On Information Coefficient and Directional Statistics

Yijian Chuan\textsuperscript{a}, Lan Wu\textsuperscript{a,b,*}

\textsuperscript{a}School of Mathematical Sciences, Peking University, Beijing, China
\textsuperscript{b}Key Laboratory of Mathematical Economics and Quantitative Finance, Peking University, Beijing, China

Abstract

Cross-sectional “Information Coefficient” (IC) is a widely and deeply accepted measure in portfolio management. In this paper, we propose that IC is a linear operator on the components of a standardized random vector of next period cross-sectional returns. From the probability perspective, IC is a linear combination of the components of a directionally projected degenerated random vector. We deduct a solution to its optimization in expectation and obtain the maximum. Their closed-form expressions are given by directional statistics in a specific condition. Simulation analysis discloses the influence of market information, such as the number of stocks, on IC. The empirical analysis of the Chinese stock market uncovers a set of interesting facts about the standardized vectors of cross-sectional returns and helps to obtain the time series of the measure in the real market. Our research discovers a potential application of directional statistics in finance, reveals the nature of the IC measure, and deepens the understanding of active portfolio management.

Keywords: cross-sectional returns, Information Coefficient, directional statistics, projected distribution, portfolio optimization

1. Introduction

In their seminal paper, Treynor and Black (1973) introduced the concept of active management, normatively demonstrating “a way to make the best possible use of information”. Based on this framework, an widely-accepted “measure”, Information Coefficient or IC as acronym, was “coined” by Ambachtsheer (1974) p. 85 to quantify the predictive ability of a strategy: “the measure […] is the correlation coefficient between the forecasted ratings and actual ratings.” Subsequently, Ambachtsheer and Farrell (1979) and Grinold (1989, 1994) contributed to the integration of IC into the process of active portfolio management. IC is extensively and intensively applied in industry, especially in active portfolio management.

Meanwhile, the fluctuation of the time series of IC was found in academia and industry. Coggin and Hunter (1983) thought that it was originated from sample errors, and used “meta-analysis” to correct it. After that, Qian and Hua (2004) re-distinguished the definition of raw IC and risk-adjusted IC, regarding the volatility of IC as “strategy risk”. Ye (2008) also investigated the impact of the variation of IC on the performance of investment specifically. Ding and Martin (2017) systematically developed a stationary
econometric model of IC for mean-variance portfolios. To sum up, the dynamic of realized IC is an issue worth of research.

The original idea of this paper is motivated from questions raised by industry: “For a running strategy, which kind of IC time series can be considered as an invalid strategy? What are the feasible statistical test methods?” To answer them, we first need to have a concrete understanding of IC, and then reveal the probability and statistical properties. The IC does measure the similarity between the series of predicted and actual values, like a metric function. However, IC does not match the definition of a correlation coefficient. And directly applying t-test to the time series of IC may lead to fallacy. In a probability view, we define IC as a linear operator of a directionally projected degenerated random vector. In active portfolio management, IC can be regarded as the weighted sum of the standardization of next period cross-sectional returns. From now on, we denote IC by $T(\omega)$, which is strictly defined in [1] of Section 2.

The standardization of next period cross-sectional returns is a projection of a degenerated component-dependent random vector on a unit sphere, which is a topic in directional statistics. Since a pioneer work of Fisher [1953], directional statistics has flourished to a certain degree. Mardia [1972]; Mardia and Jupp [2000]; Ley and Verdebout [2017, 2018] summarized and developed the main works. However, $T(\omega)$ in our paper is an unusual case in directional statistics. For one thing, the random vector to be projected on a unit sphere is degenerated in the sense of its singularity of covariance matrix. For another, derivation of the projected distribution from the original distribution is technically difficult. And the connection of the original and projected distribution is complicated. We only find some relevant results in Presnell and Rumcheva [2008].

A prime theoretical task is to obtain the maximization of the expectation of $T(\omega)$, when given the underlying distribution. We solve the optimization. It is worth emphasizing that the solution is the mean direction (MD) of a projected degenerated distribution and the maximum is just the mean resultant length (MRL). Nevertheless, as is mentioned above, it is not a trivial case, and needs some careful derivation. With the help of MD and MRL, we explore the properties of $T(\omega)$ in detail, trying to answer the questions heavily concerned by industry. Furthermore, a closed-form expression of the maximum is obtained in the case of the normal distribution of stock returns.

Due to the inherent complication of the high-dimensional projected distribution and the singularity of covariance matrix, hardly can we get intuitive sense of $T(\omega)$. Thus, numerical simulation is carried out, aiming at further exploring the probability properties of $T(\omega)$. Specifically, we investigate the effect of the change of the parameters and dimension, from which heuristic and meaningful results are derived, meeting the previous conclusion in Grinold [1994]. Based on the real market parameters, the simulation shows that the solution of the maximization of the expectation of $T(\omega)$ is not equivalent to the solution directly
generated from the population mean of the individual returns of each stock. To put it in finance, even if one has knew the means of next period returns, she could not gain the maximum \( T(\omega) \) consistently, which implies that improving the accuracy of the forecasts of expected returns does not necessarily increase the value of \( T(\omega) \). As the increase of stock number, the variance of \( T(\omega) \) decreases significantly, while the expectation of \( T(\omega) \) varies little, consistent with the previous results in [Coggin and Hunter (1983)](#). In order to explore facts about \( T(\omega) \), empirical studies based on Chinese stock market are implemented. First, the standardization of cross-sectional returns is a common and usual treatment in industry. We present a set of empirical facts emerging from the standardized vector of cross-sectional returns academically. The standardized vector is more comparable than the original one. And the standardization reveals more essential information about the market. The cross-sectional correlations between the returns of each stock decrease sharply and significantly through the standardization. Last, with the assistance of direction statistics, especially the MRL, we estimate the time series of the maximum \( T(\omega) \) in the real market with rolling windows.

