DENSITY OF CLASSICAL POINTS IN EIGENVARITIES

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Abstract. In this short note, we study the geometry of the eigenvariety parametrising $p$-adic automorphic forms for $GL_1$ over a number field, as constructed by Buzzard. We show that if $K$ is not totally real and contains no CM subfield, points in this space arising from classical automorphic forms (i.e. algebraic Grossencharacters of $K$) are not Zariski-dense in the eigenvariety (as a rigid space); but the eigenvariety possesses a natural formal scheme model, and the set of classical points is Zariski-dense in the formal scheme.

We also sketch the theory for $GL_2$ over an imaginary quadratic field, following Calegari and Mazur, emphasising the strong formal similarity with the case of $GL_1$ over a general number field.

1. Zariski-density in formal and rigid spaces

Let $A$ be a finite algebra over the formal power series ring $\mathbb{Z}_p[[T_1,\ldots,T_n]]$ for some $n \geq 0$, which is flat over $\mathbb{Z}_p$ (i.e. $p$ is not a zero-divisor in $A$). Then we have a choice of geometric objects attached to $A$: the affine scheme $\text{Spec}(A)$, and its generic fibre $\text{Spec}(A[[p]])$; the affine formal scheme $\text{Spf}(A)$; and the rigid-analytic space $(\text{Spf}(A))_{rig}$ obtained by applying Berthelot’s generic fibre construction [dJ95 §7]. We abbreviate the latter by $\text{Rig}(A)$.

Proposition 1.1. There following three sets are in canonical bijection with each other:

- Points of $\text{Spec}(A[[p]])$, i.e. maximal ideals of $A[[p]]$;
- Morphisms of formal schemes $\text{Spf}(O) \to \text{Spf}(A)$, with $O$ the ring of integers of a finite extension of $\mathbb{Q}_p$ (“rig-points” of $\text{Spf}(A)$);
- Points of $\text{Rig}(A)$.

Proof. See [dJ95 7.1.9,7.1.10].

We do not, however, obtain bijections between closed subvarieties of these geometric objects; closed subschemes of $\text{Spec}(A[[p]])$ biject with closed formal subschemes of $\text{Spf}A$ flat over $\mathbb{Z}_p$, but these correspond to a subset of the closed subvarieties of $\text{Rig}(A)$. Most of the content of the present note relates in some way or another to the following key example. If $A = \mathbb{Z}_p[T]$, then $\text{Rig}(A)$ is the rigid-analytic open unit disc, and the set of points $T$ such that $(1 + T)^p^n = 1$ for some $n \in \mathbb{N}$ is a closed subvariety of $\text{Rig}(A)$ (cut out by the $p$-adic logarithm $\log((1 + T))$, which is clearly not the analytification of any closed subvariety of $\text{Spf}(A)$.

If $P(A)$ is the common set of points from the preceding proposition, we refer to the topology on $P(A)$ whose closed subsets are given by ideals of $A[[p]]$ (or, equivalently, closed subvarieties of $\text{Spf}(A)$ flat over $\mathbb{Z}_p$) as the formal Zariski topology, and the topology whose closed subsets are given by rigid-analytic subvarieties of $\text{Rig}(A)$ as the rigid Zariski topology. As the preceding example shows, the rigid Zariski topology may be strictly finer than the formal Zariski topology.

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2. Character spaces

Let $G$ be an abelian $p$-adic analytic group; equivalently, $G$ is any group of the form $\mathbb{Z}_p^d \times H$, for $d \geq 0 \in \mathbb{Z}$ and $H$ a finite abelian group.

**Theorem 2.1.** The functor mapping an Artinian local $\mathbb{Z}_p$-algebra $A$ to the set of continuous group homomorphisms $G \to A^\times$ is pro-representable, and is represented by the formal scheme \( \hat{G} = \text{Spf} \mathbb{Z}_p[G] \), where $\mathbb{Z}_p[G]$ is the Iwasawa algebra of $G$, equipped with the canonical character $G \to \mathbb{Z}_p[G]^\times$. Moreover, the generic fibre $\hat{G}^{\text{rig}}$ of $\hat{G}$ is the rigid space constructed in [Buz04] Lemma 2 which represents the corresponding functor on the category of affinoid $\mathbb{Q}_p$-algebras.

