A compact group action which raises dimension to infinity

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1 Introduction

Interest in group actions with the property that the orbit space has greater dimension than the original space is motivated in part of P.A. Smith’s generalization of the Hilbert’s fifth problem. It asserts that among all locally compact groups only Lie group can act effectively on manifolds. It is now known as the Hilbert-Smith conjecture. The Hilbert-Smith conjecture is known to be equivalent to the conjecture that a compact topological group acting on a compact topological manifold cannot have arbitrarily small subgroups. By the work of Newman and Smith, it is sufficient to prove the special case when the topological group is the $\hat{\mathbb{Z}}_p$, where $\phi_n$ is the mod $p^n$ mapping.

C.T. Yang showed that for any $n$-manifold $M^n$, the orbit space $M/\hat{\mathbb{Z}}_p$ must have integral cohomological dimension $\dim_{\mathbb{Z}}M/\hat{\mathbb{Z}}_p = n + 2$, and more generally, for any locally compact Hausdorff cohomological $n$-dimensional space $X$ supporting an effective $\hat{\mathbb{Z}}_p$ action, $\dim_{\mathbb{Z}}X/\hat{\mathbb{Z}}_p \leq n + 3$. For locally compact finite dimensional metric space $X$, $\dim_{\mathbb{Z}}X = \dim X$, hence Yang’s results imply that if $X$ is metrizable, then $\dim X/\hat{\mathbb{Z}}_p \leq n + 3$ or $\dim X/\hat{\mathbb{Z}}_p = \infty$.

Although no effective action of $p$-adic group on manifolds has yet been constructed, there do exist $p$-adic group actions on Menger manifolds. Let $X$ be a $\mu^n$-manifold. Then there are three constructions of $\hat{\mathbb{Z}}_p$ actions on $X$.

(1) Every compact 0-dimensional metrizable group $G$ acts effectively on $X$ so that $\dim X/G = \dim X$. Hence $p$-adic group acts on Menger compacta.
This is done by Dranishnikov [4] and Mayer and Stark [8] using Pashynkov's partial product description of $\mu^n$. K. Sakai [10] has another construction.

(2) $\hat{\mathbb{Z}}_p$ acts freely on $X$ so that $\dim X/\hat{\mathbb{Z}}_p = n + 1$. This example depends on the work of Bestvina, Edwards, Mayer and Stark [8].

(3) $\hat{\mathbb{Z}}_p$ acts on $X$ so that $\dim X/\hat{\mathbb{Z}}_p = n + 2$. This is done by Mayer and Stark [8], based on a construction of Raymond and Williams [9].

In contrast, no example has been constructed where $\dim X/\hat{\mathbb{Z}}_p = \infty$.

A.N. Dranishnikov and J.E. West proved that any compact metric space $Y$ is the orbit space of a $G$-cation on a metric compactum $X$ with $\dim \mathbb{Z}_p X = 1$, where $G = \prod_{i=1}^{\infty} (\mathbb{Z}/p)$. But their paper has an error in their lemma 15.

The goal of this paper is to fix their error and generalize their group $G$ to $\hat{\mathbb{Z}}_p$.

2 The construction

Notations:

(1) For a given set $S$ of positive integers, we define $G_S = \prod_{i \in S} \mathbb{Z}/p^i$.

(2) Given a sequence $K = \{k_i\}_{i=1}^{\infty}$, we define $K^* = \{k_{2n-1}\}_{n=1}^{\infty}$.

(3) If $B = \{k_i | i \in \Lambda\}$, where $\Lambda$ is a subset of $N$ (the set of all positive integers), we define $K - B = \{k_i\}_{i \notin \Lambda}$.

(4) For the sets of integers in this paper, we will allow them to contain repeating elements, and the union is always a disjoint union. If you like, you can think them as sequences, finite or infinite.

Given a prime $p$, an infinite sequence $K$ of positive integers, and any finite dimensional simplicial complex $L$, we will define $\hat{L}(K)$.

