BRAIDED MOMENTUM IN THE Q-POINCARE GROUP

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ABSTRACT

The $q$-Poincaré group of \cite{1} is shown to have the structure of a semidirect product and coproduct $B \rtimes \widehat{SO_q}(1,3)$ where $B$ is a braided-quantum group structure on the $q$-Minkowski space of 4-momentum with braided-coproduct $\Delta p = p \otimes 1 + 1 \otimes p$. Here the necessary $B$ is not a usual kind of quantum group, but one with braid statistics. Similar braided-vectors and covectors $V(R'), V^*(R')$ exist for a general $R$-matrix. The abstract structure of the $q$-Lorentz group is also studied.

1 Introduction

Many authors have considered the possibility of $q$-deforming the Poincaré group as part of a general programme of $q$-deforming conventional quantum field theory. Such a programme, if successful, would introduce a new regularization parameter in physics\cite{2} while preserving all usual symmetries as $q$-symmetries. Reasonable candidates for the (non-commutative) algebra of ‘functions’ on $q$-Minkowski space and the $q$-Lorentz group $SO_q(1,3)$ are known\cite{3} but when it comes to the Poincaré group a problem seems to be that there are too many possibilities\cite{4,5,6,1} if one merely looks for a Hopf algebra or quantum group structure. One technique in narrowing down the possibilities has been to look for a Hopf $*$-algebra structure and/or to look for a universal $R$-matrix or quasitriangular structure. Both attempts have not been fully successful.

Instead, we would like to develop here a different structural consideration, namely that the Poincaré group should be built up from the $q$-momentum and $q$-Lorentz as a semidirect product (in analogy with the classical situation). This tells us how the complicated generators and

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relations are built up in a conceptual way from simple ones. Quantum groups such as $SO_q(1,3)$ can certainly act on other quantum groups (such as a $q$-momentum) just as in the classical case. Hence the problem is a natural and well-posed one, but again one that has so far not proven possible in a full sense (not just for the algebra but for coalgebra also) for any of the candidates above. We explain first that there are general grounds that this is not possible unless the $q$-momentum is a braided-quantum group \cite{7,8} rather than an ordinary quantum one. These objects are more like supergroups or super-quantum groups, but with bose-fermi statistics replaced by braid statistics. We then show that one of the proposed $q$-Poincaré groups, namely that in \cite{1} has just such a structure. A similar situation may prevail for \cite{4}. One may also ask about the abstract structure of the $q$-Lorentz group itself. For completeness, the paper concludes with some remarks about this.

In more detail, a braided-quantum group is a Hopf algebra with braid statistics\cite{7,8}. This means an algebra $B$ equipped with a coproduct homomorphism $B \to B \otimes B$ where $B \otimes B$ is not the usual tensor product algebra, but instead the tensor-product with braid statistics\cite{7,8}:

\[(a \otimes b)(c \otimes d) = a \Psi(b \otimes c)d \quad (1)\]

where $\Psi$ is the braided-transposition. This braiding $\Psi$ can arise in a variety of ways, but the one that concerns us comes from the universal $R$-matrix of some background quantum group.

The general grounds for needing such objects for the Poincare group are as follows. Firstly, the Poincare group should be a Hopf algebra containing the Lorentz group. Thus $U_q(so(1,3)) \subset U_q(p)$ where $p$ stands for the Poincare group Lie algebra and we wish to find its deformed enveloping algebra $U_q(p)$. In nice cases we should also be able to project $U_q(p) \to U_q(so(1,3))$ by setting the momentum generators to zero (classically, one can do this because of the semidirect structure of $p$). The projection should of course cover the inclusion. These are some minimal conditions that we might expect for the relationship between these two Hopf algebras. However, there is a theorem\cite{9}, Prop. A.2] that if these maps are Hopf algebra maps then $U_q(p) = B \triangleright U_q(so(1,3))$, as an algebra and coalgebra, for some braided-Hopf algebra $B$. This is a mathematical theorem that holds for any pair of Hopf algebras with an inclusion/projection between them and is related to a theorem of Radford\cite{10}. Thus the quantum version of the familiar semidirect product situation forces us \textit{a priori} into the braided setting, at least for
the momentum. The same theorem applies when we look in the dual form for the $q$-Poincare function algebra $P_q$. Here the usual inclusion becomes a projection $P_q \to SO_q(1,3)$ and can be expected to also cover a Hopf algebra inclusion $SO_q(1,3) \subset P_q$. From this alone we conclude that $P_q = B \times SO_q(1,3)$ for some braided-Hopf algebra $B$ of function algebra type. Moreover, the braiding $\Psi$ will be induced by the action of the universal $R$-matrix for $U_q(so(1,3))$ on the momentum and can therefore be expected to be non-trivial. We will obtain Hopf algebras of exactly this general type, the only further complication being a slight extension of the $q$-Lorentz group by a central element $g$ needed for technical reasons to do with the normalization of $R$-matrices.

2 Braided-Covectors

From the above general arguments, we are led to look for a braided-momentum group $B$ rather than a usual quantum group. In the dual function algebra description it should surely coincide as an algebra with the $q$-Minkowski space of $\mathbb{R}$. We will demonstrate this as well as a general $R$-matrix construction of which this is an example.

The basic idea is that the function-algebra of the momentum group should be nothing other than a braided-commutative (like super-commutative) plane, with trivial linear coproduct. If $R$ is the $SO_q(1,3)$ matrix, the most naive possibility is the Zamalodchikov algebra $x_1x_2 = \cdot\Psi(x_1 \otimes x_2) = x_2x_1R_{12}$, where the braiding is given by $R$. On the other hand, we have explained in $\mathbb{R}$ that this naive notion is not good when $\Psi^2 \neq \text{id}$ as is the case here. Instead, one needs some variant $R'$ when defining the notion of braided-commutativity.

**Theorem 1** Let $R$ be an invertible matrix in $M_n \otimes M_n$ obeying the QYBE and suppose that $R'$ is another invertible matrix such that

(i) $R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}$, $R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23}$

(ii) $(PR + 1)(PR' - 1) = 0$ where $P$ is the permutation matrix.

Then the braided-covectors $V^\ast(R')$ defined by generators $1, x_i$ and relations $x_2x_1R'_{12} = x_1x_2$ forms a braided-bialgebra with

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i, \quad \Psi(x_1 \otimes x_2) = x_2 \otimes x_1 R_{12}.$$ 

extended multiplicatively with braid statistics. The counit is $\epsilon(x_i) = 0$. If also
(iii) $R_{21}R'_{12} = R'_{21}R_{12}$

then $V^*(R')$ is a braided-Hopf algebra with braided-antipode $Sx_i = -x_i$ extended anti-multiplicatively with braid statistics.