**Proof.** Essentially by definition, any continuous homomorphism $G \to A^\times$ extends uniquely to a ring homomorphism $\mathbb{Z}_p[G] \to A$, and conversely any ring homomorphism $\mathbb{Z}_p[G] \to A$ gives a group homomorphism $G \to A^\times$ by composition with the canonical character (which is continuous, since $A$ is Artinian). Furthermore, $\mathbb{Z}_p[G]$ can clearly be written as an inverse limit of the quotients $(\mathbb{Z}/p^n\mathbb{Z})[G/U]$ for $U$ open in $G$, which are Artinian $\mathbb{Z}_p$-algebras. Moreover, if $G_1$ and $G_2$ are two such groups, we have

$$\mathbb{Z}_p[G_1 \times G_2] = \mathbb{Z}_p[G_1] \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[G_2];$$

the generic fibre construction commutes with fibre products, so it suffices to check that the generic fibre of $\text{Spf} \mathbb{Z}_p[G]$ agrees with Buzzard’s construction when $G$ is either $\mathbb{Z}_p$ or a finite cyclic group; both of these cases are easy. \( \square \)

Now let $K$ be a number field. We define

$$\mathcal{O}_{K,p}^\times := (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times = \prod_{v \mid p} \mathcal{O}_{K,v}^\times.$$

It is clear that $\mathcal{O}_{K,p}^\times$ is an abelian $p$-adic analytic group of dimension $d = [K : \mathbb{Q}]$; we let $W = \mathcal{O}_{K,p}^\times$. A point of $W$ is thus equivalent to a continuous homomorphism $\mathcal{O}_{K,p}^\times \to E^\times$, for $E$ some finite extension of $\mathbb{Q}_p$; we refer to these as $p$-adic weights for $K$.

Let $K_\infty^\circ$ be the identity component of $(K \otimes \mathbb{R})^\times$, and $U$ any open compact subgroup of $(\mathbb{A}_K^p)^\times$. We define

$$H(U) = K_\infty^\circ / K^\times \cdot U \cdot K_\infty^\circ.$$  

**Definition 2.2.** [Buz04] The eigenvariety for $\text{GL}_1 / K$ of tame level $U$ is the formal $\mathbb{Z}_p$-scheme $\mathcal{E}(U) = H(U)$.

The inclusion $\mathcal{O}_{K,p}^\times \hookrightarrow \mathbb{A}_K^\times$ gives a continuous map $\mathcal{O}_{K,p}^\times \to H(U)$ whose kernel is the closure in $\mathcal{O}_{K,p}^\times$ of the abelian group $\Gamma(U) = K^\times \cap \left( U \cdot \mathcal{O}_{K,p}^\times \cdot K_\infty^\circ \right)$. The cokernel of this map is finite (it is the ray class group modulo $UK_\infty^\circ$) and hence $H(U)$ is also a compact abelian $p$-adic analytic group, of dimension equal to $1 + r_2 + \delta$ where $r_2$ is the number of complex places of $K$ and $\delta$ is the defect in Leopoldt’s conjecture for $K$ at $p$.

If we write $Q(U) = \mathcal{O}_{K,p}^\times / \Gamma(U)$, then we can identify $Q(U)$ with a finite-index subgroup of $H(U)$; hence we have maps $\mathbb{Z}_p[\mathcal{O}_{K,p}^\times] \to \mathbb{Z}_p[Q(U)] \to \mathbb{Z}_p[H(U)]$, where the second map is finite and flat (and becomes étale after inverting $p$). Thus the morphism $\mathcal{E}(U) \to \mathcal{W}$ factors as a finite flat surjective map followed by the inclusion of the closed subscheme $\mathcal{W}(U) = \overline{Q(U)}$ of $\mathcal{W}$. In particular, we have the following result:

**Proposition 2.3.** Every component of $\mathcal{E}(U)$ has dimension equal to $1 + r_2 + \delta$, and maps surjectively to a component of $\mathcal{W}(U)$. 

(Note that $\mathcal{E}(U)$ is not flat over $W$ unless $r_1 + r_2 = 1$, i.e. $K$ is either $\mathbb{Q}$ or an imaginary quadratic field.)

