Global existence to the discrete Safronov–Dubovskiǐ coagulation equations and failure of mass-conservation

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Abstract. This paper presents the existence of global solutions to the discrete Safronov-Dubovskii coagulation equations for a large class of coagulation kernels satisfying \( \Lambda_{i,j} = \theta_i \theta_j + \kappa_{i,j} \) with \( \kappa_{i,j} \leq A \theta_i \theta_j \), \( \forall \ i,j \geq 1 \) where the sequence \( (\theta_i)_{i \geq 1} \) grows linearly or superlinearly with respect to \( i \). Moreover, the failure of mass-conservation of the solution is also addressed which confirms the occurrence of the gelation phenomenon.

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1. Introduction

Coagulation is one of the mechanisms that cause cluster expansion, in which clusters (or particles) grow in size through successive mergers. Some areas where coagulation phenomena play a significant role include aerosol research, raindrop creation, astrophysics (formation of planets and galaxies), and animal herding; for example, see [10, 13, 14]. Smoluchowski, a Polish physicist, made one of the first contributions in that direction when he developed an infinite system of ordinary differential equations, now known as the discrete Smoluchowski coagulation equation, to describe the time-evolution of a system of particle clusters that, due to Brownian motion, can get close enough to one another for binary coagulation of clusters to take place, see [24, 25].

The discrete Smoluchowski coagulation equation depicts the change in concentration of clusters of size \( i \) (or \( i \)-mers) at time \( t \geq 0 \), written as

\[
\frac{d \omega_i}{dt} = \frac{1}{2} \sum_{j=1}^{i-1} \Lambda_{j,i-j} \omega_{i-j} - \sum_{j=1}^{\infty} \Lambda_{i,j} \omega_j \omega_i \quad (1.1)
\]

\[
\omega_i(0) = \omega_{i,n}^0 \geq 0 \quad (1.2)
\]
for \(i \geq 1\). The coagulation kernel \(\Lambda_{i,j}\) is a nonnegative and symmetric function i.e. \(0 \leq \Lambda_{i,j} = \Lambda_{j,i} \forall i, j \geq 1\) that denotes the rate at which clusters of size \(i\) coalesce with the clusters of size \(j\) to create the larger ones.

The first term on the right-hand side of (1.1) indicates that the \(i\)-clusters are formed through binary coalescence of smaller ones, while according to the second term, they are depleted through coagulation with other clusters. In [12], Dubovskiǐ investigated a dispersed system and proposed the Safronov-Dubovskiǐ coagulation model, in which only binary collisions between particles can occur simultaneously, the mass of each particle is assumed to be proportional to some bounded kernel \(\Lambda\) of solutions to (1.3). In [9], it was shown that the classical solutions exist globally for the coagulation kernels \(j\Lambda_{i,j} \leq M, j \leq i\) and for unbounded kernel \(\Lambda_{i,j} \leq C_{\nu} h_i h_j\) with an additional assumption \(\frac{\Lambda_i}{\nu} \to 0\). Moreover, mass conservation property for \(\Lambda_{i,j} \leq C_{\nu} h_i h_j\) for \(h_i \leq \frac{h_j}{\nu}\) and uniqueness for bounded kernels, i.e., \(\Lambda_{i,j} \leq C_{\nu}, i, j \geq 1\) have been established.
solutions have been established for $\Lambda_{i,j} \leq (1 + i + j)\vartheta$ where $\vartheta \in [0, 1]$, whereas uniqueness of solutions is shown under the condition that $\Lambda_{i,j} \leq C\vartheta^i$ for $\vartheta \leq 2$. Additionally, for linear coagulation kernels and kernels fulfilling $\min\{i, j\} \Lambda_{i,j} \leq (i + j)$ for every $i, j \geq 1$, the existence of weak solutions has been demonstrated in [15] and [16] respectively, whereas the uniqueness result is identical to [8]. The results of [8] have recently been improved, and various additional interesting results have been established in [1].

Noting that the reactions given by (1.1) and (1.3) do not result in the creation or destruction of particles, hence the total mass is predicted to remain constant during the course of the time evolution. This, however, is not always the case. In [6, 7, 19, 22], it is observed that for the discrete smoluchowski coagulation equations (1.1)–(1.2), the mass conservation property fails for coagulation kernels of the form $\gamma_{i,j} \geq (ij)^{\alpha_0}$, where $\alpha_0 \in (1, 2]$. It is worth noting that gelation also occurs in the DSDC equations, and we get the same result as in the discrete Smoluchowski coagulation equation. Hence, we define the gelation time $T_{gel} \in [0, +\infty]$ of a solution $\omega = (\omega_i)_{i \geq 1}$ to the DSDC equation (1.3)–(1.4) by

$$T_{gel} = \inf \left\{ t \geq 0 \left| \sum_{i=1}^{\infty} i\omega_i(t) < \sum_{i=1}^{\infty} i\omega_i^{in} \right. \right\}.$$

In this article, we look into the existence of global solutions to the DSDC equations (1.3)–(1.4) for a class of coagulation rates of the following form.

$$\Lambda_{i,j} = \theta_i \theta_j + \kappa_{i,j}, \ i, j \geq 1,$$

with $\kappa_{i,j} \leq A\theta_i \theta_j (i, j \geq 1)$ for some $A > 0$. In this case, $(\theta_i)$ is a sequence of non-negative real numbers that may increase sub-linearly, linearly, or super-linearly with respect to $i$ (for a more detailed explanation, see assumptions (2.1)–(2.3) below).

