HYPERBOLIC STRUCTURES ON 3-MANIFOLDS, III: DEFORMATIONS OF 3-MANIFOLDS WITH INCOMPRESSIBLE BOUNDARY

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Abstract. This is the third in a series of papers constructing hyperbolic structures on Haken manifolds, analyzing the mixed case of 3-manifolds that with incompressible boundary that are not acylindrical, but are also not interval bundles. The main ingredient (beyond [Thu86] and [Thu98]) is an upper bound for the hyperbolic length of the 'window frame', that is, the boundaries of I-bundles in the manifold, using a growth rate estimate for branched hyperbolic surfaces.

This is slightly revised from the 1986 version of a preprint that circulated in the early '80's. A few figures have been added, and a few clarifications have been made in the text.

0. Introduction

In the first two parts of this series, [Thu86] and [Thu98], we analyzed hyperbolic structures on two opposite classes of 3-manifolds. In the first paper, we studied acylindrical 3-manifolds. In the second, we studied 3-manifolds of the form $S \times I$: such a manifold is really one big cylinder.

We will now study the hybrid case, that is, general 3-manifolds with incompressible boundary. We will also obtain some information about hyperbolic structures on three-manifolds whose boundary is not incompressible — provided we bound the lengths of a set of curves on $\partial M$ sufficient to intersect the boundary of any essential disk.

Recall from §7 of [Thu86] that a pared manifold is a pair $(M, P)$, where $P \subset \partial M$ is a (possibly empty) 2-dimensional submanifold with boundary such that

(a): the fundamental group of each of its components injects into the fundamental group of $M$.

(b): the fundamental group of each of its components contains an abelian subgroup with finite index.

(c): any cylinder

$$C : (S^1 \times I, \partial S^1 \times I) \to (M, P)$$

such that $\pi_1(C)$ is injective is homotopic rel boundary to $P$

(d): $P$ contains every component of $\partial M$ which has an abelian subgroup of finite index.

The terminology is meant to suggest that certain parts of the skin of $M$ have been pared off to form parabolic cusps in hyperbolic structures for $M$.

Recall also that $H(M, P)$ is the set of complete hyperbolic manifolds $M$ together with a homotopy equivalence of $(M, P)$ to $(N, \text{cusps})$, where we represent cusps by disjoint horoball neighborhoods. There are three topologies $AH(M, P)$, $GH(M, P)$ and $QH(M, P)$ on $H(M, P)$: the algebraic, geometric, and quasi-isometric or quasiconformal topologies.

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A pared manifold is *acylindrical* if $\partial M - P$ is incompressible and if every cylinder

$$C : (S^1 \times I, \partial S^1 \times I) \to (M, \partial M - P)$$

such that $\pi_1(C)$ is injective is homotopic *rel* boundary to $\partial M$. Theorem 7.1 of [Thu86] asserts that $\text{AH}(M, P)$ is compact when $(M, P)$ is acylindrical.

This is not the case if $(M, P)$ admits essential cylinders.

In order to describe what unbounded behaviour there is for $N \in \text{AH}(M, P)$, we will make use of the *characteristic submanifold* of $(M, P)$, whose theory was first developed in [JS79] and [Joh79]. The characteristic submanifold is defined provided $\partial_0(M) = \partial(M) - P$ is incompressible. The characteristic submanifold is a disjoint union of Seifert fiber spaces and interval bundles over surfaces in a 3-manifold. It is determined up to isotopy. The ends of the intervals are on $\partial M$, and in our case, we will consider only those interval bundles where the ends of the intervals are contained in $\partial_0(M)$. The characteristic submanifold has the universal property that is essential, and that any other essential Seifert fiber space or interval bundle in $M$ with the given boundary conditions is isotopic into the characteristic submanifold. The definition and development of a sufficient theory to handle embedded interval bundles and Seifert spaces is fairly elementary, and akin to the theory of the prime decomposition of a three-manifold. The theory also handles homotopy classes of maps which are not necessarily embeddings; this is more difficult. We will make free use of both types of universal properties of the characteristic submanifold, for embeddings and for homotopy classes, even though we could probably get by with only the theory for embeddings.

In the case of current interest, that of a pared manifold which admits at least one hyperbolic structure, the Seifert fiber space part of the characteristic submanifold consists of some number of thickened tori in neighborhoods of torus components of $P$, and some number of solid tori, each of which intersects $\partial_0(M)$ in a non-empty union of annuli. These annuli wind one or more times around the “long way” of the solid torus before closing. A transverse disk in a solid torus intersects the boundary annuli in two or more arcs. However, if it intersects in only two arcs, we can and will interpret the solid torus as an interval bundle, rather than as a Seifert fiber space. The transverse disks in all remaining Seifert fiber spaces intersect $\partial_0(M)$ in at least three arcs.

It is convenient for our purposes to carve out of the characteristic manifold something which we will call the *window*, which will be an interval bundle $\text{window}(M, P)$. If $M$ is made of glass, then viewed from the outside of $M$ the window is the part that one can see through without homotopic distortion or entanglement. To form the window, begin with all of the interval bundles in the characteristic submanifold. In addition, a portion of the boundary of each solid torus in the characteristic submanifold consists of essential annuli in $(M, \partial_0(M))$. Thicken each of these, to obtain additional interval bundles. Some of the new interval bundles, however, may be redundant: they may be isotopic into other interval bundles. If such absorption is possible, then eliminate interval bundles which can be absorbed, one by one. The result is the window. The ends of the intervals of the window form a subsurface of $\partial_0(M)$, the *window surface* $\text{ws}(M, P)$; it is a 2-fold covering of $\text{wb}(M, P)$, the *window base*. We may assume that $\text{window}(M, P)$ contains all of the annuli in $P$. Note that even when $M$ is orientable, the window base need not be orientable, since non-orientability of the fibration can cancel with non-orientability of the base to make the total space orientable.

The main theorem of this paper is
**Theorem 0.1** (Broken windows only). If $\Gamma \subset \pi_1(M)$ is any subgroup which is conjugate to the fundamental group of a component of $M - \text{window}(M, P)$, then the set of representations of $\Gamma$ in $\text{Isom}(\mathbb{H}^3)$ induced from $\text{AH}(M, P)$ are bounded, up to conjugacy.

Given any sequence $N_i \in \text{AH}(M, P)$, there is a subsurface with incompressible boundary $x \subset \text{wb}(M, P)$ and a subsequence $N_{i(j)}$ such that the restriction of the associated sequence of representations $\rho_{i(j)} : \pi_1(M) \to \text{Isom}(\mathbb{H}^3)$ to a subgroup $\Gamma \subset \pi_1(M)$ converges if and only if $\Gamma$ is conjugate to the fundamental group of a component of $M - X$, where $X$ is the total space of the interval bundle above $x$. Furthermore, no subsequence of $\rho_{i(j)}$ converges on any larger subgroup.

Compare with Theorem 6.2 of [Thu98].

![Figure 1](attachment:image.png)

**Figure 1.** This is a diagram for a 3-manifold obtained by gluing three thickened punctured tori to a solid torus. Its window consists of the three thickened punctured tori. Theorem 0.1 says that among all complete hyperbolic structures, the length of the core circle of the solid torus is bounded. Theorem 1.3 and the ensuing discussion gives an explicit upper bound of 18.13 (which does not pretend to be near optimal).

The distinction between the characteristic submanifold and the window is important in this statement. A good example is a three manifold obtained by thickening three surfaces with boundary, and then gluing the annuli coming from the thickened boundary components to a solid torus (figure 1). The characteristic submanifold has a part which is the solid torus, and three other parts which are the thickened surfaces. According to one reasonable definition, the entire manifold consists of characteristic submanifold (if parallel boundary faces of the characteristic manifold are pasted together). Theorem 0.1 applied to this example boils down to the assertion that the length of the core of the solid torus is bounded in $\text{AH}(M, P)$.

