STATIC SKT METRICS ON LIE GROUPS

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Abstract. An SKT metric is a Hermitian metric on a complex manifold whose fundamental 2-form \( \omega \) satisfies \( \partial \bar{\partial} \omega = 0 \). Streets and Tian introduced in [27] a Ricci-type flow that preserves the SKT condition. This flow uses the Ricci form associated to the Bismut connection, the unique Hermitian connection with totally skew-symmetric torsion, instead of the Levi-Civita connection. A SKT metric is static if the (1,1)-part of the Ricci form of the Bismut connection satisfies \( (\rho^B)^{(1,1)} = \lambda \omega \) for some real constant \( \lambda \). We study invariant static metrics on simply connected Lie groups, providing in particular a classification in dimension 4 and constructing new examples, both compact and non-compact, of static metrics in any dimension.

Introduction

Let \((M^{2n}, J, g)\) be a Hermitian manifold of real dimension \(2n\). We say that \(g\) is Strong KT (for short SKT) or pluriclosed if \( \partial \bar{\partial} \omega = 0 \). This condition is strictly related to the Bismut connection [6, 16], which is the unique Hermitian connection whose torsion tensor is totally skew-symmetric. SKT metrics were introduced in the context of type II string theory and 2-dimensional supersymmetric \(\sigma\)-models [18, 29], and they have also relations with generalized Kähler geometry [19, 14]. Moreover, Gauduchon [15] proved that for compact complex surfaces, one can find an SKT metric in the conformal class of any given Hermitian metric.

In [27, 28] Streets and Tian introduced a parabolic flow of SKT metrics defined by

\[
\frac{\partial \omega(t)}{\partial t} = - (\rho^B)^{(1,1)},
\]

where \(\rho^B\) is the Ricci form of the Bismut connection. This led to a definition of Einstein-like SKT metrics, called static. More precisely, we say that an SKT metric \(g\) on a complex manifold \((M, J)\) is static if

\[-(\rho^B)^{(1,1)} = \lambda \omega\]

for some \(\lambda \in \mathbb{R}\). It is straightforward that every Kähler-Einstein metric is static, but the link between static metric and Kähler-Einstein metric is deeper. First of all, in [27], it is shown that to any static metric with \(\lambda \neq 0\) we can associate a symplectic form that tames \(J\), called Hermitian-symplectic in [27]. In [21] it was proved that if \((M, J)\) is a compact complex surface, then the existence of a symplectic form that tames \(J\) implies the existence of a Kähler metric on \((M, J)\). Moreover, a nilmanifold, i.e. the compact quotient of a nilpotent simply connected Lie group by a discrete subgroup, endowed with an invariant complex structure \(J\) cannot admit any Kähler metric ([5, 20]), and in [11] it is proved that it cannot admit any symplectic form that tames \(J\), either. Indeed, it is still an open problem to find an example of a complex manifold admitting a symplectic form that tames \(J\), but no Kähler structures.

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Secondly, all the examples of static metrics contained in [27] are Kähler-Einstein except the Hopf manifold, that admit a static metric with \( \lambda = 0 \). In fact, we prove that on a compact Kähler manifold any static metric induces a Kähler-Einstein metric, so examples of non Kähler-Einstein static metrics on compact manifolds have to be found on non-Kähler manifolds.

Our investigation concern in particular Lie groups and compact quotients of Lie groups by discrete subgoups. The study of SKT metrics on such manifolds was developed in [13, 23, 11, 30, 31].

First, we focus on nilmanifolds, and we prove that no invariant static metrics can be found on nilmanifods toghether with invariant complex structures (with the exception of tori).

Then we classify all the invariant static metrics on simply connected Lie groups of dimension 4, obtaining that the unique 4-dimensional Lie algebra that admit a non Kähler-Einstein invariant static metric is \( \mathfrak{su}(2) \times \mathbb{R} \). This is not surprising, because it is the Lie algebra associated to the group \( S^3 \times S^1 \), that is diffeomorphic to the Hopf manifold. So the Hopf manifold admits static metrics (induced by the invariant ones), that is the same result of [27].

In the last section, we focus on static metrics with \( \lambda = 0 \). First, we prove that if \((g, J, g, D)\) is a Lie algebra together with a static metric \( g \) with \( \lambda = 0 \) and a flat Hermitian connection \( D \) then on the tangent Lie algebra \( T_D g = g \times_D \mathbb{R}^{2n} \) we can produce a static metric with \( \lambda = 0 \). Then we note that for every compact even-dimensional semisimple Lie group, the bi-invariant metric is a static metric with \( \lambda = 0 \) with respect to any compatible complex structure, so we can apply the tangent Lie algebra construction to such groups, obtaining the first examples of compact and non-compact static metrics in (real) dimension greater than 4.

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## 1. Preliminaries

We start by recalling some definitions and fixing some notation. Let \((M^{2n}, J)\) be a complex manifold of real dimension \( 2n \) and \( g \) be a Hermitian metric on \((M, J)\) with fundamental 2-form defined by \( \omega(\cdot, \cdot) = g(\cdot, J\cdot) \). By [16] there exist a unique connection \( \nabla^B \) on \( M \), called the Bismut connection, such that \( \nabla^B J = \nabla^B g = 0 \) and whose torsion 3-form

\[
c(X, Y, Z) = g(X, T^B(Y, Z))
\]

is totally skew-symmetric. It is well known that \( c = -Jd\omega \).

**Definition 1.1.** A Hermitian metric \( g \) on a complex manifold \((M, J)\) is strong Kähler with torsion or SKT if the torsion 3-form \( c \) of the Bismut connection \( \nabla^B \) is closed, i.e. \( dc = 0 \). This condition is equivalent to \( \partial \bar{\partial} \omega = 0 \).

