Lie Bracket-based Extremum Seeking Control with Attenuating Oscillations and Multi-agent Application

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Abstract—In the last decade, a control-affine Extremum Seeking Control (ESC) approach was introduced, and mainly applied on multi-agent systems. This ESC structure is compatible with single-integrator and unicycle dynamics to steer collaborative multi-agent units (such as, but not limited to vehicles) to the extremum of an objective function that we have access to its measurements, but not its expression. This approach utilized a Lie bracket approximation-system for stability characterization, however, it possesses persistent oscillations at all time. In the following years, Lie bracket-based approaches have been developed, and eventually, in a recent effort, generalized in a unifying class that under strong conditions converge asymptotically to the extremum point; nevertheless, in this class the extremum point has to be known a priori and guaranteeing vanishing control input at the extremum point requires the application of a strong condition. In this paper, we introduce a Lie bracket-based approach for control-affine ESCs, which is also compatible with multi-agent systems. The proposed ESC does not require the extremum point a priori, its oscillations attenuate structurally, and its stability is characterized by a time-dependent condition that does not require strong bounds or knowledge on the objective function compared with literature. Moreover, we show that the proposed ESC works with some control-affine cases in which the said generalized class could not. We provide a multi-agent vehicle problem to demonstrate our results numerically in direct comparison with literature.

Index Terms—Extremum Seeking Control; Adaptive Control; Lie Bracket; Control-affine Systems; Simulation-aided Proof; Kalman Filter; Gradient Estimation; Online Stability.

I. INTRODUCTION

Extremum Seeking Control (ESC) is a model free adaptive control technique that determines and stabilizes a steady state map of a dynamical system around the extremum point of an objective/cost function, that ideally, we have access to its measurements, but not its expression. In AD 2000, the classic structure of ESC [1] has been analyzed through averaging theory (refer [2] for detailed account on averaging methods) and its stability was characterized. This ESC structure has found many applications in various disciplines (refer to [3]–[8]). In addition, it also has been extended to systems with parameter uncertainties [9], constrained inputs [10] and stochastic perturbations [11]. Another ESC structure [12] also was introduced, which is more compatible with multi-agent system applications.

For the scope of this paper, we focus on Lie bracket-based ESC approaches, which often deal with problems naturally and easily expressed in control-affine formulation; this is also how multi-agent systems are often described. Durr et al. [13] first used the concept of Lie bracket-system approximation of ESC, mainly for stability characterization. They called it ’the corresponding Lie bracket system’. They showed that under certain assumptions, semi-global (local) practical uniform asymptotic stability of a control-affine ESC system follows from global (local) uniform asymptotic stability of the corresponding Lie bracket system. They also provided the multi-agent ESC structure - we are resolving in this paper with improved performance - to study single-integrator and unicycle dynamics. In the following years to [13], many research have adopted designs and structures that utilize Lie bracket-based approach for characterizing stability and providing adaptive excitation signals (amplitude and frequency wise); the reader is referred to [14], [15]. Recently, Grushkovskaya et al. [16] introduced a unifying and generalized class of ESC systems based on the Lie bracket approximation techniques. In their highly regarded work, many systems, important developed structures, including [13], [17], are considered in the generalization. They provide a rigorous proof of ESC asymptotic stability to the extremum point under strong assumptions and bounds. Through their framework, one should be able to construct control inputs that are guaranteed to vanish at the extremum point under certain condition.

Motivation: Since Durr et al. [13] introduced their Lie bracket-based ESC until the comprehensive generalization of Lie bracket-based ESCs by Grushkovskaya et al. [16], it seems that there is a trade-off dilemma with these approaches. On one hand, one can choose having an ESC that does not require the extremum point a priori [13], and in fact, functions quite well with minimal conditions on the objective function (differentiability of certain order) and much less restrictive stability conditions, but this ESC has persistent oscillations at all time including near/at the extremum point. On the other hand, one can choose having an ESC that converges asymptotically to the extremum point, but requiring it a priori and necessitate stability conditions/bounds with quite some knowledge on the mathematical characteristics of the unknown objective function [16]. We believe many control-affine ESCs, especially multi-agent ones, need to have a structure that converges to the extremum point without knowing it a priori. Moreover, the stability condition associated with this ESC needs to be possible to check for a particular problem.
without conditions/bounds on the unknown objective function that we have access only to its measurements. In another word, the motivation here is to combine the positive traits of both approaches \[13, 16\] and achieving so, with minimal downsides.

