The Dirac-Hestenes Equation for Spherical Symmetric Potentials in the Spherical and Cartesian Gauges

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Abstract
In this paper using the apparatus of the Clifford bundle formalism we show how straightforwardly solve in Minkowski spacetime the Dirac-Hestenes equation—which is an appropriate representative in the Clifford bundle of differential forms of the usual Dirac equation—by separation of variables for the case of a potential having spherical symmetry in the Cartesian and spherical gauges. We show that contrary to what is expected at a first sight, the solution of the DHE in both gauges has exactly the same mathematical difficulty.

1 Introduction
In this paper the Clifford bundle formalism is used in order to show how to solve in Minkowski spacetime the Dirac-Hestenes equation (DHE)—which is an appropriate representative in the Clifford bundle of differential forms of the usual Dirac equation—by separation of variables for the case of a potential having spherical symmetry using the Cartesian and spherical gauges¹. Our

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²See below for the precise definition of these terms.
main result is that contrary to what is expected at a first sight [2], the finding of solutions of the DHE in any one of the mentioned gauges presents exactly the same mathematical difficulty.

We hope that our approach which uses intrinsic methods and no matrix representations helps to clarify some misunderstandings appearing in the literature relative to: (i) the meaning and nature of Dirac-Hestenes spinor fields (DHSF), which are sections of the spin-Clifford bundle and their representatives in the Clifford bundle of differential forms, and (ii) the relation between the familiar Dirac equation (satisfied by covariant Dirac spinor fields) and the DHE, and its different expressions in different (spin coframe) gauges and in different coordinate charts. Section 2 present in condensed form some necessary mathematical preliminaries, whose details may be found in [9, 8, 13]. In Section 2, we discuss how to obtain solutions of the DHE in a given potential exhibiting spherical symmetry. First, solutions are obtained in detail in Section 3.1 in the spherical gauge. In Section 3.2 the DHE is presented in the Cartesian gauge. It appears, at first sight that the equation in the spherical gauge is more complicated than the DHE (for the same problem) in the Cartesian gauge. However, this is not the case. Indeed, we succeeded in putting the equations in both gauges in forms in which it becomes obvious that their solutions are easily obtained in exactly the same way. In Section 4 we present our conclusions.

2 Preliminaries

In this paper \( \mathcal{M} = (M \simeq \mathbb{R}^4, \eta, D, \tau \eta, \uparrow) \) denotes Minkowski spacetime structure\(^2\). By \( F(M) \) we denote the (principal) bundle of frames and by \( P_{SO_{1,3}}(M) \) the orthonormal frame bundle. \( P_{SO_{1,3}}(M) \) denotes the orthonormal coframe bundle. Since Minkowski spacetime is a spin manifold there exists \( P_{Spin_{1,3}}(M) \) and \( P_{Spin_{1,3}}(M) \) which are respectively the spin frame bundle and the spin coframe bundle. To continue we select the orthonormal coframe bundle and the spin coframe bundle for our considerations. We recall that sections of \( P_{SO_{1,3}}(M) \) are orthonormal coframes and that sections of \( P_{Spin_{1,3}}(M) \) are also orthonormal coframes such that two coframes differing by a \( 2\pi \) rotation are distinct and two coframes differing by a \( 4\pi \) rotation are identified. We denote in what follows by \( s : P_{Spin_{1,3}}(M) \rightarrow P_{SO_{1,3}}(M) \) the fundamental mapping present in the definition of \( P_{Spin_{1,3}}(M) \) (see [8] [13] for details). Next we introduce the Clifford bundle of differential forms \( \mathcal{C}(M, \eta) \) which is a vector bundle associated to \( P_{Spin_{1,3}}(M) \) whose section are sums of nonhomogeneous differential forms, which will be called Clifford fields. We recall that \( \mathcal{C}(M, \eta) = P_{SO_{1,3}}(M) \times_{Ad^\prime} \mathbb{R}_{1,3} \), where \( \mathbb{R}_{1,3} \simeq \mathbb{H}(2) \) is the spacetime algebra. Details of the bundle structure are as follows:

\(^2\)Note that \( \eta \in \text{sec} T_0^2 M \) is the Minkowski metric, \( D \) is the Levi-Civita connection of \( \eta \), \( \tau \eta \in \text{sec} \int_0^4 T^* M \) defines a spacetime orientation and \( \uparrow \) refers to a time orientation. Also, \( \eta \in \text{sec} T_0^2 M \) denotes the metric of the cotangent bundle. Details, may be found in [13, 12, 15].
(i) Let \( \pi_c : \mathcal{CL}(M, \eta) \to M \) be the canonical projection of \( \mathcal{CL}(M, \eta) \) and let \( \{ U_\alpha \} \) be an open covering of \( M \). There are trivialization mappings \( \psi_i : \pi_c^{-1}(U_i) \to U_i \times \mathbb{R}_{1,3} \) of the form \( \psi_i(p) = (\pi_c(p), \psi_{i,x}(p)) = (x, \psi_{i,x}(p)) \). If \( x \in U_i \cap U_j \) and \( p \in \pi_c^{-1}(x) \), then

\[
\psi_{i,x}(p) = h_{ij}(x)\psi_{j,x}(p)
\]

