On the duality between periodic orbit statistics and quantum level statistics

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Abstract - We discuss consequences of a recent observation that the sequence of periodic orbits in a chaotic billiard behaves like a poissonian stochastic process on small scales. This enables the semiclassical form factor $K_{sc}(\tau)$ to agree with predictions of random matrix theories for other than infinitesimal $\tau$ in the semiclassical limit.

1 Introduction

The spectral form factor $K(\tau)$, defined as the Fourier transform of the spectral autocorrelation function, has a central role in semiclassical analysis of chaotic systems. All spectral statistics bilinear in the density, such as spectral rigidity ($\Delta_3$) and number variance ($\Sigma_2$) may be expressed in terms of the form factor [1, 2]. The form factor is a convenient tool for semiclassical analysis while it can be expressed as a double sum over periodic orbits [1]. The approximation of the form factor thus obtained $K_{sc}(\tau)$ is called the semiclassical form factor. This paper contains some complementary results to Berry’s classical paper [1], where he shows that, in the limit of the small $\tau$, the form factor is $K_{sc}(\tau) = \tau$ for system with no time reversal symmetry and $K_{sc}(\tau) = 2\tau$ for system with time reversal symmetry. This agrees with predictions from random matrix theories [3] for the Gaussian Unitary Ensemble (GUE) and Gaussian Orthogonal Ensemble (GOE) respectively. This result is essentially the only semiclassical support for universality in level statistics for chaotic systems obtained so far. However, as we will see, the result is only obvious for smaller and smaller $\tau$ when the semiclassical limit is approached, and we need a mechanism to ensure validity up to fairly large $\tau$ (of the order of unity) for universality to be achieved.

In this paper we are going to describe the sequence of periodic orbit in a statistical language and use tools from the thermodynamic formalism of chaotic systems [4]. These tools will be worked out in section 2. The main point is assumption B, which states that the sequence of periodic orbits may be described as a poissonian process, at least on small scales. In section 3 we show that, under this assumption, Berry’s result $K = (2)\tau$ may be recovered without assuming conditional convergence of the Gutzwiller formula [5]. And even more important, this result holds for other than infinitesimal $\tau$ in the semiclassical limit. We do not discuss when assumption B breaks down and the corresponding deviation from $K = (2)\tau$ may start. In section 4 we discuss the various assumptions behind our result and how the results depend on them. In particular we
review the numerical evidence for assumption B and in section 5 we discuss related work in view of our achievements.

2 Preliminaries

The key step is the analytical continuation of the Gutzwiller formula for the level density down to the real energy axis. We are going to perform this for the Fourier transform of the level density under some simplifying assumptions.

2.1 The Fourier transform of the level density

Our starting point is Gutzwiller-Voros zeta function \[5, 6\] which for systems with two degrees of freedom reads

\[
Z(E) = \prod_p \prod_{j=0}^\infty \left( 1 - e^{iS_p(E)} |\Lambda_p|^{1/2} \Lambda_j \right),
\]

where \(\Lambda_p\) is the expanding eigenvalue of the Jacobian along the flow. To make things as transparent as possible we restrict ourselves to billiards, so that the action integral is simply \(S_p(E) = \ell_p \cdot \kappa(E)\) where the 'momentum' is \(\kappa = \sqrt{2E}\). The level density to be discussed is measured in \(\kappa\)-space and not, as is usual, in \(E\)-space. We use units such that \(m = \hbar = 1\). We have neglected the Maslov indices, this neglect is discussed in sections 2.2 and 4.

It is expected that the zeros of the zeta function approximate the quantum eigenvalues. How the the semiclassical approximation and the stationary phase approximation behind \(Z_{GV}\) affect the spectrum is not all clear. However, this is not our main concern here. What is important in this paper is that the zeta function may contain additional zeros reflecting its convergence properties.

In this paper we will use the following simplifying assumption:

Assumption A: The zeta function \(Z(\kappa)\) is entire and has zeros on the exact quantum positions \(\pm \kappa_i, i \neq 0\) and one extra zero \(\kappa_0 = i\hbar_{1/2}\) on the positive imaginary axis lying on the border of convergence. The motivation and discussion of this assumption is postponed until section 4. (The reason behind the choice of subscript of \(\hbar_{1/2}\) will be obvious in section 2.2.)

