Superconvergence of Galerkin variational integrators *  

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Abstract: We study the order of convergence of Galerkin variational integrators for ordinary differential equations. Galerkin variational integrators approximate a variational (Lagrangian) problem by restricting the space of curves to the set of polynomials of degree at most \( s \) and approximating the action integral using a quadrature rule. We show that, if the quadrature rule is sufficiently accurate, the order of the integrators thus obtained is \( 2s \).

Keywords: Lagrangian systems, Variational integrators, Geometric integration, Galerkin methods, High-order integrators

1. INTRODUCTION

Variational integrators are a class of geometric integration methods, constructed using a discrete version of Hamilton’s principle. Variational integrators are symplectic and momentum preserving, provided the discretization exhibits the same symmetries as the continuous system. This leads to more accurate results, especially in long-term simulations of conservative systems (see e.g. Marsden and West (2001); Hairer et al. (2006); Reich (1994)) but also in the simulation and optimization of dissipative systems (Kane et al. (2000); Modin and Söderlind (2011); Jiménez and Ober-Blöbaum (2018); Limebeer et al. (2020)) and controlled systems (Ober-Blöbaum et al. (2011)).

To construct higher order methods, Galerkin variational integrators were considered by Marsden and West (2001) and analyzed by Leok and Shingel (2012); Hall and Leok (2015) for optimally controlled systems. The result is provided in Ober-Blöbaum (2017).

2. OVERVIEW OF CONTINUOUS AND DISCRETE LAGRANGIAN MECHANICS

2.1 Continuous Lagrangian mechanics

We consider Lagrange functions \( \mathcal{L} : TQ \rightarrow \mathbb{R} \) on a vector space \( Q \). The action of a smooth curve \([a, b] \rightarrow Q\) is given by

\[
\mathcal{S}[q] = \int_a^b \mathcal{L}(q(t), \dot{q}(t)) \, dt.
\]

A curve \( q \) is a stationary curve of the action functional \( \mathcal{S} \) if its Gateaux derivative (often called first variation in this context),

\[
\delta \mathcal{S}[q, \delta q] = \left. \frac{\partial}{\partial \alpha} \mathcal{S}[q + \alpha \delta q] \right|_{\alpha=0},
\]

vanishes for all smooth curves \( \delta q : [a, b] \rightarrow Q \) with \( \delta q(a) = \delta q(b) = 0 \). Note that \( \delta q \) does not denote an infinitesimal, but rather a curve indicating the direction of variation.

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The stationarity condition is known as Hamilton’s principle. It can be expressed as a differential equation using integration by parts:

\[ 0 = d\mathcal{S}[q, \delta q] = \int_a^b \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \, dt = \left. \int_a^b \left( \frac{\partial L}{\partial q} \frac{d}{dt} \delta q - \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) \right|_a^b. \]

Since \( \delta q \) vanishes at the endpoints, the boundary term is zero. The integral is zero for all variations \( \delta q \) if and only if the Euler-Lagrange equation

\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \]

is satisfied.

Throughout this work we will assume that the Lagrangian is non-degenerate, i.e. \( \det \left( \frac{\partial^2 L}{\partial q \partial \dot{q}} \right) \neq 0 \). Then the Euler-Lagrange equation is a second order ODE,

\[ \ddot{q} = \left( \frac{\partial^2 L}{\partial q^2} \right)^{-1} \left( \frac{\partial L}{\partial q} - \frac{\partial^2 L}{\partial q \partial \dot{q}} \frac{d}{dt} \dot{q} \right). \]

In addition, non-degeneracy implies that the Legendre transform \( TQ \rightarrow T^*Q : (q, \dot{q}) \mapsto (q, \frac{\partial L}{\partial \dot{q}}) \) is invertible, so the Euler-Lagrange equation is equivalent to the Hamiltonian system

\[ \dot{q} = \frac{\partial H(q, p)}{\partial p}, \quad \dot{p} = -\frac{\partial H(q, p)}{\partial q} \]

with Hamiltonian

\[ H(q, p) = p\dot{q} - L(q, \dot{q}). \] (1)

It is important to note that while the Euler-Lagrange equation is a necessary condition for \( q \) to be a minimizer of the action, it is not in general a sufficient condition. In the present work we will make the assumption that solutions to the Euler-Lagrange equations are always minimizers of the action (for sufficiently small intervals of integration). In particular, this is the case for Lagrangians of mechanical type, where \( Q = \mathbb{R}^n \) and

\[ L = q^T M(q) \dot{q} - V(q), \]

where \( M \) is a positive definite \( n \times n \) matrix (Gelfand and Fomin, 1963, Chapter 5).