Since \( \nabla^B \) is a Hermitian connection, we can define the Ricci form of \( \nabla^B \) as

\[
\rho^B(X, Y) = \frac{1}{2} \sum_{k=1}^{2n} g(R^B(X, Y)e_k, J e_k),
\]

where \( \{e_i\} \) is a local orthonormal frame of the tangent bundle \( TM \) and \( R^B \) is the curvature tensor of \( \nabla^B \) defined by

\[
R^B(X, Y)Z = \nabla^B_{[X,Y]}Z - [\nabla^B_X, \nabla^B_Y]Z.
\]

In the same way, we can define the Ricci form of the Chern connection, i.e. the unique Hermitian connection such that the \((1,1)\)-part of the torsion tensor vanishes. The Ricci form of the Chern connection is related to the one of the Bismut connection by the formula \((1, 12)\)

\[
\rho^B = \rho^C + dd^\ast \omega.
\]  

(1.1)
In \cite{27,28} Streets and Tian studied the evolution equation

\[
\begin{aligned}
\frac{\partial \omega(t)}{\partial t} &= -(\rho^B)^{(1,1)} \\
\omega(0) &= \omega_0
\end{aligned}
\tag{1.2}
\]

where \((\rho^B)^{(1,1)}\) is the projection of \(\rho^B\) on the bundle of (1,1)-forms. This flow preserves the SKT condition and is elliptic on the set of SKT metrics, so short-time existence of solutions is guaranteed. Moreover, if the initial condition \(\omega_0\) is Kähler then (1.2) coincides with the Kähler-Ricci flow \cite{7}.

Equation (1.2) allows to define an Einstein-like condition for SKT metrics.

**Definition 1.2** \((\cite{27})\). A Hermitian metric \(g\) on a complex manifold \((M, J)\) is called static if it is SKT and the Ricci tensor of the Bismut connection satisfies

\[-(\rho^B)^{(1,1)} = \lambda \omega \]

for some real constant \(\lambda\).

As pointed out in \cite{27} static metrics with \(\lambda \neq 0\) carries additional structures. Indeed, if \(\lambda \neq 0\), then

\[-\frac{1}{\lambda} \rho^B(JX, X) > 0 \quad \text{and} \quad d\rho^B = 0,
\]

so \(-\frac{1}{\lambda} \rho^B\) is a symplectic form and tames the complex structures \(J\).

**Theorem 1.3.** Let \((M, J)\) be a compact complex manifold, and suppose that it admits a Kähler metric. Then the existence of a static metric is equivalent to the existence of a Kähler-Einstein metric. In particular,

- If \(g\) is a static metric with \(\lambda \neq 0\), then \(g\) is itself a Kähler-Einstein metric;
- If \(g\) is a static metric with \(\lambda = 0\), then \((M, J)\) is Calabi-Yau manifold.

**Lemma 1.4.** Let \((M, J)\) be a complex manifold and \(g\) a static metric with \(\lambda \neq 0\) such that \(\rho^B \in \Omega^{1,1}(M)\). Then \(g\) is Kähler-Einstein.

**Proof.** If \(g\) is a static metric with \(\lambda \neq 0\) and \(\rho^B \in \Omega^{1,1}(M)\), then \(\omega = -\frac{1}{\lambda} \rho^B\). But \(\rho^B = dd^*\omega + \rho^C\) is closed, so \(d\omega = 0\). \(\square\)

**Proof of Theorem 0.1.** Clearly, if \(g\) is a Kähler-Einstein metric, it satisfies the static condition.

Now suppose that \(g\) is a static metric. Since \((M, J)\) admits a Kähler metric, the \(\partial\bar{\partial}\)-lemma holds. Then there is a function \(f\) on \(M\) such that \(dd^*\omega = \partial\bar{\partial}f\), so \(dd^*\omega\) is of type \((1,1)\). Therefore \(\rho^B \in \Omega^{1,1}(M)\).

If \(\lambda \neq 0\), applying Lemma 1.4 we have that \(g\) is Kähler-Einstein.

If \(\lambda = 0\), then \(\rho^B = 0\). But in general \([\rho^B] = [\rho^C] = c_1 \in H^2(M, \mathbb{R})\), so \(c_1 = 0\), and since \((M, J)\) admits a Kähler metric it is Calabi-Yau. \(\square\)

**Remark 1.5.** Lemma 1.4 holds for any complex manifold, either compact or non-compact.

### 2. Nilmanifolds

We recall that a nilmanifold is the compact quotient of a simply connected nilpotent Lie group \(G\) by a discrete subgroup \(\Gamma\). By invariant Riemannian metric (respectively complex structure) on \(G/\Gamma\) we mean the one induced by an inner product (respectively complex structure) on the Lie algebra \(\mathfrak{g}\) of \(G\). It is well known that a nilmanifold cannot admit any Kähler metric unless it is a torus (see for example \cite{5,20}), and results about classification of SKT metrics on nilmanifolds have been found
Lemma 2.1. Let \( \mathfrak{g} \) be a nilpotent Lie algebra together with a complex structure \( J \) and a \( J \)-Hermitian SKT metric \( g \), and \( \nabla^B \) its Bismut connection. Then for any \( X, Y \in \mathfrak{g} \)

\[ \nabla^B_{X} Y^\xi = 0; \]

\[ \nabla^B_{X} Y^\perp \in \xi^\perp \text{ and } g(\nabla^B_{X} Y^\perp, Z) = -\frac{1}{2}g([Y, Z] + [Y, JZ], X^\xi); \]

\[ \nabla^B_{X} Y^\xi \in \xi^\perp \text{ and } g(\nabla^B_{X} Y^\xi, Z) = -\frac{1}{2}g([X, Z] - [X, JZ], Y^\xi). \]

Moreover, \( J\nabla^B_{JX} Y^\xi = \nabla^B_X Y^\xi \) \( (2.1) \)

\[ \nabla^B_{X} Y^\perp = \frac{1}{2}([X^\perp, Y^\perp] - [X^\perp, JY^\perp]). \]

Proof. In view of [9] we can write the Bismut connection in terms of Lie brackets as

\[ \tilde{g}(\nabla^B_Y Z) = \frac{1}{2}\left\{ \tilde{g}([X, Y] - [JX, JY], Z) - \tilde{g}([Y, Z] + [JY, JZ], X) - \tilde{g}([X, Z] - [JX, JZ], Y) \right\}. \]

Equation \( (2.2) \) can be obtained using the definition of \( \xi \). Relations \( (i), (ii), (iv) \) and the first part of \( (iii) \) comes directly by using the definition of \( \xi \). Equation \( (2.1) \) can be obtained using \( (i) \) and the integrability of \( J \).

