Synthesizing Computable Functions from Rational Specifications over Infinite Words

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The synthesis problem asks to automatically generate, if it exists, an algorithm from a specification of correct input-output pairs. In this paper, we consider the synthesis of computable functions of infinite words, for a classical Turing computability notion over infinite inputs. We consider specifications which are rational relations of infinite words, i.e., specifications defined by non-deterministic parity transducers. We prove that the synthesis problem of computable functions from rational specifications is undecidable. We provide an incomplete but sound reduction to some parity game, such that if Eve wins the game, then the rational specification is realizable by a computable function. We prove that this function is even computable by a deterministic two-way transducer.

We provide a sufficient condition under which the latter game reduction is complete. This entails the decidability of the synthesis problem of computable functions, which we proved to be \( \text{ExpTime} \)-complete, for a large subclass of rational specifications, namely deterministic rational specifications. This subclass contains the class of automatic relations over infinite words, a yardstick in reactive synthesis.

1 Introduction

Program synthesis aims at automatically generating programs from specifications. This problem can be formalized as follows. There are four parameters: two sets of input and output domains \( I, O \), a class \( S \) of relations (called specifications) from \( I \) to \( O \), and a class \( \mathcal{I} \) of (partial) functions (called implementations) from \( I \) to \( O \). Then, given a specification \( S \in \mathcal{S} \) defining the correct input/output relationships, the synthesis problem asks to check whether there exists a function \( f \in \mathcal{I} \) satisfying \( S \) in the following sense: its graph is included in \( S \) and it has the same domain as \( S \) (i.e., \( f \) is defined on \( x \in I \) iff \((x, y) \in S \) for some \( y \in O \)). Using a set-theoretic terminology, \( f \) is said to uniformize \( S \). Moreover
in synthesis, if such an $f$ exists, then the synthesis algorithm should return (a finite presentation of) it.

Program synthesis quickly turns to be undecidable depending on the four parameters mentioned before. Therefore, research on synthesis either turn to developing efficient sound but incomplete methods, see for example the syntax-guided synthesis approach [2] or bounded synthesis [20, 21], or restrict the class of specifications $S$ and/or the class of implementations $I$. A well-known example of the latter approach is reactive synthesis, where $S$ are automatic relations\(^1\) over infinite words, and $I$ are Mealy machines [8, 29, 13]. Infinite words (over a finite alphabet) are used to model infinite executions of reactive systems, and Mealy machines are used as a model of reactive systems processing bit streams.

In this paper, our goal is to synthesize, from specifications which are semantically binary relations of infinite words, stream-processing programs, which are semantically *streaming computable* functions of infinite words (just called *computable* functions in the sequel). Let us now make the computability notion we use more precise. Let $\Sigma$ and $\Gamma$ be two finite alphabets. A partial function $f : \Sigma^\omega \rightarrow \Gamma^\omega$, whose domain is denoted $\text{dom}(f)$, is said to be computable, if there exists a deterministic (Turing) machine $M$ with three tapes, a read-only one-way input tape, a two-way working tape, and a write-only output tape that works as follows: if the input tape holds an input sequence $\alpha \in \text{dom}(f)$, then $M$ outputs longer and longer prefixes of $f(\alpha)$ when reading longer and longer prefixes of $\alpha$. A definition of this machine model can be found, for instance, in [33].

**Example 1.** Over the alphabet $\Sigma = \Gamma = \{a, b, A, B\}$, consider the specification given by the relation $R_1 = \{(ux\alpha, xu\beta) \mid u\alpha, u\beta \in \{a, b\}^\omega, x \in \{A, B\}\}$. The relation $R_1$ is automatic: an automaton needs to check that the input prefix $u$ occurs shifted by one position on the output, which is doable using only finite memory. Checking that the first output letter $x$ also appears after $u$ on the input can also be done by storing $x$ in the state. Note that some acceptance condition (e.g., parity) is needed to make sure that $x$ is met again on the input. There is no Mealy machine which can realize $R_1$, because Mealy machines operate in a synchronous manner: they read one input symbol and must deterministically produce one output symbol. Here, the first output symbol which has to be produced depends on the letter $x$ which might appear arbitrarily far in the input sequence. However, $R_1$ can be uniformized by a computable function: there is an *algorithm* reading the input from left to right and which simply waits till the special symbol $x \in \{A, B\}$ is met on the input. Meanwhile, it stores longer and longer prefixes of $u$ in memory (so it needs unbounded memory) and once $x$ is met, it outputs $xu$. Then, whatever it reads on the input, it just copies it on the output (realizing the identity function over the remaining infinite suffix $\alpha$). Note that this algorithm produces a correct output under the assumption that $x$ is eventually read.

**Contributions.** We first investigate the synthesis of computable functions from *rational specifications*, which are those relations recognizable by non-deterministic finite

\(^1\)relations recognized by two-tape parity automata alternatively reading one input and one output symbol.
state transducers, i.e., parity automata over a product of two free monoids. We however show this problem is undecidable (Proposition 5). We then give an incomplete but sound algorithm in Section 3, based on a reduction to \( \omega \)-regular two-player games. Given a transducer \( T \) defining a specification \( R_T \), we show how to effectively construct a two-player game \( G_T \), proven to be solvable in \( \text{ExpTime} \), such that if Eve wins \( G_T \), then there exists a computable function which uniformizes the relation recognized by \( T \), which can even be computed by some input-deterministic two-way finite state transducer (a transducer which whenever it reads an input symbol deterministically produces none or several output symbols and either moves forward or backward on the input). It is easily seen that two-wayness is necessary: the relation \( R_1 \) of Example 1 cannot be uniformized by any deterministic device which moves only forward over the input and only uses finitely many states, as the whole prefix \( u \) has to be remembered before reaching \( x \). However, a two-way finite-state device can do it: first, it scans the prefix up to \( x \), comes back to the beginning of the input, knowing whether \( x = A \) or \( x = B \), and then can produce the output.

Intuitively, in the two-player game we construct, called unbounded delay game, Adam picks the input symbols while Eve picks the output symbols. Eve is allowed to delay her moves an arbitrarily number of steps, gaining some lookahead information on Adam’s input. We use a finite abstraction to store the lookahead gained on Adam’s moves. We show that any finite-memory winning strategy in this game can be translated into a function uniformizing the specification such that it is computable by an input-deterministic two-way transducer.

In Section 4, we provide a sufficient condition \( P \) on relations for which the game reduction is complete. In particular, we show that if a relation \( R \) satisfies \( P \), then Eve wins the game iff \( R \) can be uniformized by a computable function. A large subclass of rational relations satisfying this sufficient condition is the class of deterministic rational relations (DRAT, see [31]). Deterministic rational relations are those relations recognizable by deterministic two-tape automata, one tape holding the input word while the other holds the output word. It strictly subsumes the class of automatic relations, and, unlike for automatic relations, the two heads are not required to move at the same speed. Furthermore, when the domain of the relation is topologically closed for the Cantor distance\(^2\), we show that strategies in which Eve delays her moves at most a bounded number of steps are sufficient for Eve to win. Such a strategy can in turn be converted into an input-deterministic one-way transducer. This entails that for DRAT-specifications with a closed domain (such as for instance specifications with domain \( \Sigma^{\omega} \), i.e., total domains), if it is uniformizable by a computable function, then it is uniformizable by a function computable by an input-deterministic one-way transducer.

Based on the completeness result, we prove our main result, that the synthesis problem of computable functions from deterministic rational relations is \( \text{ExpTime} \)-complete. Hardness also holds in the particular case of automatic relations of total domain.

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\(^2\)A set \( X \subseteq \Sigma^{\omega} \) is closed if the limit, if it exists, of any sequence \( (x_i) \) of infinite words in \( X \) is in \( X \). The limit here is defined based on the Cantor distance, which, for any two infinite words \( u, v \), is 0 if \( u = v \) and otherwise \( 2^{-\ell} \) where \( \ell \) is the length of their longest common prefix.
Total versus partial domains. We would like to emphasize here on a subtle difference between our formulation of synthesis problems and the classical formulation in reactive synthesis. Classically in reactive synthesis, it is required that a controller produces for every input sequence an output sequence correct w.r.t. the specification. Consequently, specifications with partial domain are by default unrealizable. So, in this setting, the specification $R_1$ of the latter example is not realizable, simply because its domain is not total (words with none or at least two occurrences of a symbol in $\{A, B\}$ are not in its domain). In our definition, specifications with partial domain can still be realizable, because the synthesized function, if it exists, can be partial and must be defined only on inputs for which there exists at least one matching output in the specification. A well-known notion corresponding to this weaker definition is that of uniformization [27, 12, 9, 18], this is why we often use the terminology “uniformizes” instead of the more widely used terminology “realizes”. The problem of synthesizing functions which uniformize quantitative specifications has been recently investigated in [1]. In [1], it was called the good-enough synthesis problem, a controller being good-enough if it is able to compute outputs for all inputs for which there exists a least one matching output by the specification. The uniformization setting allows to formulate assumptions that the input the program receives is not any input, but belongs to some given language. Related to that, there is a number of works on reactive synthesis under assumptions on the behavior of the environment [11, 6, 28, 14, 7].

Related work. To the best of our knowledge, this work is the first contribution which addresses the synthesis of algorithms from specifications which are relations over infinite inputs, and such that these algorithms may need unbounded memory, as illustrated by the specification $R_1$ for which any infinite-input Turing-machine realizing the specification needs unbounded memory. There are however two related works, in some particular or different settings.

First, in [15], the synthesis of computable functions has been shown to be decidable in the particular case of functional relations, i.e., graphs of functions. The main contribution of [15] is to prove that checking whether a function represented by a non-deterministic two-way transducer is computable is decidable, and that computability coincides with continuity (for the Cantor distance) for this large class of functions. The techniques of [15] are different to ours, e.g., games are not needed because output symbols functionally depends on input ones, even in the specification, so, there are no choices to be made by Eve.

Second, Hosch and Landweber [25] proved decidability of the synthesis problem of Lipschitz-continuous functions from automatic relations with total domain, Holtmann, Kaiser and Thomas [23] proved decidability of the synthesis of continuous functions from automatic relations with total domain, and Klein and Zimmermann [26] proved Exp-Time-completeness for the former problem. So, we inherit the lower bound because automatic relations are particular DRAT relations, and as we show in the last section of the paper, the synthesis problem of computable functions coincides with the synthesis problem of continuous functions. We obtain the same upper bound as [26] for a more
general class of specifications, namely DRAT, and in the more general setting of specifications with partial domain. As we show, total vs. partial domains make an important difference: two-way transducers may be necessary in the former case, while one-way transducers are sufficient in the latter. [23, 26] also rely on a reduction to two-player games called delay games, but for which bounded delay are sufficient. However, our game is built such that it accounts for the fact that unbounded delays can be necessary and it also monitors the domain, which is not necessary in [23, 26] because specifications have total domain. Accordingly, the main differences between [26] and our delay games are their respective winning objectives and correctness proofs. Another difference is that our game applies to the general class of rational relations, which are asynchronous (several symbols, or none, can correspond to a single input symbol) in contrast to automatic relations which are synchronous by definition.

The work is an extended version of the conference work [19] with additional proof details.

2 Preliminaries

Words, languages, and relations. Let \( \mathbb{N} \) denote the set of non-negative integers. Let \( \Sigma \) and \( \Gamma \) denote alphabets of elements called letters or symbols. A word resp. \( \omega \)-word over \( \Sigma \) is an empty or non-empty finite resp. infinite sequence of letters over \( \Sigma \). Let \( \Sigma^*, \Sigma^+, \) and \( \Sigma^\omega \) denote the set of finite, non-empty finite, and infinite words over \( \Sigma \), respectively. Let \( \Sigma^\infty \) denote \( \Sigma^* \cup \Sigma^\omega \). The empty word is denoted by \( \varepsilon \), the length of a word by \( | \cdot | \). Usually, we denote finite words by \( u, v, w \), etc., and infinite words by \( \alpha, \beta, \gamma, \) etc. Given an (in)finite word \( \alpha = a_0 a_1 \cdots \) over \( \Sigma \) with \( a_0, a_1, \cdots \in \Sigma \), let \( \alpha(i) \) denote the letter \( a_i \), \( \alpha(i:j) \) denote the infix \( a_i a_{i+1} \cdots a_j \), \( \alpha(i) \) the prefix \( a_0 a_1 \cdots a_i \), and \( \alpha(i+1:) \) the suffix \( a_{i+1} \cdots \) for \( i \leq j \in \mathbb{N} \). Given two words \( u \in \Sigma^* \) and \( \alpha \in \Sigma^\infty \), we say that \( u \) is a prefix of \( \alpha \), denoted \( u \preceq \alpha \), if \( \alpha(i) = u \) for some \( i < |\alpha| \), and in that case let \( u^{-1} \alpha = \alpha(i:) \). We also denote by \( \text{Prefs}(\alpha) \) the set of all finite prefixes of \( \alpha \), i.e., \( \text{Prefs}(\alpha) = \{ \alpha(i) \mid i < |\alpha| \} \). For two (in)finite words \( \alpha, \beta \), let \( \alpha \wedge \beta \) denote their longest common prefix, i.e., the longest word in \( \text{Prefs}(\alpha) \cap \text{Prefs}(\beta) \) if \( \alpha \neq \beta \), otherwise \( \alpha \) if \( \alpha = \beta \).

