Weak MSO+U with Path Quantifiers over Infinite Trees

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Abstract. This paper shows that over infinite trees, satisfiability is decidable for weak monadic second-order logic extended by the unbounding quantifier \(U\) and quantification over infinite paths. The proof is by reduction to emptiness for a certain automaton model, while emptiness for the automaton model is decided using profinite trees.

This paper presents a logic over infinite trees with decidable satisfiability. The logic is **weak monadic second-order logic with \(U\) and path quantifiers** (wMSO+UP). A formula of the logic is evaluated in an infinite binary labelled tree. The logic can quantify over: nodes, finite sets of nodes, and paths (a path is a possibly infinite set of nodes totally ordered by the descendant relation and connected with respect to the child relation). The predicates are as usual in MSO for trees: a unary predicate for every letter of the input alphabet, binary left and right child predicates, and membership of a node in a set (which is either a path or a finite set). Finally, formulas can use the **unbounding quantifier**, denoted by

\[ UX \varphi(X), \]

which says that \(\varphi(X)\) holds for arbitrarily large finite sets \(X\). As usual with quantifiers, the formula \(\varphi(X)\) might have other free variables except for \(X\). The main contribution of the paper is the following theorem.

**Theorem 1.** Satisfiability is decidable for wMSO+UP over infinite trees.

Background. This paper is part of a program researching the logic MSO+U, i.e. monadic second-order logic extended with the \(U\) quantifier. The logic was introduced in [4], where it was shown that satisfiability is decidable over infinite trees as long as the \(U\) quantifier is used once and not under the scope of set quantification. A significantly more powerful fragment of the logic, albeit for infinite words, was shown decidable in [3] using automata with counters. These automata where further developed into the theory of cost functions initiated by Colcombet in [8]. Cost functions can be seen as a special case of MSO+U in the sense that decision problems regarding cost functions, such as limitedness or domination, can be easily encoded into satisfiability of MSO+U formulas. This

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encoding need not be helpful, since the unsolved problems for cost functions get encoded into unsolved problems from MSO+U.

The logic MSO+U can be used to solve problems that do not have a simple solution in MSO alone. One example (discussed later in Example 1) is the finite model problem for the two-way μ-calculus [1]. A more famous problem is the star height problem, which can be solved by a reduction to the satisfiability of MSO+U on infinite words; the particular fragment of MSO+U used in this reduction is decidable by [3]. In Section 1 we give more examples of problems which can be reduced to satisfiability for MSO+U, examples which use the fragment that is solved in this paper. An example of an unsolved problem that reduces to MSO+U is the decidability of the nondeterministic parity index problem, see [9].

The first strong evidence that MSO+U can be too expressive was given in [11], where it was shown that MSO+U can define languages of infinite words that are arbitrarily high in the projective hierarchy. In [4], the result from [11] is used to show that there is no algorithm which decides satisfiability of MSO+U on infinite trees and has a correctness proof using the axioms of ZFC. A challenging open question is whether satisfiability of MSO+U is decidable on infinite words.

The principal reason for the undecidability result above is that MSO+U can define languages of high topological complexity. Such problems go away in the weak variant, where only quantification over finite sets is allowed, because weak quantification can only define Borel languages. Indeed, satisfiability is decidable for WMSO+U over infinite words [2] and infinite trees [6]. This paper continues the research on weak fragments from [2,6]. Note that WMSO+U can, unlike MSO+U, define non Borel-languages, e.g. “finitely many a’s on every path”, which is complete for level \( \Pi_1^1 \) of the projective hierarchy. The automaton characterization of WMSO+U in this paper implies that WMSO+U definable languages are contained in level \( \Delta_2^1 \).

What is the added value of path quantifiers? One answer is given in the following section, where we show how WMSO+U can be used to solve games winning conditions definable in WMSO+U; here the use of path quantifiers is crucial. Another answer is that solving a logic with path quantifiers is a step in the direction of tackling one of the most notorious difficulties when dealing with the unbounding quantifier, namely the interaction between quantitative properties (e.g. some counters have small values) with qualitative limit properties (e.g. the parity condition). The difficulty of this interaction is one of the reasons why the boundedness problem for cost-parity automata on infinite trees remains open [9]. Such interaction is also a source of difficulty in the present paper, arguably more so than in the previous paper on WMSO+U for infinite trees [6]. One of the main contributions of the paper is a set of tools that can be used to tackle this interaction. The tools use profinite trees.

1 Notation and Some Applications

Let us begin by fixing notation for trees and parity automata. Notions of root, leaf, sibling, descendant, ancestor, parent are used in the usual sense. A tree in