A BILOCAL FIELD APPROACH TO THE LARGE-N EXPANSION OF TWO-DIMENSIONAL (GAUGE) THEORIES

Marco Cavicchi

I.N.F.N. - Sez. di Bologna and Dip. di Fisica,
Università di Bologna, Via Irnerio 46, I-40126 Bologna, Italy

Abstract

We consider a wide class of two-dimensional models as gauge theories, Gross-Neveu model, \( O(N) \) and \( CP^{N-1} \)-like models using a formalism based on the introduction of bilocal fields that permits to perform easily the large-N expansion of this set of models in a unified and general way. We mainly discuss the \( SU(N) \) gauge field theory minimally coupled to fermionic plus bosonic matter in the fundamental representation, and we obtain within the path integral approach exact equations for the particle spectrum, also in presence of renormalizable polynomial potentials. Finally, we discuss the correspondence between this new approach and the one previously used in the context of the \( O(N) \) vector models.

* E-mail: cavicchi@bo.infn.it / cavicchi@nbivax.nbi.dk
1 Introduction

One of the most interesting and still not fully understood problem of Quantum Field Theory concerns the calculation of the mass of relativistic bound states. This problem is even more complicated to solve in the case of QCD, the SU(3) gauge theory of quarks and gluons, that describes the strong interaction physics where the basic constituents are confined and perturbation theory cannot be used. The expectation is that the low energy spectrum consists of colorless mesons and baryons, but up to now the only available method seems to be based on numerical calculation using the lattice theory formulation of the theory.

The large-$N$ expansion technique proposed by 't Hooft [1] several years ago seems to be the most promising approach to obtain analytic results on the hadron spectrum. It is well known that in the limit in which the number $N$ of colours becomes very large the theory becomes much simpler, in the sense that only planar Feynman graphs survive. Furthermore, when $N \to \infty$ the theory only contains colorless, stable and noninteracting mesons with two-body decay and scattering amplitudes proportional to $\frac{1}{\sqrt{N}}$ and $\frac{1}{N}$, respectively (there is also a way of describing baryons as solitons of the effective Lagrangian in the large $N$ limit, but this will not interest us in this context).

Unfortunately, all efforts to try to solve four-dimensional large-$N$ QCD have failed up to now. The main problem is to find the semi-classical configuration, namely the master field [4], from which the action is dominated in the large-$N$ limit and whose fluctuations around the vacuum should give the particle spectrum of the theory.

The main reason of this failure is due to the fact that no method has yet been found to solve matrix models for a space time dimension $D > 2$. It is well known that the large $N$ expansion can be explicitly performed in vector models as for instance the $O(N)$ vector model (see for instance ref. [6] and references therein), the two-dimensional $CP^{N-1}$ model [7] and also in $QCD_2$ [2], [5] with matter in the fundamental representation of the gauge group where the gluon field can be eliminated by using its equation of motion.

In particular this set of models are solved by means of two slightly different methods. The vector-like models are solved in the large $N$ expansion by introducing a local composite field and by explicitly integrating over the fundamental fields (See for instance Refs.[6] and [7]). The $QCD_2$-like mod-
els are characterized by the fact that the number of components of the non-abelian gauge field goes to infinity in the large N expansion. The gauge field in two dimensions has no physical degrees of freedom and therefore can be eliminated by using its classical equation of motion. In this way one gets a non-local Coulomb interaction that is quartic in terms of the vector-like matter fields. The theory can then be solved in the large N limit by introducing a bilocal composite field as discussed in Refs. [9] and [23].

In this paper we consider a general two dimensional gauge theory with matter transforming as the fundamental representation of the gauge group (vector-like matter) and we solve it by introducing bilocal colorless fields in order to explicitly perform the large-N expansion. The introduction of a bilocal field is essential because the gluon field has a number of components that goes to \( \infty \) in the large \( N \) expansion. We also show how our formalism reduces to the usual formalism of pure vector-like models in the limit where the gauge coupling constant \( g^2 \rightarrow 0 \).

The work is organized as follows. In section 2 we write the Lagrangian for a \( SU(N) \) gauge theory minimally coupled to fermionic and bosonic matter in fundamental representation in the light-cone gauge.

In section 3 we consider the theory with only scalar fields present, at first in the free massive case, then adding with a self-interaction potential \( U(\phi^\dagger \phi) \); we solve it in the large N expansion and we show how to obtain from our solution the known results for the \( O(N) \) model in the case in which the gauge coupling \( g \) is set to 0. In the last subsection we constrain a (massless) interacting scalar field to satisfy \( \phi_i^\dagger(x)\phi_i(x) = N/2f \), to find some similarities with the \( CP^{N-1} \) model.

In section 4 we consider the theory with fermionic fields, already studied in [23], but adding current-current couplings; this kind of model is also compared with the extension of the \( CP^{N-1} \) model with quarks.

Finally in section 5 we study the full gauge theory, (fermionic plus bosonic matter in fundamental representation of the gauge group) showing how to obtain the corresponding of the 't Hooft equation [2] for all particle sectors (b-b, b-f and f-f bound states) and examining the effect of adding any renormalizable and polynomial self-interaction term. We write exact integral equations, that reduce themselves, in the case of zero gauge coupling, to the equations very similar to those previously found for the spectrum of the massive Thirring model; in this case we find the exact spectrum of states. We also examine the case of nonzero Yukawa couplings between the fermionic
and the bosonic $SU(N)$ fields.

## 2 Two-dimensional gauge theory with minimal coupling

We start with the Lagrangian

$$
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi} (i \slashed{D} - m_0) \psi + (D_\mu \phi)^\dagger (D^\mu \phi) - m_0^2 \phi^\dagger \phi \quad (2.1)
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_0 [A_\mu, A_\nu]$, $\slashed{D}_{\mu} = \gamma_\mu (\partial_\mu + ig_0 A_\mu T^a_{AB})$. All matter fields are in the fundamental representation of the gauge group $SU(N)$ and we have chosen $T^a$ to be a Hermitian matrix.

Then we move to the light-cone coordinate system $*$ and we interpret the $x^+$ coordinate as our ‘time’ coordinate. The Lagrangian is particularly simple in the light cone gauge $A_a^+ = 0$, because the nonabelian part of the field strength and his quadratic coupling with the boson field vanish:

$$
\mathcal{L}_{l.c.} = \frac{1}{2} (\partial_- A_a^+)^2 + \bar{\psi} (i \slashed{D} - m_0) \psi + \phi^\dagger (-\partial^2 - m_0^2) \phi - g_0 A_+^a J_a^- \quad (2.2)
$$

with

$$
J_a^- = \bar{\psi} T^a \gamma_\gamma \psi + i \phi^\dagger \gamma_\gamma \phi^- \quad (2.3)
$$

The equation of motion for the field $A_a^+$ does not contain any ‘time’ derivative, so we can eliminate it to get an effective (non local) Action involving only the matter fields.

$$
S_{\text{eff}} = \int d^2 x d^2 y \left\{ \bar{\psi} (i \slashed{D} - m_0) \psi + \phi^\dagger (-\partial^2 - m_0^2) \phi \right\} \delta_{xy} \frac{1}{2} g_0^2 J_a^a \frac{1}{\partial_x^2} J_a^a \quad (2.4)
$$

## 3 Bosonic theory

*The conventions are the same as in [23]
3.1 The ’t Hooft equation for the bosonic sector

We now want to study the theory in the case in which only scalar fields are present; the case with only fermions in the theory has already been treated [9, 23] with the same methods. The model without field self-interaction, sometime also called “Scalar QCD”, has been extensively studied [14] using the same techniques developed by ’t Hooft; we now want to derive the same results by our new approach. We rewrite the interaction term in the following form:

\[ S_{int} = \frac{1}{2} g_0^2 \int d^2 x d^2 y \left( \phi^\dagger(x) T^a \partial_2^{-} \phi(x) \right) G(x - y) \left( \phi^\dagger(y) T^a \partial_2^{-} \phi(y) \right) \]

(3.1)

with the Green’s function \( G \) defined by

\[-\partial^2 G(x - y) = \delta^{(2)}(x - y) \]

(3.2)

so that in momentum space \( G(k) = 1/k^2 \). In the following, for the sake of simplicity in the notation, we will use \( \partial_1 \) and \( \partial_2 \) instead of \( \partial_x^{-} \) and \( \partial_y^{-} \), respectively.

By using the relation, that is valid only for fields transforming according to the fundamental representation [23]

\[ \sum_a T^a_{AB} T^a_{CD} = \delta_{BC} \delta_{AD} - \frac{1}{N} \delta_{AB} \delta_{CD} \]

(3.3)

we see that it is convenient to introduce as in [23] a bilocal colour singlet field

\[ \sigma^{xy} \equiv \sum_A \phi_A^\dagger(y) \phi_A(x) \]

(3.4)

The \( O(1/N) \) term in eq. (3.3) is completely irrelevant to our purposes, and we will neglect it in the following.

With the help of (3.3) we can rewrite (3.1):

\[ \mathcal{L}_{int} = \frac{1}{2} g_0^2 G_{xy} \sigma^{xy} i \partial_1^+ i \partial_2^+ \sigma^{yx} \]

(3.5)

The jacobian of the change of variable from \( \phi^\dagger \phi \) to \( \sigma \) can easily be computed in the large-N limit (see [24], for example), and gives a contribution to the action that can be written formally as follows

\[ \delta S[\sigma] = -i N Tr \log \sigma \]

