Dunkl completely monotonic functions

Khaled Mehrez\(^a\) and Jamel El Kamel\(^b\)

\(^a\)Faculté de Sciences de Tunis, Université Tunis el Manar, Tunis, Tunisia; \(^b\)Département de Mathématiques, FSM, Monastir, Tunisia

**ABSTRACT**

In this paper our aim is to introduce the notion of Dunkl completely monotonic functions on \((−\sigma, \sigma), \sigma > 0\), in studying their properties and we present some examples. As application, we study the Dunkl completely monotonicity of a class of functions related to the Kummer confluent hypergeometric functions. Moreover, in the end of the paper we pose an open problem.

1. Introduction

Positive definite and completely monotonic functions play an important role in harmonic analysis, for examples, in theory of scattered data interpolation, probability theory and potential theory. The most important facts about positive definite functions are the connection with completely monotonic functions.

In classical analysis a complex valued continuous function \(f\) is said positive definite (resp. strictly positive definite) on \(\mathbb{R}\), if for every set of distinct real numbers \(x_1, x_2, \ldots, x_n\) and every set of complex numbers \(z_1, z_2, \ldots, z_n\) not all zero, the inequality

\[
\sum_{j=1}^{n} \sum_{k=1}^{n} z_j \overline{z_k} f(x_j - x_k) \geq 0 \quad (\text{resp.} > 0)
\]

holds true (see [7]).

In 1930, the class of positive definite functions is fully characterized by Bochner’s theorem [1], the function \(f\) being positive definite, if and only if, it is the Fourier transform of a positive finite Borel measure \(\mu\) on the real line \(\mathbb{R}\) :

\[
f(x) = \int_{\mathbb{R}} e^{-itx} d\mu(t).
\]
In [6], we have introduced the notion of Dunkl positive definite and strictly Dunkl positive definite functions on \( \mathbb{R}^d \). We have established the analogue of Bochner’s theorem in Dunkl setting.

A function \( f \) on \((a, b)\) is called completely monotonic on \((a, b)\), if it satisfies \( f \in C^\infty((a, b)) \) and
\[
( -1)^n f^{(n)}(x) \geq 0,
\]
for all \( n = 0, 1, 2, \ldots \) and \( a < x < b \) (see [16]).

Bernstein’s Theorem [16, p. 161], states that a function \( f : [0, \infty) \rightarrow \mathbb{R} \) is completely monotonic on \([0, \infty)\), if and only if,
\[
f(x) = \int_0^\infty e^{-tx}d\mu(t)
\]
where \( \mu \) is a nonnegative finite Borel measure on \([0, \infty)\).

In 1938, Schoenberg’s theorem [10], asserts that a function \( \varphi \) is completely monotonic on \([0, \infty)\), if and only if, \( \Phi := \varphi(\|\cdot\|^2) \) is positive definite on every \( \mathbb{R}^d \).

The paper is organized as follows. In Section 2, we present some preliminary results and notations that will be useful in the sequel. In Section 3, we give some properties of the Dunkl kernel, the Dunkl transform, and the Dunkl translation. In Section 4, we recall some results about Dunkl positive definite functions proved by the authors in [6]. In Section 5, we introduce the notion of Dunkl completely monotonic functions in studying their properties, some examples are given. As application, we study the Dunkl completely monotonicity of a class of functions related to the Kummer confluent hypergeometric functions. Moreover, at the end of the paper we pose an open problem, which may be of interest for further research.

Let us recall some classical functional spaces:

- \( C(\mathbb{R}^d) \) the set of continuous functions on \( \mathbb{R}^d \), \( C_0(\mathbb{R}^d) \) its subspace of continuous functions on \( \mathbb{R}^d \) vanishing at infinity and \( C^\infty(\mathbb{R}^d) \) its subspace of infinitely differentiable functions.
- \( \mathcal{S}(\mathbb{R}^d) \) the Schwartz space.
- \( L^p(\mathbb{R}^d, h^2_\kappa) \), \( 1 \leq p < \infty \), the space of measurable functions on \( \mathbb{R}^d \) such that
\[
\|f\|_{\kappa,p} := \left( \int_{\mathbb{R}^d} |f(x)|^p h^2_\kappa(x)dx \right)^{1/p} < \infty.
\]

Let \( \sigma > 0, M_\sigma(\mathbb{R}) \) denotes the space of nonnegative finite Borel measures on \( \mathbb{R} \) satisfying
\[
\int_0^\infty e^{\sigma|x|}d\mu(x) < \infty,
\]
and
\[
M_\infty(\mathbb{R}) = \cap_{\sigma > 0} M_\sigma(\mathbb{R}).
\]

