COMBINATORIAL THEORY OF THE SEMICLASSICAL EVALUATION OF TRANSPORT MOMENTS I: EQUVALENCE WITH THE RANDOM MATRIX APPROACH

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Abstract. To study electronic transport through chaotic quantum dots, there are two main theoretical approaches. One involves substituting the quantum system with a random scattering matrix and performing appropriate ensemble averaging. The other treats the transport in the semiclassical approximation and studies correlations among sets of classical trajectories. There are established evaluation procedures within the semiclassical evaluation that, for several linear and non-linear transport moments to which they were applied, have always resulted in the agreement with random matrix predictions. We prove that this agreement is universal: any semiclassical evaluation within the accepted procedures is equivalent to the evaluation within random matrix theory.

The equivalence is shown by developing a combinatorial interpretation of the trajectory sets as ribbon graphs (maps) with certain properties and exhibiting systematic cancellations among their contributions. Remaining trajectory sets can be identified with primitive (palindromic) factorisations whose number gives the coefficients in the corresponding expansion of the moments of random matrices. The equivalence is proved for systems with and without time reversal symmetry.

1. Introduction

Transport through a chaotic cavity is usually studied through a scattering description. For a chaotic cavity attached to two leads with \( N_1 \) and \( N_2 \) channels respectively, the scattering matrix is an \( N \times N \) unitary matrix, where \( N = N_1 + N_2 \). It can be separated into transmission and reflection subblocks

\[
S(E) = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix},
\]

which encode the dynamics of the system and the relation between the incoming and outgoing wavefunctions in the leads. The transport statistics of the cavity in question can now be expressed in terms of the subblocks of \( S(E) \). For example, the conductance is proportional to the trace \( \text{Tr} [t^{\dagger} t] \) (Landauer-Büttiker formula \([13, 28, 29]\)), while other physical properties are expressible through higher moments like \( \text{Tr} [t^{\dagger} t]^n \).

There are two main approaches to studying the transport statistics in clean ballistic systems: a random matrix theory (RMT) approach, which argues that \( S \) can be viewed as a random matrix from a suitable ensemble, and a semiclassical approach that approximates elements of the matrix \( S \) by sums over open scattering trajectories through the cavity.

It was shown by Blümel and Smilansky \([11, 12]\) that the scattering matrix of a chaotic cavity is well modelled by the Dyson’s circular ensemble of random matrices of suitable symmetry. Thus, transport properties of chaotic cavities are often treated by replacing the scattering matrix with a random one (see \([2]\) for a review). Calculating the averages over the appropriate random
matrix ensemble is a very active area with many different approaches. A partial list of recent results include the papers [24, 30, 38, 40, 43, 47, 48, 53, 54, 61]. In Section 2 we review some basic facts about integration over random matrices.

On the other hand, the semiclassical approach makes use of the following approximation for the scattering matrix elements [41, 50, 51]

\[ S_{\text{oi}}(E) \approx \frac{1}{\sqrt{N\tau_d}} \sum_{\gamma(i \to o)} A_{\gamma}(E)e^{\frac{i}{\hbar}S_{\gamma}(E)}, \]

which involves the open trajectories \( \gamma \) which start in channel \( i \) (for “input”) and end in channel \( o \) (for “output”), with their action \( S_{\gamma} \) and stability amplitude \( A_{\gamma} \). The prefactor also involves \( \tau_d \) which is the average dwell time, or time trajectories spend inside the cavity. For transport moments one considers quantities of the type

\[ M_n(X) = \left< \text{Tr} \left[ X^\dagger X \right]^n \right>_E \sim \left< \frac{1}{(N\tau_d)^n} \sum_{i,j,o} \sum_{\gamma_j(i \to o)} \prod_{j=1}^{n} A_{\gamma_j} A_{\gamma_j'}^* e^{\frac{i}{\hbar}(S_{\gamma_j} - S_{\gamma_j'})} \right>, \]

where the trace means we identify \( i_{n+1} = i_1 \) and where \( X \) is either the transmitting or the reflecting subblock of the scattering matrix. The averaging is performed over a window of energies \( E \) which is classically small but semiclassically large: the width \( \Delta E \) of the window satisfies \( \hbar/N\tau_d \ll \Delta E \ll E \). Note that we use dagger to mean conjugate-transpose of a matrix and star to denote complex conjugation.

The choice of the subblock \( X \) affects the sums over the possible incoming and outgoing channels, but not the trajectory structure which involves \( 2n \) classical trajectories connecting channels. Of these, \( n \) trajectories \( \gamma_j \), \( j = 1, \ldots, n \), contribute with positive action while \( n \) trajectories \( \gamma_j' \) contribute with negative action. In the semiclassical limit of \( \hbar \to 0 \) we require that these sums cancel on the scale of \( \hbar \) so that the corresponding trajectories can contribute consistently when we apply the averaging in (3).

The semiclassical treatment will be reviewed in Section 3. The main idea of the treatment is that, in order to achieve a small action difference, the trajectories \( \{ \gamma_j \} \), must follow the path of trajectories \( \{ \gamma_j' \} \) most of the time, deviating only in small regions called encounters. The topological configuration of encounters and trajectories’ stretches between them is described using a diagram. The task of semiclassical evaluation thus splits into two parts: evaluation of the contribution of a given diagram by integrating over all possible trajectories of given structure and enumerating all possible diagrams.

For the former task, a well established approximation emerged by extending the pioneering work of Richter and Sieber [51] by a group of physicists based mainly in Essen: S. Müller, S. Heusler, P. Braun and F. Haake, [22, 42]. Henceforth we refer to this approximation as the “Essen ansatz”. Roughly speaking, it assigns to each diagram a weight which depends on the number and type of encounters and the number of trajectory stretches between the encounters. The approximation is derived based on physically justified assumptions; a mathematical derivation remains outside reach even for the simplest chaotic systems.

Within the Essen ansatz, incremental progress has been made in evaluating various transport-related quantities, see for example [4, 6, 7, 13, 22, 27, 25, 42, 51]. In every case where a RMT prediction was available it was found to be in full agreement with the semiclassical evaluation.
This paper is devoted to proving a general theorem that implies that this will always remain the case: any semiclassical evaluation within the Essen ansatz is equivalent to the RMT evaluation.

Before we formulate our theorem, we note that the trace of any form can be expanded as a sum of products of matrix elements. For example, the trace in equation (3) expands as

\begin{equation}
\text{Tr} \left[ X^\dagger X \right]^n = \sum_{i_j, o_j} \mathcal{S}_{i_1, o_1} \mathcal{S}_{o_2, i_2} \cdots \mathcal{S}_{o_{n-1}, i_{n-1}} \mathcal{S}_{i_1, i_1}
\end{equation}

\begin{equation}
= \sum_{i_j, o_j} \mathcal{S}_{o_n, i_n} \cdots \mathcal{S}_{o_{i+1}, i_{i+1}} \mathcal{S}_{i_1, o_1} \mathcal{S}_{s_{i, i}} \cdots \mathcal{S}_{s_{i+1, i+1}} \mathcal{S}_{i_1, i_1}
\end{equation}

\begin{equation}
= \sum_{i_j, o_j} \mathcal{Z}_{i_1, o_1} \mathcal{Z}_{i_2, o_2} \cdots \mathcal{Z}_{i_n, o_n} \mathcal{Z}_{s_{i, i}} \cdots \mathcal{Z}_{s_{i+1, i+1}} \mathcal{Z}_{i_1, i_1}
\end{equation}

where X is a sub-block of the scattering matrix S, the variables \( i_1, \ldots, i_n \) run over all columns of S that appear in X and the variables \( o_1, \ldots, o_n \) run over all rows of S in X. In the last line we switched from the matrix S to its transpose \( Z = S^T \) as this will make further notation less confusing. From equation (4) it is clear that if we can evaluate the averages of products of matrix elements of S (or Z), we can have access to every linear transport moment. The same goes for the nonlinear transport moments, i.e. averages of the form \( \langle \text{Tr} \left[ X^\dagger X \right]^n \text{Tr} \left[ X^\dagger X \right]^{n_2} \cdots \rangle \).

**Theorem 1.1.** Within the Essen ansatz, the energy average of a product of the elements of the scattering matrix \( S(E) \) (or \( Z(E) = S^T(E) \)) coincides with the corresponding average in RMT,

\begin{equation}
C_E(\mathbf{a}, \mathbf{b}) \equiv \langle Z_{a_1 \sigma \tau} \cdots Z_{a_s \sigma \tau} Z_{b_1 b_1^\tau} \cdots Z_{b_l b_l^\tau} \rangle_{E} \equiv \langle U_{a_1 a_1} U_{a_2 a_2} \cdots U_{b_1 b_1} U_{b_2 b_2} \cdots U_{b_l b_l} \rangle_{\text{RMT}}.
\end{equation}

For scattering matrices with time-reversal symmetry the appropriate RMT average is over Circular Orthogonal Ensemble; without the symmetry the RMT average is over Circular Unitary Ensemble.

