Pion electromagnetic formfactor in the space-like region and $P$-phase $\delta_1^1(s)$ of $\pi\pi$ scattering from the value of the modulus of formfactor in the time-like region.

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Abstract

The problem of determining the pion electromagnetic formfactor $F(q^2)$ in the space-like region from the value of its modulus in the time-like region is solved by the formfactor analyticity. If $F(q^2)$ has no zeroes in the complex plane then the formfactor in the space-like region is determined uniquely. If $F(q^2)$ has zeroes in the complex plane $q^2$ it can be obtained in the space-like region within narrow limits using experimental data from the time-like region. The formfactor phase $\varphi(s)$ which coincides with the $P$-wave phase $\delta_1^1(s)$ of the $\pi\pi$ scattering is calculated. The value of the pion radius has been improved.

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1. Introduction.

The pion electromagnetic formfactor $F(q^2)$ is calculated theoretically only in the space-like region. For example, the formfactor has been calculated with QCD sum rules [1] and in the lattice QCD [2].

On the other hand, the main part of experimental data on formfactor are obtained in the time-like region via the reaction [3,4]

$$ e^+ e^- \rightarrow \pi^+ \pi^- $$ \hspace{1cm} (a)

At small space-like momentum transfers ($0 < Q^2 < 0.253 \text{ GeV}^2$, $Q^2 = -q^2$) the formfactor has been measured by scattering of 300 GeV pions from atomic electrons [5]

$$ \pi^+ e^- \rightarrow \pi^+ e^- $$ \hspace{1cm} (b)

Colliding–beam measurements of $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ and measurements of $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ provide direct access to $F(q^2)$. At large space-like momentum transfers formfactor $F(q^2)$ was extracted from the reaction of the electroproduction of pions from nucleons [6-9]

$$ ep \rightarrow e\pi^+ n $$

$$ en \rightarrow e\pi^- p $$ \hspace{1cm} (c)

But the presence of the nucleons and their structure complicates theoretical models and thus the determination of the pion formfactor at large $Q^2$ is model dependent. Analyticity connects values of the pion formfactor in space-like and time-like regions.

As follows from microcausality the pion electromagnetic formfactor $F(q^2)$ has the following analytical properties: 1) $F(q^2)$ is an analytical function of $q^2$ with a cut along positive $q^2$ from $q^2 = 4m^2_\pi$ to infinity. 2) On the real axis to the left of $q^2 = 4m^2_\pi$ the function $F(q^2)$ is real, and consequently, takes complex conjugate values on the upper and lower edges of the cut. 3) For complex $q^2$ with $|q^2| \rightarrow \infty$ the function $F(q^2)$ grows no faster than a finite power (in QCD formfactor decreases). 4) the function $F(q^2)$ is normalized by the condition $F(0) = 1$.

It follows from unitarity that the formfactor phase $\varphi(s) = \arctan \frac{\text{Im } F(s)}{\text{Re } F(s)}$ in the region $4m^2_\pi \leq s \leq s_{el}$ must coincide with the $p$-phase of $\pi\pi$ scattering $\delta^p_1(s)$, $s_{el} \approx 0.8 \text{ GeV}^2$.

The purpose of this work is to calculate formfactor in the space-like region from the known value of the modulus of the formfactor in the time-like region measured in [3,4]. The phase of formfactor $\varphi(s)$ which coincides with $p$-phase $\delta^p_1(s)$ of $\pi\pi$ scattering $\delta^p_1(s)$ measured in [10-12] will be also calculated.

2. Determination of the formfactor $F(s)$ in the whole complex plane from the given value of its modulus in the time-like region if formfactor has no zeroes in the complex $s$-plane.

We shall follow the method of the works [13]. If $F(s)$ has no zeroes in complex $s$-plane the function $\ln F(s)$ is an analytical function of $s$ with the cut $[s_0, \infty]$, $s_0 = 4m^2_\pi$ and, consequently, we have the formula
where the contour \( C \) contains the lower and upper edges of the cut and a large circle. It follows from Eq.\((1)\) that when \( F(s) \) has no zeroes in the complex \( s \)-plane the function \( F(s) \) is uniquely determined

\[
F(s) = F_0(s)
\]

\[
F_0(s) = \exp \left[ \frac{s \sqrt{s_0 - s}}{2 \pi} \int_{s_0}^{\infty} \frac{\ln |F(s')|^2}{\sqrt{s' - s_0 s'(s' - s)}} \right]
\]

The result of the calculation of \( F_0(s) \) is given in tables 1,2 and in Fig.1. We use for \( |F(s')| \) the phenomenological formula for formfactor obtained in \([14,15]\). This formula satisfies the following requirements: (1) formfactor has correct analytical properties; (2) at large \( Q^2 = -s \) formfactor has the asymptotic behaviour determined by QCD \([16-19]\) (taking into account the preasymptotic power correction \([20]\)); the phenomenological formula describes well the experiments \([3,4]\) \( \chi = 123 \) fitting over 120 experimental points in the interval \( 0.1296 \text{ GeV}^2 \leq s \leq 4.951 \text{ GeV}^2 \).