2. Notations and preliminaries

Let \( R \) be a fixed root system in \( \mathbb{R}^d \), \( G \) the associated finite reflexion group, and \( R_+ \) a fixed positive subsystem of \( R \), normalized so that \( \langle \alpha, \alpha \rangle = 2 \) for all \( \alpha \in R_+ \), where \( \langle x, y \rangle \) denotes the usual Euclidean inner product.
For a non zero $\alpha \in \mathbb{R}^d$, let use define the reflexion $\sigma_\alpha$ by
\[ \sigma_\alpha x = x - 2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad x \in \mathbb{R}^d. \]

Let $k$ be a nonnegative multiplicity function $\alpha \mapsto k_\alpha$ defined on $\mathbb{R}^+$ with the property that $k_\alpha = k_\beta$ where $\sigma_\alpha$ is conjugate to $\sigma_\beta$ in $G$. The weight function $h_k$ is defined by
\[ h_k(x) = \prod_{\alpha \in \mathbb{R}^+} |\langle x, \alpha \rangle|^{k_\alpha}, \quad x \in \mathbb{R}^d. \] (1)

This is a nonnegative homogeneous function of degree $\gamma_k = \sum_{\alpha \in \mathbb{R}^+} k_\alpha$, which is invariant under the reflexion group $G$.

Let $T_i$ denote Dunkl’s differential-difference operator defined in [3] by
\[ T_i f(x) = \partial_i f(x) + \sum_{\alpha \in \mathbb{R}^+} \kappa_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} (\alpha, e_i), \quad 1 \leq i \leq d, \] (2)

where $\partial_i$ is the ordinary partial derivative with respect to $x_i$, and $e_1, e_2, \ldots, e_d$ are the standard unit vectors of $\mathbb{R}^d$.

The rank-one case: in case $d = 1$, the only choice of $R$ is $R = \{ \pm \sqrt{2} \}$. The corresponding reflexion group is $G = \{ id, \sigma \}$ action on $\mathbb{R}$ by $\sigma(x) = -x$. The Dunkl operator $T := T_k$ associated with the multiplicity parameter $k \in \mathbb{C}$ is given by
\[ T_k f(x) = f'(x) + k \frac{f(x) - f(-x)}{x}. \]

Let $P_n^d$ denote the space of homogeneous polynomials of degree $n$ in $d$-variables. The operators $T_i$, $1 \leq i \leq d$ map $P_n^d$ to $P_{n-1}^d$.

The intertwining operator $V_k$ is linear operator and determined uniquely as
\[ V_k P_n^d \subset P_n^d, \quad V_k 1 = 1, \quad T_i V_k = V_k \partial_i, \quad 1 \leq i \leq d. \] (3)

According to Rösler [9], $V_k$ is a positive operator. De Jeu [2], prove that $V_k$ is an isomorphism of $C^\infty(\mathbb{R}^d)$ whose inverse is denoted by $W_k$ and admit the following integral representation.

**Theorem 1:** For $f \in C(\mathbb{R}^d)$, we have
\[ V_k f(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y), \quad x \in \mathbb{R}^d, \]
where $\mu_x$ is a probability measure on $\mathbb{R}^d$, whose carrier is in the closed ball $B(0, \| x \| )$.

The Dunkl kernel associated with $G$ and $k$ is defined by [4]: for $y \in \mathbb{C}^n$
\[ E_k(x, y) = V_k \left( e^{i\langle y, \cdot \rangle} \right)(x), \quad x \in \mathbb{R}^d. \]
\[ E_k(x, iy) = V_k \left( e^{i\langle y, \cdot \rangle} \right)(x), \quad x, y \in \mathbb{R}^d. \]
plays the role of $e^{i(x,y)}$ in the ordinary Fourier analysis.

In the rank-one case: for the group $G = \mathbb{Z}_2$, $\Re(k) > 0$ we have

$$V_kf(x) = \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(k)} \int_{-1}^{1} f(x(t)(1 - t)^{k-1}(1 + t)^k dt.$$ 

In particular, for $x, y \in \mathbb{C}, \Re(k) > 0$

$$E_k(x, y) = \frac{\Gamma(k + \frac{1}{2})}{\Gamma(\frac{1}{2}) \Gamma(k)} \int_{-1}^{1} e^{xt}(1 - t)^{k-1}(1 + t)^k dt.$$ 

$$E_k(x, y) = j_{k-\frac{1}{2}}(ixy) + \frac{x y}{(2k + 1)} j_{k+\frac{1}{2}}(ixy)$$

where for $\alpha \geq -\frac{1}{2}, j_\alpha$ is the normalized Bessel function.

