Generating a second-order topological insulator with multiple corner states by periodic driving

Ranjani Seshadri, Anirban Dutta and Diptiman Sen
Centre for High Energy Physics, Indian Institute of Science, Bengaluru 560012, India
(Dated: January 31, 2019)

We study the effects of periodic driving on a variant of the Bernevig-Hughes-Zhang (BHZ) model defined on a square lattice. In the absence of driving, the model has both topological and non-topological phases depending on the different parameter values. We also study the anisotropic BHZ model and show that, unlike the isotropic model, it has a non-topological phase which has modes localized on only two of the four edges of a finite sized square. When an appropriate term violating time-reversal symmetry is added, the edge modes get gapped and gapless states appear at the four corners of a square. When the system is driven periodically by a sequence of two pulses, multiple corner states may appear depending on the driving frequency and other parameters. We discuss to what extent the system can be characterized by topological invariants such as the Chern number and mirror winding number. We have shown that the locations of the jumps in these invariants can be understood in terms of the Floquet operator at one of the time-reversal invariant momenta.

I. INTRODUCTION

Topological insulators (TIs) have been studied extensively for the last several years. A key feature of these materials is that the bulk states are gapped but there are gapless states at the boundaries which contribute to transport and other properties at low temperatures. Furthermore, there is a bulk-boundary correspondence, namely, the bulk bands are characterized by a topological invariant (such as the Chern number for two-dimensional TIs) which is an integer, and the number of states with a particular momentum at each of the boundaries is equal to the topological invariant. Recently, a generalization of these materials called higher-order TIs has been introduced. For instance, a second-order TI in two dimensions is a system in which the bulk and edge states are both gapped but there are gapless states at the corners of the system.

Closed quantum systems driven periodically in time constitute another area that has been studied by several groups in recent years. In particular, there has been much interest in understanding the conditions under which periodic driving can generate topological phases and boundary modes. It is sometimes found that even if the time-dependent Hamiltonian lies in a non-topological phase at each instant of time, the unitary time-evolution operator for one time period (called the Floquet operator) has eigenstates which are localized near the boundaries of the system. It therefore seems interesting to investigate if higher-order TIs can also be generated by periodic driving, for example, if such a driving can generate corner states in a two-dimensional system which has no such states in the absence of driving.

In this paper, we will consider a variant of the Bernevig-Hughes-Zhang (BHZ) model which is known to have corner states for certain values of the parameters in the Hamiltonian. We will study what happens when one of the parameters is varied periodically in time and show that this can generate corner states. We will then study an anisotropic version of the BHZ model; this has a non-topological phase in which only two out of the four edges of a finite sized square has edge states which are not topologically protected. We find that this model can also have corner states, with or without driving. Interestingly, we find that driving in certain parameter regimes can generate more than one state at each corner, unlike the time-independent model which never has more than one localized state at each corner. We will investigate if the number of corner states can be understood using any topological invariants.

The plan of this paper is as follows. In Sec. II we present the Hamiltonian of an isotropic model constructed by adding an appropriate term to the BHZ Hamiltonian. This additional term is essential to have corner states. After analyzing the different symmetries of the Hamiltonian, we discuss two topological invariants called the Chern number and the mirror winding number; we find that the regions of non-zero Chern and mirror winding numbers coexist in this model. We then discuss how to numerically study edge states by looking at a ribbon which is infinitely long but has a finite width and corner states by looking at a finite sized square. In Sec. III we study the effects of periodic driving on the isotropic model. To this end, we numerically calculate the Floquet operator and find its eigenvalues and eigenstates. We show how this modifies the regions of non-zero Chern and mirror winding numbers; depending on the parameters, the magnitude of the Chern number can be larger than 1. We also find that the driving can generate corner states; these states always have Floquet eigenvalues equal to ±1. In Sec. IV we study an anisotropic model in which some of the parameters have different values in the $x$ and $y$ directions; we find that the regions of non-zero Chern and mirror winding numbers do not coincide in this model. Namely, there is a non-topological phase where the Chern number is zero but the mirror winding number is non-zero. In this phase, there are edge states on only two of the edges of a finite sized square which are not topologically protected. We again study the effects of periodic driving and show that this can generate corner
states. We find that there may be more than one state at each corner if the driving frequency is low. In Sec. III we summarize our results and point out some directions for future research. Finally, we discuss in an Appendix how the locations as well as the magnitudes of the jumps in the Chern number and mirror winding number can be understood in terms of the contributions from the time-reversal invariant momenta.

II. TIME-INDEPENDENT ISOTROPIC MODEL

In this section, we will recall the properties of a variant of the BHZ model which, in the absence of periodic driving, is known to have gapless corner states. The purpose of this is to contrast the properties of this system with that of the periodically driven system that we well study in Sec. III.

A. Bulk Hamiltonian

We will consider the following Hamiltonian for a system with periodic boundary conditions in which the momentum \( \mathbf{k} = (k_x, k_y) \) is a good quantum number:

\[
H(\mathbf{k}) = \left[ M + t_0 (\cos k_x + \cos k_y) \right] \tau^z \otimes \sigma^0
+ \Delta_1 (\sin k_x \tau^x \otimes \sigma^x + \sin k_y \tau^x \otimes \sigma^y)
+ \Delta_2 (\cos k_x - \cos k_y) \tau^y \otimes \sigma^0,
\]

where \( \tau \) and \( \sigma \) are Pauli matrices acting on the orbital and spin degrees of freedom respectively, and \( \sigma^0 \) denote \( 2 \times 2 \) identity matrices in those two spaces respectively. In Eq. (1), \( t_0 \) denotes a spin-independent but orbital-dependent hopping amplitude between nearest-neighbor sites, and \( \Delta_1 \) denotes a spin-orbit coupling. We will find later that the last term with coefficient \( \Delta_2 \) is necessary in order to have corner states, with or without periodic driving.

We first study the symmetries of the Hamiltonian in Eq. (1) for the case with \( \Delta_2 = 0 \).

