Optimal storage capacity of neural networks at finite temperatures

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Gardner’s analysis of the optimal storage capacity of neural networks is extended to study finite-temperature effects. The typical volume of the space of interactions is calculated for strongly-diluted networks as a function of the storage ratio $\alpha$, temperature $T$, and the tolerance parameter $m$, from which the optimal storage capacity $\alpha_c$ is obtained as a function of $T$ and $m$. At zero temperature it is found that $\alpha_c = 2$ regardless of $m$ while $\alpha_c$ in general increases with the tolerance at finite temperatures. We show how the best performance for given $\alpha$ and $T$ is obtained, which reveals a first-order transition from high-quality performance to low-quality one at low temperatures. An approximate criterion for recalling, which is valid near $m = 1$, is also discussed.

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I. Introduction

Recently, the tools of statistical mechanics have been extensively applied to the study of collective properties of neural networks [1]; spin glass theory has played an important role in the growth of this new field [2]. In particular, the optimal (error-free) storage capacity for recurrent networks can be obtained by calculating the typical fractional volume of the space of interactions ($\{J_{ij}\}$) satisfying the condition that for a given set of patterns each pattern is a fixed point of the deterministic (zero-temperature) dynamics

$$s_i(t + 1) = \text{sign}\left[\sum_j J_{ij} s_j(t)\right], \quad (1)$$

where $s_i(t) (= \pm 1) (i = 1, \ldots, N)$ represents the state of the $i$th neuron at time $t$, and the synaptic coupling $J_{ij}$ determines the contribution of a signal fired by the $j$th neuron to the action potential on the $i$th neuron. This approach of systematic exploration of the space of interactions, which was pioneered by Gardner [3] and reformulated in terms of canonical ensemble calculation [4], has been applied in various directions [4–11]. The Hopfield model with general continuous couplings has been found to be capable of storing at most two uncorrelated random patterns per neuron without errors and larger number of patterns for biased patterns [3]. The network with discrete (Ising-type) couplings has been also extensively investigated since the replica-symmetry theory was reported to yield wrong results for the optimal storage capacity [4, 6, 7]. The method is not limited to Hopfield-type neural networks but applicable to multilayer networks as well as to simple perceptrons [5, 11]. However, Gardner’s method is based on the concept of fixed points of the dynamics, and obviously does not work if the dynamics is stochastic (i.e., at finite temperatures). In addition, it requires perfect matching so that each pattern is inerrably recalled at every site whereas in practice one usually considers a neural network to be remembering or recalling if the overlap between the network state and one of the patterns is larger than some given value.

In this paper, we propose a scheme to define the optimal storage capacity at finite
temperatures and study its temperature dependence. We introduce the tolerance parameter
\( m \leq 1 \) in such a way that the \( m \to 1 \) limit corresponds to the perfect recall while \( (1 - m)/2 \)
measures the error allowed. We then calculate the typical fractional volume of the space of
interactions for extremely-diluted networks as a function of the storage ratio \( \alpha \), temperature
\( T \), and the tolerance parameter \( m \), which leads to the optimal storage capacity \( \alpha_c \) as a
function of \( T \) and \( m \). At zero temperature it is found that \( \alpha_c = 2 \) regardless of the tolerance
parameter \( m \). At finite temperatures, on the other hand, the optimal storage capacity
vanishes in the perfect matching limit \( (m \to 1) \) and in general increases with the tolerance.
We then discuss how the best performance is obtained for given \( \alpha \) and \( T \). We also propose
an alternative criterion for recalling, which may be regarded as a simple approximate
scheme to define the optimal storage capacity, and consider the optimal storage capacity of
dynamic model [12] as well as of the extremely-diluted model in this approximate scheme.

The contents of this paper are as follows: In Sec. II, we propose a scheme to define
the optimal storage capacity at finite temperatures together with an approximate scheme.
Section III is devoted to the calculation of the optimal storage capacity of an extremely-
diluted neural network while Sec. IV presents results of the approximate scheme. A brief
discussion is given in Sec. V.

## II. Optimal storage capacity at finite temperatures

One usually takes into account internal noise in the functioning of a neuron by extending
the deterministic evolution rule (1) to a stochastic one:

\[
P[s_i(t + 1) = \pm 1] = \frac{1}{2} \{1 \pm \tanh[\beta \sum_j J_{ij} s_j(t)]\},
\]

where the inverse temperature \( \beta \equiv 1/T \) measures the width of the threshold region,
i.e., the level of synaptic noise. The state \( s \equiv \{s_i\} \) of the network of \( N \) neurons evolves
stochastically according to Eq. (2). A given set of states of the network to be memorized
by adjusting appropriately the couplings is called the set of patterns. We now define the
overlap $M_\mu(t)$ between the network state and the $\mu$th pattern $\xi^\mu \equiv \{\xi^\mu_i\} (\mu = 1, \ldots, p)$ by

$$M_\mu(t) \equiv \frac{1}{N} \sum_{i=1}^{N} \xi^\mu_i s_i(t),$$

which also evolves stochastically along with $s(t)$. When a network is recalling pattern $\mu$, the time average of $M_\mu(t)$ over the time scale sufficiently longer than the observational time but shorter than the lifetime of the local energy minimum should be close to unity. However, since the dynamics is stochastic, it cannot be strictly unity as in the zero-temperature dynamics. Therefore, we introduce the tolerance parameter $m$ in such a way that the network is considered to be remembering the $\mu$th pattern if

$$M_\mu \equiv \frac{1}{N_t} \sum_{t=1}^{N_t} M_\mu(t) > m$$

with $N_t$ in the appropriate range as mentioned above. The quantity $(1 - m)/2$ is the maximum error allowed for the network to be qualified as functioning. It is expected that the time average $M_\mu$ is equivalent to the restricted thermal average

