TOPOLOGICAL STRUCTURE AND ENTROPY OF MIXING GRAPH MAPS

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ABSTRACT. Let \( P_G \) be the family of all topologically mixing, but not exact self-maps of a topological graph \( G \). It is proved that the infimum of topological entropies of maps from \( P_G \) is bounded from below by \( \log 3 / \Lambda(G) \), where \( \Lambda(G) \) is a constant depending on the combinatorial structure of \( G \). The exact value of the infimum on \( P_G \) is calculated for some families of graphs. The main tool is a refined version of the structure theorem for mixing graph maps. It also yields new proofs of some known results, including Blokh’s theorem (topological mixing implies specification property for maps on graphs).

1. Introduction

There is no connection between topological transitivity, and topological entropy for self-maps of general compact metric spaces. A map with positive entropy need not to be transitive, and a transitive system may have zero entropy. However, there are spaces such that every topologically transitive map on them must have positive topological entropy. For instance, on the compact interval \([0, 1]\) every transitive map has entropy at least \( \log \sqrt{2} \), and there is a transitive self-map of \([0, 1]\) with entropy equal to this bound. The similar questions about the best lower bounds for the topological entropy in various classes of transitive self-maps of a fixed space have been considered by many authors, see [1, 4, 5, 6, 7, 15, 16, 20, 24]. For more references and other results of this type, e.g., lists of known best lower bounds for the topological entropy of transitive maps on various spaces, see [5, page 341] or [4, 7, 16, 20]. The present work is motivated by the following problem:

**Problem.** Let \( G \) be a topological graph. Let \( P_G \) denote the family of all pure mixing (that is, topologically mixing, but not topologically exact) self-maps of \( G \). Find the infimum of topological entropies of maps from \( P_G \), which is hereafter denoted as \( \inf h(P_G) \).

The main result we would like to present here states that for a given graph \( G \) we have

\[
\frac{\log 3}{\Lambda(G)} \leq \inf h(P_G),
\]

where \( \Lambda(G) \) is a constant depending on the combinatorial structure of the graph \( G \). Moreover, we are able to compute \( \inf h(P_G) \) for some graphs and two infinite families of trees (defined in terms of some special structural properties).

This is a generalization of results from [16], where the pure mixing maps of the interval and the circle were considered. But it should be stressed that the methods from [16] can not be directly adapted to the more general case considered here. More precisely, they can be (after some modification) used to prove the similar results for trees, but are not applicable for graphs containing a circle and at least one branching point. The reasons are twofold: first it is harder to obtain a covering relation from containment relation if the graph contains a circle, second the interior of a connected set is no longer connected if the graph contains a branching point. The proofs in [16] heavily relies on these two facts.

To solve our problem in the new, more general context, we refine the structure theorem for pure mixing graph maps (see Theorem 6.1), and apart of the estimate for the topological entropy mentioned above, we obtain (with some additional work) new proofs of two results.
that are of general interest: Blokh’s theorem, stating that topological mixing graph maps have the specification property (see Theorem 10.8, and [25] Theorem 4.2), the main result of [25], which in turn, generalizes [14] Theorem B and C to graph maps (see Theorem 7.5).

Finally, we would like to remark that our version of the structure theorem (Theorem 6.1) for pure mixing graph maps could probably be derived along the lines of Blokh’s papers (see also the presentation of Blokh’s work in Alseda et al. [3]), but it is not a simple nor direct corollary of any theorem presented in [9, 10, 11] or [3]. To obtain the structure theorem from [9, 10, 11] or [3], one should rather rework the whole proof, and adjust it in many places. We are convinced that this approach to the structure theorem, even if succeeded, would result in less transparent and longer proof than ours.

1.1. Entropy and chaos for graph maps. Let us briefly recall one of the possible interpretations of the lower bound for topological entropy in the class $P_G$. First, note that for any fixed graph $G$, which is not a tree (contains at least one circle), the infimum of topological entropies of mixing maps is zero (see [3]).

Now, our theorem about lower bound for the topological entropy of pure mixing graph map can be rephrased in a following way. Let $G$ be any graph containing a circle. If we add (set theoretically) the family $E(G)$ of all topologically exact maps of $G$ to the family $P_G$ of all pure mixing maps then we get the family $M(G)$ of all topologically mixing maps. As we observed above in $M(G)$ we can find maps with arbitrary low entropy. By our result, if the entropy of a mixing graph map of $G$ is sufficiently small, then the map must be exact, that is, exact maps lower the entropy in the family of mixing maps. On the other hand, exact maps are regarded as more chaotic than pure mixing maps. Therefore, we can once again re-formulate our main result: adding more chaotic class of maps to the less chaotic one results in lowering topological entropy in the enlarged class. This contrasts with the common interpretation of the entropy as a quantitative measure of chaos present in the system.

But the paradox disappears if only we will treat the topological entropy as a qualitative indicator of chaos, that is, positive topological entropy is a sign of complex behavior present in the system. From this point of view the precise numerical value of the topological entropy is unimportant.

2. Basic definitions and notation

2.1. Notation and terminology. Let $(X, ho)$ be a metric space, and let $f: X \to X$ be a continuous map. In this paper letters $k, l, m, n, M, N$ will always denote integers, and by “a map” we will always mean “a continuous map”. If $A \subset X$ then we will denote the closure (interior) of $A$ by $\overline{A}$ (int$A$, respectively). An open ball with the center at $x \in X$ and radius $\varepsilon > 0$ is denoted $B(x, \varepsilon)$. Similarly, if $A$ is a subset of $X$ then $B(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon)$.

A continuum is a compact, connected metric space. An arc is a continuum homeomorphic to the interval $[0, 1]$. If $A$ is an arc and $g: [0, 1] \to A$ is a homeomorphism, then the endpoints of $A$ are $g(0)$ and $g(1)$. Clearly, the endpoints do not depend on the choice of $g$.

If $X$ is a continuum and $A \subset X$ is an arc with endpoints $x$ and $y$, then we say that $A$ is a free arc provided $A \setminus \{x, y\}$ is open in $X$.

2.2. Topological graphs. A topological graph (a graph for short) is a continuum $G$ such that there is an one dimensional simplicial complex $\mathcal{K}$ whose geometric carrier $|\mathcal{K}|$ is homeomorphic to $G$. Each such complex is called a triangulation of $G$. We say that a triangulation $\mathcal{L}$ of a graph $G$ is a subdivision of a triangulation $\mathcal{K}$ if every vertex $\mathcal{L}$ is a vertex for $\mathcal{K}$. We identify each graph $G$ with a subspace of the Euclidean space $\mathbb{R}^3$. Moreover, we assume that $G$ is endowed with the taxicab metric, that is, the distance between any two points of $G$ is equal to the length of the shortest arc in $G$ joining these points. If $G$ is a graph, and $\mathcal{K}$ is a triangulation of $G$, then every zero (one) dimensional simplex of $\mathcal{K}$ is
called a vertex (an edge) of $G$ with respect to $\mathcal{K}$. The set of all edges with respect to $\mathcal{K}$ is denoted by $\mathcal{K}^e$.

The star of a vertex $v$, denoted by $\text{St}(v)$, is the union of all the edges that contain the vertex $v$. For every $x \in G$ we define the valence of $x$, denoted $\text{val}(x)$, in the following way: if $x$ is a vertex of $G$ then $\text{val}(x)$ is equal to the number of connected components of $\text{St}(v) \setminus \{v\}$, and $\text{val}(x) = 2$ otherwise. Points $x \in G$ with $\text{val}(x) = 1$ are called endpoints of the graph $G$, and if $\text{val}(x) > 2$ we say that $x$ is a branching point. Let $\text{End}(G)$ denote the set of all endpoints of $G$.

Note that $\text{val}(x)$ is independent of the choice of triangulation. In particular, every branching point (endpoint) for some triangulation is a branching point (endpoint) for every triangulation.

Any subset of $G$ which is a graph itself is called a subgraph of $G$. The family of all subgraphs of $G$ is denoted by $\mathcal{G}(G)$, and coincides with the family of all nondegenerate subcontinua of $G$. Note that a singleton set is not a subgraph of $G$ (graphs are one-dimensional by the definition).

Following Nadler [23], we define disconnecting number of a graph $G$ as the least $n>0$ such that every subset $D$ of $G$ of cardinality $n$ disconnects $G$ (i.e., $G \setminus D$ is not connected). The disconnecting number is well defined and is denoted $D(G)$ (see [23]). Let $\Lambda(G)$ be the maximal disconnecting number among all subgraphs of $G$. It follows from [23] that $\Lambda(G) = D(G) - \chi(G) + 1$, where $\chi(G)$ is the Euler characteristic of $G$.

2.3. Topological dynamics. We refer the reader to [5] for definitions of basic concepts of the theory of dynamical systems, such as (periodic) orbit, (semi-) conjugacy etc. We call a map $f$ transitive if for every pair of nonempty open subsets $U$ and $V$ of $X$ there is an $n$ such that $f^n(U) \cap V \neq \emptyset$; we say that $f$ is totally transitive if all its iterates $f^n$ are transitive; a map $f$ is mixing if for any nonempty sets $U$ and $V$ open in $X$, there is an $N>0$ such that $f^n(U) \cap V \neq \emptyset$ for $n \geq N$; a map $f$ is exact if for any nonempty open subset $U$ of $X$ there is an $n \geq 0$ such that $f^n(U) = X$. It is well known that exactness implies mixing, and mixing implies total transitivity, but not conversely in general. In the special case of non-invertible graph maps total transitivity implies mixing (see [17] for a simple proof of that fact). The only examples of transitive graph homeomorphisms are the irrational rotations of the circle, which are even totally transitive, but not mixing. Moreover, the transitive graph maps are either totally transitive, or can be decomposed into totally transitive ones. The precise statement is presented below and its proof can be found in [2] Theorem 2.2 (with the only difference that only transitivity instead of total transitivity is claimed in [20], however the stronger conclusion follows easily from the proof presented there). Alternatively, it follows from Banks periodic decomposition theorem [8].

**Theorem 2.1.** Let $f : G \to G$ be a transitive graph map. Then exactly one of the following two statements holds:

1. $f$ is totally transitive,
2. There exist a $k > 1$ and non-degenerate connected subgraphs $G_0, \ldots, G_{k-1}$ of $G$ such that
   
   a. $G = \bigcup_{i=0}^{k-1} G_i$,
   b. $G_i \cap G_j = \text{End}(G_i) \cap \text{End}(G_j)$ for $i \neq j$,
   c. $f(G_i) = G_{(i+1) \mod k}$ for $i = 0, \ldots, k-1$,
   d. $f^i|_{G_i}$ is totally transitive for $i = 0, \ldots, k-1$.

We say that a map $f$ is pure mixing if $f$ is mixing but not exact. We recommend [19] as a source of information on transitivity.

For a definition of the topological entropy of $f$ we refer the reader again to [5]. Recall that, if $X$ is a compact space then the entropy of $f$ is a (possibly infinite) number $h(f) \in [0, +\infty]$. We will use the basic properties of the entropy such as those in [5] Section 4.1 without further reference.
Let $\hat{\gamma}(X)$ be a subclass of the class of transitive self-maps of a given compact metric space $X$. By $\inf h(\hat{\gamma}(X))$ we mean the best lower bound for the topological entropy of maps from $\hat{\gamma}(X)$, that is, $\inf h(\hat{\gamma}(X)) = \inf \{ h(f) : f \in \hat{\gamma}(X) \}$. Moreover, we say that $\inf h(\hat{\gamma}(X))$ is attainable if there exists a map $f \in \hat{\gamma}(X)$ such that $h(f) = \inf h(\hat{\gamma}(X))$.

3. Some properties of graph maps

In this section we collect some properties of graph maps, which we will use frequently in subsequent sections. The following convention will recur frequently in what follows.