The rest of the paper is organized as follows. Section 2 deducts probability properties, solving the optimization. Section 3 applies simulation to exploring the probability properties of \( T(\omega) \). Section 4 implements empirical studies in Chinese stock market. Section 5 concludes. All proofs are left in the appendix.

2. Methodology and theoretical results

2.1. Model setup

Let an \( n \)-dim orthogonal projection matrix \( P := I - \frac{1}{n}11^T \), where \( I \) is an \( n \)-dim identity matrix and \( 1 := (1, 1, \ldots, 1)^T \). Then, assume an \( n \)-dim random vector \( X \) s.t. \( P X = 0 \), \( E X := \mu \), and \( \text{cov}(X) := \Sigma \) is positive definitive. We define \( Y := \frac{P X}{\|P X\|} \) where \( \|x\| := \sqrt{\sum_{i=1}^{n} x_i^2} \). Particularly, \( Y \) is a directional projected degenerated random vector, where the degenerated means that the covariance matrix of \( P X \) is singular because of the linear projection. Next, given \( \omega \notin \langle 1 \rangle := \{ x \in \mathbb{R}^n | x \neq \alpha 1, \alpha \in \mathbb{R} \} \), the linear operator of \( Y \) is defined as

\[
T(\omega) := \left( \frac{P \omega}{\|P \omega\|} \right)^T Y.
\]

It can be represented as \( T(\omega) = \frac{\sum_{i=1}^{n} (\omega_i - \bar{\omega})(X_i - \bar{X})}{\sqrt{\sum_{i=1}^{n} (\omega_i - \bar{\omega})^2 \sum_{i=1}^{n} (X_i - \bar{X})^2}} \), where \( \bar{\omega} := \frac{1}{n} \sum_{i=1}^{n} \omega_i \) and \( \bar{X} := \frac{1}{n} \sum_{i=1}^{n} X_i \). \( T(\omega) \) is a new perspective of IC.

**Remark 1.** Two basic mathematical properties are illustrated here. First, \( \forall k > 0, \alpha \in \mathbb{R}, T(\omega) = T(k \omega + \alpha 1) \). It means that \( T(\cdot) \) is invariant of the scale and locale transformation. Thus, from now on, we restrict \( \omega \) s.t.

---

In this paper, \( Y = \frac{P X}{\|P X\|} \) is the **standardization of** \( X \), \( P X \) is the **linear projection** of \( X \) to a subspace, and \( Y = \frac{P X}{\|P X\|} \) is the **directional projection** of \( P X \) on a unit sphere. Both the two projections incur the singularity of covariance matrix. Heuristically, **standardization = linear + directional projection**.
ω^T 1 = 0 and ∥ω∥ = 1. Second, −1 ≤ T(ω) ≤ 1.

Remark 2. The motivation of the definition of T(ω) is from the alpha strategies of portfolio management in finance. In Grinold (1989, 1994), T(ω) is named as Information Coefficient (IC). In portfolio management, IC = \frac{\sum_{i=1}^{n}(\hat{r}_i - \bar{r})(r_i - \bar{r})}{\sqrt{\sum_{i=1}^{n}(\hat{r}_i - \bar{r})^2 \sum_{i=1}^{n}(r_i - \bar{r})^2}}, where \( r_i \) is the return of stock \( i \) at next period time \( t + 1 \) and \( \hat{r}_i \) is the forecasts of the return at current period time \( t \). Therefore, at time \( t \), IC is the same as \( T(\omega) \), if we regard \( \omega_i \) as \( \hat{r}_i \) and \( X_i \) as \( r_i, i = 1, 2, \cdots, n \).

Remark 3. The deduction process of the probability properties of \( T(\omega) \) is not trivial. For one thing, there is no existing result of directional statistics to use directly, particularly when the components of \( X \) are not independent. In fact, if the components of \( X \) are independent, one can easily derive a lot of the properties of \( T(\omega) \) (Mardia and Jupp, 2000). In finance, \( X \) represents the cross-sectional returns, which can be regarded as neither independent, nor with the identical distribution, according to asset pricing theory and modern portfolio theory. Therefore, we need to extend the existing work on directional statistics. For another, \( X/\|X\| \) is a main task of the projected distribution in directional statistic, while what we are concerning is \( P X/\|PX\| \). The projection matrix \( P \) brings some new issues, because of the singularity of the covariance matrix of \( PX \).

Remark 4. In finance, people concern about the properties of \( T(\omega) \) on \( \omega \). To be more specific, people care about how to forecast \( \hat{r} \) to optimize \( T(\hat{r}) \) and how to estimate the maximum of \( T(\hat{r}) \). In the sense of expectation, it is an optimization problem.

2.2. Optimization results

First, we consider an optimization problem of \( \mathbb{E} T(\omega) \) with an given distribution of \( X \). They are addressed in Theorem 1.

