Nonlinear quantum Rabi model in trapped ions

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We study the nonlinear dynamics of trapped-ion models far away from the Lamb-Dicke regime. This nonlinearity induces a blockade on the propagation of quantum information along the Hilbert space of the Jaynes-Cummings and quantum Rabi models. We propose to use this blockade as a resource for the dissipative generation of high-number Fock states. Also, we compare the linear and nonlinear cases of the quantum Rabi model in the ultrastrong and deep strong coupling regimes. Moreover, we propose a scheme to simulate the nonlinear quantum Rabi model in all coupling regimes. This can be done via off-resonant nonlinear red and blue sideband interactions in a single trapped ion, yielding applications as a dynamical quantum filter.

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I. INTRODUCTION

Proposed in 1936 by I. I. Rabi [1], the most fundamental interaction between a two-level atom and a classical light field, the semiclassical Rabi model, has played an important role in both physics and mathematics [2]. Under the rotating wave approximation (RWA), its fully quantized form, the quantum Rabi model (QRM), can be reduced into the Jaynes-Cummings model (JCM) [3], which is analytically solvable [5]. This model describes the basic interaction in trapped ions [6], superconducting circuits [7], and cavity quantum electrodynamics [8], when the systems are in the regime where the ratio of coupling strength $g$ and mode frequency $\nu$ is approximately smaller than $0.1$ [9]. On the other hand, in the ultrastrong coupling (USC) regime, $g/\nu \in (0, 1)$ [10, 11], and deep strong coupling (DSC) regime ($g/\nu > 1$) [12–14], we have to take the counter rotating term that is neglected in the JCM into account. The QRM is a fruitful physical model with applications in condensed matter, quantum optics, and quantum information processing. In fact, the QRM has been investigated in many contexts, such as quantum phase transitions [15, 16], dissipative QRM [18], generalized QRM [19–22], multiparticle QRM [23–26], and quantum thermodynamics [27], among others. Furthermore, proposals and experimental realizations of the QRM in different quantum simulators as optical lattices [28], circuit QED [29], as well as trapped ions [30, 31] have been put forward. Reference [30] introduced an analog method for the simulation of different regimes of the QRM, otherwise inaccessible to experimentation from first principles. These ideas have been recently demonstrated in the lab [31].

As one of the most controllable quantum systems, trapped ions play an important role in diverse proposals for quantum simulations [30, 31, 34, 17]. However, most of these works are based on the condition for the system to be in the Lamb-Dicke (LD) regime. In this regime, the size of the motional wavepacket of the ion is much smaller than the wavelength of the external laser driving, such that the effective coupling between the internal and the vibrational degrees of freedom, generated by the laser field, can be approximated to first order [6]. This condition can also be expressed as $\eta \sqrt{(a + a^\dagger)^2} \ll 1$, where $a^\dagger(a)$ is the creation (annihilation) operator associated to the quantum vibrational mode of the ion on a certain direction $x$ and $\eta = k\sqrt{\hbar/2M\nu}$ is the LD parameter in this direction, with $k$ the wave number of the external laser field, $M$ the mass of the ion and $\nu$ the frequency of the harmonic potential.

In this article, we study the nonlinear behaviour of a single trapped ion when it is far away from the LD regime. In the past, research beyond the LD regime was mainly focussed on the nonlinear JCM [18–51], but has also been studied for its implications in laser cooling [52–54] or for its possible applications to simulate Frack-Condon physics [55]. To set up the stage for a subsequent analysis, we first briefly review the JCM and take this as a reference to show the difference with the nonlinear JCM. The appearance of nonlinear terms in the Hamiltonian suppresses the collapses and revivals for a coherent state [12–14, 55] or for its possible applications to simulate Frack-Condon states [55]. To set up the stage for a subsequent analysis, we first briefly review the JCM and take this as a reference to show the difference with the nonlinear JCM. The appearance of nonlinear terms in the Hamiltonian suppresses the collapses and revivals for a coherent state [12–14, 55] or for its possible applications to simulate Frack-Condon states [55]. The latter could in principle be done without a precise control of pulse duration or shape, and without the requirement of a previous high-fidelity preparation of the motional ground state. Furthermore, we propose the quantum simulation of the nonlinear quantum Rabi model by simultaneous off-resonant nonlinear Jaynes-Cummings and anti-Jaynes-Cummings interactions. Finally, we also point out the possibility for the quantum Rabi model to act as a motional state filter.

