ON THE SPACES OF FIBONACCI DIFFERENCE NULL AND CONVERGENT SEQUENCES

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ABSTRACT. In the present paper, by using the band matrix \( F \) defined by the Fibonacci sequence, we introduce the sequence sequence spaces \( c_0(F) \) and \( c(F) \). Also, we give some inclusion relations and construct the bases of the spaces \( c_0(F) \) and \( c(F) \). Finally, we compute the alpha-, beta-, gamma-duals of these spaces and characterize the classes \( (c_0(F), X) \) and \( (c(F), X) \) for certain choice of the sequence space \( X \).

1. INTRODUCTION

By \( \mathbb{N} \) and \( \mathbb{R} \), we denote the sets of all natural and real numbers, respectively. Let \( \omega \) be the vector space of all real sequences. Any vector subspace of \( \omega \) is called a sequence space. Let \( \ell_\infty, c, c_0 \) and \( \ell_p \) denote the classes of all bounded, convergent, null and absolutely \( p \)-summable sequences, respectively; where \( 1 \leq p < \infty \). Moreover, we write \( bs \) and \( cs \) for the spaces of all bounded and convergent series, respectively. Also, we use the conventions that \( e = (1, 1, \ldots) \) and \( e^{(n)} \) is the sequence whose only non-zero term is 1 in the \( n \)-th place for each \( n \in \mathbb{N} \).

Let \( \lambda \) and \( \mu \) be two sequence spaces and \( A = (a_{nk}) \) be an infinite matrix of real numbers \( a_{nk} \), where \( n, k \in \mathbb{N} \). Then, we say that \( A \) defines a matrix transformation from \( \lambda \) into \( \mu \) and we denote it by writing \( A : \lambda \to \mu \), if for every sequence \( x = (x_k) \in \lambda \) the sequence \( Ax = \{A_n(x)\} \), the \( A \)-transform of \( x \), is in \( \mu \); where

\[
A_n(x) = \sum_k a_{nk}x_k \quad \text{for each} \quad n \in \mathbb{N}.
\]

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to \( \infty \). By \( (\lambda, \mu) \), we denote the class of all matrices \( A \) such that \( A : \lambda \to \mu \). Thus, \( A \in (\lambda, \mu) \) if and only if the series on the right side of (1.1) converges for each \( n \in \mathbb{N} \) and every \( x \in \lambda \), and we have \( Ax \in \mu \) for all \( x \in \lambda \). Also, we write \( A_n = (a_{nk})_{k \in \mathbb{N}} \) for the sequence in the \( n \)-th row of \( A \).

The matrix domain \( \lambda_A \) of an infinite matrix \( A \) in a sequence space \( \lambda \) is defined by

\[
\lambda_A = \{x = (x_k) \in \omega : Ax \in \lambda\}
\]
which is a sequence space.

By using the matrix domain of a triangle infinite matrix, so many sequence spaces have recently been defined by several authors, see for instance [1–7]. In the literature, the matrix domain \( \lambda_\Delta \) is called the difference sequence space whenever \( \lambda \) is a normed or paranormed sequence space, where \( \Delta \) denotes the backward difference matrix \( \Delta = (\Delta_{nk}) \) and \( \Delta' = (\Delta'_{nk}) \) denotes the transpose of the matrix \( \Delta \), the forward difference matrix, which are defined by

\[
\Delta_{nk} = \begin{cases} (-1)^{n-k} & , \quad n - 1 \leq k \leq n, \\ 0 & , \quad 0 \leq k < n - 1 \text{ or } k > n, \end{cases}
\]

\[
\Delta'_{nk} = \begin{cases} (-1)^{n-k} & , \quad n \leq k \leq n + 1, \\ 0 & , \quad 0 \leq k < n \text{ or } k > n + 1. \end{cases}
\]

for all \( k, n \in \mathbb{N} \); respectively. The notion of difference sequence spaces was introduced by Kizmaz [8], who defined the sequence spaces

\[
X(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in X\}
\]
for $X = \ell_\infty, c$ and $c_0$. The difference space $b_{vp}$, consisting of all sequences $(x_k)$ such that $(x_k - x_{k-1})$ is in the sequence space $\ell_p$, was studied in the case $0 < p < 1$ by Altay and Başar \cite{9} and in the case $1 \leq p \leq \infty$ by Başar and Altay \cite{10}, and Çolak et al. \cite{11}. Kirişçi and Başar \cite{4} have been introduced and studied the generalized difference sequence spaces

