Upstream induction and power generation of periodically surging turbines

Nathaniel J. Wei$^1$ and John O. Dabiri$^{1,2}$†

$^1$Graduate Aerospace Laboratories, California Institute of Technology, Pasadena, CA 91125, USA
$^2$Department of Mechanical and Civil Engineering, California Institute of Technology, Pasadena, CA 91125, USA

(Received xx; revised xx; accepted xx)

Floating offshore wind turbines and other energy-harvesting systems in unsteady environments exhibit streamwise translation and associated unsteady flows that affect their thrust and power generation. Similar effects may be encountered by ground-fixed turbines and tidal turbines in oscillatory streamwise inflow conditions. To characterize these dynamics, the flow velocity and pressure in the upstream induction zone of a periodically surging turbine is modeled by means of a porous disc in potential flow. The model is used in conjunction with one-dimensional axial momentum theory to predict the effects of surge motions on time-averaged power production. Additionally, an equivalence between a surging turbine in a steady inflow and a stationary turbine in an oscillatory inflow is demonstrated within the assumptions of the theoretical approach. Trends identified by the modeling framework are validated in experiments conducted with a surging-turbine apparatus in an open-circuit wind tunnel at a diameter-based Reynolds number of $Re_D = 6.3 \times 10^5$ and surge-velocity amplitudes of up to 24% of the wind speed. The streamwise flow deceleration and the amplitudes of the velocity and pressure fluctuations upstream of the turbine all increase with surge-velocity amplitude. At high tip-speed ratios, unsteady power enhancements over the steady case are observed and are found to be well-described by the present modeling approach. These results provide a description of the influence of streamwise turbine motion and inflow unsteadiness on the velocity and pressure fields upstream of the turbine that can inform tower-based wind-speed estimation and induction-control schemes in full-scale applications.

Key words:

1. Introduction

Floating offshore wind turbines (FOWTs) are an emerging technology with the potential to enable wind-energy conversion in areas of the ocean whose depths prohibit the installation of conventional fixed-bottom systems, thereby creating additional avenues for the expansion of wind power as a contributor to global energy demands. These turbines can capitalize on...
strong offshore wind resources, are by nature located close to coastal urban centers, and have fewer constraints on size and placement compared to their land-based and seafloor-mounted counterparts. Since these turbines are not fixed to the ocean floor, they can move under the influence of wind gusts and surface waves. These motions are typically small, but in certain forcing scenarios and platform configurations the velocity amplitude of the turbine motions may exceed 25% of the wind speed (Wayman 2006; Larsen & Hanson 2007; de Vaal et al. 2014). The dynamics of these moving turbines are also related by means of a reference-frame transformation to the problem of a stationary turbine in an unsteady inflow. In this vein, the oscillating-turbine system has implications for turbines mounted to airborne kites, which may undergo rapid changes in incident wind speed as they sweep through atmospheric flow gradients, tidal turbines exposed to strongly time-dependent flows, and conventional ground-fixed turbines in axial gusts. The characterization of the aerodynamics of oscillating turbines is thus of critical importance to the design and control of the next generation of wind-energy technologies.

In particular, the deceleration of the upstream flow approaching a wind turbine, or induction, affects the operation and power output of a turbine. This induction effect dictates the flow and loading conditions encountered by the blades of a turbine and is directly related to the turbine’s thrust force and efficiency (Betz 1920). The induction zone, defined roughly as the region in which the flow velocity along the turbine’s centerline is below 95% of the free-stream velocity, extends at least two turbine diameters upstream of the turbine itself (Medici et al. 2011). These reduced velocities can thus bias tower-based estimates of the true wind speed by anemometers and LiDAR systems (e.g. Larsen & Hansen 2014; Howard & Guala 2016; Simley et al. 2016; Borraccino et al. 2017; Mann et al. 2018). For floating turbines, the coupling between incident wind conditions, blade-pitch control systems, and turbine thrust can yield negatively damped (i.e. unstable) streamwise surge oscillations that increase fatigue loading on the turbine blades (Larsen & Hanson 2007; Jonkman 2008; López-Queija et al. 2022). It is therefore necessary to consider the effects of turbine motions and unsteady inflow conditions on turbine induction.

The flow deceleration upstream of a stationary horizontal-axis wind turbine has been thoroughly studied in the literature. A common modeling approach involves representing the wake of the turbine as a helical vortex line or cylindrical vortex sheet and performing Biot-Savart integration to compute the induced velocity from this wake model at any point in the domain (Johnson 1980). Evaluating this integral upstream of the turbine along its rotational axis yields a model for the induced velocity along the upstream centerline of the turbine (Medici et al. 2011):

\[
\frac{u(x)}{u_1} = 1 - a \left[ 1 + \frac{x}{R} \left( 1 + \left( \frac{x}{R} \right)^2 \right)^{-1/2} \right],
\]

(1.1)

where \(x\) is the streamwise coordinate along the axis of the turbine (originating at the turbine and positive downstream), \(u_1\) is the free-stream velocity, \(R\) is the radius of the turbine, and the induction coefficient

\[
a \equiv 1 - \frac{u(x \to 0^+)}{u_1}
\]

(1.2)

is taken from the one-dimensional (1D) axial momentum theory of Betz (1920), which uses conservation relations to derive the maximum theoretical efficiency of a turbine in steady inviscid flow. The modeling paradigm that yields Equation 1.1, often referred to as the vortex-sheet model, lends itself well to free-vortex wake simulations (Sarmast et al. 2016), and shows good agreement with experimental data (Howard & Guala 2016; Bastankhah & Porté-Agel 2017; Borraccino et al. 2017). It has also been extended to unsteady inflow conditions.
Rather than rely on assumptions regarding near-wake structure, alternative approaches model the induction effect of the turbine using potential-flow objects such as Rankine half-bodies (Araya et al. 2014; Gribben & Hawkes 2019; Meyer Forsting et al. 2021) or porous discs (Modarresi & Kirchhoff 1979). These models reflect the common practice in both numerical and experimental studies of modeling the turbine as an actuator disc.