### 2.2 Variational integrators

Lagrangian systems have a rich structure: their flows consist of symplectic maps (indeed they are equivalent to Hamiltonian systems) and symmetries of the action correspond to conserved quantities by Noether’s theorem. When approximating Lagrangian systems numerically, preserving this structure generally leads to improved numerical behaviour. Such an approach of structure-preserving discretization is known as geometric numerical integration, see Hairer et al. (2006). In the case of Lagrangian systems, the key to geometric numerical integration is to discretize the action functional instead of discretizing the Euler-Lagrange equation directly. Numerical methods obtained in this way are called variational integrators. Below we present some essential facts on variational integrators. For a more detailed discussion we refer to Marsden and West (2001).

A discrete Lagrange function on \( Q \) is a differentiable function \( L : Q \times Q \times (0, \infty) \rightarrow \mathbb{R} \). The discrete action corresponding to a discrete curve \( q = (q_0, q_1, \ldots, q_N) \) with step size \( h \) is given by

\[ \mathcal{S}_d(q) = \sum_{i=1}^N L(q_{i-1}, q_i; h). \]

We say that \( q = (q_0, q_1, \ldots, q_N) \) is critical if

\[ \frac{\partial \mathcal{S}_d(q)}{\partial \dot{q}_i} = 0 \quad \text{for } i \in \{1, \ldots, N - 1\}, \]

i.e. if the action is invariant with respect to infinitesimal variations of the interior points. This is the case if and only if \( q \) satisfies the discrete Euler-Lagrange equation

\[ D_2 L(q_{i-1}, q_i; h) + D_1 L(q_i, q_{i+1}; h) = 0 \] (2)

for \( i \in \{1, \ldots, N-1\} \), where \( D_1 \) and \( D_2 \) denote the partial derivatives of \( L \) with respect to the first and second entry.

The discrete Euler-Lagrange equation can be interpreted as equality of the two formulas for the discrete momentum,

\[ p_i = D_2 L(q_{i-1}, q_i; h) \]

and

\[ p_i = -D_1 L(q_i, q_{i+1}; h). \]

This gives a natural implementation of the discrete Euler-Lagrange equation as a one-step method

\[ \Phi_h : (q_i, p_i) \mapsto (q_{i+1}, p_{i+1}), \] (3)

which is a symplectic integrator for the Hamiltonian system (1).

### 2.3 Variational error analysis

Here and in the following, we assume that for sufficiently small \( h \) and any pair of boundary values \( (q_0, q_1) \) there exists a unique smooth minimizer \( q : [0, h] \rightarrow Q \) of the action

\[ \int_0^h L(q(t), \dot{q}(t)) \, dt, \]

subject to \( q(0) = q_0 \) and \( q(h) = q_1 \). The corresponding minimal value of the action is called the exact discrete Lagrangian and denoted by

\[ L_{\text{exact}}(q_0, q_1, h) = \min_{q \in C([0, h], Q) \atop q(0) = q_0, q(h) = q_1} \int_0^h L(q(t), \dot{q}(t)) \, dt. \]

The order of a variational integrator can be determined by comparing its discrete Lagrangian to the exact discrete Lagrangian.

**Theorem 1.** If for every smooth curve \( q(t) \) there holds that

\[ L(q(0), q(h), h) - L_{\text{exact}}(q(0), q(h), h) = O(h^\ell), \]

then the variational integrator \( \Phi_h \) defined by the discrete Lagrangian \( L \), in its symplectic form (3), is of order \( \ell \), i.e. it satisfies

\[ \Phi_h(q, p) - \varphi_h(q, p) = O(h^{\ell+1}), \]

where \( \varphi_h \) is the flow over time \( h \) of the continuous Hamiltonian system (1).

