Renormalization and energy conservation for axisymmetric fluid flows

Camilla Nobili\textsuperscript{1} and Christian Seis\textsuperscript{2}

\textsuperscript{1} Fachbereich Mathematik, Universität Hamburg, Germany. E-mail: camilla.nobili@uni-hamburg.de
\textsuperscript{2} Institut für Analysis und Numerik, Westfälische Wilhelms-Universität Münster, Germany. E-mail: seis@wwu.de

Abstract: We study vanishing viscosity solutions to the axisymmetric Euler equations with (relative) vorticity in $L^p$ with $p > 1$. We show that these solutions satisfy the corresponding vorticity equations in the sense of renormalized solutions. Moreover, we show that the kinetic energy is preserved provided that $p > 3/2$ and the vorticity is nonnegative and has finite second moments.

Contents

1 Introduction \hspace{10cm} 1
2 Renormalized solutions for linear transport equations \hspace{10cm} 7
3 Estimates on the velocity field \hspace{10cm} 10
4 Global estimates for the axisymmetric Navier–Stokes equations \hspace{10cm} 12
5 Vanishing viscosity limit. Proof of Theorem 1 \hspace{10cm} 18
6 Renormalization. Proof of Theorem 2 \hspace{10cm} 20
7 Energy conservation. Proof of Theorem 3 \hspace{10cm} 22

1 Introduction

For axisymmetric incompressible flows without swirl, the (originally three-dimensional) Navier–Stokes and Euler equations can be reduced to two-dimensional mathematical models which are obtained by assuming a cylindrical symmetry for both
the physical space variables and the velocity components. Despite this simplification, such flows are still able to describe interesting physical phenomena like the motion and interaction of toroidal vortex rings. On the mathematical level, even though two-dimensional, the (vaguely defined) degree of difficulty of analyzing solution properties lies somewhere between that of the two-dimensional planar equations and the full three-dimensional model. Indeed, as we shall see later on, axisymmetric flows\(^1\) do still feature vortex stretching and some of the standard global estimates have an unambiguous three-dimensional character. On the other hand, many of the features of the Biot–Savart kernel are typically two-dimensional even though some helpful symmetry properties are lost.

In the present work, our aim is to study renormalization and energy conservation of solutions to the Euler equations that are obtained as vanishing viscosity solutions from the axisymmetric Navier–Stokes equations. Here, renormalization is to be understood in the sense of DiPerna and Lions \[24\], that is, a solution is called renormalized if the chain rule of differentiation applies in a suitable way. We are particularly interested into solutions whose vorticity is merely \(L^p\) integrable in a sense that will be made precise later.

The analogous (though in some parts technically much simpler) studies for the two-dimensional planar equations have been conducted quite recently: As long as the vorticity is \(L^p\) integrable with exponent \(p \geq 2\), DiPerna and Lions’s theory for transport equations (combined with Calderón–Zygmund theory) ensures that the vorticity is a renormalized solution of the corresponding vorticity equation \[37\]. This fact is true regardless of the construction of the solution. If \(p \in (1, 2)\), renormalization properties are proved in \[18\] for vanishing viscosity solutions. The argument in this work relies on a duality argument and exploits the DiPerna–Lions theory. This theory, however, does not apply to the \(p = 1\) case, in which the associated velocity gradient is a singular integral of an \(L^1\) function. Instead, a stability-based theory for continuity equations proposed in \[40, 41\] can be suitably generalized in order to handle this situation and to extend the results from \[37, 18\] to the limiting case \(p = 1\); see \[17\].

Conservation of kinetic energy for vanishing viscosity solutions with \(L^p\) vorticity, \(p > 1\), is established in \[14\] for the planar two-dimensional setting (on the torus). The corresponding three-dimensional problem gained much attention in recent years, particularly in connection with Onsager’s conjecture \[38\], which states that the threshold Hölder regularity for the validity of energy conservation is the exponent \(1/3\). Energy conservation for larger Hölder exponents was proved in \[16\], see also \[25\] for partial results and \[13\] for improvements, while the sharpness of this exponent was proved in \[30\], building up on the theory developed in \[20, 21, 8, 7, 9\]. We note that Hölder-\(1/3\) regularity is guaranteed for any vorticity field in \(L^p\) with \(p \geq \frac{9}{2}\) thanks to three-dimensional Sobolev embeddings.

Before discussing our precise findings and the relevant earlier results for the axisymmetric equations, we shall introduce the mathematical model. The Euler

\(^1\)From here on we shall omit the specification without swirl for convenience.
equations for an ideal fluid in $\mathbb{R}^3$ are given by the system
\begin{align}
\partial_t u + u \cdot \nabla_x u + \nabla_x p &= 0, \\
\nabla_x \cdot u &= 0,
\end{align}
where $u = u(t,x) \in \mathbb{R}^3$ is the fluid velocity and $p = p(t,x) \in \mathbb{R}$ is the pressure. In this formulation, the (constant) fluid density is set to 1. Whenever the fluid has locally finite kinetic energy, which will be the case in the regularity framework considered in this paper, the Euler equations can be interpreted in the sense of distributions.

**Definition 1.** Let $T > 0$ and $u_0 \in L^2_{\text{loc}}(\mathbb{R}^3)^3$ be given. A vector field $u \in L^2_{\text{loc}}((0,T) \times \mathbb{R}^3)^3$ is called a distributional solution to the Euler equations \([1], (2)\) if
\begin{align}
\int_0^T \int_{\mathbb{R}^3} (\partial_t F \cdot u + \nabla_x F \cdot u \otimes u) \ dx \ dt + \int_{\mathbb{R}^3} F(t = 0) \cdot u_0 \ dx &= 0
\end{align}
for any divergence-free vector field $F \in C^\infty_c((0,T) \times \mathbb{R}^3)^3$ and
\begin{align}
\int_0^T \int_{\mathbb{R}^3} \nabla_x f \cdot u \ dx \ dt &= 0
\end{align}
for any $f \in C^\infty_c([0,T) \times \mathbb{R}^3)$.

We restrict ourself to the case of axisymmetric solutions without swirl. That is, if $(r, \theta, z)$ are the cylindrical coordinates of a point $x \in \mathbb{R}^3$, i.e., $x = (r \cos \theta, r \sin \theta, z)^T$, we shall assume that
\begin{align}
u = u(t, r, z), \quad \text{and} \quad u = u^r e_r + u^z e_z,
\end{align}
where $e_r$ and $e_z$ are the unit vectors in radial and vertical directions, which form together with the angular unit vector $e_\theta$ a basis of $\mathbb{R}^3$,
\begin{align}
e_r = (\cos \theta, \sin \theta, 0)^T, \quad e_\theta = (-\sin \theta, \cos \theta, 0)^T, \quad e_z = (0, 0, 1)^T.
\end{align}
We remark that $u^\theta = u \cdot e_\theta$ is the swirl direction of the flow, that we assume to vanish identically. Under these hypotheses on the velocity field, the vorticity vector is unidirectional, $\nabla_x \times u = (\partial_z u^r - \partial_r u^z) e_\theta$, and we write $\omega = \partial_z u^r - \partial_r u^z$. A direct computation reveals that this quantity, that we will call vorticity from here on, satisfies the continuity equation
\begin{align}
\partial_t \omega + \partial_r (u^r \omega) + \partial_z (u^z \omega) = 0
\end{align}
on the half-space $\mathbb{H} = \{(r, z) \in \mathbb{R}^2 : r > 0\}$. We remark that $\omega$ is thus a conserved quantity, because the no-penetration boundary condition $u^r = 0$ on $\partial \mathbb{H}$ comes along with the symmetry assumptions. However, opposed to the situation for the two-dimensional planar Euler equations, the vorticity is not transported by the flow, as the divergence-free condition \([2]\) becomes
\begin{align}
r^{-1} \partial_r (ru^r) + \partial_z u^z = 0
\end{align}
in cylindrical coordinates. Indeed, the continuity equation can be rewritten as a damped transport equation,

\[ \partial_t \omega + u^r \partial_r \omega + u^z \partial_z \omega = \frac{1}{r} u^r \omega, \]

where the damping term on the right-hand side describes the phenomenon of vortex stretching, \( \frac{1}{r} u^r \omega e^\theta = (\nabla x \times u) \cdot \nabla x u. \) What is transported instead is the relative vorticity \( \xi = \omega/r, \)

\[ \partial_t \xi + u^r \partial_r \xi + u^z \partial_z \xi = 0. \] (5)

We remark that the flow is entirely determined by the (relative) vorticity, as the associated velocity field can be reconstructed with the help of the Biot–Savart law in \( \mathbb{R}^3, \)

\[ u(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \times e_\theta(y) \omega(t, y) \, dy. \] (6)

A transformation into cylindrical coordinates and an analysis of the axisymmetric Biot–Savart law can be found, for instance, in [28].

Thanks to this relation, we may thus study (5), (6) instead of (1), (2). Working with the vorticity formulation has certain advantages: At least on a formal level, it is readily seen that the vorticity equation (5) preserves any \( L^p \) norm,

\[ \| \xi(t) \|_{L^p(\mathbb{R}^3)} = \| \xi_0 \|_{L^p(\mathbb{R}^3)} \quad \forall t \geq 0, \] (7)

if \( \xi_0 \) is the initial relative vorticity. This observation is crucial, for instance, in order to prove uniqueness in the case of bounded vorticity fields [19]. The drawback of working with (5) is that there is no direct way of giving a meaning to the transport term in low integrability settings (opposed to the momentum equation (1)). For instance, it is not obvious to us, how to extend common symmetrization techniques that allow for an alternative formulation of the transport nonlinearity in the planar two-dimensional setting, see, e.g., [22], [46], [6].

Whenever the product \( u \xi \) is locally integrable, we can interpret the transport equation (5) in the sense of distributions.

**Definition 2.** Let \( T > 0 \) and \( p, q \in (1, \infty) \) be given with \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( \xi_0 \in L^p_{\text{loc}}(\mathbb{H}) \) and \( u \in L^1((0, T); L^q_{\text{loc}}(\mathbb{H})) \) be such that \( r^{-1} \partial_r(r u^r) + \partial_z u^z = 0 \). Then \( \xi \in L^\infty((0, T); L^p_{\text{loc}}(\mathbb{H})) \) is called a distributional solution to the transport equation (5) with initial datum \( \xi_0 \) if

\[ \int_0^T \int_{\mathbb{H}} \xi (\partial_t f + u^r \partial_r f + u^z \partial_z f) \, r \, d(r, z) \, dt + \int_{\mathbb{H}} \xi_0 f(t = 0) r \, d(r, z) = 0 \]

for any \( f \in C^\infty_c((0, T) \times \mathbb{H}). \)

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2 We caution the reader that throughout the manuscript, we carefully distinguish between the Lebesgue spaces on the full three-dimensional space, \( L^p(\mathbb{R}^3) \), and those on the two-dimensional half-space \( L^p(\mathbb{H}) \). Notice also that the three-dimensional Lebesgue measure reduces to the weighted measure \( 2\pi r \, d(r, z) \) on \( \mathbb{H} \) when restricted to axisymmetric configurations as in (7). In particular, \( \| \xi \|_{L^1(\mathbb{R}^3)} = 2\pi \| \omega \|_{L^1(\mathbb{H})}. \)
Notice that the definition provides a distributional formulation of the continuity equation (3) in which \( \omega \) is replaced by \( r\xi \).

Simple scaling arguments show that the local integrability of the product \( u\xi \) can be expected to hold true only if \( p \geq \frac{4}{3} \). For this insight, it is crucial to observe that the Sobolev inequality

\[
\|u\|_{L^{\frac{2p}{2-p}}(H)} \lesssim \|\omega\|_{L^{p}(H)} \quad (8)
\]
is valid as in the planar two-dimensional setting, cf. [28, Proposition 2.3]. For vorticity fields with smaller integrability exponents, we propose the notion of renormalized solutions.

**Definition 3.** Let \( T > 0 \) be given. Let \( \xi_0 \in L^1_{\text{loc}}(H) \) and \( u \in L^1((0,T);L^1_{\text{loc}}(H)^3) \) be such that \( r^{-1}\partial_r(ru') + \partial_z u^z = 0 \). Then \( \xi \in L^\infty((0,T);L^1_{\text{loc}}(H)) \) is called a renormalized solution to the transport equation (5) with initial datum \( \xi_0 \) if

\[
\int_0^T \int_H \beta(\xi) (\partial_t f + u'\partial_r f + u^z\partial_z f) \, r \, d(r,z) \, dt + \int_H \beta(\xi_0)f(t=0) \, r \, d(r,z) = 0
\]
for any \( f \in C^\infty_c([0,T] \times H) \) and any bounded \( \beta \in C^1(R) \) vanishing near zero.

We remark that the notion of renormalized solutions implies the conservation of the \( L^p \) integral of vorticity in the sense of (7) via a standard approximation argument. Moreover, it is shown in [24, 3] that renormalized solutions are transported by the Lagrangian flow of the vector field \( u \) as in the smooth situation. We will further comment on this in Section 2 below. The relation between Lagrangian transport and the partial differential equations (3) and (5) was thoroughly reviewed in [4].

