The Allison–Faulkner construction of $E_8$

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Abstract. We show that the Tits index $E_{8,1}^{133}$ cannot be obtained by means of the Tits construction over a field with no odd degree extensions. The proof uses a general method coming from the theory of symmetric spaces. We construct two cohomological invariants, in degrees 6 and 8, of the Tits construction and the more symmetric Allison–Faulkner construction of Lie algebras of type $E_8$ and show that these invariants can be used to detect the isotropy rank.

Tits in [24] proposed a general construction of exceptional Lie algebras over an arbitrary field of characteristic not 2 or 3, now called the Tits construction. The inputs are an alternative algebra and a Jordan algebra, and the result is a simple Lie algebra of type $F_4$, $E_6$, $E_7$, or $E_8$, depending on the dimensions of the algebras. The construction produces, say, all real forms of the exceptional Lie algebras, and a natural question is if all Tits indices can be obtained this way. Garibaldi and Petersson in [12] showed that it is not the case for type $E_6$, namely that Lie algebras of Tits index $2 E_6^{35}$ do not appear as a result of the Tits construction. We show a similar result for type $E_8$, namely that Lie algebras of Tits index $E_{8,1}^{133}$ cannot be obtained by means of the Tits construction, provided that the base field has no odd degree extensions. The proof uses the theory of symmetric spaces and the first author’s result with Semenov and Garibaldi about isotropy of groups of type $E_7$ in terms of the Rost invariant [13].

We prefer to use a more symmetric version of Tits construction due to Allison and Faulkner. Here, the input is a so-called structurable algebra with an involution (say, the tensor product of two octonion algebras) and three constants. The Lie algebra is given by some Chevalley-like relations. The Tits construction and the Allison–Faulkner construction have a large overlap but, strictly speaking, neither one is more general than the other. The Tits construction is capable of producing Lie algebras of type $E_8$ whose Rost invariant has a nonzero three-torsion part (necessarily using a Jordan division algebra as input), but the Allison–Faulkner construction of $E_8$ cannot do this—at least when the input is a form of the tensor product of two octonion algebras, because these can always be split by a two-extension of the base field. On the other hand, the Allison–Faulkner construction is capable of producing Lie algebras of type $E_8$ with the property that the two-torsion part of their Rost invariant has symbol length 3, and this is impossible for the Tits construction (see [8, II.6]). An $E_8$ with this
The Allison–Faulkner construction of $E_8$ property would necessarily come from what we call an indecomposable bi-octonion algebra, and these are related to some unusual examples of 14-dimensional quadratic forms discovered by Izhboldin and Karpenko [16].

We produce two new cohomological invariants, one in degree 6 and one in degree 8, and show that these invariants can be used to detect the isotropy rank of either the Tits or the Allison–Faulkner construction (but unlike the results of [12] for groups of type $E_6$, we give necessary conditions only). The main tool for constructing these invariants is a calculation of the Killing form of an Allison–Faulkner construction which, under a mild condition on the base field, is near to an eight-Pfister form (a so-called Pfister neighbour).

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1 Preliminaries

Let $K$ be a field of characteristic not 2 or 3. If $q$ is a quadratic form, we write $q(x, y) = q(x + y) - q(x) - q(y)$ for the associated symmetric bilinear form. Conversely, if $b$ is a symmetric bilinear form, then $q(x) = \frac{1}{2} b(x, x)$ is the associated quadratic form. This convention agrees with [22] but differs from, say, [17, Section VII.6]. If $A$ is an algebra and $a \in A$, we denote by $L_a, R_a \in \text{End}(A)$ the left- and right-multiplication operators, respectively.

1.1 Bi-octonion algebras

A $K$-algebra with involution $(A, -)$ is called a decomposable bi-octonion algebra if it has two octonion subalgebras $C_1$ and $C_2$ that are stabilised by the involution, such that $A = C_1 \otimes_K C_2$. A bi-octonion algebra is an algebra with involution $(A, -)$ that becomes isomorphic to a decomposable bi-octonion algebra over some field extension. These are important examples of central simple structurable algebras, as defined by Allison in [1], and they are instrumental in constructing Lie algebras of type $E_8$ (see Section 1.5).

Any bi-octonion algebra $(A, -)$ is either decomposable or it decomposes over a unique quadratic field extension $E/K$. In the latter case, there exists an octonion algebra $C$ over $E$, unique up to $K$-isomorphism, from which $(A, -)$ can be reconstructed as follows. Let $i$ be the nonidentity automorphism of $E/K$, and let $^tC$ be a copy of $C$ as a $K$-algebra, but with a different $E$-algebra structure given by $e \cdot z = i(e)z$. Then $(A, -)$ is precisely the fixed point set of $^tC \otimes_E C$ under the $K$-automorphism $x \otimes y \mapsto y \otimes x$, with the involution being the restriction of the tensor product of the canonical involutions on $^tC$ and $C$ [2, Theorem 2.1]. We denote this algebra by $(A, -) = N_{E/K}(C)$.

To unify the description of both decomposable and nondecomposable bi-octonion algebras, if we consider $C = C_1 \times C_2$ as an octonion algebra over the split quadratic étale extension $K \times K$, then $N_{K\times K/K}(C)$ as defined above is just isomorphic to $C_1 \otimes_K C_2$. 
1.2 Additive and multiplicative transfer of quadratic forms

Let \( E/K \) be a quadratic étale extension and \((q, V)\) an \( n\)-dimensional quadratic space over \( E \). The additive transfer of \((q, V)\) (also known as the trace or Scharlau transfer) is the \( 2n \)-dimensional \( K \)-quadratic space \((\text{tr}_{E/K}(q), V)\) defined by \( \text{tr}_{E/K}(q)(v) = \text{tr}_{E/K}(q(v)) \) for all \( v \in V \).

Rost defined a multiplicative transfer for quadratic forms, and it has been studied by him and his students (e.g., in \([18, 26]\)) and used before to define cohomological invariants. The multiplicative transfer also appeared (independently, it seems) in an old paper of Tignol \([23]\).

If \((q, V)\) is an \( n \)-dimensional quadratic space over a quadratic étale extension \( E/K \), one defines the quadratic space \((q', V')\) where \( i \) is the nontrivial automorphism of \( E/K \), \( V' \) is a copy of \( V \) as a \( K \)-vector space but with the action of \( E \) modified by \( i \), and \( q'(v) = i(q(v)) \). The multiplicative transfer \( N_{E/K}(q) \) of \( q \) is the \( n^2 \)-dimensional \( K \)-quadratic form obtained by restricting \( q \otimes_E q \) to the \( K \)-subspace of tensors in \( V \otimes_E V \) fixed by the switch map \( x \otimes y \mapsto y \otimes x \).

In the case of a split quadratic étale extension, a quadratic form over \( K \times K \) is just a pair \((q_1, q_2)\) where \( q_1, q_2 \) are quadratic forms over \( K \) of the same dimension, and we have \( \text{tr}_{E/K}(q_1, q_2) = q_1 \perp q_2 \) and \( N_{K \times K/K}(q_1, q_2) = q_1 \otimes q_2 \).

