Abstract. Can a supercompact cardinal $\kappa$ be Laver indestructible when there is a level-by-level agreement between strong compactness and supercompactness? In this article, we show that if there is a sufficiently large cardinal above $\kappa$, then no, it cannot. Conversely, if one weakens the requirement either by demanding less indestructibility, such as requiring only indestructibility by stratified posets, or less level-by-level agreement, such as requiring it only on measure one sets, then yes, it can.

Two important but apparently unrelated results occupy the large cardinal literature. On the one hand, Laver \cite{Lav78} famously proved that any supercompact cardinal $\kappa$ can be made indestructible by $<\kappa$-directed closed forcing. On the other hand, Apter and Shelah \cite{AS97} specifically, The College of Staten Island and The CUNY Graduate Center.

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\footnote{Specifically, The College of Staten Island and The CUNY Graduate Center.}
proved that all supercompact cardinals can be preserved to a forcing extension where there is a level-by-level agreement between strong compactness and supercompactness: specifically, except in special cases known to be impossible, any cardinal $\gamma$ there is $\eta$-strongly compact if and only if it is $\eta$-supercompact.\[Can these results be combined? Specifically, we ask:\[**Open Question 1** Can a supercompact cardinal be indestructible when there is a level-by-level agreement between strong compactness and supercompactness?\]

In this article, we provide a partial answer to this question, constraining the possibilities from both above and below. But alas, our results do not settle the matter, so the question remains open. What we can prove, specifically, is that if there is a sufficiently large cardinal above the supercompact cardinal, then the answer to the question is **no**. In particular, there is at most one supercompact cardinal as in the question; more exactly, if a cardinal is indestructibly supercompact in the presence of a level-by-level agreement between strong compactness and supercompactness, then no larger cardinal $\lambda$ is $2^\lambda$-supercompact. Conversely, if the requirements in the question are weakened in any of several ways, asking either for less indestructibility, replacing it with resurrectibility or with indestructibility by stratified forcing, or for a weaker form of level-by-level agreement, demanding that it hold only on measure one sets, then the answer is **yes**. These results are summarized in the Main Theorem stated below.

**Main Theorem**

1. **There can be at most one supercompact cardinal as in the question; indeed, if $\kappa$ is indestructibly supercompact and there is a level-by-level agreement between strong compactness and supercompactness, then no cardinal $\lambda$ above $\kappa$ is $2^\lambda$-supercompact.** This same conclusion can be made if we assume only that $\kappa$ is indestructibly strong, or indestructibly $\Sigma_2$-reflecting.

2. **Conversely, relaxing the notion of indestructibility somewhat, it is relatively consistent to have in the presence of a level-by-level agreement between strong compactness and supercompactness a supercompact cardinal $\kappa$ that is indestructible by any stratified $<\kappa$-directed closed forcing and more. It follows that the supercompactness of $\kappa$ is resurrectible after any $<\kappa$-directed closed forcing.**

3. **Alternatively, by relaxing the degree of level-by-level agreement required, it is relatively consistent to have a fully indestructible supercompact cardinal with a level-by-level agreement almost everywhere between strong compactness and supercompactness.**

The precise details of these three claims—including definitions, stronger statements of the results and corollaries—appear respectively in the three sections of this article.

We will define the most important notions here. We say that a supercompact cardinal $\kappa$ is **indestructible** when it remains supercompact after any $<\kappa$-directed closed forcing. A forcing\[See the definition in the paragraph immediately preceding Observation \[\]
Observation 2

If representations of the forcing extension by $Q$, the stratification of $V$ is itself stratified. A forcing notion is $\leq \kappa$-closed if any decreasing chain of length less than or equal to $\kappa$ has a lower bound. A forcing notion is $\leq \kappa$-distributive if forcing with it adds no new sequences over the ground model of length less than $\kappa$. If the forcing notion adds no new $\kappa$-sequences, then it is $\leq \kappa$-distributive. A forcing notion is $\leq \kappa$-strategically closed if in the game of length $\kappa + 1$ in which two players alternately select conditions from it to construct a descending $\kappa$-sequence, with the second player playing at limit stages, the second player has a strategy that allows her always to continue playing. The supercompactness of $\kappa$ is resurrectible after any $\leq \kappa$-directed closed forcing if, after any such forcing $Q$ there is further $\leq \kappa$-distributive forcing $R$ such that $Q \ast R$ preserves the supercompactness of $\kappa$. (We will occasionally abuse notation and write $x$ when we should more properly write $\dot{x}$.) We say that there is a level-by-level agreement between strong compactness and supercompactness when for any two regular cardinals $\gamma \leq \eta$, the cardinal $\gamma$ is $\eta$-strongly compact if and only if it is $\eta$-supercompact, unless $\gamma$ is a measurable limit of cardinals which are $\eta$-strongly compact. We say that $\gamma$ is partially supercompact if and only if $\gamma$ is at least $\gamma^+$-supercompact. Please note that this terminology is somewhat strict, with mere measurability being insufficient.

The lottery sum $\oplus A$ of a collection $A$ of partial orderings, defined in $\text{Ham}_0(\mathbb{A})$, is the set $\{\langle Q, q \rangle : Q \in A$ and $q \in Q\} \cup \{\mathbb{1}\}$, ordered with $\mathbb{1}$ above everything and $\langle Q, q \rangle \leq \langle Q', q' \rangle$ if and only if $Q = Q'$ and $q \leq q'$ in $Q$. Intuitively, a generic object for $\oplus A$ selects a “winning” poset from $A$ and then forces with it. A forcing notion $Q$ is stratified when for any regular cardinal $\eta$ in the extension the forcing $Q$ factors in the ground model as $Q_0 \ast Q_1$, in the sense of having isomorphic complete Boolean algebras, where $|Q_0| \leq \eta$ and $\Vdash_{Q_0} Q_1$ is $\leq \eta$-distributive. It follows, as we observe below, that $Q_0$ is also stratified. A non-overlapping iteration is a forcing iteration $P$ where the forcing $P_\gamma$ at any stage $\gamma$ in $P$ is $\leq |P_\beta|$-strategically closed for any $\beta < \gamma$. A forcing notion $Q$ admits a gap at $\delta$ if it can be factored as $Q_0 \ast Q_1$, where $|Q_0| < \delta$ and $\Vdash_{Q_0} Q_1$ is $\leq \delta$-strategically closed. We denote by $\text{Cof}_\kappa$ the class of ordinals of cofinality $\kappa$, by $\text{add}(\theta_1, \theta_2)$ the canonical forcing that adds $\theta_2$ many Cohen subsets to $\theta_1$ with conditions of size less than $\theta_1$ and by $\text{coll}(\theta_1, \theta_2)$ the canonical forcing to collapse $\theta_2$ to $\theta_1$.

Observation 2 If $Q_0 \ast Q_1$ witnesses the stratification of a stratified poset $Q$ at $\eta$, then $Q_0$ is itself stratified.

Proof: Since $|Q_0| \leq \eta$, it suffices to stratify $Q_0$ only at regular $\zeta < \eta$. Let $R_0 \ast R_1$ be the stratification of $Q$ at $\zeta$, and suppose $V[G_0 \ast G_1] = V[H_0 \ast H_1]$ exhibits the equivalent representations of the forcing extension by $Q_0 \ast Q_1$ or $R_0 \ast R_1$, respectively. Since $H_0 \in V[G_0][G_1]$ and the $G_1$ forcing is $\leq \eta$-distributive, it follows that $H_0 \in V[G_0]$ and so $V[G_0] = V[H_0][G_0/H_0]$ for some (quotient) forcing generic $G_0/H_0 \subseteq Q_0/H_0$. That is, $Q_0$ factors as $R_0 \ast (Q_0/H_0)$. And since $V[H_0] \subseteq V[H_0][G_0/H_0] \subseteq V[H_0][H_1]$ and $H_1$ adds no $\zeta$-sequences over $V[H_0]$, it follows that $G_0/H_0$ also adds no $\zeta$-sequences over $V[H_0]$. So the quotient forcing is $\leq \zeta$-distributive and we have witnessed the stratification of $Q_0$ at $\zeta$. \(\square\)

The careful reader will observe that the definition we gave above for the level-by-level agreement between strong compactness and supercompactness presents two distinct depa-
tures from a full general agreement between \( \eta \)-strong compactness and \( \eta \)-supercompactness for every \( \eta \). These departures omit the cases from such a level of agreement that are known to be generally impossible. The first departure is that we only demand agreement between \( \eta \)-strong compactness and \( \eta \)-supercompactness when \( \eta \) is regular. The reason for doing so is that Magidor has proved (see [AS97, Lemma 7]) that if \( \kappa \) is supercompact and \( \eta \) is the least strong limit cardinal above \( \kappa \) of cofinality \( \kappa \), then there is an \( \eta \)-supercompactness embedding \( j : V \to M \) such that \( \kappa \) is \( \eta \)-strongly compact but not \( \eta \)-supercompact in \( M \); consequently, such counterexamples exist unboundedly often below \( \kappa \) as well. The argument also works for singular \( \eta \) of arbitrary cofinality above \( \kappa \), the basic point being that if \( \gamma \) is \( <\eta \)-strongly compact and \( \eta \) is singular with cofinality at least \( \gamma \), then \( \gamma \) is \( \eta \)-strongly compact (but needn’t be \( \eta \)-supercompact and the least such \( \gamma \) cannot be \( \eta \)-supercompact). For singular \( \eta \) of cofinality less than \( \gamma \) the question is moot because any cardinal \( \gamma \) is \( \eta \)-strongly compact or \( \eta \)-supercompact if and only if it is \( \eta^{<\gamma} \)-strongly compact and \( \eta^{<\gamma} \)-supercompact respectively, rising to the case of \( \eta^{<\gamma} \), which for such \( \eta \) is at least \( \eta^{\gamma} \). And so we restrict our attention to regular degrees of compactness.

The second restriction is to ignore the case when \( \gamma \) is a measurable limit of cardinals that are \( \eta \)-strongly compact. We do this because Menas [Men74] has shown that such cardinals are necessarily \( \eta \)-strongly compact, but not necessarily \( \eta \)-supercompact; indeed, if \( \eta \geq 2^\gamma \), then the least such \( \gamma \) cannot be \( \eta \)-supercompact. Historically, this is how Menas first showed that the notions of a strongly compact cardinal and a supercompact cardinal are not identical: the least measurable limit of strongly compact cardinals will be strongly compact but not supercompact. Later, this was improved by Magidor [Mag76] to show that in fact the least measurable cardinal can be strongly compact.

