Robust control design for path tracking of non-affine UAV

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ABSTRACT
Path tracking of Unmanned Aerial Vehicles (UAVs) with three degrees of freedom is studied in this paper with approach of dynamic sliding mode control. For this purpose, the equations of UAV are written. The difficulty and complexity of these equations is that they are non-affine with respect to control inputs. Moreover, they are not directly in the inertial coordinate system while the desired flight path is given in the inertial coordinate system. These two major problems add complexity to the design procedure. Therefore, it is necessary that equations be rewritten in the inertial coordinate system. By definition of virtual inputs; the equations convert to affine structure with respect to virtual inputs and the transformation between the real and virtual inputs has been obtained. After that, the Input/output (I-O) equations of the system are written and converted into controller canonical form. The dynamic sliding mode control law is then designed based on (I-O) equations. Optimal coefficients are also achieved numerically by considering an appropriate cost function. Finally, computer simulation is utilized to illustrate the performance of the designed controller.

1. Introduction
UAV is one of the most important nonlinear systems which has applications in civilian programmes such as rescue operation and extinguish fires. They are also applied for guarding the frontier and identification of natural disasters (Dupont, Chua, Tashrif, & Abbott, 2017). Different control methods have been studied for UAV control, like PID (Misra, Bhattacharjee, Goswami, & Ghosh, 2016; Pounds, Bersak, & Dollar, 2012; Sarhan & Qin, 2016), optimal control (Melnyk, Zhiteckii, Bogatyriv, & Pilchevsky, 2013; Nodland, Zargarzadeh, & Jagannathan, 2013), backstepping (Aznheira & Moutinho, 2008; Zheng, Zhen, & Gong, 2017), control Lyapunov function (Shafiei & Binazadeh, 2012), passivity based control (Chenarani & Binazadeh, 2017), $H_{\infty}$ control (Jiao, Du, Wang, & Xie, 2010; Kerma, Mokhtari, Abdelaziz, & Orlov, 2012), sliding mode (Castañeda, Salas-Peña, & de León-Morales, 2017; Wu, Cai, Zhao, & Wang, 2017) and dynamic sliding mode (Tavakol & Binazadeh, 2015).

Among the nonlinear robust control methods, the dynamic sliding mode technique has especial characteristics. This technique is a robust technique in stabilization of nonlinear systems with uncertainty and has greater benefits than the classical sliding mode technique (Wang, Bao, & Li, 2017 Yan, Spurgeon, & Edwards, 2005). Dynamic sliding mode controller is designed based on output information and requires no observer designing. This characteristics makes the dynamic sliding mode method as an efficient robust controller for practical implementations. However; designing the autopilot for path tracking, based on dynamic sliding mode technique for UAV with three degrees of freedom is not simply possible because of the two main problems. First, UAV equations are non-affine with respect to their inputs. Moreover, the dynamical equations are not directly in the inertial coordinate system while the desired flight path is given in the inertial coordinate system.

Robust autopilot designing for path tracking of UAV with three degrees of freedom is investigated in this paper. The main contributions of this paper are: new autopilot design for putting UAV on the predetermined desired flight path based on dynamic sliding mode technique and tuning the coefficients with the optimality approach. For this purpose, after rewriting the dynamical equations in the inertial coordinate system, the virtual input vector is defined such that the state space equations have affine structure with respect to virtual inputs. Also, the transformation between the real and virtual inputs are obtained. Then considering the error vector of position as the output vector, the (I-O) equations of the system have been written and converted into controller canonical form. The obtained canonical subsystems have suitable structures which are adequate for design of sliding surfaces. These surfaces contain coefficients which
affect the transient response and performance of the system. The optimal values of these coefficients are obtained by definition of an appropriate cost function and using a numerical algorithm. Finally, computer simulations are performed to show the performance of the designed control law.

