Localization of grouplike function and section spaces with compact domain

Claude L. Schochet and Samuel B. Smith

Abstract. By recent results in [19], the standard localization theory for function and section spaces due to Hilton-Mislin-Roitberg and Möller extend outside the CW category to the case of compact metric domain in the presence of a grouplike structure. We study applications in two cases directly generalizing the gauge group of a principal bundle. We prove an identity for the monoid $\text{aut}(\xi)$ of fibre-homotopy self-equivalences of a Hurewicz fibration $\xi$ — due to Gottlieb and Booth-Heath-Morgan-Piccinini in the CW category — in the compact case. This leads to an extended localization result for $\text{aut}(\xi)$. We also obtain an extended localization theory for groups of sections $\Gamma(\zeta)$ of a fibrewise group $\zeta$. We give two applications in rational homotopy theory.

1. Introduction

We study two generalizations of the homotopy classification problem for gauge groups. Let $X$ be a space, $G$ a connected CW topological group and $h: X \to BG$ a map. The gauge group $G(P)$ corresponding to this data is the topological group of $G$-equivariant self-maps of $P: E \to X$ where $P$ is the principal $G$-bundle induced by $h$. Fixing $G$ and $X$, the problem is that of determining the number of distinct homotopy types or, alternately, the number of $H$-homotopy types corresponding to maps in $[X, BG]$. Complete results in special cases are given by Kono [20], Crabb-Sutherland [6] and Kono-Tsukuda [21] among many others.

After rationalization, the gauge group classification problem admits a complete solution with considerable generality. By [19] Theorem D], when $X$ is a compact metric space and $G$ is a homotopy finite, connected, CW group, the rationalization of the connected component of the identity of $G(P)$ is $H$-commutative and independent of the classifying map $h: X \to BG$. When $X$ is actually a finite CW complex, this result may be deduced from an identity for the gauge group due to Gottlieb, [11], below, combined with the standard localization theory for function spaces due to Hilton-Mislin-Roitberg [17] (see [19] Theorem 5.7]). Alternately, the result for $X$ finite CW follows from corresponding localization results for section spaces due to Möller [24] and a result of Crabb-Sutherland [6] Proposition 2.2]. The proof for $X$ compact metric in [19] entails an extension of the localization results of Hilton-Mislin-Roitberg and Möller to the case of non-CW domain. In this

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paper, we expand on this analysis to develop corresponding extended localization theories for two natural generalizations of the gauge group.

The two generalizations we consider are related to two basic identities for the gauge group. First, let \( \text{map}(X, Y) \) denote the space of all continuous maps with the compact-open topology. Let \( \text{map}(X, Y; f) \) denote the path-component of a given map \( f: X \to Y \). Then, for \( X \) a finite CW complex, there is an \( H \)-equivalence

\[
G(P) \simeq \Omega \text{map}(X, BG; h)
\]

(see [12] and [1] Proposition 2.4)). Second, let \( \text{Ad}(P): E \times_G G^{\text{ad}} \to X \) denote the adjoint bundle. Here the total space is the quotient of the product by the diagonal action where \( G^{\text{ad}} = G \) is a left-\( G \) space via the adjoint action and the projection is induced by the projection \( E \to X \). This is an non-principal \( G \)-bundle. We have an isomorphism

\[
G(P) \cong \Gamma(\text{Ad}(P)),
\]

where the latter space is the group of sections with fibrewise multiplication (see [1] p.539).

Now suppose given a Hurewicz fibration \( \xi: E \to X \). We assume here and throughout that \( X \) has a distinguished, nondegenerate basepoint. By the fibre of \( \xi \), we will mean the fibre over this basepoint. We consider the monoid \( \text{aut}(\xi) \) consisting of all fibre-homotopy self-equivalences of \( \xi \) covering the identity of \( X \) topologized as a subspace of \( \text{map}(E, E) \). The monoid \( \text{aut}(\xi) \) is a natural generalization of the gauge group to the fibre-homotopy setting. Unlike the gauge group, the rational \( H \)-homotopy type of this monoid is generally a nontrivial invariant of the fibre-homotopy theory of \( \xi \) (see [11]).

By [2] Theorem 3.3, the identity (1) extends to a corresponding identity for \( \text{aut}(\xi) \) when both \( X \) and the fibre \( F \) are finite CW complexes. Our first main result extends this identity, in turn, to the case \( X \) is compact metric. Recall the universal \( F \)-fibration, for \( F \) finite CW, may be identified, up to homotopy type, as a sequence \( F \to \text{Baut}_*(F) \to \text{Baut}(F) \) where \( \text{aut}(F) \) and \( \text{aut}_*(F) \) are the monoids of free and based homotopy self-equivalences of \( F \). The spaces \( \text{Baut}(F) \) and \( \text{Baut}_*(F) \) are the Dold-Lashof classifying spaces for these monoids. (See [32, 25] for the classification theory and [13] for the identification with Dold-Lashof [8].)

**Theorem 1.** Let \( X \) be a compact metric space, \( F \) a finite CW complex and \( h: X \to \text{Baut}(F) \) a map. Let \( \xi: E \to X \) be the corresponding \( F \)-fibration. Then there is an \( H \)-equivalence

\[
\text{aut}(\xi) \simeq \Omega \text{map}(X, \text{Baut}(F); h).
\]

As a consequence, we obtain an extended localization theory for \( \text{aut}(\xi) \). Let \( \mathbb{P} \) be a collection of primes. We say a space \( Y \) is nilpotent if \( Y \) is connected, has the homotopy type of a CW complex and has a nilpotent homotopy system (see [17] Definition II.2.1]). In this case, \( Y \) admits a \( \mathbb{P} \)-localization \( \ell_{\mathbb{P}}: Y \to Y_{\mathbb{P}} \) [17] Theorem II.3A]. Given a map \( f: X \to Y \) we write \( f_{\mathbb{P}} = \ell_{\mathbb{P}} \circ f: X \to Y_{\mathbb{P}} \). Given a monoid \( G \) we write \( G_{\mathbb{P}} \) for the path component of the identity. Given a space \( Z \) with distinguished basepoint we write \( \Omega_0Z \) for the space of loops based at the basepoint. For the function space \( \text{map}(X, Y; f) \) we assume \( f \) is the basepoint.

