ON BACKGROUND DRIVING DISTRIBUTION FUNCTIONS (BDDF) FOR SOME SELFDECOMPOSABLE VARIABLES.

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Abstract. Many classical variables (statistics) are selfdecomposable. They admit the random integral representations via Lévy processes. In this note are given formulas for their background driving distribution functions (BDDF). This may be used for a simulation of those variables. Among the examples discussed are: gamma variables, hyperbolic characteristic functions, Student t-distributions, stochastic area under planar Brownian motions, inverse Gaussian variable, logistic distributions, non-central chi-square, Bessel densities and Fisher z-distributions. Found representations might be of use in statistical applications.

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In the probability or statistic course Gaussian distribution and the central limit theorem (CLT) occupy the main stage. Often as an extension stable distributions are introduced and then the class of infinitely divisible distributions (ID). But there is a subclass of selfdecomposable distributions also known as the class \( L \). Surprisingly many distributions used in statistics and other areas of applications are in \( L \). Class \( L \) distributions have a random integral representation which maybe useful in simulations. For many classical distributions in statistics we find explicite formulas for their background driving distribution (BDDF). To have a better insight in those distributions (how they are concentrated) there are also computed some their numerical values using the platform WolframAlpha.com. In Jurek and Kepczynski (2021) will be given the graphs for the BDDF computed in Section B) below.

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A). Selfdecomposability: basic formulas and relationships.

If a sequence $X_k, k = 1, 2, \ldots$ of random variables is strongly mixing, in particular, if it is stochastically independent, $a_k, k = 1, 2, \ldots$ is a sequence of positive numbers such that the triangular array $a_nX_k; 1 \leq k \leq n, n \geq 1$ is infinitesimal and $b_k, k = 1, 2, \ldots$ is a sequence of real numbers such that

$$a_n(X_1 + X_2 + \ldots + X_n) + b_n \Rightarrow X, \text{ in the weak topology, as } n \to \infty,$$

then $X$ is called selfdecomposable or Lévy class $L$ variable.

Those variables are characterized as follows:

$$X \in L \iff \forall (0 < c < 1) \exists (X_c \text{ independent of } X) \ X \overset{d}{=} cX + X_c. \quad (2)$$

See Bradley and Jurek (2014) for the case of strong mixing sequences. For the independent sequences and some historical origins of the problem cf. Gnedenko and Kolmogorov (1954), Section 29-30, or Loève (1963), Section 23 or Feller (1966), Chapter XVII.8. (The above property is used to justify the terminology selfdecomposability, selfdecomposable.)

Furthermore, from Jurek and Vervaat (1983) we have the following random integral representation for selfdecomposable variables:

$$X \text{ is a selfdecomposable iff there exists unique (in distribution) Lévy process } (Y_X(t), t \geq 0) \text{ such that }$$

$$X = \int_0^\infty e^{-t}dY_X(t), \text{ and } E[\log(1 + |Y_X(1)|)] < \infty, \quad (\ast)$$

If the identity ($\ast$) holds then to the variable $Y_X(1)$ we refer to as the BDRV (background driving random variable) of $X$. So, $Y_X(1)$ is infinitely divisible with finite log-moment; in short, $Y_X(1) \in ID_{log}$.

Similarly, if $\phi_X$ and $\psi_X$ are the characteristic functions of $X$ and $Y_X(1)$, respectively, then for $t \neq 0$

$$\psi_X(t) \equiv \psi_{YX(1)}(t) = \exp[t(\log \phi_X(t))] = \exp[t\frac{\phi_X'(t)}{\phi_X(t)}] \in ID_{log}, \quad (\ast \ast).$$

Therefore, if the identity ($\ast \ast$) holds then to the function $\psi_X$ we will refer as to the BDCF (the background driving characteristic function) of $X$; cf. Jurek (2001), Proposition 3.

We say that $X \in L$ has the factorization property if $X$ convoluted with its BDCF is in $L$ again, that is,

$$L^f := \{X \in L : \phi_X(t) \cdot \psi_X(t) \in L \} \subset L \subset ID; \quad (3)$$
Finally, if $G_X(a) := P(Y_X(1) \leq a)$ is the BDDF (background driving distribution function) of $X$ then it is given by the following formula:

$$G_X(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im \left( \exp \left[ -ia + t \frac{\phi'_X(t)}{\phi_X(t)} \right] \right) \frac{dt}{t}, \quad a \in C_{G_X}, \quad (\ast \ast \ast)$$

where $C_{G_X}$ is the set of continuity points the probability distribution function $G_X \in ID_{\log}$ (infinitely divisible with finite log-moment); cf. Jurek (2019), Proposition 1 and references therein.

If the characteristic function $\phi_X(t)$ is a real one then

$$G_X(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp \left[ t \frac{\phi'_X(t)}{\phi_X(t)} \right] \frac{\sin(ta)}{t} dt, \quad a \in C_{G_X}, \quad (\ast \ast \ast \ast)$$

is the BDDF of $X$ and $G_X(-a) = 1 - G_X(a), a \in C_{G_X}$.

Symbolically we may summarize all the above as:

$$X \in L \iff X = \int_0^\infty e^{-s} dY_X(s), \quad P(Y_X(1) \leq a) = G_X(a) \in ID_{\log}, \quad (4)$$

where $a \in C_{G_X}$ and $(Y_X(s), s \geq 0)$, is the unique (in distribution) Lévy process for $X$.

Finally, the class $L$ forms a closed (in the weak topology) convolutions semigroup of $ID$ (all infinitely divisible variables). If $X \in L$ then $aX + b \in L$, for all reals $a$ and $b$. All $X \in L$ are absolutely continuous with respect to Lebesque measure, i.e., $X \in L$ have probability densities. All non-generate class $L \subset ID$ distributions have infinite Lévy (spectral) measures in their Lévy-Kintchine representations.

The following selfdecomposable distribution are discussed below:

1. gamma; 2. chi-square; 3. log-gamma; 4. inverse gamma;
5. hyperbolic-cosine; 6. hyperbolic-sine; 7. hyperbolic-tangent;
8. via zeros of Bessel function;
9. t-Student;
10. stochastic area under planar Brownian motion;
11. exponential integral and inverse Gaussian;
12. logistic distribution;
13. non-central chi-square and Bessel $h_\nu$ distribution;
14. Fisher $z$-distribution.

Warning. All numerical values below computed via WolframAlpha are approximate not exact one, although the equality sign is used.
B). Examples of BDDF for some selfdecomposable distributions.

1). \( X := \gamma_{\alpha, \lambda} \) - the gamma random variable.

Let us recall that \( f_{\gamma_{\alpha, \lambda}}(x) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \ x \in (0, \infty), \) is the probability density function (\( \alpha > 0 \) is the shape parameter and \( \lambda > 0 \) is the rate parameter) and \( \phi_{\gamma_{\alpha, \lambda}}(t) = (1 - it/\lambda)^{-\alpha} \) is its characteristic function which is selfdecomposable by Jurek (1997), p. 97. From (\( \star \star \)) we get the BDCF \( \psi_{\gamma_{\alpha, \lambda}}(t) = \exp \alpha \left[ \frac{1}{1-it/\lambda} - 1 \right] \) (compound Poisson distribution). Finally we have (three formulae for) the BDDF:

\[
G_{\gamma_{\alpha, \lambda}}(a) = P \left( \sum_{k=1}^{N_\alpha} E_k(\lambda) \leq a \right) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp\left(-\frac{\alpha (t/\lambda)^2}{1 + (t/\lambda)^2}\right) \sin\left( ta - \frac{\alpha (t/\lambda)}{1 + (t/\lambda)^2} \right) \frac{1}{t} dt = e^{-\alpha} + e^{-\alpha} \int_0^{2\sqrt{\alpha \lambda a}} I_1(w) e^{-w^2/4\alpha} dw, \quad a \in C_{G_X} \cap (0, \infty);
\]

where \( N_\alpha \) is Poisson variable with the parameter \( \alpha \) independent of i.i.d. exponential \( E_k \) and finally \( I_1(x) \) is the (modified) Bessel function; cf. Jurek (2019), Lemma 1 and the identity (\( \star \star \star \)). Thus from (4) we may write that

\[
\gamma_{\alpha, \lambda} = \int_0^\infty e^{-t} dY_{\gamma_{\alpha, \lambda}}(t); \quad P(Y_{\gamma_{\alpha, \lambda}}(1) \leq a) = G_{\gamma_{\alpha, \lambda}}(a) \in ID_{\log}.
\]

