Chapter 1

Anderson localization and Supersymmetry

K.B. Efetov

Theoretische Physik III, Ruhr-Universität Bochum, 44780, Bochum, Germany
efetov@tp3.rub.de

The supersymmetry method for study of disordered systems is shortly reviewed. The discussion starts with a historical introduction followed by an explanation of the idea of using Grassmann anticommuting variables for investigating disordered metals. After that the nonlinear supermatrix $\sigma$-model is derived. Solution of several problems obtained with the help of the $\sigma$-model is presented. This includes the problem of the level statistics in small metal grains, localization in wires and films, and Anderson metal-insulator transition. Calculational schemes developed for studying these problems form the basis of subsequent applications of the supersymmetry approach.

1.1. Introduction

The prediction of the new phenomenon of the Anderson localization\cite{Anderson1958} has strongly stimulated both theoretical and experimental study of disordered materials. This work demonstrates the extraordinary intuition of the author that allowed him to make outstanding predictions. At the same time, one could see from that work that quantitative description of the disordered systems was not a simple task and many conclusions were based on semi-qualitative arguments. Although many interesting effects have been predicted in this way, development of theoretical methods for quantitative study of quantum effects in disordered systems was clearly very demanding.

The most straightforward way to take into account disorder is using perturbation theory in the strength of the disorder potential\cite{Efetov1982}. However, the phenomenon of the localization is not easily seen within this method and the conventional classical Drude formula for conductivity was considered in\cite{Efetov1982} as the final result for the dimensionality $d > 1$. This result is obtained after summation of diagrams without intersection of impurity lines. Diagrams with intersection of the impurity lines give a small contribution if the disorder potential is not strong, so that $\varepsilon_0 \tau \gg 1$, where $\varepsilon_0$ is the energy of the particles (Fermi energy in metals) and $\tau$ in the elastic scattering time.

Although there was a clear understanding that the diagrams with the inter-
section of the impurity lines were not small for one dimensional chains, \( d = 1 \), performing explicit calculations for those systems was difficult. This step has been done considerably later by Berezinsky\(^3\) who demonstrated localization of all states in 1\(D\) chains by summing complicated series of the perturbation theory. This result confirmed the conclusion of Mott and Twose\(^4\) about the localization in such systems made previously. As concerns the higher dimensional systems, \( d > 1 \), the Anderson transition was expected at a strong disorder but it was clear that the perturbation theory could not be applied in that case.

So, the classical Drude theory was considered as a justified way of the description of disordered metals in \( d > 1 \) and \( \varepsilon_0 \tau \gg 1 \). At the same time, several results for disordered systems could not be understood within this simple generally accepted picture.

In 1965 Gorkov and Eliashberg\(^5\) suggested a description of level statistics in small disordered metal particles using the random matrix theory (RMT) of Wigner-Dyson\(^6,7\). At first glance, the diagrammatic method of Ref.\(^2\) had to work for such a system but one could not see any indication on how the formulae of RMT could be obtained diagrammatically. Of course, the description of Ref.\(^5\) was merely a hypothesis and the RMT had not been used in the condensed matter before but nowadays it looks rather strange that this problem did not attract an attention.

The prediction of localization in thick wires for any disorder made by Thouless\(^8\) could not be understood in terms of the traditional summing of the diagrams either but, again, there was no attempt to clarify this disagreement. Apparently, the diagrammatic methods were not very widely used in that time and therefore not so many people were interested in resolving such problems.

Actually, the discrepancies were not discussed in the literature until 1979, the year when the celebrated work by Abrahams et al.\(^9\) appeared. In this work, localization of all states for any disorder already in 2\(D\) was predicted. This striking result has attracted so much attention that it was simply unavoidable that people started thinking about how to confirm it diagrammatically. The only possibility could be that there were some diverging quantum corrections to the classical conductivity and soon the mechanism of such divergencies has been discovered\(^10,12\).

It turns out that the sum of a certain class of the diagrams with intersecting impurity lines diverges in the limit of small frequencies \( \omega \to 0 \) in a low dimension \( d \leq 2 \). This happens for any weak disorder and is a general phenomenon. The corresponding contribution is represented in Fig. 1.1.

The ladder in this diagram can be considered as an effective mode usually called now “cooperon”. This mode has a form of the diffusion propagator and its contribution to the conductivity \( \sigma(\omega) \) can be written in the form

\[
\sigma(\omega) = \sigma_0 \left( 1 - \frac{1}{\pi \nu} \int \frac{1}{D_0 k^2 - i \omega / (2\pi)^d} \right),
\]

(1.1)

where \( D_0 = \frac{v_0^2 \tau}{3} \) is the classical diffusion coefficient and \( \sigma_0 = 2 e^2 \nu D_0 \) is the
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classical conductivity. The parameters $v_0$ and $\nu$ are the Fermi velocity and density
of states on the Fermi surface.

Similar contributions arise also in other quantities. Eq. (1.1) demonstrates that
in the dimensions $d = 0, 1, 2$ the correction to conductivity diverges in the limit
$\omega \to 0$. It is very important that the dimension is determined by the geometry of the
sample. In this sense, small disordered particles correspond to zero dimensionality,
$d = 0$, and wires to $d = 1$.

The contribution coming from the diffusion mode, Eq. (1.1), is conceptually
very important because it demonstrates that the traditional summation of the di-
agrams without the intersection of the impurity lines is not necessarily applicable
in low dimensionality. One can see that most important contributions come from
the diffusion modes that are obtained by summation of infinite series of diagrams
containing electron Green functions.

The cooperon contribution, Eq. (1.1), has a simple physical meaning. It is
proportional to the probability for a scattered electron wave to come back and
interfere with itself. The interference implies the quantum coherence and this
condition is achieved at low temperatures. There are many interesting effects related
to this phenomenon but discussion of these effects and experiments is beyond the
scope of this chapter.

It is also relevant to mention that the cooperon contribution is cut by an external
magnetic field, which leads to a negative magnetoresistance. At the same time,
higher order contributions can still diverge in the limit $\omega \to 0$ and these divergencies
are not avoidable provided the coherence is not lost due to, e.g., inelastic processes.

In this way, one can reconcile the hypothesis about the Wigner-Dyson level
statistics in disordered metal particles and assertion about the localization in thick
wires and 2D films with the perturbation theory in the disorder potential. The
divergences due to the contribution of the diffusion modes make the perturbation
theory inapplicable in the limit $\omega \to 0$ and therefore one does not obtain just the
classical conductivity using this approach. Of course, summing the divergent quan-
tum corrections is not sufficient to prove the localization in the low dimensional
systems and one should use additional assumptions in order to confirm the state-
ments. Usually, the perturbation theory is supplemented by the scaling hypothesis\textsuperscript{9} in order to make such far-going conclusions.

At the same time, the divergence of the quantum corrections to the conductivity makes the direct analytical consideration very difficult for small $\omega$ because even the summation of all orders of the perturbation theory does not necessarily lead to the correct result. For example, the formulae for the level-level correlation functions\textsuperscript{6,7} contain oscillating parts that cannot be obtained in any order of the perturbation theory.

All this meant that a better tool had to be invented for studying the localization phenomena and quantum level statistics. Analyzing the perturbation theory one could guess that a low energy theory explicitly describing the diffusion modes rather than single electrons might be an adequate method.

The first formulation of such a theory was proposed by Wegner\textsuperscript{15} (actually, almost simultaneously with Ref.\textsuperscript{10}). He expressed the electron Green functions in terms of functional integrals over conventional complex numbers $S(r)$, where $r$ is the coordinate, and averaged over the disorder using the replica trick. Then, decoupling the effective interaction by an auxiliary matrix field $Q$ he was able to integrate over the field $S(r)$ and represent physical quantities of interest in terms of a functional integral over the $N \times N$ matrices $Q$, where $N$ is the number of replicas that had to be put to zero at the end of the calculations. Assuming that the disorder is weak the integral over the eigenvalues of the matrix $Q$ was calculated using the saddle point approximation.

As a result, a field theory in a form of a so called $\sigma$-model was obtained. Working with this model one has to integrate over $N \times N$ matrices $Q$ obeying the constraint $Q^2 = 1$. The $\sigma$-model is renormalizable and renormalization group equations were written in Ref.\textsuperscript{15} These equations agreed with the perturbation theory of Eq. (1.1) and with the scaling hypothesis of Ref.\textsuperscript{9} However, the saddle point approximation was not carefully worked out in\textsuperscript{15} because the saddle points were in the complex plane, while the original integration had to be done over the real axis. This question was addressed in the subsequent publications\textsuperscript{16,17}.