This result was first stated in Marsden and West (2001), where the proof contained a flaw which was later fixed by Patrick and Cuell (2009). The difficulty of the proof lies in a singularity in the discrete Legendre transform when \( h \rightarrow 0 \). In a forthcoming work, we will present a new approach to this result from the perspective of modified Lagrangians (Vermeeren (2017)).
2.4 Galerkin variational integrators

An effective method to construct higher order variational integrators is to use a Galerkin discretization. To construct a Galerkin integrator, the space of smooth curves on the time interval of one step, $C^\infty([0, h], Q)$, is replaced by a finite dimensional space of polynomials $P^s([0, h], Q) = \{ q \in C^\infty([0, h], Q) \mid q \text{ a polynomial of degree at most } s \}$. We fix $s + 1$ control points $hd_0 < hd_1 < \ldots < hd_s$, where $d_0 = 0$ and $d_s = 1$. If for each of these control points a value $q(hd_i) = q_i$ is prescribed, then the polynomial $q \in P^s([0, h], Q)$ is uniquely determined. We denote by $\hat{q} : q_0, \ldots, q_s, h$ the polynomial thus obtained.

Given a continuous Lagrangian $L$, we define

$$L_p(t; q_0, \ldots, q_s, h) = L(\hat{q}(t; q_0, \ldots, q_s, h), q(t; q_0, \ldots, q_s, h)),$$

where the subscript $p$ reminds us that this is the Lagrangian evaluated on a polynomial. We would like to consider the discrete action

$$\int_0^h L_p(t; q_0, \ldots, q_s, h) \, dt.$$

However, to evaluate this integral numerically, we need a quadrature rule. We fix quadrature points $c_i \in [0, 1]$ and weights $b_i \in \mathbb{R}$, with $\sum_i b_i = 1$. We denote by $u$ the order of the corresponding quadrature rule. Then for any smooth function $f$ there holds

$$\int_0^h f(t) \, dt = h \sum_i b_i f(hc_i) = O(h^{u+1}).$$

We define the discrete Lagrangian as

$$L(q_0, q_s, h) = \min_{q_1, \ldots, q_{s-1} \in Q} \left( h \sum_i b_i L_p(hc_i; q_0, \ldots, q_s, h) \right),$$

Here we assume that there exists a unique minimizer. This is the case in particular if the Lagrangian is of minimal type and the quadrature rule is sufficiently accurate (Hall and Leok, 2015, Theorem 3.5).

The discrete Lagrangian can also be written as

$$L(q_0, q_s, h) = \min_{q \in P^s([0, h], Q), \quad q(0)=q_0, q(h)=q_s} \mathcal{S}_1[q],$$

where $\mathcal{S}_1$ denotes the internal action,

$$\mathcal{S}_1[q] = h \sum_i b_i L(q(hc_i), q(hc_i)). \quad (4)$$

For more details on the construction of Galerkin variational integrators, see for example Marsden and West (2001); Leok and Shingel (2012); Ober-Blöbaum and Saaae (2015).

3. A FEW TECHNICALITIES

Before we can prove our main result on the superconvergence of Galerkin variational integrators, we need some error estimates for polynomial interpolation.

Lemma 2. Let $q$ be smooth curve and $\hat{q}$ a family of polynomials of degree $s$, parametrized by $h$, which equals $q$ at the control points $0 = h d_0 < h d_1 < \ldots < h d_s = h$. Then for any $k \leq s$ there holds

$$\|q^{(k)} - \hat{q}^{(k)}\|_\infty = O(h^{s+1-k}),$$

where $\| \cdot \|_\infty$ denotes the maximum norm on $[0, h]$.

Proof. Since $q - \hat{q}$ has at least $s + 1$ zeros in the interval $[0, h]$, we know by the mean value theorem that $q - \hat{q}$ has at least $s$ zeros, and recursively we find that $q^{(k)} - \hat{q}^{(k)}$ has at least $s + 1 - k$ zeros. In particular, $q^{(s)} - \hat{q}^{(s)}$ has at least one zero $t_0 \in [0, h]$. Since $\hat{q}^{(s+1)} = 0$ identically, it follows that

$$\|q^{(s)} - \hat{q}^{(s)}\|_\infty \leq \|q^{(s+1)}\|_\infty \max_{t \in [0, h]} (t - t_0) = O(h).$$

