Optimal gradient continuity for
degenerate elliptic equations

by

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Abstract

We establish new, optimal gradient continuity estimates for solutions to a class of 2nd order partial
differential equations, \( L(X, \nabla u, D^2 u) = f \), whose diffusion properties (ellipticity) degenerate along the
a priori unknown singular set of an existing solution, \( S(u) := \{ X : \nabla u(X) = 0 \} \). The innovative feature
of our main result concerns its optimality – the sharp, encoded smoothness aftereffects of the operator.
Such a quantitative information usually plays a decisive role in the analysis of a number of analytic and
geometric problems. Our result is new even for the classicalequation \(|\nabla u| \cdot \Delta u = 1\). We further apply
these new estimates in the study of some well known problems in the theory of elliptic PDEs.

Keywords: Smoothness properties of solutions, optimal estimates, degenerate elliptic PDEs

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1 Introduction

Regularity theory for solutions to partial differential equations has been a central subject of research since
the foundation of the modern analysF of PDEs, back in the 18th century. Of particular interest are physical
and social phenomena that involve diffusion processes, whose mathematical models are governed by
second order elliptic PDEs.

Smoothness of weak solutions to 2nd order uniformly elliptic equations, both in divergence and in non-
divergence forms, is nowadays fairly well established. The cornerstone of the theory is a universal
modulus of continuity for solutions to the homogeneous equation: \( Lu = 0 \). This is the contents of DeGiorgi-Nash-
Moser theory for the divergence equations and Krylov-Safonov Harnack inequality for non-divergence
operators.

Despite of the profound importance of the supra-cited works, a large number of mathematical models
involve operators whose ellipticity degenerates along an a priori unknown region, that might depend on the
solution itself: the free boundary of the problem. This fact impels less efficient diffusion features for the
model near such a region and therefore the regularity theory for solutions to such equations become more
sophisticated from the mathematical view point.

The most typical case of elliptic degeneracy occurs along the singular set of an existing solution:

\[ S(u) := \{ X : \nabla u(X) = 0 \}. \]

In fact, a number of degenerate elliptic equations has its degree of degeneracy comparable to

\[ f(\nabla u)|D^2 u| \approx 1, \]  

(1.1)

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for some function \( f : \mathbb{R}^d \to \mathbb{R} \), with \( \text{Zero}(f) = \{0\} \). Thus, understanding the precise effect on the lack of smoothness impelled by the emblematic model \((1.1)\) sheds light on the underlying sharp regularity theory for a number of typical degenerate elliptic operators – see the heuristic comments in Section 5.

The main goal of this present work is to derive sharp interior regularity estimates for degenerate elliptic equations of the general form
\[
(1.2) \quad \mathcal{H}(X, \nabla u)F(X, D^2 u) = f(X), \quad B_1 \subset \mathbb{R}^d,
\]
where \( f \in L^\infty(B_1) \) and \( \mathcal{H} : B_1 \times \mathbb{R}^d \to \mathbb{R} \) degenerates as
\[
(1.3) \quad \lambda|\vec{p}|^\gamma \leq \mathcal{H}(X, \vec{p}) \leq \Lambda|\vec{p}|^\gamma,
\]
for some \( \gamma > 0 \). The 2nd order operator \( F : B_1 \times \text{Sym}(d) \to \mathbb{R} \) in equation \((1.2)\) is responsible for diffusion, i.e., \( F \) will be assumed to be a generic fully nonlinear uniformly elliptic operator: \( \lambda I_{d \times d} \leq \partial_{ij}F(X, M) \leq \Lambda\lambda I_{d \times d}. \) In Section 2 we give a more appropriate notion of ellipticity.

Regularity theory for viscosity solutions to fully nonlinear uniformly elliptic equations,
\[
F(D^2 u) = 0,
\]
has attracted the attention of the mathematical community for the last three decades or so. It is well established that solutions to the homogeneous equation is locally of class \( C^{1,\alpha_0} \) for a universal exponent \( \alpha_0 \), i.e., depending only on \( d, \lambda \) and \( \Lambda \), see for instance [2]. If no additional structure is imposed on \( F, C^{1,\alpha_0} \) is in fact optimal, see [18], [19], [20].

A quick inference on the structure of equation \((1.2)\) reveals that no universal regularity theory for such equation could go beyond \( C^{1,\alpha_0} \). In fact the degeneracy term \( \mathcal{H}(X, \nabla u) \) forces solutions to be less regular than solutions to the uniformly elliptic problem near its singular set. This particular feature indicates that obtaining sharp regularity estimates for solutions to \((1.2)\) should not follow from perturbation techniques. Indeed, it requires new ideas involving an interplay balance between the universal regularity theory for uniform elliptic equations and the degeneracy effect on the diffusion attributes of the operator coming from \((1.3)\).

In this present work we show that a viscosity solution, \( u \), to \((1.2)\) is pointwise differentiable and its gradient, \( \nabla u \), is locally of class \( C^{0,\min(\alpha_0, 1)} \), which is precisely the optimal regularity for degenerate equations of the type \((1.2)\). We further estimate the corresponding maximum regularity norm of \( u \) by a constant that depends only on universal parameters, \( \gamma, ||f||_\infty \) and \( ||u||_\infty \). Sharpness of our estimate can be verified by simple examples. We have postponed the precise statement of the main Theorem to Section 3. We highlight that the result proven in this manuscript is new even for the classical family of degenerate equations
\[
(1.4) \quad ||\nabla u||_\gamma \Delta u = 1, \quad \gamma > 0.
\]

The key, innovative feature of our main result lies precisely in the optimality of the gradient Hölder continuity exponent of a solution to the degenerate equation \((1.2)\), which in turn is an important piece of information in a number of qualitative analysis of PDEs, such as blow-up analysis, free boundary problems, geometric estimates, etc. It is quantitative bonus acquisition to the recent result in [11], where it is proven that viscosity solutions to \((1.2)\) are continuously differentiable. The logistic reasoning of the proof of our main result is inspired by recent works of the third author, [23], [24], [25], and it further uses the main crack from [11] to access a priori \( C^1 \) estimate for solutions to \((1.2)\).

