Quantum State Evolution in $C^2$ and $G_3^+$

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Abstract: Quantum mechanical qubit states as elements of two dimensional complex Hilbert space can be generalized to elements of even subalgebra of geometric algebra over three dimensional Euclidian space. The construction critically depends on generalization of formal, unspecified, complex plane to arbitrary variable, but explicitly defined, planes in 3D, and of usual Hopf fibration to special maps of the geometric algebra elements to the unit sphere in 3D generated by arbitrary unit value bivectors. Analysis of the structure of the map of the even subalgebra to the Hilbert space demonstrates that quantum state evolution in the latter gives only restricted information compared to that in geometric algebra.

Keywords: Qubits, Geometric Algebra, Clifford Translation, Berry Phase, Topological Quantum Computing

1. Introduction

Qubits, unit value elements of the Hilbert space $C^2$ of two dimensional complex vectors, can be generalized to unit value elements $\alpha + I_S \beta$ of even subalgebra $G_3^+$ of geometric algebra $G_3$ over Euclidian space $E_3$. I called such generalized qubits $S$-qubits [1].

Some minimal information about $G_3$ and $G_3^+$ is necessary. Algebraically, $G_3$ is linear space with canonical basis $\{1, e_1, e_2, e_3\}$, where 1 is unit value scalar, $\{e_i\}$ are orthonormal basis vectors in $E_3$, $\{e_i e_j\}$ are oriented, mutually orthogonal unit value areas (bivectors) spanned by $e_i$ and $e_j$ as edges, with orientation defined by rotation $e_i$ to $e_j$ by angle $\frac{\pi}{2}$, and $e_i e_2 e_3$ is unit value oriented volume spanned by ordered edges $e_1, e_2$ and $e_3$.

Subalgebra $G_3^+$ is spanned by scalar 1 and basis bivectors $\{1, e_2 e_3, e_3 e_1, e_1 e_2\}$. Variables $\alpha$ and $\beta$ in $\alpha + I_S \beta \in G_3^+$ are scalars, $I_S$ is a unit size oriented area, bivector, lying in an arbitrary given plane $S \subset E_3$. Bivector $I_S$ is linear combination of basis bivectors $e_i e_j$, see Fig.1a.

![Fig. 1. Variable complex plane in different bivector bases.](image)

It was explained in [2], [3] that elements $\alpha + I_S \beta$ only differ from what is traditionally called “complex numbers” by the fact that $S \subset E_3$ is an arbitrary, though explicitly defined, plane in $E_3$. Putting it simply, $\alpha + I_S \beta$ are “complex...
2. Parameterization of Unit Value Elements in $G_3^+$ and $C^2$ by Points of $S^3$

Let’s take a $g$-qubit:

$$\alpha + I_3 \beta = \alpha + \beta (b_1 b_3 + b_2 b_1) = \alpha + \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3, \beta_i = \beta_i,$$

$$\alpha^2 + \beta^2 = 1, b_i^2 + b_i b_j = 1$$

I will use notation $so(\alpha, \beta, S)$ for such elements. They can be considered as points on unit radius sphere in the sense that any point $(\alpha, \beta_1, \beta_2, \beta_3) \in S^3$ parameterizes a $g$-qubit.

A pure qubit state in terms of conventional quantum mechanics is two dimensional unit value vector with complex components:

$$|\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + z_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$z_1^2 + z_2^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1,$$

$$z_k = z_k^* + i z_k, k = 1, 2$$

We can also think about such pure qubit states as points on $S^3$ because:

$$S^3 \ni \{ z_1, z_2, z_3, z_4 \} \leftrightarrow \{ z = (z_1, z_2) \in C^2; |z|^2 \} = \mathbb{S}^3$$

$$z_1 + z_2 + z_3 + z_4 = 1 \quad (2.1)$$

Explicit relations, based on their parameterization by $S^3$ points, between elements $|\psi\rangle$ and elements $so(\alpha, \beta, S)$ can be established through the following construction:

Let basis bivector $B_i$ is chosen as defining the complex plane $S_i$, then we have (see multiplication rules (1.1)): $so(\alpha, \beta, S) = (\alpha + \beta B_1) + \beta_1 B_1 B_1 + \beta_2 B_2 = (\alpha + \beta B_1) + (\beta_1 + \beta_2 B_2) B_1 B_2$

