On completely homogeneous c-spaces

Sini P. ∗
Department of Mathematics, University of Calicut, Kerala, Pin-673635, India

ARTICLE INFO

Keywords:
Mathematics
C-space
Connective space
Set structure
Touching set
Homogeneous c-space
Completely homogeneous c-space

ABSTRACT

In this paper we determine completely homogeneous c-spaces. Various properties of completely homogeneous c-spaces are discussed. Relation between completely homogeneous c-space and hereditary homogeneous c-space is studied. Completely homogeneous connective spaces are also characterised.

0. Introduction

Connectivity plays a fundamental role in digital image processing and analysis. It is extensively used in image filtering and segmentation, image compression and coding, motion analysis, pattern recognition and other application. Connectivity is classically defined using either a topological or a graph theoretical framework. In topology connectivity is defined in terms of separation whereas in graph theory the same is defined in terms of paths. So the notions of topological connectivity and graph theoretical connectivity are not equivalent even though they have a wide application in digital image processing and analysis. Since discrete images are often obtained by sampling continuous scenes, compatibility of these two structures is desired.

In the late eighties, G. Matheron and J. Serra proposed a framework for connectivity known as the theory of connectivity classes. A systematic study was carried out by J. Serra [1], H. J. A. M. Heijmans [2], C. Rosenfeld [3], J. Muscat and D. Buhagiar [4], S. Dugowson [5] etc. In literature so many terminologies are used for connectivity classes. We like to follow the terminology of c-spaces used in [4].

Let X be a set. By a set structure, we mean a pair (X,A) where A ⊆ P(X). By adding various axioms we obtain various special structures-topology, connectecy, bornology etc. Though there are several different possibilities for the morphisms of a general set structure, there is only one natural choice for isomorphisms. Hence the notions of homogeneity and complete homogeneity are the same for all set structures. A set structure on X is homogeneous if for x, y ∈ X, there is an automorphism f : X → X such that f(x) = y, hereditarily homogeneous if every substructure is homogeneous and it is completely homogeneous if every bijection from X to X is an automorphism. Here we investigate these notions in the realm of c-spaces and classify completely homogeneous c-structures on a non empty set X. We study various properties of completely homogeneous c-spaces. Also we study completely homogeneous connective spaces.

1. Preliminaries

Here we introduce some basic definitions which will be used in this paper.

A c-space is an ordered pair (X, C) where C is a collection of subsets of X such that the following properties hold.

(i). ∅ ∈ C and {x} ∈ C for all x ∈ X.
(ii). If {C i : i ∈ I} be a collection of subsets in C such that ∩ i∈I C i ≠ ∅ then i∈I C i ∈ C.

Elements of C are called connected sets.

Let (X, C) be a c-space. Then X is said to be a connective space [4] if C satisfies two more conditions as given below.

(iii). Given any nonempty sets A, B ∈ C such that A ∪ B ∈ C, then there exists x ∈ A ∪ B such that {x} ∪ A ∈ C and {x} ∪ B ∈ C.
(iv). If A, B, C ∈ C are disjoint and A ∪ B ∪ i∈I C i ∈ C, then there exists J ⊆ I such that A ∪ i∈J C i ∈ C and B ∪ i∈I\J C i ∈ C X.

* Corresponding author.
E-mail address: cue3774@uoc.ac.in.

https://doi.org/10.1016/j.heliyon.2020.e05747
Received 15 November 2019; Received in revised form 24 March 2020; Accepted 10 December 2020

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A c-structure satisfying the above two conditions is called a connective structure or connectecty on X.

Let X be any set. Then \( C_X = \{ \phi \} \cup \{ \{ x \} : x \in X \} \) is a c-structure on X, called the discrete c-structure on X and is usually denoted by \( D_X \). The space \((X, D_X)\) is called the discrete c-space. Also \( C_X = P(X) \), the power set of X, is a c-structure on X, called the indiscrete c-structure on X. The space \((X, C_X)\) is called the indiscrete c-space.