**Convention (C).** Let $J$ be a free arc (e.g., an edge) in a graph $G$, and let $e$ be one of its endpoints. We identify $J$ with an interval $[0, a] \subset \mathbb{R}$, where $0 < a$ and $e$ is identified with 0. We may also assume that this identification is actually an isometry if necessary, in particular $a = \text{diam } J$. Then $J$ could be linearly ordered by the relation $\leq$ induced from $[0, a]$. It allows us to write $x < y$ to denote the relative position of points on $J$, and use usual interval notation to describe connected subsets of $J$.

The proof of the next lemma is omitted, as it is straightforward. Alternatively, it can be deduced from \[21\] Lemma 23 or \[15\] Theorem 3.11.

**Lemma 3.1.** For a map $f$ of a graph $G$ the following conditions are equivalent:

1. $f$ is mixing,
2. for every $\epsilon > 0$ and $\delta > 0$ there is an integer $N = N(\epsilon, \delta)$ such that for any subgraph $J$ of $G$ with diam $J \geq \delta$ each connected component of the set $G \setminus f^n(J)$ has diameter smaller then $\epsilon$ for every $n \geq N$.

**Definition 3.2.** Given a graph map $f$ and free arcs $I, J \subset G$ we say that $I$ covers $J$ through $f$ (or $f$-covers, for short) if there exists a free arc $K \subset I$ such that $f(K) = J$.

Properties of $f$-covering relation presented below have elementary proofs. The first five of them are adapted from \[3\] p. 590 (note that closed intervals there are arcs in our terminology).

**Lemma 3.3.** Let $I, J, K, L \subset G$ be free arcs, and let $f, g : G \to G$ be graph maps.

1. If $I$ $f$-covers $I$, then there exists $x \in I$ such that $f(x) = x$.
2. If $I \subset K, L \subset J$ and $J$ is $f$-covered by $I$, then $K$ $f$-covers $L$.
3. If $I$ $f$-covers $J$ and $J$ $g$-covers $K$, then $I$ $(g \circ f)$-covers $K$.
4. If $J$ $f(I)$, and $K_1, K_2 \subset J$ are free arcs such that $K_1 \cap K_2$ is at most one point, then $I$ $f$-covers $K_1$, or $I$ $f$-covers $K_2$.
5. If $J$ $f(I)$ is a free arc, then there exist free arcs $J_1, J_2$ such that int $J_1 \cap \text{int } J_2 = \emptyset$, $J_1 \cup J_2 = J$, and $J_1, J_2$ are $f$-covered by $I$.
6. If $K = J \cap f(I)$ contains at most one endpoint of $J$, then $K$ is $f$-covered by $I$.
7. If $S \subset G$ is a star and $J \subset f(S)$ then there are two free arcs $E_1, E_2 \subset S$ with at most one common point such that $J$ is contained in the sum of their images, equivalently, $J \subset f(E_1 \cup E_2)$.

**Lemma 3.4.** Suppose $Z = C_1 \cup \ldots \cup C_n$, where $n \geq 2$ and $C_1, \ldots, C_n$ are pairwise disjoint free arcs contained in the interior of a free arc $F \subset G$. Let $J$ be a free arc in $G$ such that $f(J)$ intersects the interior of any connected component of $F \setminus Z$. Then $J$ $f$-covers at least $n - 1$ sets among $C_1, \ldots, C_n$.

**Proof.** Note that at most one among sets $C_1, \ldots, C_n$ is not contained in $f(J)$. Let $F'$ denote the convex hull of $C_1, \ldots, C_n$ in $F$, that is, the intersection of all compact connected sets containing $C_1 \cup \ldots \cup C_n$. Clearly, $F'$ is a free arc. There are two cases to consider. First, assume that $f(J)$ contains only one endpoint of $F'$. Then use Lemma 3.3(6) and Lemma 3.4(6) to see that all $C_i$’s except at most one are $f$ covered by $J$. In the second case there are free arcs $K$ and $L$ such that $K \cup L = F' \cap f(J)$ and $K$ and $L$ have at most
one common point. Apply Lemma 3.3(5) to see that $K$ and $L$ must be $f$-covered by $J$, and observe that at most one among sets $C_1, \ldots, C_n$ is not contained in $K \cup L$. □

The proof of the next lemma is left to the reader.

**Lemma 3.5.** For each $\delta > 0$ there is a constant $\xi = \xi(\delta)$ such that if $K$ is a connected subset of $G$ with $\text{diam } K \geq \delta$ then $K$ contains a free arc $J$ with $\text{diam } J \geq \xi$.

4. **Global behavior of mixing graph maps**

By definition, a mixing map $f : G \to G$ is pure mixing if and only if there exists an open set $U \subset G$ such that $f^n(U) \neq G$ for all $n \geq 0$. We are going to prove that pure mixing of a graph map is equivalent to the existence of a special set of inaccessible points $I(f)$, which are not contained in $\text{int } f^n(U)$ for all $n \geq 0$ for any open set $U \subset G$ disjoint from $I(f)$. The set of inaccessible points has cardinality bounded above by $D^m(G)$, is forward invariant, $f(I(f)) \subset I(f)$, and all its points are periodic points of $f$. The result is implicit in [9, 3]. But our method of proof is new, and we prove in addition that for a given open set $U \subset G$ the sets $\text{int } f^n(U)$ grow in $G$ as $n \to \infty$. Moreover, if $x \in \text{int } f^n(U)$ for some $n \geq 0$, then eventually $x \in \text{int } f^k(U)$ for all sufficiently large $k$.

**Definition 4.1.** We say that a free arc $I_U \subset G$ is an *universal arc* for a map $f$ of $G$ if for any $\delta > 0$ there is an integer $M = M(\delta)$ such that $I_U$ is $f^m$-covered by any free arc $J$ of $G$ with $\text{diam } J \geq \delta$ for all $m \geq M$.

**Lemma 4.2.** Let $f : G \to G$ be a mixing map. For every free arc $F \subset G$ there is an universal arc $I_U \subset F$. Moreover,

1. there exists an integer $N_U > 0$ such that $I_U$ is $f^n$-covered by itself for all $n \geq N_U$,
   in particular $\text{int } I_U \cup \text{int } f(I_U) \cup \ldots \cup \text{int } f^{N_U}(I_U) \subset \text{int } f^k(I_U)$ for all $k \geq 0$.
2. for any subgraph $J$ of $G$ we have
   $$G \setminus \bigcup_{j=0}^{\infty} \text{int } f^j(J) \subset G \setminus \bigcup_{j=0}^{\infty} \text{int } f^j(I_U).$$

**Proof.** Let $F$ be any free arc in $G$. According to our convention (C) we may identify $F$ with $[0, 7]$. We define

$$A = [1, 2], \quad B = [3, 4], \quad C = [5, 6], \quad \text{and } I_i = (2j, 2j + 1), \quad \text{for } j = 0, \ldots, 3.$$  

Since $f$ is mixing we can find an integer $k > 0$ such that for any $E \in \{A, B, C\}$ we have $f^E(E) \cap I_i \neq \emptyset$ for any $0 \leq j \leq 3$. Using Lemma 3.4 with $Z = A \cup B \cup C$ we deduce that $A$ and $C$ must be $f^3$-covered at least two intervals among $A, B, C$. It follows that there is $I_U \in \{A, B, C\}$ which is $f^3$-covered by both, $A$ and $C$. We will show that $I_U$ is an universal arc. Fix $\delta > 0$ and some closed interval $J$ of $G$ with $\text{diam } J \geq \delta$. Let $\epsilon = \max(\text{diam } I_j : 0 \leq j \leq 1)$. Lemma 3.1(2) gives us $N = N(\epsilon, \delta)$ such that for every $n \geq N$ each connected component of $G \setminus f^n(J)$ has diameter less than $\epsilon$. We conclude that $f^n(J)$ must intersect every connected component of $F \setminus (A \cup B \cup C)$, hence Lemma 3.4 guarantees that for any $n \geq N$ some $D_n \in \{A, C\}$ must be $f^3$-covered by $J$. By the above, and Lemma 3.3(3), the free arc $I_U$ is $f^n$-covered by $J$ for any $n \geq N + k$. Therefore, we set $M(\delta) = N + k$, and $I_U$ is an universal arc for $f$ as claimed. It follows immediately that (1) holds with $N_U = N(\text{diam } I_U, \epsilon) + k$.

For the proof of (2) we fix subgraph $J$ of $G$, and we let $\delta = \text{diam } J$. Lemma 3.2(3) and the definition of $I_U$ above, give us $M = M(\xi(\delta))$ such that $f^M(I_U) \subset f^{M+1}(J)$ for all $j \geq 0$, and so

$$\bigcup_{j=0}^{\infty} \text{int } f^j(I_U) \subset \bigcup_{j=M}^{\infty} \text{int } f^j(I_U) \subset \bigcup_{j=0}^{\infty} \text{int } f^j(J).$$
Theorem 4.6.

Definition 4.3.

and characterize them with the help of the interiors of images of the universal arc.

\[ \delta > 0, \text{let} \ G \ \text{be an universal arc for} \ f \ \text{by} \]

\[ I(f) = G \ \text{inv} \int f^k(J), \]

where the second equality above follows easily from elementary properties of operations involved.

Lemma 4.4. If \( f \) is a mixing graph map and \( I_U \) is an universal arc for \( f \) then

\[ I(f) = G \ \text{inv} \int f^k(I_U), \]

and next apply Lemma 4.2.

Lemma 4.5. Let \( f \) be a mixing graph map. A free arc is an universal arc for \( f \) if and only if it is an universal arc for \( f^n \) for all \( n \geq 1 \).

Proof. It is clear that an universal arc for \( f \) is also universal for \( f^l \) for any \( l \geq 2 \). It is now enough to show that an universal arc \( I_U \) for \( f^l \), where \( l \geq 2 \) is also universal for \( f \). To this end, fix \( \delta > 0 \), let \( I_U \) be an universal arc for \( f \), and let \( M = M(\delta) \) be such that every free arc \( J \subset G \) with \( \text{diam} \ J \geq \delta \) covers \( I_U \) through \( f^m \) for all \( m \geq M \). We can also find an \( L \geq 0 \) (a multiple of \( l \)) such that \( I_U \) is \( f^L \) covered by \( I_U \). Now, every free arc \( J \subset G \) with \( \text{diam} \ J \geq \delta \) covers \( I_U \) through \( f^m \) for every \( m \geq M + L \), and the lemma follows.

Theorem 4.6. For each mixing map \( f \) on a graph \( G \) the set \( I(f) \) has the following properties:

1. For any \( \delta, \varepsilon > 0 \) there is an integer \( K = K(\varepsilon, \delta) \) such that for any subgraph \( J \) of \( G \) with \( \text{diam} \ J \geq \delta \) we have \( G \ \text{inv} \int f^k(J) \subset B(I(f), \varepsilon) \) for all \( k \geq K \).
2. The set \( I(f) \) has less than \( D^4(G) \) elements. Moreover, \( I(f) \neq \emptyset \) if and only if \( f \) is pure mixing.
3. Each point \( x \in I(f) \) is periodic for \( f \) and \( f(I(f)) = I(f) \).
4. For every \( n \geq 1 \) we have \( I(f) = I(f^n) \).

Proof. (1): Let \( I_U \) be an universal arc for \( f \). By Lemma 4.2 and Lemma 4.3 it is enough to prove that for any \( \varepsilon > 0 \) there is an integer \( n = n(\varepsilon) \) such that

\[ G \ \text{inv} \int f^n(I_U) \subset B(I(f), \varepsilon). \]

Suppose on the contrary, that there is \( \varepsilon > 0 \) such that for every \( n \geq 0 \) we can find \( x_n \notin B(I(f), \varepsilon) \cup \int f^n(I_U) \). Passing to a subsequence if necessary, we can assume that \( \bar{x} \) is the limit of the sequence \( \{x_n\} \). Clearly, \( \bar{x} \notin B(I(f), \varepsilon) \). If \( \bar{x} \in \int f^n(I_U) \) for some \( n \geq 0 \), then \( \bar{x} \in \int f^n(I_U) \subset \int f^m(I_U) \) for all \( n \) large enough, hence \( x_n \in \int f^n(I_U) \) for some \( n \),
contradicting definition of the sequence \( \{x_n\} \). But then \( \bar{x} \in G \setminus \text{int} f^n(I_U) \) for all \( n \geq 0 \), hence \( \bar{x} \in \mathcal{I}(f) \), which is a contradiction.

