Quantum operations: technical or fundamental challenge?

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Abstract
A class of unitary operations generated by idealized, semiclassical fields is studied. The operations implemented by sharp potential kicks are revisited and the possibility of performing them by softly varying external fields is examined. The possibility of using the ion traps as ‘operation factories’ transforming quantum states is discussed. The non-perturbative algorithms indicate that the results of abstract δ-pulses of oscillator potentials can become real. Some of them, if empirically achieved, could be essential to examine certain atypical quantum ideas. In particular, simple dynamical manipulations might contribute to the Aharonov–Bohm criticism of the time–energy uncertainty principle, while some others may verify the existence of fundamental precision limits of the position measurements or the reality of ‘non-commutative geometries’.

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(Some figures may appear in colour only in the online journal)

1. Introduction

One of the limitations of present-day quantum theories is the ‘passive’ evolution picture in which the physical systems (i.e., some parts of the universe) evolve under the influence of ‘the rest’, typically represented by some given (if not stationary) external conditions, and the role of active state manipulations is reduced to the choice of initial or boundary conditions. However, if the physical theories were at all created it is only because the physical systems can be actively manipulated by changing the external conditions and performing experiments.

In spite of the limited use of dynamical manipulations, the present-day quantum theories show some spectacular achievements. However, the concepts applied and questions asked are somewhat repetitive. What dominates are some idealized scenarios with pure states always described by vectors in the linear (Hilbert) spaces and the time evolution (in the absence of dissipation) always obeying the linear, unitary operations. The picture persists in the
description of composite systems and quantum field theories where the states are always represented by tensor product spaces, in a desire to conserve the linearity of basic laws at the cost of multiplying the number of variables.

Some doubts, though, persist. Can indeed all ‘gedanken states’ represented by vectors in Hilbert spaces be physically created [1]? Moreover, can all unitary operations be achieved (or at least approximated) by physical evolution? In the recent research, a lot of attention is dedicated to the finite-dimensional spin states (qubits) with hopes to develop quantum computers. However, can the problem indeed be exhausted by tensor products with complex coefficients?

One of the main problems in checking the ‘obligatory beliefs’ of quantum theories are the perturbative complications as well as the difficulties of extrapolating toward ‘the great’ and toward ‘the small’ [2]. What could help are the exact solutions, though they seldom exist. Yet, in certain areas, there appear some windows in perturbative clouds. This happened in the bound state manipulation [3, 4] (Nobel Prize 2012 for S Haroche and D Wineland), in macroscopic superpositions of Leggett [5], then in works on ‘quantum tomography’ [6–9], and last but not least in the duality links between the quantum quark and classical string dynamics [10, 11]. The purpose of this study is to show that even the well known and modest case of classical–quantum duality, for particles in time-dependent quadratic potentials, still has some unexplored consequences.

The paper is organized as follows. In section 2, the particle behavior in one-dimensional quadratic, time-dependent potentials $V(q, t) = \beta(t) \frac{q^2}{2}$ is classified in the most elementary terms. Section 3 briefly reports these classification analogues in the macroscopic world. In section 4, a class of idealized evolution effects produced by $\delta$-kicks of the quadratic potentials is presented, and section 5 outlines their optical equivalents. Their links with the arguments of Aharonov, Bohm et al against the time–energy uncertainty principle are reported in section 6. Sections 7 and 8 show how to design the soft equivalents of the singular pulse operations. Finally, section 9 discusses their possible fundamental implications.

2. Quadratic Hamiltonians: the classical–quantum structures

The quantum theories satisfy the correspondence principle, becoming classical in the macroscopic limit, as $\hbar \to 0$. However, some mathematical aspects are shared by classical and quantum theories without the need for any limiting transition. The simplest cases occur for non-relativistic time-dependent, quadratic Hamiltonians in one dimension:

$$H(t) = \frac{p^2}{2} + \beta(t) \frac{q^2}{2},$$

where $q$ and $p$ are the canonical position and momentum variables (we put for simplicity, the mass $m = 1$). Below, we shall not consider any higher space dimensions nor the deeper Hilbert space problems (carefully reviewed by Barry Simon [12]). The only challenge we attend is strictly combinatorial: how should one program the oscillations of the $c$-number amplitude $\beta(t)$ to generate some useful quantum control operations? On purely mathematical level, the problem is elementary. In classical theory and for nonsingular $\beta(t)$, the canonical equations lead to the linear evolution transformations of the canonical variables, represented by the $2 \times 2$ symplectic evolution matrices $u(t, t_0)$:

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = u(t, t_0) \begin{pmatrix} q(t_0) \\ p(t_0) \end{pmatrix}$$

given by the matrix equations

$$\frac{d}{dt} u(t, t_0) = \Lambda(t) u(t, t_0) \quad \frac{d}{dt_0} u(t, t_0) = -u(t, t_0) \Lambda(t_0)$$

$$\frac{d}{dt} u(t, t_0) = \Lambda(t) u(t, t_0) \quad \frac{d}{dt_0} u(t, t_0) = -u(t, t_0) \Lambda(t_0)$$

with
\[ A(t) = \begin{pmatrix} 0 & 1 \\ -\beta(t) & 0 \end{pmatrix} \] (4)

and
\[ u(t, \theta)u(\theta, t) = u(t, t_0) \quad u(t_0, t_0) = 1. \] (5)

In quantum theory, the corresponding evolution operators \( U(t, t_0) \) are given by
\[ i\frac{d}{dt}U(t, t_0) = H(t)U(t, t_0), \quad i\frac{d}{dt_0}U(t, t_0) = -U(t, t_0)H(t_0) \] (6)

with
\[ U(t, \theta)U(\theta, t) = U(t, t_0) \quad U(t_0, t_0) = 1 \] (7)

defining the evolution of the observables \( q \) and \( p \) in Heisenberg’s picture (‘Heisenberg’s trajectory’), given by the same symplectic matrices (3)–(5):
\[ U(t, t_0)^i\left(\begin{array}{c} q \\ p \end{array}\right)U(t, t_0) = u(t, t_0)^i\left(\begin{array}{c} q \\ p \end{array}\right), \] (8)

though now \( q \) and \( p \) are operators with \([q, p] = i\) (we put for simplicity \( \hbar = 1 \)). The exact formal correspondence of the classical and quantum cases was explored by multiple research groups, quite frequently without knowing each other. In all these approaches, a useful observation is as follows.

**Proposition 1.** In the absence of spin or any additional degrees of freedom, each unitary evolution operator \( U(t, t_0) \) in \( L^2(\mathbb{R}) \) generated by the quadratic, time-dependent Hamiltonians (1) is determined up to a phase factor by the classical motion trajectories.

**Proof.** Instead of sophisticated arguments, note simply that if two unitary operators \( U_1 \) and \( U_2 \) produce the same transformation of the canonical variables i.e., \( U_1^*qU_1 = U_2^*qU_2 \) and \( U_1^*pU_1 = U_2^*pU_2 \), then \( U_1U_2^* \) commutes with both \( q \) and \( p \). Hence, it commutes also with any function of \( q \) and \( p \), including their spectral projectors. Since in \( L^2(\mathbb{R}) \), the functions of \( q \) and \( p \) generate an irreducible algebra, then \( U_1U_2^* \) must be a c-number and since it is unitary, it can only be a phase factor, \( U_1U_2^* = e^{i\varphi} \Rightarrow U_1 = e^{i\varphi}U_2 \), where \( \varphi \in \mathbb{R} \) [12–14].

Any two unitary operators which differ only by a c-number phase factor generate the same transformation of quantum states, so we shall call them equivalent and write \( U_1 \equiv U_2 \). (The fact that they might wear different phase factors can be of interest for the linear representation theory [15, 16, 18] but does not affect the operations performed on physical states, the principal subject of our interest.)

The possibility of deducing the quantum state evolution from the transformations of the canonical variables (8) permits one to program ample families of the exact classical/quantum control operations. For their classification, the algebraic types of matrices (2)–(8) are quite essential. Since every evolution matrix \( u = u(t, t_0) \) is symplectic (\( \text{Det} u = 1 \)), its algebraic type is defined by just one invariant \( \text{Tr} u \). The characteristic equation
\[ D(\lambda) = \text{Det}(\lambda - u) = \lambda^2 - \text{Tr}(u)\lambda + 1 = 0 \] (9)

has two roots \( \lambda_{\pm} = \frac{1}{2}\text{Tr} u \pm i\sqrt{\Delta} \), where \( \Delta = 1 - \frac{1}{4}(\text{Tr} u)^2 \), permitting us to distinguish three types of evolution matrices.

