Hamiltonian Reduction of Einstein’s Equations 
without Isometries

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Abstract.
I apply the Hamiltonian reduction procedure to general spacetimes of 4 dimensions with no isometries in the (2+2) formalism and find privileged spacetime coordinates. Physical time is chosen as the area element of the two dimensional cross-section of null hypersurfaces. The physical spatial coordinates are defined by equipotential surfaces on a given spacelike hypersurface of constant physical time. The physical Hamiltonian is manifestly local and positive-definite in the privileged coordinates. The complete set of Hamilton’s equations is presented and it is found that they coincide with the Einstein’s equations written in the privileged coordinates. This shows that the Hamiltonian reduction is a self-consistent procedure.

1. Introduction
It is well-known that the true physical degrees of freedom of Einstein’s theory of gravitation reside in the conformal metric of the two dimensional spatial cross-section of null hypersurfaces[1, 2]. Using the spacetime diffeomorphism invariance, one may choose certain scalar functions in the phase space of the theory as arbitrarily specifiable spacetime coordinates, thereby eliminating unphysical degrees of freedom. In these privileged coordinates, the theory become constraint-free. This procedure is known as the Hamiltonian reduction[3, 4, 5]. Prof. Kuchar and others applied this Hamiltonian reduction to midi-superspace[6, 7, 8] which admits two commuting Killing vector fields, and showed that Einstein’s theory after Hamiltonian reduction was equivalent to cylindrical massless scalar field theory propagating in the 1+1 dimensional Minkowski spacetime.

In this article, we review the Hamiltonian reduction of spacetimes with no isometries in the framework of the (2+2) decomposition[9, 10, 11, 12, 13, 14]. The area element of the two dimensional spatial cross section of null hypersurfaces emerges as the physical time, and the physical radial coordinates are defined by equipotential surfaces on a given spacelike hypersurface of constant physical time. The reduced physical Hamiltonian turns out to be local and manifestly positive-definite in these privileged coordinates[15]. The momentum constraints are simply the defining equations of the physical momentum densities; hence, there exist no constraints to solve, and therefore, the theory becomes constraint-free. In addition, the Hamiltonian reduction is self-consistent because Hamilton’s equations of motion obtained through the Hamiltonian reduction are identical to the Ricci-flat equations in the privileged coordinates.
2. The action in the (2+2) formalism

Let us recall that the metric in the (2+2) decomposition[1, 2, 9, 10, 11] of 4 dimensional spacetimes can be written as

\[ ds^2 = 2dudv - 2hdu^2 + \tau \rho_{ab}(dy^a + A_+^a du + A_-^a dv) \times \left( dy^b + A_+^b du + A_-^b dv \right). \] (1)

The vector fields \( \hat{\partial}_\pm \) defined as

\[ \hat{\partial}_\pm := \partial_\pm - \frac{1}{2} \eta^a \partial_a, \] (2)

where

\[ \partial_+ = \partial/\partial u, \quad \partial_- = \partial/\partial v, \quad \partial_a = \partial/\partial y^a \quad (a = 2, 3), \] (3)

are horizontal vector fields orthogonal to the two-dimensional spacelike surface \( N_2 \) generated by \( \partial_a \). The inner products of the horizontal vector fields are given by

\[ \langle \partial_+, \partial_+ \rangle = -2h, \quad \langle \partial_+, \partial_- \rangle = 1, \quad \langle \partial_-, \partial_- \rangle = 0, \] (4)

which tells us that \( \hat{\partial}_- \) is a null vector field, and that \( \hat{\partial}_+ \) has a norm \(-2h\), which can be either positive, negative, or zero, depending on the sign of \( h \). In this article, we choose the sign \(-2h > 0\) so that \( v = \text{constant} \) is a spacelike hypersurface. The metric on \( N_2 \) is \( \tau \rho_{ab} \), where \( \tau \) is the area element of \( N_2 \) and \( \rho_{ab} \) is the conformal two metric with \( \det \rho_{ab} = 1 \).