(3.6)
Now, rescaling $\sigma \rightarrow N\sigma$ and sending $N \rightarrow \infty$ keeping $g_0^2 N = g^2$ fixed, we see that the action is dominated by a very simple saddle point. The action in the large-$N$ limit is then given by:

$$S/N = -iT_r \log \sigma + \int d^2x d^2y \left[ -\delta_{xy} (\partial^2_x + m^2_0)\sigma^{xy} + \frac{1}{2} g^2 G_{xy} \sigma^{yx} i \partial_1 i \partial_2 \sigma^{yx} \right]$$

in configuration space and by

$$S/N = -iT_r \log \sigma + \int \frac{d^2p}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} (p^2 - m^2_0)\sigma(p,q)(2\pi)^2 \delta^{(2)}(p + q) + \frac{g^2}{2} \int \frac{d^2p_1}{(2\pi)^2} \frac{d^2p_2}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{(2p_1 + k)_- (2p_2 - k)_-}{k_-^2} \sigma(p_1, p_2) \sigma(k - p_2, -k - p_1)$$

in momentum space.†

The master field $\sigma_0$ is the bilocal field configuration that dominates the functional integral for large $N$. It satisfies the saddle point of the action

$$\frac{\delta S}{\delta \sigma} \bigg|_{\sigma = \sigma_0} = 0;$$

As explained in Ref. [23], translational invariance imposes to the

† We must give a suitable definition of $-\partial^{-2}$ or, in momentum space, of $1/k^2$. We can easily see that the equation

$$-\partial^2_x G(x-y) = \delta^{(2)}(x-y)$$

has the general solution

$$G(x) = \delta(x^+) \left( -\frac{1}{2}|x^-| - Bx^- + A \right)$$

The terms $A$ and $Bx^-$ are the zero-modes of the operator $\partial^2$. As it has been pointed out by Einhorn [3], the $B$ term corresponds to a background colored electric field, and without loss of generality it can be taken as zero; furthermore, the constant $A$ can be gauged away, so that we can conclude that it contains no physics. In his work, ’t Hooft [2] regularized his integrals by cutting off all momenta $|p_-| < \lambda$ where $\lambda$ was a constant to send to zero at the end of all calculations. It can be demonstrated that this procedure corresponds to the choice for the constant $A$ to be equal to $1/\pi \lambda$ so that in momentum space

$$\frac{1}{k_-^2} = \frac{1}{2} \left[ \frac{1}{(k_- + i\epsilon)^2} + \frac{1}{(k_- - i\epsilon)^2} \right] + \frac{1}{\pi \lambda} (2\pi) \delta(k_-) \equiv P \frac{1}{k_-^2} + \frac{2}{\lambda} \delta(k_-)$$

In the following, for those ”historical” reasons, we will keep this regularization prescription.
master field to be local i.e.

$$\sigma_0(p, q) = \sigma_0(p)(2\pi)^2\delta^{(2)}(p + q)$$  \hspace{1cm} (3.9)

A simple algebra permits to rewrite the master field equation as follows

$$\left( p^2 - m_0^2 - g^2 \int \frac{d^2k}{(2\pi)^2} \frac{(2p_+ + k_-)^2}{k_-^2} \sigma_0(p + k) \right) \sigma_0(p) = i$$  \hspace{1cm} (3.10)

Defining

$$\delta m^2 \equiv g^2 \int \frac{d^2k}{(2\pi)^2} \sigma_0(k)$$  \hspace{1cm} (3.11)

and using the following ansatz

$$\sigma_0(p) = i(p^2 - m^2 + \Gamma(p) + ie)^{-1}$$  \hspace{1cm} (3.12)

with $m^2 = m_0^2 + \delta m^2$, we get the equation for $\Gamma(p)$

$$\Gamma(p) = -g^2 \int \frac{d^2k}{(2\pi)^2} \frac{(2p_+ + k_-)^2}{k_-^2} \frac{i}{(p + k)^2 - m^2 + \Gamma(p + k) + ie}$$  \hspace{1cm} (3.13)

that has the solution

$$\Gamma(p) = \frac{g^2}{\pi} - \frac{g^2 |p_-|}{\pi \lambda}$$  \hspace{1cm} (3.14)

In the case of scalar fields, unlike the fermionic case, we needed to renormalize the mass (the self-mass is affected by the ultraviolet divergence of the boson loop).

The particle spectrum of the theory is obtaining by studying small fluctuation around the master field. Putting $\sigma = \sigma_0 + \frac{1}{\sqrt{N}}\delta\sigma$, we find the effective action for the fluctuations by keeping only the quadratic part in $\delta\sigma$, from which we get the equation for the fluctuations:

$$\delta\sigma(p, q) = ig^2\sigma_0(p)\sigma_0(-q) \int \frac{d^2k}{(2\pi)^2} \frac{(2p + k)(2q - k)}{k_-^2} \delta\sigma(k + p, -k + q)$$  \hspace{1cm} (3.15)

The bound state equation can now be obtained by simply following the same procedure as in Refs. [4], [23]. One defines $\delta\tilde{\sigma}(r, s) = \delta\sigma(p + q, (p - q)/2)$ and
after the integration of both sides of the previous equation over $ds_+/2\pi$, the previous equation becomes (let us choose $r_- > 0$):

\[
\left[ \frac{M^2}{2|s_+-\frac{r_-}{2}|} + \frac{M^2}{2|s_-\frac{r_-}{2}|} + \frac{g^2}{\pi \lambda} - r_+ \right] \varphi(s_-) = \frac{g^2}{2\pi} \int_{s_-\frac{r_-}{2}}^{s_+ \frac{r_-}{2}} \frac{dk_- (2s_- r_- + k_-)(-2s_- r_- - k_-)}{k_-^2 2|s_-\frac{r_-}{2}| 2|s_+\frac{r_-}{2}|} \varphi(s_- + k_-)
\]

(3.16)

The $\varphi$ field is simply defined as

\[
\varphi(s_-) = \int \frac{ds_+}{(2\pi)} \delta\tilde{\sigma}(r, s)
\]

(3.17)

(The $r$-dependence has been left implicit) and by definition $r$ is the total momentum of the two-parton system.

The $\lambda$-dependent part cancels applying our regularization

\[
\int dk_- k_-^2 \varphi(s_- + k_-) = 2\lambda \varphi(s_-) + P \int dk_- k_-^2 \varphi(s_- + k_-)
\]

(3.18)

Rescaling and shifting the variables, we then easily get the ’t Hooft equation for the bosonic sector ($r^2 = 2r_- r_+$):

\[
r^2 \varphi(x) = \left[ \frac{M^2}{x} + \frac{M^2}{1-x} \right] \varphi(x) - \frac{g^2}{\pi} P \int_0^1 \frac{dy}{(y-x)^2 \frac{2x}{2(1-x)}} \varphi(y)
\]

(3.19)

### 3.2 Self-interacting scalar field

In this subsection, we want to analyze the effects of adding a potential $V(\phi^\dagger \phi)$. In $D = 2$ dimensions we can choose an arbitrary potential:

\[
V(\phi^\dagger \phi) = \sum_{n=2}^M \frac{c_k}{k!} (\phi^\dagger \phi)^k
\]

(3.20)

without losing the renormalizability of the theory.
After the introduction of the bilocal field a potential term adds to the action a local term of the form

$$\delta S = - \sum_{n=2}^{M} \frac{c_k}{k!} \int d^2x d^2y \delta_{xy} (\sigma_{xy})^k$$  \hspace{1cm} (3.21)

After the rescaling \( \sigma \rightarrow N \sigma \) we see that in the large \( N \) expansion we have to keep constant the quantities \( \bar{c}_k = c_k N^{k-1} \). For simplicity we will only study a \( \frac{1}{2}(\phi^\dagger \phi)^2 \) potential, and let it be \( v = \bar{c}_2 = c_2 N \).

The new equation for the saddle-point is quite simple:

$$\left( p^2 - m_0^2 - g^2 \int \frac{d^2k}{(2\pi)^2} \frac{(2p_+ - k_-)^2}{k^2} \sigma_0(p + k) - v \int \frac{d^2k}{(2\pi)^2} \sigma_0(k) \right) \sigma_0(p) = i$$  \hspace{1cm} (3.22)

and it can be seen that the effect of the potential term on (3.10) is just to change the definition of the mass shift \( \delta m^2 \):

$$\delta m^2 \equiv (g^2 + v) \int \frac{d^2k}{(2\pi)^2} \sigma_0(k)$$  \hspace{1cm} (3.23)

We can then easily compute, as in the previous section, the equation for the fluctuations around the master field, getting the equation previously written by Ambjørn [13]:

$$r^2 \varphi(x) = \left[ \frac{M^2}{x} + \frac{M^2}{(1-x)} \right] \varphi(x) - \frac{g^2}{\pi} P \int_0^1 \frac{dy}{(y-x)^2} \frac{(x+y)(2-x-y)}{2x2(1-x)} \varphi(y) + \frac{v}{4\pi x(1-x)} P \int_0^1 dy \varphi(y)$$  \hspace{1cm} (3.24)

that is equal to eq. (3.19) with the addition of the last term coming from the quartic potential.

If \( g = 0 \) the gluon field, that is a matrix, decouples from the matter and we are left only with vector-like fields and it is well known that one needs to introduce only a local composite field (see for instance Ref. [14]) and not a bilocal one. Therefore if we set \( g = 0 \) the two methods must agree and this is what we are going to check in the following.

Let us start from the master field equation in eq. (3.10) for the case \( g = 0 \) whose solution is given by the expression in eq. (3.12) with \( \Gamma(p) = 0 \).
Integrating both terms of eq. (3.12) over $p$ and defining

$$-i\sigma_0 \equiv \delta m^2 = v \int \frac{d^2 k}{(2\pi)^2} \sigma_0(k)$$

we get, after a Wick rotation, the following equation

$$\frac{\sigma_0}{iv} = \int \frac{d^2 p}{(2\pi)^2} \frac{1}{-p^2 + m^2 - i\sigma_0}$$

that is identical to equation (2.25) of Ref. [6] with $4f = v, K_f = K_n = 0$. The r.h.s of eq. (3.26) is divergent and must be regularized. The renormalized equation is obtained by extracting the divergent piece from $\sigma_0$ (this divergent is due to the fact that $\sigma_0$ is the vacuum expectation value of a composite local field where the two constituent fields are taken at the same point) and having it to be cancelled by the divergence appearing in the r.h.s. of eq. (3.26). After this procedure one obtains a renormalized gap equation (See eq. (3.15) of Ref. [6]). In conclusion we have shown that, if $g = 0$ the master field equation reduces to the gap equation of the vector models.

