Revisit on holographic complexity in two-dimensional gravity

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Abstract: We revisit the late-time growth rate of various holographic complexity conjectures for neutral and charged AdS black holes with single or multiple horizons in two dimensional (2D) gravity like Jackiw-Teitelboim (JT) gravity and JT-like gravity. For complexity-action conjecture, we propose an alternative resolution to the vanishing growth rate at late-time for general 2D neutral black hole with multiple horizons as found in the previous studies for JT gravity. For complexity-volume conjectures, we obtain the generic forms of late-time growth rates in the context of extremal volume and Wheeler-DeWitt volume by appropriately accounting for the black hole thermodynamics in 2D gravity.
1 Introduction

Despite the role as the earliest discovered fundamental interaction, gravity still remains as a myth by its quantum nature. General relativity reveals gravity as the manifestation of background geometry, and Ryu-Takayanagi formula [1] motivates the pursuit of bulk geometry as a dual to some quantum information on the boundary, then a natural question to ask is whether gravity in the bulk could be reconstructed from quantum information on the boundary. This triangular relation between gravity, geometry and information is the main focus recently among the high-energy physics communities.

In the context of an eternal anti-de Sitter (AdS) black hole, quantum information on the boundary could be extracted from the dubbed thermo-field double (TFD) state [2]. However, entanglement entropy alone cannot capture all the quantum information on the boundary [3, 4], nor can bulk geometry be fully reconstructed from the entanglement entropy alone [5], hence there comes the need for complexity, which continues to growth even after reaching thermal equilibrium, similar to the growth of black hole interior. This holographic insight was first formulated in the context of complexity-volume (CV) conjecture [6, 7] (see also [8, 9] for other CV proposals) and later refined in the context of complexity-action (CA) conjecture [10, 11] (see also [12–15] for the clarification of action computation).

Although there have been extensive studies on CV and CA conjectures from the bulk side (see, e.g. charged black holes [16, 17], UV divergences [18–21], subregion complexity [20, 22, 23], time-evolution [24, 25], higher derivative gravities [16, 26, 27], Einstein-Maxwell-dilaton gravities [17, 28–30], Vaidya spacetimes [31, 32], switchback effect and quenches [33, 34], and dS/FLRW boundaries [35, 36]), the difficulty to establish a convincing holographic complexity lies in the lack of unique and well-defined notion for the complexity in the field theory from the boundary side [37–62]. Note that the Lloyd bound [63] on complexity growth rate was initially founded in quantum mechanical system, it makes the two-dimensional (2D) gravity [64] be a special case to understand holographic complexity since the boundary theory could be a (super)conformal quantum mechanics (CQM) [65].

Despite the fact that the AdS$_2$/CFT$_1$ is currently poor understood [66, 67], the renewed interest on 2D gravity follows up the recent understanding [68, 69] of Sachdev-Ye-Kitaev (SYK) model [70, 71], which is conjectured to be dual to quantum gravity in two dimensions [72, 73]. As a toy model of the corresponcences between bulk 2D gravity [74] and boundary quantum mechanics (QM) [75], the 2D AdS Jackiw-Teitelboim(JT) gravity [76, 77] is found in [78] to be dual to a conformally invariant dynamics on the spacetime boundary that could be described in terms of a de Alfaro-Fubini-Furlan model [79] coupled to an external source with conformal dimension two. Later in [80], the asymptotic dynamics of 2D (A)dS JT gravity is further found to be dual to a generalized two-particle Calogero-Sutherland quantum mechanical model [81, 82]. Based on JT model, the Almheiri-Polchinski (AP) model [83] was recently introduced to study the back-reaction to AdS$_2$ since there are no finite energy excitations above the AdS$_2$ vacuum [84, 85]. A distinct feature of AP model is that the boundary time coordinate is lifted as a dynamical
variable and could be described by the 1D Schwarzian derivative action \([73, 86]\), of which
the same pattern of action also appears in the SYK model. This indicates that JT model
might arise as a holographic description of infrared limit of SYK model.

The issue of holographic complexity for 2D JT gravity has been investigated in \([87–90]\).
Naive computations of action growth rate for 2D JT gravity reduced from a near-extremal
and near-horizon limit of Reissner-Nordström (RN) black holes in higher dimensions was
found to be perplexingly vanishing at late-time, of which, however, the late-time linear
growth of the complexity could be restored by appending with an electromagnetic boundary
term \([87]\) in order to ensure the correct sign of the dilaton potential during dimensional
reduction. \([88, 89]\) proposed another restoration of the late-time linear growth of the
complexity by appropriately relating the cut-off behind the horizon with the UV cut-off
at the boundary. Recently, \([90]\) revealed an intriguing fact that the action growth rate for
charged black hole is sensitive to the ratio between the electric and magnetic charges, and
the previously identified vanishing result is due to the zero electric charged in the grand
canonical ensemble, which could be dramatically changed by adding an appropriate surface
term. In this paper, we would like to go beyond the 2D JT gravity.

The organization of the remaining parts of this paper is as follows. The holographic
complexity in terms of \(\text{CA}, \text{CV} 1.0\) as well as \(\text{CV} 2.0\), in 2D neutral and charged black
holes have been investigated in Sec. 2, Sec. 3 and Sec. 4, respectively. Sec. 5 is devoted
to conclusions. The appendix A elaborates the form of counter term and topological term
for the 2D black holes.

## 2 CA

In this section, we evaluate the late-time growth rate of holographic complexity for neutral
and charged eternal AdS\(_2\) black holes using the CA conjecture, which claims that the
complexity of a TFD state living on the boundaries is proportional to the gravitational
action evaluated on the Wheeler-DeWitt (WDW) patch

\[
\mathcal{C}_A = \frac{A_{\text{WDW}}}{\pi \hbar}.
\]

Here the coefficient \(1/\pi \hbar\) is chosen in such a way that the late-time limit \(t \equiv t_L + t_R \to \infty\)
of \(\mathcal{C}_A\) growth rate exactly saturates the Lloyd bound \([63]\)

\[
\frac{d\mathcal{C}_A}{dt} \bigg|_{t \to \infty} = 2M
\]

at least for AdS Schwarzschild black hole in \(D \geq 4\) dimensions in Einstein gravity \([11]\),
beyond which various corrections to \(2M\) at late-time are expected for other neutral AdS
black holes also with a single horizon. However, for charged AdS black holes with both
inner and outer horizons, the holographic complexity given by CA conjecture saturates at
late-time a different form \([16]\) as

\[
\frac{d\mathcal{C}_A}{dt} \bigg|_{t \to \infty} = (M - \mu Q - \Omega J)^{r_+}_{r_-},
\]

- 3 -
where the inner horizon $r_-$ emerges besides the outermost horizon $r_+$ due to the presence of conserved charges $Q$ and $J$ with corresponding chemical potentials $\mu$ and $\Omega$, respectively. The form (2.3) (see also [91]) is quite generic for charged AdS black holes with double horizons even beyond the Einstein gravity. Nevertheless, there also exists other special cases of neutral black holes with multiple horizons and charged black holes with a single horizon (for example, see [17]). All these cases mentioned above will be studied below for 2D AdS black holes beyond simple JT gravity.

2.1 Neutral black holes

We start with the neutral AdS black holes in 2D gravity with Einstein-Hilbert-dilaton action of form

$$A[g, \phi] = \frac{1}{2G} \int_M d^2x \sqrt{-g} \left( \phi R(g) + \frac{V(\phi)}{L^2} \right) + \frac{1}{G} \int_{\partial M} dx \sqrt{-h} \left( \phi K - L_{\text{ct}}^{\text{neu}}[\phi] \right), \quad (2.4)$$

where $L_{\text{ct}}^{\text{neu}}$ is a boundary counter term that renders the Euclidean on-shell action finite. With the ansatz of a linear dilaton of mass scale $\alpha$ and Schwarzschild gauge of the metric

$$\phi = \alpha r, \quad ds^2 = -f(\phi)dt^2 + f(\phi)^{-1}dr^2, \quad (2.5)$$

the corresponding equations-of-motion (EOMs)

$$R = -\frac{V'(\phi)}{L^2}, \quad (2.7)$$

$$0 = -\nabla_{\mu} \nabla_{\nu} \phi + g_{\mu\nu} \left( \nabla^2 \phi - \frac{V(\phi)}{2L^2} \right), \quad (2.8)$$

could be solved as

$$f(\phi) = -\frac{2GM}{\alpha} + j(\phi), \quad (2.9)$$

$$j(\phi) = \frac{1}{\alpha^2 L^2} \int \phi V(\phi') d\phi', \quad (2.10)$$

where $M$ is the ADM mass as shown in Appendix A in order to preserve the thermodynamic relation with the black hole temperature and Wald entropy [92] defined respectively by

$$T = \frac{f'(r_+)}{4\pi} = \frac{V(\phi_+)}{4\pi\alpha L^2}, \quad (2.11)$$

$$S = \frac{2\pi \phi_+}{G}, \quad \phi_+ \equiv \alpha r_+. \quad (2.12)$$

The boundary term is therefore computed by

$$\sqrt{-h}K = \frac{V(\phi)}{2\alpha L^2}, \quad (2.13)$$

$$L_{\text{ct}}^{\text{neu}} = \alpha \cdot \text{sgn}[j(\phi)] \sqrt{|j(\phi)|}, \quad (2.14)$$
where \(\text{sgn}[x]\) is defined by the sign of \(x\). One could refer to (2.14) in Appendix A for more details.

We next turn to a particular form of potential motivated by [93]

\[
V(\phi) = 2\phi + B + \sum_i B_i \phi^{b_i}, \quad b_i \neq 0, 1
\]

(2.15)

which becomes JT gravity when \(B = B_i = 0\). The corresponding on-shell Ricci scalar curvature \(R\) is

\[
R = -\alpha^2 f''(\phi) = -\frac{2}{L^2} + \sum_i b_i B_i \phi^{b_i-1},
\]

(2.16)

where an asymptotically AdS\(_2\) boundary corresponds to \(b_i < 1\), and a curvature singularity would appear at \(r = 0\) when there is a nonzero \(B_i\). Based on above arguments, the black hole solutions generated by (2.15) could be divided into two classes: the one with single horizon and the one with multiple horizons, which will be studied in detail below.

