Nearly optimal quantum control: an analytical approach

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Abstract
We propose nearly optimal control strategies for changing the states of a quantum system. We argue that quantum control optimization can be studied analytically within some protocol families that depend on a small set of parameters for optimization. This optimization strategy can be preferred in practice because it is physically transparent and does not lead to combinatorial complexity in multistate problems. As a demonstration, we design optimized control protocols that achieve switching between orthogonal states of a naturally biased quantum two-level system.

Keywords: quantum control, shortcuts to adiabaticity, control protocols, two-level system, sine-Gordon equation

1. Introduction
Future miniaturization technologies will face the challenge of designing elementary information-processing units that employ quantum effects and operate simultaneously at high speed and with low dissipation. Such requirements are usually contradictory to each other. Therefore, they raise the need for quantum optimal control, i.e., the design of protocols for application of time-dependent fields that lead to the desired behavior of a quantum system while minimizing some cost function\textsuperscript{[1]}. There are two major complications with control optimization at the quantum level. First, such control has to be nonlinear. This follows already from the fact that the physical characteristics of a quantum system are nonlinear functions of dynamic variables, which are the state vector components. Moreover, control fields multiply the state vector in the Schrödinger equation, so this equation is nonlinear if we treat both the state vector and control fields as dynamic variables. Even for a spin-1/2 system, conventional optimal control already becomes complex, requiring involved mathematical treatment and sometimes numerical calculations to find the optimal protocols\textsuperscript{[2–6]}. The second complication is the exponentially fast growth of the number of variables that parametrize the state vector. For example, $N$ qubits are described quantum-mechanically by a state vector with $2^N$ components. So, beyond the most elementary cases, even the numerical solution of a quantum optimization problem by the conventional methods of control theory is hard to achieve.

In this article, we argue that the complexity of the optimal quantum control problem can be considerably reduced in many applications. We propose to search for the desired optimization only within some families of control protocols that allow analytical solution of the nonstationary Schrödinger equation. If such a family is sufficiently broad, one can expect that optimization within this family will lead to a reasonable cost while the problem will be tractable.

Finding large classes of exactly solvable models with the desired properties may appear impossible at first sight. Indeed, there is even no known general analytical solution for the dynamics of a spin-1/2 system in an arbitrary time-dependent magnetic field. Higher-dimensional cases are even more complex because they require solutions of differential equations of higher than second order with time-dependent parameters, which remain poorly understood.
Contrary to such expectations, we would like to point out that it is actually not hard to generate families of time-dependent control protocols whose effects on quantum systems can be understood analytically. One possibility to do this is to apply control fields that simply compensate for terms in the Hamiltonian that are responsible for complex behavior. For example, if there are controlled couplings between some qubits, we can simply set such couplings to zero in order to make individual qubit dynamics easy to control by simple local fields for a while. It is then possible to determine and optimize costs within the family of such protocols analytically.

Another possibility to design a family of solvable Schrödinger equations is to solve an inverse problem. Namely, in the equation

$$i\frac{d\Psi}{dt} = \hat{H}(t)\Psi,$$

(1)

we can prescribe the functional form for some desired types of dynamics of $\Psi(t)$ and then treat (1) as the linear equation for unknown elements of the matrix Hamiltonian $\hat{H}(t)$. Since the number of components of the state vector scales linearly with the size of the phase space $N$, and the number of components of the matrix $\hat{H}$ scales as $N^2$, this equation can generally be solved with large redundancy. Many families of such (obtained) exactly solvable models are known [7–9]. Models generated in this way describe somewhat unnatural dynamics but this is not a drawback for quantum control purposes. After parametrizing the class of solutions of such an inverse problem, we can explore the cost function within the resulting parameter space.

In the rest of the article, we demonstrate these two strategies using the model of control of a spin-1/2 system by a time-dependent magnetic field. Instead of straightforward global optimization, we are going to use the fact that there are classes of protocols for which analytical solutions of the Schrödinger equation are known. Within such classes the optimization problem greatly simplifies, but generally the result will not coincide with a fully optimized solution. In this sense, our result is ‘nearly optimal’.