The paper is organized as follows. In Section 2 we gather the most relevant notations and known results we shall use in the paper. In Section 3 we present the main Theorem proven in this work. In Section 4 we provide a few implications the sharp estimates from Section 3 have towards the solvability of some well known open problems in the elliptic regularity theory. The proof of Theorem 3.1 is delivered in the remaining Sections 5, 6, 7 and 8.
2 Notation and preliminaries

In this article we use standard notation from classical literature. The equations and problems studied in this paper are modeled in the $d$-dimensional Euclidean space, $\mathbb{R}^d$. The open ball of radius $r > 0$ centered at the point $X_0$ is denoted by $B_r(X_0)$. Usually ball of radius $r$, centered at the origin is written simply as $B_r$. For a function $u : B_1 \to \mathbb{R}$, we denote its gradient and its Hessian at a point $X \in B_1$ respectively by

$$\nabla u(X) := (\partial_i u)_{1 \leq i \leq d} \quad \text{and} \quad D^2 u(X) := (\partial_{ij} u)_{1 \leq i, j \leq d},$$

where $\partial_i u$ and $\partial_{ij} u$ denote the $j$-th directional derivative of $u$ and the $i$-th directional derivative of $\partial_i u$, respectively.

The space of all $d \times d$ symmetric matrices is denoted by $\text{Sym}(d)$. An operator $F : B_1 \times \text{Sym}(d) \to \mathbb{R}$ is said to be uniformly elliptic if there exist two positive constants $0 < \lambda \leq \Lambda$ such that, for any $M \in \text{Sym}(d)$ and $X \in B_1$,

$$\lambda \|P\| \leq F(X, M + P) - F(X, M) \leq \Lambda \|P\|, \quad \forall P \geq 0.$$

(2.1)

Any operator $F$ satisfying the ellipticity condition (2.1) will be referred hereafter in this paper as a $(\lambda, \Lambda)$-elliptic operator. Also, following classical terminology, any constant or entity that depends only on dimension and the ellipticity parameters $\lambda$ and $\Lambda$ will be called universal. For normalization purposes, we assume, with no loss of generality, throughout the text that $F(X, 0) = 0$, $\forall X \in B_1$.

For an operator $G : B_1 \times \mathbb{R}^d \times \text{Sym}(d) \to \mathbb{R}$, we say a function $u \in C^0(B_1)$ is a viscosity super-solution to $G(X, \nabla u, D^2 u) = 0$, if whenever we touch the graph of $u$ by below at a point $Y \in B_1$ by a smooth function $\phi$, there holds $G(Y, \nabla \phi(Y), D^2 \phi(Y)) \leq 0$. We say $u \in C^0(B_1)$ is a viscosity sub-solution to $G(X, \nabla u, D^2 u) = 0$, if whenever we touch the graph of $u$ by above at a point $Z \in B_1$ by a smooth function $\phi$, there holds $G(Y, \nabla \phi(Y), D^2 \phi(Y)) \geq 0$. We say $u$ is a viscosity solution if it is a viscosity super-solution and a viscosity sub-solution. The crucial observation on the above definition is that if $G$ is non-decreasing on $M$ with respect to the partial order of symmetric matrices, then the classical notion of solution, sub-solution and super-solution is equivalent to the corresponding viscosity terms, provided the function is of class $C^2$. The theory of viscosity solutions to non-linear 2nd order PDEs is nowadays fairly well established. We refer the readers to the classical article [6].

Let us discuss now a little bit further the existing regularity theory for uniformly elliptic equations. As mentioned earlier in the Introduction, it follows from the celebrated Krylov-Safonov Harnack inequality, see for instance [2], that any viscosity solution to the constant coefficient, homogeneous equation

$$F(D^2 h) = 0,$$

(2.2)

is locally of class $C^{1, a_0}$ for a universal exponent $0 < a_0 < 1$. Hereafter in this paper, $a_0 = a_0(d, \lambda, \Lambda)$ will always denote the optimal Hölder continuity exponent for solutions constant coefficients, homogeneous, $(\lambda, \Lambda)$-elliptic equation (2.2). If no extra structural condition is imposed on $F$, $C^{1, a_0}$ is indeed the optimal regularity possible, [18], [19], [20]. However, under convexity or concavity assumption on $F$, solutions are of class $C^2$. This is a celebrated result due to Evans [7] and Krylov [15][16], independently.

For varying coefficient equations, solutions are in general only $C^{1, a_0}$ and this is the optimal regularity available, unless we impose some continuity assumption on the coefficients, i.e., on the map $X \mapsto F(X, \cdot)$, Such condition is quite natural and it is present even in the linear theory: $Lu := a_{ij}(X)D_{ij} u$. Since we aim for a universal $C^{1, a}$ estimate for solutions to equation (1.2), hereafter we shall assume a uniform continuity assumption on the coefficients of $F$, appearing in (1.2), namely

$$\sup_{|M| \leq 1} \frac{|F(X, M) - F(Y, M)|}{|M|} \leq C \omega(|X - Y|),$$

(2.3)

where $C \geq 0$ is a positive constant and $\omega$ is a normalized modulus of continuity, i.e., $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, $\omega(0^+) = 0$ and $\omega(1) = 1$. Such condition could be relaxed: it suffices some sort of VMO
condition, see [1]. We have decided to present the results of this present article under [2, 3] for sake of simplicity. For notation convenience, we will call

\[
\|F\|_{\alpha} := \inf \left\{ C > 0 : \sup_{\|M\| \leq 1} \frac{|F(X, M) - F(Y, M)|}{\|M\|} \leq C \omega(|X - Y|), \forall X, Y \in B_1 \right\}.
\]

We close this Section by mentioning that under continuity condition on the coefficients, viscosity solutions to

\[ F(X, D^2 u) = f(X) \in L^\omega(B_1) \]

are locally of class \(C^{1, \beta}\), for any \(0 < \beta < \alpha_0\), where \(\alpha_0\) is the optimal Hölder exponent for solutions to constant coefficient, homogeneous equation \(F(D^2 h) = 0\), coming from Krylov-Safonov, Caffarelli universal regularity theory. See [1], [24].

### 3 Main results

In this Section, we shall present the main result we will prove in this present work. As mentioned earlier, the principal, ultimate goal of this article is to understand the sharp smoothness estimates for functions \(u\), satisfying

\[
|\nabla u|^{\delta} \cdot |F(D^2 u)| \lesssim 1, \quad \delta > 0,
\]

in viscosity sense, for some uniformly elliptic operator \(F\). Clearly, as commented in the previous Section, even in the non-degenerate case, \(\delta = 0\), the best regularity possible is \(C^{1+\alpha_0}\). The delicate point, though, is to obtain a universal estimate, fine enough as to sense and deem the singularity appearing in RHS of Equation (3.1), along the singular set \((\nabla u)^{-1}(0)\), as \(\delta > 0\) varies.

