Achievable Rate of Private Function Retrieval from MDS Coded Databases

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Abstract—We study the problem of private function retrieval (PFR) in a distributed storage system. In PFR the user wishes to retrieve a linear combination of \( M \) messages stored in non-colluding \((N, K)\) MDS coded databases while revealing no information about the coefficients of the intended linear combination to any of the individual databases. We present an achievable scheme for MDS coded PFR with a rate that matches the capacity for coded private information retrieval derived recently, 
\[
R = (1 + R_c + R_c^2 + \cdots + R_c^{N-1})^{-1} = \frac{1}{1 - R_c},
\]
where \( R_c = \frac{c}{N} \) is the rate of the MDS code. This achievable rate is tight in some special cases.

I. INTRODUCTION

The private retrieval of information from public databases has received significant attention already for several decades from researchers in the computer science community (see, e.g., [1], [2]). While this line of work, commonly known as private information retrieval (PIR), is concerned with downloading individual messages in a private manner from databases, a recently proposed generalization of this problem [3], [4] addresses the private computation of functions of these messages. In accordance with [3] we denote this approach as private function retrieval (PFR) in the following. In PFR a user has access to a given number of databases and intends to compute a function of messages stored in these databases. This function is kept private from the databases, as they may be under the control of an eavesdropper. Both works [3], [4] characterize the fundamental information theoretic communication overhead needed to reliably compute the given function and specify the corresponding capacity and achievable rates as a function of the message size, the number of messages, and the number of databases, respectively. Further, the authors assume that the data is replicated on each database. Surprisingly, the obtained PFR capacity result is equal to the PIR capacity of [5].

However, although repetition coding adds the largest amount of redundancy and thus protects effectively against erasures, it is associated with a large storage cost. A more general way to optimally trade-off the available redundancy (or rate) versus the erasure correcting capability is given by MDS codes. In particular, for an \((N, K)\) MDS code with \( N \) code symbols and \( K \) information symbols and rate \( R_c = K/N \) \( N - K \) erasures can be recovered from any \( K \) code symbols. Coded PIR has been addressed in two different lines of work. Achievable schemes for MDS coded PIR have been presented in [6], [7] and the capacity has been established in [8]. On the other hand, in [9] linear codes with \( k \) different reconstruction sets for each code symbol have been proposed in form of so called \( k \)-server PIR.

In this paper we propose coded PFR, which to the best of our knowledge has not been addressed yet in the recent literature, with the notable exception of the parallel work in [10], which is based on a fixed \((k-)\) server PIR scheme with the inclusion of colluding databases. Our scheme is based on MDS codes which in contrast to [10] minimize the storage overhead and maximize the achievable download rate. In particular, we provide a characterization of the achievable rate of MDS coded PFR if the user wishes to compute an arbitrary linear combination of \( M \) independent equal-sized messages over some finite field \( \mathbb{F}_q \), distributed over \( N \) non-colluding MDS-coded databases. Surprisingly, our achievable rate matches the capacity for MDS coded PIR in [3]. This demonstrates that, compared to the naive scheme, where \( M \) coded messages are downloaded and linearly combined offline at the user (requiring \( M \)-times the coded PIR rate), downloading the result of the computation privately and directly from the databases does not incur any penalty in rate compared to the coded PIR case. Thus, our result strictly generalizes the achievable schemes in [3], [4] which represent special cases of our proposed PFR scheme.

II. PROBLEM STATEMENT

In the following, we use \([1 : X]\) to denote the set \(\{1, \ldots, X\}\). Similarly, \(X_{1:N} = \{X_1, \ldots, X_N\}\).