1. Time-reversal \( T \): For \( \Delta_2 = 0 \), we have the Bernevig-Hughes-Zhang (BHZ) model, a well-known example of a two-dimensional TI. The Hamiltonian is invariant under the time-reversal transformation, i.e., \( TH(\mathbf{k})T^{-1} = H(-\mathbf{k}) \), where \( T = \tau^0 \sigma^y K \), and \( K \) does complex conjugation and also transforms \( \mathbf{k} \to -\mathbf{k} \).

2. Charge conjugation \( C \): \( H(\mathbf{k}) \) has a particle-hole or charge conjugation symmetry given by \( C = \tau^y \sigma^y K \) such that \( CH(\mathbf{k})C^{-1} = -H(\mathbf{k}) \), and a chiral symmetry \( S_1 = CT = \tau^y \sigma^0 \) with \( S_1 H(\mathbf{k})S_1^{-1} = -H(\mathbf{k}) \). Therefore, the system belongs to the BDI class of the Altland-Zirnbauer classification.

There is another symmetry operator \( S_2 = \tau^x \sigma^z \) which gives \( S_2 H(\mathbf{k})S_2^{-1} = -H(\mathbf{k}) \).

3. Four-fold rotation \( C_4 \): \( H(\mathbf{k}) \) has a four-fold rotation symmetry about the \( z \)-axis, i.e., \( C_4 H(\mathbf{k})(C_4)^{-1} = H(C_4 \mathbf{k}) \), where \( C_4 = \tau^0 e^{-i(\pi/4)\sigma} \) and \( C_4 (k_x, k_y) = (k_y, -k_x) \).

4. The operators \( P_1 = \tau^z \sigma^z \) (this is basically the product of \( S_1 \) and \( S_2 \)) and \( P_2 = \tau^z \sigma^y K' \) both commute with the Hamiltonian. (Here \( K' \) does not do complex conjugation but does not transform \( \mathbf{k} \to -\mathbf{k} \). Since \( P_1 \) and \( P_2 \) anticommute with each other and \( P_2^2 = I_4 \) (the \( 4 \times 4 \) identity matrix), this explains a double degeneracy of the energy spectrum for each value of \( \mathbf{k} \); the two degenerate states have eigenvalues of \( P_1 = \pm 1 \) and are related to each other by the action of \( P_2 \).

We now consider the spectrum \( E(\mathbf{k}) \) obtained from Eq. (1) for \( \Delta_2 = 0 \). Since there is a two-fold degeneracy, for each value of \( \mathbf{k} \), there are only two distinct energy bands. The gap between the two bands is shown in Fig. 1(a) as a surface plot. For a non-zero value of \( \Delta_1 \), the spectrum is found to be gapless for \( M = 0 \), \( \pm 2 \). The region \( |M| > 2 \) is topologically trivial and the Chern number is zero as shown in Fig. 1(b). (The Chern number is calculated using the method described in Ref. 37). The regions \( -2 < M < 0 \) and \( 0 < M < 2 \), labeled as \( A \) and \( B \) in the figure, are both topological. The Chern numbers in these two regions are non-zero and have opposite signs. By studying the Hamiltonian for an infinitely long ribbon (in Sec. III B), we find there there are robust one-dimensional gapless edge modes in the topological regime as shown in Fig. 2(a).

Now switching on the \( \Delta_2 \) term, we find that the symmetries of the Hamiltonian get reduced as follows.

1. For \( \Delta_2 \neq 0 \), the time-reversal symmetry \( T \) as well as the rotational symmetry \( C_4 \) are broken. However the symmetry given by the product of the two, i.e., \( C_4 T \) is preserved. This means that \( (C_4 T) H(\mathbf{k})(C_4 T)^{-1} = H(C_4 T \mathbf{k}) \), where \( C_4 T (k_x, k_y) = (-k_y, k_x) \).

2. The charge conjugation and one of the other symmetries act as before: \( CH(\mathbf{k})C^{-1} = -H^*(\mathbf{-k}) \) and \( S_1 H(\mathbf{k})S_1^{-1} = -H(\mathbf{k}) \). The operator \( S_1 \) does not give anything simple.

3. The operator \( P_2 \) commutes with the Hamiltonian but \( P_1 \) does not.

We note that in the presence of a \( \Delta_2 \) term, the \( C_4 T \) symmetry gaps out the edge states but gapless corner states appear for certain values of \( M \). The system is then called a second-order TI.
FIG. 1: (a) Energy gap as a function of the parameters \( M \) and \( \Delta_1 \) for \( \Delta_2 = 0 \). The range of \( M \) taken here is the region where the system is topological. The band gap is zero along the lines \( M = 0, \pm 2 \). (b) Schematic phase diagram of the Hamiltonian (1) in the limit \( \Delta_2 = 0 \). The horizontal solid line shows the numerically calculated Chern number \( C \) separating the topological \( |M| < 2 \) (\( C \neq 0 \)) and non-topological regions (\( C = 0 \)). The blue and orange lines are for two degenerate states respectively, both belonging to the same band (either positive or negative energy). (c) The mirror winding number (defined in Eq. (2)) for \( \Delta_2 \neq 0 \) separating the higher-order topological insulator from the non-topological insulator.

There is a topological invariant called the mirror winding number \( \gamma_m \) which distinguishes between the higher-order topological insulator and the non-topological insulator phases. To understand this topological invariant, let us consider the mirror symmetry operator \( M_f \) which reflects the Hamiltonian \( H \) about a given line \( \Gamma \) in the Brillouin zone (BZ); the line \( \Gamma \) remains invariant under \( M_f \). This operator satisfies \( M_f^2 = -1 \) and has eigenvalues \( \pm i \). The diagonals in the BZ of the square lattice are given by \( \Gamma_{1(2)} : k_x = \pm k_y \), and we will denote the corresponding mirror symmetry operators as \( M_f^{\Gamma_{1(2)}} \) respectively. Each of these lines divides the BZ into two regions. The degenerate eigenstates of \( H(k) \) along each of these diagonals are separated using the projection operator \( P_{\pm} = (1 \pm iM_f^{\Gamma_{1(2)}})/2 \). The mirror winding number \( \gamma_{\Gamma_{1(2)}} \) along the lines \( \Gamma_{1(2)} \) is then defined as an integral of the Berry curvature along the corresponding diagonal in the BZ. This is an integral on a closed interval since the opposite ends of any diagonal are the same point (they are related to each other by a reciprocal lattice vector).

