QUANTUM CELLULAR AUTOMATA: SCHRÖDINGER AND HEISENBERG PICTURES

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ABSTRACT. Axiomatic definitions of Quantum Cellular Automata (QCA) are foundational in discrete space-time models of physics. Templates of axiomatic QCA serve as primary points of investigations concerning properties, structural results, and classes of QCA that are potentially significant for quantum algorithms and simulations. We examine, from a representation theory standpoint, the relation between the templates corresponding to the Schrödinger and the Heisenberg pictures of evolution for QCA over finite, unbounded configurations.

1. INTRODUCTION

The axiomatic definition of Quantum Cellular Automaton (QCA) proposed by Schumacher and Werner [1] encompasses fundamental physical requirements while allowing much generality. In their work, a finite dimensional $C^*$ algebra resides over each point, or cell, of an infinite lattice. These algebras are combined as an infinite incomplete tensor product algebra, called the algebra of local observables. The algebra of local observables evolves at each time step under an automorphism, called the global homomorphism, which is defined to act locally, taking each cell algebra to the tensor product of cell algebras in its neighborhood via a homomorphism. This local homomorphism of algebras is called a local transition rule, and is homogeneous, i.e., identical for all cells. This QCA definition, in particular its evolution, is in the setting of the operator algebra of observables, thus, it is in the Heisenberg picture. It has led to a classification of one-dimensional QCA by an index theory in the work of Gross, Nesme, Vogts, and Werner [2].

Arrighi, Nesme, and Werner [3], working in the Schrödinger picture, construct a QCA with a finite dimensional Hilbert space residing over each point (cell) of an infinite lattice. In this model, a QCA is described on a suitably defined Hilbert space composed over the lattice from individual Hilbert spaces of each cell, the composite being the Hilbert space of finite configurations. A typical pure state of a QCA consists of finitely many cells in "active" states in a background of cells in a quiescent state. Hence, this QCA is defined over finite, unbounded configurations. It evolves at each time step through a unitary, translation-invariant and causal operator, called the global evolution operator. Among elegant results...

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1 Infinite incomplete tensor product algebra appears in classical papers and texts [6–9] and in the study of von Neumann algebras.

2 A local condition is one that depends on a finite subset of cells.

3 Causality refers to the finite speed of information propagation.
related to this formalism, is the work of Arrighi, Nesme and Werner on localizabilitlty of quantum cellular automata (QCA) [4].

The interested reader can find in [5] a brief overview of the historical development and significance of Quantum Cellular Automata (QCA) as models of computation and for simulation of fundamental physics. In it, the authors establish the connection between a QCA and a Quantum Lattice Gas Automaton (QLGA), by giving a local condition that characterizes a QCA as a QLGA. A QLGA models multiple particles propagating on a lattice and scattering at the lattice points (called sites in this context) through interactions. The description of a QLGA in [5] illustrates the relation between the Schrödinger and the Heisenberg pictures that we will discuss.

This paper is organized as follows. Section 2 describes both the the Schrödinger and the Heisenberg templates of QCA, over finite, unbounded configurations. Section 3 expresses the correspondence between the Schrödinger and Heisenberg templates of QCA in representation theoretic terms. Section 4 applies the uniqueness part of the template correspondence to simplify the proof of the main structure theorem on QLGA in [5]. Section 5 is the conclusion.

2. Quantum Cellular Automaton

We recall the definition of Quantum Cellular Automaton (QCA) as it is defined in [3]. Informally, a QCA is a collection of cells evolving on a lattice, with the state of each cell at the end of a time step dependent on those of its neighbors prior to the step.

