The Geometry of Momentum

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To Jerry Marsden for his 60th birthday

Abstract

Although the idea of the momentum map associated with a symplectic action of a group is already contained in work of Lie, the geometry of momentum maps was not studied extensively until the 1960’s. Centering around the relation between symmetries and conserved quantities, the study of momentum maps was very much alive at the end of the 20th century and continues to this day, with the creation of new notions of symmetry. A uniform framework for all these momentum maps is still to be found; groupoids should play an important role in such a framework.

1 Introduction

The term momentum (in Italian, momento) was introduced by Galileo Galilei as the “virtue” of a moving object which keeps it moving. He expressed it as the product of weight and velocity. Galileo’s notion of momentum was an outgrowth of the medieval impetus theory of William of Occam, Jean Buridan, Nicole Oresme, and others (see, for example, Dugas [4]), which deviated from Aristotle’s view that something external to a moving (terrestrial) body was necessary to keep it in motion. (Opposition to Aristotle’s

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view had also been expressed as early as the 5th century by John (Philopon) of Alexandria.)

By now, “momentum” has a much wider meaning. Many applications of the term refer to quantities whose conservation under the time evolution of a physical system is related to some symmetry of the system. The theme of this paper (and the corresponding talk at the conference) is this relation between momentum and symmetry, especially as the meaning of the later was extended in the late 20th century. Though much of this extension was motivated by quantum mechanics, the emphasis here will be on momentum in classical mechanics. Another very important aspect of our subject which we will not treat in this paper is *symplectic reduction*, which is the simultaneous use of symmetries and conserved quantities to reduce the dimensionality of a hamiltonian system. We refer to [17] for an extensive history of reduction.

2 Momentum in the calculus of variations

Despite the original conception of momentum as proportional to velocity, we now see momentum for the most part as dual to velocity. This idea appears clearly in the calculus of variations where, given a lagrangian \( L(t, x, \dot{x}) \) for a function \( x = (x^1, ..., x^n) \) of \( t \), we define the conjugate momenta by \( p_i = \frac{\partial L}{\partial \dot{x}^i} \) and then the hamiltonian by \( H = \sum p_i \dot{x}^i - L \). One sees from the way in which the conjugate momenta transform under coordinate changes (which results in the invariance of \( H \) under such changes) that the conjugate momenta are the components of a vector \( \dot{p} \) which is dual to the velocity vector \( \dot{x} \).

Of course, if \( L = \frac{1}{2} m \sum_i (\dot{x}^i)^2 - V(x) \), then \( p_i = m \dot{x}^i \). The conflict between lower and upper indices here results from the use of orthonormal coordinates. More invariantly, we should write the first (kinetic energy) part of \( L \) as

\[
\frac{1}{2} \sum_{i,j} m_{ij} \dot{x}^i \dot{x}^j, \tag{1}
\]

where \( m_{ij} = m \delta_{ij} \). The “mass tensor” \( m_{ij} \) may then replaced in (1) by an arbitrary riemannian metric \( m_{ij}(x) \) to describe in curvilinear coordinates what Smale [21] calls a *simple mechanical system*. For such a system, the relation between momentum and velocity becomes

\[
p_i = \sum_j m_{ij} \dot{x}^j, \tag{2}
\]
the mass tensor now being interpreted as a mapping from velocity (tangent) to momentum (cotangent) vectors.

If we add the vector potential of a magnetic field to obtain

\[ L = \frac{1}{2} \sum_{i,j} m_{ij} \dot{x}^i \dot{x}^j + e \sum_i A_i(x) \dot{x}^i + V(x), \]

then the momentum and velocity are no longer proportional; instead,

\[ p_i = \sum_j m_{ij} \dot{x}^j + A_i(x). \]

To see why (3) and not the simpler relation (2) is the appropriate definition of momentum here, let us consider the example in 3-dimensional configuration space where \( m_{ij} = m \delta_{ij} \), the vector potential is \( e x^1 \dot{x}^3 \), and \( V = 0 \). This lagrangian is invariant under translations of configuration space in the direction of the \( x^3 \)-axis; if we look for conserved quantities, we find that \( p_3 = m \dot{x}^3 + ex^1 \) is a constant of motion, while the single term \( m \dot{x}^3 \) is not. (The motions are circular helices with axis in the \( x^2 \) direction.)

