Generalized Grassmannian coherent states for pseudo-Hermitian $n$-level systems

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Abstract

The purpose of this paper is to generalize fermionic coherent states for two-level systems described by pseudo-Hermitian Hamiltonian (Cherbal et al 2007 J. Phys. A: Math. Theor. 40 1835) to $n$-level systems. Central to this task is the expression of the coherent states in terms of generalized Grassmann variables. These kinds of Grassmann coherent states satisfy the bi-overcompleteness condition instead of the overcompleteness one, as it is reasonably expected because of the biorthonormality of the system. Choosing an appropriate Grassmann weight function, the resolution of identity is examined. Moreover Grassmannian coherent and squeezed states of the deformed group $SU_q(2)$ for the three-level pseudo-Hermitian system are presented.

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1. Introduction

The last decade has witnessed a growing interest in non-Hermitian Hamiltonians with real spectra [2–8]. Considering the results of various numerical studies, Bender and his collaborators [4, 5] found certain examples of one-dimensional non-Hermitian Hamiltonians that possessed real spectra. Because these Hamiltonians were invariant under $PT$ transformations, their spectral properties were linked with their $PT$-symmetry. Later Mostafazadeh introduced the notion of pseudo-Hermiticity as an alternative possible approach for a non-Hermitian operator to admit a real spectrum [9, 10].

On the other hand, Grassmannian coherent states (GCS) and their generalization [11–16] have attracted a great deal of attention in the last decade and the concept of coherent states was also introduced to $PT$-symmetric quantum mechanics [17, 18], and pseudo-Hermitian one...
Here our objective is to construct generalized Grassmannian pseudo-Hermitian coherent states (GPHCS) by introducing ladder operators for a general \( n \)-level pseudo-Hermitian system and applying the bi-orthonormal property of this system to facilitate the investigation of overcompleteness of GPHCS. We find throughout this work that many close parallels can be established between the expressions evaluated for GPHCS and the more familiar ones for boson coherent states. For boson coherent states the integration for the resolution of identity is taken over commuting complex variables, while for fermions, on the other hand, the integration is over anticommuting Grassmann numbers that have no classical analogues. This is due to the fact that, in the context of quantum field theory \[19\], fermion fields anti-commute; hence, their eigenvalues must, as noted by Schwinger, be anti-commuting numbers \[20\]. The GPHCS may be useful to describe entangled coherent states for compound systems governed by non-Hermitian Hamiltonians in quantum information theory \[21, 22\]. It is also possible to generalize the \( P \)-function, the \( Q \)-function and the Wigner function for fermions \[23\] to the pseudo-Hermitian density matrix of \( n \)-level systems \[24\].

The paper is organized as follows. In section 2, we introduce the concept of a pseudo-Hermitian operator and consider the basic spectral properties of pseudo-Hermitian Hamiltonians that have a complete biorthonormal eigenbasis. In section 3, we present the generalized Grassmannian variables based on the Majid approach \[13\] and define pseudo-Hermitian ladder operators to construct GPHCS. We show that unlike the canonical Hermitian coherent state, GPHCS satisfy the bi-overcompleteness condition instead of the overcompleteness one. We also study the stability of the GPHCS and finally following the introduced approach we construct the GPHCS for \( SU_q(2) \).

### 2. Pseudo-Hermitian Hamiltonians and biorthonormal eigenbasis

An intensive study of the Schrödinger equation with complex potentials, but with a real spectrum, was performed by different methods. The pioneer papers \[3, 4\] initiated the investigation of \( PT \)-symmetric systems and afterwards a more general class of pseudo-Hermitian models was introduced by Mostafazadeh \[9\]. Following the second approach, let \( H : {\mathcal{H}} \rightarrow {\mathcal{H}} \) be a linear operator acting in a Hilbert space \( {\mathcal{H}} \) and \( \eta : {\mathcal{H}} \rightarrow {\mathcal{H}} \) be a linear Hermitian automorphism (invertible transformation). Then the \( \eta \)-pseudo-Hermitian adjoint of \( H \) is defined by

\[
H^\# = \eta^{-1} H^\dagger \eta. \tag{2.1}
\]

\( H \) is said to be pseudo-Hermitian with respect to \( \eta \) or simply \( \eta \)-pseudo-Hermitian if \( H^\# = H \).