In the work\textsuperscript{16} the initial derivation of Ref.\textsuperscript{15} was done more carefully shifting the contours of the integration into the complex plane properly. In this way, one could reach the saddle point and integrate over the eigenvalues of matrix $Q$ coming to the constraint $Q^2 = 1$. After calculating this integral one is left with the integration over $Q$ that can be written as

$$Q = U\Lambda U^{-1}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(1.2)

where $U$ is an $2N \times 2N$ pseudo-orthogonal or pseudo-unitary matrix. This matrices vary on a hyperboloid, which corresponds to a noncompact group of the rotations. This group is quite unusual for statistical physics.
In contrast, the method of Ref.\cite{17} was based on representing the electron Green functions in a form of functional integrals over anticommuting Grassmann variables and the use of the replica trick. One could average over the disorder as well and further decouple the effective interaction by a gaussian integration over $Q$. The integration over the anticommuting variables leads to an integral over $Q$. The integral over the eigenvalues of $Q$ can be calculated using, again, the saddle point method, while the saddle points are now on the real axis. As a result, one comes to a $\sigma$-model with $Q$-fields of the form of Eq. (1.2). However, now one obtains $2N \times 2N$ matrices $U$ varying on a sphere and the group of the rotations is compact.

The difference in the symmetry groups of the matrices $Q$ of these two approaches looked rather unusual and one could only hope that in the limit $N = 0$ imposed by the replica method the results would have agree with each other.

This is really so for the results obtained in Refs.\cite{16,17} by using the renormalization group method or perturbation theory. The compact replica $\sigma$-model of Ref.\cite{17} has later been extended by Finkelstein\cite{21} to interacting electron systems. An additional topological term was added to this model by Pruiksen\cite{22} for studying the Integer Quantum Hall Effect. So, one could hope that the replica $\sigma$-models would help to solve many problems in the localization theory.

However, everything turned out to be considerably more complicated for non-perturbative calculations. Desperate attempts\cite{18} to study the level-level statistics in a limited volume and localization in disordered wires lead the present author to the conclusion that the replica $\sigma$-model of Ref.\cite{17} could not give any reasonable formulae. Calculation of the level-level correlation function using both the compact and noncompact replica $\sigma$-models was discussed later by Verbaarschot and Zirnbauer\cite{19} with a similar result. [Recently, formulae for several correlation functions for the unitary ensemble ($\beta = 2$) have nevertheless been obtained\cite{20} from the replica $\sigma$-models by viewing the replica partition function as Toda Lattice and using links with Panleve equations.]

The failure in performing non-perturbative calculations with the replica $\sigma$-models lead the present author to constructing another type of the $\sigma$-model that was not based on the replica trick. This method was called supersymmetry method, although the word “supersymmetry” is often used in field theory in a more narrow sense. The field theory derived for the disordered systems using this approach has the same form of the $\sigma$-model as the one obtained with the replica trick and all perturbative calculations are similar\cite{23}.

An attempt to calculate the level-level correlation function lead to a real surprise: the method worked\cite{24} leading in a rather simple way to the famous formulae for the level-level correlation functions known in the Wigner-Dyson theory\cite{6,7} thus establishing the relevance of the latter to the disordered systems. Since then one could use the RMT for calculations of various physical quantities in mesoscopic systems or calculate directly using the zero-dimensional supermatrix $\sigma$-model.

The calculation of the level correlations in small disordered systems followed by
the full solution of the localization problem in wires on the Bethe lattice and in high dimensionality. After that it has become clear that the supersymmetry technique is really an efficient tool suitable for solving various problems of theory of disordered metals.

By now several reviews and a book have been published where numerous problems of disordered, mesoscopic and ballistic chaotic system are considered and solved using the supersymmetry method. The interested reader can find all necessary references in those publications.

The present paper is not a complete review of all the works done using the supersymmetry method. Instead, I describe here the main steps leading to the supermatrix $\sigma$-model and first problems solved using this approach. I will try to summarize at the end what has become clear in the last almost 30 years of the development and what problems await their resolution.

1.2. Supermatrix non-linear $\sigma$-model.

The supersymmetry method is based on using both integrals over conventional complex numbers $S_i$ and anticommuting Grassmann variables $\chi_i$ obeying the anticommutation relations

$$\chi_i \chi_j + \chi_j \chi_i = 0 \quad (1.3)$$

The integrals over the Grassmann variables are used following the definition given by Berezin

$$\int d\chi_i = 0, \quad \int \chi_i d\chi_i = 1 \quad (1.4)$$

With this definition one can write the Gaussian integral $I_A$ over the Grassmann variables as

$$I_A = \int \exp \left( -\chi^+ A \chi \right) \prod_{i=1}^{N} d\chi_i^* d\chi_i = \det A, \quad (1.5)$$

which is different from the corresponding integral over complex numbers by presence of $\det A$ instead of $(\det A)^{-1}$ in the R.H.S. In Eq. (1.5), $\chi$ is a vector having as components the anticommuting variables $\chi_i$ ($\chi^+$ is its transpose with components $\chi^*$) and $A$ is an $N \times N$ matrix.

One can introduce supervectors $\Phi$ with the components $\Phi_i$,

$$\Phi_i = \begin{pmatrix} \chi_i \\ S_i \end{pmatrix} \quad (1.6)$$

and write gaussian integrals for these quantities

$$I_S = \pi^{-N} \int \exp \left( -\Phi^+ F \Phi \right) \prod_{i}^{N} d\chi_i^* d\chi_i dS^* dS = S \det F \quad (1.7)$$
In Eq. (1.7), $F$ is a supermatrix with block elements of the form

$$F_{ik} = \begin{pmatrix} a_{ik} & \sigma_{ik} \\ \rho_{ik} & b_{ik} \end{pmatrix}$$

(1.8)

where $a_{ik}$ and $b_{ik}$ are complex numbers and $\sigma_{ik}, \rho_{ik}$ are Grassmann variables. The superdeterminant (Berezinian) $SDet F$ in Eq. (1.7) has the form

$$SDet F = \det (a - \sigma b^{-1} \rho) \det b^{-1}$$

(1.9)

Another important operation is supertrace $STr$

$$STrF = Tra - Trb$$

(1.10)

Using these definitions one can operate with supermatrices in the same way as with conventional matrices. Note a very important consequence of Eq. (1.7) for supermatrices $F_0$ that do not contain the anticommuting variables and are equal to unity in the superblocks $F_{ik}$ in Eq. (1.8) ($a_{ik} = b_{ik}$). In this case one obtains

$$I_S[F_0] = 1$$

(1.11)

For such supermatrices one can write a relation that is the basis of the supersymmetry method in disordered metals

$$F_{0ik}^{-1} = \int \Phi_\alpha \Phi_\alpha^+ \exp (-\Phi^+ F \Phi) d\Phi$$

(1.12)

where $d\Phi = \pi^{-N} \prod_i^N d\chi_i^* d\chi_i dS^* dS$.

The weight denominator in the integral in Eq. (1.12) is absent and this form is analogous to what one has using the replica trick. Applying this representation to correlation functions describing disordered systems one can average over the disorder just in the beginning before making approximations. This is what is done when deriving the supermatrix $\sigma$-model and let me sketch this derivation.

Many quantities of interest can be expressed in terms of products of retarded $G^R_\varepsilon$ and advanced $G^A_\varepsilon$ Green functions of the Schrodinger equation. Using Eq. (1.12) one can write these functions as integrals over supervectors $\Phi$ (see (1113))

$$G^{R,A}_\varepsilon (y, y') = \mp i \int \Phi_\alpha (y) \Phi_\alpha^+ (y')$$

$$\times \exp \left[ i \int \Phi^+ (x) (\pm (\varepsilon - H) + i\delta) \Phi (x) \right] D\Phi^+ D\Phi$$

(1.13)

where $x$ and $y$ stand for both the space and spin variables.

The Hamiltonian $H$ in Eq. (1.13) consists of the regular $H_0$ and random $H_1$ parts

$$H = H_0 + H_1, \langle H_1 \rangle = 0$$

(1.14)

where the angular brackets $\langle \cdots \rangle$ stand for the averaging over the disorder.
The most important contribution to such quantities as conductivity and density-density correlation function is expressed in terms of a product
\[ K_{\omega} (r) = 2 \langle G_{\varepsilon - \omega}^A (r,0) G_{\varepsilon}^R (0,r) \rangle \]
where \( r \) is a coordinate and \( \omega \) is the frequency of the external electric field.

In order to express the function \( K_{\omega} (r) \) in terms of an integral over supervectors one should double the size of the supervectors. Introducing such supervectors \( \psi \) one represents the function \( K_{\omega} (r) \) in terms of a gaussian integral without a weight denominator. This allows one to average immediately this function over the random part. In the case of impurities described by a white noise disorder potential \( u (r) \) one comes after averaging to the following expression
\[ K_{\omega} (r) = 2 \int \psi_{1 \alpha}^1 (0) \psi_{1 \alpha}^2 (r) \psi_{2 \beta}^2 (0) \exp (-L) D\psi \]
where
\[ L = \int [i \bar{\psi} (\varepsilon - H_0) \psi + \frac{1}{4\pi \nu \tau} (\bar{\psi} \psi)^2 - \frac{\omega + i \delta}{2} \bar{\psi} \Lambda \psi] \, dr \]
Eq. (1.17) was obtained assuming the averages
\[ \langle u (r) u (r') \rangle = \frac{1}{2\pi \nu \tau} \delta (r - r'), \quad \langle u (r) \rangle = 0 \]
where \( \nu \) is the density of states and \( \tau \) is the elastic scattering time.

