ON THE ORTHOGONALITY OF MEASURES OF DIFFERENT SPECTRAL TYPE WITH RESPECT TO EBERLEIN CONVOLUTION

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Abstract. In this paper we show that under suitable conditions on their Fourier–Bohr coefficients, the Eberlein convolution of a measure with pure point diffraction spectra and a measure with continuous diffraction spectra is zero.

1. Introduction

Born from the discovery of quasicrystals [43], the field of Aperiodic Order is focused on the study of objects which show long range order, typically in the form of a large Bragg diffraction spectrum, but lack translational symmetry.

As introduced by Hof [22], mathematical diffraction is defined as follows. Given an object, typically a point-set \( \Lambda \) denoting the positions of atoms in an idealized solid, or more generally a translation bounded measure \( \omega \), its autocorrelation \( \gamma \) is defined as the vague limit

\[
\gamma = \lim_{n} \frac{1}{|A_n|} \omega_n * \overline{\omega_n},
\]

where \( A_n \) is a nice (van Hove) sequence and \( \omega_n \) denotes the restriction of \( \omega \) to \( A_n \). Here, when \( \Lambda \) is a pointset, we use \( \omega = \delta_\Lambda \) in (1.1) to define its autocorrelation. By eventually replacing \( A_n \) by a subsequence, the autocorrelation can always be assumed to exist [11], and is positive definite. Therefore, its Fourier transform \( \hat{\gamma} \) exists and is a positive measure [1, 15, 37]. The measure \( \hat{\gamma} \) is called the diffraction measure of \( \Lambda \) and \( \omega \), respectively. As any translation bounded measure on the second countable locally compact Abelian group(LCAG) \( \hat{G} \), it has a (unique) Lebesgue decomposition

\[
\hat{\gamma} = (\hat{\gamma})_{pp} + (\hat{\gamma})_{ac} + (\hat{\gamma})_{sc}
\]

into a pure point measure \( (\hat{\gamma})_{pp} \), a measure \( (\hat{\gamma})_{ac} \) which is absolutely continuous with respect to the Haar measure \( \theta_\hat{G} \) and a measure \( (\hat{\gamma})_{sc} \) which is singular continuous with respect to the Haar measure \( \theta_\hat{G} \).

Systems with pure point spectrum, meaning \( (\hat{\gamma})_{c} = 0 \), are now relatively well understood. Pure point diffraction was classified via dynamical spectra [24, 11, 19].

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via the almost periodicity of the autocorrelation measure \[23, 12, 20, 37\]. More recently, generalizing previous work in this direction \[45, 44, 12, 36, 20\], pure point diffraction was shown to be equivalent to mean almost periodicity of the underlying structure \[26, 27\].

The Eberlein decomposition of the autocorrelation measure plays an important role in the study of diffraction of structures. Indeed the autocorrelation $\gamma$ has an unique decomposition $\gamma = \gamma_s + \gamma_0$ into two Fourier transformable measures\[23, 37\] such that

$$\hat{\gamma}_s = (\hat{\gamma})_{pp},$$

$$\hat{\gamma}_0 = (\hat{\gamma})_c.$$

This decomposition allows us study the pure point and continuous diffraction spectrum, respectively, in the real space by studying the two components $\gamma_s$ and $\gamma_0$ of $\gamma$. This approach proved effective in the study of the diffraction spectra of measures with Meyer set support \[48, 49, 50, 51, 53\] and the diffraction of compatible random substitutions in 1 dimension \[33, 13, 46\]. For Meyer sets, one can further decompose $\gamma_0 = \gamma_{0s} + \gamma_{0a}$ in such a way that the Fourier transforms of $\gamma_{0s}$ and $\gamma_{0a}$ are $(\hat{\gamma})_{sc}$ and $(\hat{\gamma})_{ac}$, respectively \[51, 53\].

For many examples of compatible random 1-dimensional Pisot substitutions, one gets a decomposition of the generic element $\omega$ of the hull into two measures $\omega_1$ and $\omega_2$ such that the diffraction of $\omega_1$ and $\omega_2$, respectively, are the pure point and continuous diffraction spectrum, respectively, of $\omega$ \[33, 13, 46\]. A similar decomposition hold for 1-dimensional PV substitutions \[14\] and for dynamical systems of translation bounded measures \[2\]. It is the one of the goals of this paper to investigate this type of decomposition, at the level of measures and not autocorrelations, in more general settings.

We will approach this question from a different angle. If $\omega = \omega_1 + \omega_2$ is such a potential decomposition, and $A$ is any van Hove sequence, then there exist a subsequence $A'$ of $A$ so that (see \[3.2\])

$$\gamma = \gamma_1 + \gamma_2 + \omega_1 \otimes_{A'} \omega_2 + \omega_2 \otimes_{A'} \omega_1,$$

where $\gamma, \gamma_1, \gamma_2$ are the autocorrelations of $\omega, \omega_1, \omega_2$, respectively, with respect to $A'$ and $\omega_1 \otimes_{A'} \omega_2, \omega_2 \otimes_{A'} \omega_1$ denote the Eberlein convolutions (see Def. \[3.1\]). Using the relation

$$\omega_2 \otimes_{A'} \omega_1 = \omega_1 \otimes_{A'} \omega_2,$$

we obtain such a decomposition whenever when $\omega_1$ is pure point diffractive, $\omega_2$ has continuous diffraction spectrum and, the following orthogonality with respect to the Eberlein decomposition holds:

$$\omega_1 \otimes_{A} \omega_2 = \omega_2 \otimes_{A} \omega_1 = 0.$$

Note here also that for positive measures $\omega_1, \omega_2$, \(1.2\) implies that $\gamma = \gamma_1 + \gamma_2$ is equivalent to \(1.3\).
The main goal of the article is to prove that, under suitable conditions, if \( \omega_1 \) has pure point diffraction spectrum and \( \omega_2 \) has continuous diffraction spectrum, then (1.3) holds. In particular, the pure point spectrum of \( \omega = \omega_1 + \omega_2 \) is the spectrum of \( \omega_1 \) and the continuous spectrum of \( \omega \) is the spectrum of \( \omega_2 \). We prove few results in this direction.

First, in Thm. 5.4 we show that if \( \omega_1 \) has pure point diffraction spectrum, \( \omega_2 \) has continuous diffraction spectrum, and the Fourier–Bohr coefficients of \( \omega_2 \) exist uniformly then (1.3) holds. Next, in Cor. 5.7 we show that then (1.3) also holds if \( \omega_1 \) has pure point diffraction spectrum, \( \omega_2 \) has continuous diffraction spectrum, the Fourier–Bohr coefficients of \( \omega_1 \) exist uniformly and the Consistent Phase Property (CPP) (see Def. 2.12) holds for \( \omega_2 \). We continue by restricting our attention to the case when \( \omega_1 \) is Besicovitch almost periodic. In Cor. 6.8 we show that if \( \omega_1 \) is Besicovitch almost periodic with respect to \( A \), \( \omega_2 \) has continuous diffraction spectrum with respect to \( A \) and the CPP holds for \( \omega_2 \) then (1.3) holds. Finally, in Thm. 6.9 we show that if \((X,G,m)\) and \((Y,G,n)\) are dynamical systems of translation bounded measures then (1.3) holds for \( m \)-almost all \( \omega_1 \) and \( n \)-almost all \( \omega_2 \). In particular, a generalized Eberlein decomposition holds for dynamical systems of translation bounded measures such that (1.3) holds almost surely (Thm. 6.11).

The Fourier–Bohr coefficients are a central tool we use in many proofs, and along the way we prove various properties for the Fourier–Bohr coefficients of the Eberlein convolution, which are interesting by themselves. First, we show that whenever when the Eberlein convolution \( \mu \ast \tilde{\nu} \) exist, the Fourier–Bohr coefficients \( c^A_\chi(\mu \ast \tilde{\nu}) \) exist uniformly. Moreover, in Thm. 4.5 we show that, if the Fourier–Bohr coefficients \( c^A_\chi(\mu) \) exist uniformly, then

- For all \( \chi \) with \( c^A_\chi(\mu) = 0 \) we have \( c^A_\chi(\mu \ast \tilde{\nu}) = 0 \).
- For all \( \chi \) with \( c^A_\chi(\mu) \neq 0 \), the Fourier–Bohr coefficient \( c^A_\chi(\nu) \) exists and \( c^A_\chi(\mu \ast \tilde{\nu}) = c^A_\chi(\mu)c^A_\chi(\nu) \).

As an immediate consequence we get that any measure \( \mu \) with uniform Fourier–Bohr coefficients satisfies CPP, a result which was previously proven in \( \mathbb{R}^d \) by [22] and for dynamical systems of translation bounded measures in [20].

The paper is organized as follows. In Section 2 we review the basic definitions and concepts we are using in this paper, while in Section 3 we review the Eberlein convolution of measures and its properties. In Section 4 we study the connection between the Fourier–Bohr coefficients of the Eberlein convolution \( \mu \ast \tilde{\nu} \) and the Fourier–Bohr coefficients of \( \mu \) and \( \nu \). We complete the paper by providing in Section 5 and Section 6 necessary conditions for (1.3) to hold.

## 2. Preliminaries

In this section we briefly review the background information needed in this paper. Throughout the paper \( G \) is a second countable locally compact Abelian group (LCAG). We denote by \( \hat{G} \) the dual group of \( G \).

For a Borel set \( A \subseteq G \) we denote the Haar measure of \( A \) by \( |A| \).
2.1. Functions and measures. Next, we look at functions and measures on $G$. For a more detailed background we recommend [40], [38] and [39, Appendix A-C].

We denote by $C_u(G)$ the space of uniformly continuous bounded functions on $G$ and by $C_c(G)$ the subspace of $C_u(G)$ consisting of compactly supported continuous functions. $C_u(G)$ is a Banach space with respect to $\| \cdot \|_{\infty}$.

For a function $f : G \to C$ and $t \in G$ we denote by $\tilde{f}$, $f^\dagger$ and $T_t f$ the functions

$$\tilde{f}(x) = \overline{f(-x)}; \quad f^\dagger(x) = f(-x); \quad T_t f(x) = f(x-t) .$$

If at least one of $f, g \in C_u(G)$ has compact support, we can define the convolution $f \ast g \in C_u(G)$ via

$$f \ast g(t) = \int_G f(s)g(t-s)ds = \int_G f(t-r)g(r)dr .$$

Finally, for $\varphi \in C_c(G)$, its Fourier transform $\hat{\varphi}$ and inverse Fourier transform $\check{\varphi}$ are the functions on $\hat{G}$ defined by

$$\hat{\varphi}(\chi) = \int_G \overline{\chi(t)} \varphi(t)dt$$

$$\check{\varphi}(\chi) = \int_G \chi(t) \varphi(t)dt .$$

Next, we review the concept of Radon measures.

By a Radon measure we understand a linear function $\mu : C_c(G) \to C$ with the property that for each compact set $K \subseteq C$, $\mu$ is a bounded operator on the Banach space $(C(G : K), \| \cdot \|_{\infty})$. Here

$$C(G : K) := \{ \varphi \in C_c(G) : \text{supp}(\varphi) \subseteq K \} .$$

We will refer to a Radon measure simply as a measure. It is easy to see that a linear function $\mu : C_c(G) \to C$ is a measure if and only if for each compact set $K \subseteq G$ there exists a constant $C_K$ such that for all $\varphi \in C_c(G)$ with $\text{supp}(\varphi) \subseteq K$ we have

$$|\mu(\varphi)| \leq C_K \| \varphi \|_{\infty} .$$

By the Riesz–Representation Theorem [39, Thm. C4] (compare [40, Thm. 2.14] and [40, Thm. 6.19] for finite and positive measures), a Radon measure is simply a linear combination of positive regular Borel measures.

We will often write $\int_G \varphi(t)d\mu(t)$ instead of $\mu(\varphi)$.

A measure $\mu$ is called positive if for all $\varphi \in C_c(G)$ with $\varphi \geq 0$ we have $\mu(\varphi) \geq 0$.

Next, for a measure $\mu$ and $t \in G$ we denote by $\hat{\mu}$, $\mu^\dagger$, and $T_t \mu$ the measures given by

$$\hat{\mu}(\varphi) = \overline{\mu(\varphi)}; \quad \mu^\dagger(\varphi) = \mu(\varphi^\dagger); \quad T_t \mu(\varphi) = \mu(T_{-t}\varphi) \forall \varphi \in C_c(G) .$$

If $\mu$ is a measure and $\varphi \in C_c(G)$ we define

$$\varphi \ast \mu(t) = \int_G \varphi(t-s)d\mu(s) .$$

Next, given two measures $\mu, \nu$, at least one of which has compact support, we can define a new measure $\mu \ast \nu$, called the convolution of $\mu$ and $\nu$ via

$$\mu \ast \nu(\varphi) = \int_G \int_G \varphi(s+t)d\mu(s)d\nu(t) .$$
By Fubini Theorem we have $\mu \ast \nu = \nu \ast \mu$.

