Dyson gas on a curved contour

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Abstract

We introduce and study a model of a logarithmic gas with inverse temperature $\beta$ on an arbitrary smooth closed contour in the plane. This model generalizes Dyson’s gas (the $\beta$-ensemble) on the unit circle. We compute the non-vanishing terms of the large $N$ expansion of the free energy ($N$ is the number of particles) by iterating the ‘loop equation’ that is the Ward identity with respect to reparametrizations and dilatation of the contour. We show that the main contribution to the free energy is expressed through the spectral determinant of the Neumann jump operator associated with the contour, or equivalently through the Fredholm determinant of the Neumann–Poincare (double layer) operator. This result connects the statistical mechanics of the Dyson gas to the spectral geometry of the interior and exterior domains of the supporting contour.

Keywords: Dyson, logarithmic gas, loop equation, free energy

1. Introduction

The logarithmic gases were introduced by F Dyson in the seminal papers [1], where it was shown that eigenvalues of random matrices can be represented as a statistical ensemble of charged particles with 2D Coulomb (logarithmic) interaction at inverse temperature $\beta > 0$. Particles can be located either on a circle or on an interval of a straight line. The partition function of the circular Dyson’s ensemble is

$$Z_N = \prod_{j=1}^N |\Delta_N(z_j)| \prod_{j \neq i} |z_i - z_j|, \quad z_j = e^{i\theta_j},$$

(1.1)
Here
\[ \Delta_N(\{z_i\}) = \prod_{j<k} (z_j - z_k) \] (1.2)
is the Vandermonde determinant. A comprehensive description of logarithmic gases can be found in [2].

The values \( \beta = \frac{1}{2}, 1, 2 \) are special. In these cases, the logarithmic gases (on various supports) can be represented either through free fermions or through some random matrix ensembles. Furthermore, for these special values of \( \beta \) the log-gases possess an integrable structure. In particular, the \( \beta = 1 \) partition function is the tau-function of the 2D Toda lattice hierarchy (see (4.44) and (4.45) below). Positive integer values of \( \beta \) are also special. In particular, the two-point correlation function then permits a \( \beta \)-dimensional integral representation valid for any \( N \). Other values of \( \beta \) do not seem to possess special properties. Nevertheless, a few particular integrals where computed explicitly for any values of \( \beta \). In particular, the integral (1.1) equals
\[ Z_N = (2\pi)^N \frac{\Gamma(1 + N\beta)}{\Gamma^N(1 + \beta)} \]
(see [1]). This is not the case for correlation functions, though.

In this paper, we introduce a natural generalization of the Dyson ensemble (1.1): the logarithmic gas on an arbitrary closed contour (a deformed circle) and study how the partition function and various correlation functions depend on the geometry of the contour. Namely, we consider the \( \beta \)-ensemble of \( N \) particles on a simple closed contour \( \Gamma \) on a plane. An early reference on this problem is [3] (section 7). For further convenience, we place the gas into \( N \)-independent external potential \( W \). The partition function reads
\[ Z_N = \oint_{\Gamma} \cdots \oint_{\Gamma} \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{k=1}^{N} e^{W(z_k)} |dz_k|. \] (1.3)

Clearly, the dimension of \( Z_N \) is \([\text{length}]^{3N^2 + (1-\beta)N}\). The real-valued function \( W \) is defined in a strip-like neighborhood of the contour and depends on both \( z \) and \( \bar{z} \) (we write \( W = W(z) \) just for simplicity of the notation). The meaning of the potential \( W \) may be understood as a metric on the contour, \( ds = e^W|dz| \). As a functional of \( W \), the partition function is the generating functional of correlation functions of densities. A similar problem having support on the complex plane, the 2D Dyson gas, had been considered by us earlier (see [4–7]).

We are mostly interested in the large \( N \) limit, \( N \rightarrow \infty \), and in the large \( N \) expansion of the free energy. Let us introduce the small parameter
\[ \hbar = \frac{1}{N}, \] (1.4)
then in the large \( N \) limit the behavior of the partition function is of the form
\[ Z_N = N! N^{(\beta-1)N} e^F, \]
where
\[ F = \frac{F_0}{\hbar^2} + \frac{F_1}{\hbar} + F_2 + O(\hbar), \quad \hbar \rightarrow 0. \] (1.5)
The factors $e^{F_0/\beta}$ and $e^{F_1/\hbar}$ carry the dimensions $[\text{length}]^{\beta N}$ and $[\text{length}]^{(1-\beta)N}$ respectively. We are interested in the contributions that do not vanish as $\hbar \to 0$, i.e., in the first three terms $F_0$, $F_1$ and $F_2$ of the large $N$ expansion of the free energy $F$.

We obtain explicit expressions for $F_0$, $F_1$ and $F_2$ in terms of the conformal map $w_{\text{ext}}(z)$ of the exterior domain bounded by the contour $\Gamma$ to the exterior of the unit circle (normalized as $w_{\text{ext}}(\infty) = \infty$, $w'_{\text{ext}}(\infty) > 0$) and the Neumann jump operator $\hat{N}$ (which is defined below, see (2.9)). The leading contribution, $F_0$, equals the electrostatic energy of a charged 2D conductor of the shape $\Gamma$ (neutralized by a charge at infinity). The next-to-leading contribution, $F_1$, reflects the entropy and the excluded volume (the effect that the place occupied by a particle is no longer accessible for others). The most interesting quantity is $F_2$. It is shown to be closely related to the spectral theory of the interior and exterior domains. Setting $W = 0$, we have:

$$F_0 = \beta \log r,$$

$$F_1 = -(\beta - 1) \log \frac{e^r}{2\pi \beta} + \log \frac{2\pi}{\Gamma(\beta)},$$

$$F_2 = \frac{(\beta - 1)^2}{8\pi \beta} \int_{\Gamma} \log \rho_0 \hat{N} \log \rho_0 \, ds - \frac{1}{2} \log \frac{P}{2\beta} + \frac{1}{2} \log \det \hat{N}.$$  

Here $r = (w'_{\text{ext}}(\infty))^{-1}$ is the external conformal radius of the domain $D$ bounded by the contour (we also call it the conformal radius of the exterior domain $\mathbb{C} \setminus D$), $\rho_0(z) = \frac{1}{4\pi}|w'_{\text{ext}}(z)|$ is the mean density of particles in the leading order, $P$ is the perimeter of the contour $\Gamma$, $ds = |dz|$ is the line element along the contour and $\det \hat{N}$ is the regularized spectral determinant of the Neumann jump operator with the zero mode removed. The result for $F_2$ can be naturally divided into ‘classical’ part having an electrostatic nature (the first term) and the ‘quantum’ part which origin is similar to the gravitational anomaly of the free bosonic field localized on the contour. The quantum part can be also represented as the logarithm of the Fredholm determinant associated to the domain surrounded by the contour $\Gamma$. It is explicitly given by equation (4.36). We also calculate one- and two-point correlation functions of densities of the particles in the large $N$ limit.

Our main technical tool is the so-called loop equation, which is an exact relation connecting one- and two-point correlation functions. It follows from the invariance of the partition function under re-parameterizations of the contour (admissible changes of integration variables). The term ‘loop equation’ goes back to the terminology used in the theory of random matrices but long before it is known as Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy in statistical mechanics or as Ward identity w.r.t. dilatations in quantum field theory. Although this relation is to some extent tautological, it becomes really meaningful when combined with the large $N$ expansion. We show that the general form of the loop equation for contours $\Gamma$ of arbitrary shape is a condition for the boundary value of a holomorphic quantity $T(z)$ closely related to the stress-energy tensor of a Gaussian field in 2D often utilized in conformal field theory (CFT). Its mean value $\langle T(z) \rangle$ is a holomorphic function both in the interior and exterior domains with a jump across the contour $\Gamma$. The boundary condition for the mean value is equivalent to the loop equation. It states that the jump of the off-diagonal tangential-normal component $\left[ T_m(z) \right]_\Gamma = \langle T_m^+(z) \rangle - \langle T_m^-(z) \rangle$ across the contour ($z \in \Gamma$, $T^{+/-}$ is the value from inside and $T^{-/-}$ from outside) vanishes. Below we will use the notation $[\ldots]_\Gamma$ for a jump across the contour.

Another technical tool is the rule of variation of the partition function under small deformations of the contour. We show that this variation is expressed through the jump $\left[ T_m(z) \right]_\Gamma$. 

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of the normal–normal component of $\langle T(z) \rangle$ across the contour:

\[ \hbar^2 \delta \log Z_N = -\frac{1}{\pi i} \oint_\Gamma \left[ \langle T_{nn}(z) \rangle \right]_z \delta n(z) dz. \]  

(1.7)

Here $\delta n(z)$ is the infinitesimal normal displacement of the contour at the point $z \in \Gamma$. This formula, combined with the large $N$ expansion of the loop equation, allows one to find variations of $F_0$, $F_1$ and $F_2$ and then, having at hand the variations, to restore them up to constant terms which do not depend on the shape of the contour. In principle, using the iterative procedure, one can also find the higher corrections to the free energy but the calculations become rather involved. This procedure is in parallel to the one which we used for the 2D Dyson gas in [6].

The organization of the paper is as follows. In section 2 we introduce the distribution functions and correlation functions of the main observables (density of charges, the potential created by them and an analog of the stress-energy tensor). In section 3 we obtain general relations valid at a finite $N$ and express the loop equation and the variation of the partition function in terms of boundary values of $T(z)$. Section 4 is devoted to the large $N$ expansion. There we develop a systematic procedure of the large $N$ expansion and obtain explicit formulas for the first three orders of the large $N$ expansion. In section 5 we calculate correlation functions of the analogs of vertex operators. There are also six appendices. In appendix A we list a few simple but useful formulas for differential geometry of a contour. In appendices B and C we review the Green functions of Laplace operator and discuss the Neumann jump operator. In appendix D we present some facts about the double layer potentials and Fredholm eigenvalues associated with the contour $\Gamma$. Appendix E contains the technique of variation of contour integrals we used in the text. In appendix F we discuss functional determinants of the Laplace operators in interior and exterior domains.

2. Some definitions and general relations

The partition function of our model is defined by the integral (1.3). We start with some definitions and general relations.

2.1. Distribution functions

Integrating in the rhs of (1.3) over a part of the variables keeping the others fixed, one can define a set of quantities which are proportional to the probability distributions to find particles at given points. For example,

\[ R(a) = \frac{1}{Z_N} e^{W(a)} \oint_\Gamma \left[ |\Delta_{N-1}(z)|^{2\beta} \prod_{j=1}^{N-1} |a - z_j|^{2\beta} e^{W(z_j)} \right] dz_j \]  

(2.1)

is the mean density of particles while

\[ R(a, b) = \frac{\hbar(N-1)}{Z_N} [a - b]^{2\beta} e^{W(a) + W(b)} \]

\[ \times \oint_\Gamma \left[ |\Delta_{N-2}(z)|^{2\beta} \prod_{j=1}^{N-2} |a - z_j|^{2\beta} |b - z_j|^{2\beta} e^{W(z_j)} \right] dz_j \]  

(2.2)
is the pair distribution function. Obviously, $R(a, a) = 0$ which means that two particles cannot be found in one and the same point. The normalization is chosen so that the functions $R(a)$, $R(a, b)$ are dimensionless and

$$\oint_{\Gamma} R(z) |dz| = 1, \quad \oint_{\Gamma} R(a, z) |dz| = (N - 1) h R(a).$$

Let $A(z_1, \ldots, z_N)$ be an arbitrary symmetric function of the variables $z_i$; its mean value is defined as

$$\langle A \rangle = \frac{1}{Z_N} \oint_{\Gamma} \cdots \oint_{\Gamma} A(z_1, \ldots, z_N) \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{k=1}^{N} e^{W(z_k)} |dz_k|.$$

Clearly, the mean values $\langle \sum_i f(z_i) \rangle$ and $\langle \sum_{i \neq j} f(z_i, z_j) \rangle$, etc are expressed through the distribution functions as follows:

$$\langle \sum_i f(z_i) \rangle = \frac{1}{h} \oint_{\Gamma} f(z) R(z) |dz|,$$

$$\langle \sum_{i \neq j} f(z_i, z_j) \rangle = \frac{1}{h^2} \oint_{\Gamma} \oint_{\Gamma} f(z, \zeta) R(z, \zeta) |dz||d\zeta|.$$