To prove Theorem 1.1 we will put the diagrammatic method of the semiclassical approximation on a more rigorous footing, carefully describing diagrams as ribbon graphs with certain colorability properties. We define operations on the diagrams that lead to cancellations and, eventually, to the equivalence with the RMT calculation.

We note that a preliminary outline of this theorem was published in [8]. While in the process of writing up, we were notified by Marcel Novaes that he achieved a similar breakthrough although only for systems without time-reversal symmetry. His original approach uses a combinatorial identity that remains unproved [44, 45], but a new approach provides the complete equivalence [46].

It is important to mention that Theorem 1.1 does not produce any new formulae for moments such as \( M_n(X) \), it only establishes the equivalence of the two existing approaches to their evaluation. However, the description of the diagrams that we develop in the proof can be analyzed further to yield new results. This is done in the second part of this work [9], where we formalize the semiclassical evaluation of moments, re-cast it as a summation over factorizations of given permutations, and thus calculate \( M_n(X) \) for any \( n \) to several orders in the small parameter \( 1/N \).

The layout of the paper is as follows: in sections 2 and 3 we review the relevant notions and results from random matrix theory and semiclassical approximation, correspondingly. Section 4 discusses general properties of the sets of trajectories contributing on the semiclassical side. In section 5 and 6 we define and study the diagrams that classify contributing sets of orbits and relate them to the random matrix expansions for Circular Unitary and Orthogonal Ensembles,
correspondingly. We endeavored to make the paper readable to a wide variety of audiences, providing numerous examples and figures to illustrate the discussed concepts.

2. RMT prediction

It has been argued \cite{11, 12} that scattering through a chaotic cavity is described by a unitary matrix from an appropriate random matrix ensemble. In the absence of Time Reversal Symmetry (TRS) the Circular Unitary Ensemble (CUE) is used. If the TRS is broken the appropriate ensemble is the Circular Orthogonal Ensemble (COE). The final classical symmetry class involves particles with spin $\frac{1}{2}$, for which breaking spin-rotation symmetry through spin-orbit interactions leads to the Circular Symplectic Ensemble (CSE).

The CUE is the unitary group $U(N)$ endowed with the Haar measure. The averages of products of the elements of matrices $U \in U(N)$ have been studied in \cite{14, 34, 52} among other works. The main result is

\begin{equation}
\langle U_{a_1a_2} \cdots U_{a_ta_t} U_{b_1b_2}^\ast \cdots U_{b_mb_m}^\ast \rangle_{\text{CUE}(N)} = \delta_{t,s} \sum_{\sigma,\pi \in S_t} V_N^U(\sigma^{-1}\pi) \prod_{k=1}^t \delta(a_k - b_{\sigma(k)}) \delta(a_k^* - b_{(\pi(k))}) ,
\end{equation}

where $S_t$ is the symmetric group of permutations of the set $\{1, \ldots, t\}$, $\delta_{k,n} = \delta(k-n)$ is the Kronecker delta (the latter notation is used solely to avoid nesting subindices) and the coefficient $V_N^U(\sigma^{-1}\pi)$ depends only on the lengths of cycles in the cycle expansion of $\sigma^{-1}\pi$, i.e. on the conjugacy class of the permutation $\sigma^{-1}\pi$. For this reason we will refer to $V_N^U$ as the CUE class coefficients.

The COE is the ensemble of unitary symmetric matrices\footnote{Another name present in the literature is “Weingarten” function \cite{65}, even though it was probably Samuel \cite{52} who first defined the function and systematically studied it.} with a probability distribution obtained from the CUE through the mapping $W = UU^T$, where $U$ is a unitary matrix from the CUE and $U^T$ its transpose. The result analogous to (6) reads \cite{14} (see also \cite{35}),

\begin{equation}
\langle W_{a_1a_2} \cdots W_{a_ta_t} W_{b_1b_2}^* \cdots W_{b_mb_m}^* \rangle_{\text{COE}(N)} = \delta_{t,s} \sum_{\pi \in S_{2t}} V_N^O(\pi) \prod_{z \in Z_t} \delta(a_z - b_{\pi(z)}) ,
\end{equation}

where $\pi$ is a permutation on the set $Z_t = \{1, \ldots, t, \bar{1}, \ldots, \bar{t}\}$.

We mention that averaging formulae similar to (6) and (7) have recently become available for a much bigger variety of random matrix ensembles, see \cite{32} and references therein.

Both types of class coefficients used above can be calculated recursively. Namely, the class coefficients $V_N^U$ were derived by Samuel \cite{52} to satisfy $V_N^U(\emptyset) = 1$ and

\begin{equation}
NV_N^U(c_1, \ldots, c_k) + \sum_{p+q = c_1} V_N^U(p, q, c_2, \ldots, c_k) + \sum_{j=2}^k c_j V_N^U(c_1 + c_j, \ldots, \hat{c}_j, \ldots, c_k) = \delta_{c_1,1} V_N^U(c_2, \ldots, c_k) .
\end{equation}

Here $c_1, \ldots, c_k$ are the lengths of the cycles in the cycle expansion of $\sigma^{-1}\pi$. The notation $\hat{c}_j$ means that the element $c_j$ has been removed from the list. Finally, $\delta_{c_1,1}$ is the Kronecker delta.\footnote{Thus, despite the word “orthogonal” in the name, it is not the orthogonal group $O(N)$. Rather, it can be identified with $U(N)/O(N)$.}
The corresponding recursion relation for the COE class coefficients were derived by Brouwer and Beenakker [14]. They represent the permutation π in equation (7) as the product

$$\pi = T'^* e_\pi T''$$

where \( T' \) and \( T'' \) are some involutions satisfying \( T'(j) = j \) or \( T''(j) \), \( e_\pi \) is a permutation on the set \( \{1, \ldots, t\} \) and \( e_\pi \) is a permutation on the set \( \{\bar{1}, \ldots, \bar{t}\} \). Note that factorization (9) may be non-unique. What is unique is the cycle structure of the permutation on \( T \) where \( \tau = T e_\pi^{-1} T e_\pi \) or \( \tau = (1\bar{1}) \cdots (t\bar{t}) \) is used to “cast” the permutation \( e_\pi \) into a permutation acting on \( \{1, \ldots, t\} \).

It turns out that the class coefficients \( V^O_N(\pi) \) depend only on the cycle type of the permutation \( \tau \) defined above. They satisfy the recursion

$$\begin{align*}
(N + c_1) V^O_N(c_1, \ldots, c_k) + \sum_{p+q=c_1} V^O_N(p, q, c_2, \ldots, c_k) + 2 \sum_{j=2}^k c_j V^O_N(c_1 + c_j, \ldots, \hat{c_j}, \ldots, c_k) \\
= \delta_{c_1,1} V^O_N(c_2, \ldots, c_k),
\end{align*}$$

with the initial condition \( V^O_N(0) = 1 \).

In Lemma 4.13 in Section 4.2 we will give a simpler prescription for identifying the partition \( c_1, \ldots, c_k \) which corresponds to a given \( \pi \), bypassing the representation of \( \pi \) as the product in (9).

There are also expansions of \( V^O_N(\pi) \) in inverse powers of \( N \) with coefficients expressed in terms of the number of factorizations of \( \pi \) of various types: primitive factorizations [33], inequivalent factorization [5], and general factorizations [19]. It is the primitive factorizations, discussed by Matsumoto and Novak [33], that will be particularly important to us. We will give an alternative proof of their result in Section 5.3 and will extend it to the COE case in Section 6.3.

3. Semiclassical approximation

In this section we review the physical approximations involved in evaluating correlations of the type

$$C_E(\mathbf{a}, \mathbf{b}) = \langle Z_{a_10} \cdots Z_{a_s0} Z_{b_10} \cdots Z_{b_l0} \rangle E.$$  

First one employs the semiclassical approximation [41, 50, 51] from equation (2), leading to the expression

$$C_E(\mathbf{a}, \mathbf{b}) = \left( \frac{1}{(N\tau_d)^{(s+t)/2}} \sum_{\gamma_j \gamma_k} A_{\gamma_j} e^{iS_{\gamma_j}(E)} \prod_{j=1}^s A_{\gamma_j} e^{-\frac{i}{\hbar}S_{\gamma_j}(E)} \prod_{k=1}^t A_{\gamma_k} e^{-\frac{i}{\hbar}S_{\gamma_k}(E)} \right),$$

here \( N\tau_d \) is equal to the Heisenberg time.

In the second step, which we will call the “coinciding pathways approximation”, it is argued that since \( \hbar \) is very small, the energy average is a sum of oscillatory integrals, with the main contribution coming from the terms with (almost) zero phase [50, 53, 58, 59]:

$$\sum_j S_{\gamma_j}(E) - \sum_k S_{\gamma_k}(E) \lesssim \hbar.$$

To achieve this, the union of paths of \( \gamma_j \) in the phase space must be almost identical to the union of paths \( \gamma_k \). At this point, TRS starts to play a role: if it is broken then the paths of \( \gamma_k \)
Figure 1. A schematic depiction of a pair of trajectories that provide the first off-diagonal contribution. This pair requires TRS, since the loop is traversed by the trajectories in opposite directions.