$$\hat{X} = \{ x = (x_k) \in \omega : B(r, s)x \in X \},$$

where $X$ denotes any of the spaces $\ell_\infty, \ell_p, c$ and $c_0, 1 \leq p < \infty$, and $B(r, s)x = (sx_{k-1} + rx_k)$ with $r, s \in \mathbb{R} \setminus \{0\}$. Following Kirişçi and Başar \cite{4}, Sönmez \cite{13} have been examined the sequence space $X(B)$ as the set of all sequences whose $B(r, s, t)$-transforms are in the space $X \in \{\ell_\infty, \ell_p, c, c_0\}$, where $B(r, s, t)$ denotes the triple band matrix $B(r, s, t) = \{b_{nk}(r, s, t)\}$ defined by

$$b_{nk}(r, s, t) = \begin{cases} \quad r, & n = k \\ \quad s, & n = k + 1 \\ \quad t, & n = k + 2 \\ 0, & \text{otherwise} \end{cases}$$

for all $k, n \in \mathbb{N}$ and $r, s, t \in \mathbb{R} \setminus \{0\}$. Also in \cite{14, 22}, authors studied certain difference sequence spaces. Furthermore, quite recently, Kara \cite{23} has defined the Fibonacci difference matrix $\hat{F}$ by means of the Fibonacci sequence $(f_n)_{n \in \mathbb{N}}$ and introduced the new sequence spaces $\ell_p(\hat{F})$ and $\ell_\infty(\hat{F})$ which are derived by the matrix domain of $\hat{F}$ in the sequence spaces $\ell_p$ and $\ell_\infty$, respectively; where $1 \leq p < \infty$.

In this paper, we introduce the sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$ by using the Fibonacci difference matrix $\hat{F}$. The rest of this paper is organized, as follows:

In Section 2, we give some notations and basic concepts. In Section 3, we introduce sequence spaces $c_0(\hat{F})$ and $c(\hat{F})$, and establish some inclusion relations. Also, we construct the bases of these spaces. In Section 4, the alpha-, beta-, gamma-duals of the spaces $c_0(\hat{F})$ and $c(\hat{F})$ are determined and the classes $(c_0(\hat{F}), X)$ and $(c(\hat{F}), X)$ of matrix transformations are characterized, where $X$ denotes any of the spaces $\ell_\infty, f, c, f_0, c_0, bs, fs$ and $f_1$.

2. Preliminaries

A $B$-space is a complete normed space. A topological sequence space in which all coordinate functionals $\pi_k, \pi_k(x) = x_k$, are continuous is called a $K$-space. A $BK$-space is defined as a $K$-space which is also a $B$-space, that is, a $BK$-space is a Banach space with continuous coordinates. For example, the space $\ell_p$ is $BK$-space with $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$ and $c_0, c$ and $\ell_\infty$ are $BK$-spaces with $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$, where $1 \leq p < \infty$. The sequence space $\lambda$ is said to be solid (cf. \cite{24}, p. 48) if and only if

$$\tilde{\lambda} := \{(u_k) \in \omega : \exists (x_k) \in \lambda \text{ such that } |u_k| \leq |x_k| \text{ for all } k \in \mathbb{N}\} \subset \lambda.$$

A sequence $(b_n)$ in a normed space $X$ is called a Schauder basis for $X$ if for every $x \in X$ there is a unique sequence $(\alpha_n)$ of scalars such that $x = \sum_n \alpha_nb_n$, i.e., $\lim_m \|x - \sum_{n=0}^m \alpha_nb_n\| = 0$.

The alpha-, beta- and gamma-duals $\lambda^\alpha, \lambda^\beta$ and $\lambda^\gamma$ of a sequence space $\lambda$ are respectively defined by

$$\lambda^\alpha = \{ a = (a_k) \in \omega : ax = (a_kx_k) \in \ell_1 \text{ for all } x = (x_k) \in \lambda \},$$

$$\lambda^\beta = \{ a = (a_k) \in \omega : ax = (a_kx_k) \in cs \text{ for all } x = (x_k) \in \lambda \},$$

$$\lambda^\gamma = \{ a = (a_k) \in \omega : ax = (a_kx_k) \in bs \text{ for all } x = (x_k) \in \lambda \}.$$

The sequence $(f_n)$ of Fibonacci numbers defined by the linear recurrence equalities

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ with } n \geq 2.$$

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converges to the golden ratio which is important in sciences and arts. Also, some basic properties of sequences of Fibonacci numbers \cite{26} are given as
follows:
\[
\lim_{n \to \infty} \frac{f_{n+1}}{f_n} = \frac{1 + \sqrt{5}}{2} = \varphi \quad \text{(Golden Ratio)},
\]
\[
\sum_{k=0}^{n} f_k = f_{n+2} - 1 \quad \text{for each } n \in \mathbb{N},
\]
\[
\sum_{k} \frac{1}{f_k} \text{ converges},
\]
\[
f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \quad \text{for all } n \geq 1 \quad \text{(Cassini Formula)}.
\]

One can easily derive by substituting \( f_{n+1} \) in Cassini’s formula that \( f_{n-1}f_{n+1} - f_n^2 = (-1)^{n+1} \).