Investigating this example further, one finds that \( p_2 = m \dot{x}^2 \) is also a conserved quantity, while \( p_1 = m \dot{x}^1 \) is not. In fact, translations along the \( x^1 \) axis are not symmetries of the lagrangian; however, one may check directly from the equations of motion that there is a third conserved quantity, namely \( m \dot{x}^1 - e x^3 \), found most easily from the hamiltonian formalism, as follows.

The hamiltonian \( H = L - \sum_i p_i \dot{x}^i \) may be written in terms of positions and velocities as \( \frac{1}{m} \sum_i (\dot{x}^i)^2 \) or in terms of positions and conjugate momenta as \( \frac{1}{m} (p_1^2 + p_2^2 + (\frac{m}{e} - \frac{e}{m} x^1)^2) \). From this formula, one sees that \( \frac{\partial}{\partial x^2} + e \frac{\partial}{\partial p_3} \) is a symmetry; its generating hamiltonian is \( -e x^3 + p_1 = m \dot{x}^1 - e x^3 \).

Symplectic geometry gives a way to recover the proportionality of momentum and velocity, with the additional gain of “gauge invariance,” (i.e., the magnetic field and not the potential enters the equations). One introduces the coordinates \( \xi_i = m \dot{x}^i \), so that the hamiltonian has the simple form \( \frac{1}{m} \sum_i (\xi_i)^2 \), but the symplectic form \( \sum dx^i \wedge dp_i \) becomes in these coordinates \( \sum_i dx^i \wedge d\xi_i + e dx^3 \wedge dx^1 \). The \( x^i \) and \( \xi_i \) are no longer canonically conjugate variables; the extra term \( e dx^3 \wedge dx^1 \) in the symplectic form is the magnetic field. In this formulation, the symmetry of the system with respect to rotation around the \( x^2 \) axis becomes manifest.

In the language of the late 20th century, one says that the phase space of a particle in a magnetic field is no longer the cotangent bundle of configuration space; rather it is a “shifted cotangent bundle” obtained by beginning with the cotangent bundle of a principal circle bundle over configuration
space and then applying the operation of symplectic reduction. (See [3] for an exposition.)

Another link between this example and 20th century mathematics comes through the work of Noether [19], who, in the course of developing ideas of Einstein and Klein in general relativity theory, found a very general equivalence between symmetries and conservation laws in field theory (i.e. variational problems which may involve several independent variables) now known as Noether’s theorem. Noether’s work on this subject is so important that her name has become inextricably attached to the relation between symmetry and conservation laws. It would be interesting to trace the early history of this relation, perhaps even as far as ancient Greek astronomy, where the uniform circular motion of heavenly bodies might have been related to the perfection (i.e. rotational symmetry) of the celestial sphere.

In the remainder of this paper, we will concentrate on the hamiltonian point of view toward “Noether’s theorem.”

3 Hamiltonian systems with symmetry

The relation between conserved quantities and symmetries becomes very simple when expressed in terms of Poisson brackets. On a symplectic manifold $P$, the time evolution of a function $F$ under the hamiltonian flow of a function $H$ is given by the formula

$$\frac{dF}{dt} = \{F, H\}.$$  

Antisymmetry of this bracket implies immediately that $F$ is a conserved quantity for the hamiltonian flow of $H$ if and only if the hamiltonian flow of $F$ consists of symmetries of $H$. If $F$ and $G$ are both conserved quantities, then so is $\{F, G\}$, by Poisson’s theorem, which is most conveniently proved via the Jacobi identity. If $F_1, \ldots, F_k$ are conserved quantities, then so is any function $g(F_1, \ldots, F_k)$, so the process of building new conserved quantities from old ones naturally terminates if we arrive at a list $F_1, \ldots, F_k$ of functions such that, for each pair of functions in the list, $\{F_i, F_j\} = \pi_{ij}(F_1, \ldots, F_k)$ for functions $\pi_{ij}$. Lie [13] refers to such a list of functions as generating a function group, the function group itself consisting of all the $g(F_1, \ldots, F_k)$. If $F_1, \ldots, F_k$ are functionally independent, then the $\pi_{ij}$ are uniquely determined smooth functions on $\mathbb{R}^k$ and define on the set of all smooth functions on $\mathbb{R}^k$ the structure of what Lie [13] calls an abstract function group. This
is a Lie algebra structure defined by the bracket
\[
\{ F, G \} = \sum_{i,j} \pi_{ij}(\mu_1, \ldots, \mu_k) \frac{\partial F}{\partial \mu_i} \frac{\partial G}{\partial \mu_j}.
\]