Here are some numerical values:

\[
G_{\gamma_{1,1}}(0.001) = 0.135606; \quad G_{\gamma_{1,1}}(0.01) = 0.138042; \quad G_{\gamma_{1,1}}(1) = 0.394297; \quad G_{\gamma_{2,1}}(2) = 0.6035; \quad G_{\gamma_{2,1}}(3) = 0.753011; \quad G_{\gamma_{2,1}}(4) = 0.8519; \quad G_{\gamma_{2,1}}(6) = 0.95123.
\]

2). \( X := \chi^2(n) \equiv \gamma_{n/2,1/2} \) - chi-square distributions.

Recall that \( X \equiv \chi^2(n) := \sum_{m=1}^{n} Z^2_m, \) where \( Z_m \) are i. i. d. \( N(0,1) \) is called the chi-square distribution with the \( n \) degrees of freedom. Then \( \phi_{\chi^2(n)}(t) = \frac{1}{(1-2it)^{n/2}} \in L \) and its BDCF is \( \psi_{\chi^2(n)}(t) = \exp nt/2[1/(1-2it) - 1]. \) Hence by (\( \star \star \star \)) we have

\[
G_{\chi^2(n)}(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp\left(-\frac{2nt^2}{1+4t^2}\right) \sin(at - \frac{nt}{1+4t^2}) \frac{dt}{t}, \quad \text{for} \ a > 0. \quad (6)
\]
So, there exists a Lévy process \( (Y_{\chi^2(n)}(t), t \geq 0) \) such that \( Y_{\chi^2(n)}(1) \overset{d}{=} G_{\chi^2(n)} \) and

\[
\chi^2(n) = \int_0^\infty e^{-t}dY_{\chi^2(n)}(t), \quad P(Y_{\chi^2(n)}(1) \leq a) = G_{\chi^2(n)}(a) \in ID_{\log}.
\]

Note. In particular, for \( \chi^2(2) = N(0,1)^2 + \tilde{N}(0,1)^2 \) (a sum of squares of two independent standard normal rv) we have

\[
G_{\chi^2(2)}(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp[-4t^2/(1 + 4t^2)]\sin(ta - 2t/(1 + 4t^2))dt/t.
\]

**Here are some numerical values:**

\[
\begin{align*}
G_{\chi^2(2)}(1) &= 0.53013, & G_{\chi^2(2)}(3) &= 0.7477, & G_{\chi^2(2)}(5) &= 0.8686, \\
G_{\chi^2(2)}(7) &= 0.9332, & G_{\chi^2(2)}(10) &= 0.9766, & G_{\chi^2(2)}(15) &= 0.9962.
\end{align*}
\]

3. \( X := \log \gamma_{\alpha, \lambda} \) is the log-gamma variable.

Note that \( f_{\log \gamma_{\alpha, 1}}(x) = \frac{1}{\Gamma(\alpha)}e^{\alpha x - e^x}, x \in \mathbb{R} \) is the probability density of \( \log \gamma_{\alpha, 1} \). Hence \( \log \gamma_{\alpha, \lambda} = -\log \lambda + \log \gamma_{\alpha, 1} \). Furthermore, from Jurek (1997), Example c), p. 98, we have that \( \log \gamma_{\alpha, \lambda} \in L \) (is selfcomposable) and its has the characteristic function is \( \phi_{\log \gamma_{\alpha, \lambda}}(t) = \lambda^{-it}\frac{\Gamma(\alpha + it)}{\Gamma(\alpha)} \).

Using (**), log-gamma variables has the BDCF of the form

\[
\psi_{\log \gamma_{\alpha, \lambda}}(t) = \exp(-it \log \lambda + it\psi(\alpha + it)),
\]

where on the right-hand side the \( \psi(z) := \frac{d}{dz} \log \Gamma(z) \) denotes the digamma function. Finally, by (**),

\[
G_{\log \gamma_{\alpha, \lambda}}(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im(e^{-ita-it\log \lambda+it\psi(\alpha+it)})\frac{dt}{t}; \quad (7)
\]

is the BDDF of log-gamma variable. Thus we have

\[
\log \gamma_{\alpha, \lambda} = \int_0^\infty e^{-s}dY_{\log \gamma_{\alpha, \lambda}}(s), \quad P(Y_{\log \gamma_{\alpha, \lambda}}(1) \leq a) = G_{\log \gamma_{\alpha, \lambda}}(a) \in ID_{\log}.
\]

**Here are some numerical values:**

\[
\begin{align*}
G_{\log \gamma_{2,1}}(-2) &= 0.03, & G_{\log \gamma_{2,1}}(-1) &= 0.109, & G_{\log \gamma_{2,1}}(0) &= 0.3099; \\
G_{\log \gamma_{2,1}}(1) &= 0.6635, & G_{\log \gamma_{2,1}}(2) &= 0.9503.
\end{align*}
\]

For the references below let us recall that functions

\[
I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{1}{2})^{2k+\nu}}{k!(\nu + k + 1)}, \quad K_n(z) = \frac{\pi I_{-n}(z) - I_n(z)}{2 \sin(\pi n)}, \quad \nu \in \mathbb{R}, \quad z \in \mathbb{C}, \quad (8)
\]

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are called the modified Bessel functions of the first and second kind, respectively. These are solutions to the Bessel second order differential equation; cf. Gradshteyn and Ryzhik (1994), Sections 8.40, 8.43 and the formula 8.445.

4). \( X := 1/\gamma_{\alpha,\lambda} \) is the inverse - gamma variable.

From Jurek(2001), Proposition 1 with Example 1 (c) we have that random variable \( 1/\gamma_{\alpha,\lambda} \) is selfdecomposable and

\[
\begin{align*}
    f_{1/\gamma_{\alpha,\lambda}}(x) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{1}{x}\right)^{\alpha+1} e^{-\lambda/x}, \quad x > 0, \quad \text{is its pdf and,} \\
    \phi_{1/\gamma_{\alpha,\lambda}}(t) &= \frac{2}{\Gamma(\alpha)} (-i\lambda t)^{\alpha/2} K_{\alpha}(2\sqrt{-i\lambda t}), \quad t \in \mathbb{R}, \quad \text{is the characteristic function;}
\end{align*}
\]

\((K_{\alpha}(z) \) denotes the modified Bessel function\). Thus the BDDF for the inverse-gamma variable is

\[
G_{1/\gamma_{\alpha,\lambda}}(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im \left( \exp \left[ -ita - \sqrt{-i\lambda t} \frac{K_{\alpha-1}(2\sqrt{-i\lambda t})}{K_{\alpha}(2\sqrt{-i\lambda t})} \right] \right) \frac{1}{t} dt. \tag{9}
\]

This so, because by Wolframalpha,

\[
(\log K_{\alpha}(2\sqrt{-i\lambda t}))' = \frac{i\lambda}{2\sqrt{-i\lambda t}} \frac{K_{\alpha-1}(2\sqrt{-i\lambda t}) + K_{\alpha+1}(2\sqrt{-i\lambda t})}{K_{\alpha}(2\sqrt{-i\lambda t})},
\]

and hence we get

\[t(\log \phi_{1/\gamma_{\alpha,\lambda}}(t))' = \frac{\alpha}{2} - \frac{\sqrt{-i\lambda t}}{2} \frac{K_{\alpha-1}(2\sqrt{-i\lambda t}) + K_{\alpha+1}(2\sqrt{-i\lambda t})}{K_{\alpha}(2\sqrt{-i\lambda t})}.\]

Again by Wolframalpha we have the identity

\[
z \left[ \frac{K_{\alpha-1}(z) + K_{\alpha+1}(z)}{K_{\alpha}(z)} \right] = \alpha + z \frac{K_{\alpha-1}(z)}{K_{\alpha}(z)}.
\]

Applying it for \( z = 2\sqrt{-i\lambda t} \) we get, by (**), the BDCF formula \( \psi_{1/\gamma_{\alpha,\lambda}}(t) \) and then we infer the formula (9).

