Mixed-phase induced core-quakes and the changes in neutron star parameters

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ABSTRACT
We present approximate formulae describing the changes in neutron star parameters caused by the first-order phase transition to an “exotic” state (pion or kaon condensate, quark matter) resulting in formation of a mixed-phase core. The analytical formulae for the changes in radius, moment of inertia and the amount of energy released during the core-quake are derived using the theory of linear response of stellar structure to the core transformation. Numerical coefficients in these formulae are obtained for two realistic EOSs of dense matter. The problem of nucleation of the exotic phase as well as possible astrophysical scenarios leading to a core-quake phenomenon is also discussed.

Key words: dense matter – equation of state – stars: neutron

1 INTRODUCTION
After the discovery of neutron stars it became immediately clear that they are unique playgrounds to study the properties of super-dense matter in the most extreme physical conditions. On the other hand, the very existence of neutron stars, and their participation in various astrophysical phenomena, represented a major challenge for theorists. In order to construct neutron star models, one needs to know the equation of state (EOS) and other properties of matter at densities \( \gtrsim 10^{15} \text{ g cm}^{-3} \).

One of the most intriguing predictions of some theories of dense matter is a possibility of a phase transition into an “exotic” state. Several phase transitions are predicted by dense-matter theories, including pion and kaon condensation, and de-confinement of quarks (for review see e.g. Weber 1999). The most interesting case, from the point of view of its consequences for neutron star structure, corresponds to a phase transition which is of the first-order type. In such a case, equilibrium phase transition from the normal, lower density phase to the pure exotic one, occurs at a well defined pressure, \( P_0 \), and is accompanied by a density jump at the phase interface.

A first order phase transition allows for a metastability of the pure normal phase with pressure \( P > P_0 \). Therefore, a metastable core could form during neutron-star evolution (accretion, spin down), and nucleation of the exotic phase would then lead to formation of a new-phase core, accompanied by a core-quake, energy release, and other phenomena, such as shrinking of the radius and a speed up of rotation. Usually, the radius of the new-phase core is significantly smaller than the original stellar radius, and the effect of core-quake can be described within the linear response theory developed by Haensel, Zdunik & Schaeffer (1987) and Zdunik, Haensel & Schaeffer (1987); an earlier Newtonian theory was presented by Schaeffer, Haensel & Zdunik (1983). Expressions for the changes of neutron-star radius, moment of inertia, and energy release were obtained in these papers in terms of expansions in powers of the radius of the new-phase core, the numerical coefficients of the expansions being determined by the EOS of normal phase, the mass of the metastable configuration and the density jump at the phase transition.

Relaxing the condition of local (microscopic) electrical neutrality opens a possibility of coexistence of two phases of dense matter, within a range of pressures, in the form of a mixture of the two phases, lower-density (normal - N) and higher-density (superdense - S), each of them charged, and the mixture being electrically neutral on the average only (Glendenning 1991, 1992). The volume fraction occupied by the higher-density phase grows from zero at the lower pressure boundary, \( P_N^{(m)} \), of the mixed phase up to one at the upper pressure boundary, \( P_S^{(m)} \). If the surface tension at the N-S phase interface is not too large, then the mixed phase is preferred over a pure phase state (the latter is: N-phase for \( P < P_0 \) and S-phase for \( P > P_0 \)).

In this paper we study the changes of the neutron-star parameters implied by the appearance of an exotic phase which forms a mixed-phase core. Some astrophysical scenarios which could lead to formation of a mixed-phase core were already considered in literature (see e.g. Glendenning, Pei & Weber 1997).
The phase transition in stellar core results from the increase of the central pressure due to the pulsar slow-down or mass accretion on the neutron star in a binary system. In all these papers the authors assume that matter remains in equilibrium during neutron star evolution, and therefore they use an EOS of matter in ground state. However, in a first order phase transition, the new phase can appear only via nucleation. Therefore, the star acquires first a metastable core of normal phase, in which an exotic phase nucleates. Nucleation destabilizes stellar configurations, implies a core-quake, and finally new equilibrium is reached with a core of a mixed or a pure exotic phase, the actual outcome depending on the detailed kinetics of the first-order phase transition.

In the present paper we derive the formulae for the changes of neutron star parameters resulting from the formation of a mixed-phase core. These changes are proportional to the specific powers of the ratio of the mixed-phase core radius to the radius of the last stable pure N-phase configuration. We neglect the effects of rotation on neutron-star structure.

The plan of the article is as follows. In Sect. 2 our notation, construction and description of the EOS with a mixed phase segment is presented. General lowest-order formulae for the changes in neutron-star parameters are derived in Sect. 3. Numerical calculations are performed for two recent realistic EOSs of the normal phase, in which the exotic dense phase is assumed to nucleate. Results for the linear response parameters are presented in Sect. 4 where we also provide with some numerical estimates of expected effects. In Sect. 5 we briefly describe the astrophysical scenarios which could lead to a nucleation of an exotic phase and a core-quake. Some problems connected with nucleation and formation of a mixed-phase core are discussed in Sect. 6. Finally, in Sect. 7 we summarize our results and we also discuss their possible applications, and perspectives of their generalization to rotating neutron stars.