The phase of the formfactor can be found using formula

\[
\frac{1}{s' - s - i\varepsilon} = \pi i \delta(s' - s) + P \frac{1}{s' - s}
\]

Substituting Eq.\((3)\) into Eq.\((2)\) we get

\[
F_0(s) = |F(s)| e^{i\varphi_0(s)}, \quad s > s_0
\]

where the phase of the formfactor \( \varphi_0(s) \) is equal to

\[
\varphi_0(s) = -\frac{s \sqrt{s - s_0}}{2 \pi} P \int_{s_0}^{\infty} \frac{\ln |F(s')|^2}{\sqrt{s' - s_0 s'(s' - s)}} ds', \quad s > s_0
\]

Let us write the integral \((5)\) in the form more convenient for numerical calculation

\[
\varphi_0(s) = -\frac{s \sqrt{s - s_0}}{2 \pi} \left\{ \int_{s_0}^{\infty} \frac{\ln |F(s')/F(s)|^2}{\sqrt{s' - s_0 s'(s' - s)}} ds' - \frac{\pi}{s \sqrt{s_0}} \ln |F(s)|^2 \right\}
\]

The result of the calculations of \( \varphi_0(s) \) is given in the Table 1.

**3. The consideration of the formfactor with zeroes in the complex plane.**

Let us consider now the case when the formfactor \( F(s) \) has zeroes in the complex plane at \( s = s_k, k = 1, 2, \ldots \). We introduce instead of the formfactor \( F(s) \) the function \( \tilde{F}(s) \) by formula

\[
F(s) = \chi(s) \tilde{F}(s)
\]
\[ \chi(s) = \prod_{k} \chi_{k}(s) \] (7)

\[ \chi_{k}(s) = \frac{\sqrt{s_0 - s} - \sqrt{s_0 - s_{k}}}{\sqrt{s_0 - s} + \sqrt{s_0 - s_{k}}} \cdot \frac{\sqrt{s_0 - s} - \sqrt{s_0 - s_{k}}}{\sqrt{s_0 - s} + \sqrt{s_0 - s_{k}}} \]

Since to each complex zero \( s_{k} \) there corresponds a complex conjugate zero \( s_{k}^{*} \), the modulus of the function \( \chi(s) \) equals to unity on the cut. The function \( \tilde{F}(s) \) has no zeroes in the complex plane, on the cut its modulus equals to \( |F(s')| \), and it is normalized by the condition

\[ \tilde{F}(0) = \chi^{-1}(0) \geq 1 \] (8)

and, consequently, we may apply the Cauchy theorem to the function \( \frac{[\ln \tilde{F}(s)]}{s \cdot \sqrt{s - s_0}} \)

\[ \frac{1}{2\pi i} \int_{C} \frac{\ln \tilde{F}(s')ds'}{\sqrt{s' - s}(s' - s)s'} = \frac{\ln \tilde{F}(s)}{s\sqrt{s - s_0}} + \frac{\ln \tilde{F}(0)}{i(-s)\sqrt{s_0}} = \]

\[ = \frac{1}{2\pi i} \int_{C} \frac{\ln |F(s')|^2 ds'}{\sqrt{s' - s_0}s'(s' - s)} \] (9)

Therefore

\[ F(s) = \psi(s)F_{0}(s) \]

\[ \psi(s) = \chi(s)/[\chi(0)]^{1 - s/s_0} \] (10)

where \( F_{0}(s) \) is determined by Eq.(2).

It is clear from Eq.(7) that the knowledge of the modulus of the formfactor on the cut determines the value of the formfactor in the entire complex plane up to within the factor \( \psi(s) \).

