Necessary integrability conditions for evolutionary lattice equations

V.E. Adler

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Abstract

The structure of solutions is studied for the Lax equation $D_t(G) = [F,G]$ for formal power series with respect to the shift operator. It is proved that if the equation with a given series $F$ of degree $m$ admits a solution $G$ of degree $k$ then it admits, as well, a solution $H$ of degree $m$ such that $H^k = G^m$. This property is used for derivation of necessary integrability conditions for scalar evolutionary lattices.

Keywords: Volterra type lattice, higher symmetry, conservation law, integrability test

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1 Introduction

It is well known that existence of infinite sets of higher symmetries and conservation laws is a characteristic property of integrable equations. In the case of two-dimensional evolutionary equations $\partial_t(u) = f(u)$, this implies existence of formal operator series $G,R$ which satisfy the equations (see notations in section 3)

$$D_t(G) = [f_*^*,G], \quad D_t(R) + f_*^R + Rf_* = 0. \quad (1)$$

The solvability of (1) with respect to the coefficients of $G,R$ provides the necessary conditions of integrability. This approach has been applied, in the papers by Shabat et al, for classification of integrable equations, both in partial derivatives [1, 2, 3, 4, 5, 6] and differential-difference ones [7, 2, 8, 9].

*L.D. Landau Institute for theoretical physics, Ak. Semenov str. 1-A, 142432 Chernogolovka, Russian Federation. E-mail: adler@itp.ac.ru
The continuous and discrete equations have much in common, but there are differences as well.

In the continuous case, $G$ and $R$ are pseudodifferential operators, that is, Laurent series with respect to inverse powers of total derivative $D$. If we consider 1-component evolutionary equations then the coefficients of the series are scalar functions of dynamical variables $u$. The root extraction plays a very important role in the theory of such series. This operation is defined for a generic series $G = g_k D^k + g_{k-1} D^{k-1} + \ldots$, since the coefficients of the series $G^{1/k}$ are computed by explicit algebraic formulae which do not lead out from the coefficient field in consideration [12]. This property drastically simplifies the study of equations (1), because it allows us to set $\deg G = 1$, $\deg R = 0$ without loss of generality. In particular, the integrability conditions can be rewritten in the form of a sequence of conservation laws

$$D_t(\rho_j) = D(\sigma_j), \quad j = 0, 1, \ldots$$

where $\rho_j$ are explicitly expressed through $\rho_i, \sigma_i$ at $i < j$. If the left hand side belongs to the image of $D$ (which coincides with the kernel of the variational derivative) then one can find $\sigma_j$ and pass to the next step of integrability test.

The situation is more complicated in the case of the lattice equations

$$\partial_t(u_n) = f(u_{n+m}, \ldots, u_{n-m}), \quad n \in \mathbb{Z}$$

studied in the present paper. Here, $G, R$ are power series with respect to the shift operator $T$ and the root $G^{1/k}$ does not exist for a generic series $G = g_k T^k + g_{k-1} T^{k-1} + \ldots$. It is clear already from consideration of the leading coefficient which must be of a special form $g_k = h T(h) \cdots T^{k-1}(h)$ in order that the root exists. As a result, the integrability conditions are more involved and cannot be cast into the form of conservation laws, in general; equations (1) lose their effectiveness, because the degrees of the series $G, R$ are not known in advance. However, it turns out that, in fact, the integrability conditions do not depend on $k$. The goal of this paper is to prove that the general case can be reduced to $\deg G = m$, $0 \leq \deg R < m$.

In section 2, the following statement is proved: if a difference Lax equation $D_t(G) = [F, G]$ with $\deg F = m$ admits a solution $G$ of degree $k$, then there exists another solution $H = G^{m/k}$ of degree $m$. The key idea is that the coefficients of $H$ can be computed explicitly by use of equations $D_t(H) = [F, H]$ and $H^k = G^m$ simultaneously, after this it is possible to prove that each equation is fulfilled separately. In other words, extraction of the root is still possible in a certain weak sense—due to the fact that
the solutions of the Lax equation are far from being generic series and their coefficients already possess a special structure.

This observation is used in sections 3, 4 in order to formulate the necessary integrability conditions for the lattices (3). If $m > 1$ then these conditions remain more complicated comparing to the continuous case, but their form is quite suitable for testing of a given equation. Solving of the Lax equation with respect to the coefficients of $G$ amounts to the checking whether a given expression belongs to the image of an operator of the form $T^m - T^j(a)/a$ where $a$ is a fixed function, and to the computation of its preimage. In principle, both problems admit algorithmic solutions which are, however, beyond the scope of this paper. The analysis of conditions in a general form, in order to obtain classification results or to construct novel examples is, even in the case $m = 2$, a very difficult task which requires a separate study as well. It should be noted that all examples with $m > 1$ known at the moment are equivalent (up to Miura type substitutions) to the Bogoyavlensky lattices [13] and some their generalizations [14, 15, 16]. In this respect, the theory lags behind the continuous case where a number of classification results was obtained for the Burgers type equations of orders 2, 4 [17] and the KdV type equations of orders 3, 5, 7 (see references in [6]).

The integrable equations (3) at $m = 1$ (the Volterra type lattices) were classified by Yamilov [7]. In this case, the necessary integrability conditions are of the form analogous to (2):

$$D_t(\rho_j) = (T - 1)(\sigma_j), \quad \rho_j - \bar{\rho}_j = (T - 1)(s_j), \quad j = 0, 1, \ldots$$  (4)

where $\rho_j, \bar{\rho}_j$ are expressed through $\rho_i, \bar{\rho}_i, \sigma_i, s_i$ at $i < j$. The derivation of conditions (4) in papers [9, 10] was based on the assumption that the lattice (3) admits higher symmetries of orders $k, k + 1$ where $k$ is arbitrary large, which implies the existence of a series $G$ of degree 1. On the other hand, the authors noted that it were possible to derive the same conditions by use of the series $G$ of any degree, although by means of more involved computations. This is completely explained by the procedure of root extraction described above, moreover, the assumption on the symmetry of order $k + 1$ becomes redundant.

Regarding the method of derivation of concrete expressions for the conditions like (2) or (4), let us recall that the densities $\rho_j$ for the continuous Lax equations can be computed in two ways: as the residues of the fractional powers $\text{res} G^{j/k}$ (Gelfand, Dikii [18]), or as the coefficients of an expansion with respect to $\lambda$ for the logarithmic derivative of the formal Baker–Akhiezer function (Wilson [19], the idea goes up to the construction of the generating
function for the conservation laws by inversion of the Miura transformation [20]). The equivalence of both definitions was established by Wilson [19] and Flaschka [21]. A detailed description of the method based on the residues in the context of derivation of the necessary integrability conditions can be found, e.g. in [1, 2]. This method requires more involved computations comparing to the method based on the expansion of the formal \( \psi \)-function [22, 23, 6]. Both methods work in the difference setting as well [24], but only under assumption that \( \deg G = 1 \), which is an essential stipulation, as we have seen.

