MARTINGALE SOLUTIONS OF TWO AND THREE DIMENSIONAL
STOCHASTIC CONVECTIVE BRINKMAN-FORCHHEIMER EQUATIONS
FORCED BY LÊVY NOISE

MANIL T. MOHAN

Abstract. The convective Brinkman-Forchheimer equations given by

\[ \frac{\partial u}{\partial t} - \mu \Delta u + (u \cdot \nabla) u + \alpha u + \beta |u|^{r-1} u + \nabla p = f, \quad \nabla \cdot u = 0, \]

where \( \mu, \alpha, \beta > 0 \) and \( r \in [1, \infty) \) describe the motion of incompressible fluid flows in a saturated porous medium. In bounded domains (for \( d = 2, 3 \) and \( r \in [1, \infty) \)), the existence of a weak martingale solution for stochastic convective Brinkman-Forchheimer equations forced by Lévy noise consisting of a Q-Wiener process and a compensated time homogeneous Poisson random measure is established in this work. Using the classical Faedo-Galerkin approximation, a compactness method and a version of the Skorokhod embedding theorem for nonmetric spaces, we prove the existence of a weak martingale solution. For \( d = 2, r \in [1, \infty) \) and \( d = 3, r \in [3, \infty) \), we prove that the martingale solution has stronger regularity properties such as it satisfies the energy equality (Itô’s formula) and hence the trajectories are equal almost everywhere to an \( \mathbb{H} \)-valued càdlàg function defined on \([0, T]\), \( \mathbb{P} \)-a.s. Furthermore, for \( d = 2, r \in [1, \infty) \) and \( d = 3, r \in [3, \infty) \) (\( 2 \beta \mu \geq 1 \) for \( r = 3 \)), we show the pathwise uniqueness of solutions and use the classical Yamada-Watanabe argument to derive the existence of a strong solution and uniqueness in law.

1. Introduction

The convective Brinkman-Forchheimer (CBF) equations in two and three dimensional smooth bounded domains is considered in this work. The CBF equations describe the motion of incompressible fluid flows in a saturated porous medium. This model is recognized to be more accurate when the flow velocity is too large for Darcy’s law to be valid alone, and in addition, the porosity is not too small, so that the term non-Darcy models is used in the literature for these types of fluid flow models (see [31] for a discussion). Let \( \Omega \subset \mathbb{R}^d \) (\( d = 2, 3 \)) be a bounded domain with a smooth boundary \( \partial \Omega \) (at least \( C^2 \)-boundary). Let \( u(t, x) \in \mathbb{R}^d \) represent the velocity field at time \( t \) and position \( x \), \( p(t, x) \in \mathbb{R} \) denote the pressure field, \( f(t, x) \in \mathbb{R}^d \) stand for an external forcing. The deterministic CBF equations are given by

1Department of Mathematics, Indian Institute of Technology Roorkee-IIT Roorkee, Haridwar Highway, Roorkee, Uttarakhand 247667, INDIA.

e-mail: maniltmohan@ma.iitr.ac.in, maniltmohan@gmail.com.

*Corresponding author.

Key words: convective Brinkman-Forchheimer equations, Lévy noise, martingale solution, strong solution, Skorokhod representation theorem.

Mathematics Subject Classification (2010): Primary 60H15; Secondary 35R60, 35Q30, 76D05.
For uniqueness of pressure $p$, one can impose the condition $\int_\Omega p(x,t)dx = 0$ in $(0,T)$ also. The constant $\mu$ represents the positive Brinkman coefficient (effective viscosity), the positive constants $\alpha$ and $\beta$ represent the Darcy (permeability of porous medium) and Forchheimer (proportional to the porosity of the material) coefficients, respectively. Note that for $\alpha = \beta = 0$, we obtain the classical $d$-dimensional Navier-Stokes equations (NSE). Therefore, one can consider the system (1.1) as damped Navier-Stokes equations also. The absorption exponent $r \in [1, \infty)$ and the case $r = 3$ and $r > 3$ are known as the critical exponent and the fast growing nonlinearity (cf. [24]), respectively. It has been shown in Proposition 1.1, [19] that the critical homogeneous CBF equations have the same scaling as the NSE only when the permeability coefficient $\alpha = 0$ and no scale invariance property for other values of $\alpha$ and $r$. The case $r = 3$ and $\alpha = 0$ is referred to as the NSE modified by an absorption term ([1]) or the tamed NSE ([43]). The global solvability of the deterministic CBF equations (1.1) is available in the works [1, 15, 24, 37], etc and the references therein.

The global solvability of the stochastic counterpart of the problem (1.1) and related models (forced by Gaussian) in the whole space or on a torus is available in the works [3, 4, 30, 46], etc. In the paper [13], authors showed the existence and uniqueness of a strong solution to stochastic 3D tamed NSE driven by multiplicative Lévy noise with periodic boundary conditions, based on Galerkin’s approximation and a kind of local monotonicity of the coefficients. The existence and uniqueness of strong solutions for SPDEs like stochastic 3D NSE with damping, stochastic tamed 3D NSE, stochastic 3D Brinkman-Forchheimer-extended Darcy model, etc in bounded domains is established in [18]. By using classical Faedo-Galerkin approximation and compactness method, the existence of martingale solutions for stochastic 3D NSE with nonlinear damping subjected to multiplicative Gaussian noise is obtained in [29]. For a sample literature on the weak martingale solution for 2D and 3D Navier-Stokes equations and related models perturbed by Gaussian and Lévy noise, the interested readers are referred to see [6, 7, 8, 12, 34, 42, 48], etc.

For $d = 2, 3$, $r \in [3, \infty)$ ($2\beta\mu \geq 1$ for $r = 3$), the author in [38, 39] established the global existence and uniqueness of pathwise strong solutions satisfying the energy equality (Itô’s formula) for 2D and 3D SCBF equations subjected to multiplicative Gaussian and pure jump noise, respectively by exploiting a monotonicity property of the linear and nonlinear operators as well as a stochastic generalization of the Minty-Browder technique. For additive Gaussian noise, the same results on unbounded domains is obtained in [25]. The existence of random attractors is also proved in [25]. The Wentzell-Freidlin type large deviation principle for 2D and 3D SCBF equations perturbed by multiplicative Gaussian as well as pure jump noise is obtained in [40, 41], respectively. In addition to the results available in [38, 39], we prove in this paper that the martingale solution satisfying the energy equality exists for $d = r = 3$ for any $\mu, \beta > 0$.

In this work, we consider the stochastic convective Brinkman-Forchheimer (SCBF) equations perturbed by Lévy noise consisting of the Q-Wiener process and compensated time
homogeneous Poisson random measure. The SCBF equations driven by multiplicative Lévy noise are given by

\[
\begin{align*}
\frac{d}{dt}u(t) - \mu \Delta u(t) + (u(t) \cdot \nabla)u(t) + \beta |u(t)|^{r-1}u(t) + \nabla p(t) &= \sigma(t, u(t))dW(t) + \int_Z \gamma(t, u(t-), z)\tilde{\pi}(dt, dz), \quad \text{in} \ O \times (0, T), \\
\nabla \cdot u(t) &= 0, \quad \text{in} \ O \times (0, T), \\
u(t) &= 0 \quad \text{on} \ \partial O \times [0, T), \\
u(0) &= u_0 \quad \text{in} \ O,
\end{align*}
\]

(1.2)

where \(W\) is a \(Q\)-Wiener process \(\tilde{\pi}(\cdot, \cdot)\) is the compensated time homogeneous Poisson random measure, \((Z, \mathcal{Z})\) is a measurable space, and \(\sigma(\cdot, \cdot), \gamma(\cdot, \cdot, \cdot)\) are noise coefficients. The main aims of this work are two folded:

1. We first show the existence of a weak martingale solution \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), \tilde{u}, W, \tilde{\pi})\) to 2D and 3D SCBF equations for \(r \in [1, \infty)\), where \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) is a filtered probability space, \(\tilde{W}\) is a \(Q\)-Wiener process, \(\tilde{\pi}\) is a time homogeneous Poisson random measure and \(\tilde{u} = \{\tilde{u}(t)\}_{t \in [0, T]}\) is a stochastic process with trajectories in \(D(0, T; \mathbb{H}_w) \cap L^2(0, T; \mathbb{V}) \cap L^{r+1}(0, T; \mathbb{L}^{r+1})\), \(\mathbb{P}\)-a.s., satisfying an appropriate integral inequality. We use the classical Faedo-Galerkin approximation, a compactness method and a version of the Skorokhod embedding theorem for nonmetric spaces to obtain this result.

2. For \(d = 2, r \in [1, \infty)\) and \(d = 3, r \in [3, \infty)\), we prove that the martingale solution satisfies the energy equality (Itô’s formula) and hence the trajectories are equal almost everywhere to an \(\mathbb{H}\)-valued càdlàg function defined on \([0, T]\), \(\mathbb{P}\)-a.s. We prove the Itô formula by using the approximation available in [15] for functions using the elements of eigenspaces of the Stokes operator in such a way that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously. Furthermore, for \(d = 2, r \in [1, \infty)\) and \(d = 3, r \in [3, \infty)\) \((2\beta \mu \geq 1\) for \(r = 3\)), we establish the pathwise uniqueness of solutions and use the classical Yamada-Watanabe argument to derive the existence of a strong solution and hence the uniqueness in law.

We mainly follow the works [7, 42], etc to obtain the existence of a weak martingale solution and [38, 39] to derive the energy equality (Itô formula).

The organization of the paper is as follows. In section 2, we define the linear and nonlinear operators, and provide the necessary function spaces needed to obtain the global solvability results of the system (1.1). In section 3, we first provide an abstract formulation of the SCBF equations (1.2) in bounded domains. We also state our main results on the existence of weak martingale solution and the existence and uniqueness of strong solutions in the same section (Theorems 3.9 and 3.10). Furthermore, we provide the necessary functional tools like the space of càdlàg functions, deterministic compactness criterion, the Aldous condition, the Skorokhod embedding theorem, etc in the same section. One of the main results on the existence of weak martingale solution \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), \tilde{u}, W, \tilde{\pi})\) of the system (1.1) for \(d = 2, 3\) and \(r \in [1, \infty)\) (Theorem 3.9) is established in section 4 using the classical Faedo-Galerkin approximation, a compactness method and a version of the Skorokhod embedding theorem for nonmetric spaces. The section 5 is devoted for establishing the regularity properties of weak martingale solutions and the existence and uniqueness of strong solutions. For \(d = 2\),
$r \in [1, \infty)$ and $d = 3$, $r \in [3, \infty)$, we prove the global solvability results of the system (1.2). In the further analysis, the Darcy parameter $\alpha$ does not play a major role and we set $\alpha = 0$ in the rest of the paper.

2. Mathematical Formulation

This section is devoted for providing the necessary function spaces needed to obtain the global solvability results of the system (1.2). In the further analysis, the Darcy parameter $\alpha$ does not play a major role and we set $\alpha = 0$ in the rest of the paper.

2.1. Function spaces. Let $C_0^\infty(\mathbb{O}; \mathbb{R}^d)$ be the space of all infinite times differentiable functions ($\mathbb{R}^d$-valued) with compact support in $\mathbb{O} \subset \mathbb{R}^d$. Let us define

$$\mathcal{V} := \{ u \in C_0^\infty(\mathbb{O}; \mathbb{R}^d) : \nabla \cdot u = 0 \}.$$ 

Let $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{L}$ denote the closure of $\mathcal{V}$ in the Lebesgue space $L^2(\mathbb{O}) = L^2(\mathbb{O}; \mathbb{R}^d)$, Sobolev space $H^1(\mathbb{O}) = H^1(\mathbb{O}; \mathbb{R}^d)$, Lebesgue space $L^p(\mathbb{O}) = L^p(\mathbb{O}; \mathbb{R}^d)$, for $p \in (2, \infty]$, respectively. Then under some smoothness assumptions on the boundary (for instance, one can take $C^2$-boundary), we characterize the spaces $\mathbb{H}$, $\mathbb{V}$ and $\mathbb{L}$ as $\mathbb{H} = \{ u \in L^2(\mathbb{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathbb{O}} = 0 \}$, with the norm $\| u \|_{\mathbb{H}}^2 := \int_\mathbb{O} |\nabla u(x)|^2 dx$, where $n$ is the unit outward drawn normal to $\partial \mathbb{O}$, and $u \cdot n|_{\partial \mathbb{O}}$ should be understood in the sense of trace in $H^{-1/2}(\partial \mathbb{O})$ (cf. Theorem 1.2, Chapter 1, [19]), $\mathbb{V} = \{ u \in H^1_0(\mathbb{O}) : \nabla \cdot u = 0 \}$, with the norm $\| u \|_{\mathbb{V}}^2 := \int_\mathbb{O} |\nabla u(x)|^2 dx$, $\mathbb{L} = \{ u \in L^p(\mathbb{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathbb{O}} = 0 \}$, with the norm $\| u \|_{\mathbb{L}}^p := \int_\mathbb{O} |u(x)|^p dx$ and $\mathbb{L}^\infty = \{ u \in L^\infty(\mathbb{O}) : \nabla \cdot u = 0, u \cdot n|_{\partial \mathbb{O}} = 0 \}$, with the norm $\| u \|_{\mathbb{L}^\infty} = \text{ess} \sup_{x \in \mathbb{O}} |u(x)|$, respectively. Let $\langle \cdot, \cdot \rangle$ denote the inner product in the Hilbert space $\mathbb{H}$ and $\langle \cdot, \cdot \rangle$ represent the induced duality between the spaces $\mathbb{V}$ and its dual $\mathbb{V}'$ as well as $\mathbb{L}$ and its dual $\mathbb{L}'$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Remember that $\mathbb{H}$ can be identified with its dual $\mathbb{H}'$.

Note that the sum space $\mathbb{L}' + \mathbb{V}$ is well defined (see subsection 2.1, [14]), as $\mathbb{L}'$ and $\mathbb{V}$ are subspaces of the topological vector space $\mathcal{D}'(\mathbb{O})$, where $\mathcal{D}'(\mathbb{O})$ is the space of distributions on $\mathbb{O}$ with the usual topology, and the embedding $\mathbb{L}' \subset \mathcal{D}'(\mathbb{O})$ and $\mathbb{V} \subset \mathcal{D}'(\mathbb{O})$ are continuous. Furthermore, we have

$$(\mathbb{L}' + \mathbb{V})' = \mathbb{L}^p \cap \mathbb{V} \quad \text{and} \quad (\mathbb{L}^p \cap \mathbb{V})' = \mathbb{L}' + \mathbb{V},$$

where $\| u \|_{\mathbb{L} \cap \mathbb{V}} = \max \{ \| u \|_{\mathbb{L}^p}, \| u \|_{\mathbb{V}} \}$, which is equivalent to the norms $\| u \|_{\mathbb{L}^p} + \| u \|_{\mathbb{V}}$ and $\left( \| u \|_{\mathbb{L}^p}^2 + \| u \|_{\mathbb{V}}^2 \right)^{1/2}$, and

$$\| u \|_{\mathbb{L}' + \mathbb{V}} = \inf \{ \| u_1 \|_{\mathbb{L}'} + \| u_2 \|_{\mathbb{V}} : u = u_1 + u_2, u_1 \in \mathbb{L}', u_2 \in \mathbb{V} \}$$

$$= \sup \left\{ \frac{|\langle u_1 + u_2, f \rangle|}{\| f \|_{\mathbb{L}' \cap \mathbb{V}}} : 0 \neq f \in \mathbb{L}^p \cap \mathbb{V} \right\}.$$ 

It should be noted that $\mathbb{L}^p \cap \mathbb{V}$ and $\mathbb{L}' + \mathbb{V}$ are Banach spaces. Furthermore, we have the continuous embedding $\mathbb{V} \cap \mathbb{L}^p \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{H} \equiv \mathbb{H}' \hookrightarrow \mathbb{V}' \hookrightarrow \mathbb{V}' + \mathbb{L}'$, and the embedding of $\mathbb{V} \hookrightarrow \mathbb{H}$ is compact. It should be noted that $\mathbb{V} \cap \mathbb{L}^p$ and $\mathbb{V}' + \mathbb{L}'$ are separable Banach spaces. By Sobolev’s embedding, we know that $\mathbb{V} \hookrightarrow \mathbb{L}^p$ for all $p \in [2, \infty)$ in 2D and $p \in [2, 6]$ in 3D.
2.2. **Linear operator.** Let \( P_p : \mathbb{L}^p(\Omega) \to \mathbb{H} \) denote the Helmholtz-Hodge projection (\([16]\)). For \( p = 2 \), \( P_p \) becomes an orthogonal projection and for \( 2 < p < \infty \), it is a bounded linear operator. Since \( \Omega \) is of class \( \mathcal{C}^2 \), \( P \) maps \( \mathbb{H}^1(\Omega) \) into itself and is bounded (Remark 1.6, \([49]\)). Let us define

\[
\begin{align*}
Au & : = -P \Delta u, \quad u \in D(A), \\
D(A) & : = \mathbb{V} \cap \mathbb{H}^2(\Omega).
\end{align*}
\]

We know that \( A \) is a non-negative self-adjoint operator in \( \mathbb{H} \) with \( \mathbb{V} = D(A^{1/2}) \) and

\[
\langle Au, u \rangle = \|u\|_V^2, \quad \text{for all } u \in \mathbb{V} \text{ so that } \|Au\|_V \leq \|u\|_V.
\]

(2.1)

For a bounded domain \( \Omega \), the operator \( A \) is invertible and its inverse \( A^{-1} \) is bounded, self-adjoint and compact in \( \mathbb{H} \). Thus, using the spectral theorem, one can deduce that the spectrum of \( A \) consists of an infinite sequence \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots \), with \( \lambda_k \to \infty \) as \( k \to \infty \) of eigenvalues.

2.3. **Bilinear operator.** Let us define the trilinear form \( b(\cdot, \cdot, \cdot) : \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R} \) by

\[
b(u, v, w) = \int_\Omega (u(x) \cdot \nabla)v(x) \cdot w(x)dx = \sum_{i,j=1}^n \int_\Omega u_i(x) \frac{\partial v_j(x)}{\partial x_i} w_j(x)dx.
\]

If \( u, v \) are such that the linear map \( b(u, v, \cdot) \) is continuous on \( \mathbb{V} \), the corresponding element of \( \mathbb{V}' \) is denoted by \( B(u, v) \). We also denote \( B(u) = B(u, u) = \mathcal{P}[(u \cdot \nabla)u] \). An integration by parts yields

\[
\begin{align*}
\{ b(u, v, v) & = 0, \quad \text{for all } u, v \in \mathbb{V}, \\
b(u, v, w) & = -b(u, w, v), \quad \text{for all } u, v, w \in \mathbb{V}.
\end{align*}
\]

(2.2)

For \( r \in [1,3] \), using Hölder’s inequality, we have \(|\langle B(u, u), v \rangle| = |b(u, v, u)| \leq \|u\|_{L^4}^2 \|v\|_V \), for all \( v \in \mathbb{V} \) so that

\[
\|B(u)\|_{V'} \leq \|u\|_{L^4}^2, \quad \text{for all } u \in \mathbb{L}^4.
\]

and we conclude that \( B(\cdot) : \mathbb{V} \cap \mathbb{L}^4 \to \mathbb{V}' + \mathbb{L}^4 \). Furthermore, we have

\[
\|B(u) - B(v)\|_{V'} \leq (\|u\|_{\mathbb{L}^4} + \|v\|_{\mathbb{L}^4})\|u - v\|_{\mathbb{L}^4},
\]

hence \( B(\cdot) : \mathbb{V} \cap \mathbb{L}^4 \to \mathbb{V}' + \mathbb{L}^4 \) is a locally Lipschitz operator. An application of Hölder’s inequality yields

\[
|b(u, v, w)| = |b(u, w, v)| \leq \|u\|_{L^{r+1}} \|v\|_{L^{2(r+1)}} \|w\|_V,
\]

for all \( u \in \mathbb{V} \cap \mathbb{L}^{r+1}, v \in \mathbb{V} \cap \mathbb{L}^{2(r+1)}, \) and \( w \in \mathbb{V} \), so that we obtain

\[
\|B(u, v)\|_{V'} \leq \|u\|_{L^{r+1}} \|v\|_{L^{2(r+1)}}.
\]

(2.3)

Using interpolation inequality, we get

\[
|\langle B(u, u), v \rangle| = |b(u, v, u)| \leq \|u\|_{L^{r+1}} \|u\|_{L^{2(r+1)}} \|v\|_V \leq \|u\|_{L^{r+1}} \|u\|_{L^{r+3}} \|v\|_V,
\]

(2.4)

for \( r > 3 \) and all \( v \in \mathbb{V} \). Thus, one can deduce that

\[
\|B(u)\|_{V'} \leq \|u\|_{L^{r+1}} \|u\|_{L^{r+3}}.
\]

(2.5)
Using (2.3), for \( u, v \in \mathbb{V} \cap \tilde{L}^{r+1} \), we also obtain
\[
\|B(u) - B(v)\|_{\mathbb{V}} \leq \left( \|u\|_{H}^{\frac{r}{r-1}} \|u\|_{\tilde{L}^{r+1}}^{\frac{r-1}{r}} + \|v\|_{H}^{\frac{r}{r-1}} \|v\|_{\tilde{L}^{r+1}}^{\frac{r-1}{r}} \right) \|u - v\|_{\tilde{L}^{r+1}},
\]
for \( r > 3 \), by using the interpolation inequality. Therefore, the map \( B(\cdot) : \mathbb{V} \cap \tilde{L}^{r+1} \rightarrow \mathbb{V}' + \tilde{L}^{\frac{r+1}{r}} \) is locally Lipschitz.

2.4. **Nonlinear operator.** Next, we consider the nonlinear operator \( \mathcal{C}(u) := \mathcal{P}(|u|^{-1}u) \).

It can be easily seen that \( \langle \mathcal{C}(u), u \rangle = \|u\|_{\tilde{L}^{r+1}}^{r+1} \). Furthermore, for all \( u \in \tilde{L}^{r+1} \), the map is Gateaux differentiable with Gateaux derivative
\[
\mathcal{C}'(u)v = \begin{cases} 
\mathcal{P}(v), & \text{for } r = 1, \\
\mathcal{P}(|u|^{-1}v) + (r - 1)\mathcal{P}(\frac{u}{|u|} - (u \cdot v)), & \text{if } u \neq 0, \\
0, & \text{if } u = 0,
\end{cases}
\]
for \( 1 < r < 3 \), and
\[
\mathcal{P}(|u|^{-1}v) + (r - 1)\mathcal{P}(u|u|^{-r-3}(u \cdot v)),
\]
for \( r \geq 3 \).

For \( 0 < \theta < 1 \) and \( u, v \in \tilde{L}^{r+1} \), an application of Taylor’s formula yields \((38, 37)\)
\[
\langle \mathcal{C}(u) - \mathcal{C}(v), w \rangle \leq r \left( \|u\|_{\tilde{L}^{r+1}} + \|v\|_{\tilde{L}^{r+1}} \right)^{r-1} \|u - v\|_{\tilde{L}^{r+1}} \|w\|_{\tilde{L}^{r+1}},
\]
for all \( u, v, w \in \tilde{L}^{r+1} \). Thus the operator \( \mathcal{C}(\cdot) : \tilde{L}^{r+1} \rightarrow \tilde{L}^{\frac{r+1}{r}} \) is locally Lipschitz. Furthermore, for any \( r \in [1, \infty) \), we have (see \( 38\))
\[
\langle \mathcal{C}(u) - \mathcal{C}(v), u - v \rangle \geq \frac{1}{2} \|u\|_{\tilde{L}^{r+1}}^{r-1}(u - v)\|_{H}^2 + \frac{1}{2} \|v\|_{\tilde{L}^{r+1}}^{r-1}(u - v)\|_{H}^2 \geq \frac{1}{2^{r-1}} \|u - v\|_{\tilde{L}^{r+1}}^{r+1} \geq 0,
\]
for \( r \geq 1 \) and all \( u, v \in \tilde{L}^{r+1} \).

3. **Stochastic Convective Brinkman-Forchheimer equations**

In this section, we discuss the existence of weak martingale solutions for the system (1.2). Let us first provide an abstract formulation of the problem (1.2). By taking orthogonal projection \( \mathcal{P} \) onto the first equation in (1.2), we find
\[
\begin{cases} 
\text{d}u(t) + [\mu A(u(t)) + B(u(t)) + \beta \mathcal{C}(u(t))]\text{d}t \\
= \sigma(t, u(t))W(t) + \int_{Z} \gamma(t, u(t-), z)\tilde{\pi}(\text{d}t, \text{d}z),
\end{cases}
\]
for \( t \in (0, T) \), where \( u_0 \in L^{2p}(\Omega; \mathbb{H}) \), for some \( p > 1 \). Strictly speaking, one should write \( \mathcal{P}\sigma \) and \( \mathcal{P}\gamma \) for \( \sigma \) and \( \gamma \), respectively.

3.1. **Stochastic setting.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space equipped with an increasing family of sub-sigma fields \( \{\mathcal{F}_t\}_{t \geq 0} \) of \( \mathcal{F} \) satisfying the usual conditions.
3.1.1. Q-Wiener process. Firstly, we provide the definition and properties of Q-Wiener processes. Let $\mathbb{H}$ be a separable Hilbert space.

**Definition 3.1.** A stochastic process $\{W(t)\}_{t \geq 0}$ is said to be an $\mathbb{H}$-valued $\mathcal{F}_t$-adapted Q-Wiener process with covariance operator $Q$ if

(i) for each non-zero $h \in \mathbb{H}$, $\|Q^{1/2}h\|_{\mathbb{H}}^{-1}(W(t), h)$ is a standard one dimensional Wiener process,

(ii) for any $h \in \mathbb{H}$, $(W(t), h)$ is a martingale adapted to $\mathcal{F}_t$.

The stochastic process $\{W(t)\}_{t \geq 0}$ is a Q-Wiener process with covariance $Q$ if and only if for arbitrary $t$, the process $W(t)$ can be expressed as $W(t, x) = \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(x) \beta_k(t)$, where $\beta(t), k \in \mathbb{N}$ are independent one dimensional Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of $\mathbb{H}$ such that $Qe_k = \mu_k e_k$. If $W(\cdot)$ is a Q-Wiener process with $\operatorname{Tr} Q = \sum_{k=1}^{\infty} \mu_k < +\infty$, then $W(\cdot)$ is a Gaussian process on $\mathbb{H}$ and $\mathbb{E}[W(t)] = 0$, $\operatorname{Cov}[W(t)] = tQ$, $t \geq 0$. The space $\mathbb{H}_0 = Q^{1/2}\mathbb{H}$ is a Hilbert space equipped with the inner product $(\cdot, \cdot)_0$,

$$(\mathbf{u}, \mathbf{v})_0 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (\mathbf{u}, e_k)(\mathbf{v}, e_k) = (Q^{-1/2} \mathbf{u}, Q^{-1/2} \mathbf{v}), \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{H}_0,$$

where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$.

Let $\mathcal{L}(\mathbb{H})$ denote the space of all bounded linear operators on $\mathbb{H}$ and $\mathcal{L}_Q := \mathcal{L}_Q(\mathbb{H})$ represent the space of all Hilbert-Schmidt operators from $\mathbb{H}_0 := Q^{1/2}\mathbb{H}$ to $\mathbb{H}$. Since $Q$ is a trace class operator, the embedding of $\mathbb{H}_0$ in $\mathbb{H}$ is Hilbert-Schmidt and the space $\mathcal{L}_Q$ is a Hilbert space equipped with the norm $\|\Phi\|^2_{\mathcal{L}_Q} = \operatorname{Tr}(\Phi Q \Phi^*) = \sum_{k=1}^{\infty} \|Q^{1/2} \Phi^* e_k\|^2_{\mathbb{H}}$ and inner product $(\Phi, \Psi)_{\mathcal{L}_Q} = \operatorname{Tr}(\Phi Q \Psi^*) = \sum_{k=1}^{\infty} (Q^{1/2} \Psi^* e_k, Q^{1/2} \Phi^* e_k)$. For more details, the interested readers are referred to see [10].