A language resp. \( \omega \)-language \( L \) is a subset of \( \Sigma^* \) resp. \( \Sigma^\omega \), its set of finite prefixes is denoted by \( \text{Prefs}(L) \). A (binary) relation resp. \( \omega \)-relation \( R \) is a subset of \( \Sigma^* \times \Gamma^* \) resp. \( \Sigma^\omega \times \Gamma^\omega \). An \( \omega \)-relation is just called a relation when infiniteness is clear from the context. The domain \( \text{dom}(R) \) of a \( (\omega) \)-relation \( R \) is the set \( \{ \alpha \mid \exists \beta \ (\alpha, \beta) \in R \} \). It is total if \( \text{dom}(R) = \Sigma^* \) resp. \( \Sigma^\omega \). Likewise, we define \( \text{img}(R) \) the image of \( R \), as the domain of its inverse. A relation \( R \) is functional if for each \( u \in \text{dom}(R) \) there is exactly one \( v \) such that \((u, v) \in R \). By default in this paper, relations and functions are partial, i.e., are not necessarily total. Given a function \( f : \Sigma^\omega \to \Gamma^\omega \), let \( \hat{f} : \Sigma^* \to \Gamma^\infty \) denote the function defined for all \( u \in \text{Prefs}(\text{dom}(f)) \) as

\[
\hat{f}(u) = \bigwedge \{ f(ua) \mid ua \in \text{dom}(f) \},
\]

that is, the longest common prefix of all outputs of \( f \) for inputs that begin with \( u \).
Automata. A parity automaton is a tuple $A = (Q, \Sigma, q_0, \Delta, c)$, where $Q$ is a finite set of states, $\Sigma$ a finite alphabet, $q_0 \in Q$ an initial state, $\Delta \subseteq Q \times \Sigma \times Q$ a transition relation, and $c: Q \to \mathbb{N}$ is a function that maps states to priorities, also called colors. A parity automaton is deterministic if its transition relation $\Delta$ is given as a transition function $\delta$. We denote by $\delta^*$ the usual extension of $\delta$ from letters to finite words. A run of $A$ on a word $w \in \Sigma^\infty$ is a word $\rho \in Q^\infty$ of length $|w| + 1$ if $w$ is finite, otherwise of infinite length, such that $\rho(i), w(i), \rho(i + 1)) \in \Delta$ for all $0 \leq i < |w|$. A run on $\varepsilon$ is a single state. We say that $\rho$ begins in $\rho(0)$ and ends in $\rho(|w|)$ if $w$ is finite.

Given a set $E$ and a sequence $s \in E^\infty$, we let $\text{Occ}(s)$ as the set of elements of $E$ that occur in $s$, and $\text{Inf}(s)$ as the set of elements of $E$ that occur infinitely often in $s$. In particular $\text{Inf}(s) = \emptyset$ if $s$ is finite. Given a run $\rho$, we let $c(\rho) = c(\rho(0))c(\rho(1)) \cdots$ be the sequence of colors seen along that run. The run $\rho$ is accepting if $\rho \in Q^\omega$, $\rho(0) = q_0$ and $\text{max Inf}(c(\rho))$ is even. The language recognized by $A$ is the set $L(A) = \{ \alpha \in \Sigma^\omega \mid$ there is an accepting run $\rho$ of $A$ on $\alpha \}$. A language $L \subseteq \Sigma^\omega$ is called regular if $L$ is recognizable by a parity automaton.

One-way transducers. A transducer (1NFT) is a tuple $T = (Q, \Sigma, \Gamma, q_0, \Delta, c)$, where $Q$ is finite state set, $\Sigma$ and $\Gamma$ are finite alphabets, $q_0 \in Q$ is an initial state, $\Delta \subseteq Q \times \Sigma^* \times \Gamma^* \times Q$ is a finite set of transitions, and $c: Q \to \mathbb{N}$ is a parity function. It is input-deterministic (1DFT) (also called sequential in the literature) if $\Delta$ is expressed as a function $Q \times \Sigma \to \Gamma^* \times Q$. A finite non-empty run $\rho$ is a non-empty sequence of transitions of the form $(p_0, u_0, v_0, p_1)(p_1, u_1, v_1, p_2) \cdots (p_{n-1}, u_{n-1}, v_{n-1}, p_n) \in \Delta^*$. The input (resp. output) of $\rho$ is $\alpha = u_0 \cdots u_{n-1}$ (resp. $\beta = v_0 \cdots v_{n-1}$). As shorthand, we write $T: p_0 \xrightarrow{\alpha/\beta} p_n$. An empty run is denoted as $T: p \xrightarrow{\varepsilon/\varepsilon} p$ for all $p \in Q$. Similarly, we define an infinite run. A run is accepting if it is infinite, begins in the initial state and satisfies the parity condition. In this paper, we also assume that for any accepting run $\rho$, its input and output are both infinite. This can be syntactically ensured with the parity condition. The relation recognized by $T$ is $R(T) = \{ (\alpha, \beta) \mid$ there is an accepting run of $T$ with input $\alpha$ and output $\beta \}$. Note that with the former assumption, we have $R(T) \subseteq \Sigma^\omega \times \Gamma^\omega$. A relation is called rational if it is recognizable by a transducer, we denote by RAT the class of rational relations. A sequential function is a function whose graph is $R(T)$ for an input-deterministic transducer $T$.

Two-way transducers. Given $\Sigma$, let $\Sigma^\dag$ denote $\Sigma \{ \leftarrow, \rightarrow \}$ a new left-delimiter symbol. An input-deterministic two-way transducer (2DFT) is a tuple $T = (Q, \Sigma^\dag, \Gamma, q_0, \delta, c)$, where $Q$ is a finite state set, $\Sigma$ and $\Gamma$ are finite alphabets, $q_0 \in Q$ is an initial state, $\delta: Q \times \Sigma^\dag \to \Gamma^* \times \{ 1, -1 \} \times Q$ is a transition function such that for all states $q$, $\delta(q, \leftarrow) \in \Gamma^* \times \{ 1 \} \times Q$, and $c: Q \to \mathbb{N}$ is a function that maps states to colors. A two-way transducer has a two-way read-only input tape and a one-way write-only output tape. Given an input sequence $\alpha \in \Sigma^\omega$, let $\alpha(-1) = \leftarrow$, the input tape holds $\leftarrow \alpha$. We denote a transition $\delta(p, a) = (\gamma, d, q)$ as a tuple $(p, a, \gamma, d, q)$, and $\Delta$ denotes the tuple representation of $\delta$. A run of $T$ on $\alpha \in \Sigma^\omega$ is a sequence of transitions $\rho = (q_0, \alpha(i_0), \gamma_0, d_0, q_1)(q_1, \alpha(i_1), \gamma_1, d_1, q_2) \cdots \in \Delta^\omega$ such that $i_0 = 0$, and $i_{k+1} = i_k + d_k$
for all \( k \in \mathbb{N} \). The *input* of \( \rho \) is \( \alpha \) and the *output* of \( \rho \) is \( \beta = \gamma_0 \gamma_1 \cdots \). We define \( c(\rho) \) as the sequence of colors \( c(q_0)c(q_1) \cdots \); \( \rho \) is accepting if max Inf(\( c(\rho) \)) is even and it visits all positions of \( \alpha \), i.e., \( \{ i_0, i_1, \ldots \} = \mathbb{N} \). The functional \( \omega \)-relation recognized by the deterministic-two way transducer is defined as \( R(T) = \{ (\alpha, \beta) \mid \text{there is an accepting run of} \ T \text{ with input} \ \alpha \text{ and output} \ \beta \} \).

**Games.** A *game arena* is a tuple \( G = (V_0, V_1, v_0, A, E) \), where \( V = V_0 \cup V_1 \) is a set of vertices, \( V_0 \) belongs to Eve and \( V_1 \) to Adam, \( v_0 \) is an initial vertex, \( A \) is a finite set of actions, and \( E \subseteq V \times A \times V \) is a set of labeled edges such that \((v, a, v') \in E \) and \((v, a, v'') \in E \) implies that \( v' = v'' \) for all \( v \in V \) and \( a \in A \). We assume that the arena is deadlock-free (there is always at least one outgoing edge from any vertex). We use letters on edges as it is more convenient to have them at hand for the proofs, it is however not necessary. A play in \( G \) is an infinite sequence \( v_0a_0v_1a_1 \cdots \) such that \((v_i, a_i, v_{i+1}) \in E \) for all \( i \in \mathbb{N} \). Note that a play is uniquely determined by its action sequence. A *game* is of the form \( G = (G, Win) \), where \( G \) is a game arena and \( Win \subseteq V^\omega \) is a winning condition. Eve wins a play \( \alpha = v_0a_0v_1a_1 \cdots \) if \( v_0v_1 \cdots \in Win \), otherwise Adam wins.

For ease of presentation, we also write \( \alpha \in Win \). A parity condition \( Win \) is represented by a coloring function \( c : V \rightarrow \mathbb{N} \) and consists of the set of sequences \( v_0a_0v_1a_1 \cdots \) such that max Inf(\( c(v_0)c(v_1) \cdots \)) is even. A *parity game* is a game \( G = (G, Win) \) such that \( Win \) is a parity condition (given as a coloring function \( c \)).

A *strategy* for Eve resp. Adam is a function \( \sigma : (VA)^*V_0 \rightarrow A \) resp. \( \sigma : (VA)^*V_1 \rightarrow A \) such that for all \( x \in (VA)^* \) and \( v \in V_0 \) resp. \( v \in V_1 \), there exists \( v' \in V \) such that \((v, \sigma(xv), v') \in E \). A play \( v_0a_0v_1a_1 \cdots \) is consistent with a strategy \( \sigma \) for Eve resp. Adam if \( \sigma(v_0a_0 \cdots v_i) = a_{i} \) for all \( i \in \mathbb{N} \) with \( v_i \in V_0 \) resp. \( v_i \in V_1 \). A strategy \( \sigma \) for Eve is a *winning strategy* if \( \alpha \in Win \) for all plays \( \alpha \) consistent with \( \sigma \).

A *strategy automaton* for Eve is a tuple \( S = (M, V, m_0, \delta, \mu) \), where \( M \) is a finite set of (memory) states, \( V \) is the alphabet, \( m_0 \) is an initial state, \( \delta : M \times V \rightarrow M \) is the memory update function, and \( \mu : M \times V_0 \rightarrow A \) is the next action function such that for all \( v \in V_0 \) and \( m \in M \), there is \( v' \in V \) with \((v, \mu(m,v), v') \in E \). The transition function \( \delta \) is naturally extended into a function \( \delta^* : M \times V^* \rightarrow M \) over finite sequences over \( V \). A strategy automaton \( S \) defines a strategy \( \sigma_S \) as follows: for all \( x \in (VA)^* \) and \( v \in V_0 \), \( \sigma_S(xv) = \mu(\delta^*(m_0, \pi_V(x)), v) \) where \( \pi_V(x) \) is the projection of \( x \) onto \( V \).

**Problem statement.** In this section, we introduce the problem we want to solve. Let \( \Sigma, \Gamma \) be two finite alphabets. Given a relation \( R \subseteq \Sigma^\omega \times \Gamma^\omega \) and a (partial) function \( f : \Sigma^\omega \rightarrow \Gamma^\omega \), \( f \) is said to uniformize \( R \) if \( \text{dom}(f) = \text{dom}(R) \) and \((\alpha, f(\alpha)) \in R \) for all \( \alpha \in \text{dom}(R) \). We also say that \( R \) is uniformizable by \( f \) or that \( f \) is a uniformizer of \( R \). We are interested in computable uniformizers, which we now introduce.

**Definition 2 ([33] computable functions).** A function \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) is called *computable* if there exists a deterministic multi-tape machine \( M \) that computes \( f \) in the following sense, \( M \) has a read-only one-way input tape, a two-way working tape, and a write-only one-way output tape. All tapes are infinite to the right, finite to the left. For any finite
word \( w \in \Sigma^* \), let \( M(w) \) denote the output\(^3\) of \( M \) on \( w \). The function \( f \) is said to be computed by \( M \) if for all \( \alpha \in \text{dom}(f) \) and \( i \in \mathbb{N} \) there exists \( j \in \mathbb{N} \) such that \( f(\alpha):i \) is a prefix of \( M(\alpha: :j)) \).

**Remark 3.** Note that in the above definition, checking whether the infinite input belongs to the domain is not a requirement and should not be, because in general, it is impossible to do it reading only a finite prefix of the input. That is why in this definition, we assume that the input belongs to the domain of the function. It is a reasonable assumption. For instance, the inputs may have been produced by another program (e.g., a transducer) for which one has guarantees that they belong to some well-behaved (e.g., regular) language.

**Example 4.** To begin with, consider the function \( f_1: \{a,b,c\}^\omega \to \{b,c\}^\omega \) defined by \( f_1(a^nba^\omega) = b^\omega \) and \( f_1(a^nca^\omega) = c^\omega \) for all \( n \in \mathbb{N}_{\geq 1} \). It is computable by a TM which on inputs of the form \( a^nxa^\omega \) for \( x \in \{b,c\} \), outputs nothing up to reading \( x \), and then, depending on \( x \), either outputs \( c \) or \( b \) whenever it reads an \( a \) in the remaining suffix \( a^\omega \).

Consider the function \( f_2: \{a,b\}^\omega \to \{a,b\}^\omega \) defined by \( f_2(\alpha) = a^\omega \) if \( \alpha \) contains infinitely many \( a \) and \( f(\alpha) = b^\omega \) otherwise for all \( \alpha \in \{a,b\}^\omega \). It is rational but not computable, because to determine even the first output letter, an infinite lookahead is needed.

Let \( S \) be a class of relations. The \( S \)-synthesis problem asks, given a relation \( S \in S \) (finitely represented), whether there exists a computable function which uniformizes \( S \). If such a function exists, then the procedure must return a TM computing it. Our first result is an undecidability result.

**Proposition 5.** The RAT-synthesis problem is undecidable, even if restricted to the subclass of rational relations with total domain.

**Proof sketch.** We sketch a reduction from Post’s correspondence problem. Let \( u_1, \ldots, u_n \) and \( v_1, \ldots, v_n \) be a PCP instance over alphabet \( \{0,1\} \). We construct the \( \omega \)-rational relation \( R \) that contains pairs \((\alpha, \beta)\) such that \( \alpha = i_1 \cdots i_k \alpha' \) with \( i_1 \cdots i_k \in \{1, \ldots, n\}^* \) and \( \alpha' \in \{a,b\}^\omega \), and \( \beta = u_1 \cdots u_k a^\omega \) if \( \alpha' \) contains infinitely many \( a \) and otherwise if \( \alpha' \) contains finitely many \( a \), then \( \beta \neq v_1 \cdots v_k a^\omega \). If the instance of PCP has no solution, then the function \( f: i_1 \cdots i_k \alpha' \mapsto u_1 \cdots u_k a^\omega \) uniformizes \( R \), because \( u_1 \cdots u_k \neq v_1 \cdots v_k \).