(I) If \( |\text{Tr}(u)| < 2 \), then \( u \) has two complex eigenvalues \( \lambda_{\pm} = e^{\pm i\sigma} \), where \( 0 \neq \sigma \in \mathbb{R} \).

(II) If \( |\text{Tr}(u)| = 2 \), then \( u \) is in the threshold: \( \lambda_+ = \lambda_- = \pm 1 \).

(III) If \( |\text{Tr}(u)| > 2 \), then \( u \) has two real eigenvalues \( \lambda_{\pm} = e^{\pm \sigma} \), where \( 0 \neq \sigma \in \mathbb{R} \).
The classification becomes particularly relevant if the function $\beta(t)$ in (1) is periodic, $\beta(t + T) = \beta(t)$, defining a Floquet process. The (crucial) Floquet matrices $u(t_0 + T, t_0)$ then define the repeated evolution incidents. One easily shows that their types do not depend on $t_0$. Choosing $t_0 = 0$ and denoting for simplicity $u(t) = u(t, 0)$, one sees that the results of the evolution in the sequence of expanding intervals $[0, nT]$ are given by repetitions of $u(T)$, i.e., $u(nT) = u(T)^n$. Now, if $u(T)$ is in the class (I), the evolution is oscillatory. The eigenvectors of $u(T)$ define a pair of variables $A_{\pm}$ (generalizing the creation and annihilation operators), which for $t = nT$ just perform the phase rotations $U(t)^nA_{\pm}U(t) = e^{i\sigma n}\sigma A_{\pm}$. However, if $u(T)$ is in the class (III), then the equilibrium is lost: the eigenvectors of $u(T)$ define two real canonical variables $A_{\pm}$ which are multiplied by $e^{i\sigma n}$, $(0 \neq \sigma \in \mathbb{R})$ i.e., endlessly squeezed or endlessly amplified as $t = nT \to \infty$. In turn, the threshold cases (II) offer some exceptional manipulation techniques; their mechanism is simple, but consequences are not.

3. Microscopic models of the macroscopic universe

The properties of (1) also intrude into the spectral problems of the time-independent energy operators $H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ with arbitrary real potentials $V(x)$. Indeed, by looking for the real stationary solutions $\psi(x)$ of the Schrödinger equation

$$-\frac{1}{2}\frac{d^2\psi}{dx^2} + [V(x) - E]\psi(x) = 0 \quad (10)$$

for any real $E$, belonging or not to the spectrum of $H$, one sees that the pair of variables $\psi$, $\psi' = \frac{d\psi}{dx}$ is displaced along the $x$-axis according to the first-order matrix equation

$$\frac{d}{dx}\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\beta_E(x) & 0 \end{pmatrix}\begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad (11)$$

where $\beta_E(x) = 2[E - V(x)]$. So, by reinterpreting the variable $x$ as $t$, $\beta_E(x)$ as $\beta_E(t)$, then $\psi(x)$ and $\psi'(x)$ as $q(t)$ and $p = \frac{dq}{dt}$, respectively, one sees that spectral problem (10) has a classical equivalent in the form of the oscillator (1) with a time-dependent elasticity coefficient $\beta_E(t)$ [13, 19]. The spectrum and the spectral gaps of quantum system (10) correspond exactly to the stability and instability (parametric resonance) zones of the classical oscillator (1) with some consequences for macroscopic phenomena. Curiously, certain aspects of both pictures are opposite. What in the quantum case was the synonym of stability (an energy eigenstate) corresponds to an exceptional, highly unstable orbit in the classical picture. This equivalence inspired Avron and Simon to describe the structure of Saturn’s rings by spectral bands and gaps of the Schrödinger energy operator [20] (an analogy worth attention, even if astronomers opted for different models). In a different scale, the abrupt changes of the classical trajectories (11) as $E$ evolves, reflecting the behavior of the Schrödinger wave function (10), can be used in an efficient computer method to solve the spectral problems of the potential wells by observing the instabilities of classical trajectories [21].

Some intriguing cases of quantum–classical duality appear in relativistic cosmology, where certain variants of the Schrödinger equation are used to extrapolate the past cosmos evolution near the Big Bang [22–24]. If such cosmological reinterpretations were correct, then due to the exceptional character of the Schrödinger energy eigenstates, it could also reveal some instability mechanisms of Avron–Simon type in our reconstruction of the early universe (or its distant future), usually neglected in the cosmological ‘slow roll’ models [24] (see also discussions in [25, 26]). While the cosmic parameters can be studied (but not manipulated),
Figure 1. The applications of an evolution loop. The basic and perturbed evolution processes are represented by $U_0(t)$ and $U(t) = U_0(t)W(t)$, where the $W(t)$ is the evolution in the interaction frame. If for some $T$, the $U_0(T)$ closes to the evolution loop, $U_0(T) = 1$, then the whole process reduces to $W(T)$ alone, the precession operator, sensitive to manipulation programs.

our purpose below will be to discuss the parallel chapters of quantum control, which can suggest the laboratory operations.

4. The evolution controlled by sharp pulses

The exact solutions of (1) were widely explored by Lewis and Riesenfeld [23, 27, 28], then by Malkin et al [29, 30] in terms of adiabatic invariants, interrelated also with important techniques of quantum tomography [6, 7, 9]. The mathematical algorithms, though elementary, are not immediate to apply. (Anyhow, they require the solution of the second-order differential equation (10), complicated frequently by pertubative difficulties.)

As independently observed, one of the simplest ways to control the dynamical evolution consists in generating a closed evolution pattern (evolution loop) and then considering their perturbed or deformed versions. If the unperturbed evolution is represented by a certain family of unitary operators $U_0(t)$, then the perturbed evolution operators split into $U(t) = U_0(t)W(t)$, where $U_0(t)$ represents the basic dynamical process and $W(t)$ is the correction (the evolution in the interaction frame). If now at some moment $t = T$, the basic evolution closes to a loop $U_0(T) ≡ 1$, then the full evolution reduces just to the pure deformation, $U(T) ≡ W(T)$ represented in figure 1, which is, in general, much easier to manipulate using external fields. The most elementary evolution loops occur in the time-independent oscillator potentials, leading, e.g., to the non-demolishing quantum measurements [31], but do not exhaust the manipulation techniques.

The existence of non-adiabatic loops generated by time-dependent oscillator forces was noticed in 1970 by Malkin and Man’ko [32], though without elaborating the operational consequences. The possibility of driving the quantum states by $δ(t)$-pulses of the external fields was considered by Lamb Jr [33]. An extremely simple class of exact, though formal, solutions of (1) in $L^2(\mathbb{R})$ was obtained in [14, 34–37] by superposing two types of elementary operations: the incidents of free evolution and the effects of the sharp pulses of oscillator potentials. Each free evolution incident in any interval $[t_0, t_1]$ produces the unitary evolution operator $e^{-it\frac{p^2}{2}}$, where $\tau = t_1 - t_0$. In turn, the result of each sudden $δ$-kick of the quadratic potential $V(q, t) = aδ(t - t_0)\frac{q^2}{2}$ (where $a$ is the pulse amplitude) is most easily described by adopting the rectangular $δ$-model defined by $δ_\epsilon(t) = \frac{1}{\epsilon}$ in a narrow interval $[t_0, t_0 + \epsilon]$ and vanishing outside. The evolution in $[t_0, t_0 + \epsilon]$ is then induced by the constant Hamiltonian $H_\epsilon = \frac{p^2}{2} + \frac{a\epsilon}{2} q^2$. 
and the corresponding unitary operator $U_\epsilon = e^{-i[\frac{\tau}{2} \hat{q}^2 + \hat{p} \hat{p}]} = e^{-i[\frac{\epsilon}{2} \hat{q}^2 + \hat{p} \hat{p}]} \rightarrow e^{-i\frac{\epsilon}{2} \hat{q}^2}$ for $\epsilon \rightarrow 0$.

