STOCHASTIC INTEGRALS AND EVOLUTION EQUATIONS WITH
GAUSSIAN RANDOM FIELDS

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Abstract. The paper studies stochastic integration with respect to Gaussian processes and fields. It is more convenient to work with a field than a process: by definition, a field is a collection of stochastic integrals for a class of deterministic integrands. The problem is then to extend the definition to random integrands. An orthogonal decomposition of the chaos space of the random field, combined with the Wick product, leads to the Itô-Skorokhod integral, and provides an efficient tool to study the integral, both analytically and numerically. For a Gaussian process, a natural definition of the integral follows from a canonical correspondence between random processes and a special class of random fields. Some examples of the corresponding stochastic differential equations are also considered.

1. Introduction

While stochastic integral with respect to a standard Brownian motion is a well-studied object, integration with respect to other Gaussian processes is currently an area of active research, and the fractional Brownian motion is receiving most of the attention [1, 3, 4, 5, 6, 9, 10, 15, etc.] The objective of this paper is to define and investigate stochastic integrals with respect to arbitrary Gaussian processes and fields using chaos expansion.

A generalized Gaussian field $\mathcal{X}$ over a Hilbert space $H$ is a continuous linear mapping $f \mapsto \mathcal{X}(f)$ from $H$ to the space of Gaussian random variables. The corresponding chaos space $H_{\mathcal{X}}$ is the Hilbert space of square integrable random variables that are measurable with respect to the sigma-algebra generated by $\mathcal{X}(f)$, $f \in H$. The chaos expansion is an orthogonal decomposition of $H_{\mathcal{X}}$: given an orthonormal basis $\{\xi_m, m \geq 1\}$ in $H_{\mathcal{X}}$, a square integrable $H$-valued random variable $\eta$ has the chaos expansion $\eta = \sum_{m \geq 1} \eta_m \xi_m$, with $\eta_m = \mathbb{E}(\eta \xi_m) \in H$.

The definition of a generalized Gaussian field $\mathcal{X}$ already provides the stochastic integral $\mathcal{X}(f)$ for non-random $f \in H$. As a result, given the chaos expansion of a random element $\eta$ from $H_{\mathcal{X}}$, the definition of the stochastic integral $\mathcal{X}(\eta)$ requires an extension of the linearity property of $\mathcal{X}$ to linear combinations with random coefficients. An extensions of this property using the Wick product lead to the Itô-Skorokhod integral $\mathcal{X}^\circ(f)$; see Definition 4.1 below. Under some conditions, the integral coincides with

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the divergence operator (the adjoint of the Malliavin derivative) on the chaos space $H_X$.

Even for non-random $f$ there are often several ways of computing $X(f)$. It is most convenient to work with a white noise over $H$, that is, a zero-mean generalize Gaussian field such that $E(X(f)X(g)) = (f, g)_H$ for all $f, g \in H$. It turns out that, for every zero-mean Gaussian field $X$ over $H$, there exists a different (usually larger) Hilbert space $H'$ such that $X$ is a white noise over $H'$. Moreover, the space $H'$ is uniquely determined by $X$. On the other hand, every zero-mean Gaussian field $X$ over $H$ can be written in the form $X(f) = \mathfrak{B}(K^* f)$, $f \in H$, where $K^*$ is a bounded linear operator on $H$ and $\mathfrak{B}$ is a white noise over $H$, although this white noise representation of $X$ is not necessarily unique. Thus, different white noise representations of $X$ lead to different formulas for computing $X(f)$, and the chaos expansion is an efficient way for deriving those formulas. In particular, for both deterministic and random $f$, chaos expansion provides an explicit formula for $X(f)$ in terms of the Fourier coefficients of the integrand $f$.

To define stochastic integral with respect to a Gaussian process $X = X(t)$, $t \in [0, T]$, we construct a Hilbert space $H_X$ and a white noise $\mathfrak{B}$ over $H_X$ such that $X(t) = \mathfrak{B}(\chi_t)$, where $\chi_t$ is the indicator function of the interval $[0, t]$. The space $H_X$ is uniquely determined by $X$; for example, the Wiener process on $(0, T)$ has $H_X = L^2((0, T))$. Then the equality

\begin{equation}
\int_0^T f(s) \diamond dX(s) = \mathfrak{B} \circ (f), \ f \in H_{\mathfrak{B}},
\end{equation}

defines the stochastic integral with respect to $X$.

In some situations, given a Gaussian process $X = X(t)$, $t \in [0, T]$, it is possible to find a generalized Gaussian field $\mathfrak{X}$ over a Hilbert space $H$ so that $X(t) = \mathfrak{X}(\chi_t)$. Even though $\mathfrak{X}$ is not necessarily a white noise over $H$, the resulting definition of the stochastic integral,

$$
\int_0^T f(t) \diamond dX(t) = \mathfrak{X} \circ (f),
$$

coincides with the (1.1), while the space $H$ can be more convenient for computations than the space $H_X$. For example, fractional Brownian motion with the Hurst parameter bigger then $1/2$ has a rather complicated space $H_X$, but can be represented using a generalized Gaussian field over $H = L^2((0, T))$.

The paper is organized as follows. Section 2 provides background on generalized Gaussian fields, the chaos expansion, and the Wick product. Section 3 establishes a connection between Gaussian processes and fields. Section 4 investigates the Itô-Skorokhod stochastic integral. Section 5 studies the corresponding differential equations that admit a closed form solution. Section 6 studies more general stochastic evolution equations and establishes the corresponding stochastic parabolicity conditions.
The main contributions of the paper are:

1. A connection between generalized Gaussian fields over $L^2((0,T))$ and processes that are representable in the form $\int_0^t K(t,s) dW(s)$ (Theorem 3.1);
2. Chaos expansions of the Itô-Skorokhod integral (Theorem 4.2);
3. Investigation of the equation $u(t) = 1 + \int_0^t u(s) \circ dX(s)$ for a class of Gaussian random processes $X$ (Theorem 5.1).
4. A generalization of the stochastic parabolicity condition (Theorem 6.3).

In particular, we establish the following result.

**Theorem 1.1.** Let $\mathcal{X}$ be a zero-mean generalized Gaussian field over $L^2((0,T))$ and $X(t) = \mathcal{X}(\chi_t)$. Then

(a) The solution of the Itô-Skorokhod equation

$$u(t) = 1 + \int_0^t u(s) \circ dX(s)$$

is unique in the class of square integrable $\mathcal{F}^X$-measurable processes and is given by

$$u(t) = e^{\mathcal{X}(t) - \frac{1}{2} E\mathcal{X}^2(t)}.$$

(b) The Itô-Skorokhod partial differential equation

$$du(t,x) = a \int_0^t u_{xx}(s,x) ds + \sigma u_x(t,x) \circ dX(t)$$

is well-posed in $L^2(\Omega; L^2(\mathbb{R}))$ if and only if

$$at \geq \frac{\sigma^2}{2} E\mathcal{X}^2(t).$$

2. **Generalized Gaussian Fields: A Background**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $V$, a linear topological space over the real numbers $\mathbb{R}$. Everywhere in this paper, we assume that the probability space is rich enough to support all the random elements we might need.

**Definition 2.1.** (a) A generalized random field over $V$ is a mapping $\mathcal{X} : \Omega \times V \rightarrow \mathbb{R}$ with the following properties:

1. For every $f \in V$, $\mathcal{X}(f) = \mathcal{X}(\cdot, f)$ is a random variable;
2. For every $\alpha, \beta \in \mathbb{R}$ and $f, g \in V$, $\mathcal{X}(\alpha f + \beta g) = \alpha \mathcal{X}(f) + \beta \mathcal{X}(g)$;
3. If $\lim_{n \rightarrow \infty} f_n = f$ in the topology of $V$, then $\lim_{n \rightarrow \infty} \mathcal{X}(f_n) = \mathcal{X}(f)$ in probability.

(b) A generalized random field $\mathcal{X}$ is called

- zero-mean, if $E\mathcal{X}(f) = 0$ for all $f \in V$;
- Gaussian, if the random variable $\mathcal{X}(f)$ is Gaussian for every $f \in V$. 
For example, if \( W = W(t), 0 \leq t \leq T \), is a standard Brownian motion on \((\Omega, \mathcal{F}, \mathbb{P})\), then \( \mathcal{X}(f) = \int_0^T f(t)dW(t) \) is a zero-mean generalized Gaussian field over \( L_2((0,T)) \); note that

\[
E|\mathcal{X}(f_n) - \mathcal{X}(f)|^2 = \int_0^T |f_n(t) - f(t)|^2 dt.
\]

More generally, if \( \mathcal{M} \) is a bounded linear operator on \( L_2((0,T)) \), then

\[
\mathcal{X}(f) = \int_0^T (\mathcal{M}f)(t)dW(t)
\]

is a zero-mean generalized Gaussian field over \( L_2((0,T)) \). In fact, by Theorem 2.6(b) below, every zero-mean generalized Gaussian field over \( L_2((0,T)) \) can be represented in the form (2.2) with suitable \( \mathcal{M} \) and \( W \). We will also see that the fractional Brownian motion on \([0,T]\) with Hurst parameter bigger than \(1/2\) can be interpreted as a zero-mean generalized Gaussian field over \( L_2((0,T)) \).

Let \( H \) be a real Hilbert space with inner product \((\cdot, \cdot)_H\) and norm \( \| \cdot \|_H = \sqrt{(\cdot, \cdot)_H} \).

The following result is a direct consequence of the Riesz Representation Theorem.

**Theorem 2.2.** For every zero-mean generalized Gaussian field \( \mathcal{X} \) over a Hilbert space \( H \), there exists a unique bounded linear self-adjoint operator \( \mathcal{R} \) on \( H \) such that

\[
E(\mathcal{X}(f)\mathcal{X}(g)) = (\mathcal{R}f, g)_H, \ f, g \in H.
\]

**Definition 2.3.** (a) The operator \( \mathcal{R} \) from Theorem 2.2 is called the covariance operator of \( \mathcal{X} \). (b) The field \( \mathcal{X} \) is called non-degenerate if \( \mathcal{R} \) is one-to-one. (c) A white noise over \( H \) is a zero-mean generalized Gaussian field with the covariance operator equal to the identity operator.

Standard arguments from functional analysis lead to the following result.