**Theorem 2.** Let \( X \) be a simply connected compact metric space, \( F \) a finite CW complex and \( h: X \to \text{Baut}(F) \) a map. Let \( \xi: E \to X \) be the corresponding
The second generalization we consider is based on (2) in which the gauge group corresponds to a group of sections. Recall $\zeta: E \to X$ is a fibrewise group if there is a fibrewise map $m: E \times_X E \to E$ over $X$, a section $e: X \to E$ and a map $i: E \to E$ over $X$ satisfying: (i) $m$ is associative, (ii) $e$ is a two-sided unit and (iii) $i$ is an inverse with respect to the maps $m$ and $e$. The space of sections $\Gamma(\zeta)$ is then a topological group with the multiplication of sections induced by $m$. More generally, relaxing the group axioms to require identities only up to homotopy, $\zeta$ is a fibrewise grouplike space (\cite[p.62]{[5]}) and $\Gamma(\zeta)$ is a grouplike space. Our motivating example is the adjoint bundle $\text{Ad}(P)$ of a principal $G$-bundle $P$, as above. Observe that, if $P: E \to X$ is classified by a map $h: X \to BG$, then $\text{Ad}(P)$ is the pullback by $h$ of the universal $G$-adjoint bundle $EG \times_G G^{ad} \to BG$ which is, in particular, a CW fibration.

Suppose generally that $\zeta: E \to X$ is a fibrewise grouplike space with connected grouplike fibre $G$ and, further, that $\zeta$ is the pullback of a CW fibration. In this case, we may still identify a fibrewise $P$-localization $\zeta \to \zeta_{(P)}$ of $\zeta$ (see the remarks preceding Theorem [2,4] below). Our third main result extends [24, Theorem 5.3] from the case the base space is finite CW to the case of compact metric base in this context.

**Theorem 3.** Let $\zeta: E \to X$ be a fibrewise grouplike space with connected, CW grouplike fibre $G$ and base $X$ a compact metric space. Suppose $\zeta$ is the pullback of a CW fibration. Then $\Gamma(\zeta)_o$ is a nilpotent space and the map

$$\Gamma(\zeta)_o \to \Gamma(\zeta_{(P)})_o$$

induced by a fibrewise $P$-localization $\zeta \to \zeta_{(P)}$ is a $P$-localization map.

**Remark 1.1.** In many circumstances, the fibrewise $P$-localization $\zeta_{(P)}$ of a fibrewise grouplike space $\zeta: E \to X$ will itself be a fibrewise grouplike space over $X$ and the fibre map $\zeta \to \zeta_{(P)}$ will be equivariant. In this case, the equivalence of Theorem 3 is actually an $H$-equivalence. For example, this is the case for the adjoint bundle and, generally, when $\zeta$ is a CW fibration. For general compact metric $X$, the fibrewise grouplike structure for $\zeta_{(P)}$ is not assured due to the lack of uniqueness of fibrewise $P$-localization outside the CW category.

The paper is organized as follows. We prove Theorems [13, 16] in Section 2. In Section 3 we apply our results to obtain two consequences. We prove that the rationalization of $\text{aut}(\xi)_o$ is $H$-commutative and independent of the classifying map for fibrations $\xi$ with fibre satisfying a famous conjecture in rational homotopy theory. (Theorem 5.3. This result extends \cite[6.10] Theorem 4). As an application of Theorem 3 we extend \cite[19] Theorem F] on the classification of projective gauge groups (Theorem 5.6). In Section 4 we deduce based versions of our main results.

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2. Localization of Function and Section Spaces

The standard results on $P$-localization of function spaces are as follows. By Milnor [27], the path components of $\text{map}(X, Y)$ are of CW homotopy type when $X$ is a compact metric space and $Y$ is a CW complex. By Hilton-Mislin-Roitberg, when $X$ is finite CW and $Y$ is a nilpotent space then $\text{map}(X, Y; f)$ is itself a nilpotent space and

$$\text{map}(X, Y; f)_P \simeq \text{map}(X, Y_P; f_P)$$  \[17\] . Theorem II.3.11. Recall we write $f_P = \ell_Y \circ f : X \to Y_P$.

These results all hold for spaces of basepoint-preserving functions, as well. In what immediately follows, we will focus on the basepoint free case. We discuss the based case in Section 4. The preceding results also hold with alternate hypotheses: namely, $X$ may be any CW complex when $Y$ is a finite Postnikov piece (see [18]). We will not consider these alternate hypotheses here as they are not amenable to our methods.

The results of Hilton-Mislin-Roitberg were extended to section spaces by Möller [24] (see also, Scheerer [28]). Let $\xi : E \to X$ be a fibration of connected CW complexes with $X$ finite. Let $F$ be the fibre over a basepoint of $X$. If $F$ is a nilpotent space then $\xi$ then admits a fibrewise $P$-localization which is a fibre map $\xi \to \xi(P)$ over $X$ inducing $P$-localization on the fibres [26]. By [24], Theorem 5.3], under these hypotheses, the component of $\Gamma(\xi; s)$ corresponding to a given section $s : X \to E$ is a nilpotent space and

$$\Gamma(\xi; s)_P \simeq \Gamma(\xi(P); s_0)$$

where $s_0$ is the section of $\xi(P)$ induced by $s$.