2. The model

Consider the two-level system shown in figure 1(a). We assume that the system is initially in the ground state, which is separated from the excited state by a natural energy splitting $2\varepsilon$. So, the free Hamiltonian of the system is

$$\hat{H}_{\text{free}} = -\varepsilon \hat{\sigma}_z,$$

(2)

where $\hat{\sigma}_z$ is the Pauli operator. Assume that the goal is to design pulses of some control field that will lead to the definite transition from the ground state to the excited state. It is convenient to think about a two-level system in terms of a spin-1/2 in a time-dependent magnetic field. So, the general Hamiltonian is

$$\hat{H} = -\varepsilon \hat{\sigma}_z + B_{x}^{\text{ex}}(t)\hat{\sigma}_x + B_{y}^{\text{ex}}(t)\hat{\sigma}_y + B_{z}^{\text{ex}}(t)\hat{\sigma}_z,$$

(3)

where $B_{x}^{\text{ex}}$, $B_{y}^{\text{ex}}$, and $B_{z}^{\text{ex}}$ are control parameters, which we will call components of the external magnetic field. There are numerous known ways to design pulses of a magnetic field in order to flip the spin. However, a practical situation may impose conditions that favor some protocols over the others. Specifically, we will assume that the following four conditions must be met.

(i) During the spin flip, nonzero average spin polarization along the $y$-axis should be avoided by all means possible. Thus, we require that $\langle \Psi(t)|\hat{\sigma}_y|\Psi(t)\rangle = 0$ at any time.

(ii) We want to minimize the time that the spin spends not being strictly polarized along the $z$-axis. In other words, we want to flip the spin as fast as possible.

(iii) The size of the external control field that we apply should be minimized.

(iv) We are looking for protocols whose properties can be written in terms of some known special functions.

Here we would like to stress that our choice of conditions (i)–(iv) is not motivated by some immediate current experimental research. We chose them for purely illustrative reasons because they provide an example that is sufficiently nontrivial mathematically to illustrate the advantages and possible weaknesses of our approach. A more experimentally relevant problem could be to include only condition (ii) as considered in [4], or only condition (iii) as in [2]. However, our conditions look realistic for distant future applications in which large numbers of qubits will be placed in arrays, so that the phase space and energy cost for qubit dynamics during control will have to satisfy many restrictions, while shaping local pulses of control fields in 3D will not be a problem.

For definiteness, we will assume that the control protocol starts at some negative time moment so that the spin appears in the state ‘up’ by time $t = 0$, after which all external fields are switched off. The necessity of condition (i) may be justified in a real situation by the requirement to avoid unwanted coupling with ambient qubits that are placed along the $y$-axis. This condition restricts the allowed phase space for dynamics. We assume that this condition is not negotiable in the sense...
that the cost of deviations from it is formally infinite. In particular, this condition forbids the application of popular Rabi-like pulses that induce circulation of spin polarization in the xy-plane.

Condition (iv) expresses the desire to work with pulse shapes with easily characterizable properties. This means that we are not looking for a complete solution of the optimization problem in terms of some nonlinear differential equation that can be solved only numerically. Instead, we restrict our search to a specific class of protocols with known analytical solutions. This class is quite large, so we hope that the deviation of the cost of our protocol is not essentially different from the cost of a numerically exact solution of the optimization problem.

