On simple \( \mathcal{A} \)-multigraded minimal resolutions

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Abstract. Let \( \mathcal{A} \) be a semigroup whose only invertible element is 0. For an \( \mathcal{A} \)-homogeneous ideal we discuss the notions of simple \( i \)-syzygies and simple minimal free resolutions of \( R/I \). When \( I \) is a lattice ideal, the simple 0-syzygies of \( R/I \) are the binomials in \( I \). We show that for an appropriate choice of bases every \( \mathcal{A} \)-homogeneous minimal free resolution of \( R/I \) is simple. We introduce the gcd-complex \( \Delta_{\gcd}(b) \) for a degree \( b \in \mathcal{A} \). We show that the homology of \( \Delta_{\gcd}(b) \) determines the \( i \)-Betti numbers of degree \( b \). We discuss the notion of an indispensable complex of \( R/I \). We show that the Koszul complex of a complete intersection lattice ideal \( I \) is the indispensable resolution of \( R/I \) when the \( \mathcal{A} \)-degrees of the elements of the generating \( R \)-sequence are incomparable.

1. Notation

Let \( \mathcal{L} \subset \mathbb{Z}^n \) be a lattice such that \( \mathcal{L} \cap \mathbb{N}^n = \{0\} \) and let \( \mathcal{A} \) be the subsemigroup of \( \mathbb{Z}^n/\mathcal{L} \) generated by \( \{a_i = e_i + \mathcal{L} : 1 \leq i \leq n\} \) where \( \{e_i : 1 \leq i \leq n\} \) is the canonical basis of \( \mathbb{Z}^n \). Since the only element in \( \mathcal{A} \) with an inverse is 0, it follows that we can partially order \( \mathcal{A} \) with the relation

\[ c \geq d \iff \text{there is } e \in \mathcal{A} \text{ such that } c = d + e. \]

Let \( k \) be a field. We consider the polynomial ring \( R = k[x_1, \ldots, x_n] \). We set \( \text{deg}_\mathcal{A}(x_i) = a_i \). If \( x^\mathbf{v} = x_1^{v_1} \cdots x_n^{v_n} \), then we set

\[ \text{deg}_\mathcal{A}(x^\mathbf{v}) := v_1 a_1 + \cdots + v_n a_n \in \mathcal{A}. \]

It follows that \( R \) is positively multigraded by the semigroup \( \mathcal{A} \), see \cite{13}. The lattice ideal associated to \( \mathcal{L} \) is the ideal \( I_{\mathcal{L}} \) (or \( I_{\mathcal{A}} \)) generated by all the binomials \( x^{\mathbf{u}_+} - x^{\mathbf{u}_-} \) where \( \mathbf{u}_+, \mathbf{u}_- \in \mathbb{N}^n \) and \( \mathbf{u} = \mathbf{u}_+ - \mathbf{u}_- \in \mathcal{L} \). We note that if \( x^{\mathbf{u}_+} - x^{\mathbf{u}_-} \in I_{\mathcal{L}} \) then \( \text{deg}_\mathcal{A} x^{\mathbf{u}_+} = \text{deg}_\mathcal{A} x^{\mathbf{u}_-} \). Prime lattice ideals are the defining ideals of toric varieties and are called toric ideals, \cite{23}. In general lattice ideals arise in problems from diverse areas of mathematics, including toric geometry, integer programming, dynamical systems, graph theory, algebraic statistics, hypergeometric differential equations, we refer to \cite{12} for more details.

We say than an ideal \( I \) of \( R \) is \( \mathcal{A} \)-homogeneous if it is generated by \( \mathcal{A} \)-homogeneous polynomials, i.e. polynomials whose monomial terms have the same \( \mathcal{A} \)-degree.

1991 Mathematics Subject Classification. 13D02, 13D25.

Key words and phrases. Resolutions, lattice ideal, syzygies, indispensable syzygies, Scarf complex.
Lattice ideals are clearly $\mathcal{A}$-homogeneous. For the rest of the paper $I$ is an $\mathcal{A}$-homogeneous ideal. For $b \in \mathcal{A}$ we let $R[-b]$ be the $\mathcal{A}$-graded free $R$-module of rank 1 whose generator has $\mathcal{A}$-degree $b$. Let

$$(F_\bullet, \phi) : 0 \to F_p \xrightarrow{\phi_p} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to R/I \to 0,$$

be a minimal $\mathcal{A}$-graded free resolution of $R/I$. The $i$-Betti number of $R/I$ of $\mathcal{A}$-degree $b$, $\beta_{i,b}(R/I)$, equals the rank of the $R$-summand of $F_i$ of $\mathcal{A}$-degree $b$:

$$\beta_{i,b}(R/I) = \dim_k \text{Tor}_i(R/I, k)_b$$

and is an invariant of $I$, see [13]. The degrees $b$ for which $\beta_{i,b}(R/I) \neq 0$ are called $i$-Betti degrees. The minimal elements of the set $\{ b : \beta_{i,b}(R/I) \neq 0 \}$ are called minimal $i$-Betti degrees. The elements of $\text{Im} \phi_{i+1} = \ker \phi_i$ are the $i$-syzygies of $R/I$ in $F_\bullet$.

