The Curse and Blessing of Not-All-Equal in k-Satisfiability

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Abstract

We study upper bounds for the running time of algorithms for NAE-$k$-SAT and MAX-NAE-$k$-SAT approximation, as functions of $k$, the number of variables $n$ and the performance ratio $\delta$. For the first time, deterministic NAE-$k$-SAT algorithm is faster than the best $k$-SAT algorithm. The analysis relies on Linear Programming. However we do not know such improvement for randomized algorithms. As for approximation algorithms, we show that a number of MAX-$k$-SAT algorithms do not apply to MAX-NAE-$k$-SAT, including the current best one. We present a better MAX-NAE-$k$-SAT approximation algorithm, which is even faster than the best MAX-$k$-SAT approximation algorithm in a wide range of $\delta$ when $k = 3$. We also provide a tighter analysis on an existing MAX-$k$-SAT approximation algorithm, and generalize it for MAX-NAE-$k$-SAT, which is currently the best for all $k \geq 4$.

1 Introduction

Recently, there has been a growing interest in the research of NAE-$k$-SAT [SSZ16, CDL+16, GS17, XZZ+17, DMO+18]. As natural variants of the well-known $k$-SAT and MAX-$k$-SAT problems, NAE-$k$-SAT and MAX-NAE-$k$-SAT have intimate relationships to other NP-hard problems such as $k$-Coloring, Hypergraph Coloring, MAX-CUT, MAX-DI-CUT and MAX-XOR-2-SAT [GHS02]. Literally, NAE-$k$-SAT requires at least one true literal and at least one false literal in every clause for satisfaction. In the context of random satisfiability, such symmetry can be extremely convenient [ACIM01, DSS14, SSZ16]. However, whether the Not-All-Equal predicate plays a positive role in the study of worst-case upper bounds remains open.

In this paper, we study the exponential upper bounds of solving NAE-$k$-SAT exactly and approximately (MAX-NAE-$k$-SAT approximation). The latter problem focuses on the running time to achieve certain performance ratio (the ratio of the number of satisfying clauses and that under the optimal solution). The research in this area mainly concentrates on two separated lines: exponential-time exact algorithms (Table 1) and polynomial-time approximations (Table 2). Therefore, another reason motivating this work is to mingle both lines by designing exponential-time algorithm to achieve performance ratio beyond the inapproximable threshold of polynomial-time algorithms.

Why exponential upper bounds? Traditionally, problems in $P$ are well-studied. But due to the widely believed Exponential-Time Hypothesis (ETH), polynomial-time algorithms do not exist for $k$-SAT, let alone MAX-$k$-SAT and MAX-NAE-$k$-SAT [IP01]. So understanding the quality of exponential-time algorithms is important, which essentially tells us which problem is less intractable. As a strong evidence, lots of conceptual breakthroughs have been made by continuing improvements of faster exponential-time algorithms for $k$-SAT [Sch99, PPSZ05, MS11, Her14, Liu18]. In the seminar work of Håstad, it is shown that a polynomial-time algorithm with performance ratio greater than 0.875 would imply $P = NP$ (see Table 2).

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3.12573 | 1.32793 | 1.33334 | 1.30704
4.149706 | 1.49857 | 1.50001 | 1.46895
5.159888 | 1.59946 | 1.60001 | 1.56943
6.166624 | 1.66646 | 1.66667 | 1.63788

Table 1: The rounded up base $c$ in the upper bound $c^n$ of our deterministic algorithm for NAE-$k$-SAT and the corresponding upper bounds in previous results and currently fastest randomized algorithm (last column). Since a NAE-$k$-SAT instance is equivalent to a $k$-SAT instance with pairs of opposite-polarity clauses, any $k$-SAT algorithm can solve NAE-$k$-SAT within the same time in terms of the number of variables (omit the polynomial factor).

| $k$ | Lower bound | Upper bound |
|-----|--------------|-------------|
| 2   | 0.940 [LLZ02] | 0.955 [Hås01] |
| 3   | 0.875 [Zwi02] | 0.875 [Hås01] |
| 4   | 0.872 [HZ01]  | 0.875 [Hås01] |
| 5   | 0.8434 [ABZ05]| 0.875 [Hås01] |
| 6   | 0.878 [GW95]  | 0.917 [TSSW00] |
| 3   | 0.878 [Zwi98] | 0.978 [Zwi98] |
| 6   | 0.8279 [ABZ05]| 0.875 [Hås01] |

Table 2: The lower bound denotes the current best performance ratio of polynomial-time approximation algorithm; the upper bound denotes the inapproximable threshold unless $P = NP$. MAX-E-NAE-$3$-SAT restricts each clause to have exactly three literals. Assuming the Unique Game Conjecture, most upper bounds in this table can be improved (see [Aus07, Yos11] for details).

In fact, approximating MAX-SAT beyond the limitation of the current best polynomial-time algorithm remains an important open problem, and that beyond the inapproximable threshold is NP-hard [ALM98]. One should mention that there is a Linear PCP Conjecture that would imply the non-existence of a sub-exponential time algorithm for MAX-$3$-SAT with performance ratio greater than 0.875, assuming ETH [KW11]. We also refer our readers to [Kho02, MR08] for detailed surveys regarding this topic. In a word, improving the exponential upper bounds for MAX-$k$-SAT and MAX-NAE-$k$-SAT approximations is crucial as it is for $k$-SAT and NAE-$k$-SAT.

**Blessing of NAE**

Intuitively, NAE-$k$-SAT should be easier than $k$-SAT since the instance is more constrained. As an evidence, the other more constrained variants called X-SAT (exactly 1 true literal in each clause) and X-$3$-SAT can be solved in time $O(1.18^n)$ and $O(1.11^n)$ respectively [BMS05], which are much better than the fastest $k$-SAT algorithms (see Table 1). Another somewhat counterintuitive witness can be that a certain type of NAE-$3$-SAT is even in $P$ [Mor88]. However, before this work, it is unknown whether NAE-$k$-SAT algorithm has an upper bound better than $k$-SAT algorithm. It turns out that the Not-All-Equal property can be used to reduce the searching space of both branching algorithms and derandomized local search, which we vividly call it the “blessing of NAE”, leading to a better upper bound for NAE-$k$-SAT algorithm (see §4 for details). The second blessing of NAE derives from the fact that any clause in a NAE-$3$-SAT instance can be represented by a degree-2 polynomial with 3 variables, which surprisingly turns into an algorithm for MAX-NAE-$3$-SAT that runs in time $\text{poly}(n) \cdot 2^{n\omega/3}$, where $\omega$ is the matrix product exponent. Note that a length-3 clause in a 3-SAT instance does not have this property [Wil07].
Figure 3: Comparison results on MAX-NAE-$k$-SAT approximation. The $x$-axis is the performance ratio $\delta$ and the $y$-axis is the rounded up base $c$ in the upper bound $c^\delta$. “EPT(MAX-SAT)” is currently the fastest for MAX-$k$-SAT approximation, but it does not apply to MAX-NAE-$k$-SAT approximation (see §3.3). Before this work, “Hirsch” is the fastest algorithm for MAX-NAE-$k$-SAT approximation (see Theorem 3).

