Reformulated Dissipation for the Free-Stream Preserving of the Conservative Finite Difference Schemes on Curvilinear Grids

Hongmin Su · Jinsheng Cai · Shucheng Pan · Xiangyu Hu

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Abstract
In this paper, we develop a new free-stream preserving (FP) method for high-order upwind conservative finite-difference (FD) schemes on curvilinear grids. This FP method is constructed by subtracting a reference cell-face flow state from each cell-center value in the local stencil of the original upwind schemes, which effectively leads to a reformulated dissipation. It is convenient to implement this method, as it only approximates the cell-center fluxes and conservative variables before reconstructions rather than performs the FP techniques for the central and dissipation parts individually, which avoids introducing considerable complexities to the original reconstruction procedures. In addition, the proposed method removes the constraint in the traditional FP conservative FD schemes that require a consistent scheme for the metrics discretization and the central part of fluxes discretization. With this, the proposed method is more flexible in simulating the engineering problems which usually require a low-order scheme for their low-quality mesh, while the high-order schemes can be applied to approximate the flow states to improve the resolution. After demonstrating the strict FP property and the order of accuracy by two simple test cases, we consider various validation cases, including the supersonic flow around the cylinder, the subsonic flow past the three-element airfoil, and the transonic flow around the ONERA M6 wing, etc., to show that the method is suitable for a wide range of fluid dynamic problems containing complex geometries.

Shucheng Pan
shucheng.pan@nwpu.edu.cn
Hongmin Su
hongminsu@mail.nwpu.edu.cn
Jinsheng Cai
caijsh@nwpu.edu.cn
Xiangyu Hu
xiangyu.hu@tum.de

1 Department of Fluid Mechanics, School of Aeronautics, Northwestern Polytechnical University, Xi’an 710072, China
2 Department of Mechanical Engineering, Technical University of Munich, 85748 Garching, Germany
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1 Introduction

Recently, high-order numerical methods have become a powerful tool to understand the underlying mechanism in many complicated fluid mechanics problems, for example, hydrodynamic instabilities, turbulence, and aeroacoustics. Among those methods, the finite difference method (FDM) is relatively easier to implement multiple dimensions in the structured grids [26], as its dimension-by-dimension discretization does not require high-order numerical quadrature in the finite-volume method. Thus, numerous high-order FD schemes have been developed, such as the compact difference scheme [15, 30], the essentially non-oscillatory scheme (ENO) [9], the weighted essentially non-oscillatory scheme (WENO) [12], and the weighted compact nonlinear scheme (WCNS) [7]. However, when the high-order conservative FD schemes are applied on the body-fitted mesh for flows around complex geometries of the aircraft and automobiles, the violation of the geometric conservation law (GCL) [27], including the volume conservation law (VCL) [1, 2] and the surface conservation law (SCL) [34], leads to large errors that may destabilize numerical simulations or induces spurious hydrodynamic fluctuations. Specifically for the stationary curvilinear grid, this is reflected by the free-stream preservation problem, i.e. the initial uniform flow field becomes nonuniform.

Different numerical treatments have been proposed to preserve the free-stream condition for FDM in general curvilinear grids, such as the pioneering work of Thomas and Lombard [27] who used the conservative form of metrics for computing geometric transformation. Later, this conservative metric technique was applied by Visbal and Gaitonde [30] to maintain GCL for high-order compact central schemes. Then, the symmetrical conservative metric method (SCMM) was proposed by Deng et al. [6] and Abe et al. [2] independently, based on the symmetric forms of metrics derived by Vinokur and Yee [29]. In such a way, the asymmetric metric errors of the previously developed conservative metric methods [5, 27] are eliminated by preserving the coordinate-invariant property. Other variants of the SCMM are also derived in the literature, such as Liao’s cell-center version SCMM [18, 19]. And Abe et al. [2] elaborate on the geometric interpretations of metrics and Jacobian for both the conservative metric method (CMM) and SCMM. Once this SCMM is applied, a sufficient condition to preserve GCL is to apply the identical central finite difference discretization schemes for the derivatives in both the fluxes and the grid metrics [2, 6].

Straightforward extension of the above-mentioned to upwind linear schemes or nonlinear schemes cannot preserve the GCL as the different numerical schemes are required by grid metrics and fluxes, which violates the sufficient condition [2, 6]. One way to address this is to independently approximate the flow variables and grid metrics. A successful strategy to maintain GCL for nonlinear schemes is the combination of the conservative metrics and the WCNS, as demonstrated by Nonomura et al. [22]. A similar method is the alternative finite-difference form of WENO (AWENO) [4, 13, 14, 33]. Another way is to split the original upwind scheme into the central and dissipation parts (hereafter referred to as central-dissipative splitting). Then one can approximate the geometric metrics in the central part by the high-order central schemes and SCMM, which maintains the free-stream property. Then, the dissipation part is enforced to satisfy the free-stream condition by using the finite-volume-analogy approximations, for example, applying the identical metrics in the entire stencil [23], deriving a local...
difference form with every two neighboring cells [35], and replacing the transformed conservative variables with the original one [16, 17]. In addition, to satisfy the above sufficient condition, Ref. [36] introduces an offsetting term with the same WENO nonlinear weights for computing the corresponding inviscid fluxes. Nevertheless, more improvement is required when upwind linear schemes or nonlinear schemes are applied to realistic engineering problems. On the one hand, these methods require rearranging the original upwind reconstruction procedures by the summation of the central and dissipation parts, followed by applying different FP treatments for these two parts. This central-dissipative-splitting strategy, however, modifies the original formulation of the upwind schemes and may yield additional operation complexity and difficulty in implementing these algorithms. On the other hand, these methods are required to satisfy the sufficient condition provided in Refs. [6] and [2], which introduces a constraint that they must apply the unique high-order discretization for the grid metrics and the central part of the fluxes. As a result, the simulations may blow up on a low-quality grid.

In this study, we propose an efficient strategy to maintain the free-stream preserving identity by applying a reformulated upwind dissipation to the linear upwind and WENO schemes. The novelty is twofold. First, in our method, the FP property is not achieved by splitting the reconstruction operation into two parts, but by splitting the cell-center fluxes. Thus, the original schemes are not required to be modified, indicating a straightforward implementation. Second, they remove the constraint of using the same discretization for the grid metrics and fluxes, which permits using low-order schemes for grid information to enhance the robustness in low-quality meshes and using high-order schemes for flow fluxes to increase the effective resolution. The outline is organized as follows. In Sec. 2, we first introduce the Navier-Stokes (NS) equations and discuss the original linear upwind and WENO schemes on curvilinear grids based on flux vector splitting. Next, the metrics and SCL identity are also included in this section. In Sec. 3, the proposed strategy on free-stream preserving for linear upwind and WENO schemes is described in detail. Free-streaming preserving capability, order of accuracy, and robustness of the proposed method are demonstrated in Sec. 4 by a range of numerical examples, followed by a concluding remark in Sec. 5.

2 Governing Equations and Numerical Methods

2.1 Governing Equations on Stationary Curvilinear Coordinates

The compressible Navier–Stokes equations on curvilinear grids are given by

\[
\frac{\partial}{\partial t} \left( \frac{Q}{J} \right) + \frac{\partial}{\partial \xi} \left( \frac{\xi_x F + \xi_y G + \xi_z H}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{\eta_x F + \eta_y G + \eta_z H}{J} \right) \\
+ \frac{\partial}{\partial \zeta} \left( \frac{\zeta_x F + \zeta_y G + \zeta_z H}{J} \right) \\
- \frac{\partial}{\partial \xi} \left( \frac{\xi_x F_v + \xi_y G_v + \xi_z H_v}{J} \right) - \frac{\partial}{\partial \eta} \left( \frac{\eta_x F_v + \eta_y G_v + \eta_z H_v}{J} \right) \\
- \frac{\partial}{\partial \zeta} \left( \frac{\zeta_x F_v + \zeta_y G_v + \zeta_z H_v}{J} \right) = 0
\] (1)
with

\[ Q = \left( \rho \, \rho u_1 \, \rho u_2 \, \rho u_3 \, \rho E \right)^T, \]
(2)

\[ F = \left( \rho u_1 \, \rho u_1 u_1 + p \, \rho u_2 u_1 \, \rho u_3 u_1 \, (\rho E + p)u_1 \right)^T, \]
(3)

\[ G = \left( \rho u_2 \, \rho u_1 u_2 + p \, \rho u_2 u_2 \, \rho u_3 u_2 \, (\rho E + p)u_2 \right)^T, \]
(4)

\[ H = \left( \rho u_3 \, \rho u_1 u_3 + p \, \rho u_2 u_3 \, \rho u_3 u_3 \, (\rho E + p)u_3 \right)^T, \]
(5)

\[ F_v = \left( 0 \, \tau_{11} \, \tau_{12} \, \tau_{13} \, u_i \tau_{i1} - \dot{q}_1 \right)^T, \]
(6)

\[ G_v = \left( 0 \, \tau_{21} \, \tau_{22} \, \tau_{23} \, u_i \tau_{i2} - \dot{q}_2 \right)^T, \]
(7)

\[ H_v = \left( 0 \, \tau_{31} \, \tau_{32} \, \tau_{33} \, u_i \tau_{i3} - \dot{q}_3 \right)^T, \]
(8)

where \( u_1, u_2, u_3, F, G, H \) and \( F_v, G_v, H_v \) denote the velocity components, the inviscid and viscous flux vectors in \( x, y \) and \( z \) direction, respectively. \( \rho, p \) and \( E \) are the density, pressure and the total specific energy. \( t \) stands for the physical time. \( \xi, \eta, \zeta \) are the transformed coordinates on a uniform computational domain, and \( J \) is the transformed Jacobian. \( \tau_{ij} \) is the shear stress tensor

\[ \tau_{ij} = 2\mu \left( S_{ij} - \delta_{ij} \frac{S_{kk}}{3} \right), \]
(9)

\[ S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \]
(10)

and \( \dot{q}_i \) is the heat flux in direction \( i \)

\[ \dot{q}_i = -\lambda \frac{\partial T}{\partial x_i}, \]
(11)

where \( \mu \) and \( \lambda \) is the shear viscosity and thermal conductivity. The equation of state for ideal gas is

\[ p = (\gamma - 1) \rho e, \]
(12)

where the specific heat ratio is \( \gamma = 1.4 \).