An important special case is that where the functions $\pi_{ij}$ are linear. The linear functions on $\mathbb{R}^k$ are then closed under the bracket operation and form a Lie algebra $\mathfrak{g}$; the carrier $\mathbb{R}^k$ of the abstract function group is identified with the dual space $\mathfrak{g}^\ast$.

In geometric terms, a phase space with a symmetry group consists of a manifold $P$ equipped with a symplectic structure $\omega$ and a Hamiltonian action of a Lie group $G$. By the latter, we mean a symplectic action of $G$ on $P$ together with an equivariant map $J$ from $P$ to the dual $\mathfrak{g}^\ast$ of the Lie algebra of $G$ such that, for each $v \in \mathfrak{g}$, the 1-parameter group of transformations of $P$ generated by $v$ is the flow of the Hamiltonian vector field with Hamiltonian $x \mapsto \langle J(x), v \rangle$. The map $J$ is called the momentum map (or, by many authors, moment map) of the Hamiltonian action. If one is simply given a symplectic action of $G$ on $P$, any map $J$ satisfying the condition in italics above, even if it is not equivariant, is called a momentum map for the action.

The antisymmetry of the Poisson bracket now implies the Hamiltonian version of Noether’s theorem: the momentum map $J$ is invariant under the Hamiltonian flow of $H$ if and only if $H$ is invariant under the action of the component of the identity in the group $G$. We defer until later (see Sections 6 and 7) the question of how nonidentity components of a symmetry group $G$ are related to conservation laws.

The duality between symmetries and conservation laws in the Hamiltonian formalism becomes most striking in the language of Poisson geometry, which is the contemporary name for the geometry of Lie’s abstract function groups. We recall that a Poisson manifold is a manifold $M$ with a Lie algebra structure $\{ \, , \}$ on its algebra of smooth functions which is an algebra derivation in each argument. For a function $H$, the derivation $F \mapsto \{ F, H \}$ is the Hamiltonian vector field associated to $H$. If a group $G$ acts by automorphisms of a symplectic manifold $P$ in such a way that the quotient space $P/G$ inherits a manifold structure, then $P/G$ becomes a Poisson manifold through the identification of the smooth functions on $P/G$ with the $G$-invariant smooth functions on $P$.\footnote{If $P/G$ is not a manifold, we may treat it as some kind of “Poisson variety” or “Poisson space.”} The natural projection $\pi : P \to P/G$ is a Poisson map in the sense that its pullback on functions is a Lie algebra
homomorphism. If the action of $G$ is hamiltonian, its equivariant momentum map $J : P \to \mathfrak{g}^*$ is also a Poisson map, and the diagram of Poisson maps $\mathfrak{g}^* \leftarrow P \to P/G$ is called a dual pair. In symplectic terms, this means that the tangent spaces of the fibres of the two maps are symplectically orthogonal to one another. The algebras of functions on $\mathfrak{g}^*$ and $P/G$ commute with one another as subalgebras of the Lie algebra of functions on $P$, and if the fibres of the two maps in the dual pair are connected, these two subalgebras are the full commutants of one another.

As we have suggested above, many of the ideas in this section can already be found in Lie’s book [13]. (See [24] for a brief historical discussion.)