**Curse of NAE** Considering a clause in a $k$-SAT instance, whenever a literal is assigned to true, we immediately know that this clause must be satisfied no matter whatever values of other literals are. However, this is not the case in a NAE-$k$-SAT instance since we need at least two literals with different values. This is annoying because a common and powerful method in solving this kind of problem is fixing the assignments of a subset of variables to eliminate some clauses or reduce the length of the clauses by at least one, which simplifies the formula. It turns out that such curse ruins a number of existing MAX-$k$-SAT approximation algorithms when applying to MAX-NAE-$k$-SAT. We will discuss the details in §3.

The rest of the paper is organized as follows. The notations are presented in §2. In the related work in §3, we introduce: (i) why the current best randomized algorithm for $k$-SAT does not have a better upper bound for NAE-$k$-SAT; (ii) why the blessing of NAE induces a better upper bound for a deterministic algorithm; (iii) a Matrix Multiplication-based exact algorithm for MAX-NAE-(≤ 3)-SAT; (iv) a Random Walk-based algorithm for MAX-NAE-$k$-SAT approximation; (v) why the current best MAT-SAT approximation algorithms apply to MAX-$k$-SAT but not to MAX-NAE-$k$-SAT (curse of NAE). Our result on deterministic
algorithm for NAE-$k$-SAT (Theorem 7) is presented in §4, whose comparison results on $3 \leq k \leq 6$ are in Table 1. Later in §5, we present our approximation algorithms, whose upper bounds are formally stated in Theorem 14 and Theorem 15. Further remarks on them are in §5.3 and some numerical results regarding these can be found in Figure 3. Finally, we conclude this paper and provide some future work in §6.

2 Preliminaries

Let $V = \{v_i | i \in [n]\}$ be a set of $n$ Boolean variables. For all $i \in [n]$, a literal $l_i$ is either $v_i$ or $\bar{v}_i$. A clause $C$ is a set of literals and an instance $F$ is a set of clauses. The occurrence of a variable $v$ in $F$ is the total number of $v$ and $\bar{v}$ in $F$. A $k$-clause is a clause consisting of exactly $k$ literals, and $a \leq k$-clause consists of at most $k$ literals. If every clause in $F$ is a $\leq k$-clause, then $F$ is a $k$-instance. An assignment $\alpha$ of $F$ is a mapping from $V$ to $\{0, 1\}^n$. A partial assignment is the mapping restricted on $V'$ such that only variables in $V'$ are assigned. A clause $C$ is said to be satisfied by $\alpha$ if $\alpha$ assigns at least one literal to 1 in $C$, and is said to be NAE-satisfied if $\alpha$ assigns at least one literal to 1 and at least one literal to 0 in $C$. $F$ is satisfiable (resp. NAE-satisfiable) iff there exists an $\alpha$ satisfying (resp. NAE-satisfying) all clauses in $F$, and we call such $\alpha$ a satisfying assignment (resp. NAE-satisfying assignment) of $F$. The $k$-SAT problem asks to find a satisfying assignment of a given $k$-instance, and the NAE-$k$-SAT problems asks to find a NAE-satisfying assignment of it. If the context is clear, we will drop the prefix “NAE-”.

The MAX-SAT problem asks to find an assignment $\alpha$ of an instance $F$, such that the number of satisfied clauses in $F$ under $\alpha$ is maximized. If $F$ is a $k$-instance, this problem is called MAX-$k$-SAT. Similarly we can define the MAX-NAE-$k$-SAT problem, which is to find an assignment that NAE-satisfies the maximum number of clauses in a $k$-instance. By definition, a 1-clause can never be NAE-satisfied, thus when solving the NAE-$k$-SAT or MAX-NAE-$k$-SAT approximation problem for a $k$-instance $F$, we can safely assume that there is no 1-clause in $F$.

Given two assignments $\alpha, \alpha^* \in \{0, 1\}^n$, the Hamming distance $d(\alpha, \alpha^*) = \|\alpha - \alpha^*\|_1$ is the number of bits $\alpha$ and $\alpha^*$ disagree. Given instance $F$, let $s(\alpha)$ be the number of satisfied clauses in $F$ under assignment $\alpha$. The optimal assignment $\alpha^* = \arg \max_\alpha s(\alpha)$ maximizes the number of satisfied clauses in $F$. We call $\alpha$ a $\delta$-approximation assignment if $s(\alpha)/s(\alpha^*) \geq \delta$, then $\alpha^*$ is used to denote such $\alpha$. Suppose for any $k$-instance on $n$ variables, algorithm $\mathcal{A}$ outputs some $\alpha^$ (deterministically or with high probability), then $\mathcal{A}$ has performance ratio $\delta$ and $\mathcal{A}$ is a $\delta$-approximation algorithm for MAX-$k$-SAT. Further, if $\mathcal{A}$ runs in $T(n)$ time, we say that MAX-$k$-SAT has a $T(n)$-time $\delta$-approximation. Similar definitions work for MAX-NAE-$k$-SAT.

Throughout the paper, $n$ and $m$ always denote the number of variables and number of clauses in the instance respectively and assume $m = \text{poly}(n)$. Let $m_i$ be the number of $i$-clauses in a $k$-instance for all $i \in [k]$, then $m = \sum_{i \in [k]} m_i$. Define the average clause length of $F$ as $\eta = (\sum_{i \in [k]} i \cdot m_i)/m$. Random always stands for uniformly at random, and w.p. or w.h.p. stands for with probability or with high probability. We use $O^*(T(n)) = \text{poly}(n) \cdot T(n)$ to omit some polynomial factor of $n$ and use $O(T(n)) = 2^{o(n)} \cdot T(n)$ to omit sub-exponential factors of $n$.

3 Related Work

We summarize some related work in this section and show exactly where the curse and blessing of NAE are.

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1 For (NAE)-$k$-SAT, there are at most $2^k \cdot \binom{n}{\leq k}$ distinct clauses in a $k$-instance. MAX-$k$-SAT with $m = \Omega(n^k)$ has a polynomial-time approximation scheme (PTAS) [AKK99], so does MAX-NAE-$k$-SAT.
3.1 Upper Bounds for (NAE-)$k$-SAT

The fastest randomized and deterministic algorithms for $k$-SAT are summarized in Table 1. For randomized algorithms, PPSZ has been the fastest for decades, whose analysis on general case is not tight until 2014 [Her14]. Its main idea is to use bounded resolution or implication to fix the values of literals in the formula one by one; if no such literals, a literal selected in random is assigned to 0 or 1 with half probability for each. The probability of setting the correct truth value of a variable $v$ is at least the probability of $v$ appearing the last among all variables in at least one of its critical clauses in a random permutation. However, we do not know whether such probability can be greater in NAE-$k$-SAT, because the new clause with opposite polarity after the transformation has exactly the same variables. More specifically, the critical clauses of $v$ are given by a so-called critical clauses tree rooted at $v$ [PPSZ05]. But for NAE-$k$-SAT, we find out that a node in this tree can have as many as $k−1$ children, which is the same in $k$-SAT, and the depth of such tree remains the same. Therefore, we do not know an upper bound of randomized algorithm for NAE-$k$-SAT better than that for $k$-SAT.