In the present paper, we study solutions to the vorticity equation (5) in the case where the initial (relative) vorticity is unbounded, more precisely,

\[
\xi_0 \in L^1 \cap L^p(R^3) \quad (9)
\]
for some \( p \in (1, \infty) \). We are thus outside of the class of functions in which uniqueness is known to hold [19, 2]. On the positive side, existence of distributional solutions to the Euler equations (1), (2) was proved in [31] for initial vorticities satisfying (9) and under the additional assumption that the initial kinetic energy is finite, \( u_0 \in L^2(R^3)^3 \). (Notice that local \( L^2 \) bounds on the initial velocity can be deduced from the integrability assumptions on the vorticity via Sobolev embeddings, cf. (8).) For larger integrability exponents and (near) vortex sheet initial data, (crucial insights on) existence results were previously obtained in [15, 23, 39, 12, 11, 33, 32].

To the best of our knowledge, renormalized solutions (Definition 3) have not been considered in the context of the axisymmetric Euler equations.

We are particularly interested in solutions that are obtained as the vanishing viscosity limit from the Navier–Stokes equations, which are, in fact, physically meaningful approximations to the Euler equation. Hence, for any viscosity constant \( \nu > 0 \), we consider solutions \((u_\nu, p_\nu)\) to the Navier–Stokes equations

\[
\partial_t u_\nu + u_\nu \cdot \nabla_x u_\nu + \nabla_x p_\nu = \nu \Delta_x u_\nu, \quad (10)
\]

\[
\nabla_x \cdot u_\nu = 0. \quad (11)
\]
We furthermore impose fixed initial conditions, \( u_\nu(0) = u_0 \) and shall assume that \( u_\nu \) is axisymmetric, that is, \( u_\nu = u_\nu(t, r, z) \) and \( u_\nu = (u_\nu)^r e_r + (u_\nu)^z e_z \).

Instead of working with the momentum equation (10), we will mostly study its vorticity formulation, which is a viscous version of (3) (or (5)), see (18) (or (19)) below. It was shown in [28] that under the assumption (9) on the initial data, which implies that \( \omega_0 \in L^1(H) \), there exists a unique global (mild) solution \( \omega_\nu \in C_0([0, \infty) \times H) \) to the viscous vorticity equation.

Starting from this solution to the Navier–Stokes equations, our first result addresses compactness and convergence to the Euler equations.

**Theorem 1** (Compactness and convergence to Euler). Let \( u_\nu \) be the unique solution to the Navier–Stokes equations (10), (11) with initial datum \( u_0 \in L^2_{\text{loc}}(\mathbb{R}^3) \) such that the associated relative vorticity \( \xi_0 \) belongs to \( L^1 \cap L^p(\mathbb{R}^3) \) for some \( p > 1 \). Then there exist \( u \in C([0, T]; L^2_{\text{loc}}(\mathbb{R}^3)) \) with \( \nabla u \in L^\infty((0, T); L^p_{\text{loc}}(\mathbb{R}^3)^{3 \times 3}) \) and \( \xi \in L^\infty((0, T); L^1 \cap L^p(\mathbb{R}^3)) \) and a subsequence \( \{\nu_k\}_{k=0}^\infty \) such that

\[
\nu_k \rightarrow u \quad \text{strongly in } C([0, T]; L^2_{\text{loc}}(\mathbb{R}^3))
\]

and

\[
\xi_{\nu_k} \rightarrow \xi \quad \text{weakly-* in } L^\infty((0, T); L^p(\mathbb{R}^3)).
\]

Moreover, \( u \) is a distributional solution to the Euler equations (1), (2) and \( \omega = r\xi \) is the corresponding vorticity that is (in a distributional sense) related to \( u \) by the Biot–Savart law (6).

The vanishing viscosity limit was studied for finite energy solutions with mollified initial datum satisfying the bound (9) in [31]. The novelty in the above result is the kinetic energy may be unbounded. For earlier and related convergence results for non-classical solutions, we refer to [45, 33, 29, 47, 1] and references therein.

Our next statement concerns the renormalization property of the relative vorticity.

**Theorem 2** (Renormalization). Let \( u \) and \( \xi \) be the velocity field and relative vorticity, respectively, from Theorem 1. Then \( \xi \) is a renormalized solution to the transport equation (5) with velocity \( u \). In particular, it holds that

\[
\|\xi(t)\|_{L^p(\mathbb{R}^3)} = \|\xi_0\|_{L^p(\mathbb{R}^3)}
\]

and \( \xi \) is transported by the regular Lagrangian flow of \( u \) in \( \mathbb{R}^3 \).

To the best of our knowledge, in this result, renormalized solutions to the axisymmetric Euler equations are considered for the first time. We recall from the above discussion that for \( p \in (1, 4/3) \), the interpretation of the transport equation (5) as a distributional solution does not apply anymore as the transport nonlinearity is no longer integrable. In particular, while for \( p \geq 4/3 \) our result implies that distributional and renormalized solutions coincide, in the low integrability range, we show the existence of renormalized solutions. We also recall that for \( p \geq 2 \), the result in Theorem 2 is already covered in DiPerna and Lions’s original paper [24].
In section 2, we recall the theory from [24] and explain what we mean by \( \xi \) being transported by a flow. For a precise definition of regular Lagrangian flows, we refer to [3, 4].

Our final result addresses the conservation of the kinetic energy.

**Theorem 3.** Let \( p \geq \frac{3}{2} \). Suppose that the fluid has finite kinetic energy, \( u_0 \in L^2(\mathbb{R}^3)^3 \), and that \( \omega_0 \) is nonnegative and has finite impulse,

\[
\int_{\mathbb{H}} \omega_0 r^2 d(r, z) < \infty.
\]

Then the kinetic energy is preserved,

\[
\|u(t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)}.
\]

In order to show conservation of energy, the growth of vorticity at infinity has to be suitably controlled. Here, we choose a growth condition that is natural as it can be interpreted as the control of the fluid impulse. Notice that the latter is conserved by the evolution, cf. Lemma 9. This is in principle not required by our method of proving Theorem 3 and any estimate of the form \( \|r^2 \omega(t)\|_{L^1(\mathbb{H})} \lesssim \|r^2 \omega_0\|_{L^1(\mathbb{H})} \) would be sufficient. It is, however, not clear to us whether such an estimate holds true under our integrability assumptions apart from the special case considered in Lemma 6 that is, for nonnegative (or nonpositive) vorticity fields. Also, if higher order moments could be controlled, our method shows that the value of \( p \) could be lowered (at least up to \( p > \frac{6}{5} \)). See, for instance, [12] for similar results in the setting with \( p > 3 \) (and general solutions). We leave this issue for future research and consider the simplest case here.

From the result in Theorem 3, it follows that were are outside of the range in which Kolmogorov’s celebrated K41 theory of three-dimensional turbulence applies, since, similar to the case of planar two-dimensional turbulence, there cannot be anomalous diffusion.

From here on, we will simplify the notation by writing \( \nabla = (\frac{\partial}{\partial r}, \frac{\partial}{\partial z}) \), with the interpretation that \( \nabla \cdot f = \partial_r f^r + \partial_z f^z \) while \( \nabla_x \cdot f = \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 \) is the divergence with respect to a Cartesian basis. The advective derivatives \( f \cdot \nabla \) and \( f \cdot \nabla_x \) are to be interpreted correspondingly.

The remainder of the article is organized as follows: In Section 2 we recall the parts of the DiPerna–Lions theory for transport equations and explain how the results apply to the setting under consideration. In Section 3 we provide estimates for the velocity field that are essentially based on the Biot–Savart law. Section 4 contains global estimates for the axisymmetric Navier–Stokes equations, while the proof of Theorems 1, 2 and 3 are given in Sections 5, 6 and 7, respectively. This work, finally, contains an appendix in which a helpful interpolation estimate is provided.

## 2 Renormalized solutions for linear transport equations

In this section, we shall briefly recall DiPerna and Lions’s theory for linear transport equations [24] and explain how it applies to the situation at hand. We are particularly interested into well-posedness and renormalization properties of the transport
equation \([5]\), which we shall now treat as a (linear) passive scalar equation
\[
\partial_t \theta + u \cdot \nabla \theta = 0
\]
for some scalar quantity \(\theta\) and a velocity field \(u\) that does not depend on \(\theta\). Yet, we have in mind that \(u\) has its origin in the fluid dynamics problem that is considered in the main part of this paper. We shall thus continue assuming that the flow is incompressible in the sense of \([4]\), and that \(u' = 0\) on \(\partial \mathbb{H}\), which is ensured in the nonlinear setting by the Biot–Savart law \([6]\), see also the discussion in \([28]\). Working in cylindrical coordinates becomes at this point problematic as the cylindrical divergence of the velocity field \(u\) might in general be unbounded opposed to the Cartesian divergence, which vanishes identically. In order to apply the DiPerna–Lions theory, in which that boundedness is a crucial assumption, it is therefore advantageous to go back to the Cartesian formulation and rewrite \((12)\) as
\[
\partial_t \theta + u \cdot \nabla_x \theta = 0.
\]
If, in addition, \(u\) is Sobolev regular, as is the case for the axisymmetric Euler equations under the integrability assumption \([7]\) on the vorticity, the theory in \([24]\) applies. We summarize some of the main results, again formulated for the axisymmetric setting, and not aiming for the most general assumptions.

**Theorem 4** \([24]\). Let \(T > 0\) and \(p \in (1, \infty)\) be given and \(\theta_0 \in L^p(\mathbb{R}^3)\) and \(u \in L^1((0, T); W^{1,1}_{\text{loc}}(\mathbb{H}))\) be such that \(r^{-1} \partial_r (ru^r) + \partial_z u^z = 0\) and
\[
\frac{|u|}{1 + r + |z|} \in L^1((0, T) \times \mathbb{R}^3) + L^\infty((0, T) \times \mathbb{R}^3).
\]
(i) There exists a unique renormalized solution \(\theta \in L^\infty((0, T); L^p(\mathbb{R}^3))\) of the transport equation \((12)\) with initial datum \(\theta_0\).
(ii) This solution is stable under approximation in the following sense: Let \(\{\theta_0^k\}_{k \in \mathbb{N}}\) be a sequence that approximates \(\theta_0\) in \(L^p(\mathbb{R}^3)\) and \(\{u^k\}_{k \in \mathbb{N}}\) a sequence that approximates \(u\) in \(L^1((0, T); W^{1,1}_{\text{loc}}(\mathbb{H}))\) and such that \(r^{-1} \partial_r (ru^r_k) + \partial_z u^z_k = 0\). Let \(\theta^k\) denote the corresponding renormalized solution. Then \(\theta^k \to \theta\) strongly in \(C([0, T]; L^p(\mathbb{R}^3))\).
(iii) If \(q \in (1, \infty)\) is such that \(\frac{1}{p} + \frac{1}{q} \leq 1\) and \(u \in L^1((0, T); W^{1,q}_{\text{loc}}(\mathbb{H}))\), then distributional solutions are renormalized solutions and vice versa.

It has been proved in \([24, 3]\) that renormalized solutions are in fact transported by the (regular) Lagrangian flow of the vector field \(u\), and this feature carries over to the cylindrical setting. Hence, it holds that \(\theta(t, \phi(t, r, z)) = \theta_0(r, z)\), where \(\phi\) satisfies a suitably generalized formulation of the ordinary differential equation
\[
\partial_t \phi(t, r, z) = u(t, \phi(t, r, z)), \quad \phi(0, r, z) = (r, z).
\]
In terms of the vorticity, the transport identity can be rewritten as \(\omega(t, \phi(t, r, z)) = \omega_0(r, z) \phi^r(t, r, z)/r\), and thus, \(r/\phi^r(t, r, z)\) is the Jacobian. See also \([4]\) for a review of the connection between the Lagrangian and Eulerian descriptions of transport by non-smooth velocity fields.

It follows a discussion of the validity of the growth condition \([13]\) in the Euler setting.
Remark 1. In the nonlinear setting in which \( u \) can be reconstructed from \( \theta = \xi \) with the help of the Biot–Savart kernel \((6)\), the growth condition \((13)\) is automatically fulfilled provided that \( \xi \in L^\infty((0,T);L^1(\mathbb{R}^3)) \) as it is assumed in this paper. Indeed, it is proved in [28] that the axisymmetric Biot–Savart kernel satisfies similar decay estimates as the planar two-dimensional one, namely, if \( G \) is obtained from restricting the three-dimensional Biot–Savart kernel to the axisymmetric setting, so that
\[
\quad u(r, z) = \int_{\mathbb{H}} G(r, z, \bar{r}, \bar{z}) \omega(\bar{r}, \bar{z}) \, d(\bar{r}, \bar{z}),
\]
it holds that
\[
|G(r, z, \bar{r}, \bar{z})| \lesssim \frac{1}{|r - \bar{r}| + |z - \bar{z}|},
\]
 cf. [28, Eq. (2.11)]. We now denote by \( G_1 \) the restriction of \( G \) to the unit ball \( B_1(r, z) \) and set \( G_2 = G - G_1 \), and decompose \( u = u_1 + u_2 \) accordingly. Then, on the one hand, by Young’s convolution inequality,
\[
\quad \|u_1\|_{L^1(\mathbb{H})} \lesssim \|(\chi_{B_1(0)} \frac{1}{|\cdot|}) * |\omega||_{L^1(\mathbb{H})} \leq \|\chi_{B_1(0)} \frac{1}{|\cdot|}\|_{L^1(\mathbb{H})} \|\omega||_{L^1(\mathbb{H})} \lesssim \|\omega||_{L^1(\mathbb{H})}.
\]
On the other hand,
\[
\quad \|u_2\|_{L^\infty(\mathbb{R}^3)} \lesssim \|(\chi_{B_1(0)} \frac{1}{|\cdot|}) * |\omega||_{L^1(\mathbb{H})} \leq \|\omega||_{L^1(\mathbb{H})}.
\]
Using that \( \|\omega||_{L^1(\mathbb{H})} = \frac{1}{2\pi} \|\xi||_{L^1(\mathbb{R}^3)} \), we deduce \((13)\).