Despite the prevalence of work on the upstream induction region of stationary turbines, few studies have investigated the effects of turbine motion on flow properties in this region and the resulting consequences for turbine power generation. The surge oscillation mode is of particular interest because it tends to exhibit larger amplitudes relative to other degrees of freedom of motion in FOWTs (Johlas et al. 2019), and as previously mentioned, can yield unstable oscillations. The surging-turbine problem has been mostly investigated with free-vortex wake simulations that analyze turbine thrust, power, and wake characteristics, but leave the upstream induction region relatively unexplored (e.g. de Vaal et al. 2014; Wen et al. 2017). Dabiri (2020) has suggested that the introduction of streamwise unsteadiness may actually increase the efficiency of wind-energy systems if the time-averaged contribution of the unsteady velocity potential associated with the unsteadiness is nonzero. Johlas et al. (2021) independently derived a model for the power generation of a periodically surging turbine that predicts an increase in the time-averaged power with the square of the surge-velocity amplitude. Wei & Dabiri (2022) showed in recent experiments that a periodically surging turbine produces more power in the time-average than a stationary turbine in steady flow. El Makdah et al. (2019) found similar unsteady power gains for rotors in axial gusts with constant-acceleration inflow profiles. In these studies, however, the relationship between the observed power enhancements and the flow conditions upstream of the turbine has not been established in detail.

Therefore, this study investigates the flow physics upstream of a periodically surging turbine, first by extending theoretical models to include the effects of surge, and then by characterizing these dynamics in experiments. The work is structured as follows. A model for the induction and time-averaged power production of a surging turbine is derived and related to stationary turbines in periodic axial gusts in Section 2. Velocity and pressure measurements upstream of a surging-turbine apparatus are described in Section 3, and the results are compared with trends identified by the modeling framework in Section 4. Finally, the implications of the findings for surging-turbine aerodynamics and the control of turbines in unsteady flow environments are discussed in Section 5.

2. Theoretical Considerations

2.1. An induction model for a periodically surging turbine

We seek a potential-flow model for the flow in the upstream induction zone of a horizontal-axis turbine surging in the streamwise direction. We model the turbine as a porous disc, rather than applying Biot-Savart integration on vortex models of the turbine wake, so as to obtain physically interpretable expressions that do not require integration in space or time. Our approach is congruous with that of Steiros & Hultmark (2018), who extended the work of Taylor (1944) and Koo & James (1973) to predict the drag on porous plates.

We first consider a circular porous disc with radius $R$ located at streamwise coordinate $\xi = 0$, represented as a distribution of sources with a velocity potential of $\phi(r, \xi = 0) = C\sqrt{R^2 - r^2}$ for $r < R$ and $\phi(r, \xi = 0) = 0$ for $r > R$, where $C > 0$ is an arbitrary constant that represents the aggregate strength of the source distribution. Using this distribution as
a boundary condition at $\xi = 0$, we may solve the Laplace equation $\nabla^2 \phi = 0$ in cylindrical coordinates to obtain the velocity potential of a porous disc (cf. Lamb 1916; Tranter 1968):

$$\phi (r, \xi) = -\sqrt{\frac{\pi}{2}} CR^{3/2} \int_0^\infty s^{-1/2} J_{3/2}(Rs) J_0(rs) e^{-s\xi} \, ds; \quad \xi > 0. \quad (2.1)$$

Here, $J_\nu(z)$ is a Bessel function of the first kind, and $s$ is a dummy integration variable. The choice of $C = \frac{2}{\pi} V$ gives the velocity potential of a solid disc moving at axial velocity $V$ in a quiescent fluid (Lamb 1916, §102.4). More generally, the velocity $V$ represents the velocity of the disc relative to that of the fluid in the far field. For a porous disc, we may define a representative source term $a$ such that $C = \frac{2}{\pi} a V$. The choice of $a$ dictates the porosity of the disc: $a = 0$ represents a fully permeable disc, $a = 1$ yields a fully solid disc, and intermediate values ($0 < a < 1$) reduce the source strength from the solid-disc solution so that a nonzero mass flux through the disc is established. Hence, $a$ is directly analogous to the induction coefficient from 1D axial momentum theory given in Equation 1.2. Evaluating Equation 2.1 along the centerline yields

$$\phi (r = 0, \xi) = \frac{2}{\pi} a V \left[ \xi \arctan \left( \frac{R}{\xi} \right) - R \right]; \quad \xi > 0. \quad (2.2)$$

This solution is only valid for $\xi > 0$. To describe the other half of the domain as well, one might follow the ansatz of Taylor (1944) and use the even extension of $\phi$ to represent $\xi < 0$. The velocity discontinuity across the disc that this extension creates could then be removed using the base-suction correction of Steiros & Hultmark (2018). However, since in this work we are only concerned with the upstream region, we leave these derivations for future consideration.

We differentiate the velocity potential with respect to $\xi$ to obtain the streamwise velocity along the centerline:

$$u (r = 0, \xi) = \frac{2}{\pi} a V \left[ \arctan \left( \frac{R}{\xi} \right) - \frac{R \xi}{\xi^2 + R^2} \right]; \quad \xi > 0. \quad (2.3)$$

This relation emphasizes the effect of the porosity parameter (or equivalently, the induction coefficient) on the behavior of the model. For $a = 0$, the flow is everywhere unaffected by the motion of the disc. For $a = 1$, $u(r = 0, \xi \to 0^+) = V$, which satisfies the surface boundary condition for a moving solid disc.

