Coadjoint Orbits
and
Wilson Loops
in
Five Dimensional Topological Gauge Theories

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Abstract

We discussed a one-point function of a BPS Wilson loop in a supersymmetric five dimensional gauge theories defined on $M_4 \times S^1$ by using path integral expression of Wilson loops. We found that the Wilson loop gives interaction terms between charged particles and certain gauge fields on the instanton moduli space, and makes the non-charged particle charged under the gauge fields.

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1 Introduction

We discussed one-point functions of BPS Wilson loops in a five dimensional topological gauge theory defined on $M_4 \times S^1$:

$$\text{Tr}_{V} \left( \text{P exp} \left( -\sqrt{-1} \int_0^T dt (A_t + \sqrt{-1}\varphi)'X_I \right) \right).$$ \hfill (1.1)

This Wilson loops are important observable as studied in [10] and [11] for the non-commutative $U(1)$ gauge theory. It is difficult to discuss correlation functions of Wilson loops because it is cumbersome to treat the path-ordering operator, which is needed to define Wilson loops. In order to remove this complexity, we used a path integral expression of Wilson loops. The expression is an extension of relations between Kirillov’s formula of characters of a Lie group and quantum mechanics for particles whose phase spaces are coadjoint orbits. According to this path integral expression, the path-ordering operator in a Wilson loop is considered as the time-ordering operator of the quantum mechanics.

It is known [5] that by using the supersymmetry the partition function of the five dimensional topological gauge theory is expressed as a partition function of a quantum mechanics of a non-charged particle moving on an instanton moduli space. Furthermore, the quantum mechanics possesses a supersymmetry. Due to the supersymmetry, the path integral variables of the partition function of the quantum mechanics are localized to constant modes along the circle $S^1$. As a result, the partition function of the quantum mechanics becomes the index of the Dirac operator of the instanton moduli space [5]. Because the Wilson loops are BPS with respect to the supersymmetry of the topological gauge theory, this argument are applicable to correlation functions of the Wilson loops. After the variables are localized to the constant modes, the Wilson loops are considered as Chern characters of some vector bundles, and their one-point functions become the indices of the twisted Dirac operators of the vector bundles. Although it is known that indices of twisted Dirac operators are given as partition functions of quantum mechanics of charged particles [6], it is not known a direct relation between the Wilson loops and charged particles. By using the path integral expression of Wilson loops, we found that the Wilson loop gives interaction terms between a charged particle and a certain gauge field on the instanton moduli space, and makes the non-charged particle charged under the gauge fields.

$$\langle \text{Tr}_{V} \left( \text{P exp} \left( -\sqrt{-1} \int_0^T dt (A_t + \sqrt{-1}\varphi)'X_I \right) \right) \rangle$$

$$= \int_{M \times \Omega_{\lambda+\rho}} \mathcal{D}m^i \mathcal{D}\chi^i \mathcal{D}\phi^a \mathcal{D}e^a e^{-Tr \int_{M_4} c_2(AdP) e^{-S_{\text{charged}}}}.$$ \hfill (1.2)

This paper is organized as follows. In section [2] we introduce coadjoint orbits and relations between coadjoint orbits and Wilson loops. In section [3] we apply the method argued in section [2] to Wilson loops in five dimensional topological gauge theories. We denoted geometries of spaces of connections in appendix [A] in order to complement the discussion in the main part of this paper.
2 Wilson loops and coadjoint orbits

This section is a preparation for the discussion in the section \[3\] and do not possess any new results. Firstly, we introduce coadjoint orbits in \[2.1\]. In particular, the most important equation is the Kirillov formula \([2.14]\). Secondly, we denote a relation between characters and quantum mechanics on coadjoint orbits in \[2.2\]. Thirdly, we give a short derivation of the path integral expression of Wilson loops in \[2.3\].

2.1 Coadjoint orbit and Kirillov’s formula

In this subsection, we explain coadjoint actions and coadjoint orbits. For details of the things in this subsection, see Kirillov’s lecture note \([1]\) and the references in the lecture note. We take elements of Lie algebras as Hermitian matrices in order to clarify a relation between coadjoint orbits and quantum mechanics, which is denoted in the later subsections.

Let \(G\) be a real simple compact Lie group, and \(\mathfrak{g}\) be the Lie algebra of \(G\). We take \(\{X_I\}_{I=1, \ldots, \dim \mathfrak{g}}\) as a basis of the Lie algebra \(\mathfrak{g}\). The algebra is given by the relations \([X_I, X_J] = \sqrt{-1} f_{IJ}^K X_K\), where \(f_{IJ}^K\) is the structure constant. The Cartan subalgebra is denoted by \(\mathfrak{h}\).

We write the corresponding maximal torus as \(H\): \(\text{Lie}(H) = \mathfrak{h}\).

The Lie group \(G\) acts on \(\mathfrak{g}\). The action is denoted by \((\mathfrak{g}, Ad)\) and called the adjoint representation. The infinitesimal action of \(G\) gives the action of Lie algebras on \(\mathfrak{g}\). The action is denoted by \((\mathfrak{g}, ad)\) and also called the adjoint representation:

\[
\begin{align*}
ad & : \mathfrak{g} \to \text{End}(\mathfrak{g}), \\
ad(X)(Y) & := [X, Y], \quad \text{for } X, Y \in \mathfrak{g}.
\end{align*}
\]

Let \(\mathfrak{g}^*\) be the dual vector space of \(\mathfrak{g}\). The pairing is denoted by \(\langle , \rangle\):

\[
\langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}.
\]

We define the coadjoint representation \((\mathfrak{g}^*, K)\) of \(G\) as the dual representation of the adjoint representation \((\mathfrak{g}, Ad)\). In the same way, the coadjoint representation \((\mathfrak{g}^*, k)\) of \(\mathfrak{g}\) as the dual representation of \((\mathfrak{g}, ad)\):

\[
\begin{align*}
\langle K(g)F, Y \rangle & = \langle F, Ad(g^{-1})Y \rangle, \\
\langle k(X)F, Y \rangle & = -\langle F, ad(X)Y \rangle, \quad \text{for } g \in G, X, Y \in \mathfrak{g}, F \in \mathfrak{g}^*.
\end{align*}
\]

A \(G\)-orbit passing through \(F_0 \in \mathfrak{g}^*\) is called a coadjoint orbit of \(G\) and is denoted by \(\Omega_{F_0}\):

\[
\Omega_{F_0} := \{ F \in \mathfrak{g} | F = K(g)F_0, \exists g \in G \}.
\]

