The two-dimensional fretting contact with a bulk stress. 
Part I – Similar elastic materials

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Abstract. Minute relative motions along the interface of contacting machine elements, brought about by cyclic tangential loading, lead to failure of many mechanical components from fretting fatigue. From the point of view of applied mechanics, the understanding of the fretting fatigue rely on the modelling of contact problems with partial slip. The experimental simulation of fretting fatigue often implies cylindrical pads in contact with a flat specimen, leading to a plane strain contact scenario requiring the solution of the two-dimensional contact problem. The numerical solution for the latter problem is achieved in this paper for similar elastic materials by employing the technique of influence functions, derived from fundamental half-plane solutions of point forces acting on the boundary. The contact model is first divided into two parts with solutions more easily to obtain. The linear systems with the normal and shear tractions as unknowns are solved with the conjugate gradient method, which can be applied for symmetric and positive definite system matrices. The most time-consuming operations are the convolutions products assessing the displacements induced by the contact tractions. A method based on the fast Fourier transform is applied for increased computational speed without sacrificing accuracy. The proposed algorithm was benchmarked by reproducing the solution of the classic two-dimensional Cattaneo-Mindlin problem. In practical situations, other loadings besides the contact itself may induce bulk stresses within one or both of the contacting bodies. New results are obtained by introducing a bulk stress, thus replicating the conditions from fretting fatigue experiments. The advanced numerical program for the two-dimensional contact with partial slip proves itself as an efficient tool for the understanding of the fretting fatigue by numerical modelling and simulation.

1. Introduction

Prediction of fatigue life in real machine components must rely on data obtained under well-defined and well understood experimental configurations, with repeatable conditions that reduce the experiment sensitivity to material or manufacturing problems. On a theoretical level, the adoption of the Hertz framework, consisting in a simple convex contact, may be a reliable candidate due to its implicit assumption that the macroscopic geometry of contacting bodies does not affect the problem output except for the stress, strain and displacement fields. Whereas closed form solutions obtained under this idealization may be of great interest for the understanding of the underlying phenomena, many real fretting problems involve the contact of components of great diversity. Due to complexity
of the arising mathematical models, fretting contact modelling can only be achieved by a numerical method such as the finite element method. Numerical analysis may extend the repertoire of existing closed form solutions by relaxing some of the assumptions required for analytical evaluation, providing an extra generality that may lead to the design of competent tribological components.

Recent developments in semi-analytical contact modelling led to algorithms capable of fretting contact simulations with increased generality. The coupling of the normal and the tangential effects, as well as the path dependence, lack analytical support and therefore require iterative approaches, which rely on fundamental solutions for the semi-infinite solid [1-5], in the presence of coatings [6,7] or inhomogeneities [8,9]. The computational efficiency is supported by spectral methods [10,11], as well as by efficient solvers for large linear systems of equations [12,13]. The main challenge is to derive faster algorithms that allow mesh convergence studies to assess the method precision.

In this work, it is assumed that the knowledge of the stress, strain and displacement fields arising in the contacting bodies is sufficient to quantify the process of crack initiation, early development and growth. The knowledge of the size and shape of the contact region, as well as the pressure distribution, is a first step in understanding any fretting problem, whereas in a second phase, the surface damage must be quantified by finding the combination of stick and slip regions co-existing under partial slip conditions, together with the shear traction distribution. These goals are achieved via a two-dimensional semi-analytical method similar to the one used for 3D contact scenarios.

2. Model overview

The line contact problem between two parallel, infinitely long cylinders is of great significance for the experimental study of the fretting fatigue. Although in practice end effects may cause severe stress concentrators, the stress state occurring far from the ends is virtually two-dimensional, making the plane contact solution a satisfactory approximation. The deformation in the contacting cylindrical pad must be confined to a tiny area of its circumference, so that the half-plane approximation holds in the evaluation of deformation and stresses. This simplification is possible because the contact is non-conformal, i.e. the largest characteristic contact dimension is much less than the smallest characteristic radius of curvature of the contacting bodies. The stresses induced by the contact tractions are highly localized in the vicinity of the contact region, and thus are independent of other boundary conditions, as no boundaries adjacent to the contact region are assumed to weaken the contacting components.

The plane contact problem depicted schematically in figure 1 is reported to a Cartesian coordinate system with the origin in the initial point of contact, having its \(x\)-axis, referred to as the tangential direction, chosen as to separate best the contacting surfaces, and the \(z\)-axis normal to the contact patch.

