Jets, Lifts and Dynamics

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Abstract
We show that complete cotangent lifts of vector fields, their decomposition into vertical representative and holonomic part provide a geometrical framework underlying Eulerian equations of continuum mechanics. We discuss Euler equations for ideal incompressible fluid and Vlasov equations of plasma dynamics in connection with the lifts of divergence-free and Hamiltonian vector fields, respectively. As a further application, we obtain kinetic equations of particles moving with the flow of contact vector fields both from Lie-Poisson reductions and with the techniques of present framework.

Keywords: complete cotangent lift, vertical representative, diffeomorphism groups, kinetic equations of contact particles

1 Jets

Let \((\mathcal{E}, \pi, \mathcal{M})\) be a smooth bundle with coordinates \((x^a; 1 \leq a \leq \dim (\mathcal{M}) = m)\) on the base manifold \(\mathcal{M}\) and \((x^a, u^\lambda; 1 \leq \lambda \leq \text{rank } (\pi) = k)\) on the total manifold \(\mathcal{E}\). The vertical bundle associated with \(\pi\) is

\[ V_\pi = \ker T\pi = \{ \xi \in T\mathcal{E} : T\pi (\xi) = 0 \} \]  

(1)

and this is a vector subbundle of the tangent bundle \(T\mathcal{E}\). Here \(T\pi\) denotes the tangent mapping of the projection \(\pi\). Two sections \(\phi, \psi \in \mathfrak{S}(\pi)\) of the bundle \(\pi\) at a point \(x \in \mathcal{M}\) are called equivalent if their tangent mappings are equal at that point, that is, \(T_x \phi = T_x \psi\). Given a point \(x\), an equivalence class containing a section \(\phi\) is denoted by \(j_1^x \phi\). The first order jet manifold

\[ J^1 \pi = \{ j_1^x \phi : x \in \mathcal{M} \text{ and } \phi \in \mathfrak{S}(\pi) \} \]  

(2)

associated with \((\mathcal{E}, \pi, \mathcal{M})\) is the set of equivalence classes at every point \(x \in \mathcal{M}\) with induced coordinates

\[ (x^a, u^\lambda, u^\lambda_\alpha) : J^1 \pi \to \mathbb{R}^{m+k+mk} : j_1^x \phi \to \left( x^a, u^\lambda (\phi (x)), \frac{\partial \phi^\lambda}{\partial x^a} \right) \]  

(3)

We have fibrations \(\pi_0 : J^1 \pi \to \mathcal{E} : j_1^x \phi \to \phi (x)\) and \(\pi_1 : J^1 \pi \to \mathcal{M} : j_1^x \phi \to x\) of \(J^1 \pi\) on \(\mathcal{E}\) and \(\mathcal{M}\), respectively [3], [15].

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Given a differentiable map \( \rho: \mathcal{N} \to \mathcal{M} \) from a manifold \( \mathcal{N} \) to the base manifold \( \mathcal{M} \), the pull-back bundle of \( \pi \) by \( \rho \) is the triple \((\rho^*\mathcal{E}, \rho^*\pi, \mathcal{N})\) where

\[
\rho^*\mathcal{E} = \mathcal{N} \times_{\mathcal{M}} \mathcal{E} = \{(n, e) \in \mathcal{N} \times \mathcal{E} : \pi(e) = \rho(n)\}
\]

(4)
is the Whitney product and, \( \rho^*\pi = pr_1 \) is the projection to the first factor [3]. Consider the pull back bundle

\[
(\pi^*_0(T\mathcal{E}) = J^1\pi \times_{\mathcal{E}} T\mathcal{E}, \pi^*_0\tau_\mathcal{E} = pr_1, J^1\pi)
\]
of \((T\mathcal{E}, \tau_\mathcal{E}, \mathcal{E})\) by the projection \( \pi_0: J^1\pi \to \mathcal{E} \), where \( \tau_\mathcal{E} \) is the tangent bundle projection. A section of \( \pi^*_0\tau_\mathcal{E} \) is called a generalized vector field of order one [15,16]. One may regard a section of \( \pi^*_0\tau_\mathcal{E} \) as a map from \( J^1\pi \) to \( T\mathcal{E} \). We require that generalized vector fields are projectable [6].

In coordinates, a generalized vector field is

\[
\xi \left( j^1_x \phi \right) = \xi^a(x) \frac{\partial}{\partial x^a} + \xi^\lambda \left( j^1_x \phi \right) \frac{\partial}{\partial u^\lambda}_{\phi(x)}
\]

(5)
and its first order prolongation \( pr^1\xi \) is

\[
pr^1\xi = \xi + \Phi^\lambda_a \frac{\partial}{\partial u^\lambda_a}, \quad \Phi^\lambda_a = D\phi^\lambda_a \left( \xi^a - \xi^b u^\lambda_a \right) + \xi^b u^\lambda_a
\]

(6)
where \( D\phi^\lambda_a \) is the total derivative operator with respect to \( x^a \) and, \( u^\lambda_a (j^1 x^a) = \frac{\partial^2 \phi^\lambda}{\partial x^a \partial x^b} \) is an element of the second order jet bundle. Lie bracket of two first order generalized vector fields \( \xi \) and \( \eta \) is the unique first order generalized vector field

\[
[\xi, \eta]_{pro} = \left( pr^1\xi (\eta^a) - pr^1\eta (\xi^a) \right) \frac{\partial}{\partial x^a} + \left( pr^1\xi (\eta^\lambda) - pr^1\eta (\xi^\lambda) \right) \frac{\partial}{\partial u^\lambda}.
\]

(7)
If \( \xi \) and \( \eta \) are two vector fields on \( \mathcal{E} \), then \([\ , \]_{pro} reduces to the Jacobi-Lie bracket of vector fields [13].