Following [18, 17], our strategy for proving that vanishing viscosity solutions to the axisymmetric Euler equations are renormalized solutions relies on duality arguments both in the viscous and in the inviscid setting. In the latter, we quote a suitable duality theorem from DiPerna and Lions’s original work.

Lemma 1 ([24]). Let \( p, q \in (1, \infty) \) be given such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( u \) satisfy the general assumptions of Theorem 4 and let \( \theta \in L^\infty((0,T);L^p(\mathbb{R}^3)) \) be the renormalized solution to the transport equation \((12)\) with initial datum \( \theta_0 \in L^p(\mathbb{R}^3) \). Let \( \chi \in L^1((0,T);L^q(\mathbb{R}^3)) \) be given and let \( f \in L^\infty((0,T);L^q(\mathbb{R}^3)) \) be a renormalized solution of the backwards transport equation
\[
\quad - \partial_t f - u \cdot \nabla f = \chi.
\]
Then it holds
\[
\quad \int_0^T \int_{\mathbb{H}} \theta \chi r \, d(r, z) \, dt
\]
\[
\quad = \int_{\mathbb{H}} \theta(T, r, z) f(T, r, z) \, r \, d(r, z) - \int_{\mathbb{H}} \theta(0, r, z) f(0, r, z) \, r \, d(r, z).
\]
3 Estimates on the velocity field

In this section, we provide some estimates on the velocity field that turn out to be helpful in the subsequent analysis. We continue denoting by $\omega$ and $\xi$ the vorticity and relative vorticity, respectively, of a given (steady) axisymmetric velocity field $u$, that is, $\omega = \partial_z u^r - \partial_r u^z$ and $\xi = \omega / r$ independently from the Euler or Navier–Stokes background. In particular, any of the following estimates are consequences of the explicit definitions or follow from suitable properties of the Biot–Savart kernel.

Our first result is a fairly standard identity for the enstrophy, that is, the (square of the) $L^2$ norm of the velocity gradient.

**Lemma 2.** It holds that

$$\|\nabla_x u\|_{L^2(\mathbb{R}^3)} = \|\omega\|_{L^2(\mathbb{R}^3)}.$$  

We provide the argument for this standard identity for the convenience of the reader.

**Proof.** From the definition of the vorticity, we infer that

$$\frac{1}{2\pi} \|\omega\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{H}} (\partial_z u^r - \partial_r u^z)^2 r \, d(r, z)$$

$$= \int_{\mathbb{H}} (\partial_z u^r)^2 r \, d(r, z) + \int_{\mathbb{H}} (\partial_r u^z)^2 r \, d(r, z) - 2 \int_{\mathbb{H}} \partial_z u^r \partial_r u^z \, r \, d(r, z).$$

We have to identify the third term on the right-hand side: It holds that

$$-2 \int_{\mathbb{H}} \partial_z u^r \partial_r u^z \, r \, d(r, z) = \int_{\mathbb{H}} (\partial_r u^r)^2 r \, d(r, z) + \int_{\mathbb{H}} (u^r)^2 r \, d(r, z) + \int_{\mathbb{H}} (\partial_z u^z)^2 r \, d(r, z).$$

Indeed, using the no-penetration boundary condition $u^r = 0$ on $\partial \mathbb{H}$ together with the incompressibility condition [4], a multiple integration by parts reveals on the one hand that

$$\int_{\mathbb{H}} \partial_z u^r \partial_r u^z \, r \, d(r, z) = - \int_{\mathbb{H}} u^r \partial_z \partial_r u^z \, r \, d(r, z)$$

$$= - \int_{\mathbb{H}} u^r \partial_r (-\partial_z u^r - \frac{1}{r} u^r) \, r \, d(r, z)$$

$$= - \int_{\mathbb{H}} (\partial_r u^r)^2 r \, d(r, z) - \int_{\mathbb{H}} (u^r)^2 r \, d(r, z).$$

On the other hand, it holds that

$$\int_{\mathbb{H}} \partial_z u^r \partial_r u^z \, r \, d(r, z) = - \int_{\mathbb{H}} \partial_r (\partial_z u^r) u^z \, d(r, z)$$

$$= - \int_{\mathbb{H}} \partial_r \partial_z u^r u^z \, r \, d(r, z) - \int_{\mathbb{H}} \partial_z u^r u^z \, d(r, z)$$

$$= - \int_{\mathbb{H}} \partial_z (-\partial_z u^z - \frac{1}{r} u^r) u^z \, r \, d(r, z) - \int_{\mathbb{H}} \partial_z u^r u^z \, d(r, z)$$

$$= - \int_{\mathbb{H}} (\partial_z u^z)^2 r \, d(r, z).$$
It remains to notice that
\[ |\nabla u|^2 = (\partial_r u^r)^2 + \frac{1}{r^2} (u^\theta)^2 + (\partial_z u_z)^2 + (\partial_r u_z)^2 + (\partial_z u^r)^2 \quad (15) \]
to conclude the statement of the lemma. 

In the following lemma, we provide a maximal regularity estimate for the velocity gradient in terms of the relative vorticity. Our proof relies on the classical theories by Calderón, Zygmund and Muckenhoupt.

**Lemma 3.** For \( p \in (1, 2) \) it holds that
\[
\| \frac{1}{r} \nabla_x u \|_{L^p(\mathbb{R}^3)} \lesssim \| \xi \|_{L^p(\mathbb{R}^3)}. \quad (16)
\]

**Proof.** We note that in view of the Biot–Savart law (6), the velocity gradient can be represented as a singular integral of convolution type,
\[
\nabla_x u = K * (\omega e_\theta),
\]
where \( |K(x)| \sim \frac{1}{|x|^3} \). It is well-known that Calderón–Zygmund theory guarantees that
\[
\| \nabla_x u \|_{L^p(\mathbb{R}^3)} \lesssim \| \omega \|_{L^p(\mathbb{R}^3)}
\]
for any \( p \in (1, \infty) \). Our goal is to produce a weighted version of this estimate, namely
\[
\int_{\mathbb{R}^3} |\nabla_x u|^p m \, dx \lesssim \int_{\mathbb{R}^3} |\omega|^p m \, dx
\]
with \( m = m(r) = \frac{1}{r^p} \) and \( r = \sqrt{x_1^2 + x_2^2} \), which is nothing but (16). We are thus led to the theory of Muckenhoupt weights: If \( p \in (1, \infty) \) and \( m \) is in the class of Muckenhoupt weights \( A_p \) then the weighted-maximal regularity estimate (16) holds. Here, \( A_p \) is the set of nonnegative locally integrable weight functions satisfying
\[
\left( \int_B m(x) \, dx \right) \left( \int_B m(x)^{-\frac{q}{p}} \, dx \right)^{\frac{p}{q}} \leq C \quad (17)
\]
for a universal constant \( C > 0 \) and all balls \( B \) in \( \mathbb{R}^3 \), and \( q \in (1, \infty) \) with \( \frac{1}{p} + \frac{1}{q} = 1 \). This well known result can be found, for instance, in the book of Stein [44].

We thus have to show that \( m = m(r) = r^{-p} \) satisfies (17) for \( p \in (1, 2) \). For this, consider a ball in \( \mathbb{R}^3 \) with radius \( R \) and centered in a generic point \( X = (X_1, X_2, X_3) \in \mathbb{R}^3 \), i.e., \( B = B_R(X) \). We denote by \( d \) the distance of \( X \) to the \( z \)-axis, that is, \( d = \sqrt{X_1^2 + X_2^2} \). We split our argumentation into the two cases when \( d \geq 2R \) (far field) and \( d < 2R \) (near field).

Let us first consider the case where \( d \geq 2R \). Notice that we have \( d - R \leq \sqrt{x_1^2 + x_2^2} \leq d + R \) for any \( x \in B \) by the triangle inequality, and thus
\[
\frac{1}{(x_1^2 + x_2^2)^{\frac{p}{2}}} \leq \frac{1}{(d - R)^p} \quad \text{and} \quad (x_1^2 + x_2^2)^{\frac{p}{2}} \leq (d + R)^q.
\]
For \( m(x) = (x_1^2 + x_2^2)^{-\frac{q}{p}} \), we now compute
\[
\int_B m(x) \, dx \leq \frac{1}{(d - R)^p},
\]
and
\[
\left( \int_B m(x)^{-\frac{q}{p}} \, dx \right)^\frac{p}{q} \leq (d + R)^p.
\]

Making use of the fact that \( \frac{d + R}{d - R} \leq 3 \) for all \( d \geq 2R \), we deduce that
\[
\left( \int_B m(x) \, dx \right) \left( \int_B m(x)^{-\frac{q}{p}} \, dx \right)^\frac{p}{q} \leq \left( \frac{d + R}{d - R} \right)^p \leq 3^p.
\]

We now turn to the case where \( d < 2R \). We first observe that \( \sqrt{x_1^2 + x_2^2} < d + R \) and \( |x_3 - X_3| < R \) for all \( x \in B \), and we may thus bound the integral over the ball by an integral over the cylinder. Making relative transformations in cylindrical coordinates, we then have the estimates
\[
\int_B m(x) \, dx \lesssim \frac{1}{R^2} \int_0^{d+R} \frac{1}{r^{p-1}} \, dr \lesssim \frac{(d + R)^{-p}}{R^2},
\]
provided that \( p < 2 \), and
\[
\left( \int_B m(x)^{-\frac{q}{p}} \, dx \right)^\frac{p}{q} \lesssim \left( \frac{1}{R^2} \int_0^{d+R} r^{q+1} \, dr \right)^\frac{p}{q} \lesssim \left( \frac{(d + R)^{q+2}}{R^2} \right)^\frac{p}{q}.
\]

Taking the product and using that \( \frac{d + R}{R} \leq 3 \) for all \( d < 2R \), we conclude that
\[
\left( \int_B m(x) \, dx \right) \left( \int_B m(x)^{-\frac{q}{p}} \, dx \right)^\frac{p}{q} \lesssim \left( \frac{d + R}{R} \right)^{2p} \leq 3^{2p}.
\]

Hence, in either cases, we proved (17) and, thus, the proof is over. \( \blacksquare \)

4 Global estimates for the axisymmetric Navier–Stokes equations

In this section, we provide some global estimates for solutions to the Navier–Stokes equations that we turn out to be helpful later on. We start by rewriting the momentum equation (10) in terms of the vorticity \( \omega_\nu = \partial_z u_\nu^r - \partial_r u_\nu^z \) and the relative vorticity \( \xi_\nu = \omega_\nu / r \). The evolution equation for the vorticity is given by
\[
\partial_t \omega_\nu + \nabla \cdot (u_\nu \omega_\nu) = \nu \left( \Delta \omega_\nu + \frac{1}{r} \partial_r \omega_\nu - \frac{1}{r^2} \omega_\nu \right),
\]
and is equipped with homogeneous Dirichlet conditions on the boundary of the half-space, i.e. \( \omega_\nu = 0 \) on \( \partial \mathbb{H} \). It follows that the vorticity equation is conservative, as expected, because \( r^{-1} \partial_r \omega_\nu - r^{-2} \omega_\nu = \partial_r (r^{-1} \omega_\nu) \). The relative vorticity satisfies the nonconservative equation
\[
\partial_t \xi_\nu + u_\nu \cdot \nabla \xi_\nu = \nu \left( \Delta \xi_\nu + \frac{3}{r} \partial_r \xi_\nu \right)
\]
which is supplemented with homogeneous Neumann boundary conditions, \( \partial_r \xi_{\nu} = 0 \) on \( \partial \mathbb{H} \). We will mostly work with the latter equation. For initial data \( \xi_{\nu}(0) = \xi_0 \) in \( L^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \), cf. \cite{9}, well-posedness for either formulation can be inferred from the theory developed by Gallay and Šverák \cite{28}. In the following, \( \omega_{\nu} \) will always be the unique mild solution to the vorticity equation \( \text{(18)} \) in the class \( C([0, T); L^1(\mathbb{H})) \cap C((0, T); L^\infty(\mathbb{H})) \) and \( \xi_{\nu} = \omega_{\nu}/r \). We start by recalling some useful properties which can be found in various references. Yet, we provide their short proofs for the convenience of the reader. Our first concern is an \( L^p \) estimate.