Next, we review the concept of translation bounded measures, which will play a central role in the paper. Recall that each measure induces a positive measure $|\mu|$, called the total variation of $\mu$ such that for all $\varphi \in C_c(G)$ with $\varphi \geq 0$ we have (compare [38] (compare [39], Lemma C2))

$$|\mu|(\varphi) = \sup\{|\mu(\psi)| : \psi \in C_c(G), |\psi| \leq \varphi\}.$$

We can now define the concept of translation bounded measure.

**Definition 2.1.** A measure $\mu$ is called translation bounded if for all compact sets $K \subseteq G$ we have

$$\|\mu\|_K := \sup\{|\mu|(t + K) : t \in G\} < \infty.$$  

We denote the space of translation bounded measures on $G$ by $\mathcal{M}^\infty(G)$.

**Remark 2.2.** A measure $\mu$ is translation bounded if and only if for all $\varphi \in C_c(G)$ we have $\mu \ast \varphi \in C_c(G)$ [1, 37]. Equivalently, $\mu$ is translation bounded if and only if (2.1) holds for one compact set $K$ with non-empty interior [12], or equivalently, if and only if (2.1) holds for one precompact Borel set $W$ with non-empty interior [47]. Moreover any two precompact Borel sets with non-empty interior define equivalent norms on $\mathcal{M}^\infty(G)$.

All the measures we are using in this paper are translation bounded.

### 2.2. Averaging sequences and the mean

In this section we recall few basic results on averages of functions and measures.

First, let us recall that a sequence $\{A_n\}_n$ of compact sets in $G$ is called a Følner sequence if for all $t \in G$ we have

$$\lim_{n} \frac{|A_n \Delta (t + A_n)|}{|A_n|} = 0.$$  

$\{A_n\}_n$ is called a van Hove sequence if, for each compact set $K \subseteq G$, we have

$$\lim_{n \to \infty} \frac{|\partial^K A_n|}{|A_n|} = 0,$$

where the $K$-boundary $\partial^K A$ of a set $A \subseteq G$ is defined as

$$\partial^K A := \overline{(A \cap K) \setminus A \cup \{(G \setminus A) \setminus K) \cap A\}.$$  

Every van Hove sequence is a Følner sequence. Any $\sigma$-compact LCAG admits a van Hove sequence [42]. In fact, a LCAG $G$ admits a van Hove (or equivalently a Følner) sequence if and only if $G$ is $\sigma$-compact [47].

Let us emphasize here that, when working with functions, one only needs the Følner condition. When dealing with measures, the van Hove condition and Fubini give that for all $\varphi \in C_c(G)$ and $\mu \in \mathcal{M}^\infty(G)$ we have

$$\lim_{n} \frac{1}{|A_n|} \left(\mu(A_n) \int_G \varphi(t)dt - \int_{A_n} \varphi \ast \mu(t)dt\right) = 0.$$
As this relation is used frequently in the rest of the paper, to switch back and forth between averages of measures and functions, we need to use van Hove sequences.

Next, let us introduce the average (or mean) of functions.

**Definition 2.3.** Let \( f \in C_u(G) \) and \( A \) a van Hove sequence. We say that \( f \) has a well defined mean with respect to \( A \) if

\[
M_A(f) = \lim_{n} \frac{1}{|A_n|} \int_{A_n} f(t)dt
\]

exists. In this case we call \( M_A(f) \) the mean of \( f \).

\( f \) is called amenable with respect to \( A \) if

\[
\lim_{n} \frac{1}{|A_n|} \int_{x+A_n} f(t)dt
\]

exists uniformly in \( x \in G \).

Next, let us recall the following result \([16, 37] \) (compare \([26, \text{Prop. 1.3}] \) for a stronger version of this result).

**Proposition 2.4.** \([16, 37, 26] \) Let \( f \in C_u(G) \) and \( A \) be a van Hove sequence. If \( f \) is amenable with respect to \( A \) then \( f \) is amenable with respect to every van Hove sequence \( B \) and

\[ M_A(f) = M_B(f) . \]

Since amenability is independent of the choice of the van Hove sequence, we will simply call a function amenable, and make no reference to averaging sequence.

We continue by reviewing the concept of autocorrelation and diffraction for a measure.

**Definition 2.5.** Let \( \mu \in \mathcal{M}_\infty(G) \) and \( A \) be a van Hove sequence. We say that \( \mu \) has a well defined autocorrelation with respect to \( A \) if the following limit exists in the vague topology

\[
\gamma = \lim_{n} \frac{1}{|A_n|} (\mu|_{A_n}) * (\overline{\mu|_{A_n}}).
\]

In this case we call \( \gamma \) the autocorrelation of \( \mu \) with respect to \( A \).

Next, to introduce the concept of diffraction, we need first to define the Fourier transform of measures.

**Definition 2.6.** A measure \( \mu \) on \( G \) is called Fourier transformable if there exists some measure \( \nu \) on \( \widehat{G} \) such that, for all \( \varphi \in C_c(G) \) we have \( |\widehat{\varphi}|^2 \in L^2(\nu) \) and

\[
\mu(\varphi * \widehat{\varphi}) = \nu(\varphi^2).
\]

In this case \( \nu \) is called the Fourier transform of \( \mu \) and is denoted by \( \widehat{\mu} \).
Let us note here in passing that a translation bounded measure \( \mu \in \mathcal{M}_\infty(G) \) is Fourier transformable if and only if the Fourier transform of \( \mu \) as a tempered distribution is a translation bounded measure \([52]\). Moreover, in this case, the Fourier transforms of \( \mu \) in the measure and tempered distribution sense, respectively, coincide.

We can now use the following theorem to introduce the diffraction of \( \mu \).

**Theorem 2.7.** \([1, 15, 37]\) Let \( \gamma \) be an autocorrelation of \( \mu \) with respect to some van Hove sequence \( A \). Then \( \gamma \) is Fourier transformable and \( \hat{\gamma} \) is positive.

The measure \( \hat{\gamma} \) is called the diffraction of \( \mu \) (with respect to \( A \)).

**2.3. Fourier Bohr coefficients.** In this section we review the basic properties of Fourier–Bohr coefficients. For more details we refer the reader to \([26]\).

We start with the definition.

**Definition 2.8.** Let \( f \in \text{C}_u(G) \) and \( \mu \in \mathcal{M}_\infty(G) \), let \( \chi \in \hat{G} \) and let \( A \) be van Hove sequence. We say that the **Fourier–Bohr coefficient** \( c_A^\chi(f) \) exists if the following limit exists:

\[
c_A^\chi(f) := \lim_{n} \frac{1}{|A_n|} \int_{A_n} \overline{\chi(t)} f(t) dt.
\]

We further say that the Fourier–Bohr coefficient exists uniformly if the limit

\[
\lim_{n} \frac{1}{|A_n|} \int_{x+A_n} \overline{\chi(t)} f(t) dt,
\]

exists uniformly in \( x \).

Same way we say that the **Fourier–Bohr coefficient** \( c_A^\chi(\mu) \) exists if the following limit exists:

\[
c_A^\chi(\mu) := \lim_{n} \frac{1}{|A_n|} \int_{A_n} \overline{\chi(t)} d\mu(t).
\]

We further say that the Fourier–Bohr coefficient exists uniformly if the limit

\[
\lim_{n} \frac{1}{|A_n|} \int_{x+A_n} \overline{\chi(t)} d\mu(t),
\]

exists uniformly in \( x \).

The next result plays a central role in the rest of the paper.

**Lemma 2.9.** \([25, \text{Cor. 1.12}]\) Let \( f \in \text{C}_u(G) \) and \( \chi \in G \). If \( c_A^\chi(f) \) exists uniformly, then \( c_B^\chi(f) \) exists uniformly for all van Hove sequences \( B \) and

\[
c_B^\chi(f) = c_A^\chi(f)
\]

**Proof.** Note that \( c_A^\chi(f) \) exists uniformly if and only if \( \overline{\chi} f \) is amenable. As \( \overline{\chi} f \in \text{C}_u(G) \) the claim follows from Prop. \([2.3]\). \( \square \)

We will see below in Lemma \([2.11]\) that the same is true for measures.

Let us now recall the following result of \([26]\). This result will allow us transfer every property of the Fourier–Bohr coefficients from functions to measures.
Proposition 2.10. [20] Cor. 1.11] Let \( \mu \in \mathcal{M}^\infty(G) \), \( \varphi \in C_c(G) \), \( \chi \in \widehat{G} \) and let \( A \) be a van Hove sequence. Then,

(a) If the Fourier–Bohr coefficient \( c^A_\chi(\mu) \) exists (uniformly) then the Fourier–Bohr coefficient \( c^A_\chi(\mu \ast \varphi) \) exists (uniformly) and
\[
c^A_\chi(\mu \ast \varphi) = \overline{\varphi(\chi)} c^A_\chi(\mu).
\]

(b) If the Fourier–Bohr coefficient \( c^A_\chi(\mu \ast \varphi) \) exists (uniformly) and \( \overline{\varphi(\chi)} \neq 0 \) then the Fourier–Bohr coefficient \( c^A_\chi(\mu) \) exists (uniformly) and
\[
c^A_\chi(\mu) = \overline{\varphi(\chi)} c^A_\chi(\mu).
\]

As a consequence we get the following result:

Lemma 2.11. [20] Cor. 1.12] Let \( \mu \in \mathcal{M}^\infty(G) \) and \( \chi \in \widehat{G} \). If the Fourier–Bohr coefficient \( c^A_\chi(\mu) \) of \( \mu \in \mathcal{M}^\infty(G) \) exists uniformly with respect to the van Hove sequence \( A = \{A_n\} \), and \( B = \{B_n\} \) is another van Hove sequence, then the Fourier–Bohr coefficient of \( \mu \) exists uniformly with respect to \( B \) and
\[
c^B_\chi(\mu) = c^A_\chi(\mu).
\]

Due to Lemma 2.11 whenever when the Fourier–Bohr coefficient of some \( f \in C_u(G) \) or \( \mu \in \mathcal{M}^\infty(G) \), respectively, exists uniformly, we can simply write \( c_\chi(f) \) and \( c_\chi(\mu) \), respectively. We will do this in the remaining of the paper. Moreover, whenever when we write \( c_\chi(f) \) and \( c_\chi(\mu) \), respectively, it is understood that the Fourier–Bohr coefficients exist uniformly.

Since we will often need to refer to the non-zero Fourier–Bohr coefficients, we introduce the following notation.

Definition 2.12. We say that \( f \in C_u(G) \) has a well defined Fourier–Bohr spectrum with respect to \( A \) if for all \( \chi \in \widehat{G} \) the Fourier–Bohr coefficient \( c^A_\chi(f) \) exists. In this case, the Fourier–Bohr spectrum of \( f \) with respect to \( A \) is
\[
\mathcal{F}B_A(f) = \{ \chi \in \widehat{G} : c^A_\chi(f) \neq 0 \}.
\]

Same way we say that \( \mu \in \mathcal{M}^\infty(G) \) has a well defined Fourier–Bohr spectrum with respect to \( A \) if for all \( \chi \in \widehat{G} \) the Fourier–Bohr coefficient \( c^A_\chi(\mu) \) exists. In this case, the Fourier–Bohr spectrum of \( \mu \) with respect to \( A \) is
\[
\mathcal{F}B_A(\mu) = \{ \chi \in \widehat{G} : c^A_\chi(\mu) \neq 0 \}.
\]

We say that \( \mu \in \mathcal{M}^\infty(G) \) satisfies the Consistent Phase Frequency (CPP) with respect to \( A \) if the autocorrelation \( \gamma \) of \( \mu \) exists with respect to \( A \), the Fourier–Bohr spectrum is well defined with respect to \( A \) and
\[
\gamma(\{\chi\}) = |c^A_\chi(\mu)|^2 \quad \forall \chi \in \widehat{G} \quad \text{(CPP)}.
\]

If the Fourier–Bohr spectrum exists and is empty, we say that \( f \) or \( \mu \), respectively, have null Fourier–Bohr spectrum.
As usual, if the Fourier–Bohr coefficients exist uniformly, the Fourier–Bohr spectrum exists for all van Hove sequences, and is independent of the choice of the van Hove sequence. In this case we simply write $\mathcal{F}B(f)$ and $\mathcal{F}B(\mu)$, respectively, for the Fourier–Bohr spectrum.

