Research Article

Common Fixed Point Results on Generalized Weak Compatible Mapping in Quasi-Partial b-Metric Space

Pragati Gautam 1, Luis Manuel Sánchez Ruiz 2, Swapnil Verma 1, and Gauri Gupta 1

1Department of Mathematics, Kamala Nehru College (University of Delhi), August Kranti Marg, New Delhi 110049, India
2ETSID-Departamento de Matematica Aplicada CITG, Universitat Politècnica de Valencia, Valencia E-46022, Spain

Correspondence should be addressed to Luis Manuel Sánchez Ruiz; lmsr@mat.upv.es

Received 21 February 2021; Revised 14 April 2021; Accepted 31 May 2021; Published 11 June 2021

Academic Editor: Huseyin Isik

Copyright © 2021 Pragati Gautam et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The focus of this paper is to acquaint with generalized condition (B) in a quasi-partial b-metric space and to establish coincidence and common fixed point theorems for weakly compatible pairs of mapping. Additionally, with the background of quasi-partial b-metric space, the outcomes obtained are exemplified to prove the existence and uniqueness of fixed point.

1. Introduction

In the early years of 20th century, the French mathematician Fréchet [1] commenced the concept of metric space, and due to its consequences and practicable implementations, the idea has been enlarged, upgraded, and generalized in different directions. In 1922, Banach [2] introduced the very important Banach contraction principle which holds a remarkable position in the field on nonlinear analysis. One such generalization was established by Künzi et al. [3] known as quasi-partial metric space by Karapinar et al. [4, 5]. In 1993, Czerwik [6] introduced the concept of b-metric space. Later, Gupta and Gautam [7, 8] generalized quasi-partial metric space to quasi-partial b-metric space and proved some fixed point results for such spaces. Several authors [9–18] have already proved the fixed point theorem in metric space, partial metric space [19], quasi-partial metric space, quasi-partial b-metric space [7], and many different spaces. After these classical results, some researchers [20–25] introduced the distinctive concepts and used fixed point theorems to demonstrate the uniqueness of a solution of the equations in different metric spaces such as multivalued contractive type mappings, Reich–Rus–Cirić and Hardy–Rogers contraction mappings, and Chatterjea and cyclic Chatterjea contraction.

In this paper, we have introduced the generalized condition (B) in quasi-partial b-metric space to obtain coincidence and common fixed points. Moreover, some examples are given to exemplify the concept followed up with pictographic grid.

2. Preliminaries

Let us recall some definition.

Definition 1 (see [19]). A partial metric space on a nonempty set $X$ is a function $M: X \times X \rightarrow \mathbb{R}^+$ satisfying

1. $M(\tau, \tau) = M(\tau, \nu)$ (symmetry)
2. if $0 < M(\tau, \tau) = M(\tau, \nu) = M(\nu, \nu)$, then $\tau = \nu$ (indistance implies equality)
3. $M(\nu, \nu) \leq M(\tau, \nu)$, then $\tau = \nu$ (small self-distances)
4. $M(\tau, Y) + M(\nu, \nu) \leq M(\tau, \nu) + M(\nu, Y)$ (triangularity)

for all $\tau, \nu, Y \in X$. 
Definition 2 (see [4]). A quasi-partial metric on a nonempty set \(X\) is a function \(q: X \times X \rightarrow \mathbb{R}^+\) satisfying

1. If \(q(\tau, \tau) = q(\tau, \tau) = q(\nu, \nu),\) then \(\tau = \nu\) (indistancy implies equality)
2. \(q(\tau, \tau) \leq q(\tau, \nu)\) (small self-distances)
3. \(q(\tau, \nu) \leq q(\tau, \nu) + q(\nu, \nu)\) (trangularity)

for all \(\tau, \nu, \lambda \in X\). The infimum over all reals \(\rho \geq 1\) satisfying condition (30) is called the coefficient of \((X, q\rho_b)\) and represented by \(R(X, q\rho_b)\).