2.2. Density, potential and their correlation functions

The most important observables in the model are density of particles and 2D Coulomb field created by them. The density is defined as

$$\rho(z) = \bar{h} \sum_{i=1}^{N} \delta_{\Gamma}(z, z_i),$$

where $\delta_{\Gamma}(z, z')$ is the $\delta$-function on $\Gamma$ defined by the property $\oint_{\Gamma} f(z) \delta_{\Gamma}(z, a) |dz| = f(a)$ for $a \in \Gamma$ and a continuous function $f$. The field

$$\varphi(z) = -\beta h \sum_i \log |z - z_i|^2 = -2\beta \oint_{\Gamma} \log |z - \xi| \rho(\xi) |d\xi|$$

is the 2D Coulomb potential created by the particles. Equation (2.6) says that $\varphi(z)$ is the potential of a simple layer with density $\rho$. The simple layer potential $\varphi(z)$ is harmonic everywhere in the plane except for the contour $\Gamma$, where it is continuous across the boundary but has a jump of the normal derivative

$$[\partial_n \varphi(z)]_{\Gamma} = \partial_n^+ \varphi(z) - \partial_n^- \varphi(z) = 4\pi \beta \rho(z), \quad z \in \Gamma,$$

where $\partial_n^+$ and $\partial_n^-$ are normal derivatives inside and outside the contour $\Gamma$ respectively (with the unit normal vector looking outward in both cases). At infinity the field $\varphi$ behaves as

$$\varphi(z) = -2\beta \log |z| + O(|z|^{-1}).$$

Note the absence of the $O(1)$ term.
Equation (2.7) allows one to invert formula (2.6) in terms of the Neumann jump operator \( \hat{N} \), the operator that takes a function \( f \) on \( \Gamma \) to the jump of the normal derivative of its harmonic continuations to the domains inside and outside \( \Gamma \) (denoted by \( f_H \) and \( f^H \) respectively):

\[
\hat{N} f = \partial_n^+ f_H - \partial_n^- f^H.
\]  

(2.9)

See appendix C for details. We will also use the Dirichlet-to-Neumann operators \( \hat{N}^\pm \) which send a function \( f \) on \( \Gamma \) to normal derivatives of its harmonic continuations to the interior (exterior) domains:

\[
\hat{N}^+ f = \partial_n^+ f_H, \quad \hat{N}^- f = \partial_n^- f^H.
\]  

(2.10)

so that \( \hat{N} = \hat{N}^+ - \hat{N}^- \). In what follows, we denote the interior domain by \( D \), then the exterior one is \( \mathbb{C} \setminus D, \Gamma = \partial D \). Using the Neumann jump operator, we write equation (2.7) as

\[
\rho(z) = \frac{1}{2\pi} |w'_{\text{ext}}(z)| + \frac{1}{4\pi \beta} \hat{N} \varphi(z),
\]  

(2.11)

where \( w_{\text{ext}} \) is the conformal map from \( \mathbb{C} \setminus D \) onto the exterior of the unit circle such that \( w_{\text{ext}}(z) = z/r + O(1) \) as \( z \to \infty \) with a real positive \( r \) which is called the conformal radius of the exterior domain. Note that \([w'_{\text{ext}}(z)] = \partial_n^- \log |w_{\text{ext}}(z)|, z \in \Gamma\).

Correlation functions of densities can be obtained as variational derivatives of \( \log Z_N \) w.r.t. the potential. For example,

\[
\langle \rho(z) \rangle = \hbar \frac{\delta \log Z_N}{\delta W(z)}, \quad \langle \rho(z_1)\rho(z_2) \rangle_c = \hbar \frac{\delta \langle \rho(z_1) \rangle}{\delta W(z_2)} = \hbar^2 \frac{\delta^2 \log Z_N}{\delta W(z_1) \delta W(z_2)}.
\]  

(2.12)

Here \( \langle \rho(z_1)\rho(z_2) \rangle_c = \langle \rho(z_1) \rangle \langle \rho(z_2) \rangle - \langle \rho(z_1) \rangle \langle \rho(z_2) \rangle \) is the connected part of the correlation function. These formulas follow from the fact that variation of the partition function w.r.t. \( W \) inserts \( \sum_i \delta_\Gamma(z_i, z_i) \) into the integral. Clearly,

\[
\langle \rho(a) \rangle = R(a), \quad \langle \rho(a)\rho(b) \rangle = R(a, b) + \hbar \langle \rho(a) \rangle \delta_\Gamma(a, b).
\]  

(2.13)

Correlation functions of \( \varphi \)'s can be obtained from those of \( \rho \)'s using equation (2.6). An equivalent method is to consider linear response of the system to a small change of the potential \( \delta_\epsilon W(z) = \epsilon \log |z - a|^2 \) which means inserting a point-like charge \( \epsilon \to 0 \) at the point \( a \). We have the exact relations

\[
\delta_\epsilon \log Z_N = -\frac{\epsilon}{\beta \hbar} \langle \varphi(a) \rangle
\]  

(2.14)

and

\[
\delta_\epsilon \langle X \rangle = -\frac{\epsilon}{\beta \hbar} \langle X \varphi(a) \rangle_c
\]  

(2.15)

for any \( X \). In particular,

\[
\delta_\epsilon \langle \varphi(z) \rangle = -\frac{\epsilon}{\beta \hbar} \langle \varphi(z) \varphi(a) \rangle_c.
\]  

(2.16)
2.3. Stress tensor

A holomorphic traceless tensor

\[ T(z) = \beta^2 h^2 \sum_{j \neq k} \frac{1}{(z-z_j)(z-z_k)} + \beta h^2 \sum_j \frac{1}{(z-z_j)^2} + 2\beta h^2 \sum_j \frac{\partial W(z_j)}{z-z_j} \]  

(2.17)

plays an important role. We refer to it as stress tensor and utilize it in the next section. Here we express it in terms of the fields \( \varphi \) and \( \rho \):

\[ T(z) = (\partial \varphi(z))^2 + (1-\beta) h \partial_z^2 \varphi(z) + 2\beta h \int_\Gamma \frac{\partial W(\xi)}{z-\xi} |d\xi|. \]  

(2.18)

The mean value \( \langle T(z) \rangle \) is a holomorphic function for all \( z \notin \Gamma \) given by

\[ \langle T(z) \rangle = 2\beta^2 \int_\Gamma \frac{|d\xi|}{\xi-z} \int_\Gamma \frac{R(\xi, \zeta)|d\zeta|}{\xi-\zeta} + \beta h \int_\Gamma \frac{R(\xi)|d\zeta|}{\xi-\zeta} + 2\beta h \int_\Gamma \frac{\partial W(\xi)R(\xi)|d\zeta|}{z-\xi} |d\xi|, \]  

(2.19)

where the identities (2.4) are used. On the contour \( \Gamma \) it has a jump which follows from the Sokhotsky–Plemelj formula

\[ \int_\Gamma g(\zeta)|d\zeta| = \int_\Gamma g(\zeta)|d\zeta| \pm i \pi \nu(z)g(z), \]

where \( \nu(z) = -i dz/|dz| \) is the outward unit normal vector to \( \Gamma \) at the point \( z \) and \( z_\pm \) means that \( z \) approaches the contour from the interior/exterior. The second integral in (2.19) can be transformed as follows:

\[ \int_\Gamma \frac{R(\xi)|d\zeta|}{(z-\xi)^2} = i \int_\Gamma R(\xi) \nu(\zeta) d\frac{1}{\xi-z} = -i \int_\Gamma \frac{dR(\xi)\nu(\zeta)}{\xi-z} \]

\[ = -\int_\Gamma \frac{[i\partial_\xi R(\xi) + \kappa(\xi)R(\xi)]\nu(\zeta)|d\xi|}{\xi-z}. \]

Here \( \partial_\xi \) is the tangential derivative along \( \Gamma \) and \( \kappa(z) = i\nu(z)\partial_\xi \nu(z) \) is the curvature of the contour at the point \( z \). Using the notation

\[ [h(z)]_\Gamma = h(z_+)-h(z_-), \quad z \in \Gamma \]  

(2.20)

introduced in section 1 for the jump across \( \Gamma \), we have:

\[ \left[ \langle T(z) \rangle \right]_\Gamma = 2\pi \beta \nu(z) \left( 2\beta h \int_\Gamma \frac{R(z, \zeta)|d\zeta|}{z-\zeta} - 2h \partial W(z)R(z) - h \nu(z) \kappa(z) + i\partial_\xi R(z) \right). \]  

(2.21)

The first integral is convergent at \( \zeta = z \) because \( R(z, \zeta) \sim |z-\zeta|^{2\beta} \) as \( \zeta \to z \).

In what follows it will be instructive to divide the mean value of \( T \) in two parts, ‘classical’ and ‘quantum’: \( \langle T \rangle = \langle T \rangle^{(cl)} + \langle T \rangle^{(q)} \). The ‘classical’ part is obtained from (2.18) by disregarding the connected part of the pair correlation function:

\[ \langle T(z) \rangle^{(cl)} = (\langle \partial \varphi(z) \rangle)^2 + (1-\beta) h \partial_z^2 \langle \varphi(z) \rangle + 2\beta h \int_\Gamma \frac{\partial W(\xi) \langle \rho(\xi) \rangle}{z-\xi} |d\xi|. \]  

(2.22)
The ‘quantum’ contribution
\[ \langle T(z) \rangle^{[0]} = \langle (\partial \varphi(z))^2 \rangle_c \] (2.23)
is the connected part.

3. Exact relations at finite \( N \)

There are two basic relations: one is an identity for correlation functions which follow from the invariance of the partition function under re-parametrizations of the contour (the ‘loop equation’), another describes the variation of the partition function under small deformations of the contour. As we shall see, both can be expressed in terms of the jump of \( \langle T(z) \rangle \) across the contour.

3.1. Loop equation

Let us start from the obvious identity
\[
\sum_j \oint_{\Gamma} \cdots \oint_{\Gamma} \partial_j \left( \epsilon(z_j) \prod_{i k} |z_i - z_k|^{2 \alpha} \prod_m e^{W(z_m)} \right) \prod_{i=1}^N |dz_i| = 0, \tag{3.1}
\]
which expresses invariance of the partition function under re-parametrizations of the contour (changes of the integration variables). Here \( \epsilon(z) \) is an arbitrary differentiable function on \( \Gamma \). In terms of the mean value (2.3) the identity takes the form
\[
\left\langle \sum_j \left( \partial_j \epsilon(z_j) + \epsilon(z_j) \partial_j W(z_j) \right) + \beta \sum_{j \neq k} \epsilon(z_j) \partial_j \log |z_j - z_k|^2 \right\rangle = 0,
\]
which, according to (2.4), can be rewritten through the distribution functions:
\[
\oint_{\Gamma} |dz| \epsilon(z) \left( h \partial_z W(z) R(z) - h \partial_z R(z) + \beta \oint_{\Gamma} R(z, \xi) \partial_z \log |z - \xi|^2 |d\xi| \right) = 0. \tag{3.2}
\]
Since this identity holds for any \( \epsilon(z) \), it implies that
\[
\beta \oint_{\Gamma} R(z, \xi) \partial_z \log |z - \xi|^2 |d\xi| + h \partial_z W(z) R(z) - h \partial_z R(z) = 0. \tag{3.3}
\]
In fact this exact relation is the first equation of an infinite system connecting multi-point distribution functions, called the BBGKY hierarchy (after Bogoliubov, Born, Green, Kirkwood and Yvone). Note also that equation (3.3) can be obtained in a simpler way by differentiating the definition of \( R(z) \) (2.1) with respect to \( z \) along the contour \( \Gamma \) (cf [8]).

It remains to notice that equation (3.3) is equivalent to the following condition for the jump of \( \langle T(z) \rangle \) given by (2.21):
\[
\text{Im} \left\{ \nu^2(z) [\langle T(z) \rangle]_{z} \right\} = 0, \quad z \in \Gamma. \tag{3.4}
\]
In the frame with axes normal and tangential to the contour the components of the ‘stress tensor’ \( T \) read
\[
4T_{nn} = -4T_{ss} = \nu^2 T + i \nu^2 T,
4T_{sn} = i \nu^2 T - i \nu^2 T. \tag{3.5}
\]
Then the loop equation states that the tangential-normal component is continuous across the boundary:

\[
\left[\langle T_{\mu\nu}(z)\rangle \right]_{\Gamma} = 0, \quad z \in \Gamma.
\]  

(3.6)

We comment that the boundary condition is different from that used in the boundary CFT. There \(\langle T_{\mu\nu}(z)\rangle = 0\) when the argument approaches the boundary from one side.

If \(\Gamma\) is the unit circle, then \(\nu(z) = z\). Equation (3.3) acquires the form

\[
\beta \oint_{|z|=1} R(z, \xi) \frac{z + \xi}{z - \xi} + h \partial_j W(z) - h \partial_j R(z) = 0.
\]

It is equivalent to the more customary form of the loop equation

\[
\sum_j \left\{ \beta \sum_{k \neq j} \frac{z}{(z - z_j)(z - z_k)} + \frac{z}{(z - z_j)^2} + \frac{z \partial W(z_j) - z \partial W(z_j)}{z - z_j} - \frac{\beta N - \beta + 1}{z - z_j} \right\} = 0
\]

which follows from the obvious identity

\[
\sum_j \beta \prod_{k \neq j} \frac{1}{z - z_j} \prod_{m} e^{W(z_m)} \prod_{i=1}^{N} |dz_i| = 0
\]

valid for any \(z \in \mathbb{C}\) (outside of the unit circle).