**Theorem 2.4.** (a) For every zero-mean generalized Gaussian field \( \mathcal{X} \) over a Hilbert space \( H \), there exist a bounded linear operator \( \mathcal{K} \) on \( H \) and a white noise \( \mathcal{B} \) over \( H \) so that \( \mathcal{K}\mathcal{K}^* \) is the covariance operator of \( \mathcal{X} \) and, for every \( f \in H \),

\[
\mathcal{X}(f) = \mathcal{B}(\mathcal{K}^*f);
\]

as usual, \( \mathcal{K}^* \) denotes the adjoint of \( \mathcal{K} \).

(b) For every zero-mean non-degenerate generalized Gaussian field \( \mathcal{X} \) over a Hilbert space \( H \), there exists a Hilbert space \( H_{\mathcal{R}} \) such that \( H \) is continuously embedded into \( H_{\mathcal{R}} \) and \( \mathcal{X} \) extends to a white noise over \( H_{\mathcal{R}} \).

**Remark 2.5.** (a) If \( \mathcal{X} \) is non-degenerate and \( \mathcal{R} : H \rightarrow H \) is onto, then \( \mathcal{R} \) has a bounded inverse and \( H_{\mathcal{R}} = H \). (b) If \( \ker \mathcal{R} \) is non-trivial, then we can define \( H_{\mathcal{R}} \) as the closure of the factor space \( H/\ker(\mathcal{R}) \) with respect to the inner product \((\overline{f}, \overline{g})_{H_{\mathcal{R}}} = (\mathcal{R}f, g)_H\), where \( \overline{f} \) is the equivalence class of \( f \) in \( H/\ker(\mathcal{R}) \). Direct computations show that the generalized random field \( \mathcal{B} \) over \( H/\ker(\mathcal{R}) \), defined by

\[
\mathcal{B}(\overline{f}) = \mathcal{X}(f), \ f \in H,
\]

extends to a white noise over \( H_{\mathcal{R}} \).
We will now discuss several connections between generalized Gaussian fields and Gaussian processes.

Denote by $\chi_t = \chi_t(s)$ the indicator function of the interval $[0, t]$:  
\begin{equation}
\chi_t(s) = \begin{cases} 
1, & 0 \leq s \leq t; \\
0, & \text{otherwise}. 
\end{cases}
\end{equation}

With this definition, $\chi_{t_2}(s) - \chi_{t_1}(s)$ is the indicator function of the interval $(t_1, t_2]$, $t_2 > t_1$.

**Theorem 2.6.** (a) If $B$ is a white noise over $L_2(\mathbb{R})$, then $B(t) = B(\chi_t)$ is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ and, for every $f \in L_2(\mathbb{R})$, we have

\begin{equation}
B(f) = \int_{\mathbb{R}} f(s) dB(s).
\end{equation}

(b) For every zero-mean non-degenerate generalized Gaussian field $\mathcal{X}$ over $L_2((0, T))$, there exist a bounded linear operator $\mathcal{K}^*$ on $L_2(\mathbb{R})$ and a standard Brownian motion $W = W(t)$ such that, for every $f \in L_2(\mathbb{R})$,

\begin{equation}
\mathcal{X}(f) = \int_{\mathbb{R}} (\mathcal{K}^* f)(s) dW(s).
\end{equation}

**Proof.** (a) Direct computations.

(b) This follows from part (a) and from Theorem 2.4. \qed

**Remark 2.7.** While beyond the scope of this paper, a similar result is true for multi-parameter processes as well. For example, if $\mathcal{X}$ is a generalized Gaussian field over $L_2(\mathbb{R}^2)$, then the same arguments show that (2.7) holds with a Brownian sheet $W$.

From now on, we assume that the space $\mathcal{H}$ is separable, the field $\mathcal{X}$ is non-degenerate, and $\mathcal{F} = \mathcal{F}^\mathcal{X}$, the sigma-algebra generated by the random variables $\mathcal{X}(f)$, $f \in \mathcal{H}$.

**Definition 2.8.** (a) The **chaos space** generated by $\mathcal{X}$ is the collection of all square-integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. This chaos space will be denoted by $\mathcal{H}_\mathcal{X}$.

(b) The **first chaos space** generated by $\mathcal{X}$ is the sub-space of $\mathcal{H}_\mathcal{X}$, consisting of the random variables $\mathcal{X}(f)$, $f \in \mathcal{H}$. The first chaos space will be denoted by $\mathcal{H}_\mathcal{X}^{(1)}$.

It follows that $\mathcal{H}_\mathcal{X}$ is a Hilbert space with inner product $(\xi, \eta)_{\mathcal{H}_\mathcal{X}} = \mathbb{E}(\xi \eta)$, and $\mathcal{H}_\mathcal{X}^{(1)}$ is a Hilbert sub-space of $\mathcal{H}_\mathcal{X}$. Moreover, the space $\mathcal{H}_\mathcal{X}^{(1)}$ is separable: if $\{\tilde{f}_1, \tilde{f}_2, \ldots\}$ is a dense countable set in $\mathcal{H}$, then the collection of all finite linear combinations of $\mathcal{X}(\tilde{f}_i)$ with rational coefficients is a dense countable set in $\mathcal{H}_\mathcal{X}^{(1)}$.

Our next objective is to show how an orthonormal basis in $\mathcal{H}_\mathcal{X}^{(1)}$ leads to an orthonormal basis in $\mathcal{H}_\mathcal{X}$. We will need some additional constructions.

For an integer $n \geq 0$, the $n$-th **Hermite polynomial** $H_n = H_n(t)$ is defined by

\begin{equation}
H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}.
\end{equation}
Next, denote by $\mathcal{I}$ the collection of multi-indices, that is, sequences $\alpha = \{\alpha_k, k \geq 1\} = \{\alpha_1, \alpha_2, \ldots\}$ with the following properties:

- each $\alpha_k$ is a non-negative integer: $\alpha_k \in \{0, 1, 2, \ldots\}$.
- only finitely many of $\alpha_k$ are non-zero: $|\alpha| := \sum_{k=1}^{\infty} \alpha_k < \infty$.

The set $\mathcal{I}$ is countable, being a countable union of countable sets. By $\epsilon_n$ we denote the multi-index $\alpha = \{\alpha_k, k \geq 1\}$ with $\alpha_k = 1$ if $n = k$ and $\alpha_k = 0$ otherwise. For $\alpha \in \mathcal{I}$, we will use the notation $\alpha! := \alpha_1! \alpha_2! \cdots$. Let $\{\xi_1, \xi_2, \ldots\}$ be an ordered countable collection of random variables. For $\alpha \in \mathcal{I}$ define random variables $\xi_\alpha$ as follows:

$$
(2.9) \quad \xi_\alpha = \prod_{k \geq 1} \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}},
$$

where $H_{\alpha_k}$ is $\alpha_k$-th Hermite polynomial (2.8).

**Theorem 2.9.** Let $\{\xi_1, \xi_2, \ldots\}$ be an orthonormal basis in $\mathbb{H}_1^1$. Then the collection $\Xi = \{\xi_\alpha, \alpha \in \mathcal{I}\}$ is an orthonormal basis in $\mathbb{H}_X$: for every $\eta \in \mathbb{H}_X$ we have

$$
\eta = \sum_{\alpha \in \mathcal{I}} \left( \mathbb{E} (\eta \xi_\alpha) \right) \xi_\alpha, \quad \mathbb{E} \eta^2 = \sum_{\alpha \in \mathcal{I}} \left( \mathbb{E} (\eta \xi_\alpha) \right)^2.
$$

**Proof.** See [13, Theorem 2.1].

**Corollary 2.10.** Let $\mathcal{B}$ be a white noise over a separable Hilbert space $\mathcal{H}$ and let $\{m_1, m_2, \ldots\}$ be an orthonormal basis in $\mathcal{H}$. Then $\{\xi_k = \mathcal{B}(m_k), k \geq 1\}$ is an orthonormal basis in $\mathbb{H}_1^{(1)}(\mathcal{B})$ and, for every $f \in \mathcal{H}$,

$$
(2.10) \quad \mathcal{B}(f) = \sum_{k=1}^{\infty} (f, m_k)_{\mathcal{H}} \mathcal{B}(m_k).
$$

**Proof.** Note that $\mathbb{E}(\xi_k \xi_n) = \mathbb{E}(\mathcal{B}(m_k) \mathcal{B}(m_n)) = (m_k, m_n)_{\mathcal{H}}$, so the system $\{\xi_k, k \geq 1\}$ is orthonormal in $\mathbb{H}_1^{(1)}(\mathcal{B})$ if and only if $\{m_k, k \geq 1\}$ is orthonormal in $\mathcal{H}$. If $\xi \in \mathbb{H}_1^{(1)}(\mathcal{B})$, then $\xi = \mathcal{B}(f)$ for some $f \in \mathcal{H}$. By assumption, $f = \sum_{k=1}^{\infty} (f, m_k)_{\mathcal{H}} m_k$, which implies (2.10) and completes the proof.

If $\mathcal{H} = L_2((0, T))$ and $f = \chi_t$, then (2.10) becomes a familiar representation of the standard Brownian motion on $[0, T]$:

$$
(2.11) \quad W(t) = \sum_{k=1}^{\infty} \left( \int_0^t m_k(s) ds \right) \left( \int_0^T m_k(s) dW(s) \right).
$$

Now, let $\mathcal{X}$ be a zero-mean generalized Gaussian field over a separable Hilbert space $\mathcal{H}$. By (2.10) and Theorem 2.4(a), we can take a white noise representation of $\mathcal{X}$, $\mathcal{X}(f) = \mathcal{B}(\mathcal{K}^* f)$, and get an expansion of $\mathcal{X}(f)$ using an orthonormal basis in $\mathcal{H}$:

$$
(2.12) \quad \mathcal{X}(f) = \sum_{k=1}^{\infty} (\mathcal{K}^* f, m_k)_{\mathcal{H}} \mathcal{B}(m_k).
$$
Alternatively, by Theorem 2.4(b) \( \mathcal{X} \) is a white noise over the space \( \mathbf{H}_\mathcal{R} \) corresponding to the covariance operator \( \mathcal{R} \) of \( \mathcal{X} \). If \( \{\mathbf{m}_k, k \geq 1\} \) is an orthonormal basis in \( \mathbf{H}_\mathcal{R} \), then we have an equivalent expansion of \( \mathcal{X}(f) \):

\[
\mathcal{X}(f) = \sum_{k=1}^{\infty} (\mathcal{R}f, \mathbf{m}_k)_\mathbf{H} \mathcal{X} (\mathbf{m}_k).
\]

We conclude the section with a brief discussion of the Wick product, as we will need this product to define \( \mathcal{X}(f) \) for random \( f \). For more details, see [8].