The function \( f \) is clearly computable. If the PCP has a solution, no computable function uniformizes \( R \). If the integer sequence \( i_1 \cdots i_k \) is the solution, then \( u_1 \cdots u_k = v_1 \cdots v_k \).

Intuitively, for an input sequence starting with the solution, no prefix of the input sequence allows to determine whether the output must begin with \( u_1 \cdots u_k \) or is not allowed to begin with \( u_1 \cdots u_k \).

The relation \( R \) can be made complete by also allowing all “invalid” inputs together with any output, i.e., by adding all pairs \((\alpha, \beta)\) of \( \{1, \ldots, n, a, b\}^\omega \times \{1, \ldots, n, a, b\}^\omega \) where the input sequence \( \alpha \) is not of the form \( i_1 \cdots i_k \alpha' \) with \( i_1 \cdots i_k \in \{1, \ldots, n\}^* \) and \( \alpha' \in \{a,b\}^\omega \), and any output sequence \( \beta \). Any Turing machine computing \( f \) can easily be adapted to verify whether the input is valid. \( \square \)

\(^3\)The finite word written on the output tape the first time \( M \) reaches the \(|w|\)-th cell of the input tape (\( M \) is assumed to visit every input cell)
Next, we give a semi-decision procedure for solving the RAT-synthesis problem which is sound but not complete. In Section 4, we introduce a sufficient condition for completeness which yields a (sound and complete) decision procedure for large and expressive classes of rational relations.

3 Unbounded Delay Game

In this section, given a rational relation (as a transducer), we show how to construct a finite-state \( \omega \)-regular two-player game called (unbounded) delay game. We prove that if Eve wins this game then there exists a computable function which uniformizes the relation. Moreover, this function is computable by an input-deterministic two-way transducer. We analyze the complexity of solving the game, which turns out to be in \( \text{EXPTIME} \). Solving this game yields an incomplete, but sound, decision procedure for the RAT-synthesis problem.

In the game, Adam provides inputs and Eve must produce outputs such that the combination of inputs and outputs is in the relation. However, as seen in Example 1, Eve might need to wait arbitrarily long before she can safely produce output. Hence, as the game is finite, it can not store arbitrary long input words, and Eve’s actions cannot produce arbitrarily long words neither. Instead, we finitely abstract input and output words using a notion we call profiles. Informally, a profile of an input word stores the effects of the word (together with some output word) on the states of the transducer (that specifies the relation) as well as the maximal priority seen along the induced state transformation. Such profiles contain sufficient information to express a winning condition which makes sure that given the word of input symbols provided by Adam, if Eve would output concrete output words instead of their abstraction, she would produce infinitely often non-empty output words whose concatenation, together with the input word, belongs to the relation.

State transformation profiles. Let \( R \subseteq \Sigma^\omega \times \Gamma^\omega \) be a rational relation given by a transducer \( T = (Q_T, \Sigma, \Gamma, q_0^T, \Delta_T, c_T) \), and \( C_T = \text{img}(c_T) \) its set of used priorities. Let \( D = (Q_D, \Sigma, q_0^D, \delta_D, c_D) \) be a deterministic parity automaton that recognizes \( \text{dom}(R) \), and \( C_D = \text{img}(c_D) \) its set of used priorities; \( D \) can always be constructed from \( T \) by projecting away its outputs and by determinizing the resulting automaton.

Given \( u \in \Sigma^* \), its profile \( P_u \) are all the possible state transformations it induces for any output. Formally, \( P_u \subseteq Q_T \times Q_T \times C_T \) is defined as \( \{(p, q, c) \mid \text{there is } v \in \Gamma^* \text{ and there is a run } \rho \text{ of the form } T: p \xrightarrow{u/v} q \text{ with max } \text{Occ}(\rho) = c\} \). Profiles can be multiplied as \( P_1 \otimes P_2 = \{(p, r, \max\{m, n\}) \mid \exists q: (p, q, m) \in P_1, (q, r, n) \in P_2\} \). Given \( u_1, u_2 \in \Sigma^* \), it is easy to verify that \( P_{u_1u_2} = P_{u_1} \otimes P_{u_2} \), and \( P_{\varepsilon} \) is neutral for \( \otimes \).

Finite-state unbounded delay game. We now present a two-player \( \omega \)-regular game \( G_T = (G, \text{Win}) \) such that if Eve has a winning strategy, then \( R \) has a computable uniformizer. In this game, Adam’s actions are to provide input letters, letter-by-letter. Eve’s goal is to construct a sequence of state transformations \( (q_0, q_1, m_1)(q_1, q_2, m_2) \cdots \).
such that if the infinite input $\alpha \in \Sigma^\omega$ provided by Adam is in $\text{dom}(R)$, then (i) the maximal priority seen infinitely often in $(m_i)_i$ is even and (ii) $\alpha = u_0u_1 \cdots$ for some $u_i \in \Sigma^*$ such that $(q_i, q_{i+1}, m_{i+1}) \in P_u$, for all $i \geq 0$. As a consequence, all these finite runs can be concatenated to form an accepting run on $\alpha/v_0v_1 \cdots$, entailing $(\alpha, v_0v_1 \cdots) \in R$. One can then show that if Eve has a strategy to pick the state transformations while ensuring the latter property, then this strategy can be turned into a computable function, and conversely. Picking a state transformation is what we call a \textit{producing action} for Eve. Since a state transformation picked by Eve may correspond to an arbitrarily long word $u_i$, she also has an action \textit{skip} which allows her to wait before making such a producing action. Now, the difficulty for Eve is to decide when she makes producing actions, in other words, how to decompose the input $\alpha$, only based on prefixes of $\alpha$. To that end, before picking a state transformation, she may need to gather lookahead information from Adam. Consequently, the vertices of the game manipulate two consecutive profiles $P_1$ and $P_2$, with the invariant that $P_1$ is the profile of $u_i$ while $P_2$ is the profile of $u_{i+1}$, when the input played so far by Adam is $u_0 \cdots u_{i+1}$. When Eve knows enough, she picks a state transformation $(q_i, q_{i+1}, m_i)$ in $P_1$, then $P_1$ becomes $P_2$ and $P_2$ is reset to $P_\varepsilon$. The inputs of Adam up to the next producing action of Eve form the word $u_{i+2}$, and so on. The vertices of the game also store information to decide whether the input belongs to the domain of $R$ (states of $D$), the parities $m_i$, as well as the states $q_0, q_1, \ldots$. Formally, the game graph $G = (V, E)$ is composed of vertices of the form $(q, c, P_1, P_2, r) \times \{\forall, \exists\}$, where

- $q \in Q_T$, \textit{State reached on the combination of input and output sequence.}
- $c \in \{-1\} \cup C_T$, \textit{Priorities of the state transformations, $-1$ is used to indicate that no state transformation was chosen (skip action below).}
- $P_1, P_2$, \textit{Profiles obtained from the given lookahead of the input word.}
- $r \in Q_D$, \textit{State reached on the given lookahead of the input word.}

From a vertex of the form $(q, c, P_1, P_2, r, \forall)$, Adam has the following actions:

- $a \rightarrow (q, -1, P_1, P_2 \otimes P_a, \delta_D(r, a), \exists)$, for all $a \in \Sigma$.

  \textit{Adam provides the next lookahead letter and $P_2$ is updated accordingly.}

From a vertex of the form $(q, c, P_1, P_2, r, \exists)$, Eve has the following actions:

- $\text{skip} \rightarrow (q, -1, P_1, P_2, r, \forall)$, and

  \textit{Eve makes a non-producing action, i.e., she waits for further lookahead on the input.}

- $(q, q', c') \rightarrow (q', c', P_2, P_\varepsilon, r, \forall)$, where $(q, q', c') \in P_1$.

  \textit{Eve makes a producing action: a state transformation from the first lookahead profile is chosen, the state transformation is applied, and the first profile is consumed.}
The initial vertex of the game is \((q_0^T, -1, P_ε, P_ε, q_0^D, ω)\).

Let us now define \(Win \subseteq V^ω\). The condition makes sure that if the input sequence provided by Adam is in the domain of \(R\), then the sequence of state transformations can be used to build on accepting run of \(T\) on that input. \(Win \subseteq V^ω\) is the set of all plays \(γ\) satisfying the property

\[
\max \inf(\text{col}_D(γ)) \text{ is even} \rightarrow \max \inf(\text{col}_T(γ)) \text{ is even},
\]

where \(\text{col}_D(γ) = c_D(π^i(γ))\), \(\text{col}_T(γ) = π^2(γ)\), and \(π^i(γ)\) is the projection of \(γ\) onto the \(i\)-th component of each vertex. It is not difficult to see that \(Win\) is \(ω\)-regular, e.g., one can design a parity automaton for it.

We explain the intuition behind \(Win\). Our goal is to extract a computable function that uniformizes the relation from a winning strategy. Intuitively, there is a computable function that uniformizes \(R\), if every input word \(α \in \text{dom}(R)\) can be read letter-by-letter, and from time to time, a segment of output letters is produced, continuously building an infinite output word \(β\) such that \((α, β) \in R\). We relate this to \(Win\). Recall that \(R\) is defined by \(T\), and \(\text{dom}(R)\) by \(D\). Given a play \(γ\), there is a unique input word \(α \in \Sigma^ω\) that corresponds to \(γ\). Since we are looking to build a computable function \(f\) with \(\text{dom}(f) = \text{dom}(R)\), we care whether \(α \in \text{dom}(R)\). The \(ω\)-word \(\text{col}_D(γ)\) is equal to \(c(ρ_D)\), where \(ρ_D\) is the run of \(D\) on \(α\). If \(\max \inf(\text{col}_D(γ))\) is even, \(α \in L(D)\), i.e., \(α \in \text{dom}(R)\).

An output word \(β \in Γ^ω\) that corresponds to \(γ\) is only indirectly defined, instead the play defines a (possibly finite) sequence of state transformations that an output word \(β\) should induce together with \(α\). How to extract a concrete \(β\) from \(γ\) is formally defined in the proof of Theorem 6. The \(ω\)-word \(\text{col}_T(γ)\) contains the relevant information to determine whether \((α, β) \in R(T)\), i.e., \((α, β) \in R\). In particular, if \(β\) is finite, \(\max \inf \text{col}_T(γ) = -1\), that means that only finitely many producing actions have been taken. If \(\max \inf \text{col}_T(γ)\) is even, we have that \((α, β) \in R\). Thus, \(Win\) expresses that if \(α \in L(D)\), then there is some \(β \in Γ^ω\), which can be built continuously while reading \(α\) such that \((α, β) \in R\).

We make some remarks about the form of the game, in particular the use of two lookahead profiles, instead of one. Assume we would have only one profile abstracting the lookahead over Adam inputs. For simplicity, assume the specification is automatic (i.e., letter-to-letter). Suppose, so far, Adam and Eve have alternated between providing an input letter and producing an output letter (in the finite-state game, Eve producing letter(s) corresponds to the abstract action of picking state transformations), but now, she needs to wait for more inputs before she can safely output something new. Suppose that Adam has provided some more input, say the word \(u\), and Eve now has enough information about the input to be able to produce something new. Abstractly, it means that in the game, Adam has given the word \(u\) but only its profile \(P\) is stored. Eve might not be able to produce an output of the same length as \(u\) (for example, if producing the \(i\)-th output letter depends on the \((i + k)\)-th input letter). So, she cannot consume the whole profile \(P\) (i.e., pick a state transformation in \(P\)). What she has to do, is to decompose the profile \(P\) into two profiles such that \(P = P_1 \otimes P_2\) and pick a state transformation in \(P_1\), and then continue the game with profile \(P_2\) (and keep on updating...
it until she can again produce something). The problem is, firstly, that there is no unique way of decomposing \( P \) as \( P_1 \otimes P_2 \), and secondly, \( P_1 \) might not correspond to any prefix of \( u \). That is why it is needed to have explicitly the decomposition at hand in the game construction.

**From winning strategies to uniformizers.** We are ready to state our first positive result: If Eve has a winning strategy in the unbounded delay game \( G_T \), then \( R(T) \) is uniformizable by a computable function. In fact, we show a more precise result, namely, that if Eve has a winning strategy, then the relation is uniformizable by a function recognized by a deterministic two-way parity transducer. Additionally, if the domain of the relation is closed, then a deterministic one-way transducer suffices (see Section 5). Just as (one-way) transducers extend parity automata with outputs on their transitions, input-deterministic two-way transducers extend deterministic two-way parity automata with outputs. The reading tape is two-way, but the output tape is one-way. The class of functions recognizable by 2DFTs is smaller than the class of computable functions and enjoys many good algorithmic properties, e.g., decidability of the equivalence problem [3]. Note that any function recognizable by a 2DFT is computable, in the sense that it suffices to “execute” the 2DFT to get the output. So, from now on, we may freely say that a function is computable by a 2DFT.

Before stating our first theorem, we remind the reader of Remark 3. As stated, it is not necessary (nor feasible in finite time) to check whether an input stream belongs to the domain. However, since we employ the notion of uniformization to state our results, we need to properly define a domain of our uniformizer. Hence, the 2DFT we construct (in the proof of Theorem 6) has an acceptance condition corresponding to the desired domain. In order to get the output from the 2DFT in a streaming fashion, the acceptance condition can simply be ignored.

**Theorem 6.** Let \( R \) be defined by a transducer \( T \). If Eve has a winning strategy in \( G_T \), then \( R(T) \) is uniformizable by a function computable by a 2DFT.

The proof comprises two parts. First, we show that if Eve has a winning strategy in \( G_T \), then \( R(T) \) is uniformizable by a computable function \( f \), see Lemma 7. Secondly, we show that the function \( f \) is also computable by a 2DFT, see Lemma 8.

### 3.1 Extracting a Computable Function from a Winning Strategy

**Lemma 7.** If Eve has a winning strategy in \( G_T \), then \( R(T) \) is uniformizable by a computable function \( f \).

