Generalizations of Inequalities for Differentiable Co-Ordinated Convex Functions

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A generalized lemma is proved and several new inequalities for differentiable co-ordinated convex and concave functions in two variables are obtained.

1. Introduction

Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a convex function and \( a, b \in I \) with \( a < b \); we have the following double inequality:

\[
 f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \leq \frac{f(a) + f(b)}{2}. 
\] (1)

This remarkable result is well known in the literature as the Hermite-Hadamard inequality for convex mappings.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1–4]).

A modification for convex functions which is also known as coordinated convex functions was introduced as follows by Dragomir in [5].

Let us consider the bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \); a mapping \( f : \Delta \to \mathbb{R} \) is said to be convex on \( \Delta \) if the inequality

\[
 f (\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda t f (x, y) + \lambda (1 - t) f (x, w) + (1 - \lambda) t f (z, y) + (1 - t) (1 - \lambda) f (z, w)
\] (2)

holds for all \( (x, y), (z, y), (x, w), (z, w) \in \Delta \), and \( t, \lambda \in [0, 1] \).

Dragomir in [5] established the following Hadamard-type inequalities for coordinated convex functions in a rectangle from the plane \( \mathbb{R}^2 \).

Theorem 2. Suppose that \( f : \Delta = [a, b] \times [c, d] \to \mathbb{R} \) is convex on the coordinates on \( \Delta \). Then one has the inequalities as follows:

\[
 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \leq \frac{1}{(b - a)(d - c)} \int_{a}^{b} \int_{c}^{d} f(x, y) \, dy \, dx \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
\] (4)

Some new integral inequalities that are related to the Hermite-Hadamard type for coordinated convex functions are also established by many authors.
In ([6], 2008), Alomari and Darus defined coordinated s-convex functions and proved some inequalities based on this definition. In ([7], 2009), analogous results for h-convex functions on the coordinates were proved by Latif and Alomari. In ([8], 2009), Alomari and Darus established some Hadamard-type inequalities for coordinated log-convex functions.

In ([9], 2012), Latif and Dragomir obtained some new Hadamard type inequalities for differentiable coordinated convex and concave functions in two variables which are related to the left-hand side of Hermite-Hadamard type inequality for coordinated convex functions in two variables based on the following lemma.

**Lemma 3.** Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \in L(\Delta) \), then the following equality holds:

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)
- \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) \, dx
- \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) \, dy

= (b-a)(d-c)
\times \int_0^1 K(u, v) \frac{\partial^2 f}{\partial u \partial v} \, (ua+(1-u)b, vc+(1-v)d) \, du \, dv,
\]

(5)

where

\[
K(u, v) = \begin{cases} 
\begin{align*}
u, & \quad (u, v) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right], \\
u(v-1), & \quad (u, v) \in \left[0, \frac{1}{2}\right] \times \left(\frac{1}{2}, 1\right], \\
(u-1)v, & \quad (u, v) \in \left(\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \\
(u-1)(v-1), & \quad (u, v) \in \left(\frac{1}{2}, 1\right] \times \left(\frac{1}{2}, 1\right].
\end{align*}
\end{cases}
\]

(6)

**Theorem 4** (see [9]). Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( |\frac{\partial^2 f}{\partial u \partial v}| \) is convex on the coordinates on \( \Delta \) and \( \alpha, \beta \geq 1 \), then the following equality holds:

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A
\leq \frac{b-a}{4(\alpha+1)^\frac{1}{\alpha}}
\times \left(\left|\frac{\partial^2 f}{\partial u \partial v}(a, c)\right| + \left|\frac{\partial^2 f}{\partial u \partial v}(a, d)\right| + \left|\frac{\partial^2 f}{\partial u \partial v}(b, c)\right| + \left|\frac{\partial^2 f}{\partial u \partial v}(b, d)\right| \right) \times (4)^{-\frac{1}{\alpha}},
\]

(7)

where \( A \) is as given in Theorem 4.

