Semiclassical and quantum description of motion on noncommutative plane

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Abstract

We study the canonical and the coherent state quantization of a particle moving in a magnetic field on a non-commutative plane. Starting from the so called $\theta$-modified action, we perform the canonical quantization and analyze the gauge dependence of the obtained quantum theory. We construct the Malkin-Man’ko coherent states of the system in question, and the corresponding quantization. On this base, we study the relation between the coherent states and the “classical” trajectories predicted by the $\theta$-modified action. In addition, we construct different semiclassical states, making use of special properties of circular squeezed states. With the help of these states, we perform the Berezin-Klauder-Toeplitz quantization and present a numerical exploration of the semiclassical behavior of physical quantities in these states.

1 Introduction

Constructing semiclassical states for a given quantum system is an important and, in general case, an open problem in quantum theory. One can believe that for systems with quadratic Hamiltonians semiclassical (or “classical-like”) states are those ones introduced by Shrödinger in 1926, lately rediscovered and called coherent by Glauber and Sudarshan within the context of Quantum Optics, and by Klauder in the more general quantum arena. We will call them standard CS in what follows. These states were then studied by a number of authors in different contexts. Perelomov proposed so-called generalized CS for systems with a symmetry group. One can also mention some alternative constructions of CS that differ from the standard ones, see e.g.

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In addition, it was realized that constructing CS is closely related to the quantization problem \[10\].

A charged quantum particle interacting with a constant magnetic field is an important and well studied system. A family of coherent states adapted to such a system was first proposed by Malkin and Man’ko \[11\]. Afterwards, some alternative constructions were proposed in \[12, 13\]. However, all these CS are labelled not only by continuous quantum numbers, but also by a discrete quantum number. To avoid discrete quantum numbers, the authors of \[14\] have proposed new coherent states partially similar to the CS of a particle moving on a circle \[15\]. A generalization of such circular CS was proposed in our work \[16\]. Recently, quantum states of a particle in a magnetic field and on a noncommutative plane have been attracting considerable attention, see for instance \[17, 18, 19\] and \[20\]. Among the problems in the formulation of the non-commutative quantum mechanics are to implement the symmetries of the ordinary theory and the consistent interpretation of the position operator \[21\]. Some works are devoted to the search for experimental observation that can give physical evidences of the non-commutative properties of the space, or that can be used to set some limits for this non-commutative properties \[22, 23\]. The non-commutative quantum mechanics can be constructed by the quantization of a classical \(\theta\)-modified action \[24\]. In this article we will construct the standard CS for a particle in a magnetic field and on a non-commutative plane, and study the relation between these classical-like states and the “classical” trajectories predicted by the corresponding \(\theta\)-modified action. In addition, we make use of the interesting properties of the circular CS proposed in \[16\] to construct semiclassical states of this system.

The present article is organized as follows. In Section 2, we recall the formulation of classical mechanics on a non-commutative space in terms of commuting coordinates (and in terms of the so-called \(\theta\)-modified actions). In Section 3, we study such a formulation and its quantum version for a charged particle submitted to a magnetic field, and the related gauge dependence of this formulation. In Section 4, we follow the Malkin-Man’ko approach and construct coherent states for the above system on the non-commutative plane. We discuss their semiclassical properties and the corresponding quantization of physical quantities. In Section 5, we construct for the same system partially circular CS, proposed by us in \[10\], and perform the Berezin-Klauder-Toeplitz quantization based precisely on these coherent states. In addition, we present a numerical exploration of their semiclassical behavior. Finally, in Section 6, we summarize the obtained results.

2 Classical and Quantum motion on noncommutative plane

Let us start by briefly recalling the nonrelativistic quantum mechanical description of a finite-dimensional system living on a noncommutative space. Suppose
that such a system is described by coordinates $q^k$ and momenta $p_j$ operators, $k, j = 1, \ldots, d$, that obey the commutation relations

$$[q^k, q^j] = i\theta^{kj}, \quad [q^k, p_j] = i\hbar\delta^k_j, \quad [\hat{p}_k, \hat{p}_j] = 0, \quad \theta^{kj} = -\theta^{jk}, \quad (1)$$

where $\theta^{kj}$ is a real constant antisymmetric matrix. The quantum Hamiltonian $\hat{H} = H(p, q)|_{p \to \hat{p}, q \to \hat{q}}$ is constructed from the classical one $H(p, q)$ and a certain ordering is chosen. Introducing new operators, see [22],

$$\hat{x}^k = \hat{q}^k + \frac{1}{2\hbar}\theta^{kj}\hat{p}_j,$$

$$[\hat{x}^k, \hat{x}^j] = 0, \quad [\hat{p}_k, \hat{p}_j] = 0, \quad [\hat{x}^k, \hat{p}_j] = i\hbar\delta^k_j, \quad (2)$$

one can construct a path-integral representation for the matrix elements $G_x = \langle x_{\text{out}} | U(t_{\text{out}}, t_{\text{in}}) | x_{\text{in}} \rangle$ of the evolution operator $U(t, t') = \exp \left\{ -\frac{i}{\hbar}\hat{H}(t - t') \right\}$ in $x$-representation,

$$G_x = \int Dp \int_{x_{\text{in}} - \theta p/2\hbar}^{x_{\text{out}} - \theta p/2\hbar} Dq \exp \left\{ \frac{i}{\hbar}S^\theta \right\}, \quad (3)$$

where

$$S^\theta = \int dt \left[ p_jq^j - H(p, q) - \hat{p}_j\theta^{ji}\hat{p}_i/2\hbar \right], \quad (4)$$

see [24].

In quantum mechanics on commutative space, the action $S = S^\theta|_{\theta = 0}$ is just the Hamiltonian action of the classical system under consideration. In the noncommutative case this action is modified by the adding of a new term $\hat{p}_k\theta^{kj}\hat{p}_j/2\hbar$. The action $[3]$ is called $\theta$-modified Hamiltonian action of classical mechanics. As was mentioned in [25, 26], its quantization leads exactly to the commutation relations [1]. Below, we demonstrate this explicitly within the canonical quantization framework. To this end, let us treat [3] as a Lagrangian action with generalized coordinates $Q = (q, p)$ and Lagrange function $L = L(Q, \dot{Q})$,

$$L = p_jq^j - H(p, q) - \hat{p}_j\theta^{ji}\hat{p}_i/2\hbar. \quad (5)$$

Constructing the Hamiltonian formulation, we introduce the momenta

$$\pi_k = \frac{\partial L}{\partial q_k} = p_k, \quad \bar{\pi}_k = \frac{\partial L}{\partial p_k} = -\frac{\theta^{kj}}{2\hbar}\hat{p}_j, \quad (6)$$

and find the primary constraints to be $\Phi^{(1)} = (\phi, \bar{\phi}) = 0$,

$$\phi_k = \pi_k - p_k, \quad \bar{\phi}_k = \bar{\pi}_k + \frac{\theta^{kj}}{2\hbar}\hat{p}_j. \quad (7)$$

