On a conjecture on linear systems

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Abstract. In a remark to Green’s conjecture, Paranjape and Ramanan analysed the vector bundle \( E \) which is the pullback by the canonical map of the universal quotient bundle \( T_{\mathbb{P}^{g-1}}(-1) \) on \( \mathbb{P}^{g-1} \) and stated a more general conjecture and proved it for the curves with Clifford Index 1 (trigonal and plane quintics). In this paper, we state the conjecture for general linear systems and obtain results for the case of hyper-elliptic curves.

Keywords. Green’s conjecture; linear systems; hyper-elliptic curves.

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1. Introduction

Let \( C \) be a smooth projective curve of genus \( g \geq 2 \) over a field \( k \) and let \( K \) be the canonical line bundle on \( C \). In [4], Green made a conjecture which relates two aspects: Koszul cohomology (an algebraic aspect) and Clifford Index \( \gamma_C \) (a geometric aspect) of a curve. This conjecture [4] is equivalent to the following [6]: Let \( E_K \) be the pullback by the canonical map \( \Phi : C \to \mathbb{P}^{g-1} \) of universal quotient bundle on \( \mathbb{P}^{g-1} \). Then the map \( \wedge^i \Gamma(C, E_K) \to \Gamma(C, \wedge^i E_K) \) is surjective \( \forall \ i \leq \gamma_C \). Paranjape and Ramanan [6] studied the vector bundle \( E_K \) (stability properties). They also proved that all sections of \( \wedge^i E_K \) which are locally decomposable are in the image of \( \wedge^i \Gamma(E_K) \forall \ i \leq \gamma_C \). Let \( \sum_{i,K} \) be the cone of locally decomposable sections of \( \wedge^i E_K \). In [5], Hulek et al. stated a conjecture.

\textbf{Conjecture 1.1.} \( \sum_{i,K} \) spans \( \Gamma(\wedge^i E_K) \forall \ i \) and for all curves.

This is stronger than Green’s conjecture. They proved it for curves with Clifford index 1 (trigonal curves and plane quintics). Conjecture 1.1 is trivial in case of hyperelliptic curves, since \( E_K \) is the \((g - 1)\)-fold direct sum of the hyper-elliptic line bundle. Vector bundle \( E_K \) is semi-stable (even stable if \( C \) is not hyper-elliptic). In a remark to conjecture made in [5], Eusen and Schreyer [3] asked a more general question, whether \( \Gamma(\wedge^i N) \) is spanned by locally decomposable sections and holds for every (stable) globally generated vector bundle \( N \) on every curve \( C \). They gave counter examples to this more general question [3]. By broadening our view point, in this paper we state a conjecture for general
linear systems. Let $C$ be a smooth curve of genus $g \geq 2$ and let $L$ be a globally generated line bundle on $C$. The evaluation map gives rise to an exact sequence

$$0 \to E^* \to \Gamma(L)_C \to L \to 0$$

where $E^*$ is locally free of rank $h^0(L) - 1$. Let $\sum_i$ be the cone of locally decomposable sections of $\wedge^i E$. We state as follows:

**Conjecture 1.2.** $\sum_i$ spans $\Gamma(\wedge^i E) \forall i$ and for all curves.

In this paper, we prove Conjecture 1.2 in case of hyperelliptic curves for the line bundles with degree large enough and $i \leq g$.

**Theorem 1.** Let $C$ be a smooth hyper-elliptic curve of genus $g \geq 2$ and let $L$ be a globally generated line bundle on $C$ of degree $d \geq 2g + 1$ such that $H^1(L \otimes T^{-2}) = 0$, where $T$ is the hyper-elliptic line bundle on $C$. The evaluation map gives rise to an exact sequence

$$0 \to E^* \to \Gamma(L)_C \to L \to 0, \quad (1)$$

where $E^*$ is locally free of rank $h^0(L) - 1$. Let $\sum_i$ be the cone of locally decomposable sections of $\wedge^i E$. Then $\sum_i$ spans $\Gamma(\wedge^i E) \forall i \leq g$.

2. **Geometry of the hyper-elliptic curve**

Since $C$ is hyper-elliptic of genus $g \geq 2$, thus $g^1_2$ on $C$ is unique. Let $\pi : C \to \mathbb{P}^1$ be the associated 2-sheeted covering, $T := \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ is the unique $g^1_2$.

Consider the rank 2 vector bundle $W$ on $\mathbb{P}^1$, where $W := \pi^* L$. Since

$$\chi(L) = \chi(\pi^* L),$$

so

$$d + 1 - g = \text{rk}(W) \left( \frac{\deg W}{\rk W} + 1 - g_{\mathbb{P}^1} \right) = 2 \left( \frac{\deg W}{2} + 1 \right)$$

which gives

$$\deg W = d - g - 1.$$ 

Thus,

$$\det W \cong \mathcal{O}_{\mathbb{P}^1}(d - g - 1),$$

$$W \cong W^*(d - g - 1).$$

Since $\deg W = d - g - 1$, there is a unique integer $x \leq \frac{d - g - 1}{2}$ such that

$$W \cong \mathcal{O}_{\mathbb{P}^1}(x) \bigoplus \mathcal{O}_{\mathbb{P}^1}(d - g - 1 - x). \quad (2)$$

**Remark 1.**

(i) $x$ is the least integer such that

$$H^1(W(-2 - x)) = H^1(\pi^* L(-2 - x)) = H^1(L \otimes T^{-2-x}) \neq 0.$$
In particular, this implies that \( \deg(L \otimes T^{-2-x}) \leq 2g - 2 \) and thus we have

\[
\frac{d - 2g - 2}{2} \leq x \leq \frac{d - g - 1}{2}.
\]

(ii) Since \( H^1(L \otimes T^{-2}) = 0 \), thus \( x > 0 \), so we have

\[
\max\left\{ 1, \frac{d - 2g - 2}{2} \right\} \leq x \leq \frac{d - g - 1}{2}.
\]

which implies that both \( W \) and \( W(-1) \) are globally generated.

(iii) Also, \( H^1(L \otimes T^{-2}) = 0 \) implies \( H^1(L) = 0 \), thus by Riemann–Roch theorem, we have

\[
h^0(L) = d - g + 1 \tag{3}
\]

and \( \operatorname{rank}(E) = h^0(L) - 1 = d - g \geq 3 \) (since \( d \geq 2g + 1 \) and \( g \geq 2 \)).