Lemma 1 (see [6]). Let \((X, q\rho_b)\) be a quasi-partial b-metric space. Then the following hold:

1. If \(q\rho_b(\tau, \nu) = 0,\) then \(\tau = \nu\)
2. If \(\tau \neq \nu,\) then \(q\rho_b(\tau, \nu) > 0\) and \(q\rho_b(\nu, \tau) > 0\)

Lemma 2 (see [6]). Let \((X, q\rho_b)\) be a quasi-partial b-metric space and \((X, d_{q\rho_b})\) be the corresponding b-metric space. Then \((X, d_{q\rho_b})\) is complete if \((X, q\rho_b)\) is complete.

Definition 5 (see [26]). A self-mapping \(P\) on a metric space \((X, d)\) satisfies condition (B), if there exist \(\delta \in [0, 1]\) and \(\omega > 0\) such that for all \(\tau, \nu \in X,\) we have,

\[
d(Q\tau, Q\nu) \leq \delta \max \left\{d(Pr, Pu), d(Pr, Q\nu), d(Pu, Pr), d(Pr, Pu)\right\} + \frac{1}{2}\omega \left\{d(Pr, Q\nu) + d(Pu, Q\nu)\right\} + \omega \min\{d(Pr, Q\nu), d(Pr, Pu), d(Pu, Pr), d(Pr, Pu)\}.
\]

Clearly condition (B) implies generalized condition (B).

Definition 7 (see [29]). Let \(P\) and \(Q\) be self-mappings on a set \(X.\) A point \(x \in X\) is called a coincidence point of \(P\) and \(Q\)

\[
(1) A sequence \{\tau_n\} \subset X converges to \tau \in X if and only if 
q\rho_b(\tau, \tau) = \lim_{n \rightarrow \infty} q\rho_b(\tau_n, \tau) = \lim_{n \rightarrow \infty} q\rho_b(\tau_n, \tau). \quad (1)
\]

\[
(2) A sequence \{\tau_n\} \subset X is called a Cauchy sequence if and only if 
\lim_{n \rightarrow \infty} q\rho_b(\tau_n, \tau_m), \quad \lim_{n \rightarrow \infty} q\rho_b(\tau_n, \tau_m) exist. \quad (2)
\]

\[
(3) The quasi-partial b-metric space \((X, q\rho_b)\) is said to be complete if every Cauchy sequence \{\tau_n\} \subset X converges with respect to \(x_{q\rho_b}\) to a point \(\tau \in X\) such that 
q\rho_b(\tau, \tau) = \lim_{n \rightarrow \infty} q\rho_b(\tau_n, \tau_m) = \lim_{n \rightarrow \infty} q\rho_b(\tau, \tau). \quad (3)
\]

\[
(4) A mapping \(f: X \rightarrow X\) is said to be continuous at \(x_0 \in X\) if for every \(\epsilon > 0,\) there exists \(\delta > 0\) such that 
f(B(x_0, \delta)) \subset B(f(x_0), \epsilon).
\]

Definition 6 (see [27]). Let \(P\) and \(Q\) be two self-mappings on a metric space \((X,d)\). The mapping \(Q\) satisfies generalized condition (B) associated with \(P\) if there exist \(\delta \in (0, 1)\) and \(\omega \geq 0\) with \(\rho \geq 1\) such that

\[
d(Q\tau, Q\nu) \leq \delta \max\left\{d(Pr, Pu), d(Pr, Q\nu), d(Pu, Pr), d(Pr, Pu)\right\} \quad (5)
\]

\[
+ \omega \min\{d(Pr, Q\nu), d(Pr, Pu), d(Pu, Pr), d(Pr, Pu)\}. \quad (5)
\]

Clearly condition (B) implies generalized condition (B).

Definition 8 (see [30]). Let \(X\) be a nonempty set. Two mappings \(P, Q: X \rightarrow X\) are said to be weakly compatible if

\[PX = QX = w,\] where \(w\) is called a point of coincidence of \(P\) and \(Q.\)
they commute at their coincidence point, that is, if \( Pu = Qu \) for some \( u \in X \), then \( PQu = QPu \).