It is easy to obtain from Eq.(10) the asymptotic formula for the formfactor \( F(s) \) for \( s \to -\infty \). Since \( \chi(-s) \to 1 \) as \( s \to \infty \), we have

\[ |F(-s)| \to |F(s)| \exp(-[a + \ln \chi(0) + \sqrt{s/s_0}) \] (11)

where

\[ a = \frac{\sqrt{s_0}}{2\pi} \int_{s_0}^{\infty} \ln |F(s)|^2 ds \] (12)

From QCD there follows the asymptotic formula at \( Q^2 = -q^2 \to \infty \) for the pion formfactor [16-19]

\[ F(-Q^2) \to \frac{16\pi\alpha_{s}(Q^2)}{Q^2}f_{\pi}^2 \]

\[ f_{\pi} = 93 \text{ MeV} \]. Therefore
\[ a + \ln \chi(0) = 0 \quad (14) \]

If the formfactor \( F(s) \) has no zeroes in the complex plane than \( a = 0 \) and the formfactor \( F(s) \) can be determined uniquely. If the value \( a \) is small than the uncertainty of the determination of the formfactor \( F(s) \) due to the function \( \psi(s) \) will be small. The value \( a \) was calculated in the work \([15]\) from the analysis of the experiments \([3,4]\) and was found to be small:

\[ a = 0.069 \pm 0.003 \quad (15) \]

4. **Bounds on the formfactor \( F(s) \) in the space-like region.**

The formfactor \( F(s), s < 0 \) is determined up to the factor \( \psi(s) \). To determine upper and lower bounds on the value of the formfactor \( F(s) \), for \( s < 0 \) we look for the maximum and the minimum of the factor \( \psi(s) \), by fixing the value of \( s \) and changing the distribution of the zeroes \( s_k \) so that condition (14) is satisfied. Let us write the relation

\[ w_k = \left( \sqrt{s_0} - \sqrt{s_0 - s_k} \right) / \left( \sqrt{s_0} + \sqrt{s_0 - s_k} \right) \quad (16) \]

Write \( w_k \) in the form:

\[ w_k = r_k e^{i\varphi_k} \quad (17) \]

\( r_k < 1 \).

Let us write the factor \( \chi_k(s) \) in Eq.(7) in the form

\[ \chi_k(s) = \frac{(w - w_k)(w - w_k^*)}{(1 - ww_k)(1 - ww_k^*)} = \frac{w^2 - 2wr_k\cos\varphi_k + r_k^2}{1 - 2wr_k\cos\varphi_k + w^2r_k^2} \quad (18) \]

where

\[ w = \left( \sqrt{s_0} - \sqrt{s_0 - s} \right) / \left( \sqrt{s_0} + \sqrt{s_0 - s} \right) \quad (19) \]

It is obvious that the maximum \( \chi_k(s) \) is achieved if \( \cos\varphi_k = 1 \) and the minimum \( \chi_k(s) \) is achieved if \( \cos\varphi_k = -1 \) and

\[ \max \chi_k(s) = \frac{(w - r_k)^2}{(1 - wr_k)^2} \quad (20) \]

\[ \min \chi_k(s) = \frac{(w + r_k)^2}{(1 + wr_k)^2} \quad (21) \]

The value \( \chi(0) \) is equal

\[ \chi(0) = \prod_k r_k^2 \quad (22) \]

It follows from Eq.(14)

\[ \chi(0) = e^{-a} \quad (23) \]
And it follows from inequality
\[
\frac{|w| + r_1}{1 + |w| r_1} > \frac{|w| + r_2}{1 + |w| r_2}
\]
and from Eqs. (22,23) that the maximum \(\psi(s)\) is achieved if \(F(s)\) has one double real zero
\[
w_{\text{max}} = e^{-a/2}, \quad s_{\text{max}} = \frac{4e^{-a/2}}{(1 + e^{-a/2})^2}s_0 = 0.9997s_0
\]
Inequalities \(r_1 > |w|, \ r_2 > |w|, \ r_1, r_2 > |w|\)
\[
\frac{r_1 - |w|}{1 - |w| r} > \frac{r_2 - |w|}{1 - |w| r}
\]
and Esq. (22,23) prove that minimum \(\psi(s)\) is achieved if \(F(s)\) has one double real zero
\[
w_{\text{min}} = -e^{-a/2}, \quad s_{\text{min}} = -\frac{4e^{-a/2}}{(1 - e^{-a/2})^2}s_0 = -3360.31s_0
\]