In section 5, the conditions (4) are derived by use of expansion of the formal \( \psi \)-function which is much simpler than computing \( \text{Res} G^j \) and allows us to obtain explicit closed expressions for the densities in terms of the Bell polynomials. Though, the extraction of the list of integrable lattices at \( m = 1 \) requires, according to the Yamilov’s results [7, 10], just three simplest conditions which can be derived without any theory under very modest assumptions about symmetries and conservations laws. This offers hope that in the case, say, \( m = 2 \), the classification requires not too many integrability conditions as well.

2 Extraction of the root in the difference setting

Let \( \mathcal{F} \) be the field of locally analytical functions of finite number of dynamical variables \( u_n, n \in \mathbb{Z} \) and let the rule

\[
T(a(u_i, \ldots, u_j)) = a(u_{i+1}, \ldots, u_{j+1})
\]

define the action of the shift operator \( T \) on functions from \( \mathcal{F} \). The rule \( aT^ibT^j = aT^i(b)T^{i+j} \), being distributed over addition, defines a multiplication of the difference operators. The formal Laurent series with respect to the negative or positive powers of \( T \) constitute the division rings

\[
\mathcal{F}((T^{-1})) = \{ \sum_{j<+\infty} a_j T^j \mid a_j \in \mathcal{F} \}, \quad \mathcal{F}((T)) = \{ \sum_{j>-\infty} a_j T^j \mid a_j \in \mathcal{F} \}.
\]

All statements in this section are given for the series from \( \mathcal{F}((T^{-1})) \), the passage to \( \mathcal{F}((T)) \) amounts to renaming \( u_n \to u_{-n}, T \to T^{-1} \).

Let \( F = f_m T^m + \ldots \in \mathcal{F}((T^{-1})) \) be a given series of degree \( m > 0 \) and \( D_t : \mathcal{F} \to \mathcal{F} \) be a given evolutionary differentiation (that is, commuting with \( T \); this guarantees that \( D_t \) is a differentiation in \( \mathcal{F}((T^{-1})) \) as well). We are interested in solutions of the Lax equation

\[
D_t(G) = [F, G]
\]

(5)
as the series $G \in \mathcal{F}((T^{-1}))$ of degree $k > 0$. In contrast to the continuous situation, the root $G^{1/k}$ is not defined for a generic series $G$ and the study of solutions (or the obstacles for their existence) cannot be reduced to the case $k = 1$. Nevertheless, it turns out that if a solution $G$ exists then its coefficients possess a certain special structure, such that the following properties are fulfilled:

(i) equation (5) admits another solution $H$ of degree $m$, such that $H^k = G^m$. Here, $m$ may be not a minimal positive power of solutions. Thus, in the discrete setting the root extraction is possible in a weakened sense. This property is proved in theorem 2;

(ii) it is possible to choose a solution $G$ among all solutions of degree $m$ such that $G = F + o(T)$ (that is, $\deg(G - F) < 1$). It seems obvious at first sight, because (5) yields the same subset of equations for the partial sum $G_{>0} = g_n T^n + \cdots + g_1 T$ as the equation $[F, G] = 0$. However, the general solution for this subset can contain additional constant parameters comparing to $F_{>0}$, and the further equations may turn solvable only under certain choice of these constants. In principle, it may turn out that the whole set of equations for the coefficients of $G$ is solvable in $\mathcal{F}$ only for such constants that $G_{>0} \neq F_{>0}$. The fact that actually this is not the case is proved in the corollary 3.

Before we go on to the proofs, let us consider several concrete equations for the solution coefficients of the Lax equation, in two simplest examples.

**Example 1** ($m = 1, k = 2$). Let $F = f_1 T + f_0 + \ldots$ and let equation (5) admit a solution $G = g_2 T^2 + g_1 T + \ldots$ then is it possible to find a solution $H$ such that $H^2 = G$? Let us consider several first equations for the coefficients of $G$:

\[
0 = f_1 T(g_2) - T^2(f_1)g_2, \\
D_t(g_2) = f_1 T(g_1) - T(f_1)g_1 + f_0 g_2 - T^2(f_0)g_2, \\
D_t(g_1) = f_1 T(g_0) - f_1 g_0 + f_0 g_1 - T(f_0)g_1 + f_{-1} T^{-1}(g_2) - T^2(f_{-1})g_2. \\
\]

The first equation implies $g_2 = f_1 T(f_1)$ (up to a constant factor), then the second one is brought to the form

\[
(T + 1)(D_t(\log f_1)) = (T - 1)(g_1/f_1) - (T^2 - 1)(f_0)
\]

and this implies that a function $h_0 \in \mathcal{F}$ exists such that

\[
g_1/f_1 = (T + 1)(h_0), \quad D_t(\log f_1) = (T - 1)(h_0 - f_0).
\]
(Here, the properties of the difference operators with constant coefficients are used: \( \ker(T+1) = 0 \), \( \ker(T-1) = \mathbb{C} \). Now, an easy computation brings the third equation to the form

\[
(T + 1)(D_t(h_0)) = (T - 1)(g_0 - h_0^2) - (T^2 - 1)(f_1 T^{-1}(f_1))
\]

which implies that a function \( h_{-1} \in \mathcal{F} \) exists such that

\[
g_0 = h_0^2 + (T + 1)(h_{-1} T^{-1}(f_1)), \quad D_t(h_0) = (T - 1)((h_{-1} - f_{-1}) T^{-1}(f_1)).
\]

Collecting all together we obtain

\[
G = f_1 T(f_1) T^2 + f_1(T + 1)(h_0) T + h_0^2 + (T + 1)(h_{-1} T^{-1}(f_1)) + o(1)
\]

\[
= (f_1 T + h_0 + h_{-1} T^{-1})^2 + o(1)
\]

in support of the conjecture that the root can be extracted indeed.

