BAND-LIMITED LOCALIZED PARSEVAL FRAMES AND BESOV SPACES ON COMPACT HOMOGENEOUS MANIFOLDS

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Abstract. In the last decade, methods based on various kinds of spherical wavelet bases have found applications in virtually all areas where analysis of spherical data is required, including cosmology, weather prediction, and geodesy. In particular, the so-called needlets (=band-limited Parseval frames) have become an important tool for the analysis of Cosmic Microwave Background (CMB) temperature data. The goal of the present paper is to construct band-limited and highly localized Parseval frames on general compact homogeneous manifolds. Our construction can be considered as an analogue of the well-known \( \varphi \)-transform on Euclidean spaces.

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1. Introduction

In the last decade, methods based on spherical wavelets have found applications in virtually all areas where analysis of spherical data is required, including cosmology, weather prediction and geodesy (see [11], [12], [23], [53] and the references therein). In particular, they have become an important tool for the analysis of Cosmic Microwave Background (CMB) temperature data ([21], [22], [53], [51], [2], [3], [57], [8], [17], [27], [28], [56], and many other articles). In analyzing CMB temperature data, one seeks precise estimates of several parameters of the greatest interest for Cosmology and Theoretical Physics, as well as information on possible regions of non-Gaussianity, and other information as well.

In the past few years, a new kind of wavelet has found many fruitful applications in the analysis of CMB temperature data, the so-called spherical needlets, which form a Parseval (= normalized tight) frame on the sphere (see [46], [12], [33] for information about Parseval frames on Euclidean spaces). Spherical needlets were introduced in [31], [32], and then used for rigorous statistical analysis of spherical random fields in [4], [5], [26] and other articles. This analysis was particularly effective in extracting the desired consequences from CMB temperature data.

The interest in needlets on spheres can be explained by their nearly optimal space-frequency localization properties. These properties of needlets (and other localized bases, such as the “Mexican needlets” of [18]) allow one to perform frequency analysis of signals (functions), even when one only has partial information about them.

For example, the CMB models are best analyzed in the frequency domain, where the behavior at different multipoles can be investigated separately; on the other hand, partial sky coverage and other missing observations make the evaluation of spherical harmonic transforms impossible.

A recent advance in this area was the development of spin needlets on the sphere [13], [14], [15], [20], for the purpose of statistical analysis of CMB polarization, which is also expected to have very significant consequences in physics.

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In a different direction, nearly tight frames, which were smooth and highly localized in both space and frequency, were developed on general smooth compact manifolds without boundary, in [17]-[19]. These frames, as in the case of spherical needlets, were constructed from the kernels of certain functions of the Laplace-Beltrami operator. (An analogous construction had been carried out earlier for stratified Lie groups with lattice subgroups, in [15].)

In this article, we will show that on compact homogeneous manifolds, one can do better – one can arrange for the frames arising from these methods to actually be Parseval (and, of course, highly localized in space and frequency). We will also show that one can characterize Besov spaces through a knowledge of the size of frame coefficients, thereby generalizing results of [32] for the sphere. (Our results on frame characterizations of Besov spaces are closely related to those in [19].)

Our frames are a natural generalization of spherical needlets. They can be also be regarded as analogous to the well-known $\varphi$-transform [9].

In addition to the fact that one can find Parseval frames, we offer the following motivations for specializing to the case of homogeneous manifolds. First, on such manifolds, there is the possibility of finding exact formulas for the frame elements. Secondly, in theoretical physics – where many manifolds are considered – symmetry is of capital importance. Third, on homogeneous manifolds, one has the advantage that all of the frame elements at a particular scale can be obtained from each other through the group action, in the same manner as standard wavelets at a particular scale on the real line can be obtained from each other by translation.

Our frames will be band-limited, and hence smooth. One should understand that the notion of band-limitedness on a compact manifold is not canonical. Consider a (connected) compact smooth Riemannian manifold $M$, and a smooth elliptic self-adjoint positive differential operator $A$ on it. It is known that the spectrum of $A$, as an operator in the corresponding space $L^2(M)$, is discrete, nonnegative, and accumulates at infinity. Call the eigenvalues $\lambda_0 \leq \lambda_1 \leq \ldots$, where we repeat eigenvalues according to their multiplicities. The space $L^2(M)$ has an orthonormal basis $u_{\lambda_0}, u_{\lambda_1}, \ldots$ consisting of eigenfunctions of $A$.

For a fixed operator $A$ we understand the space of $\omega$-band-limited functions $E_\omega(A)$ to be the span of all eigenfunctions $u_{\lambda_j}$ such that $\lambda_j \leq \omega$.

Formally, then, there is great freedom in the notion of band-limitedness. However, for the purposes of this article, all these spaces of band-limited functions are essentially equivalent, in the sense that they give rise to the same Besov spaces $B^{p,q}_\alpha(M)$, at least if $\alpha > 0$ and $1 \leq p \leq \infty$ (see Theorem 7.5 below).

The plan of the paper is as follows. In section 2, we review some basic facts about compact homogeneous manifolds. In section 3, we discuss properties of band-limited functions associated with a second-order smooth positive elliptic differential operator $L$. It is shown in particular that if $M$ is equivariantly embedded into Euclidean space then the span of eigenfunctions of the operator $L$ is exactly the set of restrictions to $M$ of all polynomials in the ambient space. In this section we also give several equivalent definitions of Besov spaces on the manifold. In one of the definitions, we use a global modulus of continuity, constructed through use of certain vector fields on $M$. This definition is similar to the original definition of Besov spaces on Euclidean spaces and uses just the notion of smoothness. Later in the article, in Theorems 7.5 and 8.1 we describe the same spaces in terms of approximations by band-limited functions. Thus, as one of the consequences of our results, we obtain a new development of one of the oldest topics of classical harmonic analysis: the relationships between smoothness and rate of approximations by band-limited functions. Specifically, we show that there exists a complete balance between smoothness expressed in terms of modulus of continuity, and the rate of approximation by band-limited functions in all spaces $L^p(M), \ 1 \leq p \leq \infty$, as well as other equivalent definitions of Besov spaces on such manifolds.
In section 4, we describe Plancherel-Polya inequalities (Corollary 4.4) and in section 5 we obtain cubature formulas with desirable properties (Theorem 5.3). In these two sections, we do not use any special properties of homogeneous manifolds or the second-order positive elliptic differential operator $\mathcal{L}$; the results hold for any smooth compact manifold, and for any $\mathcal{L}$.

In section 6, we use the homogeneous manifold structure in an essential way to prove a crucial fact, namely that, for a particular $\mathcal{L}$, the product of two band-limited functions of the same bandwidth $\omega$ is also a band-limited function, with a certain bandwidth $C\omega$, where $C$ is independent of $\omega$. On the sphere, this property is familiar for spherical harmonics; then one may take $\mathcal{L}$ to be the spherical Laplacian, and one may take $C = 2$. This property of spherical harmonics was used crucially in the construction of spherical needlets in [31]. The generalization to homogeneous manifolds is similarly needed in our construction of band-limited Parseval frames. For more general manifolds, it is not clear how to verify this property, or even if it is true. It was conjectured in [29] that this “product” property holds for Laplace-Beltrami operators on analytic compact manifolds. If $M = G/K$ is a homogeneous manifold, we specifically take $\mathcal{L}$ to be the image of the Casimir operator under the differential of the quasiregular representation of $G$ in $L_2(M)$ (see section 2). The operator $-\mathcal{L}$ is a sum of squares of certain vector fields on $M$. In some common cases, such as compact symmetric spaces of rank one and all compact Lie groups, this operator $\mathcal{L}$ coincides with the corresponding Laplace-Beltrami operator.

A number of the results stated, and methods used, in sections 3-6, are from the articles [35]-[43]. In section 7, we review some of the results of [17] - [19], where the Laplace-Beltrami operator was used to construct nearly tight frames, which were then used to characterize Besov spaces. (The Besov space results in [19] used, in addition to results from [17] and [18], methods of Frazier-Jawerth [9] and Seeger-Sogge [45].) We argue that the results of [17] - [19] continue to hold if one uses a general $\mathcal{L}$ in place of the Laplace-Beltrami operator. The arguments of section 7 do not use any special properties of homogeneous manifolds. However, the point is that, if we are on a homogeneous manifold, we are free to use the $\mathcal{L}$ of section 2 in place of the Laplace-Beltrami operator.

Finally, in section 8, by using the results of sections 5 and 6, we construct our Parseval frames on homogeneous manifolds. By using the results of section 7, we show that they are highly localized, and that one can use them to characterize Besov spaces, for the full range of the indices. It is only in the construction of our Parseval frames that we use the results of section 6.

Let us remark that in [32], approximations by polynomials were considered on the sphere, while in this article, we consider approximations by band-limited functions. Although it is known [42] that the span of the eigenfunctions of our operator $\mathcal{L}$ is the same as the span of all polynomials when one equivariantly embeds the manifold, the relation between eigenvalues and degrees of polynomials is unknown (at least in the general case). However, it is easy to verify that for compact two-point homogeneous manifolds, the span of those eigenfunctions whose eigenvalues are not greater than a value $\ell^2$, $\ell \in \mathbb{N}$, is the same as the span of all polynomials of degree at most $\ell$. Thus, on compact two-point homogeneous manifolds, our results about approximations by band-limited functions can be reformulated in terms of approximations by polynomials.

2. Compact homogeneous manifolds

We review some very basic notions of harmonic analysis on compact homogeneous manifolds [24], Ch. II. More details on this subject can be found, for example, in [55], [58].

Let $M$, $dimM = n$, be a compact connected $C^\infty$-manifold. One says that a compact Lie group $G$ effectively acts on $M$ as a group of diffeomorphisms if:

1) every element $g \in G$ can be identified with a diffeomorphism $g : M \rightarrow M$.
of $M$ onto itself and
$$g_1 g_2 \cdot x = g_1 \cdot (g_2 \cdot x), \quad g_1, g_2 \in G, \quad x \in M,$$
where $g_1 g_2$ is the product in $G$ and $g \cdot x$ is the image of $x$ under $g$.

1) The identity $e \in G$ corresponds to the trivial diffeomorphism
$$e \cdot x = x,$$
2) for every $g \in G$, $g \neq e$, there exists a point $x \in M$ such that $g \cdot x \neq x$.

3) for every $g \in G$, $g \neq e$, there exists a point $x \in M$ such that $g \cdot x \neq x$.

A group $G$ acts on $M$ transitively if in addition to 1)- 3) the following property holds:
4) for any two points $x, y \in M$ there exists a diffeomorphism $g \in G$ such that
$$g \cdot x = y.$$ 

A homogeneous compact manifold $M$ is a $C^\infty$-compact manifold on which a compact Lie group $G$ acts transitively. In this case $M$ is necessary of the form $G/K$, where $K$ is a closed subgroup of $G$. The notation $L_p(M)$, $1 \leq p \leq \infty$, is used for the usual Banach spaces $L_p(M, dx)$, $1 \leq p \leq \infty$, where $dx$ is an invariant measure.

Every element $X$ of the (real) Lie algebra of $G$ generates a vector field on $M$, which we will denote by the same letter $X$. Namely, for a smooth function $f$ on $M$ one has 
$$X f(x) = \lim_{t \to 0} \frac{f(\exp tX \cdot x) - f(x)}{t}$$
for every $x \in M$. In the future we will consider on $M$ only such vector fields. The translations along integral curves of such vector fields $X$ on $M$ can be identified with a one-parameter group of diffeomorphisms of $M$, which is usually denoted as $\exp tX$, $-\infty < t < \infty$. At the same time, the one-parameter group $\exp tX$, $-\infty < t < \infty$, can be treated as a strongly continuous one-parameter group of operators acting on the space $L_p(M)$, $1 \leq p \leq \infty$. These operators act on functions according to the formula
$$f \to f(\exp tX \cdot x), \quad t \in \mathbb{R}, \quad f \in L_p(M), \quad x \in M.$$ 

The generator of this one-parameter group will be denoted by $D_{X,p}$, and the group itself will be denoted by
$$e^{t D_{X,p}} f(x) = f(\exp tX \cdot x), \quad t \in \mathbb{R}, \quad f \in L_p(M), \quad x \in M.$$

According to the general theory of one-parameter groups in Banach spaces [3], Ch. I, the operator $D_{X,p}$ is a closed operator on every $L_p(M)$, $1 \leq p \leq \infty$. In order to simplify notation, we will often write $D_X$ in place of $D_{X,p}$.

If $g$ is the Lie algebra of a compact Lie group $G$ then ([24], Ch. II, Proposition 6.6,) it is a direct sum $g = a + [g, g]$, where $a$ is the center of $g$, and $[g, g]$ is a semi-simple algebra. Let $Q$ be a positive-definite quadratic form on $g$ which, on $[g, g]$, is opposite to the Killing form. Let $X_1, ..., X_d$ be a basis of $g$, which is orthonormal with respect to $Q$. Since the form $Q$ is $Ad(G)$-invariant, the operator
$$-X_i^2 - X_2^2 - ... - X_d^2, \quad d = \dim G$$
is a bi-invariant operator on $G$. This implies in particular that the corresponding operator on $L_p(M)$, $1 \leq p \leq \infty$, commutes with all operators $D_j = D_{X_j}$. This operator $\mathcal{L}$, which is usually called the Laplace operator, is elliptic, and is involved in most of the constructions and results of our paper. However, as we discussed in the introduction, in many of the results prior to section 6, one could use other second order elliptic differential operators.
In the rest of the paper, the notation $\mathcal{D} = \{D_1, ..., D_d\}$, $d = \text{dim } G$, will be used for the differential operators on $L_p(M), 1 \leq p \leq \infty$, which are involved in the formula (2.2).

In some situations the operator $\mathcal{L}$ is essentially the Laplace-Beltrami operator $(-d^*d)$ of an invariant metric on $M$. This happens for example in the following cases.

1) If $M$ is a $d$-dimensional torus, and $-\mathcal{L}$ is the sum of squares of partial derivatives.