From now on, when the coadjoint orbit under consideration is definite, we omit the subscript and denote the coadjoint orbit by \(\Omega\). Let \(\text{Stab}(F) \subset G\) be the stabilizer group of the point \(F \in \mathfrak{g}\):

\[
\text{Stab}(F) := \{ g \in G | K(g)F = F \}.
\]
The coadjoint orbit $\Omega_{F_0}$ is diffeomorphic to the homogeneous manifold $G/\text{Stab}(F_0)$:

$$\Omega_{F_0} \simeq G/\text{Stab}(F_0).$$

We denote the coordinate of $\mathfrak{g}^*$ by $\{q_I\}_{I=1,\ldots,\dim \mathfrak{g}}$, and the coordinate of $\Omega$ by $\{\phi^a\}_{a=1,\ldots,\dim \Omega}$. An element $X \in \mathfrak{g}$ acts on a coadjoint orbit $\Omega$, and generates a vector field $X_\Omega \in \Gamma(\Omega, T\Omega)$. By means of the coordinate, the vector field $X_\Omega$ is given as follows:

$$X_\Omega|_F = -\sqrt{-1} \langle k(X) F, X_I \rangle \frac{\partial \phi^a}{\partial q_I} \frac{\partial}{\partial \phi^a}, \text{ at } F \in \Omega. \quad (2.8)$$

For each coadjoint orbit $\Omega$, there is a symplectic form $\omega$ defined by the following equation:

$$\omega(X_\Omega, Y_\Omega)(F) := \sqrt{-1} \langle F, [X,Y] \rangle,$$  

where $X_\Omega, Y_\Omega$ are the vector fields on $\Omega$ generated by the elements $X, Y \in \mathfrak{g}$. The symplectic form $\omega$ is called the Kirillov-Kostant two-form. The symplectic form $\omega$ is $G$-invariant. Hence, it vanishes by the action of the Lie derivative; $\mathcal{L}(X_\Omega)(\omega) = 0$ for any element $X \in \mathfrak{g}$.

Let $V_X$ be the Hamilton vector field generated by the function $X \in \mathfrak{g}$ on $\Omega$:

$$V_X := \omega^{ab} \partial_b X \partial_a. \quad (2.10)$$

Since the inverse matrix of the symplectic form is given by the following equation,

$$\omega^{ab}(F) = f^K_I(F, X^K) \frac{\partial \phi^a}{\partial q_I} \frac{\partial \phi^b}{\partial q_J}, \quad (2.11)$$

the vectors $V_{X_I}$ and $X_{\Omega_I}$ are same:

$$V_{X_I}|_F = f^K_I(F, X^K) \frac{\partial \phi^a}{\partial q_I} \frac{\partial \phi^b}{\partial q_J} \partial_a = \sqrt{-1} \langle F, [X_I, X_J] \rangle \frac{\partial \phi^a}{\partial q_J} \partial_a = X_{\Omega_I}|_F. \quad (2.12)$$

In general, $V_X = X_\Omega$ for any $X \in \mathfrak{g}$.

Every coadjoint orbit is a Kähler manifold. Then, there is a $G$-invariant metric $g_\Omega$ on each coadjoint orbit $\Omega$.

Let $\mathfrak{g}_C := \mathfrak{g} \otimes \mathbb{R} \otimes \mathbb{C}$ be the complexification of $\mathfrak{g}$. We denote the Cartan subalgebra of $\mathfrak{g}_C$ by $\mathfrak{h}_C$. Let $\lambda \in \mathfrak{h}_C^*$ be a dominant weight. The coadjoint orbit $\Omega_\lambda$ is integral. That is, the symplectic form satisfies the following conditions:

$$\int_{C_2} \omega \in 2\pi \mathbb{Z}, \quad \text{for any two cycles } C_2 \in H_2(\Omega_\lambda, \mathbb{Z}). \quad (2.13)$$

If the dominant weight $\lambda$ is in the open Weyl chamber, the coadjoint orbit $\Omega_\lambda$ is diffeomorphic to the full flag manifold $G/H$. 














































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































































The following expression of characters is the one of the most important relation in this paper. Let \( (\pi, V_{\lambda}) \) be the highest weight representation of \( g \) with the highest weight \( \lambda \). For an element \( e^{-\sqrt{-1}X} \in G \) which is sufficiently close to the identity element, the character of the element is given as follows by using Kirillov’s formula:

\[
\text{Tr}_{V_{\lambda}} \left( e^{-\sqrt{-1}X} \right) = \frac{1}{p(X)^{1/2}} \int_{F \in \Omega_{\lambda+\rho}} e^{-\sqrt{-1}(F,X)} e^{\frac{\omega}{2\pi}},
\]

(2.14)

where \( \rho \) is the half of the sum of the positive roots of \( g_C \) and \( p(X) \) is defined by the following equation:

\[
p(X) := \det \left( \frac{\sinh (ad(\sqrt{-1}X)/2)}{(ad(\sqrt{-1}X)/2)} \right).
\]

(2.15)

2.2 Quantum mechanics on coadjoint orbit and Kirillov’s formula

A partition function of a certain quantum mechanics gives a character of a Lie group:

\[
\text{Tr}_{V_{\lambda}} \left( e^{-\sqrt{-1}TX} \right) = Z_{\Omega},
\]

(2.16)

\[
Z_{\Omega} := \int D\phi Dc e^{\sqrt{-1}S_{\Omega}},
\]

(2.17)

where \( X_I \in \mathfrak{h} \) and \( T \) is the circumference of \( S^1 \) which is the time direction of the quantum mechanics. \( S_{\Omega} \) is an action whose Hamiltonian is \( X_I \). The real function \( X_I \) on the coadjoint orbit becomes a Hermitian matrix after the quantization. This is the reason why we took elements of Lie algebras as Hermitian matrices. We will describe the details of the quantum mechanics later. The relation eq.(2.16) is denoted in [3] and be proved by localizing the both hand side. We give a short proof of the relation.