Namely,

\[
\gamma_m = -\frac{1}{\pi} \int_{\Gamma_1} d\xi \frac{\partial}{\partial \xi} \langle \psi_{\xi}^+ \mid \psi_{\xi}^+ \rangle,
\]

where \( \langle \psi_{\xi}^+ \mid \psi_{\xi}^+ \rangle \) is the energy eigenstate at the momentum denoted by \( \xi \) such that \( \langle \psi_{\xi}^+ \mid \psi_{\xi}^+ \rangle = P_{\pm} \langle \psi_{\xi}^+ \rangle \).

We numerically compute the mirror winding number using a method similar to the one prescribed in Ref. [37]. The procedure is as follows. Suppose that we want to calculate the integral in Eq. (2) along the diagonal \( k_x = k_y \) from \( k = (-\pi, -\pi) \) to \( (\pi, \pi) \). We can label points on this diagonal in terms of a variable \( \xi \) lying in the range \([0, 2\pi]\) as \( k = (-\pi, -\pi) + (\xi, \xi) \). We divide the complete range of \( \xi \) into \( N \) equal intervals of size \( d\xi = 2\pi/N \), and define \( \xi_j = (j - 1/2)d\xi \) where \( j = 1, 2, \ldots, N \). The normalization condition \( \langle \psi_{\xi}^+ \mid \psi_{\xi}^+ \rangle = 1 \) implies that \( \langle \psi_{\xi}^+ \mid \partial_\xi \psi_{\xi}^+ \rangle \) is purely imaginary; this implies that the imaginary part of \( \langle \psi_{\xi}^+ \mid \partial_\xi \psi_{\xi}^+ \rangle \) is equal to \(-\langle \psi_{\xi}^+ \mid \partial_\xi \psi_{\xi}^+ \rangle \), up to terms of first order in \( d\xi \). Eq. (2) can therefore be approximated by

\[
\gamma_m = -\frac{1}{\pi} \Im \ln \left( \prod_{j=1}^{N} \langle \psi_{\xi_j}^+ \mid \psi_{\xi_j}^+ \rangle \right)
\]

if \( N \) is large. The presence of ln in the above expression implies that the winding number calculated by this method is only defined modulo 2; hence it can only take the values 0 and 1.

Fig. 1(c) shows the mirror winding number for the model described in Eq. (1) with \( \Delta_2 \neq 0 \). Comparing this with Fig. 1(b) we see that the regions of non-zero values of the Chern number and mirror winding number coexist in this model.

B. Edge states for an infinitely long ribbon

We will first study the effect of the \( \Delta_2 \) term on the edge modes in this system. To this end, we consider a strip of the material which is infinitely long in the \( x \)-direction and has a finite width (with \( N_y \) sites) in the \( y \)-direction. This means that \( k_x \) is a good quantum number and for each \( k_x \), we effectively have a chain with \( N_y \) sites which is related to its neighboring chains by factors of \( \exp(\pm ik_y) \). At each site there are four degrees of freedom corresponding to the two orbitals \( (d \text{ and } f) \) which we denote by \( \sigma \) \( = \pm 1 \) and two spins \( (\uparrow \text{ and } \downarrow) \) denoted by \( \sigma^z \) \( = \pm 1 \). The creation operator for the \( d \) orbital at the \( n_y \)-th site of the chain for spin \( s \) is denoted by \( d_{n_y,s}^\dagger \) where \( s = \uparrow, \downarrow \). Similarly, for the \( f \) orbital, the corresponding operator is \( f_{n_y,s}^\dagger \). Hence the spinor for each value of \( k_x \) has \( 4N_y \) components and the Hamiltonian \( H(k_x) \) is a \( 4N_y \times 4N_y \) matrix. By keeping the \( k_x \) part of the Hamiltonian as
shown in blue. These two bands are separated by an energy gap whose value depends on both $M$ and $\Delta_1$ consistent with Fig. 1(a). Let us first consider Fig. 2(a) where we have taken $\Delta_2 = 0$. Here the bulk gap hosts gapless edge modes (red solid lines) which go from one band to the other and are related to the Chern number of the infinite system by the bulk-boundary correspondence; we thus have a two-dimensional topological insulator. On switching on a $\Delta_2$ term, these edge modes become gapped as shown in Fig. 2(b). Since each of these edge modes lie within the same band, i.e., they do not connect the two bands, they are topologically trivial. This is consistent with the fact the Chern numbers (4a) are zero for the infinite system with $\Delta_2 \neq 0$. A useful topological invariant for $\Delta_2 \neq 0$ is the mirror winding number $\gamma_m$ which is explained in Sec. 1A. However, the bulk-boundary correspondence for this case has to be analyzed using a system which is finite in both the directions, such as a finite sized square lattice, as discussed in the next section.

C. Corner states for a finite sized square lattice

In order to find the corner states of the system, we consider a finite sized square lattice lying in the $x-y$ plane with $N_x$ and $N_y$ sites along the $x$ and $y$-directions respectively. The total number of lattice points is $N = N_x \times N_y$. Since there are two orbital ($d$ and $f$) and two spin ($\uparrow$ and $\downarrow$) degrees of freedom at each site, we arrive at a $4N_y \times 4N_y$ Hamiltonian,

\[
H = \sum_{\langle n, n' \rangle_{s = \uparrow, \downarrow}} \left[ \frac{M}{2} \left( d_{n, s}^\dagger d_{n, s} - f_{n, s}^\dagger f_{n, s} \right) \\
+ \frac{t_0}{2} \left( d_{n, s}^\dagger d_{n', s} - f_{n, s}^\dagger f_{n', s} \right) \\
+ \frac{\Delta_1}{2} \eta_{s, \bar{s}} \epsilon_{n, n'} \left( d_{n, s}^\dagger f_{n', \bar{s}} + f_{n, s}^\dagger d_{n', \bar{s}} \right) \\
+ \frac{i \Delta_2}{2} \xi_{n, n'} \left( d_{n, s}^\dagger f_{n', s} - f_{n, s}^\dagger d_{n', s} \right) + \text{H.c.} \right],
\]

where $\langle n, n' \rangle$ denotes nearest neighbors, $\epsilon_{n, n'} = i$ or 1 and $\xi_{n, n'} = \pm 1$ for nearest neighbors along the $x$ and $y$-directions respectively.

