GLC AND GLC** CONTINUOUS FUNCTIONS: A CONCEPTUAL FLAW

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Abstract: The concept of generalized locally closed sets (glc-sets), GLC**-sets followed by the notion of GLC and GLC**-continuous maps was initiated by Balachandran et al. (Generalized locally closed sets and GLC-continuous functions, Indian J. pure appl. Math 27(3): 235-244, 1996). In the present work, it has been established that the collection of glc-sets and the collection of GLC**-sets, each is exactly equal to the power set P(X) of X. Consequently, any arbitrary function with any choice of domain and range turns out to be GLC and GLC**-continuous function which is not desirable from analytic point of view.

Keywords: Topological spaces, locally closed sets, glc-set, GLC**-set, GLC-continuity, GLC**-continuity.

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1. INTRODUCTION

The idea of locally closed set was introduced by Bourbaki [2] in 1966. (see also [3]). This concept of locally closed set had been used by Ganster and Reilly [4] for defining the generalized version of continuity viz. LC-irresolute, LC-continuity and sub-LC-continuity. Balachandran et al. [1] had extended the definition of locally closed sets and initiated the notion of "Generalized locally closed set", in particular, glc-set, GLC*-set and GLC**-set. Since last few decades many topologist (cf. [4], [5], [6], [7], [8], [9], [10], [11]) are trying to explore the possibility of generalizing the classical phenomenon "continuity" of the function defined in the topological space. Following this trend Balachandran et al. [1] have also defined and explored the idea of GLC-irresolute maps and GLC-continuous maps. Extending the idea of Balachandran et al. [1], Park et al. [8] have defined semi generalized locally closed sets and locally-generalized closed sets along with SGLC-continuous functions and LGLC-continuous functions respectively. (see also [9], [10], [11]). Recently, Patil et al. [10] have further extended the concept of glc-sets and introduced the notion of g*wc-lc sets and g*w*lc sets and g*w*lc sets and have applied these concepts to define relevant different types of continuous functions.

In the present paper, authors have established that the respective collection of glc-sets, and the collection of GLC**-sets generated by the topology yield precisely the power set P(X) of X. This information leads to the conclusion that the corresponding GLC and GLC**-idea of continuity is not enhancing the class of continuous functions with some relaxed conditions however all functions with arbitrary domain and range turns out to be GLC and GLC**-continuous functions which is inadequate. In view of this observation, all the extensions turned out to be superfluous.

2. PRE-REQUISITES

The following notations have been referred throughout this work:

- \((X, \tau)\) - Topological space with topology defined on the set \(X\). 
- \(cl(A)\) - Closure of \(A\) for the subset \(A\) of \(X\) with respect to \((X, \tau)\). 
- \(int(A)\) - Interior of \(A\) for the subset \(A\) of \(X\) with respect to \((X, \tau)\). 
- \(P(X)\) - Power set of \(X\).

Definition 2.1. A subset \(B\) of \((X, \tau)\) is called g-closed [12] if \(cl(B) \subseteq G\) whenever \(B \subseteq G\) for an open set \(G\) in a topological space \((X, \tau)\). A subset \(C\) of \((X, \tau)\) is called g-open if its complement \(X - C\) is g-closed.

Example 2.1. Consider a topological space \(X = \{a, b, c, d\}\) with the topology \(\tau = \{X, \phi, \{a, b, c\}\}\), \(F_X = \{X, \phi, \{d\}\}\), where \(F_X\) is the collection of closed sets in \((X, \tau)\). Let \(A = \{a, d\}\) be a subset of \(X\). There is only one open set say \(U = X\) containing \(A\). Then it is easy to check that \(cl\{a, d\} = X\) which follows by the definition that \(cl\{a, d\} = X = U = X\). Hence \(A = \{a, d\}\) is g-closed.

Remark 2.1. It is a direct consequence from the definition of g-closed sets that every open set is g-open and every closed set is g-closed but the respective converse is not true in general.

