Multipole-expanded soft-collinear effective theory with non-abelian gauge symmetry

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Abstract

In position space the interaction terms of soft-collinear effective theory must be multipole-expanded to obtain interaction terms with homogeneous scaling behaviour. In this note we provide a manifestly gauge-invariant formulation of the theory after this expansion in the presence of non-abelian gauge fields, extending our previous result. We give the effective Lagrangian (including the Yang-Mills Lagrangian for collinear and ultrasoft gluons) and heavy-to-light transition currents to second order in the power expansion, paying particular attention to the field redefinitions that lead to the gauge symmetries of the effective Lagrangian.
1. Introduction. Soft-collinear effective theory (SCET) \[1\] is a rapidly developing framework that allows us to simplify factorization proofs for strong interaction processes which involve very energetic, nearly massless (“collinear”) particles. The effective theory is based on the observation that such processes contain at least the scales $Q$, $Q\lambda$ and $Q\lambda^2$, where $Q$ is the hard scale. Depending on the specific situation, either $Q\lambda$ or $Q\lambda^2$ is identified with the strong interaction scale, so that in any case $\lambda \ll 1$.

The possibility to classify interactions in powers of $\lambda$ makes SCET the appropriate framework to discuss power corrections to general hard processes. The effective Lagrangian and certain “currents” were systematically studied to second order in $\lambda$ in a position space formulation of SCET \[2\], which in contrast to the original hybrid momentum-position space representation does not make use of momentum “label” operators. However, in position space fields have to be Taylor-expanded in directions where they vary slowly, to obtain operators that scale with a definite power in $\lambda$. (We will refer to this as “homogeneous” scaling behaviour as opposed to operators that in addition to their leading $\lambda$ behaviour contain a series of suppressed terms.) In \[2\] this “multipole” expansion has been performed, but a manifestly gauge-invariant form of the effective Lagrangian and “currents” after multipole expansion has been derived only for abelian gauge fields. In this note we complete the construction of the homogeneous version of position-space SCET by extending the previous result to non-abelian gauge fields. The expansion will be performed to second order, but the procedure is sufficiently general to make the construction of higher order terms a straightforward exercise.

We recall \[1, 2\] that the effective theory contains a collinear quark field $\xi$, a collinear gluon field $A_c$, and ultrasoft quark and gluon fields, $q$ and $A_{us}$, respectively. The components of collinear momentum scale as $n_+ p \sim 1$, $p_+ \sim \lambda$, $n_- p \sim \lambda^2$, where $n_\pm^\mu$ are two light-like vectors, $n_+^2 = n_-^2 = 0$ with $n_+ n_- = 2$. In the position space formulation the arguments of collinear fields scale as $n_- x \sim 1$, $x_\perp \sim 1/\lambda$, $n_+ x \sim 1/\lambda^2$. The components of ultrasoft momentum are all of order $\lambda^2$ and hence ultrasoft fields vary only over large distances $x \sim 1/\lambda^2$. The components of collinear and ultrasoft gluon fields scale as the components of the corresponding momentum, while $\xi \sim \lambda$ and $q \sim \lambda^3$.

In \[2\] the effective theory was constructed in two steps. First, fields satisfying the scalings specified above were introduced and the corresponding Lagrangian was derived. This Lagrangian was then multipole-expanded. The effective Lagrangian before multipole expansion is invariant under a collinear and ultrasoft gauge symmetry (defined as gauge symmetries where the gauge transformation $U(x)$ has the same $x$ variations as collinear or ultrasoft fields) given by

\[
\begin{align*}
\text{collinear:} & \quad A_c & \to & U_c A_c U_c^\dagger + \frac{i}{g} U_c \left[ D_{us}, U_c^\dagger \right], & \xi & \to & U_c \xi, \\
& \quad A_{us} & \to & A_{us}, & q & \to & q, \\
\text{ultrasoft:} & \quad A_c & \to & U_{us} A_c U_{us}^\dagger, & \xi & \to & U_{us} \xi, \\
& \quad A_{us} & \to & U_{us} A_{us} U_{us}^\dagger + \frac{i}{g} U_{us} \left[ \partial, U_{us}^\dagger \right], & q & \to & U_{us} q.
\end{align*}
\]
Note while $A_c + A_{\text{us}}$ transforms as the gauge field $A$ in full QCD under both gauge symmetries, $\xi + q$ does not transform as the full QCD quark field $\psi$ under the collinear gauge symmetry. The reason for this is that the effective Lagrangian is obtained after integrating out the two small components of the collinear quark spinor and after applying the equation of motion for $\xi$. The applications of the equation of motion are equivalent to a redefinition of the $\xi$ field, after which the relation $\psi = F(\xi, q, A_c, A_{\text{us}})$ is non-linear and non-local. On the other hand, the gauge field equation of motion is not used, so that the relation between the effective and full gluon fields remains simple. In [2] this procedure based on a successive use of the equations of motion for $\xi$ has been used to determine the gauge-invariant effective Lagrangian including second-order power-suppressed interactions. Similarly, the relation between the QCD field and the effective fields, $F$, has been constructed to order $\lambda^2$. Then, if $\xi$ and $q$ transform according to (1), $F(\xi, q, A_c, A_{\text{us}})$ transforms as the full QCD field up to corrections of order $\lambda^3$ and higher.