The fields \( \bar{\psi} \) in Eqs. (1.16, 1.17) are conjugate to \( \psi \), the matrix \( \Lambda \) is in the space of the retarded-advanced Green functions and equals
\[ \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
The infinitesimal \( \delta \to +0 \) is added to guarantee the convergence of the integrals over the commuting components \( S \) of the supervectors \( \psi \).

The Lagrangian \( L \), Eq. (1.17) has a form corresponding to a field theory of interacting particles. Of course, physically this interaction is fictitious but this formal analogy helps one to use approximations standard for many body theories.

The first approximation done in the supersymmetry method is singling out slowly varying pairs in the interaction term. This is done writing it as
\[ L_{int} = \frac{1}{4\pi \nu \tau} \int (\bar{\psi} \psi)^2 \, dr = \frac{1}{4\pi \nu \tau} \sum_{p_1 + p_2 + p_3 + p_4 = 0} (\bar{\psi}_{p_1} \psi_{p_2}) (\bar{\psi}_{p_3} \psi_{p_4}) \]
\[ \approx \frac{1}{4\pi \nu \tau} \sum_{p_1 + p_2, q < q_0} [(\bar{\psi}_{p_1} \psi_{p_1 + q}) (\bar{\psi}_{p_2} \psi_{p_2 - q}) + (\bar{\psi}_{p_1} \psi_{p_2}) (\bar{\psi}_{p_1 + q} \psi_{p_2}) + (\bar{\psi}_{p_2} \psi_{p_2}) (\bar{\psi}_{p_1} \psi_{p_1 + q}) + (\bar{\psi}_{p_1 + q} \psi_{p_2}) (\bar{\psi}_{p_1} \psi_{p_2})] \]
where \( q_0 \) is a cutoff parameter, \( q_0 < 1/l \), where \( l \) is the mean free path.

The next step is making a Hubbard-Stratonovich transformation decoupling the products of slowly varying pairs by auxiliary slowly varying fields. The term in the
second line in Eq. (1.20) is not important and the terms in the third line are equal to each other provided one uses the form of the supervectors $\psi$ of Refs. [31,33].

After the decoupling one obtains an effective Lagrangian quadratic in the fields $\psi, \bar{\psi}$ and one can integrate out the fields $\psi, \bar{\psi}$ in Eq. (1.16) and obtain a functional integral over the supermatrix field $Q(r)$. The corresponding free energy functional $F[Q]$ takes the form

$$F[Q] = \int \left[ -\frac{1}{2} STr \ln \left( \frac{1}{2} (\omega + i\delta) A - \frac{iQ(r)}{2\tau} \right) + \frac{\pi\nu}{8\tau} STrQ^2 \right] dr$$

and physical quantities should be obtained integrating correlation functions containing $Q$ over $Q$ with the weight $\exp(-F[Q])$.

The integrals with $F[Q]$ can be simplified using the saddle point approximation. The position of the minimum of $F[Q]$ is found in the limit $\omega \to 0$ by solving the equation

$$Q = \frac{i}{\pi\nu} \left[ \left( H_0 + \frac{i}{2\tau} Q(r) \right)^{-1} \right]_{r,r}$$

One can find rather easily a coordinate independent solution of Eq. (1.2). Writing $H_0$ in a general form as

$$H_0 = \varepsilon (-i\nabla_r) - \varepsilon_0$$

and Fourier transforming the latter, one should calculate the integral over the momenta $p$. In the limit $\varepsilon_0\tau \gg 1$ one comes to the general solution

$$Q^2 = 1$$

Although the supermatrix $Q^2$ is fixed by Eq. (1.24), the supermatrix $Q$ is not. Supermatrices $Q$ of the form of Eq. (1.2) are solutions for any $8 \times 8$ supermatrices $U$ satisfying the condition $UU = 1$. With this constraint they are neither unitary nor pseudo-unitary as it was in Refs. [16,17]. Actually, they consist of both unitary and pseudo-unitary sectors “glued” by the anticommuting variables. This unique symmetry is extremely important for basic properties of many physical quantities.

The degeneracy of the minimum of the free energy functional $F[Q]$ results in the existence of gapless in the limit $\omega \to 0$ excitations (Goldstone modes). These modes formally originate from fluctuating $Q$ obeying the constraint (1.24).

In order to write the free energy functional describing the fluctuations we assume that supermatrices $Q(r)$ obeying Eq. (1.24) slowly vary in space. Assuming that $\omega$ is small, $\omega\tau \ll 1$, but finite and expanding $F[Q]$ in this quantity and gradients of $Q$ one comes to the supermatrix $\sigma$-model

$$F[Q] = \frac{\pi\nu}{8} \int STr \left[ D_0 (\nabla Q)^2 + 2i (\omega + i\delta) \Lambda Q \right] d\mathbf{r}$$
where $D_0 = v_0^2 \tau / d$ is the classical diffusion coefficient ($v_0$ is the Fermi velocity and $d$ is the dimensionality of the sample) and the $8 \times 8$ supermatrix $Q$ obeys the constraint \(1.24\).

Calculation of, e.g., the function $K_\omega (r)$, Eq. \(1.10\), reduces to calculation of a functional integral over $Q$

$$K_\omega (r) = 2 \int Q_{\omega}^{12} (0) Q_{\omega}^{21} (r) \exp \left( - F [Q] \right) DQ$$  \(1.26\)

Eqs. \(1.25, 1.26\) is a reformulation of the initial problem of disordered metal in terms of a field theory that does not contain disorder because the averaging over the initial disorder has already been carried out. The latter enters the theory through the classical diffusion coefficient $D_0$. The supermatrix $\sigma$-model, described by Eq. \(1.25\) resembles $\sigma$-models used for calculating contributions of spin waves for magnetic materials. At the same time, the noncompactness of the symmetry group of the supermatrices $Q$ makes this $\sigma$-model unique.

In order to obtain classical formulae and first quantum corrections one can parametrize the supermatrix $Q$ as

$$Q = W + \Lambda \left( 1 - W^2 \right)^{1/2}, \quad W = \begin{pmatrix} 0 & Q^{12} \\ Q^{21} & 0 \end{pmatrix}$$  \(1.27\)

and make an expansion in $W$ in Eqs. \(1.25, 1.26\). Keeping quadratic in $W$ terms both in $F [Q]$ and in the pre-exponential in Eq. \(1.26\) one has to compute Gaussian integrals over $W$. Fourier transforming the function $K_\omega$, one obtains

$$K_\omega (k) = \frac{4 \pi \nu}{D_0 k^2 - i \omega}$$  \(1.28\)

Eq. \(1.28\) is the classical diffusion propagator. Taking into account higher orders in $W$ one can compute weak localization corrections to the diffusion coefficient. The first order correction is written in Eq. \(1.1\).

The precise symmetry of $Q$ depends on the presence of magnetic or spin-orbit interactions. In analogy with symmetries of random matrix ensembles in the Wigner-Dyson theory one distinguishes between the orthogonal ensemble (both magnetic and spin orbit interactions are absent), unitary (magnetic interactions are present) and symplectic (spin-orbit interactions are present but magnetic interactions are absent).

Actually more symmetry classes are possible. They are fully classified by Altland and Zirnbauer.

In the next sections solutions of several important problems solved with the help of the $\sigma$-model, Eq. \(1.25\), will be presented.

1.3. Level statistics in small metal particles.

The first non-trivial problem solved with the supermatrix $\sigma$-model was the problem of describing the level statistics in small disordered metal particles. At first glance,
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this problem is not related to the Anderson localization. However, in the language of the $\sigma$-model the solutions of these problems is study of the field theory, Eq. (1.25), in different dimensions. The localization can be obtained in the dimensions $d = 1, 2$ and $3$, while the Wigner-Dyson level statistics can be obtained for the zero dimensional version of the $\sigma$-model.

What is the zero dimensionality of the free energy functional $F [Q]$, Eq. (1.25), can easily be understood. In a finite volume the space harmonics are quantized. The lowest harmonics corresponds to the homogeneous in the space supermatrix $Q$.

The energy of the first excited harmonics $E_1$ can be estimated as

$$E_1 = E_c / \Delta$$  \hspace{1cm} (1.29)

where energy $E_c$,

$$E_c = \pi^2 D_0 / L^2$$  \hspace{1cm} (1.30)

is usually called the Thouless energy.

The other energy scale $\Delta$,

$$\Delta = (\nu V)^{-1}$$  \hspace{1cm} (1.31)

where $V$ is the volume, is the mean level spacing.