Using the expression (2.18) for the stress tensor, we rewrite (3.6) in terms of correlation functions of the fields \(\varphi\) and \(\rho\). We have:

\[\text{Im} \left\{ \nu^2 \left( \left[ \langle (\partial \varphi)^2 \rangle \right]_{\Gamma} + h(1 - \beta) \left[ \langle (\partial^2 \varphi) \rangle \right]_{\Gamma} + 2\beta h \left[ \oint_{\Gamma} \frac{\partial W(\xi)}{z - \xi} |d\xi| \right]_{\Gamma} \right) \right\} = 0.\]

Let us consider the three terms in the lhs separately. In the first term, we separate the connected part and use equation (A6) from appendix A to write the other two:

\[2\text{Im} \left\{ \nu^2 \left[ \langle (\partial \varphi)^2 \rangle \right]_{\Gamma} \right\} = -\partial_{\rho} \left[ \langle \varphi \rangle \right]_{\Gamma}, \partial_{\varphi} \langle \varphi \rangle + 2\text{Im} \left\{ \nu^2 \left[ \langle (\partial^2 \varphi) \rangle \right]_{\Gamma} \right\}.\]

Here and below in similar calculations we take into account that the function \(\langle \varphi \rangle\) is continuous across the contour, so the jumps like \(\left[ \partial_{\rho} \langle \varphi \rangle \right]_{\Gamma}\) vanish. In the second term, we use equation (A8) from appendix A to get

\[2\text{Im} \left\{ \nu^2 \left[ \langle (\partial^2 \varphi) \rangle \right]_{\Gamma} \right\} = -\partial_{\varphi} \left[ \langle \rho \rangle \right]_{\Gamma} = -4\pi \beta \partial_{\varphi} \langle \rho \rangle.\]

In the third term we apply the Sokhotsky–Plemelj formula:

\[2\text{Im} \left\{ \nu^2 \left[ \oint_{\Gamma} \frac{\partial W(\xi)}{z - \xi} |d\xi| \right]_{\Gamma} \right\} = 2\pi \partial_{\varphi} W(z) \langle \rho(z) \rangle.\]

Combining the three terms, we bring the loop equation to the form

\[
\langle \rho(z) \rangle \left( \partial_{\varphi} \langle \varphi(z) \rangle - h \partial_{\rho} W(z) - (\beta - 1)h \partial_{\varphi} \langle \rho(z) \rangle \right) = \frac{1}{2\pi \beta} \text{Im} \left\{ \nu^2 \left[ \langle (\partial \varphi)^2 \rangle \right]_{\Gamma} \right\}
\]

(3.7)

which will be used for the large \(N\) expansion. The lhs comes from the classical part of the stress-energy tensor while the rhs is due to the ‘quantum’ fluctuations. Note that the three terms of the loop equation (3.7) have orders \(O(h^0)\), \(O(h^1)\) and \(O(h^2)\) respectively.
A comment on the relation to CFT is in order. The boundary condition there is \( \langle T_{sn} \rangle = 0 \), not \( \langle [T_{sn}]_\Gamma \rangle = 0 \). With this boundary condition, under a change of the geometry (a deformation of the contour) correlation functions transform according to the CFT rules. This is not the case in our model, and so the similarity with CFT is not complete.

### 3.2. Deformations of contour

Let us describe small deformations of the contour \( \Gamma \) by a normal displacement \( \delta n(z) \) of the point \( z \in \Gamma \) (positive for displacements in the outward direction). Below we use the rule for variation of the contour integral \( I = \oint_\Gamma f(z) dz \) given in [4] (see also appendix E). Let \( f \) be a function defined in a strip-like neighborhood of the contour, then

\[
\delta I = \oint_\Gamma \delta n(z)(\partial_n + \kappa(z)) f(z) |dz| + \oint_\Gamma \delta f(z) |dz|. \tag{3.8}
\]

Here \( \kappa(z) \) is the curvature of \( \Gamma \) at the point \( z \). The first term is due to the deformation of the contour. The second contribution is due to the variation of the function \( f \) itself (in the case if it depends on the shape of the contour).

According to (3.8), we have:

\[
\delta Z_N = \sum_j \oint_{\Gamma_j} \delta n(z_j) (\partial_n + \kappa(z_j)) \prod_{i \neq j} |z_i - z_k|^{2\beta} \prod_m e^{W_m(z_m)} \prod_{l=1}^N |dz_l|.
\]

Similarly to the derivation of equation (3.2) from identity (3.1), we write:

\[
\delta \log Z_N = \left( \sum_j \left( \kappa(z_j) + \partial_n W(z_j) + \beta \sum_{k \neq j} \partial_n \log |z_j - z_k|^2 \right) \delta n(z_j) \right)
\]

or, in terms of the mean density and the pair distribution function,

\[
\hbar^2 \delta \log Z_N = \oint_\Gamma |dz| \delta n(z) \left( \hbar \partial_n W(z) R(z) + \hbar \kappa(z) R(z) \right.
\]

\[
+ \left. \beta R(z, \xi) \partial_n \log |z - \xi|^2 |d\xi| \right).
\]

Comparing with (2.21), one can rewrite this relation in terms of the jump of \( \langle T(z) \rangle \):

\[
\hbar^2 \delta \log Z_N = -\frac{1}{2\pi \beta} \oint_\Gamma \text{Re} (\nu^2(z) \langle [T(z)]_\Gamma \rangle) \delta n(z) |dz|.
\tag{3.9}
\]

It can be written in some other forms. In terms of the normal–normal component of the stress-energy tensor we have

\[
\hbar^2 \delta \log Z_N = -\frac{1}{\pi \beta} \oint_\Gamma \langle [(T_{nn}(z))_\Gamma] \rangle \delta n(z) |dz|.
\tag{3.10}
\]

Combining (3.9) with the loop equation which says that the quantity \( \langle \nu^2(z) [T(z)]_\Gamma \rangle \) is real, we can also write

\[
\hbar^2 \delta \log Z_N = -\frac{1}{2\pi \beta} \oint_\Gamma \nu^2(z) \langle [T(z)]_\Gamma \rangle \delta n(z) |dz|.
\tag{3.11}
\]
For some particular deformations equation (3.9) gets simplified. For example, set
\[ \delta n(z) = \delta a \left( \frac{\nu(z)}{z-a} \right) \]
then a simple calculation of residues yields
\[ \beta \bar{h}^2 \delta \log Z_N = -2 \varepsilon \text{Re} \langle T(a) \rangle. \]  
(3.12)

For the variation \( \delta W \log Z_N \) caused by a small change of the potential, we obtain:
\[ h \delta W \log Z_N = \left\langle h \sum_j \delta W(z_j) \right\rangle = \oint_{\Gamma} \langle \rho(z) \delta W(z) \rangle |dz|. \]  
(3.13)

The total variation is \( \delta \log Z_N + \delta W \log Z_N \).

Let us represent the jump of \( \langle T_{nn} \rangle \) in the form similar to (3.7). We have:
\[ 4 \left[ \langle T_{nn}(z) \rangle \right]_{\Gamma} = 2 \text{Re} \left\{ \nu^2(z) \left( \left\langle \left( \partial \nu \right)^2 \right\rangle_{\Gamma} + h(1-\beta) \left\langle \left( \partial^2 \nu \right) \right\rangle_{\Gamma} \right) \right. 
+ 2 \beta \bar{h} \left[ \oint_{\Gamma} \frac{\partial W(\xi)}{z-\xi} \langle \rho(\xi) \rangle |d\xi| \right] \left. \right\} \].

In the first term, we separate the connected part and use equation (A5) from appendix A in the rest:
\[ 2 \text{Re} \left\{ \nu^2 \left[ \left\langle \left( \partial \nu \right)^2 \right\rangle \right]_{\Gamma} = \frac{1}{2} \left[ \partial_\nu \langle \nu \rangle_{\Gamma} \left( \partial_\nu^+ \langle \nu \rangle + \partial_\nu^- \langle \nu \rangle \right) \right. \right. 
+ \left. \left. 2 \text{Re} \left\{ \nu^2 \left[ \left\langle \left( \partial \nu \right)^2 \right\rangle \right]_{\Gamma} \right. \right\} \right. \}
\]

In the second term, we use equation (A7) from appendix A to get
\[ 2 \text{Re} \left\{ \nu^2 \left[ \left\langle \left( \partial \nu \right)^2 \right\rangle \right]_{\Gamma} = -\kappa \left[ \partial_\nu \langle \nu \rangle_{\Gamma} = -4\pi \beta \kappa \langle \rho \rangle \right. \right\} \right. \]

In the third term, we apply the Sokhotsky–Plemelj formula:
\[ 2 \text{Re} \left\{ \nu^2(z) \left[ \oint_{\Gamma} \frac{\partial W(\xi)}{z-\xi} \langle \rho(\xi) \rangle |d\xi| \right]_{\Gamma} \right\} = -2 \pi \partial_\nu W(z) \langle \rho(z) \rangle. \]

Combining the three terms, we obtain:
\[ 4 \left[ \langle T_{nn}(z) \rangle \right]_{\Gamma} = 2 \pi \beta \langle \rho(z) \rangle \left\{ \partial_\nu^+ \langle \nu(z) \rangle + \partial_\nu^- \langle \nu(z) \rangle - 2 h \partial_\nu W(z) \right. \right. 
+ 2(\beta - 1) h \kappa(z) \right. \right. \]
+ \left. \left. 2 \text{Re} \left\{ \nu^2(z) \left[ \left\langle \left( \partial \nu \right)^2 \right\rangle \right]_{\Gamma} \right. \right\} \right. \].
(3.14)

Similarly to (3.7), the first line comes from the classical part of the stress tensor while the second one is due to the quantum contribution.
4. Large $N$ expansion

The large $N$ (small $\bar{h}$) behavior of $\log Z_N$ is of the form (1.5). Our task is to obtain explicit expressions for the contributions to the free energy $F_0$, $F_1$ and $F_2$.

We assume that $\langle \rho(z) \rangle$ can be expanded in integer powers of $\bar{h}$:

$$\langle \rho(z) \rangle = \rho_0(z) + \bar{h}\rho_1(z) + \bar{h}^2 \rho_2(z) + \cdots$$

(4.1)

Then a similar expansion holds for $\langle \phi(z) \rangle$:

$$\langle \phi \rangle = \phi_0 + \bar{h}\phi_1 + \bar{h}^2 \phi_2 + \cdots$$

Their coefficients are related as $4\pi\beta\rho_j = [\partial_n \phi_j]_{\Gamma}$.

4.1. The leading order

In the leading order, the problem is equivalent to a 2D electrostatic problem of finding the equilibrium distribution of a charged conducting fluid. It can be easily solved by requiring the total electrostatic potential created by the charges and by the background to be constant along the contour. This will be the leading order of the regular iterative procedure based on the loop equation we develop below.

