On statistical approximation properties of $q$-Baskakov-Szász-Stancu operators

Vishnu Narayan Mishra$^{a,1}$, Preeti Sharma$^a$, Lakshmi Narayan Mishra$^{b,c}$

$^a$Department of Applied Mathematics & Humanities, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat - 395 007 (Gujarat), India
$^b$L. 1627 Awadh Puri Colony Beniganj, Phase -III, Opposite - Industrial Training Institute (I.T.I.), Ayodhya Main Road, Faizabad - 224 001, (Uttar Pradesh), India
$^c$Department of Mathematics, National Institute of Technology, Silchar - 788 010, Cachar (Assam), India

Abstract

In the present paper, we consider Stancu type generalization of Baskakov-Szász operators based on the $q$-integers and obtain statistical and weighted statistical approximation properties of these operators. Rates of statistical convergence by means of the modulus of continuity and the Lipschitz type maximal function are also established for operators.

Keywords: $q$-integers, $q$-Baskakov-Szász-Stancu operators, rate of statistical convergence, modulus of continuity, Lipschitz type maximal functions.

2000 Mathematics Subject Classification: Primary 41A10, 41A25, 41A36.

1. Introduction

In the recent years several operators of summation-integral type have been proposed and their approximation properties have been discussed. In the present paper our aim is to investigate statistical approximation properties of a Stancu type $q$-Baskakov-Szász operators. Firstly, Baskakov-Szász operators based on $q$-integers was introduced by Gupta [1] and established some approximation results. The $q$-Baskakov-Szász operators is defined as follows:

$$ D_n^{q}(f;x) = \left[ n \right]_q \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \int_0^{\left[ n \right]_q x} q^{k-1} s_{n,k}^{q}(t) f \left( \frac{t}{q^k} \right) d_q t, \quad (1.1) $$

where $x \in [0, \infty)$ and

$$ p_{n,k}^{q}(x) = \left[ \frac{n+k-1}{k} \right] q^{k(k-1)/2} \frac{x^k}{(1+x)^{n+k}}, \quad (1.2) $$

and

$$ s_{n,k}^{q}(t) = E(-[n]_q t \left[ \frac{[n]_q t}{[k]_q} \right]), \quad (1.3) $$

In case $q = 1$, the above operators reduce to the Baskakov–Szász operators [2].

Later, Mishra and Sharma [3] introduced a new Stancu type generalization of $q$-Baskakov-Szász operators is defined as

$$ D_n^{(\alpha,\beta)}(f; q; x) = \left[ n \right]_q \sum_{k=0}^{\infty} p_{n,k}^{q}(x) \int_0^{\left[ n \right]_q x} q^{k-1} s_{n,k}^{q}(t) f \left( \frac{[n]_q t q^{-k} + \alpha}{[n]_q + \beta} \right) d_q t, \quad (1.4) $$

where $p_{n,k}^{q}(x)$ and $s_{n,k}^{q}(t)$ are Baskakov and Szász basis function respectively, defined as above. The operators $D_n^{(\alpha,\beta)}(f; q; x)$ in (1.4) are called $q$-Baskakov-Szász-Stancu operators. For $\alpha = 0$, $\beta = 0$ the operators (1.4) reduce to

Email addresses: vishnunarayamishra@gmail.com, vishnu_narayanmishra@yahoo.co.in, v_n_mishra_hifi@yahoo.co.in (Vishnu Narayan Mishra), preeti.iitan@gmail.com (Preeti Sharma), lakshminarayamishra04@gmail.com (Lakshmi Narayan Mishra)

1Corresponding author
to the operators (1.1).

In the recent years, Stancu generalization of the certain operators introduced by several researchers and obtained different type of approximation properties of many operators, we refer some of the important papers in this direction as [4, 5, 6]. Recently, Mishra et al. ([14], [16]) have established very interesting results on approximation properties of various functional classes using different types of positive linear summability operators.

Before proceeding further, let us give some basic definitions and notations from \( q \)-calculus. Such notations can be found in ([7], [8]). We consider \( q \) as a real number satisfying \( 0 < q < 1 \).