It is clear from Eqs. (1.25, 1.26) that in the limit $E_c \gg \Delta, \omega$ (1.32) one may keep in these equations only the zero space harmonics of $Q$, so that this supermatrix does not depend on the coordinates. One can interpret this limit as zero dimensional one and replace the functional $F [Q]$ by the function $F_0 [Q]$,

$$F_0 [Q] = i \pi (\omega + i \delta) / 4 \Delta STr (\Lambda Q)$$  \hspace{1cm} (1.33)

The function $R (\omega)$ that determines the correlation between the energy levels is introduced as

$$R (\omega) = \left\langle \frac{\Delta^2}{\omega} \sum_{k,m} (n (\varepsilon_k) - n (\varepsilon_m)) \delta (\omega - \varepsilon_m + \varepsilon_k) \right\rangle$$  \hspace{1cm} (1.34)

It is proportional to the probability of finding two levels at a distance $\omega$.

Using the supersymmetry approach one can represent the functions $R (\omega)$ in terms of a definite integral over the supermatrices $Q$

$$R (\omega) = \frac{1}{2} - \frac{1}{2} \text{Re} \int Q_{11}^{11} Q_{11}^{22} \exp (-F_0 [Q]) dQ$$  \hspace{1cm} (1.35)

In order to calculate the integral in Eq. (1.35) one should choose a certain parametrization for the supermatrix $Q$.

It is convenient to write the supermatrix $Q$ in the form

$$Q = UQ_0 \hat{U}, \quad Q_0 = \begin{pmatrix} \cos \tilde{\theta} & i \sin \tilde{\theta} \\ -i \sin \tilde{\theta} & -\cos \tilde{\theta} \end{pmatrix}, \quad U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$  \hspace{1cm} (1.36)
where all anticommuting variables are packed in the supermatrix blocks $u$ and $v$. It is clear that the (pseudo) unitary supermatrix $U$ commutes with $\Lambda$, which drastically simplifies the integrand in Eq. (1.35).

Instead of the integration in Eq. (1.35) over the elements of the supermatrix $Q$ with the constraint (1.24) one can integrate over the elements of the matrix $\hat{\theta}$ and the matrices $u$ and $v$. Of course, it is necessary to write a proper Jacobian (Berezinian) of the transformation to these variables. The latter depends only on the elements of $\hat{\theta}$ and therefore the elements of $u$ and $v$ appear only in the pre-exponential in Eq. (1.35). The integration over the supermatrices $u$ and $v$ is quite simple and one comes to definite integrals over the elements of $\hat{\theta}$.

The number of the independent variables in the blocks $\hat{\theta}$ depends on the ensemble considered. The supermatrices $Q$ written for the unitary ensemble have the simplest structure and the blocks $\hat{\theta}$ contains only 2 variables $0 < \theta < \pi$ and $0 < \theta_1 < \infty$. The corresponding blocks $\hat{\theta}$ for the orthogonal and symplectic ensembles contain 3 independent variables. All the transformations are described in details in Refs. 31–33.

In order to get an idea about what one obtains after the integration over $u$ and $v$ in Eq. (1.36), I write here an expression for the unitary ensemble only

$$R(\omega) = 1 + \frac{1}{2} Re \int_1^{\infty} \int_{-1}^{1} \exp \{ i (x + i \delta) (\lambda_1 - \lambda) \} d\lambda_1 d\lambda \quad (1.37)$$

where $x = \pi \omega / \Delta$, $\lambda_1 = \cosh \theta_1$, and $\lambda = \cos \theta$.

So, the calculation of the level-level correlation function is reduced to an integral over 2 or 3 variables depending on the ensemble considered. The final result for the orthogonal $R_{orth}(\omega)$, unitary $R_{unit}(\omega)$, and symplectic $R_{sympl}(\omega)$ ensembles calculated using Eq. (1.35) takes the following form

$$R_{orth}(\omega) = 1 - \sin^2 \frac{x}{x^2} - \frac{d}{dx} \left( \frac{\sin x}{x} \right) \int_1^{\infty} \frac{\sin xt}{t} dt \quad (1.38)$$

$$R_{unit}(\omega) = 1 - \frac{\sin^2 x}{x^2} \quad (1.39)$$

$$R_{sympl}(\omega) = 1 - \sin^2 \frac{x}{x^2} + \frac{d}{dx} \left( \frac{\sin x}{x} \right) \int_0^{1} \frac{\sin xt}{t} dt \quad (1.40)$$

Eqs. (1.38-1.40) first obtained for the disordered metal particles agree with the corresponding formulae of the Wigner-Dyson theory obtained from the ensembles of random matrices. This agreement justified the application of the RMT for small disordered particles suggested in Ref. 5.

Actually, to the best of my knowledge, this was the first explicit demonstration that RMT could correspond to a real physical system. Its original application to nuclear physics was in that time phenomenological and confirmed by neither analytical nor numerical calculations.

A direct derivation of Eqs. (1.38-1.40) from gaussian ensembles of the random matrices using the supermatrix approach was done in the review. This allowed
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the authors to compute certain average compound-nucleus cross sections that could not be calculated using the standard RMT route.

The proof of the applicability of the RMT to the disordered systems was followed by the conjecture of Bohigas, Giannonni and Schmit about the possibility of describing by RMT the level statistics in classically chaotic clean billiards. Combination of the results for clean and disordered small systems (billiards) has established the validity of the use of RMT in mesoscopic systems. Some researches use for explicit calculations methods of RMT but many others use the supermatrix zero-dimensional $\sigma$-model (for review see, e.g. 34, 40, 41). At the same time, the $\sigma$-model is applicable to a broader class of systems than the Wigner-Dyson RMT because it can be used in higher dimensions as well. Actually, one can easily go beyond the zero dimensionality taking higher space harmonics in $F[Q]$, Eq. (1.25). In this case, the universality of Eqs. (1.38-1.39) is violated. One can study this limit for $\omega \gg \Delta$ using also the standard diagrammatic expansions of Ref. 2 and this was done in Ref. 42.

The other versions of the $\sigma$-model (based on the replica trick and Keldysh Green functions) have not shown a comparable efficiency for studying the mesoscopic systems, although the formula for the unitary ensemble, Eq. (1.39), has been obtained by these approaches 43.

The results reviewed in this section demonstrate that the development of the theory of the energy level statistics in small systems and of related phenomena in mesoscopic systems have been tremendously influenced by the ideas of the Anderson localization because important results have been obtained by methods developed for studying the latter.

1.4. Anderson localization in quantum wires

The one dimensional $\sigma$-model corresponds to quantum wires. These objects are long samples with a finite cross-section $S$ that should be sufficiently large,

$$S \nu_0^2 \gg 1, \quad (1.41)$$

where $\nu_0$ is the Fermi momentum. In other words, the number of transversal channels should be large. This condition allows one to neglect non-homogeneous in the transversal direction variations of $Q$. Of course, the inequality $\epsilon_\tau \gg 1$ should be fulfilled as before.

Then, the $\sigma$-model can be written in the form

$$F[Q] = \frac{\pi \tilde{\nu}}{8} \int \left[ D_0 \left( \frac{dQ}{dx} \right)^2 + 2i\omega \Lambda Q \right] dx, \quad (1.42)$$

where $\tilde{\nu} = \nu S$.

Again, depending of the presence of magnetic and/or spin-orbit interactions the model has different symmetries (orthogonal, unitary and symplectic). It is important to emphasize that Eq. (1.42) is not applicable for disordered chains or
thin wires where the inequality (1.41) is not fulfilled. However, the explicit solutions show that the low frequency behavior of all these systems is the same.

Computation of the correlation function $K_{\omega}(x)$, Eq. (1.26), with the one-dimensional $\sigma$-model can be performed using the transfer matrix technique. Following this method one reduces the calculation of the functional integral in Eq. (1.26) to solving an effective Schrödinger equation in the space of the elements of the supermatrix $Q$ and calculating matrix elements of $Q$ entering the pre-exponential in Eq. (1.26). This has been done in Ref. [31] and presented also in the subsequent publications [31,33].

At first glance, this procedure looks very complicated due to a large number of the elements in the supermatrices $Q$. Fortunately, the symmetries of the free energy functional $F[Q]$ in Eq. (1.42) help one again to simplify the calculations. In order to derive the transfer matrix equations one should subdivide the wire into small slices and write recursive equations taking at the end the continuous limit. Instead of this artificial subdivision it is more instructive to consider a realistic model of a chain of grains coupled by tunnelling. The free energy functional $F_J[Q]$ for such a chain can be written in the form

$$F_J[Q] = \text{STr} \left( -\sum_{i,j} J_{ij} Q_i Q_j + i \frac{(\omega + i\delta)}{4\Delta} \sum_i \Delta Q_i \right)$$  \hspace{1cm} (1.43)$$

where $J_{ij} = J$ for nearest neighbors and $J_{ij} = 0$ otherwise. The summation runs in Eq. (1.43) over the grains. The coupling constant $J$ can be expressed in terms of the matrix elements of the tunnelling from grain to grain $T_{ij}$ but at the moment this explicit relation is not important.

