PH-Pfaffian order in a translationally and rotationally invariant system

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The PH-Pfaffian topological order has been proposed as a candidate order for the $\nu = 5/2$ quantum Hall effect. The PH-Pfaffian liquid is known to be the ground state in several coupled wire and coupled stripe constructions. No translationally and rotationally invariant Hamiltonians with the PH-Pfaffian ground state have been identified so far. We propose a uniform system with a local Hamiltonian whose ground state possesses the PH-Pfaffian order.

I. INTRODUCTION

The topological order on half-integer quantum Hall plateaus has been a subject of much debate. There has long been tension between experiment and numerics. It increasingly appears that multiple topological orders are present in experimentally relevant systems. Indeed, numerical evidence exists for different topological orders on half-integer plateaus in GaAs and graphene. Some experiments even hint at different topological orders at different magnetic fields on the same $5/2$ plateau in GaAs. Such behavior differs profoundly from the intuition that builds on the properties of the simplest and best-understood quantum Hall state at $\nu = 1/3$, where the same Laughlin topological order is believed to be present in a broad range of materials and parameters. The difficulties with half-integer filling factors reflect a stronger role for composite-fermion (CF) interactions on half-integer plateaus than at most odd-denominator filling factors. Indeed, a great majority of odd-denominator states can be understood as integer quantum Hall states of CFs. Such integer states are present even for non-interacting CFs, and their interaction does not affect qualitative features, such as possible topological orders. In contrast to this picture, non-interacting CFs would not form an incompressible liquid at a half-integer filling. This agrees with the absence of the $1/2$ and $3/2$ plateaus in monolayer GaAs. At the same time, experimental evidence exists for CFs on the quantized $5/2$ plateau. This suggests that the $5/2$ plateau forms due to CF interactions. The plateau can be explained by Cooper pairing of CFs. The details of the topological order depend on the pairing channel: Different channels result in 8 possible Abelian and 8 possible non-Abelian states.

Which one or ones are present in experimentally relevant systems? Preponderance of numerical evidence points towards Pfaffian and anti-Pfaffian liquids in translationally invariant systems. Preponderance of experimental evidence suggests the PH-Pfaffian order on the $5/2$ plateau in GaAs. A possible explanation of such discrepancy comes from disorder, inevitable in any sample, but ignored in all numerical studies until a very recent paper. Weak disorder is not believed to affect topological order at $\nu = 1/3$. Strong disorder destroys the $1/3$ plateau. This behavior is the same as in the integer quantum Hall effect. At the same time, disorder can change the pairing channel in a superconductor. This suggests that disorder may change the qualitative physics of the CF superconductor at $\nu = 5/2$. Recent theoretical work does predict a complicated phase diagram in the presence of disorder with several topologically ordered phases and a gapless thermal metal. Note that a random potential is not necessary for the stabilization of the PH-Pfaffian liquid. Coupled wire constructions and a coupled stripe construction produce Hamiltonians with the PH-Pfaffian order in the ground state without any randomness. The common feature of the disorder-based approach with those constructions consists in the absence of translational and rotational symmetry.

A possible lesson might be that the PH-Pfaffian order were impossible in uniform systems. Yet, it was argued that it might be stabilized by sufficiently strong Landau level mixing (LLM) even in uniform systems. If so, a translationally invariant model should exist with the PH-Pfaffian ground state. The goal of this paper is to build such a model.

The model system is multi-component. We argue that for appropriate microscopic interactions, the components may originate from different Landau levels. This makes our model different from constructions in which wave functions of various topological orders are localized in a single Landau level, and thus LLM is ignored. This difference is consistent with recent numerical results, which suggest that the PH-Pfaffian state loses its gap after projection into the lowest Landau level. It is also consistent with the symmetry-from-no-symmetry principle, which postulates that a particle-hole symmetric topological order is only possible, if the particle-hole symmetry is broken by LLM, disorder, or another mechanism or combination of mechanisms.