For \( [n]_q = \begin{cases} 1 - q^n, & q \neq 1, \\ n, & q = 1, \end{cases} \)

and

\[ [n]_q! = \begin{cases} [n]_q[n - 1]_q[n - 2]_q\cdots[1]_q, & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases} \]

Then for \( q > 0 \) and integers \( n, k, k \geq n \geq 0 \), we have

\[ [n + 1]_q = 1 + q[n]_q \quad \text{and} \quad [n]_q + q^n[k - n]_q = [k]_q. \]

We observe that

\[ (1 + x)_n = (-x; q)_n = \begin{cases} (1 + x)(1 + qx)(1 + q^2x)\cdots(1 + q^{n-1}x), & n = 1, 2, \ldots, \\ 1, & n = 0. \end{cases} \]

Also, for any real number \( \alpha \), we have

\[ (1 + x)_\alpha = \frac{(1 + x)^\infty}{(1 + q^\alpha x)^\infty}. \]

In special case, when \( \alpha \) is a whole number, this definition coincides with the above definition.

The \( q \)-Jackson integral and \( q \)-improper integral defined as

\[ \int_0^a f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)q^n \]

and

\[ \int_0^{\infty/A} f(x) d_q x = (1 - q)a \sum_{n=0}^{\infty} f \left( \frac{q^n}{A} \right) \frac{q^n}{A}, \]

provided sum converges absolutely.

The \( q \)-analogues of the exponential function \( e^x \) (see [8]), used here is defined as

\[ E_q(z) = \prod_{j=0}^{\infty} (1 + (1 - q)q^jz) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]_q!} = (1 + (1 - q)z)_q^{\infty}, \quad |q| < 1, \]

where \( (1 - x)_q^{\infty} = \prod_{j=0}^{\infty} (1 - q^jx) \).

2. Moment estimates

**Lemma 1.** ([4]): The following hold:

1. \( D_n(1, q; x) = 1 \),
2. \( D_n(t, q; x) = x + \frac{q}{[n]_q} \).
3. \[ D_n(t^2, q; x) = \left( 1 + \frac{1}{q[n]_q} \right) x^2 + \frac{x}{[n]_q} (1 + q(q + 2)) + \frac{q^2(1 + q)}{[n]_q^2}. \]

**Lemma 2.** The following hold:
1. \[ D_n^{(\alpha, \beta)}(1; q; x) = 1, \]
2. \[ D_n^{(\alpha, \beta)}(t; q; x) = \frac{[n]_q x + q + \alpha}{[n]_q + \beta}, \]
3. \[ D_n^{(\alpha, \beta)}(t^2; q; x) = \left( \frac{[n]_q (q[n]_q + 1)}{q([n]_q + \alpha)^2} \right) x^2 + \frac{[n]_q q + 2\alpha [n]_q}{([n]_q + \alpha)^2} x + \frac{q^2(1 + q) + 2q\alpha + \alpha^2}{([n]_q + \alpha)^2}. \]

### 3. Korovkin type statistical approximation properties

The idea of statistical convergence was introduced independently by Steinhaus [8], Fast [10] and Schoenberg [11]. In approximation theory, the concept of statistical convergence was used in the year 2002 by Gadjiev and Orhan [12]. They proved the Bohman-Korovkin type approximation theorem for statistical convergence. It was shown that the statistical versions are stronger than the classical ones.

Korovkin type approximation theory has also many useful connections, other than classical approximation theory, in other branches of mathematics (see Altomare and Campiti in [13]).

Now, we recall the concept of statistical convergence for sequences of real numbers which was introduced by Fast [10] and Mishra et al. [13].

Let \( K \subseteq \mathbb{N} \) and \( K_n = \{ j \leq n : j \in K \} \). Then the *natural density* of \( K \) is defined by \( \delta(K) = \lim_n n^{-1}|K_n| \) if the limit exists, where \( |K_n| \) denotes the cardinality of the set \( K_n \).

A sequence \( x = (x_j)_{j \geq 1} \) of real numbers is said to be *statistically convergent* to \( L \) provided that for every \( \epsilon > 0 \) the set \( \{ j \in \mathbb{N} : |x_j - L| \geq \epsilon \} \) has natural density zero, i.e. for each \( \epsilon > 0 \),

\[
\lim_n \frac{1}{n} |\{ j \leq n : |x_j - L| \geq \epsilon \}| = 0.
\]

It is denoted by \( st - \lim_n x_n = L \).

In [17] Doru and Kanat defined the Kantorovich-type modification of Lupas operators as follows:

\[
\tilde{R}_n(f; q; x) = [n + 1] \sum_{k=0}^{n} \left( \int_{[k]_q}^{[k+1]_q} f(t) dt \right) \left( \begin{array}{c} n \\ k \end{array} \right) \frac{q^{-k}q^{k-1}x^{k-1/2}}{q^{(n-k)/2}(1-x+q x) \cdots (1-x+q^{n-k+1}x)}. \tag{3.1}
\]

Doru and Kanat [17] proved the following statistical Korovkin-type approximation theorem for operators (3.1).