We now apply this expression in an inertial frame containing a uniform flow with freestream velocity $u_1$, in which the disc translates at velocity $U(t)$ relative to the frame. In this frame, we define the downstream-oriented axial coordinate $x$ and the instantaneous position of the disc $x_2(t)$ as shown in Figure 1, such that $\xi = x_2 - x$ and $U(t) = \frac{dx_2}{dt}$. The velocity of the disc relative to the far-field flow velocity is thus $V = U(t) - u_1$. Applying these definitions to Equation 2.3, we arrive at the following model for the centerline velocity in the upstream induction zone ($x < x_2$):

$$u (r = 0, x, t) = u_1 + \frac{2}{\pi} a (u_1 - U(t)) \left[ \arctan \left( \frac{R}{x - x_2(t)} \right) - \frac{R(x - x_2(t))}{(x - x_2(t))^2 + R^2} \right]. \quad (2.4)$$

We reiterate that the model cannot be used to predict the velocity downstream of the porous disc, given the constraint of $\xi > 0$ on the velocity potential. Additionally, though the solution is technically valid if the turbine moves downstream faster than the wind speed, i.e. $u_1 - U(t) < 0$, we expect that the model will cease to be valid in this case because the rotor will interact with its own wake.
Wind speed, $u_1$

Flow velocity at disc plane, $u_2$

Disc position, $x_2(t)$

Flow velocity at exit plane, $u_4$

Disc Velocity, $U(t)$

Figure 1: Schematic of the parameters and control volumes referenced in Section 2. The actuator disc is located instantaneously at $x_2(t)$ and moves with velocity $U(t)$ relative to the inertial frame defined by the $x$- and $r$-axes. Circled numbers denote streamwise stations (1 through 4).

To complete the description of flow properties upstream of the surging disc, the pressure along the centerline may be modeled by substituting Equation 2.4 into the steady Bernoulli equation,

$$p(r = 0, x) = p_1 + \frac{1}{2} \rho \left( u_1^2 - u (r = 0, x, t)^2 \right),$$

(2.5)

where $\rho$ is the density of the fluid and $p_1$ is the ambient pressure in the free stream. While the expressions presented here have been confined to the centerline, Equation 2.1 can in principle be integrated at any point upstream of the porous disc. Thus, this modeling framework covers the entire upstream induction zone of a periodically surging horizontal-axis turbine. We also note that functionally similar expressions can be obtained by using the vortex-sheet model described in Section 1 in place of Equation 2.3.

### 2.2. Time-averaged power extraction by a periodically surging actuator disc

We now seek to relate the flow conditions upstream of the surging porous disc to the power that may be extracted by the disc. Equation 2.4 allows the flow velocity at the upstream side of the disc, $u(r = 0, x \rightarrow x_2^-, t) \equiv u_2$, to be derived. For $U(t) = 0$, Equation 2.4 gives $u_2/u_1 = 1 - a$, and the aforementioned interpretation of $a$ as the induction coefficient from 1D axial momentum theory holds. For $U(t) \neq 0$, we have

$$\frac{u_2}{u_1} = 1 - a \left( 1 - \frac{U(t)}{u_1} \right).$$

(2.6)

The time-varying nature of the flow velocity incident on the disc suggests that the steady-flow 1D axial-momentum analysis of Betz (1920) may be revisited to obtain a theoretical prediction for the maximum time-averaged power extracted from a periodically surging actuator disc. We consider a radially symmetric control volume, shown in Figure 1, that is split across the face of the actuator disc into upstream and downstream regions. This discontinuous control volume is defined by four streamwise stations: station 1 is located far upstream where the flow is unaffected by the presence of the disc, station 2 is located on the upstream face of the disc, station 3 is located on the downstream face of the disc, and station 4 is located far downstream of the disc where the flow properties no longer change appreciably with streamwise distance. We assume the bounding surfaces of the control volumes follow the smallest streamtube that contains the surging disc, such that no flow exits from the sides.
Furthermore, the flow-normal control surfaces at stations 2 and 3 move at the same velocity as the disc, $U(t)$, such that the volumes defined by the upstream and downstream control surfaces change with time. Finally, following Betz’s approach, we assume that the flow properties do not vary in the radial direction, so that the flow is one-dimensional. Enforcing conservation of mass in this system yields $u_1 A_1 = u_2 \pi R^2 = u_4 A_4$, where $A_1$ and $A_4$ are the cross-sectional areas of the streamtube at its inlet and outlet. Similarly, applying conservation of momentum yields $u_2 = \frac{1}{\sqrt{2}} (u_1 + u_4)$, or equivalently, $u_4 = 2u_2 - u_1$. Lastly, assuming steady flow and constant streamtube cross-sectional area across the disc gives $u_2 = u_3$, so that the flow velocity remains continuous across the disc. All of these velocities are inertial quantities, referenced to the static-observer frame shown in Figure 1.

The drag force on the disc is equal to the sum of the forces on the upstream and downstream control volumes, which can be obtained by invoking conservation of momentum:

$$\vec{F}_{cv} = \frac{\partial}{\partial t} \int_{cv} \rho \vec{u}_{cv} dV + \int_{cs} \rho \vec{u}_{f\text{lux}} \left( \vec{u}_{rel} \cdot d\vec{A} \right), \quad (2.7)$$

where $\vec{u}_{cv}$ is the flow velocity inside the control volume, $\vec{u}_{f\text{lux}}$ is the flow velocity at the control surface in a fixed frame, and $\vec{u}_{rel}$ is the flow velocity through the control surface relative to the control surface itself. In the absence of external forces on the domain, $\vec{F}_{cv}$ is equivalent to the net pressure force on the control volume. We define the upstream and downstream control volumes to be sufficiently large so that we may treat $u_{cv}$ as constant, i.e. $u_{cv} \approx u_1$ for the upstream control volume and $u_{cv} \approx u_4$ for the downstream control volume. For a streamtube with a sufficiently small expansion angle at the disc, we may assume that the change in volume due to the moving control surface is approximately cylindrical, and thus $\frac{dV}{dt} \approx \pi R^2 \frac{dx_2}{dt} = \pi R^2 U(t)$. For both control volumes, at the disc we have $u_{f\text{lux}} = u_2$. Similarly, $u_{rel} = u_2 - U$ on the moving boundaries of both control volumes. Evaluating Equation 2.7 with these quantities for each control volume and summing the results gives the drag force on the disc,

$$F_D = F_{cv, u} + F_{cv, d} = \rho \pi R^2 \left( u_1 U + u_2 (u_2 - U) \right) - \rho A_1 u_1^2 - \rho \pi R^2 \left( u_4 U + u_2 (u_2 - U) \right) + \rho A_4 u_4^2. \quad (2.8)$$