**Remark 2.13.**
(a) If $\mu$ satisfies the CPP, then $\mu$ has null Fourier–Bohr spectrum if and only if $\mu$ has continuous diffraction spectrum.
(b) We will show later in Thm. 4.7 that the uniform existence of the Fourier–Bohr coefficients implies CPP.
(c) Thm. 2.28 gives that for measures with pure point diffraction spectrum, CPP is equivalent to Besicovitch almost periodicity of the measure.

Next, we study how the Fourier–Bohr coefficients behave with respect to the basic operations on functions.

**Lemma 2.14.** Let $f \in \mathcal{C}_u(G), \chi \in \hat{G}$ and $A$ a van Hove sequence. If $c^A_\chi(f)$ exists (uniformly) then $c^A_\chi(\overline{f}), c^A_\chi(\hat{f}), c^A_\chi(f^\dagger)$ exist (uniformly) and

\[
\begin{align*}
\frac{1}{|A_n|} \int_{x+An} \chi(t)f(t)dt &= \frac{1}{|A_n|} \int_{x+An} \overline{\chi(t)}f(t)dt. \\
\end{align*}
\]

Taking the limit we get (2.3).

Next,
\[
\frac{1}{|A_n|} \int_{x-An} \chi(t)f^\dagger(t)dt = \frac{1}{|A_n|} \int_{x-An} \overline{\chi(-t)}f(-t)dt = \frac{1}{|A_n|} \int_{x+An} \overline{\chi(s)}f(s)ds
\]

Taking the limit we get (2.4).

Finally,
\[
c^A_\chi(\hat{f}) = c^A_\chi(\overline{f^\dagger}) = c^A_\chi(f^\dagger) = c^A_\chi(f),
\]

with all limits existing (uniformly) by the above.

Since, $\mu \ast \check{\varphi} = \tilde{\mu} \ast \check{\varphi}$, we can use Lemma 2.10 to transfer these relations to measures.

**Corollary 2.15.** Let $\mu \in \mathcal{C}_u(G), \chi \in \hat{G}$ and $A$ a van Hove sequence.
(a) If $c^A_\chi(\mu)$ exists then $c^A_\chi(\overline{\mu}), c^A_\chi(\hat{\mu}), c^A_\chi(\mu^\dagger)$ exist and
\[
\begin{align*}
c^A_\chi(\overline{\mu}) &= c^A_\chi(\mu) \\
c^{-A}_\chi(\hat{\mu}) &= c^A_\chi(\mu) \\
c^{-A}_\chi(\mu^\dagger) &= c^A_\chi(\mu).
\end{align*}
\]
If $c_{\chi}(\mu)$ exists uniformly then $c_{\chi}(\overline{\mu}), c_{\chi}(\mu')$ exist uniformly and
\[ c_{\chi}(\overline{\mu}) = c_{\chi}(\mu), \]
\[ c_{\chi}(\mu') = c_{\chi}(\mu). \]

2.4. Strong and weakly almost periodic functions and measures. We briefly review here the notions of strong and weakly almost periodic functions and measures. For a detailed review of these concepts and their connection to the Fourier theory for measures we recommend the review [37].

Definition 2.16. A function $f \in C_u(G)$ is called \textit{strong almost periodic} if the set $\{T_t f : t \in G\}$ has compact closure in $(C_u(G), \| \cdot \|_\infty)$. $f \in C_u(G)$ is called \textit{weakly almost periodic} if the set $\{T_t f : t \in G\}$ has compact closure in the weak topology of $(C_u(G), \| \cdot \|_\infty)$.

We denote the spaces of strong and weakly almost periodic functions by $\text{SAP}(G)$ and $\text{WAP}(G)$ respectively.

A measure $\mu \in \mathcal{M}^\infty(G)$ is called \textit{strong/weakly almost periodic}, respectively, if, for all $\varphi \in C_c(G)$ we have $\varphi \ast \mu \in \text{SAP}(G)$ or $\varphi \ast \mu \in \text{WAP}(G)$, respectively. We denote these spaces of measures by $\text{SAP}(G)$ and $\text{WAP}(G)$ respectively.

Strong almost periodicity for a function is usually called Bochner almost periodicity and is equivalent to Bohr almost periodicity. It follows immediately from the definitions that $\text{SAP}(G) \subseteq \text{WAP}(G)$ and $\text{SAP}(G) \subseteq \text{WAP}(G)$.

Let us recall the following classical result.

Proposition 2.17. [37, Prop. 4.3.11 and Prop. 4.5.9] Let $f \in \text{WAP}(G), \chi \in \widehat{G}$ and $A$ a van Hove sequence. Then $f, |f|$ and $\chi f$ are amenable. In particular, all the Fourier–Bohr coefficients of $f$ exist uniformly.

We can complete this short subsection by reviewing null weak almost periodicity.

Definition 2.18. A function $f \in C_u(G)$ is called \textit{null weakly almost periodic} if $f \in \text{WAP}(G)$ and $M_A(|f|) = 0$. We denote the space of null weakly almost periodic functions by $\text{WAP}_0(G)$.

A measure $\mu \in \mathcal{M}^\infty(G)$ is called \textit{null weakly almost periodic} if, for all $\varphi \in C_c(G)$ we have $\varphi \ast \mu \in \text{WAP}_0(G)$. We denote these space of null weakly almost periodic measures by $\text{WAP}_0(G)$.

As the following theorem emphasize, these spaces appear naturally in mathematical diffraction theory. Note that particular cases of these results have been proven in [17, 23].

Theorem 2.19. [37, Cor. 4.10.13] Let $\mu \in \mathcal{M}^\infty(G)$ be Fourier transformable. Then,

(a) $\overline{\mu}$ is pure point if and only if $\mu \in \text{SAP}(G)$.

(b) $\overline{\mu}$ is continuous if and only if $\mu \in \text{WAP}_0(G)$. 

We complete this subsection by recalling the Eberlein decomposition of these spaces, which was established in [18] for functions and [23] for measures. This decomposition is the Fourier dual to the Lebesgue decomposition of a measure into its pure point and continuous parts.

**Theorem 2.20.** [37, Thm. 4.10.10]

\[ \mathcal{W}AP(G) = \mathcal{S}AP(G) \oplus \mathcal{W}A\mathcal{P}_0(G) \]

\[ \mathcal{W}AP(G) = \mathcal{S}AP(G) \oplus \mathcal{W}A\mathcal{P}_0(G). \]

\[ \square \]

2.5. **Besicovitch almost periodicity.** In this section we review the basic properties of Besicovitch almost periodic functions and measures we will use below. Since we are interested only in functions \( f \in C_u(G) \) and measures \( \mu \in \mathcal{M}^\infty(G) \) we restrict our definitions to these cases. For a more general and detailed review of this class of functions and measures we refer the reader to [26].

**Definition 2.21.** Let \( \mathcal{A} \) be a van Hove sequence and \( 1 \leq p < \infty \). The **Besicovitch \( p \) seminorm** on \( C_u(G) \) is

\[ \| f \|_{b,p,\mathcal{A}} = \limsup_n \left( \frac{1}{|A_n|} \int_{A_n} |f(t)|^p dt \right)^{\frac{1}{p}}. \]

Let us next recall the following properties of this semi-norm from [26].

**Proposition 2.22.** [26, Lemma 1.13 and Lemma 1.16] Let \( \mathcal{A} \) be a van Hove sequence, let \( f, g \in C_u(G), t \in G \) and \( 1 < p, q < \infty \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Then,

- (a) \( \| T_t f \|_{b,p,\mathcal{A}} = \| f \|_{b,p,\mathcal{A}} \).
- (b) \( \| f \|_{b,1,\mathcal{A}} \leq \| f \|_{b,p,\mathcal{A}} \leq \| f \|_{\infty}^{p-1} \| f \|_{b,1,\mathcal{A}}. \)
- (c) \( \| f \|_{b,1,\mathcal{A}} \leq \| f \|_{b,p,\mathcal{A}} \| f \|_{b,q,\mathcal{A}}. \)

\[ \square \]

Next, we define the concept of a Besicovitch almost periodic function.

**Definition 2.23.** \( f \in C_u(G) \) is called **Besicovitch almost periodic** with respect to \( \mathcal{A} \) if for each \( \epsilon > 0 \) there exists some trigonometric polynomial \( P \) such that \( \| f - P \|_{b,1,\mathcal{A}} < \epsilon. \)

In this case we write \( f \in Bap_{\mathcal{A}}(G) \).

\( f \in C_u(G) \) is called **mean almost periodic** with respect to \( \mathcal{A} \) if for each \( \epsilon > 0 \) the set \( \{ t \in G : \| T_t f - f \|_{b,1,\mathcal{A}} < \epsilon \} \) is relatively dense. The set of mean almost periodic functions in \( C_u(G) \) is denoted by \( MAP_{\mathcal{A}}(G) \).

The next result tells us that one would get the same definition if one uses the Besicovitch \( p \)-seminorm instead.

**Proposition 2.24.** [20, Prop. 3.8] Let \( f \in C_u(G) \) and \( \mathcal{A} \) be a van Hove sequence. Then, the following are equivalent:

- (i) \( f \in Bap_{\mathcal{A}}(G) \).
(ii) There exists some $1 \leq p < \infty$ such that, for each $\varepsilon > 0$ there exists some trigonometric polynomial $P$ such that $\|f - P\|_{b,p,A} < \varepsilon$.

(iii) For all $1 \leq p < \infty$ and for each $\varepsilon > 0$ there exists some trigonometric polynomial $P$ such that $\|f - P\|_{b,p,A} < \varepsilon$.

Besicovitch almost periodic functions have well defined Fourier–Bohr spectrum:

**Proposition 2.25.** [26, Cor. 3.9] Let $f \in C_u(G)$ be Besicovitch almost periodic. Then, for all $\chi \in \hat{G}$ the Fourier–Bohr coefficient $c^A_\chi(f)$ exists.

We complete this part on Besicovitch almost periodic functions by listing the following result which will be needed in the last section of this paper. The result is not listed in [26], but it is an immediate consequence of [26, Thm. 3.19].

**Proposition 2.26.** Let $G$ be second countable LCAG, and $f \in C_u(G)$ be Besicovitch almost periodic. Then, $\mathcal{F}B_A(f)$ is at most countable set. Moreover, for each $\varepsilon > 0$ there exists some trigonometric polynomial $P = \sum_{j=1}^k c_j \chi_j$ with $\chi_1, \ldots, \chi_k \in \mathcal{F}B_A(f)$ such that $\|f - P\|_{b,2,A} < \varepsilon$.

Note here that being able to pick $\chi_1, \ldots, \chi_k \in \mathcal{F}B_A(f)$ will be essential in the last section of the paper.

Next, as usual this definition carries to measures via convolution.

**Definition 2.27.** A measure $\mu \in \mathcal{M}^\infty(G)$ is called Besicovitch almost periodic with respect to a van Hove sequence $A$ if for all $\varphi \in C_c(G)$ we have $\varphi \ast \mu \in \text{Bap}_A(G)$. In this case we write $\mu \in \text{Bap}_A(G)$.

A measure $\mu \in \mathcal{M}^\infty(G)$ is called mean almost periodic with respect to a van Hove sequence $A$ if for all $\varphi \in C_c(G)$ we have $\varphi \ast \mu \in \text{MAP}_A(G)$. In this case we write $\mu \in \text{MAP}_A(G)$.

Exactly as with functions, Besicovitch almost periodic measures have well defined Fourier–Bohr coefficients. Moreover, their Fourier–Bohr spectrum coincides with the Bragg diffraction spectrum:

**Theorem 2.28.** [26, Thm. 3.36] Let $\mu \in \mathcal{M}^\infty(G)$. Then $\mu \in \text{Bap}_A(G)$ if and only if

- The autocorrelation $\gamma$ of $\mu$ exists with respect to $A$ and $\hat{\gamma}$ is pure point.
- For each $\chi \in \hat{G}$ the Fourier–Bohr coefficient $c^A_\chi(\mu)$ exists.
- The consistent phase property (2.2) holds.

In particular, if $\mu \in \text{Bap}_A(G)$ we have

$$\mathcal{F}B_A(\mu) = \{\chi \in \hat{G} : \hat{\gamma}(\{\chi\}) \neq 0\}.$$
2.6. Dynamical systems of translation bounded measures. We complete this preliminary section by briefly reviewing the dynamical systems of translation bounded measures and their diffraction measure. For more details we recommend [11].

