Invariant means and iterates of mean-type mappings

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Dedicated to Professor Janos Aczél on the occasion of his 95th birthday.

Abstract. A classical result states that for two continuous, strict means \( M, N : I^2 \to I \) (\( I \) is an interval) there exists a unique \((M, N)\)-invariant mean \( K : I^2 \to I \), i.e. such a mean that 
\[ K \circ (M, N) = K \]
and, moreover, the sequence of iterates \(((M, N)^n)_{n=1}^{\infty}\) converge to \((K, K)\) pointwise. Recently it was proved that continuity assumption cannot be omitted in general.

We show that if \( K \) is a unique \((M, N)\)-invariant mean then, without continuity assumption, \((M, N)^n \to (K, K)\).

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1. Introduction

It is known that if \( M \) and \( N \) are continuous bivariate means in an interval \( I \), and the mean-type mapping \((M, N)\) is diagonally contractive, that is,
\[ |M(x, y) - N(x, y)| < |x - y|, \quad x, y \in I, \ x \neq y, \] then there is a unique mean \( K : I^2 \to I \) that is \((M, N)\)-invariant and, moreover, the sequence of iterates \(((M, N)^n)_{n=\in\mathbb{N}}\) of the mean-type mapping \((M, N)\) converges to \((K, K)\) ([5, Theorem 4.5 (i),(ii)]).

For the results of this type, with more restrictive assumptions, see [1]. In particular, instead of (1) it was assumed that both means are strict; in [2] it was assumed that at most one mean is not strict, and condition (1) appeared first in [4] (see also [6]). Moreover, in all these papers the uniqueness of the invariant mean was obtained under the condition that it is continuous. In [5, Theorem 4.5] it was shown that this condition can be relaxed. In Proposition 2.8 it is shown that condition (1) holds if \( M \) is right-strict in both variables.
and $N$ is left-strict in both variables (or vice versa), that is essentially weaker condition than the strictness of the means.

In Sect. 3 we give conditions under which the uniqueness of the invariant mean guarantee the relevant convergence of the sequence of iterates of the mean-type mapping (see Theorem 3.3). If $K, M, N : I^2 \to I$ are arbitrary means and $K$ is a unique $(M, N)$-invariant mean, then the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping $(M, N)$ converges to $(K, K)$ pointwise in $I^2$.

In Sect. 4 we deliver certain conditions which guarantee uniqueness of invariant means.

2. Preliminaries

Let $I \subset \mathbb{R}$ be an interval. Recall that a function $M : I^2 \to \mathbb{R}$ is called a mean in $I$, if it is internal, i.e. if

$$\min (x, y) \leq M(x, y) \leq \max (x, y), \quad x, y \in I.$$

The mean $M$ is called strict, if these inequalities are sharp unless $x = y$; symmetric if $M(x, y) = M(y, x)$ for all $x, y \in I$.

For two means $M, N : I^2 \to I$, a mean $K : I^2 \to I$ is called invariant with respect to the mean-type mapping $(M, N) : I^2 \to I^2$ or, briefly, $(M, N)$-invariant, if $K \circ (M, N) = K$ [3].

Remark 2.1. If $M, N : I^2 \to I$ are two means in an interval $I$ then, for all $n \in \mathbb{N}$, the mapping $(M_n, N_n) := (M, N)^n$ is a mean-type mapping. Moreover every $(M, N)$-invariant mean is $(M_n, N_n)$-invariant for all $n \in \mathbb{N}$.

Remark 2.2. ([3]) If $K : I^2 \to I$ is a continuous symmetric and strictly increasing in both variables, then for every mean $M : I^2 \to I$ there exists a unique function $N : I^2 \to I$ such that $K \circ (M, N) = K$, and $N$ is a mean in $I$. Moreover, the mean $N$ is called complementary to $M$ with respect to $K$.

The following result is a consequence of bivariable version of Corollary 1 in [5] (see also [1,2,4]).

Theorem 2.3. If $M, N : I^2 \to I$ are continuous means such that the mean-type mapping $(M, N)$ is diagonally-contractive, then there is a unique mean $K : I^2 \to I$ which is $(M, N)$-invariant, and the sequence of iterates $((M_n, N_n))_{n \in \mathbb{N}}$ converges (uniformly on compact subsets of $I^2$) to $(K, K)$.

