Three Point Functions in the $\mathcal{N} = 4$ Orthogonal Coset Theory

Changhyun Ahn, Hyunsu Kim and Jinsub Paeng

Department of Physics, Kyungpook National University, Taegu 41566, Korea

ahn, kimhyun, jdp2r@knu.ac.kr

Abstract

We construct the lowest higher spin-2 current in terms of the spin-1 and the spin-$\frac{1}{2}$ currents living in the orthogonal $\frac{SO(N+4)}{SO(N) \times SO(4)}$ Wolf space coset theory for general $N$. The remaining fifteen higher spin currents are determined. We obtain the three-point functions of bosonic (higher) spin currents with two scalars for finite $N$ and $k$ (the level of the spin-1 current). By multiplying $SU(2) \times U(1)$ into the above Wolf space coset theory, the other fifteen higher spin currents together with the above lowest higher spin-2 current are realized in the extension of the large $\mathcal{N} = 4$ linear superconformal algebra. Similarly, the three-point functions of bosonic (higher) spin currents with two scalars for finite $N$ and $k$ are obtained. Under the large $N$ 't Hooft limit, the two types of three-point functions in the nonlinear and linear versions coincide as in the unitary coset theory found previously.
1 Introduction

By analyzing the zero-mode eigenvalue equations for the bosonic (higher spin) currents in the extension of the large $\mathcal{N} = 4$ (non)linear superconformal algebra, its three-point functions with two scalars [1] were obtained in the context of the large $\mathcal{N} = 4$ holography [2]. Even though the corresponding three-point functions in the nonlinear and linear versions are different from each other for finite $N$ and $k$, where these two parameters characterize the $\mathcal{N} = 4$ unitary coset theory (or they correspond to two levels of the above large $\mathcal{N} = 4$ (non)linear superconformal algebra), they coincide under the large $N$ 't Hooft limit. For example, the central charge in the large $\mathcal{N} = 4$ linear superconformal algebra [2] is given by $c = 6(1 - \lambda)(N + 1)$, where the $\lambda$ is the 't Hooft coupling constant ($0 < \lambda < 1$). For fixed $\lambda$, the large $N$ 't Hooft limit is equivalent to the large $c$ limit. Note that the central charge in the nonlinear version is reduced by 3. As long as the three-point functions under the large $N$ 't Hooft limit are concerned, the higher-order effects (or subleading orders) of $1/c$ is not important, for example, in the study of marginal deformation in the Higgs phenomenon (in the context of other holographic model) [3, 4] because the leading order of $1/c$ is taken. However, we should observe the finite $N$-effect in order to see the quantum behavior (or subleading orders of $1/c$) in this large $\mathcal{N} = 4$ holography [2] (or above other holographic model).

It is natural, as raised in [1], to consider the other type of coset theory in order to observe the consistency check in the other type of large $\mathcal{N} = 4$ holography. In [5], the 16 lowest higher spin currents (one higher spin-2 current, four higher spin-$\frac{5}{2}$ currents, six higher spin-3 currents, four higher spin-$\frac{7}{2}$ currents and one higher spin-4 current) in the extension of large $\mathcal{N} = 4$ nonlinear superconformal algebra were constructed in the orthogonal coset theory for fixed $N = 4$ (and for general $k$). What is so special to the orthogonal coset theory compared to the unitary coset theory? One of the findings in [5] was that the lowest higher spin current in the $\mathcal{N} = 4$ multiplet has spin 2 and this implies that the highest higher spin current has spin 4 as above. Then we expect that we will obtain the three-point functions for the higher spin-4 current. Note that for the unitary coset theory the corresponding three-point functions were obtained for the (higher spin) currents of spins $s = 2, 3$. We did not calculate the three-point functions of spins $s$ greater than 3. We can expect the spin-dependence for the three-point functions in the unitary coset theory under the large $N$ 't Hooft limit from the results of the orthogonal coset theory because we expect that they share the common spin-behavior. Furthermore, the six higher spin-3 currents in the orthogonal coset theory transform as the adjoint of $SO(\mathcal{N} = 4)$ (we are considering the $SO(\mathcal{N} = 4)$ singlet $\mathcal{N} = 4$ multiplet) while the one higher spin-3 current in the unitary coset theory transforms as a singlet under the
SO(\(N = 4\)). In other words, the former appear in the quadratic in the fermionic coordinates in the \(N = 4\) multiplet and the latter appears in the quartic in the fermionic coordinates in the \(N = 4\) multiplet \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).}

Therefore, we should obtain the 16 lowest higher spin currents implicitly (or explicitly) for generic \(N\) in the extension of the large \(N = 4\) (non)linear superconformal algebra (in the realization of orthogonal coset theory) by generalizing the previous work in \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).} to the \(N\)-generalization. As long as the three-point functions are concerned, the several \(N\) cases are enough to determine them completely. This feature is different from the one in the bosonic coset theory \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).} (in the context of \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).} \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).}) where the explicit results for the higher spin currents (for generic \(N\)) are necessary. In this construction, the four spin-\(\frac{3}{2}\) currents in the large \(N = 4\) (non)linear superconformal algebra play an important role. We follow the procedure in \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).}, construct the zero-mode eigenvalue equations and obtain the three-point functions for finite \(N\) and \(k\) (and also under the large \(N \neq 0\) 't Hooft limit). For the unitary coset theory, the conformal dimension of a coset primary can be calculated from the quadratic Casimirs of \(su(N + 2)\) and \(su(N)\), the quantum numbers of \(u(1)\) algebras and an excitation number in \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).}. For the orthogonal coset theory, as far as we know, there is no explicit formula for the conformal dimension of a coset primary because it is rather nontrivial to obtain the correct factors in the above last two quantities. This is one of the reasons why we are interested in this particular orthogonal coset theory. See also the description of \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).} \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).} \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).} in different orthogonal coset theory.

The \(N = 4\) orthogonal coset theory we are interested in is described by the following ‘supersymmetric’ coset \footnote{Similarly, the six higher spin-2 currents in the unitary coset theory transform as the adjoint of \(SO(N = 4)\) (quadratic in the fermionic coordinates in the \(N = 4\) multiplet) while the one higher spin-2 current in the orthogonal coset theory transforms as a singlet under the \(SO(N = 4)\) (and appears in the fermionic independent term in the \(N = 4\) multiplet).}:

\[
\text{Wolf} \times SU(2) \times U(1) = \frac{SO(N + 4)}{SO(N) \times SU(2) \times U(1)}.
\] (1.1)

The fundamental currents are given by the bosonic spin-1 current \(V^a(z)\) and the fermionic spin-\(\frac{1}{2}\) current \(Q^b(z)\). The indices run over \(a, b, \ldots = 1, 2, \ldots, \frac{(N+4)(N+3)}{2}\) where the number \(\frac{(N+4)(N+3)}{2}\) is the dimension of the \(g = so(N + 4)\) algebra. For the extension of the \(N = 4\) ‘nonlinear’ superconformal algebra, the relevant coset is given by the Wolf space itself \(SO(N + 4)\). For the extension of the \(N = 4\) ‘linear’ superconformal algebra, the corresponding coset is given by the Wolf space multiplied by \(SU(2) \times U(1)\), which is equivalent to the above coset in the right hand side of (1.1).
As in [14], we can construct the explicit 16 lowest higher spin currents (which are multiple products of the above fundamental currents together with their derivatives) which are expressed in terms of the Wolf space (or Wolf space multiplied by $SU(2) \times U(1)$) coset fields. These findings will allow us to calculate the zero modes for the higher spin currents in terms of the generators of the $g = so(N + 4)$ algebra because the zero modes of the spin-1 current $V_0^a$ satisfy the defining commutation relations of the underlying finite dimensional Lie algebra $so(N+4)$. Furthermore, all the operator product expansions between the higher spin currents and the spin-$\frac{1}{2}$ current $Q^a(z)$ are determined explicitly by construction.

The minimal representations are given by two representations. See also the previous works in [15] [16] [17] [18] [19]. One minimal representation is given by $(0; v)$, where the nonnegative integer mode of the spin-1 current $V^a(z)$ in $so(N + 4)$ acting on the state $|0; v\rangle$ vanishes. Under the decomposition of $so(N + 4)$ into $so(N) \oplus su(2) \oplus su(2)$, the adjoint representation of $so(N + 4)$ can be broken into the following representations [20]:

\[
\frac{1}{2}(N + 4)(N + 3) \rightarrow (\frac{1}{2}N(N - 1), 1, 1) \oplus (1, 3, 1) \oplus (1, 1, 3) \oplus (N, 2, 2).
\]

Among these representations, the vector representation for $so(N)$ is given by $(N, 2, 2)$ \(^2\). Therefore, the representation $(0; v)$ corresponds to the representations $(N, 2, 2)$. Note that the extra $su(2)$ factor in the above branching rule comes from the one in the left hand side of (1.1). The corresponding states for the representation $(0; v)$ are given by the $-\frac{1}{2}$ mode of the spin-$\frac{1}{2}$ current $Q^a(z)$ acting on the vacuum $|0\rangle$, where the index $a$ is restricted to the $4N$ coset index \(^3\). The eigenvalue for the zero mode in the (higher spin) currents (multiple products of the above spin-1 and spin-$\frac{1}{2}$ currents) acting on this state can be obtained from the highest pole of the OPE between the (higher spin) current and the spin-$\frac{1}{2}$ current as in unitary coset theory \(^4\).

The other minimal representation is given by $(v; 0)$, where the positive half-integer mode of the spin-$\frac{1}{2}$ current $Q^a(z)$ in $so(N + 4)$ acting on the state $|v; 0\rangle$ vanishes. They are singlets with respect to $so(N)$ in the $so(N + 4)$ representation based on the vector representation. That is, the vector representation $(N + 4)$ of $so(N + 4)$ transforms as a singlet $(1, 4)_{\pm \frac{1}{2}}$ with respect to $so(N)$ under the branching $(N + 4) \rightarrow (N, 1)_0 \oplus (1, 4)_{\pm \frac{1}{2}}$ with respect to $su(N) \oplus so(4) \oplus u(1)$. The indices $0$ and $\pm \frac{1}{2}$ denote the $U(1)$ charge, which will be described later in (3.2) \(^5\). On the other hand, $(N, 1)_0$ refers to the vector representation with respect to $so(N)$, and the $su(N) \oplus u(1)$ part is described by the coset index $\lambda = (0, 0, 0, + +)$.

\(^2\)For $N = 4$, we have the breaking $28 \rightarrow (1, 3, 1, 1) \oplus (3, 1, 1, 1) \oplus (1, 1, 3, 1) \oplus (1, 1, 1, 3) \oplus (2, 2, 2, 2)$ under $su(2) \oplus su(2) \oplus su(2) \oplus su(2)$ where the $so(4)$ is replaced with the first two $su(2)$ factors.

\(^3\)We can further classify the four independent states denoted by $|0; v\rangle \rightarrow _{++} _{+-} _{-+} _{--}$ with $4N$ coset indices (See also [21]) where four linear combinations among $(_{++} _{+-} _{-+} _{--})$ refer to the $(2, 2)$ of $su(2) \times su(2)$.

\(^4\)Furthermore, the nontrivial states exist for the negative half-integer mode (as well as the $\frac{1}{2}$ mode) of the spin-$\frac{1}{2}$ current acting on the state $|0; v\rangle >$ because the action of the negative mode of the spin-$\frac{1}{2}$ current on the vacuum $|0\rangle$ is nonzero. The positive half-integer modes of the spin-$\frac{1}{2}$ current $(\frac{1}{2}, \frac{3}{2}, \cdots)$ modes acting on the state $|0; v\rangle >$ vanish.

\(^5\)In this case, the states are further classified as $|v; 0\rangle \rightarrow _{++} _{+-} _{-+} _{--}$ with explicit $su(2) \times su(2)$ double
to $so(N)$ and describes the light state $|(v;v)>$ as in unitary coset theory. For the state $|(v;0)>$, the $so(N+4)$ generator $T_a^\ast$ corresponds to the zero mode of the spin-1 current $V^a(z)$ because the zero mode of the spin-1 current satisfies the commutation relation of the underlying finite-dimensional Lie algebra $so(N+4)$. Then, the nontrivial contributions to the zero-mode (of (higher spin) currents) eigenvalue equation associated with the state $|(v;0)>$ come from the multiple product of the spin-1 current $V^a(z)$ in the (higher spin) currents. After substituting the $so(N+4)$ generator $T_a^\ast$ into the zero mode of spin-1 current $V^a_0$ in the multiple product of the (higher spin) currents, we obtain the $(N+4) \times (N+4)$ matrix acting on the state $|(v;0)>$. Then, the last $4 \times 4$ subdiagonal matrix is associated with the above $so(4) \oplus u(1)$ algebra. The eigenvalue can be obtained from each diagonal matrix element in this $4 \times 4$ matrix. Furthermore, the first $N \times N$ subdiagonal matrix provides the corresponding eigenvalues (for the higher spin currents) for the light state $|(v;v)>$, as mentioned before.

In section 2, we review the $so(N+4)$ current algebra generated by the spin-1 and the spin-$\frac{1}{2}$ currents. The 11 currents of large $\mathcal{N} = 4$ nonlinear superconformal algebra using these fundamental currents are obtained. The lowest higher spin-2 current for generic $N$ and $k$ is given. Furthermore, the remaining 15 higher spin currents can be obtained implicitly.

In section 3, the eigenvalue equations of the spin-2 stress-energy tensor are given for the above two minimal states. The eigenvalue equations of higher spin currents with spins-2, 3, and 4 for the above two minimal states are presented. The corresponding three-point functions are also described.

In section 4, the 16 currents of large $\mathcal{N} = 4$ linear superconformal algebra using the above fundamental currents are obtained. Furthermore, the 16 higher spin currents can be obtained implicitly.

In section 5, the eigenvalue equations of spin-2 stress–energy tensor are given for the above two minimal states. Next, the eigenvalue equations of higher spin currents with spins-2, 3 and 4 for the above two minimal states are given. The corresponding three-point functions are described.

In section 6, the summary of this paper is described, and future directions are explained briefly.

In Appendices $A - E$, some details in sections 2, 3, 4, 5 are presented.

We use the Thielemans package [22] in this paper.

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indices. That is the vector representation 4 breaks into $(1,2) \oplus (2,1)$ under the $su(2) \times su(2)$.

For the (higher spin) currents of the extension of the large $\mathcal{N} = 4$ linear superconformal algebra, the boldface notation is used. For the 11 currents of the large $\mathcal{N} = 4$ nonlinear superconformal algebra, the hatted notation is used.
2 The extension of the large $\mathcal{N} = 4$ nonlinear superconformal algebra

In this section, we review the $\mathfrak{so}(N + 4)$ current algebra generated by the spin-1 and the spin-$\tfrac{1}{2}$ currents. We construct the 11 currents of large $\mathcal{N} = 4$ nonlinear superconformal algebra using these fundamental currents. As far as we know, this observation is new even though the tensorial structures in the 11 currents are the same as the ones in the unitary Wolf space coset theory. We explicitly obtain the lowest higher spin-2 current for generic $N$ and $k$ by generalizing the $N = 4$ case in [5]. Furthermore, we show how the remaining 15 higher spin currents can be obtained implicitly starting from the above higher spin-2 current. Finally, the general procedure to obtain the next 16 higher spin currents is given.

2.1 The $\mathcal{N} = 1$ Kac-Moody current algebra

Let us consider the $\mathfrak{so}(N + 4)$ current algebra generated by the spin-1 and the spin-$\tfrac{1}{2}$ currents. The generators of the Lie algebra $g = \mathfrak{so}(N + 4)$ satisfy the commutation relation $[T_a, T_b] = f^c_{ab} T_c$ and some of them are given in Appendix A. The adjoint indices run over $a, b, \cdots = 1, 2, \cdots, (N+4)(N+3)/2$. The normalization for the generators is consistent with the metric $g_{ab} = \frac{1}{2} \text{Tr}(T_a T_b) = \frac{1}{2c_g} f^{ac}_{\phantom{ac}d} f_{bd}^\phantom{bd}e$ where $c_g$ is the dual Coxeter number of the Lie algebra $g = \mathfrak{so}(N + 4)$ and is given by $c_g = (N + 2)$. The operator product expansions (OPEs) between the spin-1 and the spin-$\tfrac{1}{2}$ currents are summarized as [23]

$$
V^a(z) V^b(w) = \frac{1}{(z-w)^2} k g^{ab} - \frac{1}{(z-w)} f^a_{\phantom{a}c} V^c(w) + \cdots,
$$

$$
Q^a(z) Q^b(w) = -\frac{1}{(z-w)} (k + N + 2) g^{ab} + \cdots,
$$

$$
V^a(z) Q^b(w) = + \cdots. \tag{2.1}
$$

Here $k$ is the level and a positive integer. Note that there is no singular term in the OPE between the spin-1 current $V^a(z)$ and the spin-$\tfrac{1}{2}$ current $Q^b(w)$. The $\mathcal{N} = 1$ superspace description can be obtained from (2.1). The $k$-dependence appears in the above nontrivial OPEs while the $N$-dependence appears in the OPE between the spin-$\tfrac{1}{2}$ currents. Furthermore, as we consider the multiple product of these fundamental currents, the $N$-dependence occurs from the combinations of the inverse metric $g^{ab}$ and the structure constant $f^a_{\phantom{a}c}$. 

5
2.2 The large $\mathcal{N} = 4$ nonlinear superconformal algebra

The Wolf space coset we describe is given by $[24, 25, 26]$

$$\text{Wolf} = \frac{G}{H} = \frac{SO(N+4)}{SO(N) \times SO(4)}. \quad (2.2)$$

The group indices are divided into

$$G \text{ indices} : a, b, c, \cdots = 1, 2, \cdots, \frac{1}{4}(N+4)(N+3), 1^*, 2^*, \cdots, \left(\frac{1}{4}(N+4)(N+3)\right)^*,$$

$$\frac{G}{H} \text{ indices} : \bar{a}, \bar{b}, \bar{c}, \cdots = 1, 2, \cdots, 2N, 1^*, 2^*, \cdots, 2N^*. \quad (2.3)$$

The total $4N$ coset indices in $(2.3)$ are divided into $2N$ without $*$ and $2N$ with $*$. We only consider even dimensional $G = SO(N+4)$. That is, $N = 4n$ or $N = 4n+1$ for integer $n$.

For given $(N+4) \times (N+4)$ matrix, we can associate the above $4N$ coset indices as follows:

$$\begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
\vdots & \vdots & \vdots & \vdots \\
* & * & * & * \\
\end{pmatrix}_{(N+4) \times (N+4)} \quad (2.4)$$

As described in Appendix A, for example, the generators with $2N$ coset indices have two nonzero elements located at the above $N \times 4$ and $4 \times N$ off diagonal matrices in $(2.4)$ (the other half generators with $2N$ coset indices denoted by $*$ can be obtained via the transpose of the first half generators).

As done in the unitary case of $[14]$, we would like to construct the 11 currents for generic $N$ from the data of $N = 4$ case in $[5]$. By writing the spin-$\frac{3}{2}$ currents with unknown rank-2 tensor with the coset indices as well as $SO(N = 4)$ index and using the defining OPE of the large $\mathcal{N} = 4$ nonlinear superconformal algebra with the help of $(2.1)$, we analyze each pole term in order to extract the above 11 currents explicitly.

Then we can write down the 11 currents of large $\mathcal{N} = 4$ nonlinear superconformal algebra in terms of $\mathcal{N} = 1$ Kac-Moody currents $V^a(z)$ and $Q^\bar{b}(z)$ together with the structure constant, the metric (which corresponds to the one component of above unknown rank 2 tensor with coset indices) and the three almost complex structures $h^i_{ab}(i = 1, 2, 3)$ where the index $i$ stands
for $SO(3)$ index. The three almost complex structures $(h^1, h^2, h^3)$ are antisymmetric rank-two tensors and satisfy the algebra of imaginary quaternions.\footnote{We consider two cases where $N = 4n$ and $N = 4n + 1$ cases with some integer $n$. For convenience, we only represent the almost complex structures for $N = 4n$ case. In principle, we can write down the complex structures for $N = 4n + 1$ case also.} $h^1_{\bar{a}b} h^2_{\bar{c}b} = \epsilon^{ijk} h^k_{\bar{a}c} - \delta^{ij} g_{\bar{a}b}$. The three almost complex structures using $4N \times 4N$ matrices are given by

$$
\begin{align*}
    h^1_{\bar{a}b} &= \begin{pmatrix}
    0 & 1 & 0 & 0 \\
    -1 & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 \\
    0 & 0 & -1 & 0
\end{pmatrix},
    h^2_{\bar{a}b} &= \begin{pmatrix}
    0 & i & 0 & 0 \\
    -i & 0 & 0 & 0 \\
    0 & 0 & 0 & -i \\
    0 & 0 & i & 0
\end{pmatrix},
    h^3_{\bar{a}b} &= \begin{pmatrix}
    0 & 0 & i & 0 \\
    0 & 0 & -i & 0 \\
    -i & 0 & 0 & 0 \\
    0 & -i & 0 & 0
\end{pmatrix},
\end{align*}
$$

(2.5)

where each entry in (2.5) is $N \times N$ matrix and the third almost complex structure can be written in terms of a product of other two: $h^3_{\bar{a}b} \equiv h^1_{\bar{a}c} h^2_{\bar{c}b}$. Note that $h^0_{\bar{a}b} = g_{\bar{a}b}$.

Now we obtain the final results for the 11 currents as follows:

\[
\begin{align*}
  \hat{G}^{0}(z) &= \frac{i}{(k + N + 2)} Q_{\bar{a}}^{\bar{a}}(z), \\
  \hat{G}^{i}(z) &= \frac{i}{(k + N + 2)} h^{i}_{\bar{a}b} Q^{\bar{a}} V^{b}(z), \\
  \hat{A}_{i}(z) &= (-1)^{i+1} \frac{1}{4N} f^{ab}_{\bar{c}} h^{i}_{\bar{a}b} V^{c}(z), \\
  \hat{B}_{i}(z) &= -\frac{1}{4(k + N + 2)} h^{i}_{\bar{a}b} Q^{\bar{a}} Q^{b}(z), \\
  \hat{T}(z) &= \frac{1}{2(k + N + 2)^2} \left[(k + N + 2) V^{a} V^{\bar{a}} + k Q_{\bar{a}} \partial Q^{\bar{a}} + f^{abc} Q^{\bar{a}} Q^{b} V^{c} \right](z) \\
  &\quad - \frac{1}{(k + N + 2)} \sum_{i=1}^{3} \left((-1)^{i} \hat{A}_{i} + \hat{B}_{i} \right)^{2}(z),
\end{align*}
\]

(2.6)

where the index $i (= 1, 2, 3)$ in the spin-1 currents stands for $su(2)$ adjoint index respectively. The spin-$\frac{3}{2}$ currents $\hat{G}^{\mu}(z)$ with $SO(4)$ index $\mu$ are the four supersymmetry generators, the spin-1 currents $\hat{A}_{i}(z)$ and $\hat{B}_{i}(z)$ are six spin-1 generators of $\hat{su}(2)_{k} \times \hat{su}(2)_{N}$ and the current $\hat{T}(z)$ is the spin-2 stress energy tensor. The extra factors $(-1)^{i+1}$ or $(-1)^{i}$ in (2.6) come from the sign change of the spin-1 currents.\footnote{We have the following relations between the spin-$\frac{3}{2}$ currents with double index notation where $SU(2) \times SU(2)$ symmetry is manifest and those with a single index notation where $SO(4)$ symmetry is manifest.}

\[
\begin{align*}
  \hat{G}_{11}(z) &= \frac{1}{\sqrt{2}} (\hat{G}^{1} - i \hat{G}^{2})(z), \\
  \hat{G}_{12}(z) &= \frac{1}{\sqrt{2}} (\hat{G}^{3} - i \hat{G}^{0})(z), \\
  \hat{G}_{22}(z) &= \frac{1}{\sqrt{2}} (\hat{G}^{1} + i \hat{G}^{2})(z), \\
  \hat{G}_{21}(z) &= \frac{1}{\sqrt{2}} (\hat{G}^{3} + i \hat{G}^{0})(z).
\end{align*}
\]
current, the OPEs between the spin-$\frac{3}{2}$ currents, the OPEs between the spin-1 currents and the spin-$\frac{3}{2}$ currents, the OPEs between the spin-1 currents and the OPEs between the spin-2 current and other 10 currents [28, 29, 21, 30].

2.3 The 16 lowest higher spin currents

In [5], the explicit results for the following higher spin currents (one spin-2 current, four spin-$\frac{5}{2}$ currents, six spin-3 currents, four spin-$\frac{7}{2}$ currents and one spin-4 current) for $N = 4$ were written as the fundamental spin-1 and spin-$\frac{1}{2}$ currents

$$
\begin{align*}
(2, \frac{5}{2}, \frac{5}{2}, 3) & : (T^{(2)}, T^{(\frac{5}{2})}, T^{(\frac{5}{2})}, T^{(3)}), \\
(\frac{5}{2}, 3, 3, \frac{7}{2}) & : (U^{(\frac{5}{2})}, U^{(3)}, U^{(3)}, U^{(\frac{7}{2})}), \\
(\frac{5}{2}, 3, 3, \frac{7}{2}) & : (V^{(\frac{5}{2})}, V^{(3)}, V^{(3)}, V^{(\frac{7}{2})}), \\
(3, \frac{7}{2}, \frac{7}{2}, 4) & : (W^{(3)}, W^{(\frac{7}{2})}, W^{(\frac{7}{2})}, W^{(4)}).
\end{align*}
$$

(2.7)

It is very important to obtain the lowest higher spin current from the experience in [14]. We would like to determine the above higher spin currents for generic $N$. The lowest spin in the $N = 4$ multiplet of (2.7) is given by spin-2 rather than spin-1 because there was no higher spin-1 current satisfying the primary condition and the regular conditions when $N = 4$ [5]. Can we prove this for general $N$?

Let us first try to consider the possibility of the higher spin-1 current. We can use the results in [14] in order to analyze the existence of higher spin-1 current for the orthogonal case. The ansatz for the higher spin-1 current for general $N$ is given by

$$
T^{(1)}(z) = A_a V^a(z) + B_{\bar{a} \bar{b}} Q^{\bar{a}} \bar{Q}^{\bar{b}}(z),
$$

(2.8)

where the two coefficients $A_a$ and $B_{\bar{a} \bar{b}}$ are undetermined constants. The most nontrivial constraint for the higher spin-1 current is the primary condition that the higher spin-1 current should be primary field under the stress energy tensor $\hat{T}(z)$ as follows:

$$
T^{(1)}(z) \hat{T}(w) = \frac{1}{(z-w)^2} T^{(1)}(w) + \cdots,
$$

(2.9)

where we change the order of the operators in the left hand side compared to the standard expression. Then the primary condition in (2.9) requires the following two tensor equations as follows:

$$
\frac{k}{2(k+N+2)(N+2)} A_a f_{b\bar{c}} a = B_{b\bar{c}},
$$

8
\[ A_a f^{ab}_c f^c_{bd} - \frac{k}{(N + 2)} A_a f^{a_{bc}} f^{b_{cd}} = 2(k + N + 2)A_d. \]  

(2.10)

The second equation of (2.10) is determined by the structure constant of \( g = so(N + 4) \). It is not hard to find the structure constant of \( so(N + 4) \) when \( N \) is fixed and we can test the existence of the solution. In general, there is no nontrivial \( A_a \) satisfying the second condition in the orthogonal case. Thus we obtain the trivial solution \( A_a = 0 \). Then the coefficient \( B_{\bar{a} \bar{b}} \) is also zero from the first condition in (2.10). Thus the above higher spin-1 current \( T^{(1)}(z) \) is identically zero and there is no higher spin-1 current (2.8) in the orthogonal case.

### 2.3.1 The higher spin currents of spins \( (2, \frac{5}{2}, \frac{5}{2}, 3) \)

Let us determine the first \( N = 2 \) multiplet in (2.7).