### 3. Main Results

**Definition 9.** Let \( P \) and \( R \) be two self-mappings on a quasi-partial b-metric space \((X, qpb)\).

\[
qpb((Rr, Ruv)) \leq \delta \max \left\{ qpb((Pr, Puv), qpb((Pr, Rr)), qpb((Puv, Ruv)), \frac{1}{2p} (qpb((Rr, Puv) + qpb((Pr, Ruv))) \right\} + M \min\{qpb((Pr, Rr)), qpb((Puv, Ruv)), qpb((Pr, Rr))\}.
\]

**Definition 10.** Let \( P, Q, R, S \) be four self-mappings on a quasi-partial b-metric space \((X, qpb)\).

The pair of mapping \((P, R)\) satisfies generalized condition (B) associated with \((Q, S)\) \((P, R)\) is generalized almost \((Q, S)\) contraction if there exist \( \delta \in (0, 1), \rho \geq 1 \) and \( M \geq 0 \) such that for all \( r, u \in X \), we have

\[
qpb((Rr, Su)) \leq \delta \max \left\{ qpb((Pr, Qu), qpb((Pr, Rr)), qpb((Qu, Su)), \frac{1}{2p} (qpb((Rr, Qu) + qpb((Pr, Su))) \right\} + M \min\{qpb((Pr, Rr)), qpb((Qu, Su)), qpb((Pr, Rr))\}.
\]

**Theorem 1.** Let \( P, Q, R, S \) be four self-mappings on quasi-partial b-metric space \((X, qpb)\) and if we take the mappings in pair as \((P, R)\) associated with \((Q, S)\) for all \( r, u \in X \), \( \delta \in (0, 1) \), and \( M \geq 0, \rho \geq 1 \) and

1. \( RX \subset QX \) and \( SX \subset PX \)
2. \( PX \) or \( QX \) is closed
3. \((1/\rho)(\delta + 2M) < 1\)

then the pairs \((P, R)\) and \((Q, S)\) have a coincidence point. Also \( P, Q, R, S \) have a unique common fixed point, providing that pairs \((P, R)\) and \((Q, S)\) are weakly compatible.

**Proof.** Let \( r^* \in X \). Since \( RX \subset QX \) there exists \( r_0 \in X \) such that \( v_0 = Qr_0 = Rr^* \). Suppose there exists a point \( v_1 \in Sr_0 \) corresponding to the point \( v_0 \). Also since \( SX \subset PX \) there exist \( r_1 \in X \) such that \( v_1 = Pr_1 = Sr_0 \). Going this way we get a sequence \( \{v_n\} \in X \) as

\[
v_{2m+1} = Qr_{2m+1} = Rr_{2m+1},
\]

\[
v_{2m+2} = Pr_{2m+2} = Rr_{2m+1},
\]

\[
qpb(v_{2m+1}, v_{2m+2}) = qpb(Rr_{2m+1}, Sr_{2m+1}) \]

\[
\leq \delta \max\{qpb((Pr_{2m}, Qt_{2m+1}), qpb((Pr_{2m}, Rr_{2m}), qpb((Qt_{2m+1}, Rr_{2m+1}), \frac{1}{2p} (qpb((Rr_{2m}, Qt_{2m+1}) + qpb((Pr_{2m}, Sr_{2m+1}) \right) + M \min\{qpb((Pr_{2m}, Rr_{2m}), qpb((Qt_{2m+1}, Sr_{2m+1}), qpb((Pr_{2m}, Qt_{2m+1}), qpb((Sr_{2m+1}, Rr_{2m}) \]

\[
\leq \delta \max\{qpb((v_{2m}, v_{2m+1}), qpb((v_{2m}, v_{2m+1}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}) \}
\]

\[
\frac{1}{2p} (qpb((v_{2m+1}, v_{2m+2}) + qpb((v_{2m+1}, v_{2m+2}) \right) + M \min\{qpb((v_{2m}, v_{2m+1}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}) \}
\]

\[
= \delta \max\{qpb((v_{2m}, v_{2m+1}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}) \}
\]

\[
= \delta \max\{qpb((v_{2m}, v_{2m+1}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}) \}
\]

\[
M \min\{qpb((v_{2m}, v_{2m+1}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}), qpb((v_{2m+1}, v_{2m+2}) \}
\]
This condition gives 4 cases.