$$\langle M_\mu \rangle \equiv \frac{1}{N} \sum_{i=1}^{N} \xi^\mu_i \langle s_i \rangle,$$

where the thermal measure is restricted within a single pure state (containing the configuration $\xi^\mu$) \[13\]. In the stationary state the activity $\langle s_i \rangle$ of the $i$th neuron is determined by the coupled equations

$$\langle s_i \rangle = \langle \tanh(\beta \sum_j J_{ij} s_j) \rangle = \tanh(\beta \sum_j J_{ij} \langle s_j \rangle),$$

where a mean-field approximation has been used. Such an approximation is expected to be valid for dilute networks which we mainly consider in this work. Otherwise, a reaction term may be necessary. The optimal storage capacity is given by the upper bound of the storage ratio $\alpha \equiv p/N$, where $p$ is the number of stored patterns. The problem reduces, according to Gardner \[3\], to the calculation of the typical fractional volume of the space of interactions which satisfies the following conditions:

$$\frac{1}{N} \sum_{i=1}^{N} \xi^\mu_i \tanh\left(\beta \sum_j \sqrt{N} J_{ij} \langle s_j \rangle\right) > m,$$

$$t \equiv \{t_i\} \quad (t_i = 1, \ldots, p)$$
and
\[ \sum_j (J_{ij})^2 = N \quad \text{for each } i. \] 

(4)

The condition in Eq. (4) is required to fix the scale of temperature \( T \). The optimal storage capacity at temperature \( T \) is then determined by vanishing of this fractional volume, which leads to \( \alpha_c \) as a function of \( T \) and \( m \). Application of the above scheme to an extremely-diluted neural network will be given in the following section. In this model, only an extremely small fraction of couplings among neurons are connected so that its dynamics can be solved in a rather simple manner [14, 15]. However, calculation of the fractional volume for other generic neural network is formidable due to the thermal average to be performed within one pure state. As a simple attempt, one may use the approximation \( \langle s_i \rangle \approx \xi_\mu^\mu \) and replace Eq. (3) by
\[ \frac{1}{N} \sum_{i=1}^{N} \xi_\mu^\mu \tanh \left( \frac{\beta}{\sqrt{N}} \sum_j J_{ij} \xi_\mu^\mu \right) > m, \]

(5)

which states that the network state evolved by one time step from a given pattern has overlap with that pattern greater than \( m \). This simple criterion presumably leads to results similar to those of Eq. (3) for \( m \) close to unity, where the network is expected to hover around the configuration \( \xi_\mu^\mu \) during the recalling state. The validity of this approximate scheme with regard to diluted networks is discussed in Sec. IV.

III. Extremely-diluted neural network

In this section the proposed scheme is applied to an extremely-diluted neural network, where, on the average, there are \( C (\lesssim \log N) \) connections per neuron. Such a model was first studied by Derrida et al. [14], and later its properties of the basin of attraction was studied by Gardner [15] and by Amit et al. [16]. The reason why we can implement the scheme exactly is that the dynamics of the network can be solved in a rather simple manner [14, 15]. Extending the method of Keppler and Abbott [17], we describe the time
The evolution of the overlap \( M_\mu(t) \) between pattern \( \xi_\mu \) and the network state by the one-step recursion relation

\[
M_\mu(t+1) = F_{h_\mu}[M_\mu(t)] .
\] (6)

Here the map \( F_{h_\mu}(x) \) is defined by

\[
F_{h_\mu}(x) \equiv \frac{1}{N} \sum_{i=1}^{N} \int Dz \tanh[\beta(\sqrt{1-x^2} z + h^\mu_i x)] ,
\]

where \( \beta \equiv T^{-1} \) is the inverse temperature, \( Dz \) denotes the Gaussian measure: \( Dz \equiv \exp(-z^2/2) dz/\sqrt{2\pi} \), and the subscript \( h^\mu \) denotes the \( \{h^\mu_i\} \)-dependence of the map with

\[
h^\mu_i \equiv \xi_\mu_i \frac{1}{\sqrt{C}} \sum_j J_{ij} \xi^\mu_j .
\] (7)

In the stationary state, \( M_\mu(t) \) approaches \( M_\mu(t \to \infty) = M^*_\mu \) of which value is determined by the stable fixed point solution \( x \) satisfying

\[
x = F_{h_\mu}(x)
\]

\[
|\frac{\partial}{\partial x} F_{h_\mu}(x)| \equiv |F'_{h_\mu}(x)| < 1 ,
\] (8b)

where Eq. (8b) has been imposed to guarantee its stability. For given tolerance parameter \( m \), the network is considered to be remembering the \( \mu \)th pattern if the value of the stationary overlap \( M^*_\mu \) is greater than \( m \).

Now the main quantity to calculate is the fractional volume of the space of interactions \( \{ \{J_{ij}\} \} \) for which every pattern can be remembered. The normalization condition is now given by

\[
\sum_j (J_{ij})^2 = C
\]

instead of Eq. (4). The number of the solutions of Eqs. (8) with its value greater than \( m \) is formally given by

\[
\mathcal{N}_{h_\mu} \equiv \int_m^1 dM \delta(M - F_{h_\mu}(M)) |1 - F'_{h_\mu}(M)| \theta(1 - |F'_{h_\mu}(M)|)
\] (9)
so that the fractional volume can be written as

$$V_0 = \frac{\int [\prod_{i \neq j} dJ_{ij}] \prod_i \delta \left( \sum_j (J_{ij})^2 - C \right) \prod_{\mu=1}^{\alpha C} \theta(N_{h\mu})}{\int [\prod_{i \neq j} dJ_{ij}] \prod_i \delta \left( \sum_j (J_{ij})^2 - C \right)}$$

