DIRAC SURFACES AND THREEFOLDS

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Abstract. We describe Dirac structures on surfaces and 3-manifolds. Every Dirac structure on a surface $M$ is described either by a regular 1-foliation or by a section of a circle bundle obtained as a fiberwise compactification of the line bundle $\wedge^2 TM$. Every Dirac structure on a 3-manifold $M$ is either the union of a presymplectic manifold and a foliated Poisson manifold, or the union of a Poisson manifold and a foliated presymplectic manifold.

1. Introduction

A 2-form $\omega$ on a manifold $M$ induces a skew-symmetric map $TM \rightarrow T^*M$ whose graph is an $n$-dimensional subbundle of $TM := TM \oplus T^*M$, isotropic with respect to the natural inner product on $TM$. Likewise, a bivector $\Pi$ on $M$ induces a skew-symmetric map $T^*M \rightarrow T$ whose graph is also an $n$-dimensional isotropic subbundle of $TM$. The language of Dirac geometry allows us to simultaneously generalize presymplectic geometry and Poisson geometry by taking our main geometric structure to be an isotropic $n$-dimensional subspace of $TM$, involutive with respect to the Courant bracket on $TM$. The involutivity criterion generalizes the condition of a 2-form being closed, and of a bivector being Poisson. We review the language of Dirac geometry in Section 2. In Section 3, we classify Dirac structures on surfaces. In Section 4, we classify Dirac structures on 3-manifolds.

2. Dirac Geometry

Proofs of the assertions made in this section can be found in [1], [2], or any introduction to Dirac geometry.

2.1. Linear Algebra. For a real vector space $V$ of dimension $n$, $V \oplus V^*$ has a natural split-signature inner product $\langle v + \xi, w + \eta \rangle = \frac{1}{2} \xi(w) + \eta(v)$ and natural projections $\rho : V \oplus V^* \rightarrow V$ and $\hat{\rho} : V \oplus V^* \rightarrow V^*$. A subspace $L$ of $V \oplus V^*$ is Lagrangian if it is $n$-dimensional and isotropic with respect to this inner product, and $\operatorname{Lag}(V \oplus V^*)$ is defined to be the space of all Lagrangians in $V \oplus V^*$. For $L \in \operatorname{Lag}(V \oplus V^*)$, let $\Delta := \rho(L)$ and $\hat{\Delta} := \hat{\rho}(L)$. The flags of subspaces in $V$ and $V^*$

$$0 \subset L \cap V \subset \Delta \subset V \quad \text{and} \quad 0 \subset L \cap V^* \subset \hat{\Delta} \subset V^*$$

are related by the fact that $\text{Ann}(L \cap V) = \hat{\Delta}$ and $\text{Ann}(\Delta) = L \cap V^*$. The type of $L \in \operatorname{Lag}(V \oplus V^*)$ is the pair of integers $(\dim(L \cap V), \dim(L \cap V^*))$. For example, the type of $V$ is $(n, 0)$, and the type of a Lagrangian defined as the graph of a 2-form $\omega \in \wedge^2 V^*$ is $(\dim(\ker \omega), 0)$. In most references, the type of $L$ is instead defined as the codimension of $\Delta$ in $V$, which equals the second coordinate of our definition of type. Our definition distinguishes between $L = V \subset V \oplus V^*$ and the graph of a symplectic structure, but our definition is not well-defined on general Courant algebroids that do not contain $TM$ as a subbundle. We call a Lagrangian of type $(a, b)$ even if $b$ is even, and odd otherwise. Topologically, $\operatorname{Lag}(V \oplus V^*) \cong O(n)$ ([1], Section 1.3); its two components are the even Lagrangians $\operatorname{Lag}_{\text{even}}(V \oplus V^*) \subset \operatorname{Lag}(V \oplus V^*)$ and the odd Lagrangians $\operatorname{Lag}_{\text{odd}}(V \oplus V^*) \subset \operatorname{Lag}(V \oplus V^*)$. 