3.1.2. Time homogeneous Poisson random measure. We mainly follow the works [2 5 6 21 42 44 51], etc for the basics of time homogeneous Poisson random measure. We denote $\mathbb{N} := \{0, 1, 2, \ldots\}$, $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\mathbb{R}^+ := [0, \infty)$. Let $(S, \mathcal{S})$ be a measurable space and let $M_{\bar{\mathbb{N}}}(S)$ be the set of all $\bar{\mathbb{N}}$-valued measures on $(S, \mathcal{S})$. On the set $M_{\bar{\mathbb{N}}}(S)$, we consider the $\sigma$-field $\mathcal{M}_{\bar{\mathbb{N}}}(S)$ defined as the smallest $\sigma$-field such that for all $B \in \mathcal{S}$, the map $i_B : M_{\bar{\mathbb{N}}}(S) \ni \nu \mapsto \nu(B) \in \bar{\mathbb{N}}$

is measurable.

**Definition 3.2.** Let $(Z, \mathcal{Z})$ be a measurable space. A time homogeneous Poisson random measure $\pi$ on $(Z, \mathcal{Z})$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a measurable function

$$\pi : (\Omega, \mathcal{F}) \to (M_{\bar{\mathbb{N}}}([0, t] \times Z), \mathcal{M}_{\bar{\mathbb{N}}}([0, t] \times Z))$$

such that

1. for all $B \in \mathbb{R}^+ \otimes \mathcal{Z}$, $\pi(B) := i_B \circ \pi : \Omega \to \bar{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\pi(B)]$;
2. $\pi$ is independently scattered, that is, if the sets $B_j \in \mathbb{R}^+ \otimes \mathcal{Z}, j = 1, \ldots, n$ are disjoint then the random variables $\pi(B_j), j = 1, \ldots, n$, are independent;
3. for all $U \in \mathcal{Z}$, the $\bar{\mathbb{N}}$-valued process $N(t, U)_{t \geq 0}$ defined by

$$N(t, U) := \pi((0, t] \times U), \ t \geq 0$$
is $\mathcal{F}_t$-adapted and its increments are independent of the past, that is, if $t > s \geq 0$, then $N(t, U) - N(s, U) = \pi((s, t] \times U)$ is independent of $\mathcal{F}_s$.

If $\pi$ is a time homogeneous Poisson random measure, then the formula

$$\lambda(A) := \mathbb{E}[\pi((0, 1] \times A)], \ A \in \mathcal{F}$$

defines a measure on $(Z, \mathcal{F})$ called an intensity measure of $\lambda$. Moreover, for all $T < \infty$ and all $A \in \mathcal{F}$ such that $\mathbb{E}[\pi((0, t] \times A)] < \infty$, the $\mathbb{R}$-valued process $\{\tilde{N}(t, A)\}_{t \in (0, T]}$ defined by

$$\tilde{N}(t, A) := \pi((0, t] \times A) - t\lambda(A), \ t \in (0, T]$$

is an integrable martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The random measure $d \otimes \lambda$ on $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$, where $d$ stands for the Lebesgue measure, is called a compensator of $\pi$ and the difference between a time homogeneous Poisson random measure $\pi$ and its compensator, that is,

$$\tilde{\pi} := \pi - d \otimes \lambda,$$

is called a compensated time homogeneous Poisson random measure.

Let us now explain some basic properties of the stochastic integral with respect to $\tilde{\eta}$ (see [2, 6, 21, 44], etc for more details). Let $\mathbb{H}$ be a separable Hilbert space and let $\mathcal{P}$ be a predictable $\sigma$-field on $[0, T] \times \Omega$. Let $\mathcal{L}^2_{\lambda,T}(\mathcal{P} \otimes \mathcal{F}, d \otimes \mathbb{P} \otimes \lambda; \mathbb{H})$ be the space of all $\mathbb{H}$-valued, $\mathcal{P} \otimes \mathcal{F}$-measurable processes such that

$$\mathbb{E} \left[ \int_0^T \int_Z \|\xi(t, \cdot, z)\|_{\mathbb{H}}^2 \lambda(dz) dt \right] < \infty.$$ 

If $\xi \in \mathcal{L}^2_{\lambda,T}(\mathcal{P} \otimes \mathcal{F}, d \otimes \mathbb{P} \otimes \lambda; \mathbb{H})$, then the integral process $\int_0^t \int_Z \eta(s, \cdot, z)\tilde{\pi}(ds, dz)$, is a càdlàg $\mathbb{L}^2$-integrable martingale. Moreover, the following Itô’s isometry holds for $t \in [0, T]$:

$$\mathbb{E} \left[ \left\| \int_0^T \int_Z \xi(t, \cdot, z)\tilde{\pi}(dt, dz) \right\|_{\mathbb{H}}^2 \right] = \mathbb{E} \left[ \int_0^T \int_Z \|\xi(t, \cdot, z)\|_{\mathbb{H}}^2 \lambda(dz) dt \right].$$

Since the integral $M(t) := \int_0^t \int_Z \xi(s, \cdot, z)\tilde{\pi}(ds, dz)$ is an $\mathbb{H}$-valued square integrable martingale, there exist increasing càdlàg processes so-called quadratic variation process $[M]_t$ and Meyer process $\langle M \rangle_t$ such that $[M]_t - \langle M \rangle_t$ is a local martingale (see section 1.6, [27] and section 2.2, [36]). For the process $M(t)$, it can be shown that $[M]_t = \int_0^t \|\xi(s, \cdot, z)\|_{\mathbb{H}}^2 \lambda(dz)ds$ and $\langle M \rangle_t = \int_0^t \|\xi(s, \cdot, z)\|_{\mathbb{H}}^2 \lambda(dz)ds$ (Example 2.8, [31]). Indeed, $\mathbb{E}\{\|M(t)\|_{\mathbb{H}}^2\} = \mathbb{E}\{[M]_t\} = \mathbb{E}\{\langle M \rangle_t\}$, so that we get

$$\mathbb{E} \left[ \int_0^t \int_Z \|\xi(s, \cdot, z)\|_{\mathbb{H}}^2 \pi(ds, dz) \right] = \mathbb{E} \left[ \int_0^t \int_Z \|\xi(s, \cdot, z)\|_{\mathbb{H}}^2 \lambda(dz)ds \right],$$

for all $t \in [0, T]$.

Let us now provide the assumptions satisfied by the noise coefficients $\sigma$ and $\gamma$.

**Hypothesis 3.3.** We assume that $W(t)$ is an $\mathbb{H}$-valued Q-Wiener process on the stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$; $\tilde{\pi}$ is a compensated time homogeneous Poisson random measure on a measurable space $(Z, \mathcal{F})$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a $\sigma$-finite intensity measure $\lambda$.

The noise coefficients $\sigma(\cdot, \cdot, \cdot)$ and $\gamma(\cdot, \cdot, \cdot)$ satisfy:

(H.1) The function $\sigma \in C([0, T] \times (V \cap \mathbb{L}^{r+1}); \mathcal{L}_Q(\mathbb{H}))$ and $\gamma \in \mathcal{L}^2_{\lambda,T}(\mathcal{P} \otimes \mathcal{F}, d \otimes \mathbb{P} \otimes \lambda; \mathbb{H})$;
(H.2) (Growth condition) There exist positive constants $K_1$ and $K_2$ such that for all $t \in [0, T]$ and $u \in \mathbb{H}$,
\[
\|\sigma(t, u)\|_{L^2}^2 + \int_Z \|\gamma(t, u, z)\|_{H}^2 \lambda(dz) \leq K_1(1 + \|u\|_{H}^2)
\]
and
\[
\int_Z \|\gamma(t, u, z)\|_{H}^{2p} \lambda(dz) \leq K_2(1 + \|u\|_{H}^{2p})
\]
for some $p > 1$;

(H.3) (Lipschitz condition) There exists a positive constant $L$ such that for any $t \in [0, T]$ and all $u_1, u_2 \in \mathbb{H}$,
\[
\|\sigma(t, u) - \sigma(t, v)\|_{L^2}^2 + \int_Z \|\gamma(t, u, z) - \gamma(t, v, z)\|_{H}^2 \lambda(dz) \leq L\|u_1 - u_2\|_{H}^2.
\]

**Remark 3.4.** One can fix $p = 2$ in Hypothesis (H.2).

### 3.2. Weak martingale and strong solutions

**Definition 3.5** (Weak martingale solution). A weak martingale solution of the SCBF problem (3.1) is a system $((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), \bar{u}, \bar{W}, \bar{\pi})$, where

(a) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a filtered probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, that is, a set of sub-$\sigma$-fields of $\mathcal{F}$ with $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for $0 \leq s < t < \infty$,

(b) $W$ is a Q-Wiener process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$,

(c) $\bar{\pi}$ is a time homogeneous Poisson random measure on $(Z, \mathcal{Z})$ over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with the intensity measure $\lambda$,

(d) $\bar{u} : [0, T] \times \Omega \to \mathbb{H}$ is a predictable process with $\mathbb{P}$-a.e. paths
\[
\bar{u}(\cdot, \omega) \in D([0, T]; \mathbb{H}_w) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}^{r+1})
\]
such that for all $t \in [0, T]$ and all $v \in V \cap \tilde{L}^{r+1}$, the following identity holds $\mathbb{P}$-a.s.
\[
(\bar{u}(t), v) = (u_0, v) - \int_0^t \langle \mu A \bar{u}(s) + B(\bar{u}(s)) + \beta C(\bar{u}(s)), v \rangle ds
\]
\[
+ \int_0^t (\sigma(s, \bar{u}(s))d\bar{W}(s), v) + \int_0^t \int_Z \langle \gamma(s, \bar{u}(s-), z), v \rangle \bar{\pi}(ds, dz).
\]  
\[
\text{(3.2)}
\]

It should be noted that an application of Sobolev’s inequality yields $V \subset \tilde{L}^{r+1} \subset \tilde{L}^{r+1} \subset V'$, so that $V' + \tilde{L}^{r+1} = V'$ for $d = 2, r \in [1, \infty)$ and $d = 3$ and $r \in [1, 5]$. For this case, one can replace $v \in V$ in (3.2).

Next, we present the definition of strong solutions for the SCBF equations (3.1).

**Definition 3.6.** We say that the problem (3.1) has a strong solution if and only if for every stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, and Q-Wiener processes $W(t)$ and time homogeneous Poisson random measures $\pi$ on $(Z, \mathcal{Z}(Z))$ with intensity measure $\lambda$ defined on this stochastic basis, there exists a progressively measurable process $u : [0, T] \times \Omega \to \mathbb{H}$ with $\mathbb{P}$-a.s. paths
\[
u(\cdot, \omega) \in D([0, T]; \mathbb{H}_w) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \tilde{L}^{r+1})
\]
such that for all \( t \in [0, T] \) and all \( \mathbf{v} \in \mathcal{V} \cap \ell^{r+1} \):

\[
(u(t), \mathbf{v}) = (u_0, \mathbf{v}) - \int_0^t \langle \mu A(u(s) + B(u(s)) + \beta \mathcal{C}(u(s)), \mathbf{v} \rangle ds + \int_0^t \langle \sigma(s, u(s)) dW(s), \mathbf{v} \rangle + \int_0^t \int_Z \langle \gamma(s, u(s), z), \mathbf{v} \rangle \bar{\pi}(ds, dz), \quad \mathbb{P}\text{-a.s.}
\]

(3.3)

The existence of strong solutions for the SCBF equations (3.1) satisfying the energy equality (Itô’s formula, see (3.5) below) is established in \([38, 39]\), etc. Let us now recall two important concepts of uniqueness of the solution, that is, pathwise uniqueness and uniqueness in law (cf. [21]).

**Definition 3.7.** We say that solutions of problem (3.1) are pathwise unique if and only if the following condition holds:

if \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), W, \pi, u^i)\), \(i = 1, 2\), are such solutions of the problem (3.1) that \(u^i(0) = u_0\), \(i = 1, 2\), then \(\mathbb{P}\text{-a.s. for all } t \in [0, T]\), \(u^1(t) = u^2(t), \ \mathbb{P}\text{-a.s.}\)

**Definition 3.8.** We say that solutions of problem (3.1) are unique in law if and only if the following condition holds:

if \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), W, \pi, u^i)\), \(i = 1, 2\), are such solutions of the problem (3.1) that \(u^i(0) = u_0\), \(i = 1, 2\), then \(\mathcal{L}_{\mathbb{P}^1}(u^1) = \mathcal{L}_{\mathbb{P}^2}(u^2)\).

The strong uniqueness implies weak uniqueness (cf. [36]). For \(u_0 \in \mathbb{H}\), the problem (3.1) admits a weak solution and if it has strong uniqueness property, then for any given \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) and any Brownian motion with covariance \(Q\) and any time homogeneous Poisson random measure \(\pi\) on \((Z, \mathcal{Z})\) on this stochastic basis, the problem (1.1) has a unique strong solution (cf. [36, 43, 47, 50], etc).

The main results of this work are the following:

**Theorem 3.9.** Under Hypothesis 3.3, the system (3.1) has a weak martingale solution \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), \hat{u}, W, \pi)\) in the sense of Definition 3.5. Furthermore, the solution satisfies the estimate:

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|\hat{u}(t)\|_{\mathbb{H}}^2 + \mu \int_0^T \|\hat{u}(t)\|_{\mathbb{H}}^2 dt + \beta \int_0^T \|\hat{u}(t)\|_{\ell^{r+1}}^2 dt \right] < \infty.
\]

(3.4)

**Theorem 3.10.** Let \(d = 2, r \in [1, \infty), d = 3, r \in [3, \infty) (2\beta \mu \geq 1 \text{ for } r = 3)\) and Hypothesis 3.3 be satisfied.

1. There exists a pathwise unique strong solution of the problem (3.1).
2. Moreover, if \(((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}), W, \pi, u)\) is a strong solution of problem (3.1), then for \(\mathbb{P}\text{-almost all } \omega \in \Omega\), the trajectory \(u(\cdot, \omega)\) is equal almost everywhere to a continuous \(\mathbb{H}\text{-valued function defined on } [0, T]\) satisfying the energy equality (Itô’s formula):

\[
\|u(t)\|_{\mathbb{H}}^2 + 2\mu \int_0^t \|u(s)\|_{\mathbb{H}}^2 ds + 2\beta \int_0^t \|u(s)\|_{\ell^{r+1}}^2 ds
\]

\[
= \|u_0\|_{\mathbb{H}}^2 + \int_0^t \langle \sigma(s, u(s)) dW(s), u(s) \rangle + \frac{1}{2} \int_0^t \|\sigma(s, u(s))\|_{\mathbb{H}}^2 ds
\]

\[
+ \int_0^t \|\gamma(s, u(s))\|_{\mathbb{H}}^2 \pi(ds, dz) + 2 \int_0^t \int_Z \langle \gamma(s, u(s)), z \rangle u(s) \bar{\pi}(ds, dz), \quad (3.5)
\]
where \( \vartheta \).

Furthermore, we consider that is, the functions which are right continuous and have left limits at every point \( t \in [0, T] \). The space \( \mathcal{D}([0, T]; \mathcal{E}) \) is endowed with the Skorokhod topology (c.f. \cite{36}).

A sequence \( \{v_m\} \subset \mathcal{D}([0, T]; \mathcal{E}) \) converges to \( v \in \mathcal{D}([0, T]; \mathcal{E}) \) if and only if there exists a sequence \( \{\kappa_m\} \) of homeomorphisms of \( [0, T] \) such that \( \kappa_m \) tends to the identity uniformly on \( [0, T] \) and \( v_m \circ \kappa_m \) tends to \( v \) uniformly on \( [0, T] \). The topology is metrizable by the following metric \( d_T \):

\[
d_T(u, v) := \inf_{\kappa \in \partial_T} \left\{ \sup_{t \in [0, T]} d_1(v(t), u \circ \kappa(t)) + \sup_{t \in [0, T]} |t + \kappa(t)| + \sup_{s \neq t} \left| \log \frac{\kappa(t) - \kappa(s)}{t - s} \right| \right\},
\]

where \( \partial_T \) is the set of increasing homeomorphisms of \( [0, T] \). Moreover, \((\mathcal{D}([0, T]; \mathcal{E}), d_T)\) is a complete metric space.

Let us now recall the concept of modulus of a function \( v \in \mathcal{D}([0, T]; \mathcal{E}) \), which plays the role of modulus of continuity in the space of continuous functions \( C([0, T]; \mathcal{E}) \).

**Definition 3.11** (\cite{36}). Let \( v \in \mathcal{D}([0, T]; \mathcal{E}) \) and \( \delta > 0 \) be given. A modulus for \( v \) is defined by

\[
\mathcal{W}_{[0, T], \mathcal{E}}(v, \delta) := \inf_{\Pi_{\delta}} \max_{t_j \leq s < t_{j+1}} \sup_{t \in [0, T]} d_1(v(t), v(s)),
\]

where \( \Pi_{\delta} \) is the set of all increasing sequences \( \tilde{w} = \{0 = t_0 < t_1 < \cdots < t_m = T\} \) with the following property

\[
t_{j+1} - t_j \geq \delta, \ j = 0, 1, \ldots, m - 1.
\]

For the relative compactness of a subset of the space \( \mathcal{D}([0, T]; \mathcal{E}) \), we have the following result, which is analogous to the Arzelà-Ascoli Theorem.

**Theorem 3.12** (\cite{36}). A set \( K \subset \mathcal{D}([0, T]; \mathcal{E}) \) is precompact if and only if it satisfies the following two conditions:

(a) there exists a dense subset \( J \subset [0, T] \) such that for every \( t \in J \), the set \( \{v(t), v \in K\} \) has compact closure in \( \mathcal{E} \),

(b) \( \limsup_{\delta \to 0} \mathcal{W}_{[0, T], \mathcal{E}}(v, \delta) = 0 \).

3.4. **Deterministic compactness criterion.** Let us introduce the following spaces, which are used frequently in the paper:

- \( \mathcal{D}([0, T]; \mathcal{V} + \overline{\mathcal{L}^r}) := \) the space of càdlàg functions \( v : [0, T] \to \mathcal{V} + \overline{\mathcal{L}^r} \) with the topology \( \mathcal{T}_1 \) induced by the Skorokhod metric \( d_T \),

- \( \mathcal{L}^2_{\pi}(0, T; \mathcal{V}) := \) the space \( \mathcal{L}^2(0, T; \mathcal{V}) \) with the weak topology \( \mathcal{T}_2 \),

- \( \mathcal{L}^{r+1}_{\pi}(0, T; \overline{\mathcal{L}^{r+1}}) := \) the space \( \mathcal{L}^{r+1}(0, T; \overline{\mathcal{L}^{r+1}}) \) with the weak topology \( \mathcal{T}_3 \),

- \( \mathcal{L}^2(0, T; \mathcal{H}) := \) the space of square integrable measurable functions \( v : [0, T] \to \mathcal{H} \) with the strong topology \( \mathcal{T}_4 \).

Furthermore, we consider
• \( D([0, T]; H_w) := \) the space of weakly càdlàg functions \( v : [0, T] \to H \) with the weakest topology \( T_5 \) such that for all \( \psi \in H \), the mapping \( D([0, T]; H_w) \ni v \mapsto (v(\cdot), \psi) \in D([0, T]; \mathbb{R}) \) is continuous. In particular, \( v_n \to v \) in \( D([0, T]; H_w) \) if and only if for all \( \psi \in H : (v_n(\cdot), \psi) \to (v(\cdot), \psi) \) in the space \( D([0, T]; \mathbb{R}) \).

A proof for the following result can be obtained from Theorem 2, \([42]\), which is a generalization of the results available in \([7]\). Recall that the embedding \( V \cap L^{r+1} \to L^{r+1} \) is continuous and the embedding \( V \to H \) is compact.

**Theorem 3.13** (Compactness criterion). Let \( q \in (1, \infty) \) and let

\[ Y_q := D([0, T]; V' + \mathbb{L}_{r+1}^q) \cap L_w^q(0, T; \mathbb{V}) \cap L_w^q(0, T; H) \cap D([0, T]; H_w) \]

and let \( \Upsilon \) be the supremum of of the corresponding topologies. Then a set \( K \subset Y_q \) is \( \Upsilon \)-relatively compact if the following four conditions hold:

1. \( \sup_{u \in K} \|u(t)\|_H < \infty \),
2. \( \sup_{u \in K} \int_0^T \|\tilde{u}(t)\|^q_{V'} dt < \infty \), that is, \( K \) is bounded in \( L^q(0, T; \mathbb{V}) \),
3. \( \sup_{u \in K} \int_0^T \|\tilde{u}(t)\|^q_{L_{r+1}} dt < \infty \), that is, \( K \) is bounded in \( L^q(0, T; \mathbb{L}_{r+1}) \),
4. \( \lim_{\delta \to 0} \sup_{u \in K} W_{[0, T]; V' + \mathbb{L}_{r+1}}(u; \delta) = 0 \).

It should be noted that the space \( Y_q \) is not a Polish space.

3.5. The Aldous condition. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfying the usual conditions. Let \( (\mathcal{E}, d_1) \) be a complete, separable metric space and \( \{y_n\}_{n \in \mathbb{N}} \) be a sequence of \( \mathcal{F}_t \)-adapted and \( \mathcal{E} \)-valued processes.

**Definition 3.14.** Let \( \{y_n\}_{n \in \mathbb{N}} \) be a sequence of \( \mathcal{E} \)-valued processes. The sequence of laws of these processes form a tight sequence if and only if

\[ [T] \text{ for every } \varepsilon, \zeta > 0, \text{ there exists a } \delta > 0 \text{ such that} \]

\[ \sup_{n \in \mathbb{N}} \mathbb{P}\{W_{[0, T]; \mathcal{E}}(y_n, \delta) > \zeta\} \leq \varepsilon, \]

where \( W_{[0, T]; \mathcal{E}} \) is defined in \((3.6)\).

**Definition 3.15.** A sequence \( \{y_n\}_{n \in \mathbb{N}} \) satisfies the Aldous condition in the space \( \mathcal{E} \) if and only if

\[ [A] \text{ for every } \varepsilon, \zeta > 0, \text{ there exists a } \delta > 0 \text{ such that for every sequence of } \mathcal{F}_t \text{-adapted stopping times } \{\tau_n\}_{n \in \mathbb{N}} \text{ with } \tau_n \leq T, \text{ one has} \]

\[ \sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} \mathbb{P}\{d_1(y_n(\tau_n + \theta), y_n(\tau_n)) \geq \zeta\} \leq \varepsilon. \]

A proof of the following result can be found in Theorem 2.2.2, \([22]\).

**Lemma 3.16.** The condition \([A]\) implies Condition \([T]\).

The following result provides a certain condition which guarantees that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) satisfies condition \([A]\) (see Lemma 9, \([42]\)).
Lemma 3.17. Let \((\mathbb{X}, \| \cdot \|_\mathbb{X})\) be a separable Banach space and let \(\{y_n\}_{n \in \mathbb{N}}\) be a sequence of \(\mathbb{X}\)-valued processes. Assume that for every \(\{\tau_n\}_{n \in \mathbb{N}}\) of \(\mathcal{F}_t\)-stopping times with \(\tau_n \leq T\) and for every \(n \in \mathbb{N}\) and \(\theta \geq 0\), the following holds:

\[
E\left[\|y_n(\tau_n + \theta) - y_n\|_\mathbb{X}\right] \leq C\theta^\eta,
\]

for some \(\xi, \eta > 0\) and some constant \(C > 0\). Then the sequence \(\{y_n\}_{n \in \mathbb{N}}\) satisfies the Aldous condition in the space \(\mathbb{X}\).

In view of Theorem 3.13, in order to prove that the law of \(u_n\) is tight, we require the following result:

Corollary 3.18. Let \(q \in (1, \infty)\) and let \(\{u_n\}_{n \in \mathbb{N}}\) be a sequence of càdlàg \(\mathcal{F}_t\)-adapted \(\mathcal{V} + \tilde{\mathcal{L}}^r\)-valued processes such that

(a) there exists a positive constant \(M_1\) such that

\[
\sup_{n \in \mathbb{N}} E\left[\sup_{t \in [0,T]} \|u_n(t)\|_H\right] \leq M_1,
\]

(b) there exists a positive constant \(M_2\) such that

\[
\sup_{n \in \mathbb{N}} E\left[\int_0^T \|u_n(t)\|^q_V dt\right] \leq M_2,
\]

(c) there exists a positive constant \(M_3\) such that

\[
\sup_{n \in \mathbb{N}} E\left[\int_0^T \|u_n(t)\|^q_{\tilde{\mathcal{L}}^{r+1}} dt\right] \leq M_3,
\]

(d) \(\{u_n\}_{n \in \mathbb{N}}\) satisfies the Aldous condition in \(\mathcal{V} + \tilde{\mathcal{L}}^{r+1}\).

Let \(P_n\) be the law of \(u_n\) on \(\mathcal{Y}_q\). Then for every \(\varepsilon > 0\) there exists a compact subset \(\mathcal{K}_\varepsilon\) of \(\mathcal{Y}_q\) such that

\[
P_n(\mathcal{K}_\varepsilon) \geq 1 - \varepsilon.
\]

3.6. The Skorokhod embedding theorem. We first remember the following Jakubowski’s version of the Skorokhod theorem:

Theorem 3.19 (Theorem 2, [23]). Let \((\mathcal{X}, \mathcal{T})\) be a topological space such that there exists a sequence \(\{f_k\}\) of continuous functions \(f_k : \mathcal{X} \to \mathbb{R}\) that separates points of \(\mathcal{X}\). Let \(\{X_n\}\) be a sequence of \(\mathcal{X}\)-valued random variables. Suppose that for every \(\varepsilon > 0\), there exists a compact subset \(K_\varepsilon \subset \mathcal{X}\) such that

\[
\inf_{n \in \mathbb{N}} P(\{X_n \in K_\varepsilon\}) > 1 - \varepsilon.
\]

Then there exist a subsequence \(\{X_{n_k}\}_{k \in \mathbb{N}}\), a sequence \(\{Y_k\}_{k \in \mathbb{N}}\) of \(\mathcal{X}\)-valued random variable and a \(\mathcal{X}\)-valued random variable \(Y\) defined on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

\[
\mathcal{L}(X_{n_k}) = \mathcal{L}(Y_k), \quad k = 1, 2, \ldots,
\]

and for all \(\omega \in \Omega\):

\[
Y_k(\omega) \xrightarrow{\mathcal{T}} Y(\omega) \quad \text{as} \quad k \to \infty.
\]

Let us recall the following version of Skorokhod theorem also (cf. Theorem C.1, [6])
Theorem 3.20. Let $X_1, X_2$ be two separable Banach spaces and let $\pi_i : X_1 \times X_2 \to X_i, i = 1, 2$ be the projection onto $X_i$, that is,

$$X_1 \times X_2 \ni \chi = (\chi_1, \chi_2) \to \pi_i(\chi) \in X_i.$$ 

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\chi_n : \Omega \to X_1 \times X_2, n \in \mathbb{N}$, be a family of random variables such that the sequence $\{\mathcal{L}(\chi_n), n \in \mathbb{N}\}$ is weakly convergent on $X_1 \times X_2$. Finally, let us assume that there exists a random variable $\rho : \Omega \to X_1$ such that $\mathcal{L}(\pi_1 \circ \chi_n) = \mathcal{L}(\rho)$, for all $n \in \mathbb{N}$. Then there exist a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, a family of $X_1 \times X_2$-valued random variables $\{\chi'_n, n \in \mathbb{N}\}$ on $(\Omega', \mathcal{F}', \mathbb{P}')$ and a random variable $\chi' : \Omega' \to X_1 \times X_2$ such that

(i) $\mathcal{L}(\chi'_n) = \mathcal{L}(\chi_n)$, for all $n \in \mathbb{N}$,

(ii) $\chi'_n \to \chi'$ in $X_1 \times X_2$, $\mathbb{P}'$-a.s.,

(iii) $\pi_1 \circ \chi'_n(\omega') = \pi_1 \circ \chi(\omega')$, for all $\omega' \in \Omega'$.

We need the following version of Skorokhod’s embedding theorem (see Corollary 2, [42]).