Hence we get the map:

$$so(\alpha, \beta, S) \rightarrow |\psi\rangle = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{\alpha + i \beta_1}{\beta_3 + i \beta_2}, \quad \alpha \equiv B_i$$

In the similar way, for two other selections of complex plane we get:

$$so(\alpha, \beta, S) \rightarrow |\psi\rangle = \begin{pmatrix} z_1^2 \\ z_2^2 \end{pmatrix} = \begin{pmatrix} \alpha + i \beta_2 \\ \beta_1 + i \beta_3 \end{pmatrix}, \quad l \equiv B_2$$

and $$so(\alpha, \beta, S) \rightarrow |\psi\rangle = \begin{pmatrix} z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} \alpha + i \beta_3 \\ \beta_1 + i \beta_2 \end{pmatrix}, \quad l \equiv B_3$$

So we have three different maps $so(\alpha, \beta, S) \rightarrow |\psi\rangle$.

2. Though both $so(\alpha, \beta, S)$ and $|\psi\rangle$ can be parametrized by points of $S^3$, it is not correct to say that $\{ z = (z_1, z_2) \in C^2; |z|^2 = 1 \}$ is sphere $S^3$ or $g$-qubit is sphere $S^3$. 

---

Fig. 2. Oriented basis bivectors their right screw oriented volume.
defined by explicitly declared complex planes $B_i$ satisfying (1.1):
\[
\text{so}(\alpha, \beta, S) = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 = \\
\alpha + \beta_1 B_1 + (\beta_2 B_2 + \beta_3 B_3) = \\
\alpha + \beta_1 B_1 + (\beta_2 B_2 + \beta_3 B_3) B_2
\]

There exists infinite number of options to select the triple $\{B_i\}$. It means that to recover a $\mathcal{B}$ -qubit $\text{so}(\alpha, \beta, S)$ in 3D associated with $|\psi\rangle = \begin{pmatrix} z_1 \\
 z_2 \end{pmatrix}$ it is necessary, firstly, to define which bivector $B_i$ in 3D should be taken as defining “complex” plane and then to choose another bivector $B_{ji}$, orthogonal to $B_i$. The third bivector $B_{ki}$, orthogonal to both $B_i$ and $B_{ji}$, is then defined by the first two by orientation (handedness, right screw in the used case): $I_1 B_i I_2 B_{ji} I_3 B_{ki} = I_3$.

The conclusion is that to each single element (2.1) there corresponds infinite number of elements $\text{so}(\alpha, \beta, S)$ depending on choosing of a triple of orthonormal bivectors $\{B_i, B_{ji}, B_{ki}\}$ in 3D satisfying (1.1), and associating one of them with complex plane. This allows constructing map, fibration $G^+_3 \rightarrow C^2$ restricted to unit value elements.

### 3. Fiber Bundle

Take general definition of fiber bundle as a set $(E, M, \pi, G, F)$ where $E$ is bundle (or total) space; $M$ - the base space; $F$ - standard fiber; $G$ - Lie group which acts effectively on $F$; $\pi$ - bundle projection: $\pi : E \rightarrow M$, such that each space $F_x = \pi^{-1}(x)$, fiber at $x \in M$, is homeomorphic to standard fiber $F$.

We can think about the map $\text{so}(\alpha, \beta, S) \xrightarrow{\pi} (z_1, z_2)$ as a fiber bundle. In the current case, the fiber bundle will have $\text{so}(\alpha, \beta, S) = g \in G^+_3 : |\mathcal{C}| = 1$ as total space and $\{z = (z_1, z_2) \in C^2 : |z|^2 = 1\}$ as base space. I will denote them as $G^+_3 |_{\mathcal{C}}$ and $C^2 |_{\mathcal{C}}$ respectively. The projection $\pi : G^+_3 |_{\mathcal{C}} \rightarrow C^2 |_{\mathcal{C}}$ depends on which particular $B_i$ is taken from an arbitrary triple $\{B_1, B_2, B_3\}$ satisfying (1.1) as associated complex plane of complex vectors of $C^2$ and explicitly given by (2.2), so we should write $\pi : G^+_3 |_{\mathcal{C}} \xrightarrow{\mathcal{B}} C^2 |_{\mathcal{C}}$.