**Contribution:** In this paper, we propose a Lie bracket-based ESC system compatible with ESCs expressed in control-affine form, and applicable to multi-agent systems. This ESC (i) has attenuating oscillations, (i.e. converges to the extremum point), (ii) does not require the extremum point a priori, and (iii) its stability is characterized by a simpler time-dependent condition which requires no partial/full knowledge of the expression of objective function and demands no bounds that involve the objective function over the domain of the dynamics. We note here that the proposed ESC has stability characteristics different from the control-affine ESC generalization introduced by Grushkovskaya et al. \[16\] and admit more relaxed requirement on the objective function; this will be shown in this paper by successfully resolving an example borrowed from \[16\] where they demonstrate their approach’s limitation solving it. The potential of our results is attenuating oscillations, (i.e. converges to the extremum point), (ii) does not require the extremum point a priori, and (iii) its stability is characterized by a simpler time-dependent condition which requires no partial/full knowledge of the expression of objective function and demands no bounds that involve the objective function over the domain of the dynamics. We note here that the proposed ESC has stability characteristics different from the control-affine ESC generalization introduced by Grushkovskaya et al. \[16\] and admit more relaxed requirement on the objective function; this will be shown in this paper by successfully resolving an example borrowed from \[16\] where they demonstrate their approach’s limitation solving it. The potential of our results

**II. MAIN RESULTS**

We propose an ESC system of the following form:

\[
\dot{y} = \begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{a} \end{bmatrix} = \begin{bmatrix} h(t, x) \\ J(t, z) \\ g(t, a, z) \end{bmatrix} = F(t, y),
\]

where,

\[
h(t, x) = b_0(t, x) + \sum_{i=1}^{m} b_i(t, x)\sqrt{\omega u_i(t, \omega t)},
\]

\[
J(t, z) = b_0(t, z) + \sum_{i=1}^{m} b_i(t, z)\nu_j(t),
\]

\[
g(t, a, z) = -\text{diag}[\lambda](a - J(t, z)).
\]

Equation (2) is the main body (states) of the ESC system (similar to \[13\]) in control-affine form with \(x(t_0) = x_0 \in \mathbb{R}^n\), \(\omega \in (0, \infty)\), and \(m\) is the number of control inputs. Similarly, Equation (3) is the corresponding Lie bracket system to (2) with \(\nu_j(t) = \frac{1}{2\pi} \int_{0}^{T} u_j(t, \omega t) \int_{0}^{\theta} u_i(t, \tau) d\tau d\theta \) and \(T = \frac{2\pi}{\omega}\). Throughout this paper, the notation of the objective function is \(f(x)\), and it is explicitly present in the vector fields \(b_i\).

Similar to \[13\], we assume the following on \(b_0, b_i\) and \(u_i:\)

A1. \(b_i \in C^2: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, i = 0, ..., m\).

A2. For every compact set \(C \subseteq \mathbb{R}^n\), there exist \(A_1, ..., A_6 \in [0, \infty)\) such that \(|b_i(t, x)| \leq A_1, \left|\frac{\partial b_i(t, x)}{\partial t}\right| \leq A_2, \left|\frac{\partial^2 b_i(t, x)}{\partial t^2}\right| \leq A_3, \left|\frac{\partial b_i(t, x)}{\partial x}\right| \leq A_4, \left|\frac{\partial^2 b_i(t, x)}{\partial x^2}\right| \leq A_5, \left|\frac{\partial b_i(t, x)}{\partial x^3}\right| \leq A_6\) for all \(x \in C, t \in \mathbb{R}, i = 0, ..., m; j = 1, ..., m; k = j, ..., m\).