In the limit $J \gg 1$, only small variations of the supermatrix $Q$ in space are important and the functional $F_J[Q]$, Eq. (1.43), can be approximated by $F[Q]$, Eq. (1.42). The classical diffusion coefficient $D_0$ corresponding to Eq. (1.43) takes the form

$$D_0 = \frac{4\Delta}{\nu} \sum_i J_{ij} (r_i - r_j)^2$$  \hspace{1cm} (1.44)$$

The correlation function $K_{\omega}$, Eq. (1.26), should also be taken at the discrete coordinates $r_i$ numerating the grains. Then, it can be re-written identically in the form

$$K_{\omega}(r_1, r_2) = 2\pi^2 \nu \int \Psi(Q_1) \Psi(Q_2) \frac{\partial^{12}}{\partial Q_1^{12}} \frac{\partial^{21}}{\partial Q_2^{21}} \Gamma(r_1, r_2; Q_1, Q_2) dQ_1 dQ_2$$  \hspace{1cm} (1.45)$$

where the kernel $\Gamma(r_1, r_2; Q_1, Q_2)$ is the partition function of the segment between the points $r_1$ and $r_2$. It is assumed that integration for this kernel is performed over all $Q$ except $Q_1$ and $Q_2$ at the points $r_1$ and $r_2$. So the kernel $\Gamma(r_1, r_2; Q_1, Q_2)$ depends on supermatrices $Q_1$, $Q_2$ and distances $r_2 - r_1$ (the point $r_2$ is to the right of the point $r_1$). The function $\Psi(Q)$ is the partition function of the parts of the
wire located to the right of the point $r_2$ and to the left of the point $r_1$. This function depends only on the supermatrix $Q$ at the end points $r_1$ or $r_2$.

Comparing the functions $\Psi(Q)$ at neighboring grains one comes to the following equation

$$
\Psi(Q) = \int N(Q,Q') Z_0(Q') \Psi(Q') \, dQ'
$$

(1.46)

where

$$
N(Q,Q') = \exp\left(\frac{\alpha}{4} STrQQ'\right), \quad \alpha = 8J
$$

(1.47)

$$
Z_0(Q) = \exp\left(\frac{\beta}{4} STr\Lambda Q\right), \quad \beta = -i (\omega + i\delta) \pi \Delta
$$

A similar equation can be written for the kernel $\Gamma(r,r';Q_1,Q_2)$. Comparing this function at the neighboring points $r$ and $r+1$ one obtains the recurrence equation

$$
\Gamma(r,r';Q_1,Q_2) - \int N(Q,Q'') Z_0(Q'') \Gamma(r+1,r';Q'',Q') \, dQ'' = \delta_{rr'} \delta(Q - Q')
$$

(1.48)

The $\delta$-function entering Eq. (1.48) satisfies the usual equality

$$
\int f(Q') \delta(Q - Q') \, dQ' = f(Q)
$$

(1.49)

Eqs. (1.45,1.46,1.49) reduce the problem of calculation of a functional integral over $Q(r)$ to solving the integral equations and calculation of the integrals with their solutions. In the limit $J \gg 1$ the integral equations can be reduced to differential ones. Their solution can be sought using again the parametrization (1.36). The function $\Psi(Q)$ is assumed to be a function of the elements of the block $\hat{\theta}$. Then, one obtains the differential equation for $\Psi$ in the form

$$
\mathcal{H}_0 \Psi = 0
$$

(1.50)

The explicit form of the operator $\mathcal{H}_0$ depends on the ensemble considered. The simplest equation is obtained for the unitary ensemble for which the operator $\mathcal{H}_0$ takes the form

$$
\mathcal{H}_0 = -\frac{1}{2\pi \nu D_0} \left[ \frac{1}{J_\lambda} \frac{\partial}{\partial \lambda} J_\lambda \frac{\partial}{\partial \lambda} + \frac{1}{J_{\lambda_1}} \frac{\partial}{\partial \lambda_1} J_{\lambda_1} \frac{\partial}{\partial \lambda_1} \right] - i (\omega + i\delta) \pi \nu (\lambda_1 - \lambda)
$$

(1.51)

where

$$
J_\lambda = (\lambda_1 - \lambda)^{-2}
$$

Similar equations can be written for the central part entering Eq. (1.45).

Solving these equations and substituting the solutions into Eq. (1.45) one can determine (at least numerically) the frequency dependence of the function $K_\omega(r_1,r_2)$ and, hence, of the conductivity for all frequencies in the region $\omega \tau \ll 1$ and distances $|r_1 - r_2| p_0 \gg 1$. 
The calculation becomes considerably simpler in the most interesting case of low frequencies \( \omega \ll (\tilde{\nu}^2 D_0)^{-1} \). In this limit, the main contribution into the integral in Eq. (1.45) comes from large \( \lambda_1 \gg 1 \) and the solution \( \Psi \) of Eq. (1.50) is a function of only this variable.

Introducing a new variable

\[
z = -i \omega 2 \pi^2 \tilde{\nu}^2 D_0 \lambda_1
\]

one can reduce Eq. (1.51) to the form

\[
- z \frac{d^2 \Psi (z)}{dz^2} + \Psi (z) = 0
\]

with the boundary condition

\[
\Psi (0) = 1
\]

The Fourier transformed function \( K_\omega (k) \) takes the form

\[
K_\omega (k) = \frac{4 \pi \nu A (k)}{-i \omega}, \quad A (k) = \int_0^\infty (\Phi_k (z) + \Phi_{-k} (z)) \Psi (z) dz,
\]

where the function \( \Phi_k (z) \) satisfies the following equation

\[
- \frac{d}{dz} \left( z^2 \frac{d \Phi_k (z)}{dz} \right) + ik LC \Phi_k (z) + z \Phi_k (z) = \Psi (z)
\]

with the length \( L_c \) equal to

\[
L_c = 2 \pi \nu SD_0
\]

The length \( L_c \) is actually the localization length, which will be seen from the final result. Equations (1.53-1.56) can also be obtained for the orthogonal and symplectic ensembles but with different localization lengths \( L_c \). The result can be written as

\[
L_c^{\text{symplectic}} = 2 L_c^{\text{unitary}} = 4 L_c^{\text{orthogonal}}
\]

The residue of the function \( K_\omega \) is proportional to the function \( p_\infty (r, r', \varepsilon) \) introduced by Anderson:1

\[
p_\infty (r, r', \varepsilon) = \sum_k |\phi_k (r)|^2 |\phi_k (r')|^2 \delta (\varepsilon - \varepsilon_k),
\]

where \( \phi_k (r) \) are exact eigenfunctions.

Eqs. (1.53-1.56) exactly coincide with the low frequency limit of equations derived by Berezinsky,3 provided the length \( L_c \) is replaced by the mean free path \( l \), which shows that the low frequency limit of the one dimensional systems is universal.

The exact solution of Eqs. (1.53-1.56) leads to the following expression

\[
p_\infty (x) = \frac{\pi^2 \nu}{16 L_c} \int_0^\infty \left( \frac{1 + y^2}{1 + \cosh \pi y} \right)^2 \exp \left( - \frac{1 + y^2}{4 L_c} |x| \right) y \sinh \pi y dy
\]

(1.60)
In the limit $x \gg L_c$, Eq. (1.60) reduces to a simpler form

$$p_\infty (x) \approx \frac{\nu}{4\sqrt{\pi L_c}} \left( \frac{4L_c}{|x|} \right)^{3/2} \exp \left( -\frac{|x|}{4L_c} \right) \quad (1.61)$$

The exponential form of $p_\infty (x)$ proves the localization of the wave functions and shows that the length $L_c$ is the localization length. Note, however, the presence of the pre-exponential $|x|^{-3/2}$. Due to the factor the integral over $x$ of $p_\infty (x)$ remains finite even in the limit $L_c \to \infty$. Actually, one obtains

$$\int_{-\infty}^{\infty} p_\infty (x) \, dx = \nu, \quad (1.62)$$

which proves the localization of all states.

At small $k \ll L_c^{-1}$, the function $A (k)$ in Eq. (1.55) takes the form

$$A (k) = 1 - 4\zeta (3) k^2 L_c^2$$

and the static dielectric permeability $\epsilon$ equals

$$\epsilon = -4\pi e^2 \nu \left. \frac{d^2 A (k)}{dk^2} \right|_{k=0} = 32\zeta (3) e^2 \nu L_c^2 \quad (1.63)$$

where $\zeta (x)$ is the Riemann $\zeta$-function.

All these calculations have been performed for a finite frequency $\omega$ and the infinite length of the sample. One can also consider the case of the zero frequency and a finite length $L$. A full analysis of this limit has been presented by Zirnbauer\textsuperscript{45} who calculated the average conductivity as a function of $L_c$.

There is another Fokker-Planck approach to study transport of disordered wires developed by Dorokhov, Mello, Pereyra, and Kumar\textsuperscript{46,47} (DMPK method). It can be applied also to thin wires with a small number of channels. At the same time, this method cannot be used for finite frequencies. In the case of thick wires with a large number of the channels and zero frequencies, the equivalence of the supersymmetry to the DMPK method has been demonstrated by Brouwer and Frahm\textsuperscript{48}.