**Example 2** \((m = 2, k = 2)\). Let \( F = f_2 T^2 + f_1 T + \ldots \) and equation (5) possesses a solution \( G = f_2 T^2 + g_1 T + \ldots \). Then is it possible to choose \( g_1 = f_1 \)? Assume \( g_1 \neq f_1 \), then \( F - G \) is a series of degree 1 and we obtain the following relations by repeating computations from the previous example for the equation \( D_t(G) = [F - G, G] \):

\[
G = (h_1 T + h_0 + h_{-1} T^{-1})^2 + o(1), \quad f_1 - g_1 = ch_1, \quad f_2 = g_2 = h_1 T(h_1).
\]

Therefore, if the relation \( G = H^2 \) is proved then it is possible, indeed, to obtain the solution of the form \( G + cH = f_2 T^2 + f_1 T + \ldots \) as required.

A demerit of the above computations is that the coefficients of the series \( G^{1/2} \) are found implicitly, by inversion of the operators \( T + 1 \) or \( T - 1 \). However, we can obtain them also in an explicit form—if it is known in advance that the desired series does exist. The idea is that in order to extract the root one should use both equations \( D_t(H) = [F, H] \) and \( H^2 = G \) simultaneously. This brings to the recurrent relations of the form

\[
f_1 T(h_j) - T^j(f_1) h_j = D_t(h_{j+1}) + \ldots, \quad f_1 T(h_j) + T^j(f_1) h_j = g_{j+1} + \ldots
\]

where the right hand sides contain the coefficients \( h_1, h_0, \ldots, h_{j+1} \) found on the previous steps. Subtracting yields an explicit expression for \( h_j \). After this one has to verify that each equation is fulfilled indeed. It is not obvious, but plausible, because the equations are consistent, in the sense that they imply the equation \( D_t(G) = [F, G] \) which is true by assumption.

In order to justify these heuristic arguments, in the proof of theorem 2, we will make use of the series with nonautonomous coefficients. Let us
consider an extension of the field \( \mathcal{F} \) given by the ring \( \bar{\mathcal{F}} \) with elements represented by sequences of locally analytical functions 

\[ a(n) = a(n; u_{r_n}, \ldots, u_{s_n}), \quad n \in \mathbb{Z} \]

where each function in the sequence depends on its own finite set of dynamical variables. Elements from \( \mathcal{F} \) are identified with sequences of special type 

\[ a(n) = a(u_{r_n} + j, \ldots, u_{s_n} + j) \]

By definition, multiplication of the sequences is done termwise and the operator \( T \) acts just by the shift of \( n \), that is, \( T^k(a(n)) = a(n + k) \). The multiplication in the ring 

\[ \bar{\mathcal{F}}((T^{-1})) = \{ \sum_{j<+\infty} a_j(n)T^j \mid a_j(n) = a_j(n; u_{r_j,n}, \ldots, u_{s_j,n}) \in \bar{\mathcal{F}} \} \]

is defined, as before, by the rule 

\[ a(n)T^i b(n)T^j = a(n) b(n + i)T^{i+j} \]

One can easily see that the Lax equations are always solvable in such an extension.

**Lemma 1.** Let 

\[ F(n) = f_m(n)T^m + \ldots \in \bar{\mathcal{F}}((T^{-1})), \quad m \geq 1 \quad \text{and} \quad f_m(n) \neq 0 \]

for all \( n \). Then there exists a unique series 

\[ G(n) = g_k(n)T^k + \ldots \in \bar{\mathcal{F}}((T^{-1})), \quad \text{for any degree} \ k, \ \text{which satisfies the equation (5) and the prescribed initial conditions} \ G(0), G(1), \ldots, G(m - 1). \]

**Proof.** The coefficients of \( g_j(n) \) are computed step by step at \( j = k, k-1, \ldots \). The equation for \( g_j(n) \) appears from (5) in the order of \( T^{j+m} \) and it is a recurrent relation of the form 

\[ f_m(n)g_j(n + m) - f_m(n + j)g_j(n) = \ldots \]

where the right hand side contains the members of already found sequences \( g_j(n) \) with \( i > j \). From here, all values \( g_j(n), \ n \in \mathbb{Z} \) are defined uniquely if the values \( g_j(0), \ldots, g_j(m-1) \) are given. \( \square \)

**Theorem 2.** Let series \( G, F \in \mathcal{F}((T^{-1})) \) satisfy the Lax equation 

\[ D_t(G) = [F, G] \]

and \( \deg F = m \geq 1, \deg G = k \geq 1 \). Then a series \( H \in \mathcal{F}((T^{-1})) \) of degree \( m \) exists, unique up to a factor \( 1^{1/k} \), such that 

\[ D_t(H) = [F, H] \quad \text{and} \quad H^k = G^m. \]

**Proof.** The leading coefficient of the series \( H \) satisfies the equation 

\[ f_mT^m(h_m) = T^m(f_m)h_m \]

with the general solution in \( \mathcal{F} \) of the form 

\[ h_m = \text{const} f_m. \]

The constant is determined, up to the root of 1, from comparing the leading terms in the equality 

\[ H^k = G^m. \]
Let us construct the rest coefficients as sequences $h_j(n)$ from the ring $\tilde{\mathcal{F}}$, according to lemma 1. We will prove, by induction on $j$, that there exist unique initial data $H(0), H(1), \ldots, H(m - 1)$ such that the conditions

$$(H^k - G^m)|_{n=0} = \cdots = (H^k - G^m)|_{n=m-1} = 0 \quad (6)$$

are satisfied. Assume that we have already found the coefficients $h_m(n), \ldots, h_{j+1}(n)$ such that the equation $D_t(H) = [F,H]$ is satisfied up to the order of $T^{m+j+1}$, and equations (6) are fulfilled up to the order of $T^{m(k-1)+j+1}$.

Equations for $h_j(n)$ which appear in the next orders can be written down by use of the operators $A(n) = f_m(n)T^m$ and $X(n) = h_j(n)T^j$ as follows:

$$[A(n), X(n)] = a(n)T^{m+j}, \quad n \in \mathbb{Z}, \quad (7)$$

$$A(n)^{k-1}X(n) + A(n)^{k-2}X(n)A(n) + \cdots + X(n)A(n)^{k-1} = b(n)T^{(k-1)m+j}, \quad n = 0, \ldots, m - 1 \quad (8)$$

where $a(n), b(n)$ are certain polynomials with respect to the coefficients of the series $F,G$ and coefficients $h_m, \ldots, h_{j+1}$ already found (differentiated with respect to $t$ among them). One can easily see that equation (8) is reduced by use of (7) to equations of the form $ckX(n)A(n)^{k-1} = c(n)T^{(k-1)m+j}$, that is,

$$kh_j(n)f_m(n+j) \cdots f_m(n+j+(k-2)m) = c(n), \quad n = 0, \ldots, m - 1.$$

This uniquely defines the initial data for the sequence $h_j(n)$ and completes the induction step.