The connected correlation function in the right-hand side of the loop equation (3.7) is of order $O(\bar{h}^2)$. Therefore, at the 0th step, the loop equation states that

$$\rho_0 \partial_s \phi_0 = 0$$

which means that $\phi_0$ is constant along $\Gamma$. Then

$$\rho_0(z) = \frac{1}{2\pi} |w'_{\text{ext}}(z)|, \quad z \in \Gamma.$$  

(4.2)

This is the solution to the electrostatic problem of continuous charge distribution over the contour. In this approximation discreetness of the particles is neglected. The charges form a simple layer whose electric potential $\phi_0(z) = -2\beta \int_{\Gamma} \log|z - \xi| \rho_0(\xi) d\xi|$ reads

$$\phi_0(z) = \begin{cases} 
-2\beta \log r & \text{for } z \in D \\
-2\beta \log r - 2\beta \log |w_{\text{ext}}(z)| & \text{for } z \in \mathbb{C}\setminus D.
\end{cases}$$  

(4.3)

Next, we use (3.14) to find that

$$4 \left[ \langle T_{mn} \rangle \right]_{\Gamma} \cong 2\pi\beta\rho_0 \left( \partial^+_n \phi_0 + \partial^-_n \phi_0 \right).$$  

(4.4)

Here and below $\cong$ means equality in the leading order (up to higher powers of $\bar{h}$). This should be substituted into (3.10). After some simple transformations, one can find $\delta F_0$ with $F_0$ given by

$$F_0 = \beta \log r.$$  

(4.5)

This is the electrostatic energy of the charge distribution with density (4.2).
Two-point correlation functions in the leading order can be found using the variational formulas (2.12) or simple electrostatic arguments. Explicitly, the result is:

\[
\frac{\langle \varphi(z)\varphi(\zeta) \rangle_c}{2\beta h^2} \cong \begin{cases} 
G_{\text{ext}}(z, \zeta) - G_{\text{ext}}(z, \infty) & \text{if } z, \zeta \in C\setminus D \\
G_{\text{int}}(z, \zeta) - \log \frac{|z - \zeta|}{r}, & z, \zeta \in D \\
- G_{\text{ext}}(\infty, \zeta) - \log \frac{|z - \zeta|}{r}, & z \in D, \zeta \in C\setminus D.
\end{cases}
\]  

(4.6)

Here \(G_{\text{int}}, G_{\text{ext}}\) are Green’s functions of the Dirichlet boundary problems for the Laplace operator in the interior and exterior domains respectively (see appendix B). For derivatives of the fields this gives:

\[
\frac{\langle \partial \varphi(z)\partial \varphi(\zeta) \rangle_c}{\beta h^2} \cong \begin{cases} 
\frac{w'_\text{ext}(z)w'_\text{ext}(\zeta)}{(w_{\text{ext}}(z) - w_{\text{ext}}(\zeta))^2} - \frac{1}{(z - \zeta)^2}, & z, \zeta \in C\setminus D \\
\frac{w'_\text{int}(z)w'_\text{int}(\zeta)}{(w_{\text{int}}(z) - w_{\text{int}}(\zeta))^2} - \frac{1}{(z - \zeta)^2}, & z, \zeta \in D \\
- \frac{1}{(z - \zeta)^2}, & z \in D, \zeta \in C\setminus D.
\end{cases}
\]  

(4.7)

At merging points

\[
\lim_{\zeta \to \infty} \left( \frac{w'(z)w'(\zeta)}{(w(z) - w(\zeta))^2} - \frac{1}{(z - \zeta)^2} \right) = \frac{1}{6} \{w; z\},
\]

where

\[
\{w; z\} = \frac{w''(z)}{w'(z)} - \frac{3}{2} \left( \frac{w''(z)}{w'(z)} \right)^2 = \partial^2 \log w' - \frac{1}{2} (\partial \log w')^2
\]

is the Schwarzian derivative. Then we obtain from (4.7):

\[
\langle (\partial \varphi(z))^3 \rangle_c \cong \begin{cases} 
\frac{\beta h^2}{6} \{w_{\text{ext}}; z\}, & z \in C\setminus D \\
\frac{\beta h^2}{6} \{w_{\text{int}}; z\}, & z \in D.
\end{cases}
\]  

(4.8)

4.2. General scheme of iterative solution

We start from the loop equation in the form (3.7). According to (4.8), the rhs is a priori expected to be of the order \(O(h^2)\). However, the quantity \(\Im \langle \nu^2(z)\{w; z\} \rangle\) is the tangential derivative of the curvature \(\kappa(s)\) which is continuous across the boundary:

\[
\partial_s \kappa(s) = \Im \langle \nu^2(z)\{w_{\text{int}}; z\} \rangle = \Im \langle \nu^2(z)\{w_{\text{ext}}; z\} \rangle.
\]  

(4.9)

Therefore, the rhs is actually at least \(O(h^1)\) and the loop equation simplifies to

\[
h \partial_s W(z) - \partial_s \langle \varphi(z) \rangle + (\beta - 1) h \partial_s \log \langle \rho(z) \rangle = O(h^1).
\]  

(4.10)
Integrating, we write it as
\[ \langle \varphi(z) \rangle = h W(z) + h(\beta - 1) \log \langle \rho(z) \rangle + \lambda + O(h^3), \] (4.11)
where the constant \( \lambda \) reads
\[ \lambda = -2\beta \log r - h W_H(\infty) - h(\beta - 1)(\log \langle \rho \rangle)_H(\infty). \] (4.12)

It can be found by the harmonic extension of (4.11) to the exterior of the contour. The limit \( z \to \infty \) determines the constant.

Finally, applying \( \hat{N} \) to equation (4.11), we arrive at the equation
\[ \langle \rho(z) \rangle - \rho_0(z) = \frac{h}{4\pi \beta} \hat{N} W(z) + \frac{\beta - 1}{4\pi \beta} \hat{N} \log \langle \rho(z) \rangle + O(h^3), \] (4.13)
where \( \rho_0 \) is given by (4.2). The iterations give
\[ \rho_1 = \frac{\beta - 1}{4\pi \beta} \hat{N} \log \rho_0 + \frac{1}{4\pi \beta} \hat{N} W \]
\[ \rho_2 = \left( \frac{\beta - 1}{4\pi \beta} \right)^2 \hat{N} \rho_0^{-1} \hat{N} \log \rho_0 + \frac{\beta - 1}{(4\pi \beta)^2} \hat{N} \rho_0^{-1} \hat{N} W. \] (4.14)

The \( h \)-expansion of \( \langle \rho \rangle \) implies an \( h \)-expansion of the mean value of the stress tensor \( \langle T(z) \rangle \):
\[ \langle T \rangle = T_0 + h T_1 + h^2 T_2 + \cdots. \] According to (2.18), the first few terms of the latter are:
\[ T_0 = (\partial \varphi_0)^2 \]
\[ T_1 = 2 \partial \varphi_0 \partial \varphi_1 + (1 - \beta) \partial^2 \varphi_0 + 2\beta \oint_{\Gamma} \frac{\partial W(\xi) \rho_0(\xi)}{z - \xi} |d\xi| \]
\[ T_2 = (\partial \varphi_1)^2 + (1 - \beta) \partial^2 \varphi_1 + 2 \partial \varphi_0 \partial \varphi_2 + 2\beta \oint_{\Gamma} \frac{\partial \varphi_0}{x - \xi} \rho_1(\xi) |d\xi| \]
\[ + \lim_{h \to 0} h^{-2} \langle (\partial \varphi)^2 \rangle_c. \] (4.15)

A possible iterative procedure is as follows: first use the ‘loop equation’ \( \left[ (T_{sn})_j \right]_{\Gamma} = 0 \) to find \( \varphi_j \) on \( \Gamma \), then extend \( \varphi_j \) harmonically to the complex plane (which is necessary for finding normal derivatives) and find \( F_j \) from the variation
\[ \delta F_j = -\frac{1}{\pi \beta} \oint_{\Gamma} \left[ (T_{sn})_j \right]_{\Gamma} \delta n \, ds + \oint_{\Gamma} \rho_{j-1} \delta W \, ds. \] (4.16)
Here \( \delta W \) is a change of the potential \( W \) induced by a small deformation of the contour in the case when \( W \) explicitly depends on the contour (below we discuss such example).

For the \( h \)-expansion of \( \left[ (T_{sn}) \right]_{\Gamma} \) one can also use equation (3.14). It is convenient to express the rhs in terms of \( \langle \rho \rangle \). Regrouping the terms in (3.14) and substituting the normal derivatives \( \partial_n^+ \langle \varphi \rangle \), we obtain:
\[ 4 \langle [T_{nn}]_1 \rangle_\Gamma = -4\pi \beta \langle \rho \rangle \left( \beta |w'_\text{ext}| + \hbar \partial_n (W - \frac{1}{2} (W_H + W^H)) \right) + 4\pi \beta (\beta - 1) \hbar \langle \rho \rangle \left( \frac{1}{2} \partial_n ((\log \langle \rho \rangle)_H + (\log \langle \rho \rangle)^H) + \kappa \right) + 2 \Re \left\{ \nu^2 \langle (\partial \rho)^2 \rangle \right\}_\Gamma. \] 

The first line is \( O(1) \), the second one is \( O(\hbar) \) and the third one is \( O(\hbar^2) \).

4.3. The next-to-leading order

The correction \( \rho_1 \) is given by (4.14). The corresponding correction to \( \langle \varphi \rangle \) is given by

\[ \varphi_1(z) = (\beta - 1) (\log \rho_0(z) - (\log \rho_0)_H(\infty)) + W(z) - W_H(\infty), \quad z \in \Gamma. \] 

The constant term is found from the requirement that \( \varphi_1(z) \to 0 \) as \( |z| \to \infty \). The values of \( \varphi_1 \) outside the contour are obtained by the harmonic continuations. Further, using (4.17), we find:

\[ 4 \langle [T_{nn}]_1 \rangle_\Gamma = 2\beta |w'_\text{ext}| \left( (\beta - 1) \left( \partial_n (\log \rho_0)^H + \kappa \right) + \partial_n (W_H - W) \right). \] 

This is the variation of the following simple functional:

\[ F_1 = (\beta - 1) \oint_{\Gamma} \rho_0 \log \rho_0 \, ds + \oint_{\Gamma} \rho_0 W \, ds + c_1. \] 

The first term is the ‘entropy-like’ contribution. It is due to entropy and excluded volume. The integral can be easily calculated and the result is expressed through the logarithm of the conformal radius \( r \) of the exterior domain:

\[ F_1 = - (\beta - 1) \log(2\pi r) + \oint_{\Gamma} \rho_0 W \, ds + c_1. \] 

The constant \( c_1 \) does not depend on the shape of the contour and cannot be determined from the variation. It can be found from a comparison with the exact result for the circle (see below).

4.4. Calculation of \( F_2 \)

First of all, let us separate the classical and quantum parts in \( F_2 \) according to (2.22) and (2.23):

\[ F_2 = F_2^{(c)} + F_2^{(q)}. \]

Their variations are determined by the respected parts of \( T_2 \):

\[ T_2^{(c)} = (\partial \varphi_1)^2 + (1 - \beta) \partial^2 \varphi_1 + 2 \partial \varphi_0 \partial \varphi_2 + \text{the } W \text{- dependent part}, \]

\[ T_2^{(q)} = \hbar^{-2} \langle (\partial \rho)^2 \rangle_\Gamma. \]

Note that the previously found \( F_0 \) and \( F_1 \) are in this sense classical quantities. The order \( \hbar^2 \) is the first one when the quantum correction appears.

**The classical part.** The direct expansion of (4.17) to the order \( \hbar^2 \) gives
\[
4 \left[ (T_{\text{Int}}^{(\text{cl})})_2 \right]_{\Gamma} = -4 \pi \beta \psi \left[ \frac{1}{2} |w_{\text{ext}}'| \rho_2 + 2 \pi \beta \rho_1 \left( \partial_n^+ (W_H - W) + \partial_n^- (W_H - W) \right) + 2 \pi \beta (\beta - 1) \rho_1 \left( \partial_n^+ (\log \rho_0) \hat{W} + \partial_n^- (\log \rho_0) \hat{W} + 2 \kappa \right) \right]_{\Gamma}.
\]

The final result for \( F_2^{(\text{cl})} \) is:
\[
F_2^{(\text{cl})} = \frac{1}{8 \pi \beta} \oint_{\Gamma} ((\beta - 1) \log \rho_0 + W) \hat{N} ((\beta - 1) \log \rho_0 + W) \, ds.
\] (4.23)

One can verify that the variation of this functional under small deformations of the contour is given by (4.16) with \( T_{\text{Int}} \) as in (4.22).