(1)

for \( h_{ij}(x) \in \text{Aut}(\mathbb{R}_{1,3}) \), where \( h_{ij} : U_i \cap U_j \to \text{Aut}(\mathbb{R}_{1,3}) \) are the transition mappings of \( \mathcal{CL}(M, \eta) \). We recall that every automorphism of \( \mathbb{R}_{1,3} \) is inner. Then,

\[
h_{ij}(x)\psi_{j,x}(p) = g_{ij}(x)\psi_{i,x}(p)g_{ij}(x)^{-1}
\]

(2)

for some \( g_{ij}(x) \in \mathbb{R}^*_1,3 \), the group of invertible elements of \( \mathbb{R}_{1,3} \).

(ii) As it is well known the group \( \text{SO}^e_{1,3} \) has a natural extension in the Clifford algebra \( \mathbb{R}_{1,3} \). Indeed we know that \( \mathbb{R}^*_1,3 \) (the group of invertible elements of \( \mathbb{R}_{1,3} \)) acts naturally on \( \mathbb{R}_{1,3} \) as an algebra automorphism through its adjoint representation. A set of lifts of the transition functions of \( \mathcal{CL}(M, \mathfrak{g}) \) is a set of elements \( \{ g_{ij} \} \subset \mathbb{R}^*_1,3 \) such that if\(^4\)

\[
\text{Ad} : g \mapsto \text{Ad}_g,
\]

\[
\text{Ad}_g(a) = gag^{-1}, \forall a \in \mathbb{R}_{1,3},
\]

(3)

then \( \text{Ad}_{g_{ij}} = h_{ij} \) in all intersections.

(iii) Also \( \sigma = \text{Ad}|_{\text{Spin}^e_{1,3}} \) defines a group homeomorphism \( \sigma : \text{Spin}^e_{1,3} \to \text{SO}^e_{1,3} \) which is onto with kernel \( \mathbb{Z}_2 \). We have that \( \text{Ad}_{-1} = \text{identity} \), and so \( \text{Ad} : \text{Spin}^e_{1,3} \to \text{Aut}(\mathbb{R}_{1,3}) \) descends to a representation of \( \text{SO}^e_{1,3} \). Let us call \( \text{Ad}' \) this representation, i.e., \( \text{Ad}' : \text{SO}^e_{1,3} \to \text{Aut}(\mathbb{R}_{1,3}) \). Then we can write \( \text{Ad}'_{\sigma(g)}a = \text{Ad}_g(a) = gag^{-1} \).

(iv) It is clear then, that the structure group of the Clifford bundle \( \mathcal{CL}(M, \eta) \) is reducible from \( \text{Aut}(\mathbb{R}_{1,3}) \) to \( \text{SO}^e_{1,3} \). Thus the transition maps of the principal bundle of oriented Lorentz cotetrad \( P_{\text{SO}^e_{1,3}}(M) \) can be (through \( \text{Ad}' \)) taken as transition maps for the Clifford bundle. We then have \([7]\)

\[
\mathcal{CL}(M, \eta) = P_{\text{SO}^e_{1,3}}(M) \times_{\text{Ad}'} \mathbb{R}_{1,3},
\]

(4)

i.e., the Clifford bundle is an associated vector bundle to the principal bundle \( P_{\text{SO}^e_{1,3}}(M) \) of orthonormal Lorentz coframes.

### 2.1 Clifford Fields

Recall that \( \mathcal{CL}(T_x^* M, \eta_x) \) is also a vector space over \( \mathbb{R} \) which is isomorphic to the exterior algebra \( \bigwedge T_x^* M \) of the cotangent space and \( \bigwedge T_x^* M = \bigoplus_{k=0}^4 \bigwedge^k T_x^* M \), where \( \bigwedge^k T_x^* M \) is the \( \binom{4}{k} \)-dimensional space of \( k \)-forms. There is a natural