where \( \theta(x) \) is the step function and the number of stored patterns has been scaled according to \( p \equiv \alpha C \). Here we are mainly interested in obtaining the optimal storage capacity \( \alpha_c \). If the number of stored patterns exceeds \( \alpha C \), there is no typical network \( \{J_{ij}\} \) that yields the value of the stationary overlap greater than \( m \) for all patterns. In the limit \( \alpha \to \alpha_c \), the number \( N_{h\mu} \) of stable solutions approaches zero, and we may replace the step function \( \theta(N_{h\mu}) \) by \( N_{h\mu} \). Furthermore the fractional volume vanishes only if \( N_{h\mu} = 0 \) for some \( \mu \), which implies that the replacement \( \theta(x) \to x \) would not affect the optimal storage capacity. The fractional volume to calculate is now given by

$$V = \frac{\int [\prod_{i \neq j} dJ_{ij}] \prod_i \delta \left( \sum_j (J_{ij})^2 - C \right) \prod_{\mu=1}^{\alpha C} N_{h\mu}}{\int [\prod_{i \neq j} dJ_{ij}] \prod_i \delta \left( \sum_j (J_{ij})^2 - C \right)} \hspace{1cm} (10)$$

Replacement of \( \theta(x) \) by \( x \) in general is also justified in the following sense. We may assume that, for typical \( \{\xi^\mu\} \), \( N_{h\mu} \) possesses finite, system size independent upper bound almost everywhere in the interaction space. This is reasonable since the map \( F_{h\mu}(x) \) is an average of \( N \) functions of the form

$$\int Dz \tanh[\beta(\sqrt{1-x^2} z + hx)] .$$

This function is smooth and monotonic in \( x \) with derivative having the same sign as \( h \). Therefore the average over many possible \( h \) is a sum of two parts: the monotonically increasing part from contributions of \( h > 0 \) and the monotonically decreasing part from \( h < 0 \). So in practice there are only a few solutions at most. If we denote the upper bound by \( N_0 \), we then have the identity

$$\theta(N_{h\mu}) \leq N_{h\mu} \leq N_0 \theta(N_{h\mu}) .$$
Integrating this over the interaction space, we immediately see that

$$V_0 \leq V \leq N_0^{\alpha C} V_0.$$ 

However, the fractional volumes are of the order of \(\exp(-CN)\) so that \((\log V)/CN\) is the same as \((\log V_0)/CN\) in the thermodynamic limit \(N \to \infty\).

In this work, we consider the case that every pattern \(\xi_i^\mu\) to be stored is an independently distributed random variable, taking the value \(\pm 1\) with equal probabilities. The typical fractional volume \(\tilde{V} \equiv \exp(\langle \langle \log V \rangle \rangle)\) for the random patterns involves averaging of \(\log V\) over the distribution of the random patterns \(\{\xi_i^\mu\}\), which may be obtained through the use of the well-known replica trick. To facilitate the averaging over the distribution of the random patterns, we introduce \(\delta\)-functions describing Eq. (7) with the help of the conjugate variable \(\hat{h}_i^\mu\) raised to the exponential form

$$\delta(h_i^\mu - \xi_i^\mu \sum_j \frac{J_{ij}}{\sqrt{C}} \xi_j^\mu) = \int \frac{d\hat{h}_i^\mu}{2\pi} \exp[i\hat{h}_i^\mu(h_i^\mu - \xi_i^\mu \sum_j \frac{J_{ij}}{\sqrt{C}} \xi_j^\mu)].$$

The average over the random patterns for the replicated volume \(\langle \langle V^n \rangle \rangle\) affects the exponential factor containing \(\xi_i^\mu\) in the above expression, and leads to the following:

$$\langle \langle \prod_{\alpha=1}^n \exp(-i \sum_{\mu_i} \hat{h}_i^\alpha \xi_i^\mu \sum_j \frac{J_{ij}}{\sqrt{C}} \xi_j^\mu) \rangle \rangle_{\{\xi_i^\mu\}} = \prod_{\mu} \exp\left(-\frac{1}{2} \sum_{\alpha \beta \iota} \hat{h}_i^\alpha \hat{h}_i^\beta \sum_j \frac{J_{ij}^{\alpha \beta}}{C} - \frac{1}{2} \sum_{\alpha \iota} \hat{h}_i^\alpha \sum_{\beta \iota} \hat{h}_j^\beta \frac{J_{ij}^{\alpha \beta}}{C}\right),$$

where \(\alpha\) and \(\beta\) are the replica indices and, for an extremely-diluted network, the cumulant expansion has been cut-off at the second order [8]. Following Ref. [8], we assume the second term to be independent of site \(i\), so that

$$\sum_{\beta \iota} \hat{h}_j^{\mu \beta} \frac{J_{ij}^{\alpha \beta}}{C} = \frac{1}{N} \sum_i \sum_{\beta \iota} \hat{h}_j^{\mu \beta} \frac{J_{ij}^{\alpha \beta}}{C}.$$

