Chiral stabilization of the renormalization group for flavor and color anisotropic current interactions

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Abstract

We propose an all-orders $\beta$eta function for current-current interactions in 2d with flavor anisotropy. When the number of left-moving and right-moving flavors are unequal, the $\beta$eta function has a non-trivial fixed point at finite values of the couplings. We also extend the computation to simple cases with both flavor and color anisotropy.
I. INTRODUCTION

In [1] an all-orders beta function was proposed for general Lie group $G$ anisotropic current-current interactions in two dimensions. Non-perturbative aspects of the resulting RG flows were studied in ref. 2 a strong-weak coupling duality. These beta functions generally have no non-trivial zeros, i.e. fixed points, at intermediate values of the couplings between 0 and $\infty$. Rather, interesting fixed points arise if one is attracted under renormalization group (RG) flow to sub-manifolds corresponding to a sub-group $H$ of $G$, the fixed point being the current-algebra coset $G/H$ [3] as $g \to \infty$.

Referring to $G$ anisotropy as color anisotropy, in this paper we consider anisotropy in the number of copies, or flavors, of the current algebra. The simplest form of flavor anisotropy corresponds to unequal numbers of left versus right-moving flavors $N_L$ and $N_R$. Unequal numbers of chiral flavors is equivalent to having unequal levels $k_L, k_R$ for the current algebras in the left versus right sector. Based on the work of Polyakov-Wiegmann [4], one expects a non-trivial fixed point of the beta function as $N_L, N_R \to \infty$ with $N_R - N_L$ fixed which corresponds to the WZW model at level $k = N_R - N_L$. The generalization of this fixed point when $N_L, N_R$ are not $\infty$ was proposed in [5] [6], where the phenomenon was referred to as chiral stabilization. In ref. [5] general arguments based on the preservation of the difference between left-right Virasoro central charges $c_R - c_L$ in the RG flow led to a precise identification of the infra-red (IR) fixed point as a chirally asymmetric coset; this identification was further supported by a thermodynamic Bethe ansatz analysis for the integrable cases. Evidence for these fixed points based on the beta function were not given in [4] presumably because the fixed point cannot be seen at one-loop.

We first extend the computation of the beta function in [1] to models with flavor anisotropy but isotropic in color. For equal numbers of left and right-moving flavors we do not find any non-trivial fixed points for the examples we have looked at. For unequal numbers of left-right flavors, we find the fixed points predicted in [4] [5]. The anomalous dimension of the perturbing operator in the infra-red (IR) is a strong test of the proposed beta function. We find that the beta function gives the expected result except for the well-known shift of the level $k$ by the dual Coxeter number in the affine-Sugawara construction [7]. We remark on the possible sources of this discrepancy but do not resolve it in this paper.

We also compute the beta function for color anisotropic models where the only flavor anisotropy corresponds to unequal numbers of left versus right movers, or equivalently unequal levels $k_L, k_R$. This corresponds to the models in [1] with the difference that the left and right moving levels of the current algebra are unequal. Here also we find that this chiral anisotropy stabilizes the RG flow in that it leads to fixed points at intermediate values of the coupling that are generally on color isotropic manifolds.

In the next section we propose an exact formula for the most general form of flavor anisotropy that preserves the color symmetry. In section III we consider the simplest possible chirally asymmetric case of one left-moving flavor and two right moving flavors. The most generic anisotropic model flows to a fixed point that is isotropic in the flavor couplings, the only remaining anisotropy being in the number of flavors. The model then flows to the fixed points of the kind described in [4] [5]. In section IV we extend the computation to the special case of both flavor and color anisotropy mentioned above, namely unequal levels $k_L, k_R$. Here the chirality stabilizes the flow in two ways, namely the flow to the isotropic
II. GENERAL FLAVOR ANISOTROPY

Let us denote by \( \mathcal{G}_k \) the \( \mathcal{G} \) current algebra of level \( k \) and \( S_{\mathcal{G}_k} \) the conformal WZW model with the current algebra symmetry. This theory possesses left and right-moving currents \( J^a(z) \) and \( \overline{J}^{\alpha}(\overline{z}) \) satisfying the operator product expansion (OPE)

\[
J^a(z) J^b(0) = \frac{k}{z^2} \eta^{ab} + \frac{1}{z} f^{ab}_c J^c(0) + ....
\]

and similarly for \( \overline{J} \), with \( a = 1,..\text{dim}(\mathcal{G}) \). For ordinary (bosonic) algebras we take \( \eta^{ab} = \delta^{ab}/2 \) corresponding to a normalization in the defining representation of \( tr(t^a t^b) = \delta^{ab}/2 \). For realizations based on free fermions \( J^a = \psi^\dagger t^a \psi \), the level \( k = 1 \). We will need the Casimir in the adjoint representation defined by \( \eta_{ij} f^{jk}_d = C_{adj} \delta^c_d \), where \( \eta_{ab} \delta^{bc} = \delta^a_c \).