Let’s work on a simplex first. Let $L = \Delta^m$, and $L'$ be the first barycentric subdivision of $L$. Let $\hat{L}^{(0)} = L^{(0)} = \hat{L}_0$, $\hat{L}^{(1)} = L^{(1)} = \hat{L}_1$, $K_0 = K_1 = K$, $B_0 = B_1 = \emptyset$, $N_0 = N_1 = 1$. We will inductively construct the following diagram:
Suppose $s_k : \tilde{L}_{(k)} \xrightarrow{q_k} \tilde{L}_{(k)} \xrightarrow{p_k} L^{(k)'}$ have been constructed, such that:

- (k.1) for each $(k + 1)$-simplex $\Delta$ of $L^{(k+1)}$, $s_k^{-1}(\Delta(k))$ is a closed oriented $k$-manifold,
- (k.2) the map $r_{k-1} : s_k^{-1}(L^{(k-1)'} \rightarrow \tilde{L}_{(k-1)}$ is a $N_{k-1}-1$ covering map,
- (k.3) for any simplex $\sigma \in \Delta^{(k)'}$, $s_k : s_k^{-1}(\sigma) \rightarrow \sigma$ is a $\prod_{i=1}^k N_i$ to 1 covering map,
- (k.4) for any $(k)$-simplex $\Delta$ of $L^{(k)}$, the induced map $H^1(s_k^{-1}(\Delta^{(k-1)}), \mathbb{Z}/p) \rightarrow H^1(s_k^{-1}(\Delta^{(k-1)}), \mathbb{Z}/p)$ is trivial,
- (k.5) $\pi_1(s_k^{-1}(\Delta^{(k-1)}))/\pi_1(s_k^{-1}(\Delta^{(k-1)})) \cong G_{B_{k-1}}$,
- (k.6) $G_{B_{k}}$ acts on $s_k^{-1}(L^{(k-1)'} \rightarrow \tilde{L}_{(k-1)}$ with $\tilde{L}_{(k-1)}$ as orbit space. It is a free simplicial action,
- (k.7) $\prod_{i=1}^k G_{B_{i}}$ acts on $s_k^{-1}(\Delta^{(k)})$ simplicially, and the orbit space is $\Delta^{(k)}$.
- Hence $\prod_{i=1}^k G_{B_{i}}$ acts on $L_{(k)}$ simplicially, and the orbit space is $L^{(k)'}$,
- (k.8) The map $q_k : \tilde{L}_{(k)} - s_k^{-1}(L^{(k-1)'} \rightarrow L_{(k)} \tilde{L}_{(k-1)}$ is a homeomorphism.

**Lemma 1.** $H^1(s_k^{-1}(\Delta^{(k)}))$ is a free abelian group. There exists a regular covering space $E$ of $s_k^{-1}(\Delta^{(k)})$, such that

$$\pi_1(E)/\pi_1(s_k^{-1}(\Delta^{(k)})) \cong G_{B_{k+1}}$$

Hence $G_{B_{k+1}}$ acts on $E$ as the deck transformation group.

**Proof.** $s_k^{-1}(\Delta^{(k)})$ is an oriented $k$-dimensional closed manifold, hence their $H_{k-1}$ is a free abelian group, so is its $H^1$.

Let $G$ denote $\pi_1(s_k^{-1}(\Delta^{(k)}))$, $G^1$ be the commutator subgroup of $G$, $\{g_1, g_2, \cdots, g_l\}$ be the set of free generators of $H_1(s_k^{-1}(\Delta^{(k)}))$, and $T$ be the
torsion subgroup of $H_1$. Let $f$ denote the natural map $\pi_1 \to \pi_1/G^1 \cong H_1$. Denote the first $l$ integers in $K_k$ by $n_1, n_2, \ldots, n_l$. Define $B_{k+1} = \{n_1, n_2, \ldots, n_l\}$, $N_{k+1} = N_k \times \prod_{i=1}^l p^{n_i}$. Let $G'$ denote the subgroup of $\pi_1$ generated by $f^{-1}(\{g_1^{(n_1)}, g_2^{(n_2)}, \ldots, g_l^{(n_l)}\} \cup T)$. $(G^1 \subset G')$ It is a normal subgroup of $G$. $G/G' \cong \prod_{i=1}^l Z/(p^{n_i}) = G_{B_{k+1}}$. There is a canonical regular covering space of $s_k^{-1}(\Delta(k))$, denoted as $E$, corresponding to the group $G'$. $E$ is a closed oriented $k$-manifold. $r_k : E \to s_k^{-1}(\Delta(k))$ is a $|G : G'| = \prod_{i=1}^l p^{n_i}$ covering. (This serves the proof for (k+1.2) and (k+1.3)), and $G/G' \cong G_{B_{k+1}}$ acts on $E$ as the deck transformation group, with $s_k^{-1}(\Delta(k))$ as orbit space. (This serves the proof for (k+1.5) and (k+1.6)).