The Wick product of two arbitrary elements of \( \mathbb{H}_\mathcal{X} \) can be computed by the formula

\[
\left( \sum_{\alpha \in I} c_\alpha \xi_\alpha \right) \cdot \left( \sum_{\beta \in I} d_\beta \xi_\beta \right) = \sum_{\alpha, \beta \in I} c_\alpha d_\beta \sqrt{\frac{(\alpha + \beta)!}{\alpha! \beta!}} \xi_{\alpha+\beta},
\]

where \( \alpha + \beta = \{\alpha_k + \beta_k, k \geq 1\} \) and \( \alpha! = \prod_{k \geq 1} \alpha_k! = \alpha_1! \alpha_2! \alpha_3! \cdots \). In general, there is no guarantee that, for \( \xi, \eta \in \mathbb{H}_\mathcal{X} \), the Wick product \( \xi \cdot \eta \) belongs to \( \mathbb{H}_\mathcal{X} \).

Similar to ordinary powers, we define Wick powers of a random variable \( \eta \in \mathbb{H}_\mathcal{X} \):

\[
\eta^{\circ n} = \eta \cdot \cdots \cdot \eta.
\]

Replacing ordinary powers with Wick powers in a Taylor series for a function \( f \) leads to the notion of a Wick function \( f^{\circ} \). For example, the Wick exponential \( e^{\scriptstyle \eta^{\circ}} \) is defined by

\[
e^{\scriptstyle \eta^{\circ}} = \sum_{n=1}^{\infty} \frac{\eta^{\circ n}}{n!}
\]

and satisfies \( e^{\scriptstyle (\xi + \eta)} = e^{\scriptstyle \xi} \cdot e^{\scriptstyle \eta^{\circ}} \). If \( \eta \in \mathbb{H}_\mathcal{X}^{(1)} \), then direct computations show that

\[
e^{\scriptstyle \eta^{\circ}} = e^{\eta - \frac{1}{2} \eta\eta^{\circ}}.
\]

3. Connection Between Processes and Fields

Given a zero-mean generalized Gaussian field \( \mathcal{X} \) over \( L_2((0, T)) \), we define its associated process \( X(t), t > 0 \), by

\[
X(t) = \mathcal{X}(\chi_t).
\]

Clearly, \( X(t) \) is a Gaussian process. Let \( \mathcal{K}^* \) be the operator from Theorem 2.6 and define the kernel function \( K_X = K_X(t, s) \) by

\[
K_X(t, s) = (\mathcal{K}^* \chi_t)(s).
\]

It then follows from (2.7) that

\[
X(t) = \int_0^T K_X(t, s)dW(s)
\]

for some standard Brownian motion \( W \). Let us emphasize that, while every kernel \( K(t, s) \) with minimal integrability properties can define a Gaussian process according to (3.3), only a process associated with a generalized field over \( L_2((0, T)) \) has a kernel defined according to (3.2), where \( \mathcal{K}^* \) is a bounded operator on \( L_2((0, T)) \). Recall that the definition of a generalized field (Definition 2.1) includes a certain continuity.
property, and this property translates into addition structure of the kernel function in the representation of the associated process.

Now assume that we are given a Gaussian process $X(t)$ defined by (3.3) with some kernel $K_X(t, s)$. We are not assuming that $K_X$ has the form (3.2). In what follows, we discuss sufficient conditions on $K_X(t, s)$ ensuring that $X(t)$ is the associated process of a generalized Gaussian field $\mathfrak{X}$ over $L_2((0, T))$, that is, representation (3.2) does indeed hold with some bounded linear operator $K^*$ on $L_2((0, T))$. For that, we need to recover the operator $K^*$ from the kernel $K_X(t, s)$. By linearity, if (3.2) holds and if $s_0 < s_1 < \ldots < s_N$ are points in $[0, T]$ and

$$f(s) = \sum_{k=0}^{N-1} a_k (\chi_{s_{k+1}}(s) - \chi_{s_k}(s))$$

is a step function, then

$$K^* f(s) = \sum_{k=0}^{N-1} a_k (K_X(s_{k+1}, s) - K_X(s_k, s)).$$

To extend (3.5) to continuous functions $f$, the kernel $K_X(t, s)$ must have bounded variation as a function of $t$; if this is indeed the case, then (3.5) implies that, for every smooth compactly supported function $f$ on $[0, T]$,

$$K^* f(s) = \int_0^T f(t) K_X(dt, s).$$

It now follows that if the partial derivative $\partial K_X(t, s)/\partial t$ exists and is square integrable over $[0, T] \times [0, T]$, then $K^*$, as defined by (3.6), extends to a bounded linear operator on $L_2((0, T))$.

Let us now assume that the process $X(t)$ defined by (3.3) is non-anticipating, i.e. adapted to the filtration $\{\mathcal{F}_t^W, \ 0 \leq t \leq T\}$ generated by the Brownian motion $W(s)$. Then $K_X(t, s) = 0$ for $s > t$ and (3.3) becomes

$$X(t) = \int_0^t K_X(t, s)dW(s).$$

Note that in this case we have

$$\mathbb{E}(X(t)X(s)) = \int_0^{\min(t,s)} K_X(t, \tau)K_X(s, \tau)d\tau.$$ 

For such processes, formula (3.5) and the conditions for the continuity of the corresponding operator $K^*$ must be modified as follows.

Theorem 3.1. Assume that the process $X(t)$ defined by (3.3) is non-anticipating.

(a) If $f$ is a step function (3.4), then

$$K^* f(s) = \sum_{i=0}^{N-1} \left( \chi_{s_{i+1}}(s) - \chi_{s_i}(s) \right) \left( a_i K_X(s_{i+1}, s) \right) + \sum_{k=i+1}^{N-1} a_k \left( K_X(s_{k+1}, s) - K_X(s_k, s) \right).$$
(b) If the function \( K_X(\cdot, s) \) has bounded variation for every \( s \) and \( \lim_{\delta \to 0, \delta > 0} K_X(s + \delta, s) = K_X(s^+, s) \) exists for all \( s \in (0, T) \), then

\[
(3.10) \quad K^* f(s) = K_X(s^+, s) f(s) + \int_s^T f(t) K_X(dt, s)
\]

for every continuous on \([0, T]\) function \( f \).

(c) If the function \( K_X(t, s) \) has the following properties

1. \( K_X \) is continuous and non-negative for \( 0 \leq s \leq t \leq T \), and \( \sup_{0 < t < T} K_X(t, s) \leq K_0 \);
2. For every fixed \( s_0 \), the function \( K_X(t, s_0) \) is monotone as a function of \( t \) and the partial derivative \( K^{(1)}(t, s) = \partial K_X(t, s)/\partial t \) exists for all \( 0 < s < t < T \);
3. There exists a number \( K_1 = K_1(T) \) such that

\[
(3.11) \quad \sup_{0 < t < T} \int_0^t K_X(T, s)|K^{(1)}(t, s)|ds \leq K_1^2,
\]

then the corresponding operator \( K^* \) defined by equation (3.10) is bounded on \( L^2((0, T)) \) and the operator norm \( \|K^*\| \) of \( K^* \) satisfies

\[
(3.12) \quad \|K^*\|^2 \leq (K_0 + K_1)^2.
\]

**Proof.** (a) By assumption, \( K_X(t, s) = 0 \) for \( s > t \). Fix an \( s \) such that \( s \in (s_j, s_{j+1}] \) for some \( j = 0, \ldots, N - 1 \). By (3.9) we have for this value of \( s \)

\[
K^* f(s) = \sum_{k=0}^{N-1} a_k \left( K_X(s_{k+1}, s) - K_X(s_k, s) \right) = a_j K_X(s_{j+1}, s) + \sum_{k=j+1}^{N-1} a_k \left( K_X(s_{k+1}, s) - K_X(s_k, s) \right).
\]

Since \( \chi_{s_{k+1}}(s) - \chi_{s_k}(s) \) is the indicator function of the interval \( (s_k, s_{k+1}] \), (3.9) follows.

(b) Under the additional assumptions on the kernel \( K_X \), (3.10) follows from (3.9) after passing to the limit \( \max_{j=0, \ldots, N-1} |s_{j+1} - s_j| \to 0 \).

(c) With no loss of generality, we can assume that \( K_X(s^+, s) = 0 \) and \( K^{(1)} \geq 0 \); otherwise, we replace \( K_X(t, s) \) with either \( K_X(s^+, s) - K_X(t, s) \) or \( K_X(t, s) - K_X(s^+, s) \). Let \( g \) be a smooth compactly supported function on \((0, T)\). It follows from (3.10) that

\[
K^* g(s) = \int_s^T K^{(1)}(\tau, s) g(\tau) d\tau.
\]
Then we use the Cauchy-Schwartz inequality and the properties of \( K^{(1)} \):

\[
\int_0^T \left| \int_s^T K^{(1)}(\tau, s) g(\tau) d\tau \right|^2 ds = \int_0^T \left| \int_s^T \left[ K^{(1)}(\tau, s) \right]^{1/2} \left[ K^{(1)}(\tau, s) \right]^{1/2} g(\tau) d\tau \right|^2 ds \\
\leq \int_0^T \int_s^T K^{(1)}(\tau, s) d\tau \int_s^T K^{(1)}(\tau, s) g^2(\tau) d\tau ds \\
\leq \int_0^T \left( K(\tau, s) - K(s, s) \right) \int_s^T K^{(1)}(\tau, s) g^2(\tau) d\tau ds \\
\leq \int_0^T \left( \int_0^\tau K(\tau, s) K^{(1)}(\tau, s) ds \right) g^2(\tau) d\tau \leq K_1^2(T) \| g \|^2_{L_2((0, T))}.
\]

Here are several examples of processes covered by part (c) of Theorem 3.1.