Very recently, Liu improves the upper bound of deterministic algorithm for $k$-SAT by introducing the concept of chain. Liu’s algorithm either solves the formula in desired time or produces a large enough set of chains, which can be used to boost the derandomized local search. To construct the generalized covering code for the derandomized local search, one has to prove its existence, which can be done by solving a specific linear programming. In §4, we will give an overview of Liu’s method and show a different solution to linear programming for NAE-$k$-SAT, which leads to better upper bounds (see Table 1). Note that the derandomized local search essentially derandomizes Schöning’s Random Walk [MS11]. An incremental version of Schöning’s Random Walk for MAX-$k$-SAT will be discussed in §3.3. In the context of NAE-$k$-SAT, we summarize Liu’s result in the following theorem.

**Theorem 1** ([Liu18]). There exists a deterministic algorithm for (NAE-)$k$-SAT that runs in time $O(c_k^n)$, where $\nu = \frac{\log(2k−2)−\log k−\log c_{k−1}}{\log(2^k−1)−\log(1−(\frac{n}{2k−2})^k)−k\log c_{k−1}}$, $c_3 = 3^{\frac{1}{2}\log \frac{64}{27}}$ and $c_k = (2^k−1)\nu \cdot c_{k−1}^{1−\nu k}$ for $k \geq 4$.

3.2 Exact Algorithm for MAX-(NAE-)$k$-SAT

As shown by Williams in [Wil05], MAX-2-SAT has an exact algorithm that runs in time exponentially better than $O^*(2^n)$, which relies on the non-trivial faster algorithm for Matrix Multiplication. The method “split-and-list” builds a weighted directed graph of $3 \cdot 2^n/3$ vertices and $3 \cdot 2^{2n/3}$ arcs and transforms the original problem to finding the heaviest triangle in this graph, which can be solved by Matrix Multiplication with exponential dimensions. Williams’s method also works for MAX-NAE-2-SAT and MAX-NAE-3-SAT, as noted in the blessing of NAE. For example, the polynomial $x_1 + x_2 - 2x_1x_2$ or $x_1 + x_2 + x_3 - x_1x_2 - x_1x_3 - x_2x_3$ can represent a clause of 2 or 3 literals in any NAE instance, then each monomial can be treated as same as for a binary clause when calculating the arc weight in the graph, as same as for MAX-2-SAT. Using le Gall’s fastest Matrix Multiplication, the results are presented below.

**Theorem 2** ([Wil05, Wil07, Gal14]). There exist $O^*(2^{2n/3})$-time algorithms for MAX-2-SAT, MAX-NAE-2-SAT and MAX-NAE-3-SAT, where $\omega < 2.373$ is the matrix product exponent.

This result does not apply to MAX-3-SAT or MAX-NAE-4-SAT since the longest clause in the instance is equivalent to a degree-at-least-3 polynomial, but currently Rank-3 Tensor Contraction (the generalization of Matrix Multiplication) does not have an $O(n^{4−\epsilon})$-time algorithm for any $\epsilon > 0$.

3.3 MAX-(NAE-)$k$-SAT Approximation

The up-to-date polynomial-time MAX-$k$-SAT approximations as well as their references can be found in Table 2. Hirsch gives the first algorithm to approximate MAX-$k$-SAT within $(2−\epsilon)^n$ time [Hir03]. We
call his algorithm RandomWalk (Algorithm 1), which is a variant of Schöning’s Random Walk for $k$-SAT [Sch99], modified in the following way: i) $\hat{\alpha}$ is iteratively updated as the assignment satisfying the most number of clauses so far; ii) choosing a random unsatisfied clause instead of an arbitrary one. The motivation is that we do not know the optimal assignment in advance, so there is no termination condition as for $k$-SAT (hitting an assignment satisfying all clauses). Also note that what we ask for is only an approximation, thus it remains hopeful to work out as for $k$-SAT by choosing a random unsatisfied clause to decrease the Hamming distance to a target approximation assignment. We will show that Algorithm 1 also works for MAX-NAE-$k$-SAT, and the upper bound analyzed by Hirsch can be tighten. We give the result by Hirsch in the context of MAX-NAE-$k$-SAT.

**Theorem 3 ([Hir03]).** MAX-(NAE-)k-SAT has an $O^*(\frac{2^{n-2\delta - 2} \delta}{2k})$-time $\delta$-approximation.  

| Algorithm 1: RandomWalk |
|-------------------------|
| **Input:** $k$-instance $F$ on $n$ variables |
| **Output:** assignment $\hat{\alpha}$ |
| 1: initialize $\hat{\alpha}$ as an arbitrary assignment in $\{0, 1\}^n$ |
| 2: draw $\alpha$ from $\{0, 1\}^n$ randomly |
| 3: repeat the following for $O(n)$ times: |
| 4: if $s(\alpha) > s(\hat{\alpha})$ then |
| 5: $\hat{\alpha} \leftarrow \alpha$ |
| 6: randomly choose an unsatisfied clause $C$ in $F$ |
| 7: randomly choose a variable in $C$ and change its value in $\alpha$ |
| 8: return $\hat{\alpha}$ |

There are several MAX-$k$-SAT approximation algorithms proposed by Escoffier et al., which are better than Hirsch’s algorithm [EPT14]. Now we briefly introduce them (see below for Algorithm A, B and C). Although they are originally designed for MAX-SAT approximation, they work for MAX-$k$-SAT approximation under our generalizations, with some modifications in the analysis of running time.

**Algorithm A** Let $p$, $q$ be two integers such that $p/q = (\delta - \ell)/(1 - \ell)$, where $\ell$ is the performance ratio of any given polynomial-time approximation algorithm (e.g. the algorithms which give the lower bounds in Table 2). Build $q$ subsets of variables, each one includes $np/q$ variables, where each variable appears in exactly $p$ subsets. For each subset, enumerating all possible truth assignments on its variables and run the polynomial-time approximation algorithm on the remaining formula. Return the complete assignment with maximum satisfying clauses. For MAX-$k$-SAT approximation, it can be shown that Algorithm A has performance ratio $\delta$ and runs in time $O^*(2^{n(\delta - \ell)/(1 - \ell)})$. This relies on the fact that every clause in the optimal solution contains at least one true literal. However, for MAX-NAE-$k$-SAT, as discussed in the curse of NAE, there is no guarantee of the lower bound of satisfied clauses by fixing a subset of variables. Therefore Algorithm A does not apply to MAX-NAE-$k$-SAT approximation.

**Algorithm B** Let $p$, $q$ be two integers such that $p/q = \delta$. Build $q$ subsets of variables as above. Remove from the instance the variables not in this subset and all empty clauses. Solve the remaining instance by an exact algorithm for MAX-SAT and complete this assignment with arbitrary truth values. It is not hard to see that Algorithm B for MAX-$k$-SAT approximation has performance ratio $\delta$ and runs in time $O^*(2^{n\delta})$.

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2This result follows from the idea proposed by Hirsch, however his computation and statement are wrong. This theorem is the corrected result. See §5.1 for details.
where $O^*(c^n)$ is the upper bound of the exact MAX-$k$-SAT algorithm. As discussed in the curse of NAE, Algorithm B does not work for MAX-NAE-$k$-SAT approximation for the same reason above.