(iv) We have

\[
\Gamma(W(-1)) \cong \Gamma(\pi_*(L \otimes T^{-1})) \\
\cong \Gamma(L \otimes T^{-1})
\]

Also \( H^1(L \otimes T^{-2}) = 0 \) gives \( H^1(L \otimes T^{-1}) = 0 \). Thus, by Riemann–Roch theorem, we have

\[
h^0(L \otimes T^{-1}) = d - g - 1
\]

i.e., we have

\[
h^0(W(-1)) = d - g - 1. \tag{4}
\]

Since \( W(-1) \) is globally generated and \( \Gamma(W) \cong \Gamma(L) \), we have a surjection

\[
\Gamma(L)_{\mathbb{P}^1} \to W \to 0,
\]

which is an isomorphism for sections. Since \( W(-1) \) is a globally generated bundle on \( \mathbb{P}^1 \), \( W \) is very ample, i.e., we get an inclusion

\[
\mathbb{P}(W^*) \hookrightarrow \mathbb{P}(\Gamma(L)^*) =: \mathbb{P}. \tag{5}
\]

Also we have a surjection

\[
\pi^*W \to L \to 0.
\]

In other words, we have a subbundle of \( \pi^*(W^*) \) that is isomorphic to \( L^{-1} \). This gives a morphism from \( C \) to \( \mathbb{P}(W^*) \) with the property that the pullback of \( O_W(1) \) to \( C \) is \( L \). Also the composite of this morphism with the projection \( p : \mathbb{P}(W^*) \to \mathbb{P}^1 \) is \( \pi \). Since the induced map

\[
\Gamma(\mathbb{P}(W^*), O_W(1)) \cong \Gamma(\mathbb{P}^1, W) \to \Gamma(C, L)
\]
is an isomorphism, $C$ is actually embedded in $\mathbb{P}(W^*)$. Let us denote the image of $\mathbb{P}(W^*)$ in $\mathbb{P}$ by $S$. We return to the ruled surface $p : \mathbb{P}(W^*) \to \mathbb{P}^1$. By (5), there is an embedding $\mathbb{P}(W^*) \subset \mathbb{P} = \mathbb{P}(\Gamma(L)^*)$ with hyperplane section $\tau = \mathcal{O}_W(1)$. Note that $\tau^2 = \deg W = d - g - 1$. Let $h$ be the class of a fibre of the projection $p$. Then the Picard group of $\mathbb{P}(W^*)$ is generated by $\tau$ and $h$. Since $C$ is a secant (2-section) of $\mathbb{P}(W^*)$, its class is of the form $\mathcal{O}_W(2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(m)$. To compute $m$, we note that $d = C.\tau = 2\tau^2 + m$. Thus $m = 2g - d + 2$ and we have the following proposition.

**Proposition 2.1**

There are inclusions $C \subset S \subset \mathbb{P}$ with the following properties:

(i) the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to $S$ is $\mathcal{O}_W(1)$;
(ii) the restriction of $\mathcal{O}_{\mathbb{P}}(1)$ to $C$ is $L$;
(iii) both restrictions induce isomorphisms of the corresponding linear systems;
(iv) the divisor class on $S$ defined by $C$ is $\mathcal{O}_W(2) \otimes p^*\mathcal{O}_{\mathbb{P}^1}(-d + 2g + 2)$.

**Notation.** We will use the notation $U$ for the vector space $\Gamma(L \otimes T^{-1})$, i.e. we have

$$\Gamma(W(-1)) \cong U.$$  \hfill (6)

and by (4), we have

$$\dim U = d - g - 1.$$  \hfill (7)

**3. Computation of dimensions**

In order to prove the conjecture, we want to relate the sections of $\bigwedge^i E$ to the sections of a suitable vector bundle on $\mathbb{P}^1$.

**Lemma 3.1.** Let $F$ be a vector bundle on $\mathbb{P}^1$ that is globally generated. Then the evaluation sequence is

$$0 \to \Gamma(F(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(F)_{\mathbb{P}^1} \to F \to 0.$$  \hfill (8)

**Proof.** $F$ is a sum of line bundles of degree $\geq 0$. Thus it remains to check for line bundles, which is easy. \hfill $\square$

We want to apply this lemma to $W$.

$$0 \to \Gamma(W(-1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(W)_{\mathbb{P}^1} \to W \to 0.$$  \hfill (9)

Pulling back the evaluation sequence for $W$ on $\mathbb{P}^1$ to $C$ and using (6) and the fact that $\Gamma(\pi_*L) \cong \Gamma(L)$, we get

$$0 \to U \otimes T^{-1} \to \Gamma(L)_C \to \pi^*W \to 0.$$  \hfill (9)
Also, we have a surjective map $\pi^* W \to L \to 0$. Let $Y$ be the kernel of $\pi^* W \to L \to 0$, i.e. we have

$$0 \to Y \to \pi^* W \to L \to 0,$$

$$\wedge^2(\pi^* W) \cong Y \otimes L,$$

$$\pi^*(\wedge^2 W) \cong Y \otimes L,$$

$$\pi^*(O_{\mathbb{P}^1}(d-g-1)) \cong Y \otimes L,$$

$$T^{d-g-1} \cong Y \otimes L,$$

$$Y \cong L^{-1} \otimes T^{d-g-1}.$$

Thus, we have

$$0 \to L^{-1} \otimes T^{d-g-1} \to \pi^* W \to L \to 0 \quad (10)$$

and we get a following commutative diagram

$$\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & U \otimes T^{-1} \\
\downarrow & & \downarrow \\
0 & \to & \Gamma(L)_C \\
\downarrow & & \downarrow \\
0 & \to & \Gamma(L)_C \\
\end{array} \quad (11)$$

where the left vertical map is the evaluation map.