Let us consider now the fluctuation equation for $g = 0$

$$\left[r^2 - \frac{m^2}{x(1-x)}\right] \varphi(x) = \frac{v}{4\pi} \frac{1}{x(1-x)} P \int_0^1 dy \varphi(y)$$

(3.27)

whose solution is given by

$$\varphi(x) = \frac{v}{4\pi} \frac{1}{r^2 x(1-x) - m^2} P \int_0^1 dy \varphi(y)$$

(3.28)

By integrating over $x$ we get the consistency condition

$$1 = \frac{v}{4\pi} P \int_0^1 \frac{dx}{r^2 x(1-x) - m^2} \equiv \frac{v}{4\pi} F(m, r^2)$$

(3.29)

that gives three cases, depending on the region in which we are looking for:

i) $r^2 > 4m^2$: (3.29) becomes

$$1 = \frac{v}{4\pi} \frac{4}{r^2} \frac{1}{2} \log \frac{1 + \sqrt{1 - \frac{4m^2}{r^2}}}{1 - \sqrt{1 - \frac{4m^2}{r^2}}}$$

(3.30)
or, parametrizing $r = 2m \cosh \gamma$, $\gamma > 0$:

$$\frac{\sinh 2\gamma}{2\gamma} = \frac{v}{4\pi m^2} \quad (3.31)$$

that has a unique solution iff $v > 4\pi m^2$, where we have a resonance of mass $\mu^2 = \bar{r}^2$ with $\bar{r}^2$ that satisfies (3.30).

ii) $0 < r^2 < 4m^2$: (3.29) becomes

$$1 = -\frac{v}{4\pi r^2} \sqrt{\frac{4m^2}{r^2} - 1} \arctan \frac{1}{\sqrt{\frac{4m^2}{r^2} - 1}} \quad (3.32)$$

that is, parametrizing $r = 2m \sin \theta$, $0 < \theta < \pi/2$:

$$\frac{\sin 2\theta}{2\theta} = -\frac{v}{4\pi m^2} \quad (3.33)$$

This condition can be satisfied only if $-1 < v/4\pi m^2 < 0$ (see Fig.1), and we have a bound state determined by (3.32).

iii) $r^2 < 0$. Obviously, if we find tachyonic solutions of (3.29) the theory is not defined. Writing $r^2 = -4m^2\nu^2$, $\nu > 0$ we get

$$\nu = -\frac{v}{4\pi m^2} \frac{1}{\sqrt{1 + \nu^2}} \log(\nu + \sqrt{1 + \nu^2}) \quad (3.34)$$

that admits solution iff $v/4\pi m^2 < -1$ (see Fig.2), so that this range of values for the coupling constant must be avoided.

We can find a correspondence between the approach proposed here and the methods that people used in the past. In the usual procedure for the large $N$ expansion in the context of the $O(N)$ vector model one gets a quadratic term for the fluctuations around the solution of the gap equation that can be written as (eq. (2.29) of Ref. [6]):

$$S_{\text{eff}}^{(2)} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \alpha(k) \Gamma(k^2) \alpha(-k) \quad (3.35)$$

$\Gamma(k^2)$ (not to be confused with our $\Gamma(p_-)$!) is defined by eq. (2.30) of Ref. [6]. The equation of motion for the Fourier transform of the fluctuation

\[\]
field $\alpha(x)$ gives $\Gamma(k^2) = 0$ (with $\Gamma(k)$, in the $D = 2$ case, that is given by eq. (3.30) of Ref. [3]). This condition is just equal to eq. (3.30), Wick rotated to Euclidean space and with $v = 4f$.

Another more direct way to find the connection between our approach and the one in terms of a local composite is the following. If we consider the equation of motion for the field $\delta \sigma$ obtained from the action (3.8) plus the self-interaction term, the Fourier transform of (3.21), setting $g = 0$ we have

$$\delta \sigma(p, q) = -i v \sigma_0(p) \sigma_0(-q) \int \frac{d^2k}{(2\pi)^2} \delta \sigma(k + p, -k + q)$$

(3.36)

It can be seen from (3.36) that the field $\sigma_0^{-1}(p) \delta \sigma(p, q) \sigma_0^{-1}(-q)$ depends only on $p + q$ and not by $p$ and $q$ separately. Therefore, we can identify the local field $\alpha$ of Ref. [6] with this combination of fields:

$$\alpha(p + q) = \sigma_0^{-1}(p) \delta \sigma(p, q) \sigma_0^{-1}(-q)$$

(3.37)

Substituting (3.37) into (3.36) we can eliminate $\alpha$ and we immediately get eq. (3.29) (after the introduction of a Feynman parameter and the integration over $k$).

The above discussion implies that the bilocal approach that is essential to treat large $N$ theories with matter transforming according to the fundamental representation of the gauge group (vector-like matter) reduces in the limit of zero gauge coupling constant to the standard large $N$ approach used in the pure vector models.

Finally it can be easily seen that a complete potential of the form (3.20) has just the effect to change the equation for the renormalization of mass (3.23):

$$\delta m^2 \equiv g^2 \int \frac{d^2p}{(2\pi)^2} \sigma_0(p) + \sum_{k=2}^M \frac{\bar{c}_k}{(k - 1)!} \left( \int \frac{d^2p}{(2\pi)^2} \sigma_0(p) \right)^{k-1}$$

(3.38)

Here we have to define a new renormalized scalar self-coupling to be used in the equations for the fluctuations

$$v_R \equiv V'' \left( \int \sigma_0 \right) = \sum_{k=2}^M \frac{\bar{c}_k}{(k - 2)!} \left( \int \frac{d^2p}{(2\pi)^2} \sigma_0(p) \right)^{k-2}$$

(3.39)
This procedure is in agreement with the one presented in Ref. [6].

From the above discussion, we have seen that the bilocal field formalism is a more general method than the one based on a local composite, that can be applied both to the pure vector models and to models containing a gauge field whose number of components goes to $\infty$ as $N \to \infty$.

### 3.3 A $CP^{N-1}$-like model in bilocal field approach

The model that we will discuss in this subsection is a massless scalar field, in the fundamental representation of the $SU(N)$ gauge group, minimally coupled to a gauge field in adjoint representation and subject to the constraint $\phi^i(x)\phi_i(x) = N/2f$. The main difference with respect to the $CP^{N-1}$ model is the presence of the kinetic term for the gauge field and the fact that the gauge field is not abelian, so that its number of components grows with $N$.

We therefore start with the model

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + (D_\mu \phi)^\dagger (D^\mu \phi)$$

(3.40)

The potential for the scalar field is replaced by the constraint

$$\phi^i(x)\phi_i(x) = N/2f_0$$

(3.41)

where $f_0$ is a dimensionless constant. We can express the constraint (3.41) by introducing a Lagrange multiplier in the action, so that the partition function will take the form

$$Z = \int D\sigma D\lambda e^{iS[\sigma] - i \int d^2x \lambda(x)(\sigma^x - N/2f_0)}$$

(3.42)

Where now $\lambda_x$ is a (local-field) Lagrange multiplier not to be confused with the constant previously used in the definition of $1/k^2$. After the usual rescaling $\sigma \to N\sigma$ we can obtain an action $S[\lambda, \sigma]$ that can be solved in the large-$N$ limit via a saddle point approximation.

The saddle point is obtained by imposing the two conditions:

$$\frac{\delta S}{\delta \sigma_{\sigma=\sigma_0,\lambda=\lambda_0}} = 0$$

(3.43)

$$\frac{\delta S}{\delta \lambda_{\lambda=\lambda_0,\sigma=\sigma_0}} = 0$$

(3.44)
They imply the following equations: the one for $\sigma$

$$i\sigma_0^{-1}(p, q) = p^2(2\pi)^2\delta^{(2)}(p + q) - \lambda_0(p + q) - g^2 \int \frac{d^2 k}{(2\pi)^2} \frac{(2p_ - + k_ -)^2}{k_ -^2} \sigma_0(p + k)$$

(3.45)

and

$$\int \frac{d^2 p_1}{(2\pi)^2} \sigma_0(p_1, l - p_1) = \frac{1}{2f_0} (2\pi)^2\delta^{(2)}(l)$$

(3.46)

The main difference between (3.43) and (3.44) is that $\lambda$, being a local field, has a constant vacuum expectation value. So we can write $\lambda_0(l) = \bar{\lambda} \cdot (2\pi)^2\delta^{(2)}(l)$ and the previous equations become

$$\Gamma(p) = -g^2 \int \frac{d^2 k}{(2\pi)^2} \frac{(2p_ - + k_ -)^2 - k_ -^2}{(p + k)^2 - m^2 + \Gamma(p + k) + i\epsilon}$$

(3.47)

and

$$\frac{1}{2f_0} = \int \frac{d^2 p}{(2\pi)^2} \frac{i}{p^2 - m^2 + \Gamma(p) + i\epsilon}$$

(3.48)

or

$$2\pi = \log \left( \frac{\mu_0^2}{m^2 - g^2/\pi} \right)$$

(3.49)

with the usual ansatz (3.12) for $\sigma_0$, and with $m^2 \equiv \bar{\lambda} + \delta m^2$. We needed a renormalization of $f$ ($2\pi/f_0 = 2\pi/f + \log(\Lambda^2/\mu_0^2)$) and $\mu_0^2$ is the renormalization scale. Eq. (3.49) is expression of the asymptotic freedom of the model, because the coupling constant goes to zero as the scale $\mu_0^2$ goes to infinity. Comparing (3.48) with (40) of [7], one gets some similarities; in fact, the only difference is the replacement of the mass parameter $m^2$ with $m^2 - g^2/\pi$: even in this gauge-interacting (and then nonlocal) version, the dependence on the cutoff of the coupling $f$ remains the same, and so it happens for his ‘asymptotic freedom’ property.

We then can see how the model with a potential is related to the one with a constraint. In both cases we get the same equation (compare eq. (3.47) with eq. (3.13)), but in one case the bare squared mass $m_0^2$ enters directly in the Lagrangian while in the other case is generated by the v.e.v. of the field $\lambda_x$. When the kinetic term for the gauge field is absent (as in the real $CP^{N-1}$ model) we simply have the gap-equation (3.48), and the v.e.v of the field $\lambda$.
plays the role of the physical mass $m^2$. We can now write the equation for the spectrum. It can be obtained by writing small fluctuations around the master field $\sigma = \sigma_0 + \frac{1}{\sqrt{N}} \delta \sigma, \lambda = \lambda_0 + \frac{1}{\sqrt{N}} \delta \lambda$. The equations that one obtains are practically identical to (3.24), apart a term $\delta \sigma_x \delta \lambda_x$ in coordinate space, that ensures $\delta \sigma_{xx} = 0$. It constrains the integral of the $\varphi$ field to be zero. The eqs. of motion for the fluctuations are (see eq. (3.24))

$$\left[ r^2 - \frac{M^2}{x(1-x)} \right] \varphi(x) = - \frac{g^2}{\pi} P \int_0^1 \frac{dy}{(y-x)^2} \frac{(x+y)(2-x-y)}{2x2(1-x)} \varphi(y) - \frac{\delta \lambda}{2x(1-x)}$$

(3.50)

and

$$\int_0^1 dy \varphi(y) = 0$$

(3.51)

If we define

$$F(m,r^2) \equiv P \int_0^1 dx \frac{dx}{r^2 x(1-x) - m^2}$$

(3.52)

we get, for $g = 0$, the equation $F(m,r^2) = 0$, that cannot be satisfied unless $r^2 = \infty$. If $g \neq 0$ we can just use the constraint $\int_0^1 dx \varphi(x) = 0$ to obtain $\delta \lambda$; using this equation back in the original equation we get

$$\left[ r^2 - \frac{M^2}{x(1-x)} \right] \varphi(x) = - \frac{g^2}{\pi} P \int_0^1 \frac{dy}{(y-x)^2} \frac{(x+y)(2-x-y)}{2x2(1-x)} \varphi(y) + \frac{U_\varphi(M,r^2)}{x(1-x)}$$

(3.53)

with

$$U_\varphi(M,r^2) \equiv \frac{g^2}{4\pi} \frac{1}{F(M,r^2)} P \int_0^1 \frac{dx dy}{r^2 x(1-x) - M^2} \frac{(x+y)(2-x-y)}{(y-x)^2} \varphi(y)$$

(3.54)

This equation is very similar to the one in (3.24), where now the term with $U_\varphi$ takes the place of the last term with $\frac{\nu}{4\pi}$ in eq. (3.24).
4 QCD2 with quartic fermion interaction

We can add to the Lagrangian of QCD\(_2\) a term quadratic in the fermion density \(\rho\) without destroying the renormalizability of the theory. The request of Lorentz invariance leaves us only three terms: \((\bar{\psi}\psi)^2\), \((\bar{\psi}\gamma_5\psi)^2\) and \((\bar{\psi}\gamma_\mu\psi)^2\). The resulting effective action will have a local term that is not linear in the \(\rho\)-densities.