The series $H^k - G^m$ is, for the constructed solution, a solution of the Lax equation with vanishing initial data and according to the lemma 1 it is identically zero. Therefore, equations (8) are fulfilled for all $n \in \mathbb{Z}$, not only for $n$ stated above.

Next, let us notice that the polynomials $a(n), b(n)$ and, therefore, $c(n)$, are of the same form for all $n$, as functions of the coefficients of the series $F,G,H$. This means that if $h_j(n) = p[F,G,H_{>j}]$, where $p$ is a function of a finite number of variables, then $h_j(n+1) = p[T(F), T(G), T(H_{>j})]$. Since all $f_i, g_i \in \mathcal{F}$ and $h_m = f_m \in \mathcal{F}$, hence we prove, again by induction, that all $h_j \in \mathcal{F}$. $\square$

Remark 1. It is clear from the proof that a computation of the first $r$ coefficients of $H$ requires exactly $r$ coefficients of $G$ (not taking the coefficients of $F$ into account). Indeed, the series $G$ is used only in the initial conditions (6) where the number of coefficients of both series coincides in each order of $T$.  

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It is easy to demonstrate that if the Lax equation (5) possesses at least one nontrivial solution (that is, different from $cT^0$) then its general solution is represented as a series with constant coefficients

$$G = \sum_{j<+\infty} c_j H^j$$

where $H$ is a solution of minimal positive degree $d$. This implies, in particular, that any two solutions commute and it follows from the theorem that $d$ is a divisor of $m$.

**Corollary 3.** If an equation $D_t(G) = [F, G]$ admits a nontrivial solution in $\mathcal{F}((T^{-1}))$ then it admits, as well, a solution of the form

$$G = f_m T^m + \cdots + f_1 T + g_0 + g_{-1} T^{-1} + \cdots \in \mathcal{F}((T^{-1})).$$

**Proof.** Let $G$ be a solution of degree $m$ which exists according to theorem 2. One can assume that the leading terms of $G$ and $F$ coincide, without loss of generality. Let $\text{deg}(F - G) = l$. If $l \leq 0$ then the statement is true, let us consider the case $1 \leq l < m$.

Application of theorem 2 to equation $D_t(G) = [F - G, H]$ proves that there exists a series $H \in \mathcal{F}((T^{-1}))$ of degree $l$ which satisfies equations $D_t(H) = [F - G, H]$. The series $H = h_1 T^1 + \cdots$ commutes with $G$ and $f_1 - g_1 = ch_1$. Therefore, the series $G' = G + cH$ satisfies the original equation and $\text{deg}(F - G') < l$. Repeating this arguments several times, if necessary, we come to a solution which coincides with $F$ up to the term $f_1 T$ inclusively. \hfill \Box

Notice, that the Lax equation may admit, in first $m$ orders of $T$, a solution $G_{>0}$ which contains additional parameters comparing with $F_{>0}$, even if $m$ is the minimal degree of the true solution $G$. In such a case, these parameters vanish automatically in the process of solving equations for the rest coefficients of $G$.

**Example 3.** Let us consider the lattice equation

$$u_t = f[u] = u_1 u_2 u_{-1} (u_2 - u_{-2})$$

related via the substitution $u_1 u = v$ with the modified Volterra model on the stretched lattice $v_t = v^2 (v_2 - v_{-2})$. This substitution acts on the higher symmetries as well and this guarantees (see next section) the solvability of the Lax equation with

$$F = f_* = u_1 u_2 u_{-1} T^2 + u^2 u_{-1} (u_2 - u_{-2}) T + \cdots.$$
It is easy to check that the Lax equation admits, in first two orders, solutions of degree 2 and 1:

\[ G_{>0} = F_{>0} + cuu_{-1}T, \quad H_{>-1} = uu_{-1}T + u_{1}u_{-1} - uu_{-2}, \]

moreover, \( G_{>0} = (H^2 + cH)_{>0} \). However, the equation for the third coefficient of \( H \) does not admit a solution in \( F \). For the solution \( G \), this means that the constant \( c \) must be set equal to zero when computing the fourth coefficient.

### 3 Formal symmetry

Let us recall basic notions of the symmetry approach, in the context of scalar evolutionary lattice equations

\[ \partial_t(u_n) = T^m(f(u_m, \ldots, u_{\bar{m}})). \]  

(9)

For the sake of definiteness, we will write arguments of functions in decreasing order of subscripts, moreover, we will assume (applying the reflection \( u_n \to u_{-n} \) if needed) that

\[
\begin{align*}
  f^{(m)} &\neq 0, & f^{(\bar{m})} &\neq 0, & m &\geq 1, & m &\geq \bar{m}
\end{align*}
\]

where \( f^{(j)} = \partial_j(f), \partial_j = \partial/\partial u_j \). The numbers \( m \) and \( \bar{m} \) are called order and lower order of the lattice equation. Any function \( a \in F \) gives rise to the evolutionary derivative \( \nabla_a \) and the linearization operator \( a_* \):

\[
\nabla_a = \sum_{j \in \mathbb{Z}} T^j(a) \partial_j, \quad a_* = \sum_{j \in \mathbb{Z}} a^{(j)} T^j \in F[T, T^{-1}].
\]

We will use also the notation \( D_t = \nabla_f \) for the differentiation in virtue of equation (9). The following identities are easy to prove:

\[
[\nabla_a, T] = 0, \quad (T(a))_* = Ta_*, \quad \nabla_a(b) = b_*(a), \quad (\nabla_a(b))_* = \nabla_a(b_*) + b_*a_*.
\]

The lattice equation

\[ \partial_r(u_n) = T^m(g(u_k, \ldots, u_k)) \]  

(10)

is called symmetry of equation (9) if the condition

\[
\nabla_f(g) = \nabla_g(f)
\]
holds identically with respect to \( u_j \). It means that the flows \( \partial_t, \partial_\tau \) commute (guaranteeing the existence of a common generic solution \( u_n(t, \tau) \)). The lattice equation is considered integrable if it admits symmetries of order arbitrarily large. The linearization of the latter equation brings it to the operator form

\[
\nabla f(g_*) = \nabla g(f_*) + [f_*, g_*] \tag{11}
\]

which is more convenient for the analysis. Neglecting of the term \( \nabla g(f_*) \) which is of a fixed degree in \( T \) brings to equation

\[
D_t(G) = [f_*, G]. \tag{12}
\]

Its solutions are called formal symmetries of the lattice equation \((9)\).