A Lagrangian \( L \) defines a skew form \( \epsilon \in \wedge^2 \Delta^* \) and a bivector \( \Pi \in \wedge^2 \hat{\Delta}^* \cong \wedge^2 (V/(L \cap V)) \) by the formulas
\[
\epsilon(\rho(x)) = \tilde{\rho}(x) \bigg| \Delta \quad \text{and} \quad \Pi(\tilde{\rho}(x)) = \rho(x) \bigg| \hat{\Delta} \quad \text{for all} \ x \in V \oplus V^*
\]
Conversely, any Lagrangian is uniquely specified by the pair \((\Delta, \epsilon)\), and also by the pair \((\hat{\Delta}, \Pi)\).

**Proposition 2.1.** ([1], Proposition 1.11.5) The maps \( L \mapsto (\Delta, \epsilon) \) and \( L \mapsto (\Delta^*, \Pi) \) define bijections
\[
\begin{align*}
\{ \text{Pairs } (\Delta, \epsilon) \mid \Delta \text{ a subspace of } V, \epsilon \in \wedge^2 \Delta^* \} & \leftrightarrow \text{Lag}(V \oplus V^*) \\
\{ \hat{\Delta} \text{ a subspace of } V^*, \Pi \in \wedge^2 \hat{\Delta}^* \} & \leftrightarrow \text{Lag}(\hat{\Delta} \oplus \Pi)
\end{align*}
\]

2.2. Differential Geometry. The **generalized tangent bundle** of a manifold \( M \) is the bundle \( \mathcal{T}M := TM \oplus T^*M \) endowed with the split-signature inner product \((X + \xi, Y + \eta) = \frac{1}{2} (\eta(X) + \xi(Y))\) and the **Courant bracket**
\[
[X + \xi, Y + \eta] = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} \mathcal{L}_{[X, Y]} \xi.
\]
on its space of sections. We denote by \( \text{Lag}(\mathcal{T}M) \) the \( O(n) \) bundle over \( M \) whose fiber over \( x \in M \) is \( \text{Lag}(T_x M \oplus T^*_x M) \). An **almost Dirac structure** is a Lagrangian subbundle \( E \) of \( \mathcal{T}M \) – equivalently, a section of \( \text{Lag}(\mathcal{T}M) \). Given an almost Dirac structure \( E \) on \( M \), we denote
\[
M_{(a, b)} := \{ x \in M \mid E_x \text{ has type } (a, b) \}.
\]
An almost Dirac structure \( E \) is a **Dirac structure** if it is involutive – that is, if the Courant bracket of two sections of \( E \) is again a section of \( E \). This involutivity criterion can be expressed using the language of foliated forms (or foliated poisson structures) on regions of \( M \) where the dimension of \( \Delta \) (or \( \hat{\Delta} \)) is locally constant.

**Definition 2.2.** Let \( D \) be a distribution on a manifold \( M \) given by a regular foliation. The **foliated exterior derivative**
\[
d_D : \Gamma(\wedge^k D^*) \to \Gamma(\wedge^{k+1} D^*)
\]
is given by the usual Cartan formula
\[
d_D \omega(V_0, \ldots, V_k) := \sum_i (-1)^i a(V_i) \left( \omega(V_0, \ldots, \hat{V}_i, \ldots, V_k) \right)
\]
\[
+ \sum_{i < j} (-1)^{i+j} \omega([V_i, V_j], V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_k).
\]
where \( \{V_0, \ldots, V_k\} \) are sections of \( D \). A **foliated presymplectic** form is a \( d_D \)-closed \( \omega \in \Gamma(\wedge^2 D^*) \). If \( D = \mathcal{T}M \), we recover the usual definition of a presymplectic form.

**Definition 2.3.** Let \( D \) be a distribution on a manifold \( M \) given by a regular foliation. The sheaf of **admissible** functions on \( M \) is the sheaf \( C^\infty_D \) of functions which are constant on the leaves of \( D \). A **foliated Poisson** structure is a Poisson bracket on \( C^\infty_D \).

