Direct Reconstruction of dynamical dark energy from observational Hubble Parameter data

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Abstract. Reconstructing the evolution history of the equation of state parameter $w(z)$ directly from observational data is highly valuable in cosmology, since it holds substantial clues in understanding the origin of the accelerated expansion of the Universe. Contrast to a wealth of works on reconstructing $w(z)$ from supernova data, few work pay attention to Hubble parameter data. We analyze the merit of Hubble parameter data and make an attempt on reconstructing $w(z)$ from them, using the PCA approach introduced. We find that current Hubble parameter data does well in reconstructing $w(z)$, though compared to supernova data, they are scant and their quality is much poor.

Keywords: Dark Energy: Reconstruction: Nonparametric Model — Observational Hubble Parameter Data
1 Introduction

There have been a handful of works for reconstructing dark energy from observed distance module data. To date all the existing works can fall mainly into two categories: (i) Assume a particular model about $w(z)$, for which the model equations are integrated using Eq. (2.1) and Eq. (2.2) and then compared with the observations. (ii) Characterize the data with an underlying model (or not) and reconstruct the equation of state using Eq. (2.3). In category (i), the assumed model can be of a parameterized form, of a basis representation, or of a function distribution representation. The parametric forms are the most commonly used currently, which either assume $w = \text{const.}$ or allow for a redshift variation of some form such as $w = w_0 - w_1 z/(1 + z)$, where $w_0$ and $w_1$ are constants [4, 16]. [10] utilized wavelets for detecting the redshift evolution of $w(z)$, which was followed by a Principal Component Analysis (PCA) to decorrelate the wavelet coefficients, with the advantage of detecting local features in $w$ such as bumps efficiently. [13] presented a piecewise constant description of $w(z)$ by binning redshift and estimated the corresponding band power. A PCA was also used to make the band power uncorrelated, with eigenmodes being linear combinations of redshift bins. [11] introduced a nonparametric reconstruction method by modeling the equation of state $w(z)$ with a Gaussian process and estimating the GP hyperparameters with Bayesian inference and Markov chain Monte Carlo sampling. [26] combined Gaussian process and redshift binning strategy to feature $w(z)$, in which $w$ values in individual bins are also treated as model parameters remaining to be determined. To make $w(z)$ differentiable for the calculation of the dark energy perturbations, they interpolated between the bins with narrow tanh functions. On the other hand, in category (ii), [7] developed a simple numerical method for a direct determination of the equation of state $w(z)$ from the data, which made no assumptions about the underlying cosmological model of the observational data. Their approach is model independent, but at a cost of being noisier and highly sensitive to the amount and quality of the available data. [21] smoothed supernova data over redshift using a Gaussian kernel to suppress noise before reconstruction. [18] expressed the distance modulus in a parametric form $\mu = 5 \log [z(1 + az)/(1 + bz)] + \mu_0$ and applied $\chi^2$ fitting for determining the parameters $(a, b, \mu_0)$. [6] fitted the data before reconstruction using eigenmodes which were transformed from a set of primal basis functions with PCA.

With observational data getting richer and more diverse, recently works began to study dark energy reconstruction with multiple sets of measurements. [12] extended their Gaussian process method [11] to contain a diverse set of measurements: baryon acoustic oscillations
(BAO), cosmic microwave background measurements (CMB), and supernova data (SNe), while Observational Hubble Data (OHD) were further involved by other researchers [18, 26].

So far most researches are focused on observational data acquired from SNe, CMB and BAO, especially from SNe, or their hypothetical counterparts. On the contrary, the problem of reconstructing dark energy from Hubble parameter data has got few attention of the cosmology community. Now some works have considered the role of OHD in reconstructing dark energy with combined data as stated above, and [25] has tried to use OHD solely to bear this task. Few work focus on reconstructing dark energy from OHD solely. This is partially due to the fewer number and lower quality of Hubble parameter observations compared to those of distance module data. To date the last dataset of OHD only consists of 28 observations (see Figs. 1), while the supernova cosmology project has produced 580 observations. We find that OHD, with mean relative error being 18 percent, are more uncertain than SNe data which have the mean relative error of merely 0.55 percent. Such a comparison seems depressive.