2) If the manifold $M$ is itself a group $G$ which is compact and semi-simple, then $\mathcal{L}$ is exactly the Laplace-Beltrami operator of an invariant metric on $G$ ([25], Ch. II, Exercise A4).

3) If $M = G/K$ is a compact symmetric space of rank one, then the operator $\mathcal{L}$ is proportional to the Laplace-Beltrami operator of an invariant metric on $G/K$. This follows from the fact that, in the rank one case, every second-order operator which commutes with all isometries $x \rightarrow g \cdot x$, $x \in M$, $g \in G$, is proportional to the Laplace-Beltrami operator ([25], Ch. II, Theorem 4.11).

Let us stress one more time that in the present paper we use only the properties that the operator $\mathcal{L}$ has the form (2.2) and commutes with all isometries $g : M \rightarrow g \cdot M$, $g \in G$, of $M$, and we do not explore its relation to the Laplace-Beltrami operator of the invariant metric.

Note that if $M = G/K$ is a compact symmetric space, then the number $d = \text{dim } G$ of operators in the formula (2.2) can be strictly larger than the dimension $n = \text{dim } M$. For example, on a two-dimensional sphere $S^2$ the Laplace-Beltrami operator $L_{S^2}$ can be written as

$$L_{S^2} = -(D_1^2 + D_2^2 + D_3^2),$$

where $D_i, i = 1, 2, 3$, generates a rotation in $\mathbb{R}^3$ around the coordinate axis $x_i$:

$$D_i = x_j \partial_k - x_k \partial_j,$$

where $j, k \neq i$.

3. Function spaces on compact homogeneous manifolds

The operator $\mathcal{L}$ is an elliptic differential operator which is defined on $C^\infty(M)$, and we will use the same notation $\mathcal{L}$ for its closure from $C^\infty(M)$ in $L_p(M), 1 \leq p \leq \infty$. In the case $p = 2$ this closure is a self-adjoint positive definite operator on the space $L_2(M)$. The spectrum of this operator is discrete and goes to infinity $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$. Let $u_0, u_1, u_2, ...$ be a corresponding complete system of real-valued orthonormal eigenfuctions, and let $E_\omega(\mathcal{L}), \omega > 0$, be the span of all eigenfunctions of $\mathcal{L}$, whose corresponding eigenvalues are not greater than $\omega$.

We say that a function $f \in L_p(M), 1 \leq p \leq \infty$, belongs to the Bernstein space $B_\omega^p(\mathcal{D}), \mathcal{D} = \{D_1, ..., D_d\}, d = \text{dim } G$, if and only if for every $1 \leq i_1, i_2 \leq d$, the following Bernstein inequality holds:

$$\|D_{i_1}...D_{i_k} f\|_p \leq \omega^k \|f\|_p, \quad k \in \mathbb{N}, \quad 1 \leq p \leq \infty.$$  (3.1)

We say that a function $f \in L_2(M)$ belongs to the Bernstein space $B_\omega^p(\mathcal{L})$, if and only if for every $k \in \mathbb{N}$, the following Bernstein inequality holds:

$$\|\mathcal{L}^k f\|_2 \leq \omega^k \|f\|_2, \quad k \in \mathbb{N}.$$  (3.2)

Since $\mathcal{L}$ on the space $L_2(M)$ is self-adjoint and positive-definite, there exists a unique positive square root $\mathcal{L}^{1/2}$. Thus the last inequality is equivalent to the inequality

$$\|\mathcal{L}^{k/2} f\|_2 \leq \omega^{k/2} \|f\|_2, \quad k \in \mathbb{N}.$$  (3.3)

It was shown in [22] that the Bernstein spaces $B_\omega^p(\mathcal{D}), B_\omega^p(\mathcal{L})$ are linear spaces. Moreover, it was shown in the same paper that the following equality holds:

$$B_\omega^p(\mathcal{D}) = B_\omega^p(\mathcal{D}) = B_\omega^p(\mathcal{D}), \quad \mathcal{D} = \{D_1, ..., D_d\}, \quad d = \text{dim } G,$$

which means that if the Bernstein-type inequalities (3.1) are satisfied for a single $1 \leq p \leq \infty$, then they are satisfied for all $1 \leq p \leq \infty$. 
The following embeddings were also proved in [42] which describe relations between Bernstein spaces $\mathcal{B}_\omega(\mathbb{D})$, $\mathbb{D} = \{D_1, ..., D_d\}$, $d = \dim G$, and the spaces $\mathcal{E}_\lambda(\mathcal{L})$ for $-\mathcal{L} = D_1^2 + D_2^2 + ... + D_d^2$, $d = \dim G$:

$$\mathcal{E}_\omega(\mathcal{L}) \subset \mathcal{B}_\omega(\mathbb{D}), \quad d = \dim G, \quad \omega > 0.$$  

(3.2)

$$\mathcal{B}_\omega(\mathbb{D}) \subset \mathcal{E}_{\omega^2}(\mathcal{L}) \subset \mathcal{B}_{\omega^2}(\mathbb{D}), \quad d = \dim G, \quad \omega > 0.$$  

(3.3)

These embeddings obviously imply the equality

$$\bigcup_{\omega > 0} \mathcal{B}_\omega(\mathbb{D}) = \bigcup_j \mathcal{E}_{\lambda_j}(\mathcal{L}),$$

which means that a function on $M$ satisfies a Bernstein inequality (3.1) in a norm of $L_p(M)$, $1 \leq p \leq \infty$, if and only if it is a linear combination of eigenfunctions of $\mathcal{L}$. As a consequence we have the following Bernstein-Nikolski inequality: for every $\varphi \in \mathcal{E}_\omega(\mathcal{L})$ and $1 \leq p \leq q \leq \infty$,

$$\|\mathcal{L}^k \varphi\|_q \leq C(M)\omega^{2k + \frac{d}{2} - \frac{k}{q}}\|\varphi\|_p, \quad k \in \mathbb{N}, \quad n = \dim M, \quad d = \dim G,$$

(3.4)

for a certain constant $C(M)$ which depends only on the manifold.

It is known ([58], Ch. IV) that every compact Lie group can be considered to be a closed subgroup of the orthogonal group $O(\mathbb{R}^N)$ of some Euclidean space $\mathbb{R}^N$. For a compact symmetric space $M = G/K$, where $G$ is a compact Lie group, we can identify $M$ with the orbit of a unit vector $v \in \mathbb{R}^N$ under the action of a subgroup of the orthogonal group $O(\mathbb{R}^N)$ in $\mathbb{R}^N$. In this case $K$ will be the stationary group of $v$. Such an embedding of $M$ into $\mathbb{R}^N$ is called equivariant.

We choose an orthonormal basis in $\mathbb{R}^N$ for which the first vector is the vector $v$: $e_1 = v, e_2, ..., e_N$. Let $P_m(M)$ be the space of restrictions to $M$ of all polynomials in $\mathbb{R}^N$ of degree $m$. This space is closed in the norm of $L_p(M)$, $1 \leq p \leq \infty$, which is constructed with respect to the $G$-invariant measure on $M$.

Let $T$ be the quasi-regular representation of $G$ in the space $L_p(M)$, $1 \leq p \leq \infty$. In other words, if $f \in L_p(M)$, $g \in G$, $x \in M$, then

$$\text{(T(g)f)}(x) = f(g^{-1}x).$$

(3.5)

The Lie algebra $\mathfrak{g}$ of the group $G$ is formed by those $N \times N$ skew-symmetric matrices $X$ for which $\exp tX \in G$ for all $t \in \mathbb{R}$. The scalar product in $\mathfrak{g}$ is given by the formula

$$\langle X_1, X_2 \rangle = \frac{1}{2} \text{tr}(X_1 X_2^t) = -\frac{1}{2} \text{tr}(X_1 X_2), \quad X_1, X_2 \in \mathfrak{g}.$$  

Let $X_1, X_2, ..., X_d$ be an orthonormal basis of $\mathfrak{g}$, $\dim \mathfrak{g} = d$, and $D_1, D_2, ..., D_d$ be the corresponding infinitesimal operators of the quasi-regular representation of $G$ in $L_p(M)$, $1 \leq p \leq \infty$.

The following relations were proved in [42]:

$$P_m(M) \subset B_m(\mathbb{D}) \subset E_{m^2}(\mathcal{L}) \subset B_{m^2}(\mathbb{D}), \quad d = \dim G, \quad m \in \mathbb{N},$$  

(3.6)

and

$$\bigcup_m P_m(M) = \bigcup_\omega \mathcal{B}_\omega(\mathbb{D}) = \bigcup_j \mathcal{E}_{\lambda_j}(\mathcal{L}), \quad m \in \mathbb{N},$$

(3.7)

where $P_m(M)$ is the space of restrictions to $M$ of polynomials of degree at most $m$.

Let $B(x, r)$ be a metric ball on $M$ whose center is $x$ and radius is $r$. The following important Lemma can be found in [39], [40].

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3 We would like to point out that some of the indices in our formulas (3.3), (3.4) are different from the indices in the corresponding formulas in [42].
Lemma 3.1. For any Riemannian manifold of bounded geometry $M$ there exists a natural number $N_M$, such that for any sufficiently small $\rho > 0$ there exists a set of points $\{y_\nu\}$ such that:

1. the balls $B(y_\nu, \rho/4)$ are disjoint,
2. the balls $B(y_\nu, \rho/2)$ form a cover of $M$,
3. the multiplicity of the cover by balls $B(y_\nu, \rho)$ is not greater than $N_M$.

Definition 3.2. Any set of points $M_\rho = \{y_\nu\}$ which is as described in Lemma 3.1 will be called a metric $\rho$-lattice.

To define Sobolev spaces, we fix a cover $B = \{B(y_\nu, r_0)\}$ of $M$ of finite multiplicity $N(M)$ (see Lemma 3.1)

\begin{equation}
M = \bigcup B(y_\nu, r_0),
\end{equation}

where $B(y_\nu, r_0)$ is a ball centered at $y_\nu \in M$ of radius $r_0 \leq \rho_M$, contained in a coordinate chart, and consider a fixed partition of unity $\Psi = \{\psi_\nu\}$ subordinate to this cover. The Sobolev spaces $W^k_p(M), k \in \mathbb{N}, 1 \leq p < \infty$, are introduced as the completion of $C^\infty(M)$ with respect to the norm

\begin{equation}
\|f\|_{W^k_p(M)} = \left( \sum_{\nu} \|\psi_\nu f\|^p_{W^k_p(B(y_\nu, r_0))} \right)^{1/p}.
\end{equation}

Any two such norms are equivalent.

Now we turn to Besov spaces on $M$. Suppose that $-\infty < \alpha < \infty$ and $0 < p, q \leq \infty$. We use the notation for inhomogeneous Besov spaces $B^\alpha_{p,q}$ on $\mathbb{R}^n$ from [3]. Thus, on $\mathbb{R}^n$, one takes any $\Phi \in S$ supported in the closed unit ball, which does not vanish anywhere in the ball of radius $5/6$ centered at $0$. One also takes functions $\varphi_\nu \in S$ for $\nu \geq 1$, supported in the annulus $\{\xi : 2^{-\nu} - 1 \leq |\xi| \leq 2^{\nu+1}\}$, satisfying $|\varphi_\nu(\xi)| \geq c > 0$ for $3/5 \leq 2^{-\nu}|\xi| \leq 5/3$ and also $|\partial^\gamma \varphi_\nu| \leq c_\gamma 2^{-\nu|\gamma|}$ for every multiindex $\gamma$. The Besov space $B^\alpha_{p,q}(\mathbb{R}^n)$ is then the space of $F \in S'(\mathbb{R}^n)$ such that

\begin{equation}
\|F\|_{B^\alpha_{p,q}} = \|\hat{\Phi} * F\|_{L_p} + \left( \sum_{\nu=0}^\infty (2^{\nu\alpha} \|\hat{\varphi_\nu} * F\|_{L_p})^q \right)^{1/q} < \infty.
\end{equation}

(Here we use the usual conventions if $p$ or $q$ is $\infty$. The definition of $B^\alpha_{p,q}(\mathbb{R}^n)$ is independent of the choices of $\Phi, \varphi_\nu$ ([31], page 49). Moreover, $B^\alpha_{p,q}(\mathbb{R}^n)$ is a quasi-Banach space, and the inclusion $B^\alpha_{p,q} \subseteq S'$ is continuous ([50], page 48). In particular the space $B^\alpha_{\infty,\infty}(\mathbb{R}^n) = C^\alpha(\mathbb{R}^n)$, which is the usual Hölder space if $0 < \alpha < 1$, or in general a Hölder-Zygmund space for $\alpha > 0$ ([50], page 51). It is not hard to see, by using the definition and the Fourier transform, that if $K \subseteq \mathbb{R}^n$ is compact, and if $N$ is sufficiently large, then

\begin{equation}
\{F \in C^N : \text{supp} F \subseteq K\} \subseteq B^\alpha_{p,q}
\end{equation}

where the inclusion map is continuous if we regard the left side as a subspace of $C^N$.

If $\eta : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism which equals the identity outside a compact set, then one can define $F \circ \eta$ for $F \in B^\alpha_{p,q}(\mathbb{R}^n)$, and the map $F \to F \circ \eta$ is bounded on the Besov spaces ([50], chapter 2.10). These facts then enable one to define $B^\alpha_{p,q}(M)$: let $(W_i, \chi_i)$ be a finite atlas on $M$ with charts $\chi_i$ mapping $W_i$ into the unit ball on $\mathbb{R}^n$, and suppose $\{\zeta_i\}$ is a partition of unity subordinate to the $W_i$. Then one defines $B^\alpha_{p,q}(M)$ to be the space of distributions $F$ on $M$ for which

\begin{equation}
\|F\|_{B^\alpha_{p,q}(M)} = \sum_i \|\zeta_i F \circ \chi_i^{-1}\|_{B^\alpha_{p,q}(\mathbb{R}^n)} < \infty.
\end{equation}
This definition does not depend on the choice of charts or partition of unity \((\ref{22})\).