**Theorem 1.** Let \( P := I - \frac{1}{n} 11^T \). Given a random vector \( X \) such that its covariance matrix is positive definite, \( \mathbb{P}\{PX = 0\} = 0 \), and \( \mathbb{E}_{} \frac{PX}{\|PX\|} \neq 0 \), we have:

1) The solution to the optimization problem

\[
\max_{\omega^T 1 = 0, \|\omega\| = 1} \mathbb{E} T(\omega),
\]

uniquely exists. The unique solution \( \omega^* \) is

\[
\omega^* = \frac{\mathbb{E} \frac{PX}{\|PX\|}}{\|\mathbb{E} \frac{PX}{\|PX\|}\|}.
\]

2) The value of the maximum is

\[
\mathbb{E} T(\omega^*) = \|\mathbb{E} \frac{PX}{\|PX\|}\|.
\]

\[^3\omega^T 1 = 0 \iff \omega = 0 \text{ and } \|\omega\| = 1 \iff \sum_{i=1}^{n} \omega_i^2 = 1. \text{ With the restriction, } P\omega/\|P\omega\| = \omega. \text{ One can show that the restriction provides the uniqueness.} \]
The proof is in the appendix.

Remark 5. In terms of direction statistics, the “optimal” \( \omega^* \) is the mean direction (MD) of the projected distribution \( PX/\|PX\| \), and the maximum \( \hat{\omega} \) is the mean resultant length (MRL). It is worth pointing out again that \( 1^T P X = 0 \), implying the singularity of the covariance matrix of the random vector \( PX \). Existing literature, however, concerns about the non-degenerated cases. Therefore, (3) and (4) is only pseudo-similar to general MD and MRL. We need to try new approaches.

Remark 6. When \( X \) is a 2-dim normal distribution, the results of Theorem 1 are straightforward. Specifically, assume that

\[
X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix} \right).
\]

The explicit expressions of the optimal \( \omega^* \) and \( E(\omega^*) \) are

\[
\omega^* = \frac{\sqrt{2}}{2} \cdot \text{sign} \{\mu_1 - \mu_2\} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

\[
E(\omega^*) = 2 \left[ \Phi \left( \frac{|\mu_1 - \mu_2|}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2}} \right) - \frac{1}{2} \right],
\]

where \( \Phi \) is the cumulative distribution function of the standard normal distribution.

Intuitively, we give explanations about the solution of this 2-dim case. For one thing, (5) shows that \( \omega^* \) includes two options: \( \frac{\sqrt{2}}{2}(1, -1)^T \) or \( \frac{\sqrt{2}}{2}(-1, 1)^T \). And \( \omega^* \) is only determined by the sign of \( \mu_1 - \mu_2 \), irrelevant of the exact values of \( \mu_1 \) and \( \mu_2 \). (In finance, it means that to buy the stock with higher expected return while to sell the other.) For another, (6) implies that \( E(\omega^*) \) increases with \( |\mu_1 - \mu_2| (= \sqrt{2}\|P\mu\|) \), which in finance is the dispersion of the expectation of cross-sectional returns. \( E(\omega^*) \) also increases with the correlation coefficient \( \rho \), but decreases with the volatility \( \sigma_1, \sigma_2 \). Particularly, when there is no dispersion, i.e. \( \mu_1 = \mu_2 \), \( E(\omega^*) = 0 \), which means that no active profit can be gain consistently in such a market condition. These conclusions are consistent with the common sense in active portfolio management: The more dispersed the cross-sectional expected returns are, the more active strategy profit can be gain. Likewise, the higher the correlation, the easier the statistical arbitrage.

2.3. Optimization in the case of a normal distribution

In this subsection, we impose \( X \) with the multivariate normal distribution. The main work of this subsection consists of two parts. For one thing, we transform the degenerated normal distribution of \( PX \) to a non-degenerated one. For another, we give closed-form expressions of the solution \( \omega^* \) and the corresponding maximum \( E(\omega^*) \) in a specific condition.

First, the degenerated covariance matrix of \( PX \) is settled down in the following proposition, where an orthogonal matrix \( U \) transforms the covariance matrix from \( P\Sigma P^T \) to a diagonal one.
Proposition 2. Given $X \sim N(\mu, \Sigma)$ where $\Sigma$ is positive definitive, the solution to the optimization problem (2) is

$$\omega^* = \frac{1}{\|\mathbb{E} \xi_{n-1} \| \|\xi_{n-1} \|} U \left( \frac{\mathbb{E} \xi_{n-1} \| \xi_{n-1} \|}{\|\xi_{n-1} \|} \right),$$

$$\mathbb{E} T(\omega^*) = \left\| \mathbb{E} \frac{\xi_{n-1}}{\|\xi_{n-1} \|} \right\|,$$

where $U$ and $\xi_{n-1}$ are defined as follows.

$U := VW$. $V$ is an orthogonal matrix s.t. $V^T P V = (I_{n-1} 0)$. $W := (W_{n-1})$ and $W_{n-1}$ is an orthogonal matrix s.t. $W^T (I_{n-1} 0_{n-1}) V^T \Sigma V (I_{n-1} 0_{n-1}) W_{n-1} = \Lambda_{n-1}$, where $\Lambda_{n-1}$ is a diagonal matrix. $\xi_{n-1} \sim N((I_{n-1} 0_{n-1}) U^T \mu, \Lambda_{n-1})$ is a multivariate normal distribution with independent components.

The proof is in the appendix.