**Theorem 5** (see [9]). Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( |\frac{\partial^2 f}{\partial u \partial v}| \) is convex on the coordinates on \( \Delta \) and \( q > 1 \), then the following equality holds:

\[
\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx + f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - A
\leq \frac{b-a}{16(\frac{1}{\alpha} + \frac{1}{\beta})}
\times \left(\left|\frac{\partial^2 f}{\partial u \partial v}(a, c)\right| + \left|\frac{\partial^2 f}{\partial u \partial v}(a, d)\right| + \left|\frac{\partial^2 f}{\partial u \partial v}(b, c)\right| + \left|\frac{\partial^2 f}{\partial u \partial v}(b, d)\right| \right) \times (4)^{-\frac{1}{\alpha}}.
\]

(8)

where \( A \) is as given in Theorem 4.

In ([10], 2012), analogous results which are related to the right-hand side of Hermite-Hadamard type inequality for coordinated convex functions in two variables were proved by Sarikaya et al. based on the following lemma.
Lemma 7. Let \( f: \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f / \partial u \partial v \in L(\Delta) \), then the following equality holds:
\[
f(a, c) + f(a, d) + f(b, c) + f(b, d) = \frac{1}{4} \left( \int_a^b f(x, c) \, dx + \int_a^b f(x, d) \, dx + \int_c^d f(a, y) \, dy + \int_c^d f(b, y) \, dy \right)
\]
\[
+ \frac{1}{b-a} \int_a^b f(x, y) \, dy \, dx
\]
\[
- \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] \, dx \right]
\]
\[
+ \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] \, dy \right)
\]
(11)
\[
= \frac{(b-a)(d-c)}{4}
\]
\[
\times \left( \int_0^1 (1-2u)(1-2v) \frac{\partial^2 f}{\partial u \partial v} \right)
\]
\[
\times (ua + (1-u)b, vc + (1-v)d) \, du \, dv.
\]

Theorem 8 (see [10]). Let \( f: \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f / \partial u \partial v \) is convex on the coordinates on \( \Delta \), then the following equality holds:
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|
\]
\[
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A
\]
(12)
\[
\leq \frac{(b-a)(d-c)}{16}
\]
\[
\times \left( \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right|^q \right) \times (4)^{-1} \right),
\]
where
\[
A = \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b [f(x, c) + f(x, d)] \, dx \right]
\]
\[
+ \frac{1}{d-c} \int_c^d [f(a, y) + f(b, y)] \, dy \right).
\]

Theorem 9 (see [10]). Let \( f: \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f / \partial u \partial v \) is convex on the coordinates on \( \Delta \) and \( p, q > 1, 1/p + 1/q = 1 \), then the following equality holds:
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|
\]
\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A
\]
\[
\leq \frac{(b-a)(d-c)}{4(p + 1)^2/4}
\]
\[
\times \left( \left( \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q \right) \times (4)^{-1} \right),
\]
(14)
where \( A \) is as given in Theorem 8.

Theorem 10 (see [10]). Let \( f: \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f / \partial u \partial v \) is convex on the coordinates on \( \Delta \) and \( q \geq 1 \), then the following equality holds:
\[
\left| f(a, c) + f(a, d) + f(b, c) + f(b, d) \right|
\]
\[
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx - A
\]
\[
\leq \frac{(b-a)(d-c)}{16}
\]
\[
\times \left( \left| \frac{\partial^2 f}{\partial u \partial v} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial u \partial v} (b, d) \right|^q \right) \times (4)^{-1} \right),
\]
(15)
where \( A \) is as given in Theorem 8.

In [11], Ozdemir et al. established some Simpson's inequalities for coordinated convex functions based on the following lemma.

Lemma 11. Let \( f: \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f / \partial u \partial v \in L(\Delta) \), then the following equality holds:
\[
\left( f \left( a, \frac{c+d}{2} \right) + f \left( b, \frac{c+d}{2} \right) + 4f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \right)
\]
\[
+ f \left( \frac{a+b}{2}, c \right) + f \left( \frac{a+b}{2}, d \right) \times (9)^{-1}
\]
\[+ \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{36}
- \frac{1}{6(b-a)} \int_a^b \left(f(x,c) + 4f\left(x, \frac{c+d}{2}\right) + f(x,d)\right) dx
- \frac{1}{6(d-c)} \int_c^d \left(f(a,y) + 4f\left(a + \frac{b}{2}, y\right) + f(b,y)\right) dy
+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx
= (b-a)(d-c)
\times \left( \int_0^1 p(x,u)p(y,v) \frac{\partial^2 f}{\partial u \partial v} \right.
\left. \times (ua + (1-u)b, vc + (1-v)d) du dv, \right) \tag{16}
\]