As in \[9, 10\], the eigenvalues of the pseudo-Hermitian Hamiltonian \( H \) are either real or come in complex-conjugate pairs and the following relations in the nondegenerate case hold:

\[
H^\dagger = \eta H \eta^{-1}. \tag{2.2}
\]

According to \[9\] we consider only the diagonalizable operators \( H \) with a discrete spectrum. Then, a complete biorthonormal eigenbasis \( |\psi_i\rangle, |\phi_i\rangle \) exists, i.e. a basis such that

\[
H |\psi_i\rangle = E_i |\psi_i\rangle, \quad H^\dagger |\phi_i\rangle = \bar{E}_i |\phi_i\rangle, \\
\langle \phi_i |\psi_j\rangle = \delta_{ij}, \quad \sum_i |\psi_i\rangle \langle \phi_i| = I. \tag{2.3}
\]

For a given pseudo-Hermitian \( H \) there are infinitely many \( \eta \) satisfying equation (2.2). These can however be expressed in terms of a complete biorthonormal basis \( |\psi_i\rangle, |\phi_i\rangle \) of \( H \). In
the non-degenerate case the general linear, Hermitian, invertible operator $\eta$ and its inverse satisfying equation (2.2) have the forms

$$\eta = \sum_i |\phi_i\rangle\langle\phi_i|, \quad \eta^{-1} = \sum_i |\psi_i\rangle\langle\psi_i|$$  \hspace{1cm} (2.4)

$$|\phi_i\rangle = \eta |\psi_i\rangle, \quad |\psi_i\rangle = \eta^{-1} |\phi_i\rangle,$$  \hspace{1cm} (2.5)

where we consider the non-Hermitian Hamiltonians with real spectra; hence, $\eta$ is a positive definite operator.

3. Generalized Grassmannian pseudo-Hermitian coherent states

3.1. Generalized Grassmannian variables

The basic properties of generalized Grassmann variables are discussed in [11–16]. Here we survey the properties that we shall make use of in this paper. According to [13], $\mathbb{Z}_r$-graded Grassmann algebra is generated by the variables satisfying, by definition, the following properties:

$$\theta_i \theta_j = q \theta_j \theta_i, \quad i, j = 1, 2, \ldots, i < j$$  \hspace{1cm} (3.1)

$$\theta^n = 0, \quad q = e^{\frac{2\pi i}{n}}.$$  

Analogous rules also apply for the Hermitian conjugate of $\theta$, $\theta^\dagger = \bar{\theta}$, as

$$\bar{\theta}_i \bar{\theta}_j = q \bar{\theta}_j \bar{\theta}_i, \quad i < j,$$

$$\bar{\theta}^n = 0.$$  \hspace{1cm} (3.2)

Consider the generalized Berezin’s rules of integration as

$$\int d\theta \theta^k = \int d\bar{\theta} \bar{\theta}^k = \delta_{k,n-1}.$$  \hspace{1cm} (3.3)

where $k$ is any positive integer. We also need the following relations which are necessary to compute the integral of any function over the Grassmann algebra:

$$\begin{align*}
\theta d\bar{\theta} &= q d\bar{\theta} \theta, \\
\bar{\theta} d\theta &= \bar{q} d\theta \bar{\theta}, \\
\theta d\theta &= \bar{q} d\theta \theta, \\
\bar{\theta} d\bar{\theta} &= q d\bar{\theta} \bar{\theta}, \\
\theta \bar{\theta} &= \bar{q} \theta \bar{\theta}.
\end{align*}$$  \hspace{1cm} (3.4)

Now we are ready to give a prescription for the construction of the generalized GPHCS.