### 2.1.1 Single horizon

If there is a nonzero \(B_i\), then the corresponding black hole exhibits a single horizon, namely, \(f(r)\) has one positive root, of which the Penrose diagram shares the same feature as shown in Fig.1. Without lost of generality, the left and right boundaries could be related to the Schwarzschild time by \(t_L = t, t_R = -t\), and the Eddington-Finkelstein coordinates

\[^1\text{sgn}[\text{positive}] = 1, \text{sgn}[0] = 0, \text{sgn}[\text{negative}] = -1.\]
For all of which the growth rate at late time reads

\[ \lim_{t \to +\infty} \frac{dA_{WDW}}{dt} = 2M + \sum_{i} \left( -\frac{\alpha j(\phi_c)}{G} + \frac{1}{G} L_{ct}^{\text{neu}} |\phi_{c-}| \sqrt{|f(\phi_{c-})|} \right), \]

(2.22)

For a particular choice of dilaton potential (2.15), one has

\[ j(\phi) = \phi^2 + B(\phi + B' \log(\phi)), \]

(2.23)

For all \( b_i \) within \(-1 < b_i < 1\), we have \( j(0) = 0 \), and the second term in the above action growth rate at late-time should be vanished

\[ \lim_{\phi_{c-} \to 0} \left( -\frac{\alpha j(\phi_{c-})}{G} + \frac{1}{G} L_{ct}^{\text{neu}} |\phi_{c-}| \sqrt{|f(\phi_{c-})|} \right) = 0, \]

(2.24)
which leads to the desired Lloyd bound

$$\frac{dA_{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = 2M. \quad (2.25)$$

However, if there is a $b_j \leq -1$, we have $\lim_{\phi \to 0} j(\phi) = -\infty$, and the second term in the above action growth rate at late-time becomes

$$\lim_{\phi \to 0} \left( -\frac{\alpha j(\phi)}{G} + \frac{1}{G} \left. e^[_{\text{neu}}(\phi)\sqrt{|f(\phi)|}] \right|_{\phi \to 0} \right) = -M, \quad (2.26)$$

which leads to the late-time growth rate in this cases as

$$\frac{dA_{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = M. \quad (2.27)$$

In summary, the action growth rate at late-time for neutral AdS$_2$ black holes with a single horizon is

$$\frac{dA_{\text{WDW}}}{dt} \bigg|_{t \to \infty} = \begin{cases} 2M, & f(0) = \frac{2GM}{\alpha}, \\ M, & f(0) = -\infty. \end{cases} \quad (2.28)$$

Note that the form of counter term (2.14) seems unique in order to obtain a finite value for the action growth rate at late-time in both cases of $f(0) = \frac{2GM}{\alpha}$ and $f(0) = -\infty$. The counter term induces a finite shift $-M$ from the case $f(0) = \frac{2GM}{\alpha}$ to the case $f(0) = -\infty$, which is responsible for the difference between the two cases. The physical reason behind this difference merits further study in the future.

### 2.1.2 Multiple horizons

For neutral AdS black hole in 2D gravity with multiple horizons, the Penrose diagram shares the same feature as shown in Fig. 2. Now the total change of action due to the change of WDW patch reads

$$\delta A_{\text{WDW}} = \delta A_{\text{bulk}}^{\text{WDW}} + \delta A_{\text{surf}}^{\text{WDW}} + \delta A_{\text{joint}}^{\text{WDW}}, \quad (2.29)$$

where the null surface terms can be made vanish with the choice of affine parameter for the generator of null surfaces, while the bulk and joint contributions are evaluated as

$$\delta A_{\text{bulk}}^{\text{WDW}} = \frac{1}{2G} \int_{u_0}^{u_0 + \delta t} du \int_{\rho(u_0 - v_0)}^{\rho(u_0 + v_0)} \left( -\frac{\phi V'(\phi)}{L^2} + \frac{V(\phi)}{L^2} \right) dr$$

$$- \frac{1}{2G} \int_{v_0}^{v_0 + \delta t} dv \int_{\rho(u_1 - v_2)}^{\rho(u_1 + v_2)} \left( -\frac{\phi V'(\phi)}{L^2} + \frac{V(\phi)}{L^2} \right) dr$$

$$= \delta t \frac{\alpha j(\phi)}{G} \bigg|_{\phi = \phi_B} - \delta t \frac{\phi V'(\phi)}{2G\alpha L^2} \bigg|_{\phi = \phi_B}, \quad (2.30)$$
Figure 2. Left panel: The universal feature shared by the Penrose diagrams of AdS\(_2\) black holes with multiple horizons (dashed lines), where \(r_\pm\) are the outermost and next-to-outermost horizons, respectively, and \(r_c\) corresponds to the UV-cutoff. Right panel: The change of WDW patches from the one anchored on \(t_L\) and \(t_R\) (pale blue and bright blue regions) to the one anchored on \(t_L + \delta t\) and \(t_R\) (pale blue and dark blue regions).

and

\[
\delta A_{\text{joint}}^{\text{WDW}} = \frac{1}{G} (\phi_B a_B - \phi_B a_B) + \frac{1}{G} (\phi_U a_U - \phi_U a_U) \\
= \delta t \left[ \frac{\alpha f(r) \log |f(r)|}{2} \left| \frac{r_B}{r_{UL}} \right| + \frac{\phi V(\phi)}{2\alpha^2 L^2} \left| \frac{\phi_B}{\phi_U} \right| \right],
\]

respectively. Now the growth rate of total action at late-time for neutral AdS\(_2\) black holes with multiple horizons vanishes similarly as JT gravity [87–90]

\[
\frac{dA_{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = \left( \frac{dA_{\text{WDW}}^{\text{bulk}}}{dt} + \frac{dA_{\text{WDW}}^{\text{joint}}}{dt} \right) \bigg|_{t \to +\infty} = \frac{\alpha j(\phi)}{G} \bigg|_{\phi^+_0} = 0,
\]

which will be remedied below in Sec.2.3 with similar methods as proposed in [87–89] as well as our new treatment from a dual charged black hole.
2.2 Charged black holes

We next move to the charged AdS black holes in 2D gravity with Einstein-Maxwell-dilaton action of form

\[ A = A_{\text{bulk}} + A_{\text{surf}} = \frac{1}{2G} \int_{\mathcal{M}} \left( \phi R + \frac{V(\phi)}{L^2} - \frac{G}{2} W(\phi) F^2 \right) \sqrt{-g} d^2 x + \frac{1}{G} \int_{\partial \mathcal{M}} \left( \phi K - L^{\text{ch}_{\text{ct}}} [\phi] \right) \sqrt{-h} d x . \]

(2.33)

With the ansatz of a linear dilaton and RN gauge

\[ \phi(r, t) = \alpha r, \quad ds^2 = -f(\phi) dt^2 + f(\phi)^{-1} dr^2, \]

(2.34)

the charged black hole solution could be obtained with

\[ A_t = \frac{Q k(\phi)}{\alpha}, \quad F_{tr} = - \frac{Q}{W(\phi)}, \quad F_{\mu \nu} F^{\mu \nu} = - \frac{2 Q^2}{W(\phi)^2}, \]

(2.35)

\[ f(\phi) = - \frac{2GM}{\alpha} + j(\phi) - \frac{GQ^2}{\alpha^2} k(\phi), \]

(2.36)

\[ j(\phi) = \int_{\phi}^{\phi'} \frac{V(\phi')}{\alpha^2 L^2} d\phi', \quad k(\phi) = \int_{\phi}^{\phi'} \frac{1}{W(\phi')} d\phi', \]

(2.37)

where \( Q \) is the electric charge of the system. Similar to the neutral case, the potential is taken as \( V(\phi) = 2\phi + B + \sum_i B_i \phi^b_i \) with \( b_i \neq 0, 1 \). If one further specifies \( W(\phi) = A \phi^c \), then the Ricci scalar

\[ R = - \frac{2}{L^2} - \frac{1}{L^2} \sum_i b_i B_i \phi^{b_i - 1} - \frac{cGQ^2}{A} \phi^{-1-c} \]

(2.38)

exhibits an asymptotically AdS_2 boundary provided that \( 0 \neq b_i < 1 \) and \( c \geq -1 \). A curvature singularity arises at \( \phi = 0 \) when \( \exists B_i \neq 0 \) or \( c > -1 \).