**Proof** The definition is very similar to the braided-matrices $B(R)$ introduced in [8] with $R, R'$ there. Firstly, $V^*(R')$ is by definition an associative algebra. We use here and throughout a standard compact notation where numerical suffices as in $R_{12}, x_1, x_2$ etc refer to the position in a matrix tensor product while within each tensor factor we use a standard matrix notation (so $x_1 x_2 = x_2 x_1 R_{12}$ means $x_j x_l = x_k x_l R_{i,j}^{i,j}$). We have to check that $\Psi, \Delta, S$ are well-defined when extended to products. Firstly, $\Psi$ extends to tensor products according to the rules of a braiding ($R$ generates a braided category), namely

$$\Psi(x_1 \otimes (x_2 \otimes x_3)) = (id \otimes \Psi)(\Psi(x_1 \otimes x_2) \otimes x_3) = x_2 \otimes \Psi(x_1 \otimes x_3) R_{12} = x_2 \otimes x_3 \otimes x_1 R_{13} R_{12}$$

etc. The extension to products is then in such a way that $\Psi$ is functorial with respect to the product, in the sense

$$\Psi(x_1 \otimes x_2 x_3) = (\cdot \otimes id)\Psi(x_1 \otimes (x_2 \otimes x_3)) = x_2 x_3 \otimes x_1 R_{13} R_{12}.$$ 

To see that this extension is well defined, we compute also

$$\Psi(x_1 \otimes x_3 x_2 R'_{23}) = (\cdot \otimes id)\Psi(x_1 \otimes (x_3 \otimes x_2)) R'_{23} = x_3 x_2 \otimes x_1 R_{12} R_{13} R'_{23}$$

which is consistent with the relation $x_2 x_3 = x_3 x_2 R'_{23}$ given the first of conditions (i). Hence $\Psi(x_1 \otimes ( ))$ is a well-defined map on the algebra $V^*(R')$. One can then compute in the same way from functoriality that

$$\Psi(x_1 \otimes x_2 x_3 \cdots x_a) = x_2 x_3 \cdots x_a \otimes x_1 R_{1n} \cdots R_{12}.$$ \hspace{1cm} (2)

Using this, we compute in a similar way

$$\Psi(x_1 x_2 \otimes x_3 \cdots x_a) = (id \otimes \cdot)\Psi((x_1 \otimes x_2) \otimes x_3 \cdots x_a) = (id \otimes \cdot)\Psi(x_1 \otimes x_3 \cdots x_a) \otimes x_2 R_{2a} \cdots R_{23}$$

$$= x_3 \cdots x_a \otimes x_1 x_2 R_{1a} \cdots R_{13} R_{2a} \cdots R_{23} = x_3 \cdots x_a \otimes x_2 x_1 R'_{12} R_{1a} \cdots R_{13} R_{2n} \cdots R_{23}$$

$$\Psi(x_2 x_1 R'_{12} \otimes x_3 \cdots x_a) = (id \otimes \cdot)\Psi(x_2 \otimes x_3 \cdots x_a) \otimes x_1 R_{1a} \cdots R_{13} R'_{12}$$

$$= x_3 \cdots x_a \otimes x_2 x_1 R_{2a} \cdots R_{23} R_{1a} \cdots R_{13} R'_{12}.$$
Here $R'_{12}R_{1a} \cdots R_{13}R_{2a} \cdots R_{23} = R'_{12}R_{1a}R_{2a} \cdots R_{13}R_{23}$ since matrices living in disjoint tensor factors commute. We can then repeatedly use the second of (i) to move $R'_{12}$ to the right to arrive at $R_{2a}R_{1a} \cdots R_{23}R_{13}R'_{12} = R_{23}R_{1a} \cdots R_{13}R'_{12}$. Hence $\Psi: V^*(R') \otimes V^*(R') \to V^*(R') \otimes V^*(R')$ is well defined and functorial with respect to the product. It takes the form of a transfer matrix analogous to expressions in the theory of quantum inverse scattering.

Next we extend $\Delta$ to products in such way that it is a homomorphism to the braided tensor product $\otimes$,

$$
\Delta x_1 x_2 = (x_1 \otimes 1 + 1 \otimes x_1)(x_2 \otimes 1 + 1 \otimes x_2) = x_1 x_2 \otimes 1 + 1 \otimes x_1 x_2 + x_1 \otimes x_2 + \Psi(x_1 \otimes x_2)
$$

$$
= x_1 x_2 \otimes 1 + 1 \otimes x_1 x_2 + x_1 \otimes x_2 + x_2 \otimes x_1 R_{12}
$$

$$
\Delta x_2 x_1 R'_{12} = (x_2 \otimes 1 + 1 \otimes x_2)(x_1 \otimes 1 + 1 \otimes x_1)R'_{12}
$$

$$
= x_2 x_1 R'_{12} \otimes 1 + 1 \otimes x_2 x_1 R'_{12} + x_2 \otimes x_1 R'_{12} + x_1 \otimes x_2 R_{21} R'_{12}.
$$

Hence for $\Delta$ to be well-defined we need $x_1 \otimes x_2 (R_{21} R'_{12} - 1) = x_2 \otimes x_1 (R_{12} - R_{12}')$ or $R_{21} R'_{12} - 1 = P(R - R')$ which is condition (ii). Here $P$ is the usual permutation matrix $x_1 \otimes x_2 P = x_1 \otimes x_2$.

It is trivial to see that the braiding $\Psi$ is then functorial with respect to the coproduct $\Delta: V^*(R') \to V^*(R') \otimes V^*(R')$.

Finally, for a Hopf algebra in a braided category, the antipode is braided-anti-multiplicative in the sense $S(ab) = -\Psi(Sa \otimes Sb)$. We define $S$ on products of the generators in this way. Then

$$
S(x_1 x_2) = -\Psi(Sx_1 \otimes Sx_2) = x_2 x_1 R_{12} = x_1 x_2 R_{21} R_{12}
$$

$$
S(x_2 x_1 R'_{12}) = -\Psi(Sx_2 \otimes Sx_1)R'_{12} = x_1 x_2 R_{21} R'_{12}.
$$

Thus $S$ here is well-defined by condition (iii). More generally one can compute likewise

$$
S(x_1 \cdots x_a) = (-1)^a x_a \cdots x_1 R_{12} \cdots R_{1a} R_{23} \cdots R_{2a} \cdots R_{a-1} a.
$$

□

**Example 2** If $R$ is a Hecke symmetry in the sense that it obeys

$$(PR + 1)(PR - q^2) = 0$$

(3)
(for example the standard R-matrix for all $SL_q(n)$ after a suitable scaling), then $R' = q^{-2}R$ obeys the above. Hence in this case the usual Zamalodchikov algebra $V^*(R)$ is a braided-Hopf algebra with the linear braided-coproduct.

For example, the standard bosonic quantum plane with relations $xy = q^{-1}yx$ is a braided-Hopf algebra with

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \Delta y = y \otimes 1 + 1 \otimes y, \quad Sx = -x, \quad Sy = -y$$

$$\Psi(x \otimes x) = q^2 x \otimes x, \quad \Psi(x \otimes y) = qy \otimes x, \quad \Psi(y \otimes y) = q^2 y \otimes y$$

$$\Psi(y \otimes x) = qx \otimes y + (q^2 - 1)y \otimes x.$$ 

We call this the braided-plane. The sub-braided-Hopf algebras generated by $x$ or $y$ alone are copies of the braided-line as introduced in [11].

**Example 3** If $R$ is ($q$ times) the R-matrix for $SO_q(1,3)$ in [3], we take

$$R' = P + \mu P(PR - q^{-2})(PR - q^2).$$

In this case we call $V^*(R')$ the (right handed) braided-Minkowski space $\mathbb{R}^{1,3}$. As an algebra, it coincides with a right-handed version of the $q$-Minkowski space in [3], equipped now with the linear braided-coproduct.