**Theorem 4.** If the lattice equation \((9)\) admits symmetries \((10)\) with \( k \) arbitrarily large then equation \((12)\) admits a solution \( G \in \mathcal{F}((T^{-1})) \) of the form

\[
G = f^{(m)}T^m + \cdots + f^{(1)}T + g_0 + g_{-1}T^{-1} + \cdots. \tag{13}
\]

If the lattice equation \((9)\) with \( \bar{m} < 0 \) admits symmetries \((10)\) with \(-\bar{k} \) arbitrarily large then equation \((12)\) admits a solution \( \bar{G} \in \mathcal{F}((T)) \) of the form

\[
\bar{G} = f^{(\bar{m})}T^{\bar{m}} + \cdots + f^{(-1)}T^{-1} + \bar{g}_0 + \bar{g}_1T + \cdots. \tag{14}
\]

**Proof.** It is sufficient to prove the first statement, taking the change \( u_n \to u_{n-n}, T \to T^{-1} \) into account. The series \( g_* = g^{(k)}T^k + \cdots + g^{(1)}T + o(T) \in \mathcal{F}((T^{-1})) \) satisfies equation \((5)\) in the orders \( T^{k+m}, \ldots, T^{m+1} \). The procedure of the root extraction described in theorem 2 yields, taking remark 1 into account, a series \( G \in \mathcal{F}((T^{-1})) \) such that

\[
D_t(G) = [f_*, G] + o(T^{2m-k+1}), \quad \deg G = m. \tag{15}
\]

Moreover, one can choose \( G_{>0} = (f_*)_{>0} \) (if \( k > m \)) according to corollary 3. Since \( k \) is arbitrarily large, hence equation \((5)\) is solvable in all orders of \( T \).

The conditions of solvability of equation \((12)\) with respect to the coefficients of the series \( G \) or \( \bar{G} \) serve as the necessary integrability conditions. Equation \((15)\) demonstrates that existence of a symmetry of order \( k \geq m + r \) implies that \( r \) conditions are fulfilled, for the coefficients \( g_0, \ldots, g_{-r+1} \). A symmetry with \( k \leq m \) gives no conditions because it is ‘lost’ on the background of the trivial symmetry with \( g = f \) which corresponds to the operator \( G = (f_*)_{>0} \). Analogously, if there exist symmetries with lower order conditions.
$-\bar{k} = -\bar{m} + r$ then the solvability conditions are fulfilled for the coefficients $\bar{g}_0, \ldots, \bar{g}_{r-1}$ of the series $\bar{G}$. Unfortunately, we do not know how many conditions must be checked in order to guarantee the existence of at least one symmetry, even of small order. Nevertheless, these conditions are rather convenient both for testing and classification purposes, because we write them immediately in terms of the right hand side of the equation and their form does not depend on the actual orders of symmetries which are not known in advance.

The first condition and a corollary of the $m$-th one are especially simple. Let us make use of the following simple property:

$$\text{Res}[A, B] \in \text{Im}(T^{-1}), \quad \text{Res} A := \text{coef}_{T^0} A, \quad A, B \in \mathcal{F}((T^{-1}))$$

(Indeed, $[aT^j, bT^{-j}] = (T^j - 1)(T^{-j}(a)b))$.

**Statement 5.** If the lattice equation (9) admits a symmetry (10) of order $k > 2m$ then there exist functions $\bar{\sigma}, \bar{\sigma}_1 \in \mathcal{F}$ such that

$$D_t(\log f^{(m)}) = (T^m - 1)(\bar{\sigma}), \quad D_t(f^{(0)} + \bar{\sigma}) = (T - 1)(\bar{\sigma}_1). \quad (16)$$

If (9) admits a symmetry (10) with $\bar{k} \leq 2\bar{m} < 0$ then there exist functions $\bar{\sigma}, \bar{\sigma}_1 \in \mathcal{F}$ such that

$$D_t(\log f^{(m)}) = (T^{\bar{m}} - 1)(\bar{\sigma}), \quad D_t(f^{(0)} + \bar{\sigma}) = (T - 1)(\bar{\sigma}_1). \quad (17)$$

**Proof.** According to theorem 4, equation (12) is solvable in the orders of $T^{2m}, \ldots, T^0$, moreover, we can assume $G_{>0} = (f_s)_{>0}$. Application of Res to this equation and its equivalent form

$$D_t(G)G^{-1} = f_s - G - G(f_s - G)G^{-1}$$

yields, respectively,

$$D_t(g_0) \in \text{Im}(T - 1), \quad D_t(f^{(m)})/f^{(m)} = (T^m - 1)(g_0 - f^{(0)})$$

which is equivalent to (16); equations (17) are obtained in a similar way.

In general, equation (12) in each order of $T$ is of the form

$$A_j(g_j) = b_j, \quad A_j = T^m - \frac{T^j(a)}{a}, \quad a = f^{(m)}, \quad j = 0, -1, -2, \ldots$$

where $b_j$ is a known expression which contains the coefficients of $f_s$ and $g_0, \ldots, g_{j+1}$. Thus, the integrability test for a given lattice equation amounts
to a step-by-step checking of whether \( b_j \in \text{Im} A_j \); if not then the equation is not integrable, if yes then we have to compute \( g_j \) and to go to the next condition. Notice, that equation \( A_j + m(g) = b \) is reduced to \( A_j(\tilde{g}) = \tilde{b} \) under the change \( g = T^j(a)\tilde{g} \), so that the following problems appear: to characterize the image and to compute the pre-image of the operators of the form

\[
T^m - 1, \quad T^m - \frac{T(a)}{a}, \ldots, \quad T^m - \frac{T^{m-1}(a)}{a}
\]

with a given function \( a \). The solution is well known at \( m = 1 \):

\[
\text{Im}(T - 1) = \ker E, \quad E = \frac{\delta}{\delta u} = \sum_{j \in \mathbb{Z}} T^{-j} \partial_j
\]

where \( E \) is called the Euler operator or the variational derivative, while the pre-image of \( T - 1 \) can be found by a difference version of the integration by parts algorithm or by use of the homotopy operator, see e.g. [24, 25]. At \( m > 1 \), the problem admits a constructive solution as well, although it is more complicated (in particular, the answer depends on whether \( \log a \) belongs to the image of the operator \( T^{m-d} + \cdots + T^d + 1 \) where \( d|m \)).