The thrust force is defined as the opposite of the drag force. After substituting for $A_1$, $A_4$, and $u_4$ in terms of $u_1$ and $u_2$, this can be written as

$$F_T = \rho \pi R^2 \left( -u_1 U + u_1 u_2 + u_4 U - u_2 u_4 \right) = 2\rho \pi R^2 (u_1 - u_2)(u_2 - U). \quad (2.9)$$

The instantaneous power extracted by the actuator disc can then be written as the product of the thrust force and the incident flow velocity relative to the disc at station 2,

$$P = F_T (u_2 - U) = 2\rho \pi R^2 (u_2 - U)^2 (u_1 - u_2). \quad (2.10)$$

Substituting the definition of $u_2$ from Equation 2.6 and normalizing gives a simple expression for the instantaneous coefficient of power:

$$C_p(t) \equiv \frac{P}{\frac{1}{2} \rho \pi R^2 u_1^3} = 4a(1 - a)^2 \left( 1 - \frac{U(t)}{u_1} \right)^3. \quad (2.11)$$

When $U = 0$, this expression simplifies to the Betz result, i.e. $C_{p, 0} = 4a(1 - a)^2$. To compute the time-averaged power extracted by the disc for a periodic surge-velocity waveform $U(t)$ with frequency $f$, we integrate Equation 2.11 over a single surge period,

$$\bar{C_p} = \frac{f}{2\pi} \int_0^{2\pi} 4a(1 - a)^2 \left( 1 - \frac{U(t)}{u_1} \right)^3 dt, \quad (2.12)$$
which for a sinusoidal surge-velocity waveform $U(t)/u_1 = u^* \sin(ft)$ integrates to
\[
\frac{C_p}{C_{p,0}} = 1 + \frac{3}{2}a^2. \tag{2.13}
\]
This result is identical to that derived by Wen et al. (2017) and Johlas et al. (2021) for a surging turbine. Both of these studies assume that $C_p$ is constant with respect to time and surge kinematics, and that the relevant velocity scale for the normalization of power is $u_1 - U(t)$. Thus, they both implicitly assume that the problem is equivalent to a stationary turbine with constant $C_p = C_{p,s}$ in an oscillating inflow. These assumptions allow the time-averaged power to be computed as
\[
\overline{P} = \frac{f}{2\pi} \int_0^{2\pi} \frac{1}{2} \rho \pi R^2 C_{p,s} (u_1 - U(t))^3 \, dt = \mathcal{P}_0 \left(1 + \frac{3}{2}a^2\right) \tag{2.14}
\]
for a sinusoidal surge-velocity waveform. The assumption of constant $C_p$ is analogous to the assumption of constant $a$ that is used in the present analysis, since according to the framework of Betz, $C_p$ is only dependent on $a$. It remains to be shown, however, that the problem of a surging turbine in uniform flow is equivalent to the problem of a stationary turbine in an oscillating inflow, at least within the assumptions of 1D axial momentum theory. It is to this consideration that we now turn our attention.

2.3. On the problem of a stationary turbine in an oscillating inflow

To model the power extracted from a stationary turbine in an oscillating inflow $u_1' = u_1 - U(t)$, we redefine the control volume shown in Figure 1 so that the actuator disc is located at a fixed streamwise location and the upstream and downstream control volumes do not deform as a function of time. For a stationary disc, the potential-flow induction model in Equation 2.4 gives $u_2' = u_1'(1 - a)$. Assuming that the system behaves in a quasi-steady manner allows us to treat every instance of $u_1'(t)$ as an independent steady-flow problem, and thus these primed quantities may be substituted into the relations from steady-flow 1D axial momentum theory. Using conservation of mass and momentum, the instantaneous power extracted by the stationary disc can be written as
\[
P = 2\rho \pi R^2 u_2^2 (u_1' - u_2'). \tag{2.15}
\]
Substituting for $u_1'$ and $u_2'$ and normalizing, we find that
\[
C_p(t) = 4a(1 - a)^2 \left(1 - \frac{U(t)}{u_1}\right)^3, \tag{2.16}
\]
which is identical to the coefficient of power of a surging disc in uniform flow given in Equation 2.11 above. This analysis shows that the expression for time-averaged power given by Wen et al. (2017) and Johlas et al. (2021) (Equation 2.14) does indeed satisfy the conservation relations of 1D axial momentum theory. Moreover, we see that there is no difference between an oscillating turbine in uniform flow and a stationary turbine in an oscillatory inflow in terms of thrust force and power extraction – given the assumptions of this quasi-steady analytical framework. In other words, within the present set of assumptions, the effects of turbine surge motions and inflow oscillations on turbine performance are identical. To reiterate, the key premise undergirding these conclusions is that $a$ remains constant with surge- or inflow-velocity amplitude. This premise will be interrogated empirically in the following sections.

Finally, we briefly consider potential unsteady differences between the oscillating-turbine
and oscillating-inflow cases. Brennen (1982) gives the force on a body oscillating with velocity \( W_i(t) \) in an oscillating inflow \( U_i(t) \) as

\[
F_i = -M_{ij} \frac{dW_j}{dt} + (M_{ij} + \rho V_D \delta_{ij}) \frac{dU_j}{dt}; \quad j = 1, 2, 3, \tag{2.17}
\]

where \( M_{ij} \) is the added-mass tensor of the body, \( V_D \) is the volume of the body, \( \delta_{ij} \) is the Kronecker delta operator, and the flow is assumed to be inviscid. This expression thus quantifies the influence of added-mass effects and an unsteady buoyancy force, which comes from the oscillating pressure gradient that drives the oscillating inflow (Granlund et al. 2014). We assume that neither the added-mass tensor nor the volume of the body changes as a function of time for a porous disc, and that \( U_i(t) \) and \( W_i(t) \) are periodic. Since \( U_i(t) \) and \( W_i(t) \) are periodic, the time averages over a single period of \( \frac{dU_j}{dt} \) and \( \frac{dW_j}{dt} \) are both zero. It thus follows that the time-averaged force on the body due to these two types of unsteady contributions is also zero, and therefore neither of these unsteady effects creates a theoretical difference between the time-averaged performance of an oscillating turbine and a stationary turbine in an oscillating inflow. It is possible that a fully unsteady potential-flow analysis, following that of Dabiri (2020), could reveal additional unsteady differences between these two cases (e.g. a time-dependent added-mass tensor), but this analysis remains a subject for future work.

