Abstract—New quadrotor UAV control algorithms are developed, based on nonlinear surfaces composed of tracking errors that evolve directly on the nonlinear configuration manifold, thus inherently including in the control design the nonlinear characteristics of the SE(3) configuration space. In particular, geometric surface-based controllers are developed and are shown, through rigorous stability proofs, to have desirable almost global closed loop properties. For the first time in regards to the geometric literature, a region of attraction independent of the position error is identified and its effects are analyzed. The effectiveness of the proposed ‘surface based’ controllers are illustrated by simulations of aggressive maneuvers in the presence of disturbances and motor saturation.

I. INTRODUCTION

Quadrotor unmanned aerial vehicles are characterized by a simple mechanical structure comprised of two pairs of counter rotating outrunner motors where each one is driving a dedicated propeller, resulting in a platform with high thrust-to-weight ratio, able to achieve vertical takeoff and landing (VTOL) maneuvers and operate in a broad spectrum of flight scenarios. Quadrotors have good flight endurance characteristics and acceptable payload transporting potential for a plethora of applications. Although the quadrotor UAV has six degrees of freedom, it is underactuated since it has only four inputs and can only track four commands or less.

A plethora of theoretical and experimental works regarding quadrotors exist including results demonstrating aerobatic maneuvers [1], decentralized collision avoidance for multiple quadrotors [2], safe passage schemes satisfying constraints on velocities, accelerations, and inputs [3], backstepping control laws [4], and hybrid global/robust controllers [5], [6], [7].

This work follows the geometric framework. A geometric nonlinear control system (GNCS) for a quadrotor UAV is developed directly on the special Euclidean group, thus inherently including in the control design the characteristics of the nonlinear configuration manifold, and avoiding singularities and ambiguities associated with minimal attitude representations. The key contributions of this work are: (a) An attitude and a position controller is developed based on nonlinear surfaces composed by tracking errors that evolve directly on the nonlinear configuration manifold. These controllers allow for precision pose tracking by tuning three gains per controller and are able to follow an attitude tracking command and a position tracking command. (b) In contrast to other GNCSs such as like [1], [8] - [12], rigorous stability proofs are developed and regions of attraction both with and without restrictions on the initial position/velocity error are identified. A region of attraction independent of the initial position/velocity error is desired since it introduces simplicity in trajectory design. The proposed strategies are validated in simulation in the presence of motor saturation and wind disturbances.

II. QUADROTOR KINETICS MODEL

The quadrotor studied is comprised by two pairs of counter rotating out-runner motors, see Fig. 1. Each motor drives a dedicated propeller and generates thrust and torque normal to the plane produced by the centers of mass (CM) of the four rotors. An inertial reference frame $I_R\{E_1, E_2, E_3\}$ and a body-fixed frame $I_B\{e_1, e_2, e_3\}$ are employed with the origin of the latter to be located at the quadrotor CM, which belongs to the four rotor CM plane. Vectors $e_1$ and $e_2$ are co-linear with the two quadrotor legs , see Fig. 1.

The following apply throughout the paper. The actual control input is the thrust of each propeller, which is co-linear with $e_3$. The first and third propellers generate positive thrust when rotating clockwise, while the second and fourth propellers generate positive thrust when rotating counterclockwise. The magnitude of the total thrust is denoted by $f = \sum_{i=1}^{4} f_i \in \mathbb{R}$, where $f_i$ and other system variables are defined in Table I.
The motor torques, $\tau_i$, corresponding to each propeller are assumed to be proportional to thrust,

$$\tau_i = (-1)^i b_T f_i e_3, \quad i = 1, \ldots, 4$$  \hspace{1cm} (1)

where the $(-1)^i$ term connects each propeller with the correct rotation direction (clockwise and counterclockwise). The control inputs include the total propeller thrust $f$ and moment, $b u$, given by,

$$\begin{bmatrix} f \\ u \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & d & 0 & -d \\ -d & 0 & d & 0 \\ -b_T & b_T & -b_T & b_T \end{bmatrix} F, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$  \hspace{1cm} (2)

with $F \in \mathbb{R}^4$ the thrust vector, and the $4 \times 4$ matrix to be always full rank for $d, b_T \in \mathbb{R}^+$. The spatial configuration of the quadrotor UAV is described by the quadrotor attitude and the location of its center of mass, both with respect to $I_R$. The configuration manifold is the special Euclidean group $SE(3) = \mathbb{R}^3 \times SO(3)$. The total thrust produced by the propellers, in $I_R$, is given by $R f e_3$. The equations of motion of the quadrotor are given by,

$$\dot{x} = v$$

$$m \dot{v} = -mg E_3 + R f e_3 + \delta_e$$  \hspace{1cm} (3)

$$J \dot{\omega} = b u - \omega \times J \omega + \delta_R$$  \hspace{1cm} (4)

$$\dot{R} = R S(\delta R)$$  \hspace{1cm} (5)

where $\delta_e, \delta_R$ are disturbance terms and $S(\cdot) : \mathbb{R}^3 \rightarrow SO(3)$ is the cross product map given by,

$$S(r) = [0, -r_3, r_2, r_3, 0, -r_1, r_2, r_1, 0]$$

$$S^{-1}(S(r)) = r$$  \hspace{1cm} (6)

### III. QUADROTOR TRACKING CONTROLS

Given the underactuated nature of quadrotors, in this paper two flight modes are considered:

- **Attitude Control Mode**: The controller achieves tracking for the attitude of the quadrotor UAV.
- **Position Control Mode**: The controller achieves tracking for the quadrotor CM position and a pointing attitude associated with the quadrotor yaw.