The different terms in Eq. (5) can be understood as follows. $M$ acts as a staggered chemical potential for the two orbitals. $t_0$ is the amplitude for nearest-neighbor hopping that keeps both the spin and orbital the same but differs in sign for the two orbitals. $\Delta_1$ flips both the spin and orbital degrees of freedom and also depends on the direction of hopping via $\epsilon_{n, n'}$. $\Delta_2$ describes a hopping that flips the orbital but keeps the spin the same, and this term depends on the direction of hopping through $\xi_{n, n'}$. Diagonalizing this Hamiltonian gives the $4N_y$ energy eigenvalues and eigenvectors, all of which turn out...
to be doubly degenerate.

For our numerical calculations, we have considered a square lattice with 25 sites in each direction, i.e., \(N_x = N_y = 25\). The parameter \(t_0\) is set to unity and all other parameters and the energy are expressed in units of \(t_0\). The results for \(M = 1\), \(\Delta_1 = 1\) and \(\Delta_2 = 0.1\) are shown in Fig. 3. The plot of energy eigenvalues versus the eigenvalue index is shown in Fig. 3(a). The edge states (red) and corner states (blue) are clearly separated in energy from the bulk states (black). As is clear from the inset in this figure, there are four states which are very close to zero energy. They become degenerate in the thermodynamic limit; these four-fold degenerate states are found to exist only at the corners of the square. One such state which lives at the corner labeled 3 is shown in Fig. 3(b). Similar states at zero energy exist at each of the four corners. We find that the decay length of these states is larger along the edges than along the diagonal into the bulk. Moreover, we find that the four-component spinors corresponding to these corner states have an interesting structure. We recall that the Hamiltonian in Eq. (1) changes sign under a transformation by \(S_2 = \tau^x \sigma^z\). We therefore expect the space of zero energy states to remain invariant under the action of \(S_2\); in particular, the corner states should be eigenstates of \(S_2\). We find that the states localized at corners labeled 1 and 3 have eigenvalues of +1 while those at corners 2 and 4 have eigenvalues -1.

The edge states, whose energies are shown in red in Fig. 3(a), are found to be localized along all the edges of the system. However, these are not protected topologically as explained in the analysis of the ribbon geometry in Sec. II B (they do not go from one bulk band to the other).

To recapitulate, in this section we have analyzed a topologically non-trivial two-dimensional system which, in certain parameter regimes is found to have robust zero-energy “corner modes”. These can be connected to the bulk system using a topological invariant called the mirror winding number.

III. PERIODICALLY DRIVEN ISOTROPIC MODEL

Having understood the properties of the time-independent model of a second-order TI in various parameter regimes, we now proceed to study what happens when the system is driven periodically in time. In particular, we will study the effect of varying the parameter \(M\) in the Hamiltonian. We will study the topological properties of this driven system both in momentum space (i.e., for a system with periodic boundary conditions) as well as for a finite sized square lattice using a time-evolution operator.

First, let us consider the bulk system, i.e., we work with the momentum space Hamiltonian \(H\) given in Eq. (1) with the parameter \(M\) varying in time as:

\[
M(t) = \begin{cases} 
M_1 & \text{if } 0 < t < T/4 \\
M_2 = \alpha M_1 & \text{if } T/4 < t < 3T/4 \\
M_1 & \text{if } 3T/4 < t < T 
\end{cases}
\]  

within a single time period \([0, T]\); we then continue this periodically by taking \(M(t + T) = M(t)\) for all \(t\). The
parameter $\alpha$ is the ratio of the $M$’s for the two halves of a cycle. We are interested in studying the system stroboscopically, i.e., at times $t = NT$ where $N$ runs over all integers. We will calculate the quasienergy spectrum by numerically diagonalizing the Floquet operator $U_F$ (defined in Eq. (11) below) for each momentum $k$. We find that each of the two quasienergy bands is two-fold degenerate.

The time-evolution operator is given by

$$U_F = \mathcal{F}_t e^{-i \int_0^T H(t') dt'},$$

$$= e^{-i H_1 T / 4} e^{-i H_2 T / 2} e^{-i H_3 T / 4},$$

(7)

where $T$ is the time period of the driving and $H_1$ and $H_3$ are the Hamiltonians with $M = M_1$ and $M_2$ respectively. (The symbol $\mathcal{F}_t$ denotes the time-ordered product.) Since we will study both momentum and real space systems, the Hamiltonian takes two different forms for these two cases, as will be explained below. Further, since $U_F$ is a unitary operator, its eigenvalues $\epsilon_j$ are complex numbers with unit magnitude, i.e.,

$$U_F |\psi_j\rangle = e^{-i \epsilon_j T} |\psi_j\rangle,$$

(8)

where $\epsilon_j$’s are the quasienergies and are defined modulo $2\pi / T$. We define the Floquet BZ such that $\epsilon_j \in [-\pi / T, \pi / T]$ and $|\psi_j\rangle$’s are the corresponding Floquet eigenstates.

The driving protocol described in Eq. (7) has been chosen to satisfy a particular symmetry of the Floquet operator $U_F$. We saw earlier that $S_2 = \tau^x \sigma^z$ satisfies $S_2 H S_2^{-1} = -H$; this is true in both momentum space and real space. Eq. (7) then implies that

$$S_2 U_F S_2^{-1} = U_F^{-1}.$$

(9)

This implies that if $|\psi_j\rangle$ is an eigenstate of $U_F$ with eigenvalue $e^{-i \epsilon_j T}$, then $S_2 |\psi_j\rangle$ is an eigenstate of $U_F$ with eigenvalue $e^{+i \epsilon_j T}$. Thus the eigenvalues of $U_F$ must appear in complex conjugate pairs. Next, if there is an eigenstate $|\psi_j\rangle$ with eigenvalue $e^{-i \epsilon_j T} = \pm 1$, $S_2 |\psi_j\rangle$ must be the same as $|\psi_j\rangle$. Hence $|\psi_j\rangle$ must be an eigenstate of $S_2$, and the eigenvalue must be $\pm 1$ since $(S_2)^2$ is the identity operator. We will see below that corner states always appear with $e^{-i \epsilon_j T} = \pm 1$; hence they must also be eigenstates of $S_2$ with eigenvalue $\pm 1$.

