$p$-Harmonic maps to $S^1$ and stationary varifolds of codimension two

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Abstract

We study the limiting behavior as $p \uparrow 2$ of the singular sets $\text{Sing}(u_p)$ and $p$-energy measures $\mu_p := (2-p)|du_p|^p d\text{vol}$ for families of stationary $p$-harmonic maps $u_p \in W^{1,p}(M, S^1)$ from a closed, oriented manifold $M$ to the circle. When the measures $\mu_p$ have uniformly bounded mass, we show that—up to subsequences—the singular sets $\text{Sing}(u_p)$ converge in the Hausdorff sense to the support of a stationary, rectifiable varifold $V$ of codimension 2, and the measures $\mu_p$ converge weakly in $(C^0(M))^*$ to a limit of the form

$$\mu = \|V\| + |h|^2 d\text{vol},$$

where $h$ is a harmonic one-form. For solutions on two-dimensional domains, we show moreover that the density of $V$ takes values in $2\pi \mathbb{N}$. Finally, we observe that nontrivial families $u_p$ of such maps arise naturally on any closed Riemannian manifold of dimension $n \geq 2$, via variational methods.

Mathematics Subject Classification 53C43 · 58E20

1 Introduction

In their 1995 paper [19], Hardt and Lin consider the following question: given a simply connected domain $\Omega \subset \mathbb{R}^2$ and a map $g : \partial \Omega \to S^1$ of nonzero degree, what can be said about the limiting behavior of maps

$$u_p \in W^{1,p}_g(\Omega, S^1) := \{u \in W^{1,p}(\Omega, S^1) \mid u_p|_{\partial \Omega} = g\}$$

minimizing the $p$-energy

$$\int_{\Omega} |du_p|^p = \min \left\{ \int_{\Omega} |du|^p \mid u \in W^{1,p}_g(\Omega, S^1) \right\}$$

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as \( p \in (1, 2) \) approaches 2 from below? They succeed in showing—among other things—that away from a collection \( A \) of \( |\text{deg}(g)| \) singularities, a subsequence \( u_{p_j} \) converges strongly to a harmonic map \( v \in C^1_{\text{loc}}(\Omega \setminus A, S^1) \), and the measures

\[
\mu_j = (2 - p_j)|du_j|^p/\pi(z)dz
\]

converge to the sum \( \Sigma_{a \in A} 2\pi \delta_a \) of Dirac masses on \( A \) \cite{19}. Moreover, the singular set \( A = \{a_1, \ldots, a_{|\text{deg}(g)|}\} \) minimizes a certain “renormalized energy” function \( W_g: \Omega^{\text{deg}(g)} \to [0, \infty] \) associated to \( g \), providing a strong constraint on the location of the singularities. In particular, though the homotopically nontrivial boundary map \( g \) admits no extension to an \( S^1 \)-valued map of finite Dirichlet energy—i.e, \( W_g^{1,2}(\Omega, S^1) = \emptyset \)—the limit of the \( p \)-energy minimizers as \( p \uparrow 2 \) provides us with a natural candidate for the optimal harmonic extension of \( g \) to an \( S^1 \)-valued map on \( \Omega \).

The results of \cite{19} were inspired in large part by the analysis of Bethuel, Brezis, and Hélein—contained in the influential monograph \cite{4}—of the asymptotics for minimizers \( u_{\epsilon} \) of the Ginzburg–Landau functionals

\[
E_{\epsilon}: W^{1,2}(\Omega, \mathbb{R}^2) \to \mathbb{R}, \quad E_{\epsilon}(u) = \int_{\Omega} \frac{1}{2} |du|^2 + \frac{(1 - |u|^2)^2}{4\epsilon^2}
\]

as \( \epsilon \to 0 \), with the measures

\[
\mu_{\epsilon} := \frac{|du_{\epsilon}(z)|^2}{2|\log \epsilon|} d\tau
\]

taking on the role played by the measures \( (2 - p)|du|^p/\pi(z)dz \) in the setting of \cite{19}. In recent decades, the asymptotics for critical points of the Ginzburg–Landau functionals \( E_{\epsilon} \) in higher dimensions have also been studied by a number of authors, often with an emphasis on the relationship between concentration phenomena for the measures \( \mu_{\epsilon} \) and minimal submanifolds of codimension two (see, for instance, \cite{5,6,11,25,35}, among many others). A typical result says, roughly, that if the measures \( \mu_{\epsilon} \) have uniformly bounded mass, then a subsequence converges weakly in \((C^0)^*\) as \( \epsilon \to 0 \) to a limiting measure that decomposes into two pieces: a concentrated component given by a stationary, rectifiable varifold \( V^{GL} \) of codimension two, and a diffuse measure of the form \(|h|^2 dvol\) for some harmonic one-form \( h \) (which vanishes under mild compactness assumptions).

Results of this type point to the possibility of employing variational methods for the Ginzburg–Landau functionals to produce minimal submanifolds of codimension two (see, for instance, \cite{11,35}), but to fully understand the minimal varieties produced in this way, it remains to develop an improved understanding of the concentration phenomena for \( \mu_{\epsilon} \) at small scales. In particular, the fascinating question of integrality (up to a factor of \( \pi \)) of the limiting varifold \( V^{GL} \) has been resolved only in dimension two \cite{12} and for local minimizers in higher dimensions \cite{25}.

In this paper, motivated by analogy with the Ginzburg–Landau setting, we investigate the limiting behavior as \( p \uparrow 2 \) of stationary \( p \)-harmonic maps \( u_p \in W^{1,p}(M^n, S^1) \) from an arbitrary closed, oriented manifold \( M \) to the circle. At the global level, we find that the limiting behavior of the maps \( u_p \) and their energy measures strongly resembles the asymptotics described above for solutions to the Ginzburg–Landau equations. At the smallest scales, however, the comparatively straightforward blow-up analysis for \( p \)-harmonic maps (owing to the scale-invariance of the \( p \)-energy functionals) leads us to some simpler arguments and sharper estimates than are currently available in the Ginzburg–Landau setting.
Our main result shows that the limiting behavior of the normalized energy measures \( \mu_p = (2-p)|du|^p dvol_g \) roughly mirrors that of the energy measures \( \mu_{\epsilon} \) in the Ginzburg–Landau setting, with the singular sets \( Sing(u_p) \) taking on the role of the zero sets \( u^{-1}_\epsilon(\{0\}) \).

To advertise the cleaner analysis available in the \( p \)-harmonic setting, we record also a sharp lower bound for the density of the energy concentration varifold—a consequence of the simple blow-up analysis available at points in \( Sing(u_p) \).

**Theorem 1.1** Let \( p_i \in (1, 2) \) be a sequence with \( \lim_{i \to \infty} p_i = 2 \), and let \( u_i \in W^{1,p_i}(M^n, S^1) \) be a sequence of stationary \( p_i \)-harmonic maps from a closed, oriented Riemannian manifold \( M^n \) to the circle, satisfying

\[
\sup_i (2-p_i) \int_M |du_i|^{p_i} < \infty. \tag{1.1}
\]

Then (a subsequence of) the energy measures \( \mu_i = (2-p_i)|du_i|^{p_i} dvol_g \) converge weakly in \((C^0)^*\) to a limiting measure \( \mu \) of the form

\[
\mu = \|V\| + |\tilde{h}|^2 dvol_g, \tag{1.2}
\]

where \( \tilde{h} \) is a harmonic one-form, and \( V \) is a stationary, rectifiable \((n-2)\) varifold with support given by the Hausdorff limit

\[
\text{spt}(V) = \lim_{i \to \infty} \text{Sing}(u_i)
\]

of the singular sets \( Sing(u_i) \). Moreover, the density \( \Theta_{n-2}(\|V\|, \cdot) \) satisfies

\[
\Theta_{n-2}(\|V\|, x) \geq 2\pi \text{ for } x \in \text{spt}(\|V\|). \tag{1.3}
\]

In the course of proving Theorem 1.1, we also establish the following simple compactness result for the maps.

**Theorem 1.2** Suppose that, in addition to the hypotheses of Theorem 1.1, either \( b_1(M) = 0 \) or

\[
\sup_i \|du_i\|_{L^1(M)} < \infty.
\]

Then (a subsequence of) the maps \( u_i \) converge weakly in \( W^{1,q}(M) \) for all \( q \in (1, 2) \), and strongly in \( W^{1,2}_{loc}(M \setminus \text{spt}(V)) \), to a limiting map \( v \in \bigcap_{q \in [1,2]} W^{1,q}(M, S^1) \) that is harmonic away from \( \text{spt}(V) \).

In the Ginzburg–Landau setting, Comte and Mironescu showed that the density of the limiting energy measure \( \lim_{\epsilon \to 0} \mu_{\epsilon} \) for arbitrary critical points in two-dimensional domains always takes values in \( \pi \mathbb{N} \) [12], through a somewhat delicate argument building on the quantization results of Brezis–Merle–Rivière [7] for the potential measures

\[
\frac{1}{\epsilon^2} \int \frac{1}{2(|u_\epsilon|^2(x))^2} dx
\]

in \( \mathbb{R}^2 \). In Sect. 6, we establish an analog of the Comte–Mironescu result for \( p \)-harmonic maps to \( S^1 \); in this setting, the proof is much simpler than that of [12], and closer in spirit to the arguments of [7], due to the structure of the Pohozaev identity for \( p \)-harmonic maps. For simplicity, we restrict ourselves to the following local statement, for maps on the flat disk \( D_2 \) of radius 2.

**Theorem 1.3** Let \( p_j \in (1, 2) \) with \( \lim_{j \to \infty} p_j = 2 \) and let \( u_j \in W^{1,p_j}(D_2, S^1) \) be stationary \( p_j \)-harmonic maps on the disk \( D_2 \) of radius 2, such that

\[
\sup_j (2-p_j) \int_{D_2} |du_j|^{p_j} < \infty.
\]
Then (after passing to a subsequence), on the unit disk $D_1$, the measures $\mu_j := (2 - p_j)|du_j|^p(x)dx$ converge weakly in $C^0(D_1)^*$ to a measure $\mu$ of the form

$$\mu = \sum_{j=1}^{\infty} 2\pi m_j \delta_{a_j} + |d\psi|^2(x)dx,$$

where $a_j \in D_1$, $\psi$ is a harmonic function, and, in particular, $m_j \in \mathbb{N}$.

Finally, to demonstrate that families of maps as in Theorem 1.1 arise naturally from variational methods, we combine (a special case of) Changyou Wang’s analysis of generalized Ginzburg–Landau functionals in [38] with a simple min–max construction modeled on that of [35], to show that nontrivial $p$-harmonic maps $u_p$ to $S^1$ with energy growth $E_p(u_p) \sim \frac{1}{2-p}$ exist on every closed, oriented Riemannian manifold.

**Theorem 1.4** On any closed, oriented Riemannian manifold $M^n$ of dimension $n \geq 2$, there exists a family $(1, 2) \ni p \mapsto u_p \in W^{1,p}(M, S^1)$ of stationary $p$-harmonic maps to $S^1$ for which

$$0 < \liminf_{p \to 2} (2 - p)E_p(u) \leq \limsup_{p \to 2} (2 - p)E_p(u) < \infty. \tag{1.4}$$

While our results provide further evidence of the strong link between the asymptotic behavior of $p$-harmonic $S^1$-valued maps and that of solutions to the Ginzburg–Landau equations, we highlight throughout the paper many ways in which the analysis in the $p$-harmonic map setting is simpler and cleaner than its Ginzburg–Landau analog, due to the scale-invariance of the problem (as manifested, for instance, in the simpler Pohozaev identities and monotonicity formulas). Thus, while we expect the analysis presented here to be of independent interest to those studying $p$-harmonic maps and their singularities, we hope that the asymptotic analysis of $p$-harmonic maps to $S^1$ may also serve as a valuable source of intuition in the study of energy concentration phenomena for the Ginzburg–Landau equations and related geometric pdes.

**1.1 Outline of the paper**

The paper is organized as follows.

In Sect. 2, we review some important known results about the structure of maps in $W^{1,p}(M, S^1)$ and $p$-harmonic maps to $S^1$.

In Sect. 3, we record a sharp lower bound for the $p$-energy density of a stationary $p$-harmonic map $u \in W^{1,p}(M, S^1)$ on its singular set $Sing(u)$—a simpler and sharper analog to the $\eta$-ellipticity result (cf. [5,25]) for solutions of the Ginzburg–Landau equations. We then use this to obtain $p$-independent estimates for the $p$-harmonic energy density of maps $u$ in the dual Sobolev norms $W^{-1,q} = (W^{1,q})^*$ for $q \in (1, p)$.

In Sect. 4, we employ the results of the preceding sections to estimate separately the components of the Hodge decomposition of the one-form $j\mu = u^*(d\theta)$, first globally in $L^q$ for $q \leq p$, and then in stronger norms away from $Sing(u)$.

In Sect. 5, we use these estimates, together with some standard techniques from the study of energy concentration phenomena, to complete the proofs of Theorem 1.1.

In Sect. 6, we prove Theorem 1.3, first under some compactness assumptions, using a Pohozaev-type identity, and then for the general case, by showing that the compactness assumptions hold at scales outside of which the normalized energy measures vanish.

In Sect. 7, we employ min–max arguments like those in [35] together with Wang’s results for generalized Ginzburg–Landau functionals [38] to prove Theorem 1.4. We also include a
short “Appendix”, containing the proofs of some estimates which are of use to us, but do not play a central role in the paper.

2 Preliminaries: the structure of \( W^{1,p}(M, S^1) \) and circle-valued \( p \)-harmonic maps

2.1 Topological singularities and lifting in \( W^{1,p}(M, S^1) \)

Let \( M^n \) be a compact, oriented Riemannian manifold, and consider the space \( W^{1,p}(M, S^1) \) of circle-valued Sobolev maps, realized as the collection of complex-valued maps \( u \in W^{1,p}(M, \mathbb{C}) \) satisfying \( |u| = 1 \) almost everywhere in \( M \). For each \( u \in W^{1,p}(M, S^1) \), we denote by \( ju \) the one-form

\[
ju := u^*(d\theta) = u^1 du^2 - u^2 du^1. \tag{2.1}
\]

Observe that \( |du| = |ju| \) almost everywhere on \( M \), so that \( ju \) belongs to \( L^p \). When \( u \) is smooth, the form \( ju \) is obviously closed, and it is a straightforward consequence of the Poincaré Lemma that \( u \) has a local lifting of the form \( u = e^{i\varphi} \) for some smooth, real-valued \( \varphi \).

For general \( u \in W^{1,p}(M, S^1) \), the exterior derivative \( d[ju] \) is no longer well-defined pointwise, but since \( ju \) belongs to \( L^p \), we can still make sense of \( d[ju] \) as a distribution in \( W^{-1,p} \). Namely, one defines the distributional Jacobian \( T(u) \) of \( u \) to be the \((n-2)\)-current acting on smooth \((n-2)\)-forms \( \xi \in \Omega^{n-2}(M) \) by

\[
\langle T(u), \xi \rangle := \int_M ju \wedge d\xi. \tag{2.2}
\]

The analytic and geometric properties of distributional Jacobians have been studied by a number of authors; we refer the reader to the papers [1,21,28], and the references therein for some interesting results concerning the structure of \( T(u) \). Note that for smooth, complex-valued maps, we have the pointwise relation

\[
d(u^1 du^2 - u^2 du^1) = 2du^1 \wedge du^2,
\]

and since \( u \mapsto du^1 \wedge du^2 \) defines a continuous map from \( W^{1,2}(M, \mathbb{C}) \) to the space of two-forms with values in \( L^1 \), it follows that

\[
\langle T(u), \xi \rangle = \int_M 2du^1 \wedge du^2 \wedge \xi
\]

holds for all \( u \in W^{1,2}(M, S^1) \) and \( \xi \in \Omega^{n-2}(M) \). In particular, since \( \text{rank}(du) \leq 1 \) almost everywhere, one deduces that \( T(u) = 0 \) for all \( u \in W^{1,2}(M, S^1) \). On the other hand, for \( p \in [1, 2) \), and \( k \in \mathbb{Z} \), the maps \( v_k : D_1^2 \to S^1 \) given by

\[
v_k(z) := (z/|z|)^k
\]

evidently lie in \( W^{1,p}(D, S^1) \), with nontrivial distributional Jacobian

\[
T(v_k) = 2\pi k \cdot \delta_0.
\]

Observe now that if \( u \) has the form \( u = e^{i\varphi} \) for some real-valued \( \varphi \in W^{1,p} \), then \( T(u) \) is given by

\[
\langle T(u), \xi \rangle = \int_M d\varphi \wedge d\xi,
\]
regularity theory is often available.

Proposition 2.1 [13] If \( u \in W^{1,p}(B^n, S^1) \) and \( T(u) = 0 \) in the ball \( B^n \), then \( u = e^{i\psi} \) on \( B^n \) for some \( \psi \in W^{1,p}(B^n, \mathbb{R}) \).

The significance of the lifting result for variational problems on \( W^{1,p}(M, S^1) \) is clear: away from the support of the \((n-2)\)-current \( T(u) \), an \( S^1 \)-valued solution \( u \) of some geometric p.d.e. lifts locally to a function \( \varphi \) solving an associated scalar problem, for which a stronger regularity theory is often available.

### 2.2 Weakly \( p \)-harmonic maps to \( S^1 \)

A map \( u \in W^{1,p}(M, S^1) \) for \( p \in (1, \infty) \) is called weakly \( p \)-harmonic if it satisfies

\[
\int |du|^{p-2} (du, dv) = \int |du|^{p} (u, v) \tag{2.3}
\]

for all \( v \in (W^{1,p} \cap L^\infty)(M, \mathbb{R}^2) \). Writing \( v = \varphi u + i\psi u \) in (2.3), it’s easy to see (cf., e.g., [37], Section 2) that (2.3) holds if and only if

\[
\int |du|^{p-2} (ju, d\psi) = 0 \tag{2.4}
\]

for all \( \psi \in W^{1,p}(M, \mathbb{R}) \)—i.e., when \( ju \) satisfies

\[
d^p(|ju|^{p-2} ju) = 0 \tag{2.5}
\]
distributionally on \( M \). Moreover, from the convexity of \( |\cdot|^p \), we see that (2.4) implies

\[
\int |ju + d\varphi|^p \geq \int |ju|^p
\]

for any \( \varphi \in C^\infty(M) \), so that any weakly \( p \)-harmonic map \( u \in W^{1,p}(M, S^1) \) minimizes the \( p \)-energy among all competitors of the form \( e^{i\varphi} u \).

