Cauchy-Riemann inequalities on 2-spheres of $\mathbb{R}^7$

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Abstract: We prove that an integral Cauchy-Riemann inequality holds for any pair of smooth functions $(f, h)$ on the 2-sphere $S^2$, and equality holds iff $f$ and $h$ are related $\lambda_1$-eigenfunctions. We extend such inequality to 4-tuples of functions, only valid on the $L^2$-orthogonal complement of a suitable nonzero finite dimensional space of functions. As a consequence we prove that 2-spheres are not $\Omega$-stable surfaces with parallel mean curvature in $\mathbb{R}^7$ for the associative calibration $\Omega$.

1 Introduction

In [7] we extended to submanifolds with higher codimension the variational characterization of hypersurfaces of Riemannian manifolds with constant mean curvature $H$ discovered by Barbosa, do Carmo and Eschenburg [1][2]. Given an $m$-dimensional oriented immersed submanifold $\phi : M \rightarrow \bar{M}$ of an $(m+n)$-dimensional calibrated Riemannian manifold $(\bar{M}, \bar{g})$ with a semicalibration $\Omega$ of rank $m+1$ (see [6], in [7] we denominated it by precalibration), and assuming $\phi$ has calibrated extended tangent space on a domain $D$, that is, along $D$ the vector bundle $EM = \mathbb{R}v \oplus TM$ is $\Omega$-calibrated, where $v$ is a globally defined unit normal on $D$ such that $H \in \mathbb{R}v$, then we proved that $\phi$ has constant mean curvature on $D$ if and only if $\phi$ is a critical point of the area $A_D(\phi)$ for all variations $\tilde{\phi} : [0, \varepsilon] \times D \rightarrow \bar{M}$ of $\phi$, $\tilde{\phi}(t, p) = \phi_t(p)$, $\phi_0 = \phi$, that fixes the boundary of $D$ (in case this one exists) and preserves the $\Omega$-volume $V_D(t) = \int_{[0,t]\times D} \tilde{\phi}^* \Omega$. The later condition means that $V_D(t)$ is constant on $t$, that is $V_D(t) = V_D(0) = 0$. This turns out to be equivalent to $\phi$ to be a critical point of $J_D(t) = A_D(\phi_t) + mh_D V_D(t)$, for any variation fixing the boundary of $D$, where $h_D$ is the mean value of $\|H\|$ on $D$. The second variation of $J_D(t)$ was computed, obtaining $J''_D(0) = \int_D \bar{g}(\mathcal{J}_\Omega(W), W)dM =: I_\Omega(W, W)$, where

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\( W = \frac{d}{dt}_{|t=0} \phi \) is the vector variation field for variations that preserve the \( \Omega \)-volume, and

\[
\mathcal{J}_\Omega (W) = -\Delta W - \bar{R}(W) - \bar{B}(W) + m\|H\|C_\Omega (W).
\]

This second order differential operator depends only on the normal component of \( W \) and it is the usual Jacobi operator with an extra term \( m\|H\|C_\Omega (W) \), a \( L^2 \)-self-adjoint first order differential operator defined for \( W \in C_0^\infty (NM/D) \), such that

\[
\int_D \bar{g}(C_\Omega (W), W) = \int_D \left( \sum_i \Omega (W, e_1, \ldots, \nabla W_{e_i}(0), \ldots, e_m) + (\nabla W_{e_i})(W, e_1, \ldots, e_m) \right) dM,
\]

where \( e_1, \ldots, e_m \) defines a direct o.n. frame of \( TM \). We also called by the same name the operator \( \mathcal{J}'_{\Omega, D}(W) = \mathcal{J}_\Omega (W) - \Psi_{\Omega, D}(W) \nu \), where \( \Psi_{\Omega, D}(W) \) is the linear operator:

\[
\Psi_{\Omega, D}(W) = \frac{1}{|D|} \int_D \bar{g}(\mathcal{J}_\Omega (W), \nu) dM
\]

For simplicity we assumed \( \phi \) to have parallel mean curvature, and so \( \nu \) is a parallel unit normal. We can extend the definition of \( I_\Omega (W, W) \) to the space \( H^1_{0,T} (NM/D) \) given by the \( H^1 \)-completion of the vector space generated by the set of normal vector variations \( W \) of \( \Omega \)-volume preserving variations \( \phi \) fixing the boundary of \( D \), or equivalently, it is the subspace of \( H^1_0 (NM/D) \) of the normal sections that satisfy a zero mean value property

\[
\int_D \Omega (W, e_1, \ldots, e_m) dM = 0.
\]

Then we say that \( \phi \) is \( \Omega \)-stable on \( D \) if \( I_\Omega (W, W) \geq 0 \), for all \( W \in H^1_{0,T} (NM/D) \). The space \( H^1_{0,T} (NM/D) \) is just \( H^1_{0,T} (D) \oplus H^1_0 (F) \) where \( H^1_{0,T} (D) = H^1_0 (D) \cap L^2_D (D) \) with \( L^2_D (D) = \{ f \in L^2 (D) : \int_D f dM = 0 \} \), and \( F \) is the normal subbundle orthogonal complement of \( \nu \). A Morse index theorem can be stated for submanifolds with parallel mean curvature and calibrated extended tangent space (see Remark 4.3 of [7]). If \( \tilde{M} = \mathbb{R}^{m+n} \) and \( M \) is closed, and supposing that \( \bar{\nu} \Omega = 0 \) or \( C_\Omega = 0 \) (in fact it is sufficient to assume the vanishing of \( \bar{g}(C_\Omega (\nu), \nu) \), or equivalently of \( \bar{\nu} \nu \Omega (\nu, e_1, \ldots, e_m) \)), we proved in [7] (Theorem 4.2) that under the natural integral inequality condition \( \int_M S (2 + \|H\|) dM \leq 0 \), where \( h \) and \( S \) are the height functions \( h = \bar{g}(\phi, \nu) \) and \( S = \sum_{ij} \bar{g}(\phi, B^F (e_i, e_j)) B^V (e_i, e_j) \) (\( B^V \) and \( B^F \) stand for the \( \nu \)- and \( F \)-components of the second fundamental form \( B \), respectively), if \( M \) is \( \Omega \)-stable then \( \phi \) is pseudo-umbilical, and in case \( NM \) is a trivial bundle, then \( M \) must be a sphere. So it is a fundamental question to describe for which semicalibrations \( \Omega \) are spheres \( \Omega \)-stable. If \( n = 1 \) this is completely determined, for in [11] it is proved that stable closed immersed hypersurfaces of constant mean curvature are exactly the spheres. If \( n \geq 2 \) and \( C_\Omega = 0 \) then \( m \)-spheres \( \mathbb{S}^m \) of \( \Omega \)-calibrated vector subspaces are
If either \( f \) or \( h \) is constant, or \( f \) and \( h \) are zero, the above inequality gives the short integral Cauchy Riemann inequalities holding for any pair of functions \((f, h): S^2 \to \mathbb{R}^2\)

\[
2 \int_{S^2} \phi_i \langle J \nabla f, \nabla h \rangle dM \leq \sqrt{\int_{S^2} \| \nabla f \|^2 dM} \cdot \sqrt{\int_{S^2} \| \nabla h \|^2 dM} \quad \text{for } i = 1, 2, 3 \tag{4}
\]

as we can easily see from (3) by replacing \( f \) and \( h \) by \( tf \) and \( t^{-1} h \), respectively, where \( t^2 = \| \nabla h \|_{L^2} / \| \nabla f \|_{L^2} \). We state our main results in Theorems 1.1 and 1.2:

**Theorem 1.1.** For each \( i \in \{1, 2, 3\} \) the short integral Cauchy Riemann inequality (4) holds for any smooth maps \( f, h: S^2 \to \mathbb{R} \). Furthermore, equality holds in (4) if and only if either \( f \) or \( h \) is constant, or \( f = c_j \phi_j + c_k \phi_k + c \) and \( h = -c_k \phi_j + c_j \phi_k + c' \), where \( c_j, c_k, c, c' \) are constants, and \((i, j, k)\) is a positive permutation of \((1, 2, 3)\).

The eigenvalues for the closed Dirichlet problem on the unit 2-sphere \( S^2 \) constitute an increasing sequence \( 0 = \lambda_0 < \lambda_1 < \cdots < \lambda_l < \cdots \) converging to infinity. We denote by \( E_{\lambda} \) the eigenspace of dimension \( 2l + 1 \) corresponding to the eigenvalue \( \lambda_l = l(l + 1) \) and by \( E_{\lambda}^+ \) the union of all eigenspaces \( E_{\lambda} \) with \( \lambda \geq \lambda_l \).
Theorem 1.2. The long integral Cauchy Riemann inequality holds for all \( f_\alpha \in E_{\lambda_0} \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus E_{\lambda_3}^+ \). But there exist a 4-tuple of functions \( f_\alpha \in E_{\lambda_3} \) for which (3) is not satisfied. In particular \( S^2 \) is \( \Omega \)-unstable.

In Proposition 3.2 we give one more class of functions for which (3) is satisfied, as an immediate consequence of Theorem 1.1. The index is the dimension of the largest vector space of 4-tuples of functions for which inequality (3) does not hold. The exact index will be computed somewhere later. The search of directions of instability defined by 4-tuples requires long computations. Since the functions \( f_\alpha \) can be expressed as a \( H^1 \)-sum of spherical harmonics on \( S^2 \), we express \( f_\alpha \) in a non-orthonormal sum of monomial functions. We show that the use of eigenfunctions reduces the study of the Cauchy-Riemann inequalities to consider functions \( f_\alpha \) only in one or two different eigenspaces.

We prove this by observing first that \( \int_{S^2} \phi \langle J\nabla f, \nabla h \rangle \) is a skew-symmetric functional in the three variable functions \( (\phi, f, h) \), and derive a Weitzenböck-type formula that concludes that \( \langle J\nabla \phi, \nabla f \rangle \) maps a \( \lambda_i \)-eigenfunction \( f \) into a \( \lambda_i \)-eigenfunction. Using only algebraic methods we determine that (3) holds for functions in \( E_{\lambda_i} \) where \( l = 0, 1, 2 \) or \( l \geq 6 \), but for \( l = 3 \) we need to use Mathematica and Fortran programming to diagonalize a \( 40 \times 40 \) matrix, to obtain all stable and unstable directions. The cases \( l = 4 \), and \( l = 5 \) are considerably more complicate for it correspond to diagonalize a \( 60 \times 60 \) and a \( 80 \times 80 \) matrix respectively. We do not consider these two cases here.

Recall that \( H^1_0(\mathbb{R}^m) = H^1(\mathbb{R}^m) \). Using the stereographic projection \( \sigma : \mathbb{R}^2 \to S^2 \)

\[
\sigma(w) = \left( \frac{|w|^2 - 1}{|w|^2 + 1}, \frac{2w_1}{|w|^2 + 1}, \frac{2w_2}{|w|^2 + 1} \right),
\]

Theorem 1.1 is translated into next Corollary:

Corollary 1.1. If \( f, h \in H^1(\mathbb{R}^2) \), then for \( i = 1, 2, 3 \)

\[
2 \int_{\mathbb{R}^2} \sigma_i \langle J_0 \nabla^0 f, \nabla^0 h \rangle \, dw \leq \| \nabla^0 f \|_{L^2} \| \nabla^0 h \|_{L^2}
\]

where \( J_0 \) is the canonical complex structure of \( \mathbb{R}^2 \), and \( \nabla^0 \) is the gradient operator in \( \mathbb{R}^2 \). Furthermore, equality holds if and only if \( f \) or \( h \) vanish.

