Quantum particle in Milne space

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Abstract. We present simple model of a quantum particle in the Milne space. Background spacetime includes one compact space dimension undergoing contraction to a point followed by expansion. Our model offers some insight into the nature of the cosmic singularity.

1. Introduction
The cosmological data strongly suggest that our universe emerged from a state with extremely high density of physical fields called the cosmic singularity [1]. We believe that the cosmic singularity problem is inseparable from the problem of dark energy, which seems to be the most fundamental problem of contemporary physics (see, e.g. [2]). It is attractive to assume that the singularity consists of contraction and expansion phases. This way one avoids the problem of creation of spacetime, matter, fields from ‘nothing’, and opens door for the cyclic universe models [3, 4]. The Milne space, considered recently within string/M theory scheme [5], is the simplest spacetime modelling the ‘big-crunch/big-bang’ type singularity.

Any reasonable model of the cosmic singularity should be able to describe quantum propagation of an elementary object (e.g. particle, string, membrane,..., gravitational waves) from the pre-singularity to post-singularity epoch. It is the most elementary criterion that should be satisfied. Some insight into the problem may be already achieved by studying dynamics of a test particle in low dimensional spacetime.

Results of this paper concern evolution of a particle in the two-dimensional compactified Milne space. We present three models of evolution of a particle across the singularity. Quantization is carried out by finding a self-adjoint representation of the algebra of particle’s observables. We draw conclusions, indicate improvements and suggest next steps in the last section. For extended version of this paper we recommend [6].

2. Compactified Milne space
The visualization of compactified Milne space may be presented by the isometric embedding of 2d Milne into 3d Minkowski space as follows

\[ y^0(t, \theta) = t \sqrt{1 + r^2} \]
\[ y^1(t, \theta) = rt \sin(\theta/r) \]
\[ y^2(t, \theta) = rt \cos(\theta/r) \]

and one has

\[ \frac{r^2}{1 + r^2} (y^0)^2 - (y^1)^2 - (y^2)^2 = 0, \quad r \in \mathbb{R}. \]
The induced metric on Eq.(2) \((t \neq 0)\) reads
\[ ds^2 = -dt^2 + t^2 d\theta^2, \]  
(3)
where \((t, \theta) \in \mathbb{R}^1 \times S^1\). Eq.(2) presents two cones with a common vertex at \((y^0, y^1, y^2) = (0, 0, 0)\). One term in the metric disappears/appears at \(t = 0\), thus Milne space may be used to model big-crunch/big-bang type singularity. The Milne space is locally isometric with the Minkowski space at each point, but at the vertex \(t = 0\).

3. Dynamics of a particle
Action integral of particle with mass \(m\) in compactified Milne space reads
\[ \mathcal{A} = \int d\tau L(\tau), \quad L(\tau) = \frac{m}{2e} (t^2 \dot{\theta}^2 - \dot{t}^2 - e^2) \]  
(4)
where \(\tau\) is an evolution parameter, \(e(\tau)\) denotes the 'einbein' on the world-line and 'dot' means \(d/d\tau\).

The action (4) is invariant under reparametrization with respect to \(\tau\). This gauge symmetry leads to the constraint
\[ \Phi := (p_t/t)^2 - (p_t)^2 + m^2 = 0, \]  
(5)
where \(p_t := \partial L/\partial \dot{t}\) and \(p_\theta := \partial L/\partial \dot{\theta}\) are canonical momenta. Variational principle applied to (4) gives the equations of motion of a particle
\[ \frac{d}{d\tau} p_t - \frac{\partial L}{\partial t} = 0, \quad \frac{d}{d\tau} p_\theta = 0, \quad \frac{\partial L}{\partial e} = 0. \]  
(6)
Since \(p_\theta\) is not well defined at \(t = 0\), the constraint and equations of motion are not well defined at the singularity \((S)\). However, particle’s dynamics is well defined in pre-singularity \((S_\downarrow)\) and post-singularity phases \((S_\uparrow)\). A particle can go across \(S\), since trajectories of a test particle coincide (by definition) with time-like geodesics and there is no obstacle for geodesics to reach \(S\). At the vertex \(t = 0\), there is the Cauchy problem because the \(\theta\)-dimension disappears for a moment. As the result, there are many possible particle evolutions in Milne space due to the ambiguity in passage through \(S\). In what follows we present three examples.