A c-space \((Y, C_Y)\) is said to be a sub-c-space or subspace, of the c-space \((X, C_X)\) if \( Y \subseteq X \) and \( C_Y = \{ A \subseteq C_X : A \subseteq Y \} \). Let X be any set and \( B \subseteq P(X) \). Then the intersection of all c-structures on X containing \( B \) is a c-structure and is called the c-structure generated by \( B \) and is denoted by \( (B) \). Obviously it is the smallest c-structure on X containing \( B \). Let X and Y be two c-spaces and \( f : X \to Y \) be a function. \( f \) is said to be a c-continuous function if it maps connected sets in X to connected sets in Y. \( f \) is said to be a c-isomorphism or a c-automorphism if it is bijective and both \( f \) and \( f^{-1} \) are c-continuous. X and Y are said to be c-isomorphic if there exists a c-isomorphism between them. A c-isomorphism from a c-space onto itself is called a c-automorphism.

The set of all c-automorphisms on a c-space on itself forms a group and is called the group of c-automorphisms of \((X, C_X)\).

2. Completely homogeneous c-spaces

Here we define completely homogeneous c-spaces.

**Definition 2.1.** A c-space \((X, C_X)\) is called completely homogeneous if every bijection of \( X \) is a c-automorphism.

In other words a c-space \((X, C_X)\) is said to be completely homogeneous if the group of c-automorphisms on \((X, C_X)\) is the symmetric group on X.

**Example 2.2.**

- The discrete c-space \( D_X \).
- The indiscrete c-space \( P(X) \).
- Let X be any set and \( C_X = D_X \cup \{ A \subseteq X : |A| \geq m \} \) where \( m \leq |X| \).
- Let X be an infinite set and \( C_X = D_X \cup \{ A \subseteq X : X \setminus A \text{ is finite} \} \).

**Proposition 2.3.** Every subspace of a completely homogeneous c-space is a completely homogeneous c-space.

**Proof.** Let \( Y \) be a subset of a completely homogeneous c-space and \( f \) be a permutation on \( Y \). Let \( C \subseteq C_Y \). This implies that \( C \) is connected in \( X \). Now define \( f' : X \to X \) as follows.

\[ f'(x) = \begin{cases} f(x) & \text{if } x \in Y \\ x & \text{otherwise.} \end{cases} \]

Then \( f' \) is a c-automorphism on X. So \( f'(C) \) is connected in X. Note that \( f'(C) = f(C) \) and \( f(C) \subseteq Y \). Thus \( f(C) \) is a connected set in \( C_Y \).

Since \( C \) is arbitrary, \( f \) is a c-continuous map. Similarly we can prove that \( f^{-1} \) is also a c-continuous map. Thus \( f \) is a c-automorphism on \( Y \).

This completes the proof. □

**Definition 2.4.** [6] A c-space is said to be homogeneous if for any two elements \( x, y \in X \), there exists a c-automorphism \( h \) of \((X, C_X)\) onto itself such that \( h(x) = y \).

From the definition of homogeneous c-spaces, it follows that \((X, C_X)\) is homogeneous if and only if the group of c-automorphisms on \((X, C_X)\) is a transitive permutation group on X.

A subspace of a homogeneous c-space need not be homogeneous. See the following example.

**Example 2.5.** Let \( X = \{a, b, c, d\} \) and \( C_X = D_X \cup \{ \{ a, b \}, \{c, d\}, \{ b, c \}, \{ a, d \}, \{ a, b, c \}, \{ b, c, d \}, \{a, c,d \}, \{a, b, d \}, \{ a, b, c, d \} \} \).

Here the group of all c-automorphisms, \( C(X, C_X) \) on X is \( \{ I, (a \ b \ c \ d), (a \ c \ b \ d), (a d \ c \ b), (a c \ b \ d), (a b \ c \ d), (a d \ b \ c) \} \) where \( I \) denotes the identity permutation. Clearly \( C(X, C_X) \) is a transitive permutation group. So \((X, C_X)\) is a homogeneous c-space.

Now consider \( Y = \{a, b, d\} \). The subspace c-structure on Y, \( C_Y = D_Y \cup \{ Y, \{a, b, \{a, d \}\} \} \).

Then \((Y, C_Y)\) is not homogeneous.

**Definition 2.6.** A c-space \((X, C_X)\) is said to be hereditarily homogeneous if every sub c-space of \((X, C_X)\) is homogeneous.

Note that every completely homogeneous space is hereditarily homogeneous. But the converse is not true.

