Measuring 4-local $n$-qubit observables could probabilistically solve PSPACE

Pawel Wocjan*, Dominik Janzing, Thomas Decker, and Thomas Beth

Institut für Algorithmen und Kognitive Systeme, Universität Karlsruhe, Am Fasanengarten 5, D-76131 Karlsruhe, Germany

August 1, 2003

Abstract

We consider a hypothetical apparatus that implements measurements for arbitrary 4-local quantum observables $A$ on $n$ qubits. The apparatus implements the “measurement algorithm” after receiving a classical description of $A$. We show that a few precise measurements, applied to a basis state would provide a probabilistic solution of PSPACE problems. The error probability decreases exponentially with the number of runs if the measurement accuracy is of the order of the spectral gaps of $A$.

Moreover, every decision problem which can be solved on a quantum computer in $T$ time steps can be encoded into a 4-local observable such that the solution requires only measurements of accuracy $O(1/T)$.

Provided that BQP $\neq$ PSPACE, our result shows that efficient algorithms for precise measurements of general 4-local observables cannot exist. We conjecture that the class of physically existing interactions is large enough to allow the conclusion that precise energy measurements for general many-particle systems require control algorithms with high complexity.

*e-mail: {wocjan,janzing,decker}@ira.uka.de
1 Measuring \( k \)-local observables

A characteristic feature of quantum theory is that there exists an abundance of mutually incompatible observables (described by self-adjoint operators \( A \)) for every quantum system and it is by no means obvious how to implement measurement procedures for all these quantities. On a quantum computer one could in principle measure every observable \( A \) as follows: Find a unitary transformation \( U \) which diagonalizes \( A \) with respect to the computational basis. Then \( A \) is measured by implementing \( U \) and measuring the logical state of each qubit. By identifying each binary word with the corresponding eigenvalue of \( A \) this procedure reproduces all probabilities

\[
p_j := \text{tr}(\rho P_j)
\]

correctly where \( \rho \) is the density matrix of the quantum register and \( (P_j) \) is the family of spectral projections of \( A \).

However, the implementation of the diagonalizing operation \( U \) will in general be hard. Therefore, one may restrict the attention to specific classes of observables. It is natural to consider observables with physical relevance. For example, the quantum observable “energy”, mathematically described by the self-adjoint operator \( H \) (the Hamiltonian), is certainly one of the most important observables in physics. It determines the dynamical and thermodynamic behavior of the considered quantum system. Furthermore the eigenstates of the Hamiltonian, the energy levels, are “directly” observable in many physical situations. For instance, in spectroscopy the eigenvalues of the Hamiltonian determine the frequencies of emitted or absorbed photons. Nevertheless, the determination of the energy levels in interacting many-particle systems is in general a difficult task.

To explain this more explicitly, we need to describe the class of operators which is considered. First we note that physical interaction Hamiltonians usually satisfy some locality condition in the following sense. We call an \( n \)-qubit operator \( k \)-local if it is a sum of operators which act on at most \( k \) particles non-trivially. For fundamental interactions between real physical particles one has more specific statements and may restrict the attention to pair-interactions. Nevertheless, \( k \)-local interactions among qubits are physically reasonable. They may describe effective Hamiltonians and there is not necessarily a one-to-one correspondence between qubits and physical particles (\( l \) qubits may, for instance, describe the state of one particle). The
following results show that it is in general difficult to compute the spectrum of $k$-local Hamiltonians.

The problem of determining the lowest energy value of a (classical) spin-spin interaction of Ising type is known to be NP-complete [1,2]. For interacting qubits determining the lowest energy value is even QMA-complete (“Quantum-NP”) if one allows 3-local interactions only [3,4]. Note that in these NP and Quantum-NP problems the task is not to determine the lowest eigenvalues with high precision. The demanded accuracy is only inverse polynomially in the number $n$ of interacting qubits. This has implications for the measurement procedure above:

The unitary $U$ that maps the eigenvectors of $A$ to the computational basis states is only helpful for measuring $A$ if the correspondence between computational basis states and the eigenvalues of $A$ is known. Therefore, this method would require to know the spectrum of $A$. For 2-local or 3-local observables one would need the solution of NP- and QMA-hard problems, respectively.

In this paragraph we will explain that measurements of $k$-local observables $A$ are possible up to inverse polynomial accuracy without using any knowledge on the spectrum of $A$.