Note that \( \nabla^0 h = J_0 \nabla^0 f \) if and only if \( f + ih : \mathbb{R}^2 \to \mathbb{C} \) is an holomorphic map. In this case \( f \) and \( h \) do not lie in \( H^1(\mathbb{R}^2) \) unless they are zero functions. Furthermore non-constant holomorphic maps cannot be constant in any open sets, and in particular on a set where, for some \( i, \sigma_i < 1 - \delta \) with \( \delta > 0 \) small, and so the coefficient 2 in the above inequality ( or in (4)) is expectable. Moreover, since \( \sigma_i \) are not \( L^2 \)-functions, equality only holds if \( f \) or \( h \) vanish.
2 Preliminaries

Let \( \tilde{M} \) be an \((m+n)\)-dimensional Riemannian manifold with a semicalibration \( \Omega \) of rank \((m+1)\). This means that \( \Omega \) is an \((m+1)\)-form such that \(|\Omega(u_1, \ldots, u_{m+1})| \leq 1\) for any o.n. system of vectors \( u_i \). We consider \( \phi : M \to \tilde{M} \) an \( m \)-dimensional \((m \geq 2)\), oriented, immersed submanifold with nonzero parallel mean curvature \( H = \|H\|\nu \), and calibrated extended tangent space \( H \oplus TM \), that is \( \Omega(\nu, e_1, \ldots, e_{m+1}) = 1 \) holds for a d.o.n. frame \( e_i \) of \( TM \).

Given a smooth normal section \( W \) with compact support on a domain \( D \) of \( M \), then \( C_\Omega(W) \) is defined as the normal section such that for all \( W' \in C^0(NM/\nu) \) (cf. [7])

\[
g(C_\Omega(W), W') = \sum_i \Omega(W, e_1, \ldots, \nabla^-_i W, \ldots, e_m) \tag{5}
\]

\[
+ \frac{1}{2} \left( (\nabla^-_W \Omega)(W', e_1, \ldots, e_m) + (\nabla^-_{W'} \Omega)(W, e_1, \ldots, e_m) \right)
\]

\[
- \sum_i \frac{1}{2} \nabla^-_i \Omega(W, e_1, \ldots, W', \ldots, e_m) - \sum_{i \neq j} \frac{1}{2} \Omega(W, e_1, \ldots, B(e_i, e_j)(W'), \ldots, W'_{(i)}, \ldots, e_m)
\]

where \((i)\) means the \( i \)-position. A simple computation shows that

\[
g(C_\Omega(W'), W) - g(C_\Omega(W), W') = div_M(X_{WW'}) = -\delta(\xi(W, W')) \tag{6}
\]

where \( X_{WW'} \in C^\infty(TM) \) is a vector field on \( M \) and \( \xi : \wedge^2 NM \to T^*M \) a tensor defined by

\[
g(X_{WW'}, e_i) = \Omega(W, e_1, \ldots, W_{(i)}, \ldots, e_m) = \xi(W, W')(e_i)
\]

\[
\xi(W, W')(u) = \Omega(W, W', \ast u),
\]

where \( \ast \) is the star operator on \( M \). Eq.(6) is derived by taking into consideration that \( M \) has calibrated extended space, and so

\[
\sum_i \Omega((\nabla^-_i W)_\nu, e_1, \ldots, W'_{(i)}, \ldots, e_m) = mH\|\tilde{g}(W, \nu)\tilde{g}(W', \nu)
\]

which is symmetric on \( W, W' \). Thus, \( C_\Omega \) is \( L^2 \)-self-adjoint and (2) holds. For \( n \geq 2 \), we recall Lemma 4.4 of [7]. We will give here a clearer proof.

**Lemma 2.1.** If \( n \geq 2 \), \( C_\Omega \) vanish iff (7) and (8) holds:

\[
\xi \text{ vanish} \tag{7}
\]

\[
\nabla^-_W \Omega(W', e_1, \ldots, e_m) = -\nabla^-_{W'} \Omega(W', e_1, \ldots, e_m) \tag{8}
\]

If \( n = 2 \) then (7) holds.
We get\(\nabla E\)_denotes the inclusion map, and \(r = \) zero normal covariant derivative we see the above equality (8) holds. Then we consider\(\tilde{W} = W + fW\) where \(\nabla \perp W = \nabla \perp W' = 0\) at a given point \(p\) and \(f\) is any function. From \(g(C_\Omega(W), \tilde{W}) = 0\) and (5) we conclude that \(\xi(W,W')(\nabla f) = 0\) at \(p\). Since \(f\) is arbitrary we get \(\xi = 0\).

Proof. If \(C_\Omega\) vanish, then both \(\tilde{g}\)-anti-self-adjoint and \(\tilde{g}\)-self-adjoint parts of \(g(C_\Omega(W), W')\) vanish, what means \(\text{div}(X_W W') = 0\), and by taking normal sections that at a point have zero normal covariant derivative we see the above equality (8) holds. Then we consider\(\tilde{W} = W + fW\) where \(\nabla \perp W = \nabla \perp W' = 0\) at a given point \(p\) and \(f\) is any function. From \(g(C_\Omega(\tilde{W}), \tilde{W}) = 0\) and (5) we conclude that \(\xi(W,W')(\nabla f) = 0\) at \(p\). Since \(f\) is arbitrary we get \(\xi = 0\). □

From now on we are assuming \(M\) is a \(m\)-dimensional Euclidean sphere \(S^m_r\) of radius \(r\) of a \(\Omega\)-calibrated Euclidean subspace \(\mathbb{R}^{m+1}_r\) of \(\mathbb{R}^{m+1}\), and \(\phi : S^m_r \to \mathbb{R}^{m+1} \subset \mathbb{R}^{m+1}\) denotes the inclusion map, and \(\epsilon_i, i = 1, \ldots, m + 1\), is the canonical basis of \(\mathbb{R}^{m+1}\).

We recall that the eigenvalues of \(\mathbb{S}^m_r\) for the closed Dirichlet problem are given by \(\lambda_i(r) = \frac{l(l+m-1)}{r^2}\); with \(l = 0, 1, \ldots\), and the \(\lambda_i(r)\)-eigenfunctions are of the form \(f_r(x) = f(\frac{x}{r})\) where \(f\) is a \(\lambda_i(1)\)-eigenfunction of the unit sphere \(S^m\). We omit the parameter \(r\) if \(r = 1\). Furthermore, if \(f \in E_{\lambda_i(r)}\), \(h \in E_{\lambda_i(r)}\) then

\[
\int_{\mathbb{S}^m_r} fh \, dM = 0 \text{ if } l \neq s \quad \text{and} \quad \int_{\mathbb{S}^m_r} \langle \nabla f, \nabla h \rangle \, dM = -\delta_s \lambda_i(r) \int_{\mathbb{S}^m_r} fh \, dM.
\]

There exists a \(L^2\)-orthonormal basis \(\psi_{l,\sigma}\) of \(L^2(\mathbb{S}^m_r)\) of eigenfunctions \((1 \leq \sigma \leq m_i\), where \(m_i\) denotes the multiplicity of \(\lambda_i(r)\)). The Rayleigh characterization of \(\lambda_i(r)\) is given by

\[
\lambda_i(r) = \inf_{f \in E^+_{\lambda_i(r)}} \frac{\int_{\mathbb{S}^m_r} \|\nabla f\|^2 \, dM}{\int_{\mathbb{S}^m_r} f^2 \, dM},
\]

where \(E^+_{\lambda_i(r)}\) is the \(L^2\)-orthogonal complement of the sum of the eigenspaces \(E_{\lambda_i(r)}\), \(i = 1, \ldots, l - 1\). Equality holds for \(f \in E_{\lambda_i(r)}\). Each eigenspace \(E_{\lambda_i(r)}\) is exactly composed by the restriction to \(\mathbb{S}^m_r\) of the harmonic homogeneous polynomials functions of degree \(l\) of \(\mathbb{R}^{m+1}\), and it has dimension \(m_i = (m+l)-(m+l-1)\). Thus, each eigenfunction \(\psi \in E_{\lambda_i(r)}\), is of the form \(\psi = \sum_{|a|=l} \mu_a \phi^a\), where \(\mu_a\) are some scalars and \(a = (a_1, \ldots, a_{m+1})\) denotes a multi-index of length \(|a| = a_1 + \ldots + a_{m+1} = l\) and

\[
\phi^a = \phi^{a_1}_1 \cdots \phi^{a_{m+1}}_{m+1}.
\]

From \(\nabla \phi_i = \epsilon_i^\top\) and that \(\sum_i \phi_i^2 = r^2\) we see that

\[
\left\{
\begin{array}{l}
\langle \nabla \phi_i, \nabla \phi_j \rangle = \delta_{ij} - \frac{1}{r^2} \phi_i \phi_j \\
\|\nabla \phi_i\|^2 = 1 - \frac{1}{r^2} \phi_i^2 \\
\int_{\mathbb{S}^m_r} \phi_i^2 \, dM = \frac{r^2}{m+1}|\mathbb{S}_r^m| \\
\int_{\mathbb{S}^m_r} \|\nabla \phi_i\|^2 \, dM = \lambda_1(r) \int_{\mathbb{S}^m_r} \phi_i^2 \, dM = \frac{m}{m+1}|\mathbb{S}_r^m|.
\end{array}
\right.
\]
We also denote by $\int_{S^m_r} \phi^2 dM$ any of the integrals $\int_{S^m_r} \phi_i^2 dM$, $i = 1, \ldots, m + 1$.

Note that $\lambda_1 (S^m_r) = m \| H \|^2$, $\| H \| = \frac{1}{2}$, and $|S^m_r| = r^m |S^m|$. Any smooth function $f$ on $S^m_r$ can be written as a $L^2$-convergent sum $f = \sum_l \Psi_l$, where $\Psi_l = \sum_\sigma A_{\sigma} \psi_{l, \sigma}$ is an $\lambda_l(r)$-eigenfunction and $A_{\sigma}$ are constants. This sum is in fact $H^1$-convergent to $f$ (see a proof of this in theorem 25.2 of [5], that formally holds for any compact Riemannian manifold as well). If $l = 1$ then $\phi^1, \ldots, \phi^{m+1}$ is up to a homothetic factor an $L^2$-o.n. basis of $E_{\lambda_1(r)}$. If $\tilde{f}$ is a homogeneous polynomial function of degree $l$ then, for $X, Y \in T_S^m$

$$\text{Hess} \tilde{f}(X, Y) = \text{Hess} f(X, Y) - l f(x) \langle X, Y \rangle,$$

where $\tilde{f}$ restricted to $S^m$. If $r = 1$, the Ricci tensor of $S^m$ is $(m - 1) \langle \cdot, \cdot \rangle$, and using the Reilly’s formula

$$\text{Ricci}(\nabla f, \nabla f) = (\Delta f)^2 + \frac{1}{2} \Delta (\| \nabla f \|^2) - \text{div}(\Delta f \nabla f) - \| \text{Hess} f \|^2$$

we obtain for $f \in E_{\lambda_l}$

$$\lambda_l (\lambda_l - (m - 1)) = \frac{\int_{S^m} \| \text{Hess} f \|^2 dM}{\int_{S^m} f^2 dM}$$

$$\frac{(m-1)}{m} \lambda_l (\lambda_l - m) = \frac{\int_{S^m} \| \text{Hess} f \|^2 dM}{\int_{S^m} f^2 dM} \neq 0 \quad \text{if } l \geq 2.$$

In particular $\text{Hess} f$ is a multiple of the metric if and only if $l = 1$.