**Theorem 1.** Let \( q := (q_n), 0 < q < 1, \) be a sequence satisfying the following conditions:

\[
st - \lim_n q_n = 1, \quad st - \lim_n q_n^a = a \quad (a < 1) \quad \text{and} \quad st - \lim_n \frac{1}{[n]_q} = 0, \tag{3.2}
\]

then if \( f \) is any monotone increasing function defined on \([0,1] \), for the positive linear operators \( \tilde{R}_n(f; q; x) \), then

\[
st - \lim_n \| \tilde{R}_n(f; q; \cdot) - f \|_{C[0,1]} = 0
\]

holds.

In [18] Doğru gave some examples so that \( (q_n) \) is statistically convergent to 1 but it may not convergent to 1 in the ordinary case.

Now, we consider a sequence \( q = (q_n), q_n \in (0,1) \), such that

\[
\lim_{n \to \infty} q_n = 1. \tag{3.3}
\]

The condition (3.3) guarantees that \( [n]_q \to \infty \) as \( n \to \infty \).
Theorem 2. Let $D_n^{(\alpha,\beta)}$ be the sequence of the operators (1.4) and the sequence $q = (q_n)$ satisfies (3.2). Then for any function $f \in C[0,\nu] \subset C[0,\infty)$, $\nu > 0$, we have

$$st - \lim_n \|D_n^{(\alpha,\beta)}(f;q;.) - f\| = 0,$$

(3.4)

where $C[0,\nu]$ denotes the space of all real bounded functions $f$ which are continuous in $[0,\nu]$.

**Proof.** Let $f_i = t^i$, where $i = 0, 1, 2$. Using $D_n^{(\alpha,\beta)}(1; q_n; x) = 1$, it is clear that

$$st - \lim_n \|D_n^{(\alpha,\beta)}(1; q_n; x) - 1\| = 0.$$

Now by Lemma (2)(ii), we have

$$\lim_{n \to \infty} \|D_n^{(\alpha,\beta)}(t; q_n; x) - x\| = \left\| \frac{[n]_q x + q + \alpha}{[n]_q + \beta} - x \right\| \leq \frac{(q + \alpha)x + \beta}{([n]_q + \beta)}.$$ 

For given $\epsilon > 0$, we define the following sets:

$$L = \{ k : \|D_n^{(\alpha,\beta)}(t; q_k; x) - x\| \geq \epsilon \},$$

and

$$L' = \left\{ k : \frac{(q + \alpha)}{[k]_q + \beta}x + \frac{\beta}{[k]_q + \beta} \geq \epsilon \right\}.$$

(3.5)

It is obvious that $L \subset L'$, it can be written as

$$\delta \left( \{ k \leq n : \|D_n^{(\alpha,\beta)}(t; q_k; x) - x\| \geq \epsilon \} \right) \leq \delta \left( \{ k \leq n : \frac{(q + \alpha)}{[k]_q + \beta}x + \frac{\beta}{[k]_q + \beta} \geq \epsilon \} \right).$$

By using (3.2), we get

$$st - \lim_n \left( \frac{(q + \alpha)}{[n]_q + \beta}x + \frac{\beta}{[n]_q + \beta} \right) = 0.$$

So, we have

$$\delta \left( \{ k \leq n : \frac{(q + \alpha)}{[k]_q + \beta}x + \frac{\beta}{[k]_q + \beta} \geq \epsilon \} \right) = 0,$$

then

$$st - \lim_n \|D_n^{(\alpha,\beta)}(t; q_n; x) - x\| = 0.$$

Similarly, by Lemma (2)(iii), we have

$$\|D_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\|$$

$$= \left\| \frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} \right\| x^2 + \left( \frac{(1 + q(q + 2)) [n]_q + 2\alpha [n]_q}{([n]_q + \beta)^2} \right) x + \frac{q^2(1 + q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2} - x^2$$

$$\leq \mu^2 \left[ \left( \frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} - 1 \right) + \left( \frac{(1 + q(q + 2)) [n]_q + 2\alpha [n]_q}{([n]_q + \beta)^2} \right) \nu + \left( \frac{q^2(1 + q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2} \right) \right]$$

where $\mu^2 = \max\{\nu^2, \nu, 1\} = \nu^2$.