**Proof** The Hecke-type relation for this $R$-matrix (in our normalization) is [3] $(PR + 1)(PR - q^{-2})(PR - q^2) = 0$. Hence condition (ii) in Theorem 1 is obeyed. Moreover, since $PR'$ is a polynomial in $PR$ it follows at once that conditions (i),(iii) are obeyed. To see this, it is convenient to write them in terms of $PR, PR'$: after renumbering they are $(PR)_{23}(PR)_{12}(PR')_{23} = (PR')_{12}(PR)_{23}(PR)_{12}$ and $(PR)(PR') = (PR')(PR)$. Since $PR' = PR$ and $PR' = id$ obey these, it follows that so does any polynomial in $PR, id$. Hence we have a braided-covector Hopf algebra by the theorem. The value of $\mu \neq 0$ does not change the resulting algebra but can be chosen so that $R'$ is invertible. Note that $PR' - 1$ here is proportional to the antisymmetrizer $P_A$ in the approach of [3] so that $V^*(R)$ coincides with (a right-handed version of) their $q$-Minkowski space algebra as defined by $x_1x_2P_A = 0$. □
Clearly the last proof works for any $R$-matrix obeying a polynomial identity $f(PR) = 0$. Writing the roots as $\lambda_1$ we have $(PR - \lambda_1)(PR - \lambda_2) \cdots (PR - \lambda_m) = 0$. Hence for each non-zero root $\lambda_i$ we scale $R$ so that $(PR + 1) \prod_{j \neq i} (PR + \frac{\lambda_j}{\lambda_i}) = 0$ and define $PR' = 1 + \mu \prod_{j \neq i} (PR + \frac{\lambda_j}{\lambda_i})$. Hence there is a braided-covector Hopf algebra for each non-zero root of the functional equation for $PR$ (each non-zero eigenvalue of $PR$).

Also clearly, there are plenty of variants of our construction based on slightly different conventions. For example, for the same data $R, R'$ as in Theorem 1, we can define $V(R')$ by generators $x^i$ (upper index) and relations $x_1 x_2 = R' x_2 x_1$. This forms a braided-vector Hopf algebra with linear braided-coproduct, braided-antipode and braiding,

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad S x = -x, \quad \Psi(x_1 \otimes x_2) = R_{12} x_2 \otimes x_1. \quad (4)$$

The proof that this is a braided-Hopf algebra is strictly analogous to the proof of Theorem 1.

What about the physical meaning of these braided-coproducts? Our interpretation from the point of view of non-commutative or braided geometry is that it corresponds to a group-structure on the non-commutative or braided space expressing an analog of the addition of covectors (in analogy with the meaning in the commutative case). Such a group structure means of course that we can make translations. Thus $\Delta : V^*(R') \to V^*(R') \otimes V^*(R')$ can be viewed as a braided-coaction on one copy of $B$ by another. If we denote the generators of the second copy by $p_i$ to distinguish them from $x_i$, the braided-coaction $\beta$ is

$$\beta(x_i) = x_i \otimes 1 + 1 \otimes p_i, \quad \text{i.e.,} \quad x'_i = x_i + p_i \quad (5)$$

where the second expression is a more compact notation for the coaction. This innocent expression nevertheless carries non-trivial information, for we can differentiate it to obtain the ‘braided-vector-fields’ corresponding to translation. For example, for the braided line (the 1-dimensional case) the coaction $x' = x + p$ when we remember the braid statistics $\Psi(p \otimes x) = q^2 x \otimes p$ between the two copies of the braided line, gives

$$(x^m - x^m)p^{-1}|_{p=0} = ((x+p)^m - x^m)p^{-1}|_{p=0} = \left( \sum_{r} \binom{m}{r}_q x^r p^{m-r} - x^m \right)p^{-1}|_{p=0} = [m]_q x^{m-1} \quad (6)$$

which is the usual $q$-derivative $\partial_q$ of $x^m$. We used here the notation $[m]_q = \frac{q^{m-1}}{q-1}$ and the usual $q$-binomial theorem to expand $(x \otimes 1 + 1 \otimes p)^m$ (the two terms $q^{-2}$-commute in the braided
tensor product algebra). Thus the usual $q$-derivative $\partial_q(f)(x) = \frac{f(q^2x) - f(x)}{(q^2 - 1)x}$ is the generator of braided-translations on the braided-line. From this point of view its well-known Leibnitz rule

$$\partial_q(fg) = (\partial_q f)g + L_q(f)(\partial_q g)$$

where $L_q(f)(x) = f(q^2x)$ should be viewed as analogous to the super-Leibnitz rule, with the operator $L_q$ playing the role of the usual $\pm 1$ factor.

This point of view will be developed further elsewhere. For our present purposes we note only that our braided-coproduct $\Delta$ thus replaces in our approach the study of differential calculus that characterizes the approach to the $q$-Poincaré group in [4]. We work with finite translations rather than infinitesimal ones.

3 $q$-Poincaré Group as a Semidirect Product

We have developed the braided-covectors as living in the braided category generated by an $R$-matrix (which plays the role of $\pm 1$ in the super case). On the other hand, this category is basically the category of comodules of the bialgebra algebra $A(R)$ [12] with matrix generator $t$ and relations $Rt_1t_2 = t_2t_1R$ and equipped with a certain dual quasitriangular structure $\mathcal{R} : A(R) \otimes A(R) \to \mathbb{C}$. This obeys some obvious axioms dual to those for a universal $R$-matrix in [13], and is given here by $\mathcal{R}(t_1 \otimes t_2) = R$ extended to products as a skew bialgebra-bicharacter along the lines developed in an equivalent form in [14, Sec. 3]. Thus

$$\mathcal{R}(t_1 \cdots t_a \otimes t_{a+1}) = \mathcal{R}(t_1 \otimes t_{a+1}) \cdots \mathcal{R}(t_a \otimes t_{a+1}) = R_{1a+1} \cdots R_{aa+1} \quad (7)$$

$$\mathcal{R}(t_1 \otimes t_2 \cdots t_{a+1}) = \mathcal{R}(t_{a+1} \otimes t_1) \cdots \mathcal{R}(t_2 \otimes t_1) = Ra_{a+1} \cdots R_{21} \quad (8)$$

while the expression for $\mathcal{R}(t_1 \cdots t_a \otimes t_{a+1} \cdots t_{a+b})$ is an array of $R$-matrices cf [14, Sec. 5.2]. See [15] where we also explain how it leads to the braiding $\Psi$ in the category. If $V, W$ are any two right comodules of a dual-quasitriangular Hopf algebra, written explicitly as $v \to v' = \sum v^{(1)} \otimes v^{(2)}$ and $w \to w' = \sum w^{(1)} \otimes w^{(2)}$ say, then the braiding is

$$\Psi_{V \otimes W}(v \otimes w) = \sum w^{(1)} \otimes v^{(1)} \mathcal{R}(v^{(2)} \otimes w^{(2)}). \quad (9)$$

This formula is all that we need here from the general category theory (we will give direct proofs of other relevant facts).

Proposition 4 $V^*(R')$ lives in the braided category of right $A(R)$-comodules: all its structure maps are fully covariant under the right coaction $V^*(R') \to V^*(R') \otimes A(R)$ given by $x \to x' = xt$
and the braiding $\Psi$ on $V^*(R')$ is the one inherited from this.