### 3. Experimental Methods

#### 3.1. Experimental Apparatus

To characterize the range of conditions over which the ideal-flow model holds, velocity and pressure measurements were conducted in a 2.88×2.88 m\(^2\) open-circuit fan-array wind tunnel at the Caltech Center for Autonomous Systems and Technologies (CAST). A three-bladed horizontal-axis wind turbine (Primus Wind Power AIR Silent X) with a rotor diameter of \( D = 1.17 \) m was mounted on a traverse that translated along 2-m long rails (NSK NH-series) and was actuated by a magnetic piston-type linear actuator (LinMot PS10-70x320U). A diagram of this apparatus is given in Figure 2. The hub height of the turbine was 1.97 m above the floor of the facility, and the farthest-downstream position of the turbine (defined as \( x = 0 \)) was 3.09 m downstream of the fan array. The electrical load on the turbine was provided by 10, 20, and 40-Ohm resistors (TE Connectivity TE 1000-series). A rotary torque transducer (FUTEK TRS300) and rotary encoder (US Digital EM 2) were used to measure the power produced by the turbine. The estimated blockage of the swept area of the turbine and all support structures, relative to the surface area of the fan array, was 14%. Further details regarding the dimensions and capabilities of the apparatus may be found in Wei & Dabiri (2022).

A constant-temperature hot-wire anemometry system (Dantec MiniCTA 54T42) and differential pressure transducer (MKS Baratron 398-series with Type 270B signal conditioner) were used to measure flow properties at two locations along the turbine centerline, one upstream of the turbine at \( x_u = -0.840D \) and one downstream at \( x_d = 0.810D \). The hot-wire probe was placed approximately on the centerline, while the input line of the pressure transducer was located 3.8 cm to the side. The transducer’s reference line was placed in a shielded area outside the flow of the wind tunnel. Data were collected at a sampling rate of 20 kHz and were low-pass filtered using a sixth-order Butterworth filter with a cutoff frequency of 100 Hz. The hot-wire anemometer was calibrated in the wind tunnel against a Pitot probe using the same pressure transducer. Because the facility was exposed to the atmosphere, the
temperature and relative humidity were recorded during all experiments to estimate the air density and correct the hot-wire calibration for temperature changes.

3.2. Experimental Procedure

Experiments were conducted over two nights in March 2022, in which the free-stream velocities in the wind tunnel were $u_1 = 7.79 \pm 0.10$ and $7.96 \pm 0.11 \text{ ms}^{-1}$, corresponding to an average diameter-based Reynolds number of $Re_D = 6.27 \times 10^5$. The hot-wire anemometer was calibrated at the beginning and end of each set of experiments. The turbine was operated at three tip-speed ratios, $\lambda_0 = 6.48 \pm 0.25$, $7.84 \pm 0.28$, and $8.77 \pm 0.27$, with corresponding coefficients of power of $C_{p,0} = 0.298 \pm 0.013 \approx C_{p,max}$, $0.248 \pm 0.012$, and $0.165 \pm 0.010$. The turbine was actuated in sinusoidal and trapezoidal motions (see inset, Figure 2) with an amplitude of $A = 0.3 \text{ m (0.257} D)$ and periods between $T = 1$ and $6 \text{ s}$, corresponding to nondimensional surge-velocity amplitudes between $u^* \equiv f A / u_1 = 0.039$ and $0.242$. Data were phase-averaged over 100 motion periods. Upstream and downstream flow measurements were collected in separate tests. Additionally, a series of quasi-steady flow measurements were obtained for each tip-speed ratio by placing the turbine at six equally spaced streamwise locations between $x/D = 0$ and $0.514$ and recording measurements over $120 \text{ s}$. To correct against differences in the ambient conditions between measurement sessions and facilitate
more direct comparisons, quasi-steady measurements taken at $\frac{x}{D} = 0$ on both sessions were used to scale the measured velocities and pressures from one session to match those from the other session.

To compare the analytical model with the experimental data, the parameters $u_1$ and $\alpha$ were fitted using a minimum-norm least-squares regression to the phase-averaged flow-velocity data. The wind speed was treated as a fitted parameter because the blockage of the turbine influenced the wind speed in the open test section, an effect that is well-documented in the literature (e.g. Eltayesh et al. 2019). Fitting the wind speed mitigated the effect of this source of uncertainty on the agreement between the model and data. A time-resolved prediction for the pressure was then derived by substituting the fitted model for the flow velocity into the steady Bernoulli equation (Equation 2.5). Since the absolute pressure at the reference of the pressure transducer was unknown, the mean value of this pressure prediction was set to be equal to that of the data. Lastly, due to the long length of the tubes that connected the pressure transducer to the measurement location, a first-order low-pass filter with a cutoff frequency of 2.48 Hz was inferred from the phase of the measured pressure data relative to the velocity signal. This filter was then applied to the calculated model predictions for pressure. Remaining discrepancies between the measured and modeled pressure signals could be attributed to the true filtering effect of the tubes being of higher order than the first-order filter model (Bergh & Tijdeman 1965).

4. Experimental Results

The unsteady and quasi-steady data from three selected experimental cases, all measured at $x = x_u$, are shown in Figure 3. The flow-velocity signals showed a phase lead and increased amplitude with respect to the quasi-steady measurements, effects that were well-captured by the fitted model for both sinusoidal and trapezoidal surge-velocity waveforms. The pressure predictions inferred from the model for the flow velocity showed similar agreement in terms of waveform shape, amplitude, and phase.