They are of second-class, $\det \{ \Phi^{(1)}, \Phi^{(1)} \} \neq 0$. Performing a canonical transformation to the new canonical variables (they are labeled by primes),

$$q'_k = q_k, \quad p'_k = p_k, \quad \pi'_k = \pi_k - p_k, \quad \bar{\pi}'_k = \bar{\pi}_k - q_k, \quad (8)$$
we obtain the constraints of the special form:

\[ \pi'_k = 0, \ q'_k = -\tilde{\pi}'_k - \frac{\theta^{kj}}{2\hbar} p'_j, \]

see [27]. Therefore, we can exclude \(\pi'_k = 0\) and \(q'_k\) from the consideration, in particular, from the Hamiltonian. Then the commutation relations for the rest of the variables are canonical and the new Hamiltonian reads as

\[ H'(q', p') = H\left(-\tilde{\pi}'_k - \frac{\theta^{kj}}{2\hbar} p'_j, p'_l\right), \ \{p'_j, \tilde{\pi}'_k\} = \delta_{jk}. \]

Performing one more canonical transformation \(p'_j = p_j, \tilde{\pi}'_k = -x_k\), we obtain

\[ H_\theta = H\left(x - \frac{\theta \cdot p}{2\hbar}, p\right), \ \{x_j, p_k\} = \delta_{jk}, \ \{x_j, x_k\} = \{p_j, p_k\} = 0. \quad (9) \]

It follows from the action (5) the following relations:

\[ \dot{p}_i = -\frac{\partial H(q, p)}{\partial q^i}, \ \dot{q}^i = \frac{\partial H(q, p)}{\partial p_i} + \frac{\theta^{ij}}{\hbar} \frac{\partial H(q, p)}{\partial q^j}. \quad (10) \]

Using the relation (2) we obtain the classical canonical relations for the commutative coordinates, namely

\[ \dot{p}_i = -\frac{\partial H_\theta}{\partial x^i}, \ \dot{x}^i = \frac{\partial H_\theta}{\partial p_i}. \quad (11) \]

Passing from (9) to quantum theory, we obtain

\[ \hat{H}_\theta = H\left(\hat{x} - \frac{\theta \cdot \hat{p}}{2\hbar}\right), \ \{\hat{x}_j, \hat{p}_k\} = i\hbar \delta_{jk}, \ \{\hat{x}_j, \hat{x}_k\} = 0, \ \{\hat{p}_j, \hat{p}_k\} = 0, \quad (12) \]

which corresponds to (2).

3 Charged particle in constant magnetic field on the noncommutative plane

3.1 Classical motion

Consider a classical nonrelativistic particle moving in the plane \((x^1, x^2)\) and interacting with a constant and uniform magnetic field of intensity \(B\) perpendicular to the plane. Such a field can be described by a vector potential \(A\) only \((A^0 = 0)\). The Hamiltonian of the particle is

\[ H(x, p) = \frac{1}{2m} \left[ p + \frac{e}{c} A(x) \right]^2, \ x = (x^1, x^2), \ p = (p_1, p_2). \quad (13) \]

\(^1\)The charge of an electron is \(-e\), with \(e > 0\).
In what follows, we use alternatively the Landau gauge $A_L$ and the symmetric gauge $A_S$,

$$A_L = B(0,x^1), \quad A_S = \frac{1}{2}(-Bx^2, Bx^3), \quad (14)$$

$$A_L = A_S + \nabla f, \quad f = \frac{1}{2}Bx^1x^2. \quad (15)$$

Supposing that the minimal coupling is still valid, the corresponding Hamiltonian $H(p,q)$ of (13) for the non-commutative variables $q = (q^1,q^2)$, which respect the algebra (1), can be constructed through the substitution $A(x) \rightarrow A(q)$,

$$A_L(q) = B(0,q^1), \quad A_S(q) = \frac{1}{2}(-Bq^2, Bq^1). \quad (16)$$

In the case under consideration, we can set $\theta^{kj} = \theta \varepsilon^{kj}$, $k,j = 1,2$, where $\varepsilon^{kj}$ is the Levi-Civita symbol, $\varepsilon^{12} = 1$, such that the classical equations of motion (10), in the Landau gauge (14), have the following solutions

$$q^1 = q^1_0 + R \cos(\omega t + \phi), \quad \omega = \frac{e}{cm} |B|,$$

$$q^2 = q^2_0 + \varepsilon R \sin(\omega t + \phi), \quad \varepsilon = 1 + \frac{Be}{\hbar c} \theta, \quad (17)$$

where $R$, $\phi$ and $q^0_i$ are real constants. The motion is periodic, with frequency $\omega$ (the cyclotron frequency), along an ellipse with center $(q^1_0,q^2_0)$ and eccentricity $\varepsilon$. Hence, this elliptic deformation of the circle is due to the noncommutativity parameter $\theta$ through $\varepsilon$. If we choose, instead of (14), the gauge $A_L = -B(q^2,0)$, then due to the antisymmetry of $\theta^{ik}$ the axis of the ellipse will be interchanged.

In the symmetric gauge (15) the solutions of the corresponding classical equations of motion (10) have the form

$$q^1 = q^1_0 + R \cos(\tilde{\omega} t + \phi), \quad q^2 = q^2_0 + R \sin(\tilde{\omega} t + \phi),$$

$$\tilde{\omega} = \omega |\mu_S|, \quad \mu_S = 1 - \frac{eb}{4\hbar c} \theta. \quad (18)$$

In this case the trajectory remains a circle with radius $R$ whereas the noncommutativity modifies the frequency of motion $\tilde{\omega}$: the latter differs from the cyclotron one $\omega$ by a factor that depends on the algebraic value (in particular, on the direction) of the magnetic field. In the case when $B\theta > 0$, the frequency of oscillation decreases as $|B|$ increases, and turns out to be zero for $\theta = \theta^S_c = 4\hbar c/eB$.

As was already mentioned in the literature [20, 28], the classical motion on the noncommutative plane is not gauge invariant under gradient $U(1)$ gauge transformations of the external electromagnetic field.