In agreement with the Baker formula [38]

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda [A, B] + \frac{\lambda^2}{2!} [A, [A, B]] + \frac{\lambda^3}{3!} [A, [A, [A, B]]] + \ldots,$$

both operations lead to extremely simple transformations of the canonical pair $q, p$. The free evolution incidents generate

$$e^{i\tau \hat{q}} \begin{pmatrix} q \\ p \end{pmatrix} e^{-i\tau \hat{q}} = \begin{pmatrix} q + \tau p \\ p \end{pmatrix} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

while the potential shocks

$$e^{i\tau \hat{q}} \begin{pmatrix} q \\ p \end{pmatrix} e^{-i\tau \hat{q}} = \begin{pmatrix} q \\ p - aq \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}.$$

Within this scheme, an interesting operation is performed by a pair of free evolution steps separated by an oscillator pulse:

$$e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} e^{-i\tau \hat{q}} = F_\tau.$$

It generates

$$q \rightarrow \tau p,$$

$$p \rightarrow \frac{-1}{\tau} q,$$

which might be called the *squeezed Fourier transformation*. Curiously, an equivalent operation is performed by

$$e^{i\frac{i}{2} \tau \hat{q}^2} e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} = F_\tau.$$

Henceforth, the following product of the six unitary operations yields the transformation $q \rightarrow -q$ and $p \rightarrow -p$ (the parity operator):

$$e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} \ldots e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} = P,$$

whereas the sequence of twelve unitary terms produces an evolution loop:

$$e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} \ldots e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} = 1.$$

An intriguing property of (19) is that all six free evolution exponents arise with the same signs and so do the exponents of the kick operations (a kind of non-perturbative Baker–Campbell–Hausdorff effect [38]). A more important aspect of (19) is that it contains the free evolution inversion:

$$e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} \ldots e^{-i\tau \hat{q}} e^{i\frac{i}{2} \tau \hat{q}^2} = e^{i\tau \hat{q}}.$$

If the idealized pulses could be indeed applied, the effect would be generated for every wave packet independently of its initial shape [14]. Note that in this way, the loop mechanism (19) and (20) predicted a part of the 1990 hypothesis about the quantum *time machine* [39].

The similar effects can be caused by elastic pulses with alternating signs. Their basic element might be the sequence of four operators $S = e^{-i\frac{\tau}{2} \hat{q}} e^{-i\frac{\tau}{2} \hat{p}} e^{-i\frac{\tau}{2} \hat{q}} e^{-i\frac{\tau}{2} \hat{p}}$, represented by the exponential functions of the nilpotent matrices $Q = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $Q' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$s = e^{\epsilon^2 Q'} e^{-i\epsilon Q} e^{\epsilon Q'} e^{-i\epsilon Q}$$

$$= \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + \tau a - \tau^2 a^2 & 2\tau - \tau^2 a \\ -\tau a^2 & 1 - \tau a \end{pmatrix}.$$
Figure 2. The evolution loop formed by 12 elementary evolution operators. The $\delta$-pulses of the attractive oscillator potential of amplitudes $1/\tau$ are represented by the hexagon vertices, while the six sides symbolize the $\tau$-intervals of the free evolution. Each three consecutive operators yield the squeezed Fourier operation (15)–(17). Each of the 11 operations (six consecutive oscillator kicks separated by five free evolution intervals) invert the free evolution.

The algebraic properties of $S$ depend again on $\text{Tr} s = 2 - \tau^2a^2$. If $\tau^2a^2 > 4 \Rightarrow \text{Tr} s < -2$, then the matrix $s$ is of the type (III) and the multiple pulse repetitions generate an unstable motion. However, if $0 \neq \tau^2a^2 < 4 \Rightarrow -2 < \text{Tr} s < 2$, the matrix $s$ is of type (I) and the repeated pulses yield a confined motion, including the possibility of the evolution loops. One of the simplest cases occurs for $\tau^2a^2 = 2 \Rightarrow \text{Tr} s = 0$; the Hamilton Cayley equation for $s$ then implies $s^2 = -1 \Rightarrow S^2 = -1$ and so the pulse pattern $S$, if repeated, creates the 16-step evolution loop in which the sum of the oscillator pulses cancels but the effects do not

$$\left[e^{-i\frac{\tau^2a^2}{2}}e^{-i\tau p^2/2}e^{+i\frac{\tau^2a^2}{2}}e^{-i\tau p^2/2}\right]^4 = 1.$$  \hfill (22)

An elementary algebra also permits us to predict some more general effects, such as 'time squeezing', i.e., accelerating or slowing down of the free evolution [34, 35]. Moreover, some simple, asymmetric sequences of the oscillator pulses can produce the squeezing and/or magnification of canonical variables. The most elementary of such effects are achieved by two different 'squeezed Fourier' operations, $F_\alpha$ and $F_\beta$:

$$F_\alpha F_\beta \rightarrow \begin{pmatrix} 0 & \alpha \\ -\frac{1}{\alpha} & 0 \end{pmatrix} \begin{pmatrix} 0 & \beta \\ -\frac{1}{\beta} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{\beta}{\alpha} & 0 \\ 0 & -\frac{\alpha}{\beta} \end{pmatrix} = \begin{pmatrix} -\sigma & 0 \\ 0 & -\frac{1}{\sigma} \end{pmatrix},$$  \hfill (23)

where $\sigma = \frac{\alpha}{\beta}$. Some of these phenomena were independently observed in [40, 41]. An ample collection of more general squeezing operations was described by Dodonov [42]; for the manipulation of complex Hamiltonians, see [43]. The idea of controlling the finite-dimensional qubit systems by deforming the closed dynamical processes also reappears in the recent development [44–46]. This does not yet exhaust all interesting effects.

An 'attractive repulsion'.

It turns out that some special forms of trapped motion can be created around the centers of repulsive potentials. Assume first of all that the particle obeying (1) is submitted to a sequence of $\delta$-pulses with amplitudes $\pm a$, separated by identical $\tau$-intervals of free evolution, such that $\tau^2a^2 > 4$. The Floquet matrix $s = u(2\tau)$ then has $\text{Tr} s < -2$ and the pulses are repellent.
Suppose, however, that the same pulse pattern coexists with a constant repulsive potential $V_κ(x) = -κ^2 x^2 / 2$, $κ > 0$ [47, 48].

**Proposition 2.** For adequate $a, κ, τ$, the repulsive potential $V_κ(x)$, in the presence of the repelling pulse pattern, can trap the particle.

**Proof.** The motion of the system can again be expressed by evolution matrices. The evolution generated by the repulsive $V_κ(x)$ in any $τ$-interval between two subsequent $δ$-pulses is described by the exponential matrix:

$$e^{τΛ} = \frac{e^{τΛ} + e^{-τΛ}}{2} + \frac{e^{τΛ} - e^{-τΛ}}{2Λ}. \Lambda.$$  \hspace{1cm} (24)

Here $Λ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ fulfills $Λ^2 = κ^2 1$; hence, all even powers of $Λ$ in (24) can be replaced by the corresponding powers of $κ$, leading to the well-known hyperbolic matrix:

$$e^{τΛ} = \cosh κτ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{κ}{κ} \sinh κτ \begin{pmatrix} 0 & κ \\ κ & 0 \end{pmatrix}. \hspace{1cm} (25)$$

Suppose now that (25) is intertwined with the matrices (14) of the elastic $δ$-kicks equivalent to $e^{±aQ}$. By superposing the constant repulsion acting in $[0, 2τ]$ with two opposite kicks $±a$, one thus obtains

$$u_κ(2τ) = e^{τΛ} e^{-aQ} e^{τΛ} e^{aQ}$$

$$= \begin{pmatrix} cosh 2κτ + a^2 / 2κ^2 & -a / κ \sinh 2κτ + κ \sinh κτ \\ κ - a^2 / 2κ & cosh 2κτ - a / κ \sinh 2κτ \end{pmatrix}.$$  \hspace{1cm} (26)

with

$$\text{Tr } u_κ(2τ) = 2 + \left( 4 - \frac{a^2}{κ^2} \right) \sinh^2 κτ. \hspace{1cm} (27)$$

For $κ \to 0$, this reduces to $\text{Tr } u_0(2τ) = 2 - a^2 τ^2$, so if the ‘auxiliary variable’ $y^2 = \frac{1}{4} a^2 τ^2 > 1$, then $u_0(2τ)$ is of type (III) and the particle escapes. Yet, if the motion is assisted by an additional repulsive potential, where $z = κτ$ (the second auxiliary variable) satisfies $z^2 < y^2 < z^2 (1 + \frac{1}{\sinh^2 z})$, then the particle remains trapped (see figure 3). \hspace{1cm} □

In particular, when

$$z^2 \left( 1 + \frac{1}{2 \sinh^2 z} \right) = y^2,$$

then trace (27) vanishes, the matrix $u_κ(2τ)$ satisfies $u_κ(2τ)^2 = -1 \Rightarrow u_κ(2τ)^4 = 1$, and the particle is trapped in an octagonal evolution loop represented in figure 4, where the shaded lines represent the elastic repulsion.