Now, let \( A = (a_{nk}) \) be an infinite matrix and list the following conditions:

(2.1) \[
\sup_{n \in \mathbb{N}} \sum_{k} |a_{nk}| < \infty
\]

(2.2) \[
\lim_{n \to \infty} a_{nk} = 0 \quad \text{for each } k \in \mathbb{N}
\]

(2.3) \[
\exists \alpha_k \in \mathbb{C} \ni \lim_{n \to \infty} a_{nk} = \alpha_k \quad \text{for each } k \in \mathbb{N}
\]

(2.4) \[
\lim_{n \to \infty} \sum_{k} a_{nk} = 0
\]

(2.5) \[
\exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} \sum_{k} a_{nk} = \alpha
\]

(2.6) \[
\sup_{K \in \mathcal{F}} \sum_{n} \left| \sum_{k \in K} a_{nk} \right| < \infty,
\]

where \( \mathbb{C} \) and \( \mathcal{F} \) denote the set of all complex numbers and the collection of all finite subsets of \( \mathbb{N} \), respectively.

Now, we may give the following lemma due to Stieglitz and Tietz \cite{27} on the characterization of the matrix transformations between some classical sequence spaces.

**Lemma 2.1.** The following statements hold:

(a) \( A = (a_{nk}) \in (c_0, c_0) \) if and only if (2.1) and (2.2) hold.
(b) \( A = (a_{nk}) \in (c_0, c) \) if and only if (2.1) and (2.3) hold.
(c) \( A = (a_{nk}) \in (c, c_0) \) if and only if (2.1), (2.2) and (2.4) hold.
(d) \( A = (a_{nk}) \in (c, c) \) if and only if (2.1), (2.3) and (2.5) hold.
(e) \( A = (a_{nk}) \in (c_0, \ell_{\infty}) = (c, \ell_{\infty}) \) if and only if condition (2.1) holds.
(f) \( A = (a_{nk}) \in (c_0, \ell_1) = (c, \ell_1) \) if and only if condition (2.6) holds.

3. The Fibonacci Difference Spaces of Null and Convergent Sequences

In this section, we define the spaces \( c_0(\widehat{F}) \) and \( c(\widehat{F}) \) of Fibonacci null and Fibonacci convergent sequences. Also, we present some inclusion theorems and construct the Schauder bases of the spaces \( c_0(\widehat{F}) \) and \( c(\widehat{F}) \).

Recently, Kara \cite{23} has defined the sequence space \( \ell_p(\widehat{F}) \) as follows:
\[
\ell_p(\widehat{F}) = \left\{ x \in \omega : \widehat{F}x \in \ell_p \right\}, \quad (1 \leq p \leq \infty),
\]
where \( \widehat{F} = (\widehat{f}_{nk}) \) is the double band matrix defined by the sequence \( (f_n) \) of Fibonacci numbers as follows
\[
\widehat{f}_{nk} = \begin{cases} \frac{-f_{n+1}}{f_n}, & k = n - 1, \\ \frac{1}{f_{n+1}}, & k = n, \\ 0, & 0 \leq k < n - 1 \text{ or } k > n \end{cases}
\]
for all \( k, n \in \mathbb{N} \). Also, in \cite{28}, Kara et al. have characterized some classes of compact operators on the spaces \( \ell_p(\widehat{F}) \) and \( \ell_{\infty}(\widehat{F}) \), where \( 1 \leq p < \infty \).
One can derive by a straightforward calculation that the inverse \( \hat{F}^{-1} = (g_{nk}) \) of the Fibonacci matrix \( \hat{F} \) is given by

\[
g_{nk} = \begin{cases} \frac{f_{k+1}}{f_{k+1}} f_{n+1}^{n+1}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}
\]

for all \( k, n \in \mathbb{N} \).