For convenience, we denote the fluxes in computational domain as

\[ \tilde{Q} = \frac{Q}{J}, \]
(13)

\[ \tilde{F} = \frac{\xi_x F + \xi_y G + \xi_z H}{J}, \]
\[ \tilde{G} = \frac{\eta_x F + \eta_y G + \eta_z H}{J}, \]
\[ \tilde{H} = \frac{\zeta_x F + \zeta_y G + \zeta_z H}{J}, \]
\[ \tilde{F}_v = \frac{\xi_x F_v + \xi_y G_v + \xi_z H_v}{J}, \]
\[ \tilde{G}_v = \frac{\eta_x F_v + \eta_y G_v + \eta_z H_v}{J}, \]
\[ \tilde{H}_v = \frac{\zeta_x F_v + \zeta_y G_v + \zeta_z H_v}{J}. \]
(15)
Then, the Navier–Stokes equations can be rewritten by

$$\frac{\partial \tilde{Q}}{\partial t} + \frac{\partial \tilde{F}}{\partial \xi} + \frac{\partial \tilde{G}}{\partial \eta} + \frac{\partial \tilde{H}}{\partial \zeta} - \frac{\partial \tilde{F}_v}{\partial \xi} - \frac{\partial \tilde{G}_v}{\partial \eta} - \frac{\partial \tilde{H}_v}{\partial \zeta} = 0$$

(16)

### 2.2 Discretization Methods

If not mentioned otherwise, the 3rd-order TVD Runge–Kutta method [8] is applied to perform the time integration. Our numerical methods are based on the conservative finite difference method, with the viscous fluxes and the convective fluxes being calculated by the 6th-order central difference scheme and the upwind high-order reconstruction schemes, respectively.

#### 2.2.1 Spatial Discretization of the Convective Fluxes

Without loss of generality, we consider the fluxes along the $\xi$ direction, which is indexed by $i$. In conservative FDM, let the fluxes $\tilde{F}_i$ at cell-center $i$ be the average of a numerical flux function $\tilde{F}(\xi)$,

$$\tilde{F}_i = \frac{1}{\Delta \xi} \int_{i-1/2}^{i+1/2} \tilde{F}(\xi) d\xi. \quad (17)$$

Then the derivative of $\tilde{F}_i$ can be calculated exactly by

$$\left( \frac{\partial \tilde{F}}{\partial \xi} \right)_i = \frac{\tilde{F}(i + 1/2) - \tilde{F}(i - 1/2)}{\Delta \xi}. \quad (18)$$

The numerical flux at the cell face, say $\tilde{F}(i + 1/2)$, can be reconstructed through the neighboring cell-center fluxes $\tilde{F}_{i-k+1}, \ldots, \tilde{F}_{i+k-1}$ to achieve a $(2k - 1)$th-order accuracy, such as the WENO scheme, resulting in a $(2k - 1)$th-order accuracy of $\partial \tilde{F}/\partial \xi$ as well. Therefore, the derivative of the convective fluxes can be approximated by these reconstructed cell-face numerical fluxes (denoted as $\tilde{F}_{i+1/2}, \tilde{F}_{i-1/2}$)

$$\left( \frac{\partial \tilde{F}}{\partial \xi} \right)_i = \frac{\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}}{\Delta \xi} + O(\Delta \xi^{2k-1}). \quad (19)$$

Therefore, the key to achieve the discretization of the spatial derivative in the governing equations is to approximate the cell-face numerical fluxes $\tilde{F}_{i+1/2}$ appropriately.

#### 2.2.2 The Characteristic-Wise Reconstruction Procedures

For the reconstruction of the $\tilde{F}_{i+1/2}$, a robust way is to perform reconstruction in the characteristic space. For instance, we can express the numerical flux in Eq. (18) as

$$\tilde{F}_{i+1/2} = R \cdot \left( \tilde{F}^+_{i+1/2} + \tilde{F}^-_{i+1/2} \right), \quad (20)$$

and $\tilde{F}^\pm_{i+1/2}$ is obtained by a specific upwind reconstruction scheme $UP\text{WIN}\text{D}(\cdot)$

$$\tilde{F}^+_{i+1/2} = UP\text{WIN}\text{D}(\tilde{F}^+_{i-k+1}, \tilde{F}^+_{i-k+2}, \ldots, \tilde{F}^+_{i+k-1}), \quad (21)$$

$$\tilde{F}^-_{i+1/2} = UP\text{WIN}\text{D}(\tilde{F}^-_{i+k}, \tilde{F}^-_{i+k-1}, \ldots, \tilde{F}^-_{i-k+2}), \quad (22)$$
where \( \vec{F}_m^\pm (m = i - k + 1, \cdots, i + k) \) are the transformed cell-center fluxes in the stencil \((i - k + 1, \cdots, i + k)\) of the cell ‘i’, obtained by applying the Lax-Friedrichs splitting scheme in the characteristic space,

\[
\vec{F}_m^\pm = \frac{1}{2} \left( L \vec{F}_m \pm \Lambda L \vec{Q}_m \right), m = i - k + 1, \cdots, i + k,
\]

with \( \Lambda \) being the diagonal matrix composed of the corresponding maximum eigenvalues along the stencils of the linearized Roe-average Jacobian matrix \( A_{i+1/2} = (\partial \vec{F} / \partial \vec{Q})_{i+1/2} \), \( L \) is the left matrix composed of the corresponding eigenvectors of \( A_{i+1/2} \), and \( R \) is the inverse matrix of \( L \).

The upwind reconstruction \( UP\, WIND(\cdot) \) can be the 5th-order linear upwind scheme (LU5), which results in the following positive and negative cell-face numerical fluxes \( \vec{F}_i^{\pm+} \),

\[
\begin{align*}
\vec{F}_{i+1/2}^+ &= \frac{1}{60} \left( 2 \vec{F}_{i-2}^+ - 13 \vec{F}_{i-1}^+ + 47 \vec{F}_i^+ + 27 \vec{F}_{i+1}^+ - 3 \vec{F}_{i+2}^+ \right), \\
\vec{F}_{i+1/2}^- &= \frac{1}{60} \left( -3 \vec{F}_{i-1}^- + 27 \vec{F}_i^- + 47 \vec{F}_{i+1}^- - 13 \vec{F}_{i+2}^- + 2 \vec{F}_{i+3}^- \right).
\end{align*}
\]

Also, we can consider the nonlinear upwind reconstruction for capturing discontinuities, such as the classical 5th-order WENO (WENO5) scheme [12],

\[
\tilde{h}_{i+1/2}^\pm = \sum_{k=0}^{2} \omega_k^\pm q_k^\pm,
\]

where \( \tilde{h}^\pm \) denotes each component of \( \vec{F}^\pm \). Consider \( \tilde{h}^+ \) as an example, the 3rd-order approximations for the three different sub-stencils are

\[
\begin{align*}
q_0^+ &= \frac{1}{3} \tilde{f}_{i-2}^+ - \frac{7}{6} \tilde{f}_{i-1}^+ + \frac{11}{6} \tilde{f}_i^+, \\
q_1^+ &= -\frac{1}{6} \tilde{f}_{i-1}^+ + \frac{5}{6} \tilde{f}_i^+ + \frac{1}{3} \tilde{f}_{i+1}^+, \\
q_2^+ &= \frac{1}{3} \tilde{f}_i^+ + \frac{5}{6} \tilde{f}_{i+1}^- - \frac{1}{6} \tilde{f}_{i+2}^+.
\end{align*}
\]

where \( \tilde{f}^\pm \) denotes each component of \( \vec{F}^\pm \) and the corresponding nonlinear weight \( \omega_k^+ \) is proposed to be determined by

\[
\omega_k^+ = \frac{C_k}{(\beta_k^+ + \epsilon)^n} \sum_{r=0}^{2} \frac{C_r}{(\beta_r^+ + \epsilon)^n},
\]

where \( C_0 = \frac{1}{10} \), \( C_1 = \frac{3}{5} \), \( C_2 = \frac{3}{10} \) are the optimal weights, \( \epsilon = 1.0 \times 10^{-6} \) and \( n = 2 \). The smoothness indicators are evaluated by

\[
\begin{align*}
\beta_0^+ &= \frac{1}{4} \left( \tilde{f}_{i-2}^+ - 4 \tilde{f}_{i-1}^+ + 6 \tilde{f}_i^+ \right)^2 + \frac{13}{12} \left( \tilde{f}_{i-2}^+ - 2 \tilde{f}_{i-1}^+ + \tilde{f}_i^+ \right)^2, \\
\beta_1^+ &= \frac{1}{4} \left( -\tilde{f}_{i-1}^+ + \tilde{f}_{i+1}^+ \right)^2 + \frac{13}{12} \left( \tilde{f}_{i-1}^+ - 2 \tilde{f}_i^+ + \tilde{f}_{i+1}^+ \right)^2, \\
\beta_2^+ &= \frac{1}{4} \left( -3 \tilde{f}_i^+ + 4 \tilde{f}_{i+2}^- - \tilde{f}_{i+1}^- \right)^2 + \frac{13}{12} \left( \tilde{f}_i^+ - 2 \tilde{f}_{i+1}^+ + \tilde{f}_{i+2}^+ \right)^2.
\end{align*}
\]
2.3 Geometric Conservation Law and Metrics Discretization