2 Lifts

Consider a vector field \( X \in \mathfrak{X}(\mathcal{M}) \) on \( \mathcal{M} \), and let \( \phi \) be a section of \( \pi \). The holonomic lift of \( X(x) \in T_x\mathcal{M} \) by \( \phi \) is

\[
\left( j^1_x \phi, T\phi(X(x)) \right) \in \pi^*_0(T\mathcal{E}) = J^1\pi \times_{\mathcal{E}} T\mathcal{E}.
\]

(8)
In coordinates, if \( X = X^a (x) \partial/\partial x^a \), then

\[
X^{hol} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial \phi^\lambda}{\partial x^a} \frac{\partial}{\partial u^\lambda} = X^a \frac{\partial}{\partial x^a} + X^a u^\lambda_a \left( j^1_x \phi \right) \frac{\partial}{\partial u^\lambda}.
\]

(9)
Define the holonomic part of a projectable vector field \( \xi \in \mathfrak{X}(\mathcal{E}) \) as the holonomic lift of its push forward by \( \pi \), that is

\[
H\xi = (\pi_* \xi)^{hol}.
\]

(10)
$H\xi$ is a generalized vector field of order one. Define a connection $(1;1)$ tensor
\[
\Gamma_J = dx^a \otimes \left( \frac{\partial}{\partial x^a} + u^a_\lambda \frac{\partial}{\partial u^\lambda} \right),
\]satisfying $H\xi = \Gamma_J \xi$. Then, the vertical (or evolutionary) representative
\[
V\xi = \xi - \Gamma_J (\xi) = (\xi^a - \xi^a u^\lambda_a) \frac{\partial}{\partial u^\lambda}
\]of $\xi$ is vertical valued generalized vector field of order one [13],[15],[16].

**Proposition 1** Holonomic lift is a Lie algebra isomorphism from the space of projectable vector fields in $\mathfrak{X}(\mathcal{E})$ into $J^1 \pi \times \xi T\mathcal{E}$.

**Proof.** We consider two projectable vector fields $\xi$ and $\eta$ on $\mathcal{E}$. A straightforward calculation gives
\[
[\Gamma_J (\xi), \Gamma_J (\eta)]_{pro} = [\xi^{hol}, \eta^{hol}]_{pro} = [\xi, \eta]^{hol} = \Gamma_J [\xi, \eta]
\]where $[,]_{pro}$ is the Lie bracket for generalized vector fields in Eq.(7).

On the other hand, the generalized bracket of vertical representatives satisfies
\[
[V\xi, V\eta]_{pro} = V[\xi, \eta]_{pro} + \mathcal{B}(\xi, \eta),
\]where $\mathcal{B}$ is a vertical-vector valued two-form
\[
\mathcal{B}(\xi, \eta) = [\eta^{hol}, V\xi]_{pro} - [\xi^{hol}, V\eta]_{pro}.
\]

There is, however, a class of vector fields, defined again by lifts, for which the vertical representative becomes a Lie algebra isomorphism. Let $\varphi_t : \mathcal{M} \rightarrow \mathcal{M}$ be the flow of $X$ on $\mathcal{M}$. Cotangent lift of $\varphi_t$ is a one-parameter group of diffeomorphism $\varphi_t^c$ on $T^*\mathcal{M}$ satisfying
\[
\pi_{\mathcal{M}} \circ \varphi_t^c = \varphi_t \circ \pi_{\mathcal{M}}
\]where $\pi_{\mathcal{M}}$ is the natural projection of $T^*\mathcal{M}$ to $\mathcal{M}$. The cotangent lift of the inverse flow $T^*\varphi_{-t}$ satisfies the argument in Eq.(16). Infinitesimal generator $X^c : T^*\mathcal{M} \rightarrow TT^*\mathcal{M}$ of the flow $\varphi_t^c$ is called complete cotangent lift of $X$. $X^c$ is a Hamiltonian vector field on the canonical symplectic manifold $(T^*\mathcal{M}, \Omega_{T^*\mathcal{M}} = -d\theta_{T^*\mathcal{M}})$ for the Hamiltonian function $P(X) = i_{X^c} \theta_{T^*\mathcal{M}}$ [8]. The infinitesimal version
\[
T\pi_{\mathcal{M}} \circ X^c = X \circ \pi_{\mathcal{M}}
\]of Eq.(16) gives the relation between $X$ and $X^c$ with $T\pi_{\mathcal{M}}$ being the tangent mapping of $\pi_{\mathcal{M}}$. The complete cotangent lift mapping $c^* : \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(T^*\mathcal{M})$ taking $X$ to $X^c$ is a Lie algebra isomorphism into [8],[17]
\[
[X^c, Y^c] = [X, Y]^c, \quad \forall X, Y \in \mathfrak{X}(\mathcal{M}),
\]
In Darboux’s coordinates \((x^a, y^b)\) on \(T^*M\), the complete cotangent lift of \(X = X^a(x) \partial/\partial x^a\) on \(M\) is