In view of (2.5), wherever \( u \) admits a local lifting \( u = e^{i\varphi} \) for some real-valued \( \varphi \in W^{1,p} \), we see that \( u \) is weakly \( p \)-harmonic if and only if \( \varphi \) is a \( p \)-harmonic function—i.e., a weak solution of

\[
div(|d\varphi|^{p-2} d\varphi) = 0.
\]

For \( p \in (1, 2) \), the \( C^{1,\alpha} \) regularity of \( p \)-harmonic functions was established by DiBenedetto [14] and Lewis [23] (see also [36]). Rather than using the full strength of the \( C^{1,\alpha} \) regularity, we will employ in this paper the following simpler estimates, with constants independent of \( p \), whose proof we sketch in the “Appendix”:

Proposition 2.2 Let \( B_{2r}(x) \) be a geodesic ball in some manifold \( M^n \) with \(|\sec(M)| \leq K\). If \( \varphi \in W^{1,p}(B_{2r}(x), \mathbb{R}) \) is a \( p \)-harmonic function for \( p \in [\frac{3}{2}, 2] \), then for some constant \( C(n, K) < \infty \), we have that

\[
\|d\varphi\|_{L^\infty(B_r(x))} \leq Cr^{-n} \|d\varphi\|_{L^p(B_{2r}(x))} \tag{2.6}
\]

and

\[
r^p \|\text{Hess}(\varphi)\|_{L^p(B_r(x))} \leq C \|d\varphi\|_{L^p(B_{2r}(x))}^p. \tag{2.7}
\]
Combining this with the lifting criterion of Proposition 2.1, one obtains the following partial regularity result for weakly $p$-harmonic maps to the circle:

**Corollary 2.3** Let $p \in \left[ \frac{3}{2}, 2 \right]$, and let $B_{2r}(x)$ be a geodesic ball on a manifold $M^n$ with $|\text{sec}(M)| \leq K$. If $u \in W^{1,p}(B_{2r}(x), S^1)$ is a weakly $p$-harmonic map with vanishing distributional Jacobian

$$T(u) = 0 \text{ in } B_{2r}(x),$$

then

$$\|du\|_{L^\infty(B_r(x))}^p \leq Cr^{-n}\|du\|_{L^p(B_{2r}(x))}^p$$

(2.8)

and

$$r^p\|\nabla du\|_{L^p(B_r(x))}^p \leq C\|du\|_{L^p(B_{2r}(x))}^p.$$ (2.9)

**Remark 2.4** Though Corollary 2.3 shows that weakly $p$-harmonic maps $u \in W^{1,p}(M, S^1)$ are reasonably smooth (with effective estimates) away from the support of $T(u)$, observe that the weak $p$-harmonic condition alone gives no constraint on $T(u)$ itself. Indeed, given any $v \in W^{1,p}(M, S^1)$, we can minimize $\int_M \|du\|^p$ among all maps of the form $u = e^{i\varphi}v$ to find a weakly $p$-harmonic $u$ with topological singularity

$$T(u) = d[jv + d\varphi] = dv = T(v)$$

equal to that of $v$. The problem of minimizing $p$-energy among $S^1$-valued maps with prescribed singularities in $\mathbb{R}^2$—and, more generally, among $S^{k-1}$-valued maps with prescribed singularities in $\mathbb{R}^k$—is studied in detail in [8].

### 2.3 $p$-Stationarity and consequences

A map $u \in W^{1,p}(M, S^1)$ is said to be $p$-stationary, or simply stationary, if it is critical for the energy $E_p(u)$ with respect to perturbations of the form $u_t = u \circ \Phi_t$ for smooth families $\Phi_t$ of diffeomorphisms on $M$. Equivalently, $u$ is $p$-stationary if it satisfies the inner-variation equation

$$\int_M |du|^p \text{div}(X) - p|du|^{p-2}\langle du^*du, \nabla X \rangle = 0$$

(2.10)

for every smooth, compactly supported vector field $X$ on $M$.

The most-studied class of stationary $p$-harmonic maps (for arbitrary target manifolds) are the $p$-energy minimizers, whose regularity theory for $p \neq 2$ was first investigated by Hardt–Lin [18] and Luckhaus [27], extending results of Schoen and Uhlenbeck [31] from the case $p = 2$. On the other hand, as we discuss in Sect. 7, it is also often possible to produce non-minimizing stationary $p$-harmonic maps via natural min–max constructions.

Given a stationary $p$-harmonic map $u \in W^{1,p}(M^n, S^1)$, for each geodesic ball $B_r(x) \subset M$, we define the $p$-energy density

$$\theta_p(u, x, r) := r^{p-n}\int_{B_r(x)} |du|^p.$$ (2.11)

By standard arguments, it follows from the stationary Eq. (2.10) that the density $\theta_p(u, x, r)$ is nearly monotonic in $r$: Namely, taking $X$ in (2.10) of the form

$$X = \psi \frac{1}{2} \nabla \text{dist}(x, \cdot)^2$$
for some functions $\psi \in C^\infty_c(B_r(x))$ approximating the characteristic function $\chi_{B_r(x)}$, and employing the Hessian comparison theorem to estimate the difference $\nabla X - I$ in $B_r(x)$, one obtains the following well-known estimate (cf. e.g., [18], sections 4 and 7):

**Lemma 2.5** Let $u \in W^{1,p}(M, S^1)$ be a stationary $p$-harmonic map on a manifold $M^n$ with $|\text{sec}(M)| \leq K$. Then there is a constant $C(n, K)$ such that for any $x \in M$ and almost every $0 < r < \text{inj}(M)$, we have the inequality

$$\frac{d}{dr}[e^{Cr^2} \theta_p(u, x, r)] \geq pe^{Cr^2} r^{p-n} \int_{\partial B_r(x)} |du|^p r^{-2} \frac{\partial u}{\partial v}^2. \quad (2.12)$$

In particular, $e^{Cr^2} \theta_p(u, x, r)$ is monotone increasing in $r$.

In light of the monotonicity result, it makes sense to define the pointwise energy density

$$\theta_p(u, x) := \lim_{r \to 0} \theta_p(u, x, r). \quad (2.13)$$

Perhaps the most significant consequence of Lemma 2.5 is the boundedness of blow-up sequences: Given a sequence of radii $\text{inj}(M) > r_j \to 0$, observe that the maps $u_j = u_{x,r_j} \in W^{1,p}(B^n_1(0), S^1)$ defined by

$$u_j(y) := u(\exp_y(r_j y))$$

are stationary $p$-harmonic with respect to the blown-up metrics

$$g_j(y) := r_j^{-2}([\exp_y]^* g)(r_j y)$$

on $B^n_1(0)$, with $p$-energy given by

$$E_p(u_j, B_1, g_j) = \int_{B_1(0)} |du_j|_g^p dv_g = r_j^{p-n} \int_{B_{r_j}(x)} |du|^p dv = \theta_p(u, x, r_j),$$

so it follows from Lemma 2.5 that the $p$-energies $E_p(u_j, B_1, g_j)$ are uniformly bounded from above as $r_j \to 0$.

For a local minimizer $u$ of the $p$-energy, one could then appeal to the compactness results of ([18], Section 4) to conclude immediately that a subsequence $u_{jk}$ of such a blow-up sequence converges strongly to a minimizing tangent map $u_\infty$. For $p \in (1, 2)$—the range of interest to us—it turns out that we can still obtain a strong convergence result without the minimizing assumption, but this relies on the following subtler result of [30].

**Proposition 2.6** ([30], Lemma 3.17) Let $N$ be a compact homogeneous space with left-invariant metric. For fixed $p \in (1, \infty) \setminus \mathbb{N}$, let $u_j \in W^{1,p}(B^n_2, N)$ be a sequence of maps which are stationary $p$-harmonic with respect to a $C^2$-convergent sequence of metrics $g_j \to g_\infty$ on $B_2$. If $\{u_j\}$ is uniformly bounded in $W^{1,p}(B_2, N)$, then some subsequence $u_{jk}$ converges strongly in $W^{1,p}(B_1, N)$ to a map $u_\infty$ that is stationary $p$-harmonic with respect to $g_\infty$.

**Remark 2.7** The significance of the condition $p \neq \mathbb{N}$ is that the $p$-energy has no conformally invariant dimension in this case, so that no bubbling can occur, and the proposition follows from arguments generalizing those of [24] to the case $p \neq 2$ (see [30]). The requirement that $N$ be a homogeneous space is a technical one, arising from the fact that, at present, the most general $\epsilon$-regularity theorem available for stationary $p$-harmonic maps (when $p \neq 2$) is that of [37] for homogeneous targets (see also [29,34]).
The result of Proposition 2.6 clearly applies to stationary \( p \)-harmonic maps to \( S^1 \) for any \( p \in (1, 2) \), the range of interest. In particular, it follows that for any blow-up sequence \( u_j = u_{x, r_j} \in W^{1,p}(B^n_2(0), S^1) \), \( r_j \to 0 \), we can extract a subsequence \( u_{jk} \to 0 \) such that the maps \( u_{jk} \) converge strongly in \( W^{1,p}(B^n_1(0), S^1) \) to a map \( u_{\infty} \in W^{1,p}(B^n_1(0), S^1) \) which is stationary \( p \)-harmonic with respect to the flat metric, and satisfies

\[
\theta_p(u_{\infty}, 0, r) = \theta_p(u, x)
\]

for every \( r > 0 \). Following standard arguments (see, e.g., [18]), we can then apply the Euclidean case of the monotonicity formula 2.12 (in which \( C = 0 \)) to conclude that \( u_{\infty} \) must satisfy the 0-homogeneity condition

\[
\langle du_{\infty}(x), x \rangle = 0 \text{ for a.e. } x \in \mathbb{R}^n.
\]

In the next section, we will appeal to this strong convergence to tangent maps to obtain a sharp lower bound for the density \( \theta_p(u, x) \) at singular points of \( u \), which will form the foundation for many of the estimates that follow.

### 3 Sharp \( \varepsilon \)-regularity and estimates for \( T(u) \)

#### 3.1 A sharp lower bound for energy density on \( \text{Sing}(u) \)

The analysis leading to Theorem 1.1 rests largely on the following proposition—the comparatively simple counterpart in our setting to the “\( \eta \)-compactness”/“\( \eta \)-ellipticity” results for solutions of the complex Ginzburg–Landau equations (cf. [5] Theorem 2, [25] Lemma II.7).

**Proposition 3.1** Let \( u \in W^{1,p}(M^n, S^1) \) be a stationary \( p \)-harmonic map with \( n \geq 2 \) and \( p \in (1, 2) \), and let \( x \in \text{Sing}(u) \) be a singular point. Then

\[
\theta_p(u, x) \geq c(n, p) \frac{2\pi}{2 - p},
\]

where \( c(2, p) = 1 \), and, for \( n > 2 \),

\[
c(n, p) := \int_{\mathbb{R}^{n-2}} \left( \sqrt{1 - |y|^2} \right)^{2-p} dy \to \omega_{n-2} \text{ as } p \to 2.
\]

(Here, \( \omega_m \) denotes the volume of the Euclidean unit \( m \)-ball.)

**Proof** Let \( x \in \text{Sing}(u) \); by the small energy regularity theorem (see [37], Corollary 3.2 and Theorem 2), this is equivalent to the positivity of the density \( \theta_p(u, x) > 0 \). Taking a sequence of radii \( r_j \to 0 \) and considering the blow-up sequence \( u_j = u_{x, r_j} \), we know from the discussion in Sect. 2.3 that some subsequence \( u_{jk} \) converges strongly in \( W^{1,p}_{\text{loc}}(\mathbb{R}^n, S^1) \) to a nontrivial stationary \( p \)-harmonic map

\[
v \in W^{1,p}_{\text{loc}}(\mathbb{R}^n, S^1)
\]

satisfying

\[
\theta_p(v, 0, r) = \theta_p(u, x) \text{ for all } r > 0, \quad \text{and } \frac{\partial v}{\partial r} = 0.
\]

Since the tangent map \( v \) is radially homogeneous, it follows that its restriction \( v|_{S} \) to the unit sphere defines a weakly \( p \)-harmonic map on \( S^{n-1} \).
Next, we observe that if \( n > 2 \), the restriction \( v|_S \) must again have a nontrivial singular set. Indeed, if \( v|_S \) were \( C^1 \), then since \( H^1_{dR}(S^{n-1}) = 0 \), we would have a lifting \( v|_S = e^{j\varphi} \) for some \( p \)-harmonic function \( \varphi \in W^{1,p}(S^{n-1}, \mathbb{R}) \). The only \( p \)-harmonic functions on closed manifolds are the constants, so this would contradict the nontriviality of \( v \). Thus, \( v|_S \) must have nonempty singular set on \( S^{n-1} \), and in particular, \( \text{Sing}(v) \) must contain at least one ray in \( \mathbb{R}^n \).

We proceed now by a simple dimension reduction-type argument. Fixing some singular point \( x_1 \in \text{Sing}(v) \setminus \{0\} \) of \( v \) away from the origin, a standard application of (3.2) and the monotonicity formula gives the density inequality

\[
\theta_p(v, x_1) \leq \theta_p(v, 0) = \theta_p(u, x).
\]  

(3.3)

Thus, we can take a blow-up sequence for \( v \) at \( x_1 \) to obtain a new tangent map \( v_1 \in W^{1,p}_{\text{loc}}(\mathbb{R}^n, S^1) \) satisfying

\[
0 < \theta_p(v_1, 0, r) = \theta_p(v_1, 0) \leq \theta_p(u, x) \text{ for all } r > 0.
\]  

(3.4)

This map \( v_1 \) will again be radially homogeneous (by the monotonicity formula), and from the radial homogeneity \( \frac{\partial v}{\partial r} = 0 \) of \( v \), \( v_1 \) inherits the additional translation symmetry \( (dv_1, x_1) = 0 \) in the \( x_1 \) direction. In particular, \( v_1 \) is determined by its restriction to an \((n - 2)\)-sphere in the hyperplane perpendicular to \( x_1 \), which defines a weakly \( p \)-harmonic map from \( S^{n-2} \) to \( S^1 \).

If \( n - 2 > 1 \), we can argue as before to see that \( v_1 \) must have singularities on this \((n - 2)\)-sphere, and blow up again at some point \( x_2 \in \text{Sing}(v_1) \setminus \mathbb{R}x_1 \). Carrying on in this way, we obtain finally a nontrivial stationary \( p \)-harmonic map \( v_{n-2} \in W^{1,p}_{\text{loc}}(\mathbb{R}^n, S^1) \) which is radially homogeneous, invariant under translation by some \((n - 2)\)-plane \( L^{n-2} \), and satisfies

\[
\theta_p(v_{n-2}, 0, r) \leq \theta_p(u, x) \text{ for all } r > 0.
\]  

(3.5)

By direct computation, it’s easy to see that the only weakly \( p \)-harmonic maps from \( S^1 \) to \( S^1 \) are given—up to rotations—by the identity \( z \mapsto z \) and its powers \( z \mapsto z^\kappa \) for \( \kappa \in \mathbb{Z} \). In particular, letting \( z \) denote the projection of \( x \) onto the two-plane \( L^\perp \), it follows that

\[
v_{n-2}(x) = (z/|z|)^\kappa
\]

for some \( 0 \neq \kappa \in \mathbb{Z} \). We can therefore compute

\[
\theta_p(v_{n-2}, 0, 1) = \int_{B_1^{n-2}} \int_{D_1^2} \frac{|\kappa|^p}{|z|^p} d\bar{z}dy
\]

\[
= \frac{2\pi |\kappa|^p}{2 - p} \int_{B_1^{n-2}} \left( \sqrt{1 - |y|^2} \right)^{2-p} dy
\]

\[
= \frac{2\pi |\kappa|^p}{2 - p} c(n, p).
\]

It then follows from (3.5) that

\[
\theta_p(u, x) \geq \theta_p(v_{n-2}, 0, 1) \geq c(n, p) \frac{2\pi}{2 - p},
\]  

(3.6)

as desired. \( \square \)
3.2 Consequences of Proposition 3.1 and estimates for $T(u)$

Throughout this section, let $M$ be an $n$-dimensional manifold satisfying the sectional curvature and injectivity radius bounds

$$|\text{sec}(M)| \leq k, \quad \text{inj}(M) \geq 3, \quad (3.7)$$

and let $p \in [3/2, 2)$. As a first consequence of Proposition 3.1, we employ a simple Vitali covering argument (compare, e.g., Theorem 3.5 of [30]) to obtain $p$-independent estimates for the $(n-p)$-content of the singular set $\text{Sing}(u) = spt(T(u))$ of a stationary $p$-harmonic map $u$ to $S^1$.

**Lemma 3.2** Let $u \in W^{1,p}(B_3(x), S^1)$ be a stationary $p$-harmonic map on a geodesic ball $B_3(x) \subset M$ of radius 3, satisfying the $p$-energy bound

$$E_p(u, B_3(x)) \leq \frac{\Lambda}{2 - p}. \quad (3.8)$$

For $r \leq 1$, the $r$-tubular neighborhood $N_r(\text{Sing}(u) \cap B_2(x))$ about the singular set $\text{Sing}(u)$ then satisfies a volume bound of the form

$$\text{Vol}(N_r(\text{Sing}(u)) \cap B_2(x)) \leq C(k, n) \Lambda r^p. \quad (3.9)$$

**Proof** Applying the Vitali covering lemma to the covering

$$\{B_r(y) \mid y \in \text{Sing}(u) \cap B_2(x)\}$$

of $N_r(\text{Sing}(u) \cap B_2(x))$, we obtain a finite subcollection $x_1, \ldots, x_m \in \text{Sing}(u) \cap B_2(x)$ for which

$$B_r(x_i) \cap B_r(x_j) = \emptyset \text{ when } i \neq j. \quad (3.10)$$

and

$$N_r(\text{Sing}(u) \cap B_2(x)) \subset \bigcup_{i=1}^{m} B_{5r}(x_j). \quad (3.11)$$

Now, by virtue of Proposition 3.1 and Lemma 2.5, we have for each $B_r(x_i)$ the lower energy bound

$$\frac{2\pi c(n, p)}{2 - p} \leq C(k, n) \theta_p(u, x_i, r) = C(k, n) r^{p-n} \int_{B_r(x_i)} |du|^p,$$

and from the disjointness (3.10) of $\{B_r(x_i)\}$, it follows that

$$m \frac{2\pi c(n, p)}{2 - p} \leq C(k, n) r^{p-n} \int_{N_r(\text{Sing}(u)) \cap B_2(x)} |du|^p \leq C(k, n) r^{p-n} \frac{\Lambda}{2 - p}. \quad (3.12)$$

Since $\inf_{p \in [3/2, 2)} c(n, p) > 0$, this gives us an estimate of the form

$$m \leq C(k, n) \Lambda r^{p-n}. \quad (3.13)$$

By virtue of (3.11), it then follows that

$$\text{Vol}(N_r(\text{Sing}(u) \cap B_2(x))) \leq m \cdot C'(k, n) r^n \leq C(k, n) \Lambda r^p,$$

as claimed. \qed
For analysis purposes, this volume estimate is one of the most important consequences of Proposition 3.1, and we will use it repeatedly throughout the remainder of the paper; its role should be compared with that of results like (5), Proposition 1) in the Ginzburg–Landau setting. The first application is a series of improved estimates for the distributional Jacobian \( T(u) \) of \( u \). A priori, we have only the simple estimate

\[
\| T(u) \|_{W^{-1,p}} \leq \| du \|_{L^p},
\]

since

\[
\langle T(u), \zeta \rangle = \int_M j u \wedge d\zeta \leq \| j u \|_{L^p} \| \zeta \|_{W^{1,p'}}
\]  \hspace{1cm} (3.12)

for every \( \zeta \in \Omega^{n-2}(M) \) (with \( p' := \frac{p}{p-1} \)). With Lemma 3.2 in hand, however, we are able to show that, for stationary \( p \)-harmonic \( u_p \in W^{1,p}(M, S^1) \), if \( E_p(u_p) = O(\frac{1}{\varepsilon^p}) \), then \( T(u_p) \) is in fact uniformly bounded in various norms as \( p \uparrow 2 \). We remark that, in the Ginzburg–Landau setting, similar estimates for the Jacobian in terms of a normalized Ginzburg–Landau energy are also available for arbitrary complex-valued maps (see, e.g., [20]).