The $\Omega$-stability condition for $S^m_r$ is given by the inequality $I_\Omega (f \nu + W, f \nu + W) \geq 0$ for any smooth section $W$ of $F$ and any smooth function $f \in L^2_r (S^m_r)$, where $I_\Omega$ is given by eq. (16) in [2],

$$I_\Omega (f \nu + W, f \nu + W) = I(f, f) + m \| H \| \int_{S^m_r} f^2 \tilde{g}(C_\Omega (\nu), \nu) dM$$

$$+ \int_{S^m_r} \left( \| \nabla^1 W \|^2 + m \| H \| \tilde{g}(C_\Omega (W), W) \right) dM$$

where

$$I(f, f) = \int_{S^m_r} \| \nabla f \|^2 - m \| H \|^2 \int_{S^m_r} f^2 dM \geq 0$$

and $\tilde{g}(C_\Omega (\nu), \nu) = \tilde{\nabla}_\nu \Omega (\nu, e_1, \ldots, e_m)$. Then if $C_\Omega \neq 0$, and in particular if (8) does not hold we easily have an instability factor for $S^m_r$. For $f \in E_{\lambda_1(r)}$, we have $I(f, f) = 0$ and $I_\Omega (f \nu, f \nu) \leq r \lambda_1(r) b \int_{S^m_r} f^2 dM$, where

$$b = \sup_{S^m_r} \tilde{\nabla}_\nu \Omega (\nu, e_1, \ldots, e_m).$$
We also fix a global parallel basis $W_\alpha$ of $F = \mathbb{R}^{n-1}$, where $\alpha = m+2, \ldots, m+n$. Thus, by (2)
\[
\int_{S^m_r} \bar{g}(C_\Omega(W_\alpha), W_\alpha) dM = \int_{S^m_r} \bar{\nabla}_{W_\alpha} \Omega(W_\alpha, e_1, \ldots, e_m) dM.
\]

Hence, we have the following conclusion:

**Proposition 2.1.** Suppose that $S^m_r$ lies on a $\Omega$-calibrated vector subspace $\mathbb{R}^{m+1}$ of $\mathbb{R}^{m+n}$ where $\Omega$ is a semicalibration of rank $m+1$. If $b < 0$ or if for some $\alpha \geq m+2$ we have $\int_{S^m_r} \bar{\nabla}_{W_\alpha} \Omega(W_\alpha, e_1, \ldots, e_m) dM < 0$, then $S^m_r$ is $\Omega$-unstable.

We now look for conditions for $\Omega$-stability to hold on spheres. We define a $(m-1)$-form on $\mathbb{R}^{m+n}$ by
\[
\bar{\xi}_{\alpha\beta} = \Omega(W_\alpha, W_\beta, \ldots).
\]
Then
\[
\xi(W_\alpha, W_\beta) = *\phi^* \bar{\xi}_{\alpha\beta} \in C^\infty(T^*M).
\]
These forms are co-closed if $\Omega$ is parallel, but if $\Omega$ is not parallel, the stability condition implies co-closeness of $\xi(W_\alpha, W_\beta)$ as we will recall in next theorem.

**Theorem 2.1** ([7]). Let us suppose $M = S^m_r$ is a $m$-sphere of radius $r$ of a $\Omega$-calibrated vector subspace $\mathbb{R}^{m+1}$ and that $\Omega$ is a semicalibration such that (8) holds, that is $\bar{\nabla}_W \Omega(W, e_1, \ldots, e_m) = 0$ for all $W \in NM$.

Then $M$ is $\Omega$-stable if and only if the 1-forms $\xi(W_\alpha, W_\beta)$ are co-exact, that is
\[
\xi(W_\alpha, W_\beta) = \delta \omega_{\alpha\beta}
\]
for some 2-forms $\omega_{\alpha\beta}$ on $M$ and (10), or equivalently, (11) holds $\forall f_\alpha \in C^\infty(M), \alpha = m+2, \ldots, m+n:

\[
\sum_{\alpha < \beta} -2m\|H\| \int_{S^m_r} f_\alpha \xi(W_\alpha, W_\beta)(\nabla f_\beta) \leq \sum_{\alpha} \int_M \|\nabla f_\alpha\|^2 dM, \tag{10}
\]

\[
\sum_{\alpha < \beta} -2m\|H\| \int_{S^m_r} \omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta) \leq \sum_{\alpha} \int_M \|\nabla f_\alpha\|^2 dM. \tag{11}
\]

The first inequality (10) is a direct consequence of the stability condition, while (11) is proved in Proposition 4.5 of [7] by using the Hodge theory of spheres and that
\[
\int_{S^m_r} \xi(W_\alpha, W_\beta)(f_\alpha \nabla f_\beta) dM = \int_{S^m_r} \langle \xi(W_\alpha, W_\beta), f_\alpha d f_\beta\rangle dM = \int_{S^m_r} \omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta) dM.
\]
Corollary 2.1. An $m$-sphere of an $\Omega$-calibrated Euclidean subspace $\mathbb{R}^{m+1}$ of $\mathbb{R}^{m+n}$ for which $C_{\Omega} = 0$ is $\Omega$-stable. This is the case when $n = 2$ and $\Omega$ parallel.

The condition $C_{\Omega} = 0$ is a very restrictive condition, and does not hold for most calibrations coming from special holonomy, since (7) does not hold. But the operator $C_{\Omega}$ vanish when $\Omega$ is a semicalibration defined by a Riemannian fibration of $\mathbb{R}^{m+n}$ with some $(m + 1)$-dimensional totally geodesic fibre where $\mathbb{S}_r^m$ lies [7].

Inequality (11) can be seen as the long integral Cauchy-Riemann $\Omega$-inequality for $(n - 1)$-tuples of functions on $\mathbb{S}_r^m$. If stability holds, then for each $\alpha < \beta$ fixed, and taking $f_r = 0$ for $\gamma \neq \alpha, \beta$, the short integral Cauchy-Riemann $\Omega$-inequalities hold for any pairs of functions $f, h$ on $\mathbb{S}_r^m$:

$$m\|H\| \left| \int_{\mathbb{S}_r^m} \omega_{\alpha\beta}^r(\nabla f, \nabla h) \, dM \right| \leq \|\nabla f\|_{L^2} \|\nabla h\|_{L^2} \quad \forall \alpha < \beta. \quad (12)$$

On the other hand, inequality (10) gives us a tool to determine if a sphere is $\Omega$-stable applying the Rayleigh characterization of the spectrum of $\mathbb{S}_r^m$. Let $\Theta(r) = \sup_{\alpha < \beta} \Theta_{\alpha\beta}(r)$ where $\Theta_{\alpha\beta}(r) = \sup_{\mathbb{S}_r^m} \|\xi(W_\alpha, W_\beta)\| \leq 1$. For $f_\alpha \in E_{\lambda_i(r)}^+$, by Schwartz inequality,

$$\left| \int_{\mathbb{S}_r^m} f_\alpha \xi(W_\alpha, W_\beta)(\nabla f_\beta) \, dM \right| \leq \Theta_{\alpha\beta}(r) \frac{1}{\sqrt{\lambda_i(r)}} \|\nabla f_\alpha\|_{L^2} \|\nabla f_\beta\|_{L^2}. \quad (13)$$

Using the inequality $\sum_{m+2 \leq \alpha < \beta \leq m+n} 2a_\alpha a_\beta \leq (n - 2)(a_{m+2}^2 + \ldots + a_{m+n}^2)$, for any real numbers $a_\alpha$, and $\lambda_i(r) = \frac{\lambda_i}{r^2}$ we get next Proposition:

**Proposition 2.2.** Assuming $f_\alpha \in E_{\lambda_i(r)}^+$ for all $\alpha$ then

$$- \sum_{\alpha < \beta} 2m\|H\| \int_{\mathbb{S}_r^m} f_\alpha \xi(W_\alpha, W_\beta)(\nabla f_\beta) \, dM \leq \frac{m(n - 2)}{\sqrt{\lambda_i}} \Theta(r) \sum_{\alpha} \int_{\mathbb{S}_r^m} \|\nabla f_\alpha\|^2 \, dM.$$

Consequently, $\mathbb{S}_r^m$ is $\Omega$-stable in $\mathbb{R}^{m+n}$ if $n = 2$, or if $n \geq 3$ and $\Theta(r) \leq \frac{1}{\sqrt{m(n-2)}}$.

**Corollary 2.2.** Supposing that $\xi(W_\alpha, W_\beta) = \delta \omega_{\alpha\beta}$, and $f_\alpha \in E_{\lambda_i(r)}^+$ for all $\alpha$, then

$$\sum_{\alpha < \beta} -2m\|H\| \int_{\mathbb{S}_r^m} \omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta) \, dM \leq \frac{m(n - 2)}{\sqrt{\lambda_i}} \Theta(r) \sum_{\alpha} \int_{\mathbb{S}_r^m} \|\nabla f_\alpha\|^2 \, dM.$$

Hence, if $l$ is sufficiently large such that $l(l + m - 1) \geq m^2(n - 2)^2$, the long Cauchy Riemann inequality (11) holds for functions $f_\alpha \in E_{\lambda_i(r)}^+$. 9
This estimate is in general not sharp, because it ignores the signs that $\omega_{\alpha\beta}$ can take. We also remark that for $x \in \mathbb{S}^m$, since $T_x\mathbb{S}^m = T_{rx}\mathbb{S}^m$, if $\xi(x) = \xi(rx)$ then the same holds for $\Theta_{\alpha\beta}$ and $\Theta_{\alpha\beta}(r) = \Theta_{\alpha\beta}$. In this case $\mathbb{S}^m$ is $\Omega$-stable if and only if $\mathbb{S}^m$ is so.

If $\xi(W_\alpha, W_\beta) = \delta\omega_{\alpha\beta}$ then

$$\int_{\mathbb{S}^m_{\alpha\beta}}\omega_{\alpha\beta}(\nabla f_\alpha, \nabla f_\beta)dM = \frac{1}{2}\int_{\mathbb{S}^m_{\alpha\beta}}(f_\alpha\xi(W_\alpha, W_\beta)(\nabla f_\beta) - f_\beta\xi(W_\alpha, W_\beta)(\nabla f_\alpha))dM.$$  

Applying inequality (13) to this expression we immediately deduce that:

**Proposition 2.3.** If we fix $\alpha < \beta$ and $\xi(W_\alpha, W_\beta) = \delta\omega_{\alpha\beta}$, then for any functions $f \in E^{+}_{\lambda_i}$ and $h \in E^{+}_{\lambda_j}$ we have

$$2m\|H\|\int_{\mathbb{S}^m_{\alpha\beta}}\omega_{\alpha\beta}(\nabla f, \nabla h)dM \leq m\Theta_{\alpha\beta}(r)\left(\frac{1}{\sqrt{\lambda_i(r)}} + \frac{1}{\sqrt{\lambda_j(r)}}\right)\|\nabla f\|_{L^2}\|\nabla h\|_{L^2}.$$  

## 3 The 2-sphere of $\mathbb{R}^7$

In this section we consider the unit 2-sphere $\mathbb{S}^2$ of an associative Euclidean 3-dimensional subspace $\mathbb{R}^3$ of $\mathbb{R}^7$, that is, we may assume $\mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$ is $\Omega$-calibrated by the associative calibration $\Omega$ defined in the introduction. As we have pointed out in Remark 4.4 [7], taking $W_\alpha = e_\alpha$ for $\alpha = 4, 5, 6, 7$, then

$$\check{\xi}_{45} = \check{\xi}_{67} = e_1^* = dx^1 \quad \check{\xi}_{54} = -\check{\xi}_{57} = e_2^* = dx^2 \quad \check{\xi}_{47} = \check{\xi}_{56} = -e_3^* = -dx^3.$$  

Since $\delta\omega_{\alpha\beta} = \xi(W_\alpha, W_\beta) = \ast\phi^*\check{\xi}_{\alpha\beta}$, and $\omega_{\alpha\beta} = \rho_{\alpha\beta}Vol_{\mathbb{S}^2}$ we conclude that

$$\rho_{45} = \rho_{56} = -\phi \quad \rho_{46} = -\rho_{57} = -\phi \quad \rho_{47} = \rho_{56} = \phi_3$$

(there is a misprint in [7], a wrong sign for $\check{\xi}_{56}$). Note that $Vol_{\mathbb{S}^2}(X, Y) = \langle JX, Y \rangle$. Then (3) holds iff (11) holds, and (4) holds iff (12) holds. We first prove Theorem 1.1. We will need some lemmas:

**Lemma 3.1.** On a Kähler manifold $(M, J, g)$ of real dimension $2k$, for any functions $f, h \in C^\infty(M)$ we have

$$g(J\nabla f, \nabla h) = \text{div}(hJ\nabla f) = \frac{1}{2}\text{div}(hJ\nabla f - fJ\nabla h).$$

Furthermore, if $M$ is a closed manifold, then the operator

$$\eta(\phi, f, h) := \int_M \phi g(J\nabla f, \nabla h)dM$$

is skew-symmetric in the three variables $\phi, f, h \in C^\infty(M)$.  