Figure 2. A simple pair of trajectories that do not require TRS: both loops are traversed by both trajectories in the same direction, but in a different order.

must be traversed in the same direction, while in the presence of TRS the direction of traversal becomes irrelevant.

Let us consider the case $s = t = 1$ for simplicity. The easiest way to achieve a small action difference is to let $\gamma = \gamma'$. This is known as the “diagonal approximation”, pioneered for closed systems by Berry [10] and for open systems that we consider here by Blümel and Smilansky [11]. However, it was observed in [1] that the diagonal approximation in some cases fails to predict even the leading order contribution correctly.

From analogy to disordered systems it was believed that the next off-diagonal correction would come from trajectories $\gamma$ that nearly intersect themselves, thus having a loop. The partner trajectory $\gamma'$ would run along $\gamma$ until the self-intersection, then traverse the loop in the direction opposite to $\gamma$, and in the final part it would run along $\gamma$ again. This configuration requires TRS and is schematically depicted in Fig. 1. These general ideas were given analytical form in a breakthrough work by Richter and Sieber [51] (see also [56, 55]), who calculated the correction from the diagrams shown in Fig. 1. This development paved the way for calculating higher order corrections and higher order moments. For example, one of the simplest configurations with broken TRS is presented in Fig. 2. Before we proceed, however, it is important to mention that the trajectory $\gamma$ can (and typically does) have more than one “near intersection”. The diagram of Fig. 1 is meant to represent the unique near-intersection at which the trajectories $\gamma$ and $\gamma'$ go in different directions.

The general idea is to split the set of paths into regions of two types: “links” (or “stretches”) that are traversed by exactly one $\gamma_j$ and exactly one $\gamma_k$ and “encounters” where multiple stretches meet and the trajectories $\gamma$ interconnect differently from the trajectories $\gamma'$. Using additional assumptions that all stretches are long and the sum over all possible paths a stretch can take (between two given endpoints) is well approximated by the ergodic average, Müller, Heusler,
Braun and Haake [42] formulated rules for evaluating the contribution of a given topological arrangement of trajectories. This is the third major approximation involved in the Essen ansatz.

**Definition 3.1.** The total contribution of all pairs \( \{ \gamma_j \} \) and \( \{ \gamma'_k \} \) with a given topological arrangement of links and encounters is given by a product where

- every link provides a factor \( 1/N \),
- every encounter of \( 2l \) stretches (an \( l \)-encounter) gives a factor of \( -N \),
- encounters that happen in the lead do not count (give a factor of 1).

With these approximations the problem of evaluating any transport statistic is reduced to the (hard) problem of counting all distinct topological arrangements of links and encounters, so-called “diagrams”, with their respective weights calculated according to the rules above.

**4. Restrictions on matrix coefficients**

Within the coinciding pathways approximation, links of the trajectories \( \{ \gamma_j \} \) should be in a one-to-one correspondence to links of the trajectories \( \{ \gamma'_k \} \). For the links starting or ending in channels in the lead, this implies restriction on which correlators from (11) can be non-zero. Here we describe these restrictions, first for the systems with broken TRS (the unitary case) as it is simpler and then for systems with TRS (the orthogonal case).

**4.1. The unitary case.** For the unitary case with broken TRS, assume we have a suitable configuration of trajectories \( \gamma \) and \( \gamma' \), contributing to \( C_E(a, b) \). We start from channel \( a_1 \) and follow the trajectory \( \gamma_1 \). The final stretch of \( \gamma_1 \) leads to the channel \( a_\pi(1) \). With the coinciding pathways approximation, the same stretch is traversed by a \( \gamma' \) trajectory, which we denote by \( \gamma'_{\pi(1)} \). The trajectory \( \gamma'_{\pi(1)} \), by definition, goes between channels \( b_{\pi(1)} \) and \( b_{\pi(1)} \). The final stretch of \( \gamma_1 \) must also be final for \( \gamma'_{\pi(1)} \) (since it is leading to a channel and not an encounter and since the trajectories must run in the same direction). We immediately conclude that

\[
(13) \quad a_\pi(1) = b_{\pi(1)}.
\]

We follow the trajectory \( \gamma'_{\pi(1)} \) backwards, until we are on its first stretch, about to hit channel number \( b_{\pi(1)} \). The partner of \( \gamma'_{\pi(1)} \) on this stretch is a \( \gamma \) trajectory, which we will denote \( \gamma_{\tau(1)} \). We can now conclude that

\[
(14) \quad a_{\tau(1)} = b_{\pi(1)}.
\]

We can now follow trajectory \( \gamma_2 \), finding the value of \( \pi(2) \) and \( \tau(2) \) etcetera. It is clear that the thus defined functions \( \pi \) and \( \tau \) are permutations. Since the number of end-points of the trajectories \( \gamma \) needs to be the same as the number of end-points of the trajectories \( \gamma' \), we immediately get \( s = t \) in Eq. (11). Henceforth we denote this common value by \( n \).

Further, the permutation \( \pi \) imposes \( n \) restrictions on the sets \( a \) and \( b \), namely,

\[
a_j = b_{\pi(j)}, \quad j = 1 \ldots n.
\]

Letting \( \sigma = \pi \tau^{-1} \) we can transform identities similar to (14) into

\[
a_j = b_{\sigma(j)}, \quad j = 1 \ldots n.
\]

The permutation \( \tau \) defined above and calculated from \( \sigma \) and \( \pi \) as \( \tau = \sigma^{-1} \pi \) will be called the “target permutation”.

We can summarize our discussion as a lemma.
Lemma 4.1. Let $a \in \mathbb{N}^s$, $b \in \mathbb{N}^t$. Within the “coinciding pathways approximation”, $C_E^U(a,b)$ is zero unless $s = t$ and there exist permutations $\pi, \sigma \in S_t$ such that
\begin{equation}
    a_j = b_{\sigma(j)} \quad \text{and} \quad a'_j = b_{\pi(j)}, \quad j = 1 \ldots t.
\end{equation}
Moreover, if two pairs, $(\pi_1, \sigma_1)$ and $(\pi_2, \sigma_2)$, both fulfill [15] and have the same target permutation $\tau = \sigma_1^{-1} \pi_1 = \sigma_2^{-1} \pi_2$, then their contributions to $C_E^U(a,b)$ are identical. In other words,
\begin{equation}
    C_E^U(a,b) = \delta_{t,s} \sum_{\sigma, \pi \in S_t} \Delta_U^{\tau} \prod_{j=1}^{t} \delta(a_j - b_{\sigma(j)}) \delta(a'_j - b_{\pi(j)}),
\end{equation}
where $\Delta_U^{\tau}$ is the total contribution of trajectories $\gamma$ and $\gamma'$ whose ends satisfy [15].

Remark 4.2. The permutations $\sigma$ and $\pi$ have the same role as those appearing on the right-hand side of equation [6].

Proof. What remains to be proved in Lemma 4.1 is that the contributions of $(\pi_1, \sigma_1)$ and $(\pi_2, \sigma_2)$ are identical. We will exhibit a one-to-one correspondence between the set of trajectories with ends satisfying
\begin{equation}
    a_j = b_{\sigma_1(j)} \quad \text{and} \quad a'_j = b_{\pi_1(j)}, \quad j = 1 \ldots n
\end{equation}
and trajectories with ends satisfying
\begin{equation}
    a_j = b_{\sigma_2(j)} \quad \text{and} \quad a'_j = b_{\pi_2(j)}, \quad j = 1 \ldots n.
\end{equation}
To do so we simply relabel the trajectories $\gamma'$, so that the trajectory $\gamma'_j$ becomes the trajectory $\tilde{\gamma}'_{\pi_2 \pi_1^{-1}(j)}$. Then the second identity in (17) becomes
\begin{equation}
    a_j = b_{\pi_2 \pi_1^{-1}(j)} = b_{\pi_2(j)},
\end{equation}
and the first one
\begin{equation}
    a_j = b_{\pi_2 \pi_1^{-1} \sigma_1(j)} = b_{\pi_2 \pi_1^{-1} \sigma_2(j)} = b_{\sigma_2(j)},
\end{equation}
where the equality of the target permutations was used. \hfill \Box

Example 4.3. Consider the correlator
\begin{equation}
    \langle Z_{1,2} Z_{2,1}^* \rangle_E, \quad \text{i.e.} \quad a_1 = 1, \quad a_T = 2, \quad b_1 = 2, \quad b_T = 1.
\end{equation}
For a system with broken time reversal symmetry this correlator is zero. Indeed, there are no permutations $\sigma$ and $\pi$ (on 1 element) that can satisfy equations [15].