\[
X^{c*} = X_{\mathcal{P}(X)} = X^a \frac{\partial}{\partial x^a} - y_b \frac{\partial X^b}{\partial x^a} \frac{\partial}{\partial y_a} \tag{18}
\]

with the Hamiltonian function being \(\mathcal{P}(X)(x, y) = y_b X^b(x)\). We decompose the complete cotangent lifts into vertical representative and holonomic part

\[
VX^{c*} = -(y_b \frac{\partial X^b}{\partial x^a} + X^b \frac{\partial y_a}{\partial x^a} \frac{\partial}{\partial y_a}) \quad \text{and} \quad HX^{c*} = X^a \frac{\partial}{\partial x^a} + X^a \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b}. \tag{19}
\]

where the connection in Eq.(11) has the particular form

\[
\Gamma = dx^a \otimes \left( \frac{\partial}{\partial x^a} + \frac{\partial y_b}{\partial x^a} \frac{\partial}{\partial y_b} \right). \tag{20}
\]

**Proposition 2** The mapping \(V^{c*} : \mathfrak{X}(M) \rightarrow \mathfrak{X}(T^*M) : X \mapsto VX^{c*}\) is a Lie algebra isomorphism into.

**Proof.** The vector valued two form \(\mathcal{B}\) in Eq.(16) vanishes for the complete cotangent lifts, that is, \(\mathcal{B}(X^{c*}, Y^{c*}) = 0\) for all \(X, Y \in \mathfrak{X}(M)\), therefore one has \(V [X^{c*}, Y^{c*}] = [VX^{c*}, VY^{c*}]_{\text{pro}}\) and the result

\[
V [X, Y]^{c*} = [VX^{c*}, VY^{c*}]_{\text{pro}}. \tag{21}
\]

follows from Eq.(17). \[\square\]

The last object we consider in this section is the vertical lift of one forms. Take the cotangent lift \(T^*\pi_{\mathcal{M}} : T^*M \rightarrow T^*T^*M\) of the projection \(\pi_{\mathcal{M}} : T^*M \rightarrow M\) and recall the isomorphism \(\Omega_{T^*\mathcal{M}}^2 : T^*T^*M \rightarrow TT^*M\) associated with the symplectic two-form \(\Omega_{T^*M}\) on \(T^*M\). Define the Euler vector field

\[
\mathcal{X}_E : T^*M \rightarrow TT^*M : z \mapsto \Omega_{T^*\mathcal{M}}^2 \circ T^*\pi_{\mathcal{M}}(z) \tag{22}
\]

which is vertical, that is, \(\text{image}(\mathcal{X}_E) \subset \ker(T\pi_{\mathcal{M}})\). Indeed,

\[
\langle z, T\pi_{\mathcal{M}} \circ \mathcal{X}_E (z) \rangle = \left< T^*\pi_{\mathcal{M}}(z), \Omega_{T^*\mathcal{M}}^2 \circ T^*\pi_{\mathcal{M}}(z) \right> = \Omega_{T^*\mathcal{M}}(T^*\pi_{\mathcal{M}}(z), T^*\pi_{\mathcal{M}}(z)) = 0, \tag{23}
\]

\(\forall z \in T^*\mathcal{M}\), where we used the skew-symmetry of \(\Omega_{T^*\mathcal{M}}\). \(\mathcal{X}_E\) is the unique vector field satisfying the following equalities

\[
i_{\mathcal{X}_E} \Omega_{T^*\mathcal{M}} = \theta_{T^*\mathcal{M}}, \quad \mathcal{L}_{\mathcal{X}_E} \Omega_{T^*\mathcal{M}} = -\Omega_{T^*\mathcal{M}}, \quad \mathcal{L}_{\mathcal{X}_E} \theta_{T^*\mathcal{M}} = -\theta_{T^*\mathcal{M}}, \tag{24}
\]

where \(i_{\mathcal{X}_E}\) and \(\mathcal{L}_{\mathcal{X}_E}\) are the interior product and the Lie derivative operators \[\mathcal{L}\]. Let \(\alpha \in \Lambda^1(M)\) be a one-form on \(M\). The vertical lift

\[
\alpha^{v} = \mathcal{X}_E \circ \alpha \circ \pi_{\mathcal{M}} : T^*M \rightarrow TT^*M \tag{25}
\]
of the one-form $\alpha$ is a vertical vector field on $T^*M$. The Jacobi-Lie bracket of a complete cotangent lift and a vertical lift is a vertical lift

$$[X^{cs}, \alpha^v] = (L_X \alpha)^v$$

(26)

for $X \in \mathfrak{x}(M)$ and $\alpha \in \Lambda^1(M)$ \cite{17}. In coordinates $(x^a, y_b)$ of $T^*M$, the Euler vector field is $X_E = -y_a \partial/\partial y_a$ and the vertical lift of the one-form $\alpha = \alpha_a(x) dx^a$ becomes $\alpha^v = -\alpha_a(x) \partial/\partial y_a$.

3 Dynamics

Assume that a continuum initially rests in $\mathcal{M}$, and the group $Diff(\mathcal{M})$ of diffeomorphisms acts on left by evaluation on $\mathcal{M}$

$$Diff(\mathcal{M}) \times \mathcal{M} \to \mathcal{M} : (\varphi, x) \to \varphi(x)$$

(27)

to produce the motion of particles. The right action of $Diff(\mathcal{M})$ commutes with the particle motion and constitutes an infinite dimensional symmetry group of the kinematical description. This is the particle relabelling symmetry \cite{2}.