Our goal in this work is to prove the existence of global solutions to (1.3)–(1.4) for initial data with $\sum_{i=1}^{\infty} \omega_i^{in} < \infty$ and $(\theta_i)$ growing at least linearly, i.e., $\theta_i \geq Bi$ for some $B > 0$. The paper is organized as follows. Section 2 definition of solutions and assumptions made on coagulation coefficients and provides some preliminary results. Proofs of Theorems 2.1 and 2.2 are discussed in sections 3 and 4 respectively. Finally, we have demonstrated the gelation phenomenon in Section 5.

2. Main Results and Preliminaries

To begin with, we define what we mean by a solution to (1.3)–(1.4).

**Definition 2.1.** Let $T \in (0, +\infty]$ and let $\omega^{in} = (\omega_i^{in})_{i \geq 1}$ be a sequence of non-negative real numbers. A solution $\omega = (\omega_i)_{i \geq 1}$ to (1.3)–(1.4) on $[0, T)$ is a sequence of non-negative continuous functions satisfying, for each $i \geq 1$ and $t \in (0, T)$,

1. $\omega \in C([0, T))$ and $\sum_{j=1}^{\infty} \Lambda_{i,j} \omega_j \in L^1(0, t)$,
2. and further

$$\omega_i(t) = \omega_i^{in} + \int_0^t \left( \omega_{i-1}(s) \sum_{j=1}^{i-1} j\Lambda_{i-1,j} \omega_j(s) - \omega_i(s) \sum_{j=1}^{i} j\Lambda_{i,j} \omega_j(s) - \sum_{j=i}^{\infty} \Lambda_{i,j} \omega_i(s) \omega_j(s) \right) ds$$
The coagulation rates \( \Lambda_{i,j} \) are assumed to satisfy the following assumptions, namely there exist two sequences of non-negative real numbers, \((\theta_i)\) and \((\kappa_{i,j})\), such that
\[
\Lambda_{i,j} = \theta_i \theta_j + \kappa_{i,j}.
\] (2.1)

In addition, we made the following set of assumptions on coagulation rates, which will be used in our analysis, namely
\[
\lim_{i \to \infty} \frac{\theta_i}{i} = 0 \quad \text{and} \quad \lim_{j \to \infty} \frac{\kappa_{i,j}}{j} = 0 \quad \text{for each} \ i \geq 1,
\] (2.2)
or
\[
\inf_{i \geq 1} \frac{\theta_i}{i} = B > 0 \quad \text{and} \quad \kappa_{i,j} \leq A \theta_i \theta_j \quad \text{for each} \ i,j \geq 1 (A \geq 0).
\] (2.3)

Finally, we define the Banach spaces \( Y_p \) with \( p \geq 0 \) by
\[
Y_p = \left\{ y = (y_i)_{i \geq 1}, \sum_{i=1}^{\infty} i^p |y_i| < \infty \right\},
\] equipped with the norm
\[
\|y\|_p = \sum_{i=1}^{\infty} i^p |y_i|.
\]

Due to the physical significance of the DSDC equations, we will only take into account non-negative solutions, i.e., those that remain within the non-negative cone \( Y_p^+ \) of \( Y_p \), that is,
\[
Y_p^+ = \{ y \in Y_p, \ y_i \geq 0 \ \text{for each} \ i \geq 1 \}.
\]

Our first result extends [2, Proposition 4] to the wider class of coagulation rates represented by (2.2)–(2.3), and it is as follows:

**Theorem 2.1.** Assume that the coagulation kernel satisfies (2.1) and that either (2.2) or (2.3) holds. For every \( \omega^\text{in} = (\omega^\text{in}_i)_{i \geq 1} \in Y_1^+ \), there exists at least one solution \( \omega \) to (1.3)–(1.4) on \([0, +\infty)\), such that \( \omega(t) \in Y_1^+ \) for each \( t \in [0, +\infty) \), and
\[
\|\omega(t)\|_1 \leq \|\omega^\text{in}\|_1.
\] (2.4)

The following result generalizes Theorem 2.1 to a broader class of initial data, when the coagulation rate satisfies (2.3).

**Theorem 2.2.** Assume that the coagulation kernel satisfies (2.1) and (2.3), and that \( \omega^\text{in} = (\omega^\text{in}_i)_{i \geq 1} \in Y_0^+ \). Then there exists at least one solution \( \omega \) to (1.3)–(1.4) on \([0, +\infty)\), such that \( \omega(t) \in Y_0^+ \) for each \( t \in [0, +\infty) \) and, for each \( t > 0 \),
\[
\|\omega(t)\|_1 < \infty.
\] (2.5)

**Remark 2.1.** Unfortunately, the coagulation kernel \( \Lambda_{i,j} = i + j \) is not included in either of the classes (2.2) or (2.3). Although the existence result for this kernel is available in the literature, see [1, 8, 15].
We fix $n \geq 3$, consider the following truncated system of $n$ ordinary differential equations,

$$\frac{d\omega^n_i(t)}{dt} = \omega^n_{i-1}(t) \sum_{j=1}^{i-1} j\Lambda_{i-1,j}\omega^n_j(t) - \omega^n_i(t) \sum_{j=1}^{i} j\Lambda_{i,j}\omega^n_j(t) - \sum_{j=i+1}^{n} \Lambda_{i,j}\omega^n_j(t)\omega^n_i(t), \quad i \in \mathbb{N},$$

$$\omega^n_i(0) = \omega^n_{i,0} \geq 0, \quad i \in \mathbb{N}. \quad (2.6)$$

The system (2.6), (2.7) has a unique solution $\omega^n = (\omega^n_i)_{1 \leq i \leq n}$ defined on $[0, +\infty)$, satisfying $\omega^n_i \geq 0$ for $i = 1, \cdots, n$. The following technical lemma will be helpful in the upcoming calculations.