In both gauges, the relation between the minor radius $R$ of the ellipse and the particle energy, defined as $E = H(q,p)$, reads as

$$\frac{E}{R^2} = \frac{m\omega^2}{2},$$
which holds also in the commutative case. However, for \( E = H_\theta(x,p) \), we have a \( \theta \)-dependent relation:

\[
\frac{E}{\mathcal{R}^2} = \frac{m}{2} (\mu S \omega)^2 , \quad \mathcal{R}^2 = (x^1)^2 + (x^2)^2 .
\] (19)

3.2 Quantum theory

Symmetric gauge

Passing to quantum theory, we choose first the symmetric gauge \( A_S \). Then

\[
\hat{A}_i = -\frac{B}{2} \varepsilon_{ij} \left( \hat{x}^j - \frac{\theta}{2\hbar} \varepsilon^{jk} \hat{p}_k \right) , \quad i,j,k = 1,2 ,
\]

and the quantum Hamiltonian (12) takes the form:

\[
\hat{H}_\theta = \frac{1}{2m} \left( \hat{P}^2_1 + \hat{P}^2_2 \right) , \quad \hat{m} = \frac{m}{\mu_S} , \quad \mu_S = 1 - \frac{eB}{4ch} \theta .
\] (20)

Here \( \hat{P}_i, i = 1,2 \), are components of the kinematic momentum operator,

\[
\hat{P}_i = \hat{p}_i - \frac{e\tilde{B}}{2c} \varepsilon_{ij} \hat{x}^j , \quad \left[ \hat{P}_1, \hat{P}_2 \right] = -i\hbar \frac{e\tilde{B}}{c} , \quad \tilde{B} = \frac{B}{\mu_S} .
\] (21)

Like in the classical case, for \( B\theta > 0 \), there exists a critical value \( \theta = \theta_c^S = 4ch/eB \), for which the Hamiltonian (20) does not depend on the momenta,

\[
\hat{H}_{\theta_c^S} = \frac{1}{2m} \left( \frac{e\tilde{B}}{2c} \right)^2 \left[ (\hat{x}^1)^2 + (\hat{x}^2)^2 \right] ,
\]

and its eigenvectors describes localized states.

In the general case, (20) is a Hamiltonian of one-dimensional harmonic oscillator with the spectrum

\[
E_n = \hbar \tilde{\omega} \left( n + \frac{1}{2} \right) , \quad \tilde{\omega} = \frac{e}{cm} \left| \tilde{B} \right| = \frac{e}{cm} |\mu_S B| , \quad n \in \mathbb{N} .
\] (22)

The frequency \( \tilde{\omega} \) coincide with the frequency of the classical motion (18) and, as was already mentioned, depends on the algebraic value (in particular, on the direction) of the magnetic field.

It is convenient to introduce creation \( \hat{a}^+ \) and annihilation \( \hat{a} \) operators,

\[
\hat{a} = \frac{1}{\sqrt{2m\omega}} \left[ \frac{e\tilde{B}}{2c} \hat{\tilde{z}} + 2i\hat{p}_z^* \right] , \quad \hat{a}^+ = \frac{1}{\sqrt{2m\omega}} \left[ \frac{e\tilde{B}}{2c} \hat{\tilde{z}}^* - 2i\hat{p}_z \right] ,
\]

\[
\hat{\tilde{z}} = \hat{x}^1 - i\hat{x}^2 , \quad \hat{p}_z = -i\hbar \frac{\partial}{\partial z} = \hat{p}_z^* , \quad [\hat{a}, \hat{a}^+] = 1 .
\] (23)

In terms of such operators the Hamiltonian (20) assumes the form

\[
\hat{H}_\theta = \hbar \tilde{\omega} \left( \hat{\tilde{N}} + 1 \right) , \quad \hat{\tilde{N}} = \hat{a}^+ \hat{a} .
\]
There exist additional operators of creation and annihilation,
\[
\hat{b} = \frac{1}{\sqrt{2m \omega \hbar}} \left[ \frac{e \hat{B}}{2c} \hat{z}^* + 2i\hat{p}_z \right], \quad \hat{b}^+ = \frac{1}{\sqrt{2m \omega \hbar}} \left[ \frac{e \hat{B}}{2c} \hat{z}^* - 2i\hat{p}_z \right],
\]
\[
[\hat{b}, \hat{b}^+] = 1 .
\] (24)

They commute with \(\hat{a}^+, \hat{a}, \) and \(\hat{H}_\theta\), such that it is an integral of motion,
\[
[\hat{b}, \hat{a}^+] = [\hat{b}, \hat{a}] = [\hat{b}, \hat{H}_\theta] = 0 .
\]

Thus, the eigenvectors of the Hamiltonian \(\hat{H}_\theta\) corresponding to the eigenvalues \(E_n\), are
\[
\Psi_{mn} (z, z^*) = \frac{1}{\sqrt{m! n!}} \left( \hat{b}^+ \right)^m \left( \hat{a}^+ \right)^n \psi_0 (z, z^*) ,
\] (25)

with the ground state \(\psi_0\) given by
\[
\hat{a} \psi_0 = 0 \implies \frac{\partial \psi_0}{\partial z^*} = -\frac{eB}{4} z \psi_0 \implies \psi_0 = N \exp \left[ -\frac{eB}{8\hbar} |z|^2 \right] ,
\] (26)

where \(N\) is a normalization factor. The above eigenfunctions are infinitely degenerate.

**Landau gauge** In the Landau gauge \(A_L\), the quantum Hamiltonian reads:
\[
\hat{H}_\theta = \frac{\hat{P}_1^2}{2\tilde{m}} + \frac{1}{2\tilde{m}} \left( \hat{P}_2 + \frac{eB}{c} \hat{z}^1 \right)^2,
\]
\[
\tilde{m} = \frac{m}{\mu^2}, \quad B = \frac{B}{\mu}, \quad \mu = 1 - \frac{eB}{2c\hbar} \theta .
\] (27)

Here we obtain a different critical value \(\theta = 2c\hbar/eB = 2\theta_S\).

Introducing the kinematic momentum operators \(\hat{P}_1,\) and \(\hat{P}_2,\)
\[
\hat{P}_1 = \hat{\hat{P}}_1, \quad \hat{P}_2 = \hat{\hat{P}}_2 + \frac{eB}{c} \hat{z}^1 \implies [\hat{P}_1, \hat{P}_2] = -i\hbar \frac{\hat{B}}{c} ,
\]

we write (27) as a one-dimensional harmonic oscillator Hamiltonian
\[
\hat{H}_\theta = \frac{1}{2\tilde{m}} \left[ \hat{P}_1^2 + (m\omega \hat{Q})^2 \right], \quad \hat{P} \equiv \hat{\hat{P}}_1, \quad \hat{Q} \equiv \frac{c}{eB} \hat{P}_2 ,
\]
\[
[\hat{Q}, \hat{P}] = i\hbar, \quad \omega = \frac{e}{mc} |B| ,
\] (28)

with cyclotron frequency \(\omega\). Thus, in the Landau gauge, the spectrum is
\[
E_n = \frac{eB}{mc} \left( n + \frac{1}{2} \right), \quad n \in \mathbb{N} .
\]
In this case, the annihilation operator \( \hat{a} \) from (23) is commonly changed into

\[
\hat{a} = \sqrt{\frac{m}{2}} \left( \frac{\mu}{\omega} \hat{P} + i \hat{P}_1 \right),
\]

such that

\[
\hat{H}_\theta = \hbar \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right).
\]

In the gauge under consideration, \( \hat{p}_2 \) is an integral of motion, it commutes with \( \hat{H}_\theta \). Here the eigenvectors of \( \hat{H}_\theta \) can be chosen as

\[
\Psi_{n,k_2} (z, z^*) = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n \psi_0 \left( x^1 \right) \psi_{k_2} \left( x^2 \right),
\]

with the state \( \psi_{k_2} \) and the ground state \( \psi_0 \) given by

\[
\hat{p}_2 \psi_{k_2} = k_2 \psi_{k_2} \implies \psi_{k_2} \left( x^2 \right) = \exp \left( ik_2 x^2 \right),
\]

\[
\hat{a} \psi_0 = 0 \implies \psi_0 \left( x^1 \right) = N \exp \left( -\frac{\omega}{2\hbar} \left( \frac{\mu}{\omega} k_2 + x^1 \right)^2 \right),
\]

where \( N \) is a normalization factor.