**Lemma 1.1** Let \((A, -) = N_{E/K}(C)\) for an octonion algebra \( C \) over a quadratic étale extension \( E/K \), and let \( n \) be the norm of \( C \). Then \( N_{E/K}(n) \) equals the normalised trace form \( (x, y) \mapsto \frac{1}{64} \text{tr}(L_{x+y}L_{y}) \).

**Proof** Both \( N_{E/K}(n) \) and the normalised trace form are invariant symmetric bilinear forms on \((A, -)\) in the sense that Allison defined (see \([1, \text{Theorem 17}] \) and \([2, \text{Proposition 2.2}] \)). By a theorem of Schafer \([19]\), a central simple structurable algebra has at most one such bilinear form, up to a scalar multiple. (As discussed in \([19, \text{pp. 116–117}] \), these facts are valid in characteristic 0 or \( p \geq 5 \), despite some of the original references being limited to characteristic 0.)

1.3 Lie-related triples

Let \((A, -)\) be a central simple structurable algebra over \( K \). A Lie related triple (in the sense of \([4, \text{Section 3}] \)) is a triple \( T = (T_1, T_2, T_3) \) where \( T_i \in \text{End}(A) \) and

\[
T_i\left(\overline{xy}\right) = T_j(x)y + xT_k(t)
\]

for all \( x, y \in A \) and all \((i, j, k)\) that are cyclic permutations of \((1, 2, 3)\). Define \( T \) to be the Lie subalgebra of \( \text{gl}(A) \times \text{gl}(A) \times \text{gl}(A) \) spanned by the set of related triples.

For \( a, b \in A \) and \( 1 \leq i \leq 3 \), define

\[
T_{a, b}^i = (T_1, T_2, T_3),
\]

where (taking indices mod 3):

\[
T_i = L_bL_a - L_aL_b,
\]

\[
T_{i+1} = R_bR_a - R_aR_b,
\]

\[
T_{i+2} = R_{[ab] - [aba]} + L_bL_a - L_aL_b.
\]
Let $\mathcal{T}_I$ be the subspace of $\text{End}(A)^3$ spanned by $\{T_{a,b}^i \mid a, b \in A, 1 \leq i \leq 3\}$. Since $(A, -)$ is structurable, $\mathcal{T}_I$ is a Lie subalgebra of $\mathcal{T}[4, \text{Lemma 5.4}]$. Finally, denote by $\text{Skew}(A, -) \subset A$ the $(-1)$-eigenspace of the involution, and let $\mathcal{T}'$ be the subspace of $\text{End}(A)^3$ spanned by triples of the form

$$(1.1) \quad (D, D, D) + (L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2}),$$

where $D \in \text{Der}(A, -)$ and $s_i \in \text{Skew}(A, -)$ with $s_1 + s_2 + s_3 = 0$.

**Example 1.2** Let $(C, -)$ be an octonion algebra with norm $n$. The principle of local triality holds in $\mathcal{T}_I$ in the sense that each of the projections $\mathcal{T}_I \to \text{gl}(C), (T_1, T_2, T_3) \mapsto T_i$, for $1 \leq i \leq 3$, is injective [22, Theorem 3.5.5]. The Lie algebra $\mathcal{T}_I$ is isomorphic to $\mathfrak{so}(n)$ [22, Lemma 3.5.2]. The $(i + 2)$th entry of the triple $T_{a,b}^i$ is $R_{a-b} + L_{a} - L_{b}$, and by [22, pp. 51, 54] this is the map $C \to C$ that sends

$$(1.2) \quad x \mapsto 2n(x, a) b - 2n(x, b) a.$$

**Proposition 1.3** If $(A, -)$ is a bi-octonion algebra of the form $(A, -) = N_{E/K}(C)$ for some quadratic étale extension $E/K$ and some octonion algebra $C$ over $E$, then $\mathcal{T}_I = \mathcal{T}_0 = \mathcal{T}' = \text{Lie}(R_{E/K}(\text{Spin}(n)))$, where $n$ is the norm of $C$.

**Proof** We have that $\mathcal{T}_I \subset \mathcal{T} \subset \mathcal{T}'$ and $\dim \mathcal{T}' = \dim \text{Der}(A, -) + 2 \dim \text{Skew}(A, -) = 28 + 28 = 56$ by [4, Corollary 3.5]. On the other hand, $\mathcal{T}_I$ (as an $E$-module) is precisely $\text{Lie}(\text{Spin}(n))$ [22, Theorem 3.5.5] and so $\mathcal{T}_I$ (as a $K$-vector space) is 56-dimensional and isomorphic to $\text{Lie}(R_{E/K}(\text{Spin}(n)))$.

### 1.4 Local triality

In the context of Proposition 1.3, the Lie algebra $\mathcal{T}_I$ is of type $D_4 + D_4$. Local triality holds here too: the projections $\mathcal{T}_I \to \text{gl}(A), (T_1, T_2, T_3) \mapsto T_i$ are injective for any $1 \leq i \leq 3$, and the symmetric group $S_3$ acts on $\mathcal{T}_I$ by $E$-automorphisms, where $E$ is the centroid of $\mathcal{T}_I$ (compare with [22, Section 3.5]).

### 1.5 The Allison–Faulkner construction [4, Section 4]

Let $(A, -)$ be a central simple structurable algebra and let $y = (y_1, y_2, y_3) \in K^\times \times K^\times \times K^\times$. For $1 \leq i, j \leq 3$ and $i \neq j$, define $A[ij] = \{a[ij] \mid a \in A\}$ to be a copy of $A$, and identify $A[ij]$ with $A[ji]$ by setting $a[ij] = -y_j y_i^{-1} a[ji]$. Define the vector space

$$K(A, - , y) = \mathcal{T}_I \oplus A[12] \oplus A[23] \oplus A[31]$$

and equip it with an algebra structure defined by the multiplication:

$$[a[ij], b[jk]] = -[b[jk], a[ij]] = ab[ik],$$
$$[T, a[ij]] = -[a[ij], T] = T_h(a)[ij]$$
$$[a[ij], b[ij]] = y_i y_j^{-1} T_{a,b}^i$$
for all \( a, b \in A \), \( T = (T_1, T_2, T_3) \in \mathcal{T}_F \), and \( (i j k) \) a cyclic permutation of \((1 2 3)\). Then \( K(A, -, y) \) is clearly a \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \)-graded algebra, and it is in fact a central simple Lie algebra \([4, \text{Theorems 4.1, 4.3, 4.4, and 5.5}]\).