(2): By Lemma 3.1, the diameters of components of \( G \setminus f^k(I_U) = G \setminus \text{int} f^k(J) \) tend to 0 as \( k \to \infty \). If there were at least \( D^n(G) \) points in \( I(f) \), then \( G \setminus f^k(I_U) \) would have to have at least \( D^n(G) \) components for \( k \) large enough. But by [23] Lemma 4.2, for each \( k \geq 0 \) the set \( G \setminus f^k(I_U) \) has less than \( D^n(G) \) components, since \( f^k(I_U) \) is connected, a contradiction. It is clear from Lemma 4.2, and (1) above that \( I(f) \neq \emptyset \) if and only if \( f \) is pure mixing.

(3): It is enough to show that for every \( x \in I(f) \) there is \( y \in I(f) \) such that \( f(y) = x \). Assume, contrary to our claim, that there exists a point \( x \in I(f) \) such that \( f^{-1}(x) \) is disjoint from \( I(f) \). Therefore, we can find an \( \varepsilon > 0 \) such that \( f^{-1}(x) \subset G \setminus B \) where \( B = B(I(f), \varepsilon) \).

Since \( x \notin f(B) \), the set \( U = G \setminus f(B) \) is an open neighborhood of \( x \) such that

\[
 f^{-1}(U) = f^{-1}(G \setminus f(B)) \subset G \setminus B.
\]

By (1) above, there exists \( n > 0 \) such that \( G \setminus B(I(f), \varepsilon) \) is contained in \( f^n(I_U) \). Therefore \( U \subset f^{n+1}(I_U) \), and \( x \in \text{int} f^{n+1}(I_U) \), contradicting the assumption \( x \in I(f) \).

(4): Fix \( n \geq 1 \). It is clear that \( I(f) \subset I(f^n) \). We will show that the converse inclusion also holds. Let \( I_U^1 \) and \( I_U^n \) be universal arcs for \( f \) and \( f^n \), respectively. Arguing as in proof of Lemma 4.2 and applying (1) of the same Lemma we get \( N_U^1 \) such that \( \text{int} I_U^1 \cup f(I_U^1) \cup \ldots \cup f^k(I_U^1) \subset \text{int} f^{n+k}(I_U^n) \) for \( k \geq 0 \). In particular, taking \( L > 0 \) such that \( n \cdot L > N_U^1 \), we have

\[
 \bigcup_{j=0}^{nL-N_U^1} \text{int} f^j(I_U^1) \subset \text{int}(f^n(I_U^n)) \quad \text{for every} \quad l \geq L.
\]

Summing over all \( l \geq L \) and taking complement of both sides finishes the proof. \( \square \)

5. Local behavior around inaccessible points — inaccessible sides

To establish the main theorem of the next section, we need to describe a local behavior of a map \( f \) around its inaccessible points. To do it rigorously we have to introduce some technical terminology. Let \( G \) be a graph with a fixed triangulation \( \mathcal{L} \). A canonical neighborhood of a point \( p \in G \) is an open set \( U \) such that \( n = \text{val}(p) \), and every connected component of \( U \setminus \{p\} \) is homeomorphic with \( (0,1) \). Moreover, we demand canonical neighborhoods of vertices of \( G \) to have disjoint closures. If \( f \) is pure mixing map, then without lost of generality we assume that all inaccessible points are vertices. Clearly, any point \( p \in G \) has arbitrarily small canonical neighborhoods, and there is \( \varepsilon_0 > 0 \) such that for every \( p \in G \) the open ball \( B(p, \varepsilon_0) \) is a canonical neighborhood of \( p \). We use uniform continuity of \( f \) to get an \( \varepsilon_c > 0 \) such that \( f(B(x, \varepsilon_c)) \subset B(f(x), \varepsilon_0) \). From now on we assume that with every point \( p \in G \) we associated its canonical neighborhood \( U_p = B(p, \varepsilon) \). A pair \( (p, S_p) \), where \( p \in G \) and \( S_p \) is a connected component of \( U_p \setminus \{p\} \) is called a side of \( p \). By a slightly abuse of our terminology, we will identify a side with the sole set \( S_p \), where the subscript will remind us which point in \( G \) we use as a base for our side. If \( 0 < \delta \leq \varepsilon_0 \), and \( S_p \) is a side, then a ray of length \( \delta \) in the direction \( S_p \) is a subset \( R(S_p, \delta) = S_p \cap B(p, \delta) \). Nevertheless, our analysis is local, that is, we are investigating \( f \) restricted to \( B(I(f), \varepsilon) \) for small \( \varepsilon > 0 \), our considerations will not depend of the choices we made above. If \( x, y \) are two points in a canonical neighborhood \( U_p \) of some point \( p \in G \), then we let \( (x, y) \) to denote the convex hull of \( x \) and \( y \) in \( U_p \). It is well defined since \( U_p \) is a tree.

Standing assumption: For the rest of this section we fix a graph \( G \) and a mixing map \( f : G \to G \). We also let \( I_U \) to be an universal arc for \( f \) and fix a triangulation \( \mathcal{L} \) of \( G \) such that all inaccessible points are vertices.

**Definition 5.1.** We say that a side \( S_p \) is accessible if there is \( n \geq 0 \) such that \( S_p \subset f^n(I_U) \). A side \( S_p \) is an inaccessible side if it is not accessible.
Lemma 5.2. A point \( p \in G \) is inaccessible for \( f \) (i.e., \( p \in I \)) if and only if it has an inaccessible side.

**Proof.** By Lemma 4.4 and Theorem 4.6, a point \( p \) is inaccessible for \( f \) if and only if \( p \) is not an interior point of \( f^n(I_U) \) for every \( n \geq 0 \). Equivalently, every open neighborhood of \( p \) has nonempty intersection with \( G \setminus f^n(I_U) \) for every \( n \geq 0 \). From the above and Lemma 4.2, we conclude that \( p \in I(f) \) if and only if there is a side \( S_p \) which is not contained in \( f^n(I_U) \) for every \( n \geq 0 \). \( \Box \)

Lemma 5.3. The map \( f \) has less than \( \lambda(G) \) inaccessible sides.

**Proof.** There is \( n \) such that all accessible sides and \( G \setminus B(I(f), \varepsilon_n) \) are contained in \( f^n(I_U) \). It follows that every inaccessible side contains exactly one endpoint of the subgraph \( f^n(I_U) \). To finish the proof, we conclude from 23 that a subgraph of \( G \) has less than \( \lambda(G) \) endpoints. \( \Box \)

Lemma 5.4. Let \( p \in G \). For every side \( S_p \), there is a point \( q \in G \) and a side \( S_q \) such that \( f(q) = p \) and \( f(S_q) \cap S_p \neq \emptyset \).

**Proof.** Let us choose an infinite sequence \( \{y_n\} \subset S_p \) converging to \( p \). Since \( f \) is mixing, hence surjective, there is an infinite sequence \( \{x_n\} \) such that \( f(x_n) = y_n \) for all \( n \). Passing to a subsequence if necessary, we may assume that \( x_n \) converges to some \( q \in G \), and there is a side \( S_q \) such that \( \{x_n\} \subset S_q \). By continuity \( f(q) = p \), and clearly \( f(S_q) \cap S_p \neq \emptyset \) as demanded. \( \Box \)

Lemma 5.5. If \( q \in G \) and \( S_q \) is a side such that \( f(S_q) \) intersect at least two sides of \( p = f(q) \), then every side \( S_p \) of \( p \) such that \( f(S_q) \cap S_p \neq \emptyset \) is accessible.

**Proof.** Observe that if \( x, y \in S_q \) are such that \( f(x) \) and \( f(y) \) belongs to different sides of \( p \) then there is a path in \( f(S_q) \) joining \( f(x) \) with \( f(y) \). Since \( f(S_q) \) is uniquely arcwise connected this path must contain \( p \). Then there must be a point \( q_0 \in S_q \) such that \( f(q_0) = p \). Let \( S_p \) be a side such that \( f(S_q) \cap S_p \neq \emptyset \). We can choose a point \( z \in S_q \) such that \( f(z) \in S_p \). Then \( f((z, q_0)) \) contains a ray \( R(S_p, \delta_0) \) for some \( \delta_0 > 0 \). Clearly, \( (z, q_0) \subset G \setminus B(I(f), \delta_1) \) for some \( \delta_1 > 0 \). Let \( \delta = 1/2 \min(\delta_0, \delta_1) \). By Theorem 4.6, there is an integer \( N \) such that \( f^N(I_U) \supset G \setminus B(I(f), \delta) \) for all \( n \geq N \). In particular, \( f^{N+1}(I_U) \) contains \( S_p \). \( \Box \)

Lemma 5.6. If \( q \in G \) and \( S_q \) is an accessible side then every side \( S_p \) of \( p = f(q) \) such that \( f(S_q) \cap S_p \neq \emptyset \) is accessible.

**Proof.** By Lemma 5.3 it is sufficient to consider only the case when \( f(S_q) \) intersect only one side \( S_p \). Then \( f(S_q) \) contain a ray in the direction \( S_p \) and we may proceed as in the proof of Lemma 5.5. \( \Box \)

Let \( IS \) denote the set of all inaccessible sides of points in \( G \).

**Theorem 5.7.** There is the unique bijection \( f^*: IS \rightarrow IS \) such that \( f^*(S_p) = S_q \) if and only if \( S_q \in IS \) is a side of \( q = f(p) \) such that \( f(S_p) \cap S_q \neq \emptyset \). Moreover, for every \( 0 < \varepsilon < \varepsilon_c \), there is \( \delta > 0 \) such that for every \( S_p \in IS \) if \( S_q = f^*(S_p) \), then \( f((R(S_p, \delta)) \subset R(S_q, \varepsilon)) \).

**Proof.** Let \( S_q \in IS \). By Lemma 5.4, there is a point \( q \in G \) and its side \( S_q \) such that \( f(q) = p \) and \( f(S_q) \cap S_p \neq \emptyset \). On account of Lemma 5.6, \( S_q \) must be inaccessible. It follows from Lemma 5.2 that \( q \in I(f) \). By the above, we may define a function \( g^* : IS \rightarrow IS \) such that \( g^*(S_q) = S_p \), then \( f(S_q) \cap S_p \neq \emptyset \). By Lemma 5.3, \( g^* \) must be injective, and since \( IS \) is finite, \( g^* \) is a bijection. We define \( f^* \) to be the inverse of \( g^* \). Now, if \( f^*(S_p) = S_q \), then \( f(S_p) \) is a ray in the direction \( S_q \). Moreover, \( S_q \) must be unique. Now, standard application of uniform continuity finishes the proof. \( \Box \)
Corollary 5.8. (1) If $f^*$ is a function as above, then the set $\mathcal{IS}$ consists of periodic orbits of $f^*$.
(2) For every $0 < \varepsilon < \varepsilon_0$ there is $\delta > 0$ such that for every side $S_p \in \mathcal{IS}$ there is $1 \leq m < \Lambda(G)$ such that $f^m(p) = p$ and $f^m(R(S_p, \delta)) \subset R(S_p, \varepsilon)$.
(3) For every $0 < \varepsilon < \varepsilon_0$ there is $\delta > 0$ such that for every accessible side $S_p$ of some $p \in G$ we have

$$f(R(S_p, \delta)) \subset B(f(p), \varepsilon) \cap \bigcup_{(f(p), S_p) \in \mathcal{IS}} R(S, \varepsilon).$$

Proof. Both parts follows from uniform continuity of $f$ and Lemma 5.3 and Theorem 5.7.