The model is described by the following Lagrangian:

\[
\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i \bar{\psi} \gamma^\mu \partial_\mu \psi - m_0^{(1)} \bar{\psi} \psi - m_0^{(2)} \bar{\psi} \gamma_5 \psi - f_1 (\bar{\psi} \psi)^2 - f_2 (\bar{\psi} \gamma_5 \psi)^2 - f_3 (\bar{\psi} \gamma_\mu \psi)^2
\]

(4.1)

where the \(f_i\) are three dimensionless coupling constants. In the light-cone gauge, we can eliminate the gauge field \(A_\mu\) to get a nonlocal term in the action

\[
S = \int d^2x d^2y \left\{ \left[ i \bar{\psi} \gamma^\mu \partial_\mu \psi - m_0^{(1)} \bar{\psi} \psi - m_0^{(2)} \bar{\psi} \gamma_5 \psi - f_1 (\bar{\psi} \psi)^2 \right. \right.
\]

\[
- f_2 (\bar{\psi} \gamma_5 \psi)^2 - f_3 (\bar{\psi} \gamma_\mu \psi)^2 \delta_{xy} + \frac{1}{2} g^2 J^a \frac{1}{\partial^a} J^a \right\}
\]

(4.2)

We can write the effective action in the bilocal field formalism using as master field the (bilocal) matrix \([23]\)

\[
U = \begin{pmatrix}
\rho_R & \rho_- \\
\rho_+ & \rho_L
\end{pmatrix}
\]

(4.3)

We get (see \([22]\) for the calculation of the Jacobian)

\[
S_{eff}/N = Tr(DU + i \log U) + \frac{g^2}{2} \int d^2x d^2y G_{xy} U^{12}_{xy} U^{12}_{yx}
\]

\[
- \int d^2x d^2y \left\{ \frac{a}{2} \left[ (U^{11}_{xy})^2 + (U^{22}_{xy})^2 \right] + b U^{12}_{xy} U^{21}_{xy} + c U^{11}_{xy} U^{22}_{xy} \right\} \delta_{xy}
\]

(4.4)

where
\[ D(x, y) = \delta_{xy} \left( \begin{array}{cc} -\frac{m_0^{(1)}+m_0^{(2)}}{\sqrt{2}} & i\partial_- \\ i\partial_+ & -\frac{m_0^{(1)}-m_0^{(2)}}{\sqrt{2}} \end{array} \right) \]  

(4.5)

and \( a = (f_1 + f_2)N \), \( b = 2f_3N \), \( c = (f_1 - f_2)N \) are kept fixed as \( N \to \infty \). The theory is chiral invariant if \( m_0 = a = 0 \).

We can move to Fourier space, and if we impose the saddle-point condition

\[ \frac{\delta S}{U_0^2(-q, -p)} \bigg|_{U = U_0} = 0 \]  

(4.6)

we get

\[
0 = \left( D + \frac{i}{U_0} \right)^{ij}(p, q) + g^2\delta^{ij}\delta^{12} \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2} U_0^{12}(k + p, -k + q)
- \int \frac{d^2p_1}{(2\pi)^2} \frac{d^2q_1}{(2\pi)^2} M_0^{ij}(p_1, q_1) (2\pi)^2 \delta^{(2)}(p_1 + q_1 - p - q)
\]  

(4.7)

with

\[
M_0 = \begin{pmatrix} aU_0^{11} + cU_0^{22} & bU_0^{12} \\ bU_0^{21} & aU_0^{22} + cU_0^{11} \end{pmatrix}
\]  

(4.8)

Notice that, if \( a = b + c = 0 \) \( M_0 \) is proportional to \( U_0^{-1} \).

Multiplying both sides of the saddle point equation with \( U_0^{ji}(p, q) \) and choosing the usual translationally invariant ansatz \( M_0(p, q) = (2\pi)^2 \delta(p + q)M_0(p) \) we obtain

\[
((D - A)U_0)^{ii}(p) + i\delta^{ii} + g^2\delta^{12} \int \frac{d^2k}{(2\pi)^2} U_0^{12}(p + k)U_0^{ii}(p) = 0
\]  

(4.9)

where the matrix \( A \) is given by

\[
A^{ij} = \int \frac{d^2k}{(2\pi)^2} M_0^{ij}(k)
\]  

(4.10)

If one compares eq. (4.3) with eq.(13) of Ref. \[23\] one can see that the only difference between the two is the shift:

\[
\begin{align*}
m_0L &\rightarrow m_0L + A^{11} \\
p_- &\rightarrow p_- - A^{12} \\
p_+ &\rightarrow p_+ - A^{21} \\
m_0R &\rightarrow m_0R + A^{22}
\end{align*}
\]  

(4.11)
with \( m_{0L}, m_{0R} = (m_{0}^{(1)} \mp m_{0}^{(2)})/\sqrt{2} \). The solution of the saddle point equation is given by

\[
U_0(p) = -2D_F(p') \begin{pmatrix} -m_{0L} - A_{11}^{11} & p' \\ p' + \frac{1}{2p'} \Gamma(p') & -m_{0R} - A_{22}^{22} \end{pmatrix}
\]

(4.12)

with

\[
D_F(p') = \frac{i}{2p'p' - (m_0^{(1)} - m_0^{(2)} + \sqrt{2}A_{11}^{11})(m_0^{(1)} + m_0^{(2)} + \sqrt{2}A_{22}^{22}) + \Gamma(p') + i\epsilon}
\]

(4.13)

\[
\Gamma(p_) = \frac{g^2}{\pi} = \frac{g^2 |p_-|}{\pi \lambda}
\]

(4.14)

We have changed the definition of \( \Gamma(p) \) with respect to Refs. [2] and [23], so that we have replaced \( \Gamma(p_-) \leftrightarrow -p_- \Gamma(p_-) \); the definition of the shifted momenta is \( p'_{\pm} \equiv (p - \bar{a})_{\pm} \) with \( \bar{a}_+ = A_{21}^1, \bar{a}_- = A_{12}^1 \) (see eq. (4.11)).

Now we have to enforce eq. (4.10) as consistency condition. A simple calculation shows that \((A_{22}^{22} + A_{11}^{11})/2 \equiv \delta m^{(1)}/\sqrt{2}, (A_{22}^{22} - A_{11}^{11})/2 \equiv \delta m^{(2)}/\sqrt{2}, \) with

\[
\delta m^{(1)} = (a + c) \int \frac{d^2 p}{(2\pi)^2} \frac{2i m_1}{2p_p - m^2 + \Gamma(p) + i\epsilon}
\]

(4.15)

\[
\delta m^{(2)} = (a - c) \int \frac{d^2 p}{(2\pi)^2} \frac{2i m_2}{2p_p - m^2 + \Gamma(p) + i\epsilon}
\]

(4.16)

and we have defined the renormalized masses as \( m_i \equiv m_{0}^{(i)} + \delta m^{(i)} \), and \( m^2 \equiv m_1^2 - m_2^2 \), so that we get the dependence of the coupling constants on the renormalization scale \( \mu_0^2 \):

\[
\frac{2\pi}{a + c} = \frac{m_1}{\delta m^{(1)}} \log \left( \frac{\mu_0^2}{m^2 - g^2/\pi} \right)
\]

(4.17)

\[
\frac{2\pi}{a - c} = \frac{m_2}{\delta m^{(2)}} \log \left( \frac{\mu_0^2}{m^2 - g^2/\pi} \right)
\]

A similar behaviour was already known in the context of the study of the \( CP^{N-1} \) model with quarks, (see eq. (57) of [8], for instance), and as in eq. (3.49) the only difference, due to the gauge kinetic term, is to shift the
squared mass of an amount $g^2/\pi$. Furthermore, a careful analysis shows that after regularization $A^{12} = A^{21} = 0$, due to the fact that $U^{12}(p)$ is odd in $p$.

In conclusion, the only effect on the master field of the interaction terms is to renormalize the mass, just as in the Bosonic theory and $U_0$ is given by

$$U_0(p) = \frac{-2i}{2p_+p_- - m^2 + \Gamma(p_-) + i\epsilon} \left( \frac{-m_1 - m_2}{\sqrt{2}} \frac{1}{2p_- \Gamma(p_-)} - \frac{p_-}{m_1 + m_2} \right)$$  \hspace{1cm} (4.18)

The equation for the fluctuation can be obtained as in [23], where, using this formalism, we solved QCD$_2$ with many flavors and with a chiral mass term $m^{(1)} \bar{\psi} \psi + m^{(2)} \bar{\psi} \gamma_5 \psi$. For the sake of simplicity we choose $a = c$ and $m^{(2)} = 0$, so that $m_L = m_R = m/\sqrt{2}$. Defining

$$C^{ij}(p, -q) = \left( \begin{array}{cc} -\frac{m/\sqrt{2}}{|q_-|} & 1 \\ \frac{|p_-|/2}{|p_-|/2} & -\frac{m/\sqrt{2}}{|p_-|} \end{array} \right)$$ \hspace{1cm} (4.19)

the form of the resulting equations suggests the following ansatz for the solutions

$$\phi^{ij}(r, s_-) = \theta \left( \frac{r_-}{2} - |s_-| \right) C^{ij} \left( s_- + \frac{r_-}{2}, s_- - \frac{r_-}{2} \right) \varphi(s_-)$$ \hspace{1cm} (4.20)

so that finally we get after the usual rescaling of variables the correspondent of the 't Hooft equation for QCD$_2$ with quartic fermion interaction:

$$\left[ r^2 - \frac{M^2}{x(1-x)} \right] \varphi(x) = -\frac{g^2}{\pi} P \int_0^1 \frac{dy}{(y-x)^2} \varphi(y) + \frac{a + c}{2\pi} m^2 \left( \frac{1}{x} \int_0^1 \frac{\varphi(y)}{1-y} dy + \frac{1}{1-x} \int_0^1 \frac{\varphi(y)}{y} dy \right) + \frac{b}{2\pi} m^2 \left( \frac{1}{x(1-x)} \int_0^1 \varphi(y) dy + \int_0^1 \frac{\varphi(y)}{y(1-y)} dy \right)$$ \hspace{1cm} (4.21)