For any $(k+1)$-simplex $\Delta$ of $L^{(k+1)}$, we get the covering space $E$. Let $cE$ be the cone of $E$. It is a contractible oriented $(k+1)$-manifold with boundary. There is a natural projection $s_{k+1} : cE \to \Delta$, which is a branch covering. $G_{B_{k+1}}$ acts on $cE$ in the obvious way. (This serves the proof for (k+1.7)). By (k.3), for each $\sigma \in \Delta'$, $s_{k+1} : (s_{k+1})^{-1}(\sigma) \to \sigma$ is $\prod_{i=1}^{k+1} N_i$ to 1 covering map. (This proves (k+1.3)). Let $K_{k+1} = K_k - B_{k+1}.$

**Case 1**: If $k + 2 \leq m$, then for any given a $(k+2)$-simplex $\Delta$ of $L^{(k+2)}$, it has $k+3$ faces, say $\Delta_1, \Delta_2, \ldots, \Delta_{k+3}$. For them, using the above construction, we get cones $c_1E_1, c_2E_2, \ldots, c_{k+3}E_{k+3}$.

For any $i \neq j$, $E_i$ is homeomorphic to $E_j$. $\Delta_i$ meets $\Delta_j$ along one $k$-simplex $\sigma$. We pick one homeomorphism $h_{ij} : E_i \to E_j$, such that the following diagram commute:

$$
\begin{array}{ccc}
E_i & \xrightarrow{h_{ij}} & E_j \\
\downarrow{r_k} & & \downarrow{r'_k} \\
\Delta_i & \cong & \Delta_j \\
\end{array}
$$

and $h_{ij}(r_k s_k)^{-1}(\sigma) = (r'_k s'_k)^{-1}(\sigma)$.

Denote the projection $E \to \Delta(k)$ by $Pr$. Given $\sigma_i \in E_i$ and $\sigma_j \in E_j$, if $Pr(\sigma_1) = Pr(\sigma_2) = \sigma$ and $h_{ij}(\sigma_1) = \sigma_2$, we glue the $(k+2)$-simplexes $c_i \sigma_1, c_j \sigma_2$ of $c_i E_i, c_j E_j$ along $\sigma_1$ and $\sigma_2$ via $h_{ij}$. Do the same for all the pairs.
$E_i$ and $E_j$. It can be proved easily that each $\sigma$ is glued to exactly one other $\sigma'$. So we get a $(k + 1)$-manifold. (This proves (k+1.3)).

Do the same for all $(k+2)$-simplexes of $L^{(k+2)}$, we get a $k + 1$ dimensional complex, denote it as $\tilde{L}_{(k+1)}$.

**Case 2:** If $k + 2 > m$, then $k + 1 = m$. For the unique $m$-simplex $\Delta$, we get a cone $cE$. It is the $\tilde{L}_{(m)}$.

Now we want to have the following diagram:

```
\begin{array}{cccc}
  s_{k+1}^{-1}(L^{(k)'}) & \rightarrow & \bar{L}_{(k)} & \rightarrow & \bar{L}_{(k+1)} \\
  \downarrow & & \downarrow & & \downarrow \\
  L^{(k)} & \rightarrow & \tilde{L}^{(k+1)} & \rightarrow & \tilde{L}^{(k+1)'} \\
  \downarrow & & \downarrow & & \downarrow \\
  L^{(k)'} & \rightarrow & \Delta_{m} & \rightarrow & (\Delta_{m})' \\
  \end{array}
```

The only thing missing here is $\tilde{L}^{(k+1)}$. Since we have the map $r_k \circ p_k : s_{k+1}^{-1}(L^{(k)'}) \rightarrow \bar{L}_{(k)} \rightarrow \tilde{L}^{(k)}$, we can define $\tilde{L}^{(k+1)}$ as the following: in $\bar{L}_{(k+1)}$, take any $k$-simplexes $\sigma_1, \sigma_2$, if $r_k \circ p_k(\sigma_1) = r_k \circ p_k(\sigma_2)$, then we glue them together. (This proves (k+1.8), and also serves the proof for (k+1.7)). Hence we get an inclusion: $\tilde{L}^{(k)} \rightarrow \tilde{L}^{(k+1)}$. This finishes the construction for the step $k + 1$.