Remark 7. Some interpretations about Proposition 2 are given. This proposition shows that the probability properties of $T(\omega)$, based on a projected degenerated random vector $\|P_X\|$ could be derived from the MD and MRL of $\xi_{n-1} \sim N(\cdot, \Lambda_{n-1})$, which is a projected non-degenerated random vector. In detail, if one knew the closed-form expressions of a projected non-degenerated normal distribution, she should have the closed-form expression of $\omega^*$ and $\mathbb{E} T(\omega^*)$ according to Proposition 2. Thus, the ready-made results of directional statistics in Mardia (1972); Mardia and Jupp (2000); Ley and Verdebout (2017) can be applied. However, the existing literature that we have known only gives the closed-form expression of the MD and MRL of the cases in which the covariance matrix of $\xi_{n-1}$ is a scalar matrix.

Second, we will give the closed-form expression of $\omega^*$ and $\mathbb{E} T(\omega^*)$ in a specific condition. We need to introduce a related work: Presnell and Rumcheva (2008) gives the MD and MRL of the projected normal distribution of a scalar covariance matrix. It is one of the most cutting-edge research that we have known about the closed-form expression of MD and MRL.

Proposition 3 (Eq(6) in Presnell and Rumcheva (2008)). Given an $n$-dim random vector $\xi \sim N(\nu, \lambda^2 I), \nu \neq 0, \lambda > 0$, for the projected normal distribution $\frac{\xi}{\|\xi\|}$, we have

$$\text{MD} := \frac{\nu}{\|\nu\|},$$

$$\text{MRL} := \varrho_n \left( \frac{\|\nu\|}{\lambda} \right),$$

where

$$\varrho_n (x) := \frac{\Gamma \left( \frac{n+1}{2} \right)}{\sqrt{2\Gamma \left( \frac{n+2}{2} \right)}} x M \left( \frac{1}{2} \frac{n+2}{2}, -\frac{1}{2} x^2 \right)$$

and $M(\cdot, \cdot, \cdot)$ is the confluent hypergeometric function of the first kind. $\varrho_n (x)$ is a monotonically increasing concave function.

$M$ is a solution of a confluent hypergeometric equation, which can be written as $M(a, b, z) = \sum_{n=0}^{\infty} \frac{a(n)}{b(n)} z^n$, where $a^{(0)} = 1.$
Proposition 3 deals with the case that the covariance matrix is a scalar matrix. In other words, the component of $\xi$ is i.i.d.. We expand it to a specific dependent situation in financial market in the following theorem.

**Theorem 4.** Given $X \sim N(\mu, \sigma^2 \Xi)$, $\sigma > 0$, where $\Xi$ is a correlation coefficient matrix with off-diagonal elements equaling to $\rho \in (-\frac{1}{n}, 1)$, i.e.

$$\Xi := \begin{pmatrix}
1 & \rho & \rho & \cdots & \rho \\
\rho & 1 & \rho & \cdots & \rho \\
\rho & \rho & 1 & \cdots & \rho \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \rho & \cdots & 1
\end{pmatrix},$$

the solution to the optimization (2) is

$$\omega^* = \frac{P\mu}{\|P\mu\|},$$

$$ET(\omega^*) = \varrho_{n-1} \left( \frac{\|P\mu\|}{\sigma(1 - \rho)} \right),$$

where $\varrho_{n-1}$ is defined in (7).

The proof is in the appendix.

**Remark 8.** Some interpretations about Theorem 4 are given here. First, $\sigma^2 \Xi$ is a covariance matrix widely used in financial modeling, such as one factor covariance matrix model. Second, under this specific situation, $\omega^*$ only depends on $\mu$, irrelevant of the covariance matrix of $X$, even when the components of $X$ are strongly correlated. Third, $ET(\omega^*)$ increases with the dispersion of $\mu$ and the correlation coefficient $\rho$, while decreases with volatility $\sigma$. The results are similar to that in the previous 2-dim case, consistent with the empirical common sense in finance.

3. Simulation

In this section, we aim at a better understanding of the distribution of $T(\omega)$ and the results of $ET(\omega^*)$ by simulation.

3.1. The distribution of $T(\omega)$

We simulate the sample distribution of $T(\omega^*)$ and $T(\mu)$ of $n = 10$ and $50$. To get more realistic results, the simulation parameters are estimated from the real market, i.e. the component stocks of CSI 300 Index of

\[ a(n) = a(a + 1)(a + 2) \cdots (a + n - 1). \] When $b > a > 0$, it can be represented as an integral:

\[ \frac{\Gamma(b - a)\Gamma(a)}{\Gamma(b)} M(a, b, z) = \int_0^1 e^{zt} t^{a-1} (1 - t)^{(b-a)-1} dt. \]

For detailed properties, one can refer to Abramowitz and Stegun (1964, p. 503, §13).
Chinese stock market. Specifically, we calculate the sample mean and covariance matrix of the daily simple returns from January 2017 to November 2018 as the parameters $\mu, \Sigma$. Then, Monte Carlo simulations are carried out based on them.

The simulation results, i.e. kernel smoothing empirical p.d.f.'s, are in Figure 1. The main difference between the two subfigures is $n$, i.e. the dimension or the stock number. It is easy to note the difference between the two subfigures: As $n$ grows, the volatility of $T(\omega)$ decreases uniformly regardless of the $\omega$'s. The results are in accordance with the conclusions in Coggin and Hunter (1983). More detailed discussions are in Remark 9.