![Figure 1](image.png)

**Figure 1.** OHD data and best-fit LCDM model. The red solid curves represent the best-fit fiducial LCDM model, which gives $\Omega_m = 0.29$, $\Omega_k = -0.01$, $H_0 = 67.6$ for fitting $H(z)$ data. $w_{\text{fiducial}} = -1$ is assumed.

Are Hubble data really unvalued in modeling the universe? It may be too early to give a pessimistic answer. In fact, some works have made explorative works in this problem. [17] studied the viability of constraining the cosmological parameters by using Hubble data and concluded that much fewer data points are necessary to get the comparative results against SNe data. [20] discovered that the Hubble parameter was somewhat better poised to give information about the nature of dark energy than the second derivatives. Encouraged by these pioneering works, in the present paper we attempt to reconstruct dark energy with the help of OHD solely.

The paper is structured as follows. In section 2 we analyzes the advantage of Hubble parameter data. The reconstruction method we adopt is outlined in section 3. Our results are presented in section 4, and finally we make some conclusions in section 5.
2 Merit of Hubble parameter

It is known that the relationship between the luminosity distance \( d_L(z) \) and the dark energy equation of state \( w(z) \) can be expressed as

\[
d_L(z) = \frac{c(1 + z)}{H_0(\sqrt{-\Omega})} \sin(\sqrt{-\Omega} \int_0^z d' \frac{H_0}{H_0'}),
\]

where the Hubble parameter \( H(z) \) is given by the Friedmann equation,

\[
H^2(z) = H_0^2 [\Omega_m (1 + z)^3 + \Omega_k (1 + z)^2
+ (1 - \Omega_m - \Omega_k) \exp \{3 \int_0^z \frac{1 + w(z')} {1 + z'} dz' \}],
\]

where \( H_0 = H(0) \) is the Hubble constant and \( \Omega_{m,k} \) are normalized density parameters. Eq. (2.1) and Eq. (2.2) clearly illustrate that the Hubble parameter \( H(z) \) serves as a bridge, links the equation of state \( w(z) \) and the luminosity distance \( d_L(z) \). In other words, to get \( w(z) \) from \( d_L(z) \) one must get through \( H(z) \). This means \( H(z) \) is “nearer” to \( w(z) \) than \( d_L(z) \) since to get \( w(z) \) from \( H(z) \) needs only first derivative while to get \( w(z) \) from \( d_L(z) \) requires second derivative.

Some methods reconstruct \( w(z) \) by directly reconstructing the luminosity distance curve, for which the following equation is used[6]:

\[
w(z) = \left\{ 2(1 + z)(1 + \Omega_k D^2)D'' - [(1 + z)^2 \Omega_k D^2 + 2(1 + z)\Omega_k D D' - 3(1 + \Omega_k D^2)] D' \right\} / \left\{ 3(1 + z)^2 \Omega_k + (1 + z) \Omega_m D^2 - (1 + \Omega_k D^2) D' \right\},
\]

where \( D(z) = \frac{H_0}{c(1+z)} d_L(z) \) is the normalized comoving distance and the prime denotes derivative with respect to \( z \). Eq. (2.3) provides the computing way of the inverse direction against (1), i.e. from \( D(z) \) (or equivalently \( d_L(z) \)) to \( w(z) \). Note that the second derivative of \( D(z) \) is a requisite in Eq. (2.3) even for the case of \( \Omega_k = 0 \). We can decompose Eq. (2.3) into two simpler equations by introducing the Hubble parameter \( H(z) \) as the intermediate quantity, which can be expressed as

\[
H(z) = \frac{H_0}{D'(z)} \sqrt{1 + [D(z)]^2 \Omega_k}
\]

and

\[
w(z) = \frac{1}{3} H_0^2 \frac{2 H'(1 + z) - 3 H^2 + H_0^2 \Omega_k (1 + z)^2}{3 H^2 - H_0^2 \Omega_m (1 + z)^4 - H_0^2 \Omega_k (1 + z)^2}.
\]