Theorem 3.21. Let $\mathcal{X}_1$ be a separable complete metric space and $\mathcal{X}_2$ be a topological space such there exists a sequence $\{f_k\}_{k \in \mathbb{N}}$ of continuous functions $f_k : \mathcal{X}_2 \to \mathbb{R}$ separating points of $\mathcal{X}_2$. Let $\mathcal{X} := \mathcal{X}_1 \times \mathcal{X}_2$ with the Tychonoff topology induced by the projections

$$\pi_i = \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i, i = 1, 2.$$ 

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\chi_n : \Omega \to \mathcal{X}_1 \times \mathcal{X}_2, n \in \mathbb{N}$, be a family of random variables such that the sequence $\{\mathcal{L}(\chi_n), n \in \mathbb{N}\}$ is tight on $\mathcal{X}_1 \times \mathcal{X}_2$. Finally, let us assume that there exists a random variable $\rho : \Omega \to \mathcal{X}_1$ such that $\mathcal{L}(\pi_1 \circ \chi_n) = \mathcal{L}(\rho)$, for all $n \in \mathbb{N}$. Then there exist a subsequence $\{\chi_{nk}\}_{k \in \mathbb{N}}$, a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, a family of $\mathcal{X}_1 \times \mathcal{X}_2$-valued random variables $\{\chi'_k, k \in \mathbb{N}\}$ on $(\Omega', \mathcal{F}', \mathbb{P}')$ and a random variable $\chi' : \Omega' \to \mathcal{X}_1 \times \mathcal{X}_2$ such that

(i) $\mathcal{L}(\chi'_k) = \mathcal{L}(\chi_{nk})$, for all $k \in \mathbb{N}$,

(ii) $\chi'_k \to \chi'$ in $\mathcal{X}_1 \times \mathcal{X}_2$, $\mathbb{P}'$-a.s. as $k \to \infty$,

(iii) $\pi_1 \circ \chi'_k(\omega') = \pi_1 \circ \chi(\omega')$, for all $\omega' \in \Omega'$.

4. Existence of Weak Martingale Solution

In this section, we prove Theorem 3.20 using the classical Faedo-Galerkin approximation, a compactness method and a version of the Skorokhod embedding theorem for nonmetric spaces.

4.1. Faedo-Galerkin approximation. Let $\{w_1, \ldots, w_n, \ldots\}$ be a complete orthonormal system in $\mathbb{H}$ belonging to $\text{D}(A) \subset \mathbb{V}$ and let $\mathbb{H}_n$ be the span $\{w_1, \ldots, w_n\}$. One can take $\{w_k\}_{k=1}^\infty \in \text{D}(A) \subset \mathbb{V}$ as the eigenvectors of the Stokes operator $A$. Let $\Pi_n$ denote the orthogonal projection of $\mathbb{V}'$ to $\mathbb{H}_n$, that is, $\Pi_n x = \sum_{i=1}^n \langle x, w_i \rangle w_i$. Since every element $x \in \mathbb{H}$ induces a functional $x^* \in \mathbb{V}'$ by the formula $\langle x^*, y \rangle = \langle x, y \rangle$, $y \in \mathbb{V}$, then $\Pi_n |_{\mathbb{H}}$, the orthogonal projection of $\mathbb{H}$ onto $\mathbb{H}_n$ is given by $\Pi_n x = \sum_{i=1}^n \langle x, w_i \rangle w_i$. In particular, $\Pi_n$ is the orthogonal projection from $\mathbb{H}$ onto $\text{span}\{w_1, \ldots, w_n\}$. Note that $||\Pi_n - I||_{\mathcal{L}(\text{D}(A))} \to 0$ as $n \to \infty$. We define $A^n = \Pi_n A$, $B^n(u_n) = \Pi_n B(u_n)$, $C^n(u_n) = \Pi_n C(u_n)$, $\sigma^n(\cdot, u_n) =$
\(\Pi_n\sigma(\cdot, u_n)\Pi_n\) and \(\gamma^n(\cdot, u_n, \cdot) = \Pi_n\gamma(\cdot, u_n, \cdot)\). We consider the following system of ODEs:

\[
\begin{align*}
(u_n(t), v) &= (u_0^n, v) - \int_0^t \langle \mu A^n u_n(s) + B^n(u_n(s)) + \beta \mathcal{C}^n(u_n(s)), v \rangle ds \\
&+ \int_0^t (\sigma^n(s, u_n(s))dW(s), v) + \int_0^t \int_Z (\gamma^n(s, u_n(s-), z), v)\tilde{\pi}(ds, dz),
\end{align*}
\]

for \(t \in [0, T]\) with \(u_n(0) = u_0^n = \Pi_n u_0\), for all \(v \in \mathbb{H}_n^\infty\). Since \(B^n(\cdot)\) and \(\mathcal{C}^n(\cdot)\) are locally Lipschitz (see (2.6) and (2.8)), and \(\gamma^n(\cdot, u_n, \cdot)\) is globally Lipschitz, the system (4.1) has a unique \(\mathbb{H}_n\)-valued local strong solution \(u_n(\cdot)\) and \(u_n \in L^2(\Omega; L^\infty(0, T^*; \mathbb{H}_n))\) with \(\mathcal{F}_t\)-adapted càdlàg sample paths (Theorem 6.2.3, [2], Theorem 9.1, [21]). Now we discuss the a-priori energy estimates satisfied by the solution to the system (4.1), which also implies that \(T^*\) can be extended to \(T\).

**Proposition 4.1** (Energy estimates). Under Hypothesis 3.3 let \(u_n(\cdot)\) be the unique solution of the system of stochastic ODEs (4.1) with \(u_0 \in L^2(\Omega; \mathbb{H})\). Then, we have

\[
\begin{align*}
\sup_{n \geq 1} \mathbb{E}\left[ \sup_{t \in [0, T]} \|u_n(t)\|^2_{\mathbb{H}} \right] + \mu \sup_{n \geq 1} \mathbb{E}\left[ \int_0^T \|u_n(t)\|^2_{\mathbb{H}} dt \right] + \beta \sup_{n \geq 1} \mathbb{E}\left[ \int_0^T \|u_n(t)\|^{r+1}_{\mathbb{H}^{r+1}} dt \right] \\
\leq (2\mathbb{E}[\|u_0\|^2_{\mathbb{H}}] + CK_1T)e^{CK_2T}.
\end{align*}
\]

Furthermore, for \(u_0 \in L^{2p}(\Omega; \mathbb{H})\), we obtain

\[
\sup_{n \geq 1} \mathbb{E}\left[ \sup_{t \in [0, T]} \|u_n(t)\|^2_{\mathbb{H}} \right] \leq C(\mathbb{E}[\|u_0\|^p_{\mathbb{H}}], p, K_1, K_2, T),
\]

for some \(p > 1\).

**Proof.** We prove the Theorem in the following steps.

**Step (1):** Let us first define a sequence of stopping times \(\tau_N^n\) by

\[
\tau_N^n := \inf_{t \geq 0} \{ t : \|u_n(t)\|_{\mathbb{H}} > N \},
\]

for \(N \in \mathbb{N}\). Applying the finite dimensional Itô formula to the process \(\|u_n(\cdot)\|^2_{\mathbb{H}}\), we obtain

\[
\begin{align*}
\|u_n(t \wedge \tau_N^n)\|^2_{\mathbb{H}} &= \|u_n(0)\|^2_{\mathbb{H}} - 2\int_0^{t \wedge \tau_N^n} \langle \mu A^n u_n(s) + B^n(u_n(s)) + \beta \mathcal{C}^n(u_n(s)), u_n(s) \rangle ds \\
&+ 2\int_0^{t \wedge \tau_N^n} (\sigma^n(s, u_n(s))dW(s), u_n(s)) + \int_0^{t \wedge \tau_N^n} \|\sigma^n(s, u_n(s))\|^2_{\mathbb{H}} ds \\
&+ \int_0^{t \wedge \tau_N^n} \int_Z \|u_n(s-) + \gamma^n(s, u_n(s-), z)\|^2_{\mathbb{H}} - \|u_n(s-\|^2_{\mathbb{H}} \tilde{\pi}(ds, dz) \\
&+ \int_0^{t \wedge \tau_N^n} \int_Z \|u_n(s-) + \gamma^n(s, u_n(s-), z)\|^2_{\mathbb{H}} - \|u_n(s-\|^2_{\mathbb{H}} \\
&- 2(\gamma^n(s, u_n(s-), z), u_n(s-))\| ds \\
&= \|u_0^n\|^2_{\mathbb{H}} - 2\mu \int_0^{t \wedge \tau_N^n} \|u_n(s)\|^2_{\mathbb{H}} ds - 2\beta \int_0^{t \wedge \tau_N^n} \|u_n(s)\|^{r+1}_{\mathbb{H}^{r+1}} ds
\end{align*}
\]
where we have used the fact that $\langle B^n(u_n), u_n \rangle = \langle B(u_n), u_n \rangle = 0$. It can be easily seen that $\|u_n(0)\|_{\mathbb{H}} \leq \|u_0\|_{\mathbb{H}}$. We also know that the processes $\int_0^{t \land \tau_N^n} (\sigma^n(s, u_n(s))dW(s), u_n(s))$ and $\int_Z 2(\gamma^n(s, u_n(s), z) u_n(s))d\tilde{\pi}(ds, dz)$ are martingales with zero expectation, and

$$E \left[ \int_0^{t \land \tau_N^n} \int_Z |\gamma^n(s, u_n(s), z)|^2 \tilde{\pi}(ds, dz) \right] = E \left[ \int_0^{t \land \tau_N^n} \int_Z |\gamma^n(s, u_n(s), z)|^2 \lambda(dz)ds \right].$$

(4.6)

Therefore, taking expectation in (4.5) and using Hypothesis 3.3 (H.2), we deduce that

$$E \left[ \|u_n(t \land \tau_N^n)\|_{\mathbb{H}}^2 \right] + 2\mu \int_0^{t \land \tau_N^n} \|u_n(s)\|_{\mathbb{V}}^2 ds + 2\beta \int_0^{t \land \tau_N^n} \|u_n(s)\|_{L^{r+1}}^{r+1} ds$$

$$= E \left[ \|u_0\|_{\mathbb{H}}^2 \right] + E \int_0^{t \land \tau_N^n} \left( \|\sigma^n(s, u_n(s))\|_{\mathbb{L}^2}^2 + \int_Z |\gamma^n(s, u_n(s), z)|_{\mathbb{H}}^2 \lambda(dz) \right) ds$$

$$\leq E \left[ \|u_0\|_{\mathbb{H}}^2 \right] + K_1 E \int_0^{t \land \tau_N^n} \left( \|u_n(s)\|_{\mathbb{H}}^2 + 2 + \frac{1}{2} \|u_n(s)\|_2^2 \right) ds.$$

(4.7)

An application of Gronwall’s inequality in (4.7) results to

$$E \left[ \|u_n(t \land \tau_N^n)\|_{\mathbb{H}}^2 \right] \leq (E \left[ \|u_0\|_{\mathbb{H}}^2 \right] + K_1 T) e^{K_1 T},$$

(4.8)

for all $t \in [0, T]$. As in [38] [37], one can show that $T \land \tau_N^n \to T$ as $N \to \infty$. Taking limit as $N \to \infty$ in (4.8), using the monotone convergence theorem and then substituting it in (4.7), we arrive at

$$\sup_{n \geq 1} E \left[ \|u_n(t)\|_{\mathbb{H}}^2 \right] + 2\mu \int_0^t \|u_n(s)\|_{\mathbb{V}}^2 ds + 2\beta \int_0^t \|u_n(s)\|_{L^{r+1}}^{r+1} ds$$

$$\leq (E \left[ \|u_0\|_{\mathbb{H}}^2 \right] + K_1 T) e^{2K_1 T},$$

(4.9)

for all $t \in [0, T]$.

Step (2): Next, we prove (4.2). Taking supremum from 0 to $T \land \tau_N^n$ before taking expectation in (4.5), we obtain

$$E \left[ \sup_{t \in [0, T \land \tau_N^n]} \|u_n(t)\|_{\mathbb{H}}^2 \right] + 2\mu \int_0^{T \land \tau_N^n} \|u_n(t)\|_{\mathbb{V}}^2 dt + 2\beta \int_0^{T \land \tau_N^n} \|u_n(t)\|_{L^{r+1}}^{r+1} dt$$

$$\leq E \left[ \|u_0\|_{\mathbb{H}}^2 \right] + E \int_0^{T \land \tau_N^n} \left( \|\sigma^n(t, u_n(t))\|_{\mathbb{L}^2}^2 + \int_Z |\gamma^n(t, u_n(t), z)|_{\mathbb{H}}^2 \lambda(dz) \right) dt$$

$$+ 2E \left[ \sup_{t \in [0, T \land \tau_N^n]} \left( \int_0^t (\sigma^n(s, u_n(s))dW(s), u_n(s)) \right) \right].$$
We estimate the penultimate term from the right hand side of the inequality (4.10) using Burkholder-Davis-Gundy’s (see Theorem 1, [11], [9]), Hölder’s and Young’s inequalities as

\[
2 \mathbb{E} \left[ \sup_{t \in [0, T \wedge \sigma_N^0]} \left| \int_0^t \int_Z (\gamma^n(s, u_n(s), z), u_n(s)) \pi(ds, dz) \right| \right].
\]

(4.10)

Let us now take the final term from the right hand side of the inequality (4.10) and use Burkholder-Davis-Gundy’s (Theorem 1, [20]), Hölder’s and Young’s inequalities to deduce that

\[
2 \mathbb{E} \left[ \sup_{t \in [0, T \wedge \sigma_N^0]} \left| \int_0^t \int_Z (\gamma^n(s, u_n(s), z), u_n(s)) \pi(ds, dz) \right| \right]
\leq 2\sqrt{3} \mathbb{E} \left[ \int_0^{T \wedge \sigma_N^0} \left( \int_Z \gamma^n(t, u_n(t), z) \right)^2 \pi(dt, dz) \right]^{1/2}.
\]

(4.11)

Substituting (4.12) in (4.10) and then using Hypothesis 3.3 (H.2), we find

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \sigma_N^0]} \|u_n(t)\|_{\mathcal{H}}^2 + 4\mu \int_0^{T \wedge \sigma_N^0} \|u_n(t)\|_{\mathcal{Y}}^2 dt + 4\beta \int_0^T \|u_n(t)\|_{E_{r+1}}^2 dt \right]
\leq 2 \mathbb{E} \left[ \|u_0\|_{\mathcal{H}}^2 \right] + 26K_1 \mathbb{E} \left[ \int_0^{T \wedge \sigma_N^0} \left( 1 + \|u_n(t)\|_{\mathcal{H}}^2 \right) dt \right],
\]

(4.13)

Applying Gronwall’s inequality in (4.13), we deduce that

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \sigma_N^0]} \|u_n(t)\|_{\mathcal{H}}^2 \right] \leq (2 \|u_0\|_{\mathcal{H}}^2 + 26K_1 T) e^{26K_1 T}.
\]

(4.14)

Passing \(N \to \infty\), using the monotone convergence theorem and then substituting (4.14) in (4.13), we finally arrive at (4.12).
Step (3): In order to prove (4.3), we apply finite dimensional Itô’s formula to the process \(|\|u_n(\cdot)\|_{2p}^2|\), for some \(p > 1\) to obtain

\[
\begin{align*}
\|u_n(t \wedge \tau_N^p)\|_{2p}^2 &+ 2p \mu \int_0^{t \wedge \tau_N^p} \|u_n(s)\|_{2p}^{2p-2}\|u_n(s)\|_{Y^p}^2d\sigma(s) \\
+ 2p\beta \int_0^{t \wedge \tau_N^p} \|u_n(s)\|_{2p}^{2p-2}\|u_n(s)\|_{\mathcal{L}^{p+1}}^2d\sigma(s) \\
= \|u_n(0)\|_{2p}^2 &+ 2p \int_0^{t \wedge \tau_N^p} \|u_n(s)\|_{2p}^{2p-2}\langle \sigma^n(s, u_n(s))dW(s), u_n(s) \rangle \\
+ p \int_0^{t \wedge \tau_N^p} \|u_n(s)\|_{2p}^{2p-2}\|\sigma^n(s, u_n(s))\|_{\mathcal{L}^q}^2d\sigma(s) \\
&+ 2p(p-1) \int_0^{t \wedge \tau_N^p} \|u_n(t)\|_{2p-4}^{2p-4} \text{Tr}(\langle u_n(s) \otimes u_n(s) \rangle \langle \sigma(s, u_n(s))\sigma^*(s, u_n(s)) \rangle)ds \\
&+ \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s-)+\gamma^n(s, u_n(s-), z)\|_{2p}^{2p} - \|u_n(s)\|_{2p}^2 \pi(ds, dz) \\
&+ \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s-)+\gamma^n(s, u_n(s), z)\|_{2p}^{2p} - \|u_n(s)\|_{2p}^2 \\
&+2p\|u_n(s)\|_{2p}^{2p-2}\langle \gamma^n(s, u_n(s), z), u_n(s) \rangle \lambda(dz)ds \\
&= 2p \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s)\|_{2p}^{2p-2}\langle \gamma^n(s, u_n(s), z), u_n(s) \rangle \pi(ds, dz) \\
&+ 2p \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s-)+\theta\gamma^n(s, u_n(s-), z)\|_{2p}^{2p-2}\|\gamma^n(s, u_n(s-), z)\|_{2p}^2 \\
&+ 2(p-1)\|u_n(s-)+\theta\gamma^n(s, u_n(s-), z)\|_{2p}^{2p-4} \\
&\times |\langle \gamma^n(s, u_n(s), z), \gamma^n(s, u_n(s), z) \rangle |^2 \rho(ds, dz),
\end{align*}
\]

for all \(t \in [0, T]\). Using Taylor’s formula, we obtain

\[
\begin{align*}
\int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s-)+\gamma^n(s, u_n(s-), z)\|_{2p}^{2p} - \|u_n(s)\|_{2p}^2 \pi(ds, dz) \\
+ \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s-)+\gamma^n(s, u_n(s-), z)\|_{2p}^{2p} - \|u_n(s)\|_{2p}^2 \\
-2p\|u_n(s)\|_{2p}^{2p-2}\langle \gamma^n(s, u_n(s), z), u_n(s) \rangle \lambda(dz)ds \\
= 2p \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s)\|_{2p}^{2p-2}\langle \gamma^n(s, u_n(s), z), u_n(s) \rangle \pi(ds, dz) \\
+ 2p \int_0^{t \wedge \tau_N^p} \int_Z \|u_n(s-)+\theta\gamma^n(s, u_n(s-), z)\|_{2p}^{2p-2}\|\gamma^n(s, u_n(s-), z)\|_{2p}^2 \\
+ 2(p-1)\|u_n(s-)+\theta\gamma^n(s, u_n(s-), z)\|_{2p}^{2p-4} \\
\times |\langle \gamma^n(s, u_n(s), z), \gamma^n(s, u_n(s), z) \rangle |^2 \pi(ds, dz),
\end{align*}
\]

for some \(0 < \theta < 1\). Taking supremum over time from 0 to \(T \wedge \tau_N^p\) and then taking expectation in (4.15), we get

\[
\begin{align*}
\mathbb{E}\left[\sup_{t \in [0, T \wedge \tau_N^p]} \|u_n(t)\|_{2p}^{2p}\right] &+ 2p \mu \mathbb{E}\left[\int_0^{T \wedge \tau_N^p} \|u_n(t)\|_{2p}^{2p-2}\|u_n(t)\|_{Y^p}^2dt\right] \\
+ 2p\beta \mathbb{E}\left[\int_0^{T \wedge \tau_N^p} \|u_n(t)\|_{2p}^{2p-2}\|u_n(t)\|_{\mathcal{L}^{p+1}}^2dt\right] \\
\leq \mathbb{E}\left[\|u_0\|_{2p}^{2p}\right] &+ 2p \mathbb{E}\left[\sup_{t \in [0, T \wedge \tau_N^p]} \int_0^t \|u_n(s)\|_{2p}^{2p-2}\langle \sigma^n(s, u_n(s))dW(s), u_n(s) \rangle \right].
\end{align*}
\]
\[ + p(2p - 1) \mathbb{E} \left[ \int_0^{\tau_N^\ast} \|u_n(t)\|_H^{2p-2} \|\sigma_n(t, u_n(t))\|_L^2 dt \right] \]

\[ + 2p \mathbb{E} \left[ \sup_{t \in [0, T]} \int_0^t \int_Z \|u_n(s)\|_H^{2p-2} (\gamma_n(s, u_n(s), z), u_n(s), z) \pi(ds, dz) \right] \]

\[ + 2p(2p - 1) \mathbb{E} \left[ \int_0^{\tau_N^\ast} \int_Z \|u_n(t)\|_H^{2p-2} (\gamma_n(t, u_n(t), z)) \pi(dt, dz) \right] \]

\[ = \mathbb{E} \left[ \|u_0\|_H^{2p} \right] + I_1 + I_2 + I_3 + I_4. \]  

We estimate \( I_1 \) using Burkholder-Davis-Gundy’s (see Theorem 1, [11]), Hölder’s and Young’s inequalities, and Hypothesis 3.3 (H.2) as

\[ I_1 \leq C_p \mathbb{E} \left[ \int_0^{\tau_N^\ast} \|\sigma_n(t, u_n(t))\|_L^2 \|u_n(t)\|_{4p-2} dt \right]^{1/2} \]

\[ \leq C_p \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{4p-1} \left( \int_0^{\tau_N^\ast} \|\sigma_n(t, u_n(t))\|_L^2 dt \right)^{1/2} \right] \]

\[ \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{4p} \right] + C_p \mathbb{E} \left[ \left( \int_0^{\tau_N^\ast} \|\sigma(t, u_n(t))\|_L^2 dt \right) \right] \]

\[ \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{4p} \right] + C_{p, K_1} \left( T + T^{p-1} \mathbb{E} \left[ \int_0^{\tau_N^\ast} \|u_n(t)\|_{2p} dt \right] \right). \]  

Similarly, we estimate \( I_2 \) as

\[ I_2 \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{4p} \right] + C_{p, K_1} \left( T + T^{p-1} \mathbb{E} \left[ \int_0^{\tau_N^\ast} \|u_n(t)\|_{2p} dt \right] \right). \]

For \( p = \frac{q}{2} \), using Hypothesis 3.3 (H.2), Burkholder-Davis-Gundy’s inequality (see Theorem 1, [20]), Corollary 2.4., [51], Hölder’s and Young’s inequalities, we estimate \( I_3 \) as

\[ I_3 \leq C_q \mathbb{E} \left[ \int_0^{\tau_N^\ast} \int_Z \|u_n(t)\|_{2q-2} \|\gamma_n(t, u_n(t), z)\|_H^2 \pi(dt, dz) \right]^{1/2} \]

\[ \leq C_q \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{2q-1} \left( \int_0^{\tau_N^\ast} \int_Z \|\gamma(t, u_n(t), z)\|_H^2 \pi(dt, dz) \right)^{1/2} \right] \]

\[ \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{2q} \right] + C_q \mathbb{E} \left[ \left( \int_0^{\tau_N^\ast} \int_Z \|\gamma(t, u_n(t), z)\|_H^2 \pi(dz, dt) \right) \right]^{\frac{q}{2}} \]

\[ \leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T]} \|u_n(t)\|_{2q} \right] + C_q \mathbb{E} \left[ \int_0^{\tau_N^\ast} \int_Z \|\gamma(t, u_n(t), z)\|_H^2 \lambda(dz) dt \right] \]

\[ + C_q \mathbb{E} \left[ \left( \int_0^{\tau_N^\ast} \int_Z \|\gamma(t, u_n(t), z)\|_H^2 \lambda(dz) dt \right) \right]^{\frac{q}{2}} \]
Once again for \( p = \frac{q}{2} \), we estimate \( I_4 \) using Hypothesis 3.3 (H.2), Hölder’s and Young’s inequalities as

\[
I_4 \leq C_q \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N]} \| u_n(t) \|_{H}^2 + \| u_n(t) \|_{H} \| \gamma_n(t, u_n(t), z) \|_{H}^2 \| \gamma_n(t, u_n(t), z) \|_{H}^2 \lambda(\text{d}z) \right] 
\]

\[
\leq C_q \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N]} \| u_n(t) \|_{H}^2 \int_{N} \| \gamma(t, u_n(t), z) \|_{H}^2 \lambda(\text{d}z) \right] 
\]

\[
+ C_q, K_2 \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} (1 + \| u_n(t) \|_{H}) \text{d}t \right] 
\]

\[
\leq \frac{1}{8} \mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N]} \| u_n(t) \|_{H}^2 \right] + C_q, K_2 \left( T + \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} \| u_n(t) \|_{H}^2 \text{d}t \right] \right) 
\]

\[
+ C_q, K_1 \left( T + T^{\frac{q}{2}} \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} \| u_n(t) \|_{H}^2 \text{d}t \right] \right). 
\]

Combining (4.17)-(4.20) and then substituting it in (4.16), we find

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N]} \| u_n(t) \|_{H}^{2p} \right] + 4p \mu \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} \| u_n(t) \|_{H}^{2p-2} \| u_n(t) \|_{H} \text{d}t \right] 
\]

\[
+ 4p \beta \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} \| u_n(t) \|_{H}^{2p-2} \| u_n(t) \|_{H}^{r+1} \text{d}t \right] 
\]

\[
\leq 2 \mathbb{E} \left[ \| u_0 \|_{H}^{2p} \right] + C_{p, K_2} \left( T + \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} \| u_n(t) \|_{H}^{2p} \text{d}t \right] \right) 
\]

\[
+ C_{p, K_1} \left( T + T^{p-1} \mathbb{E} \left[ \int_{0}^{T \wedge \tau_N} \| u_n(t) \|_{H}^{2p} \text{d}t \right] \right). 
\]

An application of Gronwall’s inequality in (4.21) results in

\[
\mathbb{E} \left[ \sup_{t \in [0, T \wedge \tau_N]} \| u_n(t) \|_{H}^{2p} \right] \leq \left( 2 \mathbb{E} \left[ \| u_0 \|_{H}^{2p} \right] + C_{p, K_2} T \right) e^{C_{p, K_1, K_2} T}. 
\]

Passing \( N \to \infty \), using the monotone convergence theorem, we finally arrive at (4.3). \( \square \)
Remark 4.2. In (4.5), if we take supremum over 0 to T, raise to the power $p = \frac{q}{2}$ and then take expectation, we obtain

$$2^{\frac{q}{2}} \mu^2 \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \|_{\mathcal{L}_q,1}^2 dt \right)^{\frac{q}{2}} \right] + 2^{\frac{q}{2}} \beta^2 \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \|_{\mathbb{L}^{r+1}}^2 dt \right)^{\frac{q}{2}} \right] \leq C_q \left\{ \mathbb{E}[\| u_0 \|_{\mathcal{L}_q}^2] + \mathbb{E} \left[ \left( \int_0^T \| \sigma^n(t, u_n(t)) \|_{\mathcal{L}_q}^2 dt \right)^{\frac{q}{2}} \right] \right. $$

$$+ \mathbb{E} \left[ \left( \int_0^T \int_Z \| \gamma(t, u_n(t), z) \|_{\mathbb{H}^{2/3}}^2 \pi(dt, dz) \right)^{\frac{q}{2}} \right] \right. $$

$$+ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t (\sigma^n(s, u_n(s))dW(s), u_n(s)) \right|^{\frac{q}{2}} \right] $$

$$+ \mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t \int_Z (\gamma^n(s, u_n(s), z), u_n(s)) \tilde{\pi}(ds, dz) \right|^{\frac{q}{2}} \right] \right\} =: C_q \sum_{i=1}^4 J_i. \quad (4.23)$$

Using Hypothesis 3.3 (H.2), we estimate $J_1$ as

$$\mathbb{E} \left[ \left( \int_0^T \| \sigma^n(t, u_n(t)) \|_{\mathcal{L}_q,1}^2 dt \right)^{\frac{q}{2}} \right] \leq T^{\frac{q}{2}} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0, T]} \| u_n(t) \|_{\mathcal{H}^1}^q \right] \right\}. \quad (4.24)$$