We consider models with a number of flavors. Let \( J^a_\alpha, \alpha = 1,2,..N_L, \) and \( \overline{J}^a_{\overline{\alpha}} \) denote the resulting currents where \( \alpha, \overline{\alpha} \) are the flavor indices, \( a \) are \( \mathcal{G} \) (color) indices, and in general the number of left-moving flavors \( N_L \) is not equal to \( N_R \). The perturbations of the conformal field theory we will study can be defined by the action

\[
S = S_{G_1}^{N_L,N_R} + \int \frac{d^2 x}{2\pi} \sum_{\alpha,\overline{\alpha},a} g_{\alpha\overline{\alpha}} J^a_\alpha \overline{J}^a_{\overline{\alpha}}
\]

where \( S_{G_1}^{N_L,N_R} \) is the formal action for the WZW model at level 1 with \( N_L, N_R \) numbers of chiral flavors. The above theory preserves the \( \mathcal{G} \) symmetry but introduces some anisotropy in the flavor couplings \( g_{\alpha\overline{\alpha}} \) which comprise a \( N_L \times N_R \) matrix.

The computation in [1] is easily extended to this model. Operator product expansions involving the color indices give \( C_{\text{adj}} \) at each order, and one then only has to keep track of the delta functions \( \delta^{\alpha_1 \alpha_2} \) over the flavor indices. Let 1A, 1B, 2A, 2B denote the 4 kinds of diagrams described in [1]. Define

\[
G = \frac{k^2}{16} g g^T, \quad \overline{G} = \frac{k^2}{16} g^T g
\]

where \( T \) denotes transpose. \( G \) (\( \overline{G} \)) is a \( N_L \times N_L \) (\( N_R \times N_R \)) matrix. As such, \( G^a g \) is a \( N_L \times N_R \) matrix, etc. The contributions to the \( \beta \) function from the various diagrams at \( n \)-th order in perturbation theory are

\[
\beta^{(2A)}_{g_{\alpha\overline{\alpha}}} = \frac{C_{\text{adj}}}{2} \left( G^{(n-m-1)/2} g \right)_{\alpha\overline{\alpha}} \left( G^{(m-1)/2} \right)_{\alpha\overline{\alpha}}
\]

\[
\beta^{(2B)}_{g_{\alpha\overline{\alpha}}} = \frac{C_{\text{adj}}}{2} \left( \frac{k}{4} \right)^2 g_{\alpha\overline{\alpha}} \left( G^{(m-3)/2} g \right)_{\gamma\overline{\gamma}} \left( G^{(n-m-1)/2} g \right)_{\gamma\overline{\gamma}} g_{\gamma\overline{\gamma}}
\]

\[
\beta^{(1A)}_{g_{\alpha\overline{\alpha}}} = -\frac{C_{\text{adj}}}{2} \left( \frac{k}{4} \right)^{-1} \left( G^{(m-1)/2} g \right)_{\alpha\gamma} \left( G^{(n-m)/2} g \right)_{\overline{\gamma}\overline{\alpha}} + g_{\alpha\overline{\gamma}} \left( \overline{G}^{(n-m)/2} g \right)_{\overline{\gamma}\overline{\alpha}}
\]

\[
\beta^{(1B)}_{g_{\alpha\overline{\alpha}}} = -\frac{C_{\text{adj}}}{2} \left( \frac{k}{4} \right)^{-1} \left( G^{(n-1)/2} g \right)_{\alpha\alpha} + \left( \overline{G}^{(n-1)/2} g \right)_{\overline{\alpha}\overline{\alpha}}
\]
(The \( \gamma, \bar{\gamma} \) indices are summed over.) The type 2A,B (1A,B) are for \( n \) even (odd) order, and in all cases \( m \) is odd. For type 2A, \( m = 1, 3, ..., n - 1 \) whereas for 2B \( m = 3, 5, ..., n - 1 \). For type 1A, \( m = 3, 5, ..., n - 2 \).