### 4 Malkin-Man’ko coherent states on noncommutative plane

The CS of a charged particle in a uniform magnetic field were originally constructed by Malkin and Man’ko [11]. In fact, due to the double analytic structure of the phase space, those states are the tensor products of standard CS. In the general case, the phase space is \( \mathbb{C}^2 = \{ x = (\alpha, \beta) , \alpha \in \mathbb{C}, \beta \in \mathbb{C} \} \) and the realization of such a space can be constructed as the Hilbert space

\[
L^2 \left( \mathbb{C}^2, \mu (dx) \right) = L^2 \left( \mathbb{C}, \mu (da) \right) \otimes L^2 \left( \mathbb{C}, \mu (d\beta) \right),
\]

provided with adequate measures \( \mu (da) \) and \( \mu (d\beta) \). For the specific case of the standard CS, the measure is chosen to be

\[
\mu (dx) = e^{-|\alpha|^2} \frac{d^2 \alpha}{\pi} e^{-|\beta|^2} \frac{d^2 \beta}{\pi},
\]

where \( d^2 \alpha \) and \( d^2 \beta \) are the respective Lebesgue measures on the complex planes. In this Hilbert space we can define the following orthonormal set of functions

\[
\Phi_{m,n} (x) \equiv \frac{\alpha^m \beta^n}{\sqrt{m!} \sqrt{n!}},
\]

that we put in one-to-one correspondence with the elements \( |m,n \rangle \), \( m, n \in \mathbb{N} \), of any orthonormal basis of a separable Hilbert space \( \mathcal{H} \). We now introduce the
CS corresponding to this choice of orthonormal set. They are elements of $\mathcal{H}$ defined by

$$|\alpha,\beta\rangle = |\alpha\rangle \otimes |\beta\rangle \equiv \exp\left(\frac{-|\alpha|^2 + |\beta|^2}{2}\right) \sum_{m,n} \frac{\alpha^m}{\sqrt{m!}} \frac{\beta^n}{\sqrt{n!}} |m,n\rangle . \quad (34)$$

By construction, these normalized states are labeled by points of $\mathbb{C}^2$ and form a continuous overcomplete set resolving the unity in $\mathcal{H}$. In the case of the symmetric gauge $A_S$ (16), the standard CS are constructed by choosing the states (25), eigenstates of the Hamiltonian $H_{\theta}$ (20), as an orthonormal basis. With this choice, (34) can be written as

$$|\alpha,\beta\rangle = \hat{Z} |0,0\rangle , \quad \hat{Z} = \exp \left(\alpha \hat{a}^+ - \alpha^* \hat{a}\right) \exp \left(\beta \hat{b}^+ - \beta^* \hat{b}\right) , \quad (35)$$

where the operators $\hat{a}$ and $\hat{b}$ are given by (23) and (24), and the ground state $\Psi_0$ (25), supposed normalized, is denoted here by $|0,0\rangle$. The above CS, which we have written using displacement operator [7], correspond to the CS obtained by Malkin and Man’ko.

The time evolution of these coherent states can be obtained as

$$|\alpha,\beta; t\rangle = \exp \left(-i \frac{\hat{H}_{\theta}}{\hbar} t\right) |\alpha,\beta\rangle = \exp \left(-i \frac{\hat{\omega}}{2} t\right) |\alpha \exp (-i \hat{\omega} t) , \beta\rangle . \quad (36)$$

We note here that the Hamiltonian $\hat{H}_{\theta}$ acts as the identity on the $|\beta\rangle$ part of the CS, $\hat{H}_{\theta} = \hbar \hat{\omega} \left(\hat{N}_a + 1/2\right) \otimes I$. As we will see, these states describe circular trajectories and the parameter $\alpha$ is related to the radius of the orbit, while $\beta$ is related to the center of the orbit.

For the Landau gauge one can define the semi-coherent states

$$|\alpha, k_2\rangle = \exp \left[-\frac{1}{2} |\alpha|^2\right] \exp \left(\alpha \hat{a}^+\right) |0, k_2\rangle ,$$

$$|0, k_2\rangle = |0\rangle \otimes |k_2\rangle , \quad \hat{a} |0\rangle = 0 , \quad \hat{p}_2 |k_2\rangle = k_2 |k_2\rangle , \quad (37)$$

where the operators $\hat{a}$ is now given by (28) and the ground state $\psi_0$, denoted here $|0\rangle$, is given by (31). Note that we could as well define Malkin-Man’ko coherent states for this case.

Let us study the mean value evolution of the coordinate operators for the above CS. For the symmetric gauge let us use the kinematical momentum operator (21) and introduce the centre-coordinate operator (29)

$$\hat{x}_0^i = \hat{x}^i - \frac{1}{\bar{m}\omega} \sigma^j \hat{P}_j , \quad \bar{m}\omega = \frac{m\omega}{\mu_S} , \quad \mu_S = 1 - \frac{eB}{4\hbar c} \theta , \quad i, j = 1, 2 , \quad (38)$$