As to grasp some feelings on what one should expect, let us naïvely look at the ODE

\[ u''(t) = (u')^{-\delta}, \quad u(0) = u'(0) = 0, \]

which can be simply solved for \(t \in (0, \infty)\). The solution is \(u(t) = t^{1/\delta^{\frac{1}{\delta}}}.\) After some heuristics inference, it becomes reasonable to accept that \(C^{1+\alpha_0}\) is another upper barrier for any universal regularity estimate for Equation (3.1). Thus, if no further obscure complexity interferes on the elliptic regularity theory for fully nonlinear degenerate elliptic equation, the ideal optimal regularity estimate one should hope for functions satisfying Equation (3.1) should be \(C^{1, \min\{\alpha_0, \frac{1+\alpha_0}{\delta}\}}\).

After these technical free and didactical comments, we are able to state the main result we establish in this paper.

**Theorem 3.1.** Let \(u\) be a viscosity solutions to

\[
\mathcal{H}(X, \nabla u)F(X, D^2 u) = f(X) \quad \text{in } \quad B_1.
\]

Assume \(f \in L^\omega(B_1)\), \(\mathcal{H}\) satisfies (2.3) and \(F : B_1 \times \text{Sym}(d) \rightarrow \mathbb{R}\) is uniform elliptic with continuous coefficients, i.e., satisfying (2.3). Fixed an exponent

\[ \alpha \in (0, \alpha_0) \cap \left(0, \frac{1}{1+\gamma}\right), \]

there exists a constant \(C(d, \lambda, \Lambda, \gamma, \|F\|_{\alpha_0}, \|f\|_\infty, \alpha) > 0\), depending only on \(d, \lambda, \Lambda, \gamma, \|F\|_{\alpha_0}, \|f\|_\infty\) and \(\alpha\), such that

\[ \|u\|_{C^{1, \alpha}(B_{1/2})} \leq C(d, \lambda, \Lambda, \gamma, \|F\|_{\alpha_0}, \|f\|_\infty, \alpha) \cdot \|u\|_{L^\omega}. \]

An important consequence of Theorem 3.1 is the following:
Corollary 3.2. Let $u$ be a viscosity solution to
\begin{equation}
\mathcal{H}(X, \nabla u) F(D^2 u) = f(X) \quad \text{in} \quad B_1.
\end{equation}
Assume $f \in L^\infty(B_1)$, $\mathcal{H}$ satisfies (1.3), $F$ is uniformly elliptic and concave. Then $u$ is locally in $C^{1, \frac{1}{17}}$ and this regularity is optimal.

Corollary 3.2 follows from Theorem 3.1 since solutions to concave equations are locally of class $C^{1,1}$ by Evans-Krylov Theorem.

It is interesting to understand Theorem 3.1 as a model classification for degenerate elliptic equations, linking the magnitude of the degeneracy of the operator to the optimal regularity of solutions. A quantitative, intrinsic signature of the degeneracy properties of the equation. In fact, as mentioned earlier, many classical equations have their degree of degeneracy comparable to a model equation of the form $|\nabla u|^\gamma F(D^2 u) \lesssim 1$. We shall explore this perspective within the next Section.

4 Applications and further insights

The heuristics from the “degeneracy classification” mentioned in the previous paragraph has indeed a wide range of applicability. In this intermediary Section we comment on some consequences the optimal regularity estimates stated in Section 3 have in the elliptic regularity theory.

In the sequel we shall use Theorem 3.1 and Corollary 3.2 to solve particular cases of some well known open problems. The results provided in this section give hope that decisive progress can be attempted for the general cases in the near future.

4.1 Equations from the theory of superconductivity

We start off by commenting on some applications Theorem 3.1 has to the theory of superconductivity, where fully nonlinear equations with patches of zero gradient
\begin{equation}
F(X, D^2 u) = g(X, u) \chi_{\{|\nabla u| > 0\}}
\end{equation}
governs the mathematical models. Equation (4.1) represents the stationary equation for the mean field theory of superconducting vortices when the scalar stream function admits a functional dependence on the scalar magnetic potential, see [5]. Existence and regularity properties of Equation (4.1) were studied in [3] and in [4]. The novelty to study Equation (4.1) is that one tests the equation only for touching polynomials for which $|\nabla P(X_0)| \neq 0$. It is proven in [3], Corollary 7, that solutions are locally $C^{0,\alpha}$ for some $0 < \alpha < 1$. For concave operators, it is proven, see [3] Corollary 8, that solutions are in $W^{2,p}$. An application of Alt-Caffarelli-Friedman monotonicity formula, [3] Lemma 9, gives regularity $C^{1,1}$ for the particular problem
\begin{equation}
\Delta u = cu \chi_{\{|\nabla u| > 0\}}.
\end{equation}

Equation (4.1) can be obtained as a limiting problem, as $\delta \to 0$, for the family of singular equations
\begin{equation}
|\nabla u_\delta|^\delta F(X, D^2 u_\delta) = g(X, u_\delta), \quad B_1.
\end{equation}
Indeed, it follows from Theorem 3.1 that if $u_\delta$ is a normalized solution to (4.2), for $\delta$ small enough, i.e., for
\[ \delta < 1 - \alpha_0, \]
then we can estimate
\begin{equation}
\|u\|_{C^{1,\alpha_0}_{\text{loc}}} \leq C,
\end{equation}

\begin{equation}
\|
\end{equation}
for a constant $C > 1$, that does not depend on $\delta$. In particular, estimate (4.3) gives local compactness for the family of solutions $\{u_\delta\}_{\delta > 0}$ to $(E_\delta)$. Let $u_0$ be a limiting point of such a sequence, i.e.

$$u_0 = \lim_{j \to 0} u_{\delta_j},$$

for $\delta_j = o(1)$. From (4.3), we have,

$$\nabla u_{\delta_j} \to \nabla u_0$$

locally uniformly,

$$u_0 \in C^{1,\alpha_0}_{\text{loc}}(B_1).$$

Now, fixed a regular point $Z \in B_1$ of $u_0$, i.e.,

$$|\nabla u_0(Z)| > 0.$$

Gradient convergence (4.4) and estimate (4.5) yield the existence of a small $\eta > 0$, such that

$$\inf_{B_\eta(Z)} |\nabla u_\delta| \geq \frac{1}{10}|\nabla u_0(Z)| =: c_0,$$

for all $\delta \ll 1$. Thus,

$$g(X, u_\delta) : |\nabla u_\delta|^{-\delta} \to g(X, u_0),$$

uniformly in $B_\eta(Z)$.