Using these flight modes in suitable successions, a quadrotor can perform a complex desired flight maneuver. Moreover it will be shown that each mode has stability properties that allow the safe switching between flight modes (end of Section III).

### A. Attitude Control Mode (ACM)

An attitude control system able to follow an arbitrary smooth desired orientation $R_d(t) \in SO(3)$ and its associated angular velocity $\omega_d(t) \in \mathbb{R}^3$ is developed next under the assumption that $\delta_R = 0_{3 \times 1}$.

1) **Attitude tracking errors**: For a given tracking command $(R_d, \omega_d)$ and current attitude and angular velocity $(R, \omega)$, two sets of geometric attitude tracking errors are considered. Each set consists of an attitude error function $\Psi : SO(3) \times SO(3) \rightarrow \mathbb{R}$, and an attitude error vector $e_R \in \mathbb{R}^3$, defined as follows. The first set is, $[9]:$

$$\Psi(R, R_d) = \frac{1}{2} tr[R - R_d^T R] \geq 0$$  \hspace{1cm} (7)

$$e_R(R, R_d) = \frac{1}{2} S^{-1}(R_d^T R - R^T R_d)$$  \hspace{1cm} (8)

where $tr[\cdot]$ is the trace function. The second according to $[13]$: \hspace{1cm} $\Psi(R, R_d) = 2 - \sqrt{1 + tr[R_d^T R]} \geq 0$ \hspace{1cm} (9)

$$e_R(R, R_d) = \frac{1}{2} S^{-1}(R_d^T R - R^T R_d)(1 + tr[R_d^T R])^{-\frac{1}{2}}$$  \hspace{1cm} (10)

Both $(7), (9)$ yield the angular velocity error vector, $e_\omega \in \mathbb{R}^3$,

$$e_\omega(R, R_d, \omega, \omega_d) = b \omega - R^T R_d b \omega_d$$  \hspace{1cm} (11)

For the ACM, the controller is designed to be compatible with both sets of $e_R$. This is because the first set given by $(7), (8)$ bestows excellent tracking properties to the controller if the orientation tracking error remains less than $90^\circ$ wrt. an axis-angle rotation; however for an orientation error larger than $90^\circ$, the magnitude of the attitude error vector, $(8)$, is not proportional to the orientation error and results to deteriorating performance as the state approaches the antipodal equilibrium (see $[13]$ for more details). In contrast to this, the second set given by $(9), (10)$ does not suffer from this problem but is marginally outperformed by the first set if the attitude error is less than $90^\circ$. Thus depending on the flight conditions, the user can choose which set of attitude tracking errors to use.

Note that the maximum attitude difference, that of $180^\circ$ with respect to an equivalent axis-angle rotation between $R$ and $R_d$, occurs when the rotation matrices are antipodal; then $(7)$ or $(8)$ yield $\Psi(R, R_d)=2$, i.e. 100% error. If both rotation matrices express the same attitude i.e., $R=R_d$, then $\Psi(R, R_d)=0$, i.e. 0% error. Important properties regarding $(7)-(11)$, including the associated attitude error dynamics used throughout this work are included in Proposition 1 and Proposition 2 found in Appendix A.

2) **Attitude tracking controller**: A controller is developed stabilizing $e_R, e_\omega$, to zero exponentially, almost globally under the assumption that $\delta_R = 0_{3 \times 1}$.

**Proposition 3.** For $\eta, k_R, k_\omega \in \mathbb{R}^+$, with,

$$\eta > k_R / k_\omega^2$$  \hspace{1cm} (12)

and initial conditions satisfying,

$$\Psi(R(0), R_d(0)) < 2$$  \hspace{1cm} (13)

$$\|e_\omega(0)\|^2 < 2\eta k_R (2 - \Psi(R(0), R_d(0)))$$  \hspace{1cm} (14)
and for a desired arbitrary smooth attitude \( \mathbf{R}_d(t) \in \text{SO}(3) \) in,
\[
L_2 = \{ (\mathbf{R}, \mathbf{R}_d) \in \text{SO}(3) \times \text{SO}(3) | \Psi(\mathbf{R}, \mathbf{R}_d) < 2 \} \tag{15}
\]
then, under the assumption of perfect parameter knowledge, we propose the following nonlinear surface-based controller,
\[
b_u = b_\omega \times \mathbf{J}^k \omega - \mathbf{J} \left( \frac{k_R}{k_w} \mathbf{e}_R + a_d + \eta \mathbf{s}_R \right)
\tag{16}
\]
where \( a_d \) is defined in App. A(51) and the surface \( s_R \) is,
\[
s_R = k_R \mathbf{e}_R + k_v \mathbf{e}_v
\tag{17}
\]
Then, the zero equilibrium of the quadrotor closed loop attitude tracking error \( (\mathbf{e}_R, \mathbf{e}_v) = (0, 0) \) is almost globally exponentially stable; moreover there exist constants \( \mu, \tau > 0 \) such that
\[
\Psi(\mathbf{R}, \mathbf{R}_d) < \min \{ 2, \mu e^{-\tau t} \}
\tag{18}
\]
**Proof.** See Appendix [5].
The convergence properties introduced by \( s_R \) to the developed attitude controller are analyzed at the end of Section [11] with the developed position controller.