The series is easily summed with the following result. Define
\[
g_1' = \frac{1}{1-G}, \quad G' = \frac{G}{1-G}, \quad G' = \frac{G}{1-G}
\]
(2.8)

Then
\[
\beta_{g, \bar{\alpha}} = \frac{C_{\text{adj}}}{2} \left( (g_{\alpha \bar{\alpha}}')^2 + \frac{k^2}{16} g_{\alpha \bar{\gamma}}(g_{\gamma \bar{\gamma}}')^2 g_{\gamma \bar{\gamma}} - \frac{4}{k} \left( (G'_{\alpha \gamma})^2 g_{\gamma \bar{\gamma}} + g_{\alpha \gamma}(G'_{\gamma \bar{\gamma}})^2 + (G'_{\alpha \alpha} + G'_{\alpha \bar{\alpha}}) g_{\alpha \bar{\alpha}} \right) \right)
\]
(2.9)

(In the above formula \((g_{\alpha \bar{\alpha}}')^2\) is the square of the matrix element \(g_{\alpha \bar{\alpha}}'\) rather than the \(\alpha \bar{\alpha}\) matrix element of \(g'^2\), and similarly for the \(G\) terms.)

The resulting expressions for the \(\beta\)eta function are too complicated to display here even in the case of 2 flavors. What is more important is that for the cases we have examined, the above \(\beta\)eta function with \(N_L = N_R\) has no non-trivial fixed points at intermediate values of the couplings. We expect then that when \(N_L = N_R\) the only fixed points are the trivial ones at \(g = 0\) or \(\infty\) as for the one-flavor case.

### III. CHIRAL FLAVOR ANISOTROPY

Consider the simplest case with unequal numbers of flavors, \(N_L = 1, N_R = 2, k = 1\), defined by the action
\[
S = S_{g_1}^{N_L=1,N_R=2} + \int \frac{d^2x}{2\pi} \left( g_1 J_1^a J_1^a + g_2 J_2^a J_2^a \right)
\]
(3.1)
The result eq. (2.9) reduces to

\[
\beta_{g_1} = \frac{C_{\text{adj}}}{2} \frac{2g_1(g_1 - 4)(4g_1(g_1 - 4) - g_2^2(g_2 - g_1 - 4))}{(g_1^2 + g_2^2 - 16)^2}
\]
\[
\beta_{g_2} = \frac{\beta_{g_1}}{g_1 \rightarrow g_2, g_2 \rightarrow g_1}
\]

The fixed points are \(g_1 = g_2 = g\) with \(g = g_c = 2, 4\). We computed the \(\beta\)eta function for one other example with \(N_L = 1, N_R = 3\). The only fixed points for all positive \(g\) here are \(g_{11} = g_{12} = 4/k, 2/k\) with \(g_{13} = 0\), which is equivalent to the previous example with \(N_R = 2\), and \(g_{11} = g_{12} = g_{13} = 4/k, 4/3k\). Based on these examples, we expect that one generally flows to the isotropic flavor manifold with \(g_{\alpha \bar{\alpha}} = g\) for all \(\alpha, \bar{\alpha}\) or for a decoupled subset.

When \(g_{\alpha \bar{\alpha}} = g\) for all \(\alpha, \bar{\alpha}\), the theory can be reformulated as
\[
S = S_{g_{kL} \otimes g_{kR}} + \int \frac{d^2x}{2\pi} \ g J^a J^a
\]
(3.3)
where the left-moving currents \(J^a = \sum_{\alpha} J^a_\alpha\) have level \(k_L = N_L k\) and the right moving currents have level \(k_R = N_R k\), and \(S_{g_{kL} \otimes g_{kR}}\) is the formal action for the WZW model at these levels. The \(\beta\)eta function then reduces to
\[
\beta_g = \frac{C_{\text{adj}} g^2 (1 - k_L g/4)(1 - k_R g/4)}{(1 - k_L k_R g^2/16)^2}
\] (3.4)

This beta function has two fixed points at \( g = 4/k_L \) and \( 4/k_R \). It also has two poles at \( g = \pm 4/\sqrt{k_L k_R} \), however RG flows encounter the zeros before reaching these poles. Thus the fixed point exists within the perturbative domain. When \( k_L = k_R \) the only fixed points are at \( g = 0, \infty \). If \( k_R > k_L \) and \( C_{\text{adj}} > 0 \) then the RG flows from small coupling to \( g_c = 4/k_R \).