2.2.1 Single horizon

The Penrose diagram for charged AdS_2 black hole with a single horizon is the same as the neutral case in Fig. 1, of which the change in WDW patch leads to similar change in action as

\[ \delta A_{\text{WDW}} = A_{\text{WDW}}(t_L + \delta t, t_R) - A_{\text{WDW}}(t_L, t_R) = A_{\text{dark blue&red}} - A_{\text{bright blue&red}} \]

\[ = \delta A_{\text{bulk}}^{\text{WDW}} + \delta A_{\text{surf}}^{\text{WDW}} + \delta A_{\text{joint}}^{\text{WDW}}, \]

where the contribution from the bulk term is

\[ \delta A^{\text{WDW}}_{\text{bulk}} = \frac{1}{2G} \int_{u_0}^{u_0 + \delta t} du \int_{\rho(u)}^{\rho(u+\delta t)} \frac{L^2}{(\rho - \frac{\alpha \phi}{2})^2} (\phi R + \frac{V(\phi)}{L^2} - \frac{G}{2} W(\phi) F^2) \rho^{\frac{\alpha}{2}} d\rho \]

\[ - \frac{1}{2G} \int_{v_0}^{v_0 + \delta t} dv \int_{\rho(v)}^{\rho(v+\delta v)} (\phi R + \frac{V(\phi)}{L^2} - \frac{G}{2} W(\phi) F^2) \rho^{\frac{\alpha}{2}} d\rho \]

\[ = \delta t \left( \frac{\alpha j(\phi)}{G} - \phi V(\phi) \frac{2G}{2 \alpha L^2} + \frac{Q^2 \phi}{2 \alpha W(\phi)} \right)_{\phi = -} |_{\phi = +}, \]

(2.39)
the contribution from the surface term is
\[
\delta A^{\text{WDW}}_{\text{surf}} = -\frac{1}{G} \int_{t_0}^{t_{\text{f}}} \left( \phi K - \mathcal{L}^{\text{cha}}_{\text{ct}}(\phi) \right) \sqrt{-g_{tt}} \frac{dt}{\phi = \phi_{c_-}} \\
= \delta t \left. \left( -\frac{\phi V(\phi)}{2G\alpha L^2} + \frac{\phi Q^2}{2\alpha W(\phi)} \right) \right|_{\phi = \phi_{c_-}} + \delta t \sqrt{f(\phi)} \left. \mathcal{L}^{\text{cha}}_{\text{ct}}(\phi) \right|_{\phi = \phi_{c_-}},
\]
and the contribution from the joint term is
\[
\delta A^{\text{WDW}}_{\text{joint}} = \frac{1}{G} \left( \phi_{g'} a_{g'} - \phi_{g} a_{g} \right) \\
= \delta t \left. \left( \frac{\phi V(\phi)}{2G\alpha L^2} - \frac{Q^2 \phi}{2\alpha W(\phi)} + \frac{\alpha f(\phi g)}{2G} \log \frac{-f(\phi g)}{\alpha c} \right) \right|_{\phi = \phi_{g}}.
\]
Hence the total variation of \( A^{\text{WDW}} \) is
\[
\delta A^{\text{WDW}} = \delta A^{\text{WDW}}_{\text{bulk}} + \delta A^{\text{WDW}}_{\text{surf}} + \delta A^{\text{WDW}}_{\text{joint}} \\
= \delta t \frac{\alpha j(\phi)}{G} \left. \left|_{\phi = \phi_{c_-}} \right| + \delta t \frac{\alpha j(\phi)}{G} \left. \left|_{\phi = \phi_{c_-}} \right| + \delta t \frac{\alpha f(\phi)}{2G} \log \left| \frac{f(\phi)}{\alpha c} \right| \right|_{\phi = \phi_{g}},
\]
whose growth rate at late-time reads
\[
\frac{dA^{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = 2M - \mu Q + \lim_{\phi_{c_-} \to 0} \left( -\frac{\alpha j(\phi_{c_-})}{G} + \frac{1}{G} \mathcal{L}_{\text{ct}}^{\text{cha}}[\phi_{c_-}] \sqrt{f(\phi_{c_-})} \right),
\]
with
\[
\mathcal{L}_{\text{ct}}^{\text{cha}}[\phi] = \alpha \cdot \text{sgn}[j^{\text{eff}}(\phi)] \sqrt{|j^{\text{eff}}(\phi)|} + \frac{GQ^2}{\alpha} \frac{k(\phi)}{\sqrt{f(\phi)}}
\]
and
\[
j^{\text{eff}}(\phi) \equiv j(\phi) - \frac{GQ^2}{\alpha^2} k(\phi).
\]
where \( \mu \) is the chemical potential of the charged black hole. After evaluated on the particular forms of \( V(\phi) \) and \( W(\phi) \), one has
\[
j(\phi) = \frac{1}{\alpha^2 L^2} \left( \phi^2 + B\phi + B' \log(\phi) + \sum_i \frac{B_i}{b_i + 1} \phi^{b_i+1} \right),
\]
\[
k(\phi) = A^{-1}(1 - c)^{-1} \phi^{1-c},
\]
which leads to
\[
\lim_{\phi_{c_-} \to 0} \left( -\frac{\alpha j(\phi_{c_-})}{G} + \frac{1}{G} \mathcal{L}_{\text{ct}}^{\text{neur}}[\phi_{c_-}] \sqrt{|f(\phi_{c_-})|} \right) = \begin{cases} 0, & j^{\text{eff}}(0) = 0, \\ -M, & j^{\text{eff}}(0) = -\infty. \end{cases}
\]
In summary, the action growth rate at late-time for charged AdS\(_2\) black holes with a single horizon is
\[
\frac{dA^{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = \begin{cases} 2M - \mu Q, & f(0) = \frac{2GM}{\alpha}, \\ M - \mu Q, & f(0) = -\infty. \end{cases}
\]
Here the difference in the action growth rate at late-time between the case \( f(0) = \frac{2GM}{\alpha} \) and the case \( f(0) = -\infty \) is the same as that found at the end of the section 2.1.1.
2.2.2 Multiple horizons

The Penrose diagram for charged AdS$_2$ black hole with multiple horizons shares the same features as the neutral case in Fig. 2, of which the change in WDW patch leads to similar change in action as

$$\delta A_{\text{WDW}} = A_{\text{dark blue&red}}^{\text{WDW}} - A_{\text{bright blue&red}}^{\text{WDW}} = \delta A_{\text{bulk}} + \delta A_{\text{joint}},$$

where the bulk contribution from Einstein-dilaton part is

$$\delta A_{\text{WDW}}^{\text{bulk}_1} = \frac{1}{2G} \int_{\delta v} \left( \phi R + \frac{V}{L^2} \right) \sqrt{-g} d^2 x
= \delta t \left( \frac{\alpha j(\phi)}{G} - \frac{\phi V(\phi) + Q^2 k(\phi)}{2G\alpha L^2} + \frac{Q^2}{2G} \phi W(\phi) \right) \bigg|_{\phi B}^{\phi U},$$

and the bulk contribution from Einstein-Maxwell part is

$$\delta A_{\text{WDW}}^{\text{bulk}_2} = -\frac{1}{4} \int_{\delta v} W(\phi) F^2 \sqrt{-g} d^2 x
= \delta t \left( \frac{Q^2}{2G} k(\phi) \right) \bigg|_{\phi B}^{\phi U}.$$

The contribution from the joint term is

$$\delta A_{\text{WDW}}^{\text{joint}} = \frac{1}{G} \left( \phi B a_B - \phi B a_B \right) + \frac{1}{G} \left( \phi U a_U - \phi U a_U \right)
= \delta t \left( \frac{\phi V(\phi)}{2G\alpha L^2} - \frac{Q^2}{2G} \phi W(\phi) \right) \bigg|_{\phi B}^{\phi U} + \delta t f(r) \log \frac{-f(r)}{c^2} \bigg|_{r_B}^{r_U}.$$

Hence the total variation is

$$\delta A_{\text{WDW}} = \delta A_{\text{WDW}}^{\text{bulk}_1} + \delta A_{\text{WDW}}^{\text{bulk}_2} + \delta A_{\text{WDW}}^{\text{joint}}
= \delta t \left( \frac{\alpha j(\phi)}{G} \right) \bigg|_{\phi B}^{\phi U} + \delta t f(r) \log \frac{-f(r)}{c^2} \bigg|_{r_B}^{r_U},$$

$$\frac{dA_{\text{WDW}}}{dt} \bigg|_{t\to+\infty} = \frac{\alpha j(\phi)}{G} \bigg|_{\phi B}^{\phi U} = \mu_- - \mu_+ Q.$$
is adopted for illustration, where $\phi/\phi_0 \ll 1$ for a positive $\phi_0$. For a specific solution of form

$$\phi = \frac{r}{L},$$  

$$\text{ds}^2 = -f(\phi)d\tau^2 + f(\phi)^{-1}dr^2,$$  

$$f(\phi) = -2GML + \phi^2 - \frac{\phi^3}{\phi_0}$$  

$$= -2 + \phi^2 - \frac{\phi^3}{100},$$  

with $\phi_0 = 100, GML = 1$, $f(\phi)$ exhibits three real roots

$$\phi_1 \approx 99.9800,$$  

$$\phi_+ \approx 1.4244,$$  

$$\phi_- \approx -1.4044.$$  

Since $\phi_1/\phi_0 \approx 1$ and here we have assumed $\phi/\phi_0 \ll 1$, $\phi_1$ could be neglected. The black hole could be regarded as having double horizons, whose Penrose diagram is shown in Fig. 3 and contains the feature shown in Fig. 2. Hence the action growth rate at late-time is also vanishing. To restore the linear growth of holographic complexity, we first follow the two approaches proposed in [87, 89].