Example 4.4. Consider the correlator
\begin{equation}
    \langle Z_{1,2} Z_{3,4} Z_{5,6} Z_{3,6}^* Z_{5,4}^* Z_{1,2}^* \rangle_E.
\end{equation}
The trajectories run
\begin{align*}
    \gamma_1 : 1 \rightarrow 2, \quad & \quad \gamma_2 : 3 \rightarrow 4, \quad \gamma_3 : 5 \rightarrow 6, \\
    \gamma'_1 : 3 \rightarrow 6, \quad & \quad \gamma'_2 : 5 \rightarrow 4, \quad \gamma'_3 : 1 \rightarrow 2,
\end{align*}
and the variables $a$ and $b$ have values
\begin{align*}
    a_1 &= 1, & a_T &= 2, & a_2 &= 3, & a_T &= 4, & a_3 &= 5, & a_T &= 6, \\
    b_1 &= 3, & b_T &= 6, & b_2 &= 5, & b_T &= 4, & b_3 &= 1, & b_T &= 2.
\end{align*}
Two examples of trajectory configurations contributing to the correlator above are given in Fig. 3. Note that the start and end of each trajectory (unfilled circles) are labelled by the index of the corresponding variable $a_j$ or $\bar{a}_j$ and not by its value.

From equation (15) it is clear that the only choice for $\sigma$ and $\pi$ is (in the two-line notation)

$$
\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \quad \pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.
$$

Therefore, the target permutation is $\tau = (1)(2\ 3)$ (in the cycle notation). The target permutation can be read off the diagram (see Fig. 3) by starting at $j$, then following the $\gamma$ trajectory until its end, then following the $\gamma'$ trajectory in reverse to its start. The label there is the image of $j$ under the action of $\tau$.

**Example 4.5.** For the correlator

$$
\langle Z_{1,2}Z_{3,2} Z_{1,2}^*Z_{3,2}^* \rangle_E
$$

there are two choices of the pair $(\sigma, \pi)$. The mapping $\sigma$ between the first indices is the identity permutation $id$, while the mapping between the second indices can either be $id$ or $(1\ 2)$. Therefore the target permutation $\tau$ is

$$
\tau = id \quad \text{or} \quad \tau = (1\ 2).
$$

**Remark 4.6.** Consider the correlator of the type

$$
\langle Z_{i_1,o_1}Z_{i_2,o_2} \cdots Z_{i_n,o_n}Z_{i_2,o_1}^*Z_{i_3,o_2}^* \cdots Z_{i_1,o_n}^* \rangle,
$$
which arise in evaluation of moments \cite{4}. Irrespective of the choices of the values for the indices 
\(i_1, \ldots, o_n\), one can always let 
\[ \sigma = (n \ldots 2 \ldots 1), \quad \pi = id. \]
leading to \(\tau\) being the grand cycle, \(\tau = (1 \ldots n)\). In previous papers, starting with \cite{4}, the
diagrams realizing this choice of \(\tau\) were considered as the base contribution. Correcting factors
were used to take care of other target permutations, arising, for example, when \(i_j = i_k\) for some
\(j\) and \(k\). It was argued that other target permutations corresponded to encounters happening
in a lead (which essentially removes the encounter from the diagram, see the rules of the Essen
ansatz). An encounter in the base diagram that could be moved into the lead can be seen as
“untying” the encounter. We consider this interpretation in detail in the second half of this
work \cite{9} since it provides an easier way to obtain answers for the moment generating functions.

For the proof of our main theorem, however, it is more convenient to have each diagram
representing one target permutation and to sum over the target permutation, as is already done
on the RMT side. The work by Novaes \cite{44, 45} adopts a similar strategy.

4.2. The orthogonal case. In the presence of TRS there are more possibilities to match parts
of the trajectories \(\gamma\) to \(\gamma'\), since now the “head” can be matched with the “tail”.

More precisely, the channel \(a_1\) lies at the start of the trajectory \(\gamma_1\) but must also lie on a
trajectory \(\gamma'_j\). Therefore \(a_1\) must coincide either with \(b_j\) or \(b_{\overline{j}}\). In other words,
\[ a_1 = b_{\varpi(1)}, \]
where \(\varpi(1) \in \{1, \ldots, t, \overline{1}, \ldots, \overline{t}\}\). Similarly for all other channels \(a_j\) there is a matching channel
\(b_{\varpi(j)}\), where \(j\) can be \(1, \ldots, t\) or \(\overline{1}, \ldots, \overline{t}\). This defines the permutation \(\varpi\) on \(2t\) symbols \(Z_t = \{1, \ldots, t, \overline{1}, \ldots, \overline{t}\}\). Thus the only restriction on the indices \(a_j\) and \(b_j\) is that they are equal as
multi-sets (i.e. contain the same elements the same number of times).

Notation 4.7. In the description of orthogonal trajectories, we adopt the convention that \(j\) or
\(k\) refers to the variable label that does not have the bar (correspondingly, \(\overline{j}\) is a label that does
have the bar), while \(z\) denotes a label either with or without the bar.

To understand the analogue of the target permutation in the orthogonal case, we take another
look at the unitary case. The target permutation was \(\tau = \sigma^{-1} \pi\). However, it could be argued
that \(\pi\) and \(\sigma\) act on different spaces: \(\sigma\) acts on the elements \(\{1, \ldots, t\}\) and \(\pi\) acts on \(\overline{1}, \ldots, \overline{t}\). To multiply the permutations we need to map them onto the same space, for example using the
mapping \(T : j \mapsto \overline{j}\). Then the “correct” expression for the target permutation is \(\tau = \sigma^{-1} T^{-1} \pi T\).

This correction, somewhat superfluous in the unitary case, becomes a necessity in the orthog-
nal case. The meaning of the mapping \(T\) is “propagation along the trajectory” \(\gamma\) or \(\gamma'\). We
define it as a permutation on \(Z_t\):
\[ T = (1, \overline{1}) \cdots (t, \overline{t}). \]
In particular, \(T\) is an involution, i.e. \(T^{-1} = T\). Moreover, in the orthogonal case, the per-
mutation \(\varpi\) plays the role of \(\pi\) and \(\sigma\) combined. We thus define the target permutation \(\tau\) by
\[ \tau = \varpi^{-1} T^{-1} \varpi T. \]
It acts on the ends of trajectories \(\gamma\) in the following fashion: take an end, propagate along \(\gamma\) to
the other end, find the corresponding end of a trajectory \(\gamma'\), propagate to its other end. This is
the corresponding end of the next \(\gamma\)-trajectory.
Example 4.8. Consider the correlator \[ \langle Z_{12} Z_{34} Z_{56} Z_{54}^* Z_{31}^* Z_{62}^* \rangle. \] The permutation \( \varpi \) (in two-row notation) is
\[
\varpi = \begin{pmatrix} 1 & 2 & 3 \\
\frac{1}{2} & \frac{2}{3} & \frac{3}{2}
\end{pmatrix}.
\]
Then the target permutation is \( \tau = (13 2)(2 3 1) \). Two examples of trajectory configurations contributing to the correlator above are given in Fig. 4.

Example 4.9. Consider the correlator \[ \langle Z_{12} Z_{31} Z_{23}^* Z_{11}^* \rangle. \] There are two possibilities for the permutation \( \varphi \), namely
\[
\varphi = \begin{pmatrix} 1 & 2 & 3 \\
\frac{1}{2} & \frac{2}{3} & \frac{3}{2}
\end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} 1 & 2 & 3 \\
\frac{1}{2} & \frac{2}{1} & \frac{3}{2}
\end{pmatrix}.
\]
Both of these correspond to the same target permutation \( \tau = (12)(23) \).

Note the “palindromic” symmetry of the results of Examples 4.8 and 4.9. This observation is made precise in the following Lemma.

Lemma 4.10. The orthogonal target permutation has the following properties:

1. if \( \tau(x) = y \) then \( \tau(y) = \overline{x} \),
2. \( \tau(x) \neq \overline{x} \).

Therefore the cycles on \( \tau \) come in symmetric pairs: for every cycle \((z_1 z_2 z_3 \ldots)\) there is the distinct partner cycle \((\ldots \overline{z_3} \overline{z_2} \overline{z_1})\).

Proof. Denote \( \varpi(\overline{x}) =: z \). Then we have \( y = \tau(x) = \varpi^{-1} T^{-1} \varpi(y) = \varpi^{-1}(z) \). Therefore \( \varpi(y) = \overline{z} \).

Now we can calculate \( \tau(\overline{y}) = \varpi^{-1} T^{-1} \varpi(\overline{y}) = \varpi^{-1}(z) = \overline{x} \).