The problem of obtaining an explicit minimal free resolution of $R/I$ is extremely difficult. One of the factors that make this problem hard to attack, is that given a minimal free resolution one can obtain by a change of basis a different description of this resolution. To obtain some control over this, in [10] we defined simple minimal free resolutions. We also defined and studied the gcd-complex $\Delta_{\gcd}(b)$ for a degree $b \in \mathcal{A}$. We used this complex to generalize the results in [19] and to construct the generalized algebraic Scarf complex based on the connected components of $\Delta_{\gcd}(b)$ for degrees $b \in \mathcal{A}$. When $I$ is a lattice ideal we showed that the generalized algebraic Scarf complex is present in every simple minimal free resolution of $R/I$. This current paper analyzes in more detail the notions presented in [10]. We note that the original motivation for this work came from a question in Algebraic Statistics concerning conditions for the uniqueness of a minimal binomial generating set of toric ideals.

The structure of this paper is as follows. In section 2 we discuss the notion of simple $i$-syzygies of $R/I$. The simple 0-syzygies of $R/I$ when $I$ is a lattice ideal are exactly the binomials of $I$. We also discuss the notion of a simple minimal free resolution of $R/I$. This notion requires the presence of a system of bases for the free modules of the resolution. We show that for an appropriate choice of bases every $\mathcal{A}$-homogeneous minimal free resolution of $R/I$ is simple. In section 3 we discuss the gcd-complex $\Delta_{\gcd}(b)$ for a degree $b \in \mathcal{A}$. We show that the homology of $\Delta_{\gcd}(b)$ determines the $i$-Betti numbers of degree $b$. We count the numbers of binomials that could be part of a minimal binomial generating set of a lattice ideal up to a constant multiple. In section 4 we discuss the notion of indispensable $i$-syzygies. Intrinsically indispensable $i$-syzygies are present in all $\mathcal{A}$-homogeneous simple minimal free resolutions. For the 0-step and for a lattice ideal $I_\mathcal{L}$ this means that there are some binomials of the ideal $I_\mathcal{L}$ that are part (up to a constant multiple) of all $\mathcal{A}$-homogeneous systems of minimal binomial generators of $I_\mathcal{L}$. A strongly indispensable $i$-syzygy needs to be present in every minimal free resolution of $R/I$ even if the resolution is not simple. For the 0-step and for a lattice ideal $I_\mathcal{L}$ this means that there are some elements of $I_\mathcal{L}$ that are part (up to a constant multiple) of all $\mathcal{A}$-homogeneous minimal sets of generators of $I_\mathcal{L}$, where the generators are not necessarily binomials. We consider conditions for strongly indispensable $i$-syzygies to exist. We show that the Koszul complex of a complete intersection lattice ideal $I$ is indispensable when the $\mathcal{A}$-degrees of the elements of the generating $R$-sequence are incomparable.
2. Simple syzygies

We recall and generalize the definition of a simple $i$-syzygy, see [10] Definition 3.1, to arbitrary elements of an $\mathcal{A}$-graded free module. Let $F$ be a free $\mathcal{A}$-graded module of rank $\beta$ and let $B = \{E_i : t = 1, \ldots, \beta\}$ be an $\mathcal{A}$-homogeneous basis of $F$. Let $h$ be an $\mathcal{A}$-homogeneous element of $F$:

$$h = \sum_{1 \leq t \leq \beta} c_{at} x^n E_i.$$

The $S$-support of $h$ with respect to $B$ is the set

$$S_B(h) = \{x^n E_i : c_{at} \neq 0\}.$$

We introduce a partial order on the elements of $F$:

$$h' \leq h \text{ if and only if } S_B(h') \subseteq S_B(h).$$

**Definition 2.1.** Let $F$ and $B$ be as above, let $G$ be an $\mathcal{A}$-graded subset of $F$ and let $h$ be an $\mathcal{A}$-homogeneous nonzero element of $G$. We say that $h$ is simple in $G$ with respect to $B$ if there is no nonzero $\mathcal{A}$-homogeneous $h' \in G$ such that $h' < h$.

In [10] Theorem 3.4 we showed that if $(\mathbb{F}_\bullet, \phi)$ is a minimal free resolution of $R/I$ then for any given basis $B$ of $F_i$ there exists a minimal $\mathcal{A}$-homogeneous generating set of $\ker \phi_i$ consisting of simple $i$-syzygies with respect to $B$. The proof of the next proposition is an immediate generalization of the proof of that theorem and is omitted.

**Proposition 2.2.** Let $F$ be a free $\mathcal{A}$-graded module, let $B$ be an $\mathcal{A}$-homogeneous basis of $F$ and let $G$ be an $\mathcal{A}$-graded submodule of $F$. There is a minimal system of generators of $G$ each being simple in $G$ with respect to $B$.

Given an $\mathcal{A}$-homogeneous complex of free modules $(\mathbb{G}_\bullet, \phi)$ we specify $\mathcal{A}$-homogeneous bases $B_i$ for the homological summands $G_i$. The collection of these bases forms a system of bases $\mathbb{B}$. We write $\mathbb{B} = (B_i)$ and we say that $B_i$ is in $\mathbb{B}$.

**Definition 2.3.** A based complex $(\mathbb{G}_\bullet, \phi, \mathbb{B})$ is an $\mathcal{A}$-homogeneous complex $(\mathbb{G}_\bullet, \phi)$ together with a system of bases $\mathbb{B} = (B_i)$. Let $(\mathbb{G}_\bullet, \phi, \mathbb{B})$ and $(\mathbb{F}_\bullet, \phi, \mathbb{C})$ be two based complexes, $\mathbb{B} = (B_i)$ and $\mathbb{C} = (C_i)$. We say that the complex homomorphism $\omega: \mathbb{G}_\bullet \rightarrow \mathbb{F}_\bullet$ is a based homomorphism if for each $E \in B_i$, there exists an $H \in C_i$ such that $\omega(E) = cH$ for some $c \in k^*$.