Eq. (2.5) shows that in order to calculate \( w(z) \) from \( H(z) \), only the first derivative of \( H(z) \) is needed, which indicates again that \( d_L(z) \) is more distant to \( w(z) \) than \( H(z) \). The derivative operator functions as a high-pass filter and can thus heavily amplify the observational errors. Therefore, we believe that for reconstructing \( w(z) \), \( H(z) \) is of high advantage than \( d_L(z) \) in suppressing the propagation of noises. Such an advantage of \( H(z) \) makes it desirable to do an attempt of reconstructing \( w(z) \) directly from \( H(z) \) instead of \( d_L(z) \), though the quality of \( H(z) \) data is comparatively poor. Notice that the work of [20] has given a clue about the advantage of the first derivative of data over the second derivative in reconstructing dark energy. They studied the efficacy of four different parameters: deceleration parameter,
equation of state of dark energy, the dark energy density, and the geometrical parameters $\Omega_m(z)$, which can be expressed as

$$\Omega_m(z) = \frac{\tilde{h}^2(z) - 1}{(1 + z)^3} - 1,$$

$$\tilde{h} = \frac{H(z)}{H_0},$$

in discriminating theoretical models of dark energy using supernova data, and found that the geometrical parameters $\Omega_m(z)$, a cosmological parameter constructed from the first derivative of the data, for which the theoretical models of dark energy are sufficiently distant from each other, performs the best in reconstructing dark energy from SNe data.

3 Methodology

Most existing reconstruction approaches making use of the observed distance module data can also be applied on the Hubble parameter data. Here we apply the framework proposed in [6]. We start with a set of $N$ primary basis functions $p_n(z)$, and the weighted summary function $\sum_a a_n p_n(z)$ is used to fit the observational data through weighted least square (WLS) technique, where the weighting matrix is the one with diagonal entities being the inverses of squares of errors of observations and non-diagonal entities being zeros. The principle component analysis (PCA) is implemented through calculating eigenvalues and eigenvectors of the inverse covariance matrix of the estimated parameters $a_n$. A set of optimal basis functions are then produced by transforming the primary functions through the matrix of eigenvectors. From the total $N$ transformed basis functions the best $M$ ones are selected to fit the data once more with WLS. Thus, each choice \{N, M\} will give a particular reconstruction of $w(z)$.

To tradeoff between the risk of getting a wrong $w(z)$ and risk of over-fitting, they used a combined information criteria to cull out the most suitable combinations of \{N, M\} with $N \in [2, 10]$ and $M \in [2, N]$, which is defined as:

$$\text{CIC} = (1 - s)\text{AIC} + s\text{BIC},$$

where AIC is Akaike information criteria and BIC is Bayesian criteria, defined as [15]:

$$\text{AIC} = \chi^2_{\text{min}} + 2M, \text{BIC} = \chi^2_{\text{min}} + M\ln N_d.$$

And $s$ in Eq. 3.1 is a parameter to adjust the proportion of AIC and BIC in CIC. This framework does not single out one particular combination of \{N, M\} that possesses minimum CIC, but a family of \{N, M\} reconstructions which satisfy

$$\text{CIC} = \text{CIC}_{\text{min}} + \kappa,$$

where $\kappa$ is a constant. [6] believes that $\kappa = 5$ is appropriate choice. The selected family of \{N, M\} values are then simulated through Monte Carlo calculations, which resamples the estimated parameters, according to the diagonal covariance matrix of WLS data fitting, 200 times for each \{N, M\} reconstruction. Finally, the results are bundled into a single probability distribution from which the averaged $w(z)$ and $1 - \sigma$ errors are inferred.

