Variations on Barbălat’s Lemma

Bálint Farkas and Sven-Ake Wegner

Abstract. It is not hard to prove that a uniformly continuous real function, whose integral up to infinity exists, vanishes at infinity, and it is probably little known that this statement runs under the name “Barbălat’s lemma.” In fact, the latter name is frequently used in control theory, where the lemma is used to obtain Lyapunov-like stability theorems for nonlinear and nonautonomous systems. Barbălat’s lemma is qualitative in the sense that it asserts that a function has certain properties, here convergence to zero. Such qualitative statements can typically be proved by “soft analysis,” such as indirect proofs. Indeed, in the original 1959 paper by Barbălat, the lemma was proved by contradiction, and this proof prevails in the control theory textbooks. In this short note, we first give a direct, “hard analysis” proof of the lemma, yielding quantitative results, i.e., rates of convergence to zero. This proof allows also for immediate generalizations. Finally, we unify three different versions that recently appeared and discuss their relation to the original lemma.

1. DIRECT PROOF OF BARBĂLAT’S LEMMA. In 1959, Barbălat formalized the intuitive principle that a function whose integral up to infinity exists and whose oscillation is bounded needs to be small at infinity.

Theorem 1. (Barbălat’s lemma [1, p. 269]) Suppose that \( f : [0, \infty) \to \mathbb{R} \) is uniformly continuous and that \( \lim_{t \to \infty} \int_0^t f(\tau) d\tau \) exists. Then \( \lim_{t \to \infty} f(t) = 0 \) holds.

Barbălat’s original proof, as well as its reproductions in textbooks, e.g., Khalil [5, p. 192], Popov [6, p. 211], and Slotine and Li [9, p. 124], are by contradiction. Our first aim in this note is to give a direct proof of Theorem 1 that also reveals the essence of the statement and enables us to generalize Barbălat’s lemma to vector valued functions without difficulty. This is based on the following two lemmas.

Lemma 2. Let \( f : [0, \infty) \to \mathbb{R} \) be a continuous function. We define \( S : [0, \infty) \to \mathbb{R} \) via \( S(t) := \sup_{s > t} \left| \int_s^t f(\tau) d\tau \right| \) and put \( \omega(a, b, \delta) := \sup \{ |f(x) - f(y)| \mid x, y \in (a, b) \text{ with } |x - y| \leq \delta \} \) for \( a < b \leq \infty \) and \( \delta \geq 0 \). Then \( |f(t)| \leq S(t)^{1/2} + \omega(t, t + S(t)^{1/2}, S(t)^{1/2}) \) holds for \( t \geq 0 \).

Proof. Fix \( t \geq 0 \). If \( S(t) = 0 \), then \( f(t) = 0 \) and the assertion follows immediately. Also, if \( S(t) = \infty \), then there is nothing to prove. Suppose therefore \( 0 < S(t) < \infty \), put \( s = S(t)^{1/2} > 0 \), and compute

\[
|f(t)| = \frac{1}{s} \left| \int_t^{t+s} f(\tau) d\tau \right| \\
\leq \frac{1}{s} \left| \int_t^{t+s} f(\tau) d\tau \right| + \frac{1}{s} \left| \int_t^{t+s} (f(t) - f(\tau)) d\tau \right| \\
\leq \frac{1}{s} \left| \int_t^{t+s} f(\tau) d\tau \right| + \omega(t, t + s, s)
\]

http://dx.doi.org/10.4169/amer.math.monthly.123.08.825
MSC: Primary 26A06, Secondary 26A12; 46E39
We recall that given a uniformly continuous function \( f : [0, \infty) \to \mathbb{R} \), a function \( \omega : [0, \infty) \to \mathbb{R} \) is said to be a modulus of continuity for \( f \) if \( \lim_{\tau \to 0} \omega(t) = \omega(0) = 0 \) and \( |f(t) - f(\tau)| \leq \omega(|t - \tau|) \) for all \( t, \tau \in [0, \infty) \).

**Lemma 3.** Let \( f : [0, \infty) \to \mathbb{R} \) be uniformly continuous, and let \( \omega \) be a modulus of continuity for \( f \). Consider \( S(t) = \sup_{s \geq t} |\int_s^t f(\tau)\,d\tau| \). Then we have \( |f(t)| \leq S(t)^{1/2} + \omega(S(t)^{1/2}) \) for all \( t \geq 0 \).