Consequently, we have

\[
1/\gamma_{\alpha,\lambda} = \int_0^\infty e^{-s}dY_{1/\gamma_{\alpha,\lambda}}(s), \quad P(Y_{1/\gamma_{\alpha,\lambda}}(1) \leq a) = G_{1/\gamma_{\alpha,\lambda}}(a) \in ID_{\log}.
\]

**Numerical illustrations for** \( \alpha = \lambda = 2 \).
EXAMPLE 1. Let $W_t, t \geq 0$ be a Wiener process in $\mathbb{R}^d, d \geq 3$ (the escape process), starting from zero. Then

$$L_r := \sup\{t \geq 0; ||W_t|| \leq r\}, r \geq 0,$$

and $L_r, r \geq 0$ is a process with independent increments (but not homogeneous for $d > 3$) and continuous in probability, cf. Getoor (1979).

5). $X: = \hat{C}$ - hyperbolic-cosine variable;

A random variable $\hat{C} := 2 \pi \sum_{k=1}^{\infty} \frac{1}{k} \eta_k$, (almost surely converging series), where $\eta_k$ are i.i.d. standard Laplace variables has a the characteristic function

$$\phi_{\hat{C}}(t) = \frac{t}{\cosh t} \in L, \text{ cf. Jurek (1996).}$$

Its BDCF is $\psi_{\hat{C}}(t) = \exp(-t \tanh t)$ and by (⋆⋆⋆) its BDDF is given by

$$G_{\hat{C}}(a) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \exp(-x \tanh x) \frac{\sin(ax)}{x} dx. \quad (10)$$

By the identity (4) we have the identification:

$$\hat{C} = \int_{0}^{\infty} e^{-s} dY_{\hat{C}}(s), \quad P(Y_{\hat{C}}(1) \leq a) = G_{\hat{C}}(a) \in ID_{\log}.$$  

6). $X: = \hat{S}$ - hyperbolic-sine variable;

Note that $\hat{S} := \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \eta_k$, where $\eta_k$, as above in 5), are i.i.d. Laplace r.v. and its characteristic function is $\phi_{\hat{S}}(t) := \frac{t}{\sinh t}$; (symmetric variable), cf. Jurek (1996). By (⋆⋆) its BDCF is equal to $\psi_{\hat{S}}(t) = \exp(1 - t \coth t)$ that finally, by (⋆⋆⋆), gives BDDF for $\hat{S}$ as follows

$$G_{\hat{S}}(a) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \exp(1 - x \coth x) \frac{\sin(ax)}{x} dx. \quad (11)$$

In view of (4) we have

$$\hat{S} = \int_{0}^{\infty} e^{-t} dY_{\hat{S}}(t), \quad P(Y_{\hat{S}}(1) \leq a) = G_{\hat{S}}(a) \in ID_{\log}.$$  

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Remark 1. a) Since $\Gamma(1 + it)\Gamma(1 - it) = \frac{\pi t}{\sinh(\pi t)}$ by GR 8.332 (3), from section 3), above, we infer $\text{d} \hat{S} \overset{d}{=} \log \mathcal{E}_1 - \log \mathcal{E}_2 = \log \frac{\xi_1}{\xi_2}$, where $\mathcal{E}_i, i = 1, 2,$ are i.i.d. standard exponential variables.

b) Note that the (normalized) hyperbolic-sine $\frac{2}{\pi} \frac{x}{\sinh x}, x \in \mathbb{R}$, is also a probability density function and its characteristic function is

\[ \int_{-\infty}^{\infty} e^{itx} \frac{2}{\pi^2} \frac{x}{\sinh x} \, dx = \frac{e^{\pi t}}{(1 + e^{\pi t})^2} \in L, \]

equivalently \[ \int_{0}^{\infty} e^{itx} \frac{x}{\sinh(\pi x/2)} \, dx = \frac{1}{\cosh^2(\pi t/2)} \text{; by GR page 1185(23); (note that there the Fourier transform is } 1/\sqrt{2\pi} \text{ times the characteristic function !!)}\]

7). $X := \hat{T} - \text{hyperbolic-tangent}$

Recall that a function $\phi_T(t) = \frac{\tanh t}{t}$ is called the hyperbolic-tangent characteristic function. We have that $\hat{T} \in L$ (is selfdecomposable) by Jurek (1996). By (**) it has BDCF $\phi_T(t) = \exp\left[\frac{2t}{\sinh(2t)} - 1\right] \in ID_{\log}$. It is a compound Poisson distribution $\sum_{k=1}^{N_i} \xi_k$, where $\xi_k \overset{d}{=} \hat{C} \ast \hat{S}$ where $\hat{C}$ and $\hat{S}$ are independent hyperbolic-cosine and sine variables as $\frac{2t}{\sinh(2t)} = \frac{1}{\cosh^2(\pi t/2)}$.

From (****) we have that

\[ G_T(a) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \exp\left[\frac{2t}{\sinh(2t)} - 1\right] \frac{\sin(ta)}{t} \, dt, \quad (12) \]

is the BDDF of hyperbolic-tangent random variable $\hat{T}$. Furthermore, by (4) we have

\[ \hat{T} = \int_{0}^{\infty} e^{-t} \, dY_T(t), \quad P(Y_T(1)) \leq a) = G_T(a). \]

Here are some numerical values:

$G_T(0.4) = 0.7653; G_T(1) = 0.8645; G_T(2) = 0.9528; G_T(3) = 0.9846.$

Example 2. For a Brownian Motion $(B(t), t \geq 0)$, let $\tau_1$ be the exit time of BM from the interval $[-1, 1]$, i.e., $\tau_1 := \inf\{t : |B(t)| = 1\}$.

If $g_{\tau_1} := \sup\{t < \tau_1 : B(t) = 0\}$ is the last zero of $(B(t), t \geq 0)$ before exiting $[-1, 1]$ then $B(g_{\tau_1}) \overset{d}{=} N\sqrt{g_{\tau_1}} \in L$, where $N$ is independent normal variable. Moreover, $\phi_{B(g_{\tau_1})}(t) = \tanh(t)/t$; cf. Yor (1997), Sect.18.6, p. 133; also Jurek (2001), Example 1(b).
From the above formula and the identity $(\star)$, we conclude that there is a Lévy process $(Y_{B(g(\tau_1))}(t), t \geq 0)$ such that $Y_{B(g(\tau_1))}(1) \overset{d}{=} G_T$ and

$$B(g(\tau_1)) = \int_0^{\infty} e^{-s} dY_{B(g(\tau_1))}(s),$$

where $P(Y_{B(g(\tau_1))}(1) \leq a) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \exp\left[\frac{2t}{\sinh(2t)} - 1\right] \sin(at) \ dt$.

**Remark 2.** For the hyperbolic characteristic functions we have the obvious equality: $\phi_C(t) = \phi_S(t) \cdot \phi_T(t)$. Hence we infer that on the level of their corresponding BDCF we have $\psi_C(t) = \psi_S(t) \cdot \psi_T(t)$. Finally, from (10), (11) and (12) we get $\left(1 - t \coth t\right) + \left(\frac{2t}{\sinh(2t)} - 1\right) = -t \tanh t$, that is, the formula for $G_C$ can be obtained for those of $G_S$ and $G_T$.

8). Class L distributions via zeros of Bessel function.