The initial angular velocity can be arbitrarily large by using sufficiently large gains. The region of attraction given by \( \mu \) ensures that the initial attitude error is less than 180° with respect to an axis-angle rotation for a desired \( \mathbf{R}_d \) (i.e., \( \mathbf{R}_d(t) \) is not antipodal to \( \mathbf{R}(t) \)). Consequently exponential stability is guaranteed almost globally. This is the best that one can do since it has been shown that the topology of \text{SO}(3) prohibits the design of a smooth global controller, [12].

Because \( \hat{b}_u \) is developed directly on \text{SO}(3), it avoids singularities and ambiguities associated with minimum attitude representations like Euler angles or quaternions completely. Also this controller can be applied to the attitude dynamics of any rigid body and not only on quadrotor systems.

Since attitude tracking does not depend on \( f \), the ACM is more suited for short durations of time. The thrust magnitude can be selected to achieve an additional objective compatible with the attitude tracking command, i.e. track a desired attitude command \[11, 8, 10].

Finally, despite developing \( \hat{b}_u \) under the assumption that \( \delta_R = 0 \times 1 \), its robustness properties will be tested during simulation in presence of motor saturation and wind disturbances.

### B. Position Control Mode (PCM)

Under the assumption that \( \delta_x = 0 \times 1 \), a control system is developed for the position dynamics of the quadrotor, stabilizing the tracking errors to zero asymptotically, almost globally.

1) **Position tracking errors:** For an arbitrary smooth position tracking instruction \( \mathbf{x}_d(t) \in \mathbb{R}^3 \), the tracking errors for the position and the velocity are taken as,
\[
\mathbf{e}_x = \mathbf{x} - \mathbf{x}_d, \quad \mathbf{e}_v = \mathbf{v} - \dot{\mathbf{x}}_d
\tag{19}
\]
For \( k_x, k_v \in \mathbb{R}^+ \) the position nonlinear surface is defined as,
\[
s_x = k_x \mathbf{e}_x + k_v \mathbf{e}_v
\tag{20}
\]
In the PCM, the attitude dynamics must be compatible with the desired position tracking. This results in the definition of a position-induced attitude matrix, \( \mathbf{R}_e(t) \in \text{SO}(3) \), for use as an attitude command. To define this matrix, first the desired thrust direction of the quadrotor, \( \mathbf{e}_z \), is computed by,
\[
e_{3z} = \frac{mg \mathbf{E}_3 - m k_x \mathbf{e}_e - a_s x - m \ddot{x}_d}{\| mg \mathbf{E}_3 - m k_x \mathbf{e}_e - a_s x - m \ddot{x}_d \|} \in \mathbb{S}^2, a \in \mathbb{R}^+
\tag{21}
\]
where it is assumed that by selecting \( x_d, \dot{x}_d, \ddot{x}_d \) hereafter,
\[
\| mg \mathbf{E}_3 - m k_x \mathbf{e}_e - a_s x - m \ddot{x}_d \| > 0
\]
Secondly the user defines a desired yaw direction \( \mathbf{e}_{1d} \in \mathbb{S}^2 \) of the \( e_1 \) body-fixed axis of the quadrotor such that \( \mathbf{e}_{1d} \not\parallel \mathbf{e}_{3z} \). This is used to find the position-induced heading, \( \mathbf{e}_{1h}, [8] \),
\[
\mathbf{e}_{1h} = \frac{(\mathbf{e}_{3z} \times \mathbf{e}_{1d}) \times \mathbf{e}_{3z}}{\| (\mathbf{e}_{3z} \times \mathbf{e}_{1d}) \times \mathbf{e}_{3z} \|}
\]
The position related attitude \( \mathbf{R}_e(t) \in \text{SO}(3), \hat{b}_w \omega_e(t) \in \mathbb{R}^3 \) is,
\[
\mathbf{R}_e = \begin{bmatrix} \mathbf{e}_{1h}, & \mathbf{e}_{3z} & \mathbf{e}_{1h} \end{bmatrix}, \quad \hat{b}_w \omega_e = S^{-1}(\mathbf{R}_e^T \dot{\mathbf{R}}_e) \tag{22}
\]
and the attitude dynamics are guided to follow \( \mathbf{R}_e(t), \hat{b}_w \omega_e(t) \).

2) **Position tracking controller:** Under the assumption that \( \delta_x = 0 \times 1 \), a control system is developed for the position dynamics of the quadrotor UAV, achieving almost global asymptotic stabilization of \( (\mathbf{e}_x, \mathbf{e}_v, \mathbf{e}_R, \mathbf{e}_w) \) to the zero equilibrium through the action/effect of the soon to be introduced Propositions 4 and 5.