Introducing the local order parameters

$$q_{\alpha \beta}^i \equiv \frac{1}{C} \sum_j J_{ij}^{\alpha \beta},$$

$$r_{\alpha \beta}^i \equiv \frac{1}{C} \sum_j J_{ij}^{\alpha \beta}.$$
together with their respective conjugate variables \( Q^i_{\alpha\beta} \) and \( R^i_{\alpha\beta} \), we obtain

\[
\langle \langle V^n \rangle \rangle = \frac{1}{\exp(nCN/2)} \int \prod_{a_i} dE^a_i \int \prod_{a_i < b_i} dQ^a_{\alpha\beta} dq^a_{\alpha\beta} \int \prod_{a_i} dR^a_{\alpha\beta} dr^a_{\alpha\beta} \exp(CG),
\]

where

\[
G \equiv \frac{1}{2} \sum_{a_i} E^a_i + \sum_{\alpha < \beta} Q^i_{\alpha\beta} q^i_{\alpha\beta} + \frac{1}{2} \sum_{a_i} R^i_{\alpha\beta} r^i_{\alpha\beta} + G_1 + \alpha G_2
\]

with \( G_1 \) and \( G_2 \) given by

\[
\exp(CG_1) \equiv \int \left[ \prod_{a_i \neq j} dJ^a_{ij} \right] \exp \left( -\frac{1}{2} \sum_{a_ij} E^a_i (J^a_{ij})^2 - \sum_{\alpha < \beta} Q^i_{\alpha\beta} J^\beta_{ij} - \frac{1}{2} \sum_{a_{i\beta i}} R^i_{\alpha\beta} J^\beta_{ij} J^\beta_{ji} \right),
\]

\[
\exp(G_2) \equiv \int \left[ \prod_{a_i} \frac{d\hat{h}^a_i}{2\pi} \prod_a N_{h^a} \right] \exp \left( i \sum_{a_i} \hat{h}^a_i h^a_i - \frac{1}{2} \sum_{\alpha < \beta} \hat{h}^a_i \hat{h}^\beta_i q^i_{\alpha\beta} \right. \\
\left. - \frac{1}{2N} \sum_{a_{i\beta i}} \hat{h}^a_i \hat{h}^\beta_i r^i_{\alpha\beta} - \frac{1}{2} \sum_{a_i} (\hat{h}^a_i)^2 \right).
\]

In the thermodynamic limit (\( N \to \infty \)), \( C \) also approaches infinity, albeit slowly, and \( \langle \langle V^n \rangle \rangle \) can be computed through the use of the steepest-descent method. In order to find the saddle point, we assume the replica- and site-symmetric ansatz

\[
E^a_i = E \quad R^i_{\alpha\alpha} = S \quad r^i_{\alpha\alpha} = s \quad \quad Q^i_{\alpha\beta} = Q \quad q^i_{\alpha\beta} = q \quad R^i_{\alpha\beta} = R \quad r^i_{\alpha\beta} = r \quad (\alpha \neq \beta).
\]

With this ansatz, the function \( G \) in the limit \( n \to 0 \) takes the form

\[
G = nN \left[ \frac{1}{2} (E - qQ + sS - rR) + g_1 + \alpha g_2 \right],
\]

where

\[
g_1 \equiv -\frac{1}{4} \left( \frac{Q + R}{E - Q + S - R} + \frac{Q - R}{E - Q - S + R} \right) + \log(E - Q + S - R) + \log(E - Q - S + R),
\]

\[
g_2 \equiv \frac{1}{N} \int \prod_{i=1}^{N} Dt_i \log \left[ \int \frac{dt_0}{\sqrt{2\pi/N}} \exp\left( -\frac{1}{2} N t_0^2 \right) \right] \int \prod_{i=1}^{N} Dh_i \, \mathcal{N}_H.
\]
In the above expression $\mathcal{N}_H$ is given by Eq. (9) with $h^\mu$ replaced by $H \equiv \{H_i\}$, where $H_i \equiv \sqrt{1-q} h_i - \sqrt{s-r} t_0 - \sqrt{q} t_i$. Since the saddle-point equations for the variables $E$, $S$, $Q$, and $R$ are algebraic, we can eliminate these variables and finally write the typical fractional volume in the form

$$V = \exp\left(CN\left[\frac{1}{2} \log(1-q) + \frac{1}{4} \log(1-x^2) + \frac{1}{2} \frac{q-rx}{(1-q)(1-x^2)} + \alpha g^2\right]\right),$$

where $x \equiv (s-r)/(1-q)$.

To manipulate $g^2$, we introduce the variable

$$M \equiv \frac{\partial}{\partial M} F_H(M)$$

together with its conjugate variable $\overline{\lambda}$ and use the integral representation of $\delta$-function. Noting the range of the variable $M$, we obtain

$$g^2 = \frac{1}{N} \int_{-\infty}^{\infty} \prod_{i=1}^{N} Dt_i \log\left[\int_{-\infty}^{\infty} \frac{dt_0}{\sqrt{2\pi/N}} \int_{i=0}^{1} dM \int_{-i=0}^{1} d\lambda \int_{-i=0}^{i=0} d\overline{\lambda} \frac{dM}{2\pi i/N}(1-M)^{\lambda} \right]$$

$$\times \exp[-N(\frac{1}{2}t_0^2 + \lambda M + \lambda M)] \prod_{i=1}^{N} \int Dh_i \exp[\lambda f(H_i, M) + \lambda M] \partial_M f(H_i, M)],$$

where the functions $f(H, M)$ and $\partial_M f(H, M)$ are given by

$$f(H, M) \equiv \int Dz \tanh[\beta(\sqrt{1-M^2} z + MH)]$$

$$\partial_M f(H, M) \equiv \frac{\partial}{\partial M} f(H, M).$$

Now the integration over $M$ is easily performed, and in the thermodynamic limit, the steepest-descent method yields

$$g^2 = \max_{m \leq M \leq 1} \max_{t_0} \min_{\lambda, \overline{\lambda}} \left(-\frac{1}{2}t_0^2 - \lambda M + |\overline{\lambda}| + \int Dt \log \int Dh \exp[\lambda f(H, M) + \overline{\lambda} \partial_M f(H, M)]\right),$$

(11)

where $H \equiv \sqrt{1-q} h - \sqrt{(1-q)x} t_0 - \sqrt{q} t$. In Eq. (11), we should take the minimum over $\lambda$ and $\overline{\lambda}$ rather than the maximum because the integration over $\lambda$ and $\overline{\lambda}$ runs along the imaginary axis in the complex plane, which should be deformed to pass the saddle point.
When the saddle point happens to lie on the real axis, one may conveniently sweep along
the real axis and the saddle point corresponds to the minimum point along the real axis.