2. UAV equations in three-dimensional space

UAV point mass model with three degrees of freedom is considered as follows which is a nonlinear and multivariable model (Boškovic, Chen, & Mehra, 2004).

\[
\begin{align*}
\dot{V} &= g_0 \left( \frac{T - D}{W} - \sin(\gamma) \right) \\
\dot{\gamma} &= g_0 \frac{\dot{\gamma}}{V} (\bar{n} \cos(\mu) - \cos(\gamma)) \\
\dot{\theta} &= g_0 \frac{\dot{\gamma}}{\cos(\gamma)} 
\end{align*}
\]

where airspeed \(V\), flight path \(\gamma\) and flight path-heading \(\theta\) are system state variables and thrust force \(T\), load factor \(\bar{n}\) and bank angle \(\mu\) represent control inputs. \(D\) is the representation of drag force that can be described as follows:

\[
D = (0.5 \rho S_0 C_{D0}) V^2 + \frac{2k W^2}{\rho S_0} \bar{n}^2 
\]

where, \(W, C_{D0}, k, \rho, S_0\), and \(g_0\) are UAV weight, parasite drag coefficient, induced drag coefficient, density, reference area and acceleration due to the Earth gravity constant, respectively.

3. Obtain appropriate equations for tracking

Equation (1) shows that motion components are not directly in the inertial coordinate system. In order to achieve the control target in path tracking, it is necessary to define the state variables based on the inertial coordinate system. On the other hand, the Equation (1) has non-affine structure with respect to their inputs \((T, \bar{n}, \mu)\). These are two main difficulties that will be solved in the following.

3.1. 4-1-motion equations in inertial coordinate system

The position vector of UAV is shown with \(\bar{x} = [x, y, z]^T\) where \(x, y,\) and \(z\) are longitudinal, transverse and vertical positions of UAV in the inertial coordinate system, respectively. Derivative of the position vector is the velocity vector \(\dot{\bar{x}}\). Velocity vector in the space of inertial coordinate system has three components of longitudinal speed, transverse speed and vertical speed. Therefore, the components of velocity vector are obtained as Equation (3) and are shown in Figure 1.

\[
\bar{v} = \begin{bmatrix} V \cos(\gamma) \cos(\theta) \\ V \cos(\gamma) \sin(\theta) \\ V \cos(\gamma) \end{bmatrix}
\]

Define \(\bar{q} = [V, \gamma, \theta]^T\) and \(\bar{x} = \bar{h}(\bar{q})\) where \(\bar{h}\) is a non-linear vector function of state vector \(\bar{q}\). Now by applying the chain rule in differentiation of Equation (3), one has:

\[
\bar{x} = \frac{\delta \bar{h}}{\delta \bar{q}} q
\]

Substituting Equation (1) in Equation (4), yields:

\[
\bar{x} = \begin{bmatrix} \cos(\gamma) \cos(\theta) \\ \cos(\gamma) \sin(\theta) \\ \cos(\gamma) \end{bmatrix} \begin{bmatrix} \frac{V}{\cos(\gamma)} \\ \frac{V}{\cos(\gamma)} \\ \frac{V}{\cos(\gamma)} \end{bmatrix}
\]

\[
\times \begin{bmatrix} g_0 \left( \frac{T - D}{W} - \sin(\gamma) \right) \\ g_0 \frac{\dot{\gamma}}{V} (\bar{n} \cos(\mu) - \cos(\gamma)) \\ g_0 \frac{\dot{\gamma}}{\cos(\gamma)} \end{bmatrix}
\]

\[
\bar{x} = g_0 \begin{bmatrix} \cos(\gamma) \cos(\theta) - \sin(\gamma) \cos(\theta) - \sin(\theta) \\ \cos(\gamma) \sin(\theta) - \sin(\gamma) \sin(\theta) - \cos(\theta) \\ \cos(\gamma) \end{bmatrix}
\]

\[
\times \begin{bmatrix} \frac{T - D}{W} - \sin(\gamma) \\ \frac{\bar{n} \cos(\mu) - \cos(\gamma)}{\bar{n} \sin(\mu)} \end{bmatrix}
\]
Thus, one has:
\[
\ddot{x} = g_0 \ddot{J}(\gamma, \theta) \begin{bmatrix} -\sin(\gamma) \\ -\cos(\gamma) \\ 0 \end{bmatrix} + g_0 \dot{J}(\gamma, \theta) \begin{bmatrix} T-D \over W \\ -\bar{n} \cos(\mu) \\ \bar{n} \sin(\mu) \end{bmatrix}.
\]