**The quantum part.** Using (4.8) it is straightforward to find the variation of the quantum part:
\[
\delta F_2^{(q)} = \frac{1}{12 \pi} \oint_{\Gamma} \Re \left( \nu^2 \left( \{ w_{\text{ext}} \}; z \right) - \{ w_{\text{int}} \}; z \right) \right) \, ds.
\] (4.24)

In order to understand this result, we should compare it with the variations of the regularized determinants of the Laplace operators \( \Delta = 4 \partial \bar{\partial} \) in the interior and in the exterior domains \( D, C \setminus D \) with Dirichlet boundary conditions. These determinants are given by formulas of the Polyakov–Alvarez type [9, 10], see also [11–14] and appendix E. For planar domains they are given by integrals over boundary:

\[
\log \det (-\Delta_{\text{int}}) = -\frac{1}{12 \pi} \oint_{\Gamma} \left( \psi_{\text{int}} \partial_n \psi_{\text{int}} - 2 \psi_{\text{int}} \psi_{\text{int}} \right) \, ds,
\] (4.25)

\[
\log \det (-\Delta_{\text{ext}}) = \frac{1}{12 \pi} \oint_{\Gamma} \left( \psi_{\text{ext}} \partial_n \psi_{\text{ext}} - 2 \psi_{\text{ext}} \psi_{\text{ext}} \right) \, ds,
\] (4.26)

where \( \psi_{\text{int}}(z) = \log |w_{\text{int}}'(z)|, \psi_{\text{ext}}(z) = \log |w_{\text{ext}}'(z)| \). Here \( w_{\text{int}}(z) \) is the conformal map from \( D \) to the unit disk. We note that formulas for logarithms of determinants of this type should be understood modulo some additive normalization constants in the rhs which do not depend on the shape of \( \Gamma \). Hereafter, we omit these constants for simplicity. Equivalent expressions given by integrals over the unit circle in the \( w \)-plane in terms of the inverse conformal maps \( z_{\text{int}}(w), z_{\text{ext}}(w) \) are more customary:

\[
\log \det (-\Delta_{\text{int}}) = \frac{1}{12 \pi} \oint_{|w|=1} \left( \phi_{\text{int}} \partial_n \phi_{\text{int}} + 2 \phi_{\text{int}} \right) \, |dw|,
\] (4.27)

\[
\log \det (-\Delta_{\text{ext}}) = \frac{1}{12 \pi} \oint_{|w|=1} \left( \phi_{\text{ext}} \partial_n \phi_{\text{ext}} + 2 \phi_{\text{ext}} \right) \, |dw|,
\] (4.28)

where \( \phi_{\text{int}}(w) = \log |z_{\text{int}}'(w)|, \phi_{\text{ext}}(w) = \log |z_{\text{ext}}'(w)| \). A direct calculation using the rules explained in appendix D shows that

\[
\delta \log \det (-\Delta_{\text{int}}) = \frac{1}{6 \pi} \oint_{\Gamma} \left( \Re \left( \nu^2 \{ w_{\text{int}} \}; z \right) \right) - \kappa^2 \right) \, ds |dz|,
\] (4.29)
\[ \delta \log \det (-\Delta_{\text{ext}}) = -\frac{1}{6 \pi} \oint_{\Gamma} \left( \Re \{ \nu^2 \{ w_{\text{ext}}; z \} \} - \kappa \right) \delta n |dz|. \]  

(4.30)

Hence we can write, taking into account (4.24):

\[ \delta F_2^{(q)} = -\frac{1}{2} \delta \log \det (\Delta_{\text{int}} \Delta_{\text{ext}}) \]  

(4.31)

or

\[ F_2^{(q)} = -\frac{1}{2} \log \det (-\Delta_{\text{int}}) - \frac{1}{2} \log \det (-\Delta_{\text{ext}}) + \text{const}, \]  

(4.32)

where the additive constant does not depend on the shape of the contour.

This result can be represented in yet another instructive form if one recalls the surgery (Mayer–Vietoris type) formula for the functional determinants of Laplace operators in complementary domains [14, 15]:

\[ \log \det(-\Delta_{\text{int}}) + \log \det(-\Delta_{\text{ext}}) + \log \det \hat{N} = \log P + \text{const}, \]  

(4.33)

where \( P = \oint_{\Gamma} ds \) is the perimeter of the boundary of the domain \( D \) (the length of the contour \( \Gamma \)) and \( \det \hat{N} \) is the regularized determinant of the Neumann jump operator with the zero mode removed. A simple heuristic derivation of the surgery formula is given in appendix E. This formula allows one to represent the final result for \( F_2^{(q)} \) in the form

\[ F_2^{(q)} = \frac{1}{2} \log \det \hat{N} - \frac{1}{2} \log P + c_2. \]  

(4.34)

The constant \( c_2 \) is determined below from the comparison with the result for the circle. The explicit expression of this quantity in terms the exterior and interior conformal maps are:

\[ F_2^{(q)} = \frac{1}{24 \pi} \oint_{\Gamma} \left( \psi_{\text{ext}} \partial_n \psi_{\text{ext}} - \psi_{\text{int}} \partial_n \psi_{\text{int}} + 2 \kappa (\psi_{\text{ext}} - \psi_{\text{int}}) \right) ds + \text{const} \]

\[ = \frac{1}{24 \pi} \oint_{|w|=1} \left( \phi_{\text{int}} \partial_n \phi_{\text{int}} - \phi_{\text{ext}} \partial_n \phi_{\text{ext}} + 2 (\phi_{\text{int}} - \phi_{\text{ext}}) \right) |dw| + \text{const}. \]  

(4.35)

Let us point out yet another instructive representation of \( F_2^{(q)} \). As is shown in appendix D, it is given by logarithm of the Fredholm determinant associated to the domain \( D \) (see [16] for a comprehensive treatment of Fredholm eigenvalues of the domain \( D \)). Using equation (D16) from appendix D, one can write (4.34) in the form

\[ F_2^{(q)} = -\frac{1}{2} \log \det (I + \hat{V}) + \text{const}, \]  

(4.36)

where \( I \) is the identity operator and \( \hat{V} \) is the double layer potential operator

\[ \hat{V} f(z) = \frac{1}{\pi} \oint_{\Gamma} f(\xi) \partial_n \log |\xi - z| d\xi, \quad z \in \Gamma \]  

(4.37)

also known as the Neumann–Poincaré operator. Here \( \oint_{\Gamma} \) is the principal value integral. Note that the signs in front of \( \log \det \) in (4.34) and (4.36) are opposite. The relation between the Fredholm determinant associated with a plane domain and determinants of the Laplace operators was mentioned in [17] (corollary 4.12).
4.5. Examples and particular cases

The case of no potential: \( W = 0 \). In this case, our formulas simplify:

\[
\begin{align*}
\rho_0 &= \frac{1}{2\pi} |w_{\text{ext}}'|, \\
\rho_1 &= \frac{\beta - 1}{4\pi\beta} \mathcal{N} \log |w_{\text{ext}}'|, \\
\rho_2 &= 2\pi \left( \frac{\beta - 1}{4\pi\beta} \right)^2 \frac{1}{|w_{\text{ext}}'|} \mathcal{N} \log |w_{\text{ext}}'|
\end{align*}
\]  
(4.38)

(this is obtained by iteration of equation (4.13) with \( W = 0 \)) and

\[
\begin{align*}
F_0 &= \beta \log r, \\
F_1 &= -(\beta - 1) \log(2\pi r) + c_1, \\
F_2^{(c)} &= \frac{(\beta - 1)^2}{8\pi\beta} \int r \log |w_{\text{ext}}'| \mathcal{N} \log |w_{\text{ext}}'| \, ds + \text{const.}
\end{align*}
\]  
(4.39)

The constant \( c_1 \) is given below in (4.42). The quantum part of \( F_2 \) is given by (4.34). Note that if \( W = 0 \), then contour-dependent contributions of \( F_1 \) and of the classical part of \( F_2 \) vanish at \( \beta = 1 \).

\( \Gamma \) is a circle and \( W = 0 \). In the case when \( \Gamma \) is a circle of radius \( r \) and \( W = 0 \) the integral for the partition function had been explicitly evaluated [1]:

\[
Z_N^{(0)} = \oint_{|z_1| = r} \ldots \oint_{|z_N| = r} \prod_{i < j} |z_i - z_j|^2 |dz_i| \\
= r^{N^2 + (1-\beta)N(2\pi)^N} \frac{\Gamma(1 + N\beta)}{\Gamma^N(1 + \beta)}. 
\]  
(4.40)

The conformal maps are \( w_{\text{int}}(z) = w_{\text{ext}}(z) = z/r \). The density \( \rho_0 \) is constant: \( \rho_0 = 1/(2\pi r) \). The large \( N \) expansion is of the form (1.5) with

\[
\begin{align*}
F_0 &= \beta \log r, \\
F_1 &= -(\beta - 1) \log \frac{re}{\beta} + \log \frac{2\pi}{\Gamma(\beta)}, \\
F_2 &= \log \sqrt{\beta}.
\end{align*}
\]  
(4.41)

(The expansion is made under assumptions that \( N \to \infty \) together with \( N\beta \to \infty \), so it is essential here that \( \beta \neq 0 \)). Note that for the circle we have \( \mathcal{N} = 2\mathcal{N}^+ \) (see (C6)) and \( \operatorname{det}' \mathcal{N}^+ \) is known from [18, 19]: \( \operatorname{det}'(2\mathcal{N}^+) = \pi r \), so \( F_2 \) does not depend on \( r \). Comparison with (4.39) allows one to find the constants \( c_1, c_2 \):

\[
\begin{align*}
c_1 &= (\beta - 1) \log \frac{2\pi\beta}{e} + \log \frac{2\pi}{\Gamma(\beta)}, \\
c_2 &= \frac{1}{2} \log(2\beta).
\end{align*}
\]  
(4.42, 4.43)

A complete large \( N \) expansion of (4.40) is readily available.
The case $\beta = 1$. The value $\beta = 1$ in (1.3) is special. In this case the partition function has the determinant representation

$$Z_N^{(\beta = 1)} = N! \det_{N \times N} M_{ij}, \quad 1 \leq i, j \leq N,$$

(4.44)

where $M_{ij}$ is the matrix of moments:

$$M_{ij} = \oint \Gamma z^{i-j-1} \bar{z}^{j-1} e^{W(z)} |dz|.$$

(4.45)

The proof is a simple calculation which is almost the same as for the circular Dyson ensemble.

The determinant representation implies the integrable structure. Let the potential $W$ be of the form

$$W(z) = W^0(z) + \sum_{k \geq 1} (t_k z^k + \bar{t}_k \bar{z}^k),$$

where $W^0$ is a fixed background potential and $t_k$ are varying parameters. It can be proved (see, e.g., [7]) that the partition function $Z_N$ as a function of the 'times' $t_k, \bar{t}_k$ and $N$ is the tau-function of the 2D Toda lattice hierarchy [20]. As such, it satisfies the full set of bilinear Hirota equations.

The case $W(z) = (1 - \beta) \log |w'_{\text{ext}}(z)|$. The partition function reads

$$Z_N = \oint \cdots \oint \prod_{i<j} |z_i - z_j|^{2a} \prod_{k=1}^N |w'_{\text{ext}}(z_k)|^{1-\beta} |dz_k|.$$

(4.46)

In this case the potential depends on the contour and one should use the general formula (4.16). One can see that the contour-dependent contributions to $F_1$ and to the classical part of $F_2$ vanish while the quantum part of $F_2$ is given by the same formula (4.36). Therefore, the result for $Z_N$ as $N \to \infty$ is given entirely in terms of the Fredholm determinant associated to the domain $D$:

$$Z_N = C r^{N^2} (\det(\bar{\cal I} + \bar{V}))^{-1/2} (1 + O(1/N)).$$

(4.47)

5. Correlation functions of vertex operators

Along with the density and potential the operators

$$V_\alpha(z) = \prod_{j=1}^N (z - z_j)^{\alpha}, \quad V_\alpha(z, \bar{z}) = \prod_{j=1}^N |z - z_j|^{2\alpha}$$

are of interest. Following the analogy with CFT, we call them vertex operators. One can write $V_\alpha(z, \bar{z}) = e^{-\pi \beta \phi(z)}$, $V_\alpha(z) = e^{-\pi \beta \phi(z)}$, where $\phi(z)$ is the holomorphic part of the potential $\varphi$:

$$\phi(z) = -\beta \bar{h} \sum_j \log(z - z_j), \quad \varphi(z) = \phi(z) + \overline{\phi(z)}.$$

(5.1)

Below in this section, we find correlation functions of these fields in the leading order setting the external potential $W$ to be zero and choosing $z$ in the exterior domain.
Correlation functions of $V_\alpha(z)$. Using the well known formula for the mean value of exponential function, we can write, in the large $N$ limit:

$$
\langle V_\alpha(z) \rangle = \exp \left( -\frac{\alpha}{\beta \hbar} \langle \phi(z) \rangle + \frac{\alpha^2}{2\beta^2 \hbar^2} \langle \phi^2(z) \rangle + O(\hbar) \right).
$$

(5.2)

The correlation functions of the field $\phi$ can be easily found from those of the field $\varphi$ by extracting the holomorphic part. Now we set the point $z$ in the exterior of $\Gamma$. In this case

$$
\langle \phi(z) \rangle = -\beta \log(rw_{\text{ext}}(z)) + \frac{\hbar}{2} (\beta - 1) \log(rw'_{\text{ext}}(z)) + O(\hbar^2)
$$

(the second term comes from (4.18)) and

$$
\frac{\langle \phi(z)\phi(\zeta) \rangle}{\beta \hbar} \approx \log \frac{rw_{\text{ext}}(z) - rw_{\text{ext}}(\zeta)}{z - \zeta}.
$$

Clearly, the rhs becomes $\log(rw'_{\text{ext}}(z))$ as $\zeta \to z$. Therefore, one obtains from (5.2):

$$
\langle V_\alpha(z) \rangle \approx (rw_{\text{ext}}(z))^{\alpha N}(rw'_{\text{ext}}(z))^{\beta \alpha + 1 - \beta}.
$$

(5.3)

Correlation functions of multiple products of the fields $V_\alpha$ can be obtained in a similar way. For example,

$$
\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2) \rangle \equiv \langle V_{\alpha_1}(z_1) \rangle \langle V_{\alpha_2}(z_2) \rangle \left( \frac{rw_{\text{ext}}(z_1) - rw_{\text{ext}}(z_2)}{z_1 - z_2} \right)^{\alpha_1 \alpha_2}.
$$

(5.4)

and, more generally,

$$
\left\langle \prod_{p=1}^{m} V_{\alpha_p}(z_p) \right\rangle = \prod_{p=1}^{m} \langle V_{\alpha_p}(z_p) \rangle \prod_{p < q} \left( \frac{rw_{\text{ext}}(z_p) - rw_{\text{ext}}(z_q)}{z_p - z_q} \right)^{\alpha_p \alpha_q}.
$$

(5.5)

Formulas for correlation functions of these fields for $z$ in the interior of $\Gamma$ include not only the interior conformal map $w_{\text{ext}}(z)$ but also the holomorphic part of the harmonic continuation of the function $\log |w'_{\text{ext}}(z)|$ to the interior. We do not present them here.