**Proposition 1** [8]: Let $k \geq 0$ and $y \in \mathbb{C}^d$. Then the function $f = E_k(., y)$ is the unique solution of the system

$$T_i f = \langle e_i, y \rangle f, \text{ for all } 1 \leq i \leq d, \tag{5}$$

which is real-analytic on $\mathbb{R}^d$ and satisfies $f(0) = 1$.

**Proposition 2** [5,8]: For $x, y \in \mathbb{C}^d$, $\lambda \in \mathbb{C}$

1. $E_k(x, y) = E_k(y, x)$,
2. $E_k(\lambda x, y) = E_k(x, \lambda y)$,
3. $E_k(x, y) = E_k(\overline{x}, \overline{y})$,
4. $|E_k(−ix, y)| \leq 1$,
5. $|E_k(x, y)| \leq e^{|x||y|}$.

**3. Harmonic analysis related to the Dunkl operator**

In this section, we present some properties of the Dunkl transform, the Dunkl translation and the Dunkl convolution studied and developed in great detail in [2,5,11,13].

The Dunkl transform is defined for $f \in L^1(\mathbb{R}^d, h^2_\kappa)$ by

$$D_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(−iy, y) h^2_\kappa(y) dy, \ x \in \mathbb{R}^d. \tag{6}$$

If $\kappa = 0$, then $V_k = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$, then the Dunkl transform is related closely to the Hankel transform on the real line.

**Theorem 2** [11]:

1. For $f \in L^1(\mathbb{R}^d, h^2_\kappa)$, we have $D_\kappa f \in C_0(\mathbb{R}^d)$, and

$$\|D_\kappa f\|_{C_0} \leq \|f\|_{\kappa, 1}.$$
When both \( f \) and \( D_\kappa f \) belong to \( \mathcal{L}^1(\mathbb{R}^d, h_\kappa^2) \), we have the inversion formula
\[
f(x) = c_\kappa \int_{\mathbb{R}^d} D_\kappa f(y)E_\kappa(ix, y)h_\kappa^2(y)dy.
\]

The Dunkl transform \( D_\kappa \) is an isomorphism of the Schwartz class \( \mathcal{S}(\mathbb{R}^d) \) onto itself, and \( D_\kappa^2 f(x) = f(-x) \).

The Dunkl transform \( D_\kappa \) on \( \mathcal{S}(\mathbb{R}^d) \) extends uniquely to an isometry of \( L^2(\mathbb{R}^d, h_\kappa^2) \).

If \( f, g \in L^2(\mathbb{R}^d, h_\kappa^2) \) then
\[
\int_{\mathbb{R}^d} D_\kappa f(y)g(y)h_\kappa^2(y)dy = \int_{\mathbb{R}^d} f(y)D_\kappa g(y)h_\kappa^2(y)dy.
\]

Let \( y \in \mathbb{R}^d \) be given. The Dunkl translation operator \( f \mapsto \tau_y f \) is defined in \( L^2(\mathbb{R}^d, h_\kappa^2) \) by the equation
\[
D_\kappa (\tau_y f)(x) = E_\kappa(iy, x)D_\kappa f(x), \quad x \in \mathbb{R}^d.
\]

The above definition gives \( \tau_y f \) as an \( L^2 \) function.

Let
\[
A_\kappa(\mathbb{R}^d) = \left\{ f \in \mathcal{L}^1(\mathbb{R}^d, h_\kappa^2) : D_\kappa f \in \mathcal{L}^1(\mathbb{R}^d, h_\kappa^2) \right\}.
\]

Note that \( A_\kappa(\mathbb{R}^d) \) is contained in the intersection of \( \mathcal{L}^1(\mathbb{R}^d, h_\kappa^2) \) and \( L^\infty \) and hence is a subspace of \( \mathcal{L}^2(\mathbb{R}^d, h_\kappa^2) \). For \( f \in A_\kappa(\mathbb{R}^d) \) we have
\[
\tau_y f(x) = \int_{\mathbb{R}^d} E_\kappa(ix, y)E_\kappa(-iy, \xi)D_\kappa f(\xi)h_\kappa^2(\xi)d\xi.
\]

Before stating some properties of the generalized translation operator let us mention that there is an abstract formula for \( \tau_y \) given in terms of intertwining operator \( V_k \) and its inverse \( W_k \). It takes the form [13]. For \( f \in C^\infty(\mathbb{R}^d) \) we have
\[
\tau_y f(x) = V_k^{(x)} \otimes V_k^{(y)}(W_k f(x - y)).
\]

**Theorem 3 [12]:** If \( \varphi \in A_\kappa(\mathbb{R}) \), then
\[
W_k \varphi(x) = \frac{1}{c_\kappa} \int_{\mathbb{R}^d} e^{i(x, y)} D_\kappa \varphi(y)h_\kappa^2(y)dy.
\]