**Lemma 3.3** Under the assumptions of Lemma 3.2, let \( T(u) \) denote the distributional Jacobian of \( u \). Then for any smooth \((n-2)\)-form \( \zeta \in \Omega^r_{c}(-2)(B_2(x)) \) supported in \( B_2(x) \), we have

\[
\langle T(u), \zeta \rangle \leq C(k, n) \Lambda \| \zeta \|_{L^p} \| d\zeta \|_{L^{2-p}}.
\]  \hspace{1cm} (3.13)

**Proof** Fix \( \beta \in (0, 1) \) and set \( K := \lfloor \frac{1}{2-\beta} \rfloor \geq \frac{p-1}{2-p} \). Setting \( U(r) := N_r(Sing(u) \cap B_2(x)) \), we begin with the simple estimate

\[
E_p(u, B_2(x)) \geq \int_{U(\beta) \setminus U(2-\beta \beta)} |du|^p = \sum_{j=1}^{K} \int_{U(2-\beta \beta) \setminus U(2-\beta \beta)} |du|^p,
\]

from which it follows that

\[
\int_{U(2-\beta \beta) \setminus U(2-\beta \beta)} |du|^p \leq \frac{E_p(u)}{K}.
\]  \hspace{1cm} (3.14)

for some \( j \in \{1, \ldots, K\} \). In particular, there is some scale \( s = 2^{-j} \beta \in [2^{-\frac{1}{p}} \beta, \frac{1}{2} \beta] \) for which

\[
\int_{U(2s) \setminus U(s)} |du|^p \leq \frac{E_p(u)}{K} \leq \frac{\Lambda}{2-p} \cdot \frac{2-p}{p-1} = \frac{\Lambda}{p-1}.
\]  \hspace{1cm} (3.15)

Now, let \( \psi(y) = \eta(dist(y, Sing(u))) \), where \( \eta \) is given by

\[
\eta \equiv 1 \text{ on } [0, s], \quad \eta(t) = 1 - \frac{1}{s}(t-s) \text{ for } t \in [s, 2s], \quad \text{and } \eta \equiv 0 \text{ on } [2s, \infty),
\]

so that \( \psi \equiv 1 \text{ on } U(s) \supset spt(T(u)) \) and \( \psi \equiv 0 \text{ outside } U(2s) \). For any \( \zeta \in \Omega^{n-2}_{c}(B_2(x)) \), we then have

\[
\langle T(u), \zeta \rangle = \langle T(u), \psi \zeta \rangle = \int j u \wedge d\psi \wedge \zeta + j u \wedge \psi d\zeta \leq \| d\psi \|_{L^\infty} \| j u \|_{L^p(U(2s) \setminus U(s))} \| \zeta \|_{L^{p'}(U(2s))} + \| j u \|_{L^p(U(2s))} \| d\zeta \|_{L^{p'}(U(2s))},
\]
where $p' = \frac{p}{p-1}$. Next, we note that $\|d\psi\|_{L^\infty} = \frac{1}{s}$, while

$$\|ju\|_{L^p(U(2s))} \leq E_p(u, B_2(x))^{1/p} \leq (2 - p)^{-1/p} \Lambda^{1/p},$$

and, by (3.15),

$$\|ju\|_{L^p(U(2s)) \setminus U(s))} \leq \frac{\Lambda^{1/p}}{(p - 1)^{1/p}},$$

using all of this in the preceding estimate, we then obtain

$$\langle T(u), \zeta \rangle \leq \frac{1}{s} \frac{\Lambda^{1/p}}{(p - 1)^{1/p}} \|\xi\|_{L^p(U(2s))} + (2 - p)^{-1/p} \Lambda^{1/p} \|d\xi\|_{L^p(U(2s))}, \quad (3.16)$$

Now, by Lemma 3.2, we see that

$$\|\xi\|_{L^p(U(2s))} \leq \|\xi\|_{L^\infty} Vol(U(2s))^{1-1/p} \leq C(k, n)\Lambda^{1-1/p}s^{p-1}\|\xi\|_{L^\infty}$$

and similarly

$$\|d\xi\|_{L^p(U(2s))} \leq C(k, n)\Lambda^{1-1/p}s^{p-1}\|d\xi\|_{L^\infty},$$

which we use in (3.16) to obtain

$$\langle T(u), \zeta \rangle \leq \frac{C \Lambda s^{p-2}}{(p - 1)^{1/p}} \|\xi\|_{L^\infty} + \frac{C \Lambda s^{p-1}}{(2 - p)^{1/p}} \|d\xi\|_{L^\infty}. \quad (3.17)$$

Recalling that $s$ lies in the interval $2^{\frac{1}{2-p}} \leq s \leq \frac{1}{2} \beta$, it then follows that

$$\langle T(u), \zeta \rangle \leq C(k, n)\Lambda \left(\frac{\beta^{p-2}\|\xi\|_{L^\infty}}{(p - 1)^{1/p}} + \frac{\beta^{p-1}\|d\xi\|_{L^\infty}}{(2 - p)^{1/p}}\right). \quad (3.18)$$

Finally, we observe that $\beta \in (0, 1)$ was arbitrary, so we can choose, for instance

$$\beta(p, \zeta) = \frac{(2 - p)^{1+1/p}}{(p - 1)^{1+1/p}} \frac{\|\xi\|_{L^\infty}}{\|d\xi\|_{L^\infty}}.$$ 

Indeed, in case $\beta(p, \zeta) \geq 1$, then

$$\|d\xi\|_{L^\infty} \leq \frac{(2 - p)^{1+1/p}}{(p - 1)^{1+1/p}} \|\xi\|_{L^\infty},$$

and the desired estimate follows immediately from (3.12). Otherwise, we have $\beta(p, \zeta) \in (0, 1)$, so we can plug $\beta = \beta(p, \zeta)$ into (3.18), and using the fact that $(2 - p)^{p-2}$ is uniformly bounded for $p \in (1, 2)$, we arrive an estimate of the desired form

$$\langle T(u), \zeta \rangle \leq C(k, n)\Lambda \|\xi\|^{p-1}_{L^\infty} \|d\xi\|_{L^\infty}^{2-p}. \quad (3.19)$$

By rescaling the result of Lemma 3.3, we obtain the following statement at arbitrary small scales $0 < r \leq 1$:

**Corollary 3.4** For $0 < r \leq 1$, let $u \in W^{1,p}(B_{3r}(x), S^1)$ be a stationary $p$-harmonic map with

$$\theta_p(u, x, 3r) \leq \frac{\Lambda}{2 - p}.$$
Corollary 3.6

Let $u \in \Omega_c^{n-2}(B_{2r}(x))$, we have

$$
\langle T(u), \zeta \rangle \leq C(k, n) \Lambda r^{n-p} \| \zeta \|_{L_\infty}^{p-1} \| d\zeta \|_{L_\infty}^{2-p}. 
$$

(3.20)

In particular, by virtue of energy monotonicity (Lemma 2.5), if $u \in W^{1,p}(B_3, S^1)$ satisfies $E_p(u, B_3) \leq \frac{\Lambda}{2-p}$, then (3.20) holds for all $\zeta \in \Omega_c^{n-2}(B_r(y))$, for every ball $B_r(y) \subset B_2(x)$. In the “Appendix”, we establish the following general lemma, which will imply that these estimates, together with the volume bounds of Lemma 3.2, yield $p$-independent bounds for $\| T(u) \|_{W^{-1,q}}$ for any $q \in [1, p)$:

Lemma 3.5

Let $S$ be an $(n-2)$-current in $W^{-1,p}(B_2(x))$ satisfying

$$
\langle S, \zeta \rangle \leq Ar^{n-p} \| \zeta \|_{L_\infty}^{p-1} \| d\zeta \|_{L_\infty}^{2-p} \quad \forall \zeta \in \Omega_c^{n-2}(B_r(y)),
$$

(3.21)

for every ball $B_r(y) \subset B_2(x)$. Suppose also that the $r$-tubular neighborhoods $\mathcal{N}_r(spt(S))$ about the support of $S$ satisfy

$$
Vol(\mathcal{N}_r(B_2(x) \cap spt(S))) \leq Ar^p.
$$

(3.22)

Then there is a constant $C(n, k, A)$ such that for every $1 \leq q < p$, we have

$$
\| S \|_{W^{-1,q}(B_1(x))} \leq C(n, k, A)(p-q)^{-1/q}.
$$

(3.23)

In particular, combining the results of Lemma 3.2 with Lemma 2.5 and Corollary 3.4, we find that

Corollary 3.6

Let $u \in W^{1,p}(B_3(x), S^1)$ be a stationary $p$-harmonic map with

$$
E_p(u, B_3(x)) \leq \frac{\Lambda}{2-p}.
$$

Then for every $q \in [1, p)$, we have

$$
\| T(u) \|_{W^{-1,q}(B_1(x))} \leq C(n, k, \Lambda, q). 
$$

(3.24)

Proof

By the preceding discussion, we see that

$$
\| T(u) \|_{W^{-1,q}(B_1(x))} \leq C(n, k, \Lambda)(p-q)^{-1/q}.
$$

To put this in the form (3.24), we simply separate into two cases: if $p \leq 1 + \frac{q}{2}$, then $2-p \geq \frac{2-q}{2}$, and (3.24) follows from the trivial estimate

$$
\| T(u) \|_{W^{-1,p}} \leq \| du \|_{L^p} \leq \frac{\Lambda^{1/p}}{(2-p)^{1/p}}.
$$

On the other hand, if $p > 1 + \frac{q}{2}$, then $p-q > \frac{2-q}{2}$, and so

$$
\| T(u) \|_{W^{-1,q}(B_1(x))} \leq C(n, k, \Lambda)(p-q)^{-1/q} \leq C(n, k, \Lambda, q),
$$

as claimed. 

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4 Estimates for the Hodge decomposition of $ju$

Now, let $M^n$ again be a closed, oriented Riemannian manifold, and for $p \in [3/2, 2)$, let $u \in W^{1,p}(M, S^1)$ be a stationary $p$-harmonic map with

$$E_p(u) \leq \frac{\Lambda}{2 - p}. \quad (4.1)$$

In our analysis of the global behavior of $u$, similar to the Ginzburg–Landau setting (compare, e.g., [5, 25]), the Hodge decomposition

$$ju = d\phi + d^*\xi + h \quad (4.2)$$

of $ju$ plays a central role. (For more on Hodge decomposition in the space of $L^p$ differential forms, we refer the reader to [32].) Here, $\phi \in W^{1,p}(M, \mathbb{R})$ is the function given by

$$\phi := \Delta^{-1}(\text{div}(ju)),$$

$\xi$ is the $W^{1,p}$ two-form

$$\xi := *\Delta^{-1}_H T(u),$$

and $h$ is the remaining harmonic one-form, which we can write as

$$h := \Sigma_{i=1}^k \left( \int_M \langle h_i, ju \rangle \right) h_i$$

with respect to an $L^2$-orthonormal basis $\{h_i\}_{i=1}^k$ for the space $\mathcal{H}^1(M)$ of harmonic one-forms. We remark that, in our notation, $\Delta$ denotes the negative spectrum scalar Laplacian, but $\Delta_H = dd^* + d^*d$ is the usual positive spectrum Hodge Laplacian.

We begin this section by establishing estimates which show that, for each $q \in [1, 2)$, and any sequence $u_p$ as in Theorem 1.1, the coexact component $d^*\xi$ remains bounded and the exact component $d\phi$ vanishes in $L^q$ as $p \to 2$. Once we’ve obtained these global estimates, we will show that the same behavior holds in stronger norms away from the singular sets.

For the harmonic form $h$, we need only the trivial $L^\infty$ estimate

$$\|h\|_{L^\infty(M)} \leq C(M)\|du\|_{L^1(M)} \leq C(M) \frac{\Lambda^{1/p}}{(2 - p)^{1/p}}. \quad (4.3)$$

For the exact and co-exact terms $d\phi$ and $d^*\xi$, we begin by establishing the following global estimates:

**Proposition 4.1** If $q \in (1, p)$, then

$$\|\xi\|_{W^{1,q}} \leq C(M, \Lambda, q) \quad (4.4)$$

and

$$\|\phi\|_{W^{1,q}} \leq C(M, \Lambda, q)(2 - p)^{1 - 1/p}\left|\log(2 - p)\right| \quad (4.5)$$

for some $C(M, \Lambda, q) < \infty$ independent of $p$.

**Proof** First, note that we can apply Corollary 3.6 (after some fixed rescaling) to a finite covering of $M$ by geodesic balls, to obtain the $W^{-1,q}$ estimate

$$\|T(u)\|_{W^{-1,q}(M)} \leq C(M, \Lambda, q) \quad (4.6)$$
for the distributional Jacobian $T(u)$. Since $\xi := *\Delta_H^{-1}T(u)$ by definition, it follows from the $L^q$ regularity of $\Delta_H$ and a simple duality argument that

$$\|\xi\|_{W^{1,q}} \leq C(M, q)\|T(u)\|_{W^{-1,q}} \leq C(M, \Lambda, q),$$

as desired.

To estimate $d\varphi$, we begin by observing that since $u$ is weakly $p$-harmonic, the distributional divergence

$$\text{div}(|ju|^{p-2}ju)$$

vanishes, and $\varphi$ can therefore be recast as

$$\varphi := \Delta^{-1}(\text{div}(ju - |ju|^{p-2}ju)).$$

The $L^q$ regularity of the Laplacian then gives

$$\|\varphi\|_{W^{1,q}} \leq C(q, M)\|(1 - |ju|^{p-2})ju\|_{L^q},$$

so it is enough produce an $L^q$ estimate of the desired form for $(1 - |ju|^{p-2})ju$.

To this end, we write

$$\|(1 - |ju|^{p-2})ju\|_{L^q}^q = \int |du| - |du|^{p-1}q$$

$$= \int_{|du| \leq 1} (|du|^{p-1} - |du|)^q$$

$$+ \int_{|du| \geq 1} (|du| - |du|^{p-1})^q.$$

It’s easy to check that

$$\max_{t \in [0,1]} (tp^{p-1} - t) = (2-p)(p-1)^{\frac{p-1}{2-p}},$$

so the $\{|du| \leq 1\}$ portion of the integral satisfies

$$\int_{|du| \leq 1} (|du|^{p-1} - |du|)^q \leq (2-p)^q(p-1)^{\frac{q(p-1)}{2-p}} \text{Vol}(M).$$

To estimate the $\{|du| \geq 1\}$ portion of the integral, observe that

$$1 - t^{p-2} \leq (2-p)\log(t)$$

when $t \geq 1$, so fixing some $\lambda > 1$, we split the integral again to see that

$$\int_{|du| \geq 1} (|du| - |du|^{p-1})^q \leq \int_{1 \leq |du| \leq \lambda} (2-p)^q \log(|du|)^q|du|^q$$

$$+ \int_{|du| \geq \lambda} |du|^q$$

$$\leq (2-p)^q \log(\lambda)^q \int_{1 \leq |du| \leq \lambda} |du|^q$$

$$+ \|du\|_{L^p}^q \text{Vol}(|du| \geq \lambda)^{1-q/p}$$

$$\leq (2-p)^q \log(\lambda)^q C(M)\|du\|_{L^p}^q + \|du\|_{L^p}^p \lambda^{q-p},$$
and since \( \|du\|_{L^p} \leq \Lambda^{1/p}(2 - p)^{-1/p} \), this yields
\[
\int_{\{|du|\geq 1\}} (|du| - |du|^{p-1})^q \leq C(M) \Lambda^{q/p}(2 - p)^{q - q/p} \log(\lambda)^q + \Lambda(2 - p)^{-1} \lambda^{q-p}.
\]
Taking \( \lambda = (2 - p)^{-\frac{1}{p} - \frac{q}{p-q}} \), we observe that
\[
(2 - p)^{-1} \lambda^{q-p} = (2 - p)^{q-q/p}
\]
and
\[
\log(\lambda) = \left(\frac{1}{p} + \frac{q}{p-q}\right) |\log(2 - p)|,
\]
so putting this together with the preceding inequalities, we arrive at the estimate
\[
\int ||du| - |du|^{p-1}|^q \leq \frac{C(M, \Lambda)}{(p-q)^q} (2 - p)^{q-q/p} |\log(2 - p)|^q.
\] (4.8)

Considering separately the cases \( p > 1 + \frac{q}{2} \) and \( p \leq 1 + \frac{q}{2} \) as in the proof of Corollary 3.6, and recalling that
\[
\|d\varphi\|_{L^q} \leq C(M, q) \|du\| - |du|^{p-1}\|_{L^q},
\]
we arrive at an estimate of the desired form (4.5). \( \square \)

Next, we establish estimates resembling (4.4) and (4.5) in \( W^{1,2} \) norms away from the singular set \( \text{Sing}(u) \). The simple estimates of Lemma 4.2 below by no means represent the optimal bounds of this kind, but they will suffice for the purposes of this paper.

**Lemma 4.2** Suppose now that \( p \in [\max\{q_n, 3/2\}, 2) \), where \( q_n = \frac{2n}{n+2} \) (so that \( W^{1,q_n} \hookrightarrow L^2 \) by Sobolev embedding). Letting \( r(x) := \text{dist}(x, \text{Sing}(u)) \), we have the \( L^2 \) estimates
\[
\|r(x)d^*\xi\|_{L^2(M)} \leq C(M, \Lambda) \tag{4.9}
\]
and
\[
\|r(x)d\varphi\|_{L^2(M)} \leq C(M, \Lambda)(2 - p)^{1-1/p}|\log(2 - p)|. \tag{4.10}
\]

**Proof** For \( \delta > 0 \), let \( \psi_\delta(x) \in Lip(M) \) be given by
\[
\psi_\delta(x) = \max\{0, r(x) - \delta\},
\]
so that \( \psi_\delta \equiv 0 \) on a neighborhood of \( \text{Sing}(u) = spt(T(u)) \), and \( Lip(\psi_\delta) \leq 1 \). Then \( d^*\xi \) is closed on the support of \( \psi_\delta \), and it follows that
\[
\int \psi_\delta^2 |d^*\xi|^2 = \int \langle d^*\xi, \psi_\delta^2 d^*\xi \rangle
\]
\[
= \int \langle \xi, 2\psi_\delta d\psi_\delta \wedge d^*\xi \rangle
\]
\[
\leq 2 \int |\psi_\delta d^*\xi|||\xi||.
\]
Now, since \( p \geq q_n \), we have by Sobolev embedding and Proposition 4.1 the estimate
\[
\|\xi\|_{L^2} \leq C(M)\|\xi\|_{W^{1,q_n}} \leq C(M, \Lambda);
\]
applying this in the preceding inequality, it follows that
\[
\int \psi_\delta^2 |d^\ast \xi|^2 \leq C(M, \Lambda)^2.
\]

Taking \( \delta \to 0 \) and appealing to the monotone convergence theorem, we arrive at (4.9).