Even if one objects to the S/F story about the elastic $δ$-pulses, the evolution illustrated in figure 4 is indeed an elementary equivalent of a phenomenon known for particles in Paul’s traps moving under the joint influence of two-component oscillator potentials, $β(t) = β_0 + β_1 \sin ωt$. The stability areas on the $(β_0, β_1)$-map are then determined by the Strutt diagram [49]. Each stability arm extends toward the negative values of $β_0$ (repulsive oscillators) which may be intuitively expected to accelerate the expulsion, but instead, they protect the particle against the repelling action of the oscillating field.
The effect of the repulsive potential $V(x) = -\kappa x^2$ in a repelling sequence of the elastic kicks $\pm a$ at the time moments $n\tau$, $n = 1, 2, \ldots$, on the parameter map $y = \frac{1}{2}\alpha \tau$, $z = \kappa \tau$. The lower and upper borderlines mark the stability area where the evolution is confined. The dotted curve in between marks the zeros of (27) where the motions form octagonal loops. For $y = 1.2$, the repulsive potentials with $z < 0.79$ are still too weak, but for $0.79 < z < 1.2$, they trap the motion. For $z \simeq 1.04296$, they generate an evolution loop.

The effect also shows some similarity to the behavior of a charged particle moving in crossed electric and magnetic fields in the presence of a repellent obstacle in the form of a rigid disc [50]. In this scheme, the particle ‘returns obsessively’ to the obstacle; but should the repellent disc disappear, it will escape. The question of whether an analogous effect can exist for other, non-quadratic potentials is open. Should it occur for the repulsive Coulomb fields
submerged in pulsating perturbations, it might contribute to a better understanding of some solid state effects such as the mechanism of Cooper pairs.

5. Optical equivalents: the true prehistory?

While the realistic cases of the idealized operations are still under discussion, it was recently noticed that almost all quadratic control algorithms admit a distinct interpretation. As reported by Wolf [16, 17], the evolution operations generated by the Hamiltonians (1) have some simple equivalents in the optical experiments on a 1D optical bench. In particular, the $2 \times 2$ matrix transformations (13, 14, 23) of the canonical variables $q, p$ correspond exactly to the application of some typical optical instruments (microscopes, telescopes, etc) well known in geometrical optics. In fact, supposing that $q$ is the distance from the bench axis $z$ and $p = q \sin \theta$, where $\theta$ is an angle between the light ray and the $z$-axis, matrix (13) describes the optical images formed by the congruence of light rays propagating along the bench, where $\tau$ is now the $z$-distance between the source and the image, while (14) describes the action of a thin optical lens placed on the bench, the amplitude $a$ meaning the (positive or negative) lens curvature [51]).

Both interpretations have certain gaps and advantages. Thus, what in the dynamical language is an evolution loop (convenient for the dynamical manipulations), in optical terms is the simple reproduction of an optical image. The squeezing/amplification mechanisms produced by two or more oscillator shocks in quantum mechanics are easily interpretable as the applications of the microscope (or telescope) in geometrical optics. (So, in a sense, the effects of the oscillator kicks were already known to Galileo.) Moreover, what was not so easy to predict in the dynamical language, i.e., the asymmetry of $\beta(t)$ needed to produce the squeezing, is immediately obvious at the optical level. (In fact, the symmetric apparatus could not produce amplified or reduced images.) Both dynamical and optical techniques also have their specific imperfections. In the terrain of optics, the applications of too close lenses with too big $|a|$ would mean that a part of one lens (if not the whole) must overlap with the interior of the other. More exact equivalents of the elastic pulses (14) were subsequently considered in works on optical signals in dispersive fibers [52] (for more recent studies, see also [53]). In the quantum mechanics of particles driven by potentials in ion traps, the exact application of $\delta$-kicks is practicably impossible. Even forgetting about the finite resistance of the trap walls, no infinite energy shocks can be truly engineered. Yet, despite all their imperfections, the idealized optical (or dynamical) operations might be of interest for some unfinished fundamental discussions.

6. The time–energy uncertainty?

Indeed, it is enough to recall the doubts concerning the ‘time–energy uncertainty principle’. It looks as though the ‘squeezed Fourier’ transformations can bring some new elements to the list of critical arguments. The first objections against the (too verbal) interpretations of Landau and Peierls [54] supporting this principle, appeared in the study of Aharonov and Bohm in 1961 [55]. The doubts returned in [56] and other papers. Further arguments against too dogmatic formulations were collected in Aharonov, Massar and Popescu who argued that an arbitrarily exact measurement of the energy of a quantum system can be performed in an arbitrarily short time, provided that ‘the measurement is brutal’. Their illustration was the spin measurement [57]. Seemingly, the pulse patterns of section 4 bring another application of the same argument, valid not only for the spin but also for the position–momentum states.
Indeed, suppose that the two $\delta$-pulses divided only by a very short time interval $\tau$, or alternatively, one $\delta$-pulse of the oscillator potential between two infinitesimal intervals of the free evolution, perform the ‘squeezed Fourier transformation’ (15) of a free particle propagating initially with an energy $p^2/2m$. After the operation, the unknown (classical or quantum) momentum $p$ is converted into the new particle position $\tilde{q} = \tau p$. If such a transformation could be produced, it would no longer be necessary to detect directly the particle energy, e.g., by observing its collision with a heavier micro-object [54]. It would be enough to measure its new position $\tilde{q}$. Whatever the technical difficulties, there is no fundamental law which would forbid determining the new position $\tilde{q}$ in an arbitrarily short time$^1$.

What one can still see is that our particular prescriptions of generating (15) offer an inconvenient relation between the finally measured $\tilde{q}$ and the momentum $p$ prior to the applied operation. $\tilde{q} = \tau p$ implies $\Delta \tilde{q} = \tau \Delta p$, so if the operation time $\tau$ is very short, then little errors in measuring $\tilde{q}$ will correspond to much greater errors in $p$. It looks almost like the vengeance of the time–energy uncertainty. Yet, it is not! In fact, $\tau$ is a $c$-number parameter defining the time of an external operation, valid in classical as well as in quantum theory, an authentic external time in the sense of Aharonov and Bohm [55], and $\Delta \tilde{q}$ is not limited by any universal constant. Moreover, still accepting the pulse solutions, the dynamics of the time-dependent Hamiltonians (1) can offer much better measurement methods.

The ‘Fourier microscope’. Indeed, it is enough to apply three consecutive squeezed Fourier transformations to obtain a unitary operator with a more convenient transformation matrix:

$$F_\mu F_\gamma F_\mu \rightarrow \begin{pmatrix} 0 & \mu & 0 \\ -\frac{1}{\mu} & 0 & -\frac{1}{\mu} \\ 0 & -\frac{1}{\mu} & 0 \end{pmatrix} \begin{pmatrix} 0 & \gamma & 0 \\ -\frac{1}{\gamma} & 0 & -\frac{1}{\gamma} \\ 0 & -\frac{1}{\gamma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & b & 0 \\ -\frac{1}{b} & 0 & 0 \end{pmatrix},$$

(29)

where the squeezing coefficient $b = -\frac{\mu^2}{\gamma}$ is unlimited. Thus, taking $\mu = 1$, one would obtain a ‘squeezing mechanism’ where the final position $\tilde{q} = bp$ for large $b$ leads to insignificant errors in $\Delta p = \frac{1}{b} \Delta \tilde{q}$ even if the $\tilde{q}$ measurement is far from perfect. The problem of an efficient empirical design is still open. One of the simplest ways to measure the microparticle position is to let it interact with a lattice of mesoscopic absorption centers (e.g., grains of photographic emulsion), one of which turns dark, marking the new particle position $\tilde{q}$. As the particle is projected into a lattice, the $\Delta \tilde{q}$ errors will be of the order of magnitude of the lattice distance $\Delta l$ between the neighboring mesoscopic centers, but anyhow, the initial particle momentum $p$ can be determined with an arbitrarily small error $\Delta p = \gamma \Delta l$ if $\gamma$ is small enough. While technical difficulties still exist, they do not seem to be fundamental obstacle.

The operations considered above are singular and concern the states in one spatial dimension. Their generalization in two and three dimensions, employing $\delta$-kicks as well, was considered by Fernandez [58], including the comments on the time–energy uncertainty, though the results somehow escaped attention in the noise of the research markets. Yet, all the effects described above are just idealized forms of some natural phenomena which might be of technical interest. In particular, the merits of the evolution loops were noticed again in the control problem of finite-dimensional spaces (qubits) [59] and recognized more generally in [45]. The continuous evolution affected by sequences of sharp $\delta$-kicks, under the name of decouplers, is now studied for spin systems as a promising quantum control method [44, 59].