![Figure 1: The p.d.f.'s of $T(\omega)$ w.r.t. different parameters.](image)

**Remark 9.** Some points of Figure 1 are discussed below in detail. First, there is little difference between the means of $T(\omega^*)$ in two subfigures. It seems that the $\mathbb{E}T(\omega^*)$ reveals some basic fact of the market. Second, the difference of volatility of the p.d.f.'s between the two subfigures is significant, where the p.d.f.'s of the 10 stocks is more volatile than that of 50. It shows that the dimension, or the stock number $n$, mainly impacts the variance of $T(\omega)$ rather than the expectation of $T(\omega)$ in general condition. In portfolio management, it means that the increase of the number of stocks does not impact the mean of IC, but significantly decreases the variance of IC.

### 3.2 Impact of parameters on $\omega^*$

In this subsection, we turn to the question how $\omega^*$ varies with respect to the varies of the parameters $\mu$ and $\Sigma$.

Intuitively, we will show the impact of $\Sigma$ in a 3-dim case. The way we simulate is as follows:\footnote{According to [Mardia and Jupp 2000], the projected normal distribution is determined by the “ratio” of $\mu$ and $\Sigma$, rather than the exact value. Thus, we only need to show the impact of $\Sigma$.} In detail, the red point is $\frac{\mu}{\|\mu\|}$, irrelevant to $\Sigma$. The blue asterisk and digits are all $\omega^*$'s of different parameters.
Specifically, the blue asterisk is the simulated $\omega^*$ from the original $\mu$ and $\Sigma$. The digits are from the same $\mu$, but different $\Sigma$’s: For instance, digit 2 is from $\Sigma' := \text{diag}(2,1,1)\Sigma\text{diag}(2,1,1)$, where the standard deviation of the first component is amplified by 2. And so are the digits 0.5, 3, 4, and 10. Figure 2b is the top view of Figure 2a which clearly shows the difference of $\omega^*$ generated from different $\Sigma$.

Remark 10. Some interpretations about Figure 2 are given. First, $P_{\|P\mu\|}$ is different from $\omega^*$, as the red point is different from the blue asterisk. In detail, $P_{\|P\mu\|}$ is only determined by $\mu$, while $\omega^*$ is determined by $\mu$ and $\Sigma$ jointly, so they are not the same. Second, the more heterogeneous the $\Sigma$ is, the more the difference between $P_{\|P\mu\|}$ and $\omega^*$ that is represented by digit from 1 (blue asterisk), to 2, · · · , to 10.

3.3. Comparison of the simulation and the theoretical results

As an episode, we show a simple approach to estimate $T(\omega^*)$ without simulation, and compare the approximate theoretical result with the simulation one. In this example, the usefulness of Theorem 4 is revealed.

According to Theorem 4, $\mathbb{E}T(\omega_{10}^*)$ can be calculated approximately by regarding the $\Sigma_{10}$ as the $\sigma^2 \Xi$. In details, $\|P\mu_{10}\| \approx 0.0027$, $\hat{\sigma} := \sqrt{\sum_{i=1}^{10} \sigma_i^2} \approx 0.0224$, and $\hat{\rho} := \frac{1}{10 \times 9 / 2} \sum_{1 \leq i < j \leq 10} \rho_{ij} \approx 0.1243$. Therefore, $\mathbb{E}T(\omega_{10}^*) \approx \varrho_9 \left( \frac{\|P\mu\|}{\hat{\sigma} (1 - \hat{\rho})} \right) \approx \varrho_9 (0.1363) \approx 0.0446$. It is slightly higher than the simulation real result 0.0414, but it is easy to use. In finance, the simulation results indicate that under this condition, the maximum of $T(\omega)$ in expectation is about 0.04, which is in accordance with Grinold [1994, p. 10-11]: “A reasonable IC for an outstanding (top 5%) manager forecasting the returns on 500 stocks is about 0.06. If the manager is good (top quartile), 0.04 is a reasonable number.”

In summary of this section, the simulation reveals several meaningful results. The optimal $\omega^*$ is closed to $P_{\|P\mu\|}$ generally. However, in some heteroscedastic scenarios, they can be very different. Second, the simulation based on real market condition does show that $\mathbb{E}T(\omega^*)$ cannot be very high, where 0.04 is a
reasonable one. The volatility of \( T(\omega^*) \) and \( T(\mu) \) decreases significantly with the increase of stock number \( n \). Besides, we show one can use (8) in Theorem 4 to calculate \( \mathbb{E}T(\omega^*) \) quickly at a little cost of accuracy.

4. Empirical Study

In this section, we aim at revealing some empirical facts of \( T(\omega) \), exploring the realized \( \mathbb{E}T(\omega^*) \), and trying to answer the questions that industry concerns. First, we present empirical studies on the properties of the standardized vector \( y_t := \frac{P_x x_t}{\|P_x x_t\|} \) of CSI 300 Index component stocks, investigating the impact of the cross-sectional standardization from the viewpoint of individual and overall. Specifically, the effect of the standardization in \( T(\omega) \), i.e. \( \frac{P_x x_t}{\|P_x x_t\|} \), is shown with respect to two representative stocks and the overall market in the first two subsections. Second, we show the time series of the realized \( \mathbb{E}T(\omega^*) \) and compare them with general cross-sectional index, CSSD (cross-sectional standard deviation).

Basic data of empirical research are shown here. We investigate the component stocks of CSI 300 Index at the end of 2018. The time interval of the data is 5 years from 2014 to 2018, 1220 trading days, during which some of the stocks were not traded. We abandon stocks missing more than 10% of the trading days, leaving \( n = 185 \) stocks. Missing values left are marked as 0. As a result, a matrix of \( 1220 \times 185 \) representing the returns is generated, called \( X \). \( x_t \) is the \( t \)-th column of \( X^T \) representing \( n = 185 \) returns at time \( t \). With original \( x_t \) at hand, \( y_t := \frac{P_x x_t}{\|P_x x_t\|} \) can be calculated easily.