Now, we introduce the Fibonacci difference sequence spaces \( c_0(\hat{F}) \) and \( c(\hat{F}) \) as the set of all sequences whose \( \hat{F} \)-transforms are in the spaces \( c_0 \) and \( c \), respectively, i.e.,

\[
c_0(\hat{F}) = \left\{ x = (x_n) \in \omega : \lim_{n \to \infty} \left( \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = 0 \right\},
\]

\[
c(\hat{F}) = \left\{ x = (x_n) \in \omega : \exists l \in \mathbb{C} \ni \lim_{n \to \infty} \left( \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right) = l \right\}.
\]

With the notation of (1.2), the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}) \) can be redefined as follows:

\[
c_0(\hat{F}) = (c_0)_{\hat{F}} \quad \text{and} \quad c(\hat{F}) = c_{\hat{F}}.
\]

Define the sequence \( y = (y_n) \) by the \( \hat{F} \)-transform of a sequence \( x = (x_n) \), i.e.,

\[
y_n = \hat{F}_n(x) = \begin{cases} x_0, & \text{if } n = 0, \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & \text{if } n \geq 1 \end{cases}
\]

for all \( n \in \mathbb{N} \). Throughout the text, we suppose that the sequences \( x = (x_k) \) and \( y = (y_k) \) are connected with the relation (3.2).

**Theorem 3.1.** The sets \( c_0(\hat{F}) \) and \( c(\hat{F}) \) are the linear spaces with the co-ordinatewise addition and scalar multiplication which are the BK-spaces with the norm \( \|x\|_{c_0(\hat{F})} = \|x\|_{c(\hat{F})} = \|\hat{F}x\|_\infty \).

**Proof.** This is a routine verification and so we omit the detail. \( \square \)

**Remark 3.2.** One can easily check that the absolute property does not hold on the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}) \), that is \( \|x\|_{c_0(\hat{F})} \neq \|x\|_{c(\hat{F})} \) and \( \|x\|_{c(\hat{F})} \neq \|\|x\|_{c(\hat{F})} \) for at least one sequence in the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}) \), and this says us that \( c_0(\hat{F}) \) and \( c(\hat{F}) \) are the sequence spaces of non-absolute type, where \( |x| = (|x_k|) \).

**Theorem 3.3.** The Fibonacci difference sequence spaces \( c_0(\hat{F}) \) and \( c(\hat{F}) \) of non-absolute type are linearly norm isomorphic to the spaces \( c_0 \) and \( c \), respectively, i.e., \( c_0(\hat{F}) \cong c_0 \) and \( c(\hat{F}) \cong c \).

**Proof.** Since the proof is similar for the space \( c(\hat{F}) \), we consider only the space \( c_0(\hat{F}) \). With the notation of (3.2), we consider the transformation \( T \) defined from \( c_0(\hat{F}) \) to \( c_0 \) by \( x \mapsto y = \hat{F}x \). It is trivial that the map \( T \) is linear and injective. Furthermore, let \( y = (y_k) \in c_0 \) and define the sequence \( x = (x_k) \) by

\[
x_k = \sum_{j=0}^{k} \frac{f_{k+1}}{f_{j+1}} y_j \quad \text{for all} \quad k \in \mathbb{N}.
\]

Then we have

\[
\lim_{k \to \infty} \hat{F}_k(x) = \lim_{k \to \infty} \left( \frac{f_k}{f_{k+1}} \sum_{j=0}^{k} \frac{f_{k+1}^{j+1}}{f_{j+1}} y_j - \frac{f_{k+1}}{f_k} \sum_{j=0}^{k-1} \frac{f_k^{j+1}}{f_{j+1}} y_j \right) = \lim_{k \to \infty} y_k = 0
\]

which says us that \( x \in c_0(\hat{F}) \). Additionally, we have for every \( x \in c_0(\hat{F}) \) that

\[
\|x\|_{c_0(\hat{F})} = \sup_{k \in \mathbb{N}} \left| \frac{f_k}{f_{k+1}} x_k - \frac{f_{k+1}}{f_k} x_{k-1} \right| = \sup_{k \in \mathbb{N}} |y_k| = \|y\|_\infty < \infty.
\]

Consequently we see from here that \( \hat{F} \) is surjective and norm preserving. Hence, \( \hat{F} \) is a linear bijection which shows that the spaces \( c_0(\hat{F}) \) and \( c_0 \) are linearly isomorphic. This concludes the proof. \( \square \)
Consider the sequences $\mathbf{u}$ and $\mathbf{v}$ defined by $u_k = f_{k+1}^2$ and $v_k = (-1)^{k+1}$ for all $k \in \mathbb{N}$. Then, it is clear that $u \in c_0(\hat{F})$ and $v \in \ell_\infty$. Nevertheless, $uv = \{(-1)^{k+1}f_{k+1}^2\}$ is not included in $c_0(\hat{F})$, since

$$
\hat{F}(uv) = \left\{ \frac{f_k}{f_{k+1}}(-1)^{k+1}f_{k+1}^2 - \frac{f_{k+1}}{f_k}(-1)^{k+1}f_k^2 \right\} = 2(-1)^{k+1}f_kf_{k+1}
$$

for all $k \in \mathbb{N}$. This shows that the multiplication $\ell_\infty c_0(\hat{F})$ of the spaces $\ell_\infty$ and $c_0(\hat{F})$ is not a subset of $c_0(\hat{F})$. Hence, the space $c_0(\hat{F})$ is not solid. 