**Proof.** Let \( f \) and \( \omega \) be given. We define \( \omega(t) := \omega(0, \infty, t) \), where \( \omega(0, \infty, \cdot) \) is the function defined in Lemma 2. Then, \( \omega(t) \leq \omega(\tau) \) holds for all \( t \geq 0 \). Since \( \omega(a, b, t) \leq \omega(0, \infty, t) \) holds for all \( a < b \leq \infty \), we can use Lemma 2 to obtain

\[
|f(t)| \leq S(t)^{1/2} + \omega(t, t + S(t)^{1/2}), S(t)^{1/2})
\]

\[
\leq S(t)^{1/2} + \omega(S(t)^{1/2}) \leq S(t)^{1/2} + \omega(S(t)^{1/2})
\]

as desired.

We now give the direct proof of Barbبال’s lemma.

**Proof of Theorem 1.** Let \( f \) be uniformly continuous and \( \omega \) be a modulus of continuity for \( f \). By the Cauchy criterion for indefinite integrals, we obtain (with a direct proof!) that \( S(t) \to 0 \) for \( t \to \infty \). Therefore, Lemma 3 yields \( |f(t)| \leq S(t)^{1/2} + \omega(S(t)^{1/2}) \to 0 \) for \( t \to \infty \).

Since Lemmas 2 and 3 remain true for functions with values in a Banach space \( E \) (the proofs are verbatim the same), we immediately obtain the next generalization.

**Theorem 4.** Let \( E \) be a Banach space, and suppose that \( f : [0, \infty) \to E \) is uniformly continuous such that \( \lim_{t \to \infty} \int_0^t f(\tau)\,d\tau \) exists. Then \( \lim_{t \to \infty} f(t) = 0 \) holds.

2. **BARB بال’S LEMMA IN A DIFFERENT CONTEXT.** We pointed out that all proofs of Barbبال’s lemma given in the relevant textbooks are indirect. On the other hand, there appeared recently several “alternative versions” in the literature whose proofs, or hints for a proof, are based on direct estimates. Tao [7, Lemma 1] states that \( \lim_{t \to \infty} f(t) = 0 \) holds whenever \( f \in L^2(0, \infty) \) and \( f' \in L^\infty(0, \infty) \). Desoer and Vidyasagar [3, Ex. 1 on p. 237] indicate that it is enough to require that \( f \) and \( f' \) are in \( L^2(0, \infty) \), and Teel [8, Fact 4] notes that in the latter the Lebesgue exponent 2 can be replaced by \( p \in [1, \infty) \). Here, \( f' \) can be interpreted in the sense of distributions or, equivalently, in the sense that \( f \) is absolutely continuous with the almost everywhere existing derivative being essentially bounded.

Indeed, the three results extend the classical statement that for \( 1 \leq p < \infty \) all functions in the Sobolev space \( W^{1,p}(0, \infty) \) tend to zero for \( t \to \infty \) (see, e.g., Brezis [2, Corollary 8.9]) to the “mixed Sobolev space”

\[
W^{1,p,q}(0, \infty) = \{ f \mid f \in L^p(0, \infty) \text{ and } f' \in L^q(0, \infty) \}
\]

826 © THE MATHEMATICAL ASSOCIATION OF AMERICA [Monthly 123
for $p = 2$, $q = \infty$, and $p = q \in [1, \infty)$, respectively. Our first aim in this section is to prove the following common generalization of the results of Tao [7], Desoer and Vidyasagar [3], and Teel [8].

**Theorem 5.** Let $p \in [1, \infty)$ and $q \in (1, \infty]$. Every function $f \in W^{1,p,q}(0, \infty)$ tends to zero at infinity.

Notice that our proof below shows that all three alternatives are immediate consequences of the original Barbálat lemma. The latter cannot be applied a priori, but in view of the following lemma.

**Lemma 6.** Let $p \in [1, \infty)$ and $q \in (1, \infty]$ be arbitrary. A function $f \in W^{1,p,q}(0, \infty)$ is bounded and uniformly continuous. More precisely, $f$ is $\frac{q-1}{q}$-Hölder continuous if $q < \infty$ and Lipschitz-continuous if $q = \infty$.

**Proof.** For the proof, let $q' \in [1, \infty)$ be such that $1/q' + 1/q = 1$ holds, where we use the convention $1/\infty = 0$. In particular, we read $(q - 1)/q = 1/q' = 1$ if $q = \infty$. By our assumptions we have

$$f(y) - f(x) = \int_x^y f'(s)\,ds$$

for almost every $x, y \in [0, \infty)$. Thus, $f$ can be identified with a continuous function satisfying

$$|f(x) - f(y)| \leq \left|\int_x^y f'(s)\,ds\right| \leq |x - y|^{1/q'}\|f'\|_q.$$

Here we used Hölder’s inequality for $q, q'$ with $1/q + 1/q' = 1$ so that $f$ is indeed Hölder continuous with exponent $1/q' = (q - 1)/q$. Let $r := p(q - 1)/q$. Then we have $r > 0$ and

$$\frac{d}{dx} |f(x)|^{r+1} = (r + 1)|f(x)|^{r-1} f(x) f'(x)$$

holds. For $x \in [0, \infty)$, we thus obtain

$$|f(x)|^{r+1} = |f(0)|^{r+1} + (r + 1) \int_0^x |f(x)|^{r-1} f(x) f'(x)\,ds$$

$$\leq |f(0)|^{r+1} + (r + 1)\|f\|_p^r \cdot \|f'\|_q,$$

where the last step is again an application of Hölder’s inequality for $q, q'$ with $1/q + 1/q' = 1$.