Our extensions of these localization results depend on a classical result of Eilenberg-Steenrod in [9]. Let $X$ be a compact metric space. By [9, Theorem X.10.1], there is an inverse system of finite complexes $X_j$ with structure maps $g_{ij} : X_j \to X_i$ for $i \leq j$ and compatible maps $g_j : X \to X_j$ such that the induced map

$$g : X \to \lim_{\leftarrow j} X_j$$

is a homeomorphism. Further, given any map $f : X \to Y$ for $Y$ a CW complex, there is an index $m$ and a map $f_m : X_m \to Y$ such that $f$ is homotopic to $f_m \circ g_m$. [9, Theorem X.11.9].

We apply this result to study the function space $\text{map}(X, Y; f)$, as follows. Choose $m$ for $f : X \to Y$ as above. Given $j \geq m$ define $f_j : X_j \to Y$ by setting $f_j = f_m \circ g_{mj}$. Restricting to indices $j \geq m$, we obtain a direct system $\text{map}(X_j, Y; f_j)$ with structure maps $(g_{ij})^* : \text{map}(X_i, Y; f_i) \to \text{map}(X_j, Y; f_j)$ and compatible maps

$$(g_j)^* : \text{map}(X_j, Y; f) \to \text{map}(X_j, Y; f_j)$$

both induced by precomposition. We have:

**Theorem 2.1.** ([19, Theorem 6.4]) Let $X$ be a compact metric space and $Y$ a CW complex. Then, for all $n \geq 1$, the maps $(g_j)^*$ above induce an isomorphism

$$\lim_{\leftarrow j} \pi_n(\text{map}(X_j, Y; f_j)) \cong \pi_n(\text{map}(X, Y; f)).$$  \[19\] Theorem 6.4]
This result leads to an extension of the Hilton-Mislin-Roitberg result, mentioned above, provided the space \(\text{map}(X; Y; f)\) is known \textit{a priori} to be a nilpotent space [19] Theorem 7.1. Essentially the same proof yields the following.

**Theorem 2.2.** Let \(X\) be a compact metric space. Let \(Y\) be a nilpotent space with \(\mathbb{P}\)-localization \(\ell_Y: Y \to Y_{\mathbb{P}}\). Let \(f: X \to Y\) be a given map. Then \(\ell_Y\) induces an \(H\)-equivalence

\[\Omega_\ast\text{map}(X; Y; f)_{\mathbb{P}} \simeq \Omega_\ast\text{map}(X; Y_{\mathbb{P}}; f).\]

**Proof.** Consider the commutative square

\[
\begin{array}{ccc}
\lim_j \pi_n(\text{map}(X; Y; f)) & \overset{\sim}{\longrightarrow} & \pi_n(\text{map}(X; Y; f)) \\
\downarrow & & \downarrow \\
\lim_j \pi_n(\text{map}(X; Y_{\mathbb{P}}; (f_{\mathbb{P}}))) & \overset{\sim}{\longrightarrow} & \pi_n(\text{map}(X; Y_{\mathbb{P}}; f))
\end{array}
\]

with vertical maps induced by \(\ell_Y\). That the left vertical map is a \(\mathbb{P}\)-localization map follows from [17] Theorem II.3.11. Thus the right vertical map is a \(\mathbb{P}\)-localization map, as well. Looping, we see that \(\ell_Y\) induces a weak \(H\)-equivalence \((\Omega_\ast\text{map}(X; Y; f))_{\mathbb{P}} \simeq \Omega_\ast\text{map}(X; Y_{\mathbb{P}}; f)\) and so an honest \(H\)-equivalence since the spaces are CW, again by [27].

Next let \(\xi: E \to X\) be a Hurewicz fibration over a compact metric space with a fixed section \(s: X \to E\). Suppose \(\xi\) is the pullback of a CW fibration \(\xi_0: E_0 \to B\) for some map \(h: X \to B\). Then \(\Gamma(\xi; s)\) is of CW homotopy type [19] Proposition 3.2] Given an inverse system of finite complexes \(X_j\) for \(X\) with compatible maps \(g_j: X \to X_j\) as above, choose an index \(m\) so that \(h: X \to B\) factors as \(h = h_m \circ g_m\) for some map \(h_m: X_m \to B\). Given \(j \geq m\), write \(\xi_j: E_j \to X_j\) for the pullback of the fibration \(\xi_0\) via the map \(h_j = h_m \circ g_m j: X_j \to B\). Restricting again to indices \(j \geq m\) gives a direct system of spaces of sections \(\Gamma(\xi_j)\) with structure maps \(\gamma_{ij}: \Gamma(\xi_i) \to \Gamma(\xi_j)\) and compatible maps \(\gamma_j: \Gamma(\xi_j) \to \Gamma(\xi)\) induced by \(g_{ij}\) and \(g_j\) for \(j \geq i \geq m\).

By [19] Theorem 6.5], with this set-up the induced map \(\gamma: \lim_j \Gamma(\xi_j) \to \Gamma(\xi)\) gives a bijection

\[\lim \pi_0(\Gamma(\xi_j)) \cong \pi_0(\Gamma(\xi))\]

of pointed sets. Thus we may choose an index \(m' \geq m\) and a section \(s_{m'}\) of \(\xi_{m'}\) such that \(s_{m'}\) induces a section homotopic to \(s\) via \(\gamma_{m'}\). Restricting now to indices \(j \geq m'\), let \(s_j\) be the section induced on \(\xi_j\) by \(s_{m'}\). We obtain a direct system of connected spaces \(\Gamma(\xi_j; s_j)\) with compatible maps \(\gamma_j: \Gamma(\xi_j; s_j) \to \Gamma(\xi; s)\) for \(j \geq m'\).