In summary, the two restrictions in the definition of level-by-level agreement between strong compactness and supercompactness omit exactly the cases where such an agreement is known to be impossible. The point and main contribution of [AS97] is that in all other cases, agreement is possible. In truth, however, the second restriction on level-by-level agreement will not arise in this article, and can be safely ignored, because in all the models in which we obtain level-by-level agreement here, there will be no measurable limits of cardinals with the same degree of strong compactness.

Finally, we would like to point out that a certain amount of level-by-level agreement comes for free, namely, a cardinal \( \gamma \) is \( \gamma \)-strongly compact if and only if it is \( \gamma \)-supercompact, since these both are equivalent to measurability. Therefore, since also \( \eta \)-supercompactness directly implies \( \eta \)-strong compactness for any cardinal, in order to prove the level-by-level agreement between strong compactness and supercompactness, it suffices to show that any cardinal \( \gamma \) that is \( \eta \)-strongly compact for a regular cardinal \( \eta > \gamma \) is also \( \eta \)-supercompact.

1 A Surprising Incompatibility

We begin by proving that if there is a sufficiently large cardinal above the supercompact cardinal, then the answer to Question 1 is no. This result therefore identifies a surprising incompatibility, a tension between indestructibility and the level-by-level agreement of strong
compactness and supercompactness in the presence of too many large cardinals.

**Incompatibility Theorem 3** If \( \kappa \) is an indestructible supercompact cardinal and there is a level-by-level agreement between strong compactness and supercompactness below \( \kappa \), then no cardinal \( \lambda \) above \( \kappa \) is \( 2^\lambda \)-supercompact.

Theorem 3 is due to the first author and was established by him in September 1999 during a trip to Japan. To prove this theorem, we will show that by \(<\kappa\)-directed closed forcing we can force \( \lambda \) to violate the level-by-level agreement between strong compactness and supercompactness. Since the supercompactness of \( \kappa \) will be preserved, it follows by a simple reflection argument that there must be unboundedly many violations of the level-by-level agreement below \( \kappa \); this contradicts the fact that the level-by-level agreement below \( \kappa \) is preserved to the forcing extension.

So let us begin with the following:

**Lemma 3.1** If a cardinal \( \lambda \) is \( 2^\lambda \)-supercompact, then there is a forcing extension \( V^\mathbb{P} \) in which \( \lambda \) is \( \lambda^+ \)-strongly compact but not \( \lambda^+ \)-supercompact. In \( V^\mathbb{P} \), one can arrange that \( 2^\lambda = \lambda^+ \) and \( \lambda \) has, as a measurable cardinal, trivial Mitchell rank. Consequently, \( \lambda \) will not be even \((\lambda + 2)\)-strong in \( V^\mathbb{P} \). What’s more, for any \( \delta < \lambda \), the forcing \( \mathbb{P} \) can be chosen to be \( \leq \delta \)-directed closed and with a gap below \( \lambda \).

**Proof:** Standard arguments establish that if \( \lambda \) is \( 2^\lambda \)-supercompact, then this can be preserved to a forcing extension in which \( 2^\lambda = \lambda^+ \) (one simply forces \( 2^\gamma = \gamma^+ \) with \( \text{add}(\gamma^+, 1) \)) at sufficiently many stages \( \gamma \leq \lambda \) in a reverse Easton iteration). This iteration admits a gap between any two nontrivial stages of forcing, and by starting the iteration beyond any particular \( \delta < \lambda \), we may ensure that it is \( \leq \delta \)-closed. Afterwards, we may directly force \( 2^{\lambda^+} = \lambda^{++} \) by adding a Cohen subset to \( \lambda^{++} \); since this adds no subsets to \( P_{\lambda^+} \), it therefore preserves the \( \lambda^+ \)-supercompactness of \( \lambda \). So let us assume without loss of generality that we have already performed this forcing, if necessary, and that \( 2^\lambda = \lambda^+ \) and \( 2^{\lambda^+} = \lambda^{++} \) in \( V \).

Let \( \mathbb{P} \) be the reverse Easton \( \lambda \)-iteration which at stage \( \gamma \) forces with \( Q_\gamma = \text{add}(\gamma, 1) \), provided that \( \gamma \) is above \( \delta \) and measurable in \( V \). This forcing is \( \leq \delta \)-closed and admits a gap between any two nontrivial stages of forcing. Suppose that \( G \subseteq \mathbb{P} \) is \( V \)-generic.

We claim that \( \lambda \) has trivial Mitchell rank in \( V[G] \). If not, then there would be an embedding \( j : V[G] \rightarrow M[j(G)] \) with critical point \( \lambda \) for which \( M[j(G)] \) is closed under \( \lambda \)-sequences in \( V[G] \) and \( \lambda \) is measurable in \( M[j(G)] \). By the Gap Forcing Theorem of [Ham](#), applied in \( M[j(G)] \), it follows that \( \lambda \) is measurable in \( M \) and consequently a stage of nontrivial forcing in \( j(\mathbb{P}) \). Factoring \( j(\mathbb{P}) \) as \( \mathbb{P} \ast \text{add}(\lambda, 1) \ast \mathbb{P}_{\text{tail}} \), it follows that \( j(G) \) must be \( G \ast A \ast G_{\text{tail}} \) for some \( M[G] \)-generic Cohen subset \( A \subseteq \lambda \). Since every subset of \( \lambda \) in \( V \) is in \( M \) and the forcing \( \mathbb{P} \) has size \( \lambda \), it follows that every subset of \( \lambda \) in \( V[G] \) is in \( M[G] \). In particular, every dense subset of \( \text{add}(\lambda, 1)^{V[G]} = \text{add}(\lambda, 1)^{M[G]} \) from \( V[G] \) is in \( M[G] \) and so \( A \) is actually \( V[G] \)-generic as well. Since this contradicts the fact that \( A \in V[G] \), there can be no such embedding \( j \) and so \( \lambda \) has trivial Mitchell rank in \( V[G] \). It follows that \( \lambda \) is neither \((\lambda + 2)\)-strong nor \( \lambda^+ \)-supercompact in \( V[G] \).
We claim nevertheless that $\lambda$ remains $\lambda^+$-strongly compact in $V[G]$. For this, we use a technique of Magidor, unpublished by him but expounded in [AC01, AC, Aptc, Aptd, Aptd], and [Aptd]. Let $j_0 : V \to M$ be a $\lambda^+$-supercompactness embedding generated by a normal fine measure on $P_\lambda \lambda^+$ and $h : M \to N$ an ultrapower embedding by a measure on $\lambda$ of minimal Mitchell rank in $M$, so that $\lambda$ is not measurable in $N$. Let $j = h \circ j_0$ be the combined embedding; it witnesses the $\lambda^+$-strong compactness of $\lambda$. We will lift this embedding to $j : V[G] \to N[j(G)]$ so as to witness the $\lambda^+$-strong compactness of $\lambda$ in $V[G]$.

Consider the forcing $j(P)$, factored as $P \ast P_{\lambda,h(\lambda)} \ast \text{add}(h(\lambda), 1) \ast P_{h(\lambda), j(\lambda)}$, and the forcing $j_0(P)$, factored as $P \ast \text{add}(\lambda, 1) \ast P_{\text{tail}}$. With this notation, for example, $h(P \ast \text{add}(\lambda, 1)) = P \ast P_{\lambda,h(\lambda)} \ast \text{add}(h(\lambda), 1)$.

Now let $P_{\text{term}}$ be the term forcing poset for $P_{\text{tail}}$ over $P \ast \text{add}(\lambda, 1)$ (see [For83] for the first published account of term forcing, or [Cum92, 1.2.5, p. 8]; the notion is originally due to Richard Laver). That is, $P_{\text{term}}$ consists of (sufficiently many) $P \ast \text{add}(\lambda, 1)$-names for elements of $P_{\text{tail}}$, ordered by $\tau \leq \sigma$ if and only if $1 \Vdash \tau \leq \sigma$. As in the proof of [Aptd, Lemma 3.2], a full collection of names, meaning that any name forced by $1$ to be in $P_{\text{tail}}$ is forced by $1$ to be equal to one of them, can be found of size $j(\lambda)$ in $M$, which has cardinality $\lambda^+$ in $V$. Further, since $P_{\text{tail}}$ is forced to be $\leq \lambda^+$-directed closed, it is easy to see that $P_{\text{term}}$ is $\leq \lambda^+$-directed closed in $M$, and hence also $\leq \lambda^+$-directed closed in $V$. Since $M$ has only $j(\lambda^+)$ many dense sets for $P_{\text{term}}$, and this has cardinality $\lambda^{++}$ in $V$, we may by the usual diagonalization techniques (see, e.g. [Ham94]) construct an $M$-generic filter $G_{\text{term}} \subseteq P_{\text{term}}$ in $V$. And since $h$ is the ultrapower by a measure on $\lambda$ and $P_{\text{term}}$ is $\leq \lambda$-closed, it follows that $h \upharpoonright G_{\text{term}}$ is $N$-generic for the term forcing for $P_{h(\lambda), j(\lambda)}$ over $P_{h(\lambda), 1}$ (see [For83] or [Cum92, 1.2.2, Fact 2]). Thus, we may lift the embedding $h$ to $h : M[G_{\text{term}}] \to N[h(G_{\text{term}})]$. And since $G \subseteq P$ is $V$-generic, it is also $N[h(G_{\text{term}})]$-$N$-generic, and so we may form the extension $N[h(G_{\text{term}})][G]$.

Since $\lambda$ is not measurable in $N$, the stage $\lambda$ forcing in $j(P)$ is trivial, and so $P_{\lambda,h(\lambda)}$ is $\leq \lambda$-closed in $N[G]$. Since the $h(P_{\text{term}})$ forcing is highly closed in $N$, it cannot affect closure down at $\lambda$, so the forcing $P_{\lambda,h(\lambda)}$ is $\leq \lambda$-closed in $N[h(G_{\text{term}})][G]$. Going one step more, the poset $P_{\lambda,h(\lambda)} \ast \text{add}(h(\lambda), 1)$ is $\leq \lambda$-closed in $N[h(G_{\text{term}})][G]$, has size $h(\lambda)$ there, which has size $\lambda^+$ in $V$, and $N[h(G_{\text{term}})][G]$ is closed under $\lambda$-sequences in $V[G]$. Thus, by the usual diagonalization techniques, we may construct in $V[G]$ (actually, we can do it in $M[G]$) an $N[h(G_{\text{term}})][G]$-$N$-generic $G_{\lambda,h(\lambda), 1}$ for this much of $j(P)$.