**Proof.** Let \( \sigma_3 \) be a winning strategy for Eve in \( G_T \). This implies that there exists also a finite state winning strategy for Eve, because \( G_T \) is an \( \omega \)-regular game, and \( \omega \)-regular games are finite-memory determined, by Büchi-Landweber’s Theorem.

In Algorithm 1, we give an algorithm that describes a computable function \( f : \Sigma^\omega \rightarrow \Gamma^\omega \) that uniformizes \( R \) which is constructed from a finite state winning strategy given by a strategy automaton \( S \). Intuitively, the algorithm plays the game where Adam’s
actions correspond to an input word $\alpha$ and Eve’s actions are determined by the winning strategy given by $S$. In a play, the lookahead on the input is not stored concretely, but as an abstraction in form of two profiles. When Eve takes an edge labeled with a state transformation, then the first lookahead profile is removed, the chosen state transformation is applied, the second lookahead profile takes the place of the first one, and the second lookahead profile is built anew. In the algorithm, additionally, the lookahead on the input that is used to build the two profiles is stored concretely. When Eve takes an edge labeled with a state transformation, say $(p, q, c)$, we have the information what kind of state transformation should occur for the input $u_1 \in \Sigma^*$ that currently corresponds to the first profile $P_1$. The algorithm picks an output $v_1 \in \Gamma^*$ such that $\mathcal{T} : p \xrightarrow{u_1/v_1} q$ and the maximal priority seen on this run is $c$. Then, in the game, profile $P_2$ takes the place of $P_1$ and the second profile is set to $P_\ell$. Additionally, the algorithm sets $u_1$ to $u_2$, that is, the input that currently corresponds to $P_2$, and sets $u_2$ to $\varepsilon$. The stored strings $u_1$ and $u_2$ can get arbitrarily long, but are reset from time to time, as the algorithm is built from a winning strategy, thus producing edges are taken again and again. Note that the algorithm is guaranteed to produce non-empty output strings again and again, because in Line 9 a new letter is appended to the string $u_2$ again and again, and in Line 18 the string $u_2$ is set to $u_1$ from time to time. Thus, the string $u_1$ is infinitely often not empty. In fact, $u_1$ is only empty the first time an output word is chosen.

The correctness of the algorithm can easily be seen. Assume that $\gamma$ is a play according to the winning strategy $\sigma_3$ and assume that the corresponding input word $\alpha \in \Sigma^\omega$, spelt by Adam’s actions, is part of the domain of the relation, i.e., $\alpha \in L(\mathcal{D})$. Hence, since $\max \operatorname{Inf}(\operatorname{col}_\mathcal{D}(\gamma))$ is even, also $\max \operatorname{Inf}(\operatorname{col}_\mathcal{T}(\gamma))$ is even, because $\gamma \in \text{Win}$. This means that that Eve takes infinitely many producing edges, otherwise $\max \operatorname{Inf}(\operatorname{col}_\mathcal{T}(\gamma))$ would be $-1$. Thus, the algorithm infinitely often prints a finite output word building an output word $\beta$. We have that $(\alpha, \beta) \in R$, i.e., $(\alpha, \beta) \in R(\mathcal{T})$, because $\max \operatorname{Inf}(\operatorname{col}_\mathcal{T}(\gamma))$ is even and $\beta$ was build such that there exists a run $\rho_\mathcal{T}$ of $\mathcal{T}$ on $\alpha/\beta$ such that $\max \operatorname{Inf}(\operatorname{col}_\mathcal{T}(\gamma)) = \max \operatorname{Inf}(c(\rho_\mathcal{T}))$. (Note that we assume that $\mathcal{T}$ defines a relation $R \subseteq \Sigma^\omega \times \Gamma^\omega$ (controlled with the parity condition) which guarantees that $\beta$ is build from finite words of which infinitely many are non-empty, because $\max \operatorname{Inf}(c(\rho_\mathcal{T}))$ is even.)

We now argue that algorithm yields a computable function. The algorithm handles an input sequence in a letter-by-letter fashion, the only ambiguity in the algorithm is in Line 17, that requires the algorithm to pick an output block $v_1$ such that $\mathcal{T} : p \xrightarrow{u_1/v_1} q$. We explain that this is indeed computable by a deterministic machine as described in Definition 2. Clearly, for each $(p, q, c)$ there exists a transducer, say $\mathcal{T}(p, q, c)$, that recognizes $\{ (u, v) \mid \mathcal{T} : p \xrightarrow{u/v} q \}$ with max prio is $c$. To pick some $v_1$ for a given $u_1$, a machine $M$ as in Definition 2 can work as follows. It is easily seen that the set $L = \{ v \mid \mathcal{T} : p \xrightarrow{u_1/v} q \}$ is regular, and hence computing some $v_1$ can be done by using a standard non-emptiness checking algorithm on $L$. □

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Algorithm 1 Algorithm computing a function that uniformizes $R$. The algorithm is described in the proof of Lemma 7.

Require: $\alpha \in \Sigma^\omega$, $G$ game arena, $S = (M, V, m_0, \delta, \mu)$ strategy automaton

$m \leftarrow m_0 \{\text{current state of the strategy automaton}\}$

$u_1 \leftarrow \varepsilon \{\text{first input block}\}$

$u_2 \leftarrow \varepsilon \{\text{second input block}\}$

$s_{\text{prev}} \leftarrow s_0, \text{initial vertex of } G \{\text{previous vertex in the game}\}$

$s_{\text{cur}} \leftarrow s_0 \{\text{current vertex in the game}\}$

$a \leftarrow \alpha(0) \{\text{current letter of } \alpha\}$

$\text{action} \{\text{current action of Eve}\}$

while true do

$u_2 \leftarrow u_2.a \{\text{append letter to second block}\}$

$s_{\text{cur}} \leftarrow s \text{ if } s_{\text{cur}} \xrightarrow{a} s \in E \{\text{update game vertex according to Adam’s action}\}$

$m \leftarrow \delta(m, s_{\text{cur}}) \{\text{strategy automaton is updated with Adam’s action}\}$

$s_{\text{prev}} \leftarrow s_{\text{cur}}$

$\text{action} \leftarrow \mu(m, s_{\text{cur}}) \{\text{strategy automaton yields Eve’s action}\}$

$s_{\text{cur}} \leftarrow s \text{ if } s_{\text{cur}} \xrightarrow{\text{action}} s \in E \{\text{updated game vertex is of the form } (\cdot, P_{u_1}, P_{u_2}, \cdot, \cdot)\}$

$m \leftarrow \delta(m, s_{\text{cur}}) \{\text{strategy automaton is updated according to Eve’s action}\}$

if $\text{action}$ is of the form $(p, q, c)$ \{Eve took a producing edge\} then

choose output block $v_1 \in \Gamma^*$ such that $T: p \xrightarrow{u_1/v_1} q$ with max prio $c$

$u_1 \leftarrow u_2 \{\text{first input block becomes second}\}$

$u_2 \leftarrow \varepsilon \{\text{second block is emptied}\}$

print $v_1 \{\text{produce output block}\}$

end if

$a \leftarrow \alpha.\text{nextLetter}\() \{\text{read next input letter}\}$

end while

Ensure: $\beta \in \Gamma^\infty$, if $\alpha \in \text{dom}(R)$, then $(\alpha, \beta) \in R$

3.2 Extracting a 2DFT from a Winning Strategy

We have seen that Algorithm 1 uses unbounded memory. The following lemma shows that this unbounded memory can be traded to for finite memory at the cost of having input two-wayness.

Lemma 8. If Eve has a winning strategy in $G_T$, then $R(T)$ is uniformizable by a 2DFT.

Proof. We first give an intuition how to translate a winning strategy into a 2DFT. The main idea is to use two-wayness to encode finite lookahead over the input: the reading head goes forward to gather input information, and then must return to the initial place where the lookahead was needed to transform the input letter. The difficulty is for the 2DFT to return to the correct position, even though the lookahead can be arbitrarily long.
In order to find the correct positions, we make use of a finite-state strategy automaton for Eve’s winning strategy in the following sense. A (left-to-right) run of the strategy automaton on the input word yields a unique segmentation of the input, such that segments $i$ and $i+1$ contain enough information to determine the output for segment $i$. The idea is to construct a 2DFT that simulates the strategy automaton in order to find the borders of the segments. If the 2DFT goes right, simulating a computation step of the deterministic strategy automaton is easy. Recovering the previous step of the strategy automaton when the 2DFT goes left is non-trivial, it is possible to compute this information using the Hopcroft-Ullman construction presented in [24]. We show that having the knowledge of the profiles of segments $i$ and $i+1$ is enough to deterministically produce a matching output for segment $i$ on-the-fly going from left-to-right over segment $i$ again.

Now, we present a detailed proof. Assume that Eve has a finite-state winning strategy in $G_T$. Algorithm 1 describes a computable function that uniformizes $R(T)$ built from a finite-state winning strategy given by strategy-automaton $S$. At Line 17, this algorithm makes a choice of output $v_1$. We slightly modify this choice: given two states $p$ and $q$ of the transducer recognizing $R(T)$ and a priority $c$, the relation

$$R(p,q,c) = \{(u,v) \mid T: p \xrightarrow{u/v} q \text{ and the maximal priority seen is } c\}.$$ 

is a rational relation of finite words. It is known that every rational relation on finite words has an effective rational uniformizer, see, e.g., [5]. So let $f_{(p,q,c)}$ denote such a rational function, as obtained in [5]. We modify Line 17 of the algorithm so that it chooses output $v_1 = f_{(p,q,c)}(u_1)$.

We denote by $f$ the function computed by the (modified) algorithm. We show that $f$ is recognizable by an input-deterministic two-way transducer. The main difficulty is that in the game, there is a desynchronization between the moment Eve makes a producing action (a triple $(q, q', c')$) and the interval of the play during which the input corresponding to this producing action has been provided by Adam. In Algorithm 1, this is easily solved because unbounded memory can be used to store these intervals. An input-deterministic two-way transducer must solve this by reading far enough into the input such that the producing action is known, then the transducer must deterministically go back to the input segment for which this producing action was chosen. The main difficulty lies in deterministically finding the right beginning and end points of such a segment when going back.

We provide a construction in two steps. First, we describe a simpler setting, where an input sequence is enriched by a function $f_{\text{annot}}$ with information about the corresponding play in the game consistent with the strategy defined by the strategy automaton. Those information are: vertices of the play, actions performed during the play, current state of the strategy automaton. Recall that the game alternates between Adam and Eve actions and start by an Adam action. Adam’s actions are input symbols in $\Sigma$ while Eve’s actions are either $\text{skip}$ or a triple $(q, q', c')$ of some profile. Let us denote by $A_\exists$ Eve’s actions. Then, the type of $f_{\text{annot}}$ is

$$f_{\text{annot}} : \text{dom}(R) \subseteq \Sigma^\omega \rightarrow \left(\Sigma \times ((V \times M) \times (V \times M) \times (A_\exists \times V \times M))\right)^\omega.$$
where \( V \) are the vertices of the game and \( M \) are the states of the strategy automaton. The function returns an input sequence annotated with sufficient information to reconstruct a play in the game induced by the strategy applied on this input sequence, and the run of the strategy automaton on this play. More precisely, for a single letter \( a \) the annotation contains the vertex of the game and the state of the strategy automaton, Adam’s action (which is \( a \)), the target vertex and target state induced by Adam’s action, Eve’s action as prescribed by the strategy automaton, the target vertex and target state induced by Eve’s action.

We explain how to construct an input-deterministic two-way transducer that recognizes the function \( g \) which takes an annotated input sequence \( f_{annot}(\alpha) \) and maps it to \( f(\alpha) \) for all \( \alpha \in \text{dom}(R) \). Subsequently, we explain how to modify the transducer to recognize the function \( f: \alpha \rightarrow f(\alpha) \) for all \( \alpha \in \text{dom}(R) \).

An annotated string \( \alpha' \) based on \( \alpha \) defines a unique segmentation \( u_0u_1 \ldots \) of \( \alpha \) where the endpoint of \( u_i \) is annotated with a producing action of Eve for all \( i \geq 1 \) (\( u_0 \) is defined as \( \varepsilon \)). The producing action \( (q, q', c') \) seen at the end of \( u_i \) means that for the segment \( u_{i-1} \) an output \( v_{i-1} \) must be produced such that \( T: q \xrightarrow{u_{i-1}/v_{i-1}} q' \) and the maximal priority seen is \( c' \). In the annotated sequences it is easy for an input-deterministic two-way transducer to advance to the end of the segment \( u_i \) and go back to the beginning of segment \( u_{i-1} \) for all \( i \geq 1 \) because all segment endpoints are annotated with a producing action (the intermediate positions are annotated with \( \text{skip} \)).

We now explain how an input-deterministic two-way transducer can produce the desired output \( v_{i-1} \) for the segment \( u_{i-1} \) with annotations. First, we show that for any state transformation \( (q, q', c') \), the function \( f_{(q,q',c')} \) of finite words can be computed by an input-deterministic two-way transducer.

Rational functions over finite words are recognizable by input-deterministic two-way transducers, which can be seen as follows. In [17], it is shown that a function from finite words to finite words is rational iff it is definable as an order-preserving MSO-transduction. In [16], it is shown that a function from finite words to finite words is definable as an MSO-transduction iff it is definable by an input-deterministic two-way transducer. Hence, it follows that rational functions over finite words are recognizable by input-deterministic two-way transducers.

Such a transducer can be constructed. Let \( T_{(q,q',c')} \) denote the input-deterministic two-way transducer that recognizes \( f_{(q,q',c')} \).

Consequently, when the input-deterministic two-way transducer on \( \alpha' \) has returned to the beginning of the annotated segment \( u_{i-1} \) (after it went to the endpoint of \( u_i \) to get the state transformation \( (q, q', c') \) corresponding to \( u_{i-1} \)), it can run the input-deterministic two-way transducer \( T_{(q,q',c')} \) on \( u_{i-1} \) that will produce a suitable \( v_{i-1} \). The difficulty is to stay inside of \( u_{i-1} \) for this transformation. However, the annotations in \( \alpha' \) clearly mark the endpoints of \( u_{i-2} \) and \( u_{i-1} \) (by state transformations instead of \( \text{skip} \) action), so we can run the transducer \( T_{(q,q',c')} \) designed to work on finite words because we have clear markers on the bounds of the segment \( u_{i-1} \).