**Lemma 2.1.** Let $(\psi_i)_{1 \leq i \leq n}$ be non-negative real numbers. If $t_2 \in [0, +\infty)$ and $t_1 \in [0, t_2]$, then

$$\sum_{i=1}^{n} \psi_i \omega^n_i(t_2) - \sum_{i=1}^{n} \psi_i \omega^n_i(t_1) = \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=1}^{i} j(\psi_i+1 - \psi_i)\Lambda_{i,j}\omega^n_i(s)\omega^n_j(s)ds - \int_{t_1}^{t_2} j\psi_n\Lambda_{n,j}\omega^n_n(s)\omega^n_j(s)ds$$

$$- \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=i}^{n} \psi_i \Lambda_{i,j}\omega^n_i(s)\omega^n_j(s)ds \quad (2.8)$$

We deduce the following estimates as a consequence of (2.8).

**Lemma 2.2.** Suppose that (2.1) holds. If $r \in \{1, \cdots, n\}$, $t_2 \in [0, +\infty)$ and $t_1 \in [0, t_2]$, then

$$\sum_{i=1}^{n} i\omega^n_i(t_2) - \sum_{i=1}^{n} i\omega^n_i(t_1) \leq \sum_{i=1}^{n} i\omega^n_{i,0}, \quad (2.9)$$

$$\sum_{i=1}^{n} \omega^n_i(t_2) + \frac{1}{2} \int_{t_1}^{t_2} \left| \sum_{i=1}^{n} \theta_i \omega^n_i(s) \right|^2 ds \leq \sum_{i=1}^{n} \omega^n_{i,0}, \quad (2.10)$$

$$\int_{t_1}^{t_2} \left| \sum_{i=r}^{n} \theta_i \omega^n_i(s) \right|^2 ds \leq 2 \left( \sum_{i=1}^{n} i^n \omega^n_i(t_1) \right) r^{-\eta}, \quad \text{for some } 0 < \eta < 1. \quad (2.11)$$

**Proof.** Taking $\psi_i = i$ if $i \in \{1, \cdots, n\}$ in (2.8), we get,

$$\sum_{i=1}^{n} i\omega_i(t_2) - \sum_{i=1}^{n} i\omega_i(t_1) \leq 0 \quad (2.12)$$

On replacing $t_2$ with $t_1$ and $t_1$ with 0 in (2.12), we get (2.9).

Next let $\psi_i = 1$ if $i \in \{1, \cdots, n\}$ and $\tau = 0$, we have

$$\sum_{i=1}^{n} \omega^n_i(t) - \sum_{i=1}^{n} \omega^n_i(\tau) \leq - \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=i}^{n} \Lambda_{i,j}\omega^n_i(s)\omega^n_j(s)ds = -\frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=i}^{n} \Lambda_{i,j}\omega^n_i(s)\omega^n_j(s)ds$$
Now from (2.3), using $\theta_i \theta_j \leq \Lambda_{i,j}$, it follows that
\[
\sum_{i=1}^{n} \omega_i^n(t) + \frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_i \theta_j \omega_i^n(s) \omega_j^n(s) ds \leq \sum_{i=1}^{n} \omega_i^n,
\]
which gives (2.10).

Finally, for $(i,j) \in \{1, \cdots, n\}^2$, we take $\psi_i = 1$ if $i \geq j$ and $\psi_i = i^n$ if $i < j$ for some $0 < \eta < 1$ in (2.8), we have
\[
\int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=i}^{n} i^n \Lambda_{i,j} \omega_i^n(s) \omega_j^n(s) ds \leq \sum_{i=1}^{n} \psi_i \omega_i^n(t_1) \leq \sum_{i=1}^{n} i^n \omega_i^n(t_1) \tag{2.13}
\]
Next, we notice that
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} i^n \Lambda_{i,j} \omega_i^n \omega_j^n = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} i^n \Lambda_{i,j} \omega_i^n \omega_j^n + \frac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} j^n \Lambda_{i,j} \omega_i^n \omega_j^n
\]
\[
= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min\{i, j\} \eta \Lambda_{i,j} \omega_i^n \omega_j^n.
\]
With the help of the above estimate, equation (2.13) can be rewritten as
\[
\int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min\{i, j\} \eta \Lambda_{i,j} \omega_i^n \omega_j^n \leq 2 \sum_{i=1}^{n} i^n \omega_i^n(t_1). \tag{2.14}
\]
Finally, gathering estimates from (2.3) and (2.14), we obtain
\[
r^n \int_{t_1}^{t_2} \sum_{i=r}^{n} \sum_{j=r}^{n} \theta_i \theta_j \omega_i^n(s) \omega_j^n(s) ds
\]
\[
\leq \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min\{i, j\} \eta \theta_i \theta_j \omega_i^n(s) \omega_j^n(s) ds
\]
\[
\leq \int_{t_1}^{t_2} \sum_{i=1}^{n} \sum_{j=1}^{n} \min\{i, j\} \Lambda_{i,j} \omega_i^n(s) \omega_j^n(s) ds
\]
\[
\leq 2 \sum_{i=1}^{n} i^n \omega_i^n(t_1).
\]
Therefore, (2.11) follows. □