The integrand of eq.(2.14) is a equivariantly closed differential form.

\[
(d - 2\pi \sqrt{-1}i(TV_{X_I})) \left( \frac{\omega}{2\pi} - \sqrt{-1}TX_I \right) = 0.
\]

(2.18)

Then, we can apply the localization formula for integrals of equivariantly closed forms to the integration in Kirillov’s formula. The integration is determined by the information around the zero locus of the vector field \( V_{X_I} \). We already know that the vector field \( X_{T\Omega} \) and \( V_{X_I} \) are same. Hence, the zero locus of the vector field \( V_{X_I} \) is the fixed points of the action of \( X_I \in g \). According to the following discussion, we will know that the fixed points are isolated on the coadjoint orbit \( \Omega_{\lambda+\rho} \). Let \( F \in \Omega_{\lambda+\rho} \) be a fixed point of \( X_I \)-action. We assume that there exist an element \( Y \in g \) such that the point \( K(e^{\sqrt{-1}Y})F \in \Omega_{\lambda+\rho} \) is also a fixed point of \( X_I \)-action. With this assumption, because of \( K(e^{-\sqrt{-1}Y} e^{\sqrt{-1}X_I} e^{\sqrt{-1}Y})F = F \), the element \( e^{-\sqrt{-1}Y} e^{X_I} e^{\sqrt{-1}Y} \) is in the stabilizer of \( F \): \( e^{-\sqrt{-1}Y} e^{X_I} e^{\sqrt{-1}Y} \in \text{Stab}(F) = \mathfrak{h} \). Then, \( e^{\sqrt{-1}Y} \) is in the normalizer subgroup \( N_G(H) \) of \( H \) in \( G \). Since \( N_G(H)/H \) is the Weyl group and the Weyl group is a discrete
group, \( Y \in \mathfrak{h} \) if \( Y \) is close to the origin. Therefore, \( K(e^{-\sqrt{\scriptstyle -1} Y})F = F \) and the fixed points are isolated. In this case, Kirillov’s formula becomes as follows:

\[
\text{Tr}_{V_{\lambda}} \left( e^{-\sqrt{\scriptstyle -1} T X_I} \right) = \frac{1}{p(T X_I)^{1/2}} \left( -\sqrt{\scriptstyle -1} \right)^{(\dim g - \text{rank } g)/2} \sum_{p \in \Omega_{\lambda + \rho} \atop X_{X_I}|p = 0} \frac{e^{-\sqrt{\scriptstyle -1} T X_I|p}}{\det_{ab} (T \partial_a V_{X_I}|p)^{1/2}}.
\] (2.19)

The partition function of the quantum mechanics is localized by using the method denoted in [4]. \( S_{\Omega} \) is given as follows:

\[
S_{\Omega} := \int_0^T dt \left( \theta_a \dot{\phi}^a - X_I + \frac{1}{2} \omega_{ab} \c_0^b \right).
\] (2.20)

The field contents of the quantum mechanics are real scalar fields \( \phi^a \), which correspond to a coordinate of \( \Omega_{\lambda + \rho} \), and real fermion fields \( c^a \), which correspond to the differential forms \( d\phi^a \). Furthermore, we have used a symplectic potential \( \theta_a \) for the symplectic form \( \omega \):

\[
d\theta = \omega.
\] (2.21)

The action \( S_{\Omega} \) is symmetric with respect to the following supersymmetry transformation:

\[
Q_{\Omega} \phi^a = c^a, \quad Q_{\Omega} c^a = \dot{\phi}^a - V_{X_I}^a.
\] (2.22)

Since the action is \( Q_{\Omega} \)-closed, the partition function \( Z_{\Omega} \) is independent of deformations of the action with terms which are not only \( Q_{\Omega} \)-exact but also \( Q_{\Omega} \)-closed. By deforming the action, the partition function is localized as follows:

\[
Z_{\Omega} = \mathcal{N} \sum_{p \in \Omega_{\lambda + \rho} \atop V_{X_I}|p = 0} \det_{ab} \left( \frac{T \partial_a V_{X_I}^b/2}{\sinh \left( T \partial_a V_{X_I}^b/2 \right)} \right)^{1/2} \frac{e^{-\sqrt{\scriptstyle -1} T X_I(p)}}{\det_{ab} (T \partial_a V_{X_I}|p)^{1/2}},
\] (2.24)

where \( \mathcal{N} \) is a normalization factor. By comparing eq. (2.24) with eq. (2.19), we recognize that the equation (2.16) holds if \( \mathcal{N} = (-\sqrt{\scriptstyle -1})^{\dim g - \text{rank } g} \) and the first determinant factor in eq. (2.24) is equal to \( p(-T X_I)^{-1/2} \). The determinant in \( p(-T X_I)^{-1/2} \) is taken over \( g \). Because the adjoint action of an element in \( \mathfrak{h} \) on \( \mathfrak{h} \) is trivial, the space over which the determinant in \( p(-T X_I)^{-1/2} \) is taken is reduced to \( g/\mathfrak{h} \). Furthermore by using eq. (2.8), \( p(-T X_I)^{-1/2} \) becomes the first determinant factor in eq. (2.24).

### 2.3 Path integral expression of Wilson loops

The aim of this subsection is that we write Wilson loops of gauge theories as path integrals of certain quantum mechanics by extending the correspondence between characters and partition functions (2.16). This is an old idea started from ???. The key ideas in this subsection are as follows: the path-ordering operator in a Wilson loop is considered as the time-ordering operator
of a quantum mechanics and the non-commutativity of the Lie algebra is regarded as a quantum
effect.

We consider a gauge theory on a manifold $M$. The gauge group is denoted by $G$. The
coordinate of $M$ is denoted by $x^\mu$. Let $A_\mu$ be the gauge field. For a given loop
$\gamma : [0, T] \to M$ and a given highest representation $(\pi_\lambda, V_\lambda)$ of $G$, a Wilson loop is defined by the following
equation:

$$W(\gamma, \lambda) := \text{Tr}_{V_\lambda} \left( \text{P exp} \left( -\sqrt{-1} \int_0^T dt A^I_\mu \frac{dx^\mu}{dt} X_I \right) \right). \quad (2.25)$$

If the gauge field is constant along the loop, the Wilson loop $W(\gamma, \lambda)$ can be written as a
partition function. Hence, we decompose the gauge field into a constant part and the others:

$$A^I_\mu x^\mu (\gamma(t)) = A^I_{0\mu} x^\mu + A^I_{t\mu} x^\mu (\gamma(t)). \quad (2.26)$$

With this decomposition, the Wilson loop is expanded as follows:

$$W(\gamma, \lambda) = \text{Tr}_{V_\lambda} (U(T, 0))
+ \int_0^T dt_1 \text{Tr}_{V_\lambda} (U(T, t_1) O(t_1) U(t_1, 0))
+ \int_0^T dt_1 \int_0^{t_1} dt_2 \text{Tr}_{V_\lambda} (U(T, t_1) O(t_1) U(t_1, t_2) O(t_2) U(t_2, 0))
+ \cdots, \quad (2.27)$$

where

$$U(t_1, t_2) := \exp \left( -\sqrt{-1} (t_1 - t_2) A^I_{0\mu} x^\mu X_I \right), \quad (2.28)$$

$$O(t) := \left( -\sqrt{-1} A^I_{t\mu} x^\mu X_I (\gamma(t)) \right). \quad (2.29)$$