A. System Model

In coded PFR, a user wishes to privately retrieve a linear combination of the messages stored in the databases such that the coefficients of the linear combination are kept secret from each individual database. Consider a linear distributed storage system storing \( M \) equal-sized messages on \( N \) non-colluding databases. The message \( W_m, m \in [1 : M] \), is composed from \( L \) symbols chosen independently and uniformly at random from the finite field \( \mathbb{F}_q \), with
\[
H(W_1) = \cdots = H(W_M) = L \log q, \quad (1)
\]
\[
H(W_1, \ldots, W_M) = H(W_1) + \cdots + H(W_M) = ML \log q. \quad (2)
\]
Each message is divided into \( \tilde{L} \) segments, each of \( K \) symbols, forming a \( \tilde{L} \times K \) matrix, where \( L = \tilde{L}K \). The
messages are stored using an \((N, K)\) MDS code with the full rank generator matrix defined by

\[
G = [g_1 \ g_2 \ \ldots \ g_N]_{K \times N},
\]

with \(g_n, n \in [1 : N]\), denoting the \(n\)-th column vector of \(G\). The generator matrix produces a code that can tolerate up to \(N - K\) erasures by retrieving data from any set \(K \subseteq \{1, \ldots, N\}\) databases, where \(|K| \geq K\). The encoding process for message \(W_m\) is defined as follows:

\[
[W_{m,t}]_{1 \times K} [g_1 \ g_2 \ \ldots \ g_N]_{K \times N} = [g_1^T w_{m,t} \ \ldots \ g_N^T w_{m,t}]_{1 \times N},
\]

where \(w_{m,t}, \forall m \in [1 : M], \forall t \in [1 : L]\), denotes the \(K\)-dimensional vector of symbols of the \(t\)-th segment from the message \(W_m\). The resulting \(N\) encoded symbols for each segment are then distributed over the \(N\) databases, and the code rate is given by \(R_c = \frac{K}{N}\).

Consequently, the code symbols stored at each database \(n \in [1 : N]\) are given by

\[
W_{DB_n} = \begin{bmatrix}
g_n^T w_{1,1} & g_n^T w_{1,2} & \ldots & g_n^T w_{1,L} \\
\vdots & \vdots & \ddots & \vdots \\
g_n^T w_{M,1} & g_n^T w_{M,2} & \ldots & g_n^T w_{M,L} \\
\end{bmatrix},
\]

where we use \(W[t]\) to denote the \(t\)-th column, and \(W_m(t)\) for the element of the \(m\)-th row and \(t\)-th column of the database, respectively.

In PFR, the linear combination \(\nu\) the user intends to retrieve is represented as

\[
\tilde{W}_\nu = v_\nu [W_1, \ldots, W_M]^T
\]

\[
= v_\nu (1) W_1 + \cdots + v_\nu (M) W_M
\]

\[
= [v_\nu W[1] \ \ldots \ v_\nu W[L]],
\]

where \(v_\nu\) is an \(M\)-dimensional non-zero coefficient vector of the linear combination (row vector) indexed by \(\nu\), the coefficients \(v_\nu(m), \forall m \in [1 : M]\), are chosen from the finite field \(\mathbb{F}_q\), and the addition “+” is done element-wise over the same field. We assume that the vector \(v_\nu\) is an element of the set \(V\) that contains all possible distinct \(M\)-dimensional vectors defined over \(\mathbb{F}_q\) where \(\nu \in [1 : V]\), \(|V| = \frac{q^M - 1}{q - 1}\).

For order for the user to retrieve the linear combination \(\tilde{W}_\nu\), while keeping \(\nu\) secret from each database, it generates \(N\) query matrices for the databases \(\{Q_1^{(\nu)}, \ldots, Q_N^{(\nu)}\}\). Since the query matrices are generated by the user without prior knowledge of the realizations of the stored messages, the queries must be independent of the messages,

\[
I(Q_1^{(\nu)}, \ldots, Q_N^{(\nu)}; W_1, \ldots, W_M) = 0, \ \forall \nu \in [1 : V].
\]

Upon the reception of the query \(Q_n^{(\nu)}\), the \(n\)-th database generates an answer string \(A_n^{(\nu)}\) as a deterministic function of the received query and the stored symbols from each message. Hence,

\[
H(A_n^{(\nu)} | Q_n^{(\nu)}, W_{DB_n}) = 0, \ \forall \nu \in [1 : V], \forall n \in [1 : N].
\]