Another outcome of (2.8) is the following lemma, which will help to prove Theorem 2.2.
Lemma 2.3. Assume that \((2.1)\) and \((2.3)\) hold, and let \(t \in (0, +\infty)\). Then
\[
\sum_{i=1}^{n} i \omega^n_i(t) \leq \frac{2}{B} \left( \sum_{i=1}^{n} \omega^m_i \right)^{\frac{1}{2}} t^{-\frac{1}{2}}. \tag{2.15}
\]

Proof. Using \((2.3)\), we obtain that
\[
\theta_i \geq B_i.
\]
Next, we conclude from \((2.10)\) that
\[
\int_{0}^{t} \left| \sum_{i=1}^{n} i \omega^n_i(s) \right|^2 ds \leq \frac{2}{B^2} \sum_{i=1}^{n} \omega^m_i. \tag{2.16}
\]
As a consequence, the function
\[
s \mapsto \sum_{i=1}^{n} i \omega^n_i(s)
\]
is non-increasing that follows from \((2.9)\), hence using \((2.16)\), we obtain
\[
t \left| \sum_{i=1}^{n} i \omega^n_i(t) \right|^2 \leq \frac{2}{B^2} \sum_{i=1}^{n} \omega^m_i;
\]
which shows that \((2.15)\) holds. \(\square\)

Also, we recall a part of the well-known Rellich-Kondrachov theorem \([17]\), to prove the compactness of the sequence of solutions to \((2.6)-(2.7)\), which is as follows.

**Theorem 2.3.** Suppose \(\Omega \in \mathbb{R}^n\) is a bounded open set and the boundary of \(\Omega\) is a \((n - 1)\) dimensional \(C^1\) manifold and if \(p > n\) then \(W^{1,p}\) is compactly embedded in \(C(\overline{\Omega})\).

At the end of this section, we will provide the well-posedness result of \((2.6)-(2.7)\).

**Lemma 2.4.** For each \(n \geq 3\), the system \((2.6)-(2.7)\) has a unique non-negative solution \(\omega^n = (\omega^n_i)_{1 \leq i \leq n}\) in \(C^{1}([0, +\infty), \mathbb{R}^n)\). Moreover, we have
\[
\sum_{i=1}^{n} \omega^n_i(t) \leq \sum_{i=1}^{n} i \omega^m_i, \ t \in [0, +\infty).
\]

**Proof.** The proof follows exactly the same line as given in \([2, Lemma 13]\). \(\square\)

3. **Proof of Theorem 2.1**

In this section we shall prove existence of solutions in \(Y^+_1\) of \((1.3)-(1.4)\) with initial data in \(Y^+_1\), having conditions \((2.2)\) and \((2.3)\) on the coagulation coefficients.

We omit the proof of Theorem 2.1 with the assumption \((2.2)\) on coagulation rates because it follows the same lines as that of \([2, Proposition 4]\).

Let \(i \geq 1\), for \(n > i\) and \(T \in (0, +\infty)\), from \((2.10)\) we have,
\[
\omega^n_i(t) \leq \|\omega^m_i\|_0, \ t \in [0, T], \tag{3.1}
\]
Following that, we prove the following lemma, which establishes the boundedness of \((\frac{d\omega^n_i}{dt})_{n>i}\) in \(L^2(0, T)\).
Lemma 3.1. Let $i \geq 1$ and $T \in (0, +\infty)$. There exists a constant $\Pi_i(T)$, depending on $\sum_{i=1}^{\infty} \omega_i^{n_1}$, $T$ and $i$ such that, for each $n > i$

$$\left| \frac{d\omega_i^n}{dt} \right|_{L^2(0,T)} \leq \Pi_i(T), \quad t \in [0,T]. \quad (3.2)$$

Proof. Due to (2.3), for each $i \geq 1$ and $n > i$, we get

$$\sum_{j=r_1}^{r_2} \Lambda_{i,j} \omega_j^n \leq (1 + A) \theta_i \sum_{j=r_1}^{r_2} \theta_j \omega_j^n, \quad i \leq r_1 < r_2 \leq n. \quad (3.3)$$

Now (2.6), (2.10) and (3.3) gives us

$$\left| \frac{d\omega_i^n}{dt} \right| \leq \left( \sum_{j=1}^{i-1} \Lambda_{i-1,j} \right) \left| \sum_{j=1}^{n} \omega_j^n \right|^2 + \left( \sum_{j=1}^{i-1} \Lambda_{i,j} \right) \left| \sum_{j=1}^{n} \omega_j^n \right|^2 + (1 + A) \theta_i \sum_{j=1}^{n} \theta_j \omega_j^n,$$

$$\leq \left( \sum_{j=1}^{i-1} \Lambda_{i-1,j} \right) \left| \omega_i^{n_1} \right|^2 + \left( \sum_{j=1}^{i-1} \Lambda_{i,j} \right) \left| \omega_i^{n_1} \right|^2 + (1 + A) \theta_i \left| \omega_i^{n_1} \right| \sum_{j=1}^{n} \theta_j \omega_j^n,$$

$$\left| \frac{d\omega_i^n}{dt} \right|_{L^2(0,T)} \leq T^{\frac{1}{2}} \left( \sum_{j=1}^{i-1} \Lambda_{i-1,j} + \sum_{j=1}^{i} \Lambda_{i,j} \right) \left| \omega_i^{n_1} \right|^2 + 2(1 + A) \theta_i \left| \omega_i^{n_1} \right|^3.$$ 