Here we do not need a precise definition of accuracy, we only demand that the following condition is satisfied:

**Postulate 1 (Measurement accuracy)**

A measurement with accuracy $\Delta \lambda$ has the following property: For all density matrices $\rho$ the probability to obtain an outcome in the interval $I := [\lambda_j - \Delta \lambda, \lambda_j + \Delta \lambda]$ is at least $(3/4) \text{tr}(\rho P_j)$.

Our result is not sensitive to the particular definition of accuracy. However, it is convenient to work with the formulation above.

Now we describe how to implement approximative measurements. The idea is that for every $k$-local $A$ (with $k$ constant) the corresponding time evolution $U_t := \exp(-iAt)$ (if $A$ is interpreted as a Hamiltonian $H$ of a quantum system) can be simulated efficiently in an approximative sense. Explicitly, it has been shown that the simulation of $U_t$ with elementary gates up to an error of $\epsilon$ (with respect to the operator norm) requires $O(t^2/\epsilon)$ gates [5]. Now we can choose $t$ in such a way that there is a one-to-one correspondence between the eigenvalues of $U_t$ and $A$. This is the case whenever $\|A\|t \leq \pi$. An upper bound on the norm of a $k$-local operator is easy to get. We assume without loss of generality that each $k$-local term is upper bounded by
The powers of $U := \exp(-iAt)$ are implemented as conditional gates controlled by the ancilla register.

Figure 1: Circuit for performing measurements of an observable $A$. The
powers of $U := \exp(-iAt)$ are implemented as conditional gates controlled
by the ancilla register.

the value 1. There are at most

$$\binom{n}{k} < n^k$$

$k$-local terms. Therefore, one has $\|A\| = O(n^k)$. For an appropriate
value of $t$ we can implement measurements of the “observable” $U_t$
using\footnote{Note that also a unitary operator defines in a canonical way an observable by its spectral projections if one allows complex measurement outcomes.} the quantum state estimation procedure\[6\]. We will briefly sketch the idea. In the following we drop the index $t$.

The circuit for phase estimation is shown in Fig. 1. It acts on the
registers $R_S$ and $R_A$. Controlled-$U^j$ gates are implemented in such a
way that $U^j$ is performed on $R_S$ if and only if the ancilla register $R_A$
is in a state corresponding to the binary word $j$. This can be done by
implementing controlled

$U^{2^l}$

gates which are applied if and only if the $l$th qubit of the ancilla register
is in the state $|1\rangle$. The algorithm starts with an equally weighted superposition

$$\frac{1}{\sqrt{2^m}} \sum_{j=1}^{2^m} |j\rangle,$$
of all ancilla register states where \( m \) is the number of ancilla qubits. After applying the controlled \( U^j \)-operation the discrete Fourier transform of size \( 2^m \) is applied to the ancilla register. Then the eigenvalues of \( U \) can be readout with an error of the order \( 1/2^m \). Now we consider the running time of this scheme (depending on the accuracy). Obviously, this depends on the running time for implementing the controlled-\( U_j \) operations. The natural method to implement \( U^j = \exp(-iAjt) \) is to simulate the time evolution with Hamiltonian \( A \) for the time \( tj \). The substitution of the corresponding gates by controlled gates is straightforward. However, \( j \) grows exponentially with \( m \). Consequently, this method requires exponential running time for exponential accuracy. It is likely that all possible schemes for measuring \( A \) precisely share this disadvantage.

For a black-box unitary \( U \) it is clear that exponential accuracy requires exponential time since the black-box unitary \( U \) has to be applied an exponential number of times. This can be seen by subjecting two state vectors to different unitaries \( U \) and \( \tilde{U} \) with the same eigenstates but slightly different eigenvalues. Then \( U^l \) and \( \tilde{U}^l \) can only lead to distinguishable states for large \( l \). In [7] it is described how to convert the time evolution \( \exp(-iHt) \) according to an unknown pair-interaction Hamiltonian \( H \) to a controlled-\( \exp(-iHt) \) evolution. This shows that black-box settings for unknown \( A \) do in principle make sense for energy measurements.