\[^4\text{Recall that} \quad \text{Spin}^e_{1,3} = \{ a \in \mathbb{R}^0_{1,3} : a\bar{a} = 1 \} \simeq \text{SL}(2, \mathbb{C}) \quad \text{is the universal covering group of the restricted Lorentz group} \quad \text{SO}^e_{1,3}. \quad \text{Notice that} \quad \mathbb{R}^0_{1,3} \simeq \mathbb{R}_{3,0} \simeq \mathbb{C}(2) \quad \text{the even subalgebra of} \quad \mathbb{R}_{1,3} \quad \text{is the Pauli algebra.} \]
embedding $\bigwedge T^*M \hookrightarrow \mathcal{C}(M, \eta)$ and sections of $\mathcal{C}(M, \eta)$—Clifford fields—can be represented as a sum of non-homogeneous differential forms. Let $\{e_a\} \in \sec \mathbb{P}_{SO^*_1}(M)$ (the orthonormal frame bundle) be a tetrad basis for $TU \subset TM$, i.e., $g(e_a, e_b) = \eta_{ab} = \text{diag}(1, -1, -1, -1)$ and $(a, b = 0, 1, 2, 3)$. Moreover, let $\{\varepsilon^a\} \in \sec \mathbb{P}_{SO^*_1}(M)$. Then, for each $a = 0, 1, 2, 3$, $\varepsilon^a \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, \eta)$, i.e., $\{\varepsilon^b\}$ is the dual basis of $\{e_a\}$. Finally, let $\{\varepsilon_a\}, \varepsilon^a \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, \eta)$ be the reciprocal basis of $\{\varepsilon_b\}$, i.e., $\varepsilon_a \cdot \varepsilon^b = \delta_a^b$.

Recall also that the fundamental Clifford product is generated by

$$\varepsilon^a \varepsilon^b + \varepsilon^b \varepsilon^a = 2\eta^{ab}. \quad (5)$$

If $C \in \sec \mathcal{C}(M, \eta)$ is a Clifford field, we have:

$$C = s + v_1 \varepsilon^1 + \frac{1}{2!} b_{ij} \varepsilon^i \varepsilon^j + \frac{1}{3!} t_{ijk} \varepsilon^i \varepsilon^j \varepsilon^k + p \varepsilon^5,$$  

where $\varepsilon^5 = \varepsilon^0 \varepsilon^1 \varepsilon^2 \varepsilon^3$ is the volume element and

$$s, v_i, b_{ij}, t_{ijk}, p \in \sec \bigwedge^0 T^*M \hookrightarrow \sec \mathcal{C}(M, \eta). \quad (7)$$

Next we recall the crucial result [8, 7] that in a spin manifold we have:

$$\mathcal{C}(M, \eta) = P_{\text{Spin}^*_1}(M) \times \text{Ad} \mathbb{R}_{1,3}. \quad (8)$$

### 2.2 Spinor Fields

Spinor fields are sections of associated vector bundles to the principal bundle of spinor coframes. The well known Dirac spinor fields are sections of the bundle

$$S_c(M, \eta) = P_{\text{Spin}^*_1}(M) \times_{\mu_c} \mathbb{C}^4 \quad (9)$$

$\mu_c$ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\text{Spin}^*_1 \cong \text{Sl}(2, \mathbb{C})$ in $\text{End}(\mathbb{C}^4)$ [1].

Now, we introduce the left spin-Clifford bundle, which is the following associated vector bundle:

$$\mathcal{C}^l_{\text{Spin}^*_1}(M, \eta) = P_{\text{Spin}^*_1}(M) \times_l \mathbb{R}_{1,3} \quad (10)$$

where $l$ is the representation of $\text{Spin}^*_1$ on $\mathbb{R}_{1,3}$ given by $l(a)x = ax$. Sections of $\mathcal{C}^l_{\text{Spin}^*_1}(M, \eta)$ are called left spin-Clifford fields. $\mathcal{C}^l_{\text{Spin}^*_1}(M, \eta)$ is a ‘principal $\mathbb{R}_{1,3}$-bundle’, i.e., it admits a free action of $\mathbb{R}_{1,3}$ on the right [7, 8, 13], which is denoted by $R_g$, $g \in \mathbb{R}_{1,3}$. We shall need also to consider the right real spin Clifford bundle for $M$ defined by

$$\mathcal{C}^r_{\text{Spin}^*_1}(M, \eta) = P_{\text{Spin}^*_1}(M) \times_r \mathbb{R}_{1,3}, \quad (11)$$

where $r$ is the representation of $\text{Spin}^*_1$ on $\mathbb{R}_{1,3}$ given by $r(a)x = xa$. Sections of $\mathcal{C}^r_{\text{Spin}^*_1}(M, \eta)$ are called right spin-Clifford fields. A crucial result is the proposition proved in [8] that there is a natural pairing

$$\mathcal{C}^l_{\text{Spin}^*_1}(M, \eta) \times \mathcal{C}^r_{\text{Spin}^*_1}(M, \eta) \rightarrow \mathcal{C}(M, \eta). \quad (12)$$
Such a proposition permits us to show that there is a well defined product of sections of $\mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$ by sections of $\mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$ and thus, a representation of any Clifford field by a product of appropriate sections of $\mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$ by sections of $\mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$.

The subbundle $I(M, \eta)$ of $\mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$ where the typical fiber is the ideal $I = \mathbb{R}_{1,3}e$ (see below) is called the bundle of left ideal algebraic spinor field (LIASF). Finally, we recall that there is a natural embedding $\text{Spin}^{e}_{1,3}(M) \hookrightarrow \mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$ which comes from the embedding $\text{Spin}^{e}_{1,3} \hookrightarrow \mathbb{R}_{0,1,3}$.