Combining this with the fact that $q^{(s-1)} - \hat{q}^{(s-1)}$ has a zero $t_1$ in $[0, h]$, we find

$$\|q^{(s-1)} - \hat{q}^{(s-1)}\|_\infty \leq \|q^{(s)} - \hat{q}^{(s)}\|_\infty \max_{t \in [0, h]} (t - t_1) = O(h^2).$$

Repeating this argument recursively, we obtain

$$\|q^{(k)} - \hat{q}^{(k)}\|_\infty = O(h^{s+1-k}). \quad \square$$

The following proposition contains some simple inequalities that will be useful below.

Proposition 3. For any differentiable curve $q : [0, h] \to Q$ with $\delta q(0) = \delta q(h) = 0$ there holds

(a) $\|\delta q\|_\infty \leq \frac{1}{2} \|\delta q\|_1,$

(b) $\|\delta q\|_1 \leq \frac{h^2}{2} \|\delta q\|_1,$

where $\| \cdot \|_p$ denotes the $L^p$-norm on $[0, h]$. Furthermore, for any differentiable curve $q : [0, h] \to Q$ there holds

(c) $\|q\|_1 \leq \sqrt{h} \|q\|_2.$

Proof.

(a) Let $|\delta q|$ reach its maximum in $(0, h)$ at $t_{\text{max}}$. We have

$$\|\delta q\|_1 = \int_0^{t_{\text{max}}} |\delta q| \, dt + \int_{t_{\text{max}}}^h |\delta q| \, dt \geq |\delta q(t_{\text{max}}) - \delta q(0)| + |\delta q(h) - \delta q(t_{\text{max}})| = 2|h\delta q|_\infty.$$ 

(b) We have

$$\|\delta q\|_1 = \int_0^h |\delta q(t)| \, dt \leq \int_0^h \|\delta q\|_\infty \, dt = h \|\delta q\|_\infty,$$

so the claim follows from inequality (a).

(c) This is a special case of Hölder’s inequality,

$$\|fg\|_\alpha \leq \|f\|_\alpha \|g\|_\beta,$$

with $f = 1$, $g = q$ and $\alpha = \beta = 2$. \square

4. SUPERCONVERGENCE

We now come to our main result.

Theorem 4. Let $L$ be a Galerkin discretization of a Lagrangian $\mathcal{L}$, based on polynomials of degree $s$ and a quadrature rule of degree $u$. Assume that all discrete and continuous critical curves minimize their respective actions. Then the corresponding symplectic integrator (3) is of order $\min(2s, u)$.
Proof. By Theorem 1, it suffices to show that
\[ L_{\text{exact}}(q(0), q(h); h) - L(q(0), q(h); h) = O\left(h^{\min(2s, u)+1}\right) \]
for every smooth curve \( q \).

Let \( q_{\text{EL}} \) denote the unique minimizer of the continuous action with \( q_{\text{EL}}(0) = q(0) \) and \( q_{\text{EL}}(h) = q(h) \). The subscript \( \text{EL} \) reminds us that \( q_{\text{EL}} \) satisfies the continuous Euler-Lagrange equation. Let \( \tilde{q} \in \mathcal{P}^s([0, h], Q) \) be the polynomial that agrees with \( q_{\text{EL}} \) at the control points \( 0 = \hat{h}d_0 < \hat{h}d_1 < \ldots < \hat{h}d_s = h \) and \( \tilde{q} \in \mathcal{P}^s([0, h], Q) \) the polynomial that minimizes the internal action
\[ \mathcal{E}_1[q] = h \sum_i b_i \mathcal{L}(q(hc_i), \dot{q}(hc_i)) \]
in \( \mathcal{P}^s([0, h], Q) \). Since
\[ L_{\text{exact}}(q_0, q_h; h) = \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt \]
we have to show that
\[ \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt - h \sum_i b_i \mathcal{L}(\tilde{q}(hc_i), \dot{\tilde{q}}(hc_i)) = O\left(h^{\min(2s, u)+1}\right) \]
(5)

We expand this difference as
\[ \left( \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt - \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt \right) + \left( \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt - h \sum_i b_i \mathcal{L}(\tilde{q}(hc_i), \dot{\tilde{q}}(hc_i)) \right) \]
(6)