We start with the following definition.

Definition 2.29. [11, Def. 2] A dynamical system of translation bounded measures, or simply, TMDS, we understand a pair $(X, G)$ where $X$ is a $G$-invariant and vaguely closed subset of $M^\infty(G)$ with the property that there exists some $C > 0$ and an open pre-compact set $U \subseteq G$ such that

$$X \subseteq \{ \mu \in M^\infty(G) : \|\mu\|_U \leq C \} =: M_{C,U}(G).$$

Note that $M_{C,U}(G)$ is compact and metrisable with respect to the vague topology [11, Thm. 2]. In fact, a set $X \subseteq M^\infty$ gives rise to a TMDS $(X, G)$ if and only if $X$ is $G$-invariant and vaguely compact [17].

Now, for any TMDS $(X, G)$, every $\varphi \in C_c(G)$ induces a continuous function $f_\varphi : X \to \mathbb{C}$ via

$$f_\varphi(\mu) = \varphi \ast \mu(0).$$

Given a TMDS $(X, G)$, then there exists at least a $G$-invariant ergodic measure $m$ on $X$. For any such measure $m$, we refer to $(X, G, m)$ as an ergodic TMDS. We can define the autocorrelation of any ergodic TMDS.

Theorem 2.30. [11, Thm. 5]. Let $m$ be an ergodic measure on a TMDS $(X, G)$ and let $A$ be a van Hove sequence along which the Birkhoff ergodic theorem holds. Then, there exists a measure $\gamma$ such that, for all $\varphi, \psi \in C_c(G)$ and all $t \in G$ we have

$$\gamma \ast \varphi \ast \tilde{\psi}(t) = \int_X f_\varphi(\omega)f_\psi(T_t \omega)d\mu(\omega).$$

Furthermore, for $m$-almost all $\omega \in X$, $\gamma$ is the autocorrelation of $\omega$ with respect to $A$.

Moreover, if $m$ is the unique ergodic measure on $(X, G)$ then $\gamma$ is the autocorrelation of all $\omega \in X$ with respect to $A$. \hspace{1cm} \Box$

As the uniform existence of the Fourier–Bohr coefficients will play an important role in the paper, let us note here in passing that unique ergodicity and continuity of eigenfunctions give the uniform existence of Fourier–Bohr coefficients [25, Thm. 5].

3. Eberlein convolution of measures

Here we review the basic properties of Eberlein convolution of measures and its connection to diffraction theory.

We start by reviewing the following definition, compare [26].

Definition 3.1. We say that $f, g : G \to \mathbb{C}$ have a well defined Eberlein convolution with respect to $A$ if for all $t \in G$ the function $s \to f(s)g(t-s)$ is locally integrable and the following limit exists

$$f \circ A g(t) := \lim_n \frac{1}{|A_n|} \int_{A_n} f(s)g(t-s)ds.$$
We say that $\mu, \nu \in M_\infty(G)$ have a well defined Eberlein convolution with respect to $A$ if the following vague limit exists

$$\mu \odot_A \nu = \lim_n \frac{1}{|A_n|} \mu|_{A_n} \ast \nu|_{-A_n}.$$  

**Remark 3.2.** Some authors (see for example [14]) define the Eberlein convolution as the limit

$$(3.1) \quad \lim_n \frac{1}{|A_n|} \mu \ast \nu|_{A_n}$$

Note here that Lemma 3.4 and Proposition 3.8 imply that, for all $\mu, \nu \in M_\infty(G)$, the vague limit $(3.1)$ exists if and only if $\nu \odot_A \mu$ exist, or equivalently, if $\mu \odot_{-A} \nu$ exists, and

$$\nu \odot_A \mu = \mu \odot_{-A} \nu = \lim_n \frac{1}{|A_n|} \mu \ast \nu|_{A_n}.$$  

Since we rely on results of [26, 28] we will use the convention from these papers, to avoid confusion.

Before proceeding, let us briefly discuss below where the difference between these two conventions comes from. Recall first that for $f, g \in C_c(G)$, their convolution $f \ast g$ can be defined as $\int_G f(s)g(t-s)ds$ or, equivalently $\int_G f(t-r)g(r)dr$, with the substitution $r = t - s$ showing the equality of the two integrals. Some authors prefer the first definition for the convolution, while others prefer the former definition. Since they are equivalent, the choice makes no difference.

Now, when extending to average integrals (or Eberlein convolution), one could define the Eberlein convolution as the limit of $\frac{1}{|A_n|} \int_{A_n} f(s)g(t-s)ds$, or as the limit of $\frac{1}{|A_n|} \int_{A_n} f(t-r)g(r)dr$, but this time the limits are not necessarily equal. Indeed, the substitution $r = t - s$ gives

$$\frac{1}{|A_n|} \int_{A_n} f(s)g(t-s)ds = \frac{1}{|A_n|} \int_{-A_n} f(t-r)g(r)dr.$$  

This means that the two potential definitions are not equal, but they are the same up to reflection of the van Hove sequence, or equivalently, they are the same up to changing the order of $f$ and $g$.

Whenever the Eberlein convolution has been used in the past [16, 23, 37, 29], the limits always existed uniformly in translates, which means that they are independent of choice the van Hove sequence, and hence equal under reflection of $A_n$.

In the recent years it has become clear that one need to work with Eberlein convolutions in general [3, 31, 10, 4, 6, 9, 13, 20, 28, 3] and in this case, one needs to make a choice between the two potential definitions of the Eberlein convolution. Note here that each of the two potential definitions, leads to a natural (unique) extension to the average (Eberlein) convolution of measures: $\lim_n \frac{1}{|A_n|} \mu|_{A_n} \ast \nu|_{-A_n}$ or $\lim_n \frac{1}{|A_n|} \mu|_{-A_n} \ast \nu|_{A_n}$. As pointed above, the two definitions become the same after changing the order of the measures (or equivalently a reflection of $A_n$). This shows that while one needs to make a choice for the definition, the choice makes no difference for the underlying theory but needs to be used in a consistent manner.
Let us now proceed by summarizing the most important properties of the Eberlein convolution \[26, 28\].

First, let us recall the following result, which allows us switch back and forth between the Eberlein convolution of measures and functions.

**Lemma 3.3.** Let \( \mu, \nu \in \mathcal{M}_\infty(G) \) and \( A = (A_n)_{n \in \mathbb{N}} \) a van Hove sequence. Then, the following are equivalent:

(i) The Eberlein convolution \( \mu \ast_A \nu \) exists.

(ii) For all \( \varphi, \psi \in \mathcal{C}_c(G) \) the function \( t \to (\mu * \varphi(t))(\nu * \psi(t)) \) has a well defined mean with respect to \( A \).

(iii) For all \( \varphi, \psi \in \mathcal{C}_c(G) \) the Eberlein convolution \( \varphi * \mu \ast_A (\psi * \nu) \) exists.

Moreover, in this case, for all \( \varphi, \psi \in \mathcal{C}_c(G) \) we have

\[
M(\mu * \varphi * \nu * \psi) = (\mu \ast_A \nu * \varphi * \psi)(0) = (\varphi * \mu) \ast_A (\psi * \nu)(0).
\]

**Proof.** (i) \( \Leftrightarrow \) (ii) and the equality

\[
M(\mu * \varphi * \nu * \psi) = (\mu \ast_A \nu * \varphi * \psi)(0),
\]

follow from \[26\] Prop. 1.4.

(i) \( \Rightarrow \) (ii) is a standard van Hove computation.

(iii) \( \Rightarrow \) (i) and the equality

\[
M(\mu * \varphi * \nu * \psi) = (\varphi * \mu) \ast_A (\psi * \nu)(0)
\]

follow immediately from the observation that for all \( \varphi, \psi \in \mathcal{C}_c(G) \) and \( t \in G \) we have

\[
\frac{1}{|A_n|} \int_{A_n} (\varphi * \mu)(s)(\psi * \nu)(t-s)ds = \frac{1}{|A_n|} \int_{A_n} (\varphi * \mu)(s)(\psi * \nu)(s-t)ds = \frac{1}{|A_n|} \int_{A_n} (\varphi * \mu)(s)(T_t \psi)(s)ds = \frac{1}{|A_n|} \int_{A_n} (\varphi * \mu)(s)(T_t \nu)(s)ds.
\]

□

Next, we see that the Eberlein convolution of translation bounded measures always exist along subsequences.

**Proposition 3.4.** \[28\]

Let \( \mu, \nu \in \mathcal{M}_\infty(G) \) and \( A = (A_n)_{n \in \mathbb{N}} \) a van Hove sequence in a second countable group \( G \). Then,

(a) There exists some \( C > 0 \) and precompact open set \( U \) such that

\[
\frac{1}{|A_n|} (\mu|_{A_n} * \nu|_{-A_n}) \in \{ \varphi \in \mathcal{M}_\infty(G) \mid \| \varphi \|_U \leq C \} =: \mathcal{M}_{C,U}.
\]

Moreover, \( \mathcal{M}_{C,U} \) is vaguely compact and metrisable.

(b) There exists a subsequence \( A' \) of \( A \) such that \( \mu \ast_{A'} \nu \) exists.
(c) In the vague topology we have
\[
0 = \lim_n \left( \frac{1}{|A_n|} \mu|_{A_n} * \nu|_{-A_n} - \frac{1}{|A_n|} \mu|_{A_n} * \nu \right)
\]
\[
= \lim_n \left( \frac{1}{|A_n|} \mu|_{A_n} * \nu|_{-A_n} - \frac{1}{|A_n|} \mu * \nu|_{-A_n} \right)
\]

We will often use in computations the following simple identities.

Lemma 3.5. (a) The Eberlein convolution \( f \odot_A g \) exists if and only if the Eberlein convolution of the measures \( (f \theta_G) \) and \( (g \theta_G) \) exists. Moreover, in this case we have
\[
(f \theta_G) \odot_A (g \theta_G) = (f \odot_A g) \theta_G.
\]
(b) If the Eberlein convolution \( f \odot_A g \) exists then so does \( g \odot_{-A} f \) and \( g \odot_{-A} f = f \odot_A g \).
(c) If the Eberlein convolution \( f \odot_A \tilde{g} \) exists then so does \( g \odot_A \tilde{f} \) and \( g \odot_A \tilde{f} = \tilde{f} \odot_A \tilde{g} \).
(d) If the Eberlein convolution \( \mu \odot_A \nu \) exists then so does \( \nu \odot_{-A} \mu \) and \( \nu \odot_{-A} \mu = \mu \odot_A \nu \).
(e) If the Eberlein convolution \( \mu \odot_A \tilde{\nu} \) exists then so does \( \nu \odot_A \tilde{\mu} \) and \( \nu \odot_A \tilde{\mu} = \tilde{\mu} \odot_A \tilde{\nu} \).

Proof. (a), (b) follows from \[14\] or \[26\] Prop. 1.5.
(c) is \[28\] Lemma 7.2
(d) follows from the definition.
(e) is \[28\] Lemma 8.2.

The next lemma, whose proof follows immediately from the definitions, emphasizes the relevance of the Eberlein convolution to diffraction theory.

Lemma 3.6. Let \( \mu \in \mathcal{M}^\infty(G) \). Then the autocorrelation \( \gamma \) of \( \mu \) exists with respect to \( A \) if and only if \( \mu \odot_A \tilde{\mu} \) exists. Moreover, in this case we have
\[
\gamma = \mu \odot_A \tilde{\mu}.
\]

Remark 3.7. One of the primary goals of the paper is to look at the autocorrelation of a measure \( \mu \) of the form \( \omega = a\mu + b\nu \) for some \( \mu, \nu \in \mathcal{M}^\infty(G) \) and \( a, b \in \mathbb{C} \).
By Lemma 3.4, there exists some subsequence \( \mathcal{A}' \) of \( \mathcal{A} \) such that all the convolutions below are well defined and
\[
\gamma_\omega = (a\mu + b\nu) \odot_\mathcal{A} (a\mu + b\nu)
\]
Proof.

Then, for all $t \in G$, the Eberlein convolution $(T_t f) \ast_A g$ exists and \[ (T_t f) \ast_A g = T_t(f \ast_A g). \]

Then, for all $t \in G$ we have
\[ \|T_t(f \ast_A g) - (f \ast_A g)\|_\infty = \|(T_t f - f) \ast_A g\|_\infty \leq \|T_t f - f\|_\infty \|g\|_\infty. \]

Therefore, since $f, g \in C_u(G)$ we get $f \ast_A g \in C_u(G)$.