Lemma 2.2. Let \( \mathfrak{g} \) be a nilpotent Lie algebra together with a complex structure \( J \) and a \( J \)-Hermitian SKT metric \( g \). Then

\[ g([X, JX], [Y, JY]) = \frac{1}{2}(||[X, Y]|^2 + ||[X, JY]|^2 + ||[X, JX]|^2 + ||[X, JY]|^2) \]

for every \( X, Y \in \mathfrak{g} \).

Proof. If \( X \) or \( Y \) belongs to the center, then the lemma is obviously true; so we consider the case \( X, Y \in \xi^\perp \). We can write \( c \) in terms of Lie brackets as

\[ c(X, Y, Z) = -g([JX, JY], Z) - g([JY, JZ], X) - g([JZ, JX], Y), \]

then

\[ 0 = dc(X, Y, JX, JY) = -c([X, JY], [Y, JX]) + c([X, JX], [Y, JY]) - c([X, JY], [JX, JY]) - c([Y, JX], [X, JY]) + c([Y, JY], [X, JX]) - c([JX, JY], [X, JY]) = -2g([X, JX], [Y, JY]) + ||[X, JY]|^2 + ||[X, JY]|^2 + ||[JX, JY]|^2 + ||[X, JY]|^2. \]
as required. \hfill \square

Now we are ready to prove the following

**Theorem 2.3.** Let $G/\Gamma$ a nilmanifold (not a torus) together with an invariant complex structure $J$. Then it does not admit any $J$-Hermitian invariant static metric with $\lambda = 0$.

**Proof.** Let $\mathfrak{g}$ the Lie algebra of $G$, $\tilde{J}$ the induced complex structure and $g$ a $\tilde{J}$-Hermitian SKT metric; we have $\mathfrak{g} = \xi \oplus \xi^\perp$. Choose $\{e_1, \ldots, e_{2m}\}$ and $\{f_1, \ldots, f_{2k}\}$ to be orthonormal basis respectively of $\xi^\perp$ and $\xi$ with $2m + 2k = 2n = \dim \mathfrak{g}$; then $\{e_1, \ldots, e_{2m}, f_1, \ldots, f_{2k}\}$ is an orthonormal basis of $\mathfrak{g}$. Note that $(\rho^B)^{(1,1)}(X, JX) = \rho^B(X, JX)$, so in order to prove that $(\rho^B)^{(1,1)} \neq 0$ we will show that $\rho^B(X, JX)$ is not zero for some $X \in \mathfrak{g}$.

Suppose $X \in \xi^\perp$; by definition,

$$\rho^B(X, JX) = \frac{1}{2} \left( \sum_{i=1}^{2m} g(R^B(X, JX)e_i, \tilde{J}e_i) + \sum_{j=1}^{2k} g(R^B(X, JX)f_j, \tilde{J}f_j) \right);$$

we consider the two summations separately.

- By definition of $R^B$, we obtain

$$g(R^B(X, JX)e_i, \tilde{J}e_i) = g(\nabla^B_X \nabla^B_JX e_i, \tilde{J}e_i) - g(\nabla^B_JX \nabla^B_X e_i, \tilde{J}e_i) - g(\nabla^B_{\xi, JX} e_i, \tilde{J}e_i).$$

Applying Lemma 2.1 and using the integrability of $\tilde{J}$ we have

$$g(\nabla^B_X \nabla^B_JX e_i, \tilde{J}e_i) = -g(\nabla^B_JX \nabla^B_X e_i, \tilde{J}e_i) = -\frac{1}{4} \|[X, e_i] - [JX, \tilde{J}e_i]\|^2$$

and

$$g(\nabla^B_{\xi, JX} e_i, \tilde{J}e_i) = -g([X, JX], [e_i, \tilde{J}e_i]).$$

Hence

$$g(R^B(X, JX)e_i, \tilde{J}e_i) = -\frac{1}{2} \|[X, e_i] - [JX, \tilde{J}e_i]\|^2 + g([X, JX], [e_i, \tilde{J}e_i]). \quad (2.4)$$

- Again by definition of $R^B$ and applying Lemma 2.1 and equation (2.1), we obtain

$$g(R^B(X, JX)f_j, \tilde{J}f_j) = g(\nabla^B_X \nabla^B_JX f_j, \tilde{J}f_j) - g(\nabla^B_JX \nabla^B_X f_j, \tilde{J}f_j) = \frac{1}{2} g([X, \nabla^B_JX f_j] - \tilde{J}\nabla^B_X f_j) - [JX, \tilde{J}\nabla^B_X f_j] + \nabla^B_{\xi, JX} f_j, \tilde{J}f_j) = g([X, \nabla^B_JX f_j] - [JX, \tilde{J}\nabla^B_X f_j], \tilde{J}f_j).$$

By decomposing $\nabla^B_{\xi, JX} f_j$ in components with respect to the basis $\{e_i\}$ of $\xi^\perp$ we compute that

$$\sum_{j=1}^{2k} g(R^B(X, JX)f_j, \tilde{J}f_j) = \frac{1}{2} \sum_{i=1}^{2m} \|[X, e_i] - [JX, \tilde{J}e_i]\|^2. \quad (2.5)$$

Combining equations (2.4) and (2.5) we obtain

$$\rho^B(X, JX) = \frac{1}{2} \sum_{i=1}^{2m} g([X, JX], [e_i, \tilde{J}e_i])$$

and using Lemma 2.2

$$= \frac{1}{4} \sum_{i=1}^{2m} (\|[X, e_i]\|^2 + \|[X, JX]\|^2 + \|[JX, e_i]\|^2 + \|[JX, JX]\|^2) > 0$$
since $X \in \xi^⊥$; this concludes the proof.

