ANOTHER REFINEMENT OF THE RIGHT-HAND SIDE OF THE HERMITE-HADAMARD INEQUALITY FOR SIMPLICES

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Abstract. In this paper, we establish a new refinement of the right-hand side of Hermite-Hadamard inequality for convex functions of several variables defined on simplices.

The classical Hermite-Hadamard inequality states that if \( f : I \rightarrow \mathbb{R} \) is a convex function then for all \( a < b \in I \) the inequality

\[
    f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(t) \, dt \leq \frac{f(a) + f(b)}{2}
\]

is valid. This powerful tool has found numerous applications and has been generalized in many directions (see e.g. \cite{2} and \cite{1}). One of those directions is its multivariate version:

**Theorem 1** (\cite{1}). Let \( f : U \rightarrow \mathbb{R} \) be a convex function defined on a convex set \( U \subset \mathbb{R}^n \) and \( \Delta \subset U \) be an \( n \)-dimensional simplex with vertices \( x_0, x_1, \ldots, x_n \), then

\[
    f(b_\Delta) \leq \frac{1}{\text{Vol} \Delta} \int_\Delta f(x) \, dx \leq \frac{f(x_0) + \cdots + f(x_n)}{n+1},
\]

where \( b_\Delta = \frac{x_0 + \cdots + x_n}{n+1} \) is the barycenter of \( \Delta \) and the integration is with respect to the \( n \)-dimensional Lebesgue measure.

The aim of this note is to proof a refinement of the right-hand side of \cite{1}.

Let us start with a set of definitions.

A function \( f : I \rightarrow \mathbb{R} \) defined on an interval \( I \) is called convex if for any \( x, y \in I \) and \( t \in (0, 1) \) the inequality

\[
    f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)
\]

holds.

If \( U \) is a convex subset of \( \mathbb{R}^n \), then a function \( f : U \rightarrow \mathbb{R} \) is convex if its restriction to every line segment in \( U \) is convex.

For \( n+1 \) points \( x_0, \ldots, x_n \in \mathbb{R}^n \) in general position the set \( \Delta = \text{conv} \{ x_0, \ldots, x_n \} \) is called an \( n \)-dimensional simplex. If \( K \) is a nonempty subset of the set \( N = \{0, \ldots, n\} \) of cardinality \( k \), the set \( \Delta_K = \text{conv} \{ x_i : i \in K \} \) is called a face (or a \( k-1 \)-face) of \( \Delta \). The point \( b_K = \frac{1}{k} \sum_{i \in K} x_i \) is called a barycenter of \( \Delta_K \). The barycenter of \( \Delta \) will be denoted by \( b \). By \text{card} \( K \) we shall denote the cardinality of set \( K \).

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For each \( k - 1 \)-face \( \Delta_K \) we calculate the average value of \( f \) over \( \Delta_K \) using the formula
\[
\text{Avg}(f, \Delta_K) = \frac{1}{\text{Vol}(\Delta_K)} \int_{\Delta_K} f(x) dx,
\]
where the integration is with respect to the \( k - 1 \)-dimensional Lebesgue measure (in case \( k = 1 \) this is the counting measure).

For \( k = 1, 2, \ldots, n + 1 \) we define
\[
\mathcal{A}(k) = \frac{1}{\binom{n+1}{k}} \sum_{K \subset N, \text{card } K = k} \text{Avg}(f, \Delta_K).
\]

Note that the right-hand side of the inequality (1) can be rewritten as \( \mathcal{A}(n + 1) \leq \mathcal{A}(1) \). It turns out, that

**Theorem 2.** The following chain of inequalities holds:
\[
\mathcal{A}(n + 1) \leq \mathcal{A}(n) \leq \cdots \leq \mathcal{A}(2) \leq \mathcal{A}(1).
\]

In the proof we shall use the following

**Lemma 1** ([3, Theorem 4.1]). If \( K_i = N \setminus \{i\} \) and \( b \) is the barycenter of \( \Delta \), then
\[
\text{Avg}(f, \Delta) \leq \frac{1}{n + 1} f(b) + \frac{n}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i}).
\]

**Proof of Theorem 2.** We shall prove first the inequality \( \mathcal{A}(n + 1) \leq \mathcal{A}(n) \). Let us use the notation from Lemma 1. For \( i = 0, 1, \ldots, n \) we have
\[
b_{K_i} = \frac{1}{n} \sum_{j=0}^{n} x_j = \frac{1}{n} \left( \sum_{j=0}^{n} x_j - x_i \right) = \frac{1}{n} ((n + 1)b - x_i).
\]

Summing (2) we obtain
\[
b = \frac{1}{n + 1} \sum_{j=0}^{n} b_{K_j}.
\]

Now using Lemma 1 and convexity of \( f \) applied to (3) we get
\[
\text{Avg}(f, \Delta) \leq \frac{1}{n + 1} f(b) + \frac{n}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i})
\]
\[
\leq \frac{1}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} f(b_{K_i}) + \frac{n}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i})
\]

thus, by the left-hand side of (1)
\[
\leq \frac{1}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i}) + \frac{n}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i})
\]
\[
= \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i})
\]

This shows the inequality \( \mathcal{A}(n + 1) \leq \mathcal{A}(n) \). The other inequalities follow by simple induction argument applying the same reasoning to all terms in \( \mathcal{A}(n) \) etc. \( \Box \)
Just for completeness note that similar refinement of the left-hand side of [1] can be found in [4, Corollary 2.6]. It reads as follows:

**Theorem 3.** For a nonempty subset $K$ of $N$ define the simplex $\Sigma_K$ as follows: let $A_K$ be the affine span of $\Delta_K$ and $A'_K$ be the affine space of the same dimension, parallel to $A_K$ and passing through the barycenter of $\Delta$. Then $\Sigma_K = \Delta \cap A'_K$.

For $k = 1, 2, \ldots, n + 1$ we let

$$B(k) = \frac{1}{\binom{n+1}{k}} \sum_{\substack{K \subseteq N \\text{card} K = k}} \operatorname{Avg}(f, \Sigma_K).$$

Then

$$f(b) = B(1) \leq B(2) \leq \cdots \leq B(n + 1) = \operatorname{Avg}(f, \Delta).$$

**References**

[1] Bessenyei, M: The Hermite–Hadamard inequality on simplices. American Mathematical Monthly **115**(4), 339–345 (2008)

[2] Dragomir, SS, Pearce, CEM: Selected topics on Hermite–Hadamard inequalities

[3] Nowicka, M and Witkowski, A: A refinement of the right-hand side of the Hermite-Hadamard inequality for simplices, Aequat. Math., **91** (2017), 121-128, doi:10.1007/s00010-016-0433-z,

[4] Nowicka, M and Witkowski, A: A refinement of the left-hand side of Hermite-Hadamard inequality for simplices, J. Inequal. Appl., (2015), 2015:3 73, doi:10.1186/s13660-015-0904-0,