For (4.10), we proceed similarly: with \( \psi_\delta \) defined as above, we use the equation
\[
\Delta \varphi = \text{div}((1 - |u|^{p-2})\ j \ u)
\]
(and the fact that \( \text{Lip}(\psi_\delta) \leq 1 \)) to estimate
\[
\int \psi_\delta^2 |d \varphi|^2 = \int \langle d \varphi, d(\psi_\delta^2 \varphi) \rangle - 2 \int \langle \psi_\delta d \varphi, \varphi d \psi_\delta \rangle \\
\leq \| \psi_\delta (1 - |u|^{p-2})u \|_{L^2}^2 (\| \psi_\delta d \varphi \|_{L^2} + \| \varphi \|_{L^2}) \\
+ 2 \| \psi_\delta d \varphi \|_{L^2} \| \varphi \|_{L^2}.
\]

With a few applications of Young’s inequality, it then follows that
\[
\| \psi_\delta d \varphi \|_{L^2}^2 \leq 10 (\| \psi_\delta (1 - |u|^{p-2})du \|_{L^2}^2 + \| \varphi \|_{L^2}^2) 
\tag{4.11}
\]

Now, by Sobolev embedding and Proposition 4.1, we know that
\[
\| \varphi \|_{L^2} \leq C(M) \| \varphi \|_{W^{1,q}} \leq C(M, \Lambda)(2 - p)^{2-2/p} \| \log(2 - p) \|^2, 
\tag{4.12}
\]
so all that remains is to estimate \( \| \psi_\delta (1 - |u|^{p-2})du \|_{L^2} \).

To this end, observe that the gradient estimate of Corollary 2.3 implies
\[
r(x)^p |du|^p(x) \leq C(M) \theta_p(u, x, r(x)),
\]
which together with the monotonicity of \( \theta_p(u, x, \cdot) \) yields the pointwise gradient estimate
\[
r(x)^p |du|^p(x) \leq C(M) \frac{\Lambda}{2 - p};
\]
in particular, it follows that
\[
\psi_\delta^2 |du|^2 \leq C(M, \Lambda)(2 - p)^{-2/p}. 
\tag{4.13}
\]

As in the proof of Proposition 4.1, we note that \(|(1 - |du|^{p-2})du| \leq (2 - p)|du| \) when \(|du| \leq 1\), so that
\[
\int_{\{|du| \leq 1\}} \psi_\delta^2 |(1 - |du|^{p-2})du|^2 \leq C(M)(2 - p)^2.
\]

Where \(|du| \geq 1\), we can make repeated use of the pointwise estimate (4.13), together with the fact that
\[
1 - |du|^{p-2} \leq (2 - p) \log(|du|)
\]
to find
\[
\int_{\{|du| \geq 1\}} \psi_\delta^2 (1 - |du|^{p-2})^2 |du|^2 \leq C(M, \Lambda)(2 - p)^{-2/p} \int_{\{|du| \geq 1\}} (1 - |du|^{p-2})^2
\]
\[ \leq C(M, \Lambda)(2 - p)^{2 - 2/p} \int_{|du| \geq 1} \log(|du|)^2 \]
\[ \leq C(M, \Lambda)(2 - p)^{2 - 2/p} \int \log \left( \frac{C(M, \Lambda)}{(2 - p)^{1/p} r(x)} \right)^2. \]

Splitting up the logarithm
\[
\log \left( \frac{C(M, \Lambda)}{(2 - p)^{1/p} r(x)} \right) = \log(C(M, \Lambda)) + \frac{1}{p} |\log(2 - p)| - \log(r(x)),
\]
we then see that
\[
\int_M \psi^2_\delta (1 - |du|^{p-2})^2|du|^2 \leq C(M, \Lambda)(2 - p)^{2 - 2/p} |\log(2 - p)|^2
\]
\[ + C(M, \Lambda)(2 - p)^{2 - 2/p} \int_M \log(r(x))^2. \]

Finally, it follows from the volume estimates of Lemma 3.2, that
\[ \|r(x)^{-1}\|_{L^1} \leq C(M, \Lambda), \]
so that
\[ \int_M \log(r(x))^2 \leq C(M) \int_M r(x)^{-1} \leq C(M, \Lambda). \]

In particular, it then follows that
\[ \| \psi_\delta (1 - |du|^{p-2})du \|_{L^2}^2 \leq C(M, \Lambda)(2 - p)^{2 - 2/p} |\log(2 - p)|^2, \]
which together with (4.11) and (4.12) gives
\[ \| \psi_\delta d\varphi \|_{L^2}^2 \leq C(M, \Lambda)(2 - p)^{2 - 2/p} |\log(2 - p)|^2. \] (4.14)

As before, we now take \( \delta \to 0 \) and appeal to the Monotone Convergence Theorem to arrive at the desired estimate (4.10).

\[ \square \]

5 Limiting behavior of the \( p \)-energy measure

5.1 Generalized varifolds

As before, let \( M^n \) be a closed Riemannian manifold. For \( m \in \mathbb{N} \), denote by \( A_m(M^n) \) the compact subbundle
\[ A_m(M) := \{ S \in \text{End}(TM) \mid S = S^*, \quad -nI \leq S \leq I, \quad tr(S) = m \} \] (5.1)
of the endomorphism bundle \( \text{End}(TM) \), consisting of symmetric endomorphisms with trace \( m \) and eigenvalues lying in \([-n, 1]\). In [3], Ambrosio and Soner define the space \( \mathcal{V}_m^r(M) \) of generalized \( m \)-varifolds to be the space of nonnegative Radon measures on \( A_m(M) \). Note that \( \mathcal{V}_m^r(M) \) contains the standard \( m \)-varifolds—Radon measures on the Grassmannian bundle \( G_m(M) \) (see [2,33] for an introduction)—since identifying subspaces with the associated orthogonal projections gives a natural inclusion \( G_m(M) \hookrightarrow A_m(M) \).
As with standard varifolds, for any $V \in \mathcal{V}_m$, we define the weight measure $\| V \|$ to be the pushforward $\pi_* V$ of $V$ under the projection $\pi : A_m(M) \to M$, and the first variation $\delta V$ to be the functional on $C^1$ vector fields given by

$$\delta V(X) := \int_{A_m(M)} \langle S, \nabla X \rangle dV(S). \quad (5.2)$$

A classical result of Allard (see [2], Section 5) states that any (standard) $m$-varifold $V$ whose first variation $\delta V$ is bounded in the $(C^0)^*$ sense

$$\delta V(X) \leq C \| X \|_{C^0}$$

restricts to an $m$-rectifiable varifold on the set $\{ x : \Theta^*_m(\| V \|, x) > 0 \}$ where its (upper-)$m$-dimensional density

$$\Theta^*_m(\| V \|, x) := \limsup_{r \to 0} \frac{\| V \|(B_r(x))}{\omega_m r^m}$$

is positive. In [3], this result is extended to the setting of generalized varifolds as follows.

**Proposition 5.1** ([3], Theorem 3.8(c)) Let $V \in \mathcal{V}_m(M)$ be a generalized $m$-varifold with bounded first variation

$$\delta V(X) \leq C \| X \|_{C^0} \quad (5.3)$$

and positive $m$-density

$$\Theta^*_m(\| V \|, x) > 0 \text{ for } \| V \| - a.e. \ x \in M. \quad (5.4)$$

Then there is an $m$-rectifiable (classical) varifold $\tilde{V}$ such that

$$\| \tilde{V} \| = \| V \| \text{ and } \delta \tilde{V} = \delta V. \quad (5.5)$$

In [3], this result was originally used to study concentration of energy for solutions of the parabolic Ginzburg–Landau equations, as a means for constructing codimension-two Brakke flows. In the proof of Theorem 1.1, we will use it similarly, to show that the concentrated part of $\mu$ is given by the weight measure of a stationary, rectifiable $(n - 2)$-varifold.

### 5.2 Proof of Theorems 1.1 and 1.2

As in Theorem 1.1, let $M$ be a closed, oriented Riemannian manifold, let $p_i \in (1, 2)$ with $\lim_{i \to \infty} p_i = 2$, and let $u_i \in W^{1,p_i}(M, S^1)$ be a sequence of stationary $p_i$-harmonic maps satisfying

$$\Lambda := \sup_{i \in \mathbb{N}} \int_M (2 - p_i)|d u_i|^{p_i} d v_g < \infty. \quad (5.6)$$

Passing to a subsequence, we can assume also that the $p_i$-energy measures

$$\mu_i := (2 - p_i)|d u_i|^{p_i} d v_g$$

converge in $(C^0)^*$ to a limiting measure $\mu$, and that the singular sets $\text{Sing}(u_i)$ converge in the Hausdorff metric to a limiting set

$$\Sigma = \lim_{i \to \infty} \text{Sing}(u_i).$$

Now, for each $i$, consider as in Sect. 4 the Hodge decomposition

$$j u_i = d^* \xi_i + d \varphi_i + h_i$$
of \(ju_i\), and set \(\alpha_i := d^*\xi_i + d\varphi_i\). We associate to \(ju_i, \alpha_i,\) and \(h_i\), the following \(L^1\) sections of \(\text{End}(T\mathbb{M})\):

\[
S_i := |du_i|^{p_i-2}du_i^*du_i = |ju_i|^{p_i-2}ju_i \otimes ju_i,
\]

\[
S_i^s := |\alpha_i|^{p_i-2}\alpha_i \otimes \alpha_i,
\]

and

\[
S_i^h := |h_i|^{p_i-2}h_i \otimes h_i.
\]

As we shall see, the proof of Theorem 1.1 rests largely on the following simple claim:

**Claim 5.2**

\[
\lim_{i \to \infty} (2 - p_i)\|S_i - S_i^s - S_i^h\|_{L^1} = 0.
\]

(5.7)

**Proof** For fixed \(x \in \mathbb{M}\), denoting by \(f : T^*_x\mathbb{M} \to \text{End}(T_x\mathbb{M})\) the function

\[
f(X) = |X|^{p-2}X \otimes X
\]

for \(p \in (1, 2)\), it is easy to check that

\[
|\nabla f(X)| \leq 3|X|^{p-1};
\]

as an immediate consequence, we then have

\[
|f(X + Z) - f(X)| \leq \int_0^1 3|X + tZ|^{p-1}|Z|dt
\]

\[
\leq 3(|X + Z|^{p-1} + |Z|^{p-1})|Z|
\]

for any \(X, Z \in T_x\mathbb{M}\). In particular, since \(S_i = f(ju_i) = f(\alpha_i + h_i)\), \(S_i^s = f(\alpha_i)\), and \(S_i^h = f(h_i)\) (with \(p = p_i\)) by definition, it follows that

\[
|S_i - S_i^s| \leq 3(|du_i|^{p_i-1} + |h_i|^{p_i-1})|h_i|
\]

and

\[
|S_i - S_i^h| \leq 3(|du_i|^{p_i-1} + |\alpha_i|^{p_i-1})|\alpha_i|.
\]

With this in mind, we estimate the \(L^1\) norm of \(S_i - S_i^s - S_i^h\) by splitting \(\mathbb{M} \setminus N_\delta(Sing(u_i))\) and \(\mathbb{M} \setminus N_\delta(Sing(u_i))\) for \(\delta > 0\) small, writing

\[
\|S_i - S_i^s - S_i^h\|_{L^1(\mathbb{M})} \leq \int_{N_\delta(Sing(u_i))} |S_i - S_i^s| + |S_i^h| + \int_{\mathbb{M} \setminus N_\delta(Sing(u_i))} (3|du_i|^{p_i-1}|h_i| + 4|h_i|^{p_i})
\]

\[
\leq \int_{N_\delta(Sing(u_i))} (3|du_i|^{p_i-1}|h_i| + 4|h_i|^{p_i})
\]

\[
+ \int_{\mathbb{M} \setminus N_\delta(Sing(u_i))} (3|du_i|^{p_i-1}|\alpha_i| + 4|\alpha_i|^{p_i-1}).
\]

Now, since

\[
\|h_i\|_{L^\infty} \leq C(M) \frac{\Lambda}{2 - p_i}
\]

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and, by Lemma 3.2,

$$Vol(\mathcal{N}_\delta(Sing(u_i))) \leq C(M, \Lambda) \delta^{p_i},$$

we have the simple estimate

$$\|h_i\|_{L^{p_i}(\mathcal{N}_\delta(Sing(u_i)))}^{p_i} \leq \frac{C(M, \Lambda)}{2 - p_i} \delta^{p_i}. \quad (5.8)$$

On the other hand, we know from Lemma 4.2 that

$$\int_{M \setminus \mathcal{N}_\delta(Sing(u_i))} |\alpha_i|^{p_i} \leq C(M) \left( \delta^{-2} \int_{M \setminus \mathcal{N}_\delta(Sing(u_i))} dist(x, Sing(u_i))^2 |\alpha_i|^2 \right)^{p_i/2} \leq C(M, \Lambda) \delta^{-p_i}.$$  

Returning to our estimate for $$\|S_i - S^s_i - S^h_i\|_{L^1(M)}$$, it then follows that

$$\|S_i - S^s_i - S^h_i\|_{L^1(M)} \leq \int_{\mathcal{N}_\delta(Sing(u_i))} (3 |du_i|^{p_i-1} |h_i| + 4|h_i|^{p_i})$$

$$+ \int_{M \setminus \mathcal{N}_\delta(Sing(u_i))} (3 |du_i|^{p_i-1} |\alpha_i| + 4|\alpha_i|^{p_i})$$

$$\leq C \|du_i\|_{L^{p_i}}^{p_i-1} \left( \|h_i\|_{L^{p_i}(\mathcal{N}_\delta(Sing(u_i)))} + \|\alpha_i\|_{L^{p_i}(M \setminus \mathcal{N}_\delta(Sing(u_i)))} \right)$$

$$\leq C(M, \Lambda)(2 - p_i)^{1/p_i-1} (\delta(2 - p_i)^{-1/p_i} + \delta^{-1}).$$

Multiplying by $$(2 - p_i)$$ and taking $$i \to \infty$$, we arrive at the bound

$$\limsup_{i \to \infty} (2 - p_i) \|S_i - S^s_i - S^h_i\|_{L^1(M)} \leq C(M, \Lambda) \delta;$$

since $$\delta > 0$$ was arbitrary, (5.7) follows. 

With this claim established, we next observe that measures $$\mu_i$$ can be written as

$$\mu_i = (2 - p_i)tr(S_i)dv_g,$$

and as a consequence of (5.7), we see that

$$\mu = \lim_{i \to \infty} (2 - p_i)tr(S^s_i + S^h_i)dv_g$$

$$= \lim_{i \to \infty} [(2 - p_i)|\alpha_i|^{p_i}dv_g + (2 - p_i)|h_i|^{p_i}dv_g]$$

$$= \lim_{i \to \infty} [(2 - p_i)|\alpha_i|^{p_i}dv_g + |\tilde{h}_i|^{p_i}dv_g],$$

where in the last line we’ve set

$$\tilde{h}_i := (2 - p_i)^{1/p_i} h_i.$$  

Now, since $$\{\tilde{h}_i\}$$ forms a bounded sequence in the space $$\mathcal{H}^1(M)$$ of harmonic one-forms, by passing to a further subsequence, we can assume that it converges to some limit

$$\bar{h} = \lim_{i \to \infty} \tilde{h}_i \in \mathcal{H}^1(M).$$

It’s then clear that

$$|\tilde{h}_i|^{p_i} \to |\bar{h}|^2$$
pointwise, and we can therefore write
\[ \mu = \lim_{i \to \infty} (2 - p_i)|\alpha_i|^{p_i} dv_g + |\bar{h}_i|^2 dv_g. \] (5.9)

To complete the proof of Theorem 1.1, it remains to realize the measure
\[ \nu := \lim_{i \to \infty} |\alpha_i|^{p_i} dv_g \]
as the weight measure of a stationary, rectifiable \((n - 2)\)-varifold satisfying the stated properties. To this end, we begin by remarking that, where \(|\alpha_i| > 0\), the tensor
\[ I - 2|\alpha_i|^{-2} \alpha_i \otimes \alpha_i \in \text{End}(TM) \]
belongs to \(A_{n-2}(M)\), so we can define a sequence of generalized \((n - 2)\)-varifolds \(V_i \in \mathcal{V}_{n-2}'(M)\) by
\[ \langle V_i, f \rangle := \int_M (2 - p_i)|\alpha_i|^{p_i} f(I - 2|\alpha_i|^{-2} \alpha_i \otimes \alpha_i) \text{ for } f \in C^0(A_{n-2}(M)). \] (5.10)
The associated weight measures \(\|V_i\|\) are then given by
\[ \|V_i\| := (2 - p_i)|\alpha_i|^{p_i} dv_g, \]
and since we have a uniform mass bound
\[ \sup_i \|V_i\|(M) < \infty, \]
we can pass to a further subsequence to obtain a weak limit
\[ V = \lim_{i \to \infty} V_i \in \mathcal{V}_{n-2}'(M) \]
with weight measure
\[ \|V\| = \nu. \]

We claim next that \(\delta V = 0\). To see this, let \(X\) be a \(C^1\) vector field, so that
\[ \delta V(X) = \lim_{i \to \infty} \delta V_i(X) \]
\[ = \lim_{i \to \infty} \int_M (2 - p_i)|\alpha_i|^{p_i} \langle I - 2|\alpha_i|^{-2} \alpha_i \otimes \alpha_i, \nabla X \rangle \]
\[ = \lim_{i \to \infty} \int_M (2 - p_i)|\alpha_i|^{p_i} \text{div}(X) - (2 - p_i)\langle S_i, \nabla X \rangle. \]
Appealing once more to (5.7), we then see that
\[ \delta V(X) = \lim_{i \to \infty} \int_M (2 - p_i)(|du_i|^{p_i} - |h_i|^{p_i}) \text{div}(X) - (2 - p_i)2|S_i - S_i^0, \nabla X \rangle \]
\[ = \lim_{i \to \infty} (2 - p_i) \int_M |du_i|^{p_i} \text{div}(X) - 2|du_i|^{p_i-2} du_i^* du_i, \nabla X \rangle \]
\[ - \lim_{i \to \infty} \int_M |\bar{h}_i|^{p_i} \text{div}(X) - 2(|\bar{h}_i|^{p_i-2} \bar{h}_i \otimes \bar{h}_i, \nabla X \rangle. \]
Now, it’s clear that
\[
\lim_{i \to \infty} \int_M |\vec{h}_i|^{p_i} d\nu(X) - 2\langle |\vec{h}_i|^{p_i-2}\vec{h}_i \otimes \vec{h}_i, \nabla X \rangle = \int_M |\vec{h}|^2 d\nu(X) - 2\langle \vec{h} \otimes \vec{h}, \nabla X \rangle = 0,
\]
since \(d\nu(|\vec{h}|^2 I - 2\vec{h} \otimes \vec{h}) = 0\) for harmonic \(\vec{h}\). On the other hand, we know from the \(p_i\)-stationarity of \(u_i\) that
\[
\int_M |d\alpha_i|^{p_i} d\nu(X) - p_i \langle |d\alpha_i|^{p_i-2}d\alpha_i \otimes d\alpha_i, \nabla X \rangle = 0,
\]
and consequently
\[
|\delta V(X)| = |\lim_{i \to \infty} (2 - p_i) \int_M (p_i - 2) \langle |d\alpha_i|^{p_i-2}d\alpha_i \otimes d\alpha_i, \nabla X \rangle |
\leq \lim_{i \to \infty} (2 - p_i) \Lambda |\nabla X| C^0
= 0,
\]
as claimed.

