REMARKS ON RULED SURFACES AND RANK TWO BUNDLES WITH CANONICAL DETERMINANT AND 4 SECTIONS.

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Abstract. Let $C$ be a smooth irreducible complex projective curve of genus $g$ and let $B^k(2, K_C)$ be the Brill-Noether loci parametrizing classes of (semi)stable vector bundles $E$ of rank two with canonical determinant over $C$ with $h^0(C, E) \geq k$. We show that $B^4(2, K_C)$ it has an irreducible component $B$ of dimension $3g - 13$ on a general curve $C$ of genus $g \geq 8$. Moreover, we show that for the general element $[E]$ of $B$, $E$ fits in an exact sequence $0 \to \mathcal{O}_C(D) \to E \to K_C(-D) \to 0$, with $D$ a general effective divisor of degree three, and the corresponding coboundary map $\partial : H^0(C, K_C(-D)) \to H^1(C, \mathcal{O}_C(D))$ has cokernel of dimension three.

1. Introduction and statement of the result.

Let $C$ be a smooth irreducible complex projective curve of genus $g$ and let $U_C(r, d)$ be the moduli space of (semi)stable vector bundles of rank $r$ and degree $d$ on $C$. Inside $U_C(r, d)$, consider the Brill-Noether loci $B^k_C(d)$ parametrizing classes of vector bundles $[E] \in U_C(d)$ having at least $k$ linearly independent sections. Because of their interesting geometry, Brill-Noether loci had been extensively studied and actually is an active topic of research in algebraic geometry, however some problems related with non-emptyness, dimension, irreducibility, etc., are still open. For an overview on main results on Brill-Noether theory for vector bundles on algebraic curves see ([8]). For $r = 2$ and $d = 2g - 2$, the scheme $B(2, K_C) \subset U_C(2, 2g - 2)$ which parametrizes classes of (semi)stable rank-two vector bundles $E$ with $\det(E) = \wedge^2 E = K_C$, is defined as the fiber at $K_C$ of the determinant map $\wedge^2 : U_C(2, 2g - 2) \to \text{Pic}^{2g-2}(C), E \to \wedge^2 E$. $B(2, K_C)$ is smooth and irreducible of dimension $3g - 3$. Inside $B(2, K_C)$, define Brill-Noether loci $B^k(2, K_C) := \{[E] \in B(2, K_C) : h^0(C, E) \geq k\}$. These loci have a scheme structure as degeneracy locus and is known that the expected dimension of $B^k(2, K_C)$ is given by the Brill-Noether number $\rho_{K_C}(2, k, g) := 3g - 3 - \binom{k+1}{2}$ (see [3], [11]). For a description of $B^k(2, K_C)$ for low genus we refer to e.g. ([3], [12],[13]). For $[E] \in B^k(2, K_C)$ with $h^0(C, E) = k$, the infinitesimal behavior of $B^k(2, K_C)$ at the point $[E]$ is governed by the symmetric Petri map $P_E : \text{Sym}^2(H^0(C, E)) \to H^0(C, \text{Sym}^2(E))$, that is, the tangent space to $B^k(2, K_C)$
at \([E]\) is identified with the orthogonal to the image of the symmetric \(P_E\). In ([18]) it has been proved the injectivity of the symmetric Petri map \(P_E\) on a general curve of genus \(g \geq 1\). A different approach for the injectivity of the symmetric Petri map for \(g \leq 9\) and \(k < 7\) is given in ([2]). The injectivity of the symmetric Petri map implies that on a general curve \(C\), components of the right dimension in \(B^k(2, K_C)\) are smooth.

A general result on non-emptiness and existence of components in \(B^k(2, K_C)\) of the right dimension on a general curve of genus sufficiently large is given in ([17]), where the proof uses the theory of limit linear series for higher rank. Generalizations of this result and others results about non-emptiness, irreducibility, we refer to e.g ([15], [14], [10], [19]). In particular, for \(g > 3\) and for \(k \leq 3\), non-emptiness and existence of irreducible components of \(B^k(2, K_C)\) of the right dimension on a general curve are obtained in ([5]) by studying certain determinantal locus inside the space of extensions \(\text{Ext}^1(K_C(-D_{k-1}), O_C(D_{k-1}))\), for a general effective divisor \(D_{k-1}\) of degree \(k - 1 \leq 2\). For a general curve of genus \(g \geq 5\), the existence of irreducible components in \(B^4(2, K_C)\) of the right dimension is a particular case of theorem 1.1 given in ([17]). We follow the approach as in ([5]) to show that there exists an irreducible component \(B \subset B^4(2, K_C)\) of dimension \(\rho_{K_C}(2, 4, g)\) on a general curve of genus \(g \geq 8\), moreover we show that the general vector bundle in the component \(B\) is given as an extension of suitable line bundles. In order to state our main theorem we introduce some definitions and notation.

Let \(C\) be a non-hyperelliptic curve of genus \(g \geq 3\) and let \(d\) be an integer such that \(2g - 2 \leq d \leq 4g - 4\). Let \(\delta \leq d\) a positive integer. Let \(L \in \text{Pic}^\delta(C), N \in \text{Pic}^{d-\delta}(C)\) be line bundles on \(C\); the space \(\text{Ext}^1(L, N)\) parametrizes isomorphism classes of extensions

\[0 \to N \to E \to L \to 0.\]

Any \(u \in \text{Ext}^1(L, N)\) give rise to a degree \(d\) rank-two vector bundle \(E_u\) that fits in a exact sequence

\[(u) : \quad 0 \to N \to E_u \to L \to 0.\]

To get a (semi)stable vector bundle, a necessary condition is \(2\delta - d \geq 0\). By Riemann-Roch, \(\text{Ext}^1(L, N) \simeq H^1(C, N \otimes L^\vee) \simeq H^0(C, K_C \otimes L \otimes N^\vee)^\vee\), then \(m := \dim(\text{Ext}^1(L, N)) = 2\delta - d + g - 1\) if \(L \not\simeq N\), and \(m = g\) when \(L \simeq N\).