Many interesting problems of banded random matrices\textsuperscript{49} and quantum chaos (like kicked rotor\textsuperscript{50}) can be mapped onto the 1D supermatrix $\sigma$-model. However, a detailed review of these interesting directions of research is beyond the scope of this paper.

### 1.5. Anderson localization in 2 and 2 + $\epsilon$ dimensions.

Study of localization in 2 and 2 + $\epsilon$ using the replica $\sigma$-model was started by Wegner\textsuperscript{14} using a renormalization group (RG) technique. He was able to write the RG equations for the orthogonal and unitary ensembles that could be used in 2 dimensions and extended into 2 + $\epsilon$ dimensions for $\epsilon \ll 1$. The latter was done with a hope that putting $\epsilon = 1$ at the end of the calculations one could extract at least qualitatively an information about the Anderson metal-insulator transition in 3 dimensions. Based on this calculation a conclusion about the localization at any weak
disorder in 2D was made. As concerns $2 + \epsilon$, an unstable fixed point was found, which following the standard arguments by Polyakov[51] signaled the existence of the metal-insulator transition.

The symplectic case was considered within the compact replica $\sigma$-model in Ref.[17] using the same method of RG and it was shown that the resistivity had to vanish in the limit of $\omega \rightarrow 0$. The difference between the replica $\sigma$-models used in Refs.[15,16] (noncompact) and Ref.[17] (compact) is not essential when applying the RG scheme.

Exactly the same results are obtained with the supermatrix $\sigma$-model using the RG technique[31,33,52] and let me sketch the derivation here. As usual in the RG method, one introduces a running cutoff parameter and coupling constants depending on this cutoff. The $\sigma$-model for such couplings can be written as

$$ F = \frac{1}{t} \int S Tr \left[ (\nabla Q)^2 + 2i\tilde{\omega} \Lambda Q \right] dr \tag{1.64} $$

where $\tilde{\omega} = \omega / D_0$. The bare value of $t$ equals $t = 8 (\pi \nu D_0)^{-1}$ (c.f. Eq. (1.25)).

The $\sigma$-model looks similar to classical spin $\sigma$-models considered in Ref.[51] and one can follow the RG procedure suggested in that work. Using the constraint (1.24) one can write the supermatrix $Q$ in the form

$$ Q = V \Lambda \tilde{V}, \tag{1.65} $$

where $V \tilde{V} = 1$ so that $V$ is a pseudo-unitary supermatrix.

In order to integrate over a momentum shell one can represent the supermatrix $V$ in the form

$$ V (r) = \tilde{V} (r) V_0 (r), \tag{1.66} $$

where $V_0$ is a supermatrix fast varying in space and $\tilde{V}$ is slowly varying one. These supermatrices have the same symmetry as the supermatrix $V$.

Substituting Eq. (1.66) into Eq. (1.64) one can write the free energy functional $F [Q]$ in the form

$$ F = \frac{1}{t} \int S Tr \left[ (\nabla Q_0)^2 + 2 [Q_0, \nabla Q_0] \Phi + [Q_0, \Phi]^2 + 2i\tilde{\omega} \Lambda \tilde{V} Q_0 \right] dr \tag{1.67} $$

$$ Q_0 = V_0 \Lambda \tilde{V} 0, \quad \Phi = \tilde{V} \nabla \tilde{V} = -\tilde{\Phi} $$

The next step of the RG procedure is to integrate over the fast varying matrices $Q_0$ and reduce to a functional containing only slowly varying variables $V$. After this integration the free energy $F$ in Eq. (1.67) should be replaced by energy $F$ describing the slow fluctuations

$$ \tilde{F} = - \ln \int \exp (-F) DQ_0 \tag{1.68} $$

The integration over the supermatrix $Q_0$ can be done using a parametrization (1.27) or a more convenient parametrization

$$ Q_0 = \Lambda (1 + P) (1 - P)^{-1}, \quad PA + \Lambda P = 0. \tag{1.69} $$
Integration over the fast variation means that one integrates over Fourier transformed $P_k$ with $\lambda k_0 < k < k_0$, where $k_0$ is the upper cutoff and $\lambda < 1$. As a result of the integration one comes to the same form of the functional $F$ as in Eq. (1.64).

The constant $\tilde{\omega}$ does not change under the renormalization but the new coupling constant $\tilde{t}$ can be written as

$$\tilde{t}^{-1} = t^{-1} \left( 1 + \frac{\alpha t}{8} \int_{-\lambda k_0}^{k_0} \frac{d^d k}{k^2 (2\pi)^d} \right)$$

(1.70)

The correction to the coupling constant $t$, Eq. (1.70), is written in the first order in $t$. The parameter $\alpha$ depends on the ensemble and equals

$$\alpha = \begin{cases} -1, & \text{orthogonal} \\ 0, & \text{unitary} \\ 1, & \text{symplectic} \end{cases}$$

(1.71)

Stretching the coordinates in the standard way and changing the notation for the coupling constant $t \rightarrow 2^{d+1} \pi d \Gamma(d/2) t$, where $\Gamma$ is the Euler $\Gamma$-function one obtains the RG equation for $t$

$$\beta(t) = \frac{dt}{d \ln \lambda} = (d - 2) t + \alpha t^2$$

(1.72)

where $\beta(t)$ means the Gell-Mann-Low function.

In $2D$, the solution of this equation for the coupling constant $t$ (proportional to resistivity) takes the form

$$t(\omega) = \frac{t_0}{1 + \alpha t_0 \ln (1/\omega \tau)}$$

(1.73)

For sufficiently high frequencies $\omega$ the resistivity and the diffusion coefficient $D(\omega)$ proportional to $t^{-1}(\omega)$ coincide with their bare values.

Decreasing the frequency $\omega$ results in growing the resistivity for the orthogonal ensemble until the coupling constant $t(\omega)$ becomes of the order 1. Then, the RG scheme is no longer valid because the expansion in $t$ in the R.H.S. of Eq. (1.72) is applicable only for $t \ll 1$. However, it is generally believed that $t$ diverges in the limit $\omega \to 0$ and this should mean the localization of all states with an exponentially large localization length

$$L_c \propto \exp(1/t_0)$$

(1.74)

In the symplectic ensemble the resistivity $t(\omega)$ decreases with decreasing the frequency $\omega$. This interesting result was obtained in the first order in $t_0$ by Hikami, Larkin and Nagaoka However, Eq. (1.73) means more If the bare $t_0$ is small, $t_0 \ll 1$, the effective resistivity $t(\omega)$ decays down to zero in the limit $\omega \to 0$. In this case the constant $t(\omega)$ is small for any frequency and the one loop approximation used in the derivation of Eq. (1.72) is valid for all frequencies. So, the solution for the symplectic ensembles, when used for the low frequencies, is the most reliable one obtained with the RG method.
As concerns the unitary ensemble, the first order contribution vanishes and one should calculate corrections of the second order. As a result, one comes to the following dependence of \( t(\omega) \) on the frequency
\[
t(\omega) = \frac{t_0}{(1 - t_0^2 \ln (1/\omega \tau))^{1/2}}
\]
(1.75)

One can see from Eq. (1.75) that the resistivity \( t(\omega) \) grows, as in the orthogonal ensemble, until it becomes of order 1. Again, this behavior is interpreted as localization for any disorder. The conclusions about the localization in 2D for the orthogonal and unitary ensembles were made first in Ref.15 and this agreed with the results based on using the scaling hypothesis.9

Wegner developed also theory of the Anderson metal-insulator transition in the dimensionality \( 2 + \epsilon \) for \( \epsilon \ll 1 \). One can see that the RG equation (1.72) has a fixed point \( t_c = \epsilon \), at which the Gell-Mann-Low function vanishes. At this point the total resistance of the sample does not depend on the sample size and this point should correspond to the Anderson metal-insulator transition.

Linearizing function \( \beta(t) \) near the fixed point \( t_c \), one can solve Eq. (1.72). As a result one can find a characteristic (correlation) length \( \xi \) near the fixed point
\[
\xi \sim \xi_0 \left(\frac{t_c - t_0}{t_c}\right)^{-1/y}, \quad y = -\beta'(t_c)
\]
(1.76)

where \( \xi_0 \) is the size of a sample having the entire resistance \( t_0 \). Assuming that the length \( \xi \) is the only characteristic length in the system and that the conductivity \( \sigma \) is proportional to \( t_c^{-1} \xi^{2-d} \), one can write the equation for the conductivity in the following form
\[
\sigma = A \frac{e^2}{\xi_0^{d-2} t_c} \left(\frac{t_c - t}{t_c}\right)^s, \quad s = \frac{d - 2}{y}
\]
(1.77)

The explicit values of the critical resistance \( t_c \) and the exponent \( s \) for the orthogonal and unitary ensembles equals
\[
\tilde{t}_c = \begin{cases} 
  d - 2, & \text{orthogonal} \\
  (2(d - 2))^{1/2}, & \text{unitary}
\end{cases}
\]
(1.78)

and
\[
s = \begin{cases} 
  1 \\
  1/2
\end{cases}
\]
(1.79)

Eqs. (1.76-1.79) demonstrate that the metal-insulator transition exists in any dimensionality \( d > 2 \) and the conductivity near the transition obeys a power law. Of course, this consideration is restricted by small \( \epsilon = d - 2 \) and one can use the result in 3D only qualitatively.