To formulate some simple sufficient conditions guarantying (1) let us introduce the following notions:

Definition 2.4. ([7]) Let $M : I^2 \to \mathbb{R}$ be a mean in an interval $I \subset \mathbb{R}$ and let $x_0, y_0 \in I$. The mean $M : I^2 \to \mathbb{R}$ is called:
• left-strict in the first variable at the point \( x_0 \), if for all \( t \in I \),
  \[ t < x_0 \implies M(x_0, t) < x_0; \]
• right-strict in the first variable at the point \( x_0 \), if for all \( t \in I \),
  \[ x_0 < t \implies x_0 < M(x_0, t); \]
• left-strict in the second variable at the point \( y_0 \), if for all \( t \in I \),
  \[ t < y_0 \implies M(t, y_0) < y_0; \]
• right-strict in the second variable at the point \( y_0 \), if for all \( t \in I \),
  \[ y_0 < t \implies y_0 < M(t, y_0). \]

The mean \( M \) is called \textit{left-strict in the first variable in a set} \( A \subset I \), if it is
left-strict in the first variable at every point \( x_0 \in A \); and \( M \) is called \textit{left-strict in the
first variable}, if it is left-strict in the first variable at every point. (Analogously one can introduce the remaining notions).

We adapt the convention that if a mean is left-strict (right-strict) in both variables then we call it simply \textit{left-strict (right-strict)}.

Then
\[ M \text{ is left-strict if and only if } (x \neq y \implies M(x, y) < \max(x, y)); \]
\[ M \text{ is right-strict if and only if } (x \neq y \implies M(x, y) > \min(x, y)). \]

\textit{Example 2.5.} The projective mean \( P_1 : \mathbb{R}^2 \ni (x, y) \mapsto x \), is left and right-strict
in the second variable, but it is neither left nor right strict at any point in the
first variable. Similarly, the projective mean \( P_2 : \mathbb{R}^2 \ni (x, y) \mapsto y \), is left and
right-strict in the first variable, but it is neither left nor right strict at any
point in the second variable.

The mean-type mapping \((P_1, P_2)\) coincides with the identity of \( \mathbb{R}^2 \), and of
course, the sequence of its iterates \(((P_1, P_2)^n : n \in \mathbb{N})\) converges to \((P_1, P_2)\).
Note that here \( P_1 \neq P_2 \), moreover, every mean \( K : \mathbb{R}^2 \to \mathbb{R} \) is \((P_1, P_2)\)-
invariant.

\textit{Example 2.6.} Consider the extreme mean \( \text{min} : \mathbb{R}^2 \to \mathbb{R} \). Since, \( t < x_0 \implies \min(x_0, t) < x_0 \) for every \( t \); the mean \( \text{min} \) is left-strict in the first variable
and in the second variable, but it is not right-strict in the first and the second
variable. Moreover, the extreme mean \( \text{max} : \mathbb{R}^2 \to \mathbb{R} \) is right-strict in the first
and the second variable, but it is not left-strict in the first and the second
variable.

For every \( n \in \mathbb{N} \), the \( n \)th iterate \((\min, \max)^n \) of mean-type mapping
\((\min, \max)\) has the form
\[ (\min, \max)^n(x, y) = \begin{cases} (x, y) & \text{if } x \leq y \\ (y, x) & \text{if } x > y \end{cases} \]
and, of course, the sequence of iterates of \(((\min, \max)^n : n \in \mathbb{N})\) converges.

On the other hand, one can check that a mean is invariant with respect to
the mean-type mapping \((\min, \max)\) if and only if it is symmetric.
Remark 2.7. If $M : I^2 \to \mathbb{R}$ is a strict mean then it is left and right-strict in each of the two variables.

Using the definition of one-side strict means we get the following result

**Proposition 2.8.** Let $M, N : I^2 \to I$ be two means in an interval $I$. If at every point one of the means $M$ and $N$ is strict both left and right then property (1) holds.

**Proof.** Suppose to the contrary that there exists $x, y \in I$, $x \neq y$ such that

$$|M(x, y) - N(x, y)| \geq |x - y| = |\max(x, y) - \min(x, y)|.$$

Binding this inequality with

$$\min(x, y) \leq M(x, y) \leq \max(x, y)$$

and

$$\min(x, y) \leq N(x, y) \leq \max(x, y),$$

we obtain that $M(x, y)$ and $N(x, y)$ are two different endpoints of the interval $[\min(x, y), \max(x, y)]$. Thus $\{M(x, y), N(x, y)\} = \{\min(x, y), \max(x, y)\}$.

Consequently, either

$$M(x, y) = \min(x, y) \quad \text{and} \quad N(x, y) = \max(x, y)$$

or

$$M(x, y) = \max(x, y) \quad \text{and} \quad N(x, y) = \min(x, y).$$

Assume that $x < y$ (which can be done without any loss of generality), we have that either $M(x, y) = x$ and $N(x, y) = y$ or $M(x, y) = y$ and $N(x, y) = x$.