\hspace{1cm} § It is important to notice that in the case of $a = c$ and $m^{(i)}_0 = 0$, eq. (4.17) gives the scale of the spontaneous symmetry breaking

$$\delta m^2 = g^2/\pi + \mu_0^2 \exp \left( -\frac{2\pi}{a + c} \right)$$

We have a difference with respect to the analysis made in [19] due to the gauge shift $g^2/\pi$ that appears in the correction $\delta m^2$ but that cancels out in the shifted mass $M^2 = m^2 - g^2/\pi = \delta m^2 - g^2/\pi$ giving substantially the same result.
As in section 3, we can write the solution in the case $b \neq 0$, $a + c \neq 0$, $g = 0$. The previous equation becomes (we rescale $r^2 = m^2 \mu^2$)

\[
\left[ \mu^2 - \frac{1}{x(1 - x)} \right] \varphi(x) = \\
+ \frac{a + c}{2\pi} \left( \frac{1}{x} \int_0^1 \frac{\varphi(y)}{1 - y} \, dy + \frac{1}{1 - x} \int_0^1 \frac{\varphi(y)}{y} \, dy \right) \\
+ \frac{b}{2\pi} \left( \frac{1}{x(1 - x)} \int_0^1 \varphi(y) \, dy + \int_0^1 \frac{\varphi(y)}{y(1 - y)} \, dy \right)
\]

(4.22)

that gives

\[
\varphi(x) = \frac{b}{2\pi} \frac{A + \rho B + 2B x(1 - x)}{\mu^2 x(1 - x) - 1}
\]

(4.23)

where $A \equiv \int \phi(x)$, $B \equiv \int \phi(x)/x$, $\rho = (a + c)/b$. This solution is remarkably similar to the one obtained by Fujita and Ogura [18] for the massive Thirring model [21] with $N_f = 1$; indeed, due to the equality

\[
(\gamma^\mu)_{ab}(\gamma^\mu)_{cd} = \mathbb{1}_{ad} \mathbb{1}_{bc} - (\gamma_5)_{ad}(\gamma_5)_{bc}
\]

(4.24)

in the case of $N_f = 1$ our starting Lagrangian and the one of the Thirring model coincide for $a = c = b/2 = g^*/4$ where $g^*$ is the Thirring coupling constant (there is a different sign with respect to [18] due to the anticommutation of the fermions caused by $\Sigma_a T_{ij}^a T_{kl}^a$ and a factor of 2 that follows from our definition of the fermionic densities $\rho \sim \sqrt{2} \bar{\psi} \psi$).

As in [18] from eq. (4.23) we obtain the two equations

\[
A = \frac{b}{2\pi} \left[ F(\mu^2) \left\{ A + \left( 1 + \frac{2}{\mu^2} \right) B \right\} + \frac{2}{\mu^2} B \right] \\
B = \frac{b}{2\pi} \left[ \frac{\mu^2}{2} F(\mu^2) \left\{ A + \left( 1 + \frac{2}{\mu^2} \right) B \right\} + (A + B) \log \epsilon \right]
\]

(4.25)

where $\epsilon$ is a cutoff needed to regularize the divergent equations, and $F(\mu^2) \equiv F(1, r^2/m^2)$ (see (3.29)).

We must renormalize (4.25) by adding a mass counterterm that verifies

\[
\delta \mu^2 \varphi(x) = \frac{b}{2\pi} \left[ \epsilon_1 + \frac{\epsilon_2}{x(1 - x)} \right]
\]

(4.26)
Adding (4.22) to (4.26) we get

\[ \varphi(x) = \frac{b}{2\pi} \frac{A + \rho B + \epsilon_2 + (2B + \epsilon_1) x(1-x)}{\mu^2_R x(1-x) - 1} \]  

(4.27)

where \( \mu^2_R \equiv \mu^2 + \delta \mu^2 \). In the following we will omit the suffix \( R \) for short.

From (4.25) we observe that choosing \( \epsilon_1 = -b/2\pi \cdot 2(A + B^*) \log \epsilon \), \( \epsilon_2 = -b/2\pi \cdot \rho (A + B^*) \log \epsilon \), with \( B^* = B - b/2\pi \cdot (A + B^*) \log \epsilon \) we get

\[ \varphi(x) = \frac{b}{2\pi} \frac{A + \rho B^* + 2B^* x(1-x)}{\mu^2 x(1-x) - 1} \]  

(4.28)

Now both \( A \) and \( B^* \) are finite, so we can solve the equations

\[ A = \frac{b}{2\pi} \left[ F(\mu^2) \left\{ A + \left( \rho + \frac{2}{\mu^2} \right) B^* \right\} + \frac{2}{\mu^2} B^* \right] \]  

(4.29)

\[ B^* = \frac{b}{2\pi} \frac{\mu^2}{2} F(\mu^2) \left[ A + \left( \rho + \frac{2}{\mu^2} \right) B^* \right] \]

getting

\[ \frac{b}{\pi} \left[ \rho \frac{\mu^2}{4} + 1 + \frac{b}{4\pi} \right] F(\mu^2) = 1 \]  

(4.30)

Let us first examine the case \( \rho = 1 \). We can find bound states if (4.30) has a solution for \( 0 < \mu^2 < 4 \). With the usual parametrization \( \mu = 2 \sin \theta \), \( 0 < \theta < \pi/2 \), we have to solve the equation \( (\lambda = -b/2\pi) \)

\[ \lambda^2 - \lambda (3 - \cos 2\theta) + \frac{\sin 2\theta}{2\theta} = 0 \]  

(4.31)

that, solved with respect to \( \lambda \), gives

\[ 2\lambda = 3 - \cos 2\theta \pm \sqrt{(3 - \cos 2\theta)^2 - 4 \frac{\sin 2\theta}{2\theta}} \]  

(4.32)

When \( \theta \) varies from 0 to \( \pi/2 \), \( \lambda_- \) goes from 1 to 0, whereas \( \lambda_+ \) goes from 4 to 1; this gives the allowed range for the coupling constant in order to have bound states: \( 0 < \lambda < 4 \). In Fig.3 we plot the two branches of \( 2\lambda(\theta) \), and one can note the singularity at \( \lambda = 1 \) of the function \( \theta(\lambda) \), that will be analyzed later. If \( \lambda = 1 \) \( (b = -2\pi) \) we have \( \mu^2 = 0 \) i.e. at the singularity we find
the well known massless pseudoscalar bound state of the massive Thirring model [17], that corresponds to the fundamental boson of the sine-Gordon theory [16].

Finally, we want to investigate the presence of tachyons in our model by searching a solution of (4.30) in the range $\mu^2 < 0$. Putting $\mu^2 = -4\nu^2$ we get

$$\lambda^2 - 2\lambda (1 - \nu^2) + \frac{\nu\sqrt{1 + \nu^2}}{\log(\nu + \sqrt{1 + \nu^2})} = 0 \quad (4.33)$$

The equation has solutions iff $\lambda < 0$ (see Fig.4); at $\lambda = 1 - \nu_0^2$ where $\nu_0 \approx 1.59803$ we have the ‘lightest’ tachyon, with mass $\mu_0^2 = -4\nu_0^2$; for any value of $\lambda > 0$ eq. (4.33) doesn’t have any solution, so we conclude that $\lambda > 0$ is the physical range of the coupling constant. In the tachyonic sector we have the following limits:

$$\mu^2(\lambda) \approx -e^{-\frac{1}{\lambda}} \quad \lambda \to 0^+$$
$$\mu^2(\lambda) \approx 2\lambda \quad \lambda \to -\infty \quad (4.34)$$

For a generic $\rho$ we have

$$\lambda^2 - \lambda [2 + \rho(1 - \cos 2\theta)] + \frac{\sin 2\theta}{2\theta} = 0 \quad (4.35)$$

that, solved with respect to $\lambda$, gives

$$2\lambda = 2 + \rho - \rho \cos 2\theta \pm \sqrt{(2 + \rho - \rho \cos 2\theta)^2 - 4\frac{\sin 2\theta}{2\theta}} \quad (4.36)$$

Now the allowed range of $\lambda$ goes from $1 + \rho - |1 + \rho|$ to $1 + \rho + |1 + \rho|$; when $0 < \theta < \pi/2$, $\lambda$ is always positive if $\rho > -1$, and always negative if $\rho < -1$. We can also see that for any $\rho \theta = 0$ implies $\lambda = 1$ i.e. at the critical point $\lambda = 1$ ($b = -2\pi$) we have a massless bound state.

Furthermore, if $\rho < \rho_c = -1/3$ the function $\theta(\lambda)$ has a turning point at $\theta = \theta_0 > 0$, as it can be seen from the expression of the argument of the square root in (4.36) for small $\theta$:

$$(2 + \rho - \rho \cos 2\theta)^2 - 4\frac{\sin 2\theta}{2\theta} \approx (2/3 + 2\rho)4\theta^2 \quad (4.37)$$

The behaviour of the argument of the square root in (4.36) is shown in Fig.5. We have forbidden values of $\theta$ if the function is negative. The value of the
turning point \( \theta_0 \) increases for negative \( \rho \) and covers the whole range of \( \theta \) at \( \rho = -1 \) (Fig.6); after that, the gap decreases in extension. The point \( \rho = -1 \) is very important, because it is the supersymmetric point in the full theory with fermions coupled to bosons [8]. At \( \rho = -1 \) we can only have the massless bound state at the critical point \( \lambda = 1 \), because (4.36) has no real solutions except \( \lambda = 1 \) (\( \theta = 0 \)).

We can also see that near the value \( \lambda = 1 \) the angle \( \theta \) has the behaviour

\[
\theta(\lambda) \approx \begin{cases} 
\frac{3\sqrt{5}}{\sqrt{3+\sqrt{5}}} \sqrt{1-\lambda} & \lambda \to 1^- \\
\frac{3\sqrt{5}}{\sqrt{3-\sqrt{5}}} \sqrt{\lambda-1} & \lambda \to 1^+ 
\end{cases} \quad (4.38)
\]

if \( \rho = -1/3 \), and

\[
\theta(\lambda) \approx \sqrt{\frac{1}{2\rho + 2/3}} |1-\lambda| \quad \lambda \to 1 
\]

(4.39)

if \( \rho > -1/3 \), that shows the transition at \( \rho = -1/3 \) and the singular behaviour of \( \theta(\lambda) \) near \( \lambda = 1 \).