The scaling approach developed for small \( \epsilon \) is similar to the one developed for conventional phase transitions in, e.g., spin models where one can also write \( \sigma \)-models. This method is not sensitive to whether the symmetry of the supermatrices
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$Q$ is compact or noncompact. Using this approach one comes to the conclusion that the Anderson metal-insulator transition is very similar to standard second order phase transitions.

In the next section the same problem will be considered on the Bethe lattice or in a high dimensionality. Surprisingly, the result will be very different and the peculiarity of the solution originates from the noncompactness of the group of the symmetry of the supermatrices $Q$.

1.6. Anderson metal-insulator transition on the Bethe lattice or in a high dimensionality.

It is generally difficult to find the critical point for a transition between different states and describe the critical behavior in its vicinity. The Anderson metal-insulator transition is definitely not an exception in this respect. Usually, identifying a proper order parameter one can get an idea about a transition using a mean field approximation. As concerns the Anderson transition, this is not possible. Although the $\sigma$-model, Eq. (1.25), looks very similar to spin models in a magnetic field, one cannot take an average of $Q$ with the free energy $F[Q]$, Eq. (1.25), as the order parameter because it determines the average density of states and is not related to the Anderson transition.

At the same time, the mean field approximation works very well in high dimensionality or on special structures like the Bethe lattice.

The Anderson model of the Bethe lattice was studied for the first time by Abou-Chacra, Anderson and Thouless, who proved the existence of the metal-insulator transition and found the position of the mobility edge. With the development of the supersymmetry technique it became possible to describe the critical behavior both in the metallic and insulating regime. Considering a granular model one could obtain results for the orthogonal, unitary and symplectic ensembles. Later the Anderson model has also been described.

It turned out that in all the cases the critical behavior was the same, which contrasts the results obtained within the $2 + \epsilon$ expansion. This could not be a big surprise because for most phase transitions the high dimensional results are more “universal” than those obtained in lower dimensions. However, the results for the metallic and insulating regimes did not obey the conventional scaling and this was completely unexpected.

The first attempt to solve the granular version of the supermatrix $\sigma$-model on the Bethe lattice has been undertaken in Ref. In this work correct integral equation have been written for description of critical behavior near the metal-insulator transition and the position of the mobility edge has been found. However, attempts to find a solution of this equation related to scaling properties of the $2 + \epsilon$ limit were not successful, which lead to wrong conclusions.

Studying numerically the integral equation derived in Zirnbauer found a very
unusual behavior near the critical point and presented formal reasons explaining this behavior. Finally, the density-density correlation function has been calculated for the unitary\textsuperscript{22} and orthogonal and symplectic ensembles.\textsuperscript{28} This determined the diffusion coefficient in the metallic region and localization length and dielectric permeability in the insulating one.

The form of the density-density correlation function on the Bethe lattice differs from the one on conventional lattices. Therefore the problem of the Anderson localization has been considered on such lattices in an effective medium approximation.\textsuperscript{30} The latter becomes exact on the real lattices in a high dimensionality $d \gg 1$ and the basic equations and results are similar. The derivation of the equations and the final results are shortly displayed below. A detailed discussion can be found in\textsuperscript{33}

The scheme of the derivation of the equations is similar to the one presented in Sec. 1.4 for one dimensional structures consisting of the grains. We start with Eq. (1.43) written on a $d$-dimensional lattice with $d \gg 1$ or on the Bethe lattice. Denoting by $\Psi (Q)$ the partition function of a branch of the tree structure with a fixed value $Q$ at the base and comparing it with the partition function on the neighboring site one comes to a non-linear integral equation

$$\Psi (Q) = \int N(Q, Q') Z_0 (Q') \Psi^m (Q') dQ'$$

(1.80)

where $m = 2d - 1$ for a $d$-dimensional lattice and is the branching number on the Bethe lattice. The functions $N(Q, Q')$ and $Z_0 (Q)$ have been introduced in Eq. (1.47).

The case $m = 1$ corresponds to the one-dimensional chains of the grains and Eq. (1.80) coincides with Eq. (1.46) in this limit. In this particular case equation (1.80) is linear and, as we have seen in Sec. 1.4, all states are localized for any disorder. However, at $m > 1$ the integral equation (1.80) is non-linear and has a bifurcation at a critical $\alpha_c$ corresponding to the Anderson metal-insulator transition.

The density-density correlation function $K_\omega$, Eq. (1.26), can be written in the form

$$K_\omega (r_1, r_2) = -2\pi^2 \nu_0 \int Q_{33}^{12} P_{33} (r, Q) Z (Q) \Psi (Q) dQ$$

(1.81)

where the function $P(r, Q)$ satisfies for the high dimensional lattices the following equation

$$P(r, Q) - \sum_{r'} W (r - r') \int N(Q, Q') P(r', Q') Z (Q') dQ'$$

(1.82)

$$+ m \int N_2 (Q, Q') P(r, Q') Z (Q') dQ' = \delta (r) Q^{21} \Psi (Q).$$

In Eq. (1.44) the function $N_2 (Q, Q')$ is equal to

$$N_2 (Q, Q') = \int N(Q, Q'') N(Q'', Q) Z (Q'') dQ''$$
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and

\[ W (r - r') = \begin{cases} 1, |r - r'| = 1 \\ 0, |r - r'| \neq 1 \end{cases} \]

The third term in the L.H.S. of Eq. (1.82) takes into account the fact that two segments of a broken line cannot coincide. Eqs. (1.81 1.82) are very similar to Eqs. (1.45 1.48) written for the 1D case. This is natural because in both the cases loops are absent. Their solution for a function \( \Psi (Q) \) found from Eq. (1.80) can be obtained making a spectral expansion of \( P (r, Q) \) in eigenfunctions of the integral operators entering the L.H.S. of Eq. (1.82).

In principle, this procedure is straightforward. However, solving the integral equation (1.80) is not simple because it contains a large number of the elements of the supermatrix \( Q \).

Fortunately, Eqs. (1.80 1.82) drastically simplify in the metallic regime near the metal-insulator transition and everywhere in the insulating regime provided one considers the low frequency limit \( \omega \to 0 \). The formal reason for this simplification is that the main contribution into the correlation functions comes in these cases from the region of very large values of the variables \( \lambda_1 \gg \Delta/\omega \gg 1 \). The same simplification has helped one to solve the problem of the localization in wires in Sec. 1.4.

Nevertheless, the full analysis is quite involved even for small \( \omega \). Details can be found again in Ref. and here I display only the final results.

In the insulating regime, \( \alpha < \alpha_c \), only \( \Psi = 1 \) is the solution of Eq. (1.80) in the limit \( \omega = 0 \). This solution of the simplified equation persists for all \( \alpha \) but another solution appears in the region \( \alpha > \alpha_c \). The latter solution considered as a function of \( \theta_1 = \ln (2 \lambda_1) \) has a form of a kink moving to infinity as \( \alpha \to \alpha_c \). The position \( \theta_{1c} \) of the kink depends on the distance from the critical point \( \alpha_c \) as

\[ \theta_{1c} = s (\alpha - \alpha_c)^{-1/2} \]

where \( s \) is a number of order 1. The dependence of \( \Psi (\lambda_1) \) is represented in Fig. 1.2

Only this solution should be used for \( \alpha > \alpha_c \) and this leads to a very non-trivial critical behavior of the diffusion coefficient.

The position of the critical point \( \alpha_c \) and the critical behavior have been calculated for all 3 ensembles. For large \( m \), the value \( \alpha_c \) for the orthogonal and unitary ensembles is determined by the following equations

\[ \frac{2^{1/2}}{\pi} \left( \frac{\alpha_c}{2\pi} \right)^{1/2} m \ln \frac{\alpha_c}{\alpha_c} = 1, \text{ orthogonal} \]

\[ \left( \frac{2\pi}{\alpha_c} \right)^{1/2} m \ln \frac{2}{\alpha_c} = 1, \text{ unitary} \]

One can see from Eq. (1.84) that the metallic region is broader for systems with the broken time reversal invariance. In other words, applying a magnetic field shifts the metal-insulating transition to larger values of \( \alpha_c \). This result correlates with the one, Eq. (1.78), obtained in \( 2 + \epsilon \) dimensions.
Although the position of the Anderson transition depends on the ensemble considered, the form of the correlation functions is the same.

In the insulating regime, the function \( p_\infty (r) \), Eq. (1.59), takes for \( r \gg L_c \) the following form

\[
p_\infty (r) = \text{const} \left( \frac{r}{L_c} \right)^{-d-1}
\]

where \( L_c \) is the localization length.