Quoting the theorem for \(n \geq 1\), we have:

**Theorem 2.3.** ([19] Theorem 6.5]) Let \(\xi: E \to X\) be a Hurewicz fibration over \(X\) a compact metric space with a fixed section \(s: X \to E\). Suppose \(\xi\) is the pullback of a CW fibration. Then, for all \(n \geq 1\), the maps \(\gamma_j\) above induce an isomorphism

\[\lim_j \pi_n(\Gamma(\xi_j; s_j)) \cong \pi_n(\Gamma(\xi; s)).\]
By the work of May [26], a fibration \( \xi: E \to X \) of CW complexes with nilpotent fibre \( F \) admits a \textit{fibrewise} \( \mathbb{P} \)-localization which is a fibration \( \xi(p): E_0 \to X \) and a map \( g: E \to E_0 \) over \( X \) such that \( g \) induces \( \mathbb{P} \)-localization \( F \to F_p \) on fibres. This fibrewise \( \mathbb{P} \)-localization of \( \xi \) is unique up to fibre-homotopy equivalence by Llerena [22, Theorem 6.1].

We may directly extend this construction to non-CW fibrations which are nevertheless the pullback of an appropriate CW fibration. Specifically, let \( \xi: E \to X \) be the pullback of a fibration \( \xi_0: E_0 \to B \) of CW complexes with nilpotent fibre via a map \( h: X \to B \). We take \( \xi(p) = h^{-1}(\xi_0(p)) \), the pullback of the fibrewise \( \mathbb{P} \)-localization of \( \xi_0 \). We note that uniqueness is no longer assured.

\textbf{Theorem 2.4.} Let \( \xi: E \to X \) be a fibration of spaces with nilpotent fibre and compact metric base. Suppose \( \xi \) is a pullback of a CW fibration and that, for some section \( s \) of \( \xi \), the component \( \Gamma(\xi; s) \) is a nilpotent space. Let \( \xi \to \xi(p) \) be a fibrewise \( \mathbb{P} \)-localization and \( s_0 \) the induced section. Then

\[ \Gamma(\xi; s)_p \simeq \Gamma(\xi(p); s_0). \]

\textbf{Proof.} Choose an inverse system \( X_j \) of finite complexes for \( X \) and let \( \xi_j \) be the corresponding fibrations over \( X_j \) with fibre \( F \) and compatible sections \( s_j \). Consider the commutative diagram, as in the proof of Theorem 2.2.

\[
\begin{array}{rccc}
\lim_j \pi_n(\Gamma(\xi_j; s_j)) & \xrightarrow{\cong} & \pi_n(\Gamma(\xi; s)) & \\
\downarrow & & \downarrow & \\
\lim_j \pi_n(\Gamma(\xi_j(p); (s_j)_0)) & \xrightarrow{\cong} & \pi_n(\Gamma(\xi(p); s_0)) & \\
\end{array}
\]

The horizontal maps are isomorphisms by Theorem 2.3. The left vertical map is \( \mathbb{P} \)-localization by [24, Theorem 5.3]. Thus the right vertical map is a \( \mathbb{P} \)-localization, as well. \( \square \)

Theorem 2.4 implies:

\textbf{Proof of Theorem 3} Recall we are assuming \( \zeta: E \to X \) is fibrewise grouplike over a compact metric space \( X \) with fibre \( G \) a connected CW grouplike space. Further, we assume \( \zeta \) is the pullback of CW fibration. By [19, Proposition 3.2], \( \Gamma(\zeta)_c \) is of CW type and so a nilpotent space since grouplike. The result is thus a direct consequence of Theorem 2.4. \( \square \)

We now turn to the proof of Theorem 2. Fix a compact metric space \( X \) and a finite CW complex \( F \). We write \( \xi_\infty: E_\infty \to \text{Baut}(F) \) for the universal \( F \)-fibration. (Here \( E_\infty \simeq \text{Baut}_*(F) \).) Given a map \( h: X \to \text{Baut}(F) \), let \( \xi: E \to X \) be the pullback. In [2], the authors define a fibration

\[ \xi \Box_1 \xi_\infty: E \Box E_\infty \to X \]

in which the total space is the set of all homotopy equivalences \( f_{a,b}: E_a \to (E_\infty)_b \) between fibres of \( \xi \) and fibres of \( \xi_\infty \) suitably topologized. The projection is given by \( f_{a,b} \mapsto a \in X \) and so the fibre over a basepoint of \( X \) may be identified with \( \text{aut}(F) \). Next, the authors define a fibration

\[ \Phi: \Gamma(\xi \Box_1 \xi_\infty) \to \text{map}(X, \text{Baut}(F)) \]
with fibre $\Phi^{-1}(h) \approx \text{aut}(\xi)$. We observe that the construction of $\Phi$ and the identification of the fibre both hold in our case. For the former, the key is [2] Proposition 2.1] whose proof depends on the use of exponential laws for functional fibrations which hold for $X$ compact by [3]. The identification of the fibre over $h$ with $\text{aut}(\xi)$ requires that $\xi$ be an $\mathcal{F}$-fibration in the sense of [25] Definition 2.1] where $\mathcal{F}$ is the category of “fibres” homotopic to $F$. This fact is a consequence of our assumption that $\xi$ is the pullback of a CW fibration (use [25] Lemma 3.3 and Proposition 3.4]).

**Lemma 2.5.** The space $\Gamma(\xi \Box_1 \xi_\infty)$ is contractible.

**Proof.** Observe that $\xi \Box_1 \xi_\infty$ is the pullback of the fibration

$$\xi_\infty \Box_1 \xi_\infty: E_\infty \Box E_\infty \to \text{Baut}(F)$$

by $h: X \to \text{Baut}(F)$. The fibration $\xi_\infty \Box_1 \xi_\infty$ is a CW fibration by Schön [30] since both the base $\text{Baut}(F)$ and the fibre $\text{aut}(F)$ are CW since $F$ is finite CW. Thus $\Gamma(\xi \Box_1 \xi_\infty)$ is of CW type [19] Proposition 3.2. It thus suffices to show $\Gamma(\xi \Box_1 \xi_\infty)$ is weakly contractible.