Case 1.
\[ \max\{q_p(b(v_{2m'}, v_{2m+1})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m}, v_{2m+1})) \tag{9} \]

Also,
\[ \min\{q_p(b(v_{2m}, v_{2m+1})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m}, v_{2m+1})) \tag{10} \]

which implies
\[ \rho q_p(b(v_{2m+1}, v_{2m+2}) \leq \delta q_p(b(v_{2m}, v_{2m+1})) + Mq_p(b(v_{2m}, v_{2m+1})) \]
\[ \leq (\delta + M)q_p(b(v_{2m}, v_{2m+1})) + \rho Mq_p(b(v_{2m}, v_{2m+1})) \tag{11} \]
\[ \leq \frac{\delta + M}{(1-M)\rho} q_p(b(v_{2m}, v_{2m+1})). \]

Let \( \mu_1 = ((\delta + M)/(1-M)\rho), ((\delta + M)/\rho) < 1 \) and \( M \geq 0 \), then \( \mu_1 < 1 \).
Therefore, \( q_p(b(v_{2m+1}, v_{2m+2}) \leq \mu_1 q_p(b(v_{2m}, v_{2m+1})) \).

Case 2.
\[ \max\{q_p(b(v_{2m}, v_{2m+1})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m}, v_{2m+1})) \tag{12} \]

Also,
\[ \min\{q_p(b(v_{2m}, v_{2m+2})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m+1}, v_{2m+2})) \tag{13} \]
which implies,
\[ q_p(b(v_{2m+1}, v_{2m+2}) \leq \frac{1}{\rho} (\delta q_p(b(v_{2m}, v_{2m+1})) + Mq_p(b(v_{2m}, v_{2m+1})) \]
\[ \leq \frac{\delta + M}{\rho} q_p(b(v_{2m}, v_{2m+1})). \tag{14} \]

Let \( \mu_2 = ((\delta + M)/\rho), ((\delta + M)/\rho) < 1 \) then \( \mu_2 < 1 \).
Therefore, \( q_p(b(v_{2m+1}, v_{2m+2}) \leq \mu_2 q_p(b(v_{2m}, v_{2m+1})) \).

Case 3.
\[ \max\{q_p(b(v_{2m}, v_{2m+1})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m+1}, v_{2m+2})) \tag{15} \]

Also,
\[ \min\{q_p(b(v_{2m}, v_{2m+2})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m}, v_{2m+1})) \tag{16} \]
which implies
\[ q_p(b(v_{2m+1}, v_{2m+2}) \leq \delta q_p(b(v_{2m+1}, v_{2m+2})) + Mq_p(b(v_{2m}, v_{2m+1}) \]
\[ \leq \frac{M}{\rho(1-\delta-M)} q_p(b(v_{2m}, v_{2m+1})). \tag{17} \]

Let \( \mu_3 = (M/(\rho(1-\delta-M))), ((\delta + M)/\rho) < 1 \) then \( \mu_3 < 1 \).
Therefore, \( q_p(b(v_{2m+1}, v_{2m+2}) \leq \mu_3 q_p(b(v_{2m}, v_{2m+1})) \).

Case 4.
\[ \max\{q_p(b(v_{2m}, v_{2m+1})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m}, v_{2m+1})) \tag{18} \]

Also,
\[ \min\{q_p(b(v_{2m}, v_{2m+2})), q_p(b(v_{2m+1}, v_{2m+2}))\} = q_p(b(v_{2m+1}, v_{2m+2})) \tag{19} \]
which implies
\[ q_p(b(v_{2m+1}, v_{2m+2}) \leq \delta q_p(b(v_{2m+1}, v_{2m+2}) + Mq_p(b(v_{2m}, v_{2m+1}) \]
\[ \leq \frac{M}{\rho(1-\delta)} q_p(b(v_{2m}, v_{2m+1})). \tag{20} \]

Let \( \mu_4 = (M/(\rho(1-\delta))), ((\delta + M)/\rho) < 1 \) then \( \mu_4 < 1 \).
Therefore, \( q_p(b(v_{2m+1}, v_{2m+2}) \leq \mu_4 q_p(b(v_{2m+1}, v_{2m+2})) \).
Choose \( \mu = \max\{\mu_1, \mu_2, \mu_3, \mu_4\} \Rightarrow 0 < \mu < 1 \).