At the IR fixed point we expect on general grounds that

\[
\beta_g = (2 - \Gamma_{IR})(g - g_c) + \ldots
\] (3.5)

where \( \Gamma_{IR} \) is the dimension of the irrelevant operator that the perturbation \( g J \bar{J} \) flows to in the IR. We find

\[
\Gamma_{IR} = 2 - \partial_g \beta_g (g_c) = 2 \left( 1 + \frac{C_{\text{adj}}}{k_R - k_L} \right)
\] (3.6)

In ref. [5] thermodynamic Bethe ansatz analysis and other more general arguments based on the equality of \( c_R - c_L \) and \( k_R - k_L \) in the UV and IR indicate that the flow is to the chirally asymmetric coset:

\[
G_{k_L} \otimes G_{k_R} \rightarrow \left( \frac{G_{k_L} \otimes G_{k_R - k_L}}{G_{k_R}} \right)_L \otimes (G_{k_{R - k_L}})_R
\] (3.7)

The flow arrives at the IR fixed point via the primary operator \( \phi_{\text{adj}} \) of the level \( k = k_R - k_L \) current algebra with dimension \( \Gamma_{IR} = 2(1 + C_{\text{adj}}/(k_R - k_L + \tilde{h})) \) where \( \tilde{h} \) is the dual Coxeter number of \( G \). (\( \tilde{h} = N \) for \( \text{su}(N) \).) Note that when \( k_L, k_R \to \infty \) but \( k_R - k_L \) is finite one recovers the results in [4]. Comparing with (3.6) we see that our beta function finds this fixed point except for the shift of \( k \) by the dual Coxeter number. The origin of this discrepancy is unclear, and for the remainder of this section we only discuss the possible origins of it.

The numerous tests of the beta function performed in [4] strongly indicate that the result is correct for when \( k_L = k_R = 1 \). In particular, the case of imaginary potential sine-Gordon theory has a fixed point both in the ultra-violet and infra-red, and the anomalous dimensions at both points as computed from the beta function agree with exact Bethe ansatz calculations. The dependence of \( \Gamma_{IR} \) on the difference \( k_R - k_L \) relies on all the higher order contributions to the beta function; keeping only the two-loop contribution would give \( \Gamma_{IR} = 2(1 + C_{\text{adj}}/(k_R + k_L)) \). This suggests that the beta function in [1] is not just the leading terms in \( 1/k_L, 1/k_R \).

The shift by the dual Coxeter number is the same shift that appears in the affine-Sugawara construction: \( T = J^a J^a/(k + \tilde{h}) \) which arises from proper normal-ordering [4] of the product of two currents. So one possibility is that there are additional corrections to the beta function coming from the normal-ordering of \( J \bar{J} \). However this seems unlikely since such corrections would appear at level 1. The spectrum of primary fields in the current algebra depends on the level \( k \) [3]. For example for \( \text{su}(2) \) at level \( k = 1 \) the field \( \phi_{\text{adj}} \) is not even in the spectrum and the correct result is \( \Gamma_{IR} = 4 \) corresponding to a perturbation \( T \bar{T} \) where \( T \) is the stress tensor. Since \( \Gamma_{IR} \) is not a universal function but rather takes
different forms depending on $k$ and reflecting the fusion rules, for the beta function to yield this result it would have to know about the spectrum of primary fields and would have to possess several zeros depending on the value of $k$. In the beta function was computed using the formula

$$
\langle X \rangle = F(g, \log a) \int \frac{d^2x}{2\pi} \langle J^\beta(x)X \rangle
$$

(3.8)

where $F$ is a function of the ultra-violet cutoff $a$ and $X$ is an arbitrary field or product of fields. The divergences found in are independent of which field $X$ appears in the above equation. Thus another possibility is that there are additional wavefunction renormalizations that depend on what the field $X$ is. This resolution would have the opportunity to depend intricately on the level $k$ since the spectrum of $X$ depends on $k$.