**Proposition 2.4.** ([3] Prop. 2.7, [1] Prop 2.5.3 and Cor. 2.6.3) Let \( E \) be an almost Dirac structure
\begin{enumerate}
\item If \( \Delta \) is a subbundle of \( TM \) (so \( E \) can be described as the graph of \( \epsilon \in \wedge^2 (\Delta)^* \)), then \( E \) is a Dirac structure if and only if \( \Delta \) integrates to a foliation and \( d_\Delta \epsilon = 0 \).
\item If \( E \cap TM \) is a subbundle of \( TM \) (so \( E \) can be described as the graph of \( \Pi \in \wedge^2 (\hat{\Delta})^* \)), then \( E \) is a Dirac structure if and only if \( E \cap TM \) integrates to a foliation and \( \Pi \) defines a foliated Poisson structure with \( M \) with respect to this foliation.
\end{enumerate}

For example, wherever an almost Dirac structure \( E \) is transverse to \( T^*M \subseteq TM \), \( E \) is the graph of a 2-form \( \omega \) and Proposition 2.3 states that \( E \) is Dirac structure precisely if \( \omega \) is closed. Wherever \( E \) is transverse to \( TM \), \( E \) is the graph of a bivector \( \Pi \), and \( E \) is Dirac precisely if \( \Pi \) is Poisson.
3. Dirac Structures on Surfaces

3.1. Linear Algebra. For a 2-dimensional vector space $V$, $\text{Lag}(V \oplus V^*) \cong O(2)$ consists of two circles, $\text{Lag}_e(V \oplus V^*)$ and $\text{Lag}_o(V \oplus V^*)$. The circle of even Lagrangians is covered by the maps

\[
\wedge^2 V^* \to \text{Lag}_e(V \oplus V^*) \\
\wedge^2 V \to \text{Lag}_e(V \oplus V^*)
\]

Each even Lagrangian has type $(0,0)$ except for $V$ and $V^*$, which are the images of $0 \in \wedge^2 V^*$ and $0 \in \wedge^2 V$ in the maps above, and have type $(2,0)$ and $(0,2)$, respectively. The circle of odd Lagrangians is isomorphic to $P(V)$, the projective space of lines in $V$, by

\[
P(V) \to \text{Lag}_o(V \oplus V^*) \\
L \mapsto L + \text{Ann}(L)
\]

![Diagram](image)

**Figure 1.** The topology of $\text{Lag}(V \oplus V^*)$ for $\dim(V) = 2$

3.2. Differential Geometry. Let $M$ be a surface. Then both $\text{Lag}_e(TM)$ and $\text{Lag}_o(TM)$ are circle bundles over $M$. The Maps (1) globalize to inclusions

\[
\wedge^2 TM \to \text{Lag}_e(TM) \\
\wedge^2 T^* M \to \text{Lag}_e(TM)
\]

so the circle bundle $\text{Lag}_e(TM)$ may be viewed as the fiberwise compactification of the canonical (or anticanonical) line bundle.

Let $E$ be an even almost Dirac structure. On $M_{(0,0)} \cup M_{(2,0)}$, $E$ is the graph of a 2-form and Proposition 2.4 states that $E$ is Dirac if and only if this 2-form is closed. Every 2-form on a surface is closed, so $E$ is Dirac on $M_{(0,0)} \cup M_{(2,0)}$. Similarly, $E$ is Dirac on $M_{(0,0)} \cup M_{(0,2)}$ because every bivector on a surface is Poisson. Therefore, every even almost Dirac structure on a surface is Dirac. The data of an odd Dirac structure is equivalent to the data of a regular 1-dimensional foliation on $M$. This proves

**Theorem 3.1.** Let $M$ be a surface. Even Dirac structures on $M$ are sections of the circle bundle $\text{Lag}_e(TM)$. Odd Dirac structures on $M$ correspond to regular 1-foliations on $M$.