Algorithm C  
Let $p, q$ be two integers such that $p/q = 2\delta - 1$. Build $q$ subsets of variables as above. For each subset, assign weight 2 to every clause containing only variables in this subset, and weight 1 to every clause containing at least one other variable not in this subset. Remove from the instance the variables not in this subset and all empty clauses. Solve the remaining instance using an exact algorithm for weighted MAX-$k$-SAT. Escoffier et al. show that Algorithm C for MAX-$k$-SAT approximation has performance ratio $\delta$ and runs in time $O^*(c^n(2\delta - 1))$, where $O^*(c^n)$ is the upper bound of the exact algorithm for solving weighted MAX-$(k+1)$-SAT. But for the same reason behind previous algorithms, this does not work for MAX-NAE-$k$-SAT approximation. Presently we do not know any exact algorithm for (weighted) MAX-$k$-SAT with better than $O^*(2^n)$ running time when $k \geq 3$, therefore $O^*(2^n(2\delta - 1))$ is the upper bound for Algorithm C.

By running Algorithms A, B and C together, we summarize the results of Escoffier et al. as below, which are currently the best.

**Theorem 4 ([EPT14]).** If there exists an algorithm for solving MAX-$k$-SAT exactly that runs in $O^*(c^n)$ time, MAX-$k$-SAT has an $O^*(\min(2^n(\delta - \ell)/\ell, c^n\delta, 2^n(2\delta - 1)))$-time $\delta$-approximation, where $\ell$ is the performance ratio of any given polynomial-time approximation algorithm for MAX-$k$-SAT.

Using the current best polynomial-time approximation algorithms (see Table 2) for the value $\ell$ and the exact MAX-2-SAT algorithm for value $c$ (see Theorem 2), some numerical results are illustrated in Figure 3.

### 4 Deterministic Algorithm for NAE-$k$-SAT

As mentioned in §1 and §3, we firstly transform the NAE-$k$-SAT problem to an equivalent $k$-SAT problem: for every original clause, create a copy of it and a new clause with opposite polarity in every literal. The new clause is called the conjugate of the copied clause, and we call these two clauses a conjugate pair. Two conjugate pairs are independent if they do not share variable. Solving the new $k$-instance by a deterministic $k$-SAT algorithm would give the same upper bound (omit the polynomial factor). However, as we discussed in the blessing of NAE, a conjugate pair has fewer satisfying assignments. We will show how to use this to derive a better upper bound for NAE-$k$-SAT.3

We adopt the same algorithmic framework (Algorithm 2) from previous work, but use the set of conjugate pairs instead of chains. One of the key point is how to construct generalized covering code for the subroutine DLS (for derandomized local search). This is where the linear programming comes into play. For brevity purpose, we omit the details but only use its running time analysis. The description of subroutine BR (for branching algorithm) is postponed after that. We refer our readers to [Liu18] for an integration of the method.

**Algorithm 2:** Algorithmic Framework

**Input:** $k$-instance $F$

**Output:** a satisfying assignment or Unsatisfiable

1. **BR**($F$) solves $F$ or returns a set of independent conjugate pairs $\mathcal{P}$
2. if $F$ is not solved then
3. **DLS**($F, \mathcal{P}$)
4. end if

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3As an example, the clause $\{v_1, v_2\}$ has 3 satisfying assignments, while the conjugate pair $\{\{v_1, v_2\}, \{\bar{v}_1, \bar{v}_2\}\}$ has only 2.
Lemma 5 ([Liu18]). Given $k$-instance $F$ and a set of independent conjugate pairs $\mathcal{P}$, DLS runs in time $O((\frac{2(k-1)}{k})^n \cdot \lambda^{-v_1})$, where $n' = n - kv_1$, $v_1 = |\mathcal{P}|$ and $\lambda$ is the solution to the following linear programming $\mathcal{LP}_k$ with variables $\lambda \in \mathbb{R}$, $\pi : A \mapsto [0, 1]$:

\[
\sum_{a \in A} \pi(a) = 1 \\
\lambda = \sum_{a \in A} \left( \pi(a) \cdot \left( \frac{1}{k-1} \right)^d(a,a^*) \right) \quad \forall a^* \in A \\
\pi(a) \geq 0 \quad \forall a \in A \\
A = \{0,1\}^k \backslash \{0^k, 1^k\}.
\]

There are $2^k - 1$ variables and $2^k - 1$ equality constraints in $\mathcal{LP}_k$. One can work out the determinant of the coefficient matrix to see it has full rank, so the solution is unique if feasible. In the following lemma, we provide a closed-form solution to $\mathcal{LP}_k$, which leads to the expression of an upper bound for the running time of DLS. This is different from the linear programming proposed in [Liu18] due to the difference in the solution spaces.

Lemma 6. Given integer $k \geq 3$, the solution to $\mathcal{LP}_k$ is:

\[
\lambda = \frac{k^k + (-k)^k}{(2k-2)^k - (2k-4)^k + (-2)^k}, \\
\pi(a) = \frac{(k-1)^k}{(2k-2)^k - (2k-4)^k + (-2)^k} \cdot (1 - \left( \frac{-1}{k-1} \right)^{d(a,0^k)}) \cdot (1 - \left( \frac{-1}{k-1} \right)^{d(a,1^k)}) \text{ for all } a \in A.
\]

Proof. We prove that this is a feasible solution to $\mathcal{LP}_k$ by verifying all constraints. Constraint $\pi(a) \geq 0 \ (\forall a \in A)$ is easy to verify. Multiplying $\frac{(2k-2)^k - (2k-4)^k + (-2)^k}{(k-1)^k}$ on both sides of $\sum_{a \in A} \pi(a) = 1$ to get:

\[
\text{LHS} = \sum_{a \in A} \left( 1 - \left( \frac{-1}{k-1} \right)^{d(a,0^k)} \right) \cdot \left( 1 - \left( \frac{-1}{k-1} \right)^{d(a,1^k)} \right) = \sum_{a \in A} \left( 1 - \left( \frac{-1}{k-1} \right)^{d(a,0^k)} - \left( \frac{-1}{k-1} \right)^{d(a,1^k)} + \left( \frac{-1}{k-1} \right)^k \right) = \sum_{y=1}^{k-1} \binom{k}{y} \cdot \left( 1 - \left( \frac{-1}{k-1} \right)^y - \left( \frac{-1}{k-1} \right)^{k-y} + \left( \frac{-1}{k-1} \right)^k \right) = \sum_{y=0}^{k} \binom{k}{y} \cdot \left( 1 - \left( \frac{-1}{k-1} \right)^y - \left( \frac{-1}{k-1} \right)^{k-y} + \left( \frac{-1}{k-1} \right)^k \right) = 2^k \cdot \left( 1 + \left( \frac{-1}{k-1} \right)^k \right) - 2 \cdot \left( \frac{k-2}{k-1} \right)^k = \text{RHS}.
\]

The second equality is due to the relation $d(a, 0^k) + d(a, 1^k) = k$. The third equality follows from substituting $d(a,0^k)$ with $y$ and the fact that the number of $a$ in $A$ with $d(a,0^k) = y$ is $\binom{k}{y}$. The last line is just Binomial theorem.