Let $I$ be an $\mathcal{A}$-homogeneous ideal and let $(\mathbb{F}_\bullet, \phi)$ be a minimal $\mathcal{A}$-graded free resolution of $R/I$. We let $s$ be the projective dimension of $R/I$ and let $\beta_i$ be the rank of $F_i$. For each $i$ we suppose that $B_i$ is an $\mathcal{A}$-homogeneous basis of $F_i$ and we let $\mathbb{B} = (B_0, B_1, \ldots, B_s)$.

**Definition 2.4.** ([10] Definition 3.5) Let $(\mathbb{F}_\bullet, \phi, \mathbb{B})$ be as above. We say that $(\mathbb{F}_\bullet, \phi, \mathbb{B})$ is simple if and only if for each $i$ and each $E \in B_i$, $\phi_i(E)$ is simple in $\ker \phi_{i-1}$ with respect to $B_{i-1}$.

We remark that when $I$ is a lattice ideal then for any choice of basis $B_0$, the simple 0-syzygies of $R/I$ are the binomials of $I$. It is an immediate consequence of Proposition 2.2 that one can construct a minimal simple resolution of $R/I$ with respect to a system $\mathbb{B} = (B_0, \ldots, B_s)$ starting with $B_0 = \{1\}$, see also [10] Corollary 3.6. In the next proposition we show that any minimal free resolution $(\mathbb{F}_\bullet, \phi)$ of $R/I$ becomes simple with the right choice of bases.
Proposition 2.5. Let $I$ be an $\mathcal{A}$-homogeneous ideal and let $(\mathbf{F}_*, \phi)$ be a minimal free resolution of $R/I$. There exists a system of bases $\mathbf{B}$ so that $(\mathbf{F}_*, \phi, \mathbf{B})$ is simple.

Proof. Let $C_0 = \{1\}$ and for each $i > 0$ choose a basis $C_i = \{H_{ti} : t = 1, \ldots, \beta_i\}$ of $F_i$. Let $(\mathbf{G}_*, \theta)$ be a simple minimal free resolution of $R/I$ with respect to $\mathbf{D} = (D_0, \ldots, D_s)$ where $D_0 = \{1\}$ and $D_i = \{E_{ti} : t = 1, \ldots, \beta_i\}$. Since $\mathbf{G}_*, \mathbf{F}_*$ are both minimal projective resolutions of $R/I$ there is an isomorphism of complexes $h_\bullet : \mathbf{G}_* \to \mathbf{F}_*$ that extends the identity map on $R/I$. In particular $h_0 = id_R$. For each $i$ we let $H_{ti}' = h_i(E_{ti})$ and consider the set $B_i = \{H_{ti}' : t = 1, \ldots, \beta_i\}$. We note that $B_0 = \{1\}$. It is immediate that $B_i$ is a basis for $F_i$. We claim that $(\mathbf{F}_*, \phi, \mathbf{B})$ is simple.

Indeed for $t = 1, \ldots, \beta_1$ using the commutativity of the diagram we get that $\phi_1(H_{t1}') = \phi_1(h_1(E_{t1})) = h_0(\theta_1(E_{t1})) = \theta_1(E_{t1})$. Since $\theta_1(E_{t1})$ is simple with respect to $C_0$ it follows that $\phi_1(H_{t1}')$ is simple with respect to $B_0$. For $i > 1$ and $t = 1, \ldots, \beta_i$ we have that $\phi_i(H_{ti}') = \phi_i(h_i(E_{ti})) = h_{i-1}(\theta_i(E_{ti}))$. Suppose that $\phi_i(H_{ti}')$ were not simple with respect to $B_{i-1}$. Since $h_{i-1}$ is bijective it follows that $\theta_i(E_{ti})$ is not simple with respect to $D_{i-1}$, a contradiction. 

Let $I$ be an $\mathcal{A}$-homogeneous ideal and let $(\mathbf{F}_*, \phi)$ be a minimal free resolution of $R/I$. An $i$-syzygy $h$ of $R/I$ minimal if $h$ is part of a minimal generating set of $\ker \phi_i$. By the graded version of Nakayama’s lemma it follows that $h$ is minimal if and only if $h$ cannot be written as an $R$-linear combination of $i$-syzygies of $R/I$ of strictly smaller $\mathcal{A}$-degrees. The next theorem examines the cardinality of the set of minimal $i$-syzygies of a free resolution of $R/I$.

Theorem 2.6. Let $I$ be an $\mathcal{A}$-homogeneous ideal and let $(\mathbf{F}_*, \phi)$ be a minimal free resolution of $R/I$. Let $\equiv$ be the following equivalence relation among the elements of $F_i$: $h \equiv h'$ if and only if $h = ch', c \in \mathcal{A}^*$, and let $B_i$ be a basis of $F_i$. The set of equivalence classes of the $i$-syzygies of $R/I$ that are minimal and simple with respect to $B_i$ is finite.