For the second property, we assume the contrary: \( \tau(x) = \overline{x} \). Then \( \varpi^{-1} T^{-1} \varpi(\overline{x}) = \overline{x} \) or the contradiction
\[ \varpi(\overline{x}) = \overline{\varpi(\overline{x})}. \]

The conclusion now follows. Indeed the first property implies that for every cycle \((z_1 z_2 z_3 \ldots)\) there is the cycle \((\ldots \overline{z_3} \overline{z_2} \overline{z_1})\). The fact that these are not the same cycle follows from the second property. \( \square \)
Remark 4.11. Property 1 of Lemma 4.10 can be written as \( \tau(T\tau(x)) = \overline{x} \) or, equivalently, \( T\tau T = \text{id} \). In other words, \( T\tau \) is an involution. Property 2 is equivalent to \( T\tau(x) \neq x \). Therefore, Lemma 4.10 can be reformulated as saying that \( T\tau \) consists only of cycles of length 2.

Remark 4.12. An equivalent way to express the symmetry of \( \tau \) is via the identity \( \tau(x) = \overline{\tau^{-1}(x)} \). This is similar to the definition of the hyperoctahedral group that is the subgroup of \( S_2t \) with elements satisfying \( \chi(x) = \chi(\overline{x}) \). However, our target permutations are different, since they do not form a group.

There is a close connection between the target permutation \( \tau \) and the cycle structure appearing as the true parameters of the class coefficients \( V_N^O \), see (7).

Lemma 4.13. Let the permutation\( \varpi \) be represented as the product \( T'\pi_0\pi_eT'' \), where \( T' \) and \( T'' \) are some involutions that contain only cycles of the form \( (j)(\overline{j}) \) or \( (j\overline{j}) \), \( \pi_0 \) is a permutation on the set \( \{1, \ldots, t\} \) and \( \pi_e \) is a permutation on the set \( \{\overline{1}, \ldots, \overline{t}\} \), see Section 2.

Then the cycle type of \( \tau = \varpi^{-1}T^{-1}\varpi T \) is twice that of the permutation \( T\pi_0^{-1}T\pi_0 \). Consequently, the latter cycle type does not depend on the choice of the representation.

Proof. Substituting the representation \( \varpi = T'\pi_0\pi_eT'' \) into the definition of \( \tau \) and noting that the involutions \( T', T'' \) and \( T \) commute, we get

\[
\tau = T' (T''\pi_0^{-1}\pi_e^{-1}T') T (T'\pi_0\pi_eT'') = T' (T\pi_e^{-1}T') (T\pi_0^{-1}T) \pi_0\pi_e T'' \sim (T\pi_e^{-1}T\pi_0) (T\pi_0^{-1}T\pi_e),
\]

where \( \sim \) denotes conjugate permutations. Obviously, the two parts of the last expression act on disjoint sets \( \{1, \ldots, t\} \) and \( \{\overline{1}, \ldots, \overline{t}\} \) and have the same cycle type, which explains the doubling.

A result analogous to Lemma 4.1 summarizes the discussion of this section.

Lemma 4.14. Let \( a \in \mathbb{N}^s \), \( b \in \mathbb{N}^t \). Within the “coinciding pathways approximation”, \( C^O_E(a, b) \) is zero unless \( s = t \) and there exists a permutation \( \varpi \) on \( Z_t = \{1, \ldots, t, \overline{1}, \ldots, \overline{t}\} \) such that

\[
a_z = b_{\varpi(z)}, \quad z \in Z_t.
\]

Moreover, if two permutations, \( \varpi_1 \) and \( \varpi_2 \), both fulfill (20) and have the same target permutation \( \tau = \varpi_1^{-1}T^{-1}\varpi_1T = \varpi_2^{-1}T^{-1}\varpi_2T \), then their contributions to \( C^O_E(a, b) \) are identical. In other words,

\[
C^O_E(a, b) = \delta_{t,s} \sum_{\varpi \in S_{2t}} \Delta^O(\tau) \prod_{z \in Z_t} \delta(a_z - b_{\varpi(z)}),
\]

where \( \Delta^O(\tau) \) is the total contribution of trajectories \( \gamma \) and \( \gamma' \) whose ends satisfy (20).

Proof. As before, the only part we need to prove is that the contributions of trajectories satisfying (20) with \( \varpi_1 \) and \( \varpi_2 \) are identical if the corresponding target permutations coincide. Starting with a set of trajectories described by \( \varpi_1 \) we will relabel them so that they will be described by \( \varpi_2 \). We will only relabel the partner trajectories \( \gamma' \). Note that direction-reversal is allowed.

The relabelling permutation we will denote by \( \rho \). It acts on the ends of the trajectories, namely if the original trajectory \( \gamma'_j \) run from \( b_j \) to \( b_{\overline{j}} \), the new one \( \tilde{\gamma}'_j \) runs from \( \tilde{b}_j = b_{\rho(j)} \) to \( \tilde{b}_{\overline{j}} = b_{\rho(\overline{j})} \).
We want to have \( a_z = \tilde{b}_{\wp_2(z)} \). On one hand we have \( a_z = b_{\wp_1(z)} \) (since the original trajectories agreed with \( \wp_1 \)). On the other, we have \( b_{\wp_2(z)} = b_{\rho(\wp_2(z))} \). Therefore, \( \rho \) needs to satisfy \( \rho \wp_2 = \wp_1 \), or

\[
\rho = \wp_1 \wp_2^{-1}.
\]

We now verify that \( \rho \) is a valid relabelling. Namely, if it maps the trajectory end \( j \) to the trajectory end \( k \) (or \( \bar{k} \)), then it must map the other end \( \bar{j} \) to \( \bar{k} \) (correspondingly, \( k \)). Putting it formally, \( \rho \) must satisfy \( \rho T = T \rho \). By the definition of the target permutation (and since \( T \) is an involution), we have \( \wp_2^{-1} T = \tau T \wp_2^{-1} \). Therefore,

\[
\rho T = \wp_1 \wp_2^{-1} T = \wp_1 \tau T \wp_2^{-1} = \wp_1 (\wp_1^{-1} T \wp_1) \wp_2^{-1} = T \wp_1 \wp_2^{-1} = T \rho,
\]

as desired. \( \square \)

5. Summation over the unitary diagrams

Diagrams are schematic depictions of sets of trajectories, describing which parts of trajectories \( \{\gamma_j\} \) follow which parts of trajectories \( \{\gamma'_j\} \) by using a graph whose edges correspond to stretches of \( \gamma \) and vertices correspond to encounters (see section 3).

In this section we describe how the diagrams can be encoded mathematically, operations acting within the set of diagrams and the cancellations resulting when we evaluate the contributions \( \Delta U \) as a sum over all possible diagrams. We will consider here the case of broken TRS ("unitary" diagrams) before treating TRS in section 6.

5.1. Diagrams as ribbon graphs. The drawings in Fig. 3 hint that the natural mathematical description of the topology of a set of contributing trajectories is a ribbon graph, also known as a fat graph or a map. Edges of such graphs are fattened to become strips, with two sides. At vertices, which are also fattened, the side of one edge is connected to a side of another, which prescribes a cyclic order on the stubs of the edges around the vertex. Following the sides until we return to the starting spot we trace out boundary walks of the map.

In semiclassical diagrams, each trajectory stretch corresponds to an edge of the graph. The trajectory \( \gamma \) runs along one side of the edge and \( \gamma' \) runs (in the same direction) along the other side. The vertices of degree 1 correspond to trajectories either starting from or exiting into a channel. They are labelled by the symbols 1, \ldots, \( t \) and \( \bar{1}, \ldots, \bar{t} \), which are understood to correspond to the indices of the variables \( a \) and \( b \) in the correlator \( C_E(a, b) \), equation (11). All other vertices correspond to encounters which we will refer to as internal vertices. At a vertex the trajectory can be followed by continuing along the side of the edge onto the side of the vertex and then on to the corresponding side of the next edge.

A simple ribbon graph is shown in Fig. 5. As in Figs. 3 and 5 we will continue to mark \( \gamma \) trajectories by solid lines and \( \gamma' \) trajectories by dashed lines.

We formalize the above paragraph as a definition.

Definition 5.1. The unitary diagram with the target permutation \( \tau \) is a map satisfying the following:

1. There are \( t \) vertices of degree 1 (henceforth leaves) labelled with symbols 1, \ldots, \( t \) and \( t \) leaves labelled with symbols \( \bar{1}, \ldots, \bar{t} \).
2. All other vertices have even degree greater than 2.

\[\text{Also called "germs" in the literature.}\]
Figure 5. A ribbon graph representation of the pair of the trajectories depicted in Fig. 2. To read off the trajectories, we start at the open end labelled 1 and follow the left side for $\gamma$ or the right side for $\gamma'$. The leaves (vertices of degree 1) of the graph are shown as empty circles; the internal vertex of degree 6 is represented by the filled ellipse.

(3) A portion of the boundary running from one leaf to the next is called a boundary segment. Each leaf $z$ is incident to two boundary segments, one of which is a segment running between leaves $j$ and $\overline{j}$ and the other between leaves $\tau(k)$ to $\overline{k}$ (where either $z = j = \tau(k)$ or $z = \overline{j} = \overline{k}$). The segments are given direction $j \rightarrow \overline{j}$ and $\tau(k) \rightarrow \overline{k}$ and marked by solid and dashed lines correspondingly. The following conditions are satisfied:

(a) each part of the boundary is marked exactly once,
(b) each edge is marked solid on one side and dashed on the other, both running in the same direction.