### 2.3.1 Electromagnetic boundary term

The JT-like gravity could also be obtained from dimensional reduction of four-dimensional (4D) RN black hole with action of form

$$\tilde{A}_{\text{RN}} = \frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left( \frac{1}{\ell^2} R - F_{\mu\nu}F^{\mu\nu} \right) + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{-h} \frac{1}{\ell^2} \left( K - K_0 \right)$$

$$+ \frac{1}{4\pi} \int_{\partial M} d^3x \sqrt{-h} n_\mu F^{\mu\nu} A_\nu,$$  

where $\ell^2$ is the 4D Newton constant $G_N$. After adopting an ansatz for a spherically symmetric metric

$$\text{ds}^2 = \frac{1}{\sqrt{2\Phi}} g_{ij} dx^i dx^j + 2\ell^2 \Phi d\Omega^2,$$  

the first line in the action (2.63) becomes

$$\tilde{A}_{2d} = \frac{1}{2} \int d^2x \sqrt{-g} \left( \phi R + \frac{1}{\ell^2} (2\Phi)^{-\frac{3}{2}} + \frac{\ell^2}{2} (2\Phi)^{\frac{3}{2}} F_{ij} F^{ij} \right) + \int d\Sigma \sqrt{-h} \left( \Phi K - \frac{1}{\ell^2} (2\Phi)^{\frac{3}{2}} \right).$$  

(2.65)

After further evaluated at an on-shell electromagnetic field strength, the resulting 2D action reads

$$\tilde{A}_{\text{on-shell}}^{2d} = \frac{1}{2} \int_M d^2x \sqrt{-g} \left( \Phi R + \frac{1}{\ell^2} (2\Phi)^{-\frac{3}{2}} - \frac{Q^2}{\ell^2} (2\Phi)^{-\frac{3}{2}} \right) + \int_{\partial M} d\Sigma \sqrt{-h} \left( \Phi K - \frac{1}{\ell^2} (2\Phi)^{\frac{3}{2}} \right),$$  

(2.66)
Figure 3. Left panel: Penrose diagram for the JT-like black hole (2.55) with outer and inner
horizons at $r = r_{\pm}$ (dashed lines), UV cutoff boundaries at $r = r_b$ (blue lines), and conformal
boundaries at $r = \pm \infty$. Right panel: The change of WDW patches from the one anchored on $t_L$ and
$t_R$ (pale blue and bright blue regions) to the one anchored on $t_L + \delta t$ and $t_R$ (pale blue and
dark blue regions).

where the rescaled bulk term $\mathcal{A} \equiv \tilde{\mathcal{A}}/G$ could be rewritten as

$$
\mathcal{A}_{\text{bulk}} = \frac{\phi_0}{2G} \int d^2 x \sqrt{-g} R + \frac{1}{2G} \int d^2 x \sqrt{-g} \left( \phi R + \frac{1}{L^2} V(\phi) \right) \\
= \frac{\phi_0}{2G} \int d^2 x \sqrt{-g} R + \frac{1}{2G} \int d^2 x \sqrt{-g} \left( \phi R + \frac{1}{L^2} \sum_{n=1}^{N} \frac{(-1)^{n+1}}{2^{n-2}(n-1)!} \phi^n \right)
$$

by expanding $\Phi$ around $\phi_0 = Q^2/2$ up to $N$-th order and abbreviating $L \equiv Q^3 \ell$. For
$N = 2$, the resulting bulk action

$$
\mathcal{A}_{\text{bulk}} = \frac{\phi_0}{2G} \int d^2 x \sqrt{-g} R + \frac{1}{2G} \int d^2 x \sqrt{-g} \left( \phi R + \frac{2\phi - 3\phi^2}{L^2} \right)
$$

coincides with the Eq.(2.55) as expected from dimensional reduction of 4D RN black hole.

The second line in the action (2.63) is a 4D electromagnetic boundary term suggested
in [87], which induces a 2D boundary electromagnetic boundary term

$$
A_{\text{em.bdy}}^{2d} = \frac{\ell^2}{G} \int_{\partial \mathcal{M}} dx \sqrt{-h} (2\Phi)^{\frac{3}{2}} n_i F^{ij} A_j = - \frac{Q^2}{GL} \int_{\mathcal{M}} d^2x \sqrt{-g} (2\Phi)^{-\frac{3}{2}}.
$$

(2.70)

This should also contribute to the would-be restored JT-like action by

$$
A_{\text{restored}}^{\text{JT-like}} = A_{\text{WDW}}^{\text{JT-like}} - A_{\text{WDW \ em.bdy}}
$$

when evaluated on the WDW patch. Since the first term $A_{\text{WDW \ JT-like}}$ has vanishing growth rate according to Eq.(2.32), then the growth rate of the restored action at late-time simply reads

$$
\frac{dA_{\text{restored}}^{\text{JT-like}}}{dt} \bigg|_{t \to +\infty} = - \frac{dA_{\text{WDW \ em.bdy}}}{dt} \bigg|_{t \to +\infty}.
$$

(2.71)

After expanded in terms of $\phi \sim \phi_0$ up to the second order, the contribution from the electromagnetic boundary term reads

$$
\frac{dA_{\text{WDW \ em.bdy}}}{dt} \bigg|_{t \to +\infty} = -4 \frac{\phi_0 \phi_+}{GL} + 2 \frac{\phi_+^2}{GL} - \frac{3}{2} \frac{\phi_+^3}{\phi_0 GL} + O\left(\frac{\phi_0^2 \phi_+^2}{\phi_0^2} \right).
$$

(2.72)

Therefore, the restored growth rate of $A_{\text{WDW}}$ at late-time is

$$
\frac{dA_{\text{restored}}^{\text{JT-like}}}{dt} \bigg|_{t \to +\infty} = 4 \frac{\phi_0 \phi_+}{GL} - 2 \frac{\phi_+^2}{GL} + \frac{3}{2} \frac{\phi_+^3}{\phi_0 GL} + O\left(\frac{\phi_0^2 \phi_+^2}{\phi_0^2} \right),
$$

(2.73)

which could be expressed in terms of the thermodynamic quantities

$$
T = \frac{\phi_+}{2\pi L} - \frac{3\phi_+^2}{4\pi L \phi_0}, \quad S = 2\pi \left(\phi_0 + \phi_+\right)/G, \quad M = \frac{\phi_+^2}{2GL} - \frac{\phi_+^3}{2GL \phi_0}
$$

(2.74)

as

$$
\frac{dA_{\text{restored}}^{\text{JT-like}}}{dt} \bigg|_{t \to +\infty} = 4ST + G(10M + 5S_0 T - 5ST) = 4ST + O\left(\frac{\phi_+ \phi_0^2}{\phi_0} \right)
$$

(2.75)

with $S_0 \equiv \frac{2\phi_0}{\phi_0}$. The growth rate (2.75) of the JT-like gravity is identical to the growth rate of the JT gravity given in [87] on the leading-order.

### 2.3.2 UV/IR relation for cutoff surfaces

To recover the linear action growth rate at late-time in JT gravity, [88] proposed an alternative prescription by relating the IR cutoff surface $r = r_\epsilon$ behind the horizon with the UV cutoff surface $r = r_b$ at asymptotic boundary up to the leading-order

$$
r_\epsilon = \frac{r_+^3}{r_b^2},
$$

(2.76)

and an appropriate counter term [89] should also be appreciated on the cutoff surface behind the horizon as shown with green line in Fig. 4. However, there is generally no
universal determination for the extra counter term on the $r = r_\epsilon$. Here we introduce an counter term of form $\mathcal{L}_{\text{JT-like}}^\text{ct}[\phi_0] = \frac{\sqrt{2}\phi_0}{L}$ for the JT-like gravity (2.55), so that the total action

$$A_{\text{cut-off, JT-like}} = A_{\text{JT-like}} - \frac{\phi_0}{G} \int_{\text{bdy}, r=r_\epsilon} dx \sqrt{-h} \frac{\sqrt{2}}{L}$$

(2.77)

exhibits a growth rate

$$\left. \frac{dA_{\text{cut-off, JT-like}}}{dt} \right|_{t \to +\infty} = \frac{\sqrt{2}\phi_0 \phi_+}{GL} + \frac{\sqrt{2}\phi_+^2}{2GL} + \frac{4}{4} \frac{\phi_+^3}{\phi_0 GL} + \mathcal{O}\left(\frac{\phi_+^2}{\phi_0^2}\right)$$

(2.78)

don't exhibit a growth rate. The rest of the text is not shown.
Compared with the growth rate of JT gravity in [89]

$$\frac{dA_{JT}^{WDW}}{dt} \bigg|_{t \rightarrow +\infty} = ST + \mathcal{O}(\phi_+^2),$$  \hspace{1cm} (2.80)

the different coefficients in front of $ST$ come from the different counter term $\mathcal{L}_{ct}(\phi_0)$ used.

### 2.3.3 Charged dual of neutral black hole

Apart from the previous two resolutions, we propose here a third solution by relating the neutral and charged AdS$_2$ black holes. Recall that the charged black hole has an action of form

$$A_{\text{cha}} = \frac{1}{2G} \int d^2x \sqrt{|g|} \left( \phi R + \frac{V(\phi)}{L^2} \right) - \frac{1}{4} \int d^2x \sqrt{|g|} [W(\phi) F^{\mu\nu} F_{\mu\nu} + \text{GHY term}],$$ \hspace{1cm} (2.81)

of which the EOMs and corresponding solution are showed in Eq.(A.14—A.16) and Eq.(2.35—2.37). As pointed out in [94], the metric (2.36) could also be obtained from an uncharged black hole by replacing $V(\phi)$ in (2.9) with an effective potential of the form

$$V_{\text{eff}}(\phi) \equiv V(\phi) - \frac{GQ^2L^2}{W(\phi)},$$ \hspace{1cm} (2.82)

and the charged and uncharged black hole have the same temperature and Wald entropy. In this sense, there is a dual relation between charged black holes and neutral black holes. Given a charged black hole with dilaton potential $V(\phi)$, coupling function $W(\phi)$ and electric charge $Q$, the bulk on-shell action is

$$A_{\text{cha}}^{\text{bulk}} = \frac{1}{2G} \int d^2x \sqrt{|g|} \left( \phi R + \frac{V(\phi)}{L^2} \right) - \frac{1}{4} \int d^2x \sqrt{|g|} [W(\phi) F^{\mu\nu} F_{\mu\nu} + \text{GHY term}],$$ \hspace{1cm} (2.83)

while the on-shell bulk action of dual neutral black hole with effective potential (2.82) is

$$A_{\text{neu}}^{\text{bulk}} = \frac{1}{2G} \int d^2x \sqrt{|g|} \left( \phi R + \frac{V_{\text{eff}}(\phi)}{L^2} \right) - \frac{1}{4} \int d^2x \sqrt{|g|} \left( -\frac{\phi V'(\phi)}{L^2} + \frac{V(\phi)}{L^2} - \frac{GQ^2\phi W'(\phi)}{W(\phi)^2} + \frac{GQ^2}{W(\phi)} \right),$$ \hspace{1cm} (2.84)

with flipped sign for $\frac{GQ^2}{W(\phi)}$. The difference between (2.84) and (2.83) can be rewritten as an additional electric boundary term by using the on-shell electromagnetic field strength (2.35)

$$- \int \frac{Q^2}{W(\phi)} d\phi = \frac{1}{2} \int dx \sqrt{-h} n_\mu W(\phi) F^{\mu\nu} A_\nu.$$ \hspace{1cm} (2.85)

For the neutral black hole with the effective potential (2.82), the action of the corresponding charged black hole is Eq.(2.81) plus Eq.(2.85). In this sense, the neutral black hole with
multiple horizons corresponds to the charged one with varying chemical potential in an ensemble with charged fixed.