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Proof. Let $e_i, i = 1, \ldots, 2k$ be a local o.n. frame of $TM$ with $e_{k+i} = Je_i, i = 1, \ldots, k$. Then
\[
\text{div}(hJ\nabla f) = \sum_{1 \leq i \leq 2k} g(\nabla_{e_i}(hJ\nabla f), e_i) = \sum_{1 \leq i \leq 2k} dh(e_i)g(J\nabla f, e_i) - h\text{Hess}f(Je_i, e_i) \\
= g(J\nabla f, \nabla h) - \sum_{1 \leq i \leq k} \left( h\text{Hess}f(Je_i, e_i) + h\text{Hess}f(JJe_i, Je_i) \right) = g(J\nabla f, \nabla h)
\]
where we used the fact that $\text{Hess}f$ is symmetric. The second equality follows immediately. Using the equality
\[
\text{div}(\phi hJ\nabla f) = \phi \text{div}(hJ\nabla f) + g(\nabla \phi, hJ\nabla f) = \phi \text{div}(hJ\nabla f) + h\text{div}(\phi J\nabla f)
\]
and applying Stokes we see that $\eta$ is skew-symmetric. \qed

Next lemma is a Weitzenböck type formula:

Lemma 3.2. On a Kähler manifold $(M, J, g)$ of real dimension $2k$ we have for any functions $f, h$ and o.n. frame $e_i, i = 1, \ldots, 2k$
\[
\Delta (g(J\nabla f, \nabla h)) = g(J\nabla \Delta f, \nabla h) + g(J\nabla f, \nabla \Delta h) - \text{Ricci}(\nabla f, J\nabla h) + \text{Ricci}(\nabla h, J\nabla f) \\
- \sum_{1 \leq i, j \leq 2k} 2\text{Hess} f(e_i, Je_j)\text{Hess} h(e_i, e_j).
\]

Proof. We may assume at a point $p$, $\nabla e_i(p) = 0$. Differentiating $d(g(J\nabla f, \nabla h))(e_i) = g(J\nabla f, \nabla h) + g(J\nabla f, \nabla_e \nabla h)$ with respect to $e_i$ we have at the point $p$
\[
\Delta (g(J\nabla f, \nabla h)) = \\
= \sum_i g(J\nabla e_i, \nabla f, \nabla h) + 2g(J\nabla e_i, \nabla e_i, \nabla h) + g(J\nabla f, \nabla e_i \nabla e_i \nabla h) \\
= \sum_{ij} \nabla^2_{e_i, e_j} df(e_j) dh(e_j) - 2\text{Hess} f(e_i, Je_j)\text{Hess} h(e_i, e_j) - df(\nabla^2_{e_i, e_j} dh(e_j)
\]
where $\nabla^2_{X,Y}df = \nabla_X \nabla_Y df - \nabla_{\nabla_X Y} df$, for any vector fields $X, Y, Z$. Here we use the curvature sign $R(X, Y) = -\nabla_X \nabla_Y + \nabla_Y \nabla_X + \nabla_{[X,Y]}$. Then we have (see e.g. [3] p.1234)
\[
\nabla^2_{X,Y}df(Z) = \nabla^2_{X,Z}df(Y) = \nabla_{\nabla^2_{X,Y}} df(Z) + df(R^M(X,Y)Z).
\]

Thus
\[
\sum_i \nabla^2_{e_i, e_i} df(e_j) = \sum_i \nabla^2_{e_j, e_i} df(e_i) + df(R^M(e_i, e_j)e_i) = \nabla e_j(\Delta f) + df(\text{Ricci}^M(e_j)).
\]

Replacing this equality in the above equation, and a similar one w.r.t. $h$ we obtain the formula of the lemma. \qed

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Lemma 3.3. (1) If $f \in E_{\lambda_i}$ and $h \in E_{\lambda_r}$ then $\langle J\nabla f, \nabla h \rangle \in E_{\lambda_i}$.

(2) $\langle J\nabla \phi_i, \nabla \phi_j \rangle = \phi_k$ for $(i, j, k)$ is a positive permutation of $(1, 2, 3)$.

(3) If $f \in E_{\lambda_i}$ and $h \in E_{\lambda_r}$, then

$$\Delta \langle J\nabla f, \nabla h \rangle = - (\lambda_i + \lambda_r - 2) \langle J\nabla f, \nabla h \rangle - \sum_{ij} 2 \text{Hess} f(e_i, Je_j) \text{Hess} (e_i, e_j).$$

Proof. (1) Since $\phi_1, \phi_2, \phi_3$ is a basis of $\lambda_1$-eigenfunctions, then $\text{Hess} f = - f \cdot g$. Moreover $\text{Ricci}^M = g$. Taking $e_i$ a o.n. basis that diagonalizes $\text{Hess} h$ we conclude from Lemma 3.2 that $\langle J\nabla f, \nabla h \rangle \in E_{\lambda_i}$. (2) If we consider spherical coordinates $\phi : [0, \pi] \times [0, 2\pi] \to S^2$, $\phi(\varphi, \theta) = (\sin \varphi \cos \theta, \sin \varphi \sin \theta, \cos \varphi)$, then $X = \frac{\partial}{\partial \varphi}$, $Y = \frac{1}{\sin \varphi} \frac{\partial}{\partial \theta}$ defines a d.o.n. frame of $T S^2$ and so $JX = Y$. Furthermore,

$$\nabla \phi_1 = \cos \varphi \cos \theta X - \sin \theta Y, \quad \nabla \phi_2 = \cos \varphi \sin \theta X + \cos \theta Y, \quad \nabla \phi_3 = - \sin \varphi X$$

Then we see that (2) holds. (3) is an immediate consequence of Lemma 3.2 and generalizes (1).

\[\square\]

Proposition 3.1. (1) If $f \in E_{\lambda_i}$ and $h \in E_{\lambda_r}$ with $l \neq r$ then \( \int_{S^2} \phi_l \langle J\nabla f, \nabla h \rangle dM = 0 \).

(2) \[2 \int_{S^2} \phi_k \langle J\nabla \phi_i, \nabla \phi_j \rangle = \varepsilon_{ijl} \| \nabla \phi_i \|_{L^2} \| \nabla \phi_j \|_{L^2}, \]

where $\varepsilon_{ijk} = +1, -1$, according to the signature of $(i, j, k)$ as a permutation of $(1, 2, 3)$, or zero if repeated indexes appear.

Proof. Using Lemma 3.1

$$\int_{S^2} \phi_i \langle J\nabla f, \nabla h \rangle = - \int_{S^2} h \langle J\nabla \phi_i, \nabla f \rangle dM.$$ 

By Lemma 3.3 (1) $\langle J\nabla \phi_i, \nabla f \rangle \in E_{\lambda_i}$, and so this is $L^2$-orthogonal to $h \in E_{\lambda_r}$.

(2) is an immediate consequence of Lemma 3.3 (2) and (9).

\[\square\]

Proof of Theorem 1.1.

We may assume $i = 1$. Inequality (4) is equivalent to prove that

$$4 \int_{S^2} \phi_1 \langle J\nabla f, \nabla h \rangle dM \leq \| \nabla f \|^2_{L^2} + \| \nabla h \|^2_{L^2},$$

holds $\forall f, h \in C^\infty(S^2)$. We write $f = f_0 + f_1 + f'$ and $h = h_0 + h_1 + h'$ where $f_0, h_0 \in E_{\lambda_0}$ are constants, $f_1, h_1 \in E_{\lambda_1}$, and $f', h' \in E_{\lambda_2}^+$. Then applying Proposition 3.1 (1)

$$4 \int_{S^2} \phi_1 \langle J\nabla f, \nabla h \rangle dM = 4 \int_{S^2} \phi_1 \langle J\nabla f_1, \nabla h_1 \rangle dM + 4 \int_{S^2} \phi_1 \langle J\nabla f', \nabla h' \rangle dM$$

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From Proposition 2.3 we have
\[ 4 \int_{S^2} \phi_1 \langle J \nabla f', \nabla h' \rangle dM \leq \frac{4}{\sqrt{6}} \| \nabla f' \|_{L^2} \| \nabla h' \|_{L^2} \leq \frac{2}{\sqrt{6}} (\| \nabla f' \|_{L^2} + \| \nabla h' \|_{L^2}). \]

Since \( f_1 = \sum \mu_i \phi_i, \ h_1 = \sum \sigma_j \phi_j \) where \( \mu_i, \sigma_j \) are constants, applying Proposition 3.1(2) and (9) we have
\[ 4 \int_{S^2} \phi_1 \langle J \nabla f_1, \nabla h_1 \rangle dM = 4 (\mu_2 \sigma_3 - \mu_3 \sigma_2) \int_{S^2} \phi_1^2 dM \leq (\sum_i \mu_i^2 + \sigma_i^2) \int_{S^2} \| \nabla \phi_1 \|^2 dM = \| \nabla f_1 \|_{L^2}^2 + \| \nabla h_1 \|_{L^2}^2. \]

As \( \frac{2}{\sqrt{6}} < 1 \) we conclude that (4) holds, and equality is achieved if and only if \( f' = h' = 0 \) and \( \mu_1 = \sigma_1 = 0, \mu_2 = \sigma_3, \mu_3 = -\sigma_2. \)

Next we prove Theorem 1.2. in several steps, as a consequence of Proposition 3.3, and of Lemmas 3.7, 3.8, and 3.10. Indeed, as a consequence of Proposition 3.1(1), we only need to droop our attention on 4-tuples \( (f_4, f_5, f_6, f_7) \) that have at least two components in the same eigenspace. The case we have two pairs of functions in two different eigenspaces lies in the case of Theorem 1.1 as we can easily verify. Thus we have

**Proposition 3.2.** If two elements of \( \{f_4, f_5, f_6, f_7\} \) are in \( E_{\lambda_r} \) and the other two in \( E_{\lambda_l} \) with \( r \neq l \), then the long Cauchy-Riemann inequality (3) holds.

If only three of the functions \( f_\alpha \) are in the same eigenspace, the terms involving the forth function vanish, and so we may assume the later to be zero, that is we are in the case that all functions are in the same eigenspace. This is the case we are now considering.

We denote by \( e_i, i = 1, 2, 3 \) the canonical basis of \( \mathbb{R}^3 \), and so a multi-index of non-negative integers is of the form \( a = (a_1, a_2, a_3) = \sum_i a_i e_i \). Next we recall the well known formula (see for instance [3] appendix)

**Lemma 3.4.** If \( P : \mathbb{R}^3 \to \mathbb{R} \) is an homogeneous polynomial function of degree \( l \) then
\[ \int_{S^2} P(x) dM = \frac{1}{\lambda_l} \int_{S^2} \Delta^0 P(x) dM. \]

In particular for \( |a| = a_1 + a_2 + a_3 = l \)
\[ \int_{S^2} \phi^a dM = \sum_{1 \leq i \leq 3} \frac{a_i (a_i - 1)}{l(l + 1)} \int_{S^2} \phi^{a - 2e_i} dM, \]
where the terms with \( a_i < 2 \) are considered to vanish.
Let us denote by \((O)\) and \((E)\) meaning odd and even respectively. We also say that \(a\) is \((O, E, E)\) meaning that \(a_1\) is odd and \(a_2\) and \(a_3\) are even, and so on. Since \(\int_{S^2} \phi \, dM = 0\), by induction we conclude from the previous lemma that
\[
\int_{S^2} \phi^a \, dM = 0 \quad \text{iff} \quad \exists i : a_i \text{ is } (O).
\]

Now using (9) and Lemma 3.3(2) we obtain the following two lemmas, respectively:

**Lemma 3.5.** If \(|a| = |b| = l\) then
\[
\int_{S^2} (\nabla \phi^a, \nabla \phi^b) \, dM = -l^2 \int_{S^2} \phi^{a+b} \, dM + \sum_i a_i b_i \int_{S^2} \phi^{a+b-2i} \, dM
\]
\[
= \sum_i \frac{l((a_i + b_i) - (a_i - b_i)^2) + 2a_i b_i}{2(2l + 1)} \int_{S^2} \phi^{a+b-2i} \, dM.
\]
If this does not vanish then \(a + b\) is \((E, E, E)\).

