A reducibility result for Schrödinger operator with finite smooth and time quasi-periodic potential

Jing Li\textsuperscript{a,b}

\textsuperscript{a}School of Mathematics, Shandong University, Jinan 250100, P.R. China
\textsuperscript{b}School of Mathematics and Statistics, Shandong University, Weihai 264209, P.R. China

Abstract

In the present paper, we establish a reduction theorem for linear Schrödinger equation with finite smooth and time-quasi-periodic potential subject to Dirichlet boundary condition by means of KAM technique. Moreover, it is proved that the corresponding Schrödinger operator possesses the property of pure point spectra and zero Lyapunov exponent.

Key words: Reducibility; Quasi-periodic Schrödinger operator; KAM theory; Finite smooth; Lyapunov exponent; Pure-point spectrum

MSC:35P05; 37K55; 81Q15

1. Introduction

Recently years there are many literatures to investigate the reducibility for the linear Schrödinger equation of quasi-periodic potential, of the form

\[ i\dot{u} = (H_0 + \varepsilon W(\omega t, x, -i\nabla))u, \quad x \in \mathbb{R}^d, \text{ or } x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d \]  

(1.1)

or of the more general form, where \( H_0 = -\Delta + V(x) \) or an abstract self-adjoint (unbounded) operator and the perturbation \( W \) is quasiperiodic in time \( t \) and it may or may not depend on \( x \) or/and \( V \). From the reducibility it is proved immediately that the corresponding Schrödinger operator is of the pure point spectrum property and zero Lyapunov exponent.

When \( x \in \mathbb{R}^d \), there are many interesting and important results. See [1, 2, 6, 15] and the references therein.

When \( x \in \mathbb{T}^d \) with any integer \( d \geq 1 \), there are relatively less results. In [9], it is proved that

\[ i\dot{u} = i(-\Delta + \varepsilon W(\phi_0 + \omega t, x, \omega)u), \quad x \in \mathbb{T}^d \]  

(1.2)

is reduced to an autonomous equation for most values of the frequency vector \( \omega \), where \( W \) is analytic in \((t, x)\) and quasiperiodic in time \( t \) with frequency vector \( \omega \). The reduction is made by means of Töplitz-Lipschitz property of operator and very hard KAM technique. As a special case
of $L^2$ with $d = 1$, the reduction can be automatically derived from the earlier KAM theorem for nonlinear partial differential equations, assuming $W$ is analytic in $(t,x)$. See, [11] and [12] for example.

As we know, the spectrum property depends heavily on the smoothness of the perturbation for the discrete Schrödinger operator. For example, the Anderson localization and positivity of the Lyapunov exponent for one frequency discrete quasi-period Schrödinger operator with analytic potential occur in non-perturbative sense (the largeness of the potential does not depend on the Diophantine condition. See [4], for the detail). However, one can only get perturbative results when the analytic property of the potential is weaken to Gevrey regularity (see [10]). Thus, a natural problem is that what happens when the perturbation is finite smooth in $(t,x)$.

Actually in his pioneer work, by reducibility Combescure [6] studied the quantum stability problem for one-dimensional harmonic oscillator with a time-periodic perturbation. The techniques of this paper were extended in [7] and [8], in order to deal with an abstract Schrödinger problem for one-dimensional harmonic oscillator with a time-periodic perturbation. The techniques of this paper were extended in [7] and [8], in order to deal with an abstract Schrödinger problem for one-dimensional harmonic oscillator with a time-periodic perturbation. The techni-

In the present paper, we will extend the time periodic $W$ to time quasi-periodic one. Let us consider a linear Schrödinger equation with quasi-periodic coefficient:

$$\mathcal{L}u \triangleq iu_t - xu_{xx} + Mu + \varepsilon W(\omega t,x)u = 0$$

(1.3)

subject to the boundary condition

$$u(t,0) = u(t,\pi) = 0,$$

(1.4)

where $W$ is a quasi-periodic in time $t$ with frequency $\omega$:

$$W(\omega t,x) = \mathcal{W}(\theta,x)|_{\theta = \omega t}, \quad \mathcal{W} \in C^N(\mathbb{T}^n \times [0,\pi],\mathbb{R}), \quad \mathbb{T}^n = \mathbb{R}^n/(\mathbb{Z}\mathbb{Z})^n,$$

and $W$ is also an even function of $x$.

Assume $\omega = \tau \omega_0$, where $\omega_0$ is Diophantine:

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^{n-1}}, \quad k \in \mathbb{Z}^n \setminus \{0\}$$

(1.5)

with $0 < \gamma \ll 1$, a constant, and $\tau \in [1,2]$ is a parameter. Endow $L^2[0,\pi] \times L^2[0,\pi]$ a symplectic $i du \wedge d\bar{u}$. Take $(L^2[0,\pi] \times L^2[0,\pi], i du \wedge d\bar{u})$ as phase space. Then (1.3) is a hamiltonian system with hamiltonian functional

$$H(u,\bar{u}) = \int_0^\pi \frac{1}{2}|u_t|^2 + \frac{1}{2}(Mu + \varepsilon W(\omega t,x))u\bar{u}$$

Theorem 1.1. Assume that for a.e. $x \in [0,\pi]$, the potential $W(\cdot,x)$ is $C^N$ in the variable $\theta \in \mathbb{T}^n$ with $N \geq 80n$. Then there exists $0 < \varepsilon^* = \varepsilon^*(n,\gamma) \ll \gamma \ll 1$, and exists a subset $\Pi \subset [1,2]$ with

$$\text{Measure} \Pi \geq 1 - O(\gamma)$$
and a quasiperiodic symplectic change \( u = \Psi(x, \omega t)v \) such that for any \( \tau \in \Pi \), (1.2) is changed into

\[
iv_t - v_{xx} + M_\xi v = 0, \quad v(t, 0) = v(t, \pi) = 0
\]

where \( M_\xi \) is a real Fourier multiplier:

\[
M_\xi \sin kx = (M + \xi_k) \sin kx, \quad k \in \mathbb{N},
\]

with constants \( \xi_k \in \mathbb{R} \) and \( |\xi_k| \leq \varepsilon \). Moreover, the Schrödinger operator \( L \) is of pure point spectrum property and of zero Lyapunov exponent.

**Remark 1.** We will combine the Jackson-Moser-Zehnder approximation technique (see [5], for example) and KAM technique [11] and [12], which also applies to the case dealt with in [7] and [8]. Thus our result extends theirs. We also mention [16] where the reducibility is dealt for a finite smooth and unbounded perturbation \( W \).

2. Preliminaries

2.1. Analytical Approximation Lemma

In this subsection, we cite an approximation lemma with the aim of this paper. These result can be obtained by [13] and [14].

We start by recalling some definitions and setting some new notations. Assume \( X \) is a Banach space with the norm \( \| \cdot \|_X \). First recall that \( C^\mu(T^n; X) \) for \( 0 < \mu < 1 \) denotes the space of bounded Hölder continuous functions \( f : T^n \mapsto X \) with the form

\[
\| f \|_{C^\mu, X} = \sup_{0 < |x - y| < 1} \frac{|f(x) - f(y)|}{|x - y|^\mu} + \sup_{x \in T^n} \| f(x) \|_X.
\]

If \( \mu = 0 \) then \( \| f \|_{C^\mu, X} \) denotes the sup-norm. For \( \ell = k + \mu \) with \( k \in \mathbb{N} \) and \( 0 \leq \mu < 1 \), we denote by \( C^\ell(T^n; X) \) the space of functions \( f : T^n \mapsto X \) with Hölder continuous partial derivatives, i.e., \( \partial^\alpha f \in C^\mu(T^n; X_\alpha) \) for all muti-indices \( \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \) with the assumption that \( |\alpha| : = |\alpha_1| + \cdots + |\alpha_n| \leq k \) and \( X_\alpha \) is the Banach space of bounded operators \( T : \prod_{\alpha}^{|\alpha|}(T^n) \mapsto X \) with the norm

\[
\| T \|_{X_\alpha} = \sup \{ \| T(u_1, u_2, \cdots, u_{|\alpha|}) \|_X : \| u_i \| = 1, \ 1 \leq i \leq |\alpha| \}.
\]

We define the norm

\[
\| f \|_{C^\ell} = \sup_{|\alpha| \leq \ell} \| \partial^\alpha f \|_{C^\mu, X_\alpha}.
\]