\[ q_p(b(v_{2m+1}, v_{2m+2}) \leq \mu q_p(b(v_{2m}, v_{2m+1})). \tag{21} \]

Using mathematical induction,
\[ q_p(b(v_m, v_{m+1})) \leq \mu^m q_p(b(v^*, v_b)), \tag{22} \]

which tends to 0 as \( m \) tends to \( \infty \).
So, \([v_m]\) and its subsequence is convergent.
Let \( PX \) be closed. Therefore, \( \tau \in PX \), that is, there exists \( Y \in X \) such that \( \tau = PY \), and we need to show \( \tau = RY \).
By definition,
\[ q_p(b(RY, \tau) \leq \frac{\delta + M}{\rho} q_p(b(RY, \tau)), \tag{23} \]
which is a contradiction. Hence,
\[ q_{Pb}(RY, r) = 0 \implies RY = r. \] (24)

So, \( PY = RY \), that is, \( P \) and \( R \) have a coincidence point.

Similarly, \( Q \) and \( S \) have a coincidence point.

If we also assume \( QX \) is closed, then \( (P, R) \) and \( (Q, S) \) have a coincidence point.

Since \( (P, R) \) and \( (Q, S) \) are weakly compatible, we can prove there exists a common fixed point for \( P, Q, R, S \) by contradiction.

**Example 1.** Let \( X = [0, 4] \) equipped with quasi-partial b-metric \( q_{Pb}(r, v) = |r - v| + |r| \). Let \( P, Q, R, S \) be self-mappings on quasi-partial b-metric defined by

\[
P_r = \begin{cases} \frac{r}{2} & r \in [0, 2], \\ \frac{5}{4} & r \in (2, 4], \end{cases}
\]

\[
Q_r = \begin{cases} \frac{3r}{2} & r \in [0, 2], \\ \frac{3}{2} & r \in (2, 4]. \end{cases}
\]

\[
R_r = \begin{cases} \frac{r}{6} & r \in [0, 2], \\ \frac{1}{2} & r \in (2, 4]. \end{cases}
\]

\[
S_r = \begin{cases} \frac{r}{4} & r \in [0, 2], \\ \frac{1}{4} & r \in (2, 4]. \end{cases}
\]

\[
Q(R_r, S_r) = \left\{ \frac{r - \frac{1}{2} - \frac{1}{4} + \frac{1}{2}}{\frac{1}{2}} \right\} \leq \frac{8}{5} \left\{ \frac{3}{2} - \frac{1}{4} \right\} + \frac{3}{2}. \] (27)\]

Dominance of right-hand side of equation (27) is easily visually checked in Figure 1. Thus the inequality required in Definition 10 holds for \( r, v \in [0, 2] \).

**Case 2.** For \( r \in [0, 2], v \in [2, 4] \) we have

\[
Q(R_r, S_r) = \left\{ \frac{r - \frac{1}{2} - \frac{1}{4} + \frac{1}{2}}{\frac{1}{2}} \right\} \leq \frac{8}{5} \left\{ \frac{3}{2} - \frac{1}{4} \right\} + \frac{3}{2}. \] (28)

Dominance of right-hand side of equation (28) is easily visually checked in Figure 2. Thus the inequality required in Definition 10 holds for \( r \in [0, 2], v \in [2, 4]. \)

**Case 3.** For \( r \in [2, 4], v \in [0, 2] \) we have

\[
Q(R_r, S_r) = \left\{ \frac{r - \frac{1}{2} - \frac{1}{4} + \frac{1}{2}}{\frac{1}{2}} \right\} \leq \frac{8}{5} \left\{ \frac{3}{2} - \frac{1}{4} \right\} + \frac{3}{2}. \] (29)

Dominance of right-hand side of equation (29) is easily visually checked in Figure 3. Thus the inequality required in Definition 10 holds for \( r \in [2, 4], v \in [0, 2]. \)

**Case 4.** For \( r, v \in [2, 4] \), we have

\[
Q(R_r, S_r) = \left\{ \frac{r - \frac{1}{2} - \frac{1}{4} + \frac{1}{2}}{\frac{1}{2}} \right\} \leq \frac{8}{5} \left\{ \frac{3}{2} - \frac{1}{4} \right\} + \frac{3}{2}. \] (30)

Dominance of right-hand side of equation (30) is easily visually checked in Figure 4. Thus the inequality required in Definition 10 holds for \( r, v \in [2, 4] \).