The results of this section can be summarized as follows: let $G/\Gamma$ be a nilmanifold (not a torus) endowed with an invariant complex structure $J$ and with a $J$-invariant SKT metric $g$; then, if $g$ is a static metric, it must be non-invariant and $\lambda$ must be zero. Whether such metrics exist is still not known, but a possible approach to the problem could be the following: let $g$ be a non-invariant Hermitian metric on $(G/\Gamma, J)$, with $J$ invariant. By [22] $G/\Gamma$ has a bi-invariant volume form $d\mu$, and applying the symmetrization process of [4] we can construct a new invariant $J$-Hermitian metric $\tilde{g}$ by posing

$$\tilde{g}(X, Y) = \int_{m \in M} g_m(X_m, Y_m)d\mu$$

for any left-invariant vector fields $X, Y$. Moreover, in [31] it was proved that if the metric $g$ is SKT, then $\tilde{g}$ is still SKT. Thus, if a nilmanifold admits a non-invariant static metric with $\lambda = 0$, then it induces an invariant SKT metric $\tilde{g}$. In general, however, it is not true that the Ricci form $\tilde{\rho}$ of the metric $\tilde{g}$ is obtained by the symmetrization of the Ricci tensor $\rho^B$ of $g$, so it is an open problem to check if the induced invariant metric $\tilde{g}$ is still static.

3. Static metrics in dimension 4

The study of static metrics in dimension 4 turns out to be strictly related to Kähler-Einstein metrics. Indeed, combining Lemma 4.4 of [1] and Theorem 2 of [17] we have that if $(M, J)$ is a compact complex surface and $g$ is static metric on $(M, J)$, then either $(M, J, g)$ is Kähler-Einstein or $(M, J)$ is the Hopf surface. Moreover, in [17, 27] it was proved that the Hopf surface admits a static metric with $\lambda = 0$, and the Hopf surface cannot admit any symplectic form that tames the complex structure [19, Proposition 2.24]. So

**Theorem 3.1.** Let $(M, J)$ be a compact complex surface and $g$ a static metric. Then one of the following cases occurs:

- $(M, J, g)$ is Kähler-Einstein
- $(M, J)$ is the Hopf surface and $\lambda = 0$.

In this section we prove a similar result for left-invariant static metrics on 4-dimensional simply connected Lie groups (not necessarily compact) endowed with a left-invariant complex structure.

**Theorem 3.2.** Let $(G, J, g)$ be a simply connected Lie group together with an invariant complex structure $J$ and an invariant static metric $g$. Then one of the following cases occurs:

- $(G, J, g)$ is Kähler-Einstein
- The Lie algebra $\mathfrak{g}$ of $G$ is isomorphic to $\mathfrak{su}(2) \times \mathbb{R}$ and $\lambda = 0$.

Note that the Lie algebra $\mathfrak{su}(2) \times \mathbb{R}$ plays the same role as the Hopf manifold for compact complex surfaces. This happens because $\mathfrak{su}(2) \times \mathbb{R}$ is the Lie algebra of the Lie group $S^3 \times S^1$, that is diffeomorphic to the Hopf manifold.

**Proof of Theorem 3.2.** Since we are interested in invariant structures, it is sufficient to study the induced structures on the corresponding Lie algebra. Let $\mathfrak{g}$ be a Lie algebra: the derived series of $\mathfrak{g}$ is defined by $D^1\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}], D^k\mathfrak{g} = [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}]$, and we say that $\mathfrak{g}$ is solvable if there exists an integer $s$ such that $D^s\mathfrak{g} = 0$. According to [3], a Lie algebra of dimension 4 is either solvable, isomorphic to $\mathfrak{su}(2) \times \mathbb{R}$ or isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}$. We started by considering solvable Lie algebras.

A classification of 4-dimensional solvable Lie algebras admitting a left-invariant complex structure can be found in [26], and recently Madsen and Swann in [23] gave a classification of SKT structures on
solvable Lie algebras of dimension four. With the help of a MAPLE software, we use this classification to compute directly the Ricci tensor of the Bismut connection. According to [23], we can suppose that \( J \) is defined by \( Je^1 = e^2, \ Je^3 = e^4 \), where \( \{e^1, e^2, e^3, e^4\} \) is a basis of \( \mathfrak{g}^* \). Moreover, \( \mathfrak{g} \) belongs to one of the following cases:

**Complex case:** \( \mathfrak{g} \) has structure equations

\[
\begin{align*}
d e^1 &= 0 \\
d e^2 &= a_1 e^{12} \\
d e^3 &= b_1 e^{12} + b_2 e^{13} + b_3 e^{14} - c_1 e^{23} + c_2 e^{24} \\
d e^4 &= d_1 e^{12} + d_2 e^{13} + d_3 e^{14} - f_1 e^{23} + f_2 e^{24} + h_1 e^{34},
\end{align*}
\]

where by \( e^{ij} \) we denote the wedge product \( e^i \wedge e^j \), and the real coefficients \( a_1, b_1, c_1, d_1, f_1, h_1 \) satisfy

\[
\begin{align*}
f_1 &= c_2 + d_3 - b_2 \\
a_1 c_1 - b_3 f_1 - c_2 d_2 &= 0 \\
h_1 (b_2^2 + b_3^2 + c_2^2) &= 0 \\
a_1 f_2 + b_1 h_1 - b_3 f_1 - c_2 d_2 &= 0 \\
\end{align*}
\]

In the sequel, to shorten the notation, we will denote the structure equations of \( \mathfrak{g} \) as

\[
\mathfrak{g} = (0, a_1 e^{12}, b_1 e^{12} + b_2 e^{13} + b_3 e^{14} - c_1 e^{23} + c_2 e^{24}, d_1 e^{12} + d_2 e^{13} + d_3 e^{14} - f_1 e^{23} + f_2 e^{24} + h_1 e^{34}).
\]