**Example 3.2.** Assume that \( K_X(s^+, s) = 0 \) and \( K^{(1)}(t, s) \geq 0 \). Let \( R(t, s) = \mathbb{E}(X(t)X(s)) \). By (3.8),

\[
\frac{\partial R(T, t)}{\partial t} = \int_0^t K_X(T, s) K^{(1)}(t, s) ds
\]

and therefore

\[
(3.13) \quad K_1^2(T) = \sup_{0 < t < T} \frac{\partial R(T, t)}{\partial t}.
\]

In particular, for the fractional Brownian motion \( W^H \) on \([0, T]\) with the Hurst parameter \( H > 1/2 \),

\[
R(T, t) = \frac{1}{2} \left( T^{2H} + t^{2H} - (T - t)^{2H} \right),
\]

and (see Nualart [14, Section 5.1.3]) \( W^H \) has representation (3.7) with

\[
K_X(t, s) = C_H \left( H - \frac{1}{2} \right) s^{\frac{1}{2} - H} \int_s^t (\tau - s)^{H - \frac{1}{2}} \tau^{H - \frac{1}{2}} d\tau,
\]

where

\[
C_H = \left( \frac{2H \Gamma \left( \frac{3}{2} - H \right)}{\Gamma \left( H + \frac{1}{2} \right) \Gamma(2 - 2H)} \right)^{\frac{1}{2}}
\]

and \( \Gamma \) is the Gamma-function. In this case, \( K_X(s^+, s) = 0 \) and \( K^{(1)}(t, s) \geq 0 \). By (3.13),

\[
(3.14) \quad K_1^2(T) = 2H 2^{1-2H} T^{2H-1}.
\]

The bound \( K_1(T) \) is asymptotically optimal: if \( H = 1/2 \), which corresponds to the standard Brownian motion, the right-hand side of (3.14) is equal to 1.

**Example 3.3.** Assume that

\[
K_X(t, s) = \rho((t - s)\alpha)^\alpha \chi_t(s), \quad \alpha > 0,
\]

where \( \rho \) is a non-negative, monotone, continuously differentiable function on \([0, T]\).
If \( \rho \) is non-increasing, then \( K_0 = \rho(0) \) and \( K_1^2(T) = \rho(0)(\rho(0) - \rho(T^\alpha)) \). In particular, consider the **stable Ornstein-Uhlenbeck process**

\[
(3.15) \quad dX(t) = -bX(t)dt + dW(t), \quad X(0) = 0, \ b > 0,
\]

so that \( X(t) = \int_0^t e^{-b(t-s)}dW(s) \) and \( K_X(t, s) = e^{-b(t-s)} \). For this process, \( \|K^*\| \leq 1 + \sqrt{1 - e^{-bt}} \). This bound is asymptotically optimal: as \( b \to 0 \), the process becomes \( W \), and the upper bound on \( \|K^*\| \) becomes 1.

If \( \rho \) is non-decreasing, then \( K_0 = \rho(0) \) and \( K_1^2(T) = \rho(T^\alpha)(\rho(T^\alpha) - \rho(0)) \). In particular, consider the **unstable Ornstein-Uhlenbeck process**

\[
(3.16) \quad dX(t) = bX(t)dt + dW(t), \quad X(0) = 0, \ b > 0,
\]

so that \( X(t) = \int_0^t e^{b(t-s)}dW(s) \) and \( K_X(t, s) = e^{b(t-s)} \). For this process, \( \|K^*\| \leq 1 + \sqrt{e^{bt}(e^{bt} - 1)} \). This bound is asymptotically optimal: as \( b \to 0 \), the process becomes \( W \), and the upper bound on \( \|K^*\| \) becomes 1.

The following theorem establishes a connection between a zero-mean Gaussian process and white noise.

**Theorem 3.4.** For every zero-mean Gaussian process \( X = X(t), \ t \in [0, T] \) with \( X(0) = 0 \) and the covariance function \( R(t, s) = \mathbb{E}(X(t)X(s)) \), there exist

1. a Hilbert space \( \mathcal{H}_R \) containing the indicator functions \( \chi_t \);
2. a white noise \( \mathcal{B} \) over \( \mathcal{H}_R \)

such that \( X(t) = \mathcal{B}(\chi_t) \).

**Proof.** Let the Hilbert space \( \mathcal{H}_R \) be the closure of the set of the step functions with respect to the inner product

\[
(\chi_{t_1}, \chi_{t_2})_{\mathcal{H}_R} = R(t_1, t_2).
\]

Define a generalized Gaussian field \( \mathcal{B} \) over \( \mathcal{H}_R \) by setting

\[
(3.17) \quad \mathcal{B}(\chi_t) = X(t),
\]

and then extending by linearity and continuity to all of \( \mathcal{H}_R \). With this definition, \( \mathcal{B} \) is a white noise over \( \mathcal{H}_R \). \( \square \)

By analogy with (2.3), if \( \mathcal{X} \) is a generalized Gaussian field over a Hilbert space \( \mathcal{H} \) containing \( \chi_t, \ t \in [0, T] \) and \( X(t) \) is the associated process of \( \mathcal{X} \), then \( \int_0^T f(s)dX(s) \) can be an alternative notation for \( \mathcal{X}(f) \).

If \( X(t) \) is the associated process of a zero-mean non-degenerate generalized Gaussian field \( \mathcal{X} \) over \( \mathcal{H} = L_2((0, T)) \), and \( R \) is the covariance operator of \( \mathcal{X} \), then \( R(t, s) = (R\chi_{t_1}, \chi_{t_2})_{L_2((0, T))} \) and the space \( \mathcal{H}_R \) coincides with \( \mathcal{H}_R \) from Theorem 2.4.

If \( R(t, s) = \min(t, s) \), then \( \langle \chi_{t_1}, \chi_{t_2}\rangle_{\mathcal{H}_R} = \langle \chi_{t_1}, \chi_{t_2}\rangle_{L_2((0, T))} \). That is, for the Wiener process, \( \mathcal{H}_R = L_2((0, T)) \). For the fractional Brownian motion, the space \( \mathcal{H}_R \) can be characterized using fractional derivative operators [13]. For a general Gaussian process \( X \) with covariance function \( R \), an explicit characterization of the space \( \mathcal{H}_R \) is impossible.
4. Stochastic Integration

In the definition of a generalized random field $\mathcal{X}$ over a Hilbert space $\mathbf{H}$, we consider random variables $\mathcal{X}(f)$ for non-random $f \in \mathbf{H}$. In this section, we define $\mathcal{X}(\eta)$ for $\mathbf{H}$-valued random elements $\eta$.

One possible way to proceed is to write the chaos expansion of random variables $X$ over a Hilbert space $\mathbf{H}$, and then define $\mathcal{X}(\eta)$ as a corresponding linear combination of $\mathcal{X}(\eta)$.

In the case $\mathbf{H} = L_2((0, T))$, another possibility is to take a partition $0 = t_0 < t_1 < \ldots < t_N = T$ of the interval $[0, T]$ and approximate $\eta(t)$ with a sum $\sum_{i=1}^{N} \eta(t_i^*) (\chi_{t_i} - \chi_{t_{i-1}})$, where $t_i^* \in [t_{i-1}, t_i]$, and then approximate $\mathcal{X}(\eta)$ with the corresponding linear combination of $\mathcal{X}(\chi_{t_i}) - \mathcal{X}(\chi_{t_{i-1}})$.

Either way, we need to address the following question. By definition, if $\alpha, \beta$ are real numbers and $f, g$ are elements of $\mathbf{H}$, then $\mathcal{X}(\alpha f + \beta g) = \alpha \mathcal{X}(f) + \beta \mathcal{X}(g)$. But what if $\alpha$ and $\beta$ are random variables? One possibility would be to keep the same linearity.

In the case $\mathbf{H} = L_2((0, T))$ this would imply

$$\mathcal{X} \left( \sum_{i=1}^{N} \eta(t_i^*) (\chi_{t_i} - \chi_{t_{i-1}}) \right) = \sum_{i=1}^{N} \eta(t_i^*) (X(t_i) - X(t_{i-1})), \tag{4.1}$$

where $X(t) = \mathcal{X}(\chi_t)$ is the associated process of $\mathcal{X}$. While natural, this extension of the linearity property can lead to ambiguities in the definition of the corresponding stochastic integral. Indeed, let $\mathcal{B}$ be a white noise over $L_2((0, T))$, and let $\eta(t) = \mathcal{B}(\chi_t)$. By Theorem 2.6, $\eta$ is a standard Brownian motion $W$ and, as we know, the limit of the sum $\sum_{i=1}^{N} W(t_i^*) (W(t_{i+1} - W(t_i))$ depends on the location of the points $t_i^*$.

Let us now consider an alternative to (4.1):

$$\mathcal{X} \left( \sum_{i=1}^{N} \eta(t_i^*) (\chi_{t_i} - \chi_{t_{i-1}}) \right) = \sum_{i=1}^{N} \eta(t_i^*) \diamond (X(t_i) - X(t_{i-1})). \tag{4.2}$$

This time, if we take $\eta(t) = X(t) = W(t)$, a standard Brownian motion, then the limit is the Itô integral $\int_{0}^{T} W(t) dW(t)$; it is equal to $W^2(T) - T)/2 = (W(T))^2/2$ and does not depend on the location of the points $t_i^*$.

Accordingly, we adopt the following convention: if $\mathcal{X}$ is a generalized Gaussian field over $\mathbf{H}$, then, for every $f, g \in \mathbf{H}$ and all random variables $\xi, \eta$,

$$\mathcal{X}(\xi f + \eta g) = \xi \diamond \mathcal{X}(f) + \eta \diamond \mathcal{X}(g);$$

recall that, by assumption, the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is such that $\mathcal{F}$ is generated by $\mathcal{X}$. With this convention, we proceed with the definition of the stochastic integral $\mathcal{X}(\eta)$ for $\mathbf{H}$-valued random elements $\eta$ using the chaos expansion.

There are at least three advantages of the chaos approach over time partitioning:

1. generality: spaces other than $L_2((0, T))$ can be considered;
2. possibility to use weighted chaos spaces, which eliminates many questions about convergence;
3. (2) possibility to use weighted chaos spaces, which eliminates many questions about convergence;
computational efficiency: the Wick product must only be computed for the basis elements $\xi_\alpha$.