For this, we apply Theorem [23] Following the procedure described before this result, write $X = \lim_j X_j$ with each $X_j$ a finite complex. For $j \geq m$ as above, let $\xi_j: E_j \to X_j$ be the corresponding fibration over $X_j$. Let $\xi_j \Box_1 \xi_\infty: E_j \Box E_\infty \to X_j$ the associated construction. Functorality of this construction gives fibre maps $\xi_j \Box_1 \xi_\infty \to \xi_k \Box_1 \xi_\infty$ for $j \leq k \geq m$ and compatible fibre maps $\xi \Box_1 \xi_\infty \to \xi_j \Box_1 \xi_\infty$. We thus obtain a direct system $\Gamma(\xi \Box_1 \xi_\infty)$ of sections with compatible maps $\gamma_j: \Gamma(\xi \Box_1 \xi_\infty) \to \Gamma(\xi_j \Box_1 \xi_\infty)$ satisfying the conditions of Theorem [23]. By [2] Proposition 3.1], each space $\Gamma(\xi_j \Box_1 \xi_\infty)$ is contractible. Thus

$$\pi_n(\Gamma(\xi \Box_1 \xi_\infty)) \cong \lim_j \pi_n(\Gamma(\xi_j \Box_1 \xi_\infty)) = 0,$$

for $n \geq 0$, as needed.

We obtain, as a consequence:

**Proof of Theorem 1** Let $X$ be a compact metric space, $F$ a finite CW complex and $h: B \to \text{Baut}(F)$ a map. Let $\xi: E \to X$ be the corresponding $F$-fibration. By Lemma 2.5 the connecting map

$$\delta: \Omega \text{map}(X, \text{Baut}(F); h)) \to \Phi^{-1}(h) \approx \text{aut}(\xi)$$

in the Barratt-Puppe sequence for $\Phi: \Gamma(\xi \Box_1 \xi_\infty) \to \text{map}(X, \text{Baut}(X))$ is a homotopy equivalence. By [2] Theorem 3.3], $\delta$ is a multiplicative map and so gives the desired $H$-equivalence.

Using Theorem 1 we obtain:

**Proof of Theorem 2** Recall we are assuming $X$ is a simply connected, compact metric space and $\xi: E \to X$ is the pullback of the universal $F$-fibration via $h: X \to \text{Baut}(X)$. Since $X$ is simply connected, $h$ lifts to a map $\tilde{h}: X \to \text{Baut}(X)_0$ to the universal cover. Further, we have an $H$-equivalence

$$\Omega \text{map}(X, \text{Baut}(F); h) \simeq \Omega \text{map}(X, \text{Baut}(F)_0; \tilde{h})$$

as these spaces are evidently weakly $H$-equivalent and both are CW complexes [27]. The result now follows from Theorem 2.2 and Theorem 1. □
3. Consequences in Rational Homotopy Theory

There are good algebraic models for the rational homotopy theory of the monoid \( \text{aut}(\xi) \) and the space of sections \( \Gamma(\xi) \). (See \[11\] for the former and \[16, 4\] for the latter.) However, these models require finiteness and/or nilpotence conditions on the fibration and so are not directly applicable to the case of compact domain. In our main results above, we require that \( \xi \) and \( \zeta \) occur as the pullback of a CW fibration. In this section, we observe that, in certain special circumstances, the structure of the rationalization of these CW fibrations allow for a description of the rational homotopy theory of these grouplike spaces of maps.

For the monoid \( \text{aut}(\xi) \), the CW fibration in question is the universal \( F \)-fibration. The following result is the \( \mathbb{P} \)-local version of \[2\] Proposition 6.1 in our setting.

**Theorem 3.1.** Let \( X \) be a simply connected, compact metric space and \( F \) a finite complex. Suppose the \( \mathbb{P} \)-localization of \( \text{Baut}(F)_{\circ} \) is a grouplike space. Then, for all fibrations \( \xi \) corresponding to maps in \( [X, \text{Baut}(F)] \), we have an H-equivalence

\[
\text{(aut}(\xi)_{\circ})_{\mathbb{P}} \simeq \text{map}(X, (\text{aut}(F)_{\circ})_{\mathbb{P}}; 0)
\]

where the latter is the space of null maps into the grouplike space \( (\text{aut}(F)_{\circ})_{\mathbb{P}} \) which is a grouplike space with pointwise multiplication.

**Proof.** The proof is the same as that given in \[19\] Example 4.7. We give the details for completeness. Using the homotopy inverse for \( (\text{Baut}(F)_{\circ})_{\mathbb{P}} \), we obtain a homotopy equivalence between \( \text{map}(X, (\text{Baut}(F)_{\circ})_{\mathbb{P}}; \tilde{h}_{\mathbb{P}}) \) and \( \text{map}(X, (\text{Baut}(F)_{\circ})_{\mathbb{P}}; 0) \). Looping this equivalence and applying Theorem 2 yields H-equivalences:

\[
\text{(aut}(\xi)_{\circ})_{\mathbb{P}} \simeq \Omega_{\circ}\text{map}(X, (\text{Baut}(F)_{\circ})_{\mathbb{P}}; \tilde{h}_{\mathbb{P}}) \simeq \Omega_{\circ}\text{map}(X, (\text{Baut}(F)_{\circ})_{\mathbb{P}}; 0)
\]

At the null component, we may pull \( \Omega_{\circ} \) inside to get

\[
\Omega_{\circ}\text{map}(X, (\text{Baut}(F)_{\circ})_{\mathbb{P}}; 0) \simeq \text{map}(X, \Omega_{\circ}(\text{Baut}(F)_{\circ})_{\mathbb{P}}; 0) \simeq \text{map}(X, (\text{aut}(F)_{\circ})_{\mathbb{P}}; 0),
\]

as needed. \( \square \)

The condition on \( (\text{Baut}(F)_{\circ})_{\mathbb{P}} \) is, of course, very strong and will be rarely satisfied. However, as we explain now, there is a famous class of spaces in rational homotopy theory, identified by Halperin in \[15\], that conjecturally (and in many known cases) do have this property after rationalization. In the rational case, observe that it is sufficient to check \( (\text{Baut}(F)_{\circ})_{\mathbb{Q}} \) is an H-space to apply Theorem 3.1 since, rationally, all H-spaces are grouplike (see, e.g., \[29\]).