### 1.6 Relation to the Tits–Kantor–Koecher construction

If the quadratic form \( (y_1, y_2, y_3) \) is isotropic then \( K(A, -, y) \cong K(A, -) \) where

\[
K(A, -) = \text{Skew}(A) \oplus A \oplus V_{A,A} \oplus A \oplus \text{Skew}(A)
\]

is the Tits–Kantor–Koecher construction \([3, \text{Corollary 4.7}]\). An isomorphism and its inverse are determined explicitly in \([3, \text{Theorem 2.2}]\) in the case where \(-y_1y_2^{-1} \) is a square. More generally, if \( (y_1, y_2, y_3) \) and \( (y_1', y_2', y_3') \) are similar quadratic forms, then \( K(A, -, y) \cong K(A, -, y') \) \([3, \text{Proposition 4.1}]\). In particular, if \( (A, -) \) is a bi-octonion algebra, then \( K(A, -) \) is a central simple Lie algebra of type \( E_8 \).

The range of Lie algebras of type \( E_8 \) that are of the form \( K(A, -) \) includes those with Tits index \( E_8^{91}, E_8^{66}, E_8^{28}, \) or \( E_8^0 \), and only those. We formulate a statement to this effect:

**Proposition 1.4** Let \( L \) be a Lie algebra of type \( E_8 \) corresponding to a class \( \varepsilon \in H^1(K, E_8) \). Then \( L \cong K(A, -) \) for some bi-octonion algebra \( (A, -) \) if and only if \( \varepsilon \) is in the image of \( H^1(K, \text{Spin}_{14}) \rightarrow H^1(K, E_8) \).

**Proof** Let \( L = \bigoplus_{i=-2}^2 L_i \) be the split \( E_8 \) Lie algebra with the \( \mathbb{Z} \)-grading indicated by (1.3). The subalgebra \( L_0 \) is reductive of type \( D_7 \); it is generated by a Cartan subalgebra and the root spaces of roots with \( -y_1y_2^{-1} \) a square. If \( (A, -) \) is an arbitrary bi-octonion algebra, then \( K(A, -) \) has the same \( \mathbb{Z} \)-grading as \( L \) so it must have been twisted by a cocycle coming from \( H^1(K, \text{Spin}_{14}) \). This proves the “only if” part of the statement.

For the “if” part of the statement, we prefer to make an argument using the Levi subgroup \( H \subset E_8 \) whose Lie algebra is \( L_0 \), rather than its semisimple subgroup \( \text{Spin}_{14} \). Nothing is gained or lost this way, because \( H^1(K, H) \) and \( H^1(K, \text{Spin}_{14}) \) have the same image in \( H^1(K, E_8) \); see \([25, \text{p. 657}]\). Specifically, \( H \) is the group generated by a maximal torus of \( E_8 \) and the root groups \( U_{\beta} \) where \( \beta \) has \( \alpha_1 \)-coordinate equal to zero. It acts faithfully on \( L \) by graded automorphisms, its representation on \( L_1 \) has a unique open orbit, and this orbit contains \( 1 \) (the identity in the split bi-octonions); see \([10, \text{p. 547}]\). The stabilizer of \( 1 \) is the automorphism group \( G \) of the split bi-octonion algebra \([5, \text{Corollary 8.6}]\). The map \( H^1(K, G) \rightarrow H^1(K, H) \) is surjective by the open orbit theorem from \([9, \text{pp. 28–29}]\). Consequently, any cocycle in the image of \( H^1(K, H) \rightarrow H^1(K, E_8) \) is also in the image of the map \( H^1(K, G) \rightarrow H^1(K, E_8) \) that sends the class of \( (A, -) \) to the class of \( K(A, -) \).

### 1.7 Relation to the Tits construction

Tits in \([24]\) defined the following construction of Lie algebras. Let \( C \) be an alternative algebra and \( J \) be a Jordan algebra. Denote by \( C^0 \) and \( J^0 \) the subspaces of elements of
generic trace zero and define operations $\circ$ and bilinear forms $(-,-)$ on $C^o$ and $J^o$ by the formula

$$ab = a \circ b + (a, b)1.$$  

Two elements $a$, $b$ in $J$ and $C$ define an inner derivation $(a, b)$ of the respective algebra, namely:

$$(a, b)(x) = \frac{1}{4}[[a, b], x] - \frac{3}{4}[a, b, x].$$

Then there is a Lie algebra structure on the vector space $\text{Der}(J) \otimes J^o \otimes C^o \otimes \text{Der}(C)$ defined by the formulas

$$[\text{Der}(J), \text{Der}(C)] = 0;$$
$$[B + D, a \otimes c] = B(a) \otimes c + a \otimes D(c);$$
$$[a \otimes c, a' \otimes c'] = (c, c')(a, a') + (a \circ a') \otimes (c \circ c') + (a, a')(c, c')$$

for all $B \in \text{Der}(J)$, $D \in \text{Der}(C)$, $a, a' \in J^o$, and $c, c' \in C^o$. If $(A, -) = C_1 \otimes C_2$ is a decomposable bi-octonion algebra, then $K(A, -, y)$ is isomorphic to the Lie algebra obtained via the Tits construction from the composition algebra $C_1$ and the reduced Albert algebra $\mathfrak{H}_3(C_2, y)$ [3, Remark 1.9 (c)].

**Proposition 1.5** Let $(A, -) = C_1 \otimes C_2$ be a decomposable bi-octonion algebra. Then $\mathcal{T}_f \oplus \Lambda[ij] \cong \mathfrak{so}(\langle y_i \rangle n_1 \perp \langle y_j^{-1} \rangle n_2)$, where $n_\ell$ is the norm of $C_\ell$.

**Proof** Consider the quadratic form $Q = \langle y_i \rangle n_1 \perp \langle y_j^{-1} \rangle n_2$ on the vector space $C_1 \oplus C_2$. The Lie algebra $\mathfrak{so}(Q)$ can be embedded into the Clifford algebra $C(Q)$ as the subspace spanned by elements of the form

$$[u, v]_c, \quad u, v \in C_1 \oplus C_2,$$

where $[-,-]_c$ denotes the commutator in the Clifford algebra (to avoid confusion with the commutators in $C_1$ and $C_2$). These generators satisfy the relations [17, p. 232 (30)]:

$$[[u, v]_c, [u', v']_c]_c = -2Q(u, u') [v, v']_c + 2Q(u, v') [v, u']_c$$
$$+ 2Q(v, u') [u, v']_c - 2Q(v, v') [u, u']_c.$$

If $z, z' \in C_1$ and $w, w' \in C_2$, this becomes

$$[[z, w]_c, [z', w']_c]_c = -2y_i n_1(z, z') [w, w']_c + 2y_j^{-1} n_2(w, w') [z, z']_c.$$

This implies that the 64-dimensional subspace spanned by

$$[z, w]_c, \quad z \in C_1, w \in C_2$$
generates the Lie algebra $\mathfrak{so}(Q)$. Now define a linear bijection $\theta : \mathfrak{so}(Q) \to T_I \oplus A[ij]$ by
\[
[z, z']_c \mapsto y_i T^i_{z, z'},
\]
\[
[w, w']_c \mapsto -y^{-1}_j T^i_{w, w'},
\]
\[
[z, w]_c \mapsto zw[ij]
\]
for all $z, z' \in C_1$ and $w, w' \in C_2$. By [17, p. 232 (31)] and (1.2), the restriction of $\theta$ to the subalgebra $[C_1, C_1]_c \oplus [C_2, C_2]_c \simeq \mathfrak{so}(\langle y_i \rangle n_1) \times \mathfrak{so}(\langle -y^{-1}_j \rangle n_2)$ is a homomorphism.