To maintain user privacy, the query-answer function must be identically distributed for each possible linear combination \(\nu \in [1 : V]\) from the perspective of each database \(n \in [1 : N]\). In other words, the scheme’s queries and answers strings must be independent from the desired linear combination index, therefore the following privacy constraint must be satisfied:

\[
I(A_n^{(\nu)}, Q_n^{(\nu)}, W_{DB_n}; \nu) = 0, \ \forall \nu \in [1 : V].
\]

After the user receives all answer strings from each database, the user must be able to reliably decode the desired linear combination message \(\tilde{W}_\nu\) with a probability of error \(P_e\) that goes to zero as the message size \(L\) approaches infinity. Following Fano’s inequality, this translates to the decodability constraint

\[
H(\tilde{W}_\nu | A_1^{(\nu)}, Q_1^{(\nu)}) = o(L),
\]

where \(o(L)\) represents any function of \(L\), \(f(L)\), that satisfies \(\lim_{L \to \infty} f(L)/L \to 0\).

The retrieval rate of the coded PFR scheme is characterized by the message length \(L\), the query structure \(Q\), and the query-answer function, and is defined as the ratio between the size of the desired linear combination message and the total number of downloaded symbols in bits as

\[
R = \frac{H(\tilde{W}_\nu)}{\sum_{n=1}^N H(A_n^{(\nu)})}.
\]

A rate \(R\) is said to be achievable if there exist a sequence of coded PFR schemes that satisfy the privacy and correctness constraints of (11), (12) for \(P_e \to 0\) as \(L \to \infty\).

### III. ACHIEVABLE RATE OF MDS CODED PFR

**Theorem 1.** For an \((N, K)\) coded distributed storage system with code rate \(R_c = \frac{K}{N}\), \(M\) messages and a set of \(V\) linear combinations defined over the field \(\mathbb{F}_q\), a PFR achievable rate is given as

\[
R \leq \frac{1 - R_c}{1 - R_c^M} = \left(1 + \frac{K}{N} + \frac{K^2}{N^2} + \cdots + \frac{K^{M-1}}{N^{M-1}}\right)^{-1}.
\]

**Remark 1.** This achievable rate generalizes the achievable rate of repetition coded PFR [4] which corresponds to the special case of \(K = 1\). Also, (14) is only a function of the distributed storage coding rate \(R_c\) and the number of stored independent messages \(M\), and is universal in the sense that it does not depend on the number of linear combinations \(V\) defined over the finite field \(\mathbb{F}_q\) nor on the explicit structure of the code.

**Remark 2.** If we consider each of the \(V\) linear combinations of messages in (6) as a new virtual message \(\tilde{W}_\nu\), and then apply the coded PIR scheme of [8], the scheme rate will be \(\frac{R}{1 - R_c}\) which is smaller than (14) since \(M \leq V\).

**Remark 3.** When the linear combination set \(V\) is reduced to the first \(M\) linear combinations (i.e., \(v_1 : M \in V : \{v_1 v_2 \ldots v_M\} = I_M\)), the achievable rate of (14) is tight. That is because in this setting the problem of coded PFR is reduced to coded PIR where the converse is implied from [8]. Also, we note that (14) is equivalent to the coded PIR capacity [8], which has been observed in [4] for \(K = 1\). Thus, downloading linear combinations of messages does not incur additional costs over downloading individual messages.
Remark 4. Eq. (14) is a strictly decreasing function in the number of messages \( M \) for fixed \( R_c \). As the number of messages increases \( M \to \infty \), the achievable rate approaches \( 1 - R_c \). Moreover, as \( R_c \to 1 \) in (15), \( R \to \frac{1}{\kappa} \), indicating that to maintain the privacy of the desired linear combination, the user must download all the messages and perform the computation off-line.