Suposse that \(N\) is a special line bundle. For any \(u \in \text{Ext}^1(L, N)\), consider the coboundary map induced by cohomology in the exact sequence \((u)\):

\[\partial_u : H^0(L) \to H^1(N).\]
By exactness in \((u)\) we have that 
\[ h^1(C, E_u) = h^0(N) + \dim(\text{Coker}(\partial_u)). \]
For any integer \(t > 0\), consider the degeneracy locus
\[ W_t := \{ u \in \text{Ext}^1(L, N) : \dim(\text{Coker}(\partial_u)) \geq t \}, \]
which has a natural structure of determinantal scheme, so it has expected dimension 
\[ m - t(t + h^0(L) - h^1(N)). \] If \(m > 0\) and \(W_t \neq \emptyset\), then any irreducible component 
\(\Delta_t \subseteq W_t\) is such that 
\[ \dim(\Delta_t) \geq \min\{m, m - t(t + h^0(L) - h^1(N))\}, \]
where right-hand-side is the expected dimension. Our main result is the following:

**Theorem.** Let \(C\) be a curve of genus \(g \geq 8\) with general moduli. Then \(B^4(2, K_C) \neq \emptyset\). Moreover, there exists an irreducible component \(B \subset B^4(2, K_C)\) of the expected dimension \(\rho_{K_C}(2, 4, g) = 3g - 13\) and whose general point \([E]\) fits into an exact sequence

\[ 0 \to \mathcal{O}_C(D) \to E \to K_C(-D) \to 0, \]

where,

(i).- \(D\) is a general effective divisor of degree 3, and

(ii).- \(E = E_u\) with \(u \in \Delta_3 \subseteq W_3 \subset \text{Ext}^1(K_C(-D), \mathcal{O}_C(D))\) general in \(\Delta_3\), where \(\Delta_3\) is an irreducible component of dimension \(3g - 15\) and the general element \(u \in \Delta_3\) satisfies that 
\[ \dim(\text{Cokernel}(\partial_u)) = 3. \]

**Acknowledgements.** The authors give thanks to Ciro Ciliberto, Flaminio Flamini, Elham Izadi and Christian Pauly for fruitful discussions and suggestions.

2. Preliminaries.

We give some definitions and results that we need for the proof of our main theorem. For more complete details on the topics of this section we refer the reader to e.g. [1, 5, 7, 16].

2.1. Ruled surfaces. Let \(E\) be a vector bundle over a smooth irreducible complex projective curve \(C\). The speciality of \(E\) is defined as 
\[ i(E) = h^1(C, E). \] \(E\) is said special if \(i(E) > 0\). We denote by \(\sim\) linear equivalence of divisors and by \(\equiv\) numerical equivalence of divisors.

Let \(E\) be a rank-two vector bundle over a curve \(C\) of genus \(g\). Let \(S := \mathbb{P}(E)\) be a (geometrically) ruled surface with structure map \(p : S \to C\). For any \(x \in C\) we denote by \(f_x = p^{-1}(x) \cong \mathbb{P}^1\). We denote by \(f\) a general fiber of \(p\) and by \(\mathcal{O}_S(1)\) the tautological line bundle on \(S\). We write 
\[ d = \deg(E) = \deg(\wedge^2 E) = \deg(\det(E)). \]

There is a map \(s : C \to S\) such that \(p \circ s = \text{Id}_C\) whose image we denote by \(H\) and such that \(\mathcal{O}_S(H) = \mathcal{O}_S(1)\), thus an element in the linear system \(\mathcal{O}_S(1)\) is denoted by \(H\). For any \(D \in \text{Div}(C)\) (or in \(\text{Pic}(C)\)) we denote \(f^*_D := p^*(D)\). If \(\Gamma\) is a divisor on \(S\), we set \(\deg(\Gamma) := H \cdot \Gamma\). We have that 
\[ d = \deg(E) = H^2 = \deg(H). \] We recall
that Pic \( S \) \( \simeq \mathbb{Z} [ \mathcal{O}_S(1) ] \oplus p^*(\text{Pic}(C)) \), moreover Num \( S \) \( \simeq \mathbb{Z} \oplus \mathbb{Z} \), generated by the classes \( H, f \) satisfying \( H \cdot f = 1, f^2 = 0 \) (see [7], chapter V). Any element of Pic \( S \) corresponds to a divisor on \( S \) of the form \( nH + fB, n \in \mathbb{Z}, B \in \text{Div}(C) \), as an element of Num \( F \), corresponds to \( nH + bf, b = \text{deg}(B) \). For any \( n \geq 0 \) and for any \( D \in \text{Div}(C) \), the linear system \( |nH + fD| \) if non empty, is said to be \( n\)-secant to the fibration \( p : S \to C \) since its general elements meets \( f \) at \( n \) points. An element \( \Gamma \in |H + fD| \) is called unisecant curve of \( S \) (or to the fibration \( p : S \to C \)). The irreducible unisecant of \( S \) are smooth and isomorphic to \( C \) and are called sections of \( S \).

2.2. Some properties of sections of Ruled surfaces. We recall that there is one-to-one correspondence between sections \( \Gamma \) of \( S \) and surjective maps \( E \to L \) with \( L \) a line bundle on \( C \) (see [7], chapter V), then one has an exact sequence \( 0 \to N \to E \to L \to 0 \), where \( N \in \text{Pic}(C) \). The surjection \( E \to L \) induces inclusion \( \Gamma = \mathbb{P}(L) \subset S = \mathbb{P}(E) \).

If \( L = \mathcal{O}_C(B), B \in \text{Div}(C) \) with \( b = \text{deg}(B) \), then \( b = H \cdot \Gamma \) and \( \Gamma \sim H + fM \) where \( M = L \otimes \det(E) \} \in \text{Pic}(C) \). For example, if \( 0 \to \mathcal{O}_C \to E \to \det(E) \to 0 \), the injection \( \mathcal{O}_C \to E \) gives a global section of \( E \) which corresponds to the global section of \( \mathcal{O}_S(1) \) vanishing on \( \Gamma \). For a section \( \Gamma \) that correspond to the exact sequence \( 0 \to N \to E \to L \to 0 \), we have that \( \mathcal{N}_{\Gamma/S} \), the normal bundle of \( \Gamma \) in \( S \), is such that \( \mathcal{N}_{\Gamma/S} = \mathcal{O}_\Gamma \simeq N^\vee \otimes L \). In particular \( \text{deg}(\Gamma) = \Gamma^2 = \text{deg}(L) - \text{deg}(N) \). If \( \Gamma \sim H - fD \), then \( |\mathcal{O}_S(\Gamma)| \simeq \mathbb{P}(H^0(C, E(-D))) \).

For \( \Gamma \in \text{Div}(S) \), we define \( \mathcal{O}_\Gamma(1) := \mathcal{O}_S(1) \otimes \mathcal{O}_\Gamma \). The speciality of \( \Gamma \) is defined as \( i(\Gamma) := h^1(\Gamma, \mathcal{O}_\Gamma(1)) \). \( \Gamma \) is said special if \( i(\Gamma) > 0 \).