As was shown in [11], the Einstein’s equations can be obtained from the variational principle of the following action integral:

\[ S = \int d\nu dudv \{ \pi_\tau + \pi_h \hat{h} + \pi_a \hat{A}_+^a + \pi^{ab} \hat{\rho}_{ab} - "1" \cdot C_- - "0" \cdot C_+ - \hat{\rho}_{ac}\hat{A}_{ac} \}, \] (5)

where the overdot ‘ means \( \partial_- \), and “1”, “0”, and \( A_+^a \) are Lagrange multipliers that enforce the constraints \( C_- = 0, C_+ = 0, \) and \( \pi_a = 0 \), which are given by

\[ C_- := \frac{1}{2} \pi_h \pi_\tau - \frac{h}{4} \pi_h^2 - \frac{1}{2} \pi_{hD_+ \tau} + \frac{1}{2} \pi^{ab} \pi_a \pi_b - \frac{\tau}{8h} \rho^{ab} \rho^{cd} (D_{+\rho ac} (D_{+\rho bd}) - \frac{1}{2h} \pi_a \rho_{aD_+ \pi c} \pi^{bc} - \frac{1}{2h} \pi^{ac} \pi^{bd} (D_{+\rho ac} (D_{+\rho bd}) - \tau R_{(2)} + D_+ \pi_h - \pi_a (\tau^{-1} \rho^{ab} \pi_b) = 0, \] (6)

\[ C_+ := \pi_\tau + \pi_h \hat{D}_+ \tau + \pi^{ab} \rho_{ab} - 2D_+ (h \pi_h + D_+ \tau) + 2\pi_a (h \tau^{-1} \rho^{ab} \pi_b + \rho^{ab} \pi_h) = 0, \] (7)

\[ C_a := \pi_\tau \partial_a \tau + \pi_h \partial_a h + \pi^{bc} \partial_a \rho_{bc} - 2 \partial_b (\rho_{ac} \pi^{bc}) - D_+ \pi_a - \pi_a (\tau \pi_a) = 0, \] (8)

respectively. Here, \( R_{(2)} \) is the scalar curvature of \( N_2 \), and the diff\( N_2 \)-covariant derivative[11] of a tensor density \( q_{ab} \) with weight \( s \) is defined as

\[ D_s q_{ab} := \partial_s q_{ab} - [A_s, q]_{\text{Lab}} = \partial_s q_{ab} - A_s^c \partial_a q_{cb} - q_{cb} \partial_a A_s^c - q_{ac} \partial_b A_s^c - s (\partial_s A_s^c) q_{ab}, \] (9)

where \([A_s, q]_{\text{Lab}}\) is the Lie derivative of \( q_{ab} \) along \( A_s^c := A_s^c \partial_a \). For instance, the diff\( N_2 \)-covariant derivatives of the area element \( \tau \) and the conformal metric \( \rho_{ab} \), which are a scalar density and a tensor density with weight 1 and -1 with respect to the diff\( N_2 \) transformations, respectively, are given by

\[ D_\tau \tau = \partial_\tau \tau - A_\tau^a \partial_a \tau - (\partial_a A_\tau^a) \tau, \] (10)

\[ D_\tau \rho_{ab} = \partial_\tau \rho_{ab} - A_\tau^c \partial_c \rho_{ab} - \rho_{cb} \partial_a A_\tau^c - \rho_{ac} \partial_b A_\tau^c + (\partial_a A_\tau^c) \rho_{ab}, \] (11)

On the other hand, \( h \) is a scalar under the diff\( N_2 \) transformations, whose diff\( N_2 \)-covariant derivative is given by

\[ D_\pm h = \partial_\pm h - A_\pm^a \partial_a h, \] (12)
and the $\text{diffN}_2$-covariant field strength $F_+^a$ is defined as

$$F_+^a := \partial_+ A_+^a - \partial_- A_-^a - [A_+, A_-]^a = \partial_+ A_+^a - \partial_- A_-^a - A_+^b \partial_b A_-^a + A_+^b \partial_b A_+^a.$$  

(13)