Since $V$ depends on $s$ only through $x$, it is straightforward to show that $V$ reaches its
maximum at $x = 0$ and it follows that we can set $t_0 = 0$ in Eq. (11). Since $q$ represents
the typical correlations of the solution of Eqs. (8), the typical fractional volume should
shrinks to zero as $q$ approaches unity. Accordingly, the optimal storage capacity is then
determined in this limit. When $q$ approaches unity the last term in Eq. (11) diverges as
$\sim (1 - q)^{-1}$, and we write

$$
\int Dt \log \int Dh \exp[\lambda f(H, M) + \lambda \partial_M f(H, M)] \rightarrow \frac{1}{1 - q} \int Dt \Omega_t(H_t),
$$

where the function $\Omega_t(H)$ is then given by

$$
\Omega_t(H) \equiv -\frac{1}{2}(H + t)^2 + \lambda(1 - q)f(H, M) + \lambda(1 - q)\partial_M f(H, M)
$$

and $H_t$ is the value of $H$ leading to the maximum of $\Omega_t$. Therefore $g_2$ also exhibits $(1 - q)^{-1}$
divergence:

$$
g_2 = \frac{1}{1 - q} \max_{m \leq M \leq 1} \min_{\lambda, \lambda} \left( -\lambda(1 - q)M + |\lambda(1 - q)| + \int Dt \Omega_t(H_t) \right),
$$

and the saddle-point equation over $\lambda$ reads

$$
M = \int Dt f(H_t, M), \quad (12)
$$

where the dependence on the variables $\lambda$ and $\lambda$ is implicit through $H_t(\lambda, \lambda, M; T)$. For the
minimization over $\lambda$, one should consider two cases. The first case is that the minimum
occurs at $\lambda = 0$. This happens when the absolute value of $\int Dt \partial_M f(H_t, M)$ with $\lambda = 0$ and $\lambda = \lambda_0$, where $\lambda_0$ is given by the solution of Eq. (12) with $\lambda = 0$, is less than unity.
In the other case, the minimum occurs at $\lambda \neq 0$ and the saddle point in the $(\lambda, \lambda)$ plane is
given by the solution of Eq. (12) together with the equation

$$
-\text{sign}(\lambda) = \int Dt \partial_M f(H_t, M).
$$

11
In both cases, we denote the saddle point to be \((\lambda_0, \overline{\lambda}_0)\), and write \(g_2\) in the form

\[
g_2 = -\frac{1}{2(1 - q)} \min_{m \leq M \leq 1} \alpha_0^{-1}(M; T),
\]

where

\[
\alpha_0^{-1}(M; T) \equiv \int Dt \left[ t + H_t(\lambda_0, \overline{\lambda}_0, M; T) \right]^2.
\] (13)

Combining the above, we finally obtain the typical fractional volume:

\[
\overline{V} = \exp\left(\frac{CN}{2(1 - q)} \left[ 1 - \alpha \min_{m \leq M \leq 1} \alpha_0^{-1}(M; T) \right]\right),
\]

which vanishes for

\[
\alpha > \alpha_c \equiv \max_{m \leq M \leq 1} \alpha_0(M; T). \quad (14)
\]

Interestingly, \(\alpha_0(M; T)\) also represents the maximum storage capacity for the stationary value of the overlap \(M^*_\mu\) being in the range \(M \leq M^*_\mu \leq M + \delta M\). Due to the mean-field nature of the network, the optimal storage capacity is given by the maximum value of \(\alpha_0\) for the given range of \(M\).

Since \(\alpha_0(M; T)\) defined in Eq. (13) involves minimization with respect to two variables \((\lambda, \overline{\lambda})\) in addition to the two Gaussian integrals over \(t\) and \(z\) (representing thermal average), we computed them numerically. Figure 4 shows typical behavior of \(\alpha_0(M; T)\) for several values of \(T\), with detailed behavior for \(T = 0.3\) and 0.7 displayed in the inset. There exist three types of \(M\)-dependence of \(\alpha_0(M; T)\) according to \(T\). When \(T\) is higher than \(T_1(\approx 0.566)\), the maximum capacity \(\alpha_0\) decreases monotonically with \(M\). For \(T < T_1\), \(\alpha_0\) exhibits a local minimum as well as a local maximum (as shown in the inset of Fig. 4). This local maximum (at nonzero \(M\)) is in fact the global maximum of \(\alpha_0\) for \(T\) lower than \(T_2(\approx 0.414)\) whereas \(\alpha_0\) reaches its maximum at \(M = 0\) for \(T_2 < T < T_1\).

In contrast to the naive expectation, the maximum storage capacity \(\alpha_0\) is not monotonic with \(M\) (or with the error allowed) when the temperature is lower than \(T_1\). It is of interest to note that there are two kinds of fluctuations in the dynamics: One is thermal fluctuations associated with the synaptic noise and controlled by the temperature \(T\) while the other
is dynamical fluctuations described by $\sqrt{1-M^2}$. The latter fluctuations come from the distribution of states with definite overlap $M$ and may be considered to be driven by the dynamics itself. In general thermal fluctuations randomize spin orientations and tend to decrease the capacity whereas dynamical fluctuations affect the capacity in a more or less subtle manner because the level of dynamical fluctuations depends on the overlap. At zero temperature ($T=0$) and for $M=1$ neither thermal fluctuations nor dynamical fluctuations are present. In this limit, perfect matching is allowed, leading to $\alpha_0 = 2$ similarly to Gardner’s result [3]. However, small departure from $M=1$ induces dynamical fluctuations of the potential of the neurons, so that the maximum storage capacity $\alpha_0$ decreases rapidly as $M$ is reduced. At $T=0$, as shown in Fig. 1, $\alpha_0$ reaches its maximum at $M=1$ and hence we have the optimal storage capacity $\alpha_c = 2$ regardless of the tolerance parameter $m$. At finite temperatures there always exist thermal fluctuations, which prohibit perfect matching. In this case it may be expected that allowing some error (i.e., $m < 1$) increases the capacity. On the other hand, reducing the overlap introduces dynamical fluctuations and eventually reduces the capacity if the temperature is not too high ($T < T_1$). Near $M \approx 0$, reduction of the overlap in general increases the capacity at any temperature since dynamical fluctuations favor small values of the overlap. Here we stress that one should not expect the divergence of the capacity in the limit $m \rightarrow 0$ because the trivial solution $M^*_\mu = 0$ is not included.

