ON THE LOW MACH NUMBER LIMIT FOR QUANTUM NAVIER-STOKES EQUATIONS

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Abstract. In this paper we investigate the low Mach number limit for the quantum Navier-Stokes system considered in the three-dimensional space. For general ill-prepared initial data of finite energy, we prove strong convergence of finite energy weak solutions towards weak solutions of incompressible Navier Stokes equations. Our approach relies on a careful dispersive analysis for the acoustic part, governed by the Bogoliubov dispersion relation. The a priori bounds given by the energy and the BD entropy then yield the strong convergence towards the incompressible dynamics.

1. Introduction

In this paper, we study the low Mach number limit for the Quantum-Navier-Stokes equations (QNS) posed on $(0, T) \times \mathbb{R}^3$,

\begin{equation}
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P(\rho) &= 2\nu \text{div}(\rho D u) + 2\kappa^2 \rho \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right),
\end{aligned}
\end{equation}

where the unknowns are given by the mass density $\rho$ and the velocity field of the fluid $u$. We consider a pressure given by the $\gamma$-law, i.e. $P(\rho) = \frac{1}{\gamma - 1} \rho^\gamma - 1$ with $1 < \gamma < 3$. We refer to the coefficients $\nu$ and $\kappa$ as viscosity and capillarity coefficients respectively. The energy we consider for system (1.4) is given by

\begin{equation}
E(t) = \int_{\mathbb{R}^3} \left( \frac{1}{2} \rho |u|^2 + 2\kappa^2 \nabla \sqrt{\rho} \right)^2 + \pi(\rho) \, dx,
\end{equation}

where the internal energy is given by

\begin{equation}
\pi = \pi(\rho) = \frac{\rho^\gamma - 1 - \gamma(\rho - 1)}{\gamma(\gamma - 1)}
\end{equation}

Thus, the finite energy assumption yields that

\[ \rho \to 1 \quad \text{as} \quad |x| \to \infty. \]

System (1.4) arises e.g. as a model for dissipative quantum fluids and enters the more general class of Navier-Stokes-Korteweg systems,

\[ \partial_t \rho + \text{div}(\rho \dot{u}) = 0, \]

\[ \partial_t (\rho \dot{u}) + \text{div}(\rho \dot{u} \otimes \dot{u}) + \nabla P(\rho) = 2\nu \text{div}(\mathcal{S}) + \kappa^2 \text{div}(\mathcal{K}), \]

Date: February 4, 2019.

1991 Mathematics Subject Classification. Primary: 35Q35; Secondary: 35Q30, 76Y99.

Key words and phrases. Compressible and Incompressible Navier-Stokes equation, Quantum fluids, Low Mach number limit, Acoustic Waves, Strichartz estimates, Energy estimates.
where the viscous stress tensor $\mathbb{S} = \mathbb{S}(\nabla u)$ is given by

$$\mathbb{S} = h(\rho)\mathbb{D}u + g(\rho) \text{div} \, u,$$

while the capillary (dispersive) term $K = K(\rho, \nabla \rho)$ reads

$$K = \left( \rho \text{div}(k(\rho)\nabla \rho) - \frac{1}{2}(\rho k'(\rho) - k(\rho)|\nabla \rho|^2) \right) \mathbb{I} - k(\rho)\nabla \otimes \nabla.$$  

System (1.1) is then recovered by choosing $k(\rho) = \frac{1}{\rho}$. The QNS equations can also be derived from a Chapman-Enskog expansion for the Wigner equation with a BGK term [14], see also [28] where several dissipative quantum fluid models are derived by means of a moment closure of (quantum) kinetic equations with appropriate choices of the collision terms. The inviscid counterpart of (1.4) is the Quantum Hydrodynamic system (QHD), see [5, 6, 7, 4] that arises as hydrodynamical model in superfluids [30] and Bose-Einstein condensates [38].

After a suitable rescaling (see subsection 2.1), the system (1.1) reads,

\begin{align*}
\partial_t \rho^\varepsilon + \text{div}(\rho^\varepsilon u^\varepsilon) &= 0, \\
\partial_t (\rho^\varepsilon u^\varepsilon) + \text{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \frac{1}{\varepsilon^2} \nabla P(\rho^\varepsilon) &= 2\nu \text{div}(\rho^\varepsilon \mathbb{D}u^\varepsilon) + 2\kappa^2 \rho^\varepsilon \nabla \left( \frac{\Delta \sqrt{\rho^\varepsilon}}{\sqrt{\rho^\varepsilon}} \right),
\end{align*}

with initial data

$$\rho^\varepsilon(0, x) = \rho_{\varepsilon,0},$$

$$\rho^\varepsilon(0, x) = \rho_{\varepsilon,0} u_{\varepsilon,0},$$

where $\varepsilon \ll 1$ is the scaled Mach number. Analogously, the internal energy (1.3) in the Definition (1.2) becomes

\begin{equation}
\pi^\varepsilon = \pi(\rho^\varepsilon) = \rho^\gamma - 1 - \gamma(\rho - 1) \varepsilon^2 \gamma (\gamma - 1). 
\end{equation}

In the low Mach number regime, i.e. in the limit as $\varepsilon \to 0$, the dynamics of (1.4) is formally governed by the incompressible Navier-Stokes equations,

\begin{equation}
\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u, \quad \text{div} \, u = 0.
\end{equation}

The aim of this paper is to rigorously study this limit in its full generality, i.e. by considering arbitrary finite energy initial data without imposing further regularity or smallness assumptions and in particular without being well-prepared.

The QNS system entails some mathematical difficulties due to the possible appearance of vacuum regions. Indeed, the degenerate viscosity prevents a suitable control of the velocity field in the vacuum. In particular this yields some problems in establishing the necessary compactness estimates on the convective term $\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon$. Finite energy weak solutions to (1.4) in the two and three dimensional torus were studied in [8, 33]. To the authors’ knowledge no analogue result exists for the same system in the whole space with non-trivial boundary conditions at infinity. It seems reasonable to argue that exploiting some ideas in [8, 9] it is possible to show a similar result in this framework, however this paper will focus on the low Mach number limit for (1.4) and hence we postulate the existence of such solutions (see Definition 2.1 below for more details). The analysis of the Cauchy problem associated to (1.4) is postponed to future investigations. Besides the mathematical difficulties originated from the degenerate viscosity, in this framework we also have to cope with the lack of integrability of the mass density due to non-trivial boundary conditions.
The main achievement of this paper is obtained by introducing a class of refined Strichartz estimates that allow to capture more accurately the dispersion relation for the acoustic waves. Indeed, contrarily to the classical case where the fluctuations evolve accordingly to the classical wave equation \[35, 16, 40\], here in our problem the presence of the quantum term contributes in a non-trivial way to the dispersion relation, especially at high frequencies. The dispersion relation inferred here, see formula (4.1) below, is strictly related to the Bogoliubov spectrum describing excitations in a Bose-Einstein condensate, which predicts the superfluid behavior of the gas \[13, 12, 39\]. This is somehow reminiscent of the analysis of fluctuations done when studying the quasi-neutral limit for a class of Navier-Stokes-Korteweg systems \[17, 18\].

Furthermore, in the limit we recover a weak solution of the incompressible Navier-Stokes equation \(u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T, \dot{H}^1(\mathbb{R}^3))\). We remark that we are able to obtain bounds on the gradient of the limiting solution to (1.6), even though at fixed \(\varepsilon > 0\) only a weak version of the energy inequality is available \[8, 9, 33\], see also the discussion in section 2. However, this weak version of the energy inequality will anyway yield the aforementioned natural bounds on the gradient of the velocity field in the low Mach number limit. In fact, thanks to some uniform bounds satisfied by the momentum density, we can also infer further smoothing properties for the limiting solution to (1.6), see Theorem 2.2 and Proposition 5.5 for more details. Due to the presence of an initial layer which cannot be avoided for general ill-prepared data, the weak solution enters the Leray class only if further assumptions on the initial data are made. More precisely, only for well-prepared data it is possible to show that the solutions obtained in the limiting procedure satisfy the energy inequality.

The study of singular limits for fluid dynamical equations occupies a vast portion of mathematical literature, for a more comprehensive introduction to the topic we address the reader to the monograph \[20\] and the reviews \[2, 37\]. Our method shares some similarities with \[16\] which studies the compressible Navier-Stokes equations on the whole space. Indeed, there the authors exploit some Strichartz type estimates to analyse the acoustic waves. On the other hand, for the QNS system the dispersion relation is modified and reads as in formula (4.1); thus for high frequencies the fluctuations appearing in classical fluid dynamics and in system (1.1) differ considerably. Recently, the incompressible limit for quantum Navier-Stokes equations has been investigated in \[31\]. The authors consider the system with a damping term that allows to circumvent mathematical difficulties related to the lack of control of the velocity field in the vacuum. Moreover, the initial data for \(\varepsilon > 0\) is prepared in such a way that the convergence to local strong solutions is achieved. For well-prepared data, it was shown in \[41\] that the incompressible limit of the periodic system on \(\mathbb{T}^3\) is given by the incompressible Navier-Stokes equations.

Here, we tackle the problem from a different perspective, namely we retrieve global weak solutions in the limit rather than convergence to the unique local strong solution to the limiting system. Moreover, while in \[31, 41\] the fluctuations are studied by using a wave-like dispersion as for classical fluid dynamical systems, here we consider the full dispersion relation determined by the Bogoliubov spectrum (4.1) and obtain a better control on the fluctuations. This is achieved by carrying out a refined analysis on the dispersive properties of the acoustic waves that together
with new uniform estimates enables us to study the low Mach number limit for general ill-prepared initial data without regularity or smallness assumptions and without damping. For the inviscid system, i.e. the QHD system, the low Mach number limit with ill-prepared data has been studied in [19] on the torus and on the plane in the forthcoming paper [3].

This paper is organized as follows, we introduce notations and preliminary results in Section 2. Subsequently, the needed a priori estimates are provided in Section 3. This is particularly relevant since finite energy weak solutions of (1.4) only obey a weak form of the energy inequality, for a detailed discussion see Appendix A. Section 4 is dedicated to the analysis of the acoustic waves. The strong convergence of finite energy weak solutions of (1.4) to weak solutions of (1.5) is achieved in Section 5 by means of an Aubin-Lions compactness argument. Furthermore, we investigate the regularity properties of the limit \( u \) and show that \( u \) lies in the class of Leray solutions under suitable additional assumptions. Appendix B is devoted to the proof of the dispersive estimates.

2. Preliminaries

Notations. We list the notations of function spaces and operators used in the following. We denote

- by \( \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^3) \) the space of test functions \( C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^3) \) and by \( \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3) \) the space of distributions. The duality bracket between \( \mathcal{D} \) and \( \mathcal{D}' \) is denoted by \( \langle \cdot, \cdot \rangle \).
- for \( 0 < T \leq \infty \) by \( L^p(\mathbb{R}^d) \) for \( 1 \leq p \leq \infty \) the Lebesgue space with norm \( \| \cdot \|_{L^p} \) and by \( L^p(0, T; L^q(\mathbb{R}^d)) \) the space of functions \( u : (0, T) \times \mathbb{R}^d \rightarrow \mathbb{R}^n \) with norm
  \[
  \| u \|_{L^p L^q} = \left( \int_0^T \left( \int_{\mathbb{R}^d} |u(t, x)|^q \, dx \right)^{\frac{p}{q}} \, dt \right)^{\frac{1}{p}}.
  \]
  If \( T = \infty \), we write \( L^p(0, \infty; L^q(\mathbb{R}^d)) \). Further, we denote by \( L_{p'}(0, T; L^q(\mathbb{R}^d)) \) for any \( 1 \leq p' < p \).
- the non-homogeneous Sobolev space by \( W^{k,p} = (I - \Delta)^{-\frac{s}{2}} L^p(\mathbb{R}^d) \) and \( H^{k} = W^{k,2}(\mathbb{R}^d) \). Its dual will be denoted by \( W^{-k,p'} \) with \( p' \) being the Hölder conjugate of \( p \). The homogeneous spaces are denoted by \( W^{k,p}(\mathbb{R}^d) = \bigl((-\Delta)^{-\frac{s}{2}}\bigr)^* L^p(\mathbb{R}^d) \) and \( W^{-k,p}(\mathbb{R}^d) = H^k(\mathbb{R}^d) \), and the dual space \( W^{-k,p'} \).
- by \( L^2_\infty(\mathbb{R}^d) \) the Orlicz space defined as
  \[
  L^p_\infty(\mathbb{R}^d) = \left\{ f \in L^1_{loc} : |f|\chi_{|f| \leq \frac{1}{2}} \in L^2(\mathbb{R}^d), \ |f|\chi_{|f| \geq \frac{1}{2}} \in L^p(\mathbb{R}^d) \right\},
  \]
  we refer to [11, 30] for details.
- by \( B^{s}_{q,r}(\mathbb{R}^d) \) the non-homogeneous Besov space and by \( \dot{B}^{s}_{q,r} \) the homogeneous Besov space, see [10]. We denote by \( B^{-s}_{q,r} \) the dual space of \( B^s_{q,r} \).
- by \( \mathbf{P} \) and \( \mathbf{Q} \) the Leray projectors on divergence-free and gradient vector fields respectively:
  \[
  \mathbf{Q} = \nabla \Delta^{-1} \div, \quad \mathbf{P} = \mathbf{I} - \mathbf{Q}.
  \]  
  For \( f \in W^{k,p}(\mathbb{R}^d) \) with \( 1 < p < \infty \) and \( k \) the operators \( \mathbf{P}, \mathbf{Q} \) can be expressed as composition of Riesz multipliers and are bounded linear operators on \( W^{k,p}(\mathbb{R}^d) \).
• the Fourier transform of $f$ by $\hat{f} := \mathcal{F}(f)$ and the inverse Fourier transform by $f'$.

• the frequency cut-off $P_N(f) = (\phi_N(\xi) \hat{f})'$, where $\phi$ is a smooth frequency cut-off compactly supported in $\text{supp}(\phi) \subset \{ \frac{1}{2}N \leq |\xi| \leq 2N \}$. Similarly, by $P_{\leq N}(f)$ we denote the projection on frequencies of order $|\xi| \leq N$.

• finally the symmetric part of the gradient is denoted by $D_u = \frac{1}{2}(\nabla u + (\nabla u)^T)$ and the asymmetric part by $A_u = \frac{1}{2}(\nabla u - (\nabla u)^T)$.

In what follows $C$ will be any constant independent from $\varepsilon$.

2.1. Scaling. Different scalings are reasonable see for instance the review papers [2, 37] and references therein. To recast the introduced scaling (1.4) of system (1.1), one starts writing the equations by re-scaling each length scale by its characteristic value (dimensionless scaling) and we assume the Mach number to be small. We expect the fluid to behave like an incompressible fluid on large time scales when the density is almost constant and the the velocity is small. Thus, we introduce the change of variable and unknowns,

$$t \mapsto \varepsilon t, \quad u \mapsto \varepsilon u.$$

Moreover, the viscosity and capillarity coefficients scale as

$$\nu \mapsto \varepsilon \nu, \quad \kappa \mapsto \varepsilon \kappa,$$

where

$$\nu \to \tilde{\nu} > 0, \quad \kappa \to \tilde{\kappa} > 0,$$

as $\varepsilon$ goes to 0.

**Weak solutions.** As we already mentioned in the Introduction, the degenerate viscosity prevents the velocity field to be uniquely determined in the vacuum region; indeed system (1.4) lacks bounds for $u_\varepsilon$. Consequently, in this framework (see for example [8, 33]) it turns out that the problem is best studied in terms of the more suitable variables $\sqrt{\rho_\varepsilon}$ and $\Lambda_\varepsilon = \sqrt{\rho_\varepsilon} u_\varepsilon$. In fact, this occurs also when studying the QHD system [5] and the barotropic compressible Navier-Stokes equations with degenerate viscosities [34]. Mathematically speaking, this means that whenever the symbol $\rho_\varepsilon$ appears, it should be read as $\rho_\varepsilon = (\sqrt{\rho_\varepsilon})^2$ and similarly for the momentum density $m_\varepsilon = \rho_\varepsilon u_\varepsilon = \sqrt{\rho_\varepsilon} \Lambda_\varepsilon$. At no moment neither the velocity field $u_\varepsilon$ nor its gradient $\nabla u_\varepsilon$ are defined a.e. in $\mathbb{R}^3$. For those reasons, the viscous tensor should be rather thought as

$$\rho_\varepsilon^2 D_u = \sqrt{\rho_\varepsilon} S_\varepsilon,$$

where $S_\varepsilon$ is the symmetric part of the tensor $T_\varepsilon$ defined through the following identity

$$\sqrt{\rho_\varepsilon} T_\varepsilon = \nabla (\rho_\varepsilon u_\varepsilon) - 2\nabla \sqrt{\rho_\varepsilon} \otimes \Lambda_\varepsilon,$$

in $\mathcal{D}'((0,T) \times \mathbb{R}^3)$. In this way the equation for the momentum density in (1.4) reads

$$\partial_t (\rho_\varepsilon u_\varepsilon) + \text{div} (\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon + 4\kappa^2 \nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) + \frac{1}{\varepsilon^2} \nabla P(\rho_\varepsilon) - 2\nu \text{div}(\sqrt{\rho_\varepsilon} S_\varepsilon) - \kappa^2 \nabla \Delta \rho_\varepsilon = 0.$$

In fact, at present it is not clear whether arbitrary finite energy weak solutions to (1.4) satisfy the following energy inequality,

$$E(t) + 2\nu \int_0^t \int_{\mathbb{R}^3} \rho_\varepsilon |D_u_\varepsilon|^2 dx dt' \leq E(0),$$

where $E$ is the energy functional.
see also the discussion in [11] where a similar issue is dealt with for a Navier-Stokes-Korteweg type system. As we will see in (2.5), we only assume a weak version of the energy inequality.