Because we are not restricted with the choice of $H$, we will assume that $X = B$, a white noise over $H$. Let $\{m_k, k \geq 1\}$ be an orthonormal basis in $H$. Define $\xi_k = B(m_k)$ and $\xi_\alpha, \alpha \in I$, according to (2.9).

By Theorem 2.9, every $H$-valued random element $\eta$ with $E\|\eta\|^2_H < \infty$ has chaos expansion

$$\eta = \sum_{\alpha \in J} \eta_\alpha \xi_\alpha, \eta_\alpha = E(\eta \xi_\alpha) \in H.$$  

Definition 4.1. Let $\eta$ be an $H$-valued random element with chaos expansion (4.3). The Itô-Skorokhod stochastic integral of $\eta$ with respect to $B$ is

$$\mathcal{B}^\circ(\eta) = \sum_{\alpha \in I} B(\eta_\alpha) \diamond \xi_\alpha,$$

where $\diamond$ is the Wick product.

Since every generalized Gaussian field and every Gaussian process can be represented using a white noise over a suitable Hilbert space, formula (4.4) defines stochastic integral with respect to any Gaussian process or field. We will see below that this formula also provides a chaos expansion of the integral in terms of the chaos expansion of the integrand; note that (4.4) is not a chaos expansion in the sense of (4.3). The two immediate question that are raised by the above definition and will be discussed below are (a) the convergence of the series, and (b) the dependence of the integral on the choice of the basis in $H$.

We start by deriving the chaos expansion of the integrals without investigating the question of convergence.

Theorem 4.2. Let $\eta$ be an $H$-valued random element with chaos expansion (4.3), and assume that

$$\eta_\alpha = \sum_{k=1}^\infty \eta_{\alpha,k} m_k.$$  

Then

$$\mathcal{B}^\circ(\eta) = \sum_{\alpha \in I} \left( \sum_{k=1}^\infty \sqrt{\alpha_k} \eta_{\alpha,k} m_k \right) \xi_\alpha.$$  

Proof. By (4.5) and linearity, keeping in mind that both $\eta_\alpha$ and $m_k$ are non-random,

$$\mathcal{B}(\eta_\alpha) = \sum_{k=1}^\infty \eta_{\alpha,k} m_k = \sum_{k=1}^\infty \eta_{\alpha,k} \xi_k.$$  

Therefore,

$$\mathcal{B}^\circ(\eta) = \sum_{\alpha \in I} \sum_{k=1}^\infty \eta_{\alpha,k} \xi_k \diamond \xi_\alpha = \sum_{\alpha \in I} \sum_{k=1}^\infty \sqrt{\alpha_k + 1} \eta_{\alpha,k} \xi_{\alpha+\epsilon_k};$$
recall that $\epsilon_k$ is the multi-index with the only non-zero entry, equal to one, at position $k$. By shifting the summation index, we get (4.6). Note that, for every $\alpha \in \mathcal{I}$, the inner sum in (4.6) contains finitely many non-zero terms.

Now, let us address the questions of convergence and independence of basis. The Cauchy-Schwartz inequality implies that if

$$\sum_{\alpha \in \mathcal{I}} |\alpha| \|\eta_{\alpha}\|_H^2 < \infty,$$

then $\mathfrak{B} \circ (\eta) \in \mathbb{H}_3$. Further examination of (4.6) shows that, for every $\eta$ satisfying (4.8), $\mathfrak{B} \circ (\eta) = \delta(\eta)$, where $\delta$ is the divergence operator (adjoint of the Malliavin derivative), and therefore $\mathfrak{B} \circ (\eta)$ does not depend on any arbitrary choices, such as the basis in $H$; for details, see Nualart [14] or Watanabe [16]. In particular, if $H = L^2((0, T))$, then $\mathfrak{B} \circ (\eta)$ is the It\'o-Skorokhod integral of $\eta$ in the sense of the Malliavin calculus. On the other hand, (4.6), if considered as a formal series, allows the extension of $\mathfrak{B} \circ$ to weighted chaos spaces, similar to those considered in [11, 12]; we leave this extension to an interested reader.

**Remark 4.3.** It is also possible to define

$$\mathfrak{B} \diamond (\eta) = \sum_{\alpha \in \mathcal{I}} \mathfrak{B}(\eta_{\alpha}) \cdot \xi_{\alpha},$$

where $\cdot$ is the usual product. To understand the structure of this integral, note that the Malliavin derivative $\mathbb{D}$ of $\xi_{\alpha}$ satisfies

$$\mathbb{D}\xi_{\alpha} = \sum_{k=1}^{\infty} \sqrt{\alpha_k} \xi_{\alpha-\epsilon_k} m_k;$$

this follows directly from the definition of $\mathbb{D}$ [13, Definition 1.2.1] and the relation $H'_n(x) = nH_{n-1}(x)$. On the other hand, direct computations show that

$$\mathfrak{B} \circ (\eta) = \sum_{\alpha \in \mathcal{I}} \sum_{k=1}^{\infty} \eta_{\alpha,k} \xi_k \xi_{\alpha},$$

and, using the following property of the Hermite polynomials,

$$H_1(x)H_n(x) = H_{n+1}(x) + nH_{n-1}(x),$$

we get

$$\xi_k \xi_{\alpha} = \left( \prod_{j \neq k} \frac{H_{\alpha_j}(\xi_j)}{\sqrt{\alpha_j!}} \right) \frac{H_1(\xi_k)H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} = \sqrt{\alpha_k + 1} \xi_{\alpha+\varepsilon_k} + \sqrt{\alpha_k} \xi_{\alpha-\varepsilon_k},$$

or

$$\mathfrak{B} \circ (\eta) = \sum_{\alpha \in \mathcal{I}} \left( \sum_{k=1}^{\infty} \left( \sqrt{\alpha_k} \eta_{\alpha-\epsilon_k,k} + \sqrt{\alpha_k + 1} \eta_{\alpha+\epsilon_k,k} \right) \right) \xi_{\alpha}. $$


As a result, we use (4.10) to re-write (4.11) as

\begin{equation}
B^\circ(\eta) = B^\diamond(\eta) + \sum_{\alpha \in I} (\eta_\alpha, D_\xi \eta_\alpha)H.
\end{equation}

That is, \( B^\circ \) is a sum of the Itô-Skorokhod integral plus the trace of the Malliavin derivative — a representation characteristic of the Stratonovich-type integrals [14]. In particular, if \( B \) is a white noise over \( L^2((0, T)) \) and \( W(t) = B(\chi_t) \), then \( B^\circ(W) = W^2(T) = \int_0^T W(t) \circ dW(t) \). More generally, if \( \eta = \eta(t) \) is in the domain of the Malliavin derivative, then

\begin{equation}
B^\circ(\eta) = B^\diamond(\eta) + \int_0^T D_t \eta dt,
\end{equation}

where

\[ D_t \eta = \sum_{\alpha \in I} \eta_\alpha(t) \left( \sum_{k=1}^{\infty} \sqrt{\alpha_k} \xi_{\alpha-\epsilon_k, k} \right); \]

unlike the Itô-Skorokhod integral, though, condition (4.8) is not enough to ensure the existence of \( B^\circ(\eta) \) as an element of \( H \). When \( H \) is the Hilbert space of functions on an interval \([0, T]\), square integrable with respect to a (not necessarily Lebesgue) measure \( \mu \), the sufficient conditions for the Stratonovich integrability are discussed in [14, Chapter 3].

Let \( \mathcal{X} \) be a zero-mean non-degenerate generalized Gaussian field over a separable Hilbert space \( H \). As we mentioned earlier, by Theorem 2.4(b) on page 4, \( \mathcal{X} \) is a white noise over a bigger Hilbert space \( H_R \), and then \( \mathcal{X}^\diamond(\eta) \) can be defined using (4.4). If the space \( H_R \) is difficult to describe, one can use representation (2.4) from Theorem 2.4(a) and derive an equivalent formula for the stochastic integral:

\begin{equation}
\mathcal{X}^\circ(\eta) = \mathcal{B}^\circ(K^* \eta),
\end{equation}

for every \( (\mathcal{B}, H) \)-admissible \( \eta \).

Unlike (4.4), representation (4.14) is not intrinsic: the operator \( K^* \) and the white noise \( \mathcal{B} \) are not uniquely determined by \( \mathcal{X} \). On the other hand, in many examples, such as fractional Brownian motion with the Hurst parameter bigger than \( 1/2 \), it is possible to take \( H = L^2((0, T)) \), and then (4.14) becomes more convenient than (4.4). To derive the chaos expansion of \( \mathcal{X}^\circ(\eta) \) using (4.14), fix an orthonormal basis \( \{ m_k, k \geq 1 \} \) in \( H \), define \( \xi_k = \mathcal{B}(m_k) \), and consider the corresponding orthonormal basis \( \{ \xi_\alpha, \alpha \in I \} \) in \( H_B \) constructed according to (2.9). It follows from (4.6) that

\begin{equation}
\mathcal{X}^\circ(\eta) = \sum_{\alpha \in I} \left( \sum_{k=1}^{\infty} \sqrt{\alpha_k} \tilde{\eta}_{\alpha-\epsilon_k, k} \right) \xi_\alpha,
\end{equation}

where

\[ \tilde{\eta}_{k, \alpha} = \mathbb{E}((K^* \eta, m_k)_H \xi_\alpha). \]

If \( H = L^2((0, T)) \), then (4.15) becomes

\begin{equation}
\mathcal{X}^\circ(\eta) = \sum_{\alpha \in I} \left( \sum_{k \geq 1} \sqrt{\alpha_k} \left( \int_0^T \eta_{\alpha-\epsilon_k}(t)(K m_k)(t) dt \right) \right) \xi_\alpha.
\end{equation}
where $\eta(t) = \mathbb{E}(\eta(t) \xi_\alpha)$. In this case, by analogy with the Brownian motion, $\int_0^T \eta(s) \circ dX(s)$ can be an alternative notation for $X^\circ(\eta)$, where $X(t)$ is the associated process of $\mathfrak{X}$.