Hence, the proof of lemma 3.1 is completed. \qed

As a consequence of (3.1) and Lemma (3.1), for each $i \geq 1$ and $T \in (0, +\infty)$, the sequence $(\omega_i^n)_{n>1}$ is bounded in $W^{1,2}(0, T)$. Since it follows from Theorem 2.3 that $W^{1,2}(0, T)$ is compactly embedded in $C([0, T])$. Thus, $(\omega_i^n)_{n>1}$ is relatively compact in $C([0, T])$ for each $i \geq 1$ and $T \in (0, +\infty)$. With the help of diagonal process we can identify a subsequence $(\omega_i^n)$ of $(\omega_i^n)_{n>1}$ and a sequence $\omega = (\omega_i)_{i \geq 1}$ of non-negative continuous function such that, for each $i \geq 1$ and $T \in (0, +\infty)$

$$\lim_{i \to \infty} \left| \omega_i^n - \omega_1 \right|_{C([0, T])} = 0. \quad (3.4)$$

Let $t \in [0, +\infty)$ and $r \geq 1$, then for $n_i \geq r$, it can be deduce from (2.9) that

$$\sum_{i=1}^{r} i \omega_i^{n_i}(t) \leq \left| \omega_i^{n_1} \right|_1.$$

Next, using (3.4) we allow to pass to the limit $l \to +\infty$ in the previous inequality, to obtain

$$\sum_{i=1}^{r} i \omega_i(t) \leq \left| \omega_i^{n_1} \right|_1.$$

Since the above estimate holds for any $r \geq 1$, letting $r \to +\infty$, we get

$$\sum_{i=1}^{\infty} i \omega_i(t) \leq \left| \omega_i^{n_1} \right|_1.$$
which leads to the conclusion that $\omega(t) \in Y^+_1$ and satisfies (2.4).

Thus, we may conclude from (2.9), (2.11), and (3.3) that, for $i \geq 1$, $T \in (0, +\infty)$ and $l \geq 1$,

$$
\int_0^T \left| \sum_{j=r_1}^{r_2} \Lambda_{i,j}\omega_j^n(s) \right|^2 ds \leq 4(1 + A)^2 \theta_i^2 \|\omega^{in}\|_{1,r_1^{-\eta}}, \quad i \leq r_1 < r_2 \leq n_l \tag{3.5}
$$

Now thanks to (3.4), we may pass to the limit as $l \to +\infty$ in (3.5) to obtain

$$
\int_0^T \left| \sum_{j=r_1}^{r_2} \Lambda_{i,j}\omega_j(s) \right|^2 ds \leq 4(1 + A)^2 \theta_i^2 \|\omega^{in}\|_{1,r_1^{-\eta}}, \quad i \leq r_1 < r_2. \tag{3.6}
$$

Because the right-hand side of (3.6) does not depend on $r_2$, the first conclusion of (3.6) is that for each $i \geq 1$ and $T \in (0, +\infty)$, $\sum_{j=i}^{\infty} \Lambda_{i,j}\omega_j$ belongs to $L^2(0, T)$, and

$$
\int_0^T \left| \sum_{j=r_1}^{\infty} \Lambda_{i,j}\omega_j(s) \right|^2 ds \leq 4(1 + A)^2 \theta_i^2 \|\omega^{in}\|_{1,r_1^{-\eta}}, \quad r_1 \geq i. \tag{3.7}
$$

In order to pass to the limit in (2.6), we shall show that for each $i \geq 1$,

$$
\sum_{j=i}^{n_l} \Lambda_{i,j}\omega_j^n \to \sum_{j=i}^{\infty} \Lambda_{i,j}\omega_j \quad \text{in} \quad L^2(0, T), \tag{3.8}
$$

as $l \to \infty$.

Indeed, for each $r > i$ and $n_l \geq r$, we infer from (3.5) and (3.7) that

$$
\left| \sum_{j=i}^{n_l} \Lambda_{i,j}\omega_j^n - \sum_{j=i}^{\infty} \Lambda_{i,j}\omega_j \right|_{L^2(0, T)} \leq \sum_{j=i}^{r-1} \Lambda_{i,j}|\omega_j^n - \omega_j|_{L^2(0, T)} + \sum_{j=r}^{n_l} \Lambda_{i,j}|\omega_j^n|_{L^2(0, T)} + \sum_{j=r}^{\infty} \Lambda_{i,j}|\omega_j|_{L^2(0, T)} \leq \sum_{j=i}^{r-1} \Lambda_{i,j}|\omega_j^n - \omega_j|_{L^2(0, T)} + 4(1 + A)\theta_i \|\omega^{in}\|_{1}\langle r^{-\frac{\eta}{2}}. \tag{3.9}
$$

Owing to (3.4), we may pass to the limit as $l \to +\infty$ in the preceding estimate and obtain

$$
\lim_{l \to +\infty} \left| \sum_{j=i}^{n_l} \Lambda_{i,j}\omega_j^n - \sum_{j=i}^{\infty} \Lambda_{i,j}\omega_j \right|_{L^2(0, T)} \leq 4(1 + A)\theta_i \|\omega^{in}\|_{1}\langle r^{-\frac{\eta}{2}} \tag{3.9}
$$

for any $q > i$; hence (3.8) follows.