For non-black box interactions we cannot obtain lower bounds on the measurement complexity by similar arguments since the apparatus receives a classical description of the observable to be measured. However, the result of this paper suggests that even if the interaction is known there is no efficient measurement scheme with exponential accuracy. We show that measurements of 4-local \( n \)-qubit observables \( A \) could be used to solve PSPACE-problems in polynomial time provided that the accuracy is sufficient to distinguish between the different eigenvalues of \( A \).

One may ask whether there may be any physical processes for measuring 4-local observables that do not rely on quantum circuits (consisting of elementary gates). For instance, one may guess that a measurement of the energy of a system is simpler than a measurement of an arbitrary \( k \)-local observable because energy is a specific observable determining many physical aspects of the system. But note that the quantum version of the Strong Church-Turing Thesis (compare [8, 9]) states that every problem that can be solved efficiently using
some physical process can be solved efficiently by a quantum computer.

If there existed any efficient scheme for precise measurements of 4-local observables this would imply either of the following statements:

1. The measurement process cannot efficiently be simulated on a quantum computer (in contrast to the strong quantum Church-Turing Thesis).
2. There are polynomial time algorithms to solve probabilistically PSPACE problems, i.e., PSPACE = BQP.

Assuming that both implications are unlikely, our result strongly suggests limitations for future quantum measurement technology.

The structure of the paper is as follows. In the next section we consider a class of quantum circuits with polynomial size. They could solve PSPACE problems if they were applied an exponential number of times on a polynomial number of qubits.

In Section 3 we describe how to construct a 4-local observable corresponding to this circuit in such a way that precise measurements would solve PSPACE problems in polynomial time.

2 Characterizing PSPACE by circuits

The complexity class PSPACE is usually defined with respect to the Turing machine model \[10\]. PSPACE is the class of all languages recognizable by polynomial space bounded deterministic Turing machines that halt on all inputs \[11\].

For our purposes we need a characterization of PSPACE with respect to quantum circuits. In particular, we need the result that every PSPACE language can be recognized by applying an appropriate circuit many times.

**Theorem 1 (PSPACE)**

*For every language \( L \) in PSPACE there is a polynomial-time uniformly generated family of quantum circuits \((V_l)_{l \in \mathbb{N}}\) consisting of \( s_l = \text{poly}(l) \) elementary quantum gates and acting on \( m_l = \text{poly}(l) \) many qubits. The circuit \( V_l \) decides whether an input string \( x \) of length \( l \) is an element of \( L \) in the following sense.*
There is a polynomial-time computable natural number \( r_l \) such that the \( r_l \)-fold concatenation of \( V_l \) solves the corresponding PSPACE problem, i.e.,

\[
V_l^{r_l}(|x\rangle \otimes |y\rangle \otimes |00\ldots0\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle \otimes |00\ldots0\rangle,
\]

where \( f \) is the characteristic function of \( L \), i.e., \( f(x) = 1 \) if \( x \in L \) and \( f(x) = 0 \) otherwise. The vector \( |x\rangle \) is the basis state given by the binary word \( x \in \{0,1\}^l \), the vector \( |y\rangle \) is the state of the output qubit and \( |00\ldots0\rangle \) is the initial state of \( m_l - l - 1 \) ancilla qubits.

**Proof.** In order to construct the circuit \( V_l \) corresponding to a PSPACE problem we need to have an upper bound for the required space. This is, for instance, the case for the PSPACE-complete problem QBF (Quantified-Boolean Formulas). It can be solved within the space \( O(l^2) \) where \( l \) is the length of the input. This space bound determines \( m_l \), the number of qubits.

Let \( M \) be a Turing machine that solves QBF within space \( O(l^2) \). Now we construct a quantum circuit \( V_l \) that simulates the Turing machine \( M \) for input length \( l \). Since the computational steps of a quantum circuit are unitary (thus reversible), we have to work with a reversible Turing machine \( R \) instead of \( M \) (the latter could be irreversible). Each application of the constructed circuit simulates one or two steps of \( R \).

Due to a result of Lange et al. (Theorem 3.3 in [12]) it is possible to simulate irreversible Turing machines by reversible ones without increasing the necessary space too much. More precisely, they give the simulation of a space-bounded Turing machine \( M \) by a reversible Turing machine \( R \) operating on the same space. In general, the reversible simulation by \( R \) may have an exponential time overhead. The running time overhead is not relevant here because we can derive an upper bound on the running time of the reversible machine from the required number of qubits. In the following we work with the reversible Turing machine \( R \).