In what follows, we start with a review of the PH-Pfaffian topological order. We then observe that the edge structure of a PH-Pfaffian liquid can be obtained from a two-component system. One component is made of charged fermions and the other is made of neutral bosons. In the fourth section we demonstrate that the two-component system possesses the PH-Pfaffian order in the bulk. In the final section we propose a scenario how such two-component model might be realized in a purely electronic system.

II. PH-PFAFFIAN ORDER

The anyons are labeled by their topological charge $t = 1, \sigma$, or $\psi$ and the electric charge $ne/4$, where $n$ is odd in the $\sigma$-sector and even otherwise. We will use the notation $(t, n)$. The fusion rules are
where 1 stays for vacuum and ψ is a Majorana fermion. The statistical phase, accumulated by an anyon of type \((t_1, n_1)\) while making a full counterclockwise circle around an anyon of type \((t_2, n_2)\) is

\[
\phi = \phi_{nA}(t_1, t_2, f) + \frac{\pi n_1 n_2}{4}.
\]

Finally,

\[
\phi_{nA}(\sigma, \psi, \sigma) = \phi_{nA}(\psi, \sigma, \sigma) = \pi.
\]  

The bulk statistics determines the edge Lagrangian density:\(20\):

\[
L = \frac{2}{4\pi} \partial_x \phi_c (\partial_t - v_c \partial_x) \phi_c + i\psi (\partial_t + u \partial_x) \psi,
\]

where \(\psi\) is a Majorana fermion and the charge mode \(\phi_c\) sets the charge density \(e \partial_x \phi_c / 2\pi\) on the edge. An edge excitation from the sector \((t, n)\) is created by the operator \(\exp(\imath v \phi_c / 2)\), where \(t = 1, \sigma, \psi\) acts in the neutral Majorana sector with \(\sigma\) being the twist operator. The electron operator is \(\psi \exp(\imath v \phi_c)\).

Both the thermal and electrical conductances are one half of a quantum:\(17,20\).

The edge theory differs from (5) by the opposite propagation direction of \(\phi_c\) and an additional charge integer mode \(\phi_1\) with the charge density \(e \partial_x \phi_1 / 2\pi\). The edge Lagrangian density

\[
L_{\alpha \mathrm{ Pf}} = -\frac{2}{4\pi} \partial_x \phi_c (\partial_t + v_c \partial_x) \phi_c + i\psi (\partial_t + u \partial_x) \psi + \frac{1}{4\pi} \partial_x \phi_1 (\partial_t - v_1 \partial_x) \phi_1 + w \partial_x \phi_1 \partial_x \phi_c.
\]

Edge excitations are created by the same operators as in the PH-Pfaffian state. There are two electron operators: \(\psi \exp(-2\imath \phi_c)\) and \(\exp(\imath \phi_c)\). The operator \(\exp(\imath \phi_n) = \exp(\imath(\phi_1 + 2 \phi_c))\) creates a neutral fermionic excitation in the Majorana sector \(\psi\). The electrical conductance is half a quantum, as in the PH-Pfaffian state. The thermal conductance is \(-1/2\) of a quantum\(17,18\).

It will be convenient to switch from the variables \(\phi_c\) and \(\phi_1\) to the neutral mode \(\phi_n\) and the overall charge mode \(\phi_\rho = \phi_1 + \phi_c\). The Lagrangian density becomes

\[
L_{\alpha \mathrm{ Pf}} = \frac{2}{4\pi} \partial_x \phi_\rho (\partial_t - v_\rho \partial_x) \phi_\rho + i\psi (\partial_t + u \partial_x) \psi
- \frac{1}{4\pi} \partial_x \phi_n (\partial_t - v_1 \partial_x) \phi_n + w \partial_x \phi_\rho \partial_x \phi_n.
\]

The \(\nu = 1/4\) Laughlin state is Abelian\(23\). The phase accumulated by a fundamental anyon \(\exp(\imath b)\) on a full counterclockwise circle around an identical anyon is \(\pi/2\). The fusion of \(n\) fundamental anyons yields a composite anyon \(\exp(\imath nb)\). Such anyon accumulates the phase \(mn\pi/2\) on a full circle about an anyon of type \(\exp(\imath mb)\). As a consequence, \(\exp(\imath 2ib)\) are fermions. \(\exp(\imath 4ib)\) is topologically trivial. The edge theory of the Laughlin state assumes the form

\[
L_B = \frac{4}{4\pi} \partial_x b (\partial_t - v_\rho \partial_x) b.
\]

The electrical conductance of the neutral bosons is 0. The thermal conductance equals one quantum\(15\).

