\min_{x \geq -10^{-100}}$, as done in Example 4.2. The recovered deblurred image and, relative errors and sparsity comparisons are shown in Table 2.

**Remark 4.5.** Note that, in all of the above examples, we presented the results corresponding to only two truncation values that are widely separated. Now, one take even finer truncation values to either increase the sparsity of the recovery or the accuracy of it, depending on the scenario.

### 5. Practical stopping criterion

As described above, terminating the iterations at an “appropriate instance” is the key to the regularizing effect in a (semi-) iterative regularization method, i.e., the iteration index ($m$) serves as a regularizing parameter. Where an appropriate instance is usually achieved by some *stopping criteria* for the descent or minimization process. Typically, when the error norm $\|b - b_\delta\|_2 \leq \delta$ is known, one terminates the iterations when the residue norm is less than the error norm ($\|Ax_{\delta M(\delta_1)\delta} - b_\delta\|_2 \leq \eta \delta$, for some $\eta > 1$), known as *Morozov’s discrepancy principle* (DP) [5]. However, in a practical situation the exact knowledge of $\delta$ is rarely known, only an estimate of the noise present in the data can be made. Hence, the estimation of the noise present in data is also noisy, i.e., instead of the error-norm $\delta$ we have $\delta_1$, such that $|\delta_1 - \delta| \leq \delta_1$, where $\delta_1$ is the error in estimating the error-norm $\delta$. This is crucial, since (now) the noisy DP will result in a recovered solution ($x_{\delta_1}$) which may be either overly-smoothed (or under-fitted) if $\delta_1 > \delta$, or noisy (over-fitted) if $\delta_1 < \delta$. Even if the exact $\delta$ can be estimated, it’s known that the DP condition can sometimes lead to (slightly) under-fit (or over-smoothed) solutions, especially for small $\delta$, see Example 5.1.

In this section we provide a modified discrepancy principle which is a generalization of the discrepancy principle and serves as an improvement, especially, in the absence of error-norm ($\delta$) or when a noisy approximate ($\delta_1$) of the original $\delta$ is known. Moreover, it also improves the interpretability of the stopping criterion. First, note that the DP condition is equivalent to $\|Ax_{\delta_1 \delta} - b_\delta\|_2 \leq \eta \frac{\delta}{\|b\|_2}$, i.e., terminate the iterations when the relative error in the residue is less than the relative
error in the data, instead of the residue norm less than the error-norm. The advantage of this transformation is that for $\|b\|_2 >> 1$ (which is usually the case) we have $\frac{\delta}{\|b\|_2} << 1$, and hence, $\frac{\delta}{\|b\|_2} \approx \frac{\delta}{\|b\|_2}$. Now, although, we don’t have the exact data $b$ and hence $\|b\|_2$, we can always approximate $\|b\|_2$ with $\|b_\delta\|_2$ (norm of the noisy data), since the error-norm $\delta$ is usually small. The following describes the stopping criterion algorithm:

1. First, get an estimate ($\delta_1$) of the exact noise-norm ($\delta$), and hence, an estimate of the noise relative-error ($\frac{\delta_1}{\|b_\delta\|_2}$, where $b_\delta$ is the noisy data).

2. Construct some critical points, such as $\gamma_1 = \left\lceil \frac{\delta_1}{\|b_\delta\|_2} \right\rceil \%$, where $\lceil x \rceil$ is the smallest integer greater than $x$, $\gamma_2 = \gamma_1 - 0.5\%$, $\gamma_3 = \gamma_2 - 0.5\%$ and so on.
Relative errors using different regularization methods, for 10% data noise.