We sum up the above discussion as the following Theorem:

**Theorem 4.1.** Let $u_\delta \in C^0(B_1)$ be a viscosity solution to $(E_\delta)$, with $|u_\delta| \leq 1$, $\delta \ll 1$, where $g$ is continuous w.r.t. $u$ and measurable bounded w.r.t. $X$. Assume the operator $F$ is under the hypotheses of Theorem 3.1. Then, fixed a number $\alpha < \alpha_0(d, \lambda, \Lambda)$, for $\delta$ small enough, we have

$$\|u_\delta\|_{C^{1,\alpha}(B_{4/5})} \leq C(d, \lambda, \Lambda, \alpha).$$

In particular,

$$u_\delta \to u_0 \in C^{1,\alpha}(B_{1/2}),$$

and $u_0$ is a viscosity solution to (4.1).

The advantage of Theorem 4.1 in comparison to the regularity theory developed in [3] is that it provides the asymptotically sharp $C^{1,\alpha}$ estimate in the general case of fully nonlinear operators, not necessarily concave.

**4.2 Visiting the theory of $\infty$-laplacian**

Let us now visit the theory of the $\infty$-laplacian operator, i.e.,

$$\Delta_{\infty}v := \sum_{i,j} v_{ij}v_{ij}$$

which is related to the problem of best Lipschitz extension to a given boundary datum - a nonlinear and highly degenerate elliptic operator. The theory of infinity-harmonic functions, i.e., solutions to the homogeneous PDE

$$\Delta_{\infty}h = 0,$$

has received a great deal of attention. One of the main open problems in the modern theory of PDEs is whether infinity-harmonic functions are of class $C^1$. This conjecture has been answered positively by O. Savin [22] in the plane. Evans and Savin, [8] sharpened the result to $C^{1,\alpha}$ for some small $\alpha > 0$, but still in only dimension two. Quite recently, Evans and Smart proved that infinity-harmonic functions are
everywhere differentiable regardless the dimension. Nevertheless, no continuity feature of $\nabla u$ can be inferred by their ingenuous reasoning. The famous example of the infinity-harmonic function

$$a(x, y) := x^{\frac{2}{3}} - y^{\frac{2}{3}}$$

due to Aronsson from the late 60’s sets the ideal optimal regularity theory for such a problem. That is, no universal regularity theory for infinity harmonic functions can go beyond $C^{1,\frac{1}{d}}$. Up to our knowledge, there has been no prior meaningful mathematical indication that infinity-harmonic functions should or should not have a universal $C^{1,\frac{1}{d}}$ regularity theory, other than speculation based on Aronsson’s example. Another way, though, to surmise the $C^{1,\frac{1}{d}}$ conjecture for infinity harmonic function would be by exploring the scaling properties of the equation. For instance, it one writes the infinity-laplacian as

$$\Delta u = (\nabla v)^t \cdot D^2 v \cdot \nabla v,$$

it becomes tempting to compare its degeneracy feature with

$$|\nabla u|^2 \cdot |\Delta u| \lesssim 1,$$

that has the same scaling properties as $\Delta u$ and whose solutions are locally $C^{1,\frac{1}{d}}$ regular from Corollary 3.2. Although, it is not in general true that infinity-harmonic functions satisfy (4.8), this observation sets an interesting heuristic guide.

Notice that Aronsson’s example - as many popular examples in the theory of PDEs - is a function of separable variables. In the sequel, as an application of Corollary 3.2, we show that any infinity-harmonic function with separable variables is locally of class $C^{1,\frac{1}{d}}$.

**Proposition 4.2.** Let $u: B_1 \subset \mathbb{R}^d \to \mathbb{R}$ be infinity harmonic. Assume $u$ is a function of separable variables, i.e.,

$$u(X) = \sigma_1(x_1) + \sigma_2(x_2) + \cdots \sigma_d(x_d),$$

for $\sigma_i \in C^0(B_1)$. Then $u \in C^{1,\frac{1}{d}}(B_{1/2})$.

**Proof.** Formal direct computation gives

$$0 = \Delta u = |\sigma_1(x_1)|^2 \sigma''(x_1) + |\sigma_2(x_2)|^2 \sigma''(x_2) + \cdots + |\sigma_d(x_d)|^2 \sigma''(x_d).$$

It is a manner of routine to justify the above computation using the viscosity solution machinery. We notice, however, that the $i$th term in (4.9) depends only upon the variable $x_i$. Thus, since they sum up to zero, each of them must be constant, i.e.,

$$|\sigma_i(x_i)|^2 \sigma''(x_i) = \tau_i, \quad \sum_{i=1}^d \tau_i = 0.$$

$C^{1,\frac{1}{d}}$-regularity of each $\sigma_i$ follows from Corollary 3.2 and the proof of Proposition 4.2 is concluded.  

In a number of geometrical problems, it is often that solutions behave asymptotically radial near singular points. It is therefore interesting to analyze the regularity theory for solutions that are smooth up to a possible radial singularity. More precisely, we say a function $u$ is smooth up to a possible radial singularity at a point $X_0$ if we can write, near $X_0$,

$$u(X) = \varphi(X) + \psi(|X - X_0|),$$

with $\varphi \in C^2$ and $\varphi(X) = O(|X - X_0|^2)$.

In the sequel we shall prove that functions smooth up to a possible radial singularity whose infinity-laplacian is bounded in the viscosity sense is of class $C^{1,\frac{1}{d}}$. This regularity is optimal as

$$\Delta u|X|^\frac{4}{d} = cte.$$
Theorem 4.3. Let $u \in C^0(B_1)$ satisfy
\[ \Delta u = f(X) \in L^\infty(B_1) \]
in the viscosity sense. Assume $u$ is smooth up to a possible radial singularity. Then $u \in C^{1,\frac{1}{3}}_{loc}(B_1)$.