- Construction of the lowest higher spin-2 current

  The ansatz for the higher spin-2 current based on \( N = 4 \) case [5] is given by

  \[ T^{(2)}(z) = c_1 V_a V^a(z) + c_2 \sum_{a': so(N)} V_{a'} V^{a'}(z) + c_3 \sum_{a''': so(4)} V_{a'''} V^{a'''}(z) + c_4 \sum_{i=1}^{3} \hat{A}_i \hat{A}_i(z) \]

  \[ + c_5 \sum_{i=1}^{3} \hat{B}_i \hat{B}_i(z) + c_6 Q_{\bar{a}} \partial Q^{\bar{a}}(z) + c_7 \sum_{\mu=0}^{3} h^{\mu}_{\bar{a} \bar{b}} h^{\mu}_{\bar{c} \bar{d}} f^{\bar{a} \bar{b} \bar{c} \bar{d}} e Q^{\bar{a}} Q^{\bar{b}} Q^{\bar{c}} Q^{\bar{d}}, \]  

(2.11)

where \( c_i(N, k) \) are the undetermined coefficient functions. Note that for general \( N \), we should have the different coefficients \( c_2 \) and \( c_3 \) in (2.11) even though they correspond to the subgroup in the Wolf space. Furthermore, it is nontrivial to check the tensorial structure in the \( c_7 \)-term. In the construction of the OPE between the spin-\( \frac{3}{2} \) currents in the large \( N = 4 \) linear superconformal algebra, this kind of term occurs in [1]. The indices in the almost complex structures are contracted with the ones in the structure constant and the spin-\( \frac{1}{2} \) currents. The index for spin-1 current runs over the \( so(N + 4) \) algebra. The higher spin-2 current should satisfy the following OPEs

\[ \hat{T}(z) T^{(2)}(w) = \frac{1}{(z - w)^2} 2T^{(2)}(w) + \frac{1}{(z - w)} \partial T^{(2)}(w) + \cdots, \]

\[ \phi(z) T^{(2)}(w) = + \cdots, \]  

(2.12)

where \( \phi(z) = \hat{A}_i(z), \hat{B}_i(z), F^a(z) \) and \( U(z) \). Note that by construction, the higher spin currents should commute with both the four spin-\( \frac{1}{2} \) currents \( F^a(z) \) and the spin-1 current \( U(z) \) of the large \( N = 4 \) linear superconformal algebra [6]. The requirement (2.12) determines

\footnote{From the Goddard-Schwimmer formula [28], the conditions (2.12) are equivalent to the conditions for}
every coefficient functions $c_i$ except the overall factor for $N = 4, 5, 8, 9$ cases. From those solutions, we obtain the general solution for the coefficients $c_i(N, k)$ as follows:

\[
\begin{align*}
c_1 &= -\frac{(2k^2N + k^2 + 4kN^2 + 6kN + 2k + 11N^2 - 2N - 24)}{2(k - 1)N(k + N + 2)^2}, \\
c_2 &= \frac{6(2kN + 3k + 3N + 4)}{(k - 1)N(k + N + 2)^2}, \\
c_3 &= \frac{3(k + N - 2)(2kN + 3k + 3N + 4)}{2(k - 1)(k + 2)(k + N + 2)^2}, \\
c_4 &= \frac{2(N + 2)(2k + N)}{(k + 2)(k + N + 2)^2}, \\
c_5 &= \frac{2k(2k + N)}{N(k + N + 2)^2}, \\
c_6 &= \frac{k(N + 2)(2k + N)}{N(k + N + 2)^3}, \\
c_7 &= \frac{(N + 2)(2k + N)}{4N(k + N + 2)^3}.
\end{align*}
\]

Note that the coefficients $c_2$ and $c_3$ are different in general but it is easy to see that they are the same for $N = 4$.

The appropriate choice for the overall factor of the higher spin-2 current comes from the following OPE

\[
T^{(2)}(z) T^{(2)}(w) = \frac{1}{(z - w)^4} e_1 + \frac{1}{(z - w)^2} \left[ e_2 T^{(2)} + e_3 \left( \hat{T} + \frac{1}{(k + 2)}(\hat{A}_3 \hat{A}_3 + \hat{A}_+ \hat{A}_- + i \partial \hat{A}_3) \right) \right] (w) + \frac{1}{(z - w)^2} \frac{1}{2} \partial(\text{pole-2})(w) + \cdots,
\]

where the central term or structure constants in (2.14) $e_i$ are given by

\[
\begin{align*}
e_1 &= \frac{3k(2k + N)(2kN + 3k + 3N + 4)(2k^2N + k^2 + 4kN^2 + 6kN + 2k + 11N^2 - 2N - 24)}{(k - 1)(k + 2)N(k + N + 2)^3}, \\
e_2 &= \frac{2(2k^2N + 7k^2 - 2kN^2 - 6kN - 10k - 13N^2 - 2N + 24)}{(k - 1)N(k + N + 2)^3}, \\
e_3 &= \frac{4(N + 2)(2k + N)(2k^2N + k^2 + 4kN^2 + 6kN + 2k + 11N^2 - 2N - 24)}{(k - 1)N^2(k + N + 2)^2}.
\end{align*}
\]

Because the maximum power of $k$ in the polynomial appearing in the numerators of the coefficients in (2.15) is given by 2, we could determine all the coefficients completely with the higher spin-2 current in the linear version [31],

\[
\begin{align*}
T(z) T^{(2)}(w) &= \frac{1}{(z - w)^2} 2T^{(2)}(w) + \frac{1}{(z - w)} \partial T^{(2)}(w) + \cdots, \\
\Phi(z) T^{(2)}(w) &= + \cdots,
\end{align*}
\]

where the current $\Phi(z)$ stands for the spin-1 currents $\mathbf{A}_i(z)$ and $\mathbf{B}_i(z)$, the spin-$\frac{1}{2}$ currents $\mathbf{F}^a(z)$, the spin-1 current $\mathbf{U}(z)$ of the large $\mathcal{N} = 4$ linear superconformal algebra (where the current $\mathbf{T}(z)$ is the stress energy tensor).
data of $N = 4, 5, 8, 9$ cases. We will see the three-point function with this choice of overall factor in the higher spin-2 current later.

• Construction of the other higher spin currents

Now let us determine the other three higher spin currents in the first $\mathcal{N} = 2$ multiplet in (2.7). As done in $N = 4$ case in [5], we can calculate the OPE between $\hat{G}_{21}(z)$ and $T^{(2)}(w)$ where the explicit forms are given in (2.6) with the footnote 8 and (2.11) with (2.13). Again the fundamental OPEs in (2.1) are used heavily. Three almost complex structures are given in (2.5) and the metric is also related to the following relation $h^{0}_{\bar{a}\bar{b}} = g_{\bar{a}\bar{b}}$. Then it turns out that the following nontrivial first-order pole is given by

$$\left( \hat{G}_{21} \hat{G}_{12} \right)(z) T^{(2)}(w) = \frac{1}{(z - w)} T^{(\frac{3}{2})}(w) + \cdots. \quad (2.16)$$

For $N = 4, 5, 8, 9$ cases, we have the explicit forms for the first-order pole in terms of the fundamental spin-1 and spin-$\frac{1}{2}$ currents. Even for generic $N$, we can express the explicit results for the higher spin-$\frac{5}{2}$ currents $T^{(\frac{5}{2})}(w)$ but we do not present them in this paper.

Because the higher spin-$\frac{5}{2}$ current $T^{(\frac{5}{2})}_{\pm}(w)$ is determined from the OPE (2.16), let us calculate the OPE between $\hat{G}_{21}(z)$ and this higher spin-$\frac{5}{2}$ current $T^{(\frac{5}{2})}_{\pm}(w)$ explicitly. Then we obtain the following result

$$\hat{G}_{21}(z) T^{(\frac{5}{2})}_{\pm}(w) = \frac{1}{(z - w)^2} 4T^{(2)}(w) + \frac{1}{(z - w)} \left[ \frac{1}{4}(\partial(w) + T^{(3)}) \right](w) + \cdots. \quad (2.17)$$

There are no quasiprimary fields in the first-order pole in (2.17). The numerical factor $\frac{1}{4}$ in the first term of the first-order pole is fixed by the spins of the two currents in the left hand side of the above OPE and the spin of the higher spin-2 current living in the second-order pole. Then the higher spin-3 current $T^{(3)}(w)$ can be obtained from the explicit first-order pole from the OPE $\hat{G}_{21}(z) T^{(\frac{5}{2})}_{\pm}(w)$ and subtract the derivative of the higher spin-2 current $\partial T^{(2)}(w)$, along the line of [32, 33, 34]. As before, for several $N$ case, the explicit results are found.

Therefore, the first $\mathcal{N} = 2$ multiplet in (2.7) is determined for generic $N$ completely (and implicitly).

### 2.3.2 The higher spin currents of spins $\left( \frac{5}{2}, 3, 3, \frac{7}{2} \right)$

Let us determine the second $\mathcal{N} = 2$ multiplet in (2.7). As done in (2.16), we calculate the OPE $\hat{G}_{11}(z) T^{(2)}(w)$. The spin-$\frac{3}{2}$ current $\hat{G}_{11}(z)$ is given by (2.6) with the footnote 8. The similar OPE $\hat{G}_{22}(z) T^{(2)}(w)$ can be used for other higher spin-$\frac{5}{2}$ current later. The lowest
higher spin-$\frac{5}{2}$ current $U^{(\frac{5}{2})}(w)$ of this $\mathcal{N} = 2$ multiplet can be obtained from the first-order pole of the following OPE

$$
\hat{G}_{11}(z) T^{(2)}(w) = \frac{1}{(z-w)} U^{(\frac{5}{2})}(w) + \cdots. \quad (2.18)
$$

Furthermore, from the above higher spin-$\frac{5}{2}$ current appearing in (2.18) found for generic $N$, we can calculate the OPE between the spin-$\frac{3}{2}$ currents and this higher spin-$\frac{5}{2}$ current explicitly.

$$
\left( \begin{array}{c}
\hat{G}_{21} \\
\hat{G}_{12}
\end{array} \right) (z) U^{(\frac{3}{2})}(w) = \frac{1}{(z-w)} U^{(3)}_{\pm}(w) + \cdots. \quad (2.19)
$$

There are no derivative terms or quasiprimary fields in the first-order pole of (2.19).

Because the higher spin-3 current $U^{(3)}_{\pm}(w)$ is obtained for generic $N$ from the OPE (2.19), let us calculate the OPE between $\hat{G}_{21}(z)$ and this higher spin-3 current $U^{(3)}_{\pm}(w)$ explicitly.

$$
\hat{G}_{21}(z) U^{(3)}_{\pm}(w) = \frac{1}{(z-w)^2} \left[ \frac{2(2N+3+3k)}{(N+2+k)} U^{(\frac{5}{2})}(w) \right] (w) + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial (\text{pole-2}) + U^{(\frac{7}{2})}(w) \right] (w) + \cdots. \quad (2.20)
$$

It is not difficult to obtain the $N$-dependence on the structure constant in the second-order pole of (2.20). We confirm this for $N = 4, 5, 8, 9$ as before. The numerical factor $\frac{1}{5}$ appearing in the first term of the first-order pole in (2.20) can be determined using the previous argument. There are no quasiprimary fields in the first-order pole in (2.20). Then the higher spin-$\frac{7}{2}$ current $U^{(\frac{7}{2})}(w)$ can be obtained from the explicit first-order pole from the OPE $\hat{G}_{21}(z) U^{(3)}_{\pm}(w)$ and subtract the derivative of the higher spin-$\frac{5}{2}$ current $\frac{2(2N+3+3k)}{5(N+2+k)} \partial U^{(\frac{5}{2})}(w)$.

Therefore, the second $\mathcal{N} = 2$ multiplet in (2.7) is found for generic $N$ implicitly.

### 2.3.3 The higher spin currents of spins $\left( \frac{5}{2}, 3, 3, \frac{7}{2} \right)$

Let us determine the third $\mathcal{N} = 2$ multiplet in (2.7). As done in previous subsection, we calculate the OPE $\hat{G}_{22}(z) T^{(2)}(w)$. The spin-$\frac{3}{2}$ current $\hat{G}_{22}(z)$ is given by (2.6) with the footnote 8. The lowest higher spin-$\frac{5}{2}$ current $V^{(\frac{5}{2})}(w)$ of this $\mathcal{N} = 2$ multiplet can be obtained from the first-order pole of the following OPE

$$
\hat{G}_{22}(z) T^{(2)}(w) = \frac{1}{(z-w)} V^{(\frac{5}{2})}(w) + \cdots. \quad (2.21)
$$

We can combine the two OPEs (2.18) and (2.21).
Furthermore, with the help of above higher spin-$\frac{5}{2}$ current appearing in (2.21) found for generic $N$, we can calculate the following OPE

$$\left( \hat{G}_{21} \hat{G}_{12} \right)(z) V^{(3)}(w) = \frac{1}{(z-w)} V^{(3)}_\pm(w) + \cdots.$$  \hspace{1cm} (2.22)

In this case also, we can combine the two OPEs (2.19) and (2.22).

From the higher spin-3 current $V^{(3)}_L(w)$ obtained for generic $N$ from the OPE (2.22), the OPE between $\hat{G}_{21}(z)$ and this higher spin-3 current $V^{(3)}_L(w)$ can be obtained explicitly as follows:

$$\hat{G}_{21}(z) V^{(3)}_L(w) = \frac{1}{(z-w)^2} \left[ \frac{2(3N+3+2k)}{(N+2+k)} V^{(5)}_\pm \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{1}{5} \partial \text{(pole-2)} + V^{(5)}_\mp \right](w) + \cdots.$$ \hspace{1cm} (2.23)

The $N$-dependence on the structure constant in the second-order pole of (2.23) can be confirmed for $N = 4, 5, 8, 9$ as before. This structure constant and the corresponding one in (2.20) have the $N \leftrightarrow k$ symmetry. Note the numerical factor $\frac{1}{5}$ appearing in the first term of the first-order pole. There are no quasiprimary fields in the first-order pole. Then the higher spin-$\frac{5}{2}$ current $V^{(5)}_\mp(w)$ can be obtained from the explicit first-order pole from the OPE $\hat{G}_{21}(z) V^{(3)}_L(w)$ and subtract the derivative of the higher spin-$\frac{5}{2}$ current $\frac{2(3N+3+2k)}{5(N+2+k)} \partial V^{(5)}_\pm(w)$.

Therefore, the third $\mathcal{N} = 2$ multiplet in (2.7) is found from (2.21), (2.22) and (2.23) for generic $N$ implicitly.

2.3.4 The higher spin currents of spins $(3, \frac{7}{2}, \frac{7}{2}, 4)$

Let us determine the fourth $\mathcal{N} = 2$ multiplet in (2.7). We calculate the OPE $\hat{G}_{22}(z) U^{(5)}_\mp(w)$. The spin-$\frac{5}{2}$ current $\hat{G}_{22}(z)$ is given by (2.6) with the footnote 8 and the higher spin-$\frac{5}{2}$ current $U^{(5)}_\mp(w)$ is given by (2.18). The lowest higher spin-3 current $W^{(3)}(w)$ of this $\mathcal{N} = 2$ multiplet can be obtained from the first-order pole of the following OPE

$$\hat{G}_{22}(z) U^{(5)}_\mp(w) = \frac{1}{(z-w)^2} 4T^{(2)}(w) + \frac{1}{(z-w)} \left[ \frac{1}{4} \partial \text{(pole-2)} + W^{(3)} \right](w) + \cdots.$$ \hspace{1cm} (2.24)

There are no quasiprimary fields in the first-order pole in (2.24). The numerical factor $\frac{1}{4}$ in the first term of the first-order pole is fixed by the previous description. Then the higher spin-3 current $W^{(3)}(w)$ can be obtained from the explicit first-order pole from the OPE $\hat{G}_{22}(z) U^{(5)}_\mp(w)$ and subtract the derivative of the higher spin-2 current $\partial T^{(2)}(w)$. As before, for several $N$ case, the explicit results are found.
From the higher spin-3 current $W^{(3)}(w)$ obtained for generic $N$ from the OPE (2.24), the OPE between $\hat{G}_{21}(z)$ ($\hat{G}_{12}(z)$) and this higher spin-3 current $W^{(3)}(w)$ can be obtained explicitly as follows:

\[
\left(\frac{\hat{G}_{21}}{\hat{G}_{12}}\right)(z)W^{(3)}(w) = \pm \frac{1}{(z-w)^2} \left[ \frac{(N-k)}{(N+2+k)} T^{(\frac{7}{2})}_\pm \right](w) + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial \text{(pole-2)} + W^{(\frac{7}{2})}_\pm \right](w) + \cdots. \tag{2.25}
\]

From the higher spin-$\frac{7}{2}$ current $W^{(\frac{7}{2})}_-(w)$ obtained for generic $N$ from the OPE (2.25), the OPE between $\hat{G}_{21}(z)$ and this higher spin-$\frac{7}{2}$ current $W^{(\frac{7}{2})}_-(w)$ can be obtained explicitly as follows:

\[
\hat{G}_{21}(z)W^{(\frac{7}{2})}_-(w) = \frac{1}{(z-w)^2} \left[ \frac{48(-N+k)}{5(N+2+k)} T^{(2)} \right](w) + \frac{1}{(z-w)^2} \left[ \frac{6(-N+k)}{5(N+2+k)} T^{(3)} + \frac{2(3N+4+3k)}{(N+2+k)} W^{(3)} \right](w) + \frac{16i}{(N+2+k)} (\hat{A}_3 - \hat{B}_3) T^{(2)}(w) + \frac{1}{(z-w)} \left[ \frac{1}{6} \partial \text{(pole-2)} - \frac{144(-N+k)}{((59N+88) + (30N+59)k)} \left( \hat{T}^{(2)} - \frac{3}{10} \partial^2 T^{(2)} \right) \right](w) + W^{(4)}(w) + \cdots. \tag{2.26}
\]

The various $N$-dependent structure constants appearing in (2.26) can be confirmed for $N = 4, 5, 8, 9$ as before. In particular, the nonlinear terms appear in the second- and first-order poles. In the first-order pole, the quasiprimary field of spin 4 appears. Then the higher spin-4 current $W^{(4)}(w)$ can be obtained from the explicit first-order pole from the OPE $\hat{G}_{21}(z)W^{(\frac{7}{2})}_-(w)$ and subtract both the derivative of the second-order pole with $\frac{1}{6}$ and the above quasiprimary field-term.

Therefore, the fourth $\mathcal{N} = 2$ multiplet in (2.7) is found from (2.24), (2.25) and (2.26) for generic $N$ implicitly.

### 2.4 The 16 second lowest higher spin currents

Let us denote the next 16 higher spin currents by its spin contents as follows:

\[
\begin{align*}
(3, \frac{7}{2}, \frac{7}{2}, 4) : (P^{(3)}, P^{(\frac{7}{2})}_+, P^{(\frac{7}{2})}_-, P^{(4)}), & \quad (\frac{7}{2}, 4, 4, \frac{9}{2}) : (Q^{(\frac{7}{2})}_+, Q^{(4)}_+, Q^{(\frac{7}{2})}_-, Q^{(4)}), \\
(\frac{7}{2}, 4, 4, \frac{9}{2}) : (R^{(\frac{7}{2})}_+, R^{(4)}_+, R^{(\frac{7}{2})}_-, R^{(4)}), & \quad (4, \frac{9}{2}, \frac{9}{2}, 5) : (S^{(4)}_+, S^{(\frac{9}{2})}_+, S^{(\frac{9}{2})}_-, S^{(5)}). \tag{2.27}
\end{align*}
\]
We expect that these higher spin currents in (2.27) will appear when we calculate the various OPEs between the lowest 16 higher spin currents in (2.7). In this subsection we would like to construct only four higher spin-$\frac{7}{2}$ currents only. The remaining ones will appear in Appendices B and C.

2.4.1 The four higher spin-$\frac{7}{2}$ currents

From the experience of the unitary case [35], we have the explicit OPE $T^{(2)}(z) U^{(\frac{7}{2})}(w)$ (and $T^{(2)}(z) V^{(\frac{7}{2})}(w)$) where the higher spin currents belong to the lowest $N = 4$ multiplet in the unitary coset theory. The new higher spin-$\frac{7}{2}$ currents occur in the first-order pole. This implies that we expect that we try to calculate the same OPE in the orthogonal case. It turns out that

$$T^{(2)}(z) \left( \begin{array}{c} U^{(\frac{7}{2})} \\ V^{(\frac{7}{2})} \end{array} \right) (w) = \frac{1}{(z-w)^3} c_1 \left( \begin{array}{c} \hat{G}_{11} \\ \hat{G}_{22} \end{array} \right) (w)$$

$$+ \frac{1}{(z-w)^2} \left[ \frac{1}{3} \partial \text{pole-3} + c_2 \left( \begin{array}{c} \hat{G}_{11} \\ -\hat{G}_{22} \end{array} \right) \hat{A}_3 + c_3 \left( \begin{array}{c} -\hat{G}_{21} \\ \hat{G}_{12} \end{array} \right) \hat{A}_\pm \right]$$

$$+ c_4 \left( \begin{array}{c} \hat{G}_{11} \\ -\hat{G}_{22} \end{array} \right) \hat{B}_3 + c_5 \left( \begin{array}{c} \hat{G}_{12} \\ -\hat{G}_{21} \end{array} \right) \hat{B}_\pm + c_6 \partial \left( \begin{array}{c} \hat{G}_{11} \\ \hat{G}_{22} \end{array} \right)$$

$$+ c_7 \left( \begin{array}{c} U^{(\frac{7}{2})} \\ V^{(\frac{7}{2})} \end{array} \right) \right] (w)$$

$$+ \frac{1}{(z-w)^2} \left[ \frac{2}{5} \partial \text{pole-2} - \frac{1}{20} \partial^2 \text{pole-3} + c_8 \left( \hat{T} \hat{G}_{11} - \frac{3}{8} \partial^2 \hat{G}_{11} \right) \hat{T} \hat{G}_{22} - \frac{3}{8} \partial^2 \hat{G}_{22} \right]$$

$$+ \left( \begin{array}{c} Q^{(\frac{7}{2})} \\ R^{(\frac{7}{2})} \end{array} \right) \right] (w) + \cdots \quad (2.28)$$

We have the explicit structure constants $c_1$-$c_8$ for $N = 4$ case appearing in (2.28) but we do not present them here. Note that the higher spin-$\frac{5}{2}$ currents (which appear in the left hand side of this OPE) arise at the second-order pole. The quasiprimary fields of spin-$\frac{7}{2}$ appear in the first-order pole. We can rearrange the two derivative terms in the first-order pole in order to express them in standard way where the first derivative term is written usually without the descendant term from the third-order pole [36, 37].

Similarly, we can calculate the following OPE

$$T^{(2)}(z) T^{(\frac{7}{2})}_\pm (w) = \frac{1}{(z-w)^3} c_1 \left( \begin{array}{c} \hat{G}_{21} \\ \hat{G}_{12} \end{array} \right) (w)$$

$$+ \frac{1}{(z-w)^2} \left[ \frac{1}{3} \partial \text{pole-3} + c_2 \left( \begin{array}{c} -\hat{G}_{21} \\ \hat{G}_{12} \end{array} \right) \hat{A}_3 + c_3 \left( \begin{array}{c} \hat{G}_{11} \\ -\hat{G}_{22} \end{array} \right) \hat{A}_\pm \right]$$

$$+ \cdots \quad (2.29)$$
In (2.29), the structure constants for \( N = 4 \) are known and the composite fields appearing in the right hand side look similar to the ones in (2.28).

Therefore, the four higher spin-\( \frac{7}{2} \) currents in (2.27) are determined implicitly. Once the structure constants are written in terms of \( N \) and \( k \), then we can obtain them from the first-order poles explicitly.

### 2.4.2 The remaining higher spin currents

If we would like to construct the remaining 12 higher spin currents in (2.27), then we should calculate them with the help of the spin-\( \frac{7}{2} \) currents and the known higher spin currents. In Appendix B, we present the defining OPE equations for these higher spin currents and in Appendix C, we present how they appear in the explicit OPEs between the 16 lowest higher spin currents.

### 3 Three-point functions in the extension of the large \( \mathcal{N} = 4 \) nonlinear superconformal algebra

This section describes the three-point functions with scalars for the current of spin \( s = 2 \) and the higher spin currents of spins \( s = 2, 3, 4 \) explained in previous section. The large \( \text{'t} \text{Hooft limit is defined by} \)

\[
N, k \to \infty, \quad \lambda \equiv \frac{(N + 1)}{(N + k + 2)} \text{ fixed.} \tag{3.1}
\]

As described in the introduction, there are two simplest states \(|(v; 0)\rangle \) and \(|(0; v)\rangle \) we describe. The two levels of the \( \hat{su}(2) \times \hat{su}(2) \) are given by \( k \) and \( N \) respectively.

#### 3.1 Eigenvalue equations for the spin-2 current

Let us focus on the eigenvalue equations for the stress energy tensor (2.6) acting on the above two states. We will see that the eigenvalues lead to the ones in the unitary case [1].
3.1.1 Eigenvalue equation for the spin-2 current acting on the state $|(v; 0)\rangle$

The terms containing the fermionic spin-$\frac{1}{2}$ currents $Q^a(z)$ do not contribute to the eigenvalue equation when we calculate the zero mode eigenvalues for the bosonic spin-$s$ current $J^{(s)}(z)$ acting on the state $|(v; 0)\rangle$. The zero mode of the spin-1 current $V_0^a$ satisfies the commutation relation of the underlying finite dimensional Lie algebra $g = so(N + 4)$. For the first state $|(v; 0)\rangle$, the generator $T_{a^*}$ corresponds to the zero mode $V_0^a$ as follows (See also [38]):

$$V_0^a|(v; 0)\rangle = T_{a^*}|(v; 0)\rangle.$$  

(3.2)

Then the eigenvalues are encoded in the last $4 \times 4$ diagonal matrix.

For example, we can calculate the conformal dimension of $|(v; 0)\rangle$ when $N = 4$. The explicit form for the stress energy tensor is given by (2.6). The only $Q^a(z)$-independent terms are given by the first term and the $\hat{A}_i\hat{A}_i(z)$-dependent term. Then the eigenvalue equation for the zero mode of the spin-2 current acting on the state $|(v; 0)\rangle$ leads to

$$\hat{T}_0|(v; 0)\rangle \sim \left[\frac{1}{2(k + 6)} V_a V^a - \frac{1}{(k + 6)} \sum_{i=1}^{3} \hat{A}_i\hat{A}_i\right]_0|(v; 0)\rangle$$

$$= \left[\frac{1}{2(k + 6)} \left(\sum_{a=1}^{8} T_{a^*}T_a + \sum_{a=1}^{8} T_a T_{a^*}\right)\right]_0|(v; 0)\rangle + \frac{1}{(k + 6)} t^+ (t^+ + 1)|(v; 0)\rangle$$

$$= \frac{4}{2(k + 6)}|(v; 0)\rangle + \frac{3}{4(k + 6)}|(v; 0)\rangle = \left[\frac{11}{4(k + 6)}\right]|(v; 0)\rangle,$$  

(3.3)

where $\sim$ in the first line of (3.3) means that we ignore the terms including $Q^a(z)$. In the second line, the summation over the coset indices $a = 1, 2, \cdots, 8, 1^*, 2^*, \cdots, 8^*$ is taken explicitly and we used the condition (3.2). Moreover the eigenvalue equation for the zero mode of the quadratic spin-1 currents is used where $l^+$ is the spin of the affine $\hat{su}(2)$ algebra. In the third line, we take 4 from the last $4 \times 4$ diagonal matrix.

10 The highest weight states of the large $\mathcal{N} = 4$ (non)linear superconformal algebra can be characterized by the conformal dimension $h$ and two (iso)spins $l^\pm$ of $\hat{su}(2) \oplus \hat{su}(2)$ [21].

$$\left[-\sum_{i=1}^{3} \hat{A}_i\hat{A}_i\right]_0|\text{hws}\rangle \rangle = l^+(l^+ + 1)|\text{hws}\rangle,$$

$$\left[-\sum_{i=1}^{3} \hat{B}_i\hat{B}_i\right]_0|\text{hws}\rangle \rangle = l^-(l^- + 1)|\text{hws}\rangle.$$

(3.4)

For example, in $g = so(8)$, the expressions (2.6) imply that

$$\begin{bmatrix} -\sum_{i=1}^{3} \hat{A}_i\hat{A}_i \end{bmatrix}_0 |(v; \ast)\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix} |(v; \ast)\rangle; \quad \left[-\sum_{i=1}^{3} \hat{B}_i\hat{B}_i\right]_0(z) Q^{\hat{A}^*}(w)|(v; \ast)\rangle = \frac{3}{4} Q^{\hat{A}^*}(w),$$

where each element in matrix is $4 \times 4$ block matrix and the representation $\ast = 0$ (trivial representation) or $v$ (vector representation) of $so(4)$. We can see $l^+(v; 0) = \frac{1}{2}$ (from the eigenvalues $\frac{1}{4}$ in matrix), $l^+(v; v) = 0$ (from the first 0 in matrix) and $l^-(0; v) = \frac{1}{2}$ (from the coefficient of the second order pole $\frac{1}{4}$). Then the state
From the similar calculations for \(N = 5, 8, 9\), we can obtain the \(N\)-dependence of the eigenvalue in (3.3) as follows \(^{11}\):

\[
\hat{T}_0|(v; 0) > = \left[ \frac{2N + 3}{4(k + N + 2)} \right] |(v; 0) > ,
\]

(3.5)

where the eigenvalue is the same value as the eigenvalue \(h(f; 0)\) given in unitary case \(^{2}\). We can also check that this leads to the following reduced eigenvalue equation \(\hat{T}_0|(v; 0) > = \frac{\lambda}{2}|(v; 0) > \) under the large \(N\) ’t Hooft limit (3.1).