The fundamental 2-form of the SKT metric is \( \omega = e^{12} + e^{34} \), and we obtain

\[
\rho^B(X, Y) = (a_1^2 + b_1^2 + d_1^2 + a_1 c_1 - a_1 b_3 - a_1 b_2 + h_1 e^{d_1}) \cdot e^{12} + (b_1 b_2 + d_1 d_2 + h_1 b_1) \cdot e^{13} + (b_1 b_3 + d_1 d_3 + h_1 d_3) \cdot e^{14} + (d_1 b_2 - b_1 c_1 - d_1 c_2 - d_1 d_3 - h_1 c_2 - h_1 d_3 + h_1 b_2) \cdot e^{23} + (d_1 b_3 + b_1 c_2 - d_1 c_1 + d_1 d_2 - h_1 c_1 + h_1 d_2 + h_1 b_3) \cdot e^{24} + (h_1 b_1 + h_1^2) \cdot e^{34}.
\]

If we impose that \(- (\rho^B)^{(1, 1)} = \lambda \omega \) with \( \lambda \neq 0 \), we find that \( h_1 \neq 0 \), so by (3.1) \( b_3 = c_1 = c_2 = 0 \), and the (2, 0) + (0, 2)-part of \( \rho^B \) vanishes. Then by Lemma 1.4 any static metric is Kähler-Einstein. On the other hand, imposing \( (\rho^B)^{(1, 1)} = 0 \) we find that \( d \omega = 0 \), so \( \mathfrak{g} \) must be Kähler-Einstein.

**Real case I:** \( \mathfrak{g} \) has structure equations

\[
(0, a_1 e^{12} + a_3 e^{14} - e^{23}) + b_2 e^{34}, 0, d_1 e^{12} + d_3 e^{14} - e^{23} + h_1 e^{34})
\]

where \( de^2 \) and \( de^4 \) are linearly independent and the real coefficients satisfy

\[
\begin{align*}
b_2 a_1 - b_2 d_3 + f_2 a_3 - a_3^2 &= 0 \\
d_3 a_3 - d_2 d_1 &= 0 \\
\end{align*}
\]

where \( d e^2 \) and \( d e^4 \) are linearly independent and the real coefficients satisfy

\[
\begin{align*}
\frac{b_2 a_1 + f_2 d_3 + a_2^2 + d_1^2}{t^2 - 1} \cdot e^{12} - \frac{b_2^2 + f_2^2 + b_3 a_1 + f_2 d_1}{t^2 - 1} \cdot e^{34} \\
- \frac{b_2 a_3 + d_3 f_2 + a_3 a_1 + d_3 d_1}{t^2 - 1} \cdot e^{14} + \frac{b_2 a_3 + d_3 f_2 + a_3 a_1 + d_3 d_1}{t^2 - 1} \cdot e^{23}.
\end{align*}
\]

Clearly \( (\rho^B)^{(2, 0)} + (0, 2) = 0 \), so applying Lemma 1.4 any static metric with \( \lambda \neq 0 \) is Kähler-Einstein. Moreover, if we impose \( (\rho^B)^{(1, 1)} = 0 \) we obtain that \( \mathfrak{g} \) must be abelian.
Real case II: $g$ has structure equations

$$
\begin{cases}
  de^1 = 0 \\
  de^2 = -kq^2 e^{12} - kqr(e^{14} - e^{23}) - kr^2 e^{34} \\
  de^3 = \frac{c_3q}{r} e^{12} + c_3 e^{14} \\
  de^4 = \frac{kq^3}{r} e^{12} - c_3 e^{13} + kq^2(e^{14} - e^{23}) + kqr e^{34},
\end{cases}
$$

with $q, r, k \in \mathbb{R}$ such that $q^2 + r^2 = 1$, $r > 0$ and $k \neq 0$. The fundamental 2-form of the SKT metric is $\omega = e^{12} + e^{34} + te^{14} + te^{23}$, with $t \in (-1, 1)$, and it is never Kähler. Computing $\rho^B$ we find

$$
\rho^B(X, Y) = -\frac{(c_3kq^2t + k^2q\sqrt{1 - q^2} + t\sqrt{1 - q^2}q^3 - ktq^3)}{(1 - q^2)^{3/2}(1 + t^2)} e^{12} + \frac{c_3(kq + \sqrt{1 - q^2}c_3t)}{(1 + t^2)\sqrt{1 - q^2}} e^{13} + t\sqrt{1 - q^2}kq^2 - k^2q^2 - q^2c_3^2 - \sqrt{1 - q^2}(1 + t^2)} e^{14} - \frac{k^2q(c - 32)}{e^{23} - \frac{k^2}{1 + t^2}} e^{34}.
$$

By [10], this Lie algebra does not admit any symplectic form that tames $J$, so we cannot have a static metric on $g$ with $\lambda \neq 0$. On the other hand, if $(\rho^B)^{1,1} = 0$ we must have $k = 0$, that is a contradiction. Therefore $g$ does not admit any static metric.

Real case III: $g$ has structure equations

$$
\begin{cases}
  de^1 = 0 \\
  de^2 = -k(1 + q^2) e^{12} - kqr(e^{14} - e^{23}) - kr^2 e^{34} \\
  de^3 = \frac{c_3q}{r} e^{12} - \frac{k}{2} e^{13} + c_3 e^{14} \\
  de^4 = \frac{q}{r}(kq^2 + \frac{k}{2}) e^{12} - c_3 e^{13} + (kq^2 - \frac{k}{2}) e^{14} - kq^2 e^{23} + kqr e^{34},
\end{cases}
$$

with $q, r, k \in \mathbb{R}$ such that $q^2 + r^2 = 1$, $r > 0$ and $k \neq 0$; if $c_3 = 0$ we have $g \cong \mathfrak{o}_{4,2}^+$, otherwise $g \cong \mathfrak{o}_{4,1}^\perp$, $\ker$. The fundamental 2-form of the SKT metric is $\omega = e^{12} + e^{34} + te^{14} + te^{23}$ with $t \in (-1, 1)$, and is Kähler if and only if $q = 0$. Computing $\rho^B$ we find