**Lemma 2.1.** (Jackson, Moser, Zehnder) Let \( f \in C^\ell(T^n; X) \) for some \( \ell > 0 \) with finite \( C^\ell \) norm over \( T^n \). Let \( \phi \) be a radical-symmetric, \( C^\infty \) function, having as support the closure of the unit ball centered at the origin, where \( \phi \) is completely flat and takes value \( 1 \), let \( K = \hat{\phi} \) be its Fourier transform. For all \( \sigma > 0 \) define

\[
f_\sigma(x) : = K_\sigma * f = \frac{1}{\sigma^n} \int_{T^n} K(\frac{x - y}{\sigma}) f(y) dy.
\]
Then there exists a constant $C \geq 1$ depending only on $\ell$ and $n$ such that the following holds: For any $\sigma > 0$, the function $f_\sigma(x)$ is a real-analytic function from $C^n / (\pi \mathbb{Z})^n$ to $X$ such that if $\Delta^*_\sigma$ denotes the $n$-dimensional complex strip of width $\sigma$,

$$\Delta^*_\sigma := \{ x \in C^n / (\pi \mathbb{Z})^n \mid \Im x_j \leq \sigma, \ 1 \leq j \leq n \},$$

then for $\forall \alpha \in \mathbb{N}^n$ such that $|\alpha| \leq \ell$ one has

$$\sup_{x \in \Delta^*_\sigma} \| \partial^\alpha f_\sigma(x) - \sum_{|\beta| \leq |\alpha|} \frac{\partial^\beta f(\Re x)}{\beta!} (\sqrt{-1} \Im x)^\beta \| x_\alpha \leq C \| f \|_c \sigma^{r-|\alpha|},$$

and for all $0 \leq s \leq \sigma$,

$$\sup_{x \in \Delta^*_\sigma} \| \partial^\alpha f_\sigma(x) - \partial^\alpha f_s(x) \| x_\alpha \leq C \| f \|_c \sigma^{r-|\alpha|}. \tag{2.2}$$

The function $f_\sigma$ preserves periodicity (i.e., if $f$ is $T$-periodic in any of its variable $x_j$, so is $f_\sigma$). Finally, if $f$ depends on some parameter $\xi$, then $f$ is real analytic on the complex strip $\mathbb{T}^n_\sigma$ depending only on $\xi$. If $f$ is $T$-periodic in any of its variable $x_j$, we have

$$\sup_{x \in \Delta^*_\sigma} \| \partial^\alpha f_\sigma(x) \| x_\alpha \leq C \| f \|_c \sigma^{r-|\alpha|}. \tag{2.3}$$

where $C$ denotes (different) constants depending only on $n$ and $\ell$.

For the following result, the reader can referred to [14], for detail. For ease of notation, we shall replace $\| \cdot \|_X$ by $\| \cdot \|$. Fix a sequence of fast decreasing numbers $s_\nu \downarrow 0$, $\nu \geq 0$, and $s_0 \leq \frac{1}{2}$. For a $X$-valued function $P(\phi)$, construct a sequence of real analytic functions $P^{(\nu)}(\phi)$ such that the following conclusions holds:

1. $P^{(\nu)}(\phi)$ is real analytic on the complex strip $\mathbb{T}^n_{s_\nu}$ of the width $s_\nu$ around $\mathbb{T}^n$.
2. The sequence of functions $P^{(\nu)}(\phi)$ satisfies the bounds:

$$\sup_{\phi \in \mathbb{T}^n} \| P^{(\nu)}(\phi) - P(\phi) \| \leq C \| P \|_c s^\nu_\nu, \tag{2.4}$$

where $C$ denotes (different) constants depending only on $n$ and $\ell$.

3. The first approximate $P^{(0)}$ is “small” with the perturbation $P$. Precisely speaking, for arbitrary $\phi \in \mathbb{T}^n_{s_0}$, we have

$$\| P^{(0)}(\phi) \| \leq \| P^{(0)}(\phi) - \sum_{|\alpha| \leq \ell} \frac{\partial^\alpha P(\Re \phi)}{\alpha!} (\sqrt{-1} \Im \phi)^\alpha \| + \sum_{|\alpha| \leq \ell} \frac{\partial^\alpha P(\Re \phi)}{\alpha!} (\sqrt{-1} \Im \phi)^\alpha \| \leq C \| P \|_c s^\ell_0 \sum_{m=0}^{\infty} s^m_0 \leq C \| P \|_c.$$

$$\| P^{(0)}(\phi) \| \leq C \| P \|_c \sum_{m=0}^{\infty} s^m_0 \leq C \| P \|_c \sum_{m=0}^{\infty} s^m_0 \leq C \| P \|_c.$$

$$\| P^{(0)}(\phi) \| \leq C \| P \|_c \sum_{m=0}^{\infty} s^m_0 \leq C \| P \|_c \sum_{m=0}^{\infty} s^m_0 \leq C \| P \|_c. \tag{2.5}$$
where constant $C$ is independent of $s_0$, and the last inequality holds true due to the hypothesis that $s_0 \leq \frac{1}{2}$.

(4) From the first inequality (2.3), we have the equality below. For arbitrary $\phi \in \mathbb{T}^n$,

$$P(\phi) = P^{(0)}(\phi) + \sum_{\nu=0}^{+\infty} (P^{(\nu+1)}(\phi) - P^{(\nu)}(\phi)).$$

### 2.2. Lemmas

We need the following Lemmas.

**Lemma 2.2.** [3] For $0 < \delta < 1$, $\nu > 1$, one has

$$\sum_{k \in \mathbb{Z}^n} e^{-2|k|\delta |k|^\nu} < \left( \frac{\nu}{e} \right) ^\nu \frac{(1 + e)^n}{\delta^{\nu + n}}.$$

**Lemma 2.3.** [12] If $A = (A_{ij})$ is a bounded linear operator on $\ell^2$, then also $B = (B_{ij})$ with

$$B_{ij} = \frac{|A_{ij}|}{|i - j|}, \quad i \neq j,$$

and $B_{ii} = 0$ is bounded linear operator on $\ell^2$, and $\|B\| \leq \left( \frac{\pi}{\sqrt{3}} \right) \|A\|$, where $\| \cdot \|$ is $\ell^2 \to \ell^2$ operator norm.

**Remark 2.** Lemma 2.3 holds true for the weight norm $\| \cdot \|_N$.

### 3. Main results

Consider the differential equation:

$$\mathcal{L}u = iu_t - u_{xx} + Mu + \varepsilon W(\omega t, x)u = 0$$

subject to the boundary condition

$$u(t, 0) = u(t, \pi) = 0.$$

It is well-known that the Sturm-Liouville problem

$$-y'' + My = \lambda y,$$

with the boundary condition

$$y(0) = y(\pi) = 0$$

has the eigenvalues and eigenfunctions, respectively,

$$\lambda_k = k^2 + M, \quad k = 1, 2, \ldots,$$

$$\phi_k(x) = \sin kx, \quad k = 1, 2, \ldots.$$
Write
\[ u(t, x) = \sum_{k=1}^{\infty} u_k(t) \phi_k(x). \] (3.7)

Note that \( W \) is an even function of \( x \). Write
\[ W(\omega t, x) = \sum_{k=0}^{\infty} v_k(\omega t) \phi_k(x), \] (3.8)

where
\[ \phi_k(x) = \cos 2kx, \quad k = 1, 2, \ldots. \]

Considering that
\[ W(\omega t, x) u(x) = \sum_{k=1}^{\infty} \langle W(\omega t, x) u(x), \phi_k(x) \rangle \phi_k \]

where
\[ c_{jk} = \int_0^\pi \phi_j \phi_k dx = \int_0^\pi \cos 2jx \cdot \sin lx \cdot \sin kx dx = \begin{cases} 0, & k \neq l \pm 2j, \\ \frac{\pi}{4}, & k = l \pm 2j \geq 1, \\ -\frac{\pi}{4}, & k = -l \pm 2j \geq 1. \end{cases} \] (3.9)

Then (3.1) can be expressed as
\[ \sum_{k=1}^{\infty} \left( i \dot{u}_k + \lambda_k u_k + \varepsilon \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} c_{jk} v_j u_l \right) \phi_k = 0, \]

which implies that
\[ i \dot{u}_k + \lambda_k u_k + \varepsilon \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} c_{jk} v_j u_l = 0. \] (3.10)