After producing the output \( v_{i-1} \), the two-way transducer can advance to the end of \( u_{i+1} \), obtain the producing action that should be applied to \( u_i \), go back to the beginning
of \( u_i \), apply it and so on.

The priorities that have to be seen on a run of this transducer can be derived from the annotations. Annotation \( \text{skip} \) means priority \( -1 \), and an annotation of the form \( (q, q', c') \) means priority \( c' \) must be seen. Thus, we have shown that there exists an input-deterministic two-way transducer that recognizes \( g \). Let \( T_g \) denote this transducer.

We now explain how we can get rid of the annotations. When we read an input \( \alpha \in \Sigma^\omega \), we can determine the annotations assigned by \( f_{\text{annot}} \) deterministically on-the-fly using the strategy automaton, because the strategy automaton is deterministic. Therefore we slightly have to extend the strategy automaton such that it additionally stores in its state space the current vertex (that is the input alphabet of the strategy automaton) and Eve’s chosen action (determined by the strategy automaton via it’s next move function).

Let \( \mathcal{A} \) denote this enhanced strategy automaton. It remains deterministic. The deterministic automaton \( \mathcal{A} \) has the property that when going from left-to-right in \( \alpha \), it computes the annotation in the sense that it corresponds to the information stored in the state space of \( \mathcal{A} \). So, by running \( \mathcal{A} \) from left-to-right, we have access to the annotation.

Our goal is now to extend \( T_g \) such that it computes the run of \( \mathcal{A} \) on \( \alpha \). Clearly, when going from a position \( j \) to \( j + 1 \) is is possible to compute the state of \( \mathcal{A} \) on \( j + 1 \) when the state of \( \mathcal{A} \) on \( j \) is known, however, when going from a position \( j \) to \( j - 1 \), we also want to obtain the state of \( \mathcal{A} \) that was seen on \( j - 1 \) based on the state that was seen on \( j \).

It is possible to compute this information with an input-deterministic two-way transducer using the Hopcroft-Ullman construction presented in [24]. The construction describes how to simulate a deterministic one-way automaton together with a two-way automaton. It is described for two-way automata on finite words, however, in order to obtain the state of the one-way automaton at position \( j - 1 \) when one is at position \( j \), the Hopcroft-Ullman construction only needs to re-read some positions to the left of \( j \), so the same construction can be applied in the infinite word setting.

Hence, \( T_g \) combined with \( \mathcal{A} \) using the Hopcroft-Ullman construction can be run on words \( \alpha \in \Sigma^\omega \) instead of \( f_{\text{annot}}(\alpha) \) and computes the desired function \( f \).

We like to mention that Lemma 8 could have been obtained in another way. Carton and Douéneau-Tabot [10] have shown that every computable rational function can be computed by some 2DFT. Their result provides a translation from computable rational functions to 2DFTs going through intermediate computation models. To employ their result, we need to show that the computable function \( f \) obtained from a winning strategy for Eve is rational (i.e., a non-deterministic one-way transducer recognizing \( f \) exists). The proof is similar to the proof of Lemma 8. The main difference is that we use the two-wayness to deterministically determine a priori (by obtaining some lookahead on the input) which input transformation should be realized on a finite input segment. Without two-wayness, we use non-determinism to guess and output the correct input transformation. The correctness is then verified a posteriori.

However, we decided in order to obtain Lemma 8 to give a direct construction to be self-contained and build a 2DFT with better state complexity than the 2DFT that would
result from the translations given in [10].

3.3 Complexity Analysis

**Lemma 9.** Deciding whether Eve has a winning strategy in $G_T$ is in \textsc{ExpTime}.

**Proof.** Two-player $\omega$-regular games are decidable (see, e.g., [22]). The claimed upper bound is achieved by representing the winning condition as a deterministic parity automaton, carefully analyzing its size, and then solving a parity game.

From $G_T$ we can obtain an equivalent parity game of size $|G| \cdot |W|$, where $W$ is a deterministic parity automaton that recognizes the set $\text{Win}$. First, we analyze the size of $G = (V, E)$. A vertex is an element of $Q_T \times (C_T \cup \{-1\}) \times \mathcal{P} \times \mathcal{P} \times Q_D \times \{orall, \exists\}$, where $\mathcal{P}$ is the set of all profiles. A profile is a subset of $Q_T \times Q_T \times C_T$, thus, the set of profiles is of size exponential in $|Q_T|$ and $|C_T|$. We assume that the domain automaton $D$ is obtained from $T$ by projection and determinization. A deterministic parity automaton with $2^{O(n \log n)}$ states and $O(nk)$ priorities can be constructed from a nondeterministic parity automaton with $n$ states and $k$ priorities using Safra’s construction [30]. We obtain that $D$ has a state space of size exponential in $|Q_T|$ with priorities $C_T$. It follows that $V$ and $E$ are of size exponential in $|Q_T|$ and $|C_T|$. All in all, since $|C_T| \leq |Q_T|$, $|V|$ and $|E|$ are exponential in $|Q_T|$.

Secondly, we analyze the size of $W$. Recall that $\text{Win}$ contains plays $\gamma$ that satisfy

$$\max\ \text{Inf}(\text{col}_D(\gamma)) \text{ is even } \to \max\ \text{Inf}(\text{col}_T(\gamma)) \text{ is even.}$$

We reformulate this as

$$\max\ \text{Inf}(\text{col}_D(\gamma)) \text{ is odd } \lor \max\ \text{Inf}(\text{col}_T(\gamma)) \text{ is even.}$$

A deterministic parity automaton for the second disjunct has a state set of size linear in $|Q_T|$ with priorities $C_T$. The first disjunct simply states that the input sequence $\alpha$ associated to $\gamma$ is in $\text{dom}(f)$, i.e., $\alpha \in L(D)$. A nondeterministic parity automaton for $L(D)$ has a state set of size linear in $|Q_T|$ and priorities $C_T$. Thus, a nondeterministic parity automaton for the conjunction has a state set of size linear in $|Q_T| \cdot |C_T|$. Using Safra’s construction, we obtain a deterministic variant with a state set of size exponential in $|Q_T|$ with priorities of size linear in $|Q_T| \cdot |C_T|$. Let $W$ denote this automaton with state set $W$. The alphabet of $W$ is $V$, thus, the alphabet size is exponential in $|Q_T|$.

The parity game with arena $G \times W$ has $|V| \cdot |Q_W|$ many vertices and $|E| \cdot |Q_W|$ many edges. Thus, its vertices and edges are of size exponential in $|Q_T|$. The game has priorities of size linear in $|Q_T| \cdot |C_T|$. A parity game with $n$ vertices, $m$ edges, and $k$ priorities can be solved in time $O(mn^{k/3})$, see [32]. Consequently, we obtain that our parity game is solvable in time exponential in $|Q_T| \cdot |C_T|$. Since $|C_T| \leq |Q_T|$, the overall time complexity is exponential in $|Q_T|$. \qed
3.4 Extracting a 1DFT from a Winning Strategy with Bounded Delay

We state a lemma about bounded delay. First, we introduce the notion of bounded delay 1DFT. Let $\mathcal{T} = (Q, \Sigma, \Gamma, q_0, \Delta, c)$ be a 1DFT and $K \in \mathbb{N}$. We say that $\mathcal{T}$ is $K$-delay if, intuitively, the output production is late at most $K$ steps and never ahead, i.e., for all runs $(p_0, u_0, v_0, p_1)(p_1, u_1, v_1, p_2) \ldots (p_{n-1}, u_{n-1}, v_{n-1}, p_n) \in \Delta^*$, it holds that $|u_0u_1 \ldots u_{n-1}| - |v_0v_1 \ldots v_{n-1}| \leq K$.

**Lemma 10.** Let $R$ be defined by a transducer $\mathcal{T}$. If there exists $\ell \geq 0$, such that Eve has a winning strategy in $G_{\mathcal{T}}$ with at most $\ell$ consecutive skip-moves, then $R$ is uniformizable by a function $f$ computable by a 1DFT $\mathcal{T}_f$. Moreover, if $\mathcal{T}$ is letter-to-letter, then $\mathcal{T}_f$ is $\ell$-delay.

**Proof.** Intuitively, such a strategy yields a function computable by a 1DFT, because the needed lookahead (as it is bounded) can be stored in the state space.

Let $\sigma_3$ be such a winning strategy for Eve in $G_{\mathcal{T}}$ (i.e., Eve never makes more than $\ell$ consecutive skip-moves). We can use Algorithm 1 to obtain a computable function $f$ that uniformizes $R(\mathcal{T})$. We now explain that $f$ is in fact sequential, i.e., recognizable by a 1DFT.

The construction of Algorithm 1 yields that $f$ is such that it reads a non-empty input block of length at most $\ell$, and then produces an output block (a canonical choice of size linear in $\ell$ is possible), and so on. It directly follows that $f$ is sequential, and a 1DFT $\mathcal{T}_f$ that recognizes $f$ can be effectively constructed. If additionally $\mathcal{T}$ is letter-to-letter, then the strategy, after reading a non-empty input block of length at most $\ell$, picks an output block of the same size. Therefore, $\mathcal{T}_f$ is $\ell$-delay. \(\square\)

3.5 A Note on the Incompleteness of the Decision Procedure

**Remark 11.** The converse of Theorem 6 is not true.

First, the existence of a computable uniformizer does not imply the existence of a winning strategy, otherwise, the RAT-synthesis problem would be decidable, which is a contradiction to Proposition 5. The following example strengthens this result: it shows that even if there is a 2DFT uniformizer, it does not necessarily imply the existence of a winning strategy.

**Example 12.** Consider the identity function $f: \{a, b\}^\omega \to \{a, b\}^\omega$ such that all inputs with either finitely many $a$ or $b$ are in the domain. A (badly designed) letter-to-letter transducer $\mathcal{T}$ that recognizes $f$ has five states $S, A, B, A', B'$, where $S$ is the starting state, $A, B$ (resp. $A', B'$) are used to recognize finitely many $b$ (resp. $a$), and from $S$, the first input/output letter non-deterministically either enters $A$ or $A'$. In a play in $G_{\mathcal{T}}$, at some point, Eve must make her first output choice, i.e., she starts to build a run of $\mathcal{T}$. This choice fixes whether the run is restricted to $A, B$ or $A', B'$. No matter Eve’s choice, Adam can respond with an infinite sequence of either only $a$ (for $A, B$) or $b$ (for $A', B'$), making it impossible to build an accepting run. Thus, Eve has no winning strategy, but clearly the function $f$ is computable by a 2DFT.
While in the above example, the point of failure is clearly the bad presentation of the specification, this is not the case in general. Recall the proof sketch of Proposition 5, where we provide a reduction from Post’s correspondence problem. A non-deterministic transducer constructed from a given PCP instance $u_1,v_1,\ldots,u_n,v_n$ can guess whether the input word contains infinitely many $a_i$, and accordingly either checks that the output begins with $u_{i_1}\cdots u_{i_k}$ for input sequences beginning with indices $i_1\cdots i_k$, or checks that it does not begin with a prefix equal to $v_{i_1}\cdots v_{i_k}$. As detailed in the proof sketch, if the PCP instance has no solution, there is a computable uniformization, however, using the same argumentation as in the above example, such a transducer would make it impossible to have a winning strategy. In order to have a winning strategy, the transducer must be changed such that it checks at the same time whether the output starts with $u_{i_1}\cdots u_{i_k}$ and does not start with something equal to $v_{i_1}\cdots v_{i_k}$ for input sequences beginning with $i_1\cdots i_k$. In general, depending on the PCP instance, it is not possible to make both checks in parallel.

4 A Sufficient Condition for Completeness

Theorem 6 yields a procedure which is sound but not complete for solving the RAT-synthesis problem. From a transducer $T$, construct the $\omega$-regular game $G_T$ and solve it: if Eve wins, then $T$ is uniformizable by a computable function (even computable by a 2DFT). Otherwise, nothing can be concluded.

In this section, we show that the procedure is complete for two known and expressive classes of rational relations, namely the class of automatic relations (AUT), which are for example used as specifications in Church synthesis, as well as the class of deterministic rational relations (DRAT) [31] (to be formally defined below), see Corollary 14.

To arrive at these results, we define a structural restriction on transducers that turns out to be a sufficient condition for completeness. Let $T$ be a transducer. An input (resp. output) state is a state $p$ from which there exists an outgoing transition $(p,u,v,q)$ such that $u \neq \varepsilon$ (resp. $v \neq \varepsilon$). The set of input (resp. output) states is called $Q_i$ (resp. $Q_o$).

A transducer $T$ has property $P$ if the following two conditions hold:

1. The transition set is a finite subset of $Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma^* \times Q$. In words, a transition reads one or no input letters.$^4$

2. For all words $u \in \Sigma^*$, all $v_1,v_2 \in \Gamma^*$ such that $v_1$ is a prefix of $v_2$ the following holds:

$$\text{if } T : p \xrightarrow{u/v_1} q, T : p \xrightarrow{u/v_2} r, \text{ and } q,r \text{ are input states, then } q = r.$$

The property $P$ ensures that given $\alpha \in \Sigma^\omega$ and $\beta \in \Gamma^\omega$ such that there is a run of $T$ with input $\alpha$ and output $\beta$, for each prefix $u \in \Sigma^*$ of $\alpha$ all runs $T : q_0 \xrightarrow{u/v}$ that end in an input state such that $v \in \Gamma^*$ is a prefix of $\beta$ lead to the same input state. This

$^4$We thank the anonymous reviewer for alerting us about the necessity of this condition.
implies that input prefixes together with (long enough) output prefixes are sufficient to
determine the beginning of an accepting run up to the last state reached before reading
the next input letter following the prefix. This allows us to show the following.

**Theorem 13.** Let \( R \) be defined by a transducer \( T \) with property \( P \). If \( R \) is uniformizable
by a computable function, then Eve has a winning strategy in \( G_T \).