A3. \(u_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, i = 1, ..., m\) are measurable functions. Moreover, for every \(i = 1, ..., m\) there exist constants \(L_i, M_i \in (0, \infty)\) such that \(|u_i(t, \theta) - u_i(t, \tau)| \leq L_i(t - \tau)\) for all \(t, \tau \in \mathbb{R}\) and such that \(\sup_{t, \theta \in \mathbb{R}}|u_i(t, \theta) - L_i| \leq M_i\).

A4. \(u_i(t, \cdot)\) is T-periodic, i.e. \(u_i(t, \theta + T) = u_i(t, \theta)\), and has zero average, i.e. \(\int_{t}^{t+T} u_i(t, \tau) d\tau = 0\), with \(T \in (0, \infty)\) for all \(t, \theta \in \mathbb{R}, i = 1, ..., m\).

**Remark 1:** Assumption A1 means that the vectors fields in the ESC system and corresponding Lie bracket system are smooth. Assumption A2 ensures the expressions involving the vector fields \(b_i (i = 0, 1, ..., m)\), and their derivatives are bounded uniformly in \(t\). Similarly, A3 means that the control inputs \(u_i, i = 1, ..., m\) are measurable, which is important to guarantee existence of solutions for the ESC system. Finally, assumption A4 ensures T-periodicity and zero average for the control inputs \(u_i, i = 1, ..., m\).

Equation (4) is the adaptation law that would attenuate the oscillation (excitation signals) of ESC to 0, and \(a\) is a vector containing the amplitudes of the excitation signals for every input, such that \(u_i(t, \omega t) = a_i(t)\) or \(a_i(0)\) (e.g. \(a_i(t) = a_i \sin(\omega t)\) or \(a_i \cos(\omega t)\) can be a form of control inputs). For every principal component of the diagonal matrix \(\text{diag}[\lambda]\), \(\lambda_i > 0 \in \mathbb{R}\), is a tuning parameter. Now, we introduce our Theorem.

**Theorem 1:** Let the assumptions A1-A4 be satisfied and suppose \(\exists t^* > 0\) such that \(\forall t > t^*\) and for every \(i, |J_i(t, x)| \leq 1/\lambda_i^p\) with some \(p > 1\), where \(i\) is the index of \(J\) and \(a\) components, then (i) the corresponding Lie bracket system in (3), \(\dot{z} = J\), is uniformly asymptotically stable and convergent, and (ii) \(a_i\) for every \(i\) is asymptotically convergent to 0.

*The proof can be found in the Appendix.*

**Remark 2:** The result of Theorem 1 implies that under the provided time-dependent condition, (3) is uniformly asymptotically stable. Now, as per Theorem 1.2 from \[13\] and Lemma 1 from \[16\], the local (global) uniform asymptotic stability of (3) characterizes and provides local (semi-global) practical uniform asymptotic stability for (2). The reader is referred to \[13\] for definitions on practically uniformly stable, and practically uniformly asymptotically stable systems. Furthermore, the amplitude of the excitation signals within the same condition provided in Theorem 1, simultaneously, is attenuating and asymptotically convergent to 0; i.e by the time the ESC reaches the extremum, with guaranteed practical stability, the excitation signals (oscillations) vanish. Thus, if the conditions in Theorem 1 are satisfied, then the main body of the ESC system, i.e. (2), will achieve practical stability and the amplitudes of the excitation signals vanish to 0.

**The proposed ESC in the light of the framework done by Grushkovskaya et al. \[16\]:** It is essential to discuss the proposed Lie bracket-based ESC in the context of the work done by Grushkovskaya et al. \[16\], as this work, introduced a generalization that unifies most significant Lie bracket-based approaches since Durr et al. \[13\] introduced this class of systems. Thus, a thorough and comprehensive discussion
need to be provided, through which, we highlight possible improvements to the results of Grushkovskaya et al. [16] by the proposed ESC. Here we make three brief highlights: (i) ESC stability conditions and access to the gradient for implementation between the proposed ESC and [16], (ii) Functionality of the proposed ESC in cases where [16] is limited, and (iii) Multi-agent systems between Durr et al. [13], Grushkovskaya et al. [16] and the proposed ESC.

First highlight: Our proposed ESC stability is characterized by a time-dependent condition that, in principle, is easier to verify for a given ESC system when compared to the following stability conditions introduced in [16] in a domain $D \subseteq \mathbb{R}^n$.