By the fundamental property of term forcing (see [For83] or [Cum92, 1.2.5, Fact 1]), in $N[h(G_{\text{term}})][G_{\lambda,h(\lambda), 1}]$ we may construct an $N[G_{\lambda,h(\lambda), 1}]$-$N$-generic filter $G_{h(\lambda), j(\lambda)}$ from $h(G_{\text{term}})$. Putting these filters together, let $j(G) = G_{h(\lambda), 1} * G_{h(\lambda), j(\lambda)}$ and lift the embedding to $j : V[G] \to N[j(G)]$ in $V[G]$. The attentive reader will observe that we used the term forcing only to help construct $G_{h(\lambda), j(\lambda)}$. Now that we have done so, we may discard $G_{\text{term}}$ and $h(G_{\text{term}})$.

To see that this lifted embedding witnesses the $\lambda^+$-strong compactness of $\lambda$ in $V[G]$, let $s = h(j_0 \upharpoonright \lambda^+)$. One can now easily check that $s \in N$, $s \subseteq j(\lambda^+)$, $|s| = h(\lambda^+) < j(\lambda)$ and $j \upharpoonright \lambda^+ \subseteq s$. Thus, $s$ induces a fine measure on $P_\lambda \lambda^+$ in $V[G]$, as desired.\[\square\]

\[\text{[In fact, one can show more: the lifted embedding } j \text{ itself is the ultrapower by a fine measure on } P_\lambda \lambda^+, \text{ rather than merely inducing such a measure; one need only modify } s \text{ by adding an ordinal on top from which} \]
It remains to check one last detail before the proof of Theorem 3 is complete: we need to know that λ is not one of the exceptional cases excluded from our definition of level-by-level agreement. Let us assume without loss of generality that we worked with the least λ above κ that was $2^\lambda$-supercompact. In this case, because of the level-by-level agreement, λ cannot be in $V$ a measurable limit of cardinals that are $2^\lambda$-strongly compact. By the Gap Forcing Theorem [Ham], the forcing iteration does not increase the degree of strong compactness of any cardinal, so in the extension λ is $\lambda^+$-strongly compact, but not $\lambda^+$-supercompact and not a measurable limit of cardinals that are $\lambda^+$-strongly compact. Thus, it is truly a violation of the level-by-level agreement between strong compactness and supercompactness. By reflecting this violation below κ, we contradict our assumption that there was a level-by-level agreement there, and the proof of Theorem 3 is complete.

The initial proof of Lemma 3.1 used the original form of Magidor’s method, which iteratively adds stationary non-reflecting sets; later, we saw that an appeal to the Gap Forcing Theorem allowed us to simplify this to just iterated Cohen forcing.

Before concluding this section, we would like to call attention to the fact that our proof of Theorem 3 does not fully use the hypothesis that κ is an indestructible supercompact cardinal. First, one can easily check that the proof uses only that the $2^\lambda$-supercompactness of κ is indestructible. But in fact, the reflection argument that we used to bring the violation of the level-by-level agreement at λ down below κ does not actually rely on the supercompactness of κ at all, but only on its strongness. Therefore, we have actually proved the following theorem.

**Theorem 4** If κ is an indestructible strong cardinal and there is a level-by-level agreement between strong compactness and supercompactness below κ, then no cardinal λ above κ is $2^\lambda$-supercompact. Indeed, κ need only be indestructibly $(\lambda + 2)$-strong.

What’s more, since the iteration up to λ can be arranged to be $\leq \kappa$-closed, we only need the strongness of κ to be indestructible by $\leq \kappa$-closed forcing (or even less: indestructible by $\leq \kappa^+$-closed forcing, etc.). This is interesting because [GS89] shows that any strong cardinal κ can be made indestructible by $\leq \kappa$-closed forcing. So our argument shows that this amount of indestructibility for a strong cardinal is incompatible with a level-by-level agreement between strong compactness and supercompactness if there are large enough cardinals above.

But actually, we don’t even need κ to be strong for the reflection to work. Since the violation of the level-by-level agreement at λ is witnessed in $V_{\lambda+2}$, a rank initial segment of $V$, it is enough if $V_\kappa \prec \kappa$, that is, if κ is $\Sigma_2$-reflecting. Therefore, we have proved:

**Theorem 5** If κ is an indestructible $\Sigma_2$-reflecting cardinal and there is a level-by-level agreement between strong compactness and supercompactness below κ, then no cardinal λ above κ is $2^\lambda$-supercompact.

we can definably recover the ordinal seed for the measure used in the ultrapower embedding $h$. The resulting seed $s^*$ will still induce a fine measure $\mu$ on $P_\lambda \lambda^+$, but since the hull of $s^*$ with ran($j$) is all of $M[j(G)]$, the ultrapower by $\mu$ will be precisely $j$. We refer readers to [Ham97] for an account of this seed technique and terminology.
Indeed, it is enough if the relation $V_\kappa \triangleleft_2 V_\theta$ is indestructible by $<\kappa$-directed closed forcing in $V_\theta$ for some $\theta > \lambda + 1$ (or even indestructible just by $\leq\kappa$-directed closed forcing, etc.).

2 Level-by-level agreement with near indestructibility

In this section we provide an affirmative answer to a weakened form of Question 1, showing that a supercompact cardinal can be nearly indestructible in the presence of a level-by-level agreement between strong compactness and supercompactness. Specifically, we will obtain the level-by-level agreement in a model with a supercompact cardinal $\kappa$ that is indestructible by all stratified $<\kappa$-directed closed forcing and more. Recall our definition that a poset is stratified when for every regular cardinal $\eta$ in the extension it factors in the ground model as $Q_0 \ast Q_1$, in the sense of having isomorphic complete Boolean algebras, where $|Q_0| \leq \eta$ and $\Vdash_{Q_0} Q_1$ is $\leq \eta$-distributive. It follows, as we proved in Observation 2, that $Q_0$ is also stratified. We say that $Q$ is stratified above $\kappa$ if such a factorization exists for $\eta$ above $\kappa$.

Numerous examples of stratified forcing, including Cohen forcing and collapsing posets as well as iterations of these and many others, are indicated in Corollary 7.

The proof will proceed by an iteration that we call the lottery preparation preserving level-by-level agreement, and we regard the resulting model as currently the most natural candidate for an affirmative answer to Question 1, if any exists. In particular, while we have been able to prove so far only that the supercompactness of $\kappa$ is indestructible there by any stratified $<\kappa$-directed closed forcing and a few others, we know of no obstacle to it being fully indestructible there by all $<\kappa$-directed closed forcing.

**Theorem 6** Suppose that $\kappa$ is supercompact and no cardinal is supercompact up to a cardinal $\lambda$ which is itself $2^\lambda$-supercompact. Then there is a forcing extension satisfying a level-by-level agreement between strong compactness and supercompactness in which $\kappa$ remains supercompact and becomes indestructible by any stratified $<\kappa$-directed closed forcing and more. Indeed, the supercompactness of $\kappa$ becomes indestructible by any $<\kappa$-directed closed forcing that is stratified above $\kappa$.

**Proof:** We may assume without loss of generality, by forcing if necessary, that the GCH holds and further, by forcing with the notion in [AS97] if necessary, that in $V$ there is already a level-by-level agreement between strong compactness and supercompactness. Since these forcing notions admit a very low gap, by the Gap Forcing Theorem [Ham] they do not increase the degree of supercompactness or (since the forcing notions are also mild in the sense of [Ham]) strong compactness of any cardinal. It follows that no cardinal in $V$ is supercompact up to a partially supercompact cardinal. Let $P$ be the reverse Easton support $\kappa$-iteration which begins by adding a Cohen real and then has nontrivial forcing only at later stages $\gamma$ that are inaccessible limits of partially supercompact cardinals. At such a stage $\gamma$ in $P$, assuming $V^P_\gamma$ has a level-by-level agreement between strong compactness and supercompactness, we force with the lottery sum of all $<\gamma$-directed closed posets $Q$, of size less
than the next partially supercompact cardinal, that preserve this level-by-level agreement. (Please note that this will always include the trivial poset.) The iteration $\mathbb{P}$ is an example of a modified lottery preparation of the type used in [Ham98a], [Ham00], and [Aptc]. Supposing $G \subseteq \mathbb{P}$ is $V$-generic, we will refer to the iteration $\mathbb{P}$ and the resulting model $V[G]$ as the lottery preparation preserving level-by-level agreement.

Lemma 6.1 If $\gamma$ is $\eta$-supercompact in $V$ for a regular cardinal $\eta > \gamma$, then this remains true in $V[G_\gamma]$. In particular, after the lottery preparation for preserving level-by-level agreement $V[G]$, the cardinal $\kappa$ remains fully supercompact.

Proof: (This same observation was essentially made in [Ham98a, Lemma 2.1], the modified lottery preparations resembling as they do the partial Laver preparations.) We may assume that $\gamma$ is a limit of partially supercompact cardinals, since otherwise the forcing $\mathbb{P}_\gamma$ is equivalent to forcing that is small with respect to $\gamma$ and the result is immediate by [LS67]. Let $j : V \to M$ be an $\eta$-supercompactness embedding with critical point $\gamma$ such that $\gamma$ is not $\eta$-supercompact in $M$. Since $\gamma$ is $<\eta$-supercompact in $M$ and no cardinal is supercompact up to a partially supercompact cardinal, it follows that the next partially supercompact cardinal above $\gamma$ in $M$ is at least $\eta$; further, $\eta$ itself is not even measurable in $M$ because if it were, then the $<\eta$-supercompactness of $\gamma$ would imply that $\gamma$ is $\eta$-supercompact in $M$, contrary to our assumption. In short, the next partially supercompact cardinal in $M$ above $\gamma$ is strictly above $\eta$. Thus, below a condition opting for trivial forcing at stage $\gamma$ in $j(\mathbb{P}_\gamma)$, we may factor $j(\mathbb{P}_\gamma)$ as $\mathbb{P}_\gamma \ast \mathbb{P}_{tail}$, where $\mathbb{P}_{tail}$ is $\leq\eta$-closed. Thus, by the usual diagonalization techniques, we may construct in $V[G_\gamma]$ an $M[G_\gamma]$-generic filter $G_{tail} \subseteq \mathbb{P}_{tail}$ and lift the embedding to $j : V[G_\gamma] \to M[j(G_\gamma)]$ with $j(G_\gamma) = G_\gamma \ast G_{tail}$. This embedding witnesses that $\gamma$ is $\eta$-supercompact in $V[G_\gamma]$, as desired. $\square$

Lemma 6.2 For any ordinal $\gamma$, there is a level-by-level agreement between strong compactness and supercompactness in $V[G_\gamma]$. In particular, the lottery preparation preserving level-by-level agreement $V[G]$ really does preserve the level-by-level agreement between strong compactness and supercompactness.