Applying Burkholder-Davis-Gundy’s (see Theorem 1, [20]), Hölder’s and Young’s inequalities we estimate $J_3$ as

$$\mathbb{E} \left[ \sup_{t \in [0, T]} \left| \int_0^t (\sigma^n(s, u_n(s))dW(s), u_n(s)) \right|^{\frac{q}{2}} \right] $$

$$\leq C_q \mathbb{E} \left[ \int_0^T \| \sigma^n(t, u_n(t)) \|_{\mathbb{L}_q}^2 \| u_n(t) \|_{\mathbb{H}^1}^2 dt \right]^{\frac{q}{2}} $$

$$\leq \mathbb{E} \left[ \sup_{t \in [0, T]} \| u_n(t) \|_{\mathcal{H}^1}^q \right] + C_q \mathbb{E} \left[ \left( \int_0^T \| \sigma^n(t, u_n(t)) \|_{\mathcal{L}_q}^2 \right)^{\frac{q}{2}} \right] $$

$$\leq C_q T^{\frac{q}{2}} K_1^{\frac{q}{2}} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0, T]} \| u_n(t) \|_{\mathcal{H}^1}^q \right] \right\}. \quad (4.25)$$

Making use of Remark 2.5, [51], we estimate $J_2$ as

$$\mathbb{E} \left[ \left( \int_0^T \int_Z \| \gamma^n(t, u_n(t), z) \|_{\mathbb{H}^{2/3}}^2 \pi(dt, dz) \right)^{\frac{q}{2}} \right] $$

$$\leq C_q \mathbb{E} \left[ \int_0^T \int_Z \| \gamma^n(t, u_n(t), z) \|_{\mathbb{H}^{2/3}}^2 \lambda(dz) dt \right] $$

$$+ C_q \mathbb{E} \left[ \left( \int_0^T \int_Z \| \gamma^n(t, u_n(t), z) \|_{\mathbb{H}^{2/3}}^2 \lambda(dz) dt \right)^{\frac{q}{2}} \right]$$
\[ \leq C_qK_2 \mathbb{E} \left[ \int_0^T (1 + \| u_n(t) \|_{\mathbb{H}}^q) dt \right] + C_qK_1^{\frac{q}{2}} \mathbb{E} \left[ \left( \int_0^T (1 + \| u_n(t) \|_{\mathbb{H}}^q) dt \right)^{\frac{q}{2}} \right] \]

\[ \leq C_{q,T,K_1,K_2} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| u_n(t) \|_{\mathbb{H}}^q \right] \right\}, \quad (4.26) \]

where we have used (4.26) also. Using Burkholder-Davis-Gundy's inequality (Corollary 2.4., [51]) and (4.26), we estimate \( J_4 \) as

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t (\gamma^n(s, u_n(s-), z), u_n(s)) d\pi(s, dz) \right|^q \right] \]

\[ \leq C_q \mathbb{E} \left[ \left( \int_0^T \int_Z \| \gamma^n(t, u_n(t-), z) \|^2 \| u_n(t) \|^2_{\mathbb{H}} d\pi(dt, dz) \right)^{\frac{q}{2}} \right] \]

\[ \leq \mathbb{E} \left[ \sup_{t \in [0,T]} \| u_n(t) \|^q_{\mathbb{H}} \right] + C_q \mathbb{E} \left[ \left( \int_0^T \int_Z \| \gamma^n(t, u_n(t-), z) \|^2 d\pi(dt, dz) \right)^{\frac{q}{2}} \right] \]

\[ \leq C_{q,T,K_1,K_2} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| u_n(t) \|_{\mathbb{H}}^q \right] \right\}, \quad (4.27) \]

Combining (4.25) and (4.27), and then substituting it in (4.23), we deduce that

\[ \sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \|^2_{\mathbb{V}} dt \right)^{\frac{q}{2}} \right] + \sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \|^2_{L^{r+1}} dt \right)^{\frac{q}{2}} \right] \]

\[ \leq C_{\mu,\beta,q,T,K_1,K_2} \left\{ \mathbb{E}[\| u_0 \|^q_{\mathbb{H}}] + \mathbb{E} \left[ \sup_{t \in [0,T]} \| u_n(t) \|^q_{\mathbb{H}} \right] \right\} \]

\[ \leq C(\mathbb{E}[\| u_0 \|^q_{\mathbb{H}}], \mu, \beta, q, T, K_1, K_2), \]

so that we get

\[ \sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \|^2_{\mathbb{V}} dt \right)^{p} \right] + \sup_{n \geq 1} \mathbb{E} \left[ \left( \int_0^T \| u_n(t) \|^2_{L^{r+1}} dt \right)^{p} \right] \]

\[ \leq C(\mathbb{E}[\| u_0 \|^q_{\mathbb{H}}], \mu, \beta, p, T, K_1, K_2), \quad (4.28) \]

for some \( p > 1 \).

4.2. Tightness. In order to prove tightness, we first consider the space

\[ \mathcal{Y} := D([0, T]; \mathcal{V} + \mathbb{L}^{r+1} + \mathcal{L}_w^2(0, T; \mathcal{V}) \cap \mathcal{L}_w^{r+1}(0, T; \mathbb{H}) \cap D([0, T]; \mathbb{H}_w)). \quad (4.29) \]

For each \( n \in \mathbb{N} \), the solution \( u_n(\cdot) \) of the Faedo-Galerkin approximation defines a measure \( \mathcal{L}(u_n) \) on \( (\mathcal{Y}, \mathcal{T}) \), where \( \mathcal{T} = \mathcal{T}_1 \vee \mathcal{T}_2 \vee \mathcal{T}_3 \vee \mathcal{T}_4 \vee \mathcal{T}_5 \). Using Corollary 3.18, our aim is to show that the set of measures \( \mathcal{L}(u_n) \), \( n \in \mathbb{N} \) is tight on \( (\mathcal{Y}, \mathcal{T}) \). The estimates (4.2) and (4.13) in Proposition 4.1 and (4.28) play a crucial role. Nevertheless, in order to prove tightness it is sufficient to use inequality (4.2).

Lemma 4.3. The set of measures \( \{ \mathcal{L}(u_n) : n \in \mathbb{N} \} \) is tight on \( (\mathcal{Y}, \mathcal{T}) \).
Proof. We apply Corollary 3.18 to obtain the proof. The estimates \((4.2)\) and \((4.3)\) imply the conditions (a), (b) and (c). Therefore, it is sufficient to prove that the sequence \(\{u_n\}_{n \in \mathbb{N}}\) satisfies the Aldous condition \([A]\) in the space \(V' + \mathbb{L}^{1/2}\). We use Lemma 3.17 to obtain the required result. Let \(\{\tau_n\}_{n \in \mathbb{N}}\) be a sequence of stopping times such that \(0 \leq \tau_n \leq T\). From \((4.1)\), we have

\[
u_n(t) = u_0^n - \mu \int_0^t \mu A^n u_n(s) - \int_0^t B^n(u_n(s))ds - \beta \int_0^t C^n(u_n(s))ds + \int_0^t \sigma^n(s, u_n(s))d\xi(s) + \int_0^t \gamma^n(s, u_n(s), z)\tilde{\sigma}(ds, dz) =: J^n_1 + \sum_{i=2}^6 J^n_i(t), t \in [0, T].
\]

For \(\theta > 0\), we verify that each term \(J^n_i, i = 1, \ldots, 6\) satisfies the condition \((3.7)\) in Lemma 3.17. For \(J^n_1\), it is clear that the condition \((3.7)\) is satisfied. For \(J^n_2\), we use \((4.2)\) to estimate it as

\[
\mathbb{E}[\|J^n_2(\tau_n + \theta) - J^n_2(\tau_n)\|_{V'}] = \mu \mathbb{E}\left[\left\|\int_{\tau_n}^{\tau_n + \theta} A^n u_n(s)ds \right\|_{V'}\right] \\
\leq C\mu \mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{V'}ds\right] \leq C\mu\theta^{1/2} \mathbb{E}\left[\left(\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{V'}^2 ds\right)^{1/2}\right] \\
\leq C(\mathbb{E}[\|u_0\|_{H^1}^2], K_1, T, \mu)\theta^{1/2},
\]

so that \(J^n_2\) satisfies the condition \((3.7)\) with \(\xi = 1\) and \(\eta = \frac{1}{2}\). Next, for \(d = 2, 3\) and \(r \in [1, 3]\), we consider \(J^n_3\) and estimate it using \((4.2)\), Ladyzhenskaya’s (cf. [28]) and Hölder’s inequalities as

\[
\mathbb{E}[\|J^n_3(\tau_n + \theta) - J^n_3(\tau_n)\|_{V'}] = \mathbb{E}\left[\left\|\int_{\tau_n}^{\tau_n + \theta} B^n(u_n(s))ds \right\|_{V'}\right] \\
\leq C\mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{L^2, H^1}^2 ds\right] \leq C\mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{V'}^2 ds\right]^{1/2} \left(\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{V'}^4 ds\right)^{1/4} \\
\leq C\theta^{-\frac{d}{4}} \mathbb{E}\left[\sup_{s \in [\tau_n, \tau_n + \theta]} \|u_n(s)\|_{L^2, H^1}^{2 - \frac{d}{4}} \left(\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{V'}^4 ds\right)^{\frac{d}{4}}\right] \\
\leq C\theta^{-\frac{d}{4}} \left\{\mathbb{E}\left[\sup_{t \in [0, T]} \|u_n(t)\|_{H^1}^2\right]\right\}^{1 - \frac{d}{4}} \left\{\mathbb{E}\left[\int_0^T \|u_n(s)\|_{V'}^4 ds\right]\right\}^{\frac{d}{4}} \\
\leq C(\mathbb{E}[\|u_0\|_{L^2}^2], K_1, T, \mu)\theta^{-\frac{d}{4}},
\]

which implies that the condition \((3.7)\) is satisfied with \(\xi = 1\) and \(\eta = 1 - \frac{d}{4}\). For \(d = 2, 3\) and \(r \in (3, \infty)\), using \((2.5)\), we estimate \(J^n_3\) as

\[
\mathbb{E}[\|J^n_3(\tau_n + \theta) - J^n_3(\tau_n)\|_{V'}] \leq C\mathbb{E}\left[\int_{\tau_n}^{\tau_n + \theta} \|u_n(s)\|_{L^{r+1}, H}^{\frac{r+1}{r-1}} \|u_n(s)\|_{H^1}^{\frac{r-1}{r-1}} ds\right]
\]
\[ \leq C \theta^{\frac{r-2}{r+1}} \mathbb{E} \left[ \sup_{t \in [\tau_n, \tau_n + \theta]} \| u_n(s) \|_{\mathcal{H}}^{r+1} \left( \int_{\tau_n}^{\tau_n + \theta} \| u_n(s) \|_{\mathcal{L}_r^{\gamma+1}}^2 ds \right)^{\frac{r-2}{r+1}} \right] \]

\[ \leq C \theta^{\frac{r-2}{r+1}} \left\{ \mathbb{E} \left[ \sup_{t \in [0, T]} \| u_n(t) \|_{\mathcal{H}}^2 \right]\right\}^{\frac{r-2}{r+1}} \left\{ \mathbb{E} \left[ \int_0^T \| u_n(t) \|_{\mathcal{L}_r^{\gamma+1}}^2 ds \right]\right\}^{\frac{1}{r+1}} \]

\[ \leq C (\mathbb{E} \left[ \| u_0 \|_{\mathcal{H}}^2 \right], K_1, T, \beta) \theta^{\frac{r-2}{r+1}}, \quad (4.32) \]

so that \( J_3^n \) satisfies the condition (3.7) with \( \xi = 1 \) and \( \eta = \frac{r-2}{r+1} \). Let us estimate \( J_4^n \) using (4.2) as

\[ \mathbb{E} \left[ \| J_4^n (\tau_n + \theta) - J_4^n (\tau_n) \|_{\mathcal{L}_r^{\gamma+1}}^2 \right] = \mathbb{E} \left[ \left\| \int_{\tau_n}^{\tau_n + \theta} \gamma_n (s, u_n(s)) dW(s) \right\|_{\mathcal{L}_r^{\gamma+1}}^2 \right] \]

\[ \leq C (\mathbb{E} \left[ \| u_0 \|_{\mathcal{H}}^2 \right], K_1, T, \beta) \theta^{\frac{1}{r+1}}, \quad (4.33) \]

and thus the condition (3.7) holds with \( \xi = 1 \) and \( \eta = \frac{1}{r+1} \). Using Itô’s isometry, Hypothesis 3.3 (H.2) and (L.2), we estimate \( J_5^n \) as

\[ \mathbb{E} \left[ \| J_5^n (\tau_n + \theta) - J_5^n (\tau_n) \|_{\gamma}^2 \right] \leq C (\mathbb{E} \left[ \| J_5^n (\tau_n + \theta) - J_5^n (\tau_n) \|_{\mathcal{H}}^2 \right] \]

\[ \leq C (\mathbb{E} \left[ \| u_0 \|_{\mathcal{H}}^2 \right], K_1, T, \theta), \quad (4.34) \]

therefore the condition (3.7) is satisfied with \( \xi = 2 \) and \( \eta = 1 \). Similarly, using Itô’s isometry and Hypothesis 3.3 (H.2), we estimate \( J_6^n \) as

\[ \mathbb{E} \left[ \| J_6^n (\tau_n + \theta) - J_6^n (\tau_n) \|_{\gamma}^2 \right] \leq C (\mathbb{E} \left[ \| J_6^n (\tau_n + \theta) - J_6^n (\tau_n) \|_{\mathcal{H}}^2 \right] \]

\[ \leq C (\mathbb{E} \left[ \| u_0 \|_{\mathcal{H}}^2 \right], K_1, T, \theta), \quad (4.35) \]

so that the condition (3.7) hold true with \( \xi = 2 \) and \( \eta = 1 \). Hence, an application of Lemma 3.17 implies that the sequence \( \{ u_n \}_{n \in \mathbb{N}} \) satisfies the Aldous condition in \( \mathcal{V}' + \mathcal{L}_{r+1}^\gamma \). \( \square \)
Let us now prove the main result on the existence of martingale solutions of SCBF equations. The construction of a martingale solution is based on the Skorokhod Theorem for nonmetric spaces.

**Proof of Theorem 3.9.** We prove our main result in the following steps.

**Step (1).** From Lemma 4.3 we know that the set of measures \( \{\mathcal{L}(u_n)\}_{n \in \mathbb{N}} \) is tight on the space \( (\mathcal{Y}, \mathcal{T}) \). Let \( W_n := W, \ n \in \mathbb{N} \). Therefore, the set \( \{\mathcal{L}(W_n)\}_{n \in \mathbb{N}} \) is tight on the space \( C([0, T]; \mathbb{R}) \) of real valued continuous functions on \( [0, T] \) with the standard uniform norm. Let \( \pi_n := \pi, \ n \in \mathbb{N} \). Then the set of measures \( \{\mathcal{L}(\pi_n)\}_{n \in \mathbb{N}} \) is tight on the space \( M_{\mathbb{N}}([0, T] \times \mathbb{Z}) \). Thus the set \( \{\mathcal{L}(u_n, W_n, \pi_n)\}_{n \in \mathbb{N}} \) is tight on \( \mathcal{Y} \times C([0, T]; \mathbb{R}) \times M_{\mathbb{N}}([0, T] \times \mathbb{Z}) \). Since \( L^2(0, T; \mathbb{H}) \) and \( D([0, T]; \mathbb{V} + \overline{\mathbb{V}^+}) \) are separable and completely metrizable spaces, we conclude that on each of these spaces there exists a countable family of continuous real valued mappings separating points. For the space \( L^2_w(0, T; \mathbb{V}) \), it is enough to take

\[
 f_m(u) := \int_0^T (\nabla u(t), \nabla v_m(t)) dt \in \mathbb{R}, \ u \in L^2(0, T; \mathbb{V}), \ m \in \mathbb{N},
\]

where \( \{v_m\}_{m \in \mathbb{N}} \) is a dense subset of \( L^2(0, T; \mathbb{V}) \). Then the sequence \( \{f_m\}_{m \in \mathbb{N}} \) is a continuous real valued mappings separating points of the space \( L^2_w(0, T; \mathbb{V}) \). For the space \( L^{r+1}_w(0, T, \overline{\mathbb{L}}^{r+1}) \), it is enough to consider

\[
 f_m(u) := \int_0^T (|u(t)|^{r-1}u(t), v_m(t)) dt \in \mathbb{R}, \ u \in L^{r+1}(0, T; \overline{\mathbb{L}}^{r+1}), \ m \in \mathbb{N},
\]

where \( \{v_m\}_{m \in \mathbb{N}} \) is a dense subset of \( L^{r+1}(0, T; \overline{\mathbb{L}}^{r+1}) \). Then \( \{f_m\}_{m \in \mathbb{N}} \) is a sequence of continuous real valued mappings separating points of the space \( L^{r+1}_w(0, T; \overline{\mathbb{L}}^{r+1}) \). Let \( \overline{\mathbb{H}} \subset \mathbb{H} \) be a countable and dense subset of \( \mathbb{H} \). Then for each \( h \in \overline{\mathbb{H}} \), the mapping

\[
 D([0, T]; \mathbb{H}_w) \ni u \mapsto (u(\cdot), h) \in D([0, T]; \mathbb{R})
\]

is continuous. Since \( D([0, T]; \mathbb{R}) \) is a separable complete metric space, there exists a sequence \( \{g_\ell\}_{\ell \in \mathbb{N}} \) of real valued continuous functions defined on \( D([0, T]; \mathbb{R}) \) separating points of this space. Then the mappings \( f_{h, \ell} \), \( h \in \overline{\mathbb{H}}, \ell \in \mathbb{N} \) defined by

\[
 f_{h, \ell}(u) := g_\ell((u, h)), \ u \in D([0, T]; \mathbb{H}_w)
\]

form a countable family of continuous mappings on \( D([0, T]; \mathbb{H}_w) \) separating points of this space. Therefore, by Theorem 3.21 (see Remark 2, Appendix B, [42] also), there exists a subsequence \( \{n_k\}_{k \in \mathbb{N}} \), a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and, on this space, \( \mathcal{Y} \times C([0, T]; \mathbb{R}) \times M_{\mathbb{N}}([0, T] \times \mathbb{Z}) \)-valued random variables \( (u^*, W^*, \pi^*) \), \( (\tilde{u}_k, \tilde{W}_k, \tilde{\pi}_k), \ k \in \mathbb{N} \) such that

(i) \( \mathcal{L}((\tilde{u}_k, \tilde{W}_k, \tilde{\pi}_k)) = \mathcal{L}((u_{n_k}, W_{n_k}, \pi_{n_k})) \) for all \( k \in \mathbb{N} \);

(ii) \( \tilde{u}_k, \tilde{W}_k, \tilde{\pi}_k \to (u^*, W^*, \pi^*) \) in \( \mathcal{Y} \times C([0, T]; \mathbb{R}) \times M_{\mathbb{N}}([0, T] \times \mathbb{Z}) \) with probability 1 on \( (\Omega, \mathcal{F}, \tilde{\mathbb{P}}) \) as \( k \to \infty \);

(iii) \( (W_k(\tilde{\omega}), \pi_k(\tilde{\omega})) = (W^*(\tilde{\omega}), \pi^*(\tilde{\omega})) \) for all \( \tilde{\omega} \in \tilde{\Omega} \).

We denote this sequences again by \( \{u_n, W_n, \pi_n\}_{n \in \mathbb{N}} \). Furthermore, \( W_n, \ n \in \mathbb{N} \) and \( W^* \) are \( \mathcal{Q}\)-Wiener processes, and \( \pi_n, \ n \in \mathbb{N} \) and \( \pi^* \) are time homogeneous Poisson random measures on \( (\mathbb{Z}, \mathcal{B}(\mathbb{Z})) \) with intensity measure \( \lambda \) (section 9, [3]). Using the definition of the space \( \mathcal{Y} \), we have

\[
 u_n \to u^* \text{ in } \mathcal{Y}, \tilde{\mathbb{P}}\text{-a.s.,} \quad (4.36)
\]
where \( \mathcal{Y} \) is defined in (4.29). Since the random variables \( \tilde{u}_n(\cdot, \omega) \) and \( u_n(\cdot, \omega) \) are identically distributed, we have the following estimates:

\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}_n(t) \|_{L^2}^2 \right] + \mu \sup_{n \geq 1} \mathbb{E} \left[ \int_0^T \| \tilde{u}_n(t) \|_{L^2}^2 dt \right] + \beta \sup_{n \geq 1} \mathbb{E} \left[ \int_0^T \| \tilde{u}_n(t) \|_{L_{r+1}^2}^2 dt \right] \leq (2\mathbb{E} [\| u_0 \|_{L^2}^2] + CK_1 T) e^{CK_1 T}.
\]

(4.37)

Furthermore, from (4.3) and (4.28), we infer that

\[
\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}_n(t) \|_{H^2}^{2p} \right] + \left( \int_0^T \| \tilde{u}_n(t) \|_{L^2}^2 dt \right)^p + \beta \int_0^T \| \tilde{u}_n(t) \|_{L_{r+1}^2}^2 dt \leq C \left( \mathbb{E} [\| u_0 \|_{H^2}^{2p}], p, \mu, \beta, K_1, K_2, T \right),
\]

(4.38)

for some \( p > 1 \). Using the energy estimates (4.37) and (4.38), an application of the Banach-Alaoglu theorem yields the existence of a subsequence \( \{ \tilde{u}_n \} \) (still denoted by the same symbol), which converges weakly star to \( u^* \) in \( \text{L}^{2p}(\Omega, \mathcal{F}, \mathbb{P}; \text{L}^\infty(0, T; \mathbb{H})) \), weakly to \( u^* \) in \( \text{L}^{2p}(\Omega, \mathcal{F}, \mathbb{P}; \text{L}^2(0, T; \mathbb{V})) \cap \text{L}^{r+1}(\Omega, \mathcal{F}, \mathbb{P}; \text{L}^{r+1}(0, T; \tilde{\mathbb{L}}_{r+1})) \). Moreover, we have

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| u^*(t) \|_{H^2}^{2p} \right] + \int_0^T \| u^*(t) \|_{L^2}^2 dt + \beta \int_0^T \| u^*(t) \|_{L_{r+1}^2}^2 dt \leq (2\mathbb{E} [\| u_0 \|_{H^2}^{2p}] + CK_1 T) e^{CK_1 T}.
\]

(4.39)

and

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \| u^*(t) \|_{H^2}^{2p} + \left( \int_0^T \| u^*(t) \|_{L^2}^2 dt \right)^p + \beta \int_0^T \| u^*(t) \|_{L_{r+1}^2}^2 dt \right] \leq C \left( \mathbb{E} [\| u_0 \|_{H^2}^{2p}], p, \mu, \beta, K_1, K_2, T \right).
\]

(4.40)

**Step (2).** For \( v \in \mathcal{V} \cap \tilde{\mathbb{L}}_{r+1} \), let us denote

\[
\mathcal{H}_n(u_n, W_n, \pi_n)(t) := (\tilde{u}_n^0, v) - \mu \int_0^t \langle A^n \tilde{u}_n(s), v \rangle ds - \int_0^t \langle B^n(\tilde{u}_n(s)), v \rangle ds - \beta \int_0^t \langle C^n(\tilde{u}_n(s)), v \rangle ds + \int_0^t \langle \sigma^n(s, \tilde{u}_n(s)) dW_n(s), v \rangle + \int_0^t \int_Z \langle \gamma^n(s, \tilde{u}_n(s-), z), v \rangle \tilde{\pi}_n(ds, dz),
\]

(4.41)

and

\[
\mathcal{H}(u^*, W^*, \pi^*)(t) := (u_0, v) - \mu \int_0^t \langle Au^*(s), v \rangle ds - \int_0^t \langle B(u^*(s)), v \rangle ds - \beta \int_0^t \langle C(u^*(s)), v \rangle ds + \int_0^t \langle \sigma(s, u^*(s)) dW^*(s), v \rangle + \int_0^t \int_Z \langle \gamma(s, u^*(s-), z), v \rangle \pi^*(ds, dz),
\]

(4.42)

for all \( t \in [0, T] \). Let us now prove that

\[
\lim_{n \to \infty} \| (u_n, v) - (u^*, v) \|_{L^2([0,T] \times \Omega)} = 0
\]

(4.43)
and
\[
\lim_{n \to \infty} \| \mathcal{H}(\bar{u}_n, \bar{W}_n, \pi_n, v) - \mathcal{H}(u^*, \bar{W}^*, \pi^*, v) \|_{L^2((0,T) \times \Omega)} = 0.
\] (4.44)

In order to prove (4.43), we consider
\[
\| (\bar{u}_n, v) - (u^*, v) \|_{L^2((0,T) \times \Omega)}^2 = \mathbb{E} \left[ \int_0^T |(\bar{u}_n(t) - u^*(t), v)|^2 dt \right].
\]

Since \( \bar{u}_n \to u^* \) in \( L^2(0,T; \mathbb{H}) \), \( \tilde{P} \)-a.s. (cf. (4.36)), it is immediate that
\[
\lim_{n \to \infty} \int_0^T |(\bar{u}_n(t) - u^*(t), v)|^2 dt \leq \| v \|_{\mathbb{H}}^2 \lim_{n \to \infty} \int_0^T \| \bar{u}_n(t) - u^*(t) \|^2 dt = 0, \tilde{P} \text{-a.s.} \quad (4.45)
\]

Applying Hölder’s inequality and (4.38), we obtain for some \( p > 1 \)
\[
\mathbb{E} \left[ \left( \int_0^T \| \bar{u}_n(t) - u^*(t) \|_{\mathbb{H}}^2 dt \right)^p \right] \leq C T^p \mathbb{E} \left[ \sup_{t \in [0,T]}(\| \bar{u}_n(t) \|_{\mathbb{H}}^2 + \| u^*(t) \|_{\mathbb{H}}^2) \right] \leq C(\mathbb{E} [\| u_0 \|_{\mathbb{H}}^2], K_1, K_2, T). \quad (4.46)
\]

Using (4.45), (4.46) and Vitali’s Theorem, we deduce that
\[
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |(\bar{u}_n(t) - u^*(t), v)|^2 dt \right] = 0, \quad (4.47)
\]

and hence (4.43) holds true. Let us now prove (4.44). An application of Fubini’s theorem yields
\[
\| \mathcal{H}(\bar{u}_n, \bar{W}_n, \pi_n, v) - \mathcal{H}(u^*, \bar{W}^*, \pi^*, v) \|_{L^2((0,T) \times \Omega)} \leq \int_0^T \mathbb{E} \left[ \| \mathcal{H}(\bar{u}_n, \bar{W}_n, \pi_n, v)(t) - \mathcal{H}(u^*, \bar{W}^*, \pi^*, v)(t) \|^2 \right] dt. \quad (4.48)
\]

(i). Convergence of the initial data. From (4.36), we infer that \( \bar{u}_n \to u^* \) in \( D([0,T]; \mathbb{H}_w) \) and \( u^* \) is right continuous at 0. Therefore, we get \( (\bar{u}_n(0), v) \to (u^*(0), v), \tilde{P} \)-a.s., for all \( v \in \mathbb{H} \). It can be easily seen that
\[
\lim_{n \to \infty} \mathbb{E} \left[ |(\bar{u}_n^0, v) - (u_0, v)|^2 \right] \leq \| v \|_{\mathbb{H}}^2 \lim_{n \to \infty} \| \Pi_n - I \|_{L(\mathbb{H})}^2 \mathbb{E} [\| u_0 \|_{\mathbb{H}}^2] = 0, \quad (4.49)
\]

for all \( v \in \mathbb{H} \).