The level density is usually split up into the mean density and an oscillating part \(d(\kappa) = d_0(\kappa) + d_{osc}(\kappa)\). The leading part of \(d_0\) is the Weyl term \(d_0 \sim \Lambda \kappa/2\pi\) where \(\Lambda\) is the billiard area. The Gutzwiller-Voros zeta function above is derived from the Gutzwiller formula for \(d_{osc}\). For later purposes we need the Fourier transform of the oscillating part of the level density

\[
\tilde{D}_{osc}(l) = \frac{1}{2\pi i} \int_{-\infty+iC}^{\infty+iC} e^{-i\kappa l} \frac{d}{d\kappa} \log Z(\kappa) d\kappa.
\]

We will evaluate it in two ways. First, by inserting the product representation of the zeta function. It is thus essential the the constant \(C\) in eq (2) is sufficiently large so that the contour runs in the region where the product \(\prod\) converges (and thus well above all zeros of \(Z\)). One can then exchange summation and integration and establish the result.

\[
\tilde{D}_{osc}(l) = \sum_p \ell_p \sum_{n=1}^\infty \frac{\delta(l - nl_p)}{|M_p^2 - I|^{1/2}}.
\]

Exponential divergence of this sum \(\tilde{D}_{osc}(l) \to \exp(h_{1/2}l)\), a feature to be discussed in section 2.2, is directly related to the presence of the zero \(\kappa_0 = i\hbar_{1/2}\) as can be seen from eq. (2).

Secondly we compute \(\tilde{D}_{osc}\) by means of residue calculus

\[
\tilde{D}_{osc}(l) = e^{h_{1/2}l} + \sum_{j=-\infty, j\neq 0}^\infty e^{-i\kappa_j l}.
\]
The reader may wonder why the contribution from the mean distribution \( d_0 \) has disappeared in eq (4). The reason is that we have neglected the contribution from the large semi-circle which would have yielded delta functions, and even more nasty things, associated with the Fourier transform of \( d_0 \). The tilde-sign in \( \tilde{D}_{osc} \) indicates that the result carries contribution from the extra zero \( \kappa_0 \). Without this zero we get the Fourier transform of the true level density \( D_{osc} = \sum_{j=-\infty,j\neq 0}^{\infty} e^{-i\kappa_j l} \).

We can then establish the identity
\[
D_{osc}(l) = \sum_p l_p \sum_{n=1}^{\infty} \frac{\delta(l - nl_p)}{|\Lambda_p|^\beta} \rightarrow e^{h\beta l}.
\]

We have all the way assumed that \( l > 0 \).

From now on we will tacitly replace every occurrence of \( |M_p^n - I| \) with \( |\Lambda_p| \). The errors thus induced are completely negligible in the limits we are going to explore. We must now discuss some properties of the periodic orbits sums.

### 2.2 Some properties of the set of periodic orbits

Essentially all dynamical information is encoded in the sequence of the invariants of the primitive periodic orbits \( \{l_p, \Lambda_p\} \) (the maslov indices provide some topological information though). In the asymptotic limit \( l \to \infty \) one can establish the following family of sum-rules for chaotic systems
\[
\sum_p l_p \sum_{n=1}^{\infty} \frac{\delta(l - nl_p)}{|\Lambda_p|^\beta} \rightarrow e^{h\beta l}.
\]

(From now on we will reside in the asymptotic limit of large \( l \) and write equality signs instead of arrows.) The result applies after appropriate smearing of the delta functions. The entropy-like quantity \( h \beta \) decreases with increasing \( \beta \). The special case \( \beta = 1 \) was discussed already in [8]; for bound systems one have \( h_1 = 0 \). \( h_0 \) is the topological entropy [8]. General \( \beta \) are discussed in e.g. [8, 10, 11]. Usually there are some restrictions on \( \beta \). For e.g. the Sinai billiard eq (6) is only valid provided that \( -1 < \beta \leq 1 \) [10]. In our considerations it suffices if eq. (6) is valid for \( 1/2 < \beta \leq 1 \).

We expect this property to hold for any reasonable system.