This is a hamiltonian system
\[ \left\{ \begin{array}{l}
  i \dot{u}_k = \frac{\partial H}{\partial \pi_k}, \quad k \geq 1, \\
  i \dot{\pi}_k = -\frac{\partial H}{\partial u_k}, \quad k \geq 1,
\end{array} \right. \] (3.11)

with hamiltonian
\[ H(u, \pi) = \sum_{k=1}^{\infty} \lambda_k u_k \pi_k + \varepsilon \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{j=0}^{\infty} c_{jk} v_j(\theta) u_l \pi_k. \] (3.12)

For two sequences \( x = (x_j \in \mathbb{C}, \ j = 1, 2, \ldots) \), \( y = (y_j \in \mathbb{C}, \ j = 1, 2, \ldots) \), define
\[ \langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j. \]
Then we can write
\[ H = \langle \Lambda u, \overline{u} \rangle + \varepsilon \langle R(\theta)u, \overline{u} \rangle, \] (3.13)
where
\[ \Lambda = \text{diag}(\lambda_j : j = 1, 2, \ldots), \quad \theta = \omega t, \]
\[ R(\theta) = (R_{lj}(\theta) : k, l = 1, 2, \ldots), \quad R_{lj}(\theta) = \sum_{j=0}^{\infty} c_{lj} v_{lj}(\theta). \] (3.14)

Define a Hilbert space \( h_N \) as follows:
\[ h_N = \{ z = (z_k \in \mathbb{C} : k = 1, 2, \ldots) \}. \] (3.15)

Let
\[ \langle y, z \rangle_N := \sum_{k=1}^{\infty} 2^{N} |y_k z_k|, \quad \forall y, z \in h_N. \]
\[ \| z \|_N = \langle z, z \rangle_N. \] (3.16)

Recall that
\[ \mathcal{H}(\theta, x) \subset C^{N}(\mathbb{T}_m \times [0, \pi], \mathbb{R}). \]

Note that the Fourier transformation (3.7) is isometric from \( u \in \mathcal{H}^{N}[0, \pi] \) to \( (u_k : k = 1, 2, \ldots) \in h_N \), where \( \mathcal{H}^{N}[0, \pi] \) is the usual Sobolev space. By (3.14), we have that
\[ \sup_{\theta \in \mathbb{T}_m} \| \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} R(\theta) \|_{h_{N} \rightarrow h_{N}} \leq C, \] (3.17)
where \( \| \cdot \|_{h_{N} \rightarrow h_{N}} \) is the operator norm from \( h_{N} \) to \( h_{N} \), and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), |\alpha| = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|, \alpha_j \geq 0 \) is an integer.

Actually,
\[ \partial_{\theta}^{\alpha} R(\theta) = \left( \sum_{j=0}^{\infty} C_{j} \partial_{\theta}^{\alpha} v_{lj}(\theta) : l, k = 1, 2, \ldots \right). \]

For any \( z = (z_k \in \mathbb{C} : k = 1, 2, \ldots) \in h_N \),
\[ \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} R(\theta) \right) z = \left( \sum_{|\alpha| = N} \sum_{0 \leq k \leq l} C_{j} \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} v_{lij}(\theta) \right) z_k : l, k = 1, 2, \ldots \right). \]

Thus,
\[ \left\| \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} R(\theta) \right) z \right\|_N^2 \]
\[ = \sum_{l=1}^{\infty} \left| \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} C_{j} \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} v_{lij}(\theta) \right) z_k \right|^2 \]
\[ = \sum_{l=1}^{\infty} \left| \sum_{k=1}^{\infty} C_{k} \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} v_{0lj}(\theta) \right) z_k \right|^2 \]
\[ + \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{j} \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} v_{lj}(\theta) \right) z_k \]
\[ \leq C \sum_{l=1}^{\infty} \left| \sum_{k=1}^{\infty} C_{k} \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} v_{0lj}(\theta) \right) z_k \right|^2 \]
\[ + \sum_{l=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} C_{j} \left( \sum_{|\alpha| = N} \partial_{\theta}^{\alpha} v_{lj}(\theta) \right) z_k \]
Thus, \[
\sum_{l=1}^{\infty} l^{2N} \left( \sum_{|\alpha|=N} \| C_{\alpha l k} \|_{2N} \| v_0(\theta) \|_{2N} \|^2 \right) z_k \leq C \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} v_0(\theta) \|_{2N} \sum_{l=1}^{\infty} l^{2N} |z_l|^2 \leq C \| \| z \|_{N}^2,
\]
here we used the fact that \( C_{\alpha l k} \equiv 0 \) if \( k \neq l \). Let \[
\gamma_{lj} = \frac{(\pm l \pm j)j}{l}, \quad \text{where } (\pm l \pm j)j \neq 0.
\]
Thus,
\[
\left\| \left( \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} JR^\alpha(\theta) \|_2 \right) z \right\|_{N}^2
\]
\[
= \sum_{l=1}^{\infty} l^{2N} \left\| \frac{1}{\gamma_{lj}} \sum_{j=1}^{\infty} C_{lj}(\pm l \pm j) \left( \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} v_j(\theta) \|_{2N} \right) z_k \right\|_{N}^2
\]
\[
= \sum_{l=1}^{\infty} l^{2N} \left\| \frac{1}{\gamma_{lj}} \sum_{j=1}^{\infty} C_{lj}(\pm l \pm j) \left( \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} v_j(\theta) \|_{2N} \right) z_k \right\|_{N}^2
\]
\[
= \sum_{l=1}^{\infty} l^{2N} \left\| \frac{1}{\gamma_{lj}} \sum_{j=1}^{\infty} C_{lj}(\pm l \pm j) \left( \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} v_j(\theta) \|_{2N} \right) z_k \right\|_{N}^2
\]
\[
\leq C \left( \sum_{j=1}^{\infty} \left| \frac{1}{\gamma_{lj}} \right| \sum_{j=1}^{\infty} C_{lj}(\pm l \pm j) \left( \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} v_j(\theta) \|_{2N} \right) \right) \left( \sum_{j=1}^{\infty} \left| \pm l \pm j \right|^{2N} |z_k|^2 \right) \]
\[
\leq C \sum_{j=1}^{\infty} \left| \pm l \pm j \right|^{2N} \left( \sum_{|\alpha|=N} \| \partial^\alpha_{\theta} v_j(\theta) \|_{2N} \right) \| z \|_{N}^2 \leq C \| \| z \|_{N}^2,
\]
where \( C \) is a universal constant which might be different in different places. Combing (3.18), (3.19) and (3.20), we have
\[
\sup_{\theta \in \mathbb{T}^n} \left\| \sum_{|\alpha|=N} \partial^\alpha_{\theta} R(\theta) \right\|_{h_N} \leq C. \quad (3.21)
\]

Now let us apply analytical approximation Lemma to the perturbation \( P(\theta) \). Take a sequence of real numbers \( \{ s_\nu \geq 0 \}_{\nu=0}^{\infty} \) with \( s_\nu > s_{\nu+1} \) goes fast to zero. Let \( R(\theta) = P(\theta) \). Then by (2.6) we can write,
\[
R(\theta) = R_0(\theta) + \sum_{l=1}^{\infty} R_l(\theta), \quad (3.22)
\]
where \( R_0(\theta) \) is analytic in \( \mathbb{T}^n_{s_0} \) with
\[
\sup_{\theta \in \mathbb{T}^n_{s_0}} \| R_0(\theta) \|_{h_N} \leq C, \quad (3.23)
\]
and $R_l(\theta) \ (l \geq 1)$ is analytic in $T^n_{\nu}$ with
\[
\sup_{\theta \in T^n_{\nu}} \| R_l(\theta) \|_{h_N \to h_N} \leq C l^{-1}.
\] (3.24)

3.1. Iterative parameters of domains

Let

\begin{itemize}
  \item $\epsilon_0 = \epsilon, \epsilon_{\nu} = \epsilon^{(4/3)^{1/\nu}}, \nu = 0, 1, 2, \ldots$, which measures the size of perturbation at $\nu$th step.
  \item $\nu_{\nu} = \epsilon^{1/\nu}, \nu = 0, 1, 2, \ldots$, which measures the strip-width of the analytic domain $T^n_{\nu}$, $T^n_{\nu} = \{ \theta \in C^n/(\pi Z)^n : |\text{Im} \theta| \leq \nu_{\nu} \}$.
  \item $C(\nu)$ is a constant which may be different in different places, and it is of the form $C(\nu) = C_1 2^{C_2 \nu}$, where $C_1, C_2$ are absolute constants.
  \item $K_{\nu} = 10 \nu^{-1} (\frac{4}{3})^{1/\nu} |\log \epsilon|$
  \item $\gamma_{\nu} = \frac{\nu}{\nu_{\nu}}, 0 < \gamma \ll 1$.
  \item a family of subsets $\Pi_{\nu} \subset [1, 2]$ with $[1, 2] \supset \Pi_0 \supset \cdots \supset \Pi_{\nu} \supset \cdots$, and $\text{mes} \Pi_{\nu} \geq \text{mes} \Pi_{\nu - 1} - C \gamma_{\nu - 1}$.