Correlation functions of $V_\alpha(z, \bar{z})$. Similarly to (5.2), we write

$$
\langle V_\alpha(z, \bar{z}) \rangle = \exp \left( -\frac{\alpha}{\beta \hbar} \langle \varphi(z) \rangle + \frac{\alpha^2}{2\beta^2 \hbar^2} \langle \varphi^2(z) \rangle + O(\hbar) \right)
$$

(5.6)

and make use of (4.6) and (4.18). Plugging

$$
\langle \varphi(z) \rangle = -2\beta \log |w_{\text{ext}}(z)| + \hbar(\beta - 1) \log |w'_{\text{ext}}(z)| + O(\hbar^2),
$$

$$
\langle \varphi^2(z) \rangle \approx 2\beta \hbar^2 \log \left| \frac{rw'_{\text{ext}}(z)}{1 - |w_{\text{ext}}(z)|^{-2}} \right|
$$

into (5.6), we obtain:

$$
\langle V_\alpha(z, \bar{z}) \rangle \approx |rw_{\text{ext}}(z)|^{2\alpha N} |rw'_{\text{ext}}(z)|^{\beta \alpha} \left( 1 - |w_{\text{ext}}(z)|^{-2} \right)^{-\frac{\beta^2}{4}}.
$$

(5.7)

and
\[
\langle \prod_{p=1}^{m} V_{\alpha p}(z_p, \bar{z}_p) \rangle = \prod_{p=1}^{m} \langle V_{\alpha p}(z_p, \bar{z}_p) \rangle \prod_{p<q} \frac{\rho_{\text{ext}}(z_p) - \rho_{\text{ext}}(z_q)}{(z_p - z_q)(1 - w_{\text{ext}}(z_p)w_{\text{ext}}(z_q))}
\]

These formulas match the correlation functions of vertex operators in the boundary CFT with the central charge \( c = 1 - 6 \left( \beta^{1/2} - \beta^{-1/2} \right)^2 \).

6. Concluding remarks

We have introduced and studied the statistical model of logarithmic gas (the \( \beta \)-ensemble) on a general smooth closed contour \( \Gamma \subset \mathbb{C} \) of arbitrary form. This model generalizes Dyson’s circular \( \beta \)-ensembles. We have developed a systematic procedure of the large \( N \) expansion based on an identity for correlation functions often called loop equation. Its origin is reparameterization invariance of the contour integrals representing the partition function. A combination of the loop equation with dilatation transformations of the contour allows us to compute first three contributions to the free energy which do not vanish as \( N \to \infty \), in orders \( O(N^2) \), \( O(N) \) and \( O(1) \) respectively. They are found explicitly and expressed through geometric characteristics of the contour and the spectral geometry of the interior and exterior domains. In particular, the most interesting contribution of the order of \( O(1) \) is expressed in terms of the Neumann jump operator \( \hat{N} \) which takes a function on \( \Gamma \) to the difference of normal derivatives of its harmonic continuations to the interior and exterior of the contour. The form of the final results is in parallel to that of 2D logarithmic gases where the \( O(1) \) part of the free energy is expressed through the spectral determinant of the Laplace operator of the supporting domain [5, 6].

We have also found correlation functions of the charge density and electrostatic potential in the leading order in \( N \). They are expressed through Green’s functions of the Laplace operators in the interior and exterior domains.

At last, we comment on the relation of our results to 2D CFT. First, we would like to emphasize that the holomorphic extension of the stress tensor \( T(z) \) is similar to that of CFT and the important fact that the loop equation turns out to be equivalent to a boundary condition for the holomorphic extension of the stress tensor of the logarithmic gas. The loop equation states that the tangential-normal component of the stress is continuous across the contour. This condition is to be compared to the conformal boundary condition for the stress tensor in the boundary CFT. In that case, the stress is defined only from one side of the boundary where its tangential-normal component vanishes. Besides, in CFT the stress tensor is traceless everywhere including the boundary. With the conformal boundary condition, under a change of the geometry (a deformation of the contour) correlation functions in CFT transform covariantly under conformal transformations of the domain bounded by the contour. This is not the case in our model, and so the similarity with CFT is not complete. In our case, the stress tensor \( T \) has a non-vanishing trace located exclusively on the boundary. As a result, the variation of the free energy under small deformations of the contour is expressed through the normal–normal component of \( T \). Also, it is important that the stress is well-defined not only in the large \( N \) limit but also for any finite \( N \), so one may regard it as a natural ‘discretization’ of the stress tensor of field theory when we treat the gas as a continuous media.

Like in the CFT, the \( O(1) \) part of the free energy, \( F_2 \), is expressed through the free energies of 2D Bose field. However, contrary to the CFT, in our case, the support of the Bose field are two
domains, the exterior and the interior of the contour. They are expressed through the regularized
determinants of Laplace operators in the interior and exterior domains (equation (4.32)) despite
that, the physical matter is confined to the contour.

Finally, we mention yet another interpretation of $F_2$. It is the free energy of 1D fermions $\psi$
confined on the contour with the action $\oint_{\Gamma} \bar{\psi} N \psi \, ds$.

Despite of the differences, the correlation functions of ‘vertex operators’ (exponential func-
tions of the electrostatic potential created by the charges) located in the exterior domain turn
out to be similar to the correlators of primary fields in CFT.

When this work was completed, we became aware of the recent paper [27], where similar
results for $\beta = 1$ were obtained by more rigorous mathematical methods.

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Data availability statement
All data that support the findings of this study are included within the article (and any
supplementary files).

Appendix A. Normal and tangential derivatives

Given a smooth closed curve $\Gamma$, let $\tau(z) = dz/|dz|$ be the unit tangent vector at $z \in \Gamma$
represented as a complex number pointing in positive (counterclockwise) direction. Then $\nu(z) = -i\tau(z)$ is the outward pointing unit normal vector to the curve $\Gamma$ at the point $z$. Clearly, $|\nu(z)| = 1$ for $z \in \Gamma$, i.e., $\nu(z) = \nu^{-1}(z)$. In terms of the conformal map $w(z) = w_{\text{ext}}(z)$ from the
exterior of $\Gamma$ to the exterior of the unit circle (we normalize it by the conditions $w(\infty) = \infty$ and $w'(\infty)$ is real positive) we have

$$\nu(z) = \left[ w'(z) \right] \frac{w(z)}{w'(z)}.$$  \hfill (A1)

The line element along $\Gamma$ is

$$ds = |dz| = \sqrt{dz \, d\bar{z}} = \frac{dz}{i\nu(z)} = i\nu(z) d\bar{z}.
$$

Given any function $f(z) = f(z, \bar{z})$ in the complex plane, its normal and tangential derivatives
on $\Gamma$ are:

$$\partial_\nu f(z) = \nu(z) \partial_z f(z) + \bar{\nu}(z) \partial_{\bar{z}} f(z),$$

$$\partial_s f(z) = i\nu(z) \partial_z f(z) - i\bar{\nu}(z) \partial_{\bar{z}} f(z).$$  \hfill (A2)

We write $\nu(z) = e^{i\theta}$, where $\theta$ is the angle between the normal vector and the real axis. The
curvature is given by
\( \kappa = \frac{d\theta}{ds} = iv \partial_s \nu. \) \hfill (A3)

The equivalent formula

\[ \kappa(z) = \partial_n \log \left| \frac{w(z)}{w'(z)} \right| \] \hfill (A4)

is an immediate consequence of the definition.

The Laplace operator \( \Delta = \partial^2_x + \partial^2_y = 4 \partial_z \partial_{\bar{z}} \) at a point \( z \in \Gamma \) is expressed in terms normal and tangential derivatives as

\[ \Delta = \partial^2_n + \partial^2_s + \kappa \partial_n. \]

Let \( f, g \) be any two functions in the plane defined in a strip-like neighborhood of the contour \( \Gamma \). The following formulas for \( z \in \Gamma \) are often used in the calculations:

\[ 2 \operatorname{Re} \left( v^2 \partial f \partial g \right) = \frac{1}{2} (\partial_n f \partial_n g - \partial_s f \partial_s g), \]

\[ 2 \operatorname{Im} \left( v^2 \partial f \partial g \right) = -\frac{1}{2} (\partial_n f \partial_s g + \partial_s f \partial_n g). \]

Assuming that \( \Delta f = 0 \), we also have:

\[ 2 \operatorname{Re} \left( v^2 \partial^2 f \right) = -\partial_s^2 f - \kappa \partial_n f, \]

\[ 2 \operatorname{Im} \left( v^2 \partial^2 f \right) = -\partial_s \partial_n f + \kappa \partial_s f. \]

For a function \( f = u + iv \) analytic in a vicinity of the contour the Cauchy–Riemann identities read \( \partial_n u = \partial_s v, \partial_n v = -\partial_s u \).

**Appendix B. The Green’s functions**

The Green’s function of the Dirichlet boundary value problem in a domain \( D \subset \mathbb{C} \) is a solution to the equation \( \Delta \Delta G(z, \zeta) = \Delta G(z, \zeta) G(z, \zeta) = 2\pi \delta^2(z - \zeta) \) vanishing on \( \partial D \). The Green’s function is symmetric: \( G(z, \zeta) = G(\zeta, z) \). As \( \zeta \to z \), it has the logarithmic singularity \( G(z, \zeta) = \log|z - \zeta| + \cdots \). If \( D \) is simply connected, then the Green’s function can be expressed through the conformal map \( w(z) \) from \( D \) onto the unit disk by the formula

\[ G(z, \zeta) = \log \left| \frac{w(z) - w(\zeta)}{1 - w(z)w(\zeta)} \right|. \]

Let \( \Gamma \) be a closed contour in the plane encircling the origin. Given such a contour, let \( D \) be the interior domain and \( G_{\text{int}}, G_{\text{ext}} \) be the Green’s functions in \( D \) and \( \mathbb{C} \setminus D \) respectively. We also introduce the conformal maps \( w_{\text{int}}(z) \) from \( D \) onto the unit disk and the conformal map \( w_{\text{ext}}(z) \) from \( \mathbb{C} \setminus D \) onto the exterior of the unit disk such that \( w_{\text{int}}(0) = 0, w_{\text{ext}}(\infty) = \infty \) and their derivatives at 0 and \( \infty \) are real positive.
The formulas

\[ f_H(z) = \frac{1}{2\pi} \oint_{\Gamma} f(\xi) \partial_n^+ G_{\text{int}}(z, \xi) |d\xi|, \]

\[ f^H(z) = -\frac{1}{2\pi} \oint_{\Gamma} f(\xi) \partial_n^- G_{\text{ext}}(z, \xi) |d\xi| \]  

provide solutions to the Dirichlet boundary value problems in \( D \) and \( \mathbb{C} \setminus D \), i.e., they give harmonic continuations of a function \( f \) from the contour to its interior and exterior respectively. Here \( \partial_n^\pm \) mean normal derivatives taken from inside (outside), with the normal vector looking outside in both cases. The Poisson kernel \( \partial_n G(z, \xi) \) (here \( G \) is \( G_{\text{int}} \) or \( G_{\text{ext}} \) and \( \xi \in \Gamma = \partial D \)) is expressed through the conformal map \( w(z) \) (correspondingly, \( w_{\text{int}} \) or \( w_{\text{ext}} \)) as follows:

\[
\partial_n G(z, \xi) = \frac{|w'(\xi)| (1 - |w(z)|^2)}{|w(\xi) - w(z)|^2} = |w'(\xi)| |\text{Re} \left( \frac{w(\xi) + w(z)}{w(\xi) - w(z)} \right) |
\]

\[
= |w'(\xi)| \left( \frac{w(z)}{w(\xi) - w(z)} - \frac{1}{w(\xi) - (w(z))^2} \right).
\]

The variation of the Green’s function under small deformation of the contour is given by the Hadamard formula

\[ \delta G(a, b) = \frac{1}{2\pi} \oint_{\Gamma} \partial_n G(a, \xi) \partial_n G(b, \xi) \delta n(\xi) |d\xi|, \]  

where \( \delta n(\xi) \) is the normal displacement of the boundary along the normal vector at the point \( \xi \in \Gamma \) and minus (plus) sign corresponds to the interior (exterior) problem.