### 3.1.2 Eigenvalue equation for the spin-2 current acting on the state \(|(0; v) >\)

When we calculate the eigenvalue equations for the second state \(|(0; v) >\), we use the field representation which is similar to \(^{2, 21}\):

\[
|(0; v) > = \frac{1}{\sqrt{k + N + 2}} Q^a_{\frac{1}{2}} |0 > , \quad \bar{a} = 1, 2, \cdots , 2N, 1^*, 2^*, \cdots , (2N)^* .
\]

(3.6)

We need only the coefficient of highest-order pole \(\frac{1}{(z - w)}\) in the OPE between the higher spin current \(J^{(s)}(z)\) and the spin-\(\frac{1}{2}\) current \(Q^a(w)\). The lower singular terms do not contribute to the zero mode eigenvalue equations. Let us denote the highest-order pole as follows \(^{40, 41}\):

\[
J^{(s)}(z) Q^a(w) \bigg|_{\frac{1}{(z - w)^s}} = j(s) Q^a(w),
\]

(3.7)

where \(j(s)\) stands for the corresponding coefficient of the highest order pole. Then we obtain the following eigenvalue equation for the zero mode of the spin-\(s\) current together with (3.6) and (3.7)

\[
J^{(s)}_0|(0; v) > = j(s)|(0; v) > ,
\]

(3.8)

where the explicit relation between the current and its mode is given by \(J^{(s)}(z) = \sum_{n=-\infty}^{\infty} \frac{J^{(s)}_n}{z^{n+\frac{s}{2}}}\). Therefore, in order to determine the above eigenvalue \(j(s)\), one should calculate the explicit OPEs between the corresponding (higher spin) currents and the spin-\(\frac{1}{2}\) current and read off the highest-order pole.

\(^{11}\)We can obtain the conformal dimension of light state from similar calculation

\[
\hat{T}_0|(v; v) > = \left[ \frac{2}{(k + N + 2)} \right] |(v; v) > \rightarrow \frac{2\lambda}{(N + 1)} |(v; v) > .
\]

As we expected, the conformal dimension of light state \(|(v; v) >\) vanishes in the large \(N\) ’t Hooft limit.
Let us consider the eigenvalue equation for the spin-2 current acting on the above state. Since the OPE between the spin-1 current $V^a(z)$ and the spin-$\frac{1}{2}$ current $Q^b(w)$ is regular, the terms containing $V^a(z)$ do not contribute to the highest-order pole. Therefore, the relevant terms in the spin-2 current $\hat{T}(z)$ are given by purely the spin-$\frac{1}{2}$ current-dependent terms. Then the conformal dimension of the state $|0; v\rangle$ is

$$\hat{T}_0|0; v\rangle \sim \left[ \frac{k}{2(k+N+2)^2} Q_a \partial Q^a - \frac{1}{(k+N+2)} \sum_{i=1}^{3} \hat{B}_i \hat{B}_i \right]_0 |0; v\rangle$$

$$= \frac{k}{2(k+N+2)} |0; v\rangle + \frac{1}{(k+N+2)} l^- (l^- + 1) |0; v\rangle$$

$$= \left[ \frac{(2k+3)}{4(N+k+2)} \right] |0; v\rangle . \quad (3.9)$$

In the first line of (3.9), the spin-1 current-dependent terms are ignored. In the second line, we have used the fact that the eigenvalue equation $[Q_a \partial Q^a]_0 |0; v\rangle = (k + N + 2) |0; v\rangle$ (see (3.8)) can be obtained because the highest-order pole gives the corresponding eigenvalue $Q_a \partial Q^a(z) Q^b(w) \frac{1}{(z-w)^2} = (k + N + 2) Q^b(w)$ (see (3.7)) which can be checked from the defining relation in (2.11). Furthermore, the characteristic eigenvalue equation for the affine $\hat{su}(2)$ algebra described in the footnote 10 is used. The above eigenvalue is exactly the same as the eigenvalue $h(0; f)$ described in [2]. Under the large $N$ 't Hooft limit (3.1), the eigenvalue equation implies that we have $\hat{T}_0|0; v\rangle = \frac{1}{2} (1 - \lambda) |0; v\rangle$. There exists $N \leftrightarrow k$ symmetry between the eigenvalues in (3.5) and (3.9). In the large $N$ 't Hooft limit, this is equivalent to $\lambda \leftrightarrow (1 - \lambda)$ symmetry.

### 3.2 Eigenvalue equations for the higher spin currents of spins 2, 3 and 4

Now let us consider the eigenvalue equations for the higher spin currents by following the descriptions in previous subsection.

#### 3.2.1 Eigenvalue equations for the higher spin-2 current

From the explicit expression for the higher spin-2 current $T^{(2)}_0$ (2.11) for several $N = 4, 5, 8, 9$, we obtain the eigenvalue equation for general $N$. It turns out that

$$T^{(2)}_0(v; 0) = - \left[ \frac{(2kN + k + 4N^2 - 4N - 12)}{2(k+N+2)^2} \right] |v; 0\rangle ,$$

$$T^{(2)}_0(0; v) = \left[ \frac{k(2N+1)(2k+N)}{2N(k+N+2)^2} \right] |0; v\rangle . \quad (3.10)$$
Although there is no $N \leftrightarrow k$ symmetry between the eigenvalues of the states $|(v; 0)\rangle$ and $|(0; v)\rangle$ in (3.10), there exists the $\lambda \leftrightarrow (1 - \lambda)$ symmetry (up to sign) in the large $N$ 't Hooft limit. In other words, the eigenvalue equations reduce to

$$
T_0^{(2)}|(v; 0)\rangle = -\lambda(1 + \lambda)|(v; 0)\rangle,
T_0^{(2)}|(0; v)\rangle = (1 - \lambda)(2 - \lambda)|(0; v)\rangle.
$$

(3.11)

Compared to the corresponding eigenvalue equations for the higher spin-2 current for the unitary case [1], the new last factor $(1 + \lambda)$ and $(2 - \lambda)$ in each eigenvalue occurs in (3.11) respectively. They have different $SO(4)$ representations as described in the introduction.

### 3.2.2 Eigenvalue equations for the higher spin-3 currents

In order to represent the eigenvalue equations for the higher spin-3 currents, we should classify the $|(v; 0)\rangle$ states into the following four types of column vectors

$$
|\alpha (v; 0)\rangle = (0, \cdots, 0, 1, 0, 0)^T,
|\bar{\alpha} (v; 0)\rangle = (0, \cdots, 0, 0, 1, 0)^T,
|\bar{\alpha} (v; 0)\rangle = (0, \cdots, 0, 0, 0, 1)^T.
$$

(3.12)

They have nontrivial $U(1)$ charges which will be described in section 5. On the other hand, the $|(0; v)\rangle$ states are expressed by the following forms

$$
|\alpha (0; v)\rangle = \frac{1}{\sqrt{k + N + 2}}Q^\frac{\alpha}{2}|0\rangle,
|\bar{\alpha} (0; v)\rangle = \frac{1}{\sqrt{k + N + 2}}Q^\frac{\bar{\alpha}}{2}|0\rangle.
$$

(3.13)

Now we apply the eigenvalue equations for the zero mode of the higher spin-3 currents to these states. It turns out that the eigenvalue equations for the higher spin-3 current $T^{(3)}(z)$ acting on (3.13) and (3.12) are summarized by

$$
T_0^{(3)}|\alpha (v; 0)\rangle_{\alpha \pm} = \pm \left[ \frac{2kN + k + 4N^2 - 4N - 12}{(k + N + 2)^2} \right]|(v; 0)\rangle_{\alpha \pm},
T_0^{(3)}|\alpha (0; v)\rangle_{\pm \alpha} = \pm \left[ \frac{k(2N + 1)(2k + N)}{N(k + N + 2)^2} \right]|(0; v)\rangle_{\pm \alpha}.
$$

(3.14)

where the index $\alpha$ stands for $\alpha = +, -$. The eigenvalues in (3.14) are similar to the $T_0^{(2)}$ eigenvalues in (3.10). The only overall factors are different from each other.

20
For the higher spin-3 current $W_0^{(3)}$, we have the following relations

$$W_0^{(3)}(v; 0) >_{\alpha \pm} = \pm \left[ \frac{(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^2} \right] |v; 0 >_{\alpha \pm},$$

$$W_0^{(3)}(0; v) >_{\pm \alpha} = \mp \left[ \frac{k(2N + 1)(2k + N)}{N(k + N + 2)^2} \right] |0; v >_{\pm \alpha} \cdot (3.15)$$

We can easily see that the relations (3.15) are the same as the ones in (3.14) except the overall sign.

Furthermore, under the large \(N \) 't Hooft limit (3.1), the above eigenvalue equations (3.14) and (3.15) become

$$T_0^{(3)}(v; 0) >_{\alpha \pm} = \pm 2\lambda(1 + \lambda)|(v; 0) >_{\alpha \pm},$$

$$T_0^{(3)}(0; v) >_{\pm \alpha} = \pm 2(1 - \lambda)(2 - \lambda)|(0; v) >_{\pm \alpha},$$

$$W_0^{(3)}(v; 0) >_{\alpha \pm} = \pm 2\lambda(1 + \lambda)|(v; 0) >_{\alpha \pm},$$

$$W_0^{(3)}(0; v) >_{\pm \alpha} = \mp 2(1 - \lambda)(2 - \lambda)|(0; v) >_{\pm \alpha} \cdot (3.16)$$

Compared to the unitary case, the behavior of \(\lambda(1 + \lambda)\) and \((1 - \lambda)(2 - \lambda)\) in the eigenvalues (3.16) is the same the ones in [1].

For the other remaining four higher spin-3 currents, we obtain the following nonzero results

$$\left[ U_+^{(3)} \right]_0 |(0; v) >_{\pm} = 2i \left[ \frac{(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^2} \right] |(0; v) >_{\pm} \rightarrow 4i(1 - \lambda)(2 - \lambda)|(0; v) >_{\pm},$$

$$\left[ U_-^{(3)} \right]_0 |(v; 0) >_{\pm} = 2i \left[ \frac{(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^2} \right] |(v; 0) >_{\pm} \rightarrow 4i(1 - \lambda)(2 - \lambda)|(v; 0) >_{\pm},$$

$$\left[ V_+^{(3)} \right]_0 |(v; 0) >_{\mp} = 2i \left[ \frac{(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^2} \right] |(v; 0) >_{\mp} \rightarrow 4i(1 - \lambda)(2 - \lambda)|(v; 0) >_{\mp},$$

$$\left[ V_-^{(3)} \right]_0 |(0; v) >_{\mp} = 2i \left[ \frac{(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^2} \right] |(0; v) >_{\mp} \rightarrow 4i(1 - \lambda)(2 - \lambda)|(0; v) >_{\mp},$$

(3.17)

where the large \(N \) 't Hooft limit is taken. Obviously, they are not eigenvalue equations and other relevant quantities (for example, the sum of quadratic of the triplet) can be obtained from these relations [3.17] [12]

12 More precisely, we have the following relations

$$\left[ U_+^{(3)} \right]_0 \frac{1}{\sqrt{k + N + 2}} Q_a^{0^+} >_{\pm} = -2i \left[ \frac{k(2N + 1)(2k + N)}{N(k + N + 2)^2} \right] \frac{1}{\sqrt{k + N + 2}} Q_a^{0^+} >_{\pm},$$

$$\left[ U_+^{(3)} \right]_0 \frac{1}{\sqrt{k + N + 2}} Q_a^{0^+} >_{\pm} = 2i \left[ \frac{k(2N + 1)(2k + N)}{N(k + N + 2)^2} \right] \frac{1}{\sqrt{k + N + 2}} Q_a^{0^+} >_{\pm},$$

21
3.2.3 Eigenvalue equations for the higher spin-4 current

It turns out that the eigenvalue equations of the zero mode of the higher spin-4 current \( W^{(4)}(z) \) are described as

\[
W_0^{(4)}|(v;0) > = \left[ -\frac{2(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^3(30kN + 59k + 59N + 88)} \times d_1 \right] |(v;0) > ,
\]

\[
W_0^{(4)}|0;v > = \left[ -\frac{2k(2N + 1)(2k + N)}{N(k + N + 2)^3(30kN + 59k + 59N + 88)} \times d_2 \right] |(0;v) > ,
\]

where we introduce two factors which show the \( N \leftrightarrow k \) symmetry

\[
d_1(N,k) \equiv \left( 54kN^2 + 81N^2 + 36kN^2 + 225kN + 176N + 78k^2 + 206k + 88 \right) ,
\]

\[
d_2(N,k) \equiv \left( 54k^2N + 81k^2 + 36k^2N + 225kN + 176k + 78N^2 + 206N + 88 \right) .
\]

There is no \( N \leftrightarrow k \) symmetry between the two eigenvalues in (3.18). But if we divide out the \( T_0^{(2)} \) eigenvalues (denoted by \( t^{(2)}(v;0) \) and \( t^{(2)}(0;v) \) respectively) from the \( W_0^{(4)} \) eigenvalues (denoted by \( w^{(4)}(v;0) \) and \( w^{(4)}(0;v) \) respectively), we can see the \( N \leftrightarrow k \) symmetry and the following relation satisfies \( \frac{w^{(4)}(v;0)}{t^{(2)}(v;0)} |_{N \leftrightarrow k} = -\frac{w^{(4)}(0;v)}{t^{(2)}(0;v)} \). We will see that the eigenvalues become very simple in different basis later.

Under the large \( \text{t'}Hooft \) limit (3.1), we have

\[
W_0^{(4)}|(v;0) > = -\frac{12}{5}\lambda (1 + \lambda)(2 + \lambda) |(v;0) > ,
\]

\[
W_0^{(4)}|0;v > = -\frac{12}{5}(1 - \lambda)(2 - \lambda)(3 - \lambda) |(0;v) > .
\]

There exists the \( \lambda \leftrightarrow (1 - \lambda) \) symmetry. We observe that the extra factors \( (2 + \lambda) \) and \( (3 - \lambda) \) in (3.19) are present respectively compared to the corresponding eigenvalue equations in (3.16).

Let us describe the three point functions. From the diagonal modular invariant with pairing up identical representations on the left (holomorphic) and the right (antiholomorphic) sectors [42], one of the primaries is given by \( (v;0) \otimes (v;0) \) which is denoted by \( \mathcal{O}_+ \) and the other is given by \( (0;v) \otimes (0;v) \) which is denoted by \( \mathcal{O}_- \). Then the three point functions with these two scalars are obtained and their ratios can be written as

\[
\frac{< \mathcal{O}_+ \mathcal{O}_+ T^{(2)}(2) >}{< \mathcal{O}_- \mathcal{O}_- T^{(2)}(2) >} = -\left( \frac{\lambda(1 + \lambda)}{(1 - \lambda)(2 - \lambda)} \right) , \quad \frac{< \mathcal{O}_+ \mathcal{O}_+ T^{(3)}(3) >}{< \mathcal{O}_- \mathcal{O}_- T^{(3)}(3) >} = \pm \left( \frac{\lambda(1 + \lambda)}{(1 - \lambda)(2 - \lambda)} \right) , (3.20)
\]

\[
\left[ V^{(3)}_+ \right]_0 \frac{1}{\sqrt{N + k + 2}} Q^{a + N}_{-\frac{1}{2}} |0 > = 2i \left[ k(2N + 1)(2k + N) \right] \frac{1}{N(k + N + 2)^2} \sqrt{N + k + 2} Q^{(a + N)\ast}_{-\frac{1}{2}} |0 > ,
\]

\[
\left[ V^{(3)}_+ \right]_0 \frac{1}{\sqrt{N + k + 2}} Q^{a + N}_{-\frac{1}{2}} |0 > = -2i \left[ k(2N + 1)(2k + N) \right] \frac{1}{N(k + N + 2)^2} \sqrt{N + k + 2} Q^{a\ast}_{-\frac{1}{2}} |0 > ,
\]

where the index \( a \) runs over \( a = 1, 2, \cdots, N \).
\[
\begin{align*}
\frac{\mathcal{O}_+ \mathcal{O}_+ W^{(3)}}{\mathcal{O}_- \mathcal{O}_- W^{(3)}} &= \pm \left[ \frac{\lambda(1 + \lambda)}{(1 - \lambda)(2 - \lambda)} \right], \\
\frac{\mathcal{O}_+ \mathcal{O}_+ W^{(4)}}{\mathcal{O}_- \mathcal{O}_- W^{(4)}} &= \left[ \frac{\lambda(1 + \lambda)(2 + \lambda)}{(1 - \lambda)(2 - \lambda)(3 - \lambda)} \right],
\end{align*}
\]

where the states in the three-point functions for the higher spin-3 currents are assumed from (3.16). Depending on the states, the ratios can be plus sign or minus sign. The behavior for the ratios for the three-point functions is the same as the one in the unitary case up to the overall sign. Furthermore, we see that the ratio for the three-point function for the higher spin-4 current (3.20) contains the factor \( \left[ \frac{(1 + \lambda)(2 + \lambda)(3 + \lambda)}{(1 - \lambda)(2 - \lambda)(3 - \lambda)(4 - \lambda)} \right] \) only after the analysis in the subsection 2.4 has been done. Recall that in the bosonic unitary (or orthogonal) coset theory studied in [38, 43, 17, 6], the ratios of three-point functions behave as \( \frac{1 + \lambda}{1 - \lambda} \) for the spin-2 current corresponding to the stress energy tensor, \( \frac{1 + \lambda}{1 - \lambda} \) for the higher spin-3 current, \( \frac{(1 + \lambda)(2 + \lambda)(3 + \lambda)}{(1 - \lambda)(2 - \lambda)(3 - \lambda)(4 - \lambda)} \) for the higher spin-4 current, and \( \frac{(1 + \lambda)(2 + \lambda)(3 + \lambda)(4 + \lambda)}{(1 - \lambda)(2 - \lambda)(3 - \lambda)(4 - \lambda)} \) for the higher spin-5 current. Then by shifting the \( \lambda \) appearing in the numerator as \( \lambda \rightarrow -(1 - \lambda) \), we can see the behavior of the above results in (3.20) up to sign.

Therefore, the ratios of the three-point functions can be summarized by (3.20). In order to obtain these results, the equations (3.11), (3.16), (3.19) were crucial. Not that the ratio for the three-point function for the higher spin-2 current in (3.20) has the factor \( \frac{1 + \lambda}{1 - \lambda} \) which does not appear in the unitary case [1].

## 4 The extension of the large \( \mathcal{N} = 4 \) linear superconformal algebra

We construct the 16 currents of large \( \mathcal{N} = 4 \) linear superconformal algebra using the fundamental currents as in section 2. With the lowest higher spin-2 current found in section 2, we show how the remaining 15 higher spin currents can be obtained implicitly.

### 4.1 The large \( \mathcal{N} = 4 \) linear superconformal algebra

From the \( \mathcal{N} = 4, 5, 8, 9 \) cases, we can obtain the following four spin-\( \frac{1}{2} \) currents and the spin-1 current as follows:

\[
\begin{align*}
\mathbf{F}_{11}(z) &= \frac{i}{\sqrt{2}} Q^{(2N+3)}(z), \\
\mathbf{F}_{22}(z) &= -\frac{i}{\sqrt{2}} Q^{(2N+3)*}(z), \\
\mathbf{F}_{12}(z) &= \frac{(1 - i)}{2} Q^{(2N+2)}(z), \\
\mathbf{F}_{21}(z) &= \frac{(1 + i)}{2} Q^{(2N+2)}(z), \\
\mathbf{U}(z) &= \frac{(1 + i)}{2\sqrt{2}} V^{(2N+2)}(z) + \frac{(-1 + i)}{2\sqrt{2}} V^{(2N+2)*}(z) + \frac{i}{(N + k + 2)} Q^{(2N+1)} Q^{(2N+1)*}(z)
\end{align*}
\]
\(- \frac{i}{2(N+k+2)} \left( \sum_{a=1}^{N} Q^a Q^a - \sum_{a=N+1}^{2N} Q^a Q^a \right) (z). \)

(4.1)

The corresponding so(4) generators with indices, \((2N + 1), (2N + 2)\) and \((2N + 3)\) (and their conjugates), are given in Appendix A. Note that the \(N\)-dependence in (4.1) appears in the quadratic term in the spin-\(\frac{1}{2}\) current. Furthermore, the presence of the third term in \(U(z)\) is rather new feature in the orthogonal coset theory because we do not see the quadratic term with the index living in the lower \(2 \times 2\) matrix for the unitary case.

Then from the Goddard-Schwimmer formula [28], we have

\[
T(z) = \hat{T}(z) - \frac{1}{(N+k+2)} (UU + \partial F^a F_a)(z),
\]

\[
G_a(z) = \hat{G}_a(z) - \frac{2}{(N+k+2)} \left( UF_a - \frac{1}{3(N+k+2)} \epsilon_{abcd} F^b F^c F^d + 2F^b(\alpha_{ba}^+ \hat{A}_i - \alpha_{ba}^- \hat{B}_i) \right)(z),
\]

\[
A_i(z) = \hat{A}_i(z) + \frac{1}{(N+k+2)} \alpha_{ab}^+ F^a F^b(z),
\]

\[
B_i(z) = \hat{B}_i(z) + \frac{1}{(N+k+2)} \alpha_{ab}^- F^a F^b(z), \quad a, b = 11, 12, 21, 22.
\]

(4.2)

Here the 11 currents, \(\hat{T}(z), \hat{G}_a(z), \hat{A}_i(z)\) and \(\hat{B}_i(z)\), in the nonlinear version are given in [2.6] with the footnote 8. Then the 16 currents of the large \(\mathcal{N} = 4\) linear superconformal algebra [39, 44, 45] are written in terms of the fundamental spin-1 and spin-\(\frac{1}{2}\) currents living in the orthogonal coset theory via (4.2), (4.1) and (2.6) together with the footnote 8.

### 4.2 The 16 lowest higher spin currents

As in (2.7), we present the higher spin currents with boldface notations as follows:

\[
\begin{aligned}
\left(\frac{2}{2}, \frac{5}{2}, \frac{5}{2}, 3\right) & : \ (T^{(2)}, T^{(\frac{5}{2})}_+, T^{(\frac{5}{2})}_-, T^{(3)}), \\
\left(\frac{5}{2}, 3, 3, \frac{7}{2}\right) & : \ (U^{(\frac{5}{2})}_+, U^{(3)}_+, U^{(3)}_-, U^{(\frac{7}{2})}_+), \\
\left(\frac{5}{2}, 3, 3, \frac{7}{2}\right) & : \ (V^{(\frac{5}{2})}_+, V^{(3)}_+, V^{(3)}_-, V^{(\frac{7}{2})}_+), \\
\left(3, \frac{7}{2}, \frac{7}{2}, 4\right) & : \ (W^{(3)}_+, W^{(\frac{7}{2})}_+, W^{(\frac{7}{2})}_-, W^{(4)}).
\end{aligned}
\]

(4.3)

We take the lowest higher spin-2 current \(T^{(2)}(z)\) as the one \(T^{(2)}(z)\) in the nonlinear version. From the explicit results on the 16 currents of the large \(\mathcal{N} = 4\) linear superconformal algebra in the previous subsection, we would like to construct the higher spin currents in the linear version as in section 2.
4.2.1 The higher spin currents of spins \((2, \frac{5}{2}, \frac{5}{2}, 3)\)

Let us consider the first \(\mathcal{N} = 2\) multiplet \((4.3)\). Because the nonlinear version for the appearance of the higher spin-\(\frac{5}{2}\) currents was obtained in (2.16), we calculate the similar OPEs. The following OPEs satisfy

\[
\left( \frac{G_{21}}{G_{12}} \right) (z) T^{(2)}(w) = \frac{1}{(z - w)} T^{(2)}_{\pm}(w) + \cdots. \tag{4.4}
\]

We expect that we have the extra terms for the higher spin-\(\frac{5}{2}\) currents, coming from the OPEs between the difference of the spin-\(\frac{3}{2}\) currents in the nonlinear and linear versions and the higher spin-2 current, when we compare with the ones in (2.16). However, these OPEs do not have any singular terms according to (4.2), the footnote 8 and (2.12). Therefore, we have

\[
T^{(2)}_{\pm}(w) = T^{(\frac{5}{2})}_{\pm}(w). \tag{4.5}
\]

In the second-order pole of (4.3), we can see the same expression as in (2.17) even though the left hand sides of these OPEs are different from each other. However, the first-order pole provides the new higher spin-3 current which is different from the one appearing in (2.17) in the nonlinear version.

4.2.2 The higher spin currents of spins \((\frac{5}{2}, 3, 3, \frac{3}{2})\)

Let us describe the next \(\mathcal{N} = 2\) multiplet in (4.3). Again the previous OPE (2.18) allows us to calculate the following OPE

\[
G_{21}(z) T^{(\frac{5}{2})}_{\pm}(w) = \frac{1}{(z - w)^2} 4T^{(2)}(w) + \frac{1}{(z - w)} \left[ \frac{1}{4} \partial\text{pole-2} + T^{(3)} \right](w) + \cdots. \tag{4.6}
\]

In general, we can extract the extra terms in the first-order pole in (4.6) compared to the corresponding quantity in (2.18) by noting the difference of the spin-\(\frac{3}{2}\) currents in the nonlinear and linear versions in the left hand side of the OPE. However, according to the previous analysis (there are no singular terms), we have that the corresponding higher spin-\(\frac{5}{2}\) current in linear version is the same as the one in the nonlinear version \(U^{(\frac{5}{2})}(w) = U^{(\frac{5}{2})}(w)\). The next two higher spin-3 currents can be obtained from the above higher spin-\(\frac{5}{2}\) current appearing in (4.6). It turns out that

\[
\left( \frac{G_{21}}{G_{12}} \right) (z) U^{(\frac{5}{2})}_{\pm}(w) = \frac{1}{(z - w)} U^{(3)}_{\pm}(w) + \cdots. \tag{4.7}
\]
We can see the similar nonlinear version in (2.19). Finally the last component higher spin-$\frac{7}{2}$ current can be obtained with the help of the first-order pole in (4.7) as follows:

\[
G_{21}(z) \ U_{-}^{(3)}(w) = \frac{1}{(z-w)^2} \frac{2(2N+5+3k)}{(N+2+k)} U^{(\frac{5}{2})}(w) + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial (\text{pole-2}) + U^{(\frac{5}{2})} \right] (w) + \cdots. \tag{4.8}
\]

Note that the structure constant appearing in the second-order pole in (4.8) is different from the one in (2.20). Therefore, the second $\mathcal{N} = 2$ multiplet is found for generic $N$ implicitly.

### 4.2.3 The higher spin currents of spins \( \left( \frac{5}{2}, 3, 3, \frac{7}{2} \right) \)

For the third $\mathcal{N} = 2$ multiplet, we can start with the following OPE

\[
G_{22}(z) \ T^{(2)}(w) = \frac{1}{(z-w)} V^{(\frac{5}{2})}(w) + \cdots. \tag{4.9}
\]

The corresponding nonlinear version is given by (2.21). We can easily see that the extra terms in the first-order pole in (4.9) compared to the one in (2.21) can be read off from the difference in the spin-$\frac{3}{2}$ currents in the nonlinear and linear versions. Similarly we have that the corresponding higher spin-$\frac{7}{2}$ current in the linear version is the same as the one in the nonlinear version $V^{(\frac{5}{2})}(w) = V^{(\frac{7}{2})}(w)$. Similarly we can calculate the following OPEs

\[
\left( \begin{array}{c}
G_{21} \\
G_{12}
\end{array} \right) (z) V^{(\frac{5}{2})}(w) = \frac{1}{(z-w)} V_{+}^{(3)}(w) + \cdots. \tag{4.10}
\]

Then the final higher spin-$\frac{7}{2}$ current can be determined from the first-order pole in (4.10) as follows

\[
G_{21}(z) \ V_{-}^{(3)}(w) = \frac{1}{(z-w)^2} \frac{2(3N+5+2k)}{(N+2+k)} V^{(\frac{5}{2})}(w) + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial (\text{pole-2}) + V^{(\frac{7}{2})} \right] (w) + \cdots. \tag{4.11}
\]

Again, the structure constant appearing in the second-order pole in (4.11) is different from the one in (2.23) and is the same as the one in (4.8) by $N \leftrightarrow k$ symmetry.