### 2.3 Dirac-Hestenes Spinor Fields

The importance of $\mathcal{Cl}_{\text{Spin}^{e}_{1,3}}(M, \eta)$ is that there are particular sections of this bundle that are in one-to-one correspondence with Dirac fields. This is seen as follows. Let $E_{\mu}, \mu = 0, 1, 2, 3$ be the canonical basis of $\mathbb{R}_{1,3}e$ which generates the algebra $\mathbb{R}_{1,3}$. They satisfy the basic relation $E_{\mu}E_{\nu} + E_{\nu}E_{\mu} = 2\eta_{\mu\nu}$.

We recall that $e = \frac{1}{2}(1 + E_{0}) \in \mathbb{R}_{1,3}$ is a primitive idempotent of $\mathbb{R}_{1,3}$ and

$$f = \frac{1}{2}(1 + E_{0}) \frac{1}{2}(1 + iE_{2}E_{1}) \in \mathbb{C} \otimes \mathbb{R}_{1,3}$$

is a primitive idempotent of $\mathbb{C} \otimes \mathbb{R}_{1,3}$. Now, let $I_{e} = \mathbb{R}_{1,3}e$ and $I_{C} = \mathbb{C} \otimes \mathbb{R}_{1,3}f$ be respectively the minimal left ideals of $\mathbb{R}_{1,3}$ and $\mathbb{C} \otimes \mathbb{R}_{1,3}$ generated by $e$ and $f$. Let $\phi = \phi e \in I_{e}$ and $\Psi = \Psi f \in I_{C}$. Then, any $\phi \in I_{e}$ can be written as

$$\phi = \psi e$$

with $\psi \in \mathbb{R}_{1,3}^{0}$. Analogously, any $\Psi \in I_{C}$ can be written as

$$\Psi = \psi e \frac{1}{2}(1 + iE_{2}E_{1})$$

with $\psi \in \mathbb{R}_{1,3}^{0}$.

Recall moreover that $\mathbb{C} \otimes \mathbb{R}_{1,3} \simeq \mathbb{R}_{4,1} \simeq \mathbb{C}(4)$, where $\mathbb{C}(4)$ is the algebra of the $4 \times 4$ complexes matrices. We can verify that

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

is a primitive idempotent of $\mathbb{C}(4)$ which is a matrix representation of $f$. In that it can be proved that there is a bijection between column spinors, i.e., elements of $\mathbb{C}^{4}$ (the complex 4-dimensional vector space) and the elements of $I_{C}$. 

5
Let $\Psi \in \sec Cl_{\ell_{1,3}}^\ell(M, \eta)$ be such that
\[ R_\ell \Psi = \Psi e = \Psi, e^2 = e = \frac{1}{2}(1 + E^0) \in \mathbb{R}_{1,3}. \tag{18} \]

We define a Dirac-Hestenes Spinor field (DHSF) associated with $\Psi$ as an even section $\psi$ of $Cl_{\ell_{1,3}}^\ell(M, \eta)$ such that
\[ \Psi = \psi e. \tag{19} \]

**Remark 1** An equivalent definition of a DHSF is the following. Let $Cl_{\ell_{1,3}}^\ell(M, \eta) = P_{\text{Spin}_{1,3}}(M) \times \mathbb{C} \otimes \mathbb{R}_{1,3}$ be the complex spin-Clifford bundle. Let $\Psi \in \sec Cl_{\ell_{1,3}}^\ell(M, \eta)$ be such that
\[ R_f \Psi = \Psi f = \Psi, f^2 = f = \frac{1}{2}(1 + E^0)\frac{1}{2}(1 + iE^1E^2) \in \mathbb{C} \otimes \mathbb{R}_{1,3}. \tag{20} \]

Then, a DHSF associated with $\Psi$ is an even section $\psi$ of $Cl_{\ell_{1,3}}^\ell(M, \eta) \hookrightarrow Cl_{\ell_{1,3}}^\ell(M, \eta)$ such that
\[ \Psi = \psi f. \tag{21} \]

In what follows, when we refer to a DHSF $\psi$ we omit for simplicity the wording associated with $\Phi$ (or $\Psi$). It is very important to observe that a DHSF is not a sum of even multivector fields although, under a local trivialization, $\psi \in \sec Cl_{\ell_{1,3}}^\ell(M, \eta)$ for each $x \in M$ is mapped on an even element\(^4\) of $\mathbb{R}_{1,3}$. We emphasize that a DHSF is a particular section of a spinor bundle, not of the Clifford bundle. However, and this is a very important fact, any DHSF has representatives in the Clifford bundle. This happens essentially because $P_{\text{Spin}_{1,3}}(M)$ is trivial, a fact that permits for each trivialization (i.e., choice of a spin coframe $\Xi \in \sec P_{\text{Spin}_{1,3}}(M)$ such that $s(\Xi) = \{\varepsilon^a\} \in \sec P_{\text{SO}_{1,3}}(M)$) to define a ‘unit section’ for the right spin-Clifford bundle, i.e., $1_\Xi \in \sec Cl_{\ell_{1,3}}^\ell(M, \eta)$ such that for each Dirac-Hestenes spinor field $\Psi \in \sec Cl_{\ell_{1,3}}^\ell(M, \eta)$ we have an even Clifford field $\psi_\Xi \in \sec Cl^{(0)}(M, \eta) \subset \sec Cl(M, \eta)$ such that
\[ \psi_\Xi = \Psi 1_\Xi. \tag{22} \]