Proof: Suppose inductively that the result holds below $\gamma$ and consider $V[G_\gamma]$. Since successor stages of forcing always preserve the level-by-level agreement if it exists, we may assume that $\gamma$ is a limit of stages of forcing, and hence a limit of partially supercompact cardinals. For any $\delta < \gamma$, our induction hypothesis guarantees a level-by-level agreement for $\delta$ in $V[G_{\delta+1}]$, and by the Gap Forcing Theorem [Ham], $\delta$ is neither strongly compact nor supercompact up to a partially supercompact cardinal in that model. Therefore, since the later non-trivial stages of forcing are closed beyond the next inaccessible limit of partially supercompact cardinals, which by our assumptions and the level-by-level agreement is beyond supercompact up to a partially supercompact cardinal.

\footnote{And whenever we use this terminology, we implicitly assume that the ground model $V$ has a level-by-level agreement between strong compactness and supercompactness, as well as the GCH, and that no cardinal in $V$ is supercompact up to a partially supercompact cardinal.}
the degree of strong compactness or supercompactness of $\delta$, it follows that there is a level-by-level agreement for $\delta$ in $V[G_\gamma]$. By [LS67], cardinals above $\gamma$ are not affected by small forcing or anything equivalent to small forcing, and so we need only consider the cardinal $\gamma$ itself. Suppose accordingly that $\gamma$ is $\eta$-strongly compact in $V[G_\gamma]$ for a regular cardinal $\eta > \gamma$; we will show it is also $\gamma$ itself. Suppose accordingly that the degree of strong compactness or supercompactness of $\gamma$ is stratified at regular cardinals $\eta$, we may construct in $V[G_\gamma]$, as desired.

**Lemma 6.3** If $\gamma$ is an inaccessible limit of partially supercompact cardinals, then any stratified $\lt \gamma$-directed closed forcing $Q$ over $V[G_\gamma]$ preserves the level-by-level agreement between strong compactness and supercompactness for $\gamma$ and smaller cardinals. Indeed, $Q$ need only be stratified at regular cardinals $\eta$ above $\gamma$.

**Proof:** Suppose that the result holds for cardinals below $\gamma$ (with full Boolean value) and that $Q$ is $\lt \gamma$-directed closed in $V[G_\gamma]$ and stratified for regular $\eta$ above $\gamma$. From the closure of $Q$ and the fact that no cardinal is supercompact beyond a partially supercompact cardinal in $V$ and hence also (by the Gap Forcing Theorem [Ham]) in $V[G_\gamma]$, one sees that it must preserve the level-by-level agreement between strong compactness and supercompactness for all cardinals below $\gamma$. So we consider the cardinal $\gamma$ itself.

Suppose that $\gamma$ is $\eta$-strongly compact in $V[G_\gamma]^Q$ for some regular cardinal $\eta > \gamma$. We will show that $\gamma$ is $\eta$-supercompact there as well. By the Gap Forcing Theorem [Ham], we know that $\gamma$ is $\eta$-strongly compact in $V$, and hence by the level-by-level agreement, it is $\eta$-supercompact there as well. Fix an $\eta$-supercompactness embedding $j : V \rightarrow M$ with $\gamma$ not $\eta$-supercompact in $M$. As in Lemma 6.1 it follows that the next partially supercompact cardinal of $M$ above $\gamma$ is above $\eta$. Since $Q$ is stratified above $\gamma$, we may factor $Q$ as $Q_0 \ast Q_1$, where $|Q_0| \leq \eta$ and $\Vdash_{Q_0} Q_1$ is $\leq \eta$-distributive. By cardinality considerations, forcing with $Q_1$ adds no new subsets of $P_\gamma \eta$ over $V[G_\gamma]^Q_{\leq \eta}$, and so it suffices for us to show that $\gamma$ is $\eta$-supercompact in $V[G_\gamma][g]$, where $g \subseteq Q_0$ is $V[G_\gamma]$-generic. The cardinal $\gamma$ is an inaccessible limit of partially supercompact cardinals in $M$, and so there is a lottery at stage $\gamma$ in $j(P_\gamma)$. Furthermore, since the induction hypothesis holds up to $j(\gamma)$ in $M$ and $Q_0$ is $\lt \gamma$-directed closed, has size at most $\eta$ in $M$ and, by Observation 2, is stratified above $\gamma$ (and this can be seen in $M[G_\gamma]$), it follows that $Q_0$ is allowed to appear in the stage $\gamma$ lottery of $j(P_\gamma)$. Thus, below a condition opting for $Q_0$ in this lottery, we may factor the forcing $j(P_\gamma)$ as $P_\gamma * Q_0 * P_{\text{tail}}$, where $P_{\text{tail}}$ is $\leq \eta$-closed. Now the usual diagonalization arguments apply and we may construct in $V[G_\gamma][g]$ an $M[G_\gamma][g]$-generic filter $G_{\text{tail}} \subseteq P_{\text{tail}}$ and lift the embedding to $j : V[G_\gamma] \rightarrow M[j(G_\gamma)]$ where $j(G_\gamma) = G_\gamma \ast g * G_{\text{tail}}$. Since $j \upharpoonright g \subseteq j(Q_0)$ is a directed subset of cardinality less than $j(\gamma)$ in $M[j(G_\gamma)]$, we may find a (master) condition $p$ below it. Working now below $p$ we diagonalize to construct an $M[j(G_\gamma)]$-generic object $j(g) \subseteq j(Q_0)$ and lift the embedding fully to $j : V[G_\gamma][g] \rightarrow M[j(G_\gamma)][j(g)]$, thereby witnessing the $\eta$-supercompactness of $\gamma$ in $V[G_\gamma][g]$, as desired.

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Lemma 6.4 After the lottery preparation preserving level-by-level agreement $V[G]$, the supercompactness of $\kappa$ becomes indestructible by any $<\kappa$-directed closed forcing that is stratified above $\kappa$.

Proof: Suppose that $Q$ is stratified above $\kappa$ and $<\kappa$-directed closed in $V[G]$, and assume $g \subseteq Q$ is $V[G]$-generic. Select any regular $\eta \geq |Q|$ and an $\eta$-supercompactness embedding $j : V \rightarrow M$ for which $\kappa$ is not $\eta$-supercompact in $M$. It follows as in Lemma 6.3 that $Q$ is allowed in the stage $\kappa$ lottery and the diagonalization arguments given in Lemma 6.3 allow us to lift the embedding to $j : V[G][g] \rightarrow M[j(G)][j(g)]$. This witnesses the $\eta$-supercompactness of $\kappa$ in $V[G][g]$, as desired. □

This completes the proof of the theorem. □

Corollary 7 After the lottery preparation preserving level-by-level agreement, the supercompactness of $\kappa$ becomes indestructible by:

1. the Cohen forcing $\text{add}(\kappa, 1)$,
2. the Cohen forcing $\text{add}(\theta, 1)$ for any regular $\theta$ above $\kappa$,
3. the Cohen forcing $\text{add}(\kappa, \kappa^+)$,
4. indeed, any $<\kappa$-directed closed forcing of size at most $\kappa^+$,
5. the collapse forcing $\text{coll}(\kappa, \theta)$ for any $\theta$ above $\kappa$,
6. the collapse forcing $\text{coll}(\theta_1, \theta_2)$ whenever $\kappa \leq \theta_1 \leq \theta_2$ and $\theta_1$ is regular,
7. the Lévy collapse forcing $\text{coll}(\theta_1, <\theta_2)$ whenever $\kappa \leq \theta_1 \leq \theta_2$ and $\theta_1$ is regular,
8. the forcing to add a stationary non-reflecting subset of $\theta \cap \text{Cof}_\kappa$ for any regular $\theta > \kappa$,
9. and any non-overlapping reverse Easton support iteration of these forcing notions.

Proof: These are all $<\kappa$-directed closed and either completely stratified or stratified above $\kappa$. Indeed, since $\text{add}(\theta, 1) = \text{coll}(\theta, \theta)$, we can see that cases 1, 2 and 5 are special cases of $\text{coll}(\theta_1, \theta_2)$, in case 6. And it is easy to see that this and $\text{coll}(\theta_1, <\theta_2)$, for $\kappa \leq \theta_1 \leq \theta_2$ and $\theta_1$ regular, are stratified: any regular cardinal $\eta$ in the extension must be either at most $\theta_1$ or at least $\theta_2$, and one can trivially factor the forcing. (Note that we use the $\text{gch}$ in $V$ and the fact $|\mathcal{P}| = \kappa$ in order to know that $|\text{coll}(\theta_1, \theta_2)| = \theta_2^{\theta_1} \leq \eta$ if $\eta$ is regular and at least $\theta_2$.) The cases of $\text{add}(\kappa, \kappa^+)$ or any $<\kappa$-directed closed $Q$ of size at most $\kappa^+$ are clearly stratified above $\kappa$. The conditions of the forcing to add a stationary non-reflecting subset to $\theta \cap \text{Cof}_\kappa$ are simply the bounded subsets $s \subseteq \theta \cap \text{Cof}_\kappa$ which are not stationary in their supremum nor have any initial segment stationary in its supremum, ordered by end-extension. This forcing is $<\kappa$-directed closed by simply taking unions of conditions. To see that it is stratified, we
note that for any $\eta < \theta$, the forcing is $\leq \eta$-strategically closed and hence $\leq \eta$-distributive, and for any regular $\eta \geq \theta$, by GCH in $V$ and the fact $|\mathbb{P}| = \kappa$, the forcing has size less than or equal to $\eta$. So in each case the factorization is trivial. Finally, it is easy to see that a non-overlapping iteration of these posets remains stratified above $\kappa$. □

We would like to call attention to a sense in which the hypothesis in Theorem 6, namely, that no cardinal is supercompact up to a larger cardinal $\lambda$ that is itself $2^\lambda$-supercompact, is optimal in light of Theorem 3 and Corollary 7. Specifically, Theorem 3 establishes that if one has a level-by-level agreement and a supercompact cardinal $\kappa$ that is indestructible by iterated Cohen forcing, then no larger cardinal $\lambda$ is $2^\lambda$-supercompact. Conversely, if there is no such $\lambda$ above $\kappa$, then Corollary 7 shows that this amount of indestructibility is possible. In this sense, the conclusion of Theorem 3 tightly matches the hypothesis of Theorem 6.