### 4.2.4 The higher spin currents of spins \( \left( 3, \frac{7}{2}, \frac{7}{2}, 4 \right) \)

Let us consider the final $\mathcal{N} = 2$ multiplet. As in (2.24), we calculate the following OPE with (4.6)

\[
G_{22}(z) \ U^{(\frac{5}{2})}(w) = \frac{1}{(z-w)^2} 4 T^{(2)}(w) + \frac{1}{(z-w)} \left[ \frac{1}{4} \partial (\text{pole-2}) + W^{(3)} \right] (w) + \cdots. \tag{4.12}
\]
From the first-order pole, we obtain the higher spin-3 current. Based on this result in (4.12), we can calculate the following OPEs

\[
\left( \frac{G_{21}}{G_{12}} \right)(z) W^{(3)}(w) = \pm \frac{1}{(z-w)^2 (N+2+k)} T^{(\frac{5}{2})}_\pm (w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{5} \partial (\text{pole-2}) + W^{(\frac{7}{2})}_\pm \right] (w) + \cdots ,
\]

(4.13)

which is the same form as the one in (2.25). Now the final higher spin-4 current can be obtained by considering the following OPE with (4.13)

\[
G_{21}(z) W^{(\frac{7}{2})}_\pm (w) = \frac{1}{(z-w)^3} \left[ \frac{48(-N+k)}{5(N+2+k)} T^{(2)}(w) \\
+ \frac{1}{(z-w)^2} \left[ -\frac{6(-N+k)}{5(N+2+k)} T^{(3)} + 6 W^{(3)} \right] (w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{6} \partial (\text{pole-2}) \\
- \frac{72(-N+k)}{((37N+59) + (15N+37)k)} \left( T T^{(2)} - \frac{3}{10} \partial^2 T^{(2)} \right) + W^{(4)} \right] (w) \\
+ \cdots .
\]

(4.14)

Compared to the one in (2.26), the second-order pole in (4.14) does not contain the nonlinear terms. We can see that the combination of the quasiprimary field of spin 4 and the primary higher spin-4 current (appearing in the second line of the first-order pole) can be identified with the quasiprimary field of spin 4 in [31].

4.3 The next 16 lowest higher spin currents

We can describe the next 16 higher spin currents by following the method in the subsection 2.4 in the nonlinear version.

4.4 The higher spin currents in different basis

As in the unitary case [11], we can obtain the following explicit relations where we can have the higher spin currents in the basis of [31]

\[
V^{(2)}_0(z) = T^{(2)} , \\
V^{(2),0}_{\frac{1}{2}}(z) = -\frac{i}{\sqrt{2}} \left( -T^{(\frac{5}{2})}_+ + T^{(\frac{5}{2})}_- \right) , \\
V^{(2),1}_{\frac{1}{2}}(z) = \frac{1}{\sqrt{2}} \left( U^{(\frac{5}{2})} + V^{(\frac{5}{2})} \right) , \\
V^{(2),2}_{\frac{1}{2}}(z) = \frac{i}{\sqrt{2}} \left( U^{(\frac{5}{2})} - V^{(\frac{5}{2})} \right) , \\
V^{(2),3}_{\frac{1}{2}}(z) = -\frac{1}{\sqrt{2}} \left( T^{(\frac{5}{2})}_+ + T^{(\frac{5}{2})}_- \right) ,
\]
\[ V_1^{(2), \pm 1}(z) = i \left( U_+^{(3)} - U_-^{(3)} \right), \quad V_1^{(2), \pm 2}(z) = - \left( U_+^{(3)} + U_-^{(3)} \right), \]
\[ V_1^{(2), \pm 3}(z) = \pm i \left( T^{(3)} \pm W^{(3)} \right), \]
\[ V_2^{(2), 0}(z) = i \sqrt{2} \left( W_+^{(2)} + W_-^{(2)} \right), \quad V_2^{(2), 1}(z) = - \sqrt{2} \left( U_+^{(2)} - U_-^{(2)} \right), \]
\[ V_2^{(2), 2}(z) = - i \sqrt{2} \left( U_+^{(2)} + U_-^{(2)} \right), \quad V_2^{(2), 3}(z) = - \sqrt{2} \left( W_+^{(2)} - W_-^{(2)} \right), \]
\[ V_2^{(2), 4}(z) = - 2 \left[ W^{(4)} + \frac{72}{(37N + 59) + (15N + 37)k} \left( TT^{(2)} - \frac{3}{10} \beta^2 T^{(2)} \right) \right]. \quad (4.15) \]

In doing this, Appendices \( D \) and \( E \) are necessary to check these relations explicitly. Of course, we can further reexpress the above 16 higher spin currents (4.15) in the manifest \( SO(4) \) symmetry by introducing the derivative terms as done in [46].

5 Three-point functions in the extension of the large \( \mathcal{N} = 4 \) linear superconformal algebra

As in section 3, we calculate the three-point functions for the higher spin currents (obtained in previous section) in the extension of large \( \mathcal{N} = 4 \) linear superconformal algebra.

5.1 Eigenvalue equations for the spin-2 current

Let us define the \( U \)-charge as in [21] as follows:
\[ iU_0|(v; 0) > = u(v; 0)|(v; 0) >, \quad iU_0|(0; v) > = u(0; v)|(0; v) >. \quad (5.1) \]
We obtain the eigenvalues \( u(v; 0) \) and \( u(0; v) \) in (5.1) as follows\(^{13}\):
\[ u(v; 0)_a = u(0; v)_a = - \frac{1}{2}, \quad u(v; 0)_b = u(0; v)_b = \frac{1}{2}. \quad (5.2) \]
where \( a = ++, -- \) and \( b = +-, -+ \). We need to know the value of \( u^2 \) in this section and we have \( u^2(v; 0) = u^2(0; v) = \frac{1}{4} \) for all \( (v; 0) \) and \( (0; v) \) states.

\(^{13}\)The \( U \)-charge of light state \( (v; v) \) is zero from the explicit matrix acting on the states as in the unitary case [1]. The conformal dimension for the light state in the linear and the nonlinear version is the same. That is, \( h(v; v) = h(v; v) \) for finite \( N \) and \( k \). Furthermore, the coset components of spin-1 and spin-\( \frac{1}{2} \) currents in the nonlinear version satisfy following OPEs:
\[ iU(z) \left( Q^a \right) \left( V^a \right) (w) = \frac{1}{(z - w)} \left[ - \frac{1}{2} \left( Q^a \right) \right] (w) + \cdots, \quad \tilde{a} = 1, 2, \cdots, N, (N + 1)^*, (N + 2)^*, \cdots, (2N)^*, \]
\[ iU(z) \left( Q^a \right) \left( V^b \right) (w) = \frac{1}{(z - w)} \left[ \frac{1}{2} \left( Q^a \right) \right] (w) + \cdots, \quad \tilde{b} = 1^*, 2^*, \cdots, N^*, N + 1, N + 2, \cdots, 2N. \]
We can obtain the \( U \)-charges of \(|(0; v) >_{\pm \pm} \) states from the above OPEs between \( U(z) \) and \( Q^a(w) \).
From the Goddard-Schwimmer formula [28], the following relation satisfies

\[ T_0\langle v; 0 \rangle \sim \left[ \hat{T} - \frac{1}{(k + N + 2)}UU \right]_0 \langle v; 0 \rangle > \]
\[ = \left[ h(v; 0) + \frac{1}{(k + N + 2)}u^2(v; 0) \right] \langle v; 0 \rangle > \]
\[ = \left[ \frac{(N + 2)}{2(k + N + 2)} \right] \langle v; 0 \rangle > . \]  

(5.3)

In the first line of (5.3), the spin-\( \frac{1}{2} \) current-dependent terms are ignored as in (3.3). In the second line, the result \( h(v; 0) = \frac{(2N+3)}{4(k+N+2)} \) appearing in (3.9) is substituted and the fact that \( u^2(v; 0) = \frac{1}{4} \) is used.

Because the OPE between \( F^a(z) \) and \( Q^\bar{a}(w) \) is regular, \( \partial F^aF_{\bar{a}}(z) \) term in the precise relation between the stress energy tensors in the nonlinear and linear versions (4.2) does not contribute to the eigenvalue equation. Then we obtain the zero mode eigenvalue equation of \( T(z) \) for the state \( \langle 0; v \rangle \) as follows:

\[ T_0\langle 0; v \rangle \sim \left[ \hat{T} - \frac{1}{(k + N + 2)}UU \right]_0 \langle 0; v \rangle > \]
\[ = \left[ h(0; v) + \frac{1}{(k + N + 2)}u^2(0; v) \right] \langle 0; v \rangle > \]
\[ = \left[ \frac{(k + 2)}{2(k + N + 2)} \right] \langle 0; v \rangle > . \]  

(5.4)

In the first line of (5.4), the trivial contribution described before is ignored. In the second line, the result \( h(0; v) = \frac{(2k+3)}{4(k+N+2)} \) appearing in (3.9) is substituted and the fact that \( u^2(0; v) = \frac{1}{4} \) is used. As we expect, there exists \( N \leftrightarrow k \) symmetry between the eigenvalues in (5.3) and (5.4) because the nonlinear version has this symmetry and the extra term coming from \( u^2 \) preserves this symmetry as above.

The large \( N \) limit (3.1) for (5.3) and (5.4) leads to

\[ T_0\langle v; 0 \rangle > = \frac{1}{2} \lambda \langle v; 0 \rangle >, \quad T_0\langle 0; v \rangle > = \frac{1}{2}(1 - \lambda)\langle 0; v \rangle >, \]  

(5.5)

which are exactly the same as the ones in the nonlinear version. Because there are no \( N \)-dependence in the \( U \)-charge of \( (v; 0) \) and \( (0; v) \), the second terms in (5.3) and (5.4) behave as \( \frac{1}{N} \). Therefore the second terms vanish in the large \( N \) ’t Hooft limit. We obtain the equations (5.5).
5.2 Eigenvalue equations for the higher spin currents of spins 2, 3, 4

As in the nonlinear version, we can analyze the three-point functions for the higher spin currents.

5.2.1 Eigenvalue equations for the higher spin-2, 3 currents

Because the higher spin-2 current $T^{(2)}(z)$ in the linear version is the same as the higher spin-2 current $T^{(2)}(z)$ in the nonlinear version, we have the equations (3.10) and (3.11).

Although the six higher spin-3 currents in the linear version are not the same as the corresponding higher spin-3 currents in the nonlinear version, their eigenvalues for the states $|v; 0\rangle$ and $|0; v\rangle$ are exactly the same. Then, we have (3.14) with $T^{(3)}_0$, (3.15) with $W^{(3)}_0$ and (3.16) with $T^{(3)}_0$ and $W^{(3)}_0$. We do not repeat them here.

5.2.2 Eigenvalue equations for the higher spin-4 current

For the final higher spin-4 current, the following eigenvalue equations hold

\[
W^{(4)}_0 |v; 0\rangle = \left[ -\frac{6(2kN + k + 4N^2 - 4N - 12)}{(k + N + 2)^3(15kN + 37k + 37N + 59)} \times d_3 \right] |v; 0\rangle >,
\]

\[
W^{(4)}_0 |0; v\rangle = \left[ -\frac{6k(2N + 1)(2k + N)}{N(k + N + 2)^3(15kN + 37k + 37N + 59)} \times d_4 \right] |0; v\rangle >, \tag{5.6}
\]

where we introduce two factors showing $N \leftrightarrow k$ symmetry

\[
d_3(N, k) \equiv \left( 6k^2N + 16k^2 + 9kN^2 + 55kN + 69k + 18N^2 + 64N + 59 \right),
\]

\[
d_4(N, k) \equiv \left( 6kN^2 + 16N^2 + 9kN^2 + 55kN + 69k + 18k^2 + 64k + 59 \right).
\]

In the large $N$ ’t Hooft limit, the above eigenvalue equations (5.6) lead to

\[
W^{(4)}_0 |v; 0\rangle > = -\frac{12}{5} \lambda(1 + \lambda)(2 + \lambda)|v; 0\rangle >,
\]

\[
W^{(4)}_0 |0; v\rangle > = -\frac{12}{5} (1 - \lambda)(2 - \lambda)(3 - \lambda)|v; 0\rangle >. \tag{5.7}
\]

From the explicit relations (4.15), we can write the above eigenvalue equations in the basis of $[31]$. For example, the higher spin-4 current which is a quasiprimary field $V^{(2)}_2(z)$ is given by the last equation of (4.15). Then we can calculate the following eigenvalue equation

\[
\left[ V^{(2)}_2 \right]_0 |v; 0\rangle > = -2 \left[ W^{(4)}_0 + \frac{72(N - k)}{(37N + 37k + 15Nk + 59)} \left( T_0 T^{(2)}_0 + \frac{1}{5} T^{(2)}_0 \right) \right] |v; 0\rangle >
\]

\[
= \left[ 12(2k + 3N + 5)(2kN + k + 4N^2 - 4N - 12) \right] \frac{1}{5(k + N + 2)^3} |v; 0\rangle >. \tag{5.8}
\]
Note that the factor $\left[ \frac{1}{(37N+37k+15Nk+99)} \right]$ in the quasiprimary field containing $T^{(2)}$ also appears in (5.6). Compared to the previous eigenvalue equation (5.6), the above expression (5.8) is very simple because of the contribution from the extra zero mode in the quasiprimary field. The factor $\left[ \frac{(2kN+k+4N^2-4N-12)}{(k+N+2)^2} \right]$ in (5.8) appears in (3.14) and (3.15). Then the remaining factor $\left[ \frac{(2kN+3k+5)}{(k+N+2)} \right]$ occurs in (5.8). Similarly, for other state we have the following eigenvalue equation

$$
\left[ V^{(2)}_2 \right]_0 |(0; v) > = \left[ \frac{12(2N + 3k + 5)k(2N + 1)(2k + N)}{5N(k + N + 2)^3} \right] |(0; v) > .
$$

(5.9)

The factor $\left[ \frac{k(2N+1)(2k+N)}{N(k+N+2)^2} \right]$ in (5.9) appears in (3.14) and (3.15). Then the remaining factor $\left[ \frac{(2N+3k+5)}{(k+N+2)} \right]$ occurs in (5.9). In the large $N$ ’t Hooft limit, the second and third terms in (5.8) do not contribute the eigenvalue equation. Therefore, we obtain

$$
\left[ V^{(2)}_2 \right]_0 |(v; 0) > = \frac{24}{5} \lambda (1 + \lambda)(2 + \lambda) |(v; 0) > ,
$$

$$
\left[ V^{(2)}_2 \right]_0 |(0; v) > = \frac{24}{5}(1 - \lambda)(2 - \lambda)(3 - \lambda) |(0; v) > .
$$

(5.10)

The eigenvalue equations (5.10) have common $\lambda$ dependence of (5.7). Therefore, the eigenvalue equations for the higher spin-4 currents $W^{(4)}(z)$ and $V^{(2)}_2(z)$, under the large $N$ ’t Hooft limit, are equal to each other up to the overall numerical factor.

As in the nonlinear version, the ratios of the three-point functions can be summarized by (3.20) where all the higher spin currents are replaced with the corresponding higher spin currents in the linear version.

### 6 Conclusions and outlook

In this paper, the lowest higher spin-2 current in the orthogonal $SO(N+4)/SO(N) \times SO(4)$ Wolf space coset theory for general $N$ was obtained in (2.11) and (2.13). The remaining fifteen higher spin currents were determined implicitly in the subsection 2.3. The three-point functions of bosonic (higher) spin currents with two scalars for finite $N$ and $k$ were obtained. The other type of fifteen higher spin currents together with the above lowest higher spin-2 current in the extension of the large $\mathcal{N} = 4$ linear superconformal algebra was determined implicitly in the subsection 4.2. The three-point functions of bosonic (higher) spin currents with two scalars for finite $N$ and $k$ were found. Under the large $N$ ’t Hooft limit, the two types of three-point functions in the nonlinear and linear versions coincided and their ratios were in (3.20).

Further directions can be found as follows:

- Three-point function in the bulk
It is an open problem to obtain the asymptotic symmetry algebra of the higher spin theory on the $AdS_3$ space. One of the motivations in this direction is to determine the three-point functions in the bulk theory and compare the results of this paper with them.

- The general spin $s$-dependence of the three-point function

It is a good exercise to see whether we find the above three-point function with $s = 5$ by considering the next higher spin currents and determine whether the behavior looks like that in [6]. It would be interesting to obtain the three-point functions for the higher spin-$s$ current for general $N$ and $k$.

- The operator product expansion of the 16 higher spin currents in $\mathcal{N} = 4$ superspace

It is known in [46] that the corresponding OPEs were found for the unitary coset theory. It is an open problem to obtain the similar OPEs for the orthogonal coset theory. We expect that the one single $\mathcal{N} = 4$ OPE behaves differently compared to the unitary case because the lowest $\mathcal{N} = 4$ multiplet has a superspin 2. From the experience of [46], it is enough to determine the basic 16 OPEs between the lowest higher spin-2 current and the 16 higher spin currents living in the $\mathcal{N} = 4$ multiplet. Moreover, the change of the higher spin currents is necessary to express them in $SO(4)$ symmetric way. See Appendix $D$ and $E$. Furthermore, it is an open problem to describe the $\mathcal{N} = 4$ Kac-Moody algebra which generalizes the OPEs in the subsection 2.1 and construct the $\mathcal{N} = 4$ stress energy tensor (and the higher spin $\mathcal{N} = 4$ multiplet) in terms of these $\mathcal{N} = 4$ Kac-Moody currents. See also the $\mathcal{N} = 2$ description in [47].

- An extension of small $\mathcal{N} = 4$ linear superconformal algebra

In this construction, the complete OPEs between the 16 currents (of large $\mathcal{N} = 4$ linear superconformal algebra) and the 16 lowest higher spin currents for general $N$ and $k$ should be obtained. In particular, the OPEs between the 16 lowest higher spin currents should be determined. After that we can take the appropriate limits.

- Oscillator formalism for the higher spin currents

It is an open problem to see whether we can see the oscillator formalism in an extension of the large $\mathcal{N} = 4$ linear superconformal algebra in the context of the orthogonal coset theory along the line of [2].

- The next 16 higher spin currents

We can consider the next 16 higher spin currents, where the bosonic currents contain the higher spin currents with spins 3, 4, 5. We would like to analyze the behaviors of the three-point functions to determine whether they behave as what we expect. Furthermore, the basis in [31] is more useful because the defining OPEs between the 16 currents (in the large $\mathcal{N} = 4$ linear superconformal algebra) and the next 16 higher spin currents have already been
presented. In the present paper, each eigenvalue equation for the six higher spin-3 currents in the nonlinear and linear versions has the same expression for general $N$ and $k$. It would be interesting to observe this behavior for the six higher spin-4 currents.

- Three-point functions involving the fermionic (higher spin) currents

It would be interesting (and an open problem) to explicitly obtain the three-point functions with fermionic (higher spin) currents as raised in the unitary case.

- Other approach in order to obtain the conformal dimensions of the orthogonal coset primaries

As described in the introduction, it is an open problem to obtain the conformal dimensions of the orthogonal coset primaries.

**Acknowledgments**

This work was supported by the Mid-career Researcher Program through the National Research Foundation of Korea (NRF) grant funded by the Korean government (MEST) (No. 2012-045385/2013-056327/2014-051185). CA acknowledges warm hospitality from the School of Liberal Arts (and Institute of Convergence Fundamental Studies) at Seoul National University of Science and Technology.
Appendix A  The coset generators with $so(N+4)$ algebra in complex basis

In this Appendix, we present the coset generators. Let us focus on the $N = 4n$ case. Based on the $N = 4$ case [5], we can rearrange them in order to describe the eigenvalue equations efficiently in sections 3 and 5. We describe them as follows:

$$T_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad T_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \cdots,$$

$$T_N = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}, \quad T_{N+1} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & \cdots & 0
\end{pmatrix},$$

$$T_{N+2} = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \cdots, T_{2N} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.$$

The nonzero component $-1$ for the first $N$ generators appears in the $(1, N+2)$-element, $(2, N+2)$-element, $\cdots$, and $(N, N+2)$-element, respectively. The corresponding nonzero component $1$ appears in the $(N+1, 2)$-element, $(N+1, 1)$-element, $\cdots$, $(N+1, N)$, $(N+1, N-1)$-element, respectively. The nonzero component $1$ for the last $N$ generators appears in the $(N+3, 1)$-element, $(N+3, 2)$-element, $\cdots$, and $(N+3, N)$-element, respectively. The
corresponding nonzero component $-1$ appears in the $(2, N + 4)$-element, $(1, N + 4)$-element, \(\cdots\), $(N, N + 4)$, $(N - 1, N + 4)$-element, respectively.

The remaining $2N$ coset generators can be obtained from the above coset generators by transposing. Therefore, we have the $4N$ coset generators as follows:

\[ T_1, \ T_2, \ \cdots, \ T_{2N}, \ T_1^\dagger(\equiv T_{1\ast}), \ T_2^\dagger(\equiv T_{2\ast}), \ \cdots, \ T_{2N}^\dagger(\equiv T_{2N\ast}). \]

For the $N = 4n + 1$ case, we can do the similar rearrangement but we are not presenting them here.

In the coset theory of section 4, the extra generators are located at the last $4 \times 4$ diagonal submatrix. We can generalize these for $N = 4$ \[5\] to the general $N$ as follows:

\[
T_{2N+1} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T_{2N+2} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

\[
T_{2N+3} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

(A.1)

The nonzero components in (A.1) appear in the last $4 \times 4$ matrices. The remaining $N \times N$, $N \times 4$, and $4 \times N$ matrix elements in (A.1) are trivially zero. The remaining three generators
are obtained from the action of both transposing and complex conjugate $T_{(2N+1)^*} = T_{2N+1}^\dagger$, $T_{(2N+2)^*} = T_{2N+2}^\dagger$, and $T_{(2N+3)^*} = T_{2N+3}^\dagger$. We can see that the generators $T_{2N+1}$, $T_{(2N+1)^*}$ and $-(1+i)\sqrt{2}(T_{2N+2} - iT_{2N+2}^*)$ consist of the $su(2)$ algebra. Similarly, the generators $T_{2N+3}$, $T_{(2N+3)^*}$ and $(1-i)\sqrt{2}(T_{2N+2} + iT_{2N+2}^*)$ consist of the other $su(2)$ algebra. The former is the coset generators while the latter is the subgroup generators of the coset theory.

Appendix B  The remaining next lowest higher spin currents

In section 2, the four next higher spin-$\frac{7}{2}$ currents in (2.27) were obtained. In this Appendix, the remaining 12 next higher spin currents in (2.27) are obtained. This Appendix is kind of the defining OPEs for these higher spin currents. Once these higher spin currents are determined explicitly, then we can easily describe the OPEs between the 16 lowest higher spin currents which will be studied in next Appendix C. All the structure constants appearing in the OPEs for $N = 4$ are known. We do not present them (which are rather complicated fractional functions of $k$) in this paper. For generic $N$, we expect that the structures appearing in the OPEs will be the same except the structure constants replaced by $N$-dependent expressions.

Appendix B.1  The six higher spin-4 currents and the higher spin-3 current

Recall that in the unitary case [35], the higher spin-4 current was obtained from the OPE between the particular spin-$\frac{3}{2}$ current and the higher spin-$\frac{7}{2}$ current which is the third component of the $N = 2$ multiplet (which contains the last component as the above higher spin-4 current). We can describe here similarly for the first $N = 2$ multiplet in (2.27).

Let us consider the OPE $\hat{G}_{21}(z) P^{(\frac{7}{2})}(w)$ which gives the higher spin-3 current $P^{(3)}(w)$ and the higher spin-4 current $P^{(4)}(w)$. Recall that the higher spin-$\frac{7}{2}$ current was obtained from (2.29) in the section 2. The spin-$\frac{4}{2}$ current is given in (2.6) with the footnote 8. It turns out that

$$\hat{G}_{21}(z) P^{(\frac{7}{2})}(w) = \frac{1}{(z-w)^3} \left[ c_1 \hat{A}_3 + c_2 \hat{B}_3 \right](w)$$

$$\quad + \frac{1}{(z-w)^3} \left[ -\frac{1}{2} \partial(\text{pole-4}) + c_3 T^{(2)} + c_4 \hat{T} + c_5 \hat{A}_3 \hat{B}_3 \right.\right.$$}

$$\left. \left. \quad + c_6 \left( \hat{A}_- \hat{A}_+ + \hat{A}_3 \hat{A}_3 - i \partial \hat{A}_3 \right) + c_7 \left( \hat{B}_- \hat{B}_+ + \hat{B}_3 \hat{B}_3 - i \partial \hat{B}_3 \right) \right](w)$$

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We do not present all the \(k\)-dependent structure constants \(c_1\)–\(c_{18}\). In the third-order pole, the coefficient \(-\frac{1}{2}\) in the descendant field of spin-1 current located at the fourth-order pole can be obtained from the standard procedure for given spins of the left hand side (\(h_i = \frac{3}{2}\) and \(h_j = \frac{7}{2}\)) and the spin \((h_k = 1)\) of the spin-1 current appearing in the fourth-order pole. We realize that there are no new currents in the third-order pole. There is no descendant field for the spin-2 field (appearing in the third-order pole) in the second-order pole \([35]\) and we see the presence of higher spin-3 current \(P^{(3)}(w)\) as well as two quasiprimary fields. In the first-order pole, we can calculate the numerical coefficient \(\frac{1}{6}\) (\(h_k = 3\)) described before. Furthermore, there exists the new higher spin-4 current \(P^{(4)}(w)\). In order to extract this higher spin-4 current, we should consider the correct nine quasiprimary fields. Most of these quasiprimary fields occurred in the unitary case \([35]\) where the corresponding OPE is more complicated.

Let us calculate OPE \(\hat{G}_{21}(z) Q^{(\frac{7}{2})}(w)\) which gives the higher spin current \(Q^ {+ (4)}(w)\). Again this is what we expect because the second component of \(\mathcal{N} = 2\) stress energy tensor, the spin-\(\frac{3}{2}\) current, provides the second component of the corresponding \(\mathcal{N} = 2\) multiplet containing the first component as the above higher spin-\(\frac{5}{2}\) current \([35]\). With the help of \((2.6)\) with the footnote \([8]\) and \((2.28)\), we obtain the following OPE

\[
\hat{G}_{21}(z) Q^{(\frac{7}{2})}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{B}_- \right](w)
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{2} \partial \text{pole-4} + c_2 \hat{B}_- \hat{A}_3 \right](w)
+ \frac{1}{(z-w)^2} \left[ c_3 U_+^{(3)} + c_4 \hat{A}_3 \hat{B}_- + c_5 \hat{B}_3 \hat{B}_- + c_6 \hat{B}_- T^{(2)} + c_7 \hat{B}_- \hat{T} 
+ c_8 \hat{A}_- (\hat{A}_3 \hat{A}_3 + \hat{A}_- \hat{A}_+ ) + c_9 \hat{B}_- (\hat{B}_3 \hat{B}_3 + \hat{B}_- \hat{B}_+) + c_{10} \hat{B}_- \partial \hat{A}_3 \right]
\]
In this case, the first-order pole in (B.3) gives the higher spin-4 current
and the following OPE

\begin{equation}
\tilde{G}_{12}(z) R^{(4)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \dot{B}_+ \right](w) \\
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{2} \partial \text{(pole-4)} + c_2 \dot{A}_3 \dot{B}_+ \right](w) \\
+ \frac{1}{(z-w)^2} \left[ c_3 V^{(3)} + c_4 \dot{A}_3 \partial \dot{B}_+ + c_5 \dot{B}_3 \partial \dot{B}_+ + c_6 T^{(2)} \dot{B}_+ + c_7 \dot{T} \dot{B}_+ \right] \\
+ \frac{1}{(z-w)} \left[ 16 \partial \text{(pole-2)} + c_{14} \dot{A}_3 \dot{A}_3 \partial \dot{B}_+ + c_{15} \dot{A}_3 \dot{B}_+ \partial \dot{A}_3 + c_{16} \dot{A}_- \dot{A}_3 \dot{A}_+ \dot{B}_+ \right] \\
+ \frac{1}{(z-w)} \left[ 16 \partial \text{(pole-2)} + c_{18} \dot{A}_3 \dot{B}_3 \partial \dot{B}_+ + c_{19} \dot{B}_- \dot{B}_3 \dot{A}_+ + c_{20} \dot{B}_- \dot{A}_3 \dot{B}_+ \right] \\
+ \frac{1}{(z-w)} \left[ c_{21} \dot{B}_- \dot{A}_- \partial \dot{A}_+ + c_{22} \dot{B}_- \dot{B}_- \dot{A}_3 \dot{B}_+ + c_{23} \dot{B}_- \partial^2 \dot{A}_3 + Q^{(4)}(w) \right] + \cdots.
\end{equation}

In the second-order pole of (B.2), there is no new primary field. In the first-order pole, we can see the higher spin-4 current $Q^{(4)}_+(w)$.