Now, if we choose

$$\alpha_n = \left( \frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} - 1 \right),$$

4
\[
\beta_n = \left(\frac{(1 + q(q + 2))n[q] + 2\alpha n[q]}{(n[q] + \beta)^2}\right),
\]
\[
\gamma_n = \left(\frac{q^2(1 + q) + 2\alpha\beta}{(n[q] + \beta)^2}\right),
\]

now using (3.2) we can write
\[st - \lim_{n \to \infty} \alpha_n = 0 = st - \lim_{n \to \infty} \beta_n = st - \lim_{n \to \infty} \gamma_n. \tag{3.6}\]

Now for given \(\epsilon > 0\), we define the following four sets
\[
\mathcal{U} = \{k : \|D_n^{(a,b)}(t^2; q_n; x) - x^2\| \geq \epsilon\},
\]
\[
\mathcal{U}_1 = \{k : \alpha_k \geq \frac{\epsilon}{\mu^2}\},
\]
\[
\mathcal{U}_2 = \{k : \beta_k \geq \frac{\epsilon}{\mu^2}\},
\]
\[
\mathcal{U}_3 = \{k : \gamma_k \geq \frac{\epsilon}{\mu^2}\}.
\]

It is obvious that \(\mathcal{U} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3\). Then, we obtain
\[
\delta\left(\{k \leq n : \|D_n^{(a,b)}(t^2; q_n; x) - x^2\| \geq \epsilon\}\right) \\
\leq \delta\left(\{k \leq n : \alpha_k \geq \frac{\epsilon}{\mu^2}\}\right) + \delta\left(\{k \leq n : \beta_k \geq \frac{\epsilon}{\mu^2}\}\right) + \delta\left(\{k \leq n : \gamma_k \geq \frac{\epsilon}{\mu^2}\}\right).
\]

Using (3.6), we get
\[st - \lim_{n \to \infty} \|D_n^{(a,b)}(t^2; q_n; x) - x^2\| = 0.
\]

Since,
\[
\|D_n^{(a,b)}(f; q_n; x) - f\| \leq \|D_n^{(a,b)}(t^2; q_n; x) - x^2\| + \|D_n^{(a,b)}(t; q_n; x) - x\| + \|D_n^{(a,b)}(1; q_n; x) - 1\|
\]
we get
\[
st - \lim_{n \to \infty} \|D_n^{(a,b)}(f; q_n; x) - f\| \leq st - \lim_{n \to \infty} \|D_n^{(a,b)}(t^2; q_n; x) - x^2\| \\
+ st - \lim_{n \to \infty} \|D_n^{(a,b)}(t; q_n; x) - x\| \\
+ st - \lim_{n \to \infty} \|D_n^{(a,b)}(1; q_n; x) - 1\|,
\]
which implies that
\[st - \lim_{n \to \infty} \|D_n^{(a,b)}(f; q_n; x) - f\| = 0.
\]

This completes the proof of theorem. \(\square\)

4. Weighted statistical approximation

In this section, we obtain the Korovkin type weighted statistical approximation by the operators defined in (4.4). A real function \(\rho\) is called a weight function if it is continuous on \(\mathbb{R}\) and \(\lim_{|x| \to \infty} \rho(x) = \infty, \rho(x) \geq 1\) for all \(x \in \mathbb{R}\).

Let by \(B_\rho(\mathbb{R})\) denote the weighted space of real-valued functions \(f\) defined on \(\mathbb{R}\) with the property \(|f(x)| \leq M_f \rho(x)\) for all \(x \in \mathbb{R}\), where \(M_f\) is a constant depending on the function \(f\). We also consider the weighted subspace \(C_\rho(\mathbb{R})\) of \(B_\rho(\mathbb{R})\) given by \(C_\rho(\mathbb{R}) = \{f \in B_\rho(\mathbb{R}) : f\ \text{continuous on } \mathbb{R}\}\). Note that \(B_\rho(\mathbb{R})\) and \(C_\rho(\mathbb{R})\) are Banach spaces with \(\|f\|_\rho = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}\). In case of weight function \(\rho_0 = 1 + x^2\), we have \(\|f\|_{\rho_0} = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{1 + x^2}\).