5. Stochastic Evolution Equations with Closed-Form Solutions

In this section we consider stochastic differential equations driven by a white noise $\mathfrak{B}$ or some other zero-mean generalized Gaussian random field $\mathfrak{X}$ over a Hilbert space $H$. To introduce time evolution into the stochastic integral, we use the function $\chi_t$, the indicator function of the interval $[0,t]$, and define time-dependent stochastic integrals

\begin{equation}
\mathfrak{B}_t^\circ(\eta) := \mathfrak{B}_t^\circ(\eta \chi_t), \quad \mathfrak{X}_t^\circ(\eta) := \mathfrak{X}_t^\circ(\eta \chi_t).
\end{equation}

These definitions put an obvious restriction on the Hilbert space $H$, which we call Property I: $H$ contains $\chi_t$, $t \in [0,T]$, and, for every $\eta \in H$ and every fixed $t$, the (point-wise) product $\eta \chi_t$ is defined and belongs to $H$. There is a more significant restriction on $H$, which we illustrate on the following equation:

\begin{equation}
u(t) = 1 + \mathfrak{B}_t^\circ(u), \quad 0 \leq t \leq T,
\end{equation}

where $\mathfrak{B}$ is white noise over a Hilbert space $H$ with Property I. Let us assume that the solution belongs to $H_{\mathfrak{B}}$ so that $u(t) = \sum_{\alpha \in \mathcal{I}} u_\alpha(t) \xi_\alpha$ and each $u_\alpha$ is an element of $H$. By (5.1), we can re-write (5.2) as

\begin{equation}u(t) = 1 + \mathfrak{B}_t^\circ(u \chi_t),\end{equation}

and then (4.6) implies

\begin{equation}u_\alpha(t) = 1 + \sum_{k=1}^{\infty} (u_{\alpha-\epsilon_k \chi_t}, m_k)_H.
\end{equation}

Thus, the expression $(u_{\alpha-\epsilon_k \chi_t}, m_k)_H$, as a function of $t$, must be an element of $H$, and the Hilbert space $H$ must have another special property, which we call Property II: for every $f, g \in H$, the inner product $(f \chi_t, g)_H$, as a function of $t$, is an element of $H$. The space $H_R$ corresponding to a zero-mean Gaussian process with covariance $R$ contains step functions by definition, but the point-wise multiplication is a more delicate issue, especially if $H_R$ is smaller than $L_2((0,T))$. On the other hand, the space $L_2((0,T), \mu)$, with $\mu((0,T)) < \infty$, has both Property I and Property II, which follows from the Cauchy-Schwartz inequality.

**Theorem 5.1.** If $\mathfrak{X}$ is a zero-mean generalized Gaussian field over $L_2((0,T))$, then the solution of the equation

\begin{equation}u(t) = 1 + \mathfrak{X}_t^\circ(u)
\end{equation}

is unique in $L_2((0,T); H_{\mathfrak{X}})$ and is given by

\begin{equation}u(t) = e^{\mathfrak{X}(t)},
\end{equation}

where $e^\circ$ is the Wick exponential function (2.15) and $X(t) = \mathfrak{X}(\chi_t)$ is the associated process of $\mathfrak{X}$. 

Proof. Let \( \mathfrak{X}(f) = \mathfrak{B}(\mathcal{K}^* f) \) be a white noise representation of \( \mathfrak{X} \) over \( L_2((0, T)) \). We start by establishing uniqueness of solution in \( L_2((0, T); H_B) \), which, because of the inclusion \( H_X \subseteq H_B \), is even stronger. By linearity, the difference \( Y(t) \) of two solutions of (5.5) satisfies

\[
y(t) = X \odot t (Y).
\]

If \( y(t) = \sum_{\alpha \in I} y_\alpha(t) \xi_\alpha \), then (4.16) implies

\[
y_\alpha(t) = \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t y_{\alpha-\epsilon_k}(s) \tilde{m}_k(s) ds,
\]

where \( \tilde{m}_k = \mathcal{K} m_k \). In particular, if \( |\alpha| = 0 \), then \( y_\alpha(t) = 0 \) for all \( t \). By induction on \( |\alpha| \), \( y_\alpha(t) = 0 \) for all \( \alpha \in I \): if \( y_\alpha = 0 \) for all \( \alpha \) with \( |\alpha| = n \), then, since \( |\alpha - \epsilon_k| = |\alpha| - 1 \), equality (5.7) implies \( y_\alpha = 0 \) for all \( \alpha \) with \( |\alpha| = n + 1 \).

To establish (5.6), let

\[
\tilde{M}_k(t) = \int_0^t (K m_k)(s) ds.
\]

By (2.11) and (2.12),

\[
X(t) = \sum_{k=1}^{\infty} \tilde{M}_k(t) \xi_k,
\]

and, because of the independence of \( \xi_k \) for different \( k \),

\[
e^{-\sqrt{\alpha} X(t)} = \prod_{k \geq 1} e^{-\sqrt{\alpha} \tilde{M}_k(t) \xi_k} = \sum_{\alpha \in I} \frac{\tilde{M}^\alpha(t)}{\sqrt{\alpha!}} \sqrt{\alpha} \xi_\alpha,
\]

where

\[
\tilde{M}^\alpha(t) = \prod_{k=1}^{\infty} \tilde{M}^\alpha_k(t).
\]

Similar to (5.7), we conclude that if the solution \( u = u(t) \) has the chaos expansion

\[
u_\alpha(t) = \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t u_{\alpha-\epsilon_k}(s) \tilde{m}_k(s) ds,
\]

if \( |\alpha| > 0 \). Then direct computations show that

\[
u_\alpha(t) = \frac{\tilde{M}^\alpha(t)}{\sqrt{\alpha!}}, \quad |\alpha| \geq 1,
\]

satisfies (5.8):

\[
\frac{du_\alpha(t)}{dt} = \frac{1}{\sqrt{\alpha!}} \frac{d}{dt} \prod_{k=1}^{\infty} \tilde{M}^\alpha_k(t) = \frac{1}{\sqrt{\alpha!}} \sum_{k=1}^{\infty} \alpha_k \tilde{M}^{\alpha_k-1}_k(t) \tilde{m}_k(t) \prod_{j \neq k} \tilde{M}^{\alpha_j}_j(t)
\]

\[
= \sum_{k=1}^{\infty} \sqrt{\alpha_k} \tilde{m}_k(t) \frac{\tilde{M}^{\alpha-\epsilon_k}_k(t)}{\sqrt{(\alpha - \epsilon_k)!}} = \sum_{k=1}^{\infty} \sqrt{\alpha_k} m_k(t) u_{\alpha-\epsilon_k}(t).
\]
Corollary 5.2. The solution of

$$u(t) = u_0 + \int_0^t a(s)u(s)ds + \sum_{k=1}^N X_t^\sigma(\sigma_k u_k)$$

where $X_k$ are independent generalized Gaussian random fields over $L_2((0,T))$ and $a \in L_1((0,T)), \sigma_k \in L_2((0,T))$ are non-random, is

$$u(t) = u_0 \exp \left( \int_0^t a(s)ds \right) \exp \left( \sum_{k=1}^N X_k(t) \right),$$

where $X_k(t) = \mathcal{X}_k(\sigma_k \chi_t)$.

Theorem 5.1 is a generalization of the familia result that the geometric Brownian motion $u(t) = e^{W(t) - (t/2)} = e^{\sigma W(t)}$ satisfies $u(t) = 1 + \int_0^t u(s) dW(s)$: by (5.6) and (2.10), for a class of zero-mean Gaussian processes $X = X(t)$ with covariance function $R(t,s)$, and with a suitable interpretation of the stochastic integral, the solution of the equation $u(t) = 1 + \int_0^t u(s) \diamond dX(s)$ is

$$u(t) = e^{X(t) - X(0) - \frac{1}{2}(R(t,t) - 2R(t,0) + R(0,0))}.$$  

Note that the proof works without assuming that $\mathcal{X}$ is non-degenerate or $X$ is non-anticipating. On the other hand, the special interpretation of the integral is essential: if $X$ is also a semi-martingale, then the traditional Itô integral $\int_0^T \eta(t)dX(t)$ can also be defined, and the solution of the corresponding equation $U(t) = 1 + \int_0^t u(s)dX(s)$ is the Doleans exponential

$$U(t) = e^{X(t) - X(0) - \frac{1}{2}(\mathcal{X})_t}.$$  

To see that $u$ and $U$ can be different, consider the stable Ornstein-Uhlenbeck process \( [5.1] \), for which $u(t) = \exp \left( X(t) - (1 - e^{-bt})/(2b) \right)$ and $U(t) = \exp \left( X(t) - t/2 \right)$. The reason for this difference is clear: while $U$ satisfies the equation $dU(t) = -bX(t)U(t)dt + U(t)dW(t)$, $u(t)$ satisfies $du(t) = -bX(t) \diamond u(t)dt + u(t)dW(t)$ (it is known that $dW = \circ dW$; see [3]).

As an application of Theorem 5.1 let us find the classical, square-integrable solution of the stochastic partial differential equation

$$u(t,x) = u_0(x) + a\int_0^t u_{xx}(s,x)ds + \sigma \mathcal{X}_t^\sigma(u_x(\cdot,x)), \ t \geq 0, \ x \in \mathbb{R},$$

with a smooth compactly supported initial condition $u_0$ and constant $a > 0, \sigma \in \mathbb{R}$. Let

$$\hat{u}(t,y) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{-iyx}u(t,x)dx, \ i = \sqrt{-1},$$

be the Fourier transform of $u(t,x)$. Then, by linearity,

$$\hat{u}(t,y) = \hat{u}_0(y) - ay^2 \int_0^t \hat{u}(s,y)ds + i\sigma \mathcal{X}_t(\hat{u}(\cdot,y)),$$

and therefore

$$\hat{u}(t,y) = \hat{u}_0 \exp \left( -ay^2t + \frac{1}{2}\sigma^2y^2R(t,t) + iy\sigma X(t) \right).$$
where $X(t)$ is the associated process of $\mathcal{X}$ and $R(t,s) = \mathbb{E}(X(t)X(s))$. With the notation 
\[ r(t) = at - \frac{\sigma^2}{2} R(t,t), \]
the classical solution of (5.9) becomes 
\[ u(t, x) = \bar{u}(t, x - \sigma X(t)), \]
where 
\[ \bar{u}(t, x) = \frac{1}{\sqrt{4\pi r(t)}} \int_{\mathbb{R}} \exp \left( -\frac{(x - y)^2}{4r(t)} \right) u_0(y) dy. \]
In particular, we get the following parabolicity or non-explosion condition for (5.9):
\[ (5.10) \quad at \geq \frac{\sigma^2}{2} R(t,t). \]
If $X(t) = W(t)$, then $R(t,t) = t$ and (5.10) becomes
\[ (5.11) \quad 2a \geq \sigma^2. \]
If $X(t) = W^H(t)$, $H > 1/2$, then $R(t,t) = t^{2H}$ and (5.10) becomes
\[ (5.12) \quad t^{2H-1} \leq 2a/\sigma^2; \]
this condition also appears in [7].
For the stable Ornstein-Uhlenbeck process (3.15), $R(t,t) = (1 - e^{-2bt})/(2b)$, and condition (5.10) becomes
\[ a \geq \frac{\sigma^2}{4bt} \left( 1 - e^{-2bt} \right), \]
which is equivalent to (5.11).
For the unstable Ornstein-Uhlenbeck process (3.16), condition (5.10) is
\[ (5.13) \quad at \geq \frac{\sigma^2}{4b} \left( e^{2bt} - 1 \right). \]
If (5.11) holds, then (5.13) holds for sufficiently small $t$; if (5.11) fails, so does (5.13) for all $t \geq 0$.