The fact that every Turing machine can be simulated efficiently by circuits is standard [13]. Here we need an explicit construction converting the reversible Turing machine into a circuit consisting of reversible gates.

The circuit acts on the following registers:

1. The register head encodes the internal state of the Turing machine.
2. The register `tape_index` stores the current location of the head.
3. The register `ACC` is the accumulator (temporary storage).
4. The register `tape` corresponds to a sufficiently large region of the tape that is required for computation. It consists of cell 1 to cell \( N_l \) where \( N_l \) is the space bound corresponding to the input length \( l \).

Each step of the reversible Turing machine of Lange et al. is either a moving or read-and-write transition\(^2\). A moving transition has the form \( p \rightarrow (q, \pm 1) \). That means that in state \( p \) the machine makes one step to the right (+1) (respectively to the left (−1)) and changes into state \( q \) without reading or writing any tape cell. A read-and-write transition has the form \( (p, a) \rightarrow (q, b) \) meaning that in the state \( p \) the machine overwrites the symbol \( a \) with the symbol \( b \) and changes into state \( q \) without moving the head.

Furthermore, in our construction it is determined by the state of the head whether the system performs a moving or a read-and-write operation (and not by the state of the tape). In other words, the state set \( \hat{Q} \) of \( R \) is the disjoint union of a set \( Q \) of read-and-write states, a set \( Q^{\rightarrow} \) of right-moving states and a set \( Q^{\leftarrow} \) of left-moving states.

Due to reversibility of \( R \) the moving transitions can be implemented as a unitary transformation on the registers \( \text{head} \) and \( \text{tape} \). A right-moving transition \((p \in Q^{\rightarrow})\) translates as follows:

\[
|p\rangle_{\text{head}} \otimes |i\rangle_{\text{tape-index}} \rightarrow |q\rangle_{\text{head}} \otimes |i + 1\rangle_{\text{tape-index}}. \tag{1}
\]

Analogously, a left-moving transition \((p \in Q^{\leftarrow})\) translates as follows:

\[
|p\rangle_{\text{head}} \otimes |i\rangle_{\text{tape-index}} \rightarrow |q\rangle_{\text{head}} \otimes |i - 1\rangle_{\text{tape-index}}. \tag{2}
\]

Note that the operations on the register `tape_index` are computed modulo \( N \), where \( N \) is the number of tape cells. Although the Turing machine will never move to the right when the head is at position \( N \) and never to the left when it is at position 1, this definition guarantees that eqs. (1) and (2) define unitary operators.

These transformations can be realized efficiently as a unitary transformation \( U_{\text{moving}} \) acting only on the registers \( \text{head} \) and `tape_index`.

Again, due to reversibility of \( R \) the read-and-write transitions can be realized by a unitary transformation. Let \((p, a) \rightarrow (q, b)\) be a read-and-write transition. There is a unitary transformation \( W_{r/w} \) acting

\(^{2}\text{This separation is useful in order to characterize reversibility of Turing machines} \cite{14}.\)
on the registers head and ACC realizing

$$|p\rangle_{\text{head}} \otimes |a\rangle_{\text{ACC}} \rightarrow |q\rangle_{\text{head}} \otimes |b\rangle_{\text{ACC}}.$$  

We denote by SWAP(ACC, tape[i]) the unitary operation that swaps ACC and the i-th cell of tape. We denote by $\Lambda_i(U)$ the controlled operation that performs $U$ if and only if tape_index has the value $i$. Now we define $U_{r/w}$ as the concatenation

$$\prod_{i=1}^{N} \Lambda_i(\text{SWAP(ACC, tape[i])}) W_{r/w} \prod_{i=1}^{N} \Lambda_i(\text{SWAP(ACC, tape[i])}).$$

The transformation $U = U_{r/w} U_{\text{moving}}$ is the transformation that corresponds to a moving and/or read-and-write transition of the reversible Turing machine $R$. (Note that if a read-write translation is followed by a moving transition then this $U$ performs both transitions.)

The constructed circuit $U$ does not satisfy all requirements of the theorem. The first problem is that we do not know the running time of the reversible Turing machine $R$. Consequently, we do not know how many times we have to perform the elementary circuit $U$ to obtain the solution. Even if we knew how many times we have to apply $U$, the corresponding transformation would in general change the state $|x\rangle$ and produce some garbage on the ancillas.