[6] exerted the methodology outlined above on the measured distance modulus $\mu = 5\log_{10}d_L(z) + 25$. The residuals $\mu_{\text{resid}} = \mu_{\text{data}} - \mu_{\text{fiducial}}$ are fitted, where the fiducial model is flat $\Lambda$—cold dark matter (LCDM) model with assumption of $w(z) = -1$. We refer the reader to [6] for details. Here we use this methodology to reconstruct the dark energy equation of state $w(z)$ based on Hubble parameter $H(z)$ data.
Figure 2. Reconstruction of $w$ using the simulated $H(z)$ data. The solid curves represent reconstructed $w(z)$ and the dashed curves represent 1-σ errors.

4 Results

We first test the power of simulated $H(z)$ data with the above technique. To get simulated $H(z)$ data, we use the method proposed in [25], with parameters $\Omega_m = 0.30$ and $\Omega_k = 0$ and the Hubble constant $H_0$ is set to the newest value 67.4 [? ]. This method takes the relative error, $\sigma_H/H$, as random variables which are assumed satisfying $z$-dependent Nakagami distributions $f_m(x; m, \Omega)$. By fitting OHD, we get the distribution parameters: $m = 0.63$ and $\Omega = 0.038$. The resulting measurement should have the Gaussian distribution $H_{\text{sim}} \sim N(H_{\star}, \sigma_H)$, where $H_{\star}$ is the fiducial $H$, and it gives the simulated data. A total 400 data points with $z$ evenly spaced within the range $[0, 2.0]$ are generated, with errors increasing with $z$. The quality of the simulated data set is comparable with that of observations.

The reconstruction procedure is similar to [6]. The primary basis functions are $p_n(z) = [z/(1 + z)]^{n-1}$, $n = 1 \sim N$, though other types of functions are also feasible. We can get new eigenfunctions by fitting the simulated $H(z)$ data with PCA. We find that, for a fixed $N$ the shapes of eigenfunctions vary from very smooth to very oscillatory. The fiducial model used in this procedure, which is described in Eq.(2), has parameters similar to those used in the data generation process, i.e. $\{\Omega_m, \Omega_k, H_0\} = \{0.30, 0, 67.4\}$, and $w_{\text{fiducial}} = -1$. We set $\kappa = 5$ and assess three cases $s = 0.1, 0.5$ and $1.0$, respectively. Fig.2 illustrates the reconstructed $w(z)$. On the whole, the results are quite good for all $s$ values. Meanwhile, same as stated in [6], Fig.2 shows that $s$ takes us from conservative models where $s = 1$ to more wild models where $s = 0$. This is due to the fact that models minimizing BIC criteria tend to get smooth with tight error bars, while models with low AIC values are oscillatory and have large errors.

It is expected that the number of data points should have crucial impact on the quality of reconstruction result. As shown in Fig.3, the reconstruction error drops drastically with increasing number of the simulated data points. We can see that the currently observed 28 $H(z)$ data points, with current measure quality, are really too sparse. More OHD data points, measured in future by e.g. LRG survey, are hence expected. On the other hand, 400 data points seems to be enough for improving the reconstruction quality, and increasing the number of data points further will merely leads to a marginal improvement. Further improvement on the reconstruction result need further improvement on data quality.

For comparison a hypothetical data set of $\mu(z)$ is also constructed with the method
proposed in [6], which consists of 2000 data points evenly distributed in the redshift range $z = 0.08 - 1.7$, and 300 data points in $0.03 - 0.08$ [1]. The uncertainties are composed of a constant statistical error $\sigma_{\text{mag}} = 0.15$ and systematic errors linearly drifting from $\sigma_{\text{sys}} = 0 - 0.02$. The parameters are $\{\Omega_{m}, \Omega_{k}, H_{0}\} = \{0.3, 0.0, 6.5\}$. We find that for moderate $z$ values ($z \sim 0.4 - 0.8$), both $H$-reconstructed errors and $\mu$-reconstructed errors are relatively low. When $z$ is greater than 0.6, however, the $H$-reconstructed errors are lower than $\mu$-reconstructed ones, and with the increase of $z$, the differences between them get more remarkable. Increasing $s$ also vastly magnifies the differences. Only in low $z$ region ($z < 0.4$) does $H(z)$ data produce $w(z)$ errors greater than $\mu(z)$ data. In addition, both $\mu$-reconstructed and $H$-reconstructed errors grow with decreased $z$ in low $z$ region ($z < 0.4$) and the differences between them is acceptable. Thus we conclude that, using simulated $H(z)$ data can get better result than using simulated $\mu(z)$ data. Note that the mean relative errors of $H(z)$ data are about 0.14 $\sim$ 0.16, while the relative errors of $\mu(z)$ data are about 0.003 $\sim$ 0.004, indicating the former is nearly two orders larger than the later (We compare the relative errors rather than the absolute values since $H(z)$ and $\mu(z)$ are different physical quantities). Hence, the comparison convincingly validated the merit of $H(z)$ data, as stated in section 2, in reconstructing the dark energy equation of state.