---

$^1$ What might awake some doubts is the fact that in such a measurement, the particle position $\tilde{q}$ would be used to detect its momentum $p$. Is this not in conflict with the position–momentum uncertainty? Yet, it is not, since $\tilde{q}$ and $p$ are the particle position and momentum in different time moments. In the orthodox quantum oscillator, the position measurement at some time moment $\tau$ is equivalent to the momentum measurement at $\tau - \frac{T}{2}$, where $T$ is the oscillator period (compare with the ‘non-demolishing measurement’ [31]).
(the so called ‘bam-bam-control’), though less violent methods are also considered [46, 60]. In the infinite-dimensional $L^2(\mathbb{R})$, the soft alternatives of potential kicks can be no less interesting.

7. The elementary algebraic solutions

As a matter of fact, the technique of applying nonsingular, bounded fields (e.g. in the form of rectangular steps) already allowed the design an ample family of dynamical operations including the squeezing, distorted free evolution, etc [34–36, 61]). The possibility of approaching the same effects by softly varying, differentiable fields (without any steps) was considered in [35, 62], then elaborated in computer studies for charged particles submerged in harmonic pulses [63–65]. As subsequently found, the control operations are significantly simplified if the field amplitude $\beta(t)$ is symmetric with respect to the center of the operation interval [60, 66, 67]. We shall show now that in the symmetric generation mechanisms, matrices (2)–(8), together with the corresponding ‘driving amplitude’ $\beta(t)$, can be expressed exactly (without perturbations!) in terms of a single real function which, apart from details, may be fixed at will. Without pretending to be a major mathematical discovery, it significantly facilitates the task of programming the dynamical operations. Indeed, one has the following.

**Proposition 3.** Consider a non-trivial operation interval $[-T, T]$ with the quadratic potential $\beta(t)$ and suppose $\beta(t)$ is bounded, piecewise continuous and symmetric $\beta(t) = \beta(-t)$. Whenever $u = u(t, -t)$, for $t \in [0, T]$ reaches the stability threshold with $\text{Tr} u = \pm 2$, then either $u$ or $-u$ adopts one of the forms (13) or (14), imitating the results of a simple or distorted free evolution, or else, of a sharp oscillator kick. Moreover, the evolution matrix $u(t, -t)$ and the field amplitude $\beta(t)$ in the expanding family of intervals $[-t, T]$ (where $t \leq T$) can be written explicitly in terms of $\theta(t) = u_{12}(t, -t)$, which may be chosen at will everywhere except its zero points. (In what follows, whenever there will be no reasonable doubt, we shall simplify the notation by just writing $u(t)$ instead of $u(t, -t)$ and $u(t)$ instead of $u(t, -t)$.)

**Proof.** Due to the symmetry of the Hamiltonians $H(t) = H(-t)$, the unitary evolution operators $U(t, -t)$ for the expanding intervals $[-t, T]$ satisfy

$$i\frac{d}{dt} U(t, -t) = H(t)U(t, -t) + U(t, -t)H(t).$$

Hence, the corresponding evolution matrix $u(t) = u(t, -t)$ is differentiable and fulfills

$$\frac{du}{dt} = \Lambda(t)u + u\Lambda(t).$$

Since $\Lambda(t)$ is given by (4), this becomes

$$\frac{du}{dt} = \begin{pmatrix} u_{21} - \beta u_{12} & \text{Tr} \ u \\ -\beta \text{Tr} \ u & u_{21} - \beta u_{12} \end{pmatrix} = (u_{21} - \beta u_{12})I + \text{Tr} u \begin{pmatrix} 0 & 1 \\ -\beta & 0 \end{pmatrix}.$$  

Therefore,

$$\frac{d}{dt}(u_{12}u_{21}) = \text{Tr} u(u_{21} - \beta u_{12}) = \text{Tr} u \frac{1}{2} \frac{d}{dt} \text{Tr} u = \frac{1}{4} \frac{d}{dt}(\text{Tr} u)^2$$

and integrating

$$\frac{d}{dt} \left[ u_{12}u_{21} - \frac{1}{4}(\text{Tr} u)^2 \right] = 0 \Rightarrow u_{12}u_{21} - \frac{1}{4}(\text{Tr} u)^2 = C = \text{const.}$$
To determine \( C \), it is enough to take \( t = 0 \). The initial condition \( u(0, 0) = 1 \) then tells us that \( C = -1 \), and so

\[
u_{12}u_{21} = \frac{1}{4}(\text{Tr}u)^2 - 1 \quad t \in [-T, T]. \tag{35}
\]

Hence, whenever the symmetric evolution matrix \( u(t) = u(t, -t) \) reaches the threshold values \( \text{Tr}u = \pm 2 \) (case II of our classification), (35) implies that either \( u_{12} \) or \( u_{21} \) (or both) must vanish, leading to the canonical transformations (32) which simulate the oscillator kicks, incidents of distorted free evolution, or just one of the evolution loops (cf [66, 67]), all of them with or without simultaneous parity transformation.

These facts are already a significant advantage of (32) which, in addition, provides an elementary solution of the inverse evolution problem, permitting us to reconstruct the entire evolution matrices \( u(t, -t) \) in terms of one arbitrary function \( \theta = u_{12}(t) \), which determines simultaneously the driving pulse \( \beta(t) \) in the expanding intervals \([-t, t]\).

Indeed, (32) implies \( \frac{\partial u_{11}}{\partial t} = \frac{\partial u_{22}}{\partial t} = u_{21} - \beta u_{12} \) and since the initial condition at \( t = 0 \) is \( u_{11}(0) = u_{22}(0) = 1 \), then \( u_{11} = u_{22} \) in all intervals \([-t, t] \) \((t \leq T)\). In view of (32), this means that \( u_{11}(t) = u_{22}(t) = \frac{1}{4}\text{Tr}u = \frac{1}{2}\beta(t) \). In turn, since \( u \) is simplectic, then \( \left(\frac{1}{2}\beta'\right)^2 - \theta u_{21} = 1 \) and the remaining matrix element \( \alpha = u_{21} \) is determined as

\[
\alpha = \left(\frac{1}{2}\theta'\right)^2 - 1 \tag{36}
\]

with the pulse shape \( \beta(t) \) defined in terms of \( \theta \) as well

\[
\beta = -\frac{\theta''}{2\theta} + \left(\frac{1}{2}\theta'\right)^2 - 1. \tag{37}
\]

These expressions grant that the matrix equation (32) is fulfilled, yielding an exact solution of the symmetric evolution problem for the family of the expanding intervals \([-t, t]\).

Unless stated otherwise, the subsequent remarks concern the evolution matrices in the hypothetical symmetry interval. **Observation.** Since \( \beta(t) \) in the reported algorithm is even, it is enough to postulate \( \theta \) and find \( \beta \) from (37) for \( t \geq 0 \), and then to reconstruct \( \beta \) for \( t < 0 \) by the parity argument \( \beta(-t) = \beta(t) \).

The demand to obtain a physically interpretable result with nonsingular, bounded \( \beta \)-pulse permits one to use an arbitrary twice differentiable \( \theta \) with bounded \( \theta'' \), limited only by certain auxiliary conditions. In particular, it might be of interest to look for the pulse amplitude \( \beta \) vanishing in some finite or infinite intervals on the \( t \)-axis. If not vanishing everywhere, \( \beta \) cannot be analytic on \( \mathbb{R} \); however, it can be switched on or off softly to remain continuous together with several derivatives. The simplest cases of vanishing \( \beta \) occur in sub-intervals, where \( \theta'(t) = \pm 2 \), (i.e., when the graphic of \( \theta \) on \( (\theta, t) \)-plane sticks to one of the straight lines \( \theta = \pm 2t + \text{const} \). However, this is not the only case.

**Proposition 4.** If \( \beta(t) = 0 \) in some interval \( I = [t_0, t_1] \subset \mathbb{R}_+ \), then \( \theta \) is quadratic in \( I \), i.e. \( \theta(t) = at^2 + bt + c \), with \( b^2 - 4ac = 4 \).

**Proof.** Note, that \( \theta \) can vanish only in isolated points, not in sub-intervals, and not together with \( \theta' \), otherwise the matrix \( u \) could not be simplectic. Hence, (37) implies

\[
\beta\theta^2 = -\frac{1}{2}\theta'\theta + \frac{1}{4}\theta'^2 - 1 \implies (\beta\theta^2)' = -\frac{1}{2}\theta''\theta
\]

and so

\[
\beta'\theta + 2\beta\theta' = -\frac{1}{2}\theta''. \tag{39}
\]
As a consequence, in any interval, where $\beta = 0$, there is $\theta'' = 0 \Rightarrow \theta(t) = \alpha t^2 + \beta t + c$. Moreover, by entering into (38), one sees that $b^2 - 4ac = 4$.