The above phrase is composed of two sentences. After multiplying, the result of the first sentence is as follows:
\[
g_0 \begin{bmatrix} \cos(\gamma) \cos(\theta) & -\sin(\gamma) \cos(\theta) & -\sin(\gamma) \\ \cos(\gamma) \sin(\theta) & -\sin(\gamma) \sin(\theta) & \cos(\theta) \\ \sin(\gamma) & \cos(\gamma) & 0 \end{bmatrix} \begin{bmatrix} -\sin(\gamma) \\ -\cos(\gamma) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix}.
\]

Thus the acceleration equations are obtained as follows:
\[
\ddot{x} = \begin{bmatrix} 0 \\ 0 \\ -g_0 \end{bmatrix} + g_0 \ddot{J}(\gamma, \theta) \begin{bmatrix} T-D \over W \\ -\bar{n} \cos(\mu) \\ \bar{n} \sin(\mu) \end{bmatrix}.
\]

Therefore Equation (1) has been converted into Equation (6) which show the UAV motion equations in the inertial coordinate system. Equation (6) is still non-affine with respect to control inputs (i.e. \( T, \bar{n}, \mu \)). This problem will be solved in the following.

### 3.2. 4-2-convert the equation to affine structure

The structure of Equation (6) is non-affine with respect to their control inputs. To obtain the affine structure for UAV model, the virtual input vector is defined as follows.

\[
\ddot{\bar{u}}^* = \begin{bmatrix} T-D \over W \\ \bar{n} \cos(\mu) \\ \bar{n} \sin(\mu) \end{bmatrix} = \begin{bmatrix} u_1^* \\ u_2^* \\ u_3^* \end{bmatrix}^T.
\]

Therefore
\[
\ddot{x} = \ddot{\bar{a}}_g + g_0 \ddot{J}(\gamma, \theta) \ddot{\bar{u}}^* \tag{7}
\]

where \( \ddot{\bar{a}}_g \) is as follows:

\[
\ddot{\bar{a}}_g = \begin{bmatrix} 0 & 0 & -g_0 \end{bmatrix}^T.
\]

Although, Equation (7) has the affine structure with respect to the virtual control vector \( \ddot{\bar{u}}^* \), the real inputs are \( T, \bar{n} \) and \( \mu \) and the goal is designing the real inputs.

In order to solve this difficulty, after designing \( \ddot{\bar{u}}^* \), the real inputs have been obtained from the following relations:

\[
T = Wu^*_1 + aV^2 + b \left( u^*_1 u^*_1 + u^*_2 u^*_2 \right) \over V^2; \tag{8}
\]

\[
\bar{n} = \sqrt{u^*_2 u^*_2 + u^*_3 u^*_3};
\]

\[
\mu = \tan^{-1} \left( \frac{u^*_3}{u^*_2} \right).
\]

In which, \( a = 0.5 \rho S C_{D0} \) and \( b = 2kW^2 / \rho S \).

### 4. Designing optimal dynamic sliding mode control for UAV

As stated before, the task is tracking the desired flight path by UAV. The position vector of the desired flight path is stated as \( \bar{x}_d(t) = [x_d(t), y_d(t), z_d(t)]^T \). The task is designing the virtual control law \( \ddot{\bar{u}}^* \) in such a way that \( x(t) - x_d(t) \to 0, y(t) - y_d(t) \to 0 \) and \( z(t) - z_d(t) \to 0 \).

After that the real inputs will be achieved according to Equations (8). Consequently, the error vector of position (i.e. \( \ddot{\bar{e}}(t) = \bar{x}(t) - \bar{x}_d(t) \)) is considered as the output vector of the system. The control objective is asymptotic convergence of output vector toward the zero.