To circumvent the first problem we introduce some idle cycles to guarantee that the running time is an efficiently computable function $r_l$ of the input length $l$. The second problem is solved by uncomputing the operations carried out during the computational steps and the idle cycles.

In the step $r_l/2$ (proper computational steps and the idle cycles) the solution is copied to the register solution. In the following $r_l/2$ steps we uncompute the idle cycles and the computational steps. The computational steps are uncomputed by applying $U^\dagger$ corresponding to running $R$ backwards.

Now we construct the quantum circuit $V$ that circumvents both problems as explained. Note that we drop the index $l$ in the following. The circuit operates on the registers head, ACC, tape_index, tape and the new registers solution, operation_mode, idle_counter, and counter (See Fig. 2).

The register operation_mode indicates whether the current operation is $U$, idle cycle, reverse idle cycle, or $U^\dagger$. These 4 subroutines of the whole circuit can be seen in Fig. 3.
Figure 2: Quantum circuit satisfying the requirements of Theorem 1. The gates INC and DEC increment and decrement the register counter and idle counter, respectively. The gates $b_1b_2/b'_1b'_2$ swaps the state $|b_1b_2\rangle$ and $|b'_1b'_2\rangle$. The gate 00/01 10/11 is controlled by $q_f$ and the state of the idle counter. The symbol $q_f$ represents all final states of the Turing machine $R$. The symbol $a_f$ denote the solution $f(x)$. The bit-flip on the register solution is controlled by $a_f$ and the state of the register counter.
The content of the register `counter` is incremented after each application of $U$ or `idle cycle` and decremented after each application of $U^\dagger$ or `reverse idle cycle`. Our construction uses the following upper bound on the number of necessary applications of $U$. Since the number of basis states of the register that $U$ acts on is $2^m$ it does not make sense to have $r > 2^m$. Therefore the counter is incremented until the register has the result $2^{m+1} - 1$ in order to ensure that the number of required applications of $U$ is exceeded. As soon as this number is reached the result of the computation is copied to the register `solution`, i.e., the register is incremented by 1 if and only if the answer is “true”. From this moment on the `counter` and the `idle_counter` are decremented. As soon as the `idle_counter` reaches 0 the operation mode is changed such that the concatenated application of $U^\dagger$ is started. After the appropriate number of applications the initial state of all registers are restored except from the register `solution` which is incremented by 1 if and only if the answer of the PSPACE problem is “true”.

Explicitly, one has the following rules:

1. operation mode `00`: perform the circuit $U$, increment `counter`
2. change operation mode `00` $\rightarrow$ `01` if `idle_counter` is `00` $\cdots$ `0` and `head` is in a final state
3. operation mode `01`: increment `counter` and `idle_counter`
4. increment `solution` if `operation_mode` is `01`, `counter` is `11` $\cdots$ `1` and the first tape cell is in a state indicating if the answer is true (we assume that this tape cell contains the result $f(x)$)
5. change operation mode `01` $\rightarrow$ `10` if `counter` is in `11` $\cdots$ `1`
6. operation mode `10`: decrement `counter` and `idle_counter`
7. change operation mode \( 10 \to 11 \) if \( \text{idle\_counter} \) is in \( 00 \cdots 0 \) and \( \text{head} \) is in a final state

8. operation mode \( 11 \): perform the circuit \( U^\dagger \), decrement \( \text{counter} \) and \( \text{idle\_counter} \)

9. change operation mode \( 11 \to 00 \) if \( \text{counter} \) is in \( 00 \cdots 0 \)

Note that the circuit \( V \) has the following property: applied to the initial state \( |x\rangle|0\cdots0\rangle \) the orbit length is \( r = 2(2^m+1)-1 \) if the answer is “false” and \( 2r \) whenever the answer is “true”. \( \square \)

The dependence of the orbit length on the solution is essential in the following section.

