About a Theorem of Cline, Parshall and Scott

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, let $\mathfrak{h} \subseteq \mathfrak{b}$ be respectively Cartan and Borel subalgebras of $\mathfrak{g}$, put $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, say that the roots of $\mathfrak{h}$ in $\mathfrak{n}$ are positive, let $\mathcal{W}$ be the Weyl group equipped with the Bruhat ordering, let $\mathcal{O}_0$ be the category of those BGG-modules which have the generalized infinitesimal character of the trivial module. The simple objects of $\mathcal{O}_0$ are parametrized by $\mathcal{W}$.

Say that $Y \subseteq \mathcal{W}$ is an initial segment if $x \leq y$ and $y \in Y$ imply $x \in Y$, and that $w \in \mathcal{W}$ lies in the support of $V \in \mathcal{O}_0$ if the simple object attached to $w$ is a subquotient of $V$. For such an initial segment $Y$ let $\mathcal{O}(Y)$ be the subcategory of $\mathcal{O}_0$ consisting of objects supported on $Y \subseteq \mathcal{W}$, and let $i : \mathcal{O}(Y) \to \mathcal{O}_0$ be the inclusion. Theorem 3.9 of Cline, Parshall and Scott in [7] implies the following.

**Theorem 1.** The functor $i_* : D^b(\mathcal{O}(Y)) \to D^b(\mathcal{O}_0)$ admits a left adjoint $i^*$ and a right adjoint $i!$ satisfying $i^* i_* \simeq \text{Id}_{D^b(\mathcal{O}(Y))} \simeq i! i_*$. In particular $i_*$ is a full embedding.

The purpose of this text is to give a simple proof of this Theorem and to suggest an analog for Harish-Chandra modules.

Theorem 1 above will follow from Theorem 2 below. Say that a **BGS category** is an abelian category satisfying Conditions (1) to (6) in Section 3.2 of Beilinson, Ginzburg and Soergel [1]. By Theorem 3.2.1 and Corollary 3.2.2 in [1], Theorems 1 and 2 apply to BGS categories. In [1] many natural examples of BGS categories are given, like (in the notation of [1]) the categories of BGG modules $\mathcal{O}_\lambda$ and $\mathcal{O}_q$ defined in Section 1.1, or more generally the category $\mathcal{P}(X, \mathcal{W})$ of perverse sheaves considered in Section 3.3. The fact that $\mathcal{P}(X, \mathcal{W})$ is a BGS category is viewed as obvious in [1] (and I have no doubt that it is so for algebraic geometers). Theorem 3.5.3 of [1] implies that $\mathcal{O}_q$ is of the form $\mathcal{P}(X, \mathcal{W})$, and Theorem 3.11.1 of [1] entails that $\mathcal{O}_\lambda$ is opposite to $\mathcal{O}_q$ (for some $q$). Since the axioms of BGS categories are selfopposite, $\mathcal{O}_\lambda$ is BGS.

Thank you to Bernhard Keller and Wolfgang Soergel for their interest, and to Martin Olbrich for having pointed out some mistakes in a previous version.

1 Statement

Let $A$ be a ring, $X$ a finite set and $e_\bullet = (e_x)_{x \in X}$ a family of idempotents of $A$ satisfying $\sum_{x \in X} e_x = 1$ and $e_x e_y = \delta_{xy} e_x$ (Kronecker delta) for all
The support of an $A$-module $V$ is the set $\{x \in X \mid e_x V \neq 0\}$. Let $\leq$ be a partial ordering on $X$, and for any initial segment $Y$ put

$$A(Y) := A \left/ \sum_{x \notin Y} A e_x A, \right.$$  

so that $A(Y)\text{-mod}$ is the full subcategory of $A\text{-mod}$ whose objects are supported on $Y$. (Here and in the sequel, for any ring $B$, we denote by $B\text{-mod}$ the category of $B$-modules.) The image of $e_y$ in $A(Y)$ will be still denoted by $e_y$.