\end{itemize}

For an operator-value (or a vector function) $B(\theta, \tau)$, whose domain is $(\theta, \tau) \in T^n_{\nu} \times \Pi_{\nu}$. Set
\[
\| B \|_{T^n_{\nu} \times \Pi_{\nu}} = \sup_{(\theta, \tau) \in T^n_{\nu} \times \Pi_{\nu}} \| B(\theta, \tau) \|_{h_N \to h_N},
\]
where $\| \cdot \|_{h_N \to h_N}$ is the operator norm, and set
\[
\| B \|_{T^n_{\nu} \times \Pi_{\nu}} = \sup_{(\theta, \tau) \in T^n_{\nu} \times \Pi_{\nu}} \| \partial_\tau B(\theta, \tau) \|_{h_N \to h_N}.
\]

3.2. Iterative Lemma

In the following, for a function $f(\omega)$, denote by $\partial_\omega$ the derivative of $f(\omega)$ with respect to $\omega$ in Whitney’s sense.

Lemma 3.1. Let $R_{0,0} = R_0, R_{1,0} = R_1$, where $R_0, R_1$ are defined by (3.22). Assume that we have a family of Hamiltonian functions $H_\nu$:
\[
H_\nu = \sum_{j=1}^{\infty} \lambda_j^{(v)} u_j \overline{\nu} + \sum_{l \geq 0} \epsilon_l (R_{l,v} u, \overline{\nu}), \quad v = 0, 1, \ldots, m,
\] (3.25)
where $R_{l,v}$ is operator-valued function defined on the domain $T^n_{\nu} \times \Pi_\nu$, and $\theta = \omega v$. (3.26)
\[(A1)_\nu\]
\[
\lambda_j^{(0)} = \lambda_j = f^2 + M, \quad \lambda_j^{(v)} = \lambda_j + \sum_{i=0}^{v-1} \epsilon_i \mu_i^{(i)}, \quad v \geq 1
\] (3.27)

and \(\mu_j^{(i)} = \mu_j^{(i)}(\tau) : \Pi_i \to \mathbb{R}\) with
\[
|\mu_j^{(i)}|_{\Pi_i} := \sup_{\tau \in \Pi_i} |\mu_j^{(i)}(\tau)| \leq C(i), \quad 0 \leq i \leq v-1,
\] (3.28)

\[
|\mu_j^{(i)}|_{\Pi_i}^2 := \sup_{\tau \in \Pi_i} |\partial_\tau \mu_j^{(i)}(\tau)| \leq C(i), \quad 0 \leq i \leq v-1.
\] (3.29)

\[(A2)_\nu\] \( R_{l,\nu} = R_{l,\nu}(\theta, \tau) \) is defined in \( T_m^2 \times \Pi \) with \( l \geq v \), and is analytic in \( \theta \) for fixed \( \tau \in \Pi \), and
\[
\|R_{l,\nu}\|_{T_m^2 \times \Pi} \leq C(v),
\] (3.30)

\[
\|R_{l,\nu}\|_{T_m^2 \times \Pi}^2 \leq C(v).
\] (3.31)

Then there exists a compact set \( \Pi_{m+1} \subset \Pi_m \) with
\[
mes \Pi_{m+1} \geq \mes \Pi_m - C_{\Pi_m},
\] (3.32)

and exists a symplectic coordinate changes
\[
\Psi_m : T_{\Pi_m} \times T_{\Pi_m} = T_m^2 \times \Pi_{m+1}
\] (3.33)

\[
||\Psi_m - id||_{H_{\nu} \to H_{\nu}} \leq \epsilon^{1/2}, \quad (\theta, \tau) \in T_{\Pi_m} \times \Pi_{m+1}
\] (3.34)

such that the Hamiltonian function \( H_m \) is changed into
\[
H_{m+1} = H_m \circ \Psi_m
\] (3.35)

which is defined on the domain \( T_{\Pi_{m+1}} \times T_{\Pi_{m+1}} \), and \( \lambda_j^{(m+1)} \)'s satisfy the assumptions \((A1)_{m+1}\) and \( R_{l,m+1} \) satisfy the assumptions \((A2)_{m+1}\).

3.3. Derivation of homological equations

Our end is to find a symplectic transformation \( \Psi_\nu \) such that the terms \( R_{l,\nu} \) (with \( l = v \)) disappear. To this end, let \( F \) be a linear Hamiltonian of the form
\[
F = \langle F(\theta, \tau)u, \Pi \rangle,
\] (3.36)

where \( \theta = \omega t, \quad (F(\theta, \tau))^T = F(\theta, \tau) \). Moreover, let
\[
\Psi = \Psi_m = X^{t}_{\epsilon m}F \big|_{t=1},
\] (3.37)

where \( X^{t}_{\epsilon m} \) is the flow of the Hamiltonian. Vector field \( X^{t}_{\epsilon m} \) of the Hamiltonian \( \epsilon_m F \) with the symplectic structure \( \sqrt{-du \wedge d\Pi} \). Let
\[
H_{m+1} = H_m \circ \Psi_m.
\] (3.38)
By (3.25), we write
\[ H_m = N_m + R_m, \]  
with
\[ N_m = \sum_{j=1}^{\infty} \lambda_j^{(m)} u_j \omega_j, \]
\[ R_m = \sum_{l=m}^{\infty} \epsilon_l R_{lm} = \sum_{l=m}^{\infty} \epsilon_l \langle R_{lm}(\theta)|u, \omega \rangle, \]
where \( (R_{lm}(\theta))^T = R_{lm}(\theta) \). Since the Hamiltonian \( H_m = H_m(\alpha t, u, \omega) \) depends on time \( t \), we introduce a fictitious action \( I \) constant, and let \( \theta = \alpha t \) be angle variable. Then the non-autonomous \( H_m(\alpha t, u, \omega) \) can be written as
\[ \alpha I + H_m(\theta, u, \omega) \]
with symplectic structure \( dI \Lambda d\theta + \sqrt{-1} du d\omega \). By combination of (3.36)-(3.41) and Taylor formula, we have
\begin{align*}
H_{m+1} &= H_m \circ X^1_{e_m} F \\
&= N_m + \epsilon_m \{ N_m, F \} + \epsilon_m^2 \int_0^1 (1 - \tau) \{ \{ N_m, F \}, F \} \circ X^1_{e_m} F d\tau + \epsilon_m \omega \cdot \partial \theta F \\
&\quad + \epsilon_m R_{mm} + \left( \sum_{l=m+1}^{\infty} \epsilon_l R_{lm} \right) \circ X^1_{e_{m+1}} F + \epsilon_m^2 \int_0^1 \{ R_{mm}, F \} \circ X^1_{e_m} F d\tau,
\end{align*}
where \( \{ \cdot, \cdot \} \) is the Poisson bracket with respect to \( \sqrt{-1} du d\omega \), that is
\[ \{ H(u, \omega), F(u, \omega) \} = \frac{\partial H}{\partial u} \frac{\partial F}{\partial u} - \frac{\partial F}{\partial u} \frac{\partial H}{\partial u}. \]
Let \( \Gamma_{K_m} \) be a truncation operator. For any
\[ f(\theta) = \sum_{k \in \mathbb{Z}^m} \tilde{f}(k)e^{i(k, \theta)}, \quad \theta \in \mathbb{T}^m. \]
Define, for given \( K_m > 0 \),
\[ \Gamma_{K_m} f(\theta) = (\Gamma_{K_m} f)(\theta) \triangleq \sum_{|k| \leq K_m} \tilde{f}(k)e^{i(k, \theta)}, \]
\[ (1 - \Gamma_{K_m}) f(\theta) = ((1 - \Gamma_{K_m}) f)(\theta) \triangleq \sum_{|k| > K_m} \tilde{f}(k)e^{i(k, \theta)}. \]
Then
\[ f(\theta) = \Gamma_{K_m} f(\theta) + (1 - \Gamma_{K_m}) f(\theta). \]
Let
\[ \omega \cdot \partial \theta F + \{ N_m, F \} + \Gamma_{K_m} R_{mm} = \langle [R_{mm}]|u, \omega \rangle, \]
where
\[ [R_{mm}] := \text{diag} \left( \bar{R}_{mm} \right)_{j=1,2,\ldots} \].
and $R_{nnij}(\theta)$ is the matrix element of $R_{nn}$ and $\tilde{R}_{nnij}(k)$ is the $k$-Fourier coefficient of $R_{nnij}(\theta)$. Then
\[ H_{m+1} = N_{m+1} + C_{m+1}R_{m+1}, \] (3.45)
where
\[ N_{m+1} = N_m + \varepsilon_m ([R_{nn} u, \vec{\pi}] = \sum_{j=1}^{m} \lambda_j^{(m+1)} u_j \vec{\pi}_j, \] (3.46)
\[ \lambda_j^{(m+1)} = \lambda_j^{(m)} + \varepsilon_m \tilde{R}_{nnij}(0) = \lambda_j + \sum_{l=1}^{m} e_l \mu_{j}^{(f)}, \mu_j^{(m)} := \tilde{R}_{nnij}(0). \] (3.47)