It is very useful to describe the set of periodic orbits in a statistical language. For large \( l \) the repetitions of shorter orbits are overwhelmed by the number of primitive orbits so that we may neglect the sum over \( n \) above. Let us call the density of prime orbits \( \phi(l) = \sum_p \delta(l - l_p) \). From eq. (6) we see that the mean value of this density is
\[
< \phi(l) > = e^{h_0 l / l}.
\]

For general \( \beta \) we express the sum rules in terms of the averages \(< |\Lambda_p|^{-\beta} >\>
\[
l < \phi > |\Lambda|^{-\beta} > = e^{h_0 l} < |\Lambda|^{-\beta} > = e^{h\beta l}.
\]

For instance we get the result which will be of use later
\[
< |\Lambda|^{-1} > = e^{-h_0 l}.
\]

So much for the large scale structure of the sequence \( \{l_p, \Lambda_p\} \), i.e. large smearing widths in eq (4). What about the small scale structure?

Let us order this sequence according to increasing \( l_p \) and consider the ordered sequence \( \{l_i, \Lambda_i\} \) where the integer \( i \) denote the position in the sequence. Then rescale the length variable according to
\[
\ell_i = \int_0^{\ell_i} < \phi(l') > dl',
\]
so that the mean spacing \(< \ell_i - \ell_{i-1} >\) is unity. We will now make our main assumption.

**Assumption B:**
1/ The sequence $\ell_i$ is given by a Poissonian process with unit intensity.
2/ The corresponding stabilities $\Lambda_i$ may be considered as mutually independent stochastic variables

We discuss the evidence for this assumption in sec 4.

We can now reformulate $D_{osc}$ in a purely statistical language

$$D_{osc}(l) = l \left( \phi(l) \frac{1}{\sqrt{\left| \Lambda(l) \right|}} - \frac{\phi}{\sqrt{\left| \Lambda \right|}} \right). \quad (11)$$

The introduction of phase indices (Maslov indices and symmetry indices [13]) will generally move down the leading zero $h_{1/2}$ [11] and there is a possibility that it might even cross the real $\kappa$-axis making the Gutzwiller sum conditionally convergent [12]. It is nontrivial to deduce if this really takes place for a given system. Many estimations of the position of the entropy barrier, with or without Maslovs, in the literature assumes uniform hyperbolicity of the system and are thus invalid for generic systems.

3 The form factor

The spectral form factor is defined as the Fourier transform of the spectral autocorrelation function

$$K = \frac{1}{d_0} \int_{-\infty}^{\infty} d\epsilon e^{-i\epsilon l} d_{osc}(\kappa + \epsilon/2) d_{osc}(\kappa - \epsilon/2). \quad (12)$$

It is usually regarded as a function of the dimensionless length variable $\tau = l/(2\pi d_0) = l/(A\kappa)$ with $\kappa$ as a parameter. The suggested universal behaviour of $K(\tau)$ should arise in the semiclassical limit $\kappa \to \infty$. To be meaningful the form factor needs some averaging which we will apply first at the end. The form factor can be expressed in terms of the Fourier transform of $d_{osc}(\kappa)$ according to

$$K = \frac{1}{2\pi d_0} \int_{-\infty}^{\infty} dl' \int_{-\infty}^{\infty} dl'' \delta(l - l' + l'') \cos(\kappa(l' - l'')) D_{osc}(l') D_{osc}(l''). \quad (13)$$

The derivation is straightforward, one has to use the fact that $d_{osc}(\kappa)$, and thus $D_{osc}(l)$, are real and even.

**Assuming conditional convergence**

Let us now follow Berry’s arguments a little longer [1]. If the Gutzwiller formula is conditionally convergent we can insert $\tilde{D}_{osc}$ directly instead of $D_{osc}$ into $\tilde{K}$.

$$K_{sc} = \frac{l^2}{2\pi d_0} \int \int d\ell dll' \cos(\kappa(l' - l'')) \delta(l - l' + l'') \left( \sum_i \delta(l' - l_i) \right) \left( \sum_j \delta(l'' - l_j) \right). \quad (14)$$

Keeping $\kappa$ constant and letting $l \to 0$ be small the cosine will wash away the non diagonal terms (assuming no systematic degeneracies among the $l_i$ due to e.g. time reversal symmetry) and we get

$$K_{sc} = \frac{l^2}{2\pi d_0} \left( \sum_i \delta(l - l_i) \right). \quad (15)$$

Using the sum rules in sec 2.2 we get

$$\tilde{K} = \frac{l^2}{2\pi d_0} \frac{1}{l} \exp(-h_0 l) = \frac{l}{2\pi d_0} = \tau. \quad (16)$$

This result gave rise to some enthusiasm since it agrees with predictions of random matrix theories. But one may ask two questions. First, is eq (16) true even if the Gutzwiller sum is not conditionally convergent?