The traditional Itô version of (5.9) with the Ornstein-Uhlenbeck process is
\[ du(t, x) = (au_{xx}(t, x) \pm b\sigma X(t)u_x(t))dt + \sigma u_x dW(t), \]
This equation is well-posed if and only (5.11) holds, and this condition does not depend on $b$.

We can also write the classical, square-integrable solution of
\[ (5.14) \quad u(t, x) = u_0(x) + \int_0^t a(s) u_{xx}(s, x) ds + X_s^\sigma(\sigma u_x(\cdot, x)), \quad t \geq 0, \quad x \in \mathbb{R}, \]
with a smooth compactly supported initial condition \( u_0 \) and continuous functions \( a(t), \sigma(t) \). Indeed, let
\[
A(t) = \int_0^t a(s)ds, \quad X_\sigma(t) = \mathcal{X}(\sigma \chi_t),
\]
\[
R_\sigma(t, s) = \mathbb{E}(X_\sigma(t)X_\sigma(s)), \quad r_\sigma(t) = A(t) - \frac{1}{2} R_\sigma(t, t).
\]
Then
\[
\bar{u}(t, x) = \bar{u}(t, x - X_\sigma(t)),
\]
where
\[
\bar{u}(t, x) = \frac{1}{\sqrt{4\pi r_\sigma(t)}} \int_{\mathbb{R}} \exp \left(-\frac{(x - y)^2}{4r_\sigma(t)}\right) u_0(y)dy.
\]
The parabolicity condition for (5.14) is
\[
r_\sigma(t) \geq 0,
\]
which in general cannot be simplified much further: when \( \sigma \) depends on time, there is no easy connection between \( R \) and \( R_\sigma \).

6. Chaos Solution of Stochastic Evolution Equations

Let \( \mathcal{X}_\ell, \ell \geq 1, \) be a collection of independent, zero-mean, generalized Gaussian random fields over \( L_2((0, T)) \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). By Theorem 2.6 on page 3 there is a collection \( \{W_\ell, \ell \geq 1\} \) of independent Wiener processes and \( \{K_\ell, \ell \geq 1\} \) of bounded linear operators on \( L_2((0, T)) \) such that
\[
(6.1) \quad \mathcal{X}_\ell(f) = \int_0^T (K_\ell^* f)(t)dW_\ell(t).
\]
With no loss of generality, we assume that the sigma-algebra \( \mathcal{F} \) is generated by the random variables \( W_\ell, \ell \geq 1, \) on \( [0, T] \). Note that \( X_\ell(t) = \mathcal{X}_\ell(\chi_t) \) is not necessarily adapted to filtration of the corresponding Wiener process \( W_\ell(t) \).

Introduce the following objects:

1. \( (X, H, X') \), a triple of Hilbert spaces such that \( X' \) is the dual of \( X \) relative to the inner product in \( H \). To simplify the notations, we use \( (\cdot, \cdot) \) to denote both the inner product in \( H \) and the duality between \( X \) and \( X' \).
2. \( A \), a bounded linear operator from \( L_2((0, T); X) \) to \( L_2((0, T); X') \).
3. \( M_\ell, \ell \geq 1, \) a collection of bounded linear operators from \( L_2((0, T); X) \) to \( L_2((0, T); X') \).
4. \( u_0 \in L_2(\Omega; H), f \in L_2(\Omega; L_2((0, T); X')) \), \( g_\ell \in L_2(\Omega; L_2((0, T); X')) \).
5. The Fourier cosine basis in \( L_2((0, T)) \):
\[
(6.2) \quad m_1(s) = \frac{1}{\sqrt{T}}; \quad m_k(t) = \sqrt{\frac{2}{T}} \cos \left(\frac{\pi (k - 1)t}{T}\right), \quad k > 1; \quad 0 \leq t \leq T.
\]
Also define \( \tilde{m}_k(t) = (K_\ell m_k)(t) \).
Consider the following stochastic evolution equation:

\[
\frac{du}{dt} = Au(t) + f(t) + \sum_{\ell \geq 1} \xi_{\ell}(\sigma(t))
\]

(6.5) The random variables \(\xi_{\ell}(t)\) and \(u(t)\) have a unique solution \(u(t) = u_0 + \int_0^t Au(s)ds + \int_0^t f(s)ds + \sum_{\ell \geq 1} \xi_{\ell}(\sigma(t))\).

Assume that the random element \(u_0\) and the processes \(u, f, g\) have chaos expansions

\[
\sum_{\alpha \in \mathcal{J}} u_{0,\alpha} \xi_{\alpha} = \sum_{\alpha \in \mathcal{J}} f_{\alpha}(t) \xi_{\alpha} = \sum_{\alpha \in \mathcal{J}} g_{\alpha}(t) \xi_{\alpha}.
\]

(6.6)

Substituting into (6.5) and using (4.10), we conclude that the (deterministic) functions \(u_\alpha(t)\) satisfy

\[
u_\alpha(t) = u_{0,\alpha} + \int_0^t (Au_\alpha(s) + f_\alpha(s))ds + \sum_{k,\ell} \sqrt{\alpha_{k\ell}} \int_0^t (M_{k\ell}u_{\alpha}(s) + g_{\alpha}(s))ds.
\]

(6.7)

The two special multi-indices are \((0)\) with all zero entries and \(\epsilon(ij)\) with \(\epsilon(ij) = 1\) and \(\epsilon(k\ell(ij) = 0\) otherwise. We also write \(\alpha! = \prod_{k,\ell} \alpha_{k\ell}!\).

Definition 6.1. A chaos solution of equation (6.5) is a formal series \(u(t) = \sum_{\alpha \in \mathcal{J}} u_\alpha(t)\xi_{\alpha}\), where the random variables \(\xi_{\alpha}\) are defined by (6.3) and the deterministic functions \(u_\alpha\) satisfy (6.7).

Theorem 6.2. Assume that

- for every \(U_0 \in H\) and \(F \in L_2((0,T);X')\), the deterministic evolution equation

\[
U(t) = U_0 + \int_0^t AU(s)ds + \int_0^t F(s)ds
\]

(6.8)

has a unique solution \(U \in L_2((0,T);X)\).
Then equation (6.5) has a unique chaos solution and every $u_\alpha$ is an element of $L_2((0, T); X)$.

Proof. Note that, for $|\alpha| = 0$, (6.7) is

$$u(0)(t) = u_{0}(0) + \int_{0}^{t} Au_{0}(s)ds + \int_{0}^{t} f_{0}(s)ds;$$

by assumption, equation (6.10) has a unique solution $u(0) \in L_2((0, T); X)$.

We now proceed by induction on $|\alpha|$. Assume that (6.7) has a unique solution $u_{\alpha} \in L_2((0, T); X)$ for all $\alpha$ with $|\alpha| \leq n$. Then, for $\alpha$ with $|\alpha| = n + 1$ we have $|\alpha - \epsilon(k, \ell)| = n$ so that $M_{\ell}u_{\alpha - \epsilon(k, \ell)} \tilde{m}_{k\ell} \in L_2((0, T); X')$. With only finitely many terms in the sum on the right-hand side of (6.7), the assumptions of the theorem now imply that (6.7) has a unique solution $u_{\alpha} \in L_2((0, T); X)$ for all $\alpha$ with $|\alpha| = n + 1$. \qed

In general, condition (6.9) is necessary: there is no guarantee that $f\tilde{m}_{k\ell} \in L_2((0, T); X')$ for every $f \in L_2((0, T); X')$. More information about the operator $A$ makes it possible to remove this condition; see Theorem 6.3 below.

Assume that $X_{\ell}(\chi_{\ell}) = \int_{0}^{t} K_{\ell}(t, s)dW_{\ell}(s)$. Then a sufficient condition for (6.9) to hold is

$$\text{ess sup}_{t \in (0, T)} \left( K_{\ell}(t, t) + \int_{0}^{t} \left| \frac{\partial K_{\ell}(t, s)}{\partial t} \right| ds \right) < \infty;$$

this is the case for the fractional Brownian motion with $H > 1/2$ and for the Ornstein-Uhlenbeck process (stable or unstable).

A theory of solutions of (6.5) in weighted chaos spaces can be developed in complete analogy with [11, 12]; we leave this development to an interested reader. Instead, we will investigate when the chaos solution is in fact square-integrable, that is, when no weights are necessary. This investigation will lead to an extension of condition (5.10) on page 19 to equations with variable coefficients.

Denote by $\mathcal{K}_{\ell}$ the norm of the operator $K_{\ell}$ in $L_2((0, T))$.

**Theorem 6.3.** Assume that

1. The initial condition $u_0$ and the processes $f, g_\ell$ are deterministic and

$$\sum_{\ell \geq 1} \int_{0}^{T} \mathcal{K}_{\ell}^2 \|g_\ell(t)\|_{X'} dt < \infty;$$

2. There exist positive numbers $\delta_A$ and $C_A$ such that, for all $v \in X$ and $t \in [0, T]$,

$$\|A(t)v\|_{X'} \leq C_A \|v\|_{X}, \quad (A(t)v, v) + \delta_A \|v\|_{X}^2 \leq C_A \|v\|_{H}^2.$$
(3) There exist a non-negative number $\delta_0 < \delta_A$ and a positive number $C_0$ such that, for all $v \in X$ and $t \in [0, T],\quad (6.13) \quad 2 \langle A(t)v, v \rangle + \sum_{\ell \geq 1} \mathcal{R}_\ell^2 \| M_\ell(t)v \|_H^2 + \delta_0 \| v \|_X^2 \leq C_0 \| v \|_H^2.