Since \(\nu = \|V\|\) for a generalized \((n - 2)\)-varifold \(V\) with \(\delta V = 0\), it will follow from Proposition 5.1 that \(\nu\) is indeed the weight measure of a stationary, rectifiable \((n - 2)\)-varifold, once we show that \(\nu\) satisfies
\[
\Theta^{\ast}_{n-2}(\nu, x) > 0 \text{ for all } x \in \text{spt}(\nu).
\]
In particular, to complete the proof of Theorem 1.1, it now suffices to establish the following lemma.

**Lemma 5.3** The support \(\text{spt}(\nu)\) of \(\nu\) is given by
\[
\text{spt}(\nu) = \Sigma = \lim_{i \to \infty} \text{Sing}(u_i), \tag{5.11}
\]
and for \(x \in \Sigma\), the density of \(\nu\) satisfies the lower bound
\[
\Theta^{\ast}_{n-2}(\nu, x) \geq 2\pi. \tag{5.12}
\]

**Proof** To establish (5.11), first consider \(x \in M \setminus \Sigma\), and set \(\delta = \text{dist}(x, \Sigma)\). By definition of Hausdorff convergence, it follows that
\[
\text{dist}(x, \text{Sing}(u_i)) > \frac{\delta}{2},
\]
and consequently
\[
B_{\delta/4}(x) \subset M \setminus N_{\delta/4}(\text{Sing}(u_i)),
\]
for \(i\) sufficiently large. Appealing once more to the estimates of Lemma 4.2, we then see that
\[
\nu(B_{\delta/4}(x)) \leq \lim_{i \to \infty} \inf (2 - p_i) \int_{B_{\delta/4}(x)} |\alpha_i|^{p_i}
\leq \lim_{i \to \infty} (2 - p_i) \frac{C(M, \Lambda)}{\delta^2}
= 0,
\]
so that \( x \notin \text{spt}(v) \); and since \( x \in M \setminus \Sigma \) was arbitrary, we therefore have
\[
\text{spt}(v) \subset \Sigma.
\]

Next, for \( x \in \Sigma \), we’ll show that
\[
\Theta^*_n(x, \nu) \geq 2\pi.
\]
Indeed, if \( x \in \Sigma \), then by definition there is a sequence \( x_i \in \text{Sing}(u_i) \) for which
\[
x = \lim_{i \to \infty} x_i.
\]
By Lemma 3.1, at each \( x_i \), we have
\[
\lim_{r \to 0} (2 - p_i) \int_{B_r(x_i)} |du_i|^{p_i} \geq 2\pi c(n, p_i),
\]
where \( c(n, p_i) \to \omega_{n-2} \) as \( p_i \to 2 \). In particular, fixing \( \delta > 0 \) and appealing to the monotonicity of the \( p \)-energy (Lemma 2.5), we conclude that
\[
\mu(B_\delta(x)) \geq \liminf_{i \to \infty} \mu_i(B_{\delta^2}(x_i)) \geq \lim_{i \to \infty} e^{-C(M)\delta^2} 2\pi c(n, p_i)(\delta - \delta^2)^{n-p_i} = e^{-C(M)\delta^2} 2\pi \omega_{n-2}(\delta - \delta^2)^{n-2}.
\]
Dividing through by \( \omega_{n-2}\delta^{n-2} \) and letting \( \delta \to 0 \), we arrive at the desired lower bound (5.13).

Finally, since the difference
\[
\mu - \nu = |\bar{h}|^2 dv_g
\]
clearly satisfies
\[
\lim_{\delta \to 0} \delta^{2-n} \int_{B_\delta(x)} |\bar{h}|^2 dv_g = 0,
\]
we see that (5.13) yields directly the desired density bound (5.12) for \( \nu \) on \( \Sigma \). Moreover, it follows immediately from (5.12) that \( \Sigma \subset \text{spt}(v) \), and since we’ve already shown that \( \text{spt}(v) \subset \Sigma \), this completes the proof of (5.11) as well.

With the proof of Theorem 1.1 completed, we turn our attention now to the proof of Theorem 1.2, concerning compactness of the maps. Suppose that our sequence \( u_i \in W^{1, p_i}(M, S^1) \) either satisfies the additional bound
\[
\sup_i \|du_i\|_{L(M)} \leq C,
\]
or that the first Betti number \( b_1(M) = 0 \). In either case, it follows that the harmonic component \( h_i \) of \( j u_i \) is uniformly bounded
\[
\sup_i \|h_i\|_{L_\infty} \leq C
\]
as \( i \to \infty \). Together with the \( L_q \) estimates of Proposition 4.1, this implies immediately that
\[
\limsup_{i \to \infty} \|du_i\|_{L_q} < \infty
\]
for any \( q \in [1, 2) \), so some subsequence of \( \{u_i\} \) must converge weakly in \( W^{1,q}(M, S^1) \) to some limiting map \( v \). Moreover, since (by Proposition 4.1) the exact component \( \partial_i \) of \( ju_i \) vanishes in \( L^q \) as \( i \to \infty \), it follows that

\[
d^* j u_i \to 0
\]

weakly as \( i \to \infty \), so the map \( v \) must satisfy \( d^* j v = 0 \) distributionally.

Moreover, combining (5.15) with Lemma 4.2, it follows that, away from any \( 0 < \delta \)-neighborhood \( N_\delta(\Sigma) \) of \( \Sigma \), we have

\[
\limsup_{i \to \infty} \int_{M \setminus N_\delta(\Sigma)} |du_i|^2 < \infty;
\]

and putting this together with the local \( W^{2,p} \) estimate of Corollary 2.3, we see that

\[
\limsup_{i \to \infty} \|u_i\|_{W^{2,p}(M \setminus N_\delta(\Sigma))} < \infty.
\]

Of course, for \( p > \frac{2n}{n+2} \), Rellich’s theorem gives us compactness of the embedding \( W^{2,p} \hookrightarrow W^{1,2} \); hence, since \( p_i > \frac{2n}{n+2} \) for \( i \) sufficiently large, there is indeed some subsequence of \( \{u_i\} \) which converges strongly in \( W^{1,2}(M \setminus N_\delta(\Sigma), S^1) \) to the limiting map \( v \) identified above. And since \( d^* j v = 0 \) and \( v \in W^{1,2}_{loc}(M \setminus \Sigma, S^1) \), it follows that \( v \) is indeed a strongly harmonic map in \( C^\infty_{loc}(M \setminus \Sigma, S^1) \).

\section{6 Integrality of the concentration measure in dimension 2}

To prove the integrality statement in Theorem 1.3, we first need to record suitable local versions of Theorems 1.1 and 1.2 for maps from the disk \( D_2 \). These results follow easily from the analysis of Sects. 3–5, once we fix a local variant of the Hodge decomposition of Sect. 4. To this end, choose a cutoff function \( \chi \in C^\infty(D_{\delta/3}) \) such that \( \chi \equiv 1 \) on \( D_{\delta/3} \). For a map \( u \in W^{1,p}(D_2, S^1) \), we can then define a two-form \( \xi \) and function \( \varphi \) distributionally by setting

\[
\xi := \Delta_H^{-1}(\chi d(j u))
\]

and

\[
\varphi := \Delta^{-1}(\chi \text{div}(j u)).
\]

Note that \( d\xi = 0 \), since \( \xi \) is a 2-form in dimension \( n = 2 \). In particular, it follows that

\[
dd^* \xi = \Delta_H \xi = \chi d(j u) \quad \text{and} \quad \text{div}(d \varphi) = \Delta \varphi = \chi \text{div}(j u),
\]

so writing \( j u = h + d^* \xi + d \varphi \), we then see that the remaining one-form \( h \) is strongly harmonic on \( D_{\delta/3} \) (and therefore corresponds there to the gradient of a harmonic function). Using this decomposition, we can then argue exactly as in Sects. 4 and 5 to establish the following.

\begin{proposition}
Let \( u_i \in W^{1,p_i}(D_2(0), S^1) \) be a sequence of stationary \( p_i \)-harmonic maps from \( D_2(0) \) to \( S^1 \), for which

\[
\sup_i \int_{D_2(0)} (2 - p_i)|du_i|^{p_i} < \infty.
\]

\end{proposition}
A subsequence of the measures \( \mu_i = (2 - p_i)|du_i|^p \, dv \) then converges weakly in \( (C^0(D_1))^* \) to a measure \( \mu \) of the form
\[
\mu = \sum_{j=1}^k \theta_j \delta_{a_j} + |d\psi|^2(x)dx,
\]
(6.1)
where \( \psi \) is a harmonic function, \( \{a_j\}_{j=1}^k = \Sigma \cap D_1 \) is given by the Hausdorff limit \( \Sigma \) of \( \text{Sing}(u_i) \) in \( D_1(0) \), and \( \theta_j \geq 2\pi \). Moreover, if
\[
\sup_i ||du_i||_{L^1(D_2(0))} < \infty,
\]
(6.2)
then the gradient \( d\psi \) in (6.1) vanishes, and (a subsequence of) \( \{u_i\} \) converges strongly in \( W^{1,2}_{\text{loc}}(D_{3/2}(0) \setminus \Sigma) \) to a map \( v \in C^\infty(D_{3/2} \setminus \Sigma, S^1) \) that is harmonic away from \( \Sigma \).

The proof of Theorem 1.3 begins with the observation that the desired integrality holds when the \( L^1 \) gradient bound (6.2) is in force.

**Proposition 6.2** Let \( u_i \in W^{1,p_i}(D_2(0), S^1) \) be as in Proposition 6.1, and suppose also that (6.2) holds. Passing to a subsequence, let \( v \) be the limiting map \( v = \lim_i u_i \) given by Proposition 6.1. Then the limiting measure \( \mu \) has the form
\[
\mu = \sum_{j=1}^k 2\pi \deg(v, a_j)^2 \delta_{a_j}.
\]

For \( p \)-energy minimizers with respect to a fixed boundary condition, this result follows from the analysis of [19], in which case all of the degrees \( \deg(v, a) \) are either 1 or \(-1\). It is also the immediate analog of the quantization result for 2-dimensional solutions of the Ginzburg–Landau equations in [12], though the proof in the present setting is much simpler.

The reason for the relative simplicity in our setting is the form of the Pohozaev identity. In [12], on their way to demonstrating the quantization of the energy measures \( \mu_\epsilon = \frac{|du_\epsilon(z)|^2}{|\log \epsilon|} \, dz \), Comte and Mironescu appeal to the quantization results of [4,7] for the potential measures \( \frac{W(u_\epsilon(z))}{\epsilon} \, dz \). These quantization results—though by no means trivial—can be derived in a relatively straightforward way from a Pohozaev identity that relates the integral of \( \frac{W(u_\epsilon)}{\epsilon} \) on a disk to the behavior of \( u_\epsilon \) on its boundary. It is then observed in [12] that the quantization of the potential measure gives strong constraints on the way that the degrees of the maps \( u_\epsilon \) can vary around clusters of zeroes (or “vortices”) at different scales, which ultimately give rise to the quantization of the energy measures \( \mu_\epsilon \).

In our setting, the path is much simpler, because the normalized \( (2 - p)|du|^p \) simultaneously plays the roles occupied by the energy and potential measures in the Ginzburg–Landau setting. In particular, we have the following nice Pohozaev-type identity:

**Lemma 6.3** Let \( u \in W^{1,p}(D_2(0), S^1) \) be stationary \( p \)-harmonic on \( D_2(0) \). On any annulus
\[
A_{r_1, r_2}(a) = D_{r_2}(a) \setminus D_{r_1}(a) \subset D_2(0),
\]
we then have
\[
\int_{r_1}^{r_2} \left( \int_{D_r(a)} (2 - p)|du|^p \right) dr = \int_{A_{r_1, r_2}(a)} |z - a| \left( |du|^p - p|du|^{p-2} |du| \left( \frac{z - a}{|z - a|} \right)^2 \right) dz.
\]

**Proof** The identity is simply a repackaging of the monotonicity formula in dimension two: By testing the inner variation equation
\[
\int |du|^p \, d\nu(X) - p|du|^{p-2} \langle du^*, \nabla X \rangle = 0
\]
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against vector fields of the form \( X(z) = \psi(z)(z - a) \) for test functions \( \psi \in C_c^\infty(D_r(a)) \) approximating the characteristic function \( \chi_{D_r(a)} \), we find that

\[
\int_{D_r(a)} (2 - p)|du|^p = r\int_{\partial D_r(a)} |du|^p - p|du|^{p-2}du \left( \frac{z - a}{|z - a|} \right)^2
\]

(6.3)

for almost every \( r \in [r_1, r_2] \). Integrating over \([r_1, r_2]\) then gives the desired equation. \( \square \)

With this identity in hand, we can now argue in the spirit of [4,7] to prove Proposition 6.2:

Proof. (Proof of Proposition 6.2)

Let

\[
\Sigma \cap D_1 = \{ a_1, \ldots, a_k \},
\]

so that the limiting map \( v(z) \) satisfies

\[
d^* (j v) = 0 \text{ and } d(j v) = \Sigma^k_{\ell=1} 2\pi \kappa_\ell \delta_{a_\ell},
\]

where \( \kappa_\ell = \deg(v, a_\ell) \) denotes the degree of \( v \) about \( a_\ell \). Letting \( \bar{v} \) be the map given by

\[
\bar{v}(z) := \prod^k_{\ell=1} \left( \frac{z - a_\ell}{|z - a_\ell|} \right)^{\kappa_\ell},
\]

we observe that

\[
d^* j \bar{v} = 0 \text{ and } d(j \bar{v}) = d(j v),
\]

so the difference \( j v - j \bar{v} \) is strongly harmonic. In particular, it follows that

\[
v = e^{i\varphi} \bar{v}
\]

(6.4)

for some harmonic function \( \varphi \in C^\infty(D_2(0)) \).

Now, set

\[
\delta_0 := \min\{|a_\ell - a_m| \mid 1 \leq \ell < m \leq k\},
\]

so that the density \( \theta_\ell \) of \( \mu \) at \( a_\ell \) is given by

\[
\mu(D_r(a_\ell)) = \theta_\ell
\]

for every \( r \in (0, \delta_0) \). For any \( \delta \in (0, \delta_0) \), it then follows from Lemma 6.3 that

\[
\theta_\ell = \frac{2}{\delta} \int_{\delta/2}^\delta \mu(D_r(a_\ell))dr
\]

\[
= \frac{2}{\delta} \lim_{\delta \to \infty} \int_{\delta/2}^\delta \mu_i(D_r(a_\ell))dr
\]

\[
= \frac{2}{\delta} \lim_{\delta \to \infty} \int_{A_{\delta/2,\delta}(a_\ell)} |z - a_\ell| \left( |du_i|^p - p_1 |du_i|^{p-2} |du_i| \left( \frac{z - a_\ell}{|z - a_\ell|} \right)^2 \right) dz.
\]

On the other hand, we also know that \( u_i \to v \) strongly in \( W^{1,2}(A_{\delta/2,\delta}(a_\ell)) \) and \( \|du_i\|_{L^\infty(A_{\delta/2,\delta}(a_\ell))} \) is uniformly bounded as \( i \to \infty \), so it follows that

\[
\theta_\ell = \frac{2}{\delta} \lim_{\delta \to \infty} \int_{A_{\delta/2,\delta}(a_\ell)} |z - a_\ell| \left( |du_i|^p - p_1 |du_i|^{p-2} |du_i| \left( \frac{z - a_\ell}{|z - a_\ell|} \right)^2 \right) dz
\]

\[
= \frac{2}{\delta} \int_{A_{\delta/2,\delta}(a_\ell)} |z - a_\ell|(|dv|^2 - 2|dv| \left( \frac{z - a_\ell}{|z - a_\ell|} \right)^2 dz.
\]
Since \( v = e^{i\varphi} \tilde{v} \), we can expand \( dv \) as

\[
dv(z) = v(z) \left( i \cdot d\varphi + \sum_{\ell=1}^{k} \kappa_{\ell} \frac{z - a_{\ell}}{|z - a_{\ell}|^2} p_{z - a_{\ell}} \right),
\]
where for \( 0 \neq w \in \mathbb{R}^2 \) we denote by \( p_{w} \) projection onto the line perpendicular to \( w \). In particular, if \( z \in D_{\delta}(a_{\ell}) \) for \( \delta < \frac{\delta_{0}}{2} \), then \( |z - a_{m}| > \delta_{0} - \delta > \frac{\delta_{0}}{2} \) for every \( m \neq \ell \), and it follows that

\[
|d\varphi(z) - \frac{\kappa_{\ell}}{|z - a_{\ell}|} \frac{z - a_{\ell}}{|z - a_{\ell}|} |^{2} p_{z - a_{\ell}} | \leq \|d\varphi\|_{L^{\infty}} + \Sigma_{m \neq \ell} \frac{2|\kappa_{m}|}{\delta_{0}} =: K.
\]

Combining this with the obvious estimate

\[
|d\varphi(z)| \leq K' |z - a_{\ell}| \quad \text{on} \quad D_{\delta}(a_{\ell})
\]
(where \( K' \) of course depends on \( v \)), we see that, on \( D_{\delta}(a_{\ell}) \),

\[
|d\varphi(z)|^{2} - \frac{\kappa_{\ell}^{2}}{|z - a_{\ell}|^{2}} \leq \frac{K''}{|z - a_{\ell}|}
\]
and

\[
|d\varphi(z)|^{2} \leq \frac{K''}{|z - a_{\ell}|}
\]
In particular, on the annulus \( A_{\delta/2,\delta}(a_{\ell}) \), since

\[
\int_{A_{\delta/2,\delta}(a_{\ell})} |z - a_{\ell}| \frac{\kappa_{\ell}^{2}}{|z - a_{\ell}|^{2}} = \pi \kappa_{\ell}^{2} \delta,
\]
we can apply the preceding estimates to our computation of \( \theta_{\ell} \) to conclude that

\[
|\theta_{\ell} - 2\pi \kappa_{\ell}^{2}| = \frac{2}{\delta} \int_{A_{\delta/2,\delta}(a_{\ell})} |z - a_{\ell}| \left( |d\varphi|^{2} - \frac{\kappa_{\ell}^{2}}{|z - a_{\ell}|^{2}} - 2|d\varphi| \left( \frac{z - a_{\ell}}{|z - a_{\ell}|} \right) \right)^{2} dz
\]
\[
\leq \frac{2}{\delta} \int_{A_{\delta/2,\delta}(a_{\ell})} |z - a_{\ell}| \frac{K''}{|z - a_{\ell}|}
\]
\[
= 2\pi K'' \delta.
\]
Since \( \delta > 0 \) was arbitrary, it follows finally that

\[
\theta_{\ell} = 2\pi \kappa_{\ell}^{2},
\]
which is precisely what we wanted to show. \( \square \)

Combining Proposition 6.2 with a simple contradiction argument, and scaling, we can formulate the following lemma:

**Corollary 6.4** For any \( \gamma > 0 \) and \( \Lambda < \infty \), there exists \( q(\gamma, \Lambda) \in (1,2) \) such that if \( p > q \), and \( u \in W^{1,p}(D_{2r}(x), S^{1}) \) is stationary \( p \)-harmonic with

\[
(2 - p)\theta_{p}(u, x, 2r) + \frac{1}{r} \int_{D_{2r}(x)} |du| \leq \Lambda
\]
and \( \text{Sing}(u) \cap D_{2r}(x) \subset D_{r}(x) \), then

\[
\text{dist}((2 - p)\theta_{p}(u, x, r), 2\pi \mathbb{Z}) < \gamma.
\]
We now remove the requirement of a uniform $W^{1,1}$ bound by arguing that, in the general setting of Theorem 1.3, the normalized energy measures $\mu_i$ are negligible on the complement of a collection of disks satisfying the conditions of Corollary 6.4.