**Proof**  This is the meaning of condition (i) in Theorem 1, for this is implied by and essentially implies the identity

$$R'_{12}t_1t_2 = t_2t_1R'_{12}.$$  \hspace{1cm} (10)

Firstly, note that applying the fundamental and conjugate fundamental representations $\rho^+_2(t_1) = R_{12}$ and $\rho^-_2(t_1) = R_{21}^{-1}$ to (10) gives exactly the conditions (i) in Theorem 1, while in the reverse direction, repeatedly using the conditions (i) establishes (10) in all tensor powers of these representations and hence essentially corresponds to (10) abstractly. Given this, we have $x_1t_1x_2t_2 = x_1x_1t_2t_2 = x_2x_1R'_{12}t_1t_2 = x_2x_1t_2t_1R'_{12} = x_2t_2x_1t_1R'_{12}$. This means that the algebra is covariant. Covariance of the coalgebra $\Delta$ is immediate on the generators and hence holds in general because the braiding (used in extending $\Delta$ to products) is covariant. Covariance in both cases means that the relevant structure maps are intertwiners for the quantum group action, as we have explained in detail in [16]. Another approach is to verify (10) directly without worrying about the reconstruction. For, example, it holds in Examples 2, 3 and any similar examples just because $PR'$ is a function of $PR$ and id, both of which commute with $t_1t_2$. Finally, we check that the braiding on $V^*(R')$ is indeed the one that we have used in Theorem 1. Thus, from (11) we see that the matrix coaction induces the braiding $\Psi(x_1 \otimes x_2) = x_2 \otimes x_1R(t_1 \otimes t_2) = x_2 \otimes x_1R_{12}$. \hspace{1cm} \Box

Our next task is to extend this point of view to the case where the bialgebra $A(R)$ is replaced by a quantum group function algebra $A$ (such as $SL_q(n), SO_q(n)$ etc). These are quotients of $A(R)$ by further determinant-type (and other) relations. See [12] for the standard cases but note that the procedure works more generally as we have explained in [14, Sec. 3]. In the standard cases we know the result is again dual quasitriangular since the corresponding $U_q(g)$ is quasitriangular [13], but we showed in [14, Sec. 3.2.3] that the same situation prevails in the general case where the quotienting relations and pairing are defined from a generic $R$-matrix (we showed that $U(R)$ was formally quasitriangular as a map), provided $R$ is normalized correctly. In the present setting it means that $\mathcal{R}$ in (7)-(8) etc are not consistent with the determinant-type and other relations of $A$ unless $R$ is correctly normalized (the quantum-group normalization).
In the present paper the normalization of $R$ is fixed by the requirement (ii) of Theorem 1 and it is not in general the quantum group normalization. Hence on $A$ the dual quasitriangular structure takes the form $R(t_1 \otimes t_2) = \lambda R_{12}$ where $\lambda$ is a constant that takes us to the quantum group normalization (for the standard $R$-matrices it is a power of $q$). Hence $R(t_1 \cdots t_a \otimes t_{a+1} \cdots t_{a+b})$ acquires an extra factor $\lambda^{ab}$.

This $\lambda$ then spoils the last part of the proof of Proposition 4. The situation is that $V^*(R')$ does not in general live in the braided category of representations of the dual quasitriangular Hopf algebra $A$. Instead, we must extend $A$ slightly by adjoining a single invertible group-like element $g$ commuting with $t$. The coproduct is $\Delta g = g \otimes g$. This extended Hopf algebra $\tilde{A}$ is the tensor product of $A$ with the group algebra $CZ$. We also define a dual quasitriangular structure on $CZ$ by $R(g^a \otimes g^b) = \lambda^{-ab}$ along the lines in [11] but in a dual form. This extends to a dual quasitriangular structure $\tilde{R} : \tilde{A} \otimes \tilde{A} \to C$ as the tensor product one.

**Proposition 5** Let $A(R)$ be equipped with its initial dual quasitriangular structure as in (7)-(8), and $\tilde{A}$ with the dual quasitriangular structure as described. There is a map of dual quasitriangular bialgebras $A(R) \to \tilde{A}$ defined by $t \to tg$. Moreover, $V^*(R)$ lives in the braided category of $\tilde{A}$-comodules with coaction $x \to x' = xtg$.

**Proof** The first part follows from the skew bimultiplicativity property as already exploited in (7)-(8). For example, $R(t_1 g \otimes t_2 g) = R(g \otimes t_2 g)R(t_1 \otimes t_2 g) = R(g \otimes g)R(t_1 \otimes g)R(t_1 \otimes t_2) = \lambda^{-1} \lambda R_{12} = R_{12}$, which agrees with the value on the matrix generators of $A(R)$ as in (7)-(8). For the second part, we note that $V^*(R')$ is $Z$-graded by the degree of $x_i$. Thus we have a coaction of $CZ$ by $x \to x' = xg$ (extending to products as an algebra homomorphism so $x_1 \cdots x_a \to x_1 \cdots x_a g^a$). This coaction measures the degree (or in physical terms, the scaling dimension) of any homogeneous function of the $x_i$ and $V^*(R')$ is covariant under it. Also, $A$ coacts covariantly by $x \to xt$ since $A$ is a quotient of $A(R)$ and covariance under the latter was proven in the first part of Proposition 4. Moreover the two coactions are compatible and together they give the coaction of $\tilde{A}$ as stated. The final part of the proof of Proposition 4 now goes through much as before with $\Psi(x_1 \otimes x_2) = x_2 \otimes x_1 R(t_1 g \otimes t_2 g) = x_2 \otimes x_1 R_{12}$ as required. $\square$
Throughout this paper, if we want to work with only the bialgebra $A(R)$ (or its $GL(R)$ variant obtained by inverting some elements) then we do not need to introduce $g$ and can use Proposition 4. On the other hand, if we want to work with quantum groups such as $SL_q(n)$, $SO_q(n)$ etc, we have to work with their extensions by $g$ and use Proposition 5. We give the formulae in the latter case, but for the simpler formulae for $A(R)$ the reader can simply set $g = 1, \lambda = 1$ throughout.

For example, the braided-Minkowski space $\mathbb{R}^{1,3}$ lives in the braided-category of $\widetilde{SO}_q(1,3)$-comodules, a fact which enters into many constructions by forcing us to remember the braid statistics induced by this (in the same way as we remember the anti-commuting nature of Grassmann variables). It also means that $\widetilde{SO}_q(1,3)$ acts naturally on $\mathbb{R}^{1,3}$ in a way that respects its structure as both an algebra and coalgebra. The physical meaning for the coalgebra is that when this is used to make translations as in (5), the translation is $\widetilde{SO}_q(1,3)$-covariant in the sense that one may first translate and then rotate $x' \rightarrow x'tg$ or first rotate $x$ and the displacement $p$ and then make the displacement.

Mathematically, this covariance means that one may make a semidirect product by this coaction of $\tilde{A}$ and obtain necessarily a Hopf algebra (in analogy with the classical situation for the Poincaré group). This semidirect product construction applies to any Hopf algebra in a braided category of this type\cite{17}\cite{18}, a process that we have introduced and called bosonization because it turns a braided Hopf algebra into an ordinary one.