\[A^*\] is called a solution space, which in our case is just a set of $2^k - 2$ assignments on $k$ variables.
Similarly, multiplying \(\frac{(2k-2)^k-(2k-4)^k+(-2)^k}{(k-1)^k}\) on both sides of \(\lambda = \sum_{a \in A} \left( \pi(a) \cdot \left( \frac{1}{k-1} \right)^d(a, a^*) \right)\) to get:

\[
\text{RHS} = \sum_{a \in A} \left( \left(1 - \left(\frac{-1}{k-1}\right)^{d(a,0^k)}\right) \cdot \left(1 - \left(\frac{-1}{k-1}\right)^{d(a,1^k)}\right) \cdot \left(\frac{1}{k-1}\right)^{d(a, a^*)}\right)
= \sum_{a \in \{0,1\}^k} \left( \left(1 - \left(\frac{-1}{k-1}\right)^{d(a,0^k)}\right) - \left(\frac{-1}{k-1}\right)^{d(a,1^k)} \right) \cdot \left(\frac{1}{k-1}\right)^{d(a, a^*)} \right)
= \left(\frac{k}{k-1}\right)^k + \left(\frac{-1}{k-1}\right)^k \cdot \left(\frac{k}{k-1}\right)^k
= \left(\frac{k}{k-1}\right)^k + \left(\frac{-1}{k-1}\right)^k \cdot \left(\frac{k}{k-1}\right)^k = \text{LHS}.
\]

The third equality is by noticing \(a^*\) is symmetric with respect to \(0^k\) and \(1^k\) and applying Binomial theorem. For the fourth equality to hold we need to prove that the terms in the fourth line are 0. By symmetry we only need to prove that \(\sum_{a \in \{0,1\}^k} \left(\frac{-1}{k-1}\right)^{d(a,0^k)} \cdot \left(\frac{1}{k-1}\right)^{d(a,0^k)+d(a, a^*)}\right) = 0.\) Observe that for \(a^*\) to be a NAE-assignment, there must exist \(i \in [k]\) such that the \(i\)-th bit \(a_i^* = 1\). Now we partition \(\{0,1\}^k\) into two sets \(S_0, S_1\) depending on whether the \(i\)-th bit is 1. The following is a bijection: for each \(a \in S_0\), negate the \(i\)-th bit to get \(a' \in S_1\). Then it is not hard to see that \(d(a,0^k) + d(a, a^*) = d(a', 0^k) + d(a', a^*)\) and \((-1)^{d(a,0^k)} = (-1)^{d(a',0^k)}\), so the sum of such pairs is 0. Therefore we verified all the constraints and proved the lemma.

We present algorithm BR in Algorithm 3, with parameter \(\nu\) to be fixed in Theorem 7. What is different from the branching algorithm in [Liu18] is line 3, which is another blessing of NAE. After fixing all variables in \(P\), the remaining formula is a \((k-1)\)-instance due to the maximality of \(P\). Also note that in line 4 we cannot call a deterministic NAE-(k-1)-SAT algorithm because the remaining formula not necessarily consists only conjugate pairs. For example, the conjugate pair \(\{\{v_1, v_2, v_3\}, \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}\}\) will become \(\{\bar{v}_2, \bar{v}_3\}\) after fixing \(v_1 = 1\).

**Algorithm 3: Algorithm BR**

**Input:** \(k\)-instance \(F\), parameter \(\nu\)

**Output:** a satisfying assignment or Unsatisfiable or a set of independent conjugate pairs \(P\)

1. greedily construct a maximal set of independent conjugate pairs \(P\)
2. if \(|P| < \nu n\) then
   3. for each assignment \(\alpha \in \{\{0,1\}^k \setminus \{0^k,1^k\}\}^\nu\) of \(P\) do
      4. solve the remaining formula after fixing \(\alpha\) in \(F\) by a deterministic \((k-1)\)-SAT algorithm
   5. return the satisfying assignment if satisfiable
3. end for
4. return Unsatisfiable
5. else
6. return \(P\)
7. end if

Obviously, BR runs in time \(O^\ast((2^k - 2)^\nu) \cdot c_{k-1}^n\) where \(c_{k-1}^n\) is the upper bound of a given deterministic \((k-1)\)-SAT algorithm, which is an increasing function of \(\nu\), while our computation shows that the running
time of DLS is a decreasing function of $\nu$. This immediately implies that the worst case is attained when both running time is equal, which gives the following main result.

**Theorem 7.** Given $k \geq 3$, if there exists a deterministic algorithm for $(k-1)$-SAT that runs in time $O(c_{k-1}^n)$, then there exists a deterministic algorithm for NAE-$k$-SAT that runs in time $O(c_k^n)$, where

$$c'_k = (2^k - 2)^\nu \cdot c_{k-1}^{1-k\nu}$$

and

$$\nu = \frac{\log(2^k - 2) - \log k - \log c_{k-1}}{\log(2^k - 2) + \log(k^k + (k/\log k)^k) + k\log(2k^2 - 2) - \log((2k^2 - 2)^k - (2k - 4)^k) + (-2)^k}.$$

Using the values of $c_k$ from Theorem 1, we obtain the upper bounds in Table 1.

# 5 Approximation Algorithms

In this section, we present two independent approximation algorithms: RandomWalk and ReduceSolve, which both work for MAX-$k$-SAT and MAX-NAE-$k$-SAT approximations. RandomWalk can be repeated for exponential times to get an approximation assignment w.h.p., while ReduceSolve transforms the instance to another instance with fewer variables and then solves the remaining formula.  

## 5.1 Algorithm RandomWalk

For those readers familiar with Schöning’s Random Walk [Sch99], it is obvious that Algorithm 1 works for MAX-NAE-$k$-SAT approximation: just be aware that in line 6 a NAE-unsatisfied clause is chosen randomly.

### Lemma 8 ([Hir03]).

For any $k$-instance $F$, RandomWalk returns a $\delta$-approximation assignment (resp. $\delta$-approximation NAE-assignment) of $F$ w.p. at least $(2 - 2p_8)^n$, where $p_8 = \frac{1 - \delta}{\xi(m/s(\alpha^*) - 2)}$ and $\alpha^*$ is an optimal assignment (resp. optimal NAE-assignment) of $F$.

To give the tightest bound, we need the following lemma of bounding $m, s(\alpha^*)$ by introducing the average clause length $\eta$ (see definition in §2).

### Lemma 9.

Given $k$-instance $F$, let $\eta$ be its average clause length. If $\alpha^*$ is an optimal assignment of $F$, it must be $m \leq \frac{s(\alpha^*)}{\xi}$, where $\xi = \frac{2^{k-1}(\eta + k - 2) - \eta + 1}{2^k(k-1)}$. If $\alpha^*$ is an optimal NAE-assignment of $F$, it must be $m \leq \frac{s(\alpha^*)}{\xi'}$, where $\xi' = \frac{1}{2}$ for $k = 2$ and $\xi' = \frac{2^k - 1(\eta + k - 4) - 2\eta + 4}{2^k(k+1)}$ for $k \geq 3$.