When subtracting the flow information from the governing equations by applying the uniform free-stream condition to Eq. (1), these equations are reformulated as

\[
\frac{\partial}{\partial \xi} \left( \frac{\xi_x + \xi_y + \xi_z}{J} \right) + \frac{\partial}{\partial \eta} \left( \frac{\eta_x + \eta_y + \eta_z}{J} \right) + \frac{\partial}{\partial \zeta} \left( \frac{\xi_x + \xi_y + \xi_z}{J} \right) = 0.
\]  

(29)

An sufficient condition to satisfy Eq. (29) is

\[
I_x = \left( \frac{\xi_x}{J} \right)_{\xi} + \left( \frac{\eta_x}{J} \right)_{\eta} + \left( \frac{\zeta_x}{J} \right)_{\zeta} = 0,
\]

(30)

\[
I_y = \left( \frac{\xi_y}{J} \right)_{\xi} + \left( \frac{\eta_y}{J} \right)_{\eta} + \left( \frac{\zeta_y}{J} \right)_{\zeta} = 0,
\]

\[
I_z = \left( \frac{\xi_z}{J} \right)_{\xi} + \left( \frac{\eta_z}{J} \right)_{\eta} + \left( \frac{\zeta_z}{J} \right)_{\zeta} = 0.
\]

These equations represent the consistence of vectorized computational cell surfaces in FVM as discussed in Ref. [28]. Although Eq. (30) is strictly satisfied theoretically, the numerical discretization errors from the fluxes discretization may violate these identities, which are regarded as the SCL problem by Zhang et al. [34]. Usually, the free-stream preserving condition is considered to be maintained if the discrete geometric errors do not generate spurious disturbances for the uniform flows.

Deng et al. [6] and Abe et al. [2] suggest a sufficient condition that states that any high-order finite difference scheme can maintain SCL, as long as (1) the same discretization scheme is applied to the fluxes, metrics, and Jacobian; (2) the symmetry conservative metrics and Jacobian are used, see the detailed proof in Refs. [2, 6]. The method satisfying these two conditions refers to the symmetrical conservative metric method (SCMM). Specifically, the symmetry conservative metrics can be expressed by

\[
\begin{align*}
\xi_x &= \frac{1}{2} \left[ (yz \zeta)_{\xi} - (y \zeta \zeta)_{\eta} + (y \zeta \eta)_{\zeta} - (y \zeta \eta)_{\zeta} \right], \\
\xi_y &= \frac{1}{2} \left[ (xz \zeta)_{\xi} - (x \zeta \zeta)_{\eta} + (x \zeta \eta)_{\zeta} - (x \zeta \eta)_{\zeta} \right], \\
\xi_z &= \frac{1}{2} \left[ (xz \zeta)_{\xi} - (x \zeta \xi)_{\eta} + (x \zeta \eta)_{\zeta} - (x \zeta \eta)_{\zeta} \right], \\
\eta_x &= \frac{1}{2} \left[ (yx \zeta)_{\xi} - (y \zeta \xi)_{\eta} + (y \zeta \eta)_{\zeta} - (y \zeta \eta)_{\zeta} \right], \\
\eta_y &= \frac{1}{2} \left[ (xz \xi)_{\eta} - (x \zeta \xi)_{\eta} + (x \zeta \eta)_{\zeta} - (x \zeta \eta)_{\zeta} \right], \\
\eta_z &= \frac{1}{2} \left[ (yx \xi)_{\eta} - (y \zeta \xi)_{\eta} + (y \zeta \eta)_{\zeta} - (y \zeta \eta)_{\zeta} \right], \\
\zeta_x &= \frac{1}{2} \left[ (yz \xi)_{\eta} - (y \xi \zeta)_{\eta} + (y \xi \eta)_{\zeta} - (y \xi \eta)_{\zeta} \right], \\
\zeta_y &= \frac{1}{2} \left[ (xz \xi)_{\eta} - (x \zeta \xi)_{\eta} + (x \zeta \eta)_{\zeta} - (x \zeta \eta)_{\zeta} \right], \\
\zeta_z &= \frac{1}{2} \left[ (xz \xi)_{\eta} - (x \xi \zeta)_{\eta} + (x \xi \eta)_{\zeta} - (x \xi \eta)_{\zeta} \right].
\end{align*}
\]  

(31)
and

\[
\frac{1}{J} = \frac{1}{3} \left[ \left( \frac{\xi_x}{J} + y \frac{\xi_y}{J} + z \frac{\xi_z}{J} \right)_i + \left( x \frac{\eta_x}{J} + y \frac{\eta_y}{J} + z \frac{\eta_z}{J} \right)_i + \left( x \frac{\zeta_x}{J} + y \frac{\zeta_y}{J} + z \frac{\zeta_z}{J} \right)_i \right].
\]  

(32)

With SCMM, the high-order central finite difference schemes indeed fulfill the SCL and achieve the FP property successfully. While for upwind linear and nonlinear schemes in conservative FDM, achieving the FP identity by SCMM is not straightforward since the geometric relations in Eqs. (31) and (32) usually fail to be calculated by the same upwind scheme with the flux derivatives computing, especially impracticable for the nonlinear upwind schemes.

3 Free-Stream Preserving Strategy for Upwind Schemes

In this part, we propose a novel, simple and general strategy by reformulating the upwind dissipation to fulfill the FP property for the upwind schemes. To begin with, like the existing FP methods [23, 35], the discretizations of metrics in Eqs. (31) and (32) are performed with the central schemes corresponding to the central part of the upwind schemes, for examples,

\[
(x_{i+1/2})_{2h} = \frac{1}{2} (x_i + x_{i+1}),
\]

\[
(x_{i+1/2})_{4h} = \frac{1}{12} (-x_{i-1} + 7x_i + 7x_{i+1} - x_{i+2}),
\]

\[
(x_{i+1/2})_{6h} = \frac{1}{60} (x_{i-2} - 8x_{i-1} + 37x_i + 37x_{i+1} - 8x_{i+2} + x_{i+3}),
\]

\[
(x_{i+1/2})_{8h} = \frac{1}{840} (-3x_{i-3} + 29x_{i-2} - 139x_{i-1} + 533x_i + 533x_{i+1} - 139x_{i+2} + 29x_{i+3} - 3x_{i+4}),
\]

\[
\left( \frac{\partial x}{\partial \xi} \right)_i = \frac{1}{\Delta \xi} (x_{i+1/2} - x_{i-1/2}).
\]

(33)

In the following discussions, without loss of generality, the LU5 and WENO5 schemes are considered to reconstruct the cell-face numerical fluxes with this suggested FP strategy. Unlike previous FP methods which utilize central-dissipative splitting on the reconstruction operation, the basic idea of the present FP strategy is to split the cell-center positive and negative fluxes \( \tilde{F}_m^+ \) into two parts by a special reference flow state in the local stencil, without changing the reconstruction formulation itself. With this, one part can be proved to satisfy the FP identity after the linear or nonlinear upwind reconstruction, and the remaining part achieves the FP property by a central scheme with the SCMM.

First, we denote the reference flow state \( \mathbf{Q}^*_{i+1/2} \) in the local stencil, and for brevity the subscript is omitted,

\[
\mathbf{Q}^* = \tilde{\mathbf{Q}}^*_{i+1/2} \left( \frac{1}{J} \right)_{i+1/2}.
\]

(34)

If apply the 6th-order approximations, we give

\[
\tilde{\mathbf{Q}}^*_{i+1/2} = \frac{1}{60} \left( \tilde{\mathbf{Q}}_{i-2} - 8 \tilde{\mathbf{Q}}_{i-1} + 37 \tilde{\mathbf{Q}}_i + 37 \tilde{\mathbf{Q}}_{i+1} - 8 \tilde{\mathbf{Q}}_{i+2} + \tilde{\mathbf{Q}}_{i+3} \right),
\]

(35)

and cell-face metrics

\[
\left( \frac{1}{J} \right)_{i+1/2} = \frac{1}{60} \left[ \left( \frac{1}{J} \right)_{i-2} - 8 \left( \frac{1}{J} \right)_{i-1} + 37 \left( \frac{1}{J} \right)_i + 37 \left( \frac{1}{J} \right)_{i+1} - 8 \left( \frac{1}{J} \right)_{i+2} + \left( \frac{1}{J} \right)_{i+3} \right].
\]

(36)
Thus, $Q^*$ is merely a linear combination of the original flow states, and is constant within the local stencil. Furthermore, the related fluxes of the reference conservative variables $Q^*$ are denoted as $F^*$, $G^*$ and $H^*$ which are also constant in the local stencil. It should be noted that this reference flow state is not arbitrary because it is chosen appropriately to achieve the FP identity for the upwind schemes which will be explained in detail as follows.

As both the reference flux $F^*$ and flow states $Q^*_{i+1/2}$ are local numerical variables, the Jacobian matrix ($\partial F^*/\partial Q^*$) can not be analytically derived. In addition, we emphasize that the reference flux and flow states in the reconstruction stencil ($i-2, \ldots, i+3$) are constant, and are numerical approximated by all cell-center variables in this stencil. Thus, we can follow the numerical treatment used in the original WENO method, Eq. (23), i.e. the Jacobian matrix of all cells in the stencil is approximated by the $F^*$ term by $F^*$ are denoted as $Q^*$, $F^*$ and $Q^*$, respectively.