Dualizing the diagram (11), we get

$$\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & L \otimes T^{-(d-g-1)} \\
\downarrow & & \downarrow \\
0 & \to & \pi^* W^* \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array} \quad (12)$$

The first line of (12) gives rise to an exact sequence

$$0 \to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1}U^* \otimes T^{i-1} \to \wedge^i E \to \wedge^i U^* \otimes T^i \to 0. \quad (13)$$
Since, $\pi^*W^*$ is a rank 2 bundle we only get a filtration consisting of the following two exact sequences:

$$0 \to \bigwedge^2 \pi^*W^* \otimes \bigwedge^{i-2}U^* \otimes T^{i-2} \to \bigwedge^i \Gamma(L)^*_C \to L_i \to 0,$$  \hspace{1cm} (14)

$$0 \to \pi^*W^* \otimes \bigwedge^{i-1}U^* \otimes T^{i-1} \to L_i \to \bigwedge^i U^* \otimes T^i \to 0.$$ \hspace{1cm} (15)

Since the second horizontal sequence of (12) is the pullback via $\pi$ of the dual of the sequence (8), both the above sequences come from $P^1$, i.e. there exists a vector bundle $L_i'$ on $P^1$, such that $L_i = \pi^* L_i'$ and the sequences

$$0 \to \bigwedge^2 W^* \otimes \bigwedge^{i-2}U^* \otimes O(i-2) \to \bigwedge^i \Gamma(L)^*_{P^1} \to L_i' \to 0,$$ \hspace{1cm} (16)

$$0 \to W^* \otimes \bigwedge^{i-1}U^* \otimes O(i-1) \to L_i' \to \bigwedge^i U^* \otimes O(i) \to 0$$ \hspace{1cm} (17)

are such that (14) and (15) are pullbacks of (16) and (17) respectively. Dualizing (1), we have

$$0 \to L^{-1} \to \Gamma(L)^*_C \to E \to 0.$$  

Thus the map $\bigwedge^i \Gamma(L)^*_C \to \bigwedge^i E$ is surjective. Also we have $\bigwedge^i \Gamma(L)^*_C \to L_i \to 0$. The maps $\bigwedge^i \Gamma(L)^*_C \to \bigwedge^i E \to 0$ factors through $L_i = \pi^* L_i'$. Thus we get the following commutative diagram with exact rows and columns:

$$
\begin{array}{cccccc}
0 & & & & & 0 \\
& & & & & \\
0 & \to & L \otimes T^{-(d-g-1)} \otimes \bigwedge^{i-1}U^* \otimes T^{i-1} & \to & \bigwedge^i E & \to & \bigwedge^i U^* \otimes T^i \to 0 \\
& & & & & \\
& \uparrow & & & & \uparrow \\
0 & \to & \pi^*W^* \otimes \bigwedge^{i-1}U^* \otimes T^{i-1} & \to & \pi^* L_i' & \to & \bigwedge^i U^* \otimes T^i \to 0 \\
& & & & & \uparrow & \uparrow \\
& & & & & \bigwedge^i \bigwedge^{i-1}U^* \otimes T^{i-1} & \to & \bigwedge^i \bigwedge^{i-1}U^* \otimes T^{i-1} \\
& & & & & \uparrow & \uparrow \\
& & & & & \bigwedge^1 \bigwedge^{i-1}U^* \otimes T^{i-1} & \to & \bigwedge^1 \bigwedge^{i-1}U^* \otimes T^{i-1} \\
& & & & & \uparrow & \uparrow \\
& & & & & 0 & \to & 0 \\
\end{array}
$$  \hspace{1cm} (18)

where the top horizontal sequence is (13), middle horizontal sequence is (15), left vertical sequence is obtained by dualizing (10) and tensoring it with $\bigwedge^{i-1} U^* \otimes T^{i-1}$. Now let us compute the dimensions of the spaces $\Gamma(L'_i)$ and $\Gamma(\bigwedge^i E)$ for $i \leq d - g$ ($= \text{rank } E$).

**Lemma 3.2.** When $d \geq 2g + 1$, we have

$$\dim \Gamma(L'_i) = \binom{d-g+1}{i} + \binom{d-g-1}{i-2} (d-i-g)$$

for $i \leq d - g$

**Proof.** Consider (16). Then

$$0 \to \bigwedge^2 W^* \otimes \bigwedge^{i-2}U^* \otimes O(i-2) \to \bigwedge^i \Gamma(L)^*_{P^1} \to L_i' \to 0.$$
Conjecture on linear systems

Since
\[ \det W \cong \mathcal{O}_{\mathbb{P}^1}(d - g - 1), \]
we get
\[ \det W^* \cong \mathcal{O}_{\mathbb{P}^1}(-(d - g - 1)). \]
Hence we get
\[ 0 \to \wedge^{i-2}U^* \otimes \mathcal{O}(i - d + g - 1) \to \wedge^i \Gamma(L)^* \to L_i' \to 0. \]
Since
\[ i \leq d - g, \quad h^0(\mathcal{O}(i - d + g - 1)) = 0, \]
therefore,
\[ h^0(L_i') = h^0(\wedge^i \Gamma(L)^*) + h^1(\wedge^{i-2}U^* \otimes \mathcal{O}(i - d + g - 1)). \]
By (7), we have
\[ \dim U = d - g - 1, \]
and by (3)
\[ h^0(L) = d - g + 1, \]
\[ h^1(\mathcal{O}(i - d + g - 1)) = d - i - g. \]
Thus
\[ \dim \Gamma(L_i') = \left( \binom{d - g + 1}{i} \right) + \left( \binom{d - g - 1}{i - 2} \right)(d - i - g). \]

3.1 Syzygies of the curve

The syzygies of canonically embedded curves were computed by Schreyer [8]. Based on the parallel idea, we compute the syzygies of the curve \( C \). For this, let
\[ R = \bigoplus_{i=1}^{\infty} \Gamma(C, L^i) \]
be the homogeneous coordinate ring of \( C \) with respect to \( L \) and
\[ S = \text{Sym} \Gamma(C, L) = \bigoplus_{n \geq 0} \Gamma(\mathcal{O}_{\mathbb{P}^d-g}(n)). \]
Let
\[ 0 \to F_t \to \cdots \to F_0 \to R \to 0 \quad (19) \]
be a minimal free resolution of the graded \( S \)-module \( R \). Then \( F_i = \bigoplus_j S(-j)^{\beta_{ij}} = \bigoplus_j M_{ij} \otimes S(-j) \), where \( M_{ij} \) is a \( k \)-vector space of \( \dim \beta_{ij} \) and \( S(-j) \) is the free
$S$-module with one generator in degree $j$. The resolution (19) is equivalent to the free resolution of $\mathcal{O}_C$ as an $\mathcal{O}_{Pd-g}$-module:

$$0 \to \bigoplus_j \mathcal{O}(−j)^{\delta_{d-g−1,j}} \to \cdots \to \bigoplus_j \mathcal{O}(−j)^{\delta_{0,j}} \to \mathcal{O}_C \to 0.$$ 

To find this resolution, one starts with the exact sequence

$$0 \to \mathcal{O}_S(−2τ + (d − 2g − 2)h) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

(see Proposition 2.1). The idea is to first resolve the sheaves $\mathcal{O}_S$ and $\mathcal{O}_S(−2τ + (d − 2g − 2)h)$ as $\mathcal{O}_{Pd-g}$ modules and then form a mapping cone mapping cone. The result turns out to be a minimal resolution of $\mathcal{O}_C$. Firstly, we will recall from [2] the description of the syzygies of these sheaves.