Since the Milne space has three Killing vector fields, one can find three dynamical integrals of a particle \(I_1, I_2\) and \(I_3\)
\[ I_1 = p_t \cosh \theta - p_\theta \frac{\sinh \theta}{t}, \quad I_2 = p_t \sinh \theta - p_\theta \frac{\cosh \theta}{t}, \quad I_3 = p_\theta. \]  
(7)
Observables of a particle are defined to be the dynamical integrals. Space of all particle geodesics defines the phase space. Solution to equations of motion (6) in \(S_\downarrow\) and \(S_\uparrow\) reads
\[ \theta(t) = -\sinh^{-1} \left( \frac{I_3}{mt} \right) + \tanh^{-1} \left( \frac{I_2}{I_1} \right) =: -\sinh^{-1} \left( \frac{c_1}{mt} \right) + c_2. \]  
(8)
Observables in terms of ‘basic’ observables \(c_1\) and \(c_2\) read
\[ I_1 = -m \cosh(c_2), \quad I_2 = -m \sinh(c_2), \quad I_3 = c_1. \]  
(9)
One may verify that observables satisfy the \(iso(1, 1)\) Lie algebra
\[ \{I_1, I_2\} = 0, \quad \{I_3, I_2\} = I_1, \quad \{I_3, I_1\} = I_2, \quad \{\cdot, \cdot\} := \frac{\partial \cdot}{\partial c_1} \frac{\partial}{\partial c_2} - \frac{\partial \cdot}{\partial c_2} \frac{\partial}{\partial c_1}. \]  
(10)
Since \(\{c_1, c_2\} = 1\), the variables \(c_1\) (‘momentum’) and \(c_2\) (‘position’) may be used as canonical coordinates.
4. Quantization

In what follows, by quantization we mean finding an (essentially) self-adjoint representation of the algebra of basic observables of a particle.

4.1. Simple propagation across singularity

Here we consider the following propagation: particle following spiral geodesics winding clockwise the cone \( S_1 \) continues to move along clockwise spirals in \( S_T \) (the same concerns propagation along anticlockwise spirals). Obviously, for \( p_\theta = 0 \) particle trajectories are just straight lines both in \( S_1 \) and \( S_T \). This type of propagation is similar to particle propagation in 2d de Sitter space embedded in 3d Minkowski space \([7, 8]\). The topology of phase space and the algebra of basic observables, respectively, are

\[
\Gamma = S^1 \times \mathbb{R}^3, \quad \{c_1, c_2\} = 1. \tag{11}
\]

To carry out quantization of the system, we make the redefinition \( U_2 := \exp i c_2 \) and introduce the Lie bracket

\[
<\cdot, \cdot> := \left( \frac{\partial \cdot \partial}{\partial c_1 \partial U_2} - \frac{\partial \cdot \partial}{\partial U_2 \partial c_1} \right) U_2. \tag{12}
\]

The redefinitions lead to \(<c_1, U_2> = U_2\). The reason for introducing such redefinitions is that the self-adjoint Schrödinger representation of the algebra (11) cannot exist \([9]\) in case \((c_1, c_2) \in \mathbb{R}^1 \times S^1\). It was discovered (see \([10, 11]\) and references therein) that in case canonical variables have non-trivial topology the above changes may lead to a self-adjoint representation. We have found that in our case the representation can be defined by the following mapping

\[
c_1 \rightarrow c_1 \psi(\beta) := -i \frac{d}{d\beta} \psi(\beta), \quad U_2 \rightarrow U_2 \psi(\beta) := e^{i \beta} \psi(\beta), \quad \psi \in \Omega_\lambda, \tag{13}\]

where the domain space \( \Omega_\lambda \) reads

\[
\Omega_\lambda := \{\psi \in L^2(S^1) \mid \psi \in C^\infty[0, 2\pi], \ \psi^{(n)}(2\pi) = e^{i \lambda} \psi^{(n)}(0)\}. \tag{14}\]

One may verify that the representation is essentially self-adjoint on \( \Omega_\lambda \).