It was also shown in \([31], [32]\) that the Besov space \(B^\alpha_p(M)\) is exactly the interpolation space \((L_p(M), W^r_p(M))^{K}_{\alpha/r,q}\) \(0 < \alpha < r \in \mathbb{N}, 1 \leq p, q \leq \infty\), where \(K\) is the Peetre interpolation functor.

Now we are going to describe Besov spaces in different terms. We consider the system of vector fields \(D = \{D_1, ..., D_d\}\), \(d = \text{dim} G\), on \(M = G/K\), which was described above. Since the vector fields \(D = \{D_1, ..., D_d\}\) generate the tangent space at every point of \(M\), and \(M\) is compact, it is clear that the Sobolev norm \(\ref{31}\) is equivalent to the norm

\[
\|f\|_p + \sum_{j=1}^{k} \sum_{1 \leq i_1, ..., i_j \leq d} \|D_{i_1}...D_{i_j}f\|_p, \ 1 \leq p \leq \infty.
\]

Using the closed graph theorem and the fact that each \(D_i\) is a closed operator in \(L_p(M)\), \(1 \leq p \leq \infty\), it is easy to show that the norm \(\ref{32}\) is equivalent to the norm

\[
\|f\|_{k,p} = \|f\|_p + \sum_{1 \leq i_1, ..., i_k \leq d} \|D_{i_1}...D_{i_k}f\|_p, \ 1 \leq p \leq \infty.
\]

For the same operators as above \((D_1, ..., D_d, \ d = \text{dim} G)\), let \(T_1, ..., T_d\) be the corresponding one-parameter groups of translation along integral curves of the corresponding vector fields i.e.

\[
T_j(\tau)f(x) = f(\exp \tau X_j \cdot x), \ x \in M, \tau \in \mathbb{R}, f \in L_2(M);
\]

here \(\exp \tau X_j \cdot x\) is the integral curve of the vector field \(X_j\) which passes through the point \(x \in M\). The modulus of continuity is introduced as

\[
\Omega^r_p(s,f) = \sum_{1 \leq j_1, ..., j_r \leq d} \sup_{0 \leq \tau_{j_1} \leq s} \sup_{0 \leq \tau_{j_r} \leq s} \|T_{j_1}(\tau_{j_1}) - I \)...\(T_{j_r}(\tau_{j_r}) - I\) \(f\|_{L_p(M)},
\]

where \(f \in L_p(M)\), \(r \in \mathbb{N}\), and \(I\) is the identity operator in \(L_p(M)\). We consider the space of all functions in \(L_p(M)\) for which the following norm is finite:

\[
\|f\|_{L_p(M)} + \left(\int_{0}^{\infty} (s^{-\alpha} \Omega^r_p(s,f))^q \frac{ds}{s}\right)^{1/q}, \ 1 \leq p, q < \infty,
\]

with the usual modifications for \(q = \infty\).

The following Theorem follows from general results of the second author about interpolation in spaces of representations of Lie groups \([35]-[38]\):

**Theorem 3.3.** The norm of the Besov space \(B^\alpha_p(M) = (L_p(M), W^r_p(M))^{K}_{\alpha/r,q}\) \(0 < \alpha < r \in \mathbb{N}, 1 \leq p, q \leq \infty\), is equivalent to the norm \(\ref{31}\). Moreover, the norm \(\ref{16}\) is equivalent to the norm

\[
\|f\|_{W^\alpha_p(M)} + \sum_{1 \leq j_1, ..., j_{[\alpha]} \leq d} \left(\int_{0}^{\infty} (s^{[\alpha]} - \alpha) \Omega^1_p(s, D_{j_1}...D_{j_{[\alpha]}}, f))^q \frac{ds}{s}\right)^{1/q},
\]

if \(\alpha\) is not integer \(([\alpha]\) is its integer part). If \(\alpha = k \in \mathbb{N}\) is an integer then the norm \(\ref{16}\) is equivalent to the norm (Zygmund condition)

\[
\|f\|_{W^{k-1}_p(M)} + \sum_{1 \leq j_1, ..., j_{k-1} \leq d} \left(\int_{0}^{\infty} s^{-1} \Omega^2_p(s, D_{j_1}...D_{j_{k-1}}, f))^q \frac{ds}{s}\right)^{1/q}.
\]
When $p = 2$ the first of these norms can be changed to

$$
\|f\|_{H^{\alpha}(M)} + \sum_{1 \leq j_1, \ldots, j_{\alpha} \leq d} \left( \int_0^\infty \left( s^{\alpha - \alpha \Omega_2^{1}(s, \mathcal{L}^{\alpha} f)} \right)^q \frac{ds}{s} \right)^{1/q}
$$

and the second to

$$
\|f\|_{H^{k-1}(M)} + \sum_{1 \leq j_1, \ldots, j_{k-1} \leq d} \left( \int_0^\infty \left( s^{-1} \Omega_2^{2}(s, \mathcal{L}^{(k-1)} f) \right)^q \frac{ds}{s} \right)^{1/q}. 
$$

For a function $f \in L^2(M)$ we introduce a notion of best approximation

$$
\mathcal{E}(f, \omega) = \inf_{g \in \mathcal{E}_\omega(\mathcal{L})} \|f - g\|_2 = \left( \sum_{\lambda_j \geq \omega} c_j(f)^2 \right)^{1/2},
$$

where $c_j = \langle f, u_{\lambda_j} \rangle$ are the Fourier coefficients of $f$.

A description of Besov spaces $B^\alpha_{q,1}(M)$, $\alpha > 0$, $1 \leq q \leq \infty$, in terms of the best approximation $\mathcal{E}(f, \omega)$ was given in [43], Theorems 1.1 and 1.2. We will obtain a generalization of these results in Theorem 7.5 below.

4. Plancherel-Polya (=Marcinkiewicz-Zygmund) inequalities

In this section, we again consider a compact homogeneous Riemannian manifold $M$, and the elliptic self-adjoint positive definite operator $\mathcal{L}$ on $L^2(M)$, which was introduced in [22]. However, the results of this section hold for general $M$ and $\mathcal{L}$ (if at least $M$ is smooth, compact and $\mathcal{L}$ is a positive elliptic self-adjoint second-order differential operator on $M$).

Since the operator $\mathcal{L}$ is of order two, the dimension $\mathcal{N}_\omega$ of the space $\mathcal{E}_\omega(\mathcal{L})$ is given asymptotically by Weyl’s formula [47]

$$
\mathcal{N}_\omega(M) \asymp C(M) \omega^{n/2},
$$

where $n = \text{dim} M$.

The next two theorems were proved in [39], [41], for a Laplace-Beltrami operator on a Riemannian manifold of bounded geometry, but their proofs go through for any $C^\infty$-bounded uniformly elliptic self-adjoint positive definite differential operator on $M$. In what follows the notation $n = \text{dim} M$ is used.

**Theorem 4.1.** There exist constants $C_1 = C_1(M, \mathcal{L}) > 0$ and $\rho_0(M, \mathcal{L}) > 0$, such that for any natural number $m > n/2$, any $0 < \rho < \rho_0(M, \mathcal{L})$, and any $\rho$-lattice $M_\rho = \{x_k\}$, the following inequality holds:

$$
\left( \sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} \leq C_1 \rho^{-n/2} \|f\|_{H^m(M)},
$$

for all $f \in H^m(M)$, $m > n/2$, $m \in \mathbb{N}$.

**Theorem 4.2.** There exist constants $C_2 = C_2(M, \mathcal{L}) > 0$, and $\rho_0(M, \mathcal{L}) > 0$, such that for any natural $m > n/2$, any $0 < \rho < \rho_0(M, \mathcal{L})$, and any $\rho$-lattice $M_\rho = \{x_k\}$ the following inequality holds
Corollary 4.4. There exist constants $c_1 = c_1(M, L) > 0$, $c_2 = c_2(M, L) > 0$, and $c_0 = c_0(M, L) > 0$, such that for any $\omega > 0$, and for every metric $\rho$-lattice $M_\rho = \{x_k\}$ with $\rho = c_0 \omega^{-1/2}$, the following Plancherel-Polya inequalities hold:

\[(4.2) \quad \|f\|_{L^2(M)} \leq C_2 \left\{ \rho^{n/2} \left( \sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2} + \rho^{2m} \|L^m f\| \right\}, \quad m \in \mathbb{N}, \quad m > n/2.\]

Using the constant $C_2(M, L)$ from this Theorem, we define another constant

\[(4.3) \quad c_0 = c_0(M, L) = (2C_2(M, L))^{-1/2m_0},\]

where $m_0 = 1 + n/2$, $n = \dim M$.

The previous Theorem and the Bernstein inequality imply the following Plancherel-Polya-type inequalities. Such inequalities are also known as Marcinkiewicz-Zygmund inequalities.

Theorem 4.3. There exists a constant $c_2 = c_2(M, L)$ such that for any $\omega > 0$, and for every metric $\rho$-lattice $M_\rho = \{x_k\}$ with $\rho = c_0 \omega^{-1/2}$, the following inequalities hold:

\[(4.4) \quad \rho^{-n/2} \|f\|_{L^2(M)} \leq c_2 \left( \sum_{k} |f(x_k)|^2 \right)^{1/2},\]

for all $f \in E_\omega(L)$ and $c_0$ defined in (4.3). Moreover, for the same lattice there exists a constant $c_1 = c_1(M, L, \omega)$ such that

\[(4.5) \quad c_1 \left( \sum_{k} |f(x_k)|^2 \right)^{1/2} \leq \rho^{-n/2} \|f\|_{L^2(M)}\]

Proof. Indeed, if $f \in E_\omega(M)$, and $\rho$ in (1.2) is given by $\rho = c_0 \omega^{-1/2}$ where $c_0$ was defined in (4.3), then by the Bernstein inequality

\[ C_2 \rho^{2m_0} \|L^m f\| \leq C_2 (\rho^2 \omega^{-1})^{m_0} \|f\| = \frac{1}{2} \|f\|.\]

The inequality (4.2) now implies that

\[(4.6) \quad \rho^{-n/2} \|f\|_2 \leq c_2 \left( \sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2},\]

with $c_2 = 2C_2(M, L)$ and $C_2(M, L)$ is the same as in (1.2). To prove (4.5) we apply elliptic regularity of $L$ to obtain

\[(4.7) \quad \|f\|_{L^2(M)} \leq C(M, L) \left( \|f\| + \|L^m f\| \right)\]

and then the Bernstein inequality gives

\[(4.8) \quad \|f\|_{L^2(M)} \leq C(M, L)(1 + \omega^{m/2})\|f\|.

By choosing $m_0 = 1 + n/2$ for $m$ and using Theorem 4.1 we obtain (4.5) with

\[c_1 = \{C_1(M, L)C(M, L)(1 + \omega^{m_0/2})\}^{-1}.\]
\begin{equation}
(4.9) \quad c_1 \left( \sum_k |f(x_k)|^2 \right)^{1/2} \leq \rho^{-n/2} \|f\|_{L^2(M)} \leq c_2 \left( \sum_k |f(x_k)|^2 \right)^{1/2},
\end{equation}

for all \( f \in E_\omega(L) \) and \( n = \dim M \).

The following Theorem shows that our lattices (appearing in the previous Theorems) always produce sampling sets with essentially the optimal number of sampling points (see also [40, 43]).

**Theorem 4.5.** If the constant \( c_0(M, L) > 0 \) is the same as above, then for any \( \omega > 0 \) and \( \rho = c_0 \omega^{-1/2} \), there exist \( C_1(M, L), C_2(M, L) \) such that the number of points in any \( \rho \)-lattice \( M_\rho \) satisfies the following inequalities

\begin{equation}
(4.10) \quad C_1 \omega^{n/2} \leq |M_\rho| \leq C_2 \omega^{n/2},
\end{equation}

**Proof.** According to the definition of a lattice \( M_\rho \) we have

\[
|M_\rho| \inf_{x \in M} Vol(B(x, \rho/4)) \leq Vol(M) \leq |M_\rho| \sup_{x \in M} Vol(B(x, \rho/2))
\]

or

\[
\frac{Vol(M)}{\sup_{x \in M} Vol(B(x, \rho/2))} \leq |M_\rho| \leq \frac{Vol(M)}{\inf_{x \in M} Vol(B(x, \rho/4))}.
\]

Since for certain \( c_1(M), c_2(M) \), all \( x \in M \) and all sufficiently small \( \rho > 0 \), one has a double inequality

\[
c_1(M) \rho^n \leq Vol(B(x, \rho)) \leq c_2(M) \rho^n,
\]

and since \( \rho = c_0 \omega^{-1/2} \), we obtain that for certain \( C_1(M, L), C_2(M, L) \) and all \( \omega > 0 \)

\begin{equation}
(4.11) \quad C_1 \omega^{n/2} \leq |M_\rho| \leq C_2 \omega^{n/2}.
\end{equation}

Since the inequalities (4.10) are in an agreement with Weyl’s formula (4.1), the Theorem shows that if \( \omega > 0 \) is large enough, every uniqueness set \( M_\rho \) for \( E_\omega(L) \) contains essentially the ”correct” number of points.

5. Cubature Formulas

Again we work on a compact homogeneous Riemannian manifold \( M \), and use the operator \( L \) of (2.2). However, the results of this section hold for general \( M \) and \( L \) (if at least \( M \) is smooth and compact, and \( L \) is a positive elliptic self-adjoint second-order differential operator on \( M \)).

Corollary 4.3 shows that if \( \vartheta_k \) is the orthogonal projection of the Dirac measure \( \delta_{x_k} \) on the space \( E_\omega(L) \) (in a Hilbert space \( H^{-n/2-\varepsilon}(M) \), \( \varepsilon > 0 \), which can be defined as the domain of the operator \( L^{-n/4-\varepsilon/2} \)) then there exist constants \( c_1 = c_1(M, L, \omega) > 0, c_2 = c_2(M\text{mat}, L) > 0 \), such that the following frame inequality holds

\begin{equation}
(5.1) \quad c_1 \left( \sum_k |\langle f, \vartheta_k \rangle|^2 \right)^{1/2} \leq \rho^{-n/2} \|f\|_{L^2(M)} \leq c_2 \left( \sum_k |\langle f, \vartheta_k \rangle|^2 \right)^{1/2},
\end{equation}

for all \( f \in E_\omega(L) \).