4 Symplectic groupoid actions

The “duality” between $\mathfrak{g}^*$ and $P/G$ would be more symmetrical if the fibres of $J$ could also be seen as orbits of an action. In general, $P/G$ is not the dual of a Lie algebra, but there is an object which plays the role of the second group in this picture; it is a symplectic groupoid for the Poisson manifold $P/G$. This is a groupoid whose objects are the points of $P/G$ and whose morphisms form a symplectic manifold $\Gamma(P/G)$. The symplectic structure on $\Gamma(P/G)$ is compatible with the Poisson structure on $P/G$ in the sense that target and source maps $\alpha$ and $\beta$ from $\Gamma(P/G)$ to $P/G$ are Poisson and antiPoisson (i.e. reversing the sign of Poisson brackets) respectively. The groupoid multiplication is compatible with the symplectic structure in the following sense. Writing $\Gamma$ for $\Gamma(P/G)$ and denoting by $\Gamma^{(2)} \subset \Gamma \times \Gamma$ the manifold of composable pairs, we have three natural maps from $\Gamma^{(2)}$ to $\Gamma$, namely the cartesian product projections $p_1$ and $p_2$ and the groupoid multiplication $m$. The symplectic structure $\omega$ on $\Gamma$ should then satisfy the cocycle condition $p_1^* (\omega) - m^* (\omega) + p_2^* (\omega) = 0$.

The quotient projection $\pi : P \to P/G$ may be seen as a momentum map (in an extended sense) for a symplectic groupoid action of $\Gamma(P/G)$ on $P$ whose orbits are the fibres of $J$. This means (see [13]) that we have a map $\alpha : \Gamma \times_{P/G} P \to P$ satisfying the usual “associativity property” of an action; the action being symplectic means that $p_1^* (\omega) - \alpha^* (\omega_P) + p_2^* (\omega_P) = 0$. The fibre product here is taken with respect to the maps $\beta$ and $\pi$, $\omega_P$ is the symplectic structure on $P$, and $p_1$ and $p_2$ are again the cartesian projections from $\Gamma \times_{P/G} P$ to $\Gamma$ and $P$. If we write $gx$ for $a(g, x)$, then we have the equivariance relation $J(gx) = gJ(x)$ involving the natural action of the

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We recall that a groupoid is a small category whose morphisms are all invertible; if this category has just one object, the groupoid is a group.
groupoid on its objects. We call $\pi$ the momentum map of the groupoid action.

When $\mathfrak{g}$ is the Lie algebra of $G$, a natural choice for $\Gamma(\mathfrak{g}^*)$ is the cotangent bundle $T^*G$ with its canonical symplectic structure and a groupoid structure whose source and target maps to $\mathfrak{g}^* = T^*_eG$ are given by left and right translation. Under the identification of $T^*G$ with $\mathfrak{g}^* \times G$ by left translations, the groupoid multiplication is given by $(\mu, g)(\nu, h) = \pi(\nu, gh)$. The fact that the fibres of $\pi : P \to P/G$ are the orbits of the $G$ action follows from the fact that the cotangent bundle projection is a groupoid homomorphism from $\Gamma(\mathfrak{g}^*) = T^*G$ to $G$ (considered as a groupoid with a single object), and that the symplectic groupoid action on $P$ factors through this homomorphism. For a more general Poisson manifold like $P/G$, there is no such homomorphism from $\Gamma(P/G)$ to a group, and so the symplectic groupoid is absolutely essential to the picture.

Passing from groups to groupoids leads to a new definition of a dual pair. We may define a dual pair to consist of a pair of symplectic groupoids $\Gamma_1$ and $\Gamma_2$ with underlying Poisson manifolds $Q_1$ and $Q_2$, together with commuting symplectic groupoid actions of $\Gamma_1$ and $\Gamma_2$ on a symplectic manifold $P$ having momentum maps $J_1$ and $J_2$ such that the orbits of each action are the fibres of the momentum map of the other. We note that this formulation includes the case where all the groupoids and spaces involved are discrete. The groupoid action contains both aspects of the symmetry/conservation duality, since the conserved quantity is just the momentum map of the groupoid action.

Our main point, then, is that a hamiltonian action of a Lie group should be seen as a special case of a more flexible notion of symmetry—the action of a symplectic groupoid. Such an action always comes equipped with a momentum map which is a Poisson map and which in fact determines the action of the component of the groupoid containing the identity elements. Pursuing this idea, it is natural to ask whether properties of momentum maps of hamiltonian group actions extend to the groupoid case. As a first step toward answering this question, we have established in [26] an extension of Kirwan’s nonabelian version of Atiyah, Guillemin, and Sternberg’s convexity theorem.