5. The contribution of complex zeroes into the phase of the pion formfactor \(\varphi(s)\).

The contribution of complex zeroes in the phase of the formfactor \(\varphi(s)\) is defined by the formula
\[
\delta\varphi(s) = \frac{1}{2i}\ln\psi(s), \quad s > s_0
\]
The contribution of pair complex conjugate zeroes in phase of the function \(\chi_k\) can be obtained from Eq.(18) putting \(w = (1 - 0)e^{i\theta}\)
\[
\chi_k(s) = e^{2i(\theta + \varphi_k)}
\]
where
\[
\theta = -i\ln\frac{1 + i\sqrt{s/s_0 - 1}}{1 - i\sqrt{s/s_0 - 1}} = 2\arcsin\sqrt{1 - s_0/s}
\]
\[
\theta_k = \frac{1}{2i}\ln\frac{1 - 2r_k e^{-i\theta} \cos\varphi_k + r_k^2 e^{-2i\theta}}{1 - 2r_k e^{i\theta} \cos\varphi_k + r_k^2 e^{2i\theta}}
\]
Maximum and minimum of the function \(\theta_k\) by \(\varphi_k\) are achieved if \(\cos\varphi_k = \pm 1\).
\[
\theta_k^\pm = \pm\frac{1}{i}\ln\frac{1 \mp r_k e^{-i\theta}}{1 \mp r_k e^{i\theta}} = \pm2\arcsin\frac{r_k\sin\theta}{\sqrt{1 \mp 2r_k\cos\theta + r_k^2}}
\]
The contribution of pair complex conjugate the zeroes satisfying the condition (23) has the form
\footnote{Strictly speaking, minimum and maximum \(\psi(s)\) are achieved at simple real zero. But it practically don’t change all results.}
\[ \delta \varphi^{(\pm)}(s) = 2(\theta + \theta^2 \pm \delta_0) \]  

(31)

where \( \theta_2 \) follows from Eq.(30) by changing \( r_k \rightarrow r = e^{-a/2} \) and \( \delta_0 = -\frac{9}{2} \sqrt{s/s_0} - 1. \)

In Table 2 we give the values of \( \varphi_0, \delta \varphi^{(+)}, \delta \varphi^{(-)} \) and the values of the \( P \)-phase \( \pi \pi \)-scattering \( \delta_1^1(s) \) from [11]. It is seen from Table 3 that there is a very good lower bound on the phase \( \varphi(s) \). Upper bound on the phase \( \varphi(s) \) is practically absent.

6. Improved determination of upper bound of the formfactor in the space-like region.

It can be seen from Table 3 that \( \varphi_0(s) \) within the limits of experimental errors coincides with \( \delta_1^1(s) \). This means that the value \( \delta \varphi^+(s) \) has the order of the experimental error in \( \delta_1^1(s) \). Thus we change \( \theta^{(+)k} \) on \( \theta^k \) and \( \cos \varphi^k = 1 \) on \( \cos \varphi^k = -0.96 \). If \(-0.96 \leq \cos \varphi^k \leq -0.96 \) the phase \( \varphi(s) \) coincides with \( \delta_1^1(s) \) to within the experimental errors. Improved upper bound of the formfactor in the space-like region is obtained from Eq.(18) if \( (\cos \varphi_k)_{max \cdot impr.} = -0.96 \).

The results of the calculation of \( F_0, F_{min}, F_{max}, F_{max, impr.} \) and the experimental data from ref.[5-9] are shown in Tables 1,2 and Fig. The curves \( F_0(s) \) and \( F_{min}(s) \) merge together.

7. Improved calculation of the pion radius.

The formulae for the bounds on the pion radius were obtained in [13,21,22]

\[ (r^2_\pi)_{max} = \frac{3}{2m^2_\pi} \left[ b + \frac{1}{2}(sha - a) \right] \]  

(32)

\[ (r^2_\pi)_{min} = \frac{3}{2m^2_\pi} \left[ b - \frac{1}{2}(sha + a) \right] \]  

(33)

where

\[ b = \frac{3^{3/2}}{2\pi} \int_{s_0}^{\infty} \frac{ln |F(s)|^2 ds}{s^2 \sqrt{s - s_0}} \]  

(34)

The value \( b \) was calculated in [15]

\[ b = 0.1544 \pm 0.0016 \]  

(35)

Formulae (32, 33) were obtained from the derivative of the formfactor \( F(s) \) with respect to \( s \) at \( s = 0 \)

\[ F'(0) = \left( b - \sum_k \frac{1 - w_k}{4w_k} + \frac{1}{2} ln \left| \prod_k w_k \right| \right)/s_0 \]  

(36)

and by definition

\[ r^2_\pi = 6 \cdot F'(0) \]  

(37)