By some reasons that will be explained a bit later I will use complex plane associated with $B_3$, so by (2.2) the projection is:
\[
\pi : \text{so}(\alpha, \beta, S) = \alpha + \beta_1 B_1 + \beta_2 B_2 + \beta_3 B_3 \rightarrow \begin{pmatrix} \pi + i \beta_1 \\
 \beta_2 + i \beta_3 \end{pmatrix}
\]

Then for any $z = \begin{pmatrix} x_1 + \delta_1 y_1 \\
 x_2 + \delta_2 y_2 \end{pmatrix} \in C^2 |_{\mathcal{C}}$, the fiber in $G^+_3 |_{\mathcal{C}}$ is comprised of all elements $\{F_2 = x_1 + y_1 B_1 + x_2 B_2 + y_1 B_3\}$ with an arbitrary triple of orthonormal bivectors $\{B_1, B_2, B_3\}$ in 3D satisfying (1.1). That particularly means that standard fiber is equivalent to the group of rotations of the triple $\{y_2 B_1, x_2 B_2, y_1 B_3\}$ as a whole (see Fig.3):

![Fig. 3. Two sections of a fiber: The left one received from the right by 90 degree counterclockwise rotation around vertical axis.](image)

All such rotations in $G^+_3$ are also identified by elements of $G^+_3 |_{\mathcal{C}}$ since for any bivector $B$ the result of its rotation is 3 (see, for example [4], [5]):
\[
\text{so}(\gamma, \delta, S) \rightarrow B \text{so}(\gamma, \delta, S)
\]

where
\[
\text{so}(\gamma, \delta, S) = (\gamma + \delta B_1 + \delta_2 B_2 + \delta_3 B_3)
\]
\[
\gamma - \delta_1 B_1 - \delta_2 B_2 - \delta_3 B_3
\]

So, standard fiber is identified as $G^+_3 |_{\mathcal{C}}$ and composition of rotations is:
\[
e^{-i\phi_1} (e^{-i\phi_1} B e^{i\phi_1}) e^{i\phi_2} = \begin{pmatrix} e^{i\phi_1} & e^{i\phi_2} \\
 e^{-i\phi_1} & e^{-i\phi_2} \end{pmatrix} B e^{i\phi_1} e^{i\phi_2}
\]

All that means that the fibration $G^+_3 |_{\mathcal{C}} \xrightarrow{\mathcal{B}} C^2 |_{\mathcal{C}}$ is

3 It is often convenient to write elements $\text{so}(\alpha, \beta, S)$ as exponents: $\text{so}(\alpha, \beta, S) = e^{i\phi}$. 

principal fiber bundle with standard fiber $G_3^+|\mathbb{S}^2$ and the group acting on it is the group of (right) multiplications by elements of $G^+_3|\mathbb{S}^2$.

Now the explanation why $B_3$ was taken as complex plane. As was shown in [1], [6] the variant of classical Hopf fibration

$$
C^2/\mathbb{S}^1 \to S^2; \\
\Pr(z_1, z_2) = \left(z_1z_2 + z_1z_2, i(z_1z_2 - z_1z_2), |z_1|^2 - |z_2|^2\right)
$$

can be received as one of the cases of generalized Hopf fibrations:

$$
G_3^+|\mathbb{S}^2 \to S^2; \\
so(\alpha, \beta, S) \to so(\alpha, \beta, S)\sim Bso(\alpha, \beta, S) \to S^2
$$

The basis bivector $B_1$ gives:

$$
so(\alpha, \beta, S) \to (\alpha - \beta I_3)B_1(\alpha + \beta I_3) = \\
(\alpha - \beta B_1 - \beta B_2 - \gamma B_3)B_1(\alpha + \beta B_1 + \beta B_2 + \gamma B_3) = \\
\left(\alpha^2 + \beta^2 - (\beta^2 + \gamma^2)\right)B_1 + 2(\alpha\beta_1 + \beta\beta_2)B_2 + \\
2(\beta\beta_1 - \alpha\beta_2)B_3
$$

and for other two basis bivectors:

$$
so(\alpha, \beta, S) \to 2(\beta\beta_1 - \alpha\beta_2)\left[\alpha^2 + \beta^2 - (\beta^2 + \gamma^2)\right]B_1 + 2(\alpha\beta_1 + \beta\beta_2)B_2 + \\
2(\beta\beta_1 - \alpha\beta_2)\left[\alpha^2 + \beta^2 - (\beta^2 + \gamma^2)\right]B_1 + 2(\alpha\beta_1 + \beta\beta_2)B_2 + \\
2(\beta\beta_1 - \alpha\beta_2)B_3
$$

In the literature mostly the third or the first variants are called Hopf fibrations. For my considerations it does not matter which variant to choose. I take the case (3.4) with $B_3$ as complex plane:

$$
so(\alpha, \beta, S) = \alpha + \beta B_1 + \beta B_2 + \beta B_3 = \\
\left(\alpha + \beta B_1\right) + \left(\beta + \beta B_2\right)B_2 + \\
\left(\beta + \beta B_3\right)B_2
$$

4. Tangent Spaces

We need to temporarily get back to the case of $G_2$ - geometric algebra on a plane [2].