From Jurek (2003), Theorem 1, the converging series

$$X_\nu := \sum_{k=1}^{\infty} z^{-1}_{\nu,k} \eta_k \in L,$$

where $\eta_k$ are i.i.d. Laplace rv and $z_{\nu,k}$ are zeros of the function $z^{-\nu}I_\nu(z)$ is well defined. The variable $X_\nu$ has the characteristic function $\phi_{X_\nu}(t) = (2^\nu \Gamma(\nu + 1))^{-1} \frac{\nu}{I_\nu(t)} \in \mathbb{R}$. Consequently, by $(\star \star)$, its BDCF is

$$\psi_{X_\nu}(t) = \exp[t \left(\log \phi_{X_\nu}(t)\right)'] = \exp[t \left(\nu/t - (\log I_\nu(t))'/\nu\right)]$$

$$= \exp[\nu - t(I_{\nu-1}(t)/I_\nu(t) - \nu/t) = \exp[2\nu - tI_{\nu-1}(t)/I_\nu(t)].$$

Hence, by $(\star \star \star)$,

$$G_{X_\nu}(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \exp[2\nu - tI_{\nu-1}(t)/I_\nu(t)] \frac{\sin(at)}{t} dt$$

is BDDF for $X_\nu$ and there is a Lévy process $(Y_{X_\nu}(t), t \geq 0)$ such that

$$X_\nu = \int_0^{\infty} e^{-t} dY_{X_\nu}(t), \quad P(Y_{X_\nu}(1) \leq a) = G_{X_\nu}(a) \in ID_{\log}.$$

**For an illustration here are some numerical values:**

$G_{X_2}(0.001) = 0.500757$, $G_{X_2}(0.01) = 0.5075$, $G_{X_2}(0.1) = 0.57$, 

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\[ G_{X_2}(0.5) = 0.82, \ G_{X_2}(1) = 0.95, \ G_{X_2}(2) = 0.998, \ G_{X_2}(3) = 0.99973. \]

9). \( X := T^{\nu} - \text{Student} \ t- \text{distributions}, \nu > 0. \)

A random variable \( X := T^{\nu}, \nu > 0 \) with the probability density function

\[ f_{\nu}(x) = \frac{\Gamma(\nu + 1/2)}{\sqrt{2\pi\nu} \Gamma(\nu)} (1 + \frac{x^2}{2\nu})^{-\nu-1/2}; \quad x \in \mathbb{R}, \]

is called the \textit{Student} \( t \)-distribution with \( 2\nu \) degrees of freedom. It has the characteristic function

\[ \phi_{T^{\nu}}(t) = \frac{2^{1-\nu}}{\Gamma(\nu)} (\sqrt{2\nu}|t|)^\nu K_\nu(\sqrt{2\nu}|t|) \in L, \]

cf. Jurek (2001), Example 2, p. 247. The BDCF for \( T^{\nu} \) is equal to:

\[ \psi_{T^{\nu}}(t) = \exp \left[ -|t|\sqrt{2\nu} \frac{K_{\nu-1}(\sqrt{2\nu}|t|)}{K_\nu(\sqrt{2\nu}|t|)} \right], \quad t \neq 0, \]

where \( K_\nu \) is the modified Bessel function of the second kind; cf. the end of Section 3). Thus from (\( \star \star \star \))

\[ G_{Y_{T^{\nu}}(1)}(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp \left[ -|t|\sqrt{2\nu} \frac{K_{\nu-1}(\sqrt{2\nu}|t|)}{K_\nu(\sqrt{2\nu}|t|)} \right] \frac{\sin(ta)}{t} \, dt. \quad (14) \]

Finally appealing to (4) we get

\[ T^{\nu} = \int_0^\infty e^{-t}dY_{T^{\nu}}(t), \quad P(Y_{T^{\nu}}(1) \leq a) = G_{T^{\nu}}(a) \in ID_{\log}. \]

Here are some numerical values for \( \nu = 2: \)

\[ G_{Y_{T^2}}(1)(0.02) = 0.50558, \ G_{Y_{T^2}}(1)(0.5) = 0.6253, \ G_{Y_{T^2}}(1)(1) = 0.7458, \]
\[ G_{Y_{T^2}}(1)(2) = 0.8888; \ G_{Y_{T^2}}(1)(3) = 0.9497, \ G_{Y_{T^2}}(1)(4) = 0.9756, \]
\[ G_{Y_{T^2}}(1)(10) = 0.9988 \]

Remark 3. Since \( K_\nu(z)/K_{-\nu}(z) = 1, \) therefore for Student \( t \)-distribution \( T^{1/2} \), with one degree of freedom, we recover its BDDF as

\[ G_{Y_{T^{1/2}}(1)}(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty e^{-t} \frac{\sin(ta)}{t} \, dt = \frac{1}{2} + \frac{1}{\pi} \arctan a. \]

And that is the 1-stable distribution. This is not a surprise as stable distributions are fixed points of the random integral mapping (\( \star \)), cf. Jurek and Vervaat (1983), Theorem 5.1.
10). The stochastic area under the planar Brownian motion.

a). \( \mathcal{A}_u := \int_0^u Z_s d\tilde{Z}_s - \tilde{Z}_s dZ_s, u > 0. \)

Let \((Z_1, \tilde{Z}_1)\) be the planar Brownian Motion (BM) and we define \( \mathcal{A}_u := \int_0^u Z_s d\tilde{Z}_s - \tilde{Z}_s dZ_s, u > 0. \) Then the stochastic area \( \mathcal{A}_1 \), under the graph given that \((Z_1, \tilde{Z}_1) = (1, 1)\) has the conditional characteristic function

\[
\phi_{\mathcal{A}_1}(t) := \mathbb{E}[e^{it\mathcal{A}_1}|(Z_1, \tilde{Z}_1) = (1, 1)] = \frac{t}{\sinh t} \exp[-(t \coth t - 1)]
\]

by P. Lévy (1950) or Yor (1992), p. 19.

Since \( t/\sinh t \in L \) has the factorization property (cf. (3) in section A) and \( \exp[-t(\coth t - 1)] \) is its BDCF, therefore by Iksanov, Jurek, Schreiber (2004), Proposition 1 and 3), the product \( \frac{t}{\sinh t} \exp[-(t \coth t - 1)] \), is a self-decomposable characteristic function. By (**) its BDCF is

\[
\exp(t(\log(\frac{t}{\sinh t}) \exp[-(t \coth t - 1)])) = \exp(t(\log t - \log \sinh t - t \coth t + 1)) = \exp(t(1/t - \coth t - \coth t + t/\sinh^2 t)) = \exp((1-2t \coth t + t^2/\sinh^2 t).
\]

Consequently, by the formula(* * * *), in Section A) above, we get

\[
G_{\mathcal{A}_1}(a) := P[\mathcal{A}_1 \leq a|(Z_1, \tilde{Z}_1) = (1, 1)]
\]

\[
= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp[1 - 2t \coth t + t^2/\sinh^2 t] \sin(ta) dt.
\]

Finally, by (4) we have that there is a Lévy process such that

\[
\mathcal{A}_1 = \int_0^\infty e^{-t}dY_{\mathcal{A}_1}(t), \quad P(Y_{\mathcal{A}_1}(1) \leq a) = G_{\mathcal{A}_1}(a) \in ID_{\text{log}}.
\]

Here are some numerical values:

\[
G_{\mathcal{A}_1}(0.5) = 0.649892; \quad G_{\mathcal{A}_1}(1) = 0.775697; \quad G_{\mathcal{A}_1}(1.2) = 0.8163
\]

\[
G_{\mathcal{A}_1}(1.5) = 0.86674; \quad G_{\mathcal{A}_1}(2) = 0.92558; \quad G_{\mathcal{A}_1}(3) = 0.9799
\]

b). \( \mathcal{A}_p^u := \int_0^u V_p^s d\tilde{V}_p^s - \tilde{V}_p^s dV_p^s, u > 0; \quad p > -1/2. \)

This represents the generalized Lévy stochastic area for the process:

\[
V_p^t = (V_p^t, \tilde{V}_p^t) := t^{-p} \int_0^t s^p dB_s, \quad (B_s, s \geq 0) \text{ is the planar Brownian motion.}
\]

Note that for \( p = 0 \) we get the case a), above.