Conditions (ii) and (iii) conflict with each other. In order to minimize the time of spin flip, we should apply strong control fields, but such fields induce unwanted dissipation that we also want to minimize. In order to resolve this conflict, we should quantify conditions (ii) and (iii) by introducing some cost functional $C$. We will assume the simplest form of $C$ that is consistent with the symmetry of the problem and the condition that this functional should involve not higher than second-order powers of expectations of Pauli matrices and control field components:

$$C = \int_{-\infty}^{0} \mathcal{L} \, dt = \int_{-\infty}^{0} \, dt \left[ A (\langle \Psi(t) | \hat{\sigma}_y | \Psi(t) \rangle)^2 + (B_{x}^{ex}(t))^2 + (B_{y}^{ex}(t))^2 + (B_{z}^{ex}(t))^2 \right].$$

(4)

Here the first term inside the integral penalizes all values of the state vector $|\Psi(t)\rangle$ that have nonzero expectation value of the $\hat{\sigma}_x$ operator. Since we are interested in protocols that asymptotically have $\langle \Psi(t) | \hat{\sigma}_x | \Psi(t) \rangle^2 = 0$ at $t \to -\infty$, 0, the time integral of this expression has the meaning of the effective time of spin flip. Note that starting at $t = -\infty$ does not mean that the transition takes infinite time; this is just our domain of search for protocol optimization. If the external field were the true magnetic field, the physical meaning of the rest of the cost would be merely the energy of the pulse. So, naturally, this cost favors application of smaller magnetic field values. The parameter $A$ describes the relative importance of the two cost contributions. Bigger values of $A$ favor faster protocols that require larger external field amplitudes. We will not consider other possible restrictions that have been encountered in the literature [10].

We can now quantify the problem. Our goal is to find the time-dependent external field that induces dynamics of the state vector that minimizes the functional $C$ of the form (4) under the conditions that during the evolution we have $\langle \Psi(t) | \hat{\sigma}_x | \Psi(t) \rangle = 0$ and the spin is fully polarized in opposite directions along the $z$-axis at the beginning and the end of the protocol. This minimization can be performed by the calculus of variations [11, 12], which gives Euler–Lagrange equations for the external field. Such an Euler–Lagrange approach has been used to study optimal qubit control, e.g., in [12–14].

3. Optimal square-pulse protocol

Without the natural bias (i.e., at $\varepsilon = 0$), we would be able to rotate the spin by merely applying the external field along the $y$-axis. This problem is exactly solvable and its optimization was solved in [15]. At $\varepsilon \neq 0$, the formal approach to optimization, however, would involve the introduction of Lagrange multipliers that account for evolution of the state vector and constraints. In contrast, as our first strategy, we consider a straightforward way to rotate the spin while keeping its polarization within the $xz$-plane: we just apply an external field in the $z$-direction, i.e., momentarily set $B_{y}^{ex} = \varepsilon$, in order to compensate for the intrinsic field bias. Simultaneously, we apply a constant field along the $y$-direction while keeping $B_{x}^{ex} = 0$. As a result, the spin will precess around the $y$-axis. Let $\phi$ be the angle that our spin makes with the $x$-axis, as shown in figure 1(b). For precession in this constant magnetic field we have

$$\langle \Psi(t) | \hat{\sigma}_y | \Psi(t) \rangle = \sin^2(2B_{y}^{ex} t).$$

(5)

We should switch the external field off when the spin completes rotation by an angle $\pi$. Obviously, for uniform rotation, our protocol should have a time duration

$$T_u = \pi/(2B_{y}^{ext}),$$

(6)

where the subscript ‘$u$’ stands for ‘uniform’, and the cost is

$$C_u = \int_{-T_u}^{0} \mathcal{L} \, dt = \frac{\pi^2}{4T_u} + \left( \varepsilon^2 + \frac{A}{2} \right) T_u.$$  

(7)

Using (6), this cost is minimized at $B_{y}^{ex} = \sqrt{\varepsilon^2 + \frac{A}{2}}$:

$$C_u^{\text{min}} = \pi \sqrt{\varepsilon^2 + \frac{A}{2}}.$$  

(8)

The square-pulse protocol is simple but it is expected to be strongly suboptimal because minimization is performed only within a family of field pulses parametrized by a single parameter $T_u$. In what follows, we will explore more complex strategies. First, we can improve the above strategy by allowing a more complex time dependence of $B_{y}^{ex}(t)$. Second, we can consider a different family of protocols that have nonzero values of $B_{x}^{ex}(t)$.