We may now extend the result of [2] to all orders in $\lambda$ by making a further field redefinition (affecting only terms in the action smaller than $\lambda^2$) such that $F(\xi, q, A_c, A_{\text{us}})$ transforms as the full QCD field exactly. Such a redefinition can always be made. In the approach of [2] this corresponds to further applications of the equation of motion for $\xi$, although the explicit construction can become rather difficult. After this redefinition the QCD fields are related to the effective fields by

$$\begin{align*}
A &= A_c + A_{\text{us}}, \\
\psi &= \xi + WZ^\dagger q - \frac{1}{i n_+ D} \frac{\eta_+}{2} \left( \sigma^\perp_\perp \xi + \left[ \left[ \sigma^\perp_\perp WZ^\dagger \right] q \right] \right) + \bar{\psi}_i D^\perp_{\text{us}} q_i
\end{align*}$$

(2,3)

exactly. Here $W$ and $Z$ are Wilson lines,

$$WZ^\dagger(x) \equiv P \exp \left( ig \int_{-\infty}^{0} ds n_+ A(x + sn_+) \right) \bar{P} \exp \left( -ig \int_{-\infty}^{0} ds n_+ A_{\text{us}}(x + sn_+) \right),$$

(4)

where $P$ denotes path-ordering and $\bar{P}$ reverse path-ordering. We also used the “double-bracket”, defined by

$$[[f(D)A]] \equiv f(D)A - Af(D_{\text{us}}), \quad [[Af(D)]] \equiv Af(D) - f(D_{\text{us}})A,$$

(5)

and $iD = i\sigma + g(A_c + A_{\text{us}})$, $iD_{\text{us}} = i\sigma + gA_{\text{us}}$. Then, if the effective fields transform according to (1), $A$ and $\psi$ transform as the gauge and quark field in full QCD under collinear and ultrasoft gauge transformations.

The SCET Lagrangian before multipole expansion given in [2] can now be derived very easily and extended to all orders in $\lambda$ by inserting the field redefinitions (2,3) into the QCD Lagrangian. We obtain

$$\begin{align*}
\mathcal{L} &= \bar{\xi} \left( i n_\perp D + i \sigma^\perp_\perp \frac{1}{i n_+ D} i \sigma^\perp_\perp \right) \frac{\eta_+}{2} \xi + \bar{q} i \sigma^\perp_\perp q + \bar{\xi} \left[ \left[ \sigma^\perp_\perp WZ^\dagger \right] q + \left[ \sigma^\perp_\perp WZ^\dagger q \right] \right] \\
&\quad + \bar{\xi} \frac{\eta_+}{2} \left[ \left[ \sigma^\perp_\perp WZ^\dagger \right] q \right] + \bar{\xi} \frac{\eta_+}{2} i \sigma^\perp_\perp \frac{1}{i n_+ D} \left[ \left[ \sigma^\perp_\perp WZ^\dagger \right] q \right]
\end{align*}$$

2
To derive this one must use “momentum conservation”, which means that terms with a single collinear field (multiplied by ultrasoft fields) can be dropped, since they do not contribute to the action. In [2] momentum conservation has been used to drop terms of the form $\bar{\xi} iD\gamma_{\perp}q$. This is inconvenient, because single collinear fields can be transformed into composite collinear fields by collinear gauge transformations. Manifest gauge invariance is then only restored after further applications of the equations of motion as can be seen in [2]. We can render this procedure manifestly gauge invariant by choosing a reference gauge in which momentum conservation is applied. Then, in any gauge, we first go to the reference gauge by applying the gauge transformation $U_c$, use momentum conservation in this gauge, and then transform back to the original gauge with $U_c^\dagger$. We choose collinear light-cone gauge as our reference gauge, in which case $U_c = ZW\dagger$. Then momentum conservation allows us to add or drop terms such as

$$+ \bar{q} \left[ [ZW\dagger iD] \right] \frac{n_+}{2} \xi + \bar{\tilde{q}} \left[ [ZW\dagger i\tilde{D}\gamma_{\perp}] \right] \frac{1}{in_+D} i\tilde{D}\gamma_{\perp} \frac{n_+}{2} \xi$$

$$- \bar{q} \left[ [ZW\dagger i\tilde{D}\gamma_{\perp}] \right] \frac{n_+}{2} \frac{1}{in_+D} \left[ [i\tilde{D}\gamma_{\perp}WZ\dagger] \right] q. \quad (6)$$