Thus, we can construct FP numerical fluxes $\tilde{F}_{i+1/2}^{FP}$ by

\[ \tilde{F}_{i+1/2}^{\text{FP}, +} = U PWIND \left( \tilde{F}_{i-2}^+ - \tilde{F}_{i-2}^*, \ldots, \tilde{F}_{i+2}^+ - \tilde{F}_{i+2}^* \right) + CENTRAL \left( \tilde{F}_{i-2}^*, \ldots, \tilde{F}_{i+3}^* \right) \]

\[ \tilde{F}_{i+1/2}^{\text{FP}, -} = U PWIND \left( \tilde{F}_{i+3}^- - \tilde{F}_{i+3}^*, \ldots, \tilde{F}_{i-1}^- - \tilde{F}_{i-1}^* \right) + CENTRAL \left( \tilde{F}_{i-2}^*, \ldots, \tilde{F}_{i+3}^* \right) \]
where \( \textit{UP WIND}(\cdot) \) operation, as mentioned above, is the upwind reconstruction, and \( \textit{CENTRAL}(\cdot) \) stands for the central reconstruction. After substituting the above \( \tilde{F}_{\textit{FP}, \pm} \) into Eq. (40), we can obtain the final cell-face fluxes \( \tilde{F}_{i+1/2} \), i.e.

\[
\tilde{F}_{i+1/2}^{FP} = R \cdot \left( \tilde{F}_{i+1/2}^{FP,+} + \tilde{F}_{i+1/2}^{FP,-} \right),
\]

which is free-stream preserving because the \( \textit{CENTRAL} \) portion is calculated by the SCMM while the \( \textit{UP WIND} \) portion becomes zero under the uniform flows.

Practically, Eqs. (42) and (43) can be further simplified. First, we denote that

\[
\tilde{F}_{i+1/2}^* = F^* \left( \frac{\xi x}{J} \right)_{i+1/2} + \cdots + H^* \left( \frac{\xi z}{J} \right)_{i+1/2}.
\]

With this, we have

\[
\tilde{F}_{i+1/2}^{C, \pm} = \textit{CENTRAL} \left( \tilde{F}_{i-2}^{*, \pm}, \cdots, \tilde{F}_{i+3}^{*, \pm} \right)
\]

\[
= \frac{1}{2} L \left[ F^* \left( \frac{\xi x}{J} \right)_{i+1/2} + \cdots + H^* \left( \frac{\xi z}{J} \right)_{i+1/2} \right] \pm \frac{1}{2} \Lambda L Q^* \left( \frac{1}{J} \right)_{i+1/2},
\]

which are constant vectors within the local reconstruction stencil. Here, The cell-face metric, \( (\xi x/J)_{i+1/2} \), is approximated by the central scheme employed in the \( \textit{CENTRAL}(\cdot) \) operation, say the 6th-order approximation,

\[
\left( \frac{\xi x}{J} \right)_{i+1/2} = \frac{1}{60} \left[ \left( \frac{\xi x}{J} \right)_{i-2} - 8 \left( \frac{\xi x}{J} \right)_{i-1} + 37 \left( \frac{\xi x}{J} \right)_{i} + 37 \left( \frac{\xi x}{J} \right)_{i+1} - 8 \left( \frac{\xi x}{J} \right)_{i+2} + \left( \frac{\xi x}{J} \right)_{i+3} \right].
\]

For simplicity, the \( \textit{CENTRAL} \) portion, i.e. the constant vectors \( \tilde{F}_{i+1/2}^{C, \pm} \), can be moved into the \( \textit{UP WIND} \) part without changing the final reconstructed numerical fluxes \( \tilde{F}_{i+1/2}^{FP, \pm} \). Therefore, we reformulate Eqs. (42) and (43) by

\[
\tilde{F}_{i+1/2}^{FP,+} = \textit{UP WIND} \left( \tilde{F}_{i-2}^{+} - \tilde{F}_{i-2}^{*,+} + \tilde{F}_{i+2}^{C,+}, \cdots, \tilde{F}_{i+3}^{+} - \tilde{F}_{i+2}^{*,+} + \tilde{F}_{i+1/2}^{C,+} \right),
\]

\[
\tilde{F}_{i+1/2}^{FP,-} = \textit{UP WIND} \left( \tilde{F}_{i+3}^{-} - \tilde{F}_{i+3}^{*,-} + \tilde{F}_{i+1/2}^{C,-}, \cdots, \tilde{F}_{i-1}^{-} - \tilde{F}_{i-1}^{*,-} + \tilde{F}_{i+1/2}^{C,-} \right).
\]

By comparing the above two equations with Eqs. (21) and (22), we observe that the inputs of Eqs. (21) and (22) are changed from \( \tilde{F}_{m}^{\pm} \) to \( \tilde{F}_{m}^{\pm} - \tilde{F}_{m}^{*,\pm} + \tilde{F}_{i+1/2}^{C,\pm} \) in Eqs. (48) and (49), i.e.,

\[
\tilde{F}_{m}^{\pm} \approx \tilde{F}_{m}^{FP,\pm} = \tilde{F}_{m}^{\pm} - \tilde{F}_{m}^{*,\pm} + \tilde{F}_{i+1/2}^{C,\pm}, \quad m = i-2, \ldots, i+3
\]

\[
= \frac{1}{2} L \left[ (F_m - F^*) \left( \frac{\xi x}{J} \right)_{m} + F^* \left( \frac{\xi x}{J} \right)_{i+1/2} + \cdots + (H_m - H^*) \left( \frac{\xi z}{J} \right)_{m} + H^* \left( \frac{\xi z}{J} \right)_{i+1/2} \right] \pm \frac{1}{2} \Lambda L \left[ (Q_m - Q^*) \left( \frac{1}{J} \right)_{m} + Q^* \left( \frac{1}{J} \right)_{i+1/2} \right].
\]

Note that this flux approximation is performed in the characteristic space, as each term (\( \tilde{F}_{m}^{\pm} \), \( \tilde{F}_{m}^{*,\pm} \), and \( \tilde{F}_{i+1/2}^{C,\pm} \)) involves the characteristic transformation, see Eqs. [SPSeqref7eqrefSPS].
(37), and (46). The corresponding approximation of the physical space is

\[ \tilde{F}_m \approx \tilde{F}^{FP}_m = \tilde{F}_m - \tilde{F}^*_m + \tilde{F}^*_{i+1/2} \]

\[ = (F_m - F^*) \left( \frac{\xi_x}{J} \right)_m + F^* \left( \frac{\xi_x}{J} \right)_{i+1/2} + (G_m - G^*) \left( \frac{\xi_y}{J} \right)_m + G^* \left( \frac{\xi_y}{J} \right)_{i+1/2} + (H_m - H^*) \left( \frac{\xi_z}{J} \right)_m + H^* \left( \frac{\xi_z}{J} \right)_{i+1/2}, \]

and

\[ \tilde{Q}_m \approx \tilde{Q}^{FP}_m = \tilde{Q}_m - \tilde{Q}^*_m + \tilde{Q}^*_{i+1/2} \]

\[ = (Q_m - Q^*) \left( \frac{1}{J} \right)_m + Q^* \left( \frac{1}{J} \right)_{i+1/2}. \]

In other words, the approximated fluxes and conservative variables in Eqs. (51) and (52) are first transformed to obtain the characteristic cell-center fluxes \( \tilde{F}^{FP,\pm}_m \) in Eq. (50) which are inserted into upwind reconstruction operation in Eqs. (51) and (52) latter to calculate the final FP cell-face fluxes,

\[ \tilde{\mathcal{F}}^{FP, +}_{i+1/2} = U P W I N D \left( \tilde{F}^{FP, +}_{i-2}, \tilde{F}^{FP, +}_{i-1}, \ldots, \tilde{F}^{FP, +}_{i+2} \right), \]

\[ \tilde{\mathcal{F}}^{FP, -}_{i+1/2} = U P W I N D \left( \tilde{F}^{FP, -}_{i+3}, \tilde{F}^{FP, -}_{i+2}, \ldots, \tilde{F}^{FP, -}_{i-1} \right). \]

In this way, the only difference between the original upwind FDM (not FP) and our FP method is that the cell-center fluxes have been modified (compare the above two equations with Eqs. (21) and (22)), while the upwind reconstruction algorithm itself keep invariant. This implies that the complexity of the reconstruction procedures does not increase, especially for nonlinear reconstructions.

**Remark 1** For a given uniform Cartesian grid, i.e. \( (\xi_x/J)_m = (\xi_x/J)_{i+1/2}, \ldots, (1/J)_m = (1/J)_{i+1/2} \), the present method recovers to the original upwind reconstruction schemes, as \( \tilde{F}^*_m = \tilde{F}^*_{i+1/2} \) and \( \tilde{Q}^*_m = \tilde{Q}^*_{i+1/2} \) in Eqs. (51) and (52).

**Remark 2** Unlike previous FP methods in Refs. [23, 35], this method merely modifies the cell-center fluxes, while the reconstruction scheme is not changed. This avoids the complex operations during the central-dissipative splitting of previous FP methods [23, 35]. With this, the extension to other WENO schemes is straightforward, such as the 7th-order WENO [25] and the 6th-order WENO[C] [10].

Given uniform flows, if the reconstructions of the metrics and the reference flow states keep consistent, \( \tilde{F}_m^\pm - \tilde{F}_m^{*, \pm} = 0 \) holds for any specific form of the metrics discretization, which indicates that the UPWIND portion becomes zero anyway under the uniform flows in Eqs. (42) and (43). In addition, according to Eq. (46), the central portion \( \tilde{\mathcal{F}}^{C, \pm}_{i+1/2} \) is merely determined by the metrics since \( F^*, G^*, H^* \), and \( Q^* \) are all constant vectors during the reconstruction procedures, which inherently satisfies the requirements of the SCMM. Therefore, if we compute the metrics by the 2nd-order central scheme rather than the 6th-order one in Eqs. (36) and (47), \( \tilde{\mathcal{F}}^{C, \pm}_{i+1/2} \) can still preserve the FP property. Eventually, we suggest another advantage of this FP method as follows,
Remark 3 The numerical strategy in Eqs. (48) and (49) removes the constraint that the metrics discretization must adopt the same scheme with the central part of the fluxes discretization, as long as the reference conservative quantities in Eq. (35) use the same discretization scheme with that of metrics.