Let an orthonormal basis $\{e_1, e_2\}$ is taken. It generates $G_2$ basis $\{1, e_1, e_2, e_1e_2\}$ where $e_1e_2 = e_1 \cdot e_2 + e_1 \wedge e_2$ is usual geometric product of two vectors: first member is scalar product of the vectors and second member is oriented area (bivector) swept by rotating $e_1$ to $e_2$ by the angle which is less than $\pi$.

The $G_2$ basis vectors satisfy particularly the properties:

$$
I_2 = (e_1e_2)^2 = -1, \\
I_2e_1 = -e_2 \text{ (clockwise rotation)}, \\
I_2e_2 = e_1 \text{ (counterclockwise rotation)}, \\
I_2e_1e_2 = -e_1e_2
$$

I am using notation $I_2$ for unit bivector $e_1e_2$.

For further convenience, let’s construct a matrix basis isomorphic to $\{1, e_1, e_2, e_1e_2\}$. Commonly used agreement will be that scalars are identified with scalar matrices, for example:

$$
\alpha = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}
$$

For the $G_2$ case the second order matrices will suffice to get necessary isomorphism. Let’s take

$$
e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

e_1$ and $e_2$. They satisfy the properties mentioned above, for example

$$
e_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
i_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -e_2,
$$

The operation representing scalar product gives:

$$
e_1 \cdot e_2 = \frac{1}{2}(e_1e_2 + e_2e_1) = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
$$

as it should be. If we take two arbitrary vectors expanded in basis $\{e_1, e_2\}$:

$$
\vec{a} = \alpha_1e_1 + \alpha_2e_2 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \\
\vec{b} = \beta_1e_1 + \beta_2e_2 = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}
$$

then their product is:

$$
\vec{a} \cdot \vec{b} = \begin{pmatrix} \alpha_1\beta_1 + \alpha_2\beta_2 & 0 \\ 0 & \alpha_1\beta_1 + \alpha_2\beta_2 \end{pmatrix}
$$

The first member is usual scalar product and the second one is bivector of the value equal to the area of parallelogram with
the sides $\vec{a}$ and $\vec{b}$.

Important thing to keep in mind is that multiplication of any vector by unit bivector $I_2$ rotates the vector by $\pm \frac{\pi}{2}$ (see Fig.4):

$$\vec{a}I_2 = \begin{pmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & -\alpha_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\alpha_2\epsilon_1 + \alpha_1\epsilon_2$$

(counterclockwise rotation)

$$I_2\vec{a} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & -\alpha_1 \end{pmatrix} = \alpha_2\epsilon_1 - \alpha_1\epsilon_2$$

(clockwise rotation)

This remains valid for any unit bivector of the same orientation as $I_2$. We can conclude that such multiplications give basis vectors of the tangent spaces to the original vectors.

![Fig. 4. Vector is rotated by 90 degrees when multiplied by basis bivector.](image)

This is identical to considering even elements $\alpha_1 + B\alpha_2$ corresponding to vectors and their multiplication by $B : (\alpha_1 + B\alpha_2)B = -\alpha_2 + B\alpha_1$. Elements $\alpha_1 + B\alpha_2$ and $-\alpha_2 + B\alpha_1$ are orthogonal:

$$\langle \alpha_1 + B\alpha_2, -\alpha_2 + B\alpha_1 \rangle_0 = (\alpha_1 + B\alpha_2) \cdot (-\alpha_2 + B\alpha_1)_0 = 0$$

(index 0 means scalar part).