**4. Strictly Dunkl positive definite functions**

**Definition 1:** A function \( \varphi \) of \( L^2(\mathbb{R}^d, h_\kappa^2) \) is called Dunkl positive definite (resp. strictly Dunkl positive definite) if for every finite distinct real numbers \( x_1, \ldots, x_n \), and every complex numbers \( \alpha_1, \ldots, \alpha_n \), not all zero, the inequality
\[
\sum_{j=1}^n \sum_{k=1}^n \alpha_j \overline{\alpha_k} \tau_{x_j}(\varphi)(x_k) \geq 0, \quad (\text{resp.} > 0)
\]
holds true.
Theorem 4 [6]: Let \( \varphi \in A_\kappa(\mathbb{R}^d) \), non-identically zero and Dunkl positive definite function. Then \( \varphi \) is strictly Dunkl positive definite.

Theorem 5 [6]: Let \( \varphi \in A_\kappa(\mathbb{R}^d) \). Then, \( \varphi \) is Dunkl positive definite, if and only if, there exist a nonnegative function \( \psi \in A_\kappa(\mathbb{R}^d) \) such that

\[
\varphi = D_\kappa \psi.
\] (11)

5. Dunkl completely monotonic functions

Definition 2: A function \( \varphi \) is called Dunkl completely monotonic on \( (-\sigma, \sigma) \), \( \sigma > 0 \) if \( \varphi \in C ((-\sigma, \sigma)) \) has derivatives for all orders on \( (-\sigma, \sigma) \) and

\[
(-1)^n T^n_k \varphi(x) \geq 0
\] (12)

for all \( n \in \mathbb{N} \) and \( x \in (-\sigma, \sigma) \).

For \( k = 0 \), \( T_k f = f' \), we retrieve the classical definition.

Remark 1: It’s clear that if \( \varphi \) and \( \psi \) are Dunkl completely monotonic, then \( \alpha \varphi + \beta \psi \) too, where \( \alpha \) and \( \beta \) are nonnegative constants.

Example 1: For \( y \geq 0 \), the function \( x \mapsto E_k(-x, y) \) is Dunkl completely monotonic on \( \mathbb{R} \). Indeed, for \( x, y \in \mathbb{R} \) we have:

\[
E_k(x, y) \geq 0
\]

and

\[
T_k E_k(-x, y) = -y E_k(-x, y).
\]

Thus

\[
(-1)^n T^n_k E_k(-x, y) = y^n E_k(-x, y) \geq 0; \ y \geq 0.
\]

Proposition 3: Let \( 0 < \sigma \leq \infty \) and \( \mu \) a measure on \( M_\sigma(\mathbb{R}) \). Then

\[
\varphi(x) = \int_0^{+\infty} E_k(-x, y) d\mu(y)
\]

is Dunkl completely monotonic on \([ -\sigma, \sigma ]\).

Proof: By Example 1 and since \( \mu \in M_\sigma(\mathbb{R}) \), we get

\[
(-1)^n T^n_k \varphi(x) = \int_0^{\infty} y^n E_k(-x, y) d\mu(y) \geq 0
\]

for all \( n \in \mathbb{N} \) and \( x \in (-\sigma, \sigma) \). Moreover, \( \varphi \) is continuous on \([ -\sigma, \sigma ]\), we conclude.

Proposition 4: Let \( \varphi \in C^\infty((-\sigma, \sigma)) \) and \( \varphi \) is completely monotonic function on \( (-\sigma, \sigma) \), then \( V_\kappa \varphi \) is Dunkl completely monotonic on \( (-\sigma, \sigma) \).

Proof: Since \( \varphi \) is completely monotonic on \( (-\sigma, \sigma) \), then

\[
(-1)^n \varphi^{(n)}(x) \geq 0, \ -\sigma < x < \sigma.
\]
As $V_k$ is a positive operator and satisfies

$$T_k (V_k \varphi) = V_k \left( \varphi' \right).$$

We get

$$( - 1)^n T^n_k V_k \varphi(x) = V_k \left( ( - 1)^n \varphi^{(n)}(x) \right) \geq 0, \quad -\sigma < x < \sigma.$$ 

**Proposition 5:** Let $\varphi \in C^1 ((a, b))$. If $\varphi$ is Dunkl completely monotonic on $(a, b)$, then $-T_k \varphi$ is also Dunkl completely monotonic on $(a, b)$.

**Proof:** Follows immediately by the Definition 2.

**Theorem 6:** Let $\varphi \in A_k (\mathbb{R})$ and $\mu$ is a measure on $M_\infty (\mathbb{R})$ such that:

$$\varphi(x) = \int_0^\infty V_k (f_t)(x) \text{d}\mu(t),$$

where $f_t(x) = e^{-t^2 x^2}$. Then, the function $\varphi(x)$ is Dunkl positive definite on $\mathbb{R}$ and the function $\varphi(\sqrt{|x|})$ is Dunkl completely monotonic.