Proof. With no loss of generality, we can assume $X_0 = 0$. If $u = \phi(X) + \psi(|X|)$ is smooth up to a radial singularity near the origin, then formally a direct computation yields
\[
\nabla u(X) = \nabla \phi + \psi' \frac{X}{|X|},
\]
\[
D^2 u(X) = D^2 \phi + \frac{1}{|X|^2} \psi'' X \otimes X + \psi \left[ \frac{1}{|X|} \text{Id} - \frac{1}{|X|^3} X \otimes X \right].
\]
Owing to the estimates
\[
|X|^{-2} |\phi| + |X|^{-1} |\nabla \phi| + |D^2 \phi| \leq C_1,
\]
\[
|\psi| + |\nabla \psi| \leq C_2,
\]
\[
|\Delta u| \leq C_3,
\]
we end up with
\[
(O(r) + |\psi'|^2) \cdot |\psi''| \leq C_4,
\]
which ultimately gives the desired regularity for $\psi$. Again it is standard to verify the above computation using the language of viscosity solutions.

For functions with bounded infinity-laplacian, E. Lindgren, following ideas from [9], has recently established Lipschitz estimate and everywhere differentiability.

4.3 Further degenerate elliptic equations

Another interesting example to visit is the $p$-laplacian operator, $p \geq 2$:

\[
\Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right).
\]

It appears for instance as the Euler-Lagrangian equation associated to the $p$-energy integral
\[
\int (Du)^p dX \to \min.
\]

Equations involving the the $p$-laplacian operator has received a great deal of attention for the past fifty years or so. In particular, the regularity theory for $p$-harmonic functions has been an intense subject of investigation, since the late 60’s, when Ural'tseva in [26] proved that weak solutions to the homogeneous $p$-laplacian equation

\[
\Delta_p h = 0,
\]

is locally of class $C^{1,\alpha(d,p)}$, for some $\alpha(d,p) > 0$. The sharp regularity for $p$-harmonic functions in the plane was obtained by Iwaniec and Manfredi, [13]. The precise optimal Hölder continuity exponent of the gradient of $p$-harmonic functions in higher dimensions, $d \geq 3$, has been a major open problem since then.

The $p$-laplacian operator can be written in non-divergence form, simply by passing formally the derivatives through:

\[
\Delta_p u = |\nabla|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} \Delta u.
\]
The notions of weak solutions, using its divergence structure in (4.11), and the non-divergence form in (4.12) are equivalent. 

Within the context of functions with bounded \( p \)-laplacian, the conjecture is that the optimal regularity should be \( C^{q'} \), where

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

Our next result gives a partial answer to this conjecture.

**Theorem 4.4.** Let \( p \geq 2 \) and \( u \) satisfy

\[
|\Delta_p u| \leq C,
\]

Assume \( u \) is smooth up to a possible radial singularity. Then \( u \in C^{q'}_{loc}(B_1) \).

**Proof.** Assume \( u = \varphi(X) + \psi(r) \), with \( \varphi = O(r^2), r = |X| \), has bounded \( p \)-laplacian in the viscosity sense. Direct computation implies

\[
\left( O(r) + |\psi'| \right)^{p-2} |\psi'| + \left( O(r) + |\psi'| \right)^{p-4} |\psi'|^2 |\psi''| \leq C + \tilde{C},
\]

holds in the viscosity sense, where \( \tilde{C} \) depends only on \( \varphi \), the non-singular part of \( u \). Thus, Corollary 3.2 gives

\[
u \in C^{q'}_{loc} \cong C^{q'}_{loc},
\]

and the Theorem is proven. \( \square \)

Estimates of the form \( |\nabla u|^{p-2} |D^2 u| < C \) are not rare in a number of geometric problems involving the \( p \)-laplacian operator, see for instance [71]. Let us also mention that estimates of the form \( (\varepsilon + |\nabla v|^2)^{\frac{p-2}{2}} |D^2 v| < C \) are usually obtained for bounded weak solutions to divergent form equations, \( D_i \left( A'(Dv) \right) = 0 \), see for instance [10], Chapter 8.

Let us finish this Section, by revisiting the derivation of the infinity-laplacian operator as the limit of \( p \)-laplace, as \( p \to \infty \). Let \( h \in C^0(B_1) \) be an infinity-harmonic function. For each \( p \gg 1 \), let \( h_p \) be the solution to the boundary value problem

\[
\begin{align*}
\Delta_p h_p &= 0, \text{ in } B_{3/4} \\
h_p &= h, \text{ on } B_{3/4}.
\end{align*}
\]

It is known that \( h_p \) form a sequence of equicontinuous functions and \( h_p \to h \) locally uniformly to \( h \). In particular

\[
\Delta_\infty h_p = o(1), \quad \text{as } p \to \infty.
\]

Hereafter, let us call \( h_p \) the \( p \)-harmonic approximation of the infinity-harmonic function \( h \) in \( B_{3/4} \).

**Proposition 4.5.** Let \( h \in C^0(B_1) \) be an infinity-harmonic function and \( h_p \) its \( p \)-harmonic approximation. Assume \( |\Delta_\infty h_p| = O(p^{-1}) \) as \( p \to \infty \). Then \( h \in C^{1,1/4}(B_{1/2}) \).

**Proof.** Since \( h_p \) is \( p \)-harmonic, it satisfies

\[
|\nabla h_p|^2 \Delta h_p = (2 - p) \Delta_\infty h_p.
\]

By the maximum principle,

\[
\|h_p\|_{L^\infty(B_{3/4})} \leq \|h\|_{L^\infty(B_{3/4})},
\]

From the approximation hypothesis and Corollary 3.2, we deduce

\[
\|h_p\|_{C^{1,1/4}(B_{1/2})} \leq C,
\]

for a constant \( C \) that is independent of \( p \). The proof of Proposition follows by standard reasoning. \( \square \)
Another interesting Proposition regards $p$-harmonic functions with bounded infinity-laplacian.

**Proposition 4.6.** Let $u$ be a $p$-harmonic function in $B_1 \subset \mathbb{R}^d$. Assume $\Delta_{\infty} u \in L^\infty(B_1)$. Then $u \in C^{1,\frac{1}{2}}_\text{loc}(B_{1/2})$.