2 EOS WITH A MIXED-PHASE REGION

Let us consider a general case of a first-order phase transition between the N and S phases of dense matter. As shown by Glendenning (1991, 1992), relaxing the microscopic charge-neutrality condition can make a mixed-phase state energetically preferred provided the surface tension and Coulomb contributions are sufficiently small.

Assume the thermodynamic equilibrium of a multi-component and multi-phase dense matter, neglecting the Coulomb and surface contributions to the thermodynamic quantities. The elementary constituents of the matter are hadrons (h), which may be baryons, quarks, and strongly-interacting meson (pion and kaon) condensates, as well as leptons (electrons and muons). The energy densities in both phases depend on the number densities of the matter constituents in these phases,

\[ E^N = E^N (n_h^N, n_e^N, n_\mu^N) , \quad E^S = E^S (n_h^S, n_e^S, n_\mu^S) . \]  

As the translational invariance may be broken within a phase, the number densities are actually the volume-averaged ones. We assume that the size of the region occupied by a non-uniform phase is sufficiently large compared to the characteristic length-scale of the non-uniformity. Then the volume averages within each phase are well defined.

The corresponding electric-charge densities (in the units of the elementary charge) and baryon densities (in the units of nucleon baryon charge) are given by

\[ \rho_{N} = \sum_n n_h^N q_h - n_e^N - n_\mu^N , \quad \rho_{h} = \sum_h n_h^N b_h , \]

\[ \rho_{S} = \sum_h n_h^S q_h - n_e^S - n_\mu^S , \quad \rho_{h} = \sum_h n_h^S b_h . \]  

where \( q_h \) and \( b_h \) are, respectively, the electric and baryonic charges of a hadron \( h \).

The thermodynamic equilibrium of a mixture of phases N and S, at a fixed average baryon density \( n_b \), will be calculated by minimizing the average energy density

\[ E = (1 - \chi) E^N + \chi E^S . \]  

under the condition

\[ n_b = (1 - \chi) n_b^N + \chi n_b^S , \]  

and under the constraint of average (macroscopic) electrical neutrality

\[ \rho_{N} = (1 - \chi) \rho_{N}^N + \chi \rho_{N}^S = 0 . \]  

For example, let us consider a first-order phase transition implied by kaon condensation. Let us assume that the characteristic length-scale of the hadron electric-charge inhomogeneities is much smaller than the electron and muon screening lengths (this is not always true, see e.g. Norsen & Reddy 2001). Then the electron and muon densities can be considered as uniform,

\[ n_e^N = n_e^S = n_e , \quad n_\mu^N = n_\mu^S = n_\mu . \]  

We have to determine the values of eight variables (four nucleon densities, two lepton densities, kaon density, and the volume fraction \( \chi \)) by minimizing \( E \), Eq. 1, under the constraints given by Eqs. 4 and 5. This leads to a set of non-linear equations relating the thermodynamic variables, each of these equations having a clear physical meaning. Mechanical equilibrium between the two phases requires (we remind that surface and Coulomb contributions are neglected)

\[ P^N = P^S . \]  

The strong interactions imply the equality of chemical potentials of nucleons in the two phases,

\[ \mu_n^N = \mu_n^S = \mu_n , \quad \mu_p^N = \mu_p^S = \mu_p , \]  

Finally, weak interactions involving hadrons and leptons lead to

\[ \mu_n = \mu_p + \mu_e , \quad \mu_e = \mu_{e^{-}} , \quad \mu_\mu = \mu_e . \]  

Together with Eqs. 4 and 5, we get eight equations for eight thermodynamic variables. The solution corresponds to the thermodynamic equilibrium at a fixed \( n_b \) and under the constraint of macroscopic electrical neutrality. Let us first discuss the character of the bulk equilibrium as a function of the pressure, Fig. 1. For \( P < P_N^{(m)} \), the equilibrium is realized by the pure N-phase. For \( P_N^{(m)} < P < P_S^{(m)} \), the equilibrium corresponds to a mixed m-state of both phases (the m-phase). The volume fraction occupied by
the S-phase increases monotonously with $P$, from zero at $P = P_N^{(m)}$, to one at $P = P_S^{(m)}$. For $P > P_S^{(m)}$ we have a pure S-phase. The calculations of a mixed-phase state, where kaon-condensed matter coexisted with baryon matter, were performed by Glendenning & Schaffner-Bielich (1993, 1996) and Norsen & Reddy (2001). The models of dense matter with a mixed phase of de-confined quark matter coexisting with baryon phase were constructed by Heiselberg, Pethick & Staub (1993) and Glendenning & Pethick (1992). The importance of the phase-interface (surface tension and curvature energy) effects for forming the mixed-phase state and for its spatial structure was studied by Heiselberg, Pethick & Staub (1993); Christiansen & Glendenning (1997); Christiansen, Glendenning & Schaffner-Bielich (2000) and Norsen & Reddy (2001). A mixed-phase state affects the EOS of dense matter as visualized in Fig. 2. For the sake of comparison, we show also a standard first-order transition between the pure N and S phases. For a pure N-S phase transition, the densities $n_N < n_b < n_S$ could not exist in the stellar interior because $P$ should be monotonous there to create an outward directed force which balances at each point the gravitational pull. On the contrary, the mixed-phase layer of density $n_N^{(m)} < n_b < n_S^{(m)}$ can well exist in the stellar interior, with the pressure increasing from $P_N^{(m)}$ at the top of the layer to $P_S^{(m)}$ at its bottom. Mixing of the N and S phases softens the EOS as compared to the pure N-phase EOS, but the effective softening is weaker than in the limiting case of a pure-phase transition between the N and S phases.