**Proof.** (Proof of Theorem 1.3)

Let $u_i \in W^{1, p_i}(D_2(0), S^1)$ be a sequence of stationary $p_i$-harmonic maps on $D_2$. Blowing up at one of the concentration points $a_j \in \Sigma \cap D_1$, we can assume that the measures $\mu_i = (2 - p_i) |du_i|^{p_i} dx$ converge to a multiple of the Dirac mass

$$\mu_i \to \mu = \theta \delta_0$$

at the origin. As in the beginning of the section, consider the local version of the Hodge decomposition

$$ju_i = h_i + \partial^* \xi_i + \partial \varphi_i,$$

where

$$\xi_i := \Delta^{-1}_H (\chi dju_i) = -\langle (\chi d(ju_i))(y), G(x, y) \rangle dx^1 \wedge dx^2$$

and

$$\varphi_i := \Delta^{-1}(\chi d\text{div}(ju_i)) = \langle \chi d\text{div}([1 - |ju_i|^{p-2}]ju_i)(y), G(x, y) \rangle,$$

and $G(x, y) = \frac{-1}{2\pi} \log |x - y|$ is the two-dimensional Green’s function.

Recall that since

$$\text{div}(h_i) = (1 - \chi)d\text{div}(ju_i) = 0$$

and

$$\partial^* dh_i = \partial^* ([1 - \chi]d(ju_i)),$$

$h_i$ is harmonic on the disk $D_{4/3}(0)$ (where $\chi \equiv 1$), and it follows that

$$\|h_i\|_{L^{\infty}(D_{1/2}(0))} \leq C \|h_i\|_{L^{p_i}(D_1(0) \setminus D_{3/4}(0))}. \tag{6.9}$$

Next, consider the $(2 - p_i)^{1/p_i}$-neighborhoods

$$U_i := N_{(2 - p_i)^{1/p_i}}(\text{Sing}(u_i))$$

about the singular sets $\text{Sing}(u_i)$. Our goal now is to show that

$$\lim_{i \to \infty} \mu_i(D_{1/2}(0) \setminus U_i) = 0, \tag{6.10}$$

and that $U_i$ is contained in a finite union of disks satisfying the hypotheses of Corollary 6.4.

To this end, we first observe that, by Corollary 3.4 and Lemma 8.2 of the “Appendix”, $\nabla \xi_i$ satisfies the pointwise bound

$$|\nabla \xi_i|(x) \leq \frac{C}{\text{dist}(x, \text{Sing}(u_i))} \tag{6.11}$$

for $x \in D_1(0)$. Putting this together with the volume estimate of Lemma 3.2, we find that

$$\int_{D_{1/2}(0) \setminus U_i} |\nabla \xi_i|^{p_i} \leq \sum_{j=0}^{\frac{1}{p_i} \lfloor \log_2(2 - p_i) \rfloor} \int_{N_{2^{-j}}(\text{Sing}(u_i)) \setminus N_{2^{-j-1}}(\text{Sing}(u_i))} |du_i|^{p_i} \leq \sum_{j=0}^{\frac{1}{p_i} \lfloor \log_2(2 - p_i) \rfloor} C 2^{-j p_i} \cdot 2^{(j + 1) p_i},$$

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and therefore
\[
\int_{D_{1/2}(0) \setminus U_i} |\nabla \xi_i|^p_i \leq C |\log(2 - p_i)|. \tag{6.12}
\]

For \(d\varphi_i\), the arguments in the proofs of Proposition 4.1 and Lemma 4.2 again yield the local estimates
\[
\|\varphi_i\|_{W^{1,q}(D(0))} \leq C(q)(2 - p_i)^{1 - 1/p_i} |\log(2 - p_i)| \tag{6.13}
\]
for \(q \in (1, p_i)\) and
\[
\|(2 - p_i)^{1/p_i} d\varphi_i\|^2_{L^2(D(0) \setminus U_i)} \leq C(2 - p_i)^{2 - 2/p_i} |\log(2 - p_i)|^2, \tag{6.14}
\]
respectively. In particular, rearranging (6.14) and recalling that \((2 - p_i)^{p_i-2}\) is uniformly bounded as \(i \to \infty\), we see that
\[
\int_{D(0) \setminus U_i} |d\varphi_i|^2 \leq C |\log(2 - p_i)|^2. \tag{6.15}
\]

Now, to estimate \(h_i\), we observe that
\[
\|h_i\|_{L^{p_i}(D_{1/2}(0) \setminus D_{3/4}(0))} \leq \|du_i\|_{L^{p_i}(D(0) \setminus D_{3/4}(0))} + \|d^*\xi_i\|_{L^{p_i}(D(0) \setminus D_{3/4}(0))}
\]
\[
+ \|d\varphi_i\|_{L^{p_i}(D(0) \setminus D_{3/4}(0))}
\leq (2 - p_i)^{-1/p_i} \mu_i(D_{1/2}(0) \setminus D_{3/4}(0))^{1/p_i} + \|d^*\xi_i\|_{L^{p_i}(D(0) \setminus U_i)}
\]
\[
+ \|d\varphi_i\|_{L^{p_i}(D(0) \setminus D_{3/4}(0))}
\leq (2 - p_i)^{-1/p_i} \mu_i(D_{1/2}(0) \setminus D_{3/4}(0))^{1/p_i} + C |\log(2 - p_i)|^{2/p_i}.
\]

Since \(\mu = \lim \mu_i\) vanishes on compact subsets of \(D(0) \setminus \{0\}\) by assumption, it then follows that
\[
\|h_i\|_{L^{p_i}(D_{1/2}(0) \setminus D_{3/4}(0))} \leq o\left(\frac{1}{2 - p_i}\right);
\]
and by (6.9), we therefore have
\[
\|h\|_{L^{\infty}(D_{1/2}(0))} \leq \frac{\delta_i}{2 - p_i}, \tag{6.16}
\]
where \(\lim_{i \to \infty} \delta_i = 0\).

Putting together the estimates (6.12), (6.15), and (6.16), we see finally that
\[
\lim_{i \to \infty} \mu_i(D_{1/2}(0) \setminus U_i) = \lim_{i \to \infty} (2 - p_i) \int_{D_{1/2}(0) \setminus U_i} |du_i|^{p_i}
\]
\[
\leq C \limsup_{i \to \infty} \int_{D_{1/2}(0) \setminus U_i} (2 - p_i)[|d^*\xi_i|^{p_i} + |d\varphi_i|^{p_i} + |h_i|^{p_i}]
\]
\[
\leq C \limsup_{i \to \infty} (2 - p_i)|\log(2 - p_i)|^2 + \delta_i)
\]
\[
= 0,
\]
confirming (6.10).

Next, as in the proof of Lemma 3.2, we know from a simple Vitali covering argument that
\[
U_i \subset \bigcup_{\ell=1}^{k_i} D_{3(2-p_i)^{1/p_i}}(x_i^\ell)
\]
for some \( x_1^i, \ldots, x_{k_i}^i \in \text{Sing}(u_i) \) such that
\[
D_{(2-p_i)^{1/p_i}}(x_1^i) \cap D_{(2-p_i)^{1/p_i}}(x_{m}^i) = \emptyset \text{ when } \ell \neq m,
\]
and it follows from Proposition 3.1 that
\[
[(2 - p_i)^{1/p_i}]^{p_i-2} \mu_i(U_i) \geq 2\pi k_i.
\]
In particular, \( k_i \) is uniformly bounded independent of \( i \), so passing to a subsequence, we can take \( k_i = k \) to be constant. Moreover, setting
\[
\alpha_i^{\ell m} := (2 - p_i)^{-1/p_i} \text{dist}(x_\ell^i, x_m^i),
\]
we can pass to a further subsequence for which the (possibly infinite) limits
\[
\alpha_{\ell m} = \lim_{i \to \infty} \alpha_i^{\ell m}
\]
exist. Relabeling indices if necessary, there is then some \( m_0 \leq k \) such that \( \alpha_{\ell m} = \infty \) for \( 1 \leq \ell < m \leq m_0 \), and for every \( m > m_0 \), there is some \( 1 \leq \ell \leq m_0 \) for which \( \alpha_{\ell m} < \infty \).

Now, let
\[
A := \max\{4 + 2\alpha_{\ell m} \mid \alpha_{\ell m} < \infty\},
\]
\[
r_i := A(2 - p_i)^{1/p_i},
\]
and for \( 1 \leq \ell \leq m_0 \), define the disks
\[
D_{i,\ell} := D_{r_i}(x_\ell^i).
\]
For \( i \) sufficiently large, we then see that
\[
U_i \subset D_{i,1} \cup \cdots \cup D_{i,m_0}, \quad D_{i,\ell} \cap D_{i,m} = \emptyset \text{ if } \ell \neq m,
\]
and
\[
\text{Sing}(u_i) \cap D_{2r_i}(x_\ell^i) \subset D_{r_i}(x_\ell^i).
\]
In particular, since
\[
\limsup_{i \to \infty} (2 - p_i)\theta_{p_i}(u_i, x_\ell^i, 2r_i) \leq 2\theta < \infty \quad (6.17)
\]
by the monotonicity formula, our disks will satisfy the conditions of Corollary 6.4 for some \( \Lambda \), once we show that
\[
\limsup_{i \to \infty} r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |h_i| < \infty. \quad (6.18)
\]
To establish (6.18), we consider separately the components \( h_i, d\psi_i \), and \( d^*\xi_i \) of the local Hodge decomposition. For \( h_i \), we have seen already that
\[
\|h_i\|_{L^\infty} \leq \delta_i^{1/p_i} (2 - p_i)^{-1/p_i} \leq \delta_i^{1/p_i} A_i r_i^{-1},
\]
where \( \delta_i \to 0 \), so that
\[
r_i^{-1} \int_{D_{2r_i}(x_\ell^i)} |h_i| \leq \delta_i^{1/p_i} A_i r_i^{-2} \cdot 4\pi r_i^2 \leq 1 \quad (6.19)
\]
for $i$ sufficiently large. For $d \varphi_i$, recall that, for $q < p_i$,
\[ \|d \varphi_i\|_{L^q(D_i(0))} \leq C(q)(2 - p_i)^{1 - 1/p_i}|\log(2 - p_i)|, \]
so that
\[ r_i^{-1} \int_{D_{2r_i}(x_i^i)} |d \varphi_i| \leq r_i^{-1} \|d \varphi_i\|_{L^q(D_i(0))}(4\pi r_i^2)^{1 - 1/q} \]
\[ \leq r_i^{-1} C(q)(2 - p_i)^{1 - 1/p_i}|\log(2 - p_i)| r_i^{2 - 2/q} \]
\[ \leq C(q)A^{1 - 2/q}(2 - p_i)^{1 - 2/(p_i q)}|\log(2 - p_i)|. \]
For $i$ sufficiently large, we can take $p_i > q = \frac{3}{2}$ in the estimate above, to obtain
\[ \limsup_{i \to \infty} r_i^{-1} \int_{D_{2r_i}(x_i^i)} |d \varphi_i| \leq C(q)A^{1 - 2/q} \lim_{i \to \infty} (2 - p_i)^{1 - 8/9}|\log(2 - p_i)| \]
\[ = C \lim_{i \to \infty} (2 - p_i)^{1/9}|\log(2 - p_i)| \]
\[ = 0. \]

Next, employing the pointwise gradient estimate (6.11) for $\xi_i$ with Lemma 3.2, we see that
\[ r_i^{-1} \int_{D_{2r_i}(x_i^i)} |d^* \xi_i| \leq r_i^{-1} \int_{N_{2r_i}(Sing(u_i))} |d^* \xi| \]
\[ = r_i^{-1} \Sigma_{j=0}^{\infty} \int_{N_{2^{1-j} r_i}(Sing(u_i)) \setminus N_{2^{1-j} r_i}(Sing(u_i))} |d^* \xi| \]
\[ \leq r_i^{-1} \Sigma_{j=0}^{\infty} \int_{N_{2^{1-j} r_i}(Sing(u_i)) \setminus N_{2^{1-j} r_i}(Sing(u_i))} C \frac{1}{2-j r_i} \]
\[ \leq C r_i^{p_i-2} \Sigma_{j=0}^{\infty} (2^{1-p_i})^j. \]
And since
\[ \Sigma_{j=0}^{\infty} (2^{1-p_i})^j = \frac{2^{p_i}}{2^{p_i} - 2} \to 2 \]
and
\[ r_i^{p_i-2} = A^{p_i-2} (2 - p_i) \frac{p_i-2}{p_i} \to 1 \]
as $i \to \infty$, it follows that
\[ \limsup_{i \to \infty} r_i^{-1} \int_{D_{2r_i}(x_i^i)} |d^* \xi_i| < \infty. \quad (6.20) \]
Combining this with the preceding estimates for $h_i$ and $d \varphi_i$, we see that (6.18) indeed holds.

Finally, letting
\[ \Lambda := 1 + \max_{1 \leq \ell \leq m_0} \limsup_{i \to \infty} \left[ (2 - p_i)\theta_{p_i}(u_i, x_i^i, 2r_i) + r_i^{-1} \int_{D_{2r_i}(x_i^i)} |du_i| \right], \]

and choosing an arbitrary $\gamma > 0$, it follows from Corollary 6.4 that for $i$ sufficiently large,
\[ \text{dist}((2 - p_i)\theta_{p_i}(u_i, x_i^i, r_i), 2\pi \mathbb{Z}) < \gamma; \quad (6.21) \]
in particular, we deduce that
\[
\limsup_{i \to \infty} \text{dist}((2 - p_i)\sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x^i_\ell, r_i), 2\pi \mathbb{Z}) = 0. \tag{6.22}
\]
Now, by the disjointness of the disks \( \{ D_{i,\ell} \}_{\ell=1}^{m_0} \), we know that
\[
(2 - p_i)\sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x^i_\ell, r_i) = r_i^{-2} \mu_i \left( \bigcup_{\ell=1}^{m_0} D_{i,\ell} \right),
\]
and since \( \lim_{i \to \infty} r_i^{-2} = 1 \), it follows that
\[
\lim_{i \to \infty} (2 - p_i)\sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x^i_\ell, r_i) = \lim_{i \to \infty} \mu_i \left( \bigcup_{\ell=1}^{m_0} D_{i,\ell} \right).
\]
On the other hand, since the disks \( D_{i,\ell} \) cover \( U_i \), we know from (6.10) that
\[
\lim_{i \to \infty} \mu_i(D_{1/2}(0)) = \lim_{i \to \infty} \left( \mu_i(D_{1/2}(0) \setminus \bigcup_{\ell} D_{i,\ell}) + \mu_i \left( \bigcup_{\ell} D_{i,\ell} \right) \right)
\]
\[
= \lim_{i \to \infty} \mu_i \left( \bigcup_{\ell} D_{i,\ell} \right)
\]
\[
= \lim_{i \to \infty} (2 - p_i)\sum_{\ell=1}^{m_0} \theta_{p_i}(u_i, x^i_\ell, r_i).
\]
By (6.22), it then follows that
\[
\theta = \lim_{i \to \infty} \mu_i(D_{1/2}(0)) \in 2\pi \mathbb{N},
\]
as desired. \( \square \)

**Remark 6.5** In higher dimensions—as in the Ginzburg–Landau setting—one would like to show, analogously, that for a sequence \( u_i \in W^{1,p_i}(B^0_n(0), S^1) \) of stationary \( p_i \)-harmonic maps with energy concentrating along an \((n - 2)\)-plane \( \mathcal{L}^{n-2} \), the limiting measure \( \mu \) must have the form
\[
\mu = 2\pi m \cdot \mathcal{H}^{n-2} \mathcal{L}
\]
for some \( m \in \mathbb{N} \). As in the two-dimensional case, it is possible to reduce the problem to the case where the maps \( u_i \) converge away from \( \mathcal{L} \), but the integrality question remains quite difficult after this reduction, due essentially to the interactions between distinct parallel sheets of the singular set \( \text{Sing}(u_p) \) separated by distances \( \sim t^{\frac{1}{p-2}} \) for \( t \in (0, 1) \).

### 7 Natural min–max constructions

#### 7.1 Generalized Ginzburg–Landau functionals

For \( 1 < p < 2 \) and \( \epsilon > 0 \), consider now the \( p \)-Ginzburg–Landau functionals
\[
E_{p,\epsilon} : W^{1,p}(\mathbb{R}^2) \to \mathbb{R}
\]
defined on complex-valued maps by

$$E_{p,\epsilon}(u) := \int_M (|du|^p + \epsilon^{-p} F(u)), \quad (7.1)$$

where $F(u) = (1 - |u|^2)^2$. Note that as $\epsilon \to 0$, the potential term in the energies $E_{p,\epsilon}$ penalizes deviation from the unit circle $S^1 \subset \mathbb{R}^2$, while for $S^1$-valued maps $u$, one simply recovers the $p$-energy $E_{p,\epsilon}(u) = E_p(u)$.