Let $\xi = \mathcal{O}(e_1) \oplus \mathcal{O}(e_2) \oplus \cdots \oplus \mathcal{O}(e_s)$ be a locally free sheaf of rank $s$ on $\mathbb{P}^1$, and let $p_\xi : \mathbb{P}(\xi) \to \mathbb{P}^1$ denote the corresponding $\mathbb{P}^{s−1}$ bundle. A rational normal scroll $X$ of type $S(e_1, e_2, \cdots, e_s)$ with $e_1 \geq e_2 \geq \cdots \geq e_s \geq 0$ and

$$f = e_1 + e_2 + \cdots + e_s \geq 2$$

is the image of $\mathbb{P}(\xi)$ in $\mathbb{P}^r = \mathbb{P}(H^0(\mathbb{P}(\xi), \mathcal{O}_{\mathbb{P}(\xi)}(1)))$:

$$j : \mathbb{P}(\xi) \to X \subset \mathbb{P}^r, r = f + s − 1.$$ 

The Picard group of $\mathbb{P}(\xi)$ is generated by the hyper-plane class $H = [j^*\mathcal{O}_{\mathbb{P}^r}(1)]$ and the ruling $R = [p_\xi^*\mathcal{O}_{\mathbb{P}^1}(1)]$:

$$\text{Pic } \mathbb{P}(\xi) = \mathbb{Z}H \bigoplus \mathbb{Z}R.$$ 

The intersection product is given by

$$H^s = f, H^{s−1} \cdot R = 1, R^2 = 0.$$ 

We recall from [2], the description of the syzygies of the sheaves

$$\mathcal{O}_X(aH + bR) := j_*\mathcal{O}_{\mathbb{P}(\xi)}(aH + bR), \quad a, b \in \mathbb{Z}$$

regarded as $\mathcal{O}_{\mathbb{P}^r}$- modules, at least in case $b \geq −1$. Let

$$\Phi : F \to G$$

be a map of locally free sheaves of rank $f'$ and $g'$, $f' \geq g'$, respectively on a smooth variety $V$. We recall from [1] the family of complexes $\zeta^b, b \geq −1$ of locally free sheaves on $V$, which resolve the $b$-th symmetric power of coker $\Phi$ under suitable hypothesis on $\Phi$.

Define the $j$-th term in the complex $\zeta^b$ by

$$\zeta^b_j = \begin{cases} \wedge^j F \otimes S_{b−j}G, & \text{for } 0 \leq j \leq b, \\ \wedge^{j+g'-1} F \otimes D_{j−b−1}G^* \otimes \wedge^{g'}G^*, & \text{for } j \geq b + 1 \end{cases}$$

and differential

$$\zeta^b_j \to \zeta^b_{j−1}.$$
by multiplication with $\Phi \in H^0(V, F^* \otimes G)$ for $j \neq b + 1$ and $\wedge^s \Phi \in H^0(V, \wedge^s F^* \otimes \wedge^s G)$ for $j = b + 1$ in the appropriate term of the exterior ($\wedge F$), symmetric ($SG$) or divided power ($DG$) algebra.

**PROPOSITION 3.3** [2]

$\zeta_b(a)$ for $b \geq -1$ is the minimal resolution of $O_X(aH + bR)$ as an $O_{Pr}$-module, where $\zeta_b(a) = \zeta^b \otimes O_{Pr}(a)$.

### 3.2 Minimal resolution of $O_C$

We have

$$C \subset S \subset \mathbb{P} = \mathbb{P}(\Gamma(C, L)^*)$$

$C$ is contained in a 2-dimensional rational normal scroll $S$ of type $S(e_1, e_2)$ and degree $f = e_1 + e_2 = d - g - 1 \geq 2$. $C$ is a divisor of class

$$C \sim 2H - (f - (g + 1))R \text{ on } S.$$ 

The mapping cone [8]

$$\zeta^{f-(g+1)}(-2) \to \zeta^0$$

is the minimal resolution of $O_C$ as an $O_{pd-g}$-module. We consider

$$\Phi : F \otimes O_{pd-g}(-1) \to G \otimes O_{pd-g}$$

be the map of locally free sheaves, where $F$ is a vector space of dimension $f = d - g - 1$ and $G$ is a vector space of dimension 2.

Firstly, we will compute

$$\zeta^{f-(g+1)}(-2) = \zeta^{d-2g-2} \otimes O(-2).$$

Now,

$$\zeta^{d-2g-2}_j = \begin{cases} 
\wedge^j(F \otimes O(-1)) \otimes S_{d-2g-2-j}(G \otimes O), & 0 \leq j \leq d - 2g - 2, \\
\wedge^{j+1}(F \otimes O(-1)) \otimes D_{j-d+2g+2-1}(G \otimes O)^* \otimes \wedge^2(G \otimes O)^*, & j \geq d - 2g - 1.
\end{cases}$$

Since $j + 1$ can be at most $d - g - 1$, we have

$$\zeta^{d-2g-2}_j = \begin{cases} 
\wedge^j(F \otimes O(-1)) \otimes S_{d-2g-2-j}(G \otimes O), & 0 \leq j \leq d - 2g - 2, \\
\wedge^{j+1}(F \otimes O(-1)) \otimes D_{j-d+2g+2-1}(G \otimes O)^* \otimes \wedge^2(G \otimes O)^*, & d - 2g - 1 \leq j \leq d - g - 2.
\end{cases}$$
Similarly we can compute $\zeta^0_j$:

$$
\zeta^0_j = \begin{cases} 
\wedge^j (F \otimes O(-1)) \otimes S_{0-j} (G \otimes O), & j = 0, \\
\wedge^{j+1} (F \otimes O(-1)) \otimes D_{j-1} (G \otimes O)^* \otimes \wedge^2 (G \otimes O)^*, & 1 \leq j \leq d - g - 2.
\end{cases}
$$

Since the mapping cone $\xi_f^{-(g+1)(-2)} \to \zeta^0$ is the minimal free resolution of $O_C$ as an $O_{pd-g}$-module, the exact sequence

$$
0 \to (\xi^{d-2g-2} - 2[-1] \bigoplus \zeta^0_{d-g-1}) \to (\xi^{d-2g-2} - 2[-1] \bigoplus \zeta^0)_{d-g-2} \to \cdots \to (\xi^{d-2g-2} - 2[-1] \bigoplus \zeta^0)_2 \to (\xi^{d-2g-2} - 2[-1] \bigoplus \zeta^0)_1 \to \zeta^0 \to O_C \to 0,
$$

where

$$(\xi^{d-2g-2} - 2[-1] \bigoplus \zeta^0)_p = \xi^{d-2g-2} - 2[-1] \bigoplus \zeta^0, \quad p \geq 1$$

is the minimal free resolution of $O_C$.