Proof. We will show that the number of equivalence classes of the $i$-syzygies that are simple and have $\mathcal{A}$-degree equal to an $(i + 1)$-Betti degree $b$ of $R/I$ is finite. By [10] Theorem 3.8] if $h, h' \in \ker \phi_i$ are simple with respect to $B_i$ and $S_{B_i}(h) = S_{B_i}(h')$ then $h \equiv h'$. Thus it is enough to show that there is only a finite number of candidates for $S_{B_i}(h)$ when $h \in F_i$ has $\deg_A(h) = b$. We consider the set $C = \{x^a E_i : \deg_A(x^a E_i) = b, E_i \in B_i\}$. We note that $S_{B_i}(h) \subset \mathcal{P}(C)$ where $\mathcal{P}(C)$ is the power set of $C$. The number of basis elements $E_i \in B_i$ such that $\deg_A(E_i) \leq b$ is finite. Moreover for each $E_i$ the number of monomials $x^a$ such that $\deg_A(x^a) + \deg_A(E_i) = b$ is finite. It follows that $C$ and its power set $\mathcal{P}(C)$ are finite as desired.
3. The gcd-complex

For \( b \in A \), we let \( C_b \) equal the fiber
\[
C_b := \deg_A^{-1}(b) = \deg_{\mathcal{L}}^{-1}(b) := \{ x^u : \deg_A(x^u) = b \}.
\]
Let \( I_{\mathcal{L}} \) be a lattice ideal. The fiber \( C_b \) plays an essential role in the study of the minimal free resolution of \( R/I_{\mathcal{L}} \) as is evident from several works, see [3, 9, 10, 11, 19, 20].

We denote the support of the vector \( u = (u_j) \) by \( \text{supp}(u) := \{ i : u_i \neq 0 \} \). Next we recall the definition of the simplicial complex \( \Delta_b \) on \( n \) vertices, constructed from \( C_b \) as follows:
\[
\Delta_b := \{ F \subset \text{supp}(a) : x^a \in C_b \}.
\]
\( \Delta_b \) has been studied extensively, see for example [2, 4, 5, 7, 8, 17, 18]. Its homology determines the Betti numbers of \( R/I_{\mathcal{L}} \):
\[
\beta_i(R/I_{\mathcal{L}}) = \dim_k \tilde{H}_i(\Delta_b),
\]
see [22] or [13] for a proof.

In this section we present another simplicial complex, the gcd-complex \( \Delta_{\gcd}(b) \), whose construction is based upon the divisibility properties of the monomials of \( C_b \).

**Definition 3.1.** For a vector \( b \in A \) we define the *gcd-complex* \( \Delta_{\gcd}(b) \) to be the simplicial complex with vertices the elements of the fiber \( C_b \) and faces all subsets \( T \subset C_b \) such that \( \gcd(x^a : x^a \in T) \neq 1 \).

The example below compares graphically the two simplicial complexes in a particular case.

**Example 1.** Let \( R = \mathbb{k}[a, b, c, d] \) and let \( A \) be the semigroup \( A = \langle (4, 0), (3, 1), (1, 3), (0, 4) \rangle \). For \( b = (6, 10) \) we consider the fiber \( C_{(6,10)} = \{ bc^3, ac^2d, b^2d^2 \} \) and the corresponding simplicial complexes. We see that
\[
\Delta_{\gcd}(b) \quad \Delta_b
\]
\[
\begin{array}{c}
\bullet & a \\
\bullet & bc^3 \\
\bullet & b^2d^2 \\
\end{array}
\]
\[
\begin{array}{c}
\bullet & c \\
\bullet & d \\
\bullet & b \\
\end{array}
\]

The main theorem of this section, Theorem 3.2 was proved independently in [16].

**Theorem 3.2.** Let \( b \in A \). The gcd complex \( \Delta_{\gcd}(b) \) and the complex \( \Delta_b \) have the same homology.

**Proof.** First we consider the simplicial complex \( \Delta \) with vertices the elements of the set \( S = \{ \text{supp}(a) : x^a \in C_b \} \) and faces all subsets \( T \subset S \) such that
\[
\bigcap_{\text{supp}(a) \in T} \text{supp}(a) \neq \emptyset.
\]
We define an equivalence relation among the vertices of \( \Delta_{\gcd}(b) \): we let \( x^a \equiv x^{a'} \) if and only if \( \text{supp}(a) = \text{supp}(a') \). We note that the subcomplex \( A \) of \( \Delta_{\gcd}(b) \) on the
vertices of an equivalence class is contractible. By the Contractible Subcomplex Lemma \[6\] we get that the quotient map \(\pi: |\Delta_{\gcd}(\mathbf{b})| \rightarrow |\Delta_{\gcd}(\mathbf{b})|/|\mathbf{A}| \) is a homeotopy equivalence. A repeated application of the Contractible Subcomplex Lemma yields that \(\Delta_{\gcd}(\mathbf{b})\) and \(\Delta\) have the same homology.

Next we consider the family \(\mathcal{F}\) of the facets of \(\Delta_{\mathbf{b}}\) and the corresponding nerve complex \(N(\mathcal{F})\). The vertices of \(N(\mathcal{F})\) correspond to the facets of \(\Delta_{\mathbf{b}}\), while the faces of \(\Delta_{\mathcal{F}}\) correspond to collections of facets with nonempty intersection. It follows that \(N(\mathcal{F})\) is isomorphic to \(\Delta\). By \[21\] Theorem 7.26 the two complexes \(\Delta_{\mathbf{b}}\) and \(\Delta\) have the same homology and the theorem now follows.