The surface and Coulomb effects bring positive contributions to $\mathcal{E}$ and $\mu_b$. They affect the size and shape of the structures within the mixed-phase layer. For a periodic structure, the virial theorem (see, e.g., Pethick & Ravenhall 1995 and references therein) tells us that the surface contribution is twice the Coulomb one. Both are pushing up the value of $\mu_b^{(m)}(P)$. They are particularly important at the edges of the mixed-phase region, where the droplets of one phase within the dominating one are small. It is clear that these effects increase $P_N^{(m)}$ and decrease $P_S^{(m)}$, narrowing the mixed-phase layer in the stellar interior. If sufficiently large, the surface and Coulomb contributions can entirely remove the mixed phase: the difference in $\mu_b(P)$ of the pure and mixed phases is usually small. It will occur if surface tension is greater than some critical value ($\sigma > \sigma_{\text{crit}}$) so that $\mu_b^{(m)}(P) > \mu_b^N(P)$ for $P < P_0$ and $\mu_b^{(m)}(P) < \mu_b^S(P)$ for $P > P_0$. Heiselberg, Pethick & Staub (1993) obtained $\sigma_{\text{crit}} \approx 70$ MeV fm$^{-2}$ for a transition from nucleon (N) to quark (S) matter. The actual value of the surface tension for the quark-matter droplets in baryonic medium is very poorly known, $\sigma = (10 - 100)$ MeV fm$^{-2}$.

### 3 CHANGES OF THE STELLAR PARAMETERS FOR THE MIXED-PHASE TRANSITION - LINEAR APPROACH

We assume that at a central pressure $P_c = P_{\text{crit}}$ the nucleation of the S-phase in a super-compressed core, of radius $r_N$, of configuration $\mathcal{C}$, initiates the phase transition and formation of mixed-phase core of radius $r_m$ in a new configuration $\mathcal{C}^*$, as presented on Fig. 3. Transition to a mixed phase occurring at $r_m$ is associated with a substantial drop in the adiabatic index of matter, defined as $\gamma \equiv (\rho P)^{1/2}/d\rho/dP$, from $\gamma_N$ to $\gamma_m$: mixed phase is much softer than the pure one. This is because in the mixed phase the increase of mean density is reached partly via conversion of a less dense N phase into denser S phase, and therefore requires less pressure than for a pure phase. The effect exists for any fraction of the S phase, even in the limit of the fraction of the S-phase tending to zero, and leads to a discontinuity of $\gamma$ at $\rho_m^{(m)} \equiv \rho_m$. An example of a very dramatic drop of $\gamma$ at $\rho_m$ is given in Fig. 20 of Akmal, Pandharipande & Ravenhall (1998).
...dashed line - the states which a meta-stable with respect to the transition to a mixed-phase state. For a critical central density $\rho_{\text{crit}}$ the S-phase nucleates in the super-compressed core of configuration $C$, this results in a transition $C \rightarrow C^*$ into a stable configuration with a mixed-phase core and central density $\rho_c^*$. Both configurations $C$ and $C^*$ have the same baryon number $A$.

dashes are given by Pons et al. (2000) see Fig. 1 in their article, where the transition from the solid line (no kaon condensate) to dashed line representing mixed phase is clearly accompanied by a drop in the derivative of pressure with respect to density. In Fig. 3 we show two examples of realistic EOSs with a mixed phase segment, one by Glendenning (1997), Table 9.1 and the other by Pons et al. (2004, GM-GS-U120).

As one can see, near the mixed-phase transition point the behaviour of the logarithm of pressure is linear with respect to the logarithm of baryon density; it clearly means that the polytropic approximation is valid in the small mixed-phase core regime. This is precisely the case we are interested in - we will henceforth benefit from expansion in powers of the core radius $r_m$. The validity of the $\gamma_m = \text{const.}$ approximation in the limit of small $r_m$ can be also phrased differently, using the small-$r_m$ expansions. Using techniques described in detail in Zdunik, Haensel & Schaeffer (1987), we can see that the difference between the value of $\gamma$ at $\rho_m$, and at the center of the $C^*$ configuration in Fig. 3 is quadratic in $r_m$, so that $\gamma^C = \gamma^m(\rho^C) = \gamma_m + O(r_m^2)$. As we show below, the leading changes due to a core-quake are proportional to the fifth (radius, moment of inertia) and seventh (gravitational mass) power of $r_m$. They are obtained by putting $\gamma^C \simeq \gamma_m$.