It is clear here that if the spaces $c_0(\hat{F})$ is replaced by the space $c(\hat{F})$, then we obtain the fact that $c(\hat{F})$ is not solid. This completes the proof. 

It is known from Theorem 2.3 of Jarrah and Malkowsky [25] that the domain $\lambda_T$ of an infinite matrix $T = (t_{nk})$ in a normed sequence space $\lambda$ has a basis if and only if $\lambda$ has a basis, if $T$ is a triangle. As a direct consequence of this fact, we have:

**Corollary 3.8.** Define the sequences $c^{(-1)} = \{c_k^{(-1)}\}_{k \in \mathbb{N}}$ and $c^{(n)} = \{c_k^{(n)}\}_{k \in \mathbb{N}}$ for every fixed $n \in \mathbb{N}$ by

$$
c_k^{(-1)} = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_jf_{j+1}} \quad \text{and} \quad c_k^{(n)} = \begin{cases} 0 & \text{if } 0 \leq k \leq n-1 \\ \frac{f_{k+1}^2}{f_kf_{n+1}} & \text{if } k \geq n. \end{cases}
$$

Then, the following statements hold:
(a) The sequence \( \{ c^{(n)} \}_{n=0}^{\infty} \) is a basis for the space \( c_0(\hat{F}) \) and every sequence \( x \in c_0(\hat{F}) \) has a unique representation \( x = \sum_n \hat{F}_n(x)c^{(n)}. \)

(b) The sequence \( \{ c^{(n)} \}_{n=-1}^{\infty} \) is a basis for the space \( c(\hat{F}) \) and every sequence \( z = (z_n) \in c(\hat{F}) \) has a unique representation \( z = l^{-1} + \sum_n \left[ \hat{F}_n(z) - l \right] c^{(n)}, \) where \( l = \lim_{n \to \infty} \hat{F}_n(z). \)

4. The alpha-, beta- and gamma-duals of the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}), \) and some matrix transformations

In this section, we determine the alpha-, beta- and gamma-duals of the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}), \) and characterize the classes of infinite matrices from the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}) \) to the spaces \( c_0, c, \ell_\infty, f, f_0, bs, fs, cs \) and \( \ell_1, \) and from the space \( f \) to the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}). \)

Now, we quote two lemmas required in proving the theorems concerning the alpha-, beta- and gamma-duals of the spaces \( c_0(\hat{F}) \) and \( c(\hat{F}). \)

**Lemma 4.1.** Let \( \lambda \) be any of the spaces \( c_0 \) or \( c \) and \( a = (a_n) \in \omega, \) and the matrix \( B = (b_{nk}) \) be defined by \( B_n = a_n \hat{F}_n^{-1}, \) that is,

\[
b_{nk} = \begin{cases} a_n g_{nk}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}
\]

for all \( k, n \in \mathbb{N}. \) Then, \( a \in \lambda^\beta \) if and only if \( B \in (\lambda, \ell_1). \)

**Proof.** Let \( y \) be the \( \hat{F} \)-transform of a sequence \( x = (x_n) \in \omega. \) Then, we have by (3.3) that

\[
a_n x_n = a_n \hat{F}_n^{-1}(y) = B_n(y) \quad \text{for all} \quad n \in \mathbb{N}.
\]

Thus, we observe by (4.1) that \( ax = (a_n x_n) \in \ell_1 \) with \( x \in \lambda^\beta \) if and only if \( By \in \ell_1 \) with \( y \in \lambda. \) This means that \( a \in \lambda^\beta \) if and only if \( B \in (\lambda, \ell_1). \) This completes the proof.