In [38], C-Y Wang studies the asymptotic behavior as $\epsilon \to 0$ for sequences of critical points $u_\epsilon$ of the functionals $E_{p,\epsilon}$ and more general functionals of this type, which approximate by penalization the $p$-energy of maps taking values in a submanifold $N \subset \mathbb{R}^L$ of some Euclidean space. (In the important special case $p = 2$, this analysis was developed in the earlier papers [9,10,26].) Though some of the details of [38] should be taken with care (e.g., [38] posits the existence of a convex function $\lambda \in C^\infty(\mathbb{R})$ satisfying $\lambda'(t) = 1$ for small values of $t$ and $\lambda'(t) = 0$ for $t$ large), it is not difficult to check that the results of [38] apply to the functionals $E_{p,\epsilon} : W^{1,p}(M, \mathbb{R}^2) \to \mathbb{R}$ defined by (7.1). In particular, for sequences of critical points satisfying uniform energy bounds as $\epsilon \to 0$, Wang’s analysis in [38] gives the following strong compactness theorem (compare with Proposition 2.6 for $p$-harmonic maps).

**Theorem 7.1** (cf. [38] Theorem A, Corollary B) If $p \in (1, 2)$, and $\{u_\epsilon\}$ is a sequence of critical points for $E_{p,\epsilon,\lambda}$ with $\epsilon_\epsilon \to 0$ and $\sup_i E_{p,\epsilon,\lambda}(u_\epsilon^{(i)}) < \infty$, then a subsequence of $\{u_\epsilon\}$ converges strongly in $W^{1,p}(M, \mathbb{R}^2)$ to a stationary $p$-harmonic map $u \in W^{1,p}(M, S^1)$.

As a consequence, for $p \in (1, 2)$, the functionals $E_{p,\epsilon}$ are naturally suited to the construction of stationary $p$-harmonic maps via min–max methods, in light of the following elementary lemma.

**Lemma 7.2** The generalized Ginzburg–Landau energy $E_{p,\epsilon}$ is a $C^1$ functional on $W^{1,p}(M, \mathbb{R}^2)$, with derivative

$$\langle E_{p,\epsilon}'(u), v \rangle = \int_M p(|du|^{p-2} du, dv) + \epsilon^{-p} \langle DF(u), v \rangle,$$

and satisfies the following Palais-Smale condition: if $u_j \in W^{1,p}(M, \mathbb{R}^2)$ is a sequence satisfying

$$\sup_j \|u_j\|_{W^{1,p}} \leq C < \infty \quad (7.2)$$

and

$$\lim_{j \to \infty} \|E_{p,\epsilon}'(u_j)\|_{(W^{1,p})^*} = 0, \quad (7.3)$$

then $\{u_j\}$ has a subsequence that converges strongly in $W^{1,p}$.

**Proof** The first statement is trivial. The proof of the Palais-Smale condition is also quite standard, but we include it for completeness:

For a sequence $\{u_j\}$ satisfying (7.2), we know from Rellich’s theorem that a subsequence (which we continue to denote by $\{u_j\}$) converges weakly in $W^{1,p}$ and strongly in $L^p$ to a limiting function $u \in W^{1,p}$. To confirm that the convergence is also strong in $W^{1,p}$, it is enough to show that

$$\|du\|_{L^p} \geq \limsup_j \|du_j\|_{L^p}. \quad (7.4)$$
And indeed, if the \( \{u_j\} \) also satisfies (7.3), then we see that
\[
0 = \lim_{j \to \infty} \|u_j - u\|_{W^{1,p}} \|E'_{p,e}(u_j)\|_{(W^{1,p})^*},
\]
\[
\geq \limsup_{j \to \infty} \langle E'_{p,e}(u_j), u_j - u \rangle
\]
\[
= \limsup_{j \to \infty} p \int_M (|du_j|^p - (|du_j|^{p-2} du_j, du))
\]
\[
+ \epsilon^{-p} \limsup_{j \to \infty} \int_M \langle DF(u_j), u_j - u \rangle
\]
( since \( u_j \to u \) strongly in \( L^p \))
\[
= p \limsup_{j \to \infty} \int_M (|du_j|^p - (|du_j|^{p-2} du_j, du))
\]
\[
\geq \limsup_{j \to \infty} \|du_j\|_{L^p}^p - \|du_j\|_{L^p}^{p-1} \|du\|_{L^p},
\]
from which (7.4) follows, completing the proof. \( \square \)

In the remainder of this section, we employ a simple min–max construction for the energies \( E_{p,e} \), with arguments very similar to those of [35], to prove Theorem 1.4 of the introduction. Though we focus here on \( p \)-harmonic maps to \( S^1 \) with \( p \in (1, 2) \), we remark that similar constructions can presumably be used to produce nontrivial stationary \( p \)-harmonic maps from manifolds of dimension \( n \geq k \) into arbitrary targets \( N \) with \( \pi_{k-1}(N) \neq \emptyset \), for \( p \in (1, k) \setminus \mathbb{N} \).

A detailed study of this general construction is beyond the scope of this paper, but may be an interesting direction for further investigation.

7.2 The saddle point construction and an upper bound for the energies

Consider now the collection \( \Gamma_p(M) \) of two-parameter families \( y \mapsto h_y \in W^{1,p}(M, \mathbb{R}^2) \) given by
\[
\Gamma_p(M) := \left\{ h \in C^0(D^2, W^{1,p}(M, \mathbb{R}^2)) \mid h_y \equiv y \text{ for } y \in S^1 \right\}.
\] (7.5)

For \( p \in (1, 2) \) and \( \epsilon > 0 \), we define the min–max energy levels \( c_{p,e} \) by
\[
c_{p,e}(M) := \inf_{h \in \Gamma_p(M)} \max_{y \in D^2_1} E_{p,e}(h_y),
\] (7.6)
and the limiting energy levels
\[
c_p(M) := \sup_{\epsilon > 0} c_{p,e}(M) = \lim_{\epsilon \to 0} c_{p,e}(M).
\] (7.7)

Now, we’ve observed that \( E_{p,e} \) is a \( C^1 \) functional on \( W^{1,p}(M, \mathbb{R}^2) \), which evidently vanishes on the circle of constant maps to \( S^1 \). Thus, if we can show that \( c_{p,e}(M) > 0 \), then we can apply standard results on the existence of min–max critical points (see, in particular, Chapter 4 and Corollary 5.13 in [15]) to conclude that for any minimizing sequence of families
\[
h^j \in \Gamma_p(M), \max_{y \in D^2_1} E_{p,e}(h^j_y) \to c_{p,e}(M),
\]
there exists a sequence \( v_j \in W^{1,p}(M, \mathbb{R}^2) \) for which
\[
\lim_{j \to \infty} E_{p,e}(v_j) = c_{p,e}(M), \lim_{j \to \infty} \|E'_{p,e}(u)\|_{(W^{1,p})^*} = 0,
\] (7.8)
and
\[
\lim_{j \to \infty} \text{dist}_{W^{1,p}}(v_j, h^j(D_1^2)) = 0.
\]
(7.9)

Moreover, since we can deform any minimizing sequence of families \(h^j\) to one whose maps take values in the unit disk by applying a Lipschitz retraction (similar to \([17,35]\)), we can obtain in this way a sequence \(v_j\) satisfying (7.8) and a uniform bound
\[
\sup_j \|v_j\|_{W^{1,p}} < \infty.
\]

In particular, it will then follow from Lemma 7.2 that there is indeed a critical point \(u_{p,\epsilon}\) of \(E_{p,\epsilon}\) with energy
\[
E_{p,\epsilon}(u_{p,\epsilon}) = c_{p,\epsilon}(M).
\]

To check that \(c_{p,\epsilon}(M) > 0\), we follow arguments similar to ones in \([17,35]\). Namely, for any \(h \in \Gamma_p(M)\), we note that the averaging map
\[
D_1^2 \ni y \mapsto \frac{1}{Vol(M)} \int_M h_y \in \mathbb{R}^2
\]
defines a continuous map from \(D_1^2 \to \mathbb{R}^2\) which restricts to the identity on \(S^1\), so that by elementary degree theory, there must be some \(y_0 \in D_1^2\) for which \(\int_M h_{y_0} = 0\). Now, the \(L^p\) Poincaré inequality gives us a constant \(C_p(M)\) such that
\[
\int_M |v|^p \leq C_p(M) \int_M |dv|^p
\]
whenever \(\int_M v = 0\), so by the preceding observation, for any \(h \in \Gamma_p(M)\), there is some \(y_0 \in D_1^2\) for which \(v = h_{y_0}\) satisfies
\[
E_{p,\epsilon}(v) \geq C_p(M)^{-1} \int_M |v|^p + \int_M \epsilon^{-p} F(v)
\]
\[
\geq C_p(M)^{-1} \int_{|v| \geq \frac{1}{2}} |v|^p + \int_{|v| \leq \frac{1}{2}} \epsilon^{-p} F(v)
\]
\[
\geq C_p(M)^{-1} 2^{-p} Vol \left( \left\{ |v| \geq \frac{1}{2} \right\} \right) + \epsilon^{-p} Vol \left( \left\{ |v| < \frac{1}{2} \right\} \right) \delta_0,
\]
where we’ve set \(\delta_0 := \min\{F(y) \mid |y| \leq \frac{1}{2}\} > 0\). In particular, since
\[
Vol \left( \left\{ |v| \geq \frac{1}{2} \right\} \right) + Vol \left( \left\{ |v| < \frac{1}{2} \right\} \right) = Vol(M),
\]

it follows that, for \(\epsilon < 1\),
\[
\max_{y \in D_1^2} E_{p,\epsilon}(h_y) \geq E_{p,\epsilon}(h_{y_0}) \geq \delta(p, M) > 0
\]
for any family \(h \in \Gamma_p(M)\). Taking the infimum over \(h \in \Gamma_p(M)\), we confirm that
\[
c_{p,\epsilon}(M) \geq \delta(p, M) > 0.
\]

In summary, we’ve so far established that
Lemma 7.3 For \( p \in (1, 2) \) and \( \epsilon \in (0, 1) \), there exists a critical point \( u_{p, \epsilon} \in W^{1, p}(M, \mathbb{R}^2) \) of \( E_{p, \epsilon} \) satisfying
\[
E_{p, \epsilon}(u_{p, \epsilon}) = c_{p, \epsilon}(M) \geq \delta(p, M) > 0.
\]

Next, we will establish an upper bound for the limiting energy levels \( c_p(M) = \sup_{\epsilon > 0} c_{p, \epsilon}(M) \): namely, we show that

Claim 7.4 There exists \( C(M) < \infty \) independent of \( p \) such that
\[
c_p(M) \leq \frac{C(M)}{2 - p}.
\]

The finiteness of \( c_p(M) \) will then allow us to apply Theorem 7.1 to deduce the existence of a corresponding stationary \( p \)-harmonic map, while the boundedness of \((2 - p)c_p(M)\) will provide the upper bound in Theorem 1.4.

Proof Again, the proof is very close to that of the analogous statement in Section 4 of [35], albeit somewhat simpler, since in this case we can produce a single family \( h \) lying in \( \Gamma_p(M) \) for every \( p \in (1, 2) \) and satisfying a bound
\[
\max_{y \in D^2_{1 \setminus S^1}} E_{p, \epsilon}(h_y) \leq \frac{C(M)}{2 - p}, \tag{7.11}
\]
of the desired form.

For \( y \in D^2_{1 \setminus S^1} \), define \( v_y \in \bigcap_{p \in [1, 2)} W^{1, p}(\mathbb{R}^2, S^1) \) by
\[
v_y(z) = \frac{z + (1 - |y|)^{-1} y}{|z + (1 - |y|)^{-1} y|}, \tag{7.12}
\]
and set \( v_y \equiv y \) for \( y \in S^1 \). Fix also a triangulation of \( M^n \)—that is, choose a bi-Lipschitz map \( \Phi : M \rightarrow |K| \) from \( M \) to the underlying space of a simplicial complex \( K \) in some Euclidean space \( \mathbb{R}^L \). Appyling a generic rotation, we can arrange that the projection map \( P \) from \( \mathbb{R}^L \) to the plane \( \mathbb{R}^2 \times 0 \) has full rank on the \( n \)-dimensional subspace parallel to each \( n \)-dimensional simplex \( \Delta \in K \). Denoting by \( f \in Lip(M, \mathbb{R}^2) \) the composition
\[
f(x) = (\Phi^1(x), \Phi^2(x))
\]
of the projection \( P : \mathbb{R}^L \rightarrow \mathbb{R}^2 \times 0 \) with \( \Phi : M \rightarrow \mathbb{R}^L \), we define the family
\[
h_y := v_y \circ f. \tag{7.13}
\]
Our task now is to show that \( y \mapsto h_y \) defines a continuous map \( D^2_1 \rightarrow W^{1, p}(M, \mathbb{R}^2) \), satisfying (7.11). First, since \( \Phi \) is bi-Lipschitz and \( K \) is finite, we observe that it is enough to establish this for the family
\[
F_y := v_y \circ P|_{\Delta}
onumber
\]
on each \( n \)-dimensional face \( \Delta \in K \). And since we’ve also chosen \( K \) such that the restriction \( P_\Delta \) of the projection map to \( \Delta \) has full rank, we can write
\[
P|_{\Delta} = P_0 \circ L,
onumber
\]
where \( L : \Delta \rightarrow \Delta' \subset \mathbb{R}^n \) is an invertible affine-linear map, and \( P_0 : \mathbb{R}^n \rightarrow \mathbb{R}^2 \) is simply the projection
\[
P_0(x^1, \ldots, x^n) = (x^1, x^2)
\]
onto the first two coordinates. In particular, it is enough to show that on a bounded domain \( \Omega \subset \mathbb{R}^n \), the family
\[
y \mapsto F_y(x) = v_y(x^1, x^2)
\]
is continuous in \( W^{1,p} \) for each \( p \in (1, 2) \), and satisfies
\[
\max_{y \in D^2_1} E_{p, \epsilon}(F_y) \leq \frac{C_\Omega}{2 - p}.
\]
(7.14)

This is straightforward. By direct computation, the energy \( E_{p, \epsilon}(F_y) \) on \( \Omega \) satisfies
\[
E_{p, \epsilon}(F_y) = \int_{(z,x') \in (\mathbb{R}^2 \times \mathbb{R}^{n-2}) \cap \Omega} |z + (1 - |y|)^{-1} y|^{-p} dz dx' \\
\leq \int_{x' \in P_1(\Omega)} \frac{2\pi}{2 - p} \text{diam}(\Omega)^{2-p} dx' \\
\leq \frac{C_n \text{diam}(\Omega)^{n-p}}{2 - p},
\]
where \( P_1 : \mathbb{R}^n \to \mathbb{R}^{n-2} \) denotes projection onto the last \( (n-2) \) coordinates. Thus, (7.14) holds, and we see moreover that the energy \( y \mapsto E_{p, \epsilon}(F_y) \) varies continuously in \( y \). Since the family \( y \mapsto F_y \) is obviously weakly continuous in \( W^{1,p}(\Omega, S^1) \), it follows that \( y \mapsto F_y \) is strongly continuous as well.

We conclude finally that the families \( y \mapsto h_y \) defined by (7.13) indeed belong to \( \Gamma_p(M) \), and satisfy
\[
\max_{y \in D^2_1} E_{p, \epsilon}(h_y) \leq \frac{C(M)}{2 - p},
\]
where the constant \( C(M) \) is determined by our choice of triangulation \( \Phi : M \to |\mathcal{K}| \). In particular, it follows that
\[
c_{p, \epsilon}(M) \leq \frac{C(M)}{2 - p}
\]
for every \( \epsilon > 0 \), and taking the supremum over \( \epsilon > 0 \), we therefore have
\[
c_p(M) \leq \frac{C(M)}{2 - p},
\]
(7.15)
as desired. \( \square \)

Since \( c_p(M) < \infty \), we can apply Theorem 7.1 to the min–max critical points \( u_{p, \epsilon} \) of Lemma 7.3, to conclude that

**Proposition 7.5** On every closed Riemannian manifold \( M^n \) of dimension \( n \geq 2 \), there exists for every \( p \in (1, 2) \) a stationary \( p \)-harmonic map \( u_p \in W^{1,p}(M, S^1) \) to \( S^1 \) of energy
\[
0 < E_p(u) = c_p(M) \leq \frac{C(M)}{2 - p}.
\]
7.3 Lower bounds for \( c_p(M) \)

To complete the proof of Theorem 1.4, it remains to show that the min–max energies \( c_p(M) \) satisfy a lower bound of the form

\[
    c_p(M) \geq \frac{c(M)}{2-p}.
\]

(7.16)

To achieve this, we argue as in Section 4 of [35], with Proposition 3.1 again taking on the role played by the \( \eta \)-ellipticity theorem in the Ginzburg–Landau setting.

We begin by observing that (7.16) holds for the round sphere. As discussed in the proof of Proposition 3.1, since \( b_1(S^n) = 0 \), every nontrivial weakly \( p \)-harmonic map \( u \in W^{1,p}(S^n, S^1) \) must have singularities. In particular, the stationary \( p \)-harmonic maps \( u_p \) of energy \( c_p(M) > 0 \) constructed above must have nontrivial singular set, and from Proposition 3.1 and monotonicity, it indeed follows that

\[
    \liminf_{p \to 2} (2-p) c_p(S^n) = \liminf_{p \to 2} (2-p) E_p(u_p) > 0.
\]

(7.17)

The estimate for arbitrary \( M^n \) is then an easy consequence of the following claim:

**Claim 7.6** There is a constant \( C(M^n) < \infty \) such that

\[
    c_{p,\epsilon}(S^n) \leq C(M) c_{p,\epsilon}(M)
\]

(7.18)

for every \( p \in (1, 2) \) and \( \epsilon > 0 \).

**Proof** We will construct a bounded linear map \( \Phi : W^{1,p}(M, \mathbb{R}^2) \to W^{1,p}(S^n, \mathbb{R}^2) \) that fixes the constant maps and satisfies

\[
    E_{p,\epsilon}(\Phi(u)) \leq C(M) E_{p,\epsilon}(u)
\]

for all \( u \in W^{1,p}(M, \mathbb{R}^2) \) and \( p \in (1, 2) \). For any family \( h \in \Gamma_p(M) \), we then see that \( \Phi \circ h \) defines a family in \( \Gamma_p(S^n) \), so that

\[
    c_{p,\epsilon}(S^n) \leq \max_y E_{p,\epsilon}(\Phi(h_y)) \leq C(M) \max_y E_{p,\epsilon}(h_y),
\]

and taking the infimum over \( h \in \Gamma_p(M) \) gives (7.18).