We now observe that the sums of the electric and thermal conductances of the bosonic liquid and the anti-Pfaffian liquid equal the electric and thermal conductances of the PH-Pfaffian liquid. This makes us expect that the PH-Pfaffian order should be present in a two-component system made of the anti-Pfaffian and bosonic Laughlin liquids. We start with demonstrating that the edge structure of the PH-Pfaffian liquid can be obtained from such two-component model.

We consider a two-component model with the following Lagrangian density on the edge:

\[
L = L_{\alpha \mathrm{ Pf}} + L_B + u \partial_x \phi_n \partial_x b + U \cos(2\phi_n - 4b).
\]

The cosine term is allowed in the action since it is topologically trivial and conserves the electric charge. For simplicity, we assume\(15\) that \(\bar{w} = 0\) in \(L_{\alpha \mathrm{ Pf}}\). The two counterpropagating modes \(b\) and \(\phi_n\) are gapped out if the cosine term is relevant in the renormalization group sense. After introducing a new field \(\phi_b = -2b\), the contribution to the
The Lagrangian density that depends on $\phi_n$ and $\phi_b$ becomes

$$L_{n,b} = \frac{1}{4\pi} [\partial_x \phi_b (\partial_t - v_b \partial_x) \phi_b - \partial_x \phi_n (\partial_t + v_n \partial_x) \phi_n] - \frac{u}{2} \partial_x \phi_n \partial_x \phi_b + U \cos [2 (\phi_n + \phi_b)].$$

The stability of the edge requires $|\pi u| \leq \sqrt{v_b v_n}$. The Lagrangian density $L_{n,b}$ can be diagonalized by the transformation:

$$\phi_b = \cosh \theta \tilde{\phi}_b + \sinh \theta \tilde{\phi}_n, \quad \phi_n = \sinh \theta \tilde{\phi}_b + \cosh \theta \tilde{\phi}_n,$$

$$\tanh 2\theta = -\frac{2\pi u}{v_b + v_n}. \quad \text{(14)}$$

In the new basis, the cosine term becomes

$$L_{\text{tun}} = U \cos \left[ 2 (\cosh \theta + \sinh \theta) \left( \tilde{\phi}_n + \tilde{\phi}_b \right) \right]. \quad \text{(15)}$$

Its scaling dimension can be deduced easily:

$$\Delta = 4 (\cosh 2\theta + \sinh 2\theta) = 4 \sqrt{v_b + v_n - 2\pi u} \quad \text{(16)}$$

When $\Delta < 2$, $L_{\text{tun}}$ is relevant and gaps out $\phi_b$ and $\phi_n$. This happens for

$$\frac{3(v_b + v_n)}{10\pi} < u < \frac{\sqrt{v_b v_n}}{\pi}. \quad \text{(17)}$$

The remaining two gapless modes $\phi_b$ and $\psi$ are described by the action identical to the PH-Pfaffian action $\text{(5)}$.

### IV. MODEL: VIEW FROM THE BULK

The action $\text{(10)}$ is the key to the bulk model. Indeed, $\cos(2\phi_n - 4b)$ can be represented in the form $\hat{B}\hat{B} + \hat{B}^\dagger\hat{B}^\dagger$, where $\hat{B}$ creates an excitation $B = (\psi, 0) \times \exp(2ib)$. Such an excitation is a product of two fermions and hence a boson. The edge action thus suggests the condensation of bosons $B$. The condensation results in the confinement of anyon types. As we will see, the statistics of the remaining deconfined excitations is PH-Pfaffian.