General element of $G_2$ in matrix basis \{1, $e_1$, $e_2$, $e_1e_2 = I_2$\}, isomorphic to geometrical basis \{1, $e_1$, $e_2$, $e_1e_2 = I_2$\}, is:

$$e_1 = e_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + e_2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + e_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e_4 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_6 + \alpha_4 & \alpha_2 + \alpha_3 \\ \alpha_2 - \alpha_3 & \alpha_6 - \alpha_4 \end{pmatrix}$$

This is arbitrary real valued matrix of the second order. Inversely, any matrix of second order can be uniquely mapped to the element of $G_2$, in the basis \{1, $e_1$, $e_2$, $e_1e_2 = I_2$\}:

$$\begin{pmatrix} x & y \\ z & t \end{pmatrix} = \frac{x + t}{2} + \frac{x - t}{2}e_1 + \frac{y + z}{2}e_2 + \frac{y - z}{2}e_1e_2$$

To upgrade to $G_3$, we need basis matrices of a higher order because a second order non-zero matrix orthogonal in the sense of scalar product (4.1) to both $e_1$ and $e_2$ does not exist.

It is easy to verify that the three, playing the role of three-dimensional orthonormal basis, necessary matrices can be taken as 4th-order block matrices:

$$\sigma_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix},$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ - scalar matrix corresponding to scalar 1,

$I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Easy to see that $\sigma_1$ and $\sigma_2$ are received from $e_1$ and $e_2$ by replacing scalars by corresponding scalar matrices.

By multiplications in full 4x4 form one can easily prove that block-wise multiplications are correct, the basis $\{\sigma_1, \sigma_2, \sigma_3\}$ is orthonormal, anticommutative, and $\sigma_2 \sigma_3$, $\sigma_3 \sigma_1$ and $\sigma_1 \sigma_2$ satisfy requirements (1.1).

Taking $I_3 = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix}$, multiplications give:

$$\sigma_2 \sigma_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} = I_3 \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I_3 \sigma_1 \sigma_2 = \sigma_1 \sigma_2$$

$$\sigma_3 \sigma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = I_3 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = I_3 \sigma_2 \sigma_3 = \sigma_2 \sigma_3$$

Then, since $I_3^2 = -I$, we easily get:

$$(\sigma_2 \sigma_3) (\sigma_3 \sigma_1) = -\sigma_2 \sigma_3, \quad (\sigma_2 \sigma_3) (\sigma_2 \sigma_3) = \sigma_3 \sigma_1,$$

$$(\sigma_1 \sigma_2) (\sigma_1 \sigma_2) = -\sigma_1 \sigma_2$$

that is exactly (1.1) with basis bivectors $B_1 = \sigma_2 \sigma_3$, $B_2 = \sigma_3 \sigma_1$ and $B_3 = \sigma_1 \sigma_2$.

Following all that we can write general element $g_3$ of algebra $G_3$ expanded in formal matrix basis $\{1, \sigma_1, \sigma_2, \sigma_3, I_3 \sigma_1, I_3 \sigma_2, I_3 \sigma_3, I_3 \}$ as:

$$g_3 = a_0 + a_1 \sigma_1 + a_2 \sigma_2 + a_3 \sigma_3 + a_4 I_3 \sigma_1 + a_5 I_3 \sigma_2 + a_6 I_3 \sigma_3 + a_7 I_3$$
These elements of $C^2\mid_{g_s}$ are orthogonal, in the sense of Euclidean scalar product in $C^2$: $\langle z, w \rangle = \text{Re}(\bar{z}_1w_1 + \bar{z}_2w_2)$, to each other and to the projection $\pi(\text{so}(\alpha, \beta, \beta, \beta)) = \left(\frac{\alpha + i\beta_3}{\beta_2 + i\beta_1}\right)$ of the original state in $G_3^+\mid_{g_s}$. They are the tangent space basis elements in $C^2\mid_{g_s}$ at points $z_1 = (\alpha + i\beta_1)$.

5. Hamiltonians in $C^2\mid_{g_s}$ and Their Lifts in $G_3$

Take self-adjoint (Hamiltonian) linear operator in $C^2$ in its most general form: $H = al + b\sigma_1 + c\sigma_2 + d\sigma_3 = \left(\begin{array}{cc} a+b & c-I_2d \\ c+I_2d & a-b \end{array}\right)$. Its corresponding $G_3$ element, see (4.3), is $a + I_3(b\text{Re} + c\text{Im} + d\text{Im})$ and does not belong to $G^+_3\mid_{g_s}$, though the result of its action on an element from $C^2\mid_{g_s}$ (qubit), has $\pi^{-1}$ preimage in $G^+_3\mid_{g_s}$. Let’s prove that. Take a qubit and a Hamiltonian in $C^2\mid_{g_s}$:

$$z = \left(\begin{array}{c} x_1 + iy_1 \\ x_2 + iy_2 \end{array}\right), \quad r_k = \sqrt{x_k^2 + y_k^2}, \quad \text{cos} \psi_k = \frac{x_k}{\sqrt{x_k^2 + y_k^2}}, \quad \text{sin} \psi_k = \frac{y_k}{\sqrt{x_k^2 + y_k^2}}, \quad k = 1, 2$$

$$H = \left(\begin{array}{cc} a+b & \text{Re} \\ \text{Re}^{-}\text{Im} & a-b \end{array}\right), \quad R = \sqrt{c^2 + d^2}, \quad \text{cos} \psi = \frac{c}{\sqrt{c^2 + d^2}}, \quad \text{sin} \psi = \frac{d}{\sqrt{c^2 + d^2}}$$

Then:

$$He = \left(\begin{array}{cc} a+b & \text{Re} \\ \text{Re}^{-}\text{Im} & a-b \end{array}\right) \left(\begin{array}{c} r e^{\text{Im}} \\ r e^{-}\text{Im} \end{array}\right) = \left(\begin{array}{c} (a+b) r \text{cos} \psi + R_2 \text{cos} (\psi + \psi) + \text{Re} r \text{cos} (\psi + \psi) + i ((a+b) r \text{sin} \psi + R_2 \text{sin} (\psi + \psi)) \\ (a+b) r \text{cos} \psi - R_1 \text{cos} (\psi - \psi) + \text{Re} r \text{cos} (\psi - \psi) + i ((a+b) r \text{sin} \psi - R_1 \text{sin} (\psi - \psi)) \end{array}\right)$$

For each single qubit $z = \left(\begin{array}{c} x_1 + iy_1 \\ x_2 + iy_2 \end{array}\right) \in C^2\mid_{g_s}$ its fiber (full preimage) in $G^+_3\mid_{g_s}$ is $F_z = x_1 + y_2 B_1 + x_2 B_2 + y_3 B_3$ with an
arbitrary triple of orthonormal bivectors \( \{B_1, B_2, B_3\} \) in 3D satisfying (1.1). In exponential form:

\[
F_z = x_1 + y_2 B_1 + x_3 B_3 + y_3 B_1 = \sqrt{1 - x_1^2} B_1 + \sqrt{1 - x_2^2} B_2 + \sqrt{1 - x_3^2} B_3
\]

\[
= \cos \phi + \sin \phi (b_1 B_1 + b_2 B_2 + b_3 B_3) = e^{i \phi},
\]

(5.2)

where

\[
\begin{align*}
\cos \phi &= x_1, \\
\sin \phi &= \sqrt{1 - x_1^2}, \\
b_1 &= \frac{y_2}{\sqrt{1 - x_1^2}}, \\
b_2 &= \frac{x_2}{\sqrt{1 - x_1^2}}, \\
b_3 &= \frac{y_3}{\sqrt{1 - x_1^2}}.
\end{align*}
\]

(5.3)

Using (5.2) we get from (5.3) the fiber at \( Hz \):

\[
F_{Hz} = e^{i \phi(H)},
\]

(5.4)

where

\[
\begin{align*}
\cos \phi(H) &= (a + b) r_1 \cos \psi_1 + r_2 \cos \psi_2 ; \\
I_{S(H)} &= b_1(H) B_1 + b_2(H) B_2 + b_3(H) B_3 ; \\
b_1(H) &= \frac{(a - b) r_1 \sin \psi_1 - r_2 \sin \psi_2}{\sqrt{1 - \cos^2 \phi(H)}}, \\
b_2(H) &= \frac{(a - b) r_1 \cos \psi_1 + r_2 \cos \psi_2}{\sqrt{1 - \cos^2 \phi(H)}}, \\
b_3(H) &= \frac{(a + b) r_1 \sin \psi_1 + r_2 \sin \psi_2}{\sqrt{1 - \cos^2 \phi(H)}}.
\end{align*}
\]

(5.5)

We can also explicitly write the element \( g_3(H) \in G_3^2 | \_3 \) which, acting (from the right) on \( \pi^{-1}(z) \) gives \( \pi^{-1}(Hz) \):

\[
\pi^{-1}(z) g_3(H) = \pi^{-1}(Hz)
\]

From (5.1) and (5.4):

\[
e^{i \phi} g_3(H) = e^{i \phi(H)} \Rightarrow g_3(H) = e^{-i \phi} e^{i \phi(H)}
\]

where \( I_S, \phi, I_{S(H)} \) and \( \phi(H) \) are defined by (5.3) and (5.5).