We say a space \( F \) is an \( F_{0} \)-space if \( F \) is a simply connected elliptic space (i.e., a finite complex with \( \pi_{*}(F) \otimes \mathbb{Q} \) finite-dimensional) and satisfying \( H^{\text{odd}}(F; \mathbb{Q}) = 0 \). Halperin’s conjecture (see \[15\]) translated to our setting is:

**Conjecture 3.2.** (Halperin) Let \( F \) be an \( F_{0} \)-space. Then the monoid \( \text{aut}(F)_{\circ} \) has vanishing even degree rational homotopy groups.

Examples of \( F_{0} \)-spaces include products of even-dimensional spheres and complex projective spaces for which Conjecture 3.2 is easily confirmed. Homogeneous spaces \( G/H \) of equal rank, compact pairs \( H \subseteq G \) are also \( F_{0} \)-spaces and satisfy Conjecture 3.2 by \[31\].

We note that, if \( F \) is an \( F_{0} \)-space satisfying Conjecture 3.2 then \( (\text{Baut}(F)_{\circ})_{\mathbb{Q}} \) is a rational H-space since the Sullivan model of a nilpotent space with no odd rational homotopy has trivial differential. Further, the (odd) rational homotopy
groups of aut$(F)_o$ can be, in practice, directly computed from the minimal model $(\wedge V; d)$ of $F$ using Sullivan’s identity
\[ \pi_k(\text{aut}(F)_o) \otimes \mathbb{Q} \cong H_k(\text{Der}(\wedge V; d)). \]
(See, e.g., [14].) Here the latter space is the homology of the DG vector space of negative degree derivations of $(\wedge V; d)$.

The following result extends [10] Theorem 4 and [11] Example 2.7. We write $\hat{H}^*(X; \mathbb{Q})$ for rational Čech cohomology.

**Theorem 3.3.** Let $X$ be a simply connected, compact metric space and $F$ an $F_0$-space satisfying Conjecture [23]. Let $h: X \rightarrow \text{Baut}(F)$ be a map and $\xi$ the corresponding fibration over $X$. Then:

1. The rational $H$-homotopy type of the monoid $\text{aut}(\xi)_o$ is independent of the classifying map $h$.

2. The monoid $\text{aut}(\xi)_o$ is rationally $H$-commutative and equivalent to a product of Eilenberg-Mac Lane spaces with homotopy groups given by
\[ \pi_n(\text{aut}(\xi)_o) \otimes \mathbb{Q} \cong \bigoplus_{k \geq n} \hat{H}^{k-n}(X; \mathbb{Q}) \otimes (\pi_k(\text{aut}(F)_o) \otimes \mathbb{Q}) \]
for $n \geq 1$.

**Proof.** The first statement follows directly from the preceding paragraph and Theorem 3.1. As for (2), we note $\pi_*(\text{aut}(F)_o) \otimes \mathbb{Q}$ is oddly graded and so, for degree reasons, does not admit any Samelson products. It follows that aut$(F)_o$ is rationally $H$-commutative [29]. Thus $\text{aut}(\xi)_o \approx_\mathbb{Q} \text{map}(X, \text{aut}(F)_o; 0)$ is rationally $H$-commutative, as well by [23] Theorem 4.10.

The computation of the rational homotopy groups of $\text{map}(X, \text{aut}(F); 0)$ reduces to the problem of computing homotopy groups of the space of maps into an Eilenberg-Mac Lane space. Here we have the identity of Thom [33]:
\[ \pi_q(\text{map}(X, K(\pi, n))) \cong H^{n-q}(X; \pi) \]
which holds, with ordinary cohomology when $X$ is CW. For compact spaces, the same identity holds with rational Čech cohomology by writing $X = \lim_j X_j$ and using the continuity of Čech cohomology (see the proof of [23] Theorem 5.6). \qed

We prove a related result for grouplike spaces of sections. We say a fibration $\xi: E \rightarrow X$ of connected CW complexes is **nilpotent** if the spaces $E$ and $X$ are nilpotent. In this case, $F$ is also nilpotent by [17] Theorem II.3.12 and term-by-term $P$-localization gives a fibration $\xi_F: E^P \rightarrow X^P$ with fibre $F^P$ [17] Proposition II.2.13). Note that the term-by-term $P$-localization $\xi_F: E^P \rightarrow X^P$ is not the same as the fibrewise $P$-localization $\xi_{(P)}: E^P_0 \rightarrow X$. In fact, we have:

**Lemma 3.4.** Let $\xi: E \rightarrow X$ be a nilpotent fibration with term-by-term $P$-localization $\xi_F$ and fibrewise $P$-localization $\xi_{(P)}$. Then there is a fibre-homotopy equivalence
\[ \xi_{(P)} \simeq \ell_X^{-1}(\xi_F) \]
where $\ell_X: X \rightarrow X^P$ is a $P$-localization map.