**Theorem 6** Let $V^*(R')$ be a braided-covector space as in Theorem 1 and Proposition 5. Its bosonization is the ordinary Hopf algebra $V^*(R')_{\triangleleft}\tilde{A}$. Explicitly, it has subalgebras $V^*(R')$, $\tilde{A}$ with generators $p$ and $t$ and cross-relations and coproduct

$$ pg = \lambda^{-1}gp, \quad p_1t_2 = \lambda t_2p_1R_{12}, \quad \Delta p = p \otimes tg + 1 \otimes p, \quad \Delta t = t \otimes t, \quad \Delta g = g \otimes g. $$

**Proof** We denote the generators of $V^*(R')$ now by $p_i$ as they will play the role of momentum. As explained in detail in \cite{18} Lemma 4.4, the right coaction of $\tilde{A}$ is turned by $R$ into a right action, which comes out as $p_1(t_2 \otimes t_2) = \lambda p_1R_{12}$ and $p \circ g = pR(tg \otimes g) = p\lambda^{-1}$. In fact, we have already obtained a similar action to that of $t$ (in another context) in \cite{14} Sec. 6.3 and one may check directly in the same way as there that $V^*(R')$ is covariant, while covariance under the action of $g$ is immediate. Given the covariant action, one may make a right-handed
semidirect product in a standard way. See [18, Sec. 4] for the right-handed conventions we use here. It is built on $V^*(R') \otimes \tilde{A}$ with cross relations $(p \otimes 1)(1 \otimes t) = \sum (1 \otimes t_{(1)})(p \otimes t_{(2)} \otimes 1)$ where $\Delta t = \sum t_{(1)} \otimes t_{(2)}$ is the coproduct of $\tilde{A}$. The present matrix form gives the result stated. Similarly for $(p \otimes 1)(1 \otimes g) = (1 \otimes g)(p \otimes 1)\lambda^{-1}$. Even more straightforwardly (but in a perhaps unfamiliar dual language) the coaction of $\tilde{A}$ allows us to make a cross coproduct of the coalgebra of $V^*(R')$ by the right coaction of $\tilde{A}$ as a right-handed version of the standard formulae as recalled in [14, Sec. 6]. This gives $\Delta p \otimes 1 = \sum(p_{(1)}^{(1)} \otimes 1) \otimes (1 \otimes p_{(1)}^{(2)})(p_{(2)} \otimes 1)$ which computes as stated. Here $\Delta p = \sum p_{1}\otimes p_{2}$ is the braided-coproduct and $\beta(p) = \sum p^{(1)} \otimes p^{(2)} = p \otimes t g$ is the coaction. There is also necessarily an antipode built from the antipode of $\tilde{A}$ and the braided-antipode. It is also uniquely determined from the bialgebra shown. $\Box$

This applies of course to $\mathbb{R}^{1,3}$. Its bosonization $\mathbb{R}^{1,3} \bowtie SO_q(1, 3)$ can be called the (right-handed) $q$-Poincaré group $P_q$.

There is also a left-handed version of the above constructions for which the formulae can be obtained by a symmetry principle. The left-handed Minkowski space is defined by $P_A x_1 x_2 = 0$ or $R' x_1 x_2 = x_2 x_1$ as in [3] and its bosonization gives the structure of the inhomogeneous quantum group in [1]. There are several other variants also possible. For example, the Poincaré-type Hopf algebra based on the (right-handed) braided-vectors $V(R')$ in [3] is

**Proposition 7** The bosonization of $V(R')$ is a semidirect product Hopf algebra $V(R') \bowtie \tilde{A}$ with cross relations and coproduct

$$\begin{align*}
x g &= \lambda g x, \quad x_1 t_2 = \lambda^{-1} t_2 R_{12}^{-1} x_1, \\
\Delta x^i &= x^j \otimes g^{-1} S t^i_j + 1 \otimes x^i, \quad \Delta t = t \otimes t, \quad \Delta g = g \otimes g.
\end{align*}$$

**Proof** Here $V(R')$ lives in the braided category of $\tilde{A}$-comodules by the right coaction $x \rightarrow g^{-1} t^{-1} x$ as an element of $V(R') \otimes \tilde{A}$. Its bosonization follows the same steps as in the proof of Theorem 6. Thus $\mathcal{R}$ turns the right coaction of $\tilde{A}$ into a right action $x_1 \triangleleft t_2 = \mathcal{R}(g^{-1} t_1^{-1} \otimes t_2) x_1 = \lambda^{-1} R_{12}^{-1} x_1$ and $x \triangleleft g = \mathcal{R}(g^{-1} t^{-1} \otimes g) = \lambda x$. The cross relations then come out from $(x_1 \otimes 1)(1 \otimes t_2) = (1 \otimes t_2)(x_1 \triangleleft t_2 \otimes 1)$ and similarly for $(x \otimes 1)(1 \otimes g)$. The semidirect coproduct by the right coaction $\beta(x^i) = \sum x^{(i(1)} \otimes x^{(i(2)} = x^j \otimes g^{-1} S t^i_j$ is then computed in the standard way. Note that $\Delta^{op}$ (the opposite coproduct) gives an equally good Hopf algebra which in the present example has the matrix form

$$\Delta^{op} x = x \otimes 1 + g^{-1} t^{-1} \otimes x. \quad (11)$$

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Theorem 6 and its variants such as in Proposition 7 have many applications, allowing us to develop the \( q \)-Poincaré group \( P_q \) along standard lines for a semidirect product. The most important is that it coacts covariantly on \( q \)-Minkowski space:

**Corollary 8** The semidirect product \( V^*(R') \rtimes \tilde{A} \) coacts covariantly on \( V^*(R') \). Denoting the generators of the latter by \( x_i \), the coaction is \( x'_i = xtg + p \).

**Proof** This is ensured by a general feature of bosonization that the (co)representations of the original braided-Hopf algebra correspond to the usual (co)-representations of the bosonization. The theorem (for modules) is [17, Thm. 4.2], which we use now in a dual form for comodules. The result, which can also be verified directly in our present setting, is

\[
\beta(x) = (\beta_{\tilde{A}} \otimes \text{id})\beta(x) = (\beta_{\tilde{A}} \otimes \text{id})(x \otimes 1 + 1 \otimes p) = x \otimes 1 \otimes t g + 1 \otimes p \otimes 1
\]

where \( \beta_{\tilde{A}} \) is the coaction of \( \tilde{A} \) on \( V^*(R') \). The result is written compactly as stated. The general theory of bosonization ensures not only that the result is a right coaction but that it is an algebra homomorphism (\( V^*(R') \rtimes \tilde{A} \) coacts covariantly). In our case, we can see it directly as

\[
\begin{align*}
x'_1 x'_2 &= (x_1 t_1 g + p_1)(x_2 t_2 g + p_2) = x_1 x_2 t_1 t_2 g^2 + p_1 p_2 + p_1 x_2 t_2 g + x_1 t_1 g p_2 \\
x'_2 x'_1 R'_{12} &= (x_2 t_2 g + p_2)(x_1 t_1 g + p_1) R'_{12} = x_2 x_1 R'_{12} t_1 t_2 g^2 + p_2 p_1 R'_{12} + x_2 t_2 g p_1 R'_{12} + p_2 x_1 t_1 g R'_{12}
\end{align*}
\]

The first two terms are clearly equal (using Proposition 5 for the first term) while

\[
\begin{align*}
p_2 x_1 t_1 g R'_{12} &= x_1 p_2 t_1 g R'_{12} = x_1 t_1 g p_2 R_{21} R'_{12} = x_1 t_1 g p_2 + x_1 t_1 g p_2 P(R - R') = x_1 t_1 g p_2 + x_2 t_2 g p_1 R_{12} - x_2 t_2 g p_1 R'_{12}
\end{align*}
\]

using in turn the cross-relations in Theorem 6, the condition (ii) in Theorem 1 and the action of the usual permutation \( P \). Using again the cross-relations in Theorem 6 we see that the expressions for \( x'_1 x'_2 \) and \( x'_2 x'_1 R'_{12} \) coincide. It is trivial to see that the linear braided-coproduct \( \Delta_0 \) on \( V^*(R) \) is likewise covariant. □

Similarly in the other conventions \( V(R') \rtimes \tilde{A} \) etc. As a second application, let us note that the dual of a semidirect product as an algebra and coalgebra is again a semidirect product as an algebra and coalgebra (the product and coproduct become interchanged). Details are in [18].