Inclusion of quadratic correction in the expression for $\gamma_m$ contributes to the higher order terms, i.e., seventh power of $r_m$ in the case of radius and moment of inertia, and ninth power of $r_m$ in the case of gravitational mass.

...to the mixed-phase segment $\rho_N^\text{m}$. The configurations calculated with these two EOSs are identical up to $r_m = 35 \text{ fm}$; the configuration with such central density is denoted by $C_0$, and will be treated as a “reference configuration”.

When the central density exceeds $\rho_{N}^\text{m}(\rho_c^*)$, the models based on these EOSs are different, due to the appearance of a softer mixed-phase core in configurations corresponding to the mixed-phase segment of the EOS. For a fixed baryon number $A$, greater than a baryon number $A_0$ of the reference configuration $C_0$, we compare the global parameters, such as mass-energy, radius, and moment of inertia. Their difference corresponds to the changes in these parameters implied by the phase transition in the stellar core.

From now on we will restrict ourselves to a linear response of neutron star to the appearance of the mixed phase core. The calculation is based on expressing the change in the density profile, due to the presence to a small core, as the combination of two independent solutions of the linearly perturbed equations of stellar structure (Haensel, Zdunik & Schaeffer 1987). The presence of a denser phase in the core changes the boundary condition at the...
phase transition pressure $P_N^{(m)}$ and allows us to determine the numerical coefficients in the expression for the density profile change. The leading term in the perturbation of the boundary condition at the edge of the new phase results from the mass excess due to the lower stiffness, and higher density of the new phase as compared to those of the N-phase.

Let us compare basic expressions obtained when the new core consists of a pure S-phase, considered by Haensel, Zdunik & Schaeffer (1986) and Zdunik, Haensel & Schaeffer (1987), and in the present case of a mixed-phase core. For a pure S-phase core the phase transition is accompanied by density jump $\rho_N \to \rho_S$ which leads to the lowest-order expression for the core-mass excess (with respect to the pure N-phase configuration),

$$\delta m_{\text{core}} = \frac{4}{3} \pi (\rho_S - \rho_N) r_S^3 + O(r_S^5),$$

(10)

where powers $(r_S)^l$ with $l > 3$ have been neglected. In the case of a mixed-phase core, considered in the present paper, the lowest-order expression vanishes, because there is no density jump at the core radius.

Let us introduce the notation appropriate for description of the change of the properties of matter at the N-phase – m-phase transition point. The transition occurs at $P = P_m \equiv P_N^{(m)}$. Density does not change and is equal $\rho_m \equiv \rho_N^{(m)}$, while the adiabatic index changes from $\gamma_N$ on the N-side to a lower value $\gamma_m$ on the m-side.

To the lowest order in $r_m$, the mass excess in our case is due to the difference in the stiffness (adiabatic index) of the matter in N-phase and m-phase. Expanding term by term, we get

$$\delta m_{\text{core}} = \frac{4}{3} \pi \rho_m (1 + x_m)(1 + 3x_m)(\kappa_m^2 - \kappa_N^2) r_m^3 + O(r_m^5),$$

(11)

where $x_m$ is the ratio of pressure $P_m$ and the energy density $\rho_mc^2$ at the phase transition point, $x_m = P_m/\rho_mc^2$. The parameters $\kappa_N^2$, $\kappa_m^2$ are defined by:

$$\kappa_N^2 = 4\pi G \rho_N/m_N c^2, \quad \kappa_m^2 = 4\pi G \rho_m/v_m^2,$$

(12)

where $v_N$ and $v_m$ denote the sound velocity on the both sides of the core boundary. Let us remind that sound velocity $v$ is related to the adiabatic index by $v^2 = \gamma P/\rho + P/c^2$.

The main parameter of the linear theory - the radius of the core of mixed phase $r_m$ is connected with the density range of metastability which can be achieved by the N phase of matter, i.e., the difference between $\rho_{\text{crit}}$ and $\rho_N^{(m)}$, via relation (see Zdunik, Haensel & Schaeffer 1987):

$$\Delta \rho_{\text{crit}} := \frac{\rho_{\text{crit}} - \rho_N^{(m)}}{\rho_N^{(m)}} = \frac{1}{6} \kappa_N^2 (1 + x_m)(1 + 3x_m) r_m^2.$$

(13)

Strictly speaking, the lowest order works only for sufficiently small value of $r_m$. From Eq. (11) and Eq. (12) we have

$$\delta m_{\text{core}} = \frac{4}{3} \pi \rho_m (1 + x_m)(1 + 3x_m)^2 \times$$

$$\times \frac{4\pi G \rho_m}{\gamma_N P_m^2} (\gamma_N/\gamma_m - 1) r_m^3 + O(r_m^5)$$

(14)

The new phase manifests itself in the boundary conditions by the presence of the prefactor $(\gamma_N/\gamma_m - 1)$, which acts similarly as the prefactor $(\rho_S/\rho_N - 1)$ in the case of a first order phase transition from pure N-phase to pure S-phase.