4. Protocols with time-dependent $B_{y}^{ex}$

Let us now assume that we again apply the field with $B_{x}^{ex} = \varepsilon$ and $B_{z}^{ex} = 0$ during some time $T_0$. Next, we treat $T_0$ as a free parameter and we also allow an arbitrary time dependence of $B_{y}^{ex}(t)$. The only constraint on the transverse field is that

$$2 \int_{-T_0}^{0} B_{y}^{ex}(t) \, dt = \pi.$$

We have then

$$\langle \Psi(t) | \hat{\sigma}_y | \Psi(t) \rangle = \sin^2 \left( \int_{-T_0}^{t} 2B_{y}^{ex}(t') \, dt' \right).$$  

(9)
To simplify this expression, we will introduce the phase variable
\[
\phi(t) \equiv -\pi/2 + 2\int_{-T_0}^{t} B_y(s') \, ds',
\]
in terms of which the cost functional now reads
\[
C_{\text{SG}}[\phi(t)] = \int_{-T_0}^{0} \left\{ \frac{\dot{\phi}^2}{4} + \varepsilon^2 + A \cos^2(\phi) \right\} \, dt,
\]
where the dot denotes differentiation with respect to \( t \) and the subscript ‘SG’ stands for ‘sine-Gordon’ because equation (10) coincides with a Lagrangian of the classical mechanical motion of a particle along the coordinate \( \phi \) in the sine-Gordon potential \(-A \cos^2(\phi)\). The term with \( \varepsilon^2 \) in (10) should be kept because it penalizes protocols with longer duration \( T_0 \).

Varying the cost functional over \( \phi(t) \), one can find the equation of motion. Instead of solving this equation, we invoke the classical mechanical analogy and say that the energy \( \varepsilon \) of the particle motion is conserved:
\[
\frac{\dot{\phi}^2}{4} - \varepsilon^2 - A \cos^2 \phi = \varepsilon = \text{const.} \tag{11}
\]
Hence,
\[
\dot{\phi} = 2\sqrt{\varepsilon^2 + A \cos^2 \phi + \varepsilon}. \tag{12}
\]

Formally, the duration of the protocol \( T_0 \) should also be optimized. However, by varying cost over this parameter we find the value of energy that cannot be used to satisfy the boundary conditions. Instead, we will treat \( \varepsilon \) as an additional parameter for optimization. Integrating over time, we find (definitions of special functions are provided in the appendix):
\[
\phi = \arcsin \left[ \sin \left( 2\sqrt{\varepsilon^2 + A + \varepsilon (t + T_0)} \right) \frac{A}{\varepsilon^2 + A + \varepsilon} \right]. \tag{13}
\]
and using the boundary conditions, \( \phi(t = -T_0) = -\pi/2 \) and \( \phi(t = 0) = \pi/2 \), we obtain
\[
T_0(\varepsilon) = \frac{1}{\sqrt{\varepsilon^2 + A + \varepsilon}} K \left( \frac{A}{\varepsilon^2 + A + \varepsilon} \right), \tag{14}
\]
where \( K(m) \) is the complete elliptic integral of the first kind (see appendix).

Substituting (12)–(14) into (10) we find
\[
C_{\text{SG}} = -\varepsilon T_0(\varepsilon) + 2\sqrt{\varepsilon^2 + A + \varepsilon} E \left( \frac{A}{\varepsilon^2 + A + \varepsilon} \right), \tag{15}
\]
where \( E(m) \) is the complete elliptic integral of the second kind. Finally, we should minimize the cost over the remaining free parameter \( \varepsilon \). Differentiating (15) with respect to \( \varepsilon \), we find that the minimum is achieved at \( \varepsilon = 0 \), so
\[
\phi(t) = \arcsin \left[ \sin \left( 2\sqrt{\varepsilon^2 + A (t + T_0)} \right) \frac{A}{\varepsilon^2 + A} \right],
\]
\[
\pi/2 \geq \phi \geq -\pi/2, \tag{16}
\]
where \( T_0 \) is given by
\[
T_0 = \frac{1}{\sqrt{\varepsilon^2 + A}} K \left( \frac{A}{\varepsilon^2 + A} \right).
\]
and the minimal cost within this family of protocols is given by
\[
C_{\text{min}} = 2\sqrt{\varepsilon^2 + A} E \left( \frac{A}{\varepsilon^2 + A} \right) \tag{18}
\]