The next theorem, however, shows that in another sense Theorem 6 and Corollary 7 are not optimal; there is more indestructibility than we have claimed so far. Specifically, while each of the posets mentioned in Corollary 7 preserves the GCH, and indeed, a straightforward factor argument shows that any stratified forcing whatsoever preserves the GCH, our preparatory forcing in fact makes the supercompactness of $\kappa$ indestructible by $\text{add}(\kappa, \kappa^+)$, which of course does not preserve the GCH.

**Corollary 8** After the lottery preparation preserving level-by-level agreement, the supercompactness of $\kappa$ becomes indestructible by $\text{add}(\kappa, \kappa^+)$.\n
**Proof:** We reiterate our implicit assumption for this preparation that the ground model $V$ satisfies the GCH and a level-by-level agreement between strong compactness and supercompactness, and that no cardinal in $V$ is supercompact up to a partially supercompact cardinal there.

We begin by showing that for any cardinal $\gamma \leq \kappa$ the forcing $\text{add}(\gamma, \gamma^+)$ over $V[G_{\gamma}]$ preserves the level-by-level agreement between strong compactness and supercompactness. Suppose inductively that this holds (with full Boolean value) for all cardinals below $\gamma$. Since $\text{add}(\gamma, \gamma^+)$ adds no small sets, it does not affect the level-by-level agreement between strong compactness and supercompactness for cardinals below $\gamma$, and since the forcing is itself small with respect to larger cardinals, it preserves such agreement above $\gamma$ by [LS67]. So it remains only to check the agreement right at $\gamma$. Accordingly, suppose that $\gamma$ is $\lambda$-strongly compact for some regular cardinal $\lambda > \gamma$ in $V[G_{\gamma}][g]$, where $g \subseteq \text{add}(\gamma, \gamma^+)$ is $V[G_{\gamma}]$-generic; we aim to show that $\gamma$ is also $\lambda$-supercompact there.

The Gap Forcing Theorem [Ham] implies that $\gamma$ is $\lambda$-strongly compact in $V$ and hence also $\lambda$-supercompact, by the level-by-level agreement there, witnessed by an embedding $j : V \rightarrow M$. We may assume that $\gamma$ is not $\lambda$-supercompact in $M$. We will lift this embedding to $V[G_{\gamma}][g]$, thereby witnessing the $\lambda$-supercompactness of $\gamma$ there.

We note first that $\gamma$ must be a limit of partially supercompact cardinals, since otherwise the forcing $\mathbb{P}_\gamma * \text{add}(\gamma, \gamma^+)$ is equivalent to small forcing followed by $<\gamma$-closed forcing that adds a subset to $\gamma$; but by [Ham98], all such forcing destroys the measurability of $\gamma$, contrary to our assumption that $\gamma$ is $\lambda$-strongly compact in $V[G_{\gamma}][g]$. And since these
smaller cardinals are fixed by \( j \), it follows now that \( \gamma \) is an inaccessible limit of partially supercompact cardinals in \( M \), and hence a nontrivial stage of forcing in \( j(\mathbb{P}_\gamma) \).

By elementarity the induction hypothesis holds up to \( j(\gamma) \) in \( M \), and so the forcing \( \text{add}(\gamma, \gamma^{++}) \), which is the same in \( V[G_\gamma] \) and \( M[G_\gamma] \) since \( \gamma^+ \leq \lambda \), preserves the level-by-level agreement between strong compactness and supercompactness over \( M[G_\gamma] \). It is therefore allowed to appear in the stage \( \gamma \) lottery of \( j(\mathbb{P}_\gamma) \). Below a condition opting for this poset in that lottery, therefore, the forcing \( j(\mathbb{P}_\gamma) \) factors as \( \mathbb{P}_\gamma \ast \text{add}(\gamma, \gamma^{++}) \ast \mathbb{P}_{\text{tail}} \), where \( \mathbb{P}_{\text{tail}} \) is the part of the forcing at stages beyond \( \gamma \). Since the next partially supercompact cardinal in \( M \) must be beyond \( \lambda \), it follows that \( \mathbb{P}_{\text{tail}} \) is \( \lambda \)-closed in \( M[G_\gamma][g] \). Further, \( M[G_\gamma][g] \) is closed under \( \lambda \)-sequences in \( V[G_\gamma][g] \) and the number of dense subsets of \( \mathbb{P}_{\text{tail}} \) in \( M[G_\gamma][g] \) is at most \( |j(\gamma)|^V \leq (\gamma^+)^M = \lambda^+ \). Therefore, we may simply lift up these dense sets in \( V[G_\gamma][g] \), diagonalize to construct an \( M[G_\gamma][g]-\)generic filter \( G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}} \) and lift the embedding to \( j : V[G_\gamma] \rightarrow M[j(G_\gamma)] \) with \( j(G_\gamma) = G_\gamma \ast g \ast G_{\text{tail}} \). It remains to lift the embedding through the forcing \( \text{add}(\gamma, \gamma^{++}) \).

If \( \gamma^{++} \leq \lambda \), and this is the easy case, the usual master condition argument allows us to lift the embedding. Specifically, since \( g \in M[j(G_\gamma)] \) one uses \( j^* \gamma^{++} \) to see that \( j^* g \) is also in \( M[j(G_\gamma)] \), and since this is a directed subset of \( j(\text{add}(\gamma, \gamma^{++})) \) of size less than \( j(\gamma) \), there is a condition \( p \), called the master condition, below it. Diagonalizing below this condition, one builds an \( M[j(G_\gamma)]\)-generic filter \( g^* \subseteq j(\text{add}(\gamma, \gamma^{++})) \) and lifts the embedding fully to \( j : V[G_\gamma][g] \rightarrow M[j(G_\gamma)][j(g)] \), with \( j(g) = g^* \), as desired.

For the only remaining case, the hard case, assume \( \lambda = \gamma^+ \). In this case \( j^* g \) will not be in \( M[j(G_\gamma)] \), and there will be no master condition. Nevertheless, with care we will still be able to construct a generic filter extending \( j^* g \). (This technique appears in [AS97], p. 119-120 and [Apt99], p. 555-556, and is similar to a technique involving strong cardinals in [Ham94], p. 277–278.) We will construct an \( M[j(G_\gamma)]\)-generic filter \( g^* \subseteq j(\text{add}(\gamma, \gamma^{++})) \) in \( V[G_\gamma][g] \) such that \( j^* g \subseteq g^* \), and then lift the embedding fully to \( j : V[G_\gamma][g] \rightarrow M[j(G_\gamma)][j(g)] \), with \( j(g) = g^* \), thereby witnessing the \( \lambda \)-supercompactness of \( \gamma \) in \( V[G_\gamma][g] \).

As a first step towards this, suppose that \( A \subseteq j(\text{add}(\gamma, \gamma^{++})) \) is a maximal antichain in \( M[j(G_\gamma)] \) and \( r \in j(\text{add}(\gamma, \gamma^{++})) \) is a condition that is compatible with every element of \( j^* g \). We will find a condition \( r^+ \leq r \) that decides \( A \) while remaining compatible with \( j^* g \). Since \( \text{add}(\gamma, \gamma^{++}) \) is \( \gamma^+ \)-c.c., it follows that the antichain \( A \) has size at most \( j(\gamma) \) in \( M[j(G_\gamma)] \). Since also the usual supercompactness arguments show that \( j^* \gamma^{++} \) is unbounded in \( j(\gamma^{++}) \), it follows that \( A \subseteq j(\text{add}(\gamma, \alpha)) \) for sufficiently large \( \alpha < \gamma^{++} \). Fix such an \( \alpha \) such that also \( r \in j(\text{add}(\gamma, \alpha)) \) and let \( q = \cup \{ j^* (g \cap \text{add}(\gamma, \alpha)) \} \). Since \( q \) has size at most \( \gamma^+ \), it is in \( M[j(G_\gamma)] \) and is a (master) condition for \( j(\text{add}(\gamma, \alpha)) \), which is a complete subposet of \( j(\text{add}(\gamma, \gamma^{++})) \). Since \( r \) is compatible with every element of \( j^* g \), we know that \( r \) and \( q \) are compatible. Choose \( r^+ \in j(\text{add}(\gamma, \alpha)) \) below \( r \) and \( q \) and deciding \( A \). We claim that \( r^+ \) remains compatible with \( j^* g \). We see this, consider any condition \( j(p) \) for \( p \in g \). Since \( p \) is a partial function from \( \gamma \times \gamma^{++} \) into \( 2 \), we may split it into two pieces \( p = p_0 \cup p_1 \) where \( \text{dom}(p_0) \subseteq \gamma \times \alpha \) and \( \text{dom}(p_1) \subseteq \gamma \times [\alpha, \gamma^{++}) \). It follows that \( j(p) = j(p_0) \cup j(p_1) \), with the key point being that the domain of \( j(p_1) \) is disjoint from the domain of any element of \( j(\text{add}(\gamma, \alpha)) \). In particular, \( r^+ \) is compatible with \( j(p_1) \). Since also \( r^+ \leq q \leq j(p_0) \), it follows
that $r^+$ is compatible with $j(p)$, as we claimed.