4.1. Components of vector \( y_t \)

We show the difference between the representative components of \( y_{it} \) and their corresponding \( x_{it} \) with basic descriptive statistics and primitive time series analysis, where \( i = 1, 2 \). The descriptive statistics of the representative two are in Table [1] and the discussions about them are in Remark [11].

Remark 11. Two points of Table [1] are discussed. First, the descriptive statistics of \( x_i \) and \( y_i \) are significantly different, where the standardization increases the absolute value of sample means individually. Also, the standard deviation of \( y_i \) are larger than that of \( x_i \) significantly. The sample coefficient of variance decreases through the standardization. Second, we interpret some facts. The range of \( y_{it} \), worth emphasizing here, is wider than that of \( x_{it} \) restricted by the 10% price limit. Thus, the standardization reveals the hidden dynamic information.

Figure 3 shows the time series of the two representives: \( x_{it} \) and \( y_{it} \), \( i = 1, 2 \). Basic time series analysis shows that the classical hypothesis tests of autoregression and moving average are not significant, and under the BIC, ARMA(0, 0) is chosen. The discussions are in Remark [12].

Remark 12. Two points about Figure 3 are illustrated below. First, more information is displayed in the time series of \( y_{it} \) than that of \( x_{it} \). In detail, these dark circles, representing \( x_{it} \approx 0.1 \), are in a line along

---

\({}^6\) The range of \( x_i \) is defined as \( \sup_{0 \leq t, s \leq T} |x_{it} - x_{is}| \).
Table 1: Descriptive statistics.\textsuperscript{a}

| Sample Statistics | 000001.SZ | 600018.SH |
|-------------------|-----------|-----------|
| \(x_{1t}\) mean  | 0.0005    | 0.0004    |
| \(y_{1t}\) mean  | -0.0022   | -0.0026   |
| \(x_{2t}\) mean  | 0.0004    | 0.0004    |
| \(y_{2t}\) mean  | -0.0026   | -0.0026   |
| \(x_{1t}\) std   | 0.0205    | 0.0235    |
| \(y_{1t}\) std   | 0.0508    | 0.0649    |
| \(x_{2t}\) std   | 0.0235    | 0.0649    |
| \(y_{2t}\) std   | 0.0649    | 0.0649    |
| \(x_{1t}\) CV\textsuperscript{b} | 41.3871   | 65.0917   |
| \(y_{1t}\) CV\textsuperscript{b} | 22.9746   | 25.3970   |
| \(x_{2t}\) CV\textsuperscript{b} | 65.0917   | 25.3970   |
| \(y_{2t}\) CV\textsuperscript{b} | 25.3970   | 25.3970   |
| \(x_{1t}\) min\textsuperscript{c} | -0.1002   | -0.0999   |
| \(y_{1t}\) min\textsuperscript{c} | -0.1810   | -0.2509   |
| \(x_{2t}\) min\textsuperscript{c} | -0.0999   | -0.2509   |
| \(y_{2t}\) min\textsuperscript{c} | -0.2509   | -0.2509   |
| \(x_{1t}\) max\textsuperscript{c} | 0.1000    | 0.1007    |
| \(y_{1t}\) max\textsuperscript{c} | 0.3274    | 0.4661    |
| \(x_{2t}\) max\textsuperscript{c} | 0.1007    | 0.4661    |
| \(y_{2t}\) max\textsuperscript{c} | 0.4661    | 0.4661    |

\textsuperscript{a} This is the descriptive statistics about two representative stocks: The one is 000001.SZ, Ping An Bank Co., Ltd., the other is 600018.SH, Shanghai International Port (Group) Co., Ltd. For simplicity, let \(x_{1t}, y_{1t}\) represent the returns and the standardized “returns” of 000001.SZ, and \(x_{2t}, y_{2t}\) 600018.SH.

\textsuperscript{b} CV represents the coefficient of variance, i.e. \(\sigma/|\mu|\), a measure free from scaling.

\textsuperscript{c} The minimum and maximum of \(x_{1t}\) and \(x_{2t}\) are resulted from the price limit system, \(\pm10\%\), in Chinese stock market, leading to the inefficacy of the information expression of daily simple returns.

Figure 3: The time series of \(x_{it}\) and \(y_{it}\), \(i = 1, 2\). In detail, the dark solid lines represent \(x_{it}\) while the light dotted ones \(y_{it}\). Besides, the dark circles indicate the daily up limit of \(x_{it}\), while the light crosses are the corresponding in \(y_{it}\). Some discussion about the figures are in Remark 12.

x-axis, but the light crosses, representing the corresponding \(y_{it}\), are jagged. There are times of the same dark circles but high light crosses: They can be interpreted as the condition that almost all other stocks are mild while only this stock profit sharply. Second, it is easy to see similar volatility clustering in Figure 3. The standardization does not change the property of volatility clustering.
4.2. Overall $y_t$

In this subsection, we turn to the overall property of $y_t$.

To begin with, we show the histograms of the means of $x_{it}$ and $y_{it}$, i.e. $\bar{x}_i$ and $\bar{y}_i$, $i = 1, 2, \ldots, n$ of $n = 185$ stocks together in Figure 4, for the purpose of supplementing the results of the two representatives. It is easy to conclude that, for most stocks, $\bar{y}_i$ is of larger range than $\bar{x}_i$.