Secondly, we note that the number of periodic orbits contained in one period of the cosine in eq (14) is $\sim \phi / \kappa \sim \exp(h_0 A\kappa \tau)/(A\kappa^2 \tau)$. To make this estimate we have assumed that the
smearing width is \( \Delta l \approx < \phi >^{-1} \) which is the smallest conceivable choice. In the semiclassical limit \( (\kappa \rightarrow \infty) \) the argument leading to eq (14) is brutally violated for other than infinitesimal \( \tau \). If the predictions of random matrix theories are correct one expects eq (16) to hold up to \( \tau \sim 1 \).

Now to the second question. Is the result \( K = \tau \) correct for finite \( \tau \), and if the answer is yes, why?

The generic case

The key lies in the stochastic nature of the periodic orbits as formulated in assumption B. First we insert the general formula for \( D_{osc}(l) \):

\[
K_{sc} = \frac{l^2}{2\pi d_0} \int \int dld'' \cos(\kappa(l - l''))\delta(l - \frac{l' + l''}{2}) \cdot (\sum_i \delta(l' - l_i) \frac{1}{\sqrt{|\Lambda_i|}} - < \phi > < \frac{1}{\sqrt{|\Lambda|}} >) (\sum_j \delta(l'' - l_j) \frac{1}{\sqrt{|\Lambda_j|}} - < \phi > < \frac{1}{\sqrt{|\Lambda|}} >) .
\] (17)

We clearly see that we are dealing with correlation in the sequence \( \{l_i, \Lambda_i\} \) and, according to assumption B, there are no correlations at all. Therefore only the diagonal terms will contribute. In order to avoid delta functions in the resulting form factor we smear it

\[
\bar{K}_{sc} = \frac{1}{\Delta} \int_{l}^{l+\Delta} K(l') dl',
\] (18)

which now may be expressed as

\[
\bar{K}_{sc} = \frac{l^2}{2\pi d_0 \Delta} \int_{l}^{l+\Delta} dl' \left\{ \left( \sum_i \delta(l' - l_i) \frac{1}{\sqrt{|\Lambda_i|}} - < \phi > < \frac{1}{\sqrt{|\Lambda|}} > \right) \right\}^2 .
\] (19)

The integral in this expression is just the variance of the sum of \( 1/\sqrt{|\Lambda|} \) in a window of a poissonian process,

\[
\bar{K} = \frac{l^2}{2\pi d_0 \Delta} V_\Delta \left( \sum \frac{1}{\sqrt{|\Lambda|}} \right) .
\] (20)

The calculation of this variance is an elementary exercise in probability theory (see Appendix) and the result is \( V_\Delta = \lambda \Delta < 1/\sqrt{|\Lambda|} > \). The intensity \( \lambda \) equals the mean density of prime orbits \( \lambda = < \phi(l) > = \exp(h_{0l})/l \) and according to the sum rules in sec 2.2 we have

\[
< 1/\sqrt{|\Lambda|} > = < 1/\Lambda > = \exp(-h_{0l}).
\]

We thus get our final result

\[
\bar{K}_{sc} = \frac{l^2}{2\pi d_0 \Delta} \frac{\exp(h_{0l}) \Delta}{l} \exp(-h_{0l}) = \frac{1}{2\pi d_0} \exp(-h_{0l}) = \tau ,
\] (21)

which is the same as for conditionally convergent systems but the result now holds for other than infinitesimal \( \tau \) in the semiclassical limit.

It is straightforward to generalize to the systematic degeneracies of periodic orbit exhibited by time reversible systems, and we will not discuss it.

4 Motivations for our basic assumptions

Our result relies on a series of assumptions and approximations. Some of them may be removed or modified without altering the result. In this section we motivate our assumptions and discuss the extent to which the results depend on them.