Let \( M_\rho = \{x_k\}, \ k = 1, ..., N(M_\rho), \) be a \( \rho \)-lattice on \( M \) (see Lemma 3.1). We construct the Voronoi partition of \( M \) associated to the set \( M_\rho = \{x_k\}, \ k = 1, ..., N(M_\rho) \). Elements of this partition will be denoted as \( M_{j, \rho} \). Let us recall that the distance from each point in \( M_{j, \rho} \) to \( x_j \) is less than or equal to its distance to any other point of the family \( M_\rho = \{x_k\}, \ k = 1, ..., N(M_\rho) \).
Some properties of this cover of \( M \) are summarized in the following Lemma, which follows easily from the definitions.

**Lemma 5.1.** The sets \( M_{k,\rho} \), \( k = 1, \ldots, N(\rho) \), have the following properties:

1) they are measurable;
2) they are disjoint;
3) they form a cover of \( M \);
4) there exist positive \( a_1 \), \( a_2 \), independent of \( \rho \) and the lattice \( M_{\rho} = \{x_k\} \), such that

\[
 a_1 \rho^n \leq \mu(M_{k,\rho}) \leq a_2 \rho^n. \tag{5.2}
\]

Our next goal is to prove the following fact.

**Theorem 5.2.** Say \( \rho > 0 \), and let \( \{M_{k,\rho}\} \) be the disjoint cover of \( M \) which is associated with a \( \rho \)-lattice \( M_{\rho} \). If \( \rho \) is sufficiently small then for any sufficiently large \( K \in \mathbb{N} \) there exists a \( C(K) > 0 \) such that for all smooth functions \( f \) the following inequality holds:

\[
 \left| \sum_{\nu} \sum_{x_k \in M_{\rho}} \psi_{\nu} f(x_k) \mu(M_{k,\rho}) - \int_M f(x)dx \right| \leq C(K) \rho^{n/2+1/\beta} \|(I + L)^{\beta/2} f\|_2, \tag{5.3}
\]

where \( C(K) \) is independent of \( \rho \) and the \( \rho \)-lattice \( M_{\rho} \).

**Proof.** We start with the Taylor series

\[
 \psi_{\nu} f(y) - \psi_{\nu} f(x_k) = \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha (\psi_{\nu} f)(x_k)(y - x_k)^\alpha + \sum_{|\alpha| = m} \frac{1}{\alpha!} \int_0^1 t^{m-1} \partial^\alpha \psi_{\nu} f(x_k + t\theta) \partial^\alpha dt, \tag{5.4}
\]

where \( f \in C^\infty(\mathbb{R}^d) \), \( y \in B(x_k, \rho/2) \), \( x = (x^{(1)}, \ldots, x^{(d)}), \ y = (y^{(1)}, \ldots, y^{(d)}), \ \alpha = (\alpha_1, \ldots, \alpha_d), \ (x - y)^\alpha = (x^{(1)} - y^{(1)})^{\alpha_1} \ldots (x^{(d)} - y^{(d)})^{\alpha_d}, \ \tau = \|x - x_i\|, \ \theta = (x - x_i)/\tau \).

We are going to use the following inequality, which easily follows from Lemma 6.19 in [1], and which is essentially the Sobolev imbedding theorem:

\[
 |(\psi_{\nu} f)(x_k)| \leq C_{n,m} \sum_{0 \leq j \leq m} |\rho|^{-j/p} \|(\psi_{\nu} f)\|_{W^j_{2}(B(x_k, \rho))}, \quad 1 \leq p \leq \infty, \tag{5.5}
\]

where \( m > n/p \), and the functions \( \{\psi_{\nu}\} \) form the partition of unity which we used to define the Sobolev norm in [3.9]. Using (5.5) for \( p = 1 \) we obtain that the following inequality

\[
 \left| \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha (\psi_{\nu} f)(x_k)(y - x_k)^\alpha \right| \leq C(n,m) |\rho|^{j/n} \sum_{1 \leq |\alpha| \leq m} \sum_{0 \leq j \leq m} |\rho|^{-j-n} \|(\psi_{\nu} f)\|_{L^1(B(x_k, \rho))}, \quad m > n, \tag{5.6}
\]

for some \( C(n,m) \geq 0 \). Since, by the Schwarz inequality,

\[
 \|(\partial^\alpha (\psi_{\nu} f))\|_{L^1(B(x_k, \rho))} \leq C(n) \rho^{n/2} \|(\partial^\alpha (\psi_{\nu} f))\|_{L^2(B(x_k, \rho))}, \tag{5.7}
\]

some properties of this cover of \( M \) are summarized in the following Lemma, which follows easily from the definitions.
we obtain the following estimate, which holds for small $\rho$:

(5.8) \[ \sup_{y \in B(x_k, \rho)} \left| \sum_{1 \leq |\alpha| \leq m-1} \frac{1}{\alpha!} \partial^\alpha (\psi_x f)(x_k)(x_k - y)^\alpha \right| \leq C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{\frac{1}{2} - n/2} \| \partial^\beta (\psi_x f) \|_{L^2(B(x_k, \rho))}, \quad m > n. \]

Next, using the Schwarz inequality and the assumption that $m > n = \dim \mathbf{M}$, $|\alpha| = m$, we obtain
\[
\left| \int_{B(x_k, \rho)} t^{m-1} \partial^\alpha \psi_x f(x_k + t\theta) \theta^\alpha dt \right| \leq \int_0^\tau t^{m-n/2 - 1/2} \| t^{n/2 - 1/2} \partial^\alpha \psi_x f(x_k + t\theta) \| dt \leq C \left( \int_0^\tau t^{2m-n-1} \right)^{1/2} \left( \int_0^\tau t^{n-1} | \partial^\alpha \psi_x f(x_k + t\theta) |^2 dt \right)^{1/2} \leq C \tau^{m-n/2} \left( \int_0^\tau t^{n-1} | \partial^\alpha \psi_x f(x_k + t\theta) |^2 dt \right)^{1/2}, \quad m > n.
\]

We square this inequality, and integrate both sides of it over the ball $B(x_k, \rho/2)$, using the spherical coordinate system $(\tau, \theta)$. We find
\[
\int_{B(x_k, \rho)} \left| \psi_x f(y) - \psi_x f(x_k) \right|^2 d\tau d\theta \leq \int_0^\rho \int_0^2 t^{2m-n} \left| \partial^\alpha (\psi_x f)(x_k + t\theta) \theta^\alpha dt \right|^2 \tau^{n-1} d\theta d\tau \leq C(m, n) \int_0^\rho \int_0^2 t^{n-1} \left( \int_0^\tau t^{m-1} | \partial^\alpha (\psi_x f)(x_k + t\theta) \theta^\alpha dt \right)^2 \tau^{n-1} d\theta d\tau \leq C(m, n) \int_0^\rho \int_0^2 \tau^{m-n/2} \left( \int_0^\tau t^{n-1} | \partial^\alpha (\psi_x f)(x_k + t\theta) |^2 \tau^{n-1} d\theta d\tau \right) dt \leq C(m, n, \rho)^{2|\alpha|} \int_0^\rho | \partial^\alpha (\psi_x f) \|_{L^2(B(x_k, \rho))}^2 dt,
\]

where $\tau = \| x - x_k \| \leq \rho/2$, $m = |\alpha| > n$. Let $\{M_{k, \rho}\}$ be the Voronoi cover of $\mathbf{M}$ which is associated with a $\rho$-lattice $M_{\rho}$ (see Lemma 5.1). From here we obtain

(5.9) \[ \int_{B(x_k, \rho)} | \psi_x f(y) - \psi_x f(x_k) | dx \leq C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{\frac{1}{2} + n/2} \| \partial^\beta (\psi_x f) \|_{L^2(B(x_k, \rho))} + \sum_{|\alpha| = m} \frac{1}{\alpha!} \int_{B(x_k, \rho)} \left| \int_0^\tau t^{m-1} \partial^\alpha \psi_x f(x_k + t\theta) \theta^\alpha dt \right| \leq C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{\frac{1}{2} + n/2} \| \partial^\beta (\psi_x f) \|_{L^2(B(x_k, \rho))} + \rho^{n/2} \sum_{|\alpha| = m} \frac{1}{\alpha!} \left( \int_0^\tau t^{m-1} \partial^\alpha \psi_x f(x_k + t\theta) \theta^\alpha dt \right)^{1/2} \tau^{n-1} \tau^{1/2} \leq C(n, m) \sum_{1 \leq |\beta| \leq 2m} \rho^{\frac{1}{2} + n/2} \| \partial^\beta (\psi_x f) \|_{L^2(B(x_k, \rho))}.
\]
Next, we have the following inequalities
\[
\sum_\nu \sum_{x_k \in M_\rho} \psi_\nu f(x_k) \mu M_{k,\rho} - \int_M f(x) dx = \\
- \sum_\nu \left( \sum_k \int_{M_{k,\rho}} \psi_\nu f(x) dx - \sum_k \psi_\nu f(x_k) \mu M_{k,\rho} \right) \leq \\
\sum_\nu \sum_k \left| \int_{M_{k,\rho}} \psi_\nu f(x) dx - \psi_\nu f(x_k) \mu M_{k,\rho} dx \right| \\
\leq C(n, m) \rho^{n/2} \sum_\nu \sum_{x_k \in M_\rho} \sum_{1 \leq |\beta| \leq 2m} \rho^{|\beta|} \| \partial^\beta (\psi_\nu f) \|_{L_2(B(x_k, \rho))},
\]
where \( m > n \). Using the definition of the Sobolev norm and elliptic regularity of the operator \( I + L \), where \( I \) is the identity operator on \( L_2(M) \), we obtain the inequality (5.3). \( \square \)

Now we are going to prove existence of cubature formulas which are exact on \( E_\omega(M) \), and have positive coefficients of the "right" size.

**Theorem 5.3.** There exists a positive constant \( a_0 \), such that if \( \rho = a_0(\omega + 1)^{-1/2} \), then for any \( \rho \)-lattice \( M_\rho \), there exist strictly positive coefficients \( \lambda_{x_k} > 0, x_k \in M_\rho \), for which the following equality holds for all functions in \( E_\omega(M) \):
\[
\int_M f dx = \sum_{x_k \in M_\rho} \lambda_{x_k} f(x_k).
\]

Moreover, there exist constants \( c_1, c_2 \), such that the following inequalities hold:
\[
c_1 \rho^n \leq \lambda_{x_k} \leq c_2 \rho^n, \quad n = \dim M.
\]

**Proof.** By using the Bernstein inequality, and our Plancherel-Polya inequalities (4.9), and assuming that
\[
\rho < \frac{1}{2\sqrt{\omega + 1}}
\]
we obtain from (5.3) the following inequality:
\[
\left| \sum_\nu \sum_{x_k \in M_\rho} \psi_\nu f(x_k) \mu M_{k,\rho} - \int_M f(x) dx \right| \leq C_1 \rho^{n/2} \sum_{|\beta|=1}^K (\rho \sqrt{1 + \omega})^{|\beta|} \| f \|_2 \leq \\
C_2 \rho^n \left( \rho \sqrt{1 + \omega} \right) \left( \sum_{x_k \in M_\rho} |f(x_k)|^2 \right)^{1/2},
\]
where \( C_2 \) is independent of \( \rho \in (0, (2\sqrt{\omega + 1})^{-1}) \) and the \( \rho \)-lattice \( M_\rho \).

Let \( R_\omega(L) \) denote the space of real-valued functions in \( E_\omega(L) \). Since the eigenfunctions of \( L \) may be taken to be real, we have \( E_\omega(L) = R_\omega(L) + iR_\omega(L) \), so it is enough to show that (5.11) holds for all \( f \in R_\omega(L) \).

Consider the sampling operator
\[
S : f \to \{ f(x_k) \}_{x_k \in M_\rho},
\]
which maps \( R_\omega(L) \) into the space \( \mathbb{R}^{|M_\rho|} \) with the \( \ell^2 \) norm. Let \( V = S(R_\omega(L)) \) be the image of \( R_\omega(L) \) under \( S \). \( V \) is a subspace of \( \mathbb{R}^{|M_\rho|} \), and we consider it with the induced \( \ell^2 \) norm. If \( u \in V \),
denote the linear functional $y \to (y, u)$ on $V$ by $\ell_u$. By our Plancherel-Polya inequalities \[\text{(5.15)}\], the map
\[\{f(x_k)\}_{x_k \in M_p} \to \int_M f dx\]
is a well-defined linear functional on the finite dimensional space $V$, and so equals $\ell_v$ for some $v \in V$, which may or may not have all components positive. On the other hand, if $w$ is the vector with components $\{\mu(M_{k,\rho})\}$, $x_k \in M_p$, then $w$ might not be in $V$, but it has all components positive and of the right size
\[a_1 \rho^n \leq \mu(M_{k,\rho}) \leq a_2 \rho^n,\]
for some positive $a_1$, $a_2$, independent of $\rho$ and the lattice $M_p = \{x_k\}$. Since, for any vector $u \in V$ the norm of $u$ is exactly the norm of the corresponding functional $\ell_u$, inequality \[\text{(5.15)}\] tells us that
\[\|Pw - v\| \leq \|w - v\| \leq C_2 \rho^n \left(\rho \sqrt{1 + \omega}\right),\]
where $P$ is the orthogonal projection onto $V$. Accordingly, if $z$ is the real vector $v - Pw$, then
\[v + (I - P)w = w + z,\]
where $\|z\| \leq C_2 \rho^n \left(\rho \sqrt{1 + \omega}\right)$. Note, that all components of the vector $w$ are of order $O(\rho^n)$, while the order of $\|z\|$ is $O(\rho^{n+1})$. Accordingly, if $\rho \sqrt{1 + \omega}$ is sufficiently small, then $\lambda := w + z$ has all components positive and of the right size. Since $\lambda = v + (I - P)w$, the linear functional $y \to (y, \lambda)$ on $V$ equals $\ell_v$. In other words, if the vector $\lambda$ has components $\{\lambda_{x_k}\}$, $x_k \in M_p$, then
\[\sum_{x_k \in M_p} f(x_k)\lambda_{x_k} = \int_M f dx\]
for all $f \in R_\omega(\mathcal{L})$, and hence for all $f \in E_\omega(\mathcal{L})$, as desired.

\[\Box\]

6. ON THE PRODUCT OF EIGENFUNCTIONS OF THE CASIMIR OPERATOR $\mathcal{L}$ ON COMPACT HOMOGENEOUS MANIFOLDS

In this section, we will use the assumption that $M$ is a compact homogeneous manifold, and that $\mathcal{L}$ is the operator of \[\text{(2.2)}\] in an essential way.