The first term in eq.\,(2.27) is written as the partition function of quantum mechanics whose
Hamiltonian is $A^I_{0\mu} x^\mu X_I (\phi)$. The integrand of the second term is a correlation function of a
operator $O(t_1)$ inserted in the time $t_1$. The other terms are also considered as integrals of
correlation functions of operators $O(t_i)$. The $r + 1$-th term is written as follows:

$$\int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{r-1}} dt_r \text{Tr}_{V_\lambda} (U(T, t_1) O(t_1) U(t_1, t_2) O(t_2) \cdots U(t_{r-1}, t_r) O(t_r) U(t_r, 0))
= \int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{r-1}} dt_r \int_{\Omega_{\lambda+\rho}} \mathcal{D}\phi \mathcal{D}c \left( \prod_{i=1}^r O(t_i) \right) e^{\sqrt{-1} S_{\Omega}}. \quad (2.30)$$

Since operators are treated as classical quantities in a path integral, for any pairs of time $t_i$ and
$t_j$ the operators $O(t_i)$ and $O(t_j)$ commute in the path integral. Then eq.\,(2.30) is simplified as follows:

$$\int_0^T dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{r-1}} dt_r \int_{\Omega_{\lambda+\rho}} \mathcal{D}\phi \mathcal{D}c \left( \prod_{i=1}^r O(t_i) \right) e^{\sqrt{-1} S_{\Omega}}
= \frac{1}{r!} \int_0^T dt_1 \int_0^T dt_2 \cdots \int_0^T dt_r \int_{\Omega_{\lambda+\rho}} \mathcal{D}\phi \mathcal{D}c \left( \prod_{i=1}^r O(t_i) \right) e^{\sqrt{-1} S_{\Omega}}. \quad (2.31)$$

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By applying the above argument, we arrive at the following path integral expression of the Wilson loop:

\[
W(\gamma, \lambda) = \int_{\Omega_{\lambda+\rho}} \mathcal{D}\phi \mathcal{D}c e^{\sqrt{-1}S_\Omega}, \tag{2.33}
\]

where \(S_\Omega\) is an action with a time-dependent Hamiltonian

\[
\tilde{S}_\Omega := \int_0^T dt \left( \theta_a \dot{\phi}^a - A^I_\mu \dot{\gamma}(t) X_I(\phi(t)) + \frac{1}{2} \omega_{ab} c^a \dot{c}^b \right). \tag{2.34}
\]

The equation (2.33) is nothing but the equalities of partition functions of quantum mechanics with a time-dependent Hamiltonian.

3 Coadjoint orbits and Wilson loops in 5D supersymmetric gauge theories

In this section, we apply the relation between Wilson loops and quantum mechanics to Wilson loops of certain supersymmetric five dimensional gauge theories defined on a five dimensional manifold \(M_4 \times S^1\). The Wilson loops which we will consider are the following ones wrapping around the circle in the fifth direction:

\[
\text{Tr}_{V_\lambda} \left( \text{P exp} \left( -\sqrt{-1} \int_0^T dt (A_5 + \sqrt{-1}\varphi)^I X_I \right) \right), \tag{3.1}
\]

where \(A_5\) is the fifth component of the gauge field and \(\varphi\) is a scalar field. The five dimensional gauge theories which we will consider is five dimensional topological non-abelian gauge theories defined on a Cartesian product of a general four dimensional manifold \(M_4\) and a circle \(S^1\).

The minimally coupled \(\mathcal{N} = 1\) supersymmetric five dimensional gauge theory defined on a flat manifold \(\mathbb{R}^4 \times S^1\) is one of the five dimensional topological gauge theories.

It is known that the four dimensional component of the action of the topological gauge theories can be integrated [5]. The resulting action is an action of a particle moving on instanton moduli space \(\mathcal{M}\). The coordinate \(t\) if the fifth direction is treated as a time variable of the quantum mechanics. The particle is not charged under any gauge groups. We will show that the Wilson loops give interaction terms of charged particles and gauge fields. Therefore, we showed a direct relation between Wilson loops and charged particles.

At first we describe shortly the known relation between the five dimensional gauge theory and a particle moving on instanton moduli space \(\mathcal{M}\) in subsection 3.1. In subsection 3.2, we will show that the interaction terms of a charged particle and gauge field are coming from the Wilson loop.
3.1 5D supersymmetric gauge theory and quantum mechanics on instanton moduli space

The minimally coupled $\mathcal{N} = 1$ supersymmetric five-dimensional gauge theory is treated as a topological gauge theory by the following method. We introduce the standard flat metric on the five-dimensional Euclidean manifold $\mathbb{R}^4 \times S^1$. The coordinate of $S^1$ is denoted by $t$. The circumference of the circle $S^1$ is $T$. By regarding the $\mathcal{N} = 1$ supersymmetric gauge theory as a $\mathcal{N} = 2$ supersymmetric four-dimensional gauge theory depending on $t$, we twist the theory like the Donaldson-Witten theory. By this procedure the theory can be seen as a five-dimensional topological gauge theory.

We denote a coordinate of $M_4$ by $x^\mu$. The field contents of the topological gauge theory is as follows: a gauge field $A_\mu$ and $A_t$, an adjoint real scalar field $\varphi$, a real scalar fermion field $\eta$, a real one-form fermion field $\psi_\mu dx^\mu$ and a real self-dual fermion field $\frac{1}{2}\xi_{\mu\nu} dx^\mu \wedge dx^\nu$. Here we have defined the self-duality by using the Hodge star operator $\ast$ defined on the four-dimensional manifold $M_4$. There is a scalar supercharge $Q_{YM}$. Because the supersymmetry transformations caused by $Q_{YM}$ are not closed on shell, we add an auxiliary self-dual field $H_{\mu\nu}$ to close the supersymmetry transformations on shell. With the auxiliary field, $(A, \psi, A_t + \sqrt{-1}\varphi)$, $(A_t - \sqrt{-1}\varphi, \eta)$ and $(\xi_{\mu\nu}, H_{\mu\nu})$ form multiplets under the supercharge $Q_{YM}$. The most important multiplet is $(A, \psi, A_t + \sqrt{-1}\varphi)$ whose supersymmetry transformations are:

$$Q_{YM} A = \psi,$$
$$Q_{YM} \psi = d_A (A_t + \sqrt{-1}\varphi) - \partial_t A,$$  \hspace{1cm} (3.2)
$$Q_{YM} (A_t + \sqrt{-1}\varphi) = 0.$$