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II. JAYNES-CUMMINGS MODELS IN TRAPPED IONS

The Hamiltonian describing a laser-cooled two-level ion trapped in a harmonic potential and driven by a monochromatic laser field can be expressed as ($\hbar = 1$)

$$H = \frac{\omega_0}{2} \sigma_z + \nu a^\dagger a + \frac{\Omega}{2} \sigma_x [e^{i(\eta a^\dagger a + \omega_0 t + \phi)} + \text{H.c.}],$$

(1)

where $\omega_0$ is the two-level transition frequency, $\sigma_z, \sigma_x$ are Pauli matrices associated to this two-level system, $\Omega$ is the Rabi frequency, $\omega$ is the driving laser frequency, and $\phi$ is the phase of the laser field.

In the Lamb-Dicke regime, moving to an interaction picture with respect to $H_0 = \frac{\omega_0}{2} \sigma_z + \nu a^\dagger a$, and after the application of the so-called optical RWA, the Hamiltonian in Eq. (1) can be written as

$$H_{\text{int}}^{\text{LD}} = \frac{\Omega}{2} \sigma_x [1 + i\eta (ae^{i\omega t} + a^\dagger e^{-i\omega t})] e^{i(\phi - \delta t)} + \text{H.c.},$$

(2)

where $\delta = \omega - \omega_0$ is the laser detuning and the condition $\eta \ll 1$ allows to keep only zero and first order terms in the expansion of $\exp[i\eta (a + a^\dagger)]$. When $\delta = -\nu$ and $\Omega \ll \nu$, after applying the vibrational RWA, the dynamics of such a system is described by Jaynes-Cummings Hamiltonian, $H_{\text{JC}} = ig(\sigma^+ a - \sigma^- a^\dagger)$, where $g = \eta \Omega/2$ and $\phi = 0$. This JCM is analytically solvable and generates population exchange between states $|\downarrow, n\rangle \leftrightarrow |\uparrow, n-1\rangle$ with rate $\Omega_{n,n-1} = \eta \Omega \sqrt{n}$. On the other hand, when the detuning is chosen to be $\delta = \nu$, the effective model is instead described by the anti-JCM $H_{\text{aJC}} = ig(\sigma^+ a^\dagger - \sigma^- a)$, which generates population transfer between states $|\downarrow, n\rangle \leftrightarrow |\uparrow, n+1\rangle$ with rate $\Omega_{n,n+1} = \eta \Omega \sqrt{n+1}$.

When the trapped-ion system is beyond the Lamb-Dicke regime, the simplification of the exponential term described above is not justified and Eq. (2) reads

$$H_{\text{int}} = \frac{\Omega}{2} \sigma_x e^{i\eta (ae^{i\omega t} + a^\dagger e^{-i\omega t}) - i\delta t} + \text{H.c.},$$

(3)

When $\delta = -\nu$ and $\Omega \ll \nu$, after applying the vibrational RWA, the effective Hamiltonian describing the system is given by the nonlinear Jaynes-Cummings model [LS], which can be expressed as

$$H_{\text{nJC}} = ig(\sigma^+ f_1(\hat{n})a - \sigma^- a^\dagger f_1(\hat{n})],$$

(4)

where the nonlinear function $f_1$ [LS] is given by

$$f_1(\hat{n}) = e^{-\eta^2/2} \sum_{l=0}^{\infty} \frac{(-\eta^2)^l}{l!(l+1)!} \hat{n}^l a^l,$$