In the first case, putting $x_0 = x$, $t := y$ we have $x_0 < t$ and $M(x_0, t) = x_0$ so, in view of Definition 2.4, $M$ is not right-strict in the first variable at the point $x_0$. Moreover, putting $t := x$, $y_0 := y$ we have $t < y_0$ and $N(t, y_0) = y_0$, so $N$ is not left-strict in the second variable at $y_0$.

We omit similar considerations in the second case. □

### 2.1. Weakly contractive mean-type mappings

In this section we deal with some relaxation of the assumption (1). For two means $M, N : I^2 \to I$, the mean-type mapping $(M, N)$ is *weakly contractive* if for every elements $x, y \in I$ with $x \neq y$ there is a positive integer $n = n(x, y)$ such that

$$|M_n(x, y) - N_n(x, y)| < |x - y|.$$

(2)

Next lemma provide a complete characterization of weak-contractivity in terms of $(M_2, N_2)$.

**Lemma 2.9.** Let $M, N : I^2 \to I$ be means. Mean-type mapping $(M, N)$ is weakly contractive if and only if mean-type mapping $(M_2, N_2)$ is contractive.
Proof. We prove that if inequality (2) is satisfied for some triple \((x, y, n)\) then it is also valid for \((x, y, 2)\). As \(M\) and \(N\) are means we get

\[ |M(x, y) - N(x, y)| \leq |x - y|. \]

Therefore

either \(|M(x, y) - N(x, y)| < |x - y|\) or \(|M(x, y) - N(x, y)| = |x - y|\).

In the first case we obtain validity of (2) for the triple \((x, y, 1)\) which, by mean property, implies its validity for \((x, y, 2)\).

In the second case we get, by mean property,

\[ \{M(x, y), N(x, y)\} = \{x, y\}. \]

If \((M, N)(x, y) = (x, y)\) then \((M, N)^n(x, y) = (x, y)\) for all \(n \in \mathbb{N}\) contradicting the assumption. Thus we obtain \(M(x, y) = y\) and \(N(x, y) = x\). Applying the same argumentation to the pair \((y, x)\) we get that either

\[ |M_2(x, y) - N_2(x, y)| = |M(y, x) - N(y, x)| < |x - y|, \]

which implies (2) for the triple \((x, y, 2)\) or

\[ (M_2, N_2)(x, y) = (M, N)(y, x) = (x, y). \]

However in the second subcase we get

\[ (M_n, N_n)(x, y) = \begin{cases} (y, x) & \text{if } n \text{ is odd,} \\ (x, y) & \text{if } n \text{ is even,} \end{cases} \]

which implies that (2) is valid for no \(n \in \mathbb{N}\) contradicting the assumption.

As the converse implication is trivial, the proof is complete \(\square\)

**Proposition 2.10.** Let \(M, N : I^2 \to I\) be two means in an interval \(I\). If all the following conditions are valid:

(i) \(M\) is right-strict in the first variable or \(N\) is left-strict in the second variable,

(ii) \(M\) is left-strict in the first variable or \(N\) right-strict in the second variable,

(iii) \(M\) is left-strict in the second variable or \(M\) is right-strict in the second variable or \(N\) is left-strict in the first variable or \(N\) is right-strict in the first variable,

then the mapping \((M, N)\) is weakly contractive.

**Proof.** Fix \(x, y \in I, x < y\). It suffices to prove that

\[ \{(M, N)^n(x, y) : n \in \mathbb{N}\} \not\subset \{(x, y), (y, x)\}. \]

Suppose to the contrary that \(\{(M, N)^n(x, y) : n \in \mathbb{N}\} \subset \{(x, y), (y, x)\}\).

Applying property (i) we obtain that \((M, N)(x, y) \neq (x, y)\). In view of property (ii) we get \((M, N)(y, x) \neq (y, x)\).

The only remaining case is that \((M, N)(x, y) = (y, x)\) and \((M, N)(y, x) = (x, y)\). Equivalently,

\[ M(x, y) = y, \quad N(x, y) = x, \quad M(y, x) = x, \quad \text{and} \quad N(x, y) = y. \]
However these equalities cannot be simultaneously valid as they are excluded by consecutive alternatives in (iii), respectively.

The case \( x > y \) is completely analogous.