From the curves of $\alpha_0$ it is straightforward to get the optimal storage capacity $\alpha_c$ defined by Eq. (14) for given tolerance parameter and temperature. Typical behavior of $\alpha_c$ as a function of $m$ is shown in Fig. 2 at several temperatures. At temperatures higher than $T_1 (\approx 0.566)$, $\alpha_0$ is a monotonic decreasing function of $m$. Consequently, we have $\alpha_c(m;T) = \alpha_0(M = m;T)$, and the curves of $\alpha_c$ are identical to those of $\alpha_0$. For $T < T_1$, there appears a plateau on which the optimal storage capacity is constant over some range of the tolerance parameter. (See Fig. 2. The boundary of this region is displayed by the dotted line.) For $T < T_2 (\approx 0.414)$, it is interesting to note that $\alpha_0$ reaches its maximum
near $M \approx 1$ and that a large value of the overlap is mostly favored.

Consider a problem that we want to store and recall $\alpha C$ random patterns in the network at temperature $T$ at the best performance, that is, we want the stationary overlap as large as possible. When $\alpha$ is small, one can easily find a set of couplings ($\{J_{ij}\}$) that yields the stationary value of the overlap near unity. As $\alpha$ increases, it becomes more difficult to find such a set of couplings. In general the quality of performance will deteriorate with the storage ratio $\alpha$. Since $\alpha_0(M;T)$ is the maximum storage capacity with the stationary overlap $M$ at temperature $T$, the best performance $M_p(\alpha;T)$ for given storage ratio $\alpha$ and temperature $T$ is determined by the largest value of $M$ for which $\alpha_0(M;T)$ is greater than $\alpha$. This implies $\alpha = \alpha_c(M_p;T)$ and the curve of the best performance also corresponds to the optimal storage capacity. Therefore Fig. 2 also represents curves of the best performance with the abscissa and the ordinate denoting $M_p$ and $\alpha$, respectively. As the number of stored patterns increases, there occurs a first-order transition from good performance to poor performance at temperatures not too high ($T < T_1$). Interestingly, at low temperatures ($T < T_2$) the network near saturation naturally favors high-quality performance; there are no networks yielding low-quality performance.

IV. Analysis using approximate criterion

In this section, we study the proposed scheme with the approximate criterion given by Eq. (5) instead of Eq. (3) because it is very difficult to solve the dynamics of neural networks in general. In fact the calculation even with the approximation is not easy and we implement the calculation only for extremely-diluted neural networks. The validity of the approximation will be tested against the result of Sec. III. The fractional volume $V$ of the space of interactions ($\{J_{ij}\}$) satisfying Eqs. (3) and (5) is given by

$$V = \int \prod_{i \neq j} dJ_{ij} \prod_{\mu=1}^{\alpha C} \theta \left( \frac{1}{N} \sum_{i=1}^{N} \xi_i^\mu \tanh \left( \frac{\beta}{\sqrt{C}} \sum_j J_{ij} \xi_j^\mu \right) - m \right) \prod_i \delta \left( \sum_j (J_{ij})^2 - C \right)$$
\[
\times \left\{ \int [\prod_{i \neq j} dJ_{ij}] \prod_{i} \delta \left( \sum_{j} (J_{ij})^2 - C \right) \right\}^{-1}.
\]

Although there is no restriction on the correlations between \( J_{ij} \) and \( J_{ji} \), different sites \( i \) and \( j \) are not decoupled because of Eq. (5); thereby it is not easy to calculate the typical fractional volume \( \mathcal{V} \equiv \exp(\langle \langle \log V \rangle \rangle) \), which involves the average \( \langle \langle \rangle \rangle \) of \( \log V \) over the distribution of the random patterns \( \{ \xi^\mu \} \). Nevertheless the calculation can be performed for an extremely-diluted network as discussed in the previous section. In this case the cumulant expansion can be cut-off as before at the second order.

Following procedure similar to that in Sec. III, we obtain the typical fractional volume in the form

\[
\mathcal{V} = \exp \left( C N \left[ \frac{1}{2} \log(1 - q) + \frac{1}{4} \log(1 - x^2) + \frac{1}{2} \frac{q - r x}{(1 - q)(1 - x^2)} + \alpha g_2 \right] \right), \tag{15}
\]

where the function \( g_2 \) in this case is given by

\[
g_2 = \frac{1}{N} \int_{-\infty}^{\infty} \prod_{i=1}^{N} D t_i \log \left[ \prod_{i=1}^{\infty} \frac{d t_0}{\sqrt{2\pi/N}} \exp \left( -\frac{1}{2} N t_0^2 \right) \right. \\
\times \left. \int \prod_{i=1}^{N} D h_i \theta \left( \frac{1}{N} \sum_{i=1}^{N} \tanh \left[ \beta \left( \sqrt{1 - q} h_i - \sqrt{s - r t_0} - \sqrt{q t_0} \right) \right] - m \right) \right] .
\]