Lemma 6 enables us to employ the original Barbálat lemma to prove Theorem 5.

**Proof of Theorem 5.** By Lemma 6, the function $f$ is bounded and uniformly continuous, hence so is $|f|^p$. Indeed, we have by the mean-value theorem

$$\left||f(x)|^p - |f(y)|^p\right| \leq \sup_{|t| \leq \|f\|_{\infty}} p|t|^{p-1} \cdot |f(x) - f(y)| = p\|f\|_{\infty}^{p-1}\|f(x) - f(y)\|.$$  \hfill (1)
This inequality implies the asserted uniform continuity of $|f|^p$. By assumption, we can apply Barbălat’s lemma and obtain the statement.

Tao’s formulation [7, 3rd paragraph on p. 698] might erroneously establish the impression that his alternative [7, Lemma 1] uses a weaker assumption than Barbălat’s lemma, but has the same conclusion. Our second aim in this section is to illustrate that this is not the case. The following example of a function that satisfies the assumptions of Barbălat’s lemma but not those of [7, Lemma 1] combines two effects. The difference between Lebesgue and improper Riemann integral on the one hand and that between uniform continuity and having a bounded derivative on the other.

**Example 7.** Let $f(x) = 0$ for $x \in [0, 2)$ and $f(x) = (-1)^n f_n(x)$ for $x \in [n, n+1)$ with $n \geq 2$ and

$$f_n(x) = \begin{cases} (x - n)^{\frac{1}{2}}, & x \in [n, n + \frac{1}{2}n^{\frac{1}{2}}), \\ (n + n^{-\frac{1}{2}} - x)^{\frac{1}{2}}, & x \in [n + \frac{1}{2}n^{-\frac{1}{2}}, n + n^{-\frac{1}{2}}), \\ 0, & x \in [n + n^{-\frac{1}{2}}, n+1), \end{cases}$$

i.e., $f$ looks as follows.

![Graph of function](image)

Straightforward computations show that $\lim_{t \to \infty} \int_0^t f(\tau) \, d\tau = \frac{\sqrt{2}}{3} \sum_{n=2}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ exists and that $f$ is uniformly continuous. On the other hand $f \notin L^2(0, \infty)$ and $f' \notin L^\infty(a, \infty)$ for any $a > 0$.

For given $1 \leq p < \infty$, the function $f$ in Example 7 can easily be modified such that $f \notin L^p(a, \infty)$ holds. With some additional work, it is also possible to construct a single $f$ such that $f \notin L^p(a, \infty)$ is true for all $1 \leq p < \infty$. Finally, in all these cases, $f$ can also be changed into a $C^\infty$–function; $|f'|$ is then bounded on any finite interval but unbounded at infinity. Thus, for every $p \in [1, \infty)$ and $q \in (1, \infty]$ there is a function $f$ to which Barbălat’s lemma can be applied but which fails the assumptions of Theorem 5.

Concerning the other direction, it is easy to construct a function $f$ that satisfies the condition of [7, Lemma 1], i.e., $f \in L^2(0, \infty)$ and $f' \in L^\infty(0, \infty)$, but whose improper Riemann integral $\int_0^\infty f(t) \, dt$ does not exist. Thus, Tao’s alternative is incomparable with the original Barbălat lemma. The same is true for the statement of Theorem 5 whenever $p \neq 1$. Only in the case $p = 1$, Theorem 5 is a special case of Barbălat. Let $f \in W^{1,q}$ with $q \in (1, \infty]$. By Lemma 3, $f$ is uniformly continuous and bounded. Therefore, $f \in L^1(0, \infty)$ implies that the improper Riemann integral $\int_0^\infty f(t) \, dt$ exists.

### 3. Rates of Convergence

In this section, we use the methods of Section 1 to derive estimates for the speed of decay in the previous versions of Barbălat’s lemma so as to make the statement quantitative. We start with a modification of Theorem 1.

**Theorem 8.** Suppose that for $f: [0, \infty) \to \mathbb{R}$ the limit $\lim_{t \to \infty} \int_0^t f(\tau) \, d\tau$ exists, and that $f$ is Hölder continuous of order $\alpha \in (0, 1]$, i.e., $\omega(\tau) = c \tau^\alpha$ is a modulus of
continuity for a constant $c \geq 0$. Then we have $|f(t)| \leq (1 + c)S(t)^{\alpha/(1+\alpha)}$ for $t \geq 0$, where $S(t) = \sup_{t \geq 1} |\int_t^1 f(\tau) d\tau|$.