**Proof.** We prove this by probabilistic argument. It is easy to see that a random assignment $\alpha$ satisfies $\sum_{i \in [k]} \frac{2^i - 1}{2^i} m_i$ clauses in expectation, thus $\alpha$ satisfies this number of clauses with positive probability, so it must be $s(\alpha^*) \geq \sum_{i \in [k]} \frac{2^i - 1}{2^i} m_i$. Since $\eta = (\sum_{i \in [k]} i \cdot m_i)/m$, we can eliminate $m_1$ by $\eta$. By substitution, multiplication and rearranging we have:

$$\frac{s(\alpha^*)}{m} \geq \sum_{i=1}^{k} \left( \frac{2^i - 1(\eta + i - 2) - \eta + 1}{2^i(k-1)} m_i \right) \geq \min_{2 \leq i \leq k} \left( \frac{2^i - 1(\eta + i - 2) - \eta + 1}{2^i(k-1)} \right) = \frac{2^k - 1(\eta + k - 2) - \eta + 1}{2^k(k-1)}.$$

Suppose the polynomial-time RandomWalk succeeds w.p. $p$, then one can repeat it for $O^*(1/p)$ times to get an $O^*(1/p)$-time randomized algorithm with high probability of success. If no ambiguity, we still call such repeating algorithm RandomWalk.
The last equality can be shown by induction on $i$.

The statement for NAE-assignment can be proved in a similar way: since there is no 1-clause, we know that $s(\alpha^*) \geq \sum_{i=2}^{k} \frac{2^{i-2}m_i}{2^i}$; eliminating $m_2$ by $\eta$ for $k \geq 3$, we proved this lemma.

Based on Lemma 9, one can easily show that $s(\alpha^*) \geq m/2$ for MAX-(NAE-)k-SAT, thus Lemma 8 immediately implies Theorem 3.

Our key observation is that although the probability of success of RandomWalk is an increasing function of $\eta$, a random guess is actually not too bad when $\eta$ is small: there are many short clauses in the formula, then the optimal solution cannot satisfy too many clauses, and a random guess should not be too far away from it. Indeed, a random guess (line 2 of Algorithm 1) already yields an arbitrarily good approximation with non-negligible probability. To show this, we first take a detour to focusing on a special subformula.

Definition 10. Given $k$-instance $F$ and an arbitrary optimal (NAE)-assignment $\alpha^*$ of $F$, define the maximal (NAE)-satisfiable subformula of $F$ as $G$ being a $k$-instance consisting of all (NAE)-satisfied clauses of $F$ under $\alpha^*$.

Clearly $G$ has $n$ variables, because otherwise assigning a variable outside $G$ can (NAE-)satisfy more clauses of $F$. Analogous to what we defined for $F$, let $w_i$ be the number of $i$-clauses in $G$ for all $i \in [k]$ and $w = \sum_{i \in [k]} w_i = s(\alpha^*)$, then the average clause length is $\theta = (\sum_{i \in [k]} i \cdot w_i)/w$. We also need the following definition.

Lemma 11. Given a maximal (NAE)-satisfiable subformula $G$ of $F$, let $B_\tau(G)$ be the set of all variables whose occurrences are upper bounded by $\tau$, we have that for any $\lambda > 0$, it must be $|B_\tau(G)| \geq (1 - \frac{\theta}{\lambda})n$, where $\tau = \frac{\lambda w}{n}$.

Proof. The proof is by a probabilistic argument. Let $X$ be occurrence of a random variable in $G$, the average occurrence $E[X] = (\sum_{i \in [k]} i \cdot w_i)/n$. Thus by Markov’s inequality it must be $\forall \lambda > 0, \Pr[X \geq \lambda w/n] \leq E[X]/(\lambda w/n) = \theta/\lambda$, which is $\Pr[X < \lambda w/n] \geq 1 - \theta/\lambda$. So there exists at least $(1 - \theta/\lambda)n$ variables, whose occurrences are upper bounded by $\lambda w/n$.

Lemma 12. For any $k$-instance $F$, RandomWalk returns a $\delta$-approximation (NAE)-assignment of $F$ w.p. at least $2^{-\frac{\theta}{1 - \delta + \theta}}n$, where $\theta$ is the average clause length of a maximal (NAE)-satisfiable subformula of $F$.

Proof. The probability of success of RandomWalk is lower bounded by random guess (line 2 of Algorithm 1). By Lemma 11 with $\lambda = 1 - \delta + \theta$, within $G$ there exist $(1 - \theta/(1 - \delta + \theta))n$ variables whose occurrences are upper bounded by $(1 - \delta + \theta)w/n$, and we call these variables sub-$\tau$ variables, since they constitute a subset of $B_\tau(G)$. Note that the total occurrences of sub-$\tau$ variables in $G$ is at most:

$$(1 - \theta/(1 - \delta + \theta))n \cdot (1 - \delta + \theta)w/n = (1 - \delta)w.$$

This is the maximal number of clauses sub-$\tau$ variables can appear. We have that at least $\delta w$ clauses do not contain any sub-$\tau$ variable. So no matter how to change the assignments of sub-$\tau$ variables in $\alpha^*$ and let $\alpha_\delta$ be the altered $\alpha^*$, $\alpha_\delta$ still satisfies at least $\delta w$ clauses. Now randomly guessing an $\alpha$ from $\{0, 1\}^n$, it agrees with some $\alpha_\delta$ w.p. at least $2^{-n\theta/(1 - \delta + \theta)}$, because the number of variables which are not sub-$\tau$ variable is $n\theta/(1 - \delta + \theta)$. The conclusion follows immediately. It is easy to see the choice of $\lambda$ is optimal for maximizing the lower bound of this probability.

A tighter upper bound for MAX-(NAE-)k-SAT relies on the following correlation of $\eta$ and $\theta$.

Lemma 13. Given $k$-instance $F$ ($k \geq 2$), let $\eta, \theta > 1$ be the average clause length of $F$ and its maximal satisfiable subformula respectively, we have that $\frac{2}{\eta - 1} \geq \frac{1}{\eta - 1} + \frac{1}{k - 1} \frac{2^{k-1}-1}{2^k-1}$. Let $\theta'$ be the average clause length of the maximal NAE-satisfiable subformula of $F$, we have that $\theta' = 2$ when $k = 2$ and $\frac{2}{\eta - 2} \geq \frac{1}{\eta - 2} + \frac{1}{k - 2} \frac{2^{k-2}-1}{2^k-1}$ when $k \geq 3$.
Proof. Using \( \eta = (\sum_{i \in [k]} i \cdot m_i)/m \) and \( \theta = (\sum_{i \in [k]} i \cdot w_i)/w \) to eliminate \( m_1 \) and \( w_1 \), and by \( w \geq \sum_{i \in [k]} \frac{2^i - 1}{2^i} m_i \) (see proof of Lemma 9), we have:

\[
\sum_{i=2}^{k} (\frac{i - \theta}{\theta - 1} + 1) w_i \geq \sum_{i=2}^{k} \left( \frac{1}{2} \cdot \frac{i - \eta}{\eta - 1} + \frac{2^i - 1}{2^i} \right) m_i.
\]

Using the fact that \( \forall i \in [k], w_i \leq m_i \) for the left-hand side, and rearranging the right-hand side, it becomes:

\[
\sum_{i=2}^{k} i - \frac{1}{\theta - 1} m_i \geq \sum_{i=2}^{k} \left( \frac{i - 1}{2\eta - 2} + \frac{1}{2} - \frac{1}{2i} \right) m_i.
\]