Now we calculate using (1.4) that
\[
\theta([[z, w], [z', w']], \cdot) = \theta(-2y_i n_1(z, z')\langle w, w' \rangle [z, z']_c + 2y^{-1}_j n_2(w, w')\langle z, z' \rangle [z, z']_c)
= -2y_i n_1(z, z')\theta([w, w']_c) + 2y^{-1}_j n_2(w, w')\theta([z, z']_c)
= 2y_i y^{-1}_j (n_1(z, z') T^i_{w, w'} + n_2(w, w') T^i_{z, z'})
\]
(1.5)

Meanwhile, we have
\[
[[z, w], [z', w']] = [zw[ij], z'w'[ij]] = y_i y^{-1}_j T^i_{zw, z'w'}
\]
(1.6)

To complete the proof that $\theta$ is an isomorphism, we show that the triples (1.5) and (1.6) are equal. It suffices to compare the $i$th entries of each triple (by §1.4). After recalling that
\[
L_x L_{x'} + L_{x'} L_x = L_{x'x} = n_1(x, x') \text{id}
\]
for all $x \in C_\ell$ [22, Lemma 1.3.3 (iii)], the $i$th entry of (1.5) is
\[
2y_i y^{-1}_j (n_1(z, z') L_{zw} - L_{zw} L_{z'} + n_2(w, w') L_{z} - L_{z} L_{z'})
= 2y_i y^{-1}_j (L_{zw} L_{z'} + L_{z} L_{zw} - L_{z} L_{zw} + n_2(w, w') L_{z} - L_{z} L_{zw} - L_{z} L_{zw} L_{z'})
= 2y_i y^{-1}_j (2L_{zw} L_{z'} L_{zw} - 2L_{z'} L_{zw} L_{z} + n_2(w, w') L_{z} - L_{z} L_{zw} - L_{z} L_{zw} L_{z'}).
\]

In the last line, we have used (multiple times) the fact that $C_1$ and $C_2$ commute and associate with each other in $A$. Using this fact a few more times, the $i$th entry of (1.6) is just
\[
4y_i y^{-1}_j (L_{zw} L_{z'} L_{zw} - L_{zw} L_{z'} + n_2(w, w') L_{zw} - L_{zw} L_{z'} L_{zw}).
\]

\[\blacksquare\]

2 The Killing form of $K(A, -, y)$

By our convention, the Killing form (as a quadratic form) on a Lie algebra $L$ is the form $x \mapsto \frac{1}{2} \text{tr}(\text{ad}_x^2)$. For any quadratic form $q = \langle x_1, \ldots, x_n \rangle$, the Killing form of $\mathfrak{so}(q)$ is
\[
(2 - n) \lambda^2(q),
\]
where $\lambda^2(q) = \bigwedge_{i<j} (x_i x_j)$ [9, Exercise 19.2].

**Lemma 2.1** Let $A = N_{E/K}(C)$ as before, and let $\rho_{ij} : R_{E/K}(\text{Spin}(n)) \to \text{GL}(A[ij])$ be the representation lifted from the representation of $T_I$ in $A[ij]$. Every quadratic form $q$ on $A$ invariant under this action of $R_{E/K}(\text{Spin}(n))$ is a scalar multiple of
the multiplicative transfer $N_{E/F}(n)$ (equivalently, a scalar multiple of the trace form $(x, y) \mapsto \text{tr}(L_{x}y + y_{x}))$.

**Proof** We can extend scalars from $F$ to $E$, and then $q_{E}$ is a quadratic form on $A_{E} = C \otimes_{E} \mathcal{O} C$, which is invariant under the action of $R_{E/F}(\text{Spin}(n)) \times F E = \text{Spin}(n) \times \text{Spin}(n)$. Then clearly $q_{E}$ decomposes as $q_{1} \otimes q_{2}$, for some $\text{Spin}(n)$-invariant form $q_{1}$ on $C$ and some $\text{Spin}(n)$-invariant form $q_{2}$ on $\mathcal{O} C$. This implies $q_{1} \simeq (\lambda_{1}) n$ and $q_{2} \simeq (\lambda_{2}) n$ for certain scalars $\lambda_{i} \in E^{\times}$, and therefore, $q_{E} = (\lambda_{1}\lambda_{2}) n \otimes \mathcal{O} C$. However, since $(q_{E}, A_{E})$ is extended from $(q, A)$ and $n \otimes \mathcal{O} C (1 \otimes 1) = 1$, we have $\lambda_{1}\lambda_{2} \in K^{\times}$. Therefore, $q = q_{E} |_{A} = (\lambda_{1}\lambda_{2}) N_{E/F}(n)$. 

We can now calculate the Killing form of $K(A, -, y)$ in the case where $(A, -)$ is a bi-octonion algebra.

**Proposition 2.2** If $(A, -) = N_{E/K}(C)$, then the Killing form on $K(A, -, y)$ is

\[
\text{(2.2)} \quad (-30) \left( \text{tr}_{E/F}(\lambda^{2}(n)) \downarrow \langle y_{1}y_{2}^{1-1}, y_{2}y_{3}^{1-1}, y_{3}y_{1}^{1-1} \rangle N_{E/F}(n) \right).
\]

**Proof** Let $\kappa$ be the Killing form of $K(A, -, y)$. If $x, y \in K(A, -, y)$ are from different homogeneous components in the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-grading, then $\text{ad}_{x} \text{ad}_{y}$ shifts the grading and consequently $\kappa(x, y) = \text{tr} \text{ad}_{x} \text{ad}_{y} = 0$.

Let $\tau$ be the Killing form of $\mathcal{T}_{I}$. The Killing form of $\text{Lie}(\text{Spin}(n))$ is $(-6) \lambda^{2}(n)$; see (2.1). Since $\mathcal{T}_{I} \simeq \text{Lie}(R_{E/F}(\text{Spin}(n)))$ by Proposition 1.3, we have

\[
\tau = \text{tr}_{E/F}((-6) \lambda^{2}(n)) = (-6) \text{tr}_{E/F}(\lambda^{2}(n)).
\]

There is an automorphism of $K(A, -, y) \otimes_{K} K^\text{alg}$ that swaps the two simple subalgebras of $\mathcal{T}_{I} \otimes_{K} K^\text{alg}$, and this implies $\kappa_{|\mathcal{T}_{I}}$ is a scalar multiple of $\tau$; say $\kappa_{|\mathcal{T}_{I}} = \langle \phi_{0} \rangle (-6) \text{tr}_{E/F}(\lambda^{2}(n))$ for some $\phi_{0} \in K^{\times}$.