### 4 Formal conservation law

The symmetric case \( \bar{m} = -m \) is of most interest, because only such type of lattice equations may admit higher order conservation laws. Let us recall that a function \( \rho \in \mathcal{F} \) is called a density of conservation law for the lattice equation (9) if there exists a function \( \sigma \in \mathcal{F} \) such that

\[
\nabla f(\rho) = (T - 1)(\sigma).
\]

A density is called trivial if \( \rho \in \text{Im}(T - 1) \) and two densities are called equivalent if their difference is trivial. In order to factor out the trivial conservation laws, let us apply the Euler operator to (18), this yields the equation for \( r = E(\rho) = \rho^\dagger \):

\[
\nabla f(r) + f^\dagger r + r^\dagger f = 0
\]

where \( \dagger : \mathcal{F}(T^{\pm 1}) \rightarrow \mathcal{F}(T^{\mp 1}) \) denotes the conjugation \( (aT^j)^\dagger = T^{-j}a \). Application of linearization once again yields the operator equation

\[
\nabla f(r^\ast) + f^\dagger r^\ast + r^\ast f = \sum_{m \geq 1, j \geq \bar{m}} T^{-j}(r f^{(i,j)}) T^{i-j} = 0.
\]
Notice, that the operator $r_*$ is symmetric, $r_*=r^\dagger_*$. In particular, $r$ depends on a symmetric set of variables: $r=r(u_k,\ldots,u_{-k})$, $r^{(\pm k)}\neq 0$. The number $k$ is called the order of the conservation law. It is easy to see that the degrees with respect to $T$ of the four terms in equation (19) are equal to $k$, $k-\tilde{m}$, $k+m$ and $M\leq m-\tilde{m}$, respectively. This implies that equation (9) with $\tilde{m}\neq -m$ can not possess conservation laws of order $k>\min(m,-\tilde{m})$. So, we assume that $\tilde{m}=m$ in what follows, that is, the lattice equation is of the form

$$\partial_t(u_n)=T^n(f(u_m,\ldots,u_{-m})), \quad f^{(\pm m)}\neq 0.$$ (20)

The last sum in (19) is of a fixed degree with respect to $T$. Neglecting it brings to equation

$$D_t(R)+f^\dagger R+Rf_*=0$$ (21)

and its solution in the form of a series from $\mathcal{F}((T^{-1}))$ or $\mathcal{F}((T))$ is called a formal conservation law for the lattice equation (20). Equation (21) is invariant with respect to the conjugation, so we may restrict ourselves with consideration of series from $\mathcal{F}((T^{-1}))$.

**Theorem 6.** Let the lattice equation (20) admits conservation laws of order $k$ arbitrarily large. Then equation (12) admits solutions of the form

$$G=f^{(m)}T^m+\cdots+f^{(1)}T+g_0+g_{-1}T^{-1}+\cdots \in \mathcal{F}((T^{-1})), \quad \tilde{G}=f^{(-m)}T^{-m}+\cdots+f^{(-1)}T^{-1}+\tilde{g}_0+\tilde{g}_1T+\cdots \in \mathcal{F}((T))$$ (22)

and equation (21) admits a solution of the form

$$R=r_lT^l+r_{l-1}T^{l-1}+\cdots \in \mathcal{F}((T^{-1})), \quad 0\leq l<m$$ (24)

such that

$$\tilde{G}^\dagger R=-RG.$$ (25)

**Proof.** Let us consider operators $r'_l$, $r_*$ corresponding to the conservation laws of orders $k'>k>m$ as series from $\mathcal{F}((T^{-1}))$. It follows from (19) that these series satisfy equation (21) in the first $k-m$ orders of $T$. The extraction of the root from the series $r_*^{-1}r'_l$ of degree $k'-k>0$ brings to a series of degree $m$ which also satisfies equation (12) in first $k-m$ orders, that is up to the terms $o(T^{3m-k+1})$. Since $k$ is arbitrarily large, hence equation (12) is solvable for all orders of $T$ and we come to a solution of the form (22), taking the corollary 3 into account.

Conservation laws of orders $k=qm+l$, $0\leq l<m$ constitute an infinite set at least for one value of $l$. For the corresponding series $r_*$, the series
The constructed series $G, R \in \mathcal{F}((T^{-1}))$ allow to obtain the series $\bar{G} = -(RGR^{-1})^\dagger \in \mathcal{F}((T))$ which satisfies equation (12). Moreover,

$$(\bar{G}^\dagger)_{>0} = -RG_{>0}R^{-1} = -R(f^*_s)_{>0}R^{-1} = (f^*_s)_{>0}$$

according to (21), therefore $\bar{G} < 0 = (f^*_s) < 0$. \qed

It is clear from comparing with theorem 4 (at $\bar{m} = -m$) that assumption about existence of an infinite set of conservation laws brings to more restrictive integrability conditions than assumption about the higher symmetries:

| conservation laws | higher symmetries |
|------------------|------------------|
| ↓                | ↓                |
| $G, R$           | $G, G$           |

A weak point of the conditions which follow from the equation for $R$ is that the degree $l = \deg R$ is not known in advance, so we have to inspect the values $l = 0, \ldots, m - 1$. In particular, we come to the following statement instead of 5.

**Statement 7.** If the lattice equation (20) admits two conservation laws of orders $k' > k > 3m$ then there exist functions $\sigma, \sigma_1, s, s_1 \in \mathcal{F}$ and an integer $l, 0 \leq l < m$ such that

$$D_t(\log f^{(m)}) = (T^m - 1)(\sigma), \quad D_t(f^{(0)} + \sigma) = (T - 1)(\sigma_1), \quad (26)$$

$$\log(-T^l(f^{(m)}/f^{(-m)})) = (T^m - 1)(s), \quad D_t(s) + 2f^{(0)} = (T - 1)(s_1). \quad (27)$$

**Proof.** In the notations of theorem 6, the series $G, \bar{G}, R$ constructed from $r^*_s, r_s$ satisfy equations (12), (21) in first $2m + 1$ orders. Equations (26) are proved like in statement 5. First equation (27) follows from (21) in the leading order, with the function $s = \log(f^{(-m)}/r_l)$. Multiplication of (21) by $R^{-1}$ and applying Res results in $D_t(\log r_l) + 2f_0 \in \text{Im}(T - 1)$ which is equivalent to the second equation (27). \qed

5 The lattices of order 1

The integrability conditions simplify drastically for the first order lattice equations

$$\partial_t(u_n) = T^n(f(u_1, u, u_{-1})), \quad f^{(\pm 1)} \neq 0. \quad (28)$$
In this case, the Lax equation for $G = f^{(1)}T + g_0 + g_{-1}T^{-1} + \ldots$ turns out to be equivalent to a sequence of conservation laws (possibly trivial) defined by certain recurrent relations. In order to write them down we will use the polynomials

$$P_0 = 1, \quad P_1(x_1) = x_1, \quad P_2(x_1, x_2) = x_2 + \frac{x_1^2}{2},$$

$$P_3(x_1, x_2, x_3) = x_3 + x_1 x_2 + \frac{x_1^3}{6} \ldots$$

defined by the generating function

$$P_0[x] + P_1[x] \lambda + P_2[x] \lambda^2 + \ldots = \exp(x_1 \lambda + x_2 \lambda^2 + x_3 \lambda^3 + \ldots).$$

These polynomials are well known in the representation theory of infinite-dimensional Lie algebras (see e.g. [26, 27]) and are related to the complete exponential Bell polynomials $Y_k$ by the change $k!P_k(x_1, \ldots, x_k) = Y_k(x_1, \ldots, k!x_k)$.