Appendix C. The Neumann jump operator

Given a smooth closed contour \( \Gamma \) in the plane, one defines the Neumann jump operator as

\[ \hat{N} f(z) = \partial_n^+ f_H(z) - \partial_n^- f^H(z), \quad z \in \Gamma. \]

Since

\[
\oint_{\Gamma} f \partial_n^+ g_H ds = \oint_{\Gamma} g \partial_n^+ f_H ds, \quad \oint_{\Gamma} f \partial_n^- g^H ds = \oint_{\Gamma} g \partial_n^- f^H ds
\]

by virtue of Green’s theorem, the operator \( \hat{N} \) is symmetric (self-adjoint) with respect to the scalar product \( \oint_{\Gamma} fg \) ds of functions \( f, g \) on \( \Gamma \).

The Neumann jump operator has a zero mode: a constant function. In the space of functions \( f \) such that \( \oint_{\Gamma} f ds = 0 \) the operator \( \hat{N} \) is non-degenerate. The kernel of the inverse operator in this space is \( -\frac{1}{2\pi} \log |z - \xi| \). Indeed, assuming that \( \oint_{\Gamma} f ds = 0 \), let us define \( g(z) \) by

\[ g(z) = -\frac{1}{2\pi} \oint_{\Gamma} \log |z - \xi| f(\xi) |d\xi|, \quad z \in \Gamma, \]

then \( \hat{N} g(z) = f(z) \) by the property of potentials of a simple layer.
More precisely, let \( \hat{K} \) be the operator defined by

\[
\hat{K} f(z) = -\frac{1}{2\pi} \oint_{\Gamma} \log|z - \xi| f(\xi) d\xi
\]

on the space of all functions on \( \Gamma \) (the simple layer operator). It is not difficult to see that the following relations hold true:

\[
\hat{K} \hat{N} f = f - \frac{1}{2\pi} \oint_{\Gamma} |w'| f d\xi,
\]

\[
\hat{N} \hat{K} f = f - \frac{1}{2\pi} \oint_{\Gamma} |w'| f d\xi,
\]

where \( w = w_{\text{ext}} \), and

\[
\hat{K} |w'| = \log(1/r).
\]

The first one easily follows from the properties of the simple layer potential. Let us outline the proof of the second one. The left-hand side of it equals

\[
-\frac{1}{2\pi} \oint_{\Gamma} \log|\xi - z| \partial_n^+ f_H(\xi) d\xi + \frac{1}{2\pi} \oint_{\Gamma} \log|\xi - z| \partial_n^- f_H^H(\xi) d\xi.
\]

Each integral here is a simple layer potential. Using the fact that the simple layer potential is continuous across the contour, we can think that in the first integral \( z \) approaches the contour from the exterior \((z_-)\) while in the second one \( z \) approaches the contour from the interior \((z_+)\). Then the harmonic continuation of \( \log|\xi - z_-| \) to the interior is the function \( \log|\xi - z_-| \) itself while the harmonic continuation of \( \log|\xi - z_+| \) to the exterior is \( \log|\xi - z_+| - \log|\nu(\xi)| \). Therefore, using Green’s theorem, we can transform the left-hand side of (C4) as follows:

\[
-\frac{1}{2\pi} \oint_{\Gamma} f(\xi) \partial_n^+ \log|\xi - z_-| d\xi + \frac{1}{2\pi} \oint_{\Gamma} f(\xi) \partial_n^- \log|\xi - z_+| d\xi
\]

The first two integrals on the right-hand side are double-layer potentials. The properties of the double layer potentials are outlined in appendix D. The main property is that the double layer potential is discontinuous across the contour (see (D1)). Equation (D1) states that the difference of the first two integrals in the right-hand side above (the jump of the double layer potential across the contour) is equal to \( f(z) \). Finally, the second equation in (B1) states that the last integral provides the harmonic continuation of the function \( f \) to infinity and thus equals \( f_H^H(\infty) \). Therefore, we see that the left-hand side of (C4) is \( f(z) - f_H^H(\infty) \) which is the same as the right-hand side.

In the case when \( \Gamma \) is a circle

\[
\partial_n^+ f_H(z) = -\partial_n^- f_H^H(z),
\]
and
\[ \hat{\mathcal{N}} f = 2 \partial_n^+ f_H = 2 \hat{\mathcal{N}}^+ f. \]  
\(\text{(C6)}\)

**Appendix D. Double layer potentials and Fredholm eigenvalues of planar domains**

**Double layer potentials.** Given a function \( f \) on \( \Gamma \), one can construct the double layer potential defined by
\[ V_f(z) = \frac{1}{\pi} \oint_{\Gamma} f(\xi) \partial_{n_\xi} \log |\xi - z| \, d\xi. \]

It is harmonic everywhere in the plane except for the contour \( \Gamma \) where it has a jump. Its behavior across the boundary is given by
\[ V_f(z) = \begin{cases} f(z) + \frac{1}{\pi} \oint_{\Gamma} f(\xi) \partial_{n_\xi} \log |\xi - z| \, d\xi, & z \to z_+ \\ -f(z) + \frac{1}{\pi} \oint_{\Gamma} f(\xi) \partial_{n_\xi} \log |\xi - z| \, d\xi, & z \to z_- \end{cases} \]
\(\text{(D1)}\)

where \( z \to z_\pm \) means that one approaches the point \( z \in \Gamma \) from inside and outside respectively and \( \oint_{\Gamma} \) is the principal value integral. Using the Green’s theorem, we can represent (D1) as simple layer potentials:
\[ V_f(z_+) = \frac{1}{\pi} \oint_{\Gamma} \log |\xi - z| \partial_n f^H(\xi) \, d\xi + 2f^H(\infty), \]
\[ V_f(z_-) = \frac{1}{\pi} \oint_{\Gamma} \log |\xi - z| \partial_n^+ f_H(\xi) \, d\xi. \]
\(\text{(D2)}\)

Let us introduce the operator \( \hat{\mathcal{V}} \) acting in the space of functions on \( \Gamma \) in the following way:
\[ \hat{\mathcal{V}} f(z) = \frac{1}{\pi} \oint_{\Gamma} f(\xi) \partial_{n_\xi} \log |\xi - z| \, d\xi, \quad z \in \Gamma. \]
\(\text{(D3)}\)

Note that it is not self-adjoint. In this notation
\[ V_f(z_+) = (\hat{\mathcal{I}} + \hat{\mathcal{V}}) f(z), \quad V_f(z_-) = -(\hat{\mathcal{I}} - \hat{\mathcal{V}}) f(z). \]

Applying the Neumann jump operator \( \hat{\mathcal{N}} \) to both sides of (D2) and using the properties of the simple layer potentials, we have:
\[ \hat{\mathcal{N}}(\hat{\mathcal{I}} + \hat{\mathcal{V}}) f = -2 \partial_n f^H, \quad \hat{\mathcal{N}}(\hat{\mathcal{I}} - \hat{\mathcal{V}}) f = 2 \partial_n^+ f_H \]
\(\text{(D4)}\)

or, in the operator form,
\[ \hat{\mathcal{N}}(\hat{\mathcal{I}} + \hat{\mathcal{V}}) = -2 \hat{\mathcal{N}}^-, \quad \hat{\mathcal{N}}(\hat{\mathcal{I}} - \hat{\mathcal{V}}) = 2 \hat{\mathcal{N}}^+. \]
\(\text{(D5)}\)
For completeness, we present here some other properties of the simple and double layer potentials (see [21]). The adjoint operator $\hat{V}^\dagger$ acts as

$$\hat{V}^\dagger f(z) = \frac{1}{\pi} \oint_{\Gamma} f(\xi) \partial_n \log |\xi - z| |d\xi|, \quad z \in \Gamma. \tag{D6}$$

Equation (D5) imply the following values of normal derivatives of simple layer potential:

$$-2\partial^+_n \hat{K}f(z) = -f(z) + \hat{V}^\dagger f(z),$$
$$-2\partial^-_n \hat{K}f(z) = f(z) + \hat{V}^\dagger f(z). \tag{D7}$$

Summing the first relation in (D5) with the adjoint second one, we get

$$\hat{N}\hat{V} = \hat{V}^\dagger \hat{N}. \tag{D8}$$

Taking the difference of the relations (D5), we obtain the operator relation

$$\hat{N}^+ (\hat{I} + \hat{V}) = -\hat{N}^- (\hat{I} - \hat{V}), \tag{D9}$$

which is equivalent to

$$\partial^+_n V f(z) = \partial^-_n V f(z) \tag{D9}$$

(normal derivative of the double layer potential is continuous across the boundary). With the help of (D2), the continuity relation (D9) can be also represented in the operator form as follows:

$$\hat{N}^+ \hat{K} \hat{N}^- \hat{N}^+ = \hat{N}^- \hat{K} \hat{N}^+ \hat{N}^- \tag{D10}.$$ 

The proof of this operator relation is based on (C3) and (C4).

**Fredholm eigenvalues of the domain $D$.** Eigenvalues $\lambda_\alpha$ of the operator $\hat{V}$ are called Fredholm eigenvalues of the domain $D$ [16]. The eigenvalue equation can be written in the form

$$\frac{1}{\pi} \oint_{\Gamma} \phi_\alpha(\xi) \partial_n \log |\xi - z| |d\xi| = \lambda_\alpha \phi_\alpha(z). \tag{D11}$$

It is easy to see that the constant function $\phi_0 = 1$ is the eigenfunction with the eigenvalue $\lambda_0 = 1$. For $\alpha \neq 0$, one can show [16] that if $\lambda_\alpha$ is an eigenvalue, then $-\lambda_\alpha$ is also an eigenvalue and $|\lambda_\alpha| < 1$. Moreover, estimates for the eigenvalues [22–24] imply that for smooth enough contours $\sum_\alpha |\lambda_\alpha| < \infty$. The Fredholm determinant associated to the domain $D$, or rather to the contour $\Gamma$, is

$$\det(\hat{I} + \hat{V}) = \prod_\alpha (1 + \lambda_\alpha). \tag{D12}$$

For smooth enough contours, the infinite product is convergent and no regularization is required (see [22, 23]). We also have

$$\det(\hat{I} + \hat{V}) = 2 \det'(\hat{I} - \hat{V}), \tag{D13}$$

where prime in the right-hand side means, as usual, that the zero eigenvalue is removed.

**Determinant of the Neumann jump operator and Fredholm determinant.** Equation (D5) imply that

$$\det' \hat{N} \det(\hat{I} + \hat{V}) = 2 \det(-2\hat{N}^-), \tag{D14}$$

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or
\[
\det' \hat{N} \det'(\hat{I} - \hat{V}) = \det'(2\hat{N}^+),
\]
(D15)

where the modified determinants \( \det' \hat{N} , \det'(2\hat{N}^\pm) \) are regularized by means of the \( \zeta \)-functions (see appendix F). As is proved in [18], \( \det'(2\hat{N}^+) = -\det'(-2\hat{N}^-) = \frac{1}{2} P \), where \( P = \oint_\Gamma ds \) is length of the contour \( \Gamma \). Therefore, the regularized determinant of the Neumann jump operator is expressed through the Fredholm determinant of the domain \( \mathcal{D} \) as follows:

\[
\log \det' \hat{N} = -\log \det(\hat{I} + \hat{V}) + \log P.
\]
(D16)

The formulas (D14) and (D15) require some justification. We regularize the determinant of the left-hand side of (D5) as

\[
\lim_{\mu \to 0} \det \left( (\hat{N}_\mu + \varepsilon \hat{P}_0)(\hat{I} + \hat{V}) \right),
\]

where \( \hat{N}_\mu = \hat{N} + \mu \hat{I} \) and \( \hat{P}_0 \) is the rank one projector on constant functions acting as

\[
\hat{P}_0 f = \frac{1}{P} \oint_\Gamma f \, ds.
\]

Since \( \hat{P}_0 \) is a rank one operator, we have

\[
\det \left( (\hat{N}_\mu + \varepsilon \hat{P}_0)(\hat{I} + \hat{V}) \right) = \det(\hat{N}_\mu)(\hat{I} + \hat{V}) \left( 1 + \varepsilon \text{tr} (\hat{N}_\mu^{-1} \hat{P}_0) \right).
\]

In our case, the operators \( \hat{N}_\mu \) and \( \hat{I} + \hat{V} \) do not have a multiplicative anomaly [25, 26], i.e. \( \det (\hat{N}_\mu)(\hat{I} + \hat{V}) = \det \hat{N}_\mu \det(\hat{I} + \hat{V}) \). Therefore,

\[
\lim_{\mu \to 0} \det \left( (\hat{N}_\mu + \varepsilon \hat{P}_0)(\hat{I} + \hat{V}) \right) = \varepsilon \det' \hat{N} \det(\hat{I} + \hat{V}).
\]

We can write the first equation in (D5) in the form

\[
(\hat{N}_\mu + \varepsilon \hat{P}_0)(\hat{I} + \hat{V}) = -2\hat{N}^- + \mu(\hat{I} + \hat{V}) + \varepsilon \hat{P}_0(\hat{I} + \hat{V}).
\]
(D17)

The determinant of the right-hand side can be calculated as above with the result

\[
\lim_{\mu \to 0} \det \left( -2\hat{N}^- + \mu(\hat{I} + \hat{V}) + \varepsilon \hat{P}_0(\hat{I} + \hat{V}) \right) = 2\varepsilon \det'(-2\hat{N}^-).
\]

This yields (D14).