(3.3)

We have used a covariant derivative operator $d_A := d + A^I ad(\sqrt{-1}X_I)$ acting on $g$-valued differential forms. The action consists of the standard action $S_{YM}$ and the following topological term $S_{top}$:

$$S_{top} = \frac{\theta \sqrt{-1}}{T} \int_{S^1} dt \left( \frac{-1}{8\pi^2} \right) \int_{M_4} \text{Tr}_g (F \wedge F),$$  \hspace{1cm} (3.4)

where $\theta$ is a coupling constant. The standard action $S_{YM}$ can be written as a $Q_{YM}$-exact term up to a topological term:

$$S_{YM} + S_{top} = \int_{M_4 \times S^1} \{Q, W\} + \tau \int_{S^1} dt \left( \frac{-1}{8\pi^2} \right) \int_{M_4} \text{Tr}_g (F \wedge F),$$  \hspace{1cm} (3.5)

where $\tau := \frac{g_{YM}}{\theta} + \frac{4\pi^2}{g_{YM}}$. Due to this property of the action, correlation functions of BPS observables are independent of deformations of the action with terms which are not only $Q_{YM}$-exact but also $Q_{YM}$-closed. From now on, to make expressions simple we set $g_{YM} = 1$. We deform the action except the topological term as follows:

$$\int_{M_4 \times S^1} \hat{d}^5x \text{Tr}_g \left\{ Q_{YM}, F_{\mu\nu} \xi^{\mu\nu} - \frac{1}{2} \psi^\mu (D_\mu (A_t - \sqrt{-1}\varphi) - \partial_t A_\mu) \right\}.$$  \hspace{1cm} (3.6)

With this deformed action, the equations of motions of the multiplets $(A_t - \sqrt{-1}\varphi, \eta)$ and $(\xi_{\mu\nu}, H_{\mu\nu})$ give restrictions on the multiplet $(A, \psi, A_t + \sqrt{-1}\varphi)$. The equations of motion of the
multiplet \((\xi_{\mu\nu}, H_{\mu\nu})\) are as follows:

\[
F^{(+)} = 0, \quad (d_A \psi)^{(+)} = 0,
\]

where the symbol \((+)\) means the self-dual component. According to this conditions, the domain of the path integral is restricted to the moduli space of anti-instantons and tangent vectors on them. By integrating out \((A_t - \sqrt{-1}\varphi, \eta)\), we obtain the following equations:

\[
d^*_A \psi = 0, \quad d^*_A (A_t + \sqrt{1} \varphi) - d^*_A \partial_t A - 2\sqrt{-1} \psi \wedge \psi = 0.
\]

(3.8)

(3.9)

We have used the adjoint operator \(d^*_A\) of \(d_A\):

\[
d^*_A = - * d_A *\]

for \(g\)-valued one-forms.

After the integration of the multiplets, the deformed action becomes as follows:

\[
S_{SQM} := \int_0^T dt \left( \frac{1}{2} G_{ij} \dot{m}^i \dot{m}^j + \frac{1}{2} \chi^i \left( G_{ij} \partial_t + \dot{m}^k G_{il} \Gamma^l_{kj} \right) \chi^j \right).
\]

(3.11)

In order to derive the above action, we have used the following equation for the Levi-Civita connection \(\Gamma^k_{ji}\):

\[
G_{ik} \Gamma^k_{ji}(m) = \int_{M_4} \text{Tr}_g \left( b_i \wedge * \nabla_j b_l \right),
\]

(3.12)

where \(\nabla_i := \partial_i + \epsilon_i^j ad(\sqrt{-1}X_l)\) is the \(\mathcal{A}^* / G_0\) direction of the covariant derivative of the universal bundle \(\mathcal{E}\). See Appendix A for details.

### 3.2 One-point function of Wilson loop and quantum mechanics of charged particle

We discuss a one-point function of a Wilson loop and show that the Wilson loop gives the interaction terms of a charged particle and a gauge field.

Let \(\gamma_{x_0} : [0, T] \to M_4 \times S^1\) be a path on \(M_4 \times S^1\) such that \(\gamma_{x_0}(t) = (x_0, t) \in M_4 \times S^1\) for \(t \in [0, T]\). For a given highest representation \((\pi_\lambda, V_\lambda)\), we define the following Wilson loop:

\[
W(\gamma_{x_0}, \lambda) = \text{Tr}_{V_\lambda} \left( \text{P exp} \left( - \sqrt{-1} \int_0^T dt (A_t + \sqrt{-1}\varphi) X_I \right) \right).
\]

(3.13)

According to the equation (2.33), we write the Wilson loop \(W(\gamma_{x_0}, \lambda)\) as a partition function of a quantum mechanics on a coadjoint orbit:

\[
W(\gamma_{x_0}, \lambda) = \int_{\Omega_{\lambda+\rho}} \mathcal{D}\phi^a \mathcal{D}c^a \exp \left( \sqrt{-1} S_{\text{Wilson}} \right).
\]

(3.14)
where

\[ S_{\text{Wilson}} := \int_0^T dt \theta_t \dot{\phi}^a - (A_t + \sqrt{-1}\varphi) I X_I + \frac{1}{2} \omega_{ab} c^a c^b. \] (3.15)

Because of the supersymmetry transformations (3.3), the Wilson loop \( W(\gamma_{x_0}, \lambda) \) is BPS:

\[ Q_{YM} W(\gamma_{x_0}, \lambda) = 0. \] (3.16)

Furthermore, the Wilson loop consists of \( A_t + \sqrt{-1}\varphi \), which was not integrated out in subsection 3.1. Therefore, we can apply the argument in subsection 3.1. By combining the previous argument and the path integral expression of the Wilson loop, the one-point function of the Wilson loop becomes as follows:

\[ \langle W(\gamma_{x_0}, \lambda) \rangle = \int D\phi_{YM} e^{-S_{\text{stop}} - S_{YM}} \text{Tr}_V \left( \text{P exp} \left( -\sqrt{-1} \int_0^T dt (A_t + \sqrt{-1}\varphi) I X_I \right) \right) \] (3.17)

\[ = \int \mathcal{D}m^i \mathcal{D}\chi^i \int_{\Omega_{\lambda+\rho}} \mathcal{D}\phi^n \mathcal{D}c^a \exp \left( -\tau \int \frac{1}{8\pi^2} \int_{\mathcal{M}_4} \text{Tr}_\mathcal{B} (F \wedge F) - S_{\text{SQM}} + \sqrt{-1} S_{\text{Wilson}} \right). \] (3.18)

Hence, the effect of the Wilson loop \( W(\gamma_{x_0}, \lambda) \) is concentrated on the Hamiltonian \((A_t + \sqrt{-1}\varphi) I (x_0, m, \chi) X_I(\phi)\).