3.2. Coherent states

Now we will develop a formalism to construct the generalized GCS for pseudo-Hermitian $n$-level systems. Let us introduce the generalized coherent states as the eigenstates of the annihilation operator $b$ which is defined as

$$b := \sum_{i=0}^{n-1} \sqrt{\rho_{i+1}} |\psi_i\rangle \langle\phi_{i+1}|.$$  \hspace{1cm} (3.5)

Using equation (2.3), it is straightforward to check that the action of the annihilation operator $b$ on the $|\psi_i\rangle$ eigenstates is

$$b |\psi_i\rangle = \sqrt{\rho_i} |\psi_{i-1}\rangle.$$  \hspace{1cm} (3.6)
so $|\psi_0\rangle$ is the vacuum state, and the annihilation operator $b$ has the nilpotency degree of order $n$, i.e. $b^n = 0$. Let us have the following quantization relations between the biorthonormal eigenstate \{ $|\psi_i\rangle$, $|\phi_i\rangle$, $i = 0, 1, 2, \ldots, 3n - 1$ \} and the generalized Grassmannian variables $\theta$, $\bar{\theta}$:

\[
\begin{align*}
\theta |\psi_i\rangle &= q^{-i}|\psi_i\rangle \theta \\
\bar{\theta} |\psi_i\rangle &= \bar{q}^{-i}|\psi_i\rangle \bar{\theta} \\
\theta |\phi_i\rangle &= q^{-i}|\phi_i\rangle \theta \\
\bar{\theta} |\phi_i\rangle &= \bar{q}^{-i}|\phi_i\rangle \bar{\theta}
\end{align*}
\]  

(3.7)

Considering equations (2.4), (3.7) it is easy to check that

\[
[\theta, \eta] = [\theta, \eta^{-1}] = [\bar{\theta}, \eta] = [\bar{\theta}, \eta^{-1}] = 0.
\]  

(3.8)

The generalized GPHCS denoted by $|\theta\rangle_n$ by definition are the eigenstates of the annihilation operator $b$ with the eigenvalues given by the label of the coherent states, so we must find the GPHCS $|\theta\rangle_n$ such that

\[
b|\theta\rangle_n = \theta|\theta\rangle_n,
\]  

(3.9)

where the eigenvalue $\theta$ is a complex generalized Grassmannian variable. We write generically

\[
|\theta\rangle_n = \sum_{i=0}^{n-1} \alpha_i \theta^i |\psi_i\rangle;
\]  

(3.10)

considering equation (3.9) we get

\[
|\theta\rangle_n = \sum_{i=0}^{n-1} \frac{\alpha_i^{(n+i)}}{\sqrt{\rho_i}} \theta^i |\psi_i\rangle,
\]  

(3.11)

where \{ $\rho_i! := \rho_0 \rho_1 \rho_2 \ldots \rho_{3n} \rho_n = 1$ \} and $\rho_i$s in general are the complex variables.

As it is usual for Hermitian systems, in order to express the states $|\theta\rangle_n$ in terms of the vacuum state we use the Hermitian conjugate of the annihilation operator $b^\dagger$. But it is important to note that here we are dealing with the pseudo-Hermitian system, so as a result of this fact it is easy to show that $b^\dagger$ is not the creation operator for the $|\psi_i\rangle$ eigenbasis. To overcome this problem we need to introduce the new operator $b^\sharp$, which is the $\eta$ pseudo-Hermitian of the $b^\dagger$, as

\[
b^\sharp := n^{-1} b^\dagger \eta = \sum_{i=0}^{n-1} \sqrt{\rho_{i+1}} |\psi_{i+1}\rangle \langle \psi_i|,
\]  

(3.12)

such that

\[
b^\sharp |\psi_i\rangle = \sqrt{\rho_{i+1}} |\psi_{i+1}\rangle;
\]  

(3.13)

so one can see that for the pseudo-Hermitian system, biorthonormal system, instead of $b^\dagger$, $b^\sharp$ is the creation operator. Using equation (3.13) we get

\[
|\theta\rangle_n = \sum_{i=0}^{n-1} \frac{\alpha_i^{(n+i)}}{\rho_i} \theta^i (b^\sharp)^i |\psi_0\rangle,
\]  

(3.14)

and considering the following $q$-commutator relations

\[
[b^\sharp, \theta]_q = 0 \quad [b^\sharp, \bar{\theta}]_q = 0
\]  