We will use this resolution to compute the $\dim \Gamma(\wedge^i E)$. For this, consider (19), the minimal free resolution of $R$ and recalling the results from [7], we have

$$M_{p, p+q} = \operatorname{coker}(\wedge^{p+1} V \otimes \Gamma(C, K \otimes L^{g-1}) \to \Gamma(C, \wedge^p E^* \otimes L^q)),$$

where $M_{p, p+q} = \langle \text{Tor}^S_p (C, R) \rangle_{p+q}$ and $\dim(\text{Tor}^S_p (C, R))_{p+q} = \beta_{p, p+q}$. Since $H^1(L) = 0$, so we have

$$M_{p, p+2} \approx H^1(\wedge^{p+1} E^* \otimes L),$$

$$M^*_{p, p+2} \approx H^0(\wedge^{p+1} E \otimes L^{-1} \otimes K).$$

**Lemma 3.4.** When $d \geq 2g + 1$, we have

$$
\dim \Gamma(\wedge^i E) = \begin{cases} 
\binom{d-g+1}{i} + \binom{d-g-1}{i-2} (d - i), & 2 \leq i \leq g - 1, \\
\binom{d-g+1}{i} + \binom{d-g-1}{i-2} (d - i), & g \leq i \leq d - g.
\end{cases}
$$

**Proof.** Consider (13),

$$0 \to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \to \wedge^i E \to \wedge^i U^* \otimes T^i \to 0,$$

i.e.,

$$0 \to L \otimes T^{-(d-g-i)} \otimes \wedge^{i-1} U^* \to \wedge^i E \to \wedge^i U^* \otimes T^i \to 0.$$
Thus, 

\[ h^0(\wedge^i E) = [h^0(L \otimes T^{-(d-g-i)}) - h^1(L \otimes T^{-(d-g-i)})] \left( \begin{array}{c} d - g - 1 \\ i - 1 \end{array} \right) \]

\[ + [h^0(T^i) - h^1(T^i)] \left( \begin{array}{c} d - g - 1 \\ i \end{array} \right) + h^1(\wedge^i E) \]

\[ = \left( \begin{array}{c} d - g - 1 \\ i - 1 \end{array} \right) (g - d + 2i + 1) + \left( \begin{array}{c} d - g - 1 \\ i \end{array} \right) (1 + 2i - g) \]

\[ + h^1(\wedge^i E). \]

Now, 

\[ H^1(\wedge^i E) = H^1(\wedge^{d-g-i} E^* \otimes L) \quad (\text{rank } E = d - g) \]

\[ = H^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K)^*. \]

Thus, 

\[ h^1(\wedge^i E) = h^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K). \]

Since 

\[ M_{p,p+2}^* \approx H^0(\wedge^{p+1} E \otimes L^{-1} \otimes K), \]

we have 

\[ M_{d-g-i-1}^* = H^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K) \]

and 

\[ \dim M_{d-g-i-1,d-g-i+1}^* = h^0(\wedge^{d-g-i} E \otimes L^{-1} \otimes K) = h^1(\wedge^i E). \]

In order to compute \( h^1(\wedge^i E) \), i.e. \( \dim M_{d-g-i-1,d-g-i+1}^* \), let us look at degree \((d - g - i + 1)\) component of the term \((\zeta^{d-2g-2}(-2)[-1] \oplus \xi^0)_{d-g-i-1} = \zeta^{d-2g-2}_{d-g-i-2}(-2) \oplus \xi^0_{d-g-i-1}\) in the minimal free resolution of \( \mathcal{O}_C \).

For \( i \leq g - 1, d - g - i + 2 \geq d - 2g - 1 \), we have 

\[ \dim M_{d-g-i-1,d-g-i+1}^* = \dim[\wedge^{d-g-i-1}(F \otimes \mathcal{O}(-1)) \otimes D_{g-i-1} \]

\[ \times (G \otimes \mathcal{O})^* \otimes \wedge^2(G \otimes \mathcal{O})^* \otimes \mathcal{O}(-2)] \]

\[ = \left( \begin{array}{c} d - g - 1 \\ i \end{array} \right) (g - i), \]

since the contribution of term \( \zeta^0_{d-g-i-1} \) in degree \((d - g - i + 1)\) in the minimal free resolution of \( \mathcal{O}_C \) is zero.

For \( i \geq g \), 

\[ \dim M_{d-g-i-1,d-g-i+1}^* = 0, \]
since the contribution of both terms $\xi^{d-g-i-1}_d$ and $\xi^{d-2g-2}_{d-g-i-2}$ in degree $(d-g-i+1)$ in the minimal free resolution of $O_C$ is zero. Thus for $i < g$,

$$h^0(\wedge^i E) = \left(\frac{d-g-1}{i-1}\right)(g-d+2i+1) + \left(\frac{d-g-1}{i}\right)(1+2i-g) + h^1(\wedge^i E)$$

$$= \left(\frac{d-g-1}{i-1}\right)(g-d+2i+1) + \left(\frac{d-g-1}{i}\right)(1+i) - \left(\frac{d-g-1}{i}\right)(g-i) + \left(\frac{d-g-1}{i}\right)(g)$$

$$= \left(\frac{d-g-1}{i-1}\right)(g-d+2i+1) + \left(\frac{d-g-1}{i}\right)(1+i) + \left(\frac{d-g+1}{i}\right) - \left(\frac{d-g+1}{i}\right)$$

$$= \left(\frac{d-g+1}{i}\right) + \left(\frac{d-g-1}{i-2}\right)(d-i-g),$$

and for $i \geq g$,

$$h^0(\wedge^i E) = \left(\frac{d-g+1}{i}\right) + \left(\frac{d-g-1}{i-2}\right)(d-i-g) + \left(\frac{d-g-1}{i}\right)(i-g).$$

Thus, we have

$$\dim \Gamma(\wedge^i E) = \left\{\begin{array}{ll}
\left(\frac{d-g+1}{i}\right) + \left(\frac{d-g-1}{i-2}\right)(d-i-g), & 2 \leq i \leq g-1, \\
\left(\frac{d-g+1}{i}\right) + \left(\frac{d-g-1}{i-2}\right)(d-i-g) + \left(\frac{d-g-1}{i}\right)(i-g), & g \leq i \leq d-g.
\end{array}\right.$$
where $\alpha'_i$’s are induced by diagram (18).