(1)

Now we prove that \( \frac{k - 1}{2\eta - 2} \leq \frac{k - 1}{2\eta - 2} + \frac{1}{2} - \frac{1}{2k} \). Assume for contradiction that \( \frac{k - 1}{2\eta - 2} < \frac{k - 1}{2\eta - 2} + \frac{1}{2} - \frac{1}{2k} \), it must be:

\[
\frac{k - 2}{\theta - 1} < (\frac{k - 1}{2\eta - 2} + \frac{1}{2} - \frac{1}{2k}) \frac{k - 2}{k - 1} = \frac{k - 2}{2\eta - 2} + \frac{1}{2} - \frac{1}{2^{k-1}} \frac{2^{k-2} + \frac{k - 2}{k - 1}}{2^k - \frac{1}{2^k}} \leq \frac{k - 2}{2\eta - 2} + \frac{1}{2} - \frac{1}{2^{k-1}}.
\]

The last inequality holds for any \( k \geq 2 \). We can continue this process to get \( \frac{k - 1}{2\eta - 2} < \frac{k - 1}{2\eta - 2} + \frac{1}{2} - \frac{1}{2k} \) for all \( 2 \leq i \leq k \). This is a contradiction since all coefficients of \( m_i \) in the left-hand side of (1) are strictly smaller than those in the right-hand side, so inequality (1) does not hold unless \( k = 1 \) or \( m_i = 0 \) for all \( 2 \leq i \leq k \), which makes \( F \) a 1-instance. As a result, we have \( \frac{k - 1}{\theta - 1} \geq \frac{k - 1}{2\eta - 2} + \frac{1}{2} - \frac{1}{2k} \), which is \( \frac{\theta - 1}{\theta - 1} \geq \frac{\theta - 1}{\theta - 1} + \frac{k - 1}{2\eta - 2} \) as we stated. Similar argument would give us the result for NAE in this lemma, and the case \( k = 2 \) follows directly from the fact that there is no 1-clause.

Since \( \theta \) is upper bounded by some function of \( \eta \), we have that the worst case is achieved when the equalities in Lemma 13 are attained. \(^6\) Now the probability in Lemma 8 is an increasing function of \( \eta \), while the probability in Lemma 12 is a decreasing function of \( \eta \), thus the worst case is when they are equal. So we have our main result on MAX-(NAE-\( k \))-SAT as the following.

**Theorem 14.** MAX-\( k \)-SAT has an \( O^*(\gamma^n) \)-time \( \delta \)-approximation, where \( \gamma \) satisfies the following equation system \( \mathcal{M}_k(\delta) \):

\[
\begin{align*}
\xi &= \frac{2^{k-1}(\eta + k - 2) - \eta + 1}{2^{k}(k - 1)} \\
\gamma &= 2^{\frac{\theta - 1}{\theta - 1}} = 2^{\frac{2}{k - \xi \delta k}} - \frac{2\xi - 2\xi \delta}{k - \xi \delta k} \\
\frac{2}{\theta - 1} &= \frac{1}{\eta - 1} + \frac{1}{k - 1} \frac{2^{k-1} - 1}{2^{k-1}}
\end{align*}
\]

where integer \( k \geq 2 \) and constant \( \delta \in [0, 1] \) are given, and \( \gamma, \xi, \theta, \eta \) are variables of \( \mathcal{M}_k(\delta) \).

MAX-NAE-k-SAT has an \( O^*(\gamma^n) \)-time \( \delta \)-approximation, where \( \gamma' \) satisfies the following equation system \( \mathcal{M}'_k(\delta) \):

\[
\begin{align*}
\xi' &= \frac{2^{k-1}(\eta + k - 4) - 2\eta + 4}{2^{k}(k - 2)} \text{ for } k \geq 3 \text{ and } \xi' = \frac{1}{2} \text{ for } k = 2 \\
\gamma' &= 2^{\frac{\theta' - 1}{\theta' - 1}} = 2^{\frac{2\xi' - 2\xi' \delta}{k - \xi' \delta k}} - \frac{2\xi' - 2\xi' \delta}{k - \xi' \delta k} \\
\frac{2}{\theta' - 2} &= \frac{1}{\eta - 2} + \frac{1}{k - 2} \frac{2^{k-2} - 1}{2^{k-2}}
\end{align*}
\]

where integer \( k \geq 2 \) and constant \( \delta \in [0, 1] \) are given, and \( \gamma', \xi', \theta', \eta \) are variables of \( \mathcal{M}'_k(\delta) \).  

\(^6\)Note that \( \eta = k \) gives \( \theta > k \) under the equality condition, which is infeasible. But as we will see in the equation system, the solution always has \( \eta \) less than \( k/2 \), so we omit the feasibility condition.
Observe that by monotonicity with respect to $\eta$, a binary search solves $M_k(\delta)$ and $M'_k(\delta)$ to arbitrary precisions in reasonable time.

5.2 Algorithm ReduceSolve

Our second algorithm (Algorithm 4) is reducing the formula to another formula with fewer variables and solving it by an exact algorithm for MAX-(NAE-)$k$-SAT. The high-level idea is the following: deliberately choose variables with low occurrence, such that these variables can be fixed without falsifying too many clauses, then solving the reduced formula still yields a good approximation.

**Algorithm 4: ReduceSolve**

**Input:** $k$-instance $F$ on $n$ variables, parameter $t$

**Output:** assignment $\hat{\alpha}$

1: initialize $V_e \leftarrow \emptyset$, $F_e = \emptyset$

2: for $i \leftarrow 1$ to $t$ do

3: choose the variable $v$ in $F$ with the lowest occurrence

4: $V_e \leftarrow V_e \cup \{v\}$

5: for every clause $C$ of $F$ containing $v$ do

6: $F_e \leftarrow F_e \cup C$

7: eliminate $C$ from $F$

8: solve $F$ by an exact algorithm for MAX-(NAE-)$k$-SAT to get $\alpha^{(1)}$

9: $\alpha^{(2)} \leftarrow$ a random partial assignment on $V_e$

10: return $\hat{\alpha} \leftarrow \alpha^{(1)} \cup \alpha^{(2)}$

Line 9 (NAE-)satisfies at least half of the clauses in $F_e$ in expectation, which is not hard to see since every clause in $F_e$ contains at least one variable from $V_e$. If using the method of conditional probabilities (see Chapter 16 in [AS16]), it is guaranteed to find an $\alpha^{(2)}$ in polynomial time (NAE-)satisfying at least half of the clauses, which makes our algorithm deterministic, provided that the exact algorithm is deterministic (the current best one is).

**Theorem 15.** If there exists an algorithm for solving MAX-$k$-SAT exactly that runs in $O^*(c^n)$ time, then MAX-$k$-SAT has an $O(c^{n(1-2(1-\delta)\xi^{1/2})})$-time $\delta$-approximation where $\xi = \frac{2^k-1+(\eta+k-2)-(\eta+1)}{2^k(k-1)}$ and $\eta$ is the average clause length of $F$.