**Example 2.7.** Let \( X = \mathbb{Z} \) and \( X' = 2\mathbb{Z} \). Define a c-space on X as follows.

\[ C_X = D_X \cup \{ C \subseteq X : C \cap X' \text{ is a cofinite subset of } X' \} \]

Let \( x, y \in X \) and \( f \) be the function on X which maps \( x \to y \) and \( y \to x \) and keeping all other elements fixed. Let \( C \subseteq X \) be nondegenerate. Since the symmetric difference \( C \triangle f(C) \) is finite, we have that \( C \) is connected if and only if \( f(C) \) is connected. So \( f \) is a c-automorphism on X which maps \( x \to y \). Thus \((X, C_X)\) is a homogeneous c-space. Let \( Y \subseteq X \) and \( Y' = Y \cap \mathbb{Z} \). If \( Y' \) is cofinite in \( X' \), then the subspace c-structure on X by \( C_Y = D_Y \cup \{ C \subseteq C_Y : C \cap Y' \text{ is a cofinite subset of } Y' \} \). Clearly \((Y, C_Y)\) is a homogeneous c-space as before. Otherwise \( C_Y = D_Y \), which is also homogeneous. Thus \((X, C_X)\) is a hereditarily homogeneous c-space.

Define a map \( f : X \to X \) as \( f(x) = x + 1 \). Since \( 2\mathbb{Z} \subseteq C_X \) and \( f(2\mathbb{Z}) = 2\mathbb{Z} + 1 \not\in C_X \), \( f \) is not a c-automorphism on X. So \((X, C_X)\) is not a completely homogeneous c-space.

We now prove two important properties of completely homogeneous c-spaces.

For proving the subsequent theorems, we need the following set theoretic results.

Let P and Q be two subsets of set X and \( |P| = |Q| \), it does not necessarily follow that there exists a bijection of X which maps P onto Q. In order to exist such a function, we must also have that \( |X \setminus P| = |X \setminus Q| \). If X is an infinite set, it is possible to choose P and Q such that \( P \cup Q = X \), \( P \cap Q = \emptyset \) and \( |P| = |Q| = |X| \) since for any infinite cardinal number \( a \), we have \( a + a = a \).

For \( A \subseteq X \), let \( \Phi_A(X) \) denote the pair \( ([A], [X \setminus A]) \) and \( \Phi_X = (\Phi_A(X) : A \subseteq X) \).

Using the natural order on cardinal numbers, we can define an order on \( \Phi_X \) as follows. If X is a finite set with \( |X| = n \). The order on \( \Phi_X \) is\((0, n) < (1, n-1) < (2, n-2) < \ldots < (n-1, 1) < (n, 0)\).

If X is an infinite set with \( |X| = k \). The order on \( \Phi_X \) is

\((0, k) < (1, k) < (2, k) < \ldots < (k, k) < \ldots < (k, k) < \ldots < (k, 1) < (k, 0)\)

where \( k < k \). In both cases we may identify \((k, k)\) with \( \lambda \) for \( k \leq k \). Thus \( \Phi_X \) is an ordered set.

Let A, B \( \subseteq X \). There is a bijection f from X to X such that \( f(A) = B \) if and only if \( \Phi_A(A) = \Phi_B(B) \), and in this case, every bijection from A
to $B$ can be extended to a bijection from $X$ to $X$. Hence, a set structure $\mathcal{A}$ is completely homogeneous if and only if for every $A \in \mathcal{A}$ and every $B \subseteq X$ the equality $\phi_X(A) = \phi_X(B)$ implies $B \in \mathcal{A}$. In particular, if $\mathcal{A}$ is completely homogeneous and $A \in \mathcal{A}$ such that $|A| < |X|$ then $B \in \mathcal{A}$ for all $B \subseteq X$ such that $|B| = |A|$. We summarize these observations in the following theorem.

**Theorem 2.8.** Let $X$ be any set. Then the completely homogeneous set structures on $X$ are exactly the structures $\mathcal{A}_K := \{A \subseteq X : \phi_X(A) \in K\}$ for $K \subseteq \Phi_X$.

**Definition 2.9.** Let $A \subseteq \mathcal{P}(X)$ is said to form an upper family if $B \in A$ and $B \subseteq C$ implies that $C \in A$.

**Proposition 2.10.** Let $X$ be any set and $A \subseteq \mathcal{P}(X)$ be an upper family. Then $\mathcal{C}_X = D_X \cup A$ is a c-space.

**Proof.** Proof is trivial. □

The c-space in Proposition 2.10 is called an upper c-space and $\mathcal{C}_X$ is an upper c-structure on $X$.