The $\text{diffN}_2$-covariant derivatives of the conjugate momenta $\pi_\tau, \pi_h, \pi_a,$ and $\pi^{ab}$, which are tensor densities of weights 0, 1, 1 and 2, respectively, are given by

$$D_\pm \pi_\tau = \partial_\pm \pi_\tau - A_+^a \partial_\pm \pi_a,$$

(14) $$D_\pm \pi_h = \partial_\pm \pi_h - A_+^c \partial_\pm \pi_a - (\partial_\pm A_+^c) \pi_h,$$

(15) $$D_\pm \pi_a = \partial_\pm \pi_a - A_+^c \partial_\pm \pi_a - \pi_a \partial_\pm A_+^c - (\partial_\pm A_+^c) \pi_a,$$

(16) $$D_\pm \pi^{ab} = \partial_\pm \pi^{ab} - A_+^c \partial_\pm \pi^{ab} + \pi^{cb} \partial_\pm A_+^a + \pi^{ac} \partial_\pm A_+^b - 2(\partial_\pm A_+^c) \pi^{ab}.$$  

(17)

The conformal two metric $\rho_{ab}$, being a metric of unit determinant, satisfies the condition

$$\rho^{bc} \partial_\pm \rho_{bc} = \rho^{bc} \partial_a \rho_{bc} = \rho^{bc} D_\pm \rho_{bc} = 0.$$  

(18)

3. Hamiltonian reduction

Let us define a potential function $R$ and its conjugate momentum $\pi_R$ as

$$\partial_+ R := -h \pi_h, \quad \pi_R = -\partial_+ \ln(-h),$$  

(19)

respectively[13, 14]. The transformation from $(h, \pi_h)$ to $(R, \pi_R)$ is a canonical transformation, as it changes the action integral by total derivatives only. If we impose the constraints $C_+ = 0$ and $C_a = 0$ and choose the Lagrange multiplier $A_-^a = 0$, then the action in (5) becomes

$$S = \int dvdu^2 y \{ \pi_\tau \dot{\pi}_\tau + \pi_R \dot{R} + \pi_a \dot{A}_+^a + \pi^{ab} \dot{\rho}_{ab} - C_- \} + \text{total derivatives},$$  

(20)

where the constraint $C_- = 0$ in the new variables is given by

$$C_- = -\frac{1}{2h} \pi_\tau \partial_- R - \frac{1}{4h^2} (\partial_- R)^2 + \frac{1}{2h} (D_\tau) (\partial_+ R) - \frac{1}{h} (D_\tau) (\partial_- R) + \frac{1}{h} (\partial_+ R) \partial_+ \ln(-h) - \frac{1}{h} (\partial_+ R) A_+^a \partial_a \ln(-h) - \tau R^{(2)} + \frac{1}{2\tau^2} \rho_{ab} \pi_a \pi_b - \partial_a (\tau^{-1} \rho_{ab} \pi_b) - \frac{1}{2h} \rho_{ab} \rho_{cd} \pi^{ac} \pi^{bd} - \frac{\tau}{8h} \rho_{ab} \rho_{cd} (D_+ \rho_{ac}) (D_+ \rho_{bd}) - \frac{1}{2h} \pi^{ac} D_+ \rho_{ac} = 0.$$  

(21)

The function $h$ still appears in (21), but as $h$ is related to $R$ and $\pi_R$ by (19), the constraint function $C_-$ given by (21) may be viewed as a function of the new variables $(\tau, Q^\pm; \pi_\tau, \Pi_f)$, where $Q^\pm = (R, A_+^a, \rho_{ab})$ and $\Pi_f = (\pi_\tau, \pi_a, \pi^{ab})$. Notice that the first term in (21) is linear in $\pi_\tau$, and that all the remaining terms are independent of $\pi_\tau$. Thus, the equation of motion for $\tau$ is given by

$$\dot{\tau} = \int dvdu^2 y \frac{\partial C_-}{\partial \pi_\tau} = -\frac{1}{2h} \partial_+ R.$$  

(22)