6. Structure theorem for pure mixing graph maps

There is a natural way to provide examples of pure mixing map of a circle: (1) Start with an interval map such that, either both endpoints are fixed and at least one of them is inaccessible, or both endpoints are inaccessible and form a single cycle of length two.
(2) Identify the endpoints of the interval to obtain a circle. After the identification we still have a well defined map, with the same number of inaccessible sides as at the beginning.

We will have a well defined map, with the same number of inaccessible sides as at the beginning.
As it was noted in [14] and elaborated in [16] all pure mixing maps of the circle may be regarded as a result of applying this procedure to some pure mixing interval map. We will extend this result to a pure mixing map of an arbitrary graph.

From the previous section we see that for a mixing graph map $f$ and a point $x \in G$ either the set of all preimages of $x$ is dense in $G$, or $f^{-1}(x) = \{y\}$ where $x, y \in \mathcal{I}(f)$, and all sides of points lying on the (finite) orbit of $x$ are inaccessible. Moreover, given a pure mixing graph map $f$ as above, we may construct a new graph $G'$ by detaching inaccessible sides from points of $\mathcal{I}(f)$ (we keep the space compact by adding some additional points). Since inaccessible sides are mapped onto inaccessible sides in a one-to-one way, the map $f$ lifts in a natural way to a new map $g : G' \to G$. Any inaccessible point for $g$ is an endpoint of $G'$ having now only finite number of inaccessible preimages.

The proof is only a formalization of the procedure described above.

Theorem 6.1 (Structure Theorem). Let $f : G \to G$ be a pure mixing graph map. Then there exist a graph $G'$, a pure mixing map $g : G' \to G'$, and a continuous surjection $\pi : G' \to G$ such that:

(1) The map $f$ is factor of $g$ via $\pi$, that is $\pi \circ g = f \circ \pi$. Moreover, $\pi$ is one-to-one on $G' \setminus \mathcal{I}(g)$ and $\pi(\mathcal{I}(g)) = \mathcal{I}(f)$.
(2) If $e \in \mathcal{I}(g)$ then $e$ is an endpoint of $G'$ and $g^{-1}(e) = \{e'\}$ for some $e' \in \mathcal{I}(f)$. Moreover, $\mathcal{I}(g)$ has less then $\Lambda(G)$ elements.

Proof. Let $\mathcal{L}$ be a triangulation of $G$. By the definition of canonical neighborhood each side $S_p \in \mathcal{IS}$ is contained in exactly one edge of $\mathcal{L}$. Let $X$ denote the disjoint union of edges of $\mathcal{L}$, so $X$ is homeomorphic with $[0, 1] \times \{1, \ldots, l\}$, for some $l > 0$. Any edge of $G$ can be now identified with a component of $X$.

There is a unique equivalence relation $\mathcal{R}$ on $X$ such that $G$ is the identification space (quotient space) $G = X/\mathcal{R}$. The relation $\mathcal{R}$ is called the incidence relation and informs us which endpoints are to be attached to form $G$.

Obviously, we can have $x \mathcal{R} y$ for $x \neq y$ only if $x, y$ are endpoints of some components of $X$. We can view equivalence classes with respect to $\mathcal{R}$, denoted by $[x]_{\mathcal{R}}$ as elements of $G$. Moreover, if the class $[x]_{\mathcal{R}}$ has more than one element then it represents a vertex $\nu$ of $G$ and consists of $\text{val}(\nu)$ points of $G$. Let $\mathcal{IS}$ denote the set of all inaccessible sides of points in $G$. Then sides from $\mathcal{IS}$ are in a one-to-one correspondence with a subset of endpoints of components of $X$. Therefore, we may write $\mathcal{IS} \subset X$ by convenient abuse of notation. We define $\mathcal{R} = X \setminus \mathcal{IS}$, and call a point $x \in X$ regular if and only if $x \in \mathcal{R}$. 

□
We define a new relation \( R' \subset R \) by declaring \( xR'y \) if and only if \( xRy \) and either, both \( x \) and \( y \) are regular, or \( x = y \). Clearly, \( R' \) is an equivalence relation, and \( [x]_{R'} \subset [x]_R \) for \( x \in X \). Moreover, the space \( G' = X/R' \) is easily seen to be homeomorphic with the space obtained by removing from each inaccessible side for \( f \) on \( G \) a tiny ray lying on that side. Hence, \( G' \) is a graph.

Now, we will define a map \( g': G' \to G' \). First, note that if \( x \) is a regular point such that \( [x]_R \notin \mathcal{I}(f) \) then \( [x]_{R'} = [x]_R \). And if we denote \( [y]_R = f([x]_R) \) then also \([y]_{R'} = [y]_R \). This is true, because every side of \([y]_R \) is accessible. In that case, we define \( g([x]_{R'}) = f([x]_R) = [y]_{R'} \).

If \( x = v \in IS \), that is, \( x \) is an endpoint representing some inaccessible side of a point \( v = [x]_R \in G \), then there is a point \( y \in X \) representing a side \( f'(S_v) \), where \( f' \) is a map defined in Theorem 5.7. In particular, \( y \in IS \). Then we have \([x]_{R'} = [x] \), \([y]_{R'} = [y] \) and we may define \( g([x]_{R'}) = [y]_{R'} \).

It remains to consider the case when \( x \) is a regular point, and \([x]_R \in f^{-1}(\mathcal{I}(f)) \). In that case all sides of \([x]_R \) are accessible, hence \( v = f([x]_R) \) must have accessible and inaccessible sides. Therefore \( v = [y]_R \) for some regular point \( y \). Clearly, if \( y' \) is another regular point such that \( v = [y']_R \) then \( yR'y' \) by the definition of \( R' \). It follows that we may define \( g([x]_{R'}) = [y]_{R'} \).

Now continuity easily follows from Theorem 5.7 and Corollary 5.8. Other points are also easy to see.

7. TRANSITIVITY AND ENTROPY OF PURE MIXING GRAPH MAPS

With the structure theorem at hand we can now study the topological entropy of pure mixing graph maps. In this (and next sections) we will utilize the structure theorem and other tools.

To estimate the topological entropy of pure mixing graph maps we need the notion of a loose horseshoe from [16]. Recall that an \( s \)-horseshoe for \( f \) is a free arc \( J \) contained in the domain of \( f \), and a collection \( C = \{A_1, \ldots, A_s\} \) of \( s \geq 2 \) nonempty compact subsets of \( J \) fulfilling the following three conditions: (a) each set \( A \in C \) is an union of finite number of arcs, (b) the interiors of the sets from \( C \) are pairwise disjoint, (c) \( J \subset f(A) \) for every \( A \in C \).

If the union of elements of \( C \) is a proper subset of \( J \), or \( J \) is a proper subset of \( f(A) \) for some \( A \in C \) then we say that a horseshoe \((J, C)\) is loose. The following lemma is adapted from [15] and summarizes results of [16, Section 4.2]. It is easy to see that the assumption that the graph is an interval \([0, 1]\) or a circle was inessential there, and the result holds for arbitrary graph.

**Lemma 7.1** ([16]). If a transitive graph map \( f \) has a loose \( s \)-horseshoe then \( h(f) > \log s \).

The next two theorems provide a lower bounds of topological entropy of pure mixing graph map. The first of these facts comes from [10, Proposition 4.2] for the tree maps and with the weak inequality. Later Baldwin in [6] observed that the inequality is in fact strict. Here we present a variant for that result which is valid for graph maps.

**Theorem 7.2.** Let \( f \) be a transitive map of a graph \( G \). If \( e \) is an endpoint of \( G \) such that \( f^{-1}(e) = \{e\} \), then \( e \) is an accumulation point of fixed points of \( f \) and \( h(f) > \log 3 \).

**Proof.** We identify the edge containing \( e \) with the unit interval \([0, 1]\) with \( e = 0 \), and we use the induced ordering \(<\) and write about intervals, etc. In addition, we agree to write \( x < y \) for any \( x \in [0, 1] \) and \( y \in G \setminus [0, 1] \). We can find an \( 0 < \epsilon < 1 \) such that \( f([0, \epsilon]) \subset [0, 1] \).

Observe that for any \( 0 < \epsilon < \epsilon_0 \) we have:

\((\circledast)\): If \( \epsilon' = \min f(G \setminus [0, \epsilon]) \), then \( 0 < \epsilon' < \epsilon \) (since \( \epsilon' \geq \epsilon \) would imply that \( G \setminus [0, \epsilon] \) is a proper invariant set with nonempty interior contradicting transitivity, and \( \epsilon' = 0 \) would contradict \( f^{-1}(e) = \{e\} \)).
\textbf{(**):} If $e'' = \max f([0,e])$, then $e < e''$ (as $e'' \leq e$ would imply that $[0,e]$ is invariant contradicting transitivity).

We will use (\*) and (**) all the time without any reference. To prove that $e$ is an accumulation point of fixed points of $f$ it is enough to show that for any $0 < e < \min f(G \setminus [0,e_0])$ we have $f(a) > a$ and $f(b) < b$ for some $0 < a < b < e$. To see it let $e' = \min f(G \setminus [0,s))$. Then there is $b \in [e',e)$ such that $f(b) < b$. There also must be $0 < a < b$ such that $f(a) = \max f([0,b]) > b$.

For the proof that $h(f) > \log 3$, let $0 < a < \min f(G \setminus [0,e_0])$ be a fixed point of $f$, and set $b = \min f(G \setminus [0,a))$. We have $0 < b < a$, and since $e$ is an accumulation point of fixed points of $f$ there are fixed points $s$ and $t$ of $f$ such that $0 < s < b$, $s < t$ and $f([0,b]) = f(z)$ for some $s < z \leq b$. Moreover, we may assume that there is no fixed point in $(s,t)$, so $f(x) > x$ for all $x \in (s,t)$. It follows that $t < a$ and there must be fixed points $u$ and $v$ of $f$ such that $t \leq u < v \leq a$ and $f(G \setminus [0,s)) = f(w)$ for some $u < w < v$. Again we may assume that $f(y) < y$ for all $y \in (u,v)$. There are fixed points $p$ and $q$ such that for some $p < r < q$ we have $f(r) = \max f([0,v]) = \max f([p,q])$. Clearly, we have $s \leq p < q \leq u$, since $\max f([0,s]) \leq \max f([s,t])$ and $\max f([0,v]) \leq \max f([u,v])$. Therefore, $[p,r]$, $[r,v]$, $[w,v]$ form a loose 3-horseshoe for $f$, as $f(r) > v$, and $f(w) < s$. \hfill $\Box$

**Theorem 7.3.** If $f$ is a pure mixing map of a graph $G$, then

1. $h(f) > (1/\Lambda(G)) \cdot \log 3$,
2. there exists $0 < m < \Lambda(G)$ such that $f^m$ has infinitely many fixed points.

**Proof.** By Theorem 6.1 $f$ is a factor of a map $g : G' \to G'$ such that $g^{-1}(I(g)) = I(g)$. Moreover, $I(g)$ has less than $\Lambda(G)$ elements, so there is $0 < m < \Lambda(G)$ such that $g^{-m}(e) = [e]$ for some endpoint of $G'$. By Theorem 7.2 we see that $h(g^m) > \log 3$. As $g^m$ is an extension of $f^m$ via finite-to-one semiconjugacy, we have $h(f^m) = h(g^m) > \log 3$. Moreover, $g^m$ has infinitely many fixed points, and so does $f^m$. \hfill $\Box$

As a direct consequence we get the following lower bound for $\inf(h(P_G))$ for a given graph $G$.

**Corollary 7.4.** If $G$ is a graph, then $\inf(h(P_G)) \geq (1/\Lambda(G)) \cdot \log 3$.

As the second corollary we get the following theorem which is contained in [25] as Theorem 4.1. Here, $\Fix(f^k)$ denotes the set of all fixed points of $f^k$.