Now thanks to (3.4) and (3.3), it is easy to show that for $i \geq 1$, we have

$$
\lim_{l \to +\infty} \left| \sum_{j=1}^{i-1} j\Lambda_{i-1,j}\omega_{i-1,j} \omega_j^n - \sum_{j=1}^{i-1} j\Lambda_{i-1,j}\omega_{i-1,j} \omega_j \right|_{L^2(0, T)} = 0, \tag{3.10}
$$
and
\[
\lim_{l \to \infty} \left| \sum_{j=1}^{i} j \Lambda_{i,j} \omega_i^n \omega_j^n - \sum_{j=1}^{i} j \Lambda_{i,j} \omega_j \right|_{L^2(0,T)} = 0. \tag{3.11}
\]

Finally, using (3.8), (3.10), and (3.11), it is straightforward to pass to the limit in the \(i\)-th equation of (2.6) as \(l \to +\infty\), and infer that \(\omega\) is a solution to (1.3)–(1.4) on \([0, +\infty)\). Thus, the proof of Theorem 2.1 is complete.

4. Proof of Theorem 2.2

Owing to the fact that the bounds (3.10) and (3.11) depend only on the \(Y0\)-norm of \(\omega^n\), the same argument that is used to prove Theorem 2.1 leads to the conclusion that there exists a subsequence \((\omega^n_i)\) of \((\omega^n_i)_{n \geq i}\), and a sequence \(\omega = (\omega_i)_{i \geq 1}\) of non-negative continuous functions, such that, for each \(i \geq 1\) and \(T \in (0, +\infty)\),
\[
\lim_{l \to \infty} \|\omega^n_i - \omega_i\|_{C([0,T])} = 0. \tag{4.1}
\]

Let \(t \in (0, +\infty)\) and \(r \geq i\). If \(n_t \geq r\), we can deduce from (2.10) and (2.11) that
\[
\sum_{i=1}^{r} \omega^n_i(t) \leq \|\omega^n\|_0 \quad \text{and} \quad \sum_{i=1}^{r} i \omega^n_i(t) \leq \frac{2}{B} \|\omega^n\|_0^\frac{1}{2} t^{-\frac{1}{2}}.
\]

With the help of (4.1), we let \(l \to +\infty\) in the previous estimates to obtain
\[
\sum_{i=1}^{r} \omega_i(t) \leq \|\omega^n\|_0 \quad \text{and} \quad \sum_{i=1}^{r} i \omega_i(t) \leq \frac{2}{B} \|\omega^n\|_0^\frac{1}{2} t^{-\frac{1}{2}}.
\]

Since the aforementioned estimate holds for any \(r \geq 1\), we have
\[
\|\omega(t)\|_0 \leq \|\omega^n\|_0 \quad \text{and} \quad \|\omega(t)\|_1 \leq \frac{2}{B} \|\omega^n\|_0^\frac{1}{2} t^{-\frac{1}{2}}; \tag{4.2}
\]
hence (2.5) follows.

Next, let \(i \geq 1\) and \(T \in (0, +\infty)\). For \(r \geq i\) and \(n_t \geq r\), (2.10) and (3.3) yields
\[
\int_0^T \left| \sum_{j=i}^{r} \Lambda_{i,j} \omega^n_j(s) \right|^2 ds \leq 2(1 + A)^2 \theta_i^2 \|\omega^n\|_0^2.
\]
Using (4.1), we first pass to the limit as \(l \to +\infty\), and then let \(r \to +\infty\) to argue that
\[
\sum_{j=i}^{\infty} \Lambda_{i,j} \omega_j \in L^2(0,T). \tag{4.3}
\]

We next consider \(t_1 > 0\) and \(T \in (t_1, +\infty)\). It is clear from (4.2) that \(\omega(t_1) \in Y_1^+\). Next, following the calculations that we performed to get (3.5)–(3.8), we can now use (2.11), (3.3), and (3.4), and for each \(i \geq 1\), to obtain
\[
\lim_{l \to \infty} \left| \sum_{j=i}^{n_l} \Lambda_{i,j} \omega^n_j - \sum_{j=i}^{\infty} \Lambda_{i,j} \omega_j \right|_{L^2(0,T)} = 0. \tag{4.4}
\]
We are now ready to verify that \( \omega \) is a solution to (1.3)–(1.4) on \([0, +\infty)\). Let \( i = 1 \) and \( n_i \geq i \). Using (2.10), we have

\[
\omega_i^{\mu}(t_2) = \omega_i^{\mu}(t_1) + \int_{t_1}^{t_2} \left( \omega_{i-1}(s) \sum_{j=1}^{i-1} j \Lambda_{i-1,j} \omega_j(s) - \omega_i(s) \sum_{j=1}^{i} j \Lambda_{i,j} \omega_i(s) \omega_j(s) \right) ds,
\]

for each \( t_2 \in (0, +\infty) \) and \( t_1 \in (0, t_2) \). Now thanks to (1.1) and (4.4), we may pass to the limit as \( l \to +\infty \), and obtain

\[
\omega_i(t_2) = \omega_i(t_1) + \int_{t_1}^{t_2} \left( \omega_{i-1}(s) \sum_{j=1}^{i-1} j \Lambda_{i-1,j} \omega_j(s) - \omega_i(s) \sum_{j=1}^{i} j \Lambda_{i,j} \omega_i(s) \omega_j(s) \right) ds
\]

for each \( t_2 \in (0, +\infty) \) and \( t_1 \in (0, t_2) \).