An element of the tangent space $T_{\varphi}Diff(\mathcal{M})$ at $\varphi \in Diff(\mathcal{M})$ is a map $V_{\varphi} : \mathcal{M} \to T\mathcal{M}$ called the material velocity field and satisfies $\tau_M \circ V_{\varphi} = \varphi$. In particular, the tangent space $T_{id_M}Diff(\mathcal{M})$ at the identity $id_M \in Diff(\mathcal{M})$ is the space $\mathfrak{x}(\mathcal{M})$ of smooth vector fields on $\mathcal{M}$. The Lie algebra of $Diff(\mathcal{M})$ is $\mathfrak{x}(\mathcal{M})$ with minus the Jacobi-Lie bracket of vector fields \cite{8}.

The dual space $\mathfrak{x}^*(\mathcal{M}) \simeq \Lambda^1(\mathcal{M}) \otimes Den(\mathcal{M})$ of the Lie algebra is the space of one-form densities on $\mathcal{M}$. The pairing between $\alpha \otimes d\mu \in \mathfrak{x}^*(\mathcal{M})$ and $X \in \mathfrak{x}(\mathcal{M})$ is given by

$$\langle \alpha \otimes d\mu, X \rangle = \int_{\mathcal{M}} \langle \alpha(x), X(x) \rangle d\mu(x) .$$

(28)

The pairing inside the integral is the natural pairing of finite dimensional spaces $T_x\mathcal{M}$ and $T^*_x\mathcal{M}$. The coadjoint action is

$$ad^*_X : \mathfrak{x}^*(\mathcal{M}) \to \mathfrak{x}^*(\mathcal{M})$$

$$\alpha \otimes d\mu \to L_X (\alpha \otimes d\mu) = (L_X \alpha + (div_{d\mu} X) \alpha) \otimes d\mu$$

(29)

$\forall X \in \mathfrak{x}(\mathcal{M})$ and hence the Lie-Poisson equations on $\mathfrak{x}^*(\mathcal{M})$ are

$$\dot{\alpha} = -L_X \alpha - (div_{d\mu} X) \alpha,$$

(30)

where $div_{d\mu} X$ denotes the divergence of the vector field $X$ with respect to the volume form $d\mu$.

In terms of vertical lifts, the dynamics in Eq. (30) is generated by the vector field $(L_X \alpha + (div_{d\mu} X) \alpha)^v$. For the divergence free vector fields, if $\alpha = y_a dx^a$, then the Lie-Poisson equations are generated by

$$(L_X (y_a dx^a))^v = V X^{cs} (x^a, y_a).$$

(31)
3.1 Ideal incompressible fluid

For an ideal incompressible fluid in a bounded compact region \( Q \subset \mathbb{R}^3 \) the configuration space is the group \( \text{Diff}_{vol}(Q) \) of volume preserving diffeomorphisms on \( Q \). The Lie algebra \( \mathfrak{x}_{\text{div}}(Q) \) of \( \text{Diff}_{vol}(Q) \) is the algebra of divergence free vector fields parallel to the boundary of \( Q \) and, the dual space \( \mathfrak{x}^*_{\text{div}}(Q) \) is the space

\[
\mathfrak{x}^*_{\text{div}}(Q) = \{ [\mathcal{Y}] \otimes d^3q \in (\Lambda^1(Q)/d\mathcal{F}(Q)) \otimes \text{Den}(Q) \},
\]

of one-form modulo exact one-form densities on \( Q \). Here, \([\mathcal{Y}] = \{ \mathcal{Y} + d\tilde{p} : \tilde{p} \in \mathcal{F}(Q) \}\) denotes the equivalence class containing \( \mathcal{Y} \) and the volume three for \( d^3q \) is the Euclidean volume on \( \mathbb{R}^3 \)[2],[11].

Let \((x^a, \mathcal{Y}_b)\) be induced coordinates and \( X^a = X^a \partial/\partial x^a \) be a divergence free vector field. The complete cotangent lift of \( X \) is

\[
X^{c*} = X^a \frac{\partial}{\partial x^a} - \mathcal{Y}_b \left( \frac{\partial X^b}{\partial x^a} \right) \frac{\partial}{\partial \mathcal{Y}_a},
\]

and its vertical representative becomes

\[
VX^{c*} = \left( -\mathcal{Y}_b \frac{\partial X^b}{\partial x^a} - X^a \frac{\partial \mathcal{Y}_b}{\partial x^a} \right) \frac{\partial}{\partial \mathcal{Y}_a}.
\]

Equations of motion for the dynamics generated by \( VX^{c*} \) are

\[
\frac{\partial [\mathcal{Y}]}{\partial t} = -\mathcal{L}_X [\mathcal{Y}].
\]

For a generic element \( \mathcal{Y} + d\tilde{p} \in [\mathcal{Y}] \), Eq.(33) becomes Euler’s equations for ideal fluid, that is \( \partial \mathcal{Y}/\partial t + \mathcal{L}_X \mathcal{Y} = 0 \). If the dual space \( \mathfrak{x}^*_{\text{div}}(Q) \) is identified with exact two forms by \([\mathcal{Y}] \rightarrow d\mathcal{Y} = \omega \in \Lambda^2(Q)\), then Eq.(34) becomes the Euler’s equation in vorticity form \( \partial \omega/\partial t + \mathcal{L}_X \omega = 0 \).