We start with the first term of (6). From Lemma 2 we know that \( q - \tilde{q} = O(h^{s+1}) \) and \( \dot{q} - \dot{\tilde{q}} = O(h^s) \), hence
\[ \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt - \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt = \int_0^h \left( \frac{\partial L(q_{\text{EL}}, \dot{q}_{\text{EL}})}{\partial q} (q_{\text{EL}} - \tilde{q}) + \frac{\partial L(q_{\text{EL}}, \dot{q}_{\text{EL}})}{\partial \dot{q}} (\dot{q}_{\text{EL}} - \dot{\tilde{q}}) + O(h^{2s}) \right) \, dt \]
\[ = \int_0^h \left( \left( \frac{\partial L(q_{\text{EL}}, \dot{q}_{\text{EL}})}{\partial q} - \frac{d}{dt} \frac{\partial L(q_{\text{EL}}, \dot{q}_{\text{EL}})}{\partial q} \right) (q_{\text{EL}} - \tilde{q}) + O(h^{2s}) \right) \, dt \]
\[ + \left( \frac{\partial L(q_{\text{EL}}, \dot{q}_{\text{EL}})}{\partial \dot{q}} (\dot{q}_{\text{EL}} - \dot{\tilde{q}}) \right) \bigg|_0^h. \]

The boundary term vanishes because \( \dot{q}(0) = q_{\text{EL}}(0) \) and \( \dot{\tilde{q}}(h) = q_{\text{EL}}(h) \). Furthermore, \( q_{\text{EL}} \) solves the Euler-Lagrange equation, so we find
\[ \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt - \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt = \int_0^h O(h^{2s}) \, dt \]
(7)

To bound the second term of (6) we follow the arguments of (Hall and Leok, 2015, Theorem 3.3). Since \( \tilde{q} \) is the minimizing element of \( \mathcal{P}^s([0, h], Q) \), we have
\[ h \sum_i b_i \mathcal{L}(\tilde{q}(hc_i), \dot{\tilde{q}}(hc_i)) \leq h \sum_i b_i \mathcal{L}(\tilde{q}(hc_i), \dot{\tilde{q}}(hc_i)) \]
\[ \leq \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt + O(h^{u+1}) \]
On the other hand, since \( q_{\text{EL}} \) minimizes the continuous action, there holds
\[ h \sum_i b_i \mathcal{L}(\tilde{q}(hc_i), \dot{\tilde{q}}(hc_i)) \]
\[ = \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt + O(h^{u+1}) \]
\[ = \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt + O(h^{u+1}) \]
\[ = \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt + O(h^{2s+1} + h^{u+1}), \]
where the last line follows from (7). Combining both inequalities we find
\[ \int_0^h L(\tilde{q}, \dot{\tilde{q}}) \, dt - h \sum_i b_i \mathcal{L}(\tilde{q}(hc_i), \dot{\tilde{q}}(hc_i)) = O(h^{\min(2s, u)+1}). \]
(8)

Equations (7) and (8) together imply the desired result (5). \( \square \)

5. CONVERGENCE OF THE GALERKIN CURVES

Theorem 4 states that for a sufficiently accurate quadrature rule, the one-step method obtained by Galerkin discretization has order \( 2s \), twice the degree of polynomials used. If we compare the polynomial approximations to the exact solution at arbitrary times (away from the mesh points), we find an error of order \( s \), the same as the degree of polynomials. This halving of the order was also observed in (Hall and Leok, 2015, Section 3.4). Below we prove this claim under a coercivity assumption (9). This assumption is satisfied in particular for mechanical Lagrangians, as shown for the case of a constant mass matrix in Hall and Leok (2015). We will provide a more general proof in a forthcoming publication.