The maximum value of \( F'(0) \) is reached when the function \( F(s) \) has one negative zero \( (s_1)_{max} \), so that \( (w_1)_{max} = -e^{-a} \) and then \( (s_1)_{max} = -839.8s_0, (r^2_\pi)_{max} = (0.463 \pm 0.005)fm^2 \). The
minimum value of \( F'(0) \) is reached when the function \( F(s) \) has one positive zero \( (s_1)_{\text{min}} \), so that \( (w_1)_{\text{min}} = e^{-a} \) and then \( (s_1)_{\text{min}} = 0.9988s_0 \), \( (r_2^2)_{\text{min}} = 0.256\text{fm}^2 \). This zero gives in the phase \( \varphi(s) \) the additional term \( \sim 180^\circ \) what is inconsistent with the experimental data on \( \delta_1^1(s) \) [11]. The minimum of \( F'(0) \) which is consistent with the experimental data of \( \delta_1^1(s) \) is reached when formfactor \( F(s) \) has two complex conjugate zeroes \( (w_1)_{\text{min.impr.}} = r_1 e^{i\varphi}, \quad (w_1^*)_{\text{min.impr.}} = r_1 e^{-i\varphi} \), and \( (r_1)_{\text{min.impr.}} = e^{-a/2}, \quad (\cos\varphi_1)_{\text{min.impr.}} = -0.96, \quad (s_1)_{\text{min.impr.}} = (46.23 + 11.33i)s_0 \).

We have obtained:

\[
F'_{\text{min.impr.}}(0) = b - \frac{a}{2} + 0.96sh\frac{a}{2} = 0.1530 \pm 0.0016
\] (38)

and

\[
(r_2^2)_{\text{min.impr.}} = (0.4623 \pm 0.0048)\text{fm}^2
\] (39)

Taking into account the closeness of \( (r_2^2)_{\text{max}} \) and \( (r_2^2)_{\text{min.impr.}} \) we have obtained:

\[
r_2^2 = (0.463 \pm 0.005)\text{fm}^2
\] (40)

This value of \( r_2^2 \) is slightly larger than those obtained in [3, 5]

\[
r_2^2 = (0.422 \pm 0.003 \pm 0.013)\text{fm}^2 \quad [3]
\]

\[
= (0.439 \pm 0.008)\text{fm}^2 \quad [5]
\] (41)

This disagreement is due to the fact that the authors of [3,5] used models, which give the underestimated value of \( r_2^2 \) [15].

I thank V.L.Morgunov and V.A.Novikov for useful discussions.
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The result of the calculations of the pion formfactor $F(-Q^2)$ in the space-like region $0 \leq Q^2 \leq 0.253 \text{ GeV}^2$