To investigate the dissipation of the present FP scheme and show the difference between it and the original scheme, taking the LU5 scheme as an example, we suggest rewriting them to the summation of central and dissipation parts. Using Eq. (24) to replace \( \tilde{F}_{i+1/2}^\pm \) in Eq. (20) results in

\[
\tilde{F}_{i+1/2}^O = R \cdot \left( \tilde{F}_{i+1/2}^+ + \tilde{F}_{i+1/2}^- \right)
\]

\[
= \frac{1}{60} R \cdot \left( 2 \tilde{F}_{i-2}^+ - 13 \tilde{F}_{i-1}^1 + 47 \tilde{F}_i^+ + 27 \tilde{F}_{i+1}^+ - 3 \tilde{F}_{i+2}^+ \right) + \frac{1}{60} R \cdot \left( -3 \tilde{F}_{i-1}^- + 27 \tilde{F}_i^- + 47 \tilde{F}_{i+1}^- - 13 \tilde{F}_{i+2}^- + 2 \tilde{F}_{i+3}^- \right). 
\]  

(55)

Then, substituting \( \tilde{F}_m^\pm \) defined in Eq. [SPSeqref7eqrefSPS] into the above Eq. (55) yields an alternative form of the original LU5 scheme,

\[
\tilde{F}_{i+1/2}^O = \frac{1}{60} \left( \tilde{F}_{i-2} - 8 \tilde{F}_{i-1} + 37 \tilde{F}_i + 37 \tilde{F}_{i+1} - 8 \tilde{F}_{i+2} + \tilde{F}_{i+3} \right)
\]

\[+
\frac{1}{60} \sum_{s=1}^{5} R^s \lambda^s L^s \left( \tilde{Q}_{i-2} - 5 \tilde{Q}_{i-1} + 10 \tilde{Q}_i - 10 \tilde{Q}_{i+1} + 5 \tilde{Q}_{i+2} - \tilde{Q}_{i+3} \right),
\]

(56)

where \( s \) indexes the component of \( \tilde{Q} \), \( L^s \) and \( R^s \) stand for the \( s \)-th row of \( L \) and the \( s \)-th column of \( R \), respectively. \( \lambda^s \) is the \( s \)-th element of the diagonal matrix \( \Lambda \), i.e. \( \lambda^s = \Lambda_{s,s} \).

Note that the central part is free-stream preserving if the symmetry conservative metrics (Eqs. [SPSeqref7eqrefSPS] and [SPSeqref6eqrefSPS]) are used, with the derivatives in these metrics being computed by the 6th-order central schemes listed in Eq. (33), as both the two conditions required by the SCMM are achieved, as discussed in Sec. 2.3. In addition, we rewrite Eq. (55) by replacing \( \tilde{F}_m^\pm \) with \( \tilde{F}_m^{FP,\pm} \) to obtain the FP LU5,

\[
\tilde{F}_{i+1/2}^{FP} = R \cdot \left( \tilde{F}_{i+1/2}^{FP,+} + \tilde{F}_{i+1/2}^{FP,-} \right)
\]

\[
= \frac{1}{60} R \cdot \left( 2 \tilde{F}_{i-2}^{FP,+} - 13 \tilde{F}_{i-1}^{FP,+} + 47 \tilde{F}_i^{FP,+} + 27 \tilde{F}_{i+1}^{FP,+} - 3 \tilde{F}_{i+2}^{FP,+} \right) + \frac{1}{60} R \cdot \left( -3 \tilde{F}_{i-1}^{FP,-} + 27 \tilde{F}_i^{FP,-} + 47 \tilde{F}_{i+1}^{FP,-} - 13 \tilde{F}_{i+2}^{FP,-} + 2 \tilde{F}_{i+3}^{FP,-} \right). 
\]  

(57)

Then, an alternative form of this FP LU5 scheme is derived by substituting the approximation in Eq. (50) into Eq. (57),

\[
\tilde{F}_{i+1/2}^{FP} = \frac{1}{60} \left( \tilde{F}_{i-2} - 8 \tilde{F}_{i-1} + 37 \tilde{F}_i + 37 \tilde{F}_{i+1} - 8 \tilde{F}_{i+2} + \tilde{F}_{i+3} \right) + \Delta \tilde{F}^* 
\]

\[+
\frac{1}{60} \sum_{s=1}^{5} R^s \lambda^s L^s \left[ \left( \tilde{Q}_{i-2} - \tilde{Q}_{i-1}^g \right) - 5 \left( \tilde{Q}_{i-1} - \tilde{Q}_{i-1}^g \right) + 10 \left( \tilde{Q}_i - \tilde{Q}_i^g \right) 
\]

\[-10 \left( \tilde{Q}_{i+1} - \tilde{Q}_{i+1}^g \right) + 5 \left( \tilde{Q}_{i+2} - \tilde{Q}_{i+2}^g \right) - \left( \tilde{Q}_{i+3} - \tilde{Q}_{i+3}^g \right) \right],
\]

(58)
which can be further simplified by

$$\tilde{F}_{i+1/2}^P = \tilde{F}_{i+1/2}^O + \Delta \tilde{F}^* + \frac{1}{60} \left( \frac{1}{J^6} \frac{\partial J}{\partial \xi} \right) \sum_{s=1}^{5} R^s \lambda^s L^s Q^* \Delta \xi^5 + O(\Delta \xi^6). \tag{59}$$

Here $\Delta \tilde{F}^*$ is defined by

$$\Delta \tilde{F}^* = \frac{1}{60} \left( \tilde{F}^*_{i-2} - 8 \tilde{F}^*_{i-1} + 37 \tilde{F}^*_{i} + 37 \tilde{F}^*_{i+1} - 8 \tilde{F}^*_{i+2} + \tilde{F}^*_{i+3} \right) - \tilde{F}^*_{i+1/2}. \tag{60}$$

which equals zero if the cell-face metrics, say $(\xi_x / J)_{i+1/2}$, are calculated by the 6th-order central scheme, according to the definition of $\tilde{F}_{m}^*$ $(m = i - 2, \cdots, i + 3)$ and $\tilde{F}_{i+1/2}^*$ in Eqs. (45) and (38), respectively. Therefore, the present FP numerical fluxes $\tilde{F}_{i+1/2}^P$ is essentially an approximation of the original fluxes $\tilde{F}_{i+1/2}^O$ by adding an extra 5th-order truncation errors in term of $Q^*$. Meanwhile, the LU5 and WENO5 (see Appendix) with the present FP method only appropriately reformulate the dissipation without altering the central part of the original upwind schemes.

As a conclusion, the proposed FP method for upwind schemes is implemented by a simple approximation of fluxes and conservative variables, i.e. Equations (51) and (52), before the reconstruction procedures. We consider that our method has two advantages over previous central-dissipative-splitting FP methods [16, 23, 35, 36]:

1. it alleviates the implementation difficulty resulting from modifying the original reconstruction schemes and applying different FP techniques, as we merely change the inputs of the reconstruction schemes from $\tilde{F}_m^\pm$ (see Eq. (24)) to $\tilde{F}_m^{FP,\pm}$ (see Eqs. (53) and (54)), with $m = i - 2, \cdots, i + 3$;
2. it removes the constraint that the metrics discretization scheme is required to be the same one applied to the central part of the fluxes discretization in the previous FP methods, indicating a more flexible way to calculate the geometric metrics and fluxes and also increase robustness as well.

4 Numerical Tests on Curvilinear Grids

Several problems, including the free-stream, isotropic vortex convection, double Mach reflection, subsonic flow past the 30P30N three-element airfoil and transonic flow around the three-dimensional (3D) ONERA M6 wing, etc. are conducted to check the performance of the proposed FP method on the curvilinear grids. The global Lax-Friedrichs flux splitting is employed for the double Mach reflection and supersonic flow past a cylinder while other verifications are performed with the local one. In addition, the geometric metrics are obtained by the 2nd- to 8th-order central schemes. In the following, WENO5 and WENO7 are the original 5th-order and 7th-order WENO schemes [12, 25]. WENOUC6 denotes the 6th-order WENOUC6 scheme [10]. Besides, the WENO schemes with the proposed FP strategy are denoted by WENOX-MY, which presents the Xth-order WENO together with Yth-order metrics and Jacobian, respectively. For example, WENO7-M6 means the 7th-order WENO and 6th-order metrics and Jacobian, and LU5-M6 denotes the suggested 5th-order linear upwind scheme coupled with the 6th-order metrics and Jacobian. For comparison, the latest FP method is reproduced by the authors, denoted by WENO5-FP2019 [35]. In addition, we also adopt 2nd-order metrics to the WENO5 scheme for the method in Ref. [35], which is denoted by WENO5-FP2019-M2.
Fig. 1 The wavy and random grids at a resolution of $21 \times 21$

4.1 Simple Test Cases

4.1.1 Free-Streaming Problem

First, we consider the wavy and random grids to test the free-stream preservation, as shown in Fig. 1. The wavy case is defined in the domain $(x, y) \in [-10, 10] \times [-10, 10]$ by

$$x_{i,j} = x_{\text{min}} + \Delta x_0 \left[ (i - 1) + A_x \sin \left( \frac{n_{xy} \pi (j - 1) \Delta y_0}{L_y} \right) \right]$$

$$y_{i,j} = y_{\text{min}} + \Delta y_0 \left[ (j - 1) + A_y \sin \left( \frac{n_{yx} \pi (i - 1) \Delta x_0}{L_x} \right) \right],$$

(61)

where $L_x = L_y = 20$, $x_{\text{min}} = -L_x/2$, $y_{\text{min}} = -L_y/2$, $A_x \Delta x = 0.6$, $A_y \Delta y = 0.6$, and $n_{xy} = n_{yx} = 8$. And the grid points of the random case are generated in a random direction with 20% of the original Cartesian grid size. A coarse grid resolution of $21 \times 21$ is chosen here to highlight the differences between the original WENO and the proposed FP WENO schemes, for both the wavy and random grids.