3. Algebraic points

Let $\kappa$ be a $p$-adic weight for $K$. We say that $\kappa$ is algebraic if we can write

$$\kappa(x) = \prod_i \sigma_i(x)^{n_i}$$

where $\sigma_1, \ldots, \sigma_d$ are the ring homomorphisms $\mathcal{O}_{K,p} \to \overline{\mathbb{Q}}_p$ arising from the $d$ embeddings $K \to \overline{\mathbb{Q}}_p$, and $n_i \in \mathbb{Z}$. We say $\kappa$ is parallel if $\kappa$ factors through the norm map $N_{K/\mathbb{Q}}$ (extended $\mathbb{Z}_p$-linearly to a ring homomorphism $\mathcal{O}_{K,p} \to \mathbb{Z}_p$). Note that an algebraic weight is parallel if and only if the $n_i$ are all equal.

If $\kappa$ is algebraic in the above sense when restricted to some open neighbourhood of the identity, we say $\kappa$ is locally algebraic; this is equivalent to the existence of a factorisation $\kappa = \varepsilon \kappa'$ where $\kappa'$ is algebraic and $\varepsilon$ has finite order. Similarly, if $\kappa$ is a $p$-adic weight which becomes parallel when restricted to some open neighbourhood of the identity, we say $\kappa$ is locally parallel.

Let us fix an isomorphism between $\mathbb{C}$ and $\overline{\mathbb{Q}}_p$. Then there is a bijection between algebraic Grössencharacters of $K$ (of level containing $U$) and points of $\mathcal{E}(U)$ whose projection to $W$ is locally algebraic. This maps a Grössencharacter of infinity-type $x \mapsto \prod_i \sigma_i(x)^{n_i}$ to a locally algebraic character with the same algebraic part.

We make the following assumption, which will remain in force for the remainder of this section:

**Assumption.** The field $K$ contains no CM subfield.

**Theorem 3.1** ([Weil54]). If the above assumption holds, then the infinity-type of every algebraic Grössencharacter of $K$ is parallel (i.e. factors through the norm map $N_{K/\mathbb{Q}}$).

If $\kappa$ is a locally algebraic weight, we define $c(\kappa)$ to be the smallest integer $r \geq 0$ such that $\kappa$ is algebraic when restricted to $1 + p^r \mathcal{O}_{K,p}$.

**Proposition 3.2.** For any $N < \infty$, there is a 1-dimensional closed formal subscheme of $W$ that contains every locally algebraic weight $\kappa \in \mathcal{W}(U)$ with $c(\kappa) \leq N$.

**Proof.** If $\kappa \in \mathcal{W}(U)$ is locally algebraic, then by Weil’s theorem it must be of the form $x \mapsto \varepsilon(x) N_{K/\mathbb{Q}}(x)^k$ for some $k \in \mathbb{Z}$ and finite-order $\varepsilon$. Since the subgroup $\Gamma(U)(1+p^N \mathcal{O}_{K,p})$ has finite index in $\mathcal{O}_{K,p}^\times$, there are only finitely many candidates for $\varepsilon$. Hence the locally algebraic weights with $c(\kappa) \leq N$ are contained in the union of finitely many translates of the 1-dimensional subscheme $\mathcal{W}_0 \subseteq \mathcal{W}$ parametrising parallel weights (which is simply the space of characters of $N_{K/\mathbb{Q}}(\mathcal{O}_{K,p}^\times) \subseteq \mathbb{Z}_p^\times$).

We assume henceforth that $K$ is not totally real, so $\mathcal{W}(U)$ has dimension $1 + r_2 + \delta > 1$. It follows that the locally algebraic weights with $c(\kappa) \leq N$ are not dense in the formal Zariski topology of $\mathcal{W}(U)$. In particular, for a fixed coefficient field $E$ which is discretely valued, the set of $E$-valued finite-order characters is finite (since $E$ contains finitely many $p$-power roots of unity) and thus the locally algebraic $E$-valued weights are not formally Zariski-dense in $\mathcal{W}(U)$.