Examples 2.5 and 2.7 are upper c-spaces generated by $\{[a, b], [c, d], [b, c], [a, d]\}$ and cofinite subsets of $\mathbb{Z}$ respectively.

**Remark 2.11.** An upper c-space always satisfies axiom (iii). But it needs not satisfy axiom (iv). For example, let $X = \{a, b, c, d\}$ and $\mathcal{C}_X = D_X \cup \{[a, b, c], X\}$. If we take $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$, then $A \cup B \cup C \in \mathcal{C}_X$. Note that no two element subsets are connected in $\mathcal{C}_X$. So $\mathcal{C}_X$ is not a connective structure on $X$.

In an upper c-space, if at least one of the sets $A, B, \{C_i : i \in I\}$ is nondegenerate, then it satisfies condition (iv) in the definition of a connective space. For, if $A$ or one of the $|C_i : i \in I|$ is nondegenerate, take $J = I$, and if $B$ is nondegenerate, take $J = \emptyset$.

Now we characterise connective upper c-spaces.

**Theorem 2.12.** Let $(X, \mathcal{C}_X)$ be an upper c-space. Then $\mathcal{C}_X$ is connective if and only if for every connected set $C$ and distinct points $a, b \in C$, $C \setminus \{a\}$ or $C \setminus \{b\}$ is connected.

**Proof.** Assume that $(X, \mathcal{C}_X)$ is a connective space. Let $C$ be a connected set and $a, b \in C$ with $a \neq b$. Let $A = \{a\}$, $B = \{b\}$ and $\{C_i : i \in I\}$ be the collection of all singletons in $C \setminus \{a, b\}$. Then $A \cup B \cup \bigcup_{i \in I} C_i = C$. Since $(X, \mathcal{C}_X)$ is a connective space, there exists $J \subseteq I$ such that $A \cup \bigcup_{i \in J} C_i$ and $B \cup \bigcup_{i \notin J} C_i$ are connected. Since $\mathcal{C}_X$ is an upper c-structure on $X$ and at least one of $J, I \setminus J$ is nonempty, $C \setminus \{a\}$ or $C \setminus \{b\}$ is connected.

Conversely assume that the condition in the theorem holds. Since $\mathcal{C}_X$ is an upper c-space, $\mathcal{C}_X$ satisfies axiom (iii). It also satisfies axiom (iv) whenever at least one of the sets $A, B, C_i$ is nondegenerate. Let $A, B, C_i$ be singleton sets and $A \cup B \cup \bigcup_{i \in I} C_i \in \mathcal{C}_X$. Then by our assumption we get $J = \emptyset$ or $J = I$ such that $A \cup \bigcup_{i \in J} C_i$ and $B \cup \bigcup_{i \in I \setminus J} C_i$ are connected. So $\mathcal{C}_X$ is a connective structure on $X$. □

**Definition 2.13.** [4] Let $(X, \mathcal{C}_X)$ be a c-space and $A \subseteq X$. Then a point $x \in X$ is said to touch the set $A$ if there is a nonempty subset $C \subseteq A$ such that $\{x\} \cup C$ is connected. The set of all points touching the set $A$ is denoted by $t(A)$.

**Proposition 2.14.** Let $(X, \mathcal{C}_X)$ be a c-space. Then $\mathcal{C}_X$ is an upper c-structure on $X$ if and only if $t(C) = X$ for all nondegenerate $C \in \mathcal{C}_X$.

**Proof.** First suppose that $\mathcal{C}_X$ is an upper c-structure on $X$ and $C$ is a nondegenerate connected set. Then for any $x \in X \setminus C$, $\{x\} \cup C$ is connected and hence $t(C) = X$.

Now assume that $t(C) = X$ for all nondegenerate $C \in \mathcal{C}_X$. Let $A$ be a nondegenerate connected set. Since $t(A) = X$, then for each $x \in X$, there is a nonempty subset $C \subseteq X$ such that $\{x\} \cup C$ is connected. Note that $(\{x\} \cup C) \cap \mathcal{C}_X$ is nonempty. So $\{x\} \cup A$ is connected for each $x \in X$ and hence every superset of $A$ is connected. □

**Definition 2.15.** Let $(X, \mathcal{C}_X)$ be a c-space and $A$ be a subset of $X$. Then $A$ is said to be c-dense in $X$ if $t(A) = X$.