**Lemma 3.6.**

1. \[
\int_{S^2} \phi_1 \langle J \nabla \phi^a, \nabla \phi^b \rangle \, dM = (a_1 b_2 - a_2 b_1) \int_{S^2} \phi^{a+b+\varepsilon_2+\varepsilon_3} \, dM + (-a_1 b_3 + a_3 b_1) \int_{S^2} \phi^{a+b+\varepsilon_2-\varepsilon_3} \, dM + (a_2 b_3 - a_3 b_2) \int_{S^2} \phi^{a+b+2\varepsilon_1-\varepsilon_2-\varepsilon_3} \, dM.
\]
If this does not vanish then \(a + b\) is \((E, O, O)\).

2. \[
\int_{S^2} \phi_2 \langle J \nabla \phi^a, \nabla \phi^b \rangle \, dM = (a_2 b_3 - a_3 b_2) \int_{S^2} \phi^{a+b+\varepsilon_1-\varepsilon_3} \, dM + (-a_2 b_1 + a_1 b_2) \int_{S^2} \phi^{a+b-\varepsilon_1+\varepsilon_3} \, dM + (a_3 b_1 - a_1 b_3) \int_{S^2} \phi^{a+b-\varepsilon_1+2\varepsilon_2-\varepsilon_3} \, dM.
\]
If this does not vanish then \(a + b\) is \((O, E, O)\).

3. \[
\int_{S^2} \phi_3 \langle J \nabla \phi^a, \nabla \phi^b \rangle \, dM = (a_3 b_1 - a_1 b_3) \int_{S^2} \phi^{a+b-\varepsilon_1+\varepsilon_2} \, dM + (-a_3 b_2 + a_2 b_3) \int_{S^2} \phi^{a+b+\varepsilon_1-\varepsilon_2} \, dM + (a_1 b_2 - a_2 b_1) \int_{S^2} \phi^{a+b-\varepsilon_1-2\varepsilon_2+\varepsilon_3} \, dM.
\]
If this does not vanish then \(a + b\) is \((O, O, E)\).

Note that since \(|a| = |b| = l\) then
\[
-a_1 b_3 + a_3 b_1 = a_1 b_2 - a_2 b_1 + l(b_1 - a_1)
\]
\[
a_2 b_3 - a_3 b_2 = a_1 b_2 - a_2 b_1 + l(a_2 - b_2).
\]
Proposition 3.3. If \( f_4, f_5, f_6, f_7 \) are all elements of \( E_{\lambda_l} \) where \( l \geq 6 \), then the long Cauchy-Riemann inequality (3) holds.

Proof. We have \( \Theta(1) = 1 \) and \( m = 2, n = 5 \). Therefore, \( m^2(n-2)^2 = 36 \leq \lambda_6 = 6 \times 7 \). From Corollary 2.2, we conclude that (3) holds for \( l \geq 6 \).

We now have to consider the case \( l \leq 5 \). We write \( f_{\alpha} = \sum_{|d|=l} A_{\alpha}^d \phi^d \), where \( A_{\alpha}^d \) are constants. This summation is not \( L^2 \)-orthogonal, in general, since eigenfunctions \( \psi_{l,\sigma} \) are usually a sum of linearly independent monomials. We have using Lemma 3.6

\[
\begin{align*}
4 \int_{S^2} \phi_1 (\langle J\nabla f_4, \nabla f_5 \rangle + \langle J\nabla f_6, \nabla f_7 \rangle) dM \\
+ 4 \int_{S^2} \phi_2 (\langle J\nabla f_4, \nabla f_6 \rangle - \langle J\nabla f_5, \nabla f_7 \rangle) dM \\
- 4 \int_{S^2} \phi_3 (\langle J\nabla f_4, \nabla f_7 \rangle + \langle J\nabla f_5, \nabla f_6 \rangle) dM = \\
\sum_{a+b=(E,O,O)} 4 (A_5^d A_5^a + A_6^d A_6^a) \int_{S^2} \phi_1 (\langle J\nabla \phi^a, \nabla \phi^b \rangle) dM \\
+ \sum_{a+b=(O,E,O)} 4 (A_6^d A_6^a - A_5^d A_5^a) \int_{S^2} \phi_2 (\langle J\nabla \phi^a, \nabla \phi^b \rangle) dM \\
- \sum_{a+b=(O,O,E)} 4 (A_5^d A_5^a + A_6^d A_6^a) \int_{S^2} \phi_3 (\langle J\nabla \phi^a, \nabla \phi^b \rangle) dM.
\end{align*}
\]

We will divide the proof into several lemmas.

Lemma 3.7. If \( l = 1 \) and set \( A_i^\alpha = A_{i,e}^\alpha \), that is, \( f_{\alpha} = A_1^\alpha \phi_1 + A_2^\alpha \phi_2 + A_3^\alpha \phi_3 \), then (4) holds. Furthermore, given \( f_4 \) and \( f_5 \) with arbitrary coefficients \( A_i^4 \) and \( A_i^5 \) respectively, then equality holds for a 4-tuple \( (f_4, f_5, f_6, f_7) \) iff

\[
\begin{align*}
f_6 &= (A_5^3 + A_5^4) \phi_1 + A_6^5 \phi_2 + A_6^5 \phi_3 \\
f_7 &= (A_3^2 - A_2^3) \phi_1 - (A_2^5 + A_5^6) \phi_2 + (A_5^5 + A_2^5) \phi_3.
\end{align*}
\]

Proof. Using Lemmas 3.6 and 3.3(2)

\[
(15) + (16) + (17) = \left( 4 (A_2^4 A_3^5 - A_2^3 A_2^5 + A_2^6 A_3^7 - A_3^6 A_2^7) \right. \\
-4 (A_1^4 A_3^5 - A_1^3 A_4^5 - A_1^5 A_2^7 + A_3^5 A_2^7) \\
\left. -4 (A_1^4 A_2^7 - A_2^5 A_3^7 + A_1^5 A_5^6 - A_2^5 A_1^6) \right) \int_{S^2} \phi^2 dM.
\]

Since \((-a+b+c)^2 \geq 0\) and \((a+b+c)^2 \geq 0\) for any real numbers \(a, b, c\), we have

\[
\begin{align*}
2ac + 2ab - 2bc &\leq a^2 + b^2 + c^2 \quad \text{with equality iff } a = b + c \\
-2ac - 2ab - 2bc &\leq a^2 + b^2 + c^2 \quad \text{with equality iff } a = -b - c
\end{align*}
\]

(18)
Applying these inequalities, we have
\[(15) + (16) + (17) = \left(4(A_2^4A_3^5 + A_2^4A_1^7 - A_3^5A_1^7) + 4(-A_3^4A_2^5 + A_3^4A_1^6 - A_2^5A_1^6)\right.\]
\[\left. + 4(A_2^6A_3^5 + A_2^6A_1^7 - A_3^6A_1^7) + 4(-A_3^6A_2^7 - A_3^6A_1^6 - A_2^6A_1^6)\right) \int_{S^2} \phi^2 dM\]
\[\leq \left((A_3^5)^2 + (A_3^7)^2 + (A_1^7)^2 + (A_2^5)^2 + (A_2^6)^2 + (A_3^7)^2 + (A_1^5)^2 + (A_3^6)^2 + (A_2^7)^2 + (A_1^4)^2\right) \int_{S^2} \phi^2 dM,\]
with equality iff
\[
\begin{align*}
A_2^4 &= A_3^5 + A_1^7, & A_1^6 &= A_3^4 + A_2^5, \\
A_3^5 &= A_5^4 + A_2^6, & A_2^7 &= -A_4^5 - A_5^6.
\end{align*}
\] (19)

On the other hand, using the last equality of (9) with \(\lambda_1 = 2\) we see that
\[
\sum_{\alpha} \|\nabla f\alpha\|_{L^2}^2 = (\sum_{\alpha} (2(A_\alpha^2)^2) \int_{S^2} \phi^2 dM.
\]
Thus, \((15) + (16) + (17) \leq \sum_{\alpha} \|\nabla f\alpha\|_{L^2}^2\) with equality iff (19) holds, and the lemma is proved. \(\square\)

Next we consider the case \(l \geq 2\).

\[(15) = \sum_{a+b=(E,O,O)} 4(A_a^4A_b^5 + A_a^6A_b^7).\] (20)
\[\cdot ((a_1b_2 - a_2b_1) \int_{S^2} \phi^{a+b-\varepsilon_2+\varepsilon_3} + (-a_1b_3 + a_3b_1) \int_{S^2} \phi^{a+b+\varepsilon_2-\varepsilon_3} + (a_2b_3 - a_3b_2) \int_{S^2} \phi^{a+b+2\varepsilon_1-\varepsilon_2-\varepsilon_1})\]

\[(16) = \sum_{a+b=(O,E,O)} 4(A_a^4A_b^6 - A_a^5A_b^7).\] (21)
\[\cdot ((a_2b_3 - a_3b_2) \int_{S^2} \phi^{a+b+\varepsilon_1+\varepsilon_2} + (-a_2b_1 + a_1b_2) \int_{S^2} \phi^{a+b-\varepsilon_1+\varepsilon_2} + (a_3b_1 - a_1b_3) \int_{S^2} \phi^{a+b-\varepsilon_1+2\varepsilon_2-\varepsilon_1})\]

\[(17) = \sum_{a+b=(O,O,E)} 4(-A_a^5A_b^6 - A_a^4A_b^7).\] (22)
\[\cdot ((a_3b_1 - a_1b_3) \int_{S^2} \phi^{a+b+\varepsilon_1+\varepsilon_2} + (-a_3b_2 + a_2b_3) \int_{S^2} \phi^{a+b+\varepsilon_1-\varepsilon_2} + (a_1b_2 - a_2b_1) \int_{S^2} \phi^{a+b-\varepsilon_1-2\varepsilon_2+\varepsilon_1})\]

Note that the above terms are such that \(a \neq b\) (otherwise \(a+b = (E,E,E)\)), and are skew-symmetric on \((a,b)\). We also have

\[
\sum_{\alpha} \|\nabla f\alpha\|^2 = \sum_{ab} \left(\sum_{\alpha} A_{\alpha a}A_{\alpha b} \left(\sum_{i} \frac{l((a_i+b_i) - (a_i-b_i))^2 + 2a_ib_i}{2(l+1)} \int_{S^2} \phi^{a+b-2\varepsilon_i}\right)\right)\] (23)
Lemma 3.8. If \( l = 2 \), then (4) holds with equality if and only if \( f_{\alpha} = 0 \) \( \forall \alpha \).