$$
\rho^B(X, Y) = -\frac{1}{4} \frac{-8kq^6 + 4k^2q^4 + 8c_3\sqrt{1 - q^2}kq^3t + k^2q^2 - 4c_3^2q^2 + 12c_3kq\sqrt{1 - q^2}t - 6k^2}{(1 + q^2)(t^2 - 1)} e^{12} - \frac{1}{4} \frac{8c_3kq^3 - 4kqc_3}{(t^2 - 1)\sqrt{1 - q^2}} e^{13} - \frac{1}{4} \frac{8c_3kq^3 - 4kqc_3 - 4\sqrt{1 - q^2}c_3^2t + 3tk^2\sqrt{1 - q^2}}{(t^2 - 1)(1 - q^2)} e^{14} - \frac{8c_3kt\sqrt{1 - q^2} + 16t^2k^2q^3 - 8k^2q^5 - 10ktq^2 - 4c_3q^2 + 4q^2c_3^2}{\sqrt{1 - q^2}(1 - q^2)c_3} \cdot e^{23} + \frac{1}{2} \frac{kq\sqrt{8t^2q^4 - 4q^4 + 2t^2q^2 - 4t^2 + 1}}{\sqrt{1 - q^2}(t^2 - 1)} e^{34}.
$$

Imposing that $-(\rho^B)^{1,1} = \lambda \omega$, we find that $q = t = 0$, so $d\omega = 0$. Thus $g$ is static if and only if it is Kähler-Einstein.

This concludes the proof in the solvable case.
The non-solvable 4-dimensional Lie algebras $\mathfrak{su}(2) \times \mathbb{R}$ and $\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}$ have structure equations

\[
\mathfrak{su}(2) \times \mathbb{R} = (\mathfrak{e}^{23}, \mathfrak{e}^{13}, -\mathfrak{e}^{12}, 0) \\
\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R} = (\mathfrak{e}^{32}, \mathfrak{e}^{13}, \mathfrak{e}^{12}, 0).
\]

From a more general result in [24] we have that the only complex structures on these algebras are defined in both cases by

\[
J \mathfrak{e}^1 = \mathfrak{e}^2, \quad J \mathfrak{e}^3 = -p \cdot \mathfrak{e}^3 + (1 + p^2) \cdot \mathfrak{e}^4.
\]

All the metrics compatible with those complex structures have the form

\[
\omega = m_{11} \mathfrak{e}^{12} + m_{32} \mathfrak{e}^{13} + \frac{m_{31} + pm_{32}}{1 + p^2} \mathfrak{e}^{14} - m_{31} \mathfrak{e}^{23} + \frac{m_{32} - pm_{31}}{1 + p^2} \mathfrak{e}^{24} + m_{44} \mathfrak{e}^{34}. \tag{3.3}
\]

Both these Lie algebras are semisimple, thus by Theorem 8 of [8] they cannot admit any invariant symplectic structure. So no invariant static metric with $\lambda \neq 0$ can be found on these algebras. Moreover, since they are unimodular, every 3-form is closed, then every $J$-compatible metric is SKT.

We study the two cases separately:

- $\mathfrak{su}(2) \times \mathbb{R}$. Let $g$ be a Hermitian metric whose fundamental 2-form satisfy (3.3). Then

\[
(\rho^B)^{(1,1)}(X,Y) = -\frac{2 m_{31}^2 - 2 m_{32}^2 + (1 + p^2) m_{44} m_{11} - m_{44}^2 (1 + p^2)^2}{-m_{31}^2 - m_{32}^2 + m_{44} m_{11} p^2 + m_{44} m_{11}} \cdot \mathfrak{e}^{12} \\
+ \frac{1}{2} \frac{1}{2} m_{44}^2 m_{32} m_{11} m_{31} + m_{31} m_{32} + m_{31} m_{11}} \cdot \mathfrak{e}^{14} - \mathfrak{e}^{23}.
\]

As said before, this Lie algebra can only admit static metric with $\lambda = 0$; imposing the vanishing of $(\rho^B)^{(1,1)}$ we obtain that $m_{11} = (1 + p^2) m_{44}$ and $m_{31} = m_{32} = 0$, so every metric in the form

\[
\omega = m_{44} (1 + p^2) \mathfrak{e}^{12} + m_{44} \mathfrak{e}^{34} \tag{3.4}
\]

is static with $\lambda = 0$. Therefore every complex structure on $\mathfrak{su}(2) \times \mathbb{R}$ admits a compatible static metric with $\lambda = 0$. More in general, for these metrics we have $p_B = 0$.

- $\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}$. Let $g$ be a Hermitian metric whose fundamental 2-form satisfy (3.3). Then

\[
(\rho^B)^{(1,1)}(X,Y) = -\frac{2 m_{31}^2 - 2 m_{32}^2 + (1 + p^2) m_{44} m_{11} + m_{44}^2 (1 + p^2)^2}{-m_{31}^2 - m_{32}^2 + m_{44} m_{11} p^2 + m_{44} m_{11}} \cdot \mathfrak{e}^{12} \\
+ \frac{1}{2} \frac{1}{2} m_{44}^2 m_{32} m_{11} m_{31} + m_{31} m_{32} + m_{31} m_{11}} \cdot \mathfrak{e}^{13} + \mathfrak{e}^{24} \\
+ \frac{1}{2} m_{44}^2 m_{32} m_{11} m_{31} + m_{31} m_{32} + m_{31} m_{11}} \cdot \mathfrak{e}^{14} - \mathfrak{e}^{23}.
\]

Again, this algebra can only admit static metric with $\lambda = 0$, and imposing that $(\rho^B)^{(1,1)} = 0$ we obtain that $m_{11} = -(1 + p^2) m_{44}$ and $m_{31} = m_{32} = 0$; but $m_{11} m_{44} \leq 0$, that is a contradiction because $g$ is positive definite. Then $\mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}$ does not admit any static metric.
4. TANGENT LIE ALGEBRAS AND COMPACT SEMISIMPLE LIE GROUPS