Let us determine $\phi_{0}$. The grading on $K(A, -, y)$ makes it a sum of four $\mathcal{T}_{I}$-modules. For $T, S \in \mathcal{T}_{I}$ and $a \in A$,

\[
[T, [S, a[ij]]] = T_{k}(S_{k}(a))[ij].
\]

Therefore,

\[
\kappa(T, S) = \text{tr} \text{ad}_{T} \text{ad}_{S} = \tau(T, S) + \text{tr}(T_{1}S_{1}) + \text{tr}(T_{2}S_{2}) + \text{tr}(T_{3}S_{3}).
\]

The trace forms of the irreducible representations $\mathcal{T}_{I} \rightarrow \text{gl}(A)$, $T \mapsto T_{t}$ for $1 \leq t \leq 3$ are all equal (despite them being inequivalent representations) and so $\text{tr}(T_{1}S_{1}) = \text{tr}(T_{2}S_{2}) = \text{tr}(T_{3}S_{3})$ for all $T, S \in \mathcal{T}_{I}$. Moreover, $\text{tr}(T_{1}S_{1})$ is a scalar multiple of $\tau(T, S)$.

To determine the ratio between $\text{tr}(T_{1}S_{1})$ and $\tau(T, S)$, we can assume $A = C_{1} \otimes C_{2}$ is decomposable, and consider the subalgebra $\mathfrak{so}(n_{1}) \subset \mathfrak{so}(n_{1}) \times \mathfrak{so}(n_{2}) \simeq \text{Lie}(R_{E/F}(\text{Spin}(n)))$, where $n_{k}$ is the norm on $C_{k}$. It is well-known that the Killing form $\kappa_{1}$ on $\mathfrak{so}(n_{1})$ is $6 (= 8 - 2)$ times the trace form of its vector representation $\mathfrak{so}(n_{1}) \rightarrow \text{gl}(C_{1})$, while the trace form of the representation $\mathfrak{so}(n_{1}) \rightarrow \text{gl}(C_{1} \otimes C_{2})$ is clearly eight times the trace form of the vector representation. But $\kappa_{1}$ is equal to the restriction of the Killing form $\tau$ on $\mathfrak{so}(n_{1}) \times \mathfrak{so}(n_{2})$, so this means that (if $T \in \mathcal{T}_{I}$
belongs to the $\mathfrak{so}(n_1)$ subalgebra we have $\text{tr}(T_i^2) = 8 \text{tr}(T_i|_{C_i^2}) = \frac{8}{6} \kappa_i(T) = \frac{8}{6} \tau(T)$.

In conclusion, $\phi_0 = 5$, so $\kappa|_{\mathbb{T}_i} = (-30) \text{tr}_{E/K}(\Lambda^2(n))$.

The restriction $\kappa|_{A[ij]}$ is an invariant form under the action of $R_{E/K}(\text{Spin}(n))$, which means it is proportional to $N_{E/K}(n)$, by Lemma 2.1. Say $\kappa|_{A[ij]} = (\phi_{ij})N_{E/K}(n)$.

To determine the $\phi_{ij}$, it suffices to calculate $\kappa(1[ij])$, since $\kappa(1[ij]) = \phi_{ij}N_{E/K}(n)$ (1) $= \phi_{ij}$. By definition $\kappa(1[ij])$ is half the trace of $\text{ad}_{1[ij]}^2$. The graded components of $K(A, -, y)$ are invariant under $\text{ad}_{1[ij]}^2$, so we work out the trace separately for each of these components.

For all $b \in A$, we have

$$[1[ij], [1[ij], b[jk]]] = [1[ij], b[ik]] = -\gamma_i\gamma_j^{-1}[1[ij], b[ik]] = -\gamma_i\gamma_j^{-1}b[jk],$$

so $\text{ad}_{1[ij]}^2|_{A[ki]} = -\gamma_i\gamma_j^{-1} \text{id}$, and $\text{tr}(\text{ad}_{1[ij]}^2|_{A[ki]}) = -64\gamma_i\gamma_j^{-1}$. Similarly, for all $b \in A$,

$$[1[ij], [1[ij], b[ik]]] = (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})[1[ij], [1[ij], b[ik]]] = (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})[1[ij], \bar{b}[jk]] = (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})[\bar{b}[ik]] = (-\gamma_i\gamma_j^{-1})(-\gamma_k\gamma_i^{-1})(-\gamma_l\gamma_i^{-1})b[ki] = -\gamma_i\gamma_j^{-1}b[ki],$$

so $\text{ad}_{1[ij]}^2|_{A[ki]} = -\gamma_i\gamma_j^{-1} \text{id}$, and $\text{tr}(\text{ad}_{1[ij]}^2|_{A[ki]}) = -64\gamma_i\gamma_j^{-1}$. In contrast, for all $b \in A$,

$$[1[ij], [1[ij], b[ij]]] = [1[ij], \gamma_i\gamma_j^{-1}T_{1,b}^i] = -\gamma_i\gamma_j^{-1}(T_{1,b}^i)k(1) = -\gamma_i\gamma_j^{-1}(R_{b-\bar{b}} + L_b - L_{\bar{b}})(1) = 2\gamma_i\gamma_j^{-1}(b - \bar{b}).$$

Therefore, $\text{ad}_{1[ij]}^2|_{A[ij]}$ has a 50-dimensional kernel $\{a[ij] | \bar{a} = a\}$ and a 14-dimensional eigenspace $\{a[ij] | \bar{a} = -a\}$ with eigenvalue $-4\gamma_i\gamma_j^{-1}$. This proves that $\text{tr}(\text{ad}_{1[ij]}^2|_{A[ij]}) = -56\gamma_i\gamma_j^{-1}$.

Now if $T = (T_1, T_2, T_3) \in \mathbb{T}_i$, then

$$[1[ij], [1[ij], T]] = [1[ij], -T_k(1)][ij] = -\gamma_i\gamma_j^{-1}T_{1,T_k(1)}. $$

We can use (1.1) to write $T = (D, D, D) + (L_{s_2} - R_{s_3}, L_{s_3} - R_{s_1}, L_{s_1} - R_{s_2})$ for some unique $D \in \text{Der}(A, -)$ and $s_j \in \text{Skew}(A, -)$ such that $s_1 + s_2 + s_3 = 0$. Note that the kth entry of $T$ is $L_{s_k} - R_{s_k}$. Then $T_k(1) = D(1) + L_{s_k}(1) - R_{s_k}(1) = s_i - s_j$, so $\text{ad}_{1[ij]}^2(T) = -\gamma_i\gamma_j^{-1}T_{1,T_k(1)}$ is the triple whose kth entry is

$$-\gamma_i\gamma_j^{-1}(R_{1T_k(1)-T_{1}(1)} + L_{T_k(1)}L_1 - L_1L_{T_k(1)}) = -\gamma_i\gamma_j^{-1}(R_{2(s_i-s_j)} + L_{2(s_i-s_j)}) = -2\gamma_i\gamma_j^{-1}(L_{s_i} - R_{s_j} - (L_{s_j} - R_{s_i})).$$