Now we are ready to prove our main result as follows:
Theorem 3. Let $\mathcal{D}_n^{(\alpha,\beta)}$ be the sequence of the operators (1.4) and the sequence $q = (q_n)$ satisfies (3.2). Then for any function $f \in C_B[0, \infty)$, we have

$$st - \lim_{n \to \infty} \|D_n^{(\alpha,\beta)}(f; q_n; \cdot) - f\|_{\rho_0} = 0.$$ 

Proof. By Lemma (2)(iii), we have $\mathcal{D}_n^{(\alpha,\beta)}(t^2; q_n; x) \leq Cx^2$, where $C$ is a positive constant, $\mathcal{D}_n^{(\alpha,\beta)}(f; q_n; x)$ is a sequence of positive linear operators acting from $C_q[0, \infty)$ to $B_q[0, \infty)$.

Using $\mathcal{D}_n^{(\alpha,\beta)}(1; q_n; x) = 1$, it is clear that

$$st - \lim_{n} \|\mathcal{D}_n^{(\alpha,\beta)}(1; q_n; x) - 1\|_{\rho_0} = 0.$$ 

Now, by Lemma (2)(ii), we have

$$\|\mathcal{D}_n^{(\alpha,\beta)}(t; q_n; x) - x\|_{\rho_0} = \sup_{x \in [0, \infty)} \frac{|\mathcal{D}_n^{(\alpha,\beta)}(t; q_n; x) - x|}{1 + x^2} \leq \frac{(q + \alpha) + \beta}{[n]_q + \beta}.$$ 

Using (3.2), we get

$$st - \lim_{n} \left(\frac{(q + \alpha) + \beta}{[n]_q + \beta}\right) = 0,$$

then

$$st - \lim_{n} \|\mathcal{D}_n^{(\alpha,\beta)}(t; q_n; x) - x\|_{\rho_0} = 0.$$ 

Finally, by Lemma (2)(iii), we have

$$\|\mathcal{D}_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2\|_{\rho_0} \leq \left(\frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} - 1\right) \sup_{x \in [0, \infty)} \frac{x^2}{1 + x^2}$$

$$+ \left(\frac{1 + q(q + 2)}{([n]_q + 2\alpha[n]_q)}\right) \sup_{x \in [0, \infty)} \frac{x}{1 + x^2} + \frac{q^2(1 + q + 2\alpha + \alpha^2)}{([n]_q + \beta)^2}.$$ 

Now, if we choose

$$\alpha_n = \left(\frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} - 1\right),$$

$$\beta_n = \left(\frac{1 + q(q + 2)}{([n]_q + 2\alpha[n]_q)}\right),$$

$$\gamma_n = \left(\frac{q^2(1 + q + 2\alpha + \alpha^2)}{([n]_q + \beta)^2}\right),$$

then by (3.2), we can write

$$st - \lim_{n \to \infty} \alpha_n = 0 = st - \lim_{n \to \infty} \beta_n = st - \lim_{n \to \infty} \gamma_n.$$ 

(4.1)

Now for given $\epsilon > 0$, we define the following four sets:

$$S = \{ k : \|\mathcal{D}_n^{(\alpha,\beta)}(t^2; q_k; x) - x^2\|_{\rho_0} \geq \epsilon \},$$

$$S_1 = \{ k : \alpha_k \geq \frac{\epsilon}{3} \},$$

$$S_2 = \{ k : \beta_k \geq \frac{\epsilon}{3} \},$$

$$S_3 = \{ k : \gamma_k \geq \frac{\epsilon}{3} \}.$$
This completes the proof of the theorem. It is obvious that \( S \subseteq S_1 \cup S_2 \cup S_3 \). Then, we obtain
\[
\delta \left( \{ k : n : \| D_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2 \|_{\rho_0} \geq \epsilon \} \right)
\leq \delta \left( \{ k \in S \} \right) + \delta \left( \{ k \in S_2 \} \right) + \delta \left( \{ k \in S_3 \} \right).
\]
Using (4.1), we get
\[
st - \lim_{n \to \infty} \| D_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2 \|_{\rho_0} = 0.
\]
Since
\[
\| D_n^{(\alpha,\beta)}(f; q_n; x) - f \|_{\rho_0} 
\leq \| D_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2 \|_{\rho_0} + \| D_n^{(\alpha,\beta)}(t; q_n; x) - x \|_{\rho_0} + \| D_n^{(\alpha,\beta)}(1; q_n; x) - 1 \|_{\rho_0},
\]
we get
\[
st - \lim_{n \to \infty} \| D_n^{(\alpha,\beta)}(f; q_n; x) - f \|_{\rho_0} \leq st - \lim_{n \to \infty} \| D_n^{(\alpha,\beta)}(t^2; q_n; x) - x^2 \|_{\rho_0}
+ st - \lim_{n \to \infty} \| D_n^{(\alpha,\beta)}(t; q_n; x) - x \|_{\rho_0}
+ st - \lim_{n \to \infty} \| D_n^{(\alpha,\beta)}(1; q_n; x) - 1 \|_{\rho_0},
\]
which implies that
\[
st - \lim_{n \to \infty} \| D_n^{(\alpha,\beta)}(f; q_n; x) - f \|_{\rho_0} = 0.
\]
This completes the proof of the theorem. \( \square \)