Consider a $2n$-dimensional Lie algebra $\mathfrak{g}$ and a flat connection $D$ of $\mathfrak{g}$. We define the tangent Lie algebra $(T_D \mathfrak{g} = \mathfrak{g} \ltimes D \mathbb{R}^{2n}, [\cdot, \cdot])$ with the Lie bracket

$$(\{X_1, X_2\}, (Y_1, Y_2))_D = ((X_1, Y_1), D X_1 Y_2 - D Y_1 X_2).$$

Additionally, if $(J, g)$ is a Hermitian structure on $\mathfrak{g}$ and if $D$ is Hermitian, i.e. $D g = DJ = 0$, then on $T_D \mathfrak{g}$ we can define a complex structure $J(X_1, X_2) = (J X_1, JX_2)$ and a $\bar{J}$-Hermitian metric $\tilde{g}$ such that $(\mathfrak{g}, 0)$ and $(0, \mathfrak{g})$ are orthogonal (see [2]).

**Theorem 4.1.** Let $(\mathfrak{g}, J, g)$ be a Hermitian Lie algebra and $D$ a Hermitian flat connection. Then $g$ is a $J$-Hermitian static metric with $\lambda = 0$ if and only if $\tilde{g}$ is a $\bar{J}$-Hermitian static metric with $\lambda = 0$.

**Proof.** By [10, Proposition 3.1], $(T_D \mathfrak{g}, \bar{J}, \tilde{g})$ is SKT if and only if $(\mathfrak{g}, J, g)$ is SKT, so we only have to prove that

$$(\rho^B)^{(1,1)} = 0 \iff (\tilde{\rho}^B)^{(1,1)} = 0,$$

where $\rho^B$ is the Ricci tensor of the Bismut connection $\nabla^B$ of $(T_D \mathfrak{g}, \bar{J}, \tilde{g})$. The Bismut connections of $(T_D \mathfrak{g}, \bar{J}, \tilde{g})$ and $(\mathfrak{g}, J, g)$ are related by

$$\tilde{g}(\nabla^B_{(X_1, X_2)}(Y_1, Y_2), (Z_1, Z_2)) = g(\nabla^B_{X_1} Y_1, Z_1) + g(D X_1 Y_2, Z_2),$$

so the curvature $\bar{R}^B$ of $\nabla^B$ is given by $\bar{R}^B = (R^B, R^D) = (R^B, 0)$ since $D$ is flat. Hence, the Ricci tensors of $\nabla^B$ and $\tilde{\nabla}^B$ are equal, as well as their $(1,1)$ parts.

Let $(M, J, g)$ be a complex manifold with an SKT metric such that the Bismut connection $\nabla^B$ has trivial holonomy. Then clearly $-(\rho^B)^{(1,1)} = 0$, i.e. $g$ is a static metric with $\lambda = 0$. It is well known that this condition holds if $M$ is a Lie group and $g$ a bi-invariant metric, that is a metric which is both left-invariant and right-invariant. Let $\tilde{g}$ be the induced bi-invariant metric on the Lie algebra $\mathfrak{g}$, then it satisfies

$$\tilde{g}([X, Y], Z) = -\tilde{g}(Y, [X, Z]).$$

By using this equation and the integrability of the complex structure in (2.2) and (2.3) we obtain that $\nabla^B_X Y = 0$ for every $X, Y \in \mathfrak{g}$, so $\text{Hol}(\nabla^B) = 0$, and $c(X, Y, Z) = -\frac{1}{2}g([X, Y], Z)$. Then, applying (4.2) and the Jacobi identity we obtain $dc = 0$, so $\tilde{g}$ is SKT.

Since the work of Samuelson and Wang [25, 32], it has been known that every compact even-dimensional Lie group $G$ admits a left-invariant complex structure $J_L$ and a right-invariant one $J_R$. Moreover, if $G$ is semisimple, the bi-invariant metric $g_K$ induced by the Killing form is compatible with both $J_L, J_R$.

**Proposition 4.2.** Let $G$ be a compact, even-dimensional semisimple Lie group. Then it admits a static metric with $\lambda = 0$.

**Remark 4.3.** The Lie algebra $\mathfrak{su}(2) \times \mathbb{R}$ considered in the former section is contained in this class, and we can obtain a bi-invariant metric by setting $m_{44} = 1$ and $p = 0$ in (3.4).

In view of this proposition, if we find a flat Hermitian connection $D$ on a $2n$-dimensional compact semisimple Lie group $G$ whose Lie algebra is $\mathfrak{g}$, then we can construct a static metric with $\lambda = 0$ on the tangent Lie algebra $T_D \mathfrak{g}$.

**Example 4.4.** For every $2n$-dimensional compact semisimple Lie group $G \cong G_0 \times S^1$, where $G_0$ is a $(2n - 1)$-dimensional compact semisimple Lie group, we can construct a flat Hermitian connection $D$ ([10, Proposition 3.4]).
If we consider the Lie algebra \( \mathfrak{su}(2) \times \mathbb{R} \) together with the complex structure \( J e^1 = e^2, J e^3 = e^4 \) and the \( J \)-Hermitian bi-invariant metric \( g = \sum e^i \otimes e^i \), we can define the Hermitian connection
\[
D_{e_i} Y = 0 \quad i = 1, 2, 3 \quad D_{e_4} Y = J Y
\]
for any \( Y \in \mathfrak{g} \). The corresponding tangent Lie algebra has structure equation
\[
T_D \mathfrak{g} = (-f^{23}, f^{13}, -f^{12}, 0, -f^{16}, f^{45}, -f^{48}, f^{47}),
\]
where \( f^i = (e^i, 0) \) for \( i = 1, \ldots, 4 \) and \( f^j = (0, e^j) \) for \( j = 5, \ldots, 8 \), and the induced Hermitian structure \((\tilde{J}, \tilde{g})\) is given by
\[
\tilde{J} f^{2i-1} = f^{2i}, \quad i = 1, 4 \quad \tilde{g} = \sum_{i=1}^8 f^i \otimes f^i.
\]

Note that for any other compact semisimple Lie group this result cannot be applied since for every non-abelian simple Lie algebra \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\). However, this is not the only way to construct flat Hermitian connections, as shown in the next example.