**Proof.** The proof follows by similar reasoning as in the proof of Proposition 4.5. We omit the details. \qed

We leave as an open problem whether Proposition 4.6 holds true without the extra assumption on the boundedness of the infinity-laplacian. We further conjecture that if $\alpha(d,p)$ is the optimal (universal) Hölder continuity exponent for $p$-harmonic functions, then

$$\alpha(d,p) > \frac{1}{3} + o(1), \quad \text{as } p \to \infty.$$ 

## 5 Universal compactness

From this Section on we start delivering the proof for the main sharp regularity estimate announced in Section 3, namely, Theorem 3.1. In this first step, we obtain a universal compactness device to access the optimal regularity theory for solutions to Equation (1.2). The proof we shall present here uses the main technical tool obtained in the recent work of Imbert and Silvestre, [11].

**Lemma 5.1.** Let $\vec{q} \in \mathbb{R}^d$ be an arbitrary vector and $u \in C(B_1)$, a viscosity solution to

$$(E_{\vec{q}}) \quad |\vec{q} + \nabla u|^7 F(X,D^2u) = f(X),$$

satisfying $\|u\|_{L^\infty(B_1)} \leq 1$. Given $\delta > 0$, there exists $\varepsilon > 0$, that depends only upon $d, \lambda, \Lambda,$ and $\gamma,$ such that if

$$\|M\|^{-1} \cdot \|F(X,M) - F(0,M)\|_{L^\infty(B_1)} \leq \varepsilon,$$

then we can find a function $h$, solution to a constant coefficient, homogeneous, $(\lambda, \Lambda)$-uniform elliptic equation

$$\vec{q}(D^2 h) = 0, \quad B_{1/2}$$

such that

$$\|u - h\|_{L^\infty(B_{1/2})} \leq \delta.$$ 

**Proof.** Let us suppose, for the sake of contradiction, that the thesis of the Lemma fails. That means that we could find a number $\delta_0 > 0$ and sequences, $F_j(X,M), f_j, \vec{q}_j$ and $u_j$, satisfying

$$F_j(X,M) \text{ is } (\lambda, \Lambda)\text{-elliptic,}$$

$$\|M\|^{-1} \cdot \|F(X,M) - F(0,M)\|_{L^\infty(B_1)} = o(1),$$

$$\|f_j\|_{L^\infty(B_1)} = o(1),$$

however,

$$\sup_{B_{1/2}} |u_j - h| \geq \delta_0,$$

for any $h$ satisfying a constant coefficient, homogeneous, $(\lambda, \Lambda)$-uniform elliptic equation (5.2).

Initially, arguing as in [11], the sequence $u_j$ is pre-compact in $C^0(B_{1/2})$-topology. In fact, as in [11], Lemma 4, there is a universally large constant $A_0 > 0$, such that, if for a subsequence $\{\vec{q}_k\}_{k \in \mathbb{N}}$, there holds,

$$|\vec{q}_k| \geq A_0, \quad \forall k \in \mathbb{N},$$
then, the corresponding sequence of solutions, \( \{ u_{j} \}_{j \in \mathbb{N}} \), is bounded in \( C^{0,1}(B_{2/3}) \). If 
\[
|\tilde{q}_{j}| < A_{0}, \quad \forall j \geq j_{0},
\]
then, by Harnack inequality, see [12], \( \{ u_{j} \}_{j \geq j_{0}} \) is bounded in \( C^{0,\beta}(B_{2/3}) \) for some universal \( 0 < \beta < 1 \).

From the compactness above mentioned, up to a subsequence, \( u_{j} \to u_{\infty} \) locally uniformly in \( B_{2/3} \). Our ultimate goal is to prove that the limiting function \( u_{\infty} \) is a solution to a constant coefficient, homogeneous, \((\lambda, \Lambda)\)-uniform elliptic equation (5.2). For that we also divide our analysis in two cases. 

If |\( \tilde{q}_{j} \)| bounded, we can extract a subsequence of \( \{ \tilde{q}_{j} \} \), that converges to some \( \tilde{q}_{\infty} \in \mathbb{R}^{d} \). Also, by uniform ellipticity and (5.5), up to a subsequence \( F_{j}(X, \cdot) \to \tilde{F}(\cdot) \), and 
\[
|\tilde{q}_{\infty} + \nabla u_{\infty}|^{2} \tilde{F}(D^{2}u_{\infty}) = 0,
\]
Arguing as in [11], Section 6, we conclude that \( u_{\infty} \) is a solution to a constant coefficient, homogeneous elliptic equation, which contradicts (5.8).

If |\( \tilde{q}_{j} \)| is unbounded, then taking a subsequence, if necessary, |\( \tilde{q}_{j} \)| → ∞. In this case, define \( \tilde{e}_{j} = \tilde{q}_{j} / |\tilde{q}_{j}| \) and then \( u_{j} \) satisfies 
\[
|\tilde{e}_{j} + \nabla u_{j}|^{|\gamma|} F_{j}(X, D^{2}u_{j}) = f_{j}(X) / |\tilde{q}_{j}|^{|\gamma|}.
\]
Letting \( j \to \infty \) and taking another subsequence, if necessary, we also end up with a limiting function \( u_{\infty} \), satisfying \( \tilde{F}_{\infty}(D^{2}u_{\infty}) = 0 \) for some \((\lambda, \Lambda)\)-uniform elliptic operator, \( \tilde{F}_{\infty} \). As before, this gives a contradiction to (5.8). The Lemma is proven. \( \square \)

6 Universal flatness improvement

In this Section, we deliver the core sharp oscillation decay that will ultimately imply the optimal \( C^{1, \alpha} \) regularity estimate for solutions to Equation (1.2). The first task is a step-one discrete version of the aimed optimal regularity estimate. This is the contents of next Lemma.

**Lemma 6.1.** Let \( \tilde{q} \in \mathbb{R}^{d} \) be an arbitrary vector and \( u \in C(B_{1}) \) a normalized, i.e., |\( u | \leq 1 \), viscosity solution to

\[
(E_{\tilde{q}}) \quad |\tilde{q} + \nabla u|^{\gamma} F(X, D^{2}u) = f(X).
\]

Given \( \alpha \in (0, \alpha_{0}) \cap (0, \frac{1}{\gamma + 1}] \), there exist constants \( 0 < \rho_{0} < 1/2 \) and \( \varepsilon_{0} > 0 \), depending only upon \( d, \lambda, \Lambda, \gamma \) and \( \alpha \), such that if

\[
(6.1) \quad \|M\|^{-1} \cdot \|F(X, M) - F(0, 0, M)\|_{L^{\infty}(B_{1})} + \|f\|_{L^{\infty}(B_{1})} \leq \varepsilon_{0},
\]
then there exists an affine function \( \ell(X) = a + \bar{b} \cdot X \), such that
\[
\sup_{B_{\rho_{0}}} |u(X) - \ell(X)| \leq \rho_{0}^{1+\alpha}.
\]
Furthermore,
\[
|a| + |\bar{b}| \leq C(d, \lambda, \Lambda),
\]
for a universal constant \( C(d, \lambda, \Lambda) \) that depends only upon dimension and ellipticity constants.