5 Poisson Lie groups and beyond

We have seen that the cotangent bundle of a Lie group $G$ is a rather special symplectic groupoid in that it admits a homomorphism to a group. This
homomorphism, the cotangent bundle projection $\pi$, is a covering morphism in the sense of [4]; this means that its fibres are bisections of the groupoid, i.e. they project diffeomorphically to the space of objects under the source and target maps. Finally, $\pi$ is a Poisson map when $G$ carries the zero Poisson structure. This is equivalent to the fact that the fibres of $\pi$ are lagrangian submanifolds of the cotangent bundle.

The properties of the symplectic groupoids $T^*G$ listed in the paragraph above nearly characterize these examples, the only exceptions being the quotients of cotangent bundles by lattices of bi-invariant (and hence closed) 1-forms. On the other hand, if we drop the requirement that the fibres of a covering morphism $\pi$ from a symplectic groupoid $\Gamma$ to a group $G$ be lagrangian, we obtain a much larger class of symplectic groupoids which are still closely connected to groups. The group $G$ must now carry a nonzero Poisson structure in order for $\pi$ to be a Poisson map; compatibility of the group structure with the Poisson structure is expressed by the condition that the multiplication $G \times G \to G$ be a Poisson map. A symplectic groupoid action of $\Gamma$ on $P$ still corresponds to an action of the group $G$ on $P$. The manifold of objects of $\Gamma$, which is the target of the momentum maps of such actions, turns out to be a group itself; it is called the dual of the Poisson-Lie group $G$ and is denoted $G^*$. (The group structure on $G^*$ is essentially encoded in the Poisson structure on $G$, and vice-versa.) The momentum map is equivariant with respect to an action of $G$ on $G^*$ known as the dressing action. Since the fibres of $\pi$ are not lagrangian, the elements of $G$ no longer act on $P$ by symplectic transformations. Rather, the action as a whole is a Poisson action in the sense that the map $G \times P \to P$ is a Poisson map. (The dressing action is also a Poisson action.) The original groupoid $\Gamma$ is simultaneously a symplectic groupoid for both $G$ and $G^*$; as a symplectic manifold, it is known as the Heisenberg double of the pair $(G, G^*)$ of dual Poisson Lie groups. What we have described here is essentially Lu’s momentum map theory [16] for Poisson Lie group actions.

Having dealt with compatibility conditions for symplectic structures on groupoids and Poisson structures on groups, one might ask for a common framework for both. This is provided by the notion of Poisson groupoid [2]. Since Poisson structures correspond to possibly degenerate bivector fields, one might hope for a theory which also includes degenerate 2-forms. This is provided by the theory of Dirac structures [8]. These are subbundles of a direct sum $TP \oplus T^*P$ which are maximal isotropic for a natural symmetric bilinear form and which are closed under a bracket discovered by Courant [8] and which has become the prototype for an object known as a Courant algebroid [14].
Computations on moduli spaces of flat connections in gauge theory have led to a yet more general notion of hamiltonian symmetry. Here, the 2 form $\omega_P$ on phase space is neither closed nor nondegenerate, but these “defects” are compensated for by the presence of an auxiliary structure on the group, just as the noninvariance of the symplectic structure on a Poisson $G$-space is compensated for by the Poisson structure on the group. In the theory of quasi-hamiltonian $G$-spaces, introduced in [1], the symmetry group $G$ acting on $P$ now carries a bi-invariant inner product and, hence, a bi-invariant 3-form $\phi$ defined by $\phi(u, v, w) = \langle [u, v], w \rangle$. The momentum map is now group-valued, i.e. we have $J : P \to G$ with the property that $d\omega_P = J^*\phi$. As in the ordinary hamiltonian case, $\omega_P$ is $G$-invariant, and $J$ is now equivariant with respect to the adjoint action of $G$ on itself. The relation between the momentum map and the infinitesimal generators of the symmetry group is now more complicated. For $v \in \mathfrak{g}$, the corresponding vector field $v_P$ on $P$ satisfies the condition that,

$$i_{v_P}\omega_P = J^*\langle \frac{1}{2}(\theta + \overline{\theta}), v \rangle,$$

where $\theta$ and $\overline{\theta}$ are the left-invariant and right-invariant Maurer-Cartan forms on $G$. Note that, since $\omega_P$ may be degenerate, this condition does not necessarily determine the action uniquely for a given momentum map.