As a result, all postulates of Theorem 1 are satisfied \((\delta = 4/5, p = 2 \geq 1, M = 0)\) and 0 is a unique common fixed point of \( P, Q, R, S \).

If \( P = Q \) and \( R = S \), we get a corollary.

**Corollary 1.** Let \( P \) and \( S \) be self-mappings on quasi-partial b-metric space \((X, q_{Pb})\). If for all \( r, v \in X, P \) satisfies the following conditions:

(1) \( SX \subseteq PX \)

(2) \( PX \) is closed

(3) \((\delta + 2M)/p < 1\)

then \( P \) and \( S \) have a coincidence point. Also \( P \) and \( S \) have a common fixed point if \((P, S)\) are weakly compatible.

**Proof.** Taking \( P = Q \) and \( R = S \) in Theorem 1, the above result can be obtained.

**Theorem 2.** Let \( P, Q, R, S \) be self-mappings on a quasi-partial b-metric space \((X, q_{Pb})\). If the pair \((P, R)\) is associated with \((Q, S)\) and satisfies

The point 0 is a coincidence point of these mapping. Furthermore, \( PR0 = RP0 = 0 \) and \( SQ0 = QS0 = 0 \), that is, the two pairs \((P, R)\) and \((Q, S)\) are weakly compatible.

**Case 1.** For \( r, v \in [0, 2] \), we have

\[
P_X = \left\{ 0, \frac{1}{2} \right\} \cup \left\{ \frac{5}{4} \right\}
\]

\[
Q_X = \left\{ 0, \frac{3}{2} \right\},
\]

\[
S_X = \left\{ 0, \frac{1}{4} \right\} \subseteq PX
\]

\[
R_X = \left\{ 0, \frac{1}{6} \right\} \cup \left\{ \frac{1}{2} \right\}.
\]

The point 0 is a coincidence point of these mapping. Furthermore, \( PR0 = RP0 = 0 \) and \( SQ0 = QS0 = 0 \), that is, the two pairs \((P, R)\) and \((Q, S)\) are weakly compatible.
Figure 1: Dominance of right-hand side of equation (27) is visually checked for \( \tau, \nu \in [0, 2] \).

Figure 2: Dominance of right-hand side of equation (28) is visually checked for \( \tau \in [0, 2], \nu \in [2, 4] \).

Figure 3: Dominance of right-hand side of equation (29) is visually checked for \( \tau \in [2, 4], \nu \in [0, 2] \).
Let Example 2.

Case 1. For $\tau, v \in [0, 2]$, we have

\[
Q(R_\tau, S_\tau) = \left\{ \tau \left( \frac{2\tau}{5} - 1 \right) \right\} + \left| \frac{\tau}{5} \right| \leq \frac{10}{9} \left( |2 - 1| + |2\tau| \right).
\]  

(35)

Dominance of the right-hand side of equation (35) is easily visually checked in Figure 6. Thus the inequality required in theorem holds for $\tau, v \in [0, 2]$.

Case 2. For $\tau \in [0, 2], v > 2$ we have

\[
Q(R_\tau, S_\tau) = \left\{ \tau \left( \frac{2\tau}{5} - 1 \right) \right\} + \left| \frac{\tau}{5} \right| \leq \frac{10}{9} \left( |2 - 1| + |2\tau| \right).
\]  

(36)

Case 3. For $\tau > 2, v \in [0, 2]$, we have

\[
Q(R_\tau, S_\tau) = \left\{ 2 - \frac{2\tau}{5} \right\} + |2\tau| \leq \frac{10}{9} \left( |4 - 2\tau| + |4\tau| \right).
\]  

(36)
Dominance of the right-hand side of equation (36) is easily visually checked in Figure 7. Thus, the inequality required in the theorem holds for $\tau > 2, \nu \in [0, 2]$.