**Lemma 4.2.** [33] Theorem 3.1] Let \( C = (c_{nk}) \) be defined via a sequence \( a = (a_k) \in \omega \) and the inverse matrix \( V = (v_{nk}) \) of the triangle matrix \( U = (u_{nk}) \) by

\[
c_{nk} = \begin{cases} \sum_{j=k}^{n} a_j v_{jk}, & 0 \leq k \leq n, \\ 0, & k > n \end{cases}
\]

for all \( k, n \in \mathbb{N}. \) Then, for any sequence space \( \lambda, \)

\[
\lambda^\gamma_U = \{ a = (a_k) \in \omega : C \in (\lambda, \ell_\infty) \},
\]

\[
\lambda^\beta_U = \{ a = (a_k) \in \omega : C \in (\lambda, c) \}.
\]

Combining Lemmas 2.1, 4.1 and 4.2, we have;

**Corollary 4.3.** Consider the sets \( d_1, d_2, d_3 \) and \( d_4 \) defined as follows:

\[
d_1 = \left\{ a = (a_k) \in \omega : \sup_{K \in F} \sum_n \left| \sum_{k \in K} \frac{f_{n+1}}{f_k f_{k+1}} a_n \right| < \infty \right\},
\]

\[
d_2 = \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{j=k}^{n} \frac{f_{j+1}}{f_k f_{k+1}} a_j \right| < \infty \right\},
\]

\[
d_3 = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{j=k}^{n} \frac{f_{j+1}}{f_k f_{k+1}} a_j \text{ exists for each } k \in \mathbb{N} \right\},
\]

\[
d_4 = \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} \sum_{j=k}^{n} \frac{f_{j+1}}{k f_k f_{k+1}} a_j \text{ exists} \right\}.
\]

Then, the following statements hold:

(a) \( \{ c_0(\hat{F}) \}^\alpha = \{ c(\hat{F}) \}^\alpha = d_1. \)

(b) \( \{ c_0(\hat{F}) \}^\beta = d_2 \cap d_3 \) and \( \{ c(\hat{F}) \}^\beta = d_2 \cap d_3 \cap d_4. \)
(c) \{c_0(\hat{F})\}^\gamma = \{c(\hat{F})\}^\gamma = d_2.

**Theorem 4.4.** Let \( \lambda = c_0 \) or \( c \) and \( \mu \) be an arbitrary subset of \( \omega \). Then, we have \( A = (a_{nk}) \in (\lambda, c, \mu) \) if and only if

(4.2) \[ D^{(m)} = \left( d^{(m)}_{nk} \right) \in (\lambda, c) \text{ for all } n \in \mathbb{N}, \]
(4.3) \[ D = (d_{nk}) \in (\lambda, \mu), \]

where \( d^{(m)}_{nk} = \left\{ \frac{f_{k+1}^2}{k f_{k+1}} a_{nj}, \quad 0 \leq k \leq m, \quad k > m \right\} \), and \( d_{nk} = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_{j} f_{j+1}} a_{nj} \) for all \( k, m, n \in \mathbb{N} \).

**Proof.** To prove the theorem, we follow the similar way due to Kirişçi and Başar [4]. Let \( A = (a_{nk}) \in (\lambda, c) \) and \( x = (x_k) \in \lambda, c \). Recalling the relation (4.2), we have

(4.4) \[ \sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} a_{nk} \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_{j} f_{j+1}} y_j = \sum_{k=0}^{m} b_{nk} y_k = D^{(m)}(n)(y) \]

for all \( m, n \in \mathbb{N} \). Since \( Ax \) exists, \( D^{(m)} \) must belong to the class \( (\lambda, c) \). Letting \( m \to \infty \) in the equality (4.4), we obtain \( Ax = Dy \) which gives the result \( D \in (\lambda, \mu) \).

Conversely, suppose the conditions (4.2), (4.3) hold and take any \( x \in \lambda, c \). Then, we have \( (d_{nk})_{k \in \mathbb{N}} \in \lambda, c \) which gives together with (4.2) that \( A_n = (a_{nk})_{k \in \mathbb{N}} \in \lambda, c \) for all \( n \in \mathbb{N} \). Thus, \( Ax \) exists. Therefore, we derive by the equality (4.4) as \( m \to \infty \) that \( Ax = Dy \), and this shows that \( A \in (\lambda, c) \).

By changing the roles of the spaces \( \lambda, c \) and \( \lambda, \mu \) in Theorem 4.4, we have:

**Theorem 4.5.** Suppose that the elements of the infinite matrices \( A = (a_{nk}) \) and \( B = (b_{nk}) \) are connected with the relation

(4.5) \[ b_{nk} := -\frac{f_{n+1}}{f_n} a_{n-1,k} + \frac{f_n}{f_{n+1}} a_{nk} \]

for all \( k, n \in \mathbb{N} \) and \( \mu \) be any given sequence space. Then, \( A \in (\mu, \lambda, c) \) if and only if \( B \in (\mu, \lambda, c) \).