B1. There exists an $x^* \in D$ such that $\nabla f(x^*) = 0$, $\nabla f(x) \neq 0$ for all $x \in D \setminus \{x^*\}$; $f(x^*) = \hat{f}^* \in \mathbb{R}$, $f(x) > f(x^*)$ for all $x \in D \setminus \{x^*\}$, where $\hat{f}^*$ is the cost function evaluated at the extremum point $x^*$.

B2. There exist constants $\gamma_1, \gamma_2, \kappa_1, \kappa_2, \mu$ and $m_1 \geq 1$ such that for all $x \in D$,

$$\gamma_1|x - x^*|^{2m_1} \leq \tilde{f}(x) \leq \gamma_2|x - x^*|^{2m_1},$$

$$\kappa_1\tilde{f}(x)^2 - \frac{\mu}{m_1} \leq \left\|\nabla f(x)\right\|^2 \leq \kappa_2\tilde{f}(x)^2 - \frac{\mu}{m_1},$$

where $\left\|\cdot\right\|$ is the $l_2$ norm, and $\tilde{f}(x) = f(x) - f^*$.

B3. The functions $b_{si}(\tilde{f}(x)) \in C^2(D \setminus \{x^*\}; \mathbb{R})$, $D \subseteq \mathbb{R}^n$; the functions $L_{b_{si}}(\tilde{f}(x))$, $L_{b_{si}}(\tilde{f}(x)) \in C(D; \mathbb{R})$, for all $s, p, q \in \{1, 2\}$ and $i, j, l = 1, n$, where Lie derivative is denoted as $L_q f(x)$ and defined as $L_q f(x) = \lim_{s \to 0} \frac{f(x + s q(x - f(x)))}{s}$, and the notation $b_{uv}$ is used to represent an element in the vector field $b$ where $v$ represents the index for vector field (similar to $i$ in (2)) and $w$ is the index for the position of element within $b_i$.

B4. The functions $b_{si}(\tilde{f}(x))$ are Lipschitz on each compact set from $D$, and

$$\alpha_1\tilde{m}^2(x) \leq L_{b_{si}}(\tilde{f}(x)) \leq \alpha_2\tilde{m}^2(x),$$

$$\left|b_{si}(\tilde{f}(x))\right| \leq M_{\tilde{m}^3}(x),$$

$$\left\|L_{b_{si}} L_{b_{si}}(\tilde{f}(x))\right\| \leq H\tilde{m}^4(x)$$

for all $x \in D$, $s, p, q = \frac{1}{2}, i, j, l = \frac{1}{n}$ with $m_2 \geq 1/m_1 - 1$, $m_3 = (m_2 + 1)/2$, $m_4 = 3(1 + m_2)/2 - 1/m_1$, and some $\alpha_1, \alpha_2, M > 0, H > 0$.

We note that, for conditions B1-B4 to be verified for a given ESC system, one would need some information on the objective function, its gradient and the Lie brackets involving the gradient and its derivative, in addition to the extremum point. The bounds provided in conditions B2-B4 are over the domain of the dynamics, which we believe, is harder to obtain through estimation/approximation methods applied on the measurements of the objective function, such as and not limited to Kalman Filter, among others. Our stability condition on the other hand is time-dependent and is verified by a time-dependent bound, which can be synchronized with the time progression of the system. We believe, this condition can be easier in stability analysis for a particular system of interest; it can be, possibly, applied in a simulation-aided basis. Multi-agent systems, can benefit greatly from proposed ESC stability condition as it provides an online stability verification analysis instead of conditions on the spacial variables/domain. However, our condition, to be implemented, still requiring access to accurate gradient estimations. We believe the full potential of our proposed ESC will be realized on building on the introduced results in this paper by approximating and estimating the gradient accurately so that the adaptation laws can work efficiently and within the condition of Theorem 1 or an expanded version of it incorporating a relevant estimation theory. A reason to believe that the introduced results can go in this direction, may be noted from the development of gradient estimators based on measurements of the objective function during the ESC oscillations (see [13] for gradient estimation through Kalman Filter based on measurements of the objective function); these estimations are not accurate without the ESC excitation, otherwise the literature of ESC would have ceased and would be replaced by gradient estimators providing a gradient decent system. A supportive example of our discussion here can be found in the recent work [20], where an attenuating adaptation law was developed to be coupled with the classic structure of ESC [1] utilizing Kalman Filter for gradient estimation based on the normally accessible measurements of the objective function. We also emphasize that while the ESC in [16] involves Lie brackets in the stability analysis only, condition B4 is required for guaranteeing vanishing control inputs at the extremum point, i.e. one cannot utilize their class of ESC system and guarantee attenuating oscillations without testing condition B4, which involves Lie bracket operations involving the gradient.