This shows $\ker(\text{ad}_{1[ij]}^2|_{\mathbb{T}_i})$ is the 42-dimensional subspace of $\mathbb{T}_i$ whose kth projection is

$$\{ D + L_s - R_{s} \mid D \in \text{Der}(A, -), s \in \text{Skew}(A, -) \}.$$. 
And the subspace of $\mathcal{T}_I$ whose $k$th projection is
$$\{ L_s + R_s \mid s \in \text{Skew}(A, -) \}$$
is a 14-dimensional eigenspace of $\text{ad}_{[i,j]}[\tau_i]$ with eigenvalue $-4y_i y_j^{-1}$. This proves that $\text{tr}(\text{ad}_{[i,j]}[\tau_i]) = -56y_i y_j^{-1}$. Therefore,
$$\phi_{ij} = \kappa(1[i,j]) = \frac{1}{2} \text{tr}(\text{ad}_{[i,j]}[\tau_i]) = -32y_i y_j^{-1} - 32y_i y_j^{-1} - 28y_i y_j^{-1} - 28y_i y_j^{-1} = -120y_i y_j^{-1},$$
and we can simplify to get (2.2) because 30 is in the same square class as 120.

If $\text{char}(K) = 5$, then the Killing form on $E_8$ is zero. However, if $(A, -) = N_{E/K}(C)$ then the symmetric bilinear form on $K(A, -)$ associated to
$$\text{tr}_{E/K}(\lambda^2(n)) \perp \langle y_1 y_2^{-1}, y_2 y_3^{-1}, y_3 y_4^{-1} \rangle N_{E/K}(n) \tag{2.3}$$
is nondegenerate and Lie invariant. This can be proved in at least two ways: one can factor out $-30$ in the Killing form of the Chevalley Lie algebra of type $E_8$ defined over $\mathbb{Z}$, extend the new bilinear form to the split $E_8$ over $K$, and then twist it to get the form (2.3) on $K(A, -)$. This form is clearly invariant and nondegenerate (its radical is a nonzero ideal and $E_8$ is a simple Lie algebra in all characteristics). Alternatively, one use the hint from [11, Exercise 27.21 (2)]: lift the Killing form of $K(A, -)$ to the ring of Witt vectors, divide by $-30$ up there, and reduce modulo 5 to get (2.3).

**Lemma 2.3** Let $(A, -) = N_{E/K}(C)$, and let $\kappa'$ be a nondegenerate Lie invariant bilinear form on $K(A, -)$. If $-1$ is a sum of two squares in $K$, then $\kappa' \in I^6(K)$ and there is a unique 64-dimensional form $q \in I^6(K)$ such that $q + \kappa' \in I^6(K)$.

**Proof** Since $\kappa'$ is unique up to a scalar multiple, we can assume without loss of generality that
$$\kappa' = \text{tr}_{E/K}(\lambda^2(n)) \perp \langle y_1 y_2^{-1}, y_2 y_3^{-1}, y_3 y_4^{-1} \rangle N_{E/K}(n).$$
The assumption that $-1$ is a sum of two squares is equivalent to the identity $4 = 0$ in the Witt ring $W(K)$. This assumption implies that $\text{tr}_{E/K}(\lambda^2(n)) = 0$ [9, Lemma 19.8] and also that $N_{E/K}(n) \in I^6(K)$ [18, 26, Satz 2.16 (ii)], hence $\kappa' \in I^6(K)$. Setting $q = N_{E/K}(n)$ yields
$$q + \kappa' = \{1, y_1 y_2^{-1}, y_2 y_3^{-1}, y_3 y_4^{-1}\} N_{E/K}(n) = \langle -y_1 y_2^{-1}, -y_2 y_3^{-1} \rangle N_{E/K}(n) \in I^6(K).$$

The uniqueness of $q$ follows from the Arason–Pfister Hauptsatz.

Let $Q(*) \subset R(*) \subset H^1(*, E_8)$ be the functors $\text{Fields}_{/K} \to \text{Sets}$ such that for all fields $F/K$:

1. $Q(F)$ is the set of isomorphism classes of Lie algebras of type $E_8$ that are isomorphic to $K(A, -)$ for some bi-octonion algebra $(A, -)$ over $F$ and some $\gamma = (\gamma_1, \gamma_2, \gamma_3) \in (K^*)^3$; i.e. $Q(F)$ is the image of the Allison–Faulkner construction
$$H^1(F, (G_2 \times G_2 \times \mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})^3) \to H^1(F, E_8).$$
Recall from 1.7 that \( Q(\ast) \) contains all Lie algebras of type \( E_8 \) that are obtainable using the Tits construction from a reduced Albert algebra and an octonion algebra. Whereas, \( R(\ast) \) strictly contains all Lie algebras of type \( E_8 \) that are obtainable using the Tits construction from an Albert algebra (even a division algebra) and an octonion algebra. Any cohomological invariant \( Q(\ast) \rightarrow \bigoplus_{i \geq 0} H^i(\ast, \mathbb{Z}/2\mathbb{Z}) \) can be extended uniquely to a cohomological invariant \( R(\ast) \rightarrow \bigoplus_{i \geq 0} H^i(\ast, \mathbb{Z}/2\mathbb{Z}) \) [9, Section 7].

By applying the quadratic form invariants \( e_n : I^n(\ast) \rightarrow H^n(\ast, \mathbb{Z}/2\mathbb{Z}) \) for \( n = 6 \) and 8, we obtain cohomological invariants of the Tits construction and the Allison–Faulkner construction.

**Corollary 2.4** If \(-1\) is a sum of two squares in \( K \), then there exist nontrivial cohomological invariants

\[
\begin{align*}
  h_6 : R(\ast) & \rightarrow H^6(\ast, \mathbb{Z}/2\mathbb{Z}), \\
  h_8 : R(\ast) & \rightarrow H^8(\ast, \mathbb{Z}/2\mathbb{Z}),
\end{align*}
\]

such that if \((A, -) = N_{E/F}(C)\), then

\[
\begin{align*}
  h_6(K(A, -, y)) & = e_6(N_{E/F}(n)), \\
  h_8(K(A, -, y)) & = (-\gamma_1\gamma_2^{-1}) \cup (-\gamma_2\gamma_3^{-1}) \cup e_6(N_{E/F}(n)).
\end{align*}
\]