Having computed the Floquet operator $U_F(k)$, we investigate if there is any correspondence between the bulk states and the corner states that appear due to the driving. To choose an appropriate time period for studying the Floquet problem, we have to compare the time period $T$ with an intrinsic time scale of the system, which, in this case can be taken to be $1 / t_0$. For fast driving, i.e., when the time period $T < < 1 / t_0$, the system does not have time to respond to the changing Hamiltonian and the properties are not very different from the static system. On the other hand, the system has interesting behavior for intermediate driving frequency, i.e., when $T$ is comparable to $1 / t_0$.

To study the intermediate frequency regime, we fix $T = 1$ and numerically find the eigenvalues and eigenvectors of $U_F(k)$ for different values of $\alpha$. We then use these eigenvectors to compute the topological invariants. The results for this are shown in Fig. 4. The two topological invariants i.e., the Chern number and mirror winding number are shown for the cases $\Delta_2 = 0$ and $\Delta_2 = 0.1$ in Figs. 4(a) and 4(b) respectively. Once again, the mirror winding number $\gamma_m$ is calculated modulo 2.

In the Fig 4(a) we have shown the Chern number for the case $\Delta_2 = 0$. The locations and magnitudes of the jumps in the Chern number can be understood by studying the time-evolution operator at some special points in the BZ. The Chern number changes abruptly when $U_F(k)$ becomes equal to $I_4$ at one of the four time-reversal invariant momenta $(0, 0)$, $(0, \pi)$, $(\pi, 0)$ and $(\pi, \pi)$. The magnitude and sign of the jump is determined by the momentum point where the time-evolution operator becomes $I_4$. A detailed explanation of this is given in Appendix A.

Now we switch on the $\Delta_2$ term, and compute the mirror winding number $\gamma_m$. Again, the jumps in $\gamma_m$ can be understood by the same method as discussed in Appendix A.

For the finite sized system on a $25 \times 25$ lattice, we find that corner states only appear when the mirror winding number of the bulk system is non-zero, implying that there is a non-trivial second-order bulk boundary correspondence. The time-evolution operator $U$ for the lattice model is given by the time-ordered product given in Eq. (7) with the Hamiltonian $H$ of the form given in Eq. (8) with the parameter $M$ given by Eq. (6) for the two halves of the cycle, i.e., $H_1 = H_{|M=M_1}$ and $H_3 = H_{|M=M_2}$. For a finite sized square lattice, $U_F$ is a $4N \times 4N$ matrix where $N = N_x \times N_y$. The fac-
tor of 4 is from the two spin and two orbital degrees of freedom at each lattice site. Diagonalizing this Hamiltonian gives $4N$ quasienergy eigenvalues denoted by $\epsilon_j$ in Eq. (8) and each eigenvector $|\psi_j\rangle$ is a $4N$-component spinor.

![Diagram of quasienergy and corner state](image)

**FIG. 5:** (a) Quasienergy spectrum and (b) corner state of the system driven as given by Eq. (9). The red dots close to $(-1,0)$ in (a) are the quasienergies corresponding to the corner states. The corner state shown in (b) is obtained by an appropriate superposition of these four degenerate states. Here the results are shown for the case when the ratio of the values of $M$ in the two halves of the time period is $\alpha = 2$, $M_1 = 2.5$, $\Delta_1 = 1$ and $\Delta_2 = 0.1$

Since each eigenvalue $\exp(-i\epsilon_j T)$ is a complex number of unit magnitude, it can be represented as a point on the unit circle. Figure 5(a) shows a plot of $\cos(\epsilon_j T)$ versus $\sin(\epsilon_j T)$. The points marked in red (shown more clearly in the inset) correspond to the eigenstates which are localized at one of the corners. One such state is shown in Fig. 5(b). These corner states are four-fold degenerate and lie close to $\sin(\epsilon_j T) = 0$. For this particular choice of parameters, these quasienergies lie at $\cos(\epsilon_j T) = -1$. They may also lie at $\cos(\epsilon_j T) = +1$ for a different set of parameters. For any choice of parameters, as long as we are in the higher-order topological sector, i.e., the mirror winding number is non-zero, these corner states exist and are always separated from the bulk and edge states by a finite gap which is proportional to $\Delta_2$.

It is interesting to note that corner states can appear even when both $M_1$ and $M_2$ are larger than 2; we can see from Fig. 4(b) that there are many such ranges of values of $M_1$ and $M_2$ where the mirror winding number is non-zero and there are corner states. Thus the periodic driving can generate corner states even when the instantaneous Hamiltonian does not have corner states at any time; as Fig. 4(c) shows, the time-independent Hamiltonian has no corner states if $M > 2$.

**IV. ANISOTROPIC MODEL**

We now consider a variation of the model discussed so far by making the hopping amplitude $t_0$ anisotropic, i.e., taking the hoppings along $x$ and $y$-directions to be $t_x$ and $t_y$ respectively. In the case of the static system, the Hamiltonian in Eq. (1) becomes

$$H(k) = (M + t_x \cos k_x + t_y \cos k_y) \tau^z \otimes \sigma^0 + \Delta_1 (\sin k_x \tau^x \otimes \sigma^x + \sin k_y \tau^z \otimes \sigma^y) + \Delta_2 (\cos k_x - \cos k_y) \tau^y \otimes \sigma^0.$$  (10)

This Hamiltonian has the same symmetries as the one in Eq. (1) except for the ones which involve the $C_4$ transformation.