Now, recall that $\tau = \tau(v, u, y^a)$. If we solve this equation for $v$, then $v$ may be viewed as a function of $(\tau, u, y^a)$ and consequently, $(R, A_+^a, \rho_{ab})$ may be regarded as functions of $(\tau, u, y^a)$. Therefore,

$$\dot{R} = \dot{\tau} \partial_+ R, \quad \dot{A}_+^a = \dot{\tau} \partial_+ A_+^a, \quad \dot{\rho}_{ab} = \dot{\tau} \partial_+ \rho_{ab},$$

C_+ = -\frac{2h}{\partial_+ R} \dot{C}_-,  

(23)
Figure 1. On the $R = \text{constant}$ “equipotential” surface $N_2$ on $\Sigma_{\tau}$, $Y^a$ are introduced such that $Y^a = \text{constant}$ is normal to $N_2$ at each point $p$ on $\Sigma_{\tau}$. Then the “shift” vector $A_+^a$ becomes zero at $p$.

because $\partial u / \partial v = \partial y^a / \partial v = 0$. Then, the action in (20) becomes

$$S = \int d\tau d\nu d^2y d\tau \{ \pi_\tau + \pi_R \partial_\tau R + \pi_a \partial_\tau A_+^a + \pi^{ab} \partial_\tau \rho_{ab} + \left( \frac{2h}{\partial_\tau R} \right) C_- \}
= \int d\tau d\nu d^2y \{ \pi_\tau \partial_\tau R + \pi_a \partial_\tau A_+^a + \pi^{ab} \partial_\tau \rho_{ab} - C_{(1)} \}
= \int d\tau d\nu d^2y \{ \sum_l \Pi_l \partial_\tau Q^I - C_{(1)} \},$$

where we replaced $d\nu d\tau$ by $d\tau$ in the second line, and $C_{(1)}$ is defined as

$$C_{(1)} = -\left( \frac{2h}{\partial_\tau R} \right) C_- - \pi_\tau.$$

Notice that if we impose the constraint $C_- = 0$, then $C_{(1)}$ reduces to

$$C_{(1)} = -\pi_\tau.$$

The second step in the Hamiltonian reduction consists of identifying arbitrarily specifiable coordinates $u$ and $y^a$ as

$$u = R, \quad y^a = Y^a$$

such that the “shift” vector $A_+^a$ is zero, i.e.,

$$A_+^a = 0.$$
Hamilton’s equations of motion follow from the variational principle of the action integral in (24):

$$\partial_\tau Q^I = \int_{\Sigma_\tau} du^2 y^2 \frac{\delta C(1)}{\delta I} \bigg|_{u=R, y^a=Y^a},$$

$$\partial_\tau \Pi_I = -\int_{\Sigma_\tau} du^2 y^2 \frac{\delta C(1)}{\delta Q^J} \bigg|_{u=R, y^a=Y^a},$$

where $Q^I = (R, A^a, \rho_{ab})$, $\Pi_I = (\pi_R, \pi_a, \pi^{ab})$, and $\Sigma_\tau$ is a spacelike hypersurface defined by $\tau = \text{constant}$.

4. Main results
In the following, we present the main results of this article[13, 14]. The spacetime metric in these privileged coordinates becomes

$$ds^2 = -4hdRd\tau - 2hdR^2 + \tau \rho_{ab}dY^a dY^b.$$  (33)

Einstein’s constraint equations

1. $C_- = 0 \Rightarrow \tau_a = -H + 2\partial_R \ln(-h)$ (physical Hamiltonian)  (34)
2. $C_+ = 0 \Rightarrow \tau_a = -\pi^{ab}\partial_R \rho_{ab}$ (physical momentum)  (35)
3. $C_0 = 0 \Rightarrow \tau^{-1}\pi_a = -\pi^{bc}\partial_a \rho_{bc} + 2\partial_b(\pi^{bc}\rho_{ac}) - \tau \partial_a (H + \pi_R)$ (physical momentum)  (36)