5. Rates of statistical convergence

In this section, by using the modulus of continuity, we will study rates of statistical convergence of operators (1.4) and Lipschitz type maximal functions are introduced.

Lemma 3. Let \( 0 < q < 1 \) and \( a \in [0, b q] \), \( b > 0 \). The inequality
\[
\int_a^b |t - x| d \omega t \leq \left( \int_a^b |t - x|^2 d \omega t \right)^{1/2} \left( \int_a^b d \omega t \right)^{1/2}
\]
(5.1)
is satisfied.

Let \( C_B(0, \infty) \), the space of all bounded and continuous functions on \( [0, \infty) \) and \( x \geq 0 \). Then, for \( \delta > 0 \), the modulus of continuity of \( f \) denoted by \( \omega(f; \delta) \) is defined to be
\[
\omega(f; \delta) = \sup_{|t - x| \leq \delta} |f(t) - f(x)|, \ t \in [0, \infty).
\]
It is known that \( \lim_{\delta \to 0} \omega(f; \delta) = 0 \) for \( f \in C_B(0, \infty) \) and also, for any \( \delta > 0 \) and each \( t, x \geq 0 \), we have
\[
|f(t) - f(x)| \leq \omega(f; \delta) \left( 1 + \frac{|t - x|}{\delta} \right).
\]
(5.2)
Theorem 4. Let \((q_n)\) be a sequence satisfying (5.2). For every non-decreasing \(f \in C_B[0, \infty)\), \(x \geq 0\) and \(n \in \mathbb{N}\), we have
\[
|D_n^{(\alpha, \beta)}(f; q_n; x) - f(x)| \leq 2\omega(f; \sqrt{\delta_n(x)}),
\]
where
\[
\delta_n(x) = \left( \frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} + 1 - \frac{2[n]_q}{[n]_q + \beta} \right)x^2 + \left( \frac{[n]_q + q^2[n]_q - 2\alpha\beta - 2q\beta}{([n]_q + \beta)^2} \right)x + \frac{q^2(1 + q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2}.
\]

Proof. Let \(f \in C_B[0, \infty)\) be a non-decreasing function and \(x \geq 0\). Using linearity and positivity of the operators \(D_n^{(\alpha, \beta)}\) and then applying (5.2), we get for \(\delta > 0\)
\[
|D_n^{(\alpha, \beta)}(f; q_n; x) - f(x)| \leq D_n^{(\alpha, \beta)}(|f(t) - f(x)|; q_n; x)
\]
\[
\leq \omega(f, \delta) \{ D_n^{(\alpha, \beta)}(1; q_n; x) + \frac{1}{\delta} D_n^{(\alpha, \beta)}(|t - x|; q_n; x) \}.
\]
Taking \(D_n^{(\alpha, \beta)}(1; q_n; x) = 1\) and using Cauchy-Schwartz inequality, we have
\[
|D_n^{(\alpha, \beta)}(f; q_n; x) - f(x)| \leq \omega(f, \delta) \left\{ 1 + \frac{1}{\delta} \left( D_n^{(\alpha, \beta)}((t - x)^2; q_n; x)^{1/2} D_n^{(\alpha, \beta)}(1; q_n; x)^{1/2} \right) \right\}
\]
\[
\leq \omega(f, \delta) \left[ 1 + \frac{1}{\delta} \left\{ \left( \frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} + 1 - \frac{2[n]_q}{[n]_q + \beta} \right)x^2 + \left( \frac{[n]_q + q^2[n]_q - 2\alpha\beta - 2q\beta}{([n]_q + \beta)^2} \right)x + \frac{q^2(1 + q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2} \right\}^{1/2} \right].
\]
Taking \(q = (q_n)\), a sequence satisfying (5.2) and choosing \(\delta = \delta_n(x)\) as in (5.3), the theorem is proved. \(\square\)

Now we will give an estimate concerning the rate of approximation by means of Lipschitz type maximal functions.