Let \( \Gamma_1 \) be a reducible unisecant such that \( H \cdot \Gamma_1 = n \), then there exist a section \( \Gamma_0 \subset S \) and an effective divisor \( A \) of degree \( a \) on \( C \) such that \( \Gamma_1 = \Gamma_0 + fA \) with \( H \cdot \Gamma_0 = n - a \). It can be shown that there exists a line bundle \( L \in \text{Pic}(C) \) of degree \( n - a \) such that \( 0 \to N \otimes \mathcal{O}_C(-A) \to E \to L \oplus \mathcal{O}_A \to 0 \), and \( i(\Gamma_1) = h^1(C, L \oplus \mathcal{O}_A) = h^1(C, L) = i(\Gamma_0) \), where \( \Gamma_0 \subset \Gamma_1 \) the unique section (see [16]).

2.3. Hilbert scheme of unisecants and Quot-Scheme. For any \( n \in \mathbb{N} \), denote by \( \text{Div}^{1,n}(S) \) the Hilbert scheme of unisecants curves of \( S \) which are of degree \( n \) with respect to \( \mathcal{O}_S(1) \). Since elements of \( \text{Div}^{1,n}(S) \) correspond to quotients of \( E \), therefore \( \text{Div}^{1,n}(S) \) can be endowed with a natural structure of Quot-scheme (see [16], section 4.4), and one has an isomorphism

\[ \Phi_{1,n} : \text{Div}^{1,n}(S) \to \text{Quot}_{E,n+a-g+1}^C, \quad \Gamma \to \{ E \to L \oplus \mathcal{O}_A \}, \]

and there is the universal quotient \( \mathcal{Q}_{1,\delta} \) \( \xrightarrow{\pi} \text{Div}^{1,n}(S) \) and a morphism \( \text{Proj}(\mathcal{Q}_{1,\delta}) \xrightarrow{\pi} \text{Div}^{1,n}(S) \). The isomorphism \( \Phi_{1,n} \) allows to identify tangent spaces \( H^0(\Gamma, \mathcal{N}_{\Gamma/S}) \simeq T_{[\Gamma]}(\text{Div}^{1,n}(S)) \simeq \text{Hom}(N \otimes \mathcal{O}_C(-A), L \oplus \mathcal{O}_A) \) and obstruction spaces.
$H^1(\Gamma, \mathcal{N}_{T/S}) \simeq \text{Ext}^1(N \otimes \mathcal{O}_C(-A), L \otimes \mathcal{O}_A)$ (see [16], section 4.4). We define

$$S^{1,n} := \{ \Gamma \in \text{Div}^{1,n}(S) : R^1\pi_* (\mathcal{O}_{\mathcal{Q},1}(\mathcal{F}))(1)) \Gamma \neq 0 \}.$$ 

This support is the scheme that parametrizes degree $n$, special unisecants of $S$.

**Definition 2.1.** Let $\Gamma \in \text{Div}^{1,n}(S)$. We say that:

(i).- $\Gamma$ is linearly isolated if $\dim(|\mathcal{O}_S(\Gamma)|) = 0$;

(ii).- $\Gamma$ is algebraically isolated if $\dim(\text{Div}^{1,n}(S)) = 0$.

Let $\Gamma$ be a special unisecant of $S$. Assume that $\Gamma \in \mathcal{F}$, where $\mathcal{F} \subset \text{Div}^{1,n}(S)$ is a subscheme.

(iii).- $\Gamma$ is specially unique in $\mathcal{F}$, if $\Gamma$ is the only special unisecant in $\mathcal{F}$.

(iv).- $\Gamma$ is specially isolated in $\mathcal{F}$, if $\dim(\mathcal{F} \cap S^{1,n}) = 0$.

When $\mathcal{F} = |\mathcal{O}_S(\Gamma)|$, $\Gamma$ is said to be linearly specially unique in case (iii) and linearly specially isolated in case (iv).

When $\mathcal{F} = \text{Div}^{1,n}(S)$, $\Gamma$ is said to be algebraically specially unique in case (iii) and algebraically specially isolated in case (iv).

**2.4. A result on deformation theory.** Let $Y$ be a smooth projective variety and $j : X \subset Y$ be a closed smooth subvariety. Let $\mathcal{I}_X \subset \mathcal{O}_X$ be the ideal sheaf of $X$.

We have an inclusion of tangent sheaves $\mathcal{T}_X \subset \mathcal{T}_Y|_X$ and a restriction morphism $\mathcal{T}_Y \to \mathcal{T}_Y|_X$. Let $\mathcal{T}_Y < X > \subset \mathcal{T}_Y$ the image inverse of $\mathcal{T}_X$ under the restriction morphism. The sheaf $\mathcal{T}_Y < X >$ is called the sheaf of germs of tangent vectors to $Y$ which are tangents to $X$. This is a coherent sheaf of rank $\dim(Y)$ on $Y$. We have a restriction map $\mathcal{T}_Y < X > \to \mathcal{T}_X$ giving the exact sequence $0 \to \mathcal{T}_Y(-X) \to \mathcal{T}_Y < X > \to \mathcal{T}_X \to 0$, where $\mathcal{T}_Y(-X)$ is the vector bundle of tangent vectors of $Y$ vanishing along $X$. Let $H^1(Y, \mathcal{T}_Y < X >) \xrightarrow{H^1(R)} H^1(X, \mathcal{T}_X)$ be the map in cohomology induced by the above exact sequence. The map $H^1(R)$ associates to a first-order deformation of $(Y, X)$ the corresponding first-order deformation of $X$. We have (see [16], Proposition 3.4.17):

First-order deformations of the pair $(Y, X)$. The infinitesimal deformations of the pair $(Y, X)$ (equivalently of the closed embedding $j$) are controlled by the sheaf $\mathcal{T}_Y < X >$, that is,

(i).- The obstructions lie in $H^2(Y, \mathcal{T}_Y < X >),$

(ii).- First-order deformations are parametrized by $H^1(Y, \mathcal{T}_Y < X >)$ and the space $H^0(Y, \mathcal{T}_Y < X >)$ parametrizes infinitesimal automorphisms.

**2.5. Extensions of line bundles and the Segre invariant.** Let $E$ be a vector bundle over a curve $C$. The slope of $E$ is defined as $\mu E := \frac{\text{deg}(E)}{\text{rank}(E)}$. A vector bundle $E$ is stable (respectively semistable) if for all subvector bundle $F \subset E$ we have that $\mu(F) < \mu(E)$ (respectively $\mu(F) \leq \mu(E)$).
For a rank two vector bundle $E$, the Segre invariant $s(E)$ of $E$ is defined as $s(E) := \deg(E) - 2(\max\{\deg(N)\})$, where the maximum is taken among all sub-line bundles $N$ of $E$. The bundle $E$ is stable (resp. semi-stable) if $s(E) > 0$ (resp $s(E) \geq 0$). For any $A \in \text{Pic}(C)$, one has $s(E) = s(E \otimes A)$.

Let $\delta, d$ be positive integers such that $\delta \leq d$. Consider $L \in \text{Pic}^{\delta}(C), N \in \text{Pic}^{d-\delta}(C)$.