We can see more clearly what’s happening if we investigate the tachyonic sector. The coupling constant is given by

\[
\lambda = 1 - \rho \nu^2 \pm \sqrt{(1 - \rho \nu^2)^2 - \frac{\nu \sqrt{1 + \nu^2}}{\log(\nu + \sqrt{1 + \nu^2})}} \quad (4.40)
\]

In Fig.7 we can see the behaviour of the argument of the square root in (4.40): the function \( \nu(\lambda) \) has a turning point \( \nu_0 > 0 \) (for example \( \nu_0 \approx 1.59803 \) if \( \rho = 1 \)) that disappears at \( \rho \leq \rho_c = -1/3 \), the critical point seen before.

The special case \( \rho = 0 \) contains no tachyons for any value of \( \lambda \), since it can be seen that (4.40) has no real solutions with nonzero \( \nu \).

Thus, it has turned out that in the tachyonic region \( \lambda \) has always the opposite sign of \( \rho \): we have tachyons in our theory if \( \rho \lambda < 0 \); since \( \lambda = -b/2\pi \), \( \rho = (a + c)/b \) we can conclude that the presence of tachyons depends only on \( a + c \): we have tachyons in the theory if \( a + c > 0 \). Our theory of self-coupled fermions is well defined provided that the coefficient of \( (\bar{\psi}\psi)^2 \) in our starting lagrangian (4.1) is positive (i.e. \( f_1 < 0 \)).

In conclusion, in the free-tachyon sector we have a bound state for
$2(1 + \rho) < \lambda < 0$ if $\rho < -1$ and for $0 < \lambda < 2(1 + \rho)$ if $\rho \geq 0$. The mass of the bound state is given by $M = 2m_F \sin \theta(\lambda)$, with $\theta(\lambda)$ given by (4.36). If $\rho = -1$ the only allowed value for the coupling constant is $\lambda = 1$, for which the tachyon disappears and we only have a massless bound state; in the range $-1 < \rho < 0$ we cannot have bound states without having also tachyons, so we won’t examine this range of the parameter $\rho$ (and the point $\rho_c = -1/3$) far in detail.

We finally observe that an interesting case is $b = 0$, $a + c = 2f_1N \neq 0$, that corresponds to the Gross-Neveu model [19] if one identifies $f_1 \leftrightarrow -g^2/2$, $g^2$ being the Gross-Neveu coupling constant (not to be confused with the gauge coupling), as used in [19]. One can see that the previous calculations lead to

$$1 = \frac{a + c}{4\pi} \mu^2 F(\mu) \tag{4.41}$$

that admits a bound state (note that such a bound state is never massless) if $a + c < 0$, given by the solution of $(\mu = 2\sin \theta)$

$$1 = -\frac{a + c}{\pi} \theta \tan \theta \tag{4.42}$$

If $a + c > 0$ equation (4.41) has a solution in the tachyonic sector and the theory does not admit a ground state [19] [20]. This confirms the results obtained before.
5 Complete gauge theory

Finally, now that we’ve got experienced in using this formalism, we are ready to solve the full gauge theory (2.1):

\[
\mathcal{L} = - \frac{1}{4} F_\mu^a F^{a\mu} + \bar{\psi}(i\slashed{\partial})\psi + (D_\mu \phi)\dagger(D^\mu \phi) - \tilde{V}(\phi^\dagger \phi) - \tilde{W}_1(\phi^\dagger \phi)\bar{\psi}\psi
\]

\[
- \tilde{W}_2(\phi^\dagger \phi)\bar{\psi}\gamma_5 \psi - F_1(\phi^\dagger \phi)(\bar{\psi}\psi)^2 - F_2(\phi^\dagger \phi)(\bar{\psi}\gamma_5 \psi)^2 - F_3(\phi^\dagger \phi)(\bar{\psi}\gamma_{\mu} \psi)^2
\]

where we require the potentials to be polynomials in $\phi^\dagger \phi$. Using the light cone gauge we are able to remove $A_\mu^a$ from the Lagrangian:

\[
\mathcal{L} = \bar{\psi}(i\slashed{\partial})\psi + \phi^\dagger (-\partial^2)\phi - \tilde{V}(\phi^\dagger \phi) - \tilde{W}_1(\phi^\dagger \phi)\bar{\psi}\psi - \tilde{W}_2(\phi^\dagger \phi)\bar{\psi}\gamma_5 \psi
\]

\[
- F_1(\phi^\dagger \phi)(\bar{\psi}\psi)^2 - F_2(\phi^\dagger \phi)(\bar{\psi}\gamma_5 \psi)^2 - F_3(\phi^\dagger \phi)(\bar{\psi}\gamma_{\mu} \psi)^2 + \frac{1}{2} g^2 J_\mu J_\mu^a \frac{1}{\partial^2} J^a
\]

This is our (nonlocal) starting Lagrangian, with $J^a$ that is given by (2.3):

\[
J^a_\mu = \bar{\psi} T^a \gamma_\mu \psi + i \phi^\dagger T^a \gamma_\mu \phi
\]

5.1 The ’t Hooft equation for fermion-boson bound states

We begin discussing the case

\[
\tilde{V}(\phi^\dagger \phi) = m_{0B}^2 \phi^\dagger \phi
\]

\[
\tilde{W}_1(\phi^\dagger \phi) = m_0^F
\]

\[
\tilde{W}_2(\phi^\dagger \phi) = 0
\]

\[
F_i(\phi^\dagger \phi) = 0
\]

First, we eliminate the unnecessary degree of freedom $\psi_L$ from the Lagrangian (its kinetic term only contain $\partial_-$), and in the following we will define $\psi \equiv 2^{\frac{1}{2}} \psi_R$ (this is to remove boring $\sqrt{2}$ terms out the Lagrangian);
then, to rewrite the whole interaction Lagrangian, we have to define four bilocal fields:

\[ \rho_{xy} \equiv \sum_{A} \bar{\psi}_{A}(y)\psi_{A}(x) \]  
\[ \sigma_{xy} \equiv \sum_{A} \phi_{A}^{\dagger}(y)\phi_{A}(x) \]  
\[ \chi_{xy} \equiv \sum_{A} \phi_{A}^{\dagger}(y)\psi_{A}(x) \]  
\[ \bar{\chi}_{xy} \equiv \sum_{A} \bar{\psi}_{A}(y)\phi_{A}(x) \]

In terms of the previous bilocal the interaction Lagrangian becomes:

\[ L_{\text{int}} = \frac{g^2}{2} G_{xy}(\rho_{yx} \rho_{xy} + \sigma_{yx} \sigma_{xy} - \chi_{yx} \bar{\chi}_{xy} + \bar{\chi}_{yx} \chi_{xy}) \]  
\[ \text{(5.9)} \]

Now, it is a bit more difficult to get the Jacobian of the change of variables

\[ \mathcal{D} \bar{\psi} \mathcal{D} \psi \mathcal{D} \phi^{\dagger} \mathcal{D} \phi \rightarrow \mathcal{D} \rho \mathcal{D} \sigma \mathcal{D} \bar{\chi} \mathcal{D} \chi \]

it is expressed by:

\[ J[\rho, \sigma, \bar{\chi}, \chi] = \int \mathcal{D} \bar{\psi} \mathcal{D} \psi \mathcal{D} \phi^{\dagger} \mathcal{D} \phi \delta(\rho - \bar{\psi}\psi)\delta(\sigma - \phi^{\dagger}\phi)\delta(\bar{\chi} - \bar{\psi}\phi)\delta(\chi - \phi^{\dagger}\psi) \]

\[ \text{(5.10)} \]

By using auxiliary fields to exponentiate the \( \delta \)'s, we can calculate the Jacobian in the large-N limit:

\[ J = \exp(-N \text{STr} \log U) \]

\[ \text{(5.11)} \]

with

\[ U = \begin{pmatrix} \rho_{xy} & -\chi_{xy} \\ \bar{\chi}_{xy} & -\sigma_{xy} \end{pmatrix} \]

\[ \text{(5.12)} \]

STr is the Supertrace, defined as

\[ \text{STr} \left( \begin{array}{cc} m_{1} & \mu_{2} \\ \mu_{1} & m_{2} \end{array} \right) \equiv \text{Tr}(m_{1}) - \text{Tr}(m_{2}) \]

\[ \text{(5.13)} \]

with \( m_{i} \) and \( \mu_{i} \) that are matrices with commuting and anticommuting elements, respectively. This explains the choice of the – signs in (5.12).
By rescaling all fields by $N$, keeping $g^2N = g^2$ constant we get the full action of the model:

$$S/N = Tr_{pq} Str(DU + i \log U)$$

$$+ \frac{1}{2} g^2 \int \frac{d^2p_1}{(2\pi)^2} \frac{d^2p_2}{(2\pi)^2} \frac{d^2k}{(2\pi)^2} \frac{c_{ij}(p_1, p_2, k)}{k^2} U^{ij}(p_1, p_2) U^{ji}(k - p_2, -k - p_1)$$

(5.14)

with

$$D^{ij}(p, q) = \begin{pmatrix} p_+ - \frac{m_0^2}{2p_-} & 0 \\ 0 & p^2 - m_0^2 \end{pmatrix} (2\pi)^2 \delta(2)(p + q)$$

(5.15)

$$c_{ij}(p, q; k) = \begin{pmatrix} 1 & -(2q_- - k_-) \\ (2p_+ + k_-) & (2p_- + k_-)(2q_- - k_-) \end{pmatrix}$$

(5.16)

The saddle point equation is obtained by imposing

$$\frac{\delta S}{\delta U^{ij}(-q, -p)} \bigg|_{U = U_0} = 0$$

(5.17)

that gives

$$U_0^{ij}(p, q) = U_0^{ji}(p) \delta^{ij} \delta^{(2)}(p + q)$$

(5.18)

with

$$U_0^{11}(p) = \rho_0(p) = \frac{-2ip_-}{2p_+ p_- - M_F^2 - \frac{\alpha}{\pi} |p| + i\epsilon}$$

(5.19)

and (note the sign due to the Supertrace)

$$- U_0^{22}(p) = \sigma_0(p) = \frac{i}{2p_+ p_- - M_B^2 - \frac{\alpha}{\pi} |p| + i\epsilon}$$

(5.20)

The equation for the bound states can now be trivially obtained. We start from the equation for the fluctuations

$$(1 - 2\delta^{ij}) \delta U^{ij}(p, q) = -ig^2 U_0^{ij}(p) U_0^{ji}(-q) \int \frac{d^2k}{(2\pi)^2} \frac{c_{ij}(-q, -p; k)}{k^2} \delta U^{ij}(k + p, -k + q)$$

(5.21)
that is the generalized equation for the 't Hooft "blob", which has a leg made of a particle of type \( i \) and the other one made of a particle of type \( j \). The \( \delta U^{11} \) component leads to the 't Hooft equation [2], already obtained within this formalism in [23]; the \( \delta U^{22} \) component gives (3.19). Finally the other two components \( \delta U^{12} \) or \( \delta U^{21} \) give the equation for the scalar-fermion bound states:

\[
 r^2 \varphi(x) = \left[ \frac{M_B^2}{x} + \frac{M_F^2}{(1-x)} \right] \varphi(x) - \frac{g^2}{\pi} P \int_0^1 dy \frac{dy}{(y-x)^2} \frac{(x+y)}{2x} \varphi(y) \quad (5.22)
\]

recently written by K. Aoki [26].