(3.15)
the coherent state derived above can be written in a compact form as
\[
|\theta\rangle_n = \sum_{i=0}^{n-1} \frac{(b^\dagger \theta)^i}{\rho_i!} |\psi_0\rangle =: e_{q}(b^\dagger \theta) |\psi_0\rangle
\]  
(3.16)
where the \(q\)-commutator and the generalized \(q\)-exponential function are defined as
\[
[A, B]_q := AB - qBA,
\]
\[
e_{q}^{x} := \sum_{n=0}^{\infty} \frac{x^n}{\rho_n!}.
\]
(3.17)
One must note that it is possible to construct another family of GPHCS, \(|\tilde{\theta}\rangle_n\), in terms of \(|\phi_i\rangle\). These coherent states are the eigenbasis of the operator \(\tilde{b}\) which annihilates the dual states \(|\phi_i\rangle\):
\[
\tilde{b} |\tilde{\theta}\rangle_n = \theta |\tilde{\theta}\rangle_n,
\]
(3.18)
where the explicit form of the operator \(\tilde{b}\) is defined as
\[
\tilde{b} = \eta b \eta^{-1} = \sum_{i=0}^{n-1} \sqrt{\rho_{i+1}} |\phi_i\rangle \langle \psi_{i+1}|;
\]
(3.19)
then the coherent state corresponding to the dual states \(|\phi_i\rangle\) can be obtained as follows:
\[
|\tilde{\theta}\rangle_n = \sum_{i=0}^{n-1} \frac{q^{i(i+1)}}{\rho_i!} \theta^i |\phi_i\rangle.
\]
(3.20)
Now in order to determine \(|\tilde{\theta}\rangle\) in terms of \(|\phi_i\rangle\) let us have the following definition:
\[
b^{\varepsilon} := \eta^{-1} b^\dagger \eta^i, \quad \text{where} \quad \eta^{-1} = \eta;
\]
(3.21)
according to equations (3.19) and (3.21) we find \(\tilde{b}^{\varepsilon} = b^\dagger\), and then \(|\tilde{\theta}\rangle_n\) is
\[
|\tilde{\theta}\rangle_n = \sum_{i=0}^{n-1} \frac{q^{i(i+1)}}{\rho_i!} \theta^i (\tilde{b}^{\varepsilon})^i |\phi_0\rangle.
\]
(3.22)
The \(q\)-commutation relation between \(\theta\) and \(\tilde{b}^{\varepsilon}\) is
\[
[\theta, \tilde{b}^{\varepsilon}]_q = 0;
\]
(3.23)
then equation (3.22) reduces to the following form:
\[
|\tilde{\theta}\rangle_n = \sum_{i=0}^{n-1} \frac{(\tilde{b}^{\varepsilon})^i |\phi_0\rangle = e_{q}(\tilde{b}^{\varepsilon}) |\phi_0\rangle}.
\]
(3.24)
which is the \(q\)-exponential form of \(|\tilde{\theta}\rangle_n\). Now we have all the ingredient to prove the completeness of the GPHCS which is the task of the next subsection.

3.3. Resolution of identity

Let us examine whether or not the GPHCS \(|\theta\rangle_n\) and \(|\tilde{\theta}\rangle_n\) satisfy the overcompleteness property. By introducing a weight function
\[
w(\theta, \tilde{\theta}) = \sum_{i, j=0}^{n-1} \alpha_{ij} \theta^i \tilde{\theta}^j,
\]
(3.25)
and considering equations (3.1)–(3.7), (3.15) and (3.23) it is clear that neither the integral \(|\theta\rangle \langle \theta|\) nor the integral of \(|\tilde{\theta}\rangle \langle \tilde{\theta}|\), against the measure \(d\theta d\tilde{\theta} w(\theta, \tilde{\theta})\), is unnormalized, i.e.
\[
\int d\theta d\tilde{\theta} w(\theta, \tilde{\theta}) |\theta\rangle \langle \theta| \neq I, \quad \int d\theta d\tilde{\theta} w(\theta, \tilde{\theta}) |\tilde{\theta}\rangle \langle \tilde{\theta}| \neq I.
\]
(3.26)
In order to overcome this impasse and realize the resolution of identity it is necessary to consider equation (2.3), i.e. the biorthonormal nature of the system. Then it is reasonable to check the integrals $\langle \psi | \hat{b} \hat{b}^\dagger | \psi \rangle$ against the measure $d\bar{\theta} d\theta \, w(\theta, \bar{\theta})$. Choosing the proper weight function we can realize the resolution of identity as
\[
\int d\bar{\theta} d\theta \, w(\theta, \bar{\theta}) |\theta \rangle \langle \bar{\theta}| = \int d\bar{\theta} d\theta \, w(\theta, \bar{\theta}) |\bar{\theta} \rangle \langle \theta| = I.
\] (3.27)