**Proof:** For $t > 0$, the function $x \mapsto f_t(x) = e^{-t^2 x^2}$ is positive definite on $\mathbb{R}$. Bochner’s theorem implies that

$$f_t(x) = \int_\mathbb{R} e^{-ity} \text{d}\mu(y)$$

where $\mu$ is a finite nonnegative Borel measure on $\mathbb{R}$.

Then,

$$V_k (f_t)(x) = \phi_t(x) = \int_\mathbb{R} \left( \int_\mathbb{R} e^{-itz} \text{d}\mu(y) \right) \text{d}\mu_x(z).$$

Since the measures $\mu$ and $\mu_x$ are bounded, we have

$$\phi_t(x) = \int_\mathbb{R} E_k (-itx, y) \text{d}\mu(y).$$

Now, we will prove that the function $x \mapsto \phi_t(x)$ is Dunkl positive definite on $\mathbb{R}$, for all $t > 0$. In fact from the formula (10) and Theorem 3, we have
\[ \tau_x \phi_t(y) = \int_{\mathbb{R}} \int_{\mathbb{R}} W_k(\phi_t)(\eta - \xi) d\mu_x(\eta) d\mu_y(\xi) \]
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} W_k(f_t)(\eta - \xi) d\mu_x(\eta) d\mu_y(\xi) \]
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} f_t(\eta - \xi) d\mu_x(\eta) d\mu_y(\xi) \]
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\eta} e^{ix\xi} d\mu_x(\eta) d\mu_y(\xi) \]
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it\eta} e^{ix\xi} d\mu_x(\eta) \left[ \int_{\mathbb{R}} e^{ix\xi} d\mu_y(\xi) \right] d\mu(z) \]
\[ = \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} e^{-it\eta} e^{ix\xi} d\mu_x(\eta) \right] \left[ \int_{\mathbb{R}} e^{ix\xi} d\mu_y(\xi) \right] d\mu(z) \]
\[ = \int_{\mathbb{R}} E_k(-itx, z) E_k(ity, z) d\mu(z) \]
\[ = \int_{\mathbb{R}} E_k(itx, z) E_k(ity, z) d\mu(z), \]

which implies that for every finite distinct real numbers \(x_1, x_2, \ldots, x_n\) and every complex numbers \(\alpha_1, \alpha_2, \ldots, \alpha_n\) not all zero, we get
\[
\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \bar{\alpha_k} \tau_{x_j} \phi_t(x_k) = \int_{\mathbb{R}} \left| \sum_{j=1}^{n} \alpha_j E_k(itx_j, z) \right|^2 d\mu(z) \geq 0.
\]

Hence, the function \(x \mapsto \phi_t(x)\) is Dunkl positive definite on \(\mathbb{R}\), for all \(t > 0\).

Next, we define the function
\[
\Phi(x) = \int_{0}^{\infty} \phi_t(x) d\mu(t).
\]

Since the function \(\Phi \in A_k(\mathbb{R})\), we have for every finite distinct real numbers \(x_1, x_2, \ldots, x_n\) and every complex numbers \(\alpha_1, \alpha_2, \ldots, \alpha_n\) not all zero
\[
\sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \bar{\alpha_l} \tau_{x_j} \Phi(x_l) = \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \bar{\alpha_l} \int_{\mathbb{R}} E_k(-ix_j, \xi) E_k(ix_l, \xi) D_k \Phi(\xi) h_k^2(\xi) d\xi
\]
\[= \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \bar{\alpha_l} \int_{\mathbb{R}} E_k(-ix_j, \xi) E_k(ix_l, \xi) \]
\[\times \left[ \int_{\mathbb{R}} E_k(-is, \xi) \Phi(s) h_k^2(s) ds \right] h_k^2(\xi) d\xi
\]
\[= \int_{0}^{\infty} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \bar{\alpha_l} \int_{\mathbb{R}} E_k(-ix_j, \xi) E_k(ix_l, \xi) \]
\[\times \left[ c_k \int_{\mathbb{R}} E_k(-is, \xi) \phi_t(s) h_k^2(s) ds \right] \times h_k^2(\xi) d\xi d\mu(t)
\]
\[= \int_{0}^{\infty} \sum_{j=1}^{n} \sum_{l=1}^{n} \alpha_j \bar{\alpha_l} \int_{\mathbb{R}} E_k(-ix_j, \xi) E_k(ix_l, \xi) \]
\[\times D_k(\phi_t)(\xi) h_k^2(\xi) d\xi d\mu(t)\]
\[
\int_0^\infty \sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} \tau_{x_j} \phi_t(x_l) \, d\mu(t) \geq 0.
\]