Next, by Lemma \[ the Eberlein convolution of the measures $(f \theta_G)$ and $(g \theta_G)$ exists. Moreover, in this case we have
\[ (f \theta_G) \ast_A (g \theta_G) = (f \ast_A g) \theta_G \in WAP(G). \]
by Prop. 3.8. Since \( f \ast_A g \in C_u(G) \), it follows that \( f \ast_A g \in \text{WAP}(G) \) [37, Prop. 4.10.5].

The last claim follows from [37] Prop. 4.3.11 and Prop. 4.5.9.

At this point, we can relate the Fourier–Bohr coefficient of the Eberlein convolution to the coefficient of the Dirac measure at that point in its Fourier transform.

**Corollary 3.10.**

(a) Let \( f, g \in C_u(G) \) and \( A = \{A_n\} \) a van Hove sequence. If the Eberlein convolution \( f \ast_A g \) exists, then for each \( \chi \in \hat{G} \) the Fourier–Bohr coefficients \( c_\chi(f \ast_A g) \) exist uniformly and

\[
(f \ast_A g) \ast (\chi) = c_\chi(f \ast_A g).
\]

(b) Let \( \mu, \nu \in M^\infty(G) \) and \( A = \{A_n\} \) a van Hove sequence. If the Eberlein convolution \( \mu \ast_A \nu \) exists, then for each \( \chi \in \hat{G} \) the Fourier–Bohr coefficients \( c_\chi(\mu \ast_A \nu) \) exist uniformly and

\[
(\mu \ast_A \nu)(\chi) = c_\chi(\mu \ast_A \nu).
\]

**Proof.** (a) By Lemma 3.9 we have \( f \ast_A g \in \text{WAP}(G) \). Therefore, by [37] Prop. 4.3.11 and Prop. 4.5.9 the Fourier–Bohr coefficients \( c_\chi(f \ast_A g) \) exist uniformly.

Since \( (f \ast_A g) \ast_B = (f \ast_B g \ast_B) \), \( f \ast_A g \ast_B \) is Fourier transformable by Lemma 3.9. Therefore, by [37] Thm. 4.10.14 we have

\[
(f \ast_A g)(\chi) = c_\chi(f \ast_A g).
\]

(b) The uniform existence of the Fourier–Bohr coefficient follows from Prop. 2.10, Lemma 3.3 and (a).

Since \( \mu \ast_A \nu \) is Fourier transformable by Prop. 3.8, the last relation follows from [37] Thm. 4.10.14.

Finally, let us recall the following result from [28], which will play an essential role in our paper.

**Theorem 3.11.** [28] Let \( \mu, \nu \in M^\infty(G) \) and \( A \) a van Hove sequence along which \( \mu \ast_A \nu \) exists. If \( \mu \in \text{MAP}_A(G) \) then \( \mu \ast_A \nu \in \text{SAP}(G) \).

By combining this with Lemma 3.5 we get.

**Corollary 3.12.** Let \( \mu, \nu \in M^\infty(G) \) and \( A \) a van Hove sequence along which \( \mu \ast_A \nu \) exists. If \( \nu \in \text{MAP}_A(G) \) then \( \mu \ast_A \nu \in \text{SAP}(G) \).

4. Fourier–Bohr coefficients of Eberlein convolution

We start by covering the following result which relates the Fourier–Bohr coefficient of \( f \ast_A g \) to the Fourier–Bohr coefficients of \( f, g \). Particular versions of this result can be found in [16, 23, 37, 29, 26], and Prop. 4.11 below generalizes all these results.

Before proving the result, let us emphasize that while the uniform existence of the Fourier–Bohr coefficient \( c_\chi(f \ast g) \) follows from Lemma 3.9, the proof of this result relies on long and technical results, which cover most of the review [37].
Proposition 4.1. Let $f, g \in C_u(G)$, $\chi \in \widehat{G}$ and $A$ be a van Hove sequence such that

(a) The Eberlein convolution $f \ast_A g$ exists.
(b) The Fourier–Bohr coefficient $c_A^\chi(f)$ exists.
(c) The Fourier–Bohr coefficient $c_A^\chi(g)$ exists uniformly.

Then, the Fourier–Bohr coefficient $c_A^\chi(f \ast g)$ exists uniformly and satisfies

$$c_A^\chi(f \ast g) = c_A^\chi(f)c_A^\chi(g).$$

Proof. Let $\varepsilon > 0$. Then, by (b) and (c) there exists an $N_\varepsilon$ so that, for all $n, m > N_\varepsilon$ and all $x \in G$ we have

$$\frac{1}{|A_m|} \int_{A_n} \int_{x + s + A_n} \chi(u)g(u)\chi(s)f(s)du ds - c_A^\chi(f)c_A^\chi(g) \leq \frac{\varepsilon}{4\|f\|_\infty + 1}.$$

In particular, for all $m, n > N_\varepsilon$, we have for all $x \in G$

$$\left| \frac{1}{|A_m|} \int_{A_n} \int_{x + s + A_n} \chi(u)g(u)\chi(s)f(s)du ds - c_A^\chi(f)c_A^\chi(g) \right| \leq \frac{\varepsilon}{4\|f\|_\infty + 1}.$$

Thus, for all $m, n > N_\varepsilon$ and $x \in G$ we have

$$\left| \frac{1}{|A_m|} \int_{A_n} \int_{x + s + A_n} \chi(u)g(u)\chi(s)f(s)du ds - c_A^\chi(f)c_A^\chi(g) \right| \leq \frac{\varepsilon}{4\|f\|_\infty + 1}.$$
Recall that for all \( t \in G \) we have
\[
f \otimes_A g(t) = \lim_{m} \frac{1}{|A_m|} \int_{A_m} f(s)g(t-s)ds.
\]
Define
\[
h_m(t) := \left( \frac{1}{|A_m|} \int_{A_m} f(s)g(t-s)ds \right) \chi(t)1_{x+A_n}(t).
\]
Then, \( h_m(t) \) converges pointwise to \( f \otimes_A g(t)\chi(t)1_{x+A_n}(t) \) and is bounded by \( \|f\|_\infty 1_{x+A_n} \in L^1(G) \). Therefore, by the dominated convergence theorem we have
\[
\lim_{m} \frac{1}{|A_n|} \int_{x+A_n} h_m(t)dt = \frac{1}{|A_n|} \int_{x+A_n} \chi(t)f \otimes_A g(t)dt.
\]
Therefore, there exists some \( N(n,x,\epsilon) \) such that, for all \( m > N(n,x,\epsilon) \) we have
\[
(4.2) \quad \left| \frac{1}{|A_n|} \int_G h_m(t)dt - \frac{1}{|A_n|} \int_{x+A_n} \chi(t)f \otimes_A g(t)dt \right| < \frac{\epsilon}{2}.
\]
Note here that for all \( m \) we have
\[
\frac{1}{|A_m|} \int_{x+A_n} h_m(t)dt = \frac{1}{|A_m|} \int_{x+A_n} \chi(t) \frac{1}{|A_n|} \int_{A_n} f(s)g(t-s)dsdt
\]
\[
= \frac{1}{|A_m|} \frac{1}{|A_n|} \int_{x+A_n} \int_{A_m} \chi(s)f(s)\chi(t-s)g(t-s)dsdt
\]
\[
= \frac{1}{|A_m|} \frac{1}{|A_n|} \int_{x+A_n} \chi(s)f(s)\left( \int_{x+A_n} \chi(t-s)g(t-s)dt \right)ds
\]
\[
= \frac{1}{|A_m|} \frac{1}{|A_n|} \int_{x+A_n} \chi(s)f(s)\left( \int_{x+s+A_n} \chi(u)g(u)du \right)ds.
\]
Therefore for all \( m > N(n,x,\epsilon) \) we have by (4.2)
\[
(4.3) \quad \left| \frac{1}{|A_n|} \frac{1}{|A_m|} \int_{A_m} \chi(s)f(s)\left( \int_{x+s+A_n} \chi(u)g(u)du \right)ds - \frac{1}{|A_n|} \int_{x+A_n} \chi(t)f \otimes_A g(t)dt \right| < \frac{\epsilon}{2}.
\]
Now, pick one \( m > \max\{N(n,x,\epsilon),N_\epsilon\} \). By Combining (4.1) and (4.3) we get that for this \( n,m,x \) we have
\[
\left| \frac{1}{|A_n|} \int_{x+A_n} \chi(t)f \otimes_A g(t)dt - c^A_\chi(f)c_\chi(g) \right|
\]
\[
\leq \left| \frac{1}{|A_n|} \frac{1}{|A_m|} \int_{A_m} \int_{x+s+A_n} \chi(u)g(u)\chi(s)f(s)du ds - c^A_\chi(f)c_\chi(g) \right|
\]
\[
+ \left| \frac{1}{|A_n|} \frac{1}{|A_m|} \int_{A_m} \chi(s)f(s)\left( \int_{x+s+A_n} \chi(u)g(u)du \right)ds - \frac{1}{|A_n|} \int_{x+A_n} \chi(t)f \otimes_A g(t)dt \right|
\]
\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Therefore, for all \( n > N_\epsilon \) and all \( x \in G \) we have
\[
\left| \frac{1}{|A_n|} \int_{x+A_n} \chi(t)f \otimes_A g(t)dt - c^A_\chi(f)c_\chi(g) \right| < \epsilon,
\]
which proves the claim.

As an interesting consequence we get that, if \( a_\chi(g) \) exist uniformly and is non-zero, the existence of \( f \otimes_A g \) implies the existence of the Fourier–Bohr coefficient \( c_\chi^A(f) \).

**Theorem 4.2.** Let \( f, g \in C_\mu(G), \chi \in \hat{G} \) and \( A \) be a van Hove sequence such that the Eberlein convolution \( f \otimes_A g \) exists and the Fourier–Bohr coefficient \( c_\chi(g) \) exists uniformly. Then, \( c_\chi(f \otimes_A g) \) exist uniformly and

(a) If \( c_\chi(g) = 0 \) then
\[
c_\chi(f \otimes g) = 0.
\]

(b) If \( c_\chi(g) \neq 0 \) then the Fourier–Bohr coefficient \( c_\chi^A(f) \) exists and
\[
c_\chi(f \otimes_A g) = c_\chi^A(f) c_\chi(g).
\]

**Proof.** The uniform existence of \( c_\chi(f \otimes_A g) \) follows from Proposition 4.1

(a) Consider the sequence
\[
a_n = \frac{1}{|A_n|} \int_{A_n} \chi(t) f(t) dt.
\]
Since \( f \in C_\mu(G) \), \( a_n \) is bounded in \( \mathbb{C} \), and hence has a convergent subsequence \( a_{k_n} \).

Let \( B_n := A_{k_n} \) and set \( B = \{ B_n \}_n \). Since \( B \) is a subsequence of \( A \), and the Fourier–Bohr coefficient \( c_\chi^B(f) \) exists, by Proposition 4.1 we get
\[
c_\chi(f \otimes_A g) = c_\chi(f \otimes_B g) = c_\chi^B(f) c_\chi(g) = 0.
\]

(b) Consider again the sequence
\[
a_n = \frac{1}{|A_n|} \int_{A_n} \chi(t) f(t) dt.
\]
We want to show that \( a_n \) converges to \( c := \frac{c_\chi(f \otimes g)}{c_\chi(g)} \).

Since \( f \in C_\mu(G) \), \( a_n \) is bounded in \( \mathbb{C} \), and therefore it is a subset of a compact metric set. Therefore, to prove that \( a_n \) converges to \( c \) it suffices to show that any convergent subsequence of \( a_n \) converges to \( c \).

Now, we repeat the argument in (a). Let \( a_{k_n} \) be a subsequence of \( a_n \) which converges to some \( c' \). Let \( B_n := A_{k_n} \) and set \( B = \{ B_n \}_n \). Then, the Fourier–Bohr coefficient \( c_\chi^B(f) \) exists and \( c_\chi^B(f) = c' \). Again by Proposition 4.1 we get
\[
c_\chi(f \otimes_A g) = c_\chi^B(f) c_\chi(g) = c' \cdot c_\chi(g).
\]
Since \( c_\chi(g) \neq 0 \), we get \( c' = c \), which proves the claim.

We can now take advantage of Lemma 3.5(c) to flip the assumption on the uniform existence of Fourier–Bohr coefficients to the first variable. We start by showing the following version of the result in Theorem 4.2 which will allow us do flip.