**Lemma 4.** It holds that

\[
\| \xi_{\nu} \|_{L^\infty((0,T);L^p(\mathbb{R}^3))} \leq \| \xi_0 \|_{L^p(\mathbb{R}^3)}. \tag{20}
\]

**Proof.** We can perform a quite formal computation as solutions can be assumed to be smooth by standard approximation procedures. A direct calculation yields

\[
\frac{d}{dt} \frac{1}{p} \int_\mathbb{H} |\xi_{\nu}|^p r \, d(r,z) = \nu \int_\mathbb{H} |\xi_{\nu}|^{p-2} \xi_{\nu} \Delta \xi_{\nu} \, r \, d(r,z) + 3 \nu \int_\mathbb{H} |\xi_{\nu}|^{p-2} \xi_{\nu} \partial_r \xi_{\nu} \, d(r,z),
\]

where we made use of the no-penetration boundary conditions on the velocity field \( u \) to eliminate the advection term. The Cartesian Laplacian \( \Delta_x = \Delta + \frac{1}{r} \partial_r \) is coercive, because

\[
\int_\mathbb{H} |\xi_{\nu}|^{p-2} \xi_{\nu} \left( \Delta \xi_{\nu} + \frac{1}{r} \xi_{\nu} \right) \, r \, d(r,z) = - (p - 1) \int_\mathbb{H} |\xi_{\nu}|^{p-2} |\nabla \xi_{\nu}|^2 \, r \, d(r,z) \leq 0
\]

as can be seen by an integration by parts. Another integration by parts reveals that the first order term is nonpositive and can thus be dropped,

\[
\int_\mathbb{H} |\xi_{\nu}|^{p-2} \xi_{\nu} \partial_r \xi_{\nu} \, d(r,z) = \frac{1}{p} \int_\mathbb{H} \partial_r |\xi_{\nu}|^p \, d(r,z) = - \frac{1}{p} \int_{\partial \mathbb{H}} |\xi_{\nu}|^p \, d(r,z) \leq 0.
\]

A combination of the previous estimates yields

\[
\frac{d}{dt} \frac{1}{p} \int_\mathbb{H} |\xi_{\nu}|^p \, r \, d(r,z) + \nu (p - 1) \int_\mathbb{H} |\xi_{\nu}|^{p-2} |\nabla \xi_{\nu}|^2 \, r \, d(r,z) \leq 0, \tag{21}
\]

and an integration in time yields the desired estimate \( \text{(20)}. \)  

Our next estimate quantifies integrability improving features of the advection-diffusion equation \( \text{(19)} \) by suitably extending the estimates on the \( L^p \) norm established in the previous lemma to any \( q \in [p, \infty) \).

**Lemma 5.** For any \( q \in [p, \infty], \) it holds that

\[
\| \xi_{\nu}(t) \|_{L^q(\mathbb{R}^3)} \lesssim \left( \frac{1}{\nu t} \right)^{\frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right)} \| \xi_0 \|_{L^p(\mathbb{R}^3)} \quad \forall t > 0. \tag{22}
\]
Proof. Our proof is a small modification of the argument of Feng and Šverák in [26, Lemma 3.8], where the case \( p = 1 \) is considered. We define \( E_q(t) = \| \xi_\nu(t) \|_{L^q(\mathbb{R}^3)}^q \) for some \( q \in [p, \infty) \) and claim that

\[
\frac{d}{dt} E_q(t)^{-\frac{3}{q}} \geq \nu \left( \int_{\mathbb{R}^3} |\xi_\nu|^{\frac{q}{2}} \, dx \right)^{-\frac{q}{2}}. \tag{23}
\]

Let us postpone the proof of this estimate a bit and explain first how it implies (22). Notice that, by interpolation of Lebesgue spaces, it is enough to show (22) for exponents \( q > p \). Notice that all constants can be chosen uniformly in \( q \). We have thus proved (22) for \( q = 2\bar{q} \), which settles the case where \( q = 2^k p \).

If \( q = \infty \), we may now simply take the limit in (22) and use the convergence of the Lebesgue norms, \( \| \cdot \|_{L^\infty} = \lim_{q \to \infty} \| \cdot \|_{L^q} \).

It remains to provide the argument for (23). We start by recalling that

\[
- \frac{d}{dt} E_q(t) \geq \frac{q-1}{q} \nu \int_{\mathbb{R}^3} |\xi_\nu|^{q-2} |\nabla \xi_\nu|^2 \, dx \sim \frac{q-1}{q} \nu \int_{\mathbb{R}^3} |\nabla |\xi_\nu|^{\frac{q}{2}}|^2 \, dx.
\]

Notice that the constants in the estimate can be chosen independently of \( q \) as \( q > 1 \), and can thus be dropped. We estimate the right-hand-side with the help the 3D Nash inequality \( \| f \|_{L^2(\mathbb{R}^3)} \lesssim \| f \|_{L^2(\mathbb{R}^3)}^{2/5} \| \nabla f \|_{L^2(\mathbb{R}^3)}^{3/5} \), and obtain

\[
- \frac{d}{dt} E_q(t) \geq \nu \left( \int_{\mathbb{R}^3} |\xi_\nu|^{\frac{q}{2}} \, dx \right)^{-\frac{q}{2}} \left( \int_{\mathbb{R}^3} |\xi_\nu|^q \, dx \right)^{\frac{q}{2}} ,
\]

which can be rewritten as (23). \( \blacksquare \)

We also note that the fluid impulse is conserved along the viscous flow.

Lemma 6. Suppose that \( r^2 \omega_0 \in L^4(\mathbb{H}) \). Then

\[
\int_{\mathbb{H}} \omega_\nu(t)r^2 \, d(r, z) = \int_{\mathbb{H}} \omega_0 r^2 \, d(r, z).
\]
This identity can be seen in several ways, see, for instance [28, Lemma 6.4] for a proof that is based on the symmetry properties of the Biot–Savart kernel and applies to our regularity setting. We omit the proof and remark only that
\[
\int_{\mathbb{R}^3} u \, dx = \frac{1}{2} \int_{\mathbb{R}^3} \omega e_\theta \times x \, dx = -\frac{1}{2} \left( \int_{\mathbb{H}} \omega r^2 d(r, z) \right) e_z,
\]
whenever \( u \) is an axisymmetric vector field and \( \omega \) the associated vorticity. The conservation of momentum follows immediately from the Euler equations (1), (2).

The last global estimate concerns the energy balance law, for which we assume that the initial kinetic energy is bounded.

**Lemma 7.** Suppose that \( \| u_0 \|_{L^2(\mathbb{R}^3)} < \infty \). Then
\[
\| u_\nu(t) \|^2_{L^2(\mathbb{R}^3)} + \nu \int_0^t \| \nabla_x u_\nu \|^2_{L^2(\mathbb{R}^3)} \, dt = \| u_0 \|^2_{L^2(\mathbb{R}^3)} \quad \text{for all} \; t > 0. \tag{24}
\]

It is a classical result by Leray that for any divergence-free initial datum \( u_0 \) in \( L^2(\mathbb{R}^3) \), there exists a weak solution to the Navier–Stokes equations (10), (11) satisfying the energy inequality
\[
\| u_\nu(t) \|^2_{L^2(\mathbb{R}^3)} + \nu \int_0^t \| \nabla_x u_\nu \|^2_{L^2(\mathbb{R}^3)} \, dt \leq \| u_0 \|^2_{L^2(\mathbb{R}^3)}, \tag{25}
\]
\cf. [36]. Whether there is an energy equality (24) for such solutions is an important open problem. There are various conditions available in the literature under which an equality can be established, most notably, Serrin’s condition \( u \in L^q((0, T); L^p(\mathbb{R}^d)) \) with \( \frac{2}{p} + \frac{2}{q} \leq 1 \) or Shinbrot’s criterion \( \frac{2}{p} + \frac{2}{q} \leq 1 \) and \( p \geq 4 \), \cf. [42, 43]. We refer to [15] for an extension of the previous results to a larger class of function spaces and to [5] for a recent improvement based on assumptions on the gradient of the velocity.

It is not difficult to see, that we can construct mild solutions in the setting of [28] that satisfy the inequality (25), and thus, thanks to the uniqueness in that setting, our solutions do as well. We remark that in [10] Buckmaster and Vicol construct weak solutions for the three-dimensional Navier for which the energy inequality is not automatically achieved. Unfortunately, it is not obvious how to check Serrin’s or Shinbrot’s integrability conditions to ensure an energy equality in the axisymmetric setting. The problem is the appearance of weights as, for instance, in (16) and in suitable Sobolev inequalities. For this reason, we provide a proof of (24) that is tailored to our needs but still mimics the original arguments in [42, 43].

**Proof.** Thanks to the well-posedness result in [28], we may suppose that (25) holds true in our setting. In particular, we deduce
\[
u_\nu \in L^\infty((0, T); L^2(\mathbb{R}^3)^3) \quad \text{and} \quad \nabla_x u_\nu \in L^2((0, T); L^2(\mathbb{R}^3)^{3\times3}). \tag{26}
\]
In addition, thanks to the \( L^p \) bound on the vorticity in Lemma 4 and the weighted maximal regularity estimate in Lemma 3, it holds that
\[
\frac{1}{r} \nabla_x u_\nu \in L^\infty((0, T); L^p(\mathbb{R}^3)^{3\times3}). \tag{27}
\]
By standard density arguments, we may thus find a sequence \( \{u^\delta_\nu \}_{\delta \downarrow 0} \) of divergence-free functions in \( C^\infty_c((0, T); C^\infty_c(\mathbb{R}^3)^3) \) that converges towards \( u_\nu \) in \( L^2((0, T); H^1_0(\mathbb{R}^3)^3) \) and stays bounded in all the spaces in which \( u_\nu \) is contained. We furthermore denote by \( \eta^\varepsilon \) a standard mollifier on \( \mathbb{R} \). Because

\[
F(t, x) = \int_0^T \eta^\varepsilon(t - \tau) u^\delta_\nu(\tau, x) \, d\tau
\]

is an admissible test function in the definition of distributional solution of the Navier–Stokes equations, we find that

\[
\begin{align*}
\int_0^T \int_{\mathbb{R}^3} &\eta^\varepsilon(T - \tau) u^\delta_\nu(\tau, x) \cdot u_\nu(T, x) \, dx \, d\tau \\
= &\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(-\tau) u^\delta_\nu(\tau, x) \cdot u_\nu(0, x) \, dx \, d\tau \\
+ &\int_0^T \int_0^T \int_{\mathbb{R}^3} \frac{d\eta^\varepsilon}{dt}(t - \tau) u^\delta_\nu(\tau, x) \cdot u_\nu(t, x) \, dx \, d\tau \, dt \\
- &\int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(t - \tau) \nabla_x u^\delta_\nu(\tau, x) : u_\nu(t, x) \otimes u_\nu(t, x) \, dx \, d\tau \, dt \\
- &\nu \int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(t - \tau) \nabla_x u^\delta_\nu(\tau, x) : \nabla_x u_\nu(t, x) \, dx \, d\tau \, dt.
\end{align*}
\]

In a first step, we send \( \delta \) to zero with \( \varepsilon > 0 \) fixed. The convergence is obvious for all but the nonlinear term. It is enough to show that the nonlinear term vanishes when \( u^\delta_\nu \) is replaced by \( v^\delta = u^\delta_\nu - u_\nu \). Performing an integration by parts, we can throw the derivative on one of the \( u_\nu(t, x) \). Hölder’s inequality then yields

\[
\begin{align*}
&\left| \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon(\tau - t) \nabla_x v^\delta(\tau, x) : u_\nu(t, x) \otimes u_\nu(t, x) \, dx \, d\tau \, dt \right| \\
\leq &\int_0^T \| \eta^\varepsilon \ast v^\delta \|_{L^4(\mathbb{R}^3)} \| u_\nu \|_{L^4(\mathbb{R}^3)} \| \nabla_x u_\nu \|_{L^2(\mathbb{R}^3)} \, dt,
\end{align*}
\]

where by \( * \) we denote the convolution-type operation between \( \eta^\varepsilon \) and \( v^\delta \). We now have to make use of the interpolation inequality in Lemma [13] in the appendix and notice that \( |\nabla u| \leq |\nabla_x u| \) for any axisymmetric velocity field \( u \). We find that

\[
\begin{align*}
&\int_0^T \| \eta^\varepsilon \ast v^\delta \|_{L^4(\mathbb{R}^3)} \| u_\nu \|_{L^4(\mathbb{R}^3)} \| \nabla_x u_\nu \|_{L^2(\mathbb{R}^3)} \, dt \\
\lesssim &\int_0^T \| \eta^\varepsilon \ast v^\delta \|_{L^2(\mathbb{R}^3)} \| \eta^\varepsilon \ast \nabla_x v^\delta \|_{L^2(\mathbb{R}^3)} \| \frac{1}{r} \eta^\varepsilon \ast \nabla_x v^\delta \|_{L^p(\mathbb{R}^3)} \\
&\times \| u_\nu \|_{L^2(\mathbb{R}^3)} \| \nabla_x u_\nu \|_{L^2(\mathbb{R}^3)} \| \frac{1}{r} \nabla_x u_\nu \|_{L^p(\mathbb{R}^3)} \, dt.
\end{align*}
\]
Using Hölder’s and Young’s convolution inequality, we then infer that

\[
\int_0^T \| \eta^\varepsilon * v^\delta \|_{L^1(\mathbb{R}^3)} \| u^\nu \|_{L^1(\mathbb{R}^3)} \| \nabla_x u^\nu \|_{L^2(\mathbb{R}^3)} \, dt \\
\lesssim \| v^\delta \|_{L^\infty(\mathbb{R}^3)} \left( \frac{1}{r} \nabla_x v^\delta \right)_{L^2(\mathbb{R}^3)} \| \nabla_x u^\nu \|_{L^2(\mathbb{R}^3)} \\
\times \| u^\nu \|_{L^\infty(\mathbb{R}^3)} \left( \frac{3}{2} \nabla_x u^\nu \right)_{L^2((0,T) \times \mathbb{R}^3)}
\]

From (26) and (27) and the assumptions on \( v^\delta \), we deduce that the right-hand side in the above estimate is vanishing as \( \delta \to 0 \). Passing to the limit in the weak formulation of the Navier–Stokes equations above thus yields

\[
\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (T - \tau) u^\nu(\tau, x) \cdot u^\nu(T, x) \, dx \, d\tau \\
= \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (-\tau) u^\nu(\tau, x) \cdot u_0(x) \, dx \, d\tau \\
- \int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (t - \tau) \nabla_x u^\nu(\tau, x) : u^\nu(t, x) \otimes u^\nu(t, x) \, dx \, d\tau \, dt \\
- \nu \int_0^T \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (t - \tau) \nabla_x u^\nu(\tau, x) : \nabla_x u^\nu(t, x) \, dx \, d\tau \, dt.
\]

Notice that the term that involved the time derivative on \( \eta^\varepsilon \) dropped out by imposing that \( \eta^\varepsilon \) is an even function.