Let $K' = \cup_{i=1}^{m} B_i$. We have $G_{K'}$ acts on $\hat{L}$ with $L'$ as orbit space.

For general $m$-complex, we define $\hat{L}(K)$ to be the pull back:

```
\begin{array}{cccc}
  \hat{L} & \overset{\text{homeo}}{\longrightarrow} & \hat{L}' & \longrightarrow & \hat{\Delta} \\
  \downarrow & & \downarrow & & \downarrow \\
  L & \overset{1st \text{ subdivision}}{\longrightarrow} & L' & \longrightarrow & \Delta_{m} \overset{\text{homeo}}{\longrightarrow} (\Delta_{m})' \\
  \end{array}
```

**Lemma 2.** For each $(k + 1)$-simplex $\Delta$ of $L^{(k+1)}$, the induced map $q_k^*: H^1(s_{k}^{-1}(\Delta^{(k)}), \mathbb{Z}/p) \rightarrow H^1(E = s_{k+1}^{-1}(\Delta^{(k)}), \mathbb{Z}/p)$ is trivial.
Proof. By the universal coefficient theorem for homology, we have: $0 \to H_1(C) \otimes \mathbb{Z}/p \to H_1(C; \mathbb{Z}/p) \to \text{Tor}(H_0(C), \mathbb{Z}/p) \to 0$. In the case $H_0(C)$ is free, we have $H_1(C) \otimes \mathbb{Z}/p \cong H_1(C, \mathbb{Z}/p)$.

Using the relation between $\pi_1$ and $H_1$, we get the following commutative diagram:

$$
\begin{array}{cccc}
\pi_1(E) & \rightarrow & H_1(E) & \rightarrow \oplus_{\mathbb{Z}/p} H_1(E; \mathbb{Z}/p) \\
\downarrow & & \downarrow & \\
\pi_1(s_k^{-1}(\Delta(k))) & \rightarrow & H_1(s_k^{-1}(\Delta(k))) & \rightarrow \oplus_{\mathbb{Z}/p} H_1(s_k^{-1}(\Delta(k)); \mathbb{Z}/p)
\end{array}
$$

The image of $\pi_1(E)$ in $\pi_1(s_k^{-1}(\Delta(k)))$ is $G'$, while $G'$'s image in $H_1(s_k^{-1}(\Delta(k)); \mathbb{Z}/p)$ is 0. The map $\pi_1(E) \to H_1(E) \to H_1(E; \mathbb{Z}/p)$ is onto. Hence we know the map $q_{k*}: H_1(E; \mathbb{Z}/p) \to H_1(s_k^{-1}(\Delta(k)); \mathbb{Z}/p)$ is 0.

For any chain complex $C$, by the universal coefficient theorem for cohomology, we have: $0 \to \text{Ext}(H_0(C), \mathbb{Z}/p) \to H^1(C; \mathbb{Z}/p) \to \text{Hom}(H_1(C, \mathbb{Z}/p) \to 0$. In the case $H_0(C)$ is free, we have $H^1(C; \mathbb{Z}/p) \cong \text{Hom}(H_1(C), \mathbb{Z}/p)$. And we have the following commutative diagram:

$$
\begin{array}{cccc}
H^1(E; \mathbb{Z}/p) & \rightarrow & \text{Hom}(H_1(E), \mathbb{Z}/p) \\
\downarrow & & \downarrow (q_{k*})^* \\
H^1(s_k^{-1}(\Delta(k)); \mathbb{Z}/p) & \rightarrow & \text{Hom}(H_1(s_k^{-1}(\Delta(k))), \mathbb{Z}/p)
\end{array}
$$

Since $q_{k*} = 0$, we know $q_{k*}^* = 0$. \qed

Lemma 3. Given any space $M$ and any commutative diagram:

$$
\begin{array}{cccc}
L_0 & \rightarrow & L_1 & \rightarrow \cdots & \rightarrow & L_m \\
| & \uparrow & | & \uparrow & \cdots & \uparrow \\
| & f_0 & | & f_1 & \cdots & f_m \\
M & \rightarrow & M & \rightarrow \cdots & \rightarrow & M
\end{array}
$$

it uniquely determines a map $f': \hat{L} \to M$ such that, for any $x \in \hat{L}_k$, $f'(s_k^{-1}(L(k-1)) = f_k(x)$, where $q_k$ is the projection $\hat{L}_k \to \hat{L}_k$. 