Let $u \in \text{Ext}^1(L, N)$ be and $(u) : 0 \to N \to E \to L \to 0$ the corresponding exact sequence. Tensoring the exact sequence $(u)$ by $N^\vee$ we consider $\mathcal{E}_u := E \otimes N^\vee$ which fits in an extension $(e) : 0 \to \mathcal{O}_C \to \mathcal{E}_u \to L \otimes N^\vee \to 0$ and $s(E_u) = s(\mathcal{E}_u)$. Take $\mathbb{P} := \mathbb{P}(\text{Ext}^1(L, N))$, by Serre duality we have that $\mathbb{P} \simeq \mathbb{P}(H^0(K_C \otimes L \otimes N^\vee)^\vee)$. If $\deg(L-N) = 2\delta - d \geq 2$, then $\dim(\mathbb{P}) \geq g \geq 3$ and the map $\phi : \phi_{|K_C \otimes L \otimes N^\vee|} : C \to \mathbb{P}$ is a morphism. Set $X := \phi(C)$. For any positive integer $h$ denote by $\text{Sec}_h(X)$ the $h^{th}$-secant variety of $X$, defined as the closure of the union of all linear subspaces $<\phi(D)> \subset \mathbb{P}$, for all effective general divisors of degree $h$. One has that $\dim(\text{Sec}_h(X)) = \min\{\dim(\mathbb{P}), 2h-1\}$. We have the following result (see \[9\]).

**Proposition 2.1.** (Lange-Narashiman) Let $2\delta - d \geq 2$. For any integer $\sigma$ such that $\sigma \equiv 2\delta - d(\mod 2)$ and $4d - 2\delta \leq \sigma \leq 2\delta - d - 1$, one has

$$s(\mathcal{E}_u) \geq \sigma \iff e \not\in \text{Sec}_{\left\lfloor \frac{\sigma + 2\delta - d}{2} \right\rfloor}(X).$$

Let $t \geq 1$ be any integer. For $g \geq 2$ consider a integer $d$ such that $2g-2 \leq d \leq 4g-4$.

Let $\delta \leq d$ be a positive integer such that $2\delta - d \geq 0$. Let $L \in \text{Pic}^{\delta}(C)$ be a special and effective, consider a special line bundle $N \in \text{Pic}^{d-\delta}(C)$. Let $u \in \text{Ext}^1(L, N)$ be and $(u) : 0 \to N \to E \to L \to 0$ the corresponding exact sequence. In order to get $E = E_u$ (semi)stable a necessary condition is $2\delta - d \geq 0$. With these hypothesis we have that $h^0(L), h^1(L), h^0(N) \geq 0, h^1(N) > 0$. Consider the locus $\mathcal{W}_t = \{u \in \text{Ext}^1(L, N) : \partial_u : H^0(L) \to H^1(N) \text{ has Cokernel of dimension } \geq t\}$.

**Alternative description of $\mathcal{W}_t$.** Let $u \in \text{Ext}^1(L, N)$ be and consider the cup product map $\cup : H^0(L) \otimes H^1(N \otimes L^\vee) \to H^1(N)$. We have that $\partial_u(s) = s \cup u$. By Serre duality, $\cup$ is equivalent to consider the multiplication map

$$\mu : H^0(L) \otimes H^0(K_C \otimes N^\vee) \to H^0(K_C \otimes L \otimes N^\vee).$$

Note that when $N = L$, $\mu$ coincides with the Petri map $\mu_L$ for line bundles. For any subspace $W \subseteq H^0(K_C \otimes N^\vee)$ consider the multiplication map

$$\mu|_W := \mu_W : H^0(L) \otimes W \to H^0(K_C \otimes L \otimes N^\vee),$$

there is a commutative diagram
Assume also that $2\deg(\text{componentes } \Delta t) \geq 0$. Let $0 \to W \to H^0(L) \to H^0(K_C \otimes L) \to H^0(K_C \otimes L \otimes N^\vee) \to 0$ be an exact sequence on a general curve $C$ such that $\deg(N) = d - \delta > 0$, $\deg(L) = \delta > 0$ and $h^0(L) \cdot h^1(L) > 0$, $h^0(N) \cdot h^1(N) > 0$. Assume also that $2\delta - d \geq 1$. By Brill-Noether theory for line bundles (see [1], chapter IV and V), the Brill-Noether varieties $W_{d-\delta}^{-1}(C)$ and $W_{d-\delta}^{-g}(C)$ are irreducible, generically smooth, of dimension $\rho(g, \ell - 1, \delta) = g - \ell (g - \delta + \ell - 1)$.
and \( \rho(g, n-1, d-\delta) = g - n(g - (d - \delta) + n - 1) \) respectively. Let \( \mathcal{N}_0 \to C \times \text{Pic}^{d-\delta}(C) \) and \( \mathcal{L}_0 \to C \times \text{Pic}^{\delta}(C) \) be the Poincaré line bundles (see [1], p. 166) and let \( \mathcal{L} \to C \times W_{d-\delta}^{g-1}(C), \mathcal{N} \to C \times W_{d-\delta}^{n-1}(C) \) be the restrictions of \( \mathcal{L}_0, \mathcal{N}_0 \) to the Brill-Noether locus. Set \( \mathcal{Y} := \text{Pic}^{d-\delta}(C) \times W_{d-\delta}^{g-1}(C), \ Z := W_{d-\delta}^{n-1}(C) \times W_{d-\delta}^{g-1}(C) \subset \mathcal{Y} \). They are both irreducible, of dimension \( \dim(\mathcal{Y}) = g + \rho(g, \ell - 1, \delta), \dim(Z) = \rho(g, \ell - 1, \delta) + \rho(g, n - 1, d - \delta) \). Consider the projections

\[
\pi_{12} : C \times \mathcal{Y} \to C \times \text{Pic}^{d-\delta}(C), \ \pi_{13} : C \times \mathcal{Y} \to C \times W_{d-\delta}^{g-1}(C), \ \pi_{23} : C \times \mathcal{Y} \to \mathcal{Y}.
\]

We define \( \mathcal{E} := R^1(\pi_{23})_* (\pi_{12}^*(\mathcal{N}) \otimes \pi_{13}^*(\mathcal{L}^\vee)) \). In our numerical hypothesis we have that \( \mathcal{E} \to \mathcal{Y} \) is a vector bundle of rank \( m = 2\delta - d + g - 1 \) (see [1], p. 176-180). Consider the projective bundle morphism \( \gamma : \mathbb{P}^k(E) \to \mathcal{Y} \), where \( y = (N, L) \in \mathcal{Y} \), the fiber \( \gamma^{-1}(y) = \mathbb{P}(\text{Ext}^1(N, L)) = \mathbb{P} \). We have that \( \dim(\mathbb{P}(\mathcal{E})) = \dim(\mathcal{Y}) + m - 1 \), and \( \dim(\mathbb{P}(\mathcal{E})|_Z) = \dim(Z) + m - 1 \). Since (semi)stability is an open condition with the conditions on \( g, d \) and \( \delta \), there is an open, dense subset \( \mathbb{P}(E)^0 \subset \mathbb{P}(\mathcal{E}) \) and a morphism \( \pi_{d,\delta} : \mathbb{P}(E)^0 \to U_C(d) \).