In our paper, we do not use the notation \( \Lambda \) instead of \( \sqrt{\rho}u \) for the sake of consistency with the literature regarding (quantum) Navier-Stokes equations. The definition of finite energy weak solutions will therefore be given in terms of the mathematical unknowns \( \sqrt{\rho} \) and \( \sqrt{\rho}u \) instead of the physical unknowns of density \( \rho \) and momentum \( m \). We recall that under suitable assumptions on the mass density \( \rho \) the quantum pressure term can be alternatively rewritten as

\[
2\rho \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) = \text{div} (\rho \nabla^2 \log \rho) = \nabla \Delta \rho - 4\text{div}(\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}).
\]

**Definition 2.1.** A pair \((\rho, u)\) with \( \rho \geq 0 \) is said to be a finite energy weak solution of the Cauchy Problem (1.4) if

(i) integrability conditions
\[
\sqrt{\rho} \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3); \quad \sqrt{\rho}u \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3);
\]
\[
\nabla \sqrt{\rho} \in L^2_{\text{loc}}((0, T) \times \mathbb{R}^3);
\]

(ii) continuity equation
\[
\int_{\mathbb{R}^3} \rho \phi_0(0) + \int_0^T \int_{\mathbb{R}^3} \rho \phi_t + \sqrt{\rho} \sqrt{\rho}u \nabla \phi = 0,
\]
for any \( \phi \in C_c^{\infty}([0, T) \times \mathbb{R}^3) \).

(iii) momentum equation
\[
\int_{\mathbb{R}^d} \rho u \phi(0) + \int_0^T \int_{\mathbb{R}^d} \sqrt{\rho} \sqrt{\rho}u \phi_t + (\sqrt{\rho} \otimes \sqrt{\rho}u) \nabla \phi + \frac{1}{\varepsilon} \rho \text{div} \psi
\]
\[
- 2\nu \int_0^T \int_{\mathbb{R}^d} \nabla \sqrt{\rho} \sqrt{\rho}u \nabla \psi - 2\nu \int_0^T \int_{\mathbb{R}^d} (\nabla \sqrt{\rho} \otimes \sqrt{\rho}u) \nabla \phi
\]
\[
+ \nu \int_0^T \int_{\mathbb{R}^d} \sqrt{\rho} \nabla \sqrt{\rho}u \Delta \phi + \nu \int_0^T \int_{\mathbb{R}^d} \sqrt{\rho} \nabla \sqrt{\rho}u \nabla \psi
\]
\[
- 4\kappa^2 \int_0^T \int_{\mathbb{R}^d} (\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) \nabla \psi + 2\kappa^2 \int_0^T \int_{\mathbb{R}^d} \sqrt{\rho} \nabla \sqrt{\rho} \nabla \psi = 0,
\]
for any \( \phi \in C_c^{\infty}([0, T) \times \mathbb{R}^3) \).

(iv) there exists a tensor \( T \in L^2((0, T) \times \mathbb{R}^3) \) satisfying identity (2.2) in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \) such that the following energy inequality holds for a.e. \( t \in [0, T] \),

\[
E(t) + 2\nu \int_0^t \int_{\mathbb{R}^3} |S| dx dt \leq E(0),
\]

where \( S \) is the symmetric part of \( T \), i.e. \( S = T_{\text{sym}} \).

(v) Let \( \mu = \nu - \sqrt{\nu^2 - \kappa^2} \) and for \( 0 < c < \mu \) define
\[
B_\varepsilon(t) = \int_{\mathbb{R}^3} \frac{1}{2} \left| \sqrt{\rho}u + 2c \nabla \sqrt{\rho} \right|^2 + \pi \varepsilon + \kappa^2 \left| \nabla \sqrt{\rho} \right|^2 dx.
\]
Then the Bresch-Dessjards entropy inequality holds for a.e. \( t \in [0, T] \),

\[
B_\varepsilon(t) + c \int_0^t \frac{1}{2} |A_\varepsilon|^2 \, dx \, ds + C \int_0^t \int_{\mathbb{R}^3} \left| \nabla \sqrt{\rho_\varepsilon} \right|^2 \, dx \, ds + \frac{c\gamma}{2\varepsilon^2} \int_0^t \int_{\mathbb{R}^3} \left| \nabla \sqrt{\rho_\varepsilon} \right|^2 \, dx \, ds
\leq \int_{\mathbb{R}^3} \frac{1}{2} \left| \sqrt{\rho_{\varepsilon,0} u_{\varepsilon,0}} + 2\nu \nabla \sqrt{\rho_{\varepsilon,0}} \right|^2 + \frac{\sigma}{\varepsilon} \left| \nabla \sqrt{\rho_{\varepsilon,0}} \right|^2 \, dx,
\]

where \( A_\varepsilon = T_\varepsilon^{sym} \), with \( T_\varepsilon \) defined as in the previous point.

The definition of finite energy weak solutions including the energy and BD entropy inequalities as stated in points (iv) and (v) is motivated by the following. Recently, in [8] the authors proved global existence of weak solutions on \( \mathbb{T}^d \) with \( d = 2, 3 \) satisfying the energy inequality \( E(t) \leq E(0) \). While for smooth solutions of (1.4) with \( \rho_\varepsilon > 0 \) the tensor \( S_\varepsilon \) is equivalent to \( \sqrt{\rho_\varepsilon} \mathbf{D} u_\varepsilon \), this information may not be recovered for finite energy weak solutions. In the Appendix A we will show that the finite weak solutions constructed in [8] as limit of smooth approximating solutions indeed satisfy the energy inequality (2.5), with \( S_\varepsilon \) defined as the symmetric part of the tensor in (2.2). We stress here that despite the fact that the energy-energy dissipation holds in this weaker sense, it will anyway provide the necessary bounds we will need in our analysis. The same argument also applies for the BD-entropy in (2.6).

Main result. Let us specify the assumptions on the initial data for the system (1.4). Let \( \nu > \kappa \). We consider initial data \((\rho_{\varepsilon,0}, u_{\varepsilon,0})\) such that

\[
\|\nabla \sqrt{\rho_\varepsilon}\|_{L^2(\mathbb{R}^3)} \leq C, \quad \|\sqrt{\rho_\varepsilon} u_{\varepsilon,0}\|_{L^2(\mathbb{R}^3)} \leq C, \quad \|\pi_\varepsilon(0)\|_{L^1(\mathbb{R}^3)} \leq C,
\]

where is \( C \) independent on \( \varepsilon > 0 \). Furthermore, we assume that

\[
\sqrt{\rho_\varepsilon^0 u_{\varepsilon,0}^0} \rightharpoonup u_0 \quad \text{in} \quad L^2(\mathbb{R}^3).
\]

With this definition at hand, we now state the main Theorem characterising the low Mach number regime for (1.4).

**Theorem 2.2.** Let \( 1 < \gamma < 3 \), let \((\rho_\varepsilon, u_\varepsilon)\) be a finite energy weak solution of (1.4) with initial data satisfying (2.7) and (2.8) and let \( 0 < T < \infty \) be an arbitrary time. Then \( \rho_\varepsilon - 1 \) converges strongly to 0 in \( L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^4(0, T; H^1(\mathbb{R}^3)) \) for any \( 0 \leq s < 1 \). For any subsequence (not relabeled) \( \sqrt{\rho_\varepsilon} u_\varepsilon \) converging weakly to \( u \) in \( L^\infty(0, T; L^2(\mathbb{R}^3)) \), then \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \) is a global weak solution to the incompressible Navier-Stokes equation (1.6) with initial data \( u|_{t=0} = P(u_0) \) and \( \sqrt{\rho_\varepsilon} u_\varepsilon \) converges strongly to \( u \) in \( L^2(0, T; L^{6/5}(\mathbb{R}^3)) \).

Moreover, \( Q(\rho_\varepsilon u_\varepsilon) \) converges strongly to 0 in \( L^2(0, T; L^q(\mathbb{R}^3)) \) for any \( 2 < q < \frac{9}{4} \). Finally the limiting solution \( u \) also satisfies \( u \in L^{\frac{4}{s-1}}(0, T; H^s(\mathbb{R}^3)) \), for \( 0 \leq s \leq \frac{\nu}{\gamma} \).

**Remark 2.3.** Let us remark that in order for the limiting function \( u \) to satisfy the energy inequality, i.e. to be a Leray weak solution [22], stronger assumptions on the initial data \((\rho_{\varepsilon,0}, u_{\varepsilon,0})\) are needed. Indeed the initial total energy for the compressible system in general does not converge, as \( \varepsilon \to 0 \), to the initial energy for (1.4), which would be given by \( \frac{1}{\gamma} \int |P u_0|^2 \). The excess energy determines an initial layer which
cannot be avoided for ill-prepared data. On the other hand, if we require
\(\sqrt{\rho_0^\varepsilon} \to u_0 = P(u_0)\) strongly in \(L^2(\mathbb{R}^3)\),
\[\begin{align*}
\pi_\varepsilon(\rho_0^\varepsilon) &\to 0 \quad \text{strongly in } L^1(\mathbb{R}^3), \\
\nabla \sqrt{\rho_0^\varepsilon} &\to 0 \quad \text{strongly in } L^2(\mathbb{R}^3)
\end{align*}\]
then the following Proposition holds true.

**Proposition 2.4.** Under the same assumptions of Theorem 2.2, let \(\rho_0^\varepsilon, u_0^\varepsilon\) further satisfy (2.9). Then the limiting solution \(u\) to (1.6) satisfies the energy inequality
\[\int_{\mathbb{R}^3} \frac{1}{2} |u(t)|^2 \, dx + \nu \int_0^t \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt' \leq \int_{\mathbb{R}^3} \frac{1}{2} |u_0|^2 \, dx,
\]
for almost every \(t \in [0, T]\).

### 3. Uniform estimates

In this Section, we start our analysis on the low Mach number limit by inferring some uniform estimates for finite energy weak solutions to (1.4). In our framework we need to take into account the non trivial boundary conditions for the mass density. For this reason, we provide some estimates on the quantities \(\sqrt{\rho_0^\varepsilon - 1}\) and \(\rho_0^\varepsilon - 1\). Furthermore, the lack of control for \(\nabla u_\varepsilon\) in the vacuum will be compensated by the bounds inferred from (2.5) and (2.6).

We shall use repeatedly the following observation, see for instance Theorem 4.5.9 in [27]: if \(f \in D'(\mathbb{R}^d)\) with \(\nabla f \in L^p(\mathbb{R}^d)\) for \(p < d\), then there exists a constant \(c\) such that \(f - c \in L^{p^*}(\mathbb{R}^d)\), where \(p^*\) is the critical Sobolev exponent. The condition \(p < d\) is sharp.

Firstly, we state some a priori bounds on the initial data.

#### 3.1. Initial data of finite energy

If the initial data \((\rho_0^0, u_0^0)\) is assumed to be of finite energy, i.e. \(E(\rho_0^0, u_0^0) < \infty\), then
\[\nabla \sqrt{\rho_0^\varepsilon} \in L^2(\mathbb{R}^3), \quad \sqrt{\rho_0^\varepsilon} u_0^\varepsilon \in L^2(\mathbb{R}^3), \quad \frac{\rho_0^\varepsilon - 1 - \gamma (\rho_0^\varepsilon - 1)}{\varepsilon^2 \gamma (\gamma - 1)} \in L^1(\mathbb{R}^3).\]
This implies additional bounds which we summarize.

**Lemma 3.1.** If the initial data \((\rho_0^0, u_0^0)\) is of finite energy, then there exists \(C > 0\) independent from \(\varepsilon > 0\) such that
\[\begin{align*}
(i) \quad &\|\rho_0^\varepsilon - 1\|_{L^2} \leq C \varepsilon^\beta, \quad \text{where} \quad \beta = \beta(\gamma) = \begin{cases} \frac{2}{(6-\gamma)} & \gamma < 2; \\
1 & \gamma \geq 2; \end{cases} \\
(ii) \quad &\sqrt{\rho_0^\varepsilon} - 1 \in H^1(\mathbb{R}^3) \quad \text{and in particular for } 2 \leq p < 6 \text{ we have } \|\sqrt{\rho_0^\varepsilon} - 1\|_{L^p} \leq C \varepsilon^\alpha(p), \quad \text{where} \\
&\alpha(p) = \begin{cases} \frac{2(6-p)}{p(6-\gamma)} & \gamma < 2; \\
\frac{2p}{(6-\gamma)} & \gamma \geq 2; \end{cases} \\
(iii) \quad &\rho_0^\varepsilon u_0^\varepsilon \in L^2(\mathbb{R}^3) + L^\frac{4}{s}(\mathbb{R}^3). \quad \text{In particular } \rho_0^\varepsilon u_0^\varepsilon \in H^{-s}(\mathbb{R}^3) \quad \text{with } s > \frac{1}{2}.\end{align*}\]
Proof. We prove the first statement following the method in [13]. By convexity of the function $s \to s^γ - 1 - γ(s - 1)$ for $γ > 1$ and the fact that the internal energy, as defined in [13], satisfies $π(\rho_0^2) \in L^1(\mathbb{R}^3)$ one concludes that

$$\int_{\mathbb{R}^3} |\rho_0^2 - 1|^2 1_{|\rho_0^2| \leq \frac{1}{2}} + |\rho_0^2 - 1|^γ 1_{|\rho_0^2 - 1| > \frac{1}{2}} dx \leq C\varepsilon^2$$

and when $γ \geq 2$ one has

$$\int_{\mathbb{R}^d} (\rho_0 - 1)^2 dx \leq C\varepsilon^2.$$

Upon observing that

$$\sqrt{\rho_0} - 1 = \frac{\rho_0 - 1}{1 + \sqrt{\rho_0}} \leq (\rho_0 - 1),$$

we obtain,

$$\int_{\mathbb{R}^3} |\sqrt{\rho_0^2} - 1|^2 1_{|\rho_0^2| \leq \frac{1}{2}} + |\sqrt{\rho_0^2} - 1|^γ 1_{|\rho_0^2 - 1| > \frac{1}{2}} dx \leq C\varepsilon^2,$$

in particular $\sqrt{\rho_0^2} - 1 \in L^p_2$. From what we said before, the bound $\nabla \sqrt{\rho_0} \in L^2(\mathbb{R}^3)$ implies there exists $c > 0$ such that $\sqrt{\rho_0^2} - c \in L^6(\mathbb{R}^3)$. It is easy to conclude that necessarily $c = 1$ since $\sqrt{\rho_0^2} - 1 \in L^2(\mathbb{R}^3)$ and by interpolation with (3.4) we have

$$\int_{\mathbb{R}^3} |\sqrt{\rho_0^2} - 1|^p 1_{|\rho_0^2 - 1| \leq \frac{1}{2}} + |\sqrt{\rho_0^2} - 1|^p 1_{|\rho_0^2 - 1| > \frac{1}{2}} dx \leq C\varepsilon^\alpha(p),$$

for any $2 \leq p \leq 6$, where $\alpha(p) = \frac{2(6 - p)}{p(6 - γ)}$ for $γ < 2$ and $\alpha(p) = \frac{6 - p}{2p}$ for $γ ≥ 2$. In particular,

$$\|\sqrt{\rho_0^2} - 1\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^\alpha,$$

where

$$\alpha(2) = \begin{cases} \frac{4}{6 - γ} & γ < 2, \\ 1 & γ ≥ 2. \end{cases}$$

Therefore, $\sqrt{\rho_0^2} - 1 \in H^1(\mathbb{R}^3)$. Next we show that for $1 < γ < 2$, one has

$$\|\rho_0^2 - 1\|_{L^2(\mathbb{R}^3)} \leq C\varepsilon^{2\alpha(4)},$$

In view of (3.3), it is sufficient to show that the second term in (3.3) is bounded by $C\varepsilon^\gamma$. To that end we observe that for $|z - 1| ≥ \frac{1}{2}$, one has

$$(z^2 - 1)^2 ≤ 25(z - 1)^4.$$ 

Hence,

$$\int_{\mathbb{R}^3} |\rho_0^2 - 1|^2 1_{|\rho_0^2 - 1| > \frac{1}{2}} dx \leq \int_{\mathbb{R}^3} |\sqrt{\rho_0^2} - 1|^4 1_{|\rho_0^2 - 1| > \frac{1}{2}} dx \leq C\varepsilon^{4\alpha(4)}.$$ 

Since for $1 < γ < 2$, we have $2\alpha(4) = \frac{2γ}{6 - γ}$, the statement (i) follows. Finally, to prove (iii) we notice that if the initial data is of finite energy then we obtain

$$\sqrt{\rho_0^2} - 1 \in H^1(\mathbb{R}^3), \quad \sqrt{\rho_0^2} u_0^0 \in L^2(\mathbb{R}^3),$$

which allow us to conclude

$$\rho_0^2 u_z^0 = \sqrt{\rho_0^2} u_0^0 + (\sqrt{\rho_0^2} - 1)\sqrt{\rho_0^2} u_0^0 \in L^2(\mathbb{R}^3) + L^2(\mathbb{R}^3).$$

□
3.2. Uniform estimates on the solution. By Definition 2.4, the finite energy weak solution \((\rho_\varepsilon, u_\varepsilon)\) we consider satisfies the energy inequality 2.5 and the BD entropy type inequality 2.6 that imply the following a priori estimates listed below.