Differently than $\beta$, the function $\theta$ cannot vanish in non-trivial intervals, but its vanishing at some particular points might be of interest. Below, $t = 0$ is the symmetry center of $\beta$. Obviously, $u(0, 0) = 1 \Rightarrow \theta(0) = 0, \theta'(0) = 2$ and $\alpha(0) = u_{21}(0) = 0 \Rightarrow \theta''(0) = 0$. If, moreover, $\theta$ is three times differentiable at $t = 0$, then its third derivative $\theta'''$ defines the value of the driving amplitude $\beta(0) = -\frac{1}{8} \theta'''(0)$ at the symmetry point (all this following from (39), proposition 4).

Apart from $t = 0$, $\theta(t)$ can have other null points in which the consistency conditions are more relaxed, corresponding to distinct dynamical effects. In consequence, the types of operations defined by $\theta(t)$ in any symmetry interval $I = [-T, T]$ can be deduced just from very simple data including $\theta$ and its derivatives. In particular, the ‘distorted free evolution’, the ‘squeezed Fourier’ and the simulation of a sudden $\delta$-kick of the oscillator potential obey the propositions 5, 6 and 7.

**Proposition 5.** If $\theta(T) \neq 0$ and $\theta'(T) = \pm 2$, then the matrix $u = u(T, -T)$ adopts one of the characteristic shapes of the deformed free evolution, i.e.,

$$\theta'(T) = 2 \Rightarrow u = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}, \quad \theta'(T) = -2 \Rightarrow u = \begin{pmatrix} -1 & \tau \\ 0 & -1 \end{pmatrix}$$

with the ‘distorted time’ $\tau = \theta(T)$ or $\tau = -\theta(T)$, respectively. Moreover, if $\theta''(T) = 0$, then the driving amplitude $\beta(t)$ vanishes softly, $\beta(T) = \beta'(T) = 0$, in the limits of the operation interval.

**Proof.** Indeed, the diagonal terms of $u$ in both cases are $u_{11} = u_{22} = \text{Tr} u = \pm 1$. Hence, if $\theta(T) \neq 0$, the nominator of (36) vanishes and so $\alpha = u_{21} = 0$ implying (40). Note also that whenever the evolution yields the second matrix of (40), then its repetition $u^2$ in $[-T, 3T]$ will give $u^2 = \begin{pmatrix} 1 & -2 \tau \\ 0 & 1 \end{pmatrix}$; so if $\tau$ in (40) was positive, then $u^2$ generates the free evolution inversion with the effective time $-2\tau$.

**Proposition 6.** If $\theta(T) \neq 0$, $\theta'(T) = 0$ and $\theta''(T) = -2$, then the matrix $u$ produces the squeezed Fourier transformation (29) with $b = \theta(T)$. If in addition $\theta'''(T) = 0$, then the driving amplitude $\beta(t)$ vanishes softly (with the continuous first derivative) outside the operation interval.

**Proof.** Indeed, if $\theta'(T) = 0$, then the corresponding matrix $u$ has zeros on the diagonal, with $u_{12} = b = \theta(T)$. Moreover, if $\theta''(T) = -2 \Rightarrow \theta'(T) = -\frac{2}{3}$, then (38) implies that $\beta(T) = 0$, and if $\theta'''(T) = 0$, then also $\beta'(T) = 0$.

**Proposition 7.** When $\theta(T) = 0$ for $T \neq 0$, the nonsingularity of $\beta$ and $\alpha$ at $t = T$ requires $\theta'(T) = \pm 2$. Then, $\beta(T) = -\frac{1}{4} \text{sgn} [\theta'(T)] \theta'''(T)$ and $\alpha(T) = \frac{1}{4} \theta'''(T)$. In particular, if $\theta'''(T) = 0$ and $\theta''(T) = -2a \neq 0$, then $\beta(t)$ vanishes at the extremes $\pm T$ and $u(T)$ simulates the effect of the $\delta$-pulse $\beta(t) = ab(t)$.

**Proof.** Indeed, due to nonsingularity and vanishing of $\theta$ at $t = T$, equation (38) implies $\frac{1}{4} \theta^2 - 1 = 0 \Rightarrow \theta'(T) = \pm 2$. Hence, (39) reduces to $\pm 4 \beta(T) = -\frac{1}{4} \theta'''(T)$. $\alpha(T)$ in (36) is the nonsingular limit of a fraction in which both nominator and denominator tend to 0 for $t \to T$; equation (38) showing that $\alpha = \frac{1}{2} \theta''(t) + \beta(t)\theta(t) \to \frac{1}{2} \theta''(T) = \alpha(T) = -a$ for $t \to T$. 

\[\Box\]
Figure 5. The polynomial \( \theta_1(t) = 2t - \frac{41}{4}t^3 + \frac{93}{4}t^5 + \frac{71}{4}t^7 + \frac{19}{4}t^9 \) with \( \theta'_1(0) = 2 \), \( \theta'_1(1) = -2 \) yields parity-free evolution. If squared, it inverts the free evolution. In turn, \( \theta_2(t) = 2t - \frac{1}{32}(-105t^3 + 189t^5 - 135t^7 + 35t^9) \) with \( \theta'_2(0) = \theta'_2(1) = 2 \) and \( \tau = \theta_2(1) = 2.5 \) yields the free evolution acceleration.

The elementary models. While the best choice of \( \theta(t) \) from the point of view of the laboratory techniques is still an open problem, one can note the existence of very simple models showing how the dynamical operations (40) and (29) could be achieved in either a fast or slow way. The choice of \( \theta(t) = 2\sin(\omega t) \) of course, brings no novelty, leading to \( \beta(t) = \omega^2 = \text{const} \) of the traditional, time-independent oscillator (1). However, already the simple polynomial modification

\[
\theta(t) = 2t - \theta_3t^3 - \theta_5t^5 - \theta_7t^7 - \theta_9t^9
\]

is sufficient to design the examples of all operations of propositions 5–7 in either short or long intervals \([-T, T]\). Thus, to generate the incidents of ‘distorted free evolution’, one needs

\[
\theta(T) = b, \quad \theta'(T) = \pm 2, \quad \theta''(T) = 0, \quad \theta'''(T) = 0
\]

leading to a systems of four linear equations for the ‘new amplitudes’ \( \Theta_k = \theta_kT^k \), \( k = 3, 5, 7 \) and 9, where the matrix \( A \) is the same for all operation intervals. The cases \( \theta'(T) = \pm 2 \) of (42) yield \( \nu = 0 \) or \( \nu = 4 \) of the inverted or accelerated free evolution (see figure 5):

\[
A\Theta = A \begin{pmatrix} \Theta_3 \\ \Theta_5 \\ \Theta_7 \\ \Theta_9 \end{pmatrix} = \begin{pmatrix} -b + 2T \\ \nu T \\ 0 \\ 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 3 & 5 & 7 & 9 \\ 3 & 10 & 21 & 36 \\ 1 & 10 & 35 & 84 \end{pmatrix}.
\]

The operations of the ‘distorted time’ (40) could not occur for the constant oscillator potentials \( \beta_0 \), though they can be generated by \( \beta(t) \) in the form of finite steps (see the maps in [34–36]). They also occur in harmonic fields, though the results require a computer study [65–67]. Here, they can be given by the exact formula (43). In turn, the squeezed Fourier transformation (29) is generated by time-independent oscillator potentials, but to obtain a large \( b \), it needs a long time of waiting. Here, it requires only

\[
\theta(T) = b, \quad \theta'(T) = 0, \quad \theta''(T) = -\frac{2}{b}, \quad \theta'''(T) = 0.
\]
Figure 6. One of typical shapes of $\theta(t) = 2t - \frac{1}{4}t^3 - 2t^5 - \frac{7}{4}t^7 + \frac{1}{2}t^9$ permitting generation of the 'squeezed Fourier' in $[-1, 1]$ for $b = 2$ and $T = 1$ in (45).

Figure 7. The quantitative difference between two pulses $\beta_1$ and $\beta_2$ explains the qualitative difference between the inverted and accelerated free evolution. In turn, $\beta_3$ produces the squeezed Fourier operation represented in figure 6.

where $0 \neq b \in \mathbb{R}$. Within the polynomial model (41), it means

$$A\Theta = A \begin{pmatrix} \Theta_3 \\ \Theta_5 \\ \Theta_7 \\ \Theta_9 \end{pmatrix} = \begin{pmatrix} -b + 2T \\ 0 \\ 2T^2 \\ \frac{b}{0} \end{pmatrix}$$

leading to a family of $\theta(t)$ illustrated in figure 6.