The proof divides into two main steps. The first step is to show that the total length of geodesics representing $\partial \text{wb}(M, P)$ in three-manifolds $N \in \text{AH}(M, P)$ is bounded. This will be proven in §1, by a growth rate argument which actually gives reasonable and concrete bounds for this total length.

The second main step is to show that a certain relative deformation space, that of hyperbolic structures on $M - \text{window}$ relative to $P$ and window, is compact. This is quite parallel to the proof of the main theorem of part I of this series, [Thu86]. It will be a corollary of a more general theorem we will prove, which asserts that if $(M, \partial M)$ is a three-manifold with boundary, and if $X \subset \partial M$ is a sufficiently complicated collection of curves on $\partial M$, then when $X$ is held bounded, so is $M$. See Theorem 3.1 for the formal statement. This relative boundedness theorem will make use of the relative uniform injectivity of pleated surfaces, proven in §2.
We will construct a certain branched 2-manifold, which describes the boundary and how some parts of it are homotopically identified with other parts.

Branched 2-manifolds are an analogue of train tracks, in one dimension higher. We will make use of them in two forms: in the form of non-Hausdorff 2-manifolds, and also in the form of the Hausdorffifications of these 2-manifolds which are standard 2-complexes.

It is easy to construct many examples of non-Hausdorff manifolds: if you begin with a standard manifold, and then identify an open set of one by a homeomorphism to an open set in another, the result is still always a manifold, since each point has a neighborhood homeomorphic to a subset of $\mathbb{R}^n$. However, the identification space is typically not Hausdorff, since there are likely to be pairs of points on the frontiers of the open sets such that all neighborhoods of one intersect all neighborhoods of the other in the quotient topology. Non-Hausdorff manifolds arise frequently as quotient spaces of dynamical systems.

We will construct a branched manifold $BS(M, P)$ associated with a pared manifold having incompressible boundary, which either is a quotient space of $\partial_0(M)$, or at least has the image of $\partial_0(M)$ as a dense subset. This branched manifold admits a hyperbolic structure such that the branch points lie on a finite number of closed geodesics. There will be a map $BS(M, P) \to M$ defined up to homotopy, and the inclusion of $\partial_0(M)$ into $M$ factors up to homotopy through this map.

**Figure 2.** A certain 2-complex $KS(M, P)$ is associated with an atoroidal pared 3-manifold $(M, P)$, roughly described by starting with $\partial_0(M)$ and collapsing the $I$-directions of its windows (characteristic $I$-bundles), and collapsing the transverse disks of the essential solid torus Seifert fiber spaces. *Above,* an essential solid torus having three annuli on its boundary that are part of $\partial_0(M)$ and three annuli that cut through the interior of $M$. The uppermost interior annulus adjoins a ‘wide’ part of window$(M, P)$, while the other two attach to portions of $M$ that are relatively acylindrical—the adjoining windows are just thin slivers of glass. *Below,* a cross-section of KS, showing the collapsing maps from $\partial_0(M)$.

First, we construct the Hausdorff version. Begin with a characteristic submanifold $W$ for $M$. Throw out any components of $W$ which are neighborhoods of cusps. Adjust each solid torus in the Seifert fiber part of $W$ so that instead of intersecting $\partial M$ in a union of annuli, it intersects in a union of circles. Similarly, adjust any solid torus which is a component of the interval bundle part so that any annulus of intersection with $\partial M$ is collapsed to a circle. Squeeze together any pairs of annuli in the frontier of $W$ which are parallel in $M$. Now
collapse each solid torus to a circle, and collapse each of the interval bundles to its base. The result is a 3-complex \( K(M, P) \), and the image of \( \partial_0(M) \) in \( K(M, P) \) is a 2-complex \( KS(M, P) \).

\( KS \) is a 2-manifold except along certain branch curves, some which come from the solid tori in \( W \) and others from annuli in the frontiers of the interval bundles. The components of \( KS \) cut by these curves are surfaces of negative Euler characteristic. Choose a length for each branch curve, and extend this choice to give a metric on \( KS \) that is hyperbolic except at the branch curves, which are geodesics.

**Figure 3.** This is the branched surface \( BS(M, P) \) which locally isometrically embeds in \( KS(M, P) \), from figure 2. Each branch curve \( \alpha \subset KS(M, P) \) is resolved to a number of copies: in this example, there are 3 copies of \( \alpha \) coming from the components \( \partial_0(M) \) that maps to \( \alpha \), and 5 copies for the pairs of non-adjacent normal directions to \( \alpha \). In a more general example, a cross-section disk could rotate when it is isotoped once around the solid torus, so that \( \alpha \) is replaced by some covering space of \( \alpha \), but this does not change the basic principle that \( BS(M, P) \) actually branches.

We will construct a branched surface \( BS \) from \( KS(M, P) \) by replacing each branch curve \( \alpha \) by a probably disconnected covering space \( m(\alpha) \), with a neighborhood on \( BS \) that locally embed in in \( KS(M, P) \). For each point \( t \in \alpha \), an element in its pre-image is determined by specifying a pair of sheets of \( KS(M, P) \) at \( t \); these sheets might be permuted by a rotation when you go once around \( \alpha \). We want to create as much branching as we can without violating a certain homotopy injectivity principle to be described below, in order to get tighter bounds for the geometry of hyperbolic structures on \( (M, P) \).

As a start, we can put the union of all circles on \( \partial_0(M) \) that map to \( \alpha \) into \( m(\alpha) \). In cases coming from a window base that is an annulus or Moebius band, these boundary circles are not enough to create actual branching. But we can add to \( m(\alpha) \), in addition to the boundary circles, one circle for each pair of sheets of \( KS(M, P) \) along \( \alpha \) that are non-adjacent with respect to circular order around \( \alpha \). We define \( BS(M, P) \) by taking the union of the the complement of the branch circles in \( KS(M, P) \) with the annular neighborhoods of \( m(\alpha) \), identifying them on open sets. This quotient space \( BS(M, P) \) is a non-Hausdorff surface, with a hyperbolic structure inherited from \( KS(M, P) \).

We need to construct a “fundamental cycle” for \( BS \), which can be defined as a locally finite singular 2-cycle which has positive degree at every point. A locally finite 2-cycle is homologous to one which is transverse to any particular point, so the degree makes sense even at the branch curves. Many non-Hausdorff manifolds without boundary do not admit such. For a 1-dimensional example, a non-recurrent train track on a surface determines a non-Hausdorff 1-manifold with no fundamental cycle.

In the present case, we can start with the fundamental cycle of \( \partial_0(M) \), which pushes forward to a cycle on \( BS \). It does not have positive degree quite everywhere: it has zero
degree at each of the circles which bridge between non-adjacent sheets around a branch curve $\alpha$ of $KS$. A local adjustment can easily be made so that all these degrees are positive.

Geodesics make perfect sense on a non-Hausdorff hyperbolic surface: they are paths which have local behavior modeled on a straight line in the hyperbolic plane. Unlike in the Hausdorff case, a tangent vector does not have a unique geodesic extension. If you construct a geodesic in a certain direction, you may eventually come to a branch point; at such a place, there may be several possible extensions of the geodesic. Our surface $BS$ satisfies the property that every geodesic has an extension to a bi-infinite geodesic; we will call a non-Hausdorff hyperbolic surface of this type a \textit{complete} surface.

A non-Hausdorff hyperbolic surface still has a developing map, defined by analytic continuation, that maps some covering space to the hyperbolic plane.