To see how the linear growth rate at late-time is restored for neutral AdS$_2$ black hole with multiple horizons, we start with following action

$$ A = \frac{1}{2G} \int d^2x \sqrt{|g|} \left( \Phi R + \frac{(2\Phi)^{-\frac{3}{2}} - G^2 \lambda^2 (2\Phi)^{-\frac{3}{2}}}{\ell^2} \right) + \text{GHY term}, \quad (2.86) $$

where the JT gravity could be induced from keeping the first order expansion of the dilaton around $\Phi = \phi_0 = \left(\frac{G\lambda}{2}\right)^2$, namely,

$$ A \approx \frac{\phi_0}{2G} \int d^2x \sqrt{-g} R + \frac{1}{2G} \int d^2x \sqrt{-g} \phi \left( R + \frac{2}{\ell^2} \right) + \text{High order terms}, \quad (2.87) $$

with $L^2 = (G\lambda)^3 \ell^2$. The dual charged black hole of (2.86) could be identified by defining

$$ V(\Phi) = (2\Phi)^{-\frac{1}{2}}, \quad W(\Phi) = \frac{\ell^2}{G}(2\Phi)^{\frac{3}{2}}, \quad Q = \lambda, \quad (2.88) $$

and the corresponding action reads

$$ A_{\text{dual}} = \frac{1}{2G} \int d^2x \sqrt{|g|} \left( \Phi R + \frac{1}{\ell^2} (2\Phi)^{-\frac{1}{2}} \right) - \frac{1}{4} \int d^2x \sqrt{|g|} \frac{\ell^2}{G}(2\Phi)^{\frac{3}{2}} F_{\mu\nu} F_{\mu\nu} + \text{GHY term} + \frac{1}{2} \int dx \sqrt{-h} \frac{\ell^2}{G}(2\Phi)^{\frac{3}{2}} n_{\mu} F_{\mu\nu} A_{\nu}. \quad (2.89) $$

According to Eq.(2.54), the late-time action growth rate of the dual charged black hole (2.89) without the electric boundary term (2.85) is

$$ \frac{dA_{\text{dual}}}{dt} \bigg|_{t \to +\infty} = \mu - \lambda - \mu + \lambda, \quad \mu_{\pm} = -\frac{\lambda k(\Phi_{\pm})}{\alpha}. \quad (2.90) $$

After expanding $\Phi_{\pm}$ in (2.90) around $\phi_0$, $\Phi_{\pm} = \phi_0 \pm \phi_+$, one can obtain the restored growth rate for JT gravity

$$ \frac{dA_{\text{restored}}}{dt} \bigg|_{t \to +\infty} = 2G\lambda^2 \frac{\phi_+}{\alpha L^2} + \mathcal{O}(\phi_+^2) = 4ST + \mathcal{O}(T^2). \quad (2.91) $$

This is the same as found in [87].

3 CV 1.0

In this section, we investigate in the context of 2D gravity the CV 1.0 conjecture [6, 7], which claims a proportionality between the complexity of the TFD state living on the boundaries and the volume of extremal/maximal time slice anchored at the boundary times $t_L$ and $t_R$ as shown in Fig. 5, namely,

$$ \mathcal{C}_V = \max \left[ \frac{V}{G\ell} \right], \quad (3.1) $$

where the characteristic scale $\ell$ is set by the AdS$_2$ radius $L$ for simplicity. Due to the symmetry of TFD state under boundary time shift $t_L \to t_L + \Delta t$, $t_R \to t_R - \Delta t$, the volume should only depend on the total boundary time $t = t_L + t_R$, where a symmetric setup $t_L = t_R$ could be adopted for convenience. Following the technique from [25], we can similarly compute the growth rate of extremal volume for 2D gravity as shown below.
3.1 Growth rate of extremal volume

Armed with the in-falling Eddington-Finkelstein coordinate $v = t + r^*(r)$, the metric becomes $ds^2 = -f(r)dv^2 + 2dvdr$, and hence the volume of the extremal surface parameterized by $r(\lambda)$ and $v(\lambda)$ could be computed by

$$V = \int d\lambda L(r, \dot{r}, \dot{v}), \quad L(r, \dot{r}, \dot{v}) \equiv \sqrt{-f(r)\dot{v}^2 + 2\dot{v}\dot{r}},$$

(3.2)

where the independence of $L$ on $v$ gives rise to a conserved quantity $E$ defined by

$$E = -\frac{\partial L}{\partial \dot{v}} = \frac{f\dot{v} - \dot{r}}{\sqrt{-f\dot{v}^2 + 2\dot{v}\dot{r}}}. \quad (3.3)$$

On the other hand, the reparametrization invariance of volume with respect to the choice of $\lambda$ implies that $-f\dot{v}^2 + 2\dot{v}\dot{r} = 1$, therefore, the extremal surface could be determined by

$$E = f(r)\dot{v} - \dot{r}, \quad (3.4)$$

$$\dot{r}^2 = f(r) + E^2, \quad (3.5)$$

and the maximal volume becomes

$$V = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\dot{r}} = 2\int_{r_{\text{min}}}^{r_{\text{max}}} \frac{dr}{\sqrt{f(r) + E^2}}, \quad (3.6)$$
where \( r_{\text{min}} \) is determined by the condition \( \dot{r} = 0 \), namely, \( f(r_{\text{min}}) + E^2 = 0 \). To further evaluate the maximal volume, there is a trick by first noting that

\[
t_R + r^*(\infty) - r^*(r_{\text{min}}) = \int_{r_{\text{min}}}^{\infty} dv = \int_{r_{\text{min}}}^{\infty} \frac{dr}{r} = \int_{r_{\text{min}}}^{\infty} \frac{f(r) \dot{r} - \dot{r}}{f(r)r} + \frac{1}{f(r)}
\]

Then it is easy to see that

\[
\frac{\mathcal{V}}{2} + E(t_R + r^*_\infty - r^*_\text{min}) = \int_{r_{\text{min}}}^{\infty} dr \left[ \frac{1}{\sqrt{f(r) + E^2}} + \frac{E}{f(r)\sqrt{f(r) + E^2}} + \frac{1}{f(r)} \right]
\]

which, after taken time derivative with respect to \( t_R \), gives rise to

\[
\frac{1}{2} \frac{d\mathcal{V}}{dt_R} + \frac{dE}{dt_R}(t_R + r^*_\infty - r^*_\text{min}) + E = \int_{r_{\text{min}}}^{\infty} dr \frac{dE}{dt_R} \left[ \frac{E}{f(r)\sqrt{f(r) + E^2}} + \frac{1}{f(r)} \right]
\]

namely,

\[
\frac{1}{2} \frac{d\mathcal{V}}{dt_R} = -E.
\]

When written with \( t = t_L + t_R = 2t_L = 2t_R \), one finally arrives at

\[
\frac{d\mathcal{V}}{dt} = -E = \sqrt{-f(r_{\text{min}})}.
\]

### 3.2 Growth rate at late-time

To evaluate the late-time behavior of growth rate of the extremal volume, one has to specify \( r_{\text{min}} \), which is defined by the larger root of \( f(r_{\text{min}}) + E^2 = 0 \) with two positive roots meeting at the extremum of \( f(r) \) at late-time.

**Neutral cases** For neutral black hole, \( f(r) = -2GM/\alpha + j(\phi) \), hence \( f'(r_{\text{min}}) = 0 \) leads to \( j'(\phi_{\text{min}}) = 0 \), namely, \( V(\phi_{\text{min}}) = 0 \). For one of the particular choice of potential (2.15), \( V(\phi) = 2\phi + B \), this gives rise to \( \phi_{\text{min}} = -B/2 \). After inserting the definition of

\[
j(\phi) = \int d\phi \frac{V(\phi)}{\alpha^2 L^2} = \frac{\phi^2 + B \phi}{\alpha^2 L^2},
\]

the growth rate at late-time becomes

\[
\left. \frac{d\mathcal{V}}{dt} \right|_{\text{t} \to \infty} = \sqrt{\frac{2GM}{\alpha} - j(\phi_{\text{min}})} = \sqrt{\frac{2GM}{\alpha} + \frac{B^2}{4\alpha^2 L^2}} \]

\[
= \sqrt{\frac{\phi_+^2 + B \phi_+}{\alpha^2 L^2} + \frac{B^2}{4\alpha^2 L^2}} = \frac{2\phi_+ + B}{2\alpha L} \equiv 2\pi LT,
\]

where in the second line we have used \( f(\phi_+) = 0 \) and \( T = \frac{2\phi_+ + B}{4\pi \alpha L^2} \). This coincides with the result [87] from JT gravity when setting \( B = 0 \).
Charged cases  For charged black hole, $f'(r_{\text{min}}) = 0$ leads to $j'(\phi_{\text{min}}) = GQ^2k'(\phi_{\text{min}})/\alpha^2$ by noting that

$$f(r) = -\frac{2GM}{\alpha} + j(\phi) - \frac{GQ^2}{\alpha^2}k(\phi), \quad j'(\phi) = \frac{V(\phi)}{\alpha^2 L^2}, \quad k'(\phi) = \frac{1}{W(\phi)},$$

(3.17)

namely, $V(\phi_{\text{min}}) = GQ^2L^2/W(\phi_{\text{min}})$. For our particular choice, $V(\phi) = 2\phi$, $W(\phi) = A$, this gives rise to $\phi_{\text{min}} = GQ^2L^2/(2A)$. Since $j(\phi) = \phi^2/(\alpha^2 L^2)$ and $k(\phi) = \phi/A$, the growth rate at late-time becomes

$$\frac{dV}{dt}\bigg|_{t\to\infty} = \sqrt{\frac{2GM}{\alpha} - j(\phi_{\text{min}}) - \frac{GQ^2}{\alpha^2}k(\phi_{\text{min}})} = \sqrt{\frac{2GM}{\alpha} + \frac{G^2Q^4L^2}{4\alpha^2 A^2}}$$

$$= \sqrt{\frac{\phi_+^2}{\alpha^2 L^2} - \frac{GQ^2\phi_+}{\alpha^2 A} + \frac{G^2Q^4L^2}{4\alpha^2 A^2}} = \frac{\phi_+}{\alpha L} - \frac{GQ^2L}{2\alpha A} = 2\pi LT,$$

(3.18)

where in the second line we have used $f(\phi_+) = 0$ and $T = \frac{1}{4\pi\alpha L^2} \left(2\phi_+ - \frac{GQ^2L^2}{A}\right)$.