Theorem 5. Assume that there exists a \( C > 0 \) such that for every continuous critical curve \( q_{\text{EL}} \) of the action \( \mathcal{S} \) and for any variation \( \delta q \), vanishing at the endpoints, there holds
\[ \mathcal{S}[q_{\text{EL}} + \delta q] - \mathcal{S}[q_{\text{EL}}] \geq C \|\delta q\|^2. \]
(9)

Then, for sufficiently small \( h \), the polynomial \( \tilde{q} \) of degree \( s \) minimizing the discrete action satisfies
\[ \|\tilde{q} - q_{\text{EL}}\| = O\left(h^{\min(s, \frac{u}{2})+1}\right) \]
and
\[ \|\tilde{q} - q_{\text{EL}}\| = O\left(h^{\min(s, \frac{u}{2})+2}\right) \]
where \( \| \cdot \|_p \) denotes the \( L^p \)-norm on \([0, h]\).

Proof. From the proof of Theorem 4 we know that
\[ \mathcal{S}[\tilde{q}] - \mathcal{S}[q_{\text{EL}}] = \int_0^h L(\tilde{q}, \dot{\tilde{q}}) - \int_0^h L(q_{\text{EL}}, \dot{q}_{\text{EL}}) \, dt \]
\[ = L(q(0), q(h); h) - L_{\text{exact}}(q(0), q(h); h) + O(h^{u+1}) \]
\[ = O(h^{\min(2s, u)+1}). \]
From (9), with \( \delta q = \tilde{q} - q_{EL} \), it now follows that
\[
C \| \tilde{q} - q_{EL} \|_2^2 = O \left( h^{\min(2s,u)+1} \right)
\]
and hence
\[
\| \tilde{q} - q_{EL} \|_2 = O \left( h^{\min(s,\frac{u}{4}) + \frac{1}{2}} \right).
\]
By Hölder’s inequality (Proposition 3(c)) it follows that
\[
\| \tilde{q} - q_{EL} \|_\infty = O \left( h^{\min(s,\frac{u}{4}) + 1} \right).
\]
Finally, since \( q_{EL}(0) = \tilde{q}(0) \) and \( q_{EL}(h) = \tilde{q}(h) \), it follows from Proposition 3(a) that
\[
\| \tilde{q} - q_{EL} \|_\infty = O \left( h^{\min(s,\frac{u}{4}) + 1} \right)
\]
and from Proposition 3(b) that
\[
\| \tilde{q} - q_{EL} \|_1 = O \left( h^{\min(s,\frac{u}{4}) + 2} \right).
\]

6. POSSIBLE EXTENSION TO FORCED SYSTEMS

Lagrangian systems with external forces are an important extension of the theory, especially towards the study of optimal control problems. Variational integrators for systems with external forces were presented in Marsden and West (2001), along with a brief argument suggesting variational error analysis is possible in this case too. This is studied in detail in a recent preprint by Fernández et al. (2021). In the forced case order estimates can be obtained by comparing the discrete Lagrangian to the exact discrete Lagrangian and the discrete forces to the exact discrete forces. Alternatively, a forced system can be embedded into a Lagrangian system without external forces of higher dimension. This approach to variational error analysis of forced systems was taken by De Diego and de Almagro (2018).