Assume that, for any pair $(Y, y)$ where $Y$ is an initial segment and $y$ a maximal element of $Y$, the module $M_y := A(Y)e_y$ does not depend on $Y$, but only on $y$. This is equivalent to the requirement that $A(Y)e_y$ be supported on

$$\{x \in X \mid x \leq y\}.$$  

If $(V_\gamma)_{\gamma \in \Gamma}$ a family of $A$-modules, let $\langle V_\gamma \rangle_{\gamma \in \Gamma}$ denote the class of those $A$-modules which admit a finite filtration whose associated graded object is isomorphic to a product of members of the family.

Assume that, for any $x \in X$, the module $A e_x$ belongs to $\langle M_y \rangle_{y \in X}$.

**Theorem 2.** Let $Y \subset X$ be an initial segment and $i_* : D^b(A(Y)\text{-mod}) \to D^b(A\text{-mod})$ the induced functor. Then $i_*^! := R \text{Hom}_A(A(Y), ?)$ is a right adjoint to the functor $i_*$ from $D^b(A(Y)\text{-mod})$ to $D^b(A\text{-mod})$ and we have $i_*^! i_* \simeq \text{Id}_{D^b(A(Y)\text{-mod})}$. In particular $i_*$ is a full embedding. If the right flat dimension of $A(Y)$ over $A$ is finite, then $i^* := A(Y) \otimes_A ?$ is a left adjoint to $i_*$ satisfying $i_*^! i_* \simeq \text{Id}_{D^b(A(Y)\text{-mod})}$.

2 Proof

**Proof that Theorem 2 implies Theorem 1.** In view of BGG [3] it suffices to check that the Verma module $M_x$, with $x \in \mathcal{W}$, is projective into $\mathcal{O}(\mathcal{W}_{\geq x})$ (obvious notation). Let $V$ be in $\mathcal{O}(\mathcal{W}_{\geq x})$ and, for any $\lambda \in h^*$, let $V^\lambda$ be the corresponding weight subspace of $V$. Letting $\rho$ be the half sum of the positive roots and putting $\lambda := -x \rho - \rho$ we have
\[ \text{Hom}_g(M_x, V) \simeq H^0(n, V^\lambda) \subset V^\lambda. \]

It suffices to show that this inclusion is an equality. Otherwise there would be a weight \( \mu \) satisfying

\[ \mu > \lambda, \quad V^\mu \neq 0, \quad nV^\mu = 0. \]

Letting \( L_y \) be a simple quotient of \( U(g)V^\mu \), we would have

\[ -y\rho - \rho = \mu > \lambda = -x\rho - \rho \]

and thus (see for instance Lemma 7.7.2 in Dixmier [8]) \( y > x \), which is impossible. □

Lemma 3. Let \( A \) be a ring, \( I \) a left projective idempotent twosided ideal and \( B \) the quotient ring \( A/I \). Then \( i^! := \text{R Hom}_A(B, ?) \) is a right adjoint to the functor \( i_* \) from \( D^b(B\text{-mod}) \) to \( D^b(A\text{-mod}) \) and we have \( i^! i_* \simeq \text{Id}_{D^b(B\text{-mod})} \). In particular \( i_* \) is a full embedding. If the right flat dimension of \( B \) over \( A \) is finite, then \( i^* := B \otimes_A \) is a left adjoint to \( i_* \) satisfying \( i^* i_* \simeq \text{Id}_{D^b(B\text{-mod})} \).

Proof. The Lemma follows from Theorem 3.1 and Proposition 3.6 of Cline, Parshall and Scott in [6]. □

Let us go back to the setting of Theorem 2.

Lemma 4. For any \( x, y \in X \) with \( x \) maximal there is a nonnegative integer \( n \) and an exact sequence \( (Ae_x)^n \to Ae_y \to V \) such that \( V \in \langle M_z \rangle_{z < x} \). In particular \( e_x V = 0 \).