We can define a space $GF$ on which the geodesic flow exists, as the set of pairs $GF = \{(X, l)\}$, where $l$ is a bi-infinite unit-speed geodesic on $BS$, and $X$ is a unit tangent vector to $l$. Give $GF$ the compact-open topology. There is a continuous map $p : GF \to T_1(BS)$, which forgets $l$. The preimage of a point $X$ is a totally-disconnected set, most likely a Cantor set—one can think of $p^{-1}(x)$ as the sequence of decisions $l$ has to make each time it comes to a choice of branches. The space $GF$ is Hausdorff: even when $X_1$ and $X_2$ are tangent vectors along the branch locus which do not have disjoint neighborhoods, any two elements $(X_1, l_1)$ and $(X_2, l_2)$ have neighborhoods which are disjoint, distinguished from each other in the Cantor set direction.

The map $GF \to T_1(S)$ is not a fibration, because the particular choices confronting a geodesic which starts out at a tangent vector $X$ depends on $X$. For example, some geodesics never cross branch lines, or cross only a finite number of branch lines, or cross branch lines in forward time but not in backward time.

A fundamental cycle $Z$ for a branched surface determines an invariant transverse measure for the geodesic flow, by the following stipulations:

(a): the total transverse measure of the preimage of any point $p \in BS$ is the degree of $Z$ at $p$, and is distributed among the tangent vectors $X$ in proportion to Lebesgue measure on the circle.

(b): Among the elements of $GF$ over a given tangent vector $X \in T_1(BS)$ such that they traverse the same tangent vectors for time $0 \leq t < t_0$ or $0 \geq t > t_0$, the probability for the possible choices of tangent vector $X_{t_0}$ at time $t_0$ is proportional to the degree of $Z$ at the base point of $X_{t_0}$.

This transverse measure gives rise also to a measure on $GF$, defined locally as the product of Lebesgue measure along geodesics with the transverse measure.

\textbf{Proposition 1.1} (Exponential map injective). Let $BS(M, P)$ be the covering space induced from the universal covering $\hat{M}$ of $M$. Let $x \in BS(M, P)$ be an arbitrary point. The exponential “map” at $x$ is injective, in the sense that if $l_1$ and $l_2$ are any two geodesic intervals beginning at $x$ which have the same endpoints, then they coincide.

\textbf{Proposition 2.1} is a related statement of a more topological form.

\textbf{Proof of 1.1.} If there are geodesics $l_1$ and $l_2$ on $BS(\hat{M}, P)$ which have identical endpoints, but are not identical, push them forward to $KS(M, P)$. Pick a maximal interval of the image of $l_1$ whose interior does not intersect the image of $l_2$. This configuration gives rise to a rectangle in $(M - W)$, with two of its sides on $\partial M$ and the other two on the characteristic
Figure 4. Geodesics on non-Hausdorff surfaces can coincide for a time until they reach a branch point, then separate. The space GF is the global description of this behavior, consisting of all possible unit tangent vectors to BS(M, P) together with a bi-infinite geodesic through the vector. Proposition 1.2 limits the rate of branching by the growth of volume in H^3. This limit in turn limits the total length of the branch locus.

submanifold W. Every such rectangle can be deformed, rel W, into W, by the theory of the characteristic submanifold. (See [Joh79] or [JS79].) This contradicts the fact that l_1 is a hyperbolic geodesic.

The locus of endpoints of all geodesics of length R emanating from a point x ∈ BS is a branched 1-manifold C_R(x). In an ordinary hyperbolic surface, this curve is the sphere of radius R, and its total length is 2π sinh R. For a hyperbolic structure on BS, C_R(x) will tend to grow faster the more the surface branches. To make a formal statement, let β ⊂ BS be any component of the branch locus. For any fundamental cycle Z, there is a constant value degree(β) to the degree of Z along β. Choose an arbitrary orientation for β, so we can talk about its left side and its right side. The degree of Z just to the left of β is also a constant, dl(β), and the degree just to the right is another constant dr(β).

**Proposition 1.2** (Growth proportional to branching). Let Z as above be a fundamental cycle for BS, and let g be a hyperbolic structure on BS. Let Y be the set of branch curves on BS(M, P). For each β ∈ Y, define

\[ l(β) = -\log \left( \frac{\text{degree}(β)}{\text{dl}(β)} \right) - \log \left( \frac{\text{degree}(β)}{\text{dr}(β)} \right). \]

Then the average A(R), averaged over x ∈ BS, of length(C_R(x) degree_x(Z), is at least

\[ 2π \sinh(R) \exp \left( R \sum_{β ∈ Y} \frac{0.5 \text{length}_g(β) \text{degree}(β)}{\text{mass}(Z)} l(β) \right) \]

**Remark.** We are really making an estimate for the entropy of the geodesic flow with respect to the measure determined by Z. The quantity l(β) is minus the log of the probability that a geodesic coming along on the right chooses to cross β, plus a similar term for the left. In the summation, this is multiplied by the ratio of the flux of the geodesic flow through either side of β to the total measure of GF.

**Proof of 1.2.** For each x in BS and for each R > 0, we will define a certain function F_{(x,R)} on the part GF_x of GF above x, whose integral is the total length of C_R(x). F_{(x,R)} will be constant over each set of (X, l) ∈ GF_x such that the geodesics l agree for time 0 ≤ t ≤ R.
These conditions determines $F$: the value $F_{(x,R)}(X,l)$ is $2\pi \sinh(R)/\text{degree}_x(Z)$ times the product, over all choices which $l$ makes during time $0 \leq t \leq R$, of the reciprocal of the probability of making that choice.

The average $A(R)$ of $C_R(x)\text{degree}_x(Z)$, as $x$ ranges over BS, is the average of $F_{(x,R)}(X,l)\text{degree}_x(Z)$. The products in the formula for $F_{(x,R)}(X,l)$ can be thought of as exp of the sum of minus the logarithms of the probabilities. Since exp is convex upward, $A(R)$ is greater than or equal to $2\pi \sinh(R) \exp(B(R))$, where $B(R)$ is the expected value, among all geodesics of length $R$ in BS, of the sum of minus the logarithms of the probabilities of the choices it makes.

There is an exact formula for $B(R)$. Each branch curve $\beta$ on BS contributes $-\log(\text{degree}(\beta))/dl(\beta)$ each time a geodesic flows through from the left, and the total volume of the flow through $\beta$ in time $R$ is $.5R\text{length}(\beta)\text{degree}(\beta)$. There is a symmetrical formula for flow from the right. The contribution to $B(R)$ coming from flow through $\beta$ is therefore

$$\frac{.5R\text{length}_g(\beta)}{\text{area}(BS)}l(\beta),$$

so

$$A(R) \geq 2\pi \sinh(R) \exp \left( \sum_{\beta} \frac{.5R\text{length}_g(\beta)\text{degree}(\beta)}{\text{area}(Z)}l(\beta) \right).$$

The proposition follows. \qed

**Theorem 1.3** (Window frame bounded). For any pared manifold $(M, P)$ such that $\partial_0(M)$ is incompressible, there is a constant $C$ such that among all elements $N \in \mathcal{AH}(M, P)$, the length in $N$ of $\partial wb(M, P)$ is less than $C$.

**Proof of 1.3.** The surface area of a sphere of radius $R$ in $\mathbb{H}^3$ is $4\pi \sinh^2(R)$, so both the volume of the ball of radius $R$ and the area of the sphere of radius $R$ have exponent of growth $2R$. (The exponent of growth of a function $f(R)$ is $\limsup \log(f)$; in this case, the lim sup is the limit.)

We can represent the surface BS as a pleated surface in $N$, by mapping all the branch curves to their corresponding geodesics (if they have geodesics), and then extending to the remaining parts of BS, which are ordinary surfaces with boundary. If some of the branch curves are parabolic elements, so that they have zero length, then we can similarly construct a pleated surface based on BS minus a union of branch geodesics; it is still a branched surface of a similar type. This imparts a complete hyperbolic structure of finite area to BS or to some subsurface $Q$.