IV. COLOR AND FLAVOR ANISOTROPY

The most general model with flavor and color anisotropy corresponds to a perturbation

$$
\sum_A g_{A\alpha}^\alpha d_{ab}^A J_a^\alpha J_b^\beta
$$

where $g_{A\alpha}^\alpha$ are couplings and $d_{ab}^A$ are fixed quadratic forms that characterize the color anisotropy. It is possible to find the beta function for this case however the formulas are rather cumbersome. In this section we limit ourselves to $g_{A\alpha}^\alpha = g_A$ for all $\alpha, \bar{\alpha}$. As in the previous section the theory is then defined by

$$
S = S_{Gi_L \otimes G_{k_R}} + \int \frac{d^2x}{2\pi} \sum_A g_A d_{ab}^A J_a^\alpha J_b^\beta
$$

(4.1)

where $J^a (J^\beta)$ has level $k_L = N_L k$ ($k_R = N_R k$). The beta function in is easily generalized to the case $k_L \neq k_R$ by keeping track of the $k$ terms in $JJ$ versus $JJ$ OPE’s. As in define the RG data $C, \tilde{C}, D$ in terms of the OPE’s

$$
\mathcal{O}_A(z, \bar{z})\mathcal{O}_B(0) \sim \frac{1}{z^2} \sum_C C_{C}^{AB} \mathcal{O}_C(0)
$$

(4.2)

$$
T^A(z)\mathcal{O}_B(0) \sim \frac{1}{z^2} \left( 2k_L D_{C}^{AB} + \tilde{C}_{C}^{AB} \right) \mathcal{O}_C(0)
$$

(4.3)

where $T^A = d_{ab}^A J^a J^b$. Let $D(g)$ be the matrix of couplings

$$
D(g)_A^B = \frac{\sqrt{k_L k_R}}{2} \sum_C g_C D_{B}^{AC}
$$

(4.4)

and let

1This is actually reminiscent of the dependence of the phase structure of Yang-Mills theory on the numbers of chiral flavors and colors. See e.g. [9] [10].
\[ g' = g(1 - D^2(g))^{-1} \] (4.5)

Then the result is
\[
\beta_g = -\frac{1}{2} C(g', g')(1 + D^2) + \frac{k_L + k_R}{2\sqrt{k_L k_R}} \left( C(g'D, g'D)D - \tilde{C}(g'D, g) \right)
\] (4.6)

As an example we work out the \(su(2)\) case. Let us normalize the currents as
\[
J_3(z)J_3(0) \sim \frac{k_L}{2} \frac{1}{z^2}, \quad J_5(z)J^\pm(0) \sim \pm \frac{1}{z} J^\pm(0), \quad J^+(z)J^-(0) \sim \frac{k_L}{2} \frac{1}{z^2} + \frac{1}{z} J_3(0)
\] (4.7)

and consider the action
\[
S = S_{su(2)_L \otimes su(2)_R} + \int \frac{d^2x}{2\pi} \left( g_1(J^+\mathcal{J} - J^-\mathcal{J}) + g_2 J_3\mathcal{J}_3 \right)
\] (4.8)

The RG data was computed in \([1]\): \(C_{11}^{12} = C_{11}^{21} = -1, C_{21}^{11} = -2, \tilde{C}_{11}^{11} = \tilde{C}_{21}^{11} = 1, \tilde{C}_{22}^{12} = 2, D_{11}^{11} = D_{22}^{22} = 1/2\). The result is
\[
\beta_{g_1} = \frac{16g_1 (k_L k_R g_1^2 g_2 + 16g_2 - 2(k_L + k_R)(g_1^2 + g_2^2))}{(k_L k_R g_2^2 - 16)(k_L k_R g_1^2 - 16)}
\]
\[
\beta_{g_2} = \frac{16g_1^2 (k_R g_2 - 4)(k_L g_2 - 4)}{(k_L k_R g_1^2 - 16)^2}
\] (4.9)

The zeros of the above \(\beta\)eta function are (i) \(g_1 = g_2 = 4/k_R, 4/k_L\) and (ii) \(g_1 = -g_2 = 4/k_R, 4/k_L\). Thus one can flow to the \(su(2)\) color isotropic manifolds \(g_1 = \pm g_2\) and then to the fixed points of the kind described in section III. When \(k_R = k_L\) there are regions that are attracted to the isotropic manifolds, but then flow off to \(\infty\) \([2]\) away from the isotropic line. Here, one instead reaches the fixed point before reaching the pole, and the flow is again chirally stabilized.

V. DISCUSSION

When one of the levels \(k_L\) or \(k_R\) equals zero, the model is related to the important problem of the Kondo lattice \([3]\). Thus some of the models we are considering here can be viewed as intermediate between bulk and boundary perturbations. It would be interesting to generalize our computation of the \(\beta\) function to the case of purely boundary interactions, such as in the Kondo model. This could have some applications to the problem of tachyon condensation in string theory \([11]\).

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