Deconfined excitations braid trivially with $B$. Hence, the only non-trivial deconfined excitation of the Bose-liquid is $\exp(2ib)$. The deconfined excitations of the anti-Pfaffian liquid are $(\psi, 2n)$ and $(1, 2n)$. The attachment of any number of bosons $B$ does not change the superselection sector of an excitation. Thus, $\exp(2ib)$ and $\psi$ can be identified.

What about deconfined anyons that combine topological excitations of the Bose and anti-Pfaffian subsystems? First, we can combine any number of deconfined excitations in the Bose and anti-Pfaffian sectors. This yields anyons of the types $(t, 2n) \times \exp(2nib)$, where $t = 1, \psi$. By attaching $(n - m)$ $B$-particles, any such anyon can be reduced to the standard type $(t', 2n) \times \exp(2nib)$, where $t' = 1, \psi$ is not necessarily the same as $t$. In addition to products of deconfined excitations of the two subsystems, deconfined excitations exist in the $\sigma$ sector: $(\sigma, 2n+1) \times \exp([2n+1]ib)$. Without loss of generality we can set $n = m$ since attaching $(n - m)$ bosons $B$ changes $(\sigma, 2n+1) \times \exp([2n+1]ib)$ into $(\sigma, 2n+1) \times \exp([2n+1]ib)$. Thus, all superselection sectors can be labeled as $(t,n) \times \exp(inb)$.

We will now observe that all deconfined anyons can be identified with excitations of a PH-Pfaffian liquid. We identify $(t,n) \times \exp(inb)$ with the $(t,n)$ anyon of the PH-Pfaffian order. All fusion rules are satisfied after such identification. The non-Abelian part of the braiding phase is also correct. The Abelian part of the mutual braiding phase of the anyons $(t_1, n_1) \times \exp(in_1b)$ and $(t_2, n_2) \times \exp(in_2b)$ is now the sum of the anti-Pfaffian contribution $-n_1n_2\pi/4$ and the Laughlin contribution $n_1n_2\pi/2$. This gives the correct PH-Pfaffian value.

### V. CONCLUSIONS

The above model demonstrates that the PH-Pfaffian topological order can be obtained in a uniform system. All other known models with that order break the translational and rotational symmetry either because of impurities or because the models consist of coupled wires or stripes. Since our model combines fermions in the anti-Pfaffian state with neutral bosons, its most natural realization would come from cold atoms. The model may seem disconnected from the physics of the $5/2$ plateau in semiconductors, where only fermions are present. We propose a scenario that makes a connection with a purely fermionic system. We assume that electrons are present in four spin-resolved Landau levels. Electrons in one level exhibit the anti-Pfaffian order and form one of the two subsystems we need. The electrical conductance of the anti-Pfaffian subsystem is one half of a quantum. The other three partially and fully filled Landau levels contribute two quanta to the electrical conductance, as necessary for the total conductance of $5/2$. One Landau level is fully filled. The sum of the filling factors of the other two is $1$. Thus, the number of the holes in one of those Landau levels equals the number of the electrons in the other. We assume that all holes from one level combine with the electrons from the other level to form neutral bosons. The bosons form the Laughlin $\nu = 1/4$ state, provided that their two-body interaction favors the relative angular momentum $+4$, where the plus emphasizes that only one sign of the angular momentum along the $z$-axis is favored. An appropriate choice of the interaction between the bosons and the fermions in the anti-Pfaffian state yields the desired model system.

The boson interaction breaks the time-reversal symmetry. This property is not shared by the Coulomb interaction in realistic samples. The time-reversal symmetry is broken instead by the external magnetic field to which neutral bosons do not minimally couple. Even if the interaction with the magnetic field is the only contribution to the microscopic Hamiltonian that breaks the time-reversal symmetry, it is possible that additional symmetry-breaking interactions are generated in the effective low-energy Hamiltonian. Of course,
it may well be that this does not happen for realistic Coulomb interactions. The point of our model is to show that the PH-Pfaffian order is possible without breaking the translational symmetry. More research is needed to understand if the PH-Pfaffian order could be stabilized in realistic semiconductor heterostructures in the absence of random impurities.

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