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Proof. For any $x \in \tilde{L}$, there is a $k \geq 0$, such that $x \in \tilde{L}^{(k)} - \tilde{L}^{(k-1)}$. The map $q_k : \tilde{L}^{(k)} - s_k^{-1}(L^{(k-1)})' \to \tilde{L}^{(k)} - \tilde{L}^{(k-1)}$ is a homeomorphism. We define $f'(x) = f_k(q_k^{-1}(x))$. This well defines the map $f' : \tilde{L} \to M$.

Lemma 4. Given any subcomplex $K$ of $L$, integers $l > m \geq k \geq 0$, any map $f : \tilde{K} \cup \tilde{L} \to B^l(\mathbb{Z}/p)$, there exists an extension $\tilde{f} : \tilde{L} \to B^l(\mathbb{Z}/p)$.

Proof. Without lose of generality, we suppose $L = \Delta^m$. Since $B^l(\mathbb{Z}/p)$ is path connected, if in addition $k = 0$, the map $f$ can be extended to an $f : \tilde{K} \cup \tilde{L} \to B^l(\mathbb{Z}/p)$. The following diagram defines $\tilde{f} : q_2^{-1}(K(2)) \cup s_2^{-1}(L(1))' \to B^l(\mathbb{Z}/p)$.

![Diagram](attachment:image.png)

We will construct the map by induction. Suppose we have $\tilde{f} : q_2^{-1}(K(2)) \cup s_2^{-1}(L(1))' \to B^l(\mathbb{Z}/p)$. Take any $(k+1)$-simplex $\Delta \in L$, where $f$ is not defined on $\Delta$, $\tilde{f}$ is defined only on $s_{k+1}^{-1}(\Delta^{(k)})$.

![Diagram](attachment:image.png)

Since the map $r_k : E \to s_k^{-1}(\Delta^{(k)})$ induces a trivial map between $H^1(\mathbb{Z}/p)$, the map $\tilde{f} : E \to (B^l(\mathbb{Z}/p))$ is null homotopic. Hence $\tilde{f}$ lifts to a map $E \to E^l(\mathbb{Z}/p)$. On the other hand, $\pi_i(E^l(\mathbb{Z}/p)) = 0$ for $i = 1, 2, \ldots, m$, we get an extension $\tilde{f} : cE \to E^l(\mathbb{Z}/p) \to B^l(\mathbb{Z}/p)$. Do this for all the $(k+1)$-simplexes of $L$, we have a $\tilde{f} : \tilde{L}^{(k+1)} \to B^l(\mathbb{Z}/p)$.
By induction, we get a map $\bar{L}_{(m)} \to B^l(\mathbb{Z}/p)$, and we have the commutative diagram:

$$
\begin{array}{cccc}
\pi_1^{-1}(L^{(0)}) & \pi_2^{-1}(L^{(1)}) & \cdots & s_m^{-1}(L^{(m-1)}) \\
\downarrow & \downarrow & & \downarrow \\
\bar{L}_{(0)} & \bar{L}_{(1)} & \cdots & \bar{L}_{(m)} \\
\downarrow & \downarrow & & \downarrow \\
B^l(\mathbb{Z}/p) & B^l(\mathbb{Z}/p) & \cdots & B^l(\mathbb{Z}/p) \\
\end{array}
$$

This defines the map $\bar{L} \to B^l(\mathbb{Z}/p)$. $
$

Given a finite dimensional simplicial complex $L$ and an infinite sequence $K$ of positive integers, we can construct a space $X$ as following:

Let $K_1 = K^*$, $L_1 = \hat{L}(K_1)$, $B_1 = K_1'$, $K_2 = (K - B_1')^*$. Inductively, let $L_k = \hat{L}_{k-1}(K_k)$, $B_k = B_{k-1} \cup K_k'$, $K_{k+1} = (K - B_k')^*$. (The way I define the $K_i's$ satisfies tow conditions: (1) each $K_i$ is an infinite sequence, (2) $\cup B_i = K$.)