The following is now immediate:

**Corollary 3.3.** Let \(I_\mathcal{L}\) be a lattice ideal.

\[
\beta_{i,\mathbf{b}}(R/I_\mathcal{L}) = \dim_k \tilde{H}_i(\Delta_{\gcd}(\mathbf{b})).
\]

The connected components of \(\Delta_{\gcd}(\mathbf{b})\) were used in \[10\] to determine certain complexes associated to a simple minimal free \(A\)-homogeneous resolution of \(R/I_\mathcal{L}\), see \[10\] Definitions 4.7 and 5.1. In Theorem 3.5 below we use the complex \(\Delta_{\gcd}(\mathbf{b})\) to determine the number of equivalence classes of minimal binomial generators of \(I_\mathcal{L}\). First we prove the following lemma:

**Lemma 3.4.** For \(\mathbf{b} \in \mathbf{A}\), let \(I_{\mathcal{L},\mathbf{b}}\) be the ideal generated by all binomials of \(I_\mathcal{L}\) of \(A\)-degree strictly smaller than \(\mathbf{b}\). Let \(G(\mathbf{b})\) be the graph with vertices the elements of \(C_\mathbf{b}\) and edges all the sets \(\{x_\mathbf{u}, x_\mathbf{v}\}\) whenever \(x_\mathbf{u} - x_\mathbf{v} \in I_{\mathcal{L},\mathbf{b}}\). A set of monomials in \(C_\mathbf{b}\) forms the vertex set of a component of \(G(\mathbf{b})\) if and only if it forms the vertex set of a component of \(\Delta_{\gcd}(\mathbf{b})\).

**Proof.** We note that if \(x_\mathbf{u}, x_\mathbf{v}\) belong to the same component of \(\Delta_{\gcd}(\mathbf{b})\) then there exists a sequence of monomials \(x_\mathbf{u} = x_\mathbf{u}_1, x_\mathbf{u}_2, \ldots, x_\mathbf{u}_s = x_\mathbf{v}\) such that \(d = \gcd(x_\mathbf{u}_1, x_\mathbf{u}_2) \neq 1\). Therefore

\[
x_\mathbf{u} - x_\mathbf{u}_2 = d(\frac{x_\mathbf{u}_1}{d} - \frac{x_\mathbf{u}_2}{d}) \in I_{\mathcal{L},\mathbf{b}}.
\]

It follows that \(x_\mathbf{u} - x_\mathbf{v} \in I_{\mathcal{L},\mathbf{b}}\) and \(x_\mathbf{u}, x_\mathbf{v}\) belong to the same component of \(G(\mathbf{b})\).

For the converse we note that the binomials of degree \(\mathbf{b}\) in \(I_{\mathcal{L},\mathbf{b}}\) are spanned by binomials of the form \(x_\mathbf{a}(x_\mathbf{r} - x_\mathbf{s})\) where \(x_\mathbf{a} \neq 1\). Moreover any such binomial determines an edge from \(x_\mathbf{a}^{x_\mathbf{r}+s}\) to \(x_\mathbf{a}^{x_\mathbf{r}+s}\) in \(\Delta_{\gcd}(\mathbf{b})\). Thus if \(x_\mathbf{u}, x_\mathbf{v}\) lie in the same component of \(G(\mathbf{b})\) then any minimal expression of \(x_\mathbf{u} - x_\mathbf{v}\) as a sum of binomials \(x_\mathbf{a}(x_\mathbf{r} - x_\mathbf{s})\) results in a path from \(x_\mathbf{u}\) to \(x_\mathbf{v}\) in \(\Delta_{\gcd}(\mathbf{b})\). \(\square\)

The graph \(G(\mathbf{b})\) was first introduced in \[9\] to determine the number of different binomial generating sets of a toric ideal \(I_\mathcal{L}\). The results stated for toric ideals in \[9\] hold more generally for lattice ideals with identical proofs. We choose an ordering of the connected components of \(\Delta_{\gcd}(\mathbf{b})\) and let \(t_i(\mathbf{b})\) be the number of vertices of the \(i\)-th component of \(\Delta_{\gcd}(\mathbf{b})\).

**Theorem 3.5.** Let \(I_\mathcal{L}\) be a lattice ideal and consider the equivalence relation on \(R\) of Theorem 2.7. The cardinality of the set \(T\) of equivalence classes of the minimal binomials of \(I_\mathcal{L}\) is given by

\[
|T| = \sum_{\mathbf{b} \in \mathbf{A}} \sum_{i \neq j} t_i(\mathbf{b})t_j(\mathbf{b}).
\]
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Proof. In the course of the proof of [9, Theorem 2.6] applied to the lattice ideal \( I_L \) it was shown that the minimal binomials of \( A \)-degree \( b \) are the difference of monomials that belong to different connected components of \( G(b) \). Lemma 3.3 and a counting argument finishes the proof. \( \Box \)

We remark that if \( b \in A \) is not a 1-Betti degree of \( R/I_L \), then there is no minimal binomial generator of \( A \)-degree \( b \). It follows that \( \Delta_{\gcd}(b) \) has exactly one connected component. The nontrivial contributions to the formula of Theorem 3.5 come from the 1-Betti degrees of \( R/I_L \).

4. Indispensable syzygies

In this section we discuss the notion of indispensable complexes that first appeared in [10, Definition 3.9]. Intrinsically an indispensable complex of \( R/I \) is a based complex \((F\bullet, \phi, B)\) that is “contained” in any based simple minimal free resolution of \( R/I \).

The indispensable binomials of a lattice ideal \( I_L \) are the binomials that appear in every minimal system of binomial generators of the ideal up to a constant multiple. They were first defined in [15] and their study was originally motivated from Algebraic Statistics; see [1, 14, 15, 24] for a series of related papers.

Theorem 4.1. Let \( I_L \) be a lattice ideal. The indispensable binomials of \( I_L \) occur exactly in the minimal \( A \)-degrees \( b \) such that \( \Delta_{\gcd}(b) \) consists of two disconnected vertices.

Proof. This theorem was proved in [9] for toric ideals. The same proof applies to lattice ideals. \( \Box \)

An immediate consequence of Theorem 4.1 is the following:

Corollary 4.2. Let \( I_L \) be a lattice ideal and \( S \) a minimal system of \( A \)-homogeneous (not necessarily binomial) generators of \( I_L \). If \( f \) is an indispensable binomial of \( I_L \) then there is a \( c \in k^* \) such that \( cf \in S \).