We solve \((A_t + \sqrt{-1}\varphi) I \) as a function of \( m^i \) and \( \chi^i \). It is known that \((A_t + \sqrt{-1}\varphi) I \) is related to the curvature of the universal bundle [5]. At first, we solve \( \psi_\mu \) as a function of \( m^i \) and \( \chi^i \). The equations for \( \psi_\mu \) (3.7) and (3.8) are solved by using \( b_i \) defined in eq.(A.14):

\[ \psi_\mu = b_i \chi^i. \] (3.20)

By comparing eq.(3.9) with eq.(A.25), \( A_t + \sqrt{-1}\varphi \) is solved as follows:

\[ A_t + \sqrt{-1}\varphi = \dot{m}^i \epsilon_i - \frac{1}{2} F_{ij} \chi^i \chi^j. \] (3.21)

By substituting eq.(3.21) into eq.(3.19), the one-point function of the Wilson loop becomes as follows:

\[ \langle W(\gamma_{x_0}, \lambda) \rangle = \int_{\mathcal{M}_4 \times \Omega_{\lambda+\rho}} \mathcal{D}m^i \mathcal{D}\chi^i \mathcal{D}\phi^n \mathcal{D}c^a e^{-T\tau \int_{\mathcal{M}_4} c_2(AdP) e^{-S_{\text{charged}}}}, \] (3.22)

where \( c_2(AdP) \) is the second Chern class of the vector bundle \( AdP \) defined in [A.1] and \( S_{\text{charged}} \) is defined as follows:

\[ S_{\text{charged}} := \int_0^T dt \frac{1}{2} G_{ij} \dot{m}^i \dot{m}^j - \sqrt{-1} \theta_t \phi^a + \dot{m}^i \epsilon_i (x_0, m) \sqrt{-1} X_I \]

\[ + \frac{1}{2} \chi^i (G_{ij} \partial_t + \dot{m}^k G_{ik} \Gamma_{kj}^l - F_{ij} (x_0, m) \chi^k \chi^j \sqrt{-1} X_I) \chi^j - \sqrt{-1} \frac{1}{2} \omega_{ab} c^a c^b. \] (3.23)
\( S_{\text{charged}} \) is nothing but an action for a charged particle denoted in \([6]\) except the imaginary unit \( \sqrt{-1} \). The reason of the appearance of the imaginary unit is the following difference of the treatment of \( S^1 \): in one hand \( S^1 \) is considered as a part of the Euclidean space in the gauge theory, in the other hand \( S^1 \) is regarded as the time of the quantum mechanics on the coadjoint orbit.

There is a general theory that a partition function of a charged particle gives a index of a twisted Dirac operator of a vector bundle \([6]\). To use the theory, we need to determine the vector bundle under consideration. The charged particle which we are considering is a particle moving on the instanton moduli space \( \mathcal{M} \) and is charged with respect to \( G \) as the representation \( \pi_\lambda \). Furthermore, the local connection one-form \( \epsilon_i \) and the curvature \( F_{ij} \) are evaluated at a fixed point \( x_0 \in \mathcal{M}_4 \). Hence, the appropriate vector bundle is \( \iota_{x_0}^* (\mathcal{E}_\lambda) \), which is defined in subsection \( \text{A.2} \) restricted to the instanton moduli space \( \mathcal{M} \): \( \iota_{x_0}^* (\mathcal{E}_\lambda)|_\mathcal{M} \). Therefore, we arrive at the following fact that the one-point function of the Wilson loop is the index of the vector bundle \( \iota_{x_0}^* (\mathcal{E}_\lambda)|_\mathcal{M} \):

\[
\langle \text{Tr}_{V_\lambda} \left( \text{P exp} \left( -\sqrt{-1} \int_0^T dt (A_t + \sqrt{-1} \varphi) X_I \right) \right) \rangle = \int_{\mathcal{M}} \hat{A}(\mathcal{M}) ch \left( \iota_{x_0}^* (\mathcal{E}_\lambda)|_{\mathcal{M}} \right) = \text{ind}_{\iota_{x_0}^* (\mathcal{E}_\lambda)|_{\mathcal{M}}} (D) \tag{3.25}
\]

4 Discussion

We expect that this method helps understanding of the emergence of geometries by condensations of Wilson loops \([10]\).

Acknowledgment

I thank Tudor Dimofte for his notification. Y. N. is supported by JSPS-IHES-EPDI Fellowship.

A Geometries of spaces of connections

In this appendix we describe geometries of spaces of connections. In \( \text{A.1} \) we discuss geometries of a gauge-orbit space. The instanton moduli space \( \mathcal{M} \) are considered as a subspace of the gauge-orbit space. In \( \text{A.2} \) we discuss vector bundles over the gauge-orbit space. In this appendix, we consider elements of a Lie algebra \( \mathfrak{g} \) as anti-Hermitian matrices to simplify notations.

A.1 Geometries on \( \mathcal{A}^*/G_0 \)

We describe geometries of a gauge-orbit space, which are used the main part of this paper. Almost all of the things in this subsection are written in \([7]\) and \([8]\).
For a real compact Lie group $G$, there is a $G$-principal bundle $P \to M_4$. We denote the adjoint bundle by $AdP := P \times_{ad} g$. In general, for a vector bundle $\mathcal{V} \to M_4$, let $\Omega^r(M_4, \mathcal{V})$ be the space of $\mathcal{V}$-valued $r$-forms:

$$\Omega^r(M_4, \mathcal{V}) := \Gamma(M_4, \wedge^r T^* M_4 \otimes \mathcal{V}).$$

(A.1)

The space of connection of $AdP$ are denoted by $\mathcal{A}$. The space $\mathcal{A}$ is an affine space modeled on $\Omega^1(M_4, AdP)$.