6. Clifford Translations

Suppose, action on \( z \) in \( C^2 | \_3 \) is multiplication by an exponent: \( C_{\psi}^I(z) = e^{i \psi z} \), often called Clifford translation, that's:

\[
C_{\psi}^I(z) = \left( \begin{array}{c} r_1 e^{i \psi_1} \\ r_2 e^{i \psi_2} \end{array} \right) = \left( \begin{array}{c} r_1 \cos(\psi + \psi_1) + i r_1 \sin(\psi + \psi_1) \\ r_2 \cos(\psi + \psi_2) + i r_2 \sin(\psi + \psi_2) \end{array} \right)
\]

Then the fiber of \( C_{\psi}^I(z) \) in \( G_3^2 | \_3 \) is:

\[
F_{C_{\psi}^I(z)} = r_1 \cos(\psi + \psi_1) + r_2 \sin(\psi + \psi_2) B_1 + r_2 \cos(\psi + \psi_2) + r_1 \sin(\psi + \psi_1) B_2\]

\[
+ \left[ r_1 \cos(\psi + \psi_1) + r_2 \sin(\psi + \psi_2) B_3 \right] B_3 =
\]

\[
r_1 e^{i \psi} e^{i \psi_1} B_1 + r_2 e^{i \psi} e^{i \psi_2} B_2 = e^{i \psi} F_z
\]

In other words, Cliffords translations in \( \mathbb{C}^2 | \_3 \) are equivalent to multiplications of fibers in \( G_3^2 | \_3 \) by standard fiber elements with bivector part \( B_1 \) (which is associated with formal imaginary unit \( i \)) and the same phase \( \psi \).

In commonly used quantum mechanics all states from the set \( C_{\psi}^I(z) \) are considered as identical to the state \( z \), they have the same values of observables. That is not commonly true for \( \mathcal{B} \)-qubits. Only in the case when all the observables common plane, say \( I_S \), (unspecified in the model)6 is the same as the plane associated with complex plane \( \mathbb{C} \) in \( Cl_{\psi}^I(z) \), the \( \mathcal{B} \)-qubit \( e^{i \psi} F_z \) leaves any such observable \( e^{i \phi} \) unchanged:

\[
F_z e^{i \psi} \equiv F_z e^{i \phi} F_z = F_z e^{i \phi} F_z
\]

Suppose angle \( \psi \) in \( Cl_{\psi}^I(z) \) is varying. Then the tangent to Clifford orbit is

\[
\frac{\partial}{\partial \psi} C_{\psi}^I(z) = \frac{\partial}{\partial \psi} e^{i \psi z} = i e^{i \psi z} = i C_{\psi}^I(z).
\]

It is orthogonal to \( C_{\psi}^I(z) \) in the sense of usual scalar product in \( \mathbb{C}^2 \) : \( \langle u, v \rangle = \text{Re}(u_1 v_1 + u_2 v_2) \), and remains on \( \mathbb{C}^2 | \_3 \) : \( | C_{\psi}^I(z) \rangle = 1 \) (translational velocity of Clifford translation is one).

For any \( z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 | \_3 \) define \( w = \begin{pmatrix} z_1 \\ -z_2 \end{pmatrix} \in \mathbb{C}^2 | \_3 \). The elements \( iz, iw \) and \( iw \) are orthogonal to \( z \) and pairwise orthogonal. That means they span tangent space and the plane spanned by \( \{ w, iw \} \) is orthogonal to the Clifford orbit plane spanned by \( \{ z, iz \} \).