The field $\psi_\Xi$, which is a nonhomogeneous sum of even differential forms (and which looks like a superfield) is said to be the representative of a Dirac-Hestenes spinor field (or of a Dirac spinor field) in the Clifford bundle.

### 2.4 Dirac and Dirac-Hestenes Equations

Using $\psi_\Xi$ we can write a representative of the Dirac equation satisfied by a DHSF $\Psi$ in interaction with an electromagnetic field $A \in \sec \int M \hookrightarrow Cl_{\ell_{1,3}}^\ell(M, \eta) \subset \mathbb{R}_{1,3}$.\(^4\)

Note that it is meaningful to speak about even (or odd) elements in $Cl_{\ell_{1,3}}^\ell(M)$ since $\text{Spin}_{1,3} \subseteq \mathbb{R}_{1,3}$.\(^4\)
sec $\mathcal{C}(M, \eta)$ in the Clifford bundle. First we recall that if $E^\mu$, $\mu = 0, 1, 2, 3$ is the canonical basis of $\mathbb{R}^{1,3} \rightarrow \mathbb{R}_{1,3}$ than the Dirac equation for a DHSF is

$$\partial^s \Psi E^{21} + m \Psi E^0 - q A \Psi = 0. \quad (23)$$

where $\partial^s$ is the (spin) Dirac operator action on sections of $\mathcal{C}(M, \eta)$. We have in an arbitrary gauge $\Xi$ with $s(\Xi) = \{\varepsilon^a\}$ that

$$\partial^s \Psi = \varepsilon^a D_{\varepsilon a} \Psi = \varepsilon^a (\partial_{\varepsilon a} \Psi + \frac{1}{2} \omega_{\varepsilon a} \Psi) \quad (24)$$

where $D_{\varepsilon a}$ is the spinor covariant derivative, $\partial_{\varepsilon a}$ is the spin-Pfaff derivative (details on $\partial_{\varepsilon a}$ which are not going to be used anymore in this paper may be found in [8]) and $\omega_{\varepsilon a}$ is the $\bigwedge^2 T^*M$ connection 1-form in the gauge $\Xi$ evaluated at the vector field $e_a \in \text{sec} TM$.

The representative of the Dirac equation (Eq.23) in the Clifford bundle in the gauge $\Xi$ called the DHE is (taking into account that $\partial^s$ is represented in $\mathcal{C}(M, \eta)$ by the operator $\partial^s_{\varepsilon a}$)

$$\partial^s \Psi = \varepsilon^a D_{\varepsilon a} \Psi = \varepsilon^a (\partial_{\varepsilon a} \Psi + \frac{1}{2} \omega_{\varepsilon a} \Psi) \quad (25)$$

where $\partial = \varepsilon^a D_{\varepsilon a}$

is the Dirac operator acting on sections of the Clifford bundle. The action of the covariant derivative of a Clifford field $C \in \text{sec} \mathcal{C}(M, \eta)$ is given by the notable formula (see, e.g., [8]),

$$D_{\varepsilon a} C = \partial_{\varepsilon a} C + \frac{1}{2} [\omega_{\varepsilon a}, C], \quad (27)$$

where $\partial_{\varepsilon a}$ is the Pfaff derivative of form fields, i.e., taking into account Eq.26,

$$\partial_{\varepsilon a} C = e_a(s) + e_a(v_i)\varepsilon^i + \frac{1}{2!} e_a(h_{ij})\varepsilon^i\varepsilon^j + \frac{1}{3!} e_a(t_{ijk})\varepsilon^i\varepsilon^j\varepsilon^k + e_a(p)\varepsilon^5. \quad (28)$$

We need also to recall that the relation of the $\bigwedge^2 T^*M$ connection 1-forms in two different gauges $\Xi$ and $\Xi'$ related by $S \in \text{sec} \text{Spin}^c_{1,3}(M) \hookrightarrow \mathcal{C}(M, \eta)$ is given by [8]

$$\Xi'\omega X = S \Xi'\omega X S^{-1} + (D_X S) S^{-1}, \quad (29)$$

where $X \in \text{sec} TM$.