We will show that the standardization almost eliminates the cross-sectional correlation between each stock. In detail, two figures of the heat maps of the cross-sectional correlation coefficient matrix between each stock are plotted in Figure 5: The left one is that of $x$ before the standardization, and the right one $y$ after the standardization.

Remark 13. We give some explanations about Figure 5. It is difficult to ignore the difference that $\text{corr}(x)$ is greater than $\text{corr}(y)$. In fact, by simple calculation, the mean of sample correlation coefficients of $x$ is $\text{mean}(\text{corr}(x)) \approx 0.3855$, while that of $y$ is $\text{mean}(\text{corr}(y)) \approx -0.0034$. In addition, the standard deviation of sample correlation coefficients of $x$, $\text{std}(\text{corr}(x)) \approx 0.1043$, is close to that of $y$, $\text{std}(\text{corr}(y)) \approx 0.1109$. As a consequence, the standardization from $x$ to $y$ eliminates the cross-sectional correlation significantly.

Also, we show that the standardization preserves basic time series properties for each stock individually. Specifically, calculate the correlation coefficient between the time series of $x_{it}$ and $y_{it}$, i.e. before and after the standardization. Then we get $n = 185$ correlation coefficients, and their histogram is Figure 6. It is easy to see that all of them are significantly positive. Therefore, we conclude that the standardization does preserve some basic properties.

4.3. The time series of $T(\omega^*)$

Last, we show the time series properties about $T(\omega^*)$ in Figure 7.

We use the rolling calculation $\|\bar{y}(T)\|_t := \left\| \frac{1}{T} \sum_{s=t-T+1}^{t} y_s \right\|$ as an estimator of $T(\omega^*)$, where $T$ is the length of the rolling period and $t$ is the time. In order to compare the time series properties, the CSSD of
Figure 5: The heat maps of the cross-sectional correlation coefficient matrix of $x_i$ and $y_i$: two 185-by-185 matrix. The two subfigures are plotted separately, where (a) is the correlation coefficients of $x_i$, while (b) $y_i$. For the purpose of clear demonstration, these two correlation matrix are processed after permutation along axises, resulting in that the $(i, j)$-th element in (a) is not necessary that in (b). But the permutation does not affect the results. What’s more, due to the symmetry of correlation matrix, (a) just shows the upper triangular one, while (b) lower.

Figure 6: The histogram of the correlation coefficients between $x_i$ and $y_i$. Rather than the cross-sectional correlation coefficients in Figure 5, we plot the correlation coefficients of the standardization from $x$ to $y$, where the vertical line marked 0.7219 is the sample mean of the $n = 185$ correlation coefficients.

the rolled stock returns is also calculated\(^7\). Easily, taking $T = 20$ and $n = 185$, the time series of $T(\omega^*)$ and CSSD\(_t\) are in Figure 7.

**Remark 14.** This remark interprets Figure 7. To begin with, we provide some descriptive statistics: The sample mean of $T(\omega^*)$ is 0.2224 and the standard deviation is 0.0323. For CSSD, the sample mean is 0.0046

\(^7\)CSSD\(_t\) := $\frac{1}{\sqrt{n}} || P\tilde{e}_t || = \sqrt{\frac{1}{n} \sum_{i=1}^n \left( \tilde{e}_{it} - \bar{\tilde{e}}_t \right)^2 }$, where $\bar{\tilde{e}}_t$ := $\frac{1}{T} \sum_{s=t-T+1}^t x_s$ and $\bar{x}_t := \frac{1}{n} \sum_{i=1}^n \tilde{x}_{it}$. 

13
and the standard deviation is 0.0014. Furthermore, it is difficult to ignore the mean-reverting property of \( T(\omega^*) \). Last but not least, there is some co-movement between \( T(\omega^*) \) and CSSD. In fact, the correlation coefficient of these two time series is about 0.4574. Moreover, the most correlated time periods of them are about January 2015, when the China stock market suffered the notorious meltdown crash.

In summary of this section, the standardization from \( x \) to \( y \) brings some insights, and the estimate of \( T(\omega) \) reveals new information too. First, \( x_{it} \)’s are standardized into \( y_{it} \)’s, improving the comparability between different stocks. In our specific study, the volatility of \( y_{it} \) of the representative stocks is greater than that of \( x_{it} \). Second, on the one hand, the overall empirical study shows that the standardization eliminates the cross-sectional correlation significantly. On the other hand, the standardized returns inherited a lot of information from the original ones. Last, from the viewpoint of financial market, the estimates of \( T(\omega^*) \) are of mean-reverting property, and they are positively correlated with CSSD particularly in some extreme scenarios.

5. Conclusion

In this paper, we focus on the cross-sectional measure IC from a probability perspective, and define it as \( T(\omega) \) mathematically: \( T(\omega) \) is a linear combination of the components of \( PX/\|PX\| \). In the definition, \( PX/\|PX\| \) is the standardized random vector generated by linear and directional projections, which represents the standardization of next period cross-sectional returns. Theoretically, we solve the optimization that maximizes \( T(\omega) \) in expectation and give the maximum. We prove that the solutions could be expressed as the MD and MRL in directional statistics. Therefore, some closed-from expressions under particular condition are given. According to the theoretical results, the measure IC is influenced by the dispersion and correlation of cross-sectional returns \( X \) positively, while volatility negatively. These conclusions are
consistent with those of finance literature.