The lattice is typically $\mathbb{Z}^n$. Associated with each cell is a finite dimensional complex Hilbert space, the cell Hilbert space, $W$. The state of a cell can be quiescent, given by a chosen element $|q\rangle \in W$, or if it is any other element of $W$, it is active. The Hilbert space of QCA is called the Hilbert space of finite configurations, and is defined through first constructing an orthonormal basis for it over the lattice ($\mathbb{Z}^n$) as a whole, and then completing it under an appropriate norm. An element of this basis is specified by a tensor product, over the lattice, of individual cells’ states. Each cell’s state is drawn from the same orthonormal basis $B$ of $W$ containing the quiescent state $|q\rangle$. Only finitely many tensor factors (cell states) are allowed to be in active states, while the rest are in quiescent state. This basis is called the set of finite configurations, denoted by $C_q$:

\[ C_q = \{ \bigotimes_{x \in \mathbb{Z}^n} |c_x\rangle : |c_x\rangle \in B, \text{ all but finite } |c_x\rangle = |q\rangle \} \tag{1} \]

The Hilbert space of finite configurations, denoted by $H_q$, is then the completion of the span of $C_q$ under the inner-product norm on $C_q$ induced from the inner product on $W$.

The state of a QCA is a density operator on $H_q$, a positive trace class operator with trace 1. For the measurements that only concern a finite subset $D \subset \mathbb{Z}^n$ of cells, the density operator can be restricted to $D$. This restriction is obtained by a partial trace over cells (tensor factors) not in $D$. For that, let us write the Hilbert space $H_q$ as a tensor product of two Hilbert spaces, \[ H_q = \bigotimes_{j \in D} W_j \otimes H_D, \]

\[ H_D \]

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4We refer to both quantum cellular automaton and quantum cellular automata by QCA.

5We refer to both quantum lattice gas automaton and quantum lattice gas automata by QLGA.

6Completion in the inner product norm induced by those on the two tensor factors is assumed.
where $\mathcal{H}_D$ is the Hilbert space complement to the tensor factors in $D$. Let $\{|w_\alpha\rangle\}$ be some orthonormal basis of $\mathcal{H}_D$. The restricted density operator is calculated by first writing $\rho$ as

$$\rho = \sum_{\alpha, \beta} \rho_{\alpha, \beta} \otimes \langle w_\alpha | w_\beta \rangle$$

where $\rho_{\alpha, \beta} \in \text{End}(\bigotimes_{j \in D} W_j)$. Then $\rho$ restricted to $D$ is

$$\rho|_D := \sum_\alpha \rho_{\alpha, \alpha}.$$

The state of a cell in a QCA after a time step of evolution depends on the state of its neighboring cells prior to the time step. Since the evolution is spatially homogeneous, the positions of the neighbors of any cell with respect to the cell can be given by the same finite neighborhood $\mathcal{N} \subset \mathbb{Z}^n$. The cells in the neighborhood of cell $z \in \mathbb{Z}^n$, denoted by $\mathcal{N}_z$, are given by simply translating $\mathcal{N}$ to $z$.

$$\mathcal{N}_z = z + \mathcal{N} = \{z + k : k \in \mathcal{N}\}$$

From the templates of QCA under consideration, we first describe the Schrödinger template (on the lattice $\mathbb{Z}^n$). It is a slight generalization of the definition in [3] with respect to spatial homogeneity, i.e., translation-invariance. Let $W$ be a finite dimensional Hilbert space, $\mathcal{N}$ be a neighborhood, and $|q\rangle$ be some vector in $W$.

**Definition 2.1.** The Schrödinger template of a QCA is given by the Hilbert space of finite configurations $\mathcal{H}_q$, and a unitary operator $R$ on $\mathcal{H}_q$. $R$ is called the global evolution operator, and is required to be:

(i) *Projectively translation-invariant:* A translation operator $\tau_z$, for some $z \in \mathbb{Z}^n$, is a linear operator on $\mathcal{H}_q$, defined on a basis element $|c\rangle = \bigotimes_{j \in \mathbb{Z}^n} |c_j\rangle \in \mathcal{C}_q$ as:

$$\tau_z : \bigotimes_{j \in \mathbb{Z}^n} |c_j\rangle \mapsto \bigotimes_{j \in \mathbb{Z}^n} |c_{j+z}\rangle$$

$R$ is projectively translation-invariant if, for all $z \in \mathbb{Z}^n$, $\tau_z R \tau_z^{-1} = e^{i\theta_z} R$, where $\theta_z$ depends on $z$.