Assuming that \( V_p^1 = (V_p^1, \tilde{V}_p^1) = (1, 1) \) then for the conditional stochastic area \( \mathcal{A}_p^u \) we have

\[
\phi_{\mathcal{A}_p^u}(t) := \mathbb{E}[e^{it\mathcal{A}_p^u}|V_p^1 = (1, 1)] = (2^\nu \Gamma(\nu + 1))^{-1} \frac{|t|^{\nu}}{I_{\nu}(\nu)} \exp\left[ - |t| \frac{I_{\nu+1}(|t|)}{I_{\nu}(|t|)} \right]
\]

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where \( I_\nu(z) \) (cf. the end of Section 3) is the modified Bessel function and \( \nu := p + 1/2 > 0 \); cf. Biane and Yor (1987) or Yor (1989) or Duplantier (1989); also Jurek (2003).

Since \( (2t^{\nu}(\nu+1))^{-1} \frac{\nu t}{L(t)} \in L_f \subset L \) (see Iksanov, Jurek, Schreiber (2004), Propositions 1 and 3), then using Theorem 1 in Jurek (2003) we have that \( \phi_{\mathcal{H}_1}(t) \in L \). Hence

\[
\log \phi_{\mathcal{H}_1}(t) = -\log(2t^{\nu}(\nu+1)) + \nu \log t - \log I_{\nu}(t) - t \frac{I_{\nu+1}(t)}{I_{\nu}(t)}.
\]

From Wolframalpha.com we have the identities:

\[
(i) \quad (\log I_{\nu}(t))' = \frac{I_{\nu-1}(t)}{I_{\nu}(t)} - \frac{\nu}{t}; \quad (ii) \quad 2\nu - t \frac{I_{\nu-1}(t) + I_{\nu+1}(t)}{I_{\nu}(t)} = -2t \frac{I_{\nu+1}(t)}{I_{\nu}(t)}
\]

\[
(iii) \quad \left( \frac{I_{\nu+1}(t)}{I_{\nu}(t)} \right)' = \frac{1}{2} \left( \frac{1}{\nu+2(t)/I_{\nu}(t)} - (I_{\nu+1}(t)/I_{\nu}(t))^2 - I_{\nu-1}(t)I_{\nu+1}(t)/I_{\nu}^2(t) \right)
\]

Hence we get the derivative:

\[
(\log \phi_{\mathcal{H}_1}(t))' = \frac{\nu}{t} - \left( \frac{I_{\nu-1}(t)}{I_{\nu}(t)} - \frac{\nu}{t} \right) - (I_{\nu+1}/I_{\nu} + t(I_{\nu+1}/I_{\nu})')
\]

\[= 2\nu/t - I_{\nu-1}/I_{\nu} - I_{\nu+1}/I_{\nu} - t(I_{\nu+1}/I_{\nu})'
\]

\[= 2\nu/t - I_{\nu-1}/I_{\nu} - I_{\nu+1}/I_{\nu} - t/2(1 + I_{\nu+2}/I_{\nu} - (I_{\nu+1}/I_{\nu})^2 - I_{\nu-1}I_{\nu+1}/I_{\nu}^2),
\]

and consequently by (**) we get the BDCF

\[
\psi_{\mathcal{H}_1}(t) = \exp[t(\log \phi_{\mathcal{H}_1}(t))']
\]

\[= \exp[2\nu - t \frac{I_{\nu-1}(t) + I_{\nu+1}(t)}{I_{\nu}(t)} - \frac{t^2}{2} (1 + \frac{I_{\nu+2}(t)}{I_{\nu}(t)} - (\frac{I_{\nu+1}(t)}{I_{\nu}(t)})^2 - \frac{I_{\nu-1}(t)I_{\nu+1}(t)}{I_{\nu}^2(t)})]
\]

\[= \exp[-2t \frac{I_{\nu+1}(t)}{I_{\nu}(t)} - \frac{t^2}{2} (1 + \frac{I_{\nu+2}(t)}{I_{\nu}(t)} - (\frac{I_{\nu+1}(t)}{I_{\nu}(t)})^2 - \frac{I_{\nu-1}(t)I_{\nu+1}(t)}{I_{\nu}^2(t)})],
\]

where we used the identity (ii). Hence we have the formula for BDDF as follows:

\[
G_{\mathcal{H}_1}(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \left( \exp[-2t \frac{I_{\nu+1}(t)}{I_{\nu}(t)}] \right) \frac{\sin(ta)}{t} dt.
\]
From the identification (4), from section A) we have
\[ A_{1}^{p} = \int_{0}^{\infty} e^{-t}dY_{A_{1}^{p}}(t), \quad P(Y_{A_{1}^{p}}(1) \leq a) = G_{A_{1}^{p}}(a) \in ID_{log}. \]

Some numerical values for \( \nu = 2 \equiv p = 3/2 \):
\[ G_{A_{1}^{3/2}}(0.1) = 0.5420; \quad G_{A_{1}^{3/2}}(0.3) = 0.6239; \quad G_{A_{1}^{3/2}}(0.5) = 0.700; \]
\[ G_{A_{1}^{3/2}}(1) = 0.84422; \quad G_{A_{1}^{3/2}}(2) = 0.97553; \quad G_{A_{1}^{3/2}}(3) = 0.997414. \]

Remark 4. By some (tedious) calculation one may check directly that we have equal two BDDF \( G_{A_{1}} = G_{A_{1}^{0}} \), i.e, by putting \( \nu = 1/2 \) in (16) we get equality (14). Similarly, we have the equality on the level of their corresponding characteristic functions \( \phi_{A_{1}}(t) = \phi_{A_{1}^{0}}(t), t \in \mathbb{R} \).

11). Some integral functionals of Wiener Process.

a). Exponential integrals.

For the process \( (W_{t} + \alpha t, t \geq 0) \), (a parameter \( \alpha > 0 \)) and the variable \( X := \int_{0}^{\infty} e^{-W_{t} - \alpha t}dt \), Urbanik (1992) in Example 3.3 (on p.309), proved
\[ X \overset{d}{=} 1/\gamma_{2\alpha,2}, \text{ which has pdf } \frac{4^\alpha}{\Gamma(2\alpha)} x^{-2\alpha-1} e^{-2/x}, x > 0 \]
Thus
\[ \int_{0}^{\infty} e^{-W_{t} - \alpha t}dt \overset{d}{=} \int_{0}^{\infty} e^{-s}dY_{1/\gamma_{2\alpha,2}}(s), \quad Y_{1/\gamma_{2\alpha,2}}(1) \overset{d}{=} G_{1/\gamma_{2\alpha,2}}. \]
The BDDF for the inverse gamma was given in 4). above.

Remark 5. Note that if we define \( \tau_{s} := \inf(t > 0 : W_{t} + \alpha t > s) \) then \( (\tau_{s}, s \geq 0) \) is the inverse Gaussian (Lévy) process.

b). Inverse Gaussian variable \( IG(\lambda, \mu) \).