3.2 Collisionless plasma

We take \( \mathcal{M} \) to be cotangent bundle \( T^*Q \) of \( Q \subset \mathbb{R}^3 \) in which the plasma particles move. The configuration space of collisionless nonrelativistic plasma is the group

\[
\text{Diff}_{can}(T^*Q) = \{ \varphi \in T^*Q : \varphi^*\Omega_{T^*Q} = \Omega_{T^*Q} \}
\]

of all canonical diffeomorphisms where \( \Omega_{T^*Q} \) is the canonical symplectic two form on \( T^*Q \)[4],[9],[10]. We assume that, the Lie algebra of \( \text{Diff}_{can}(T^*Q) \) is the space of globally Hamiltonian vector fields \( \mathfrak{x}_{\text{ham}}(T^*Q) \) with minus the Jacobi-Lie bracket so that the equations

\[
[X_h, X_f]_{JL} = -X_{[h,f]_{\Omega_{T^*Q}}} \]

describe a Lie algebra isomorphism

\[
h \rightarrow X_h : (\mathcal{F}(T^*Q), \{ , \}_{\Omega_{T^*Q}}) \rightarrow (\mathfrak{x}_{\text{ham}}(T^*Q), -[ , ]_{JL}), \]

between \( \mathfrak{x}_{\text{ham}}(T^*Q) \) and the space of smooth functions \( \mathcal{F}(T^*Q) \) modulo constants endowed with the (nondegenerate) canonical Poisson bracket \( \{ , \}_{\Omega_{T^*Q}} \).
Proposition 3  The dual space of the Lie algebra $\mathfrak{x}_{\text{ham}}(T^*Q)$ of Hamiltonian vector fields is

$$\mathfrak{x}^*_{\text{ham}}(T^*Q) = \{ \Pi_{id} \otimes d\mu \in \Lambda^1(T^*Q) \otimes \text{Den}(T^*Q) : \text{div}_{T^*Q} \Pi_{id}^\sharp \neq 0 \}. \quad (38)$$

With this definition of the dual space the $L_2$-pairing of the Lie algebra and its dual becomes nondegenerate provided we take the volume form to be the symplectic one $d\mu = \Omega_{T^*Q}$ in

$$\int_{T^*Q} \left\langle X_h(z), \Pi_{id}(z) \right\rangle d\mu(z) = -\int_{T^*Q} \left\langle dh, \Pi_{id}^\sharp \right\rangle d\mu = -\int_{T^*Q} i_{\Pi_{id}^\sharp} (dh) d\mu = \int_{T^*Q} \text{div}_{T^*Q} \Pi_{id}^\sharp d\mu,$$  

$$\int_{T^*Q} \left\langle dh, \Pi_{id}^\sharp \right\rangle d\mu = \int_{T^*Q} \text{div}_{T^*Q} \Pi_{id}^\sharp d\mu,$$  

where we use the musical isomorphism $\Omega_{T^*Q}^\sharp: \Pi_{id} \to \Pi_{id}^\sharp$ induced from the symplectic two-form $\Omega_{T^*Q}$ and apply integration by parts [8, internet supplement]. The dual of the Lie algebra isomorphism in Eq. (37) is

$$\Pi_{id}(z) \to \text{div}_{T^*Q} \Pi_{id}^\sharp(z) \quad (40)$$

and it is a momentum map. In Darboux’s coordinates $z = (q^i, p_i)$ on $T^*Q$, we have $\Omega_{T^*Q} = dq^i \wedge dp_i$ and we take $\Pi_{id} = \Pi_i (q^i) dq^i + \Pi^i (z) dp_i$. Then, the momentum map

$$f(z) = \text{div}_{T^*Q} \Pi_{id}^\sharp(z) = \frac{\partial \Pi_i^\sharp (z)}{\partial q^i} - \frac{\partial \Pi_i (z)}{\partial p_i} \quad (41)$$

defines the plasma density function.

In the induced coordinates $(q^i, p_j; \Pi_i, \Pi^j)$ on $T^*T^*Q$, consider the Hamiltonian function $h = (1/2m) \delta^{ij}p_ip_j + e\phi(q)$ which is the energy of a charged particle on $Q$ [9]. The corresponding Hamiltonian vector field is

$$X_h(z) = \frac{1}{m} \delta^{ij}p_i \frac{\partial}{\partial q^j} - e \frac{\partial \phi}{\partial q^i} \frac{\partial}{\partial p_i}. \quad (42)$$