Let us calculate OPE $\tilde{G}_{12}(z) R^{(4)}(w)$ which gives the higher spin current $R^{(4)}_-(w)$. The third component of $N = 2$ stress energy tensor provides the third component of the corresponding $N = 2$ multiplet containing the first component as the above higher spin-$\frac{7}{2}$ current [35]. With the help of (2.7) with the footnote [8] and (2.28), we obtain the following OPE

\begin{equation}
\tilde{G}_{12}(z) R^{(4)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \dot{B}_+ \right](w) \\
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{2} \partial \text{(pole-4)} + c_2 \dot{A}_3 \dot{B}_+ \right](w) \\
+ \frac{1}{(z-w)^2} \left[ c_3 V^{(3)} + c_4 \dot{A}_3 \partial \dot{B}_+ + c_5 \dot{B}_3 \partial \dot{B}_+ + c_6 T^{(2)} \dot{B}_+ + c_7 \dot{T} \dot{B}_+ \right] \\
+ \frac{1}{(z-w)} \left[ 16 \partial \text{(pole-2)} + c_{14} \dot{A}_3 \dot{A}_3 \partial \dot{B}_+ + c_{15} \dot{A}_3 \dot{B}_+ \partial \dot{A}_3 + c_{16} \dot{A}_- \dot{A}_3 \dot{A}_+ \dot{B}_+ \right] \\
+ \frac{1}{(z-w)} \left[ 16 \partial \text{(pole-2)} + c_{18} \dot{A}_3 \dot{B}_3 \partial \dot{B}_+ + c_{19} \dot{B}_- \dot{B}_3 \dot{A}_+ + c_{20} \dot{B}_- \dot{A}_3 \dot{B}_+ \right] \\
+ \frac{1}{(z-w)} \left[ c_{21} \dot{B}_+ \dot{A}_- \partial \dot{A}_+ + c_{22} \dot{B}_+ \partial^2 \dot{A}_3 + R^{(4)}_-(w) \right] + \cdots.
\end{equation}

In this case, the first-order pole in (B.3) gives the higher spin-4 current $R^{(4)}_-(w)$.

Now we can consider the other spin-$\frac{3}{2}$ current in the left hand side of (B.2). Then we obtain the following OPE

\begin{equation}
\tilde{G}_{12}(z) Q^{(4)}_+(w) = \frac{1}{(z-w)^4} \left[ c_1 \dot{A}_+ \right](w) \\
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{2} \partial \text{(pole-4)} + c_2 \dot{B}_3 \dot{A}_+ \right](w) \\
+ \frac{1}{(z-w)^2} \left[ c_3 \left( 4 \dot{A}_3 \dot{A}_3 \dot{A}_+ + 2i \dot{A}_3 \partial \dot{A}_+ + \dot{A}_- \dot{A}_+ \dot{A}_+ \right) + \bar{Q}^{(3)}_+ \right](w)
\end{equation}
In the second-order pole of (B.4), there exists a new higher spin-3 current \( \bar{Q}^{(3)}(w) \). It is not clear at the moment how this appears in the different \( \mathcal{N} = 4 \) multiplet. In the OPE \( T^{(2)}(z) U^{(3)}_{-}(w) \) in next Appendix C, we also observe the appearance of this higher spin-3 current \( \bar{Q}^{(3)}(w) \). The first-order pole in (B.4) gives the higher spin-4 current \( \tilde{Q}^{(4)}(w) \).

Now we can consider the other spin-\( \frac{3}{2} \) current in the left hand side of (B.3). Then we obtain the following OPE

\[
\hat{G}_{21}(z) R^{(2)}_{\frac{3}{2}}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{A}_- \right](w) \\
+ \frac{1}{(z-w)^2} \left[ -\frac{1}{2} \partial \text{pole-4} + c_2 \hat{B}_- \hat{A}_- \right](w) \\
+ \frac{1}{(z-w)^2} \left[ c_3 \left(-6\hat{A}_3 \hat{A}_- + 4i\hat{A}_3 \partial \hat{A}_- + \hat{A}_- \hat{A}_- \hat{A}_+ \right) + \tilde{R}^{(3)}_+ \right](w) \\
+ \frac{1}{(z-w)^2} \left[ \frac{1}{6} \partial \text{pole-2} + c_4 \hat{A}_- \hat{B}_+ \hat{B}_- + c_5 \hat{A}_- \partial^2 \hat{B}_3 \\
+ c_6 \hat{B}_3 \hat{A}_- \partial \hat{B}_3 + c_7 \hat{B}_3 \hat{B}_3 \partial \hat{A}_- + c_8 \hat{B}_- \hat{B}_3 \hat{A}_+ \hat{B}_+ + \tilde{R}^{(4)}_+ \right](w) + \cdots \tag{B.5}
\]

In the second-order pole of (B.5), there exists a new higher spin-3 current \( \tilde{R}^{(3)}_+(w) \). It is not clear at the moment, as before, how this appears in the different \( \mathcal{N} = 4 \) multiplet. In the OPE \( T^{(2)}(z) V^{(3)}_+(w) \) in next Appendix C, we also observe the appearance of this higher spin-3 current \( \tilde{R}^{(3)}_+(w) \). The first-order pole in (B.5) gives the higher spin-4 current \( \tilde{R}^{(4)}_+(w) \).

Recall that the OPE between the spin-\( \frac{3}{2} \) current \( \hat{G}_{22}(z) \) and the higher spin current living in the lowest component of \( \mathcal{N} = 2 \) multiplet gives the other higher spin current which belongs to the lowest component of other \( \mathcal{N} = 2 \) multiplet. Let us consider the OPE \( \hat{G}_{22}(z) Q^{(2)}(w) \) which gives the higher spin-4 current \( S^{(4)}(w) \) with the help of section 2. We obtain the following OPE

\[
\hat{G}_{22}(z) Q^{(2)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{A}_3 + c_2 \hat{B}_3 \right](w) \\
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{2} \partial \text{pole-4} + c_3 T^{(2)} + c_4 T + c_5 \hat{A}_3 \hat{B}_3 \\
+ c_6 \left( \hat{A}_- \hat{A}_+ + \hat{A}_3 \hat{A}_3 - i\partial \hat{A}_3 \right) + c_7 \left( \hat{B}_- \hat{B}_+ + \hat{B}_3 \hat{B}_3 - i\partial \hat{B}_3 \right) \right](w)
\]
\[
+ \frac{1}{(z - w)^2} \left[ c_8 (T^{(3)} - W^{(3)}) + c_9 \hat{A}_3 \hat{T} + c_{10} \hat{B}_3 T^{(2)} + c_{11} \hat{B}_3 \hat{T} \right. \\
+ c_{12} \left( \hat{A}_3 \hat{A}_3 \hat{B}_3 + \hat{A}_- \hat{B}_3 \hat{A}_+ - i \hat{B}_3 \partial \hat{A}_3 \right) + c_{13} \hat{B}_3 \partial \hat{B}_3 \\
+ c_{14} \left( \hat{B}_3 \hat{B}_3 \hat{B}_3 + \hat{B}_- \hat{B}_3 \hat{B}_+ \right) + c_{15} \hat{B}_- \partial \hat{B}_3 + c_{16} \hat{B}_+ \partial \hat{B}_- \\
+ c_{17} \left( 4 \hat{A}_3 \partial \hat{A}_3 + 2 \hat{A}_- \partial \hat{A}_+ + 2 \hat{A}_+ \partial \hat{A}_- - (\hat{G}_{11} \hat{G}_{22} + \hat{G}_{21} \hat{G}_{12} - 2 \partial \hat{T}) \right) \\
+ c_{18} \partial^2 \hat{B}_3 + P^{(3)} \right] (w) \\
+ \frac{1}{(z - w)} \left[ \frac{1}{6} \partial (\text{pole-2}) + c_{19} \left( \hat{T} \hat{A}_- \hat{A}_+ - \frac{3}{2} \partial \hat{A}_- \partial \hat{A}_+ - \frac{1}{2} i \partial \hat{T} \hat{A}_3 \right) \right. \\
+ c_{20} \left( \hat{T} \hat{B}_- \hat{B}_+ - \frac{3}{2} \partial \hat{B}_- \partial \hat{B}_+ - \frac{1}{2} i \partial \hat{T} \hat{B}_3 \right) + c_{21} \left( \hat{T} \partial T^{(2)} - \frac{3}{10} \partial^2 T^{(2)} \right) \\
+ c_{22} \left( \hat{T} \hat{T} - \frac{3}{10} \partial^2 \hat{T} \right) + c_{23} \left( \hat{T} \hat{A}_3 \hat{A}_3 - \frac{3}{10} \partial^2 (\hat{A}_3 \hat{A}_3) \right) \\
+ c_{24} \left( \hat{T} \hat{A}_3 \hat{B}_3 - \frac{3}{10} \partial^2 (\hat{A}_3 \hat{B}_3) \right) + c_{25} \left( \hat{T} \hat{B}_3 \hat{B}_3 - \frac{3}{10} \partial^2 (\hat{B}_3 \hat{B}_3) \right) \\
+ c_{26} \left( \hat{T} \partial \hat{A}_3 - \frac{1}{2} \partial \hat{T} \hat{A}_3 - \frac{1}{4} \partial^3 \hat{A}_3 \right) \\
+ c_{27} \left( \hat{T} \partial \hat{B}_3 - \frac{1}{2} \partial \hat{T} \hat{B}_3 - \frac{1}{4} \partial^3 \hat{B}_3 \right) + P^{(4)} + S^{(4)} \right] (w) + \cdots. 
\]

Note that the higher spin-3 current \(P^{(3)}(w)\) and the higher spin-4 current \(P^{(4)}(w)\) appeared in (B.1). The quasiprimary fields appearing in the first-order pole \((B.6)\) occurred in the OPE \((B.1)\).

Therefore, the six higher spin-4 currents and the higher spin-3 current in \((2.27)\) are determined.

**Appendix B.2  The four higher spin-9/2 currents**

Recall that the OPE \((B.1)\) provides the relation between the third component \(P^{(3)}_-(\hat{T})(w)\) and the fourth component \(P^{(4)}(w)\) which live in the first \(\mathcal{N} = 2\) multiplet in \((2.27)\). Now we can apply this description to the second \(\mathcal{N} = 2\) multiplet of \((2.27)\). Then we consider the OPE \(\hat{G}_{21}(z) Q^{(4)}_-(w)\) where the third component of the above \(\mathcal{N} = 2\) multiplet is taken with the same spin-3/2 current. It turns out that

\[
\hat{G}_{21}(z) Q^{(4)}_-(w) = \frac{1}{(z - w)^4} \left[ c_1 \hat{G}_{11} \right] (w) \\
+ \frac{1}{(z - w)^3} \left[ - \frac{1}{3} \partial (\text{pole-4}) + c_2 U^{(3)} + c_3 \hat{B}_- \hat{G}_{12} + c_4 \hat{G}_{11} \hat{A}_3 \right]
\]

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\[ + \quad c_5 \hat{G}_{11} \hat{B}_3 + c_6 \hat{G}_{21} \hat{A}_+ + c_7 \partial \hat{G}_{11} \right] (w) \\
+ \quad \frac{1}{(z-w)^z} \left[ c_8 \left( \hat{T} \hat{G}_{11} - \frac{3}{8} \partial^2 \hat{G}_{11} \right) + \tilde{Q}^{(7)} \right] (w) \\
+ \quad \frac{1}{(z-w)^{\frac{1}{2}}} \left( \frac{1}{7} \partial \text{pole-2} + c_9 \left( \hat{T} U^{(\frac{7}{2})} - \frac{1}{4} \partial^2 U^{(\frac{7}{2})} \right) \right) \\
+ \quad c_{10} \left( -\frac{1}{4} \hat{B}_- \partial^2 \hat{G}_{12} + \partial \hat{B}_- \partial \hat{G}_{12} - \frac{1}{2} \partial^2 \hat{B}_- \hat{G}_{12} + \frac{i}{60} \partial^3 \hat{G}_{11} \right) \\
+ \quad c_{11} \left( \hat{T} \partial \hat{G}_{11} - \frac{3}{4} \hat{T} \hat{G}_{11} - \frac{1}{5} \partial^3 \hat{G}_{11} \right) \\
+ \quad c_{12} \left( \frac{1}{2} \hat{G}_{11} \partial^2 \hat{A}_3 + \partial \hat{G}_{11} \partial \hat{A}_3 - \frac{1}{4} \partial^2 \hat{G}_{11} \hat{A}_3 + \frac{1}{30} \partial^3 \hat{G}_{11} \right) \\
+ \quad c_{13} \left( \frac{1}{2} \hat{G}_{21} \partial^2 \hat{A}_+ + \partial \hat{G}_{21} \partial \hat{A}_+ - \frac{1}{4} \partial^2 \hat{G}_{21} \hat{A}_+ + \frac{1}{15} \partial^3 \hat{G}_{11} \right) \\
+ \quad c_{14} \left( -\frac{1}{4} \partial^2 \hat{G}_{11} \hat{B}_3 + \partial \hat{G}_{11} \partial \hat{B}_3 - \frac{1}{2} \hat{G}_{11} \partial^2 \hat{B}_3 - \frac{i}{30} \partial^3 \hat{G}_{11} \right) + \tilde{Q}^{(7)} \right] (w) \\
+ \quad \cdots \right. \\
(B.7) 

In the third-order pole of (B.7), the coefficient \(-\frac{1}{3}\) in the descendant field of spin-\(\frac{3}{2}\) current located at the fourth-order pole can be obtained from the standard procedure for given spins of the left hand side \((h_i = \frac{3}{2} \text{ and } h_j = 4)\) and the spin \((h_k = \frac{3}{2})\) of the spin-\(\frac{3}{2}\) current appearing in the fourth-order pole. There is no descendant field for the spin-\(\frac{5}{2}\) field (appearing in the third-order pole) in the second-order pole \((h_k = \frac{5}{2})\). In the second-order pole of (B.7), there exists a new higher spin-\(\frac{7}{2}\) current \(\tilde{Q}^{(\frac{7}{2})}(w)\). In the OPE \(T_+^{(\frac{7}{2})}(z) U_+^{(3)}(w)\) in next Appendix C, we also observe the appearance of this higher spin-\(\frac{7}{2}\) current \(\tilde{Q}^{(\frac{7}{2})}(w)\). In the first-order pole, the coefficient \(\frac{1}{3}\) in the descendant field of spin-\(\frac{7}{2}\) current located at the second-order pole \((h_k = \frac{7}{2})\) can be obtained according to previous analysis. There are various quasiprimary fields. Two of them contain the stress energy tensor and the remaining four of them do not contain the stress energy tensor. 

Now we can apply the above description to the third \(\mathcal{N} = 2\) multiplet of (2.27). Then we consider the OPE \(\hat{G}_{21}(z) \hat{R}_{\hat{4}}(w)\) where the third component of the above \(\mathcal{N} = 2\) multiplet is taken with the same spin-\(\frac{3}{2}\) current. It turns out that we obtain 

\[
\hat{G}_{21}(z) \hat{R}_{\hat{4}}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{G}_{22} \right] (w) \\
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{3} \partial \text{pole-4} + c_2 V^{(\frac{7}{2})} + c_3 \hat{A}_3 \hat{G}_{22} + c_4 \hat{A}_- \hat{G}_{12} + c_5 \hat{B}_3 \hat{G}_{22} \right] (w) \\
+ \frac{1}{(z-w)^2} \left[ c_6 \hat{G}_{21} \hat{B}_+ + c_7 \partial \hat{G}_{22} \right] (w)
\]
In the second-order pole of (B.8), there exists a new higher spin-$\frac{7}{2}$ current $\tilde{R}^{(\frac{7}{2})}(w)$. In the OPE $T^{(\frac{7}{2})}(z) V^{(3)}(w)$ in next Appendix C, we also observe the appearance of this higher spin-$\frac{7}{2}$ current $\tilde{R}^{(\frac{7}{2})}(w)$. In the first-order pole, there are various quasiprimary fields which can be analyzed before.

Recall that the OPE (B.2) provides the relation between the first component $Q^{(\frac{2}{3})}(w)$ and the second component $Q^{(4)}(w)$ which live in the second $\mathcal{N} = 2$ multiplet in (2.27). Now we can apply this description to the fourth $\mathcal{N} = 2$ multiplet of (2.27). Then we consider the OPE $\hat{G}_{21}(z) S^{(4)}(w)$ where the first component of the above $\mathcal{N} = 2$ multiplet is taken with the same spin-$\frac{3}{2}$ current. Then we obtain

\[
\hat{G}_{21}(z) S^{(4)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{G}_{21} \right](w) \\
+ \frac{1}{(z-w)^3} \left[ -\frac{1}{3} \partial (\text{pole-4}) + c_2 T^{(\frac{7}{2})}_+ + c_3 \hat{B}_- \hat{G}_{22} + c_4 \hat{G}_{11} \hat{A}_- \\
+ c_5 \hat{G}_{21} \hat{A}_3 + c_6 \hat{G}_{21} \hat{B}_3 + c_7 \partial \hat{G}_{21} \right](w) \\
+ \frac{1}{(z-w)^2} \left[ c_8 \left( \hat{T} \hat{G}_{21} - \frac{3}{8} \partial^2 \hat{G}_{21} \right) + \tilde{S}_+^{(\frac{7}{2})} \right](w) \\
+ \frac{1}{(z-w)} \left[ \frac{1}{4} \partial (\text{pole-2}) + c_9 \left( \hat{T} T^{(\frac{7}{2})}_+ - \frac{1}{4} \partial^2 T^{(\frac{7}{2})}_+ \right) \\
+ c_{10} \left( \hat{T} \partial \hat{G}_{21} - \frac{3}{4} \partial \hat{T} \hat{G}_{21} - \frac{1}{5} \partial^3 \hat{G}_{21} \right) \\
+ c_{11} \left( -\frac{1}{4} \hat{B}_- \partial^2 \hat{G}_{22} + \partial \hat{B}_- \partial \hat{G}_{22} - \frac{1}{2} \partial^2 \hat{B}_- \hat{G}_{22} - \frac{i}{60} \partial^3 \hat{G}_{21} \right) \right].
\]
In the second-order pole of (B.9), there exists a new higher spin-$\frac{7}{2}$ current $\tilde{S}^{(\tilde{7})}_+(w)$. In the first-order pole, there are various quasiprimary fields which can be analyzed before. The first-order pole in (B.9) gives the higher spin-$\frac{9}{2}$ current $S^{(9)}_+(w)$.

Recall that the OPE (B.4) provides the relation between the first component $Q^{(\tilde{7})}_+(w)$ and the third component $Q^{(4)}_-(w)$ which live in the second $\mathcal{N} = 2$ multiplet in (2.27). Now we can apply this description to the fourth $\mathcal{N} = 2$ multiplet of (2.27). Then we consider the OPE $\hat{G}_{12}(z) S^{(4)}_+(w)$ where the first component of the above $\mathcal{N} = 2$ multiplet is taken with the same spin-$\frac{3}{2}$ current. Then we obtain

$$
\hat{G}_{12}(z) S^{(4)}_+(w) = \frac{1}{(z - w)^3} \left[ c_1 \hat{G}_{12} \right](w) + \frac{1}{(z - w)^3} \left[ -\frac{1}{3} \partial (\text{pole-4}) + c_2 T^{(\tilde{7})}_+ + c_3 \hat{A}_3 \hat{G}_{12} + c_4 \hat{A}_+ \hat{G}_{22} + c_5 \hat{B}_3 \hat{G}_{12} + c_6 \hat{G}_{11} \hat{B}_+ + c_7 \partial \hat{G}_{12} \right](w) + \frac{1}{(z - w)^2} \left[ c_8 \left( \hat{T} \hat{G}_{12} - \frac{3}{8} \partial^2 \hat{G}_{12} \right) + \tilde{S}^{(\tilde{7})}_- \right](w) + \frac{1}{(z - w)} \left[ \frac{1}{4} \partial (\text{pole-2}) + c_9 \left( \hat{T} T^{(\tilde{7})}_- - \frac{1}{4} \partial^2 T^{(\tilde{7})}_- \right) \right] + \frac{1}{4} \hat{A}_+ \partial^2 \hat{G}_{22} + \partial \hat{A}_+ \partial \hat{G}_{22} - \frac{1}{2} \partial^2 \hat{A}_+ \hat{G}_{22} + \frac{i}{60} \partial^3 \hat{G}_{12} + \frac{1}{4} \hat{A}_3 \partial^2 \hat{G}_{12} + \partial \hat{A}_3 \partial \hat{G}_{12} - \frac{1}{2} \partial^2 \hat{A}_3 \hat{G}_{12} - \frac{i}{120} \partial^3 \hat{G}_{12} + \frac{1}{4} \hat{B}_3 \partial^2 \hat{G}_{12} + \partial \hat{B}_3 \partial \hat{G}_{12} - \frac{1}{2} \partial^2 \hat{B}_3 \hat{G}_{12} - \frac{i}{120} \partial^3 \hat{G}_{12} + \frac{1}{4} \partial^2 \hat{G}_{11} \hat{B}_+ + \partial \hat{G}_{11} \partial \hat{B}_+ - \frac{1}{2} \hat{G}_{11} \partial^2 \hat{B}_+ - \frac{i}{15} \partial^3 \hat{G}_{12} + \frac{1}{4} \hat{T} \partial \hat{G}_{12} - \frac{3}{4} \partial \hat{T} \hat{G}_{12} - \frac{1}{5} \partial^3 \hat{G}_{12} + \tilde{S}^{(\tilde{7})}_- \right](w) + \cdots.
$$

In the second-order pole of (B.10), there exists a new higher spin-$\frac{7}{2}$ current $\tilde{S}^{(\tilde{7})}_+(w)$. The first-order pole in (B.10) gives the higher spin-$\frac{9}{2}$ current $S^{(9)}_+(w)$. 

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Therefore, the four higher spin-$\frac{9}{2}$ currents in $[2.27]$ are determined.

**Appendix B.3  The higher spin-5 current**

Let us consider the following OPE

$$\hat{G}_{21}(z) S_{-}^{(3)}(w) = \frac{1}{(z-w)^5} \left[ c_1 \hat{A}_3 + c_2 \hat{B}_3 \right](w) + \frac{1}{(z-w)^4} \left[ - \partial (\text{pole-5}) \right] + c_3 T^{(2)} + c_4 \hat{T} + c_5 \hat{A}_3 \hat{A}_3 + c_6 \hat{A}_3 \hat{B}_3 + c_7 \hat{A}_- \hat{A}_+ + c_8 \hat{B}_3 \hat{B}_3 + c_9 \hat{B}_- \hat{B}_+ + c_{10} \hat{A}_3 + c_{11} \partial \hat{B}_3 \right](w)$$

$$+ \frac{1}{(z-w)^3} \left[ - \frac{1}{4} \partial (\text{pole-4}) - \frac{1}{12} \partial^2 (\text{pole-5}) \right] + c_{12} T^{(3)} + c_{13} W^{(3)} + c_{14} \hat{A}_3 T^{(2)} + c_{15} \hat{A}_3 \hat{T} + c_{16} \hat{A}_3 \hat{A}_3 \hat{A}_3 + c_{17} \hat{A}_3 \hat{A}_3 \hat{B}_3 + c_{18} \hat{A}_3 \hat{B}_3 \hat{B}_3 + c_{19} \hat{A}_3 \partial \hat{A}_3 + c_{20} \hat{A}_3 \partial \hat{B}_3 + c_{21} \hat{A}_- \hat{A}_3 \hat{A}_+ + c_{22} \hat{A}_- \hat{B}_3 \hat{A}_+ + c_{23} \hat{A}_- \partial \hat{A}_+ + c_{24} \hat{A}_- \partial \hat{A}_- + c_{25} \hat{B}_3 T^{(2)} + c_{26} \hat{B}_3 \hat{T} + c_{27} \hat{B}_3 \hat{B}_3 \hat{B}_3 + c_{28} \hat{B}_3 \partial \hat{A}_3 + c_{29} \hat{B}_3 \partial \hat{B}_3 + c_{30} \hat{B}_- \hat{A}_3 \hat{B}_+ + c_{31} \hat{B}_- \hat{B}_3 \hat{B}_+ + c_{32} \hat{B}_- \partial \hat{B}_+ + c_{33} \hat{B}_+ \partial \hat{B}_- + c_{34} \hat{G}_{11} \hat{G}_{22} + c_{35} \hat{G}_{21} \hat{G}_{12} + c_{36} \partial \hat{T} + c_{37} \partial^2 \hat{A}_3$$

$$+ c_{38} \partial^2 \hat{B}_3 + c_{39} \left( \hat{T} \hat{A}_3 - \frac{1}{2} \partial^2 \hat{A}_3 \right) + c_{40} \left( \hat{T} \hat{B}_3 - \frac{1}{2} \partial^2 \hat{B}_3 \right) + c_{41} P^{(3)} \right](w)$$

$$+ \frac{1}{(z-w)^2} \left[ c_{42} \left( \hat{T} \hat{\partial} \hat{A}_3 - \frac{1}{2} \partial \hat{T} \hat{A}_3 - \frac{1}{4} \partial^2 \hat{A}_3 \right) + c_{43} \left( \hat{T} \hat{\partial} \hat{B}_3 - \frac{1}{2} \partial \hat{T} \hat{B}_3 - \frac{1}{4} \partial^2 \hat{B}_3 \right) \right] + c_{44} \left( \hat{T} \hat{T}^{(2)} - \frac{3}{10} \partial^2 \hat{T}^{(2)} \right) + c_{45} \left( \hat{T} \hat{T} - \frac{3}{10} \partial^2 \hat{T} \right)$$

$$+ c_{46} \left( \hat{T} \hat{A}_3 \hat{A}_3 - \frac{3}{10} \partial^2 \hat{A}_3 \hat{A}_3 \hat{A}_3 \right) + c_{47} \left( \hat{T} \hat{B}_3 \hat{B}_3 - \frac{3}{10} \partial^2 \hat{B}_3 \hat{B}_3 \hat{B}_3 \right)$$

$$+ c_{48} \left( \hat{T} \hat{A}_- \hat{A}_+ - \frac{3}{2} \hat{\partial} \hat{A}_- \hat{\partial} \hat{A}_+ + \frac{i}{2} \hat{\partial} \hat{T} \hat{A}_3 \right) + c_{49} \left( \hat{T} \hat{B}_3 \hat{B}_3 - \frac{3}{10} \partial^2 \hat{B}_3 \hat{B}_3 \right)$$

$$+ c_{50} \left( \hat{T} \hat{B}_- \hat{B}_+ - \frac{3}{2} \hat{\partial} \hat{B}_- \hat{\partial} \hat{B}_+ - \frac{i}{2} \hat{\partial} \hat{T} \hat{B}_3 \right) + \hat{\bar{S}}^{(4)} \right](w)$$

$$+ \frac{1}{(z-w)} \left[ \frac{1}{8} \partial (\text{pole-2}) \right] + c_{51} \left( \hat{T} T^{(3)} - \frac{3}{14} \partial^2 T^{(3)} \right) + c_{52} \left( \hat{T} W^{(3)} - \frac{3}{14} \partial^2 W^{(3)} \right)$$