To identify the weight function we replace the explicit form of $|\theta \rangle$ and $|\bar{\theta} \rangle$ from equations (3.16) and (3.24) into equation (3.27) and we get
\[
\int d\bar{\theta} d\theta \, w(\theta, \bar{\theta}) |\theta \rangle \langle \bar{\theta}| = \int d\bar{\theta} d\theta \sum_{k,l=0}^{n-1} c_{k,l} \theta^k \bar{\theta}^l \sum_{i,j=0}^{n-1} q^{(i+1)} q^{(j+1)} \delta_{k+i,n-1} \delta_{l+j,n-1} |\psi_i \rangle \langle \phi_j| \]
\[
= \sum_{k,l=0}^{n-1} c_{k,l} \sum_{i,j=0}^{n-1} q^{(i+1)} q^{(j+1)} \rho_{i,n-1} \rho_{j,n-1} |\psi_i \rangle \langle \phi_j|,
\] (3.29)

which in turn by the completeness of the biorthonormal basis $\sum_{i=0}^{n-1} |\psi_i \rangle \langle \phi_i| = I$ of the pseudo-Hermitian system leads to the following constrain on the $c_{i,j}$ coefficients:
\[
c_{i,j} = \rho_{i,n-1} \bar{\rho}_{j,n-1} \bar{q}^{(i+1)} q^{(j+1)} \delta_{i,j}.
\] (3.30)

Thus the weight function must be equal:
\[
w(\theta, \bar{\theta}) = \sum_{i=0}^{n-1} q^{(i+1)} \rho_{n-i-1} |\theta^i \bar{\theta}^i|.
\] (3.31)

Finally we find that the continuous set of $|\theta \rangle$ and $|\bar{\theta} \rangle$ produces the system of biorthonormal coherent states which provide a resolution of identity (bi-overcompleteness) for the pseudo-Hermitian system.

### 3.4. Time evolution of GPHCS

In this section we shall discuss the time evolution of the GPHCS. The coherent states remain coherent for all the times provided that the time evolution of the initial state $|\theta, 0 \rangle_n = |\theta \rangle_n$, $|\theta, t \rangle_n$, managed by Hamiltonian, is also an eigenstate of the lowering operator $b$:
\[
b|\theta, t \rangle_n = \theta(t) |\theta, t \rangle_n, \quad \text{where} \quad |\theta, t \rangle_n = e^{-iHt} |\theta \rangle_n.
\] (3.32)

Recalling $|\theta \rangle_n$ from equation (3.11) one can write
\[
|\theta, t \rangle_n = \sum_{k=0}^{n-1} \frac{\theta^k}{\sqrt{\rho_k}} e^{-i E_k t} |\psi_k \rangle,
\] (3.33)

to get the proper solution assuming
\[
E_k = -(n-k-2)E.
\]

Then one can express the evolved coherent states as
\[
|\theta, t \rangle_n = e^{(n-2)E} |\theta(t) \rangle_n, \quad \text{where} \quad \theta(t) = e^{-iE \theta};
\] (3.34)

thus, the evolved coherent states are actually the remaining eigenstates of the annihilation operator, which manifests the stability of the time evolution of the CS $|\theta \rangle$. Similarly one can show that the $|\theta, t \rangle$’s are also stable and $|\theta \rangle$ and $|\bar{\theta}, t \rangle$ satisfy the resolution of identity as
\[
\int d\bar{\theta} d\theta \, w(\theta, \bar{\theta}) |\bar{\theta}, t \rangle \langle \theta| = \int d\bar{\theta} d\theta \, w(\theta, \bar{\theta}) |\theta, t \rangle \langle \bar{\theta}| = I.
\]
3.5. $SU_q(2)$ deformed pseudo-Hermitian coherent states