| $n$ | $Q^2$ | $F^2(-Q^2)_{\text{Exp}}$ | $F^2_0(-Q^2)$ | $F^2_{\text{min}}(-Q^2)$ | $F^2_{\text{max}}(-Q^2)$ | $F^2_{\text{max,impr.}}(-Q^2)$ |
|-----|-------|------------------|---------------|------------------|------------------|------------------|
| 0   | 0     | 1                | 1             | 1                | 1                | 1                |
| 1   | 0.015 | 0.944 ± 0.007    | 0.943         | 0.943            | 0.966            | 0.944            |
| 2   | 0.017 | 0.921 ± 0.006    | 0.935         | 0.935            | 0.961            | 0.936            |
| 3   | 0.019 | 0.933 ± 0.006    | 0.928         | 0.928            | 0.957            | 0.929            |
| 4   | 0.021 | 0.926 ± 0.006    | 0.921         | 0.921            | 0.952            | 0.922            |
| 5   | 0.023 | 0.914 ± 0.007    | 0.914         | 0.914            | 0.948            | 0.915            |
| 6   | 0.025 | 0.905 ± 0.007    | 0.907         | 0.907            | 0.943            | 0.908            |
| 7   | 0.027 | 0.898 ± 0.008    | 0.900         | 0.900            | 0.938            | 0.901            |
| 8   | 0.029 | 0.884 ± 0.008    | 0.894         | 0.894            | 0.934            | 0.895            |
| 9   | 0.031 | 0.884 ± 0.009    | 0.887         | 0.887            | 0.929            | 0.888            |
| 10  | 0.033 | 0.890 ± 0.009    | 0.881         | 0.881            | 0.925            | 0.882            |
| 11  | 0.035 | 0.866 ± 0.010    | 0.871         | 0.871            | 0.917            | 0.872            |
| 12  | 0.037 | 0.876 ± 0.011    | 0.868         | 0.868            | 0.916            | 0.869            |
| 13  | 0.039 | 0.857 ± 0.011    | 0.861         | 0.861            | 0.911            | 0.862            |
| 14  | 0.042 | 0.849 ± 0.009    | 0.852         | 0.852            | 0.905            | 0.854            |
| 15  | 0.046 | 0.837 ± 0.009    | 0.840         | 0.840            | 0.896            | 0.842            |
| 16  | 0.050 | 0.830 ± 0.010    | 0.828         | 0.828            | 0.887            | 0.830            |
| 17  | 0.054 | 0.801 ± 0.011    | 0.816         | 0.816            | 0.878            | 0.818            |
| 18  | 0.058 | 0.800 ± 0.012    | 0.805         | 0.805            | 0.870            | 0.807            |
| 19  | 0.062 | 0.809 ± 0.012    | 0.793         | 0.793            | 0.861            | 0.795            |
| 20  | 0.066 | 0.785 ± 0.014    | 0.782         | 0.782            | 0.853            | 0.784            |
| 21  | 0.070 | 0.785 ± 0.015    | 0.772         | 0.772            | 0.844            | 0.775            |
| 22  | 0.074 | 0.777 ± 0.016    | 0.761         | 0.761            | 0.836            | 0.764            |
| 23  | 0.078 | 0.769 ± 0.017    | 0.751         | 0.751            | 0.828            | 0.754            |
| 24  | 0.083 | 0.757 ± 0.010    | 0.738         | 0.738            | 0.818            | 0.741            |
| 25  | 0.089 | 0.715 ± 0.016    | 0.724         | 0.724            | 0.806            | 0.727            |
| 26  | 0.095 | 0.724 ± 0.018    | 0.710         | 0.710            | 0.795            | 0.714            |
| 27  | 0.101 | 0.680 ± 0.017    | 0.696         | 0.696            | 0.783            | 0.700            |
| 28  | 0.107 | 0.696 ± 0.019    | 0.683         | 0.683            | 0.772            | 0.687            |
| 29  | 0.103 | 0.688 ± 0.020    | 0.670         | 0.670            | 0.761            | 0.674            |
| 30  | 0.119 | 0.676 ± 0.021    | 0.657         | 0.657            | 0.750            | 0.661            |
| 31  | 0.125 | 0.665 ± 0.023    | 0.645         | 0.645            | 0.740            | 0.650            |
| 32  | 0.131 | 0.651 ± 0.024    | 0.633         | 0.633            | 0.729            | 0.638            |
| 33  | 0.137 | 0.646 ± 0.027    | 0.621         | 0.621            | 0.719            | 0.626            |
| 34  | 0.144 | 0.616 ± 0.023    | 0.608         | 0.608            | 0.708            | 0.613            |
| 35  | 0.153 | 0.654 ± 0.023    | 0.592         | 0.592            | 0.693            | 0.597            |
1. $F_0(-Q^2)$ is free from complex zeroes formfactor.
2. The formfactor $F_{\text{min}}(-Q^2)$ has complex zeroes so that $F(-Q^2)$ is minimal.
3. The formfactor $F_{\text{max}}(-Q^2)$ has complex zeroes so that $F(-Q^2)$ is maximal.
4. The formfactor $F_{\text{max.impr}}(-Q^2)$ has complex zeroes so $F(-Q^2)$ is maximal and the phase of the formfactor $\varphi(s)$ coincides with $p$-phase $\delta_1(s)$ of the $\pi\pi$-scattering ref.[11].
5. $F_{\text{exp}}(-Q^2)$ is the experimental value of the formfactor from ref [5]. $F_{\text{exp}}(Q^2)$ is measured by scattering 300 GeV pions from the electrons of a liquid hydrogen target.