Fix a fundamental cycle $Z$ on BS, as described. It defines also a fundamental cycle on any of the subsurfaces $Q$ which might arise. From proposition 1.2, we see that the exponential growth rate of $A(R)$ is $1$ plus a positive linear combination of the lengths of the curves $\beta$. (If some branch curves were removed from $\beta$, then they are not included in the linear combination, but the inequality still holds since their length is zero.)

If $Q$ is compact, the argument is direct. In that case, there is an upper bound to the area of intersection of $\tilde{Q}$ with a ball of radius say $1$ in $\mathbb{H}^3$. The volume of a ball of radius $R$ in $\mathbb{H}^3$ is $\int_0^R 4\pi \sinh^2(t) dt$, so it grows with exponential growth rate $2R$. Therefore, the image of the exponential map at any point in $\tilde{Q}$ can have area at most a constant times $\exp(2R)$, so the total length of the branch set must be bounded.

In general, each cusp of $Q$ is contained in a subsurface bounded by branch curves, with no branch curves in the interior. Construct horoball neighborhoods of the cusps. If $x \in \tilde{Q}$
is any point, and if $x$ is in a horoball $H$ embedded in $\tilde{Q}$ which covers one of the horoball neighborhoods of cusps, then the area of intersection of the ball of radius $R$ about $x$ with $H$ is less than $2\pi \sinh(R)$. Whether or not $x$ is in a horoball, if $H'$ is one of the horoballs not containing $x$, then the area of the intersection of the ball of radius $R$ about $x$ with $H'$ is no more than a constant $\left(\frac{e}{e-1}\right)$ times the area of intersection with an outer ring of $H'$ of width 1.

Consider $Q$ minus smaller horoball neighborhoods of the cusps, shrunk a distance of 1. This subset $K$ of $Q$ is compact, so there is a supremum to the ratio of volume in $N$ with area of intersection with $K$. Therefore, the length of the intersection of the ball of radius $R$ about any point in $\tilde{Q}$ with $K$ is no more than a bounded multiple of $\exp(2R)$. By 1.2, this implies that the total length of the branch set is bounded.

A more careful analysis would show that the geodesic flow for $Q$ is ergodic, and that the area of the image of the exponential map in $\tilde{Q}$ at any two points has a bounded ratio (with bound depends on the pair of points, though), so that in particular, the exponential growth rate of area is independent of $x$.

Note that if the hyperbolic structure is turned into a metric on the Hausdorffification of $Q$, then the growth rate of the ball of radius $R$ in the metric might be considerably more than the growth rate of the image of the exponential map. This is hard to exploit, however, because it is hard to know what cover of $Q$ is induced from the universal cover of $M$. The exponential map, in effect, picks out a subset of the fundamental group of the universal covering of the fundamental group of $Q$ which injects no matter which 3-manifold it arose from.

As an example of how the constants work out in Theorem 1.3, consider the three-manifold of figure 1 obtained by gluing three thickened punctured tori to a solid torus. In this case, BS has three branch curves, each of degree 1, and the rest of the surface has degree 2. For any hyperbolic structure, the three branch curves have the same length, $a$. If $\beta$ is any of these curves, then $l(\beta) = 2 \log(2) \approx 1.38629$. The exponential growth rate of $A(R)$ is therefore $1 + 1.5 \frac{\text{length}(\beta) l(\beta)}{(12\pi)} \approx 1 + .05516 \text{length}(\beta)$. Consequently, $\text{length}(\beta) < 18.1294$. If $n$ punctured tori are glued to a solid torus, then there are $n(n-1)/2$ branch curves in BS, and we can choose $Z$ to give each of them degree $1/(n-1)$, and the rest of the surface degree 1. Then $l(\beta) = 2 \log(n-1)$, mass($Z$) = $2\pi n$, so the exponential growth rate of $A(R)$ is

$$1 + \frac{(n-1) \log(n-1)}{8\pi} \text{length}(\beta),$$

so $\text{length}(\beta) < (8\pi)/(n-1) \log(n-1))$. For $n = 4$, the length is less than 7.6256; for $n = 7$, the length is less than 2.3378.

It would be interesting to work out extensions of this growth rate argument to more general contexts.

The argument certainly applies directly in one dimension lower, to train tracks on surfaces, and gives an estimate for the minimum length of a train track, over all hyperbolic structures on its surface and all maps of the train track into the surface; this is not very exciting. It also applies to faithful representations of a surface group in an arbitrary homogeneous space.

What could be very useful would be the extension of this analysis to general incompressible branched surfaces in a 3-manifold which carries at least one surface with positive weights. The problem is that it does not seem possible to make a hyperbolic pleated surface to represent a branched surface which has vertices where branch curves cross. It can be given a
hyperbolic structure in the complement of such points, but the exponential map is no longer injective. Perhaps by estimating the entropy for the geodesic flow and correcting for the rate of occurrence of multiple counting because of geodesics which go in opposite ways around vertices where positive curvature is concentrated, an estimate could be worked out.

A related problem is to extend the growth rate estimates in terms of the length of the branch locus to the case of hyperbolic branched surfaces with geodesic boundary. The estimate should involve the length of the boundary, and degrade as the boundary becomes longer. In the case of a Hausdorff hyperbolic surface with geodesic boundary, the growth rate was studied by Patterson and Sullivan; the exponent of growth is the same as the Hausdorff dimension of the limit set for the group.

2. Relative injectivity of pleated surfaces

A 3-manifold which admits essential cylinders decomposes into its window, some miscellaneous solid tori, and an “acylindrical” part $\text{Acyl}(M)$. It is not really acylindrical, however, once the windows are removed: it is just that cylinders in $M$ cannot cross $\text{Acyl}(M)$ in an essential way.

To express this in a general way, let $N$ be a 3-manifold, $f : S \to N$ a compact surface mapped into $N$, and $X \subset S$ a system of non-trivial and homotopically distinct simple closed curves on $S$ including all components of $\partial(S)$. Then $(S, X, f)$ is \emph{incompressible} in $N$ if

(a): there is no compressing disk for $(S, f)$ whose boundary is a curve on $S$ which intersects $X$ in one or fewer points.

The triple $(S, X, f)$ is \emph{doubly incompressible} in $N$ if in addition

(b): there are no essential cylinders with boundary in $S - X$,

(c): there is no compressing disk for $(S, f)$ whose boundary is a curve on $S$ which intersects $X$ in two or fewer points, and

(d): Each maximal abelian subgroup of $\pi_1(S - X)$ is mapped to a maximal abelian subgroup of $\pi_1(N)$.

There is also a weaker form of (d); $(S, X, f)$ is \emph{weakly doubly incompressible} if it satisfies (a), (b), (c), and

(d1): Each maximal cyclic subgroup of $\pi_1(S - X)$ is mapped to a maximal cyclic subgroup of $\pi_1(N)$.

This weakening of (d) allows for simple closed curves on $S$ to be homotopic to a $(\mathbb{Z} + \mathbb{Z})$-cusp of $N$. The significance of this is that in geometric limits, cyclic subgroups can turn into $\mathbb{Z} + \mathbb{Z}$.

\begin{proposition} \label{prop:acylindrical} \textbf{(Acylindrical part doubly incompressible).} The triple $$(\text{Acyl}(M), \partial \text{wb}(M) \cap \text{Acyl}(M), \subset)$$ is doubly incompressible.
\end{proposition}

Compare \cite{Thu86}.

\begin{proof}[Proof of \ref{prop:acylindrical}] This is part of the basic theory of the characteristic submanifold. See \cite{John79} or \cite{JS79}.

We will extend the results of §5 of \cite{Thu86} to apply to doubly incompressible triples $(S, X, f)$.
\end{proof}
**Theorem 2.2** (Relative injectivity). Let $N$ be a hyperbolic 3-manifold, and $(S, X, f)$ weakly doubly incompressible in $N$. If $\lambda$ is any maximal lamination on $S$ containing all curves in $X$ as leaves, and if $f_\lambda : P_\lambda \to N$ is a $\lambda$-pleated surface, then $\lambda$ injects into $\mathbb{P}(N)$.