Hence an immediate corollary of our approach is
**Corollary 9** The dual of the $q$-Poincaré group in Theorem 6 or in [4] is also a semidirect product as an algebra and coalgebra $B \rtimes \tilde{H}$ where $B$ is dual to $V^*(R')$ and $\tilde{H}$ dual to $\tilde{A}$.

**Proof** This follows at once by dualizing the algebras and coalgebras in Theorem 6 in a standard way. Another more conceptual route to the same result is to dualise $V^*(R')$ in the braided category and bosonize the resulting braided-Hopf algebra $B$ according to the original bosonization formulae in [17, Thm. 4.1]. Here the categorical dual has the opposite algebra and opposite coalgebra to the usual dual, but has the advantage that it remains a Hopf algebra in the same braided category. The resulting $B$ is then viewed as a Hopf algebra in the category of left $\tilde{H}$-modules (rather than right $\tilde{A}$-comodules) and the bosonized algebra is the semidirect product by this action of $\tilde{H}$ and the bosonized coalgebra the semidirect product by the left coaction $\beta(b) = R_{21} \triangleright b$. □

For example, we obtain

$$U_q(p) = B \rtimes U_q(\text{so}(1, 3))$$

(12)

as a well-defined Hopf algebra, where $B$ is the braided-momentum group dual to $\mathbb{R}^{1,3}$ in Example 3 and $U_q(\text{so}(1, 3))$ is the quantum group dual to $SO_q(1, 3)$, acting by $q$-rotations and scaling.

As noted in the proof, there are two ways to compute this dualization. The first is to dualise the algebra and coalgebra of $\mathbb{R}^{1,3}$ and of $\mathbb{R}^{1,3}_c \rtimes SO_q(1, 3)$ a standard way, while another more natural way is to dualise $\mathbb{R}^{1,3}$ in the category and then bosonize it. Details will be given elsewhere. It should also be mentioned that this $U_q(p)$ does not appear to be a Hopf $\ast$-algebra in the sense of [19], but whether these are physically the right axioms in the first place (or whether something more general is needed as it seems for $U_q(p)$) is not clear: if $\ast$ does not respect the coproduct, it only means that the tensor product of unitary representations is not necessarily unitary. This is caused by the effective ‘interaction’ due to the braiding $\Psi$ and is not necessarily unphysical. See [18] for some initial results about $\ast$-structures on braided groups and semidirect products.

Finally, let us mention that the above constructions can be mixed to give various hybrid Poincaré-like ordinary Hopf algebras. For example, we can bosonize both $V(R')$ and $V^*(R')$ (and not their duals as in Corollary 9) but by the extended quantum enveloping algebra $\tilde{H}$ (and not by its corresponding quantum function algebra $\tilde{A}$ as in Theorem 6 or Proposition 7) to obtain ordinary Hopf algebras $V(R') \rtimes \tilde{H}$ and $V^*(R') \rtimes \tilde{H}$. To describe these, it is convenient
to write the quantum enveloping algebra $H$ (such as $U_q(so(1,3))$) with FRT-type generators $I^\pm = \{I^\pm_i\}$ as a quotient of a matrix bialgebra. Here the FRT approach is not limited to the standard $R$-matrices, as we have demonstrated in Sec. 3. Its extension $\tilde{H}$ is simply the tensor product by the Hopf algebra $(\mathbb{C}Z)^\ast = C(\mathbb{Z})$ (functions on $\mathbb{Z}$) with formal quasitriangular structure $R(a,b) = \lambda^{-ab}$. There are several ways to describe this Hopf algebra, for example as the group algebra of a circle via the Fourier transform. For our purposes a convenient description is to choose a basis $\{\delta_a\}$ of $\delta$-functions. Also useful is powers of $\gamma = \sum_a \lambda^a \delta_a$. Then the quasitriangular structure and the duality pairing with $\mathbb{C}Z$ are

$$\mathcal{R} = \sum_{a,b} \lambda^{-ab} \delta_a \otimes \delta_b = \sum_a \delta_a \otimes \gamma^{-a}, \quad < \gamma^a, g^b > = \lambda^{ab}. \quad (13)$$

Here $\Delta \delta_a = \delta_b \otimes \delta_{a-b}$ while $\Delta \gamma = \gamma \otimes \gamma$ is group-like.

**Proposition 10** Denoting the generators of $V(R')$ by $p^i$, its bosonization by the corresponding extended quantum enveloping algebra $\tilde{H}$ is as follows. $V(R') \ltimes \tilde{H}$ has cross relations and coproduct

$$\gamma p = \lambda^{-1} p \gamma, \quad I^+_2 p_1 = \lambda^{-1} R_{12}^{-1} p_1 I^+_2, \quad I^-_2 p_1 = \lambda R_{21} p_1 I^-_2, \quad \Delta p = p \otimes 1^- \gamma \otimes p, \quad \Delta I^\pm = I^\pm \otimes I^\pm.$$

**Proof** We use here the original form of the bosonization theorem by quantum enveloping algebras. The left action of $I^\pm$ is computed from the right coaction of $\tilde{A}$. For example, on $V(R')$ it is $I^+_2 > \triangleright p_1 = < g^{-1} St_1, I^+_2 > p_1 = < St_1, I^+_2 > p_1 = \lambda^{-1} R_{12}^{-1} p_1$ and $I^-_2 \triangleright p_1 = < g St_1, I^-_2 > p_1 = \lambda R_{21} p_1$ which then immediately gives the semidirect product algebra $(1 \otimes I^+_2)(p_1 \otimes 1) = \sum I^+_2(1) \triangleright p_1 \otimes I^+_2(2) = (I^+_2 \triangleright p_1 \otimes 1)(1 \otimes I^+_2)$ in the form stated. The action of $\gamma$ is $\gamma \triangleright p = < g^{-1} St, \gamma > p = < g^{-1}, \gamma > p = \lambda^{-1} p$ giving the semidirect product stated. For the coalgebra we make the semidirect coproduct by the coaction $\beta(p) = \mathcal{R}_{21} \triangleright p = \sum \mathcal{R}^{(2)} < \mathcal{R}^{(1)}, g^{-1} St > p = 1^- \sum \gamma^{-a} < \delta_a, g^{-1} > \otimes p = 1^- \gamma \otimes p$. Here we used $\sum \mathcal{R}^{(2)} < S \mathcal{R}^{(1)}, t > = 1^-$ for the part in $H$ and [13] for the part in $C(\mathbb{Z})$. Writing this coaction as $\beta(p) = \sum p^{(1)} \otimes p^{(2)}$, the resulting semidirect coproduct is $\Delta(p \otimes 1) = (p^{(1)} \otimes 1)(1 \otimes p^{(2)})^{(1)} \otimes (p^{(2)}^{(2)} \otimes 1) = p \otimes 1 + 1^- \gamma \otimes p$ as stated. □

There are analogous formulae for $V^\ast(R') \ltimes \tilde{H}$. For example, we have

$$\mathbb{R}^{1,3} \ltimes U_q(so(1,3)) \quad (14)$$
as a Hopf algebra that can be perhaps also be considered as some kind of hybrid $q$-Poincare group. There are plenty of other possibilities. For example the braided-tensor product of two braided-Hopf algebras is also a braided-Hopf algebra, which can then likewise be bosonized to obtained an ordinary Hopf algebra, for example $\left( R^{1,3} \otimes R^{1,3} \right) \vartriangleright U_q(\mathfrak{so}(1, 3))$ etc. The point we wish to make is that there is a genuine and rich braided linear algebra\cite{16} which can be used as freely as familiar classical linear algebra (but remembering the braid statistics) for constructions involving vectors and matrices.