Second highlight: We claim that our ESC can work for control-affine ESCs in cases where the generalized class in [16] may not. Given this, it is not clear if the proposed ESC in this paper can be characterized by the unifying and generalized class in [16]; the ESC in [13] is claimed by [16] to be included in this paper can be characterized by the unifying and generalized class in [16]; the ESC in [13] is claimed by [16] to be included in the unification, but this seems not to be partially/fullly the case with our ESC. This could be due to its newly developed structure or because simply the proposed ESC has different stability characteristics that is more relaxed compared with B1-B4. Example 1 below is provided to support this argument.

**Example 1.** This simple ESC system in control-affine form was used in [16] to demonstrate that with the failure to satisfy condition B4, the ESC will have non vanishing oscillations at the extremum point. We resolved the same example by our proposed ESC [1], and it works effectively. The equation of the system is $\dot{x} = f_1(x)u_1(t) + u_2(t)$ with $f_1(x) = 2(x - x^*)^2$, $x \in \mathbb{R}$, $x^* = 1$, $u_1 = \sqrt{\omega} \cos(\omega t)$, $u_2 = \sqrt{\omega} \sin(\omega t)$, $\omega = 8$. The corresponding Lie bracket system is $\dot{z} = -2(z - z^*)$. One easily can solve the Lie bracket differential equation for $z(t)$ and show that $J = -2(z(t) - z^*)$ satisfies Theorem 1, see Figure 1.

Third highlight: In the next section, we will reproduce the results of the multi-agent problems using single-integrator and unicycle dynamics similar to Durr et al. [13], and discuss each problem in detail to show how our proposed ESC system works effectively on multi-agent problems.
the amplitudes of the excitation signals are function of vehicle and initial condition is and additional degree of freedom (x(2)) with that of [16] (persistently oscillating).

moving to the next section, we make an important note that, even though the structure of [13] has been included in the generalization of the work in [16], due to the conditions and the assumptions imposed on the objective function, the problems provided in [13] could not be solved by [16] to achieve vanishing oscillation. The significance of this will be that, the literature of Lie bracket-based ESCs have developed in characteristics that seem not compatible, or to say the least, limiting, to multi-agent ESCs. In order to demonstrate our claim, the objective function for the third vehicle in [13] (provided in detail in the next section) is analyzed within the cycle) stability if solved by the generalized ESC approach (5) below.

III. SIMULATION RESULTS FOR TWO MULTI-AGENT PROBLEMS

To illustrate the main results of this paper via simulation, we will use two problems of three-agent vehicle system similar to [13]. Each agent moves in a two-dimensional plane, and there are two extremum seeking loops representing each dimension (xi and yi, i = 1, 2, 3 is the vehicle index). In addition, a high pass filter (Gi(s) = s/(s + hi)) is introduced to improve transient behavior of the system and it introduces an additional degree of freedom (x(2)). We aim at reproducing the results of [13], for both single-integrator and unicycle dynamics, as well as solving the same problem with our proposed ESC system (4). The following gives apply for both problems. First, second and third agents are assigned the maps f1 = −(x1 − 1)2/2 − (y1 − 1)2/2 + x2 2 + y2 2 + e−(x2 2+y2 2)+10, f2 = (x2 + 1)2/2 − (y2 + 1)2/2 + sin(x1 + y1) − 10, and f3 = −(x3 + 1)2/2 − 3(y3 − 1) − 2/2 + 10, respectively. Moreover, the following parameters are also used in both problems h1 = h2 = h3 = 1, c1 = c2 = c3 = 0.3, and the initial condition is x0 = y0 = [2, −2, 0, −2, 2, 0, −1, 2, 5, 0]T and a0 = [1, 1, 1, 1, 1]T. Finally, the control inputs for each vehicle i are ui1 = sin(ωi t) and u2i = cos(ωi t). Note that the amplitudes of the excitation signals are function of t in the main ESC states (body) but they are constant (equivalent to their initial condition) through the corresponding Lie bracket system.