In [19], Lenze introduced a Lipschitz type maximal function as
\[
f_{\alpha}(x, y) = \sup_{t > 0, t \neq x} \frac{|f(t) - f(x)|}{|t - x|^\alpha}.
\]
In [20], the Lipschitz type maximal function space on \(E \subset [0, \infty)\) is defined as follows
\[
\tilde{V}_{\alpha,E} = \{ f = \sup(1 + x)^\alpha f_{\alpha}(x, y) \leq M \frac{1}{(1 + y)^\alpha}; x \geq 0 \text{ and } y \in E \},
\]
where \(f\) is bounded and continuous on \([0, \infty)\), \(M\) is a positive constant and \(0 < \alpha \leq 1\). Also, let \(d(x, E)\) be the distance between \(x\) and \(E\), that is,
\[
d(x, E) = \inf\{|x - y|; y \in E\}.
\]

Theorem 5. If \(D_n^{(\alpha, \beta)}\) be defined by (1.4), then for all \(f \in \tilde{V}_{\alpha,E}
\]
\[
|D_n^{(\alpha, \beta)}(f; q_n; x) - f(x)| \leq M(\delta_n^+ + d(x, E)), \tag{5.3}
\]
where
\[
\delta_n(x) = \left( \frac{[n]_q(q[n]_q + 1)}{q([n]_q + \beta)^2} + 1 - \frac{2[n]_q}{[n]_q + \beta} \right)x^2 + \left( \frac{[n]_q + q^2[n]_q - 2\alpha\beta - 2q\beta}{([n]_q + \beta)^2} \right)x + \frac{q^2(1 + q) + 2q\alpha + \alpha^2}{([n]_q + \beta)^2}. \tag{5.4}
\]
Proof. Let \( x_0 \in \bar{E} \), where \( \bar{E} \) denote the closure of the set \( E \). Then we have
\[
| f(t) - f(x) | \leq | f(t) - f(x_0) | + | f(x_0) - f(x) |.
\]
Since \( D_n^{(\alpha,\beta)} \) is a positive and linear operators, \( f \in \tilde{V}_{\alpha,E} \) and using the above inequality
\[
| D_n^{(\alpha,\beta)}(f; q_n; x) - f(x) | \leq D_n^{(\alpha,\beta)}(| f(t) - f(x_0) |; q_n; x) + (| f(x_0) - f(x) |) D_n^{(\alpha,\beta)}(1; q_n; x)
\]
\[
\leq M \left( D_n^{(\alpha,\beta)}(|t - x_0|^\alpha; q_n; x) + |x - x_0|^\alpha D_n^{(\alpha,\beta)}(1; q_n; x) \right). \tag{5.5}
\]
Therefore, we have
\[
D_n^{(\alpha,\beta)}(|t - x_0|^\alpha; q_n; x) \leq D_n^{(\alpha,\beta)}(|t - x|^\alpha; q_n; x) + |x - x_0|^\alpha D_n^{(\alpha,\beta)}(1; q_n; x).
\]
Now, we take \( p = \frac{2}{\alpha} \) and \( q = \frac{2}{(2-\alpha)} \) and by using the Hölder’s inequality, one can write
\[
D_n^{(\alpha,\beta)}(|t - x|^\alpha; q_n; x) \leq D_n^{(\alpha,\beta)} \left( (t - x)^2; q_n; x \right)^{\alpha/2} \left( D_n^{(\alpha,\beta)}(1; q_n; x) \right)^{(2-\alpha)/2}
\]
\[
+ |x - x_0|^\alpha D_n^{(\alpha,\beta)}(1; q_n; x)
\]
\[
= \delta_n^\frac{2}{\alpha} + |x - x_0|^\alpha.
\]
Substituting this in (5.5), we get (5.3).
This completes the proof of the theorem. □

Remark 1. Observe that by the conditions in (3.2),
\[
st - \lim_{n} \delta_n = 0.
\]
By (3.2), we have
\[
st - \lim_{n} \omega(f; \delta_n) = 0.
\]
This gives us the pointwise rate of statistical convergence of the operators \( D_n^{(\alpha,\beta)}(f; q_n; x) \) to \( f(x) \).

Remark 2. If we take \( E = [0, \infty) \) in Theorem 3, since \( d(x, E) = 0 \), then we get for every \( f \in \tilde{V}_{\alpha,[0,\infty)} \)
\[
| D_n^{(\alpha,\beta)}(f; q_n; x) - f(x) | \leq M \delta_n^\frac{2}{\alpha}
\]
where \( \delta_n \) is defined as in (5.3).