**Proposition 4.3.** Let \( f, g \in C_\mu(G), \chi \in \hat{G} \) and \( A \) be a van Hove sequence such that the Eberlein convolution \( f \otimes_A g \) exists and the Fourier–Bohr coefficient \( c_\chi(g) \) exists uniformly. Then, \( c_\chi(f \otimes_A g) \) exist uniformly and
(a) If \( c_\chi(g) = 0 \) then 
\[ c_\chi(f \ast_A \hat{g}) = 0. \]
(b) If \( c_\chi(g) \neq 0 \) then the Fourier–Bohr coefficient \( c_\chi^A(f) \) exists and 
\[ c_\chi(f \ast_A \hat{g}) = c_\chi^A(f) c_\chi(g). \]

Proof. Since the Fourier–Bohr coefficients \( c_\chi(g) \) exist uniformly, by Lemma 2.14 so do the Fourier–Bohr coefficients \( c_\chi(\tilde{g}) \) and 
\[ c_\chi(\hat{g}) = \overline{c_\chi(g)}. \]

The claim follows now from Theorem 4.2. \( \square \)

By combining this result with Lemma 3.5 (c), we get:

**Corollary 4.4.** Let \( f, g \in C_u(G) \), \( \chi \in \hat{G} \) and \( A \) be a van Hove sequence such that the Eberlein convolution \( f \ast_A \hat{g} \) exists and the Fourier–Bohr coefficient \( c_\chi(f) \) exists uniformly. Then, \( c_\chi(f \ast A \hat{g}) \) exist uniformly and

(a) If \( c_\chi(f) = 0 \) then 
\[ c_\chi(f \ast_A \hat{g}) = 0. \]
(b) If \( c_\chi(f) \neq 0 \) then the Fourier–Bohr coefficient \( c_\chi^A(g) \) exist and 
\[ c_\chi(f \ast_A \hat{g}) = c_\chi(f) c_\chi^A(g). \]

\( \square \)

Next, let \( \mu, \nu \in \mathcal{M}^\infty(G) \) and \( \chi \in \hat{G} \). By [15, 37] there exists some \( \varphi \in C_c(G) \) such that \( \varphi(\chi) \neq 0 \). Combining Prop 4.3 and Cor. 4.1 for \( \mu * \varphi \) and \( \nu * \varphi \) with Prop 2.10 and Prop. 3.3 we get the following results.

**Theorem 4.5.** Let \( \mu, \nu \in \mathcal{M}^\infty(G) \), \( \chi \in \hat{G} \) and \( A \) be a van Hove sequence such that the Eberlein convolution \( \mu \ast_A \hat{\nu} \) exists and the Fourier–Bohr coefficient \( c_\chi(\nu) \) exists uniformly. Then, \( c_\chi(\mu \ast_A \hat{\nu}) \) exist uniformly and

(a) If \( c_\chi(\nu) = 0 \) then 
\[ c_\chi(\mu \ast_A \hat{\nu}) = 0. \]
(b) If \( c_\chi(\nu) \neq 0 \) then the Fourier–Bohr coefficient \( c_\chi^A(\mu) \) exist and 
\[ c_\chi(\mu \ast_A \hat{\nu}) = c_\chi^A(\mu) c_\chi(\nu). \]

\( \square \)

**Theorem 4.6.** Let \( \mu, \nu \in \mathcal{M}^\infty(G) \), \( \chi \in \hat{G} \) and \( A \) be a van Hove sequence such that the Eberlein convolution \( \mu \ast_A \hat{\nu} \) exists and the Fourier–Bohr coefficient \( c_\chi(\mu) \) exists uniformly. Then, \( c_\chi(\mu \ast_A \hat{\nu}) \) exist uniformly and

(a) If \( c_\chi(\mu) = 0 \) then 
\[ c_\chi(\mu \ast_A \hat{\nu}) = 0. \]
(b) If \( c_\chi(\mu) \neq 0 \) then the Fourier–Bohr coefficient \( c_\chi^A(\nu) \) exist and 
\[ c_\chi(\mu \ast_A \hat{\nu}) = c_\chi(\mu) c_\chi^A(\nu). \]

\( \square \)
This result has some interesting consequences for diffraction.

We start by showing that the uniform existence of the Fourier–Bohr coefficient implies that the intensity of the Bragg peak at that point is the square of the absolute value of the Fourier–Bohr coefficient. For $G = \mathbb{R}^d$ this result is well known, see [21]. For uniquely ergodic dynamical systems the result also follows from [23 Thm. 5].

In particular, the uniform existence of all Fourier–Bohr coefficients implies the CPP.

**Theorem 4.7.** Let $\mu \in M^\infty(G)$ let $A$ be a van Hove sequence, and let $\gamma$ be the autocorrelation of $\mu$ with respect to $A$. If for some $\chi \in \widehat{G}$ the Fourier–Bohr coefficient $c_\chi(\mu)$ exists uniformly, then we get

$$\widehat{\gamma}(\{\chi\}) = |c_\chi(\mu)|^2.$$  

In particular, if all Fourier–Bohr coefficients of $\mu$ exist uniformly, then $\mu$ satisfies CPP [22].

It follows from Thm. 4.7 that, under the assumption that the Fourier–Bohr coefficients exist uniformly, null Fourier–Bohr spectrum is equivalent to null pure point diffraction spectrum.

**Corollary 4.8.** Let $\mu \in M^\infty(G)$ be a measure. If for all $\chi \in \widehat{G}$ the Fourier–Bohr coefficients $c_\chi(\mu)$ exist uniformly, then, the following are equivalent.

(i) $\mu$ has null Fourier–Bohr spectrum with respect one van Hove sequence $A$.

(ii) $\mu$ has null Fourier–Bohr spectrum with respect all van Hove sequences.

(iii) There exists an autocorrelation $\gamma$ of $\mu$ such that $\overline{\gamma}_{pp} = 0$.

(iv) For every autocorrelation $\gamma$ of $\mu$ we have $\overline{\gamma}_{pp} = 0$.

Let us emphasize here that systems with a single Bragg peak at $t = 0$ are not considered to have null Fourier–Bohr spectrum.

Let us also note in passing that if the Fourier–Bohr spectrum exists uniformly, then the Bragg diffraction spectrum is independent of the choice of the van Hove sequence.

**Corollary 4.9.** Let $\mu \in M^\infty(G)$ be a measure such that for all $\chi \in \widehat{G}$ the Fourier–Bohr coefficients $c_\chi(\mu)$ exist uniformly. Let $\gamma_1, \gamma_2$ be autocorrelations of $\mu$ with respect to the van Hove sequences $A$ and $B$, respectively. Then

$$\overline{(\gamma_1)}_{pp} = \overline{(\gamma_2)}_{pp}.$$  

Moreover, if $\gamma$ is the autocorrelation of $\mu$ with respect to some van Hove sequence $A$, the Bragg diffraction spectrum $B_A(\mu)$ does not depend on $A$ and satisfies

$$\mathcal{F}B(\mu) = B_A(\mu) := \{\chi \in \widehat{G} : \widehat{\gamma}(\{\chi\}) \neq 0\}.$$  

□
The following example shows that in this situation, the continuous diffraction spectrum can depend on the van Hove sequence, even if the Fourier–Bohr spectrum exists uniformly.

**Example 4.10.** Consider the Thue-Morse measure $\omega$ [7, Sect.4.6]. Define

$$\mu = \delta_Z + \omega|_{(0,\infty)},$$

that is $\mu = \sum_{n\in\mathbb{Z}} c(n)\delta_n$ where

$$c(n) = \begin{cases} 1 & \text{if } n \in \mathbb{Z} \text{ and } n < 0 \\ 1 + \omega(n) & \text{if } n \in \mathbb{Z} \text{ and } n \geq 0 \end{cases}$$

Now, the Fourier–Bohr coefficients of $\omega$ exist uniformly. Moreover, since $\omega$ has (singular) continuous diffraction spectrum [7, Thm. 10.1], all the Fourier–Bohr coefficients are zero.

It follows immediately that the Fourier–Bohr coefficients $c_{\chi}(\mu)$ exist uniformly and

$$c_{\chi}(\mu) = \begin{cases} 1 & \text{if } \chi \in \mathbb{Z} \\ 0 & \text{otherwise}. \end{cases}$$

In particular, if the autocorrelation $\gamma_A$ of $\mu$ exists with respect to some van Hove sequence $A$ then

$$(\hat{\gamma}_A)_{pp} = \delta_Z.$$ 

Now, consider the van Hove sequences $B_n = [-n^2,n]$ and $C_n = [-n,n]$. Then, an easy computation shows that the autocorrelations $\gamma_B$ and $\gamma_C$ of $\mu$ exist with respect to these sequences and

$$\gamma_B = \delta_Z + \gamma_{TM},$$

$$\gamma_C = \delta_Z,$$

where $\gamma_{TM}$ is the measure from [7, Thm. 10.1].

In particular, we have

$$(\hat{\gamma}_B)_c = \hat{\gamma}_{TM} \neq 0$$

$$(\hat{\gamma}_C)_c = 0.$$ 

5. On orthogonality with respect to Eberlein convolution

In this section we establish the first orthogonality with respect to the Eberlein convolution. The key for this result will be Prop. 5.1 and Prop. 5.3 below.

**Proposition 5.1.** Let $\mu, \nu \in M^\infty$ and $A$ a van Hove sequence so that

- $\mu \otimes_A \tilde{\nu}$ exists.
- The Fourier–Bohr coefficients $c_{\chi}(\mu)$ exist uniformly for all $\chi \in \hat{G}$ and satisfy

$$c_{\chi}(\mu) = 0 \quad \forall \chi \in \hat{G}.$$

Then, $\mu \otimes_A \tilde{\nu}$ is a continuous measure.
Proof. By Thm. 4.5 \( c_\chi(\mu \boxdot A \tilde{\nu}) \) exist and
\[
c_\chi(\mu \boxdot A \tilde{\nu}) = 0 \quad \forall \chi \in \hat{G}.
\]
Then, by Cor. 3.10 we have
\[
\widehat{\mu \boxdot A \tilde{\nu}}(\{\chi\}) = c_\chi(\mu \boxdot A \tilde{\nu}) = 0 \quad \forall \chi \in \hat{G}.
\]
□

As an immediate consequence we get the following result.

**Corollary 5.2.** Let \( \mu \in \mathcal{M}^\infty(G) \) and \( A \) a van Hove sequence so that

- The autocorrelation \( \gamma \) of \( \mu \) exists and \( \widehat{\gamma} \) is continuous.
- The Fourier–Bohr coefficients \( c_\chi(\mu) \) exist uniformly for all \( \chi \in \hat{G} \).

Let \( \nu \in \mathcal{M}^\infty(G) \) be any measure and \( A' \) be any subsequence of \( A \) such that \( \mu \boxdot A' \tilde{\nu} \) exists. Then \( \mu \boxdot A' \tilde{\nu} \) is a continuous measure.

**Proof.** By Cor. 4.8 \( \mu \) has null Fourier–Bohr spectrum. The claim follows from Prop. 5.1. □

Next, we prove the following result, which complements Prop. 5.1.

**Proposition 5.3.** Let \( \mu, \nu \in \mathcal{M}^\infty \) and \( A \) a van Hove sequence so that

- \( \mu \boxdot \tilde{\nu} \) exists.
- The Fourier–Bohr coefficients \( c_\chi(\mu) \) exist uniformly for all \( \chi \in \hat{G} \).
- For all \( \chi \in \mathcal{F}(\mu) \), the Fourier–Bohr coefficient \( c^A_\chi(\nu) \) exist satisfy
\[
c^A_\chi(\nu) = 0.
\]

Then, \( \mu \boxdot A \tilde{\nu} \) is a continuous measure.

**Proof.** By Thm. 4.6 \( c_\chi(\mu \boxdot \tilde{\nu}) \) exist and
\[
c_\chi(\mu \boxdot A \tilde{\nu}) = 0 \quad \forall \chi \in \hat{G}.
\]
Then, by Cor. 3.10 we have
\[
\widehat{\mu \boxdot A \tilde{\nu}}(\{\chi\}) = c_\chi(\mu \boxdot A \tilde{\nu}) = 0 \quad \forall \chi \in \hat{G}.
\]
□

We can now prove the following result, which is one of the main results in the paper.