**Theorem 7.5.** Let $f : G \to G$ be a graph map with $\# \Fix(f^k) < \infty$ for each $k \geq 1$. If $f$ is transitive, then it is strongly transitive, that is, for every non-empty open set $U \subset G$ there exists $n > 0$ such that

$$G = U \cup f(U) \cup \ldots \cup f^n(U).$$

**Proof.** If $f$ is an irrational rotation then the result is well-known. Assume that $f$ is non-invertible. By Theorem 2.1 it is enough to prove that if $f$ is totally transitive then it must be exact. But it is well known (see [17] for a simple proof of that fact) that every totally transitive, and non-invertible graph map must be mixing. By Theorem 7.3 a mixing graph map with $\# \Fix(f^k) < \infty$ for each $k \geq 1$ must be exact. \hfill $\Box$

We recall that a map $f : X \to Y$ is monotone if $X$ and $Y$ are topological spaces, $f$ is continuous, and for each point $y \in Y$ its preimage $f^{-1}(y)$ is connected. If $X$ is a tree and there is a finite set $P \subset X$ such that for each connected component $C$ of $X \setminus P$ the map $f|_P : C \to Y$ is monotone, then we say that $f$ is $P$-monotone. We say that a tree map $f$ piecewise monotone if $f$ is $P$-monotone with respect to some finite $P \subset T$. If $f$ is a $P$-monotone map then the closures of connected components of $T \setminus P$ are called $P$-basic intervals for $f$.

Theorem 7.5 is a generalization of the following well-known result (see [25] for more comments).
Corollary 7.6. Let \( f : G \rightarrow G \) be a piecewise monotone graph map. If \( f \) is transitive then it is strongly transitive, in particular a totally transitive piecewise monotone graph map must be exact.

Proof. It is easy to see that every transitive and piecewise monotone map fulfills the assumption of Theorem 7.5 (by transitivity, images of nondegenerate continua remain nondegenerate).

8. Topological entropy of transitive tree maps

In this section we collect some technical results which we will use in the next section. We will also need some additional terminology, which we recall below.

Let \( f : T \rightarrow T \) be a tree map. We say that \( f \) is linear on a set \( S \subset T \) if there is a constant \( \alpha \) such that \( d(f(x), f(y)) = \alpha d(x, y) \) for all \( x, y \in S \) (here, as always \( d \) denotes the taxicab metric on \( T \)). If a \( P \)-monotone map \( f \) is linear on every \( P \)-basic interval then we call it \( P \)-linear or piecewise linear if there is no need to single out \( P \). A \( P \)-monotone map \( f \) is Markov map if \( P \subset T \) contains all vertices of \( T \) and \( f(P) \subset P \). In the above situation we call \( f \) a \( P \)-Markov map for short. If \( f \) is a \( P \)-Markov map, then the Markov graph of \( f \) with respect to \( P \) \( (P \)-Markov graph of \( f \) for short) is then defined as a directed graph with the set of \( P \)-basic intervals as a set of vertices and with the set of edges defined by the \( f \)-covering relation, that is, there is an edge from a \( P \)-basic interval to a \( P \)-basic interval in the Markov graph \( (I \rightarrow J) \) if and only if \( J \) is \( f \)-covered through \( f \) by \( I \). A path (of length \( n \)) in a graph \( G \) is any sequence \( I_0, I_1, \ldots, I_n \) of vertices of \( G \) such that there exists an edge \( I_{m-1} \rightarrow I_m \) for each \( m = 1, \ldots, n \). A cycle of length \( n \) is any path \( I_0, I_1, \ldots, I_n \) of length \( n \) such that \( I_0 = I_n \). Graph is strongly connected if for any pair of its vertices \( I, J \) there is a path \( I_0, I_1, \ldots, I_n \) in \( G \) with \( I = I_0 \) and \( J = I_n \). If \( I_1, \ldots, I_n \) is an enumeration of the set of \( P \)-basic intervals then the incidence matrix of \( f \) with respect to \( P \) is a \( n \times n \) matrix \( A = [a_{ij}] \) with \( a_{ij} = 1 \) if \( I_i \rightarrow I_j \), and \( a_{ij} = 0 \) otherwise. The spectral radius of a square complex matrix is defined as the largest absolute value of its eigenvalues.

There are also few results we would like to recall for later reference. The following Lemma comes from [6 Corollary 1.11].

Lemma 8.1. Suppose \( T \) is a tree and \( f : T \rightarrow T \) is \( P \)-Markov and \( P \)-linear with respect to some \( P \) containing all nodes of \( T \). Then \( f \) is transitive if and only if the \( P \)-Markov graph of \( f \) is strongly connected and is not a graph of a cyclic permutation.

The following lemma is well-known. We restate it in a suitable form.

Lemma 8.2. Let \( f : T \rightarrow T \) be a Markov tree map. If \( \mathcal{G} \) is any Markov graph of \( f \) and \( A = [a_{ij}]_{\mathbb{R}} \) is the corresponding incidence matrix with a spectral radius \( \rho \geq 0 \), then \( h(f) = \log \rho \) if \( \rho > 0 \), and \( h(f) = 0 \) otherwise. Moreover, if there is \( s > 0 \) such that for every vertex \( v \) of \( \mathcal{G} \) the number of directed paths of length \( n > 0 \) starting at \( v \) in \( \mathcal{G} \) is bounded from the above by \( s^n \), then \( h(f) \leq \log s \).

If in addition \( \mathcal{G} \) is strongly connected and there are \( t > 0 \) and \( n > 0 \) such that for some vertex \( v \) of \( \mathcal{G} \) the number of paths of length \( n > 0 \) starting at \( v \) in \( \mathcal{G} \) is bounded below by \( t^n \), then \( \log t \leq h(f) \).

Proof. The connection between the spectral radius of the incidence matrix and topological entropy is well known (see [6 Proposition 1.4]). It is easy to see that the number of paths of length \( n > 0 \) starting at \( v \) in \( \mathcal{G} \) is equal to the row-sum of row \( v \) in \( A^n \). Moreover, the incidence matrix of a graph is irreducible if and only if the graph is strongly connected. The bounds on the spectral radius \( \rho \) of irreducible square matrix \( B = [b_{ij}]_{i,j=1}^m \) comes from well known formula (e.g. see [22 Exercise 4.2.3., p.111]):

\[
\min_{i=1,...,m} \sum_{j=1}^m b_{ij} \leq \rho \leq \max_{i=1,...,n} \sum_{j=1}^m b_{ij}.
\]
Theorem 8.3. Let $T$ be a tree, and $f : T \to T$ be transitive and Markov. Let $z$ be a fixed point of $f$, and let $T'$ be the tree which is obtained from $T$ by attaching an arc to $z$ at one of the endpoints of the arc. Then for every $\epsilon > 0$, there is a transitive Markov map $f' : T' \to T'$ such that $h(f') < h(f) + \epsilon$ and both ends of the new arc are fixed for $f'$. Furthermore, $f'$ can be defined so that, in addition,

(\star): If some subset of the endpoints of $T$ forms an $f$-cycle $C$, then $C$ is also a periodic orbit of $f'$.

Let $(X, x_0)$ and $(Y, y_0)$ be topological pointed spaces (spaces with distinguished basepoints). The wedge sum of $(X, x_0)$ and $(Y, y_0)$ (denoted as $(X, x_0) \wedge (Y, y_0)$) is the quotient of the disjoint union of $X$ and $Y$ by the identification $x_0 \sim y_0$. The $m$-th wedge power $(X, x_0)^{\wedge m}$ is defined as wedge sum of $m$ copies of $(X, x_0)$.

Lemma 8.4. Let $X$ be a topological space and $f : X \to X$ be a transitive map. If $x_0$ is a fixed point of $f$, then for every $m \geq 2$ there is a transitive, but not totally transitive map $F : (X, x_0)^{\wedge m} \to (X, x_0)^{\wedge m}$ such that $h(F) = h(f)/m$. Moreover, $x_0$ is the unique fixed point of $F$, and all other periodic orbits of $f$ are periodic points of $F$ with $m$ times longer primary periods (formed by the $m$-times copy of points of the orbit of $f$). Furthermore, if $X$ is a tree and $f$ is a Markov map, then the same hold for $(X, x_0)^{\wedge m}$ and $F$ respectively.

Proof. Let us identify a disjoint union of $m$ copies of $X$ with $X \times \{0, \ldots, m-1\}$. Define a map $F' : X \times \{0, \ldots, m-1\} \to X \times \{0, \ldots, m-1\}$ by

$$F'(x, i) = \begin{cases} (x, i+1), & \text{for } i = 0, \ldots, m-2, \\ (f(x), 0), & \text{for } i = m-1. \end{cases}$$

It is easy to see that $F'$ induces a quotient map $F$ on $(X, x_0)^{\wedge m}$ with the desired properties. The rest of the proof is now straightforward.

Note, that if $m \geq 2$, then the map constructed in Lemma 8.4 is not totally transitive. To remove this problem we need the following Lemma which gives a general method of constructing exact Markov maps from transitive but not totally transitive Markov maps and enables us to control the entropy. The proof of the upper bound for entropy follows the ideas of [6].

Lemma 8.5. Let $f$ be a transitive $P$-Markov and $P$-linear map of a tree $T$, which is not totally transitive, then for every $\epsilon > 0$, there is a transitive (hence, exact) $P'$-Markov and $P'$-linear map $f' : T \to T$ such that $h(f') < h(f) + \epsilon$. Furthermore, $f'$ can be defined so that, if some subset of the endpoints of $T$ form an $f$-cycle $C$, then $C$ is also a periodic orbit of $f'$.

Proof. Let $f : T \to T$ be a transitive, but not totally transitive $P$-Markov and $P$-linear tree map. We denote the $P$-Markov graph of $f$ by $G$. Fix $\epsilon > 0$ and let $r, s > 1$ be such that $h(f) < \log r < \log s < h(f) + \epsilon$. By Theorem 2.1, $f$ has the unique fixed point $p \in T$ and we can enumerate the closures of connected components of $T \setminus \{p\}$ by $T_0, T_1, \ldots, T_{n-1}$ with $n \geq 1$ in such a way that $f(T_j) = T_{j+1}$ for $j = 0, \ldots, n-1$, and $f(T_{n-1}) = T_0$ hold. Without loss of generality we may assume that $p \in P$. Moreover, for each $j = 0, 1, \ldots, n-1$ the map $f|_{T_j} : T_j \to T_j$ is a totally transitive (thus, exact) piecewise linear Markov map, in particular, there is a point $q \neq p$ such that $f(q) = p$. Without loss of generality we may assume that $q \in T_0$. Let $[x, y]$ be a $P$-basic interval in $T_0$ containing $q$. If $q \not\in P$, then either $f|_{[x, y]}$ would be not monotone, or $f(T_0)$ would intersect $T_j$ with $j \neq 1$. Hence $q \in P$, and...
without loss of generality we may that assume $x = q$. For $j = 0, 1, \ldots, n - 1$ let $I_j = [p, z_j]$ denote the $P$ basic interval in $T_j$ containing $p$.

Let us choose $\lambda > 0$ such that $3\lambda < s^2$, for all $L \geq \lambda$, where $r$ and $s$ are as above. Then there is a point $w_0$ in the interior of $[p, z_0]$ and $L \geq \lambda$ such that for $k = 0, \ldots, L - 1$ and $j = (k \mod n)$ we have $w_j = f^k(w_0) \in \text{int} I_j$, and $f^j(w_0) \in P$.