The last term in (6) and an additional term $\bar{\xi} WZ\dagger iD\gamma_{\perp}q + \text{h.c.}$, contained in the double brackets, have not been given in [2], because they are $\lambda^4$ corrections. With the addition of these terms the effective Lagrangian is exact to all orders in $\lambda$, and this may be considered an advantage of this formalism. (The pure gluon Lagrangian is at this stage the same as in full QCD.) However, the individual terms in the Lagrangian do not have a simple $\lambda$ scaling due to the presence of $A_c$ and $A_{\text{us}}$ in the covariant derivative and in $W$, although their largest component is easily determined. The inhomogeneity is inevitable at this stage, because the gauge transformations are not homogeneous. The inhomogeneity arises form the presence of $A_{\text{us}}$ in the collinear transformation of $A_c$, and from the multiplication of collinear fields with the ultrasoft function $U_{\text{us}}(x)$. However, the exact Lagrangian (6) serves as a starting point for the multipole expansion.

2. Homogeneous gauge transformations. The effective theory should be constructed such that every term has a simple (homogeneous) scaling behaviour. In addition to expanding quantities such as $ZW\dagger$ and $(in_+D)^{-1}$ in (6) one must account for the fact that momentum is not conserved at collinear-ultrasoft interaction vertices. To be specific, when an incoming collinear line with momentum $p$ absorbs an ultrasoft momentum $k$, the outgoing collinear line has momentum $p + \frac{1}{2}(n_-k)n_+$. The components of $k$ small relative to those of $p$ are neglected in the propagator; the corresponding terms in the expansion of the full propagator are part of interaction vertices. In position space this corresponds to the Taylor-expansion of ultrasoft fields around $x_- \equiv \frac{1}{2}(n_+x)n_-$, whenever they multiply
collinear fields, since in such a product the $x$ variations are dominated by the variations of collinear fields. (The procedure is analogous to the familiar multipole expansion in atomic physics, and in non-relativistic effective field theory \[3\], where the role of light-front time $x_-$ is taken by real time $t$.) We therefore perform the “light-front multipole expansion” \[2\]

$$
\phi_{\text{us}}(x) = \phi_{\text{us}}(x_-) + \left[ x_\perp \partial \phi_{\text{us}} \right](x_-) + \frac{1}{2} n_- x \left[ n_+ \partial \phi_{\text{us}} \right](x_-) + \frac{1}{2} \left[ x_\mu \partial_\mu \partial^\nu \phi_{\text{us}} \right](x_-) + \mathcal{O}(\lambda^3 \phi_{\text{us}}).
$$

(8)
of all ultrasoft fields, where $x_- = \frac{1}{2} (n_+ x) n_-$. The expanded effective Lagrangian is homogeneous in $\lambda$, but it is obviously no longer invariant term by term under the gauge transformations \(1\), since they mix different orders in $\lambda$.

For abelian gauge fields it was shown \[2\] that the effective Lagrangian does assume a gauge invariant form after several applications of the equation of motion for $\xi$, after which the collinear quark field $\xi$ transforms with $U_{\text{us}}(x_-)$ under ultrasoft transformations $U_{\text{us}}(x)$. The applications of the equation of motion for $\xi$ are equivalent to the field redefinition

$$
\xi(x) = \exp \left( ig \int_C dy \mu A^\mu_{\text{us}}(y) \right) \hat{\xi}(x),
$$

(9)

where $C$ denotes a straight path from $x_-$ to $x$. The new field $\hat{\xi}(x)$ has the homogeneous transformation $\hat{\xi}(x) \to U_{\text{us}}(x_-) \hat{\xi}(x)$, guaranteeing the term-by-term gauge invariance of the multipole-expanded Lagrangian. The abelian case is simple, because collinear gauge transformations are homogeneous in $\lambda$ for abelian gauge fields and $A_c$ does not transform under ultrasoft gauge transformations. Hence, for abelian fields the only inhomogeneity in \(1\) comes from $U_{\text{us}}$ when it multiplies the collinear field $\xi$.