We will now discuss some properties of the diagrams that follow from the basic definition above.

The boundary segments marked solid correspond to the $\gamma$-trajectories and dashed correspond to $\gamma'$-trajectories. The boundary walks of the map can be read by alternatingly following $\gamma$ and $\gamma'$ (in reverse) trajectories. Then each edge is traversed in opposite directions on the two sides, which means the graph is orientable [60, Chap. X]. The labels of the leaves are arranged in a special way: a trajectory $\gamma$ starts at a vertex $j$ and ends at $\overline{j}$. The trajectory $\gamma'$ that immediately follows it, starts (when read in reverse) at $\overline{j}$ and ends at $\tau(j)$, where $\tau$ is the target permutation of the diagram in question, see the discussion of Section 4.1. To summarize,

**Lemma 5.2.** A unitary diagram satisfies the following properties:

1. The map is orientable.
2. Each boundary walk passes through a non-zero even number of leaves. Their labels form the sequence of the form
   
   $j, \overline{j}, \tau(j), \overline{\tau(j)}, \tau^2(j), \ldots, \overline{\tau^f(j)},$
   
   where $\tau$ is the target permutation of the diagram and $\tau^{f+1}(j) = j$. This establishes a one-to-one correspondence between the boundary walks and the cycles of $\tau$.

5.2. Operations on diagrams. We now define some operation on diagrams which will later allow us to find cancellations of their contributions evaluated according to Definition 3.1.

The first operation is tying together two leaves. Given two leaves $j$ and $k$ (with no bars) we form a new vertex of degree 4 as shown in Fig. 6. Note that the trajectory $\gamma$ connected to the
vertex $j$ is still connected to $j$ after tying, whereas the $\gamma'$ trajectory becomes connected to $k$. Tying two leaves together creates an internal vertex and two edges.

The reversal of this operation is untying. It can only be performed if there are two leaves (without bars) attached on the opposite sides of the vertex of degree 4 that is to be untied.

Lemma 5.3. Consider a diagram with the target permutation $\tau$. If we tie the leaves $j$ and $k$ together, the target permutation of the modified diagram is $(jk)\tau$. Untying a 4-vertex with leaves $j$ and $k$ directly attached to its opposite sides also results in the target permutation $(jk)\tau$.

Proof. The target permutation is read off a diagram by following a $\gamma$-trajectory (solid, black lines in the figures) from a leaf $i$ to the next leaf (which must be $j$) and then following the $\gamma'$-trajectory (dashed, red lines) in reverse until the next leaf which is the leaf $\tau(i)$. When the leaves $j$ and $k$ are tied together, the $\gamma$-trajectories are not changed, but the $\gamma'$-trajectories that originated from $j$ and $k$ are switched around. Thus the target permutation of the modified diagram acts as the permutation $\tau$ followed by the interchange of $j$ and $k$. Hence the multiplication by $(jk)$ on the left.

Untying the vertex with leaves $j$ and $k$ is thus equivalent to multiplying by the inverse of $(jk)$, which is the same as multiplying by $(jk)$ itself. \hfill \Box

This operation of tying can be generalized to the case of two leaves with bars and to the case of more than two leaves (of the same kind – either all with or all without bars). The reversal, untying, will only work on a vertex of degree $2m$ if there are $m$ leaves of the same kind attached to it. In terms of the target permutations, the tying of several leaves without bars is equivalent to multiplying by the cycle of length $m$. The tying of leaves with bars is multiplication by the cycle on the right and untying is multiplication by the inverse cycle on the appropriate side. The precise order of the elements in the cycle has to agree with the spatial arrangement of the leaves around the cycle. However, we will not need these operations in our proofs and thus omit the precise details here. We will need these generalisations however to obtain moment generating functions as in the accompanying paper [9], so we provide full details there.

The second operation we will need is contracting an edge. While any edge connecting two (internal) vertices in a diagram can be contracted to produce another diagram, we will only apply this operation in the precise circumstances described below.

In order to contract an edge going between vertices $x$ and $y$, see Fig. 7 we require that the vertex $x$ has degree 4 and that the edge attached to $x$ opposite the edge $(x, y)$ is coming directly from the leaf number 1. The vertex $y$ has no restrictions, in particular it can be of any even degree.
The operation of contracting an edge leads to the diagram with a larger vertex, shown on the right of Fig. 7. We can reverse the operation if we have a vertex of an even degree larger than 4 and an edge connecting the vertex to leaf 1. As the figure suggests, the new vertex inherits the edge to 1 and the two neighbouring edges. This operation is called splitting the vertex. The following lemma is obvious.

**Lemma 5.4.** The operations of contracting an edge or splitting the vertex do not change the target permutation.

In some cases considered later the diagram will have no leaf with number 1. In such a case the leaf with the smallest number will play the role of leaf 1.

**5.3. Cancellations among unitary diagrams.** The result of Lemma 4.1 states that the semiclassical correlator $C^U_E(a, b)$ can be expressed as

$$C^U_E(a, b) = \delta_{t,s} \sum_{\sigma, \pi \in S_t} \Delta^U(\tau) \prod_{j=1}^{l} \delta(a_j - b_{\sigma(j)}) \delta(a_j - b_{\pi(j)}) ,$$

where $\Delta^U(\tau)$ is the sum of contributions of diagrams with the target permutation $\tau = \sigma^{-1}\pi$. The contribution of every diagram is evaluated according to the rules in Definition 3.1. Namely, each internal vertex of a diagram gives a factor of $(-N)$ and each edge (including the ones leading to the leaves) a factor of $1/N$. Denoting by $D^U_{v,e}(\tau)$ the number of diagrams with the target permutation $\tau$, $v$ internal vertices and $e$ edges, we have the following statement, which finished the proof of Theorem 1.1 in the case of broken time-reversal symmetry.

**Theorem 5.5.** The total contribution of the unitary diagrams with target permutation $\tau$ is

$$\Delta^U(\tau) := \sum_{v,e} D^U_{v,e}(\tau) \frac{(-1)^v}{N^{e-v}} = V^U_N(\tau).$$

**Proof.** The proof of the theorem has two parts. In the first part we exhibit cancellations among the diagrams. In the second part, the diagrams that survive cancellations will be shown to satisfy the same recursion as $V^U_N(\tau)$ with the same initial conditions.

The cancellation is based on the operations described in Section 5.2. We define an involution $P$ on the set of diagrams which changes $v$ by one but leaves $(e - v)$ invariant. This means that the two diagrams that are images of each other under $P$ produce contributions to the sum in
that are equal in magnitude but opposite in sign. The sum therefore reduces to the sum over the fixed points of $P$.

The involution $P$ is described by the following algorithm:

1. Find the leaf with minimal number $j$ (without bar).
2. If it is attached to a vertex of degree 4 whose opposite edge ends in a leaf, untie the node and return to step 1. If the untying produces an edge which directly connects two leaves ($k$ and $\bar{k}$), this edge is removed from consideration.
3. If the leaf $j$ is attached to a node of degree 4 whose opposite edge leads to another internal vertex, contract the edge.
4. Otherwise, the leaf $j$ is attached to a vertex of degree 6 or higher. We split this vertex into two by inserting a new edge.
5. Reverse all operations performed in step 2.

The algorithm is illustrated by an example in Figs. 8 and 9. The original diagram is Fig. 8(a) and the result is in Fig. 9(g). However, if we start with Fig. 9(g) and apply the algorithm, we would use step 3 instead of step 4 and arrive at Fig. 8(a).

The algorithm preserves the target permutation (see Lemma 5.4), therefore both diagrams contribute to the same sum. Note that at most one of the operations in steps 3 and 4 are performed. These operations are the inverses of each other and provide the required difference in the number $v$ while leaving $(e - v)$ invariant. If the steps 3 or 4 are never reached, by virtue of the diagram untying completely, the corresponding contribution does not cancel and has to be counted.

Consider a diagram that unties completely. Recording the untyings, as suggested by Lemma 5.3, we get

$$ (s_v r_v) \cdots (s_2 r_2) (s_1 r_1) \tau = id, \quad s_j < r_j, $$

where the identity permutation on the right corresponds to the resulting “empty” diagram. The condition in step 1 of the algorithm ensures that if $j < k$ then $s_j \leq s_k$. Given the sequence of untyings, we can reconstruct the diagram uniquely, therefore the diagrams that survive the cancellations are in one-to-one correspondence with the factorizations of $\tau$ into a product of
transpositions satisfying
\[ \tau = (s_1 r_1) (s_2 r_2) \cdots (s_v r_v), \quad s_j < r_j, \quad s_j \leq s_{j+1}. \]
These are known as primitive factorizations\(^4\) that are counted by monotone single Hurwitz numbers \([21, 33]\). It is known \([33]\) that the number of such factorizations provides the coefficients in the asymptotic expansion of the class coefficients \(V_{\mathcal{U}}^N(\tau)\). Here we provide a basic alternative proof of this fact that will be easy to generalize to the orthogonal case.