Proof. It is enough to repeat the proof of Lemmas 2 and 6 but with $s = S(t)^{1/(1+\alpha)}$. 

Next, we specialize to the situation of Theorem 5.

Corollary 9. Let $f \in W^{1,p,q}(0, \infty)$ for some $p \in [1, \infty)$ and $q \in (1, \infty]$. Then we have $|f(t)|^p \leq (1 + \|f\|_{p^{-1}} \|f'\|_q)S(t)^{(q-1)/(2q-1)}$ for $t \geq 0$, where $S(t) = f_t^\infty |f(\tau)|^{\rho} d\tau$.

Proof. By Lemma 6, under our assumptions, $f$ is bounded, and $\tau \mapsto \|f'\|_q \tau^{(q-1)/q}$ is a modulus of continuity for $f$. As in equation (1) in the proof of Theorem 5, we conclude that $\omega(\tau) = p\|f\|_{p^{-1}} \|f'\|_q \tau^{(q-1)/q}$ is a modulus of continuity of $|f|^p$. It is therefore enough to apply Theorem 8 to the latter function and to $\alpha = (q-1)/q$ to obtain the assertion because then $\alpha/(1+\alpha) = (q-1)/(2q-1)$.

We point out that Corollary 9 contains the quantitative versions of the results of Tao [7, Lemma 1], Desoer and Vidyasagar [3, Ex. 1 on p. 237], and Teel [8, Fact 4].

We finish this short note by illustrating how Barbát’s lemma and its variations are typically used in the control theoretic literature, for instance, to obtain (asymptotic) stability of solutions of ordinary differential equations. The next example is taken from Hou, Duan, and Guo [4, Example 3.1].

Example 10. Consider the system

$$
\begin{align*}
\dot{e}(t) &= -e(t) + \theta(t)\omega(t), \\
\dot{\theta}(t) &= -e(t)\omega(t),
\end{align*}
$$

with $\omega$ bounded and continuous. For some solution $(e, \theta)$ of this equation, we define $V = e^2 + \theta^2$ and compute $\dot{V}(t) = -2e^2(t) \leq 0$ for every $t \geq 0$. Thus, $V$ and therefore $e$ and $\theta$ are bounded. We put $f := 2e^2$, and in view of the positivity of $V$ and $f$, we obtain

$$
\int_0^R f(t) dt = \int_0^R -\dot{V}(t) dt = -V(R) + V(0) \leq V(0)
$$

so that $f \in L^1(0, \infty)$. Since $\dot{f}(t) = 4\dot{e}(t)e(t) = -4e(t)^2 + 4\theta(t)e(t)\omega(t)$ holds and $\theta, e, \omega$ are bounded, we conclude $\dot{f} \in L^\infty(0, \infty)$. That is $f \in W^{1,1,\infty}(0, \infty)$, and by Theorem 5, it follows $f(t) \to 0$ for $t \to \infty$. From the definition of $f$, we finally obtain that also $e(t) \to 0$ for $t \to \infty$.

ACKNOWLEDGMENT. The authors would like to thank the referees and the editor for their careful work and their valuable comments. Bálint Farkas was supported by the Hungarian Research Fund, OTKA 100461.

REFERENCES

1. I. Barbát, Systèmes d’équations différentielles d’oscillations non Linéaires, Rev. Math. Pures Appl. 4 (1959) 267–270.
100 Years Ago This Month in The American Mathematical Monthly
Edited by Vadim Ponomarenko

Miss ELEANORA HARRIS, head of the department of mathematics at the Hutchinson, Kan., high school, and secretary of the Kansas Association of Mathematics Teachers, has been markedly successful in conducting a mathematics club for the advanced students in the high school. They have discussed the practical uses of graphs, of the slide rule, and of determinants of the second and third orders. They have studied the history of our numerals, of logarithms, and of the Pythagorean theorem. And they have debated the question whether one year of algebra and one year of geometry should not be required of all students for graduation from the high school.

The first annual meeting of the Association will take place on Friday and Saturday, December 29, 30, 1916, at Columbia University, New York City. The selection of the place of the meeting was determined by the fact that the American Association for the Advancement of Science meets in New York during holiday week, thus bringing together large numbers representing most of the national scientific societies of the country. In particular, the American Mathematical Society will hold its annual meeting at Columbia University on Wednesday and Thursday, December 27, 28, so that the juxtaposition of dates for the Association and the Society will make it convenient for mathematicians to attend both meetings.

—Excerpted from “Notes and News” 23 (1916) 357–362.