4 CV 2.0

We next turn to CV 2.0 conjecture proposed in [8] that the holographic complexity of eternal black hole should be proportional to the spacetime volume of the whole WDW patch

$$C = \frac{PV_{\text{WDW}}}{\hbar},$$

(4.1)

of which the late-time limit certainly approaches the product of thermodynamics pressure $P$ and thermodynamic volume $V_{\text{th}}$

$$\lim_{t\to+\infty} \frac{dC}{dt} = \frac{PV_{\text{th}}}{\hbar},$$

(4.2)

for AdS black hole with a single horizon, or

$$\lim_{t\to+\infty} \frac{dC}{dt} = \frac{P(V_{\text{th}}^+-V_{\text{th}}^-)}{\hbar},$$

(4.3)

for AdS black hole with multiple horizons. Here $V_{\text{th}}^+$ and $V_{\text{th}}^-$ is the thermodynamic volume defined by the outer horizon $r_+$ and inner horizon $r_-$, respectively. Eq.(4.2) and Eq.(4.3) obey the Lloyd bound in many cases as shown in [8][95][96]. In this section, we would like to investigate CV 2.0 for eternal AdS$_2$ black holes.

4.1 Growth rate at late-time

The WDW patch in the 2D AdS black holes with single horizon and multiple horizons as shown in Fig. 1 and the Fig. 2, respectively.
Single-horizon For 2D AdS black hole with a single horizon, the change in spacetime volume of WDW patch could be directly computed by

\[ \delta V_{\text{WDW}} = \delta V_{\text{dark blue}} - \delta V_{\text{bright blue}} \]

\[ = \int_{u_0}^{u_0 + \delta t} d\mu \int_{\epsilon}^{\mu} \rho\left(\frac{u - (\epsilon + \delta t)}{2}\right) d\mu - \int_{v_0}^{v_0 + \delta t} d\nu \int_{\nu}^{\nu + \delta t} \rho\left(\frac{\mu - \nu}{2}\right) d\mu \]

\[ = \delta t(r_B - r_c) + O(\delta t), \quad (4.4) \]

of which the late-time limit under \( t \to +\infty, r_B \to r_h \) and \( r_c \to 0 \) gives rise to a growth rate as

\[ \frac{dV_{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = r_h. \quad (4.5) \]

Multiple horizons For 2D AdS black hole with multiple horizons, the change in spacetime volume of WDW patch could be directly computed by

\[ \delta V_{\text{WDW}} = V_{\text{dark blue}} - V_{\text{bright blue}} \]

\[ = \int_{u_0}^{u_0 + \delta t} d\mu \int_{\epsilon}^{\mu} \rho\left(\frac{u - (\epsilon + \delta t)}{2}\right) d\mu - \int_{v_0}^{v_0 + \delta t} d\nu \int_{\nu}^{\nu + \delta t} \rho\left(\frac{\mu - \nu}{2}\right) d\mu \]

\[ = \delta t(r_B - r_U) + O(\delta t), \quad (4.6) \]

of which the late-time limit under \( t \to +\infty, r_B \to r_+ \) and \( r_U \to r_- \) gives rise to a growth rate as

\[ \frac{dV_{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = r_+ - r_. \quad (4.7) \]

In summary, one has

\[ \frac{dV_{\text{WDW}}}{dt} \bigg|_{t \to +\infty} = \begin{cases} r_h, & \text{single-horizon black holes,} \\ r_+ - r_-, & \text{multiple-horizon black holes.} \end{cases} \quad (4.8) \]

4.2 Thermodynamic volume

To see whether Eq.(4.8) is compatible to Eq.(4.2) and Eq.(4.3), one first turns to the thermodynamic volume of black hole chemistry [97–103], which treats the cosmological constant \( \Lambda \) as the thermodynamic pressure [97–100] so that the thermodynamic volume \( V_{\text{th}} \) plays a role in the extended first law of black hole thermodynamics

\[ dM = TdS + \mu dQ + V_{\text{th}} dP + \ldots. \quad (4.9) \]

Following the works [97, 104], we will identify the late-time growth rate of WDW volume with the thermodynamic volume for JT and JT-like (JT+constant potential) black holes.

Before that, there is a subtlety in the definition of entropy in 2D gravity. As argued in [104] that the Smarr relation for 2D neutral spinless black holes should be modified as

\[ G_2 M = 2TS_2 - 2V_{\text{th}} P_2. \quad (4.10) \]
where the entropy defined by [104]

\[ S_2 = \lim_{D \to 2} S_D = \lim_{D \to 2} \frac{w_D - 2}{4} \left( \frac{r_+}{r_D} \right)^{D-2} = \lim_{D \to 2} \left( \frac{1}{2} + \frac{D - 2}{2} \tilde{S}_{BH} \right) \]  

seems to have \( S_2 = \frac{1}{2} \) for the black holes having two horizon points as its “profile boundary” with area \( A_{\text{bdy}} = 1 \) for each point. However, as argued in [94], it is reasonable to redefine

\[ S_2 = \lim_{D \to 2} \left( \frac{1}{4} + \frac{D - 2}{4} \tilde{S}_{BH} \right) \]  

for black hole with only one horizon as its “profile boundary”.

4.2.1 JT black hole

For locally AdS\(_2\) spacetime, like JT gravity, Eq.(4.10) can be derived from Komar integral relation [105, 106]

\[ \int_{\partial \Sigma} dS_{ab} \left( \nabla^a \xi^b - \Lambda_2 w^{ab} \right) = 0, \]  

where \( \Lambda_2 = 1/L^2 \) is the cosmological constant for 2D spacetime, \( \partial \Sigma \) is the boundary of the spacelike hyper-surface \( \Sigma \), \( dS_{ab} = dS \cdot [\hat{n}_a n_b] \), \( n_b \) and \( \hat{n}_a \) is the unit normal to \( \Sigma \) and \( \partial \Sigma \), respectively, and \( w^{ab} \) is the so called killing potential

\[ \xi^b = \nabla_a w^{ab}. \]  

According to [104][97], Eq.(4.13) is equal to

\[ \frac{\hat{n}_a n_b}{4\pi} \left( \nabla^a \xi^b - \Lambda_2 w^{ab}_{\text{AdS}} \right) \bigg|_{r \to +\infty} = \frac{\hat{n}_a n_b}{4\pi} \nabla^a \xi^b \bigg|_{r = r_+} + \frac{\Lambda_2}{4\pi} \left( \hat{n}_a n_b w^{ab}_{\text{AdS}} \bigg|_{r \to +\infty} - \hat{n}_a n_b w^{ab}_{\text{AdS}} \bigg|_{r \to -\infty} \right), \]  

where \( w^{ab}_{\text{AdS}} \) is the Killing potential for pure AdS\(_2\), and Eq.(4.15) could be regarded as Eq.(4.10) after appreciating Eq.(4.12) and the following definitions

\[ P_2 = \frac{\Lambda_2}{8\pi}, \]  

\[ M = \frac{1}{4\pi} \hat{n}_a n_b \left( \nabla^a \xi^b - \Lambda_2 w^{ab}_{\text{AdS}} \right) \bigg|_{r \to +\infty}, \]  

\[ V_{\text{th}} = - \left( \hat{n}_a n_b w^{ab}_{\text{AdS}} \bigg|_{r \to +\infty} - \hat{n}_a n_b w^{ab}_{\text{AdS}} \bigg|_{r \to -\infty} \right). \]  

Therefore, the volume of JT gravity is given by evaluating Eq.(4.18)

\[ V_{\text{JT}+} = r_+, \]  

then according to[8], the volume defined by \( r_- \) is

\[ V_{\text{JT}+} = r_- \]  

which leads to

\[ \lim_{t \to +\infty} \frac{dC}{dt} = \frac{P(V_{\text{th}} - V_{\text{th}})}{\hbar}. \]  

The late-time growth rate in JT gravity satisfies Eq. (4.3).
4.2.2 JT-like black holes

**Neutral case** For asymptotically AdS neutral black holes with dilaton potential (2.15), the behavior of $R$ is

$$\lim_{\phi \to +\infty} R(\phi) = \lim_{\phi \to +\infty} \frac{V'(\phi)}{L^2} = -\frac{2}{L^2}. \quad (4.21)$$