(ii). Convergence of the linear term. From (4.36), we have \( \bar{u}_n \to u^* \) in \( L^2_w(0,T; \mathbb{V}) \), \( \tilde{P} \)-a.s., so that
\[
\lim_{n \to \infty} \int_0^t \langle A\bar{u}_n(s), v \rangle ds = \lim_{n \to \infty} \int_0^t \langle \nabla \bar{u}_n(s), \nabla v \rangle ds = \int_0^t \langle \nabla u^*(s), \nabla v \rangle ds = \int_0^t \langle A u^*(s), v \rangle ds,
\]

\( \tilde{P} \)-a.s., for all \( v \in \mathbb{V} \). For \( 1 < p \leq 2 \), using Hölder’s inequality, we get
\[
\mathbb{E} \left[ \left( \int_0^t \langle A\bar{u}_n(s), v \rangle ds \right)^p \right] \leq \| v \|^p_{\mathbb{V}} \mathbb{E} \left[ \left( \int_0^t \| \bar{u}_n(s) \|_{\mathbb{V}} ds \right)^p \right] \leq \| v \|^p_{\mathbb{V}} t^{p \frac{p}{2}} \mathbb{E} \left[ \left( \int_0^t \| \bar{u}_n(s) \|_{\mathbb{V}}^2 ds \right) \right] \leq \| v \|^p_{\mathbb{V}} t^{p \frac{p}{2}} \mathbb{E} \left[ \left( \int_0^t \| \bar{u}_n(s) \|_{\mathbb{V}}^2 ds \right) \right] \leq C(\mathbb{E} [\| u_0 \|_{\mathbb{H}}^2], K_1, T) \| v \|^p_{\mathbb{V}} < \infty,
\]

(4.51)
Let us now consider the properties of the operator $B(\cdot)$ and the dominated convergence theorem, one can conclude that for all $t \in [0, T]$, using (4.50), (4.51) and Vitali’s theorem, we deduce

$$\lim_{n \to \infty} \mathbb{E} \left[ \left| \int_0^t \langle A\bar{u}_n(s) - A\bar{u}^*(s), \nu \rangle \, ds \right|^2 \right] = 0,$$

for all $t \in [0, T]$. Using the fact that $\sup_{n \geq 1} \mathbb{E} \left[ \int_0^T \|u_n(t)\|_{L^2}^2 \, dt \right] \leq C(\mathbb{E}[\|u_0\|_{L^2}^2], K_1, \mu, T) < \infty$, and the dominated convergence theorem, one can conclude that

$$\lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \left| \int_0^t \langle A\bar{u}_n(s) - A\bar{u}^*(s), \nu \rangle \, ds \right|^2 \right] \, dt = 0, \quad (4.52)$$

for all $\nu \in \mathcal{V}$. Let us now consider the nonlinear terms.

(iii). Convergence of the Navier-Stokes nonlinearity. Using (2.22), we obtain for all $t \in [0, T]$ and $n \in \mathbb{N}$

$$\int_0^t \langle B^n(\bar{u}_n(s)) - B(\bar{u}^*(s)), \nu \rangle \, ds = \int_0^t \langle B(\bar{u}_n(s)), (\Pi_n - I)\nu \rangle \, ds - \int_0^t \langle B(\bar{u}_n(s) - \bar{u}^*(s), \nu), \bar{u}_n(s) \rangle \, ds \quad \text{Case (1):} \quad d = 2, 3 \quad \text{and} \quad r \in [1, 3].$$

Let us first consider the case $d = 2$ and $r \in [1, 3]$. Using the properties of the operator $B(\cdot)$, and Ladyzhenskaya’s and Hölder’s inequalities, we have

$$|I_1^n(t)| \leq \|(\Pi_n - I)\nu\|_{\mathcal{V}} \int_0^t \|\bar{u}_n(s)\|_{L^4}^2 \, ds \leq C t^{\frac{4-d}{4}} \|\Pi_n - I\|_{L(\mathcal{V})} \|\nu\|_{\mathcal{V}} \sup_{s \in [0, t]} \|\bar{u}_n(s)\|_{L^4}^2 \left( \int_0^t \|\bar{u}_n(s)\|_{L^2}^2 \, ds \right)^{\frac{d}{8}}. \quad (4.54)$$

Since $\|\Pi_n - I\|_{L(\mathcal{V})} \to 0$, $\bar{u}_n \to \bar{u}^*$ in $L^2(0, T; \mathbb{H}) \cap L^2_w(0, T; \mathcal{V})$, $\mathbb{P}$-a.s. and $\bar{u}_n(\cdot)$ satisfies the energy estimate (4.37), we immediately obtain

$$I_1^n(t) \to 0, \quad \mathbb{P}\text{-a.s. as } n \to \infty. \quad (4.55)$$

Let us now consider $I_2^n(t)$. Using Ladyzhenskaya’s and Hölder’s inequalities, we get

$$|I_2^n(t)| \leq \|\nu\|_{\mathcal{V}} \int_0^t \|\bar{u}_n(s) - \bar{u}^*(s)\|_{L^4} \|\bar{u}_n(s)\|_{L^4} \, ds \leq C \|\nu\|_{\mathcal{V}} \int_0^t \|\bar{u}_n(s) - \bar{u}^*(s)\|_{L^4}^\frac{4-d}{4} \|\bar{u}_n(s) - \bar{u}^*(s)\|_{L^2}^\frac{d}{2} \|\bar{u}_n(s)\|_{L^4}^\frac{4-d}{4} \|\bar{u}_n(s)\|_{L^2}^\frac{d}{2} \, ds \leq C t^{\frac{4-d}{8}} \|\nu\|_{\mathcal{V}} \sup_{s \in [0, t]} \|\bar{u}_n(s)\|_{L^4}^\frac{4-d}{4} \left( \int_0^t \|\bar{u}_n(s) - \bar{u}^*(s)\|_{L^2}^2 \, ds \right)^{\frac{d}{8}} \times \left( \int_0^t \|\bar{u}_n(s)\|_{L^2}^2 \, ds \right)^{\frac{d}{8}} \left( \int_0^t \|\bar{u}_n(s)\|_{L^2}^2 \, ds \right)^{\frac{d}{8}}. \quad (4.56)$$
Since $\mathbf{u}_n \to \mathbf{u}^*$ in $L^2(0, T; \mathbb{H}) \cap L^2_{\text{loc}}(0, T; \mathbb{V})$, $\bar{\mathbb{P}}$-a.s. and $\mathbf{u}_n(\cdot)$ satisfies the energy estimate (4.37), we obtain

$$I_2^n(t) \to 0, \bar{\mathbb{P}}\text{-a.s. as } n \to \infty. \quad (4.57)$$

A calculation similar to (4.56) yields that

$$I_3^n(t) \to 0, \bar{\mathbb{P}}\text{-a.s. as } n \to \infty. \quad (4.58)$$

The convergences (4.55), (4.57) and (4.58) imply

$$\lim_{n \to \infty} \int_0^t \langle B^n(\mathbf{u}_n(s)), \mathbf{v} \rangle ds = \int_0^t \langle B(\mathbf{u}^*(s)), \mathbf{v} \rangle ds, \bar{\mathbb{P}}\text{-a.s.}, \quad (4.59)$$

for all $\mathbf{v} \in \mathbb{V}$. For some $p > 1$ and $n \in \mathbb{N}$, we use Ladyzhenskaya’s and Hölder’s inequalities to get

$$\mathbb{E} \left[ \left( \int_0^t \| B^n(\mathbf{u}_n(s)) \|_{\mathbb{V}} ds \right)^p \right] \leq C \| \mathbf{v} \|_{\mathbb{V}}^p \mathbb{E} \left[ \left( \int_0^t \| \mathbf{u}_n(s) \|_{\mathbb{V}}^p \| \mathbf{u}_n(s) \|_{\mathbb{H}}^{2 - \frac{2}{p}} ds \right)^p \right]$$

$$\leq C t^{\frac{4 - d}{4}} \mathbb{E} \left[ \sup_{s \in [0, t]} \| \mathbf{u}_n(s) \|_{\mathbb{V}}^\frac{4}{p} \right] \left\{ \mathbb{E} \left[ \left( \int_0^t \| \mathbf{u}_n(s) \|_{\mathbb{H}}^2 ds \right) \right] \right\}^{\frac{d}{4}}. \quad (4.60)$$

Using (4.59) and (4.60), an application of Vitali’s theorem gives

$$\lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t \langle B^n(\mathbf{u}_n(s)) - B(\mathbf{u}^*(s)), \mathbf{v} \rangle ds \right)^2 \right] = 0, \quad (4.61)$$

for all $t \in [0, T]$ and $\mathbf{v} \in \mathbb{V}$. Since

$$\mathbb{E} \left[ \left( \int_0^t \| B^n(\mathbf{u}_n(s)) \|_{\mathbb{V}} ds \right)^2 \right]$$

$$\leq C t^{\frac{4 - d}{4}} \left\{ \mathbb{E} \left[ \sup_{s \in [0, t]} \| \mathbf{u}_n(s) \|_{\mathbb{V}}^\frac{4}{p} \right] \right\} \left\{ \mathbb{E} \left[ \left( \int_0^t \| \mathbf{u}_n(s) \|_{\mathbb{H}}^2 ds \right) \right] \right\}^{\frac{d}{4}}$$

$$\leq C (\mathbb{E} [\| \mathbf{u}_0 \|_{\mathbb{H}}^4], K_1, K_2, T),$$

an application of dominated convergence theorem yields

$$\lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \left( \int_0^t \langle B^n(\mathbf{u}_n(s)) - B(\mathbf{u}^*(s)), \mathbf{v} \rangle ds \right)^2 \right] dt = 0, \quad (4.62)$$

for all $\mathbf{v} \in \mathbb{V}$.

**Case (2):** $d = 2, 3$ and $r \in (3, \infty)$. For $d = 2, 3$ and $r \in (3, \infty)$, we estimate $|I_1^n(t)|$ using (2.5) and Hölder’s inequality as

$$|I_1^n(t)| \leq \| (\Pi_n - I) \mathbf{v} \|_{\mathbb{V}} \int_0^t \| \mathbf{u}_n(s) \|_{\mathbb{V}}^{\frac{r + 3}{r + 1}} \| \mathbf{u}_n(s) \|_{\mathbb{H}}^{\frac{r - 3}{r + 1}} ds$$

$$\leq t^{\frac{r - 3}{r + 1}} \| (\Pi_n - I) \|_{\mathbb{L}(\mathbb{V})} \| \mathbf{v} \|_{\mathbb{V}} \sup_{s \in [0, t]} \| \mathbf{u}_n(s) \|_{\mathbb{H}}^{\frac{r + 1}{r + 1}} \left( \int_0^t \| \mathbf{u}_n(s) \|_{\mathbb{H}}^{r + 1} ds \right)^{\frac{1}{r + 1}}. \quad (4.63)$$
Since \( \| \Pi_n - I \|_{L(V)} \to 0 \), \( \bar{u}_n \to u^* \) in \( L^2(0, T; \mathbb{H}) \cap L^{r+1}_w(0, T; \tilde{L}^{r+1}) \), \( \bar{u} \)-a.s. and \( \bar{u}_n(\cdot) \) satisfies the energy estimate (4.37), we obtain (4.55). Once again using (2.3), interpolation and H"older’s inequalities, we estimate \( |I_2^n(t)| \) as

\[
|I_2^n(t)| \leq \| v \|_V \int_0^t \| \bar{u}_n(s) - u^*(s) \|_{\mathbb{H}^{2(r+1)}}^2 \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}} ds
\]

\[
\leq \| v \|_V \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}} \| \bar{u}_n(s) - u^*(s) \|_{\mathbb{H}^{r+1}}^2 ds
\]

\[
\leq t^{1/2} \| v \|_V \left( \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}}^{r+1} ds \right)^{1/2} \left( \int_0^t \| \bar{u}_n(s) - u^*(s) \|_{\mathbb{H}^{r+1}}^2 ds \right)^{1/2}
\]

\[
\times \left( \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}}^{r+1} ds + \int_0^t \| u^*(s) \|_{\tilde{L}^{r+1}}^{r+1} ds \right)^{1/2} \tag{4.64}
\]

Since \( \bar{u}_n \to u^* \) in \( L^2(0, T; \mathbb{H}) \cap L^{r+1}_w(0, T; \tilde{L}^{r+1}) \), \( \bar{u} \)-a.s. and \( \bar{u}_n(\cdot) \) satisfies the energy estimate (4.37), one can easily obtain (4.57). The convergence (4.58) can be established in a similar way. The convergence (4.59) follows from (4.55), (4.57) and (4.58).

For some \( 1 < p \leq 2 \), we use (2.5) to find

\[
\mathbb{E} \left[ \left( \int_0^t \langle B^n(\bar{u}_n(s)), v \rangle ds \right)^p \right] \leq \| v \|^p_2 \mathbb{E} \left[ \left( \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}}^{r+1} \| \bar{u}_n(s) \|_{\mathbb{H}^{r+1}}^{r+1} ds \right)^p \right]
\]

\[
\leq t^{p(r-2)/(r-1)} \| v \|^p_2 \mathbb{E} \left[ \sup_{s \leq [0,t]} \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}}^{r+1} \left( \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}}^{r+1} ds \right)^{p(r-3)/(2r-1)} \right]
\]

\[
\leq t^{p(r-2)/(r-1)} \| v \|^p_2 \left\{ \mathbb{E} \left[ \sup_{s \leq [0,t]} \| \bar{u}_n(s) \|_{\mathbb{H}^{r+1}}^2 \right] \right\}^{p(r-3)/(2r-1)} \tag{4.65}
\]

\[
\leq C(\mathbb{E} [\| u_0 \|_{\mathbb{H}^{r+1}}^2], K_1, \beta, T),
\]

for all \( v \in V \). Using (4.59) and (4.65), an application of Vitali’s theorem yields (4.61) and the convergence (4.62) follows.

(iv). Convergence of the Forchheimer nonlinearity. Let us now consider the Forchheimer nonlinearity as

\[
\int_0^t \langle \mathbb{E}^n(\bar{u}_n(s)) - \mathbb{E}(u^*(s)), v \rangle ds
\]

\[
= \int_0^t \langle \mathbb{E}^n(\bar{u}_n(s)), (\Pi_n - I)v \rangle ds + \int_0^t \langle \mathbb{E}(\bar{u}_n(s)) - \mathbb{E}(u^*(s)), v \rangle ds := J_1^n(t) + J_2^n(t). \tag{4.66}
\]

Note that Sobolev’s embedding implies that \( D(A) \subset \tilde{L}^{r+1} \subset \mathbb{H} \) and \( D(A) \) is dense in \( \tilde{L}^{r+1} \). We estimate \( |J_1^n(t)| \) as

\[
|J_1^n(t)| \leq \| (\Pi_n - I)v \|_{\tilde{L}^{r+1}} \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}} ds
\]

\[
\leq Ct^{r/(r+1)} \| (\Pi_n - I) \|_{L(D(A))} \| v \|_{D(A)} \left( \int_0^t \| \bar{u}_n(s) \|_{\tilde{L}^{r+1}}^{r+1} ds \right)^{r/(r+1)}, \tag{4.67}
\]
for all \( v \in D(A) \). Since \( \| \Pi_n - 1 \|_{L(D(A))} \to 0 \), \( \bar{u}_n \to u^* \) in \( L_{w}^{r+1}(0, T; \mathbb{L}^{r+1}) \), \( \bar{u}_n(\cdot) \) satisfies the energy estimate (4.37), we have
\[
J_1^n(t) \to 0, \quad \bar{P}-a.s. \quad a.s. \quad n \to \infty.
\] (4.68)

For \( d = 2, 3 \) and \( r \in [1, 3] \), using Taylor’s formula and Hölder’s and interpolation inequalities, we estimate \( |J_2^n(t)| \) as
\[
|J_2^n(t)| \leq \| v \| \lesssim \int_0^t \| C(\bar{u}_n(s)) - C(u^*(s)) \|_1 ds
\]
\[
\leq C\| v \|_{D(A)} \int_0^t \left( \int_0^1 C'(\theta \bar{u}_n(s) + (1 - \theta)u^*(s)) d\theta \right) \| \bar{u}_n(s) - u^*(s) \|_H ds
\]
\[
\leq C\| v \|_{D(A)} \int_0^t \left( \| \bar{u}_n(s) \|_{\mathcal{L}_2^{r-1}} + \| u^*(s) \|_{\mathcal{L}_2^{r-1}} \right)^{r-1} \| \bar{u}_n(s) - u^*(s) \|_H ds
\]
\[
\leq C\| v \|_{D(A)} \left( \int_0^t \| \bar{u}_n(s) - u^*(s) \|_{\mathcal{L}_2^{r+1}}^2 ds \right)^{1/2}
\]
\[
\times \left[ \left( \int_0^t \| \bar{u}_n(s) \|^{2(r-1)}_{\mathcal{L}_2^{r-1}} ds \right)^{1/2} + \left( \int_0^t \| u^*(s) \|^{2(r-1)}_{\mathcal{L}_2^{r-1}} ds \right)^{1/2} \right]
\]
\[
\leq C\| v \|_{D(A)} \left( \int_0^t \| \bar{u}_n(s) - u^*(s) \|_{\mathcal{L}_2^{r+1}}^2 ds \right)^{1/2}
\]
\[
\times \sup_{s \in [0, t]} \| \bar{u}_n(s) \|_{\mathcal{L}_H}^{2(r-1)} \left( \int_0^t \| \bar{u}_n(s) \|_{r+1}^{r+1} ds \right) \rightarrow \frac{2(r-1)}{r+1}
\]
\[
+ \sup_{s \in [0, t]} \| u^*(s) \|_{\mathcal{L}_H}^{2(r-1)} \left( \int_0^t \| u^*(s) \|_{r+1}^{r+1} ds \right) \rightarrow \frac{2(r-1)}{r+1}.
\] (4.69)

For \( r \in [1, 2] \), one can cease the calculations in the penultimate step in (4.69) as \( |O| < \infty \) and \( \| u \|_{\mathcal{L}_2^{r-1}} \leq \| O \|^{2-r} \| u \|_{\mathcal{L}_H}^{2(r-1)} \). Since \( \bar{u}_n \to u^* \) in \( L^2(0, T; \mathbb{L}_H) \cap L_{w}^{r+1}(0, T; \mathbb{L}^{r+1}) \), \( \bar{u}_n(\cdot) \) and \( u^*(\cdot) \) satisfy the energy estimate (4.37), one can obtain
\[
J_2^n(t) \to 0, \quad \bar{P}-a.s. \quad a.s. \quad n \to \infty.
\] (4.70)

For \( d = 2, 3 \) and \( r \in [3, \infty) \), we estimate \( |J_2^n(t)| \) using Taylor’s formula and Hölder’s and interpolation inequalities as
\[
|J_2^n(t)|
\]
\[
\leq C\| v \|_{D(A)} \left( \int_0^t \| \bar{u}_n(s) - u^*(s) \|_{\mathcal{L}_H}^{2(r-1)} \right)^{1/2} \left( \int_0^t \| \bar{u}_n(s) - u^*(s) \|_{r+1}^{r+1} ds \right)^{1/2}
\]
\[
\times \left[ \left( \int_0^t \| \bar{u}_n(s) \|_{r+1}^{r+1} ds \right)^{1/2} + \left( \int_0^t \| u^*(s) \|_{r+1}^{r+1} ds \right)^{1/2} \right].
\] (4.71)
Since $\mathbf{u}_n \to \mathbf{u}^*$ in $L^2(0, T; \mathbb{H}) \cap L^4_{\text{w}}(0, T; \mathbb{R}^{r+1})$, $\mathbb{P}$-a.s. and $\mathbf{\bar{u}}_n(\cdot)$ and $\mathbf{u}^*(\cdot)$ satisfy the energy estimate (4.37), one can conclude (4.70). The convergences (4.68) and (4.70) imply

$$
\lim_{n \to \infty} \int_0^t \langle C^n(\mathbf{u}_n(s)), \mathbf{v} \rangle ds = \int_0^t \langle C(\mathbf{u}^*(s)), \mathbf{v} \rangle ds, \quad \mathbb{P}\text{-a.s.,}
$$

for all $\mathbf{v} \in D(A)$. Since $D(A)$ is dense in $\mathbb{L}^{r+1}$, we can find a sequence $\mathbf{v}_m \in D(A)$ such that $\|\mathbf{v}_m - \mathbf{v}\|_{\mathbb{L}^{r+1}} \to 0$ as $m \to \infty$, for all $\mathbf{v} \in \mathbb{L}^{r+1}$. Thus, the convergence (4.72) holds true for all $\mathbf{v} \in \mathbb{L}^{r+1}$. For $1 < p \leq 1 + \frac{1}{r}$, let us now consider

$$
\mathbb{E} \left[ \left\| \int_0^t \langle C^n(\mathbf{\bar{u}}_n(s)), \mathbf{v} \rangle ds \right\|_p^p \right] \leq \|\mathbf{v}\|_{\mathbb{L}^{r+1}}^p \mathbb{E} \left[ \left( \int_0^t \|\mathbf{\bar{u}}_n(s)\|_{\mathbb{L}^{r+1}}^{r+1} ds \right)^{\frac{p}{r+1}} \right]
$$

$$
\leq C(\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{H}}^4], K_1, K_2, \beta, T),
$$

for all $\mathbf{v} \in \mathbb{L}^{r+1}$. Using (4.72) and (4.73), an application of Vitali’s theorem implies

$$
\lim_{n \to \infty} \mathbb{E} \left[ \left( \int_0^t \langle C^n(\mathbf{\bar{u}}_n(s)) - C(\mathbf{u}^*(s)), \mathbf{v} \rangle ds \right)^2 \right] = 0,
$$

for all $t \in [0, T]$ and $\mathbf{v} \in \mathbb{L}^{r+1}$. Since

$$
\mathbb{E} \left[ \left\| \int_0^t \langle C^n(\mathbf{\bar{u}}_n(s)) \rangle ds \right\|_{\mathbb{L}^{r+1}}^2 \right] \leq C t^{\frac{2}{r+1}} \left\{ \mathbb{E} \left[ \left( \int_0^t \|\mathbf{\bar{u}}_n(s)\|_{\mathbb{L}^{r+1}}^{r+1} ds \right)^2 \right] \right\}^{\frac{r+1}{r+2}}
$$

$$
\leq C(\mathbb{E}[\|\mathbf{u}_0\|_{\mathbb{H}}^4], K_1, K_2, \beta, T),
$$

an application of the dominated convergence theorem yields

$$
\lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \left( \int_0^t \langle C^n(\mathbf{\bar{u}}_n(s)) - C(\mathbf{u}^*(s)), \mathbf{v} \rangle ds \right)^2 \right] dt = 0,
$$

for all $\mathbf{v} \in \mathbb{L}^{r+1}$.

(v). Convergence of the noise terms. Let us now move to the convergence of the noise terms. We first consider

$$
R^n(t) := \int_0^t (\sigma^n(s, \mathbf{u}_n(s))) d\mathbf{W}^*(s) - \sigma(s, \mathbf{u}^*(s)) d\mathbf{W}^*(s), \mathbf{v})
$$

$$
= \int_0^t (\sigma(s, \mathbf{\bar{u}}_n(s))) d\mathbf{W}^*(s), (\Pi_n - I)\mathbf{v} + \int_0^t ((\sigma(s, \mathbf{\bar{u}}_n(s)) - \sigma(s, \mathbf{u}^*(s))) d\mathbf{W}^*(s), \mathbf{v})
$$

$$
= \sum_{j=1}^{\infty} \int_0^t (\sigma(s, \mathbf{\bar{u}}_n(s))) e_j, (\Pi_n - I)\mathbf{v}) d\beta^*_j(s)
$$

$$
+ \sum_{j=1}^{\infty} \int_0^t ((\sigma(s, \mathbf{\bar{u}}_n(s)) - \sigma(s, \mathbf{u}^*(s))) e_j, \mathbf{v}) d\beta^*_j(s)
$$

$$
= R^n_1(t) + R^n_2(t),
$$

(4.76)
where \{\beta_t\}_{t=1}^\infty is a sequence of one-dimensional Brownian motion. For \(v \in \mathbb{H}\), we estimate the quadratic variation of the process \(R^n\) using Hypothesis \([3,3]\) \((H.2)\) as

\[
[R^n_t] = \sum_{j=1}^\infty \int_0^t \|\sigma(s, \bar{u}_n(s))e_j, (\Pi_n - I)v\|^2 ds \\
\leq \|\Pi_n - I\|_{L(\mathbb{H})}^2 \|v\|^2 \int_0^t \|\sigma(s, \bar{u}_n(s))\|^2_{L(Q)} ds \\
\leq K_1 t \|\Pi_n - I\|_{L(\mathbb{H})}^2 \|v\|^2 \sup_{s \in [0,t]} (1 + \|\bar{u}_n(s)\|^2_{\mathbb{H}}).
\]

Since \(\|\Pi_n - I\|_{L(\mathbb{H})} \rightarrow 0\), \(\bar{u}_n \rightarrow u^*\) in \(L^2(0, T; \mathbb{H})\), \(\mathbb{P}\)-a.s., and \(\bar{u}_n(\cdot)\) satisfies the energy estimate \([1,3,7]\), we deduce that for all \(t \in [0, T]\) and \(v \in \mathbb{H}\),

\[
[R^n_t] \rightarrow 0, \text{ \(\mathbb{P}\)-a.s. as } n \rightarrow \infty. \tag{4.77}
\]

Using Hypothesis \([3,3]\) \((H.3)\), we estimate the quadratic variation of the process \(R^n\) as

\[
[R^n_t] = \sum_{j=1}^\infty \int_0^t \|((\sigma(s, \bar{u}_n(s)) - \sigma(s, u^*(s)))e_j, v\|^2 ds \\
\leq \|v\|^2 \int_0^t \|\sigma(s, \bar{u}_n(s)) - \sigma(s, u^*(s))\|^2_{L(Q)} ds \\
\leq L \|v\|^2 \int_0^t \|\bar{u}_n(s) - u^*(s)\|^2_{\mathbb{H}} ds,
\]

for \(v \in \mathbb{H}\). Since \(\bar{u}_n \rightarrow u^*\) in \(L^2(0, T; \mathbb{H})\), \(\mathbb{P}\)-a.s., we infer that for all \(t \in [0, T]\) and \(v \in \mathbb{V}\),

\[
[R^n_t] \rightarrow 0, \text{ \(\mathbb{P}\)-a.s. as } n \rightarrow \infty. \tag{4.78}
\]

Using \((4.77)\) and \((4.78)\), one can show that the quadratic variation of the process \(R^n\)

\[
[R^n_t] = \sum_{j=1}^\infty \int_0^t \|((\sigma(s, \bar{u}_n(s))e_j, (\Pi_n - I)v) + ((\sigma(s, \bar{u}_n(s)) - \sigma(s, u^*(s)))e_j, v\|^2 ds \\
\leq 2 \sum_{j=1}^\infty \int_0^t \|((\sigma(s, \bar{u}_n(s))e_j, (\Pi_n - I)v)\|^2 ds + 2 \sum_{j=1}^\infty \int_0^t \|((\sigma(s, \bar{u}_n(s)) - \sigma(s, u^*(s)))e_j, v\|^2 ds \\
= 2[R^n_1] + 2[R^n_2] \rightarrow 0, \text{ \(\mathbb{P}\)-a.s. as } n \rightarrow \infty. \tag{4.79}
\]

Furthermore, for some \(p > 1\), we see that for every \(t \in [0, T]\) and \(n \in \mathbb{N}\)

\[
\mathbb{E}\{[R^n_1]^p \} = \mathbb{E}\left[\left(\sum_{j=1}^\infty \int_0^t \|((\sigma(s, \bar{u}_n(s))e_j, (\Pi_n - I)v)\|^2 ds \right)^p \right] \\
\leq \|\Pi_n - I\|_{L(\mathbb{H})}^{2p} \|v\|_{L(Q)}^{2p} \mathbb{E}\left[\left(\int_0^t \|\sigma(s, \bar{u}_n(s))\|^2_{L(Q)} ds \right)^p \right].
\]
\[ \leq C \|v\|_{H^2}^{2p} K_1 \mathbb{E} \left[ \left( \int_0^t (1 + \|\hat{u}_n(s)\|_{H^2}^2) ds \right)^p \right] \]

\[ \leq C_p t^p \|v\|_{H^2}^{2p} \left\{ 1 + \mathbb{E} \left[ \sup_{s \in [0,t]} \|\hat{u}_n(s)\|_{H^2}^{2p} \right] \right\} \]

\[ \leq C \left( \mathbb{E} \left[ \|u_0\|_{H^2}^{2p} \right], p, K_1, K_2, T \right), \quad (4.80) \]

where we have used (4.38). Thus, by using (4.77) and (4.80), an application of Vitali’s theorem yields

\[ \lim_{n \to \infty} \mathbb{E}\{[R^n_t]_t\} = 0 \quad \text{for all } v \in \mathbb{H}. \quad (4.81) \]

For some \( p > 1 \), once again using Hypothesis 3.3 (H.3), (4.38) and (4.40), we get

\[ \mathbb{E}\{[R^n_t]_t\} = \mathbb{E} \left[ \left( \sum_{j=1}^\infty \int_0^t |(\sigma(s, \hat{u}_n(s)) - \sigma(s, u^*(s))) e_j, v) |^2 ds \right)^p \right] \]

\[ \leq \|v\|_{H^2}^{2p} \mathbb{E} \left[ \left( \int_0^t \|\sigma(s, \hat{u}_n(s)) - \sigma(s, u^*(s))\|^2_{L_2} ds \right)^p \right] \]

\[ \leq \|v\|_{H^2}^{2p} L^p \mathbb{E} \left[ \left( \int_0^t \|\hat{u}_n(s) - u^*(s)\|^2_{L^2} ds \right)^p \right] \]

\[ \leq C_p \|v\|_{H^2}^{2p} L^p \left\{ \mathbb{E} \left[ \sup_{s \in [0,t]} \|\hat{u}_n(s)\|_{H^2}^{2p} \right] + \mathbb{E} \left[ \sup_{s \in [0,t]} \|u^*(s)\|_{H^2}^{2p} \right] \right\} \]

\[ \leq C \left( \mathbb{E} \left[ \|u_0\|_{H^2}^{2p} \right], p, K_1, K_2, T \right), \quad (4.82) \]

for all \( n \in \mathbb{N} \). Using (4.78) and (4.82) and Vitali’s theorem, we arrive at

\[ \lim_{n \to \infty} \mathbb{E}\{[R^n_t]_t\} = 0 \quad \text{for all } v \in \mathbb{H}. \quad (4.83) \]

Thus, using Itô’s isometry, (4.81) and (4.83), we deduce that

\[ \mathbb{E} \left[ \left( \int_0^t \left| \frac{\sigma^n(s, \hat{u}_n(s)) - \sigma(s, u^*(s))}{\hat{u}_n(s)} \right| dW^*(s), v \right|^2 \right] \]

\[ = \mathbb{E}\{[R^n_t]_t\} \leq 2 \mathbb{E}\{[R^n_t]_t\} + 2 \mathbb{E}\{[R^n_t]_t\} \to 0 \quad \text{as } n \to \infty. \quad (4.84) \]

Once again an application of Itô’s isometry and (4.37) gives

\[ \mathbb{E} \left[ \int_0^t \left| \frac{\sigma^n(s, \hat{u}_n(s))}{\hat{u}_n(s)} \right| dW^*(s), v \right|^2 \]

\[ \leq \|v\|_{H^2}^2 \mathbb{E} \left[ \int_0^t \|\sigma(s, \hat{u}_n(s))\|^2_{L_2} ds \right] \leq K_1 \|v\|_{H^2}^2 t \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \|\hat{u}_n(s)\|_{H^2}^2 \right] \right\} \]

\[ \leq C \left( \mathbb{E} \left[ \|u_0\|_{H^2}^2 \right], K_1, T \right) \|v\|_{H^2}^2. \quad (4.85) \]

By (4.84), (4.85) and the dominated convergence theorem, we finally have

\[ \lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \left( \int_0^t \left| \frac{\sigma^n(s, \hat{u}_n(s)) - \sigma(s, u^*(s))}{\hat{u}_n(s)} \right| dW^*(s), v \right|^2 \right] dt = 0, \quad (4.86) \]
for all \( \mathbf{v} \in \mathbb{H} \).