4 \ q-Lorentz Group as a Double Cross Product

In the above we have assumed that the $q$-Lorentz group $SO_q(1, 3)$ is given as an $R$-matrix quantum group such as in \cite{3}. We conclude by mentioning that a version of this also has an abstract structure by which it is built up from smaller factors. Here again the necessary abstract mathematical constructions have been introduced previously by the author\cite{20,21}, as a double cross product. This is a generalization of a Hopf algebra semidirect product to the situation when both Hopf algebras act on each other. Thus, if $H_1$ acts on $H_2$ from the left by $\triangleright$ and $H_2$ acts back from the right by $\triangleleft$ in a compatible way, the result is a double cross product Hopf algebra $H_1 \bowtie H_2$. This coincides with $H_1 \otimes H_2$ as a coalgebra but has product

\begin{equation}
(a \otimes b)(c \otimes d) = \sum a(b_1)\triangleright c_1(b_2)\triangleleft c_2)d, \quad a, c \in H_1, \quad b, d \in H_2.
\end{equation}

where $\Delta a = \sum a_1 \otimes a_2$ etc. See \cite[Sec. 3.2]{20} for details. Here $H_1 \bowtie H_2$ contains $H_i$ as sub-Hopf algebras, with cross relations $(1 \otimes b)(c \otimes 1) = \sum (b_1)\triangleright c_1(1 \otimes b_2)\triangleleft c_2)$. We have introduced this construction some years ago and shown that every Hopf algebra which factorizes into two sub-Hopf algebras in a certain sense, is such a double cross product (this is a quantum analog of a Manin triple).

For example, we have shown that if $H_i$ act on the same space and as such generate a bigger Hopf algebra, it is a double cross product. This was the strategy behind the $q$-Lorentz group in \cite{3} where the quantum function algebra $SO_q(1, 3)$ is realized as two copies of $SL_q(2)$ with matrix generators $t, s$ say and cross relations

\begin{equation}
R_{12}s_1t_2 = t_2s_1R_{12}
\end{equation}
Here $R$ denotes the $SL_q(2)$ $R$-matrix. In fact, we had already studied exactly this Hopf algebra or bialgebra in [21, Thm 3.2] (previously to [3]) where we obtained it for a general $R$-matrix as the double cross product $A(R) \bowtie \bowtie A(R)$ (or a Hopf algebra quotient of it) coming from actions $s_1 \triangleright t_2 = R_{12}^{-1} t_2 R_{12}$ and $s_1 \triangleleft t_2 = R_{12}^{-1} s_1 R_{12}$. We did not study its $*$-structure or dual quasitriangular structure there.

This $A(R) \bowtie \bowtie A(R)$ (including the $q$-Lorentz group in the form $SL_q(2) \bowtie \bowtie SL_q(2)$) is an example of the following abstract construction for a dual-quasitriangular Hopf algebra $(A, R)$.

**Theorem 11** Let $(A, R)$ be a dual quasitriangular Hopf algebra. Then there is a double cross product Hopf algebra $A \bowtie \bowtie A$ built on $A \otimes A$ as a coalgebra and with product

$$(a \otimes b)(c \otimes d) = \sum R^{-1}(b(1) \otimes c(1))ac(2) \otimes b(2)dR(b(3) \otimes c(3)), \quad a, b, c, d \in A$$

This contains $A \otimes 1, 1 \otimes A$ as sub-Hopf algebras. The actions giving rise to this double cross product are induced by the left and right adjoint coactions, turned into left and right actions by $R$ as

$$b \triangleright c = \sum R(Sb \otimes c(1), Sc(3))c(2), \quad c \triangleleft b = \sum b(2)R((Sb(1))b(3) \otimes c).$$

**Proof** This is a case of [21, Thm 1.7] with $R \in (A \otimes A)^*$ used as an anti-self-duality pairing of $A$ with itself. We include here a direct proof for completeness. Thus, $A$ coacts on $A$ by the right adjoint coaction $b \mapsto \sum b(2) \otimes (Sb(1))b(3)$, this becomes converted by $R$ to a right action $\triangleright$ (just as in the proof of Theorem 6 above and [18, Lemma 4.4]). Similarly $A$ coacts on $A$ by the left adjoint coaction which becomes converted by $R^{-1} = R \circ (S \otimes \text{id})$ to the left action $\triangleleft$. One can check using the axioms of a dual-quasitriangular structure that these actions are compatible in the way needed in [21, Sec. 3.2] [21]. Hence we have a double cross product Hopf algebra. The product (15) can then be computed as

$$\sum b(1) \triangleright c(1) \otimes b(2) \triangleleft c(2) = \sum R^{-1}(b(1) \otimes c(1), Sc(1), Sc(3))c(1,2) \otimes b(2,2)dR((Sb(2,1))b(3,2) \otimes c(2))$$

$$\quad = \sum R^{-1}(b(1,1) \otimes c(1,1))R^{-1}(b(1,2) \otimes Sc(3))c(1,2) \otimes b(2,2)dR((Sb(2,1))b(3,1) \otimes c(2,1))R(b(2,3) \otimes c(2,2))$$

$$\quad = R^{-1}(b(1) \otimes c(1))R^{-1}(b(2) \otimes Sc(3))c(2) \otimes b(4)dR((Sb(3) \otimes c(4))R(b(5) \otimes c(5))$$

where we used that the dual-quasitriangular structure $R$ is a skew Hopf-bicharacter. This means that it is anti-multiplicative in its second input, with the result that $R^{-1}$ is multiplica-
tive in the sense $\mathcal{R}^{-1}(a \otimes bc) = \sum \mathcal{R}^{-1}(a_{(1)} \otimes b)\mathcal{R}^{-1}(a_{(2)} \otimes c)$. Using this again we can cancel $\mathcal{R}^{-1}(b_{(2)} \otimes Sc_{(3)})\mathcal{R}(Sb_{(3)} \otimes c_{(4)})$ and obtain the result stated in the theorem. \[\square\]

The resulting structure in Theorem 11 also works for $(A, \mathcal{R})$ a dual quasitriangular bialgebra. This was the basis behind the example $A(\mathcal{R}) \bowtie A(\mathcal{R})$ in [21]. On the other hand, the abstract form in Theorem 11 also makes clear a connection with an independent construction in [22] for the quantum double of a factorizable quantum group $H$ (such as $U_q(sl_2)$) which gives $D(H)$ as isomorphic to $H \otimes H$ as an algebra and a doubly-twisted coalgebra. They denoted the latter $H \otimes_{\mathcal{R}} H$ (the twisted square) and it is easy to see that it is exactly the dual Hopf algebra to our double cross product $A \bowtie A$ with $A$ dual to $H$. Thus, the dual of the quantum double of a factorizable quantum group, is isomorphic to the double cross product $A \bowtie A$. We have already shown in [23, Example 4.6] that the quantum double $D(H)$ for any $H$ is a double cross product $H^{\text{cop}} \bowtie H$ by mutual coadjoint actions. Thus both quantum doubles and their duals (in nice cases) are double cross products. In fact, the formulae for $D(H)$ and $A \bowtie A$ are strictly analogous, $\mathcal{R}$ playing the role of a self-pairing as mentioned above.