3 Constructing the observable

In this section we construct a family of observables \( (A_l) \) in such a way that the spectral properties of \( A_l \) reflect the length of the orbit \( (V_j^l|x\rangle \otimes |00\cdots0\rangle)_{j\in\mathbb{N}} \) for inputs of length \( l \). The idea to construct Hamiltonians corresponding to quantum circuits already appeared in [15]. In this article, the purpose was to show that a closed quantum (Hamiltonian) system can in principle implement a circuit without any external control operations. Similar constructions were also used in the context of complexity theory in order to show that determining the spectrum of physical Hamiltonians may be computationally hard [3, 4]. However, their constructions deal with quantum circuits of polynomial size. The whole sequence of gates is in some sense encoded into the Hamiltonian. The solution of a NP or QMA problem is then reflected in the least eigenvalue of the Hamiltonian. The fact that the determination of the least eigenvalue encompasses NP or QMA even if only inverse polynomial accuracy is required is due to the polynomial length of the program. Here we have typically an exponential number of applications and the solution of the problem is therefore encoded in the “hyperfine structure” of the spectrum.

Let \( V \) be a quantum circuit as in Theorem 11 and \( s \) be its size, i.e., the number of elementary two-qubit gates. We need a register \( \text{clock} \) indicating which gate is applied. It consists of \( s_l \) qubits. The allowed states of the register \( \text{clock} \) are of the form \( |0\cdots010\cdots0\rangle \) indicating which gate of \( V \) is applied currently. We denote by \( V_j \) the elementary gates of \( V \) (in contrast to the preceding section where the index denoted the input length).
We first define the forward-time operator

\[ F = V_1 \otimes |1\rangle_2 \langle 0|_2 \otimes |0\rangle_1 \langle 1|_1 + V_2 \otimes |1\rangle_3 \langle 0|_3 \otimes |0\rangle_2 \langle 1|_2 + \vdots + V_s \otimes |1\rangle_1 \langle 0|_1 \otimes |0\rangle_s \langle 1|_s. \]

The operators \( V_j \) operate on all registers of the preceding section. The operators \( |0\rangle_i \langle 1|_i \) and \( |1\rangle_i \langle 0|_i \) are annihilation and creation operators, respectively, on the \( i \)th qubit of the clock.

We denote the linear span of the vectors

\[ F^j \Psi_0 \] for \( j \in \mathbb{N} \)

with \( |\Psi_0\rangle := |x\rangle \otimes |00\cdots 0\rangle \otimes |100\cdots 0\rangle \) as \( \mathcal{O} \). All states of this orbit are orthogonal until one has recurrence to the initial state \( |\Psi_0\rangle \). This can be seen as follows: If the register clock is in an allowed state there is only one summand of \( F \) that is relevant. Its action on the clock is simple since it moves the 1 to the next qubit. Therefore it is clear that the first \( s - 1 \) states are orthogonal. The whole circuit \( V \) is a classical logical operation which permutes basis states. Therefore the state \( F^s |\Psi_0\rangle \) is either orthogonal to \( |\Psi_0\rangle \) or both states coincide.

Along the same line we argue that all states of the orbit are orthogonal until a state coincides with the initial state. Hence \( F \) acts as a cyclic shift on \( \mathcal{O} \).

The dimension of \( \mathcal{O} \) is \( 2sr \) if \( f(x) = 1 \) and \( sr \) if \( f(x) = 0 \). We denote the dimension by \( d \).

Let \( \omega \) be a primitive complex \( d \)-th root of unity. The eigenvalues of \( F \) restricted to \( \mathcal{O} \) are

\[ \omega^0, \omega^1, \omega_2, \ldots, \omega^{d-1}. \]

Furthermore, the initial state vector \( |\Psi_0\rangle \) is a superposition of all eigenvectors of \( F \) restricted to \( \mathcal{O} \) with equal weights. All this follows from properties of the cyclic shift operator.

The backward-time operator is defined as the adjoint of \( F \). The observable \( A \) is defined as the sum of the forward and backward time operators, i.e., \( A := (F + F^\dagger)/2 \). It is 4-local since each \( V_j \) is 2-local and is coupled to a 2-local propagator. The dynamics of the clock may be interpreted as a propagation of a spin-wave. Note that the idea of our construction is not to implement the quantum circuit \( V \).
by the autonomous time evolution \( \exp(-iAt) \). The aim is rather to obtain an observable such that its spectral properties correspond to the orbit length of the circuit.

Since \( F \) and \( F^\dagger \) commute on \( \mathcal{O} \) the eigenvalues of \( A \) restricted to \( \mathcal{O} \) are \((\omega^j + \bar{\omega}^j)/2 = \cos(2\pi j/d)\). The non-real eigenvalues are 2-fold degenerated. Only the eigenvalues 1 and \(-1\) have multiplicity 1. In a hypothetical energy measurement applied to the initial state vector \( |\Psi_0\rangle \) one would obtain all 2-fold degenerated eigenvalues with probability \( 2/d \) each and the non-degenerated eigenvalues with probability \( 1/d \). Note that only the first case is relevant for large \( d \) since there are at most two non-degenerated values.