Single-Integrator Dynamics. Our proposed ESC system is in (5) below, with ω1 = 10, ω2 = 15, ω3 = 20, λx1 = λx2 = λx3 = λy1 = λy2 = λy3 = 0.033.

\[
\begin{align*}
\dot{x}_i &= c_i (f_i(x) - x_i h_i) \sqrt{\omega_i u_{1i}(\omega_i t)} + a_{x1} \sqrt{\omega_i u_{2i}(\omega_i t)}, \\
\dot{y}_i &= -c_i (f_i(x) - x_i h_i) \sqrt{\omega_i u_{2i}(\omega_i t)} + a_{y1} \sqrt{\omega_i u_{1i}(\omega_i t)}, \\
\dot{x}_e_i &= -x_e_i h_i + f_i(x), \\
\dot{z}_x &= \frac{1}{2} (c_i a_0 \nabla z_i f_i(z) - c_i^2 \nabla z_i f_i(z) (f_i(x) - z_i h_i)), \\
\dot{z}_y &= \frac{1}{2} (c_i a_0 \nabla z_i f_i(z) + c_i^2 \nabla z_i f_i(z) (f_i(z) - z_i h_i)), \\
\dot{z}_e &= -z_e_i h_i + f_i(z), \\
\dot{a}_x &= -\lambda_{12} (a_{x1} - J_{x1}), \\
\dot{a}_y &= -\lambda_{12} (a_{y1} - J_{y1}).
\end{align*}
\]

Now, we compare our simulation results with that of reference [13]. Figure 2 can see that the ESC system is approximated by the corresponding Lie bracket system, but clearly oscillations are persistent near/at the extremum point. However, by using our proposed ESC, we are able to attenuate the oscillations to zero by the time the agents arrive at the extremum, as seen in Figure 3. This is further illustrated in Figure 4 by plotting the trajectories of x and y coordinates of vehicle-2 vs. time. Additionally, we include a verification of Theorem 1 by plotting |J22| and |J32| vs. time (for vehicle-2), and with a viable upper bound in the form 1/t^1+ε (we choose ε = 0.05). These plots are found in Figure 5.

Unicycle Dynamics. Our proposed ESC system is given in (6) below. We note here that C1i and S1i refer to cos(Ωi t) and sin(Ωi t), with Ω1 = 1, Ω2 = 2, Ω3 = 3 and ω1 = 50, ω2 = 55, ω3 = 60. The tuning attenuating parameters are λx1 = λx2 = λx3 = λy1 = λy2 = λy3 = 0.033.
\[ \lambda_{x2} = \lambda_{x3} = \lambda_{y1} = \lambda_{y2} = \lambda_{y3} = 0.0166. \]
\[ \dot{x}_i = (c_i f_i(x) - x_{ei} h_i) \sqrt{\omega_i} u_{j1}(\omega_i t) + a_{x1} \sqrt{\omega_i} u_{j2}(\omega_i t) C_i, \]
\[ \dot{y}_i = (c_i f_i(x) - x_{ei} h_i) \sqrt{\omega_i} u_{j1}(\omega_i t) + a_{y1} \sqrt{\omega_i} u_{j2}(\omega_i t) S_i, \]
\[ \dot{x}_{ei} = -x_{ei} h_i^t + f^t(x), \]
\[ \dot{z}_{xi} = \frac{1}{2} (c_i a_{0i} \nabla_{z_{xi}} f_i(z) C_i^2 + c_i a_{0i} \nabla_{z_{yi}} f_i(z) C_i S_i), \]
\[ \dot{z}_{yi} = \frac{1}{2} (c_i a_{0i} \nabla_{z_{xi}} f_i(z) S_i^2 + c_i a_{0i} \nabla_{z_{yi}} f_i(z) S_i), \]
\[ \dot{z}_{ei} = -z_{ei} h_i + f_i(x), \]
\[ \dot{a}_{xi} = -\lambda_{ix}(a_{xi} - J_{xi}), \]
\[ \dot{a}_{yi} = -\lambda_{iy}(a_{yi} - J_{yi}). \]