| n  | $Q^2$ | $F_0^2(-Q^2)_{\text{Exp}}$ | $F_0^2(-Q^2)$ | $F_{\text{min}}^2(-Q^2)$ | $F_{\text{max}}^2(-Q^2)$ | $F_{\text{max.impr}}^2(-Q^2)$ |
|----|-------|----------------------------|----------------|--------------------------|--------------------------|-----------------------------|
| 36 | 0.163 | 0.563 ± 0.024              | 0.575          | 0.575                    | 0.677                    | 0.581                       |
| 37 | 0.173 | 0.534 ± 0.038              | 0.558          | 0.558                    | 0.662                    | 0.564                       |
| 38 | 0.183 | 0.586 ± 0.034              | 0.542          | 0.542                    | 0.648                    | 0.548                       |
| 39 | 0.193 | 0.544 ± 0.036              | 0.527          | 0.527                    | 0.634                    | 0.533                       |
| 40 | 0.203 | 0.529 ± 0.040              | 0.513          | 0.513                    | 0.620                    | 0.520                       |
| 41 | 0.213 | 0.616 ± 0.048              | 0.499          | 0.499                    | 0.607                    | 0.506                       |
| 42 | 0.223 | 0.487 ± 0.049              | 0.486          | 0.485                    | 0.594                    | 0.493                       |
| 43 | 0.233 | 0.417 ± 0.058              | 0.473          | 0.473                    | 0.581                    | 0.480                       |
| 44 | 0.243 | 0.593 ± 0.074              | 0.460          | 0.460                    | 0.569                    | 0.468                       |
| 45 | 0.253 | 0.336 ± 0.073              | 0.449          | 0.448                    | 0.558                    | 0.457                       |
Table 2

The results of the calculations of the pion formfactor $F(-Q^2)$ in the space-like region $0.18 \leq Q^2 \leq 9.77\, \text{GeV}^2$.

| $Q^2/\text{GeV}^2$ | Ref. [6] | Ref. [7] | Ref. [8] | Ref. [9] |
|---------------------|----------|----------|----------|----------|
| $F_0(-Q^2)$         | $(F_0(-Q^2))_{\text{Exp}}$ | $(F_0(-Q^2))_{\text{Exp}}$ | $(F_0(-Q^2))_{\text{Exp}}$ | $(F_0(-Q^2))_{\text{Exp}}$ |
| $F_{\text{min}}(-Q^2)$ | $F_{\text{min}}(-Q^2)$ | $F_{\text{min}}(-Q^2)$ | $F_{\text{min}}(-Q^2)$ | $F_{\text{min}}(-Q^2)$ |
| $F_{\text{max}}(-Q^2)$ | $F_{\text{max}}(-Q^2)$ | $F_{\text{max}}(-Q^2)$ | $F_{\text{max}}(-Q^2)$ | $F_{\text{max}}(-Q^2)$ |
| $F_{\text{max,impr.}}(-Q^2)$ | $F_{\text{max,impr.}}(-Q^2)$ | $F_{\text{max,impr.}}(-Q^2)$ | $F_{\text{max,impr.}}(-Q^2)$ | $F_{\text{max,impr.}}(-Q^2)$ |
| 0.18 | 0.850 ± 0.044 | 0.740 | 0.740 | 0.807 | 0.744 |
| 0.29 | 0.634 ± 0.029 | 0.639 | 0.639 | 0.719 | 0.646 |
| 0.40 | 0.570 ± 0.016 | 0.562 | 0.562 | 0.649 | 0.571 |
| 0.79 | 0.384 ± 0.014 | 0.391 | 0.391 | 0.483 | 0.407 |
| 1.19 | 0.238 ± 0.017 | 0.295 | 0.295 | 0.383 | 0.315 |
| 0.62 | 0.445 ± 0.016 | 0.452 | 0.452 | 0.543 | 0.465 |
| 1.07 | 0.309 ± 0.019 | 0.319 | 0.319 | 0.409 | 0.338 |
| 1.20 | 0.269 ± 0.012 | 0.293 | 0.293 | 0.381 | 0.313 |
| 1.31 | 0.242 ± 0.015 | 0.274 | 0.274 | 0.361 | 0.295 |
| 1.20 | 0.262 ± 0.014 | 0.293 | 0.293 | 0.381 | 0.313 |
| 2.01 | 0.154 ± 0.014 | 0.191 | 0.191 | 0.270 | 0.216 |
| 1.22 | 0.290 ± 0.030 | 0.290 | 0.289 | 0.378 | 0.310 |
| 1.20 | 0.294 ± 0.019 | 0.293 | 0.293 | 0.381 | 0.313 |
| 1.71 | 0.238 ± 0.020 | 0.221 | 0.229 | 0.303 | 0.245 |
| 3.30 | 0.102 ± 0.023 | 0.118 | 0.117 | 0.184 | 0.146 |
| 1.99 | 0.179 ± 0.021 | 0.193 | 0.193 | 0.272 | 0.218 |
| 3.99 | 0.004 ± 0.678 | 0.096 | 0.095 | 0.157 | 0.124 |
| 1.18 | 0.256 ± 0.026 | 0.297 | 0.297 | 0.385 | 0.317 |
| 1.94 | 0.193 ± 0.025 | 0.198 | 0.197 | 0.277 | 0.223 |
| 3.33 | 0.086 ± 0.033 | 0.117 | 0.116 | 0.183 | 0.145 |
| 6.30 | 0.059 ± 0.030 | 0.056 | 0.055 | 0.105 | 0.083 |
| 9.77 | 0.070 ± 0.019 | 0.032 | 0.030 | 0.068 | 0.056 |