Then the chaos solution of (6.3) satisfies
\begin{equation}
(6.14) \quad \sup_{0 \leq t < T} \mathbb{E} \| u(t) \|_H^2 + \delta_0 \int_0^T \mathbb{E} \| u(t) \|_X^2 dt \leq C(C_A, \delta_A, C_0, T) \left( \| u_0 \|_H^2 + \int_0^T \| f(t) \|_X^2 dt \right)
+ \sum_{\ell \geq 1} \mathcal{R}_\ell^2 \int_0^T \| g_\ell(t) \|_X^2 dt).
\end{equation}

Proof. By assumptions of the theorem, (6.7) takes the form
\begin{equation}
(6.15) \quad u_{\ell(ij)}(t) = \int_0^t A\epsilon_{\ell(ij)}(s)ds + \int_0^t \left( M_j u_0(s) + g_j(s) \right) \tilde{m}_{ij}(s)ds, \quad |\alpha| = 1;
\end{equation}
\begin{equation}
(6.16) \quad u_\alpha(t) = \int_0^t A\epsilon_{\ell(ij)}(s)ds + \sum_{k,\ell=1}^{\infty} \sqrt{\alpha_{k\ell}} \int_0^t M_{\ell,\kappa_{\alpha}}(s) \tilde{m}_{k\ell}(s)ds, \quad |\alpha| > 1.
\end{equation}

By (6.12), the operator $A$ generates a semi-group $\Phi_{t,s}$, and the solution of (6.8) is
\begin{equation}
(6.17) \quad U(t) = \Phi_{t,0} U_0 + \int_0^t \Phi_{t,s} F(s)ds.
\end{equation}

By induction on $|\alpha|$ we conclude that if $|\alpha| = n$, \{$(k_1, \ell_1), \ldots, (k_n, \ell_n)$\} is the characteristic set of $\alpha$, and $\mathcal{P}_n$ is the set of all permutations of \{1, 2, \ldots, n\}, then
\begin{equation}
(6.18) \quad u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_n} \Phi_{t, s_1} M_{\sigma(1)} \cdots \Phi_{s_n, s_1} M_{\sigma(n)} u_0(s_1)
+ g_{\ell(1)}(s_1) \left( K_{\ell(1)} m_{\sigma(1)}(s_1) \cdots \sum_{\kappa_{\sigma(1)}} (K_{\ell(1)} m_{\sigma(1)})(s_1) ds_n, \right.
\end{equation}
where $ds^n = ds_1 \cdots ds_n$. We then re-write (6.19) as
\begin{equation}
(6.19) \quad \bar{m}_\alpha(s) = \frac{1}{\sqrt{\alpha! \cdot n!}} \sum_{\sigma \in \mathcal{P}_n} \left( K_{\ell_1} h_{\kappa_1}(s_{\sigma(1)}), \ldots, K_{\ell_n} h_{\kappa_n}(s_{\sigma(n)}) \right).
\end{equation}
From (6.17) and the definition of the function $G$, we conclude that

$$\sum_{\alpha \in \mathcal{J}} \|u_{\alpha}(t)\|_H^2 \leq \sum_{\ell_1, \ldots, \ell_n=1}^{\infty} \left( \prod_{j=1}^{n} \mathcal{R}_{\ell_j}^2 \right) \int_0^t \int_0^{s_n} \cdots \int_0^{s_2}$$

(6.20)

$$\left\| \Phi_{t,s_n} M_{\ell_n} \cdots \Phi_{s_2,s_1} \left( M_{\ell_1} u_0(s_1) + g_{\ell_1}(s_1) \right) \right\|_H^2 ds^n.$$

Similarly,

$$\sum_{\alpha \in \mathcal{J}} \int_0^t \|u_{\alpha}(t)\|_{\mathcal{X}}^2 ds \leq \sum_{\ell_1, \ldots, \ell_n=1}^{\infty} \left( \prod_{j=1}^{n} \mathcal{R}_{\ell_j}^2 \right) \int_0^t \int_0^{s_n} \cdots \int_0^{s_2}$$

(6.21)

$$\left\| \Phi_{s,s_n} M_{\ell_n} \cdots \Phi_{s_2,s_1} \left( M_{\ell_1} u_0(s_1) + g_{\ell_1}(s_1) \right) \right\|_{\mathcal{X}}^2 ds^n ds.$$

For $n \geq 1$, denote by $F_n^H(t)$ and $F_n^X(t)$ the right-hand sides of (6.20) and (6.21), respectively. For $n = 0$, define $F_0^H(t) = \|u_0(t)\|_H^2$, $F_0^X(t) = \int_0^t \|u_0(s)\|_{\mathcal{X}}^2 ds$. Then

$$\mathbb{E}\|u(t)\|_H^2 \leq \sum_{n=0}^{\infty} F_n^H(t), \quad \mathbb{E}\|u(t)\|_{\mathcal{X}}^2 dt \leq \sum_{n=0}^{\infty} F_n^X(T).$$

For brevity, introduce the notation

$$\mathbb{I} = \|u_0\|_H^2 + \int_0^T \|f(t)\|_{\mathcal{X}}^2 dt + \sum_{\ell \geq 1} \mathcal{R}_{\ell}^2 \int_0^T \|g_{\ell}(t)\|_{\mathcal{X}}^2 dt.$$

Then assumption (6.12) implies

$$\sup_{0 < t < T} F_0^H(t) + \delta_0 F_0^X(T) \leq C_1(C_A, \delta_A, T) \mathbb{I}.$$  

(6.23)

For $n \geq 1$, we find using (6.13) that

$$\frac{dF_n^H(t)}{dt} + \delta_0 \frac{dF_n^X(t)}{dt} \leq C_0 F_n^H(t)$$

$$+ \sum_{\ell_1, \ldots, \ell_n \geq 1} \left( \prod_{j=1}^{n} \mathcal{R}_{\ell_j}^2 \right) \int_0^t \int_0^{s_n} \cdots \int_0^{s_2}$$

(6.24)

$$\left\| M_{\ell_n} \Phi_{t,s_n-1} M_{\ell_{n-1}} \cdots \Phi_{s_2,s_1} \left( M_{\ell_1} u_0(s_1) + g_{\ell_1}(s_1) \right) \right\|_H^2 ds^{n-1}$$

$$- \sum_{\ell_1, \ldots, \ell_{n+1} \geq 1} \left( \prod_{j=1}^{n+1} \mathcal{R}_{\ell_j}^2 \right) \int_0^t \int_0^{s_n} \cdots \int_0^{s_2}$$

$$\left\| M_{\ell_{n+1}} \Phi_{t,s_n} M_{\ell_n} \cdots \Phi_{s_2,s_1} \left( M_{\ell_1} u_0(s_1) + g_{\ell_1}(s_1) \right) \right\|_H^2 ds^n.$$

Then, after summation in $n$ and integration in time,

$$\sum_{n=1}^{N} (F_n^H(t) + \delta_0 F_n^X(t)) \leq C_0 \int_0^T \sum_{n=1}^{N} F_n^H(s) ds + C_2(C_A, \delta_A, T) \mathbb{I}.$$
for every $N \geq 1$. Applying Gronwal’s inequality, we get (6.14) from (6.22). Theorem 6.3 is proved. □

**Example 6.4.** Let $X$ be the Sobolev space $H^1(\mathbb{R}^d)$ and $H = L^2(\mathbb{R}^d)$. Consider the following equation driven by a single fractional Brownian motion $W^H$ with $H \geq 1/2$ [there should be no difficulty distinguishing between $H$ a space and $H$ a Hurst parameter]:

$$du(t, x) = \sum_{i,j=1}^{d} a_{ij}(t, x) \frac{\partial^2 u(t, x)}{\partial x_i \partial x_j} dt + \sum_{i=1}^{d} \sigma_i(t, x) \frac{\partial u(t, x)}{\partial x_i} \circ dW^H(t), \ 0 < t \leq T.$$ 

In this case, $K^2 = 2H^{2-2H}T^{2H-1}$; see (3.14) on page 10. Condition (6.13) becomes

$$\delta_0 |y|^2 \leq \sum_{i,j=1}^{d} \left( a_{ij}(t, x) - H^{2-2H}T^{2H-1}\sigma_i(t, x)\sigma_j(t, x) \right)y_i y_j \leq C_0 |y|^2$$

for all $t \in [0, T]$ and all $x, y \in \mathbb{R}^d$. Let us now compare (6.25) with (5.12) on page 19 if $a, \sigma$ are constants and $d = 1$. If $H = 1/2$, then (6.25) becomes (5.11), which is (3.12) with $H = 1/2$ (recall that $\delta_0$ can be zero). If $H > 1/2$, then (6.25) becomes $2a/\sigma^2 \geq H^{2-2H}T^{2H-1}$, which is slightly stronger than (3.12) because $1 < H^{2-2H} < 1.07$ for $1/2 < H < 1$.

**Example 6.5.** Let $X(t) = X(\chi_t)$ and consider the equation

$$(6.26) \quad du(t, x) = au_{xx} dt + \sigma u_x \circ dX(t), \ x \in \mathbb{R}, \ 0 < t \leq T,$$

with constant $a, \sigma$. Condition (6.13) in this case is

$$(6.27) \quad a \geq \frac{\sigma^2}{2} K^2.$$ 

Recall condition (5.10) on page 19, which was derived from the closed-form solution of equation (6.26) and is both necessary and sufficient for (6.26) to have a square-integrable solution. We conclude that, for equations with constant coefficients, (6.27) should imply (5.10), but not necessarily the other way around. As a result, comparison of (5.10) and (6.27) produces a lower bound on the operator norm $K$ for the field $X$ in terms of the covariance function $R$ of the associated process $X$:

$$K \geq \sup_{0 < t < T} \frac{R(t, t)}{t}.$$ 

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