First assumption A. The presence of a leading zero \( \kappa_0 = ih_{1/2} \) such that \( h_{1/2} > 0 \) (no Maslovs) has already been discussed. In the general case it is reasonable to assume the presence of several zeros not associated with any quantum state. Their contribution is naturally included into \( < \phi > < |\Lambda|^{-1/2}(l) > \) giving rise to oscillatory and exponentially decreasing corrections.
The assumption that the semiclassical zeros equals the quantum eigenvalues is not crucial for our results. It was mainly introduced for computational and notational convenience. However, the expected failure of the semiclassical eigenvalues to be real will have consequences for the large $\tau$ behaviour of the semiclassical form factor \cite{20}, see section 5!

There is one example for which assumption A is exactly fulfilled and that is compact billiards on surfaces of constant negative curvature. These systems are special having zeros on the exact quantum positions. But there is also a zero on the imaginary $\kappa$ axis, right on the border of convergence \cite{14}. Indeed there is a zero on the border of convergence of each $j$-factor in the zeta function (1).

Now to assumption B. In ref \cite{15} the authors pursued the original idea to consider the spectrum of lengths $l_j$ of the prime cycles and do level statistics à la quantum chaos. Their system was a touching three-disk billiard. The first step is to unfold the spectrum, cf section 2.2, yielding the sequence $\ell_j$. Then they studied the level spacing distribution, which was found to be an exponential to very high accuracy, and spectral rigidity, which was found to agree with $\Delta_3(L) = L/15$. This is consistent with the sequence $\ell_j$ being given by a poissonian process and motivates assumption B. The three-disk billiard, as well as any generic Euclidean chaotic billiard, has almost certainly an infinite symbolic dynamics. It would be nice to know if the greater regularity among the cycles for a finite symbolic dynamics still exhibits this kind of randomness.

We do not attempt to explain this random feature, we only offer the following hand-waving argument. Neighbours in the sequence $\{l_j\}$ need not be close in phase space and there is therefore no reason for correlations. In a small proximity to a given length $l$ there may be many periodic orbits. The sequence of periodic orbit could thus, perhaps, be viewed upon as the random superposition of many sequences, each sequence corresponding to a topologically distinct family of periodic orbits. This would give rise to the poissonian nature.

Assumption B2, concerning the the stabilities, is in the same spirit as B1. If the lengths are uncorrelated so should the eigenvalues be.

The restriction to billiards is mostly for convenience. We find it highly unlikely that a smooth potential would exhibit complete chaos. Our calculations would be easily modified for a chaotic smooth potential, if existing, which is homogeneous; the 'momentum' variable $\kappa$ being some other power of energy $E$. If one chooses to consider non homogeneous potentials one must perform the Fourier transform with respect to $\bar{h}$ instead of $\kappa$. We won’t speculate about this case since it would take us to far from the case where assumption B has been verified.

\section{Discussion}

It is interesting to note that the semiclassical form factor $K_{sc}(\tau)$ in the region $0 < \tau < 1$ depend on both the very small scale structure and the very coarse structure on the sequence of periodic orbits. In this paper we have focused on the deep asymptotic limit $\kappa \to \infty$. For finite energies one may have to consider pre-asymptotic behaviour and power law corrections of the periodic orbit sum rules involved. This is discussed in refs \cite{19,4}. In these papers we did not correct for the divergence of the trace formula but this procedure is readily justified from the present results.

This pre-asymptotic behaviour extend considerably the non universal regime in spectral statistics derivable form the form factor as compared to the regime discussed by Berry. Such non-universal regimes have been observed in several numerical experiments \cite{17,18,4,4}. In ref \cite{4} we also discuss the role of marginally stable orbits which plays a major role for moderate energies.

The semiclassical status of the large $\tau$ limit is much more obscure. It is clear that the quantum form factor approaches unity in this limit $K \to 1$ provided that there are no systematic degeneracies in the spectrum. In \cite{24} Keating demonstrates that, since we cannot expect the trace formula to produce poles exactly on the real axis, the semiclassical form factor should diverge exponentially. If the imaginary part of the poles is much less than the mean spacing, as is indicated by \cite{21}, this exponential take off should not occur until fairly large $\tau$ and one can still hope for saturation of the semiclassical form-factor.

Some evidence of saturation is presented in ref. \cite{23} for the hyperbola billiard and other systems.
The exponential collapse of $K_{sc}(\tau)$ reported in [23] appears to be due to neglect of the extra zero(s) and illustrates the hazard of inserting diverging series into the form-factor. This is particularly dangerous since the expression thus obtained need not be divergent.