Now we iterate this idea to construct the $M[j(G_\gamma)]$-generic filter $g^* \subseteq j(\text{add}(\gamma, \gamma^{++}))$. Since the forcing $j(\text{add}(\gamma, \gamma^{++}))$ has size $j(\gamma^{++})$ and is $j(\gamma^{++})$-c.c., it has $j(\gamma^{++})$ many maximal antichains in $M[j(G_\gamma)]$. Since $|j(\gamma^{++})|^{\gamma^+} = \gamma^{++}$, we may enumerate these antichains in a sequence $\langle A_\beta \mid \beta < \gamma^{++} \rangle$ in $V[G_\gamma][g]$. We now define a descending sequence $\langle r_\beta \mid \beta < \gamma^{++} \rangle$ of conditions in $j(\text{add}(\gamma, \gamma^{++}))$, each of which is compatible with every element of $j^* g$. At successor stages, if $r_\beta$ is defined, we employ the argument of the previous paragraph to select a condition $r_{\beta+1} \leq r_\beta$ that decides the antichain $A_\beta$ and remains compatible with $j^* g$. At limit stages $\eta$, let $r_\eta = \bigcup \{ r_\beta \mid \beta < \eta \}$. Because $M[j(G_\gamma)]$ is closed under $\gamma^+$ sequences in $V[G_\gamma][g]$, we know that $r_\eta \in M[j(G_\gamma)]$, and it clearly remains compatible with every element of $j^* g$.

Let $g^*$ be the filter generated by the conditions $\{ r_\beta \mid \beta < \gamma^{++} \}$. Since these conditions decide every maximal antichain, $g^*$ is $M[j(G_\gamma)]$-generic. And since $j^* g \subseteq g^*$, we may lift the embedding to $j : V[G_\gamma][g] \to M[j(G_\gamma)][j(g)]$, where $j(g) = g^*$, thereby witnessing the $\gamma^+$-supercompactness of $\gamma$ in $V[G_\gamma][g]$, as desired.

The careful reader will observe that we have actually proved that if $\gamma$, a limit of partially supercompact cardinals, is $\lambda$-supercompact in $V$ for some regular cardinal $\lambda$ above $\gamma$, then this is preserved by forcing over $V[G_\gamma]$ with $\text{add}(\gamma, \gamma^{++})$. In particular, the case $\gamma^* = \kappa$ shows that the supercompactness of $\kappa$ in $V[G]$ is indestructible by $\text{add}(\kappa, \kappa^{++})$, just as the theorem states, and so the proof is complete. □

**Corollary 9** After the lottery preparation preserving level-by-level agreement, the supercompactness of $\kappa$ becomes indestructible by $\text{add}(\theta, \theta^+)$ for any regular cardinal $\theta \geq \kappa$.

**Proof:** We use a similar argument for this Corollary. First we claim for any $\gamma \leq \kappa$ that forcing with $\text{add}(\theta, \theta^+)$ over $V[G_\gamma]$, where $\theta \geq \gamma$ is a regular cardinal below the next partially supercompact cardinal above $\gamma$, preserves the level-by-level agreement between strong compactness and supercompactness. Suppose inductively that this holds below $\gamma$ and that $g \subseteq \text{add}(\theta, \theta^+)$ is $V[G_\gamma]$-generic. It is easy to see that the level-by-level agreement below or above $\gamma$ is not affected, so suppose that $\gamma$ is $\lambda$-strongly compact for some regular cardinal $\lambda > \gamma$ in $V[G_\gamma][g]$; we will show that $\gamma$ is $\lambda$-supercompact there as well. By the Gap Forcing Theorem [Ham] we know that $\gamma$ is $\lambda$-strongly compact and hence $\lambda$-supercompact in $V$.

If $\lambda < \theta$, then since $\text{add}(\theta, \theta^+)$ adds no new subsets to $P_\gamma \lambda$, the $\lambda$-supercompactness of $\gamma$ in $V[G_\gamma]$ is trivially preserved to $V[G_\gamma][g]$. So we may assume $\theta \leq \lambda$. In this case it follows that $\gamma$ is a limit of partially supercompact cardinals, since otherwise the forcing $P_\gamma * \text{add}(\theta, \theta^+)$ would be equivalent to small forcing followed by $<\gamma^*$-closed forcing adding a subset to $\theta$, which by [HS98] would destroy the $\theta$-strong compactness of $\gamma$, contrary to our assumption that $\gamma$ is $\lambda$-strongly compact in $V[G_\gamma][g]$. So, let $j : V \to M$ be a $\lambda$-supercompactness embedding such that $\gamma$ is not $\lambda$-supercompact in $M$. Since the induction hypothesis holds up to $j(\gamma)$ in $M$, the forcing $\text{add}(\theta, \theta^+)$ appears in the stage $\gamma$ lottery of $j(P_\gamma)$, and below a condition opting for this poset in that lottery we may factor the iteration as $P_\gamma * \text{add}(\theta, \theta^+) * P_{\text{tail}}$, where $P_{\text{tail}}$ is $\leq \lambda$-closed in $M[G_\gamma][g]$. And the diagonalization argument of Corollary 8 shows
how to lift this embedding to \( j : V[G_\gamma] \to M[j(G_\gamma)] \), where \( j(G_\gamma) = G_\gamma * g * G_{\text{tail}} \) for some generic filter \( G_{\text{tail}} \subseteq P_{\text{tail}} \) constructed in \( V[G_\gamma][g] \). It remains to lift the embedding through the forcing \( \text{add}(\theta, \theta^+) \).

If \( \theta^+ \leq \lambda \), this can be done with the usual master condition argument, and we omit the details. The remaining case, the hard case of \( \theta = \lambda \), proceeds as in the hard case of Corollary \( \text{Corollary 10} \) specifically, one first shows as before that if \( r \in j(\text{add}(\theta, \theta^+)) \) is compatible with \( j^* g \) and \( A \subseteq j(\text{add}(\theta, \theta^+)) \) is a maximal antichain in \( M[j(G_\gamma)] \), then there is a stronger condition \( r^+ \leq r \) deciding \( A \) and still compatible with \( j^* g \). For this, one uses the fact that \( j^* \theta^+ \) is unbounded in \( j(\theta^+) \) and consequently \( A \) is contained in \( j(\text{add}(\theta, \alpha)) \) for sufficiently large \( \alpha < \theta^+ \). By counting antichains and iterating this argument, we once again construct a descending sequence of conditions that eventually meet every maximal antichain of \( j(\text{add}(\theta, \theta^+)) \) in \( M[j(G_\gamma)] \). The filter \( g^* \) generated by these conditions is therefore \( M[j(G_\gamma)] \)-generic and extends \( j^* g \), so we may lift the embedding to \( j : V[G_\gamma][g] \to M[j(G_\gamma)][j(g)] \), where \( j(g) = g^* \), thereby witnessing the \( \lambda \)-supercompactness of \( \gamma \) in \( V[G_\gamma][g] \), as desired.

Finally, we observe that we have actually proved that if \( \gamma \) is a limit of partially supercompact cardinals and is \( \lambda \)-supercompact in \( V \), then forcing with \( \text{add}(\theta, \theta^+) \) over \( V[G_\gamma] \) for any \( \theta \leq \lambda \) preserves the \( \lambda \)-supercompactness of \( \gamma \). In particular, the case \( \gamma = \kappa \) shows that the full supercompactness of \( \kappa \) is indestructible by \( \text{add}(\theta, \theta^+) \) for any regular \( \theta \geq \kappa \).

We regret that we have not been able to generalize these results to \( \text{add}(\kappa, \kappa^{++}) \) or \( \text{add}(\theta, \theta^{++}) \) for \( \theta > \kappa \). Nevertheless, because the results of this section show that after the lottery preparation preserving level-by-level agreement the supercompact cardinal \( \kappa \) becomes indestructible by a great variety of forcing notions, and we know of no specific \( \kappa \)-directed closed forcing notion which does not preserve the supercompactness of \( \kappa \) over this model, we regard it as currently the most natural candidate for a positive answer to Question \( \text{Question 10} \), if any exists. The key questions then become:

**Question 10** After the lottery preparation preserving level-by-level agreement, is the supercompact cardinal \( \kappa \) fully indestructible? If not, for which posets does it become indestructible?

To conclude this section, we will show that the lottery preparation preserving level-by-level agreement makes the supercompact cardinal resurrectible after any \( \kappa \)-directed closed forcing. Recall that a supercompact cardinal \( \kappa \) is said to be resurrectible if after any \( \kappa \)-directed closed forcing \( Q \) there is further \( \kappa \)-distributive forcing \( R \) such that \( Q * R \) preserves the supercompactness of \( \kappa \). Thus, even if \( Q \) destroys the supercompactness of \( \kappa \), it is recovered by further forcing with \( R \).

**Corollary 11** After the lottery preparation preserving level-by-level agreement, the supercompactness of \( \kappa \) is resurrectible after any \( \kappa \)-directed closed forcing.

**Proof:** The point here is that indestructibility by \( \text{coll}(\kappa, \theta) \) implies resurrectibility by any \( \kappa \)-directed closed forcing \( Q \) of size \( \theta \), assuming \( \theta^{\kappa} = \theta \). This is true because such a poset \( Q \) completely embeds into the collapse poset (this can be seen by observing that \( Q \times \text{coll}(\kappa, \theta) \)...
is $\kappa$-closed, collapses $\theta$ to $\kappa$ and has size $\theta$, and noting that there is only one forcing notion with these features). Thus, we may view $\text{coll}(\kappa, \theta)$ as $Q \ast R$, where $R$ is $\text{coll}(\kappa, \theta)$, and so even if $Q$ happens to destroy the supercompactness of $\kappa$, further forcing with $R$ amounts altogether to forcing with $Q \ast R = \text{coll}(\kappa, \theta)$, which preserves the supercompactness of $\kappa$.\qed

The previous argument actually provides a stronger kind of resurrectibility than we claimed. Namely, define that $\kappa$ is $\theta$-resurrectible if for any $\kappa$-closed $Q$ there is $\theta$-distributive forcing $R$ such that $Q \ast R$ preserves the supercompactness of $\kappa$. Thus, for example, $\kappa$ is resurrectible if and only if it is $\kappa$-resurrectible. We now define that $\kappa$ is strongly resurrectible if it is $\theta$-resurrectible for every $\theta \geq \kappa$. Since any $\kappa$-closed forcing $Q$ embeds completely into $\text{coll}(\theta, |\mathbb{Q}|)$, and the supercompactness of $\kappa$ is indestructible by such forcing, we have actually proved:

**Corollary 12** After the lottery preparation preserving level-by-level agreement, the supercompactness of $\kappa$ becomes strongly resurrectible.

If indeed all one desires is resurrectibility, then actually there is no need as in Theorem 6 for restricting the possibility of large cardinals above the supercompact cardinal in question. Specifically, we claim the following:

**Theorem 13** If $\kappa$ is the least supercompact cardinal—whether or not there are large cardinals above $\kappa$—there is a forcing extension, preserving all supercompact cardinals, with a level-by-level agreement between strong compactness and supercompactness, in which the supercompactness of $\kappa$ becomes strongly resurrectible.