| Methods   | Constraint: $x \geq 0$ | $x \geq 0$ | Constraint: $x \geq -10^{-100}$ | $x \geq 0$ |
|-----------|------------------------|-----------|-------------------------|-----------|
| IRel1     | 0.2937                 | 0         | 0.2937                  | 0         |
| IRh tv    | 0.2138                 | 53634     | 0.3769                  | 0         |
| IReg l    | 0.2250                 | 0         | 0.2250                  | 0         |
| IRfista   | 0.2096                 | 41189     | 0.2096                  | 0         |
| IRmrnsd   | 0.1972                 | 0         | 0.1972                  | 0         |
| IRenrich  | 0.2249                 | 0         | 0.2249                  | 0         |
| IRrn      | 0.2375                 | 7389      | 0.2466                  | 0         |
| IRhy. fg mres | 0.2937           | 0         | 0.2937                  | 0         |
| IRhy. gm res| 0.2334              | 0         | 0.2334                  | 0         |
| IRhy. lsqr| 0.2736                 | 0         | 0.2736                  | 0         |

Truncated-gradients:

| $\xi_{m,\lambda} = \xi_{m}, \lambda = 0$, no truncation | 0.2105     | 34698     | 0.2105     | 0         |
| $\xi_{m,\lambda} = \xi_{m}, \lambda = 10^{-5}$ | 0.2082     | 43096     | 0.2082     | 25975     |
| $\xi_{m,\lambda} = \xi_{m}, \lambda = 2 \times 10^{-5}$ | 0.2002    | 52670     | 0.2002     | 45751     |
| $\lambda_m = \alpha \% \|\xi_m\|_\infty$, $\alpha = 10$ | 0.2107    | 47050     | 0.2107     | 28446     |
| $\lambda_m = \alpha \% \|\xi_m\|_\infty$, $\alpha = 40$ | 0.2043    | 57149     | 0.2043     | 55556     |
| $k = 32768 (\approx 50\% \times n)$ | 0.2094     | 40584     | 0.2094     | 15302     |
| $k = 6554 (\approx 10\% \times n)$ | 0.2023     | 54725     | 0.2023     | 48526     |
| $k \lambda_m$, $k = 32768$, $\lambda_m = 40\% \|\xi_m\|_\infty$ | 0.2043    | 57149     | 0.2043     | 55556     |
| $k \lambda_m$, $k = 6554$, $\lambda_m = 10\% \|\xi_m\|_\infty$ | 0.2023    | 54725     | 0.2023     | 48526     |

Table 1. Image deblurring, Examples 4.1 and 4.2

Typically, $\gamma_1$, $\gamma_2$ and $\gamma_3$ are enough to get a good recovery, but one can opt for more $\gamma_i$’s and/or with finer spacings.

3. Perform the minimization for certain number of iterations and captures the recoveries at those critical points, i.e., recoveries ($x_{\delta M(\gamma_i)}^{\lambda}$) at $M(\gamma_i)$, corresponding to $\gamma_i$, such that $\frac{\|Ax_{\delta M(\gamma_i)}^{\lambda} - b\|_2}{\|b\|_2} \% \leq \eta \gamma_i$.

4. Now, setting $x_{\delta M(\gamma_1)}^{\lambda}$ as the base recovery, one can compare (heuristically) the other recoveries with it and select appropriately, such as the most sparse recovery close to $x_{\delta M(\gamma_1)}^{\lambda}$ etc., see Example 5.1.
A GRADIENT-THRESHOLDING ALGORITHM FOR SPARSE REGULARIZATION

Relative errors using different regularization methods, for 10% data noise.