**Lemma 3.2.** If \((\rho_\varepsilon, u_\varepsilon)\) is a finite energy weak solution of (1.3), then there exists \(C > 0\) independent from \(\varepsilon > 0\) such that

(i) such that \(\|\rho_\varepsilon - 1\|_{L^\infty(\mathbb{R}^4; L^2(\mathbb{R}^3))} \leq C\varepsilon^\beta\), for \(\beta(\gamma)\) defined as in (3.1).

(ii) \(\|\frac{1}{\sqrt{\varepsilon}} \nabla \rho_\varepsilon^\frac{1}{2}\|_{L^2(\mathbb{R}^4; L^2(\mathbb{R}^3))} \leq C\),

(iii) \(\sqrt{\rho_\varepsilon} - 1 \in L^p(\mathbb{R}^4; H^1(\mathbb{R}^3))\) and in particular for \(2 \leq p < 6\) and for \(\alpha(p)\) defined as in (3.2) it holds \(\|\sqrt{\rho_\varepsilon} - 1\|_{L^p(\mathbb{R}^4; L^p(\mathbb{R}^3))} \leq C\varepsilon^\alpha(p)\),

(iv) \(\|\nabla^2 \sqrt{\rho_\varepsilon}\|_{L^2(\mathbb{R}^4; L^2(\mathbb{R}^3))} \leq C\),

(v) for any \(0 \leq s < 2\) and \(2 \leq p < \frac{4}{3}\), there exists \(0 < \beta(p, s) < 2\) such that \(\|\sqrt{\rho_\varepsilon} - 1\|_{L^p(\mathbb{R}^4; H^s(\mathbb{R}^3))} \leq C\varepsilon^\beta\), Moreover, for \(1 < s \leq 2\), \(\|\sqrt{\rho_\varepsilon} - 1\|_{L^{4\gamma}(\mathbb{R}^4; H^s(\mathbb{R}^3))} \leq C\). In particular, \(\sqrt{\rho_\varepsilon} - 1 \in L^2(\mathbb{R}^4; L^\infty(\mathbb{R}^3))\).

(vi) \(\|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^\infty(\mathbb{R}^4; L^3(\mathbb{R}^3))} \leq C\).

**Proof.** The first and the third statement are proven similarly to Lemma 3.1 exploiting the fact that \(\pi_\varepsilon \in L^\infty(\mathbb{R}^4; L^1(\mathbb{R}^3))\). The remaining statements except the fifth are direct consequences of inequalities (2.4) and (2.6). Statement (v) follows by interpolation of (ii) and (iv). For \(0 < s < 2\) and \(0 < \theta < 1\) there exists \(\beta(\gamma)\) such that \(\|\sqrt{\rho_\varepsilon} - 1\|_{L^\infty(\mathbb{R}^4; L^2(\mathbb{R}^3))} \leq C\varepsilon^\beta\), where \(s = 2\theta\) and \(p\) such that \(p \leq \frac{2}{s}\). In particular if \(s > \frac{3}{4}\) this yields for any \(2 \leq p < \frac{4}{3}\) that \(\sqrt{\rho_\varepsilon} - 1\) converges strongly to \(0\) in \(L^p(0, T; L^\infty(\mathbb{R}^3))\). By interpolation between the bounds \(\sqrt{\rho_\varepsilon} - 1 \in L^\infty(\mathbb{R}^4; H^1(\mathbb{R}^3))\) and \(\nabla^2 (\sqrt{\rho_\varepsilon} - 1) \in L^2(\mathbb{R}^4; L^2(\mathbb{R}^3))\), one may infer the slightly stronger bound \(\sqrt{\rho_\varepsilon} - 1 \in L^\frac{2s}{s+2}(\mathbb{R}^4; H^s(\mathbb{R}^3))\) for \(1 < s \leq 2\). \(\Box\)

3.3. Bounds on density fluctuation \(\sigma_\varepsilon\) and momentum \(m_\varepsilon u_\varepsilon\). Next, we provide bounds on the density fluctuation \(\sigma_\varepsilon := \frac{\rho_\varepsilon - 1}{\rho_\varepsilon}\).

**Lemma 3.3.** If \((\rho_\varepsilon, u_\varepsilon)\) is a finite energy weak solution of (1.3), then for any \(0 < T < \infty\), \(\sigma_\varepsilon\) satisfies

(i) \(\sigma_\varepsilon^0 \in L^2(\mathbb{R}^3)\) with \(m = \min\{2, \gamma\}\);

(ii) \(\sigma_\varepsilon \in L^\infty(\mathbb{R}^4; L^2(\mathbb{R}^3))\); 

(iii) \(\varepsilon \nabla \sigma_\varepsilon \in L^{\infty}(\mathbb{R}^4; L^2(\mathbb{R}^3)) + L^4(0, T; L^2(\mathbb{R}^3))\) and \(\varepsilon \sigma_\varepsilon \in L^4(0, T; H^1(\mathbb{R}^3))\); 

(iv) \(\varepsilon \nabla^2 \sigma_\varepsilon \in L^2(0, T; L^2(\mathbb{R}^3)) + L^\frac{4}{3}(0, T; L^\frac{4}{3}(\mathbb{R}^3))\) and \(\varepsilon \nabla^2 \sigma_\varepsilon \in L^\frac{8}{3}(0, T; H^\frac{8}{3}(\mathbb{R}^3))\);

(v) In particular, if \(\gamma = 2\), then \(\sigma_\varepsilon \in L^2(0, T; H^1(\mathbb{R}^3))\).

All the previous bounds are uniform in \(\varepsilon > 0\).

**Proof.** The first bound follows from

\[
\int_{\mathbb{R}^4} |\rho_\varepsilon^0 - 1|^2 1_{|\rho_\varepsilon^0 - 1| \leq \frac{1}{2}} + |\rho_\varepsilon^0 - 1|^\gamma 1_{|\rho_\varepsilon^0 - 1| > \frac{1}{2}} \, dx \leq C\varepsilon^2;
\]

and similarly the second.

The third bound follows by observing \(\varepsilon \nabla \sigma_\varepsilon = 2\sqrt{\rho_\varepsilon} \nabla \sqrt{\rho_\varepsilon}\) and applying the bounds for \(\sqrt{\rho_\varepsilon} - 1\) of Lemma 3.2. In particular,

\[
\|\varepsilon \nabla \sigma_\varepsilon\|_{L^4(0, T; L^2)} \leq \|\nabla \sqrt{\rho_\varepsilon}\|_{L^4(0, T; L^2)} + \|\sqrt{\rho_\varepsilon} - 1\|_{L^4(0, T; L^\infty)} \|\nabla \sqrt{\rho_\varepsilon}\|_{L^\infty(0, T; L^2)}.
\]
Hence, we conclude that \( \varepsilon \sigma_\varepsilon \in L^4(0,T;L^6(\mathbb{R}^3)) \). By interpolation with \( \sigma_\varepsilon \in L^\infty(L^2 + L^7(\mathbb{R}^3)) \), it follows \( \varepsilon \sigma_\varepsilon \in L^4(\mathbb{R}^3;L^2(\mathbb{R}^3)) \). We remark that if \( \gamma \geq 2 \), then \( \sigma_\varepsilon \in L^\infty(0,\infty;L^2(\mathbb{R}^3)) \) and the interpolation is not needed. Thus, \( \varepsilon \sigma_\varepsilon \in L^4(0,T;H^1(\mathbb{R}^3)) \). Similarly for,
\[
\varepsilon \nabla^2 \sigma_\varepsilon = 2 \sqrt{\rho_\varepsilon} \nabla^2 \sqrt{\rho_\varepsilon} + 2 |\nabla \sqrt{\rho_\varepsilon}|^2,
\]
we conclude exploiting again the bounds on \( \sqrt{\rho_\varepsilon} - 1 \). Moreover, for \( \gamma = 2 \), the estimate \( \frac{1}{\varepsilon} \nabla \sqrt{\rho_\varepsilon}^2 \) allows us to conclude that \( \sigma_\varepsilon \in L^2(0,T;H^1(\mathbb{R}^3)) \).

**Corollary 3.4.** If \( (\rho_\varepsilon, u_\varepsilon) \) is a finite energy weak solution of \( H(\mathbb{R}^3) \), then for any \( 0 < T < \infty \),
\[
\rho_\varepsilon u_\varepsilon \in L^4(0,T;L^2(\mathbb{R}^3)),
\]
Proof. It is sufficient to write \( \rho_\varepsilon u_\varepsilon = \sqrt{\rho_\varepsilon} u_\varepsilon + (\sqrt{\rho_\varepsilon} - 1) \sqrt{\rho_\varepsilon} u_\varepsilon \) and to see that
\[
\|\rho_\varepsilon u_\varepsilon\|_{L^4(0,T;L^2(\mathbb{R}^3))} \leq C\|\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^\infty(0,T;L^2)} \left( T^\frac{3}{4} + \|\sqrt{\rho_\varepsilon} - 1\|_{L^4-L^\infty} \right)
\]
\[
\leq C' \left( 1 + T^\frac{3}{4} \right),
\]
since \( \sqrt{\rho_\varepsilon} - 1 \in L^4(0,T;L^\infty(\mathbb{R}^3)) \) from Lemma 3.2.

The Corollary 3.4 together with the a priori bounds on \( \sqrt{\rho_\varepsilon} - 1 \) and \( S_\varepsilon \) allow us to prove a stronger estimate on \( \rho_\varepsilon u_\varepsilon \) by splitting high and low frequencies.

**Proposition 3.5.** If \( (\rho_\varepsilon, u_\varepsilon) \) is a finite energy weak solution of \( H(\mathbb{R}^3) \), then for any \( 0 < T < \infty \) and for any \( 0 \leq s \leq \frac{1}{2} \) and \( 1 \leq p < \frac{4}{3+4s} \),
\[
(3.6)
\rho_\varepsilon u_\varepsilon \in L^p(0,T;H^s(\mathbb{R}^3)),
\]
where the bound is uniform in \( \varepsilon > 0 \). In particular for any \( 0 \leq s_1 < \frac{1}{2} \), one has \( \rho_\varepsilon u_\varepsilon \in L^2(0,T;H^{s_1}(\mathbb{R}^3)) \).

The following Lemma is necessary for the proof of Proposition 3.5.

**Lemma 3.6.** There exist \( f_1 \in L^2(\mathbb{R}^3;W^{1,\frac{3}{2}}(\mathbb{R}^3)) \) and \( f_2 \in L^{\frac{3}{4}}(0,T;H^1(\mathbb{R}^3)) \) such that
\[
\rho_\varepsilon u_\varepsilon = f_1 + f_2, \quad \text{a.e. in } \mathbb{R}^3.
\]
Proof. From \( (2.2) \), the following identity holds in \( D' \),
\[
\nabla (\rho_\varepsilon u_\varepsilon) = \sqrt{\rho_\varepsilon} T_\varepsilon + \nabla \sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon} u_\varepsilon = T_\varepsilon + (\sqrt{\rho_\varepsilon} - 1) T_\varepsilon + 2 \nabla \sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon} u_\varepsilon,
\]
and from the bounds of Lemma 3.2 we deduce that \( \nabla (\rho_\varepsilon u_\varepsilon) \in L^2(\mathbb{R}^3;L^{\frac{3}{2}}(\mathbb{R}^3)) \) + \( L^{\frac{3}{4}}(0,T;L^2(\mathbb{R}^3)) \). Indeed,
\[
\nabla (\sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon} u_\varepsilon) \leq C \nabla \sqrt{\rho_\varepsilon} \|L^2 L^2 \| \|\rho_\varepsilon u_\varepsilon\|_{L^\infty L^2},
\]
\[
\|((\sqrt{\rho_\varepsilon} - 1) T_\varepsilon)\|_{L^\frac{3}{4} L^2} + \|T_\varepsilon\|_{L^\frac{3}{4} L^2} \leq C_T (1 + \|\sqrt{\rho_\varepsilon} - 1\|_{L^4-L^\infty}) \|T_\varepsilon\|_{L^\frac{3}{4} L^2},
\]
Therefore there exist \( g_1, g_2 \) such that
\[
\nabla (\rho_\varepsilon u_\varepsilon) = g_1 + g_2 \quad \text{a.e. in } \mathbb{R}^3 \text{ and } g_1 \in L^2(\mathbb{R}^3;L^{\frac{3}{2}}(\mathbb{R}^3)), \quad g_2 \in L^{\frac{3}{4}}(0,T;L^2(\mathbb{R}^3))
\]
We show that \( g_1, g_2 \) can always be chosen such that they are gradient fields. If not, then given a decomposition \( g_1, g_2 \), we observe that
\[
(3.7) \quad 0 = P(\nabla (\rho_\varepsilon u_\varepsilon)) = P(g_1) + P(g_2),
\]
where \( P \) denotes the Leray projector onto divergence free vector fields. We define \( \tilde{g}_1 := Q(g_1) \) and \( \tilde{g}_2 := g_2 + P(g_1) \). From (3.7) this implies that
\[
\tilde{g}_2 = g_2 + P(g_1) = g_2 - P(g_2) = Q(g_2).
\]
Hence, there exist \( \tilde{f}_1, \tilde{f}_2 \) such that
\[
\tilde{g}_1 = \nabla \tilde{f}_1, \quad \tilde{g}_2 = \nabla \tilde{f}_2, \quad \text{a.e. in } \mathbb{R}^3,
\]
and \( \nabla (\rho \varepsilon u_\varepsilon) = \tilde{g}_1 + \tilde{g}_2 \). We recall again that for any distribution \( f \) such that \( \nabla f \in L^p(\mathbb{R}^d) \) with \( p < d \), there exists a \( c \in \mathbb{R} \) such that \( f - c \in L^{p^*}(\mathbb{R}^d) \) with the Sobolev exponent \( p^* = \frac{dp}{d-p} \). Thus, there exist real numbers \( c_1, c_2 \) such that
\[
\tilde{f}_1 - c_1 \in L^2(\mathbb{R}^+; L^3(\mathbb{R}^3)), \quad \text{and} \quad \tilde{f}_2 - c_2 \in L^{6^*}(0, T; L^6(\mathbb{R}^3)).
\]
Define,
\[
f_1 = \tilde{f}_1 - c_1 - \mathbb{P} \leq 1 (\tilde{f}_1 - c_1), \quad f_2 = \tilde{f}_2 - c_2 + \mathbb{P} \leq 1 (\tilde{f}_1 - c_1).
\]
We claim that \( f_1 \in L^{6^*}(\mathbb{R}^3) \). Indeed, since \( \tilde{B}^0_{6^*, \frac{d}{2}} \hookrightarrow L^{6^*} \), we have
\[
\| f_1 \|_{L^{6^*}} \leq C \| f_1 \|_{B^{0}_{6^*, \frac{d}{2}}} = \left( \sum_{j > 0} 2^{-j} \| 2^j f_1 \|_{L^{6^*}} \right) \frac{2^j}{\omega_j} \leq C \| \nabla f_1 \|_{L^{6^*}},
\]
where we used in the last step that \( L^{6^*} \hookrightarrow \tilde{B}^0_{6^*, \frac{d}{2}} \). We conclude that \( f_1 \in L^6(\mathbb{R}^+; W^{1,6^*}(\mathbb{R}^3)) \).

Next, we check that \( f_2 \in L^{6^*}(0, T; H^1(\mathbb{R}^3)) \). An application of Bernstein’s inequality gives
\[
\| \mathbb{P} \leq N (\tilde{f}_1 - c_1) \|_{L^6} \leq C N^{1/2} \| \mathbb{P} \leq N (\tilde{f}_1 - c_1) \|_{L^3},
\]
therefore \( f_2 \in L^6 \). Again by Bernstein’s inequality, we control
\[
\| \nabla f_2 \|_{L^2} \leq C \left( \| \nabla (\tilde{f}_2 - c_2) \|_{L^2} + C \| \nabla (\mathbb{P} \leq 1 (\tilde{f}_1 - c_1)) \|_{L^{6^*}} \right).
\]
Thus \( f_2 \in L^6 \) and \( \nabla f_2 \in L^2 \). Since
\[
\nabla (\rho \varepsilon u_\varepsilon - f_1 - f_2) = 0 \quad \text{a.e. in } \mathbb{R}^3,
\]
we infer
\[
\rho \varepsilon u_\varepsilon - f_1 - f_2 = C \quad \text{a.e. in } \mathbb{R}^3.
\]
The Sobolev embedding yields that since \( f_1 \in L^2(\mathbb{R}^+; W^{1,6^*}(\mathbb{R}^3)) \) also \( f_1 \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^3)) \), thus by Corollary 3.3
\[
f_2 + C = \rho \varepsilon u_\varepsilon - f_1 \in L^2(0, T; L^2(\mathbb{R}^3)).
\]
This implies \( f_2 + C \in L^{6^*}(0, T; H^1(\mathbb{R}^3)) \) and in particular \( f_2 + C \in L^{6^*}(0, T; L^6(\mathbb{R}^3)) \), again by Sobolev embedding. We recover that necessarily \( C = 0 \). Finally,
\[
\rho \varepsilon u_\varepsilon = f_1 + f_2 \quad \text{a.e. in } \mathbb{R}^3,
\]
where \( f_1 \in L^2(0, T; W^{1,6^*}(\mathbb{R}^3)) \) and \( f_2 \in L^{6^*}(0, T; H^1(\mathbb{R}^3)) \). \( \Box \)

The statement of Proposition 3.5 follows by interpolation.
To perform this analysis we use identity (2.4) and rewrite system (1.4) as
\[ \text{see (4.8) below.} \]
(4.2)
(4.6)
where we observe that by Lemma 3.1,
\[ \text{so that (4.2) reads} \]
(4.5)
(4.3)
waves.