Proof. So we have \( a, b \) running over

\[
\{(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)\}.
\]

In (20) = (15) we have to consider the following terms: The terms with \( a + b = (2,1,1) \) that are given by \( a = (2,0,0) \) with \( b = (0,1,1) \), \( a = (1,1,0) \) with \( b = (1,0,1) \), and vice versa. The terms with \( a + b = (0,3,1) \) that are given by \( a = (0,2,0) \) with \( b = (0,1,1) \) and vice versa. The terms with \( a + b = (0,1,3) \) that are given by \( a = (0,0,2) \) with \( b = (0,1,1) \) and vice versa. Thus,

\[
(\text{15}) = 
\begin{align*}
&= 4(A^4_{(200)}A^5_{(011)} - A^4_{(011)}A^5_{(200)}) + A^6_{(200)}A^7_{(011)} - A^6_{(011)}A^7_{(200)}) \cdot \left(2 \int_{S^2} \phi_1^2 \phi_2 - 2 \int_{S^2} \phi_1^2 \phi_2^2\right) \\
&+ 4(A^4_{110}A^5_{(101)} - A^4_{(101)}A^5_{110}) + A^6_{110}A^7_{(101)} - A^6_{(101)}A^7_{110}) \cdot \left(- \int_{S^2} \phi_1^2 \phi_2^2 - \int_{S^2} \phi_1^2 \phi_2^2 + \int_{S^2} \phi_1^4\right) \\
&\quad + 4(A^4_{020}A^5_{(011)} - A^4_{(011)}A^5_{020}) + A^6_{020}A^7_{(011)} - A^6_{(011)}A^7_{020}) \cdot \left(2 \int_{S^2} \phi_1^2 \phi_2^2\right) \\
&\quad + 4(A^4_{002}A^5_{011}) - A^4_{(011)}A^5_{002}) + A^6_{002}A^7_{011)} - A^6_{(011)}A^7_{002}) \cdot \left(-2 \int_{S^2} \phi_1^2 \phi_2^2\right).
\end{align*}
\]

Using lemma 3.4, we have \( \int_{S^2} \phi_1^2 \phi_2^2 dM = \frac{1}{3} (1 + 2 \delta_{ij}) \int_{S^2} \phi^2 dM\). Therefore,

\[
(\text{15}) = 
\begin{align*}
&\left[\frac{4}{3} \left(A^4_{110}A^5_{(101)} - A^4_{(101)}A^5_{110}) + A^6_{110}A^7_{(101)} - A^6_{(101)}A^7_{110})\right) \\
&\quad + \frac{4}{3} \left(A^4_{020}A^5_{(011)} - A^4_{(011)}A^5_{020}) + A^6_{020}A^7_{(011)} - A^6_{(011)}A^7_{020})\right) \\
&\quad - \frac{4}{3} \left(A^4_{002}A^5_{011}) - A^4_{(011)}A^5_{002}) + A^6_{002}A^7_{011)} - A^6_{(011)}A^7_{002})\right]\right] \cdot \left(\int_{S^2} \phi^2\right).
\end{align*}
\]

In (21) = (16) we have the following terms: The terms with \( a + b = (3,0,1) \) that are given by \( a = (2,0,0) \) with \( b = (1,0,1) \) and vice versa. The terms with \( a + b = (1,2,1) \) that are given by \( a = (0,2,0) \) with \( b = (1,0,1) \), and \( a = (1,1,0) \) with \( b = (0,1,1) \), and vice versa. The terms with \( a + b = (1,0,3) \) that are given by \( a = (1,1,0) \) with \( b = (0,1,1) \) and vice versa. Thus,

\[
(\text{16}) = 
\begin{align*}
&= 4(A^4_{(200)}A^6_{(101)} - A^4_{(101)}A^6_{(200)}) - A^5_{(200)}A^7_{(101)} + A^5_{(101)}A^7_{(200)}) \cdot \left( - \int_{S^2} \phi_1^2 \phi_2^2\right) \\
&\quad + 4(A^4_{110}A^6_{(101)} - A^4_{(101)}A^6_{110}) - A^5_{110}A^7_{(101)} + A^5_{(101)}A^7_{110}) \cdot \left( \int_{S^2} \phi_1^2 \phi_2^2 + \int_{S^2} \phi_1^2 \phi_2^2 - \int_{S^2} \phi_1^4\right) \\
&\quad + 4(A^4_{020}A^6_{(011)} - A^4_{(011)}A^6_{020}) - A^5_{020}A^7_{(011)} + A^5_{(011)}A^7_{020}) \cdot \left(2 \int_{S^2} \phi_1^2 \phi_2^2 - 2 \int_{S^2} \phi_2^2 \phi_2^2 \right) \\
&\quad + 4(A^4_{002}A^6_{(011)} - A^4_{(011)}A^6_{002}) - A^5_{002}A^7_{011}) + A^5_{(011)}A^7_{002}) \cdot \left(2 \int_{S^2} \phi_1^2 \phi_2^2 \right)
\end{align*}
\]

that is,

\[
(\text{16}) = 
\begin{align*}
&\left[-\frac{8}{9} \left(A^4_{(200)}A^6_{(101)} - A^4_{(101)}A^6_{(200)}) - A^5_{(200)}A^7_{(101)} + A^5_{(101)}A^7_{(200)})\right) \\
&- \frac{4}{9} \left(A^4_{110}A^6_{(101)} - A^4_{(101)}A^6_{110}) - A^5_{110}A^7_{(101)} + A^5_{(101)}A^7_{110})\right) \\
&+ \frac{8}{9} \left(A^4_{002}A^6_{(011)} - A^4_{(011)}A^6_{002}) - A^5_{002}A^7_{011}) + A^5_{(011)}A^7_{002})\right]\right] \cdot \left(\int_{S^2} \phi^2\right).
\end{align*}
\]
In (22) = (17) we have the following terms: The terms with \( a + b = (3,1,0) \) that are given by \( a = (2,0,0) \) with \( b = (1,1,0) \) and vice versa. The terms with \( a + b = (1,3,0) \) that are given by \( a = (0,2,0) \) with \( b = (1,1,0) \) and vice versa. The terms with \( a + b = (1,1,2) \) that are given by \( a = (1,0,1) \) with \( b = (0,1,1) \), and by \( a = (0,0,2) \) with \( (1,1,0) \), and vice versa. Therefore,

\[
\begin{align*}
(17) = & \quad 4(-A^5_{(200)}A^6_{(110)} + A^5_{(110)}A^6_{(200)} - A^4_{(200)}A^7_{(110)} + A^4_{(110)}A^7_{(200)}) \cdot \left( \int S^2 \phi_1^2 \phi_2^2 \right) \\
& + 4(-A^5_{(020)}A^6_{(110)} + A^5_{(110)}A^6_{(020)} - A^4_{(020)}A^7_{(110)} + A^4_{(110)}A^7_{(020)}) \cdot \left( -2 \int S^2 \phi_2^2 \phi_3^2 \right) \\
& + 4(-A^5_{(101)}A^6_{(011)} + A^5_{(011)}A^6_{(101)} - A^4_{(101)}A^7_{(011)} + A^4_{(011)}A^7_{(101)}) \cdot \left( - \int S^2 \phi_2^2 \phi_3^2 - \int S^2 \phi_1^2 \phi_3^2 + \int S^2 \phi_3^4 \right) \\
& + 4(-A^5_{(002)}A^6_{(110)} + A^5_{(110)}A^6_{(002)} - A^4_{(002)}A^7_{(110)} + A^4_{(110)}A^7_{(002)}) \cdot \left( \int S^2 \phi_2^2 \phi_3^2 - \int S^2 \phi_1^2 \phi_3^2 \right),
\end{align*}
\]

that is

\[
(17) = \frac{3}{2} \left( -A^5_{(200)}A^6_{(110)} + A^5_{(110)}A^6_{(200)} - A^4_{(200)}A^7_{(110)} + A^4_{(110)}A^7_{(200)} \right) - \frac{3}{2} \left( -A^5_{(020)}A^6_{(110)} + A^5_{(110)}A^6_{(020)} - A^4_{(020)}A^7_{(110)} + A^4_{(110)}A^7_{(020)} \right) + \frac{3}{2} \left( -A^5_{(101)}A^6_{(011)} + A^5_{(011)}A^6_{(101)} - A^4_{(101)}A^7_{(011)} + A^4_{(011)}A^7_{(101)} \right) \cdot \left( \int S^2 \phi^2 \right).
\]

On the other hand,

\[
\begin{align*}
\Sigma \alpha \| \nabla f_\alpha \|^2 &= \left( \frac{3}{2} \sum \alpha \left( (A^\alpha_{(200)})^2 + (A^\alpha_{(020)})^2 + (A^\alpha_{(002)})^2 \right) \right. \\
& \quad - \frac{3}{2} \sum \alpha \left( A^\alpha_{(200)}A^\alpha_{(020)} + A^\alpha_{(200)}A^\alpha_{(002)} + A^\alpha_{(200)}A^\alpha_{(002)} \right) \\
& \quad + \frac{3}{2} \sum \alpha \left( (A^\alpha_{(110)})^2 + (A^\alpha_{(101)})^2 + (A^\alpha_{(011)})^2 \right) \right) \cdot \left( \int S^2 \phi^2 \right).
\end{align*}
\]

Thus, we have to show that (15) + (16) + (17) \( \leq (24) \). This can be shown by proving that the following four inequalities hold

\[
\begin{align*}
4A^5_{(110)}A^5_{(101)} - 8A^6_{(011)}A^7_{(020)} + 8A^6_{(011)}A^7_{(002)} - 8A^5_{(101)}A^7_{(200)} - 4A^4_{(110)}A^7_{(011)} + 8A^5_{(110)}A^7_{(020)} + 8A^4_{(110)}A^7_{(200)} - 8A^4_{(110)}A^7_{(020)} - 4A^5_{(101)}A^7_{(011)} + 4A^7_{(200)}A^7_{(020)} + 4A^7_{(200)}A^7_{(020)} + 4A^7_{(200)}A^7_{(020)} \\
\leq 8(A^7_{(200)})^2 + 8(A^7_{(020)})^2 + 8(A^7_{(002)})^2 + 6(A^5_{(110)})^2 + 6(A^5_{(101)})^2 + 6(A^5_{(011)})^2, \\
-4A^4_{(101)}A^5_{(110)} - 8A^6_{(020)}A^7_{(011)} - 8A^6_{(020)}A^7_{(011)} + 8A^4_{(101)}A^6_{(200)} + 4A^4_{(110)}A^6_{(011)} - 8A^4_{(101)}A^6_{(002)} + 8A^5_{(110)}A^6_{(020)} - 8A^5_{(110)}A^6_{(020)} - 4A^4_{(101)}A^6_{(011)} + 4A^6_{(200)}A^6_{(020)} + 4A^6_{(200)}A^6_{(002)} + 4A^6_{(020)}A^6_{(020)} \\
\leq 8(A^6_{(011)})^2 + 8(A^6_{(020)})^2 + 8(A^6_{(002)})^2 + 6(A^4_{(101)})^2 + 6(A^5_{(110)})^2 + 6(A^5_{(011)})^2, \\
4A^5_{(110)}A^7_{(101)} - 8A^4_{(011)}A^5_{(020)} + 8A^4_{(011)}A^5_{(002)} + 8A^4_{(011)}A^5_{(002)} + 4A^4_{(011)}A^5_{(110)} - 8A^5_{(020)}A^5_{(110)} - 8A^5_{(020)}A^5_{(110)} + 4A^5_{(011)}A^5_{(011)} + 4A^5_{(011)}A^5_{(011)} + 4A^5_{(011)}A^5_{(011)} + 4A^5_{(011)}A^5_{(011)} \\
\leq 8(A^5_{(011)})^2 + 8(A^5_{(020)})^2 + 8(A^5_{(002)})^2 + 6(A^4_{(101)})^2 + 6(A^5_{(110)})^2 + 6(A^5_{(110)})^2,
\end{align*}
\]
Proof. If \( l \)

\( \text{Lemma 3.10.} \) For any real numbers \( A, B, C, X, Y, Z \), we have

\[
2AB + 2AC + 2BC + 2XY + 2XZ + 2YZ + 4((A, B, C), (X - Y, Y - Z, Z - X))
\leq 3(A^2 + B^2 + C^2) + 4(X^2 + Y^2 + Z^2),
\]

with equality iff \( (A, C, B) \) and \( (X - Y, Y - Z, Z - X) \) are collinear.