#### 4.1. I-O equations for UAV motion

Conversion of equations into the controllable canonical form is the next step for implementation of dynamic sliding mode method. For this purpose, input/output model of system should be obtained. Define output vector as follows:

\[
\ddot{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} x(t) - x_d(t) \\ y(t) - y_d(t) \\ z(t) - z_d(t) \end{bmatrix}.
\]

By differentiation from each output and obtaining \( n_i \) (where \( n_i \) is the relative degree related to \( i \)th output for \( i = 1, 2, 3 \)), one has:

\[
\dot{y}_1 = \ddot{x} - \dot{x}_d \\
\dot{y}_2 = \ddot{y} - \dot{y}_d
\]

In Equation (6), \( \ddot{x}, \ddot{y}, \ddot{z} \) are elements of \( \ddot{x} \) vector. Thus, \( u_1^* \) appears in \( \ddot{y}_1 \) and therefore, \( n_1 = 2 \) is the relative degree of the first output. By differentiating the second and third output, one has:

\[
\dot{y}_2 = \ddot{y} - \dot{y}_d \\
\dot{y}_3 = \ddot{z} - \dot{z}_d
\]

and

\[
\ddot{y}_3 = \ddot{z} - \dot{z}_d
\]

Considering Equation (6), \( u^*_2 \) appears in \( \ddot{y}_2 \) and \( u^*_3 \) appears in \( \ddot{y}_3 \), hence, \( n_2 = n_3 = 2 \).

#### 4.2. Controller canonical equations

To transfer I-O equations to controllable canonical form, the following state variables are define:

\[
\xi_1^{(1)} = x - x_d
\]
\[ \xi_2^{(1)} = \dot{x} - \dot{x}_d \]
\[ \xi_1^{(2)} = y - y_d \]
\[ \xi_2^{(2)} = \dot{y} - \dot{y}_d \]
\[ \xi_1^{(3)} = z - z_d \]
\[ \xi_2^{(3)} = \dot{z} - \dot{z}_d \]

Now, the I-O equations may be rewritten in the following three subsystems which have the controllable canonical structure:

First subsystem:
\[
\begin{align*}
\dot{\xi}_1^{(1)} &= \phi_1(\xi_1^{(1)}, u^*, t) + \Delta_1 \\
\dot{\xi}_2^{(1)} &= \phi_2(\xi_2^{(1)}, u^*, t) + \Delta_2
\end{align*}
\] (10)

Second subsystem:
\[
\begin{align*}
\dot{\xi}_1^{(2)} &= \phi_1(\xi_1^{(2)}, u^*, t) + \Delta_1 \\
\dot{\xi}_2^{(2)} &= \phi_2(\xi_2^{(2)}, u^*, t) + \Delta_2
\end{align*}
\] (11)

Third subsystem:
\[
\begin{align*}
\dot{\xi}_1^{(3)} &= \phi_1(\xi_1^{(3)}, u^*, t) + \Delta_1 \\
\dot{\xi}_2^{(3)} &= \phi_2(\xi_2^{(3)}, u^*, t) + \Delta_2
\end{align*}
\] (12)

where \( \Delta_i \)s represent uncertain terms due to parametric uncertainties or external disturbances. It is assumed that the upper bound of \( |\Delta_i| \) is known and

\[ |\Delta_i| \leq \rho_i \| \xi_i^{(i)} \| + I_i \quad i = 1, 2, 3; \rho_i \geq 0; I_i \geq 0 \] (13)

and \( \xi^{(i)} = [\xi_1^{(i)}, \xi_2^{(i)}]^T \). Since system (10) has three inputs \( (u_1^*, u_2^*, u_3^*) \), and three subsystem, thus three sliding surfaces should be designed. The sliding surfaces are designed such that the reduced-order model of each subsystem (which is the motion equations on the related sliding surface) be asymptotically stable.

\[
\begin{align*}
s_1 &= \sum_{j=1}^{2} a_1^{(1)} \xi_j^{(1)} - a_2^{(1)} \xi_2^{(1)} \\
s_2 &= \sum_{j=1}^{2} a_1^{(2)} \xi_j^{(2)} - a_2^{(2)} \xi_2^{(2)} \\
s_3 &= \sum_{j=1}^{2} a_1^{(3)} \xi_j^{(3)} - a_2^{(3)} \xi_2^{(3)}
\end{align*}
\] (14)