Definition 2.2. Let \(S\) be a subset of a topological space \((X, \tau)\). \(S\) is said to be generalized locally closed (glc-set) [1] if there exists g-open set \(G\) and g-closed set \(F\) such that \(S = G \cap F\). The collection of all generalized locally closed set is denoted by GLC (cf. [1]).
Example 2.2. Consider a topological space \( X = \{a, b, c, d\} \) with the topology \( \tau = \{X, \emptyset, \{b, c, d\}, \{a, c, d\}, \{c, d\}\} \), \( F_X = \{\tau, X, \{a\}, \{b\}, \{a, b\}\} \). In view of Definition 2.1, the collection of g-closed sets \( \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\} \) and the collection of g-open sets \( \{\emptyset, X, \{b, c, d\}, \{a, c, d\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}, \{c\}\} \). We now show that \( A = \{b, c\} \subseteq X \) is a glc-set.

Claim: \( A = \{U \subseteq X : U \) is g-open and \( V \) is g-closed\}.

We now consider \( U = \{b, c, d\} \) a g-open set and \( V = \{a, b, c\} \) a g-closed set. Then,
\[ U \cap V = \{b, c, d\} \cap \{a, b, c\} = \{b, c\} \]
is a glc-set. It may be verified easily that the collection of all glc-sets is exactly equal to \( P(X) \).

Remark 2.2. It is clear that every g-closed set is glc-set and every g-open set is glc-set.

Definition 2.3. Consider a subset \( S \) of a topological space. Then \( S \subseteq GLC** \) if \( S \subseteq G \cap F \) for any open set \( G \) and a g-closed set \( F \) of \( (X, \tau) \) respectively (cf.[1]).

Definition 2.4. Let \( (X, \tau) \) and \( (Y, \sigma) \) be two topological spaces. A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be GLC-continuous (resp. GLC**-continuous) if \( f^{-1}(V) \subseteq GLC \) (resp. \( f^{-1}(V) \subseteq GLC** \)) for each \( V \subseteq \sigma \) (cf.[1]).

Definition 2.5. A function \( : (X, \tau) \to (Y, \sigma) \) is said to be GLC-irresolute (resp. GLC**-irresolute) if \( f^{-1}(V) \subseteq GLC \) (resp. \( f^{-1}(V) \subseteq GLC** \)) for each \( V \subseteq \sigma \) (cf.[1]).

Claim:

\[ A = G_c \cap G_o \]  \hspace{1cm} (3.1)

where \( G_c \) and \( G_o \) are g-closed and g-open sets in \((X, \tau)\) respectively. Since, \( A \) is not g-closed, there exists at least one index \( \beta \in J \) such that

\[ A \subseteq U_\beta \text{ but } cl(A) \nsubseteq U_\beta \]  (cf. Definition 2.1) Then

Either

\[ D(A) \subseteq CU_\beta \]  \hspace{1cm} (3.2)

Or

\[ S(\neq \emptyset) \subseteq D(A) \text{ and } S \subseteq CU_\beta \text{ such that } A \cup D(A) \subseteq U_\beta \]  \hspace{1cm} (3.3)

Consider the set \( A \cup CU_\beta \)

Claim: \( A \cup CU_\beta \) is g-closed.

There exists a family \( \{U_\alpha\}_{\alpha \in J} \) of open sets such that

\[ A \cup CU_\beta \subseteq U_\alpha \text{ for } \alpha \in J \]

3. MAIN RESULT

We are now set to state the main result of this paper.

Theorem 3.1. Let \((X, \tau)\) be the topological space and GLC and GLC** be the collection of sets described in the Definition 2.2 and 2.3 respectively. Then

\[ GLC \cong GLC** \cong P(X) \]

where \( P(X) \) is the power set of \( X \).

Proof. Let \( X \) be any non empty set \( \tau = \{\emptyset, X, \{U_\alpha\}_{\alpha \in J}\} \) be the topology on \( X \). Let \( A \) be any non empty proper subset of \( X \).