Case 4. For $\tau, \nu > 2$, we have

$$Q(R\tau, S\nu) = ||2 - 1| + |2|| \leq \frac{30}{9}. \tag{37}$$

Dominance of the right-hand side of equation (37) is easily visually checked in Figure 8. Thus, the inequality required in the theorem holds for $\tau, \nu > 2$.

As a result, all postulates of Theorem 2 are satisfied ($\delta = (5/9)$, $\rho = 2 \geq 1$, and $M = 0$), and 0 is a unique common fixed point of $P, Q, R, S$.

If $P = Q$ and $R = S$, we get a corollary.

**Corollary 2.** Let $P$ and $S$ be self-mappings on quasi-partial $b$-metric space $(X, q_{pb})$. If for all $\tau, \nu \in X$, the pair of mapping $(P, S)$ satisfies

$$q_{pb}(R\tau, S\nu) \leq 3\left[\max(q_{pb}(P\tau, R\tau), q_{pb}(P\tau, S\nu), q_{pb}(Q\nu, S\nu), q_{pb}(P\tau, S\nu), q_{pb}(R\tau, Q\nu))\right]$$

$$+ M \min(q_{pb}(P\tau, R\tau), q_{pb}(Q\nu, S\nu), q_{pb}(P\tau, S\nu), q_{pb}(Q\nu, R\tau)), \tag{38}$$
and $P$ satisfies the following conditions:

(1) $SX < PX$

(2) $(\delta + 2M)/\rho < 1$

then the pair $(P, S)$ has a coincidence point. Also $P$ and $S$ have a common fixed point if $(P, S)$ are weakly compatible.

Proof. Taking $P = Q$ and $R = S$ in Theorem 2, the above result can be obtained.

4. Conclusion

This paper expounds a new notion in quasi-partial $b$-metric space which is generalized condition (B) that helped to demonstrate coincidence and common fixed point for two weakly compatible pairs of self-mappings. The incentive behind using quasi-partial $b$-metric space is the fact that the distance from point $x$ to point $y$ may be different to that from $y$ to $x$, and the self-distance of a point need not always be zero; also the distance between two points $x$ and $z$ is not equal to the sum of the two distances having a point $y$ in between $x$ and $z$. Furthermore, the results acquired are validated by exploratory examples.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. M. Fréchet, “Sur quelques points du calcul fonctionnel,” Rendiconti del Circolo Matematico di Palermo, vol. 22, no. 1, pp. 1–72, 1906.

[2] S. Banach, “Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,” Fundamenta Mathematicae, vol. 3, pp. 133–181, 1922.

[3] H.-P. A. Künzi, H. Pajoohesh, and M. P. Schellekens, “Partial quasi-metrics,” Theoretical Computer Science, vol. 365, no. 3, pp. 237–246, 2006.

[4] E. Karapınar, “Generalizations of Caristi Kirk’s theorem on partial metric spaces,” Fixed Point Theory and Applications, vol. 2011, p. 256, Article ID 4, 2011.

[5] E. Karapınar, M. Erhan, and O. Ali, “Fixed point theorems on quasi-partial metric spaces,” Mathematical and Computer Modelling, vol. 57, pp. 2442–2448, 2013.

[6] S. Czerwik, “Contractation mappings in b-metric spaces,” Acta Mathematica et Informatica Universitatis Ostraviensis, vol. 1, no. 5–11, 2011.

[7] A. Gupta and P. Gautam, “Quasi partial b-metric spaces and some related fixed point theorems,” Fixed Point Theory and Applications, vol. 18, 2015.

[8] A. Gupta and P. Gautam, “Topological structure of quasi-partial b-metric spaces,” International Journal of Pure and Applied Mathematics, vol. 17, no. 8–18, 2016.

[9] L. B. Ciric, “A generalization of Banach principle,” Proceedings of the American Mathematical Society, vol. 45, pp. 727–730, 1972.

[10] S. K. Chatterjea, “Fixed point theorems,” Comptes rendus de l’Académie bulgare des Sciences, vol. 25, pp. 727–730, 1972.

[11] R. Kannan, “Some results on fixed points,” Bulletin of the Calcutta Mathematical Society, vol. 60, pp. 71–76, 1968.