Denote by \(p_v^{U}(\tau)\) the number of primitive factorizations of the target permutation \(\tau\) into \(v\) transpositions. We have shown that
\[ \Delta^{U}(\tau) = \sum_v p_v^{U}(\tau) \frac{(-1)^v}{N^{e-v}}, \]
where the number of edges \(e\) can be related to the other quantities as follows. First, the permutation \(\tau\) is a permutation on \(t\) elements. The completely untied graph with \(2t\) leaves contains \(t\) edges. Each tying increases the number of edges by 2 and the number of internal vertices by one. Since each of the \(v\) transpositions corresponds to a tying, we end up with \(e = t + 2v\) edges.

The quantity \(p_v^{U}(\tau)\) depends only on the cycle structure of \(\tau\) so let \(c_1, \ldots, c_k\) be the lengths of the cycles. Without loss of generality, we take \(\tau = (12 \ldots c_1) \cdots (t-c_k+1 \ldots t)\). Consider the term \((s_1 r_1)\) on the left of a primitive factorization of \(\tau\). Without this term, it is also a factorization, but of the permutation \((s_1 r_1)\tau\) and into \(v-1\) factors.

For \(s_1 = 1\), we will now investigate the cycle structure of the permutation \((1 r_1)\tau\). If \(r_1\) belongs to the first cycle of \(\tau\) (i.e. the one that contains 1) it splits into two, of lengths \(q\) and \(r\) with \(q + r = c_1\). Otherwise, if \(r_1\) belongs to cycle number \(j > 1\), the first cycle joins with it to form a cycle of length \(c_1 + c_j\).

Finally, it can happen that \(s_1 \neq 1\), but only if the element 1 does not appear in any transposition of the factorization (equivalently, the leaf 1 is attached directly to the leaf \(1\)). This can happen only if 1 is in cycle of its own within the permutation \(\tau\), that is if \(c_1 = 1\). In this case,

\[^4\]The traditional definitions are slightly different from ours in ordering the second elements of the transpositions.
the given primitive factorization is also a factorization of the permutation on \( t - 1 \) element with cycle lengths \( c_2, \ldots, c_k \). Altogether we have
\[
(26) \quad p_v^U(c_1, \ldots, c_k) = \delta_{c_1,1}p_v^U(c_2, \ldots, c_k) + \sum_{q+r=c_1} p_{v-1}^U(q, r, c_2, \ldots, c_k) + \sum_{j=2}^k c_jp_{v-1}^U(c_1 + c_j, \ldots, \hat{c}_j, \ldots).
\]
The notation \( \hat{c}_j \) again means that \( c_j \) is removed from the lengths of the cycles. We would like to substitute this recursion into (25) and we need to calculate the power of \( N - 1 \) at which the terms of (26) would enter the sum in \( \Delta^U \). On the left, we have \( e - v = (t + 2v) - v = t + v \). The first term on the right corresponds to permutations on \( t - 1 \) elements, hence the power is \( t + v - 1 \). The other two terms on the right have the same \( t \), but a reduced number of internal vertices (factors in the factorization), also resulting in \( t + v - 1 \). Altogether, we get
\[
(27) \quad N\Delta^U(c_1, \ldots, c_k) = \delta_{c_1,1}\Delta^U(c_2, \ldots, c_k) - \sum_{q+r=c_1} \Delta^U(q, r, c_2, \ldots, c_k) - \sum_{j=2}^k c_j\Delta^U(c_1 + c_j, \ldots, \hat{c}_j, \ldots),
\]
which exactly mirrors the recursion relations for \( V_N^U \) given in (8). \( \square \)

6. Summation over the orthogonal diagrams

Building on the results for systems with broken TRS, we now turn to the case of TRS. We first develop the mathematical description of the “orthogonal” diagrams before summing their contributions.

6.1. Orthogonal diagrams as ribbon graphs. As before, in the description of orthogonal diagrams, variables \( j \) or \( k \) refer to the leaf label that does not have the bar (correspondingly, \( \bar{j} \) is a label that does have the bar), while \( z \) denotes a label either with or without the bar.

The conditions that make a valid orthogonal diagram are almost identical to the unitary case. The only significant difference is that trajectories \( \gamma \) and \( \gamma' \) do not have to run in the same direction.

Definition 6.1. The orthogonal diagram with the target permutation \( \tau \) is a locally orientable map satisfying the following:

1. There are \( t \) leaves labelled with symbols \( 1, \ldots, t \) and \( t \) leaves labelled with symbols \( \overline{1}, \ldots, \overline{t} \).
2. All other vertices have even degree greater than 2.
3. Each leaf is incident to two boundary segments, one of which runs between labels \( z_1 \) and \( \overline{z_1} \) and is marked solid, and the other runs between labels \( \tau(z_2) \) and \( \overline{z_2} \) and is marked dashed. Each edge is marked solid on one side and dashed on the other.

Since \( \tau \) does not preserve the two “halves” of the set \( Z_t = \{1, \ldots, t, \overline{1}, \ldots, \overline{t}\} \), there can be boundary segments running between \( j \) and \( k \) or between \( \overline{j} \) and \( \overline{k} \) (see Fig. 4 for some examples). Thus there is no natural way to assign direction to the boundary segments. However, when we consider moment generating functions \([9]\) we will again have a natural direction and for this reason we retain the directional arrows in the figures of orthogonal diagrams.

The consequence of dropping the direction requirement in Definition 6.1 is that we now need both orientable and (globally) non-orientable diagrams. Therefore, when drawn on a plane, some edges of the map might have twists in them: going around a graph on a closed walk can...
Figure 10. An example of tying two leaves in an orthogonal diagram. Part (a) shows the original diagram; leaves 1 and 2 are to be tied. In part (b) one of the edges received a twist to align properly. The result is shown in part (c). According to Lemma 6.2, the target is transformed as $(1 \ 2 \ 3)(3 \ 2 \ 1)(2 \ 1) = (1 \ 2 \ 3)(3 \ 2 \ 1)$.

bring you back on the reverse side of the edge, see Figure 4(b) for an example. The last property of Lemma 5.2 still applies to orthogonal diagrams. In fact, the face labels fit the pattern

$\tau(z)$, $\tau(\tau(z))$, $\tau^2(z)$, $\ldots$, $\tau^f(z)$,

independently of the direction chosen for the boundary. This is because $\tau(\tau(z)) = z$ due to Lemma 4.10.

Given an orthogonal diagram, we can read off the target permutation in the following fashion. To determine $\tau(z)$, we find the leaf labelled $z$ and go from it along the boundary segment marked solid, until we arrive at the next leaf (which must be marked $z$, according to Def. 5.1). From there we follow the dashed boundary segment. The leaf we arrive at is the leaf $\tau(z)$. Note that the same prescription applies to unitary diagrams as well, if we let $z$ be the labels $\{1, \ldots, t\}$ only.

6.2. Operations on diagrams. The two types of operations for unitary diagrams described in Section 5.2 can be defined for orthogonal diagrams in a similar fashion.

In fact, the operation of contracting an edge is the exactly same. Namely, if we have a vertex of degree 4 with the leaf labelled 1 adjacent to it, we can contract the edge that is opposite the leaf 1.

The generalization of the tying/untying operation is a little more exciting. We can tie any two leaves together (that is with or without bars). Before tying, the edges need to be arranged as in Fig. 6; this might require adding a twist to one of them, see Fig. 10.

Lemma 6.2. Consider a diagram with the target permutation $\tau$. If we tie the leaves $z_1$ and $z_2$ together, the target permutation of the modified diagram is $(z_1 \ z_2) \tau(\overline{z_1} \overline{z_2})$. Untying a 4-vertex with leaves $z_1$ and $z_2$ directly attached to its opposite sides also results in the target permutation $(z_1 \ z_2) \tau(\overline{z_1} \overline{z_2})$.

Proof. According to the rules of reading off the target permutation, the result of $\tau$ is the endpoint of a dashed boundary segment. After tying the leaves, the dashed segment that was finishing at
z_1 now finishes at z_2 (see Figure 6) and vice versa. This means that z_1 and z_2 must be switched after the original \( \tau \) is applied.

Similarly, the image of \( \overline{\tau} \) is computed by following the solid segment to \( z_1 \) and then the dashed segment to the leaf that is the result \( \tau(\overline{\tau}) \). After the tying operation, the dashed segment coming out of \( z_1 \) is actually the segment that was previously coming out of \( z_2 \) and therefore leading to \( \tau(\overline{\tau}) \). To account for this change, we need to switch \( \overline{\tau} \) and \( \overline{\tau} \) before we apply \( \tau \).

The untying operation is equivalent to multiplying by the inverses of the transpositions, which are the transpositions themselves.