This remains true if degeneracies of the pleated surface are allowed, where closed curves of $\lambda$ map to cusps of $N$.

If the recurrent part of $\lambda$ consists only of closed curves, then at least a degenerate pleated surface always exists which represents $S \to f X$.

**Proof of 2.2.** Let $\rho$ be the recurrent part of $\lambda$. According to 5.5 of [Thu86], the map $\rho \to \mathbb{P}(N)$, restricted to non-degenerate leaves, is a covering map to its image, and it extends to a map on a small neighborhood of $\rho$ in $S$ which is a covering map to its image.

It follows easily from the weak double incompressibility of $(S, X)$ that this can only be the trivial covering, so that at least $\rho$ embeds in $\mathbb{P}(N)$: condition (d1) guarantees that a closed leaf can map only as a trivial covering to its image, condition (b) guarantees that no two closed leaves are mapped to a single closed leaf in the image, and condition (b) also prevents components of $\rho$ with more complicated neighborhoods to map by non-trivial coverings.

Each end of each leaf $l$ of $\lambda$ is asymptotic with either one or two leaves of $\rho$ — one if it is asymptotic with a closed leaf, two if it is asymptotic with a non-compact leaf. We will refer to these two types of ends as type 1 ends and type 2 ends.

Suppose $l_1$ has the same image in $\mathbb{P}(N)$ as $l_2$. Since $\rho$ embeds, it follows that $l_1$ and $l_2$ are asymptotic (on $S$) at both ends.

A closed loop can be formed, by bridging between the two leaves at their two ends along short arcs. If an end is of type 2, then the short arc will not intersect any closed leaves of $\rho$. If an end is of type 1, then the arc can be chosen so that it intersects a closed leaf of $\rho$ in at most one point. In particular, the closed loop on $S$ intersects $X$ in at most two points.

It follows from condition (c) that the image of the loop is null-homotopic in $N$. It follows from hyperbolic geometry that $l_1 = l_2$.

There remains still the possibility that a leaf $l$ of $\lambda - \rho$ could have image in $\mathbb{P}(N)$ which is a circle. Such a situation would force both ends of $l$ to be of type 2, since the set of identifications under $\lambda \to \mathbb{P}(N)$ is closed. This also forces the image of $l$ to coincide with the image of the circle at either end — which must be a single circle — and it forces the closed leaf to be non-degenerate. Construct a loop on $S$ which follows along $l$, crosses a short bridge to the closed leaf, then unwinds on the closed leaf, finally crossing a bridge to the other end of $l$. Since the amount of unwinding is adjustable, it can be chosen so that the entire configuration maps to a null-homotopic curve in $N$. This contradicts incompressibility, (a), since a small homotopy of the loop makes it intersect a closed leaf of $\lambda$ in at most one point.

To prove that a possibly degenerate pleated surface always exists provided $\rho$ consists only of closed leaves, we must show that each leaf $l$ of $\lambda - \rho$, when pushed forward to $N$ by a continuous map, can be straightened out to a geodesic without changing the asymptotic behavior of its ends. In $\mathbb{H}^3$, the closed leaves at either end of $l$ either are covered by geodesics, or they map to cusps. If one of the curves is parabolic and one hyperbolic, the endpoints are automatically distinct, so $l$ can be straightened. If the end curves are both hyperbolic, then it follows from incompressibility of $(S, X)$, as above, that the endpoints are distinct.

Finally, suppose that $l$ is asymptotic to parabolic curves at both ends. Map $l$ to $N$ and lift to $\mathbb{H}^3$. If the two endpoints are the same, then $l$ can be retracted into an arbitrarily small neighborhood of a cusp. Form a subsurface of $S$ from a neighborhood of $l$ together
with the circle or circles it is asymptotic to. Choose two non-commuting elements of the fundamental group of this surface which are represented by loops, such that one loop (say, parallel to one of the closed leaves) does not intersect \(X\), and the other loop intersects \(X\) in at most one point. (Make it from \(l\), with a short bridge between its ends if they are asymptotic, or a bridge to the second closed leaf, a traversal of this leaf, a bridge back to \(l\), and a return journey along \(l\) back to the basepoint, otherwise.) The commutator of the two loops is null-homotopic in \(N\); it intersects \(X\) in at most two points, violating (c).

**Theorem 2.3** (Relative uniform injectivity). Let \(S\) be a compact surface, and \(X\) a collection of non-trivial, homotopically distinct simple closed curves on \(S\) which includes all boundary components. Let \(B\) and \(\epsilon_0\) be positive constants. Among all pleated surfaces \(f : S \to N\) (\(N\) a variable hyperbolic manifold) pleated along laminations \(\lambda\) containing \(X\), where \((S, X, f)\) is doubly incompressible and the total length of \(X\) in \(N\) is less than \(B\), the associated maps

\[
g : \lambda \to \mathbb{P}(N)
\]

are uniformly injective on the \(\epsilon_0\)-thick part of \(S\). That is, for every \(\epsilon > 0\) there is a \(\delta > 0\) such that for any such \(S, N, \lambda\) and \(f\) and for any two points \(x, y \in \lambda\) whose injectivity radii are greater than \(\epsilon_0\), if \(d(x, y) \geq \epsilon\) then \(d(g(x), g(y)) \geq \delta\).

The same statement holds true when degenerate pleated surfaces are allowed, in which certain closed leaves of \(\lambda\) may be parabolic.

The uniform injectivity theorem of [Thu86] is the special case of this when \(X = \partial S\), and where the constant \(B = 0\) — that is, where any boundary curves are parabolic.

**Proof of 2.3.** As in [Thu86], the main step of the proof is to establish that the geometric limits of surfaces which satisfy the hypotheses are at least weakly doubly incompressible.

Consider a sequence \(f_i : S \to N_i\) of pleated surfaces, pleated along \(\lambda_i \supset X\), such that \((S, X, f_i)\) are doubly incompressible, and for which the total length of the curves of \(X\) in \(N_i\) is less than \(B\). Let \(g_i\) be the metric induced on \(S\) by \(f_i\).

For each \(i\), let \(B_i\) be a collection of points, one in each component of the \(\epsilon_0\)-thin part of \(S\) with respect to \(g_i\), and let \(E_i\) be a collection of orthonormal frames at the elements of \(B_i\).

There is some subsequence such that sequence of hyperbolic surfaces \(S_i\) defined by the metric \(g_i\) on \(S\) with respect to the collection of base frames \(E_i\) converges to a geometric limit. By this we mean that, first, if the universal cover of \(S\) is developed into \(\mathbb{H}^2\), where any of the frames \(e_i \in E_i\) is sent to some fixed base frame in \(\text{hy}^2\), then the sequence of images of \(\pi_1(S)\) converge in the Hausdorff topology for \(\text{Isom hy}^2\): this is the geometric limit from the point of view of \(e_i\). Second, the Hausdorff limit of the image of \(E_i\) from the point of view of \(e_i\) must exist, that is, any other frames which stay within a bounded distance of \(e_i\) should converge. When these conditions are met, there is a well-defined, possibly disconnected, geometric limit with respect to \(E_i\).

We can pass to a further subsequence so that the system of geodesics \(X\) and the laminations \(\lambda_i\) also converges geometrically, that is, in the Hausdorff topology when they are developed onto \(\mathbb{H}^2\) using frames in \(E_i\) to get started.

Let \(R'\) be the topological surface which is the geometric limit. On \(R'\), denote the limiting hyperbolic metric \(g'\), curve system \(Y'\) and lamination \(\lambda'\).