**Proof.** Let \( z = (z_k) \in \mu \). Then, by taking into account the relation (4.5) one can easily derive the following equality

\[ \sum_{k=0}^{m} b_{nk} z_k = \sum_{k=0}^{m} \left( -\frac{f_{n+1}}{f_n} a_{n-1,k} + \frac{f_n}{f_{n+1}} a_{nk} \right) z_k \]

which yields as \( m \to \infty \) that \( (Bz)_n = [\hat{F}(Az)]_n \). Therefore, we conclude that \( Az \in \lambda, c \) whenever \( z \in \mu \) if and only if \( Bz \in \lambda \) whenever \( z \in \mu \).

This step completes the proof. \( \square \)

Prior to giving some natural consequences of Theorems 4.4 and 4.5, we give the concept of *almost convergence*. The shift operator \( \hat{P} \) is defined on \( \omega \) by \( \hat{P}_n(x) = x_{n+1} \) for all \( n \in \mathbb{N} \). A Banach limit \( L \) is defined on \( \ell_\infty \), as a non-negative linear functional, such that \( L(Px) = L(x) \) and \( L(e) = 1 \). A sequence \( x = (x_k) \in \ell_\infty \) is said to be almost convergent to the generalized limit \( l \) if all Banach limits of \( x \) coincide and are equal to \( l \) \[32\], and is denoted by \( f - \lim x_k = l \). Let us define \( t_{mn}(x) \) via a sequence \( x = (x_k) \) by

\[ t_{mn}(x) = \frac{1}{m+1} \sum_{k=0}^{m} x_{n+k} \quad \text{for all } m, n \in \mathbb{N}. \]

Lorentz \[32\] proved that \( f - \lim x_k = l \) if and only if \( \lim_{m \to \infty} t_{mn}(x) = l \), uniformly in \( n \). It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. By \( f_0, f \) and \( f_s \), we denote the spaces of almost null and almost convergent sequences and series, respectively. Now, we can give the following two lemmas characterizing the strongly and almost conservative matrices:
Lemma 4.6. \( A = (a_{nk}) \in (f, c) \) if and only if (2.1), (2.3) and (2.5) hold, and
\[
\lim_{n \to \infty} \sum_k \Delta(a_{nk} - \alpha_k) = 0
\]
also holds, where \( \Delta(a_{nk} - \alpha_k) = a_{nk} - \alpha_k - (a_{nk+1} - \alpha_{k+1}) \) for all \( k, n \in \mathbb{N} \).

Lemma 4.7. \( A = (a_{nk}) \in (c, f) \) if and only if (2.1) holds, and
\[
\exists \alpha_k \in \mathbb{C} \ni f - \lim_{n \to \infty} a_{nk} = \alpha_k \text{ for each fixed } k \in \mathbb{N};
\]
\[
\exists \alpha \in \mathbb{C} \ni f - \lim_{n \to \infty} \sum_k a_{nk} = \alpha.
\]

Now, we list the following conditions;
\[
\sup_{m \in \mathbb{N}} \sum_{k=0}^{m} |d_{mk}^{(n)}| < \infty
\]
\[
\exists d_{nk} \in \mathbb{C} \ni \lim_{m \to \infty} d_{nk}^{(n)} = d_{nk} \text{ for each } n, k \in \mathbb{N}
\]
\[
\sup_{n \in \mathbb{N}} \sum_k |d_{nk}| < \infty
\]
\[
\exists \alpha_k \in \mathbb{C} \ni \lim_{n \to \infty} d_{nk} = \alpha_k \text{ for each } k \in \mathbb{N}
\]
\[
\sup_{N, K \in \mathcal{F}} \left| \sum_{n \in \mathbb{N}} \sum_{k \in K} d_{nk} \right| < \infty
\]
\[
\exists \beta_n \in \mathbb{C} \ni \lim_{m \to \infty} \sum_{k=0}^{m} d_{nk}^{(n)} = \beta_n \text{ for each } n \in \mathbb{N}
\]
\[
\exists \alpha \in \mathbb{C} \ni \lim_{n \to \infty} \sum_k d_{nk} = \alpha
\]

It is trivial that Theorem 4.4 and Theorem 4.5 have several consequences. Indeed, combining Theorems 4.4, 4.5 and Lemmas 2.1, 4.6 and 4.7, we derive the following results:

Corollary 4.8. Let \( A = (a_{nk}) \) be an infinite matrix and \( a(n, k) = \sum_{j=0}^{n} a_{jn} \) for all \( k, n \in \mathbb{N} \). Then, the following statements hold:

(a) \( A = (a_{nk}) \in (c_0(\hat{F}), c_0) \) if and only if (4.9), (4.10), (4.11) hold and (4.12) also holds with \( \alpha_k = 0 \) for all \( k \in \mathbb{N} \).