Let us now consider
\[
S^n(t) := \int_0^t \int_Z \left( \gamma^n(s, \bar{u}_n(s), z) - \gamma(s, \mathbf{u}^*(s), z), \mathbf{v} \right) \pi^*(ds, dz)
\]
\[
= \int_0^t \int_Z \left( \gamma(s, \bar{u}_n(s), z), (\Pi_n - I) \mathbf{v} \right) \pi^*(ds, dz)
\]
\[
+ \int_0^t \int_Z \left( \gamma(s, \bar{u}_n(s), z) - \gamma(s, \mathbf{u}^*(s), z), \mathbf{v} \right) \pi^*(ds, dz)
\]
\[
= S^n_1(t) + S^n_2(t).
\]

(4.87)

For \( \mathbf{v} \in \mathbb{H} \), using Hypothesis 3.3 (H.2), we estimate the Meyer process of \( S^n_1(t) \) as
\[
\langle S^n_1 \rangle_t = \int_0^t \int_Z \left| \gamma(s, \bar{u}_n(s), z), (\Pi_n - I) \mathbf{v} \right|^2 \lambda(dz) ds
\]
\[
\leq \| (\Pi_n - I) \mathbf{v} \|_2^2 \int_0^t \int_Z \| \gamma(s, \bar{u}_n(s), z) \|^2 \lambda(dz) ds
\]
\[
\leq K_1 \| \Pi_n - I \|_{L(\mathbb{H})} \| \mathbf{v} \|_2^2 \int_0^t \left( 1 + \| \mathbf{u}_n(s) \|_2^2 \right) ds
\]
\[
\leq K_1 t \| \Pi_n - I \|_{L(\mathbb{H})} \| \mathbf{v} \|_2^2 \sup_{s \in [0, t]} \left( 1 + \| \mathbf{u}_n(s) \|_2^2 \right).
\]

Since \( \| \Pi_n - I \|_{L(\mathbb{H})} \to 0 \), \( \bar{u}_n \to \mathbf{u}^* \) in \( L^2(0, T; \mathbb{H}) \), \( \mathbb{P} \)-a.s., and \( \bar{u}_n(\cdot) \) satisfies the energy estimate (4.37), we deduce that for all \( t \in [0, T] \) and \( \mathbf{v} \in \mathbb{H} \),
\[
\langle S^n_1 \rangle_t \to 0, \text{ \( \mathbb{P} \)-a.s. as } n \to \infty.
\]

(4.88)

Using Hypothesis 3.3 (H.3), we estimate the Meyer process of \( S^n_2(t) \) as
\[
\langle S^n_2 \rangle_t = \int_0^t \int_Z \left| \left( \gamma(s, \bar{u}_n(s), z), (\Pi_n - I) \mathbf{v} \right) + \left( \gamma(s, \bar{u}_n(s), z) - \gamma(s, \mathbf{u}^*(s), z), \mathbf{v} \right) \right|^2 \lambda(dz) ds
\]
\[
\leq 2 \int_0^t \int_Z \left| \gamma(s, \bar{u}_n(s), z), (\Pi_n - I) \mathbf{v} \right|^2 \lambda(dz) ds
\]
\[
+ 2 \int_0^t \int_Z \left| \left( \gamma(s, \bar{u}_n(s), z) - \gamma(s, \mathbf{u}^*(s), z), \mathbf{v} \right) \right|^2 \lambda(dz) ds
\]
\[
= 2 \langle S^n_1 \rangle_t + 2 \langle S^n_2 \rangle_t \to 0, \text{ \( \mathbb{P} \)-a.s. as } n \to \infty.
\]

(4.89)

Using (4.88) and (4.89), one can show that the Meyer process of \( S^n \)
\[
\langle S^n \rangle_t = \int_0^t \int_Z \left| \left( \gamma(s, \bar{u}_n(s), z), (\Pi_n - I) \mathbf{v} \right) + \left( \gamma(s, \bar{u}_n(s), z) - \gamma(s, \mathbf{u}^*(s), z), \mathbf{v} \right) \right|^2 \lambda(dz) ds
\]
\[
\leq 2 \int_0^t \int_Z \left| \gamma(s, \bar{u}_n(s), z), (\Pi_n - I) \mathbf{v} \right|^2 \lambda(dz) ds
\]
\[
+ 2 \int_0^t \int_Z \left| \left( \gamma(s, \bar{u}_n(s), z) - \gamma(s, \mathbf{u}^*(s), z), \mathbf{v} \right) \right|^2 \lambda(dz) ds
\]
\[
= 2 \langle S^n_1 \rangle_t + 2 \langle S^n_2 \rangle_t \to 0, \text{ \( \mathbb{P} \)-a.s. as } n \to \infty.
\]
Moreover, for some $p > 1$, we obtain that for every $t \in [0, T]$ and $n \in \mathbb{N}$

$$
\mathbb{E}[\langle (S_n^n)_{n,t} \rangle] = \mathbb{E}\left[ \left( \int_0^t \int_Z |(\gamma(s, \bar{u}_n(s), z), (\Pi_n - I)v)|^2 \lambda(dz)ds \right)^p \right]
$$

$$
\leq \|(\Pi_n - I)v\|^2_{L^p}\mathbb{E}\left[ \left( \int_0^t \int_Z \|\gamma(s, \bar{u}_n(s), z)\|^2_\lambda(dz)ds \right)^p \right]
$$

$$
\leq CK^p_{\Pi}v^2\mathbb{E}\left[ \left( \int_0^t (1 + \|\bar{u}_n(s)\|_H^2)ds \right)^p \right]
$$

$$
\leq Ct^pK^p_{\Pi}v^2\mathbb{E}\left[ \left( 1 + \mathbb{E}\left[ \sup_{s \in [0,t]} \|\bar{u}_n(s)\|_{H}^{2p} \right] \right)^p \right]
$$

$$
\leq C\left( \mathbb{E}\left[ \|u_0\|^{2p}_{H} \right], p, K_1, K_2, T \right),
$$

(4.91)

where we have used (4.38). Thus, by using (4.38) and (4.91), an application of Vitali’s theorem gives

$$
\lim_{n \to \infty} \mathbb{E}[\langle (S_n^n)_{n,t} \rangle] = 0 \text{ for all } v \in H.
$$

(4.92)

Similarly, for some $p > 1$, using (4.38) and (4.40), we find that

$$
\mathbb{E}[\langle (S^{n}_2)_{n,t} \rangle] = \mathbb{E}\left[ \left( \int_0^t \int_Z |(\gamma(s, \bar{u}_n(s), z) - \gamma(s, u^*(s), z)), v)|^2 \lambda(dz)ds \right)^p \right]
$$

$$
\leq \|v\|^2_{L^p}\mathbb{E}\left[ \left( \int_0^t \int_Z \|\gamma(s, \bar{u}_n(s), z) - \gamma(s, u^*(s), z)\|^2_\lambda(dz)ds \right)^p \right]
$$

$$
\leq L^p\|v\|^2_{L^p}\mathbb{E}\left[ \left( \int_0^t \|\bar{u}_n(s) - u^*(s)\|_H^2ds \right)^p \right]
$$

$$
\leq C_p\|v\|^2_{L^p}L^p\left\{ \mathbb{E}\left[ \sup_{s \in [0,t]} \|\bar{u}_n(s)\|_{H}^{2p} \right] + \mathbb{E}\left[ \sup_{s \in [0,t]} \|u^*(s)\|_{H}^{2p} \right] \right\}
$$

$$
\leq C\left( \mathbb{E}\left[ \|u_0\|^{2p}_{H} \right], p, K_1, K_2, T \right),
$$

(4.93)

for all $n \in \mathbb{N}$. Using (4.89) and (4.93) and Vitali’s theorem, we arrive at

$$
\lim_{n \to \infty} \mathbb{E}[\langle (S^{n}_2)_{n,t} \rangle] = 0 \text{ for all } v \in H.
$$

(4.94)

Therefore, using Itô’s isometry, (4.92) and (4.94), we deduce that

$$
\mathbb{E}\left[ \left( \int_0^t \int_Z \left( (\gamma_n(s, \bar{u}_n(s), z) - \gamma(s, u^*(s), z)), v \right) \bar{\pi}^*(ds, dz) \right)^2 \right]
$$

$$
= \mathbb{E}[\langle (S^n)_{n,t} \rangle] \leq 2\mathbb{E}[\langle (S^n)_{n,t} \rangle] + 2\mathbb{E}[\langle (S^n)_{n,t} \rangle] \to 0 \text{ as } n \to \infty.
$$

(4.95)

Once again an application of Itô’s isometry and (4.37) gives

$$
\mathbb{E}\left[ \left( \int_0^t \int_Z \left( (\gamma_n(s, \bar{u}_n(s), z), v) \bar{\pi}^*(ds, dz) \right)^2 \right] = \mathbb{E}\left[ \left( \int_0^t \int_Z (\gamma_n(s, \bar{u}_n(s), z), v)^2 \lambda(dz)ds \right) \right]
$$

$$
= \mathbb{E}[\langle (S^n)_{n,t} \rangle].
$$

(4.96)
Using (4.95) and (4.96), an application of the dominated convergence theorem yields

\[ \lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \left( \| \gamma^n(s, \tilde{u}_n(s-), z) - \gamma(s, u^*(s-), z) \| \rho(z) \right) \rho(z) \; dz \right] \; ds = 0, \quad \mathbb{P}\text{-a.s.,} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \]  

Using (4.95) and (4.96), an application of the dominated convergence theorem yields

\[ \lim_{n \to \infty} \int_0^T \mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{R}^d} \left( \| \gamma^n(s, \tilde{u}_n(s-), z) - \gamma(s, u^*(s-), z) \| \rho(z) \right) \rho(z) \; dz \right) \; ds \right] \; dt = 0, \quad (4.97) \]

for all \( v \in \mathbb{H} \).

**Step (3).** Since \( u_n(\cdot) \) is the solution of Faedo-Galerkin approximation (4.1), for all \( t \in [0, T] \)

\[ (u_n(t), v) = \mathcal{H}_n(u_n, W_n, \pi_n, v)(t), \quad \mathbb{P}\text{-a.s.,} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \]

In particular, we have

\[ \int_0^T \mathbb{E} \left[ \left( u_n(t), v - \mathcal{H}_n(u_n, W_n, \pi_n, v)(t) \right)^2 \right] \; dt = 0, \quad \mathbb{P}\text{-a.s.,} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \]

Since \( \mathcal{L}(u_n, W_n, \pi_n) = \mathcal{L}(\tilde{u}_n, \tilde{W}_n, \tilde{\pi}_n) \), we obtain

\[ \int_0^T \mathbb{E} \left[ \left( \tilde{u}_n(t), v - \mathcal{H}_n(\tilde{u}_n, \tilde{W}_n, \tilde{\pi}_n, v)(t) \right)^2 \right] \; dt = 0, \quad \mathbb{P}\text{-a.s.,} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \]

Furthermore, from the equations (4.43) and (4.44), we deduce

\[ \int_0^T \mathbb{E} \left[ \left( u^*(t), v - \mathcal{H}_n(u^*, W^*, \pi^*, v)(t) \right)^2 \right] \; dt = 0, \quad \mathbb{P}\text{-a.s.,} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \]

Hence, for almost all \( t \in [0, T] \) and \( \mathbb{P} \)-almost all \( \omega \in \Omega \)

\[ (u^*(t), v) - \mathcal{H}_n(u^*, W^*, \pi^*, v)(t) = 0, \quad \text{for all} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \]

That is, for almost all \( t \in [0, T] \) and \( \mathbb{P} \)-almost all \( \omega \in \Omega \), we have

\[ (u^*(t), v) - (u_0, v) + \mu \int_0^t \langle A u^*(s), v \rangle \; ds + \int_0^t \langle B(u^*(s)), v \rangle \; ds \]

\[ + \beta \int_0^t \langle c(u^*(s)), v \rangle \; ds - \int_0^t \langle \sigma(s, u^*(s)) \rangle \; ds - \int_0^t \langle \sigma(s, u^*(s)) \rangle \; ds \]

\[ - \int_0^t \int_{\mathbb{R}^d} \langle \gamma(s, u^*(s-), z), v \rangle \tilde{\pi}^*(ds, dz) = 0, \quad \text{for all} \quad v \in \mathbb{V} \cap \mathbb{H}^r. \quad (4.98) \]

Since \( u^* \) is a \( \mathbb{F} \)-valued random variable, in particular \( u^* \in \mathcal{D}([0, T]; \mathbb{H}_w) \), that is, \( u^* \) is weakly càdlàg. Hence the function on the left-hand side of the above equality is càdlàg with respect to \( t \). Since two càdlàg functions are equal for almost all \( t \in [0, T] \) must be equal for all \( t \in [0, T] \), we infer that (4.98) is satisfied for all \( t \in [0, T] \). Putting \( u^* := \tilde{u}, \quad W^* := \tilde{W} \) and \( \pi^* := \tilde{\pi} \), we obtain that the system \( ((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P}), \tilde{u}, \tilde{W}, \tilde{\pi}) \) is a martingale solution of the system (3.1), which completes the proof. \( \square \)
5. Existence and Uniqueness of Strong Solution

For \(d = 2\), \(r \in [1, \infty)\) and \(d = 3\), \(r \in [3, \infty)\), the martingale solution of convective Brinkman-Forchheimer equations obtained in Theorem 3.9 has stronger regularity properties. We prove that \(\mathbb{P}\text{-a.s.},\) the trajectories are equal almost everywhere to an \(H\)-valued càdlàg function defined on \([0, T]\). Moreover, for \(d = 2\), \(r \in [1, \infty)\) and \(d = 3\), \(r \in [3, \infty)\) \((2\mu \geq 1\) for \(r = 3\)), by showing the pathwise uniqueness, we use the classical Yamada-Watanabe argument to show the existence of a strong solution and uniqueness in law. The existence and uniqueness of a strong solution using global monotonicity property of the linear and nonlinear argument to show the existence of a strong solution and uniqueness in law. The existence and uniqueness of a strong solution using global monotonicity property of the linear and nonlinear operators and a stochastic generalization of the Minty-Browder technique is established in the works [38, 39], etc.

**Proposition 5.1.** Let \(d = 2\), \(r \in [1, 3]\) and Hypothesis 3.3 be satisfied. Let 
\[
(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}, \bar{\mu}, W, \bar{\pi})
\]
be a martingale solution for the problem (3.1) such that
\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \|\bar{u}(t)\|_H^2 + \mu \int_0^T \|\bar{u}(t)\|_V^2 \, dt + \beta \int_0^T \|\bar{u}(t)\|_{L^{r+1}}^{r+1} \, dt \right] < +\infty. \tag{5.1}
\]
Then, for \(\mathbb{P}\text{-almost all } \bar{\omega} \in \bar{\Omega}\), the trajectory \(\bar{u}(\cdot; \omega)\) is equal almost everywhere to a continuous \(H\)-valued function defined on \([0, T]\). Moreover, \(\bar{u}(\cdot)\) satisfies the following Itô formula (energy equality):
\[
\|\bar{u}(t)\|_H^2 + 2\mu \int_0^t \|\bar{u}(s)\|_V^2 \, ds + 2\beta \int_0^t \|\bar{u}(s)\|_{L^{r+1}}^{r+1} \, ds = \|\bar{u}_0\|_H^2 + \int_0^t (\sigma(s, \bar{u}(s)) \, dW(s), \bar{u}(s)) + \frac{1}{2} \int_0^t \|\sigma(s, \bar{u}(s))\|_L^2 \, ds + \int_0^t \|\bar{\gamma}(s, \bar{u}(s), z)\|_H^2 \, ds \, dz + 2 \int_0^t \int_{\mathbb{Z}} \bar{\gamma}(s, \bar{u}(s), z) \, \bar{\pi}(ds, dz), \quad \text{for all } t \in [0, T], \mathbb{P}\text{-a.s.} \tag{5.2}
\]

**Proof.** If \(\bar{u}\) is a martingale solution of the problem (3.1), then in particular \(\bar{u} \in D([0, T]; H_\omega) \cap L^2(0, T; \mathbb{H}) \cap L^{r+1}(0, T; \mathbb{H}^{r+1})\), \(\mathbb{P}\text{-a.s.}\) and
\[
\bar{u}(t) = u_0 - \int_0^t [\mu A\bar{u}(s) + B(\bar{u}(s)) + \beta C(\bar{u}(s))] \, ds + \int_0^t \sigma(s, \bar{u}(s)) \, dW(s)
\]
\[
+ \int_0^t \int_{\mathbb{Z}} \bar{\gamma}(s, \bar{u}(s), z) \, \bar{\pi}(ds, dz), \quad \text{in } \mathbb{V}', \tag{5.3}
\]
since \(\mathbb{V}' \subset \mathbb{H}^{r+1}\). Let us consider the following Stokes equations
\[
\bar{y}(t) = -\mu \int_0^t A\bar{y}(s) \, ds + \int_0^t \sigma(s, \bar{u}(s)) \, dW(s) + \int_0^t \int_{\mathbb{Z}} \bar{\gamma}(s, \bar{u}(s), z) \, \bar{\pi}(ds, dz), \quad \text{in } \mathbb{V}', \tag{5.4}
\]
in \(\mathbb{V}'\). Since \(A : \mathbb{V} \to \mathbb{V}'\) and Hypothesis 3.3 is also satisfied, by the standard existence results for the stochastic Stokes system (cf. [36, 44]), we infer that the system (5.4) has a
unique progressively measurable solution \( \tilde{y} \) such that \( \tilde{y} \in D([0, T]; \mathbb{H}) \cap L^2(0, T; V) \), \( \mathbb{P} \)-a.s.

and

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| \tilde{y}(t) \|_{\mathbb{H}}^2 + \mu \int_0^T \| \tilde{y}(t) \|_V^2 dt \right] \leq C(K_1, T) < \infty. \tag{5.5}
\]

For \( d = 2 \) and \( r \in [1, 3] \), an application of Gagliardo-Nirenberg interpolation inequality yields

\[
\int_0^T \| \tilde{y}(t) \|_{L^{r+1}}^{r+1} dt \leq C \int_0^T \| \tilde{y}(t) \|_H^2 \| \tilde{y}(t) \|_V^{-1} dt \leq C T^{\frac{3-r}{r}} \sup_{t \in [0, T]} \| \tilde{y}(t) \|_H^2 \left( \int_0^T \| \tilde{y}(t) \|_V^2 dt \right)^{\frac{1}{2}}. \tag{5.6}
\]

Let us define

\[
v(t) := \tilde{u}(t) - \tilde{y}(t), \quad t \in [0, T].
\]

For \( \mathbb{P} \)-almost all \( \omega \in \tilde{\Omega} \) the function \( v = v(\cdot, \omega) \) is a weak solution of the following deterministic equation (cf. \cite{26}):

\[
\begin{aligned}
\frac{dv(t)}{dt} &= -\mu A v(t) - B(v(t) + \tilde{y}(t)) - \beta \mathcal{C}(v(t) + \tilde{y}(t)), \\
v(0) &= u_0.
\end{aligned} \tag{5.7}
\]

Let \( \bar{\omega} \in \bar{\Omega} \) be such that \( u(\cdot, \bar{\omega}) \in D([0, T]; \mathbb{H}_w) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \widetilde{L}^{r+1}) \), and \( \tilde{y}(\cdot, \bar{\omega}) \in D([0, T]; \mathbb{H}) \cap L^2(0, T; V) \cap L^{r+1}(0, T; \widetilde{L}^{r+1}) \) be the unique weak solution of the system \( (5.7) \), whose existence and uniqueness is ensured by Theorem 3.7, \cite{26}. By the uniqueness, we obtain for almost all \( t \in [0, T] \)

\[
\bar{v}(t) = \bar{u}(t) - \bar{y}(t).
\]

Let us put

\[
\bar{u}(t) = \tilde{v}(t) + \bar{y}(t), \quad t \in [0, T].
\]

Then \( \bar{u} \in D([0, T]; \mathbb{H}) \) and \( \bar{u}(t) = \bar{u}(t) \) for almost all \( t \in [0, T] \), which completes the proof.

The Itô formula \( (5.2) \) can be established in a similar way as in Theorem 3.6, \cite{39}. One can prove the Itô formula in the following way also. Since \( \bar{v}(\cdot) \) satisfies the energy equality (cf. Theorem 3.7, \cite{26}), we have

\[
\| \bar{v}(t) \|_{\mathbb{H}}^2 = \| u_0 \|_{\mathbb{H}}^2 - 2 \int_0^t \langle \mu A \bar{v}(s) + B(\bar{v}(s) + \bar{y}(s)) + \beta \mathcal{C}(\bar{v}(s) + \bar{y}(s)), \bar{v}(s) \rangle ds, \tag{5.8}
\]

for all \( t \in [0, T] \). An application of Itô’s formula to the process \( \| \tilde{y}(\cdot) \|_{\mathbb{H}}^2 \) yields

\[
\| \tilde{y}(t) \|_{\mathbb{H}}^2 = -2\mu \int_0^t \langle A \bar{y}(s), \tilde{y}(s) \rangle ds + 2 \int_0^t \langle \sigma(s, \bar{u}(s)) d\bar{W}(s), \tilde{y}(s) \rangle + \int_0^t \| \sigma(s, \bar{u}(s)) \|_{L_2}^2 ds + \int_0^t \int_Z \| \gamma(s, \bar{u}(s), z) \|_{\mathbb{H}}^2 \pi(ds, dz) + 2 \int_0^t \int_Z (\gamma(s, \bar{u}(s), z), \bar{y}(s)) \tilde{\pi}(ds, dz), \tag{5.9}
\]

for all \( t \in [0, T] \), \( \mathbb{P} \)-a.s. Using Itô’s product formula, we also have

\[
(\bar{v}(t), \tilde{y}(t)) = -\int_0^t \langle \mu A \bar{v}(s) + B(\bar{v}(s) + \tilde{y}(s)) + \beta \mathcal{C}(\bar{v}(s) + \tilde{y}(s)), \tilde{y}(s) \rangle ds
\]
for all \( t \in [0, T], \mathbb{P}\text{-a.s.} \). Using the fact that \( \bar{u}(t) = \tilde{v}(t) + \bar{y}(t), \ t \in [0, T] \), and combining the above equations, we obtain

\[
\| \bar{u}(t) \|_{H^2}^2 = \| \tilde{v}(t) \|_{H^2}^2 + \| \bar{y}(t) \|_{H^2}^2 + 2(\tilde{v}(t), \bar{y}(t)) \\
= \| u_0 \|_{H^2}^2 - 2\mu \int_0^t \langle \mu A \tilde{u}(s) + B(\tilde{u}(s)) + \beta \mathcal{C}(\tilde{u}(s)), \tilde{u}(s) \rangle \, ds + \int_0^t \| \sigma(s, \tilde{u}(s)) \|_{L^q}^2 \, ds \\
+ 2 \int_0^t (\sigma(s, \tilde{u}(s)) \, dW(s), \bar{y}(s)) + \int_0^t \| \gamma(s, \bar{u}(s)), z \|_{H^1}^2 \, ds, dz \\
+ 2 \int_0^t \int_Z (\gamma(s, \bar{u}(s)), \bar{y}(s)) \, dz, d\bar{u}(s, d\bar{y}(s)), \mathbb{P}\text{-a.s.,} \quad (5.11)
\]

for all \( t \in [0, T] \), which completes the proof of (5.2). \( \square \)

**Remark 5.2.** The above mentioned method won’t work for \( d = 2, 3 \) and \( r \in (3, \infty) \) (for \( d = 3 \) case \( r = 3 \) also), since we need \( \bar{y} \in L^{r+1}(\Omega; L^{r+1}), \mathbb{P}\text{-a.s.} \) for the solvability of the system (5.7). Under Hypothesis 3.3, it is not possible to obtain \( \bar{y} \in L^{r+1}(\Omega; L^{r+1}(0, T; L^{r+1})) \). Therefore, we use the methods available in [38, 39], etc., to show the Itô formula (5.2). For the completeness of the paper, we provide a detailed proof of the next proposition.