We finally send \( \varepsilon \) to zero and may thus choose \( \varepsilon < T \) from here on. Notice first that

\[
\nu \int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (t - \tau) \nabla_x u^\nu(\tau, x) : \nabla_x u^\nu(t, x) \, dx \, d\tau \, dt \\
\rightarrow \nu \int_0^T \| \nabla_x u^\nu \|_{L^2(\mathbb{R}^3)}^2 \, dt
\]

thanks to standard convergence properties of the mollifier. For the convergence of the end-point integrals, we make use of the fact that our solutions are continuous in time with respect to the weak topology in \( L^2(\mathbb{R}^3) \), see, e.g., [43, Corollary 3.2]. Because \( \eta^\varepsilon \) is chosen even, Lebesgue’s convergence theorem then yields

\[
\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (T - \tau) u^\nu(\tau, x) \cdot u^\nu(T, x) \, dx \, d\tau \rightarrow \frac{1}{2} \| u^\nu(T) \|_{L^2(\mathbb{R}^3)}^2, \\
\int_0^T \int_{\mathbb{R}^3} \eta^\varepsilon (-\tau) u^\nu(\tau, x) \cdot u_0(x) \, dx \, d\tau \rightarrow \frac{1}{2} \| u_0 \|_{L^2(\mathbb{R}^3)}^2.
\]

It remains to argue that the nonlinear term is vanishing. Notice first that

\[
\int_0^T \int_{\mathbb{R}^3} u^\nu_{\delta} \cdot (u^\nu \cdot \nabla_x u^\nu_{\delta}) \, dx \, dt = \frac{1}{2} \int_0^T \int_{\mathbb{R}^3} u^\nu \cdot \nabla_x |u^\nu_{\delta}|^2 \, dx \, dt = 0
\]

for any \( \delta \) if \( u^\nu_{\delta} \) is defined as above. This identity carries over to the limit \( \delta \to 0 \) as can be seen by using the same kind of estimates that we used above in order to
control the nonlinear term. We may thus rewrite the nonlinear term above as

$$\int_0^T \int_{\mathbb{R}^3} (u_\nu - \eta^\varepsilon \ast u_\nu) \cdot (u_\nu \cdot \nabla_x u_\nu) \, dxdt,$$

and, by applying the same kind of estimates again, we observe that this term vanishes as $\varepsilon \to 0$ by the convergence properties of the mollifier.

5 Vanishing viscosity limit. Proof of Theorem 1

In this section, we turn to the proof of Theorem 1. The compactness argument is based on the a priori estimate (20) on the relative vorticity and local estimates on the velocity field. The latter are provided by the following two lemmas.

Lemma 8. For any $R > 0$ and any $p_* \in (1, p] \cap (1, 2)$, there exists a constant $C(R)$ such that

$$\|u_\nu\|_{L^\infty((0,T); W^{1,p_*(B_R(0)))} \leq C(R) \left( \|\xi_0\|_{L^p(\mathbb{R}^3)} + \|\omega_0\|_{L^1(\mathbb{H})} \right).$$  (28)

where $B_R(0)$ is the ball in $\mathbb{R}^3$ centered in the origin.

Proof. By standard interpolation between Lebesgue spaces, we may without loss of generality assume that $p = p_*$. The bound on the gradient is an immediate consequence of the maximal regularity estimate in Lemma 3 and formula (15),

$$\|\nabla u_\nu\|_{L^\infty((0,T); L^p(B_R))} \lesssim R^{p-1} \|\nu \partial_\nu^{-1} \nabla u_\nu\|_{L^\infty((0,T); L^p(\mathbb{H}))} \lesssim R^{p-1} \|\xi_0\|_{L^p(\mathbb{R}^3)},$$

where $B_R = B^\mathbb{H}_R(0)$ denotes an open ball of radius $R$ centered at 0 in the half-space $\mathbb{H}$. Notice that it is enough to show the statement of the lemma for $W^{1,p_*}(B_R)$ equipped with $d(r,z)$ instead of $W^{1,p_*}(B_R(0))$ equipped with $dx$.

In order to deduce an estimate on the velocity field itself, we first invoke the Poincaré estimate for mean-zero functions and the previous bound to observe that

$$\|u_\nu\|_{L^p(B_R)} \lesssim R \|\nabla u_\nu\|_{L^p(B_R)} + R^{\frac{2}{p} - 2} \|u_\nu\|_{L^1(B_R)}$$

$$\lesssim R^p \|\xi_0\|_{L^p(\mathbb{R}^3)} + R^{\frac{2}{p} - 2} \|u_\nu\|_{L^1(B_R)}$$  (29)

uniformly in time. It remains to bound the $L^1$ norm of $u$. For this purpose, we make use of the decay behavior of the Biot–Savart kernel. In [28], the authors show that the decay of the axisymmetric Biot–Savart kernel is identical (in scaling) to that of the planar Biot–Savart kernel, that is, if we rewrite (6) as

$$u_\nu(r, z) = \int_{\mathbb{H}} K(r, z, \bar{r}, \bar{z}) \omega_\nu(\bar{r}, \bar{z}) \, d(\bar{r}, \bar{z}),$$

then the kernel $K$ obeys the estimate

$$|K(r, z, \bar{r}, \bar{z})| \lesssim \frac{1}{|r - \bar{r}| + |z - \bar{z}|}.$$  (28)
We thus have and write
\[
\int_{B_R} |u_\nu(r, z)| d(r, z) \lesssim \int_{B_R} \int_{\mathbb{H}} \frac{\left|\omega_\nu(\bar{r}, \bar{z})\right|}{|r - \bar{r}| + |z - \bar{z}|} d(\bar{r}, \bar{z}) d(r, z)
\]
\[
= \int_{B_R} \int_{B_{2R}(r, z)} \frac{\left|\omega_\nu(\bar{r}, \bar{z})\right|}{|r - \bar{r}| + |z - \bar{z}|} d(\bar{r}, \bar{z}) d(r, z)
\]
\[
+ \int_{B_R} \int_{\mathbb{H}\setminus B_{2R}(r, z)} \frac{\left|\omega_\nu(\bar{r}, \bar{z})\right|}{|r - \bar{r}| + |z - \bar{z}|} d(\bar{r}, \bar{z}) d(r, z).
\]

For the near-field, we use Fubini’s theorem, Young’s convolution estimate and Lemma 9 to deduce
\[
\int_{B_R} \int_{B_{2R}(r, z)} \frac{\left|\omega_\nu(\bar{r}, \bar{z})\right|}{|r - \bar{r}| + |z - \bar{z}|} d(\bar{r}, \bar{z}) d(r, z) \leq \int_{B_{3R}} \frac{1}{|r| + |z|} d(r, z) \int_{B_{2R}} |\omega_\nu| d(r, z)
\]
\[
\lesssim R\|\omega_\nu\|_{L^1(\mathbb{H})} \leq R\|\xi_0\|_{L^1(\mathbb{R}^3)}.
\]

For the far-field, we simply observe that the kernel is bounded below, and thus
\[
\int_{B_R} \int_{\mathbb{H}\setminus B_{2R}(r, z)} \frac{\left|\omega_\nu(\bar{r}, \bar{z})\right|}{|r - \bar{r}| + |z - \bar{z}|} d(\bar{r}, \bar{z}) d(r, z) \lesssim \int_{B_{3R}} \int_{\mathbb{H}\setminus B_{2R}(r, z)} |\omega_\nu(\bar{r}, \bar{z})| d(\bar{r}, \bar{z})
\]
\[
\lesssim R\|\omega_\nu\|_{L^1(\mathbb{H})} \leq R\|\xi_0\|_{L^1(\mathbb{R}^3)}.
\]

Plugging the previous bounds into (29) yields (28) as desired.

**Lemma 9.** For any $R > 0$, it holds that
\[
\|\partial_t u_\nu\|_{L^2((0, T), W^{1,1}(B_R(0)))} \leq C(R) \left(\|\xi_0\|_{L^p(\mathbb{R}^3)} + \|\omega_0\|_{L^1(\mathbb{H})}\right),
\]
where $W^{1,1}_\sigma(B_R(0))^3$ is the Banach space that is dual to the space of divergence-free vector fields in $W^{1,\infty}_0(B_R(0))^3$.

The proof of this estimate is fairly standard. We sketch the argument for the convenience of the reader.

**Proof.** Let $F$ be a divergence-free vector field in $W^{1,\infty}_0(B_R(0))^3$. Then
\[
(u_\nu \cdot \nabla u_\nu, F)_{W^{1,1}_\sigma(B_R(0)) \times W^{1,\infty}_0(B_R(0))} = -\int_{B_R(0)} u_\nu \otimes u_\nu : \nabla F dx
\]
\[
\leq \|u_\nu\|_{L^2(B_R(0))}^2 \|F\|_{W^{1,\infty}_0(B_R(0))},
\]
and a similar bound holds for the dissipation term $-\nu \Delta u_\nu$. The statement thus follows directly from the momentum equation and Lemma 8.

We are now in the position to prove the compactness result.
Proof of Theorem 7. Thanks to Lemmas 8 and 9, the sequence of velocity fields \(\{u_\nu\}_{\nu \downarrow 0}\) satisfies the hypotheses of the Aubin–Lions Lemma, and thus, for any \(R > 0\), there exists a subsequence that converges strongly in \(C([0,T];L^2(B_R(0)))\). By applying a diagonal sequence argument, this convergence carries over to the space \(C([0,T];L^2(K))\) for any compact \(K\) in \(\mathbb{R}^3\). Hence, there exists a subsequence (not relabelled) and a vector field \(u \in C([0,T];L^2_{\text{loc}}(\mathbb{R}^3))\) such that

\[ u_\nu \rightarrow u \quad \text{strongly in } C([0,T];L^2_{\text{loc}}(\mathbb{R}^3)). \]

It is readily checked that \(u\) is a distributional solution to the Euler equations (1), (2).

Moreover, from the a priori estimate on the relative vorticity in Lemma 4, we deduce that there exists a function \(\xi \in L^\infty((0,T);L^p(\mathbb{R}^3))\) such that, upon taking a further subsequence,

\[ \xi_n \rightarrow \xi \quad \text{weakly-⋆ in } L^\infty((0,T);L^p(\mathbb{R}^3)). \]

We finally notice that the velocity field \(u\) and the vorticity \(\omega = r\xi\) are related by the Biot–Savart law that holds true in the sense of distributions. ■

6 Renormalization. Proof of Theorem 2

In this section, we provide the argument for the renormalization property of the relative vorticity obtained as the vanishing viscosity solution of the Navier–Stokes equations in Theorem 1. Our approach is based on the duality formula in Lemma 1 established in [24] and follows closely the argumentation from [18, 17].

In a first step, we show a compactness result for a backwards advection-diffusion equation, that is, as we will see, dual to the vorticity formulation (19) of the Navier–Stokes equations.

Lemma 10. Let \(q \in (2,\infty)\) and \(\chi \in L^1((0,T);L^q(\mathbb{R}^d))\) be given. Let \(f_\nu\) denote the unique solution in the class \(L^\infty((0,T);L^q(\mathbb{R}^3))\) with \(\nabla|f_\nu|^q \in L^2((0,T);L^2(\mathbb{R}^3))\) to the backwards advection-diffusion equation

\[ -\partial_t f_\nu - u_\nu \cdot \nabla f_\nu = \chi + \nu \left( \Delta f_\nu - \frac{1}{r} \partial_r f_\nu \right) \]

in \(\mathbb{H}\) with final datum \(f_\nu(T) = 0\) and homogeneous Dirichlet boundary conditions on \(\partial\mathbb{H}\). Then there exists a subsequence \(\{\nu_k\}_{k \in \mathbb{N}}\) (the same as in Theorem 1) such that

\[ f_{\nu_k} \rightarrow f \quad \text{weakly-⋆ in } L^\infty((0,T);L^q(\mathbb{R}^3)), \]

where \(f\) is the unique solution to the backwards transport equation (14).