$$+ c_{53} \left( \hat{T} \hat{A}_3 T^{(2)} - \frac{1}{2} \partial^2 \hat{A}_3 T^{(2)} - \frac{3}{10} \hat{A}_3 \partial^2 T^{(2)} \right)$$
\[ + c_{54} \left( \dot{T} \dot{A}_3 \dot{T} - \frac{1}{2} \partial^2 \dot{A}_3 \dot{T} - \frac{3}{10} \dot{A}_3 \partial^2 \dot{T} \right) + c_{55} \left( \dot{T} \dot{A}_3 \dot{A}_3 \dot{A}_3 - \frac{9}{4} \partial \dot{A}_3 \dot{A}_3 \partial \dot{A}_3 \right) \\
+ c_{56} \left( \dot{T} \dot{A}_3 \dot{A}_3 \dot{B}_3 - \frac{3}{2} \dot{A}_3 \partial \dot{A}_3 \partial \dot{B}_3 - \frac{3}{4} \partial \dot{A}_3 \partial \dot{A}_3 \dot{B}_3 \right) \\
+ c_{57} \left( \dot{T} \dot{A}_3 \dot{B}_3 \dot{B}_3 - \frac{3}{2} \partial \dot{A}_3 \dot{B}_3 \partial \dot{B}_3 - \frac{3}{4} \dot{A}_3 \partial \dot{B}_3 \partial \dot{B}_3 \right) \\
+ c_{58} \left( \dot{T} \dot{A}_3 \partial \dot{A}_3 - \frac{1}{2} \partial \dot{T} \dot{A}_3 \dot{A}_3 - \frac{1}{2} \partial \dot{A}_3 \partial^2 \dot{A}_3 - \frac{1}{6} \partial^2 \dot{A}_3 \dot{A}_3 \right) \\
+ c_{59} \left( \dot{T} \dot{A}_3 \partial \dot{B}_3 - \frac{1}{2} \partial \dot{T} \dot{A}_3 \dot{B}_3 - \frac{1}{2} \partial \dot{A}_3 \partial^2 \dot{B}_3 - \frac{1}{6} \dot{A}_3 \partial^2 \dot{B}_3 \right) \\
+ c_{60} \left( - \frac{3}{4} \partial \dot{A}_3 \partial \dot{A}_3 \partial \dot{A}_+ + \dot{T} \dot{A}_3 \dot{A}_+ \dot{A}_3 + \frac{3}{4} \partial \dot{A}_3 \partial \dot{A}_3 \partial \dot{A}_+ - \frac{3}{4} \partial \dot{A}_3 \partial \dot{A}_3 \dot{A}_+ \right) \\
- \frac{i}{2} \partial \dot{T} \dot{A}_3 \dot{A}_3 + \frac{i}{2} \partial \dot{T} \dot{A}_3 \dot{A}_+ + \frac{i}{4} \partial^2 \dot{A}_3 \partial \dot{A}_3 - \frac{i}{4} \partial^2 \dot{A}_3 \partial \dot{A}_+ + \frac{1}{10} \partial^2 \dot{T} \dot{A}_3 \\
- \frac{i}{24} \partial \dot{A}_3 \dot{A}_3 + \frac{i}{24} \partial \dot{A}_3 \dot{A}_+ \right) \\
+ c_{61} \left( - \frac{i}{24} \dot{A}_3 \partial^2 \dot{B}_3 - \frac{3}{4} \partial \dot{A}_3 \partial \dot{B}_3 \partial \dot{A}_+ + \dot{T} \dot{A}_3 \dot{B}_3 \dot{A}_+ + \frac{i}{4} \partial \dot{A}_3 \partial^2 \dot{B}_3 - \frac{3}{4} \partial \dot{A}_3 \partial \dot{B}_3 \partial \dot{A}_+ \\
- \frac{3}{4} \partial \dot{A}_3 \partial \dot{B}_3 \dot{A}_+ - \frac{i}{2} \partial \dot{T} \dot{A}_3 \dot{B}_3 \right) \\
+ c_{62} \left( \dot{A}_3 \partial \dot{A}_3 \partial \dot{A}_+ + \frac{1}{3} \partial^2 \dot{A}_3 \partial \dot{A}_+ - \partial \dot{A}_3 \partial \dot{A}_3 \partial \dot{A}_+ - \frac{1}{3} \dot{A}_3 \partial \dot{A}_3 \partial^2 \dot{A}_+ \right) \\
+ c_{63} \left( \dot{T} \partial \dot{A}_3 \partial \dot{A}_+ - \frac{1}{2} \partial \dot{T} \dot{A}_3 \dot{A}_+ - \frac{1}{2} \partial \dot{A}_3 \partial \dot{A}_3 \partial \dot{A}_+ + \frac{i}{10} \partial \dot{A}_3 \partial \dot{A}_3 \partial \dot{A}_+ \right) \\
+ c_{64} \left( \dot{T} \dot{B}_3 \partial^2 \dot{T} - \frac{1}{2} \partial^2 \dot{B}_3 \partial^2 \dot{T} - \frac{3}{10} \dot{B}_3 \partial^2 \dot{T} \right) \\
+ c_{65} \left( \dot{T} \dot{B}_3 \dot{T} - \frac{1}{2} \partial^2 \dot{B}_3 \dot{T} - \frac{3}{10} \dot{B}_3 \partial^2 \dot{T} \right) + c_{66} \left( \dot{T} \dot{B}_3 \dot{B}_3 \dot{B}_3 - \frac{9}{4} \partial \dot{B}_3 \partial \dot{B}_3 \partial \dot{B}_3 \right) \\
+ c_{67} \left( \dot{T} \dot{B}_3 \partial \dot{A}_3 - \frac{1}{2} \partial \dot{T} \dot{A}_3 \dot{B}_3 - \frac{1}{2} \partial \dot{A}_3 \partial \dot{B}_3 - \frac{1}{6} \partial \dot{A}_3 \dot{B}_3 \right) \\
+ c_{68} \left( \dot{T} \dot{B}_3 \partial \dot{B}_3 - \frac{1}{2} \partial \dot{T} \dot{B}_3 \dot{B}_3 - \frac{1}{2} \partial \dot{B}_3 \partial \dot{B}_3 - \frac{1}{6} \partial \dot{B}_3 \dot{B}_3 \right) \\
+ c_{69} \left( - \frac{i}{24} \dot{A}_3 \partial^3 \dot{B}_3 - \frac{3}{4} \partial \dot{A}_3 \partial \dot{A}_3 \partial \dot{B}_3 + \dot{T} \dot{A}_3 \dot{B}_3 \dot{A}_+ + \frac{i}{4} \partial \dot{A}_3 \partial^2 \dot{B}_3 - \frac{3}{4} \partial \dot{A}_3 \dot{B}_3 \partial \dot{B}_3 \\
- \frac{3}{4} \partial \dot{A}_3 \dot{B}_3 \dot{B}_3 + \frac{i}{2} \partial \dot{T} \dot{A}_3 \dot{B}_3 \right) \\
+ c_{70} \left( - \frac{3}{4} \dot{B}_3 \partial \dot{B}_3 \partial \dot{B}_3 - \frac{i}{2} \partial \dot{T} \dot{B}_3 \partial \dot{B}_3 - \frac{3}{4} \partial \dot{B}_3 \dot{B}_3 \partial \dot{B}_3 \right) \\
- \frac{3}{4} \partial \dot{B}_3 \partial \dot{B}_3 \dot{B}_3 + \frac{i}{2} \partial \dot{T} \dot{B}_3 \dot{B}_3 + \frac{3i}{4} \partial^2 \dot{B}_3 \partial \dot{B}_3 \\
- \frac{i}{4} \partial \dot{B}_3 \partial \dot{B}_3 + \frac{1}{10} \partial \dot{A}_3 \dot{B}_3 \dot{B}_3 + \frac{i}{8} \partial \dot{B}_3 \dot{B}_3 + \frac{i}{24} \partial \dot{B}_3 \dot{B}_3 \dot{B}_3 \\
+ c_{71} \left( \dot{B}_3 \partial \dot{B}_3 \partial \dot{B}_3 + \frac{1}{3} \partial \dot{B}_3 \dot{B}_3 \dot{B}_3 - \partial \dot{B}_3 \partial \dot{B}_3 \dot{B}_3 - \frac{1}{3} \dot{B}_3 \dot{B}_3 \partial \dot{B}_3 \right) \]
\[ c_{72} \left( \hat{T} \partial \hat{B}_- \hat{B}_+ - \frac{1}{2} \partial \hat{T} \hat{B}_- \hat{B}_+ - \frac{1}{2} \partial^2 \hat{B}_- \partial \hat{B}_+ + \frac{i}{10} \partial^2 \hat{B}_3 - \frac{1}{6} \partial^3 \hat{B}_- \hat{B}_+ \right) \\
+ c_{73} \left( \hat{T} \hat{G}_{11} \hat{G}_{22} - \hat{T} \partial \hat{T} - \frac{2iN}{3(N + 2 + k)} \hat{T} \partial^2 \hat{A}_3 + \frac{2i}{3(N + 2 + k)} \hat{T} \partial^2 \hat{B}_3 - \partial \hat{G}_{11} \partial \hat{G}_{22} \right) \\
- \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_3 \hat{A}_3 = 2 \cdot \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_3 \hat{B}_3 - \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_- \hat{A}_+ \\
- \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{B}_3 \hat{B}_3 - 1 \cdot \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{B}_- \hat{B}_+ + \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_3 \\
+ \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{B}_3 + \frac{1}{6} \partial^3 \hat{T} + \frac{iN}{10(N + 2 + k)} \partial^4 \hat{A}_3 - \frac{ik}{10(N + 2 + k)} \partial^4 \hat{B}_3 \\
+ c_{74} \left( \hat{T} \hat{G}_{21} \hat{G}_{12} - \hat{T} \partial \hat{T} + \frac{2iN}{3(N + 2 + k)} \hat{T} \partial^2 \hat{A}_3 + \frac{2i}{3(N + 2 + k)} \hat{T} \partial^2 \hat{B}_3 - \partial \hat{G}_{21} \partial \hat{G}_{12} \right) \\
- \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_3 \hat{A}_3 + \frac{2}{(N + 2 + k)} \partial \hat{T} \hat{A}_3 \hat{B}_3 - \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_- \hat{A}_+ \\
- \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{B}_3 \hat{B}_3 - \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{B}_- \hat{B}_+ + \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_3 \\
+ \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{B}_3 + \frac{1}{6} \partial^3 \hat{T} - \frac{iN}{10(N + 2 + k)} \partial^4 \hat{A}_3 - \frac{ik}{10(N + 2 + k)} \partial^4 \hat{B}_3 \\
+ c_{75} \left( \hat{T} \partial \hat{T}^{(2)} - \partial \hat{T} \hat{T}^{(2)} - \frac{1}{6} \partial^3 \hat{T}^{(2)} \right) \\
+ c_{76} \left( \hat{T} \partial^2 \hat{A}_3 - \frac{3}{2} \partial \hat{T} \partial \hat{A}_3 + \frac{3}{10} \partial^2 \hat{A}_3 - \frac{3}{20} \partial^4 \hat{A}_3 \right) \\
+ c_{77} \left( \hat{T} \partial^2 \hat{B}_3 - \frac{3}{2} \partial \hat{T} \partial \hat{B}_3 + \frac{3}{10} \partial^2 \hat{B}_3 - \frac{3}{20} \partial^4 \hat{B}_3 \right) \\
+ c_{78} \left( \hat{T} \hat{P}^{(3)} - \frac{3}{14} \partial^2 \hat{P}^{(3)} \right) + S^{(5)} \right) (w) + \cdots. \tag{B.11} \]

In the fourth-order pole, the coefficient \(-1\) in the descendant field of spin-1 current located at the fifth-order pole can be obtained from the standard procedure for given spins of the left hand side \((h_i = \frac{3}{2} \text{ and } h_j = \frac{9}{2})\) and the spin \((h_k = 1)\) of the spin-1 current appearing in the fifth-order pole. There is no descendant field for the spin-3 field (appearing in the third-order pole) in the second-order pole \((h_k = 3)\). Furthermore, there exists a new higher spin-4 current \(\tilde{S}^{(4)}(w)\). In the OPE \(T^{(2)}(z) W^{(4)}(w)\) in next Appendix C, we also observe the appearance of this higher spin-4 current \(\tilde{S}^{(4)}(w)\). In the first-order pole, the coefficient \(\frac{1}{8}\) in the descendant field of spin-4 current located at the second-order pole \((h_k = 4)\) can be obtained according to previous analysis. There are also various quasiprimary fields. Two of them have their \(N\)-dependence on the coefficient functions. The first-order pole in \([B.11]\) gives the higher spin-5 current \(S^{(5)}(w)\).

Therefore, the higher spin-5 current in \((2.27)\) is determined.
Appendix C  The next higher spin currents appearing in the OPEs between the lowest higher spin currents

As described before, in section 2, the four next higher spin-\( \frac{7}{2} \) currents in (2.27) were obtained and in Appendix B, the remaining 12 next higher spin currents in (2.27) were obtained. In this Appendix, we would like to see them in the OPEs between the 16 lowest higher spin currents. All the structure constants appearing in the OPEs for \( N = 4 \) are known. We do not present them (which are rather complicated fractional functions of \( k \)) in this paper.

In subsection 2.4, we have seen the four higher spin-\( \frac{7}{2} \) currents. We will see how the remaining higher spin currents appear in the right hand side of OPEs between the 16 higher spin currents. The lowest higher spin-3 current will appear at the end of this Appendix. Then we can start with the higher spin-4 currents.

- The higher spin-4 current in the OPE \( T^{(2)}(z) U_-^{(3)}(w) \)

Let us consider the following OPE

\[
T^{(2)}(z)U_-^{(3)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{A}_+ \right](w) + \frac{1}{(z-w)^2} \left[ c_2 U_-^{(3)} + c_3 \hat{A}_3 \partial \hat{A}_+ + c_4 \hat{A}_+ \hat{T} + c_5 \hat{A}_+ \hat{A}_3 \hat{A}_3 + c_6 \hat{A}_+ \hat{A}_+ \hat{A}_- \right. \\
+ c_7 \hat{A}_+ \hat{B}_3 \hat{B}_3 + c_8 \hat{A}_+ \hat{B}_+ \hat{B}_- + c_9 \hat{A}_+ \partial \hat{A}_3 + c_{10} \hat{A}_+ \partial \hat{B}_3 + c_{11} \hat{B}_3 \partial \hat{A}_+ \right. \\
+ c_{12} \hat{G}_{11} \hat{G}_{12} + c_{13} \partial^2 \hat{A}_+ - \tilde{Q}_-^{(3)} \right](w) + \frac{1}{(z-w)} \left[ \frac{1}{3} \partial \text{pole-2} + c_{14} \left( \hat{T} \partial \hat{A}_+ - \frac{1}{2} \partial \hat{T} \hat{A}_+ - \frac{1}{4} \partial^2 \hat{A}_+ \right) + \tilde{Q}_-^{(4)} + Q_-^{(4)} \right](w) + \ldots \tag{C.1}
\]

In the first-order pole of (C.1), the coefficient \( \frac{1}{3} \) in the descendant field of spin-3 current located at the second-order pole can be obtained from the standard procedure for given spins of the left hand side (\( h_i = 2 \) and \( h_j = 3 \)) and the spin (\( h_k = 3 \)) of the spin-3 current appearing in the second-order pole. There exists a new higher spin-4 current \( \tilde{Q}_-^{(4)}(w) \).

- The higher spin-4 current in the OPE \( T^{(2)}(z) V_+^{(3)}(w) \)

We calculate the following OPE

\[
T^{(2)}(z)V_+^{(3)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{A}_- \right](w) + \frac{1}{(z-w)^2} \left[ c_2 V_+^{(3)} + c_3 \hat{A}_3 \partial \hat{A}_- + c_4 \hat{A}_- \hat{T} + c_5 \hat{A}_- \hat{A}_3 \hat{A}_3 + c_6 \hat{A}_- \hat{B}_3 \hat{B}_3 \right.
\]
\[\begin{align*}
&+ c_7 \hat{A}_- \hat{B}_+ \hat{B}_- + c_8 \hat{A}_- \partial \hat{A}_3 + c_9 \hat{A}_- \partial \hat{B}_3 + c_{10} \hat{A}_+ \hat{A}_- \hat{A}_- + c_{11} \hat{B}_3 \partial \hat{A}_- \\
&+ c_{12} \hat{G} \hat{G} + c_{13} \partial^2 \hat{A}_- - \hat{R}^{(3)}_+(w) \\
&+ \frac{1}{(z - w)} \left[ \frac{1}{3} \partial^3(\text{pole-2}) + c_{14} \left( \hat{T} \partial \hat{A}_- - \frac{1}{2} \partial \hat{T} \hat{A}_- - \frac{1}{4} \partial^3 \hat{A}_- \right) + \hat{R}^{(4)}_+ + \hat{R}^{(4)}_+ \right](w) \\
&+ \cdots \quad \text{(C.2)}
\end{align*}\]

Again the first-order pole of \((\text{C.2})\) contains the new higher spin-4 current \(\hat{R}^{(4)}_+(w)\).

- The higher spin-4 current in the OPE \(T_+(z) U^{(\hat{\mathcal{g}})}(w)\)

Let us calculate the following OPE

\[T_+(z) U^{(\hat{\mathcal{g}})}(w) = \frac{1}{(z - w)^4} \left[ c_1 \hat{B}_- \right](w) + \frac{1}{(z - w)^3} \left[ \frac{1}{2} \partial^3(\text{pole-4}) + c_2 \hat{A}_3 \hat{B}_- \right](w) + \frac{1}{(z - w)^2} \left[ \frac{1}{12} \partial^2(\text{pole-4}) - \frac{1}{12} \partial^2(\text{pole-3}) + c_3 U^{(3)}_+ + c_4 \hat{A}_3 \hat{A}_3 \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{16} \hat{A}_3 \hat{A}_3 \hat{B}_- + c_{17} \hat{A}_3 \hat{A}_3 \hat{B}_- + c_{18} \hat{A}_3 \hat{B}_3 \hat{B}_- + c_{19} \hat{A}_3 \hat{B}_- T \right] + \frac{1}{(z - w)} \left[ \frac{1}{12} \partial^2(\text{pole-4}) - \frac{1}{12} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{20} \hat{A}_3 \hat{B}_- \partial \hat{A}_3 + c_{21} \hat{A}_3 \hat{B}_- \partial \hat{B}_3 + c_{22} \hat{A}_3 \hat{B}_- \hat{B}_- + c_{23} \hat{A}_3 \hat{B}_- \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{24} \hat{A}_- \hat{B}_- \partial \hat{A}_- + c_{25} \hat{A}_- \hat{A}_3 \hat{B}_- + c_{26} \hat{A}_- \hat{B}_- \hat{B}_- + c_{27} \hat{A}_- \hat{B}_- \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{28} \hat{B}_- \partial \hat{A}_- + c_{29} \hat{B}_- \hat{B}_- \hat{B}_- + c_{30} \hat{B}_- \hat{B}_- \hat{B}_- + c_{31} \hat{B}_- \hat{B}_- \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{32} \hat{B}_- \hat{B}_- \partial \hat{B}_- + c_{33} \hat{B}_- \hat{B}_- \hat{B}_- + c_{34} \hat{B}_- \hat{B}_- \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{35} \hat{B}_- \hat{B}_- \hat{B}_- + c_{36} \hat{B}_- \hat{B}_- \hat{B}_- + c_{37} \hat{B}_- \hat{B}_- \hat{B}_- + c_{38} \hat{B}_- \hat{B}_- \hat{B}_- + c_{39} \hat{B}_- \hat{B}_- \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{40} \hat{B}_- \partial^2 \hat{B}_- + c_{41} \hat{B}_- \partial^T + c_{42} \hat{B}_- \partial^2 \hat{A}_3 + c_{43} \hat{B}_- \hat{B}_- \hat{B}_- \right] + \frac{1}{(z - w)} \left[ \frac{1}{10} \partial^3(\text{pole-4}) - \frac{1}{10} \partial^2(\text{pole-3}) + \frac{1}{2} \partial(\text{pole-2}) \right] + c_{44} \hat{B}_- \hat{B}_- \hat{B}_- + c_{45} \hat{G}_1 \hat{T} + c_{46} \hat{G}_1 \hat{G}_1 \hat{B}_- + c_{47} \hat{G}_1 \hat{U}^{(3)}(w) + \cdots \quad \text{(C.3)}
\end{align*}\]

In the third-order pole of \((\text{C.3})\), the coefficient \(\frac{1}{2}\) in the descendant field of spin-1 current located at the fourth-order pole can be obtained from the standard procedure for given spins.
of the left hand side \((h_i = \frac{5}{2} \text{ and } h_j = \frac{5}{2})\) and the spin \((h_k = 1)\) of the spin-1 current appearing in the fourth-order pole.

- The higher spin-4 current in the OPE \(T_\pm^{(4)}(z) V^{(\frac{7}{2})}(w)\)

Similarly we have the following OPE

\[
T_\pm^{(4)}(z) V^{(\frac{7}{2})}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{B}_+ \right](w) + \frac{1}{(z-w)^3} \left[ \frac{1}{2} \partial (\text{pole-4}) + c_2 \hat{A}_3 \hat{B}_+ \right](w)
\]

\[
+ \frac{1}{(z-w)^2} \left[ - \frac{1}{12} \partial^2 (\text{pole-4}) + \frac{1}{2} \partial (\text{pole-3}) + c_3 V^{(3)} + c_4 \hat{A}_3 \hat{A}_3 \hat{B}_+ \right]
\]

\[
+ c_5 \hat{A}_3 \partial \hat{B}_+ + c_6 \hat{A}_+ \hat{A}_- \hat{B}_+ + c_7 \hat{B}_3 \partial \hat{B}_+ + c_8 \hat{B}_+ T^{(2)} + c_9 \hat{B}_+ \hat{T}
\]

\[
+ c_{10} \hat{B}_+ \hat{B}_3 \hat{B}_3 + c_{11} \hat{B}_+ \hat{B}_+ \hat{B}_- + c_{12} \hat{B}_+ \partial \hat{A}_3 + c_{13} \hat{B}_+ \partial \hat{B}_3
\]

\[
+ c_{14} \hat{G}_{22} \hat{G}_{12} + c_{15} \partial^2 \hat{B}_+ \right](w)
\]

\[
+ \frac{1}{(z-w)} \left[ \frac{1}{120} \partial^3 (\text{pole-4}) - \frac{1}{10} \partial^2 (\text{pole-3}) + \frac{1}{2} \partial (\text{pole-2})
\]

\[
+ c_{16} \hat{A}_3 \hat{A}_3 \partial \hat{B}_+ + c_{17} \hat{A}_3 \hat{B}_3 \partial \hat{B}_+ + c_{18} \hat{A}_3 \hat{B}_+ \hat{T} + c_{19} \hat{A}_3 \hat{B}_+ \partial \hat{A}_3
\]

\[
+ c_{20} \hat{A}_3 \hat{B}_+ \partial \hat{B}_3 + c_{21} \hat{A}_3 \partial^2 \hat{B}_+ + c_{22} \hat{A}_- \hat{B}_+ \partial \hat{A}_+ + c_{23} \hat{A}_- \hat{B}_+ \partial \hat{A}_+ + c_{27} \hat{B}_3 \hat{V}^{(3)}
\]

\[
+ c_{28} \hat{B}_3 \hat{B}_3 \partial \hat{B}_+ + c_{29} \hat{B}_3 \hat{G}_{22} \hat{G}_{12} + c_{30} \hat{B}_3 \partial^2 \hat{B}_+ + c_{31} \hat{B}_+ T^{(3)}
\]

\[
+ c_{32} \hat{B}_+ \hat{W}^{(3)} + c_{33} \hat{B}_+ \hat{B}_3 \partial \hat{A}_3 + c_{34} \hat{B}_+ \hat{B}_3 \partial \hat{B}_3 + c_{35} \hat{B}_+ \hat{B}_- \partial \hat{B}_+
\]

\[
+ c_{36} \hat{B}_+ \hat{B}_+ \partial \hat{B}_- + c_{37} \hat{B}_+ \hat{G}_{11} \hat{G}_{22} + c_{38} \hat{B}_+ \hat{G}_{12} \hat{G}_{21} + c_{39} \hat{B}_+ \partial \hat{T}^{(2)}
\]

\[
+ c_{40} \hat{B}_+ \partial \hat{T} + c_{41} \hat{B}_+ \partial^2 \hat{A}_3 + c_{42} \hat{B}_+ \partial^2 \hat{B}_3 + c_{43} \hat{G}_{12} \hat{V}^{(4)}
\]

\[
+ c_{44} \hat{G}_{21} \partial \hat{G}_{22} + c_{45} \hat{G}_{22} \hat{G}_{12} \hat{V}^{(4)} + c_{46} \hat{G}_{22} \partial \hat{G}_{12} + c_{47} \partial \hat{B}_+ T^{(2)}
\]

\[
+ c_{48} \partial \hat{B}_+ \hat{T} + c_{49} \partial \hat{B}_+ \partial \hat{B}_3 + c_{50} \partial \hat{V}^{(3)} + c_{51} \partial^2 \hat{B}_+ + R^{(4)}(w)
\]

\[
+ \ldots .
\]

We can also describe the numerical factors in the derivative terms in \([C,4]\) as before.

Therefore, we have seen the four higher spin-4 currents \(Q^{(4)}_{\pm}(w)\) and \(R^{(4)}_{\pm}(w)\) in \([2.27]\). The remaining two higher spin-4 currents will appear at the end of this Appendix.

- The higher spin-\(\frac{9}{2}\) current in the OPE \(T^{(2)}(z) W^{(\frac{7}{2})}_{\pm}(w)\)

Let us consider the following OPE

\[
T^{(2)}(z) W^{(\frac{7}{2})}_{\pm}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{G}_{21} \right](w)
\]

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+ \frac{1}{(z - w)^3} \left[ c_2 T^\frac{5}{2}_+ + c_3 \hat{A}_3 \hat{G}_{21} + c_4 \hat{A}_- \hat{G}_{11} + c_5 \hat{B}_3 \hat{G}_{21} + c_6 \hat{B}_- \hat{G}_{22} + c_7 \partial \hat{G}_{21} \right] (w)

+ \frac{1}{(z - w)^2} \left[ \frac{1}{5} \partial (\text{pole-3}) + c_8 W^\frac{3}{2}_+ + c_9 \hat{A}_4 T^\frac{7}{2}_+ + c_{10} \hat{A}_3 \hat{A}_3 \hat{G}_{21} + c_{11} \hat{A}_3 \hat{B}_3 \hat{G}_{21} + c_{12} \hat{A}_3 \hat{B}_- \hat{G}_{22} + c_{13} \hat{A}_3 \partial \hat{G}_{21} + c_{14} \hat{A}_- U^\frac{7}{2}_- + c_{15} \hat{A}_- \hat{A}_3 \hat{G}_{11} + c_{16} \hat{A}_- \hat{B}_3 \hat{G}_{11} + c_{17} \hat{A}_- \hat{B}_- \hat{G}_{12} + c_{18} \hat{A}_- \partial \hat{G}_{11} + c_{19} \hat{A}_+ \hat{A}_- \hat{G}_{21} + c_{20} \hat{B}_3 T^\frac{5}{2}_+ + c_{21} \hat{B}_3 \hat{B}_3 \hat{G}_{21} + c_{22} \hat{B}_3 \partial \hat{G}_{21} + c_{23} \hat{B}_- \hat{V}^\frac{7}{2}_+ + c_{24} \hat{B}_- \hat{B}_3 \hat{G}_{22} + c_{25} \hat{B}_- \partial \hat{G}_{22} + c_{26} \hat{B}_+ \hat{B}_- \hat{G}_{21} + c_{27} \hat{G}_{11} \partial \hat{A}_- + c_{28} \hat{G}_{21} T^2 + c_{29} \hat{G}_{21} \hat{T} + c_{30} \hat{G}_{21} \partial \hat{A}_3 + c_{31} \hat{G}_{21} \partial \hat{B}_3 + c_{32} \hat{G}_{22} \partial \hat{B}_- + c_{33} \partial T^\frac{7}{2}_+ + c_{34} \partial^2 \hat{G}_{21} + c_{35} P^\frac{5}{2}_+ \right] (w)

+ \frac{1}{(z - w)} \left[ -\frac{1}{42} \partial^2 (\text{pole-3}) + \frac{2}{7} \partial (\text{pole-2}) + c_{36} \left( \hat{T} T^\frac{5}{2}_+ - \frac{1}{4} \partial^2 T^\frac{5}{2}_+ \right) \right.

+ c_{37} \left( \partial \hat{B}_- \partial \hat{G}_{22} - \frac{1}{2} \partial^2 \hat{B}_- \hat{G}_{22} - \frac{1}{4} \hat{B}_- \partial^2 \hat{G}_{22} - \frac{i}{60} \partial^4 \hat{G}_{21} \right)

+ c_{38} \left( \partial \hat{G}_{21} \partial \hat{A}_3 - \frac{1}{4} \partial^2 \hat{G}_{21} \hat{A}_3 - \frac{1}{2} \hat{G}_{21} \partial^2 \hat{A}_3 - \frac{i}{30} \partial^3 \hat{G}_{21} \right)

+ c_{39} \left( \partial \hat{G}_{21} \partial \hat{B}_3 - \frac{1}{4} \partial^2 \hat{G}_{21} \hat{B}_3 - \frac{1}{2} \hat{G}_{21} \partial^2 \hat{B}_3 - \frac{i}{30} \partial^3 \hat{G}_{21} \right)

+ c_{40} \left( \partial \hat{G}_{11} \partial \hat{A}_- - \frac{1}{4} \partial^2 \hat{G}_{11} \hat{A}_- - \frac{1}{2} \hat{G}_{11} \partial^2 \hat{A}_- + \frac{i}{15} \partial^3 \hat{G}_{21} \right)

+ c_{41} \left( \hat{T} \partial \hat{G}_{21} - \frac{3}{4} \partial \hat{T} \hat{G}_{21} - \frac{1}{5} \partial^2 \hat{G}_{21} \right) + \hat{S}^\frac{7}{2}_+ + S^\frac{7}{2}_+ \right] (w) + \cdots \quad (C.5)

There is no descendant field for the spin-$\frac{3}{2}$ field (appearing in the fourth-order pole) in the third-order pole $(h_k = \frac{3}{2})$. In the second-order pole of $(C.5)$, the coefficient $\frac{1}{7}$ in the descendant field of spin-$\frac{3}{2}$ current located at the third-order pole can be obtained from the standard procedure for given spins of the left hand side $(h_i = 2$ and $h_j = \frac{7}{2})$ and the spin $(h_k = \frac{5}{2})$ of the spin-$\frac{5}{2}$ current appearing in the third-order pole. In the first-order pole, the coefficient $\frac{2}{7}$ in the descendant field of spin-$\frac{5}{2}$ current located at the second-order pole $(h_k = \frac{7}{2})$ can be obtained according to previous analysis. Note that there exists a new higher spin-$\frac{7}{2}$ current $\hat{S}^\frac{7}{2}_+ (w)$ which belongs to other $\mathcal{N} = 4$ multiplet. We have seen the various quasiprimary fields appearing in the first-order pole before.