5.2 Solving the most general Lagrangian

Our method works also in the case of an arbitrary potential of the form

\[
\tilde{V}(\sigma) = m_0^2 B \sigma + \sum_{n=2} v_n \frac{\sigma^n}{n! N^{n-1}}
\]

\[
\tilde{W}_i(\sigma) = m_0^{(i)} + \sum_{n=1} w_n^{(i)} \frac{\sigma^n}{n! N^n}
\]

\[
F_i(\sigma) = \sum_{n=0} f_n^{(i)} \frac{\sigma^n}{n! N^{n+1}}
\]

(5.23)

Here the crucial difference is that we cannot remove anymore \( \psi_L \) as before and therefore we have to introduce a 3 \( \times \) 3 bilocal field matrix:

\[
U = \begin{pmatrix}
\rho_R & \rho_- & -\chi R \\
\rho_+ & \rho_L & -\chi L \\
\chi L & \chi R & -\sigma
\end{pmatrix}
\]

(5.25)

As a consequence we get

\[
D(p, q) = \begin{pmatrix}
-\frac{m_0^{(1)} + m_0^{(2)}}{\sqrt{2}} & p_- & 0 \\
-\frac{m_0^{(1)} - m_0^{(2)}}{\sqrt{2}} & 0 & 0 \\
0 & 0 & p^2 - m_0^2
\end{pmatrix} (2\pi)^2 \delta^{(2)}(p + q)
\]

(5.26)

and the gauge part now couples the fermionic and the bosonic sectors.
We can also define

\[ \tilde{V}(N\sigma) = N \left[ m_{0B}^2 \sigma + V(\sigma) \right] \]
\[ \tilde{W}_i(N\sigma) = m_i^{(i)} + W_i(\sigma) \]
\[ \mathcal{A}(\sigma) = N[F_1(N\sigma) + F_2(N\sigma)] \]
\[ \mathcal{B}(\sigma) = 2NF_3(N\sigma) \]
\[ \mathcal{C}(\sigma) = N[F_1(N\sigma) - F_2(N\sigma)] \] (5.27)

and the $L/R$ $\rho$-densities are defined (apart a $\sqrt{2}$ factor) as in [23].

If we impose the saddle point condition (5.17) to the resulting effective action with the usual translationally invariant ansatz we get (defining as in (4.10)

\[ A_{ij} \equiv \int M_{ij} \]

\[ (1 - 2\delta^{ij})(D - A + \frac{i}{U_0})^{ij}(p) + g^2 \int \frac{d^2k}{(2\pi)^2} \frac{H^{ij}(U_0; p, -p; k)}{k^2} = 0 \] (5.29)

The self-interacting part can be written as

\[ \mathcal{V}(\sigma, \rho) = V(\sigma) + W_1(\sigma) \left( \frac{\rho^{11} + \rho^{22}}{\sqrt{2}} \right) + W_2(\sigma) \left( \frac{\rho^{11} - \rho^{22}}{\sqrt{2}} \right) \]
\[ + \frac{1}{2} \mathcal{A}(\sigma) \left[ \left( \rho^{11} \right)^2 + \left( \rho^{22} \right)^2 \right] + \mathcal{B}(\sigma)\rho^{12}\rho^{21} + \mathcal{C}(\sigma)\rho^{11}\rho^{22} \] (5.30)

the gauge coupling part is given by the matrix $H^{ij}$ that has four nonzero elements:

\[ H^{23}(U; p, q; k) = (2p + k)U^{13}(k - q, -k - p) \]
\[ H^{21}(U; p, q; k) = U^{12}(k - q, -k - p) \]
\[ H^{31}(U; p, q; k) = -(2q - k)U^{32}(k - q, -k - p) \]
\[ H^{33}(U; p, q; k) = (2p + k)(2q - k)U^{33}(k - q, -k - p) \] (5.31)

and the matrix $M_0$ is given by

\[ M_0 = \begin{pmatrix}
\frac{W_1 + W_2}{\sqrt{2}} + A\rho_0^{11} + C\rho_0^{22} & B\rho_0^{12} & 0 \\
B\rho_0^{21} & \frac{W_1 - W_2}{\sqrt{2}} + A\rho_0^{22} + C\rho_0^{11} & 0 \\
0 & 0 & -M_0^{33}
\end{pmatrix} \] (5.32)
with \( M^{33}_0 = \delta \mathcal{V}(\sigma, \rho) / \delta \sigma \).

If \( U_0 \) is translationally invariant it is easy to see that \( f M_0 \) is a constant. It can be demonstrated that only the diagonal terms of the matrix \( A^{ij} \) are nonzero, so that we can absorb them in a mass renormalization. The master field \( U^{ij}_0 \) is block diagonal:

\[
U_0 = \begin{pmatrix}
\rho_0^{11} & \rho_0^{12} & 0 \\
\rho_0^{21} & \rho_0^{22} & 0 \\
0 & 0 & -\sigma_0
\end{pmatrix}
\]

(5.33)

For \( i, j = 1, 2 \) we have

\[
\rho^{ij}_0(p) = -2D_F(p) \begin{pmatrix}
-\frac{m_0^{(1)} - m_0^{(2)}}{\sqrt{2}} - A^{11} & p_- \\
p_+ + \frac{1}{2p_-} \Gamma(p_-) & -\frac{m_0^{(1)} + m_0^{(2)}}{\sqrt{2}} - A^{22}
\end{pmatrix}
\]

(5.34)

with

\[
D_F(p) = \frac{i}{2p_+ p_- - (m_0^{(1)} - m_0^{(2)} + \sqrt{2} A^{11})(m_0^{(1)} + m_0^{(2)} + \sqrt{2} A^{22}) + \Gamma(p_-) + i\epsilon}
\]

and

\[
-U^{33}_0(p) \equiv \sigma_0(p) = \frac{i}{2p_+ p_- - (m_0^{33} + A^{33}) + \Gamma(p_-) + i\epsilon}
\]

(5.35)

The constants \( A^{ij} = \int M^{ij}_0 \) are defined by using (5.32); one can see that \( A^{12} = A^{21} = 0 \) as before; the remaining constants must be regularized and then re-absorbed in a mass renormalization:

\[
\sqrt{2} A^{11} = \delta m^{(1)} - \delta m^{(2)} \\
\sqrt{2} A^{22} = \delta m^{(1)} + \delta m^{(2)} \\
A^{33} = \delta m^{2}_B
\]

(5.37)

so that, after the introduction of the renormalized masses \( m^2_B = m^2_{0B} + \delta m^2_B \), \( m_i = m^{(i)} + \delta m^{(i)} \), \( m^2_F = m^2_1 - m^2_2 \) and of the shifted masses \( M^2_a = m^2_a - g^2 / \pi \), \( a = B, F \) we can write the propagators in a more familiar form:
\[
\rho_0^{ij}(p) = \frac{-2i}{2p_+p_- - M_F^2 - g^2|p_-|/\pi\lambda + i\epsilon} \left( \frac{-m_1 - m_2}{\sqrt{2}} \Gamma(p_-) - \frac{p_-}{\sqrt{2}} \right) \tag{5.38}
\]

\[
\sigma_0(p) = \frac{i}{2p_+p_- - M_B^2 - g^2|p_-|/\pi\lambda + i\epsilon} \tag{5.39}
\]

It is now easy to get the equations for the fluctuations of the master field, by applying the same methods as in the previous section. Because of the presence of higher than quadratic terms in the densities appearing in the lagrangian, as in the \(O(N)\) theory, we have to renormalize all the constants that couple two densities i.e. the scalar coupling \(v_2\), the Yukawa couplings \(w_1^{(1)}\) and \(w_1^{(2)}\) of \((\ref{eq:5.24})\) and the constants \(a_0\), \(b_0\), \(c_0\) that did not need to be renormalized in the version of the model \((QCD_2\) with quartic interaction term) studied in the previous section. This is our renormalization choice:

\[
v \equiv V''(\Lambda_B) + \frac{1}{2} A''(\Lambda_B) \left[ \left( \Lambda_{F_{11}}^1 \right)^2 + \left( \Lambda_{F_{22}}^2 \right)^2 \right] + C''(\Lambda_B) \Lambda_{F_{11}} \Lambda_{F_{22}}^2
\]

\[
\frac{w_1 + w_2}{\sqrt{2}} \equiv \frac{W_1'(\Lambda_B) + W_2'(\Lambda_B)}{\sqrt{2}} + A'(\Lambda_B) \Lambda_{F_{11}}^1 + C'(\Lambda_B) \Lambda_{F_{22}}^2 \tag{5.40}
\]

\[
\frac{w_1 - w_2}{\sqrt{2}} \equiv \frac{W_1'(\Lambda_B) - W_2'(\Lambda_B)}{\sqrt{2}} + A'(\Lambda_B) \Lambda_{F_{22}}^2 + C'(\Lambda_B) \Lambda_{F_{11}}^1
\]

\[
a \equiv A(\Lambda_B) \quad b \equiv B(\Lambda_B) \quad c \equiv C(\Lambda_B)
\]

Here the prime means derivative with respect to \(\sigma\), and we defined three (infinite) constants \(\Lambda_B = \int \sigma_0\), \(\Lambda_{F_{ii}} = \int \rho_0^{ii}\).