Let $P' = P \cup \{w_j : j = 0, \ldots, L - 1\}$, and let $G'$ be the Markov graph of $f$ with respect to $P'$. We define $f' : T \to T$ by modifying $f$ on $[x, y]$ only, that is, we put $f'(z) = f(z)$ for all $z \in T \setminus [x, y]$. Next, we choose two points $x', y'$ in $[x, y]$ with $x = x' < y' < y$, then we set $f'(y') = w_0$ and $f'(x') = p$, and extend $f'$ to $[x, x']$ by making it linear on $[x, y']$ and $[y', x']$. We identify $[x', y]$ with $[x, y]$ by a linear homeomorphism $\psi$ such that $\psi(x') = x$ and for $z \in [x', y]$ we define $f'(z) = f(\psi(z))$. Then $f'$ is a continuous map of $T$, which is $P'$-Markov and $P'$-linear with respect to $P' = P'' \cup \{x', y'\}$. Note that if some subset of the endpoints of $T$ form an $L$-cycle $C$, then $C$ is also a periodic orbit of $f'$ since $f'|_{P'} = f|_{P'}$.

We claim that $f'$ is totally transitive. First, observe that $f'(T_0)$ contains $f'([x, y])$, and hence it intersects interiors of both, $T_0$ and $T_1$. As $p$ is the unique fixed point of $f'$, and $T_0, \ldots, T_{n-1}$ are no longer invariant for $(f')^{\nu}$ it follows from Theorem 2.1 that $f'$ is totally transitive provided $f'$ is transitive.

To show that $f'$ is transitive we consider the $P'$-Markov graph of $f'$, denoted by $G'$. By Lemma 8.1 it is enough to prove that $G'$ is strongly connected, since it is clear that it is not a graph of a cyclic permutation. We will say that a $P'$-basic interval $J$ of $f'$ is an “old” one if $J$ is, either a $P''$-basic interval of $f$, or $J = [x', y]$. With this nomenclature $G'$ has two “new” vertices, that is $P''$-basic intervals $J' = [x, y]$ and $J'' = [y', x']$. Observe that the subgraph of $G'$ given by the set of old $P'$ basic intervals together with all edges between them is isomorphic to $G$. Moreover, if any old $P'$-basic interval $f'$-covered $[x, y]$, then it also $f'$-covers $[x', y]$, $J'$, and $J''$, and the last two intervals $f'$-cover an old $P'$ basic interval $[w_0, p]$. It easily follows that $G'$ is strongly connected and thus $f'$ is transitive. The proof of the claim is completed.

To estimate the topological entropy of $f'$ we fix a vertex $v$ of $G'$ and provide a bound on the number of paths of length $L$ in $G'$ starting at $v$. By the definition of $f'$ and our choice of $L$ every path of length $L$ in $G'$ can pass at most once through $J'$ or $J''$. Moreover, to every path $\alpha$ of length $L$ in $G$ corresponds, either exactly one path in $G'$ if $\alpha$ does not contain $[x, y]$, or exactly three paths in $G'$ otherwise. By Lemma 8.2, there are at most $3^L$ paths of length $L$ starting at any fixed vertex of $G$, and we conclude that there are at most $3^L$ paths of length $L$ in $G'$ starting at $v$. Using Lemma 8.2 and by our choice of $s$ we have that $h(f')^{\nu} \leq \log(3^L) < \log \frac{3^L}{2},$ hence $h(f')^{\nu} < \log s < h(f) + \varepsilon.$

The last lemma shows how to construct pure mixing examples from exact Markov tree maps with the topological entropy as small as possible.

**Lemma 8.6.** Let $f : T \to T$ be an exact Markov tree map. If $O$ is a single periodic orbit for $f$ with $m > 0$ elements contained in the set of endpoints of $T$ then for every $\varepsilon > 0$, there is a pure mixing map $f' : T \to T$ such that $h(f') < \max\{h(f), \log 3/m\} + \varepsilon$ and $O = I(f')$ is a periodic orbit of $f'$.

**Proof.** Fix any $\varepsilon > 0$. Let $O$ be a single periodic orbit of $f$ with $m > 0$ elements contained in the set of endpoints of $T$. Set $\eta = \max\{h(f), \log 3/m\}$. Choose $r, s > 1$ such that $\eta < \log r < \log s < \eta + \varepsilon$, and fix $L > 0$ such that $3r^L < s^2$. Take any $P \subset T$ such that $f$ is Markov with respect to $P$. Choose any $o_0 \in O$ and for $j = 1, 2, \ldots, m - 1$ put $o_j = f^j(o_0)$, and let $I_j = [o_j, z_j]$ denote the $P$ basic interval containing the endpoint $o_j$. Reasoning as in the proof of Lemma 8.5 we can find a $P$-basic interval $[x, y]$ such that $f(x) = o_0$ and $x \neq o_{m-1}$. Since $f$ is exact, it cannot collapse any of intervals $I_j$, and thus $[x, y] \neq I_j$ for any $j$. We order $[x, y]$ such that $x < y$ and we choose two points $x', y'$ in $[x, y]$ with...
that for each \( O \) there are points \( \{ f' \} \) such that for each \( \alpha \) we have \( (\alpha, f') = \alpha \). Hence (\( \alpha, f' \)) is linearly to \( T \). Then \( \alpha \) there is an isometry \( \psi_0: \alpha \to \alpha_0 \) such that for each \( \alpha \) we have \( \alpha \). We obtained a new tree \( T' \), which is homeomorphic with the original \( T \), and \( T \subset T' \). Now, to finish the proof it is enough to construct \( f': T' \to T' \) such that the entropy bounds hold and \( \alpha' = \alpha_0, \ldots, \alpha_{f'j-1} \) is a non-accessible periodic orbit of \( f' \). First set \( f'(x) = f(x) \) for each \( x \in T \setminus \{ x, y \} \). Observe that there is \( u_0 \in T_0 \) such that for \( j = 1, \ldots, mL \) and \( l = j \) mod \( m \) we have

\[
f'(\{ \alpha_0, f' \}(u_0)) \subset I_1, \quad \text{and} \quad (f')^{mL}(u_0) = z_0.
\]

Next, we choose a sequence \( \{ \alpha_i \} \subset \{ \alpha_0, \ldots, \alpha_{f'j-1} \} \) such that for each \( j = 1, 2, \ldots \) we have

\[
a_j < \alpha_j \quad \text{and} \quad a_0 = \alpha_{f'j-1} \quad \text{and} \quad \lim_{j \to \infty} \alpha_j = \alpha_{f'j-1}.
\]

Set \( \alpha_j = \varphi_{m-1}(\alpha_j) \) for \( j = 0, 1, \ldots \), and find infinite sequences \( \{ b_j \}_{j=0}^\infty, \{ c_j \}_{j=0}^\infty, \{ d_j \}_{j=0}^\infty \) such that for each \( j = 1, 2, \ldots \) we have

\[
a_j < c_{j-1} < b_j < c_j < e_{j} < d_j < e_{j-1} < a_j.
\]

For \( k = 0, 1, \ldots, m - 1 \) and \( x \in [a_0, a_{f'j-1}] \) set \( f'(x) = \varphi_k(x) \). For \( j = \infty \) put \( f'(\alpha_j) = \alpha_j \) and for each \( j = 1, 2, \ldots \) we define \( f'(\alpha_j) = \alpha_j \), \( f'(c_{j-1}) = f'(c_j) = \alpha_{j-1} \), \( f'(e_{j-1}) = f'(e_j) = \alpha_j \), and extend \( f' \) linearly to \( \{ a_j, c_{j-1}, c_j, e_j, d_j \} \) and \( \{ e_{j-1}, a_{j-1} \} \). Now for each \( j = 1, 2, \ldots \) there are points \( u_j \in \{ \alpha_j, \alpha_{j-1} \} \) and \( w_{j-1} \in \{ \alpha_j, \alpha_{j-1} \} \) such that the following conditions hold:

\[
(\varpi): \text{for each } k = 1, \ldots, L \text{ we have}
\]

\[
(f')^{mL}(\{ \alpha_j, u_j, w_{j-1} \}) \subset \{ a_0, a_1, \ldots, a_{f'j-1} \} \quad \text{and} \quad (f')^{mL}(\{ w_{j-1}, \alpha_{j-1} \}) \subset \{ e_{j-1}, a_{j-1} \}.
\]

Hence \( f' \) is well defined on intervals \( \{ \alpha_j, u_j \}, \{ w_{j-1}, \alpha_{j-1} \} \) for \( i = 1, \ldots, mL - 1 \).

\[
(\dagger): \text{we have}
\]

\[
(f')^{mL}(u_j) = c_{j-1}, \quad \text{and} \quad (f')^{mL}(w_{j-1}) = e_{j-1}.
\]

We have defined two sequences \( \{ u_j \}_{j=0}^\infty \) and \( \{ w_{j-1} \}_{j=0}^\infty \). For \( j = 1, 2, \ldots \) we define

\[
f'(b_j) = u_{j-1} \quad \text{and} \quad f'(d_j) = w_j,
\]

and extend \( f' \) linearly to \( \{ c_{j-1}, b_j \}, \{ b_j, c_j \}, \{ e_j, d_j \} \) and \( \{ d_j, e_{j-1} \} \). Finally, we set \( f'(x') = \alpha_0, f'(y') = \alpha_{j-1} \), and extend \( f' \) linearly to \( [x', y'] \). We identify \( [x', y'] \) with \( [x, y] \) by a linear homeomorphism \( \psi \) such that \( \psi(x') = x \) and \( \psi(y') = y \). Then \( f' \) is a continuous map of \( T' \). Reasoning as in \( \dagger \) one gets that \( f' \) is a pure mixing map, and it is clear that \( O' = \{ a_0, \ldots, a_{f'j-1} \} \) is a non-accessible periodic orbit of \( f' \).

To complete the topological entropy of \( f' \), for each \( j = 1, 2, \ldots \) we define a sequence of maps \( \{ f'_j \}_{j=1}^\infty \) such that \( h(f'_j) \to h(f') \) as \( j \to \infty \), and for each \( j = 1, 2, \ldots \) we have

\[
h(f'_j) \leq \eta + \varepsilon.
\]

To this end set \( Q'_0 = \{ \alpha_j, \alpha_{j-1} \} \) and \( Q'_k = \varphi_{k-1}(Q'_{k-1}) \). Let

\[
\Omega_k = \bigcup_{j=1}^m Q_k \quad \text{and} \quad S_j = \bigcup_{k=0}^{m-1} Q_k.
\]

Next, we define a sequence of linear Markov maps \( f'_j: T' \to T' \) by

\[
f'_j(x) = \begin{cases} f'(x), & \text{for } x \in T \cup S_j, \\ \varphi_k(x), & \text{for } x \in \{ a_0, a_1 \} \setminus \Omega_k \text{ for some } k = 0, \ldots, m - 1. \end{cases}
\]
Lemma 9.1. Let \( n \in \mathbb{N} \). Moreover, all endpoints of \( T \), and the entropy function is lower semicontinuous, we get

\[
\limsup_{j \to \infty} h(f_j^n) = \sup_{j \to \infty} h(f_j^n) \leq h(f^n).
\]

Since \( f_j^n \) converges uniformly to \( f^n \) on \( T' \), and the entropy function is lower semicontinuous, we get

\[
h(f^n) \leq \liminf_{j \to \infty} h(f_j).
\]

To finish the proof we need to show that \( h(f_j^n) \leq \eta + \varepsilon \) for every \( j = 1, 2, \ldots \). But by the way \( f_j^n \) is defined it is a linear Markov map on \( T' \), so Lemma 8.2 applies, and the upper bound can be obtained by counting paths in the Markov graph of \( f_j^n \) in a similar way as in the proof of Theorem 8.3. The details are left to the reader. \( \square \)

9. Examples

In this section we construct a few examples of pure mixing graph maps which will prove that the lower bounds for \( \inf(h(P_{f_j})) \) derived from Corollary 7.4 are in some cases equal to the infimum, hence we solve our main Problem in these cases.

Recall that an \( n \)-star is a tree \( T_n = ([0, 1], 0)^\times n \), where \( n \geq 1 \).