We remark that renormalized solutions to advection-diffusion equations have been considered, for instance, in [24, 35, 27].
Proof. We start with an a priori estimate. A direct computation reveals that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} |f_\nu|^q \, dx = - \int_{\mathbb{R}^3} |f_\nu|^{q-2} f_\nu \chi \, dx + \nu(q-1) \int_{\mathbb{R}^3} |f_\nu|^{q-2} |\nabla f_\nu|^2 \, dx \\
\geq - \|f_\nu\|_{L^q(\mathbb{R}^3)}^{q-1} \|\chi\|_{L^q(\mathbb{R}^3)},
\]
and thus, by a Gronwall argument and our choice of the final datum,
\[
\|f_\nu\|_{L^\infty((0,T); L^q(\mathbb{R}^3))} \leq \|\chi\|_{L^1((0,T); L^q(\mathbb{R}^3))}.
\]
Hence, there exists a subsequence \(\{\nu_k\}_{k \in \mathbb{N}}\) that can be chosen as a subsequence of the one found in Theorem 1 and an \(\tilde{f} \in L^\infty((0,T); L^q(\mathbb{R}^3))\) such that
\[
f_{\nu_k} \to \tilde{f} \quad \text{weakly-\star in} \quad L^\infty((0,T); L^q(\mathbb{R}^3)).
\]
Since at the same time
\[
u_k \to u \quad \text{strongly in} \quad L^2((0,T); L^2_{loc}(\mathbb{R}^3)),
\]
and \(q \geq 2\), we find in the limit that \(\tilde{f}\) solves the backward advection equation (14), and thus, \(\tilde{f} = f\) by uniqueness. In particular, the convergence result holds true for the subsequence from Theorem 1.]

We finally turn to the proof of the renormalization property.

Proof of Theorem 2. Let \(\chi \in C_c^\infty((0,T) \times \mathbb{H})\) be given and \(f_\nu\) a solution to the backwards advection-diffusion equation considered in Lemma 10. From the statement of the lemma, it follows that \(\{f_{\nu_k}\}_{k \in \mathbb{N}}\) converges to \(f\) weakly-\* in \(L^\infty((0,T); L^q(\mathbb{R}^3))\) for any \(q \in (2, \infty)\). By using the advection-diffusion equation, this convergence can be upgraded to hold in \(C((0,T], L^q_{weak}(\mathbb{R}^3))\) for any \(q \in (2, \infty)\), that is,
\[
\sup_{[0,T]} \int_{\mathbb{R}^3} (f_{\nu_k}(t) - f(t)) \zeta \, dx \to 0 \quad \forall \zeta \in L^{\tilde{q}}(\mathbb{R}^3),
\] (30)
where \(1/q + 1/\tilde{q} = 1\). Indeed, it is not difficult to obtain this result for smooth test functions and the full statement is obtained by standard approximation procedures.

Upon a standard approximation argument, \(f_\nu\) can be considered as a test function in the distributional formulation of the vorticity formulation (19) of the Navier–Stokes equations. Thus
\[
\int_{\mathbb{R}^3} f_\nu(0) \xi_0 \, dx = \int_0^T \int_{\mathbb{R}^3} \xi_\nu \left( \partial_t f_\nu + u_\nu \cdot \nabla f_\nu + \nu \left( \Delta f_\nu - \frac{1}{r} \partial_r f_\nu \right) \right) \, dx \, dt \\
= - \int_0^T \int_{\mathbb{R}^3} \xi_\nu \chi \, dx \, dt.
\]
As a consequence of Theorem 1, Lemma 10 and (30), we can pass to the limit in this identity and find
\[
\int_{\mathbb{R}^3} f(0) \xi_0 \, dx + \int_0^T \int_{\mathbb{R}^3} \xi_\chi \, dx \, dt = 0.
\]
On the other hand, because \( u \) satisfies the assumptions of Theorem 4, see also Remark 1, there exists a unique distributional solution \( \tilde{\xi} \in L^\infty(0, T; L^p(\mathbb{R}^3)) \) to the transport equation (5) with \( u \) being the given solution to the Euler equations and with initial datum \( \xi_0 \). By Lemma 1, we then find that

\[
\int_{\mathbb{R}^3} f(0)\xi_0 \, dx + \int_0^T \int_{\mathbb{R}^3} \tilde{\xi} \chi \, dxdt = 0,
\]

and thus,

\[
\int_0^T \int_{\mathbb{R}^3} (\xi - \tilde{\xi}) \chi \, dxdt = 0.
\]

Because \( \chi \) was arbitrarily fixed, we infer that \( \tilde{\xi} = \xi \) almost everywhere, and thus, \( \xi \) coincide almost everywhere with the renormalized solution \( \tilde{\xi} \). ■

7 Energy conservation. Proof of Theorem 3

We now prove Theorem 3. Throughout this section, we thus suppose that \( \omega_\nu \) is nonnegative and has finite impulse. Moreover, we assume that \( p > \frac{3}{2} \) as in the assumption of the theorem. Notice that by interpolation between Lebesgue spaces, we may always suppose that \( p \in \left( \frac{3}{2}, 2 \right) \), which we will do from here on.

One of the main ingredients of the proof is the convergence of the kinetic energy that is established in the following lemma.

Lemma 11. Let \( \{\nu_k\}_{k \in \mathbb{N}} \) be the subsequence found in Theorem 7. Then it holds that

\[
\lim_{k \to \infty} \|u_{\nu_k}(t)\|_{L^2(\mathbb{R}^3)} = \|u(t)\|_{L^2(\mathbb{R}^3)}
\]

for any \( t \in [0, T] \).

Proof. We have already seen in Theorem 4 that \( u_{\nu_k} \) converges to \( u \) strongly in \( C(0, T; L^p(\mathbb{R}^3)) \). We have to turn this result into a global convergence result. In fact, it is enough to show that

\[
\sup_k \|u_{\nu_k}(t)\|_{L^2(\mathbb{R}^3 \setminus B_R(0))} \to 0 \quad \text{as} \quad R \to \infty.
\]

Indeed, if (31) holds true, given \( \varepsilon > 0 \), we can find a radius \( R \geq 1 \) such that

\[
\sup_k \|u_{\nu_k}(t, \cdot + h)\|_{L^2(\mathbb{R}^3 \setminus B_{2R}(0))} \leq \varepsilon \quad \text{for any} \quad |h| \leq 1.
\]

Moreover, thanks to the strong convergence in \( B_{2R}(0) \), we have that

\[
\sup_k \|u_{\nu_k}(t) - u_{\nu_k}(t, \cdot + h)\|_{L^2(\mathbb{R}^3 \setminus B_R(0))} \leq \varepsilon \quad \text{for} \quad |h| \quad \text{sufficiently small.}
\]

Combining both estimates, we find that

\[
\sup_k \|u_{\nu_k}(t) - u_{\nu_k}(t, \cdot + h)\|_{L^2(\mathbb{R}^3)} \leq 3\varepsilon \quad \text{for} \quad |h| \quad \text{sufficiently small.}
\]

22
By Riesz’ compactness criterion, the latter result together with \([31]\) and the standard energy estimate \([25]\) imply strong convergence in \(L^2(\mathbb{R}^3)\) for \(\tau \in [0, T]\).

We now give the argument for \([31]\). For notational convenience, we write \(u\) and \(\nu\) instead of \(u_{\nu_k}\) and \(\nu_k\). We consider a smooth cut-off function \(\eta_R\) that is 1 in \(B_R = B_R(0)\) and 0 outside \(B_{2R} = B_{2R}(0)\). Testing the Navier–Stokes equations with \((1 - \eta_R)^2 u\) and integrating by parts yields

\[
\frac{d}{dt} \frac{1}{2} \int (1 - \eta_R)^2 |u|^2 \, dx + \nu \int (1 - \eta_R)^2 |\nabla u|^2 \, dx = \int (\eta_R - 1) \nabla \eta_R \cdot u |u|^2 \, dx + 2 \int (\eta_R - 1) \nabla \eta_R \cdot u \nu |u| \, dx + 2 \nu \int (1 - \eta_R)(\nabla \eta_R \cdot \nabla) u \cdot u \, dx. \tag{32}
\]

The error term in \([34]\) is quite easily estimated. Indeed, using the Cauchy–Schwarz inequality together with the elementary inequality \(2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2\), we can absorb the gradient term in \([34]\) in the second term in \([32]\) and we are left with an error term of the form \(\frac{\nu}{R^2} \|u\|_{L^2(\mathbb{R}^3)}^2\). In view of the energy inequality for the Navier–Stokes equations, this term is obviously vanishing as \(R \to \infty\) uniformly in \(t\).

As a next step, we address the first error term in \([33]\). Using the properties of the cut-off function, this term is bounded as follows:

\[
\int (\eta_R - 1) \nabla \eta_R \cdot u |u|^2 \, dx \lesssim \frac{1}{R} \int_{B_{2R} \setminus B_R} |u|^3 \, dx \lesssim \int_{B_{2R} \setminus B_R} |u|^3 \, d(r, z). \tag{35}
\]

Here, we have used the same notation for both the ball in \(\mathbb{R}^3\) and the one in \(\mathbb{H}\). It should be clear from the situation, which one is considered. We now use the Sobolev embedding in two dimensions and find

\[
\int_{B_{2R} \setminus B_R} |u|^3 \, d(r, z) \lesssim \left( R^{-\frac{\alpha}{2}} \int_{B_{2R} \setminus B_R} |u|^\alpha \, d(r, z) + \int_{B_{2R} \setminus B_R} |\nabla u|^\alpha \, d(r, z) \right)^\frac{2}{\alpha}. \tag{36}
\]

Moreover, thanks to local Calderon–Zygmund estimates (which we can perform on the level of the standard Biot–Savart kernel in \(\mathbb{R}^3\)), we find that

\[
\int_{B_{2R} \setminus B_R} |\nabla u|^\frac{\alpha}{2} \, d(r, z) \lesssim R^{-\frac{\alpha}{2}} \int_{B_{3R} \setminus B_{\frac{3R}{2}}} |u|^\frac{\alpha}{2} \, d(r, z) + \int_{B_{3R} \setminus B_{\frac{3R}{2}}} |\omega|^\frac{\alpha}{2} \, d(r, z). \tag{36}
\]

With regard to the first term in \([36]\), we notice that by Jensen’s inequality and the energy inequality, it holds that

\[
R^{-\frac{\alpha}{2}} \int_{B_{3R} \setminus B_{\frac{3R}{2}}} |u|^\frac{\alpha}{2} \, d(r, z) \lesssim R^{-\frac{\alpha}{2}} \left( \int_{B_{3R} \setminus B_{\frac{3R}{2}}} |u|^2 \, d(r, z) \right)^\frac{\alpha}{5} \lesssim R^{-1} \|u\|_{L^2(\mathbb{R}^3)}^\frac{\alpha}{5} \lesssim R^{-1} \|u\|_{L^2(\mathbb{R}^3)}^\frac{\alpha}{5},
\]

23
and, thus, the first term in (36) vanishes as $R \to \infty$, uniformly in $t$. For the second term in (36), we appeal to Hőlder’s inequality,

$$
\int_{B_{3R} \setminus B_2} |\omega|^\frac{6}{5} d(r, z) \leq \left( \int_{B_{3R} \setminus B_2} |\omega|^p d(r, z) \right)^\frac{1}{3(p-1)} \left( \int_{B_{3R} \setminus B_2} |\omega| d(r, z) \right)^\frac{5p-6}{3(p-1)}.
$$

We can easily smuggle in some weights,

$$
\int_{B_{3R} \setminus B_2} |\omega|^\frac{6}{5} d(r, z) \lesssim R^{-\frac{9p-12}{5(p-1)}} \| \xi \|_{L_p(\mathbb{R}^3)}^{\frac{6}{p}} \| \omega \|_{L^2(\mathbb{R})}^{\frac{5p-6}{3(p-1)}}.
$$

It remains to observe that the exponent on $R$ is negative because $p > \frac{3}{2} > \frac{4}{3}$. Combining the above estimates, we conclude that

$$
\lim_{R \to \infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} |u|^3 dx \sim \lim_{R \to \infty} \int_{B_{2R} \setminus B_R} |u|^3 d(r, z) = 0
$$

uniformly in $t$, and thus, in view of (35), the first term in (33) vanishes.