### 2.1 Comparison with invariants of \( G_2 \times F_4 \)

Since \( R(K) \) contains the image of the Tits construction \( H^1(K, G_2 \times F_4) \rightarrow H^1(K, E_8) \), there are unique cohomological invariants

\[
\begin{align*}
  h_i^* : H^i(\ast, G_2 \times F_4) & \rightarrow H^i(\ast, \mathbb{Z}/2\mathbb{Z}) \quad i = 6, 8,
\end{align*}
\]

such that \( h_i^*(C, J) = h_i(L) \) where \( L \) is the Lie algebra of type \( E_8 \) constructed from the octonion algebra \( C \) and Albert algebra \( J \). The cohomological invariants of \( G_2 \) and \( F_4 \) are classified [11]. The unique nontrivial invariant \( e_3 \) of \( G_2 \) assigns an octonion algebra \( C \) to the class \( e_3(C) = (a_1) \cup (a_2) \cup (a_3) \), where \( \langle a_1, a_2, a_3 \rangle \) is the norm of \( C \). The unique nontrivial mod 2 invariants \( f_3, f_5 \) of \( F_4 \) assign a reduced Jordan algebra \( \mathcal{H}_3(C, \gamma) \) to the classes \( f_3(\mathcal{H}_3(C, \gamma)) = e_3(C) \), and \( f_5(\mathcal{H}_3(C, \gamma)) = (-\gamma_1\gamma_2^{-1}) \cup (-\gamma_2\gamma_3^{-1}) \cup e_3(C) \), respectively (see [11, Section 22] and [22, p. 118]). Comparing with Corollary 2.4 and using the fact that \( e_6(N_{K_{x/K}(n_1, n_2)}) = e_6(n_1 \otimes n_2) = e_3(n_1) \cup e_3(n_2) \) yields

\[
\begin{align*}
  h_i^*(C_1, \mathcal{H}_3(C_2, \gamma)) = h_i(K(C_1 \otimes C_2, -)) = e_3(C_1) \cup f_{i-3}(\mathcal{H}_3(C_2, \gamma))
\end{align*}
\]

for all pairs of octonion algebras \( C_1, C_2 \) and scalars \( \gamma_1, \gamma_2, \gamma_3 \). If two invariants with values in \( H^i(\ast, \mathbb{Z}/2\mathbb{Z}) \) agree up to odd-degree extensions, then they are equal, so it follows that

\[
\begin{align*}
  h_6^*(C, J) & = e_3(C) \cup f_3(J), \\
  h_8^*(C, J) & = e_3(C) \cup f_5(J)
\end{align*}
\]

for all octonion algebras \( C \) and Albert algebras \( J \).
3 Isotropy of Tits construction

In this section, we continue to assume that the base field $K$ is of characteristic not 2 or 3.

3.1 Generalities on symmetric spaces

We use the basics of the theory of symmetric spaces over arbitrary fields; we refer to [15] for the generalities. Let $G$ be a (connected) split reductive algebraic group over a field $K$ and $\sigma$ be an involution on $G$ (that is an automorphism of order 2). Then the fixed point subgroup $H = G^\sigma$ has a reductive connected component $H^\circ$; in the case when $\sigma$ is from $G(K)$ and the commutator subgroup of $G$ is simply connected, $H$ is connected and has the same rank as $G$ (see [14, Théorème 3.1.5]). We state some facts about its normalizer in the lemma below.

**Lemma 3.1**

1. $N_G(H) = N_G(H^\circ)$;
2. $g \in G$ belongs to $N_G(H)$ if and only if $\sigma(g)g^{-1}$ belongs to the center of $G$;
3. If $\sigma$ is from $G(K)$ and $T$ is a $\sigma$-stable split maximal torus in $G$, then the map $N_H(T)/T \to H/H^\circ$ is surjective.

**Proof** The first two items are from [15, Corollary 1.3], and the third is [14, Lemme 3.1.4]. Note that $T$ as above always exists by [15, Proposition 2.3].

A torus $S$ in $G$ (not necessary maximal) is called $\sigma$-split if $\sigma(t) = t^{-1}$ for all $t \in S$. In the particular case $S = \mathbb{G}_m$, $S$ defines two opposite parabolic subgroups in $G$; they are also called $\sigma$-split and are characterized by the fact that $\sigma$ sends a $\sigma$-split parabolic subgroup to an opposite parabolic subgroup. Possible types of $\sigma$-split maximal parabolic subgroups correspond to the white vertices on the Satake diagram of $(G, \sigma)$, see [21, Lemma 2.9 and 2.11].

The quotient variety $G/H$ is called a symmetric space. It is known to be spherical, that is for any parabolic subgroup $P$ in $G$, $H$ acts on $G/P$ with a finite number of orbits. In particular, there is an open orbit; it consists of all $\sigma$-split parabolic subgroups of the same type as $P$ (provided they exist).

Let us state a general lemma that will be applied to the case of $E_8$ below.

**Lemma 3.2** Let $G$ be a split adjoint semisimple group over an infinite field $K$ of characteristic not 2, $H = G^\sigma$ be the fixed point subgroup of an involution $\sigma$ on $G$, $P$ be a parabolic subgroup of $G$, $C$ be the stabilizer of a point from the open orbit of the action of $H$ on $G/P$, $[\xi]$ be an element from $H^1(K, G)$. Assume that the twisted form $\xi G$ contains (over the base field $K$) a parabolic subgroup $P'$ of the same type as $P$ and a subgroup $H'$ that is conjugate to $H$ over a separable closure of $K$. Then $[\xi]$ comes from some $[\zeta] \in H^1(K, C)$ such that $\zeta H$ is isomorphic to $H'$. 
3.2 \( E_8/P_8 \) as a compactification of \( D_8/N(A_7) \)

Let \( G \) be the split group of type \( E_8 \) over \( K \) and \( \sigma \) be the involution whose fixed point subgroup is \( D_8 \) obtained by erasing vertex 1 from the extended Dynkin diagram:

```
  . 1 ------ 3 ------ 4 ------ 5 ------ 6 ------ 7 ------ 8 ---- 0
                                 . 2
```

More precisely, \( \sigma \) is the inner automorphism defined by \( \omega_1'(\alpha_i) \), where \( \alpha_i \) are the fundamental roots and \( \omega_i' \) are coweights defined by \( \omega_i'(\alpha_i) = \delta_{ij} \).

All vertices of the Satake diagram for the symmetric space \( E_8/D_8 \) are white; in particular, there is a parabolic subgroup \( P \) of type \( P_8 \) such that \( \sigma(P) \) is opposite to \( P \). It is not difficult to construct such a parabolic subgroup directly: it is defined by \( S = \mathbb{G}_m \) which is the image of \( \alpha_1^\vee \) in the maximal torus \( T \) (note that \( \alpha_1^\vee \) is Weyl-conjugate to \( \omega_8^\vee \) and so it has type \( P_8 \) indeed).

**Lemma 3.3** The stabilizer of a point from the open orbit of the action of \( D_8 \) on \( E_8/P_8 \) is \( N(A_7) \), the normalizer of the maximal subgroup of type \( A_7 \) in the simply connected group of type \( E_7 \).

**Proof** To check the claim we may pass to the algebraic closure of \( K \). The stabilizer of the point corresponding to \( P \) is \( L^0 \), where \( L = P \cap \sigma(P) \). It contains the \( A_7 \) subgroup generated by root subgroups corresponding to \( \pm \alpha_2, \pm \alpha_4, \ldots, \pm \alpha_8 \) and \( \pm \alpha_0 \) (where \( \alpha_0 \) stands for the negative maximal root). Since \( A_7 \) is maximal in the commutator subgroup \( E_7 \) of \( L \) and \( L \) is an almost direct product of \( E_7 \) and the \( \sigma \)-split torus \( S \), we see that the connected component of \( L^0 \) is \( A_7 \).