By following the same procedure as outlined in Sec. II A we find the topological invariants, i.e., the Chern number and the mirror winding number, corresponding to the cases with $\Delta_2 = 0$ and $\Delta_2 \neq 0$ respectively. The results are shown in Fig. 6.

In the isotropic case, as discussed in Sec. II A at $M = 0$ there is a transition from one topologically non-trivial phase to another (regions labeled $A$ and $B$ in Fig. 1(b)). However, on introducing an anisotropy by taking $t_x \neq t_y$, the two topologically non-trivial regions are separated by an intermediate region where the Chern number is zero. (The boundaries of all the regions can be found by finding the values of $M$ where the energy eigenvalues of Eq. (10), with $\Delta_2 = 0$, vanish at one the four time-reversal invariant momenta). For the case $t_x = 1$ and $t_y = 2$, we see in Fig. 6(a) that the intermediate phase, labeled as $II$, lies in the region $|M| \leq 1$, whereas the phases labeled $I$ and $III$ are topologically non-trivial with Chern numbers $C = \pm 1$ respectively. The width $W_M$ of phase $II$ depends on the hoppings as $W_M = 2|t_x - |t_y||$.

![Diagram of Chern number and mirror winding number](image)

**FIG. 6:** Between the two topologically non-trivial regions $I$ and $III$, there is a region $II$ where the Chern number of the system is zero. The width of this region is given by $2|t_x - |t_y||$. In (a) we have taken $t_x = 1$ and $t_y = 2$ and have set $\Delta_2 = 0$ here. In (b) we show the mirror winding number for the same system but with $\Delta_2 = 0.3$.

On studying the Hamiltonian on a square lattice with $t_x \neq t_y$, we find that the edge states in the topological and non-topological phases behave very differently from each other. In regions $I$ and $III$, the edge states are present along all the edges of the system and are topologically protected. However, in region $II$, the edge states exist only on the edges parallel to the $x$-direction (when $|t_x| > |t_y|$) and are not topologically protected, even though they are gapless (if $\Delta_2 = 0$) and they lie inside the bulk gap. This is consistent with the value of the Chern number which is zero in region $II$. If $|t_y| < |t_x|$, the edge states are found only on the edges parallel to
the $y$–direction.

For $\Delta_2 = 0$, the different phases of the system can be distinguished from each other by the Chern number as shown in Fig. 6(a). If $\Delta_2 \neq 0$, the Chern number is ill-defined in all the phases, but they can be distinguished from each other by the mirror winding number. Fig. 6(b) shows the mirror winding number $\gamma_m$ for the anisotropic system with $\Delta_2 = 0.3$. We see that $\gamma_m = 1$ for $|M| \leq 3$. The mirror winding number does not differentiate between the phases $I$, $II$ and $III$ which are topologically different from one another as shown by the Chern number. We thus see that the regions of non-zero Chern and mirror winding numbers are not identical in the anisotropic model.

![FIG. 7: Edge states for phases corresponding to regions I and II for a system with the same parameters as in Fig. 6. In region I, since the Chern number is non-zero, the edge states exist along all edges of the sample. However, in region II where the Chern number is zero, the edge states exist only along the edges parallel to the $x$–direction. We have taken $t_x = 1, t_y = 2, \Delta_1 = 1$ and $\Delta_2 = 0.$](image)

The corner states are obtained by diagonalizing the Hamiltonian in real space on a square lattice. The expression for this Hamiltonian is similar to Eq. (5) with the hoppings along $x$ and $y$–directions being $t_x$ and $t_y$ respectively. Figure 7 shows the edge states in regions I and II for $\Delta_2 = 0$. We see that in region I, the edge states lie on all the edges while in region II, they only lie on the edges parallel to the $x$–direction (for $|t_x| < |t_y|$). This is consistent with the Chern numbers in these two regions. Fig. 8 shows the energy eigenvalues (the corner states lie at zero energy) and a corner state in region $II$ for $\Delta_2 = 0.3$.

We now study the effect of periodically driving the system by varying the parameter $M$ between two values $M_1$ and $M_2$ both of which lie in the “non-topological regime”, i.e., in region $II$ of Fig. 6. For simplicity we choose $M_2 = \alpha M_1$ as defined earlier in Eq. (6) and set $M_1 = -0.9$. The choice of the value of $M_1$ is arbitrary and the range of $\alpha$ is chosen to be $-1 < \alpha < 1$.

![FIG. 9: Topological invariants calculated from the Floquet operator of the Hamiltonian in Eq. (10) as a function of $\alpha$ for $M_1 = 0.9$, $\Delta_1 = 1$, $T = 3$, $t_x = 1$ and $t_y = 2$. In (a) we have set $\Delta_2 = 0$ and plotted the Chern number calculated from the Floquet eigenstates showing jumps at certain values of $\alpha$ indicating topological transitions. In (b) we show the mirror winding number for the system with $\Delta_2 = 0.3$; this shows a jump at one value of $\alpha$.](image)

In Fig. 9 (a), the Chern number is calculated for the anisotropic Hamiltonian in Eq. (10) for $\Delta_2 = 0$. The jumps in the Chern number at some specific values of the parameter $\alpha$ signify topological transitions and can be understood as discussed earlier. We note that $U_F(k) = \pm 1$ at these points and the magnitude of the Chern number can exceed 1 unlike the static system.

For $\Delta_2 \neq 0$, we compute the mirror winding number $\gamma_m$ from the wave functions of the quasienergy states of the Floquet operator. The mirror winding number also jumps at some specific values of $\alpha$ signifying second-order topological transitions. Unlike the Chern number the mirror winding number only jumps between 0 and 1 as we can see in Fig. 9 (b).

Finally we consider a finite sized square with $30 \times 30$ sites and periodically drive the anisotropic Hamiltonian. The corner states appear in this system by varying the parameter values for which the mirror winding number of the Hamiltonian in Eq. (10) is non-zero. For low-frequency driving, i.e., when the time period of driving is large compared to the time scale $1/t_0$, we sometimes find more than 4 corner states in the system which become de-
generate at zero quasienergy in the thermodynamic limit. Fig. 10 shows that for some values of the parameters, there are a total 12 corner states (3 at each corner), all lying at zero quasienergy in the thermodynamic limit.