**Proposition 5.3.** Let \( d = 2, 3, r \in [3, \infty) \) and Hypothesis 3.3 be satisfied. Let

\[
((\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \mathbb{P}), \bar{\tilde{u}}, \bar{W}, \bar{\pi})
\]

be a martingale solution for the problem (4.1) such that (5.1) be satisfied. Then, for \( \mathbb{P}\text{-almost } \) all \( \bar{\omega} \in \bar{\bar{\Omega}} \), the trajectory \( \bar{\tilde{u}}(\cdot; \bar{\omega}) \) is equal almost everywhere to a continuous \( \mathbb{H} \)-valued function defined on \( [0, T] \) and \( \bar{\tilde{u}}(\cdot) \) satisfies Itô’s formula (5.2).

**Proof.** For \( d = 2, 3, r \in [3, \infty) \), let us now establish the energy equality (Itô’s formula) satisfied by \( \bar{\tilde{u}}(\cdot) \). We use an approximation of \( \bar{\tilde{u}}(\cdot) \) such that the approximations are bounded and converge in both Sobolev and Lebesgue spaces simultaneously (cf. [15] for such a construction in bounded domains and [19] for periodic domains).

**Step (1).** *Approximation:* We approximate \( \bar{u}(t) \), for each \( t \in (0, T) \) using the finite-dimensional space spanned by the first \( n \) eigenfunctions of the Stokes operator as (Theorem 4.3, [15])

\[
\bar{u}_n(t) := \Pi_{1/n} \bar{u}(t) = \sum_{\lambda_j < n^2} e^{-\lambda_j/n}(\bar{u}(t), e_j)e_j. \quad (5.12)
\]

Note that \( D(A) \subset \mathbb{H}^2(\Omega) \subset L^p(\Omega), \) for all \( p \in [2, \infty) \). Let us discuss the properties of the approximation given in (5.12). The authors in [15, 39] proved that such an approximation satisfies for every \( t \in [0, T], \mathbb{P}\text{-a.s.}:

1. \( \bar{u}_n(t) \to \bar{u}(t) \) in \( \mathbb{H}^1_0 \) with \( \| \bar{u}_n(t) \|_{\mathbb{H}^1} \leq C \| \bar{u}(t) \|_{\mathbb{H}^1}, \)
2. \( \bar{u}_n(t) \to \bar{u}(t) \) in \( L^p(\Omega) \) with \( \| \bar{u}_n(t) \|_{L^p} \leq C \| \bar{u}(t) \|_{L^p}, \) for any \( p \in (1, \infty), \)
3. \( \bar{u}_n(t) \) is divergence free and zero on \( \partial\Omega \).
The constant \( C \) appearing in (1) and (2) is absolute. Since \( \tilde{u} \in L^2(\bar{\Omega}; L^\infty(0, T; \mathbb{H})) \cap L^2(\bar{\Omega}; L^2(0, T; V)) \cap L^{r+1}(\bar{\Omega}; L^{r+1}(0, T; \bar{\mathbb{L}}^{r+1})) \), we also obtain

\[
\tilde{u}_n \rightarrow \tilde{u} \in L^2(\bar{\Omega}; L^\infty(0, T; \mathbb{H})) \cap L^2(\bar{\Omega}; L^2(0, T; V)) \cap L^{r+1}(\bar{\Omega}; L^{r+1}(0, T; \bar{\mathbb{L}}^{r+1})),
\]

(5.13)

by an application of the dominated convergence theorem.

In order to complete the proof of energy equality, we use an approximation available in [17] also. Let \( \zeta(t) \) be an even, positive, smooth function with compact support contained in the interval \((-1, 1)\), such that \( \int_{-\infty}^{\infty} \zeta(s)ds = 1 \). Let us denote by \( \zeta^h \), a family of mollifiers related to the function \( \zeta \) as

\[
\zeta^h(s) := \frac{1}{h} \zeta\left(\frac{s}{h}\right), \quad \text{for } h > 0.
\]

In particular, we get \( \int_0^h \zeta^h(s)ds = \frac{1}{2} \). For any function \( v \in L^p(0, T; \mathbb{X}) \), where \( \mathbb{X} \) is a Banach space, for \( p \in [1, \infty) \), we define its mollification in time \( v^h(\cdot) \) as

\[
v^h(s) := (v \ast \zeta^h)(s) = \int_0^T v(\tau)\zeta^h(s - \tau)d\tau, \quad \text{for } h \in (0, T).
\]

From Lemma 2.5, [17], we know that this mollification has the following properties. For any \( v \in L^p(0, T; \mathbb{X}) \), \( v^h \in C^k([0, T]; \mathbb{X}) \), for all \( k \geq 0 \) and

\[
\lim_{h \to 0} \|v^h - v\|_{L^p(0, T; \mathbb{X})} = 0. \tag{5.14}
\]

It should be noted that \( v^h \in \tilde{V}_T = \{v \in C^0_c(0 \times [0, T]; \mathbb{R}^n) : \nabla \cdot v(\cdot, t) = 0\} \) (which is dense in \( L^2(0, T; V) \) and \( L^{r+1}(0, T; \bar{\mathbb{L}}^{r+1}) \), cf. [17]) and \( v^h(x, T) = 0 \). Since \( \tilde{u} \in L^2(\bar{\Omega}; L^\infty(0, T; \mathbb{H})) \cap L^2(\bar{\Omega}; L^2(0, T; V)) \cap L^{r+1}(\bar{\Omega}; L^{r+1}(0, T; \bar{\mathbb{L}}^{r+1})) \), we obtain

\[
\tilde{u}^h \rightarrow \tilde{u} \in L^2(\bar{\Omega}; L^2(0, T; \mathbb{H})) \cap L^2(\bar{\Omega}; L^2(0, T; V)) \cap L^{r+1}(\bar{\Omega}; L^{r+1}(0, T; \bar{\mathbb{L}}^{r+1})),
\]

(5.15)

for some \( p > 1 \) and any \( q \in [1, \infty) \), where \( \tilde{u}^h = \tilde{u} \ast \zeta^h \).  

**Step (2). Itô’s formula:** For some time \( t_1 > 0 \), let us set

\[
\tilde{u}^h_n(t) = \int_0^{t_1} \zeta^h(t - s)\tilde{u}_n(s)ds =: (\zeta^h \ast \tilde{u}_n)(t),
\]

with the parameter \( h \) satisfying \( 0 < h < T - t_1 \) and \( h < t_1 \), where \( \zeta^h \) is the even mollifier given above. It can be easily seen that \( \tilde{u}^h_n \in \tilde{V}_T, \tilde{P}\text{-a.s.} \). Note that \( \tilde{u}^h_n(\cdot) \) satisfies the following Itô stochastic differential:

\[
\tilde{u}^h_n(t) = \tilde{u}^h_n(0) + \int_0^t (\zeta^h \ast \tilde{u}_n)(s)ds. \tag{5.16}
\]

Applying Itô’s product formula to the process \( (\tilde{u}^h_n(\cdot), \tilde{u}(\cdot)) \), we find

\[
(\tilde{u}^h_n(t), \tilde{u}(t)) = (\tilde{u}(0), \tilde{u}^h_n(0)) - \int_0^t (\tilde{u}^h_n(s), \mu A\tilde{u}(s) + B(\tilde{u}(s)) + \beta C(\tilde{u}(s)))ds
\]

\[
+ \int_0^t (\tilde{u}^h_n(s), \sigma(s, \tilde{u}(s))dW(s)) + \int_0^t \int_Z (\tilde{u}^h_n(s-), \gamma(s, \tilde{u}(s-), z))d\pi(ds, dz)
\]

\[
+ \int_0^t (\tilde{u}(s), (\zeta^h \ast \tilde{u}_n)(s))ds + [\tilde{u}^h_n, \tilde{u}]_t, \tag{5.17}
\]
where \([\bar{u}^h_n, \bar{u}]\) is the covariance process between the processes \(\bar{u}_n^h(\cdot)\) and \(\bar{u}(\cdot)\). Using stochastic Fubini’s theorem (Lemma A.1.1, [32]), we find

\[
[\bar{u}^h_n, \bar{u}]_t = \int_0^t \int_0^s \zeta^h(t-s) \left( \int_0^s \Pi_{1/n} \sigma(\tau, \bar{u}(\tau)) dW(\tau) \right) ds
+ \int_0^t \zeta^h(t-s) \left( \int_0^s \int_0^s \Pi_{1/n} \gamma(\tau, \bar{u}(\tau), z) \bar{\pi}(d\tau, dz) \right) ds,
\]

\[
\int_0^t \sigma(\tau, \bar{u}(\tau)) dW(\tau) + \int_0^t \int_0^s \gamma(\tau, \bar{u}(\tau), z) \bar{\pi}(d\tau, dz)
\]

and hence

\[
[\bar{u}^h_n, \bar{u}]_{t_1} = \int_0^{t_1} \int_0^s \zeta^h(t_1-s) ds \left( \Pi_{1/n} \sigma(\tau, \bar{u}(\tau)), \sigma(\tau, \bar{u}(\tau)) \right)_{\mathcal{L}_Q} d\tau
+ \int_0^{t_1} \int_0^s \zeta^h(t_1-s) ds \left( \Pi_{1/n} \gamma(\tau, \bar{u}(\tau), z), \gamma(\tau, \bar{u}(\tau), z) \right) \bar{\pi}(d\tau, dz).
\]

(5.18)

Since the function \(\zeta^h\) is even in \((-h, h)\), we obtain \(\dot{\zeta}^h(r) = -\dot{\zeta}^h(-r)\). Change the order of integration yields (see [19])

\[
\int_0^{t_1} (u(s), \dot{\zeta}^h * u(s)) ds = \int_0^{t_1} \int_0^{t_1} \dot{\zeta}^h(s-\tau)(u(s), u(\tau)) ds d\tau
= -\int_0^{t_1} \int_0^{t_1} \dot{\zeta}^h(\tau-s)(u(s), u(\tau)) ds d\tau = -\int_0^{t_1} \int_0^{t_1} \dot{\zeta}^h(\tau-s)(u(s), u(\tau)) d\tau ds
= - \int_0^{t_1} \int_0^{t_1} \dot{\zeta}^h(s-\tau)(u(\tau), u(s)) ds d\tau = 0.
\]

(5.19)

Thus, from (5.17), we immediately have

\[
(\bar{u}^h_n(t_1), \bar{u}(t_1)) = (\bar{u}(0), \bar{u}^h_n(0)) - \int_0^{t_1} \langle \bar{u}^h_n(s), \mu A \bar{u}(s) + B(\bar{u}(s)) + \beta C(\bar{u}(s)) \rangle ds
+ \int_0^{t_1} \langle \bar{u}^h_n(s), \sigma(s, \bar{u}(s)) dW(s) \rangle + \int_0^{t_1} \int_0^{t_1} \langle \bar{u}^h_n(s-), \gamma(s, \bar{u}(s-), z) \rangle \bar{\pi}(ds, dz)
+ \int_0^{t_1} \left( \int_0^{t_1} \zeta^h(t_1-s) ds \right) \left( \Pi_{1/n} \sigma(\tau, \bar{u}(\tau)), \sigma(\tau, \bar{u}(\tau)) \right)_{\mathcal{L}_Q} d\tau
+ \int_0^{t_1} \left( \int_0^{t_1} \zeta^h(t_1-s) ds \right) \left( \Pi_{1/n} \gamma(\tau, \bar{u}(\tau), z), \gamma(\tau, \bar{u}(\tau), z) \right) \bar{\pi}(d\tau, dz).
\]

(5.20)
Case (1). Limit as $n \to \infty$. Let us first take the limit as $n \to \infty$. Using (5.13), it can be easily seen that
\[
\mathbb{E}[(\tilde{u}_n^h(t_1), \tilde{u}(t_1)) - (\tilde{u}_n^h(t_1), \tilde{u}(t_1))] \leq \mathbb{E}[[\tilde{u}_n^h(t_1) - \tilde{u}_n^h(t_1), \tilde{u}(t_1)]_{t_1}^t] \leq \mathbb{E}[\|\tilde{u}_n^h(t_1) - \tilde{u}_n^h(t_1)\|_{H}^2 + \|\tilde{u}(t_1)\|_{H}^2]^{1/2} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.21}
\]
Similarly, we obtain
\[
\mathbb{E}[(\tilde{u}_n^h(0), \tilde{u}(0)) - (\tilde{u}_n^h(0), \tilde{u}(0))] \to 0 \quad \text{as} \quad n \to \infty. \tag{5.22}
\]
Let us consider
\[
\mathbb{E} \left[ - \int_0^{t_1} \langle \tilde{u}_n^h(s), A\tilde{u}(s) \rangle \, ds + \int_0^{t_1} \langle \tilde{u}_n^h(s), A\tilde{u}(s) \rangle \, ds \right] 
\leq \mathbb{E} \left[ \int_0^{t_1} (\nabla \tilde{u}_n^h(s) - \nabla \tilde{u}_n^h(s), \nabla \tilde{u}(s)) \, ds \right] 
\leq \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) - \tilde{u}_n^h(s) \|_{v}^2 \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}(s) \|_{v}^2 \, ds \right]^{1/2} \to 0 \quad \text{as} \quad n \to \infty, \tag{5.23}
\]
by using (5.13). For the Navier-Stokes nonlinearity, for $d = 2, 3$ and $r = 3$, we have
\[
\mathbb{E} \left[ - \int_0^{t_1} \langle \tilde{u}_n^h(s), B(\tilde{u}(s)) \rangle \, ds + \int_0^{t_1} \langle \tilde{u}_n^h(s), B(\tilde{u}(s)) \rangle \, ds \right] 
\leq \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}(s) \|_{L^4}^2 \| \tilde{u}_n^h(s) - \tilde{u}_n^h(s) \|_{v} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}(s) \|_{L^4}^4 \, ds \right]^{1/2} \to 0 \quad \text{as} \quad n \to \infty. \tag{5.24}
\]
For the case $d = 2, 3$ and $r \in (3, \infty)$, we use (2.4) and Hölder’s inequality to estimate the Navier-Stokes nonlinearity as
\[
\mathbb{E} \left[ - \int_0^{t_1} \langle \tilde{u}_n^h(s), B(\tilde{u}(s)) \rangle \, ds + \int_0^{t_1} \langle \tilde{u}_n^h(s), B(\tilde{u}(s)) \rangle \, ds \right] 
\leq \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}(s) \|_{L^{r+1}} \| \tilde{u}_n^h(s) - \tilde{u}_n^h(s) \|_{v} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}(s) \|_{L^{r+1}} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) \|_{L^{r+1}} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) \|_{L^{r+1}} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) \|_{L^{r+1}} \, ds \right] \to 0 \quad \text{as} \quad n \to \infty. \tag{5.25}
\]
We estimate the Forchheimer nonlinearity using (5.13) as
\[
\mathbb{E} \left[ - \int_0^{t_1} \langle \tilde{u}_n^h(s), C(\tilde{u}(s)) \rangle \, ds + \int_0^{t_1} \langle \tilde{u}_n^h(s), C(\tilde{u}(s)) \rangle \, ds \right] 
\leq \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) - \tilde{u}_n^h(s) \|_{L^{r+1}} \| \tilde{u}(s) \|_{L^{r+1}} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) \|_{L^{r+1}} \, ds \right] \mathbb{E} \left[ \int_0^{t_1} \| \tilde{u}_n^h(s) \|_{L^{r+1}} \, ds \right] \to 0 \quad \text{as} \quad n \to \infty. \tag{5.26}
\]
Applying Hypothesis [3.3] (H.3), Burkholder-Davis-Gundy’s and Hölder’s inequalities, we find

$$
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \left| \int_0^t (\tilde{u}_n^h(s), \sigma(s, \tilde{u}(s)) dW(s)) - \int_0^t (\tilde{u}_n^h(s), \sigma(s, \tilde{u}(s)) dW(s)) \right| \right] \\
\leq \sqrt{3K_1 T} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}_n^h(t) - \tilde{u}^h(t) \|^2_{\mathcal{H}} \right] \right\}^{1/2} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}(t) \|^2_{\mathcal{H}} \right] \right\}^{1/2} \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
$$

(5.27)

Once again an application of Burkholder-Davis-Gundy’s and Hölder’s inequalities yield

$$
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \int_Z (\tilde{u}_n^h(s-), \tilde{u}^h(s-), \gamma(s, \tilde{u}(s-), z)) \tilde{\pi}(ds, dz) \right] \\
\leq \sqrt{3K_1 T} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}_n^h(t) - \tilde{u}^h(t) \|^2_{\mathcal{H}} \right] \right\}^{1/2} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}(t) \|^2_{\mathcal{H}} \right] \right\}^{1/2} \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
$$

(5.28)

Let us now consider

$$
\begin{align*}
\mathbb{E} \left[ \int_0^{\tau^1} \left( \int_0^{t_1} \gamma^h(t_1 - s) ds \right) \left( \Pi_{1/n} \sigma(\tau, \tilde{u}(\tau)), \sigma(\tau, \tilde{u}(\tau)) \right) \mathcal{L}_Q - \| \sigma(\tau, \tilde{u}(\tau)) \|^2_{\mathcal{L}_Q} \right] d\tau \\
\leq \| \Pi_{1/n} - I \|_{\mathcal{L}(H)} \mathbb{E} \left[ \int_0^{\tau^1} \left( \int_0^{t_1} \gamma^h(t_1 - s) ds \right) \| \sigma(\tau, \tilde{u}(\tau)) \|^2_{\mathcal{L}_Q} d\tau \right] \\
\leq \| \Pi_{1/n} - I \|_{\mathcal{L}(H)} \mathbb{E} \left[ \int_0^{\tau^1} \| \sigma(\tau, \tilde{u}(\tau)) \|^2_{\mathcal{L}_Q} d\tau \right] \\
\leq K_1 T \| \Pi_{1/n} - I \|_{\mathcal{L}(H)} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}(t) \|^2_{\mathcal{H}} \right] \right\} \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
$$

(5.29)

Finally, we have

$$
\begin{align*}
\mathbb{E} \left[ \int_0^{\tau_1} \left( \int_0^{\tau_1} \gamma^h(s, z) ds \right) \left( \Pi_{1/n} \gamma(\tau, \tilde{u}(\tau-), z), \gamma(\tau, \tilde{u}(\tau-), z) \right) \tilde{\pi}(d\tau, dz) \right] \\
\leq \| \Pi_{1/n} - I \|_{\mathcal{L}(H)} \mathbb{E} \left[ \int_0^{\tau_1} \left( \int_0^{\tau_1} \gamma^h(s, z) ds \right) \| \gamma(\tau, \tilde{u}(\tau-), z) \|^2_{\mathcal{H}} \tilde{\pi}(d\tau, dz) \right] \\
\leq \| \Pi_{1/n} - I \|_{\mathcal{L}(H)} \mathbb{E} \left[ \int_0^{\tau_1} \| \gamma(\tau, \tilde{u}(\tau), z) \|^2_{\mathcal{H}} \lambda(dz) d\tau \right] \\
\leq K_1 T \| \Pi_{1/n} - I \|_{\mathcal{L}(H)} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| \tilde{u}(t) \|^2_{\mathcal{H}} \right] \right\} \\
\rightarrow 0 \text{ as } n \rightarrow \infty.
\end{align*}
$$

(5.30)
Using the convergences \((5.21)-(5.30)\), along a subsequence (still denoted by the same symbol), one can pass limit \(n \to \infty\) in \((5.20)\) to obtain

\[
(u^h(t_1), u(t_1)) = (u^h(0), u(0)) - \int_0^{t_1} \langle \dot{u}^h(s), \mu A \dot{u}(s) + B(\dot{u}(s)) + \beta \mathcal{C}(\dot{u}(s)) \rangle \, ds \\
+ \int_0^{t_1} \langle \dot{u}^h(s), \sigma(s, \dot{u}(s)) \, dW(s) \rangle + \int_0^{t_1} \int_0^Z \langle u^h(s-), \gamma(s, \dot{u}(s-), z) \rangle \, d\bar{\mathcal{P}}, d\, dz \\
+ \int_0^{t_1} \left( \int_0^t \zeta^h(t_1-s) \, ds \right) \| \sigma(\tau, \dot{u}(\tau)) \|^2_{L^2_Q} \, d\tau \\
+ \int_0^{t_1} \int_0^Z \left( \int_0^t \zeta^h(t_1-s) \, ds \right) \| \gamma(\tau, \dot{u}(\tau-), z) \|^2_{H} \, d\tau, d\, dz, \ \bar{P}\text{-a.s.}
\]

**Case (2).** Limit as \(h \to 0\). Let us now pass \(h \to 0\). Since \(\bar{u}^h \to u \in L^2(\bar{\Gamma}; L^2(0, T; \mathcal{H})) \cap L^{r+1}(\bar{\Omega}; L^{r+1}(0, T; \bar{\mathcal{L}}^{r+1}))\), the convergence

\[
\mathbb{E} \left[ - \int_0^{t_1} \langle \dot{u}^h(s), \mu A \dot{u}(s) + B(\dot{u}(s)) + \beta \mathcal{C}(\dot{u}(s)) \rangle \, ds \\
+ \int_0^{t_1} \langle \dot{u}(s), \mu A \dot{u}(s) + B(\dot{u}(s)) + \beta \mathcal{C}(\dot{u}(s)) \rangle \, ds \right] \to 0 \text{ as } h \to 0
\]

follows similarly as in \((5.23)-(5.26)\). The fact that \(\int_0^{t_1} ((\bar{u}(s) \cdot \nabla) \bar{u}(s), \bar{u}(s)) \, ds = 0\) can be obtained in a similar way as in Theorem 4.1, [17]. Using Itô’s isometry the fact that \(\bar{u}^h \to u\) in \(L^4(\Omega; L^4(0, T; \mathcal{H}))\), we get

\[
\mathbb{E} \left[ \left| \int_0^{t_1} (\bar{u}^h(s) - \bar{u}(s), \sigma(s, \bar{u}(s)) \, dW(s)) \right|^2 \right] \\
\leq \mathbb{E} \left[ \int_0^{t_1} \| \dot{u}^h(s) - \bar{u}(s) \|^2_{H^1} \| \sigma(s, \bar{u}(s)) \|^2_{L^2_Q} \, ds \right] \\
\leq \left\{ \mathbb{E} \left[ \int_0^{t_1} \| \dot{u}^h(s) - \bar{u}(s) \|^4_{H^1} \, ds \right] \right\}^{1/2} \left\{ \mathbb{E} \left[ \int_0^{t_1} \| \sigma(s, \bar{u}(s)) \|^4_{L^2_Q} \, ds \right] \right\}^{1/2} \\
\leq K_1 \sqrt{2T} \left\{ \mathbb{E} \left[ \int_0^{t_1} \| \dot{u}^h(s) - \bar{u}(s) \|^4_{H^1} \, ds \right] \right\}^{1/2} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \| \bar{u}(t) \|^4_{H^1} \right] \right\}^{1/2} \\
\leq C \left( \mathbb{E} \left[ \| u_0 \|^2_{H^1} \right], K_1, K_2, T \right) \left\{ \mathbb{E} \left[ \int_0^{t_1} \| \dot{u}(s) - \bar{u}(s) \|^4_{H^1} \, ds \right] \right\}^{1/2} \to 0 \text{ as } h \to 0.
\]

Using the fact that \(\int_0^h \zeta^h(s) \, ds = \frac{1}{2}\), we estimate

\[
\mathbb{E} \left[ \int_0^{t_1} \left( \int_0^t \zeta^h(t_1-s) \, ds \right) \| \sigma(\tau, \bar{u}(\tau)) \|^2_{L^2_Q} \, d\tau \right] \\
= \mathbb{E} \left[ \int_0^h \zeta^h(s) \left( \int_0^{t_1} \| \sigma(\tau, \bar{u}(\tau)) \|^2_{L^2_Q} \, d\tau - \int_{t_1-s}^{t_1} \| \sigma(\tau, \bar{u}(\tau)) \|^2_{L^2_Q} \, d\tau \right) \, ds \right] \\
= \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \| \sigma(\tau, \bar{u}(\tau)) \|^2_{L^2_Q} \, d\tau \right] - \mathbb{E} \left[ \int_0^h \zeta^h(s) \int_{t_1-s}^{t_1} \| \sigma(\tau, \bar{u}(\tau)) \|^2_{L^2_Q} \, d\tau \, ds \right]
\]
\[
\rightarrow \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \| \sigma(\tau, \bar{u}(\tau)) \|_{LQ}^2 \, d\tau \right] \text{ as } h \to 0,
\]
where we have used the continuity of the integral in the final term and the fact that
\[
\mathbb{E} \left[ \int_0^{t_1} \| \sigma(\tau, \bar{u}(\tau)) \|_{LQ}^2 \, d\tau \right] \leq K_1 T \left\{ 1 + \mathbb{E} \left[ \sup_{\tau \in [0, t_1]} \| \bar{u}(\tau) \|_{H}^2 \right] \right\} < \infty.
\]
It should be noted that at a countable number of points only jump occurs. Let us first assume that \( t_1 \) is a point in \((0, T)\), where a jump does not occur. Using Itô’s isometry and the fact that \( \bar{u}^h \to \bar{u} \) in \( L^4(\Omega; L^4(0, T; \mathbb{H})) \), we obtain
\[
\mathbb{E} \left[ \int_0^{t_1} \int_Z (\bar{u}^h(s) - \bar{u}(s), \gamma(s, \bar{u}(s), z)) \, \pi(ds, dz) \right]^2 \leq \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \| \bar{u}^h(s) - \bar{u}(s) \|^4_{H} \, ds \right]^{1/2} \left\{ 1 + \mathbb{E} \left[ \sup_{t \in [0, T]} \| \bar{u}(t) \|^4_{H} \right] \right\}^{1/2} \to 0 \text{ as } h \to 0.
\] (5.35)

Once again, using the fact that \( \int_0^{h} \zeta^h(s) \, ds = \frac{1}{2} \), we estimate
\[
\mathbb{E} \left[ \int_0^{t_1} \int_Z \left( \int_0^{t_1} \zeta^h(t_1 - s) \, ds \right) \gamma(\tau, \bar{u}(\tau), z) \, \pi(d\tau, dz) \right] \\
= \mathbb{E} \left[ \int_0^{t_1} \zeta^h(t_1 - s) \int_0^s \int_Z \gamma(\tau, \bar{u}(\tau), z) \, \pi(d\tau, dz) \, ds \right] \\
= \mathbb{E} \left[ \int_0^{t_1} \zeta^h(s) \left( \int_0^{t_1} \int_Z \gamma(\tau, \bar{u}(\tau), z) \, \pi(d\tau, dz) - \int_{t_1-s}^{t_1} \int_Z \gamma(\tau, z) \, \pi(d\tau, dz) \right) \, ds \right] \\
= \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \int_Z \gamma(\tau, \bar{u}(\tau), z) \, \pi(d\tau, dz) \right] \\
- \mathbb{E} \left[ \int_0^{t_1} \zeta^h(s) \int_0^{t_1} \int_Z \gamma(\tau, \bar{u}(\tau), z) \, \pi(d\tau, dz) \, ds \right] \\
\to \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \int_Z \gamma(\tau, \bar{u}(\tau), z) \, \pi(d\tau, dz) \right] \text{ as } h \to 0,
\] (5.36)
since jumps are not occurring at \( t_1 \) and the fact that
\[
\mathbb{E} \left[ \int_0^{t_1} \int_Z \| \gamma(\tau, \bar{u}(\tau), z) \|^2 \, \pi(d\tau, dz) \right] \leq K_1 T \left\{ 1 + \mathbb{E} \left[ \sup_{\tau \in [0, t_1]} \| \bar{u}(\tau) \|^2_{H} \right] \right\} < \infty.
\]
Using the convergences (5.32)-(5.36) in (5.31), along a subsequence, one can conclude that
\[
- \int_0^{t_1} \langle \bar{u}(s), \mu A \bar{u}(s) + B(\bar{u}(s)) + \beta C(\bar{u}(s)) \rangle \, ds
\]
\[ + \frac{1}{2} \int_0^{t_1} \| \sigma(s, \tilde{u}(s)) \|^2_{\mathbb{L}^2} ds + \frac{1}{2} \int_0^{t_1} \| \gamma(s, \tilde{u}(s), z) \|^2_{\mathbb{L}^2} \tilde{\pi}(ds, dz) \]
\[ + \int_0^{t_1} (\tilde{u}(s), \sigma(s, \tilde{u}(s)) d\tilde{W}(s)) + \int_0^{t_1} \int_{\mathbb{Z}} (\tilde{u}(s), \gamma(s, \tilde{u}(s), z)) \tilde{\pi}(ds, dz) \]
\[ = \lim_{h \to 0} (\tilde{u}^h(t_1), \tilde{u}(t_1)) - \lim_{h \to 0} (\tilde{u}(0), \tilde{u}^h(0)), \mathbb{P}\text{-a.s.} \] (5.37)

Making use of the $\mathbb{H}$-weak continuity (form right, since $\tilde{u} \in D([0, T]; \mathbb{H}_w)$, $\mathbb{P}$-a.s.) of $u(\cdot)$ at zero and the fact that $\zeta(t) = \zeta(-t)$ and $\int_0^h \zeta^2(s) ds = \frac{h}{2}$, we find
\[ (\tilde{u}(0), \tilde{u}^h(0)) = \int_0^{t_1} \zeta^2(s)(\tilde{u}(0), \tilde{u}(s) - \tilde{u}(0)) ds \]
\[ = \frac{1}{2} \| \tilde{u}(0) \|^2_{\mathbb{H}} + \int_0^h \zeta^2(s)(\tilde{u}(0), \tilde{u}(s) - \tilde{u}(0)) ds \]
\[ \to \frac{1}{2} \| \tilde{u}(0) \|^2_{\mathbb{H}} \text{ as } h \to 0, \mathbb{P}\text{-a.s.} \] (5.38)

Since jumps does not occur at $t_1$, using the fact that $\tilde{u}(\cdot)$ is $\mathbb{H}$-weakly continuous in time at $t_1$, we deduce that
\[ (\tilde{u}(t_1), \tilde{u}^h(t_1)) = \int_0^{t_1} \zeta^2(s)(\tilde{u}(t_1), \tilde{u}(t_1) - s) ds \]
\[ = \frac{1}{2} \| \tilde{u}(t_1) \|^2_{\mathbb{H}} + \int_0^h \zeta^2(s)(\tilde{u}(t_1), \tilde{u}(t_1) - s) ds \]
\[ \to \frac{1}{2} \| \tilde{u}(t_1) \|^2_{\mathbb{H}} \text{ as } h \to 0, \mathbb{P}\text{-a.s.} \] (5.39)

Therefore, from (5.37), we have following the Itô formula:
\[ \| \tilde{u}(t_1) \|^2_{\mathbb{H}} + 2\mu \int_0^{t_1} \| \tilde{u}(s) \|^2_{\mathbb{H}} ds + 2\beta \int_0^{t_1} \| \tilde{u}(s) \|^2_{\mathbb{H}} ds \]
\[ = \int_0^{t_1} \| \sigma(s, \tilde{u}(s)) \|^2_{\mathbb{L}^2} ds + \int_0^{t_1} \int_{\mathbb{Z}} \| \gamma(s, \tilde{u}(s), z) \|^2_{\mathbb{L}^2} \tilde{\pi}(ds, dz) \]
\[ + 2 \int_0^{t_1} (\tilde{u}(s), \sigma(s, \tilde{u}(s)) d\tilde{W}(s)) + 2 \int_0^{t_1} \int_{\mathbb{Z}} (\tilde{u}(s), \gamma(s, \tilde{u}(s), z)) \tilde{\pi}(ds, dz), \] (5.40)

for a.e. $t_1 \in [0, T]$, $\mathbb{P}$-a.s.