The complete cotangent lift of $X_h$ and its decomposition into vertical representative and holonomic part are

$$X_{h}^{c*} = X_h - \delta^{ij} \frac{1}{m} \Pi_i \frac{\partial}{\partial \Pi^j} + e\Pi^j \frac{\partial^2 \phi}{\partial q^i \partial q^j} \frac{\partial}{\partial \Pi^i},$$

$$HX_{h}^{c*} = X_h + X_h(\Pi_i) \frac{\partial}{\partial \Pi_i} + X_h(\Pi^j) \frac{\partial}{\partial \Pi^j},$$

$$VX_{h}^{c*} = \left( e\Pi^j \frac{\partial^2 \phi}{\partial q^i \partial q^j} - X_h(\Pi^j) \right) \frac{\partial}{\partial \Pi_i} - \left( \frac{1}{m} \Pi_j \delta^{ij} + X_h(\Pi^j) \right) \frac{\partial}{\partial \Pi^j}. \quad (43)$$
where $X_h(\Pi_i)$ denotes the action of $X_h$ on $\Pi_i$. Since, Hamiltonian vector fields are divergence free, the Lie-Poisson equations

$$\dot{\Pi}_i = -X_h(\Pi_i) + e\frac{\partial^2 \phi}{\partial q^i \partial q^j} \Pi^j$$

$$\dot{\Pi}^i = -X_h(\Pi^i) - \frac{1}{m} \delta^j_\Pi \Pi_j$$

are generated solely by $VX_h^*$. These are Vlasov equations in the momentum variables \[4\]. For the density formulation, we make back-substitution of the plasma density function $f(z) = \text{div} \Omega T\ast Q \Pi \circ\text{id}$ into Eqs. (44) and obtain the Vlasov equation

$$\frac{\partial f}{\partial t} + \delta_{ij} \frac{p_i}{m} \frac{\partial f}{\partial q_j} - e \frac{\partial \phi}{\partial q_i} \frac{\partial f}{\partial p_i} = 0.$$ 

(45)

### 3.3 Contact flows in 3D

Let $M$ be a three dimensional manifold with a contact one form $\sigma \in \Lambda^1(M)$ satisfying $d\sigma \wedge \sigma \neq 0$. A contact form determines a contact structure which, locally is the kernel of the contact form $\sigma$. A diffeomorphism on $M$ is called a contact diffeomorphism if it preserves the contact structure. We denote the group of contact diffeomorphisms by $\text{Diff}_{\text{con}}(M)$. A vector field on a contact manifold $(M, \sigma)$ is called a contact vector field if it generates a one-parameter group of contact diffeomorphisms \[1\],\[12\].

In Darboux’s coordinates $(x, y, z)$ on $M$, we take the contact form to be $\sigma = xdy + dz$. For a real valued function $K = K(x, y, z)$ on $M$, there corresponds a contact vector field

$$X_K = \left(\frac{\partial K}{\partial y} - x \frac{\partial K}{\partial z}\right) \frac{\partial}{\partial x} - \frac{\partial K}{\partial x} \frac{\partial}{\partial y} + \left(-K + x \frac{\partial K}{\partial x}\right) \frac{\partial}{\partial z},$$

(46)

on $M$ satisfying the identities

$$i_{X_K} \sigma = -K \quad \text{and} \quad i_{X_K} d\sigma = dK - (i_{R_\sigma} dK) \sigma,$$

(47)

where $R_\sigma = \partial / \partial z$ is the Reeb vector field of $\sigma$. $R_\sigma$ is the unique vector field satisfying $i_{R_\sigma} \sigma = 1$ and $i_{R_\sigma} d\sigma = 0$. The divergence $\text{div}_dX_K$ of $X_K$ with respect to the volume form $d\mu = d\sigma \wedge \sigma$ can be computed to be $\text{div}_dX_K = -2R_\sigma K$.

Contact Poisson (or Lagrange) bracket of two smooth functions on $M$ is defined by

$$\{L, K\}_c = \frac{\partial L}{\partial x} \frac{\partial K}{\partial y} - \frac{\partial L}{\partial y} \frac{\partial K}{\partial x} + \frac{\partial K}{\partial x} \left(L - z \frac{\partial L}{\partial x}\right) - \frac{\partial L}{\partial z} \left(K - z \frac{\partial K}{\partial x}\right),$$

(48)

\forall L, K \in \mathcal{F}(M). The identity $[X_K, X_L]_{JL} = -X_{\{K, L\}_c}$ establishes an isomorphism between Lie algebras $(\mathfrak{X}_{\text{con}}(M), -[\cdot, \cdot]_{\text{JL}})$ and $(\mathcal{F}(M), \{\cdot, \cdot\}_c)$. Following result gives a precise definition of the linear algebraic dual of $\mathfrak{X}_{\text{con}}(M)$.
Proposition 4  The dual space of the algebra $\mathfrak{X}_\text{con}(\mathcal{M})$ of contact vector fields is 
\[ \mathfrak{X}^*_\text{con}(\mathcal{M}) = \left\{ \alpha \otimes d\mu \in \Lambda^1(\mathcal{M}) \otimes \text{Den}(\mathcal{M}) : d\alpha \wedge \sigma - 2\alpha \wedge d\sigma \neq 0 \right\} \tag{49} \]
where $\sigma$ is the contact form on $\mathcal{M}$ and $d\mu = d\alpha \wedge \sigma$.