We give the formal proof of this result in Section 4.1 and provide a proof sketch first
to not interrupt the flow of this section.

**Proof sketch.** In fact, we explain how to construct a winning strategy from a continuous
(a computable function is always continuous, see Section 4.1) uniformizer
\( f \) of \( R \). Given \( \alpha \in \text{dom}(R) \) and \( f(\alpha) \), we show that it is possible to decompose the input \( \alpha \) into \( u_0u_1\cdots \)
and the output \( f(\alpha) \) into \( v_0v_1\cdots \) such that there exists an accepting run
\( T : q_0 \xrightarrow{u_0/v_0} q_1 \xrightarrow{u_1/v_1} q_2 \cdots \) where each \( q_i \) is an input state for \( i > 0 \). Moreover, this decomposition
and run can be determined in a unique way and on-the-fly, in the sense that a factor
\( u_i/v_i \) only depends on the factors \( u_0/v_0, \ldots, u_{i-1}/v_{i-1} \). This makes it possible for Eve
to pick a corresponding state transformation sequence \((q_0, q_1, c_0)(q_1, q_2, c_1)\cdots \) which is
only dependent on the so far seen actions of Adam spelling \( u_0u_1\cdots \). The main idea
to determine the \( u_i \) is to look at the indices \( j \) for which the longest common prefix of
the sets \( S_j = \{ f(\alpha(\cdot j)\beta) \mid \alpha(\cdot j)\beta \in \text{dom}(R) \} \) strictly increases. Given \( u_i \), the output
\( v_0v_1\cdots v_i \) is any common prefix of the sets \( S_j \), such that a run \( T : q_i \xrightarrow{u_i/v_i} \) is defined and
its target is an input state. The fact that \( T \) has property \( P \) guarantees that each of these
runs has the same target, thus, the next state transformation \((q_i, q_{i+1}, c_i) \) is uniquely
determined.

We formally introduce AUT and DRAT. A relation is deterministic rational if it is
recognized by a transducer where \( Q_i \) and \( Q_o \) partition its state space, and its transition
relation \( \Delta \) is a function \((Q_i \times \Sigma \times \{\varepsilon\} \to Q) \cup (Q_o \times \{\varepsilon\} \times \Gamma \to Q) \). It is automatic
if additionally \( \Delta \) strictly alternates between \( Q_i \) and \( Q_o \) states. It is easy to see that
every DRAT-transducer (and a fortiori every AUT-transducer) satisfies the property \( P \).
In general, given any transducer \( T \), we do not know if it is decidable whether \( T \) has
property \( P \).

**Main result.** We now state our main result: Asking for the existence of a uniformization
which is computable by a Turing machine or computable by an input-deterministic
two-way transducer (2DFT), are equivalent questions, as long as specifications are DRAT
relations. Moreover, these questions are decidable.

**Corollary 14.** Let \( R \) be defined by a DRAT-transducer \( T \). The following are equivalent:

1. \( R \) is uniformizable by a computable function.
2. \( R \) is uniformizable by a function computable by a 2DFT.
3. Eve has a winning strategy in \( G_T \).
Note that the above result also holds for the slightly more general case of relations given by transducers with property $\mathcal{P}$.

**Theorem 15.** The AUT- and DRAT-synthesis problems are ExpTime-complete.

**Proof.** Membership in ExpTime directly follows from Lemma 9 and Corollary 14. In [26] it was shown that this problem is ExpTime-hard in the particular case of automatic relations with total domain, so the lower bound applies to our setting. \qed

### 4.1 From Continuous Functions to Strategies

In the upcoming proofs, we make use of the notion of continuity instead of computability, as this simplifies the proofs. We first recall its definition. A function is called *continuous* if

$$\forall \alpha \in \text{dom}(f) \ \forall i \in \mathbb{N} \ \exists j \in \mathbb{N} \ \forall \beta \in \text{dom}(f) : |\alpha \land \beta| \geq j \rightarrow |f(\alpha) \land f(\beta)| \geq i. \ (2)$$

It is easy to see that every computable function is also continuous.

**Example 16.** Consider the function $f_1$ of Example 4. $f_1$ is continuous, because the $i$-th output symbol only depends on the $\max(i, n + 1)$ first input symbols. Consider the function $f_2$ of Example 4. The function $f_2$ is clearly rational, but it is not continuous. We verify that $f_2$ is not continuous, let $\alpha_n$ denote $a^n b^w$, we have that $|\alpha_n \land a^w| = n$ and $|f_2(\alpha_n) \land f_2(a^w)| = 0$ for all $n \in \mathbb{N}$. Thus, $f_2$ is not continuous.

We introduce an auxiliary function used in the proof of Theorem 13 that yields the “best” (meaning most beneficial for Eve) priority associated to a profile and two states.

**Definition 17 (best).** We define the function $\text{best}: \mathcal{P} \times Q_T \times Q_T \rightarrow c_T$, where $\mathcal{P}$ denotes the set of all profiles. We let

$$\text{best}(P, p, q) = \begin{cases} 
\max\{c \mid (p, q, c) \in P, \ c \text{ is even}\} & \text{if there is an even } c' \text{ s.t. } (p, q, c') \in P \\
\min\{c \mid (p, q, c) \in P, \ c \text{ is odd}\} & \text{if there is no even } c' \text{ s.t. } (p, q, c') \in P
\end{cases}$$

The meaning of $\text{best}(P, p, q) = c$ is that it for an input word $u$ that has profile $P$, the best priority Eve can achieve when going from $p$ to $q$ via some $u/v$ is $c$. This $c$ is the maximal even priority if there is some, while $c$ is the minimal odd priority if there is no even priority that she can achieve.

As already mentioned in the proof sketch of Theorem 13, we show a stronger result, namely, the following result.

**Theorem 18.** Let $R$ be defined by a transducer $T$ with property $\mathcal{P}$. If $R$ is uniformizable by a continuous function $f$, then Eve has a winning strategy in $G_T$. \[22\]
Proof. To begin with, we describe how to inductively factorize an input word $\alpha \in \Sigma^\omega$. Let $u_0 = \varepsilon$, and let $u_0 \cdots u_{i-1}$ be the already defined factors of $\alpha$, let $\alpha = u_0 \cdots u_{i-1} \alpha'$, and let

$$u_i := \alpha'(i),$$

(3)

where $k \in \mathbb{N}_{\geq 1}$ is chosen such that $\alpha'(i)$ is the shortest non-empty prefix of $\alpha'$ such that there is a prefix of $f(u_0 \cdots u_{i-1} \alpha'(i))$, say $v$, that satisfies $\Delta_T^i(q_0^T, u_0 \cdots u_{i-1}/v) \cap Q_i^T \neq \emptyset$ (let $Q_i^T$ denote the set of input states of the transducer $T$ as defined at the beginning of Section 4). In words, the input $u_0 \cdots u_i$ provides enough information to determine a run of $T$ that consumes the input $u_0 \cdots u_{i-1}$. If such a prefix of $\alpha'$ does not exist, let $u_i = \alpha'$, that is, the infinite remainder of $\alpha$.

Let $\alpha_1, \alpha_2 \in \Sigma^\omega$, and let $u_1^\alpha \cdots$ and $u_2^\alpha \cdots$ denote their factorizations built according to Eq. (3). Note that $u_i^\alpha = u_i^\beta$ for all $i$ such that $|u_i^\alpha| \leq |\alpha_1 \wedge \alpha_2|$. Furthermore, given an input word $\alpha$, the factorization according to Eq. (3) can be determined on-the-fly reading the input word letter-by-letter. If $\alpha \in \text{dom}(f)$, its factorization has infinitely many factors, because $f$ is continuous meaning that $f$ grows over time ($f$ has been defined in Section 2).

In the remainder of the proof we are only interested in plays where the input word $\alpha$ spelt by Adam is in $L(D)$, i.e., $\text{dom}(R)$, because all other plays automatically satisfy $\text{Win}$, so there is nothing to show for plays whose corresponding input word $\alpha \notin L(D)$. Since $f$ uniformizes $R$, we have that $\text{dom}(f) = \text{dom}(R)$, so we can safely consider only plays where its corresponding input word $\alpha \in \text{dom}(f)$.

The idea of the proof is that we use the sequence of input blocks $u_0u_1\cdots$ of $\alpha$ built as in Eq. (3) to obtain the actions of Eve. Her actions pick state transformations, our goal is to pick state transformations by defining a sequence of output blocks $v_0/v_1\cdots$ and pick state transformations that are induced by $u_0/v_0 \cdot u_1/v_1 \cdots$. Formally, let $v$ be the shortest prefix of $f(u_0 \cdots u_i)$ such that $\Delta_T^i(q_0^T, u_0 \cdots u_{i-1}/v) \cap Q_i^T \neq \emptyset$ and define

$$v_{i-1} = (v_0 \cdots v_{i-2})^{-1}v.$$  

(4)

We define a strategy that satisfies the following invariant for all $i \in \mathbb{N}$:

**Lemma 19.** If there exists a run $\rho$ of $T$ on $u_0 \cdots u_{i-1}/v_0 \cdots v_{i-1}$ of the form

$$T: \begin{array}{l}
q_0^T \xrightarrow{u_0/v_0} q_1 \in Q_i^T \\
q_1 \xrightarrow{u_1/v_1} q_2 \in Q_i^T \\
\quad \vdots \\
q_{i-1} \xrightarrow{u_{i-1}/v_{i-1}} q_i \in Q_i^T,
\end{array}$$

then the sequence of chosen producing edges in the play after the input $u_0 \cdots u_i$ has the label sequence

$$(q_0^T, q_1, c_0)(q_1, q_2, c_1) \cdots (q_{i-1}, q_i, c_{i-1}),$$

where the edge labeled with $(q_j, q_{j+1}, c_j) \in P_u$ with $c_j = \text{best}(P_u, q_j, q_{j+1})$ (see Definition 17) is taken after the input $u_0 \cdots u_{j+1}$ for all $0 \leq j \leq i - 1$. Furthermore, after the input $u_0 \cdots u_i$ the play is in the vertex

$$(q_{i-1}, -1, P_{u_{i-1}}, P_u, \delta_{D}^u(q_0^T, u_0 \cdots u_i), \emptyset)).$$

23
We show the invariant holds by induction. For $k = 0$, the invariant trivially holds. For the step $k−1 \rightarrow k$, assume that Adam’s actions have spelt $u_0 \cdots u_k$, and for $v_0 \cdots v_{k−2}$ the invariant holds. Let $v_{k−1}$ as in Eq. (4), let $\rho_{k−1}$ be a run of the form $q_{k−1} \xrightarrow{u_{k−1}/v_{k−1}} q_k \in Q_1^T$ which must exist by choice of $v_{k−1}$. The invariant yields that after input $u_0 \cdots u_{k−1}$ the play is in the vertex
\[
(q_k−2, −1, P_{uk−2}, P_{uk−1}, \delta^D_\rho(q_0^D, u_0 \cdots u_{k−1}), \exists)
\]
and then Eve takes the producing edge labeled with $(q_{k−2}, q_{k−1}, c_{k−2})$. Thus, after further input $u_k$, the play is in the vertex
\[
(q_k−1, −1, P_{uk−1}, P_{uk}, \delta^D_\rho(q_0^D, u_0 \cdots u_k), \exists).
\]
Since $\rho_{k−1}$ is of the form $\mathcal{T}: q_{k−1} \xrightarrow{u_{k−1}/v_{k−1}} q_k$, the priority $\text{best}(P_{uk−1}, q_{k−1}, q_k)$ is defined, let $c_{k−1}$ denote this priority. Thus, this vertex has a producing edge with label $(q_{k−1}, q_k, c_{k−1}) \in P_{uk−1}$. Eve’s action is to take this edge. This concludes the proof that the invariant holds for $k$.

It is left to show the above defined strategy is winning. Recall the condition for a play $\gamma$ such that it is winning for Eve.

$$\text{max Inf}(\text{col}_D(\gamma)) \text{ is even } \rightarrow \text{max Inf}(\text{col}_T(\gamma)) \text{ is even.}$$

Consider a play $\gamma$ according to the above defined strategy, let $\alpha$ be the input word associated to $\gamma$, assume that $\alpha \in \text{dom}(f)$, and let $u_0 \cdots$ be its factorization according to Eq. (3). By definition of the game graph, the sequence $\text{col}_D(\gamma)$ is exactly the sequence $c_D(\rho_D)$, where $\rho_D$ is the run of $D$ on $\alpha$. Since $\alpha \in L(D)$, we have that $\text{max Inf}(\text{col}_D(\gamma))$ is even. Thus, we have to show that $\text{max Inf}(\text{col}_T(\gamma))$ is even.

Recall that $\text{col}_T(\gamma)$ is the sequence of colors obtained from the chosen state transformation functions in the play, where $−1$ is the color when no state transformation function is chosen. Since $\alpha \in \text{dom}(f)$ and $f$ is continuous, the factorization according to Eq. (3) that was used to build the strategy has infinitely many factors. Hence, infinitely many producing edges are taken, meaning that $\text{max Inf}(\text{col}_T(\gamma)) \in C_\mathcal{T}$. Furthermore, we also defined infinitely many $v_i$. Let $\beta = v_0v_1 \cdots$, clearly $(\alpha, \beta) \in R$, because $f(\alpha) = \beta$.

Thus, there exists an accepting run of $\mathcal{T}$ on $\alpha/\beta$. Consider the run $\rho$ of $\mathcal{T}$ on $\alpha/\beta$ of the form $\rho_0\rho_1 \cdots$ build according to Lemma 19. We show that every accepting run $\rho'$ of $\mathcal{T}$ on $\alpha/\beta$ has a factorization of the form:

$$\mathcal{T}: q_0^T \xrightarrow{u_0/v_0^0} q_1 \in Q_i^T \xrightarrow{u_1/v_1^1} q_2 \in Q_i^T \cdots \xrightarrow{u_{i−1}/v_{i−1}^i} q_{i−1} \xrightarrow{u_{i−1}/v_{i−1}^i} q_i \in Q_i^T.$$

Assume there is a run $\rho'$ that does not have this property.