From this it is clear that in the non-abelian case we must find new collinear fields $\hat{\xi}$ and $\hat{A}_c$, such that the Lagrangian expressed in terms of the new field variables is invariant under the homogenized version of the gauge symmetries, given by

\begin{align*}
\text{collinear:} \\
n_+ \hat{A}_c & \to U_c n_+ \hat{A}_c U^\dagger_c + \frac{i}{g} U_c \left[ n_+ \partial, U^\dagger_c \right], \\
\hat{A}_\perp & \to U_c \hat{A}_\perp U^\dagger_c + \frac{i}{g} U_c \left[ \partial_\perp, U^\dagger_c \right], \\
n_- \hat{A}_c & \to U_c n_- \hat{A}_c U^\dagger_c + \frac{i}{g} U_c \left[ n_- D_{\text{us}}(x_-), U^\dagger_c \right], \\
A_{\text{us}} & \to A_{\text{us}}, & q & \to q,
\end{align*}

(10)

\begin{align*}
\text{ultrasoft:} \\
\hat{A}_c & \to U_{\text{us}}(x_-) \hat{A}_c U^\dagger_{\text{us}}(x_-), & \hat{\xi} & \to U_{\text{us}}(x_-) \hat{\xi}, \\
A_{\text{us}} & \to U_{\text{us}} A_{\text{us}} U^\dagger_{\text{us}} + \frac{i}{g} U_{\text{us}} \left[ \partial, U^\dagger_{\text{us}} \right], & q & \to U_{\text{us}} q.
\end{align*}

It is easily checked that every term now has the same scaling in $\lambda$ as was required. Since the transformations of ultrasoft fields are unaltered, no redefinition of these fields is needed.
In (10) fields and gauge transformations without argument are taken at \( x \) as in (1), while other arguments are given explicitly.

Note that the collinear Wilson line

\[
W_c(x) \equiv P \exp \left( ig \int_{-\infty}^{0} ds n_+ \hat{A}_c(x + sn_+) \right)
\]

has the simple transformations

\[
W_c \to U_c W_c, \quad W_c \to U_{us}(x-) W_c U_{us}^\dagger(x-),
\]

because the arguments of collinear fields in the path-ordered product correspond to the same \((x + sn_+)_- = x_-\). The other objects with simple transformation properties under (10) are \( \hat{\xi}, q, F_{us}^{\mu\nu}, in_+ \hat{D}_c, i \hat{D}_{\perp c}, in_- \hat{D} \) (but not \( in_- \hat{D}_c \)) and \( i D_{us}^\mu \). (The “hat” indicates that the covariant derivative contains \( \hat{A}_c \), not \( A_c \).) The multipole-expanded Lagrangian will be composed of these objects.

3. Field redefinitions. To find the field variables that lead to the new gauge symmetries, we first fix the collinear gauge symmetry by choosing light-cone gauge for \( A_c \) and \( \hat{A}_c \), such that \( n_+ A_c = n_+ \hat{A}_c = 0 \). We then define

\[
\xi = R \hat{\xi}, \quad A_c = R \hat{A}_c R^\dagger,
\]

where now

\[
R(x) = P \exp \left( ig \int_C dy \mu A^\mu_{us}(y) \right)
\]

with \( C \) a straight path from \( x_- \) to \( x \). Since the distance from \( x_- \) to \( x \) is at most of order \( 1/\lambda \), whereas \( A_{us} \) varies only over distances of order \( 1/\lambda^2 \), we can expand \( R \) in \( \lambda \) using

\[
\int_C dy \mu A^\mu_{us}(y) = \int_0^1 ds (x - x_-)_\mu A^\mu_{us}(x_- + s(x - x_-)) = x_{\perp \mu} A^\mu_{us}(x_-) + \frac{1}{2} n_- x n_+ A_{us}(x_-) + \frac{1}{2} x_{\perp \mu} x_{\perp \nu} [\partial^\nu A^\mu_{us}](x_-) + \ldots,
\]

where after the second equality the first term is of order \( \lambda \) and the other two of order \( \lambda^2 \), and all fields are evaluated at \( x_- \) (after derivatives are taken). It is straightforward to see that the new fields \( \hat{\xi}, \hat{A}_c \) have the required transformations (10) under the ultrasoft gauge symmetry.