![Figure 1. Schematic of a 2D fretting contact problem.](image_url)
As the contact is established and a normal force is transmitted, the bodies deform elastically, surface particles within each body are compressed along the normal direction, so that the initial point of contact evolves into a contact region $\Gamma_C$ accommodating a mutual contact pressure $p(x)$. The static force equilibrium in the normal direction yields:

$$ W = \int_{\Gamma_C} p(x) dx. \quad (1) $$

The equation of the separation $h(x)$ between the two bounding surfaces, comparing the contact geometry before and after the deformation accompanying the initial contact point evolving into a contact surface, results as:

$$ h(x) = h_0(x) + u(x) - \delta, \quad (2) $$

where $h_0(x)$ is the initial separation, i.e., the contact geometry in undeformed state, $u(x)$ the composite normal displacement (i.e., the sum of displacements of corresponding points on the surfaces of the contacting bodies), and $\delta$ the rigid-body approach, i.e. the normal approach of points within the bodies distant to the contact region. If a problem computational domain $\Gamma \supset \Gamma_C$ is considered, it will be divided into regions of contact and regions of non-contact, which are a priori unknown. The contact status of every particle on the contacting surfaces is governed by the boundary conditions providing complementarity requirements for pressure and surface separation:

$$ \begin{cases} 
    p(x)h(x) = 0, & x \in \Gamma, \\
    p(x) > 0, & x \in \Gamma_C, \\
    h(x) > 0, & x \in \Gamma - \Gamma_C.
\end{cases} \quad (3) $$

Relations (3) imply that only compressive normal tractions are assumed in this contact model, i.e. $p(x) < 0, x \in \Gamma$, is not allowed. Whereas this assumption may be too strong in case of rubber, metallic materials exhibit little adhesion effects, as the actual contact area, established between the peaks of the inherent surface microtopography (i.e. roughness), is much smaller than the apparent contact area. From a computational point of view, this assumption is required to guarantee the convergence of the employed minimization scheme, as discussed in [14]. The boundary conditions (3) also imply that the contacting bodies are considered impenetrable in the frame of Linear Theory of Elasticity, i.e. $h(x) < 0, x \in \Gamma$, is also not allowed.

Fretting occurs when a contact is subjected to minute tangential relative displacements over at least part of the interface. The latter displacements may be the result of a mismatch in the elastic properties of the contacting materials in the presence of a normal load alone, or the outcome of an express tangential force. The latter case is considered, for simplicity, in this paper, implying that the contacting bodies are assumed to have similar elastic properties. The relative tangential displacements induced by the tangential force will be resisted by friction, giving rise to the shearing tractions $q(x)$, acting in opposite directions over the surfaces of the contacting bodies. The static force equilibrium in the tangential direction yields:

$$ Q = \int_{\Gamma_C} q(x) dx. \quad (4) $$

In a fretting contact, the applied tangential force $Q$ is usually smaller than the limiting friction force, and therefore no macroscopic relative tangential motion (i.e., sliding) occurs between the contacting bodies, i.e. $|Q| < \mu F$, where $\mu$ is the frictional coefficient. The global stick regime applies to points in the contacting bodies distant to the contact region, whereas on the contact surface specific points may experience micro-sliding, or slip, accompanying the tangential deformation. A Coulombian
friction law is assumed for each surface particle within the contact rather than for the rigid body as a whole. The assumption of full stick on the contact area leads \[15,16\] to an infinite \(\mu\) required to prevent slip at the contact edges, where the shear tractions approach infinity while the contact pressure falls to zero. Consequently, a slip \(s(x)\) is expected to occur under any level of tangential loading, and the contact area will be divided into regions of stick \(\Gamma_S\) and regions of slip \(\Gamma_C - \Gamma_S\), which are a priori unknown. It should be noted that in the stick regions, the relative displacement of surface particles remains constant and equal to the value attained when entering the stick zone. Although a quasi-static contact model is considered here, the existence of slip is conditioned by a variation in the level of the tangential load. However, the model does not depend explicitly on time, as in the case of viscoelastic contacting materials. An incremental application of the tangential load is sufficient to properly simulate the loading history. The stick or slip status of any particle on the contact area, corresponding to the last loading increment, is governed by the complementarity conditions:

\[
\begin{aligned}
|q(x)| < \mu p(x), & \quad s(x) = 0, \quad x \in \Gamma_S, \\
|q(x)| = \mu p(x), & \quad s(x) > 0, \quad x \in \Gamma_C - \Gamma_S.
\end{aligned}
\] (5)