The polynomials (41) are, of course, not unique models, but they permit us to construct easily the sharp and soft alternatives of the 'time machine', 'Fourier microscope' etc. Figure 7 compares the shapes of the corresponding driving amplitudes $\beta(t)$. 
Finally, the $\delta$-pulses $\beta(t) = a\delta(t)$, discussed in so many papers, admit now the soft polynomial models (41) with boundary conditions obeying proposition 7 (see figure 8), i.e,

$$A = \begin{pmatrix} 2T & 4T & 0 \\ 4T & aT^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} \theta_3 \\ \theta_5 \\ \theta_7 \end{pmatrix} = \begin{pmatrix} \frac{23}{8T^2} + \frac{3a}{4T} \\ \frac{3}{8T^2} + \frac{2a}{T^3} \\ -\frac{7}{8T^6} - \frac{7a}{4T^5} \\ -\frac{3}{8T^8} - \frac{a}{2T^7} \end{pmatrix}.$$

The above results offer some progress in describing a class of nonsingular dynamical operations on the traditional QM level. It looks as though the dynamical state transformations described above can occur under the influence of softly varying external fields, perhaps in standing wave equivalents of the laser beam traps (or in some variants of the ‘structured fields’ [53]). Below, we shall try to see whether they can be achieved for charged particles in traditional ion traps.

8. The quasistatic approximation

The description of quantum states in ion traps depends on multiple idealizations which may fail at various points. Thus, the commonly used oscillator potentials in the Penning and Paul traps neglect the granular structure of the trap walls (which are never exactly smooth!). Moreover, the traps are typically designed to maintain the charged particles to investigate their internal degrees rather than to manipulate the position and momentum states. In some cases (including the mass spectroscopy), the trap is just a system of parallel metal bars [68], so the charged particle can be efficiently trapped without obeying the Mathieu equations. Moreover, even for almost perfect hyperbolic walls, there are still multiple problems concerning quadratic Hamiltonians (1).

The most important ones are the limitations of non-relativistic approach. In reality, any potential pulse applied to the trap surface needs some time to propagate over the trap walls and to penetrate into its interior. Moreover, even if the delay is negligible, the too fast field changes can awake the radiative pollution, usually neglected in the trap descriptions.

The cases of dynamical transformations generated with some precision are not indeed many. The results for the spin (qubit) states seem promising [69–72], though usually limited to finite-dimensional Hilbert spaces. A notable success was achieved by the technique of
inducing Rabi rotations between pairs of bound states of hydrogen-like atoms (the ‘pink states’ [73]). Nonetheless, the paradoxes of quantum mechanics are still showing some intriguing possibilities [5]. Progress is also noted in ‘quantum tomography’ [6] (see, e.g., the ‘robust’ results in [9]). Our algorithms of section 7, while simplifying part of the problem, are still far from this level. Yet, they may suggest a new advance in the traditional domain.

Instead of just keeping the ions to monitor their internal structure, it could be as interesting to inject a single charged particle into the variable trap fields and manipulate its position–momentum state to check with some more details the traditional motion laws of quantum mechanics. The fields in the trap should be coherent, without inducing the sharp absorption/emission effects (i.e., slowly varying, long waves, formed by clouds of tiny quanta). The laser cooling [74] could help at the preliminary stage, but not during the proper evolution experiments, which should not be interrupted (‘decoupled’?) by sudden kicks [44, 46].

To avoid packet reflections from the walls, the trap should be wider than the traditional results in [9]). Our algorithms of section 7, while simplifying part of the problem, are still far from this level. Nonetheless, the paradoxes of quantum mechanics are still showing some intriguing possibilities [77]. To avoid this, it is easier to work in the Lorentz gauge, where the ‘currents’ [77].) To avoid this, it is easier to work in the Lorentz gauge, where the ‘currents’ [77]. To avoid packet reflections from the walls, the trap should be wider than the traditional results [75]. But how ample should it be? The answer depends on the relativistic corrections.

In the relativistic theories, a useful tool is the traditional Einstein–Infeld–Hoffmann (EIH) method [76] of developing the physical data into the increasing powers of \( \frac{1}{c} \) (Newtonian, post-Newtonian, etc). In our case, if the interval \([-T, T]\) in propositions 3–7 is too short, the functions \( \theta \) and \( \beta \) oscillate fast, so the EIH terms will show the insufficiency of quadratic formula (1). However, an inverse option of using soft and slowly changing fields in longer time intervals might reduce the difficulty.

Indeed, imagine that the trap surface \( \Sigma \) is a perfectly conducting (metallic) shell, composed of a number of disjoint leaves, \( \Sigma = \Sigma_1 \cup \ldots \cup \Sigma_n \), surrounding a certain operation domain \( \Omega \subset \mathbb{R}^3 \). Consider then a static charge distribution \( \rho \) on \( \Sigma \) creating a scalar potential \( \phi \), constant on every connected leaf \( \Sigma_j \subset \Sigma \) and harmonic in \( \Omega \), corresponding to a solution of the traditional electrostatic problem. Assume now that the charge density on \( \Sigma \) starts to evolve as \( \rho(x) \rightarrow \rho(x, t) = \beta(t)\rho(x) \), where \( \beta(t) \) is a certain continuous field amplitude with \( \beta(0) = 1 \). If such a homogeneous change of the charge density could be produced all over \( \Sigma \), then, in agreement with the linearity of the theory, the local change of the scalar potential \( \phi \) on \( \Sigma \) would be \( \phi(x') \rightarrow \phi(x', t) = \beta(t)\phi(x') \). However, what will be the response of the rest of the field in \( \Omega \)?

In the Coulomb gauge, it would just be an instantaneous effect of the changing surface density \( \rho(x, t) \) given by \( \int_\Sigma \frac{\rho(x, t) d\Sigma x'}{|x - x'|} = \beta(t) \rho(x) \). However, the Coulomb gauge is not convenient to describe the field propagation. (It requires the non-local ‘transversal currents’ [77].) To avoid this, it is easier to work in the Lorentz gauge, where the ‘absolute time’ \( t \) in the Coulomb integral must be replaced by the retarded time \( t_r = t - \frac{|x - x'|}{c} \) integrated over all sources [77, 78]. Then, one can observe that for \( \rho(x, t) = \beta(t)\rho(x) \), the \( \frac{1}{c} \)-contributions unexpectedly simplify, resembling the effect noticed by Griffiths [78]. The amplitude \( \beta \) does not even need to be analytic, if only \( \beta(t + h) = \beta(t) + h\beta'(t) + \frac{1}{2}h^2\beta''(t) + o(h^3) \), then \( \phi(x, t) \) inside of \( \Omega \) reduces to

\[
\phi(x, t) = \int_\Sigma \frac{\beta(t - \frac{|x - x'|}{c})\rho(x')}{|x - x'|} d\Sigma x' = \beta(t)\phi(x) - \beta'(t) \frac{Q}{c} + \frac{1}{2} \beta''(t) \frac{1}{c^2} \int_\Sigma |x - x'|\rho(x') d\Sigma x' + \ldots, \tag{47}
\]

where \( Q \) is the initial static charge on \( \Sigma \) and the last term represents the first non-trivial correction of the delay mechanism. In addition, if the operation takes place in a traditional ion trap, where \( \Sigma \) splits into several disjoint leaves with opposite charges, then the \( \frac{1}{c} \)-terms completely cancel and the last term of (47) is the only \( \phi \)-correction of the \( \frac{1}{c} \) EIH-order.
In Paul’s description, the time dependence was periodic and the dimensionless time coordinate was $\frac{\omega t}{2 \pi}$. If, however, the time dependence of $\beta(t)$ is arbitrary, then it is convenient to describe the changes of $\rho$ on $\Sigma$ in terms of a fixed time unit $\tau$ independent of any frequency. It can be done by replacing $t_{\tau} \rightarrow t_{\tau}/\tau$ leading to the correction

$$
\delta \phi = \frac{1}{2} \beta'' \left( \frac{t}{\tau} \right) \left( \frac{1}{\tau} \right)^2 \int_{\Sigma} \frac{|x - x'| \rho(x')}{|x - x'|} dx' + \ldots,
$$

which can be also written as

$$
\delta \phi = \frac{1}{2} \beta'' \left( \frac{t}{\tau} \right) \left( \frac{1}{\tau} \right)^2 \int_{\Sigma} \frac{\epsilon(x, \mathbf{x}') \rho(x')}{|x - x'|} dx',
$$

where the dimensionless amplitude in the integral of (48) is

$$
\epsilon(x, \mathbf{x}') = \frac{|x - x'|^2}{(\epsilon \tau)^2}.
$$

The real size of the trap is of course not infinite, but it seems that way to perform the operations of section 7; it can be much wider than $r_0 \approx 5 \text{ mm}$ as assumed in Paul’s report, particularly if the time unit $\tau$ is not too small. In fact, by assuming $\tau = 10^{-2} \text{ s}$ then $\tau_0 = 5 \text{ cm}$, and the distances $|x - x'|$ limited by $R = 30 \text{ cm}$ in the real trap laboratory, we would end up with a tolerant estimation

$$
\epsilon(x, \mathbf{x}') \leq \frac{R^2}{(\epsilon \tau)^2} \approx 10^{-14}.
$$