which are integral of motion

$$[\hat{x}_0^i, \hat{H}_{\theta}] = [\hat{x}_0^2, \hat{H}_{\theta}] = 0 , \quad [\hat{x}_0^1, \hat{x}_0^2] = \frac{i\hbar}{\bar{m}\omega} .$$
From (38) and (23) we have
\[ \hat{x}^1 - \hat{x}_0^1 = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^+) , \quad \hat{x}^2 - \hat{x}_0^2 = i \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} - \hat{a}^+) . \]
From the fact that \( |\alpha, \beta\rangle \) is an eigenvector of \( \hat{a} \) with eigenvalue \( \alpha \), we easily derive:
\[ \langle \alpha, \beta | (\hat{x}^1 - \hat{x}_0^1) |\alpha, \beta \rangle = \sqrt{2} \frac{\hbar}{m\omega} \tilde{\omega} \text{Re} (\alpha) , \]
\[ \langle \alpha, \beta | (\hat{x}^2 - \hat{x}_0^2) |\alpha, \beta \rangle = -\sqrt{2} \frac{\hbar}{m\omega} \text{Im} (\alpha) . \]
Writing \( \alpha = \hbar^{-1/2} R e^{-i\phi} \), with \( R \) and \( \phi \) real constants, and using the time-dependence of \( |\alpha, \beta; t\rangle \) we finally get
\[ \langle \alpha, \beta; t | (\hat{x}^1 - \hat{x}_0^1) |\alpha, \beta; t \rangle = \sqrt{2} \frac{R}{m\omega} \cos (\tilde{\omega} t + \phi) , \]
\[ \langle \alpha, \beta; t | (\hat{x}^2 - \hat{x}_0^2) |\alpha, \beta; t \rangle = \sqrt{2} \frac{R}{m\omega} \sin (\tilde{\omega} t + \phi) , \]
where the frequency of oscillation \( \tilde{\omega} \) concords with the classical expression (18).
Alternatively, for the Landau gauge we have from (37),
\[ \langle \alpha, k_2; t | \hat{x}^1 |\alpha, k_2; t \rangle = -\sqrt{\frac{m\hbar}{2\omega}} \text{Re} (\alpha) - \frac{\mu}{\omega} k_2 , \]
\[ \langle \alpha, k_2 | \hat{p}_1 |\alpha, k_2 \rangle = \sqrt{\frac{m\omega\hbar}{2}} \text{Im} (\alpha) , \]
with \( \mu \) given by (28). Writing \( \alpha = \hbar^{-1/2} R e^{-i\phi} \), with \( R \) and \( \phi \) real constants, and using the time-dependence of \( |\alpha \rangle \) we get
\[ \langle \alpha, k_2 | \hat{x}^1 |\alpha, k_2 \rangle = -\sqrt{\frac{m}{2\omega}} 2 R \cos (\omega t + \phi) - \frac{\mu}{\omega} k_2 . \]
Once again, the frequency of oscillation \( \omega \) concords with the classical expression (17).
As expected, by computing the mean values \( \langle \hat{f} \rangle = \langle \alpha, \beta | \hat{f} |\alpha, \beta \rangle \) and the dispersion \( \left( \Delta \hat{f} \right)^2 = \langle \hat{f}^2 \rangle - \langle \hat{f} \rangle^2 \), we see that the above CS saturate the uncertainty relations,
\[ \Delta \hat{x}_i = \sqrt{\frac{\mu \hbar}{2B |e|}} , \quad \Delta \hat{p}_i = \sqrt{\frac{\hbar B |e|}{2c\mu}} , \quad \Delta \hat{x}_i \Delta \hat{p}_i = \frac{\hbar}{2} , \]
where \( i = 1, 2 \) and \( \mu = \mu_S \) is given by (21) for the symmetric gauge and for the Landau gauge (the semi-coherent states) \( i = 1 \) and \( \mu \) is given by (28).
We recall that standard CS \( |\alpha, \beta\rangle \) resolve the unity operator in the Hilbert space spanned by the eigenfunctions \( |m, n\rangle \):
\[ \int_C \int_C \frac{d^2\alpha}{\pi^2} \frac{d^2\beta}{\pi^2} |\alpha, \beta\rangle \langle \alpha, \beta| = I . \]
Hence, they allow a quantization à la “Berezin-Klauder” or “anti-Wick” quantization \[30\] of the classical quantities \(f(\alpha, \beta)\) through the correspondence

\[
f(\zeta, \bar{\zeta}) \mapsto \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{d^2\alpha d^2\beta}{\pi^2} f(\alpha, \beta) |\alpha, \beta\rangle \langle \alpha, \beta| \equiv \hat{f}.
\] (39)

For mild conditions on \(f\), this linear map produces a well-defined operator \(\hat{f}\) in \(\mathcal{H}\). Starting from the classical quantities \((x_1, x_2)\), with \(z = x_1 - ix_2\), using the explicit form of \(\hat{a}\) and \(\hat{b}\) (\(\hat{a}^+\) and \(\hat{b}^+\)) in (23), and the fact that \(\alpha\) and \(\beta\) (\(\alpha^*\) and \(\beta^*\)) are their respective eigenvalues, we see that these complex parameters are related with the classical quantities by

\[
x_1 = \sqrt{\frac{2\hbar}{\omega m}} (\text{Re} \alpha + \text{Re} \beta),\quad x_2 = \sqrt{\frac{2\hbar}{\omega m}} (\text{Im} \beta - \text{Im} \alpha),
\]

\[
p_1 = \sqrt{\frac{\hbar \omega m}{2}} (\text{Im} \alpha + \text{Im} \beta),\quad p_2 = \sqrt{\frac{\hbar \omega m}{2}} (\text{Re} \alpha - \text{Re} \beta).
\] (40)

Using the decomposition (34) it is a matter of simple calculation to prove that (39), for \(f(\alpha, \beta) = \alpha\) and \(f(\alpha, \beta) = \beta\), give

\[
\alpha \mapsto \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{d^2\alpha d^2\beta}{\pi^2} \alpha |\alpha, \beta\rangle \langle \alpha, \beta| = \hat{a}, \quad \beta \mapsto \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{d^2\alpha d^2\beta}{\pi^2} \beta |\alpha, \beta\rangle \langle \alpha, \beta| = \hat{b},
\]

and, in the same way, we have \(\alpha^* \mapsto \hat{a}^+\), \(\beta^* \mapsto \hat{b}^+\). The above expressions allows perform the CS quantization of the classical quantities (40),

\[
x_1 \mapsto \hat{x}^1 = \sqrt{\frac{2\hbar}{\omega m}} (\hat{a} + \hat{a}^+ + \hat{b} + \hat{b}^+)\quad p_1 \mapsto \hat{p}_1 = \frac{i}{2} \sqrt{\frac{\hbar \omega m}{2}} (\hat{a}^+ - \hat{a} - \hat{b} + \hat{b}^+),
\]

\[
x_2 \mapsto \hat{x}^2 = \frac{i}{2} \sqrt{\frac{\hbar}{2\omega m}} (a - a^+ - \hat{b} + \hat{b}^+)\quad p_2 \mapsto \hat{p}_2 = \frac{1}{2} \sqrt{\frac{\hbar \omega m}{2}} (\hat{a} + a^+ - \hat{b} - \hat{b}^+).
\] (41)

Using the above relations we can see explicitly that the definition of a classical variable in the form (2),

\[
q^k = x^k - \frac{1}{2\hbar} \theta^{kj} p_j \mapsto \hat{x}^k - \frac{1}{2\hbar} \theta^{kj} \hat{p}_j,
\]

reproduces the adequate quantum theory with non-commuting coordinates for the position operator where \([\hat{q}^1, \hat{q}^2] = i\theta\).

5 Circular-coherent states on the noncommutative plane

In this section we construct a different CS family for the problem of a charged particle in a non-commutative plane. Due to the nature of the behavior of a
charged particle in a uniform magnetic field, our approach will make use of the coherent states for the motion of a quantum particle on a circle. In the work [14] the authors propose the construction of CS for a particle in a uniform magnetic field by precisely using the CS for the circle. The latter are constructed from the angular momentum operator \( \hat{J} \) and the unitary operator \( \hat{U} \) that represents the position of the particle on the unit circle. These operators obey the commutation relations [15],

\[
[\hat{J}, \hat{U}] = U, \quad [\hat{J}, \hat{U}^+] = -\hat{U}^+ . \tag{42}
\]

The introduction of these CS permits to avoid the problem of the infinite degeneracy present in the approach followed by Man’ko and Malkin, and, in addition, takes into account the momentum part of the phase space. Consequently, the so obtained CS offer a better way to compare the quantum behavior of the system with the classical trajectories in the phase space. In the present approach, we give a generalization, or a squeezed version, of these circular CS for the particle in the magnetic field and on a noncommutative plane.