Without loss of generally, considering \( a_1^{(1)} = a_2^{(2)} = a_2^{(3)} = 1 \), then:
\[
\begin{align*}
s_1 &= a_1^{(1)} \xi_1^{(1)} + \xi_2^{(1)} \\
s_2 &= a_1^{(2)} \xi_1^{(2)} + \xi_2^{(2)} \\
s_3 &= a_1^{(3)} \xi_1^{(3)} + \xi_2^{(3)}
\end{align*}
\] (15)

Now, the following equation which is the reaching condition will be solved to get the control law \( \bar{u}^* \) (see more details about the reaching condition in (Edwards & Spurgeon, 2002)).

\[
\dot{s} = -K \bar{s} - K_0 s_i \Theta + \Delta
\] (16)

where \( s = [s_1, s_2, s_3]^T \), \( \Delta = [\Delta_1, \Delta_2, \Delta_3]^T \), \( K = [k_0] \in R^{3 \times 3} \), \( K_0 = diag(k_0) \in R^{3 \times 3} \), \( s_i = [s_i(s_1), s_i(s_2), s_i(s_3)]^T \) and \( sat_s(s_i) \) is defined as follows:

\[
sat_s(s_i) = \begin{cases} 
1 & s_i > \varepsilon \\
\frac{s_i}{\varepsilon} & |s_i| \leq \varepsilon \\
-1 & s_i < \varepsilon 
\end{cases}
\] (17)

This results in

\[
\bar{u}^* = J^{-1}(\gamma, \theta) \left( \frac{1}{g_0} \Theta + \bar{u}_d - \bar{u}_c - \bar{a}_g \right)
\] (18)

where

\[
\begin{align*}
\bar{u}_c &= \begin{bmatrix} a_1^{(1)} s_2^{(1)} \\
a_1^{(2)} s_2^{(2)} \\
a_1^{(3)} s_2^{(3)}
\end{bmatrix} \\
\bar{u}_d &= \begin{bmatrix} -k_{11} s_1 - k_{12} s_2 - k_{13} s_3 - k_{01} sat_s(s_1) \\
-k_{21} s_1 - k_{22} s_2 - k_{23} s_3 - k_{02} sat_s(s_2) \\
-k_{31} s_1 - k_{32} s_2 - k_{33} s_3 - k_{03} sat_s(s_3)
\end{bmatrix};
\end{align*}
\] (19)

where the free parameters \( k_{ij}, k_{0i} \) and \( a_1^{(i)} \) (for \( i = 1, 2, 3 \) and \( j = 1, 2, 3 \)) will be chosen with an optimal numerical solver (like PSO or genetic toolbox of Matlab) such that the following cost function be minimized.

\[
J = \int_0^T (\bar{y}^T Q \bar{y} + \bar{u}^T R \bar{u}) \, dt
\] (20)

5. Simulations

In this section computer simulations are done to show the performance of the proposed control law. The parameters are selected as \( \rho = 1.2251 \text{ kg/m}^3 \), \( S_0 = 37.16 \text{ m}^2 \), \( W = 14.515 \text{ kg} \), \( k = 0.1 \) and \( C_{D0} = 0.02 \).
Figure 2. Time responses of desired and actual position vector.

Figure 3. Desired and actual flight paths.
Firstly, the motion equations of UAV were written. Then the dynamical model was rewritten in the inertial coordinate system. Moreover, by define of virtual input vector, the equations was converted to the affine structure with respect to virtual control inputs and the transformation between the real inputs and virtual inputs was given. By definition of output vectors as the difference between the current position in the inertial system and the desired one, the I-O equations were achieved and rewritten in canonical controller form which was adequate for designing sliding surfaces. Furthermore, dynamic sliding mode control law has been calculated based on the designed sliding surfaces and the desired reaching law. Additionally, the appropriate cost function was given and free coefficients in control law were obtained by optimality approach. Computer simulations shown the performance of the proposed control law.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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