IV. PROOF OF THEOREM 1

A. Query generation

The generation of the queries is shown in Algorithm 1. Let \( B \in [1 : V] \) be the block indicator and \( R \in [1 : K] \) be the repetition indicator, respectively. Let the \( v \)-sum be the combination of \( v \) distinct elements out of \( V \) elements. Since we have \( \binom{V}{v} \) different combinations, we denote each different combination as a type of the \( v \)-sum. Let the components of these combinations be symbols of the \( V \) virtual messages. As mentioned above, we generate the query set for each database in blocks, where a block represents a group of all \( \binom{V}{v} \) types of \( v \)-sums for all \( v \in [1 : V] \), resulting in \( V \) blocks in total. To this end, we let the size of the dependent virtual messages to be \( L = KN^V \) (i.e., \( \tilde{L} = N^V \)).

For a desired linear combination \( v \in [1 : V] \) we use the notation \( Q^{[v]}(DB_B) \) to indicate the query set of the database \( DB_B \in [1 : N] \). This set is composed from all \( V \) disjoint subsets \( Q^{[v]}_{B,R}(DB_B) \) generated for each block \( B \) and repetition \( R \). We require \( K^{V-B}(N-K)^{B-1} \) distinct instances of each type of \( v \)-sum for every set \( Q^{[v]}_{B,R}(DB_B) \). Each block and repetition subset is further subdivided into two subsets: the first subset \( Q^{[v]}_{B,R}(DB_B,M) \) consists of all \( v \)-sum types with symbols from the desired linear combination, and the second subset \( Q^{[v]}_{B,R}(DB_B,T) \) contains only \( v \)-sum types with symbols from undesired linear combinations. The query sets for all databases are generated by Algorithm 1 with the following procedures.

1) Index assignment: In the MDS-coded PIR scheme [3], the user privately applies a random permutation over the coded symbols of each message independently. The goal is to make the coded symbols queried from each database to appear to be chosen randomly and independently from the desired message. However, for the PFR problem the linear function is computed element-wise, thus there is a dependency across the symbols with the same index, which must be maintained under a permutation. To this end, we modify the permutation to be fixed across all messages. Let \( \pi(\cdot) \) be a random permutation function over \( [1 : L] \). We use the notation \( U_{\nu}(t) \), where

\[
U_{\nu}(t) \triangleq \sigma_t \tilde{W}_{\nu}(\pi(t)) = \sigma_t V_{\nu} W[\pi(t)],
\]

(16)
to indicate the permuted message symbol from the virtual message \( \tilde{W}_{\nu} \). The random variable \( \sigma \) is used to indicate the sign assigned to each individual virtual message symbol, \( \sigma_t \in \{+1, -1\} \). Both \( \sigma_t \) and \( \pi \) are randomly selected privately and uniformly by the user.

2) Block \( B = 1 \): This block is described by Steps 3 to 10 of Algorithm 1, where we have \( v = 1 \) for the \( v \)-sum.

Initialization: In the initialization step, the user queries the first database \( DB_1 = 1 \) for \( K^{V-1} \) distinct symbols from the desired linear combination \( U_{\nu}(i) \). This is done by calling the function \( \text{new}(U_{\nu}) \) that will select a symbol from message \( U_{\nu} \) with a new index \( i \) each time it is called (Step 6).

Database symmetry: Database symmetry is obtained via the “For” loop in Step 3, resulting in a total number of \( NK^{V-1} \) symbols over all databases.

Message symmetry: In Step 7, to maintain message symmetry, the user ask each database for the same number of distinct symbols of all other linear combinations \( U_{\nu}(i) \), \( \theta \in \{1, \ldots, V\} \setminus \{\nu\} \), resulting in a total number of \( NVK^{V-1} \) symbols. As a result, the query sets for each database are symmetric with respect to all linear combination vectors in \( [1 : V] \). We associate the symbols of undesired messages in \( K \) groups \( G \in [1 : K] \) to be exploited as distinct side information for different rounds of the scheme as shown in Step 7.