The relation between momentum and symmetry becomes even looser in Karshon’s theory of “abstract momentum maps” [12]. Here, $G$ is a torus acting on $P$, and $J$ is an equivariant map from $P$ to $\mathfrak{g}^*$. There is no longer a 2-form on $P$. Instead, the action and the momentum map are related by the condition that, along the fixed point set of any subgroup $H \subseteq G$, the image of $J$ must lie in a subspace parallel to $h^\perp \subseteq \mathfrak{g}^*$; i.e. the component of the momentum map given by each element of $h$ is a conserved quantity for the restricted action. Remarkably, this weak relation is sufficient to prove versions of many results usually associated with ordinary momentum maps.

6 Universal momentum maps

There have been at least two quite different approaches to the idea of a “universal momentum map.” Evans and Lu [10] construct for each Poisson Lie group a Poisson space which is the target of a momentum map for every Poisson action of the given group. Ortega and Ratiu [20], on the other hand, give preeminence to the symmetry/conservation relationship and, after fixing a particular action, find a space whose function algebra
consists of the quantities which are conserved for all invariant hamiltonian systems. We now describe these two approaches. Details omitted here will appear in [5].

The construction by Evens and Lu may be seen as an extension of that of Chu [6]. (Also see [11].) In this earlier work, one has a Lie group $G$ acting on a manifold $P$ carrying a closed (but possibly degenerate) 2-form $\omega$ which is preserved by the action. For each $x$ in $P$, we may pull back $\omega$ by the map $g \mapsto gx$ from $G$ to $P$ to get a closed, left-invariant 2-form $K(x)$ on $G$. Such forms correspond to 2-cocycles with constant real coefficients for the Lie algebra $g$; i.e. $K$ is a (smooth) map from $P$ to $Z^2(g)$. The target $g^*$ of usual momentum maps is the same as the space $C^1(g)$ of Lie algebra cochains, and the coboundary operator is a natural map $\delta : g^* \to Z^2(g)$. It turns out that $Z^2(g)$ is the dual of a Lie algebra, and that $\delta$ is a Poisson map. If $J : P \to g^*$ is an equivariant momentum map in the usual sense, then the composition $\delta \circ J$ is equal to $K$. Notice that the map $K$ exists and is canonically associated to every symplectic action, while $J$ may not exist and is defined only up to an additive constant (which is in the kernel of $\delta$). On the other hand, $K$ may be much "cruder" than $J$. For instance, when $g$ is abelian, $\delta$ is the zero map, and so $K$ is constant and equal to zero (i.e. the orbits of a hamiltonian action are isotropic), even though $J$ may be nontrivial.

If a momentum map $J : P \to g^*$ is not equivariant, then $\delta \circ J$ differs from $K$ by a constant element $b$ of $Z^2(g)$. If one adds the cocycle $b$ to the Poisson structure on $g^*$ as a constant term, then $J$ becomes a Poisson map and is equivariant with respect to an affine action of $G$ on $g^*$ whose linear part is the coadjoint action. The translational part of this action is a 1-cocycle on $G$ with values in $g^*$; the derivative of this cocycle at the identity of $G$ may be identified with $b$. (See [22] or [24].) One may view $g^*$ with this affine Poisson structure as a Poisson submanifold of the dual of a central extension of $g$. Although one could use the 1-dimensional extension associated to $b$, it is interesting to use the universal central extension $X(g)$ of $g$ by $Z^2(g)^*$ in which the bracket $[x, y]$ of two elements of $g$ is the linear functional on $Z^2(g)$ given by $b \mapsto b(x, y)$. There is a natural Poisson map $\hat{\delta}$ from $X(g)^*$ to $Z^2(g)$. (It is not just the restriction map.) Any momentum map $J$ for a symplectic action of $G$ on $P$ yields an equivariant Poisson map $\hat{J} : P \to X(g)^*$ with $\hat{\delta} \circ \hat{J} = K$. Although $\hat{J}$ is not completely determined by the action, it does provide, for any action admitting a momentum map, a Poisson map which lifts $K$ and which is as "refined" as $J$.