In the symmetric gauge, let us introduce the centre-coordinate, as in (38), operators

\[
\hat{x}_1^0 = \hat{x}^0 - \frac{1}{\tilde{m}\omega} \hat{P}_2, \quad \hat{x}_2^0 = \hat{x}^2 + \frac{1}{\tilde{m}\omega} \hat{P}_1 , \tag{43}
\]

which are integral of motion, \([H_\theta, \hat{x}_i^0] = 0\), and the relative motion coordinates,

\[
\hat{r}_1 = \hat{x}^1 - \hat{x}_0^1 = \frac{1}{\tilde{m}\omega} \hat{P}_2, \quad \hat{r}^2 = \hat{x}^2 - \hat{x}_0^2 = -\frac{1}{\tilde{m}\omega} \hat{P}_1 . \tag{44}
\]

Next let us define the operators

\[
\hat{r}_{0\pm} = \hat{x}_0^1 \pm i\hat{x}_0^2, \quad \hat{r}_{\pm} = \hat{r}_1 \pm i\hat{r}^2 = \frac{1}{\tilde{m}\omega} \left( \hat{P}_2 \mp i\hat{P}_1 \right) . \tag{45}
\]

They obey the commutation rules

\[
[\hat{r}_{0+}, \hat{r}_{0-}] = 2\frac{\hbar}{\tilde{m}\omega}, \quad [\hat{r}_{+}, \hat{r}_{-}] = -2\frac{\hbar}{\tilde{m}\omega}, \quad [\hat{r}_{0\pm}, \hat{r}_{\pm}] = 0 . \tag{46}
\]

We now define the angular momentum operator \( \hat{J} \), which is just proportional to the Hamiltonian [20],

\[
\hat{J} = \hat{r}_1 \hat{P}_2 - \hat{r}_2 \hat{P}_1 = \frac{2}{\tilde{\omega}} \hat{H}_\theta = \tilde{m}\omega \hat{r}_+ \hat{r}_- + \hbar = \tilde{m}\omega \hat{r}_- \hat{r}_+ - \hbar . \tag{47}
\]

The above expression coincides with the classical one [19]. Due to the rules,

\[
[J, \hat{r}_{0\pm}] = 0, \quad [J, \hat{r}_{\pm}] = \pm 2\hbar \hat{r}_\pm , \tag{47}
\]

the operator \( \hat{J} \) can be identified as the generator of rotations about the axis passing through the classical point \((x_0^1, x_0^2)\) and perpendicular to the \((x^1, x^2)\) plane. The nonunitary operator \( \hat{r}_- \) is the counterpart of the unitary operator \( \hat{U} \).
in (42), which describes to a certain extent the angular position of the particle on a circle. Actually, the factorization of $\hat{r}_-$, where, in the present case,
\[
\hat{r}_- = R_- \hat{V} , \quad \hat{R}_- = \sqrt{\hat{r}_- \hat{r}_-^\dagger} 
\]
allows to view $\hat{V}$ as a unitary operator related to $\hat{U}$.

The symmetries and the integrability of the model can be encoded into the two independent Weyl-Heisenberg algebras issued from the rules (47), one for the center of the circular orbit and the other for the relative motion. They allow one to construct the Fock space (25), with orthonormal basis \{\ket{m,n} ≡ \ket{m} \otimes \ket{n}, m,n \in \mathbb{N}\}, as repeated actions of the raising operators $\hat{r}_0^-$ and $\hat{r}_+$,
\[
\hat{r}_0^- |m\rangle = \sqrt{\frac{2 \hbar (m+1)}{\tilde{m} \tilde{\omega}}} |m+1\rangle , \quad \hat{r}_+ |n\rangle = \sqrt{\frac{2 \hbar (n+1)}{\tilde{m} \tilde{\omega}}} |n+1\rangle .
\]

On the other hand, we have
\[
\hat{r}_0^+ |m\rangle = \sqrt{\frac{2 \hbar m}{\tilde{m} \tilde{\omega}}} |m-1\rangle , \quad \hat{r}_- |n\rangle = \sqrt{\frac{2 \hbar n}{\tilde{m} \tilde{\omega}}} |n-1\rangle ,
\]
and the eigenvalue equation
\[
\hat{J} \ket{m,n} = (2n+1) \hbar |m,n\rangle .
\]

The circular CS $|z_0,\zeta\rangle$, as they were proposed in [14] (although with some notational differences), are constructed in the Hilbert space spanned by the orthonormal basis \{\ket{m,n}\} as solutions to the eigenvalue equations:
\[
\hat{r}_0^+ |z_0,\zeta\rangle = z_0 |z_0,\zeta\rangle , \quad \hat{Z} |z_0,\zeta\rangle = \zeta |z_0,\zeta\rangle , \quad z_0,\zeta \in \mathbb{C},
\]
where the operator $\hat{Z}$ is defined by
\[
\hat{Z} = e^{\frac{\hbar}{4} \left( \frac{\hat{J}}{\hbar} + 1 \right) \hat{r}_-}.
\]

The projection of the CS (51) in this Fock basis reads as
\[
\langle m,n | \zeta, z_0 \rangle = e^{-\frac{z_0^2}{2}} \tilde{m} \tilde{\omega} \frac{\tilde{\zeta}^m}{\sqrt{\mathcal{E}(\zeta^2)} \sqrt{m! n!} e^{-\frac{1}{4} n(n+1)}},
\]

\[
\tilde{m} \tilde{\omega} = \sqrt{\frac{\hbar}{\mu S}} , \quad \tilde{\zeta} = \sqrt{\frac{\tilde{m} \tilde{\omega}}{2 \hbar}} \zeta,
\]

where, for notational convenience, we have introduced the dimensionless variables
\[
\tilde{z}_0 = \sqrt{\frac{\tilde{m} \tilde{\omega}}{2 \hbar}} z_0 , \quad \tilde{\zeta} = \sqrt{\frac{\tilde{m} \tilde{\omega}}{2 \hbar}} \zeta.
\]

The normalization factor involves the function
\[
\mathcal{E}(t) = \sum_{n=0}^{\infty} e^{-n(n+1)} \frac{t^n}{n!},
\]

13
We can generalize the above formulation and, consequently, obtain a squeezed version of the CS \((51)\), defining the CS of the charge in a uniform magnetic field as the eigenvector of the commuting operators \(\hat{r}_0+\) and \(\hat{Z}_\lambda\) \([10]\),

\[
\hat{r}_0+ |z_0, \zeta\rangle = z_0 |z_0, \zeta\rangle , \quad \hat{Z}_\lambda |z_0, \zeta\rangle = \zeta |z_0, \zeta\rangle ,
\]