In ([5], sections 6,7) authors study the image and fibers of the map \( \pi_{d,\delta} \) under certain numerical conditions on \( d, \delta \) and in a more general context with respect to the line bundles \( L \) and \( N \). They give in a different way a proof on the existence of irreducible and regular components in Brill-Noether loci \( B_3^d(C) \) and \( B^k(2, K_C), k \leq 3 \). Some of such components are the (dominant) image under \( \pi_{d,\delta} \) by certain degeneracy loci in \( \mathbb{P}(\mathcal{E}) \) satisfying similar conditions to the good components of definition 2.2, such loci are called by the authors Total good components (see [5], definition 6.13). Following this philosophy, for \( k = 4 \) and for specific numerical conditions we found an irreducible (not good) component \( \mathbb{P}(\Delta_3) \subset \mathbb{P}(E) \) of dimension \( 3g - 13 \) that satisfies the conditions of the main theorem and fill-up an irreducible component \( B \subset B^4(2, K_C) \) of dimension \( 3g - 13 \). We prove this in the next section.

3. PROOF OF THE THEOREM.

For the following lemma we adapt an argument of Lazarsfeld (see [6], Theorem 1.1).

**Lemma 3.1.** Let \( C \) be a non-hyperelliptic curve of genus \( g \geq 8 \) and let \( D = q_1 + q_2 + q_3 \) be a general effective divisor of degree 3 on \( C \). There exist a rank two vector bundle \( \mathcal{F} \) on \( C \) with the following properties:

- (i).- \( \det(\mathcal{F}) = K_C(-D), h^0(\mathcal{F}) = 3 \) and \( \mathcal{F} \) is globally generated,
- (ii).- \( h^0(\mathcal{F}^\vee) = 0 \).

**Proof.** (i).- Consider \( p_1, \ldots, p_{g-5} \) general points on \( C \). Note that the line bundle \( A := K_C(-p_1 - \cdots - p_{g-5}) \) is of degree \( g + 3 \), free of base points with \( h^0(C, A) = 5 \). \( M_i := K_C \otimes A^i \) is a line bundle with only one section and \( h^0(C, M_i(-q_i)) = 0, i = 1, 2, 3 \). The line bundle \( M_2 := A(-D) \) is such that \( |M_2| = g_1 \) is free of base points.
Consider the extension map \( \text{Ext}^1(M_2, M_1) \xrightarrow{\beta} \text{Hom}(H^0(M_2), H^1(M_1)) \) which sends an extension

\[
((e) : 0 \to M_1 \to E \to M_2 \to 0)
\]
to the coboundary map

\[
(\partial_e : H^0(M_2) \to H^1(M_1) = H^0(A)^\vee).
\]

Any non-trivial extension in \( \text{Kernel}(\beta) \) satisfies that \( h^0(\mathcal{F}) = 3 \) and \( \det(\mathcal{F}) = \mathcal{F}^\vee = K_C(-D) \). Note that \( \text{Ext}^1(M_2, M_1) \cong H^1(M_1 \otimes M_2^*) \cong H^0(A^2(-D))^\vee \) and \( \text{Hom}(H^0(M_2), H^1(M_1)) \cong H^0(M_2)^\vee \otimes H^0(A)^\vee \).

We prove that \( \text{Kernel}(\beta) \) is a vector space of dimension \( g - 5 \): The map \( \beta \) is dual to the multiplication map \( m_D : H^0(M_2) \otimes H^0(A) \to H^0(M_2 \otimes A) = H^0(A^2(-D)) \).

By the base point free pencil trick applied to \( M_2, \ker(m_D) = H^0(C, \mathcal{O}_C(D)) \), then \( \dim \ker(m_D) = 1 \) and \( m_D \) has cokernel of dimension \( h^0(A^2(-D)) - 9 = g - 5 = \dim \text{Ker}(\beta) \).

Now consider an extension \(( (e) : 0 \to M_1 \to \mathcal{F} \to M_2 \to 0) \) in \( \text{Kernel}(\beta) \) and suppose that \( \mathcal{F} \) is not globally generated, then the three sections of \( \mathcal{F} \) generates a subsheaf \( \mathcal{F}_1 \) of \( \mathcal{F} \) fitting into an exact sequence \( 0 \to M_1(-B) \to \mathcal{F}_1 \to M_2 \to 0 \), for some divisor \( B \) over \( C \). For any \( y \in (C - \{p_1, \ldots, p_{g-5}\}) \), \( h^0(M_2(-y)) = 0 \) and \( h^0(M_2(-y)) = 1 \), then \( h^0(\mathcal{F}(y)) \leq 1 \). By Riemann-Roch we have that \( h^0(\mathcal{F} - h^0(\mathcal{F}^\vee \otimes K_C) = -3 \), that is, \( h^0(K_C \otimes \mathcal{F}^\vee) = 6 \), this implies that \( h^0(\mathcal{F}(y)) = -5 + h^0(K_C(y) \otimes \mathcal{F}^\vee) \geq 1 \), then \( h^0(\mathcal{F}(y)) = 1 \), then \( \mathcal{F} \) is globally generated at \( (C - \{p_1, \ldots, p_{g-5}\}) \), that is, \( \mathcal{F} \) is globally generated away the points \( \{p_1, \ldots, p_{g-5}\} \), this implies that \( B \subset \text{supp}(\{p_1, \ldots, p_{g-5}\}) \). Thus, if \( \mathcal{F} \) comes from an element \( e \in \text{Ker}(\beta) \) that fails to be generated by global sections, then there is a point \( x \) among the points \( \{p_1, \ldots, p_{g-5}\} \) and a subsheaf \( \mathcal{F}_2 \) given by the following extension \( 0 \to M_1(-x) \to \mathcal{F}_2 \to M_2 \to 0 \) so that \( e \) is induced from this extension and such extension is surjective on global sections. Since \( \text{Hom}(H^0(M_2), H^1(M_1(-x))) \cong \text{Hom}(H^0(M_2), H^0(A(x))) \), such extensions are parametrized by elements in

\[
\text{Kernel} [\text{Ext}^1(M_2, M_1(-x)) \to \text{Hom}(H^0(M_2), H^0(A(x))],
\]