The last inequality holds because the function \( x \mapsto \phi_t(x) \) is Dunkl positive definite and the measure \( \mu \) is nonnegative. Thus the function \( \Phi \) is Dunkl positive definite. Now, we will prove that the function \( \varphi(\sqrt{|x|}) \) is Dunkl completely monotonic on \( \mathbb{R} \). From (13) we obtain

\[
\varphi(\sqrt{|x|}) = \int_0^\infty \phi_t(\sqrt{|x|}) \, d\mu(t) = \int_0^\infty V_k(e^{-t^2|x|}) \, d\mu(t)
\]

\[
= \int_0^\infty V_k(e^{-t|x|}) \, dv(t) = \int_0^\infty E_k(-t, |x|) \, dv(t).
\]

(14)

By using the Proposition 3 and (14) we conclude that the function \( x \mapsto \varphi(\sqrt{|x|}) \) is Dunkl completely monotonic on \( \mathbb{R} \). So, the proof of Theorem 6 is complete.

Lemma 1: Let \( \varphi \in A_k(\mathbb{R}^d) \) be a radial function. If \( \varphi \) is Dunkl positive definite function, then \( D_k \varphi \) is even.

Proof: For \( \varphi \in A_k(\mathbb{R}^d) \), we get

\[
D_k \varphi(x) = c_k \int_{\mathbb{R}^d} E_k(-ix, y) \varphi(y) h^2_k(y) \, dy.
\]

Thus,

\[
D_k \varphi(-x) = c_k \int_{\mathbb{R}^d} E_k(ix, y) \varphi(y) h^2_k(y) \, dy
\]

\[
= \int_{\mathbb{R}^d} E_k(-ix, y) \overline{\varphi(y)} h^2_k(y) \, dy
\]

\[
= D_k \varphi(x).
\]

Finally, Corollary 1 in [6] completes the proof.

Lemma 2: Let \( \varphi \in A_k(\mathbb{R}) \). If \( \varphi \) is Dunkl positive definite, then the function \( W_k \varphi \) is strictly positive definite on \( \mathbb{R} \).

Proof: For \( \varphi \in A_k(\mathbb{R}) \), by means of Theorem 3, we have

\[
W_k \varphi(x) = \int_{\mathbb{R}} e^{ixy} D_k \varphi(y) h^2_k(y) \, dy.
\]

(15)

Since \( \varphi \) is Dunkl positive definite function on \( \mathbb{R} \), we obtain that the function \( D_k \varphi \) is nonnegative. Thus, for every finite distinct real numbers \( x_1, \ldots, x_n \) and every complex numbers \( \alpha_1, \ldots, \alpha_n \) not all zero, we have

\[
\sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} W_k \varphi(x_j - x_l) = \int_{\mathbb{R}} \left| \sum_{j=1}^n \alpha_j e^{ix_jy} \right|^2 D_k \varphi(y) h^2_k(y) \, dy \geq 0.
\]
which implies that the function $W_k\varphi$ is positive definite on $\mathbb{R}$. Now, suppose that the function $W_k\varphi$ is not strictly positive definite, then there exist distinct reals points $x_1, x_2, \ldots, x_n$ and complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero such that

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \overline{\alpha_k} W_k \varphi(x_j - x_k) = 0.$$ 

Thus

$$\int_{\mathbb{R}} \left| \sum_{j=1}^{n} \alpha_j e^{ix_j t} \right|^2 D_k \varphi(t) h_k^2(t) dt = 0.$$ 

Since $\varphi$ is Dunkl positive definite and belongs to $A_k(\mathbb{R})$, we have $D_k \varphi$ is nonnegative continuous function. Then

$$\left| \sum_{j=1}^{n} \alpha_j e^{ix_j t} \right| D_k \varphi(t) = 0.$$ 

Moreover, since $D_k \varphi$ is non-identically zero, then there exist an open subset $U \subset \mathbb{R}$ such that

$$D_k \varphi(t) \neq 0, \quad \forall t \in U.$$ 

Thus,

$$\sum_{j=1}^{n} \alpha_j e^{ix_j t} = 0, \quad \forall t \in U.$$ 

From Lemma 6.7 in [15, p. 72], we get

$$\alpha_j = 0, \quad \forall j \in \{1, \ldots, n\}.$$ 

Then, we deduce that the function $W_k \varphi$ is strictly positive definite function. \hfill \Box

**Theorem 7:** Let $\varphi \in A_k(\mathbb{R})$ be a real function and Dunkl positive definite. For $k > 0$, we consider the function $\varphi_k(x) = D_k \varphi(x)|x|^{2k+1}$. If $\varphi_k$ is convex such that $\lim_{|x| \to \infty} \varphi_k(x) = 0$. Then

1. $W_k \varphi$ is even and nonnegative.
2. The function $\varphi(\sqrt{|x|})$ is Dunkl completely monotonic on $\mathbb{R}$.