For \( \lambda > 0, \mu > 0 \), random variables with probability density functions \( f_{IG(\lambda, \mu)} \) and characteristic functions \( \phi_{IG(\lambda, \mu)}(t) \) given as
\[ f_{IG(\lambda, \mu)}(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{(x - \mu)^2}{2\mu^2x}\right]; \quad \phi_{IG(\lambda, \mu)}(t) = \exp\left[\frac{\lambda}{\mu} \left(1 - \sqrt{1 - \frac{2\mu^2it}{\lambda}}\right)\right]. \]
are called the inverse Gaussian distributions. By Halgreen (1979) we have that they are selfdecomposable. Since \((\log \phi_{IG(\lambda,\mu)}(t))' = i\mu(1 - 2\lambda^{-1}t)^{-1/2}\), thus its BDDF is \(\psi_{IG(\lambda,\mu)}(t) = \exp[i\mu t(1 - 2\mu^2\lambda^{-1}it)^{-1/2}]\). Consequently,

\[
G_{IG(\lambda,\mu)}(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im((-\lambda + i\mu \sqrt{1 - 2\mu^2\lambda^{-1} it}) dt), \tag{17}
\]

is the BDDF for \(IG(\lambda, \mu)\). So from the identification (4) we get

\[
IG(\lambda, \mu) = \int_0^\infty e^{-t}dY_{IG(\lambda, \mu)}(t), \quad P(Y_{IG(\lambda, \mu)}(1) \leq a) = G_{IG(\lambda, \mu)}(a) \in ID_{\log}.
\]

Here are some numerical values for \(\lambda = \mu = 1:\)

- \(G_{IG(1,1)}(-5) = 0.00; \quad G_{IG(1,1)}(-3) = 0.04; \quad G_{IG(1,1)}(-2) = 0.23;\)
- \(G_{IG(1,1)}(-1) = 0.55; \quad G_{IG(1,1)}(0.1) = 0.77; \quad G_{IG(1,1)}(0) = 0.79;\)
- \(G_{IG(1,1)}(0.5) = 0.87; \quad G_{IG(1,1)}(1) = 0.91;\)
- \(G_{IG(1,1)}(2) = 0.96; \quad G_{IG(1,1)}(3) = 0.98; \quad G_{IG(1,1)}(5) = 0.99.\)

Similarly we have for \(\mu = 1\) and \(\lambda = 2:\)

- \(G_{IG(2,1)}(-3) = 0.007; \quad G_{IG(2,1)}(-2) = 0.14; \quad G_{IG(2,1)}(-1) = 0.54;\)
- \(G_{IG(2,1)}(-0.5) = 0.72; \quad G_{IG(2,1)}(0) = 0.85; \quad G_{IG(2,1)}(0.5) = 0.926;\)
- \(G_{IG(2,1)}(1) = 0.9638; \quad G_{IG(2,1)}(2) = 0.9914.\)

\[\]

**c). Quadratic functional of BM.**

For the Brownian process \((W_t, t \geq 0)\) starting from zero, the drift parameter \(b\), the staring starting point \(a\) and an independent of \(W\) a standard normal variable \(N\), the variable

\[
Q(a, b) \equiv Q(a, b, W) := N(\int_0^1 (W_s + bs + a)^2 ds)^{1/2}
\]

is selfdecomposable.

To see this, let us take two Brownian motions \(W\) and \(\tilde{W}\) and two standard normal variables \(N\) and \(\tilde{N}\) that are all independent. Then for \(0 < c < 1\) we have \(N \overset{d}{=} cN + \sqrt{1 - c^2}\tilde{N}\) and hence

\[
Q(a, b) = N(\int_0^1 (W_s + bs + a)^2 ds)^{1/2} \overset{d}{=} (cN + \sqrt{1 - c^2}\tilde{N})(\int_0^1 (W_s + bs + a)^2 ds)^{1/2} \overset{d}{=} cQ + \sqrt{1 - c^2}\tilde{Q}, \tag{18}
\]

with \(Q\) and \(\tilde{Q}\) independent variables. This, with the characterization (2), proves the selfdecomposable property of \(Q(a, b)\).
From by Wenocur (1986) or from Yor (1992), p.19, formula (2.9) we have the characteristic function

$$\phi_Q(a,b)(t) = \mathbb{E}[\exp(itN(\int_0^1 (W_s + bs + a)^2 ds)^{1/2})] = \mathbb{E}[\exp(-\frac{t^2}{2} \int_0^1 (W_s + bs + a)^2 ds)]$$

$$= (cosh t)^{-1/2} \exp[-\frac{b^2}{2}(1 - \frac{\tanh t}{t}) - ba(1 - \frac{1}{\cosh t}) - \frac{a^2}{2} t \tanh t)]$$

$$= (cosh t)^{-1/2} \exp[-\frac{a^2}{2} t \tanh t)] \exp[\frac{b^2}{2}(\frac{\tanh t}{t} - 1)] \exp[ba(\frac{1}{\cosh t} - 1)].$$

**Remark 6.** In the formula above for $\phi_Q(a,b)$ we have:

(i) $(cosh t)^{-1/2} \in L$ has the factorization property (see (3) in the section A) and $\exp[-\frac{1}{2} t \tanh t]$ is its BDCF (which in fact, is $s$-selfdecomposable distribution; cf. Iksanov, Jurek and Schreiber (2004), p.1367).

(ii) $\exp[\frac{b^2}{2}(\frac{\tanh t}{t} - 1)]$ and $\exp[ba(\frac{1}{\cosh t} - 1)]$ are compound Poisson of the form $\sum_{k=1}^{N_t} \xi_k$. In the first case, $\alpha = \frac{b^2}{2}$ and $\xi_k$ are i.i.d. with hyperbolic tangent and in the second case, $\alpha = ba > 0$ and $\xi_k$ are i.i.d. hyperbolic cosine variables.

We have that

$$(\log \phi_Q(t))' = (-\frac{1}{2} \log \cosh t - \frac{a^2}{2} t \tanh t + \frac{b^2}{2}(\frac{\tanh t}{t} - 1) + ab(\frac{1}{\cosh t} - 1))'$$

$$= -\frac{1}{2} \tanh t - \frac{a^2}{2}(\tanh t + t \frac{1}{\cosh^2 t}) + \frac{b^2}{2}(\frac{1}{\cosh^2 t} - 1 - t^{-2} \tanh t)$$

$$- ab(\cosh t)^{-2} \sinh t$$

$$= -\frac{1}{2} \tanh t(1 + a^2 + b^2 t^{-2}) + \frac{1}{2 \cosh^2 t}(-a^2 t + b^2 t^{-1} - 2ab \sinh t).$$

Hence, the BDCF is

$$\psi_Q(a,b)(t) = \exp[-\frac{1}{2} \tanh t((1 + a^2) t + b^2 t^{-1}) + \frac{1}{2 \cosh^2 t}(-a^2 t^2 + b^2 - 2ab \sinh t)]$$

and finally by (** **) we get its BDDF

$$G_Q(a,b)(x) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \exp[-\frac{1}{2} \tanh t((1 + a^2) t + b^2 t^{-1})]$$

$$+ \frac{1}{2 \cosh^2 t}(-a^2 t^2 + b^2 - 2ab \sinh t)] \frac{\sin(t x)}{t} dt. \quad (19)$$
In summary, by (4) in Section A), we have that

\[ Q(a, b) = \int_{0}^{\infty} e^{-s}dY_{Q(a, b)}(s), \quad P(Y_{Q(a, b)}(1) \leq u) = G_{Q(a, b)}(u) \in ID_{\log} \]

**Here are some numerical values for** \( a = 1 \) **and** \( b = 2 \):

\[
Q(1, 2)(0.01) = 0.501664, \quad Q(1, 2)(0.1) = 0.516603, \quad Q(1, 2)(1) = 0.648221, \\
Q(1, 2)(2) = 0.763609, \quad Q(1, 2)(3) = 0.849722, \quad Q(1, 2)(4) = 0.908518, \\
Q(1, 2)(5) = 0.966382. 
\]

12). **Logistic distribution** \( l(a, b) \).