If there exists an algorithm for solving MAX-NAE-$k$-SAT exactly that runs in $O^*(c^n)$ time, then MAX-NAE-$k$-SAT has an $O(c^{n(1-2(1-\delta)\xi'^{1/2})})$-time $\delta$-approximation where $\xi' = \frac{2^k-1+(\eta+k-4)-(\eta+4)}{2^k(k-2)}$ for $k \geq 3$ and $\xi' = \frac{1}{2}$ for $k = 2$.

**Proof.** We analyze Algorithm 4. In the following, step $i$ corresponds to the loop variable $i$ in line 2-7. Let $F^{(i)}$ be the remaining formula after elimination in step $i$, and let $m^{(i)}$ be the number of clauses in $F^{(i)}$. Clearly there are $n-i$ variables in $F^{(i)}$. Taking the average we have that the lowest occurrence is upper bounded by $\frac{km^{(i)}}{n-i-1}$. The number of clauses in $F^{(i+1)}$ is $m^{(i+1)} \geq m^{(i)} - \frac{km^{(i)}}{n-i-1}$, because the lowest-occurrence variable appears in at most $\frac{km^{(i)}}{n-i-1}$ clauses. Expanding until $m^{(0)} = m$, the following must hold for the number of clauses in $F^{(t)}$:

$$m^{(t)} \geq m \cdot \prod_{i=1}^{t} \left(1 - \frac{k}{n-i}\right).$$

It is possible that there are less than $n-i$ variables in step $i$ when a variable’s all occurrences are eliminated by other variable, then we still think that there are $n-i$ variables with some variable’s occurrence being 0, and the analysis still holds.

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Using the fact $1 - y = \exp(-y - o(y))$ for $y \to 0$, we have:

$$m^{(t)} \geq m \cdot \exp(-\sum_{i=1}^{t} \frac{k}{n-i} - o(\sum_{i=1}^{t} \frac{k}{n-i})).$$

(2)

Note that $\sum_{i=1}^{t} \frac{1}{n-i} = \ln \frac{n}{n-t} + O(\frac{1}{n})$. Now assuming $t = \Theta(n)$ we have $\ln \frac{n}{n-t} = \Theta(1)$, so (2) becomes:

$$m^{(t)} \geq m \cdot \exp(-k \ln \frac{n}{n-t} - o(1)) = m \cdot (\frac{n}{n-t})^{-k} \cdot (1 - o(1)).$$

(3)

Let $x < 1$ be a parameter to be fixed later. Observe that if $m^{(t)} \geq (2x - 1)m$, we eliminated at most $(2 - 2x)m$ clauses in the first $t$ steps, and at most $(1 - x)m$ clauses are unsatisfied. As a result, to obtain an assignment falsifying at most $(1 - x)m$ clauses in $F$, by (3) it is sufficient to have:

$$m \cdot (\frac{n}{n-t})^{-k} \cdot (1 - o(1)) \geq (2x - 1)m \Leftrightarrow t \leq (1 - (2x - 1)^{1/k} - o(1))n.$$

Choosing $t = (1 - (2x - 1)^{1/k} - o(1))n$, which is $t = \Theta(n)$ as we assumed for (3), we have that the variables in the remaining formula $F^{(t)}$ is at most $n - t = (2x - 1)^{1/k}n + o(n)$.

Now we fix parameter $x$. If at most $(1 - \delta)s(\alpha^*)$ clauses in the maximal satisfiable subformula are unsatisfied, we definitely have a $\delta$-approximation assignment. In the worst case, all $(1 - x)m$ clauses we falsified are from the maximal satisfiable subformula, which gives $(1 - x)m \leq (1 - \delta)s(\alpha^*)$ to meet the condition. By Lemma 9 it suffices to have $x = 1 - (1 - \delta)\xi$, where $\xi = \frac{2^{h-1}(\eta+k-2)-\eta+1}{2^{k}(k-1)}$ and $\eta$ is the average clause length of $F$.

Finally we solve the remaining formula $F^{(t)}$ by an exact algorithm in time $O^*(c^{n(1 - 2(1 - \delta)\xi)^{1/k} + o(n)}) = O(c^{n(1 - 2(1 - \delta)\xi)^{1/k}})$ (line 8). If we find an optimal assignment $\alpha^{(1)}$ for $F^{(t)}$, by union with $\alpha^{(2)}$ we get a $\delta$-approximation assignment, because $\alpha^{(1)}$, $\alpha^{(2)}$ are on disjoint variables. Therefore we proved the theorem for MAX-$k$-SAT. Substitute $\xi$ with $\xi'$ from Lemma 9 and we obtain the result for MAX-NAE-$k$-SAT.

### 5.3 Remarks

Using Theorem 2 for the value $c$ in ReduceSolve, some numerical results by running RandomWalk and ReduceSolve together are presented in Figure 3. As we discussed before Theorem 14, the probability of success by random guess is a decreasing function of the average clause length $\eta$, while the probability of success by only RandomWalk ignoring the random initial assignment is an increasing function of $\eta$. Now observe that the upper bound of ReduceSolve is a decreasing function of $\eta$, thus running RandomWalk and ReduceSolve together might yield a better upper bound. However due the existence of faster MAX-NAE-(≤ 3)-SAT approximation algorithm, our calculation shows that ReduceSolve is always faster than RandomWalk, even in the worst case of $\eta = 2$. On the contrary, because of the absence of exact algorithm faster than $2^n$ for $k \geq 4$, RandomWalk is faster than ReduceSolve. It is possible that faster Matrix Multiplication algorithm or faster exact algorithm for MAX-NAE-$k$-SAT is found, then combining RandomWalk and ReduceSolve would yield an algorithm with upper bound tighter than either of them.

### 6 Summary

We have proved a better upper bound for NAE-$k$-SAT by introducing the concept of conjugate pairs and then (i) solving a Linear Programming for the analysis of the derandomized local search; (ii) using a branching algorithm with $2^k - 2$ branches to transform to a $(k - 1)$-SAT problem with fewer variables. The result is better than the current fastest deterministic algorithm for $k$-SAT, due to the blessing of NAE. We also
illustrated why such blessing does not help the current best randomized $k$-SAT algorithm PPSZ. It would be intriguing to pursue a nature explanation to such bottleneck of PPSZ for NAE-$k$-SAT.

On the other hand, we showed that there is a curse of NAE to the exponential-time approximation algorithms, ruining the current best algorithms which rely on the polynomial-time approximation algorithms or exact algorithms. Two algorithms RandomWalk and ReduceSolve are presented, which work for both MAX-$k$-SAT and MAX-NAE-$k$-SAT approximations. We provided a tighter analysis on RandomWalk by leveraging the probabilities of two seemingly unrelated events using the average clause length. The algorithm ReduceSolve proposed in this work reduces the variables in the formula and then solves it by an exact algorithm, which benefits from another blessing of NAE: an algorithm for MAX-NAE-$(\leq 3)$-SAT exponentially better than the $O(2^n)$-time brute-force search. It is worthy noting that RandomWalk and ReduceSolve can be combined to achieve a possibly better upper bound, since the probability of success by only Random Walk ignoring the random initial assignment is an increasing function of $\eta$, while the upper bound of ReduceSolve is a decreasing function of $\eta$. However this is not the case in our result dues to numerical reasons. Our results can be improved if faster Matrix Multiplication or faster exact algorithm for MAX-NAE-$k$-SAT is found.

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