We construct this map \( \Phi \) as follows. First, denote by \( S^+_n \) the northern hemisphere \( S^+_n = \{(x^1, \ldots, x^{n+1}) \in S^n \mid x^{n+1} \geq 0\} \), and consider the reflection map

\[
    R : W^{1,p}(S^+_n, \mathbb{R}^2) \to W^{1,p}(S^n, \mathbb{R}^2)
\]

given by

\[
    (Ru)(x^1, \ldots, x^{n+1}) = u(x^1, \ldots, x^n, |x^{n+1}|).
\]

\( R \) is clearly a bounded linear map which fixes the constants, and has the effect of doubling \( E_{p,\epsilon} \)—i.e.,

\[
    E_{p,\epsilon}(Ru, S^n) = 2 E_{p,\epsilon}(u, S^+_n)
\]

for every \( u \in W^{1,p}(S^+_n) \).

Next, since \( S^+_n \) is a topological ball, we can choose some smooth \( f : S^+_n \hookrightarrow M \) which is a diffeomorphism onto its image. Fixing such an \( f \), we see that the pullback map

\[
    P_f : W^{1,p}(M, \mathbb{R}^2) \to W^{1,p}(S^+_n, \mathbb{R}^2)
\]

\( Springer \)
given by

\[(P_f u) = u \circ f\]

is another bounded linear map that fixes the constant maps, and satisfies

\[E_{p, \epsilon}(P_f u, S^n_+) \leq C(M)E_{p, \epsilon}(u, M).\]

In particular, taking \(\Phi := R \circ P_f\) gives a map \(\Phi : W^{1, p}(M, \mathbb{R}^2) \rightarrow W^{1, p}(S^n, \mathbb{R}^2)\) satisfying the desired properties, confirming the claim. \(\square\)

Finally, taking the supremum over \(\epsilon > 0\) in (7.18), we see that

\[c_p(S^n) \leq C(M)c_p(M).\]

Combining this with (7.17), it follows finally that

\[
\liminf_{p \to 2} (2 - p)c_p(M) \geq C(M)^{-1} \liminf_{p \to 2} (2 - p)c_p(S^n) > 0,
\]

as desired. In particular, putting this together with the conclusion of Proposition 7.5, we arrive finally at the result of Theorem 1.4.

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## 8 Appendix

### 8.1 Proof of Proposition 2.2

In this short section, we demonstrate the independence from the parameter \(p \in [\frac{3}{2}, 2]\) of some standard estimates for \(p\)-harmonic functions (namely, the Lipschitz and \(W^{2, p}\) estimates discussed in Proposition 2.2). This is simply a matter of keeping track of \(p\) in the estimates of [14,23], but we give some details in the interest of completeness.

Let \(B_2(x)\) be a geodesic ball in a manifold \(M^n\) satisfying the sectional curvature bound

\[|\text{sec}(M)| \leq k,\]

and let \(\varphi \in W^{1, p}(B_2(x), \mathbb{R})\) be a \(p\)-harmonic function on \(B_2(x)\) for \(p \in [\frac{3}{2}, 2]\). Recall that, by the convexity of the \(p\)-energy functional, \(\varphi\) must be the unique minimizer for the \(p\)-energy with respect to its Dirichlet data.

For \(\epsilon > 0\), we consider as in [23] the perturbed \(p\)-energy functionals

\[F_\epsilon(\psi) = \int (\epsilon + |d\psi|^2)^{p/2},\]

and let \(\varphi_\epsilon \in W^{1, p}(B_2(x))\) minimize \(F_\epsilon(\psi)\) with respect to the condition \(\psi - \varphi \in W^{1, p}_0(B_2(x))\). Setting

\[\gamma_\epsilon := (\epsilon + |d\varphi_\epsilon|^2)^{1/2},\]

we then have that

\[\text{div}(\gamma_\epsilon^{p - 2}d\varphi_\epsilon) = 0,\]

as desired. In particular, putting this together with the conclusion of Proposition 7.5, we arrive finally at the result of Theorem 1.4.
and by standard results on quasilinear equations of this form (see, e.g., Chapter 4 of [22]), it follows that \( \varphi_\varepsilon \) is a smooth, classical solution of (8.1). Moreover, since \( \varphi \) is the unique \( p \)-energy minimizer with respect to its Dirichlet data, we know that \( \varphi_\varepsilon \to \varphi \) strongly in \( W^{1,p}(B_2) \) as \( \varepsilon \to 0 \). The task now (as in [14,23]) is to establish estimates of the form given in (2.2) for the perturbed solutions \( \varphi_\varepsilon \), and pass them to the limit \( \varepsilon \to 0 \).

As in [23], we observe now that, for \( \varphi_\varepsilon \) solving (8.1), the energy density \( \gamma_\varepsilon^p \) satisfies the divergence-form equation

\[
div(A_\varepsilon \nabla (\gamma_\varepsilon^p)) = p \gamma_\varepsilon^{p-2} [(A_\varepsilon, Hess(\varphi_\varepsilon)^2) + Ric(d\varphi_\varepsilon, d\varphi_\varepsilon)],
\]

where \( Hess(\varphi_\varepsilon)^2 \) denotes the composition

\[
Hess(\varphi_\varepsilon)^2(X, Y) = tr(Hess(\varphi_\varepsilon)(X, \cdot)Hess(\varphi_\varepsilon)(Y, \cdot)),
\]

and

\[
A_\varepsilon := I + (p - 2)\gamma_\varepsilon^{-2} d\varphi_\varepsilon \otimes d\varphi_\varepsilon.
\]

In particular, it follows that

\[
div(A_\varepsilon \nabla (\gamma_\varepsilon^p)) \geq p(p - 1)\gamma_\varepsilon^{p-2} |Hess(\varphi_\varepsilon)|^2 - C(n, k)\gamma_\varepsilon^p.
\]

Now, since \( |\nabla \gamma_\varepsilon| \leq |Hess(\varphi_\varepsilon)| \), when we integrate (8.4) against a test function \( \psi \in C_0^\infty(B_2(x)) \) with \( \psi \equiv 1 \) on \( B_1(x) \) and \( |\nabla \psi| \leq 2 \), we find that

\[
\int B_2 \psi^2 p(p - 1)\gamma_\varepsilon^{p-2} |Hess(\varphi_\varepsilon)|^2 \leq \int B_2 2p\psi |d\psi| \gamma_\varepsilon^{p-1} |\nabla \gamma_\varepsilon| + C(k, n)\gamma_\varepsilon^p
\]

\[
\leq \int B_2 4p\gamma_\varepsilon^{p/2} (\psi \gamma_\varepsilon^{p/2} |Hess(\varphi_\varepsilon)|) + C(k, n)\gamma_\varepsilon^p,
\]

and an application of Young’s inequality yields

\[
p(p - 1) \int B_2 \psi^2 \gamma_\varepsilon^{p-2} |Hess(\varphi_\varepsilon)|^2 \leq \frac{C(k, n)}{(p - 1)} \int B_2 \gamma_\varepsilon^p.
\]

In particular, since Hölder’s inequality gives

\[
\int |Hess(\varphi_\varepsilon)|^p \leq \left( \int |\gamma_\varepsilon^{p-2} |Hess(\varphi_\varepsilon)|^2 \right)^{p/2} \left( \int \gamma_\varepsilon^p \right)^{2-p/2},
\]

it follows that

\[
\|d\varphi_\varepsilon\|_{W^{1,p}(B_1)}^p \leq \frac{C(k, n)}{(p - 1)^2} \int B_2 \gamma_\varepsilon^p,
\]

and since \( p \in \left[ \frac{3}{2}, 2 \right] \), we can rewrite this as

\[
\|d\varphi_\varepsilon\|_{W^{1,p}(B_1)}^p \leq C(k, n) \int B_2 \gamma_\varepsilon^p.
\]

To obtain \( L^\infty \) estimates for \( \gamma_\varepsilon \), we can apply Moser iteration (see, e.g., [16], Chapter 8) to (8.4). Since the eigenvalues of

\[
A_\varepsilon = I + (p - 2)\gamma_\varepsilon^{-2} d\varphi_\varepsilon \otimes d\varphi_\varepsilon
\]

are bounded between \( p - 1 \) and 1, and we are working with \( p \in \left[ \frac{3}{2}, 2 \right] \), it is easy to see that the resulting estimate has the desired form

\[
\|d\varphi_\varepsilon\|_{L^\infty(B_1)}^p \leq \|\gamma_\varepsilon\|_{L^\infty(B_1)}^p \leq C(k, n) \int B_2 \gamma_\varepsilon^p.
\]
Finally, since $\varphi_\epsilon \to \varphi$ strongly in $W^{1, p}(B_2(x))$, we have that
\[
\lim_{\epsilon \to 0} \int_{B_2} \gamma^p_\epsilon = \int_{B_2} |d\varphi|^p,
\]
and it follows from (8.7) and (8.6) that
\[
\|d\varphi\|^p_{L^\infty(B_1)} \leq \liminf_{\epsilon \to 0} \|d\varphi_\epsilon\|^p_{L^\infty(B_1)} \leq C(k, n) \int_{B_2} |d\varphi|^p, \tag{8.8}
\]
and
\[
\|d\varphi\|^p_{W^{1, p}(B_1)} \leq \liminf_{\epsilon \to 0} \|d\varphi_\epsilon\|^p_{W^{1, p}(B_1)} \leq C(k, n) \int_{B_2} |d\varphi|^p. \tag{8.9}
\]
Proposition 2.2 then follows by scaling.

8.2 Proof of Lemma 3.5

In this section, we prove Lemma 3.5, which we employed in the proof of Corollary 3.6. For convenience, we restate the lemma here:

**Lemma 8.1** Let $B_2(x)$ be a geodesic ball in a manifold $M^n$ of sectional curvature $|\sec(M)| \leq k$ and injectivity radius $\text{inj}(M) \geq 3$. Let $S$ be an $(n - 2)$-current in $W^{1, p}(B_2(x))$ satisfying, for some constant $A$,
\[
\langle S, \zeta \rangle \leq Ar^{-p} \|\zeta\|^p_{L^\infty} \|d\zeta\|_{L^\infty}^{2-p} \quad \forall \zeta \in \Omega_c^{n-2}(B_r(y)), \tag{8.10}
\]
for every ball $B_r(y) \subset B_2(x)$. Suppose also that the $r$-tubular neighborhoods $N_r(spt(S))$ about the support of $S$ satisfy
\[
\text{Vol}(B_2(x) \cap N_r(spt(S))) \leq Ar^p. \tag{8.11}
\]
Then there is a constant $C(n, k, A)$ such that for every $1 < q < p$, we have
\[
\|S\|_{W^{1, q}(B_1(x))} \leq C(n, k, A)(p - q)^{-1/q}. \tag{8.12}
\]

Let’s begin now by making some simple reductions. First, let’s assume that $|\sec(M)| \leq 1$, since the result for a general sectional curvature bound $|\sec(M)| \leq k$ then follows from a scaling and covering argument. We may then apply the Rauch comparison theorem to conclude that the given metric $g$ on $B_2(x)$ is uniformly equivalent to the flat one $g_0$, with
\[
C(n)^{-1} g_0 \leq g \leq C(n)g_0
\]
for some constant $C(n)$; and as a consequence, we see that it will suffice to establish the lemma in the flat case. Next, we note that every $(n - 2)$-current $S$ in $B_2^n(0) \subset \mathbb{R}^n$ is described by a finite collection of scalar distributions $S_{ij}$, where
\[
\langle S_{ij}, \varphi \rangle := \langle S, *\varphi dx^i \wedge dx^j \rangle.
\]
Thus, it is enough to show that Lemma 8.1 holds with a scalar distribution $f$ in place of the $(n - 2)$-current $S$. Our first step in proving this is then the following observation:

**Lemma 8.2** For $p \in (1, 2)$, let $f \in W^{-1, p}(B_2^n(0))$ be a distribution on $B_2^n(0)$ satisfying the estimate
\[
\langle f, \varphi \rangle \leq Ar^{n-p} \|\varphi\|^p_{L^\infty} \|d\varphi\|_{L^\infty}^{2-p} \quad \forall \varphi \in C_c^\infty(B_r(x)) \tag{8.13}
\]
for every ball \( B_r(x) \subset B_2^n \). Fixing a cutoff function \( \chi \in C^\infty_c(B_{5/3}(0)) \) such that \( \chi \equiv 1 \) on \( B_{4/3}(0) \), set

\[
w(x) := \langle (\chi f)(y), G(x - y) \rangle,
\]

where \( G \) is the \( n \)-dimensional Euclidean Green's function. We then have for \( x \in B_1(0) \setminus spt(f) \) a pointwise gradient estimate of the form

\[
|dw(x)| \leq C_n A \cdot \text{dist}(x, spt(f))^{-1}.
\]

**Proof** For \( x \in B_1 \setminus spt(f) \), we observe that the pointwise derivatives \( w_i(x) := \partial_i w(x) \) are well-defined, and given by

\[
w_i(x) := \partial_i w(x) = c_n \langle (\chi f)(y), |x - y|^{-n}(x - y)_i \rangle,
\]

where \( c_n \) is a dimensional constant.

To establish (8.14), first choose a function \( \zeta \in C^\infty_c([\frac{3}{4}, \frac{3}{2}]) \) satisfying

\[
\zeta \equiv 1 \text{ on } \left[ \frac{3}{4}, \frac{3}{2} \right] \text{ and } |\zeta'| \leq 10,
\]

and for \( j \in \mathbb{Z} \), set

\[
\zeta_j(t) := \zeta(2^{-j} t).
\]

Defining

\[
\eta_j(t) := \frac{\zeta_j(t)}{\sum_{k \in \mathbb{Z}} \zeta_k(t)}.
\]

it's easy to see that the functions \( \eta_j \) satisfy

\[
\begin{align*}
spt(\eta_j) & \subset (2^{j-1}, 2^{j+1}), \\
\sum_{j \in \mathbb{Z}} \eta_j(t) & = 1,
\end{align*}
\]

and

\[
|\eta'_j| \leq 30 \cdot 2^{-j}.
\]

Given \( x \in B_1^n \setminus spt(f) \), let \( m = \lceil - \log_2 \text{dist}(x, spt(f)) \rceil \), so that

\[
2^{1-m} \geq \text{dist}(x, spt(f)) \geq 2^{-m}.
\]

Writing

\[
w_i(x) = c_n \langle (\chi f)(y), |x - y|^{-n}(x - y)_i \rangle
\]

\[
= c_n \langle (\chi f)(y), \sum_{j \in \mathbb{Z}} \eta_j(|x - y|)|x - y|^{-n}(x - y)_i \rangle,
\]

and observing that

\[
1 - \sum_{j = -m}^{2} \eta_j(|x - y|) = 0
\]

when \( y \in spt(\chi f) \subset B_{4}(x) \setminus B_{2-m}(x) \), it follows that

\[
w_i(x) = c_n \langle (\chi f)(y), \sum_{j = -m}^{2} \eta_j(|x - y|)|x - y|^{-n}(x - y)_i \rangle
\]

\[
= c_n \sum_{j = -m}^{2} \langle (\chi f)(y), \eta_j(|x - y|)|x - y|^{-n}(x - y)_i \rangle.
\]

Setting

\[
\varphi_j(y) := \chi(y) \eta_j(|x - y|)|x - y|^{-n}(x - y)_i,
\]
we can then use (8.15)-(8.17) to see that
\[ spt(\varphi_j) \subset B_{2^{j+1}}(x), \]
\[ \|\varphi_j\|_{L^\infty} \leq 2^{(j-1)(1-n)}, \]
and
\[ \|d\varphi_j\|_{L^\infty} \leq C_n 2^{-n(j-1)}. \]

By (8.13), it therefore follows that
\[
|\langle f, \varphi_j \rangle| \leq A(2^{j+1})^{n-p} \|\varphi_j\|_{L^\infty}^{-p} \|d\varphi_j\|_{L^\infty}^{-2-p}
\leq C_n A(2^{j+1})^{n-p} \cdot 2^{(j-1)(1-n)(p-1)} \cdot 2^{-n(2-p)(j-1)}
\leq C'_n A 2^{-j}.
\]

Summing from \( j = -m \) to \( j = 2 \), we obtain finally
\[
|w_i(x)| = |c_n \Sigma_{j=-m}^2 \langle f, \varphi_j(y) \rangle|
\leq C_n A \Sigma_{j=-m}^2 2^{-j}
\leq C_n A 2^m
\leq 2C_n A \cdot dist(x, spt(f))^{-1},
\]
giving the desired estimate (8.14).

**Corollary 8.3** Let \( f \in W^{-1,p}(B^n_2(0)) \) be as in Lemma 8.2, satisfying
\[
\langle f, \varphi \rangle \leq A r^{n-p} \|\varphi\|_{L^\infty}^{-p} \|d\varphi\|_{L^\infty}^{-2-p} \quad \forall \varphi \in C_c^\infty(B_r(x)) \tag{8.18}
\]
for every ball \( B_r(x) \subset B_2(0) \). In addition, suppose that the tubular neighborhoods \( N_r(spt(f)) \) about the support of \( f \) satisfy the volume bound
\[
Vol(N_r(spt(f))) \leq Ar^p. \tag{8.19}
\]
Then there is a constant \( C(n, A) < \infty \) depending only on \( n \) and \( A \) such that for every \( q \in (1, p) \), we have the estimate
\[
\|f\|_{W^{-1,q}(B_1(0))} \leq C(n, A)(p - q)^{-1/q}. \tag{8.20}
\]

**Proof** By Lemma 8.2, there exists a function \( w \in W^{1,p}(B^n_2(0)) \) satisfying
\[
\Delta w = f \text{ on } B_{4/3}(0)
\]
and
\[
|dw(x)| \leq \frac{C_n A}{dist(x, spt(f))} \tag{8.21}
\]
for \( x \in B_1(0) \setminus spt(f) \). For any \( \varphi \in C_c^\infty(B_1(0)) \) and \( q \in (1, p) \), we then have
\[
\langle f, \varphi \rangle = \langle \Delta w, \varphi \rangle
= -\int \langle dw, d\varphi \rangle
\leq \|dw\|_{L^q(B_1(0))} \|d\varphi\|_{L^{q'}}.
\]
while, by (8.21) and (8.19), we see that
\[ \int_{B_1(0)} |dw|^q \leq C_n A^q \int_{B_1(0)} \text{dist}(x, \text{spt}(f))^{-q} \]
(since \( \text{dist}(x, \text{spt}(f)) \leq 3 \) on \( B_1(0) \)) \leq 4C_n A^q \int_{B_1(0)} [\text{dist}(x, \text{spt}(f))^{-q} - 4^{-q}] \]
(by the fundamental theorem of calculus) \[ = CA^q \int_{B_1(0)} \int_0^4 qr^{-q-1} dr \]
(by Fubini) \[ \leq CA^{q+1} \int_0^4 r^{p-q-1} dr \]
\[ \leq CA^{q+1} \frac{1}{p-q}. \]
Thus, we indeed have
\[ \langle f, \phi \rangle \leq C(n, A)(p-q)^{-1/q} \|d\phi\|_{L^q}, \]
the desired \( W^{-1,q} \) estimate. \qed

As remarked previously, Lemma 8.1 now follows by applying Corollary 8.3 to the scalar component distributions of the \((n-2)\)-current \( S \).

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