---

5 It is called “translation” because does not change distance between elements.

6 In the \( G_3^2 \) model observables are also elements of \( G_3^2 \), see [1], [6].
The vector of translational velocity rotates while moving in the orbit plane. Both \( w \) and \( iw \) also rotates with the same unit value angular velocity since
\[
\frac{\partial}{\partial \psi} Cl_{\psi}(w) = \frac{\partial}{\partial \psi} e^{i\psi w} = e^{i\psi w} Cl_{\psi}(iw)
\]
and
\[
\frac{\partial}{\partial \psi} Cl_{\psi}(iw) = \frac{\partial}{\partial \psi} e^{i\psi iw} = ie^{i\psi iw} = -Cl_{\psi}(w).
\]
To make the planes of rotations clearly identified, that is difficult to do with formal imaginary unit, let’s lift Clifford translation to \( G_3^+|\mathbb{S}^3 \) using (6.1). Translational velocity, similar to the \( C^2|\mathbb{S}^2 \) case, is
\[
\frac{\partial}{\partial \psi} F_{Cl_{\psi}(z)} = \frac{\partial}{\partial \psi} e^{i\psi w} F_z = B_1 F_{Cl_{\psi}(z)}
\]
and is orthogonal to \( F_{Cl_{\psi}(z)} \):
\[
\{ F_{Cl_{\psi}(z)}, B_2 F_{Cl_{\psi}(z)} \}_0 = \{ F_{Cl_{\psi}(z) B_{Cl_{\psi}(z)}^{-1}}, B_2 \}_0 = (B_3)_0 = 0.
\]
Two other components of the tangent space, orthogonal to \( F_{Cl_{\psi}(z)} \) and \( B_1 F_{Cl_{\psi}(z)} \) at any point of the orbit, are \( B_1 F_{Cl_{\psi}(z)} \) and \( B_2 F_{Cl_{\psi}(z)} \). Their velocities while moving along Clifford orbit are:
\[
\frac{\partial}{\partial \psi} (B_1 F_{Cl_{\psi}(z)}) = \frac{\partial}{\partial \psi} (B_1 e^{B_2 F_z} F_z) = B_1 B_2 F_{Cl_{\psi}(z)} = B_2 F_{Cl_{\psi}(z)}
\]
(derivative of \( B_1 F_{Cl_{\psi}(z)} \) is orthogonal to \( B_1 F_{Cl_{\psi}(z)} \) and looking in the direction \( B_2 F_{Cl_{\psi}(z)} \))
\[
\frac{\partial}{\partial \psi} (B_2 F_{Cl_{\psi}(z)}) = \frac{\partial}{\partial \psi} (B_2 e^{B_2 F_z} F_z) = B_2 B_1 F_{Cl_{\psi}(z)} = -B_1 F_{Cl_{\psi}(z)}
\]
(derivative of \( B_2 F_{Cl_{\psi}(z)} \) is orthogonal to \( B_2 F_{Cl_{\psi}(z)} \) and looking in the direction \( -B_1 F_{Cl_{\psi}(z)} \)).

These two equations explicitly show that the two tangents, orthogonal to Clifford translation velocity, rotate in moving plane \( \{ B_1 F_{Cl_{\psi}(z)}, B_2 F_{Cl_{\psi}(z)} \} \) with the same unit value rotational velocity. Interesting to notice that the triple of the translational velocity and two rotational velocities has orientation opposite to the triple of tangents: if the tangents \( \{B_1 F_{Cl_{\psi}(z)}, B_2 F_{Cl_{\psi}(z)}, B_3 F_{Cl_{\psi}(z)} \} \) have right screw orientation, the speed triple \( \{-B_1 F_{Cl_{\psi}(z)}, B_2 F_{Cl_{\psi}(z)}, B_3 F_{Cl_{\psi}(z)} \} \) is left screw.

If a fiber, \( G \)-qubit, makes full circle in Clifford translation:
\[
F_z \rightarrow F_{Cl_{\psi}(z)} = e^{i\psi F_z}, \quad 0 \leq \psi \leq 2\pi,
\]
both \( B_1 F_{Cl_{\psi}(z)} \) and \( B_2 F_{Cl_{\psi}(z)} \) also make full rotation in their own plane by \( 2\pi \).

This is special case of the \( G \)-qubit Berry phase incrementing in a closed curve quantum state path.

7 Clearly, the three \( B_1 F_{Cl_{\psi}(z)} \) are identical to earlier considered tangents \( T_i \).

747 Alexander Soiguine: Quantum State Evolution in \( C^2 \) and \( G_3^+ \)

7. Conclusions

Evolution of a quantum state described in terms of \( G_3^+ \) gives more detailed information about two state system compared to the \( C^2 \) Hilbert space model. It confirms the idea that distinctions between “quantum” and “classical” states become less deep if a more appropriate mathematical formalism is used. This paradigm spreads from trivial phenomena like tossed coin experiment [7] to recent results on entanglement and Bell theorem [8] where the former was demonstrated as not exclusively quantum property.

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