Proof.  This follows from the requirement that the pairing between $\mathfrak{X}_\text{con}(\mathcal{M})$ and $\mathfrak{X}^*_\text{con}(\mathcal{M})$ be nondegenerate. We compute
\[
\int_{\mathcal{M}} \langle \alpha, X_K \rangle d\mu = \int_{\mathcal{M}} \alpha \wedge i_{X_K} d\sigma \wedge \sigma + \int_{\mathcal{M}} (i_{X_K} \sigma) \alpha \wedge d\sigma
= \int_{\mathcal{M}} \alpha \wedge (dK - (i_{R_K} dK)) \wedge \sigma - \int_{\mathcal{M}} K \alpha \wedge d\sigma
= \int_{\mathcal{M}} K \left( d\alpha \wedge \sigma - 2\alpha \wedge d\sigma \right), \tag{50}
\]
where we use the identities in Eq. (47) at the second step. □

A geometric definition of density of contact particles can be achieved by considering the Lie algebra isomorphism $\mathcal{F}(\mathcal{M}) \to \mathfrak{X}_\text{con}(\mathcal{M}) : K \to X_K$ the dual of which is a momentum map 
\[ \mathfrak{X}^*_\text{con}(\mathcal{M}) \to \text{Den}(\mathcal{M}) : \alpha \to d\alpha \wedge \sigma - 2\alpha \wedge d\sigma. \tag{51} \]
and defines a real valued function $L$ on $\mathcal{M}$
\[ Ld\sigma \wedge \sigma = d\alpha \wedge \sigma - 2\alpha \wedge d\sigma. \tag{52} \]
In coordinates, let $\alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz \in \mathfrak{X}^*_\text{con}(\mathcal{M})$ and recall $\sigma = xdy + dz$. Then,
\[ L(x, y, z) = -\frac{\partial \alpha_x}{\partial y} + \frac{\partial \alpha_y}{\partial x} - x \frac{\partial \alpha_z}{\partial x} + x \frac{\partial \alpha_x}{\partial z} - 2\alpha_z. \tag{53} \]

The dual space $\mathfrak{X}^*_\text{con}(\mathcal{N})$ admits the Lie-Poisson bracket
\[ \{ \delta \zeta, \delta \rho \} (\alpha) = -\int_{\mathcal{M}} \left\langle \alpha, \left[ \delta \zeta, \delta \rho \right]_{J_{LP}} \right\rangle d\mu = -\int_{\mathcal{M}} \left\langle \alpha, [X_H, X_K]_{J_{LP}} \right\rangle d\mu, \tag{54} \]
where $\delta \zeta, \delta \rho \in \mathcal{F}(\mathfrak{X}^*_\text{con}(\mathcal{M}))$ and $\delta \zeta/\delta \alpha = X_H, \delta \rho/\delta \alpha = X_K \in \mathfrak{X}^*_\text{con}(\mathcal{M})$. The Hamiltonian operator $J_{LP}(\alpha)$ associated to the Lie-Poisson bracket in Eq. (54) is defined by
\[ \{ \delta \zeta, \delta \rho \} (\alpha) = -\int_{\mathcal{M}} \left\langle X_H, J_{LP}(\alpha) X_K \right\rangle d\mu \tag{55} \]
and a direct computation gives

Proposition 5  The Hamiltonian differential operator associated to the Lie-Poisson bracket in Eq. (54) is
\[
J_{LP}(\alpha) = -\begin{pmatrix}
\alpha_x \frac{\partial}{\partial x} + \alpha_y \frac{\partial}{\partial y} + \alpha_z \frac{\partial}{\partial z} & \alpha_x \frac{\partial}{\partial x} & \alpha_y \frac{\partial}{\partial y} & \alpha_z \frac{\partial}{\partial z} \\
\alpha_x \frac{\partial}{\partial x} & \alpha_y \frac{\partial}{\partial y} + \alpha_y \frac{\partial}{\partial y} & \alpha_y \frac{\partial}{\partial y} & \alpha_z \frac{\partial}{\partial z} \\
\alpha_x \frac{\partial}{\partial x} & \alpha_y \frac{\partial}{\partial y} & \alpha_y \frac{\partial}{\partial y} & \alpha_z \frac{\partial}{\partial z} \\
\alpha_x \frac{\partial}{\partial x} & \alpha_y \frac{\partial}{\partial y} & \alpha_y \frac{\partial}{\partial y} & \alpha_z \frac{\partial}{\partial z}
\end{pmatrix}, \tag{56}
\]
\[ \partial/\partial x \cdot \alpha y = \alpha_y \partial/\partial x + \partial \alpha_y / \partial x. \]  
Assuming \( \delta \mathbf{R} / \delta \alpha = X_K \), the Lie-Poisson equations on \( X^*_\text{con} (\mathcal{M}) \) are

\[ \dot{\alpha} = J_{LP} (\alpha) X_K = - \text{ad}^*_{X_K} \alpha = - \mathcal{L}_{X_K} \alpha - \text{(div}\, d\mu \text{)} X_K \alpha. \]  

(57)

The Lie-Poisson bracket on the dual space \( \text{Den} (\mathcal{M}) \) of \( \mathcal{F} (\mathcal{M}) \), as defined by Eq.(51), is

\[ \{ \mathfrak{H}, \mathfrak{K} \} (L) = \int_{\mathcal{M}} L \left\{ \frac{\delta \mathfrak{H}}{\delta L}, \frac{\delta \mathfrak{K}}{\delta L} \right\} c \, d\mu = \int_{\mathcal{M}} L \{ H, K \} c \, d\mu, \]  

(58)

where \( \delta \mathfrak{H} / \delta L = H, \delta \mathfrak{K} / \delta L = K \in \mathcal{F} (\mathcal{M}) \) and \( d\mu = d\sigma \wedge \sigma \).