\[ C_{m+1}R_{m+1} = \varepsilon_m (1 - \Gamma_{K_{mn}}) R_{nn} \]
\[ + \varepsilon_m^2 \int_0^1 (1 - \tau) \{ [N_m, F], F \} \circ X^\tau_{\tau, F} d\tau \]
\[ + \varepsilon_m^2 \int_0^1 [R_{nn}, F] \circ X^\tau_{\tau, F} d\tau \]
\[ + \left( \sum_{l=m+1}^{\infty} e_l R_{im} \right) \circ X^\tau_{\tau, F}. \] (3.51)

The equation (3.43) is called homological equation. Developing the Poisson bracket $\{N_m, F\}$ and comparing the coefficients of $u_i \vec{\pi}_j (i, j = 1, 2, \ldots)$, we get
\[ \omega \cdot \partial_\theta F(\theta, \tau) + \sqrt{-1} \{ F(\theta, \tau) \Lambda^{(m)} - \Lambda^{(m)} F(\theta, \tau) \} = \Gamma_{K_{mn}} R_{nn}(\theta) - [R_{nn}], \] (3.52)
where
\[ \Lambda^{(m)} = \text{diag} (\lambda_j^{(m)} : j = 1, 2, \ldots), \] (3.53)
and we assume $\Gamma_{K_{mn}} F(\theta, \tau) = F(\theta, \tau)$. Write $F_{ij}(\theta)$ is the matrix elements of $F(\theta, \tau)$. Then (3.52) can be rewritten as:
\[ \omega \cdot \partial_\theta F_{ij}(\theta) - \sqrt{-1} (\lambda_j^{(m)} - \lambda_j^{(m)}) F_{ij}(\theta) = \Gamma_{K_{mn}} R_{nnij}(\theta), \quad i \neq j, \] (3.54)
\[ \omega \cdot \partial_\theta F_{ii}(\theta) = \Gamma_{K_{mn}} R_{nnii}(\theta) - \tilde{R}_{nnii}(0), \] (3.55)
where $i, j = 1, 2, \ldots$.

3.4. Solutions of the homological equations

Lemma 3.2. There exists a compact subset $\Pi_{m+1} \subset \Pi_m$ with
\[ \text{mes} (\Pi_{m+1}) \geq \text{mes} \Pi_m - C \gamma_m \] (3.56)
such that for any $\tau \in \Pi_{m+1}$ (Recall $\omega = \tau \partial_\theta$), the equation (3.54) has a unique solution $F(\theta, \tau)$, which is defined on the domain $^\tau_{\tau_{m+1}} \times \Pi_{m+1}$, with
\[ \|F(\theta, \tau)\|^\tau_{\tau_{m+1} \times \Pi_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{2(2m+3)}{N}}, \] (3.57)
\[ \|F(\theta, \tau)\|^\tau_{\tau_{m+1} \times \Pi_{m+1}} \leq C(m+1)\varepsilon_m^{-\frac{4(2m+3)}{N}}. \] (3.58)
Proof. By passing to Fourier coefficients, (3.54) can be rewritten as
\[
(\langle k, \omega \rangle + \lambda_i^{(m)} - \lambda_j^{(m)})^2 F_{ij}(k) = \sqrt{-1} \tilde{R}_{mnij}(k),
\]
where \(i, j = 1, 2, \ldots, k \in \mathbb{Z}^n \) with \(|k| \leq K_m\). Recall \( \omega = \tau \omega_0 \). Let
\[
A_k = |k|^{n+3},
\]
and let
\[
Q_{ki j}^{(m)} = \left\{ \tau \in \Pi_m \mid |\langle k, \omega \rangle + \lambda_i^{(m)} - \lambda_j^{(m)}| < \frac{(|i - j| + 1)\gamma_m}{A_k} \right\},
\]
where \(i, j = 1, 2, \ldots, k \in \mathbb{Z}^n \) with \(|k| \leq K_m\), and \(k \neq 0\) when \(i = j\). Let
\[
\Pi_{m+1} = \Pi_m \setminus \bigcup_{|k| \leq K_m} \bigcup_{i \neq j} Q_{ki j}^{(m)}.
\]
Then for any \(\tau \in \Pi_{m+1}\), we have
\[
|\langle k, \omega \rangle + \lambda_i^{(m)} - \lambda_j^{(m)}| \geq \frac{(|i - j| + 1)\gamma_m}{A_k}.
\]
Recall that \(R_{mn}(\theta)\) is analytic in the domain \(\mathbb{T}^n_{\gamma_m}\) for any \(\tau \in \Pi_m\),
\[
|R_{mnij}(k)| \leq C(m) e^{-\gamma_m|k|}.
\]
It follows
\[
|\tilde{F}_{ij}(k)| = \frac{|\tilde{R}_{mnij}(k)|}{|\langle k, \omega \rangle + \lambda_i^{(m)} - \lambda_j^{(m)}|} \leq \frac{A_k}{\gamma_m(|i - j| + 1)} \cdot |\tilde{R}_{mnij}(k)|
\]
\[
\leq \frac{|k|^{n+3}}{\gamma_m(|i - j| + 1)} \cdot C(m) e^{-\gamma_m|k|}.
\]
Therefore, by (3.63), we have
\[
\sup_{\theta \in \mathbb{T}^n_{\gamma_m} \times \Pi_{m+1}} |F_{ij}(\theta, \tau)|
\]
\[
\leq \frac{C(m)}{\gamma_m(|i - j| + 1)} \sum_{|k| \leq K_m} |k|^{n+3} e^{-(s_m - \gamma_m)|k|}
\]
\[
\leq \left( \frac{n + 3}{e} \right)^{n+3} (1 + e) \left( \frac{2}{s_m - s_m'} \right)^{2n+3} \frac{C(m)}{\gamma_m(|i - j| + 1)} \text{ (by Lemma 7.2)}
\]
\[
\leq \frac{1}{s_m - s_m'} \cdot \frac{C \cdot C(m)}{\gamma_m(|i - j| + 1)}
\]
\[
\leq \frac{C \cdot C(m)}{\gamma_m(|i - j| + 1)}
\]
where \(C\) is a constant depending on \(n\), \(s_m' = s_m - \frac{s_m - s_m}{4}\).
By Lemma 3.2 and the Remark 2, we have

$$\| F(\theta, \tau) \|_{\mathcal{V}_{m+1}^\tau} \leq C \cdot C(m) \gamma_m^{-1} \epsilon_m^{-\frac{2(2m+3)}{N}} \leq C(m+1) \epsilon_{m}^{-\frac{2(2m+3)}{N}}.$$  \hfill (3.64)

It follows $s_m' > s_{m+1}$ that

$$\| F(\theta, \tau) \|_{\mathcal{V}_{m+1}^\tau} \leq \| F(\theta, \tau) \|_{\mathcal{V}_{m}^\tau} \leq C(m+1) \epsilon_{m}^{-\frac{2(2m+3)}{N}}.$$  \hfill (3.65)