where
\[p(x,u) = \begin{cases} (u-\frac{1}{6}), & t \in [0, \frac{1}{2}] \\ (u-\frac{5}{6}), & t \in \left(\frac{1}{2}, 1\right] \end{cases}, \tag{17}
\]
\[p(y,v) = \begin{cases} (v-\frac{1}{6}), & s \in [0, \frac{1}{2}] \\ (v-\frac{5}{6}), & s \in \left(\frac{1}{2}, 1\right] \end{cases}. \]

**Theorem 12** (see [11]). Let \( f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \) is convex on the coordinates on \( \Delta \), then the following equality holds:
\[\left| \left| f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \right| \right|
\]
\[+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx - A \leq \left(\frac{5}{72}\right)^2 (b-a)(d-c)
\]
\[\times \left( \frac{\partial^2 f}{\partial u \partial v}(a,c) + \frac{\partial^2 f}{\partial u \partial v}(a,d) + \frac{\partial^2 f}{\partial u \partial v}(b,c) + \frac{\partial^2 f}{\partial u \partial v}(b,d) \right), \tag{18}\]

where
\[A = \frac{1}{6(b-a)} \int_a^b \left(f(x,c) + 4f\left(x, \frac{c+d}{2}\right) + f(x,d)\right) dx
+ \frac{1}{6(d-c)} \int_c^d \left(f(a,y) + 4f\left(a + \frac{b}{2}, y\right) + f(b,y)\right) dy. \tag{19}\]

For recent results and generalizations concerning Hermite-Hadamard type inequality for differentiable coordinated convex functions see ([12], 2012) and the references given therein.

In this paper, a generalized lemma is proved and several new inequalities for differentiable coordinated convex and concave functions in two variables are obtained.

**2. Lemmas**

To establish our results, we need the following lemma.

**Lemma 13.** Let \( f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a,b] \times [c,d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial u \partial v} \in L(\Delta) \) and \( \lambda \in [0,1] \), then the following equality holds:
\[\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) dy dx
+ (1-\lambda)^2 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + \lambda(1-\lambda)\left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right)\right]
\]
\[- \frac{1}{2(b-a)} \times \int_a^b \left( \lambda f(x,c) + 2(1-\lambda) f\left(x, \frac{c+d}{2}\right) + \lambda f(x,d) \right) dx
\]
\[- \frac{1}{2(d-c)} \times \int_c^d \left( \lambda f(a,y) + 2(1-\lambda) f\left(\frac{a+b}{2}, y\right) + \lambda f(b,y) \right) dy
\]
\[= (b-a)(d-c)
\times \left( \int_0^1 M(u,v) \frac{\partial^2 f}{\partial u \partial v} \right.
\left. \times (ua + (1-u)b, vc + (1-v)d) du dv, \right) \tag{20} \]
where
\[ M(u, v) = \begin{cases} 
(u - \frac{\lambda}{2})(v - \frac{\lambda}{2}), \\
(u, v) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \\
(u - \frac{\lambda}{2})(v - (1 - \frac{\lambda}{2})), \\
(u, v) \in (\frac{1}{2}, 1] \times [0, \frac{1}{2}] \\
(u - (1 - \frac{\lambda}{2}))(v - \frac{\lambda}{2}), \\
(u, v) \in (\frac{1}{2}, 1] \times (\frac{1}{2}, 1]. 
\end{cases} \]