Superpotential $\ln(-h)$

4. $\partial_\tau \ln(-h) = H - \tau^{-1}$  (37)
5. $-\partial_\tau \ln(-h) = \pi_R$  (38)
6. $-\partial_a \ln(-h) = \tau^{-1}\pi_a$  (39)

Integrability conditions

7. $\partial_R (\tau^{-1}\pi_a) = \partial_a \pi_R$  (40)
8. $\partial_\tau \pi_R = -\partial_R H$  (41)
9. $\partial_\tau (\tau^{-1}\pi_a) = -\partial_a H$  (42)

In the above equations, $H$ is defined by

$$H = \frac{1}{\tau} \rho_{ab} \rho_{cd} \pi^{ac} \pi^{bd} + \frac{1}{4} \rho^{ab} \rho^{cd} (\partial_R \rho_{ac})(\partial_R \rho_{bd}) + \pi^{ac} \partial_R \rho_{ac} + \frac{1}{2} \frac{1}{2\tau}.$$  (43)

Topological censorship

10. $\tau R_{(2)} = \frac{1}{2} \tau^{-2} \rho^{ab} \pi_a \pi_b - \partial_a (\tau^{-1} \rho^{ab} \pi_b)$  (44)

Einstein’s evolution equations I

11. $\frac{\partial \pi_a}{\partial \tau} = 2\tau^{-1} \pi_a + (\pi^{bc} + \frac{\tau}{2} \rho^{bd} \rho^{ac} \partial_R \rho_{bc})(\partial_R \rho_{bc}) - \partial_b(2\pi^{bc} \rho_{ac} + \tau \rho^{bc} \partial_R \rho_{ac})$  (45)
12. $\frac{\partial \pi}{\partial \tau} = \frac{1}{2} \tau^{-2} + \tau^{-2} \rho_{ab} \rho_{cd} \pi^{ac} \pi^{bd} - \frac{1}{4} \rho^{ab} \rho^{cd} (\partial_R \rho_{ac})(\partial_R \rho_{bd}) - 2\tau^{-2} \partial_a(h \rho^{ab} \pi_b)$  (46)
The evolution equations of $\rho_{ab}$ and $\pi^{ab}$ can be found from the reduced action principle

$$S^* = \int dR d^2 Y \{ \pi^{ab} \partial_\tau \rho_{ab} - C^*_{(1)} \},$$

where $C^*_{(1)}$ is the restriction of $C_{(1)}$ to the coordinates $u = R$ and $y^a = Y^a$:

$$C^*_{(1)} := C_{(1)}|_{u=R, y^a=Y^a} = -\pi_\tau$$

$$= H - 2\partial_R \ln(-h).$$

**Einstein’s evolution equations II**

13. $\frac{\partial \rho_{ab}}{\partial \tau} = \int dR d^2 Y \frac{\delta C^*_{(1)}}{\delta \pi^{ab}}$,  

14. $\frac{\partial \pi^{ab}}{\partial \tau} = -\int dR d^2 Y \frac{\delta C^*_{(1)}}{\delta \rho_{ab}}$.

It can be shown that, by straightforward calculations, the whole set of equations summarized in Section 4 is identical to the vacuum Einstein’s equations $R_{AB} = 0$ in the privileged coordinates where the metric is given by (33). Thus, the whole procedure of the Hamiltonian reduction is self-consistent, even though the final theory is written in the privileged coordinates.

Finally, I would like to mention that the integral of (44) over a closed two surface $N_2$ becomes

$$\int_{N_2} d^2 Y \tau^{-2} \rho^{ab} \pi_a \pi_b = 16\pi (1 - g) \geq 0,$$

where $g$ is the genus of $N_2$. This identity states that, as long as the out-going null hypersurface forms a congruence of null geodesics which admits a cross section, the spatial topology of that null hypersurface is either a two sphere or a torus. This is a remarkably simple proof of topological censorship, as it does not rely on assumptions such as global hyperbolicity, asymptotic conditions, energy conditions, and so on, either inside or outside the null hypersurface, which are normally assumed in the literature[16, 17, 18].

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