- $\alpha_1$ is injective. Consider the left vertical sequence of (18),

$$
0 \to L^{-1} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \to \pi^* W^* \otimes \wedge^{i-1} U^* \otimes T^{i-1} \\
\to L \otimes T^{-(d-g-1)} \otimes \wedge^{i-1} U^* \otimes T^{i-1} \to 0.
$$

This gives rise to

$$
0 \to \Gamma(L^{-1} \otimes T^{i-1}) \otimes \wedge^{i-1} U^* \to \Gamma(W^*(i-1)) \otimes \wedge^{i-1} U^* \\
\alpha_1 \Gamma(L \otimes T^{-(d-g-1)+(i-1)}) \otimes \wedge^{i-1} U^* \to \ldots.
$$

Since

$$
\Gamma(W^* \otimes O_{\mathbb{P}^1}(i-1)) \cong \Gamma(W \otimes O_{\mathbb{P}^1}(-d+g+i)) \\
\cong \Gamma(\pi_* L \otimes O_{\mathbb{P}^1}(-d+g+i)) \\
\cong \Gamma(\pi_* (L \otimes \pi^* O_{\mathbb{P}^1}(-d+g+i))) \\
\cong \Gamma(L \otimes \pi^* O_{\mathbb{P}^1}(-d+g+i)) \\
\cong \Gamma(L \otimes T^{-d+g+i}).
$$

Thus $\alpha_1$ is injective.

- $\alpha_3$ is injective by definition. Hence $\alpha_2$ is injective and since both the spaces have the same dimension for $i \leq g$, the map $\Gamma(L^i) \to \Gamma(\wedge^i E)$ is an isomorphism. □

4. Construction of a subbundle of $E$

We want to prove the conjecture for hyperelliptic curves of genus $g$. We shall first do this for $i = 2$. The main point is to construct sufficiently many locally decomposable sections that are not globally decomposable.

Consider $p : \mathbb{P}(W^*) \to \mathbb{P}^1$, the natural projection. For every $a \in \mathbb{P}^1$, the fibre $l_a = p^{-1}(a)$ is a secant of the curve $C$. Let $W^*_a$ be the fibre of $W^*$ at $a$. Since $W$ is globally generated, we have $\Gamma(W)_{\mathbb{P}^1} \to W \to 0$. Hence $W = \pi_* L$ and $\Gamma(\pi_* L) \cong \Gamma(L)$ and thus we have $\Gamma(L)_{\mathbb{P}^1} \to W \to 0$, which gives $W^* \to \Gamma(L^*)_{\mathbb{P}^1}$ and we can identify $\Gamma(L)^*$ with $\Gamma(E)$. Also, we have

$$
0 \to E^* \to \Gamma(L)_C \to L \to 0,
$$

which gives $\Gamma(L^*)_C \to E$, i.e. a map $\Gamma(E)_C \to E$. Thus we get a map

$$(W_a)^*_C \to \Gamma(E)_C \to E$$

which is composite of the inclusion of $W_a^*$ in $\Gamma(E)$ and the evaluation map.

Let $F(a)$ be the subbundle of $E$ generated by the image of $W_a^*$. A section of $\Gamma(E)$ is non-zero at every point of $C$ if it corresponds to a point of $\mathbb{P}(\Gamma(L)^*)$ not on the curve $C$, while a section corresponding to a point say $x \in C$ vanishes exactly at $x$. Hence the map $W_a^* \to F(a)$ is an isomorphism outside $C \cap l_a$ but has rank 1 over $C \cap l_a$. The induced map $\wedge^2 W_a^* \to \wedge^2 F(a)$ has simple zeros exactly over $C \cap l_a$. Hence $F(a)$ has rank 2 and $\wedge^2 F(a) = T$. The vector bundle $F(a)$ has $W_a^*$ as its space of sections i.e. $\dim \Gamma(F(a)) = 2$. On the other hand, $\dim \Gamma(\wedge^2 F(a)) = \dim \Gamma(T) = 2$. Thus we get a 2-dimensional subspace of $\Gamma(\wedge^2 E)$ consisting of locally decomposable sections of
which only the 1-dimensional subspace $\wedge^2\Gamma(F(a)) \subset \Gamma(\wedge^2 F(a))$ consists of globally decomposable sections.

The next step is to globalize this construction, i.e. to vary the point $a$. We consider the graph inclusion $\Gamma \subset C \times \mathbb{P}^1$ given by the map $\pi$. This divisor belongs to the line bundle $p_1^* T \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(1)$, where $p_1$ and $p_2$ are the natural projections to $C$, resp. $\mathbb{P}^1$. The direct image by $p_2$ of the bundle morphism $p_2^* W^* \to \Gamma(E)_{C \times \mathbb{P}^1}$ yields the map $W^* \to \Gamma(E)_{\mathbb{P}^1}$, and hence a map $\wedge^2 W^* \to \wedge^2 \Gamma(E)_{\mathbb{P}^1}$.

On the other hand, the bundle homomorphism $p_2^* W^* \to \Gamma(E)_{C \times \mathbb{P}^1} \to p_1^* E$ fails to be injective precisely over $\Gamma$. Thus, we get a morphism $p_2^* (\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \to p_1^* (\wedge^2 E)$. Taking direct image by $p_2$ gives a morphism $\wedge^2 W^* \otimes \Gamma(T) \otimes \mathcal{O}(1) \to \Gamma(\wedge^2 E)_{\mathbb{P}^1}$. For every $a \in \mathbb{P}^1$ this induces a map $\Gamma(T) \to \Gamma(\wedge^2 E)_{\mathbb{P}^1}$ and this gives exactly the space of locally decomposable sections described above.