### 3 Spherical Symmetric Solutions of the DHE

It is supposed that when the potential $A$ has spherical symmetry, as it is the case, e.g., in a hydrogen atom, that it is mathematically more simple to solve
the Dirac equation or the DHE in the Cartesian gauge than in the spherical
gauge. As will be shown below the mathematical difficult involved in solving
the DHE in any one of these gauges is exactly the same one. To proceed, we define
precisely some terms. Let \( \{x^\mu\} \) be global coordinate functions for
\( M \) in Einstein-Lorentz coordinate gauge, i.e., \( e_0 = \partial/\partial x^0 \in \text{sec } TM \) is an inertial reference
frame and \( \{x^\mu\} \) is a naturally adapted coordinate system to \( e_0 \) (nacs|e_0), the
coordinate functions \( x^i, i = 1, 2, 3 \) being the Cartesian coordinate functions of
the 3-dimensional rest space of \( e_0 \). Let \( \{x'^0 = x^0, x'^i\} \) be spherical coordinate
functions naturally adapted to \( e_0 \), i.e., \((x'^1, x'^2, x'^3) = (r, \theta, \varphi) \) are the usual
spherical coordinate functions of the 3-dimensional rest space of \( e_0 \) relative to
a given space point \( \mathbf{12} \mathbf{13} \).

We have now, the following two sections\(^5\) of \( \{e_\mu\}, \{\bar{e}_\mu\} \in \text{sec } P_{SO_{1,3}}(M):\)

\[
e_\mu = \partial/\partial x^\mu, \quad e'_0 = \partial/\partial x^0, \quad e'_1 = \frac{\partial}{\partial r}, \quad e'_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e'_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}. \tag{30}\]

The corresponding dual frames are the sections \( \{\gamma^\mu\} \) and \( \{\gamma'^\mu\} \) of \( P_{SO_{1,3}}(M) \), with

\[
\gamma^\mu = dx^\mu, \quad \gamma'^0 = dx^0, \quad \gamma'^1 = dr, \quad \gamma'^2 = r d\theta, \quad \gamma'^3 = r \sin \theta d\varphi. \tag{31}\]

Let \( \Xi, \Xi' \) be two sections of \( P_{Spin_{1,3}}(M) \) such that

\[
s(\Xi) = \{\gamma^\mu\}, \quad s(\Xi') = \{\gamma'^\mu\}. \tag{32}\]

The spin coframes \( \Xi, \Xi' \) are called respectively Cartesian and the spherical
gauges. Recall that \( \omega_{e_\mu} = 0 \), but some of the \( \omega_{e'_\mu} \) are non null (see below).
We introduce yet another Cartesian gauge \( \Xi_o \) and another spherical gauge \( \Xi_s \)
(which are convenient for doing calculations) by

\[
s(\Xi_o) = \{\Gamma^\mu\}, \quad \Gamma^\mu = U \gamma^\mu U^{-1}, \quad U = e^{\gamma^2 \frac{\pi}{4}}, \tag{33}\]

and

\[
s(\Xi_s) = \{\vartheta^\mu\}, \quad \vartheta^\mu = \Omega \Gamma^\mu \Omega^{-1}, \tag{34}\]

where \( \Omega \in \text{sec } Spin_{1,3}(M) \mapsto \text{sec } Cl(M, \eta) \) is given by

\[
\Omega = \exp(\gamma^{12} \frac{\gamma^2}{2}) \exp(\gamma{31} \frac{\gamma}{2}). \tag{35}\]

\(^5\)Note that \( \{e_\mu\} \) is a section of \( F(M) \) which also belongs to \( P_{SO_{1,3}}(M) \).
The dual basis of \( \{ \Gamma^\mu \} \in \sec P_{SO_1,3}^* (M) \) is \( \{ e_\mu \} \in \sec P_{SO_1,3}^* (M) \) with

\[
e_0 = \partial / \partial x^0, \quad e_1 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_2 = -\frac{\partial}{\partial r}, \quad e_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi}.
\] (36)

To simplify the writing of formulas we denote in what follows the representatives of a DHSF satisfying the DHE in the gauges \( \Xi, \Xi_0 \) and \( \Xi_s \) by

\[
\psi_{\Xi} := \psi_c, \quad \psi_{\Xi_0} := \psi_o = \psi_c U^{-1},
\]

\[
\psi_{\Xi_s} := \psi_s = \psi_o \Omega^{-1}.
\] (37)

It is important for what follows to take into account that \( \psi_c, \psi_o \) and \( \psi_s \) are even sections of the Clifford bundle. Then, each Clifford field can be expressed in any arbitrary coordinate chart of \( M \), and as usual (sloppy notation) we denote a given coordinate expression of a Clifford field by the same symbol.

### 3.0.1 Spherical Gauge

We now investigate the solution of the DHE for \( A = V(r) \vartheta^0 = V(r) \gamma^0 \) in the spherical gauge \( \Xi_s \). In this case, the DHE is

\[
\vartheta^\mu \left( \partial_{\vartheta^\mu} \psi_s + \frac{1}{2} \omega_{\vartheta^\mu} \right) \vartheta^{13} + m \vartheta^0 - q A \vartheta^s = 0,
\] (38)

where \( \omega_{\vartheta^\mu} \in \sec \bigwedge^2 T^* M \hookrightarrow \sec C(M, \eta) \) is given by [8]

\[
\omega_{\vartheta^\mu} = 2(\partial_{\vartheta^\mu} \Omega) \Omega^{-1},
\] (39)

where \( \Omega \) is given by Eq. (35).