We now continue to assume collinear light-cone gauge for \( A_c \), but we restore collinear gauge invariance for the new fields. To this end, we note that if \( \hat{A}_c \) is not in light-cone gauge, a gauge transformation \( U_c = W_c^\dagger \) will transform it to this gauge. Hence, we should replace in (13) \( \hat{\xi} \) by \( W_c^\dagger \hat{\xi} \) and \( g A_{\perp c} \) by \( W_c^\dagger [i \hat{D}_c W_c] \) etc. (since according to (10) these are the quark and gluons fields in collinear light-cone gauge). We then find the following collinear field redefinitions:

\[
\xi = R W_c^\dagger \hat{\xi}, \quad g A_{\perp c} = R \left( W_c^\dagger i \hat{D}_{\perp c} W_c - i \partial_{\perp} \right) R^\dagger, \quad gn_- A_c = R \left( W_c^\dagger in_- \hat{D} W_c - in_- D_{us}(x_-) \right) R^\dagger.
\]
Recall that the fields without hats on the left-hand side are still in light-cone gauge. The quantities on the right-hand side are expressed entirely in terms of the new collinear gluon field $\hat{A}_c$. It is straightforward to verify that the new fields have the required collinear and ultrasoft transformations (10). In particular, when the fields on the right-hand side transform according to the collinear gauge symmetry (10), the expressions in (16) remain invariant as they should, because the collinear gauge symmetry is fixed for the fields on the left-hand side.

4. The multipole-expanded quark Lagrangian. For the remainder of this paper we use the following notation: we drop the “hat” on the new fields, since the multipole-expanded SCET Lagrangian contains only these fields. Collinear fields without argument will be understood to be evaluated at $x$, but ultrasoft fields without arguments are always evaluated at $x_- = \frac{1}{2}(n_+ x) n_-$. Furthermore, derivatives on ultrasoft fields operate on the field before setting $x = x_-; \text{derivatives enclosed in square brackets operate only inside the bracket.}$

The field redefinitions (16) are inserted into (6) taken in collinear light-cone gauge (where $WZ^\dagger = 1$) and the resulting expression is expanded in $\lambda$. To see the sort of terms that arise, we consider the collinear Lagrangian given by the first term in (6), which takes the form

$$\mathcal{L} = \bar{\xi} n_- D^{\frac{\eta_+}{2}} \xi + \bar{\xi} W_c \left( R^\dagger iD^- D_{us}(x) R - in_- D_{us} \right) W_c^{\frac{\eta_+}{2}} \xi$$

$$+ \bar{\xi} \left( i\partial_{\perp c} + W_c \left( R^\dagger i\partial_{\perp us}(x) R - i\partial_{\perp} \right) W_c^{\dagger} \right) \frac{1}{in_+ D_{us}(x)} RW_c^{\dagger}$$

$$\left( i\partial_{\perp c} + W_c \left( R^\dagger i\partial_{\perp us}(x) R - i\partial_{\perp} \right) W_c^{\dagger} \right) \frac{\eta_+}{2} \xi$$

when expressed in the new field variables. (Note that with our conventions $in_- D$ contains the collinear gauge field at $x$ and the ultrasoft gauge field at $x_-).$ From this and similar manipulations of the terms in the Lagrangian with the ultrasoft quark field, we see that we need the expansion of $(R^\dagger iD^- D_{us}(x) R - in_- D_{us}), (R^\dagger i\partial_{\perp us}(x) R - i\partial_{\perp}), R^\dagger (in_+ D_{us}(x))^{-1} R$, $R^\dagger q(x)$ and $i\partial_{\perp} R^\dagger q(x)$ in $\lambda$.

The expansion is constructed most easily by choosing the special gauge

$$(x - x_-)_{\mu} A^\mu_{us}(x) = \left( x_{\perp \mu} + \frac{1}{2} (n_+ x) n_{+\mu} \right) A^\mu_{us}(x) = 0,$$

which by analogy with Fock-Schwinger or fixed-point gauge we will refer to as fixed-line gauge (as $x_-$ depends on $x$ through $n_+ x$). If $A^\mu_{us}(x)$ does not satisfy the gauge condition, we can always choose $U_{us}(x) = R^\dagger$ to transform the field to fixed-line gauge. In fixed-line gauge the gauge field can be represented in terms of the field strength tensor by the relations

$$n_- A_{us}(x) - n_- A_{us} = \int_0^1 ds (x - x_-)_{\mu} n^\mu F^\mu_{us}(y(s)).$$
we can use the ultrasoft gauge transformation in fixed-line gauge. The relation with the expressions above is now established, since

\[ g_{\mu\nu} \rightarrow \partial_{\mu} + \nabla_{\mu} \]

This converts \( F^\mu_{\nu} \) to \( (\partial_{\mu} + \nabla_{\mu}) \partial_{\nu} \). The expansion in terms of covariant derivatives is obtained most directly from the Taylor-expansion one returns to the general gauge, which becomes trivial since \( R(x_\perp) = 1 \). After this expansion one returns to the general gauge, which becomes trivial since \( R(x_\perp) = 1 \). After this expansion every single term has a homogeneous scaling behaviour in \( \lambda \).