The definition of the map $K$ is extended to Poisson actions of a Poisson Lie group $G$ in the construction of Evens and Lu [14]. Their target is the
space $\mathcal{L}(D\mathfrak{g})$ of lagrangian subalgebras of the Drinfeld double $D\mathfrak{g}$ of the Lie bialgebra $\mathfrak{g}$, the Lie algebra of the double group $DG$. The Drinfeld double is a Lie algebra structure on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ together with the invariant inner product $((x, \xi), (y, \eta)) \mapsto \xi(y)\eta(x)$; a lagrangian subalgebra is a subalgebra which is maximal isotropic with respect to the inner product. (This is another example of a Dirac structure in a Courant algebroid.) Evens and Lu show that $\mathcal{L}(D\mathfrak{g})$ has the structure of an algebraic variety which is stratified by Poisson manifolds such that there is a dense subalgebra of continuous functions on $\mathcal{L}(D\mathfrak{g})$ whose restrictions to the strata are smooth and which is closed under the stratum-wise Poisson bracket. Given a Poisson action of $G$ on $\mathcal{P}$, they construct a natural equivariant map $EL: \mathcal{P} \to \mathcal{L}(D\mathfrak{g})$ which is a smooth Poisson map into a single stratum. Using the theory of Dirac structures [8][14][15], it is possible to construct the equivariant map $EL$ for arbitrary Poisson actions, though it is generally not continuous (though it is in fact smooth when $\mathcal{P}$ is symplectic). Nevertheless, $EL$ can be shown to be a Poisson map in a certain sense. When $G$ has the zero Poisson structure, the double $D\mathfrak{g}$ is the semidirect product of $\mathfrak{g}$ with $\mathfrak{g}^*$ via the coadjoint representation. Among the maximal isotropic subspaces of $\mathfrak{g} \oplus \mathfrak{g}^*$ are the graphs of antisymmetric mappings $\mathfrak{g} \to \mathfrak{g}^*$, which correspond to skew symmetric bilinear forms, or 2-cochains with real coefficients, on $\mathfrak{g}$. It turns out that the subspace is a subalgebra if and only if the cochain is a cocycle. (The corresponding subgroup is the graph of the $\mathfrak{g}^*$-valued cocycle on $G$ connected with the affine action discussed above.) Thus, the target space $Z^2(\mathfrak{g})$ is contained in $\mathcal{L}(D\mathfrak{g})$ (as an open submanifold). The Poisson structure on $Z^2(\mathfrak{g})$ induced from the Evens-Lu structure on $\mathcal{L}(D\mathfrak{g})$ turns out to be identical to that which occurs in the construction of Chu, and the maps $EL$ and $K$ coincide when $\mathcal{P}$ is symplectic.

Continuing with the case where $G$ has the zero Poisson structure, we also find among the lagrangian subalgebras of $D\mathfrak{g}$ the sums $\mathfrak{h} \oplus \mathfrak{h}^\perp$, where $\mathfrak{h}$ is an arbitrary subalgebra of $\mathfrak{g}$ and $\mathfrak{h}^\perp$ is its annihilator in $\mathfrak{g}^*$. These occur as the values of $EL$ when the manifold $\mathcal{P}$ carries the zero Poisson structure: $EL(p) = \mathfrak{g}_p \oplus \mathfrak{g}_p^\perp$, where $\mathfrak{g}_p$ is the Lie algebra of the isotropy subgroup $G_p$ of $p$ in $G$. (The corresponding subgroup of the double group is the conormal bundle of $G_p$.) Thus, in the absence of any Poisson structure at all, we still have a “momentum map” which essentially assigns to each point in a $G$-manifold its isotropy submanifold. The level sets of this momentum map, or isotropy types, are invariant under any $G$-equivariant map.