- The higher spin-$\frac{7}{2}$ current in the OPE $T^2(z) W^\frac{5}{2}_- (w)$
Similarly we consider the following OPE

\[ T^{(2)}(z) W_-^{(\frac{3}{2})}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{G}_{12} \right](w) \]

\[ + \frac{1}{(z-w)^3} \left[ c_2 T^{\frac{3}{2}} + c_3 \hat{A}_3 \hat{G}_{12} + c_4 \hat{A}_4 \hat{G}_{22} + c_5 \hat{B}_3 \hat{G}_{12} + c_6 \hat{B}_+ \hat{G}_{11} + c_7 \partial \hat{G}_{12} \right](w) \]

\[ + \frac{1}{(z-w)^2} \left[ \frac{1}{3} \partial \text{pole-3} + c_8 W_-^{(\frac{1}{2})} + c_9 \hat{A}_3 T^{(\frac{3}{2})} + c_{10} \hat{A}_3 \hat{A}_3 \hat{G}_{12} + c_{11} \hat{A}_3 \hat{B}_3 \hat{G}_{12} \right. \]

\[ + c_{12} \hat{A}_3 \hat{B}_+ \hat{G}_{11} + c_{13} \hat{A}_3 \partial \hat{G}_{12} + c_{14} \hat{A}_4 \hat{V}^{(\frac{5}{2})} + c_{15} \hat{A}_+ \hat{A}_3 \hat{G}_{22} \]

\[ + c_{16} \hat{A}_+ \hat{A}_2 \hat{G}_{12} + c_{17} \hat{A}_- \hat{B}_3 \hat{G}_{22} + c_{18} \hat{A}_4 \hat{B}_+ \hat{G}_{21} + c_{19} \hat{A}_- \partial \hat{G}_{22} \]

\[ + c_{20} \hat{B}_- \hat{T}^{(\frac{3}{2})} + c_{21} \hat{B}_- \hat{B}_3 \hat{G}_{12} + c_{22} \hat{B}_3 \partial \hat{G}_{12} + c_{23} \hat{B}_- \hat{U}^{(\frac{5}{2})} \]

\[ + c_{24} \hat{B}_- \hat{B}_+ \hat{G}_{11} + c_{25} \hat{B}_- \hat{B}_- \hat{G}_{12} + c_{26} \hat{B}_+ \partial \hat{G}_{11} + c_{27} \hat{G}_{11} \partial \hat{B}_+ \]

\[ + c_{28} \hat{G}_{12} T^{(2)} + c_{29} \hat{G}_{12} \hat{T} + c_{30} \hat{G}_{12} \partial \hat{A}_3 + c_{31} \hat{G}_{12} \partial \hat{B}_3 \]

\[ + c_{32} \hat{G}_{22} \partial \hat{A}_+ + c_{33} \partial T^{(\frac{3}{2})} + c_{34} \partial^2 \hat{G}_{12} + c_{35} P^{(\frac{1}{2})} \] \( (w) \)

\[ + \frac{1}{(z-w)^{\frac{1}{2}}} \left[ - \frac{1}{42} \partial^2 \text{pole-3} + \frac{2}{7} \partial \text{pole-2} \right] \]

\[ + c_{36} \left( \hat{T} T^{(\frac{3}{2})} - \frac{1}{4} \partial^2 T^{(\frac{3}{2})} \right) \]

\[ + c_{37} \left( \partial \hat{A}_+ \partial \hat{G}_{22} - \frac{1}{2} \partial^2 \hat{A}_+ \hat{G}_{22} - \frac{1}{4} \hat{A}_+ \partial^2 \hat{G}_{22} + \frac{i}{60} \hat{A}_+ \partial^3 \hat{G}_{12} \right) \]

\[ + c_{38} \left( \partial \hat{A}_- \partial \hat{G}_{12} - \frac{1}{4} \hat{A}_- \partial^2 \hat{G}_{12} - \frac{1}{2} \hat{A}_- \partial^2 \hat{G}_{12} - \frac{i}{120} \hat{A}_- \partial^3 \hat{G}_{12} \right) \]

\[ + c_{39} \left( \partial \hat{B}_- \partial \hat{G}_{12} - \frac{1}{4} \hat{B}_- \partial^2 \hat{G}_{12} - \frac{1}{2} \hat{B}_- \partial^2 \hat{G}_{12} - \frac{i}{120} \hat{B}_- \partial^3 \hat{G}_{12} \right) \]

\[ + c_{40} \left( \partial \hat{G}_{11} \partial \hat{B}_+ - \frac{1}{4} \partial^2 \hat{G}_{11} \hat{B}_+ - \frac{1}{2} \hat{G}_{11} \partial^2 \hat{B}_+ - \frac{i}{15} \partial^3 \hat{G}_{12} \right) \]

\[ + c_{41} \left( \hat{T} \partial \hat{G}_{12} - \frac{3}{4} \partial \hat{T} \hat{G}_{12} - \frac{1}{5} \partial^3 \hat{G}_{12} \right) + \tilde{S}^{(\frac{5}{2})} + S^{(\frac{7}{2})} \] \( (w) + \ldots \) \( \text{(C.6)} \)

Note that there exists a new higher spin-\( \frac{9}{2} \) current \( \tilde{S}^{(\frac{7}{2})} \) \( (w) \) in the first-order pole of \( \text{(C.6)} \) which belongs to other \( \mathcal{N} = 4 \) multiplet. There are various quasi-primary fields.

- The higher spin-\( \frac{9}{2} \) current in the OPE \( T^{(\frac{7}{2})}_+(z) U_-^{(3)}(w) \)

Let us consider the OPE

\[ T^{(\frac{7}{2})}_+(z) U_-^{(3)}(w) = \frac{1}{(z-w)^4} \left[ c_1 \hat{G}_{11} \right](w) \]

\[ + \frac{1}{(z-w)^3} \left[ \frac{1}{3} \partial \text{pole-4} + c_2 U^{(\frac{7}{2})} + c_3 \hat{A}_3 \hat{G}_{11} + c_4 \hat{A}_+ \hat{G}_{21} + c_5 \hat{B}_3 \hat{G}_{11} \right] \]

\[ + \ldots \]
In the third-order pole of (C.7), the coefficient \( \frac{1}{3} \) in the descendant field of spin-\( \frac{3}{2} \) current located at the fourth-order pole can be obtained from the standard procedure for given spins of the left hand side \( (h_i = \frac{5}{2} \text{ and } h_j = 3) \) and the spin \( (h_k = \frac{3}{2}) \) of the spin-\( \frac{3}{2} \) current appearing in the fourth-order pole. In the second-order pole, the coefficient \( \frac{2}{7} \) in the descendant field of spin-\( \frac{3}{2} \) current located at the third-order pole can be obtained. In the first-order pole, the coefficient \( \frac{2}{7} \) in the descendant field of spin-\( \frac{7}{2} \) current located at the second-order pole \( (h_k = \frac{7}{2}) \) can be obtained according to previous analysis. Note that there exists a new higher spin-\( \frac{9}{2} \) current \( \overline{Q}^{(\frac{9}{2})} (w) \) in the first-order pole of (C.7) which belongs to other \( \mathcal{N} = 4 \) multiplet. There are various quasiprimary fields.

- The higher spin-\( \frac{9}{2} \) current in the OPE \( T^{(\frac{9}{2})} (z) V^{(3)}_+ (w) \)
Furthermore, we have the following OPE

\[
T^{(\frac{9}{2})}_{\pm} (z) V^{(3)}_+ (w) = \frac{1}{(z - w)^4} \left[ c_1 \hat{G}_{22} \right] (w) + \frac{1}{(z - w)^3} \left[ \frac{1}{3} \partial \text{pole-4} + c_2 V^{(\frac{9}{2})} + c_3 \hat{A}_{3} \hat{G}_{22} + c_4 \hat{A}_{-} \hat{G}_{22} + c_5 \hat{B}_{3} \hat{G}_{22} \right. \\
+ c_6 \hat{B}_{+} \hat{G}_{21} + c_7 \partial \hat{G}_{22} \left. \right] (w) + \frac{1}{(z - w)^2} \left[ - \frac{1}{20} \partial^2 \text{pole-4} + \frac{2}{5} \partial \text{pole-3} \right. \\
+ c_8 V^{(\frac{9}{2})} + c_9 \hat{A}_{3} V^{(\frac{9}{2})} + c_{10} \hat{A}_{3} \hat{A}_{3} \hat{G}_{22} + c_{11} \hat{A}_{3} \hat{B}_{3} \hat{G}_{22} + c_{12} \hat{A}_{3} \hat{B}_{+} \hat{G}_{21} \\
+ c_{13} \hat{A}_{3} \partial \hat{G}_{22} + c_{14} \hat{A}_{-} T^{(\frac{9}{2})} + c_{15} \hat{A}_{-} \hat{A}_{3} \hat{G}_{12} + c_{16} \hat{A}_{-} \hat{B}_{3} \hat{G}_{12} \\
+ c_{17} \hat{A}_{-} \hat{B}_{+} \hat{G}_{11} + c_{18} \hat{A}_{-} \partial \hat{G}_{12} + c_{19} \hat{A}_{+} \hat{A}_{-} \hat{G}_{22} + c_{20} \hat{B}_{3} \hat{B}_{3} \hat{G}_{22} \\
+ c_{21} \hat{B}_{3} \partial \hat{G}_{22} + c_{22} \hat{B}_{+} T^{(\frac{9}{2})} + c_{23} \hat{B}_{+} \hat{B}_{3} \hat{G}_{21} + c_{24} \hat{B}_{+} \hat{B}_{-} \partial \hat{G}_{22} \\
+ c_{25} \hat{B}_{+} \partial \hat{G}_{21} + c_{26} \hat{G}_{12} \partial \hat{A}_{-} + c_{27} \hat{G}_{21} \partial \hat{B}_{+} + c_{28} \hat{G}_{22} T^{(2)} \\
+ c_{29} \hat{G}_{22} \partial \hat{T} + c_{30} \hat{G}_{22} \partial \hat{A}_{3} + c_{31} \hat{G}_{22} \partial \hat{B}_{3} + c_{32} \partial V^{(\frac{9}{2})} \\
+ c_{33} \partial^2 \hat{G}_{22} + c_{34} \hat{B}_{3} V^{(\frac{9}{2})} + \tilde{R}^{(\frac{9}{2})} - 4 R^{(\frac{9}{2})} \right] (w) + \frac{1}{(z - w)} \left[ \frac{1}{210} \partial^3 \text{pole-4} + \frac{1}{14} \partial^2 \text{pole-3} + \frac{3}{7} \partial \text{pole-2} \right. \\
+ c_{35} \left( \hat{T} V^{(\frac{9}{2})} - \frac{1}{4} \partial^2 V^{(\frac{9}{2})} \right) \\
+ c_{36} \left( \partial \hat{A}_{-} \partial \hat{G}_{12} - \frac{1}{2} \partial^2 \hat{A}_{-} \hat{G}_{12} - \frac{1}{4} \hat{A}_{-} \partial^2 \hat{G}_{12} + \frac{i}{60} \partial^3 \hat{G}_{22} \right) \\
+ c_{37} \left( \partial \hat{G}_{22} \partial \hat{A}_{3} - \frac{1}{4} \partial^2 \hat{G}_{22} \hat{A}_{3} - \frac{1}{2} \hat{G}_{22} \partial^2 \hat{A}_{3} - \frac{i}{30} \partial^3 \hat{G}_{22} \right) \\
+ c_{38} \left( \partial \hat{G}_{22} \partial \hat{B}_{3} - \frac{1}{4} \partial^2 \hat{G}_{22} \hat{B}_{3} - \frac{1}{2} \hat{G}_{22} \partial^2 \hat{B}_{3} + \frac{i}{30} \partial^3 \hat{G}_{22} \right) \\
+ c_{39} \left( \partial \hat{G}_{21} \partial \hat{B}_{+} - \frac{1}{4} \partial^2 \hat{G}_{21} \hat{B}_{+} - \frac{1}{2} \hat{G}_{21} \partial^2 \hat{B}_{+} + \frac{i}{15} \partial^3 \hat{G}_{22} \right) \\
+ c_{40} \left( \hat{T} \partial \hat{G}_{22} - \frac{3}{4} \partial \hat{T} \hat{G}_{22} - \frac{1}{5} \partial^3 \hat{G}_{22} \right) + \tilde{R}^{(\frac{9}{2})} + R^{(\frac{9}{2})} \left. \right] (w) + \cdots . \quad (C.8)
\]

Note that there exists a new higher spin-$\frac{9}{2}$ current $\tilde{R}^{(\frac{9}{2})}(w)$ in the first-order pole of \((C.8)\) which belongs to other $\mathcal{N} = 4$ multiplet. There are various quasi-primary fields.

- The higher spin currents of spins $s = 3, 4, 5$ in the OPE $T^{(2)}(z) W^{(4)}(w)$

Let us consider the final OPE

\[
T^{(2)} (z) W^{(4)} (w) = \frac{1}{(z - w)^5} \left[ c_1 \hat{A}_{3} + c_2 \hat{B}_{3} \right] (w)
\]
\[\begin{align*}
&+ \frac{1}{(z - w)^4} \left[ -\frac{1}{2} \partial (\text{pole-5}) + c_3 T^{(2)} + c_4 \hat{T} + c_5 \hat{A}_3 \hat{A}_3 + c_6 \hat{A}_3 \hat{B}_3 + c_7 \hat{A}_- \hat{A}_+ \\
&+ c_8 \hat{B}_3 \hat{B}_3 + c_9 \hat{B}_- \hat{B}_+ + c_{10} \partial \hat{A}_3 + c_{11} \partial \hat{B}_3 \right] (w) \\
&+ \frac{1}{(z - w)^3} \left[ c_{12} T^{(3)} + c_{13} W^{(3)} + c_{14} \hat{A}_3 T^{(2)} + c_{15} \hat{A}_3 \hat{T} + c_{16} \hat{A}_3 \hat{A}_3 \hat{A}_3 \\
&+ c_{17} \hat{A}_3 \hat{A}_3 \hat{B}_3 + c_{18} \hat{A}_3 \hat{B}_3 \hat{B}_3 + c_{19} \hat{A}_3 \hat{B}_+ \hat{B}_- + c_{20} \hat{A}_3 \partial \hat{A}_3 + c_{21} \hat{A}_3 \partial \hat{B}_3 \\
&+ c_{22} \hat{A}_- \partial \hat{A}_+ + c_{23} \hat{A}_+ \hat{A}_- \hat{A}_3 + c_{24} \hat{A}_+ \hat{A}_- \hat{B}_3 + c_{25} \hat{A}_+ \partial \hat{A}_- + c_{26} \hat{B}_3 T^{(2)} \\
&+ c_{27} \hat{B}_3 \hat{T} + c_{28} \hat{B}_3 \hat{B}_3 \hat{B}_3 + c_{29} \hat{B}_3 \partial \hat{A}_3 + c_{30} \hat{B}_3 \partial \hat{B}_3 + c_{31} \hat{B}_- \partial \hat{B}_+ \\
&+ c_{32} \hat{B}_+ \hat{B}_- \hat{B}_3 + c_{33} \hat{B}_+ \partial \hat{B}_- + c_{34} \hat{G}_{11} \hat{G}_{22} + c_{35} \hat{G}_{12} \hat{G}_{21} + c_{36} \partial \hat{T} \\
&+ c_{37} \partial^2 \hat{A}_3 + c_{38} \partial^2 \hat{B}_3 + c_{39} P^{(3)} \right] (w) \\
&+ \frac{1}{(z - w)^2} \left[ \frac{1}{6} \partial (\text{pole-3}) \\
&+ c_{40} \left( \hat{T} \partial \hat{A}_3 - \frac{1}{2} \partial \hat{T} \hat{A}_3 - \frac{1}{4} \partial^2 \hat{A}_3 \right) + c_{41} \left( \hat{T} \partial \hat{B}_3 - \frac{1}{2} \partial \hat{T} \hat{B}_3 - \frac{1}{4} \partial^2 \hat{B}_3 \right) \\
&+ c_{42} \left( \hat{T} T^{(2)} - \frac{3}{10} \partial^2 T^{(2)} \right) + c_{43} \left( \hat{T} \hat{T} - \frac{3}{10} \partial \hat{T} \right) \\
&+ c_{44} \left( \hat{T} \hat{A}_3 \hat{A}_3 - \frac{3}{10} \partial^2 (\hat{A}_3 \hat{A}_3) \right) + c_{45} \left( \hat{T} \hat{A}_3 \hat{B}_3 - \frac{3}{10} \partial^2 (\hat{A}_3 \hat{B}_3) \right) \\
&+ c_{46} \left( \hat{T} \hat{A}_- \hat{A}_+ - \frac{3}{2} \partial \hat{A}_- \partial \hat{A}_+ - \frac{i}{2} \partial \hat{T} \hat{A}_3 \right) + c_{47} \left( \hat{T} \hat{B}_3 \hat{B}_3 - \frac{3}{10} \partial^2 (\hat{B}_3 \hat{B}_3) \right) \\
&+ c_{48} \left( \hat{T} \hat{B}_- \hat{B}_+ - \frac{3}{2} \partial \hat{B}_- \partial \hat{B}_+ - \frac{i}{2} \partial \hat{T} \hat{B}_3 \right) \\
&+ P^{(4)} + \tilde{P}^{(4)} + S^{(4)} + \tilde{S}^{(4)} \right] (w) \\
&+ \frac{1}{(z - w)} \left[ -\frac{1}{56} \partial^2 (\text{pole-3}) + \frac{1}{4} \partial (\text{pole-2}) \\
&+ c_{49} \left( \hat{T} T^{(3)} - \frac{3}{14} \partial^2 T^{(3)} \right) + c_{50} \left( \hat{T} W^{(3)} - \frac{3}{14} \partial^2 W^{(3)} \right) \\
&+ c_{51} \left( \hat{T} \hat{A}_3 T^{(2)} - \frac{1}{2} \partial^2 \hat{A}_3 T^{(2)} - \frac{3}{10} \hat{A}_3 \partial^2 T^{(2)} \right) \\
&+ c_{52} \left( \hat{T} \hat{A}_3 \hat{T} - \frac{1}{2} \partial^2 \hat{A}_3 \hat{T} - \frac{3}{10} \hat{A}_3 \partial^2 \hat{T} \right) + c_{53} \left( \hat{T} \hat{A}_3 \hat{A}_3 \hat{A}_3 - \frac{9}{4} \partial \hat{A}_3 \hat{A}_3 \partial \hat{A}_3 \right) \\
&+ c_{54} \left( \hat{T} \hat{A}_3 \hat{A}_3 \hat{B}_3 - \frac{3}{2} \hat{A}_3 \partial \hat{A}_3 \partial \hat{B}_3 + \frac{3}{4} \partial \hat{A}_3 \partial \hat{A}_3 \hat{B}_3 \right) \\
&+ c_{55} \left( \hat{T} \hat{A}_3 \hat{B}_3 \hat{B}_3 - \frac{3}{2} \partial \hat{A}_3 \hat{B}_3 \partial \hat{B}_3 + \frac{3}{4} \hat{A}_3 \partial \hat{B}_3 \partial \hat{B}_3 \right) \\
&+ c_{56} \left( \hat{T} \hat{A}_3 \partial \hat{A}_3 - \frac{1}{2} \partial \hat{T} \hat{A}_3 \hat{A}_3 - \frac{1}{2} \partial \hat{A}_3 \partial^2 \hat{A}_3 - \frac{1}{6} \partial \hat{T} \hat{A}_3 \hat{A}_3 \right) 
\end{align*}\]
\[ + \ c_{57} \left( \hat{T} \hat{A}_3 \partial \hat{B}_3 - \frac{1}{2} \hat{T} \hat{A}_3 \hat{B}_3 - \frac{1}{2} \partial \hat{A}_3 \partial^2 \hat{B}_3 - \frac{1}{6} \hat{A}_3 \partial^3 \hat{B}_3 \right) \]
\[ + \ c_{58} \left( - \frac{3}{4} \hat{A}_\perp \partial \hat{A}_3 \partial \hat{A}_+ + \hat{T} \hat{A}_\perp \hat{A}_3 \hat{A}_+ - \frac{3}{4} \partial \hat{A}_\perp \hat{A}_3 \partial \hat{A}_+ - \frac{3}{4} \partial \hat{A}_\perp \partial \hat{A}_3 \hat{A}_+ \right) \]
\[ - \ i \frac{1}{2} \partial \hat{T} \hat{A}_3 \hat{A}_3 + \ i \frac{1}{2} \partial \hat{T} \hat{A}_- \hat{A}_+ + \ i \frac{1}{4} \partial^2 \hat{A}_3 \partial \hat{A}_3 - \ i \frac{1}{4} \partial^2 \hat{A}_- \partial \hat{A}_+ + \ \ i \frac{1}{10} \partial^2 \hat{T} \hat{A}_3 \]
\[ - \ i \frac{1}{24} \partial^3 \hat{A}_3 \hat{A}_3 + \ i \frac{1}{24} \partial^3 \hat{A}_- \hat{A}_+ \right) \]
\[ + \ c_{59} \left( - \ i \frac{1}{24} \hat{A}_3 \partial^3 \hat{B}_3 - \frac{3}{4} \hat{A}_\perp \partial \hat{B}_3 \partial \hat{A}_+ + \hat{T} \hat{A}_\perp \hat{B}_3 \hat{A}_+ + \ i \frac{1}{4} \partial \hat{A}_3 \partial^2 \hat{B}_3 - \frac{3}{4} \partial \hat{A}_\perp \hat{B}_3 \partial \hat{A}_+ \right) \]
\[ + \ c_{60} \left( \hat{A}_3 \partial \hat{A}_\perp \partial \hat{A}_+ + \frac{1}{3} \partial^2 \hat{A}_3 \partial \hat{A}_- \hat{A}_+ - \partial \hat{A}_3 \partial \hat{A}_- \hat{A}_+ - \frac{1}{3} \hat{A}_3 \partial \hat{A}_- \partial^2 \hat{A}_+ \right) \]
\[ + \ c_{61} \left( \hat{T} \partial \hat{A}_\perp \hat{A}_+ - \frac{1}{2} \partial \hat{T} \hat{A}_- \hat{A}_+ - \frac{1}{2} \partial^2 \hat{A}_- \partial \hat{A}_+ + \ i \frac{1}{10} \partial^2 \hat{T} \hat{A}_3 - \frac{1}{6} \partial^3 \hat{A}_\perp \hat{A}_+ \right) \]
\[ + \ c_{62} \left( \frac{1}{2} \partial^2 \hat{T} \hat{B}_3 \hat{T} - \ i \frac{1}{2} \partial^2 \hat{B}_3 \hat{T} - \frac{3}{10} \hat{B}_3 \partial^2 \hat{T} \right) + \ c_{63} \left( \frac{1}{2} \partial^2 \hat{B}_3 \hat{B}_3 + \ i \frac{1}{2} \partial \hat{B}_3 \partial \hat{B}_3 \right) + \ c_{64} \left( \hat{T} \hat{B}_3 \hat{B}_3 \hat{B}_3 - \frac{9}{4} \partial \hat{B}_3 \hat{B}_3 \partial \hat{B}_3 \right) \]
\[ + \ c_{65} \left( \hat{T} \hat{B}_3 \partial \hat{A}_3 - \frac{1}{2} \partial \hat{T} \hat{A}_3 \hat{B}_3 - \frac{1}{2} \partial^2 \hat{A}_3 \partial \hat{B}_3 - \frac{1}{6} \partial^3 \hat{A}_3 \hat{B}_3 \right) \]
\[ + \ c_{66} \left( \hat{T} \hat{B}_3 \partial \hat{B}_3 - \frac{1}{2} \partial \hat{T} \hat{B}_3 \hat{B}_3 - \frac{1}{2} \partial \hat{B}_3 \partial^2 \hat{B}_3 - \frac{1}{6} \partial^3 \hat{B}_3 \hat{B}_3 \right) \]
\[ + \ c_{67} \left( - \ i \frac{1}{24} \hat{A}_3 \partial^3 \hat{B}_3 - \frac{3}{4} \hat{A}_\perp \partial \hat{B}_3 \partial \hat{A}_3 \hat{B}_+ + \hat{T} \hat{B}_- \hat{A}_3 \hat{B}_+ + \ i \frac{1}{4} \partial \hat{A}_3 \partial^2 \hat{B}_3 - \frac{3}{4} \partial \hat{A}_- \hat{B}_3 \partial \hat{B}_+ \right) \]
\[ + \ c_{68} \left( - \ i \frac{1}{2} \partial \hat{B}_- \partial \hat{B}_3 \partial \hat{B}_+ - \ i \frac{1}{2} \partial \hat{T} \hat{B}_3 \partial \hat{B}_3 + \ i \frac{1}{2} \partial \hat{T} \hat{B}_3 \partial \hat{B}_3 - \frac{3}{4} \partial \hat{B}_- \partial \hat{B}_3 \partial \hat{B}_+ \right) \]
\[ - \ i \frac{1}{4} \partial^2 \hat{B}_- \partial \hat{B}_+ + \ i \frac{10}{24} \partial^2 \hat{T} \hat{B}_3 + \ i \frac{8}{24} \partial^3 \hat{B}_3 \hat{B}_3 + \ i \frac{8}{24} \partial^3 \hat{B}_- \hat{B}_+ \right) \]
\[ + \ c_{69} \left( \hat{B}_3 \partial \hat{B}_- \partial \hat{B}_+ + \frac{1}{3} \partial^2 \hat{B}_3 \hat{B}_- \hat{B}_+ - \partial \hat{B}_3 \partial \hat{B}_- \hat{B}_+ - \frac{1}{3} \hat{B}_3 \partial \hat{B}_- \partial^2 \hat{B}_+ \right) \]
\[ + \ c_{70} \left( \hat{T} \partial \hat{B}_- \hat{B}_+ - \frac{1}{2} \partial \hat{T} \hat{B}_- \hat{B}_+ - \frac{1}{2} \partial^2 \hat{B}_- \partial \hat{B}_+ + \ i \frac{10}{24} \partial^2 \hat{T} \hat{B}_3 - \frac{1}{6} \partial^3 \hat{B}_- \hat{B}_+ \right) \]
\[ + \ c_{71} \left( \hat{T} \hat{G}_{11} \hat{G}_{22} - \hat{T} \partial \hat{T} - \ i \frac{2i \hat{N}}{3(N + 2 + k)} \hat{T} \partial^2 \hat{A}_3 + \ i \frac{2i \hat{N}}{3(N + 2 + k)} \hat{T} \partial^2 \hat{B}_3 + \ i \frac{2i \hat{N}}{3(N + 2 + k)} \partial \hat{G}_{11} \partial \hat{G}_{22} \right) \]
\[ + \ c_{72} \left( \hat{T} \hat{A}_3 \hat{A}_3 + \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_3 \hat{A}_3 + \frac{2}{(N + 2 + k)} \partial \hat{T} \hat{A}_3 \hat{B}_3 + \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{A}_- \hat{A}_+ \right) \]
\[- \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{B}_3 \hat{B}_3 - \frac{1}{(N + 2 + k)} \partial \hat{T} \hat{B}_- \hat{B}_+ + \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_3 \]
\[+ \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{B}_3 + \frac{1}{6} \partial^3 \hat{T} + \frac{iN}{10(N + 2 + k)} \partial^4 \hat{A}_3 - \frac{ik}{10(N + 2 + k)} \partial^4 \hat{B}_3 \]
\[+ c_{72} \left( \hat{T} \partial \hat{G}_{21} \hat{G}_{12} - \hat{T} \partial \hat{T} + \frac{2iN}{3(N + 2 + k)} \hat{T} \partial^2 \hat{A}_3 + \frac{2ik}{3(N + 2 + k)} \hat{T} \partial^2 \hat{B}_3 \right) \]
\[\partial \hat{G}_{21} \partial \hat{G}_{12} \]
\[- \frac{1}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_3 \hat{A}_3 + \frac{2}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_3 \hat{B}_3 - \frac{1}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_- \hat{A}_+ \]
\[- \frac{1}{(N + 2 + k)} \partial \hat{T} \partial \hat{B}_3 \hat{B}_3 - \frac{1}{(N + 2 + k)} \partial \hat{T} \partial \hat{B}_- \hat{B}_+ + \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{A}_3 \]
\[+ \frac{i}{(N + 2 + k)} \partial \hat{T} \partial \hat{B}_3 + \frac{1}{6} \partial^3 \hat{T} - \frac{iN}{10(N + 2 + k)} \partial^4 \hat{A}_3 - \frac{ik}{10(N + 2 + k)} \partial^4 \hat{B}_3 \]
\[+ c_{73} \left( \hat{T} \partial \hat{T}^{(2)} - \partial \hat{T} \hat{T}^{(2)} - \frac{1}{6} \partial^3 \hat{T}^{(2)} \right) \]
\[+ c_{74} \left( \hat{T} \partial^2 \hat{A}_3 - \frac{3}{2} \partial \hat{T} \partial \hat{A}_3 + \frac{3}{10} \partial^2 \hat{T} \hat{A}_3 - \frac{3}{20} \partial^4 \hat{A}_3 \right) \]
\[+ c_{75} \left( \hat{T} \partial^2 \hat{B}_3 - \frac{3}{2} \partial \hat{T} \partial \hat{B}_3 + \frac{3}{10} \partial^2 \hat{T} \hat{B}_3 - \frac{3}{20} \partial^4 \hat{B}_3 \right) \]
\[+ c_{76} \left( \hat{T} \partial P^{(3)} - \frac{3}{14} \partial^2 P^{(3)} \right) + S^{(5)} + \tilde{S}^{(9)} \right](w) + \cdots. \tag{C.9} \]

In the fourth-order pole of (C.9), the coefficient \(-\frac{1}{2}\) in the descendant field of spin-1 current located at the fifth-order pole can be obtained from the standard procedure for given spins of the left hand side \((h_i = 2, h_j = 4)\) and the spin \((h_k = 1)\) of the spin-1 current appearing in the fifth-order pole. There is no descendant field for the spin-2 field (appearing in the fourth-order pole) in the third-order pole \((h_k = 2)\). In the second-order pole, the coefficient \(\frac{1}{6}\) in the descendant field of spin-3 current located at the third-order pole \((h_k = 3)\) can be obtained according to previous analysis. There are new higher spin-4 currents \(\tilde{P}^{(4)}(w)\) and \(\tilde{S}^{(4)}(w)\) (appeared in Appendix B). In the first-order pole, the coefficient \(\frac{1}{4}\) in the descendant field of spin-4 current located at the second-order pole \((h_k = 4)\) can be obtained similarly. Note that there exists a new higher spin-5 current \(\tilde{S}^{(5)}(w)\) in the first-order pole of (C.9) which belongs to other \(\mathcal{N} = 4\) multiplet. In particular, the correct presence of various quasiprimary fields is very important to obtain the final higher spin-5 current which is the highest higher spin current in (2.27). Two of the quasiprimary fields have the explicit \(N\)-dependence in their expressions.