(ii) *Causal relative to some neighborhood $\mathcal{N}$:* $R$ is causal relative to the neighborhood $\mathcal{N}$ if for all $z \in \mathbb{Z}^n$ and every pair $\rho, \rho'$ of density operators on $\mathcal{H}_q$ satisfying

$$\rho|_{\mathcal{N}_z} = \rho'|_{\mathcal{N}_z},$$

the operators $R\rho R^\dagger, R\rho' R^\dagger$ satisfy

$$R\rho R^\dagger|_z = R\rho' R^\dagger|_z$$

The state of the QCA evolves by conjugation by $R$. If the state of QCA is $\rho$ before a time step, then after the time step it is $R\rho R^\dagger$.

Since global phase factors have no effect on measurements, we are justified in the following definition.

**Definition 2.2.** Schrödinger templates (Definition 2.1) given by $(\mathcal{H}_q, R)$ and $(\mathcal{H}_q, R')$ are equivalent if $R = e^{i\theta} R'$ for some $\theta \in \mathbb{R}$. The equivalence class of $R$ will be denoted by $[R]$. 
Using this equivalence, the Schrödinger template of a QCA can be expressed as a pair \((H_q, [R])\), where \(R\) satisfies the conditions above.

The Heisenberg template of a QCA is described over an algebra of operators called the \textit{Infinite (incomplete) Tensor Product Algebra} (ITPA), a subalgebra of \(B(H_q)\) (bounded linear operators on \(H_q\)). This algebra is constructed from operators deemed to be \textit{local}, acting as identity on all but a finite set of cells.

**Definition 2.3.** For a finite subset \(D \in \mathbb{Z}^n\), the algebra of operators local upon \(D\) is \(\mathcal{A}_D = \bigotimes_{j \in D} \text{End}(W) \otimes \bigotimes_{k \in \mathbb{Z}^n \setminus D} \mathbb{I}\).

**Remark 2.4.** We write \(\mathcal{A}_z\) to mean \(\mathcal{A}_{\{z\}}\), the \textit{cell algebra} of operators local upon one tensor factor \(z\).

To construct the ITPA, we take an ascending chain of finite subsets \(\{D_k \subset \mathbb{Z}^n : |D_k| < \infty\}_{k \in \mathbb{N}}\) whose union is the entire lattice, i.e.,

\[
D_0 \subsetneq D_1 \subsetneq \ldots
\]

such that: \(\mathbb{Z}^n = \bigcup_{k \in \mathbb{N}} D_k\). The algebra of operators obtained by taking the corresponding union of ascending chain of local algebras is the ITPA, denoted \(\mathcal{Z}_W\).

\[
\mathcal{Z}_W = \bigcup_k \mathcal{A}_{D_k}.
\]

We note that \(\mathcal{Z}_W\) is a \(*\)-algebra since each of its constituent cell algebras, \(\mathcal{A}_z = \text{End}(W)\), is a \(*\)-algebra with the adjoint map on operators as its \(*\)-involution,

\[
*: \text{End}(W) \to \text{End}(W)
\]

\[
a \mapsto a^\dagger.
\]

This map extends to a \(*\)-involution on \(\mathcal{Z}_W\).

The following useful result (Theorem 3.9 from [5]) says that conjugation of \(\mathcal{Z}_W\) by \(R\) is an automorphism of \(\mathcal{Z}_W\).

**Theorem 2.5.** Let \(R\) be a unitary and causal map on \(H_q\) relative to some neighborhood \(\mathcal{N}\). Then for every \(x \in \mathbb{Z}^n\),

\[
\mathcal{A}_x \subset \text{span}(\prod_{k \in \mathcal{N}} R^\dagger \mathcal{A}_{x-k} R)
\]

In particular \(\mathcal{Z}_W = R^\dagger \mathcal{Z}_W R\).

Let this automorphism be \(\gamma_{[R]}\),

\[
\gamma_{[R]} : \mathcal{Z}_W \to \mathcal{Z}_W
\]

\[
b \mapsto R^\dagger b R
\]

\(\gamma_{[R]}\) is a \(*\)-automorphism as it commutes with the \(*\)-involution of \(\mathcal{Z}_W\) (adjoint map).