From Ushakov (1999), p. 298 (or W.Feller p. 52) random variable with the probability distribution function

\[
p(x; a, b) = \frac{\pi}{b\sqrt{3}} \frac{\exp[-\frac{\pi}{\sqrt{3}}(\frac{x-a}{b})]}{(1 + \exp[-\frac{\pi}{\sqrt{3}}(\frac{x-a}{b})])^2}; \quad -\infty < x < \infty \quad (a \in \mathbb{R}, \ b > 0);
\]

which gives the mean-value \( a \), the variance \( b^2 \), and the characteristic function

\[
\phi_{l(a, b)}(t) = e^{ita} \Gamma(1 - ict)\Gamma(1 + ict), \quad \text{where} \ c := \frac{b\sqrt{3}}{\pi},
\]

is called the **logistic distribution**. It is selfdecomposable by 3), and the fact that \( L \) is a convolution semigroup; cf. last paragraph in the section A). Since \( \Gamma(1 - ict)\Gamma(1 + ict) = \frac{\pi ct}{\sinh(\pi ct)}, \ t \in \mathbb{R} \), by Gradsteyn-Ryzhik 8332.3 we get

\[
\phi_{l(a, b)}(t) = e^{ita} \frac{\pi ct}{\sinh(\pi ct)}.
\]

Thus by (\(*\)), the BDCF for the logistic \( l(a, b) \) distribution is

\[
\psi_{l(a, b)}(t) = \exp[ita + 1 - \pi ct \coth(\pi ct)].
\]

Finally, the BDDF for \( l(a, b) \) is:

\[
G_{l(a, b)}(u) = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \Re(\exp(-itu + itu + 1 - \pi ct \coth(\pi ct))) \frac{1}{t} dt, \ u \in \mathbb{R}.
\]

Comp. with the hyperbolic-sine function above in 6).

By the identification formula (4) we have

\[
l(a, b) = \int_{0}^{\infty} e^{-t}dY_{l(a, b)}(t), \quad P(Y_{l(a, b)}(1) \leq u) = G_{l(a, b)}(u) \in ID_{\log}.
\]

**Here are some values for** \( l(0, 1) \):
The Bessel function $I_{\nu}(x)$ appearing in probability density functions.

(a). Non-central chi-square distribution.

The non-central chi-square distribution $\chi^2_c(k)$, with $k$-degrees of freedom ($k>0$) and the non-centrality exponent $c$ ($c>0$), has the probability density

$$
\frac{1}{2} e^{-(x+c)/2} \left( \sqrt{\frac{x}{c}} \right)^{k/2-1} I_{k/2-1}(\sqrt{cx}), \quad x>0;
$$

This is a Poisson mixture of the gamma densities $\gamma_{ak,1}$ for specified shape parameters $a_k$ and scales equal 1; see Feller (1966), p. 58, formula (7.3). By WolframAlpha.com $E[\chi^2_c(k)] = c + k$, $\text{Var}[\chi^2_c(k)] = 2(2c + k)$, and the characteristic function $\phi_{\chi^2_c(k)}(t)$ is of the form

$$
\phi_{\chi^2_c(k)}(t) = \exp\left[ \frac{itc}{1 - 2it} \right] \sqrt{\frac{x}{c}} e^{-x/2} dx
$$

It corresponds to the sum of two independent ID variable: the first one is the compound Poisson with the Levy-Khintchine formula

$$
\exp\left[ \int_0^\infty (e^{itx} - 1) \frac{1}{2} e^{-x/2} dx \right]
$$

(not in L as its Levy (spectral) measure is finite !) and the second one is the chi-square with $k/2$ degrees of freedom and it is in L,(see 2) above) with its Levy-Khintchine-formula

$$
\frac{1}{(1 - 2it)^{k/2}} = \exp\left[ \int_0^\infty (e^{itx} - 1) \frac{k}{2} e^{-x/2} dx \right]
$$

All in all, the non-central $\chi^2_c(k)$ -distribution is infinitely divisible and its Lévy (spectral) measure

$$
M(dx) := \frac{1}{2} e^{-x/2} dx + \frac{k}{2} e^{-x/2} dx = \left[ \frac{1}{2} e^{-x/2}(c + k/x) \right] dx
$$

has a density such that the function $x\left[ \frac{1}{2} e^{-x/2}(c + k/x) \right]$ is decreasing on the positive half-line. Consequently, the non-central $\chi^2_c(k)$-distribution is selfdecomposable; Jurek and Mason (1993), p. 94, or Jurek (1997), p. 96.
Since
\[
\left( \log \left( \frac{\exp \left( \frac{it\lambda}{1-2it} \right)}{(1-2it)^{k/2}} \right) \right)' = (\lambda/2)\left( \frac{1}{1-2it} - 1 \right)' - k/2(\log(1-2it))'
\]
\[
= \lambda/2 \frac{2i}{(1-2it)^2} - \frac{k/2}{1-2it} = i\left( \frac{\lambda}{(1-2it)^2} + \frac{k}{1-2it} \right),
\]
from (⋆⋆) we have the BDCF \( \psi_{\chi_2(k)}(t) = \exp[it(\frac{c}{(1-2it)\tau} + \frac{k}{1-2it})] \), and finally, the BDDF of \( \chi_2(k) \) is
\[
G_{\chi_2(k)}(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im \left( \exp \left[ -ita + it\left( \frac{c}{(1-2it)^2} + \frac{k}{1-2it} \right) \right] \right) \frac{dt}{t} \tag{21}
\]

**Here are some numerical values** when \( c = 1, k = 2 \) AND the above integral is from \( t=0 \) to \( t=10 \)!!
\[
G_{\chi_2(1)}(2) = 0.4729; \quad G_{\chi_2(2)}(2) = 0.55157; \quad G_{\chi_2(4)}(2) = 0.709; \quad G_{\chi_2(8)}(2) = 0.88; \quad G_{\chi_2(10)}(2) = 0.93; \quad G_{\chi_2(15)}(2) = 0.98
\]

(b). Bessel \( h_\nu \) - density.

For \( \nu > 0, h_\nu(x) := e^{-x\frac{\nu I_0(x)}{x}}, 0 < x < \infty, \) is a probability density function of random variable \( \tilde{h}_\nu, \) (Note that for \( \nu = r \in \{1,2,...\} \) it is the probability density of the first passage time of symmetric random walk trough \( r \).)

For the characteristic function for \( t \in \mathbb{R} \) we have
\[
\phi_{h_\nu}(t) := \int_0^\infty e^{itx} h_\nu(x) dx = [1 - it - \sqrt{(1-it)^2 - 1}]^\nu
\]
\[
= \exp \nu \left[ it \int_0^\infty \frac{e^{-x} I_0(x)}{1 + x^2} dx + \int_0^\infty \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) \frac{e^{-x} I_0(x)}{x} dx \right], \tag{22}
\]
see Feller (1966), p.414 and 476 or Ushakov (1999), p. 283, for the first equality. For the second one see a proof below.

From (22), multiplying the density of the Lévy (spectral) measure by \( x \) we get decreasing function \( \nu I_0(x)e^{-x} \). Therefore, similarly as for the non-central chi-square above, we get that Bessel density \( h_\nu \) ( the random variable \( \tilde{h}_\nu \)) is selfdecomposable.

Since \( (\log(1-it - \sqrt{(1-it)^2 - 1}))' = i/\sqrt{-t(t+2i)} \) therefore, by (⋆⋆), the BDCF for the selfdecomposable Bessel density \( h_\nu \) is
\[
\psi_{\tilde{h}_1}(t) = \exp(it/\sqrt{-t(t+2i)}) = \exp i\sqrt{\frac{-t}{t+2i}} = \exp i\sqrt{\frac{2}{2-it}} - 1 \in \text{ID}_{\text{log}}
\]
and by (⋆⋆⋆) the BDDF is given as
\[
G_{\tilde{h}_\nu}(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im(\exp[-ita + i\nu \sqrt{\frac{1}{1-it/2} - 1}]) \frac{dt}{t}, \quad a \in C_{\nu}. \tag{23}
\]

Hence by (4), from section A), we have
\[
\tilde{h}_\nu = \int_0^\infty e^{-t} dY_{\tilde{h}_\nu}(t), \quad P(Y_{\tilde{h}_\nu}(1) \leq a) = G_{\tilde{h}_\nu}(a) \in ID_{\log}.
\]

Remark 7. Note that if $\mathcal{E}(2)$ denotes the exponential distribution with the parameter 2 then the compound Poisson $\exp(\mathcal{E}(2))$ has the characteristic function $\exp(\frac{1}{1-it/2} - 1)$ and its logarithm is under the square root in (23).