Each component of \(S - X\) is incompressible in \(N_i\). Since the boundary components of each such component have bounded length with respect to \(g_i\), there is a non-elementary subgroup of \(\pi_1(S) - X\) based at any point \(x \in B_i\) generated by loops through \(x\) of bounded
length. Therefore, the injectivity radius of $N_i$ at the image of $x$ is bounded above zero. The image of each $e_i \in E_i$ in $N_i$ can be extended uniquely to an orthonormal frame $f_i$ in $N_i$, and there is a further subsequence so that the $N_i$ with collection of base frames \{f\}, converges geometrically, to a hyperbolic manifold $N$ and so that the maps $f_i : S \to N_i$ also converge geometrically to a $\lambda'$-pleated surface $f' : R' \to N$.

It may be that there are pairs of cusps of $R'$ which came from the two sides of a sequence of geodesics on $S$ which either were degenerate, or grew shorter and shorter in the sequence. Form a new topological surface $R$ by joining the two cusps together along a simple closed curve; we can think of $f$ as a degenerate pleated surface for $R$, if we extend $\lambda$ by adding a closed leaf for each parabolic curve we adjoin, and spinning the ends of leaves which tend to these cusps around the new closed leaves. Label the new curves as belonging to $Y$ according to whether they were limits of elements of $X$, possibly after passing to a subsequence. Pick a homotopy class $f$ of maps of the new surface $R$ into $N$, which agrees with the previous homotopy class $f'$ on $R'$. For each degenerate curve, this involves a free choice of a power of the Dehn twist about the curve, and if the cusp is a $\mathbb{Z} + \mathbb{Z}$ cusp, the choice of how the annulus wraps around the torus.

We verify the conditions for double incompressibility of the limiting pleated surface, with curve system $Y$.

Condition (a) is contained in condition (c) that there are no compressing disks which intersect $Y$ in two or fewer points. Suppose there were an essential disk for $f : R \to N$ with boundary a curve on $R$ meeting $Y$ in two or fewer points. The map $f$ is approximated by a map of $R$ to the $N_i$, for sufficiently high $i$. The approximation at least restricted to $R'$ factors (up to homotopy) through an embedding $j_i$ of $R$ as a subsurface of $S$ with incompressible boundary. This factorization extends over the degenerate curves of $R$ as well, since the annulus of $R$ and the annulus of $S$ both map into the thin set of $N_i$; the fundamental group of this component of the thin set is cyclic, and generated by the core curve of the annulus, so the two annuli must be homotopic rel boundary. The compressing disk would be approximated by a compressing disk for $S$ whose boundary intersects $X$ in only two points, contradicting the hypothesis that $(S, X, f_i)$ is doubly incompressible.

Condition (b) says that there should be no essential annulus for $f : R \to N$ whose boundary is disjoint from $X$. Any essential annulus would be approximable by an annulus for $f_i : S \to N$, for $i$ sufficiently high. We need to check that the approximating annulus is essential, that is, that its two boundary components are not homotopic on $S$. If the length of either of the boundary curves on $R$ is greater than zero, then that boundary curve can be represented by a geodesic on $R$. Two geodesics are isotopic on $R$ if and only if they coincide; the same criterion carries over to the approximating surfaces, so an essential annulus which has a boundary curve of positive length would carry over to an essential annulus for the approximations, contradicting the hypotheses. If there were an essential annulus whose boundary components have zero length on $R$, then when it is carried over to the approximations, it would still essential for otherwise we would have added a degenerate curve joining the two cusps in question when we formed $R$. Again, this contradicts the hypothesis that $(S, X, f_i)$ is doubly incompressible.

Condition (d1) also persists in a geometric limit. Let $H$ be a maximal cyclic subgroup of the fundamental group of $R - Y$. The embedding $j_i$ of $R$ as an incompressible subsurface of $S$ for high $i$ carries $H$ to a maximal cyclic subgroup of $\pi_1(S - X)$. Therefore, its image is
a maximal cyclic subgroup of $\pi_1(N)_i$. If $\alpha$ is any element of $\pi_1(N)$ such that some power is a cyclic generator of the image of $H$ in $\pi_1(N)$, then $\alpha$ is approximated in $N_i$ by an element with a power approximately, and therefore exactly, equal to the cyclic generator of the image of $H$ in $N_i$; the power can only be 1.

We have established that any sequence of pleated surfaces satisfying the hypotheses of the theorem, has a subsequence converging to a weakly doubly incompressible pleated surface in a hyperbolic 3-manifold. Theorem 2.3 follows now from 2.2.

3. Relative boundness for $AH(M)$

The relative uniform injectivity theorem of the last section has a direct application to the boundness of deformations of a hyperbolic 3-manifold when the total length of a sufficiently complicated system of curves on its boundary is held bounded.

**Theorem 3.1** (Relative boundedness). Let $M$ be a 3-manifold, and $X$ a collection of non-trivial, homotopically distinct curves on $\partial M$ such that $X, \subset \partial M$ is doubly incompressible. Then for any constant $A > 0$, the subset of $AH(M)$ such that the total length of $X$ not exceeding $A$ is compact.

Note that by setting $X = \emptyset$ we obtain 1.2 of [Thu86]. By letting $X$ be the set of core curves of the parabolic annuli of a pared manifold and setting $A = 0$ we obtain 7.1 of [Thu86].

**Proof of 3.1.** The proof is much the same as the proof of the main theorem of [Thu86], but we will go through the details for the sake of completeness.

We may as well assume that $\partial M$ is non-empty. Let $Z \supset X$ be a collection of curves which has at least one element on each boundary component of $M$. Choose a triangulation $\tau$ of $M$, with one vertex on each element of $Z$ and such that each element of $Z$ is formed by one edge of $\tau \cap \partial M$. Choose an orientation for each element of $Z$.

Denote the closed unit ball $\mathbb{H}^3 \cup \mathbb{S}^2_\infty = \mathbb{H}^3$. For any element $N$ of $AH(M)$, an ideal simplicial map $f_N : M \to N$ can be defined in the standard way: let $\tilde{f}_N : \tilde{M} \to \tilde{N}$ by map a vertex $v$ of $\tilde{\tau}$ to the positive fixed point at infinity for the covering transformation which generates forward motion along the component of $v$ on $\tilde{Z}$, and extend $f_N$ to a piecewise-straight map.

There is a canonical factorization of $f_N|\partial M$ as an ideal simplicial map $i_N$ to a possibly degenerate hyperbolic structure on $\partial M$, followed by a possibly degenerate pleated surface $p_N$. The hyperbolic structure $g(N)$ on $\partial M$ is a metric on $\partial M$ minus any torus components, and possibly with certain curves deleted whose two sides become cusps in the hyperbolic structure.

Associated with $f_N$ is a certain infinity subcomplex $\iota_N$ of $\tau$, consisting of those simplices such that any copy of them in $\tilde{M}$ is mapped to a single point by $\tilde{f}_N$. The degenerate simplices for an ideal simplicial map are those simplices of which at least one edge is contained in $\iota$. Thus, a degenerate triangle collapses either to a line, or to a point at infinity. A degenerate 3-simplex collapses to an ideal triangle, a line, or a point. A 3-simplex is not called degenerate if it merely flattens to a quadrilateral, or maps with reversed orientation.

Let $N(i)$ be any infinite sequence of elements of $AH(M)$ such that the total length of $X$ is bounded by $A$. We will extract a convergent subsequence. For notational simplicity, each time we pass to a subsequence, we implicitly relabel it by the full sequence of positive integers.
First, observe that $\tau_{N(i)} = \tau$ does not depend on $i$, for it can be determined homotopically: a simplex is in $\tau$ if and only if it is homotopic, rel vertices, either to a torus boundary component of $M$, or to a curve in the collection $Z$.