Near the transition the localization length \( L_c \) grows in a power law

\[
L_c = \frac{\text{const}}{(\alpha_c - \alpha)^{1/2}}
\]

In this regime there is another interesting region of \( 1 \ll r \ll L_c \) where the function \( p_\infty (r) \) decays in a power law

\[
p_\infty (r) = \text{const} \left( \frac{r}{L_c} \right)^{-d-1}
\]

Remarkably, Eq. (1.85) obtained for \( d \gg 1 \) properly describes also the one-dimensional wires (c.f. Eq. (1.61)).

The integral of \( p_\infty (r) \) over the volume is convergent for all \( \alpha \leq \alpha_c \) and remains finite in the limit \( \alpha \to \alpha_c \), indicating that the wave functions at the transition point decay rather fast. At the same time, all moments of this quantity diverge in this limit. The second moment determines the electric susceptibility \( \kappa \),

\[
\kappa \delta_{\alpha \beta} = e^2 \int r_\alpha r_\beta p_\infty (r) d^4r
\]

Near the transition calculation of the integral in Eq. (1.88) leads to the result

\[
\kappa = 4\pi^2 \nu c L_c
\]

where \( c \) is a coefficient.
This equation shows that the susceptibility in the critical region is proportional to the localization length $L_c$ and not to $L_c^2$ as it would follow from the one-parameter scaling and results obtained in $2 + \epsilon$ dimensions. The unusual dependence of the susceptibility $\kappa$ on $L_c$ in Eq. (1.89) arises formally from the anomalous exponent $(d + 2)/2$ in the power law behavior of the pre-exponent in Eq. (1.85).

As concerns the metallic regime one comes for the real lattices to the diffusion propagator, Eq. (1.28), at all $\alpha > \alpha_c$. However, except the limit $\alpha \gg 1$, the diffusion coefficient $D$ obtained now is different from the classical diffusion coefficient $D_0$. Its behavior in the critical region $\alpha - \alpha_c \ll \alpha_c$ is especially interesting. This is not a power law behavior as one could expect from the one parameter scaling. Instead, the diffusion coefficient decays near the transition exponentially

$$D = \text{const} \exp \left[ -s (\alpha - \alpha_c)^{-1/2} \right] (\alpha - \alpha_c)^{3/2}$$

(1.90)

This is a very unusual behavior. Formally, it follows from the non-compact symmetry of the supermatrices $Q$. For any compact symmetry one would obtain in the same approximation a power law dependence of the diffusion coefficient on $\alpha - \alpha_c$.

The exponential decay of the diffusion coefficient $D$, Eq. (1.90), follows from the shape of the function $\Psi$, Fig. 1.2. The position of the kink $\lambda_H = \cosh \theta_H$, Eq. (1.86), goes to infinity as $\lambda_H \propto \exp \left[ s (\alpha - \alpha_c)^{1/2} \right]$ and this results in the form (1.90) of the diffusion coefficient.

The same results, Eqs. (1.85-1.90), have been obtained later for the Anderson model on the Bethe lattice and this completed the study of this model started in Ref. 55. The agreement of the results obtained for the Anderson model and granulated $\sigma$-model is, of course, not accidental because, the critical behavior is formed by long time correlations and the result should not be sensitive to short distance structures. As it has been discussed previously, the low frequency behavior of wires and strictly one dimensional chains is also described by identical equations.

The exponential decay of the diffusion coefficient was interpreted in Ref. 33 in terms of tunnelling between quasi-localized states. This may happen provided the wave function is concentrated in centers with a large distance

$$\zeta \propto (\alpha - \alpha_c)^{-1/2}$$

(1.91)

between them. The decay of the amplitudes of the wave functions in the single center is fast as it can be seen from the fast decay in Eq. (1.87). Then, the tunnelling leads to an overlap between the wave functions of the different centers and to formation of a conduction band with an effective bandwidth $\Gamma$,

$$\Gamma \propto \Delta \exp (-a\zeta)$$

(1.92)

where $a$ is a coefficient.

The exponential decay of the diffusion coefficient $D$, Eq. (1.90), can follow quite naturally from such a picture. Of course, the picture implies the existence of
weakly overlapping centers of the localization near this transition. A well established
multifractality of wave functions at the transition (for a recent review, see, e.g. [36])
may point out on this strong inhomogeneity near the transition.

Another indication in favor of the presented picture comes from the fact that
the solution $\Psi$ of Eq. (1.80) loses the sensitivity to the existence of the transition
at frequencies $\omega \gtrsim \Gamma$. This can be seen from a more detailed analysis of Eq. (1.80).
The interpretation in terms of formation of a very narrow conduction band near the
transition with the bandwidth $\Gamma$ is consistent with this property of the solution $\Psi$.

The fixed point found in $2 + \epsilon$ for small $\epsilon$ corresponds to a weak disorder and
the strong inhomogeneities are not seen in this approach. One cannot speak of a
narrow conduction band near the transition in $2 + \epsilon$ dimensions within this picture.

In principle, centers of (quasi) localization exist in $2D$ and can be described in
the framework of $\sigma$-model (for a review, see Refs. [33, 35]). However, the idea about
these centers of the (quasi) localizations is not incorporated in the conventional $2 + \epsilon$
scheme. So, the standard continuation of the results obtained for small $\epsilon$ to $\epsilon = 1$
may result in losing an important information.

The non-trivial form of the function $\Psi$ (see e.g. Fig. 1.2) has led the present
author to the idea [30] that this function might play the role of an order parameter for
the Anderson transition. It was guessed that a Laplace transform of this function
could be related to a conductance distribution. This idea has been further developed
in Ref. [56] where a functional in an extended space was constructed such that its
minimum was reached at the function $\Psi (Q)$. This resembles the Landau theory of
phase transitions but the role of the order parameter is played by a function.

The concept of the function order parameter was also discussed in later works
on the Bethe lattice [57].

1.7. Discussion.

In this paper the basics of the supersymmetry method has been presented. It
is explained how the non-linear supermatrix $\sigma$-model is derived and it is shown
how one can calculate within this model. It is demonstrated how one comes to
the Wigner-Dyson statistics in a limited volume and how one obtains Anderson
localization in disordered wires. Renormalization group scheme is explained in 2
and $2 + \epsilon$ dimensions for small $\epsilon$, renormalization group equations are written and
solved. It is shown how one solves the problem of the Anderson metal-insulator
transition on the Bethe lattice and high dimensionality.

From the technical point of view all this was a demonstration how one can
calculate in the dimensions $d = 0$, $d = 1$, $d = 2$, and $d \gg 1$. Due to the lack
of the space the present paper is not a complete review of the application of the
supersymmetry technique and many interesting works are not mentioned. However,
the calculational schemes presented here have been used in most of the subsequent
works. So, having read this paper one can get an idea on how one can work in all
situations where the supersymmetry method is useful.

This is a chapter in the book devoted to 50 years of the Anderson localization and I tried to describe shortly how one of the directions of the field was developing in 1980s after the second most important work on the Anderson localization[^23] has been published. Many of the authors of the present volume entered this field motivated by this publication. I hope that the development of the supersymmetry method has been useful in solving several interesting problems of the Anderson localization.

Although the supersymmetry method proved to be an adequate method for studying disordered systems (at least, without electron-electron interaction), several very important problems have not been solved so far. In spite of the common believe that all states are localized in disordered films (orthogonal and unitary ensembles), the solution for the two-dimensional $\sigma$-model has not been found in the limit of low frequencies. The problem of the integer quantum Hall effect has not been solved either, although the idea about instantons[^22] was very useful for the understanding of this phenomenon. The problem of describing the critical behavior near the transition between the Hall plateaus still awaits its resolution.

One more interesting problem is to understand the critical behavior near the Anderson transition.

Of course, a lot of information comes from numerical simulations but solving the 2D problem analytically would be really a great achievement. As concerns the Anderson transition in 3D, the hope to solve it exactly is not realistic because even a simpler Ising model has not been solved in spite of numerous attempts. However, in the conventional theory of phase transition one can start with a mean field theory justifiable in high dimensions, determine the upper critical dimension and then make an expansion near this dimensionality.

Unfortunately, until now a similar procedure has not been found for the Anderson transition, although the supermatrix $\sigma$-model resembles spin models for which this procedure is standard. This concerns also the 2D case, where conventional spin $\sigma$-models are solvable. However, the well developed methods like the Bethe Ansatz or methods of the conformal field theory do not work here.

The formal reason of the failure of these approaches for studying the supermatrix $\sigma$-model is that the group of the symmetry of the supermatrices $Q$ is not compact. These supermatrices consist of a block varying on a sphere and another one with elements on the hyperboloid. The latter part of $Q$ is formally responsible for the localization but its presence leads to difficulties when applying the well developed methods. It is clear that the importance of the noncompact symmetry is not fully appreciated.

I can only express my hope that these problems will be resolved in the next 50 years and the book devoted to 100 years of the Anderson localization will contain the complete theory of this phenomenon.
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