So a c-space is an upper c-space if and only if each nondegenerate connected set is c-dense in $X$.

Now we study the relation between upper c-spaces and completely homogeneous c-spaces. We have the following theorem.

**Theorem 2.16.** Every completely homogeneous c-space is an upper c-space.

**Proof.** Let $C \subseteq \mathcal{C}_X$ and $|C| > 1$ and $B$ be a superset of $C$. Now choose two points $x$ in $C$ and $y$ in $B \setminus C$. Now define a function $f : X \to X$ such that $f(x) = y$, $f(y) = x$ and $f$ fixes all other points in $X$. Since $(X, \mathcal{C}_X)$ is a completely homogeneous c-space $f$ is a c-automorphism on $X$. It follows that $f(C) = (C \setminus \{x\}) \cup \{y\} \in \mathcal{C}_X$. Since $C \cap f(C) = C \setminus \{x\}$ and $|C| > 1$, $C \cap f(C)$ is non-empty and hence $C \cup f(C) = C \cup \{y\}$ is connected. Note that $B = \bigcup_{b \in C} (\mathcal{C}_X \cup \{b\})$. This implies that $B$ is connected. So if $(X, \mathcal{C}_X)$ is a completely homogeneous c-space and $C \subseteq \mathcal{C}_X$ is nondegenerate, then $B \in \mathcal{C}_X$ for all $B \subseteq X$ such that $C \subseteq B$. Hence the proof. □

In a completely homogeneous c-space, each nondegenerate set is c-dense in $X$.

Note that an upper c-space need not be even homogeneous. For example, let $X = \{a, b, c\}$ and $\mathcal{C}_X = D_X \cup \{[a, b], X\}$. Then $\mathcal{C}_X$ is an upper c-space but not homogeneous. A homogeneous space need not be an upper space. For example, $X = \{a, b, c, d\}$ and $\mathcal{C}_X = D_X \cup \{[a, b], [c, d], X\}$. Then $\mathcal{C}_X$ is a homogeneous c-space but not an upper c-space.

**Theorem 2.17.** Let $X$ be any set and $\mathcal{A}_X$ be a completely homogeneous set structure on $X$. Then $\mathcal{A}_X$ is a c-space if and only if $\mathcal{A}_X$ satisfies the following two conditions.

1. $(0, 1] \subseteq K$
2. $K \setminus (0, 1]$ is an upper subset of $\Phi_X$.

**Proof.** Let $K : = \{A \subseteq X : \phi_X(A) \in K\}$ for $K \subseteq \Phi_X$ be a completely homogeneous set structure on $X$. Assume that $\mathcal{A}_X$ is a c-structure on $X$. Then $\emptyset \in \mathcal{C}_X$ and $\{x\} \in \mathcal{C}_X$ for all $x \in X$ and so $(0, 1] \subseteq K$. Since $\mathcal{A}_X$ is a completely homogeneous c-space, it is an upper c-space and hence $K \setminus (0, 1]$ must be an upper subset of $\Phi_X$.

Conversely assume that $(0, 1] \in K$ and $K \setminus (0, 1]$ upper subset of $\Phi_X$. Then clearly $D_X \subseteq \mathcal{A}_X$ and $\mathcal{A}_X \setminus D_X$ is an upper family. This implies that $\mathcal{A}_X$ is a c-space. □

**Remark 2.18.** Let $X$ be a finite set. Then $\mathcal{C}_X$ is a completely homogeneous c-structure on $X$ if and only if $\mathcal{C}_X$ is one of the following.

(a) $D_X$.
(b) $D_X \cup \{B \subseteq X : |B| \geq n\}$, $1 < n \leq |X|$.

Now using Theorem 2.17, we can deduce two properties of completely homogeneous c-structures on an infinite set $X$.

**Remark 2.19.** Let $\mathcal{C}_X$ be a completely homogeneous c-structure on an infinite set $X$ and $C \subseteq \mathcal{C}_X$ such that $|C| = |X|$.

(a) If $|X \setminus C| = |X|$, then $B \in \mathcal{C}_X$ for all $B \subseteq X$ such that $|B| = |C|$. 

Definition 2.20. [8] The successor of a cardinal $m$ is the least cardinal greater than $m$. A cardinal is said to be a limit cardinal if it is not the successor of a cardinal.