**Proof:** From Lemma 1, we conclude that the function $W_k \varphi$ is even and we have

$$W_k \varphi(x) = \int_{0}^{\infty} \cos(xy) D_k \varphi(y)|y|^{2k+1} dy.$$ 

Since the function $\varphi_k(y) = D_k \varphi(y)|y|^{2k+1}$ is convex downwards on $[0, \infty)$ and $\lim_{|x| \to \infty} \varphi_k(x) = 0$, we deduce by Lemma 1 in [17], that the function $W_k \varphi$ is nonnegative. By Lemma 2, the function $W_k \varphi$ is strictly positive definite on $\mathbb{R}$, nonnegative and radial. From Theorem 7.14 in [15], we conclude that the function $x \mapsto W_k \varphi(\sqrt{|x|})$ is completely monotonic on $\mathbb{R}$. From Proposition 4 we conclude that the function $\varphi(\sqrt{|x|})$ is Dunkl completely monotonic on $\mathbb{R}$. \hfill \Box
6. Applications

Theorem 8: Let $p > 0$ and $k > -1/2$. Let

$$
\varphi_{k,p}(x) = \frac{\Gamma(k + \frac{1}{2})e^{-\frac{x^2}{4p}}}{2p^{k+\frac{1}{2}}} + \frac{\Gamma(k + 1)x}{2(2k + 1)p^{k+1}} \cdot \, \, _1F_1 \left( k + 1; k + \frac{3}{2}; \frac{x^2}{4p} \right),
$$

where $_1F_1$ is the Kummer confluent hypergeometric function. Then $\varphi_{k,p}$ is Dunkl completely monotonic on $\mathbb{R}$.

Proof: Let $d\mu(t) = e^{-pt^2} t^{2k+1} dt$ where $p > 0$, we obtain for all $\sigma > 0$

$$
\int_0^\infty e^{\sigma t} d\mu(t) = \int_0^\infty e^{t(\sigma - pt)} t^{2k+1} dt < +\infty,
$$

which implies that the measure $\mu \in M_\infty(\mathbb{R})$. From the Sonine formula [14, p. 394], we have

$$
\int_0^\infty J_k(xt)e^{-pt^2} t^{k+1} dt = \frac{x^k e^{-\frac{x^2}{4p}}}{(2p)^{k+1}},
$$

where $x, p, k$ complex numbers such that $\Re(p) > 0$, $\Re(k) > -1$ and $J_k$ stands for the Bessel function of the first kind. We change in the above Sonine formula $x$ by $ix$, we get:

$$
\int_0^\infty j_{k-\frac{1}{2}}(ixt)e^{-pt^2} t^{k+\frac{1}{2}} dt = \frac{\Gamma(k + \frac{1}{2})e^{-\frac{x^2}{4p}}}{2p^{k+\frac{1}{2}}} = I_{k,p}(x). \quad (16)
$$

On the other hand, using (4), we have

$$
\int_0^\infty E_k(-x,t)d\mu(t) = \int_0^\infty \left( j_{k-\frac{1}{2}}(ixt) + \frac{x}{2k + 1} t j_{k+\frac{1}{2}}(ixt) \right) d\mu(t) = I_{k,p}(x) + J_{k,p}(x),
$$

where

$$
J_{k,p}(x) = \frac{x}{2k + 1} \int_0^\infty j_{k+\frac{1}{2}}(ixt) d\mu(t) = \frac{x}{2k + 1} \int_0^\infty j_{k+\frac{1}{2}}(ixt)e^{-pt^2} t^{2k+1} dt.
$$

Now, we calculate the function $J_{k,p}$. From the integral representation

$$
\int_0^\infty t^{m+1}J_k(xt)e^{-pt^2} dt = \frac{x^k \Gamma(1 + m + \frac{k}{2})}{2^{k+1}(\sqrt{p})^{k+m+2} \Gamma(k+1)} \times _1F_1 \left( 1 + \frac{m}{2} + \frac{k}{2}; k + 1; -\frac{x^2}{4p} \right).
$$

We have

$$
\int_0^\infty t^{k+m+1}j_k(ixt)e^{-pt^2} dt = \frac{\Gamma(1 + m + \frac{k}{2})}{2(\sqrt{p})^{k+m+2}} \times _1F_1 \left( 1 + \frac{m}{2} + \frac{k}{2}; k + 1; \frac{x^2}{4p} \right).
$$

Let $m = k - 1$, we obtain:

$$
\int_0^\infty t^{2k}j_k(ixt)e^{-pt^2} dt = \frac{\Gamma(k + \frac{1}{2})}{2(\sqrt{p})^{2k+1}} \times _1F_1 \left( k + \frac{1}{2}; k + 1; \frac{x^2}{4p} \right).
$$
Hence
\[ J_{k,p}(x) = \frac{\Gamma(k+1)x}{2(2k+1)p^{k+1}} \times_1 F_1 \left( k + 1; k + \frac{3}{2}; \frac{x^2}{4p} \right). \]
Finally, by Proposition 3 we conclude that the function \( \varphi_{k,p} \) is Dunkl completely monotonic on \( \mathbb{R} \).