With these results it is easy to write down the multipole-expanded SCET Lagrangian
to any order in $\lambda$. To order $\lambda^2$ the result takes the form

$$\mathcal{L} = \bar{\xi} \left( \int_{\gamma_{\perp}} D \bar{c} \right) \left( \frac{1}{m c} \frac{1}{m_{\perp} D} \frac{1}{m_{\perp} D} \right) \frac{1}{2} \gamma_{\perp} \xi + \bar{q}(x) i D_{\perp} c(x) q(x) + \mathcal{L}^{(1)}_{\xi} + \mathcal{L}^{(2)}_{\xi} + \mathcal{L}^{(1)}_{\xi} + \mathcal{L}^{(2)}_{\xi}, \quad (26)$$

where the power-suppressed interaction terms are given by

$$\mathcal{L}^{(1)}_{\xi} = \frac{\xi (x_{\perp}^\mu n_{\perp}^\nu W_c g F^{\mu\nu} W_{\perp}^c)}{2} \frac{\gamma_{\perp}}{2} \xi,$$

$$\mathcal{L}^{(2)}_{\xi} = \frac{\xi (x_{\perp}^\mu n_{\perp}^\nu W_c g F^{\mu\nu} W_{\perp}^c)}{2} \frac{\gamma_{\perp}}{2} \xi + \frac{\xi (x_{\perp}^\mu n_{\perp}^\nu W_c g F^{\mu\nu} W_{\perp}^c)}{2} \frac{\gamma_{\perp}}{2} \xi,$$

$$\mathcal{L}^{(1)}_{\xi} = \bar{q} W_{\gamma_{\perp}}^c \bar{D}_{\perp} c - \bar{q} \xi \bar{D}_{\perp} c W_c q,$$

$$\mathcal{L}^{(2)}_{\xi} = \frac{\xi (x_{\perp}^\mu n_{\perp}^\nu W_c g F^{\mu\nu} W_{\perp}^c)}{2} \frac{\gamma_{\perp}}{2} \xi + \frac{\xi (x_{\perp}^\mu n_{\perp}^\nu W_c g F^{\mu\nu} W_{\perp}^c)}{2} \frac{\gamma_{\perp}}{2} \xi,$$

For abelian gauge fields these expressions coincide with those given in [4]. (The final result (27-31) for the non-abelian case has already been presented in [4].) In this form every term in the Lagrangian scales as a single power in $\lambda$, which can be determined from the scaling rules for fields, coordinates and derivatives. We note that this Lagrangian is exact, i.e. its coefficients are not modified by radiative corrections, neither do radiative corrections induce new operators [3].

5. The Yang-Mills Lagrangian. It remains to perform the $\lambda$ expansion of the pure Yang-Mills Lagrangian. Recall that in the first version of position space SCET, which is invariant under the gauge symmetry (1), the Yang-Mills part of the Lagrangian is the same as in QCD with $A$ replaced by $A_{c} + A_{us}$. Before the field redefinition (16) we can rearrange the Yang-Mills Lagrangian as

$$\mathcal{L}_{YM} = -\frac{1}{2} \text{tr} \left( G^{\mu\nu}_c G_{\mu\nu}^c \right) - \text{tr} \left( G^{\mu\nu}_c F_{\mu\nu}^{us}(x) \right) - \frac{1}{2} \text{tr} \left( F_{\mu\nu}^{us}(x) F_{\mu\nu}^{us}(x) \right) \quad (32)$$

with the definition

$$G^{\mu\nu}_c = [D_{\mu\nu}^{us}(x), A_{c}^\mu] - [D_{\mu\nu}^{us}(x), A_{c}^\mu] - i g [A_{c}^\mu, A_{c}^\mu]. \quad (33)$$

The first two terms of (32) are products of collinear and ultrasoft fields which must be multipole-expanded. The third term is the ultrasoft Yang-Mills Lagrangian, which contributes a leading power term to the action.
As before we go to light-cone gauge for the collinear fields, insert the field redefinitions (10) and expand the result in $\lambda$ choosing ultrasoft fixed-line gauge in an intermediate step. Including terms of order $\lambda^2$ the result is

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2} \text{tr} \left( F_{\mu\nu}^c F_{\mu\nu}^c \right) - \frac{1}{2} \text{tr} \left( F_{\mu\nu}^{\text{us}}(x) F_{\mu\nu}^{\text{us}}(x) \right) + \mathcal{L}_{\text{YM}}^{(1)} + \mathcal{L}_{\text{YM}}^{(2)},$$