Therefore, we have observed the presence of the next 16 lowest higher spin currents in the right hand side of the OPEs between the 16 lowest higher spin currents.
Appendix D  The complete OPEs between the 16 currents and the 16 lowest higher spin currents for generic $N$

In this Appendix, we describe the complete OPEs between the 16 currents (of large $N = 4$ linear superconformal algebra) and the 16 lowest higher spin currents for generic $N$ from the results of $N = 4, 5, 8, 9$. Except the few cases, these are linear.

Appendix D.1  The OPEs between the spin-$\frac{1}{2}$ currents and the 16 lowest higher spin currents

We perform the various OPEs between the four spin-$\frac{1}{2}$ currents, $F_{11}(z) \equiv F_{11}(z)$, $F_{22}(z) \equiv F_{22}(z)$, $F_{12}(z) \equiv F_{12}(z)$, and $F_{21}(z) \equiv F_{21}(z)$, and the 16 higher spin currents obtained previously as follows:

\[
\left( \frac{F_{11}}{F_{22}} \right) (z) T^{(3)}(w) = \frac{1}{(z-w)} \left( \frac{U^{(\frac{5}{2})}_-}{-V^{(\frac{5}{2})}_-} \right) (w) + \cdots,
\]

\[
\left( \frac{F_{11}}{F_{22}} \right) (z) V^{(3)}_-(w) = \frac{1}{(z-w)} T^{(\frac{5}{2})}_\pm (w) + \cdots,
\]

\[
\left( \frac{F_{11}}{F_{22}} \right) (z) V^{(3)}_+(w) = \frac{1}{(z-w)} T^{(\frac{5}{2})}_\mp (w) + \cdots,
\]

\[
\left( \frac{F_{11}}{F_{22}} \right) (z) W^{(3)}_\pm (w) = \pm \frac{1}{(z-w)^2} \left[ 4 T^{(2)}(w) + \frac{1}{(z-w)} \left[ \mp \partial T^{(2)} - W^{(3)} \right] (w) + \cdots, \right.
\]

\[
\left( \frac{F_{11}}{F_{22}} \right) (z) W^{(3)}_\mp (w) = \mp \frac{1}{(z-w)^2} \left[ 4 T^{(2)}(w) + \frac{1}{(z-w)} \left[ \mp \partial T^{(2)} - W^{(3)} \right] (w) + \cdots, \right.
\]

\[
\left( \frac{F_{11}}{F_{22}} \right) (z) W^{(4)}_\pm (w) = \frac{1}{(z-w)^2} \left[ 5 \left( \frac{U^{(\frac{5}{2})}_-}{V^{(\frac{5}{2})}_-} \right) 
+ \frac{36(-N+k)}{(37N+59) + (15N+37)k} \left( \frac{F_{11}}{F_{22}} \right) T^{(2)} \right] (w)
\]

\[
- \frac{1}{(z-w)} \left[ \partial \left( \frac{U^{(\frac{5}{2})}_-}{V^{(\frac{5}{2})}_-} \right) 
+ \frac{36(-N+k)}{(37N+59) + (15N+37)k} \partial \left( \frac{F_{11}}{F_{22}} \right) T^{(2)} \right] (w) + \cdots,
\]

\[
\left( \frac{F_{12}}{F_{21}} \right) (z) U^{(3)}_+ (w) = - \frac{1}{(z-w)} \left( \frac{U^{(\frac{5}{2})}_+}{V^{(\frac{5}{2})}_+} \right) (w) + \cdots,
\]

\[
\left( \frac{F_{12}}{F_{21}} \right) (z) V^{(3)}_+ (w) = \frac{1}{(z-w)} \left( \frac{U^{(\frac{5}{2})}_+}{V^{(\frac{5}{2})}_+} \right) (w) + \cdots,
\]
Let us perform the various OPEs between the spin-1 current, \(U(z) \equiv U(z)\), and the 16 higher spin currents as follows:

\[
U(z) \left( \frac{U^{(\frac{5}{2})}}{V^{(\frac{3}{2})}} \right) (w) = \frac{1}{(z-w)^2} \left( \frac{U^{(\frac{5}{2})}}{-V^{(\frac{3}{2})}} \right) (w) + \cdots,
\]

\[
U(z) \left( \frac{W^{(\frac{7}{2})}}{W^{(\frac{5}{2})}} \right) (w) = \mp \frac{1}{(z-w)^2} T^{(\frac{5}{2})} (w) + \cdots,
\]

\[
U(z) \left( \frac{W^{(\frac{7}{2})}}{V^{(\frac{3}{2})}} \right) (w) = \mp \frac{1}{(z-w)^2} \left( \frac{V^{(\frac{3}{2})}}{-U^{(\frac{5}{2})}} \right) (w) + \cdots,
\]

\[
U(z) \left( \frac{V^{(\frac{7}{2})}}{U^{(\frac{5}{2})}} \right) (w) = \mp \frac{1}{(z-w)^2} T^{(\frac{5}{2})} (w) + \cdots.
\]
\[ U(z) \mathbf{W}^{(4)}(w) = - \frac{1}{(z-w)^3} \mathbf{T}^{(2)}(w) \]
\[ + \frac{1}{(z-w)^2} \left[ 2 \mathbf{T}^{(2)} + \frac{72(-N+k)}{((37N+59) + (15N+37)k)} U \mathbf{T}^{(2)} \right](w) + \ldots. \]  

Again, by introducing the quasiprimary field of spin 4, the above nonlinear terms disappear.

The OPEs between the three spin-1 currents, \( A_\pm(z) \equiv A_\pm(z) \) and \( A_3(z) \equiv A_3(z) \), and the 16 higher spin currents are

\[
A_\pm(z) \left( \mathbf{V}^{(\frac{5}{2})} \mathbf{U}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)} \mathbf{I} \left( \mathbf{U}^{(\frac{5}{2})}, \mathbf{V}^{(\frac{5}{2})} \right)(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{T}^{(3)} \mathbf{U}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)} \left( \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})} \right)(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)} \mathbf{T}^{(3)}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{U}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \left( \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})} \right)(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{V}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \left( \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})} \right)(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{W}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \left( \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})} \right)(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{W}^{(3)} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{T}^{(3)}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{U}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{V}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{W}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{W}^{(3)} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{T}^{(3)}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{U}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{V}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{W}^{(\frac{5}{2})} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{U}^{(\frac{5}{2})} \mathbf{V}^{(\frac{5}{2})}(w) + \ldots,
\]
\[
A_\pm(z) \left( \mathbf{W}^{(3)} \mathbf{W}^{(3)} \right)(w) = \pm \frac{1}{(z-w)^2} \mathbf{T}^{(3)}(w) + \ldots,
\]
\[ A_3(z) \mathbf{W}^{(2)}_\pm (w) = \frac{1}{(z-w)^2} \frac{i(12N+25+13k)}{5(N+2+k)} \mathbf{T}_\pm^{(2)}(w) \pm \frac{1}{(z-w)} \frac{i}{2} \mathbf{W}^{(2)}_\pm (w) + \ldots, \]
\[ A_3(z) \mathbf{W}^{(4)}(w) = \frac{1}{(z-w)^2} \left[ 3i \mathbf{T}^{(3)} + 3i \mathbf{W}^{(3)} + \frac{72(-N+k)}{(37N+59) + (15N+37k)} A_3 \mathbf{T}^{(2)} \right](w) + \ldots. \]

The OPEs between the other three spin-1 currents, \( B_\pm(z) \equiv B_\pm(z) \) and \( B_3(z) \equiv B_3(z) \), and the 16 higher spin currents obtained are

\[ B_\pm(z) \mathbf{T}^{(2)}_\pm (w) = \mp \frac{1}{(z-w)} i \left( \mathbf{V}^{(2)}_\mp \mathbf{U}^{(2)}_\mp \right)(w) + \ldots, \]
\[ B_\pm(z) \mathbf{T}^{(3)}(w) = \frac{1}{(z-w)} i \left( \mathbf{V}^{(3)}_\mp \mathbf{U}^{(3)}_\mp \right)(w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(2)}_\mp \\ \mathbf{V}^{(2)}_\pm \end{pmatrix}(w) = \pm \frac{1}{(z-w)} i \mathbf{T}^{(2)}_\pm (w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(3)}_\pm \\ \mathbf{V}^{(3)}_\pm \end{pmatrix}(w) = \mp \frac{1}{(z-w)} i \mathbf{T}^{(3)}(w) + \frac{1}{(z-w)} i \mathbf{W}^{(3)}(w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(2)}_\pm \\ \mathbf{V}^{(2)}_\pm \end{pmatrix}(w) = \mp \frac{1}{(z-w)} i \mathbf{T}^{(2)}_\pm (w) + \ldots, \]
\[ B_\pm(z) \mathbf{W}^{(4)}(w) = \frac{1}{(z-w)^2} \left[ \mp 6i \left( \mathbf{V}^{(3)}_\mp \mathbf{U}^{(3)}_\mp \right) + \frac{72(-N+k)}{(37N+59) + (15N+37k)} B_\pm \mathbf{T}^{(2)} \right](w) + \ldots, \]
\[ B_\pm(z) \mathbf{T}^{(3)}(w) = \mp \frac{1}{(z-w)} i \mathbf{T}^{(3)}(w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(3)}_\mp \\ \mathbf{V}^{(3)}_\mp \end{pmatrix}(w) = \pm \frac{1}{(z-w)^2} 2i \mathbf{T}^{(2)}(w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(2)}_\pm \\ \mathbf{V}^{(2)}_\mp \end{pmatrix}(w) = \pm \frac{1}{(z-w)^2} i \left( \mathbf{U}^{(2)}_\pm \mathbf{V}^{(2)}_\mp \right)(w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(3)}_\mp \\ \mathbf{V}^{(3)}_\mp \end{pmatrix}(w) = \pm \frac{1}{(z-w)} i \left( \mathbf{U}^{(3)}_\pm \mathbf{V}^{(3)}_\mp \right)(w) + \ldots, \]
\[ B_\pm(z) \begin{pmatrix} \mathbf{U}^{(2)}_\pm \\ \mathbf{V}^{(2)}_\pm \end{pmatrix}(w) = \mp \frac{1}{(z-w)^2} i \left( \mathbf{U}^{(2)}_\pm \mathbf{V}^{(2)}_\pm \right)(w) + \ldots, \]
\[ B_3(z) \mathcal{W}^{(3)}_{\pm}(w) = -\frac{1}{(z-w)^2} \frac{i(13N + 25 + 12k)}{5(N + 2 + k)} \mathcal{T}^{(3)}_{\pm}(w) \pm \frac{1}{(z-w)2} \mathcal{W}^{(5)}_{\pm}(w) + \cdots, \]

\[ B_3(z) \mathcal{W}^{(4)}(w) = \frac{1}{(z-w)^2} \left[ -3i \mathcal{T}^{(3)} + 3i \mathcal{W}^{(3)} + \frac{72(-N + k)}{(37N + 59 + 15N + 37k)} B_3 \mathcal{T}^{(2)} \right](w) + \cdots. \tag{D.4} \]

As done before, the nonlinear terms appearing in (D.2), (D.3) or (D.4) can be removed by introducing the extra quasiprimary field of spin-4 in the expression of the higher spin-4 current. See also the subsection 4.4. Via explicit field identifications between the fields in this paper and those in [31], we have checked that the above OPEs (D.2), (D.3) and (D.4) are the same as the ones in [31].

**Appendix D.3  The OPEs between the spin-\(\frac{3}{2}\) currents and the 16 lowest higher spin currents**

The OPEs between the four spin-\(\frac{3}{2}\) currents currents, \(G_{11}(z) \equiv G_{11}(z), G_{22}(z) \equiv G_{22}(z), G_{12}(z) \equiv G_{12}(z)\) and \(G_{21}(z) \equiv G_{21}(z)\), and the 16 higher spin currents obtained are

\[
\begin{align*}
\left(G_{11} \ G_{22}\right)(z) \mathcal{T}^{(2)}(w) & = \frac{1}{(z-w)} \left( \mathcal{U}^{(\frac{5}{2})}_{\pm} \mathcal{V}^{(\frac{5}{2})}_{\mp}\right)(w) + \cdots, \\
\left(G_{11} \ G_{22}\right)(z) \mathcal{T}^{(3)}_{\pm}(w) & = -\frac{1}{(z-w)} \left( \mathcal{U}^{(3)}_{\pm} \mathcal{V}^{(3)}_{\mp}\right)(w) + \cdots, \\
\left(G_{11} \ G_{22}\right)(z) \mathcal{T}^{(4)}_{\pm}(w) & = -\frac{1}{(z-w)} \left( \mathcal{U}^{(3)}_{\mp} \mathcal{V}^{(3)}_{\pm}\right)(w) + \cdots, \\
\left(G_{11} \ G_{22}\right)(z) \mathcal{T}^{(5)}(w) & = \pm \frac{1}{(z-w)^2} \left( \frac{(-N + k)}{(N + 2 + k)} \mathcal{U}^{(\frac{5}{2})}_{\pm} \mathcal{V}^{(\frac{5}{2})}_{\mp}\right)(w) \\
& \quad + \frac{1}{(z-w)2} \left[ \frac{1}{5} \partial (\text{pole-2}) + \mathcal{W}^{(5)}_{\mp}\right](w) + \cdots, \\

\end{align*}
\]
\[
\begin{align*}
\left( \begin{array}{c} G_{11} \\ G_{22} \end{array} \right) (z) & \left( \begin{array}{c} V^\left( \frac{7}{2} \right) \\ U^\left( \frac{7}{2} \right) \end{array} \right) (w) \\
& = \pm \frac{1}{(z - w)^2} \left[ \frac{1}{5} \partial (\text{pole-2}) \pm \frac{48(-N + k)}{5(N + 2 + k)} T^2(w) \\
& + \frac{1}{(z - w)^2} \left( 6 T^3 - \frac{6(-N + k)}{5(N + 2 + k)} W^3 \right) (w) \\
& + \frac{1}{(z - w)} \left[ \frac{1}{6} \partial (\text{pole-2}) \mp W^4 \right] \right] (w) + \cdots,
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c} G_{11} \\ G_{22} \end{array} \right) (z) & W_+^3 (w) \\
& = \pm \frac{1}{(z - w)^2} \left( \frac{12(2N + 5 + 3k)}{5(N + 2 + k)} (U_+^3) \right) (w) + \frac{1}{(z - w)} \left[ \frac{1}{6} \partial (\text{pole-2}) \right] (w) + \cdots,
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c} G_{11} \\ G_{22} \end{array} \right) (z) & W_-^3 (w) \\
& = \mp \frac{1}{(z - w)^2} \left( \frac{12(3N + 5 + 2k)}{5(N + 2 + k)} (U_-^3) \right) (w) + \frac{1}{(z - w)} \left[ \frac{1}{6} \partial (\text{pole-2}) \right] (w) + \cdots,
\end{align*}
\]

\[
\begin{align*}
\left( \begin{array}{c} G_{11} \\ G_{22} \end{array} \right) (z) & W_+^4 (w) \\
& = \pm \frac{1}{(z - w)^2} \left( \frac{12(-N + k)(41N + 70 + (12N + 41)k)}{5((37N + 59) + (15N + 37)k)} \left( \begin{array}{c} U^\left( \frac{7}{2} \right) \\ V^\left( \frac{7}{2} \right) \end{array} \right) \right) (w) \\
& + \frac{1}{(z - w)^2} \left[ \pm 7 \left( \begin{array}{c} U^\left( \frac{7}{2} \right) \\ V^\left( \frac{7}{2} \right) \end{array} \right) \right] \left[ \frac{216(-N + k)}{5((37N + 59) + (15N + 37)k)} \partial \left( \begin{array}{c} U^\left( \frac{7}{2} \right) \\ V^\left( \frac{7}{2} \right) \end{array} \right) \right] + \cdots.
\end{align*}
\]
\[
\begin{align*}
\left( \frac{G_{12}}{G_{21}} \right) (z) T^{(2)}(w) &= \frac{1}{(z-w)} T^{(2)}_\pm(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) T^{(3)}_\pm(w) &= \frac{1}{(z-w)^2} \frac{4}{3} T^{(2)}(w) + \frac{1}{(z-w)} \left[ \partial T^{(2)} \mp T^{(3)} \right](w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) T^{(3)}(w) &= \pm \frac{1}{(z-w)^2} \frac{5}{3} T^{(5)}_\pm(w) + \frac{1}{(z-w)} \frac{1}{5} \partial \text{pole}(2)(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) U^{(3)}_\pm(w) &= \frac{1}{(z-w)} \left( \frac{U^{(3)}_+}{U^{(3)}_-} \right)(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) V^{(3)}_\pm(w) &= \frac{1}{(z-w)} \left( \frac{V^{(3)}_+}{V^{(3)}_-} \right)(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) U^{(3)}_+(w) &= \frac{1}{(z-w)^2} \frac{2}{3} (3N + 5 + 2k) \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) \\
&\quad + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial \text{pole}(2) + \frac{1}{5} \frac{12}{3} \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) V^{(3)}_+(w) &= \pm \frac{1}{(z-w)^2} \frac{12}{5} (3N + 5 + 2k) \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) \\
&\quad + \frac{1}{(z-w)} \frac{1}{6} \partial \text{pole}(2)(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) W^{(3)}(w) &= \mp \frac{1}{(z-w)^2} \frac{(N-k)}{(N+2+k)} \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) \\
&\quad + \frac{1}{(z-w)} \left[ \frac{1}{5} \partial \text{pole}(2) + \frac{1}{5} \frac{12}{3} \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) + \cdots, \\
\left( \frac{G_{12}}{G_{21}} \right) (z) W^{(3)}_\pm(w) &= \pm \frac{1}{(z-w)^2} \frac{48}{5} \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) \\
&\quad + \frac{1}{(z-w)^2} \left[ \frac{6}{5} \left( \frac{U^{(3)}_+}{V^{(3)}_+} \right)(w) + \frac{1}{6} \partial \text{pole}(2) \mp W^{(4)} \right](w) + \cdots,
\end{align*}
\]
\[
\begin{pmatrix} G^{12} \\ G_{21} \end{pmatrix}(z) \mathcal{W}^{(4)}(w) = \\
\pm \frac{72(-N+k)}{(37N+59+(15N+37)k)} \left[ T T^{(2)} - \frac{3}{10} \partial^2 T^{(2)} \right] (w) + \ldots ,
\]

\[
+ \frac{1}{(z-w)^2} \left[ \pm 7 W^{(2)}_+ - \frac{216(-N+k)}{5(37N+59+(15N+37)k)} \partial T^{(2)}_+ \right] (w)
\]

\[
+ \frac{1}{(z-w)} \left[ \pm \partial W^{(2)}_+ + \frac{72(-N+k)}{(37N+59+(15N+37)k)} T^{(2)} \partial \left( \begin{array}{c} G^{12} \\ G_{21} \end{array} \right) \right] (w)
\]

\[
+ \frac{1}{(z-w)} \left[ \pm \partial W^{(2)}_+ + \frac{72(-N+k)}{(37N+59+(15N+37)k)} T^{(2)} \partial \left( \begin{array}{c} G^{12} \\ G_{21} \end{array} \right) \right] (w)
\]

\[
+ \frac{36(-N+k)}{(37N+59+(15N+37)k)} T^{(2)} \partial \left( \begin{array}{c} G^{12} \\ G_{21} \end{array} \right) \right] (w)
\]

\[
- \frac{18(-N+k)}{5(37N+59+(15N+37)k)} \partial^2 T^{(2)}_+ \right] (w) + \ldots .
\]

The nonlinear terms appearing in (D.5) can be removed by adding the extra quasiprimary field of spin-4 to the higher spin-4 current. See also the subsection 4.4. Via the explicit field identifications between the fields in this paper and those in [31], the above OPEs (D.5) are the same as the ones in [31].

**Appendix E** The OPEs between the 16 currents and the 16 higher spin currents in component approach with different basis

Let us present the description of [31] as follows:

\[
G^a(z) V^{(s)}_0(w) = \frac{1}{(z-w)} V^{(s),a}_\frac{1}{2}(w) + \ldots ,
\]

\[
A^{\pm,i}(z) V^{(s),a}_{\frac{1}{2}}(w) = \frac{1}{(z-w)} \alpha^{\pm,i}_{ab} V^{(s),b}_{\frac{1}{2}}(w) + \ldots ,
\]

\[
G^a(z) V^{(s),b}_{\frac{1}{2}}(w) = \frac{1}{(z-w)^2} 2s \delta^{ab} V^{(s)}_0(w)
\]

\[
+ \frac{1}{(z-w)} \left[ \alpha^{+,i}_{ab} V^{(s),+,i}_1 + \alpha^{-,i}_{ab} V^{(s),-,i}_1 + \delta^{ab} \partial V^{(s)}_0 \right] (w) + \ldots ,
\]

\[
Q^a(z) V^{(s),+;i}_1(w) = \pm \frac{1}{(z-w)} 2 \alpha^{+,i}_{ab} V^{(s),b}_{\frac{1}{2}}(w) + \ldots ,
\]

\[
A^{\pm,i}(z) V^{(s),\pm;J}_1(w) = \frac{1}{(z-w)^2} 2s \delta^{ij} V^{(s)}_0(w) + \frac{1}{(z-w)} \epsilon^{ijk} V^{(s),\pm,k}_1(w) + \ldots ,
\]

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\[ G^a(z) V_1^{(s),\pm,i}(w) = \frac{1}{(z-w)^2} 4 (s + \gamma_{\pm}) \alpha_{ab}^{\pm,i} V_1^{(s),b}(w) \]
\[ + \frac{1}{(z-w)} \left[ \frac{1}{2s+1} \partial \text{(pole-2)} + \alpha_{ab}^{\pm,i} V_1^{(s),b} \right] (w) + \cdots, \]
\[ U(z) V_1^{(s),a}(w) = -\frac{1}{(z-w)^2} 2 V_1^{(s),a}(w) + \cdots, \]
\[ Q^a(z) V_1^{(s),b}(w) = \frac{1}{(z-w)^2} 4 s \delta_{ab} V_0^{(s)}(w) \]
\[ + \frac{1}{(z-w)} 2 \left[ \alpha_{ab}^{\pm,i} V_1^{(s),+i} + \alpha_{ab}^{+,i} V_1^{(s),-i} - \delta_{ab} \partial V_0^{(s)} \right] (w) + \cdots, \]
\[ A^{\pm,i}(z) V_1^{(s),a}(w) = \pm \frac{1}{(z-w)^2} \left[ \frac{8s(2(s+1)+4\gamma_{\pm})}{2s+1} \right] \alpha_{ab}^{\pm,i} V_1^{(s),b}(w) \]
\[ + \frac{1}{(z-w)} \alpha_{ab}^{\pm,i} V_1^{(s),b}(w) + \cdots, \]
\[ G^a(z) V_2^{(s),b}(w) = -\frac{1}{(z-w)^3} \left[ \frac{16s(s+1)(2\gamma - 1)}{2s+1} \right] \delta_{ab} V_0^{(s)}(w) \]
\[ - \frac{1}{(z-w)^2} \frac{8s+1}{2s+1} \left[ (s + \gamma_{\pm}) \alpha_{ab}^{\pm,i} V_1^{(s),+i} - (s + \gamma_{-}) \alpha_{ab}^{-i} V_1^{(s),-i} \right] (w) \]
\[ + \frac{1}{(z-w)} \left[ \frac{1}{2(s+1)} \partial \text{(pole-2)} + \delta_{ab} V_2^{(s)} \right] (w) + \cdots, \]
\[ U(z) V_2^{(s)}(w) = \frac{1}{(z-w)^3} 8 s V_0^{(s)}(w) - \frac{1}{(z-w)^2} 4 \partial V_0^{(s)}(w) + \cdots, \]
\[ Q^a(z) V_2^{(s)}(w) = -\frac{1}{(z-w)^2} 2 (2s+1) V_1^{(s),a}(w) + \frac{1}{(z-w)} 2 \partial V_1^{(s),a}(w) + \cdots, \]
\[ A^{\pm,i}(z) V_2^{(s)}(w) = \pm \frac{1}{(z-w)^2} 2(s+1) V_1^{(s),\pm,i}(w) + \cdots, \]
\[ G^a(z) V_2^{(s)}(w) = \frac{1}{(z-w)^3} \left[ \frac{16s(s+1)(2\gamma - 1)}{2s+1} \right] V_1^{(s),a}(w) \]
\[ + \frac{1}{(z-w)^2} (2s+3) V_1^{(s),a}(w) + \frac{1}{(z-w)} \partial V_1^{(s),a}(w) + \cdots, \]
\[ T(z) V_2^{(s)}(w) = -\frac{1}{(z-w)^4} \left[ \frac{24s(s+1)(2\gamma - 1)}{2s+1} \right] V_0^{(s)}(w) \]
\[ + \frac{1}{(z-w)^2} (s+2) V_2^{(s)}(w) + \frac{1}{(z-w)} \partial V_2^{(s)}(w) + \cdots, \]
\[ T(z) V_0^{(s)}(w) = \frac{1}{(z-w)^2} s V_0^{(s)}(w) + \frac{1}{(z-w)} \partial V_0^{(s)}(w) + \cdots, \]
\[ T(z) V_2^{(s),a}(w) = \frac{1}{(z-w)^2} (s + \frac{1}{2}) V_2^{(s),a}(w) + \frac{1}{(z-w)} \partial V_2^{(s),a}(w) + \cdots, \]
\[ T(z) V_{1}^{(s),\pm,i}(w) = \frac{1}{(z-w)^2} (s+1) V_{1}^{(s),\pm,i}(w) + \frac{1}{(z-w)} \partial V_{1}^{(s),\pm,i}(w) + \cdots, \]
\[ T(z) V_{\frac{3}{2}}^{(s),a}(w) = \frac{1}{(z-w)^2} (s + \frac{3}{2}) V_{\frac{3}{2}}^{(s),a}(w) + \frac{1}{(z-w)} \partial V_{\frac{3}{2}}^{(s),a}(w) + \cdots. \] (E.1)

Here the two parameters are introduced as follows: \( \gamma_+ = \gamma = \frac{k^-}{k^+ + k^-} \) and \( \gamma_- = 1 - \gamma = \frac{k^+}{k^+ + k^-} \) where \( k^+ = k + 1 \) and \( k^- = N + 1 \). From the OPEs in (E.1), the higher spin-s, \((s + \frac{1}{2})\), \((s + 1)\), \((s + \frac{3}{2})\) currents are primary fields under the stress energy tensor \( T(z) \). Note that the higher spin-(s + 2) current \( V_{2}^{(s)}(w) \) is not a primary current because there is a fourth-order pole term. We can consider the extra composite field \( T^{(s)} T(w) \) in order to make the above higher spin-(s + 2) current transforming as a primary field. See also the subsection 4.4. We can analyze what has been done in [48] in order to see the explicit relations between the higher spin currents in Appendix D and those in Appendix E. The final expressions are given in the subsection 4.4.

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