With the help of these definitions, we can write the form of the matrix \(\delta \tilde{M}\), obtained by expanding the potential \((\ref{eq:5.30})\) up to second order in the fluctuations (compare with \((\ref{eq:5.32})\)):

\[
\delta \tilde{M} = \begin{pmatrix}
    a\delta U^{11} + c\delta U^{22} + \frac{w_1 + w_2}{\sqrt{2}} \delta U^{33} & b\delta U^{12} \\
    b\delta U^{21} & a\delta U^{22} + c\delta U^{11} + \frac{w_1 - w_2}{\sqrt{2}} \delta U^{33} \\
    0 & 0 \\
    0 & \delta \tilde{M}^{33}
\end{pmatrix} \tag{5.41}
\]
with
\[
\delta \tilde{M}^{33} = v\delta U_{33} + \frac{w_1 + w_2}{\sqrt{2}} \delta U_{11} + \frac{w_1 - w_2}{\sqrt{2}} \delta U_{22}
\] (5.42)

Our final result is the equations for the fluctuations in the most general case of a 2-dimensional gauge theory:

\[
(1 - 2\delta^{i3})i\delta U^{kl}(p, q) = g^2 U_0^{ki}(p) U_0^{jl}(-q) \int \frac{d^2k}{(2\pi)^2} \frac{H^{ij}(\delta U; p, q; k)}{k_-^2}
- U_0^{ki}(p) U_0^{jl}(-q) \int \frac{d^2k}{(2\pi)^2} \delta \tilde{M}^{ij}(k + p, -k + q)
\] (5.43)

These equations are the generalization of the ones already seen; the effect of the interacting potentials is to add inhomogeneous terms in the integral equations of motion. If one writes the entire set of equations, one can see that the independent degrees of freedom are the Fermion-Fermion and Boson-Boson fields \( \rho = \bar{\psi} \gamma_5 \psi \) and the Fermion-Boson bound state \( \delta \chi_R = \bar{\psi} \gamma_5 \phi \). Since this last one is not subject to the interactions given by the additional potentials \( V, W_i \) and \( F_i \), it does not develop an inhomogeneous term in his equations of motion. These are the full equations for the Fermion-Fermion and Boson-Boson bound states in the case of no chiral \( \bar{\psi} \gamma_5 \psi \) term \( m_R = m_L \equiv m_F, w_1 = w, w_2 = 0 \) (for the equation of the Fermion-Boson bound state, see (3.22)):

\[
\left[ r^2 - \frac{M_F^2}{x(1-x)} \right] \varphi_F(x) = -\frac{g^2}{\pi} \frac{d}{dy} \left( \frac{dy}{(y-x)^2} \varphi_F(y) \right)
+ \frac{a + c}{2\pi} m_F^2 \left( \frac{1}{x} \int_0^1 \frac{\varphi_F(y)}{1-y} dy + \frac{1}{1-x} \int_0^1 \frac{\varphi_F(y)}{y} dy \right)
+ \frac{b}{2\pi} m_F^2 \left( \frac{1}{x(1-x)} \int_0^1 \varphi_F(y) dy + \int_0^1 \frac{\varphi_F(y)}{y(1-y)} dy \right)
+ \frac{w}{2\pi} m_F \frac{m_F}{x(1-x)} \int_0^1 \varphi_B(y) dy
\] (5.44)
and

\[
\left[ r^2 - \frac{M_B^2}{x(1-x)} \right] \varphi_B(x) = -\frac{g^2}{\pi} P \int_0^1 \frac{dy}{(y-x)^2} \frac{(x+y)(2-x-y)}{4x(1-x)} \varphi_B(y) \\
+ \frac{1}{4x(1-x)} \left( \frac{v}{\pi} \int_0^1 \varphi_B(y) dy + \frac{w}{2\pi m_F} \int_0^1 \frac{\varphi_F(y)}{y(1-y)} dy \right)
\]

(5.45)

For \( g = 0 \) we can give the solution of those equations: let us define

\[
\rho = \frac{a+c}{b} \quad \sigma = \frac{w}{m_F b} \quad \eta = \frac{wm_F}{v}
\]

(5.46)

\[
A = \int_0^1 \varphi_F(y) dy \quad B = \int_0^1 \frac{\varphi_F(y)}{y} dy \quad C = \int_0^1 \varphi_B(y) dy
\]

(5.47)

so that

\[
\varphi_B(x) = \frac{v}{4\pi r_B^2 x(1-x) - m_B^2} \left( \frac{C + \eta B}{x} \right)
\]

\[
\varphi_F(x) = \frac{bm_F^2}{2\pi} \frac{A + \rho B + \sigma C + 2B(1-x)}{r_F^2 x(1-x) - m_F^2}
\]

(5.48)

We then get a new set of equations that after the renormalization of \( B \) (see (4.25)) are given by:

\[
A = \frac{b}{2\pi} \left[ F(\mu_F) \left\{ A + \sigma C + \left( \rho + \frac{2}{\mu_F^2} \right) B^* \right\} + \frac{2}{\mu_F^2} B^* \right]
\]

\[
B^* = \frac{b}{2\pi} \frac{\mu_F^2}{2} F(\mu_F) \left\{ A + \sigma C + \left( \rho + \frac{2}{\mu_F^2} \right) B^* \right\}
\]

\[
C = \frac{v}{4\pi m_B^2} F(\mu_B) (C + \eta B^*)
\]

(5.49)

If \( 1 - v/4\pi m_B^2 F(\mu_B) = 0 \) then \( B^* = 0 \); the only solution that doesn’t imply also \( A = C = 0 \) requires \( \mu_F^2 = 0 \) and

\[
A(1-\lambda) = -\frac{wm_F}{2\pi} C
\]

(5.50)

where we defined \( \lambda = -b/2\pi \).
If \(1 - \nu / 4\pi m_B^2 F(\mu_B) \neq 0\) we can solve the matrix equations by obtaining
\[
A = 2\mu_F^2 (1 - \lambda) B^* \tag{5.51}
\]
\[
\left( \frac{\sin 2\theta_B}{2\theta_B} + \frac{\nu}{4\pi m_B} \right) C = -\frac{wm_F}{4\pi m_B^2} B^* \tag{5.52}
\]
\[
\lambda^2 - \lambda \left[ 2 + \rho (1 - \cos 2\theta_F) \right] + \frac{\sin 2\theta_F}{2\theta_F} + \frac{w}{2\pi m_F} (1 - \cos 2\theta_F) \left( \frac{C}{B^*} \right) = 0 \tag{5.53}
\]
where we made the usual parametrization \(\mu_a = 2\sin \theta_a, 0 < \theta_a < \pi/2, a = F, B\).

The equations depend on two constants, that we can fix by normalizing \(\varphi_F\) and \(\varphi_B\); requiring \(A = C = 1\) we get
\[
1 = \left[ \frac{\nu}{4\pi m_B^2} + \frac{wm_F}{8\pi m_B^2} \frac{\mu_F^2}{1 - \lambda} \right] F(\mu_B) \tag{5.54}
\]
\[
1 = -2\lambda F(\mu_F) \left[ \frac{\mu_F^2}{4} - \frac{w}{4\pi m_F} \left( 1 - \frac{1}{\lambda} \right) + 1 - \frac{\lambda}{2} \right] \tag{5.55}
\]
The second equation can be solved as a function of \(\lambda\)
\[
2\lambda = 2 - \frac{w}{2\pi m_F} + \rho (1 - \cos 2\theta_F)
\]
\[
\pm \sqrt{\left( 2 - \frac{w}{2\pi m_F} + \rho (1 - \cos 2\theta_F) \right)^2 - 4 \left( \frac{\sin 2\theta_F}{2\theta_F} - \frac{w}{2\pi m_F} \right)} \tag{5.56}
\]
We will not proceed further in solving those equations, but we can give a perturbative estimate of the shift in \(\theta_a\) (\(w\) small):
\[
2\delta \theta_B = -\frac{wm_F}{8\pi m_B^2} \frac{1}{1 - \lambda} \frac{1 - \cos 2\theta_F}{h(\theta_B)} \tag{5.57}
\]
\[
2\delta \theta_F = -\frac{w}{2\pi m_F} \frac{1 - \lambda}{h(\theta_F) - \lambda \rho \sin 2\theta_F} \tag{5.58}
\]
where \(h(x) = d/dx(\sin 2x/2x), \theta_a = \bar{\theta}_a + \delta \theta_a\).
6 Conclusions

We have shown the validity and the flexibility of the bilocal field method recently proposed \[23\]; we have reproduced via the path-integral approach all the known results about models as $O(N)$ and massive Gross-Neveu model in the large-$N$ limit, and we have generalized the 't Hooft equations for $QCD_2$ and scalar $QCD_2$ to the case of self-interacting fields.

Our formalism is a kind of generalization of the one used in the context of the $O(N)$ models \[6\], but also applies to all cases in which the field is a ‘vector’ in the colour index, independently if the vector index belongs to an internal symmetry group (such as $O(N)$) or to a gauge group.

We could also conclude that the two-dimensional gauge models with matter in the fundamental representation are as trivial, from the point of view of the large-$N$ expansion, as the $O(N)$ and $CP^{N-1}$ models, as we just have to pay the price of introducing a bilocal colour-singlet field instead of a local one. This is not the case of matter in the adjoint representation, in which one can construct an infinite number of multilocal fields that all contribute for large $N$. Such a structure enlarges the difficulties but gives a richer spectrum of states, both in the case of $QCD_2$ with fermionic \[24\] or bosonic \[25\] matter in adjoint representation.

Acknowledgements
I would like to thank Nordita for the hospitality given to me during the first six months of 1993.
I also want to thank P. Di Vecchia for many stimulating discussions and for having carefully read this manuscript.

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Fig. 1 Graphic resolution of the equation $x = a \cdot \sin(x)$ for $a > 1$, $a < 1$
Fig. 2 Graphic resolution of the equation $x = a \cdot f(x)$ for $a > 1, a < 1$

$f(x) = (1+x^2)^{-1/2} \cdot \log(x + \sqrt{1+x^2})$
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9401086v1
Fig. 3 Coupling constant as a function of the mass angle $\alpha$ (eq. (4.32))

In the bound state region $M = 2m \sin(\alpha/2)$
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9401086v1
Fig. 4 Coupling constant as a function of the $z$ variable (see eq. (4.33))

In the tachyon region $M^2 = -4m^2z^2$
This figure "fig1-4.png" is available in "png" format from:

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Fig.5 Plot of the argument of the square root in eq.(4.36) for \( \rho = 1, -1/3, -3/5, -1, -5 \)
This figure "fig1-5.png" is available in "png" format from:

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Fig. 6 Plot of the argument of the square root in eq. (4.36) for \( \rho = -1 \)
This figure "fig1-6.png" is available in "png" format from:

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Fig. 7 Plot of the argument of the square root in eq. (4.40) for rho = -1, -1/3, 0, 1
This figure "fig1-7.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9401086v1
This figure "fig1-8.png" is available in "png" format from:

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