Recall from [Thu79], [Thu97] or [Thu86] that to each edge of each non-degenerate simplex is associated an edge invariant in $\mathbb{C} - \{0, 1, \infty\}$. The edge invariants of opposite edges of a three-simplex are equal, and the three values tend to 0, 1 and $\infty$ if the shape of the simplex degenerates. We may pass to a subsequence of $N(i)$ such that the edge invariants of each edge of each nondegenerate 3-simplex of $\tau$ converge in $\hat{\mathbb{C}}$.

Following §3 of [Thu86], we define a “bad complex” $B$ for the sequence $N(i)$, which is formed as the union of all 1-simplices (which are bad only because they do not yield information), all degenerate simplices, and within each degenerating 3-simplex, a quadrilateral attached along the four edges whose edge invariants are tending to 0 and $\infty$. The good submanifold $G$ for the sequence $N(i)$ is obtained by deleting a regular neighborhood of $B$, and the “cutting surface” $\kappa$ for $\{N\}(i)$ is defined as the internal boundary of $G$, $\kappa = \partial G - \partial M$.

An ideal triangle in $N(i)$ serves like a base frame: a certain “view” of $\tilde{\mathcal{N}}(i)$ is obtained by sending its three vertices to 0, 1, and $\infty$. If $\alpha$ is any path in $G$ beginning and ending on a triangle of $\tau$, it defines a sequence of views of $\tilde{\mathcal{N}}(i)$, which are related by a sequence of isometries of $\mathbb{H}^3$. The submanifold $G$ was defined in such a way that this sequence of isometries converges. Note in particular that when a path in $G$ enters a degenerating 3-simplex through one triangle and exits through the other accessible triangle, the isometry that relates the two views converges to the identity. Therefore, the sequence of representations $\rho(i)$ for $\pi_1(M)$ based at any point $p \in G$ converges when restricted to $\pi_1((G, p))$.

As in the earlier paper, we will augment $G$ together with its map to $M$, in several steps $g_i : G_i \to M$, in such a way that the sequence of composed representations $G_i \to M \to \text{Isom}(\mathbb{H}^3)$ continues to converge, until finally the map to $M$ is a proper map (one which takes boundary to boundary) of degree 1. We start with $G_0$ be $G$, and $g_0$ the inclusion.

The key point is that the surface $\kappa$ is in a certain sense becoming smaller and smaller in $N(i)$, as the sequence progresses, so that we can think of it as if it were only 1-dimensional. If we proceed far out in the sequence, we can homotope the map of $\kappa$ to $N(i)$ so that it lies close to the 1-skeleton of the image ideal 3-simplices, so that the area of the image of $\kappa$ tends to zero, and it geometrically it looks like a 1-complex. In fact, we will analyze in terms of a map to a certain graph $\gamma$ defined by this geometry.

Toward this end, define a subsurface $\kappa_0 \subset \kappa$ to consist of the intersection of a regular neighborhood of $\iota$ with $\kappa$, together with a diagonal band on each side of the twisted quadrilateral of $B$. The diagonal bands are to be arranged to “indicate” vertices of ideal simplices which are converging together, from the point of view of its component of $G\cap$ the simplex; the two diagonal bands on opposite sides of a twisted quadrilateral of $B$ run in different directions. Each component of $\kappa_0$ will be collapsed to a point to form the vertices of $\gamma$.

The intersection of $\kappa - \kappa_0$ with any 3-simplex is a rectangle, with two opposite sides on 2-faces of the simplex. Form a foliation $F$ transverse of $\kappa - \kappa_0$ so that $\partial \kappa_0 - \partial M$ consists of leaves of $F$, and the leaves of $F$ are transverse to the non-degenerate triangles of $\tau$, by foliating each rectangle with a product foliation. The leaf space of any rectangle is homeomorphic to the unit interval $[0, 1]$. The parametrizations can be chosen consistently, so that the leaf space of a rectangle is identified by an isometry to the leaf space of an adjoining rectangle. When this is done, the leaves of $F$ are all compact, since a leaf enters a rectangle at most once.
The graph $\gamma$ is defined by collapsing each component of $\kappa_0$ to a point, and each leaf of $F$ to a point. Let $p : \kappa \to \gamma$ be the collapsing map.

Proposition 3.7 of [Thu86] shows (by an easy argument) that the fundamental group of $p^{-1}$ of any edge or vertex of $\gamma$ has image in $\pi_1(M)$ which is abelian. Define $\gamma_0$ to consist of those cells $\beta$ of $\gamma$ such whose associated group (the image in $\pi_1(M)$ of $\pi_1(p^{-1}(\beta))$) is nontrivial. Proposition 3.8 of [Thu86] concludes that the image of the fundamental group of $p^{-1}$ of any component of $\gamma_0$ in $\pi_1(M)$ is abelian: this is deduced from the nature of commutative subgroups of Isom($\mathbb{H}^3$).

Let $\gamma_1 = \gamma - \gamma_0$. An edge $e$ of $\gamma_1$ has one of two types: either $p^{-1}(e)$ is a cylinder, or it is a rectangle. Form the first enlargement $G_0$ of $G_0 = G$ by attaching a 2-handle to each cylinder of the form $p^{-1}(e)$, $e$ an edge of $G_1$. The extension $g_1$ of $g_0$ over $G_1 - G_0$ is easy to make over the 2-handles, since their attaching curves are null-homotopic in $M$.

The next enlargement $G_2$ of $G_1$ is formed by attaching a semi-2-handle to each rectangle of the form $p^{-1}(e)$, for all edges $e$ in $\gamma_1$. By definition, a semi-2-handles is a copy of $D^2 \times [0, 1]$, where $D^2$ is the intersection of the unit disk in $\mathbb{R}^2$ with the closed upper-half-plane. It is attached along the round part of its boundary to the rectangle in question. We need to make the extension over the semi-2-handles in such a way that $[-1, 1] \times [0, 1]$ maps to $\partial M$.

The ends of the leaves of the foliation $F$ in $p^{-1}(e)$ are on $\partial M$, near two edges of $G_1$ triangles. These leaves define a sequence of views of changes of viewpoint in $N(i)$, whose composition converges to a transformation sending the triangle at one end to the triangle at the other end, while preserving the edge which is near the leaves. If we use the factorization of $f_{N(i)}|\partial M$ as $p_{N(i)} \circ i_{N(i)}$, we see that the corresponding triangles of $\partial M - \lambda$ are mapped by the $\lambda$-pleated map $p_{N(i)}$ close to each other in $N$. Since the change of view between the two triangles converges as $i \to \infty$ and since the thick part of an ideal triangle of $\partial M - \lambda$ is always in the thick part of $\partial M$, we can choose $\epsilon_0$ sufficiently small that two points $x_1$ and $x_2$ on these two leaves and in the $\epsilon_0$-thick part of $\partial M$ (with respect to $g(N(i))$) are mapped in $\mathbb{P}N(i)$ so their distance goes to zero with $i$.

We can now apply Theorem 2.3, to conclude that the distance between $x_1$ and $x_2$ goes to zero on $\partial M$ as measured by the metric $g(N(i))$. In particular, for each sufficiently high $i$, there is a well-defined homotopy class of arcs joining $x_1$ to $x_2$ on $\partial M$, and this arc is homotopic to the arc joining $x_1$ to $x_2$ on $\kappa - \kappa_0$.

Does the homotopy class depend on $i$? Since the total length of $X$ is bounded by $A$, no component of $X$ can pass between the leaf of $x_1$ and the leaf of $x_2$ while they are close together — these leaves are infinite, and they remain close for a long time, if $i$ is high. The arc $\alpha(i)$ joining $x_1$ with $x_2$ for $N(i)$ does not cross any component of $X$. The loops $\alpha(j) \cup \alpha(i)$ are null-homotopic in $M_i$, so they must also be null-homotopic on $\partial M$, by the incompressibility of $(\partial M, X, \subset)$. Therefore, the homotopy class $\alpha(i)$ is independent of $i$.