The following cases have been considered:

Case 1. \( A \subseteq U_\alpha \) (\( \neq X \)) for all \( \alpha \in J \) and \( U_\alpha \in \tau \) implies \( A \subseteq X \) only. It is clear that \( cl(A) \subseteq X \). Hence, \( A \) is g-closed.

Referring Remark 2.2, we conclude that \( A \) is glc-set.

Case 2. \( A \subseteq U_\alpha \) for some \( \alpha \in J \) and \( CA \subseteq X \) but \( CA \nsubseteq U_\alpha \) for each \( \alpha \in J \) where \( C \) stands for the complement of \( A \) in \( X \). It is obvious that \( cl(CA) \subseteq X \). Hence \( CA \) is g-closed which implies that \( A \) is g-open. In view of Remark 2.2, the set \( A \) is glc again.

Case 3. \( A \subseteq U_\alpha \) for some \( \alpha \in J \) and \( CA \subseteq U_\delta \) for some \( \delta \in J \) where \( U_\alpha, U_\delta \in \tau \). Let if possible that \( A \) is neither g-open nor g-closed.
Since \( A \subseteq A \cup CU_\beta \), the collection \( \{U_\alpha^*\}_{\alpha \in J} \) is a sub-collection of \( \{U_\alpha\}_{\alpha \in J} \) of open sets containing \( A \).

i) Consider \( \beta \in J \) such that \( A \cup CU_\beta \subseteq U_\beta^* \) and \( U_\beta \subseteq U_\beta^* \).

- If \( U_\beta^* = X \), then \( cl(A \cup CU_\beta) \subseteq X \) and \( A \cup CU_\beta \) is g-closed.
- If \( U_\beta^* (\neq \emptyset, X) \), then

\[
cl(A \cup CU_\beta) = cl(A) \cup cl(CU_\beta) = A \cup D(A)_s \cup S \cup CU_\beta \quad (\because CU_\beta \text{ is closed})
\]

\[
= A \cup D(A)_s \cup CU_\beta \subseteq U_\beta^* \quad (\because CU_\beta \subseteq U_\beta^* ; U_\beta \subseteq U_\beta^* \text{ and using (3.3)})
\]

Therefore \( A \cup CU_\beta \) is g-closed.

ii) We next consider \( \alpha(\neq \beta) \in J \) such that \( A \cup CU_\beta \subseteq U_\alpha^* \) for \( U_\alpha \subseteq U_\alpha^* \) and \( cl(A) \subseteq U_\alpha \)

\[
cl(A \cup CU_\beta) = cl(A) \cup cl(CU_\beta) = cl(A) \cup CU_\beta \subseteq U_\alpha^*
\]

Thus, \( A \cup CU_\beta \) is g-closed.

**Claim:** \( A = (A \cup CU_\beta) \cap U_\beta \) where \( G_c = A \cup CU_\beta \) is g-closed and \( G_o = U_\beta \) is g-open (cf. Remark 2.1). Consider

\[
G_c \cap G_o = (A \cup CU_\beta) \cap U_\beta = (A \cap U_\beta) \cup (CU_\beta \cap U_\beta)
\]

\[
= A
\]

Hence, (3.1) holds and finally we conclude that \( A \) is glc-set. Since, \( A \) was arbitrary subset of \( X \), every subset of \( X \) is glc-set. Thus, the collection GLC is precisely equal to \( P(X) \).

Since \( G_o = U_\beta \) (open), it is direct by the Definition 2.3 that

\[
\text{GLC}^{**} \cong P(X)
\]

This completes the proof.

4. CONCLUSION

- In view of Definitions 2.4 and 2.5, each function \( f \) defined from \( (X, \tau) \) to \( (Y, \sigma) \) turns out to be GLC-continuous (irresolute) and GLC**-continuous (irresolute) which is not acceptable as a generalization of the classical concept of continuity in Topology.

- All generalizations of GLC-set and GLC**-set turned out to be stagnant and finally not desirable.

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