Let $X$ be the inverse limit of $p_{k+1} : L_{k+1} \to L_k$. The group $G = \prod_{i \in K} \mathbb{Z}/p^i$ acts on $L$, and the quotient space is $L$.

**Theorem 1.** For any compact metric space $L$, and any prime $p$, $L$ is the orbit space of a $G$-action on a metric compactum $X$ with $\dim_{\mathbb{Z}/p} X = 1$, where $G$ is a compact group.

**Proof.** If $L$ is a finite dimensional simplicial complex, the $X$ is the space we construct above.

To prove that $\dim_{\mathbb{Z}/p} X \leq 1$, it is enough to show that the restriction map $i^* : H^1(X; \mathbb{Z}_p) \to H^1(A; \mathbb{Z}_p)$ is surjective for any closed subset $A \subset X$. It is equivalent to show that any map $\phi : A \to K(\mathbb{Z}_p; 1)$ extends to $X$. We can use of the fact that $K(\mathbb{Z}_p; 1)$ is a neighborhood retract to relate $\phi$ to maps of $|L_i|$ into $K(\mathbb{Z}_p; 1)$. Consider the infinite mapping cylinder $M_\infty$ of the inverse system $\{p_i : |L_{i-1}| \leftarrow |L_i|\}_{i=3}^\infty$, that is, let $M_i$ be the mapping cylinder of $p_i$, and for each $i$, identify the copy of $|L_{i-1}|$ in $M_i$ with the domain end of $M_{i-1}$. Let $C_j = \cup_{i=1}^\infty M_i$. Then the collapses generate an inverse system $C_i \leftarrow C_j$ containing $|L_{j-1}| \leftarrow |L_j|$ with inverse limit $Z = M_\infty \cup X$. Now $A \subset Z$ and $\phi : A \to K(\mathbb{Z}_p; 1)$ extends to a map $\Phi : U \to K(\mathbb{Z}_p; 1)$ where $U$ is an open neighborhood of $A$ in $Z$.

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The collapses of the mapping cylinders $M_i$ are connected to the identities of the $M_i$ by one parameter families of retractions which extend to give a one-parameter family of retractions $r_t : Z \to Z$ of $Z$, $0 \leq t \leq 1$ with $r_1$ the identity and $r_0$ the retraction onto $|L_n|$ and such that for $t_n = (n-1)/n$, $r_{t_n}$ is the projection onto $C_n$. By compactness, there is an $n$ such that $r_t(A) \subset U$ for all $t \geq t_n - 1$, so Borsuk's homotopy extension theorem implies that $\phi$ extends to $X$ provided that $\Phi \circ r_n$ extends to $|L_n|$. For $n$ sufficiently large, the union $T$ of all subcomplexes of $L_n$ of the form $p^{-1}_n(\sigma)$ for simplices $\sigma$ of $|L_{n-1}|$ that $\sigma \cap r_{t_n}(A) \subset U$. Let $\Psi = \Phi|_T : T \to K(\mathbb{Z}_p; 1)$. By lemma, $\Psi$ can be extend $\Phi : |L_n| \to K(\mathbb{Z}_p; 1)$. Thus, $\phi$ extends to $X$, hence $\dim \mathbb{Z}_p X \leq 1$.

To prove the general case, we first construct a space $X$ over the Hilbert cube as follows:

For $n = 1$, let $X_1 = I$, $K$ be any sequence of positive integers. $K_1 = K^*$, $B_1 = \emptyset$. For each $n$ consider the projection $I^{n+1} \to I^n$ and the following diagram:

\[
\begin{array}{cccc}
X_n & \xrightarrow{p'_{n+1}} & X_{n+1} & \xrightarrow{p_{n+1}} & X_{n+1} \\
q_n & & q_{n+1} & & \\
I^n & \xrightarrow{q_n} & I^{n+1} & \xrightarrow{q_{n+1}} & I^{n+1}
\end{array}
\]

where $q_n$ is assumed by induction, $X_{n+1}$ is a pull back, $X_{n+1} = \widetilde{X}_{n+1}(K_{n+1})$, $K_{n+1} = (K - B_n)^*$, $B_{n+1} = B_n \cup K'_{n+1}$. Let: $q_{n+1} = p_{n+1} \circ p''_{n+1} : X_{n+1} \to I^{n+1}$. Let $G_n = G_{K_n} \times G_{n-1}$.