According to Eq.(4.21), the Komar integral relation Eq.(4.13) should be modified as follows

$$\int_{\partial \Sigma} dS r_n b_a \nabla^a \xi^b = \frac{\Lambda}{2} \int_{\partial \Sigma} dS n_b V'(\phi) w^{ab} - \frac{\Lambda}{2} \int_{\Sigma} dV n_b (\nabla_a V'(\phi)) w^{ab}. \quad (4.22)$$

then Eq.(4.22) could be rewritten as

$$\frac{n_a n_b}{4\pi} \left( \nabla^a \xi^b - \Lambda_2 w^{ab}_{AdS} \right) \bigg|_{r \to +\infty} = \frac{n_a n_b}{4\pi} \nabla^a \xi^b \bigg|_{r \to r_+} + \frac{\Lambda}{4\pi} \frac{1}{2} \left[ n_a n_b V'(\phi) w^{ab} \right]_{r \to r_+}^{|+\infty} - n_a n_b w^{ab}_{AdS} \bigg|_{r \to +\infty} - \frac{1}{2} \int_{r_+}^{+\infty} d\nu n_b \partial_a V'(\phi) w^{ab} \bigg). \quad (4.23)$$

With Eq.(4.23), the thermal volume (4.18) here is

$$V_{th} = -\left( \frac{1}{2} \frac{n_a n_b V'(\phi) w^{ab}}{4\pi} \right)_{r \to r_+}^{+\infty} - \frac{n_a n_b w^{ab}_{AdS}}{4\pi} \bigg|_{r \to +\infty} - \frac{1}{2} \int_{r_+}^{+\infty} d\nu n_b \partial_a V'(\phi) w^{ab} \bigg). \quad (4.24)$$

Setting $\xi^a = (\frac{\partial}{\partial t})^a$ and $w^{ab}_{AdS} = r$, Eq.(4.24) becomes

$$V_{th}^+ = \frac{V(\phi_+) - B}{2\alpha}. \quad (4.25)$$

As a consistent check, the thermal volume (4.25) of JT-like could reduce to Eq.(4.19) by setting $V(\phi) = 2\phi$. One can see that the late-time growth rate in JT+constant potential gravity satisfies Eq. (4.3).

**Charged case** For the charged case, we define the effective dilaton potential (2.82) introduced in Sect.2.3.3 as an “effective volume”. The potential (4.25) then becomes the effective volume

$$V_{th}^{eff} = \frac{V_{th}^{eff}(\phi_+) - B}{2\alpha}. \quad (4.26)$$

which contains the contribution of $\mu Q$ as seen from

$$P_2 \cdot V_{th}^{eff} = \frac{\Lambda}{8\pi} \left( \frac{V_{th}^{eff}(\phi_+) - B}{2\alpha} \right) = \frac{\Lambda}{8\pi} \left( \frac{V(\phi_+) - B - \frac{G Q^2 L^2}{W(\phi_+)}}{2\alpha} \right) = P_2 \cdot V_{th}^+ - \frac{G Q^2}{16\pi \alpha W(\phi_+)} \cdot (4.27)$$

Here the second term of Eq.(4.27) is proportional to $\mu Q$ term with $\mu$ defined by the coefficient of the last term in (4.27), which is an extended version of [104].
5 Conclusions

In this paper, the complexity growth has been investigated in terms of various holographic complexity conjectures in generic 2D eternal AdS black holes, which are proposed to be dual to the thermal field double states. For CA conjecture in the context of 2D neutral black holes with double event horizons in JT-like gravity when obtained by dimensional reduction from 4D AdS-RN black hole, it should contain an extra contribution from an electromagnetic boundary term that reproduces the the linear late-time growth rate instead of the vanishing result without it. This salvation is similar to the CA case in JT gravity as first found in [87]. A second proposal involving with a relation for the UV/IR cutoff surfaces is also checked to reproduce the non-vanishing growth rate at late-time. In addition, we propose a third resolution by explicitly working out the charged dual of a neutral black hole and recurring the vanishing growth, which is consistent with the approach proposed in [87] for JT gravity.

In the second part of this paper, the late-time growth rates of holographic complexity in terms of CV 1.0 and CV 2.0 are also studied. For CV 1.0, the obtained form of late-time growth rate is universal regardless of neutral or charged black holes with single horizon or multiple horizons. However, it is generally difficult to rewrite it with consistent thermodynamic quantities. For CV 2.0 proposed in [8] that the late-time growth rate of complexity in CV2.0 is equal to the form of $P V_{th}$ with pressure $P$ associated with the cosmology constant and $V_{th}$ regarded as the thermal volume, the late-time growth rate for JT-like gravity with constant scalar potential satisfies the form of

$$P \left( \frac{V_{th}^+ - V_{th}^-}{\hbar} \right)$$

after properly accounting for the thermodynamic first law and Smarr relation.

To close this section, we summarize the main results in the present paper as follows.

| Conjecture | CA | CV 1.0 | CV 2.0 |
|------------|----|--------|--------|
| Neutral Single horizon Neutral | $2M/\pi \hbar$ | $M/\pi \hbar$ | $\frac{r_h^*}{\pi \hbar L^2}$ |
| U(1)Charged | $(2M - \mu Q)/\pi \hbar$ | $(M - \mu Q)/\pi \hbar$ | $\sqrt{-f(\phi_{min})/GL}$ |
| Neutral Multiple horizons Dual to charged case$^1$ | $(\mu_+ Q - \mu_- Q)/\pi \hbar$ | |
| U(1)Charged | | | $\frac{r_+ - r_-}{\pi \hbar L^2}$ |

where

a: $f(\phi)$ is finite at the singularity,

b: $f(\phi)$ is divergent at the singularity,
†: See Subsect. 2.3.3 for more detail,

*: See Subsect. 4.2 for more details. In particular, the late-time growth rate in JT and JT-like gravity with constant scalar potential satisfies the form of \( \frac{P(v_k^+-v_k^-)}{h}\).

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Appendices

A On-shell action

In this appendix, we fix the counter term and topological term used for 2D gravity.

A.1 counter term

A.1.1 Neutral black holes

We start with the Euclidean action of neutral black holes

\[
\mathcal{A}_E = -\frac{1}{2G} \int_M d^2x \sqrt{g} \left( \phi R + \frac{V(\phi)}{L^2} \right) - \frac{1}{G} \int_{\partial M} d\sqrt{h} K + \frac{1}{G} \int_{\partial M} d\sqrt{h}\mathcal{L}_{\text{ct}}^{\text{neu}}(\phi),
\]

where \( M \) is a 2D spacetime region outside of the black hole with boundary \( \partial M \). The corresponding EOMs are

\[
R = -\frac{V'(\phi)}{L^2},
\]

\[
0 = -\nabla_\mu \nabla^\mu \phi + g_{\mu\nu} \left( \nabla^2 \phi - \frac{V(\phi)}{2L^2} \right),
\]

whose solution under linear dilaton and Schwarzschild gauge is Eq.(2.9). Consider that

\[
\lim_{\phi \to +\infty} j(\phi) = +\infty,
\]

the boundary counter term \( \mathcal{L}_{\text{ct}}^{\text{neu}}(\phi) \) should be of form

\[
\mathcal{L}_{\text{ct}}^{\text{neu}}(\phi) = \frac{1}{L} \left( \int V(\phi) d\phi = \alpha \sqrt{j(\phi)}, \right)
\]

(A.1)

(A.2)

(A.3)

(A.4)
as we will see below with correct form of free energy corresponding to the on-shell action consisting of following terms. The bulk part of the on-shell action reads

$$A_{\text{on-shell}}^{\text{bulk}} = \frac{1}{2G} \int_\mathcal{M} \, d^2x \sqrt{g} \left( \phi R + \frac{V(\phi)}{L^2} \right) \bigg|_{\phi_{\text{max}}} = -\frac{\beta \alpha j(\phi)}{G} \bigg|_{\phi_{\text{max}}} + \frac{\beta V(\phi)}{2\alpha GL^2} \bigg|_{\phi_{\text{max}}} .$$

(A.5)

The contribution from GHY term reads

$$A_{\text{on-shell}}^{\text{GHY}} = -\frac{1}{G} \int_{\partial \mathcal{M}} \, dx \sqrt{h} \phi K \bigg|_{\phi_{\text{max}}} = -\frac{\beta V(\phi)}{2\alpha GL^2} \bigg|_{\phi_{\text{max}}} .$$

(A.6)

where the extrinsic curvature $K$ on $\partial \mathcal{M}$ is

$$K = \frac{V(\phi)}{2\alpha L^2 \sqrt{f(r)}} .$$

(A.7)

with setting the boundary $\partial \mathcal{M}$ to $r = r_{\text{max}}$. The contribution from boundary counter term reads

$$A_{\text{on-shell}}^{\text{ct}} = \frac{1}{G} \int_{\partial \mathcal{M}} \, dx \sqrt{h} \mathcal{L}^{\text{ct}}_{\text{neu}} (\phi) \bigg|_{\phi_{\text{max}}} = \frac{\beta \alpha j(\phi)}{G} \left[ \sqrt{1 - C_1 j(\phi)} \bigg|_{\phi_{\text{max}}} \right] \approx -\frac{\alpha \beta C_1}{2G} + \frac{\beta \alpha j(\phi)}{G} \bigg|_{\phi_{\text{max}}} \quad (\text{for } \phi_{\text{max}} \to \infty).$$

(A.8)

After summing over above contributions, the total on-shell action is

$$A_{\text{on-shell}}^E = -\frac{\beta \phi_+ V(\phi_+)}{2\alpha GL^2} + \frac{\alpha \beta j(\phi_+)}{2G} .$$

(A.9)

and the corresponding free energy reads

$$\mathcal{F} = \frac{A_{\text{on-shell}}^E}{\beta} = -\frac{\phi_+ V(\phi_+)}{2\alpha GL^2} + \frac{\alpha j(\phi_+)}{2G} .$$

(A.10)