In this section, we will derive the coherent states for the modified $SU(2)$, i.e. $SU_q(2)$, following the technique developed in the previous section. For this purpose, consider a three-level pseudo-Hermitian system; we first need to start with these quantization relations:

\[
\begin{align*}
\theta |\psi_0\rangle &= q |\psi_0\rangle \theta, & \langle \psi_0 | \bar{\theta} &= q \bar{\theta} \langle \psi_0 | \\
\theta |\psi_1\rangle &= |\psi_1\rangle \theta, & \langle \psi_1 | \bar{\theta} &= \bar{\theta} \langle \psi_1 | \\
\theta |\psi_2\rangle &= q |\psi_2\rangle \theta, & \langle \psi_2 | \bar{\theta} &= \bar{q} \bar{\theta} \langle \psi_2 | \\
\bar{\theta} |\psi_0\rangle &= q |\psi_0\rangle \bar{\theta}, & \theta \langle \psi_0 | &= q \langle \psi_0 | \theta \\
\bar{\theta} |\psi_1\rangle &= |\psi_1\rangle \bar{\theta}, & \theta \langle \psi_1 | &= \langle \psi_1 | \theta \\
\bar{\theta} |\psi_2\rangle &= \bar{q} |\psi_2\rangle \bar{\theta}, & \theta \langle \psi_2 | &= \bar{q} \langle \psi_2 | \theta.
\end{align*}
\] (3.35)

The same relations hold between $|\phi_i\rangle$ and $\theta$ and $\bar{\theta}$. Let us define the operators $b$, $b^\sharp$ and $b_z$ as

\[
\begin{align*}
b &:= \sqrt{\rho_1} |\phi_1\rangle \langle \phi_1| + \sqrt{\rho_2} |\phi_2\rangle \langle \phi_2|, \\
b^\sharp &:= \eta^{-1} b^\eta := \sqrt{\rho_1} |\phi_1\rangle \langle \phi_1| + \sqrt{\rho_2} |\phi_2\rangle \langle \phi_2|, \\
b_z &:= [b, b^\sharp] := b b^\sharp - q b^\sharp b.
\end{align*}
\]

Using the explicit form of the operators $b$, $b^\sharp$ and $b_z$ we will try to find the conditions which make it possible that these operators be the generators of the $su_q(2)$ Lie algebra. To do so consider the commutation relation between $b$ and $b_z$:

\[
[b_z, b] = (p_1 - q p_2 + q^2 p_1) \sqrt{\rho_1} |\phi_1\rangle \langle \phi_1| + (p_2 - q p_1 + q^2 p_2) \sqrt{\rho_2} |\phi_2\rangle \langle \phi_2|.
\] (3.36)

To get the proper solution we have

\[
(p_1 - q p_2 + q^2 p_1) = (p_2 - q p_1 + q^2 p_2),
\]
or equivalently

\[
(1 + q + q^2) (p_1 - p_2) = 0 \Rightarrow q = e^{i \theta}.
\] (3.37)

With the above restriction equation (3.36) reduces to the following form:

\[
[b_z, b] = (p_1 - q p_2 + q^2 p_1) b = (p_2 - q p_1 + q^2 p_2) b.
\] (3.38)

The same condition, equation (3.37), is required for the commutator of $b^\sharp$ and $b_z$ to satisfy the $su_q(2)$ algebra; then we have

\[
\begin{align*}
[b, b^\sharp] &= b_z, \\
[b_z, b^\sharp] &= (p_1 - q p_2 + q^2 p_1) b = (p_2 - q p_1 + q^2 p_2) b \\
[b^\sharp, b_z] &= (p_1 - q p_2 + q^2 p_1) b^\sharp = (p_2 - q p_1 + q^2 p_2) b^\sharp.
\end{align*}
\] (3.39)