**Proof.** For a \( \delta > 0 \) to be chosen a posteriori, let \( b \) be a solution to a constant coefficient, homogeneous, \((\lambda, \Lambda)\)-uniform elliptic equation that is \( \delta \)-close to \( u \) in \( L^{\infty}(B_{1/2}) \). The existence of such a function is the thesis of Lemma (5.1) provided \( \varepsilon_{0} \) is chosen small enough, depending only on \( \delta \) and universal parameters. Since our choice for \( \delta \) - later in the proof - will depend only upon universal parameters, we will conclude that the choice of \( \varepsilon_{0} \) is too universal.
From normalization of \( u \), it follows that \( \|h\|_{L^\infty(B_{1/2})} \leq 2 \); therefore, from the regularity theory available for \( h \), see for instance [2], Chapters 4 and 5, we can estimate

\begin{align}
(6.2) \quad \sup_{B_r} |h(X) - (\nabla h(0) \cdot X + h(0))| & \leq C(d, \lambda, \Lambda) \cdot r^{1+\alpha_0} \quad \forall r > 0, \\
(6.3) \quad |\nabla h(0)| + |h(0)| & \leq C(d, \lambda, \Lambda),
\end{align}

for a universal constant \( 0 < C(d, \lambda, \Lambda) \). Let us label

\begin{equation}
\ell(X) = \nabla h(0) \cdot X + h(0).
\end{equation}

It readily follows from triangular inequality that

\begin{equation}
\sup_{B_{\rho_0}} |u(X) - \ell(X)| \leq \delta + C(d, \lambda, \Lambda) \cdot \rho_0^{1+\alpha_0}.
\end{equation}

Now, fixed an exponent \( \alpha < \alpha_0 \), we select \( \rho_0 \) and \( \delta \) as

\begin{align}
(6.6) \quad \rho_0 & := \rho_0^{\alpha_0 - \alpha} \sqrt{\frac{1}{2C(d, \lambda, \Lambda)}}, \\
(6.7) \quad \delta & := \frac{1}{2} \left( \frac{1}{2C(d, \lambda, \Lambda)} \right)^{\frac{1+\alpha}{\alpha_0 - \alpha}},
\end{align}

where \( 0 < C(d, \lambda, \Lambda) \) is the universal constant appearing in (6.2). We highlight that the above choices depend only upon \( d, \lambda, \Lambda \) and the fixed exponent \( 0 < \alpha < \alpha_0 \). Finally, combining (6.2), (6.5), (6.6) and (6.7), we obtain

\begin{equation}
\sup_{B_{\rho_0}} |u(X) - \ell(X)| \leq \frac{1}{2} \left( \frac{1}{2C(d, \lambda, \Lambda)} \right)^{\frac{1+\alpha}{\alpha_0 - \alpha}} + C(d, \lambda, \Lambda) \cdot \rho_0^{1+\alpha} \cdot \rho_0^{\alpha_0 - \alpha} \\
= \frac{1}{2} \rho_0^{1+\alpha} + \frac{1}{2} \rho_0^{1+\alpha} \\
= \rho_0^{1+\alpha},
\end{equation}

and the Lemma is proven. \( \square \)

In the sequel, we shall iterate Lemma 6.1 in appropriate dyadic balls as to obtain the precise sharp oscillation decay of the difference between \( u \) and affine functions \( \ell_k \).

**Lemma 6.2.** Under the conditions of the previous lemma, there exists a sequence of affine functions \( \ell_k(X) := a_k + \tilde{b}_k \cdot X \) satisfying

\begin{equation}
|a_{k+1} - a_k| + \rho_0^{k} |\tilde{b}_{k+1} - \tilde{b}_k| \leq C_0 \rho_0^{(1+\alpha)k},
\end{equation}

such that

\begin{equation}
\sup_{B_{\rho_0}} |u(X) - \ell_k(X)| \leq \rho_0^{k(1+\alpha)},
\end{equation}

where \( \alpha \) is a fixed exponent within the range

\begin{equation}
\alpha \in (0, \alpha_0) \cap \left( 0, \frac{1}{1 + \gamma} \right)
\end{equation}

and \( C_0 \) is a universal constant that depends only on dimension and ellipticity.
Proof. We argue by finite induction. The case $k = 1$ is precisely the statement of Lemma 6.1. Suppose we have verified (6.9) for $j = 1, 2, \ldots, k$. Define the rescaled function

$$v(X) := \frac{(u - \ell_k)(\rho_0^k X)}{\rho_0^{k(1+\alpha)}}.$$ 

It readily follows from the induction assumption that $|v| \leq 1$. Furthermore, $v$ satisfies

$$\left| \rho_0^{-k\alpha} b_k + \nabla v \right| \gamma F_k(X, D^2 v) = f_k(X),$$

where

$$f_k(X) = \rho_0^{k(1-\alpha(1+\gamma))} f(\rho_0^k X)$$

and

$$F_k(X, M) := \rho_0^{k(1-\alpha)} f(\rho_0^k X, \frac{1}{\rho_0^{k(1-\alpha)}} M).$$

It is standard to verify that the operator $F_k$ is $(\lambda, \Lambda)$-elliptic. Also, the $\omega$-norm of the corresponding coefficient oscillation of $F_k$, as defined on (2.3), hereafter called $\beta_k$, does not increase. Also, one easily estimate

$$\| f_k \|_{L^\omega(B_1)} \leq \rho_0^{k(1-\alpha(1+\gamma))} \| f \|_{L^\omega(B_{\rho_0})}.$$ 

Due to the sharpness of the exponent selection made in (6.10), namely $\alpha \leq \frac{1}{1+\gamma}$, we conclude $(F_k, f_k)$ satisfies the smallness assumption (6.1), from Lemma 6.1.