Applying $\partial_t$ to both sides of (3.59), we have

$$\left( -\langle k, \omega \rangle + \lambda^{(m)} - \lambda^{(m)} \right) \partial_t \hat{F}_{ij}(k) = \sqrt{-1} \partial_t \hat{R}_{nnij}^{(k)}(\check{\tau} + \epsilon_m) + (\ast).$$  \hfill (3.66)

Recalling $|k| \leq K_m = 10 \gamma_m^{-1} (\frac{1}{4}) |\log \epsilon_m|$, and using (3.67) and (3.68) with $v = m$, and using (3.69), we have, on $\tau \in \Pi_{m+1}$,

$$|\partial_t \hat{F}_{ij}(k) | \leq C(m) e^{-s_m |k|}.$$  \hfill (3.67)

According to (3.61), (3.65), (3.67) and (3.68), we have

$$|\partial_t \hat{F}_{ij}(k) | \leq \frac{A_k}{\gamma_m(|i-j| + 1)} \cdot C \cdot C(m) \gamma_m^{-1} \epsilon_m^{-\frac{2(2m+3)}{N}} e^{-s_m |k|} \text{ for } i \neq j.$$  \hfill (3.69)

Note that $s_m > s_m' > s_{m+1}$. Again using Lemma 2.2 and Lemma 2.3, we have

$$\| F(\theta, \tau) \|_{\mathcal{V}_{m+1}^\tau} = \| \partial_t F(\theta, \tau) \|_{\mathcal{V}_{m+1}^\tau} \leq C^2 \cdot C(m) \gamma_m^{-2} \epsilon_m^{-\frac{4(2m+3)}{N}} \leq C(m+1) \epsilon_{m}^{-\frac{4(2m+3)}{N}}.$$  \hfill (3.70)

The proof of the measure estimate (3.66) will be postponed to section 3.7. This completes the proof of Lemma 3.2.

### 3.5. Coordinate change $\Psi$ by $e_m F$

Recall $\Psi_m = \Psi_{e_m F} |_{\tau=1}$, where $\Psi_{e_m F}$ is the flow of the Hamiltonian $e_m F$, vector field $X_{e_m F}$ with symplectic $\sqrt{-\text{Id} du \wedge \Pi}$. So

$$\sqrt{-\text{Id} u} = e_m \frac{\partial F}{\partial \Pi} - \sqrt{-\text{Id} \Pi} = e_m \frac{\partial F}{\partial \Pi} \hat{\theta} = \omega.$$  \hfill (3.71)

More exactly,

$$\begin{cases} \sqrt{-\text{Id} u} = e_m F(\theta, \tau) u, \ \theta = \omega \tau, \\ -\sqrt{-\text{Id} \Pi} = e_m F(\theta, \tau) \Pi, \ \theta = \omega \tau, \\ \hat{\theta} = \omega. \end{cases}$$  \hfill (3.72)
Let \( z = \left( \begin{array}{c} u \\ \Pi \end{array} \right) \),

\[
B_m = \left( \begin{array}{cc}
-\sqrt{-1}F(\theta, \tau) & 0 \\
0 & \sqrt{-1}F(\theta, \tau)
\end{array} \right).
\]

Recall \( \theta = \omega t \). (3.71)

Then
\[
\frac{dz(t)}{dt} = \varepsilon_m B_m(\theta) z, \quad \dot{\theta} = \omega.
\]

(3.72)

Let \( z(0) = z_0 \in h_N \times h_N, \theta(0) = \theta_0 \in T_{m+1}^{\eta} \) be initial value. Then
\[
\begin{cases}
\dot{z}(t) = z_0 + \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)z(s)ds, \\
\theta(t) = \theta_0 + \omega t.
\end{cases}
\]

(3.73)

By Lemmas 3.2 in Section 7,
\[
\|B_m(\theta)\|_{\mathcal{T}_{m+1}^\eta \times \Pi_{m+1}^\eta} \leq C(m+1)\varepsilon_m \frac{2(2m+3)}{\eta^3},
\]

(3.74)

\[
\|B_m(\theta)\|_{\mathcal{T}_{m+1}^\eta \times \Pi_{m+1}^\eta} \leq C(m+1)\varepsilon_m \frac{4(2m+3)}{\eta^3}.
\]

(3.75)

It follows from (3.73) that
\[
z(t) - z_0 = \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)z_0 ds + \int_0^t \varepsilon_m B_m(\theta_0 + \omega s)(z(s) - z_0) ds.
\]

Moreover, for \( t \in [0, 1], \|z_0\|_N \leq 1,
\[
\|z(t) - z_0\|_N \leq \varepsilon_m C(m+1)\varepsilon_m \frac{4(2m+3)}{\eta^3} + \int_0^t \varepsilon_m \|B_m(\theta_0 + \omega s)\| \|z(s) - z_0\|_N ds,
\]

(3.76)

where \( \|\cdot\| \) is the operator norm from \( h_N \times h_N \to h_N \times h_N \).

By Gronwall’s inequality,
\[
\|z(t) - z_0\|_N \leq C(m+1)\varepsilon_m \frac{4(2m+3)}{\eta^3} \cdot \exp \left( \int_0^t \varepsilon_m \|B_m(\theta_0 + \omega s)\| ds \right) \leq \varepsilon_m^{1/2}.
\]

(3.77)

Thus,
\[
\Psi^m_{m+1} \colon \mathcal{T}_{m+1}^\eta \times \Pi_{m+1} \rightarrow \mathcal{T}_m^\eta \times \Pi_m,
\]

(3.78)

and
\[
\|\Psi^m - id\|_{h_N \rightarrow h_N} \leq \varepsilon_m^{1/2}.
\]

(3.79)

Since (3.72) is linear, so \( \Psi^m_{m+1} \) is linear coordinate change. According to (3.73), construct Picard sequence:
\[
\begin{cases}
z_0(t) = z_0, \\
z_{j+1}(t) = z_0 + \int_0^t \varepsilon_m B(\theta_0 + \omega s)z_j(s)ds, \quad j = 0, 1, 2, \ldots.
\end{cases}
\]

By (3.79), this sequence with \( t = 1 \) goes to
\[
\Psi^m(z_0) = z(1) = (id + P_m(\theta_0))z_0.
\]

(3.80)
where \( id \) is the identity from \( h_N \times h_N \to h_N \times h_N \), and \( P(\theta_0) \) is an operator form \( h_N \times h_N \to h_N \times h_N \) for any fixed \( \theta_0 \in T_{sm+1}^n, \tau \in \Pi_{m+1}^n \), and is analytic in \( \theta_0 \in T_{sm+1}^n \), with
\[
\| P_m(\theta_0) \|_{T_{sm+1}^n \times \Pi_{m+1}^n} \leq \varepsilon_m^{1/2}.
\] (3.81)

Note that (3.72) is a Hamiltonian system. So \( P_m(\theta_0) \) is a symplectic linear operator from \( h_N \times h_N \) to \( h_N \times h_N \).

### 3.6. Estimates of remainders

The aim of this section is devoted to estimate the remainders:
\[
R_{m+1} = (3.48) + \ldots + (3.51).
\]

- **Estimate of (3.48).**
  
  By (3.41), let
  \[
  \tilde{R}_{mm} = \tilde{R}_{mm}(\theta) = \left(\begin{array}{cc}0 & \frac{1}{2} R_{m,m}(\theta) \\ \frac{1}{2} R_{m,m}(\theta) & 0\end{array}\right),
  \]
  then
  \[
  R_{mm} = \langle \tilde{R}_{mm} \left(\begin{array}{c}u \\ \tau\end{array}\right), \left(\begin{array}{c}u \\ \tau\end{array}\right)\rangle.
  \]
  So
  \[
  (1 - \Gamma_{K_m}) R_{mm} \triangleq \langle (1 - \Gamma_{K_m}) \tilde{R}_{mm} \left(\begin{array}{c}u \\ \tau\end{array}\right), \left(\begin{array}{c}u \\ \tau\end{array}\right)\rangle.
  \]
  By the definition of truncation operator \( \Gamma_{K_m} \),
  \[
  (1 - \Gamma_{K_m}) \tilde{R}_{mm} = \sum_{|k| > K_m} \tilde{R}_{mm}(k)e^{i \langle k, \theta \rangle}, \quad \theta \in T_{sm}^n, \quad \tau \in \Pi_m.
  \]
  Since \( \tilde{R}_{mm} = \tilde{R}_{mm}(\theta) \) is analytic in \( \theta \in T_{sm}^n \),
  \[
  \sup_{(\theta, \tau) \in T_{sm+1}^n \times \Pi_{m+1}^n} \| (1 - \Gamma_{K_m}) \tilde{R}_{mm} \|_{L_N \to N_N} \leq \sum_{|k| > K_m} \| \tilde{R}_{mm}(k) \|_{L_N} e^{2|k|_{sm+1}} \leq \frac{C^2(m)e^{-2K_m(sm+1)/|k|}}{\varepsilon_m} \leq \frac{C^2(m)e^{2}_m}{\varepsilon_m}.
  \]

That is,
\[
\| (1 - \Gamma_{K_m}) \tilde{R}_{mm} \|_{T_{sm+1}^n \times \Pi_{m+1}^n} \leq \varepsilon_m C(m).
\]
Thus,
\[
\| \varepsilon_m (1 - \Gamma_{K_m}) \tilde{R}_{mm} \|_{T_{sm+1}^n \times \Pi_{m+1}^n} \leq \varepsilon_m^2 C(m) \leq \varepsilon_{m+1} C(m+1).
\] (3.82)


• Estimate of (3.50).