Proof. Let \( f \) be an indispensable binomial of \( I_L \) and \( b = \deg_A f \). Since \( H_1(\Delta_{\gcd}(b)) = 1 \) there is a unique element \( f' \) in \( S \) of \( A \)-degree \( b \). Since \( C_b \) is a set with exactly two elements it follows that \( f' \) is a binomial. Since \( I_L \) contains no monomials, it follows that \( f' = cf \) for some \( c \in k^* \). \( \Box \)

It is clear that if \((F\bullet, \phi, B)\) is a minimal free resolution of \( R/I \) and \( f \) is an indispensable binomial, then there exists an element \( E \in B_1 \) and a \( c \in k^* \) such that \( \phi_1(E) = cf \). We let the indispensable 0-syzygies of \( R/I \) to be the indispensable binomials of \( I_L \). We extend the definition for \( i \geq 0 \):

Definition 4.3. Let \((F\bullet, \phi, B)\) be a based complex. We say that \((F\bullet, \phi, B)\) is an indispensable complex of \( R/I \) if for each based minimal simple free resolution \((G\bullet, \theta, C)\) of \( R/I \) where \( C_0 = \{1\} \), there is an injective based homomorphism \( \omega : (F\bullet, \phi, B) \to (G\bullet, \theta, C) \) such that \( \omega_0 = id_R \). If \( B = (B_j) \) and \( E \in B_{i+1} \) we say that \( \phi_{i+1}(E) \in F_i \) is an indispensable \( i \)-syzygy of \( R/I \).

It follows immediately from the definition that an indispensable \( i \)-syzygy of \( R/I \) is simple. Moreover if \((F\bullet, \phi, B)\) is an indispensable complex of \( R/I \) and \((G\bullet, \theta, W)\) is a minimal simple free resolution of \( R/I \) then the based homomorphism of Definition 4.3 is unique, up to rearrangement of the bases elements of the same \( A \)-degree.
and constant factors. In [10] Theorem 5.2 we showed that if \( I_L \) is a lattice ideal then the generalized algebraic Scarf complex is an indispensable complex.

The next theorem examines when the Koszul complex of a lattice ideal generated by an \( R \)-sequence of binomials is indispensable. Let \( I \) be an ideal generated by an \( R \)-sequence \( f_1, \ldots, f_s \) and let \((\mathbf{K}_*, \phi)\) be the Koszul complex on the \( f_i \). We denote the basis element \( e_{j_1} \wedge \cdots \wedge e_{j_t} \) of \( K_i \) by \( e_J \) where \( J \) is the ordered set \( \{j_1, \ldots, j_t\} \) and let \( \text{sgn}[j_k, J] = (-1)^{k+1} \). For each \( j \in J \) we write \( J_j \) for the set \( J \setminus \{j\} \). The canonical system of bases \( \mathbf{B} = (B_0, \ldots, B_s) \) consists of the following: \( B_0 = \{1\} \), \( B_1 = \{e_i : i = 1, \ldots, s\} \) where \( \phi(e_i) = f_i \) and \( B_i = \{e_J : J = \{j_1, \ldots, j_t\}, 1 \leq j_1 < \cdots < j_t \leq s\} \) where

\[
\phi_i(e_J) = \sum_{j \in J} \text{sgn}[j, J] f_j e_{J_j}.
\]

In [10] Example 3.7 it was shown that \((\mathbf{K}_*, \phi, \mathbf{B})\) is a simple minimal free resolution of \( R/I \).

**Theorem 4.4.** Let \( I_L = (f_1, \ldots, f_s) \) be a lattice ideal where \( \{f_i : i = 1, \ldots, s\} \) is an \( R \)-sequence of binomials such that \( B_i = \text{deg}_A(f_i) \) are incomparable. Let \((\mathbf{K}_*, \phi)\) be the Koszul complex on the \( f_i \) and let \( \mathbf{B} \) be the canonical system of bases of \( \mathbf{K} \). Then \((\mathbf{K}_*, \phi, \mathbf{B})\) is an indispensable complex of \( R/I_L \).

**Proof.** Let \( f_i = x^{b_i} - x^{c_i} \). We note that if \( e_J \in B_i \) then

\[
\text{deg}_A(e_J) = \sum_{i \in J} \text{deg}_A f_i
\]

and \((\mathbf{K}_*, \phi)\) is \( \mathcal{A} \)-homogeneous. The incomparability assumption on the degrees of the \( f_i \) shows that each \( B_i \) is minimal and that \( \beta_{i, B_i}(R/I) = 1 \). It follows that \( f_i \) is an indispensable binomial, see [9] Corollary 3.8. We also note that for each \( i \), \( C_{b_i} \) consists of exactly two monomials.