Proof. Since
\[
\iint_0^1 M(u, v) \frac{\partial^2 f}{\partial u \partial v} 
\times (ua + (1 - u)b, vc + (1 - v)d) du dv 
= \iint_0^{1/2} \iint_0^1 \frac{\partial^2 f}{\partial u \partial v} 
\times (ua + (1 - u)b, vc + (1 - v)d) du dv 
+ \iint_0^{1/2} \int_0^1 \frac{\partial^2 f}{\partial u \partial v} 
\times (ua + (1 - u)b, vc + (1 - v)d) du dv 
+ \int_0^{1/2} \int_0^1 \frac{\partial^2 f}{\partial u \partial v} 
\times (ua + (1 - u)b, vc + (1 - v)d) du dv 
+ \int_0^{1/2} \int_0^1 \frac{\partial^2 f}{\partial u \partial v} 
\times (ua + (1 - u)b, vc + (1 - v)d) du dv,
\]
thus, by integration by parts, it follows that
\[
\iint_0^{1/2} \frac{\partial^2 f}{\partial u \partial v} 
\times (ua + (1 - u)b, vc + (1 - v)d) du dv 
= \int_0^{1/2} \frac{\partial f}{\partial u} \left( v - \frac{\lambda}{2} \right) 
\times \left( v - \frac{\lambda}{2} \right) \frac{\partial f}{\partial u} 
= \int_0^{1/2} \frac{\partial f}{\partial u} \left( v - \frac{\lambda}{2} \right) 
\times \left( v - \frac{\lambda}{2} \right) \frac{\partial f}{\partial u} 
= \frac{(1 - \lambda)^2}{4(c - d)(a - b)} f(\frac{a + b + c + d}{2}) 
+ \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f(\frac{b, c + d}{2}) 
- \frac{1 - \lambda}{2(c - d)(a - b)} \int_0^{1/2} f((ua + (1 - u)b, c + d) du 
+ \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f(\frac{a + b}{2}, d) 
+ \frac{\lambda^2}{4(c - d)(a - b)} f(b, d).
\[-\frac{\lambda}{2(c - d)(a - b)} \int_0^{1/2} f(ua + (1 - u)b, d) \, du - \frac{1 - \lambda}{2(c - d)(a - b)} \int_0^{1/2} f\left(\frac{a + b}{2}, vc + (1 - v)d\right) \, dv \]
\[-\frac{\lambda}{2(c - d)(a - b)} \int_0^{1/2} f(b, vc + (1 - v)d) \, dv + \frac{1}{(c - d)(a - b)} \int_0^{1/2} f(ua + (1 - u)b, vc + (1 - v)d) \, dv.\]

Similarly, we can get
\[\int_0^{1/2} \int_{1/2}^1 \left(u - \frac{\lambda}{2}\right) \left(v - \left(1 - \frac{\lambda}{2}\right)\right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv = \frac{(1 - \lambda)^2}{4(c - d)(a - b)} f\left(\frac{a + b + c + d}{2}\right) + \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f\left(\frac{a + d}{2}\right) + \frac{1 - \lambda}{2(c - d)(a - b)} \int_{1/2}^1 f\left(ua + (1 - u)b, \frac{c + d}{2}\right) \, du \]
\[+ \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f\left(\frac{a + b}{2}, c\right) + \frac{\lambda^2}{4(c - d)(a - b)} f(b, c) - \frac{\lambda}{2(c - d)(a - b)} \int_0^{1/2} f(ua + (1 - u)b, c) \, du - \frac{1 - \lambda}{2(c - d)(a - b)} \int_{1/2}^1 f\left(\frac{a + b}{2}, vc + (1 - v)d\right) \, dv \]
\[+ \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f\left(\frac{a}{2}, c\right) + \frac{1}{(c - d)(a - b)} \int_0^{1/2} f\left(\frac{a + b}{2}, vc + (1 - v)d\right) \, dv + \frac{\lambda}{2(c - d)(b - a)} \int_0^{1/2} f(a, vc + (1 - v)d) \, dv + \frac{1}{2(c - d)(b - a)} \int_{1/2}^1 f(ua + (1 - u)b, vc + (1 - v)d) \, dv,\]
\[\int_{1/2}^1 \int_{1/2}^1 \left(u - \left(1 - \frac{\lambda}{2}\right)\right) \left(v - \left(1 - \frac{\lambda}{2}\right)\right) \frac{\partial^2 f}{\partial u \partial v} \times (ua + (1 - u)b, vc + (1 - v)d) \, du \, dv = \frac{(1 - \lambda)^2}{4(c - d)(a - b)} f\left(\frac{a + b + c + d}{2}\right) + \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f\left(\frac{a + c + d}{2}\right) + \frac{1 - \lambda}{2(c - d)(a - b)} \int_{1/2}^1 f\left(ua + (1 - u)b, \frac{c + d}{2}\right) \, du \]
\[+ \frac{\lambda(1 - \lambda)}{4(c - d)(a - b)} f\left(\frac{a + c}{2}, d\right) + \frac{\lambda^2}{4(c - d)(a - b)} f(a, c) - \frac{\lambda}{2(c - d)(a - b)} \int_0^{1/2} f\left(\frac{a + b}{2}, c\right) \, du - \frac{1 - \lambda}{2(c - d)(a - b)} \int_{1/2}^1 f\left(\frac{a + b}{2}, vc + (1 - v)d\right) \, dv \]
\[+ \frac{\lambda}{2(c - d)(a - b)} \int_0^{1/2} f\left(\frac{a}{2}, c\right) \, dv + \frac{\lambda}{2(c - d)(b - a)} \int_0^{1/2} f\left(\frac{a}{2}, vc + (1 - v)d\right) \, dv + \frac{1}{2(c - d)(b - a)} \int_{1/2}^1 f(a, vc + (1 - v)d) \, dv + \frac{1}{2(c - d)(b - a)} \int_{1/2}^1 f\left(\frac{a}{2}, \frac{c + d}{2}\right) \, dv.\]
Remark 14. Applying Lemma 13 for $\lambda = 0, 1, 1/3$, we get the results of Lemmas 3, 7, and 11, respectively.