Any vector fields on $\mathcal{A}$ are described as follows:

$$\int_{M_4} dx \alpha^I(x; A) \frac{\delta}{\delta A^I}(x) \in T_A \mathcal{A},$$

(A.2)

where $\alpha : \mathcal{A} \to \Omega^1(M_4, AdP)$. From now on, to simplify notations we will omit the integral symbol and the functional derivative in vector fields.

The metric $g_A$ on $\mathcal{A}$ are defined as follows:

$$g_A(\alpha, \beta) := \int_{M_4} Tr (\alpha \wedge * \beta), \quad \text{for } \alpha, \beta \in \Gamma(\mathcal{A}, T \mathcal{A}).$$

(A.3)

Let $\mathcal{G}$ and $\text{Lie}(\mathcal{G}) = \Gamma(M_4, AdP)$ be the gauge transformation group and its Lie algebra. An element $\gamma \in \mathcal{G}$ acts on $A \in \mathcal{A}$ as follows:

$$\gamma \cdot A = \gamma A \gamma^{-1} + \gamma d \gamma^{-1}.$$ 

(A.4)

Similarly, $\Lambda \in \text{Lie}(\mathcal{G})$ acts on $A \in \mathcal{A}$ as follows:

$$\Lambda \cdot A = A - d_A \Lambda,$$

(A.5)

where $d_A := d + ad(A)$. Then, the vector field $\Lambda_A \in \Gamma(\mathcal{A}, T \mathcal{A})$ corresponding to this action becomes as follows:

$$\Lambda_A|_A = d_A \Lambda.$$

(A.6)

We denote the stabilizer group of $A \in \mathcal{A}$ by $\text{Stab}(A)$: $\text{Stab}(A) := \{ \gamma \in \mathcal{G} | \gamma \cdot A = A \}$. A connection $A$ is called irreducible when the stabilizer group is the center of $G$:

$$\text{Stab}(A) = C(G).$$

(A.7)

The space of irreducible connections is denoted by $\mathcal{A}^*$. We consider the following subgroup of $\mathcal{G}$:

$$\mathcal{G}_0 = \{ \gamma \in \mathcal{G} | \gamma(\infty) = e \}.$$ 

(A.8)

Since the stabilizer group of $A \in \mathcal{A}^*$ is a subgroup of $G$ and $\mathcal{G}_0$ does not contains the global gauge transformation $\bar{G}$, $\mathcal{G}_0$ action on $\mathcal{A}$ is free. Therefore, the space $\mathcal{A}^*$ can be considered as a $\mathcal{G}_0$-principal bundle: $\mathcal{A}^* \to \mathcal{A}^*/\mathcal{G}_0$. 
We can consider a connection of $\mathcal{A}^* \to \mathcal{A}^*/\mathcal{G}_0$. The vertical vectors are defined as vectors caused by the gauge transformations:

$$V_A\mathcal{A}^* := \text{Im}(d_A).$$

(A.9)

The horizontal vectors are defined as the orthogonal vectors of the vertical vectors with respect to the metric $g_A$:

$$H_A\mathcal{A} := \text{Ker}(d_A^*).$$

(A.10)

We denote the projection operator from tangent vectors to horizontal vectors by $P_H$:

$$P_H|_A = 1 - d_A \frac{1}{d_A^*d_A}d_A^*.$$  

(A.11)

According to the decomposition of tangent vectors, the connection one-form $\Theta \in \Gamma(\mathcal{A}^*, T^*\mathcal{A} \otimes \Omega^1(M_4, AdP))$ is given by the following equation:

$$\Theta|_A = \int_{M_4} dy G_A(\cdot, y)d_A^*|y d_A A(y),$$

(A.12)

where $G_A(x, y)$ is the Green function of the Laplacian $d_A^*d_A$ on $\Omega^0(M_4, AdP)$.

We identify the tangent vector space at $A \in \mathcal{A}^*/\mathcal{G}_0$ and the horizontal vector space at $A \in \mathcal{A}^*$: $T_A\mathcal{A}^*/\mathcal{G}_0 \simeq T_A\mathcal{A}^*$. A metric $g_{\mathcal{A}^*/\mathcal{G}_0}$ of $\mathcal{A}^*/\mathcal{G}_0$ is defined by the following equation:

$$g_{\mathcal{A}^*/\mathcal{G}_0}([\alpha], [\beta]) := g_{\mathcal{A}^*}(P_H\alpha, P_H\beta)(A), \quad \text{for } \alpha, \beta \in T_A\mathcal{A}^*.$$  

(A.13)

We take an open set $U \subset \mathcal{A}^*/\mathcal{G}_0$, fix a local section $\sigma \in \Gamma(U, \mathcal{A}^*)$ and introduce a coordinate $\{m^i\}$ on $U$. We write a vector $b_i$ as an abbreviation of the following vector:

$$b_i(x, m) := P_H \circ \sigma\left(\frac{\partial}{\partial m^i}|_m\right).$$

(A.14)

With these notations the metric $g_{\mathcal{A}^*/\mathcal{G}_0}$ is written as follows:

$$G_{ij}(m) = \int_{x \in M_4} \text{Tr}_g(b_i \wedge *b_j)(x, m).$$

(A.15)

Furthermore, we write the local connection one-form on $U$ by $\epsilon \in \Gamma(U, T^*U \otimes \text{Lie}(\mathcal{G}))$:

$$\epsilon = \epsilon_i dm^i := \sigma^*(\Theta).$$

(A.16)

From the decomposition of the vector field $\sigma^*(\frac{\partial}{\partial m^i})$, we obtain the following equation:

$$\frac{\partial A}{\partial m^i}(x; m) = b_i(x, m) + d_A\epsilon_i(x, m).$$

(A.17)

If we regard the curvature $F_A = dA + A \wedge A$ as a map from $\mathcal{A}$ to $\Omega^2(M_4, AdP)$, the instanton moduli space $\mathcal{M}$ are considered as a subspace of $\mathcal{A}^*/\mathcal{G}_0$ determined by the condition $F^+(+) = 0$. In this subspace we need to impose the following condition on the tangent vectors $b_i$:

$$(d_A b_i)^{(+) = 0}.$$  

(A.18)
A.2 Vector bundles over $\mathcal{A}^*/\mathcal{G}_0$

We consider a vector bundle called the universal bundle over $M_4 \times \mathcal{A}^*/\mathcal{G}_0$ [2]. Furthermore, we give a definition of a certain vector bundle over $\mathcal{A}^*/\mathcal{G}_0$ which is needed to the discussion of this paper.