The fitted values of the induction coefficient $\alpha$ across all experiments, shown in Figure 4a, followed an increasing trend with surge-velocity amplitude $u^*$. A similar increasing trend was observed in the amplitude of the time-varying signal of the pressure difference between the upstream and downstream measurement locations, $p_{ud}(t) \equiv p(x = x_{u}, t) - p(x = x_{d}, t)$, which is plotted in Figure 4b. In these plots, we denote amplitudes of fluctuations in time-varying quantities with a circumflex, e.g. $\hat{p}_{ud}$, to distinguish them from magnitudes of mean values. The fore-aft pressure difference can be understood as an analogue for turbine thrust, and in steady flow, higher thrust is associated with a larger velocity deficit upstream of the turbine and hence a higher induction coefficient. Therefore, we may qualitatively explain the increasing trend of the induction coefficient $\alpha$ as a function of the surge-velocity amplitude $u^*$ as follows. If the thrust amplitude increases with $u^*$, as evidenced by the linearly increasing amplitude of the time-varying fluctuations of $p_{ud}$, then the amplitude of the time-varying flow-velocity fluctuations upstream of the turbine, $\hat{u}$, should increase as well. This is indeed observed in Figure 5a. Equation 2.4 shows that, for fixed surge kinematics $x_2(t)$ and $U(t)$, an increase in the amplitude of $u$ necessarily implies an increase in the induction coefficient $\alpha$. Thus, we see that the increasing trend in $\alpha$ as a function of $u^*$ shown in Figure 4a is consistent with the increasing thrust amplitude suggested in Figure 4b. Furthermore, integrating the expression for thrust given in Equation 2.9 over a sinusoidal surge-velocity waveform, as done previously for $C_p(t)$, yields a time-averaged thrust enhancement of

$$\frac{F_T}{F_{T,0}} = 1 + \frac{1}{2} u^*^2,$$

(4.1)
Figure 3: Phase-averaged velocity and pressure profiles for (a,b) a sinusoidal surge-velocity waveform with $\lambda_0 = 6.48$, (c,d) a trapezoidal waveform with $\lambda_0 = 6.48$, and (e,f) a trapezoidal waveform with $\lambda_0 = 8.77$. All surge-velocity waveforms had $u^* = 0.242$. The solid red lines represent unsteady measurements, the blue squares represent quasi-steady measurements, and the correspondingly colored dotted and dashed lines show the model results.

an increasing function with $u^*$. This increasing time-averaged thrust as a function of $u^*$ is also consistent with the trend between $a$ and $u^*$ shown in Figure 4a, because, as previously discussed, higher mean-thrust states are associated with higher induction coefficients. We cannot directly confirm these arguments in the absence of thrust measurements, but they suggest that the behavior of $a$ is not unexpected within the present analytical framework.
Figure 4: (a) Regression-fitted induction-coefficient values, plotted against surge-velocity amplitude. A linear fit is given as an orange dotted line. (b) Amplitude of the difference of the pressures measured upstream and downstream of the turbine, 
\[
  \rho_{ud} \equiv p(x = x_u) - p(x = x_d),
\]
plotted against surge-velocity amplitude. In these and the following figures, circles represent sinusoidal surge-velocity waveforms, while diamonds represent trapezoidal waveforms.

Figure 5: (a) Amplitude and (b) phase of the measured flow velocity at \( x = x_u \), plotted against surge-velocity amplitude. Model trends are given as orange dotted lines. Error bars are plotted on every fourth point for clarity.

To incorporate the dependence of \( a \) on \( u^* \) shown in Figure 4a into the model, a linear fit was performed over all values of \( a \), regardless of tip-speed ratio, and this fit for \( a \) was used to compute models for the velocity and pressure for arbitrary surge-velocity amplitudes. These models, plotted in Figures 5 and 6, demonstrated the same qualitative behavior as the measured data. The influence of tip-speed ratio was not modeled, so some deviations as a function of \( \lambda \) that are not accounted for in the model trends are visible in the phase results. While the model is semi-empirical due to its reliance on the linear fit from Figure 4a, the qualitative agreement in amplitude, phase, and waveform shape suggest that the assumptions undergirding the potential-flow model were reasonable within the upstream induction zone.

Finally, the time-averaged power measurements from these experiments were plotted against the surge-velocity amplitude \( u^* \) in Figure 7 for the three tested tip-speed ratios.
The data from the highest two tip-speed ratios increased with $u^*$, while the power produced at the lowest tip-speed ratio (near the optimal operating condition of the turbine) remained close to its steady value. As a reference, the model for time-averaged power given in Equation 2.13 was evaluated at $\lambda = 0.330 = \langle \lambda \rangle$, the average of the fitted induction coefficients from Figure 4a. This was plotted as a dashed blue line in Figure 7. The model captured the general trend of the data from the two higher tip-speed ratios, but overpredicted the power enhancements observed in the data at higher surge-velocity amplitudes. To account for the dependence of the induction coefficient $a$ on the surge-velocity amplitude $u^*$, the model for the time-averaged power was modified with a first-order expansion of $a$ about $a = \langle a \rangle$,

$$a \approx \langle a \rangle + \frac{\partial a}{\partial u^*} u^*, \quad (4.2)$$

where $\frac{\partial a}{\partial u^*} = 0.324$ is the slope of the linear fit shown in Figure 4a above. The adjusted model, plotted as an orange dashed line in Figure 7, showed closer agreement with the data. These results suggest that accounting for the dependence of turbine induction on surge velocity is important for capturing the time-averaged power production of periodically surging turbines.

5. Discussion and Conclusions

In this work, an analytical model for the flow properties in the upstream induction zone of a periodically surging wind turbine has been presented and demonstrates good agreement with experimental measurements at surge-velocity amplitudes of up to 24% of the wind speed. While similar theoretical tools have been widely applied to the analysis of wind turbines in steady flow, a major contribution of this work is the extension of these methods to unsteady flow contexts. The porous-disc model is also similar in principle to the actuator-disc models often used in numerical simulations of large wind farms (e.g. Calaf et al. 2010; Stevens & Meneveau 2017), and thus this study could inform modifications of existing actuator-disc simulations for surging-turbine or dynamic-inflow conditions. This may be particularly useful for large-eddy simulations (LES) of floating offshore turbine arrays, where the analytical turbine model can help to parameterize the coupling between turbine inflow conditions, sea-surface waves, and floating-platform dynamics. Additionally, better predictions of near-wake flow properties could be obtained by applying the base-suction
Figure 7: Time-averaged power, normalized by steady power, plotted against surge-velocity amplitude. The model from Equation 2.13, evaluated at $a = \langle a \rangle = 0.330$, is plotted as a dashed blue line. The orange dotted line shows the adjusted model, plotted with a first-order expansion for $a$ as a function of $u^*$ (Equation 4.2), which agrees more closely with the data taken at the two higher tip-speed ratios than the uncorrected model.