By induction, we have an action of $G_n$ on $\widetilde{X}_n$, with orbit map $q_n$, so we get an induced action of $G_{n+1}$ on $\widetilde{X}_{n+1}$ with orbit map $q_{n+1}$. $\dim \mathbb{Z}_p \widetilde{X}_{n+1} = 1$. Now form the inverse limits:

\[
\begin{array}{cccccc}
\widetilde{X}_1 & \xrightarrow{q_1} & \widetilde{X}_2 & \xrightarrow{q_2} & \widetilde{X}_3 & \cdots & \xrightarrow{q} X \\
I^1 & \xrightarrow{q_1} & I^2 & \xrightarrow{q_2} & I^3 & \cdots & \xrightarrow{q} I^\infty
\end{array}
\]

We now have an action of infinite product $\prod G_{K_n} \cong G_K$ on $X$ with orbit map $q$.

Using a similar construction as before, i.e. the infinite mapping cylinder, we can prove $\dim \mathbb{Z}_p X \leq 1$. Actually, each $\dim \mathbb{Z}_p \widetilde{X}_k = 1$, so the extension clearly exists.

To complete the proof, let $L$ be any compact metric space, embed $L$ in $I^\infty$, and let $X = q^{-1}(L)$. 

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Note:
(1) If we choose $K = \{1, 1, 1, \cdots \}$, then $G = \prod \mathbb{Z}/p$. This fixes the error of A.N.Dranishnikov and J.E.West’s paper.
(2) If we choose $K = \{1, 2, 3, \cdots \}$, then $G = \prod_{i=1}^{\infty} \mathbb{Z}/p^i$. On the other hand, $\hat{\mathbb{Z}}_p$ is a subgroup of $G$, so we get a $\hat{\mathbb{Z}}_p$ group action on $\hat{L}$. I guess the quotient space has dimension infinity, but I haven’t proven it yet.

**Theorem 2.** Given any integer $n \geq 2$, any prime $p$, any sequence $K$ of positive integers, there is an action of $G = G_K$ on a compact, two-dimensional, metric space $X$ such that $\dim X/G = n$, moreover, $\dim X \times X = 0$.

**Proof.** This follows the same proof as the proof of Theorem A of [D].

**Corollary 2.1.** For any $p, K$ as above, there is an action of $G = G_K$ on a compact, two-dimensional, metric space $X_\infty$ such that $\dim X_\infty/G = \infty$ and $\dim X_\infty \times X_\infty = 3$.

**References**

[1] S.M. Ageev *Classifying spaces for free actions and the Hilbert-Smith conjecture*, Russian, Acad. Sci. Sbornik Math. 75 (1993), 137-144.

[2] M. Bestvina, *Characterizing k-dimensional universal Mneger compacta*, Mem. Amer. Math. Soc. #380 71 (1988).

[3] A.N. Dranishnikov, J.E. West, *Compact group actions that raise dimension to infinity*, Topology and its applications 80 (1997) 101-114.

[4] A.N. Dranishnikov, *On free actions of zero-dimensional compact groups*, Izv. Akad. Nauk USSR 32 (1998), 217-232.

[5] R. Edwards, *Some remarks on the Hilbert-Smith Conjecture*, Proceedings of the 4th Annual Western Workshop in geometric topology 1987, Oregon State Univ. (1987),1-8.

[6] R.J. Daverman, *Problems about finite-dimensional manifolds*, Open Problems in topology (1990), 431-455.

[7] D. Montgomery, L. Zippin, *Topological Transformation groups*, Interscience Publishers, Inc. New York, 1955.
[8] J.C. Mayer and C.W. Stark, *Group actions on Menger manifolds* preprint.

[9] F. Raymond, and R.F. Williams, *Examples of p-adic transformation groups*, Ann. Math, 78 (1963), 92-106.

[10] K. Sakai, *Free actions of zero-dimensional compact groups on Menger manifolds*, preprint.

[11] H. Torunczyk, *On CE-images of the Hilbert cube and characterization of Q-manifolds*, Fund. Math. 106 (1980), 31-40.

[12] C.T. Yang, *p-adic transformation groups*, Michigan Math. J, vol. 7 (1960) 201-218.