\section{Uniqueness of invariant means and convergence of iterates of mean-type mappings}

\textbf{Definition 3.1.} Let \( M, N : I^2 \rightarrow I \) be means in an interval \( I \). Define the diagonal basin \( A_{M,N} \subset I^2 \) by

\[
\{(x, y) \in I^2 : ((M_n, N_n)(x, y))_{n=1}^{\infty} \text{ converges to some point on the diagonal}\}.
\]

\textbf{Theorem 3.2.} Let \( M, N : I^2 \rightarrow I \) be means in an interval \( I \). Then \( A_{M,N} \) is the maximal subset of \( I^2 \) such that all \((M, N)\)-invariant means are equal to each other.

\textbf{Proof.} Fix \( x, y \in I \). Let \( a = (a_n) \) be a sequence which is a shuffling of \( M_n \) and \( N_n \), i.e.

\[
a = (x, y, M_1(x, y), N_1(x, y), M_2(x, y), N_2(x, y), M_3(x, y), N_3(x, y), \ldots).
\]

By \([6, \text{p. 413}]\) we obtain that \( L(x, y) := \liminf_{n \rightarrow \infty} a_n \) and \( U(x, y) := \limsup_{n \rightarrow \infty} a_n \) are the smallest and the biggest \((M, N)\)-invariant means, respectively.

It is necessarily and also sufficient to prove that \( L(x, y) = U(x, y) \) if and only if \((x, y) \in A_{M,N}\).

If \( L(x, y) = U(x, y) \) then we obtain that the sequence \( a \) converges, whence \((M_n(x, y))_{n \in \mathbb{N}}\) and \((N_n(x, y))_{n \in \mathbb{N}}\) converge to a common limit, i.e. \((x, y) \in A_{M,N}\).

Conversely, if \((x, y) \in A_{M,N}\) then we get

\[
\lim_{n \rightarrow \infty} M_n(x, y) = \lim_{n \rightarrow \infty} N_n(x, y),
\]

which implies that the sequence \( a \) is convergent, and therefore \( L(x, y) = U(x, y) \). As \( L \) and \( U \) is the smallest and the greatest \((M, N)\)-invariant mean, we have that all \((M, N)\)-invariant means have the same value at the point \((x, y)\).

\textbf{Theorem 3.3.} Let \( K, M, N : I^2 \rightarrow I \) be arbitrary means. \( K \) is a unique \((M, N)\)-invariant mean if and only if the sequence of iterates \(((M, N)^n)_{n \in \mathbb{N}}\) of the mean-type mapping \((M, N)\) converges to \((K, K)\) pointwise in \( I^2 \).
Proof. For \( x, y \in I \). Let \( a \) be a sequence which is a shuffling of \( M_n \) and \( N_n \), i.e
\[
a = (x, y, M_1(x, y), N_1(x, y), M_2(x, y), N_2(x, y), M_3(x, y), N_3(x, y), \ldots).
\]
By [6] we obtain that
\[
L(x, y) := \liminf_{n \to \infty} a_n, \quad U(x, y) := \limsup_{n \to \infty} a_n
\]
are the smallest and the biggest \((M, N)\)-invariant means, respectively.

\((\Rightarrow)\) As \( K \) is a unique \((M, N)\)-invariant mean we get \( K = L = U \). It implies that the sequence \((a_n)\) is convergent to \( K(x, y) \). Therefore both \( M_n \) and \( N_n \) converge to \( K \) pointwise on \( I^2 \). In particular its Cartesian product \((M, N)^n = (M_n, N_n)\) converges to \((K, K)\) pointwise on the same set.

\((\Leftarrow)\) We know that the sequence \((a_n)\) is convergent to \( K(x, y) \), in particular \( K(x, y) = L(x, y) = U(x, y) \). As \( x, y \) were taken arbitrarily we have \( K = L = U \). It implies that the smallest and the biggest \((M, N)\)-invariant mean coincide providing the uniqueness. \(\square\)

4. Mean-type mappings with diagonally-contractive iterates and invariant means

**Theorem 4.1.** If \( M, N : I^2 \to I \) are continuous means such that \((M, N)\) is weakly contractive then there exists a unique mean \( K : I^2 \to I \) which is \((M, N)\)-invariant and, moreover, the sequence of iterates \(((M, N)^n)_{n \in \mathbb{N}}\) of the mean-type mapping \((M, N)\) converges (uniformly on compact subsets of \( I^2 \)) to \((K, K)\).

**Proof.** In view of Theorem 3.3 we get that \((M_2, N_2)\) is a contractive mean-type mapping, which is also continuous. Applying the result from [5] (see the beginning of this paper) we obtain that there exists a unique \((M_2, N_2)\)-invariant mean. As every \((M, N)\)-invariant mean is also \((M_2, N_2)\)-invariant we get that there exists at most one \((M, N)\)-invariant mean.