6.1 Forced Galerkin integrators

Forced Lagrangian systems are defined by the Lagrange-d’Alembert principle
\[
\delta \int_a^b L(q, \dot{q}) \, dt + \int_a^b f(q, \dot{q}) \delta q \, dt = 0.
\]
The corresponding (forced) Euler-Lagrange equation is
\[
\frac{\partial L(q, \dot{q})}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} + f(q, \dot{q}) = 0.
\]
(10)
The discrete Lagrange-d’Alembert principle requires a discrete Lagrangian \( L(q_{k-1}, q_k; h) \) and discrete forces \( F^\pm(q_{k-1}, q_k; h) \). It reads
\[
\sum_k L(q_k, q_{k+1}) + \sum_k (F^-(q_k, q_{k+1}) + F^+(q_{k-1}, q_k)) \delta q_k = 0
\]
and yields the equations
\[
D_1 L(q_k, q_{k+1}) + D_2 L(q_{k-1}, q_k) + F^-(q_k, q_{k+1}) + F^+(q_{k-1}, q_k) = 0.
\]
Galerkin integrators for forced systems are constructed as follows (see Campos et al. (2015)). As in Section 2.4 we denote by \( \hat{q}(\cdot; q_0, \ldots, q_s, h) \) the polynomial of degree at most \( s \) defined by its values \( q_0, \ldots, q_s \) at control points
\[
\hat{q}(t; q_0, \ldots, q_s, h) = \sum_{k=0}^s \hat{q}_k(t; q_0, \ldots, q_s, h)
\]
and
\[
\hat{q}_k(t; q_0, \ldots, q_s, h) = \frac{1}{h^k} \sum_{\nu=0}^h \hat{q}(t; q_0, \ldots, q_s, h, t_k).
\]
(11)
where \( h = h_d < h_1 < \ldots < h_d_s \), where \( d_0 = 0 \) and \( d_s = 1 \). We define
\[
L_p(t; q_0, \ldots, q_s, h) = L \left( \hat{q}(t; q_0, \ldots, q_s, h), \dot{\hat{q}}(t; q_0, \ldots, q_s, h) \right)
\]
and
\[
f_p(t; q_0, \ldots, q_s, h) = f \left( \hat{q}(t; q_0, \ldots, q_s, h), \dot{\hat{q}}(t; q_0, \ldots, q_s, h) \right).
\]
Consider quadrature points \( c_i \in [0,1] \) and weights \( b_i \in \mathbb{R} \). Given \( q_0 \) and \( q_s \), we impose
\[
\delta \sum_i b_i L_p(hc_i; q_0, \ldots, q_s, h) + \sum_i b_i f_p(hc_i; q_0, \ldots, q_s, h) \frac{\partial \hat{q}(hc_i; q_0, \ldots, q_s, h)}{\partial q_k} = 0, \quad i = 1, \ldots, s-1.
\]
where \( \delta \) stands for arbitrary variations of the interior control values \( q_1, \ldots, q_{s-1} \). This gives us \( s-1 \) equations
\[
\sum_i b_i \frac{\partial L}{\partial q_k}(hc_i; q_0, \ldots, q_s, h) + \sum_i b_i \frac{\partial f}{\partial q_k}(hc_i; q_0, \ldots, q_s, h) \frac{\partial \hat{q}(hc_i; q_0, \ldots, q_s, h)}{\partial q_k} = 0, \quad i = 1, \ldots, s-1
\]
(12)
Equations (12)–(13) are the discrete Lagrangian and discrete forces defining the Galerkin integrator with polynomials of degree \( s \) and quadrature rule given by \((b_i, c_i)\).

6.2 Obstruction to proving superconvergence

In the presence of external forces, the exact discrete Lagrangian depends not just on the Lagrangian, but also on the external forces. It is obtained by evaluating the action over the interval \([0, h]\) on the solution \( q_{EL} \) of the forced Euler-Lagrange equation (10). Hence the estimate for the first term in (6) becomes
\[
\int_0^h L(q_{EL}, \dot{q}_{EL}) \, dt - \int_0^h \tilde{L}(\tilde{q}, \dot{\tilde{q}}) \, dt
\]
\[= \int_0^h \left( \left( \frac{\partial L(q_{EL}, \dot{q}_{EL})}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L(q_{EL}, \dot{q}_{EL})}{\partial \dot{q}} \right)(q_{EL} - \tilde{q}) + O(h^{2s}) \right) \, dt
\]
\[= - \int_0^h f(q_{EL}, \dot{q}_{EL})(q_{EL} - \tilde{q}) \, dt + O(h^{2s+1}). \]
(14)
For generic forces $f$ we have $f(q_{EL}, \dot{q}_{EL}) = O(1)$ and $q_{EL} - \tilde{q} = O(h^*)$, so we can only estimate (14) by $O(h^{s+1})$. We expect a similar estimate for the difference between the exact discrete forces

$$F^\text{exact}(q_0, q_s; h) = \int_0^h f(q_{EL}, \dot{q}_{EL}) \frac{\partial q_{EL}}{\partial q_0},$$

and their numerical approximations. This means that we cannot prove superconvergence using the forced analogue of variational error analysis.

6.3 Possible workaround

In the previous subsection we observed that our proof of superconvergence fails in the presence of external forces. This is because our proof requires that the dynamics are given by Hamilton’s principle rather than the Lagrange-d’Alembert principle. Still, we expect a superconvergence result to hold for forced systems too. This expectation is based on well-understood low-order methods (e.g. the midpoint rule and Störmer-Verlet method are Galerkin integrators based on linear polynomials, but they are second order methods) as well as preliminary numerical observations for higher-order methods.