Let \((\mathbf{G}_*, \theta, \mathbf{W})\) be a simple minimal resolution of \( R/I_L \) where \( \mathbf{W} = (W_0, \ldots, W_s) \) and \( W_0 = \{1\} \). We let \( \omega_0 = \text{id}_R : K_0 \to G_0 \). We prove that there is a based isomorphism \( \omega : (\mathbf{K}_*, \phi, \mathbf{B}) \to (\mathbf{G}_*, \theta, \mathbf{W}) \) which extends \( \omega_0 \) by showing that if \( \omega_i : K_i \to G_i \) has been defined for \( i \leq k \) then \( \omega_{k+1} \) can be constructed with the desired properties. Thus we assume that for each basis element \( e_J \) of \( B_k \) there exists \( c_J \in \mathbb{K}^* \) and \( H_J \in W_k \) such that \( \omega_k(e_J) = c_J H_J \). We note that if \( e_L \in B_{k+1} \) then \( \omega_k \phi_{k+1}(e_L) \).

i.e.

\[
\sum_{j \in L} \text{sgn}[j, L] f_j c_{L,j} H_{L,i}
\]

is a simple \( k \)-syzygy with respect to \( W_k \). This follows as in the proof of [9] Corollary 3.8. We will define \( \omega_{k+1} : K_{k+1} \to G_{k+1} \) by specifying its image in the basis elements \( e_L \) of \( B_k \) so that the following identity holds:

\[
\theta_{k+1} \omega_{k+1}(e_L) = \omega_k \phi_{k+1}(e_L)\,.
\]

Since \( \omega_k \phi_{k+1}(e_L) \) is a \( k \)-syzygy, it follows that

\[
\omega_k \phi_{k+1}(e_L) = \sum_{i=1}^{t} \theta_{k+1}(p_i H_i)
\]

where \( H_i \in W_{k+1} \) and \( \text{deg}_A(p_i H_i) = \text{deg}_A(e_L) \). We will show that \( t = 1 \). First we notice that for some \( i \)

\[
S_{W_k}(\theta_{k+1}(p_i H_i)) \cap S_{W_k}(\omega_k(\phi_{k+1}(e_L))) \neq \emptyset.
\]
Without loss of generality we can assume that this is the case for \( i = 1 \) and we write \( H \) in place of \( H_1 \). Moreover we can assume that

- \( \mathcal{K} = \{1, \ldots, k + 1\} \) and that
- \( x^{a_1}H_{k+1} \in S_{W_z}(\theta_{k+1}(H)) \cap S_{W_z}(\omega_k(\phi_{k+1}(e_L))) \).

Let \( q_L = \) be the coefficient of \( H_{k+1} \) in \( \theta_{k+1}(H) \). We have that \( \deg_A(p_1q_L) = b_1 \). We will show that \( p_1q_L \) is a constant multiple of \( f_1 \). For \( t \in L_1 \) we write \( L_{1,t} \) for the set \( L_1 \setminus \{t\} \). Since \( \theta_{1,0}(H) = 0 \) the coefficient of \( H_{L_{1,t}} \), in \( \theta_{k+1}(H) \) must be zero for any \( t \in L_1 \). The contributions to this coefficient come from the differentiation of the term of \( \theta_{k+1}(H) \) involving \( H_{L_{1,t}} \), and all other terms of \( \theta_{k+1}(H) \) involving \( H_{L_{1,t}} \)

where \( L' \setminus \{t'\} = L_{1,t} \). Let \( X \) be the set consisting of such \( L' \) and let \( q' \) be the coefficient of \( H_{L_{1,t}} \) when \( L' \in X \). We get

\[
0 = \text{sgn}[t, L_1]q_L + \sum_{L' \in X} q' \text{sgn}[t', L']f_{t'}.
\]

Since \( f_1, \ldots, f_s \) is a complete intersection it follows that \( q_L \in \langle f_L : L' \in X \rangle \). Therefore \( b_1 \geq \deg_A(q_L) \geq \deg_A(f_L) \) for at least one \( t' \). By the incomparability of the degrees of the \( f_i \) it follows that \( t' = 1 \), and \( q_L \) is a constant multiple of \( f_1 \) and thus \( p_1 \in k^* \). Moreover we have shown that for each \( t \) in \( L_1 \) there is a term in \( \theta_{k+1}(H) \) involving \( H_{L_{1,t}} \). By a degree consideration it follows that the coefficient of this term has degree \( b_1 \) and thus repeating the above steps we can conclude that the coefficient of this term is a constant multiple of \( f_1 \). It follows that

\[
S_{W_z}(\omega_k(\phi_{k+1}(e_L))) \subset S_{W_z}(\theta_{k+1}(H)).
\]

Since \( \theta_{k+1}(H) \) is simple it follows that

\[
S_{W_z}(\omega_k(\phi_{k+1}(e_L))) = S_{W_z}(\theta_{k+1}(H)),
\]

and \( \theta_{k+1}(H) = c \omega_k(\phi_{k+1}(e_L)) \) where \( c \in k^* \). We let \( H_L = H \) and \( c_L = c^{-1} \). It follows that the homomorphism \( \omega_{k+1} : K_{k+1} \rightarrow G_{k+1} \) defined by setting

\[
\omega_{k+1}(e_L) = c_L H_L
\]

has the desired properties.

Next we consider strongly indispensable complexes.

**Definition 4.5.** Let \( (F, \phi, B) \) be a based complex. We say that \( (F, \phi, B) \) is a strongly indispensable complex of \( R/I \) if for every based minimal free resolution \( (G, \theta, C) \) of \( R/I \), (not necessary simple) with \( C_0 = \{1\} \), there is an injective based homomorphism \( \omega : (F, \phi, B) \rightarrow (G, \theta, C) \) such that \( \omega_0 = id_R \). If \( B = (B_j) \) and \( E \in B_i \) we say that \( \phi_{i+1}(E) \in F_i \) is a strongly indispensable \( i \)-syzygy of \( R/I \).