Considering equation (3.9), the coherent states in question are

\[
|\theta\rangle = b |\theta\rangle
\]
and $|\theta\rangle$ can be found as

\[
|\theta\rangle = |\psi_0\rangle + \frac{\bar{q}}{\sqrt{\rho_1}} \theta |\psi_1\rangle + \frac{1}{\sqrt{\rho_1 \rho_2}} \theta^2 |\psi_2\rangle = \left(1 + \frac{\bar{q}}{\sqrt{\rho_1}} \theta b^\sharp + \frac{1}{\sqrt{\rho_1 \rho_2}} \theta^2 b^\sharp b\right) |\psi_0\rangle
\] (3.40)

where in the second equality in equation (3.40) we have used the operator $b^\sharp$ to introduce $|\theta\rangle$ in terms of $|\psi_0\rangle$ and finally based on the definition of the $q$-exponential function, equation (3.17), we have

\[
|\theta\rangle = e_q^{(b^\sharp \bar{\theta})} |\psi_0\rangle.
\]
To get the dual space coherent states we recall $\tilde{b}$:

$$\tilde{b} = \eta b \eta^{-1} = \sqrt{\rho_1} |\phi_0\rangle \langle \psi_1| + \sqrt{\rho_2} |\phi_1\rangle \langle \psi_2|;$$

the dual space coherent states are the eigenstates of $\tilde{b}$:

$$\tilde{b}|\tilde{\theta}\rangle = \theta|\tilde{\theta}\rangle.$$

Then the dual GPHCS states are

$$|\tilde{\theta}\rangle = |\phi_0\rangle + \sqrt{\rho_1} \theta|\phi_1\rangle + \frac{1}{\sqrt{\rho_1 \rho_2}} \theta^2 |\phi_2\rangle;$$

and using the definition of the $q$-exponential function, equation (3.17), we get

$$|\tilde{\theta}\rangle = e_q^{(\tilde{b}^\dagger \tilde{\theta})}|\psi_0\rangle.$$ (3.41)

3.6. Resolution of identity

This section treats the overcompleteness of the generalized Grassmannian coherent state for a three-level system. Considering equation (3.25) the weight function of a three-level system is

$$w(\theta, \bar{\theta}) = \sum_{i,j=0}^{2} \alpha_{ij} \theta^i \bar{\theta}^j.$$ (3.42)

Taking into account the pseudo-Hermiticity of the system, both of the following integrals do not satisfy the resolution of identity, i.e.

$$\int d\bar{\theta} \int d\theta w(\theta, \bar{\theta}) |\theta\rangle \langle \theta| \neq I, \quad \int d\bar{\theta} \int d\theta w(\theta, \bar{\theta}) |\bar{\theta}\rangle \langle \bar{\theta}| \neq I$$

but like that of the GPHCS, equation (3.27), we can get the overcompleteness for the integrals:

$$\int d\bar{\theta} \int d\theta w(\theta, \bar{\theta}) |\theta\rangle \langle \bar{\theta}| = \int d\bar{\theta} \int d\theta w(\theta, \bar{\theta}) |\theta\rangle \langle \theta| = I$$

with the following proper weight function:

$$w(\theta, \bar{\theta}) = \rho_1 \rho_2 + \frac{\rho_1 \theta \bar{\theta} + \theta^2 \bar{\theta}^2}{q}.$$ (3.43)

3.7. Stability

Turning to the results of section 3.4 we will study the stability of generalized GCS of $SU_q(2)$. As mentioned in section 3.4, time evolution of coherent states governed by Hamiltonian and the evolved coherent states will be coherent states if they remain as the eigenstates of the annihilation operator, i.e. equation (3.32). Now considering equation (3.32) and the explicit form of $SU_q(2)$ coherent states, i.e. equation (3.40), the evolved coherent state is

$$|\theta, t\rangle = e^{-iE_0 t} |\psi_0\rangle + \frac{\bar{q}}{\sqrt{\rho_1}} e^{-iE_1 t} \theta |\psi_1\rangle + \frac{1}{\sqrt{\rho_1 \rho_2}} e^{-iE_2 t} \theta^2 |\psi_2\rangle.$$ (3.44)

Taking into account that $E_0 = -E, E_1 = 0$ and $E_2 = E$, we put $\theta(t) = e^{-iEt} \theta$ and rewrite the above equation in the form

$$|\theta, t\rangle = e^{iEt} \left( |\psi_0\rangle + \frac{\bar{q}}{\sqrt{\rho_1}} \theta(t) |\psi_1\rangle + \frac{1}{\sqrt{\rho_1 \rho_2}} \theta^2(t) |\psi_2\rangle \right) = e^{iEt} |\theta(t)\rangle$$

which manifests the stability of the time evolution of the CS $|\theta\rangle$. 
3.8. Squeezed states