Based on the analysis of section 2, the reconstruction $w(z)$ from $\mu(z)$ data can be split into two procedures: first reconstruct $H(z)$ from $\mu(z)$ and second from $H(z)$ to $w(z)$. One may imagine that the error propagation and amplification taken in the first procedure will induce errors of the reconstructed $H(z)$ similar in value to, or at least at the same order as, errors of the simulated $H(z)$ data, so they would produce comparable errors for the resultant $w(z)$. However, this is not the case. We find that, compared to the errors of simulated $H(z)$ data, the reconstructed $H(z)$ errors are quite small. This is fairly surprising! We give the explanation as follows: First note both simulated and reconstructed $H(z)$ data are noisy, with errors measuring the extent of noise. Because we use the first $M < N$ eigenfunctions, which encompass the dominant features in the data, to fit the data, and throw away the higher ones which contain noise-induced oscillations, noises are suppressed in the resultant $w(z)$. However, this is the case for $w(z)$ reconstruction from the simulated $H(z)$ data but not the case for the reconstruction from the reconstructed $H(z)$ data. By resampling the parameters in $\mu(z)$ data fitting, noises are to a large extent suppressed in the first procedure.
of $w(z)$ reconstruction from $\mu(z)$ data, resulting in relatively low $H(z)$ errors, while the second procedure essentially maintains all the noise-induced features inherited from the first procedure. In a word, it is the noise-suppression effect in the reconstruction from simulated $H(z)$ data, which is lacked in the reconstruction from reconstructed $H(z)$ data, that makes the former gets better $w(z)$ reconstructions, even though the quality of simulated $H(z)$ data is much poor than the reconstructed ones. In addition, the error curves of the reconstructed $H(z)$ data does not show the turn-up nature (grow with decreased $z$) that exists in the final $w(z)$ in low $z$ region, which indicates that the decrease of errors of $\mu$-reconstructed $w(z)$ with increasing $z$ in low $z$ region comes from the second $H - w$ procedure rather than the first $\mu - H$ procedure, since otherwise such a decrease should also occur in errors of reconstructed $H(z)$ data for small $z$.

Then we use the simulated $H(z)$ data to reconstruct two types of evolving $w(z)$. One is a standard slow evolution, expressed as $w = \frac{1}{4}[-1 + erf(ln(z^2 - 1))]$; the other evolves more drastically, modeled by $w = -1 + 0.31 sin[12ln(1 + z)]$. The reconstruction result is shown in Fig.4 and 5. We set $s = 0.2$ and $\{\Omega_m, \Omega_k, H_0\} = \{0.30, 0.0, 67.4\}$. We can see again that the simulated $H(z)$ data outperforms, in most cases, the simulated SNe data. For the second type of $w(z)$, not only the errors constructed by the former are much lower than the later, but also the former constructed $w(z)$ is more faithful with the underlying $w(z)$. It seems that the simulated $H(z)$ data perform better for more oscillating evolving dark energy.