3) Side-information exploitation: In Steps 11 to 20, we generate the blocks \( B \in [2 : V] \) by applying two subroutines “Exploit-SI” and “M-Sym”, respectively. We first use the subroutine “Exploit-SI” to generate queries for new symbols of the desired linear combination \( U_{\nu} \) by combining these symbols with different side information groups from the previous block associated with \( N - K \) neighboring databases, as shown in Step 13. This is required by our proposed MDS coded scheme to ensure privacy and is in contrast to [4], where the side information of previous blocks from all databases is utilized.

Then, the subroutine “M-Sym” is used to generate side information to be exploited in the following blocks. This subroutine select symbols of undesired messages to generate \( v \)-sums that enforce symmetry in the block queries. For example in \( B = 2 \), if we have the queries \( U_{\nu}(i) + U_2(j) \), and \( U_{\nu}(i) + U_2(j) \in Q^{[v]}_{2,R}(DB_2,M) \), this subroutine will generate \( U_2(l) + U_3(i) \). As a result, we can show that the symmetry over the linear combinations and databases is maintained. By the end of this step we have in total \( N \binom{V}{V} K^{V-B}(N-K)^{B-1} \) queries for each block from all databases.

4) Generation of further query rounds: We require further query rounds to obtain \( K \) linear equations for each coded symbol to be able to decode. To this end, we circularly shift the order of the database at each repetition. The shift is done for the initial block, \( B = 1 \), in Steps 22 to 25. However, for the following blocks we only rotate the indices of desired messages \( U_{\nu} \) and combine them with new groups of side information from the neighboring databases from the first round as seen in Steps 26 to 33. This rotation and side information exploitation for \( B \in [2 : V] \) is done using the subroutine “Reuse-SI” (omitted in the interest of space).

5) Query set assembly: Finally, in Steps 35 to 37, we assemble each query set from the queries disjoint subsets obtained in the previous blocks and rounds.

Remark 5. Note that the proposed scheme significantly differs from the one presented in [4] in terms of how the side information is exploited due to coding. In particular, we distribute the side information over \( K \) rounds such that every database is queried for each message and linear combination only once.
Algorithm 1: Query set generation algorithm

Input: \( \nu, K, N, M, \) and \( V \).
Output: \( Q^{[\nu]}(1), \ldots, Q^{[\nu]}(N) \)

1. Initialize: All query sets are initialized as a null set \( Q^{[\nu]}(1), \ldots, Q^{[\nu]}(N) \leftarrow \emptyset \), the block counter \( B = 1 \), and repetition counter \( R = 1 \). Let number of neighboring databases \( Nb = N - K \)

2. Let repetition \( R_B = K^{V-B} \) \((N - K)^{B-1} \) \(\forall B \in [1 : V]\)

3. For first database block \( DB_1 = 1 : N \) do
4. For side information group \( G = 1 : K \) do

5. For repetition group \( RG = 1 : (R_B/K) \) do

6. \( Q^{[\nu]}_{1,R}(DB_1, M) \leftarrow \{u_0, u_\nu = \text{new}(U_0)\}

7. \( Q^{[\nu]}_{1,R}(DB_1, I_G) \leftarrow \{\text{new}(U_1), \ldots, \text{new}(U_{\nu})\}\setminus \{u_\nu\}

8. End For (repeat within the same SI group)
9. End For (repetition for SI groups)
10. End For

11. For block \( B = 2 : V \) do
12. For \( DB_B = 1 : N \) do

13. \( Q^{[\nu]}_{B,R}(DB_B, M) \leftarrow \text{Exploit-SI}(Q^{[\nu]}_{B-1,R}(DB_B + 1, I_{Nb}), \ldots, Q^{[\nu]}_{B-1,R}(DB_B + Nb, I_B))\)

14. For side-information group \( G = 1 : K \) do

15. For \( RG = 1 : (R_B/K) \) do

16. \( Q^{[\nu]}_{B,R}(DB_B, I_G) \leftarrow \text{M-Sym}(Q^{[\nu]}_{B,R}(DB_B, M), \ldots)\)