4.2. Constrained propagation across singularity

Now we take into account that \( S_1 \) and \( S_T \) have the (clockwise and anticlockwise) rotational symmetry quite independently. Apart from this we assume that the singularity \( S \) may ‘change’ the clockwise type geodesics into anticlockwise ones, and vice-versa. From mathematical point of view such case is allowed because at \( S \) the space dimension disappears, thus \( p_\theta \) is not well defined there, so it may have different signs in \( S_1 \) and \( S_T \). Therefore, the space of geodesics has ‘reflection’ type of symmetry independently in \( S_1 \) and \( S_T \). The last symmetry is of discrete type, so it is not the isometry of the compactified Milne space. The phase space \( \Gamma \) has now the topology \( S^1 \times \mathbb{R}^1 \times S^1 \times \mathbb{Z}_2 \). The algebra of basic observables reads

\[
\langle c_1, U_2 \rangle = U_2, \quad \langle c_1, U_3 \rangle = \varepsilon U_3, \quad \langle U_2, U_3 \rangle = 0, \quad \varepsilon = \pm 1, \tag{15}\]

where

\[
\langle \cdot, \cdot \rangle := \left( \frac{\partial \cdot \partial}{\partial c_1 \partial U_2} - \frac{\partial \cdot \partial}{\partial U_2 \partial c_1} \right) U_2 + \left( \frac{\partial \cdot \partial}{\partial c_1 \partial U_3} - \frac{\partial \cdot \partial}{\partial U_3 \partial c_1} \right) U_3. \tag{16}\]

Mapping of observables into operators is defined by

\[
c_1 \rightarrow \hat{c}_1 \psi(\beta) f_\varepsilon \varphi(\alpha) := -i \frac{d}{d\beta} \psi(\beta) f_\varepsilon \varphi(\alpha), \quad \hat{c}_1 \psi(\beta) f_\varepsilon \varphi(\alpha) := \varepsilon f_\varepsilon \psi(\beta) f_\varepsilon \varphi(\alpha), \quad \varepsilon = \pm 1, \tag{17}\]

\[\hat{c}_1 \psi(\beta) f_\varepsilon \varphi(\alpha) := \varepsilon f_\varepsilon \psi(\beta) f_\varepsilon \varphi(\alpha), \quad \varepsilon = \pm 1, \tag{17}\]
\[
U_2 \rightarrow \hat{U}_2 \psi(\beta)f_\epsilon \varphi(\alpha) := e^{i\beta \psi(\beta)f_\epsilon \varphi(\alpha)}, \\
U_3 \rightarrow \hat{U}_3 \psi(\beta)f_\epsilon \varphi(\alpha) := e^{i\beta \epsilon \psi(\beta)f_\epsilon \varphi(\alpha)} := e^{i\beta \epsilon \psi(\beta)f_\epsilon e^{i\alpha} \varphi(\alpha)},
\]
where the domain space is \(\Omega_\lambda \otimes E \otimes \Omega_\lambda\) (\(E\) denotes Hilbert space spanned by eigenvectors of \(\hat{\epsilon}\)). One may check that the representation is essentially self-adjoint.

4.3. Free propagation across singularity

Finally, we assume that there is no connection at all between trajectories in the upper and lower parts of the Milne space. For instance, spiral type geodesic winding the cone in \(S^1\) may be ‘turned’ by \(S\) into straight line in \(S^1\), and vice-versa. In addition we propose that either in \(S^1\) or in \(S^1\) (but not in both) \(p_0\) may equals zero. Obviously, the present case also includes transitions of spiral geodesics into spiral ones, and straight line into straight line geodesics. Now, the topology of phase space reads \(\Gamma = (S^1 \times \mathbb{R}^1) \times (S^1 \times \mathbb{R}^1)\). The algebra of basic observables is defined by

\[
\{c_1, c_2\} = 1, \quad \{c_1, c_3\} = 1, \quad \{c_i, c_j\} = 0, \quad i = 1, 2; \quad j = 3, 4
\]

where

\[
\{\cdot, \cdot\} := \frac{\partial}{\partial c_1} \frac{\partial}{\partial c_2} + \frac{\partial}{\partial c_3} \frac{\partial}{\partial c_3} - \frac{\partial}{\partial c_2} \frac{\partial}{\partial c_1} - \frac{\partial}{\partial c_3} \frac{\partial}{\partial c_4}.
\]

Redefinitions (of non-vanishing brackets) lead to

\[
\langle c_1, U_2 \rangle = U_2, \quad \langle c_4, U_3 \rangle = U_3.
\]