correction of Steiros & Hultmark (2018) to the region downstream of the surging porous disc, which might further enhance the utility of this modeling framework for LES applications. Finally, this work establishes that a comprehensive model for the induction coefficient of a surging turbine can be used to infer behaviors in the thrust and power production of the turbine. A control-volume analysis shows that the time-averaged power increases quadratically relative to the steady case with surge-velocity amplitude, enhancements which are diminished if the dependence of the induction coefficient on surge-velocity amplitude are taken into account as well. These predictions appear to be more valid for turbines at high tip-speed ratios; Wei & Dabiri (2022) have suggested that blade-level aerodynamics such as flow separation may be responsible for decreases in time-averaged power at lower tip-speed ratios. More detailed flow measurements are required to verify and quantify these effects. Future work may also uncover a first-principles relationship between $a$ and $u^*$, instead of the semi-empirical approach and associated arguments presented in this work. To validate such a model, direct thrust-force measurements would be necessary, a task that would be prohibitively difficult in our facility given the large inertial forces experienced by the turbine in our apparatus.

This work has several implications for full-scale wind-energy systems in real-world flow conditions. First, the analytical model for flow properties upstream of a surging turbine can be used in conjunction with nacelle-mounted LiDAR units for improved load control and wind-speed estimation in floating offshore applications. The same principles can be applied to stationary turbines in gusty environments and kite-mounted aerial turbines. Secondly, these analytical and experimental results reinforce the evidence collected by Wen et al. (2017), El Makdah et al. (2019), Johlas et al. (2021), and Wei & Dabiri (2022) that streamwise unsteadiness (either in the flow or in the turbine itself) can lead to increases in power extraction above the reference steady case. The present investigations suggest that quasi-steady processes can explain some of these enhancements, but further gains may be possible by leveraging fully unsteady effects, such as unsteady velocity potentials with fore-aft asymmetry across the turbine plane (Dabiri 2020). These results imply that floating offshore turbines, tidal turbines, and other energy-harvesting systems in unsteady flow environments may potentially produce more power than steady-flow models might predict, giving these arrays a strong advantage in terms of energy density. Finally, the conclusion that the induction
coefficient, thrust amplitude, and time-averaged thrust of a surging turbine all increase with surge-velocity amplitude is critical for the control of floating offshore turbines, since variations in turbine thrust can drive oscillations in the floating platforms and increase blade and support-structure fatigue. Control schemes that can anticipate the increase in thrust and induction coefficient with increased surge motions could potentially damp out unwanted oscillations more effectively and extend the operational lifespan of offshore turbines.

Acknowledgements. The authors would like to thank Konstantinos Steiros for sharing resources that helped with the derivation of the analytical model in this work. The authors also appreciate the assistance and safety supervision of Malaika Cordeiro, Matt Fu, and Peter Gunnarson during the experiments.

Funding. This work was supported by the National Science Foundation (grant number CBET-2038071) and the Caltech Center for Autonomous Systems and Technologies. Nathaniel Wei was supported by a National Science Foundation Graduate Research Fellowship.

Declaration of interests. The authors report no conflict of interest.

Data availability statement. The data that support the findings of this study are available upon request.

Author ORCID. N. J. Wei, https://orcid.org/0000-0001-5846-6485; J. O. Dabiri, https://orcid.org/0000-0002-6722-9008

REFERENCES

ARAYA, DANIEL B., CRAIG, ANNA E., KINZEL, MATTHIAS & DABIRI, JOHN O. 2014 Low-order modeling of wind farm aerodynamics using leaky Rankine bodies. Journal of Renewable and Sustainable Energy 6 (6), 063118.

BASTANKHAH, M. & PORTÉ-AGÉL, F. 2017 Wind tunnel study of the wind turbine interaction with a boundary-layer flow: Upwind region, turbine performance, and wake region. Physics of Fluids 29 (6), 065105, publisher: American Institute of Physics.

BERGH, H. & TUDEMAN, H. 1965 Theoretical and experimental results for the dynamic response of pressure measuring systems. Tech. Rep. NLR-TR F. 238. Nationaal Lucht- en Ruimtevaartlaboratorium.

BETZ, ALBERT 1920 Das Maximum der theoretisch möglichen Ausnützung des Windes durch Windmotoren. Zeitschrift für das gesamte Turbinenwesen 26, 307–309.

BORRACCINO, ANTOINE, SCHLIFE, DAVID, HAIZMANN, FLORIAN & WAGNER, ROZENN 2017 Wind field reconstruction from nacelle-mounted lidar short-range measurements. Wind Energy Science 2 (1), 269–283, publisher: Copernicus GmbH.

BRENNEN, C. E. 1982 A Review of Added Mass and Fluid Inertial Forces. Tech. Rep. CR 82.010. Naval Civil Engineering Laboratory, Port Hueneme, CA, USA.

CALAF, MARC, MENEVEAU, CHARLES & MEYERS, JOHAN 2010 Large eddy simulation study of fully developed wind-turbine array boundary layers. Physics of Fluids 22 (1), 015110.

CHATTOT, JEAN-JACQUES 2014 Actuator Disk Theory—Steady and Unsteady Models. Journal of Solar Energy Engineering 136 (3).

DABIRI, JOHN O. 2020 Theoretical framework to surpass the Betz limit using unsteady fluid mechanics. Physical Review Fluids 5 (2), 022501, publisher: American Physical Society.

EL. MAKDAH, ADNAN M., RIZZANTE, SACHA, ZHANG, KAI & RIVAL, DAVID E. 2019 The influence of axial gusts on the output of low-inertia rotors. Journal of Fluids and Structures 88, 71–82.

ELTAYESH, ABDELGALIL, HANNA, MAGDY BASSILY, CASTELLANI, FRANCESCO, HUZAYYIN, A. S., EL-BATSH, HESHAM M., BURLANDO, MASSIMILIANO & BECCHETTI, MATTEO 2019 Effect of Wind Tunnel Blockage on the Performance of a Horizontal Axis Wind Turbine with Different Blade Number. Energies 12 (10), 1988.