17. End For (repeat within the same SI group)
18. End For (repetition for SI groups)
19. End For (repeat for each database)
20. End For (repeat for each block)

21. For query round \( R = 2 : K \) do
22. For \( DB_1 = 1 : N \) do

23. \( Q^{[\nu]}_{1,R}(DB_1, M) \leftarrow Q^{[\nu]}_{1,R-1}(DB_1 - 1, M)\)

24. \( Q^{[\nu]}_{1,R}(DB_1, I_G) \leftarrow Q^{[\nu]}_{1,R-1}(DB_1 - 1, I_G)\)

25. End For (initializing rounds)
26. End For (block \( B = 2 : V \))
27. For \( DB_B = 1 : N \) do
28. For side information group \( G = 1 : K \) do

29. \( Q^{[\nu]}_{B,R}(DB_B, I_G) \leftarrow Q^{[\nu]}_{B,R-1}(DB_B - 1, I_G)\)

30. \( Q^{[\nu]}_{B,R}(DB_B, M) \leftarrow \text{Reuse-SI}(Q^{[\nu]}_{B,R}(DB_B, I_G), Q^{[\nu]}_{B-1,1}(DB_B + 1, I_{Nb+B-1}), \ldots, Q^{[\nu]}_{B-1,1}(DB_B + Nb, I_B))\)

31. End For (SI groups)
32. End For (repeating for each database)
33. End For (repeating for each block)

34. For \( DB_B = 1 : N \) do
35. For \( R = 1 : K \) do

36. \( Q^{[\nu]}(DB_B) \leftarrow \bigcup_{B=1}^{V} \bigcup_{R=1}^{K} (Q^{[\nu]}_{B,R}(DB_B, M) \cup Q^{[\nu]}_{B,R}(DB_B, I_G))\)

37. End For (assembling the query sets)

B. Sign assignment and redundancy elimination

We carefully assign an alternating sign \( \sigma_t \in [+1, -1] \) to each symbol in the query set, based on the desired linear combination index \( \nu \). The intuition behind the sign assignment is to introduce a uniquely solvable linear equation system from the different \( \nu \)-sum types. By obtaining such an equation system in each block, the user can opt from downloading these queries, compute them off-line, and thus reduce the download rate. Based on this insight we can state the following lemma.

Lemma 1 (\(H\)). For all \( \nu \in [1 : V] \), each database \( n \in [1 : N] \), and based on the side information available from the neighboring databases, there are \( {V-M \choose \nu} \) redundant \( \nu \)-sum types out of all possible types \( {V \choose \nu} \) in each block \( v \in [1 : V - M] \) of the query sets.

Lemma 1 is also applicable when the desired linear function is performed over MDS-coded databases due to the fact that each MDS-coded symbol is itself a linear combination. That is, the MDS code can be seen as an inner code and the desired linear function as an outer “code” with respect to the databases. Hence, the redundancy resulting from the linear dependencies between messages is also present under MDS coding and we can extend Lemma 1 to our scheme. We now make the final modification to our PFR query sets. We first directly apply the sign assignment \( \sigma_t \), then remove the redundant \( \nu \)-sum types from each block \( B \in [1 : V] \). Finally, we generate the query matrices \( Q^{[\nu]}(DB_B) \) using a one-to-one mapping function \( f \), for which \( Q^{[\nu]}(DB_B) \) is the preimage.

Proof. The proof of optimality for arbitrary \( N, K, M, V \) and \( \nu \) follows from the structure of the query and Lemma 1. The achievable rate is given as:

\[
R = \frac{KNV}{\nu} \sum_{\nu=1}^{V} \left( {V \choose \nu} - {V-M \choose \nu} \right)V^{-\nu}(N-K)^{\nu-1}
\]

\[
= \frac{N^{V-K}(N-K)}{N^{V-M-K}(N-K)^{\nu}} \sum_{\nu=1}^{V} \left( {V \choose \nu} - {V-M \choose \nu} \right)V^{-\nu}(N-K)^{\nu-1}
\]

\[
= \frac{N^{V-K} - N^{V-M-K}}{N^{V-K} - N^{V-M-K}} \sum_{\nu=1}^{V-M} \left( {V-M \choose \nu} \right)V^{-\nu}(N-K)^{\nu-1}
\]

where (a) follows from the definition of the PFR rate \(E\); (b) follows from the fact that the second term of the summation in the denominator is equal to zero for \( \nu > V - M \) and consequently we can change the upper bound of the summation; and the first term of the summation follows from the binomial theorem.