It should be noted that in the slip zones, the direction of the shear traction opposes the direction of the increment of relative surface tangential displacement induced by the last loading increment. The geometrical condition of deformation for the same load increment relates slip to the relative tangential displacement \(v(x)\) and the rigid-body tangential translation \(\omega\), measured between points within the contacting bodies distant to the contact region:

\[
s(x) = v(x) - \omega, \quad x \in \Gamma_C.
\] (6)

3. Algorithm description

The difficulty in solving the system of equations and inequalities (1) - (6) stems from two facts: (1) the shape and size of the contact area and of the stick area are a priori unknown and keep changing with the load level, and (2) assessment of surface displacement \(u(x)\) and \(v(x)\) requires knowledge of both contact tractions \(p(x)\) and \(q(x)\). In other words, the contact problems in the normal and in the tangential directions are coupled and cannot be solved independently. The latter difficulty is avoided in this paper by adopting the assumption of elastic similarity of contacting bodies, which decouples the effects of the normal and shear loadings and allows for a separate analysis. Indeed, with this assumption, the pressure-induced tangential displacements of two corresponding particles will be the same, thus there will be no relative slip to generate frictional tractions. Moreover, as the shear tractions act in opposite directions on the contacting surfaces, corresponding points move in the normal direction by the same amount so that the separation \(b(x)\), and consequently the contact pressure distribution, remain unchanged.

With this simplifications, equations (1) - (3) build the contact model in the normal direction, aiming to find the contact area and the pressure distribution, whereas the set (4) - (6) governs the stick area and the shear tractions distribution. As pressure is needed as a limit to the shear tractions according to equations (5), the solution of the normal contact problem must be obtained first.

In order to find the contact area and the stick area, the two sets of equations and inequalities are discretized and a semi-analytical approach is adopted. The latter method assures that a solution is achieved for real engineering components, which have arbitrary profiles in the contact region, but can be represented by elastic half-spaces. Consequently, superposition of effects applied to fundamental solutions (i.e., the Green’s functions) of stresses and displacements generated in the semi-infinite plane, assisted by a numerical discretization, lead to solutions of great generality, applicable to prescribed, but otherwise arbitrary, half-plane loads. This versatile displacement computation technique is required as the contact area \(\Gamma_C\) and the stick area \(\Gamma_S\) must be determined by trial-and-
error, in an iterative approach that requires repeated computation of displacement for prescribed, but otherwise arbitrary contact tractions.

The numerical treatment of the continuous model is conducted by splitting the chosen computational domain, expected to include the contact area, into finite regions over which a prescribed traction element is taken to act. In this manner, the continuous distribution of tractions is substituted by a discrete set of traction elements, whose spatial localization is achieved by choosing a related set of representative points (usually the centers of the boundary elements), also referred to as ‘matching points’ [17]. The traction elements are then iterated until the boundary conditions are satisfied for all the matching points. From a computational point of view, it is convenient to assume a stepwise distribution (i.e., piecewise uniform) of traction, consisting in non-overlapping columns of uniform tractions raised on a uniform rectangular mesh. The latter discretization may have difficulties in predicting asymptotic decrease to zero at the edge of the contact, but can be conveniently treated with the aid of spectral methods [10,11]. This discretization implies that all continuous problem parameters are substituted by discrete values, e.g. $p_i$ is the discrete counterpart of $p(x)$ when $x$ is located inside the $i^{th}$ boundary element. The condition for any discretization technique is that an analytical solution for the displacement induced by a traction element in each matching point exists. In this case, the closed-form solution of a line loading of the elastic half-space by a uniform distribution of tractions is readily available [17], and can be used to derive the influence coefficients $K_{i-j}$, expressing the contribution of the traction in the boundary element $i$ to the effect (displacement or stress) in the matching point $j$. Superposition of effects in the frame of linear theory of elasticity leads to the following pressure-induced displacement equation:

$$u(x) = \int_{-\infty}^{\infty} p(x) g(t-x) dt,$$

where $g(x)$ is the relevant Green’s function, i.e., the Boussinesq solution for a unit point force acting normally on the half-plane boundary. Equation (7) is a continuous linear convolution, and its discrete counterpart is the discrete cyclic convolution:

$$u_i = \sum_{j=1}^{N} p_j \cdot K_{i-j+N} H_{i-j}, \quad i = 1 \cdots N,$$

where $H(x)$ is the Heaviside unit step function, and $N$ the number of boundary elements. The most efficient computation of the convolution (8) is achieved via the Discrete Convolution Fast Fourier Transform (DCFFT) method [10]. By applying the latter technique, the order of computations is reduced from $O(N^2)$, corresponding to direct summation, to an improved $O(N \log N)$. The reduction comes from the fact that, in the frequency domain, convolution is calculated as an element-wise product between the Fourier transforms of the convolution members, in just $O(N)$ operations. The transfer to and from the frequency domain is achieved via the fast Fourier transform (FFT) algorithm, whose computational complexity dictates the overall DCFFT method complexity.

Additional difficulty arises in application of this technique to the line contact (plain strain) as the displacement in the plane contact problem is undefined to the extent of an arbitrary constant. As opposed to the 3D case, in which the influence coefficients expressing the displacement induced by a rectangular uniform pressure are unique, as shown by the Love [18]. As suggested in [17], a datum point should be chosen to circumvent this limitation. However, in this paper, a different method is adopted. It is first proven that the magnitude of the residual in solving the linear system is independent of the datum point chosen in 2D displacement calculation. Let $u^{(1)}$ and $u^{(2)}$ be two displacement fields induced by the same pressure distribution, but calculated with respect to different datum points, i.e.

$$u^{(1)} = u^{(2)} + C,$$

where $C$ is a constant.
where $C$ is a constant relating the chosen references. The rigid-body approach calculated with each of the displacement fields results from equation (2), considered only for the boundary elements in contact:

$$\delta^{(i)} = \frac{h_i + u_i^{(i)}}{\text{card}(A)}, \quad i = 1 \ldots N, \quad j = 1, 2. \quad (10)$$

Here, \(\text{card}(A)\) denotes the cardinal of the set \(\{i \in [1,N] : p_i > 0\}\), i.e. the number of contacting patches, and superscripts are used to identify the considered datum point. By plugging equation (9) into relation (10), one can see that \(\delta^{(1)} = \delta^{(2)} + C\), which, based on relation (2), leads to \(h^{(1)} = h^{(2)}\), i.e. the residual in the linear system solution is independent of the chosen datum point. In other words, the contact solver only needs relative displacements of surface points in order to derive the contact area and the pressure distribution, and therefore the original solver for the 3D case can be applied to line contact without additional modification. The same feature can be observed for the contact in the tangential direction, whose surface of separation is described by equation (6), and the residual is the vanishing stick on the stick regions.

The solution of the normal contact problem is thus achieved based on the 3D contact solver advanced by Polonsky and Keer [12] for the rough contact of elastic bodies. The algorithm strategy consists in solving the system resulted from digitization of equation (2), having the magnitude of the pressure elements as unknowns. The latter system is essentially linear in the nodal pressures, and consequently can be solved using dedicated numerical methods, such as the Conjugate Gradient Method (CGM). The contact area (i.e., the size of the system), which also iterated during system solution, results from the stabilization of two apparently opposing trends: boundary elements for which the pressure element results negative at the current iteration are excluded from the contact area, whereas patches with negative gap are reincluded. Any modification in the size of the system leads to a reset of the conjugate directions used in the residual minimization. The contact area and the pressure distribution are thus iterated simultaneously, and convergence is achieved when pressure is stabilized from one iteration to the next, with respect to the imposed load.

The solution of the tangential contact problem, described by the set of equations and inequalities (4)-(6), can be obtained in the same manner as for its normal counterpart. An iterative approach is employed to derive the slip or stick status of any boundary element from the contact area, starting from a full stick regime. The shear tractions on the stick region and the stick region itself are derived simultaneously. The shear tractions on the stick area result as the solution of the system of equations resulted from digitization of equation (11), having as residual the relative slip distances corresponding to the last tangential load increment, as the slip distances vanish on the stick region. The latter region is also derived by the stabilization of two opposing trends: the boundary elements for which the Coulomb friction’s law is not verified are excluded from the current stick region, whereas the surface patches with relative slip distances not opposing the shear tractions are reincluded. The static force equations is also enforced in a correction outside the CGM core.