1. $F_0(-Q^2)$ is free from complex zeroes formfactor.  
2. The formfactor $F_{\text{min}}(-Q^2)$ has complex zeroes so that $F(-Q^2)$ is minimal.  
3. The formfactor $F_{\text{max}}(-Q^2)$ has complex zeroes so that $F(-Q^2)$ is maximal.  
4. The formfactor $F_{\text{max,impr.}}(-Q^2)$ has complex zeroes, so $F(-Q^2)$ is maximal and the phase of $\varphi(s)$ coincides with $p$-phase $\delta_1^p(s)$ of the $\pi\pi$-scattering [ref.11].  
5. $F_{\text{exp}}(-Q^2)$ is the experimental value of the formfactor from ref.[6-9]. $F(-Q^2)_{\text{Exp}}$ has been obtained from the electroproduction of pions from nucleons $ep \rightarrow e\pi^+n$ by extrapolation.
Table 3

The results of the calculations of the phase $\varphi(s)$ of the pion formfactor.

| $\sqrt{s}/GeV$ | $\varphi_0(s)/\text{deg.}$ | $\delta\varphi^+(s)/\text{deg.}$ | $\delta\varphi^-(s)/\text{deg.}$ | $\delta_1^1(s)/\text{deg.}$ |
|---------------|-----------------------------|--------------------------------|--------------------------------|-----------------------------|
| 0.51          | 9.86                        | 351.35                         | -0.0020                         | 9.3 ± 0.7                   |
| 0.53          | 11.31                       | 351.15                         | -0.0023                         | 10.4 ± 0.6                  |
| 0.55          | 12.95                       | 350.94                         | -0.0026                         | 13.1 ± 0.8                  |
| 0.57          | 14.83                       | 350.72                         | -0.0029                         | 13.5 ± 0.7                  |
| 0.59          | 17.02                       | 350.49                         | -0.0033                         | 17.6 ± 0.8                  |
| 0.61          | 19.64                       | 350.26                         | -0.0037                         | 19.4 ± 0.8                  |
| 0.63          | 22.84                       | 350.02                         | -0.0041                         | 20.9 ± 0.8                  |
| 0.65          | 26.70                       | 349.78                         | -0.0045                         | 25.5 ± 0.7                  |
| 0.67          | 31.71                       | 349.53                         | -0.0050                         | 32.1 ± 0.7                  |
| 0.69          | 38.31                       | 349.28                         | -0.0055                         | 37.5 ± 0.5                  |
| 0.71          | 47.07                       | 349.03                         | -0.0060                         | 46.1 ± 0.9                  |
| 0.73          | 58.65                       | 348.52                         | -0.0071                         | 73.0 ± 2.3                  |
| 0.75          | 73.28                       | 348.52                         | -0.0071                         | 73.0 ± 2.3                  |
| 0.79          | 106.12                      | 348                            | -0.0084                         | 113.3 ± 1.9                 |
| 0.81          | 117.09                      | 347.74                         | -0.0091                         | 118.1 ± 1.1                 |

1. $\varphi_0(s)$ is phase of the pion formfactor free from complex zeroes.
2. $\delta\varphi^+(s)$ is the maximal contribution of complex zeroes in the phase of the pion formfactor.
3. $\delta\varphi^-(s)$ is the minimal contribution of complex zeroes in the phase of the pion formfactor.
4. $\delta_1^1(s)$ is the $P$-phase of the $\pi\pi$-scattering [ref.11].
Figure 1: 1. $F_0(-Q^2)$ is free from complex zeroes formfactor. 2. The formfactor $F_{\text{min}}(-Q^2)$ has complex zeroes so that $F(-Q^2)$ is minimal. 3. The formfactor $F_{\text{max}}(-Q^2)$ has complex zeroes so that $F(-Q^2)$ is maximal. 4. The formfactor $F_{\text{max.impr}}(-Q^2)$ has complex zeroes so $F(-Q^2)$ is maximal and the phase of the formfactor $\varphi(s)$ coincides with $p$-phase $\delta_1(s)$ of the $\pi\pi$-scattering ref.[11]. 5. The experimental value of the formfactor from ref [5-9].