(b) \( A = (a_{nk}) \in (c_0(\hat{F}), c_0) \) if and only if (4.9), (4.10), (4.11) hold and (4.12) also holds with \( \alpha_k = 0 \) for all \( k \in \mathbb{N} \) with \( a(n, k) \) instead of \( a_{nk} \).

(c) \( A = (a_{nk}) \in (c_0(\hat{F}), c) \) if and only if (4.9), (4.10), (4.11) and (4.12) hold.

(d) \( A = (a_{nk}) \in (c_0(\hat{F}), cs) \) if and only if (4.9), (4.10), (4.11) and (4.12) hold with \( a(n, k) \) instead of \( a_{nk} \).

(e) \( A = (a_{nk}) \in (c_0(\hat{F}), \ell_\infty) \) if and only if conditions (4.9), (4.10) and (4.11) hold.

(f) \( A = (a_{nk}) \in (c_0(\hat{F}), bs) \) if and only if conditions (4.9), (4.10) and (4.11) hold with \( a(n, k) \) instead of \( a_{nk} \).

(g) \( A = (a_{nk}) \in (c_0(\hat{F}), \ell_1) \) if and only if (4.9), (4.10) and (4.13) hold.

(h) \( A = (a_{nk}) \in (c_0(\hat{F}), bv_1) \) if and only if (4.9), (4.10) and (4.13) hold with \( a_{nk} - a_{n-1,k} \) instead of \( a_{nk} \).

Corollary 4.9. Let \( A = (a_{nk}) \) be an infinite matrix. Then, the following statements hold:

(a) \( A = (a_{nk}) \in (c(\hat{F}), \ell_\infty) \) if and only if (4.9), (4.10), (4.11) and (4.13) hold.

(b) \( A = (a_{nk}) \in (c(\hat{F}), bs) \) if and only if (4.9), (4.10), (4.11) and (4.13) hold with \( a(n, k) \) instead of \( a_{nk} \).

(c) \( A = (a_{nk}) \in (c(\hat{F}), c) \) if and only if (4.9), (4.10), (4.11), (4.12), (4.14) and (4.15) hold.

(d) \( A = (a_{nk}) \in (c(\hat{F}), cs) \) if and only if (4.9), (4.10), (4.11), (4.12), (4.14) and (4.15) hold with \( a(n, k) \) instead of \( a_{nk} \).
(e) $A = (a_{nk}) \in (c(F), c_0)$ if and only if (4.9), (4.10), (4.11), (4.12) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (4.14) and (4.15) also hold with $\alpha = 0$.

(f) $A = (a_{nk}) \in (c(F), c \alpha_0)$ if and only if (4.9), (4.10), (4.11), (4.12) hold with $\alpha_k = 0$ for all $k \in \mathbb{N}$, (4.14) and (4.15) also hold with $\alpha = 0$ with $a(n, k)$ instead of $a_{nk}$.

(g) $A = (a_{nk}) \in (c(F), c_1)$ if and only if (4.9), (4.10), (4.13) and (4.14) hold.

(h) $A = (a_{nk}) \in (c(F), b_{n1})$ if and only if (4.9), (4.10), (4.13) and (4.14) hold with $a_{nk} - a_{n-1,k}$ instead of $a_{nk}$.

**Corollary 4.10.** $A = (a_{nk}) \in (c(F), f)$ if and only if (4.9), (4.10), (4.14) and (4.15) hold, and (4.11), (4.12) also hold with $d_{nk}$ instead of $a_{nk}$.

**Corollary 4.11.** $A = (a_{nk}) \in (c(F), f_0)$ if and only if (4.9), (4.10), (4.14) and (4.15) hold, and (4.11), (4.12) also hold with $d_{nk}$ instead of $a_{nk}$ and $\alpha_k = 0$ for all $k \in \mathbb{N}$.

**Corollary 4.12.** $A = (a_{nk}) \in (c(F), f_s)$ if and only if (4.9), (4.10), (4.11), (4.12), (4.14) and (4.15) hold with $a(n, k)$ instead of $a_{nk}$ and (4.11), (4.12) hold with $d(n, k)$ instead of $d_{nk}$.

**Corollary 4.13.** $A = (a_{nk}) \in (f, c(F))$ if and only if (2.7), (2.8), (2.9) and (4.10) hold with $b_{nk}$ instead of $a_{nk}$, where $b_{nk}$ is defined by (4.3).

**Corollary 4.14.** $A = (a_{nk}) \in (f, c_0(F))$ if and only if (2.7) and (2.8) hold, (2.9) and (4.10) also hold with $b_{nk}$ instead of $a_{nk}$ and $\alpha_k = 0$ for all $k \in \mathbb{N}$, where $b_{nk}$ is defined by (4.3).

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