For a sufficiently smooth pointing direction \( \mathbf{e}_{1d}(t) \in \mathbb{S}^2 \) and a sufficiently smooth position tracking instruction \( \mathbf{x}_d(t) \in \mathbb{R}^3 \) the following position controller is defined,
\[
f(\mathbf{x}_d, \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_d) = (mg \mathbf{E}_3 - m k_x \mathbf{e}_e - a_s x - m \ddot{x}_d)^T \mathbf{R}_e \mathbf{E}_3 \tag{23a}
\]
\[
b_u(\mathbf{R}_x, \hat{b}_w \omega_e) = b_w \omega_e \times \hat{b}_u - \mathbf{J} \hat{b}_u \mathbf{J} \left( \frac{k_R}{k_w} \mathbf{e}_R + a_d + \eta \mathbf{s}_R \right) \tag{23b}
\]
where \( s_R, a_d \), are given by \[17\], App. A(51), and \( \dot{e}_R \) is given by App. A(42) if \{17\}, \[8\] are used and is given by App. A(47) if \{19\}, \[10\] are used. The desired attitude matrix that is used in all the components of \( \hat{b}_u \) is given by \[22\].

The utilization of nonlinear surfaces resulted to the thrust feedback expression, \[23a\], comprised by three gains. However \[23a\] can be scaled to a PD form as in [1]. Since \[23a\] is paired with the newly developed attitude controller \[23b\], it forms a new PCM controller of improved closed loop response w.r.t. [1], see Sect. [V] and its behavior/closed-loop stabilization properties are investigated next.

The closed loop system defined by \[3\] under the action of \[23a\]-\[23b\] is shown to achieve almost global asymptotic stabilization of \( (\mathbf{e}_x, \mathbf{e}_v, \mathbf{e}_R, \mathbf{e}_w) \) to the zero equilibrium by the combined action of Propositions 4 and 5. Specifically \[23b\] drives \( \mathbf{R}(t) \) asymptotically track \( \mathbf{R}_e(t) \) and combined with \[23a\], asymptotic position tracking is achieved. The first result of exponential stability for a sub-domain of the quadrotor closed loop position dynamics is presented next.
We define
\[ x = \theta \] can be achieved by introducing bounds on the initial position/velocity error is less than \( 180 \) degrees.

The region of attraction in comparison to the regions in \([1], [8]\) is not antipodal to \( \Psi(0) \) and \( \Psi(x(0)) \).

Then two new regions of attraction are produced involving larger initial attitude errors and are given by \( \Psi(0), \Psi(x(0)) \leq \psi_p \leq \psi_p - 1 \)

and \( \psi_p < 1 \) and \( \psi_p - \Psi(0) \leq \psi_p \leq \psi_p - 1 \)

Therefore, the zero equilibrium of the closed loop error \( k_{R}, k_{w} \in \mathbb{R}^{+} \), such that,

\[ \lambda_{\text{min}}(W_{3}) > \frac{\|P_{2}\|^{2}}{4\mu k_{R} k_{w}} \]

then the zero equilibrium of the closed loop error \( \Psi(0) \leq \psi_p \leq \psi_p - 1 \) for the closed loop system is exponentially stable in the domain given by \( \Psi(0), \Psi(x(0)) \leq \psi_p \leq \psi_p - 1 \)

and \( \psi_p < 1 \) and \( \psi_p - \Psi(0) \leq \psi_p \leq \psi_p - 1 \)

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Finally, the proposition that follows shows that the structure of the position controller is characterized by almost global exponential attractiveness. This compensates for the reduced position/velocity free region of attraction and introduces greater freedom to the user in regards to control objectives, since the region of attraction does not depend explicitly on the initial position/velocity error. If the quadrotor initial states are outside of \([24]\), with respect to the initial attitude, Proposition 3 still applies due to the action of \( [23] \). Thus the attitude state enters \([24]\) in finite time \( t^{*} \). The result regarding the position mode is stated next.

Proposition 5. For initial conditions satisfying \( [14] \), and \( \psi_p < 1 \) and \( \psi_p - \Psi(0) \leq \psi_p \leq \psi_p - 1 \)

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is showcased in Fig. 2 where the quadrator response is shown during an attitude maneuver (Fig. 2a), and a position maneuver (Fig. 2b). In both cases, the same simulation is repeated but with larger gains \( \eta \), (a), resulting in faster reaching times, see black solid lines in Fig. 2a and 2b. In Fig. 2a by doubling \( \eta \), the reaching time from \( t_{s_R}=0.169 \) improves to \( t_{s_R}=0.099 \) and in Fig. 2b by increasing \( a \) by four, the reaching time from \( t_{s_x}=1.999 \) improves to \( t_{s_x}=0.569 \). As a result, the strict algebraic relation to the gains imposed by the proposed controller design, introduces "sliding like" closed loop dynamics, see description in Fig. 2 and allows for finer control on the convergence rate to the zero equilibrium by using the insights gained by the Lyapunov analysis. Also the sliding behavior is achieved here without the signum function; thus chattering is avoided.