In terms of the black hole temperature and entropy (2.11), Eq.(A.10) could be rewritten as

$$\mathcal{F} = -TS + \frac{\alpha C_1}{2G} .$$

(A.11)

where $\frac{\alpha C_1}{2G}$ is the ADM energy $M$ in terms of the Hamiltonian analysis of the generic 2D theory as referred in [107]. Hence $f(r)$ could be expressed as

$$f(r) = -\frac{2GM}{\alpha} + \frac{1}{\alpha^2 L^2} \int_{\phi_h}^{\phi_{\text{max}}} V(\phi) d\phi .$$

(A.12)

Note that Eq.(2.14) is consistent with Eq.(A.4) since $j(\phi) > 0$ when $\phi > \phi_h$. A more rigorous argument has been given in Ref.[94] for the counter term taking the form of (A.4).
A.1.2 Charged black holes

We next turn to the Euclidean action of charged black holes

\[ A_E = -\frac{1}{2G} \int_M d^2 x \sqrt{g} \left( \phi R + \frac{V(\phi)}{L^2} - \frac{G}{2} W(\phi) F^2 \right) - \frac{1}{G} \int_{\partial M} dx \sqrt{h} \phi K \]
\[ + \frac{1}{G} \int_{\partial M} dx \sqrt{h} L_{ct}^{\text{cha}}(\phi), \quad (A.13) \]

which yields the following EOMs

\[ R + \frac{1}{L^2} V'(\phi) - \frac{G}{2} W'(\phi) F^2 = 0, \quad (A.14) \]
\[ \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \left( \nabla^2 \phi - \frac{V'(\phi)}{2L^2} \right) = -\frac{GW(\phi)}{4} g_{\mu\nu} F^2 + GW(\phi) F_{\mu\rho} F^\rho = 0, \quad (A.15) \]
\[ \nabla_\mu (W(\phi) F^{\mu\nu}) = 0, \quad (A.16) \]

with solution given by Eq. (2.35)–(2.37). Consider that

\[ \lim_{\phi \to +\infty} \left( j(\phi) - \frac{GQ^2}{\alpha^2} k(\phi) \right) = +\infty, \]

then the boundary counter term \( L_{ct}^{\text{cha}}(\phi) \) should be of form

\[ L_{ct}^{\text{cha}}(\phi) = \alpha \sqrt{j(\phi) - \frac{GQ^2}{\alpha^2} k(\phi)} + \frac{GQ^2}{\alpha} \frac{k(\phi)}{\sqrt{|f(\phi)|}} \quad (A.17) \]

as we will see below with correct form of free energy corresponding to the on-shell action by substituting \( R \) and \( F^2 \) in Eq. (A.13) with Eq. (A.14) and Eq. (2.35) respectively. The bulk on-shell action reads

\[ A_{\text{bulk}} = -\frac{1}{2G} \int_M d^2 x \sqrt{g} \left( R + \frac{V(\phi)}{L^2} - \frac{G}{2} W(\phi) F^2 \right) \]
\[ = \beta \phi V(\phi) \bigg|_{\phi = \phi_{\text{max}}} - \frac{\beta \alpha j(\phi)}{G} \bigg|_{\phi = \phi_{\text{max}}} - \frac{\beta Q^2 \phi}{2a W(\phi)} \bigg|_{\phi = \phi_{\text{max}}}. \quad (A.18) \]

The contribution from GHY term is

\[ A_E^{\text{GHY}} = -\frac{1}{G} \int_{\partial M} dx \sqrt{h} \phi K \]
\[ = -\frac{\beta \phi V(\phi)}{2G a L^2} \bigg|_{\phi = \phi_{\text{max}}} + \frac{\beta Q^2 \phi}{2a W(\phi)} \bigg|_{\phi = \phi_{\text{max}}}. \quad (A.19) \]

where the scalar extrinsic curvature \( K \) on \( \partial M \) is

\[ K = \frac{V(\phi)}{L^2} - \frac{GQ^2}{W(\phi)} \frac{\phi_{\text{max}}}{\sqrt{f(r)}}. \quad (A.20) \]
The contribution from counter term is
\[
A_{ct}^E = \frac{1}{G} \int_{\partial M} dx \sqrt{h} \mathcal{L}_{ct}^{\text{cha}}(\phi)
= -\beta M + \frac{\beta \alpha j(\phi)}{G} \bigg|_{\phi_{\text{max}}}. \tag{A.21}
\]

After summing over above contributions, then total on-shell action is
\[
A_{\text{on-shell}}^E = A_{\text{bulk}}^E + A_{\text{GHY}}^E + A_{ct}^E
= -\frac{\beta}{2G\alpha L^2} \phi V \bigg|_{\phi_+} + \frac{\beta Q^2 \phi}{2\alpha W} \bigg|_{\phi_+} + \frac{\beta \alpha j(\phi)}{G} \bigg|_{\phi_+} - \beta M, \tag{A.22}
\]
and the corresponding free energy \(\mathcal{F}\) reads
\[
\mathcal{F} = \frac{A_{\text{on-shell}}}{\beta} = -\frac{1}{2G\alpha L^2} \phi_+ V(\phi_+) + \frac{Q^2 \phi_+}{2\alpha W(\phi_+)} + \frac{Q^2}{\alpha} k(\phi_+) + M. \tag{A.23}
\]

Since the temperature and the entropy of the black hole are defined by
\[
T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi \alpha L^2} \left( V(\phi_+) - \frac{GL^2 Q^2}{W(\phi_+)} \right), \tag{A.24}
\]
\[
S = -\partial_T \mathcal{F} = \frac{2\pi \phi_+}{G}, \tag{A.25}
\]
the \(\mathcal{F}\) could be rewritten as
\[
\mathcal{F} = -TS + M - \mu Q, \tag{A.26}
\]
where
\[
\mu = -\frac{Qk(\phi_+)}{\alpha}, \tag{A.27}
\]
is the chemical potential.

### A.2 Topological term

When deriving the JT(-like) gravity action from the 4D RN black hole action, there is an extra topological term
\[
A_{\text{top}} = \frac{\phi_0}{2G} \int_M d^2 x \sqrt{-g} R + \frac{\phi_0}{G} \int_{\partial M} dx \sqrt{-h} K, \tag{A.28}
\]
where \(q\) is the charge of the corresponding RN black hole. For neutral black holes, the on-shell action contribution from the topological term is
\[
A_{\text{top}}^{\text{neu}} = -\frac{\phi_0}{2G} \int_M d^2 x \sqrt{g} R - \frac{\phi_0}{G} \int_{\partial M} dx \sqrt{-h} K
= -\frac{\beta \phi_0 V(\phi_h)}{2G\alpha L^2}. \tag{A.30}
\]
while for charged cases, the topological term is

$$A_{\text{top}}^{\text{cha}} = \frac{\phi_0}{2G} \int_M d^2x \sqrt{g} R - \frac{\phi_0}{G} \int_{\partial M} dx \sqrt{h} K + \beta \phi_0 V(\phi_h) - \frac{\beta}{2G} \phi_0 Q^2.$$  \hspace{1cm} (A.31)

We will see below the above definitions reproducing the correct form of free energy.

### A.2.1 Neutral black holes

For neutral black holes with the on-shell Euclidean action of form

$$A_{\text{total}} = -\frac{\phi_0}{2G} \int_M d^2x \sqrt{g} R - \frac{\phi_0}{G} \int_{\partial M} dx \sqrt{h} K - \frac{1}{2G} \int_M d^2x \sqrt{g} \left( \phi R + \frac{V(\phi)}{L^2} \right) - \frac{1}{G} \int_{\partial M} dx \sqrt{h} \left( \phi K - L_{\text{ct}}^{\text{neu}}(\phi) \right),$$  \hspace{1cm} (A.32)

the on-shell action $A_{\text{total}}$ from Eq.(A.30) and Eq.(A.9) reads

$$A_{\text{on-shell}}^{\text{total}} = \frac{\beta \phi_0 V(\phi_h)}{2GoL^2} - \frac{\beta}{2G} \phi_0 V(\phi_h),$$  \hspace{1cm} (A.33)

then the total free energy $F_{\text{total}}$ is

$$F_{\text{total}} = \frac{A_{\text{on-shell}}^{\text{total}}}{\beta} = -\frac{\phi_0 V(\phi_h)}{2GoL^2} + \frac{\beta}{2G} \phi_0 V(\phi_h) = -ST + M.$$  \hspace{1cm} (A.34)

where the entropy $S$ is

$$S = \frac{2\pi (\phi_h + \phi_0)}{G}.$$  \hspace{1cm} (A.35)

### A.2.2 Charged black holes

For charged black holes with the on-shell Euclidean action of form

$$A_{\text{total}} = -\frac{\phi_0}{2G} \int_M d^2x \sqrt{g} R - \frac{\phi_0}{G} \int_{\partial M} dx \sqrt{h} K - \frac{1}{2G} \int_M d^2x \sqrt{g} \left( \phi R + \frac{V(\phi)}{L^2} \right) + \frac{1}{4} \int_M d^2x \sqrt{g} W(\phi) F^2 - \frac{1}{G} \int_{\partial M} dx \sqrt{h} \left( \phi K - L_{\text{ct}}^{\text{cha}}(\phi) \right),$$  \hspace{1cm} (A.36)

the on-shell action from Eq.(A.22) and Eq.(A.31) reads

$$A_{\text{on-shell}}^{\text{total}} = -\frac{\beta}{2GoL^2} \phi + \phi_0) V \mid_{\phi_h} + \frac{\beta Q^2 (\phi + \phi_0)}{2aW} \mid_{\phi_h} + \frac{\beta \alpha}{G} j(\phi_h) + \frac{\alpha \beta C_1}{2G},$$  \hspace{1cm} (A.37)

then the free energy $F_{\text{total}}$ is

$$F_{\text{total}} = \frac{A_{\text{on-shell}}^{\text{total}}}{\beta} = -ST + M - \mu Q,$$  \hspace{1cm} (A.38)

where the entropy $S$ is

$$S = \frac{2\pi (\phi_h + \phi_0)}{G}.$$  \hspace{1cm} (A.39)
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