A potential way to remedy our proof is the observation that forced systems can also be described by Hamilton’s principle if we double the dimension and introduce a variable $Q$, which in the end we will require to be a copy of $q$ (see Galley (2013)). In particular, we consider the extended Lagrangian

$$L^e(q, Q, \dot{q}, \dot{Q}) = L(Q, \dot{Q}) - L(q, \dot{q}) + \frac{1}{2}(f(Q, \dot{Q}) + f(q, \dot{q}))(Q - q).$$

Taking variations with respect to $Q$ we find the Euler-Lagrange equation

$$\frac{\partial L(Q, \dot{Q})}{\partial Q} - \frac{d}{dt} \frac{\partial L(Q, \dot{Q})}{\partial \dot{Q}} + \frac{1}{2}f(Q, \dot{Q}) + \frac{1}{2}f(q, \dot{q}))(\dot{Q} - \dot{q}) = 0.$$

When we impose $Q = q$ this equation reduces to the familiar forced Euler-Lagrange equation (10). The same conclusion holds for variations with respect to $q$. As pointed out by De Diego and de Almagro (2018), this observation can be used to apply variational error analysis to forced systems. In our present context, we need to show that the forced Galerkin integrator (11) is equivalent to a Galerkin integrator for the extended system (16).

Consider the Galerkin integrator for (16) defined by the Lagrangian

$$L^p_p(t; q_0, Q_0), \ldots, (q_s, Q_s), h)$$

and the quadrature rule with points $c_i$ and weights $b_i$. By Theorem 4 this integrator is of order $2s$ if the quadrature rule is sufficiently accurate and if all critical curves minimize the action. Varying $Q_k$ (or $q_k$) for some $k$ with $0 < k < s$, and then imposing $Q_\ell = q_\ell$ for all $0 < \ell < s$, leads to the internal equations (11). As before, we assume that these uniquely determine $q_1 = Q_1, \ldots, q_{s-1} = Q_{s-1}$ as functions of $q_0, Q_0, q_s, Q_s$, allowing us to define the extended discrete Lagrangian

$$L^e(q_0, Q_0, q_s, Q_s, h) = \sum_i h b_i L^p_p(h c_i; (q_0, Q_0), \ldots, (q_s, Q_s), h).$$

Its discrete Euler-Lagrange equations, evaluated on $Q_0 = q_0, Q_s = q_s$, are equivalent to the discrete Lagrange-d’Alembert principle for (12)–(13). Hence the forced Galerkin integrator defined by (12)–(13) is of order $2s$.

The attentive reader may have noticed a problem with the argument above. To apply Theorem 4 to Galerkin integrator for the extended system, we need to property that critical curves of the action are minimizers, but this does not hold for Lagrangians of the form (15). In the proof of Theorem 4 we used this assumption to show that the minimizing polynomial $\tilde{q}$ is close to the polynomial interpolating the continuous solution $\hat{q}$ and hence to estimate the second term in (6). However, it is plausible that even without this assumption the difference $q - \tilde{q}$ will be small in a generic case. We currently do not have a precise statement of this claim, so this proof is left to be finished in future work.

7. CONCLUSION AND OUTLOOK

Following the approach of Hall and Leok (2015), but with stronger error bounds obtained from the calculus of variations, we have shown that Galerkin variational integrators exhibit superconvergence: given a suitably accurate quadrature rule, the order of such an integrator is twice the degree of polynomials used to construct it. In our presentation here we have relied heavily on established results concerning variational error analysis and kept technical details to a minimum. Since some of these details are worthy of attention, we plan to continue this topic in a forthcoming paper, where we will

- Present a new proof of Theorem 1 based on modified Lagrangians.
- Show that mechanical Lagrangians (with a possibly position-dependent mass matrix) and their discretizations satisfy a coercivity condition as assumed in Theorem 5. From this condition it also follows that critical curves are minimizers.
- Provide numerical experiments to illustrate Theorems 4 and 5, beyond what is already available in the work of Ober-Blöbaum and Sachse (2015).

An important additional topic for future work is to turn the arguments sketched in 6.3 into a rigorous proof of superconvergence in the presence of external forces.

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