We can see clearly that, when the unicycle dynamics is solved by the ESC in [13], the same persistent oscillations issue, we commented on earlier with the single integrator problem, still existent as seen in Figure 6. On the other hand, if we resolve the problem using our proposed ESC system, the results are improved and the oscillations vanish at the extremum as can be seen in Figure 7. This simulation comparison can be illustrated further in Figure 8 where vehicle-1 simulation of x and y coordinates is compared between the approach of [13] and our proposed system. Finally, Figure 9 demonstrates the verification of Theorem 1 for vehicle-1.

**Remark 3:** The above two multi-agent problems are based on the commonly used structures for multi-agent systems in ESC literature (e.g. [13]). They suffer from persistent oscillation around the extremum point. By examining the ESC literature in multi-agent systems, we see the persistent oscillation issue is an unresolved one except in the approaches where vanishing control input (asymptotic stability) is provided such as [16]. However, as we discussed earlier, failure to satisfy the condition B4 (demonstrated in the Appendix) shows that the control inputs do not vanish at the extremum point through the recent generalized ESC in [16]. On the other hand, our proposed ESC works effectively as demonstrated in this section of the paper. To the best of our knowledge, we believe the given results are the first in literature that improved significantly the single integrator and unicycle dynamics multi-agent ESCs since [13].

**IV. Conclusion and Future Work**

The proposed ESC system in this paper shows promising results to control-affine structured ESCs, and particularly, multi-agent systems ESC such as single-integrator and unicycle dynamics. Important aspects of the proposed ESC that (1) it has time-dependant stability characteristics that makes it work in cases, other Lie bracket-based ESCs may not, and (2) it does not require the extremum point a priori or strong bounds...
Our proposed ESC

Now, let the evaluation of the integral rearranging, we have

\[ J_i - z_i(0) = \int_0^t J_i d\tau = \int_0^t J_i d\tau + \int_t^{t^*} J_i d\tau. \]  

(7)

Now, let the evaluation of the integral \( \int_0^t J_i d\tau = z_i^* \),

\[ z_i(t) = z_i(0) + z_i^* + \int_t^{t^*} J_i d\tau \]  

but,  

(8)

A. Proof of Theorem 1

Following Remark 1, it is clear that \( J \) is Riemann integrable. By applying \( \int_0^t (\cdot) d\tau \) on both sides of every component of \( J \) in (3), \( J_i \) with \( |J_i| \leq 1/t^p \forall t \in (t^*, \infty) \) with \( p > 1 \), and rearranging, we have

\[ z_i(t) - z_i(0) = \int_0^t J_i d\tau = \int_0^t J_i d\tau + \int_t^{t^*} J_i d\tau. \]  

(7)

Now, let the evaluation of the integral \( \int_0^t J_i d\tau = z_i^* \),

\[ z_i(t) = z_i(0) + z_i^* + \int_t^{t^*} J_i d\tau \]  

but,

(8)

APPENDIX

Now from (11) and (12).

\[ |l_i - m_i| \leq |z_i(t; l_i) - z_i(t; m_i)| + \int_{l_i}^{m_i} |J_i| d\tau \]  

(13)

By letting \( \int_{l_i}^{m_i} |J_i| d\tau = M \), then from (14), we have for every \( \epsilon > 0 \) a \( \delta = f(\epsilon) = \epsilon + M > 0 \) such that

\[ |l_i - m_i| < \delta \quad \text{with} \quad |z_i(t; l_i) - z_i(t; m_i)| < \epsilon \quad \forall \epsilon > 0 \]  

(14)

The (\( \epsilon, \delta \)) stability condition is proved in (14), see section 4.3 in [21]. Condition (14) and (10) finishes the proof of first part of Theorem 1. Now, (3) can be rewritten as: \( \dot{a}_i + \lambda_i a_i = \lambda_i J_i \).