3. Main Results

Theorem 15. Let $f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. If $|\partial^2 f / \partial u \partial v|$ is convex on the coordinates on $\Delta$ and $\lambda \in [0, 1]$, then the following equality holds:

$$\left| \frac{1}{(b-a)(d-c)} \int_a^b f(x, y) \, dx \right| + \frac{1}{2} \left[ \phi(x, c) + \phi(x, d) \right] - A \leq \frac{(a+b, c+d)}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right]$$

where

$$A = \frac{1}{2} \int_a^b \left( f(x, c) + 2(1-\lambda) f \left( x, \frac{c+d}{2} \right) \right) \, dx + \frac{1}{2(d-c)} \int_c^d \left( f \left( \frac{a+b}{2}, y \right) + f \left( \frac{a+b}{2}, y \right) \right) \, dy$$

Proof. From Lemma 13, we obtain

$$\int_a^b \int_c^d \frac{1}{(b-a)(d-c)} f(x, y) \, dy \, dx$$

Multiplying both sides by $(b-a)(d-c)$ and using the change of the variable $x = ua + (1-u)b$ and $y = vc + (1-v)d$, which completes the proof.
\[ + \frac{\lambda^2}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \leq (b - a)(d - c) \]

\[ \times \left\| M(u, v) \right\| \]

\[ \times \left| \frac{\partial^2 f}{\partial u \partial v} (ua + (1 - u)b, vc + (1 - v)d) \right| du dv. \]

(28)

Because \( \frac{\partial^2 f}{\partial u \partial v} \) is a convex function on the coordinates on \( \Delta \), then one has

\[ \left( \frac{1}{(b - a)(d - c)} \right) \int_a^b \int_c^d f(x, y) dy dx \]

\[ + (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \]

\[ + \frac{\lambda^2}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \]

\[ \leq (b - a)(d - c) \]

\[ \times \left\| M(u, v) \right\| \]

\[ \times \left| \frac{\partial^2 f}{\partial u \partial v} (ua + (1 - u)b, vc + (1 - v)d) \right| du dv. \]

(29)

On the other hand, we have

\[ \int_0^1 |M(u, v)| u v du dv \]

\[ = \int_0^1 |M(u, v)| u (1 - v) du dv \]

\[ = \int_0^1 |M(u, v)| (1 - u) v du dv \]

\[ = \int_0^1 |M(u, v)| (1 - u)(1 - v) du dv, \]

\[ \int_0^1 |M(u, v)| u v du dv \]

\[ = \int_{1/2}^{1/2} \left| u - \frac{\lambda}{2} \right| \left( v - \frac{\lambda}{2} \right) uv du dv \]

\[ + \int_{1/2}^{1/2} \left| u - \frac{\lambda}{2} \right| \left( v - \frac{1 - \lambda}{2} \right) uv du dv \]

\[ + \int_{1/2}^{1/2} \left| u - \frac{1 - \lambda}{2} \right| \left( v - \frac{1 - \lambda}{2} \right) uv du dv \]

\[ = \left( \frac{2\lambda^2 - 2\lambda + 1}{8} \right)^2, \]

(30)

which completes the proof. \( \Box \)

Remark 16. Applying Theorem 15 for \( \lambda = 0, 1, 1/3 \), we get the results of Theorems 4, 8, and 12, respectively.