| Methods  | Constraint: | Relative error | Sparsity (# 0s) | Constraint: | Relative error | Sparsity |
|----------|-------------|----------------|-----------------|-------------|----------------|----------|
| IRell1   | $x \geq 0$  | 0.2285         | 0               | $x \geq -10^{-100}$ | 0.2285         | 0        |
| IRhtv    | $x \geq -10^{-100}$ | 0.3462         | 23454           | 0.4236      | 0              |          |
| IRregls  | $x \geq 0$  | 0.1717         | 0               | 0.1717      | 0              |          |
| IRfista  | $x \geq -10^{-100}$ | 0.1767         | 13590           | 0.1767      | 0              |          |
| IRemn    | $x \geq 0$  | 0.1717         | 0               | 0.1717      | 0              |          |
| IRmrnsd  | $x \geq -10^{-100}$ | 0.1618         | 1               | 0.1618      | 1              |          |
| IRenrich | $x \geq 0$  | 0.1717         | 12644           | 0.1717      | 0              |          |
| IRirn    | $x \geq -10^{-100}$ | 0.1756         | 15699           | 0.3840      | 0              |          |
| IRhy. fgmres | $x \geq -10^{-100}$ | 0.2285         | 0               | 0.2285      | 0              |          |
| IRhy. gmres | $x \geq -10^{-100}$ | 0.1918         | 0               | 0.1918      | 0              |          |
| IRhy. lsqr | $x \geq -10^{-100}$ | 0.2539         | 0               | 0.2539      | 0              |          |

| Truncated-gradients: | |
|----------------------|------|
| $\xi_{m,\lambda} = \xi_m$, $\lambda = 0$, no truncation | 0.1700 | 12644 | 0.1700 | 0 |
| $\xi_{m,\lambda} = \xi_m \chi_{|\xi_m| > |\lambda|}$, $\lambda = 10^{-5}$ | 0.1695 | 11460 | 0.1695 | 3747 |
| $\xi_{m,\lambda} = \xi_m \chi_{|\xi_m| > |\lambda|}$, $\lambda = 3 \times 10^{-5}$ | 0.1673 | 27440 | 0.1673 | 25961 |
| $\lambda_m = \alpha \% \|\xi_m\|_\infty$, $\alpha = 10$ | 0.1739 | 21765 | 0.1739 | 19343 |
| $\lambda_m = \alpha \% \|\xi_m\|_\infty$, $\alpha = 40$ | 0.1927 | 39592 | 0.1927 | 39015 |
| $k = 32768 (\approx 50\% \times n)$ | 0.1709 | 23945 | 0.1709 | 14491 |
| $k = 6554 (\approx 10\% \times n)$ | 0.2265 | 39393 | 0.2265 | 39128 |
| $k\lambda_m$, $k=32768$, $\lambda_m = 40\% \|\xi_m\|_\infty$ | 0.1709 | 23945 | 0.1709 | 14491 |
| $k\lambda_m$, $k=6554$, $\lambda_m = 10\% \|\xi_m\|_\infty$ | 0.1739 | 21765 | 0.1739 | 19343 |

Table 2. Image deblurring, Examples 4.3 and 4.4

Example 5.1. In this example we compare the results obtained by using the modified DP (as discussed in §5) over the simple discrepancy principle. We considered the Hubble telescope image (from Example 4.3) but with a smaller relative error in the added noise, i.e., $\frac{\|\delta\|_2}{\|b\|_2} = 3.4242\%$, for $\delta = 0.098591$. The corresponding $\gamma_i$ values are $\gamma_1 = 4\%$, $\gamma_2 = 3.5\%$, $\gamma_3 = 3\%$ and $\gamma_4 = 2.5\%$. We performed two sets of minimization: (1) Constrained minimization ($x_i \geq 0$) for truncated and not-truncated gradients, (2) Unconstrained minimization ($x_i > -\infty$), for 100 iterations each. One can notice, from Table 3, that using the modified DP one can recover a regularized sparse solution in the absence of the exact error-norm ($\delta$), even with better sparsity and/or accuracy than the simple DP-solution.
Discrepancy principle vs. Modified discrepancy principle.