**Proof.** Using the uniqueness theorem for fibrewise localization [22] Theorem 6.1) we obtain a fibrewise map $\xi_{(P)} \rightarrow (\ell_X)^{-1}(\xi_F)$ over $X$. By the 5-lemma and [7] Theorem 3.3], this is a fibre homotopy equivalence. \qed
As a direct consequence, we have:

**Theorem 3.5.** Let \( \zeta: E \to X \) be a fibrewise grouplike space over a compact metric space \( X \) with fibre \( G \) connected CW. Suppose \( \zeta \) is the pullback of nilpotent fibration \( \xi \) with \( \mathbb{P} \)-localization \( (\xi)_\mathbb{P} \) fibre-homotopically trivial. Then

\[
\left( \Gamma(\zeta)_\mathbb{P} \right)_\pi \simeq \text{map}(X, G_\mathbb{P}; 0).
\]

**Proof.** By the remarks preceding the proof of Theorem 3, the fibrewise localization of \( \zeta \) may be taken as the pullback of the fibrewise localization \( (\xi)_\mathbb{P} \) of \( \xi \). Since the latter is the pullback of \( \xi_\mathbb{P} \) which is, by hypothesis, fibre-homotopically trivial, we see \( (\zeta)_\mathbb{P} \) is fibre-homotopically trivial, as well. Now use Theorem 3. \( \square \)

We apply this last result to a natural generalization of the gauge group. Given a group \( G \) let \( \text{Pad}(G) = G/Z(G) \) denote the projectification. Given a map \( h : X \to \text{BP}G \) write \( \text{Pad}(h) : E \to X \) for the corresponding principal \( \text{PG} \)-bundle and \( \text{Pad}(\eta) : E \times \text{PG}G_\text{ad} \to \text{BP}G \) the associated \( G \)-bundle where \( \text{PG} \) acts on \( G_\text{ad} = G \) by the adjoint action. This is a fibrewise group. We call the group of sections \( \Gamma(\text{Pad}(h)) \) the *projective gauge group*. When \( G = U(n) \), the projective gauge group corresponds to the group of unitaries of a complex matrix bundle (see [19], Example 3.7).

The classification of rational H-types of projective gauge groups for \( G \) a compact, connected Lie group and \( X \) a compact metric space is given by [19], Theorem F. The following result extends this to all connected Lie groups but at the expense of the H-structure.

**Theorem 3.6.** Let \( X \) be a compact metric space and \( G \) a connected Lie group. Then:

1. The rational homotopy type of \( \Gamma(\text{Pad}(h)) \) is independent of \( h \in [X, \text{BP}G] \).

2. The rational homotopy groups of \( \Gamma(\text{Pad}(h)) \) are given by

\[
\pi_n(\Gamma(\text{Pad}(h))) \otimes \mathbb{Q} \cong \bigoplus_{k \geq n} H^{k-n}(X; \mathbb{Q}) \otimes (\pi_k(G) \otimes \mathbb{Q}),
\]

for \( n \geq 1 \).

**Proof.** We note that \( \text{Pad}(h) \) is the pullback by \( h \) of the universal projective adjoint bundle \( \text{Pad}(\eta) : E \text{PG} \times \text{PG}G_\text{ad} \to \text{BP}G \). Here \( \eta : E \text{PG} \to \text{BP}G \) is the universal principal \( \text{PG} \)-bundle. By [19] Lemma 5.11, the total space \( E \text{PG} \times \text{PG}G_\text{ad} \) is a nilpotent space. (This result assumes \( G \) is compact Lie. However, all that is used there is that \( Z(G) \) have vanishing higher rational homotopy groups, which is true for \( G \) connected Lie.) Thus \( \text{Pad}(\eta) \) is nilpotent since \( \text{BP}G \) is simply connected. Below we show the rationalization of \( \text{Pad}(\eta) \) is fibre-homotopically trivial. The result (1) then follows from Theorem 3.5. The result (2) is proved using the arguments given in the proof of Theorem 3.3 (2).

To show \( \text{Pad}(\eta) : (E \text{PG} \times \text{PG}G_\text{ad})_\mathbb{Q} \to (\text{BP}G)_\mathbb{Q} \) is fibre homotopically trivial, we first show the total space is an H-space. Let \( E \text{PG} \times \text{PG}G_\text{ad} \) denote the total space of the universal \( G \)-adjoint bundle. Then \( E \text{PG} \times \text{PG}G_\text{ad} \) is also a nilpotent space. In fact, we have

\[
E \text{PG} \times \text{PG}G_\text{ad} \simeq \text{map}(S^1, BG; 0)
\]
the free loop space of the classifying space (see, e.g., [19] Lemma 9.1). Since $BG$ is simply connected, $\text{map}(S^1, BG; 0)$ is nilpotent by [17] Theorem II.3.11. Further, by this last result again,

$$(EG \times_G G^{\text{ad}})_Q \simeq \text{map}(S^1, (BG)_Q; 0).$$

Now since $G$ is a connected Lie group, it has oddly graded rational homotopy groups. It follows, as above, that $(BG)_Q$ is an $H$-space. Thus $\text{map}(X, (BG)_Q; 0)$ is an $H$-space, as well. Next, by [19] Lemma 5.9, the natural map

$$\pi: EG \times_G G^{\text{ad}} \to EPG \times_{PG} G^{\text{ad}}$$

induces a surjection on rational homotopy groups. (Again, the lemma is stated for $G$ compact Lie but the proof uses only that $Z(G)$ is rationally aspherical.) Thus, by [23] Lemma 5.8, since $EG \times_G G^{\text{ad}}$ is a rational $H$-space so is $EPG \times_{PG} G^{\text{ad}}$. We have shown that $\text{Pad}(\eta_{PG})$ is a sectioned, nilpotent fibration of rational $H$-spaces with simply connected base. By Lemma 3.7 below, $(\text{Pad}(\eta_{PG}))_Q$ is fibre-homotopically trivial, as needed.

**Lemma 3.7.** Let $\xi: E \to B$ be a fibration of nilpotent spaces with $E, B$ and the fibre $F$ each a rational $H$-space of finite type. Suppose $B$ is simply connected and the linking homomorphism $\partial: \pi_k(B) \to \pi_{k-1}(E)$ is trivial after rationalization for all $k \geq 2$. Then $\xi_Q$ is fibre homotopically trivial.