Note that \( d \) depends on the solution of the PSPACE problem. Explicitly, the possible measurement results are

1. either
   \[
   \cos(2\pi j/(2sr)), \quad j = 0, \ldots, 2sr - 1
   \]
2. or
   \[
   \cos(2\pi j/(sr)), \quad j = 0, \ldots, sr - 1
   \]

depending on whether \( f(x) = 1 \) or \( f(x) = 0 \).

Note that a perfect energy measurement can distinguish between the two cases even after few samples: after applying the function “arccos” we obtain values with distance \( 2\pi/d \) and all values occur with equal probability (if the non-degenerated values are neglected). Then it is easy to distinguish between the two cases \( d = rs \) and \( d = 2rs \).

Now we examine what accuracy is sufficient to distinguish between the two cases. For doing so, we will restrict our attention to those measurement values which are between \( 1/\sqrt{2} \) and \(-1/\sqrt{2}\). This means that half of the measurement outcomes have to be ignored because the probability to obtain an outcome in this interval is about \( 1/2 \). These values correspond to angles in the interval \([\pi/4, 3\pi/4]\) and \([5\pi/4, 7\pi/4]\).

In the following we assume that we have obtained a measurement value in this interval. For each outcome \( E \) we chose \( j \) such that \(|\arccos(E) - 2\pi j/(2rs)|\) is minimal. If \( f(x) = 1 \) then the probability of obtaining an odd value for \( j \) is at least \( 1/2 \cdot 3/4 = 3/8 \). If \( f(x) = 0 \) then the probability of obtaining an even value for \( j \) is at least \( 1 \cdot 3/4 = 3/4 \). Therefore, the probability of odd value is at most \( 1/4 \). This difference in probability allows to distinguish between the two cases. It is obvious that the error probability decreases exponentially with the number of measurements.
Note that the observable $A$ has spectral gaps that are considerably smaller than the required accuracy. This can already be seen if we consider the $A$-invariant subspace $O$. The distance of the largest eigenvalue $1$ and the second largest eigenvalue $\cos(2\pi/d)$ of $H$ is approximatively given by $(2\pi/d)^2$ since the derivative of the cosinus function at $0$ is $0$.

Note that the required accuracy is directly connected with an upper bound on the running time $T$. In our setting the running time is the number $r$ of necessary applications of the circuit $V$ times the number $s$ of gates of $V$.

In the construction of the preceding section we obtained the upper bound on $rs$ from the required space. More generally, whenever we know that $r$ applications of $V$ are sufficient we need a measurement with accuracy of the order $1/(rs)$ to determine the solution of the PSPACE problem.

This discussion proves the following theorem:

**Theorem 2 (Measurement precision vs. running time)**

Let $\{A_i\}$ be a family of 4-local observables corresponding to the family $\{V_i\}$ of quantum circuits in Theorem 1. Then every measurement in the sense of Definition 2 could be used to solve PSPACE problems in polynomial time whenever the accuracy is of the order of the spectral gaps of $H$. It is even sufficient to have an measurement error $1/T_i$, where $T_i$ is the running time of the algorithm based on the circuit $V_i$.

### 4 Conclusions

We have shown that every apparatus which implements precise measurements of 4-local $n$-qubit observables would solve PSPACE problems. This conclusion does only hold for exponentially small errors of the measurement. On the other hand, we have argued that algorithms which measures with inverse-polynomial accuracy can be implemented efficiently. Provided that PSPACE problems cannot be solved efficiently, i.e., $\text{PSPACE} \neq \text{BQP}$, the complexity of measurements depend on the required accuracy. The statement that exponential accuracy has stronger computational power is also well-known in classical analog computational models [16, 17].

One may ask why one should try to measure general 4-local observables. A possible motivation to develop a complexity theory of
measurements is that some proposals for quantum algorithms use joint observables on the quantum register [18].