Notice that as \( \omega \in C([0, t_2]) \) and \( \sum_{j=1}^{r} \Lambda_{i,j} \omega_j \) belongs to \( L^1(0, t_2) \) by (3.3) we may let \( t_1 \to 0 \) and deduce that \( \omega_i \) satisfies the \( i \)-th equation of (1.3) in the sense of definition given in (2.1). As a result, \( \omega \) is a solution to (1.3)–(1.4) on \([0, +\infty)\), completing the proof of Theorem 2.1.

5. Gelation Phenomenon in equation (1.3)–(1.4)

**Theorem 5.1.** Assume that \( \Lambda_{i,j} \) satisfies (2.1) and

\[
\Lambda_{i,j} \geq C(i,j)^{\frac{\kappa}{2}}, \quad (i, j) \in \mathbb{N}^2,
\]

for some \( \kappa_0 \in (1, 2) \) and \( C > 0 \). Consider \( \omega^{in} \in Y_1^+ \), \( \omega^{in} \neq 0 \), and denote \( \omega = (\omega_i)_{i \geq 1} \) a solution to (1.3)–(1.4). Then \( T_{gel} < +\infty \).

**Proof.** For \( t_2 \geq t_1 \geq 0 \), we multiply (1.3) by \( \phi_i \) and take the sum over \( i \) from 1 to \( q \). After a simple calculation, we have

\[
\sum_{i=1}^{r} \phi_i \omega_i(t_2) - \sum_{i=1}^{r} \phi_i \omega_i(t_1) = \int_{t_1}^{t_2} \left[ \sum_{i=1}^{r} \sum_{j=1}^{i} j \phi_{i+1} \Lambda_{i,j} \omega_i(s) \omega_j(s) - \sum_{i=1}^{r} \sum_{j=1}^{i} (j \phi_i + \phi_j) \Lambda_{i,j} \omega_i(s) \omega_j(s) \right. \\
- \sum_{i=r+1}^{\infty} \sum_{j=1}^{r} \phi_j \Lambda_{i,j} \omega_i(s) \omega_j(s) \right] ds.
\]

(5.2)

As \( \omega = (\omega_i)_{i \geq 1} \in Y_1^+ \) given by Theorem 2.1, hence it follows from above inequality that for \( \phi_i = i \), the function \( t \to \sum_{i=1}^{\infty} i \omega_i(t) \) is non-increasing, i.e.,

\[
\sum_{i=1}^{\infty} i \omega_i(t_2) \leq \sum_{i=1}^{\infty} i \omega_i(t_1).
\]

(5.3)

Now we consider another sequence \( \Phi = (\Phi_i)_{i \geq 1} \) decaying sufficiently rapidly and each solution \( \omega = (\omega_i)_{i \geq 1} \) of (1.3)–(1.4), we have,

\[
\sum_{i=1}^{\infty} \Phi_i \omega_i(t_2) - \sum_{i=1}^{\infty} \Phi_i \omega_i(t_1) = \int_{t_1}^{t_2} \sum_{i=1}^{\infty} \sum_{j=1}^{i} [j(\Phi_{i+1} - \Phi_i) - \Phi_j] \Lambda_{i,j} \omega_i(s) \omega_j(s) ds.
\]
Let $\Phi_i = \min(i, r)$, then for $i \geq j \geq (r - 1)$, we have $[j(\Phi_i + 1 - \Phi_i) - \Phi_j] \leq -r$ and $[j(\Phi_i + 1 - \Phi_i) - \Phi_j] \leq 0$, otherwise. Since

$$\sum_{i=1}^{\infty} \Phi_i \omega_i(t_2) - \sum_{i=1}^{\infty} \Phi_i \omega_i(t_1) = \int_{t_1}^{t_2} \left( \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \sum_{i=r}^{\infty} \sum_{j=1}^{\infty} \right) [j(\Phi_i + 1 - \Phi_i) - \Phi_j] \Lambda_{i,j} \omega_i(s) \omega_j(s) ds.$$

Hence

$$\sum_{i=1}^{\infty} \Phi_i \omega_i(t_2) - \sum_{i=1}^{\infty} \Phi_i \omega_i(t_1) \leq -\frac{r}{2} \int_{t_1}^{t_2} \sum_{i=r}^{\infty} \sum_{j=r}^{\infty} \Lambda_{i,j} \omega_i(s) \omega_j(s) ds.$$

$$\int_{t_1}^{t_2} \sum_{i=r}^{\infty} \sum_{j=r}^{\infty} \Lambda_{i,j} \omega_i(t) \omega_j(t) dt \leq \frac{2}{r} \sum_{i=1}^{\infty} i \omega_i(t_1). \quad (5.4)$$

Next, consider

$$j = \sum_{i=1}^{\infty} i^{-(\kappa_0 + 1)} < \infty$$

since $\kappa_0 + 1 > 2$. With the help of the Holder’s inequality, (5.1), and (5.4), it follows for $t_2 \geq t_1 \geq 0$ that

$$\int_{t_1}^{t_2} \left( \sum_{k=1}^{\infty} k^{-\frac{n_0}{2}} \sum_{i=k}^{\infty} i^{\frac{n_0}{2}} \omega_i(s) \right)^2 ds$$

$$\leq j \int_{t_1}^{t_2} \sum_{k=1}^{\infty} k^{-\frac{n_0}{2}} k^\frac{1}{2} \left( \sum_{i=k}^{\infty} i^{\frac{n_0}{2}} \omega_i(s) \right)^2 ds$$

$$\leq \frac{j}{C} \sum_{k=1}^{\infty} k^{-\frac{n_0}{2}} k^\frac{1}{2} \int_{t_1}^{t_2} \sum_{i=k}^{\infty} \sum_{j=k}^{\infty} \Lambda_{i,j} \omega_i(s) \omega_j(s) ds$$

$$\leq \frac{2j^2}{C} M_1(t_1).$$

However

$$\sum_{k=1}^{\infty} k^{-\frac{n_0}{2}} \sum_{i=k}^{\infty} i^{\frac{n_0}{2}} \omega_i(s) = \sum_{i=1}^{\infty} \left( \sum_{k=1}^{i} k^{-\frac{n_0}{2}} \right) i^{\frac{n_0}{2}} \omega_i(s) \geq \sum_{i=1}^{\infty} i \omega_i(s)$$