Projecting onto irrotational vector fields we obtain the system describing acoustic waves.

\[ \text{By Lemma 3.6, we have that} \]
\[ \text{Proof of Proposition 3.5.} \]

\[ \text{This section is devoted to the analysis of the acoustic waves in the system. For highly subsonic flows they undergo rapid oscillations in time, so that one expects the acoustic waves to converge weakly to 0. Furthermore, we will see that the dispersion relation satisfied by the fluctuations around the incompressible flow is not given by the classical waves but by the (scaled) Bogoliubov dispersion relation [13], which in our system reads} \]
\[ (4.1) \]
\[ \omega(\xi) = \frac{1}{\varepsilon} \sqrt{|\xi|^2 + \varepsilon^2 \lambda^2 |\xi|^4}, \]
see (4.8) below.

To perform this analysis we use identity (2.4) and rewrite system (1.4) as
\[ (4.2) \]
\[ \begin{cases} \partial_t \rho_\varepsilon + \text{div}(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t (\rho_\varepsilon u_\varepsilon) + \text{div}(\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon) + \frac{1}{\varepsilon^2} \text{div}(\rho_\varepsilon \rho_\varepsilon) = 2 \nu \text{div}(\rho_\varepsilon D u_\varepsilon) - 4 \kappa^2 \text{div}(\nabla \sqrt{\rho_\varepsilon} \otimes \nabla \sqrt{\rho_\varepsilon}) + \kappa^2 \nabla \Delta \rho_\varepsilon, \end{cases} \]
where we recall that the term \( \rho_\varepsilon D u_\varepsilon \) should be interpreted as in (2.3). We notice that, by using (1.5) we can write
\[ \frac{1}{\varepsilon^2} \nabla \text{div}(\rho_\varepsilon) = \frac{1}{\gamma \varepsilon^2} \nabla \rho_\varepsilon = \frac{1}{\varepsilon} \nabla \sigma_\varepsilon + (\gamma - 1) \nabla \pi_\varepsilon, \]
so that (4.2) reads
\[ (4.3) \]
\[ \begin{cases} \partial_t \sigma_\varepsilon + \frac{1}{\varepsilon^2} \text{div}(m_\varepsilon) = 0, \\ \partial_t m_\varepsilon + \frac{1}{\varepsilon^2} \nabla (1 - \kappa^2 \varepsilon^2 \Delta) \sigma_\varepsilon = F_\varepsilon, \end{cases} \]
and
\[ (4.4) \]
\[ F_\varepsilon = \text{div}(-\rho_\varepsilon u_\varepsilon \otimes u_\varepsilon - 4 \kappa^2 \nabla \sqrt{\rho_\varepsilon} \otimes \nabla \sqrt{\rho_\varepsilon} + 2 \nu \rho_\varepsilon D u_\varepsilon - (\gamma - 1) \pi_\varepsilon I). \]
Projecting onto irrotational vector fields we obtain the system describing acoustic waves
\[ (4.5) \]
\[ \begin{cases} \partial_t \sigma_\varepsilon + \frac{1}{\varepsilon^2} \text{div}(Q m_\varepsilon) = 0, \\ \partial_t Q m_\varepsilon + \frac{1}{\varepsilon^2} \nabla (1 - \kappa^2 \varepsilon^2 \Delta) \sigma_\varepsilon = Q(F_\varepsilon). \end{cases} \]
The initial datum for (4.5) is given by
\[ \sigma_\varepsilon \big|_{t=0} = \frac{\rho_\varepsilon \big|_{t=0} - 1}{\varepsilon}, \quad m_\varepsilon \big|_{t=0} = \rho_\varepsilon u_\varepsilon \big|_{t=0}, \]
where we observe that by Lemma 3.1
\[ (4.6) \]
\[ \sigma_\varepsilon,0 \in H^{-\frac{3}{2}}(\mathbb{R}^3), \quad m_\varepsilon,0 \in H^{-\frac{1}{2}}(\mathbb{R}^3). \]
The main result of this section shows the strong convergence to 0 of the acoustic waves.
Theorem 4.1. Let \((\rho_\varepsilon, u_\varepsilon)\) be a finite energy weak solution of \((1.4)\). Then, for any \(0 < T < \infty\)

(i) the density fluctuations \(\rho_\varepsilon - 1\) converge strongly to 0 in \(C(0,T; L^2(\mathbb{R}^3))\) and \(L^1(0,T; H^s(\mathbb{R}^3))\) for any \(s \in (-\frac{3}{2}, 1)\),

(ii) If \(\gamma = 2\), then \(\sigma_\varepsilon\) converges strongly to 0 in \(L^2(0,T; L^q(\mathbb{R}^3))\) for any \(2 < q < 6\),

(iii) for any \(2 < q < \frac{9}{4}\) there exists \(\delta > 0\) such that \(Q(m_\varepsilon)\) converges strongly to 0 in \(L^2(0,T; B^0_{q,2}(\mathbb{R}^3))\).

In order to infer estimates on \((\sigma_\varepsilon, Qm_\varepsilon)\) by studying \((4.5)\), we derive Strichartz estimates for a symmetrization of the linearised system \((4.5)\) that will ultimately imply the convergence of \((\sigma_\varepsilon, Qm_\varepsilon)\). More precisely, we define

\[
\tilde{\sigma}_\varepsilon := (1 - \varepsilon^2 \kappa^2 \Delta)^{\frac{1}{2}} \sigma_\varepsilon, \quad m_\varepsilon := (-\Delta)^{-\frac{1}{2}} \text{div} m_\varepsilon,
\]

and check that if \((\sigma_\varepsilon, m_\varepsilon)\) is a solution of \((4.5)\) then \((\tilde{\sigma}_\varepsilon, m_\varepsilon)\) satisfies the symmetrised system

\[
\begin{cases}
\partial_t \tilde{\sigma}_\varepsilon + \frac{i}{\varepsilon} (-\Delta)^{\frac{1}{2}} (1 - \kappa^2 \varepsilon^2 \Delta)^{\frac{1}{2}} \tilde{m}_\varepsilon = 0, \\
\partial_t \tilde{m}_\varepsilon - \frac{i}{\varepsilon} (-\Delta)^{\frac{1}{2}} (1 - \kappa^2 \varepsilon^2 \Delta)^{\frac{1}{2}} \tilde{\sigma}_\varepsilon = \tilde{F}_\varepsilon,
\end{cases}
\]

where \(\tilde{F}_\varepsilon = (-\Delta)^{-\frac{1}{2}} \text{div} F_\varepsilon\). Hence, the linear evolution is characterised by the unitary semigroup \(e^{-itH_\varepsilon}\), where

\[
H_\varepsilon = \frac{1}{\varepsilon} \sqrt{(-\Delta)(1 - (\varepsilon \kappa)^2 \Delta)}
\]

is a self-adjoint operator with Fourier multiplier given by \((4.1)\). In what follows, we are going to provide a class of Strichartz estimates for the linear propagator \(e^{-itH_\varepsilon}\) which will yield a control of some mixed space-time norms of \((\tilde{\sigma}_\varepsilon, m_\varepsilon)\) in terms of the (scaled) Mach number. An interpolation argument exploiting the \textit{a priori} estimates introduced in Section 3 gives the final result. The Strichartz estimates are stated in the framework of Besov spaces, for the sake of conciseness we postpone their proof to the appendix [3].

Before stating the next Proposition, we recall that a pair of Lebesgue exponents \((p, q)\) is called Schrödinger admissible if \(2 \leq p, q \leq \infty\) and \(\frac{2}{p} + \frac{3}{q} = \frac{3}{2}\).

Proposition 4.2. Let \(\varepsilon > 0\), fix \(\alpha > 0\) arbitrarily small and let \((p, q)\), \((p_1, q_1)\) be two admissible pairs. Then the following estimates hold true

\[
\|e^{itH_\varepsilon} f\|_{L^p_t B^0_{q,2}} \leq C \varepsilon^\alpha \|f\|_{B^{\frac{3}{2}}_{2,2}},
\]

\[
\left\| \int_0^t e^{i(t-s)H_\varepsilon} F(s) ds \right\|_{L^p_t B^0_{q,2}} \leq C \varepsilon^\alpha \|F\|_{L^{p_1}_t B^{q_1}_{q_1,2}}.
\]

Proposition 4.2 will be proved in Appendix [3] in fact it will be a consequence of the more general Proposition [11.9].

Let us remark that the case \(\varepsilon = 1\) was already studied in [23], where the authors infer dispersive estimates for the propagator \(e^{-itH_1}\) in order to study scattering properties for the Gross-Pitaevskii equation. In our case we need to keep track of the \(\varepsilon\)-dependence of the estimates, in order to show the convergence to zero of the acoustic part. However, since \(H_\varepsilon = H_\varepsilon(\sqrt{-\Delta})\) is a non-homogeneous function of \(\sqrt{-\Delta}\), it is not possible to obtain a decay in \(\varepsilon\) by simply rescaling the estimates in [23]. This is for example different from what happens for classical fluids [10].
where the wave-like acoustic dispersion yields the convergence to zero by scaling
the estimates and by considering the fast dynamics for the fluctuations.

On the other hand here we can exploit that the Strichartz estimates associated
to the operator \( (4.5) \) are slightly better than the ones for the Schrödinger operator
close to the Fourier origin. This fact can also be seen in \( [23] \) for \( H_1 \). By exploiting
this regularizing effect, the estimates stated in Proposition \( (4.2) \) somehow improve a
class of similar estimates inferred in \( [11] \) in another context (the linear wave regime
for the Gross-Pitaevskii equation). Indeed the authors of \( [11] \) consider \( H_ε \) in two
different regimes: for low frequencies below the threshold \( \frac{1}{2} \), the operator behaves
like the wave operator, while above the threshold it is Schrödinger-like. In this way
the low frequency part experiences a derivative loss, due to the wave-type dispersive
estimates inferred.

Here we do not split \( H_ε \) in low and high frequencies, nevertheless we prove the
convergence to zero of the acoustic part by only losing a small amount of derivatives.

In order to apply the estimates \( (4.9), (4.10) \) to \( (4.7) \), we first need to bound its
right hand side in suitable spaces.

**Lemma 4.3.** If \((ρ_ε, u_ε)\) is a finite energy solution to \( (1.4) \), then one has,
(i) \( F_{ε,1} = \text{div} (ρ_ε u_ε \otimes u_ε + 4κ^2 \nabla \sqrt{ρ_ε} \otimes \nabla \sqrt{ρ_ε}) + (γ - 1) \nabla π_ε \in L^{∞}(0, T; B_{2,2}^{-s}(\mathbb{R}^3)) \),
for \( s > \frac{5}{2} \).
(ii) \( F_{ε,2} = 2ν \text{div} (ρ_ε D u_ε) \in L^{∞}(0, T; B_{2,2}^{-1}(\mathbb{R}^3)) \).

**Proof.** We shall use repeatedly the embeddings \( L^{1}(\mathbb{R}^3) ⊆ H^{-s}(\mathbb{R}^3) = B_{2,2}^{-s}(\mathbb{R}^3) \)
valid for \( s > \frac{5}{2} \). For the first statement, we observe that \( (ρ_ε u_ε \otimes u_ε + 4κ^2 (\nabla \sqrt{ρ_ε} \otimes \nabla \sqrt{ρ_ε}) + π_ε \in L^{∞}(0, T; L^{1}(\mathbb{R}^3)) \),
and thus \( F_{ε,1} \in L^{∞}(0, T; H^{-s}(\mathbb{R}^3)) \) for \( s > \frac{5}{2} \). In particular this implies \( F_{ε,1} \in L^{∞}(0, T; B_{2,2}^{-s}(\mathbb{R}^3)) \) for \( q ≥ 2 \) and \( s > \frac{5}{2} \).

Regarding the second statement, we observe that \( \|\sqrt{ρ_ε} S_ε\|_{L^{∞}(0, T; L^{2})} \leq C T \left(\|S_ε\|_{L^{∞}(0, T; L^{2})} + \|\sqrt{ρ_ε} - 1\|_{L^{2}(0, T; L^{∞})} + \|\sqrt{ρ_ε} S_ε\|_{L^{∞}(0, T; L^{2})}\right) \),
and thus \( F_{ε,2} \in L^{∞}(0, T; H^{-1}(\mathbb{R}^3)) \).

**Remark 4.4.** Here, we need to use Strichartz estimates in non-homogeneous spaces.
This is due to the fact that \( L^{1} \) has no embedding in any homogeneous Besov space
but \( B_{0}^{1,∞} \).

By combining the dispersive estimates of Proposition \( (4.2) \) and the bounds in
Lemma \( 4.3 \) we can then infer the convergence to zero of \((σ_ε, Q m_ε)\).

**Proposition 4.5.** Let \((σ_ε, m_ε)\) be solution of \( (1.4) \) with initial data \((σ_ε,0, m_ε, 0)\).
Then for any \( s \in \mathbb{R} \), and \( α > 0 \) arbitrarily small and for any admissible pairs
\((p,q),(p_1,q_1)\), the following estimate holds true
\begin{equation}
(4.11) \quad \| (σ_ε, Q(m_ε)) \|_{L^{p}_{t} B^{−s}_{q,2}} \leq C T \left( ε^α \| (σ_ε,0, m_ε, 0)\|_{H^s} + ε^α \| F_{ε} \|_{L^{p_1}_{t} B^{−\frac{s}{2}}_{q_1,2}} \right).
\end{equation}

Moreover, assume that \((ρ_ε, u_ε)\) is a finite energy weak solution of \( (1.4) \) and \((σ_ε, Q m_ε)\)
the respective solution of acoustic waves, then for any Schrödinger admissible pair
\((p,q)\) and any \( s > 5/2 \)
\begin{equation}
(4.12) \quad \| (σ_ε, Q(m_ε)) \|_{L^{p}_{t} B^{−s}_{q,2}} \leq C T ε^α.
\end{equation}
The condition $s > 5/2$ is due to the low regularity of the nonlinearity in (4.3).

Proof. The inequality (4.11) follows from (4.9), (4.10) and the observation that for any $s \in \mathbb{R}$ and $1 < q \leq \infty$ one has that

$$\|\sigma_\varepsilon\|_{L^p_t B^s_{q,2}} \leq C \|\tilde{\sigma}_\varepsilon\|_{L^p_t B^s_{q,2}},$$

and

$$\|Qm_\varepsilon\|_{L^p_t B^s_{q,2}} \leq C \|\tilde{m}_\varepsilon\|_{L^p_t B^s_{q,2}}.$$ \hfill (4.13)

Indeed, to check (4.13) we define the operator $T(f) = (1 - \varepsilon^2\kappa^2\Delta)^{-\frac{1}{2}} f$. By means of Bernstein inequalities and straightforward estimates on the derivatives of the symbol of $T$, we conclude that $T : B^s_{q,2} \to B^s_{q,2}$ is bounded for any $s \in \mathbb{R}$.