**Theorem 5.4.** Let \( \mu, \nu \in \mathcal{M}^\infty(G) \) and \( A \) a van Hove sequence with the following properties:

- The autocorrelation \( \gamma_\mu \) of \( \mu \) exists with respect to \( A \) and \( \widehat{\gamma_\mu} \) is pure point.
- The autocorrelation \( \gamma_\nu \) of \( \nu \) exists with respect to \( A \) and \( \widehat{\gamma_\nu} \) is continuous.
- The Fourier–Bohr coefficients \( c_\chi(\nu) \) exist uniformly for all \( \chi \in \hat{G} \).
Then, the following Eberlein convolutions exist and
\[ \mu \ast_{A} \tilde{\nu} = \nu \ast_{A} \tilde{\mu} = 0. \]

In particular, for all \(a, b \in \mathbb{C}\) the autocorrelation \(\gamma_{\omega}\) of \(\omega = a\mu + b\nu\) exists with respect to \(A\) and
\[
(\tau_{\omega})_{pp} = |a|^2 \gamma_{\mu},
(\tau_{\omega})_{cc} = |b|^2 \gamma_{\nu}.
\]

Proof. By Lemma 3.5(e), it suffices to show that \(\mu \ast_{A} \tilde{\nu}\) exists and is zero. By Prop 3.4 it suffices to show that for any subsequence \(B\) of \(A\) for which \(\mu \ast_{B} \tilde{\nu}\) exists we have
\[ \mu \ast_{B} \tilde{\nu} = 0. \]

Let \(B\) be any subsequence of \(A\) such that \(\mu \ast_{B} \tilde{\nu}\) exists.

Next, since the Fourier–Bohr coefficients \(c_{\chi}(\mu)\) exist uniformly for all \(\chi \in \hat{G}\). Therefore, by Prop. 5.1 we have \(\mu \ast_{B} \tilde{\nu} \in \mathcal{WAP}_{0}(G)\).

Moreover, by [26], since \(\gamma_{\mu}\) is pure point, we have \(\mu \in \mathcal{MAP}(G)\), and hence \(\mu \ast_{B} \tilde{\nu} \in \mathcal{SAP}(G)\) by Theorem 3.11.

Therefore, by Thm. 2.20 we have
\[ \mu \ast_{B} \tilde{\nu} \in \mathcal{SAP}(G) \cap \mathcal{WAP}_{0}(G) = \{0\}. \]

The last claim follows from (3.2). \(\square\)

Similarly we get the following result. Note here that since the Consistent Phase Property may not hold for \(\nu\), we cannot relate the null Fourier–Bohr spectrum of \(\nu\) with its diffraction spectra.

**Proposition 5.5.** Let \(\mu, \nu \in \mathcal{M}^{\infty}(G)\) and \(A\) a van Hove sequence with the following properties:

- The autocorrelation \(\gamma_{\mu}\) of \(\mu\) exists with respect to \(A\) and \(\gamma_{\mu}\) is pure point.
- The Fourier–Bohr coefficients \(c_{\chi}(\mu)\) exist uniformly for all \(\chi \in \hat{G}\). For all \(\chi \in \mathcal{FB}(\mu)\) the Fourier–Bohr coefficient \(c_{\chi}^{A}(\nu)\) exists and
  \[ c_{\chi}^{A}(\nu) = 0. \]

Then, the following Eberlein convolutions exist and
\[ \mu \ast_{A} \tilde{\nu} = \nu \ast_{A} \tilde{\mu} = 0. \]

Proof. The proof is identical to the one of Thm. 5.4. \(\square\)

As an immediate consequence we get the following results.

**Corollary 5.6.** Let \(\mu, \nu \in \mathcal{M}^{\infty}(G)\) and \(A\) a van Hove sequence with the following properties:

- The autocorrelation \(\gamma_{\mu}\) of \(\mu\) exists with respect to \(A\) and \(\gamma_{\mu}\) is pure point.
- The Fourier–Bohr coefficients \(c_{\chi}(\mu)\) exist uniformly for all \(\chi \in \hat{G}\).
- \(\nu\) has null Fourier–Bohr spectrum with respect to \(A\).
Then, the following Eberlein convolutions exist and
\[ \mu \odot_A \tilde{\nu} = \nu \odot_A \tilde{\mu} = 0. \]
\[ \square \]

**Corollary 5.7.** Let \( \mu, \nu \in M^\infty(G) \) and \( A \) a van Hove sequence with the following properties:

- The autocorrelation \( \gamma_\mu \) of \( \mu \) exists with respect to \( A \) and \( \hat{\gamma}_\mu \) is pure point.
- The autocorrelation \( \gamma_\mu \) of \( \mu \) exists with respect to \( A \) and \( \hat{\gamma}_\mu \) is pure point.
- The Fourier–Bohr coefficients \( c_\chi(\mu) \) exist uniformly for all \( \chi \in \hat{G} \).
- The measure \( \nu \) satisfies the CPP (2.2).

Then, the following Eberlein convolutions exist and
\[ \mu \odot_A \tilde{\nu} = \nu \odot_A \tilde{\mu} = 0. \]
\[ \square \]

Let us complete the section by showing that any measure \( \mu \) which satisfies the conditions of Prop. 5.5 is Besicovitch almost periodic. We will show in the next section that we can relax the uniform existence of the Fourier–Bohr coefficients in Prop. 5.5 to the existence of the Fourier–Bohr coefficients and CPP holding.

**Lemma 5.8.** Let \( \mu \) be a measure so that autocorrelation \( \gamma_\mu \) of \( \mu \) exists with respect to \( A \) and \( \hat{\gamma}_\mu \) is pure point. If the Fourier–Bohr coefficients \( c_\chi(\mu) \) exist uniformly for all \( \chi \in \hat{G} \), then \( \mu \in B_{\text{ap}} A(G) \).

**Proof.** By Theorem 4.7 the consistent phase property holds. Then, by [26, ] we have \( \mu \in B_{\text{ap}} A(G) \).
\[ \square \]

By using the approach of [14] in the next section we will prove in Prop. 6.7 that if \( \mu \in B_{\text{ap}} A(G) \) and \( c_\chi^A(\nu) \) exists and is zero for all \( \chi \in F_B A(\mu) \) then
\[ \mu \odot_A \tilde{\nu} = \nu \odot_A \tilde{\mu} = 0. \]

Lemma 5.8 implies then that Prop. 5.5 is a particular case of Prop. 6.7.

6. **Besicovitch almost periodicity**

The goal of this section is to extend Prop. 5.5 from Besicovitch almost periodic measures uniform Fourier–Bohr coefficients to arbitrary Besicovitch almost periodic measures. In particular, Prop. 6.7 will give an alternate proof for Prop. 5.5 which does not rely on Thm. 3.11.

The more general version of the result proven in Prop. 6.7 will allow us to show that, given two ergodic TMDS, one with pure point and the other with continuous diffraction spectrum, respectively, almost surely all measures in the first dynamical system are orthogonal with respect to the Eberlein convolution to the measures in the second system.

Let us start by recalling the following result. Since [14] uses a slightly different definition, and requires that \( A \) is symmetric and \( G = \mathbb{R}^d \), we include the proof (which works in general) for completion.
Lemma 6.1. [14] Lemma 4.1] Let $f \in C_u(G)$. Then $c^A_\chi(f)$ exists if and only if $f \otimes_A \chi$ exists. Moreover, in this case

$$(f \otimes_A \chi)(x) = \chi(x)c^A_\chi(f).$$

Proof. For all $n$ and all $x \in G$ we have

$$\frac{1}{|A_n|} \int_{A_n} f(s)\chi(x-s)ds = \chi(x) \frac{1}{|A_n|} \int_{A_n} f(s)\overline{\chi(s)}ds.$$

□

As an immediate consequences we get:

Corollary 6.2. [14] Cor. 4.4] Let $f \in C_u(G)$ with null Fourier–Bohr spectrum with respect to $A$. Then, for all trigonometric polynomials $P$ the following Eberlein convolutions exist and

$$(f \otimes_A \tilde{P}) = (P \otimes_A \tilde{f}) = 0.$$

□

Next, let us note that for $\chi \in \hat{G}$ and $t \in G$ we have

$$\tilde{\chi}(t) = \overline{\chi(-t)} = \chi(t).$$

Because of this relation, we can more generality show that the Eberlein convolution of $f$ with trigonometric polynomials which correspond to null Fourier–Bohr coefficients exists and is zero.

Corollary 6.3. Let $f \in C_u(G)$, $A$ a van Hove sequence, $\chi_1, \ldots, \chi_k \in \hat{G}$ and $c_1, \ldots, c_k \in \mathbb{C}$. Let $P = c_1 \chi_1 + \ldots + c_k \chi_k$.

If for each $1 \leq j \leq k$ the Fourier–Bohr coefficient $c^A_{\chi_j}(f)$ exists and $c^A_{\chi_j}(f) = 0$ then, the following Eberlein convolutions exist and

$$(f \otimes_A \tilde{P}) = (P \otimes_A \tilde{f}) = 0.$$

□

Next, we will use the argument below few times in the remaining of the paper. In order to avoid repetition, we list the result as an auxiliary Lemma.

Lemma 6.4. Let $f,g \in C_u(G)$ and $A$ be a van Hove sequence. Assume that for each $\epsilon > 0$ there exists some $h \in C_u(G)$ so that

(a) $\|g - h\|_{b,2,A} < \epsilon$.

(b) The Eberlein convolution $f \otimes_A \tilde{h}$ exists and

$$\|f \otimes_A \tilde{h}\|_\infty < \epsilon.$$

Then, $f \otimes_A \tilde{g}$ exists and

$$f \otimes_A \tilde{g} = 0.$$
Proof. Let \( t \in G \) be arbitrary. Let \( \epsilon > 0 \).

Let \( h \in C_\alpha(G) \) be so that \( f \ast_A \hat{h} \) exists and

\[
\|g - h\|_{b,2,A} < \frac{\epsilon}{2 + 2\|f\|_\infty}
\]

\[
\|f \ast_A \hat{h}\|_\infty < \frac{\epsilon}{2 + 2\|f\|_\infty}.
\]

Then, by Prop. 2.22 we have,

\[
\|T_tg - T_th\|_{b,2,A} < \frac{\epsilon}{2 + 2\|f\|_\infty}.
\]

Therefore, there exists some \( N_1 \) such that, for all \( n > N_1 \) we have

\[
\left( \frac{1}{|A_n|} \int_{A_n} |g(s-t) - h(s-t)|^2 \, ds \right)^{\frac{1}{2}} \leq \frac{\epsilon}{1 + \|f\|_\infty}.
\]

Next, since \( |f \ast_A \hat{h}(t)| < \frac{\epsilon}{2 + 2\|f\|_\infty} \), there exists some \( N_2 \) such that, for all \( n > N_2 \) we have

\[
\left| \frac{1}{|A_n|} \int_{A_n} f(s) \overline{h(s-t)} \, ds \right| < \frac{\epsilon}{1 + \|f\|_\infty}.
\]

Let \( N = \max\{N_1, N_2\} \). Then, for all \( n > N \) we have

\[
\left| \frac{1}{|A_n|} \int_{A_n} f(s) \overline{g(t-s)} \, ds \right| = \left| \frac{1}{|A_n|} \int_{A_n} f(s) \overline{g(s-t)} \, ds \right|
\]

\[
\leq \left| \frac{1}{|A_n|} \int_{A_n} f(s) \overline{h(s-t)} \, ds \right| + \left| \frac{1}{|A_n|} \int_{A_n} f(s) \left( \overline{h(s-t)} - \overline{g(s-t)} \right) \, ds \right|
\]

\[
< \frac{\epsilon}{1 + \|f\|_\infty} + \|f\|_\infty \left( \frac{1}{|A_n|} \int_{A_n} |h(s-t) - g(s-t)|^2 \, ds \right)^{\frac{1}{2}}
\]

Cauchy-Schwarz

\[
\leq \frac{2\epsilon}{2 + 2\|f\|_\infty} + \|f\|_\infty \left( \frac{1}{|A_n|} \int_{A_n} |h(s-t) - g(s-t)|^2 \, ds \right)^{\frac{1}{2}}
\]

\[
< \frac{\epsilon}{1 + \|f\|_\infty} + \|f\|_\infty \frac{\epsilon}{1 + \|f\|_\infty} = \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we get

\[
\lim_{n} \frac{1}{|A_n|} \int_{A_n} f(s) \overline{g(t-s)} \, ds = 0.
\]

Since \( t \in G \) is arbitrary, this proves the claim.

By combining Cor. 6.2 with Lemma 6.4 we get the following result.