Proof. This follows from the projectivity of \( M_x = Ae_x \). □

Proof of Theorem 2. Assume \( Y = X \setminus \{ x \} \) where \( x \) is maximal. Put \( e := e_x \), \( I := AeA \) and \( B := A(Y) = A/I \). By the previous Lemma there is a nonnegative integer \( n \) and an exact sequence \( (Ae)^n \to A \to V \) with \( IV = 0 \). Letting \( J \subset A \) be the image of \( (Ae)^n \to A \), we have \( J = IJ \subset I \subset J \), and thus \( I = J \). Lemma 3 applies, proving the Theorem for the particular initial segment \( Y \). Lemma 4 shows that \( (B, Y, (e_y)_{y \in Y}) \) satisfies the assumptions of Theorem 2; and an obvious induction shows the existence of some right adjoint \( i^! \) to \( i_* \), which is a full embedding satisfying \( i^! i_* \simeq \text{Id}_{D^b(A(Y)\text{-mod})} \). Then the Theorem follows from Theorem 3.1 of [6]. □
3 Harish-Chandra modules

Let $G$ be a connected semisimple Lie group with finite center, let $K$ be a maximal compact subgroup, let $Z$ be the center of the complexified enveloping algebra, let $I$ be the annihilator of the trivial module $\mathbb{C}$ in $Z$, let $\hat{Z}$ and $\hat{I}$ be the respective $I$-adic completions of $Z$ and $I$, and let $\mathcal{H}_0$ be the $\hat{Z}$-category of those Harish-Chandra modules having the generalized infinitesimal character of $\mathbb{C}$.

Theorem 5. There is a (uniquely determined up to isomorphism) $\hat{Z}$-algebra $A$ and a finite set $X$ satisfying

- $\mathcal{H}_0$ is equivalent, as $\hat{Z}$-category, to the category $A$-fd of finite dimensional $A$-modules,
- $A$ is finitely generated over $\hat{Z}$,
- $A/\text{rad}(A)$ is isomorphic to the algebra of $\mathbb{C}$-valued functions on $X$,
- $A$ is selfopposite,
- the global dimension of $A$ is $\dim G/K$,
- the inclusion of $A$-fd into $A$-mod is compatible with Ext-calculus.

Proof. Let $n$ be a positive integer and $\mathcal{H}_n$ the full subcategory of $\mathcal{H}_0$ whose objects are killed by $\hat{I}^n$. This category comes with a selfduality and is equivalent to $A_n$-fd where $A_n$ is a finite dimensional $(\hat{Z}/\hat{I}^n)$-algebra that is equipped with an anti-involution and $A_n/\text{rad}(A_n)$ is isomorphic to the algebra of $\mathbb{C}$-valued functions on $X$. We have $A_{n+1}/\hat{I}^n A_{n+1} = A_n$ and $A_{n+1}/\hat{I} A_{n+1} = A_1$; moreover the projections $A_{n+1} \to A_n$ commute with the anti-involutions. Let $A$ be the limit of the $A_n$. Since $Z$ is a symmetric algebra over a finite dimensional vector space, it is noetherian, and so is $\hat{Z}$ by Proposition III.3.4.8 of Bourbaki in [5], implying that $A$, being finitely generated over $\hat{Z}$, is itself noetherian. Section I.5.5 of Borel-Wallach [4] entails that the global dimension of $A$-fd is $\dim G/K$ and that the inclusion of $A$-fd into $A$-mod is compatible with Ext-calculus. The claim about the global dimension of $A$ now follows from statements 12 and 14 of Eilenberg in [9]. □

I hope there is always a partial ordering on $X$ which satisfies the assumptions of Theorem 2.
4 Example