The inequality (4.12), follows from observing that the projection on the gradient part $Q$ is given by a matrix valued Fourier multiplier $m(\xi) = \frac{\xi_i \xi_j}{|\xi|^2}$ while the change of variables $(-\Delta)^{-\frac{1}{2}} \text{div}$ corresponds to the multiplier $\frac{\xi_i}{|\xi|}$. Hence, the operator $m_\varepsilon \mapsto Qm_\varepsilon$ is a Fourier multiplier of degree 0. Therefore for any $1 \leq q, r \leq \infty$ and $s \in \mathbb{R}$ one has

$$\|Qm_\varepsilon\|_{B^s_{q,r}} \leq C \|m_\varepsilon\|_{B^s_{q,r}}.$$ \hfill (4.14)

Similarly, it is easy to check that

$$\|\tilde{m}_{\varepsilon,0}\|_{B^s_{q,r}} \leq C \|m_{\varepsilon,0}\|_{B^s_{q,r}}.$$ and that

$$\|\tilde{\sigma}_{\varepsilon,0}\|_{B^s_{q,r}} \leq C \left( \|P_{\leq 0} (\sigma_{\varepsilon,0})\|_{B^s_{q,r}} + \|P_{> 0} (\varepsilon \nabla \sigma_{\varepsilon,0})\|_{B^s_{q,r}} \right),$$ where we recall that $\sigma_{\varepsilon,0} \in B^s_{2,2}$ for $s < -\frac{3}{2}$ and $\varepsilon \nabla \sigma_{\varepsilon,0} \in H^{-\frac{3}{2}}$ uniformly in $\varepsilon$. It remains to prove (4.12). From (4.0), we have that $\sigma_{\varepsilon,0}, m_{\varepsilon,0} \in H^2$ if $\tilde{s} = s - s_1 + s_0 < -\frac{3}{2}$ and $s - s_1 + s_0 < -\frac{3}{2}$ provided $\gamma > \frac{3}{2}$. Lemma 4.3 yields $F_{\varepsilon} = F_{\varepsilon,1} + F_{\varepsilon,2}$ with $F_{\varepsilon,1} \in L^\infty(0, \infty; B^\frac{-3}{2}_{q,2}(\mathbb{R}^3))$ for $s > -\frac{3}{2}$ and $F_{\varepsilon,2} \in L^2(0, T; B^\frac{-3}{2}_{q,2}(\mathbb{R}^3))$. Hence, for $(p_1, q_1) = (\infty, 2)$ we obtain

$$\|F_{\varepsilon}\|_{L^{p_1}(0, T; B^\frac{-3}{2}_{q_1,2})} \leq C_T,$$

provided $s > \frac{5}{2}$. Thus for any admissible pair $(p, q)$ and $s > \frac{5}{2}$, one has that

$$\|(\sigma_{\varepsilon, Q(m_{\varepsilon}))\|_{L^p_t B^{s-\frac{3}{2}}_{q,2}} \leq C_T \varepsilon^{\sigma_0} \left( \|\sigma_{\varepsilon,0}\|_{B^{s}_{q,2}} + \|F_{\varepsilon}\|_{L^1 B^\frac{-3}{2}_{q,2}} \right) \leq C_T \varepsilon^{\sigma_0}.$$ This completes the proof. \hfill \square

Proof of Theorem (4.1). The convergence of $\rho_{\varepsilon} - 1 = \varepsilon \sigma_{\varepsilon}$ towards 0 follows from the uniform bounds established in Lemma 3.3. The first statement is immediate and the second statement follows from the bound $\varepsilon \sigma_{\varepsilon} \in L^4(\mathbb{R}_+; H^1(\mathbb{R}^3))$ and $\sigma_{\varepsilon} \in L^\infty(0, T; H^{-\gamma}(\mathbb{R}^3))$ for $s > \frac{5}{2}$. By interpolation $\sigma_{\varepsilon} \to 0$ in $L^2(0, T; H^s(\mathbb{R}^3))$ for any $s \in (-\frac{3}{2}, 1)$. For the sake of completeness, we state that for any $2 \leq q \leq 6$,

$$\|\varepsilon \sigma_{\varepsilon}\|_{L^p B^{s-\frac{3}{2}}_{q,2}} \leq C \varepsilon^{(1-\theta)s_0},$$

where $s$ and $p$ are such that

$$s = \theta \left( \frac{9}{4} + \frac{5}{2q} \right) - \frac{11}{4} + \frac{1}{2q}, \quad \frac{1}{p} = \frac{\theta}{2} \left( \frac{3}{q} - 1 \right) + \frac{3}{2} \left( 1 - \frac{1}{q} \right).$$
To obtain a bound on $Qm_\varepsilon$, we interpolate between the \textit{a priori} bound \cite{30}, i.e. $m_\varepsilon \in L^{\frac{3}{\theta - s}}(0, T; H^s(\mathbb{R}^3))$ for any $0 \leq s \leq \frac{1}{2}$, and the inequality \cite{11,22},
\[\|\|\sigma_\varepsilon, Q(m_\varepsilon)\|\|_{L^p_tB^{1-s}\varepsilon_2} \leq C_T\varepsilon^\delta,\]
valid for any $\delta > \frac{s}{2}$ and for any $\delta > 0$. Bound \cite{30} implies that $m_\varepsilon \in L^{\frac{3}{\theta - 3s}}(0, T; B^{s-3(\frac{s}{2} - \frac{r}{q})}_q(\mathbb{R}^3))$ for any $q \geq 2$ and $0 \leq s \leq \frac{1}{2}$. By interpolation, we have that for $r = \theta(-\delta - \delta) + (1 - \theta)(s - 3(\frac{s}{2} - \frac{r}{q}))$ it holds
\[\|Q(m_\varepsilon)\|_{L^p_tB^{s}_{q,2}} \leq \left(\|Q(m_\varepsilon)\|^{\theta}_{B^{s}_{q,2}} \|Q(m_\varepsilon)\|^{1-\theta}_{B^{s}_{q,2}}\right)_{L^p_t}\]
\[\leq \|Q(m_\varepsilon)\|^{\theta}_{L^p_tB^{s}_{q,2}} \|Q(m_\varepsilon)\|^{1-\theta}_{L^p_tB^{s}_{q,2}}.\]
Hence, choosing $\delta = \frac{1}{2}\left(\frac{1}{2} - \frac{r}{q}\right)$, we look for $(\theta, s, r, q)$ such that $0 < \theta < 1$, $0 < s \leq \frac{1}{2}$, $2 < q \leq 6$ and moreover
\[\theta > \frac{\frac{1}{q} - \frac{1}{2q}}{s + \frac{3}{2q} + \frac{1}{2}} \iff r > 0,\]
as well as
\[\frac{1}{p} = \theta \left(\frac{s}{2} + \frac{3}{2q} - \frac{1}{2}\right) + \frac{3}{2} \left(\frac{1}{2} - \frac{1}{q}\right).\]
We compute that for $2 < q < \frac{3}{2}$ there exists $\delta > 0$ and $(\theta, s, r)$ such that the above requirements are met and moreover $r > \delta$ and $p \geq 2$. We find that
\[\|Qm_\varepsilon\|_{L^2(0,T;B^{s}_{q,2})} \leq C_T\varepsilon^{(1-\theta)\alpha}.\]
Finally, we show the bound for $\sigma_\varepsilon$ provided $\gamma = 2$. By Lemma \cite{23}, $\sigma_\varepsilon \in L^2(0, T; H^1(\mathbb{R}^3))$. Interpolation with \cite{11,22} yields that $\sigma_\varepsilon \to 0$ in $L^2(0, T; L^6(\mathbb{R}^3))$. \hfill \square

5. Convergence to the limiting system

The uniform bound $\rho_\varepsilon u_\varepsilon \in L^{\frac{3}{\theta - s}}(0, T; H^s(\mathbb{R}^3))$ for $0 \leq s \leq \frac{1}{2}$ shown in Proposition \cite{30} implies that up to passing to a subsequence there exists $u \in L^{\frac{3}{\theta - s}}(0, T; H^s(\mathbb{R}^3))$ such that $\rho_\varepsilon u_\varepsilon \to u$. We decompose $\rho_\varepsilon u_\varepsilon = m_\varepsilon = Q(m_\varepsilon) + P(m_\varepsilon)$ by means of the Helmholtz projection operator and analyse the convergence of the incompressible part $P(m_\varepsilon)$.

**Proposition 5.1.** Under the assumptions of Theorem \cite{24}, $P(m_\varepsilon)$ converges strongly to $u$ in $L^2(0, T; L^2(\mathbb{R}^3))$ as $\varepsilon$ goes to 0. Further, $m_\varepsilon$ converges strongly to $u$ in $L^2(0, T; L^2(\mathbb{R}^3))$.

**Proof.** From \cite{30,24}, we have $P(m_\varepsilon) \in L^p(0, T; H^s(\mathbb{R}^3))$ for $0 \leq s \leq \frac{1}{2}$ and $1 \leq p < \frac{4}{1+4s}$. Thus, there exists $\hat{u} \in L^p(0, T; H^s(\mathbb{R}^3))$ such that $P(m_\varepsilon) \rightharpoonup \hat{u}$ weakly in $L^p(0, T; H^s(\mathbb{R}^3))$. Moreover, from
\[\partial_t P(m_\varepsilon) + P(\text{div} (m_\varepsilon \otimes u_\varepsilon)) = 2\nu P(\text{div}(\sqrt{\rho_\varepsilon}S_\varepsilon)) + \kappa^2 P(\text{div}(\nabla \sqrt{\rho_\varepsilon} \otimes \nabla \sqrt{\rho_\varepsilon})),\]
with $S_\varepsilon$ defined in \cite{24}, we conclude that $\partial_t P(m_\varepsilon) \in L^2(0, T; H^{-s}(\mathbb{R}^3))$ for any $s > \frac{3}{2}$. Indeed, it suffices to observe that from the energy bounds of Lemma \cite{30} we have $\nabla \sqrt{\rho_\varepsilon} \in L^\infty(0, T; L^2(\mathbb{R}^3))$, $S_\varepsilon \in L^2(0, T; L^2(\mathbb{R}^3))$ and $\sqrt{\rho_\varepsilon} u_\varepsilon \in L^\infty(0, T; L^2(\mathbb{R}^3))$. In the virtue of the Aubin-Lions Lemma, we infer from $P(m_\varepsilon) \in L^p(0, T; H^s(\mathbb{R}^3))$
with $0 \leq s \leq \frac{1}{2}$ and $1 \leq p < \frac{4}{1+\delta}$ and from $\partial_t P(m_\varepsilon) \in L^2(0, T; H^{-s}(\mathbb{R}^3))$ for $s > \frac{5}{2}$ that if $\tilde{u}$ is the weak limit of $P(m_\varepsilon)$ then $P(m_\varepsilon) \rightharpoonup \tilde{u}$ strongly in $L^2(0, T; L^2_{loc}(\mathbb{R}^3))$.

We recall that $m_\varepsilon \rightharpoonup u$ weakly in $L^p(0, T; H^s(\mathbb{R}^3))$. In order to conclude that the sequence $\{m_\varepsilon\}$ converges strongly to $u$ in $L^2(0, T; L^2_{loc}(\mathbb{R}^3))$, i. e. $u = \tilde{u}$, it remains to show that $Q(m_\varepsilon) \rightharpoonup 0$ strongly in $L^2(0, T; L^2_{loc}(\mathbb{R}^3))$.

From Theorem 4.1, one has that $Q(m_\varepsilon)$ converges strongly to $0$ in $L^2(0, T; B^\delta_{q,2}(\mathbb{R}^3))$ for some $q > 2$ and $\delta > 0$. We notice that $B^\delta_{q,2}(\mathbb{R}^3)$ is continuously embedded in $L^q$, thus $Q(m_\varepsilon) \rightharpoonup 0$ in $L^2(0, T; L^q(\mathbb{R}^3))$ for some $q > 2$ and therefore in particular in $L^2(0, T; L^2_{loc}(\mathbb{R}^3))$. \hfill $\Box$

The strong convergence of $\sqrt{\rho_\varepsilon} - 1$ provided by Lemma 3.2 allows us to infer the strong convergence of $\sqrt{\rho_\varepsilon} u_\varepsilon$ to $u$.

**Corollary 5.2.** Under the assumptions of Theorem 2.2, $\sqrt{\rho_\varepsilon} u_\varepsilon$ converges strongly to $u$ in $L^2(0, T; L^2_{loc}(\mathbb{R}^3))$.

**Proof.** It suffices to consider for any compact $K \subset \mathbb{R}^3$,

\[
\|\sqrt{\rho_\varepsilon} u_\varepsilon - u\|_{L^2(0, T; L^2(K))} \leq \|\rho_\varepsilon u_\varepsilon - u\|_{L^2(0, T; L^2(K))} + \|(1 - \sqrt{\rho_\varepsilon})\sqrt{\rho_\varepsilon} u_\varepsilon\|_{L^2(0, T; L^2(K))} \\
\leq \|\rho_\varepsilon u_\varepsilon - u\|_{L^2(0, T; L^2(K))} + C\|1 - \sqrt{\rho_\varepsilon}\|_{L^q(0, T; L^q(\mathbb{R}^3))}\|\sqrt{\rho_\varepsilon} u_\varepsilon - u\|_{L^2(0, T; L^2(K))} \\
\leq C\left(\|\rho_\varepsilon u_\varepsilon - u\|_{L^2(0, T; L^2(K))} + \varepsilon^\beta\right),
\]

for some $\beta > 0$, where we used the convergence provided by Lemma 3.2 in the last step. \hfill $\Box$

The obtained compactness enables us to pass to limit in the weak formulation of the equations.

**Lemma 5.3.** Under the assumptions of Theorem 2.2, the limit function $u$ is a weak solution of (1.8) with initial data $u|_{t=0} = P(u_0)$.

**Proof.** Let $\phi \in C^\infty_c([0, T) \times \mathbb{R}^3; \mathbb{R}^3)$. We infer from Lemma 3.2 and Corollary 5.2 that passing to the limit $\varepsilon \to 0$ in the continuity equation

\[
\int_{\mathbb{R}^3} \rho_\varepsilon \phi_t(0) + \int_0^T \int_{\mathbb{R}^3} \rho_\varepsilon \phi_t + \sqrt{\rho_\varepsilon} \sqrt{\rho_\varepsilon} u_\varepsilon \nabla \phi = 0,
\]

yields $\text{div } u = 0$ in $D'(([0, T) \times \mathbb{R}^3))$. We consider the weak formulation of the momentum equation projected onto divergence free vector fields, let $\psi \in C^\infty_c([0, T) \times \mathbb{R}^3; \mathbb{R}^3)$ such that $\text{div } \psi = 0$, and consider

\[
\int_{\mathbb{R}^3} \rho_\varepsilon \psi_t(0) + \int_0^T \int_{\mathbb{R}^3} \sqrt{\rho_\varepsilon} \sqrt{\rho_\varepsilon} u_\varepsilon \psi_t + \left(\sqrt{\rho_\varepsilon} u_\varepsilon \otimes \sqrt{\rho_\varepsilon} u_\varepsilon\right) \nabla \psi \\
- 2\nu \int_0^T \int_{\mathbb{R}^3} \left(\sqrt{\rho_\varepsilon} u_\varepsilon \otimes \nabla \sqrt{\rho_\varepsilon}\right) \nabla \psi - 2\nu \int_0^T \int_{\mathbb{R}^3} \left(\nabla \sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon} u_\varepsilon\right) \nabla \psi \\
+ \nu \int_0^T \int_{\mathbb{R}^3} \sqrt{\rho_\varepsilon} \sqrt{\rho_\varepsilon} u_\varepsilon \Delta \psi - 4\kappa^2 \int_0^T \int_{\mathbb{R}^3} \left(\nabla \sqrt{\rho_\varepsilon} \otimes \nabla \sqrt{\rho_\varepsilon}\right) \nabla \psi = 0,
\]

(5.1)
Invoking Lemma 3.2 and Corollary 5.2 one concludes that the 6.3 converges to
\[
\int_{\mathbb{R}^3} P(u_0) \psi(0) + \int_0^T \int_{\mathbb{R}^3} u \psi_t + (u \otimes u) \nabla \psi + \nu \int_0^T \int_{\mathbb{R}^3} u \Delta \psi = 0.
\]
Moreover we used that \( \rho_{\varepsilon,0} u_{\varepsilon,0} \) converges weakly to \( u_0 \) in \( L^2_{\text{loc}}(\mathbb{R}^3) \) as consequence from (2.8) and Lemma 5.1. We conclude by recalling that \( \psi \) is divergence free. Therefore, there exists a function \( p \) defined on \((0, T) \times \mathbb{R}^3\) such that \( u \) is solution of
\[
\partial_t u + u \cdot \nabla u + \nabla p = 2\nu \Delta u, \quad \text{div} \, u = 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^3),
\]
with initial data \( P(u_0) \), where we recall that by (2.8) we assumed \( \sqrt{\rho_{\varepsilon} u_{\varepsilon}} \to u_0 \) in \( L^2(\mathbb{R}^3) \).

As we already said, at fixed \( \varepsilon > 0 \) the finite energy weak solutions \((\rho_{\varepsilon}, u_{\varepsilon})\) to (1.4) satisfy a weak version of the energy inequality due to the degenerate viscosity, namely
\[
E(t) + 2\nu \int_0^t |S_x|^2 \, dsdx \leq E(0),
\]
where \( S_x \) is given by (2.1). We remark that in fact in the limit as \( \varepsilon \to 0 \) it is possible to recover the usual energy dissipation. More precisely, the uniform boundedness of \( S_x \in L^2(0, T; L^2(\mathbb{R}^3)) \) only yields that \( S_x \to S \) weakly in \( L^2((0, T) \times \mathbb{R}^3) \). In the next Proposition we show that in fact we have \( S = \frac{1}{2}D u \). Moreover, by assuming the initial data to be well-prepared then by the convergence of the total energy at initial time we can also show that the limit function \( u \) is indeed a Leray solution.

**Proposition 5.4.** Under the assumptions of Theorem 2.2, let \( S_x \) be as defined in (2.1) then
\[
S_x \to D u \quad \text{in} \quad L^2((0, T) \times \mathbb{R}^3).
\]
Consequently, the limiting u solution to (1.1) satisfies \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)) \). If additionally, \((\rho_{\varepsilon}^0, u_{\varepsilon}^0)\) satisfies (2.4), then \( u \) is a Leray solution of (1.0), i.e. it satisfies (2.10).