Combining the previous inequalities yields

$$\int_{t_1}^{t_2} \left( \sum_{i=1}^{\infty} i \omega_i(s) \right)^2 ds \leq DM_1(t_1) \quad (5.5)$$

for some constant $D$ depending only on $\kappa_0$ and $C$. Next, we deduce from (5.3) and (5.5) that

$$\int_{0}^{\infty} M_1(s)^2 ds \leq D \| \omega^{in} \|_1. \quad (5.6)$$
From (5.3) and (5.6), it can be inferred that the total mass \( M_1 \) is a non-increasing, non-negative function that also belongs to \( L^2(0, +\infty) \). Hence

\[
\lim_{t \to \infty} M_1(t) = 0
\]

which clearly indicates that \( T_{\text{gel}} < +\infty \) since \( M_1(0) > 0 \). □

Next, we take into account the case when \( \kappa_0 = 2 \); in this case, we can explicitly calculate gelation time.

**Proposition 5.1.** Assume that \( \Lambda_{i,j} \) satisfy (2.1) and

\[
\Lambda_{i,j} \geq \zeta_{ij},
\]

for \( i, j \geq 1 \) and where constant \( \zeta \) is a positive real number.

Consider \( T \in (0, +\infty) \) and \( \omega^m \in Y_1^+ \) and assume that (1.3)–(1.4) has a solution \( \omega \in Y_1^+ \). Then

\[
\lim_{t \to \infty} \|\omega(t)\|_1 = 0
\]

**Proof.** Under the given assumptions (5.7), existence of the solution is given by Theorem 2.1. Let \( t_2 \geq t_1 \geq 0 \) and taking \( \phi_i = 1 \) in (5.2), and passing to the limit as \( r \to \infty \) with the help of (2.10), we have

\[
\sum_{i=1}^{\infty} \omega_i(t_2) - \sum_{i=1}^{\infty} \omega_i(t_1) \leq - \frac{1}{2} \int_{t_1}^{t_2} \sum_{i,j=1}^{\infty} \Lambda_{i,j} \omega_i(s) \omega_j(s) ds
\]

(5.9)

Using the lower bound in (5.7), we finally arrive at

\[
\sum_{i=1}^{\infty} \omega_i(t_2) + \frac{\zeta}{2} \int_{t_1}^{t_2} \|\omega(s)\|_1^2 dt \leq \sum_{i=1}^{\infty} \omega_i(t_1)
\]

(5.10)

Now consider \( t \in (0, +\infty) \). We know from (5.3) that \( t \to \|\omega(t)\|_1 \) is not increasing, and we infer from the foregoing estimate (with \( t_1 = 0 \) and \( t_2 = t \)) that

\[
\frac{\zeta t}{2} \|\omega(t)\|_1^2 \leq \sum_{i=1}^{\infty} \omega_i^m \leq \sum_{i=1}^{\infty} i \omega_i^m = \|\omega^m\|_1
\]

Thus

\[
\|\omega(t)\|_1 \leq \left( \frac{2 \|\omega^m\|_1}{\zeta t} \right)^{\frac{1}{2}}, \quad t \in (0, +\infty),
\]

that concludes the proof of the Proposition. □

Now recall (4.2), it can be easily noticed that we have proved a much stronger result than (2.8), namely
Proposition 5.2. Assume that (2.1) and (2.3) hold, and that \( \omega^{in} = (\omega)_{i \geq 1} \in Y_0^+ \). Then there exists at least one solution \( \omega \) to (1.3)–(1.4) on \([0, +\infty)\) such that \( \omega(t) \in Y_0^+ \) for each \( t \in [0, +\infty) \), and, for each \( t > 0 \),

\[
\|\omega(t)\|_1 \leq \frac{2}{B}\|\omega^{in}\|_1^{\frac{1}{2}} t^{-\frac{1}{2}}.
\]  

(5.11)

According to the above result, gelation happens for coagulation rates satisfying (2.3), even when the initial total mass is infinite.

Finally, in the following section, we will demonstrate that as time increases to infinity, the number of particles decreases to zero.

6. Appendix

From (5.9), it follows that

\[
\sum_{i=1}^{\infty} \omega_i(t_1) + \frac{1}{2} \int_{t_1}^{t_2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \Lambda_{i,j} \omega_i(s) \omega_j(s) ds \leq \sum_{i=1}^{\infty} \omega_i(t_1)
\]

which implies that

\[
\int_{t_1}^{t_2} \left( \sum_{i=1}^{\infty} \omega_i(s) \right)^2 ds \leq \frac{2}{C} \sum_{i=1}^{\infty} \omega_i(t_1)
\]

We can see from (5.9) that the total number of particles \( M_0 \) is a non-increasing and non-negative function of time that also belongs to \( L^2(0, +\infty) \). Therefore

\[
\lim_{t \to \infty} M_0(t) = 0.
\]

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