Let
\[ S_m = \begin{pmatrix} 0 & \frac{1}{2}F(\theta, \tau) \\ \frac{1}{2}F(\theta, \tau) & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & -\sqrt{-1}id \\ \sqrt{-1}id & 0 \end{pmatrix}. \]

Then we can write
\[ F = (S_m(\theta) \begin{pmatrix} u \\ \frac{1}{2}F(\theta, \tau) \end{pmatrix}, \begin{pmatrix} u \\ \frac{1}{2}F(\theta, \tau) \end{pmatrix}) = (S_m \xi_c, \zeta) = \begin{pmatrix} u \\ \frac{1}{2}F(\theta, \tau) \end{pmatrix}. \]

Then
\[ \varepsilon_m^2 \{R_{mm}, F\} = 4 \varepsilon_m^2 \langle \tilde{R}_{mm}(\theta) \rangle S_m(\theta) u, u \rangle. \tag{3.83} \]

Noting \( T_{m_m}^n \times \Pi_m \supset T_{m+1}^n \times \Pi_{m+1} \). By (3.30) and (3.31) with \( l = m, n = m, \)
\[ \| \tilde{R}_{mm}(\theta) \|_{T_{m+1}^n \times \Pi_{m+1}} \leq \| \tilde{R}_{mm}(\theta) \|_{T_{m}^n \times \Pi_{m}} \leq C(m), \tag{3.84} \]
\[ \| \tilde{R}_{mm}(\theta) \|_{\mathcal{F}_{T_{m+1}^n \times \Pi_{m+1}}} \leq C(m). \tag{3.85} \]

Let \( \tilde{S}_m(\theta) = \mathcal{J} S_m(\theta) \). Then by Lemmas 3.2 in Section 7, we have
\[ \| \tilde{S}_m(\theta) \|_{T_{m+1}^n \times \Pi_{m+1}} \leq C(m+1) \varepsilon_m^{-2/(2n+3)}, \tag{3.86} \]
\[ \| \tilde{S}_m(\theta) \|_{\mathcal{F}_{T_{m+1}^n \times \Pi_{m+1}}} \leq C(m+1) \varepsilon_m^{4/(2n+3)} \tag{3.87} \]

and
\[ \| \tilde{R}_{mm} S_m \|_{T_{m+1}^n \times \Pi_{m+1}} = \| \tilde{R}_{mm} \tilde{S}_m \|_{T_{m+1}^n \times \Pi_{m+1}} \leq C(m)C(m+1) \varepsilon_m^{-2/(2n+3)}. \tag{3.88} \]

Note that the vector field is linear. So, by Taylor formula, one has
\[ (3.50) = \varepsilon_m^2 \langle \tilde{R}_m^{*}(\theta) u, u \rangle, \]
where
\[ \tilde{R}_m^{*}(\theta) = \sum_{j=1}^{\infty} \frac{4^j \varepsilon_m^{j-1}}{j!} \tilde{R}_{mm} \tilde{S}_{m} \cdots \tilde{S}_{m}. \]

By (3.84) and (3.86),
\[ \| \tilde{R}_m^{*}(\theta) \|_{T_{m+1}^n \times \Pi_{m+1}} \leq \sum_{j=1}^{\infty} C(m)C(m+1) \varepsilon_m^{j-1} \varepsilon_m^{-2/(2n+3)} \frac{j!}{j!} \leq C(m)C(m+1) \varepsilon_m^{-2/(2n+3)}. \]

By (3.85) and (3.87),
\[ \| \tilde{R}_m^{*}(\theta) \|_{\mathcal{F}_{T_{m+1}^n \times \Pi_{m+1}}} \leq C(m)C(m+1) \varepsilon_m^{-4/(2n+3)}. \]
Thus,

\[
\|e_m^2 \tilde{R}_m\|_{\mathcal{T}_m} \leq C(m)C(m + 1)\varepsilon_m^2 \frac{2(2n+3)}{\nu} \leq C(m + 1)\varepsilon_{m+1}, \tag{3.89}
\]

and

\[
\|e_m^2 \tilde{R}_m\|_{\mathcal{T}_m} \leq C(m)C(m + 1)\varepsilon_m^2 \frac{4(2n+3)}{\nu} \leq C(m + 1)\varepsilon_{m+1}. \tag{3.90}
\]

- Estimate of \(3.49\)

By (3.43),

\[
\left\{ N_m, F \right\} = \langle [R_{nm}]u, \mathfrak{F} \rangle - H_{nm} - \omega \cdot \partial_\theta F \Delta R_{nm}. \tag{3.91}
\]

Thus,

\[
3.49 = e_m^2 \int_0^1 (1 - \tau) \{ R_{nm}^*F \} \circ X_{\varepsilon_m}^\tau d\tau. \tag{3.92}
\]

Note \(R_{nm}^*\) is a quadratic polynomial in \(u\) and \(\mathfrak{F}\). So we write

\[
R_{nm}^* = \langle \mathcal{R}_m(\theta, \tau)z, z \rangle, \quad z = \left( \begin{array}{c} u \\ \mathfrak{F} \end{array} \right). \tag{3.93}
\]

By (3.28) and (3.29) with \(l = \nu = m\), and using (3.86) and (3.87),

\[
\| \mathcal{R}_m \|_{\mathcal{T}_m} \leq C(m)\varepsilon_m \frac{2(2n+3)}{\nu}, \quad \| \mathcal{R}_m \|_{\mathcal{T}_m} \leq C(m)\varepsilon_m \frac{4(2n+3)}{\nu}, \tag{3.94}
\]

where \(\| \cdot \|\) is the operator norm in \(h_N \times h_N \to h_N \times h_N\). Recall \(F = \langle S_m(\theta, \tau)u, u \rangle\). Set

\[
\| \mathcal{R}_m \langle S_m \rangle = 2 \mathcal{R}_m \langle S_m \rangle = 2 \mathcal{R}_m \mathcal{J} S_m. \tag{3.95}
\]

Using Taylor formula to (3.92), we get

\[
3.49 = \frac{e_m^2}{2!} \left\{ \{ R_{nm}^*F \}, F \right\} + \ldots + \frac{e_m^j}{j!} \left\{ \ldots \{ R_{nm}^*F \}, \ldots \right\} + \ldots 
\]

\[
= \left\langle \left( \sum_{j=2}^{\infty} \frac{e_m^j}{j!} \left\{ \ldots \{ \mathcal{R}_m, \tilde{S}_m \}, \ldots \right\} \right) u, u \right\rangle 
\]

\[
\Delta = (\mathcal{R}^{*+}(\theta, \tau)u, u). \tag{3.96}
\]

By (3.86), (3.94) and (3.95), we have

\[
\| \mathcal{R}^{*+}(\theta, \tau) \|_{\mathcal{T}_m} \leq \sum_{j=2}^{\infty} \frac{1}{j!} \| \mathcal{R}_m(\theta, \tau) \|_{\mathcal{T}_m} \leq \sum_{j=2}^{\infty} \frac{C(m)}{j!} \left( C(m + 1)\varepsilon_m^{\frac{2(2n+3)}{\nu}} \right)^j 
\]

\[
\leq C(m + 1)\varepsilon_m^{4/3} = C(m + 1)\varepsilon_{m+1}. \tag{3.97}
\]