**Proof.** In virtue of Lemma 5.3 the limit function \( u \) is a weak solution of (1.6) with initial data \( u|_{t=0} = P(u_0) \). Next, we show that \( S_x \to D u \) in \( L^2(0, T; L^2(\mathbb{R}^3)) \). From Lemma 3.2 one has that \( S_x \to S \) weakly in \( L^2((0, T) \times \mathbb{R}^3) \). Moreover, \( \sqrt{\rho_{\varepsilon} S_{\varepsilon}} \to S \) in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \). Indeed, let us write \( \sqrt{\rho_{\varepsilon} S_{\varepsilon}} = S_x + (\sqrt{\rho_{\varepsilon}} - 1)S_x \). The second term converges to \( 0 \) in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \) since \( \sqrt{\rho_{\varepsilon}} - 1 \to 0 \) strongly in \( L^\infty(0, T; L^4(\mathbb{R}^3)) \) for \( 2 \leq q < 6 \) from Lemma 3.2. On the other hand, from (2.1) we infer that \( \sqrt{\rho_{\varepsilon} S_{\varepsilon}} \to D u \) in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \). Indeed, from Proposition 5.1 we have \( \nabla (\rho_{\varepsilon} u_{\varepsilon}) \to \nabla u \) in \( \mathcal{D}'((0, T) \times \mathbb{R}^3) \) and from \( \nabla \sqrt{\rho_{\varepsilon}} \to 0 \) in \( L^2((0, T) \times \mathbb{R}^3) \) by Lemma 5.2 it follows \( \nabla \sqrt{\rho_{\varepsilon}} \otimes \sqrt{\rho_{\varepsilon}} u_{\varepsilon} \to 0 \) in \( L^2((0, T) \times \mathbb{R}^3) \). Thus \( S = D u \in L^2((0, T; L^2(\mathbb{R}^3)) \). We observe that for \( u \in H^1(\mathbb{R}^3) \) such that \( \text{div} \, u = 0 \), one has
\[
\int_{\mathbb{R}^3} |\nabla u|^2 \, dx = 2 \int_{\mathbb{R}^3} |D u|^2 \, dx.
\]
Finally, by lower semi-continuity we conclude that
\[
\int_{\mathbb{R}^3} \frac{1}{2} |u|^2 \, dx + \nu \int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \, dt
\leq \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^3} \frac{1}{2} \rho_\varepsilon |u_\varepsilon|^2 + \kappa^2 |\nabla \sqrt{\rho_\varepsilon}|^2 \, dx + 2\nu \int_0^T \int_{\mathbb{R}^3} |S_\varepsilon|^2 \, dx \, dt
\leq \int_{\mathbb{R}^3} \frac{1}{2} \rho_0^0 |u_0|^2 + \kappa^2 |\nabla \sqrt{\rho_0^0}|^2 + \pi_0^0 \, dx.
\]
Thus, \( u \in L^\infty(0, T; L^2(\mathbb{R}^3)) \) and \( \nabla u \in L^2(0, T; L^2(\mathbb{R}^3)) \). The additional uniform estimate is provided by Proposition 3.5. In order to conclude (2.10), it remains to show that,
\[
\int_{\mathbb{R}^3} \frac{1}{2} \rho_\varepsilon^0 |u_\varepsilon|^2 + \kappa^2 |\nabla \sqrt{\rho_\varepsilon^0}|^2 + \pi_\varepsilon^0 \, dx \to \int_{\mathbb{R}^3} \frac{1}{2} |u_0|^2 \, dx.
\]
If the initial data satisfies (2.9), the proof is complete. \( \square \)

Finally, we stress that, since the bounds obtained in Proposition 3.5 are uniform in \( \varepsilon > 0 \), they are also inherited by the solution to (1.6) obtained in the limit. Next Proposition proves the last statement of Theorem 2.2.

**Proposition 5.5.** Let \( u \) be the solution to (1.6) obtained in the limit. Then for any \( 0 < T < \infty \) it satisfies \( u \in L^p(0, T; H^s(\mathbb{R}^3)) \), with \( 0 \leq s \leq \frac{1}{2} \) and \( 1 \leq p < \frac{11}{11 + 4s} \).

**Appendix A. Energy and BD entropy inequality**

This Section is devoted to discussing Definition 2.1 of finite energy weak solutions for (1.4). More precisely we are concerned with the weaker version for the energy, BD entropy respectively, dissipation terms appearing in (2.5), (2.6) respectively. In [8, 33] the system (1.4) is studied in the two and three dimensional torus and they prove the existence of finite energy weak solutions for the QNS system. However it is not clear whether such solutions satisfy the energy inequality (2.3) and in general it is not true. On the other hand it is possible to show that the weaker version (2.5), and analogously (2.6), holds true. As it will be clear from the discussion below, this is due to the lack of control for the velocity field and its gradient in the vacuum region. For this reason when we pass to the limit the sequence of approximating solutions we can only infer \( \rho_\varepsilon |\nabla u_\varepsilon|^2 \to |S_\varepsilon|^2 \), see below for more details. Our setup in this paper is different, as we consider the full three dimensional space with non-trivial boundary conditions for the mass density. In fact, in this framework no existence result for finite energy weak solutions is available at the moment in the literature. This will in fact be subject of a forthcoming paper. However we think it could be useful for the reader to see how to prove that the finite energy weak solutions constructed in [8] satisfy (2.5) and (2.6).

We shall consider initial data \( (\rho_0^0, u_0^0) \) of finite energy, i.e. \( E(\rho_0^0, u_0^0) \leq +\infty \) satisfying the following assumptions,
\begin{align}
\rho^0 &\geq 0 \quad \text{in} \quad \mathbb{T}^3, \\
\rho &\in L^1(\mathbb{T}^3) \cap L^7(\mathbb{T}^3), \\
\nabla \sqrt{\rho} &\in L^2(\mathbb{T}^3),
\end{align}
(A.1)
Theorem A.1. Let \( u_0 = 0 \) on \( \{ \rho^0 = 0 \} \),
\[
\sqrt{\rho^0} u^0 \in L^2(T^3) \cap L^{2+}(T^3).
\]

**Theorem A.1.** Let \( d = 3 \). Let \( \nu, \kappa \) and \( \gamma \) positive such that \( \kappa^2 < \nu^2 < \frac{2}{3} \kappa^2 \) and \( 1 < \gamma < 3 \). Then for any \( 0 < T < \infty \) there exists a finite energy weak solution \( (\rho_\delta, u_\delta) \) of (A.1) on \( (0, T) \times T^3 \) with initial data \( (\rho^{0\delta}, u^{0\delta}) \) of finite energy satisfying (A.1) and (A.2). In particular, \( (\rho_\delta, u_\delta) \) satisfies (2.5) and (2.6). Moreover for a.e. \( 0 \leq s < t < T \) one has
\[
E(t) + \int_s^t \|S_\delta(t')\|^2 dt' \leq E(s).
\]

Firstly, we recall needed uniform estimates and compactness results obtained in [8] and secondly, we show Proposition A.7 and Proposition A.8 that imply Theorem A.1. The weak solution provided in [8] is obtained as limit of a sequence of approximating solutions \( \{ (\rho^{\delta}, u^{\delta}) \}_\delta \) satisfying the following system.

\[
\begin{cases}
\partial_t \rho^{\delta}_\varepsilon + \text{div} \rho^{\delta}_\varepsilon u^{\delta}_\varepsilon = 0 \\
\partial_t (\rho^{\delta}_\varepsilon u^{\delta}_\varepsilon) + \text{div} (\rho^{\delta}_\varepsilon u^{\delta}_\varepsilon \otimes u^{\delta}_\varepsilon) + \nabla ((\rho^{\delta}_\varepsilon)^\gamma + P_\delta(\rho^{\delta}_\varepsilon)) + \tilde{p}_\delta(\rho^{\delta}_\varepsilon) u^{\delta}_\varepsilon = \kappa^2 \text{div} K_\delta + 2\nu \text{div}(S_\delta),
\end{cases}
\]

with initial data
\[
\begin{aligned}
\rho^{\delta}_\varepsilon(0, x) = \rho^{\delta,0}_\varepsilon(x), \\
\rho^{\delta}_\varepsilon u^{\delta}_\varepsilon(0, x) = \rho^{\delta,0}_\varepsilon(x) u^{\delta,0}_\varepsilon(x).
\end{aligned}
\]

The approximating viscosity term is defined as
\[
S_\delta = h_\varepsilon(\rho_\varepsilon)D u_\varepsilon + g_\varepsilon(\rho_\varepsilon) \text{ div } u_\varepsilon I,
\]

with
\[
\begin{aligned}
h_\delta = \rho^{\delta,0}_\varepsilon + \delta(\rho^{\delta}_\varepsilon)^\gamma + \delta(\rho^{\delta}_\varepsilon)^\gamma, \\
g_\delta = \rho^{\delta,0}_\varepsilon h^{\delta}_\varepsilon(\rho^{\delta}_\varepsilon) - h_\delta(\rho^{\delta}_\varepsilon).
\end{aligned}
\]

The approximating dispersive term reads
\[
\begin{aligned}
\text{div } K_\delta = 2\rho^{\delta,0}_\varepsilon \nabla \left( h^{\delta}_\varepsilon(\rho^{\delta}_\varepsilon) \text{ div } h^{\delta}_\varepsilon(\rho^{\delta}_\varepsilon) \nabla \sqrt{\rho^{\delta}_\varepsilon} \right).
\end{aligned}
\]

**A.0.1. Initial data.** Next, we specify the initial data for which the system (A.4) is considered. Given initial data \( (\rho^{0\delta}, u^{0\delta}) \) of finite energy satisfying (A.1) and (A.2) one may construct a sequence of smooth initial data \( (\rho^{0\delta}, u^{0\delta}) \) such that
\[
\begin{aligned}
\rho^{\delta,0}_\varepsilon \rightarrow \rho^{0}_\varepsilon \quad &\text{strongly in } L^1(T^d), \\
\{ \rho^{\delta,0}_\varepsilon \}_\delta \quad &\text{uniformly bounded in } L^1 \cap L^7(T^d), \\
\{ h^{\delta}_\varepsilon(\rho^{\delta}_\varepsilon) \nabla \sqrt{\rho^{\delta}_\varepsilon} \}_\delta \quad &\text{uniformly bounded in } L^2 \cap L^{2+\eta}(T^d), \\
\delta(\rho^{\delta}_\varepsilon) \nabla \sqrt{\rho^{\delta}_\varepsilon} \rightarrow \nabla \sqrt{\rho^{0}_\varepsilon} \quad &\text{strongly in } L^2(T^d), \\
\{ \sqrt{\rho^{\delta}_\varepsilon(0)} u^{\delta}_\varepsilon \}_\delta \quad &\text{uniformly bounded in } L^2 \cap L^{2+\eta}(T^d), \\
\rho^{\delta,0}_\varepsilon u^{\delta}_\varepsilon \rightarrow \rho^{0}_\varepsilon u^{0}_\varepsilon \quad &\text{in } L^1(T^d), \\
f_\delta(\rho^{\delta}_\varepsilon) \rightarrow 0 \quad &\text{strongly in } L^1(T^d).
\end{aligned}
\]
In virtue of Theorem 6 in [8], there exists a global smooth solution to the Cauchy problem (A.4) equipped with initial data as specified as in (A.7).

**Proposition A.2.** Let \( \nu, \kappa > 0 \) such that \( \kappa^2 < \nu^2 < \frac{3}{2} \kappa^2 \) and \( \gamma \in (1,3) \). Then, for \( \delta > 0 \) sufficiently small, there exists a global smooth solution of (A.4) with initial data (A.5).

### A.1. A priori bounds

We provide the uniform estimates that will be needed subsequently, for their proof we refer the reader to [8]. These bounds are obtained from the energy equality and the Bresch-Desjardins entropy inequality for the system (A.4). The energy functional for the approximating system (A.4) is defined for \( t \in [0,T) \) as

\[
E_\delta(t) = \int_{\mathbb{R}^d} \frac{1}{2} \rho_\delta^2 |u_\delta|^2 + \kappa^2 |h_\delta'(\rho_\delta)| \nabla \sqrt{\rho_\delta^2}|^2 + \frac{1}{\varepsilon^2(\gamma - 1)} \rho_\delta^\gamma + \frac{1}{\varepsilon^2} f_\delta(\rho_\delta^2) \, dx.
\]

Firstly, we recall the energy inequality for the system (A.4).

**Lemma A.3.** Let \((\rho_\delta^2, u_\delta^2)\) be a global smooth solution of (A.4). Then for any \(0 \leq s < t \leq T\) and \((\rho_\delta^2, u_\delta^2)\),

\[
E_\delta(t) + 2\nu \int_s^t \int_{\mathbb{R}^d} h_\delta(\rho_\delta) |D u_\delta|^2 + g_\delta(\rho_\delta) |\nabla u_\delta|^2 + \tilde{p}_\delta(\rho_\delta) |u_\delta|^2 \, dx \, dt = E_\delta(s)
\]

Secondly, given a smooth solution \((\rho_\delta^2, u_\delta^2)\) of (A.4), we introduce the effective velocity \(v^\delta = u_\delta^2 + c\nabla \phi_\delta(\rho_\delta^2)\), for some suitable constant \(c\). Then \((\rho_\delta^2, v^\delta)\) is a smooth solution of the viscous Euler system

\[
\begin{align*}
\partial_t \rho_\delta^2 + \text{div}(\rho_\delta^2 v^\delta) &= c \Delta h_\delta(\rho_\delta^2), \\
\partial_t (\rho_\delta^2 v^\delta) + \text{div}(\rho_\delta^2 v^\delta \otimes v^\delta) + \nabla (\rho_\delta^2) \gamma + \lambda \nabla p_\delta(\rho_\delta^2) - c \Delta (h_\delta(\rho_\delta^2)v^\delta) + \tilde{p}(\rho_\delta^2) v^\delta \\
-2(\nu - c) \text{div}(h_\delta(\rho_\delta^2)D v^\delta) - 2(\nu - c) \nabla (g_\delta(\rho_\delta^2) \text{div} v^\delta) - \tilde{\kappa}^2 \text{div} K_\delta &= 0,
\end{align*}
\]

where the function \(\phi\) is defined as in [8]

\[
\begin{align*}
\mu &= \nu - \sqrt{\nu^2 - \kappa^2}, \\
\tilde{\kappa}^2 &= \kappa^2 - 2\nu c + c^2, \\
\tilde{\lambda} &= (\mu - c) / \mu.
\end{align*}
\]

The Bresch-Desjardins entropy is defined as

\[
B_\delta(t) = \int_{\mathbb{R}^d} \frac{1}{2} \rho_\delta^2 |v^\delta|^2 + \frac{(\rho_\delta^2) \gamma}{\varepsilon^2(\gamma - 1)} + \frac{\lambda}{\varepsilon^2} f_\delta(\rho_\delta^2) + 2\tilde{\kappa}^2 \left| h_\delta'(\rho_\delta^2) \nabla \sqrt{\rho_\delta^2} \right|^2.
\]

By Proposition 2 in [8], any smooth solution \((\rho_\delta^2, v^\delta)\) of (A.10) satisfies the related energy inequality.

**Lemma A.4.** Let \((\rho_\delta^2, u_\delta^2)\) be a global smooth solution of (A.4). Given \(c \in (0, \mu)\), the pair \((\rho_\delta^2, v^\delta)\) is a smooth solution of (A.10) and the BD entropy inequality is
satisfied,
(A.12)
\[
B_\delta(t) + c \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^\delta)|Av|^2 \, dx \, dt + (2\nu - c) \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^\delta)|Dv|^2 + g_\delta(\rho_\varepsilon^\delta)|\text{div} \, v|^2 \, dx \, dt
\]
\[
+ \int_0^t \int_{\mathbb{T}^d} \rho_\varepsilon^\delta |\nabla \rho_\varepsilon^\delta| \, dx \, dt + \frac{c'\gamma}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^\delta) |\nabla \rho_\varepsilon^\delta|^2 (\rho_\varepsilon^\delta)^{\gamma - 2} \, dx \, dt + c\lambda \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^\delta) |\nabla \rho_\varepsilon^\delta|^2 f_\delta''(\rho_\varepsilon^\delta) \, dx \, dt
\]
\[
+ c\kappa^2 \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^\delta) |\nabla \phi_\delta(\rho_\varepsilon^\delta)|^2 \, dx \, dt + c\kappa^2 \int_0^t \int_{\mathbb{T}^d} g_\delta(\rho_\varepsilon^\delta) |\Delta \phi_\delta(\rho_\varepsilon^\delta)|^2 \, dx \, dt
\]
\[
\leq B_\delta(0)
\]

We summarise the needed a priori bounds that are consequences of Lemma A.3 and Lemma A.4.