Lemma 9.1. Let \( n > 1 \) and let \( T_n \) be a star with \( n \) endpoints. Then for every \( \varepsilon > 0 \), there is

1. an exact Markov map \( F_\varepsilon: T_n \to T_n \) such that \( (\log 3)/n \leq h(F_\varepsilon) < (\log 3)/n + \varepsilon \).
2. a pure mixing map \( G_\varepsilon: T_n \to T_n \) such that \( (\log 3)/n \leq h(G_\varepsilon) < (\log 3)/n + \varepsilon \).

Moreover, all endpoints of \( T_n \) form a single periodic orbit of \( F_\varepsilon \) and \( G_\varepsilon \).

Proof. First observe that for each \( n \geq 2 \) there is a transitive Markov map \( f_n \) of an \( n \)-star \( T_n \) such that all endpoints of \( T_n \) form a single cycle for \( f_n \) and \( h(f_n) = \log 3/\varepsilon \). It is a consequence of Lemma 8.4 applied to the 3-horseshoe map given by \( f(x) = |1 - |1 - 3x|| \) for \( 0 \leq x \leq 1 \). Then we can apply Lemmas 8.5 and 8.6 to finish the proof. \( \square \)

Let \( T \) be a tree. We say that a point \( p \in T \) is a central root of \( T \) if both connected components of \( T \setminus \{p\} \) are homeomorphic to each other. The full binary tree \( B_n \) with \( 2^n \) endpoints can be defined inductively. Let \( B_1 = [0, 1] \). Note that \( 1/2 \in [0, 1] \) is the central root of \( B_1 \). Assume that we have defined \( B_n \) and \( z_n \in B_n \) is a central root of \( B_n \). Let \( T' = (B_n, z_n) \setminus ([0, 1], 0) \) and let \( z_{n+1} \) denote the endpoint \( 1 \in T' \). We define \( B_{n+1} = (T', z_{n+1}) \). Clearly, \( z_{n+1} \in B_{n+1} \) is a central root of \( B_{n+1} \) and thus \( B_{n+1} \) has \( 2^{n+1} \) endpoints.

Lemma 9.2. Let \( n \geq 1 \) and let \( B_n \) be a complete binary tree with \( 2^n \) endpoints. Then for every \( \varepsilon > 0 \), there is

1. an exact Markov map \( F_\varepsilon: B_n \to B_n \) such that \( (\log 3)/2^n \leq h(F_\varepsilon) < (\log 3)/2^n + \varepsilon \) and there is a fixed point of \( F_\varepsilon \) which is a central root for \( B_n \).
2. a pure mixing map \( G_\varepsilon: B_n \to B_n \) such that \( (\log 3)/2^n \leq h(G_\varepsilon) < (\log 3)/2^n + \varepsilon \).

Moreover, all endpoints of \( B_n \) form a single periodic orbit of \( F_\varepsilon \) and \( G_\varepsilon \).

Proof. First note that the second part of the theorem follows from the first and Lemma 8.6. We will prove the first part by induction on \( n \). For \( n = 1 \), note that \( B_1 = [0, 1] \), and consider the piecewise linear Markov map \( f: [0, 1] \to [0, 1] \) given by

\[
f(x) = \begin{cases} 
1 - 3x, & \text{for } 0 \leq x < 1/6, \\
3x, & \text{for } 1/6 \leq x < 1/3, \\
2 - 3x, & \text{for } 1/3 \leq x < 1/2, \\
1 - x, & \text{for } 1/2 \leq x \leq 1. 
\end{cases}
\]
It is clear that \( f \) is transitive, but not totally transitive, and \( h(f) = \log \sqrt{3} \). Given any \( \varepsilon > 0 \) we can apply Lemma 8.5 to get a piecewise linear Markov and exact map \( F_{\varepsilon} : [0, 1] \rightarrow [0, 1] \). Note that \( z = 1/2 \) is a fixed point of \( F_{\varepsilon} \) which is a central root for \( B_2 \) and the endpoints of \( \{0, 1\} \) form a two cycle for \( F_{\varepsilon} \).

Now assume that the theorem holds for \( n \geq 1 \), fix an \( \varepsilon > 0 \), and let \( G : B_n \rightarrow B_n \) be an exact map provided by induction hypothesis for \( \varepsilon/3 \). Let \( z \) be a fixed point of \( G \) which is a central root for \( B_n \). Apply Theorem 8.3 to obtain an exact piecewise linear Markov map \( G' : T' \rightarrow T' \) with \( h(G') < h(G) + \varepsilon/3 \), where \( T' \) is \( B_n \) with an arc \([z, z_0]\) attached to \( z \), that is, \( T' = (B_n, z) \cup ([z, z_0], z) \). Note that the 2\(^n\) endpoints of \( T' \) other than \( z_0 \) form a cycle for \( G' \) and \( z \) and \( z_0 \) are fixed for \( G' \). Observe that the 2\(^n\) wedge power \( (T', z_0)^{2^n} \) is just \( B_{n+1} \). Now we can apply Lemma 8.3 to get a transitive map \( G'' : B_{n+1} \rightarrow B_{n+1} \) with \( h(G'') = h(G') \). Applying Lemma 8.5 to \( G'' \) and \( \varepsilon/3 \) we get the map \( F_{\varepsilon} : B_{n+1} \rightarrow B_{n+1} \) with all desired properties.

A \( \sigma \)-graph, \( \theta \)-graph, 8-graph are spaces homeomorphic to the symbol representing the Greek letter sigma, theta, and the figure eight, respectively. A dumbbell is a graph homeomorphic to the following subset of a complex plane \( \mathbb{C} : C_{-2} \cup I \cup C_2 \), where \( C_{\omega} = \{z \in \mathbb{C} : |z - \omega| = 1\} \) and \( I \) is a line segment joining \( z = -1 \) with \( z = 1 \).

**Theorem 9.3.** Let \( \mathcal{P}_G \) denote the family of all pure mixing maps of a given graph \( G \).

1. If \( T_n \) is an n-star, \( n \geq 2 \), then
   \[ \inf(h(\mathcal{P}_T)) = \log 3/n. \]

2. If \( B_n \) is a full binary tree with \( 2^n \) endpoints, \( n \geq 1 \), then
   \[ \inf(h(\mathcal{P}_B)) = \log 3/2^n. \]

3. If \( G_{\sigma} \) is a sigma graph, then
   \[ \inf(h(\mathcal{P}_{\sigma})) = \log 3/2. \]

4. If \( G_{\theta} \) is a theta graph, then
   \[ \log 3/4 \leq \inf(h(\mathcal{P}_{\theta})) \leq \log 3/3. \]

5. Let \( G_8 \) be a figure-eight graph, and \( G_d \) be the dumbbell graph, then
   \[ \inf(h(\mathcal{P}_{G_8})) = \inf(h(\mathcal{P}_{G_d})) = \log 3/4. \]

**Proof.** Part (1) and (2) follow from Lemmas 9.1 and Lemma 9.2, respectively.

To prove (3) we fix \( \varepsilon > 0 \) and take an exact map of the interval with the endpoints forming a cycle of length two and entropy smaller than \( \log \sqrt{3} + \varepsilon/2 \). By Theorem 8.3 there is an exact Markov map on the 3-star, with the entropy smaller than \( \log \sqrt{3} + \varepsilon \) and two out of three endpoints of the 3-star form a cycle for that map. By Lemma 8.6 we can find a pure mixing map of the 3-star with the entropy smaller than \( \log \sqrt{3} + \varepsilon \) and two out of three endpoints of the 3-star still form a cycle which is inaccessible for that map. Identifying those two endpoints we get a pure mixing map of the sigma graph with topological entropy at most \( \log \sqrt{3} + \varepsilon \). On the other hand it is easy to see that a pure mixing map of the sigma graph can have at most two inaccessible sides since by Corollary 8.8 they have to form a cycle. Therefore its topological entropy is greater than \( \log \sqrt{3} \).

To see (4) fix \( \varepsilon > 0 \) and take a pure mixing map of the 3-star \( T_3 \) with topological entropy smaller than \( \log \sqrt{3}/3 + \varepsilon \) for which endpoints of \( T_3 \) form an inaccessible three cycle. Then we identify these endpoints to get a theta graph, and the proof of the upper bound for the infimum is complete. The lower bound comes from Theorem 7.3.

The last point, (5) follows from Lemma 9.2 and 9.1, respectively. To see this observe that identifying endpoints in the binary tree \( B_2 \) or 4-star in the appropriate way we get the dumbbell graph, and the figure-eight graph, respectively. □
We conjecture that the upper bound in the (4) above is actually the infimum, that is, \( \inf(h(\mathcal{P}_G)) = \log 3/3 \). Note that the bound from Corollary 7.4 is not always the best possible. It is an interesting question to find the formula for \( \inf(h(\mathcal{P}_G)) \) depending on the combinatorial structure of \( G \).

10. **Mixing implies specification property for graph maps**

In this section we present an alternative proof of the fact that every mixing graph map has the specification property. The result was originally proved by A. Blokh. Our approach extends ideas of the proof presented by J. Buzzi in \([13]\) in the context of compact interval. In order to carry out with the demonstration, we recall some terminology.

**Definition 10.1.** Let \( n > 0 \) be an integer and let \( \varepsilon > 0 \). The **closed Bowen ball** is the set

\[
B_n(x, \varepsilon) = \{y \in G : \rho(f^i(x), f^i(y)) \leq \varepsilon \text{ for } i = 0, \ldots, n\}.
\]

By \( B^e_n(x, \varepsilon) \) we denote the connected component of \( B_n(x, \varepsilon) \) containing \( x \).

**Lemma 10.2.** Let \( f \) be a mixing map of a graph \( G \). Assume that \( \alpha, \varepsilon > 0 \) are positive real numbers. Then there are an integer \( N = N(\alpha, \varepsilon) > 0 \) and positive real number \( \delta = \delta(\varepsilon) > 0 \) such that for every \( y \in G \) and every integer \( n \geq 0 \) there is a point \( z = z(y, \varepsilon, n) \in G \) for which

\[
z \in B_n(y, \varepsilon) \quad \text{and} \quad B(z, \delta) \subset \left( \bigcup_{y \in G} B(f^i(y), \varepsilon) \right) \setminus \{y\} \text{ hold for every } x \in G \text{ and } k \geq N.
\]

**Proof.** Fix \( \alpha, \varepsilon > 0 \). First, assume that \( f \) is exact. It is easy to see that for each \( \delta > 0 \) there is an integer \( N > 0 \) such that \( f^N(B(x, \alpha)) = G \) for each \( x \in G \) and \( k \geq N \). Set \( z = y \) and the proof for the first case is finished.

Consider the second case, when \( f \) is pure mixing. By the uniform continuity and compactness, it is enough to prove the assertion of the Lemma for a map \( f^m \) for some \( m > 0 \). Using Theorem 6.1 we can find the integer \( m \), graph \( G' \), and pure mixing map \( g : G' \to G' \), which is an extension of \( f^m \) and \( g^{-1}(e) = \{e\} \) for each \( e \in I(g) \). It is sufficient to prove that the Lemma holds for \( g \). To simplify further our notation we can assume that \( g \) has the unique inaccessible endpoint \( e_0 \). The other cases can be handled analogously. Using our convention (C) we isometrically identify the edge containing \( e_0 \) with an interval \([0, a]\) with \( e_0 = 0 \) and \( a > 0 \). We fix \( \alpha, \varepsilon > 0 \) and assume that \( \varepsilon < a \). By continuity and Theorem 6.2 we can find fixed points \( 0 < p < p' < \varepsilon \) such that \( g([0, p')) \subset [0, \varepsilon] \) and \( g([0, p]) \subset [0, \varepsilon] \). Let \( \delta = p/2 \). Theorem 6.4 allows us to find an integer \( N > 0 \) such that \( G' \setminus [0, \varepsilon] \subset g^N(J) \) for each subgraph \( J \) of \( G' \) with \( \text{diam } J \geq \alpha \) and \( k \geq N \). In particular, \( G' \setminus [0, \varepsilon] \subset g^N(B(x, \alpha)) \) for every \( x \in G' \) and \( k \geq N \). Fix \( y \in G' \) and let us agree that \( \min \emptyset = \infty \). Consider

\[
m = \min\{k : g^k(y) \in G' \setminus [0, p]\}.
\]

We have the following cases:

- **Case 1:** \( m = 0 \). Then we set \( z = y \).
- **Case 2:** \( 0 < m < \infty \). Observe that \( g^k(y) \in [0, p] \) for \( k = 0, 1, \ldots, m - 1 \). Furthermore, \( g^m(y) \in [p, p'] \). But \( g([p, p']) \subset [0, a] \) and since \( p, p' \) are fixed points for \( g \), we see that \( \{p, p'\} \) is \( g^m \)-covered by itself. Therefore, we can find \( z \in \{p, p'\} \) such that \( g^m(z) = g^m(y) \).
- **Case 3:** \( m = \infty \). Then it is enough to take \( z = p \).