(34)

where we define the collinear field strength tensor $F_{\mu\nu}^c$ through its components

$$gn_+ n_- F_{\mu\nu}^c \equiv \left[ n_+ D_c, n_- D \right], \quad gF_{\mu\nu}^c \equiv \left[ D_{\mu\perp}, iD_{\nu\perp}^c \right],$$

$$gn_+ F_{\mu\nu}^c \equiv \left[ n_+ D_c, iD_{\nu\perp}^c \right], \quad gn_- F_{\mu\nu}^c \equiv \left[ n_- D, iD_{\nu\perp}^c \right].$$

(35)

This definition almost coincides with the standard one except that it contains $n_- D$ rather than $n_- D_c$, which is related to the presence of $A_{\text{us}}$ in the collinear transformation of $n_- \hat{A}_c$ in (10). The first and second order power-suppressed gluon self-interactions are given by

$$\mathcal{L}_{\text{YM}}^{(1)} = \text{tr} \left( n_+^\mu F_{\mu\nu\perp}^c W_c [ x_\perp n_\sigma F_{\rho\sigma}^{\text{us}}, W]\left[ iD_{\nu\perp}^c W \right] \right) - \text{tr} \left( n_+^{\mu\nu\perp} W_c n^\rho F_{\rho\mu\perp} W \right),$$

$$\mathcal{L}_{\text{YM}}^{(2)} = \frac{1}{2} \text{tr} \left( n_+^\mu F_{\mu\nu\perp}^c W_c [ x_\perp n_\sigma F_{\rho\sigma}^{\text{us}}, W]\left[ iD_{\nu\perp}^c W \right] \right) - \frac{1}{2} \text{tr} \left( n_+^{\mu\nu\perp} W_c [ x_\perp n_\sigma F_{\rho\sigma}^{\text{us}}, W]\left[ iD_{\nu\perp}^c W \right] \right)$$

$$+ \text{tr} \left( F_{\mu\nu\perp}^c W_c [ x_\perp F_{\rho\mu\perp}, W]\left[ iD_{\nu\perp}^c W \right] \right) + \frac{1}{2} \text{tr} \left( n_+^{\mu\nu\perp} W_c n_\sigma F_{\rho\sigma}^{\text{us}} W \right) - \text{tr} \left( F_{\mu\nu\perp}^c W_c F_{\rho\mu\perp}^{\text{us}} W \right) - \text{tr} \left( n_+^{\mu\nu\perp} W_c n_\sigma F_{\rho\sigma}^{\text{us}} W \right) \right).$$

(36)

(37)

This together with the quark Lagrangian completes the construction of the soft-collinear effective Lagrangian including second-order power corrections. The Yang-Mills effective Lagrangian is not renormalized by hard fluctuations so that the expressions given here again hold to all orders in perturbation theory. One can now specify gauge fixing conditions for the collinear and ultrasoft gauge symmetries and derive the corresponding ghost Lagrangians according to the standard procedure.

6. The heavy-to-light transition current. For completeness we also give the result for the representation of colour-singlet currents $\bar{\psi}\Gamma Q$ in the effective theory, where $\Gamma$ is an arbitrary Dirac matrix. These currents appear in weak decays of heavy quarks $Q$ into light quarks. The matching to SCET is relevant when the light quark carries large momentum of order of the heavy quark mass. In the effective theory we introduce a heavy quark field $h_v$ labelled by the meson velocity. The residual $x$ variations of this field are identical to those of the ultrasoft light quark field, so that $h_v$ must be Taylor-expanded around $x_-$ in products with collinear fields. The derivation of the SCET current to order $\lambda^2$ and
its gauge-invariant expression after multipole-expansion for abelian gauge fields have been given in [3]. With the method presented here we find for the non-abelian case

\[
\left[ \bar{\psi}(x) \Gamma Q(x) \right]_{\text{QCD}} = e^{-i m_\perp x} \left\{ J^{(A0)} + J^{(A1)} + J^{(A2)} + J^{(B1)} + J^{(B2)} \right\}
\]

with

\[
J^{(A0)} = \bar{\xi} \Gamma W_c h_v,
\]

\[
J^{(A1)} = \bar{\xi} \Gamma W_c x_\perp D^\mu_{\text{us}} h_v - \bar{\xi} i D^\perp_{\text{us}} \left( i n_+ \bar{D}_c \right)^{-1} \frac{\eta^+}{2} \Gamma W_c h_v,
\]

\[
J^{(A2)} = \bar{\xi} \Gamma W_c \left( \frac{1}{2} n_\perp x n_+ D^\mu_{\text{us}} h_v + \frac{1}{2} x_\perp x_\nu D^\mu_{\text{us}} D^\nu_{\text{us}} h_v + \frac{i D^\perp_{\text{us}}}{2 m} h_v \right)
\]