The initial condition is uniform with a $M = 0.5$ flow in $x$ direction,

$$u = 0.5, v = 0, p = 1, \rho = \gamma,$$

(62)

where $\gamma = 1.4$ is the specific heat ratio. The time step is 0.2, and the average ($L_2$) and maximal ($L_{\infty}$) errors of the velocity components $v$ for the two grids are estimated at $t = 20$. As shown in Table 1, both the $L_2$ and $L_{\infty}$ errors of the original WENO5 and WENO7 schemes have been significantly reduced to almost the machine zero by our proposed FP method (WENO5-M6, WENO7-M6, WENO7-M8), indicating that our method indeed preserves the free-streaming identity.
Fig. 2 The vorticity magnitude contours ranging from 0.0 to 0.006 of the 2D moving vortex on both the wavy and random grids at $t = 40$
Table 1 The $L_2$ and $L_\infty$ errors of the $v$ component on the wavy and random grids at a resolution of $21 \times 21$

| Method     | Wavy grid $L_2$ error | Wavy grid $L_\infty$ error | Random grid $L_2$ error | Random grid $L_\infty$ error |
|------------|-----------------------|-----------------------------|-------------------------|----------------------------|
| WENO5      | $2.45 \times 10^{-2}$ | $4.72 \times 10^{-2}$       | $1.29 \times 10^{-2}$   | $4.41 \times 10^{-2}$      |
| WENO7      | $1.03 \times 10^{-2}$ | $1.98 \times 10^{-2}$       | $1.57 \times 10^{-2}$   | $5.08 \times 10^{-2}$      |
| WENO5-M6   | $5.58 \times 10^{-16}$| $2.05 \times 10^{-15}$      | $7.60 \times 10^{-16}$  | $2.70 \times 10^{-15}$     |
| WENO7-M6   | $5.88 \times 10^{-16}$| $1.92 \times 10^{-15}$      | $8.35 \times 10^{-16}$  | $2.54 \times 10^{-15}$     |
| WENO7-M8   | $4.90 \times 10^{-16}$| $1.94 \times 10^{-15}$      | $6.20 \times 10^{-16}$  | $1.96 \times 10^{-15}$     |

4.1.2 Isotropic Vortex

The two-dimensional (2D) moving isotropic vortex problem on wavy and random grids is considered to evaluate the vortex preservation property. In this case, the fluid is treated as an ideal gas with the specific heat ratio $\gamma = 1.4$. And the initial condition is the superposition of an isotropic vortex (centered at $(x_c, y_c) = (0, 0)$) and a uniform flow of Mach 0.5, with the perturbations of the velocity, temperature, and entropy being

$$\begin{align*}
(\delta u, \delta v) &= \epsilon \tau e^{\alpha(1-r^2)}(\sin \theta, -\cos \theta), \\
\delta T &= -\frac{(\gamma - 1)\epsilon^2}{4\alpha \gamma} e^{2\alpha(1-r^2)}, \\
\delta S &= 0,
\end{align*}$$

(63)

where $\alpha = 0.204$, $\tau = r/r_c$, and $r = \sqrt{(x-x_c)^2 + (y-y_c)^2}$. Here $r_c = 1.0$ and $\epsilon = 0.02$ denote the vortex core length and strength, respectively. $T = p/\rho$ is the temperature, and $S = p/\rho^\gamma$ is the entropy. The periodic boundary condition is imposed, and the vortex moves back to the original position at $t = 40$.

As shown in Fig. 2, the vorticity magnitude contours of the original WENO5 scheme do not maintain the vortex structure on a relatively coarse resolution ($21 \times 21$) for both the wavy and random grids in Fig. 1, due to the free-stream violation errors. While this vortex exhibits in the numerical results of the proposed FP schemes. Next, we assess the grid convergence rate of the proposed schemes on the wavy grids with the resolution of $21 \times 21$, $41 \times 41$, $81 \times 81$, $161 \times 161$ and $321 \times 321$. And the time step $\Delta t$ of each grid is decreased until the $L_2$ and $L_\infty$ errors are invariant to eliminate the time integration errors, as suggested by Refs. [23] and [35]. The $L_2$ and $L_\infty$ errors of $v$ on those wavy grids are listed in Table 2, as well as the corresponding convergence rates.

The numerical results indicate that the schemes with consistent flux reconstruction and metric reconstruction, i.e. the order of the central part of the flux reconstruction method is the same as the order of the metric reconstruction method, including the LU5-M6, WENO5-M6, and WENO7-M8, show their corresponding formal order of accuracy with the resolution increase. On the other hand, for WENO5-M2, WENO5-M4, and WENO7-M4, when the resolution is increased, the obtained order of accuracy is dominated by the metric reconstruction portion whose order is lower than that of the flux reconstruction. However, we note here that for coarse and moderate grid resolutions, from $21 \times 21$ to $81 \times 81$, these schemes have approximately the same accuracy as that of WENO5-M6 and WENO7-M8. Thus, we conclude that the discretization errors of the metrics do not have a significant effect on the accuracy of the under-resolved simulations.
Table 2  The $L_2$ and $L_\infty$ errors of the $v$ component and the corresponding convergence rates on the wavy grid

| Method       | Grid size | $L_2$ error | Convergence rate | $L_\infty$ error | Convergence rate |
|--------------|-----------|-------------|------------------|------------------|-----------------|
| LU5-M6       | 21 $\times$ 21 | $1.83 \times 10^{-3}$ | –                | $1.32 \times 10^{-2}$ | –               |
|              | 41 $\times$ 41   | $3.53 \times 10^{-4}$ | 2.37             | $3.32 \times 10^{-3}$ | 1.99            |
|              | 81 $\times$ 81   | $1.69 \times 10^{-5}$ | 4.38             | $1.60 \times 10^{-4}$ | 4.38            |
|              | 161 $\times$ 161 | $8.77 \times 10^{-7}$ | 4.27             | $1.08 \times 10^{-5}$ | 3.89            |
|              | 321 $\times$ 321 | $3.02 \times 10^{-8}$ | 4.86             | $4.06 \times 10^{-7}$ | 4.73            |
| WENO5-M2     | 21 $\times$ 21   | $2.12 \times 10^{-3}$ | –                | $1.53 \times 10^{-2}$ | –               |
|              | 41 $\times$ 41   | $4.75 \times 10^{-4}$ | 2.16             | $4.28 \times 10^{-3}$ | 1.84            |
|              | 81 $\times$ 81   | $1.63 \times 10^{-5}$ | 4.86             | $1.22 \times 10^{-4}$ | 5.13            |
|              | 161 $\times$ 161 | $4.29 \times 10^{-6}$ | 1.93             | $4.13 \times 10^{-5}$ | 1.56            |
|              | 321 $\times$ 321 | $1.12 \times 10^{-6}$ | 1.94             | $1.01 \times 10^{-5}$ | 2.03            |
| WENO5-M4     | 21 $\times$ 21   | $2.35 \times 10^{-3}$ | –                | $1.66 \times 10^{-2}$ | –               |
|              | 41 $\times$ 41   | $5.40 \times 10^{-4}$ | 2.12             | $4.94 \times 10^{-3}$ | 1.75            |
|              | 81 $\times$ 81   | $1.72 \times 10^{-5}$ | 4.97             | $1.65 \times 10^{-4}$ | 4.90            |
|              | 161 $\times$ 161 | $8.71 \times 10^{-7}$ | 4.30             | $1.07 \times 10^{-5}$ | 3.95            |
|              | 321 $\times$ 321 | $2.99 \times 10^{-8}$ | 4.86             | $3.98 \times 10^{-7}$ | 4.75            |
| WENO5-M6     | 21 $\times$ 21   | $2.42 \times 10^{-3}$ | –                | $1.70 \times 10^{-2}$ | –               |
|              | 41 $\times$ 41   | $5.47 \times 10^{-4}$ | 2.15             | $5.02 \times 10^{-3}$ | 1.75            |
|              | 81 $\times$ 81   | $1.74 \times 10^{-5}$ | 4.97             | $1.68 \times 10^{-4}$ | 4.90            |
|              | 161 $\times$ 161 | $8.77 \times 10^{-7}$ | 4.31             | $1.08 \times 10^{-5}$ | 3.96            |
|              | 321 $\times$ 321 | $3.02 \times 10^{-8}$ | 4.86             | $4.06 \times 10^{-7}$ | 4.73            |
| Method      | Grid size | $L_2$ error | Convergence rate | $L_\infty$ error | Convergence rate |
|------------|-----------|-------------|-----------------|------------------|-----------------|
| WENO7-M4   | 21 × 21   | 2.59 × 10^{-3} | –               | 1.75 × 10^{-2}  | –               |
|            | 41 × 41   | 6.42 × 10^{-4} | 2.01            | 5.64 × 10^{-3}  | 1.63            |
|            | 81 × 81   | 6.67 × 10^{-6} | 6.59            | 7.01 × 10^{-5}  | 6.33            |
|            | 161 × 161 | 5.34 × 10^{-8} | 6.96            | 8.35 × 10^{-7}  | 6.39            |
|            | 321 × 321 | 1.43 × 10^{-9} | 5.22            | 1.64 × 10^{-8}  | 5.67            |
| WENO7-M6   | 21 × 21   | 2.67 × 10^{-3} | –               | 1.79 × 10^{-2}  | –               |
|            | 41 × 41   | 6.54 × 10^{-4} | 2.03            | 5.73 × 10^{-3}  | 1.64            |
|            | 81 × 81   | 6.81 × 10^{-6} | 6.59            | 7.24 × 10^{-5}  | 6.31            |
|            | 161 × 161 | 5.19 × 10^{-8} | 7.04            | 7.59 × 10^{-7}  | 6.58            |
|            | 321 × 321 | 4.51 × 10^{-10}| 6.85            | 7.01 × 10^{-9}  | 6.76            |
| WENO7-M8   | 21 × 21   | 2.70 × 10^{-3} | –               | 1.80 × 10^{-2}  | –               |
|            | 41 × 41   | 6.55 × 10^{-4} | 2.04            | 5.73 × 10^{-3}  | 1.65            |
|            | 81 × 81   | 6.82 × 10^{-6} | 6.59            | 7.24 × 10^{-5}  | 6.31            |
|            | 161 × 161 | 5.19 × 10^{-8} | 7.04            | 7.59 × 10^{-7}  | 6.58            |
|            | 321 × 321 | 4.49 × 10^{-10}| 6.85            | 6.89 × 10^{-9}  | 6.78            |
Fig. 3 The density contours ranging from 1.25 to 21.5 of the double Mach reflection problem with the grid (961 × 241 cells) disturbed by 5% randomness.