The following Theorem \[\text{6.1}\] plays a crucial role in our construction of Parseval frames in section 8. Note that some parts of the proof of this Theorem can be found in the papers \[\text{[39], [42]}\].

**Theorem 6.1.** If $M = G/K$ is a compact homogeneous manifold and $\mathcal{L}$ is defined as in \[\text{(2.2)}\], then for any $f$ and $g$ belonging to $E_\omega(\mathcal{L})$, their product $fg$ belongs to $E_{4d_\omega}(\mathcal{L})$, where $d$ is the dimension of the group $G$.

**Proof.** First, we are going to show that a function $f \in L_2(M)$ belongs to the space $E_\omega(\mathcal{L})$ if and only if there exists a constant $C(f, \omega)$ such that the following Bernstein inequality is satisfied for all natural $k$
\[\|\mathcal{L}^k f\| \leq C(f, \omega)\omega^k\|f\|,\]
The fact that the above Bernstein inequality holds true for any $f \in E_\omega(\mathcal{L})$ with $C(f, \omega) = 1$ is obvious. Conversely, assume that
\[\lambda_m \leq \omega < \lambda_{m+1}.\]
If a vector $f$ belongs to the space $E_\omega(\mathcal{L})$ and the Fourier series
\[f = \sum_{j=0}^\infty c_j w_j,\]
Using formula (2.2) one can easily verify that for any natural\[ k \]
contains terms with \[ j \geq m + 1, \] then\[ \lambda_{m+1}^{2k} \sum_{j=m+1}^{\infty} |c_j|^2 \leq \sum_{j=m+1}^{\infty} |\lambda_j^k c_j|^2 \leq \|L^k f\|^2 \leq C^2 \omega^{2k} \|f\|^2, \quad C = C(f, \omega), \]
which implies
\[ \sum_{j=m+1}^{\infty} |c_j|^2 \leq C^2 \left( \frac{\omega}{\lambda_{m+1}} \right)^{2k} \|f\|^2. \]
In the last inequality the fraction \( \omega/\lambda_{m+1} \) is strictly less than 1 and \( k \) can be any natural number.
This shows that the series (6.2) does not contain terms with \( j \geq m + 1 \), i.e. the function \( f \) belongs to \( E_{\omega}(L) \).
Now, since every smooth vector field on \( M \) is a differentiation of the algebra \( C^\infty(M) \), one has that for every operator \( D_j, 1 \leq j \leq d \), the following equality holds for any two smooth functions \( f \) and \( g \) on \( M \):
\[ D_j(fg) = fD_jg + gD_jf, \quad 1 \leq j \leq d. \]
Using formula (2.2) one can easily verify that for any natural \( k \in \mathbb{N} \), the term \( L^k (fg) \) is a sum of \( d^k \), \( (d = \text{dim}G) \), terms of the following form:
\[ D_{j_1} \ldots D_{j_k} (fg), \quad 1 \leq j_1, \ldots, j_k \leq d. \]
For every \( D_j \) one has
\[ D_j^2(fg) = f(D_j^2 g) + 2(D_j f)(D_j g) + g(D_j^2 f). \]
Thus, the function \( L^k (fg) \) is a sum of \( (4d)^k \) terms of the form
\[ (D_{i_1} \ldots D_{i_m} f)(D_{j_1} \ldots D_{j_{2k-m}} g). \]
This implies that
\[ |L^k (fg)| \leq (4d)^k \sup_{0 \leq m \leq 2k} \sup_{x,y \in M} |D_{i_1} \ldots D_{i_m} f(x)| |D_{j_1} \ldots D_{j_{2k-m}} g(y)|. \]
Let us show that the following inequalities hold:
\[ \|D_{i_1} \ldots D_{i_m} f\|_2 \leq \omega^{m/2} \|f\|_2 \]
and
\[ \|D_{j_1} \ldots D_{j_{2k-m}} g\|_2 \leq \omega^{(2k-m)/2} \|g\|_2 \]
for all \( f, g \in E_{\omega}(L) \). First, we note that the operator
\[ -L = D_1^2 + \ldots + D_d^2 \]
commutes with every \( D_j \) (see the explanation before the formula (2.2) ). The same is true for \( L^{1/2} \). But then
\[ \|L^{1/2} f\|_2^2 = \langle L^{1/2} f, L^{1/2} f \rangle = \langle L f, f \rangle = \sum_{j=1}^{d} < D_j^2 f, f > = \sum_{j=1}^{d} < D_j f, D_j f > = \sum_{j=1}^{d} \|D_j f\|_2^2, \]
and also
\[ \|L f\|_2^2 = \|L^{1/2} L^{1/2} f\|_2^2 = \sum_{j=1}^{d} \|D_j L^{1/2} f\|_2^2 = \]
which means that the following inequalities hold

\[ \sum_{j=1}^{d} \|L_j^{1/2}D_j f\|_2^2 = \sum_{j,k=1}^{d} \|D_j D_k f\|_2^2. \]

From here by induction on \( s \in \mathbb{N} \) one can obtain the following equality:

\[ \|L_n^{1/2}f\|_2^2 = \sum_{1 \leq i_1, \ldots, i_s \leq d} \|D_{i_1} \ldots D_{i_s} f\|_2^2, \quad s \in \mathbb{N}, \]

which implies the estimates (6.6) and (6.7). For example, to get (6.6) we take a function \( f \) from \( E_\omega(\mathcal{L}) \), an \( m \in \mathbb{N} \) and do the following

\[ \|D_{i_1} \ldots D_{i_m} f\|_2 \leq \left( \sum_{1 \leq i_1, \ldots, i_m \leq d} \|D_{i_1} \ldots D_{i_m} f\|_2 \right)^{1/2}. \]

(6.9)

\[ \|L_n^{m/2}f\|_2 \leq \omega^m/2 \|f\|_2. \]

In a similar way we obtain (6.7).

In terminology of the paper [12], it means that if \( f \) and \( g \) belong to \( E_\omega(\mathcal{L}) \) they also belong to \( B^2_{\psi\infty}(\mathbb{D}) \), where \( \mathbb{D} = \{D_1, \ldots, D_d\} \). But it was shown in [42], Theorem 3.3, that \( B^2_{\psi\infty}(\mathbb{D}) = B^\infty_{\psi\infty}(\mathbb{D}) \) which means that the following inequalities hold

\[ |D_{i_1} \ldots D_{i_m} f| \leq \omega^m/2 \|f\|_\infty \]

(6.10)

and similarly

\[ |D_{j_1} \ldots D_{j_{2k-m}} g| \leq \omega^{(2k-m)/2} \|g\|_\infty. \]

(6.11)

Thus, for \( f, g \in E_\omega(\mathcal{L}) \) we obtain the estimate

\[ |D_{i_1} \ldots D_{i_m} f| \|D_{j_1} \ldots D_{j_{2k-m}} g| \leq \omega^k \|f\|_\infty \|g\|_\infty. \]

Now, by using (6.5) we arrive at the following estimate:

\[ \|L^k(fg)\| \leq (\|f\|_\infty \|g\|_\infty) \left(4d \omega\right)^k. \]

We square both sides of this inequality and integrate over the compact manifold \( M \). We find that, for the constant \( C(M, f, g) = \sqrt{\text{Vol}(M)} \|f\|_\infty \|g\|_\infty \), the following inequality holds for all \( k \in \mathbb{N} \)

\[ \|L^k(fg)\| \leq C(M, f, g) \left(4d \omega\right)^k. \]

According to previous steps of the proof, this implies that the product \( fg \) belongs to \( E_{4d \omega}(\mathcal{L}) \). The Theorem is proved.

**Remark 6.2.** The last part of the Theorem can be proved without referring to the paper [12].

Indeed, the formula (6.10) along with the formula (6.6) imply the estimate

\[ \|L^k(fg)\|_2 \leq (4d)^k \sup_{0 \leq m \leq 2k} \|D_{i_1} \ldots D_{i_m} f\|_2 \|D_{j_1} \ldots D_{j_{2k-m}} g\|_\infty \leq \]

(6.12)

\[ (4d)^k \omega^{m/2} \|f\|_2 \sup_{0 \leq m \leq 2k} \|D_{j_1} \ldots D_{j_{2k-m}} g\|_\infty. \]

Using the Sobolev embedding Theorem and elliptic regularity of \( \mathcal{L} \), we obtain for every \( s > \frac{\dim M}{2} \)

\[ \|D_{j_1} \ldots D_{j_{2k-m}} g\|_\infty \leq C(M) \|D_{j_1} \ldots D_{j_{2k-m}} g\|_{H^s(M)} \leq \]

(6.13)

\[ C(M) \left\{ \|D_{j_1} \ldots D_{j_{2k-m}} g\|_2 + \|L^s/2 D_{j_1} \ldots D_{j_{2k-m}} g\|_2 \right\}, \]
where $H^s(M)$ is the Sobolev space of $s$-regular functions on $M$. Since the operator $L$ commutes with each of the operators $D_j$, the estimate (2) gives the following inequality:

$$\|D_{j_1} \ldots D_{j_{2k-m}} g\|_\infty \leq C(M) \left\{ \omega^{k-m/2} \|g\|_2 + \omega^{k-m/2+s} \|g\|_2 \right\} \leq$$

(6.14) $$C(M) \omega^{k-m/2} \left\{ \|g\|_2 + \omega^s \|g\|_2 \right\} = C(M, g, \omega, s) \omega^{k-m/2}, \quad s > \frac{\dim M}{2}.$$ 

Finally we have the following estimate:

(6.15) $$\| L^k(fg) \|_2 \leq C(M, f, g, \omega, s)(4d\omega)^k, \quad s > \frac{\dim M}{2}, \quad k \in \mathbb{N},$$

which leads to the same result that was obtained above.

7. Results on General Manifolds

In this section, we explain some general results on compact manifolds. We start afresh in our notation.

Let $(M, g)$ be a smooth, connected, compact Riemannian manifold without boundary with $(\text{[24]})$ Riemannian measure $\mu$. Let $L$ be a smooth, positive, second order elliptic differential operator on $M$, whose principal symbol $\sigma_2(L)(x, \xi)$ is positive on $\{(x, \xi) \in T^*M : \xi \neq 0\}$. For $x, y \in M$, let $d(x, y)$ denote the geodesic distance from $x$ to $y$.

In $\text{[17]}$ and $\text{[19]}$, the first author and Azita Mayeli proved a number of general results about the kernels of $f(t^2L)$ (for $f \in \mathcal{S}(\mathbb{R}^+)$) and about frames constructed from such kernels, in the Besov space framework – in the special case in which $L$ was $\Delta$, the Laplace-Beltrami operator on $M$. In this section, we review some of these results, and argue that they generalize to the situation in which $L$ is general. (In $\text{[17]}$ – $\text{[19]}$, it was assumed that the manifold was orientable, but this hypothesis was not actually used and may be dropped.)

First, we have:

**Theorem 7.1.** (Near-diagonal localization) Say $f \in \mathcal{S}(\mathbb{R}^+)$ (the space of restrictions to the non-negative real axis of Schwartz functions on $\mathbb{R}$). For $t > 0$, let $K_t(x, y)$ be the kernel of $f(t^2L)$. Then:

(a) Say $f(0) = 0$. Then for every pair of $C^\infty$ differential operators $X$ (in $x$) and $Y$ (in $y$) on $M$, and for every integer $N \geq 0$, there exists $C_{N, X, Y}$ as follows. Suppose $\deg X = j$ and $\deg Y = k$. Then

(7.1) $$t^{n+j+k} \left| \left( \frac{d(x, y)}{t} \right)^N X Y K_t(x, y) \right| \leq C_{N, X, Y}$$

for all $t > 0$ and all $x, y \in M$.

(b) For general $f$, the estimate (7.1) at least holds for $0 < t \leq 1$.

This was proved in section 4 of $\text{[17]}$, in the special case in which $L = \Delta$, the Laplace-Beltrami operator on $M$. (A similar result to part (a) had been proved earlier in $\text{[31]}$ and $\text{[32]}$ in the special case where $M$ was a sphere and $f$ had compact support away from the origin.) The arguments in $\text{[17]}$ used certain properties of $\Delta$, which we shall now argue are shared by general $L$. Once this is observed, the proofs in $\text{[17]}$ go through just the same as in $\text{[17]}$, and will not be repeated here.

Let us then list the properties of $L$ which were used in section 4 of $\text{[17]}$ in the special case $L = \Delta$, and verify that they hold for general $L$.

1. For $\lambda > 0$, let $N(\lambda)$ denote the number of eigenvalues of $L$ which are less than or equal to $\lambda$ (counted with respect to multiplicity). Then for some $c > 0$, $N(\lambda) = c\lambda^{n/2} + O(\lambda^{(n-1)/2})$.

2. $\sqrt{L}$ is a positive elliptic pseudodifferential operator on $M$ of order 1.
(3) If \( p(\xi) \in S^m_r(\mathbb{R}) \) (an ordinary symbol of order \( m \) on \( \mathbb{R} \), depending only on the “dual variable” \( \xi \)), then \( p(\sqrt{T}) \in OPS^m_{1,0}(\mathcal{M}) \).

(4) Say \( h \in S(\mathbb{R}) \) is even, and satisfies \( \text{supp } h \subseteq (-1, 1) \), and let \( K^h_t(x,y) \) be the kernel of \( h(t\sqrt{L}) \). Then for some \( C > 0 \), if \( d(x,y) > C|t| \), then \( K^h_t(x,y) = 0 \).