Remark 3. By using (4.1), It is easy to verify that
\[
st - \lim_{n} \delta_n = 0.
\]
That is, the rate of statistical convergence of operators (1.4) to the function \( f \) are estimated by means of Lipschitz type maximal functions.

Acknowledgment
The authors would like to express their deep gratitude to the anonymous learned referee(s) and the editor for their valuable suggestions and constructive comments, which resulted in the subsequent improvement of this research article. Special thanks are due to our great Masters and friend academicians Prof. Abdel-shafi Obada, Editor in Chief, Prof. A.M. El-Sayed, Associate Editor of JOEMS for kind cooperation, smooth behavior during communication and for their efforts to send the reports of the manuscript timely. The second and third authors P. Sharma and L.N. Mishra acknowledge the MHRD, New Delhi, India for supporting this research article. The authors declare that there is no conflict of interests regarding the publication of this research article.
References:

[1] V. Gupta, A Note on $q$-Baskakov-Szász Operators, Lobachevskii Journal of Mathematics, Vol. 31(4) (2010) 359–366.

[2] V. Gupta, G.S. Srivastava, Simultaneous approximation by Baskakov-Szász type operators, Bull. Math. De la Soc. Sci. Math. de Roumanie (N. S.) 37 (85)(3-4) (1993) 73–85.

[3] V.N. Mishra, P. Sharma, A short note on approximation properties of $q$-Baskakov-Szász-Stancu operators, Southeast Asian Bull. Math., Vol. (38)(2014) pp. 1–15.

[4] D.K. Verma, P.N. Agrawal, Approximation by Baskakov-Durrmeyer-Stancu operators based on $q$-integers. Lobachevskii J. Math. 34 (2013), no. 2, 187–196.

[5] İcioğlu, Gürhan, Ram N. Mohapatra, Approximation properties by $q$-Durrmeyer-Stancu operators. Anal. Theory Appl. 29 (2013), no. 4, 373–383.

[6] V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators, J. Inequal. Appl. Vol. 2013 (2013) Art ID 586.

[7] T. Ernst, The History of $q$-Calculus and a New Method, U.U.D.M. Report 2000:16, Department of Mathematics, Uppsala University, Sweden, 2000.

[8] V.G. Kac, P. Cheung, Quantum Calculus, Universitext, Springer-Verlag, New York, 2002.

[9] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math. 2 (1951) 73–74.

[10] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.

[11] I.J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959) 361–375.

[12] A.D. Gadjiev, C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math. 32 (2002) 129–138.

[13] F. Altomare, M. Campiti, Korovkin-type approximation theory and its applications, Walter de Gruyter & Co., de Gruyter Stud. Math. 17 (1994) Berlin.

[14] L.N. Mishra, V.N. Mishra, K. Khatri, Deepmala, On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)$ – class by matrix $(C^1.N_p)$ Operator of conjugate series of its Fourier series, Appl. Math. Comput., Vol. 237 (2014), 252-263. doi: 10.1016/j.amc.2014.03.085.

[15] V.N. Mishra, K. Khatri, L.N. Mishra, Statistical approximation by Kantorovich-type discrete $q$-Beta operators, Adv. Differ. Equ. Vol. 2013 (2013) Art ID 345.

[16] V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, Trigonometric approximation of periodic Signals belonging to generalized weighted Lipschitz $W^r(L_r, \xi(t)), (r \geq 1)$ – class by Nörlund-Euler $(N, p_n)(E, q)$ operator of conjugate series of its Fourier series, Journal of Classical Analysis, Volume 5, Number 2 (2014), 91-105. doi:10.7153/jca-05-08.

[17] O. Doğru, K. Kanat, On statistical approximation properties of the Kantorovich type Lupaș operators. Math. Comput. Model. 55 (2012), 1610–1621.

[18] O. Doğru, On statistical approximation properties of Stancu type bivariate generalization of $q$-Balazs-Szabados operators, in: Proc. Int. Conference on Numer. Anal. and Approx. Th., Cluj-Napoca, Romania, 2006.

[19] B. Lenze, Bernstein-Baskakov-Kantorovich operators and Lipschitz-type maximal functions, in: Approx. Th., Kecskemé, Hungary, Colloq. Math. Soc. János Bolyai 58 (1990) 469–496.

[20] A. Aral, O. Doğru, Bleimann, Butzer and Hahn operators based on the $q$-integers, J. Inequal. Appl. 2007, Art 79410.