(55)

where

\[
\hat{Z}_\lambda = \exp \left[ \frac{\lambda}{4} \left( \frac{j}{\hbar} + 1 \right) \right] \hat{r}_- = \sum_{n \geq 1} e^{\frac{\lambda}{4} \sqrt{n}} |n-1\rangle \langle n| .
\]

(56)

The operator \(\hat{Z}_\lambda\) coincides with \(\hat{Z}\) from \((51)\) for \(\lambda = 2\), and with just \(\hat{r}_-\) for \(\lambda = 0\), i.e., the case in which we have the tensor product of standard coherent states, called in this context the Malkin-Man’ko CS \([11]\). For an arbitrary \(\lambda \geq 0\), \(\hat{Z}_\lambda\) controls the dispersion relations of the angular momentum \(\hat{J}\) and of the “position operator” \(\hat{r}_-\). Note the expressions in terms of the number operator \(\hat{N}\) and the resulting commutation rule,

\[
[\hat{Z}_\lambda, \hat{Z}_\lambda^\dagger] = 2 \frac{2}{\hbar \omega} \partial_\lambda e^{\lambda (\hat{N}+1)} , \\
\hat{Z}_\lambda^\dagger \hat{Z}_\lambda = 2 \frac{2}{\hbar \omega} \partial_\lambda \left[ \sinh(\lambda/2) e^{\lambda (\hat{N}+1)/2} \right] .
\]

(57)

We will call the states defined in \((55)\) \(\lambda\)-coherent states (\(\lambda\)-CS). In the Fock basis \((50)\) these CS read as \([16]\)

\[
|z_0, \zeta\rangle = e^{-\frac{|z_0|}{2}} \sum_{m,n} \frac{z_0^m}{\sqrt{m!}} \frac{\zeta^n}{\sqrt{n!}} e^{-\frac{\lambda n(n+1)}{2}} |m,n\rangle ,
\]

(58)

where, for a fixed value of \(\lambda\) in \((55)\), the normalization function \(\mathcal{E}_\lambda (t)\) is a kind of generalized “exponential”

\[
\mathcal{E}_\lambda (t) = \sum_{n=0}^{\infty} e^{-\frac{\lambda n(n+1)}{2}} \frac{t^n}{n!} \equiv \sum_{n=0}^{\infty} \frac{t^n}{x_n!} ,
\]

(59)

where

\[
x_n = e^{n\lambda} n \equiv \partial_\lambda e^{n\lambda} , \quad x_n! = x_1 x_2 \cdots x_n , \quad x_0! = 1 .
\]

(60)

The complex numbers \(z_0\) and \(\zeta\) parameterize, respectively, the position of the centre of the circle and the classical phase space of the circular motion. As was shown in \([14]\) these CS have some properties that made them more suitable to describe the classical behavior of a charged particle in a magnetic field, in comparison with the Malkin-Man’ko CS \([11]\). Besides, as we show in the next section, our generalization with the \(\lambda\) parameter can be explored to improve in an appreciable way these interesting characteristics.

The \(\lambda\)-CS \(|z_0, \zeta\rangle\) \((55)\) are the tensor product of the states \(|z_0\rangle\) and \(|\zeta\rangle\), where the first one is the standard CS described in the previous section. So, in order
to perform the Berezin-Klauder-Toeplitz quantization using our CS, we concentrate only on the states $|\zeta\rangle$. For convenience, we put $\hbar = \tilde{m}\tilde{\omega}/2 = 1$, and so $\tilde{\zeta} = \zeta$. Then, in the basis $\{|n\rangle\}$, the latter admits the decomposition

$$
|\zeta\rangle = \frac{1}{\sqrt{\mathcal{E}_\lambda(|\zeta|^2)}} \sum_{n=0}^{\infty} \frac{\zeta^n}{\sqrt{x_n!}} |n\rangle, \quad x_n = e^{n\lambda} n. \quad (61)
$$

As was shown in [16] the CS states $|\zeta\rangle$ resolve the unity operator in the Hilbert space spanned by the kets $|n\rangle$,

$$
\int \mathcal{C} \varpi_\lambda \left(|\zeta|^2\right) \frac{d^2\zeta}{\pi} \mathcal{E}_\lambda(|\zeta|^2) |\zeta\rangle \langle \zeta| = I,
$$

where the weight function $\varpi_\lambda$ is given under the form of the Laplace transform,

$$
\varpi_\lambda(t) = \frac{e^{-\lambda/2}}{\sqrt{2\pi\lambda}} \int_0^{+\infty} du \exp \left(-e^{-\lambda/2}tu\right) e^{-\frac{(\ln u)^2}{2\lambda}} = \frac{e^{-\lambda/2}}{\sqrt{2\pi\lambda}} \mathcal{L} \left[e^{-\frac{(\ln u)^2}{2\lambda}}\right] \left(e^{-\lambda/2}t\right). \quad (62)
$$

The corresponding CS quantization of functions on the complex plane, the phase space for the relative motion, is the map

$$
f(\tilde{\zeta}, \tilde{\zeta}) \mapsto \int \mathcal{C} \frac{d^2\zeta}{\pi} \varpi_\lambda \left(|\zeta|^2\right) \mathcal{E}_\lambda \left(|\zeta|^2\right) |\zeta\rangle \langle \zeta| \overset{def}{=} \hat{f}. \quad (62)
$$

Using the fact that the weight function $\varpi_\lambda$ solves the following moment problem [16],

$$
\int_0^\infty t^n \varpi_\lambda(t) \, dt = n! \exp \left\{ \frac{\lambda n(n+1)}{2} \right\}, \quad \lambda \geq 0,
$$

it is easy to obtain the quantization of the variable $\zeta$,

$$
\zeta \mapsto \hat{\zeta} = \int \mathcal{C} \frac{d^2\zeta}{\pi^2} \varpi_\lambda \left(|\zeta|^2\right) \mathcal{E}_\lambda \left(|\zeta|^2\right) \zeta \, |\zeta\rangle \langle \zeta| = \sum_n \exp \left[ \frac{\lambda}{2} n \right] \sqrt{n} |n-1\rangle \langle n| = \hat{Z}_\lambda. \quad (63)
$$