Proof. Note first that

\[
\langle (A, C, B), (X - Y, Y - Z, Z - X) \rangle = \langle (X, Z, Y), (A - B, B - C, C - A) \rangle,
\]

and \( 2|\langle u, v \rangle| \leq |u|^2 + |v|^2 \) with equality iff \( u, v \) are collinear. Then

\[
4\langle (A, C, B), (X - Y, Y - Z, Z - X) \rangle =
2\langle (A, C, B), (X - Y, Y - Z, Z - X) \rangle
+ 2\langle (X, Z, Y), (A - B, B - C, C - A) \rangle
\leq A^2 + B^2 + C^2 + (X - Y)^2 + (Y - Z)^2 + (Z - X)^2
+ X^2 + Y^2 + Z^2 + (A - B)^2 + (B - C)^2 + (C - A)^2.
\]

But this is just (26). \( \square \)

\( \text{Lemma 3.10.} \) If \( l = 3 \) the functions

\[
f_4 = 3\phi_3^3 + 4\phi_1\phi_3 + 4\phi_2^2\phi_3 \quad f_5 = -4\phi_2^3 - 4\phi_2\phi_3^2 - 3\phi_1^2\phi_2
\]

\[
f_6 = -\phi_1^3 - \phi_1\phi_2^2 - 2\phi_1\phi_3^2 \quad f_7 = \phi_1\phi_2\phi_3
\]

do not satisfy the inequality (3).
Proof. We will show how we have found such functions. We have
\[
\int_{S^2} \phi^{(6,0,0)} dM = \frac{3}{5} \int_{S^2} \phi^2 dM \\
\int_{S^2} \phi^{(4,2,0)} dM = \frac{3}{5} \int_{S^2} \phi^2 dM \\
\int_{S^2} \phi^{(2,2,2)} dM = \frac{1}{5} \int_{S^2} \phi^2 dM.
\]

The multi-powers \(a, b\) are running all over
\[
\{(300), (030), (003), (210), (201), (120), (012), (102), (111)\}.
\]

The terms with \(a + b = b + a = (E, O, O)\) are given by (1,1,1) + (3,0,0), (0,0,3) + (0,3,0), (2,0,1) + (0,3,0), (0,2,1) + (0,3,0), (2,1,0) + (0,3,0), (0,1,2) + (0,0,3), (2,0,1) + (2,1,0), (0,2,1) + (2,1,0), (0,1,2) + (2,0,1), (1,1,1) + (1,2,0), (1,1,1) + (1,0,2). Thus,
\[
(15) =
\]
\[
= 4A^4_{(111)}A^5_{(300)} - A^4_{(300)}A^7_{(111)} + A^6_{(111)}A^4_{(300)} - A^6_{(300)}A^7_{(111)} \cdot (-3 \int_{S^2} \phi_1^2 \phi_2^2 + 3 \int_{S^2} \phi_1^4 \phi_2^2) \\
+ 4A^4_{(003)}A^5_{(030)} - A^4_{(030)}A^7_{(003)} + A^6_{(003)}A^7_{(030)} - A^6_{(030)}A^7_{(003)} \cdot (-9 \int_{S^2} \phi_1^2 \phi_2^2 \phi_3^2 - \int_{S^2} \phi_1^4 \phi_2^2) \\
+ 4A^4_{(201)}A^5_{(030)} - A^4_{(030)}A^7_{(201)} + A^6_{(201)}A^7_{(030)} - A^6_{(030)}A^7_{(201)} \cdot (6 \int_{S^2} \phi_1^2 \phi_2^2 \phi_3^2 - 3 \int_{S^2} \phi_1^4 \phi_2^2) \\
+ 4A^4_{(021)}A^5_{(030)} - A^4_{(030)}A^7_{(021)} + A^6_{(021)}A^7_{(030)} - A^6_{(030)}A^7_{(021)} \cdot (-3 \int_{S^2} \phi_1^2 \phi_2^4) \\
+ 4A^4_{(012)}A^5_{(003)} - A^4_{(003)}A^7_{(012)} + A^6_{(012)}A^7_{(003)} - A^6_{(003)}A^7_{(012)} \cdot (3 \int_{S^2} \phi_1^2 \phi_3^4) \\
+ 4A^4_{(201)}A^5_{(210)} - A^4_{(210)}A^7_{(201)} + A^6_{(201)}A^7_{(210)} - A^6_{(210)}A^7_{(201)} \cdot (2 \int_{S^2} \phi_1^4 \phi_3^2 + 2 \int_{S^2} \phi_1^4 \phi_2^2 - \int_{S^2} \phi_1^6) \\
+ 4A^4_{(021)}A^5_{(210)} - A^4_{(210)}A^7_{(021)} + A^6_{(021)}A^7_{(210)} - A^6_{(210)}A^7_{(021)} \cdot (-4 \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2 + 2 \int_{S^2} \phi_1^2 \phi_2^4 \phi_2^2 - \int_{S^2} \phi_1^4 \phi_2^4) \\
+ 4A^4_{(012)}A^5_{(201)} - A^4_{(201)}A^7_{(012)} + A^6_{(012)}A^7_{(201)} - A^6_{(201)}A^7_{(012)} \cdot (-2 \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2 + 4 \int_{S^2} \phi_1^2 \phi_2^4 \phi_2^2 + \int_{S^2} \phi_1^4 \phi_3^2) \\
+ 4A^4_{(111)}A^5_{(120)} - A^4_{(120)}A^7_{(111)} + A^6_{(111)}A^7_{(120)} - A^6_{(120)}A^7_{(111)} \cdot (j \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2 + \int_{S^2} \phi_1^2 \phi_2^4 \phi_2^2 - 2 \int_{S^2} \phi_1^4 \phi_2^4) \\
+ 4A^4_{(111)}A^5_{(102)} - A^4_{(102)}A^7_{(111)} + A^6_{(111)}A^7_{(102)} - A^6_{(102)}A^7_{(111)} \cdot (-j \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2 - \int_{S^2} \phi_1^2 \phi_2^4 \phi_2^2 + 2 \int_{S^2} \phi_1^4 \phi_2^4)
\]

The terms with \(a + b = b + a = (O, E, O)\) are given by (0,0,3) + (3,0,0), (2,0,1) + (3,0,0), (0,2,1) + (3,0,0), (1,1,1) + (3,0,0), (1,2,0) + (0,0,3), (1,0,2) + (0,0,3), (2,1,0) + (1,1,1) + (2,1,0), (1,2,0) + (2,0,1), (1,0,2) + (2,0,1), (1,2,0) + (0,2,1), (1,1,1) + (0,1,2). Hence,
\[
(16) =
\]
\[
= 4A^4_{(003)}A^6_{(300)} - A^4_{(300)}A^6_{(003)} - A^5_{(003)}A^7_{(300)} + A^5_{(300)}A^7_{(003)} \cdot (9 \int_{S^2} \phi_1^2 \phi_2^2 \phi_3^2) \\
+ 4A^4_{(201)}A^6_{(300)} - A^4_{(300)}A^6_{(201)} - A^5_{(300)}A^7_{(201)} + A^5_{(201)}A^7_{(300)} \cdot (3 \int_{S^2} \phi_1^4 \phi_3^2) \\
+ 4A^4_{(021)}A^6_{(300)} - A^4_{(300)}A^6_{(021)} - A^5_{(300)}A^7_{(021)} + A^5_{(021)}A^7_{(300)} \cdot (6 \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2 + 3 \int_{S^2} \phi_1^2 \phi_2^4 \phi_2^2) \\
+ 4A^4_{(111)}A^6_{(300)} - A^4_{(300)}A^6_{(111)} - A^5_{(300)}A^7_{(111)} + A^5_{(111)}A^7_{(300)} \cdot (-3 \int_{S^2} \phi_1^2 \phi_3^4 + 3 \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2) \\
+ 4A^4_{(120)}A^6_{(300)} - A^4_{(300)}A^6_{(120)} - A^5_{(300)}A^7_{(120)} + A^5_{(120)}A^7_{(300)} \cdot (6 \int_{S^2} \phi_1^2 \phi_2^4 \phi_3^2 - 3 \int_{S^2} \phi_1^2 \phi_2^4 \phi_2^2)
\]

20
\[
+4(A_{(111)}^4 A_{(210)}^6 - A_{(210)}^4 A_{(111)}^6 - A_{(111)}^7 A_{(210)}^7 + A_{(210)}^7 A_{(111)}^7) \cdot (-\int_{S_\alpha} \phi_1^4 \phi_2^3 - \int_{S_\alpha} \phi_1^2 \phi_2^3 \phi_3^2 + 2 \int_{S_\alpha} \phi_1^2 \phi_2^4) \\
+4(A_{(120)}^4 A_{(201)}^6 - A_{(201)}^4 A_{(120)}^6 - A_{(120)}^5 A_{(201)}^7 + A_{(201)}^5 A_{(120)}^7) \cdot (2 \int_{S_\alpha} \phi_1^4 \phi_2^2 - 4 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 - \int_{S_\alpha} \phi_1^2 \phi_2^4) \\
+4(A_{(102)}^4 A_{(201)}^6 - A_{(201)}^4 A_{(102)}^6 - A_{(102)}^5 A_{(201)}^7 + A_{(201)}^5 A_{(102)}^7) \cdot (3 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 - \int_{S_\alpha} \phi_2^4) \\
+4(A_{(102)}^4 A_{(120)}^6 - A_{(120)}^4 A_{(102)}^6 - A_{(102)}^5 A_{(120)}^7 + A_{(120)}^5 A_{(102)}^7) \cdot (2 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 + 2 \int_{S_\alpha} \phi_2^4 \phi_3^2 - \int_{S_\alpha} \phi_2^4 \phi_3^3) \\
+4(A_{(111)}^4 A_{(120)}^6 - A_{(120)}^4 A_{(111)}^6 - A_{(111)}^5 A_{(120)}^7 + A_{(120)}^5 A_{(111)}^7) \cdot (\int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 + 2 \int_{S_\alpha} \phi_2^4 \phi_3^2 - \int_{S_\alpha} \phi_2^4 \phi_3^3)
\]

The terms with \(a + b = b + a = (O, O, E)\) are given by \((0, 3, 0) + (3, 0, 0) + (3, 0, 0), (0, 1, 2) + (3, 0, 0), (1, 2, 0) + (0, 3, 0), (1, 0, 2) + (0, 3, 0), (1, 1, 1) + (0, 0, 3), (1, 2, 0) + (2, 1, 0), (1, 1, 1) + (2, 0, 1), (1, 1, 1) + (0, 2, 1), (0, 1, 2) + (1, 2, 0) + (0, 1, 2). \) Therefore

\[(17) = \]

\[
= 4(-A_{(030)}^4 A_{(300)}^7 + A_{(300)}^4 A_{(030)}^7 - A_{(030)}^5 A_{(300)}^6 + A_{(300)}^5 A_{(030)}^6) \cdot (-9 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2) \\
+4(-A_{(210)}^4 A_{(703)}^7 + A_{(703)}^4 A_{(210)}^7 - A_{(210)}^5 A_{(703)}^6 + A_{(703)}^5 A_{(210)}^6) \cdot (-3 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2) \\
+4(-A_{(012)}^4 A_{(300)}^7 + A_{(300)}^4 A_{(012)}^7 - A_{(012)}^5 A_{(300)}^6 + A_{(300)}^5 A_{(012)}^6) \cdot (6 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 - 3 \int_{S_\alpha} \phi_2^4 \phi_3^2) \\
+4(-A_{(120)}^4 A_{(703)}^7 + A_{(703)}^4 A_{(120)}^7 - A_{(120)}^5 A_{(703)}^6 + A_{(703)}^5 A_{(120)}^6) \cdot (3 \int_{S_\alpha} \phi_2^4 \phi_3^2) \\
+4(-A_{(012)}^4 A_{(300)}^7 + A_{(300)}^4 A_{(012)}^7 - A_{(012)}^5 A_{(300)}^6 + A_{(300)}^5 A_{(012)}^6) \cdot (-6 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 + 3 \int_{S_\alpha} \phi_2^4 \phi_3^2) \\
+4(-A_{(111)}^4 A_{(703)}^7 + A_{(703)}^4 A_{(111)}^7 - A_{(111)}^5 A_{(703)}^6 + A_{(703)}^5 A_{(111)}^6) \cdot (-3 \int_{S_\alpha} \phi_2^4 \phi_3^2 + 3 \int_{S_\alpha} \phi_1^2 \phi_3^4) \\
+4(-A_{(120)}^4 A_{(710)}^7 + A_{(710)}^4 A_{(120)}^7 - A_{(120)}^5 A_{(710)}^6 + A_{(710)}^5 A_{(120)}^6) \cdot (-3 \int_{S_\alpha} \phi_2^4 \phi_3^2) \\
+4(-A_{(111)}^4 A_{(701)}^7 + A_{(701)}^4 A_{(111)}^7 - A_{(111)}^5 A_{(701)}^6 + A_{(701)}^5 A_{(111)}^6) \cdot (\int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 + \int_{S_\alpha} \phi_2^4 \phi_3^2 - 2 \int_{S_\alpha} \phi_1^2 \phi_3^4) \\
+4(-A_{(111)}^4 A_{(721)}^7 + A_{(721)}^4 A_{(111)}^7 - A_{(111)}^5 A_{(721)}^6 + A_{(721)}^5 A_{(111)}^6) \cdot (-\int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 - \int_{S_\alpha} \phi_1^2 \phi_2^4 \phi_3^2 + 2 \int_{S_\alpha} \phi_1^2 \phi_3^4) \\
+4(-A_{(102)}^4 A_{(712)}^7 + A_{(712)}^4 A_{(102)}^7 - A_{(102)}^5 A_{(712)}^6 + A_{(712)}^5 A_{(102)}^6) \cdot (2 \int_{S_\alpha} \phi_1^2 \phi_2^2 \phi_3^2 - 4 \int_{S_\alpha} \phi_1^2 \phi_2^4 \phi_3^2 - \int_{S_\alpha} \phi_2^4 \phi_3^4)
\]