The cardinals $0, \aleph_0, \aleph_n$ etc. are limit cardinals. An uncountable cardinal $\aleph_n$ will be a limit cardinal if and only if $\gamma$ is a limit ordinal [8].

In the following theorem we characterize completely homogeneous $c$-structures on an infinite set.

Theorem 2.21. Let $X$ be an infinite set and $|X| = k$. Then $C_X$ is a completely homogeneous $c$-structure on $X$ if and only if $C_X$ is one of the following.

(a) $D_X$.
(b) $D_X \cup \{B \subseteq X : |B| \geq n\}$, $1 < n \leq k$.
(c) $D_X \cup \{B \subseteq X : |X \setminus B| \leq n\}$ where $n < k$.
(d) $D_X \cup \{B \subseteq X : |X \setminus B| < n\}$ where $n \leq k$ and $n$ is a limit cardinal.

Proof. It is clear that the above $c$-spaces are completely homogeneous. Now we show that all completely homogeneous $c$-structures on $X$ are of the above form. Assume that $C_X$ is a completely homogeneous $c$-structure on $X$. Then $C_X = A_X$ for some $K \subseteq D_X$ such that $K \subseteq \{0, 1\}$ and $K$ is an upper set. If $K = \{0, 1\}$, then $C_X = D_X$. Otherwise $K \subseteq \{0, 1\} \neq \emptyset$ and is an upper set. Since the set of all cardinal numbers is well ordered, $(\lambda : (\lambda, \nu) \in K \setminus \{0, 1\})$ has a least element and let it be $\lambda$. If $\lambda < k$, then $\nu = k$. In this case $C_X = D_X \cup \{B \subseteq X : |B| \geq \lambda\}$. Now consider the case $\lambda = k$. In this case let $S = \{\nu : (\lambda, \nu) \in K \setminus \{0, 1\}\}$. Clearly $|X|$ is an upper bound of $S$ and $S$ is a bounded set. Let $n = \sup S$. We consider two cases. First assume that $m \in S$. In this case $C_X = D_X \cup \{B \subseteq X : |X \setminus B| \leq m\}$ where $m \leq |X|$. If $m \notin S$, then $m$ is a limit cardinal and it follows that $C_X = D_X \cup \{B \subseteq X : |X \setminus B| < n\}$ where $m \leq |X|$. This completes the proof.

Let $(X, A_X)$ be a completely homogeneous connective space. Here we show that if $K$ contains a nondegenerate finite type, then $K = \Phi_X$. That is, the $c$-space is indiscrete. Using Theorem 2.12, we can easily prove the following result.

Proposition 2.22. Let $X$ be a set and $C_X$ be a completely homogeneous $c$-structure on $X$. Then $C_X$ is connective if and only if for any $C \in C_X$, $C \setminus \{x\} \in C_X$ for all $x \in X$.

Proof. Let $C_X$ be a completely homogeneous $c$-structure on $X$. By Theorem 2.16, $C_X$ is an upper structure on $X$. Then from Theorem 2.12, $C_X$ is connective if and only if for every connected set $C$ and distinct points $a, b \in C$, $C \setminus \{a\}$ or $C \setminus \{b\}$ is connected. Since $C_X$ is completely homogeneous, the transposition $(a, b)$ is a $c$-automorphism. So if $C \setminus \{a\}$ is connected, then $C \setminus \{b\}$ is also connected and vice versa. Hence both $C \setminus \{a\}$ and $C \setminus \{b\}$ are connected. Also if $|C| \\leq 1$, then clearly $C \setminus \{x\} \in C_X$ for every $x \in X$. Thus $C_X$ is connective if and only if for any $C \in C_X$, $C \setminus \{x\} \in C_X$ for all $x \in X$.

Corollary 2.23. The only completely homogeneous connective $c$-space on a finite set $X$ is the discrete $c$-space $P_X$.

Proposition 2.24. Let $X$ be any set with $|X| = k$ and $C_X = D_X \cup \{B \subseteq X : |B| \geq n\}$, $1 < n \leq k$. Then $C_X$ is a connective structure if and only if $n = 2$ or $n$ is an infinite cardinal.

Proof. Assume that $C_X$ is a connective space. Then by Proposition 2.22, if $n$ is a finite cardinal, then $n = 2$. So either $n = 2$ or $n$ is an infinite cardinal. Conversely if $n = 2$ then $C_X = P(X)$. Obviously $P(X)$ is a connective space. If $n$ is an infinite cardinal, then $C_X$ is connective again by Proposition 2.22.