Our simulation analysis supplements the theoretical results. It shows the distributions of $T(\omega^*)$ and $T(\mu)$, and the impact of stock number $n$: The more stocks, the less dispersive the $T(\omega)$. Furthermore, we simulate $\omega^*$ and $ET(\omega^*)$ by different parameters $\mu$ and $\Sigma$. By the comparison of the theoretical results and the simulation, we give an estimate of $ET(\omega^*)$ in general condition.

The empirical studies reveal that the standardization $\frac{y_t}{\|y_t\|}$ is a unique and useful tool to excavate more information about whole market. The time series $y_{it}$ is bounded and mean-reverting. Meanwhile, $y_{it}$ keeps most information of $x_{it}$: $x_t$ and $y_t$ are highly correlated, and $y_t$ maintains the similar volatility clustering property of $x_t$. Moreover, the cross-sectional correlation between $y_{it}$ and $y_{jt}$ is eliminated by the standardization, which is no longer statistically significant. Last, the estimate of $T(\omega^*)$ is correlated with CSSD particularly in some extreme scenarios. These results in Chinese stock market were rarely founded and revealed.

Acknowledgments

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

References

Abramowitz, M., Stegun, I.A., 1964. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards.

Ambachtsheer, K.P., 1974. Profit potential in an “almost efficient” market. The Journal of Portfolio Management 1, 84–87. doi:10.3905/jpm.1974.408485

Ambachtsheer, K.P., Farrell, J.L., 1979. Can active management add value? Financial Analysts Journal 35, 39–47. doi:10.2469/faj.v35.n6.39

Coggin, T.D., Hunter, J.E., 1983. Problems in measuring the quality of investment information: the perils of the information coefficient. Financial Analysts Journal 39, 25–33. doi:10.2469/faj.v39.n3.25

Ding, Z., Martin, R.D., 2017. The fundamental law of active management: redux. Journal of Empirical Finance 43, 91 – 114. doi:10.1016/j.jempfin.2017.05.005

Fisher, R.A., 1953. Dispersion on a sphere. Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 217, 295–305. doi:10.1098/rspa.1953.0064

Grinold, R.C., 1989. The fundamental law of active management. The Journal of Portfolio Management 15, 30–37. doi:10.3905/jpm.1989.409211
Appendix: Proofs

Proof of Theorem 1. We rewrite the expression of $T(\omega)$ to the square loss function, and thus by Lagrange multiplier, we prove the first part of the theorem. The proof of the second part follows directly.

With the restriction $\omega \in (1)^\perp \cap S^{n-1} := \{x \in \mathbb{R}^n | x^T 1 = 0, \|x\| = 1\}$, we have

$$ET(\omega) = \mathbb{E} \omega^T \frac{PX}{\|PX\|},$$

$$= \mathbb{E} \left[ -\frac{1}{2} \left( 1 - 2\omega^T \frac{PX}{\|PX\|} + 1 \right) \right] + 1,$$

$$= -\frac{1}{2} \mathbb{E} \left\| \omega - \frac{PX}{\|PX\|} \right\|^2 + 1.$$

Consequently,

$$\max_{\omega \in (1)^\perp \cap S^{n-1}} ET(\omega) \iff \min_{\omega \in (1)^\perp \cap S^{n-1}} \mathbb{E} \left\| \omega - \frac{PX}{\|PX\|} \right\|^2.$$

By Lagrange multiplier, we can prove the solution to the optimization (9) is (3). Furthermore, with (3) in the expression, the maximum (4) is gotten.
Proof of Theorem 2. Theorem 2 is a straightforward corollary of following Lemma 5, and one can prove the lemma by algebra matrix.

Lemma 5. Given $X \sim N(\mu, \Sigma)$, $\Sigma > 0$, there exist a matrix $U$ such that

1. $U$ is orthogonal matrix: $U^T U = U U^T = I$,
2. $U^T P U = \begin{pmatrix} I_{n-1} & 0 \\ 0^T & 0 \end{pmatrix}$,
3. $(I_{n-1} \ 0_{n-1}) U^T \Sigma U \begin{pmatrix} I_{n-1} \\ 0_{n-1}^T \end{pmatrix} =: \Lambda_{n-1}$ is an $(n - 1)$-dim positive definitive diagonal matrix, i.e. $\Lambda_{n-1} = \text{diag}\{\lambda_1, \cdot\cdot\cdot, \lambda_{n-1}\}$, $\lambda_i > 0, i = 1, 2, \cdot\cdot\cdot, n - 1$.

Furthermore,

$$\frac{P X}{\|P X\|} = U \begin{pmatrix} \xi_{n-1} \\ 0 \end{pmatrix} = U \begin{pmatrix} \xi_{n-1} \\ 0 \end{pmatrix},$$

(10)

where

$$\xi_{n-1} := (I_{n-1} \ 0_{n-1}) U^T X,$$

(11)

$$\sim N(\begin{pmatrix} I_{n-1} & 0_{n-1} \end{pmatrix} U^T \mu, \Lambda_{n-1}),$$

(12)

Proof of Proposition 4. Define $U$ as follows,

$$U_{ij} = \begin{cases} \frac{n-j}{\sqrt{(n-j)(n-j+1)}}, & i = j \neq n; \\ -\frac{1}{\sqrt{(n-j)(n-j+1)}}, & i < j \neq n; \\ \frac{1}{\pi}, & j = n; \\ 0, & \text{otherwise.} \end{cases}$$

Then, by Lemma 5 and Theorem 2 one can prove it by tedious calculation. \qed