In order to rewrite the equation $D_t(G) = [f_*, G]$ in the form of conservation laws, we make use of the fact that it serves as the compatibility condition for equations

$$G(\psi) = \lambda \psi, \quad D_t(\psi) = f_*(\psi).$$

Let us consider expansions with respect to $\lambda$ of the ratios

$$p = T(\psi)/\psi, \quad q = D_t(\psi)/\psi,$$

this brings to equations

$$f^{(1)}p + g_0 - \lambda + \frac{g_{-1}}{T^{-1}(p)} + \frac{g_{-2}}{T^{-1}(p)T^{-2}(p)} + \ldots = 0,$$

$$q = f^{(1)}p + f^{(0)} + \frac{f^{(-1)}}{T^{-1}(p)},$$

$$D_t(p)/p = (T - 1)(q).$$

It is easy to see that equation (31) defines an invertible change between the coefficients of the series $G$ and $p = p_{-1}\lambda + p_0 + p_1\lambda^{-1} + \ldots$:

$$p_{-1} = \frac{1}{f^{(1)}}, \quad p_0 = -\frac{g_0}{f^{(1)}}, \quad p_1 = -\frac{g_{-1}T^{-1}(f^{(1)})}{f^{(1)}}, \quad \ldots,$$

so that $G \in \mathcal{F}((T^{-1}))$ if and only if $p \in \mathcal{F}((\lambda^{-1}))$. Moreover, (32) implies that $q = \lambda - \sigma_0 - \sigma_1\lambda^{-1} - \ldots \in \mathcal{F}((\lambda^{-1}))$ and a solution of equations (29) (a
formal Baker–Akhiezer function) is constructed by integration of equations (30) as a series of the form

$$\psi(n) = a(n)\lambda^n(1 + a_1(n)\lambda^{-1} + a_2(n)\lambda^{-2} + \ldots)$$

with coefficients in a certain extension of the field $\mathcal{F}$. One more change

$$p = \frac{\lambda}{f(1)} \exp(-\rho_1\lambda^{-1} - \rho_2\lambda^{-2} - \ldots)$$

brings equations (32) and (33) to relations

$$D_t(\rho_j) = (T - 1)(\sigma_j), \quad j \geq 0,$$

$$\rho_0 = \log f(1), \quad \rho_1 = f(0) + \sigma_0,$$

$$P_{j+1}(\rho_0) + f^{(-1)}T^{-1}(f^{(1)}P_{j-1}(\rho)) + \sigma_j = 0, \quad j > 0.$$  (37)

The existence of a formal symmetry $G \in \mathcal{F}((T^{-1}))$ is equivalent to solvability of equations (35) with respect to $\sigma_j \in \mathcal{F}$, moreover, the densities $\rho_{j+1}$ are explicitly found from (37):

$$\rho_2 = f^{(-1)}T^{-1}(f_1) + \frac{1}{2}\rho_1^2 + \sigma_1,$$

$$\rho_3 = f^{(-1)}T^{-1}(f_1\rho_1) + \rho_1\rho_2 - \frac{1}{6}\rho_1^3 + \sigma_2,$$

$$\rho_4 = f^{(-1)}T^{-1}(f_1(\rho_2 + \frac{1}{2}\rho_1^2)) + \rho_1\rho_3 + \frac{1}{2}\rho_2^2 - \frac{1}{2}\rho_1^2\rho_2 + \frac{1}{24}\rho_1^4 + \sigma_3, \ldots$$

In order to write analogously the consequences from existence of the second formal symmetry $\bar{G} \in \mathcal{F}((T))$, we consider a function $\bar{\psi}$ which satisfies equations

$$\bar{G}^t(\bar{\psi}) = -\lambda \bar{\psi}, \quad D_t(\bar{\psi}) = -f^{(1)}(\bar{\psi}).$$

It is easy to check, as before, that the ratios

$$\bar{p} = T(\bar{\psi})/\bar{\psi}, \quad \bar{q} = D_t(\bar{\psi})/\bar{\psi}$$

can be expanded into series with respect to $\lambda$, of the form

$$\bar{p} = -\frac{\lambda}{T(f^{(-1)})} \exp(-\bar{p}_1\lambda^{-1} - \bar{p}_2\lambda^{-2} - \ldots), \quad \bar{q} = \lambda - \bar{\sigma}_0 - \bar{\sigma}_1\lambda^{-1} - \ldots$$

with coefficients $\bar{p}_j, \bar{\sigma}_j \in \mathcal{F}$, and that these ratios satisfy equations

$$\bar{q} = -T(f^{(-1)}\bar{p}) - f^{(0)} - \frac{T^{-1}(f^{(1)})}{T^{-1}(\bar{p})}, \quad D_t(\bar{p})/\bar{p} = (T - 1)(\bar{q}).$$

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This implies the same recurrent relations (35), (37) for functions \( \bar{\rho}_j, \bar{\sigma}_j \) as for \( \rho_j, \sigma_j \), but with the initial data

\[
\bar{\rho}_0 = \log(-T(f^{(-1)})), \quad \bar{\rho}_1 = -f^{(0)} + \sigma_0
\]  

(38) instead of (36). If lattice equation (28) admits, in addition, a formal conservation law \( R = r_0 + r_{-1}T^{-1} + \ldots \in \mathcal{F}((T^{-1})) \) then it follows from equations (21), (25) that one can take \( \bar{\psi} = R(\psi) \). Then the series

\[
s = \log(\bar{\psi}/\psi) = s_0 + s_1 \lambda^{-1} + \ldots \in \mathcal{F}((\lambda^{-1}))
\]

satisfies the equations

\[
\log \bar{p} - \log p = (T - 1)(s), \quad D_t(s) = \bar{q} - q
\]

that is, both density sequences are related by equations

\[
\rho_j - \bar{\rho}_j = (T - 1)(s_j), \quad j \geq 0.
\]  

(39)