4. Results and discussions

Investigations of fretting fatigue employs experimental rigs in which one or both contacting bodies are subjected to bulk stresses other than the contact itself. A difference between the bulk stresses applied in the $x$-direction will generate a mismatch in the strains that will affect the contact configuration, so that direct superposition of the bulk stresses to the contact stresses computed in the absence of the bulk stresses would not be accurate. An additional term must be added in the tangential separation equation, accounting for the relative supplementary tangential displacement due to the mismatch in the bulk stresses. Under conditions of plane strain, the modified version of equation (6) results as:

$$s(x) = v(x) - \omega - \sigma x (1 - \nu^2)/E, \quad x \in \Gamma_C, \quad (11)$$
with $E$ and $\nu$ the Young modulus and the Poisson’s ratio of the contacting materials, respectively.

The effect of a tensile bulk stress $\sigma$ present in the flat lower contacting body, on the contact tractions and on the $\sigma_{xx}$ stress component, is depicted in figure 2. The Hertz pressure $p_H$ and the Hertz contact radius $a_H$ are used as normalizers for stresses and coordinates, respectively. In all simulations, the ratio $Q/(\mu H)$ is kept constant at 0.5. By solving the normal contact problem, a well-known semi-elliptic Hertz pressure distribution is obtained. The solution of the tangential contact problem subsequently provides the distribution of the shear tractions.

The central stick zone is bordered by the two local maximums of the shear tractions $q(x)$, whereas the outer regions are in slip. In the latter regions, the shear tractions follow the Hertz pressure profile. For vanishing $\sigma$, a symmetrical distribution of the shear tractions is predicted, matching perfectly the Cattaneo-Mindlin solution [15,16]. When the ratio $\sigma/(\mu p_H)$ is increased, the bulk tension shifts the position of the stick zone toward the leading edge of the contact. An analytical solution for this scenario is available [19], but only applies to a stick zone not reaching the contact edge. The numerical predictions agree well with the aforementioned closed-form relation:

$$q(x) = \begin{cases} \mu p_H \sqrt{1-(x/a_H)^2}, & c \leq |x| \leq a_H, \\ \mu p_H \sqrt{1-(x/a_H)^2} - c/a_H \sqrt{1-(x+e)/c}^2, & |x| < c, \end{cases}$$  \tag{12}

where $c/a_H = \sqrt{1-Q/(\mu H)}$ and $e = a_H \sigma/(4p_H)$. The latter parameter $e$ accounts for the shifting of the stick zone of radius $c$.

Further increase of the bulk stress results in reverse slip occurring at the leading edge of the contact for the cases when $\sigma/(p_H) > 1$. Under these circumstances, there exist no analytical solution and the contact problem can only be solved numerically.

![Figure 2](image)

**Figure 2.** The effect of a bulk tensile stress on (a) shear tractions; (b) $\sigma_{xx}$ stress component.

The shifting of the stick region with increasing tensile bulk stress has a significant effect on the stress field, particularly on the distribution of the $\sigma_{xx}$ stress component on the contacting surface. The latter stress is increased considerably at the trailing edge of the contact, i.e. at $x/a_H = -1$, where cracks are likely to initiate [20]. The tensile bulk stress thus has a detrimental effect on the service life of the contacting element, as it mainly affects the site of crack nucleation in a fretting contact.
5. Conclusions

Fretting fatigue experiments may rely on experimental rigs consisting in cylindrical pads in contact with a flat specimen subjected to a bulk stress. The arising 2D contact problem admits an analytical solution under strong assumptions that limit its applicability. A numerical solution is advanced in this paper that extends the problem generality and allows for solutions relevant in practical engineering applications.

The discrete model of the contact problem is split into two parts: the normal frictionless contact is solved first, and the shear tractions are subsequently derived based on the pressure solution obtained in the first step. In both cases, a similar algorithm is employed, consisting in a modified conjugate gradient method applied to linear systems having the contact tractions as unknowns. The iterative search of the contact parameters is supported by the numerical solution for displacement computation, relying on fundamental solutions for the elastic half-plane. Acceleration of displacement computation is provided by spectral methods and the fast Fourier transform.

The simulations reflect the influence of the bulk stress on the contact tractions and stresses. With increasing tensile bulk stress, the central stick region is moved toward the leading edge of the contact, until the edge is reached and reversed slip occurs making the analytical solution unviable. The shifts reflect in the magnitude of the $\sigma_{xx}$ stress riser at the trailing edge of the contact, which is expected to favour crack initiation.

6. References

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