We have shown that $v$ is under the hypotheses of Lemma 6.1 which ensures the existence in the affine function $\tilde{\ell}(X) := a + \tilde{b} \cdot X$ with $|a| + |\tilde{b}| \leq C(d, \lambda, \Lambda)$, such that

$$\sup_{B_{\rho_0}} |v(X) - \tilde{\ell}(X)| \leq \rho_0^{1+\alpha}.$$ 

In the sequel, we define the $(k + 1)$th approximating affine function, $\ell_{k+1}(X) := a_{k+1} + \tilde{b}_{k+1} \cdot X$, where the coefficients are given by

$$a_{k+1} := a_k + \rho_0^{(1+\alpha)k} a \quad \text{and} \quad \tilde{b}_{k+1} := \tilde{b}_k + \rho_0^{\alpha k} \tilde{b}.$$ 

Rescaling estimate (6.12) back, we obtain

$$\sup_{B_{\rho_0^{k+1}}} |u(X) - \ell_{k+1}(X)| \leq \rho_0^{(k+1)(1+\alpha)}$$

and the proof of Lemma 6.2 is complete. \qed

7 Smallness regime

In this Section we comment on the scaling features of the equation that allow us to reduce the proof of Theorem 3.1 to the hypotheses of Lemma 6.1 and Lemma 6.2.

Let $v \in C(B_1)$ be a viscosity solution to

$$\mathcal{H}(X, \nabla v) F(X, D^2 v) = f(X),$$

where $\mathcal{H}$ satisfies (1.3) and $F$ is a $(\lambda, \Lambda)$-elliptic operator with continuous coefficients, i.e., satisfying (2.3). Fix a point $Y_0 \in B_{1/2}$ define $u : B_1 \to \mathbb{R}$ as

$$u(X) := \frac{v(\eta X + Y_0)}{\tau},$$

where $\tau$ is a suitable positive constant.
for parameters \( \eta \) and \( \tau \) to be determined. We readily check that \( u \) solves
\[
\mathcal{H}_{\eta, \tau}(X, \nabla u) F_{\eta, \tau}(X, D^2 u) = f_{\eta, \tau}(X),
\]
where
\begin{align}
F_{\eta, \tau}(X, M) &:= \frac{\tau}{\eta^2} F \left( \eta X + Y_0, \frac{\eta^2}{\tau} M \right) \\
\mathcal{H}_{\eta, \tau}(X, \bar{p}) &:= \left( \frac{\tau}{\eta} \right)^{\gamma} H \left( \eta X + Y_0, \frac{\eta}{\tau} \bar{p} \right) \\
f_{\eta, \tau}(X) &:= \frac{\eta^{\gamma+2}}{\tau^{\gamma+1}} f(\eta X + Y_0).
\end{align}

Easily one verifies that \( F_{\eta, \tau} \) is uniformly elliptic with the same ellipticity constants as the original operator \( F \), i.e., it is another \((\lambda, \Lambda)\)-elliptic operator. Also \( \mathcal{H}_{\eta, \tau} \) satisfies the degeneracy condition (1.3), with the same constants. Let us choose
\[
\tau := \max \left\{ 1, \| v \|_{L^\infty(B_1)} \right\},
\]
thus, \( |u| \leq 1 \) in \( B_1 \). Now, for the universal \( \varepsilon_0 \) appearing in the statement of Lemma 6.1 choose
\[
\eta := \min \left\{ 1, \lambda \left( \varepsilon_0 \| f \|_{L^\infty} \right)^{\frac{1}{\gamma+2}}, \omega^{-1} \left( \frac{\varepsilon_0}{C} \right) \right\}.
\]

With these choices, \( u \) is under the assumptions of Lemma 6.1.

The above reasoning certifies that in order to show Theorem 3.1, it is enough to work under the smallness regime requested in the statement of Lemma 6.1. Once established the desired optimal regularity estimate the normalized function \( u \), the corresponding estimate for \( v \) follows readily.

# 8 Sharp local regularity

In this Section we conclude the proof of Theorem 3.1. From the conclusions delivered in Section 7, it suffices to show the aimed \( C^{1, \alpha} \) estimate at the origin for a solution \( u \) under the hypotheses of Lemma 6.1 and Lemma 6.2. For a fixed exponent \( \alpha \) satisfying the sharp condition (6.10), we will establish the existence of an affine function
\[
\ell_*(X) := a_* + \tilde{b}_* \cdot X,
\]
such that
\[
|\tilde{b}_*| + |a_*| \leq C,
\]
and
\[
\sup_{B_r} |u(X) - \ell_*(X)| \leq Cr^{1+\alpha}, \quad \forall r \ll 1,
\]
for a constant \( C \) that depends only on \( d, \lambda, \Lambda, \gamma \) and \( \alpha \).

Initially, we notice that it follows from (6.6) that the coefficients of the sequence of affine functions \( \ell_k \) generated in Lemma 6.2, namely \( \tilde{b}_k \) and \( a_k \), are Cauchy sequences in \( \mathbb{R}^d \) and in \( \mathbb{R} \), respectively. Let \( \tilde{b}_* \) and \( a_* \) be the limiting coefficients, i.e.,
\begin{align}
\lim_{k \to \infty} \tilde{b}_k &=: \tilde{b}_* \in \mathbb{R}^d \\
\lim_{k \to \infty} a_k &=: a_* \in \mathbb{R}.
\end{align}

It also follows from the estimate obtained in (6.3) that
\begin{align}
|a_* - a_k| &\leq \frac{C_0}{1 - \rho_0^{1+\alpha}} \\
|\tilde{b}_* - \tilde{b}_k| &\leq \frac{C_0}{1 - \rho_0^{\alpha}}.
\end{align}
Now, fixed a $0 < r < \rho_0$, we choose $k \in \mathbb{N}$ such that 
\begin{equation*}
\rho_0^{k+1} < r \leq \rho_0^k.
\end{equation*}

We estimate 
\begin{equation*}
\sup_{B_r} |u(X) - \ell(X)| \leq \sup_{B_{\rho_0^k}} |u(X) - \ell(X)| + \sup_{B_{\rho_0^{k+1}}} |\ell(X) - \ell(X)| \\
\leq \rho_0^{k(1+\alpha)} + \frac{C_0}{1 - \rho_0^k} \rho_0^{k(1+\alpha)} \\
\leq \frac{1}{\rho_0^{\alpha}} \left[ 1 + \frac{C_0}{1 - \rho_0^k} \right] \rho^{1+\alpha},
\end{equation*}
and the proof of Theorem 3.1 is finally complete. \qed

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