If $M$ is orientable, $\text{Lag}_e(TM)$ is trivial, so $\Gamma(M, \text{Lag}_e(TM)) \cong \text{Map}(M, S^1)$.

**Corollary 3.2.** Let $M$ be an orientable surface. The path components of the space of even Dirac structures are classified by $H^1(M; \mathbb{Z})$.

A necessary and sufficient condition for the existence of a regular 1-foliation on a surface is that the Euler characteristic of the surface vanishes.

**Corollary 3.3.** The only closed surfaces that admit an odd Dirac structure are the torus and the Klein bottle.
4. Dirac Structures on 3-manifolds

4.1. Linear Algebra. Let $V$ be a 3-dimensional real vector space, and let $Gr(k, V)$ denote the Grassmannian of $k$-dimensional planes in $V$. Then $\text{Lag}(V \oplus V^*) \cong O(3)$ is diffeomorphic to two copies of $\mathbb{R}P^3$. $\text{Lag}_e(V \oplus V^*)$ consists of Lagrangians of type $(1, 0), (3, 0),$ and $(1, 2)$. The Lagrangians of type $(1, 0)$ and $(1, 2)$ are precisely the ones for which $\dim(\Delta) = 2$. By Proposition 2.1, these Lagrangians are classified by pairs $(\hat{\Delta}, \Pi)$, where $\hat{\Delta} \in \text{Gr}(2, V^*)$ and $\Pi \in \wedge^2 \hat{\Delta}^*$. The space of all such pairs is the total space of a real line bundle over $\text{Gr}(2, V^*) \cong \mathbb{R}P^2$ with projection $(\hat{\Delta}, \Pi) \mapsto \hat{\Delta}$. The zero section of this bundle corresponds to Lagrangians of type $(1, 2)$. Similarly, the Lagrangians of type $(1, 0)$ and $(3, 0)$ are precisely the Lagrangians for which $\Delta = V$, and are classified by elements $\epsilon \in \wedge^2 V$. This set is a 3-ball whose zero corresponds to the Lagrangian $V$.

The space $\text{Lag}_0(V \oplus V^*)$ consists of Lagrangians of type $(0, 1), (0, 3),$ and $(2, 1)$. The Lagrangians of type $(0, 1)$ and $(2, 1)$ are classified by pairs $(\Delta, \epsilon)$, where $\Delta \in \text{Gr}(2, V)$ and $\epsilon \in \wedge^2 \Delta^*$. This set is the total space of a real line bundle over $\text{Gr}(2, V) \cong \mathbb{R}P^2$ whose zero section corresponds to Lagrangians of type $(2, 1)$. The Lagrangians of type $(0, 1)$ and $(0, 3)$ are classified by elements $\epsilon \in \wedge^2 V^*$, a 3-ball whose zero corresponds to $V^*$.

![Figure 2. The topology of $\text{Lag}(V \oplus V^*)$ for $\dim(V) = 3$.](image)

4.2. Differential Geometry. Let $E$ be a Dirac structure on a 3-manifold. We can use Proposition 2.1 and the discussion above to describe the Dirac geometry of different regions of $M$ using the language of foliated presymplectic and foliated Poisson structures

$M_{(1,0)} \cup M_{(1,2)}$: Foliated Poisson

$M_{(0,1)} \cup M_{(2,1)}$: Foliated Presymplectic

$M_{(1,0)} \cup M_{(3,0)}$: Presymplectic

$M_{(0,1)} \cup M_{(0,3)}$: Poisson

This is summarized in the following theorem.

**Theorem 4.1.** Let $M$ be a three dimensional manifold.

1. Every even Dirac structure on $M$ is the union of a presymplectic manifold and a foliated Poisson manifold. These manifolds are glued along the region $M_{(1,0)}$.

2. Every odd Dirac structure on $M$ is the union of a Poisson manifold and a foliated presymplectic manifold. These manifolds are glued along the region $M_{(0,1)}$.

**References**

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