**Proof.** By [15] Theorem 4.6, $\xi$ is a rational fibration as in [15] Definition 4.5. In general, this means that $\xi_Q$ admits a Koszul-Sullivan model which is a sequence

$$(\wedge V_3; d_3) \overset{P}{\longrightarrow} (\wedge V_2; d_2) \overset{J}{\longrightarrow} (\wedge V_1; d_1).$$

Here $(\wedge V_j; d_j)$ is a free DG algebra over $\mathbb{Q}$ for $j = 1, 2, 3$ and a Sullivan model for $F, E, B$, respectively. The maps $P$ and $J$ are Sullivan models for the projection and fibre inclusion. The models $(\wedge V_1; d_1)$ and $(\wedge V_3; d_3)$ are minimal, meaning $d_j(V_j)$ is contained in the decomposables of $\wedge V_j$ for $j = 1, 3$. Since $B$ and $F$ are assumed to be rational $H$-spaces, this forces $d_3 = d_1 = 0$. The model $(\wedge V_2; d_2)$ is not, in general, minimal. However, by [15] Theorem 4.12, the vanishing of the rational linking homomorphism implies $(\wedge V_2; d_2)$ is a minimal model for $E$, as well. Since $H^*(E; \mathbb{Q}) \cong H^*(\wedge V_2; d_2)$ is free we conclude $d_2 = 0$. Now we may directly obtain a DG algebra map $J: \wedge (V_2; 0) \to \wedge (V_3; 0)$ with $J \circ P = 1_{\wedge V_2}$. The maps $J, J$ induce an isomorphism $A: \wedge (V_2; 0) \to (\wedge V_1; 0) \otimes (\wedge V_3; 0)$ of DG algebras satisfying $\pi_2 \circ A = P$ where $\pi_2: (\wedge V_1; 0) \otimes (\wedge V_3; 0) \to (\wedge V_3; 0)$ is the projection. Using the correspondence between (homotopy classes of) maps between minimal DGAs and maps between rational spaces, we obtain $\alpha: F_Q \times B_Q \to E_Q$, the needed fibre homotopy equivalence. \qed

4. The Based Case

We deduce versions of our main results in the basepoint preserving cases. First, let $\xi: E \to X$ be a fibration over a based space $X$ and let $F$ be the fibre over a fixed basepoint. We denote by $\text{aut}^F(\xi)$ the submonoid of $\text{aut}(\xi)$ consisting of equivalences inducing the identity on $F$. This is the natural based version of $\text{aut}(\xi)$. Note that $\text{aut}^F(\xi)$ is the fibre over the identity map $1_F$ of the restriction map $\text{res}: \text{aut}(\xi) \to \text{aut}(F)$. We have the following based version of the identity (1) in...
our setting. Write $\text{map}_*(X,Y)$ for the space of basepoint preserving maps from $X$ to $Y$.

**Theorem 4.1.** Let $X$ be a compact metric space, $F$ a finite CW complex and $h: B \to \text{Baut}(F)$ a map. Let $\xi: E \to X$ be the corresponding $F$-fibration. Then there is an $H$-equivalence

$$\text{aut}^F(\xi) \simeq \Omega\text{map}_*(X, \text{Baut}(F); h).$$

**Proof.** First, note that, by [30], $\text{aut}^F(\xi)$ is of CW type since $\text{aut}(\xi)$ and $\text{aut}(F)$. Now compare the long exact homotopy sequence of the restriction fibration above to that of loops on the evaluation fibration $\omega: \text{map}(X, \text{Baut}(F); h) \to \text{Baut}(F)$ with fibre $\text{map}_*(X, \text{Baut}(F); h)$. The result follows from Theorem 1 and the 5-lemma. \qed

As before, this gives a corresponding localization result for $\text{aut}^F(\xi)_o$:

**Theorem 4.2.** Let $X$ be a compact metric space, $F$ a finite CW complex and $h: B \to \text{Baut}(F)$ a map. Let $\xi: E \to X$ be the corresponding $F$-fibration. Then $\text{aut}^F(\xi)_o$ is a nilpotent space and we have an $H$-equivalence

$$\left(\text{aut}^F(\xi)_o\right)_P \simeq \Omega\text{map}_*(X, (\text{Baut}(F)_o)_P; (\tilde{h})_P).$$

\qed

Next, given a fibration $\zeta: E \to X$ write $\Gamma_*(\zeta)$ for the space of basepoint preserving sections of $\zeta$. We have:

**Theorem 4.3.** Let $\zeta: E \to X$ be a fibrewise grouplike space with connected, CW grouplike fibre $G$ and base $X$ a connected compact metric space. Suppose $\zeta$ is the pullback of a CW fibration. Then $\Gamma_*(\zeta)_o$ is a nilpotent space and the map

$$\Gamma_*(\zeta)_o \to \Gamma_*(\zeta_\mathbb{P})_o$$

induced by a fibrewise $\mathbb{P}$-localization $\zeta \to \zeta_\mathbb{P}$ is a $\mathbb{P}$-localization map.

**Proof.** In this case, the relevant evaluation fibration $\omega: \Gamma(\zeta)_o \to E$ with fibre $\Gamma_*(\zeta)_o$ is of CS type since $E$ is not well-behaved under fibrewise localization. However, we may repeat the entire argument in the based case to achieve the needed result. We note that Möller’s theorem [24, Theorem 5.3], for $X$ a finite complex, is proved in the based setting. The rest of the argument thus proceeds as before. \qed

Finally, we remark that our applications Theorems 3.3 and 3.6 also hold in the respective based settings but with ordinary Čech cohomology replaced by reduced Čech cohomology in the rational homotopy calculations.

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Department of Mathematics, Wayne State University, Detroit MI 48202
E-mail address: claude@math.wayne.edu

Department of Mathematics, Saint Joseph’s University, Philadelphia PA 19131
E-mail address: smith@sju.edu