Let $\mathbb{K}$ be a commutative ring, let $z$ be an element of $\mathbb{K}$, let $A$ be the $\mathbb{K}$-algebra of the quiver

$$
\begin{array}{c}
\begin{array}{c}
| \quad f \\
\downarrow \\
\begin{array}{c}
\quad a \quad \varepsilon \\
\quad b \quad e
\end{array}
\end{array}
\end{array}
$$

modulo the relations

$$0 = ab = bc = ac - zf = b^2 + ca - ze.$$

If $\mathbb{K} := \mathbb{C}[[z]]$, where $z$ is an indeterminate, and $\mathcal{H}_0$ is the category of those Harish-Chandra modules over $SL(2, \mathbb{C})$ which have the generalized infinitesimal character of the trivial module, then $\mathcal{H}_0$ is equivalent to $A$-fd (see Gelfand-Ponomarev [10]).

Put $X := \{e, f\}$ with the ordering $e < f$. By Bergman’s Diamond Lemma [2] the set $\{e, f, a, b, c, b^2\}$ is a $\mathbb{K}$-basis of $A$, and we have

- $eAe = \mathbb{K} e \oplus \mathbb{K} b \oplus \mathbb{K} b^2$,  
- $eAf = \mathbb{K} c$,  
- $fAe = \mathbb{K} a$,  
- $fAf = \mathbb{K} f$,

$$M_e = Ae/AfAe = Ae/Aa = Ae/(\mathbb{K} a \oplus \mathbb{K} b^2), \quad M_f = Af.$$ 

The sequence $M_f \twoheadrightarrow Ae \twoheadrightarrow M_e$, where the first arrow is right multiplication by $a$, being exact, $Ae$ belongs to $\langle M_x \rangle_{x \in X}$, and $A$ satisfies the assumptions of Theorem 2.

Denoting by dim $R$ the global dimension of a ring $R$ and setting $B := A/AfA$, we have $1 + \dim \mathbb{K} \leq \dim A \leq 2 + \dim \mathbb{K}$. Indeed the isomorphism $B \simeq \mathbb{K}[t]$, where $t$ is an indeterminate, implies $\dim B = 1 + \dim \mathbb{K}$ (see for instance Theorem 4.3.7 in [11]), and the claim follows from the proof of Lemma 3 and the spectral sequence

$$\text{Ext}^p_B(V, \text{Ext}^q_A(B, W)) \Longrightarrow \text{Ext}^{p+q}_A(V, W).$$

In particular the functor $i^*$ of Theorem 2 exists if $\dim \mathbb{K} < \infty$. 

5
References

[1] Beilinson A., Ginzburg V., Soergel W.; Koszul duality patterns in representation theory, *J. Am. Math. Soc.* 9 (1996) No.2, 473-527.

[2] Bergman G.; The diamond lemma for ring theory, *Adv. in Math.* 29, no. 2, 178–218 (1978).

[3] Bernstein I.N., Gelfand I.M., Gelfand S.I.; Category of g-modules, *Funct. Anal. Appl.* 10 (1976) 87-92.

[4] Borel A., Wallach N.; *Continuous cohomology, discrete subgroups, and representations of reductive groups*, American Mathematical Society, 2000.

[5] Bourbaki N.; *Algèbre commutative*, Chapitres 1 à 4, Hermann, Paris, 1961.

[6] Cline E., Parshall B., Scott L.; Algebraic stratification in representation categories, *J. Algebra* 117 (1988), no. 2, 504-521.

[7] Cline E., Parshall B., Scott L.; Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.* 391 (1988) 85-99.

[8] Dixmier J.; *Algèbres enveloppantes*, Gauthier-Villars, Paris 1974.

[9] Eilenberg S.; Homological dimension and syzygies, *Ann. of Math.* 64 (1956) 328-336. (Errata in *Ann. of Math.* 65 (1957) 593.)

[10] Gelfand I. M., Ponomarev V. A.; The category of Harish-Chandra modules over the Lie algebra of the Lorentz group. (Russian) *Dokl. Akad. Nauk SSSR* 176 (1967) 243–246.

[11] Weibel Ch.; *An introduction to homological algebra*, Cambridge University Press, 1995.