Quantization of the algebra of basic observables reads

\[
\hat{c}_1 \psi(\beta) \varphi(\alpha) := -i \frac{d}{d\beta} \psi(\beta) \varphi(\alpha), \quad \hat{U}_2 \psi(\beta) \varphi(\alpha) := e^{i\beta} \psi(\beta) \varphi(\alpha),
\]
\[
\hat{c}_1 \psi(\beta) \varphi(\alpha) := \psi(\beta) \left( -i \frac{d}{d\alpha} \right) \varphi(\alpha), \quad \hat{U}_3 \psi(\beta) \varphi(\alpha) := \psi(\beta) e^{i\alpha} \varphi(\alpha).
\]

The representation is essentially self-adjoint on \(\Omega_\lambda \otimes \Omega_\lambda\).

4.4. Ambiguity of quantization

Since \(\Omega_\lambda\) is labelled by \(0 \leq \lambda < 2\pi\), there are \(\infty\)-many unitarily non-equivalent representations. It means that there are \(\infty\)-many quantum systems corresponding to one classical system. Such quantization would not make much sense. Fortunately, one can remove this ambiguity by imposition of time (\(T\)) reversal invariance upon the algebra of observables [8].

In case of propagation considered in subsection A the algebra is time-reversal invariant because

\[
\hat{T} \hat{c}_1 \hat{T}^{-1} = -\hat{c}_1, \quad \hat{T} \hat{U}_2 \hat{T}^{-1} = \hat{U}_2^{-1},
\]

where \(\hat{T}\) denotes an anti-unitary operator corresponding to the transformation \(T\). The first equation in (25) results from the correspondence principle between classical and quantum physics, because \(c_1\) has interpretation of momentum of a particle. The assumed form of \(\hat{U}_2\) and anti-unitarity of \(\hat{T}\) lead to the second equation in (25). The formal reasoning at the level of operators should be completed by the corresponding one at the level of the domain space \(\Omega_\lambda\) of the algebra. It is clear that \(\Omega_\lambda\) may be spanned by eigenfunctions of \(\hat{c}_1\), i.e.

\[
f_{m,\lambda}(\beta) := (2\pi)^{-1/2} \exp i\beta(m + \lambda/2\pi), \quad m = 0, \pm 1, \pm 2, \ldots.
\]

Following step-by-step the method of the imposition of the \(\hat{T}\)-invariance upon dynamics of a test particle in de Sitter’s space, presented in Sec.(4.3) of [8], leads to the result: \(\lambda = 0\) for ‘bosons’ and \(\lambda = \pi\) for ‘fermions’. One can apply the same method (with some small modification) to remove the ambiguity of quantization considered in the remaining two subsections.
5. Conclusions

Milne space represents simple, but non-trivial model of the universe with big-crunch/big-bang type singularity. Propagation of the quantum test particle is well defined mathematically (up to the Cauchy problem at \( t = 0 \)). We have analysed three ways of particle’s transition across the singularity, but more cases are possible. Many situations should be taken into account, because at the singularity the equations defining classical dynamics are not well defined. Owing to the Cauchy problem, the singularity acts as a ‘generator of uncertainty’ in particle’s propagation from pre- to post-singularity era.

Extension of our model to higher dimensional Milne space may be carried out (as it was done in case of de Sitter space [12]). However, the result would be basically similar to the one presented in the present paper, in the context of revealing the nature of the cosmic singularity, due to the Cauchy problem at \( t = 0 \) which cannot be avoid in any dimension. Much more promising would be replacement of a test particle by a physical one. Real particle may collapse into a black hole at the singularity, modify the spacetime there, or both. Work is in progress.

Our model concerns point-like objects. Next step would be examination of dynamics of extended objects like strings or membranes. It was recently shown [14, 13] that a test string in the zero-mode state twisted around the shrinking dimension propagates smoothly and uniquely across the Milne space singularity. However, for drawing firm conclusions about the physics of the problem one should also examine the non-zero string modes and go beyond the semi-classical approximation. It would be also very interesting to examine propagation of gravitational waves in the compactified Milne space. The upcoming experiments to detect gravity waves (see, e.g. [15]) may bring soon new observational data on the very early universe calling for interpretation. Finally, one should extend analysis to generalizations of the Milne space [16, 17, 18].

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