where \( \text{Ext}^1(M_2, M_1(-x)) \cong H^0(A^2(x-D))^\vee \). Since \( h^0(A) = 5 \) and \( x \in \text{supp}(\{p_j\}_{j=1}^{g-5}) \), then \( h^0(A(x)) = 6 \). By the base point free pencil trick applied to \( M_2, \) note that \( H^0(C, \mathcal{O}_C(x + D)) \) is the kernel of the map \( H^0(M_2) \otimes H^0(A(x)) \to H^0(A^2(x-D)) \) and \( h^0(C, \mathcal{O}_C(x + D)) = 1 \), then the cokernel has dimension \( h^0(A^2(x-D)) - 11 = g - 6 \). This implies that the extensions in \( \text{Kernel}(\beta) \) which fail to be generated by global sections have codimension at least 1 in \( \text{Kernel}(\beta) \), so for a general extension in \( \text{Kernel}(\beta) \), the corresponding vector bundle \( \mathcal{F} \) satisfies that \( \mathcal{F} \) is globally generated, \( h^0(\mathcal{F}) = 3 \) and \( \mathcal{F}^\vee = K_C(-D) \).
(ii). We have the identification $\mathcal{F} \simeq \mathcal{F}^\vee \otimes K_C(-D)$. Setting $H = H^0(C, \mathcal{F})$, since $\mathcal{F}$ is generated by global sections there is an exact sequence

$$0 \to K_C^\vee(D) \to H \otimes \mathcal{O}_C \to \mathcal{F} \to 0,$$

take $V = H^\vee$ and dualize to get $0 \to \mathcal{F}^\vee \to V \otimes \mathcal{O}_C \to K_C(-D) \to 0$, thus $h^0(\mathcal{F}^\vee \otimes K_C(-D)) = h^0(V) = 3$. We have that $h^0(\mathcal{F}^\vee) = 0$, otherwise if $0 \neq s \in H^0(\mathcal{F}^\vee)$, then $Z(s) := \{\text{zeroes of } s\}$ defines a subsheaf $R \subset \mathcal{F}^\vee$ of degree $\geq 0$, then $h^0(R \otimes K_C(-D)) \geq \deg(R \otimes K_C(-D)) - g + 1 \geq g - 4$. Since $\mathcal{F}^\vee \otimes K_C(-D) \simeq E$, then $R \otimes K_C(-D) \to \mathcal{F}^\vee \otimes K_C(-D) \simeq \mathcal{F}$, then $h^0(\mathcal{F}) \geq g - 4$ which is a contradiction since $g \geq 8$. $\square$

**Lemma 3.2.** Let $\mathcal{G} := G(3, H^0(C, K_C(-D)))$ be the Grassmannian of 3–planes in $H^0(C, K_C(-D))$. For $V \in \mathcal{G}$ general, the map

$$\mu_V : V \otimes H^0(C, K_C(-D)) \to H^0((K_C(-D))^2)$$

has kernel of dimension 3.

**Proof.** Note that $\wedge^2 V \subset \ker(\mu_V)$, then $\dim \ker(\mu_V) \geq 3, \forall V \in \mathcal{G}$. Let $\Sigma_3 := \{V \in \mathcal{G} : \dim \ker(\mu_V) = 3\}$. We are going to show that $\Sigma_3 \neq \emptyset$ and that $\dim(\Sigma_3) = \dim(\mathcal{G})$.

Consider the vector bundle $\mathcal{F}$ constructed in lemma 3.1. Since $\mathcal{F}$ is generated by global sections we have the exact sequence

$$0 \to (K_C(-D))^\vee \to H^0(\mathcal{F}) \otimes \mathcal{O}_C \to \mathcal{F} \to 0. \quad (1)$$

Dualizing (1) and since $\text{Ext}^1(\mathcal{F}, K_C(-D)^\vee) = 0$ we have an exact sequence

$$0 \to \mathcal{F}^\vee \to H^0(\mathcal{F})^\vee \otimes \mathcal{O}_C \to K_C(-D) \to 0. \quad (2)$$

Tensoring (2) by $K_C(-D)$ we have

$$0 \to \mathcal{F}^\vee \otimes K_C(-D) \to H^0(\mathcal{F})^\vee \otimes K_C(-D) \to (K_C(-D))^2 \to 0. \quad (3)$$

We know that $h^0(\mathcal{F}^\vee) = 0$, also a non-zero section $\tau \in H^0(\mathcal{O}_C)$ induces an isomorphism $H^0(\mathcal{F}^\vee) \otimes H^0(\mathcal{O}_C) \simeq H^0(\mathcal{F})^\vee$, hence taking cohomology in (2) we have an injective map $H^0(\mathcal{F})^\vee \to H^0(K_C(-D))$. Let $V := \iota(H^0(\mathcal{F})^\vee)$, then $V \subset H^0(K_C(-D))$ has dimension 3. Since $\mathcal{F}^\vee \otimes \det(\mathcal{F}) \simeq \mathcal{F}$, then $H^0(\mathcal{F}^\vee \otimes K_C(-D)) \simeq H^0(\mathcal{F})$, so we take cohomology in (3) to obtain

$$0 \to H^0(C, \mathcal{F}) \to H^0(\mathcal{F})^\vee \otimes H^0(K_C(-D)) \to H^0((K_C(-D))^2) \to \cdots. \quad (4)$$
From (4) we have that
\[ H^0(C,\mathcal{F}) \simeq \ker(H^0(\mathcal{F})^\vee \otimes H^0(K_C(-D))) \to H^0((K_C(-D))^2)). \] (5)

Since \( H^0(\mathcal{F})^\vee \otimes H^0(K_C(-D)) \simeq V \otimes H^0(K_C(-D)) \), from (5) we have \( H^0(\mathcal{F}) \simeq \ker(\mu_V) \), then \( V \simeq H^0(C,\mathcal{F}) \), so \( \Sigma_3 \neq \emptyset \). By upper semicontinuity of the function \( \mathbb{G} \to \mathbb{Z}, V \to \dim \ker(\mu_V) \), we have that for the general \( V \in \mathbb{G} \), \( \dim \ker(\mu_V) = 3 \), then \( \dim (\Sigma_3) = \dim(\mathbb{G}) \). \( \Box \)

**Proof of the Theorem:**

**Step 1.-** By Lemma 3.2 and the alternative description of \( \mathcal{W}_t \) we have that \( \mathcal{W}_3 \neq \emptyset \), so the expected dimension of \( W_3 \) is \( 3g - 18 \). Let \( \mathbb{P} := \mathbb{P}((H^0((K_C(-D))^2))^\vee) \simeq \mathbb{P}(\text{Ext}^1(K_C(-D), O_C(D))) \), we denote by \( \pi_u \) the hyperplane in \( \mathbb{P} \) defined by \( \{ u = 0 \} \subset H^0((K_C(-D))^2) \), where \( u \) corresponds to the extension

\[(u): 0 \to O_C(D) \to E \to K_C(-D) \to 0.\]