5. Optimal shortcut to adiabaticity

A bigger class of analytically tractable protocols can be obtained by solving the inverse problem to the nonstationary Schrödinger equation. Many techniques to do this have been developed for spin-1/2 systems [7–9]. Here we will focus on the class of such solutions called shortcuts to adiabaticity [9, 16–20]. This class covers not only spin-1/2 but also many practically interesting multistate situations [21].

Let \( \hat{H}_0(t) \) be some time-dependent Hamiltonian and \( |u(t)\rangle \) be one of the instantaneous eigensates of this Hamiltonian, which depends continuously on \( t \). Note that \( |u(t)\rangle \) is generally not a solution of the nonstationary Schrödinger equation with \( \hat{H}_0(t) \). The Hamiltonian of a shortcut to adiabaticity is obtained by adding a counterterm \( \hat{H}_{ct}(t) \) such that the evolution of the state vector with the Hamiltonian \( \hat{H}(t) = \hat{H}_0(t) + \hat{H}_{ct}(t) \) does follow the path of eigenstates \( |u(t)\rangle \). It turns out that there is a formal expression for such a counterterm [9, 19, 20]:
\[
\hat{H}_{ct} = \frac{i}{2} \left( |\partial_t u(t)\rangle \langle u(t)| - |u(t)\rangle \langle \partial_t u(t)| \right). \tag{19}
\]
In other words, the instantaneous eigenvector \( |u(t)\rangle \) of \( \hat{H}_0(t) \) becomes the exact solution of the nonstationary Schrödinger equation (1) with the Hamiltonian \( \hat{H}(t) = \hat{H}_0(t) + \hat{H}_{ct}(t) \), where the counterterm is given by (19).

For the following discussion, it will be convenient to introduce a new vector \( \mathbf{B} \) with absolute value \( B \) and components \( B_i \equiv B \cos(\phi) = B_i^{\text{ext}} \); \( B_1 = 0 \); \( B_2 = B \sin(\phi) = B^{\text{ext}} - \varepsilon \). Note that \( B \) and \( \phi \) generally depend on time, although later we may not explicitly write out this time dependence. If now we choose
\[
\hat{H}_0(t) = B_2(t) \hat{\sigma}_z + B_1(t) \hat{\sigma}_y, \tag{20}
\]
then, as desired, we will have evolution of the eigensate \( |u(t)\rangle \) such that spin polarization will always be directed along the field \( \mathbf{B} \), which is restricted to the \( xz \)-plane. For such a spin-1/2 Hamiltonian, the counterterm is [18, 19]
\[
\hat{H}_{ct}(t) = \frac{1}{2} \frac{d}{dt} \left( \arctan \frac{B_1}{B_2} \hat{\sigma}_y = -\frac{B_2 B_1 - B_1 B_2}{2[B_1]^2 + [B_2]^2} \hat{\sigma}_y, \tag{21}
\]
which can also be expressed as
\[
\hat{H}_{ct} = -\frac{1}{2} \dot{\phi} \hat{\sigma}_y. \tag{22}
\]
i.e., it is induced by the magnetic field \( B_{y,ct} = -\dot{\phi}/2 \) directed along the \( y \)-axis. The boundary conditions in terms of the angle \( \phi \) read

\[
\phi(-\infty) = -\frac{\pi}{2}, \quad \phi(0) = \frac{\pi}{2}.
\]

We will also assume that the spin rotates counterclockwise. The cost function (4) now has the Lagrangian

\[
L = \frac{1}{4} \dot{\phi}^2 + B^2 + 2B\varepsilon \sin \phi + \varepsilon^2 + A\cos^2 \phi.
\]