![Quasienergies](image1)

**FIG. 10:** Multiple corner states on a square of size 30 × 30 sites with periodic driving. The system is driven by varying \( M \) from \(-0.9\) to \(0.9\) with a frequency \( T = 3\). In (a) we have plotted the quasienergies of the bulk, edge and corner states calculated from the Floquet operator. There are twelve corner states close to zero energy which are degenerate in the thermodynamic limit. By taking appropriate superpositions among these states, we find three states at each corner, as shown in (b), (c) and (d). The corner states decay exponentially into the bulk. We have taken \( t_x = 1, t_y = 2, \Delta_1 = 1 \) and \( \Delta_2 = 0.3\).

V. DISCUSSION

We first summarize the results obtained in this work. We considered a variant of the BHZ model in which there is an additional term which breaks the \( \mathcal{C}_4 \) and \( \mathcal{T} \) symmetries separately but is invariant under their product. This term is known to produce gapless states localized at the corners of a finite sized square lattice. The system can be characterized by two topological invariants called the Chern number and mirror winding number; the corner states appear when the mirror winding number is non-zero. We then studied the effect of periodic driving in this model. We find that the driving can generate corner states in certain ranges of parameters where the corresponding time-independent Hamiltonian has no such states. We have shown that these parameter ranges can again be characterized by the mirror winding number. Next, we have studied an anisotropic version of the model where the hopping amplitudes in the \( x \) and \( y \) directions are different. An interesting feature of this model is that there is a non-topological phase in which the Chern number is zero and there are states localized along only two of the four edges of a finite sized square; this phase has corner states and a non-zero mirror winding number. Periodic driving of this anisotropic model can also generate corner states. For the periodically driven models, we sometimes find that there can be more than one state localized at each corner of the system. We have shown that the locations of the jumps in the Chern number and mirror winding number can be understood as the points where the Floquet operator becomes proportional to the identity matrix at one of the time-reversal invariant momenta; these jumps signify topological transitions.

We have so far not found a topological invariant which can predict the number of states at each corner for the periodically driven system. The mirror winding number is not useful here since it is only defined modulo 2. Hence an important direction for future study would be to look for a topological invariant which can predict an arbitrary number of corner states, and, if possible, also predict the number of Floquet eigenvalues equal to \( \pm 1 \) and the eigenvalues of \( \tau^x \sigma^z \). For instance, an interesting topological invariant to look at would be the winding number of the one-dimensional states which are localized at one particular edge of an infinitely long ribbon with a finite width; these states would be labeled by a momentum \( k_x \) (as shown in Fig. 2), and an expression similar to Eq. (2) could then be used to calculate the winding number. At first sight, such a numerical calculation appears challenging because a finite width allows hybridization between the states at the opposite edges and we would therefore generally find states which are superpositions of states localized at the two edges. However, if these can be projected to one of the edges in a way which varies smoothly with \( k_x \), it may be possible to calculate the corresponding winding number. This may provide a way of understanding the number of corner states.

Acknowledgments

A.D. and R.S. thank Adhip Agarwala for useful discussions. A.D. acknowledges funding from SERB, DST, India for NPDF Research Grant No. PDF/2016/001482.
Appendix A: Contributions to Chern and mirror winding numbers from various momenta

The jumps in the Chern number shown in Fig. 4(a) can be understood by studying the time-evolution operators at some specific points in the Brillouin zone. These are the points where

\[ U_F(k_\pm) = \pm I_4. \]  

(A1)

First, let us study the case when \( \Delta_2 = 0 \). Consider the Hamiltonian at the momenta \( k_+ = (0, 0) \) and \( k_- = (\pi, \pi) \). At these two momenta the \( \Delta_1 \) and \( \Delta_2 \) terms vanish. Therefore the Hamiltonians in the two halves of the cycle, \( H_1 \) and \( H_2 \) commute with each other and the time-evolution operator given in Eq. (7) can be simplified to

\[ U_F(k_\pm) = e^{-i(M_1 + \alpha M_1 \pm 4t_0)}/(2\pi)^3. \]  

(A2)

This, in combination with Eq. (A1) and the fact that we have fixed \( t_0 = 1 \), gives the condition in Eq. (A1) if

\[ (M_1 + \alpha M_1 \pm 4) T/2 = n \pi, \]  

(A3)

where \( n \) is an integer. For all our calculations we have set \( M_1 = 2.5 \) and \( T = 1 \).

For \( k_+ \), the first few values of \( \alpha \) obtained from Eq. (A3) are given by 2.427, 4.940, 7.453 and 9.966. Similarly, for \( k_- \), \( \alpha \) is found to be 3.113, 5.627 and 8.140. These are precisely the points in Fig. 4(a) where the Chern number jumps; all the jumps are by \( \pm 1 \) since there is a contribution from either \( k_+ \) or \( k_- \), but not both.

Similarly, the points \( k_1 = (0, \pi) \) and \( k_2 = (\pi, 0) \) also contribute. In this case Eq. (7) reduces to

\[ U_F(k_{1,2}) = e^{-i(M_1 + \alpha M_1 \pm 4t_0)}/(2\pi)^3, \]  

which gives the condition in Eq. (A1) if

\[ (M_1 + \alpha M_1) T/2 = n \pi, \]  

(A5)

where \( n \) is an integer. This gives the values of \( \alpha \) as 1.512, 4.027, 6.540 and 9.053. Note that the contributions at these values of \( \alpha \) come from both \( k_1 \) and \( k_2 \). This explains the jumps of \( \pm 2 \) in the Chern number at these points.