We consider a projection $\pi_1 : M_4 \times \mathcal{A}^* \to M_4$ and a pullback bundle $\pi_1^*(AdP) \to M_4 \times \mathcal{A}^*$. At a point $(x, A) \in M_4 \times \mathcal{A}^*$, we define a covariant derivative of the pullback bundle by $d_A$ along $M_4$ direction and $d_A$ along $\mathcal{A}^*$ direction. The group $\mathcal{G}_0$ cause a bundle map on $\pi_1^*(AdP)$. Then we can define a vector bundle $E$ called the universal bundle as follows:

\[
\begin{array}{c}
\pi_1^*(AdP) \\ \downarrow \\
M_4 \times \mathcal{A}^* \\ \downarrow \\
\pi_1^*(AdP)/\mathcal{G}_0 \\
E
\end{array}
\]  

(A.19)  

We consider a projection $\pi_1 : M_4 \times \mathcal{A}^* \to M_4$ and a pullback bundle $\pi_1^*(AdP) \to M_4 \times \mathcal{A}^*$. At a point $(x, A) \in M_4 \times \mathcal{A}^*$, we define a covariant derivative of the pullback bundle by $d_A$ along $M_4$ direction and $d_A$ along $\mathcal{A}^*$ direction. The group $\mathcal{G}_0$ cause a bundle map on $\pi_1^*(AdP)$. Then we can define a vector bundle $E$ called the universal bundle as follows:

\[
\begin{array}{c}
\pi_1^*(AdP) \\ \downarrow \\
M_4 \times \mathcal{A}^* \\ \downarrow \\
\pi_1^*(AdP)/\mathcal{G}_0 \\
E
\end{array}
\]  

(A.20)  

The covariant derivative on $E$ is given by the following equation:

\[
d^\mathcal{E}_{A+\epsilon} = d + A + dm \wedge (\partial_i + \epsilon_i).
\]  

(A.22)  

The curvature becomes as follows [3]:

\[
F^\mathcal{E}(\partial_{i}, \partial_{j})(x, m) = -2 \left( \frac{1}{d_A^* d_A} [b_{i}^{\mu}, b_{j}^{\mu}] \right)(x, m).
\]  

(A.25)  

We apply the construction of the universal bundle to other associated vector bundles over $M_4$. For a given highest representation $(\pi_\lambda, V_\lambda)$ of $G$, we define a vector bundle $\mathcal{E}_\lambda := P \times_{\pi_\lambda} V_\lambda$.

By using the projection $\pi_1 : M_4 \times \mathcal{A}^* \to M_4$, we define a pullback bundle $\pi_1^*(\mathcal{E}_\lambda) \to M_4 \times \mathcal{A}^*$. By dividing the vector bundle $\pi_1^*(\mathcal{E}_\lambda)$ by $\mathcal{G}_0$, we obtain a vector bundle $\mathcal{E}_\lambda \to M_4 \times \mathcal{A}^*/\mathcal{G}_0$, whose typical fiber is $V_\lambda$. Furthermore, by using the following inclusion map $\iota_{x_0}$

\[
\begin{array}{c}
\iota_{x_0} : \mathcal{A}^*/\mathcal{G}_0 \\ (m) \mapsto (x_0, m)
\end{array}
\]  

(A.26)  

(A.27)  

we define a vector bundle $\iota_{x_0}^* \mathcal{E}_\lambda$ over $\mathcal{A}^*/\mathcal{G}_0$. The curvature of $\iota_{x_0}^*(\mathcal{E}_\lambda)$ at $m \in \mathcal{A}^*/\mathcal{G}_0$ is given by the following equation:

\[
\frac{1}{2} \pi_\lambda \left( F^\mathcal{E}(\partial_{i}, \partial_{j})(x_0, m) \right) dm \wedge dm.
\]  

(A.28)  

References

[1] A. A. Kirillov, “Lectures on the orbit method,” American Mathematical Society, 2004

[2] A. A. Kirillov, “Characters of unitary representations of Lie groups: Reduction theorems,” Funkcional. Anal. i Prilozhen. 3 (1969), no.1, 36-47; English transl. in Functional Anal. Appl. 3 (1969).
[3] A. Alekseev, L. Faddeev and S. Shatashvili, “Quantization of symplectic orbits of compact Lie groups by means of the functional integral” J. Geom. Phys. 5, 3, (1989); A. Alekseev, and S. Shatashvili, “Path integral quantization of the coadjoint orbits of the Virasoro group and 2-d gravity” Nucl. Phys. B 323, 719, (1989); A. J. Niemi and P. Pasanen, “Orbit geometry, group representations and topological quantum field theories,” Phys. Lett. B 253, 349 (1991). M. Stone, “Supersymmetry and the quantum mechanics of spin,” Nucl. Phys. B314, 557, (1998).

[4] M. Blau, E. Keski-Vakkuri and A. J. Niemi, “Path integrals and geometry of trajectories,” Phys. Lett. B 246, 92 (1990).

[5] N. Nekrasov, “Five dimensional gauge theories and relativistic integrable systems,” Nucl. Phys. B 531, 323 (1998) [arXiv:hep-th/9609219].

[6] A. Hietamaki, A. Y. Morozov, A. J. Niemi and K. Palo, “Geometry of N=1/2 supersymmetry and the Atiyah-Singer index theorem,” Phys. Lett. B 263, 417 (1991).

[7] D. Groisser and T. H. Parker, “The riemannian geometry of the Yang-Mills moduli space,” Commun. Math. Phys. 112 (1987) 663.

[8] S. K. Donaldson and P. B. Kronheimer, “The geometry of four-manifolds” Oxford Univ, 1997.

[9] M. F. Atiyah and I. M. Singer, “Dirac Operators Coupled To Vector Potentials,” Proc. Nat. Acad. Sci. 81 (1984) 2597.

[10] T. Nakatsu and K. Takasaki, “Melting Crystal, Quantum Torus and Toda Hierarchy,” Commun. Math. Phys. 285, 445 (2009) [arXiv:0710.5339 [hep-th]].

[11] T. Nakatsu, Y. Noma and K. Takasaki, “Integrable Structure of 5d $\mathcal{N}=1$ Supersymmetric Yang-Mills and Melting Crystal,” Int. J. Mod. Phys. A 23, 2332 (2008) [arXiv:0806.3675 [hep-th]]. T. Nakatsu and K. Takasaki, “Integrable structure of melting crystal model with external potentials,” [arXiv:0807.4970 [math-ph]].

[12] E. Witten, “Topology Changing Amplitudes in (2+1)-Dimensional Gravity,” Nucl. Phys. B 323, 113 (1989).