To analyze GNCSs consisting of different structure and strategies, a criterion is needed for a commensurate comparison of their performance. To this end the Root-Mean-Square (RMS) of the thrusts is used as a criterion, given by,

\[
f_{\text{RMS}}(t) = \sqrt{\frac{1}{T} \int_{0}^{T} (f_i(t))^2 \, dt}
\]

Specifically we use (33) to calculate the RMS control effort difference, \( \Delta f_{\text{RMS}}(t) \), given by,

\[
\Delta f_{\text{RMS}}(t) = f_{\text{RMS}}^{\text{proposed}}(t) - f_{\text{RMS}}^{\text{benchmark}}(t)
\]

tune our developed GNCS such that (34) is negative during the simulation at all times so that the benchmark controller has equal or larger control authority. By comparing the controller performance, if the developed GNCS produces the least error with less control effort it is deemed superior. The system parameters were taken from a real quadrotor described in [15]:

\[
J = \begin{bmatrix} 0.0181, 0, 0; 0, 0.0196, 0; 0, 0, 0.0273 \end{bmatrix} kg m^2
\]

\[
m = 1.225 kg, d = 0.23 m, b_y = 0.0121 m
\]

and the actuator constraints, see [15], are given by:

\[
f_{i,\text{min}} = 0[N], f_{i,\text{max}} = 6.9939[N]
\]

The wind profile shown in Fig. 4d is used in conjunction with the drag equation, \( [16] \), with the drag coefficient and reference area matrices of the quadrotor to be given by,

\[
C_D=\text{diag}(0.2,0.22,0.5), A_D=\text{diag}(0.0907,0.0907,0.4004)m^2
\]

The torque due to wind is calculated by assuming that the disturbance force is applied at 0.04e3. Finally all simulations were conducted using fixed-step integration with \( dt=1\times10^{-3}s \).

A. Geometric-NCS comparison

For this comparison, the GNCS in [1] was selected as a benchmark since it is the first quadrotor GNCSs developed directly on SE(3), it demonstrates remarkable results in aggressive maneuvers, and to validate the claims of Sect. III-B2. The controllers use the first set of error vectors given by \( \{0, 0\} \), and no saturation/disturbances are included, to conclude controller competence. The gains were tuned using (34) as follows. First the attitude gains were tuned for a desired pitch command of 90° followed by tuning the position gains for a desired \( x_d=[1; 1; 1] \). Tuning the attitude controller first, ensures that during the PCM, the attitude controller embedded in the position control loop will produce homogeneous control effort. Also the gains must be compliant to (12), (28).

The developed controller gains are:

\[
k_\omega=150, k_R=5625, \eta=0.8
\]

\[
k_\nu=59.82, k_z=894.62, a=0.5071
\]

The benchmark controller \( [1] \) parameters used are:

\[
k_\omega = [2.1720, 0; 0; 2.3520, 0; 0, 3.2760]
\]

\[
k_R=[65.16, 0; 0; 70.56, 0; 0, 98.28], k_\nu=38.71, k_z=375.61
\]
The initial conditions (IC’s) are: \( x(0) = v(0) = \dot{\omega}(0) = 0_{3 \times 1}, R(0) = I \). The results are presented in Fig. 3.

Examing Fig. 3b the effectiveness of (16) (solid black line: 1) with respect to the benchmark controller (dashed blue line: 2) in performing attitude maneuvers is demonstrated as \( \Psi \) converges to zero faster and with less control effort, see Fig. 3a inner plot. The quadrotor response for a position command to \( x_d = [1; 1; 1][cm] \) is shown in Fig. 3c, 3d. Examining Fig. 3(d), it is clear that the developed position controller (23a, 23b) performs equally well with the benchmark controller. However the attitude error during the position maneuver is negotiated better by the developed position controller as \( \Psi \) converges to zero faster and with a smaller overall error, \( \Psi < 0.1198 \), an important prevalence. In Fig. 3a the value of, (34), is displayed for both the attitude (inner plot), and position (outer plot), maneuvers. Notice that the benchmark controller underperforms despite using more control effort, see Fig. 3a.

The flight scenario, to be achieved through the concatenation of the two flight modes, is described next:

(a) \((t < 4)\): Position Mode: Translation from the IC’s to \( x_d = [0; 1; 0], v_d = [0; 0; 7] \), \( e_{1d} = [1; 0; 0] \) using smooth polynomials of eighth degree (SP8th).

(b) \((4 \leq t < 4.4)\): Attitude Mode: The quadrotor performs a 180° pitch maneuver, i.e. goes inverted. \( R_d(t) \) was designed by defining the pitch angle using SP8th.

(c) \((4.4 \leq t < 4.9)\): Attitude Mode: The quadrotor recovers from its inverted state to \( R_d(t) = I \), i.e. point to point command.

(d) \((4.9 \leq t < 10)\): Position Mode: Translation to \( x_d = [-1; 1.5; 10] \), \( e_{1d} = [1; 0; 0] \) using SP8th with IC’s equal to the values of the states of the quadrotor at the end of the attitude mode.