Let \( J_G := \{(W, \pi) \in \mathbb{G} \times \mathbb{P} : \ker(\mu_W) \subset \pi\} \) and let \( \pi_1 : J_G \to \mathbb{G}, \pi_2 : J_G \to \mathbb{P} \) the projections to the first and second factor respectively. From lemma 3.2 we have that for \( W \in \mathbb{G} \) general, \( \dim(\ker(\mu_W)) = 3g - 12 \). The fiber of \( \pi_1 \) over a general element \( V \in \mathbb{G} \), is isomorphic to the linear system of hyperplanes in \( \mathbb{P} \) passing through the general subspace \( \mathbb{P}(\ker(\mu_V)) \), then the general fiber of \( \pi_1 \) is irreducible and of dimension \((h^0((K_C(-D))^2) - 1) - (3g - 12) = 2 \). On the other hand, since \( \Sigma_3 \subset \pi_1(J_G) \) then \( J_G \) dominates \( \mathbb{G} \) through \( \pi_1 \), there exists a unique component \( J_3 \) of \( J_G \) dominating \( \mathbb{G} \) through \( \pi_1 \) and \( \dim(J_3) = 2 + \dim(\mathbb{G}) = 3g - 16 \). By Serre duality, \( \partial_u : H^0(C, K_C(-D)) \to H^1(O_C(D)) \) is symmetric, that is, \( \partial_u = \partial_u^\vee \), then \( \ker(\partial_u) = \ker(\partial_u^\vee) = (\ker(\partial_u))^\perp \), in particular \( \ker(\partial_u) \) is uniquely determined, so the general fiber of the map \( \pi_2|_{J_3} : J_3 \to \mathbb{P} \) is irreducible and zero-dimensional, then \( \pi_2(J_3) \subset \mathbb{P}(W_3) \subset \mathbb{P} \) is irreducible of dimension \( 3g - 16 \). Note that \( \pi_2(J_3) \) give rise to the existence of a not good component \( \Delta_3 \subset \mathcal{W}_3 \) of dimension \( 3g - 15 \) with \( \Delta_3 = \pi_2(J_3) \) such that for the general element \( u \in \Delta_3 \) we have that \( \partial_u : H^0(C, K_C(-D)) \to H^1(C, O_C(D)) \) has cokernel of dimension 3.

Consider the image \( X \) of the map \( C \to \mathbb{P} \) defined by the (very ample) linear system \( |(K_C(-D))^2| \). Note that for \( \epsilon \in \{ 0, 1 \} \) and for \( \sigma = g - 6 - \epsilon > 0 \) we have that \( 3g - 16 = \dim(\mathbb{P}(\Delta_3)) > \dim \text{Sec}^2_{3g - 16 - \epsilon}(\mathbb{C}) \), then by Lange-Narashiman we have that for general \( u \in \Delta_3 \), vector bundles coming from extensions in \( \Delta_3 \) are stable vector bundles. This shows in particular that \( B^4(C, K_C) \neq \emptyset \) and describes some points in it.

**Step 2.- Unobstructed sections.** Let \( u \in \Delta_3 \) be a general extension and let \( E = E_u \) be the vector bundle that fits in the extension \( u : 0 \to O_C(D) \to E \to K_C(-D) \to 0 \). Let \( \Gamma = \Gamma_u \) be the section corresponding to the quotient \( E \to K_C(-D) \). Let \( S = \mathbb{P}(E) \) be and \( p : S \to C \) the structure map. Let \( c \) be the class of
\( \mathcal{O}_S(1) \) in \( \text{Pic}(S) \) (or \( \text{Num}(S) \simeq H^2(S, \mathbb{Z}) \)). The tangent bundle \( T_S \) fits in a exact sequence

\[
0 \to T_{S/C} \to T_S \to p^*T_C \to 0, \tag{6}
\]

where \( T_{S/C} := \text{Ker}(T_S \to p^*(T_C)) \) is the relative tangent sheaf (the sheaf of tangent vectors along the fibers of \( p \)). \( T_{S/C} \) is dual to the relative canonical sheaf \( \omega_{S/C} \) and \( \omega_{S/C} = \mathcal{O}_S(-2c) \otimes p^*(\text{det}(E)) = \mathcal{O}_S(-2c) \otimes p^*(K_C) \), then \( T_{S/C} = \mathcal{O}_S(2c) \otimes p^*(T_C) \). On the other hand we have that \( \mathcal{N}_{\Gamma/S} \simeq K_C(-2D) \), then \( h^0(\Gamma, \mathcal{N}_{\Gamma/S}) = g - 6 \) and \( h^1(\Gamma, \mathcal{N}_{\Gamma/S}) = 1 \). By \( [10] \), p. 177, eq(3.56) We have

\[
0 \to T_S(-\Gamma) \to T_S < \Gamma \to \mathcal{T}_\Gamma \to 0. \tag{7}
\]

Tensoring the exact sequence (7) by \( \mathcal{O}_S(-\Gamma) \) we have

\[
0 \to T_{S/C}(-\Gamma) \to T_S(-\Gamma) \to p^*(T_C) \otimes \mathcal{O}_S(-\Gamma) \to 0. \tag{8}
\]

Recall that \( \Gamma \sim \mathcal{O}_S(1) - f_D \) (see section 2.1), since \( K_S \equiv -2c + (4g - 4)f \), then \( K_S(\Gamma) \equiv -c + a_0f \) for some \( a_0 \in \mathbb{Z} \), since \( T_{S/C} \simeq \mathcal{O}_S(2c) \otimes p^*(T_C) \), we have that \( K_S(\Gamma) \otimes (T_{S/C})^\vee \equiv -3c + af \) for some integer \( a \), in particular \( -3c + af \) is non-effective, then \( h^2(T_{S/C}(-\Gamma)) = h^0(K_S(\Gamma) \otimes (T_{S/C})^\vee) = 0 \). Similary we have that \( K_S(\Gamma) - p^*(T_C) \equiv -c + bf \) for some integer \( b \), then it is non-effective, so \( h^2(p^*(T_C) \otimes \mathcal{O}_S(-\Gamma)) = h^0(K_S(\Gamma) \otimes (p^*(T_C))^\vee) = 0 \), from (8) we have that \( h^2(T_S(-\Gamma)) = 0 \), and from (7) we deduce that \( h^2(T_S < \Gamma >) = 0 \), then \( (\Gamma, S) \) is unobstructed (see section 2.4), that is, the first-order infinitesimal deformations of the closed embedding \( \Gamma \hookrightarrow S \) are unobstructed with \( S \) not fixed, in particular \( \Gamma \) is unobstructed in \( S \) fixed and \( \Gamma \) varies in a \( g - 6 \)-dimensional family.