Developing along the direction of Arrighi, Nesme, and Werner [3], we state a theorem from [3] deriving the counterpart of causality in the Heisenberg template from that in the Schrödinger template. To state the theorem, we define the reflected neighborhood of cell \(z\), \(\mathcal{V}_z\),

\[
\mathcal{V}_z = z - \mathcal{N}
\]

**Theorem 2.6** (Structural Reversibility). Let \(R : H_q \to H_q\) be a unitary operator and \(\mathcal{N}\) a neighborhood. Then the following are equivalent:
(i) \( R \) is causal relative to the neighborhood \( \mathcal{N} \).
(ii) For every operator \( A_z \) local upon cell \( z \), \( R^\dagger A_z R \) is local upon \( \mathcal{N}_z \).
(iii) \( R^\dagger \) is causal relative to the neighborhood \( \mathcal{V} \).
(iv) For every operator \( A_z \) local upon cell \( z \), \( RA_z R^\dagger \) is local upon \( \mathcal{V}_z \).

Based on the constructs just introduced, the Heisenberg template of a QCA (on the lattice \( \mathbb{Z}^n \)) can be stated. Let \( W \) be a finite dimensional Hilbert space, and let \( \mathcal{N} \subset \mathbb{Z}^n \) be a neighborhood.

**Definition 2.7.** The Heisenberg template of a QCA is given by the ITPA \( Z_W (3) \), and a \( * \)-automorphism \( \gamma \) of \( Z_W \). \( \gamma \) is required to be:

(i) **Translation-invariant**: A translation operator \( \mu_z \), for some \( z \in \mathbb{Z}^n \), is a linear operator on \( Z_W \), defined on a basis element \( b = \bigotimes_{j \in \mathbb{Z}^n} b_j \in Z_W \) as:

\[
\mu_z : \bigotimes_{j \in \mathbb{Z}^n} b_j \mapsto \bigotimes_{j \in \mathbb{Z}^n} b_{j+z}
\]

\( \gamma \) is translation-invariant if \( \mu_z \gamma \mu_z^{-1} = \gamma \) for all \( z \in \mathbb{Z}^n \).

(ii) **Causal relative to some neighborhood \( \mathcal{N} \)**: For every element \( A_z \in Z_W \) local upon \( z \), \( \gamma(A_z) \) is local upon \( \mathcal{N}_z \).

At each time step of evolution, \( b \in Z_W \) evolves to \( \gamma(b) \).

Summarizing, the Heisenberg template can be given as a pair \( (Z_W, \gamma) \) satisfying the conditions above. As \( \gamma_{[R]} (4) \) is a specific example of the automorphism \( \gamma \) in the Heisenberg template, we need to show that it is translation-invariant in the Heisenberg template sense. This follows because \( R \) is projectively translation-invariant, and because for all \( b \in Z_W \),

\[
\mu_z(b) = \tau_z b \tau_z^{-1},
\]

where \( \mu_z \) is the translation operator in the Heisenberg template, and \( \tau_z \) is the translation operator in the Schrödinger template (Definition 2.1 (i)). Consequently,

\[
\mu_z \gamma_{[R]} \mu_z^{-1}(b) = \tau_z R^\dagger \tau_z^{-1} b \tau_z R \tau_z^{-1} = e^{-i\theta_z} R^\dagger b e^{i\theta_z} R = R^\dagger b R = \gamma_{[R]}(b).
\]

So \( \gamma_{[R]} \) is translation-invariant.

### 3. Relating the Schrödinger and Heisenberg Templates of QCA

We are now ready to address the relation between the Schrödinger and Heisenberg templates. From the previous section, it is clear that given a global evolution operator \( [R] \) on \( \mathcal{H}_q \), a \( * \)-automorphism \( \gamma_{[R]} \) of \( Z_W (3) \) always exists (as in eq. (4)). This implies that given a Schrödinger template of QCA (Definition 2.1), we can always obtain the Heisenberg template (Definition 2.7).

Given the Heisenberg template of a QCA, the condition that determines if a dynamically equivalent Schrödinger template exists is best phrased in the language of representations.
The obvious action of $Z_W$ on the basis $C_q$ \[1\] of $H_q$, 

$$Z_W \times C_q \mapsto C_q$$

$$\left( \bigotimes_i A_i, \bigotimes_i |c_i\rangle \right) \mapsto \bigotimes_i A_i(|c_i\rangle),$$

extends to an irreducible representation on $H_q$. We denote this canonical representation $(\pi, H_q)$.