Another motivation is that one is interested in measurements for physically relevant joint observables like energy. The reader may object that the specific interactions constructed in this paper are rather unphysical for several reasons:

1. Most interactions in nature are pair-interactions and not 4-local.
2. Our construction uses long-range interactions among distant qubit quadruples.
3. Interactions in natural many-particle systems have typically high symmetry. For instance, the interactions in solid states respect the translational invariance of the lattice.
4. There exist only a few fundamental interactions in physics.

We have already argued that pair-interactions between particles may correspond to $k$-local terms if some qubits encode the physical state of one particle. This refutes the first objection.

We conjecture that the solution of PSPACE problems would even be possible if the class of observables was restricted to those which appear as Hamiltonians of real many-particle systems. This conjecture is supported by the following ideas:

Quantum cellular automata like the Hamiltonian dynamical system constructed in [19] are also computationally universal. Hamiltonians for those types of cellular automata have the property that every cell interacts only with some cells in its neighborhood. By merging some cells together to one cell we can always obtain a Hamiltonian with pair-interactions among qudits. This seems to indicate that neither the symmetry nor locality assumptions on the interactions prevents the Hamiltonian from corresponding to computationally universal networks. Due to the fact that computers exist it is clear that the structure of the fundamental interactions is general enough to allow universal systems. Therefore we guess that spectral properties of more realistic Hamiltonians encode PSPACE problems in a similar way as in our paper.

Acknowledgements

Thanks to Jörn Müller-Quade and Markus Grassl for helpful discussions. This work has been supported by grants of the Landesstiftung
Baden-Württemberg (project “Kontinuierliche Modelle der Quanteninformationsverarbeitung”).

References

[1] F. Barahona. On the computational complexity of Ising spin models. *J. Phys. A: Math. Gen.*, 15:3241–3253, 1982.

[2] P. Wocjan and Th. Beth. The 2-local Hamiltonian problem encompasses NP. [quant-ph/0301087](http://arxiv.org/abs/quant-ph/0301087).

[3] A. Kitaev, A. Shen, and M. Vyalyi. *Classical and Quantum Computation*, volume 47. Am. Math. Soc., Providence, Rhode Island, 2002.

[4] J. Kempe and O. Regev. 3-local Hamiltonian is QMA-complete. [quant-ph/0302079](http://arxiv.org/abs/quant-ph/0302079), 2003.

[5] S. Lloyd. Universal quantum simulators. *Science*, 273:1073, 1996.

[6] R. Cleve, A. Ekert, C. Macchiavello, and M. Mosca. Quantum algorithms revisited. *Proc. Roy. Soc. London A*, 454:339–354, 1998. see also [quant-ph/9708016](http://arxiv.org/abs/quant-ph/9708016).

[7] D. Janzing. Quantum algorithm for measuring the energy of n qubits with unknown pair-interactions. *Quant. Inform. & Comp.*, vol 2, No.3, p.198–207 2002.

[8] P. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. [quant-ph/9508027](http://arxiv.org/abs/quant-ph/9508027v2), 1996.

[9] M. Nielsen and I. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[10] J. E. Hopcroft and J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley series in computer science. Addison-Wesley, 1979.

[11] M. Garey and D. Johnson. *Computers and Intractability*. Freeman and Company, New York.

[12] K.-J. Lange, P. McKenzie, and A. Tapp. Reversible space equals deterministic space. *Journal of Computer and System Sciences*, 60:354–367, 2000.

[13] J. Savage. *Models of Computation*. Addison Wesley, 2000.
[14] C. H. Bennett. Time/space trade-offs for reversible computation. *SIAM J. Computing*, 18(4):766–776, 1989.

[15] J. Kempe and O. Regev. 3-local Hamiltonian is QMA-complete. *Quantum Computation and Information*, 3(3):258–264, 2003.

[16] J. Hartmanis and J. Simon. On the power of multiplication in random access machines. In *Proceedings of the 15th Annual Symposium on Switching and Automata Theory*, IEEE Computer Society, pages 13–23, Long Beach, CA, 1974.

[17] A. Vergis, K. Steiglitz, and B. Dickinson. The complexity of analog computation. *Math. Comput. Simulation*, (28):91–113, 1986.

[18] M. Ettinger and P. Hoyer. A quantum observable for the graph isomorphism problem. *LANL preprint* quant-ph/9901029.

[19] N. Margolus. Parallel quantum computation. In W. Zurek, editor, *Complexity, Entropy, and the Physics of Information*. Addison Wesley Longman, 1990.