At first (and eventually, second) sight Eq. (38) is more difficult to solve than the corresponding equation in the Cartesian gauge (Eq. (56) below) because the \( \vartheta^\mu \) are variable covector fields and some of the \( \omega_{\vartheta^\mu} \neq 0 \). However, let us analyze the term \( \vartheta^\mu (\partial_{\vartheta^\mu} \Omega) \Omega^{-1} \vartheta^s \). We have

\[
\vartheta^\mu (\partial_{\vartheta^\mu} \Omega) \Omega^{-1} \vartheta^s = \Omega^\mu \Omega^{-1} (\partial_{\vartheta^\mu} \Omega) \Omega^{-1} \vartheta^s = -\Omega^\mu (\partial_{\vartheta^\mu} \Omega^{-1}) \Omega \vartheta^s.
\] (40)

Now, since

\[
\Gamma^0 (\partial_{\vartheta^0} \Omega^{-1}) \Omega \vartheta^s = 0,
\]

\[
\Gamma^1 (\partial_{\vartheta^1} \Omega^{-1}) \Omega \vartheta^s = \frac{\gamma^1}{2r} \cot \theta \vartheta^s - \frac{\gamma^3}{2r} \vartheta^s,
\]

\[
\Gamma^2 (\partial_{\vartheta^2} \Omega^{-1}) \Omega \vartheta^s = 0,
\]

\[
\Gamma^3 (\partial_{\vartheta^3} \Omega^{-1}) \Omega \vartheta^s = -\frac{\gamma^1}{2r} \cot \theta \vartheta^s + \frac{\gamma^3}{2r} \vartheta^s,
\] (41)
the term
\[ \Gamma^\mu (\partial_\alpha \Omega^{-1}) \Omega \psi_s = 0, \]
and Eq. (42) becomes
\[ \vartheta^\mu \partial_\mu \psi_s \vartheta^{13} + m \psi_s \vartheta^0 - q A \psi_s = 0. \]

Writing
\[ \psi_s = \psi_{s1}(r, \theta) e^{(n \varphi - E \varphi) \vartheta^{13}}, \]
where \( n \in \mathbb{Z} \) we can separate Eq. (43), once \( T \) we recall that the Pfaff derivatives \( \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial \theta} \), (which in the following we write simply as \( \frac{\partial}{\partial r} \) and \( \frac{\partial}{\partial \theta} \)) act only on the components of the Clifford fields (see Eq. (28)) We get a trivial equation in the \( \varphi \) variable and the following equation for \( \psi_{s1} \),
\[ (\vartheta_{30} \frac{\partial}{\partial r} + \vartheta_{10} \frac{\partial}{\partial \theta}) \psi_{s1} \vartheta_{13} + \frac{n \vartheta_{20}}{r \sin \theta} \psi_{s1} + (E - V) \psi_{s1} = -m \vartheta^0 \psi_{s1} \vartheta^0, \]
where \( n \in \mathbb{Z} \).
Next we write
\[ \psi_{s1}(r, \theta) = \vartheta^{12} \mathcal{J}(r) \vartheta^{13} \zeta(\theta) + \mathcal{J}(r) \tilde{\zeta}(\theta), \]
and get
\[ \sin \theta \left( \frac{d \zeta(\theta)}{d \theta} + \gamma^{13} \kappa \zeta(\theta) \right) - \lambda \tilde{\zeta}(\theta) = 0 \]
for the angular component, where \( \kappa \in \mathbb{R} \) is the separation constant. Eq. (46) has the general solution [5]
\[ \zeta_{p\lambda}(\theta) = b \sin |\lambda| \theta \exp(\vartheta^{13} \theta / 2|\vartheta^{13} | \lambda \sin \theta \mathcal{C}_{\lambda-1}^{0} \cos \theta) + (p + 2 |\lambda|) \mathcal{C}_{p}^{0}(\cos \theta) \]
where \( \mathcal{C}_{p}^{a} \) are the Gegenbauer polynomials defined by
\[ (p+1)C_{p+1}^{a}(z) - 2(p+a)zC_{p}^{a}(z) + (p+2a+1)C_{p-1}^{a}(z) = 0, \quad C_{-1}^{a}(z) = 0, \quad C_{0}^{a}(z) = 1, \]
where \( p \in \mathbb{N}, \ a \in \mathbb{R}^+ \) and
\[ b = \frac{2 |\lambda| \Gamma(|\lambda|)}{4\pi \sqrt{\Gamma(p+2|\lambda|+1)}}. \]