On the other hand

\[
\sum_{\alpha} \| \nabla f_{\alpha} \|_{L_2}^2 =
\]

\[
(\int_{S_\alpha} \phi^2 \cdot dM) \cdot \left( \sum_{\alpha} \frac{2 \times 6}{5 \times 7} (A_{(300)}^4)^2 + (A_{(030)}^4)^2 + (A_{(003)}^6)^2 \right) \\
+ \sum_{\alpha} \frac{11 \times 2}{5 \times 7} (A_{(210)}^4)^2 + (A_{(201)}^4)^2 + (A_{(201)}^4)^2 + (A_{(212)}^4)^2 + (A_{(120)}^5)^2 + (A_{(1,0,2)}^5)^2 \\
+ \sum_{\alpha} \frac{1 \times 12}{5 \times 7} (A_{(111)}^4)^2 - \sum_{\alpha} \frac{5 \times 7}{5 \times 7} (A_{(210)}^4 A_{(012)}^4 + A_{(201)}^4 A_{(210)}^4 + A_{(201)}^4 A_{(212)}^4) \\
- \sum_{\alpha} \frac{12 \times 5}{5 \times 7} (A_{(300)}^4 A_{(120)}^4 + A_{(300)}^4 A_{(120)}^4 + A_{(030)}^4 A_{(210)}^4 + A_{(030)}^4 A_{(212)}^4 + A_{(030)}^4 A_{(212)}^4 + A_{(030)}^4 A_{(212)}^4)
\]

(27)

Then (15) + (16) + (17) \leq (27) to hold for all possible coefficients \(A_{(\alpha)}^4\) is equivalent to two linearly independent inequalities:
and

\[
\begin{align*}
-36A^4_{(003)}A^5_{(030)} &= -12A^4_{(201)}A^5_{(030)} - 36A^4_{(021)}A^5_{(030)} - 36A^4_{(003)}A^5_{(012)} - 12A^4_{(201)}A^5_{(210)} - 4A^4_{(021)}A^5_{(210)} \\
-4A^4_{(021)}A^5_{(012)} &= 8A^4_{(210)}A^5_{(111)} - 8A^5_{(102)}A^4_{(111)} + 36A^4_{(003)}A^5_{(030)} + 36A^4_{(201)}A^5_{(300)} + 12A^4_{(021)}A^5_{(030)} \\
+12A^4_{(003)}A^5_{(120)} &= 8A^5_{(210)}A^5_{(111)} - 4A^4_{(201)}A^5_{(120)} - 12A^4_{(201)}A^5_{(102)} + 12A^4_{(021)}A^6_{(120)} + 4A^4_{(210)}A^6_{(102)} \\
-8A^5_{(012)}A^5_{(111)} + 36A^5_{(030)}A^5_{(030)} + 36A^5_{(210)}A^5_{(300)} + 12A^5_{(012)}A^6_{(300)} + 36A^5_{(030)}A^6_{(120)} + 12A^5_{(030)}A^6_{(012)} \\
-12A^5_{(210)}A^6_{(120)} &= 8A^4_{(201)}A^7_{(111)} + 8A^4_{(021)}A^7_{(111)} + 4A^4_{(012)}A^7_{(120)} + 4A^4_{(210)}A^7_{(102)} \\
&= 54 \left( A^4_{(003)}A^5_{(030)} + A^5_{(300)}A^5_{(030)} + A^5_{(030)}A^6_{(120)} + A^5_{(030)}A^6_{(012)} + A^4_{(201)}A^5_{(030)} + A^6_{(030)}A^7_{(030)} \right) \\
+ &12 \left( A^4_{(012)}A^5_{(012)} + A^5_{(030)}A^5_{(012)} + A^5_{(030)}A^5_{(012)} + A^4_{(003)}A^4_{(012)} + A^4_{(003)}A^4_{(021)} \right) \\
&\leq 54 \left( A^4_{(003)}A^5_{(030)} + A^5_{(300)}A^5_{(030)} + A^5_{(030)}A^6_{(120)} + A^5_{(030)}A^6_{(012)} + A^4_{(201)}A^5_{(030)} + A^6_{(030)}A^7_{(030)} \right) \\
+ &12 \left( A^4_{(012)}A^5_{(012)} + A^5_{(030)}A^5_{(012)} + A^5_{(030)}A^5_{(012)} + A^4_{(003)}A^4_{(012)} + A^4_{(003)}A^4_{(021)} \right) \\
&\leq 22 \left( A^4_{(201)}A^5_{(201)} + A^5_{(012)}A^5_{(012)} + A^5_{(012)}A^5_{(012)} + A^6_{(120)}A^7_{(012)} + A^6_{(012)}A^7_{(012)} \right)
\end{align*}
\]
Now we consider only (28). We define the polynomial function on 10 variables

\[
F[a, b, c, x, y, z, w, u, v, g] := \\
54(a^2 + b^2 + c^2) + 22(x^2 + y^2 + z^2 + w^2 + u^2 + v^2) + 12g^2 \\
+ 36ab - 36ac - 36bc + 36yb + 36az - 36xc \\
- 36wc - 36bu + 12xb + 12xw - 12yc - 12au + 12xv \\
- 12yu - 12zc - 12bv + 12wu - 12cu - 12cv - 12bw \\
- 12bz - 12ax - 12ay - 8g(u - v + w - x + y - z) \\
+ 4yw + 4xz - 4xu - 4yv - 4zu - 4wv - 4xy - 4zw - 4uv
\]

The inequality (28) is equivalent to say \( F[a, b, c, x, y, w, u, v] \geq 0 \) where

\[
a = A_{(003)}^4, \quad b = A_{(030)}^5, \quad c = A_{(300)}^6, \quad g = A_{(111)}^7, \quad x = A_{(201)}^4, \quad y = A_{(021)}^4, \quad z = A_{(012)}^5, \quad w = A_{(210)}^5, \quad u = A_{(120)}^6, \quad v = A_{(102)}^6.
\]

Using the Mathematica programming we see that \( F \) has only a critical point, that is at 0, where \( F \) vanishes. Then we compute the Hessian of \( F \), giving a \( 10 \times 10 \) matrix. Using a "Diag" package in Fortran 77 programming (see [4]), we are able to obtain the eigenvalues of Hess\( F \) at the origin, obtaining

\[
\begin{align*}
193.95260118883090 \\
111.22289635621148 \\
123.94135950568288 \\
-3.6844648605223074 \\
64.82632587231515 \\
39.493408799262383 \\
5.8125282188336085 \\
26.731868670430774 \\
34.522364101735334 \\
15.181112147219768
\end{align*}
\]

Then we see there is a negative eigenvalue, what shows that there is a direction given by the eigenvector corresponding to this negative eigenvalue where \( F \) might be negative. Using the same Fortran programming we obtain the eigenvectors

\[
\begin{align*}
(0.50859017, 0.56532378, 0.57392096, 0.15439833, 0.15001239, 0.12873210, 0.9321734758, 0, -0.14536746, 0.105746386, 0.15834592, -0.03) \\
(-0.70249306, 0.29669516, -0.22772120, 0.28225840, 0.32578962, 0.38748263, 0.16652599, -0.872224667, -0.03, 2.929873805, -0.03, -5.7370482, -0.02) \\
(2.63951560, -0.02, 0.52362035, 0.5103903, -0.24485076, 0.2105906, -0.11992104, -0.38731071, -0.38258444, -0.16648765, 2.33700155, -0.03) \\
(-0.35473784, -0.43567216, -0.301160685, -0.02, 0.34413141, 0.44414138, -0.46720073, -0.28186146, -0.7498440473, -0.02, -0.2198649, 0.12437580) \\
(-0.21201306, 2.41075175, -0.02, -2.94440636, -0.02, -0.44169835, 0.19279750, -0.15999765, 0.32263096, 0.36103837, -0.57445451, -0.40224205) \\
(-0.17222347, -0.21619180, 1.17006858, -0.02, 0.76854558, 0.03, 0.1679725, 0.69109876, 9.10242003, -0.02, -0.21071519, 0.11895024, 2.85076642, -0.02) \\
(-0.2078084, 0.23610482, 0.30748260, 0.3301515, 0.23524015, 4.49400444, 0.02, 0.3326664, 0.52470301, 0.19533343, 0.22929391) \\
(-0.14729467, 0.13044269, -0.22190370, -7.04675358, -0.2, 0.01745129, -0.2, 0.1880081, -0.68157240, 0.61403888, -6.7778742, 0.11935459) \\
(0.1003429, -0.94355331, -0.02, -7.49166773, -0.02, -0.4487484, 0.38166992, -0.19261905, -5.88942769, 5.71153059, -0.02, 0.72842332, -0.23787165) \\
(-4.73928140, -0.02, -1.38729246, -0.02, -0.14742221, -0.45555392, 5.70707316, 0.02, -0.21413765, 0.5200747656, -2.15775514, -0.2, -0.71975838, 0.02, 0.83399977)
\]

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where $E - 0k$ means that we have to multiply the number by $10^{-k}$. Then the eigenvector corresponding to the negative eigenvalue is $S = [a, b, c, x, y, z, w, u, v, g]$ where

\[
\begin{align*}
    a &= 0.35473784 & b &= -0.43567216 & c &= -0.0401160688 \\
    x &= 0.34413141 & y &= 0.44414138 & z &= -0.46720073 \\
    w &= -0.28168146 & u &= -0.0749844407 & v &= -0.21986490 \\
    g &= 0.12437580
\end{align*}
\]

We take the vector $[3, -4, -1, 3, 4, -4, -3, -1, -2, 1]$ in a neighbourhood of $10S$, and verify that $F[3, -4, -1, 3, 4, -4, -3, -1, -2, 1] = -138 < 0$, what proves our lemma.

\[\square\]

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References

[1] J.L. Barbosa and M. do Carmo: Stability of minimal surfaces and eigenvalues of the Laplacian. Math. Z., 173(1) (1980), 1328.

[2] J.L. Barbosa, M. do Carmo, and J. Eschenburg: Stability of hypersurfaces of constant mean curvature in Riemannian manifolds. Math. Z., 197(1) (1988), 123138.

[3] S. Brendle: Blow-up phenomena for the Yamabe equation. J. Amer. Math. Soc. 21 (2008), no. 4, 951979.

[4] T. Hahn: Routines for the diagonalization of complex matrices. arXiv:physics/0607103v2[physics.comp-ph]

[5] J. Jost.: Postmodern Analysis, Universitext, Springer Berlin (1998)

[6] D. Pumberger and T. Riviere: Uniqueness of tangent cones for semicalibrated integral 2-cycles”, Duke Math. J., 152 (3) (2010), 441-480.

[7] I.M.C. Salavessa: Stability of submanifolds with parallel mean curvature in calibrated manifolds, Bull. Braz. Math. Soc. NS, 41(4) (2010), 495-530.

[8] I.M.C. Salavessa and A. Pereira do Vale: Transgression forms in dimension 4, IJGMMP, 3(4-5) (2006), 1221-1254.