Simulation results of the maneuver are illustrated in Fig. 4 where the duration that the attitude mode is utilized is illustrated by the magenta colored intervals. The percentage attitude error using (9) is shown in Fig. 4a. It is observed that up to \( t = 4.4 \), i.e. the beginning of the quadrotor recovery from the inverted position, the quadrotor attitude error is maintained below 5% (below 9° wrt. an axis-angle rotation). During the recovery interval \((4.4 < t < 4.9)\), despite the large attitude error of 77.64% introduced by the step command, the quadrotor successfully converges to the desired orientation undeterred by the disturbances due to wind and motor saturations, see Fig. 4c, 4d. The position response is shown in Fig. 4b. During the position mode, i.e. \( t < 4 \) and \( t > 4.9 \), the states track the reference trajectories effectively, see Fig. 4b, 4d. At the position mode interval, \( ||e_x|| \) (not shown here due to space) increases above 0.06m, to 0.5m, only between \( 3 < t < 4 \) where the wind increases rapidly, see Fig. 4b for the wind profile. The effect of the wind at the same interval is evident also by the noisy motor thrusts, see Fig. 4c at \( 3 < t < 4 \). A simulation conducted in the absence of wind, not shown due to space, showed that the noisy behavior in Fig. 4c is eradicated and \( ||e_x|| < 0.06 \) throughout the position mode interval. Concluding, the effectiveness of the proposed GNCSs in performing precise trajectory tracking maneuvers (attitude/position) and recovery maneuvers in the presence of motor saturations and disturbances was shown. The safe switching between flight modes, stated at the end of Section III-B was also demonstrated.

V. CONCLUSION AND FUTURE WORK

In this paper, new controllers for a quadrotor unmanned micro aerial vehicle were developed, based on nonlinear
surfaces and employing tracking errors that evolve directly on the nonlinear configuration manifold, inherently including in the control design the nonlinear characteristics of the SE(3) configuration space. Through rigorous stability proofs, the developed controllers were shown to have desirable closed loop properties that are almost global. A region of attraction, independent of the position error, was produced and analyzed for the first time, wrt. the geometric literature. The effectiveness of the developed GNCS was validated by numerical simulations of aggressive maneuvers, in the presence of motor saturations and disturbances due to wind.

Our future work will include experimental trials and an investigation of the developed GNCS robustness properties.

APPENDIX A

The attitude tracking errors associated with the attitude error functions studied in [9, 13], and related properties are summarized next.

**Proposition 1.** Employing \{7, 8\}, for a given tracking command \( \mathbf{R}_d \) and current attitude \( \mathbf{R} \), the following hold:

(i) \( \Psi \) is locally positive-definite about \( \mathbf{R} = \mathbf{R}_d \) and,

\[
\| e_R(\mathbf{R}, \mathbf{R}_d) \|^2 = (2 - \Psi(\mathbf{R}, \mathbf{R}_d))\Psi(\mathbf{R}, \mathbf{R}_d) \tag{35}
\]

(ii) A lower bound of \( \Psi \) is given as follows,

\[
\frac{1}{2} \| e_R(\mathbf{R}, \mathbf{R}_d) \|^2 \leq \Psi(\mathbf{R}, \mathbf{R}_d) \tag{36}
\]

(iii) Let \( \psi \in \mathbb{R}^+ \). If \( \Psi(\mathbf{R}, \mathbf{R}_d) < \psi < 2 \), then the upper bound of \( \Psi \) is given by,

\[
\Psi(\mathbf{R}, \mathbf{R}_d) \leq \frac{1}{2 - \psi} \| e_R(\mathbf{R}, \mathbf{R}_d) \|^2 \tag{37}
\]

(iv) The left-trivialized derivative of \( \Psi \) is given by,

\[
T^e_{\mathbf{L}_R}(\mathbf{D}_R \Psi(\mathbf{R}, \mathbf{R}_d)) = e_R \tag{38}
\]

(v) The critical points of \( \Psi \), where \( e_R = 0 \), are \( \{ \mathbf{R}_d \} \cap \{ \mathbf{R}_d \exp(\pi S(s)) | s \in \mathbb{S}^2 \} \).

As to \{9, 11\}, the attitude error vector is well defined in [15]. Thus for a tracking command \( \mathbf{R}_d \) and current attitude \( \mathbf{R} \),

(vi) \( \Psi \) is locally positive-definite about \( \mathbf{R} = \mathbf{R}_d \).

(vii) In [15] the left-trivialized derivative of \( \Psi \) is given by,

\[
T^e_{\mathbf{L}_R}(\mathbf{D}_R \Psi(\mathbf{R}, \mathbf{R}_d)) = e_R \tag{39}
\]

(viii) The critical points of \( \Psi \), where \( e_R = 0 \), are \( \{ \mathbf{R}_d \} \cap \{ \mathbf{R}_d \exp(\pi S(s)) | s \in \mathbb{S}^2 \} \) and there exists only one critical point \( \{ \mathbf{R}_d \} \) in [15].

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Proof of Proposition 1. See [9] for statements (i)-(v). See [13] for statements (vi)-(ix).

The associated attitude error dynamics of (7)-(11) to be used in the subsequent control design are given next.