Some numerical values for $G_{\tilde{h}_{10}}(\cdot)$:
- $G_{\tilde{h}_{10}}(1) = 0.0091$,  $G_{\tilde{h}_{10}}(5) = 0.03430$,  $G_{\tilde{h}_{10}}(8) = 0.09192$,
- $G_{\tilde{h}_{10}}(10) = 0.1272$,  $G_{\tilde{h}_{10}}(200) = 0.717063$,  $G_{\tilde{h}_{10}}(900) = 0.868878$,
- $G_{\tilde{h}_{10}}(1200) = 0.882203$,  $G_{\tilde{h}_{10}}(1500) = 0.911412$.

A proof of the second equality in (22). Since $(\Phi_{\nu/\nu}(t))^{1/n} = \phi_{\nu/n}(t)$ we have that
\[
n(\phi_{\nu/n}(t) - 1) = n((\phi_{\nu}(t))^{1/n} - 1) \to \log \phi_{\nu}(t), \quad \text{as} \quad n \to \infty,
\]
where $h_{\nu/n}(x) = e^{-x \nu/n I_{\nu/n}(x)}$, $0 < x < \infty$. Thus
\[
n(\phi_{\nu/n}(t) - 1) = n(\int_0^\infty (e^{itx} - 1)e^{-x \nu/n I_{\nu/n}(x)} dx)
= it \int_0^\infty \frac{x}{1 + x^2} e^{-x \nu/n I_{\nu/n}(x)} dx + \int_0^\infty (e^{itx} - 1) \left(\frac{itx}{1 + x^2} e^{-x \nu/n I_{\nu/n}(x)} dx \right)
\to it \nu \int_0^\infty \frac{e^{-x I_0(x)} dx}{1 + x^2} + \int_0^\infty (e^{itx} - 1) \left(\frac{itx}{1 + x^2} e^{-x I_0(x)} dx \right),
\]
which concludes the proof of the second equality in (22).

From (22) we have two ways to the expression for $t(\log \phi_{\nu}(t))'$ which leads to the equality:

**Corollary 1.** For $t > 0$ we have
\[
\frac{1}{t} \sqrt{\frac{-t}{t + 2i}} = \int_0^\infty e^{-x I_0(x)} dx + \int_0^\infty (e^{itx} - \frac{1}{1 + x^2}) e^{-x I_0(x)} dx
= \int_0^\infty e^{itx} e^{-x I_0(x)} dx. \tag{24}
\]
14). Fisher z-distribution of the dispersion proportion.
For positive integers \( m_1 \geq 1 \) and \( m_2 \geq 1 \), degrees of freedom, the \( z(m_1, m_2) \)-distribution (introduced by R. A. Fisher in 1924) has the probability density

\[
f_{z(m_1, m_2)}(x) := \frac{2m_1^{m_1/2}m_2^{m_2/2}\Gamma\left(\frac{m_1+m_2}{2}\right)}{\Gamma\left(\frac{m_1}{2}\right)\Gamma\left(\frac{m_2}{2}\right)} \frac{e^{m_1x}}{(m_2 + m_1 e^{2x})^{(m_1+m_2)/2}}
\]

with the expected value \( \frac{1}{2} \left( \frac{1}{m_2} - \frac{1}{m_1} \right) \) and the variance \( \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) \), and the characteristic function

\[
\phi_{z(m_1, m_2)}(t) = \left( \frac{m_2}{m_1} \right)^{it/2} \frac{\Gamma\left(\frac{m_1+it}{2}\right)\Gamma\left(\frac{m_2-it}{2}\right)}{\Gamma\left(\frac{m_1}{2}\right)\Gamma\left(\frac{m_2}{2}\right)}, \quad (25)
\]

cf. Ushakov (1999), p.309. Changing the parametrization \( \alpha_1 := m_1/2, \alpha_2 := m_2/2 \) and then using formulas from 3) we infer

\[
\phi_{z(2\alpha_1, 2\alpha_2)}(t) = \left( \frac{\alpha_2}{\alpha_1} \right)^{it} \frac{\Gamma(\alpha_1 + it)\Gamma(\alpha_2 - it)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} = \phi_{\log^{\frac{\gamma_2}{\alpha_1}}\gamma_1}(t) \phi_{-\log^{\frac{\gamma_2}{\alpha_2}}\gamma_2}(t) \in L,
\]

that is, Fisher z-distributions are selfdecomposable. Furthermore, their BDCF are \( e^{it[\log(\alpha_2/\alpha_1) + \psi(\alpha_1 + it) - \psi(\alpha_2 - it)]} \).

Finally, for the BDDF we have

\[
G_{z(2\alpha_1, 2\alpha_2)}(a) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \Im\left( e^{-i\alpha_1-\alpha_1t+\psi(\alpha_1+i\alpha_1)+\psi(\alpha_2-i\alpha_2)} \right) \frac{dt}{t}, \quad (26)
\]

As in previous cases, by (4), we have that

\[
2z(2\alpha_1, 2\alpha_2) = \int_0^\infty e^{-t}dY_{z(2\alpha_1, 2\alpha_2)}(t), \quad P(Y_{z(2\alpha_1, 2\alpha_2)}(1) \leq a) = G_{z(2\alpha_1, 2\alpha_2)}(a) \in IG_{\log}.
\]

Here are some values for z-distribution with \( \alpha_1 = 1 \) and \( \alpha_2 = 2 \):

- \( G_{z(2,4)}(-5) = 0.02727 \); \( G_{z(2,4)}(-3) = 0.1022 \); \( G_{z(2,4)}(-2) = 0.18881 \);
- \( G_{z(2,4)}(-1) = 0.3296 \); \( G_{z(2,4)}(-0.01) = 0.522759 \); \( G_{z(2,4)}(0) = 0.524879 \);
- \( G_{z(2,4)}(0.01) = 0.5270 \); \( G_{z(2,4)}(0.1) = 0.5461 \); \( G_{z(2,4)}(0.5) = 0.63107 \);
- \( G_{z(2,4)}(1) = 073103 \); \( G_{z(2,4)}(2) = 0.8818 \); \( G_{z(2,4)}(3) = 0.9582 \);
- \( G_{z(2,4)}(4) = 0.987396 \); \( G_{z(2,4)}(5) = 0.996587 \).

As a by-product of the above we get a relation between the z-distribution and the log-gamma:
Corollary 2. For two independent gamma random variables $\gamma_{\alpha_1, \alpha_1}$ and $\tilde{\gamma}_{\alpha_2, \alpha_2}$ we have

$$2 \zeta(2\alpha_1, 2\alpha_2) \overset{d}{=} \log \gamma_{\alpha_1, \alpha_1} + (-\log \tilde{\gamma}_{\alpha_2, \alpha_2}) = \log \frac{\gamma_{\alpha_1, \alpha_1}}{\tilde{\gamma}_{\alpha_2, \alpha_2}}.$$ 

Remark 8. The classical Fisher - Snedecor distribution $F(d_1, d_2) \overset{d}{=} \frac{\chi^2(d_1)/d_1}{\chi^2(d_2)/d_2}$ (a ratio of two independent chi-square distributions with degrees of freedom $d_1$ and $d_2$, respectively) has the probability density

$$f_{F(d_1, d_2)}(x) = \frac{1}{B(d_1/2, d_2/2)} (d_1/d_2)^{d_1/2} x^{d_1/2 - 1} (1 + d_1/d_2 x)^{-d_1+d_2/2}, \ x > 0;$$

where $B(x, y)$ is the beta function. Thus it is a Fisher z-distribution with appropriately changed variable $x$ and parameters.

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