#1 is a sharp form of Weyl’s theorem, which is true for any second order elliptic differential operator on \( \mathcal{M} \) whose principal symbol is positive on \( \{ (x,\xi) \in T^*\mathcal{M} : \xi \not= 0 \} \). (47, Corollary 4.2.2). (Actually, weaker forms of Weyl’s theorem would suffice for the arguments in [17].)

#2 was used implicitly in [17] (specifically, in the use of #3). It follows from Theorem 2 of Seeley [44], as Seeley himself pointed out in that article. That theorem tells us, in particular, that if \( S \) is a classical positive invertible elliptic pseudodifferential operator of order \( k > 0 \) on \( \mathcal{M} \), whose principal symbol is positive on \( \{ (x,\xi) \in T^*\mathcal{M} : \xi \not= 0 \} \), then \( \sqrt{S} \) is a classical positive elliptic pseudodifferential operator on \( \mathcal{M} \) of order \( k/2 \). To apply this theorem to obtain #2, one lets \( P \) be the projection onto the null space of \( L \), which is a finite-dimensional space of smooth functions. Thus \( P \) has a smooth kernel. Then one notes that \( \sqrt{L} = \sqrt{L^\ast + P} - P \).

#3 is an immediate consequence of the main theorem of Strichartz [48]. In fact, that theorem tells us, that if \( S \) is a self-adjoint elliptic operator in \( OPS^m_{1,0}(\mathcal{M}) \), then \( p(S) \in OPS^m_{1,0}(\mathcal{M}) \).

#4 is a consequence of the finite speed of propagation property of the wave equation. With no claim of originality, we now explain this in some detail. In this discussion, all differential operators and functions will be taken to be smooth, without further comment.

Suppose that \( L_1 \) is a second-order differential operator on an open set \( V \) in \( \mathbb{R}^n \), that \( L_1 \) is elliptic, and in fact that, for some \( c > 0 \), its principal symbol \( \sigma_2(L_1)(x,\xi) \geq c^2|\xi|^2 \), for all \( (x,\xi) \in V \times \mathbb{R}^n \). Suppose that \( U \subseteq \mathbb{R}^n \) is open, and that \( \overline{U} \subseteq V \). Then if \( \text{supp } F, G \subseteq K \subseteq U \), where \( K \) is compact, then any solution \( u \) of

\[
\begin{align*}
(\frac{\partial^2}{\partial t^2} + L_1)u &= 0 \\
u(0,x) &= F(x) \\
u_t(0,x) &= G(x)
\end{align*}
\]

on \( U \) satisfies \( \text{supp } u(t,\cdot) \subseteq \{ x : \text{dist } (x,K) \leq |t|/c \} \).

(This is a special case of Theorem 4.5 (iii) of [49]. In that reference, \( V = \mathbb{R}^n \). But we can always extend \( L_1 \) from \( U \) to an operator on all of \( \mathbb{R}^n \) satisfying the hypotheses, by letting \( L'_1 = \psi L_1 + c^2(1-\psi)\Delta \) for a cutoff function \( \psi \in C_c^\infty(V) \) which equals 1 in a neighborhood of \( \overline{U} \).)

It is an easy consequence of this that a similar result holds on manifolds. With \( L \) as before, let us look at the problem

\[
\begin{align*}
(\frac{\partial^2}{\partial t^2} + L)u &= 0 \\
u(0,x) &= F(x) \\
u_t(0,x) &= G(x)
\end{align*}
\]

on \( \mathcal{M} \). The first thing to note is that the problem has a unique solution in any open \( t \)-interval about zero. Namely, if \( F = \sum_k a_k \varphi_k \) and \( G = \sum_k b_k \varphi_k \), where the \( \varphi_k \) are an orthonormal basis
of eigenfunctions of $L$, with corresponding eigenvalues $\lambda_k$, then the solution is
\[
u(x,t) = \sum [a_k \cos(\sqrt{\lambda_k} t) + b_k \sin(\sqrt{\lambda_k} t)] \varphi_k(x),
\]
where we interpret $\sin(\sqrt{-\lambda_k} t)$ as $t$ if $\lambda_k = 0$. Note also that
\[
(7.8) \quad u = \cos(t \sqrt{L}) F \text{ is the solution if } G \equiv 0.
\]
We then **claim** that there is a $C > 0$, depending only on $M$ and $L$, such that if $\text{supp } F,G \subseteq K \subseteq M$, then the solution $u$ satisfies $\text{supp } u(t,\cdot) \subseteq \{ x : d(x,K) \leq C|t| \}$, where now $d$ is geodesic distance. This is proved as follows:

- It is enough to show that, for some $\delta > 0$, the result is true whenever $|t| < \delta$. For, suppose that this is known. It suffices then to show that if, for some $T > 0$, the result is true whenever $|t| < T$, then it is also true whenever $|t| < T + \delta$. For this, say $T \leq t < T + \delta$, and select $t_0 > T$ with $t - t_0 < \delta$. By assumption, $\text{supp } u(t_0,\cdot) \subseteq K' := \{ x : d(x,K) \leq C t_0 \}$, and thus also $\text{supp } u(t,\cdot) \subseteq K'$. We clearly have that $u(t,x) = v(t - t_0,x)$, where $v$ is the solution of

\[
(7.9) \quad \left(\frac{\partial^2}{\partial t^2} + L\right)v = 0
\]
\[
(7.10) \quad v(0,x) = u(t_0,x)
\]
\[
(7.11) \quad v_t(0,x) = u_t(t_0,x)
\]

Thus
\[
\text{supp } u(t,\cdot) = \text{supp } v(t - t_0,\cdot) \subseteq \{ x : d(x,K') \leq C(t - t_0) \} \subseteq \{ x : d(x,K) \leq C t \}
\]
as claimed. Similarly if $-T \geq t \geq -T - \delta$.

- It suffices to show that, for some $\delta, \epsilon > 0$, the result is true whenever $|t| < \delta$, and the supports of $F$ and $G$ are both contained in an open ball $B$ of radius $\epsilon$. For, we could then cover $M$ by a finite number of such open balls, and choose a partition of unity $\{ \zeta_j \}$ subordinate to this covering. If we let $f_j, g_j = (\zeta_j f, \zeta_j g)$, and if we let $u_j$ be the solution with data $f_j, g_j$ in place of $f, g$, then surely $u = \sum_j u_j$. Then surely $\text{supp } u(t,\cdot) \subseteq \{ x : d(x,K) \leq C|t| \}$ as desired.

- To find appropriate $\delta, \epsilon$, one need only cover $M$ with balls $B_k$ of some radius $\epsilon$, for which the balls $B_k'$ with the same centers and radius $2\epsilon$ are charts, on which, if we use local coordinates, the geodesic distance is comparable to the Euclidean distance. The existence of a suitable $\delta, C$ now follows at once from the aforementioned result for the wave equation on open subsets of $\mathbb{C}^n$. This proves the “claim”.

To prove #4, it suffices to write (for some $c$)
\[
(7.12) \quad h(t \sqrt{L}) F = c \int_{-1}^1 \hat{h}(s) \cos(st \sqrt{L}) F ds
\]
for any $F \in C^\infty(M)$. (This is easily verified by using the eigenfunction expansion of $F$ and the Fourier inversion formula.) #4 follows at once from (7.8) and the “claim”.

Thus we have Theorem [7.1] for general $L$.

We turn now to Besov spaces. For the rest of this section, we fix $a > 1$. We also fix $\alpha, p, q$ with $-\infty < \alpha < \infty$ and $0 < p, q \leq \infty$. We let $B^a_{pq}$ be the Besov space of section 3.
We fix a finite set \( P \) of real \( C^\infty \) vector fields on \( M \), whose elements span the tangent space at each point. We also fix a spanning set of the differential operators on \( M \) of degree less than or equal to \( J \) (for any fixed \( J \)):

\[
\mathcal{P}^J = \{ X_1 \ldots X_M : X_1, \ldots, X_M \in P, 1 \leq M \leq J \} \cup \{ \text{the identity map} \}.
\]

The following results were obtained in Lemmas 2.4, 3.2 and 3.3 of [19], again in the special case \( L = \Delta \). In the present article, as we shall see, the technical restrictions on \( l \) and \( M \) in Lemmas 7.3 and 7.4 below will end up playing no role, so the reader is advised not to pay undue attention to them.

**Lemma 7.2.** Say \( l, M \) are integers with \( l \geq 0 \) and \( M > n \). Then there exists \( C > 0 \) as follows.

Say \( \sigma, \nu \in \mathbb{R} \) with \( \sigma \geq \nu \).

Say \( x_0 \in M \), and suppose that \( \varphi_1 = L^j \Phi \), where \( \Phi \in C^{2l}(M) \) satisfies:

\[
|\Phi(y)| \leq (1 + \alpha \nu d(y, x_0))^{-M}.
\]

Also suppose \( x_1 \in M \), that \( \varphi_2 \in C^{2l}(M) \), and that for all \( y \in M \),

\[
|L^j \varphi_2(y)| \leq (1 + \alpha \nu d(y, x_1))^{n-M}.
\]

Then,

\[
\left| \int_M (\varphi_1 \varphi_2)(y) d\mu(y) \right| \leq Ca^{-\sigma n} \left( 1 + \alpha \nu d(x_0, x_1) \right)^{n-M}.
\]

**Lemma 7.3.** Fix \( b > 0 \). Also fix an integer \( l \geq 1 \) with

\[
l > \max(n(1/p - 1)_+ - \alpha, \alpha).
\]

where here \( x_+ = \max(x, 0) \). Fix \( M \) with \((M - 2l - n)p > n + 1 \) if \( 0 < p < 1 \), \( M - 2l - n > n + 1 \) otherwise.

Then there exists \( C > 0 \) as follows.

Say \( j \in \mathbb{Z} \). Write \( M \) as a finite disjoint union of measurable subsets \( \{ E^j_k : 1 \leq k \leq N_j \} \).

Suppose:

\[
\text{the diameter of each } E^j_k \text{ is less than or equal to } ba^{-j}.
\]

For each \( k \) with \( 1 \leq k \leq N_j \), select any \( \tilde{x}^j_k \in E^j_k \).

Suppose that, for each \( j \geq 0 \), and each \( k \),

\[
\tilde{\varphi}^j_k = (a^{-2j} L)^j \tilde{\Phi}_k^j,
\]

where \( \tilde{\Phi}_k^j \in C^\infty(M) \) satisfies the following conditions:

\[
|X \tilde{\Phi}_k^j(y)| \leq a^{j(\text{deg } X + n)}(1 + a \nu d(y, \tilde{x}_k^j))^{-M} \text{ whenever } X \in \mathcal{P}^M.
\]

Then, for every \( F \) in the inhomogeneous Besov space \( B^{a_0}_p(M) \), if we let

\[
\tilde{s}_{j,k} = (F, \tilde{\varphi}^j_k),
\]

then

\[
\left( \sum_{j=0}^{\infty} a^{j0q} \left[ \sum_k \mu(E^j_k) |\tilde{s}_{j,k}|^p \right]^{q/p} \right)^{1/q} \leq C \| F \|_{B_p^{a_0}}.
\]
Lemma 7.4. Fix $b > 0$. Also fix an integer $l \geq 1$ with
\[ 2l > n(1/p - 1) + \alpha. \]
where here $x_+ = \max(x, 0)$. Fix $M$ with $(M - n)p > n + 1$ if $0 < p < 1$, $M - n > n + 1$ otherwise. If $0 < p < 1$, we also fix a number $\rho > 0$. Then there exists $C > 0$ as follows.

Say $j \in \mathbb{Z}$. Select sets $E^j_k$ and points $\tilde{x}^j_k$ as in Lemma 7.3. If $0 < p < 1$, we assume that, for all $j, k$,
\[ \mu(E^j_k) \geq \rho a^{-jn}. \]
Suppose that, for each $j \geq 0$, and each $k$, $\tilde{\varphi}^j_k = (a^{-2j}L)^j\tilde{\Phi}^j_k$, where $\tilde{\Phi}^j_k \in C^\infty(M)$ satisfies the following conditions:
\[ |X\tilde{\Phi}^j_k(y)| \leq a^{i(\deg X + n)} (1 + a^d(y, \tilde{x}^j_k))^{-M} \text{ whenever } X \in \mathcal{P}^d. \]

Suppose that $\{\tilde{s}_{j,k} : j \geq 0, 1 \leq k \leq N_j\}$ satisfies
\[ \left( \sum_{j=0}^\infty a^{j\alpha q} \left( \sum_k \mu(E^j_k)|\tilde{s}_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty. \]
Then $\sum_{j=0}^\infty \sum_k \mu(E^j_k)\tilde{s}_{j,k}\tilde{\varphi}^j_k$ converges in $B^{\alpha q}_{p,q}(M)$, and
\[ \left( \sum_{j=0}^\infty \sum_k \mu(E^j_k)|\tilde{s}_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty. \]

Again, these three results were proved in [19] in the special case in which $L = \Delta$. The arguments in [19] used certain properties of $\Delta$, which we shall now argue are shared by general $L$. Once this is observed, the proofs in [19] go through just the same as in [19], and will not be repeated here.

Let us then list the properties of $L$ which were used in the proofs of these lemmas in [19] in the special case $L = \Delta$, and verify that they hold for general $L$.