Whatever happens to the form factor we can of course not expect assumption B to hold throughout the spectrum and a deviation form $K_{sc}(\tau) = (2)\tau$ should be expected. If the expected saturation indeed takes place it is, of course, highly desirable to understand its classical origin and its manifestation by the periodic orbits and the connection to the fact that the system is bound.

The reader may think that our assumption B contradicts the concept of action repulsion discussed by Argaman et.al. [22]. However, if the periodic orbit correlations proposed in ref. [22] really occur this repulsion is not an effect acting among neighbours in the sequence $\{l_i\}$ of cycles but over vast distances, the name action repulsion is thus not very appropriate.

Much of the present work on the semiclassical trace formula is concerned with taming the diverging trace formula and the computation of corrections in order to obtain accurate results for the bottom part of the spectrum. Many of these corrections disappear in the semiclassical limit. In this paper we took the opposite point of view. We tried to relate asymptotic (=semiclassical) properties of the spectrum to the asymptotic behaviour of the periodic orbits. As both are conveniently expressed in a statistical language this approach aims at unveiling the duality between periodic orbit statistics and level statistics.

Acknowledgements

This work was supported by the Swedish Natural Science Research Council (NFR) under contract no. F-FU 06420-303.

References

[1] M. V. Berry, Proc. R. Soc. Lond. A 400 (1985) 229.
[2] F. J. Dyson and M. L. Mehta, J. Math. Phys. 4 (1963) 701.
[3] M. L. Mehta, Random matrices and the statistical theory of energy levels, Academic Press, New York (1967).
[4] C. Beck, F. Schlögl, Thermodynamics of chaotic systems, Cambridge Nonlinear Science Series 4, Cambridge (1993).
[5] M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics, Springer, New York (1990).
[6] A. Voros, J. Phys. A 21, 685 (1988).
[7] J. H. Hannay and A. M. Ozorio de Almeida, J. Phys. A 17, 3429, (1984).
[8] R. Artuso, E. Aurell and P. Cvitanović, Nonlinearity 3, 325; 3 361, (1990).
[9] P. Cvitanović and P.E. Rosenqvist, in G.F. Dell’Antonio, S. Fantoni and V.R. Manfredi, eds., From Classical to Quantum Chaos, Soc. Italiana di Fisica Conf. Proceed. 41.
[10] P. Dahlqvist, Approximate zeta functions for the Sinai billiard and related systems, Nonlinearity, to appear. pp. 57-64 (Ed. Compositori, Bologna 1993).
[11] P. Dahlqvist, Periodic orbit asymptotics for intermittent Hamiltonian systems, in Proc. Los Alamos Center for Nonlinear Science - Quantum Complexity in Mesoscopic systems, May -94, Physica D, to appear.
[12] R. Aurich, J. Bolte, C. Matthies, M. Sieber and F. Steiner, Physica D 63, 71, (1993).
[13] P. Cvitanović and B. Eckhardt, Nonlinearity 6 (1993) 277.
[14] D. A. Hejhal, Duke Math. J., 43, 441, (1976).
Appendix

Åke Svensson owns a small shop in the old town of Stockholm. Customers arrive at the shop according to a poissonian process with intensity \( \lambda \). The amount of money paid by one customer is considered as a stochastic variable \( x \) with probability distribution \( f(x) \). \( x \) belonging to different customers are mutually independent.

During time \( T \) the amount of cash received by Åke is \( X_T \). What is the variance of \( X_T \)?

The distribution of arrivals during time \( T \) in a poissonian process is

\[
p_n = e^{-\lambda T} \frac{(\lambda T)^n}{n!} ,
\]

so the distribution of the variable \( X \) is

\[
F(X) = \sum_{n=0}^{\infty} p_n f^{*n}(X) ,
\]

where \( f^{*n} \) is the n-fold convolution of \( f \). The mean and variance in such a convolution are additive so we have

\[
< x_n > = n < x >
\]

\[
< x_n^2 > = n < x^2 > + n(n-1) < x >^2 .
\]

A short calculation now yields the mean and variance of \( X_T \)

\[
< X_T > = \lambda T \cdot < x >
\]

\[
V(X_T) \equiv < X_T^2 > - < X_T >^2 = \lambda T \cdot < x^2 > .
\]