The fact that in our case the density stays continuous, results in the linear-response effects which are proportional to a power of the core radius greater by two than in the case of the transition to a pure S-phase.

On the basis of the above analytical considerations, we expect that the relative changes in stellar parameter $Q$ contain a prefactor $(\gamma_N/\gamma_m - 1)$ and are proportional to $r_m^5$ in the case of $Q = R, I$ and to $r_m^3$ in the case of mass-energy $E = Mc^2$. Summarizing, we expect the following form of the lowest-order expressions:

$$\delta Q \equiv \frac{Q^* - Q}{Q_0} \approx - (\gamma_N/\gamma_m - 1) \cdot \beta Q \cdot (r_m)^l,$$

(15)

where $r_m \equiv r_m/R_0$, and $l = 5$ for radius $R$ and moment of inertia $I$, and $l = 7$ for the energy $E$. The coefficients $\beta Q$ are functionals of the reference configuration, $\beta Q = \beta Q[C_0]$.

### 4 Changes in Stellar Parameters for Realistic EOSs

In order to explore how a realistic neutron star will react on the appearance of a mixed-phase core, to describe the N-phase we used two recent EOSs: SLy of Douchin & Haensel (2001) and FPS EOS of Pandharipande & Ravenhall (1989). Both EOSs describe in unified way (i.e., starting from a single effective nuclear Hamiltonian) both the crust and the core of neutron star. They assume that neutron star core is composed of neutrons, protons, electrons and muons.

In the actual calculations, we approximated the mixed-phase softening by replacing the mixed-phase segment of the EOS starting at $n_m = n_N^{(m)}$, by a polytrope with $\gamma_m < \gamma_N$. As we explained in Sect. 3, this is equivalent to neglecting the r-dependence of $\gamma$ within the mixed-phase core, and it does not change the leading terms of the small $r_m$-expansions. Also, as shown in Figs. 4 this approximation is excellent for not too large mixed-phase core.

We constructed a large set of EOSs with a mixed-phase segment, using many values of the phase-transition density $n_m$ and several values of $\gamma_m$. In this way we were able to study the linear response of neutron star to the appearance of a small mixed-phase core for a wide choice of the mixed-phase parameters.

Note again that the coefficients $\beta Q$ depend only on the properties of the reference configuration $C_0$ - the EOS of the mixed-phase (here approximated by a polytrope) enters merely via $\gamma_m$ in the factor $(\gamma_N/\gamma_m - 1)$ in Eq. (15).

We apply the arguments presented in Sect. 3 to the EOS constructed in the manner described above. By changing the value of $\rho_m$, we obtain a family of reference configurations $C_0$, and by comparing $C$ and $C^*$ we extract the values of the linear response coefficients $\beta Q[C_0]$.

### 4.1 Results for SLy EOS of the N-phase

The values of $\beta Q[C_0]$ are plotted in Fig. 6. An important parameter relevant for the linear response to a perturbation of equilibrium configuration is the adiabatic index of the N-phase EOS. We plot its value at the center of the reference configuration $C_0$ versus $M[C_0]$ and its central baryon number density in Fig. 6. As it can be seen on Fig. 6 for a wide astrophysically interesting range of neutron-star masses, the...
values of the functionals $\beta_Q[C_0]$ stay almost constant. The plateau values for SLy EOS are $\beta_I^{\text{plat}} = 2.0$, $\beta_R^{\text{plat}} = 0.8$, and $\beta_E^{\text{plat}} = 0.05$ for the moment of inertia, radius and mass-energy, respectively ($\beta_I^{\text{plat}} = 2.4$, $\beta_R^{\text{plat}} = 1.0$, and $\beta_E^{\text{plat}} = 0.05$ for FPS EOS, see next sub-section for details).

The values of $\beta_Q$ increase rapidly with decreasing mass below 0.4 $M_\odot$. This is due to the specific generic features of the realistic EOS of neutron star crust (Haensel [2001]), which results in the existence of the minimum neutron star mass $M_{\text{min}}$ (see Haensel, Zdunik & Douchin [2002], and references therein). For $M \rightarrow M_{\text{min}}$ neutron stars become more and more “soft” with respect to the fundamental mode of radial perturbation, and become unstable at $M_{\text{min}}$. Therefore, at the same mass-excess $\delta m_{\text{core}}$ the effect of neutron star “shrinking” due to an increased gravitational pull by the mixed-phase core grows, and very rapidly non-linear effects become important.