| Unconstrained, $x_i > -\infty$ | index (m) | $\gamma_i$ | $\frac{||Ax_m^\delta - b\delta||_2}{||b\delta||_2}$ | $\frac{||\hat{x}_m^\delta - x^\delta||_2}{||x^\delta||_2}$ | sparsity |
|---------------------------------|-----------|-----------|---------------------------------|---------------------------------|----------|
| $\lambda_m = 0$                | 5         | 4         | 3.8453                          | 15.9971                         | 0        |
|                                | 7         | 3.5       | 3.4889                          | 15.3097                         | 0        |
|                                | 8         | 3.4242    | 3.3836                          | 15.0990                         | 0        |
|                                | 16        | 3         | 2.9915                          | 14.6378                         | 0        |
|                                | 64        | 2.5       | 2.4998                          | 18.1620                         | 0        |
| $\lambda_m = 10\%||\xi_m||_\infty$ | 5         | 4         | 3.8490                          | 16.1104                         | 2280     |
|                                | 7         | 3.5       | 3.4713                          | 15.3842                         | 1640     |
|                                | 8         | 3.4242    | 3.3693                          | 15.2295                         | 1539     |
|                                | 15        | 3         | 2.9908                          | 14.6846                         | 958      |
|                                | 53        | 2.5       | 2.4951                          | 18.3960                         | 36       |
| $\lambda_m = 40\%||\xi_m||_\infty$ | 9         | 4         | 3.9741                          | 17.9874                         | 18697    |
|                                | 14        | 3.5       | 3.4152                          | 16.9030                         | 16884    |
|                                | 14        | 3.4242    | 3.4152                          | 16.9030                         | 16884    |
|                                | 43        | 3         | 2.9714                          | 16.2881                         | 9390     |

| Constrained, $x_i \geq 0$      | index (m) | $\gamma_i$ | $\frac{||Ax_m^\delta - b\delta||_2}{||b\delta||_2}$ | $\frac{||\hat{x}_m^\delta - x^\delta||_2}{||x^\delta||_2}$ | sparsity |
|---------------------------------|-----------|-----------|---------------------------------|---------------------------------|----------|
| $\lambda_m = 0$                | 5         | 4         | 3.8607                          | 15.9490                         | 5791     |
|                                | 8         | 3.5       | 3.4160                          | 14.8190                         | 10601    |
|                                | 8         | 3.4242    | 3.4160                          | 14.8190                         | 10601    |
|                                | 26        | 3         | 2.9991                          | 13.7672                         | 18355    |
| $\lambda_m = 10\%||\xi_m||_\infty$ | 5         | 4         | 3.8767                          | 15.9377                         | 11887    |
|                                | 8         | 3.5       | 3.4915                          | 15.0107                         | 10732    |
|                                | 10        | 3.4242    | 3.3720                          | 14.6331                         | 11177    |
|                                | 31        | 3         | 2.9990                          | 13.7393                         | 19756    |
| $\lambda_m = 40\%||\xi_m||_\infty$ | 10        | 4         | 3.8407                          | 17.6140                         | 31385    |
|                                | 19        | 3.5       | 3.4318                          | 16.8342                         | 29986    |
|                                | 21        | 3.4242    | 3.3981                          | 16.6935                         | 29877    |

Table 3. Relative errors using the modified DP, Example 5.1.

Remark 5.2. Note that, in our iterative algorithm both the parameters: (1) the iteration index $(m)$ and (ii) the truncation values $(\lambda, \lambda_m$ or $k)$ serve as the regularization parameter as well as the sparsity parameter. That is, in a regularized setting $(m \leq M_0$, for some appropriate $M_0 \in \mathbb{N})$, the sparsity of the recovery decreases with increasing $m$ values (and/or decreasing truncation values) and vice-versa, where as, the accuracy (relative error) of the recovery increases with increasing $m$ values (and/or decreasing truncation values) and vice-versa.

6. Conclusion

In this paper we presented a very simple iterative regularization method that motivates sparsity in the reconstruction. A sparse regularized solution of an inverse problem is achieved by opting a descent direction, during the minimization process,
that motivates sparsity. We provided many such descent directions that serve the purpose. We prove the convergence of this algorithm to a regularized (sparse) solution when dealing with a noisy data. We also provided an efficient and effective (practical) stopping criterion to terminate the iterations during the descent process, when dealing with a noisy data, without the knowledge of the exact error-norm.

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