Strongly indispensable complexes are indispensable. This is a strict inclusion as \([10]\), Example 6.5] shows. When \( I_C \) is a lattice ideal, the algebraic Scarf complex \([19]\), Construction 3.1], is shown to be “contained” in the minimal free resolution of \( R/I_C \), \([19]\) Theorem 3.2], and is a strongly indispensable complex. Moreover as follows from Corollary \([12]\), the strongly indispensable 0-syzgies of \( R/I_C \) coincide with the indispensable 0-syzgies of \( R/I_C \) and are the indispensable binomials of \( I_C \). For higher homological degrees this is no longer the case. First we note the following:

**Theorem 4.6.** Let \( I_C \) be a lattice ideal and let \( (F, \phi, B) \) be a strongly indispensable complex for \( R/I_C \). Let \( B = (B_j) \), \( E \in B_{i+1} \) and \( \deg_A(E) = b \). Then \( \dim_k \tilde{H}_i(\Delta_{gcd}(b)) = 1 \) and \( b \) is a minimal \( i \)-Betti degree of \( R/I_C \).
Proof. Suppose that \( \dim_k \tilde{H}_i(\Delta_{gcd}(b)) > 1 \) or that there is an \( i \)-Betti degree \( b' \) such that \( b' < b \). Let \((G_\bullet, \theta, C)\) be a minimal resolution of \( R/I_L \) where \( C = (C_i) \), let \( \omega : (F_\bullet, \phi, B) \longrightarrow (G_\bullet, \theta, C) \) be the based homomorphism of Definition 4.5 and suppose that \( \omega(E) = cH \) where \( H \in C_{i+1} \) and \( c \in k^* \). By our assumptions there exists \( H' \in C_{i+1} \) such that \( H' \neq H \) and \( \deg_{A}(H) \leq b \). Let \( x^a \in C_{b-b'} \). By replacing \( H \) with \( H + x^a H' \) we get a new basis \( C'_i+1 \) of \( G_{i+1} \) and a new system of bases \( C' = (C'_j) \), where \( C'_j = C_{j-1} \) for \( j \neq i+1 \). Let \( \omega' : (F_\bullet, \phi, B) \longrightarrow (G_\bullet, \theta, C') \) be the based homomorphism of Definition 4.5 \( \omega_j = \omega'_j \) for \( j \leq i \). Let \( H'' \in C_{i+1} \) be such that \( \omega'_{i+1}(E) = c'H'' \) where \( c' \in k^* \). Thus

\[
\theta_{i+1}(c'H'') = \theta_{i+1}(\omega'_{i+1}(E)) = \omega'_{i+1}(E) = \omega_{i+1}(E) = \theta_{i+1}(\omega_{i+1}(E)) = c\theta_{i+1}(cH).
\]

It follows that \( c'H'' - cH \in \ker \theta_{i+1} \). If \( H'' \neq H + x^a H' \) then \( H'' \in C_{i+1} \) and we get a direct contradiction to the minimality of \((G_\bullet, \theta, C)\). If \( H'' = H + x^a H' \) then \( \theta_{i+1}(c'H - c\omega_{i+1}(E)) = 0 \). Examination of the two cases when (a) \( c' \neq c \), and (b) \( c' = c \), leads again to a contradiction of the minimality of the resolution \((G_\bullet, \theta, C)\). 

Theorem 4.6 shows that the two conditions

1. \( \dim_k \tilde{H}_i(\Delta_{gcd}(b)) = 1 \)
2. \( b \) is a minimal \( i \)-Betti degree

are necessary for the existence of a strongly indispensable \( i \)-syzygy in \( A \)-degree \( b \). The following example shows that these conditions are not sufficient for the existence of an indispensable \( i \)-syzygy and consequently of a strongly indispensable \( i \)-syzygy.

Example 2. Consider the lattice ideal \( I_L = \langle f_1, f_2 \rangle \) where \( f_1 = x_1 - x_2 \), \( f_2 = x_2 - x_3 \) and \( \deg_{A} f_i = 1 \). Let \((K_\bullet, \phi)\) be the Koszul complex on the \( f_i \). By considering the \( i \)-Betti numbers for \( i = 1, 2 \) it is immediate that \( \dim_k H_2(\Delta_2) = 1 \) and \( 2 \) is a minimal 2-Betti degree. However there is no indispensable complex of length greater than 0, since the generators of \( I_L \) are not indispensable binomials.

Generic lattice ideals are characterized by the condition that the binomials in a minimal generating set have full support, [19]. In this case the Scarf complex is a minimal free resolution of \( R/I_L \) and each of the Betti degrees of \( R/I_L \) satisfy the conditions of Theorem 4.6. We finish this section by giving the strongest result for the opposite direction of Theorem 4.6.

Theorem 4.7. Let \( I_L \) be a lattice ideal. The \( A \)-homogeneous minimal free resolution \((F_\bullet, \phi, B)\) of \( R/I_L \) is strongly indispensable if and only if for each \( i \)-Betti degree \( b \) of \( R/I_L \), \( b \) is a minimal \( i \)-Betti degree and \( \dim_k \tilde{H}_i(\Delta_{gcd}(b)) = 1 \).

Proof. One direction of this theorem follows directly from Theorem 4.6. For the other direction we assume that \( b \) is minimal whenever \( b \) is an \( i \)-Betti degree and that \( \dim_k \tilde{H}_i(\Delta_{gcd}(b)) = 1 \) for all \( i \). Let \((G_\bullet, \theta, D)\) be a minimal free resolution of \( R/I_L \). By assumption the \( A \)-degrees of the elements of \( D_i \) are distinct and incomparable. It follows that the \( A \)-homogeneous isomorphism \( \omega : F \longrightarrow G \) that extends \( \id_R : F_0 \longrightarrow C_0 \) is a based homomorphism.

Acknowledgment
The authors would like to thank Ezra Miller for his essential comments on this manuscript.
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