GRANLUND, K., MONNIER, B., OL, M. & WILLIAMS, D. 2014 Airfoil longitudinal gust response in separated vs. attached flows. Physics of Fluids 26 (2), 027103, publisher: American Institute of Physics.

GRIBBEN, BRIAN J & HAWKES, GRAHAM S 2019 A potential flow model for wind turbine induction and wind farm blockage. Tech. Rep.. Frazer-Nash Consultancy.

HOWARD, KEVIN B. & GUALA, MICHELE 2016 Upwind preview to a horizontal axis wind turbine: a wind tunnel and field-scale study. Wind Energy 19 (8), 1371–1389.

JOHLS, HANNAH M., MARTÍNEZ-TOSSAS, LUIS A., CHURCHFIELD, MATTHEW J., LACKNER, MATTHEW A. &
2021 Floating platform effects on power generation in spar and semisubmersible wind turbines. *Wind Energy* 24 (8), 901–916.

Johlas, H. M., Martínez Tossas, L. A., Schmidt, D. P., Lackner, M. A. & Churchfield, M. J. 2019 Large eddy simulations of floating offshore wind turbine wakes with coupled platform motion. *Journal of Physics: Conference Series* 1256 (1), 012018, publisher: IOP Publishing.

Johnson, Wayne 1980 *Helicopter Theory*. Princeton University Press.

Jonkman, Jason 2008 Influence of Control on the Pitch Damping of a Floating Wind Turbine. In *46th AIAA Aerospace Sciences Meeting and Exhibit, Aerospace Sciences Meetings* NREL/CP-500-42589. American Institute of Aeronautics and Astronautics.

Koo, J.-K. & James, David F. 1973 Fluid flow around and through a screen. *Journal of Fluid Mechanics* 60 (3), 513–538, publisher: Cambridge University Press.

Lamb, Sir Horace 1916 *Hydrodynamics*, 4th edn. Cambridge: University Press.

Larsen, Gunner Chr & Hansen, Kurt S. 2014 Full-scale measurements of aerodynamic induction in a rotor plane. *Journal of Physics: Conference Series* 555, 012063, publisher: IOP Publishing.

Larsen, T. J. & Hansen, T. D. 2007 A method to avoid negative damped low frequent tower vibrations for a floating, pitch controlled wind turbine. *Journal of Physics: Conference Series* 75, 012073, publisher: IOP Publishing.

López-Queijia, Javier, Robles, Eider, Jugo, Josu & Alonso-Quesada, Santiago 2022 Review of control technologies for floating offshore wind turbines. *Renewable and Sustainable Energy Reviews* 167, 112787.

Mann, Jakob, Peña, Alfredo, Trolldborg, Niels & Andersen, Søren J. 2018 How does turbulence change approaching a rotor? *Wind Energy Science* 3 (1), 293–300, publisher: Copernicus GmbH.

Medici, D., Ivanell, S., Dahlberg, J.-Å. & Alfredsson, P. H. 2011 The upstream flow of a wind turbine: blockage effect. *Wind Energy* 14 (5), 691–697.

Meyer Forsting, A, Rathmann, Os, Laan, MP van der, Trolldborg, N, Gribben, B, Hawkes, G & Branlard, E 2021 Verification of induction zone models for wind farm annual energy production estimation. *Journal of Physics: Conference Series* 1934 (1), 012023.

Modarresi, K. & Kirchhoff, R. H. 1979 The Flow Field Upstream Of A Horizontal Axis Wind Turbine. *Wind Energy Center Reports*.

Sarmast, Sasán, Segalini, Antonio, Mikkelsen, Robert F. & Ivanell, Stefan 2016 Comparison of the near-wake between actuator-line simulations and a simplified vortex model of a horizontal-axis wind turbine. *Wind Energy* 19 (3), 471–481.

Simley, Eric, Angelou, Nikolaos, Mikkelsen, Torben, Söholm, Mikael, Mann, Jakob & Pao, Lucy Y. 2016 Characterization of wind velocities in the upstream induction zone of a wind turbine using scanning continuous-wave lidars. *Journal of Renewable and Sustainable Energy* 8 (1), 013301, publisher: American Institute of Physics.

Steirios, K. & Hultmark, M. 2018 Drag on flat plates of arbitrary porosity. *Journal of Fluid Mechanics* 853, publisher: Cambridge University Press.

Stevens, Richard J.A.M. & Meneveau, Charles 2017 Flow Structure and Turbulence in Wind Farms. *Annual Review of Fluid Mechanics* 49 (1), 311–339.

Taylor, G. I 1944 The aerodynamics of porous sheets. *Tech. Rep.* 2237. Aeronautical Research Council (Great Britain), London, England.

Tranter, Clement John 1968 *Bessel Functions with Some Physical Applications*. Hart Publishing Company.

de Vaal, J. B., Hansen, M. O. L. & Moan, T. 2014 Effect of wind turbine surge motion on rotor thrust and induced velocity. *Wind Energy* 17 (1), 105–121.

Wayman, E. N. (Elizabeth N.) 2006 Coupled dynamics and economic analysis of floating wind turbine systems. Thesis, Massachusetts Institute of Technology, accepted: 2007-01-10T16:56:13Z.

Wei, Nathaniel J. & Dabiri, John O. 2022 Phase-averaged dynamics of a periodically surging wind turbine. *Journal of Renewable and Sustainable Energy* 14 (1), 013305, publisher: American Institute of Physics.

Wen, Binrong, Tian, Xinliang, Dong, Xinghan, Peng, Zhike & Zhang, Wenming 2017 Influences of surge motion on the power and thrust characteristics of an offshore floating wind turbine. *Energy* 141, 2054–2068.

Yu, Wei, Tavernier, Delphine, Ferreira, Carlos, van Kuik, Gis A. M. & Schepers, Gerard 2019 New dynamic-inflow engineering models based on linear and nonlinear actuator disc vortex models. *Wind Energy* 22 (11), 1433–1450.