**Proposition 3.3.** The closure of the locally algebraic weights in the rigid Zariski topology of $\mathcal{W}(U)$ is a closed rigid subvariety of $\mathcal{W}(U)$ of dimension 1. However, this set is dense in the formal Zariski topology of $\mathcal{W}(U)$.
Proof. Let $u_1, \ldots, u_{d-1}$ be a $\mathbb{Z}_p$-basis for the torsion-free part of the subgroup

$$C = \left\{ x \in \mathcal{O}_{K,p}^\times : N_{K/Q}(x) = 1 \right\}.$$

The functions $\kappa \mapsto \log(\kappa(u_i))$ are analytic functions on $\mathcal{W}^\rig$. Moreover, the derivatives of these functions are linearly independent at the origin, and hence anywhere (since they are homomorphisms of rigid-analytic group varieties). Thus they cut out a reduced rigid subvariety of $\mathcal{W}^\rig$ of dimension 1. I claim that every locally algebraic point of $\mathcal{W}(U)$ lies in this subvariety. Indeed, suppose $\kappa$ is such a point, with residue field $E$. Then $\kappa(C)$ must be finite, since the algebraic part of $\kappa$ is trivial on $C$. Therefore $\kappa(u_1), \ldots, \kappa(u_{d})$ must be roots of unity in $E^\times$; as the subgroup of $C$ generated by the $u_i$ is pro-$p$, these must be $p$-power roots of unity. Hence they are zeros of the $p$-adic logarithm.

On the other hand, the even powers of the norm character $\mathcal{O}_{K,p}^\times \to \mathbb{Z}_p^\times$ are clearly in $\mathcal{W}(U)$, and the closure of these (in either the formal or the rigid Zariski topology) is a formal subscheme of $\mathcal{W}$ of dimension 1; so the dimension of the rigid Zariski closure of the locally algebraic weights in $\mathcal{W}(U)$ is exactly 1.

For the second statement, since $\mathcal{W}(U) = Q(U)$ is affine, it suffices to check that there is no nonzero element of $\mathbb{Z}_p[Q(U)]$ whose image under any locally constant character is zero. This is clear since $\mathbb{Z}_p[Q(U)]$ is by construction the inverse limit of the $\mathbb{Z}_p$-group rings of the finite quotients of $Q(U)$. \hfill \Box

We now lift these statements to $\mathcal{E}(U)$. If $\chi$ is a point of $\mathcal{E}(U)$, we say $\chi$ is locally algebraic if its image $\kappa \in \mathcal{W}(U)$ is so (equivalently, if it corresponds to an algebraic Grössencharacter of $K$); if this is the case, we define $c(\chi) = c(\kappa)$, which is the smallest power of $p$ divisible by the $p$-part of the conductor of the corresponding Grössencharacter.

**Proposition 3.4.** For any $N < \infty$, the set of points $\chi \in \mathcal{E}(U)$ with $c(\chi) < N$ (or with values in a given coefficient field $E$) is contained in a finite union of 1-dimensional closed subschemes of $\mathcal{E}(U)$. The set of all locally algebraic points is not contained in any proper closed subscheme of $\mathcal{E}(U)$, but is contained in a 1-dimensional closed subvariety of the generic fibre $\mathcal{E}(U)^\rig$.

It follows that a rigid-analytic function on $\mathcal{E}(U)^\rig$ is not determined by its values at locally algebraic characters, but that a bounded rigid-analytic function is determined by these values.

### 4. Sketch of the GL$_2$ theory

We now suppose $K$ is an imaginary quadratic field in which $p$ splits, and $\mathfrak{R}$ an integral ideal of $\mathcal{O}_K$ prime to $p$. For integers $a, b \geq 2$, we let $S_{a,b}(\Gamma_1(\mathfrak{P}^r))$ denote the space of cuspidal cohomological automorphic forms for $GL_2/K$ of weight $(a, b)$ and level $\Gamma_1(\mathfrak{P}^r)$; equivalently, this is the space $H^1_{\text{par}}(Y_1(\mathfrak{P}^r), V_{a,b})$ where $Y_1(\mathfrak{P}^r)$ is the appropriate arithmetic quotient of $GL_2(\mathbb{A}_K)$ and $V_{a,b}$ is the locally constant sheaf on $Y_1(\mathfrak{P}^r)$ corresponding to the algebraic representation of $\text{Res}_{K/Q} GL_2$ of highest weight $(a, b)$. This is a finite-dimensional vector space over $K$.

We fix a choice of prime $p \nmid p$, and hence an embedding $K \hookrightarrow \mathbb{Q}_p$. For a locally constant character $\chi$ of $\mathcal{O}_{K,p}^\times$ of conductor $c$, with values in a finite extension $E$ of $\mathbb{Q}_p$, we let $S_{a,b}(\Gamma_1(\mathfrak{P}^r), \chi)_{\text{ord}}$ denote the subspace of $S_{a,b}(\Gamma_1(\mathfrak{P}^r)) \otimes_K E$ of forms on which the diamond operators act via $\chi$ and which are ordinary at $p$ and $\overline{\mathfrak{P}}$.