**Theorem 3.1** (Template Correspondence). A Heisenberg template of a QCA (Definition 2.7) given by $(Z_W, \gamma)$ admits a Schrödinger template (Definition 2.1) if and only if there is a Hilbert space of finite configurations $H_q$ and a unitary operator $R$ on $H_q$ that intertwines the representations $(\pi, H_q)$ and $(\pi \gamma, H_q)$ of $Z_W$, 

$$\pi(b)R = R\pi(\gamma(b)) \quad \forall b \in Z_W.$$ 

That is 

$$R^\dagger \pi(b)R = \pi(\gamma(b)) \quad \forall b \in Z_W. \quad (5)$$

The Schrödinger template is given by $(H_q, [R])$, and is unique.

**Proof.** $R$ is unique up to a phase factor by Schur’s Lemma [11] as the representations $(\pi, H_q)$ and $(\pi \gamma, H_q)$ are irreducible. This implies there is a unique $[R]$. We also need to show:

(i) $[R]$ is projectively translation-invariant: Observe that for all $b \in Z_W$ and all $z \in \mathbb{Z}^n$, 

$$\pi(\mu_z(b)) = \tau_z \pi(b) \tau_z^{-1},$$

which implies 

$$\pi(\mu_z \gamma \mu_z^{-1}(b)) = \tau_z \pi(\gamma \mu_z^{-1}(b)) \tau_z^{-1}$$

$$= \tau_z R^\dagger \pi(\mu_z^{-1}(b)) R \tau_z^{-1}$$

$$= \tau_z R^\dagger \tau_z^{-1} \pi(b) \tau_z R \tau_z^{-1}.$$ 

Since $\gamma$ is translation-invariant, 

$$\pi(\gamma(b)) = \pi(\mu_z \gamma \mu_z^{-1}(b))$$

$$= \tau_z R^\dagger \tau_z^{-1} \pi(b) \tau_z R \tau_z^{-1}.$$ 

This implies that $\tau_z R \tau_z^{-1}$ intertwines the representations $(\pi, H_q)$ and $(\pi \gamma, H_q)$. But $R$ is the unique intertwiner up to a phase factor, so 

$$\tau_z R \tau_z^{-1} = e^{i\theta_z} R.$$ 

for some $\theta_z \in \mathbb{R}$. Thus $[R]$ is projectively translation-invariant.

(ii) If $\gamma$ is causal relative to the neighborhood $\mathcal{N}$, then $[R]$ is causal relative to the same neighborhood $\mathcal{N}$: This follows by the equivalence of Theorem 2.6(i) and (ii). \hfill \qed
4. Application of template correspondence to the structure theorem of quantum lattice gas automata

Using the result on template correspondence from the previous section, we simplify the proof of the main theorem in [5], Theorem III.16, by which a local condition on a QCA characterizes a QLGA.

Recall the definition of a quantum lattice gas automaton (QLGA) as given in [5]. A QLGA is defined as a Schrödinger template. It is based on a particular type of Hilbert space of finite configurations and an evolution operator that consists of particles propagating and scattering on interacting with lattice sites and each other. We take the definition of a QLGA from [5], defined on a lattice $\mathbb{Z}^n$, with a neighborhood $N$. It is defined to be:

(i) The cell Hilbert space is $W = \bigotimes_{z \in N} V_z$, for some finite-dimensional vector spaces $\{V_z\}_{z \in N}$.

(ii) The quiescent state $|q\rangle$ is a simple product:

$$|q\rangle = \bigotimes_{z \in N} |q_z\rangle,$$

where $|q_z\rangle \in V_z$.

(iii) A Hilbert space of finite configurations $H_C$ defined in terms of $W = \bigotimes_{z \in N} V_z$ and $|q\rangle$.