Of course, there are no interesting $G$-invariant hamiltonian systems when $\mathcal{P}$ has the zero Poisson structure, since all hamiltonian vector fields are zero; nevertheless, this example provides a nice lead-in to the construction
of Ortega and Ratiu [20], since their construction is motivated in part by the fact that conventional momentum maps may miss the “conservation of isotropy”.

Ortega and Ratiu begin with a group \( G \) acting by automorphisms on a Poisson manifold \( M \). Let \( \mathcal{E} \) be the set of hamiltonian vector fields of \( G \)-invariant functions defined on \( G \)-invariant open subsets of \( M \). The (local) flows of the vector fields in \( \mathcal{E} \) form a pseudogroup \( \mathcal{G} \) of Poisson diffeomorphisms between open subsets of \( M \), and we may form the momentum space \( M/\mathcal{G} \). In general, \( M/\mathcal{G} \) is not a manifold, but it is a “smooth Poisson space” in the sense that, if we define the smooth functions on \( M/\mathcal{G} \) to be the \( \mathcal{G} \)-invariant smooth functions on \( M \), the space of these functions forms a Poisson algebra. Ortega and Ratiu’s optimal momentum map is simply the natural projection map \( J : M \to M/\mathcal{G} \). Immediately from its definition, it has the property of being being conserved by the flow of any \( G \)-invariant hamiltonian. Since the elements of \( \mathcal{G} \) commute with \( G \), \( G \) acts in a natural way on the momentum space, and the optimal momentum map is equivariant. It is universal in the sense that, given any equivariant Poisson map \( K : M \to P \) which is conserved by the flows of all \( G \)-invariant hamiltonians, there is a unique equivariant Poisson map \( \phi : M/\mathcal{G} \to P \) such that \( K = \phi \circ J \). The optimal momentum map can be nontrivial even if a momentum map in the usual sense does not exist. For hamiltonian actions, it may be more refined than the usual momentum map \( J \). For instance, in the case of a proper hamiltonian action, each level set of \( J \) is a connected component of the intersection of a level set of \( J \) with an isotropy type.

We remark that the optimal momentum map is very closely related to the Poisson Γ structure introduced by Condevaux, Dazord, and Molino [7] in a paper whose title is essentially the same as that of this one.

7 The Baer groupoid

We can forge closer links among some of the concepts in this paper with the aid of the Baer groupoid \( BG \) associated to a group \( G \). The set of objects of this groupoid is the set \( SG \) of subgroups of \( G \); a morphism from \( H \) to \( K \) is a subset of \( G \) which is at the same time a right coset \( Ka \) and a left coset \( bH \). For such a morphism, it follows that \( bH = aH \), so that \( K = aHa^{-1} \), and \( K \) and \( H \) are conjugate. The product of two morphisms is the usual product of subsets in a group.

Thus, the \( BG \) orbits in \( SG \) are the conjugacy classes, and the isotropy subgroup \( H \in SG \) is isomorphic to the quotient \( NH/H \), where \( NH \) is the
normalizer of $H$ in $G$. 3

Any action of a group $G$ on a space $P$ induces an action of $BG$ on $P$. The momentum map of this action assigns to each $x \in P$ the isotropy subgroup $G_x$, and the coset $aG_x$ maps $x$ to $ax$. The level sets of this momentum map are exactly the isotropy types, so we are close to the Evens–Lu momentum map. If the action is hamiltonian, we can extend the $BG$ action on $P$ to an action of the product groupoid $BG \times T^*G$ with object space $SG \times g^*$. The momentum map of this extended action has as level sets the intersection of the level sets of $J : P \to g^*$ with the isotropy types. The connected components of these level sets are, for proper actions, precisely the level sets of the optimal momentum map. Except for this question of connectedness, we have recovered the optimal momentum map using a natural construction involving a momentum space which depends only on the group and not on the particular (proper) action under consideration.

Further development of these ideas looks like a good project for the early 21st century.

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3When $G$ is a compact connected Lie group, the restriction of $BG$ to the set of maximal tori is a transitive groupoid which might be called the Weyl groupoid, since its isotropy groups are the Weyl groups. A possible advantage of this groupoid over the individual Weyl groups is that it admits a nontrivial action of the automorphism group of $G$. 13
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