**Proof:** We may assume, by forcing with the poset of $[\mathbb{AS}97]$ if necessary, that there is a level-by-level agreement between strong compactness and supercompactness in $V$ and that the GCH holds there (note that the $[\mathbb{AS}97]$ forcing preserves all supercompact cardinals and by the Gap Forcing Theorem $\mathbb{Ham}$ creates no new ones). Because $\kappa$ is supercompact, it is a limit of strong cardinals (see $[\mathbb{AC}$, Lemma 2.1 and the subsequent remark]). Furthermore, since $\kappa$ is the least supercompact cardinal in $V$, no cardinal below $\kappa$ is supercompact up to a strong cardinal cardinal (lest it be fully supercompact). Let $P$ be the reverse Easton support $\kappa$-iteration with nontrivial forcing only at stages $\gamma < \kappa$ that are inaccessible limits of strong cardinals. At such a stage $\gamma$, the forcing is the lottery sum of all $\text{coll}(\theta_1, \theta_2)$, where $\gamma \leq \theta_1 \leq \theta_2$, each $\theta_i$ is regular, and $\theta_2$ is less than the next strong cardinal above $\gamma$ (plus trivial forcing). Suppose that $G \subseteq P$ is $V$-generic.

The usual lifting arguments (see e.g. $\mathbb{Ham00}$) establish that $\kappa$ remains supercompact in $V[G]$ and furthermore that the supercompactness of $\kappa$ becomes indestructible there by further forcing with $\text{coll}(\theta_1, \theta_2)$ whenever $\kappa \leq \theta_1 \leq \theta_2$ and each $\theta_i$ is regular. (Note: the possibility of strong cardinals above $\kappa$ is irrelevant here, since for any $\theta$ one can use an embedding $j : V \to M$ for which $\kappa$ is not $\theta$-supercompact in $M$; consequently, the next strong cardinal in $M$ above $\kappa$ is above $\theta$, which is all that is needed in the lifting argument). It follows as in Corollary $[\mathbb{I}]$ that the supercompactness of $\kappa$ is strongly resurrectible in $V[G]$.\[16]
We now argue that the model $V[G]$ retains the level-by-level agreement between strong compactness and supercompactness. Since the forcing $\mathbb{P}$ has size $\kappa$, we know that the level-by-level agreement for cardinals above $\kappa$ holds easily by [LS67]. It remains only to consider cardinals $\gamma$ below $\kappa$. Accordingly, suppose that $\gamma$ is $\theta$-strongly compact in $V[G]$ for some regular cardinal $\theta > \gamma$; we aim to show it is also $\theta$-supercompact there. By the Gap Forcing Theorem [Ham], we know that $\gamma$ is $\theta$-strongly compact in $V$, and hence also $\theta$-supercompact there. Fix a $\theta$-supercompactness embedding $j : V \rightarrow M$ for which $\gamma$ is not $\theta$-supercompact in $M$. It follows that the next strong cardinal in $M$ above $\gamma$ is above $\theta$.

We first treat the case in which $\gamma$ is a limit of strong cardinals, that is, when $\gamma$ is a stage of forcing in $\mathbb{P}$. In the stage $\gamma$ lottery, the generic $G$ selected some winning poset $Q$, and below a condition deciding this we may factor the forcing $\mathbb{P}$ as $\mathbb{P}_{\gamma} \ast Q \ast \mathbb{P}_{\text{tail}}$ where $Q$ is either trivial forcing or $\text{coll}(\theta_1, \theta_2)$ for some $\gamma \leq \theta_1 \leq \theta_2$. Since $\mathbb{P}_{\text{tail}}$ is closed beyond the next strong cardinal above $\gamma$, it does not affect the $\theta$-supercompactness of $\gamma$, and so it suffices for us to show that $\gamma$ is $\theta$-supercompact in $V[G_{\gamma}][g]$, where $g \subseteq Q$ is $V[G_{\gamma}]$-generic. If $Q$ is trivial, or if $\theta_2 < \theta$, then the usual lifting arguments allow us to lift the embedding $j : V \rightarrow M$ to $j : V[G_{\gamma}][g] \rightarrow M[j(G_{\gamma})][j(g)]$, thereby witnessing the $\theta$-supercompactness of $\gamma$ in $V[G_{\gamma}][g]$. So we may assume $\theta < \theta_2$. Since $\theta$ is a cardinal in $V[G]$, it follows that $\theta \leq \theta_1$. If $\theta < \theta_1$, then the forcing $Q$ adds no new subsets to $P, \theta$, and so the $\theta$-supercompactness of $\gamma$ in $V[G_{\gamma}]$ (from the case of trivial $Q$ above) is preserved to $V[G_{\gamma}][g]$. So we have reduced to the case that $\theta = \theta_1$. Since $\gamma$ is $\theta_1$-strongly compact in $V[G]$, it is also $\theta_2$-strongly compact there, and hence $\theta_2$-supercompact in $V$. One may therefore employ the usual arguments to lift a $\theta_2$-supercompactness embedding from $V$ to $V[G_{\gamma}][g]$, as desired.

We now treat the case that $\gamma$ is not a limit of strong cardinals. Let $\delta < \gamma$ be the supremum of the strong cardinals below $\gamma$. If $\delta$ is not inaccessible, then the forcing $\mathbb{P}$ factors as $\mathbb{P}_{\delta} \ast \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\delta}$ is small relative to $\gamma$ and $\mathbb{P}_{\text{tail}}$ is closed beyond the next strong cardinal above $\gamma$. Such forcing must preserve the $\theta$-supercompactness of $\gamma$. We may therefore assume alternatively that $\delta$ is inaccessible, and hence a stage of forcing. The generic $G$ selected a winning poset $Q$ in the stage $\delta$ lottery, and below a condition deciding this we may factor $\mathbb{P}$ as $\mathbb{P}_{\delta} \ast Q \ast \mathbb{P}_{\text{tail}}$. The forcing $\mathbb{P}_{\text{tail}}$ is closed beyond $\theta$, and does not affect the $\theta$-supercompactness of $\gamma$. Thus, it suffices for us to see that $\gamma$ is $\theta$-supercompact in $V[G_{\delta}][g]$, where $g \subseteq Q$ is $V[G_{\delta}]$-generic. The forcing $Q$ is either trivial or $\text{coll}(\theta_1, \theta_2)$ for some $\delta \leq \theta_1 \leq \theta_2$. If $Q$ is trivial or $\theta_2 < \gamma$, then the forcing is small relative to $\gamma$, and the result is immediate. So assume $\gamma \leq \theta_2$. Since $\gamma$ is a cardinal, it must be that $\gamma \leq \theta_1$. If in addition $\theta < \theta_1$, then $\gamma < \theta_1$ and we may ignore the forcing $Q$, since it does not destroy the $\theta$-supercompactness of $\gamma$ in $V[G_{\delta}]$, a small forcing extension. What remains is the case $\gamma \leq \theta_1 \leq \theta$. Here, as in Corollary 3, we make a key use of the main result of [HS98]: the forcing $\mathbb{P}_{\delta} \ast \text{coll}(\theta_1, \theta_2)$ is small forcing followed by $\prec \gamma$-closed forcing that adds a new subset to $\theta_1$. By [HS98], such forcing necessarily destroys the $\theta_1$-strong compactness of $\gamma$, contrary to our assumption that $\gamma$ is $\theta$-strongly compact in $V[G]$, and hence in $V[G_{\delta}][g]$. So the proof is complete.

Because of this argument, the limitation identified above the supercompact cardinal in Theorem 3 does not engage for resurrectibility.
3 Indestructibility with near level-by-level agreement

In this section, we will prove that full indestructibility is compatible with a level-by-level agreement between strong compactness and supercompactness almost everywhere. To make this notion of almost everywhere precise, let us define that a set $A \subseteq \kappa$ is \textit{large} with respect to supercompactness if for every $\theta \geq \kappa$ there is a $\theta$-supercompactness embedding $j : V \rightarrow M$ with $\kappa \in j(A)$. That is, the set is large in virtue of having measure one with respect to all these induced normal measures.\footnote{We warn the reader that the collection of large subsets of a supercompact cardinal do not form a filter. Specifically, if $M_\theta$ is the collection of normal measures induced by $\theta$-supercompactness embeddings with critical point $\kappa$, then the collection of large sets is precisely $\cap_\theta (\cup M_\theta)$. Since $M_\theta \subseteq M_\lambda$ whenever $\lambda \leq \theta$, the collections $M_\theta$ are eventually equal, and the collection of large sets is simply the union of the measures in this stabilized eventual value for $M_\theta$. Since the proof of Example 14 shows that each $M_\theta$ has many measures, this means that the collection of large sets is the union of a great number of normal measures, and therefore it cannot be a filter.}

To illustrate this notion, define that a measurable cardinal $\gamma$ is \textit{robust} if whenever $\gamma$ is $\theta$-supercompact, then there is a $\theta$-supercompactness embedding $j : V \rightarrow M$ with $\text{cp}(j) = \gamma$ such that $\gamma$ is $\theta$-supercompact in $M$. Equivalently, $\gamma$ is robust if it has nontrivial Mitchell rank in every degree of supercompactness that it exhibits. Conversely, we could define that $\gamma$ is \textit{precarious} when it is $\eta$-supercompact with trivial Mitchell rank for some $\eta$; these cardinals necessarily lose bit of their supercompactness in their most supercompact ultrapowers. Thus, by definition, the robust cardinals are exactly the non-precarious measurable cardinals.

\textbf{Example 14} There are many robust cardinals below any supercompact cardinal; indeed, the collection of robust cardinals is large with respect to supercompactness.\footnote{The set of precarious cardinals is also large with respect to supercompactness, since by a suitable choice of supercompactness embedding—using a measure of Mitchell rank 1—one can arrange that $\kappa$ has trivial Mitchell rank in $M$ for its largest degree of supercompactness.}

\textbf{Proof:} The basic point is that if a cardinal $\kappa$ is $2^{\theta^{<\kappa}}$-supercompact, then it has very high Mitchell rank in $\theta$-supercompactness. To see this, suppose $j : V \rightarrow M$ is a $2^{\theta^{<\kappa}}$-supercompactness embedding. Since all the $\theta$-supercompactness measures from $V$ are in $M$, it follows that $\kappa$ is $\theta$-supercompact in $M$. Thus, for the induced $\theta$-supercompactness factor embedding $j_0 : V \rightarrow M_0$, the cardinal $\kappa$ is $\theta$-supercompact in $M_0$. So the Mitchell rank is at least 1 in $V$, and so it is at least 1 in $M$ and therefore also in $M_0$; so it is at least 2 in $V$, and so at least 2 in $M$ and therefore at least 2 in $M_0$, and so on cycling around up to $\theta$ and beyond.