Theorem 17. Let \( f : \Delta \subseteq \mathbb{R}^2 \rightarrow \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( \partial^2 f / \partial u \partial v \) is convex on the coordinates on \( \Delta \) and \( q > 1 \), one gets the following inequality:

\[ \left| \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) dy dx \right| \]

\[ + (1 - \lambda)^2 f \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \]

\[ + \left( \frac{\lambda}{4} \left[ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right] - A \right) \]

\[ \leq \left( \frac{2\lambda^2 - 2\lambda + 1}{8} \right)^2. \]
where \( \lambda \in [0, 1] \) and \( A \) is as given in Theorem 15 and \( 1/p + 1/q = 1 \).

**Proof.** From Lemma 13, we obtain

\[
\begin{align*}
\frac{1}{(b-a)(d-c)} & \int_a^b \int_c^d f(x, y) \, dy \, dx \\
& + (1-\lambda)^2 \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) - A \right] \\
& \leq (b-a)(d-c) \left( \int_0^1 |M(u, v)|^p \, du \, dv \right)^{1/p} \\
& \times \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} \right|^q \, du \, dv \right)^{1/q}.
\end{align*}
\]

(33)

By using the well-known Hölder inequality for double integrals, then one has

\[
\begin{align*}
\frac{1}{(b-a)(d-c)} & \int_a^b \int_c^d f(x, y) \, dy \, dx \\
& + (1-\lambda)^2 \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) - A \right] \\
& \leq (b-a)(d-c) \left( \int_0^1 |M(u, v)|^p \, du \, dv \right)^{1/p} \\
& \times \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} \right|^q \, du \, dv \right)^{1/q}.
\end{align*}
\]

(34)

We note that

\[
\begin{align*}
\int_0^1 |M(u, v)|^p \, du \, dv \\
& = \int_0^{1/2} \left( \left| u - \frac{\lambda}{2} \right| \left( \frac{1}{2} - \frac{\lambda}{2} \right) \right|^p \, du \, dv \\
& \quad + \int_0^{1/2} \int_0^{1/2} \left( \left| u - \frac{\lambda}{2} \right| \left( \frac{1}{2} - \frac{\lambda}{2} \right) \right|^p \, du \, dv \\
& \quad + \int_0^{1/2} \int_0^{1/2} \left( \left| u - \frac{\lambda}{2} \right| \left( \frac{1}{2} - \frac{\lambda}{2} \right) \right|^p \, du \, dv \\
& \quad + \int_0^{1/2} \int_0^{1/2} \left( \left| u - \frac{\lambda}{2} \right| \left( \frac{1}{2} - \frac{\lambda}{2} \right) \right|^p \, du \, dv \\
& = \left( \frac{\lambda^{p+1} + (1-\lambda)^{p+1}}{(p + 1)2^p} \right)^2.
\end{align*}
\]

(35)

Hence, it follows that

\[
\begin{align*}
\frac{1}{(b-a)(d-c)} & \int_a^b \int_c^d f(x, y) \, dy \, dx \\
& + (1-\lambda)^2 \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) - A \right] \\
& \leq (b-a)(d-c) \left( \int_0^1 |M(u, v)|^p \, du \, dv \right)^{1/p} \\
& \times \left( \int_0^1 \left| \frac{\partial^2 f}{\partial u \partial v} \right|^q \, du \, dv \right)^{1/q}.
\end{align*}
\]

(36)

\( \square \)

**Remark 18.** Applying Theorem 17 for \( \lambda = 0, 1 \), we get the results of Theorems 5 and 9, respectively.

**Theorem 19.** Let \( f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \). If \( |\partial^2 f/\partial u \partial v|^q \) is convex on the coordinates on \( \Delta \) and \( q \geq 1 \), then

\[
\begin{align*}
\frac{1}{(b-a)(d-c)} & \int_a^b \int_c^d f(x, y) \, dy \, dx \\
& + (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)
\end{align*}
\]