Here, we may exclude the functions \( \bar{\sigma}_j = \sigma_j - D_t(s_j) \) from consideration, because \( \bar{\rho}_j \) can be found from the recurrent relations

\[
\bar{\rho}_0 = \log(-T(f^{(-1)})), \quad \rho_1 - \bar{\rho}_1 = 2f^{(0)} + D_t(s_0), \\
P_{j+1}[-\bar{\rho}] - P_{j+1}[-\bar{\rho}] + f^{(-1)}T^{-1}(f^{(1)}(P_{j-1}[\bar{\rho}] - P_{j-1}[\rho])) = D_t(s_j), \quad j > 0.
\]

It was already mentioned in the Introduction that the classification of integrable lattice equations (28) requires, according to [7], only three simplest conditions (1 of the type (35) and 2 of the type (39), cf. also with statement 7) which can be cast into the form

\[
D_t(\log f^{(1)}) \in \text{Im}(T - 1),
\]

\[
\log(-f^{(1)}/f^{(-1)}) = (T - 1)(s), \quad D_t(s) + 2f^{(0)} \in \text{Im}(T - 1).
\]

These conditions can be derived under assumptions that the equation admits a symmetry of order \( k \geq 2 \) and a conservation law of order \( k' \geq 3 \), or that it admits a pair of conservation laws of orders \( k' > k \geq 3 \). The analysis of these conditions is a rather tedious task which brings to a finite list of equations. One can prove by inspection that all of them admit an infinite set of higher symmetries (and conservation laws, except for several degenerate cases like the linear equation \( \partial_t(u_n) = u_{n+1} - u_{n-1} \) which does not admit conservation laws of order \( > 0 \), but equation (21) admits the solution \( R = 1 \)).
For the sake of completeness, let us prove the statement on the equivalence of the constructed conservation laws (35) with the standard definition through the residues of the powers of formal symmetry. The proof follows, essentially, to Kupershmidt [24, ch. IX.3] and it is an adaptation for the discrete case of the proof by Flaschka [21].

**Statement 8.** Let the lattice equation (28) admits the formal symmetry 
\[ G = f^{(1)} T + \ldots \in \mathcal{F}((T^{-1})) \] and the quantities \( \rho_j \) are defined by relations (31), (34), then

\[ \text{Res} G^j - j \rho_j \in \text{Im}(T - 1), \quad j = 1, 2, \ldots . \]  

**Proof.** Equations (29), (30) for the \( \psi \)-function imply

\[ T(\psi) = p \psi = \sum p_j \lambda^{-j} \psi = \sum p_j G^{-j}(\psi) \]

from where the identity

\[ T = p_{-1} G + p_0 + p_1 G^{-1} + p_2 G^{-2} + \ldots \]  

follows which is equivalent to relation (31) between the series \( G \) and \( p \). Let us introduce the notations

\[ G_j = (G^j)_{\geq 0}, \quad A_j = (G^j)_{< 0}, \quad \epsilon_j = \text{coef}_{T^{-1}} G^j \]

(in particular, \( \epsilon_{-1} = 1/T^{-1}(f^{(1)}), \epsilon_0 = 0 \)). Right multiplication of (41) by \( G^j \) and neglecting of the negative powers of \( T \) result in

\[ T G_j + T(\epsilon_j) = p_{-1} G_{j+1} + p_0 G_j + \cdots + p_j G_0, \quad j = -1, 0, 1, \ldots . \]

which is equivalent to equation (a difference version of the Cherednik formula [29])

\[ (T - p) G = -T(E) \]  

for the generating series

\[ G = 1 + \lambda^{-1} G_1 + \lambda^{-2} G_2 + \ldots , \quad E = \epsilon_{-1} \lambda + \epsilon_0 + \epsilon_1 \lambda^{-1} + \ldots . \]

Application of Res yields

\[ \text{Res } G = 1 + \lambda^{-1} \text{Res } G + \lambda^{-2} \text{Res } G^2 + \ldots = \frac{T(E)}{p} . \]

Next, let us denote

\[ \alpha_j = A_j(\psi)/\psi = \frac{\epsilon_j}{T^{-1}(p)} + \frac{a_{j-2}}{T^{-1}(p)T^{-2}(p)} + \ldots = \epsilon_j T^{-1}(f^{(1)}) \lambda^{-1} + \ldots . \]
\[ G_j(\psi)/\psi = \lambda^j - \alpha_j, \quad TG_j(\psi)/\psi = p(\lambda^j - T(\alpha_j)). \]

Let us apply the identity (42) to \( \psi(\mu) \) and divide the result by \( \psi(\mu) \), this yields

\[
-T(E(\lambda)) = (T - p(\lambda)G(\lambda)(\psi(\mu)))/\psi(\mu)
= \sum_{j \geq 0} \lambda^{-j} \left( p(\mu)(\mu^j - T(\alpha_j(\mu))) - p(\lambda)(\mu^j - \alpha_j(\mu)) \right)
= \frac{\lambda(p(\mu) - p(\lambda))}{\lambda - \mu} + \sum_{j \geq 0} \lambda^{-j} \left( p(\lambda)\alpha_j(\mu) - p(\mu)T(\alpha_j(\mu)) \right).
\]

Division by \(-p(\lambda)\) and passage to the limit \( \mu \to \lambda \) brings to equation

\[
\frac{T(E)}{p} = \lambda \partial_\lambda (\log p) + \sum_{j \geq 0} \lambda^{-j}(T - 1)(\alpha_j)
\]
and the statement follows from comparison with (34) and (43).

In conclusion, notice that an analog of the substitution \( p = T(\psi)/\psi \) can be defined for the lattice equations of order \( m \) as well: one can prove that an equation \( G(\psi) = \lambda \psi \) with the series \( G \in \mathcal{F}((T^{-1})) \) of order \( m \) is equivalent to equation

\[
(T^m - p_{m-1}T^{m-1} - \cdots - p_0)(\psi) = 0, \quad p_j \in \mathcal{F}((\lambda^{-1})).
\]

In the matrix notations, equations (29) are replaced by

\[
T(\Psi) = P\Psi, \quad D_t(\Psi) = Q\Psi
\]
where

\[
\Psi = \begin{pmatrix} \psi \\ \vdots \\ T^{m-1}(\psi) \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} p_0 & \cdots & p_{m-1} \end{pmatrix}
\]
and elements of the matrix \( Q \) are expressed through \( f(j) \) and \( p_j \). Equation (33) is replaced with

\[
D_t(P) = T(Q)P - PQ
\]
which implies, in particular, that \( \log p_0 \) serves as a generating series for the densities of conservation laws. However, in the context of the integrability conditions this equation leads to rather cumbersome relations, and, apparently, it gives no advantages comparing with the straightforward solving of the Lax equation for the formal symmetry.
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