We say that a locally algebraic weight $\kappa : x \mapsto x^d \mathfrak{P}^e \varepsilon(x) \in \mathcal{W}$, with $\varepsilon$ of finite order, is *arithmetical* if $a, b \geq 2$. 
Theorem 4.1 ([Hid94, Theorem 3.2]). There exists a finitely-generated $\mathbb{Z}_p[\mathcal{O}_{K,p}^\times]$-module $\mathbb{H}$ such that for any arithmetical character $\kappa$ as above,

$$S_{a,b}(\Gamma_1(\mathfrak{p}, \varepsilon)) \cong \mathbb{H} \otimes_{\mathbb{Z}_p[\mathcal{O}_{K,p}^\times], \kappa} E.$$ 

The sheaf $\mathbb{H}$ on $\mathcal{W}$ corresponds to the pushforward of $\mathcal{O}(\mathcal{E}(U))$ to $\mathcal{W}$ in the $\text{GL}_1$ theory. Hida has given a characterisation of its geometry analogous to proposition 2.3 above:

Theorem 4.2 ([Hid94, Theorem 6.2]). The support of $\mathbb{H}$ is an equidimensional subscheme of $\mathcal{W}$ of dimension 1.

We also have an obstruction to the existence of locally algebraic points arising from archimedean considerations, analogous to Theorem 3.1:

Theorem 4.3 ([Har87, 3.6.1]). $S_{a,b}(\Gamma_1(\mathfrak{p}, \varepsilon))$ is zero unless $a = b$.

Hence any arithmetical weight lying in $\text{Supp} \mathbb{H}$ is locally parallel, and thus contained in a translate by some locally constant character of the 1-dimensional subscheme $\mathcal{W}_0$ parametrising parallel weights.

Theorem 4.4 ([CM09, Lemma 8.8]). There exist pairs $(K, \mathfrak{N}, \mathfrak{p})$ such that $\text{Supp} \mathbb{H}$ has nonempty intersection with, but does not contain, the component of $\mathcal{W}_0$ containing the character $x \mapsto (\mathfrak{N}/\mathbb{Q} x)^2$.

We deduce that in such cases, $\text{Supp} \mathbb{H}$ has irreducible components $X$ of dimension 1 such that for any $N < \infty$, the set of arithmetical weights $\kappa \in X$ with $c(\kappa) < N$ is finite; in particular, there are finitely many arithmetical weights in $X(E)$ for any given field $E$. Furthermore, the set of all arithmetical weights is not dense in the rigid Zariski topology of $X$. However, since the set of locally parallel weights is not formally Zariski-closed in $\mathcal{W}$, one cannot rule out the possibility that the set of all arithmetical locally algebraic weights in $X$ is infinite (and hence dense in $X$ for the formal Zariski topology).

Remark 4.5. It is asserted in [CM09, Theorem 8.9] that there are components $X$ which contain only finitely many arithmetical weights, but the proof given therein relies on the assertion that the intersection of $X$ with the set of locally parallel weights is formally Zariski-dense in $X$ (and hence must be either finite or all of $X$); this is false as the set of all locally parallel weights is not formally Zariski-closed in $\mathcal{W}$, and the rigid space $\mathcal{W}^{\text{rig}}$ is not quasicompact. Similarly, the arguments of Theorem 7.1 of op.cit. do not show that the Galois-theoretic deformation space constructed therein has finitely many specialisations which are Hodge-Tate with parallel Hodge-Tate weights, but rather the weaker statement that for any $N$ it has finitely many specialisations $V$ which have parallel Hodge-Tate weights and for which the Weil-Deligne representation $D_{\text{pst}}(V)$ has Artin conductor $< N$ (in particular, it has finitely many crystalline specialisations of parallel weight).

Remark 4.6. The essential difference between the $\text{GL}_1$ and $\text{GL}_2$ cases is that in the latter we lack an explicit description of the subscheme $\text{Supp} \mathbb{H}$. Thus in the former case we can show that every component of $\mathcal{E}(U)$ contains infinitely many points corresponding to classical automorphic forms, while in the latter case we cannot rule out the existence of reducible components of $\text{Supp} \mathbb{H}$ containing only finitely many such points – we merely assert that the existence of such components has not been proven.

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