Lemma A.5. Let \((\rho_\varepsilon^\delta, u_\varepsilon^\delta)\) be a smooth solution of (A.4) with initial data satisfying \(\rho_\varepsilon^\delta > 0\) and assumptions in (A.12). Then there exists \(C > 0\) independent from \(\delta\) such that

\[
\sup_t \int \rho_\varepsilon^\delta |u_\varepsilon^\delta|^2 \, dx \leq C, \quad \sup_t \int |h_\delta(\rho_\varepsilon^\delta) \sqrt{\rho_\varepsilon^\delta}|^2 \, dx \leq C
\]

(A.13)

\[
\sup_t \int (\rho_\varepsilon^\delta (\rho_\varepsilon^\delta) \gamma) \, dx \leq C, \quad \int \int h_\delta(\rho_\varepsilon^\delta) |Du_\varepsilon^\delta|^2 \, dx \, dt \leq C
\]

(A.14)

\[
\sup_t \int f_\delta(\rho_\varepsilon^\delta) \, dx \leq C, \quad \int \int |\tilde{\rho}(\rho_\varepsilon^\delta)| |u_\varepsilon^\delta|^2 \, dx \, dt.
\]

(A.15)

In particular,

\[
\sup_t \int \left| \nabla \sqrt{\rho_\varepsilon^\delta} \right|^2 \, dx \leq C, \quad \int \int \rho_\varepsilon |Du_\varepsilon^\delta|^2 \, dx \, dt.
\]

(A.16)

Moreover, we recall the following convergence results from [8].

Lemma A.6. Let \((\rho_\varepsilon^\delta, u_\varepsilon^\delta)\) be a smooth solution of (A.4). Then

\[
h_\delta(\rho_\varepsilon^\delta) - \rho_\varepsilon^\delta \to 0 \quad \text{strongly in} \quad L^1((0, T) \times \mathbb{T}^d),
\]

\[
h_\delta(\rho_\varepsilon^\delta) \sqrt{\rho_\varepsilon^\delta} \to \sqrt{\rho_\varepsilon} \quad \text{strongly in} \quad L^2((0, T) \times \mathbb{T}^d),
\]

\[
h_\delta(\rho_\varepsilon^\delta) \nabla \sqrt{\rho_\varepsilon^\delta} \to \nabla \sqrt{\rho_\varepsilon} \quad \text{strongly in} \quad L^2((0, T) \times \mathbb{T}^d),
\]

(A.17)

\[
(\rho_\varepsilon^\delta)^{\gamma} \to \rho_\varepsilon^{\gamma} \quad \text{strongly in} \quad L^1((0, T) \times \mathbb{T}^d),
\]

\[
p(\rho_\varepsilon^\delta) \to 0 \quad \text{strongly in} \quad L^1((0, T) \times \mathbb{T}^d),
\]

\[
\tilde{\rho}(\rho_\varepsilon^\delta) \to 0 \quad \text{strongly in} \quad L^1((0, T) \times \mathbb{T}^d),
\]

\[
\sqrt{\rho_\varepsilon^\delta} u_\varepsilon^\delta \to \nabla \sqrt{\rho_\varepsilon} u_\varepsilon \quad \text{strongly in} \quad L^2((0, T) \times \mathbb{T}^d),
\]

Given the construction of the sequence of approximating solutions done in [8], in what follows we show that finite energy weak solutions \((\rho_\varepsilon, u_\varepsilon)\) obtained as limit of \((\rho_\varepsilon^\delta, u_\varepsilon^\delta)\) satisfy (2.2). The key-point consists in observing that, the energy dissipation for the approximating system (A.4) is bounded from below by

\[
\int_0^t \int_{\mathbb{T}^d} \rho_\varepsilon^\delta |Du_\varepsilon^\delta|^2 \, dx \, dt \leq \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^\delta) |Du_\varepsilon^\delta|^2 + g_\delta(\rho_\varepsilon^\delta) \, |\text{div} \, u_\varepsilon^\delta|^2 + \tilde{\rho}_\delta(\rho_\varepsilon^\delta) |u_\varepsilon^\delta|^2 \, dx \, dt,
\]

(A.18)
Indeed, by definition $\gamma > 1$ and

$$h_\delta(\rho_\varepsilon^d) \geq \rho_\varepsilon^d \geq 0, \quad g_\delta(\rho_\varepsilon^d) = \frac{1}{8} \varepsilon(\rho_\varepsilon^d) + \varepsilon(\gamma - 1)(\rho_\varepsilon^d)^\gamma.$$

Given a vector valued function $u$ such that $\nabla u \in L^2(T^d)$, it holds

$$\| \text{div } u \|_{L^2(T^d)} \leq C \sqrt{d} \| Du \|_{L^2(T^d)}.$$

Therefore,

$$\int_0^t \int_{T^d} h_\delta(\rho_\varepsilon^d)|Du_\varepsilon^d|^2 + g_\delta(\rho_\varepsilon^d)|\text{div } u_\varepsilon^d|^2 dx dt \geq \int_0^t \int_{T^d} \rho_\varepsilon^d|Du_\varepsilon^d|^2 + \varepsilon \left( (\rho_\varepsilon^d)^\gamma + (\rho_\varepsilon^d)^\gamma \right) |Du_\varepsilon^d|^2 - \frac{3}{8} \varepsilon(\rho_\varepsilon^d)^2 |Du_\varepsilon^d|^2 dx dt \geq \int_0^t \int_{T^d} \rho_\varepsilon^d|Du_\varepsilon^d|^2 dx dt.$$

Observing that $\hat{p}(\rho_\varepsilon^d)|u_\varepsilon^d|^2 \geq 0$, we conclude (A.13).

**Proposition A.7.** Let $(\rho_\varepsilon, u_\varepsilon)$ be a finite energy weak solution of (1.4) on $(0, T) \times T^3$ obtained as limit of a sequence $\{(\rho_\varepsilon^d, u_\varepsilon^d)\}$ smooth solution of (A.4). Then $(\rho_\varepsilon, u_\varepsilon)$ satisfies (2.5).

**Proof.** First, we observe that from (A.7) we conclude that

$$\int_{T^d} \frac{1}{2} \rho_\varepsilon^d |u_\varepsilon^d|^2 + \frac{k^2}{2} h_\delta(\rho_\varepsilon^d)|\nabla \rho_\varepsilon^d|^2 + \frac{1}{\varepsilon^2(\gamma - 1)}(\rho_\varepsilon^d)^\gamma + \frac{1}{\varepsilon^2} f_\delta(\rho_\varepsilon^d) dx \to \int_{T^d} \frac{1}{2} \rho_\varepsilon^d|u_\varepsilon^d|^2 + \frac{k^2}{2} |\nabla \rho_\varepsilon^d|^2 + \frac{1}{\gamma - 1}(\rho_\varepsilon^d)^\gamma dx.$$

Next, we notice that since $\sqrt{\rho_\varepsilon^d} Du_\varepsilon^d \in L^2(0, T; L^2(T^d))$, there exists $S_\varepsilon \in L^2(0, T; L^2(T^d))$ such that $\sqrt{\rho_\varepsilon^d} Du_\varepsilon^d \to S_\varepsilon$ weakly in $L^2(0, T; L^2(T^d))$ as $\delta \to 0$. By exploiting the lower semi-continuity of the energy functional, we conclude

$$\int_{T^d} \frac{1}{2} \rho_\varepsilon |u_\varepsilon|^2 + \frac{k^2}{2} |\nabla \rho_\varepsilon|^2 + \frac{1}{\gamma - 1} \rho_\varepsilon^d dx + 2\nu \int_0^t \int_{T^d} |S_\varepsilon|^2 dx dt \leq \liminf_{\delta \to 0} E_\delta(t) +$$

$$\leq \liminf_{\delta \to 0} E_\delta(t) + 2\nu \int_0^t \int_{T^d} h_\delta(\rho_\varepsilon^d)|Du_\varepsilon^d|^2 + g_\delta(\rho_\varepsilon^d)|\text{div } u_\varepsilon^d|^2 + \hat{p}(\rho_\varepsilon^d)|u_\varepsilon^d|^2 dx dt \leq \liminf_{\delta \to 0} E_\delta(0) = E(0).$$

It remains to check that $S_\varepsilon$ satisfies (2.4). From Lemma (A.6) we conclude that $\sqrt{\rho_\varepsilon^d} Du_\varepsilon^d \to \sqrt{\rho_\varepsilon} S_\varepsilon$ in $L^1((0, T) \times T^d)$. We are left to show that

$$\sqrt{\rho_\varepsilon} S_\varepsilon = (\nabla(\rho_\varepsilon u_\varepsilon)) - \nabla \sqrt{\rho_\varepsilon} \otimes \sqrt{\rho_\varepsilon} u_\varepsilon)^{sym} \text{ in } D'((0, T) \times T^d).$$

Let $\phi \in D((0, T) \times T^d)$ and consider

$$\langle \sqrt{\rho_\varepsilon^d} Du_\varepsilon^d, \phi \rangle = \langle (\nabla(\rho_\varepsilon^d u_\varepsilon^d))^{sym}, \phi \rangle - 2\langle \left( \sqrt{\rho_\varepsilon^d} \otimes \sqrt{\rho_\varepsilon^d} u_\varepsilon^d \right)^{sym}, \phi \rangle.$$
From \(\rho_\varepsilon^B u_\varepsilon^B \to \rho_x u_x\) in \(L^1((0, T) \times \mathbb{T}^d)\) as well as \(\nabla \sqrt{\rho_\varepsilon^B} \to \nabla \sqrt{\rho_x}\) and \(\sqrt{\rho_\varepsilon^B} u_\varepsilon^B \to \sqrt{\rho_x} u_x\) both strongly in \(L^2((0, T) \times \mathbb{T}^d)\) we conclude that

\[
(RHS) \to \nabla (\rho_x u_x)^{sym} - 2 \left( \nabla \sqrt{\rho_x^\varepsilon} \otimes \sqrt{\rho_x^\varepsilon} \right)^{sym}
\]

in \(D'((0, T) \times \mathbb{T}^d)\).

\[\Box\]

**Proposition A.8.** Let \((\rho_x, u_x)\) be a finite energy weak solution of (2.2) on \((0, T) \times \mathbb{T}^3\) obtained as limit of a sequence \(\{(\rho_\varepsilon^B, u_\varepsilon^B)\}\) smooth solution of (A.4). Then \((\rho_x, u_x)\) satisfies (2.6).

**Proof.** We denote by \(A_\varepsilon\) the weak \(L^2(0, T; L^2(\mathbb{T}^d))\) of \(\sqrt{\rho_\varepsilon^B} u_\varepsilon^B\). Similarly to the considerations made for \(S_\varepsilon\), we show that \(\sqrt{\rho_x} A_\varepsilon = \sqrt{\rho_x} T_x^{sym}\) with \(T_x\) as defined in (2.2). Thus,

\[
\|A_\varepsilon\|_{L^2(0, T; L^2(\mathbb{T}^d))} \leq \liminf_{\delta \to 0} \|\sqrt{\rho_\varepsilon} A_\varepsilon\|_{L^2(0, T; L^2(\mathbb{T}^d))} \leq \liminf_{\delta \to 0} \|\sqrt{h_\delta(\rho_\varepsilon^B)} A_\varepsilon\|_{L^2(0, T; L^2(\mathbb{T}^d))}
\]

Further, we have that \(\nabla^2 \sqrt{\rho_\varepsilon^B}\) converges weakly to \(\nabla^2 \sqrt{\rho_x}\) in \(L^2((0, T; L^2(\mathbb{T}^d))\) from Lemma A.6, thus by Lemma 5.3 in [8] we conclude that

\[
\|\nabla^2 \sqrt{\rho_x}\|_{L^2(0, T; L^2(\mathbb{T}^d))} \leq \liminf_{\delta \to 0} C \int_0^T \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^B) \left| \nabla^2 \phi_\delta(\rho_\varepsilon^B) \right|^2 + g_\delta(\rho_\varepsilon^B) \left| \Delta \phi_\delta(\rho_\varepsilon^B) \right|^2 dx dt.
\]

Moreover,

\[
c\gamma \int_0^t \int_{\mathbb{T}^d} \left| \nabla \rho_\varepsilon^B \right|^2 \rho_\varepsilon^B \leq \liminf_{\delta \to 0} c\gamma \int_0^t \int_{\mathbb{T}^d} h_\delta(\rho_\varepsilon^B) \left| \nabla \rho_\varepsilon^B \right|^2 (\rho_\varepsilon^B)^{\gamma/2} dx dt.
\]

By observing that \(B_\varepsilon(0) \to B(0)\) and exploiting lower semi-continuity of norms, we infer (2.6).

**Appendix B. Strichartz estimates for the acoustic wave system.**

The main purpose of this appendix is to give a proof of Proposition 4.2 (see Proposition B.8 below), that is we want to study the dispersive properties satisfied by solutions to system (4.7). Even if the paper only studies the three dimensional setting for the sake of completeness the whole analysis is carried out in the general \(d\)-dimensional setting \((d \geq 2)\).

As already mentioned, for \(\varepsilon = 1\), the dispersive analysis associated to the operator \(H_1\) has been carried out in [23, 24, 25]. In this paper, we need to carefully track down the \(\varepsilon\)-dependence on the estimates as the (scaled) Mach number \(\varepsilon\) not only determines a time scale but also a frequency threshold such that the operator behaves differently. This is due to the non-homogeneity of the dispersion relation and is opposite to the analysis of low Mach number limit in classical fluid dynamics where the Mach number \(\varepsilon\) only determines the time scale. The dispersive analysis for non-homogeneous symbols has been investigated in more general framework also in [20, 21, 22].

**B.1. Dispersive estimate.** In what follows we are going to prove the \(L^\infty - L^1\) dispersive estimate associated for the semigroup \(e^{it H_x}\). For the convenience of the reader, we recall the stationary phase estimate in (2.2).

**Proposition B.1** ([23]). Let \(\phi(r) \in C^\infty(0, \infty)\) satisfy the following.

(i) \(\phi'(r), \phi''(r) > 0\) for all \(r > 0\).
\( (ii) \) \( \phi'(r) \sim \phi'(s) \) and \( \phi''(r) \sim \phi''(s) \) for all \( 0 < s < r < 2s \).

\( (iii) \) \( |\phi^{(k+1)}(r)| \lesssim \frac{\phi^{(k)}(r)}{r^k} \) for all \( r > 0 \) and \( k \in \mathbb{N} \).

Let \( \chi(r) \) be a dyadic cut-off function with support around \( r \sim R \) and that satisfies

\[
|\chi^{(k+1)}(r)| \lesssim R^{-k}.
\]

These estimates are supposed to hold uniformly for \( r \) and \( R \), but may depend on \( k \). Then if

\[
I_\phi(t, x, R) := \int_{\mathbb{R}^d} e^{it\phi(|\xi|)+ix\cdot \xi} \, d\xi
\]

we have

\[
(B.1) \quad \sup_{x \in \mathbb{R}^d} |I_\phi(t, x, R)| \lesssim t^{-\frac{d}{2}} \left( \frac{\phi'(R)}{R} \right)^{-\frac{d-1}{2}} (\phi''(R))^{-\frac{1}{2}}
\]

Several observations are in order. We define

\[
(B.2) \quad h(r) = \det(\text{Hess}(\phi(r))),
\]

and exploiting that \( \phi \) is a radial function we compute

\[
h(r) = \left( \frac{\phi'(r)}{r} \right)^{d-1} \phi''(r),
\]

so that the right hand side of (B.1) involves \( h(R)^{-1/2} \). Furthermore, from Proposition 2 in [13] it follows that the dispersive estimate (B.1) is sharp in the sense that there exists \( t_0 \) and \( R_0 \) such that for all \( |t| > |t_0| \) and \( R > R_0 \) there exists \( C > 0 \) such that

\[
(B.3) \quad \sup_{x \in \mathbb{R}^d} |I_\phi(t, x, R)| \geq Ct^{-\frac{d}{2}} \left( \frac{\phi'(R)}{R} \right)^{-\frac{d-1}{2}} (\phi''(R))^{-\frac{1}{2}}.
\]

In [23], the estimate (B.1) has been applied to the pseudo-differential operator \( H_1 \), i.e. \( \phi(r) = r\sqrt{1 + \kappa^2 r^2} \). We remark that the dispersive estimate for the symbol \( \omega \) defined by the Bogliubov dispersion relation (4.1) can be obtained from estimate (B.1) by a rescaling. Indeed, by defining

\[
\phi_\varepsilon(r) = \frac{1}{\varepsilon^2} \phi(\varepsilon r).
\]

We have that \( \omega(r) = \phi_\varepsilon(r) \) and

\[
(B.4) \quad I_{\phi_\varepsilon}(t, x, R) := \int_{\mathbb{R}^d} e^{ix\xi + it\phi_\varepsilon(r)} \chi(r) d\xi = \varepsilon^{-d} I_\phi(t, x, C_\varepsilon R).
\]

Since the symbol \( \phi \) is non-homogeneous this rescaling affects the support of frequencies from being of order \( R \) to being of order \( \varepsilon R \) in addition to change the time scale. Finally, to track down the \( \varepsilon \)-dependence in the dispersive estimate it is enough to study the properties of the Hessian matrix of \( \phi_\varepsilon \).