Finally, we get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \wedge^2 W^* & \longrightarrow & \wedge^2 W^* \otimes \Gamma(T) \otimes \mathcal{O}_{\mathbb{P}^1}(1) & \longrightarrow & \wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \wedge^2 \Gamma(E)_{\mathbb{P}^1} & \longrightarrow & \Gamma(\wedge^2 E)_{\mathbb{P}^1} & \longrightarrow & D^2_{\mathbb{P}^1} & \longrightarrow & 0
\end{array}
\]

where $D^2_{\mathbb{P}^1} := \Gamma(\wedge^2 E)_{\wedge^2 \Gamma(E)}$ and the top horizontal row is the evaluation sequence for $\mathcal{O}_{\mathbb{P}^1}(1)$, which is

\[0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0.\]

Tensoring with $\wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(1)$, we get

\[0 \to \wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to \wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) \to 0.\]

We have to show that the locally decomposable sections constructed above together with $\wedge^2 \Gamma(E)$ generate $\Gamma(\wedge^2 E)$. For this, we consider the map $\wedge^2 W^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) \to D^2_{\mathbb{P}^1}$.

We want to show that this map is injective (as a bundle map) and that the resulting rational curve in $\mathbb{P}(D^2)$ is the rational normal curve of degree $d - g - 3$ (recall that $\dim D^2 = d - g - 2$). This is sufficient since the rational normal curve of degree $n$ in $\mathbb{P}^n$ spans $\mathbb{P}^n$.

Our aim is to do this by entirely reducing the problem to computations on $\mathbb{P}^1$, respectively $\mathbb{P}^1 \times \mathbb{P}^1$.

**Lemma 4.1 [5].** Let $\mathcal{O}_{\mathbb{P}^1}(-n) \to \Gamma(\mathcal{O}_{\mathbb{P}^1}(n))_{\mathbb{P}^1}$ be a non-zero $SL_2(\mathbb{C})$-equivariant morphism. Then this morphism defines an embedding of $\mathbb{P}^1$ into $\mathbb{P}(\Gamma(\mathcal{O}_{\mathbb{P}^1}(n)))$ as a rational normal curve of degree $n$.

We return to the bundle $W$. Sequence (16) gives for $i = 2$ the following sequence:

\[0 \to \wedge^2 W^* \otimes \mathcal{O}(\Gamma) \to L'_2 \to 0\]

Consider $\mathbb{P}^1 \times \mathbb{P}^1$ together with projections $q_1$ and $q_2$ respectively. Taking pullback of (21) via $q_1$ and $q_2$, we get a map $q_2^* \wedge^2 W^* \to q_1^* L'_2$ that vanishes along the diagonal $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$. Hence, we get a morphism

\[q_2^* \wedge^2 W^* \otimes \mathcal{O}(\Delta) \to q_1^* L'_2.\]
Applying $q_2^*$, we get a map
\[ \wedge^2 W^* \otimes \Gamma(O(1)) \otimes O(1) \to \Gamma(L'_2)_{\mathbb{P}^1}. \]

This gives rise to a commutative diagram
\[
\begin{array}{c}
0 \\
\downarrow \\
\wedge^2 W^* \\
\downarrow \\
\wedge^2 W^* \otimes \Gamma(O(1)) \otimes O(1) \\
\downarrow \\
\wedge^2 W^* \otimes O(2) \\
\downarrow \\
0
\end{array}
\quad \begin{array}{c}
0 \\
\downarrow \\
\wedge^2 \Gamma(W)^{\ast}_{\mathbb{P}^1} \\
\downarrow \\
\Gamma(L'_2)_{\mathbb{P}^1} \\
\downarrow \\
H^1(\wedge^2 W^*) = \Gamma(\wedge^2 W \otimes O(-2))^* \\
\downarrow \\
0
\end{array}
\]

(22)

where the left-hand column is the Euler sequence on $\mathbb{P}^1$ twisted by $\wedge^2 W^* \otimes O(1)$, the right hand column comes from (21) and the map $\wedge^2 W^* \to \wedge^2 \Gamma(W)^*$ is the natural one. This diagram is $SL_2(\mathbb{C})$ equivariant, where $SL_2(\mathbb{C})$ acts on $\mathbb{P}^1$ in the usual way and on $\mathbb{P}^1 \times \mathbb{P}^1$ by the diagonal action. In particular the morphism $\wedge^2 W^* \otimes O(2) \to \Gamma(\wedge^2 W \otimes O(-2))^*$ is $SL_2(\mathbb{C})$ equivariant, by Lemma 4.2, it defines an embedding of $\mathbb{P}^1$ into $\mathbb{P}(\Gamma(\wedge^2 W \otimes O(-2))^*)$ as a rational normal curve of degree $d - g - 3$.

**Lemma 4.2** [5]. Diagram (22) gives rise to a commutative and exact diagram
\[
\begin{array}{c}
0 \\
\downarrow \\
\Gamma(\wedge^2 W \otimes O(-2)) \\
\downarrow \\
\Gamma(\wedge^2 W \otimes O(-2)) \otimes \Gamma(O(1)) \\
\downarrow \\
0
\end{array}
\quad \begin{array}{c}
\Gamma(L'_2)^* \\
\downarrow \\
\wedge^2 \Gamma(W) \\
\downarrow \\
0
\end{array}
\]

(23)

**PROPOSITION 4.3**

$\Gamma(\wedge^2 E)$ is generated by locally decomposable sections.

**Proof.** We have constructed maps $p_2^*(\wedge^2 W^*) \otimes O(\Gamma) \to p_1^*(\wedge^2 E)$ on $C \times \mathbb{P}^1$ and $q_2^* \wedge^2 W^* \otimes O(\Delta) \to q_1^* L'_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the diagram
\[
\begin{array}{c}
C \xrightarrow{p_1} C \times \mathbb{P}^1 \\
\downarrow \pi \quad \downarrow \pi \times id \\
\mathbb{P}^1 \quad \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{q_3} \mathbb{P}^1
\end{array}
\]

(24)
Pulling the morphism $q_2^* \wedge^2 W^* \otimes \mathcal{O}(\Delta) \rightarrow q_1^* L'_2$ on $\mathbb{P}^1 \times \mathbb{P}^1$ back to $C \times \mathbb{P}^1$, we get a morphism $p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \rightarrow p_1^*(\pi^* L'_2)$. By construction the diagram,

$$
p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \rightarrow p_1^*(\pi^* L'_2) \\
\| \\
p_2^*(\wedge^2 W^*) \otimes \mathcal{O}(\Gamma) \rightarrow p_1^*(\wedge^2 E)
$$