**Step 3.- Specially isolated sections.** Since the quotient \( E \to K_C(-D) \) corresponds to \( \Gamma \), to show that there are only finitely many sections corresponding to special quotients like these, that is, to show that \( \Gamma \) it is a specially isolated linear section, we need to show that the family of such corresponding quotients (which vary in a \( g - 6 \) dimensional family) do not not intersect in positive dimension the 3-dimensional family of special line bundles in \( W_{g-5}^4(C) \simeq W_3^0(C) \).

The isomorphism \( T_{[\Gamma]}(\text{Div}^{1,2g-5}(S)) \simeq H^0(\Gamma, \mathcal{N}_{\Gamma/S}) \simeq H^0(C, K_C(-2D)) \) allows us to identify \( H^0(\Gamma, \mathcal{N}_{\Gamma/S}) \hookrightarrow H^0(C, K_C) \) as a subspace of \( H^0(C, K_C) \) and by Serre duality we consider \( V_1 := H^0(\Gamma, \mathcal{N}_{\Gamma/S})^\vee \hookrightarrow H^1(C, \mathcal{O}_C) \) as a subspace of \( H^1(C, \mathcal{O}_C) \). Denote by \( N = \mathcal{O}_C(D) = \text{Kernel} (E \to K_C(-D) \to 0) \), and let \( V_2 = T_{[\Gamma]}(W_3^0(C)) \) be the tangent space of \( W_3^0(C) \) at \( N \). Note that \( h^0(C, N) = 1 = h^0(C, N^2) \) and the image of the Petri map \( \mu_N : H^0(C, N) \otimes H^0(C, K_C \otimes N^\vee) \to H^0(C, K) \) is \( H^0(C, K_C \otimes N^\vee) \), then \( V_2 = (\text{Image}(\mu_N))^\perp \subset H^1(C, \mathcal{O}_C) \). With these identifications of tangent spaces, \( \Gamma \) is specially isolated if \( V_1 \cap V_2 = \{0\} \) in \( H^1(C, \mathcal{O}_C) \). Let \( s, s^2 \)
be sections of $H^0(C, N)$ and $H^0(C, N^2)$ respectively. Consider the cup product maps $H^1(C, \mathcal{O}_C) \cup \rightarrow H^1(C, N), H^1(C, \mathcal{O}_C) \cup \rightarrow H^1(C, N^2)$. Note that $V_2 = \text{Ker}(\cup s), T_{N^2}(W^0_D) = \text{Ker}(\cup s^D) = (\text{Im}(\mu_N)') = (H^0(K_C(-2D)))$, and $V_1$ can be identified with the image of $\cup s^2$. By Serre duality we have that $\phi \cup s = 0$ if and only if $\phi \cup s, \omega > \phi, \mu_N(s \otimes \omega)$ for every $s \in H^0(C, N)$ and for every $\omega \in H^0(C, K_C(-N))$. From this description we have that $V_1 \cap V_2 = \{0\}$, then $\Gamma$ is specially isolated, moreover $\Gamma$ is linearly isolated, thus $\dim(H_\omega < \phi \cap V) = 0$, then there are at most finitely many sections corresponding to special quotients $E \rightarrow K_C(-D)$. By section 2.1 we have that $\Gamma \sim H - f_D$, this implies that $h^0(C, E_u(-D)) = 1$.

**Step 4.** The map $\P(\Delta_3) \rightarrow B^4(2, K_C)$ is generically injective. Consider the map $\pi : \P(\Delta_3) \rightarrow B^4(2, K_C)$ which sends an extension

$$((u) : 0 \rightarrow \mathcal{O}_C(D) \rightarrow E_u \rightarrow K_C(-D) \rightarrow 0) \rightarrow [E_u].$$

For a general element $[E_u] \in \pi(\P(\Delta_3)) \subset B^4(2, K_C)$, we have that $\pi^{-1}([E_u])$ correspond to the extensions $u' \in \P(\Delta_3)$ that induce exact sequences $0 \rightarrow \mathcal{O}_C(D) \rightarrow E_{u'} \rightarrow K_C(-D) \rightarrow 0$ and a diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{O}_C(D) & \overset{\iota_1}{\rightarrow} & E_u & \rightarrow K_C(-D) & \rightarrow 0 \\
& \phi & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{O}_C(D) & \overset{\iota_2}{\rightarrow} & E_{u'} & \rightarrow K_C(-D) & \rightarrow 0
\end{array}
$$

with an isomorphism $\phi : E_u \rightarrow E_{u'}$ of stable bundles. The maps $\phi \circ \iota_1$ and $\iota_2$ determine two non-zero sections $\sigma_1 \neq \sigma_2$ in $H^0(E_u(-D))$. For step 3, we have that for the general extension $u \in \Delta_3$, the corresponding section $\Gamma$ of the quotient line bundle $E_u \rightarrow K_C(-D)$ is linearly isolated, then $h^0(C, E_u(-D)) = 1$, so for some scalar $\lambda \neq 0$ we have that $\phi \circ \iota = \lambda \iota_2$, so by [5], Lemm 4.5 on proportional extensions we have that $E_u = E_{u'}$, i.e $u, u'$ are proportional in $\Delta_3$, then $\pi$ is generically injective. In particular for $u, u' \in \Delta_3$ general points, the general vector bundles $E_u, E_{u'}$ cannot be isomorphic.

**Step 5. A global moduli map.** Given a special and effective line bundle $L \in \text{Pic}(C)$, the condition of canonical determinant forces that the kernel $N$ of the quotient map $E \rightarrow L$ is isomorphic to $K_C \otimes L'$ because we require that $K_C \simeq \det(E) = L \otimes N$, then $N$ is uniquely determined by $L$. For $D$ a general effective divisor of degree three, $L = K_C(-D)$ depends of $\rho(L) := \rho(g, g - 4, 2g - 5) = 3$ parameters. From steps (1)-(3) and the construction in 2.6 we obtain an irreducible component $\P(\Delta_3) \subset \P(E)$ of dimension $3g - 16 + \rho(L) = 3g - 13$, where a point in $\P(\Delta_3)$ corresponds to the datum of a pair $(y, u)$ with $y = (\mathcal{O}_C(D), K_C(-D))$, $D$ a general and effective divisor of degree 3 and $u \in \P(\Delta_3)$ an extension as in step (1). From steps (2)-(4) we have that the global moduli map $\pi_{2g-2, 2g-5, 3}(\P(\Delta_3))$:
\( \mathbb{P}(\Delta_3) \to B^4(2, K_C) \) fill-up an irreducible component \( B \subset B^4(2, K_C) \) of dimension 
\( \rho_{K_C}(2, 4, g) = 3g - 13. \)

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