We finally turn to the term that involves the pressure, that is, the second term in (36). Using the properties of the cut-off function and Hőlder’s inequality, we observe that

$$
\int (\eta_R - 1) \nabla \eta_R \cdot up \, dx \lesssim \left( \frac{1}{R} \int_{B_{2R} \setminus B_R} |u|^3 \, dx \right)^\frac{1}{3} \left( \frac{1}{R} \int_{B_{2R} \setminus B_R} |p|^3 \, dx \right)^\frac{2}{3}.
$$

In view of (37), it is enough to show that the pressure term is bounded, in the sense that

$$
\frac{1}{R} \int_{B_{2R} \setminus B_R} |p|^3 \, dx \lesssim 1
$$

uniformly in $R$, $\nu$, and $t$. To establish this estimate, we recall that $p$ solves the Poisson equation $-\Delta p = \nabla^2 : u \otimes u$, and thus, we have that $p = \sum_{ij} \partial_i \partial_j G \ast (u_i u_j)$, where $G$ is the Newtonian potential in $\mathbb{R}^3$, $G(z) = \frac{1}{4\pi |z|}$. Let us write $f = G \ast (u_i u_j)$. The localized Calderón–Zygmund estimates yield

$$
R^{-\frac{5}{3}} \| \nabla^2 f \|_{L^2(B_{3R} \setminus B_R)} \lesssim R^{-\frac{2}{3}} \| u_i u_j \|_{L^2(B_{3R} \setminus B_R)} + R^{-\frac{5}{3}} \| \nabla f \|_{L^2(B_{3R} \setminus B_R)}.
$$

The first term is controlled thanks to (37). (Notice that the exact value of the radii in (37) is not of importance but the scaling in $R$.) Now, since for any $m \in \left( \frac{3}{2}, \infty \right)$, it holds that

$$
R^{-\frac{5}{3}} \| \nabla f \|_{L^2(B_{3R})} \lesssim R^{\frac{5}{6} - \frac{3}{m}} \| \nabla f \|_{L^m(\mathbb{R}^3)}
$$

by Hőlder’s inequality, and $R^{\frac{5}{6} - \frac{3}{m}} \to 0$ as $R \to \infty$ for $m < 9$, it suffices to show that $\| \nabla f \|_{L^m(\mathbb{R}^3)}$ is bounded for some $m \in \left( \frac{3}{2}, 9 \right)$. Notice first that the standard Sobolev inequality in $\mathbb{R}^3$ yields

$$
\| \nabla f \|_{L^m(\mathbb{R}^3)} \lesssim \| \nabla^2 f \|_{L^2(\mathbb{R}^3)}
$$
as long as $s = \frac{3m}{34 + m} \in [1, 3)$. By our choice of $m$, we have to restrict the range of admissible $s$ to the interval $(1, \frac{9}{4})$. Now we use the maximal regularity properties of the Laplacian, in the sense that

$$
\| \nabla^2 f \|_{L^q(\mathbb{R}^3)} \lesssim \| \Delta f \|_{L^q(\mathbb{R}^3)} \lesssim \| u \|_{L^{2s}(\mathbb{R}^3)}^2.
$$

In order to estimate the velocity field in $L^{2s}$, we use the Sobolev inequality in three dimensions and Calderón–Zygmund estimates for the gradient of the Biot–Savart kernel,

$$
\| u \|_{L^{2s}(\mathbb{R}^3)} \lesssim \| \nabla u \|_{L^{\frac{6s}{2s+3}}(\mathbb{R}^3)} \lesssim \| \omega \|_{L^{\frac{6s}{2s+3}}(\mathbb{R}^3)}.
$$

Let us now write $q = \frac{6s}{2s+3}$ and notice that $q \in (\frac{6}{5}, \frac{27}{15})$ by our choice of $s$. Recall from our line of proof that we have to show the boundedness of $\| \omega \|_{L^q(\mathbb{R}^3)}$ for some value of $q$ in the above interval. This is achieved via interpolation of the estimates in Lemmas 4 and 6. Indeed, setting $\theta = \frac{q-1}{p-1}$ for some $q \in (\frac{6}{5}, \frac{3}{2})$ whose explicit value we will specify in a moment, we have that

$$
\| \omega \|_{L^q(\mathbb{R}^3)}^q = \int |r \xi|^{(1-\theta)+\theta p} r \, d(r, z)
$$

$$
= \int r^{1+\theta(p-1)} |\xi|^{1-\theta} |\xi|^{\theta p} r \, d(r, z)
$$

$$
\leq \left( \int |\xi|^{\frac{1+\theta(p-1)}{1-\theta}} r \, d(r, z) \right)^{1-\theta} \left( \int |\xi|^p r \, d(r, z) \right)^{\theta}
$$

$$
= \| \xi r^{\frac{q(p-1)}{p-q} \|_{L^{1-\theta}(\mathbb{R}^3)} \| \xi \|_{L^p(\mathbb{R}^3)}.}
$$

where for the second to last inequality we used Hölder estimate with exponents $1/\theta$ and $\frac{1}{1-\theta}$ and in the last identity we used the definition of $\theta$. We may now determine $q$ by requiring that $\frac{q(p-1)}{p-q} = 2$, which yields $q = \frac{2p}{p+1}$. It remains to notice that this choice of $q$ is admissible because $\frac{2p}{p+1} \in (\frac{6}{5}, \frac{4}{3})$ for any $p \in (\frac{3}{2}, 2)$ which we may assume as explained in the introduction to this section. By using the a priori bounds in Lemmas 4 and 6 we conclude that (38) holds uniformly in $\nu$ and $t$. We have thus established Lemma 11.

With these preparations, we are now in the position to prove Theorem 3. Our short proof is strongly inspired by [14].

**Proof of Theorem 3.** In order to prove conservation of energy, we choose a subsequence as in Theorem 1, which we will not relabel for notational convenience, and recall the energy identity in Lemma 7 which we rewrite as

$$
0 \geq \| u_{\nu}(t) \|_{L^2(\mathbb{R}^3)}^2 - \| u_0 \|_{L^2(\mathbb{R}^3)}^2 = -2\nu \int_0^t \| \nabla_x u_{\nu}(s) \|_{L^2(\mathbb{R}^3)}^2 \, ds.
$$

Thanks to Lemmas 2, 6 and 5, we observe that

$$
\| \nabla_x u_{\nu}(s) \|_{L^2(\mathbb{R}^3)}^2 = \| r \xi_{\nu}(s) \|_{L^2(\mathbb{R}^3)}^2 \leq \| r^{2} \xi_{\nu}(s) \|_{L^1(\mathbb{R}^3)} \| \xi_{\nu}(s) \|_{L^\infty(\mathbb{R}^3)} \lesssim \left( \frac{1}{\nu s} \right)^{\frac{3}{2p}},
$$

25
and thus, the energy identity implies that
\[ 0 \geq \|u_\nu(t)\|_{L^2(\mathbb{R}^3)}^2 - \|u_0\|_{L^2(\mathbb{R}^3)}^2 \geq -C(\nu t)^{1 - \frac{3}{2p}}, \]
because \( p > \frac{3}{2} \). Sending \( \nu \) to zero, we conclude that
\[ \lim_{\nu \to \infty} \|u_\nu(t)\|_{L^2(\mathbb{R}^3)} = \|u_0\|_{L^2(\mathbb{R}^3)}, \]
and the statement of the theorem follows upon applying Lemma 11, in which the convergence of the kinetic energy is established. ■

Appendix: Two auxiliary inequalities

We conclude this paper with two auxiliary inequalities, that are weighted versions of standard Sobolev and interpolation inequalities.

**Lemma 12.** Let \( 1 \leq s \leq t < \infty \) and \( \alpha, \beta \in \mathbb{R} \) be such that
\[ \frac{2 + \alpha}{t} = \frac{2 - s + \beta}{s} \quad \text{and} \quad \alpha + t > 0. \]
Then
\[ \left( \int_{\mathbb{H}} |f|^t r^\alpha d(r, z) \right)^{\frac{1}{t}} \lesssim \left( \int_{\mathbb{H}} |\nabla f|^{s \beta} d(r, z) \right)^{\frac{1}{s}}, \]
for any \( f \in C^\infty_c(\mathbb{H}) \).

This estimate is proved, for instance, in [34]. We recall the argument for completeness.

**Proof.** Step 1. We first treat the special case \( s = 1 \), and thus
\[ \frac{2 + \alpha}{t} = 1 + \beta. \]
We set \( \gamma = \frac{\alpha}{t} \) and let \( g \in C^\infty_c(\mathbb{H}) \) be defined by \( g(r, z) = f(2r, z) \) and \( A = [R, 2R] \times \mathbb{R} \) and \( B = [R, 4R] \times \mathbb{R} \) be two subsets of \( \mathbb{H} \) for some \( R > 0 \) fixed. By Hölder’s inequality, we then have that
\[ \int_A (f - g)^2 r^{2\gamma} d(r, z) \leq \int_R^{2R} \|r^{\gamma}(f - g)\|_{L^1(dz)} \|r^{\gamma}(f - g)\|_{L^\infty(dz)} dr. \]
We now use the embedding \( W^{1,1} \subset L^\infty \), that holds true in one space dimension, in each variable. On the one hand, using the embedding in \( r \) (in form of the fundamental theorem of calculus), we have
\[ \sup_{r \in (R, 2R)} \|r^{\gamma}(f - g)\|_{L^1(dz)} \leq \sup_{r \in (R, 2R)} r^{\gamma} \int_r^{2r} \|\partial_r f(\rho)\|_{L^1(dz)} d\rho \leq \int_B |\nabla f| r^{\gamma} d(r, z). \]
On the other hand, it holds that
\[
\int_0^{2R} \| r^\gamma (f - g) \|_{L^\infty(dz)} \lesssim \int_R^{2R} r^\gamma \| \partial_z (f - g) \|_{L^1(dz)} \, dr \lesssim \int_B |\nabla f| r^\gamma \, d(r,z),
\]
where we have used the triangle inequality and a rescaling argument in the last inequality. Combining the previous three estimates, we find that
\[
\left( \int_A (f - g)^2 r^{2\gamma} \, d(r,z) \right)^{\frac{1}{2}} \lesssim \int_B |\nabla f| r^\gamma \, d(r,z).
\tag{39}
\]

Our next goal is the Hardy-type inequality
\[
\int_A |f - g| r^\gamma \, d(r,z) \lesssim \int_B |\nabla f| r^{\gamma+1} \, d(r,z),
\tag{40}
\]
which holds true provided that \( \gamma > -1 \), and thus \( \alpha + t > 0 \). It can be established as follows. Using the fundamental theorem again, we observe that
\[
\int_A |f - g| r^\gamma \, d(r,z) \leq \int_R^{2R} r^\gamma \int_r^{2r} \| \partial_r f(\rho) \|_{L^1(d\rho)} \, d\rho \, dr \lesssim R^{\gamma+1} \int_B |\partial_r f| \, d(r,z),
\]
which implies (40) because the prefactor \( R^{\gamma+1} \) can be smuggled into the integrand.

Towards the weighted Sobolev inequality with \( s = 1 \), we set \( A_k = [2^k, 2^{k+1}] \times \mathbb{R} \) and \( B_k = [2^k, 2^{k+2}] \times \mathbb{R} \) and estimate with the help of the triangle inequality
\[
\left( \int_{\mathbb{H}} |f - g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} \leq \sum_{k \in \mathbb{Z}} \left( \int_{A_k} |f - g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}}
\]
Interpolation between Lebesgue spaces and an application of (39) and (40) yields
\[
\left( \int_{A_k} |f - g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} \leq \left( \int_{A_k} |f - g|^\beta r^\gamma \, d(r,z) \right)^{\frac{1}{t}} \left( \int_{A_k} (f - g)^2 r^{2\gamma} \, d(r,z) \right)^{\frac{1-\beta+\gamma}{2}}
\leq \left( \int_{B_k} |\nabla f|^\beta r^\gamma \, d(r,z) \right)^{\frac{1}{t}} \left( \int_{B_k} |\nabla f|^\gamma \, d(r,z) \right)^{1-\beta+\gamma}
\sim \int_{B_k} |\nabla f|^\beta \, d(r,z),
\]
because \( \beta - \gamma = \frac{2}{t} - 1 \in [0, 1] \) for \( t \in [1, 2] \). Summation over \( k \) yields
\[
\left( \int_{\mathbb{H}} |f - g|^t r^\alpha \, d(r,z) \right)^{\frac{1}{t}} \leq C \int_{\mathbb{H}} |\nabla f|^\beta \, d(r,z)
\]

for some universal constant $C$. It remains to apply the triangle inequality and a change of variables to the effect that
\[
\left( \int_{\mathbb{H}} |f|^4 r^\alpha d(r, z) \right)^\frac{1}{3} \leq \left( \int_{\mathbb{H}} |g|^4 r^\alpha d(r, z) \right)^\frac{1}{3} + C \int_{\mathbb{H}} |\nabla f|^\beta d(r, z)
\]
\[
= \frac{1}{2^{\frac{1}{3}}} \left( \int_{\mathbb{H}} |f|^4 r^\alpha d(r, z) \right)^\frac{1}{3} + C \int_{\mathbb{H}} |\nabla f|^\beta d(r, z).
\]

We can absorb the first term on the right-hand side by the left-hand side because $\alpha + 1 > 0$ and obtain
\[
\left( \int_{\mathbb{H}} |f|^{\frac{4\alpha}{2+\alpha}} r^\alpha d(r, z) \right)^\frac{1}{3} \lesssim \int_{\mathbb{H}} |\nabla f|^\beta d(r, z).
\]

**Step 2.** The general case $s \geq 1$ follows from the special case $s = 1$. Indeed, choosing $f = |h|^{(1+\beta)/r}$ in \[(41),\] we find
\[
\left( \int_{\mathbb{H}} |f|^4 r^\alpha d(r, z) \right)^\frac{1}{3} \lesssim \left( \int_{\mathbb{H}} |f|^{(1+\beta)/r} d(r, z) \right)^\frac{1}{3} \left( \int_{\mathbb{H}} |\nabla f|^{p r^{1-p}} d(r, z) \right)^\frac{1}{3}.
\]
and the statement follows with the help of Hölder’s inequality. ■

We finally provide an interpolation inequality.