Proposition 2. The error dynamics of (7)-(11) satisfy:

\[ \dot{\Psi}(R, R_d) = e^T_R e_\omega \]  \hspace{1cm} (41)

\[ \dot{e}_R = E(R, R_d) e_\omega \]  \hspace{1cm} (42)

\[ E(R, R_d) = \frac{1}{2} \{ tr[R^T R_d] I - R^T R_d \} \]  \hspace{1cm} (43)

\[ \|E(R, R_d)\| \leq 1 \]  \hspace{1cm} (44)

\[ \|\dot{e}_R\| \leq \frac{1}{2} \|e_\omega\| \]  \hspace{1cm} (45)

Employing (41), (42), the following hold:

\[ \dot{\Psi}(R, R_d) = e^T_R e_\omega \]  \hspace{1cm} (46)

\[ \dot{e}_R = E(R, R_d) e_\omega \]  \hspace{1cm} (47)

\[ E(R, R_d) = \frac{\{ tr[R^T R_d] I - R^T R_d + 2 e_R e^T_R \}}{2(1 + tr[R^T_d R])} \]  \hspace{1cm} (48)

\[ \|\dot{e}_R\| \leq \frac{1}{2} \|e_\omega\| \]  \hspace{1cm} (49)

The time derivative of (11) is given by,

\[ \dot{e}_\omega = b_\omega a_d \]  \hspace{1cm} (50)

\[ = J^{-1}(b_\omega b_\omega \times J^b) + a_d \]  \hspace{1cm} (51)

\[ a_d = S^b b_\omega R^T R_d b_\omega - R^T R_d b_\omega d \]  \hspace{1cm} (52)

Proof of Proposition 2. See [10], [9], for (41)-(45). See [13], for (46), (49). See [13], or [9], for (50), (51).

Appendix B

Proof of Proposition 3. We employ a sliding methodology in (15) by defining the nonlinear surface in terms of the attitude configuration errors (7), (8) or (9), (10) and apply Lyapunov analysis.

(a) Lyapunov candidate: We define,

\[ V = \frac{1}{2k_\omega} s^T_R s_R + 2\eta k_R k_\omega \Psi \]  \hspace{1cm} (53)

\[ \text{Differentiating } (52) \text{ and substituting } (16) \text{ we get,} \]

\[ V = -\eta k_R k_\omega z_R W_3 z_R, \quad W_3 = \begin{bmatrix} k^2_R & 0 \\ 0 & k^2_\omega \end{bmatrix} \]  \hspace{1cm} (54)

where \( z_R = [\|e_R\|; \|e_\omega\|] \).

(b) Boundedness of \( \Psi(R, R_d) \): We define the Lyapunov function,

\[ V = \frac{1}{2} e^T_R e_\omega + \eta k_R \Psi \]  \hspace{1cm} (55)

\[ V \leq (\eta k_R - \frac{k^2_R}{k_\omega}) \|e_\omega\|^2 \leq 0 \]  \hspace{1cm} (56)

Equations (54), (55) imply that \( V'(t) \leq V(t) \), \( V(t) \geq 0 \).

Applying (14) we obtain,

\[ \eta k_R \Psi(R(t), R_d(t)) \leq V(t) \leq V(0) < 2\eta k_R \psi_a \]  \hspace{1cm} (57)

implying that the attitude error function is bounded by,

\[ \Psi(R(t), R_d(t)) \leq \psi_a < 2, \forall t \geq 0 \]  \hspace{1cm} (58)

Proof of Proposition 4. A sliding methodology is utilized through the definition of the surface in terms of the error vectors defined in (19), followed by Lyapunov analysis. The position mode necessitates analysis of the coupled attitude and position dynamics. Thus the preceding analysis of the attitude mode, is utilized here to characterize the properties of the closed loop system under the action of the controllers with the difference that \( R_d(t) \) is substituted with \( R_x(t) \). This is because differentiation of the Lyapunov function \( V \) in (53), parametrized by \( R_x \), gives the same result for \( V \) as in (53) and thus it can be considered in (53).

(a) Boundedness of \( e_R(R, R_x) \): The assumptions of Proposition 4 imply compliance to Proposition 3 by replacing \( R_d \) with \( R_x \). Thus the properties of (16) still apply in this analysis. Resultantly by replacing \( R_d \) with \( R_x \), (55) still holds and equation (29) in (54) leads to,

\[ \eta k_R \Psi(R(t), R_x(t)) \leq V(t) \leq V(0) < 2\eta k_R \psi_a \]  \hspace{1cm} (59)
signifying that the attitude error function is bounded by,
\[ \Psi(R(t), R_x(t)) \leq \psi_p < 1, \forall t \geq 0 \quad (68) \]
(b) Position Error Dynamics: The analysis that follows is developed in the following domain,
\[ D = \{ (e_x, e_v, e_R, e_w) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \} \]
\[ \Psi(R, R_x) \leq \psi_p < 1 \quad (69) \]

**Proposition A.** For initial conditions in (69), the cosine function is given by \((R_x e_3)^T R e_3 \) and the following holds,
\[ (R_x e_3)^T R e_3 \geq 1 - \Psi(R, R_x) > 0 \quad (70) \]

The sine of the angle between \(R e_3\) and \(R_x e_3\) is given by \((R_x e_3)^T R e_3 - R_x e_3\) and using (35),
\[ \|(R_x e_3)^T R e_3\| R e_3 - R_x e_3 \| \leq \Pi e_3 \quad (71) \]
where for \(\{7, 8\}\) it holds that,
\[ \|e_R\| = \sqrt{\Psi(2 - \Psi)} \leq \sqrt{\Psi_p (2 - \Psi)} = \theta < 1 \quad (72) \]

while for \(\{9, 10\}\), (see (13),
\[ \|e_R\| = \sqrt{\Psi(1 - \Psi/4)} \leq \sqrt{\Psi_p (1 - \Psi/4)} = \theta < 1 \quad (73) \]