C. Correctness (decodability)

To prove correctness, we show that the user can obtain the desired linear combination \( W_v \) from the answers retrieved from \( N \) databases. From the query answers \( A_i^{[\nu]} \), we group the \( K \) identical queries from different rounds and databases. Each group will result in \( K \) linearly independent equations that can be uniquely solved. We decode, block by block, starting from block one, which we directly decode and obtain \( KN((V-M)^{\nu})V^{-\nu}(N-K)^{\nu-1} \) decoded symbols. Now,
using these symbols we regenerate \((V_M - V)\) redundant symbols according to Lemma 1 and obtain \(K N(v) K V^v (N - K)^{v-1}\) symbols in total. Out of these queries there are \(K N(v) - (V - 1)K V^v (N - K)^{v-1}\) symbols from \(\tilde{W}_v\).

Next, for blocks \(B \in [2 : V]\), we use the symbols obtained in the previous block \(B - 1\) to remove the side information associated with the desired linear combination symbols of the current block \(B\), then the operations from the first block (decode and retrieve redundancy) are repeated. As a result, we obtain a total number of symbols equal to \(K N(\sum_{v=1}^{V} (v - 1)K V^v (N - K)^{v-1}) = K N\) denoting precisely the number of symbols in \(\tilde{W}_v\).

**D. Privacy**

Privacy is guaranteed by preserving an equal number of requests for any linear combination \(\tilde{W}_v\), where the requests are symmetric from the perspective of the accessed virtual messages. As the MDS code can be seen as an outer code, the arguments in [3], [4] apply here as well. In particular, each database is queried with precisely the same \(v\)-sum type components, i.e., \(U_v(t)\), which ensures symmetry. This can be seen from Step 16 in Algorithm 1 where the same subroutine “M-Sym” is used for each block and database. By selecting a permutation \(\tau(t)\) and a sign assignment \(\sigma_t\) uniformly at random, queries for code symbols are permuted in the same way over all databases. With other words, for any \(U_v(t) = \sigma_v U_v W[\tau(t)]\), there exist \(\sigma_v, \tau(t)\) such that \(Q[0] DB_1D_2 DB_3

| \(i,\bar{i}\)  | \(DB_1\)     | \(DB_2\)     | \(DB_3\)     |
|---------|------------|------------|------------|
| (1, 1)  | \(g_1^{(4)}\) | \(g_2^{(4)}\) | \(g_3^{(4)}\) |
| (1, 2)  | \(g_1^{(5)}\) | \(g_2^{(5)}\) | \(g_3^{(5)}\) |
| (1, 3)  | \(g_1^{(6)}\) | \(g_2^{(6)}\) | \(g_3^{(6)}\) |

**VI. OUTER BOUND FOR THE SPECIAL CASE \(V = 2\)**

In the special case of \(V = 2, M\) independent messages, and any \((N, K)\) MDS code an outer bound for the coded PFR problem is obtained by combining the independent of answer strings from any \(K\) databases [8] Lemma 2] with [4]. Thus, we can show that the retrieval rate is upper bounded as \(R \leq \frac{K H(\omega_1, \omega_2)+H(\omega_1)(N-K)}{N H(\omega_1)}\), where the joint distribution of \((\omega_1, \omega_2)\) is the joint distribution of \((\tilde{W}_1, \tilde{W}_2, x)\) for all \(x \in [1 : \tilde{L}]\) selected id with respect to the symbols of the messages.

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