Proposition 2.25. Let $X$ be an infinite set and $C_X = D_X \cup \{B \subseteq X : |X \setminus B| \leq n\}$ where $n \leq |X|$. Then $C_X$ is a connective structure on $X$ if and only if $n$ is an infinite cardinal.

Proof. Assume that $C_X$ is a connective space and $B \in C_X$ with $|X \setminus B| = n$. Then by Proposition 2.22, $B \setminus \{x\} \in C_X$ for all $x \in B$. If $n$ is a finite cardinal, then $|X \setminus B| = n$, which is not possible. Hence $n$ must be an infinite cardinal. Converse part is obvious.

Now we can list completely homogeneous connective $c$-spaces.

Theorem 2.26. Let $X$ be any set. Then the completely homogeneous connective structures on $X$ are exactly the following.

(a) $D_X$.
(b) $D_X \cup \{B \subseteq X : |B| \geq n\}$ where $n = 2$ or is an infinite cardinal, $n \leq |X|$.
(c) $D_X \cup \{B \subseteq X : |X \setminus B| \leq n\}$ where $n$ is an infinite cardinal, $n \leq |X|$.
(d) $D_X \cup \{B \subseteq X : |X \setminus B| < n\}$ where $n \leq |X|$ and $n$ is a limit cardinal.

Proof. We use Theorem 2.21 and restrict the conditions (b) and (c) using Proposition 2.24 and 2.25, respectively. For (d), we observe that for a limit cardinal $n$, we have the implication $|X \setminus B| < n \Rightarrow |X \setminus B| < n$ and we use Proposition 2.22.

Definition 2.27. [4] Let $X$ be a $c$-space and $A \subseteq X$. Then $A$ is said to be $t$-closed if it contains all of its touching points. That is, $t(A) = A$. The $t$-closure of a set $A$ is defined to be the smallest $t$-closed set containing $A$ and is denoted by $\bar{A}$.

The definition of $C_X$ separation axiom in connection with connective spaces can be found in [4].

Definition 2.28. [4] A $c$-space $(X, C_X)$ is said to be $C_1$ if distinct points of $X$ are disconnected.

That is, a $c$-space $(X, C_X)$ is $C_1$ if and only if $\{x\} \neq \{x\}$ for all $x \in X$.

Proposition 2.29. A completely homogeneous connective $c$-space $(X, C_X)$ is either indiscrete or $C_1$.

Proof. Assume that $(X, C_X)$ is not $C_1$. Then there exists $x \in X$ such that $\{x\} \neq \{x\}$. Let $y \in (\{x\}) \setminus \{x\}$. Then $\{x, y\}$ is a connected set. Since $C_X$ is a completely homogeneous $c$-space, $C_X$ is indiscrete.

Declarations

Author contribution statement

Sini P.: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data availability statement

Data will be made available on request.
Declaration of interests statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

Acknowledgements

The valuable suggestions from Dr. Ramachandran. P. T., Professor, Department of Mathematics, University of Calicut during the preparation of this paper are greatly acknowledged. The author is thankful to the referees and editors for their valuable comments and suggestions which improved this paper.

References

[1] J. Serra, Connectivity on complete lattices, J. Math. Imaging Vis. 9 (1998) 231–251.
[2] H.J.A.M. Heijmans, Connected morphological operators for binary images, Comput. Vis. Image Underst. 73 (1) (1999) 99–120.
[3] A. Rosenfeld, Connectivity in digital pictures, J. Assoc. Comput. Mach. 17 (1) (1970) 146–160.
[4] J. Muscat, D. Buhagiar, Connective spaces, Mem. Fac. Sci. Shimane Univ., Ser. B, Math. Sci. 39 (2006) 1–12.
[5] S. Dugowson, On connectivity spaces, Cah. Topol. Géom. Différ. Catég. 51 (4) (2010) 282–315.
[6] P.K. Santhosh, Some Problems on c-Spaces, PhD thesis, University of Calicut, 2015.
[7] R.R. Stoll, Set Theory and Logic, Dover Publications Inc., New York, 1961.
[8] K. Devlin, The Joy of Sets: Fundamentals of Contemporary Set Theory, Springer-Verlag, New York, 1993.