\( \square \)
\[
\frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A \\
\leq (b-a)(d-c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^2 \\
\times \left( \left\{ \frac{\partial^2 f}{\partial u \partial v} (a,c) \right\}^q + \left\{ \frac{\partial^2 f}{\partial u \partial v} (a,d) \right\}^q + \left\{ \frac{\partial^2 f}{\partial u \partial v} (b,c) \right\}^q + \left\{ \frac{\partial^2 f}{\partial u \partial v} (b,d) \right\}^q \right) \times (4)^{-1/2},
\]

where \( \lambda \in [0, 1] \) and \( A \) is as given in Theorem 15.

**Proof.** From Lemma 13, we obtain

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx \right| + (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \\
+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A \\
\leq (b-a)(d-c) \\
\times \left( \int_0^1 |M(u,v)| \right) \\
\times \left\{ \frac{\partial^2 f}{\partial u \partial v} (ua + (1-u)b, vc + (1-v)d) \right\}^q \\
\times (4)^{-1/2},
\]

By using the well-known power mean inequality for double integrals, then one has

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx \right| + (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \\
+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A \\
\leq (b-a)(d-c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^{2-(2/q)} \\
\times \left( \int_0^1 |M(u,v)| \right) \\
\times \left\{ \int_0^b \frac{\partial^2 f}{\partial u \partial v} (u(1-v) + v(1-u), c + (1-v)d) \right\} \right\}^q \\
\times (4)^{-1/2}.
\]

(37)

Because \( |\frac{\partial^2 f}{\partial u \partial v}|^q \) is a convex function on the coordinates on \( \Delta \), then one has

\[
\left\{ \frac{\partial^2 f}{\partial u \partial v} (ua + (1-u)b, vc + (1-v)d) \right\}^q \\
\leq \left\{ u \frac{\partial^2 f}{\partial u \partial v} (a,c) \right\}^q + u(1-v) \frac{\partial^2 f}{\partial u \partial v} (a,d) \\
+ (1-u) \frac{\partial^2 f}{\partial u \partial v} (b,c) \\
+ (1-u)(1-v) \frac{\partial^2 f}{\partial u \partial v} (b,d) \right\}^q.
\]

(40)

Thus, it follows that

\[
\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y) \, dy \, dx \right| + (1-\lambda)^2 f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \\
+ \frac{\lambda^2}{4} \left[ f(a,c) + f(a,d) + f(b,c) + f(b,d) \right] - A \\
\leq (b-a)(d-c) \left( \frac{2\lambda^2 - 2\lambda + 1}{4} \right)^{2-(2/q)} \\
\times \left( \int_0^1 |M(u,v)| \right) \\
\times \left\{ \int_0^b \frac{\partial^2 f}{\partial u \partial v} (u(1-v) + v(1-u), c + (1-v)d) \right\} \right\}^q \\
\times (4)^{-1/2}.
\]

(41)
On the other hand, we obtain
\[
\begin{align*}
\oint_0^1 |M(u, v)| &\times\left\{u v |M_uu v (a, c)| ^q + u (1 - v) |M_uu v (a, d)| ^q \\
+ (1 - u) v |M_uu v (b, c)| ^q \\
+ (1 - u) (1 - v) |M_uu v (b, d)| ^q \right\} du dv \\
= \left\{\frac{\partial^2 f}{\partial u^2 v} (a, c) \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (a, d) \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (b, c) \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (b, d) \right\} \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (a, c) \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (a, d) \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (b, c) \right\} \left\{\frac{\partial^2 f}{\partial u^2 v} (b, d) \right\} \right\}.
\end{align*}
\]

Thus, we get the following inequality:
\[
\left|\frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) dy dx \right|
+ (1 - \lambda)^2 f\left(\frac{a + b}{2} , \frac{c + d}{2}\right)
+ \frac{\lambda^2}{4} \left[f(a, c) + f(a, d) + f(b, c) + f(b, d)\right] - A
\leq (b - a)(d - c) \left(2\lambda^2 - 2\lambda + 1\right)^2
\times\left(\left|\frac{\partial^2 f}{\partial u^2 v} (a, c)\right|^q + \left|\frac{\partial^2 f}{\partial u^2 v} (a, d)\right|^q \\
+ \left|\frac{\partial^2 f}{\partial u^2 v} (b, c)\right|^q + \left|\frac{\partial^2 f}{\partial u^2 v} (b, d)\right|^q \right)^{1/q},
\]
which completes the proof.