Similarly, we have $\hat{\bar{\zeta}} = \hat{\bar{Z}}_\lambda^\dagger$. Let us now quantize the classical observable $|\zeta|^2$ that, as it will be discussed in the next section, represents a $\lambda$-deformation of the classical relative angular momentum. One obtains:

$$
|\zeta|^2 \mapsto \int \mathcal{C} \frac{d^2\zeta}{\pi^2} \varpi_\lambda \left(|\zeta|^2\right) \mathcal{E}_\lambda \left(|\zeta|^2\right) |\zeta|^2 \, |\zeta\rangle \langle \zeta| = \sum_n (n+1) \exp \left[ \lambda(n+1) \right] |n\rangle \langle n| = \hat{Z}_\lambda \hat{Z}_\lambda^\dagger = \partial_\lambda e^{\lambda(N+1)}. \quad (64)
$$
Therefore, restoring the units,
\[
\zeta \mapsto \hat{\zeta} = \sqrt{\frac{2\hbar}{\tilde{m}\tilde{\omega}}} \exp \left[ \frac{\lambda}{2} (\hat{a}^+ \hat{a} + 1) \right] \hat{a}, \quad \bar{\zeta} \mapsto \hat{\bar{\zeta}} = \sqrt{\frac{2\hbar}{\tilde{m}\tilde{\omega}}} \exp \left[ \frac{\lambda}{2} (\hat{a}^+ \hat{a} + 1) \right] ,
\]
with the operator \( \hat{a} \) and \( \hat{a}^+ \) given in (23). Repeating the procedure of the previous section, for the standard CS \( |z_0\rangle \), we obtain,
\[
z_0 \mapsto r_{0-} = \sqrt{\frac{2\hbar}{\tilde{m}\tilde{\omega}}} \hat{b}, \quad z^*_0 \mapsto r_{0+} = \sqrt{\frac{2\hbar}{\tilde{m}\tilde{\omega}}} \hat{b}^+ ,
\]
with the operator \( \hat{b} \) and \( \hat{b}^+ \) given in (24). Using the relations (41) and the transformation (2) we see that this CS quantization reproduces the non-commutative relation (1).

5.0.1 Numerical analysis

Following [15] a criterion to test the closeness of the introduced \( \lambda \)-CS (61) to the classical phase space, is verifying how expectation value of the angular momentum operator approaches the respective classical quantity. It can be done by the evaluation of the relative error \( e \),
\[
e (\lambda, l) = \frac{|\langle \hat{J} \rangle_\zeta - l|}{l} ,
\]
with the expectation value of the angular momentum, in the units \( \hbar = \tilde{m}\tilde{\omega} / 2 = 1 \), given by
\[
\langle \hat{J} \rangle_\zeta = \langle \zeta | \hat{J} | \zeta \rangle = \frac{1}{\mathcal{E}_\lambda(|\zeta|^2)} \sum_{n=0}^{+\infty} |\zeta|^{2n} \frac{(2n + 1)}{n!} e^{-\frac{\lambda}{2} n(n+1)} .
\]
The parameter \( \zeta \) is related to the classical angular momentum \( l = \tilde{m}\tilde{\omega} r^2 = 2r^2 \) (where \( r \) is the classical radius) by
\[
|\zeta|^2 = \frac{l}{2} \exp \left( \frac{\lambda l}{2} \right) .
\]
As observed in [14] the error computation (67) by using Kowalski-Rembielevski CS shows that the approximate equality \( \langle \hat{J} \rangle_\zeta \simeq l \) does not hold for arbitrary small \( l \), being good only for \( l > 1 \). From Fig. 1 we see that, for our \( \lambda \)-CS, this approximation can be improved, for \( l \leq 1 \), by making \( \lambda \) increase.
Fig.1-Error function $e$ as a function of $\lambda$ for $|\zeta| = 1$.

Fig.2-Error function $e$ as a function of $l$ for $\lambda = 2$ (solid line), $\lambda = 4$ (dashed line) and $\lambda = 6$ (dotted line).

But this behavior is not shared by arbitrary values of $l > 1$, as can be seen in Figure 2: the error starts to oscillate as $l$ increases and, occasionally, we can have $e(\lambda, l) > e(\lambda', l)$ with $\lambda > \lambda'$ for $l > 1$. The approximation is better near integer values assumed by $l$. This can be related to the fact that, in the construction of the CS for a particle on a circle, as mentioned in [14], the angular momentum can assume only integer values in the boson case.

5.0.2 Harmonic oscillator phase space

In view of the commutation rule (57) that illustrates a sort of “$\lambda$” deformation of the harmonic oscillator, it is natural to consider the quantized version (64) of the classical observable $|\zeta|^2$ as a Hamiltonian $\hat{H} = |\zeta|^2 = (\hat{N} + 1) \exp(\lambda(\hat{N} + 1))$ ruling the time evolution of quantum states. We thus investigate the time evolution of the quantized version $\hat{Z}_\lambda$, as found in (63), of the classical phase space point $\zeta \equiv (q + ip)/\sqrt{2}$, comparing it with the phase space circular classical trajectories. This time evolution is well caught through its mean value in coherent states $|\zeta\rangle$ (lower symbol) [16]:

$$\zeta(t) \equiv \langle \zeta | e^{-i\hat{H}t} \hat{\zeta} e^{i\hat{H}t} | \zeta \rangle$$

$$= \frac{\zeta}{\mathcal{E}_\lambda(|\zeta|^2)} \sum_{n=0}^{+\infty} \frac{|\zeta|^{2n}}{x_n!} \exp \left[ -i (x_{n+2} - x_{n+1}) t \right],$$

(68)

with $x_n = ne^\lambda$. Setting the initial state $\zeta = 1$, we plot the phase-space ($\Re \zeta \times \Im \zeta$) for different values of $\lambda$. For $\lambda = 0$ we obtain a circle, as is expected for the standard coherent states. For $\lambda \neq 0$ the trajectories are confined between two circles. The general behavior can be see in Figure 3, where we set $\lambda = 2$ (the circular CS of [14]).
The dependence of the internal radius $r_{\text{int}}$ with the value of $\lambda$ can be viewed in Figure 4. The internal radius decreases from 1 (at $\lambda = 0$) up to almost 0.05 (for $\lambda \sim 0.3$), which represents the most squeezed version of our $\lambda$-CS. After that, it starts to increase again. For $\lambda > 6$ the trajectories become circular again, with a period proportional to $e^{-\lambda}$.

6 Conclusion

We have studied a $\theta$-modified classical action for a charged particle in a magnetic field. The canonical quantization of this model yields a quantum mechanics for this non-commutative space. The classical theory and the quantum theory are not gauge-invariant and obey some peculiarities for critical values of the non-commutative parameter $\theta$. These values depend on the gauge. In the symmetric gauge, the energy of the quantum particle depends on the relation between the respective signs of $B$ and $\theta$. Thus, if we assume that these quantities are independent, the difference between the energy levels will change if we invert the direction of the magnetic field. We have constructed the standard CS of particles in the $\theta$-modified quantum theory and have shown that the mean values of the position operator coincide with the “classical” trajectories of the $\theta$-modified classical theory. In addition, we have constructed a family of non-standard circular CS parameterized by $\lambda \geq 0$ and have used such states to perform the Berezin-Klauder-Toeplitz quantization. As a result, we have reproduced the $\theta$-modified quantum theory in the symmetric gauge. With the help of numerical explorations, we have shown to what extent the mean values of some physical quantities depend on the choice of the parameter $\lambda$.

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