The “numerical plateau” values $\beta_Q^{\text{plat}}$ deserve an additional comment. The plateau region extends within $0.5M_\odot \lesssim M[C_0] \lesssim 0.9M_{\text{max}}$. It is easy to verify that our plateau values of $\beta_Q$ multiplied by $\simeq 3$ are very close to the linear response coefficients for the pure S-phase core, $\alpha_Q$, obtained for a medium-stiff EOS of the N-phase by Haensel, Zdunik & Douchin [2002]. Clearly, apart from some normalization constant, the linear response coefficient does not depend on the nature of the new-phase core, which enters only via $\partial Q/\partial m_{\text{core}}$ and the numerical factors multiplying $(\gamma_N/\gamma_{\text{init}} - 1)(\bar{r}_m)^3$ for a mixed-phase core. This result from the fact that the change of the stellar parameters $Q$ depends on the mass excess in the core via the relation (leading term): $\delta Q \simeq (\partial Q/\partial m)\delta m_{\text{core}}$ and the numerical factors multiplying $(\gamma_N/\gamma_{\text{init}} - 1)(\bar{r}_m)^3$ or $(\rho_S/\rho_N - 1)(\bar{r}_S)^3$ in $\delta m_{\text{core}}$ in the case of second and first order transition respectively differ by $\frac{\kappa_S^2}{\kappa_S^2} R^2$ which is approximately $1/3$.

4.2 Results for FPS EOS of the N-phase

The values of $\beta_Q[C_0]$ are plotted in Fig. 6 where they can be easily compared with those obtained for the SLy EOS. The plateau values are $\beta_I^{\text{plat}} = 2.4$, $\beta_R^{\text{plat}} = 1.0$, and $\beta_E^{\text{plat}} = 0.05$. As we see, they depend rather weakly on the EOS of the N-phase, provided it is realistic. The $M_0$ range of the plateau region, $(0.6 - 1.6) M_\odot$, is significantly narrower than for the SLy EOS. One can try to point out reasons for this. First, on the high-mass side, the difference results from the fact that $M_{\text{SLy}}^{\text{max}} = 1.79 M_\odot$, to be compared with $M_{\text{FPS}}^{\text{max}} = 2.05 M_\odot$. Otherwise, the quasi-constancy is valid, similarly as for the SLy EOS, up to about 0.9 $M_{\text{max}}$, which seems to be generic. “Softness” of hydrostatic equilibrium at $M$ close to $M_{\text{max}}$ with respect to radial perturbations is the general-relativistic feature, and does not depend on the EOS of the N-phase. On the contrary, on the low-mass side, the difference in the $\beta_Q$ behavior reflects the differences between the EOSs at sub-nuclear densities. This difference between the SLy and FPS EOSs was studied, albeit in a different context, by Haensel, Zdunik & Douchin [2002]. In the relevant stellar mass range, the FPS stars are less bound and therefore, with decreasing mass, its $\beta$ coefficients start to increase earlier than for the SLy EOS.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.png}
\caption{The linear response parameters $\beta_Q$, versus the mass $M_0$ of the reference configuration $C_0$. The N-phase is described by the SLy EOS (solid lines, $M_{\text{max}} = 2.05 M_\odot$) and, for comparison, by FPS EOS (dashed lines, $M_{\text{max}} = 1.79 M_\odot$).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig6.png}
\caption{The value of the adiabatic index at the center of non-rotating configuration for the SLy EOS (upper panel) and FPS EOS (lower panel), plotted against the gravitational mass (bottom horizontal axis), and baryon number density $n_b$ at the star center (upper horizontal axis). Vertical dotted line marks the maximum allowable mass ($M_{\text{max}} = 2.05 M_\odot$ for the SLy EOS, and $M_{\text{max}} = 1.79 M_\odot$ for the FPS EOS).}
\end{figure}
4.3 Estimating changes in stellar parameters in a core-quake

Let us now calculate the changes in stellar parameters. First, the estimates will be obtained for the SLy EOS of the N-phase. Consider $c_0$, configuration of mass $1.4 M_\odot$ calculated for the SLy EOS. It has $R \equiv R_0 = 11.73$ km, $I \equiv I_0 = 1.37 \times 10^{45}$ g cm$^2$, and a central adiabatic index $\gamma_N \approx 2.94$. Using the plateau values of the $\beta_0$ coefficients, we can rewrite the formulae for the changes in neutron-star parameters in a form suitable for making numerical estimates. For the stellar radius we get,

$$\Delta R \simeq -4.23 \times 10^{-5} \cdot (2.94/\gamma_m - 1)(r_m/1 \text{ km})^3 \text{ km}. \quad (16)$$

Assume $\gamma_m = 1.5$. Then, a 1 km core implies shrinking by about 4 cm, a very small star-quake indeed, but still larger by more than an order of magnitude than that associated with macro-glitches in the pulsar timing. However, the rise of the star-quake amplitude with $r_m$ is very steep and for $r_m = 4$ km we get $\Delta R = 42$ m, rather impressive even by the terrestrial standards.