Since the Lagrangian (24) does not depend explicitly on \( \dot{B} \), variation over \( B \) produces an algebraic constraint:

\[
\frac{\partial L}{\partial B} = 2B + 2\varepsilon \sin \phi = 0,
\]

from which we find

\[ B = -\varepsilon \sin \phi. \] (26)

At this point, we encounter a complication. By definition, \( B \) is the absolute value of the field vector, so it is always non-negative. Hence, constraint (26) can be satisfied only for \( \sin \phi \leq 0 \). Since negative values of \( B \) are impossible and larger positive values of \( B \) are unfavorable at \( \sin \phi > 0 \), we have to construct our protocol so that at \( \phi \leq 0 \) condition (26) is satisfied and at \( \phi > 0 \) we just set \( B = 0 \). (Recall that \( B = 0 \) does not mean that the whole field is zero; it is the field in the \( xz \)-plane that vanishes.) Let us now consider these two stages of the protocol separately.

**Case** \(-\pi/2 \leq \phi < 0\). In this region, the field is given by equation (26). Substituting (26) into (24), we obtain

\[
\mathcal{L}_{\phi<0} = \frac{\dot{\phi}^2}{4} + \varepsilon^2 \cos^2 \phi + A\cos^2 \phi.
\]

This is a Lagrangian of the classical motion of a particle with mass \( m = 1 \) in a potential \( V(\phi) = -\varepsilon^2 \cos^2 \phi - A\cos^2 \phi \). The corresponding equation for energy conservation reads

\[
\frac{\dot{\phi}^2}{4} - (\varepsilon^2 + A)\cos^2 \phi = e.
\]

Energy \( e \) is found here from the observation that at \( t \to -\infty \) we must have \( \phi \to 0 \), and consequently \( e = 0 \). So,

\[
\phi = 2\sqrt{\varepsilon^2 + A} \cos \phi,
\]

from which we find

\[
\phi(t) = 2 \arctan \frac{1}{2} \sqrt{\varepsilon^2 + A (t + T)} = \frac{\pi}{2},
\]

\[
= \arctan \{ \sinh[2\sqrt{\varepsilon^2 + A (t + T)]} \}, \quad \phi < 0.
\]

At this point, we treat \( T \) as a free parameter, whose value we will obtain self-consistently later.

The field components \( B_x \), \( B_t \) and \( B_{y,ct} \) are found as

\[
B_x = B \cos \phi = -\varepsilon \sin \phi \cos \phi = -\varepsilon \frac{a}{1 + a^2}, \quad (31)
\]

\[
B_t = B \sin \phi = -\varepsilon \sin^2 \phi = -\varepsilon \frac{a^2}{1 + a^2}, \quad (32)
\]

\[
B_{y,ct} = -\dot{\phi}/2 = -\frac{\sqrt{\varepsilon^2 + A}}{\sqrt{1 + a^2}},
\]

\[
a = \sinh[2\sqrt{\varepsilon^2 + A (t + T)]}].
\]

We plot them in figure (b). We can also calculate the cost of rotating the spin from \( \phi = -\pi/2 \) to \( \phi = 0 \) along this protocol:

\[
C_{\phi<0} = \int_{-\infty}^{-T} L_{\phi<0} dt = \frac{1}{2} \int_{-\infty}^{-T} \dot{\phi}^2 dt = \sqrt{\varepsilon^2 + A}.
\]

![Figure 2](image-url)
Case $\pi/2 \geq \phi > 0$. For such values of the rotation angle we have $B = 0$, and the Lagrangian (24) reads

$$L_{\phi > 0} = \frac{1}{4} \dot{\phi}^2 + \varepsilon^2 + A \cos^2 \phi.$$  \hfill (35)