with the same notations as in Sec. III. Using the integral representation of the \( \theta \)-function

\[
\theta \left( \frac{1}{N} \sum_{i=1}^{N} \tanh h_i - m \right) = \int_{m}^{1} dM \int_{-\infty}^{+\infty} \frac{d \lambda}{2\pi i/N} \exp \left( -N \lambda (M - \frac{1}{N} \sum_{i=1}^{N} \tanh h_i) \right)
\]

and noting the range of the integral variable \( M \), we obtain

\[
g_2 = -m \lambda - \frac{1}{2} t_0^2 + \int D t \log \left( \int D h \exp \{ \lambda \tanh \left[ \beta \left( \sqrt{1 - q} h - \sqrt{(1 - q) x} t_0 - \sqrt{q} t \right) \right] \} \right),
\]

where \( t_0 \) and \( \lambda \) are to be determined by the saddle-point equations. Since \( \mathcal{V} \) depends on \( s \) only through \( x \), it is straightforward to show that \( \mathcal{V} \) reaches its maximum at \( x = 0 \) and \( t_0 = 0 \).

The optimal storage capacity can be determined according to the condition that the typical fractional volume shrinks to zero, which happens as \( q \) approaches unity. In this
limit, the typical fractional volume given by Eq. (15) has the leading term:

$$V = \exp \left( CN \left[ \frac{1}{2} \log(1 - q) - \alpha m \lambda + \frac{\alpha}{1 - q} \int Dt \Omega_t(H_t) \right] \right),$$

where $\Omega_t(H) \equiv -\frac{1}{2}(H + t)^2 + \lambda(1 - q) \tanh(\beta H)$ in this case and $H_t$ is again the value of $H$ leading to the maximum of $\Omega_t$. Note that $H_t$ depends on $\lambda(1 - q)$ and $T$ in addition to $t$. Thus, in the limit $q \rightarrow 1$ and $\lambda \rightarrow \infty$ with $\lambda(1 - q)$ fixed, the saddle-point equations read

$$\alpha_c = \left( \int Dt \left\{ t + H_t[\lambda(1 - q); T] \right\}^2 \right)^{-1}$$

$$m = \int Dt \tanh \left( \beta H_t[\lambda(1 - q); T] \right),$$

where $H_t$ is, by definition, given by the solution of the equation

$$\Omega'_t(H) \equiv -(H + t) + \lambda(1 - q) \beta[1 - \tanh^2(\beta H)] = 0.$$  \hspace{1cm} (17)

Equation (17) has a unique root unless $\lambda(1 - q) > (3\sqrt{3}/4)T^2$ and $t_- < t < t_+$. [In this range Eq. (17) has three roots.] Here $t_\pm$ are defined to be

$$t_\pm \equiv \Omega'_{t=0}[H = -T \tanh^{-1}\left\{ \frac{2}{\sqrt{3}} \cos\left( \frac{\pi \pm \phi}{3} \right) \right\}]$$

with

$$\phi \equiv \cos^{-1}\left( \frac{3\sqrt{3}T^2}{4\lambda(1 - q)} \right).$$

At zero temperature it is straightforward to compute $H_t$ and to write the optimal storage capacity in the form

$$\alpha_c = \left( \int_0^{\sqrt{2} \text{erf}^{-1}(m)} Dt \; t^2 \right)^{-1}$$

while at finite temperatures Eqs. (16) can be solved numerically.

Figure 3 displays the optimal storage capacity $\alpha_c$ as a function of the tolerance parameter $m$ at various temperatures. The overall dependence of $\alpha_c$ on $m$ is qualitatively different from that obtained in Sec. III. In particular $\alpha_c$ diverges as $m \rightarrow 0$. Our approximate criterion loses its validity near $m = 0$ as it should be. However, $m$-dependence of $\alpha_c$ near $m = 1$ is not too disparate from that of Fig. 2 at finite temperatures. The $\alpha_c$ curves at
various temperatures shown in Fig. 3 are reproduced to expose the detailed behavior near $m = 1$ in Fig. 4, which, for comparison, also displays the corresponding curves obtained in Sec. III. At given temperature the two curves indeed coincide with each other in the limit $m \to 1$. Therefore we conclude that the approximate scheme based on Eq. (5) is valid for $m$ close to unity.

It is of interest to apply the above scheme to the dynamic model of neural networks [12], where a neuron is forced to have state $s_i = -1$ during the refractory period. As a consequence, Gardner’s method cannot be applicable even at zero temperature. In the dynamic model, equations describing the time evolution of relevant physical quantities in general assume the form of differential-difference equations due to the retardation in interactions. In particular, the activity $\langle s_i(t) \rangle$ of the $i$th neuron at time $t$ and the overlap $M_\mu(t)$ between the network state and the $\mu$th pattern at time $t$ are determined by the differential-difference equations

$$ \frac{d}{dt} \langle s_i(t) \rangle = \left( \frac{1}{2} - a \right) - \left( \frac{1}{2} + a \right) \langle s_i(t) \rangle + \frac{1}{2} \left( 1 - \langle s_i(t) \rangle \right) \tanh \left( \frac{\beta}{\sqrt{N}} \sum_j J_{ij} \langle s_i(t-1) \rangle \right) $$

$$ \frac{d}{dt} M_\mu(t) = -\left( \frac{1}{2} + a \right) M_\mu(t) + \frac{1}{2N} \sum_{i=1}^{N} \xi_\mu^i \left( 1 - \langle s_i(t) \rangle \right) \tanh \left( \frac{\beta}{\sqrt{N}} \sum_j J_{ij} \langle s_i(t-1) \rangle \right), $$

respectively. Here $a$ represents the ratio of the refractory period to the time duration of the action potential. In the stationary state, the overlap $M_\mu$ takes the form

$$ M_\mu = \frac{4a}{1 + 2a} \frac{1}{N} \sum_{i=1}^{N} \frac{\xi_\mu^i \tanh \left( \frac{\beta}{\sqrt{N}} \sum_j J_{ij} \langle s_i \rangle \right)}{1 + 2a + \tanh \left( \frac{\beta}{\sqrt{N}} \sum_j J_{ij} \langle s_i \rangle \right)}. \tag{18} $$