Map the semi-2-handle to $M$ so that the portion $[-1, 1] \times [0, 1]$ of its boundary covers a regular neighborhood of $\alpha$, to obtain the extension $g_2$ of $g_1$ over $G_2$.

Define $\kappa_2$ to be the portion of $\partial G_2$ which we have not mapped to $\partial M$. There is one component of $\kappa_2$ for each component of $\gamma_0$ union the vertices of $\gamma$ — the 2-handles and semi-2-handles have served to sever $\gamma$ along the edges associated to a trivial group. The image of the fundamental group of any component of $\kappa_2$ in $\pi_1(M)$ is abelian. Furthermore, $\partial \kappa_2$ does not intersect $X$.

We form $G_3$ by attaching three-manifolds to the components of $\kappa_2$. A component $C$ of $\kappa_2$ has one of three types. First, it may be a closed surface whose fundamental group has
image in $\pi_1(M)$ isomorphic to $\mathbb{Z} + \mathbb{Z}$. In this case, it is homologous to a cusp, and we can attach a three-manifold $M_C$ whose boundary is $C \cup T^2$, with a map to $M$ having the same image fundamental group, and realizing this homology. Second, it may be a closed surface whose fundamental group has image isomorphic to $\mathbb{Z}$. From the acylindricity of $(\partial M, X, \subset)$ it follows that $C$ is homologous to in $H_2(M, \partial M)$ to $\partial M$, by a 3-chain represented by a manifold whose image fundamental group is also $\mathbb{Z}$. We attach such a manifold $M_C$.

Now we have $G_3$ with a proper map $g_3 : G_3 \to M$. We will analyze the geometry of $\partial g_3 : \partial G_3 \to \partial M$ in conjunction with the possibly degenerate hyperbolic metric $g_{N(i)}$ on $\partial M$ and the pleated map $p_{N(i)}$ to $N(i)$ to show that $\partial g_3$ and hence $g_3$ have degree one.

The original good submanifold $G = G_0$ intersects every non-degenerate triangle of $\tau$ in the complement of a regular neighborhood of its boundary. We may isotope so that this regular neighborhood is as small as we like. The intersection of $G$ with $\partial M$ then takes up most of the area of the possibly degenerate hyperbolic structure for $\partial M$.

The boundary of $G_1$ was exactly the same as the boundary of $G_0$. The boundary behavior of $G_2$ was modified, on account of the semi-2-handles. These had the effect of joining sides of ideal triangles in pairs, which by the relative uniform injectivity theorem were close and nearly parallel. Only small bridges were added between sides of triangles, so the map of $\partial G_2$ to $\partial M$ has degree one at most image points, as measured by the metric $g_{N(i)}$.

The boundary curves of a component $C$ of $\kappa_2$ are formed by a sequence of segments which cycle through the following types:

(i): a long stretch inside an ideal triangle and nearly parallel to one of its sides,
(ii): a jump and U-turn to another triangle,
(iii): a long stretch in the reverse direction nearly parallel to one of its sides, and
(iv): a U-turn near a vertex of the triangle.

These curves can be chosen so that they are geodesics except at the U-turns, where they have geodesic curvature alternately $\pm \pi$. Therefore, their net geodesic curvature is small.

If $C$ is of the type which actually intersects $\partial M$, it lifts to a cover of $M$ with infinite cyclic fundamental group. Each component of $\partial C$ is homologous to a standard representative on $\partial M$: contained in a geodesic, if there is a geodesic representing $\mathbb{Z}$, otherwise contained in one of the curves where the metric degenerates. If we knew that the boundary components of $C$ were simple curves on $\partial M$, or if we knew that they lifted to simple curves in the $\mathbb{Z}$ cover, we could conclude that each of the curves of $\partial C$ is homologous to a standard representative of its class with a chain of small net area, by an application of the Gauss-Bonnet theorem.

However, the boundary components of $C$ are not necessarily simple curves, and the form of the Gauss Bonnet theorem which gives a formula for the area of chains on a surface with boundary a non-embedded union of curves must involve information about the amount of topological turning of the boundary curves.

One solution to this difficulty is given in §6 of [Thu86], by constructing a geometric limit of the gluing patterns for the triangles of $\partial M - \lambda$, and observing that the curves of $\partial C$ are simple curves on the limit surface. Here is an alternate method:

Each curve $\alpha$ of $\partial C$ is embedded except in the short bridges from triangle to triangle. These bridges cannot cross the thick part of any other ideal triangle, so they may intersect other portions of $\alpha$ only when it is near a vertex of some ideal triangle. Truncate each ideal
triangle in a neighborhood of each vertex, before constructing $\alpha$, making sure to cut off a large enough neighborhood so that it does not pass under any of the bridges. Now, when $\alpha$ is constructed, it is embedded.

It follows that the part of $\partial M_C$ which is mapped to $\partial M$ has net area near 0. The degree of $g_3|\partial G_3$ is an integer $\approx 1$, therefore $= 1$. Some component of $G_3$ must then map with positive degree. The sequence of representations restricted to the image of the fundamental group of any component of $G_3$ converges, by construction, so it follows from an application of 4.3 of [Thu86] that the sequence of representations of $\pi_1(M)$ converges, and hence that the sequence $N(i)$ converges to a limit in $AH(M)$.

4. Proof that only windows break

We can now obtain the main theorem by logically assembling the previous results. Here is the statement once more:

**Theorem 0.1.** If $\Gamma \subset \pi_1(M)$ is any subgroup which is conjugate to the fundamental group of a component of $M - \text{window}(M, P)$, then the set of representations of $\Gamma$ in $\text{Isom}(\mathbb{H}^3)$ induced from $AH(M, P)$ are bounded, up to conjugacy.

Proof of 0.1. Let $M$ be a 3-manifold with incompressible boundary. Consider any sequence \(\{N_i\} \in AH(M, P)\) Theorem 1.3 asserts that the total length of $\partial wb(M, P)$ is uniformly bounded.

Proposition 2.1 tells us that the triple

\[(\text{Acyl}(M), \partial wb(M) \cap \text{Acyl}(M), \subset)\]

is doubly incompressible. Theorem 3.1 implies that the image of the restriction

\[AH(M) \to AH(\text{Acyl}(M))\]

has a compact closure.

The analogous restriction map for any solid torus component of $M - \text{window}(M)$ (which is not a part of $\text{Acyl}(M)$) also has a compact closure, since this is just a question of the boundedness of the length of a component of $\partial wb(M)$.

This proves the first paragraph of the theorem. For the second paragraph, we apply Theorem 6.2 of [Thu98] to find a subsequence such that each for each component $S$ of $\partial M$, there is a subsurface $c(S) \subset S$ with incompressible boundary, such that

(a): For each component $c(S)_i$ of $c(S)$, the sequence of representations of its fundamental group converges up to conjugacy.

(b): If $\Gamma$ is any nontrivial subgroup of $\pi_1(c(S))$ such that for some subsequence of the subsequence, its sequence of representations converges up to conjugacy, then $\Gamma$ is conjugate to a subgroup of $\pi_1(c(S)_i)$ for some $i$.

It follows from (b) that the subsurface $c(S)$, up to isotopy, includes $S_1 = S - \text{window}(M) \cap S$. The window is “transparent”, so that any closed curve in $c(S) - S_1$ can be pushed
through to the opposite side of the window, giving a homotopy class of curves on a boundary component $S'$ of $M$ which must be contained in $c(S')$. Also, the sequence of representations of $\pi_1(M \cup c(S))$ must converge; this intersects the fundamental group of $\partial M$ based on the opposite side of a window in a large subgroup. These constraints, along with lemma 4.3 of [Thu86], yield the description given in the second paragraph of the statement.

\begin{thebibliography}{99}

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