To begin with, we assume that the run $\rho'$ can not be factorized such that input word $\alpha$ is split into factors $u_0v_1 \cdots$. This problem can not occur, because Item (1) of property $\mathcal{P}$ ensures the possiblity of such a factorization. Item (1) ensures that after reading an
input letter a state is reached. Thus, any input factorization can be used to obtain a factorization of the run. Hence, assume that the factorization \( \rho'_0 \rho'_1 \cdots \) is picked such that it respects the input factorization, but some reached state is different. Towards a contradiction, pick the first \( i \) such that \( \rho'_i \) of the form \( q_i u_i/v_i \to p_{i+1} \) with \( p_{i+1} \neq q_{i+1} \).

However, since either \( v_0 \cdots v_i \) is a prefix of \( v'_0 \cdots v'_i \) or vice versa, because \( T \) satisfies Item (2) of property \( P \), it is implied that if \( T : q_i u_i/v_i \to q_{i+1} \) and \( T : q_i u_i/v'_i \to p_{i+1} \), then \( p_{i+1} = q_{i+1} \). We have a contradiction.

Hence, pick any accepting run \( \rho' \) of \( T \) on \( \alpha/\beta \). Since \( \rho' \) is accepting, \( \max \text{Inf}(c(\rho')) \) is even. Recall that we have to show that \( \max \text{Inf}(\text{col}(\gamma)) \) is even. Recall that \( \text{col}(\gamma) = c_0 c_1 \cdots \) according to Lemma 19. We have chosen \( c_i \) as best \( (P_{u_i}, q_i, q_{i+1}) \). Consider the factorization \( \rho'_0 \rho'_1 \cdots \) of \( \rho' \) as above. Clearly, the color \( c_i \) is at least as good for Eve as the maximal color that occurs on \( \rho'_i \), i.e., \( \max \text{Occ}(c(\rho'_i)) \). Thus, since \( \max \text{Inf}(c(\rho')) \) is even, we also have that \( \max \text{Inf}(\text{col}(\gamma)) \) is even which concludes the proof.

5 The Special Case of Closed Domains

We turn to the setting of closed domains and show that bounded delay suffices.

**Lemma 20.** Let \( R \) with \( \text{dom}(R) \) closed be defined by a transducer \( T \) with property \( P \). If \( R \) is uniformizable by a computable function, then there exists a computable \( \ell \geq 0 \) (computable from \( T \)) such that Eve has a winning strategy in \( G_T \) with at most \( \ell \) consecutive skip-moves.

Intuitively, the reason why bounded lookahead suffices in the setting of closed domains is that (basically at each point of time during a play) Adam's moves describe a series of longer and longer finite input words that “converge” to a valid infinite input word from the domain. Hence, Eve can not wait arbitrarily long to make producing moves, as such a play describes a valid infinite input sequence and a finite output sequence. We give the formal proof in the upcoming sections.

Together with Theorem 13 and Lemma 10, we obtain the following corollary.

**Corollary 21.** Let \( R \) with \( \text{dom}(R) \) closed be defined by a transducer \( T \) with property \( P \). The following are equivalent:

1. \( R \) is uniformizable by a computable function.
2. \( R \) is uniformizable by a function computable by a 1DFT.
3. Eve has a winning strategy in \( G_T \).

Moreover, if \( T \) is letter-to-letter, then \( R \) is uniformizable by a computable function iff it is uniformizable by a function computable by an \( \ell \)-delay 1DFT for some computable \( \ell \).

We highlight two facts regarding closed domains.

**Remark 22.** The set of infinite words over a finite alphabet is closed, i.e., every total domain is closed.
Remark 23. Furthermore, it is decidable whether a domain (e.g., given by a Büchi automaton) is closed.

It is a well-known fact that the topological closure of a Büchi language is a Büchi language (one can trim the automaton and declare all states to be accepting) and therefore one can check closedness by checking equivalency with its closure.

5.1 Uniform Continuity

In the upcoming proofs, we heavily rely on the notion of uniform continuity. First, we recall what is meant by uniform continuity. Intuitively, it means that the amount of input letters that have to be seen before further output letters can be determined is independent of the concrete input sequence. Formally, a function \( f : \Sigma^\omega \to \Gamma^\omega \) is uniformly continuous if

\[
\forall i \in \mathbb{N} \ \exists j \in \mathbb{N} \ \forall \alpha \in \text{dom}(f) \ \forall \beta \in \text{dom}(f) : |\alpha \land \beta| \geq j \to |f(\alpha) \land f(\beta)| \geq i. \tag{5}
\]

The function \( m : \mathbb{N} \to \mathbb{N} \) which associates to any \( i \), some \( j \) satisfying the above equation is called a modulus of continuity. It guarantees that to get \( i \) output symbols, it suffices to read at most \( m(i) \) input symbols.

We state two easy to see facts about (uniformly) continuous function used in the remainder.

Remark 24. If \( f \) is continuous, i.e., \( f \) satisfies Eq. (2), we have

\[
\forall u \in \text{Prefs(dom}(f)) \ \forall i \in \mathbb{N} \ \exists j \in \mathbb{N} \ \forall u' \in \text{Prefs(dom}(f)) : |u \land u'| \geq j \to |\hat{f}(u)| \geq i. \tag{6}
\]

If \( f \) is uniformly continuous, i.e., \( f \) satisfies Eq. (5), we have

\[
\forall i \in \mathbb{N} \ \exists j \in \mathbb{N} \ \forall u \in \text{Prefs(dom}(f)) : |u| \geq j \to |\hat{f}(u)| \geq i. \tag{7}
\]

To prove Lemma 20 we make an important connection between closed domains and uniformly continuous functions.

It is a well-known fact that the set \( \Sigma^\omega \) is a compact space when equipped with the Cantor distance defined in Footnote 2. A closed subset of a compact space is compact, see [4]. Hence, a closed domain \( D \subseteq \Sigma^\omega \) is a compact space. We arrive at the following remark as a consequence of König’s Lemma, or equivalently of the fact that continuous functions on a compact space are uniformly continuous as stated by the Heine-Cantor theorem.

Remark 25. Assume \( R(T) \) has a closed domain. If \( R(T) \) is uniformizable by a continuous function, then \( R(T) \) is uniformizable by a uniformly continuous function.

5.2 Finite-state Bounded Delay Game

We show that if \( R(T) \) is uniformizable by a uniformly continuous function \( f \), then Eve has a winning strategy in the corresponding finite-state delay game (based on \( T \) with property \( P \)) with bounded delay.
We show that the bound on the necessary lookahead can be based on their profiles, and provide an adapted game. Given a profile $P$, let $L(P)$ denote the set $\{u \in \Sigma^* \mid P_u = P\}$. Let $\mathcal{P}_{\text{fin}}$ be the set of profiles whose associated languages are finite, let $L = \bigcup_{P \in \mathcal{P}_{\text{fin}}} L(P)$, and let $\ell$ be the length of the longest word in $L$. We show that a lookahead of $\ell$ suffices. We change the game used to model the continuous setting to (implicitly) reflect this bound, i.e., Eve is only allowed to delay to pick skip at most $\ell$ times in a row.

**Finite-state bounded delay game.** Let $G^\text{uni} = (G, \text{Win})$ be the infinite-duration turn-based two-player game obtained by removing from $G_T$ (see Page 10) all edges $(q, c, P_1, P_2, r, \exists) \xrightarrow{\text{skip}} (q, -1, P_1, P_2, r, \forall)$ such that $L(P_2)$ is infinite.

**Remark 26.** The game $G^\text{uni}$ can be obtained from $G_T$ in $\text{EXPTime}$.

To see this one has to realize that for each profile $P$ one has to check whether $|L(P)| \neq \infty$. The number of profiles is exponential in $T$ as is the size of an automaton for $L(P)$. Checking whether $|L(P)| \neq \infty$ can be done in polynomial time in the automaton for $L(P)$.

**Remark 27.** Let $\sigma_3$ be a winning strategy for Eve in $G^\text{uni}$, clearly it is also a winning strategy for Eve in $G_T$ since $G^\text{uni}$ is obtained from $G_T$ by adding edge constraints. Furthermore, if Eve plays according to $\sigma_3$ she has at most $\ell$ consecutive skip-moves.

### 5.3 From Uniformly Continuous Functions to Strategies with Bounded Delay

**Lemma 28.** Let $R$ with $\text{dom}(R)$ closed be defined by a transducer with property $\mathcal{P}$. If $R$ is uniformizable by a uniformly continuous function $f$, then Eve has a winning strategy in $G^\text{uni}_f$.

**Proof.** To begin with, we recall some definitions made in the paragraph preceding the definition of $G^\text{uni}_f$, see Page 26. Given a profile $P$, $L(P)$ denotes the set $\{u \in \Sigma^* \mid P_u = P\}$. The set $\mathcal{P}_{\text{fin}}$ is the set of profiles whose associated languages are finite, $L = \bigcup_{P \in \mathcal{P}_{\text{fin}}} L(P)$, and $\ell$ is the length of the longest word in $L$. Since the game $G^\text{uni}_f$ is the game $G_T$ where all edges $(q, c, P_1, P_2, r, \exists) \xrightarrow{\text{skip}} (q, -1, P_1, P_2, r, \forall)$ with $L(P_2)$ infinite have been removed, Eve never uses skip more than $\ell$ times in a row.

We describe how to inductively factorize an input word $\alpha \in \Sigma^\omega$. Let $u_0 = \varepsilon$, and let $u_0 \cdots u_{i-1}$ be the already defined factors of $\alpha$, let $\alpha = u_0 \cdots u_{i-1} \alpha'$, and let

$$u_i := \alpha'(\varepsilon: k),$$

(8)

where $k \in \mathbb{N}_{\geq 1}$ is the smallest number such that $P_{u_i} \notin \mathcal{P}_{\text{fin}}$, in other words, $L(P_{u_i})$ is infinite. Note that such a prefix always exists since $L$ is finite. Moreover, this implies that $|u_i| \leq \ell$ for all $i \in \mathbb{N}$.

Let $\alpha_1, \alpha_2 \in \Sigma^\omega$, and let $u^0_0 \cdots$ and $u^0_2 \cdots$ denote their factorizations built according to Eq. (8). Note that $u^1_i = u^0_i$ for all $i$ such that $|u^0_i \cdots u^1_i| \leq |\alpha_1 \land \alpha_2|$. Furthermore, given
an input word $\alpha$, the factorization according to Eq. (8) can be determined on-the-fly reading the input sequence letter-by-letter.

Recall that $f$ uniformizes $R = R(T)$, thus, $\text{dom}(f) = \text{dom}(R) = L(D)$. Note that if the prefix $u_0 \cdots u_t$ of the input word $\alpha$ from a play can not be completed to a word from $\text{dom}(f)$, we do not care how the strategy behaves from this point on, because then the play automatically belongs to $\text{Win}$. So in the remainder of the proof we only consider plays where Adam spells a word that belongs to $\text{dom}(f)$.

We design a strategy of Eve that behaves as follows. For all $i \in \mathbb{N}_{\geq 1}$, after Adam’s actions have spelt $u_0 \cdots u_i$, Eve’s action is to take a producing edge. In between factors, Eve’s action is to skip. This ensures that for all $i \in \mathbb{N}_{\geq 1}$, after Adam’s actions have spelt $u_0 \cdots u_i$ and Eve has picked an action, the play is in a vertex of the form

$$(q_{i-1}, -1, P_{u_{i-1}}, P_{u_i}, \delta'_D(q_0^D, u_0 \cdots u_i), \exists)$$

Note that since $L(P_{u_i})$ is infinite, as ensured by the factorization according to Eq. (8), Eve cannot use skip and must take a producing edge. Furthermore, in between factors, Eve is indeed allowed to use the skip action.

Now we explain how we use the sequence of input blocks $u_0 u_1 \cdots$ of $\alpha$ built as in Eq. (8) to obtain the actions of Eve. Her actions pick state transformations by defining a sequence of output blocks $v_0 v_1 \cdots$ and pick state transformations that are induced by $u_0 / v_0 \cdot u_1 / v_1 \cdots$.

As described at the beginning of this proof, after a certain amount of input, Eve must make a producing action. Generally, given two prefixes $u \leq uu' \in \text{Prefs}(\text{dom}(f))$, it can be the case that $f(u) = f(uu')$. However, if $u'$ is such that $L(P_{u'})$ is infinite, then we can pick an arbitrary long $u''$ that has the same profile as $u'$. We have that $uu'' \in \text{Prefs}(\text{dom}(f))$, because having the same profile means that they induce the same behavior in $T$. Since $f$ is uniformly continuous, $f(u) \leq f(uu'')$ if $u''$ is long enough. The idea is to use $f(uu'')$ instead of $f(uu')$ to determine Eve’s next action.

Formally, in order to choose output blocks $v_0, v_1, \ldots$, we inductively build blocks $\bar{u}_0, \bar{u}_1, \ldots$, where $\bar{u}_i$ is based on $u_i$ for all $i \in \mathbb{N}$. The idea is that, since each $u_i$ is such that $L(P_{u_i})$ is infinite for $i \in \mathbb{N}_{\geq 1}$, we can pick some $\bar{u}_i$ that has the same profile as $u_i$ and such that there is a prefix $\bar{v}$ of the output

$$\hat{f}(\bar{u}_0 \cdots \bar{u}_i)$$

that satisfies $\Delta_T^*(q_0^T, \bar{u}_0 \cdots \bar{u}_{i-1} / \bar{v}) \cap Q^T_i \neq \emptyset$ ($Q^T_i$ denotes the set of input states of $T$ as defined at the beginning of Section 4). This is possible if $\bar{u}_0 \cdots \bar{u}_i$ can be completed to an $\omega$-word from $\text{dom}(f)$ because of Eq. (7). This makes it possible to pick output blocks for all previous input blocks. We let $\bar{v}$ be the shortest prefix of $\hat{f}(\bar{u}_0 \cdots \bar{u}_i)$ such that $\Delta_T^*(q_0^T, \bar{u}_0 \cdots \bar{u}_{i-1} / \bar{v}) \cap Q^T_i \neq \emptyset$ and define

$$\bar{v}_{i-1} = (\bar{v}_0 \cdots \bar{v}_{i-2})^{-1} \bar{v},$$

for all $i \in \mathbb{N}_{\geq 1}$. Based on $\bar{u}_j$, $\bar{v}_j$, and $u_j$, we pick $v_j$, which is explained further below.