Based on the definition of the standard squeezing operator we define the generalized Grassmannian pseudo-Hermitian squeezing operator as follows:

\[ S(\theta) = \exp\left[ \frac{1}{2}(\theta b^\dagger - \overline{\theta} b^2) \right]; \]  

(3.44)

it is easy to check that for a three-level system, \( SU_\rho(2) \) algebra, the operators \( b^3 \) and \( b^\dagger_3 \) are zero; then the squeezing operator is reduced to

\[ S(\theta) = I + \frac{1}{2}(\theta b^\dagger - \overline{\theta} b^2) - \frac{\bar{q}}{4} \bar{\theta} (b^\dagger b^2 + q b^3 b^3). \]  

(3.45)

The generalized Grassmannian pseudo-Hermitian squeezed states by definition are obtained from the application of \( S(\theta) \) on the ground state, \( |\psi_0\rangle \), i.e.

\[ |\xi\rangle = S(\theta)|\psi_0\rangle. \]  

(3.46)

Using equation (3.45) we have

\[ |\xi\rangle = |\psi_0\rangle + \frac{\sqrt{\rho_1 \rho_2}}{2} \theta |\psi_2\rangle - \frac{\rho_1 \rho_2}{4} \theta \overline{\theta} |\psi_0\rangle, \]

\[ = \left( 1 - \frac{\rho_1 \rho_2}{4} \theta \overline{\theta} \right) |\psi_0\rangle + \frac{\sqrt{\rho_1 \rho_2}}{2} \theta |\psi_2\rangle. \]  

(3.47)

Similar to the case of GPHCS, there is another family of generalized Grassmannian pseudo-Hermitian squeezed states which can be obtained from the action of \( \eta \) on \( |\xi\rangle \), equation (3.41); then

\[ |\tilde{\xi}\rangle = \eta |\xi\rangle = \left( 1 - \frac{\rho_1 \rho_2}{4} \theta \overline{\theta} \right) |\phi_0\rangle + \frac{\sqrt{\rho_1 \rho_2}}{2} \theta |\phi_2\rangle. \]  

(3.48)

Finally we define the operator \( \tilde{S}(\theta) \) in terms of \( S(\theta) \) as follows:

\[ \tilde{S}(\theta) = \eta S(\theta) \eta^{-1}; \]  

(3.49)

using equation (3.45), it is possible to rewrite \( \tilde{S}(\theta) \) as

\[ \tilde{S}(\theta) = I + \frac{\sqrt{\rho_1 \rho_2}}{2} (\theta |\psi_2\rangle \langle \psi_0| - \overline{\theta} |\phi_0\rangle \langle \phi_2|) - \frac{\rho_1 \rho_2}{4} \theta \overline{\theta} (|\phi_2\rangle \langle \psi_0| + q |\phi_0\rangle \langle \psi_2|), \]

and considering the fact that for the \( SU_\rho(2) \) algebra the operators \( b^\dagger c^\dagger \) and \( b^\dagger \) are zero we can get the exponential form of \( \tilde{S}(\theta) \) as

\[ \tilde{S}(\theta) = \exp\left[ \frac{1}{2}(\theta \tilde{b}^\dagger \tilde{c}^\dagger - \overline{\theta} \tilde{b}^\dagger \tilde{b}^\dagger) \right]. \]

4. Conclusion

Generalized GCS associated with the pseudo-Hermitian lowering operator, which annihilate the eigenbasis of the \( n \)-level pseudo-Hermitian Hamiltonian \( H \) are constructed. Taking to account the bi-orthonormal nature of the pseudo-Hermitian system, it is possible to prepare two families of the coherent states. Meanwhile the resolution of identity is discussed and it is explored that the resulted coherent states satisfy the bi-overcompleteness condition instead of the overcompleteness one, i.e. their bi-overcompleteness inherited from the bi-orthonormality of the pseudo-Hermitian eigenbasis. Furthermore as a special case the generalized GCS of \( q \)-deformed \( SU(2) \), \( SU_\rho(2) \), for a three-level system were studied in detail and finally the squeezed states of this three-level system were introduced. Finally we note that the construction of GPHCS outlined here may also be extended to compound systems of the fermionic and Grassmannian density matrix in quantum information theory.
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