4.2 Inviscid Wall-Bounded Flows

4.2.1 Double Mach Reflection

Now we test the shock-capturing capability of the proposed numerical methods by the double Mach problem [31] with the following initial condition

\[
(\rho, u, v, p) = \begin{cases} 
(1.4, 0, 0, 1.0) & \text{if } x - y \tan(\pi/6) \geq 1/6, \\
(8.0, 7.1447, -4.125, 116.5) & \text{if } x - y \tan(\pi/6) < 1/6.
\end{cases}
\]  

The grids resolution is 961 × 241, with the randomness of the grid points in Figs. 3 and 4 being 5% and 20%, respectively. The computation is performed until \( t = 0.2 \) with a CFL number of 0.6.

As shown in Fig. 3a–d, although the standard WENO5 and WENO7 schemes perform well on the uniform Cartesian grid, they exhibit large errors (as indicated by the spurious oscillations) in the grid with 5% randomness, even in the smooth regions. This issue can be
addressed by using WENO5-M2 and WENO5-M6, as shown in Fig. 3e and g which agree with the result of the WENO5-FP2019 method [35], indicating that spurious oscillation in Fig. 3c, d arises from the free-stream violation errors of the original WENO5 and WENO7 schemes. However, WENO5-FP2019 is not able to maintain the FP property if the low-order schemes are used for the metrics, especially for low-quality meshes. Here, we can reduce the scheme used for the metrics in WENO5-FP2019 from the 6th order to the 2nd order, termed WENO5-FP2019-M2. As shown in Fig. 3h, this treatment introduces additional errors, implying the FP property of the WENO5-FP2019 has been destroyed, i.e. this method can’t remove the constraint of the SCMM. Besides, the numerical results of WENOCU6-M6 and WENO7-M8 indicate that the present FP method can also perform well on shock capturing and free-stream preserving when it is extended to other upwind schemes, as given in Fig. 3i and j. We also notice that increasing the randomness from 5% to 20% does not affect the numerical results in the FP methods, although larger errors are observed in the original schemes, as shown in Fig. 4. However, when we increase the randomness of the grid to 25%, the numerical simulations with WENO5-M6 blow up, while WENO5-M2 and WENO5-M4 methods still obtain reasonable numerical results, as shown in Fig. 5. In addition, like Ref. [35], we list the computational costs in Table 3. Compared to WENO5-FP2019, as expected, our WENO5-M6 achieves additional efficiency of 4%.
Table 3 The computational costs for simulating the double Mach problem on the 5% randomized grid

| Schemes           | WENO5 | WENO5-M6 | WENO5-FP2019 |
|-------------------|-------|----------|--------------|
| Computational costs | 100%  | 113%     | 117% [35]    |

4.2.2 Supersonic Flow Past a Cylinder

The supersonic flow past a cylinder [12] is solved to demonstrate the potential application of the present schemes on supersonic flows with curved walls. The initial condition is a uniform flow state with $M = 2$. The slip wall boundary condition is imposed at the cylinder surface, and the supersonic inflow and outflow boundary conditions are employed at the left and right boundaries, respectively, with the grid generated by

$$
x = \left( R_x - (R_x - 1)\eta' \right) \cos \left( \theta(2\xi' - 1) \right),
$$
$$
y = \left( R_y - (R_y - 1)\eta' \right) \sin \left( \theta(2\xi' - 1) \right),
$$
$$
\xi' = \frac{\xi - 1}{i_{\text{max}} - 1}, \xi = i + 0.2\phi_i,
$$
$$
\eta' = \frac{\eta - 1}{j_{\text{max}} - 1}, \eta = j + 0.2\sqrt{1 - \phi_i^2},
$$

where $\theta = 5\pi/12$, $R_x = 3$, $R_y = 6$, and $\phi_i \sim U(0, 1)$. The number of grid points is $i_{\text{max}} = 61$ and $j_{\text{max}} = 81$. The free-stream pressure and density are set to $p = 1$ and $\rho = \gamma$, respectively. Our simulations are conducted until a steady state is reached after 5000 steps, with a time step of $\Delta t = 0.005$.

As shown in Fig. 6b–d, the grid randomness in Fig. 6a leads to significant errors in the original WENO schemes, even in the smooth flow regions. However, our methods capture the bow shock sharply, and generate smooth contours away from the shock, without generating spurious free-streaming violations, as demonstrated by the pressure contours in Fig. 6e–h. The computational costs of the original WENO and two FP WENO schemes listed in Table 4 show that our method achieves 14% speedup compared with the WENO5-FP2019.

4.2.3 Transonic Flow Past a NACA0012 Airfoil

Now we test the application of the proposed schemes on predicting the aerodynamic problems by considering the transonic flow past a NACA0012 airfoil. The simulations are performed on a coarse grid with a resolution of $160 \times 32$, with two setups: 1) Mach number $M = 0.8$, angle of attack $AOA = 1.25^\circ$ (case 1); 2) $M = 0.85$, $AOA = 1.0^\circ$ (case 2). The Mach-number contours in Fig. 7 show that the WENO5-M2, WENO5-M4, and WENO-M6 schemes achieve similar flow structures in both two cases. And the corresponding pressure coefficient distributions along the airfoil surface of those schemes essentially coincide with each other, see Fig. 8. This observation indicates that the order of the metric discretization does not significantly affect the overall prediction accuracy on such a coarse resolution, which is in agreement with the conclusion in Sec. 4.1.2. Clearly, in comparison with the FVM based on 2nd-order MUSCL reconstruction and Roe scheme (hereafter referred to as MUSCL-ROE-FVM), improved agreement with the high-resolution ($1280 \times 177$ cells) reference MUSCL-ROE-FVM solution is achieved by the present WENO schemes, especially on the upper airfoil surface, with the shockwave being captured more sharply.
Fig. 6 The pressure contours ranging from 1.2 to 5.4 of the supersonic flow past a cylinder

Table 4 The computational costs for computing the supersonic flow past a cylinder

| Schemes | WENO5 | WENO5-M6 | WENO5-FP2019 |
|---------|-------|----------|--------------|
| Computational costs | 100% | 117% | 133% [35] |
Fig. 7 The Mach number contours ranging from 0.172 to 1.325 of NACA0012 airfoil for case 1 (the left column) and case 2 (the right column)
4.3 Viscous Wall-Bounded Flows

4.3.1 Laminar Vortex Shedding from a Circular Cylinder

After testing the performance of the proposed method on inviscid flows, now we consider the 2D laminar subsonic flows past the circular cylinder at $Re_D = 150$ [21] to validate the accuracy of the FP schemes on resolving unsteady vortex shedding in viscous flows. The simulations are performed on a mesh with 177400 cells, and a far-field boundary with $M = 0.2$ and $T = 277.78K$ is located 200 diameters from the center of the cylinder, as illustrated in Fig. 9. To reduce the computational costs, the implicit dual-time-stepping algorithm with Lower-Upper Symmetric-Gauss-Seidel (LUSGS) [11, 32] is employed to the simulations, with $\Delta t = 0.001s$ for physical time marching and a CFL number of 20 for sub-iteration. In Fig. 10, the Mach number contours indicate that the vortex shed from the cylinder in the wake is well resolved by the WENO5-M2 and WENO5-M6 schemes, and is significantly dissipated in the 2nd-order MUSCL-ROE-FVM result. As listed in Table 5, compared to experimental data in Ref. [21], both the predicted averaged drag coefficient $C_D$ and lift Strouhal number $St$ of our schemes show smaller error than of the MUSCL-ROE-FVM. In addition, the WENO5-M2 scheme achieves approximately the same results as those in the WENO5-M6 scheme, both the vortex structures in the wake and statistic data ($C_D$ and $St$), which indicates the order of metrics and Jacobian has no significant influences on the under-resolved simulations, as in Sects. 4.1.2 and 4.2.3.