**Appendix E. Variation of contour integrals**

Here we present the rules of variation of contour integrals over the simple closed curve \( \Gamma \) under small deformations of the curve. The variation of the curve is described by the normal displacement \( \delta n(\mathbf{z}) \).

Consider a contour integral of a general form \( \oint_\Gamma F(f, \partial_n f) \, ds \), where \( F \) is a fixed function and the function \( f \) may depend on the contour. Calculating the linear response to the deformation of the contour, one should vary all items in the integral independently and add the results. There are four things to be varied: the symbol \( \oint_\Gamma \), the normal derivative \( \partial_n \), the line element \( ds \) and the
function $f$. By variation of $\oint_\Gamma$, we mean integration of the old function over the new contour, which gives the contribution

$$\oint_\Gamma \delta n \partial_n F \, ds.$$  

The change of the slope of the normal vector results in

$$\delta \frac{\partial}{\partial n} = - \partial_s (\delta n) \frac{\partial}{\partial s}.$$  

The rescaling of the line element gives

$$\delta ds = \kappa \delta n \, ds.$$  

The variation of the function $f$ is case-dependent. The procedure is clarified by a few examples which are used in the main text.

**Example 1.**

$$\delta \oint_\Gamma f(z) \, ds = \oint_\Gamma \delta n(z) (\partial_n + \kappa(z)) f(z) \, ds.$$  

(E1)

The result holds provided the function $f$ itself does not change. If it does, one should add the term $\oint_\Gamma \delta f(z) \, ds$.

**Example 2.**

$$\delta \oint_\Gamma f(z) \partial_n g(z) \, ds = \oint_\Gamma \delta n(z) (\partial_n f(z) \partial_n g(z)) \, ds + \oint_\Gamma \delta n(z) f(z) \partial_n g(z) \kappa(z) \, ds$$

$$- \oint_\Gamma \partial_s (\delta n(z)) f(z) \partial_s g(z) \, ds,$$

where the functions $f, g$ are fixed. Taking the last integral by parts and using the formula for the Laplace operator evaluated on the contour $\partial_n^2 + \partial_s^2 + \kappa \partial_n = \Delta$, we can represent the result in the invariant form:

$$\delta \oint_\Gamma f(z) \partial_n g(z) \, ds = \oint_\Gamma \delta n(z) ((\nabla f, \nabla g) + f \Delta g) \, ds.$$  

(E2)

Here $(\nabla f, \nabla g) = \partial_n f \partial_n g + \partial_s f \partial_s g$. The same result can be obtained in an easier way by reducing the contour integral to a 2D integral by means of the Green theorem.

**Example 3.** Let us vary the integrals $\oint_\Gamma f(z) \partial^+_n g_H(z) \, ds$, $\oint_\Gamma f(z) \partial^-_n g^H(z) \, ds$, where $f, g$ are assumed to be contour-independent functions. However, the functions $g_H, g^H$ change when we change the contour. We have:

$$\partial^+_n \delta g_H = \partial^+_n (\partial^+_n (g - g_H) \delta n)_H, \quad \partial^-_n \delta g^H = \partial^-_n (\partial^-_n (g - g^H) \delta n)^H.$$  

(E3)
(this follows from the Hadamard variational formula). Adding this to the result of example 2, we get:

\[ \delta \oint_{\Gamma} f(z) \partial_n^+ g_H(z) \, ds = \oint_{\Gamma} \delta n (\nabla f, \nabla g) \, ds - \oint_{\Gamma} \delta n \, \partial_n^- (f - f_H) \partial_n^+ (g - g_H) \, ds, \]  

(E4)

\[ \delta \oint_{\Gamma} f(z) \partial_n^- g^H(z) \, ds = \oint_{\Gamma} \delta n \, \partial_n^- (f - f^H) \partial_n^- (g - g^H) \, ds. \]  

(E5)

As a simple corollary, we obtain:

\[ \delta \oint_{\Gamma} f(\hat{N}) g \, ds = \oint_{\Gamma} \delta n \, \partial^- (f - f^H) \partial^- (g - g^H) \, ds. \]  

(E6)

**Example 4.** Here we show how to find variation of the external conformal radius \( r \) of the domain \( D \). The external conformal radius is defined as

\[ r = (w'(\infty))^{-1}, \]  

(E7)

where \( w(z) \) is the conformal map from \( \mathbb{C} \setminus D \) onto the exterior of the unit circle such that \( w(\infty) = \infty \) and \( w'(\infty) \) is real positive. The integral representation of \( \log r \) is

\[ \log r = -\frac{1}{2\pi} \oint_{\Gamma} \log |w'(z)| \, |w'(z)| \, |dz|. \]  

(E8)

A direct calculation using the Hadamard variational formula yields

\[ \delta \log r = \frac{1}{2\pi} \oint_{\Gamma} |w'(z)|^2 \delta n(z) |dz|. \]  

(E9)

### Appendix F. Functional determinants

Given an elliptic operator \( A \), one can define the \( \zeta \)-function \( \zeta_A(s) \) as

\[ \zeta_A(s) = \text{tr}'(A^{-s}) = \sum_i \lambda_i^{-s}, \]  

(F1)

where prime means summing over nonzero eigenvalues \( \lambda_i \) only. The regularized determinant is defined as

\[ \text{det}' A = e^{-\zeta_A(0)}. \]  

(F2)

As a consequence,

\[ \text{det}'(cA) = c^{\zeta_A(0)} \text{det}' A \]  

(F3)

for any constant \( c \).

Given a compact domain \( D \subset \mathbb{C} \), one can consider the Gaussian path integral

\[ \int [D\chi] \exp \left( -\int_D |\nabla \chi|^2 \, d^2 z \right) = \int [D\chi] \exp \left( \int_D \Delta \chi \, d^2 z \right) \]  

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Let \( \partial \) be the object introduced by this definition. The 'surgery formula' is justified by the 'obstacle formula' in the plane is not flat and (see (F2)). More generally, the definition of the determinant extends to the case when the metric in the plane is not flat and \( \Delta \) is the Laplace–Beltrami operator in this metric.

The determinant (F5) was calculated in [10] in terms of the conformal factor of the metric. For Laplace operators in non-compact exterior domains, a different approach is required. It was developed in [14], where the determinant \( \log \det(-\Delta_D) \) of the Laplace operator in \( \mathbb{C} \backslash D \) was defined in terms of scattering on the compact 'obstacle' \( D \). Here we will not go into details and only mention that the term 'determinant' for the object introduced by this definition is justified by the 'surgery formula' proved in [14]. Here \( \hat{N} \) is the Neumann jump operator on \( \Gamma = \partial D \) and \( P \) is the perimeter of \( \partial D \).
Let us give a simple heuristic explanation of the surgery formula using formal Gaussian path integrals over scalar fields. We start with the path integral

\[ Z_{\mathcal{C}} = \int [D\chi] \exp \left( -\int_{\mathcal{C}} |\nabla \chi|^2 \, d^2 z \right) \]

which does not depend on the contour and write

\[ \int_{\mathcal{C}} |\nabla \chi|^2 \, d^2 z = \int_{\mathcal{D}} |\nabla \chi|^2 \, d^2 z + \int_{\mathcal{C}\setminus\mathcal{D}} |\nabla \chi|^2 \, d^2 z. \]

Let \( f(z), z \in \Gamma = \partial \mathcal{D} \) be the function \( \chi(z) \) restricted to the contour \( \Gamma \). We can represent the field \( \chi \) as

\[ \chi(z) = \begin{cases} f_{\text{in}}(z) + \chi_{\text{in}}(z), & z \in \mathcal{D}, \\ f_{\text{out}}(z) + \chi_{\text{out}}(z), & z \in \mathcal{C}\setminus\mathcal{D}, \end{cases} \]

where \( \chi_{\text{in}}, \chi_{\text{out}} \) are fields in the domains \( \mathcal{D} \) and \( \mathcal{C}\setminus\mathcal{D} \) respectively with Dirichlet boundary conditions. A simple computation using the Green theorem shows that

\[ \int_{\mathcal{C}} |\nabla \chi|^2 \, d^2 z = \int_{\mathcal{D}} |\nabla \chi_{\text{in}}|^2 \, d^2 z + \int_{\mathcal{C}\setminus\mathcal{D}} |\nabla \chi_{\text{out}}|^2 \, d^2 z + \oint_{\Gamma} f\hat{N} f \, ds \]

and, therefore,

\[ Z_{\mathcal{C}} = \int [D\chi_{\text{in}}][D\chi_{\text{out}}][Df] \exp \left( -\int_{\mathcal{D}} |\nabla \chi_{\text{in}}|^2 \, d^2 z - \int_{\mathcal{C}\setminus\mathcal{D}} |\nabla \chi_{\text{out}}|^2 \, d^2 z - \oint_{\Gamma} f\hat{N} f \, ds \right) \]

\[ = CP^{1/2}(\det(-\Delta_{\mathcal{D}}))^{-1/2}(\det(-\Delta_{\mathcal{C}\setminus\mathcal{D}}))^{-1/2}(\det\hat{N})^{-1/2} \]

with some contour-independent constant \( C \). This gives a ‘physical’ explanation of (F8). The appearance of the factor \( P^{1/2} \) is due to the zero mode of the operator \( \hat{N} \).

As is pointed out in [14], the problem of defining the determinant of exterior Laplacian could be solved by placing the contour on the sphere \( S^2 \). Let \( g_0 \) be a metric in the plane of the form

\[ g_0(z) = \frac{g_{\text{flat}}(z)}{f(|z|^2)}, \quad z \in \mathcal{C}, \]

where \( g_{\text{flat}} \) is the standard flat metric and the function \( f(x) \) is equal to 1 for \( x \leq R \) and \( f(x) = x^2 \) for \( x \geq R' (R' > R) \). In other words, \( g_0 \) is the same as the standard metric on the disk \( B_{R,0} \) of radius \( R \) centered at the origin while its behavior at infinity means that \( g_0 \) extends to a smooth metric on \( S^2 \) regarded as the compactification of \( \mathcal{C} \) by adding the point at infinity. This compactifies \( \mathcal{C}\setminus\mathcal{D} \) to a domain \( \mathcal{D}' \subset S^2 \). We assume that \( \partial \subset \mathcal{D} \) and \( \mathcal{D} \subset B_{R,0} \). We can compare (F8) with the surgery formula from [15] applied for \( S^2 \) with the metric \( g_0 \):

\[ \log \det(-\Delta_{\mathcal{D}}) + \log \det(-\Delta_{\mathcal{D}' \setminus g_0}) + \log \det\hat{N} = \log \det(-\Delta_{S^2, g_0}) + \log P + \text{const.} \]  

(F9)

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In particular, we can specialize it for the case when $D$ is the unit disk $U$ with the boundary $\partial U = S^1$ (the unit circle):

$$\log \det(-\Delta_U) + \log \det(-\Delta_{U^c, g_0}) + \log \det' \hat{\mathcal{N}}_{S^1} = \log \det(-\Delta_{S^2, g_0}) + \text{const.} \quad (F10)$$

Here $U^c$ is the complement to the unit disk in the metric $g_0$. Note that $\log \det(-\Delta_U)$ and $\log \det' \hat{\mathcal{N}}_{S^1}$ are universal constants. Now, summing (F8) and (F10) and subtracting (F9), we obtain the relation

$$\log \det(-\Delta_{\mathbb{C} \setminus D}) = \log \det(-\Delta_{\mathbb{C}^c, g_0}) - \log \det(-\Delta_{U, g_0}) + \text{const.} \quad (F11)$$

In order to find the quantity in the rhs, one can apply the Polyakov–Alvarez formula (F6) with $D_0 = U^c$ with the background reference metric $g_0$ and $\sigma(w) = \log |z_{\text{ext}}(w)| := \phi_{\text{ext}}(w)$. Again, $\phi_{\text{ext}}$ is harmonic in $U^c$ and the first term in the rhs of (F6) reduces to a contour integral. Tending $R \to \infty$, we see that the limit of the second term is 0 and we obtain the result

$$\log \det (-\Delta_{\mathbb{C} \setminus D}) = \frac{1}{12\pi} \oint_{|w|=1} (\phi_{\text{ext}} \partial_\theta \phi_{\text{ext}} + 2\phi_{\text{ext}}) |dw| + \text{const} \quad (F12)$$

(cf (4.28)).

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