(iv) A propagation operator relative to the neighborhood $N$, $\sigma : H_C \mapsto H_C$, defined on the basis of $H_C$ to be

$$\sigma : \bigotimes_{x \in \mathbb{Z}^n} \bigotimes_{z \in N} |k_x(z)\rangle \mapsto \bigotimes_{x \in \mathbb{Z}^n} \bigotimes_{z \in N} |k_{x+z}(z)\rangle$$

(v) A local collision operator $F$, which is a unitary operator on the site Hilbert space $W = \bigotimes_{z \in N} V_z$, such that $F$ fixes $|q\rangle$ (an eigenvector with eigenvalue one): $F|q\rangle = |q\rangle$. The global collision operator $\hat{F} : H_C \mapsto H_C$, is the application of $F$ at every cell, defined as

$$\hat{F} = \bigotimes_{x \in \mathbb{Z}^n} F$$

(vi) A global evolution operator $R$ consisting of propagation $\sigma$ followed by the collision $\hat{F}$:

$$R = \hat{F}\sigma$$

A state of the QLGA is a vector in the Hilbert space of finite configurations.

First, let us recall the terminology of [5], but generalized to an arbitrary Heisenberg template automorphism $\gamma$. The patch of propagated image $\gamma(A_z)$ on $A_x$, is

$$D_{z,x} = \gamma(A_z) \cap A_x$$

We restate Theorem III.10 of [5], as we will refer to it.

**Theorem 4.1** (Theorem III.10 in [5]). Suppose that $R$ is the global evolution of a QCA with neighborhood $N$. Then $A_x = \text{span}(\prod_{y \in N} D_{x-y,x})$ if and only if there exists an isomorphism of vector spaces:

$$S : W \longrightarrow \bigotimes_{z \in N} V_z$$

for some vector spaces $\{V_z\}_{z \in N}$. Under the isomorphism $S$, for each $y \in N$:

$$D_{x-y,x} \cong \text{End}(V_y) \bigotimes_{z \in N, z \neq y} 1_{V_z}$$
We state and prove the direction of the main theorem in [5], Theorem III.16, in which a local condition on the Heisenberg template of a QCA implies that it has a Schrödinger template. Moreover, it is a QLGA (upto a global isomorphism).

**Theorem 4.2** (Theorem III.16 in [5]). Let \((\mathcal{Z}_W, \gamma)\) be the Heisenberg template of a QCA with neighborhood \(\mathcal{N}\), satisfying: \(\mathcal{A}_x = \text{span}(\prod_{y \in \mathcal{N}} D_{x-y,x})\). Then it admits a Schrödinger template which is a QLGA.

**Proof.** The condition \(\mathcal{A}_x = \text{span}(\prod_{y \in \mathcal{N}} D_{x-y,x})\), implies, by Theorem 4.1, the existence of \(S\) (unitary under an appropriate inner product choice) giving the isomorphism (10) and (11). We replace \(W\) with \(\tilde{W} = S(W)\) and take the quiescent state \(|\tilde{q}\rangle \in \tilde{W} = \bigotimes_{z \in \mathcal{N}} V_z\) to be a product,

\[
|\tilde{q}\rangle = \bigotimes_{z \in \mathcal{N}} |\tilde{q}_z\rangle, \quad \text{where } |\tilde{q}_z\rangle \in V_z.
\]

Construct the Hilbert space of finite configurations \(\mathcal{H}_q\) from \(\tilde{W}\) and quiescent state \(|\tilde{q}\rangle\). This is isomorphic to the Hilbert space of finite configurations constructed from \(W\) and the quiescent state \(|q\rangle = S^{-1}(|\tilde{q}\rangle)\), \(\mathcal{H}_q\). The map \(\tilde{S}\) on \(\mathcal{H}_q\),

\[
\tilde{S} = \bigotimes_{x \in \mathbb{Z}^n} S,
\]

(12)
gives that isomorphism, i.e., \(\mathcal{H}_q = \tilde{S}(\mathcal{H}_q)\).