**Corollary B.2.** Let \( \phi_\varepsilon(r) = \frac{1}{\varepsilon^2} r\sqrt{1 + (\varepsilon k)^2 r^2}, \) \( R > 0 \) be given and let \( \chi(r) \in C_\varepsilon(0, \infty) \) be as in Proposition [B.1]. Then there exists a constant \( C > 0 \) such that

\[
(B.5) \quad \sup_{x \in \mathbb{R}^d} |I_{\phi_\varepsilon}(x, t, R)| \leq Ct^{-\frac{d}{4}} \left( \frac{\phi_\varepsilon'(R)}{R} \right)^{-\frac{d-1}{2}} \phi_\varepsilon''(R)^{-\frac{1}{2}}
\]
In particular, this implies there exists $C > 0$ independent from $\varepsilon$ such that,

$$\sup_{x \in \mathbb{R}^d} |I_{\phi_{\varepsilon}}(x, t, R)| \leq C t^{-\frac{d}{2}} h(\varepsilon R)^{-\frac{1}{2}} \leq C' t^{-\frac{d}{2}}$$

**Proof.** For fixed $\varepsilon > 0$, the assumptions of Proposition B.1 on $\phi_{\varepsilon}(r)$ are met and (B.5) follows. By using (B.4) and (B.1) we obtain

$$\sup_{x \in \mathbb{R}^d} |I_{\phi_{\varepsilon}}(t, x, R)| = \varepsilon^{-d} \sup_{x \in \mathbb{R}^d} |I_{\phi_{\varepsilon}}(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varepsilon R)|$$

$$\leq C \varepsilon^{-\frac{d}{2}} \left( \frac{\varepsilon^2 (\varepsilon R)}{\varepsilon R} \right)^{-\frac{d-1}{2}} (\phi''(\varepsilon R))^{-\frac{1}{2}} = C \varepsilon^{\frac{d}{2}} h(\varepsilon R)^{-\frac{1}{2}}.$$

To conclude the second estimate, it is enough to show that for $d \geq 2$ there exists $C > 0$ such that $h(r)^{-\frac{1}{2}} \leq C$ uniformly on $(0, \infty)$. This will be proved in the next Lemma B.3.

In virtue of the dispersive estimate (B.6), we reduced the problem of tracking the $\varepsilon$-dependence to the study of the (scaled) function $h(r)$ as defined in (B.2).

**Lemma B.3.** Let $h$ be defined as in (B.2). There exists $C > 0$ such that for any $\lambda \in [0, \infty],

$$0 \leq h(\lambda)^{-\frac{1}{2}} \leq C \frac{1}{\kappa^2} \left( \frac{\kappa \lambda}{\sqrt{1 + (\kappa \lambda)^2}} \right)^{\frac{d-2}{2}}$$

For $d = 2$, there exists $C > 0$ such that for any $\lambda \in [0, \infty],

$$0 \leq h(\lambda)^{-\frac{1}{2}} \leq C.$$

**Proof.** This follows from immediate computations.

The estimate in (B.6) implies that the operator $H_{\varepsilon}$ has the same dispersive properties as the Schrödinger operator. As a consequence (B.6) would yield Schrödinger type dispersive estimates for frequency localized functions. However, from (B.7) we shall infer that in fact, for $d > 2$, we can derive better estimates, due to the regularizing effect of $\frac{\varepsilon^2 (\varepsilon R)}{\varepsilon R}$ when $\varepsilon r$ is small. This has already been pointed out in [23] for the operator $e^{itH}$ and is explained by a different curvature of the geometric surface $|\xi| \sqrt{1 + |\xi|^2}$ with respect to $|\xi|^2$. We reformulate this observation in the next Corollary.

**Corollary B.4.** Let $d \geq 2$ and let $\phi_{\varepsilon}(r) = \frac{1}{\varepsilon r} \sqrt{1 + (\varepsilon \kappa)^2 r^2}$ and $R > 0$ be given and let $\chi(r) \in C_c(0, \infty)$ be as in Proposition B.1. Then there exists a constant $C > 0$ such that for any $0 < t \leq T$ one has,

$$\sup_{x \in \mathbb{R}^d} |I_{\phi_{\varepsilon}}(t, x, R)| \leq C t^{-\frac{d}{2}} \left( \frac{\varepsilon \kappa R}{\sqrt{1 + (\varepsilon \kappa R)^2}} \right)^{\delta},$$

for any $0 \leq \delta \leq \frac{d-2}{2}$.

This motivates to define the pseudo-differential operator $K_{\varepsilon}$ corresponding to the Fourier multiplier

$$m(\varepsilon \xi) = \frac{(\varepsilon \xi \xi)}{\sqrt{1 + (\varepsilon \kappa |\xi|)^2}}, \quad \text{as} \quad K_{\varepsilon} := \frac{\sqrt{-\varepsilon^2 \Delta}}{\sqrt{1 - \varepsilon^2 \Delta}}.$$
In particular, this allows us to gain the factor $\varepsilon^d$ in the dispersive estimate \((4.10)\), at the expense of a factor $R^d$ corresponding to a loss of derivatives.

### B.2. Strichartz estimates

Next, we infer the needed Strichartz estimates from the dispersive estimate \((B.8)\). The Strichartz estimates follow from abstract results, see [29]. Here, the main interest consists in highlighting the $\varepsilon$-dependence of these estimates. To that end, we perform the $TT^\ast$-argument, see for instance [21]. First, we recall the definition of admissible exponents.

**Definition B.5.** We say the pair of exponents $(p, q)$ is Schrödinger admissible if $2 \leq p, q \leq \infty$, 
\[
\frac{2}{p} + \frac{d}{q} = \frac{d}{2}
\]
and $(p, q, d) \neq (2, \infty, 2)$.

The first step consists in showing a pointwise in time estimate.

**Lemma B.6** (Pointwise estimate). For fixed $\varepsilon > 0$ and $R > 0$, let $f \in L^1(\mathbb{R}^d)$ such that $\text{supp}(f) \subset \{ \xi \in \mathbb{R}^d : \frac{1}{2} R \leq |\xi| \leq 2R \}$. The following estimate holds for any $2 \leq q \leq \infty$:
\[
\| e^{itH_s} f \|_{L^q(\mathbb{R}^d)} \leq C t^{-d(\frac{1}{2} - \frac{1}{q})} \| U_\varepsilon^{\delta(1 - \frac{d}{2})} f \|_{L^{q'}(\mathbb{R}^d)},
\]
and consequently
\[
\| e^{itH_s} f \|_{L^q(\mathbb{R}^d)} \leq C t^{-d(\frac{1}{2} - \frac{1}{q})} (\varepsilon R)^{\delta(1 - \frac{d}{2})} \| f \|_{L^{q'}(\mathbb{R}^d)}.
\]

**Proof.** The operator $e^{itH_s}$ is unitary on $L^2$ therefore there exists $C_1 > 0$ not depending on $\varepsilon, R$ so that
\[
\| e^{itH_s} f \|_{L^2(\mathbb{R}^d)} \leq C_1 \| f \|_{L^2},
\]
Furthermore, Corollary [B.3] guarantees that there exists $C_2 > 0$ not depending on $\varepsilon, R$
\[
\| e^{itH_s} f \|_{L^\infty(\mathbb{R}^d)} \leq C_2 t^{\frac{d}{2}} \| U_\varepsilon^{\delta} f \|_{L^1(\mathbb{R}^d)}.
\]

By a standard interpolation argument we conclude the proof. Estimate \((B.10)\) follows from
\[
\left( \frac{\varepsilon R}{\sqrt{1 + \varepsilon R^2}} \right)^{\delta(1 - \frac{d}{2})} \leq (\varepsilon R)^{\delta(1 - \frac{d}{2})}.
\]

Next, we show Strichartz estimates localized in frequencies on dyadic blocks.

**Lemma B.7.** For $d \geq 2$, $\varepsilon, R > 0$ and $\frac{d-2}{2} > \delta > 0$, let $f \in L^2(\mathbb{R}^d)$ and $F \in L^p(0, T; L^{q'})$ such that $\text{supp}(\hat{f}), \text{supp}(\hat{F}(t)) \subset \{ \xi \in \mathbb{R}^d : \frac{1}{2} R \leq |\xi| \leq 2R \}$ Then for $m(r)$ defined as in \((B.9)\), there exists a constant $C > 0$ independent from $T, \varepsilon$ such that for any $(p, q), (p_1, q_1)$ admissible pairs,
\[
\| e^{itH_s} f \|_{L^p_x L^{q'}_t} \leq C m(\varepsilon R)^{\delta(\frac{d}{2} - \frac{1}{q})} \| f \|_{L^2},
\]
\[
\| \int_{\mathbb{R}} e^{-itH_s} F(t) dt \|_{L^2} \leq C m(\varepsilon R)^{\delta(\frac{d}{2} - \frac{1}{q})} \| F \|_{L^p_x L^{q'}_t}.
\]
Moreover,

\[
\left\| \int_{\mathbb{R}} e^{i(t-s)H_\varepsilon} F(s) ds \right\|_{L^p_t L^q_x} \leq C^2 m(\varepsilon R)^{\delta \left(1 - \frac{r}{q} - \frac{1}{p} \right)} \| F \|_{L^p_t L^q_x},
\]

(B.13)

\[
\left\| \int_{s<t} e^{i(t-s)H_\varepsilon} F(s) ds \right\|_{L^p_t L^q_x} \leq C^2 m(\varepsilon R)^{\delta \left(1 - \frac{r}{q} - \frac{1}{p} \right)} \| F \|_{L^p_t L^q_x}.
\]

(B.14)

**Proof.** Given (B.8) and considering the fact that \( e^{itH_\varepsilon} \) is an isometry on \( L^2(\mathbb{R}^d) \), we observe that Theorem 1 of [29] applies. We notice that the constants in the estimates (B.11) are identical as coming from an abstract duality argument. □

We remark that for \( \varepsilon = 1 \), we recover the Strichartz estimates provided by Theorem 2.1. in [23].

**Proposition B.8.** Let \( d \geq 2 \), \( \varepsilon > 0 \), fix \( \frac{d-2}{2} > \delta > 0 \) and let \( m \) be defined as in (B.9). Then there exists a constant \( C > 0 \) independent from \( T, \varepsilon \) such that for any \((p,q)\), \((p_1,q_1)\) admissible pairs,

\[
\| e^{itH_\varepsilon} f \|_{L^p_t B^0_{q_1}} \leq C \| U_\varepsilon^{\delta \left(\frac{d}{2} - \frac{q_1}{q} \right)} f \|_{L^2}
\]

Moreover for any \((p_1,q_1)\) admissible, we have

\[
\left\| \int_{s<t} e^{i(t-s)H_\varepsilon} F(s) ds \right\|_{L^p_t B^0_{q_1}} \leq C \| U_\varepsilon^{\delta \left(1 - \frac{r}{q} - \frac{1}{p} \right)} f \|_{L^p_t B^0_{q_1}}.
\]

(B.15)

**Proof.** By the scaling \( t' = \frac{t}{N^2} \) and \( x' = \frac{x}{N} \), for \( N \in \mathbb{N} \), we achieve that \( P_N(e^{itH_\varepsilon} f)(t',x') \) is spectrally supported in the annulus \( \{ \xi \in \mathbb{R}^d : \frac{1}{2} \leq |\xi| \leq 2 \} \). Therefore, we infer from (B.11) that

\[
\| P_N(e^{itH_\varepsilon} f) \|_{L^p_t L^q_x} = N^{\frac{1}{q} + \frac{2}{q}} \| P_1(e^{itH_\varepsilon} f) \|_{L^p_t L^q_x} \leq CN^{\frac{1}{q} + \frac{2}{q}} m(\varepsilon) \| P_1(f) \|_{L^2} \leq C m(\varepsilon N) R^{\delta \left(\frac{d}{2} - \frac{1}{q} \right)} \| P_N(f) \|_{L^2}
\]

for any admissible pair \((p,q)\). Similarly, the bound (B.13) implies that for admissible pairs \((p,q)\) and \((p',q')\), we have

\[
\left\| P_N \left( \int_{s<t} e^{i(t-s)H_\varepsilon} F(s) ds \right) \right\|_{L^p_t L^q_x} \leq C^2 m(\varepsilon R)^{\delta \left(1 - \frac{1}{q} - \frac{1}{p} \right)} \| P_N(F) \|_{L^p_t L^q_x}.
\]

Hence, given an admissible pair \((p, q)\), we compute

\[
\| e^{itH_\varepsilon} f \|_{L^p_t B^0_{q_1}} \leq \| N^s \| P_N(e^{itH_\varepsilon} f) \|_{L^p_t L^q_x} \|_{L^2} \leq C \| N^s m(\varepsilon) R^{\delta \left(\frac{d}{2} - \frac{1}{q} \right)} \| P_N(f) \|_{L^2} \|_{L^2} \leq C \| U_\varepsilon^{\delta \left(\frac{d}{2} - \frac{1}{q} \right)} f \|_{B^0_{q_1}},
\]
where we have used Minkowski inequality in the first and third inequality and (B.11) in the second. Similarly, we proceed for (B.12). Indeed,

\[
\left\| \int_{\mathbb{R}} e^{i(t-s)H_x} F(s) ds \right\|_{L^p B_{q,2}^{s-\frac{3}{2}}} = N^s \left\| \mathcal{P}_N \left( \int_{\mathbb{R}} e^{i(t-s)H_x} F(s) ds \right) \right\|_{L^p B_{q,2}^{s}} \leq C \left\| N^s \mu_N \delta \left( \frac{1}{2} - \frac{3}{2} \right) \| \mathcal{P}_N F \|_{L^p L^q_{\xi}} \right\|_{L^2} \leq C \left\| U_\varepsilon \delta \left( \frac{1}{2} - \frac{3}{2} \right) F \right\|_{L^p L^q_{\xi} B_{q,2}^{s-\frac{3}{2}}},
\]

\[\square\]

The final estimates follows upon observing that the presence of the operator \( U_\varepsilon \) may be exploited to gain a factor \( \varepsilon \) on the RHS of the estimates. Further, for our purpose we need the Strichartz estimates to hold in non-homogeneous Besov spaces.

**Proposition B.9.** Fix \( \varepsilon > 0, \) fix \( \frac{d-2}{2} > \delta > 0 \) and \( s \in \mathbb{R}. \) Let \( d \geq 2. \) There exists a constant \( C > 0 \) independent from \( T, \varepsilon \) such that for any \((p,q)\) admissible pair, the following hold true.

\[(B.16) \quad \| e^{itH_x} f \|_{L^p_t L^{\delta(s)\frac{d}{2}}_{\xi} B_{q,2}^{s-\frac{3}{2}}} \leq C \varepsilon^{\delta(s)\frac{d}{2}} \| f \|_{B_{q,2}^{s-\frac{3}{2}}}, \]

Moreover for any \( (p_1,q_1) \) admissible, we have

\[(B.17) \quad \left\| \int_{s<t} e^{i(t-s)H_x} F(s) ds \right\|_{L^p B_{q,2}^{\delta(s)}} \leq C \varepsilon^{\delta(s)} \left\| f \right\|_{L^p B_{q,2}^{\delta(s)}}. \]

**Proof.** From the Definition (B.3), one has \( \mu_\varepsilon (\xi) \leq C \varepsilon |\xi|. \) For \( \varepsilon \geq 1, \) this yields the following bound for the Fourier multiplier \( U_\varepsilon^\alpha, \)

\[\| U_\varepsilon^\alpha f \|_{B_{q,2}^{s}} \leq C \varepsilon^\alpha \| f \|_{B_{q,2}^{s+\alpha}}. \]

Hence, we conclude

\[(B.18) \quad \| e^{itH_x} f \|_{L^p B_{q,2}^{\delta(s)}} \leq C \varepsilon^{\delta(s)\frac{d}{2}} \| f \|_{B_{q,2}^{\delta(s)\frac{d}{2}}}. \]

Moreover for any \((p_1,q_1)\) admissible, we have

\[(B.19) \quad \int_{s<t} \left\| e^{i(t-s)H_x} F(s) ds \right\|_{L^p B_{q,2}^{\delta(s)}} \leq C \varepsilon^{\delta(s)\frac{d}{2}} \left\| f \right\|_{L^p B_{q,2}^{\delta(s)\frac{d}{2}}}. \]

For \( s > 0, \) one has that \( B_{q,r}^s \) is continuously embedded in \( \dot{B}_{q,r}^s \) if \( q \) is finite with

\[\| f \|_{\dot{B}_{q,r}^s} \leq \frac{C}{|s|} \| f \|_{B_{q,r}^s}, \]

while for \( s < 0 \) the space \( \dot{B}_{q,r}^s \) is continuously embedded in \( B_{q,r}^s \) with

\[\| f \|_{B_{q,r}^s} \leq \frac{C}{|s|} \| f \|_{\dot{B}_{q,r}^s}. \]
Using these embeddings and applying the estimates to \((1 - \Delta)^{\frac{\delta}{2}} f\) with \(\tilde{s} = s + \delta \left(1 - \frac{1}{q} - \frac{1}{q_1}\right)\), we obtain from (B.18) that
\[
\left\| e^{it\mathcal{H}_s} f \right\|_{L^p_t B^\beta_{q,2}} \leq C \epsilon \left(1 - \frac{1}{q} - \frac{1}{q_1}\right) \left\| f \right\|_{B^\beta_{q,2}}.
\]
Similarly, from (B.19), we conclude
\[
(B.20) \quad \left\| \int_{s < t} e^{i(t-s)\mathcal{H}_s} F(s) ds \right\|_{L^p_t B^0_{q,2}} \leq C \epsilon \left(1 - \frac{1}{q} - \frac{1}{q_1}\right) \left\| F \right\|_{L^p_t B^0_{q,2}}.
\]
Applying (B.20) to \((1 - \Delta)^{\frac{\tilde{s}}{2}} F\) with \(\tilde{s} = -s - \delta \left(1 - \frac{1}{q} - \frac{1}{q_1}\right)\) yields (B.17) \(\square\)

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