This Lagrangian is the same as the one that we studied in section 4. The only difference is that the initial boundary conditions are now $\phi(t = -T) = 0$. Using the results of the previous section and the symmetry of evolution with Lagrangian (35), we find that

$$T = \frac{T_0}{2} = \frac{1}{2 \sqrt{\varepsilon^2 + A}} \left( \frac{A}{\varepsilon^2 + A} \right),$$  \hfill (36)

$$\phi(t) = \arcsin \left[ \sin \left( 2 \sqrt{\varepsilon^2 + A} (t + T) \right) \frac{A}{\varepsilon^2 + A} \right], \quad \phi > 0,$$  \hfill (37)

and the cost of the second part of the protocol is

$$C_{\phi > 0} = \sqrt{\varepsilon^2 + A} E \left( \frac{A}{\varepsilon^2 + A} \right).$$  \hfill (38)

Finally, we combine the two costs to find

$$C_{\min} = C_{\phi > 0} + C_{\phi < 0} = \sqrt{\varepsilon^2 + A} + \sqrt{\varepsilon^2 + A} E \left( \frac{A}{\varepsilon^2 + A} \right).$$  \hfill (39)

Figure 2 summarizes the dependence of the angle $\phi$ on time as given by equations (30) and (37), as well as the components of the field that induce this spin rotation.

Let us now compare costs of optimal protocols in equation (8) for uniform rotation, equation (18), for fully compensated bias $B_1 = \varepsilon$ but flexible time dependence of $B_1(t)$, and the cost (39) of the protocol that used shortcuts to adiabaticity. In terms of the dimensionless ratio $\gamma = \varepsilon^2/A$, these costs can be written as

$$C_{\min} = \sqrt{A} \sqrt{\gamma} + 1 \left[ 1 + E \left( \frac{1}{1 + \gamma} \right) \right],$$  \hfill (40)

$$C_{u,G}^\text{min} = 2 \sqrt{A} \sqrt{\gamma} + 1 \left[ 1 + E \left( \frac{1}{1 + \gamma} \right) \right],$$  \hfill (41)

$$C_{u,G}^\text{min} = \pi \sqrt{A} \sqrt{\gamma} + \frac{1}{2}.$$  \hfill (42)

We find that ratios of these costs depend only on $\gamma$. Figure 3 presents the ratios $C_{\min}^\text{min} / C_{u,G}^\text{min}$ and $C_{\min}^\text{min} / C_{u,G}^\text{min}$ versus $\gamma$. It shows that these two ratios are smaller than 1 for any $\gamma$. Thus, for any choice of parameters $A$ and $\varepsilon$, the protocol using shortcuts to adiabaticity has a lower cost than the protocols based on instantaneous compensation. For small and large $\gamma$, the limits of the two ratios are

$$\lim_{\gamma \to 0} \frac{C_{\min}^\text{min}}{C_{u,G}^\text{min}} = \frac{2 \sqrt{2}}{\pi} \approx 0.900,$$  \hfill (43)

$$\lim_{\gamma \to \infty} \frac{C_{\min}^\text{min}}{C_{u,G}^\text{min}} = \frac{1}{2} + \frac{1}{\pi} \approx 0.818,$$  \hfill (44)

Figure 3. Comparison of costs of different protocols. The ratios $C_{\min}^\text{min} / C_{u,G}^\text{min}$ and $C_{\min}^\text{min} / C_{u,G}^\text{min}$ are plotted as functions of $\gamma (\gamma = \varepsilon^2/A)$ from $\gamma = 0$ to $\gamma = 2$. The protocol that uses shortcuts to adiabaticity always has a smaller cost.

6. Conclusions

We have demonstrated the possibility of a mostly analytical approach to quantum control problems. Instead of pursuing numerically exact but complex and nontransparent solutions we explored the possibility of restricting the optimization problem to some sufficiently broad classes of analytically tractable protocols. By construction, such protocols are generally suboptimal. The precise cost function, however, is unknown in most practical situations. There is no reason then to know the numerically exact solution of an optimization problem. It is much more desirable for practically useful protocols to be physically meaningful and analytically tractable. Such a control can be relatively easily planned and adjusted upon receiving additional information about performance. This simplicity is what our approach can provide.