Let the representation \((\tilde{\pi}, \mathcal{H}_q)\) of \(\mathcal{Z}_W\) be defined by

\[
\tilde{\pi}(b) = \tilde{S}\pi(b)\tilde{S}^{-1}, \quad b \in \mathcal{Z}_W
\]

Define a propagation operator \(\sigma\) as in (7).

\[
\sigma : \bigotimes_{x \in \mathbb{Z}^n} \bigotimes_{z \in \mathcal{N}} |k_x(z)\rangle \mapsto \bigotimes_{x \in \mathbb{Z}^n} \bigotimes_{z \in \mathcal{N}} |k_{x+z}(z)\rangle
\]

Let the algebra of \(\mathbb{A}_x = SA_xS^{-1} = \text{End}(\tilde{W})\). Then \(\tilde{\pi}\gamma(\mathbb{A}_x) = \sigma^{-1}\tilde{\pi}(\mathbb{A}_x)\sigma\). This implies \(\sigma\tilde{\pi}\gamma(\mathbb{A}_x)\sigma^{-1}\) is an automorphic image of \(\tilde{\pi}(\mathbb{A}_x)\). But \(\tilde{\pi}(\mathbb{A}_x) = \mathbb{A}_x = \text{End}(\tilde{W})\). Thus there is a unitary map \(F\) on \(\tilde{W}\) (Schur’s Lemma applied to \(\mathbb{A}_x\) action on \(\tilde{W}\)), such that \(\sigma\tilde{\pi}\gamma(\mathbb{A}_x)\sigma^{-1} = F^{-1}\tilde{\pi}(\mathbb{A}_x)F\) (considering index \(x\)). By translation invariance of \(\gamma\), the same \(F\) works for every cell. This cell-wise automorphism is achieved by \(\hat{F}\) constructed from \(F\) as in (8),

\[
\hat{F} = \bigotimes_{x \in \mathbb{Z}^n} F,
\]

where \(F\) is required to fix \(|\tilde{q}\rangle\), i.e., \(F|\tilde{q}\rangle = |\tilde{q}\rangle\), so that \(\hat{F}\) is defined on \(\mathcal{H}_q\). As \(\mathbb{A}_x, x \in \mathbb{Z}^n\), generate \(\mathcal{Z}_W\),

\[
\sigma\tilde{\pi}\gamma(b)\sigma^{-1} = \hat{F}^{-1}\tilde{\pi}(b)\hat{F}
\]

for all \(b \in \mathcal{Z}_W\). Rewriting the above relation, we obtain

\[
\tilde{\pi}\gamma(b) = \sigma^{-1}\hat{F}^{-1}\tilde{\pi}(b)\hat{F}\sigma.
\]

Substituting the definition of \(\tilde{\pi}\),

\[
\tilde{S}\pi\gamma(b)\tilde{S}^{-1} = \sigma^{-1}\hat{F}^{-1}\tilde{S}\pi(b)\tilde{S}^{-1}\hat{F}\sigma.
\]

This implies

\[
\pi\gamma(b) = \tilde{S}^{-1}\sigma^{-1}\hat{F}^{-1}\tilde{S}\pi(b)\tilde{S}^{-1}\hat{F}\sigma.
\]
Thus \[ R = \tilde{S}^{-1} \hat{F} \sigma \tilde{S} \]
intertwines \( (\pi, \mathcal{H}_q) \) and \( (\pi \gamma, \mathcal{H}_q) \). By Theorem 3.1 [R] is the unique such global evolution operator. The Schrödinger template of this QCA is \( (\mathcal{H}_q, [R]) \). Indeed it is a QLGA, as up to a global isomorphism of \( \tilde{S} \), it is composed of the propagation operator \( \sigma \) followed by the collision operator \( \hat{F} \). □

5. Conclusion

The discussion in this paper frames the relation between the Schrödinger and Heisenberg templates of QCA in the broader sense of identifying classes of representations of ITPA. Among important constructions of QCA expressed in the Heisenberg picture are the Clifford QCA (CQCA) and their generalizations, by Schlingemann, Vogts, and Werner in [10]. A fundamental case in which the Heisenberg template, an automorphism of the ITPA, admits a Schrödinger template, i.e., a global evolution operator on a Hilbert space of finite configurations, is that of QLGA [5], discussed in the previous section. More complex QCA in the Schrödinger template are constructed in [12]. These QCA are composed of QLGA, but unlike QLGA, have no particle description at the time scale at which the dynamics are homogeneous.

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8ITPA is encountered in discussions of hyperfinite \( II_1 \) factor, such as in Jones and Sunder [9]. In this paper, the representations of ITPA being alluded to are the *-algebra representations in the specific context of QCA.
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