\[
- \bar{\xi} \Gamma \frac{1}{i n_+ D_c} \left[ i n_- D W_c - W_c i n_- D_{\text{us}} \right] h_v - \bar{\xi} \frac{i D^\perp_{\text{us}}}{2 m} \left( i n_+ \bar{D}_c \right)^{-1} \frac{\eta^+}{2} \Gamma W_c x_\perp D^\mu_{\text{us}} h_v,
\]

\[
J^{(B1)} = -\bar{\xi} \Gamma \frac{\eta^+}{2 m} \left[ i D_{\text{us}} \Gamma W_c \right] h_v,
\]

\[
J^{(B2)} = -\bar{\xi} \Gamma \frac{\eta^+}{2 m} \left[ i D_{\text{us}} \Gamma W_c \right] x_\perp D^\mu_{\text{us}} h_v - \bar{\xi} \Gamma \frac{\eta^+}{2 m} \left[ i n_- D W_c - W_c i n_- D_{\text{us}} \right] h_v
\]

\[
- \bar{\xi} \Gamma \frac{1}{i n_+ D_c} \left[ i D_{\text{us}}^\perp \Gamma W_c \right] h_v + \bar{\xi} \frac{i D^\perp_{\text{us}}}{2 m} \left( i n_+ \bar{D}_c \right)^{-1} \frac{\eta^+}{2} \Gamma W_c x_\perp D^\mu_{\text{us}} h_v,
\]

which is identical to the result of [3] derived there for the abelian case. Recall that derivatives operate on ultrasoft fields before \( x = x_\perp \) is set, and that derivatives with square brackets act only within the brackets. Contrary to the effective Lagrangian the expansion of the current may be corrected by perturbative effects, although some relations between the various terms in the expansion hold due to reparameterisation invariance.

**Conclusion.** In this note we completed the construction of soft-collinear effective theory in position space in terms of operators with homogeneous power counting in the expansion parameter \( \lambda \). We first gave the exact SCET Lagrangian before multipole expansion and then showed how the multipole-expansion (necessary to make the operators homogeneous in \( \lambda \)) can be constructed in a manifestly gauge invariant form to any order in \( \lambda \). The key ingredient in this construction was to find the collinear field redefinitions, after which the fields transform under homogeneous gauge transformations, and to work out the \( \lambda \) expansion in the intermediate fixed-line gauge. Since the SCET Lagrangian is not renormalized by hard interactions, this allows us in principle to derive the effective Lagrangian to any desired accuracy. For heavy-to-light transition currents the result has been derived to second order in \( \lambda \) and at tree-level. This extends the results of [3], where the manifestly gauge-invariant form of the multipole expansion has only been given for abelian gauge fields, and where the construction was restricted to second order.

**Note added.** When this work was completed, an article by Pirjol and Stewart appeared [4], which addresses the gauge-invariant formulation of power-suppressed interactions in
the hybrid momentum-position space version of SCET. The detailed comparison of our results with this paper is complicated by two facts: first, we do not know the gauge transformations under which the theory is presumed to be invariant. (The transformations given in Table II of the second reference of \[1\] are obviously not appropriate, because they are not homogeneous in $\lambda$, similar to (1)). Second, although the authors now also use the term “multipole expansion”, no multipole expansion as described around (8) is performed in the hybrid representation. To see how the two formulations are related, we note that in the position representation used in the present note the power-suppressed terms in the effective Lagrangian do not contain any two-point vertices (propagator corrections). The explicit factors of $x$ in interaction vertices that come from the multipole expansion act as derivatives on propagators in momentum space and the sum of such terms reconstructs the full QCD propagators attached to collinear-ultrasoft vertices order by order in $\lambda$. On the other hand the hybrid representation does not contain multipole-expanded interaction vertices, but the effective Lagrangian contains power-suppressed two-point interactions \[8\]. Multiple insertions of these interactions (combined with the fact that collinear lines are assigned the full ultrasoft momentum absorbed at a collinear-ultrasoft vertex in the hybrid representation) also lead to the reconstruction of the full propagators. When this dictionary is applied, the expressions for the Lagrangian and heavy-to-light current given in \[3\] seem to agree with those of \[4,5\] and those given here except for the Yang-Mills Lagrangian, where we were unable to establish a relation. We wish to note further that contrary to the claim made in \[5\] the fact that no new operators in the SCET Lagrangian are generated by radiative corrections has already been proven in \[6\] (Section 3.4, after Eq. (63)). The arguments based on reparameterisation invariance given in \[5\] therefore provide an alternative derivation of this result.

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