Similarly,

\[
\| \mathcal{R}^{*+} \|_{\mathcal{T}_m} \leq C(m + 1)\varepsilon_{m+1}. \tag{3.98}
\]
Estimate of (3.51)

\[ (3.51) = \sum_{l=m+1}^{\infty} \epsilon_l (R_{lm} \circ X_{lm}^1). \]  

Write

\[ R_{lm} = \langle \tilde{R}_{lm} (\theta) u, u \rangle. \]

Then, by Taylor formula:

\[ R_{lm} \circ X_{lm}^1 = R_{lm} + \sum_{j=1}^{\infty} \frac{1}{j!} (\tilde{R}_{lmj} u, u), \]

where

\[ \tilde{R}_{lmj} = 4^j \tilde{R}_{lm}(\theta) \underbrace{S_m(\theta) \ldots S_m(\theta)}_{j\text{-fold}} \varepsilon_m. \]

By (3.30), (3.31), \[ \| \tilde{R}_{lm} \|_{T_n^m \times \Pi_m} \leq C(l), \quad \| \tilde{R}_{lm} \|_{T_n^m \times \Pi_m} \leq C(l). \]

Combining the last inequalities with (3.86) and (3.87), one has

\[ \| \tilde{R}_{lmj} \|_{T_n^m \times \Pi_m} \leq C^2(m) (\epsilon_m \epsilon_m^{-\frac{1}{2n+3}})^j, \]

and

\[ \| \tilde{R}_{lmj} \|_{T_n^m \times \Pi_m} \leq C^2(m) (\epsilon_m \epsilon_m^{-\frac{1}{2n+3}})^j. \]

Thus, let

\[ \tilde{R}_{l,m+1} := R_{lm} + \sum_{j=1}^{\infty} \frac{1}{j!} \tilde{R}_{lmj}, \]

then

\[ (3.51) = \sum_{l=m+1}^{\infty} \epsilon_l \langle \tilde{R}_{l,m+1} u, u \rangle \]

and

\[ \| \tilde{R}_{l,m+1} \|_{T_n^m \times \Pi_m} \leq C^2(m) \leq C(m+1), \quad \| \tilde{R}_{l,m+1} \|_{T_n^m \times \Pi_m} \leq C^2(m) \leq C(m+1). \]
As a whole, the remainder \( R_{m+1} \) can be written as
\[
C_{m+1}R_{m+1} = \sum_{l=m+1}^{\infty} \varepsilon_l \langle R_l, \psi \rangle u_1 \Pi_l, \quad \nu = m + 1,
\]
where \( R_l \) satisfies (3.30) and (3.31) with \( \nu = m + 1, l \geq m + 1 \). This shows that the Assumption \((A2)_\nu\) with \( \nu = m + 1 \) holds true.

By (3.47),
\[
\mu_j^{(m)} = \frac{\varepsilon_l}{\Pi_l}(0).
\]

In (3.30) and (3.31), we have
\[
|\mu_j^{(m)}|_{\Pi_m} \leq |R_{mmjj}(\theta, \tau)| \leq C(m),
\]
\[
|\mu_j^{(m)}|_{\Pi_m} \leq |\partial_{\nu} R_{mmjj}(\theta, \tau)| \leq C(m).
\]

This shows that the Assumption \((A1)_\nu\) with \( \nu = m + 1 \) holds true.

### 3.7. Estimate of measure

In this section, \( C \) denotes a universal constant, which may be different in different places. Now let us return to (3.60)
\[
Q_{ki}^{(m)} \triangleq \left\{ \gamma \in \Pi_m \left| -\langle k, \omega \rangle \tau + \lambda_i^{(m)} - \lambda_j^{(m)} \right| < \frac{(i-j+1)\gamma m}{A_k} \right\}. \tag{3.101}
\]

**Case 1.** If \( i = j \), one has \( k \neq 0 \).

At this time,
\[
Q_{kii}^{(m)} = \left\{ \gamma \in \Pi_m \left| \langle k, \omega \rangle \tau < \frac{\gamma m}{A_k} \right\} \right\}. \tag{3.102}
\]

It follows
\[
|\langle k, \omega \rangle| < \frac{\gamma m}{|k|^{n+1}} \tau < \frac{\gamma}{2^{|k|^{n+1}}}. \tag{3.103}
\]

Recall \( |\langle k, \omega \rangle| > \frac{\gamma}{|k|^{n+1}} \). Then
\[
\text{mes}Q_{kii}^{(m)} = 0. \tag{3.103}
\]

**Case 2.** \( i \neq j \).

If \( Q_{kij}^{(m)} = \varnothing \), then \( \text{mes}Q_{kij}^{(m)} = 0 \). So we assume \( Q_{kij}^{(m)} \neq \varnothing \), in the following. Then \( \exists \tau \in \Pi_m \) such that
\[
-\langle k, \omega \rangle \tau + \lambda_i^{(m)} - \lambda_j^{(m)} < \frac{(i-j+1)\gamma m}{A_k}. \tag{3.104}
\]

It follows from (3.27) and (3.28) that
\[
\lambda_i^{(m)} - \lambda_j^{(m)} = i^2 - j^2 + O(\varepsilon_0) \geq \frac{2}{3}|i^2 - j^2|. \tag{3.105}
\]

When \( |i| \geq C|k| \gg |\langle k, \omega \rangle| \) or \( |j| \geq C|k| \gg |\langle k, \omega \rangle| \). By (3.105), one has
\[
|\langle k, \omega \rangle + \lambda_i^{(m)} - \lambda_j^{(m)}| \geq \frac{2}{3}|i+j| |i-j| - |\langle k, \omega \rangle| \geq \frac{1}{2}|i+j| |i-j|.
\]
which implies $Q_{kij}^{(m)} = \emptyset$, then

$$mesQ_{kij}^{(m)} = 0. \tag{3.106}$$

Now assume $|i| < C|k|$ and $|j| < C|k|$.

Note that

$$-(k, \omega) + i^2 - j^2 = -(k, \omega_0) + \frac{i^2 - j^2}{\tau}$$

and

$$\left| \frac{d}{d\tau} \left( -(k, \omega_0) + \frac{i^2 - j^2}{\tau} \right) \right| = \frac{|i^2 - j^2|}{\tau^2} \geq \frac{1}{4} |i^2 - j^2|. \tag{3.107}$$

It follows that

$$mesQ_{kij}^{(m)} \leq \frac{8}{|i^2 - j^2|} \left( \frac{|i^2 - j^2| + 1}{A_k} \gamma_m + C_1 \epsilon_0 \right). \tag{3.108}$$

Then

$$mes \bigcup_{|k| \leq K_m} \bigcup_{i \leq C|k|} \bigcup_{j \leq C|k|} Q_{kij}^{(m)} \leq \sum_{|k| \leq K_m} C\gamma_m \sum_{i=1}^{C|k|} \sum_{j=1}^{i+j} \frac{1}{A_k} \leq \sum_{|k| \leq K_m} \frac{C|k|^2 \gamma_m}{A_k} \leq C\gamma_m. \tag{3.109}$$

Combining (3.103), (3.106) and (3.109), we have

$$mes \bigcup_{|k| \leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{kij}^{(m)} \leq C\gamma_m. \tag{3.110}$$

Let

$$\Pi_{m+1} = \Pi_m \setminus \bigcup_{|k| \leq K_m} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} Q_{kij}^{(m)}.$$

Then we have proved the following Lemma 3.3.

**Lemma 3.3.**

$$mes\Pi_{m+1} \geq mes\Pi_m - C\gamma_m.$$

4. **Proof of Theorems**

Let

$$\Pi_\infty = \bigcap_{m=1}^{\infty} \Pi_m,$$

and

$$\Psi_\infty = \lim_{m \to \infty} \Psi_0 \circ \Psi_1 \circ \cdots \circ \Psi_m.$$
By (3.33) and (3.34), one has
\[ \Psi_\infty : T^n \times \Pi_\infty \to T^n \times \Pi_\infty, \]
\[ ||\Psi_\infty - id|| \leq \epsilon_0^{1/2}, \]
and, by (3.35),
\[ H_\infty = H \circ \Psi_\infty = \sum_{j=1}^{\infty} \lambda_j^\infty Z_j \mathbf{Z}_j, \]
where
\[ \lambda_j^\infty = \lim_{m \to \infty} \lambda_j^{(m)}. \]

By (3.27) and (3.28), the limit \( \lambda_j^\infty \) does exist and
\[ \lambda_j^\infty = j^2 + M + O(\epsilon_0). \]

This completes the proof of Theorem 1.1.

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