- In the proof of Lemma 7.2 in the special case $L = \Delta$ (which was Lemma 2.4 in [19]), the only property used of $L$ was that it was a smooth, second-order partial differential operator, which satisfied $(LF,G) = (F,LG)$ for all $F, G \in C^2(M)$.
- In the proof of Lemma 7.3 in the special case $L = \Delta$ (which was Lemma 3.2 in [19]), the only properties of $L$ were used that it was a smooth, second-order partial differential operator, and that Lemma 7.2 above holds.
- In the proof of Lemma 7.4 in the special case $L = \Delta$ (which was Lemma 3.3 in [19]), again these properties of $L$ were used: it is a smooth, second-order partial differential operator, and Lemma 7.2 above holds. In addition, the following result of Seeger-Sogge [45] was used:

Choose $\beta_0 \in C^\infty_0((1/4,16))$, with the property that for any $s > 0$, $\sum_{\nu=-\infty}^\infty \beta_0^2(2^{-2\nu} s) = 1$. For $\nu \geq 1$, define $\beta_\nu \in C^\infty_0((2^{2\nu-2} s, 2^{2\nu+1}))$, by $\beta_\nu(s) = \beta_0(2^{-2\nu}s)$. Also, for $s > 0$, define the smooth function $\beta_{-1}(s)$ by $\beta_{-1}(s) = \sum_{\nu=-\infty}^{s-4} \beta(2^{-2\nu} s)$. (Note that $\beta_{-1}(s) = 0$ for $s \geq 4$). Then (45), for $F \in C^\infty(M)$, $\|F\|_{B^{\alpha q}_{p,q}}$ is equivalent to the $F^\nu$ norm (actually a quasi-norm if $0 < q < 1$) of the sequence $\{2^{\nu a} \|\beta_\nu(L)F\|_p : -1 \leq \nu \leq \infty\}$.

(Note that the notation of [19] is slightly different from that of [45]; what [19] calls $\beta_{k-1}(s^2)$, is called $\beta_k(s)$ in [45].) By Theorem 4.1 of [45], the result does hold for general $L$, and in fact would hold if we only knew that $L = P^2$ for some first-order elliptic, positive, classical pseudodifferential operator on $M$. Of course, we do know that our $L$
satisfies this condition (see our comments on Seeley’s work above, in our discussion of point
#2, following Theorem 7.1).

Thus we do indeed have Lemmas 7.3 and 7.4 for general L. In the next section, we will put this
to use in the case where M is a compact homogeneous manifold.

To conclude this section, we shall continue to work on our general M, and show how Theorem
7.1 and the Seeger-Sogge characterization of Besov spaces can be used to obtain a description
of Besov spaces in terms of best approximations by band-limited functions. This result gives a
generalization of a part of Theorem 1.1 of [43], where such a description was given in the case
p = 2 for manifolds of bounded geometry. Our arguments are analogous to those of [32], Proposition 5.3,
where the case in which M is the sphere was dealt with.

We need to make a few observations first. In the situation of Theorem 7.1 (a), it is easy to see
from eigenfunction expansions that \( f(t^2L) \) maps distributions on M to distributions on M. We have:

\[
(7.22) \quad \text{If } 1 \leq p \leq \infty, \text{ then } f(t^2L) : L_p(M) \to L_p(M), \text{ with norm bounded independent of } t.
\]

Indeed, say that \( K_i \) is the kernel of \( f(t^2L) \); it suffices to observe that for some \( C > 0 \),
\[
\int |K_i(x,y)|\,dx \leq C \quad \text{for all } y, \quad \text{and} \quad \int |K_i(x,y)|\,dy \leq C \quad \text{for all } x.
\]

This however is evident from (7.1) with \( j = k = 0 \), since by (21) of [17], for any \( N > n \) there is a \( C_N \) such that
\[
\int [1 + d(x,y)]^{-N}\,dy \leq C_N t^n \quad \text{for all } x.
\]

Suppose next that \( \alpha > 0 \) and \( 1 < p \leq \infty, 0 < q < \infty \). Then, on M,

\[
(7.23) \quad B^\alpha_p \subseteq L_p, \text{ and for some } C > 0, ||F||_{L_p} \leq C||F||_{B^\alpha_p} \text{ for all } F \in B^\alpha_p.
\]

Indeed, recalling our original definition of Besov spaces in section 2, we see that it is enough to
prove this on \( \mathbb{R}^n \). Choose the \( \Phi, \varphi_\nu \), in (8.10) in such a manner that \( \Phi + \sum_{\nu=0}^{\infty} \varphi_\nu = 1 \) pointwise,
so that this is true in \( S' \) as well. From this, if \( F \in S' \) is such that \( \sum_{\nu=0}^{\infty} ||\varphi_\nu * F||_{L_p} < \infty \), then
\( F \in L_p \), and \( F = \Phi * F + \sum_{\nu=0}^{\infty} \varphi_\nu * F \) in \( L_p \). But the absolute convergence of \( \sum_{\nu=0}^{\infty} ||\varphi_\nu * F||_{L_p} \)
follows easily if \( F \in B^\alpha_p \), by Hölder’s inequality if \( q \geq 1 \), or directly if \( 0 < q < 1 \). This shows that
\( B^\alpha_p \subseteq L_p \), and the inequality \( ||F||_{L_p} \leq C||F||_{B^\alpha_p} \) follows similarly.

The argument that \( \sum_{\nu=0}^{\infty} ||\varphi_\nu * F||_{L_p} \) converges absolutely may be adapted to M. Let the \( \beta_\nu \)
be as in the Seeger-Sogge result described above. A similar argument, using their characterization
of Besov spaces, shows that, assuming \( \alpha > 0 \) and \( 1 < p \leq \infty, 0 < q < \infty \), one has:

\[
(7.24) \quad \text{If } F \in B^\alpha_p(M), \text{ then } \sum ||\beta_\nu(L)F||_p < \infty.
\]

We let \( E_\omega(L) \) denote the span of all eigenfunctions of \( L \) with eigenvalue less than or equal to
\( \omega \). Let \( \mathcal{D}' \) denote the space of distributions on M. We note:

\[
(7.25) \quad \text{If } j \geq 0 \text{ and } F \in \mathcal{D}'(M), \text{ then } \sum_{\nu=j+1}^{\infty} \beta^2_\nu(L)F \text{ converges in } \mathcal{D}'(M), \text{ and } F - \sum_{\nu=j+1}^{\infty} \beta^2_\nu(L)F \in E_{2^{j+4}}(L).
\]

Indeed, the convergence of the series in \( \mathcal{D}' \) follows from an examination of the eigenfunction
expansion of a smooth function. Next, let \( G = F - \sum_{\nu=j+1}^{\infty} \beta^2_\nu(L)F \). By the properties of the \( \beta_\nu \),
note that one has that \( \sum_{\nu=j+1}^{\infty} \beta^2_\nu(s) = 1 \) for \( s \geq 2^{j+4} \). If \( u_\nu \) is an eigenfunction of \( L \) with
eigenvalue \( \lambda \), we then see that \( G(u_\nu) = 0 \) if \( \lambda \geq 2^{j+4} \). Let \( \{u_i\} \) be an orthonormal basis for \( E_{2^{j+4}}(L) \),
consisting of real-valued eigenfunctions, and say \( G(u_i) = a_i \). Then \( G - \sum_i a_i u_i \) annihilates all
eigenfunctions of \( L \), so it must be zero, as needed.

For \( 1 \leq p \leq \infty \), if \( F \in L_p \), we let

\[
\mathcal{E}(F, \omega, p) = \inf_{G \in \mathcal{E}(L)} \|F - G\|_p.
\]

We then have:

**Theorem 7.5.** Say \( \alpha > 0 \), \( 1 \leq p \leq \infty \), and \( 0 < q < \infty \). Then \( F \in B_p^{\alpha q} \) if and only if \( F \in L_p \) and

\[
\|F\|_{B_p^{\alpha q}} := \|F\|_{L_p} + \left( \sum_{j=0}^{\infty} (2^{\alpha j} \mathcal{E}(F, 2^{2j}, p))^q \right)^{1/q} < \infty.
\]

Moreover,

\[
\|F\|_{B_p^{\alpha q}} \sim \|F\|_{B_p^{\alpha q}}.
\]

**Proof.** Let the \( \beta_{nu} \) be as above.

We first show, for \( F \in B_p^{\alpha q} \), that \( \|F\|_{B_p^{\alpha q}} \leq C \|F\|_{B_p^{\alpha q}} \). Because of (7.28), it is enough to show that

\[
\left( \sum_{j=2}^{\infty} (2^{\alpha j} \mathcal{E}(F, 2^{2j}, p))^q \right)^{1/q} \leq C \|F\|_{B_p^{\alpha q}}.
\]

But by (7.26), (7.27), and (7.28), for \( j \geq 2 \) we have

\[
\mathcal{E}(F, 2^{2j}, p) \leq \| \beta_{2j}^2(L)F \|_p \leq C \sum_{\nu=j}^{\infty} \| \beta_{\nu}(L)F \|_p < \infty.
\]

If one recalls the Seeger-Sogge characterization of \( B_p^{\alpha q} \) and the assumption that \( \alpha > 0 \), and if one uses a standard argument, one does find \( \|F\|_{B_p^{\alpha q}} \leq C \|F\|_{B_p^{\alpha q}} \). (One introduces an operator on \( \ell^q(\mathbb{N}) \) with an appropriate kernel, and invokes Proposition 3.1 of [19] to show that this operator is bounded on \( \ell^q \).)

For the converse, say \( \nu \geq 0 \). We simply note that if \( G \in E_{2^{2\nu}-2}(L) \), then \( \beta_{\nu}(L)G = 0 \). Thus, by (7.22), if \( F \in L_p \), then \( \| \beta_{\nu}(L)(F) \|_p = \| \beta_{\nu}(L)(F - G) \|_p \leq C \| F - G \|_p \). Accordingly,

\[
\| \beta_{\nu}(L)(F) \|_p \leq C \mathcal{E}(F, 2^{2\nu+2}, p) \quad \text{for} \quad \nu \geq 1; \quad \text{and} \quad \| \beta_{\nu}(L)(F) \|_p \leq C \| F \|_p \quad \text{for all} \ \nu.
\]

From this, we find at once that \( \|F\|_{B_p^{\alpha q}} \leq C \|F\|_{B_p^{\alpha q}} \). □

## 8. Parseval frames and Besov spaces

We now revert to the notation of sections 1 through 6. We modify the construction of “needlets” in [31], to produce a Parseval frame on \( M \).

Say \( a > 1 \). Choose a function \( f \in C_c^\infty \), supported in the interval \([a^{-2}, a^2]\) such that

\[
\sum_{j=-\infty}^{\infty} |f(a^{-2}j)|^2 = 1
\]

for all \( s > 0 \).

(For example, we could choose a smooth function \( \Phi \) on \( \mathbb{R}^+ \) with \( 0 \leq \Phi \leq 1 \), with \( \Phi \equiv 1 \) in \([0, a^{-2}]\) and with \( \Phi = 0 \) in \([a^2, \infty)\), and let \( f(t) = |\Phi(t/a^2) - \Phi(t)|^{1/2} \) for \( t > 0 \).)
Recalling (2.2), we note that the eigenspace for $L$ corresponding to the eigenvalue $\lambda_0 = 0$ is the space of constant functions, since the $D_j$ span the tangent space at each point. Let $P$ be the projection in $L_2(M)$ onto the space of constant functions. We now apply the spectral theorem. By [17], Lemma 2.1(b), we have

$$(8.2) \sum_{j=-\infty}^{\infty} |f|^2(a^{-2j}L) = I - P,$$

where the sum converges strongly on $L_2(M)$. (This is, in fact, easily seen, if one diagonalizes $L$.)

Say now $F \in L_2(M)$. We apply (8.2) to $F$ and take the inner product with $F$. We find

$$(8.3) \sum_{j=-\infty}^{\infty} \|f(a^{-2j}L)F\|^2 = \|(I - P)F\|^2$$

Expand $F = \sum_m A_m u_m$ in terms of our eigenfunctions of $L$. Then $f(a^{-2j}L)F = \sum_m f(a^{-2j}L)A_m u_m \in E_{a^{2j+4}}(L)$, since $f(a^{-2j}L)u_m = 0$ if $\lambda_m \geq a^{2j+4}$. Also $\langle f(a^{-2j}L)F \rangle \in E_{a^{2j+4}}(L)$, so by Theorem 6.1, the product of these two functions, $\|f(a^{-2j}L)F\|^2$ is in $E_{4da^{2j+4}}(L)$. Putting

$$(8.4) \rho_j = a_0(4da^{2j+4} + 1)^{-1/2}$$

we now find from the cubature formula that

$$(8.5) \|f(a^{-2j}L)F\|^2 = \sum_{k=1}^{N_j} b_k^j |\langle f(a^{-2j}L)F \rangle (x_k^j)\|^2,$$

where $x_k^j \in M_{\rho_j}$, $(k = 1, \ldots, N_j = N(M_{\rho_j}))$, and

$$(8.6) b_k^j \sim \rho_j^n;$$

in the sense that the ratio of these quantities is bounded above and below by positive constants.

Now, for $t > 0$, let $K_t$ be the kernel of $f(t^2L)$, so that, for $F \in L_2(M)$,

$$(8.7) [f(t^2L)F](x) = \int_M K_t(x, y)F(y)d\mu(y).$$

For $x, y \in M$, we have

$$(8.8) K_t(x, y) = \sum_m f(t^2\lambda_m)u_m(x)\overline{\sigma}_m(y).$$

Corresponding to each $x_k^j$ we now define the functions

$$(8.9) \varphi_k^j(y) = K_{a^{-j}}(x_k^j, y) = \sum_m f(a^{-2j}\lambda_m)\overline{\sigma}_m(x_k^j)u_m(y),$$

$$(8.10) \phi_k^j = \sqrt{b_k^j} \varphi_k^j.$$

From (8.3), (8.5), (8.7), (8.9) and (8.10), we find that for all $F \in L_2(M)$,

$$(8.11) \|\langle F, \phi_k^j \rangle\|^2.$$

Note that, by (8.3) and (8.10) and the fact that $f(0) = 0$, each $\phi_k^j \in (I - P)L_2(M)$.

Thus the $\phi_k^j$ form a Parseval frame (i.e. normalized tight frame) for $(I - P)L_2(M)$. Note also that each $\phi_k^j$ is a finite linear combination of eigenfunctions of $L$, hence
is smooth. Moreover, since $f$ vanishes on $[a^4, \infty)$, we have $\phi_k^j \equiv 0$ once $a^{-2j} \lambda_1 \geq a^4$. Thus, for some $\Omega$ (specifically $\Omega = [(\log_a \lambda_1)/2] - 1$, where $[\cdot]$ is the greatest integer function), we have

$$\phi_k^j \equiv 0 \text{ if } j < \Omega.$$  

Note that, by (8.4), for $j \geq \Omega$, we have

$$\rho_j \sim a^{-j},$$

in the sense that the ratio of these quantities is bounded above and below by positive constants.