**Proof of Proposition A.** See [10], [9].

Equation (70) is used by adding and subtracting \((R_x e_3)^T R e_3\) to obtain,
\[ m x = -m \frac{k_z}{k_v^2} e_v - a x + X + m \dot{x}_d \quad (74) \]
where \(f \in \mathbb{R}, \ X \in \mathbb{R}^3 \) are given by,
\[ f = \|U\| (R_x e_3)^T R e_3 \quad (75) \]
\[ X = \|U\| \left( (R_x e_3)^T R e_3 - R_x e_3 \right) \quad (76) \]
\[ U = m g E_3 - m k_x \frac{k_z}{k_v^2} e_v - a x + m \dot{x}_d \quad (77) \]

Then by taking the time derivative of (19), the error dynamics of \(e_v\) are given by,
\[ m \dot{e}_v = -m \frac{k_z}{k_v^2} e_v - a x + X \quad (78) \]

(c) Translational dynamics Lyapunov candidate: We define,
\[ V_x = \frac{m}{2k_v} s_x^T s_x + a k_x k_v e_v^T e_v \quad (79) \]

Differentiating (79) and substituting (78) we get,
\[ \dot{V}_x = s_x^T (-a x + X) + 2a k_x k_v e_v^T e_v \quad (80) \]

Using (71)/(73), a bound of \(X\) is given by,
\[ \|X\| \leq (B + (a k_x + m k_x^2) \|e_v\| + a k_x \|e_x\|) \|e_R\| \]
\[ \leq (B + (a k_x + m k_x^2) \|e_v\| + a k_x \|e_x\|) \|e_R\| \theta \quad (81) \]

Defining \(z_x = ||e_x||; ||e_v||\), using (81) in (80) we arrive,
\[ \dot{V}_x \leq -z_x^T \Pi_1 z_x + z_x^T \Pi_2 z_R \quad (82) \]
and by (27), \(\Pi_1\) is positive definite.

(d) Lyapunov candidate for the complete system: We define,
\[ \dot{V}_g = V_z + V \quad (83) \]

and using (86)/(37) or (40), (83) is bounded as follows,
\[ z_R^T W_1 z_R + z_R^T \Pi_3 z_x \leq \dot{V}_g \leq z_R^T W_1 z_R + z_R^T \Pi_3 z_x \quad (84) \]
\[ \Pi_3 = \left[ ak_x k_v + \frac{m k_z^2}{k_v} \right] \Pi_1 \left[ \frac{a k_x k_v + m k_z^2}{k_v} \right] \Pi_1 \quad (85) \]

and both \(\Pi_3, \Pi_4\) matrices are positive definite. By replacing \(R_x\) with \(R_x\) and differentiating we arrive again in (53). Using (53) and (82) the derivative of (83) is,
\[ \dot{V}_g \leq -z_x^T \Pi_1 z_x + z_x^T \Pi_2 z_R - \eta z_x W_3 z_R \quad (86) \]

Moreover (28) ensures that (86) is negative definite. Thus the zero equilibrium of the tracking errors of the complete system dynamics is exponentially stable in (24). A region of attraction is given by the domain (24), and (29).

(f) Alternative regions of exponential stability: The Lyapunov analysis above was developed in (24) without restrictions on the initial position/velocity error. This resulted to a complicated Lyapunov analysis and a reduced region of exponential stability. Instead if we restrict our analysis to,
\[ D_p = \{ (e_x, e_v, e_R, e_w) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \} \]
\[ \Psi(0) \psi_p < 1, \|e_v\| < \theta, \|e_x(0)\| < e_{\text{max}} \} \quad (87) \]

and bound the third order error terms that arise during the analysis using \(e_{\text{max}}\) then (24), is given by
\[ \Pi_1 = \left[ ak_x^2 (1 - \theta) \right] \quad (88) \]
\[ \Pi_2 = \left[ \frac{B k_x}{ak_x + \frac{m k_z^2}{k_v}} e_{\text{max}} \right] \quad (89) \]

Alternatively a restriction on the initial velocity error results to domain,
\[ D_v = \{ (e_x, e_v, e_R, e_w) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \} \]
\[ \Psi(0) \psi_p < 1, \|e_v\| < \theta, \|e_v(0)\| < e_{v_{\text{max}}} \} \quad (90) \]

then similarly using \(e_{v_{\text{max}}}\) to bound the third order error terms, \(\Pi_1\) is given by (88) and (89) changes to,
\[ \Pi_2 = \left[ \frac{B k_x}{2ak_x + \frac{m k_z^2}{k_v}} e_{v_{\text{max}}} \right] \quad (91) \]

were in both cases (27) is given by
\[ \theta < \min \left( \frac{a k_x^2}{ak_x^2 + m k_z^2} \right) \quad (92) \]

Note that the Lyapunov analysis continues in the same manner as in Appendix C with (88), (89), (92) (corresponding to (87)) and (88), (91), (92) (corresponding to
being utilized instead of (26), (27). It should be noted that (22) signifies a larger basin than (27) but a restriction on the initial position/velocity error is introduced and this might not be desirable in some instances.