On the other hand, applying some general results from [6], we have that (without any assumptions for the means \( M \) and \( N \)) there exists at least one \((M, N)\)-invariant mean. Therefore there exists a unique \((M, N)\)-invariant mean; say \( K : I^2 \to I \).

We can now apply Theorem 3.3 to obtain that \(((M, N)^n)_{n \in \mathbb{N}}\) converges to \((K, K)\) pointwise on \( I^2 \). Moreover, in view of [5, Corollary 4.4 (ii)], this convergence is uniform on every compact subset of \( I^2 \). \(\square\)

**Remark 4.2.** Let us emphasize that the continuity assumption in the previous theorem cannot be omitted; see [6, section 3.1] for the details.
Theorem 4.3. Let $M, N : I^2 \to I$ be (not necessarily continuous) means and $c \in [0,1)$. If for every point $(x, y) \in I^2$, $x \neq y$, there is a positive integer $n = n(x, y)$ such that

$$|M_n(x, y) - N_n(x, y)| < c|x - y|,$$

where $(M_n, N_n) := (M, N)^n$, then there is a unique mean $K : I^2 \to I$ that is $(M, N)$-invariant and, moreover the sequence of iterates $((M, N)^n)_{n \in \mathbb{N}}$ of the mean-type mapping $(M, N)$ converges (uniformly on compact subsets of $I^2$) to $(K, K)$.

Proof. Take an arbitrary $(x, y) \in I^2$. Since $M$ and $N$ are means in $I$, for every $n \in \mathbb{N}$, we have

$$\min (x, y) \leq M(x, y) \leq \max (x, y), \quad \min (x, y) \leq N(x, y) \leq \max (x, y),$$

hence

$$\min (x, y) \leq \min (M(x, y), N(x, y))$$

$$\leq M(M(x, y), N(x, y)) \leq \max (M(x, y), N(x, y)) \leq \max (x, y),$$

and

$$\min (x, y) \leq \min (M(x, y), N(x, y))$$

$$\leq N(M(x, y), N(x, y)) \leq \max (M(x, y), N(x, y)) \leq \max (x, y),$$

whence

$$\min (x, y) \leq \min (M_1(x, y), N_1(x, y)) \leq \min (M_2(x, y), N_2(x, y))$$

$$\leq \max (M_2(x, y), N_2(x, y)) \leq \max (M_1(x, y), N_1(x, y))$$

$$\leq \max (x, y),$$

and, by induction, for every $n \in \mathbb{N}$,

$$\min (x, y) \leq \min (M_n(x, y), N_n(x, y)) \leq \min (M_{n+1}(x, y), N_{n+1}(x, y))$$

$$\leq \max (M_{n+1}(x, y), N_{n+1}(x, y)) \leq \max (M_n(x, y), N_n(x, y))$$

$$\leq \max (x, y).$$

Thus the sequence $(\min (M_n(x, y), N_n(x, y)) : n \in \mathbb{N})$ is increasing and the sequence $(\max (M_n(x, y), N_n(x, y)) : n \in \mathbb{N})$ is decreasing; and both are convergent. Clearly, if $x = y$ or $M_k(x, y) = N_k(x, y)$ for some positive integer $k$, then $M_n(x, y) = N_n(x, y)$ for all $n \geq k$, and consequently we have

$$\lim_{n \to \infty} M_n(x, y) = \lim_{n \to \infty} N_n(x, y).$$

Assume that

$$\lim_{n \to \infty} \min (M_n(x, y), N_n(x, y)) < \lim_{n \to \infty} \max (M_n(x, y), N_n(x, y)).$$

In particular, we have $M_n(x, y) \neq N_n(x, y)$ for every $n \in \mathbb{N}$. By the assumption, for every $n \in \mathbb{N}$ there is $k = k_n \in \mathbb{N}$ such that
\[|M_{kn}(M_n(x,y),N_n(x,y)) - N_{kn}(M_n(x,y),N_n(x,y))| < c|M_n(x,y) - N_n(x,y)|\]
equivalently
\[|M_{n+k_n}(x,y) - N_{n+k_n}(x,y)| < c|M_n(x,y) - N_n(x,y)|,
\]
which leads to a contradiction. Thus
\[
\lim_{n \to \infty} \min (M_n(x,y),N_n(x,y)) = \lim_{n \to \infty} \max (M_n(x,y),N_n(x,y)),
\]
and consequently,
\[
\lim_{n \to \infty} M_n(x,y) = \lim_{n \to \infty} N_n(x,y).
\]
Finally, we can define \(K\) as a (common) pointwise limit of sequences \((M_n)\) and \((N_n)\), as they are equal. \(\square\)

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