By general frame theory, if $F \in L_2(M)$, we have

$$(8.14) \quad (I - P)F = \sum_{j=1}^{\infty} \sum_{k} (F, \phi_k^j) \phi_k^j = \sum_{j=1}^{\infty} \sum_{k} b_k^j(F, \varphi_k^j) \varphi_k^j,$$

with convergence in $L_2$.

We now explain how to characterize Besov spaces on $M$ by using our Parseval frames. We let $B_{p,0}^{\sigma}(M)$ be the space of distributions $F$ in the Besov space $B_{p,0}(M)$, for which $F(1) = 0$. We claim:

**Theorem 8.1.** With the $\varphi_k^j$ as above, for some $C > 0$ we have:

(a) Suppose that \( \{s_k^j : j \geq \Omega, 1 \leq k \leq N_j\} \) satisfies

$$(8.15) \quad \left( \sum_{j=1}^{\infty} a^{jq(\alpha - n/p)} \left( \sum_{k} |s_k^j|^p \right)^{q/p} \right)^{1/q} < \infty.$$  

Then

$$\sum_{j=1}^{\infty} \sum_{k} a^{-nq} s_k^j \varphi_k^j \text{ converges in } B_{p,0}^{\alpha q}(M),$$

and

$$(8.17) \quad \left\| \sum_{j=1}^{\infty} \sum_{k} a^{-nq} s_k^j \varphi_k^j \right\|_{B_{p,0}^{\alpha q}} \leq C \left( \sum_{j=1}^{\infty} a^{jq(\alpha - n/p)} \left( \sum_{k} |s_k^j|^p \right)^{q/p} \right)^{1/q}.$$  

(b) Suppose $F \in B_{p,0}^{\alpha q}(M)$. Then

$$(8.18) \quad \left( \sum_{j=1}^{\infty} a^{jq(\alpha - n/p)} \left( \sum_{k} |(F, \varphi_k^j)|^p \right)^{q/p} \right)^{1/q} < \infty.$$  

Moreover, the expression in (8.18) defines a quasi-norm on $B_{p,0}^{\alpha q}(M)$ which is equivalent to the usual quasi-norm on this space. (If $1 \leq p, q \leq \infty$, these quasi-norms are in fact norms.)

(c) Let $\alpha_0 = 1/\sqrt{\mu(M)}$. Say $F \in B_{p,0}^{\alpha q}(M)$. Then

$$(8.19) \quad F = F(\alpha_0) + \sum_{j=1}^{\infty} \sum_{k} b_k^j(F, \varphi_k^j) \varphi_k^j,$$

with the convergence of the right side being in $B_{p,0}^{\alpha q}(M)$. (Here $F(\alpha_0)$ means the distribution $F$ applied to the constant function $\alpha_0$.)

(d) Let $b_{p,0}^{\alpha q}$ denote the quasi-Banach spaces of sequences $\{s_k^j\}$ $(j \geq \Omega, 1 \leq k \leq N_j)$ satisfying (8.15). Then there are well-defined bounded operators $\tau : B_{p,0}^{\alpha q}(M) \to b_{p,0}^{\alpha q}$ and $\sigma : b_{p,0}^{\alpha q} \to B_{p,0}^{\alpha q}(M)$, given by $\tau(F) = \{ (F, \varphi_k^j) \}$, $\sigma(\{s_k^j\}) = \sum_{j=1}^{\infty} \sum_{k} b_k^j s_k^j \varphi_k^j$ (with convergence in $B_{p,0}^{\alpha q}(M)$); and on $B_{p,0}^{\alpha q}(M)$, $\sigma \circ \tau = id.$
Proof. For each \( j \geq \Omega \), let \( E^j_k = M^j_k = M_{k,p_j} \) be the disjoint cover of Lemma \( 5.1 \).

We are going to show that we can apply Lemmas \( 7.3 \) and \( 7.4 \) with

\[
\hat{\varphi}^j_k = \varphi_k^{j+\Omega}, \quad \hat{E}^j_k = E_k^{j+\Omega}, \quad \hat{x}^j_k = x_k^{j+\Omega}, \quad L = \mathcal{L}.
\]

Choose \( l \in \mathbb{N} \) satisfying \( (7.14) \) (and hence \( (7.19) \) as well). Define \( f_t(s) = f(s)/s^l \), so that \( f_t \) is another \( C^\infty \) function with support in \([a^{-2}, a^2]\). We have \( f(s) = s^l f_t(s) \), and for any \( t > 0 \), \( f(t^2 \mathcal{L}) = (t^2 \mathcal{L})^l f_t(t^2 \mathcal{L}) \). If \( K^j_l(x,y) \) is the kernel of \( f_t(t^2 \mathcal{L}) \), then an examination of \( (8.18) \) and the corresponding equation for \( K^l_t \) shows that

\[
(8.20) \quad \mathcal{K}_t(x,y) = (t^2 \mathcal{L} y) \mathcal{K}_t(x,y),
\]

where \( \mathcal{L}_y \) means \( \mathcal{L} \) applied in the \( y \) variable. Put

\[
(8.21) \quad \Phi^j_k(y) = \mathcal{K}_t(x_k^j, y);
\]

then

\[
(8.22) \quad \phi^j_k = (a^{-2j} \mathcal{L})^j \Phi^j_k.
\]

Set \( \hat{\Phi}^j_k = a^{-2j \Omega} \Phi^j_k + \Omega \); then we have \( (7.10) \). Let us check that the other hypotheses of Lemmas \( 7.3 \) and \( 7.4 \) hold as well.

Since each \( M_{k,p_j} \) is contained in a ball of radius \( p_j/2 \), since \( (8.13) \) and \( (5.2) \) hold, and since \( E^j_k = M^j_k = M_{k,p_j} \), we see that \( (\mathcal{R}10) \) and \( (8.20) \) hold for the \( \hat{E}^j_k \) (for some \( b, \rho > 0 \)). Moreover, by \( (8.21) \) and Theorem \( 7.1 \), \( (7.17) \) holds for \( \Phi^j_k \), up to a multiplicative constant (independent of \( j \) and \( y \)). (For this, the values of \( l \) and \( M \) are irrelevant.)

Thus we may avail ourselves of the conclusions of Lemmas \( 7.3 \) and \( 7.4 \). Note, by \( (8.13) \) and \( (5.2) \), that

\[
(8.23) \quad \mu(E^j_k) \sim a^{-jn}.
\]

Let

\[
(8.24) \quad c^j_k = a^{jn} \mu(E^j_k);
\]

then the set \( \{c^j_k\} \) is bounded above and below by positive constants.

For (a), say that \( (\mathcal{R}10) \) holds. We find that

\[
(8.25) \quad \left( \sum_{j=0}^{\infty} a^{j q a} \left[ \sum_k \mu(E^j_k) \left| (s^j_k/c^j_k)^p \right| \right]^{q/p} \right)^{1/q} \sim \left( \sum_{j=0}^{\infty} a^{j q (\alpha-n/p)} \left[ \sum_k |s^j_k|^p \right]^{q/p} \right)^{1/q} < \infty,
\]

where now \( \sim \) means that the ratio of the quantities is bounded above and below by positive constants independent of the particular collection of \( \{s^j_k\} \). Noting that \( \mu(E^j_k) (s^j_k/c^j_k) = a^{-jn} s^j_k \), we now see that part (a) of the theorem follows at once from Lemma \( 7.4 \).

For (b), suppose first that \( F \in B^{q,a}_p(M) \). The sum in \( (\mathcal{R}10) \) is less than or equal to

\[
C \left( \sum_{j=0}^{\infty} a^{j q a} \left[ \sum_k \mu(E^j_k) (F, \varphi^j_k)^p \right]^{q/p} \right)^{1/q} ,
\]

for some \( C \), which is less than or equal to \( C \|F\|_{B^{q,a}_p} \) for some (other) \( C \), by Lemma \( 7.3 \). To complete the proof of (b), we must obtain the reverse inequality for \( F \in B^{q,a}_p(M) \).
Before doing that, let us prove (c). By (8.14), (8.19) holds for \( F \in C^\infty(M) \), with convergence in \( L_2 \). Note next that if \( F \in B_p^{\alpha q}(M) \), the right side of (8.19) does converge to some element, say \( T(F) \), in \( B_p^{\alpha q}(M) \). Indeed, to see this, by (a), we need only check that

\[
\left( \sum_{j=\Omega}^{\infty} a^{jq(\alpha-n/p)} \left[ \sum_k |b^j_{\alpha_k}(F, \varphi^j_k)|^p \right]^{q/p} \right)^{1/q} < \infty.
\]

But, by (8.6) and (8.13), this quantity is less than or equal to

\[
C \left( \sum_{j=\Omega}^{\infty} a^{jq(\alpha-n/p)} \left[ \sum_k |(F, \varphi^j_k)|^p \right]^{q/p} \right)^{1/q}
\]

for some \( C \), which (by the part of (b) that we have shown), is less than or equal to \( C\|F\|_{B_p^{\alpha q}} \) for some (other) \( C \). Thus, by (a), the right side of (8.19) does converge to some element \( T(F) \in B_p^{\alpha q}(M) \), and moreover, the map \( T: B_p^{\alpha q} \rightarrow B_p^{\alpha q} \) is bounded. Next note that, if \( F \in C^\infty(M) \), then \( T(F) = F \). Indeed, the right side of (8.19) converges to \( F \) in \( L_2 \), hence in the sense of distributions. But it converges to \( T(F) \) in \( B_p^{\alpha q} \), hence also to \( T(F) \) in the sense of distributions. Thus \( T(F) = F \) as claimed. Finally, \( C^\infty \) is dense in \( B_p^{\alpha q} \) (for instance, by Theorem 7.1 (a) of [9]; the constructions in that paper show that the building blocks can be taken to be smooth). Since \( T \) is bounded, we must have \( T(F) = F \) for all \( F \in B_p^{\alpha q} \). This proves (c).

Now we complete the proof of (b). For \( F \in B_{p,0}^{\alpha q}(M) \), we have, from (c) and then (a), that

\[
\|F\|_{B_p^{\alpha q}} = \left\| \sum_{j=\Omega}^{\infty} \sum_k b^j_{\alpha_k}(F, \varphi^j_k) \varphi^j_k \right\|_{B_p^{\alpha q}} \leq C \left( \sum_{j=\Omega}^{\infty} a^{jq(\alpha-n/p)} \left[ \sum_k |b^j_{\alpha_k}(F, \varphi^j_k)|^p \right]^{q/p} \right)^{1/q},
\]

from which

\[
\|F\|_{B_p^{\alpha q}} \leq C \left( \sum_{j=\Omega}^{\infty} a^{jq(\alpha-n/p)} \left[ \sum_k |(F, \varphi^j_k)|^p \right]^{q/p} \right)^{1/q}.
\]

This proves (b).

Finally, for (d), it is clearly enough to reformulate (a) by showing that in (8.16) and (8.17), we can replace the sum \( \sum_{j=\Omega}^{\infty} \sum_k a^{-nj} s^j_k \varphi^j_k \) by \( \sum_{j=\Omega}^{\infty} \sum_k b^j_{\alpha_k} s^j_k \varphi^j_k \). (Then (d) will follow at once from this, (b) and (c)). But this reformulation of (a) is clear from (8.6) and (8.13), which imply that \( b^j_{\alpha_k} \sim a^{-nj} \), and from (a), applied with \( b^j_{\alpha_k} a^{-nj} s^j_k \) in place of \( s^j_k \). \( \square \)

We close by noting the relation of our frames to the group action and to dilations of the underlying quadratic form. Standard wavelets on the real line have the property that wavelets on the same scale may be obtained from each other by translation, while wavelets on different scales may be obtained from each other by appropriate translations and dilations. As we shall argue, something similar happens on homogeneous manifolds, at least up to constant multiples. This discussion is in large part adapted from [17] and [18].

Let \( T \) be the quasi-regular representation of \( G \) on \( L^2(M) \) (see (3.3)); this is a unitary representation, which commutes with the self-adjoint operator \( L \). Consequently, as operators on \( L^2(M) \), \( f(\Delta) \) commutes with elements of \( G \) for any bounded Borel function \( f \) on \( \mathbb{R} \), and in particular, if \( f \in \mathcal{S}(\mathbb{R}) \), which we now assume.

Recall (8.9). Fix \( j \). We claim that

\[
(8.26) \quad \text{if } gx^j_k = x^j_{k'}, \text{ then } T(g) \varphi^j_k = \varphi^j_{k'}.
\]
Indeed, if \( g \in G \), \( F \in L^2(M) \), \( x \in M \) and \( t > 0 \), we have
\[
\int_M K_t(gx, gy)F(y)dy = \int_M K_t(gx, y)F(g^{-1}y)d(y) = [f(t^2L)(T(g)L)](gx) = T(g)([f(t^2L)(F)])(gx);
\]
but this is just \([f(t^2L)(F)](x) = \int_M K_t(x, y)F(y)dy\), so
(8.27)
\[
K_t(gx, gy) = K_t(x, y)
\]
for all \( x, y \in M \). This, together with \( \text{(8.9)} \), implies \( \text{(8.20)} \) at once. Thus, for any fixed \( j \), we can obtain all of the \( \varphi_{j,k} \) by applying elements of the group \( G \) to any one of them. (For example, on the sphere, for any fixed \( j \), all of the \( \varphi_{j,k} \) are rotates of each other.) This is then true as well for the frame elements \( \phi_k \), up to constant multiples (recall \( \text{(8.10)} \)).

As far as different scales are concerned, there is a dilation in the background. Recall the discussion leading to \( \text{(2.2)} \). When we pass from the kernel of \( f(L) \) to the kernel of \( f(t^2L) \), we are replacing the \( D_j \) by \( tD_j \), or equivalently replacing the \( X_j \) by \( tX_j \), or equivalently replacing the quadratic form \( Q \) by its dilate \( Q/t^2 \).

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