[12] S. Reich, “Kannan’s fixed point theorem,” Bollettino dell’Umane Matematica Italiana, vol. 4, no. 1–11, 1971.

[13] L. J. Ciric, B. Samet, H. Aydi, and C. Vetro, “Common fixed points of generalized contractions on partial-metric spaces and an application,” Applied Mathematics and Computation, vol. 218, pp. 2398–2406, 2011.

[14] T. Kamran, M. Samreen, and Q. U. L. Ain, “A generalization of b-metric space and some fixed point theorems,” Mathematics, vol. 5, pp. 1–7, 2017.

[15] T. Abdeljawad, K. Abodayeh, and N. Mlaiki, “On fixed point generalizations to partial b-metric spaces,” Journal of Computational and Applied Mathematics, vol. 19, pp. 883–891, 2015.

[16] T. Zamrescu, “Fixed point theorems in metric spaces,” Archiv der Mathematik, vol. 23, pp. 292–298, 1972.

[17] T. G. Bhaskar and V. Lakshmikantham, “Fixed point theorems in partially ordered metric spaces and applications,” Nonlinear Analysis, vol. 65, no. 1379–1393, pp. 4341–4349, 2006.

[18] T. L. Hicks, “Fixed point theorems for quasi-metric spaces,” Japanese Journal of Mathematics, vol. 33, no. 2, pp. 231–236, 1988.

[19] S. G. Matthews, “Partial metric topology,” Annals of the New York Academy of Sciences, vol. 728, pp. 183–197, 1994.

[20] P. Gautam, V. N. Misra, and K. Negi, “Common fixed point theorems for cyclic Reich-Rus-Ciric contraction mappings in quasi-partial b-metric space,” Annals of Fuzzy Mathematics and Informatics, vol. 20, no. 2, pp. 149–156, 2020.

[21] P. Gautam, V. N. Misra, V. Narayan Misra, R. Ali, and S. Verma, “Interpolative Chatterjea and cíclico Chatterjea contraction on quasi-partial $b$-$b$-metric space,” AIMS Mathematics, vol. 6, no. 2, pp. 1727–1742, 2021.

[22] V. N. Misra, L. M. Sánchez Ruiz, P. Gautam, and S. Verma, “Interpolative Reich–Rus–Ciric and Hardy–Rogers contraction on quasi-partial $b$-metric space and related fixed point results,” Mathematics, vol. 8, no. 1598, pp. 1–11, 2020.

[23] P. Gautam, L. M. Sánchez Ruiz, and S. Verma, “Fixed point of interpolative Reich–Rus–Ciric contraction mapping on rectangular quasi-partial $b$-metric space,” Symmetry, vol. 13, no. 32, pp. 1–8, 2021.

[24] P. Gautam and S. Verma, “Fixed point via implicit contraction mapping on quasi-partial $b$-metric space,” The Journal of Analysis, 2021.

[25] P. Gautam, S. Verma, M. De La Sen, and S. Sundriyal, “Fixed point Results for $\omega$-interpolative chatterjea type contraction in quasi-partial $b$-metric space,” International Journal of Analysis and Applications, vol. 19, no. 2, pp. 280–287, 2021.

[26] G. V. R. Babu, M. L. Sandhi, and M. V. R. Kameshwari, “A note on a fixed point theorem of Berinde on weak contractions,” Carpathian Journal of Mathematics, vol. 24, no. 1, pp. 8–12, 2008.

[27] M. Abbas, G. V. R. Babu, and G. N. Alemayehu, “On common fixed points of weakly compatible mappings satisfying generalized condition (B),” Filomat, vol. 25, no. 2, pp. 9–19, 2013.

[28] M. Abbas and D. Ilic, “Common fixed points of generalized almost nonexpansive mappings,” Filomat, vol. 24, no. 3, pp. 11–18, 2010.

[29] A. Tomar, S. Beloul, R. Sharma, and S. Upadhyay, “Common fixed point theorems via generalized condition (B) in quasi-
partial metric space and applications,” *Demonstratio Mathematica*, vol. 50, no. 1, pp. 278–298, 2017.

[30] G. Jungck and B. E. Rhoades, “Fixed point for set valued functions without continuity,” *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 3, pp. 227–238, 1998.