4.3.2 Subsonic Flow Past a Three-Element Airfoil (30P30N)

In this section, the subsonic flow past a three-element airfoil (30P30N) is simulated to demonstrate the capability and accuracy of the present FP schemes in simulating aerodynamic problems of complex geometries on turbulent conditions. Here, the far-field boundary condition with $M = 0.2$ and $AOA = 8^\circ$ is imposed on the boundary of the computational domain which is calculated by a mesh containing $10^5$ cells, as shown in Fig. 11. The steady solution is obtained by the implicit time marching with LUSGS and the SST turbulence model [20].
Fig. 9 The block topology (a) and the enlarged view of the grid points (b) of the circular cylinder.

Fig. 10 The Mach number contours (right: the enlarged view) ranging from 0 to 0.26 of the laminar vortex shedding from a circular cylinder at $Re_D = 150$. 

(a) MUSCL-ROE-FVM

(b) WENO5-M2

(c) WENO5-M6
Table 5 The experimental and simulated averaged drag coefficient ($C_D$) and lift Strouhal number ($St$) of the laminar vortex shedding from a circular cylinder

|       | Exp | MUSCL-ROE-FVM | WENO5-M2 | WENO5-M6 |
|-------|-----|---------------|----------|----------|
| $C_D$ | 1.34 | 1.312         | 1.334    | 1.334    |
| $St$  | 0.179−0.182 | 0.172       | 0.182    | 0.182    |

Fig. 11 The block topology (a) and the enlarged view of the grid points (b) of the 30P30N three-element airfoil

As illustrated in Fig. 12a–d, WENO5-M2, WENO5-M6 and LU5-M6 produce similar continuous and smooth Mach-number contours around the 30P30N airfoil. The slip flows at the tail of the slat are thinner in these FP schemes, and at the tail of the flap, we observe vortex shedding which is absent in the 2nd-order MUSCL-ROE-FVM. In addition, as in Fig. 12e, the pressure coefficients distributions along the airfoil surface in the present schemes match the experimental data in Ref. [3] very well, while being significantly underestimated on the upper surface in the 2nd-order MUSCL-ROE-FVM results.

4.3.3 Transonic Flow Pass the ONERA M6 Wing

In addition to the 2D test cases above, we also select the transonic flow around the ONERA M6 wing to validate the robustness of our schemes in predicting turbulent flows in three dimensions. We perform the simulations under a condition of $M = 0.84$, and $AOA = 3.06^\circ$, with LUSGS for time advancing and a mesh of 294912 cells for spatial discretization. The Reynolds number of $Re_l = 1.172 \times 10^7$ is sufficiently large that the SST turbulence model is invoked, where $l = 0.64607m$ is the mean aerodynamic chord.

The pressure contours in Fig. 13 indicate that both the WENO5-M2 and WENO5-M6 schemes obtain a better result than MUSCL-ROE-FVM, as the flow filed near the shock-wave is more clearly captured. As illustrated in Fig. 14, compared to MUSCL-ROE-FVM, the pressure distributions of WENO5-M2 and WENO5-M6 are in better agreement with the experimental results [24] at the 6 selected cross-sections along the spanwise direction. Again, we emphasize that the order of metrics does not affect the predicting accuracy in this
Fig. 12 The Mach number contours ranging from 0.05 to 0.5 in different schemes (a–d) and the pressure coefficient distributions on the surface of 30P30N airfoil (e)
Fig. 13 The pressure contours ranging from $1.2 \times 10^5$ Pa to $4.6 \times 10^5$ Pa of the ONERA M6 wing at four different spanwise cross-sections

3D problem, as demonstrated by the qualitative comparison in Fig. 13 and the quantitative comparison in Fig. 14, between WENO5-M2 and WENO5-M6.

5 Conclusion

The proposed free-stream preserving method in this work suggests reformulating the dissipation of the upwind conservative finite-difference schemes to eliminate the free-stream preserving violation errors on curvilinear grids. In particular, to construct the reformulated
Fig. 14 The pressure coefficient distributions at 6 cross-sections along the spanwise direction of the ONERA M6 wing surface

dissipation, the reference variables are subtracted from the cell-center conservative variables and fluxes for the linear upwind schemes and the WENO schemes, respectively. In such a way, the proposed numerical approach and its algorithmic formulation have the following main properties: (1) avoid the complex operations to rearrange the original upwind scheme to a summation of the central and dissipation parts and then employ different FP techniques for them as the previous FP methods; (2) this method is more flexible in practical engineering problems, as the metrics and fluxes can be computed in an inconsistent manner, with the free-steaming condition being preserved. The simple test cases demonstrate that the proposed method strictly preserves the free-stream condition on stationary curvilinear grids, with a formal high-order accuracy. The accuracy and robustness of this method are assessed in various fluid dynamics problems involving complex geometries, such as the NACA0012 airfoil, the 30P30N airfoil, and the ONERA M6 wing, where the shockwaves, vortex structures, and turbulent flows are well predicted. In addition, it can be concluded from a range of complex validation cases that the accuracy of metrics discretization shows no significant effect on under-resolved numerical simulations, which are usually encountered in practical problems where low-quality and coarse-resolution grids are widely used.

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Data Availability The datasets generated in this study are available from the corresponding author on reasonable request.
Conflict of interest  The authors declare that they have no conflict of interest.

Appendix

The cell-face numerical fluxes $\bar{F}_{i+1/2}^O$ reconstructed by the original WENO5 scheme can be expressed by the summation of the central and dissipation part [23, 35]. Substituting Eq. [SPSeqref7eqrefSPS] into Eq. (25), and then into Eq. (20) gives

$$\bar{F}_{i+1/2}^O = R \left( \bar{F}_{i+1/2}^+ + \bar{F}_{i+1/2}^- \right)$$

$$= \frac{1}{60} \left( \bar{F}_{i-2} - 8 \bar{F}_{i-1} + 37 \bar{F}_i + 37 \bar{F}_{i+1} - 8 \bar{F}_{i+2} + \bar{F}_{i+3} \right)$$

$$- \frac{1}{60} \sum_s R^s \left[ (20 \omega_0^+ - 1) \hat{f}_{i,1}^{s,+} - (10 \omega_0^+ + 10 \omega_1^+ - 5) \hat{f}_{i,2}^{s,+} + \hat{f}_{i,3}^{s,+} \right]$$

$$+ \frac{1}{60} \sum_s R^s \left[ (20 \omega_0^- - 1) \hat{f}_{i,1}^{s,-} - (10 \omega_0^- + 10 \omega_1^- - 5) \hat{f}_{i,2}^{s,-} + \hat{f}_{i,3}^{s,-} \right],$$

where

$$\hat{f}_{i,r+1}^{s,+} = \hat{f}_{i,r+1}^{s,+} - 3 \hat{f}_{i,r}^{s,r} + 3 \hat{f}_{i,r-1}^{s,r} - \hat{f}_{i,r-2}^{s,r}, \quad r = 0, 1, 2$$

$$= \frac{1}{2} L^s \left( \bar{F}_{i+r+1} - 3 \bar{F}_{i+r} + 3 \bar{F}_{i+r-1} - \bar{F}_{i+r-2} \right)$$

$$+ \frac{1}{2} \lambda^s L^s \left( \bar{Q}_{i+r+1} - 3 \bar{Q}_{i+r} + 3 \bar{Q}_{i+r-1} - \bar{Q}_{i+r-2} \right).$$

Similarly, the cell-face numerical fluxes $\bar{F}_{i+1/2}^{FP}$ obtained with the proposed FP WENO5 scheme can be given by applying the approximations suggested in Eqs. (51) and (52), i.e.

$$\bar{F}_{i+1/2}^{FP} = R \left( \bar{F}_{i+1/2}^{FP,+} + \bar{F}_{i+1/2}^{FP,-} \right)$$

$$= \frac{1}{60} \left( \bar{F}_{i-2} - 8 \bar{F}_{i-1} + 37 \bar{F}_i + 37 \bar{F}_{i+1} - 8 \bar{F}_{i+2} + \bar{F}_{i+3} \right) + \Delta \bar{F}^*$$

$$- \frac{1}{60} \sum_s R^s_{i+1/2} \left[ (20 \omega_0^+ - 1) \hat{f}_{i,1}^{s,+} - (10 \omega_0^+ + 10 \omega_1^+ - 5) \hat{f}_{i,2}^{s,+} + \hat{f}_{i,3}^{s,+} \right]$$

$$+ \frac{1}{60} \sum_s R^s_{i+1/2} \left[ (20 \omega_0^- - 1) \hat{f}_{i,1}^{s,-} - (10 \omega_0^- + 10 \omega_1^- - 5) \hat{f}_{i,2}^{s,-} + \hat{f}_{i,3}^{s,-} \right],$$

where

$$\hat{f}_{i,r+1}^{s,+} = \hat{f}_{i,r+1}^{s,+} - 3 \hat{f}_{i,r}^{s,r} + 3 \hat{f}_{i,r-1}^{s,r} - \hat{f}_{i,r-2}^{s,r}, \quad r = 0, 1, 2$$

$$= \frac{1}{2} L^s_{i+1/2} \left( \bar{F}_{i+r+1} - 3 \bar{F}_{i+r} + 3 \bar{F}_{i+r-1} - \bar{F}_{i+r-2} \right)$$

$$+ \frac{1}{2} \lambda^s L^s_{i+1/2} \left( \bar{Q}_{i+r+1} - 3 \bar{Q}_{i+r} + 3 \bar{Q}_{i+r-1} - \bar{Q}_{i+r-2} \right)$$

$$- \frac{1}{2} L^s_{i+1/2} \left( \bar{Q}_{i+r+1}^* - 3 \bar{Q}_{i+r}^* + 3 \bar{Q}_{i+r-1}^* - \bar{Q}_{i+r-2}^* \right).$$
and $\Delta F^\ast$ is given by Eq. (60).

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