**Lemma 6.5.** Let \( f, g \in C_\alpha(G) \) and \( A \) a van Hove sequence be so that \( f \in \mathcal{B}^2_A(G) \).
Assume that for all \( \chi \in \mathcal{F}B_A(f) \) the Fourier–Bohr coefficient \( c^A_\chi(g) \) exists and \( c^A_\chi(g) = 0 \).

Then, the following Eberlein convolutions exist and

\[
f \ast_A \hat{g} = g \ast_A \hat{f} = 0.
\]
Proof. Let \( \epsilon > 0 \). Then, by Prop. 2.26 there exists some \( c_1, \ldots, c_k \in \mathbb{C} \) and \( \chi_1, \ldots, \chi_k \in \mathcal{FB}_A(f) \) such that, with \( P = \sum_{j=1}^{m} c_j \chi_j \), we have

\[
\| f - P \|_{b_{2,A}} < \epsilon.
\]

Then, by Cor. 6.3 the following Eberlein convolution exists and

\[
g \odot \tilde{P} = 0.
\]

Therefore, by Lemma 6.4, \( g \odot_A \tilde{f} \) exists and

\[
g \odot_A \tilde{f} = 0.
\]

The claim follows now from Prop. 3.5 (c).

As an immediate consequence we get:

**Corollary 6.6.** Let \( f, g \in \mathcal{C}(G) \) and \( A \) a van Hove sequence. If \( f \in \mathcal{Bap}_A^2(G) \) and \( g \) has null Fourier–Bohr spectrum with respect to \( A \) then the following Eberlein convolutions exist and

\[
f \odot_A \tilde{g} = g \odot_A \tilde{f} = 0.
\]

We can now prove the following result. The technique used below is similar to the one in [13, Thm. 4.5]

**Proposition 6.7.** Let \( A \) be a van Hove sequence in \( G \) and \( \mu \in \mathcal{Bap}_A(G) \cap \mathcal{M}^\infty(G) \).

If \( \nu \) is any measure with the property that for all \( \chi \in \mathcal{FB}_A(\mu) \) the Fourier Bohr coefficient \( a_A^\chi(\nu) \) exists and \( a_A^\chi(\nu) = 0 \), then, the following Eberlein convolutions exists and

\[
\mu \odot_A \tilde{\nu} = \nu \odot_A \tilde{\mu} = 0.
\]

In particular, (1.3) holds for all \( \nu \in \mathcal{M}^\infty(G) \) which have null Fourier–Bohr spectrum with respect to \( A \).

**Proof.** By Lemma 6.5 for all \( \varphi, \psi \in \mathcal{C}(G) \) the Eberlein convolution \( (\varphi \ast \nu) \odot_A \overline{(\mu \ast \psi)} \) exists and

\[
(\varphi \ast \nu) \odot_A \overline{(\mu \ast \psi)} = 0.
\]

The claim follows now from Lemma 3.3.

As an immediate consequence we get:

**Corollary 6.8.** Let \( A \) be a van Hove sequence in \( G \) and \( \mu \in \mathcal{Bap}_A(G) \cap \mathcal{M}^\infty(G) \).

If \( \nu \) is any measure with the property that the autocorrelation \( \gamma_\nu \) exists with respect to \( A \), \( \gamma_\nu \) is continuous and the CPP holds for \( \nu \), then

\[
\mu \odot_A \tilde{\nu} = \nu \odot_A \tilde{\mu} = 0.
\]

This result has an interesting consequence for (TMDS).
**Theorem 6.9.** Let \((X, G, m)\) and \((Y, G, n)\) be two ergodic (TMDS), and let \(\gamma\) and \(\eta\) be the autocorrelations of \((X, m)\) and \((Y, n)\), respectively. Let \(A\) be any van Hove sequence along which the ergodic theorem holds.

Assume that \(\widehat{\gamma}\) is a pure point measure and \(\widehat{\eta}\) is a continuous measure.

Then, there exists sets \(X \subseteq X, Y \subseteq Y\) with the following properties:

(a) \(m(X) = n(Y) = 1\).

(b) For all \(\mu \in X\) the autocorrelation of \(\mu\) with respect to \(A\) is \(\gamma\).

(c) For all \(\nu \in Y\) the autocorrelation of \(\nu\) with respect to \(A\) is \(\eta\).

(d) \(X \subseteq \mathcal{B}_{\mathcal{A}}(G)\).

(e) For every \(\mu \in X\) and \(\nu \in Y\) the following Eberlein convolutions exist and \(\mu \circledast_{\mathcal{A}} \nu = \nu \circledast_{\mathcal{A}} \mu = 0\).

(f) For all \(\mu \in X_1, \nu \in Y_2\) and \(a, b \in \mathbb{C}\) the autocorrelation \(\gamma_\omega\) of \(\omega = a\mu + b\nu\) exists with respect to \(A\) and

\[
\left(\widehat{\gamma_\omega}\right)_{pp} = |a|^2 \widehat{\gamma},
\]

\[
\left(\widehat{\gamma_\omega}\right)_c = |b|^2 \widehat{\eta}.
\]

**Proof.** By [11, Thm. 5], there exists sets \(X_1 \subseteq X, Y_1 \subseteq Y\) of full measure such that, for all \(\mu \in X_1\), the autocorrelation of \(\mu\) with respect to \(A\) is \(\gamma\) and for all \(\nu \in Y_1\), the autocorrelation of \(\nu\) with respect to \(A\) is \(\eta\).

Next, since \((X, m, G)\) has pure point diffraction spectrum, the set \(X_2 = X \cap \mathcal{B}_{\mathcal{A}}(G)\) has full measure in \(X\) by [20, Cor. 6.11] or [27, Thm. 3.7].

Now, for each \(\chi \in \widehat{G}\), by [25, Thm. 5] there exists a set \(Y_\chi \subseteq Y\) of full measure such that \(c_\chi^A(\nu) = 0\) for all \(\nu \in Y_\chi\). We would like to chose \(Y_2 := \cap_{\chi \in \widehat{G}} Y_\chi\), but we do not know that this has full measure. We get around this issue by restricting to the pure point spectrum of \((X, G, m)\), which is a countable set by second countability of \(G\).

Let \(B := \{\chi \in \widehat{G} : \widehat{\gamma}(\{\chi\}) \neq 0\}\).

Note here that for all \(\mu \in X_1 \cap X_2\), by Thm 2.28 we have \(\mathcal{F}B_{\mathcal{A}}(\mu) = B\).

Define

\[
X := X_1 \cap X_2 \\
Y := Y_1 \cap (\cap_{\chi \in B} Y_\chi).
\]

The definition of \(Y\) implies that for all \(\chi \in B\) and \(\nu \in Y\) the Fourier–Bohr coefficient \(a_\chi^A(\nu)\) exists and \(a_\chi^A(\nu) = 0\).

Then \((a)-(d)\) hold.

(e) Let \(\mu \in X\) and \(\nu \in Y\). Then, \(\mathcal{F}B_{\mathcal{A}}(\mu) = B\) and hence, By Prop. 6.7 we have \(\nu \circledast_{\mathcal{A}} \mu = 0\).

Lemma 3.5 (e) completes this step.
(f) follows from \[3.2\].

Under the assumptions of Thm. 6.9, if furthermore \((X, G)\) is uniquely ergodic then \(\gamma\) is the autocorrelation of all \(\omega \in X\) [11, Thm. 5]. Moreover, the continuity of the eigenfunctions imply in this case that the CPP holds [25, Thm. 5]. In fact, in this case, all measures in \(X\) must be Weyl and hence Besicovitch almost periodic [27, Thm. 6.15].

Therefore, by repeating the proof of Thm. 6.9 we get:

**Proposition 6.10.** Let \((X, G, m)\) and \((Y, G, n)\) be two ergodic (TMDS), and let \(\gamma\) and \(\eta\) be the autocorrelations of \((X, m)\) and \((Y, n)\), respectively. Let \(A\) be any van Hove sequence along which the ergodic theorem holds.

Assume that \(\hat{\gamma}\) is a pure point measure, that \(\hat{\eta}\) is a continuous measure, that \(X\) is uniquely ergodic and has continuous eigenfunctions.

Then, there exists a set \(Y \subseteq Y\) with the following properties:

(a) \(m(X) = n(Y) = 1\).
(b) For all \(\mu \in X\) the autocorrelation of \(\mu\) with respect to \(A\) is \(\gamma\).
(c) For all \(\nu \in Y\) the autocorrelation of \(\nu\) with respect to \(A\) is \(\eta\).
(d) For every \(\mu \in X\) and \(\nu \in Y\) the following Eberlein convolutions exist and
   \[\mu \otimes_A \nu = \nu \otimes_A \mu = 0.\]
(e) For all \(\mu \in X, \nu \in Y\) and \(a, b \in \mathbb{C}\) the autocorrelation \(\gamma_\omega\) of \(\omega = a\mu + b\nu\) exists with respect to \(A\) and
   \[\gamma_\omega(p) = |a|^2 \hat{\gamma}\]
   \[\gamma_\omega(c) = |b|^2 \hat{\eta}.\]

Finally, by combining Theorem 6.9 with [2] we get the following result.

**Theorem 6.11.** Let \((X, G, m)\) be an ergodic TMDS with autocorrelation \(\gamma\), and \(A\) a van Hove sequence along which the ergodic theorem holds. Then, there exists two TB \((X_{pp}, G, m_{pp}), (X_c, G, m_c)\), two Borel factor mappings

\[\pi_{pp}: X \to X_{pp}\]
\[\pi_c: X \to X_c\]

and some set \(X' \subseteq X\) of full measure such that

(a) The diffraction of \((X_{pp}, G, m_{pp})\) is \((\hat{\gamma})_{pp}\).
(b) The diffraction of \((X_c, G, m_c)\) is \((\hat{\gamma})_c\).
(c) For all \(\omega \in X'\) we have
   \[\omega = \pi_{pp}(\omega) + \pi_c(\omega)\].
(d) For all \(\omega \in X'\) we have \(\pi_{pp}(\omega) \in \text{Bap}_A(G)\).
For all $\omega \in \mathcal{X}'$ we have
\[ \pi_{pp}(\omega) \otimes_A \pi_c(\omega) = \pi_c(\omega) \otimes_A \pi_{pp}(\omega) = 0. \]

For all $\omega \in \mathcal{X}'$ and all $a, b \in \mathbb{C}$, the autocorrelation $\gamma_\vartheta$ of $\vartheta = a\pi_{pp}(\omega) + b\pi_c(\omega)$ exists with respect to $\mathcal{A}$ and
\[ (\hat{\gamma}_{\vartheta})_{pp} = |a|^2 (\hat{\gamma})_{pp}, \]
\[ (\hat{\gamma}_{\vartheta})_c = |b|^2 (\hat{\gamma})_c. \]

**Proof.** By [2, Thm. 4.1] there exists two TB $(\mathcal{X}_{pp}, G, m_{pp}), (\mathcal{X}_c, G, m_c)$, two Borel factor mappings
\[ \pi_{pp} : \mathcal{X} \to \mathcal{X}_{pp}, \]
\[ \pi_c : \mathcal{X} \to \mathcal{X}_c \]
and a set $Z \subseteq \mathcal{X}$ of full measure such that
- The diffraction of $(\mathcal{X}_{pp}, G, m_{pp})$ is $(\hat{\gamma})_{pp}$.
- The diffraction of $(\mathcal{X}_c, G, m_c)$ is $(\hat{\gamma})_c$.
- For all $\omega \in Z$ we have
  \[ \omega = \pi_{pp}(\omega) + \pi_c(\omega). \]

Next, let $X \subseteq \mathcal{X}_{pp}$ and $Y \subseteq \mathcal{X}_c$ be the sets of full measure given by Thm. 6.9. Since $\pi_{pp}$ and $\pi_c$ are Borel factor maps, the set $\pi_{pp}^{-1}(X), \pi_c^{-1}(Y)$ have full measure in $\mathcal{X}$ and hence so does
\[ \mathcal{X}' = Z \cap (\pi_{pp}^{-1}(X)) \cap (\pi_c^{-1}(Y)). \]

The claims follow immediately. \qed

**Remark 6.12.** Under the conditions of Thm. 6.11 if there exists a weak model set $\Lambda(W)$ (see the monograph [2] or [32, 54, 55] for definitions and properties) such that each measure $\omega \in \mathcal{X}$ is supported inside a translate of $\Lambda(W)$, then, by [2 Thm. 5.10], $\mathcal{X}'$ can be chosen so that $\pi_{pp}(\omega), \pi_c(\omega)$ are supported inside a translate of $\Lambda(W)$ for all $\omega \in \mathcal{X}'$.

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