**Example 4.5.** Let us consider \( G = S^3 \times S^3 \). The associated Lie algebra \( \mathfrak{g} \) is
\[
\mathfrak{su}(2) \times \mathfrak{su}(2) = (-e^{23}, e^{13}, -e^{12}, -e^{56}, e^{46}, -e^{45}).
\]
Clearly \([\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}\), so we cannot apply Proposition 3.4 of \([10]\). However, the linear connection \( D \) defined by
\[
D_{e_1} = \begin{bmatrix}
0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
D_{e_2} = \begin{bmatrix}
0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0
\end{bmatrix},
\]
\[
D_{e_3} = \begin{bmatrix}
0 & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} \\
0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0 \\
0 & 0 & \frac{1}{2} & 0 & -\frac{1}{\sqrt{2}} & 0
\end{bmatrix},
\]
\[
D_{e_4} = D_{e_5} = D_{e_6} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
is flat and Hermitian, so we can apply Theorem 4.1 and construct the tangent Lie algebra \( T_D \mathfrak{g} \) over \( \mathfrak{su}(2) \times \mathfrak{su}(2) \).

**References**

1. B. Alexandrov and S. Ivanov, *Vanishing theorems on Hermitian manifolds*, Differential Geom. Appl. **14** (2001), no. 3, 251–265.
2. M.L. Barberis, A. Fino, *New strong HKT manifolds arising from quaternionic representations*, Math. Z. **267** (2011), no. 3-4, 717–735.
3. L. Bérard-Bergery, *Les espaces homogènes riemanniens de dimension 4*, Riemannian geometry in dimension 4 (Paris, 1978/1979), Textes Math., vol. 3, CEDIC, 1981, pp. 40–60.
4. F. A. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), no. 1, 1–40.
5. C. Benson and C. S. Gordon, *Kähler and symplectic structures on nilmanifolds*, Topology **27** (1988), 513–518.
6. J. M. Bismut, *A local index theorem for non-Kähler manifolds*, Mathematische Annalen **284** (1989), no. 4, 681–699.
7. H. D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math. **81** (1985), no. 2, 369–372.
[8] B. Y. Chu, *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. **197** (1974), 145–159.
[9] I. G. Dotti and A. Fino, *HyperKähler torsion structures invariant by nilpotent Lie groups*, Classical Quantum Gravity **19** (2002), no. 3, 551–562.
[10] N. Enrietti, and A. Fino, *Special Hermitian metrics and Lie groups*, Preprint (2010).
[11] N. Enrietti, A. Fino, and L. Vezzoni, *Tamed symplectic forms and SKT metrics*, Preprint arXiv:1002.3099 (2011), to appear in J. Symplectic Geom.
[12] A. Fino and G. Grantcharov, *Properties of manifolds with skew-symmetric torsion and special holonomy*, Adv. Math. **189** (2004), no. 2, 439–450.
[13] A. Fino, M. Parton, and S. Salamon, *Families of strong KT structures in six dimensions*, Comment. Math. Helv. **79** (2002), no. 2, 317–340.
[14] A. Fino and A. Tomassini, *Non Kähler solvmanifolds with generalized Kähler structure*, J. Symplectic Geom. **7** (2009), no. 2, 1–14.
[15] P. Gauduchon, *La 1-forme de torsion d’une variété Hermitienne compacte*, Math. Ann. **267** (1984), 495–518.
[16] P. Gauduchon, *Hermitian connections and Dirac operators*, Boll. Un. Mat. Ital. B (7) **11** (1997), no. 2, suppl., 257–288.
[17] P. Gauduchon and S. Ivanov, *Einstein-Hermitian surfaces and Hermitian Einstein-Weyl structures in dimension 4*, Math. Z. **226** (1997), no. 2, 317–326.
[18] S. J. Gates, C. M. Hull, and M. Röcek, *Twisted multiplets and new supersymmetric nonlinear sigma models*, Nucl. Phys. B **248** (1984), 157–186.
[19] M. Gualtieri, *Generalized Kähler geometry*, Preprint arXiv:1007.3485v1 (2010).
[20] K. Hasegawa, *Minimal models of nilmanifolds*, Proc. Amer. Math. Soc. **106** (1989), no. 1, 65–71.
[21] T.-J. Li, W. Zhang, *Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds*, Comm. Anal. Geom., **17** (2009), no. 4, 651–683.
[22] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293–329.
[23] T. Madsen and A. Swann, *Invariant strong KT geometry on four-dimensional solvable Lie groups*, J. Lie Theory **21** (2011), no. 1, 055–070.
[24] A. Rezaei-Aghdam and M. Sephid, *Complex and biHermitian structures on four dimensional real Lie algebras*, J. Phys. A: Math. Theor. **43** (2010), no. 32, 325210.
[25] H. Samelson, *A class of complex-analytic manifolds*, Portugaliae Math. **12** (1953), 129–132.
[26] J. E. Snow, *Invariant complex structures on four-dimensional solvable real Lie groups*, Manuscripta Math. **66** (1990), no. 4, 397–412.
[27] J. Streets and G. Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Not. IMRN (2010), no. 16, 3101–3133.
[28] J. Streets and G. Tian, *Regularity results for pluriclosed flow*, Preprint arXiv:1008.2794v1 (2010).
[29] A. Strominger, *Superstrings with torsion*, Nuclear Phys. B **274** (1986), 253–284.
[30] A. Swann, *Twisting Hermitian and Hypercomplex geometries*, Duke Math. J. **155** (2010), no. 2, 403–431.
[31] L. Ugarte, *Hermitian structures on six-dimensional nilmanifolds*, Transform. Groups **12** (2007), no. 1, 175–202.
[32] H. Wang, *Complex parallisable manifolds*, Proc. Amer. Math. Soc. **5** (1954), 771–776.