To prove the claim, now, let $\theta$ be any strong limit cardinal above $\kappa$ and let $j : V \rightarrow M$ be a $\theta$-supercompactness embedding by a measure with trivial Mitchell rank (or just make $j(\kappa)$ as small as possible). It follows that $\kappa$ is not $\theta$-supercompact in $M$; but by the closure of $M$ it is $<\theta$-supercompact there. Therefore, since $\theta$ is a strong limit cardinal, the basic point in the previous paragraph shows that it exhibits nontrivial Mitchell rank in $M$ for every degree of supercompactness below $\theta$. So it is robust in $M$. And since this is true for...
arbitrarily large $\theta$, it follows that the set of robust cardinals below $\kappa$ is large with respect to supercompactness. □

While the argument in the example verifies the robustness of a cardinal by having it exhibit a limit degree of supercompactness, this is not at all the only way to be robust, for a cardinal $\kappa$ could be only $\kappa^+$-supercompact, for example, and still be robust by simply having nontrivial Mitchell degree for $\kappa^+$-supercompactness. The collection of cardinals with a largest degree of supercompactness and nontrivial Mitchell rank for that degree of supercompactness is also large with respect to supercompactness; this can be seen by using $\theta$-supercompactness embeddings $j : V \rightarrow M$ for $\theta$ a successor cardinal arising from a $\theta$-supercompactness measure with Mitchell rank at least 2.

Let us now turn to the main theorem of this section:

**Theorem 15** Suppose that $\kappa$ is supercompact and no cardinal is supercompact up to a larger cardinal $\lambda$ which is $2^\lambda$-supercompact. Then there is a forcing extension in which $\kappa$ becomes indestructibly supercompact and level-by-level agreement holds between strong compactness and supercompactness on a set that is large with respect to supercompactness.

**Proof:** As in Theorem 8, we may assume without loss of generality, by forcing if necessary, that the GCH holds and further, by forcing with the notion in [AS97] if necessary, that in $V$ there is already a level-by-level agreement between strong compactness and supercompactness. Thus, we also have that no cardinal is supercompact up to a partially supercompact cardinal. Let $\mathbb{P}$ be the reverse Easton support $\kappa$-iteration which adds a Cohen real and then has nontrivial forcing only at cardinals $\gamma$ that are inaccessible limits of partially supercompact cardinals. At such a stage $\gamma$, the stage $\gamma$ forcing $Q_\gamma$ is the lottery sum of all $<\gamma$-directed closed $Q$ of size less than $\theta_\gamma$, the least cardinal such that $\gamma$ is not $\theta_\gamma$-supercompact in $V$.

Suppose that $G \subseteq \mathbb{P}$ is $V$-generic, and consider the model $V[G]$.

First, we claim that $\kappa$ is indestructibly supercompact in $V[G]$. It suffices to argue that if $Q \in V[G]$ is $<\kappa$-directed closed in $V[G]$ and $H \subseteq Q$ is $V[G]$-generic, then $\kappa$ is supercompact in $V[G][H]$. Fix any regular cardinal $\theta > |Q|$ and a $\theta$-supercompactness embedding $j : V \rightarrow M$ so that $\kappa$ is not $\theta$-supercompact in $M$. Since $\kappa$ will necessarily be $<\theta$-supercompact in $M$, the forcing $Q$ will appear in the stage $\kappa$ lottery of $j(\mathbb{P})$. Below a condition opting for $Q$ in this lottery, the forcing $j(\mathbb{P})$ factors as $\mathbb{P} \ast Q \ast \mathbb{P}_{\text{tail}}$, where $\mathbb{P}_{\text{tail}}$ is the remainder of the iteration. Note that since no cardinal is supercompact beyond a partially supercompact cardinal, no cardinal above $\kappa$ is partially supercompact. Thus, the next nontrivial stage of forcing in $j(\mathbb{P})$ is beyond $\theta$, and so $\mathbb{P}_{\text{tail}}$ is $\leq \theta$-closed in $M[G][H]$. Therefore, by the usual diagonalization techniques, we may construct in $V[G][H]$ an $M[G][H]$-generic filter $G_{\text{tail}} \subseteq \mathbb{P}_{\text{tail}}$ and lift the embedding to $j : V[G] \rightarrow M[j(G)]$ with $j(G) = G \ast H \ast G_{\text{tail}}$. After this, we find a master condition below $j"H$ in $M[j(G)]$ and again by diagonalization construct an $M[j(G)]$-generic $j(H) \subseteq j(Q)$ in $V[G][H]$. This allows us to lift the embedding fully to $j : V[G][H] \rightarrow M[j(G)][j(H)]$, which witnesses the $\theta$-supercompactness of $\kappa$ in $V[G][H]$, as desired.

Second, we claim that in $V[G]$ there is a level-by-level agreement between strong compactness and supercompactness at the cardinals $\gamma$ that are robust in $V$. To see this, on
the one hand, the main results of [HS98] show that since the forcing $\mathbb{P}$ is mild and admits a very low gap (see [Ham] for the relevant definitions), it does not increase the degree of strong compactness or supercompactness of any cardinal. On the other hand, we will now argue that it fully preserves all regular degrees of supercompactness of every partially supercompact cardinal $\gamma$ that is robust in $V$. Suppose that $\gamma$ is robust and $\theta$-supercompact in $V$ for some regular cardinal $\theta$ above $\gamma$. A simple reflection argument shows that $\gamma$ must be a limit of partially supercompact cardinals, and therefore is a nontrivial stage of forcing. The generic filter $G$ opted for some particular forcing $Q$ in the stage $\gamma$ lottery. By increasing $\theta$ if necessary, we may assume that $|Q| \leq \theta$. Since no cardinal is supercompact beyond a partially supercompact cardinal, as in Lemma 6.1, the next nontrivial stage of forcing beyond $\gamma$ is well beyond $\theta$. It therefore suffices to argue that $\gamma$ is $\theta$-supercompact in $V[G_{\gamma+1}]$. Fix any $\theta$-supercompactness embedding $j : V \to M$ with critical point $\gamma$ such that $\gamma$ is $\theta$-supercompact in $M$ (this is where we use the robustness of $\gamma$). It follows that $Q$ appears in the stage $\gamma$ lottery of $j(\mathbb{P})$, and we may lift the embedding to $j : V[G_{\gamma+1}] \to M[j(G_{\gamma+1})]$ just as we did in the previous paragraph with $\kappa$. Thus, $\gamma$ remains $\theta$-supercompact in $V[G_{\gamma+1}]$ and hence in $V[G]$. So we retain the level-by-level agreement between strong compactness and supercompactness for the partially supercompact cardinals $\gamma$ that are robust in $V$.

Now let us argue that the collection $A$ of partially supercompact cardinals that are robust in $V$ remains large with respect to supercompactness in $V[G]$. The point is that if $j : V \to M$ is a $\theta$-supercompactness embedding with critical point $\kappa$, then below a condition opting for trivial forcing in the lottery at stage $\kappa$, we may lift the embedding to $j : V[G] \to M[j(G)]$. In particular, if $\kappa \in j(A)$ for the original embedding, then this remains true for the lifted embedding. Indeed, this argument shows that our forcing preserves every set that is large with respect to supercompactness in the ground model.

We would like to point out that in fact in $V[G]$ we have much more level-by-level agreement than just at the robust cardinals. In particular, for any cardinal $\gamma$ that is a limit of partially supercompact cardinals, it happens that the generic filter chooses a poset $Q$ with $|Q|^+ \leq \theta$, and this happens on a large set since we can arrange it at stage $\kappa$ in $j(\mathbb{P})$, then one may apply the lifting argument of the proof using an embedding witnessing enough supercompactness of $\gamma$.

The problematic cardinals are exactly the precarious cardinals $\gamma$ for which the forcing opts for $Q$ at stage $\gamma$ of the largest possible size. This case is difficult because $Q$ will be too large to appear in the stage $\gamma$ lottery of $j(\mathbb{P})$, and so one may not use any ordinary lifting argument to preserve the supercompactness of $\gamma$. Despite our many attempts, there seems to be no easy solution to this annoying problem: restricting the posets that appear in the stage $\gamma$ lottery on the $V$-side simply causes a corresponding restriction on the $j$-side, and it seems that there are always these borderline posets that are allowed in the $\mathbb{P}$ lottery but not in the $j(\mathbb{P})$ lottery.

Even for these problematic cardinals $\gamma$, though, it will often happen that nevertheless the level-by-level agreement between strong compactness and supercompactness is preserved, because the forcing $Q$ opted for in the stage $\gamma$ lottery is, say, equivalent to smaller forcing,
or does not add sets below \(|Q|\) but destroys the \(|Q|\)-supercompactness of \(\gamma\) (so that both strong compactness and supercompactness drop evenly). Nevertheless, we know of no way to ensure the level-by-level agreement at all cardinals below \(\kappa\), so Question \(\[\]\) remains open.

The theorems of this section, as well as the previous section, suggest that a positive answer to Question \(\[\]\) is possible. Further, since the cardinals below \(\kappa\) at which the level-by-level agreement holds exhibit themselves some degree of indestructibility, the results suggest the intriguing possibility that one could have a supercompact cardinal and a level-by-level agreement between strong compactness and supercompactness in the presence of universal indestructibility: every partially supercompact cardinal \(\gamma\) is fully indestructible by \(<\gamma\)-directed closed forcing.

**Question 16** Is universal indestructibility consistent with level-by-level agreement if there is a supercompact cardinal?

Universal indestructibility, like the level-by-level agreement between strong compactness and supercompactness in Theorem \(\[\]\), is by itself incompatible with large cardinals above a supercompact cardinal. Specifically, in [AH99] we show that when universal indestructibility holds, then no cardinal is supercompact beyond a measurable cardinal. The question is, does this affinity indicate a compatibility of the two notions?

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