**Lemma 13.** Let $p \in (1, 2]$ and $\lambda = \frac{3p-3}{2p-6}$. Then
\[
\left( \int_{\mathbb{H}} |f|^4 r d(r, z) \right)^\frac{1}{3} \lesssim \left( \int_{\mathbb{H}} |f|^2 r d(r, z) \right)^\frac{1}{3} \left( \int_{\mathbb{H}} |\nabla f|^{p r^{1-p}} d(r, z) \right)^{\frac{1}{3}}
\]
for any $f \in C^\infty_c(\mathbb{H})$.

**Proof.** *Step 1:* It is enough to prove that
\[
\left( \int_{\mathbb{H}} |f|^4 r d(r, z) \right)^\frac{1}{3} \lesssim \left( \int_{\mathbb{H}} |f|^2 r d(r, z) \right)^\frac{1}{3} \left( \int_{\mathbb{H}} |\nabla f|^{q r^{1-p}} d(r, z) \right)^\frac{1}{3},
\]
where $\beta = \frac{5p-4}{7p-6}$ and $q = \frac{16p-12}{7p-6}$.

Indeed, the statement in \[(42),\] immediately follows from \[(43),\] and Hölder’s inequality. Let $a$ and $b$ be Hölder dual exponents given by $a = \frac{4(1-\lambda)}{\lambda} = \frac{7p-4}{2p-6}$ and $b = \frac{4(1-\lambda)}{4(1-\lambda)-q} = \frac{7p-4}{2}$. We write and estimate
\[
\int_{\mathbb{H}} |\nabla f|^q r^\beta d(r, z) = \int_{\mathbb{H}} (|\nabla f|^2 r)^{\frac{q}{2}} |\nabla f|^{\frac{1-\beta}{2} + \frac{q}{2}} d(r, z)
\]
\[
\leq \left( \int_{\mathbb{H}} (|\nabla f|^2 r)^{\frac{q}{2}} d(r, z) \right)^{\frac{1}{2}} \left( \int_{\mathbb{H}} |\nabla f|^{\frac{1-\beta}{2} + \frac{q}{2}} d(r, z) \right)^{\frac{1}{2}}
\]
\[
= \left( \int_{\mathbb{H}} |\nabla f|^2 r d(r, z) \right)^{\frac{1}{2}} \left( \int_{\mathbb{H}} |\nabla f|^{p r^{1-p}} d(r, z) \right)^{\frac{1}{2}}.
\]
Now, plugging the resulting estimate into (43) yields (42).

Step 2. The interpolation inequality (43) follows from the weighted Sobolev inequality from Lemma 12 in the formulation

\[
\left( \int_{\mathcal{H}} |f|^r r \, d(r, z) \right)^\frac{1}{t} \lesssim \left( \int_{\mathcal{H}} |\nabla f|^s r^\alpha \, d(r, z) \right)^\frac{1}{s},
\]

where \( t = \frac{16p - 12}{7p - 6} \), \( s = \frac{16p - 12}{13p - 10} \) and \( \alpha = \frac{11p - 10}{13p - 10} \) via a Ladyshenskaya-type argument. Notice that \( t, s \) and \( \alpha \) satisfy the dimensional condition

\[
t = \frac{3s}{2 - s + \alpha}.
\]

Indeed, substituting \(|f|^\frac{1}{t} \) for \( f \) in (44) implies that

\[
\left( \int_{\mathcal{H}} |f|^4 r \, d(r, z) \right)^\frac{1}{t} \lesssim \left( \int_{\mathcal{H}} |f|^{(\frac{4}{t} - 1)s} |\nabla f|^s r^\alpha \, d(r, z) \right)^\frac{1}{s}.
\]

We now use Hölder’s inequality with dual exponents \( a = \frac{13p - 10}{6p - 6} \) and \( b = \frac{13p - 10}{7p - 4} \) and get, since \( r^\alpha = r^{\frac{1}{2} - \frac{1}{s} - 1 + \frac{1}{b}} \), that

\[
\int_{\mathcal{H}} |f|^{(\frac{4}{t} - 1)s} |\nabla f|^s r^\alpha \, d(r, z) \leq \left( \int_{\mathcal{H}} |f|^{(\frac{4}{t} - 1)s} r \, d(r, z) \right)^\frac{1}{s} \left( \int_{\mathcal{H}} |\nabla f|^s r^{\alpha - 1 + \frac{1}{b}} \, d(r, z) \right)^\frac{1}{b}
\]

\[
= \left( \int_{\mathcal{H}} |f|^2 r \, d(r, z) \right)^\frac{1}{s} \left( \int_{\mathcal{H}} |\nabla f|^q r^\beta \, d(r, z) \right)^\frac{1}{b}.
\]

Combining the previous two estimates, it is straightforward to deduce (43). This completes the proof.

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References

[1] ABÉ, K. Vanishing viscosity limits for axisymmetric flows with boundary. Preprint arXiv:1806.04811, 2018.

[2] ABIDI, H., HMIDI, T., AND KERAANI, S. On the global well-posedness for the axisymmetric Euler equations. Math. Ann. 347, 1 (2010), 15–41.
[3] Ambrosio, L. Transport equation and Cauchy problem for $BV$ vector fields. *Invent. Math. 158*, 2 (2004), 227–260.

[4] Ambrosio, L., and Crippa, G. Continuity equations and ODE flows with non-smooth velocity. *Proc. Roy. Soc. Edinburgh Sect. A 144*, 6 (2014), 1191–1244.

[5] Berselli, L., and Chiodaroli, E. On the energy equality for the 3D Navier-Stokes equations. Preprint arXiv:1807.02667, 2018.

[6] Bohun, A., Bouchut, F., and Crippa, G. Lagrangian solutions to the 2D Euler system with $L^1$ vorticity and infinite energy. *Nonlinear Anal. 132* (2016), 160–172.

[7] Buckmaster, T. Onsager’s conjecture almost everywhere in time. *Comm. Math. Phys. 333*, 3 (2015), 1175–1198.

[8] Buckmaster, T., De Lellis, C., Isett, P., and Székelyhidi, Jr., L. Anomalous dissipation for 1/5-Hölder Euler flows. *Ann. of Math. (2) 182*, 1 (2015), 127–172.

[9] Buckmaster, T., De Lellis, C., and Székelyhidi, Jr., L. Dissipative Euler flows with Onsager-critical spatial regularity. *Comm. Pure Appl. Math. 69*, 9 (2016), 1613–1670.

[10] Buckmaster, T., and Vicol, V. Nonuniqueness of weak solutions to the navier-stokes equation. Preprint arXiv:1709.10033, 2017.

[11] Chae, D., and Imanuvilov, O. Y. Existence of axisymmetric weak solutions of the 3-D Euler equations for near-vortex-sheet initial data. *Electron. J. Differential Equations* (1998), No. 26, 17.

[12] Chae, D., and Kim, N. Axisymmetric weak solutions of the 3-D Euler equations for incompressible fluid flows. *Nonlinear Anal. 29*, 12 (1997), 1393–1404.

[13] Cheskidov, A., Constantin, P., Friedlander, S., and Shvydkoy, R. Energy conservation and Onsager’s conjecture for the Euler equations. *Nonlinearity 21*, 6 (2008), 1233–1252.

[14] Cheskidov, A., Filho, M. C. L., Lopes, H. J. N., and Shvydkoy, R. Energy conservation in two-dimensional incompressible ideal fluids. *Comm. Math. Phys. 348*, 1 (2016), 129–143.

[15] Cheskidov, A., and Luo, X. Energy equality for the Navier-Stokes equations in weak-in-time Onsager spaces. Preprint arXiv:1802.05785, 2018.

[16] Constantin, P., E, W., and Titi, E. S. Onsager’s conjecture on the energy conservation for solutions of Euler’s equation. *Comm. Math. Phys. 165*, 1 (1994), 207–209.
[17] Crippa, G., Nobili, C., Seis, C., and Spirito, S. Eulerian and Lagrangian solutions to the continuity and Euler equations with $L^1$ vorticity. SIAM J. Math. Anal. 49, 5 (2017), 3973–3998.

[18] Crippa, G., and Spirito, S. Renormalized solutions of the 2D Euler equations. Comm. Math. Phys. 339, 1 (2015), 191–198.

[19] Danchin, R. Axisymmetric incompressible flows with bounded vorticity. Uspekhi Mat. Nauk 62, 3(375) (2007), 73–94.

[20] De Lellis, C., and Székelyhidi, J. L. The Euler equations as a differential inclusion. Ann. of Math. (2) 170, 3 (2009), 1417–1436.

[21] De Lellis, C., and Székelyhidi, Jr., L. Dissipative continuous Euler flows. Invent. Math. 193, 2 (2013), 377–407.

[22] Delort, J.-M. Existence de nappes de tourbillon en dimension deux. J. Amer. Math. Soc. 4, 3 (1991), 553–586.

[23] Delort, J.-M. Une remarque sur le problème des nappes de tourbillon axisymétriques sur $\mathbb{R}^3$. J. Funct. Anal. 108, 2 (1992), 274–295.

[24] DiPerna, R. J., and Lions, P.-L. Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. 98, 3 (1989), 511–547.

[25] Eyink, G. L. Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Phys. D 78, 3-4 (1994), 222–240.

[26] Feng, H., and Šverák, V. R. On the Cauchy problem for axi-symmetric vortex rings. Arch. Ration. Mech. Anal. 215, 1 (2015), 89–123.

[27] Figalli, A. Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. J. Funct. Anal. 254, 1 (2008), 109–153.

[28] Gallay, T., and Šverák, V. Remarks on the cauchy problem for the axisymmetric navier-stokes equations. Confluentes Mathematici 7, 2 (2015), 67–92.

[29] Hmidi, T., and Zerguine, M. Inviscid limit for axisymmetric Navier-Stokes system. Differential Integral Equations 22, 11-12 (2009), 1223–1246.

[30] Isett, P. A proof of Onsager’s conjecture. Ann. of Math. (2) 188, 3 (2018), 871–963.

[31] Jiu, Q., Wu, J., and Yang, W. Viscous approximation and weak solutions of the 3D axisymmetric Euler equations. Math. Methods Appl. Sci. 38, 3 (2015), 548–558.

[32] Jiu, Q., and Xin, Z. On strong convergence to 3-D axisymmetric vortex sheets. J. Differential Equations 223, 1 (2006), 33–50.
[33] JIU, Q. S., AND XIN, Z. P. Viscous approximations and decay rate of maximal vorticity function for 3-D axisymmetric Euler equations. *Acta Math. Sin. (Engl. Ser.)* 20, 3 (2004), 385–404.

[34] KOCH, H. *Non-Euclidean singular integrals and the porous medium equation.* Habilitation thesis, Universität Heidelberg, Germany, 1999.

[35] LE BRIS, C., AND LIONS, P.-L. Renormalized solutions of some transport equations with partially $W^{1,1}$ velocities and applications. *Ann. Mat. Pura Appl. (4) 183*, 1 (2004), 97–130.

[36] LERAY, J. Sur le mouvement d’un liquide visqueux emplissant l’espace. *Acta Math.* 63, 1 (1934), 193–248.

[37] LOPES FILHO, M. C., MAZZUCATO, A. L., AND NUSSENZVEIG LOPES, H. J. Weak solutions, renormalized solutions and enstrophy defects in 2D turbulence. *Arch. Ration. Mech. Anal.* 179, 3 (2006), 353–387.

[38] ONSAGER, L. Statistical hydrodynamics. *Nuovo Cimento (9) 6*, Supplemento, 2 (Convegno Internazionale di Meccanica Statistica) (1949), 279–287.

[39] SAINT RAYMOND, X. Remarks on axisymmetric solutions of the incompressible Euler system. *Comm. Partial Differential Equations* 19, 1-2 (1994), 321–334.

[40] SEIS, C. A quantitative theory for the continuity equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34, 7 (2017), 1837–1850.

[41] SEIS, C. Optimal stability estimates for continuity equations. *Proc. Roy. Soc. Edinburgh Sect. A* 148, 6 (2018), 1279–1296.

[42] SERRIN, J. The initial value problem for the Navier-Stokes equations. In *Nonlinear Problems (Proc. Sympos., Madison, Wis., 1962).* Univ. of Wisconsin Press, Madison, Wis., 1963, pp. 69–98.

[43] SHINBROT, M. The energy equation for the Navier-Stokes system. *SIAM J. Math. Anal.* 5 (1974), 948–954.

[44] STEIN, E. M. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[45] UKHOVSKII, M. R., AND IUDOVICH, V. I. Axially symmetric flows of ideal and viscous fluids filling the whole space. *J. Appl. Math. Mech.* 32 (1968), 52–61.

[46] VECCHI, I., AND WU, S. J. On $L^1$-vorticity for 2-D incompressible flow. *Manuscripta Math.* 78, 4 (1993), 403–412.

[47] WU, G. Inviscid limit for axisymmetric flows without swirl in a critical Besov space. *Z. Angew. Math. Phys.* 61, 1 (2010), 63–72.