Apart from the scalar potential $\phi(x, t)$, what matters is the vector potential $\mathbf{A}(x, t)$ created by the external currents $J_{\text{ext}}(x, t)$ needed to feed the variable charge density $\rho(x, t)$. What we know about them is only that when arriving at the surface $\Sigma$, must they obey $J_{\text{ext}}(x, t) = \beta(t) J_{\text{ext}}(x)$ to assure the charge density accumulating as $\beta(t) \rho(x)$. The vector potential $\mathbf{A}(x, t)$ enters into the dynamical equations in two places: (1) by contributing to the expression for the electric field, $E = -\nabla \phi - \frac{\epsilon A}{\mu_0}$ and (2) by defining the magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$ (both depending on the geometry of the external currents, but contributing to the motion equations with $\frac{1}{\mu_0}$-EIH terms only).

In case of cylindric traps with variable potentials $\phi(x, t) = \beta(t) (\frac{x^2}{2} - \frac{z^2}{2})$, one might imagine a particle crossing the trap in the $z$-direction with the velocity $v_z = \frac{\beta}{\omega}$ defined well enough to feel the influence of $\beta(t)$ in some given operation interval, with the perpendicular momenta $p_x, p_y$ not too high, so that the partial states on the $x$-and $y$-plane would perform the evolution processes described by the quadratic Hamiltonians (1).2

The exact control of the initial and final operation moments and of the initial and final particle state are still absent, so our description is an intuitive rather than empirical design. What it implies, however, is that all previously described operations can indeed occur in the trap interiors, thus confirming the reality of the squeezed Fourier, the retarded, advanced or inverted free evolution incidents, as well as the simulation of the positive or negative pulses of the oscillator potential [14, 18, 34–36, 41, 58].

The unsolved problems, however, are twofold: (1) how to produce the pulse amplitudes $\beta(t)$ depending arbitrarily on time? and (2) how to assure that they will appear at the points of the trap surface simultaneously? Should one imagine a dense net of cables running from a common source of voltage to the net of points on the surface $\Sigma$? Should these technologies be achieved (or at least approximated), and can it be expected that the ion traps of the internal diameters 5 cm (or more) instead of 5 mm could become the efficient ‘wave packet laboratories’?

2 Comparatively, in order to understand why the pulsating, homogeneous electric fields permit us to predict so exactly the Rabi rotations, one has to remember that a typical hydrogen-like atom (e.g. rubidium) is of the size of 1 Å, while the red light wave has the longitude of 7000 Å; so the irradiated atom ‘does not see’ the space dependence of the light wave, it just reacts to the homogeneous, oscillating electric field, causing the observed Rabi effect.
9. Fundamental problems

Apart from the purely technical challenge, quantum state manipulations can give us a hint about the validity limits of our theories, including the controversies about fundamental problems such as the time–energy uncertainty, which might still deserve some comments.

_Does the 'time operator' exist?_ One of the reasons which could support the time–energy uncertainty principle was the idea that the _time_ and _energy_ are a pair of canonically conjugate spacetime observables ($x^0$, and $p_0$) whose quantum equivalents should therefore obey $[\textbf{t, E}] = i\hbar \Rightarrow \Delta t \Delta E \geq \frac{\hbar}{2}$. The idea seemed verbally plausible [79] and the collection of difficulties was not immediately noticed. Yet, it was never obvious why the probability of the ‘time of arrival’ should be normalized in time. (The particle may arrive at the detector many times or never.) The most dangerous paradox, though, was the _theorem of Pauli_. If the hypothetical time and energy operators $\textbf{t, E}$ are self-adjoint and fulfill $[\textbf{t, E}] = i\hbar$, then both must have continuous, translation invariant spectra. In particular, $\textbf{E}$ cannot have the lower bound against all known facts concerning the energy operators!

In an effort to avoid Pauli’s paradox, Kijowski [80] considered the one-dimensional simplified model replacing the free energy operator $\textbf{E} = \frac{p^2}{2m}$ by $\tilde{\textbf{E}} = \frac{p^2}{2m}$ with an infinite spectrum covering $\mathbb{R}$ and then constructed the canonically conjugated ‘time’ $\tilde{\textbf{t}}$. (An idea followed implicitly in a trend of papers [81, 82].) However, the physical interpretation became artificial. In order to avoid trouble, the formalism should be limited either to the wave packets localized to the left from the detector and moving to the right (the ‘right movers’) or vice versa. Yet, to restrict the problem to the ‘left component’ is not the same as to consider the ‘right movers’ (and vice versa). Moreover, the obtained probability distribution excluded the interference of both (left and right) components at the detection point against the basic principles of quantum theory (the difficulty discovered by Leavens [83–89]). In spite of subsequent works on POVM measures and some new but still partial results [90–92], it becomes obvious that the idea of the time operator has too many gaps to grant the time–energy uncertainty (except if some additional limitations of quantum measurements proved real [93]).

Meanwhile, the counter-arguments of Aharonov, Bohm and other authors [39, 55–57] grow stronger. In terms of our proposals, it is not even necessary to assure that the transformation $p \rightarrow \tilde{q}$ (squeezed Fourier) of section 7 can be performed arbitrarily fast. It is enough that it will be achieved with sufficient precision within a finite operation time (2T in our models). Indeed, according to the time–energy uncertainty, the energy measurement confined to a finite time interval $[-T, T]$ can never be arbitrarily good. If the new particle position $\tilde{q}$ will admit a measurement precise enough, then the measurement of the initial (free) energy $E = \frac{p^2}{2m} = \frac{1}{2} \frac{\tilde{p}^2}{\tilde{m}}$ can become too exact for the _time–energy_ uncertainty (following the suggestion in [57]). Although the practical solution is still missing, what matters are not the technical difficulties, but the absence of any universal barrier. So, is the barrier indeed absent?

_Is there a minimal distance?_ The techniques described in section 7 could also provide some insight into the quantum structure ‘in the little’, e.g. to check some ideas about the universal limits which could forbid too small distances or too great momenta [94, 95]. It includes, in particular, the hypothesis that the position measurements by the Heisenberg microscope cannot shrink the wave packets beyond some minimal size, related to the Planck distance [96]. However, one has to remember that the arbitrarily narrow wave packets are just the linear combinations of much wider ones (the Fourier transforms, wavelets, etc). So, if the evolution laws are indeed linear (the discussions seem not yet concluded; see [69] but also [97]) and if only the most insignificant squeezing of all position variables $q \rightarrow \kappa q, p \rightarrow \frac{1}{\kappa} p, |\kappa| < 1$ could be performed (and of course repeated), it would violate any limits of the wave packet’s compression and similarly abolish any upper bound of the packet momentum. Hence, if such
limits exist, they should be indirectly observed by failures of the squeezed Fourier or ordinary coordinate squeezing, even at mesoscopic levels.

Quite similar remarks concern the hypothesis of non-commutative geometries of the space coordinates (assumed sometimes to prevent the singularity creation below the Planck scale [95, 98]). The simplest case of $[x, y] = iv$, (with $0 \neq v \in \mathbb{R}$), if $x$ and $y$ were interpreted as the coordinates of some micro-object, could not survive even the most insignificant squeezing of both $x$ and $y$.

Some less destructive implications might still exist without challenging the spacetime structure. Indeed, if the squeezing/amplification can be generated for the continuous space variables, then it might help to observe some traditional quantum effects. In fact, the quantum mechanical particle–wave duality is difficult to observe for beams of heavy particles, with very narrow wave fronts running almost along the classical trajectories. If, however, the coordinate amplification can be engineered, expanding the wave fronts without affecting the coherence, then it could help to detect the heavy particle interference or else the duality limitations if they exist.

Apart from these particular problems, the techniques of the dynamical state transformations might still check some more general mysteries. In spite of their successes, it seems frustrating that all quantum theories were constructed by just multiplying ad infinitum the same linear state-observable structure, while leaving the basic paradoxes almost forgotten. Until now, the scheme never failed. But, how ample is indeed the orthodox truth?

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