By restricting the search for optimal protocols to a class of solutions of some exactly solvable model, we automatically resolve the problem of combinatorial complexity of

$$\lim_{\gamma \to 0} \frac{C_{\min}^\text{min}}{C_{u,G}^\text{min}} = \frac{2 \sqrt{2}}{\pi} \approx 0.900,$$  \hfill (43)

$$\lim_{\gamma \to \infty} \frac{C_{\min}^\text{min}}{C_{u,G}^\text{min}} = \frac{1}{2} + \frac{1}{\pi} \approx 0.818.$$  \hfill (44)
quantum systems because state vector components no longer have to be treated as independent parameters. As an additional bonus, such a restriction leads to control field pulses whose shapes can be written in terms of known special functions, and consequently can be easily characterized. Numerical solution may be needed only for a small number of saddle-point equations for control parameters.

Using the spin-1/2 example, we demonstrated two basic possibilities for choosing the family of solutions for optimization. One is based on simple compensation of terms in the Hamiltonian that prevented an analytical solution from existing in the first place. The other approach is based on using families of solutions obtained by solving an inverse problem, i.e., assuming some functional form for the evolution of the state vector (a shortcut to adiabaticity in our case) and then finding the family of Hamiltonians that produces this evolution. By restricting the search for optimal control among such classes of analytically solvable Schrödinger equations, we were able to express the cost functional only in terms of control parameters.

The worked-out example revealed not only the simplicity of our approach but also potential problems that may become important in more complicated situations. For example, protocols that are based on straightforward removal of unwanted couplings are simple, and generally need only control of the local degrees of freedom. We showed, however, that more complex protocols can be substantially less costly. We also found that, interestingly, it was impossible to build the optimal protocol fully within the family of shortcuts to adiabaticity because equations of motion drove one of the parameters beyond its range of definition. We resolved this problem by combining different types of protocols in order to describe different time intervals. The resulting protocol, however, had the best performance among all strategies that we explored. It is expected that similar properties of optimized control protocols will be generally found in multistate problems too. So, it seems that more complex models will still require such manual resolution of complications.

Finally, we note that we left many characteristics of our approach unstudied. For example, in addition to extending it to more complex quantum problems, it should be useful to explore the robustness of our protocols with respect to uncertainty of parameters, e.g., using the integral nullification method introduced in [22]. It should also be interesting to compare our protocols with ones based on more complex (than shortcuts to adiabaticity) classes of solvable models that can be derived by inverting the Schrödinger equation [7].

Appendix. Elliptic integrals and Jacobi elliptic functions

Here we list special functions that we have used in the text, namely, the elliptic integrals and Jacobi elliptic functions.

The incomplete elliptic integral of the first kind, \( F(\phi|m) \), with modulus \( m \) is defined as

\[
F(\phi|m) = \int_0^{\phi} \frac{1}{\sqrt{1 - m \sin^2 \theta}} \, d\theta.
\]  

(A1)

The complete elliptic integral of the first kind, \( K(m) \), is defined as

\[
K(m) = F\left(\frac{\pi}{2}|m\right) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - m \sin^2 \phi}} \, d\phi.
\]  

(A2)

The incomplete elliptic integral of the second kind, \( E(\phi|m) \), is defined as

\[
E(\phi|m) = \int_0^{\phi} \sqrt{1 - m \sin^2 \theta} \, d\theta.
\]  

(A3)

and the complete elliptic integral of the second kind, \( E(m) \), is defined as

\[
E(m) = E\left(\frac{\pi}{2}|m\right) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \phi} \, d\phi.
\]  

(A4)

The Jacobi elliptic functions are the inverses of the incomplete elliptic integrals of the first kind [23]. If \( u = F(\phi|m) \), then the Jacobi elliptic function \( sn(u, m) \) is defined as [23]

\[
sn(u|m) = \sin \phi.
\]  

(A5)

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