**Proposition 6** The Hamiltonian operator \( J_{LP} (L) \) for the Lie Poisson bracket in Eq.(54) is

\[ J_{LP} (L) = X_L + \left( 4L + \partial L / \partial z \right) \frac{\partial}{\partial z}, \]  

(59)

and the Lie-Poisson equation on \( \text{Den} (\mathcal{M}) \) becomes

\[ \dot{L} = - \{ L, K \}_c - 2 \text{div}\, d\mu (X_K) L. \]  

(60)

**Proof.** The verification of the Hamiltonian operator in Eq.(59) is a straightforward calculation which follows directly from the definition of the Lie-Poisson bracket in Eq.(54). To obtain the Lie-Poisson equation we compute the coadjoint action negative of which is the required equation. By definition

\[ \langle \text{ad}^*_{X_K} L, H \rangle = \langle L, \text{ad}_K H \rangle = \langle L, \{ K, H \} \rangle_c \]

\[ = \int_N L \{ H, K \}_c d\mu = - \int_N L \left( X_K (H) + \frac{\partial K}{\partial z} H \right) d\mu \]

\[ = \int_N \left( X_K (L) + \text{div}\, d\mu (X_K) L - \frac{\partial K}{\partial z} L \right) H d\mu \]

\[ = \int_N \{ L, K \}_c - \frac{\partial K}{\partial z} L + \text{div}\, d\mu (X_K) L - \frac{\partial K}{\partial z} H d\mu \]

\[ = \int_N \{ \{ L, K \}_c + 2 \text{div}\, d\mu (X_K) L \} H d\mu, \]  

(61)

where we use integration by parts at the third step and the identities

\[ \{ H, K \}_c = X_K (H) + \frac{\partial K}{\partial z} H = - X_H (K) - \frac{\partial H}{\partial z} K \]  

(62)

at the second and fourth steps. \( \blacksquare \)

The equation of motion \( \dot{L} = - \text{ad}^*_{X_K} L \) is the kinetic equation of contact particles in density formulation.

**Proposition 7** The Hamiltonian differential operators \( J_{LP} (\alpha) \) in Eq.(56) and \( J_{LP} (L) \) in Eq.(59) are related by

\[ HJ_{LP} (L) K = - X_H J_{LP} (\alpha) X_K \mod \text{(div)}. \]  

(63)
We now obtain dynamics of contact particles by the methods of previous sections. Let \((\mathcal{M}, \sigma)\) be a contact manifold and consider the contact vector field \(X_K\) in Eq.(46). Its complete cotangent lift is

\[
X^*_c = X_K + \left( \Upsilon \frac{\partial K}{\partial z} + \Psi \frac{\partial K}{\partial x} \right) \frac{\partial}{\partial \alpha_x} + (\Phi + \Psi) \left( \frac{\partial K}{\partial y} \frac{\partial}{\partial \alpha_y} + \frac{\partial K}{\partial z} \frac{\partial}{\partial \alpha_z} \right)
\]

(64)

where we use the following abbreviations

\[
\Upsilon = \alpha_x \left( 1 + x \frac{\partial}{\partial x} \right), \quad \Psi = \alpha_y \frac{\partial}{\partial y} - \alpha_z \frac{\partial}{\partial x} - x \alpha_x \frac{\partial}{\partial y}, \quad \Phi = x \alpha_z \frac{\partial}{\partial z} + \alpha_z
\]

(65)

and the induced coordinates \((x, y, z, \alpha_x, \alpha_y, \alpha_z)\) on \(T^*N\). \(X^*_c\) is a canonically Hamiltonian vector field. The vertical representative \(VX^*_c\) of \(X^*_c\) is

\[
VX^*_c = \left( \Upsilon \frac{\partial K}{\partial z} + \Psi \frac{\partial K}{\partial x} - X_K(\alpha_x) \right) \frac{\partial}{\partial \alpha_x} + \left( (\Phi + \Psi) \frac{\partial K}{\partial y} - X_K(\alpha_y) \right) \frac{\partial}{\partial \alpha_y}
\]

\[
+ \left( (\Phi + \Psi) \frac{\partial K}{\partial z} - X_K(\alpha_z) \right) \frac{\partial}{\partial \alpha_z}.
\]

(66)

with \(X_K(\alpha_x)\) denoting the action of \(X_K\) on \(\alpha_x\). To obtain the equations of motion for the momentum variables, one needs to add the divergence term, that is,

\[
\dot{\alpha} = VX^*_c(\alpha) - (\text{div}_{\partial \mu} X_K) \alpha.
\]

(67)

It can be checked that Eq.(67) and Eq.(57) are equal. In coordinates, the system of equations in Eq.(67) takes the form

\[
\dot{\alpha}_x = \Upsilon \frac{\partial K}{\partial z} + \Psi \frac{\partial K}{\partial x} - X_K(\alpha_x) + 2 \frac{\partial K}{\partial z} \alpha_x
\]

\[
\dot{\alpha}_y = (\Phi + \Psi) \frac{\partial K}{\partial y} - X_K(\alpha_y) + 2 \frac{\partial K}{\partial z} \alpha_y
\]

\[
\dot{\alpha}_z = (\Phi + \Psi) \frac{\partial K}{\partial z} - X_K(\alpha_z) + 2 \frac{\partial K}{\partial z} \alpha_z.
\]

(68)

Substituting \(L\) in Eq.(53) to the system of Eqs.(68) we obtain the evolution of the density of contact particles as given by Eq.(60).

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