The radial equation is
\[ -\vartheta^{3} \frac{d}{dr} \mathcal{J}(r) \vartheta^{13} + \left( \vartheta^{13} \frac{\kappa}{r} + \vartheta^0 (E - V) \right) \mathcal{J}(r) + m \mathcal{J}(r) \vartheta^0 = 0 \]
that can be decomposed writing
\[ \mathcal{J}(r) = \mathcal{J}_0(r) - \vartheta^{23} \mathcal{J}(r), \]
as

\[ \frac{d}{dr} J_1(r) + \frac{\kappa}{r} J_1(r) + (V - m - E) J_0(r) = 0, \]  
(53)

\[ \frac{d}{dr} J_0(r) - \frac{\kappa}{r} J_0(r) + (E - m - V) J_1(r) = 0. \]  
(54)

These are the well known radial equations for the Dirac equation solution concerning the hydrogen atom [14], whose solutions are well known.

### 3.1 Cartesian Gauge

We now investigate how to solve the DHE for \( A = V(r)\gamma^0 \) in a Cartesian gauge. First, the DHE in the gauges \( \Xi \) and \( \Xi_0 \) are respectively

\[ \gamma^\mu \partial_\mu \psi \gamma^{21} + m \psi \gamma^0 - qA \psi = 0, \]  
(55)

\[ \Gamma^\mu \partial_\mu \psi_o \gamma^{13} + m \psi_o \gamma^0 - qA \psi_o = 0. \]  
(56)

Taking into account the (obvious) operator identity

\[ \Gamma^\mu \partial_\mu = \vartheta^\mu \partial_\mu, \]  
(57)

we can write Eq.(56) as

\[ \vartheta^\mu \partial_\mu \psi_o \gamma^{13} + m \psi_o \gamma^0 - qA \psi_o = 0, \]  
(58)

or

\[ \Omega \Gamma^\mu \Omega^{-1} \partial_\mu \psi_o \gamma^{13} + m \psi_o \gamma^0 - qA \psi_o = 0 \]  
(59)

which, after introducing

\[ \psi = \Omega^{-1} \psi_o, \ A' = \Omega^{-1} A \Omega \]  
(60)

becomes,

\[ \Gamma^\mu \partial_\mu \psi \gamma^{13} - \Gamma^\mu (\partial_\mu \Omega^{-1}) \Omega \psi + m \psi \gamma^0 - qA' \psi = 0. \]  
(61)

which taking into account that according to Eq.(42) \( \Gamma^\mu (\partial_\mu \Omega^{-1}) \Omega \psi = 0 \) can be easily be solved by separation of variables by writing

\[ \psi = \psi_1(r, \theta) e^{(\gamma^\mu \vartheta^\mu - Et) \gamma^{13}}. \]  
(62)

We have a trivial differential equation in the \( \varphi \) variable and the following equation for \( \psi_1 \),

\[ (\gamma_3 \partial_r + \gamma_{10} \frac{\partial}{\partial \theta}) \psi_1 \gamma_{13} + \frac{n \gamma_{20}}{r \sin \theta} \psi_1 + (E - V) \psi_1 = -m \gamma^0 \psi_1 \gamma^0. \]  
(63)

which can be solved in exactly the same way that Eq.43 has been solved once we take into account that the \( \{ \gamma^\mu \} \) and the \( \{ \vartheta^\mu \} \) satisfy the same algebraic relations. We obviously get the same spectrum, as it may be.
Remark 2 An equation like Eq. (63) has been used by Krüger [5] and also Daviau [3]. However those authors arrive at that equation using what to us seems to be a completely ad hoc argument (also used by Hestenes and Lasenby, Doran and Gull [6]) which involves: (i) a confusion between active local Lorentz transformations and transformations relating the different expressions of the representatives of a DHSF in different gauges and (ii) a supposed change of the Dirac operator under an active change of a Lorentz gauge transformation generated by $\Omega$. Both assumptions are nonsequitur and produce misunderstandings. The concept of active Lorentz gauge transformations of a DHSF and the DHE have been discussed in a thoughtful way in [10, 11]. Fortunately, Eq. (63) is a fidedigne one, for otherwise the interesting results found by Krüger and Daviau should be considered wrong.

4 Conclusions

In this paper we showed how to solve (in Minkowski spacetime) the DHE by separation of variables for the case of a potential having spherical symmetry in two different ways, i.e., using the Cartesian and spherical gauges. We show that contrary to what is expected at a first sight, the solution of the DHE in any one of those gauges presents exactly the same mathematical difficulty.

We also clarified some misunderstandings appearing in the literature related to the meaning and nature of DHSF, the DHE and its different expressions in different (spin coframe) gauges and why the use of different coordinate charts does not imply change of gauge. We conjecture that "tricks" analogous to the ones used in this paper can be used to solve with the same mathematical difficulties the DHE with potentials exhibiting some others symmetries, both in the Cartesian gauge and also in the gauge exhibiting the symmetry of the potential. We will discuss this issue in another paper.

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