We now consider the case when \( \Delta_2 \neq 0 \). The behavior at the momenta \( k_\pm \) is the same as in the case \( \Delta_2 = 0 \), because at these points the \( \Delta_2 \) term vanishes, and the Hamiltonians for the two halves of the cycle continue to commute. However, this is not true at the momenta \( k_{1,2} \) since the \( \Delta_2 \) term survives at these momenta. The time-evolution operator must therefore be written as

\[ U_F(k_{1,2}) = U_1 U_2. \]  

(A6)

We would again like this to be equal to \( \pm I_4 \) as in Eq. (A1). However, this would imply that \( U_1 \) and \( U_2 \) must commute. This contradicts the fact that \( U_1 \) and \( U_2 \) do not commute at \( k_{1,2} \). This contradiction implies that these two momenta cannot contribute to a change in any topological invariant. Indeed we see in Fig. 4(b) that the mirror winding number jumps when \( U_F(k) = \pm I_4 \) at \( k = k_{1,2} \).

We can summarize the results presented here as follow. For \( \Delta_2 = 0 \), there are jumps in the Chern number whenever \( U_F(k) = \pm I_4 \) at any of the four momenta \( (0, 0) \), \( (\pi, \pi) \), \( (0, \pi) \) and \( (\pi, 0) \). For \( \Delta_2 \neq 0 \), there are jumps in the mirror winding number whenever \( U_F(k) = \pm I_4 \) at any of the two momenta \( (0, 0) \) and \( (\pi, \pi) \).

1 M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
2 X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
3 W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Science 357, 61 (2017), and Phys. Rev. B 96, 245115 (2017).
4 Y. Peng, Y. Bao, and F. von Oppen, Phys. Rev. B 95, 235143 (2017).
5 J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Phys. Rev. Lett. 119, 246401 (2017); Luka Trifunovic, Piet W. Brouwer, Phys. Rev. X 9, 011012 (2019).
6 Z. Song, Z. Fang, and C. Fang, Phys. Rev. Lett. 119, 246402 (2017).
7 M. Ezawa, Phys. Rev. Lett. 120, 026801 (2018).
8 V. Dwivedi, C. Hickey, T. Eschmann, and S. Trebst, Phys. Rev. B 98, 054432 (2018).
9 G. van Miert and C. Ortix, Phys. Rev. B 98, 081110(R) (2018).
10 F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, Science Advances 4, 0346 (2018).
11 B. Y. Xie, H. F. Wang, H.-X. Wang, X. Y. Zhu, J.-H. Jiang, M. H. Lu, and Y. F. Chen, Phys. Rev. B 98, 205147 (2018).
12 D. Calugaru, V. Juricic, and B. Roy, Phys. Rev. B 99, 041301(R) (2019).
13 Byungmin Kang, Ken Shiozaki, Gil Young Cho, arXiv:1812.06999.
14 Raquel Queiroz, Ady Stern, Gil Young Cho, arXiv:1807.04141.
15 Chen-Hsuan Hsu, Peter Stano, Jelena Klinovaja, Daniel Loss, Phys. Rev. Lett. 121, 196801 (2018).
16 Akishi Matsugtani, Haruki Watanabe, Phys. Rev. B 98, 205129 (2018).
17 J. Dziarmaga, Adv. Phys. 59, 1063 (2010).
18 A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Rev. Mod. Phys. 83, 863 (2011).
19 A. Dutta, G. Aeppli, B. K. Chakrabarti, U. Divakaran, T. F. Rosenbaum, and D. Sen, Quantum phase transitions in transverse field spin models: from statistical physics to quantum information (Cambridge University Press, Cambridge, 2015).
20 L. D’Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. 65, 239 (2016).
21 T. Oka and H. Aoki, Phys. Rev. B 79, 081406 (2009); T. Kitagawa, T. Oka, A. Brataas, L. Fu, and E. Demler, Phys. Rev. B 84, 235108 (2011).
22 T. Kitagawa, E. Berg, M. Rudner, and E. Demler, Phys. Rev. B 82, 235114 (2010).
23 Z. Gu, H. A. Fertig, D. P. Arovas, and A. Auerbach, Phys. Rev. Lett. 107, 216601 (2011).
24 N. H. Lindner, G. Refael, and V. Galitski, Nature Phys. 7, 490 (2011);
25 E. S. Morell and L. E. F. Foa Torres, Phys. Rev. B 86, 125449 (2012).
26 B. Dóra, J. Cayssol, F. Simon, and R. Moessner, Phys. Rev. Lett. 108, 056602 (2012).
27 A. Kundu, H. Fertig, and B. Seradjeh, Phys. Rev. Lett. 113, 236803 (2014).
28 M. Thakurathi, A. A. Patel, D. Sen, and A. Dutta, Phys. Rev. B 88, 155133 (2013); M. Thakurathi, K. Sengupta, and D. Sen, Phys. Rev. B 89, 235434 (2014).
29 Y. T. Katan and D. Podolsky, Phys. Rev. Lett. 110, 016802 (2013).
30 Z. Hua-Xin, W. Tong-Tong, G. Jin-Song, L. Shuai, S. Ya-Jun, and L. Gui-Lin, Chinese Phys. Lett. 31, 030503 (2014).
31 M. S. Rudner, N. H. Lindner, E. Berg, and M. Levin, Phys. Rev. X 3, 031005 (2013); F. Nathan and M. S. Rudner, New J. Phys. 17, 125014 (2015).
32 D. Carpentier, P. Delplace, M. Fruchart, and K. Gawedzki, Phys. Rev. Lett. 114, 106806 (2015).
33 M. Thakurathi, D. Loss, and J. Klinovaja, Phys. Rev. B 95, 155407 (2017).
34 B. Mukherjee, A. Sen, D. Sen and K. Sengupta, Phys. Rev. B 94, 155122 (2016); B. Mukherjee, P. Mohan, D. Sen and K. Sengupta, Phys. Rev. B 97, 205415 (2018).
35 B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, Science 314, 1757 (2006).
36 A. Altland and M. Zirnbauer Phys. Rev. B 55, 1142 (1997).
37 T. Fukui, Y. Hatsugai, and H. Suzuki, J. Phys. Soc. Jpn. 74, 1674 (2005).
38 B. Huang and W. Vincent Liu, arXiv:1811.00555.
39 M. Rodriguez-Vega, A. Kumar, and B. Seradjeh, arXiv:1811.04808.
40 Y. Peng and G. Refael, arXiv:1811.11752.