We show that for the input block sequence $u_0 \cdots u_i$, with $\bar{u}_0 \cdots \bar{u}_{i-1}$ and $\bar{v}_0 \cdots \bar{v}_{i-1}$ defined as described above, we can pick an output block sequence $v_0 \cdots v_{i-1}$ such that the following claim is satisfied for all $i \in \mathbb{N}$:
Lemma 29. If there is a run $\bar{\rho}$ of $\mathcal{T}$ on $\bar{u}_0 \cdots \bar{u}_{i-1} / \bar{v}_0 \cdots \bar{v}_{i-1}$ of the form

$$
\mathcal{T}: \begin{array}{cccc}
q_0 & \rightarrow & q_1 & \rightarrow & \cdots & \rightarrow & q_{i-1} & \rightarrow & q_i \\
\bar{u}_0 / \bar{v}_0 & \bar{u}_1 / \bar{v}_1 & \cdots & \bar{u}_{i-1} / \bar{v}_{i-1}
\end{array}
$$

then there is a run $\rho$ of $\mathcal{T}$ on $u_0 \cdots u_{i-1} / v_0 \cdots v_{i-1}$ of the form

$$
\mathcal{T}: \begin{array}{cccc}
q_0 & \rightarrow & q_1 & \rightarrow & \cdots & \rightarrow & q_i \\
\bar{u}_0 / \bar{v}_0 & \bar{u}_1 / \bar{v}_1 & \cdots & \bar{u}_{i-1} / \bar{v}_{i-1}
\end{array}
$$

and the sequence of chosen producing edges in the play after the input $u_0 \cdots u_i$ has the label sequence

$$(q_i^0, q_0, c_0)(q_1, q_2, c_1) \cdots (q_{i-1}, q_i, c_{i-1}),$$

where $c_j = \text{best}(P_{u_j}, q_j, q_{j+1})$ (see Definition 17) and the edge labeled with $(q_j, q_{j+1}, c_j)$ is taken after the input $u_0 \cdots u_{j+1}$ for all $0 \leq j \leq i - 1$. Furthermore, after the input $u_0 \cdots u_i$ the play is in the vertex

$$(q_{i-1}, -1, P_{u_{i-1}}, P_{u_i}, \delta_D(q_0^D, u_0 \cdots u_i), \exists).$$

We show the that the claim holds by induction. For $k = 0$, the claim is trivially true. For the step $k - 1 \rightarrow k$, assume that Adam’s actions have spelt $u_0 \cdots u_k$, we have already defined $\bar{u}_0 \cdots \bar{u}_{k-1}$ (which yields the definition of $\bar{v}_0 \cdots \bar{v}_{k-2}$), and $v_0 \cdots v_{k-2}$ such that the invariant is satisfied. Clearly, after $u_0 \cdots u_k$, the play is in

$$(q_{k-1}, -1, P_{u_{k-1}}, P_{u_k}, \delta_D(q_0^D, u_0 \cdots u_k), \exists).$$

We now define $\bar{u}_k$, $\bar{v}_{k-1}$ and $\bar{v}_{k-1}$ as follows. Let $\bar{u}_k$ such that Eq. (9) is satisfied. Since the profiles of $\bar{u}_0 \cdots \bar{u}_{k-1}$ and $u_0 \cdots u_{k-1}$ are the same, it follows that $\bar{u}_0 \cdots \bar{u}_{k-1} \in \text{Prefs}(\text{dom}(f))$, because we assumed that $u_0 \cdots u_{k-1} \in \text{Prefs}(\text{dom}(f))$. This yields the definition of $\bar{v}_{k-1}$ as given in Eq. (10). Pick some $\bar{\rho}_k$ of the form $\mathcal{T}: q_k \xrightarrow{\bar{u}_k / \bar{v}_k} q_{k+1} \in Q^T$, and let $c_k = \text{best}(P_{u_k}, q_k, q_{k+1})$, we have $(q_k, q_{k+1}, c_k) \in P_{u_k}$. Since $\bar{u}_k$ and $u_k$ have the same profile, we also have $(q_k, q_{k+1}, c_k) \in P_{u_k}$, which implies that we can pick $\bar{v}_k \in \Gamma^*$ such that $\bar{\rho}_k$ is of the form $\mathcal{T}: q_k \xrightarrow{u_k / v_k} q_{k+1}$ and $c_k = \text{max Occ}(c(\bar{\rho}_k))$. Furthermore, $(q_k, q_{k+1}, c_k) \in P_{u_k}$ implies that Eve can take a producing edge with label $(q_k, q_{k+1}, c_k)$. Hence, the claim is satisfied for $k$.

It is left to argue that the strategy is a winning strategy for Eve. Recall that if $u_0 u_1 \cdots \in \text{dom}(f)$, then $\bar{u}_0 \bar{u}_1 \cdots \in \text{dom}(f)$ and $(\bar{u}_0 \bar{u}_1 \cdots, f(\bar{u}_0 \bar{u}_1 \cdots)) \in R$. The same argument as in the proof of Theorem 18 yields that the strategy is winning. The main point is to prove that every accepting run $\bar{\rho}'$ of $\mathcal{T}$ on $\bar{u}_0 \bar{u}_1 \cdots / f(\bar{u}_0 \bar{u}_1 \cdots)$ can be factorized into the form

$$
\mathcal{T}: \begin{array}{cccc}
q_0 & \rightarrow & q_1 & \rightarrow & \cdots & \rightarrow & q_i \\
\bar{u}_0 / \bar{v}_0 & \bar{u}_1 / \bar{v}_1 & \cdots & \bar{u}_{i-1} / \bar{v}_{i-1}
\end{array}
$$

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with the same intermediate states as both the runs $\rho$ and $\bar{\rho}$ inductively constructed in order to define the strategy that satisfies Lemma 29. In order to prove that this factorization of $\bar{\rho}'$ exists, we use that $T$ has property $P$ exactly as we have done in the proof of Theorem 13. Since $\bar{\rho}'$ is accepting and each $c_i$ is at least as good as $\max \text{Occ}(c(\bar{\rho}'_i))$, it is easy to see that the strategy is winning.

Finally, we are able to prove Lemma 20. The statement is a direct consequence of Remark 25 together with Lemma 28 and Remark 27.

6 Discussion

Continuous functions. We have shown that checking the existence of a computable function uniformizing a relation given by transducer with property $P$ is decidable (a consequence of Theorems 6 and 13 and Lemma 9). The proofs of Theorems 6 and 13 use another notion, which is easier to manipulate mathematically than computability, namely that of continuity. These notions are closely related. If a function $f : \Sigma^\omega \rightarrow \Gamma^\omega$ is computable, it is also continuous. This is not difficult to see when comparing the definitions of computable and continuous functions. The converse does not hold because the continuity definition does not have any computability requirements (see [15] for a counter-example). However, regarding synthesis, the two notions coincide:

**Theorem 30.** Let $R$ be defined by $T$ with property $P$. The following are equivalent:

1. $R$ is uniformizable by a continuous function.
2. $R$ is uniformizable by a computable function.

**Proof.** Indeed, any computable uniformizer is continuous. Theorem 13 states that if there exists a computable uniformizer, then there exists a winning strategy in the delay game. However, in the proof of this theorem, we show a stronger statement: If there exists a continuous uniformizer, then there exists a winning strategy in the delay game. Such a strategy can be assumed to have finite-memory (as finite-memory suffices to win games with $\omega$-regular conditions). We have shown in the proof of Lemma 7 how to translate a finite-state winning strategy into an algorithm (a Turing machine) that computes a function $f$ which uniformizes the relation.

Further undecidability results. Proposition 5 states that it is undecidable whether a rational relation is uniformizable by a computable function. As we have seen in Section 4, restricting the class of specifications (to DRAT) yields decidability. Another approach to obtain decidability is to change the class of desired implementations, and not the class of specifications. The next proposition shows that this is not fruitful.

**Proposition 31.** It is undecidable whether a rational relation $R$ is uniformizable by a sequential resp. letter-to-letter sequential function, even if $R$ has total domain.
Proof. To prove Proposition 5, we sketched a reduction from Post’s correspondence problem. The reduction from Post’s correspondence problem yields the undecidability for computable as well as sequential functions but not for letter-to-letter sequential functions. Hence, we now give a slightly more complicated proof that also yields the desired result for letter-to-letter sequential functions. Similar ideas have been used in [9] to prove that uniformizability of rational functions by sequential functions of finite words is undecidable.

We show all undecidability results by reduction from the halting problem for Turing machines. Let $M$ be a deterministic Turing machine, and let $R_M$ be the relation that contains pairs $(\alpha, \beta)$ of the form $\alpha = c_0$c_1$\cdots c_n$\$\alpha'$ and $\beta = c_0'$c_1'$\cdots c_n'$\$\beta'$, where $c_0, \ldots, c_n, c'_0, \ldots, c'_n$ code configurations of $M$ in the usual way, $c_0$ is the initial configuration, $c_n$ is a halting configuration, $\alpha'$ and $\beta' \in \{a, b\}^\omega$, and either $\beta = \alpha$ if $\alpha'$ contains infinitely many $a$, or there is some $i \in \{0, \ldots, n-1\}$ such that $\text{succ}(c_i) \neq c'_{i+1}$, where $\text{succ}(c_i)$ is the successor configuration of $c_i$ if $\alpha'$ contains finitely many $a$. We call inputs of the form $c_0$c_1$\cdots c_n$\$\alpha'$ a valid encoding.

We make the relation $R_M$ total by adding all pairs $(\alpha, \beta)$, where $\alpha$ is not a valid encoding (we do not care about the outputs then).

We argue that $R_M$ is $\omega$-rational. Clearly, a transducer can guess whether the input is a valid encoding and verify this. If the input is not a valid encoding, the transducer must find an encoding error, and has nothing to verify regarding the output.

If the input is a valid encoding, the transducer can guess and verify whether the input contains infinitely many $a$. If it guesses infinitely many $a$, it has to verify that input and output are equal. If it guesses finitely many $a$, it reads the first configuration of the output, and can then read $c_i$ and $c'_{i+1}$ in parallel for all $i \in \{0, \ldots, n-1\}$ and check whether $\text{succ}(c_i) \neq c'_{i+1}$.

We show that $R_M$ is uniformizable by a continuous function, a computable function, a sequential function, resp. a letter-to-letter sequential function iff $M$ does not halt on the empty tape.

Assume $M$ does not halt. Let $f_{id}$ be the identity function. Clearly, $f_{id}$ is continuous, computable, sequential, and synchronous sequential. It is easy to verify that $f_{id}$ uniformizes $R_M$. Consider an infinite input word $\alpha$. We only consider the interesting case that $\alpha$ is a valid encoding. If $\alpha$ contains infinitely many $a$, clearly $(\alpha, f_{id}(\alpha)) \in R_M$. If $\alpha$ contains finitely many $a$, there has to be some $i \in \{0, \ldots, n-1\}$ such that $\text{succ}(c_i) \neq c'_{i+1}$. This is true, because $M$ does not halt, thus, the configuration sequence $c_0$c_1$\cdots c_n$ has an error, i.e., there is some $i$ such that $\text{succ}(c_i) \neq c_{i+1}$. Hence, since $c_0$c_1$\cdots c_n$ = $c'_0$c'_1$\cdots c'_n$, we obtain $\text{succ}(c_i) \neq c'_{i+1}$.

Assume $M$ does halt. Towards a contradiction, assume that $f$ is a function that uniformizes $R_M$ that is either continuous, computable, sequential, or synchronous sequential. Note that every synchronous sequential function is sequential, every sequential function is computable and every computable function is continuous. We arrive at a contradiction by using only properties of continuous functions. Since $M$ halts, there exists a configuration sequence starting with the initial configuration that ends with a halting configuration that has no errors. Let $u$ denote this configuration sequence. Consider $uu^\omega \in \text{dom}(f)$. According to Eq. (2), there exists some $j \in \mathbb{N}$ such that for all $\beta \in \text{dom}(f)$ holds that
|uaω ∧ β| ≥ j implies that |f(uaω) ∧ f(β)| ≥ |u|. Consider ua1bω ∈ dom(f), we have
f(uaω) = uaω and f(ua1bω) = u′β′ with |u| = |u′| and u ≠ u′, because u′ must begin with a configuration sequences such that there is some i ∈ {0, ..., n − 1} such that
|uaω ∧ ua1bω| ≥ j, but |f(uaω) ∧ f(ua1bω)| < |u|, which is a contradiction.

7 Conclusion and Future Work

We investigated the synthesis of algorithms (a.k.a. Turing machine computable functions) from rational specifications. While undecidable in general, we have proven decidability for DRAT (and a fortiori AUT). Furthermore, we have shown that the whole computation power of Turing machines is not needed, two-way transducers are sufficient (and necessary). As TMs reading heads are read-only left-to-right, converting a 2DFT into a TM requires that the TM stores longer and longer prefixes of the input in the working-tape for later access. This is the only use the TM needs to make of the working tape. 

This is a naive translation, and sometimes the working tape can be flushed (some prefixes of the input may possibly not be needed anymore). More generally, it is an interesting research direction to fine-tune the class of functions targeted by synthesis with respect to some constraints on the memory, including quantitative constraints.

Related to the latter research direction is the following open question: is the synthesis problem of functions computable by input-deterministic one-way (a.k.a. sequential) transducers from deterministic rational relations decidable? It is already open for automatic relations. We have shown that if a rational relation with closed domain is uniformizable by a computable function, then it also uniformizable by a sequential function. However, closedness is not a necessary condition: e.g., the function which maps any aωxcω for x ∈ {#, $}, is sequential; a sequential transducer just has to erase the aω part, but its domain is not closed. This problem is interesting because sequential transducers only require bounded memory to compute a function (in contrast to two-way transducers that require access to unboundedly large prefixes of the input).

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