Using the integrating factor \( e^{\lambda_i t} \),

\[ \int_0^t \frac{d}{dt} (a_ie^{\lambda_i t}) d\tau = \int_0^t \lambda_i J_i e^{\lambda_i t} d\tau. \]  

(15)

Applying \( |\cdot| \) and bounds on \( J_i \) on (15) and taking \( \lim_{t \to \infty} \),

\[ 0 \leq \lim_{t \to \infty} |a_i(t)| \leq \lim_{t \to \infty} \left| \int_0^t \lambda_i J_i e^{\lambda_i t} d\tau \right| e^{-\lambda_i t} \]

\[ + \lim_{t \to \infty} \left| \int_0^t \lambda_i J_i e^{\lambda_i t} d\tau \right| / e^{\lambda_i t} + \lim_{t \to \infty} |a_i e^{\lambda_i t} + \lambda_i J_i t e^{\lambda_i t}|. \]

Now, by using Taylor expansion of \( e^{\lambda_i t} \) in \( t \), we get

\[ 0 \leq \lim_{t \to \infty} |a_i(t)| \leq \lim_{t \to \infty} J_i \frac{1+\lambda_i t + (\lambda_i t)^2/2! + \ldots}{1 + \lambda_i t + (\lambda_i t)^2/2! + \ldots}. \]  

(16)
Let the value of integral in (16) evaluated at \( t^* \) be \( c \), then by writing (16) in a polynomial coefficient form,

\[
0 \leq \lim_{t \to \infty} \left| a_i(t) \right| \leq \lambda \lim_{t \to \infty} \frac{c + a_1t^{1-p} + a_2t^{2-p} + \ldots}{c_0 + c_1t + c_2t^2 + \ldots}.
\]

Finally, using Squeeze theorem

\[
\lim_{t \to \infty} \left| a_i(t) \right| = 0.
\]

(17)

This proves the second claim of Theorem 1.

B. Checking Condition B4 for the multi-agent system in (13)

Without lose of generality, we limit our analysis here to the objective function (map) assigned to vehicle-3 using the single-integrator and unicycle dynamics (see section III in this paper or reference [13]). We analyze the said objective function vs. the bounds in condition B4. The expression for the objective function for vehicle-3 is \( f_3 = -(x_3 + 1)^2/2 - 3(y_3 - 1)^2/2 + 10 \). The extremum points are \( x_{3e} = -1 \) and \( y_{3e} = 1 \), which correspond to the optimal value of cost function as \( f_3 = 10 \). Thus, \( f_3 = f_3 - f_3 = -(x_3 + 1)^2/2 - 3(y_3 - 1)^2/2 \).

Now, each element of the vector field \( b \) in (5) are

\[
\begin{align*}
    b_{11}(\tilde{f}_3) &= c_3(f_3(x) - x_{3e}h_3), \\
    b_{21}(\tilde{f}_3) &= a_3, \\
    b_{12}(\tilde{f}_3) &= a_3, \\
    b_{22}(\tilde{f}_3) &= -c_3(f_3(x) - x_{3e}h_3), \\
    b_{13}(\tilde{f}_3) &= 0, \\
    b_{23}(\tilde{f}_3) &= 0.
\end{align*}
\]

As per condition B4, the following inequality should be satisfied for all \( x \in D \)

\[
|b_{ai} (\tilde{f}(x))| \leq M_{f^{m3}} (x),
\]

where \( s \in \{1, 2\} \), and \( i \in \{1, 2, 3\} \). Let us analyze \( b_{12}(\tilde{f}_3) = a_3 \) as an element of the vector field, and the extremum point \( x = -1, y = 1 \) as the point of reference. Since \( M_{f^{m3}} = 0 \) at the reference point for any \( M > 0 \) and \( m_3 \), the inequality becomes \( |a_3| \leq 0 \). However, \( a_3 = constant > 0 \) is the amplitude of the excitation signal and is defined to be a positive constant [13], which establishes a contradiction. Thus, the condition B4 is not satisfied for vehicle-3 of the multi-agent system in (13).

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