Remark 20. Applying Theorem 19 for \( \lambda = 0, 1 \), we get the result of Theorems 6 and 10, respectively.

**Theorem 21.** Let \( f: \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d \).

If \( |\partial^2 f/\partial u^2 v|^q \) is concave on the coordinates on \( \Delta \) and \( q > 1 \), then
\[
\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right|
+ (1 - \lambda)^2 f\left(\frac{a + b}{2} , \frac{c + d}{2}\right) + \frac{\lambda^2}{4}
\times\left[f(a, c) + f(a, d) + f(b, c) + f(b, d)\right] - A
\leq \frac{(b-a)(d-c)}{4} \left(\frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{p+1}\right)^{2/p}.
\]

Proof. Similarly as in Theorem 17, because \( |\partial^2 f/\partial u^2 v|^q \) is a concave function on the coordinates on \( \Delta \), by the reversed direction of (4), we get
\[
\left|\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \right|
+ (1 - \lambda)^2 f\left(\frac{a + b}{2} , \frac{c + d}{2}\right)
\times\left[f(a, c) + f(a, d) + f(b, c) + f(b, d)\right] - A
\leq \frac{(b-a)(d-c)}{4} \left(\frac{\lambda^{p+1} + (1 - \lambda)^{p+1}}{p+1}\right)^{2/p},
\]
which yields the desired result.

**Conflict of Interests**

The author has declared that no conflict of interests exists.
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References

[1] S. S. Dragomir, "Hermite-Hadamard's type inequalities for operator convex functions," *Applied Mathematics and Computation*, vol. 218, no. 3, pp. 766–772, 2011.

[2] S. S. Dragomir, "Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces," *Linear Algebra and Its Applications*, vol. 436, no. 5, pp. 1503–1515, 2012.

[3] A. El Farissi, "Simple proof and refinement of Hermite-Hadamard inequality," *Journal of Mathematical Inequalities*, vol. 4, no. 3, pp. 365–369, 2010.

[4] X. Gao, "A note on the Hermite-Hadamard inequality," *Journal of Mathematical Inequalities*, vol. 4, no. 4, pp. 587–591, 2010.

[5] S. S. Dragomir, "On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane," *Taiwanese Journal of Mathematics*, vol. 5, no. 4, pp. 775–788, 2001.

[6] M. Alomari and M. Darus, "The Hadamard's inequality for s-convex function of 2-variables on the co-ordinates," *International Journal of Mathematical Analysis*, vol. 2, no. 13–16, pp. 629–638, 2008.

[7] M. A. Latif and M. Alomari, "On Hadamard-type inequalities for h-convex functions on the co-ordinates," *International Journal of Mathematical Analysis*, vol. 3, no. 33–36, pp. 1645–1656, 2009.

[8] M. Alomari and M. Darus, "On the Hadamard's inequality for log-convex functions on the coordinates," *Journal of Inequalities and Applications*, vol. 2009, Article ID 283147, 2009.

[9] M. A. Latif and S. S. Dragomir, "On some new inequalities for differentiable co-ordinated convex functions," *Journal of Inequalities and Applications*, vol. 2012, article 28, 2012.

[10] M. Z. Sarıkaya, E. Set, M. E. Özdemir, and S. S. Dragomir, "New some Hadamard's type inequalities for co-ordinated convex functions," *Tamsui Oxford Journal of Information and Mathematical Sciences*, vol. 28, no. 2, pp. 137–152, 2012.

[11] M. E. Özdemir, A. O. Akdemir, H. Kavurmaci, and M. Avci, "On the Simpson's inequality for co-ordinated convex functions," Classical Analysis and ODEs, http://arxiv.org/abs/1101.0075.

[12] M. E. Özdemir, H. Kavurmaci, A. O. Akdemir, and M. Avci, "Inequalities for convex and s-convex functions on $\Delta = [a, b\times c, d]$, " *Journal of Inequalities and Applications*, vol. 2012, article 20, 2012.
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