Clearly, in any case \( z \in B_n(y, \varepsilon) \) for all \( n \geq 0 \) and \( B(I(g), \delta) \cap B(z, \delta) = \emptyset \).

**Lemma 10.3.** Let \( \alpha > 0 \). Then there is a constant \( \gamma = \gamma(\alpha) \) such that for every map \( g : G \to G \) if \( J \subset G \) is a star with \( \text{diam } g(J) \geq \alpha \), then there is a free arc \( g(J) \subset J \) which contains two subarcs \( J_1, J_2 \) with at most one common point such that \( g(J_i) \) is also a free arc \( g \)-covered by \( J_i \) with \( \text{diam } g(J_i) \geq \gamma \) for \( i = 1, 2 \).
Proof. By Lemma 10.4, there is a constant \( \xi = \xi(\alpha) \) such that \( g(J) \) contains a free arc \( K \) with \( \text{diam} K \geq \xi \). We conclude from Lemma 10.3 that \( K \) is contained in \( f(E_1 \cup E_2) \) where \( E_1, E_2 \) are edges of \( S \). Clearly, one of these edges, say \( E_1 \), must \( g \)-cover a free arc \( K' \) with \( \text{diam} K' \geq \xi/3 \). We set \( y = \xi/6 \) and write \( K' = K_1 \cup K_2 \), where \( K_1, K_2 \) are free arcs with at most one common point, and \( \text{diam} K_i \geq y \). Now, an application of Lemma 3.3 finishes the proof.

Lemma 10.4. Let \( f : G \to G \) be a mixing graph map. If \( 0 < \epsilon < \frac{1}{2} \text{diam} G \) and \( \delta > 0 \) then there is an \( N = N(\epsilon, \delta) > 0 \) such that \( B(x, \epsilon) \subset B(x, \delta) \) for all \( x \in G \) and all \( n \geq N \).

Proof. If the conclusion of the Lemma does not hold, then, using compactness of \( G \), we could find a point \( x \in G \) and \( \epsilon > 0 \) such that the set

\[
B = \bigcap_{k=0}^{\infty} B^k(x, \epsilon)
\]

would have a non-empty interior. Then \( \text{diam} f^n(B) \leq 2\epsilon < \text{diam} G \) for all \( n \) contradicting topological mixing.

Lemma 10.5. If \( f : G \to G \) is a mixing graph then for every \( \epsilon > 0 \) there is a constant \( \zeta = \zeta(\epsilon) \) such that

\[
0 < \zeta \leq \text{diam} f^n(B^\epsilon(x, \epsilon))
\]

for every \( n \) and \( x \in G \).

Proof. Fix \( \epsilon > 0 \). Without loss of generality we may assume that \( \epsilon < \frac{1}{2} \text{diam} G \). Let \( \mathcal{K}(\epsilon) \) be a family of subgraphs of \( G \) with diameter at least \( \epsilon \). Then \( \mathcal{K}(\epsilon) \) is a closed subset of a hyperspace of subcontinua of \( G \). Moreover, \( \Phi_n = \text{diam} f^n \) is a continuous function on \( \mathcal{K}(\epsilon) \) for all \( n \geq 0 \). Observe that there is an \( N \) such that \( \text{diam} f^n(J) \geq 1/2 \text{diam} G \) for all \( J \in \mathcal{K}(\epsilon) \), and \( n \geq N \). Therefore

\[
\beta(\epsilon) := \inf \{ \text{diam} f^n(J) : J \in \mathcal{K}(\epsilon) \} = \min_{0 \leq j \leq N} \Phi_j(\mathcal{K}(\epsilon)) > 0,
\]

as no subgraph of \( G \) can be mapped by any \( f^n \) onto a point.

Fix \( x \in G \). We claim that \( \text{diam} f^n(B^\epsilon_n(x, \epsilon)) \geq \beta(\epsilon) \) defined above. We have \( f^{n+1}(B^\epsilon_{n+1}(x, \epsilon)) \subset f f^n(B^\epsilon_n(x, \epsilon)) \) for all \( n \). Furthermore, note that:

(\dag): If \( f^{n+1}(B^\epsilon_{n+1}(x, \epsilon)) \neq f f^n(B^\epsilon_n(x, \epsilon)) \) then there is \( y \in B^\epsilon_n(x, \epsilon) \) with

\[
\rho(f^{n+1}(x), f^{n+1}(y)) \geq \epsilon,
\]

hence \( \text{diam} f^{n+1}(B^\epsilon_{n+1}(x, \epsilon)) \geq \epsilon \).

(\ddag): If \( f^{n+1}(B^\epsilon_n(x, \epsilon)) = f f^n(B^\epsilon_n(x, \epsilon)) \) and \( \text{diam} f^n(B^\epsilon_n(x, \epsilon)) \geq \epsilon \) then

\[
\text{diam} f^{n+1}(B^\epsilon_{n+1}(x, \epsilon)) \geq \beta(\epsilon)
\]

by the definition of \( \beta(\epsilon) \).

Applying (\dag) or (\ddag), accordingly, we get:

(\ddagger): If \( \text{diam} f^n(B^\epsilon_n(x, \epsilon)) \geq \epsilon \) for some \( n \geq 0 \) then \( \text{diam} f^{n+k}(B^\epsilon_n(x, \epsilon)) \geq \beta(\epsilon) \) for every \( k \geq 0 \) by the definition of \( \beta(\epsilon) \).

Now, \( B(x, \epsilon) = B^\epsilon_0(x, \epsilon) \), and we proceed by induction.

The following definition was introduced by Bowen in [12].

Definition 10.6. We say that a continuous map \( f : X \to X \) acting on a compact metric space \((X,d)\) has the specification property if for every \( \epsilon > 0 \) there exists an integer \( M = M(\epsilon) \) such that for any \( s \) there exists \( x_1, x_2, \ldots, x_s \in X \), for any integers \( a_1 \leq b_1 < a_2 \leq b_2 < \ldots < a_s \leq b_s \), with \( a_i - b_{i-1} \geq M \), for \( 2 \leq i \leq s \) and for any integer \( p \) with \( p \geq M + b_s - a_1 \) there exists a point \( x \in X \) with \( f^p(x) = x \) such that \( d(f^p(x), f^p(x_i)) < \epsilon \) for \( a_i \leq n \leq b_i \), and \( 1 \leq i \leq s \).

Remark 10.7. Without loss of generality we can take $a_1 = 0$ and $p = M + b_1$ in the above definition.

Theorem 10.8 (Blokh). Every mixing graph map has the specification property.

Proof. Let $f : G \to G$ be a mixing graph map and fix $\varepsilon > 0$. Our first task is to find an suitable integer $N > 0$ as in the definition of specification. Without lost of generality we may assume that $\varepsilon < (1/2) \text{diam } G$. Set $\alpha = \text{diam } I_U$, where $I_U$ is an universal arc for $f$. Let $N_1 = N(\varepsilon/2, \alpha/2)$ and $\delta = \delta(\varepsilon/2)$ be given by Lemma 10.2 for $\varepsilon/2$ and $\alpha$ as above. We may also assume that the open ball $B(x, \delta)$ is a canonical neighborhood of $x$ for each $x \in G$. We plug $\delta$ and $\varepsilon/2$ into Lemma 10.4 to get an $N_2 = N(\varepsilon/2, \delta)$ such that $B'_i(x, \varepsilon/2) \subset B(x, \delta)$ for all $x \in G$ and all $n \geq N_2$. As a direct consequence of Lemma 10.5 and Lemma 10.3 we can find a constant $\beta > 0$ such that for every $x \in G$ and every $n > 0$ the following condition holds

\[ \text{(\star)}: \text{ there is a free arc } J' \subset B'_i(x, \varepsilon/2) \text{ containing two free arcs } J_1 \text{ and } J_2 \text{ with at most one common point and two free arcs } K_1 \text{ and } K_2 \text{ with diam } K_i \geq \beta \text{ for } i = 1, 2 \text{ and such that } K_i \text{ is } f^n \text{ covered by } J_i \text{ for } i = 1, 2. \]

For $\beta > 0$ as defined above we can find an $N_3 > 0$ such that if $n \geq N_3$, then $I_U$ is $f^n$ covered by each closed interval $K$ with diam $K \geq \beta$.

We claim that $N = N_1 + N_2 + N_3$ will fulfill the definition of the specification property. For the proof of our claim we assume to simplify the notation that $s = 2$ and we choose any $x_1, x_2 \in G$, and integers $0 = a_1 \leq b_1 < a_2 \leq b_2$, with $a_2 - b_1 \geq N$. Finally, we fix any $p \geq b_2 + N$. For $i = 1, 2$ let $y_i = f^n(x_i)$ and $n(i) = b_i - a_i + N_2$. Let $z_i = z(y_i, \varepsilon/2, n(i)) \in B(n(i))(y_i, \varepsilon/2)$ be provided by Lemma 10.2. By our choice of $N_2$ we conclude that $B_i = B'_{n(i)}(z_i, \varepsilon/2) \subset B(z_i, \delta)$ for $i = 1, 2$. Let us denote the free arcs constructed in $B_i$ in (\star) above by $J_i$ and $J_i'$. By Lemma 10.2 we have $B(z_i, \delta) \subset f^n(I_U)$ for each $i = 1, 2$ and $k \geq N_1$. Applying Lemma 5.3 we see that for $i = 1, 2$ and $k$ as above $I_U$ must $f^k$ cover at least one free arc $I_i(k) \in \{ I_1, I_2 \}$. Let $I_1 = I_1(k(1))$, where $k(1) = a_2 - b_1 - N_2 + N_3$, and $I_2 = I_2(k(2))$, where $k(2) = p - (b_2 + N_2 + N_3)$. Appealing again to the condition (\star) we see that each $I_i$ covers through $f^{m(i)}$ an interval $K_i$ with diam $K_i \geq \beta$. This in turn implies that each $K_i$ must $f^{N_i}$-cover $I_U$. In conclusion, we get

\[ I_1 f^{m(1)} K_1 f^{p(1)} I_U f^{m(2)} K_2 f^{p(2)} I_U f^{p(3)} f^{m(3)} f f^{m(4)} f f^{m(5)} K f f^{m(6)} f f^{m(7)} f f^{m(8)} f \]

where $I f f f$ denotes here that $I$ $f$-covers $K$. It follows that $I_1$ is $f^n$ covered by itself, since $p = n(1) + k(1) + n(2) + k(2) + 2N_3$. Therefore there is a $p$-periodic point $q \in I_1$ such that $q \in B'_{n(1)}(z_1, \varepsilon/2)$ and $z_1 \in B(n(1))(y_1, \varepsilon/2)$. Moreover, $r = f^{m(3)}(q) \in I_2$, hence $r \in B'_{n(2)}(z_2, \varepsilon/2)$ and $z_2 \in B(n(2))(y_2, \varepsilon/2)$. This finishes the proof.

\[ \square \]

Acknowledgements

This work was supported by the Polish Ministry of Science and Higher Education from sources for science in the years 2010-2011, grant no. IP2010 029570.

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