Expression for the fractional change in $I$ is

$$\Delta I/I_0 \simeq -9 \times 10^{-6} \cdot (2.94/\gamma_m - 1)(r_m/1 \text{ km})^3. \quad (17)$$

Assume as before that $\gamma_m = 1.5$. Formation of a 1 km core implies speed-up of the neutron star rotation by $\Delta \Omega/\Omega \simeq -\Delta I/I \approx 10^{-5}$, one order of magnitude larger than in the biggest pulsar macro-glitches. If the core radius is 4 km, then $\Delta \Omega/\Omega \approx -\Delta I/I \approx 9 \times 10^{-5}$, a very distinct feature of a neutron-star core-quake.

Let consider now the energy release. We have

$$\Delta E \simeq 4.1 \times 10^{45} \cdot (2.94/\gamma_m - 1)(r_m/1 \text{ km})^7 \text{ erg}. \quad (18)$$

which means that formation of a 1 km core with $\gamma_m = 1.5$ will release a total energy of $4 \times 10^{45}$ erg. Corresponding amount of released energy for a core with $r_m = 2$ km and $r_m = 4$ km is $5 \times 10^{47}$ erg and $6.4 \times 10^{49}$ erg, respectively. It may be worthwhile to compare it with energies associated with a few known astrophysical processes involving neutron stars. For example, the energy associated with Soft Gamma Repeaters (SGR) outbursts, which are now believed to be caused by an extremely strong magnetic field, is $\sim 10^{44} - 10^{45}$ erg. The Vela pulsar macro-glitches release from $10^{38}$ to $10^{42}$ erg. The thermonuclear Type I X-ray bursts involve energies $10^{36} - 10^{39}$ erg, but in the case of the so-called super-bursts the released energy can be as high as $10^{44}$. By adjusting the core radius, the energies associated with core-quaques can be made similar to those released in phenomena listed above. It should be however noted that the core radius is not a free parameter in our approach but is determined by the range of metastability possible in the considered physical situation which is rather poorly known and model dependent. The leading terms in Eqs. (16, 17) for the radius $r_m = 1$ km correspond to the degree of metastability, which is usually called “over-compression” of the order of 0.5%, i.e $\Delta \rho_{\text{crit}} \simeq 0.005$, see Eq. (18).

Up to now, we performed all numerical estimates for the SLy EOS of the N-phase. For the same values of $r_m$ and $\gamma_m$, the values of $\Delta R$ etc., for the FPS model of the N-phase, will be somewhat different, but these differences are of no practical importance. What is really crucial, is the strong dependence of $\Delta R$, $\Delta I$, and especially of $\Delta E$, on the mixed-phase core radius.

5 ASTROPHYSICAL SCENARIOS

One can consider several astrophysical scenarios of the formation of a mixed-phase core in neutron star. In all cases we are dealing with a two step process. The first step consists in the nucleation of the S-phase in the metastable N-phase medium, studied by Haensel & Schaeffer (1982); Muto & Tatsumi (1993); Hida & Sato (1997, 1998). The second step consists in the relaxation of the system to a stable mixed state in which N-phase coexists with a S-phase.

The first scenario is connected with neutron star birth, in which proton-neutron star is formed in gravitational collapse of a massive stellar core. Central core of a proto-neutron star has a relatively low density, compared to central density of neutron star in which it will transform eventually (after a few minutes). This is due to large lepton fraction (30-40%) resulting from neutrino trapping. The diffusion of neutrinos and resulting deleptonization of matter is connected with softening of the neutron-star core EOS and matter compression to higher density. Simultaneously, the core is heated (up to $30-60$ MeV/$k_B$) by the deposition of the energy of strongly degenerate electron neutrinos via their down-scattering on the constituents of dense medium. The optimal conditions for nucleation of the S-phase are just after deleptonization (i.e., $\sim 10$-20 seconds after the bounce ending the collapse phase). The thermal fluctuations at $T \sim 40$ MeV/$k_B$ are expected to be sufficient to overcome energy barriers, resulting from the Coulomb and surface effects, and to form a mixed-phase core in thermodynamic equilibrium.

Central compression of matter during slow-down of pulsar rotation is a second scenario to be considered. Let the pulsar rotation frequency be $\nu$. The effect of rotation on the pressure distribution is to lowest order $\propto \nu^2$. Therefore, the rate of the quasi-static increase of central pressure is $P_c \propto -\nu$. The central-core temperature is $< 10^9$ K, so that nucleation can proceed only via quantum fluctuation in super-compressed core. Nucleation induces pressure deficit, collapse of the central core accompanied by matter heating, and energy release. During these violent processes involving matter flow, N and S-phase mix forming a mixed-phase core.