Since $\xi_\mu^i \sum_j J_{ij} \xi_\mu^j$ tends to be positive for typical types of interactions, we may, in the extreme-dilution limit, make an approximation in Eq. (18) as

$$ M_\mu = \frac{1}{N} \sum_{i=1}^{N} \frac{4a \tanh \left( \frac{\beta}{\sqrt{N}} \sum_j \xi_\mu^i J_{ij} \xi_\mu^j \right)}{(1 + 2a)^2 - \tanh^2 \left( \frac{\beta}{\sqrt{N}} \sum_j \xi_\mu^i J_{ij} \xi_\mu^j \right)}. $$
The optimal storage capacity of the dynamic model is now determined by Eqs. (16) except for that the \( \tanh x \) function is replaced by \( 4a \tanh x/(1 + 2a)^2 - \tanh^2 x \). Unlike the Hopfield model which discretizes the time, the dynamic model takes into account the existence of relevant time scales, and consequently, displays the overlap \( M_\mu = 1/(1 + a) \) in the case of perfect recalling. This is reflected in the equation corresponding to Eq. (16b). For comparison with the Hopfield model, therefore, we rescale the tolerance parameter \( m \) by \( \tilde{m} \equiv (1 + a)m \), and finally get

\[
\alpha_c(\tilde{m}) = \left( \int_0^{\sqrt{2}\text{erf}^{-1}(\tilde{m})} Dt t^2 \right)^{-1}
\]

at zero temperature.

V. Discussion

We have proposed a new method to study the optimal storage property of neural networks at finite temperatures and investigated the optimal storage capacity \( \alpha_c \) for an extremely-diluted network, as a function of temperature \( T \) and the tolerance parameter \( m \). At zero temperature, it has been found that \( \alpha_c = 2 \) regardless of the tolerance parameter whereas at finite temperature \( \alpha_c \) vanishes in the perfect matching limit \( (m \to 1) \), in general increasing with the tolerance. The best performance for given storage ratio \( \alpha \) and temperature has been also obtained. At low temperatures \( (T < T_1 \approx 0.566) \) the network exhibits a first-order transition from high-quality performance to low-quality performance as the number of stored random patterns is increased. High-quality performance seems to be naturally favored by extremely-diluted networks if the level of noise is not so high. We have also studied an approximate scheme, which yields qualitatively different results except near \( m = 1 \). The crude approximation \( \langle s_i \rangle \approx \xi_i^\mu \) used in Eq. (16) has been found not good in the whole range of \( m \); for \( m \) close to unity, however, it can be regarded as an accurate approximation.
Instead of the proposed criterion for recalling, one may consider a slightly different criterion: The time average of $\xi^\mu_i s_i(t)$ for each site $i$ should be greater than $m$. In the same spirit as in Eq. (5), one may consider the problem of calculating the typical fractional volume of the space of interactions satisfying Eq. (4) and

$$\xi^\mu_i \tanh(\frac{\beta}{\sqrt{N}} \sum_j J_{ij} \xi^\mu_j) > m$$

(19)

for each $i$. The problem is then equivalent to Gardner’s problem with her parameter $\kappa$ given by $\kappa = T \tanh^{-1} m$, which implies that the required basin of attraction grows larger with the level of synaptic noise and with the accuracy of recalling. At zero temperature this leads to the optimal capacity $\alpha_c = 2$ regardless of $m$, whereas at finite temperatures $\alpha_c$ increases with the tolerance. Despite their resemblance, the behavior of the optimal storage capacity with the criterion (19) is qualitatively different from that of (5). Although the argument of tanh in both cases is a sum over many sites $j$, it should be strongly correlated with $\xi^\mu_i$ if the network ($\{J_{ij}\}$) is to function as associative memory. In general, the overlap on site $i$, given by the r.h.s. of Eq. (19), will vary from site to site according to some distribution with finite variance. The criterion (5) in the large $N$ limit requires the overlap averaged over that distribution be greater than $m$ while that of (19) demands that the overlap be greater than $m$ for each site.

Finally, there are several points for further investigation. Since we have assumed the replica- and site-symmetric ansatz in the calculation of the typical fractional volume, its stability against replica-symmetry-breaking should be checked. It is also of interest to extend our results to the fully-connected networks and other types of networks.

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Figure Captions

Fig. 1 Typical behavior of the maximum storage capacity $\alpha_0$ as a function of the stationary overlap $M^*$ at various temperatures: From top to bottom, $T = 0, 0.3, 0.5,$ and $0.7$, respectively. Detailed behavior at $T = 0.3$ and $0.5$ is displayed in the inset.

Fig. 2 Typical behavior of the optimal storage capacity $\alpha_c$ as a function of the tolerance parameter $m$ at various temperatures: From top to bottom, $T = 0.3, 0.5,$ and $0.7$, respectively. The dotted curve shows the boundary of the region in which $\alpha_c$ is constant.

Fig. 3 The optimal storage capacity $\alpha_c$ as a function of the tolerance parameter $m$ at $T = 0, 0.3, 0.5, 0.7,$ and $2.0$ when the alternative approximate criterion given by Eq. (5) is used.

Fig. 4 Detailed behavior of the optimal storage capacity $\alpha_c$ for the tolerance parameter $m$ near unity. Solid lines and dotted lines are results of Eqs. (3) and (5), respectively.