**Remark 6.3.** The operation \((z_1 z_2) \tau(\overline{z_1 \overline{z_2}}) =: \bar{\tau}\) preserves the properties described in Lemma 4.10. Indeed, according to Remark 4.11, we need to show that \( T \bar{\tau} \) consists only of cycles of length 2. Denoting \((\overline{z_1 \overline{z_2}}) =: q\), we have \((z_1 z_2) = TqT \) and therefore \( T \bar{\tau} = TTqT \tau q = qT \tau q \sim T \tau \). Since the operation of conjugation does not affect the cycle type, \( T \bar{\tau} \) has the same cycle lengths as \( T \tau \).

### 6.3. Cancellations among the orthogonal diagrams

Similar to the unitary case, comparing Lemma 4.14 with the RMT result (7), we see that we need to compare the coefficients \( \Delta^O(\tau) \) and \( V_N^O(\bar{\tau}) \) (where \( \tau = \bar{\tau}^{-1}T \bar{\tau}T \)). Moreover, according to Lemma 4.13, both coefficients in fact depend on the set of numbers \( c_1, \ldots, c_k \), which are the lengths of half the cycles in the cycle representation of \( \tau \) (each cycle has its "mirror" image in the representation, see Lemma 4.10) only one length per pair appears in the list).

The coefficient \( \Delta^O(\tau) \) is the sum of contributions of orthogonal diagrams with the target permutation \( \tau \). The contribution of every diagram is evaluated according to the rules in Definition 6.1. Denoting by \( D^O_{v,e}(\tau) \) the number of orthogonal diagrams with the target permutation \( \tau \), \( v \) internal vertices and \( e \) edges, we have the following statement.

**Theorem 6.4.** The total contribution of the orthogonal diagrams with target permutation \( \tau \) is

\[
\Delta^O(\tau) := \sum_{v,e} D^O_{v,e}(\tau) \frac{(-1)^v}{N^{e-v}} = V_N^O(\tau).
\]

**Proof.** The cancellation algorithm is completely analogous to the unitary case. Namely, we fix a linear ordering of the set \( Z_t \) and untie, while it is possible, the vertex adjacent to the leaf with the minimal label of the leaves still present in the diagram. We remove any edges that directly connect two leaves from the consideration.

If at any point it becomes impossible to untie the vertex adjacent to the minimal leaf, it is either because the vertex is of a degree higher than 4 or because the edge opposite the minimal leaf is not going to a leaf. Then we correspondingly split the vertex or contract the edge. After this we re-tie previously untied vertices. The new diagram has the contribution that cancels the contribution of the original diagram.

The only diagrams that survive this process are those that untie to an empty diagram. Recording every step according to Lemma 6.2, we get

\[
(s_v r_v) \cdots (s_2 r_2) (s_1 r_1) \tau(\overline{s_1 \overline{r_1}}) (\overline{s_2 \overline{r_2}}) \cdots (\overline{s_v \overline{r_v}}) = id,
\]

where \( s_j < r_j \) and \( s_j \leq s_{j+1} \). Note that the choice of the node to untie is unique at each step, so the surviving diagrams are in one-to-one correspondence with the “palindromic” primitive factorizations

\[
\tau = (s_1 r_1) (s_2 r_2) \cdots (s_v r_v) (\overline{s_v \overline{r_v}}) \cdots (\overline{s_2 \overline{r_2}}) (\overline{s_1 \overline{r_1}}), \quad s_j < r_j, \quad s_j \leq s_{j+1}.
\]
We now need to understand the number $p^O_v(\tau) = p^O_v(c_1, \ldots, c_k)$ of such factorizations which provide us with the semiclassical contribution.

\[(31) \Delta^O(\tau) := \sum_v p^O_v(\tau) \frac{(-1)^v}{N^{e-v}}.\]

To do this, we derive a recursion similar to (26). The number of factorizations of $\tau$ with $2v$ factors is equal to the number of factorization of $(s_1 r_1) \tau (s_1^T r_1^T)$ with $2(v-1)$ factors, summed over all possible choices of $(s_1 r_1)$. As before, we treat the case $s_1 \neq \min(Z_t)$ separately ($s_1$ and $s_1^T$ must then be cycles of their own).

Without loss of generality we assume that $s_1$ appears in cycle number 1. There are $2c_j$ possibilities for $r_1$ to appear in the cycle number $j > 1$ or its “mirror” image, cycle number $2k - j + 1$. This cycle joins the first cycle in the result of the multiplication $(s_1 r_1) \tau (s_1^T r_1^T)$. If $r_1$ belongs to the first cycle, it splits into two parts. Finally, if $r_1$ belongs to the mirror image of the first cycle, the product is $(s_1 r_1) (s_1 a \ldots b \tau c \ldots d)(\bar{a} \ldots \bar{c} r_1 \bar{b} \ldots \bar{d} \tau^T)(\bar{c} \ldots \bar{d} r_1 \bar{b} \ldots \bar{a} s_1^T)(s_1 \tau^T s_1^T), \]

therefore the cycle lengths do not change. In the latter case, $r_1$ has $c_1$ possibilities, including $s_1^T$.

The cases above give rise to the following terms,

\[(32) p^O_v(c_1, \ldots, c_k) = \delta_{c_1,1} p^O_v(c_2, \ldots, c_k) + \sum_j 2c_j p^O_{v-1}(c_1 + c_j, \ldots, \hat{c}_j, \ldots) + \sum_{q+r=c_1} p^O_{v-1}(q, r, c_2, \ldots, c_k) + c_1 p^O_{v-1}(c_1, \ldots, c_k).\]

Taking the generating function with respect to $1/N$ according to the middle expression in (28), we recover recursion (10), which completes the proof.

7. Conclusions

While the RMT approach described in Sec. 2 in terms of the recursive class coefficients is computationally inefficient, it turned out to be useful for establishing the equivalence between semiclassics and RMT. The equivalence is established at the level of moments of scattering matrix elements which immediately implies equivalence of moments (both linear and non-linear) of matrix subblocks. The result is proved for moments of COE and CUE (corresp. with and without time-reversal symmetry), but, since there is a simple formula connecting COE and CSE moments, the equivalence extends immediately to CSE moments (corresp. systems with spin-orbit interactions) as well.

Furthermore, since all the moments agree, indirectly we obtain the joint probability density of the transmission eigenvalues of $t^T t$ semiclassically. In fact, current RMT approaches start from this joint probability density (which follows the Jacobi ensemble [2, 20]) and obtain linear and non-linear transport moments through a variety of different methods [24, 30, 38, 40, 43, 47, 48, 53, 54, 61].

Once the semiclassical diagrams have been expressed in terms of ribbon graphs, the most important step is showing that the contributions of the vast majority of diagrams cancel. We identified pairs of diagrams whose contributions cancel exactly, leaving only the diagrams that correspond to primitive factorisations. These were then shown to match the RMT class coefficients. While possibly the simplest cancellation algorithm, it is certainly not unique. In fact, in an earlier version of our proof we considered a cancellation algorithm which reduced the set
of diagrams (with broken TRS) to those that can be put into correspondence with inequiva-
lent factorisations [5] (those are the fully untieable diagrams, when more general untyings are
allowed).

The cancellation we used relies strongly on the product of the semiclassical edge and vertex
contribution from Definition 3.1 being exactly $-1$. This is the case for the correlators of the
subblocks of the scattering matrix we considered here, but no longer holds for energy-dependent
correlators, superconducting or tunneling leads, Wigner delay times and other physically relevant
questions. In these cases semiclassical results are known for low moments or up to a given
order in inverse channel number [6, 7, 25, 26] and agree with perturbative RMT expansions [3, 14, 15, 36, 37, 39, 57] but a general proof of the equivalence is lacking. For the Wigner delay
times, the transport matrix follows an inverse Wishart distribution [15] and correlators of such
matrices have been expressed [31] in a form similar to the correlators in Sec. 2. A strategy
similar to the one used here might then be fruitful. Starting from the joint probability density
of the delay times, the linear moments, as well as the moments of the mean time delay, are also
available from RMT [30, 38, 40].

Another case where the edge and vertex contributions do not allow direct cancellation is
when we consider Ehrenfest time effects. Below the Ehrenfest time the quantum and classical
propagation are fairly similar while for longer times wave interference dominates. This wave
interference is incorporated in the semiclassical approximation on top of the underlying classical
motion which can then be used to obtain the typical dependence of quantum transport on the
Ehrenfest time. In particular results are known for low orders in a perturbative expansion in the
inverse channel number for low moments [16, 17, 23, 49, 62, 63, 66, 67] and for all moments at
leading order [64]. Although this question is outside the range of applicability of RMT, attempts
have been made to phenomenologically treat Ehrenfest time effects using “effective” RMT. This
approach provides the correct results for some quantities but importantly not for all (notably the
weak localisation correction to the conductance) and its validity must then be checked against
semiclassical approaches. It is here, where RMT answers are not available in principle, that our
classification of the semiclassical diagrams (continued in [9]) can become very useful.

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