**Remark 2:** For \( p > 0 \), the function
\[ \phi_p(x) = \sqrt{\pi}e^{-\frac{x^2}{4p}} + \frac{x}{p} \times_1 F_1 \left( 1; \frac{3}{2}; \frac{x^2}{4p} \right) \]
is completely monotonic on \( \mathbb{R} \). In particular, for \( p = \frac{1}{4} \), the function
\[ \phi_0(x) = 2\sqrt{\pi}e^{-\frac{x^2}{4}} \left( 1 + \frac{\gamma(\frac{1}{2},x^2)}{\sqrt{\pi}} \right) \]
\[ = 2\sqrt{\pi}e^{-\frac{x^2}{4}} \left( 1 + \text{erf}(x) \right), \]
where \( \gamma(a,z) \) and \( \text{erf}(z) \) are respectively the incomplete gamma and error functions, is completely monotonic on \( \mathbb{R} \).

**7. Open problem**

Finally, motivated by the results in Section 5, we pose the following problem (Bernstein’s theorem in Dunkl setting): Let \( \varphi \in C^\infty((-\sigma,\sigma)) \), \( \sigma > 0 \). Then \( \varphi \) is Dunkl completely monotonic on \((-\sigma,\sigma)\) if and only if there exists a nonnegative measure \( \nu \in M_\sigma(\mathbb{R}) \) such that
\[ \varphi(x) = \int_0^\infty E_k(-x,y)d\nu(y). \]

We note that the Proposition 3, is a necessary condition of this problem. Furthermore, in this case, by using Theorem 6, we prove the Schoenberg’s theorem in Dunkl setting.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**ORCID**

Khaled Mehrez [http://orcid.org/0000-0001-9948-3636]

**References**

[1] S. Bochner, *Integral Transform and their Application*, Applied Mathematical Sciences Vol. 25, Springer-verlag, New York, 1930.
[2] M.F.E. De Jeu, *Dunkl operators*, thesis, Leiden University, 1994.
[3] C.F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. 311 (1989), pp. 167–183.
[4] C.F. Dunkl, *Integral kernels with reflexion group invariance*, Can. J. Math. 43 (1991), pp. 1213–1227.

[5] C.F. Dunkl, *Hankel transform associated to finite reflexion groups*, Contemp. Math. 138 (1992), pp. 123–138.

[6] J. El Kamel and K. Mehrez, *Dunkl positive definite functions*, Tamsui Oxf. J. Info. Math. Sci. 30(1) (2017), pp. 1–23.

[7] M. Ky Fan, *Les fonctions définies positives et les fonctions complètement monotones*, Memorial Sciences Mathématiques, Paris, 1950.

[8] M. Rösler, *Dunkl operator: Theory and applications*, in *Orthogonal Polynomials and Special Functions* (Leuven, 2002), Lecture Notes in Maths Vol. 1817, E. Koelink and W. Van Assche, eds., Berlin, Springer, 2003, pp. 93–135.

[9] M. Rösler, *Positivity of the Dunkl’s intertwining operator*, Duke Math. J. 98 (1999), pp. 445–463.

[10] I.J. Schoenberg, *Metric spaces and completely monotone functions*, Ann. Math. 39 (1938), pp. 811–841.

[11] S. Thangavelu and Y. Xu, *Convolution operator and maximal function for the Dunkl transform*, J. Anal. Math. 97 (2005), pp. 25–55.

[12] C. Torossian, *Une application des opérateurs de Dunkl au théorème de restriction de Chevalley*, C. R. Acad. Sci. Paris 318 (1994), pp. 895–898.

[13] K. Trimèche, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Integral Transforms Spec. Funct. 13 (2002), pp. 17–38.

[14] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

[15] H. Wendland, *Scattered Data Approximations*, Cambridge University Press, Cambridge, 2005.

[16] D.V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, NJ, 1946.

[17] V.P. Zastavnyi, *Extension of a function from the exterior of an interval to a positive-definite function on the entire axis and an approximation characteristic of the class $W^{r,\beta}_M$*, Ukrainian Math. J. 55(7) (2003), pp. 1189–1197.