It is known (and can be deduced from Lemma 3.1) that \( A_7 \) has index 2 in \( N(A_7) \), so it remains to present an element from \( L^0 \) not lying in \( A_7 \). Consider any lifting \( \tilde{w}_0 \) of the longest element in the Weyl group of \( E_7 \). Note that \( \tilde{w}_0 \) normalizes \( A_7 \) but cannot belong to \( L^0 \), otherwise the fixed point subgroup \( E_7^\sigma \) would be not connected. Lemma 3.1 implies that \( \sigma(\tilde{w}_0) = \tilde{w}_0 \alpha_1^\vee (-1) \), for the second factor is the only nontrivial element in the center of \( E_7 \). Now \( \tilde{w}_0 \alpha_1^\vee(i) \), where \( i \) is a square root of \(-1\), is an element from \( L^0 \cap N(A_7) \) not belonging to \( A_7 \).

One can show that \( N(A_7) \) is an extension of \( \mathbb{Z}/2\mathbb{Z} \) by \( SL_6/\mu_2 \), which is split if and only if \(-1\) is a square in \( K \).
**Lemma 3.4** Let $[\xi] \in H^1(K, \text{PGO}_{2n})$ be in the image of $H^1(K, \text{GL}_n / \mu_2 \rtimes \mathbb{Z}/2\mathbb{Z})$. Then there exists a quadratic field extension $E/K$ such that the orthogonal involution corresponding to $\xi_E$ is hyperbolic.

**Proof** Consider the following short exact sequence:

$$
H^1(K, \text{GL}_n / \mu_2) \to H^1(K, \text{GL}_n / \mu_2 \rtimes \mathbb{Z}/2\mathbb{Z}) \to H^1(K, \mathbb{Z}/2\mathbb{Z}),
$$

and take $E/K$ corresponding to the image in $H^1(K, \mathbb{Z}/2\mathbb{Z})$ of $[\zeta]$ in $H^1(K, \text{GL}_n / \mu_2 \rtimes \mathbb{Z}/2\mathbb{Z})$ whose image in $H^1(K, \text{PGO}_{2n})$ is $[\xi]$. Passing to $E$ we see that $[\xi_E]$ comes from $H^1(E, \text{GL}_n / \mu_2)$ and so produces a hyperbolic involution.

**Theorem 3.5** Let $K$ be a two-special (that is with no odd degree extensions) field of characteristic not 2 and 3, $L$ be a Lie algebra of type $E_8$ over $K$ obtained via the Tits construction. Then the group corresponding to $L$ is not of Tits index $E_{8,1}^{133}$.

**Proof** Assume the contrary. Obviously the base field is infinite, for there are only split groups of type $E_8$ over finite fields. Let $L$ be obtained via the Tits construction from $C_i$ and $\mathcal{H}_3(C_2, \gamma)$ for some octonion algebras $C_i$ and $C_2$, i.e., is $K(A, -, \gamma)$ for $(A, -) = C_i \otimes C_2$. Denote by $[\xi]$ the class corresponding to $L$ in $H^1(K, E_8)$. By Proposition 1.5 $L$ contains a Lie subalgebra of type $D_8$, namely $so((\gamma_1) n_1 \perp (-\gamma_j^{-1}) n_2)$, and so the corresponding group contains a subgroup $H'$ of type $D_8$ with the same Lie algebra (see [7, Exposé XXII, Corollaire 5.3.4]), that is corresponding to the quadratic form $(\gamma_1) n_1 \perp (-\gamma_j^{-1}) n_2$.

Applying Lemma 3.2 to the case $G = E_8$, $H = D_8$ and $H'$ as above, we see that $[\xi]$ comes from some $[\zeta] \in H^1(K, N(A_7))$ such that $H'$ is isomorphic to $\zeta D_8$. Now the image of $N(A_7)$ in $\text{PGO}_{16}$ normalizes $\text{SL}_8 / \mu_2$ and so is contained in $\text{GL}_8 / \mu_2 \rtimes \mathbb{Z}/2\mathbb{Z}$. Applying Lemma 3.4, we see that the quadratic form $(\gamma_1) n_1 \perp (-\gamma_j^{-1}) n_2$ becomes hyperbolic over a quadratic field extension $E/K$. It follows that $e_3(n_1) + e_3(n_2)$ is trivial over $E$, hence $n_1 - n_2$ belongs to $I^4$ and so is hyperbolic over $E$. Now $n_1 - n_2$ is divisible by the discriminant of $E$ and so $e_3(n_1) + e_3(n_2)$ is a sum of two symbols with a common slot. But the Rost invariant of the anisotropic kernel of type $E_7$ is $e_3(n_1) + e_3(n_2)$, and applying [13, Theorem 10.18] we see that this group must be isotropic, a contradiction.

Note that [9, Appendix A] provides an example of a strongly inner group of type $E_7$ over a two-special field, hence an example of a group of Tits index $E_{8,1}^{133}$ over such a field, which is not obtained via the Tits construction.

**Corollary 3.6** Suppose $K$ is a field such that $-1$ is a sum of two squares, and let $L$ be a Lie algebra over $K$ of type $E_8$ obtained via the Tits construction.

1. If $h_8(L) \neq 0$ then $L$ is anisotropic.
2. If $-1$ is a square in $K$ and $h_8(L) \neq 0$ then $L$ has $K$-rank $\leq 1$.

**Proof** It suffices to prove both items in case $K$ is two-special.

(i) Suppose $L$ is isotropic. We can assume that $L$ does not have Tits index $E_{8,1}^{133}$ by Theorem 3.5. Using [6, Table 10] we see that $L$ corresponds to a class in the image of
\( H^1(K, \text{Spin}_{14}) \to H^1(K, E_8) \), which implies by Proposition 1.4 that it is isomorphic to \( K(A, -) \cong K(A, - \cdot (1, -1, 1)) \) for some bi-octonion algebra \( A \). Then clearly \( h_8(L) = 0 \).

(ii) Suppose \( L \) has \( K \)-rank \( \geq 2 \). Then \( L \) corresponds to a class in the image of \( H^1(K, \text{Spin}_{12}) \to H^1(K, E_8) \). Its anisotropic kernel is a subgroup of \( \text{Spin}(q) \) for some 12-dimensional form \( q \) belonging to \( I^3(K) \), and by a well-known theorem of Pfister (see [9, Theorem 17.13]) \( q \) is similar to \( n_1 - n_2 \) for a pair of three-Pfister forms \( n_i \) with a common slot, say \( n_i = \langle \langle x, y_i, z_i \rangle \rangle \). If \( C_i \) is the octonion algebra corresponding to \( n_i \) then we have \( L \cong K(C_1 \otimes C_2, -) \), and since \(-1\) is a square,

\[
h_6(L) = e_6(\langle \langle x, y_1, z_1, x, y_2, z_2 \rangle \rangle) = (-1) \cup (x) \cup (y_1) \cup (y_2) \cup (z_1) \cup (z_2) = 0.
\]

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