(25)

commutes where the map $p_1^*(\pi^* L'_2) \rightarrow p_1^* \wedge^2 E$ is the pullback via $p_1$ of the corresponding map in diagram (18). Pushing this down via $\pi \times id$ to $\mathbb{P}^1 \times \mathbb{P}^1$ leads to the commutative diagram

$$
q_2^* \wedge^2 W^* \otimes \mathcal{O}(\Delta) \rightarrow q_1^* L'_2 \\
q_2^* \wedge^2 W^* \otimes \mathcal{O}(\Delta) \rightarrow q_2^* \wedge^2 W^* \otimes (\pi \times id)_* \mathcal{O}(\Gamma) \rightarrow q_1^* (\pi_* \pi^* L'_2) \\
q_2^* \wedge^2 W^* \otimes \mathcal{O}(\Delta) \rightarrow q_2^* \wedge^2 W^* \otimes (\pi \times id)_* \mathcal{O}(\Gamma) \rightarrow q_1^* (\pi_* \wedge^2 E)
$$

(26)

Now taking $q_2^*$ of the outermost square, we get

$$
\wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \Gamma(\mathbb{P}^1, L'_2) \\
\wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \Gamma(C, \wedge^2 E)
$$

(27)

where the right-hand vertical map is an isomorphism from Proposition 4.3. Thus in order to compute the diagram

$$
\wedge^2 W^* \rightarrow \wedge^2 \Gamma(W)^*_{\mathbb{P}^1} \\
\wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \Gamma(\wedge^2 E)
$$

(28)

we can compute

$$
\wedge^2 W^* \rightarrow \wedge^2 \Gamma(W)^*_{\mathbb{P}^1} \\
\wedge^2 W^* \otimes \Gamma(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes \mathcal{O}_{\mathbb{P}^1}(1) \rightarrow \Gamma(L'_2)^*_{\mathbb{P}^1}
$$

(29)

and the result follows from Lemma 4.2. \(\square\)

5. Proof of the main result

Here we prove Theorem 1. We shall first show that for $2 \leq i \leq d - g$, there is a natural epimorphism

$$
\wedge^{i-2} \Gamma(W)^* \otimes \Gamma(L'_2)_{\mathbb{P}^1} \rightarrow \Gamma(L'_i)_{\mathbb{P}^1} \rightarrow 0.
$$
Setting $i = 2$ in (16), we get

$$0 \to \wedge^2 W^* \to \wedge^2 \Gamma(L)^*_{\mathbb{P}^1} \to L'_2 \to 0.$$  

Twisting with $\wedge^{i-2} \Gamma(W)^*$, we get an exact sequence

$$0 \to \wedge^{i-2} \Gamma(W)^* \otimes \wedge^2 W^* \to \wedge^{i-2} \Gamma(W)^* \otimes \wedge^2 \Gamma(W)^* \to \wedge^{i-2} \Gamma(W)^* \otimes L'_2 \to 0$$

(since $\Gamma(L) \cong \Gamma(W)$) Combining this with (16), we get

$$0 \to \wedge^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}(i - 2) \otimes \wedge^2 W^* \to \wedge^{i-2} \Gamma(W)^* \to L'_i \to 0 \quad \text{(30)}$$

Here the middle vertical map is the canonical one and the left hand vertical map is given by taking $\wedge^{i-2}$ of the dual evaluation sequence

$$0 \to W^* \to \Gamma(W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \to \Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to 0.$$

Taking $\wedge^{i-2}$ of the above sequence, we get

$$0 \to \wedge^2 W^* \otimes \wedge^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i - 2) \to \wedge^{i-2} \Gamma(W)^* \to F_{i-2} \to 0,$$

$$0 \to F_{i-2} \to \wedge^{i-2} \Gamma(W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \to \wedge^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i - 2) \to 0.$$

Tensoring the above sequence with $\wedge^2 W^*$, we get

$$0 \to F_{i-2} \otimes \wedge^2 W^* \to \wedge^{i-2} \Gamma(W)^* \otimes \mathcal{O}_{\mathbb{P}^1} \otimes \wedge^2 W^*$$

$$\to \wedge^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}_{\mathbb{P}^1}(i - 2) \otimes \wedge^2 W^* \to 0.$$

Taking the associated cohomology sequence of (30), we get the following commutative diagram:

$$
\begin{array}{cccc}
0 & \to & \wedge^{i-2} \Gamma(W)^* \otimes \wedge^2 \Gamma(W)^* & \to \wedge^{i} \Gamma(W)^* \\
\downarrow & & \downarrow & \downarrow \\
\wedge^{i-2} \Gamma(W)^* \otimes \Gamma(L'_2) & \to & \Gamma(L'_i) & \to 0 \\
\downarrow & & \downarrow & \\
H^1(\wedge^{i-2} \Gamma(W)^* \otimes \wedge^2 W^*) & \to & H^1(\wedge^{i-2} \Gamma(W(-1))^* \otimes \mathcal{O}(i - 2) \otimes \wedge^2 W^*) & \to 0 \\
\downarrow & & \downarrow & \vdots \\
0 & & \vdots & \\
\end{array}
$$

(31)

Here $W$ is a rank 2 vector bundle on $\mathbb{P}^1$ of degree $d - g - 1$.

$$\det W \cong \mathcal{O}(d - g - 1),$$

$$\det W^* \cong \mathcal{O}(-d + g + 1),$$

i.e.

$$\wedge^2 W^* \cong \mathcal{O}(-d + g + 1).$$

Since $2 \leq d - g - 1$. Therefore $\Gamma(\wedge^2 W) = 0$,
The top horizontal map is clearly surjective. The bottom horizontal map is surjective since $H^2$ vanishes on $\mathbb{P}^1$. By standard diagram chasing the middle horizontal map must be surjective thus giving our first claim.

By construction the natural diagram

\[
\begin{array}{ccc}
\wedge^{i-2}\Gamma(W)^* \otimes \Gamma(L'_2) & \rightarrow & \wedge^{i-2}\Gamma(E) \otimes \Gamma(\wedge^2 E) \\
\downarrow & & \downarrow \\
\Gamma(L'_i) & \rightarrow & \Gamma(\wedge^i E)
\end{array}
\] (32)

commutes.

Since $\Gamma(W)^* \cong \Gamma(L)^*$ and $\Gamma(L)^*$ can be identified with $\Gamma(E)$, by Proposition 3.5, the horizontal maps are isomorphisms for $i \leq g$. Hence the natural map $\wedge^{i-2}\Gamma(E) \otimes \Gamma(\wedge^2 E) \rightarrow \Gamma(\wedge^i E)$ is surjective for $i \leq g$ and our claim follows from Proposition 4.3.

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