A third scenario involves neutron star accreting matter in a close binary system. Central pressure increases due to the gravity of accumulated layers of accreted matter, at a rate $P_c \propto M$. The core temperature remains below $10^9$ K (see e.g. Miralda-Escudé, Haensel & Paczynski 1990), and therefore quantum fluctuations are crucial for the nucleation process to start. Then the pressure deficit triggers core collapse, accompanied by matter heating and flow, and a mixed-phase core forms.

6 SOME PROBLEMS CONNECTED WITH NUCLEATION OF A MIXED NORMAL-EXOTIC PHASE

Let us assume that the phase transition from a pure N-phase to a pure S-phase is the first-order one, at $P = P_0$ (Fig.
Let us further assume, that within the pressure interval $P_N^{(m)} < P < P_S^{(m)}$ the thermodynamic equilibrium of dense matter is realized in the form of the mixed NS phase.

Consider a neutron star, where the central pressure $P_c$ increases in time due to a mass accretion in a binary system or due to a spin-down of neutron-star rotation. The central pressure increases on a characteristic timescale $t_{\text{cool}} = P_c / \dot{P}_c$. Let us assume that initially the stellar core consisted of a pure N phase. In this case the internal temperature does not exceed $10^{10}$ K, the thermal fluctuations are negligible and the S phase can nucleate only via quantum fluctuations on a certain timescale $\tau$. However, as long as $P_c < P_0$, the nucleation of the S-phase in the quantum regime is impossible ($\tau = \infty$). The actual nucleation will start at some $P_c = P_{\text{nucle}} > P_0$, by a formation of a single droplet. If the expansion rate of the first drop is larger than the formation rate of other droplets in the core, then the pure S phase will fill the central core with $P \leq P_0$. The S-phase core will then coexist with the outer layer of the N phase, with a baryon density drop $n_S - n_N$ at the interface. Such a scenario seems likely in the case of the quark-matter nucleation (Iida & Satô 1997, 1998).

The central temperature of a proto-neutron star stays high, $T_c > 10^{11}$ K, for some ten seconds. This may be enough to complete the nucleation of the S phase below $P_0$ and mix it with the N phase, achieving the thermodynamic equilibrium. It is not excluded, however, that the core will have no time to evolve to thermodynamic equilibrium in the low-temperature limit, because of a too rapid cooling to $T \approx 10^{10}$ K. It is therefore possible that the final mixed state with some spatial structure will be quite different from the strict ground state of the core at $T = 0$; the mixed phase can remain “frozen” in some metastable state in the core or its part.

Finally, let us mention two difficulties in forming a mixed phase in kpton-condensed neutron-star cores (Norsen 2002). It is difficult to nucleate kaon condensate because of the slowness of weak interaction processes; very high temperature $T > 10^{11}$ K and low kaon effective masses are required for the condensation due to thermal fluctuations. This may happen only in massive newborn neutron stars, with particularly high central temperatures and densities, where a mixed-phase core could be formed. Medium-mass neutron stars have insufficient central density to nucleate the kaon condensate at their birth. On the other hand, high-mass neutron stars, which gain their mass by accretion, can remain in a metastable state forever, because their internal temperature is too low for nucleating kaon condensate in their cores.

7 CONCLUSIONS

Formation of a mixed-phase core implies shrinking of neutron star radius, spin-up, and is associated with energy release. Changes in neutron star parameters strongly depend on the size of the mixed-phase core. The decrease of stellar radius, $\Delta R$, and of the moment of inertia, $\Delta I$, are proportional to the fifth power of the core radius. The total energy released during the core-quake, $\Delta E$, is proportional to the seventh power of the core radius. These powers are higher by two than the powers of the core radius in the formulæ expressing the changes of the stellar parameters implied by the formation of a pure high-density phase core.

Apart of the core radius, the formulæ involve also the drop of the adiabatic index at the mixed-phase core edge, and a numerical coefficient determined by the normal-phase configuration. In the stellar mass range $0.5 M_\odot \lesssim M \lesssim 0.9 M_{\text{max}}$ the numerical coefficients for $\Delta R$, $\Delta \Gamma$ and $\Delta E$ depend rather weakly on $M$ and can be replaced by their “plateau value”.

This paper completes the study, within the linear response theory, of the effect of phase transitions in the neutron star core on the stellar structure. It extends the methods initiated in the 1980s for a single (pure) dense phase core (Schaeffer, Haensel & Zdunik 1983; Haensel, Zdunik & Schaeffer 1986; Zdunik, Haensel & Schaeffer 1987) to the case of a mixed-phase core. Our results are presented in the form of formulæ which are convenient to make numerical estimates, and which become very precise for small mixed-phase cores, when the brute-force calculations are impossible because of limited numerical precision.

The final state reached in a phase transition in neutron-star core depends on the kinetics of this process. Even if the mixed state is the stable final configuration, it does not mean that it is reached, because a one-phase core may form instead. Mixed-state is particularly difficult to be formed if a significant strangeness production is necessary in the dense-phase nucleation process. At temperatures and densities characteristic of the cores of accreting neutron stars, the mixed-phase core may never form.

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