In particular, the $q$-Lorentz group in the form $SL_q(2) \bowtie SL_q(2)$ is also the dual of the quantum double of $U_q(sl_2)$ as proposed in [23]. In algebraic terms the latter is built on the algebra $U_q(sl_2) \otimes SL_q(2)$ with a doubly-twisted coproduct. The isomorphism $SL_q(2) \bowtie SL_q(2) \rightarrow D(U_q(sl_2))^*$ is

$$t \otimes 1 \mapsto SL^- \otimes t, \quad 1 \otimes s \mapsto SL^+ \otimes t.$$  

The same formulae hold for any dual-factorizable quantum function algebra $A$ of matrix type.

The abstract picture here is that for any $(A, \mathcal{R})$ the Hopf algebra $A \bowtie A$ in Theorem 11 comes equipped with Hopf algebra maps $\pi_1 : A \bowtie A \rightarrow H^{\text{cop}}$ (the dual $H$ of $A$ but with the opposite coalgebra) and $\pi_2 : A \bowtie A \rightarrow A$,

$$\pi_1(a \otimes b) = [(a \otimes \text{id})(\mathcal{R})][(\text{id} \otimes b)(\mathcal{R}^{-1})], \quad \pi_2(a \otimes b) = ab \quad (17)$$

and a Hopf algebra map $(\pi_1 \otimes \pi_2) \circ \Delta : A \bowtie A \rightarrow D(H)^*$ where $D(H)^*$ is built on $H^{\text{cop}} \otimes A$ as an algebra. In the dual-factorizable case this becomes an isomorphism according to results in [22] and the projections become dual to the the usual inclusions of $H$, $H^{\text{cop}}$ in $D(H)$. [22] also proved that that twisted square is quasitriangular (corresponding to the usual quasitriangular structure on $D(H)$). The dual version of this is
**Proposition 12** A ▷◁ A in Theorem 11 is dual quasitriangular with

\[ R((a \otimes b) \otimes (c \otimes d)) = \sum R^{-1}(d \otimes (ab)(1))R((ab)(2) \otimes c) \]

**Proof** This is by direct computation using the axioms of a dual quasitriangular structure and the structure of A ▷◁ A in Theorem 11. Note that the canonical inclusions \( A \subset A \bowtie A \supset A \) allow one to pull-back the dual quasitriangular structure on \( A \bowtie A \) to ones on each factor. The left factor recovers its initial dual quasitriangular structure, while the right factor recovers its inverse transpose one. □

For example, we conclude that \( A(R) \bowtie A(R) \) is dual-quasitriangular. The value of the dual quasitriangular structure on the generators is

\[ R((t_1 \otimes s_2) \otimes (t_3 \otimes s_4)) = R^{-1}_{41}R^{-1}_{42}R_{13}R_{23} \tag{18} \]

where we used the multiplicativity properties of \( R^{-1} \), \( R \) to write the dual quasitriangular structure on \( A \bowtie A \) equally well as a product of four copies of \( R \), and then evaluated. This also makes the connection with [22] transparent. On the other hand, because \( A(R) \bowtie A(R) \) is dual quasitriangular, we have in particular that

\[ \Lambda^K_B \Lambda^I_A R^A_J R^B_L = R^I_A K_B \Lambda^A_J \Lambda^B_L \tag{19} \]

where \( I = (i_0, i_1), J = (j_0, j_1) \) etc are multi-indices and

\[ \Lambda^{(i_0, i_1)}(j_0, j_1) = t^{i_0}_{j_0} \otimes s^{i_1}_{j_1}, \quad R^{(i_0, i_1)}(j_0, j_1) = R^{-1}_{k_1} a^{i_0}_{k_0} b^{i_1}_{k_1} R^{-1}_{l_1} a^{i_1}_{l_1} c R_{j_0}^{k_0} d R_{j_1}^{l_0} R_{j_0}^{l_0}. \tag{20} \]

From this we conclude at once that there is a bialgebra homomorphism \( A(R_{\infty}) \to A(R) \bowtie A(R) \) given by \( t^I_J \mapsto \Lambda^I_J = t^{i_0}_{j_0} \otimes s^{i_1}_{j_1} \), where \( R_{\infty} \) denotes the composite \( R \)-matrix in (20). For example, it means that \( SL_q(2) \bowtie SL_q(2) \) contains a quotient of the FRT bialgebra for the corresponding composite \( R \)-matrix. Note that there are several \( R \)-matrices that can be built up from four copies of an initial one, and the one in [24] appears different from the one used in [3] for a similar purpose and obtained by twistor considerations. Another easy abstract property of \( A \bowtie A \), motivated by [3] is

**Proposition 13** If \( A \) is a Hopf \( \ast \)-algebra over \( \mathbb{C} \) and \( R \) is of real type in the sense \( \overline{R(a \otimes b)} = R(b^\ast \otimes a^\ast) \) then \( A \bowtie A \) is a Hopf \( \ast \)-algebra with \( (a \otimes b)^\ast = b^\ast \otimes a^\ast \).
Proof. This is by direct computation. Thus,
\[
((a \otimes b)(c \otimes d))^* = \sum R(Sb_{(1)} \otimes c_{(1)})(ac_{(2)} \otimes b_{(2)}d)^*R(b_{(3)} \otimes c_{(3)})
\]
\[
= R(c^*_{(1)} \otimes S^{-1}(b^*_{(1)}))d^*b^*_{(2)} \otimes c^*_{(2)}a^*R(c^*_{(3)} \otimes b^*_{(3)}) = (d^* \otimes c^*)(b^* \otimes a^*)
\]
so that * on \( A \bowtie A \) is an anti-algebra homomorphism. We used here the assumed reality property of \( R \) and the various definitions. The last equality uses the invariance \( R \circ (S \otimes S) = R \) to write \( R(c^*_{(1)} \otimes S^{-1}(b^*_{(1)})) = R(Sc^*_{(1)} \otimes b^*_{(1)}) \). That * commutes with the tensor coalgebra structure of \( A \bowtie A \) and the other axioms for a Hopf *-algebra are straightforward. \( \square \)

For example, \( SU_q(2) \) at real \( q \) obeys the reality condition for its \( R \) and the last proposition gives the *-structure on \( SU_q(2) \bowtie SU_q(2) \) as

\[
t^i \!{}^*_j = Ss^j \!{}^i, \quad s^i \!{}^*_j = St^j \!{}^i,
\]
which is of the form used in \[18\].

In this way, we see that several constructions in the literature regarding the \( q \)-Lorentz group are related through the theory of double cross products. Other complicated quantum groups can surely likewise be understood by means of such semidirect or double-semidirect product constructions. For example, the Poincaré groups in \[5\] can perhaps be understood in terms of the general extension theory of Hopf algebras and bicrossproducts\[21\]. Also, there are signs that the twistor construction motivating \[3\] can be understood in terms of braided tensor products as in \[16\]. These are topics for further work.

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