Finally we test the performance of OHD. The newest and most comprehensive observational data set of $H(z)$, which consists of 28 data points, are collected from [2, 3, 5, 19, 23, 27] and one can find a copy from [8]. The most recently measured $H_0$, with value 67.4 ± 1.4, is also added into this data set, making up 29 data points in total. To make a comparison, the data set of distance module $\mu(z)$ is also acquired. The last distance module measurements, which consist of 580 usable data points, were produced by Supernova Cosmology Project [24] and can be downloaded directly from http://supernova.lbl.gov/Union/. The parameters of LCDM model are: $\{\Omega_m, \Omega_k, H_0\} = \{0.29, -0.01, 67.6\}$, and $w_{\text{fiducial}} = -1$. Interestingly, the most likely reconstruction favors a transition from $w < -1$ at low redshift to $w > -1$ at higher redshift, a behavior that is consistent with the quintom model which allows $w$ to cross -1 [9]. And we find that the quality of two reconstructions are comparable in most $z$ region. Roughly
Figure 5. Reconstruction of evolving dark energy of the second type using the simulated $H(z)$ data (a) and error comparison (b). In (b), the red line denotes $w$ errors constructed by simulated $H(z)$ data and the blue line corresponds to simulated SNe data.

Figure 6. Reconstruction of $w$ using the observational $H(z)$ data. The solid curves represent reconstructed $w(z)$ and the dashed curves represent 1-$\sigma$ errors.

speaking, the reconstruction from OHD performs better than the reconstruction from $\mu(z)$ for relatively large $z$ values and $s > 0.1$. Concerning about the much poorer quality and much fewer observational data points of $H(z)$ compared to $\mu(z)$ this result is impressive.

It is shown from Fig.6 that the shapes of the curves $s = 0.1$ and $s = 0.5$ are quite similar as same as the error curves. This behavior reveals that the reconstructions obtained from $H(z)$ data is more “stable” than those from $\mu(z)$, in the sense that they are less dependent on $s$ values. Here we can give an intuitive interpretation for it. The parameter $s$ in fact controls $M$ values and the number of $\{N, M\}$ combinations. The smaller $s$ is, with higher probability are larger $M$s included and the bigger is the set of $\{N, M\}$ combinations. Therefore, smaller $s$ results in emerging of larger $M$ in the set of $\{N, M\}$ combinations, which leads to more precise data fitting and more noisy features reserved. Since $\mu$ is more “distant” from $w$ than $H$, even little noise included in the SNe data fitting curves may induce comparatively large oscillations of the resultant $w(z)$. On the contrary, the noisy features entering the eigenmodes due to dropped $s$ will not lead the resultant $w(z)$ more oscillating significantly. Hence, from
this aspect the merit of Hubble parameter data in reconstructing dark energy equation of state is established more solidly.

5 Conclusions

We made an attempt in this paper in reconstructing dark energy equation of state $w(z)$ from Hubble parameter data using the method proposed by [6]. We first made a comparation on the simulated data, with both classes of simulations being of observational quality. The results showed that the simulated $H(z)$ data can produce more conservative $w(z)$ than simulated $\mu(z)$ data. Then the performance of OHD was evaluated against SNe data. We found that the errors of two types of reconstructed $w(z)$, from OHD and $\mu(z)$ data, are comparable. We also concerned that the number of data points may impact the quality of reconstruction, then we found 400 data points seemed to be enough for improving the reconstruction quality for OHD. Further improvement require the improvement of data quality.

We have shown that the Hubble parameter dataset alone is potentially capable to be used to reconstruct the dark energy equation of state just as current SN1a datasets. Finally, the future CMB observation programs, such as the Atacama Cosmology Telescope, may be able to identify more than 2000 passively evolving galaxies up to $z \sim 1.5$ via the Sunyaev-Zel’dovich effect, and their spectra can be analyzed to yield age measurements that will yield approximately 1000 $H(z)$ determinations with 15 percent error [22]. This ensures a future dataset outperforming current SN1a datasets. Combined with future BAO observations, it is reasonable to expect that the Hubble parameter data will do a great job in the exploring the expansion history of the universe.

It should be pointed out that the results and the analysis in this paper are method-specific, since we consider just one reconstruction method. Whether similar results and conclusions can be obtained by using other reconstruction methods, such as Gaussian process modeling [11], is still unknown.

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