Let $t_1$ be a point in $(0, T)$ where a jump occurs. Let $\tilde{t}_1$ be the point in $(0, T)$ where the jump occurred before $t_1$ (take $\tilde{t}_1 = 0$, if the first jump occurs at $t_1$). Let $\tilde{u}(t_1-)$ denote the left limit of $u(\cdot)$ at the point $t_1$. From (5.31), we have
\[ (\tilde{u}^h(t_1), \tilde{u}(t_1)) = (\tilde{u}^h(0), \tilde{u}(0)) - \int_0^{t_1} (\tilde{u}^h(s), \mu A \tilde{u}(s) + B(\tilde{u}(s)) + \beta \mathcal{C}(\tilde{u}(s))) ds \]
\[ + \int_0^{t_1} (\tilde{u}^h(s), \sigma(s, \tilde{u}(s)) d\tilde{W}(s)) + \int_0^{t_1} \left( \int_{\tau}^{t_1} \zeta^2(t_1 - s) ds \right) \| \sigma(\tau, \tilde{u}(\tau)) \|^2_{\mathbb{L}^2} d\tau \]
\[ + \int_0^{\tilde{t}_1} \int_{\mathbb{Z}} (\tilde{u}^h(s), \gamma(s, \tilde{u}(s), z)) \tilde{\pi}(ds, dz) \]
+ \left(\mathbf{u}^h(t_1-), \mathbf{u}(t_1) - \mathbf{u}(t_1-)) \right) - \int_{t_1}^{t_1} \int_{\mathbb{R}} (\mathbf{u}^h(s), \gamma(s, \mathbf{u}(s), z)) \lambda(dz)ds \\
+ \int_0^{t_1} \zeta^h(t_1 - s) \int_s^t \|\gamma(\tau, \mathbf{u}(\tau-), z)\|_{\mathbb{H}^2}^2 (d\tau, dz)ds,
\end{align}

where we have used the fact that \( u(t_1)-u(t_1-)=\gamma(t_1, u(t_1-), u(t_1)-u(t_1-)) \), \( \lambda u(t_1)-u(t_1-), z \) (see Section 4.3.2, \cite{2}) and stochastic Fubini’s Theorem for compensated Poisson random measures (cf. Lemma A.1.1, \cite{32}). It should be noted that the convergences (5.32)-(5.34) hold true in this case also. Similar to (5.35), one can show that

\begin{equation}
\mathbb{E}\left[ \int_0^{t_1} \int_{\mathbb{R}} (\mathbf{u}^h(s-) - \mathbf{u}(s-), \gamma(s, \mathbf{u}(s-), z)) \mathbb{P}(ds, dz) \right] \rightarrow 0 \quad \text{as} \quad h \rightarrow 0.
\end{equation}

Once again using the fact that \( \int_0^h \zeta^h(s)ds = \frac{1}{2} \) and the \( \mathbb{H} \)-weak continuity at \( t_1- \), we find

\begin{align}
\left| \int_0^{t_1} \zeta^h(s)(\mathbf{u}(t_1), \mathbf{u}(t_1) - \mathbf{u}(t_1-))ds - \int_0^{t_1} \zeta^h(s)(\mathbf{u}(t_1), \mathbf{u}(t_1) - \mathbf{u}(t_1-))ds \right|
=
\int_0^{t_1} \zeta^h(s)(\mathbf{u}(t_1), \mathbf{u}(t_1) - \mathbf{u}(t_1-))ds \rightarrow 0 \quad \text{as} \quad h \rightarrow 0, \quad \mathbb{P}\text{-a.s.}
\end{align}

Thus, from (5.32), we have

\begin{equation}
(\mathbf{u}^h(t_1), \mathbf{u}(t_1)) \rightarrow \frac{1}{2} \|\mathbf{u}(t_1)\|_{\mathbb{H}^2}^2 - \frac{1}{2} (\mathbf{u}(t_1), \mathbf{u}(t_1) - \mathbf{u}(t_1-)) \quad \text{as} \quad h \rightarrow 0 \quad \mathbb{P}\text{-a.s.}
\end{equation}

Let us now explain the convergence of \( (\mathbf{u}^h(t_1-), \mathbf{u}(t_1) - \mathbf{u}(t_1-)) \). A calculation similar to (5.43) gives

\begin{align}
(\mathbf{u}^h(t_1-), \mathbf{u}(t_1) - \mathbf{u}(t_1-))
= \int_0^{t_1} \zeta^h(s)(\mathbf{u}(t_1) - \mathbf{u}(t_1-))ds \\\n= \frac{1}{2} (\mathbf{u}(t_1) - \mathbf{u}(t_1-)) + \int_0^{t_1} \zeta^h(s)(\mathbf{u}(t_1) - \mathbf{u}(t_1-))ds \\\n\rightarrow \frac{1}{2} (\mathbf{u}(t_1) - \mathbf{u}(t_1-)) - \frac{1}{2} \|\mathbf{u}(t_1) - \mathbf{u}(t_1-)\|_{\mathbb{H}^2}^2, \quad \text{as} \quad h \rightarrow 0, \quad \mathbb{P}\text{-a.s.}
\end{align}

As there are no jumps in \( (\mathbf{u}(t_1), \mathbf{u}(t_1-)) \), using Hölder’s inequality, we get

\begin{align}
\mathbb{E}\left[ \int_0^{t_1} \int_{\mathbb{R}} (\mathbf{u}^h(s) - \mathbf{u}(s), \gamma(s, \mathbf{u}(s), z)) \mathbb{P}(ds, dz) \right] \\leq\n\frac{1}{2} \left\{ \mathbb{E}\left[ \int_0^{T} \|\mathbf{u}(t) - \mathbf{u}(t)\|_{\mathbb{H}^2}^2 \right] \right\}^{1/2} \mathbb{E}\left[ \int_0^{T} \left( \int_{\mathbb{R}} \gamma(t, \mathbf{u}(t), z) \mathbb{P}(dz) \right)^2 \right]^{1/2} \\
\leq K_1 \sqrt{2T} \left\{ \mathbb{E}\left[ \int_0^{t_1} \|\mathbf{u}^h(s) - \mathbf{u}(s)\|^4_{\mathbb{H}} \right] \right\}^{1/2} \left\{ 1 + \sup_{t \in [0, T]} \|\mathbf{u}(t)\|_{\mathbb{H}^4} \right\}^{1/2} \\
\leq C\left( \mathbb{E}\left[ \|\mathbf{u}_0\|_{\mathbb{H}^4} \right], K_1, K_2, T \right) \left\{ \mathbb{E}\left[ \int_0^{T} \|\mathbf{u}^h(t) - \mathbf{u}(t)\|^4_{\mathbb{H}} dt \right] \right\}^{1/2}
\end{align}
Since a jump occurs at the point \( t_1 \) and the jumps are isolated, a calculation similar to (5.36) yields

\[
\mathbb{E} \left[ \int_0^{t_1} \zeta^h(t_1 - s) \int_0^s \int_Z \| \gamma(\tau, u(\tau -), z) \|_{\mathbb{H}}^2 \pi(d\tau, dz) ds \right] \\
= \frac{1}{2} \mathbb{E} \left[ \int_0^{t_1} \int_Z \| \gamma(\tau, u(\tau -), z) \|_{\mathbb{H}}^2 \pi(d\tau, dz) \right] \\
- \mathbb{E} \left[ \int_0^h \zeta^h(s) \int_0^{t_1 - s} \int_Z \| \gamma(\tau, u(\tau -), z) \|_{\mathbb{H}}^2 \pi(d\tau, dz) ds \right] \\
\to \frac{1}{2} \mathbb{E} \left[ \| \bar{u}(t_1) - \bar{\bar{u}}(t_1) \|_{\mathbb{H}}^2 \right] \text{ as } h \to 0. \tag{5.47}
\]

Combining the convergences (5.42)-(5.47), substituting it in (5.41) and then taking the limit along a subsequence as \( h \to 0 \), we find

\[
\frac{1}{2} \| \bar{u}(t_1) \|_{\mathbb{H}}^2 = \frac{1}{2} \| \bar{u}(0) \|_{\mathbb{H}}^2 - \int_0^{t_1} \langle \bar{u}(s), \mu A \bar{u}(s) + B(\bar{u}(s)) + \beta C(\bar{u}(s)) \rangle ds \\
+ \int_0^{t_1} \langle \bar{u}(s), \sigma(s, u(s)) \rangle d\bar{W}(s) + \frac{1}{2} \int_0^{t_1} \| \sigma(s, u(s)) \|_{\mathbb{L}_2}^2 ds \\
+ \frac{1}{2} \int_0^{t_1} \int_Z \| \gamma(s, u(s -), z) \|_{\mathbb{H}}^2 \pi(ds, dz) + \int_0^{t_1} \int_Z \langle \bar{u}(s -), \gamma(s, u(s -), z) \rangle \bar{\pi}(ds, dz), \tag{5.48}
\]

\( \bar{\mathbb{P}} \)-a.s. and the Itô formula (5.40) holds true for all \( t_1 \in [0, T] \). \( \square \)

Let us now prove the pathwise uniqueness of solutions of the problem (3.1) for \( d = 2, r \in [1, \infty) \), \( d = 3 \), \( r \in [3, \infty) \) (\( 2\beta \mu \geq 1 \) for \( r = 3 \)).

**Proposition 5.4.** Let \( d = 2, r \in [1, \infty) \), \( d = 3 \), \( r \in [3, \infty) \) (\( 2\beta \mu \geq 1 \) for \( r = 3 \)) and Hypothesis [3.3] be satisfied. If \( \bar{u}_1 \) and \( \bar{u}_2 \) are two solutions of the problem (3.1) defined on the same filtered probability space \( (\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \bar{\mathbb{P}}) \), then \( \bar{\mathbb{P}} \)-a.s. for all \( t \in [0, T] \), \( \bar{u}_1(t) = \bar{u}_2(t) \).

**Proof.** Let \( \bar{u}_1(\cdot) \) and \( \bar{u}_2(\cdot) \) be two solutions of the problem (3.1) defined on \( (\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \geq 0}, \bar{\mathbb{P}}) \). For \( N > 0 \), let us define

\[
\tau^1_N = \inf_{0 \leq t \leq T} \left\{ t : \| \bar{u}_1(t) \|_{\mathbb{H}} \geq N \right\}, \quad \tau^2_N = \inf_{0 \leq t \leq T} \left\{ t : \| \bar{u}_2(t) \|_{\mathbb{H}} \geq N \right\} \text{ and } \tau_N = \tau^1_N \wedge \tau^2_N.
\]

Then, one can show that \( \tau_N \to T \) as \( N \to \infty \), \( \bar{\mathbb{P}} \)-a.s. (cf. [38, 39]).

We define \( \bar{z}(\cdot) = \bar{u}_1(\cdot) - \bar{u}_2(\cdot), \bar{\sigma}(\cdot, \cdot) = \sigma(\cdot, \bar{u}_1(\cdot)) - \sigma(\cdot, \bar{u}_2(\cdot)) \) and \( \bar{\gamma}(\cdot, \cdot) = \gamma(\cdot, \bar{u}_1(\cdot), \cdot) - \gamma(\cdot, \bar{u}_2(\cdot), \cdot) \). Then, \( \bar{z}(\cdot) \) satisfies the system:

\[
\begin{align*}
\text{d}\bar{z}(t) &= -[\mu A \bar{z}(t) + B(\bar{u}_1(t)) - B(\bar{u}_2(t)) + \beta(\bar{C}(\bar{u}_1(t)) - \bar{C}(\bar{u}_2(t)))]dt \\
&\quad + \bar{\sigma}(t) d\bar{W}(t) + \int_Z \bar{\gamma}(t, z) d\bar{\pi}(dt, dz), \\
\bar{z}(0) &= \bar{w}_0,
\end{align*}
\tag{5.49}
\]

for a.e. \( t \in [0, T] \) in \( V' + \bar{\mathbb{L}}^{r+1} \).
Case (1): $d = 2$ and $r \in [1, 3]$. Let us first consider the case $d = 2$ and $r \in [1, 3]$. We apply Itô’s formula to the process \( \| \mathbf{z} (\cdot) \|_{H}^{2} \) to find

\[
\| \mathbf{z} (t) \|_{H}^{2} + 2 \mu \int_{0}^{t} \| \mathbf{z} (s) \|_{V}^{2} \, ds = \| \mathbf{z} (0) \|_{H}^{2} - 2 \int_{0}^{t} \langle B (\mathbf{u}_1 (s)) - B (\mathbf{u}_2 (s)), \mathbf{z} (s) \rangle \, ds
\]

\[
- 2 \beta \int_{0}^{t} \langle C (\mathbf{u}_1 (s)) - C (\mathbf{u}_2 (s)), \mathbf{z} (s) \rangle \, ds
\]

\[
+ \int_{0}^{t} \left( \| \mathbf{\hat{\sigma}} (s) \|_{L_{Q}}^{2} + \int_{\mathbb{Z}} \| \mathbf{\hat{\gamma}} (s, z) \|_{H}^{2} \lambda (dz) \right) \, ds + M_{t},
\]

where \( M_{t} \) is the local martingale given by

\[
M_{t} = 2 \int_{0}^{t} \langle \mathbf{\hat{\sigma}} (s) dW (s), \mathbf{z} (s) \rangle + \int_{0}^{t} \int_{\mathbb{Z}} \left( \| \mathbf{\hat{\gamma}} (s, z) \|_{H}^{2} + 2 \langle \mathbf{\hat{\gamma}} (s, z), \mathbf{z} (s) \rangle \right) \tilde{\pi} (ds, dz).
\]

Using Hölder’s, Ladyzhenskaya’s and Young’s inequalities, we estimate \( \langle B (\mathbf{u}_1) - B (\mathbf{u}_2), \mathbf{z} \rangle \) as

\[
\langle B (\mathbf{u}_1) - B (\mathbf{u}_2), \mathbf{z} \rangle = \langle B (\mathbf{z}, \mathbf{u}_2), \mathbf{z} \rangle = - \langle B (\mathbf{z}, \mathbf{z}), \mathbf{u}_2 \rangle \leq \| \mathbf{z} \|_{V} \| \mathbf{z} \|_{L_{4}} \| \mathbf{u}_2 \|_{L_{4}}
\]

\[
\leq 2^{1/4} \| \mathbf{z} \|_{V}^{3/2} \| \mathbf{z} \|_{H}^{1/2} \| \mathbf{u}_2 \|_{L_{4}} \leq \frac{\mu}{2} \| \mathbf{z} \|_{V}^{2} + \frac{27}{16 \mu^{3}} \| \mathbf{u}_2 \|_{L_{4}} \| \mathbf{z} \|_{H}^{2}.
\]

Applying (2.9) and (5.51) in (5.50), we get

\[
\| \mathbf{z} (t) \|_{H}^{2} + \mu \int_{0}^{t} \| \mathbf{z} (s) \|_{V}^{2} \, ds + \frac{\beta}{2^{r-2}} \int_{0}^{t} \| \mathbf{z} (s) \|_{L^{r+1}}^{r+1} \, ds
\]

\[
\leq \| \mathbf{z} (0) \|_{H}^{2} + \frac{27}{8 \mu^{3}} \int_{0}^{t} \| \mathbf{u}_2 (s) \|_{L_{4}}^{4} \| \mathbf{z} (s) \|_{H}^{2} \, ds + \int_{0}^{t} \left( \| \mathbf{\hat{\sigma}} (s) \|_{L_{Q}}^{2} + \int_{\mathbb{Z}} \| \mathbf{\hat{\gamma}} (s, z) \|_{H}^{2} \lambda (dz) \right) \, ds + \tilde{M}_{t},
\]

where \( \tilde{M}_{t} \) is the local martingale

\[
\tilde{M}_{t} = 2 \int_{0}^{t} \mathbf{e}^{- \frac{1}{4} (\mathbf{s})} (\mathbf{\hat{\sigma}} (s) dW (s), \mathbf{z} (s)) + \int_{0}^{t} \mathbf{e}^{- \frac{1}{4} (\mathbf{s})} \int_{\mathbb{Z}} \left( \| \mathbf{\hat{\gamma}} (s, z) \|_{H}^{2} + 2 \langle \mathbf{\hat{\gamma}} (s, z), \mathbf{z} (s) \rangle \right) \tilde{\pi} (ds, dz).
\]

Taking expectation in (5.53), and then using Hypothesis 3.3 (H.3), we deduce

\[
\mathbb{E} \left[ e^{- \frac{1}{4} (\mathbf{s})} \| \mathbf{z} (t \wedge \tau_{N}) \|_{H}^{2} + \mu \int_{0}^{t \wedge \tau_{N}} e^{- \frac{1}{4} (\mathbf{s})} \| \mathbf{z} (s) \|_{V}^{2} \, ds + \frac{\beta}{2^{r-2}} \int_{0}^{t \wedge \tau_{N}} e^{- \frac{1}{4} (\mathbf{s})} \| \mathbf{z} (s) \|_{L^{r+1}}^{r+1} \, ds \right]
\]

\[
\leq \mathbb{E} \left[ \| \mathbf{z} (0) \|_{H}^{2} \right] + L \mathbb{E} \left[ \int_{0}^{t \wedge \tau_{N}} \| \mathbf{z} (s) \|_{H}^{2} \, ds \right].
\]
An application of Gronwall’s inequality in (5.54) yields
\[ \mathbb{E}[e^{-\theta(t \wedge \tau_N)}\|\bar{z}(t \wedge \tau_N)\|^2_{\mathbb{H}}] \leq \mathbb{E}[\|\bar{z}(0)\|^2_{\mathbb{H}}] e^{LT}. \] (5.55)
Thus the initial data \( \bar{u}_1(0) = \bar{u}_2(0) = u_0 \) leads to \( \bar{z}(t \wedge \tau_N) = 0, \mathbb{P}\)-a.s. But the fact that \( \tau_N \to T \) as \( N \to \infty \) provide \( \bar{z}(t) = 0 \) and hence \( \bar{u}_1(t) = \bar{u}_2(t) \) for all \( t \in [0, T] \), \( \mathbb{P}\)-a.s., and the uniqueness follows.

**Case (2):** \( d = 2, 3 \) and \( r \in (3, \infty) \). Let us now consider the case \( d = 2, 3 \) and \( r \in (3, \infty) \). Using Hölder’s and Young’s inequalities, we estimate \( \langle B(\bar{u}_1) - B(\bar{u}_2), \bar{z} \rangle = -B(\bar{z}, \bar{z}), \bar{u}_2 \rangle \) as
\[ |\langle B(\bar{z}, \bar{z}), \bar{u}_2 \rangle| \leq \|\bar{z}\|_V \|\bar{u}_2 \bar{z}\|_H \leq \frac{\mu}{2} \|\bar{z}\|^2_{V} + \frac{1}{2\mu} \|\bar{u}_2 \bar{z}\|^2_{H}. \] (5.56)
Taking the term \( \|\bar{u}_2 \bar{z}\|^2_{H} \) from (5.56) and using Hölder’s and Young’s inequalities, we estimate
\[
\int_0^T |\bar{u}_2(x)|^2 \|\bar{z}(x)\|^2 dx = \int_0^T |\bar{u}_2(x)|^2 \|\bar{z}(x)\|^2 x^{1-r} \|\bar{z}(x)\|^2 \frac{2(r-3)}{r-1} dx \\
\leq \left( \int_0^T |\bar{u}_2(x)|^r \|\bar{z}(x)\|^2 dx \right)^{\frac{2}{r}} \left( \int_0^T \|\bar{z}(x)\|^2 dx \right)^{1-\frac{2}{r}} \\
\leq \beta \mu \left( \int_0^T |\bar{u}_2(x)|^r \|\bar{z}(x)\|^2 dx \right) + \frac{r-3}{r-1} \left( \frac{2}{\beta \mu (r-1)} \right)^{\frac{2}{r}} \left( \int_0^T \|\bar{z}(x)\|^2 dx \right),
\] (5.57)
for \( r > 3 \). Using (5.57) in (5.56), we deduce
\[ |\langle B(\bar{z}, \bar{z}), \bar{u}_2 \rangle| \leq \frac{\mu}{2} \|\bar{z}\|^2_{V} + \frac{\beta}{r-1} \|\bar{u}_2 \bar{z}\|^2_{H} + \hat{\gamma} \|\bar{z}\|^2_{H}, \] (5.58)
where \( \hat{\gamma} = \frac{r-3}{2\beta \mu (r-1)} \left( \frac{2}{\beta \mu (r-1)} \right)^{\frac{2}{r}} \). From (2.9), we deduce
\[ \beta \langle \mathcal{C}(\bar{u}_1) - \mathcal{C}(\bar{u}_2), \bar{z} \rangle \geq \frac{\beta}{2} \|\bar{u}_1 \|_{\mathbb{H}}^2 + \frac{\beta}{2} \|\bar{u}_2 \|^2_{\mathbb{H}}. \]
Thus, using the above two estimates and (2.9) in (5.50), we infer that
\[
\|\bar{z}(t \wedge \tau_N)\|^2_{\mathbb{H}} + \mu \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|^2_{V} ds + \beta \frac{1}{2^r} \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|_{L^{r+1}}^{r+1} ds \\
\leq \|\bar{z}(0)\|^2_{\mathbb{H}} + 2\hat{\gamma} \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|^2_{H} ds + \int_0^{t \wedge \tau_N} \left( \|\mathcal{S}\|^2_{\mathbb{H}} + \int_Z \|\mathcal{Y}(s, z)\|^2_{\mathbb{H}} \lambda(dz) \right) ds + M_t.
\] (5.59)
Taking expectation in (5.59), and then using Hypothesis 3.3 (H.3), we obtain
\[
\mathbb{E} \left[ \|\bar{z}(t \wedge \tau_N)\|^2_{\mathbb{H}} + \mu \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|^2_{V} ds + \beta \frac{1}{2^r} \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|_{L^{r+1}}^{r+1} ds \right] \\
\leq \mathbb{E} \left[ \|\bar{z}(0)\|^2_{\mathbb{H}} + \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|^2_{\mathbb{H}} ds \right].
\] (5.60)
Applying Gronwall’s inequality in (5.60), we arrive at
\[ \mathbb{E}[\|\bar{z}(t \wedge \tau_N)\|^2_{\mathbb{H}}] \leq \mathbb{E}[\|\bar{z}(0)\|^2_{\mathbb{H}}] e^{(L+2\hat{\gamma})t}. \] (5.61)
Thus the initial data \( \bar{u}_1(0) = \bar{u}_2(0) = u_0 \) leads to \( \bar{z}(t \wedge \tau_N) = 0, \mathbb{P}\)-a.s. Using the fact that \( \tau_N \to T, \mathbb{P}\)-a.s., implies \( \bar{z}(t) = 0 \) and hence \( \bar{u}_1(t) = \bar{u}_2(t) \), \( \mathbb{P}\)-a.s., for all \( t \in [0, T] \).
Case (3): $d = r = 3$ with $2 \beta \mu \geq 1$. For the case $d = 3$, $r = 3$ and $2 \beta \mu \geq 1$, the estimate
\[
|\langle B(\bar{z}, \bar{z}), \bar{u}_2 \rangle| \leq \|\bar{u}_2 \bar{z}\|_\mathbb{H} \|\bar{z}\|_V \leq \theta \mu \|\bar{z}\|^2_V + \frac{1}{4 \mu \theta} \|\bar{u}_2 \bar{z}\|^2_\mathbb{H},
\] (5.62)
for some $0 < \theta \leq 1$, helps us to obtain
\[
\mathbb{E} \left[ \|\bar{z}(t \wedge \tau_N)\|^2_\mathbb{H} + 2 \mu (1 - \theta) \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|^2_V ds + \left( \beta - \frac{1}{2 \mu \theta} \right) \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|_{r+1}^{r+1} ds \right]
\leq \mathbb{E} \left[ \|\bar{z}(0)\|^2_\mathbb{H} \right] + L \mathbb{E} \int_0^{t \wedge \tau_N} \|\bar{z}(s)\|^2_\mathbb{H} ds,
\] (5.63)
and the pathwise uniqueness follows.

**Proof of Theorem 3.10.** By Theorem 3.9, we infer the existence of a martingale solution and by Proposition 5.4, we know that the solution is pathwise unique. Therefore, assertions (1) and (3) of Theorem 3.10 follow from Theorems 2, [43] and Theorem 8, [50]. Assertion (2) is a direct consequence of Propositions 5.1 and 5.3.

**Acknowledgments:** M. T. Mohan would like to thank the Department of Science and Technology (DST), India for Innovation in Science Pursuit for Inspired Research (INSPIRE) Faculty Award (IFA17-MA110). The author would also like to thank Prof. J. C. Robinson, University of Warwick for useful discussions and providing the crucial reference [15].

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