(5)

with $a^l a^\dagger = \hat{n}!/(\hat{n} - l)!$. The dynamics of this model can also be solved analytically, and as the linear JCM, yields to population exchange between states $|\downarrow, n\rangle \leftrightarrow |\uparrow, n-1\rangle$, but in this case with a rate $\tilde{\Omega}_{n,n-1} = |f_1(n-1)|\Omega_{n,n-1} = \eta \Omega \sqrt{n} f_1(n-1)|$, where $f_1(n)$ corresponds to the value of the diagonal operator $f_1$ evaluated on the Fock state $|n\rangle$, i.e., $f_1(n) = \langle f_1(\hat{n}) \rangle_n$. If the detuning in Eq. (3) is chosen to be $\delta = \nu$, and $\Omega \ll \nu$, then the application of the vibrational RWA yields the nonlinear anti-JCM,

$$H_{\text{nJC}} = ig(\sigma^+ a^\dagger f_1(\hat{n}) - \sigma^- f_1(\hat{n}) a],$$

(6)

which, as the linear anti-JCM, generates population exchange between states $|\downarrow, n\rangle \leftrightarrow |\uparrow, n+1\rangle$ with rate $\tilde{\Omega}_{n,n+1} = |f_1(n)|\Omega_{n,n+1} = \eta \Omega \sqrt{n+1} |f_1(n)|$. The nonlinear function $f_1$ depends on the LD parameter $\eta$ and on the Fock state $|n\rangle$ on which it is acting. The LD regime is then recovered when $\eta \sqrt{n} \ll 1$. In this regime, $\langle f_1(n) \rangle_n \approx 1$, and thus the dynamics are the ones that correspond to the linear models.

Beyond the LD regime the nonlinear function $f_1$, which has an oscillatory behaviour both in $n \in \mathbb{N}$ and $\eta \in \mathbb{R}$, needs to be taken into account. In Fig. 1, we plot the logarithm of the absolute value of $f_1(n, \eta)$ for different values of $n$ and $\eta$, where the green regions represent lower values of log $(|f_1(n, \eta)|)$, i.e., values for which $f_1 \approx 0$. This oscillatory behaviour can also be seen in Fig. 2, where we plot the value of $f_1$ as a function of the Fock state number $n$ for $\eta = 0.5$. For this specific case, we can see that the function is close to zero around $n = 14$ and $n = 48$, meaning that for $\eta = 0.5$, the rate of population exchange between $|\downarrow, 15\rangle \leftrightarrow |\uparrow, 14\rangle$ and $|\downarrow, 49\rangle \leftrightarrow |\uparrow, 48\rangle$ states on the nonlinear JCM will vanish. The same will happen to the exchange rate between $|\downarrow, 14\rangle \leftrightarrow |\uparrow, 15\rangle$ and $|\downarrow, 48\rangle \leftrightarrow |\uparrow, 49\rangle$ states for the nonlinear anti-JCM.
We observe approximate collapses and revivals for an initial coherent state with an average number of photons of $|\alpha|^2 = 30$ by evolving with the JCM, as shown in Ref. [56], see Fig. 2a. Here, we plot the initial coherent state with an average number of photons $|\alpha|^2 = 30$ by evolving with the JCM, as shown in Ref. [56], see Fig. 2a. Here, we plot the average of $\sigma_z$ versus time for a coherent initial state $|\alpha = \sqrt{30}\rangle$ after (a) linear JC and (b) nonlinear JC evolution, both with the same coupling strength $g$ and $\eta = 0.5$ for the nonlinear case. As shown in (a), there exists an approximate collapse and subsequent revival in the JCM dynamics, while for the nonlinear JCM this is not the case.

In other words, one can obtain large final Fock states from state $|\downarrow, 0\rangle$ to state $|\uparrow, 1\rangle$, while at the same time, the depolarizing noise transfers population from $|\uparrow, 1\rangle$ to $|\downarrow, 1\rangle$. The simultaneous action of both processes will “heat” the motional state, progressively transferring the population of the system from one Fock state to the next one. Eventually, all the population will be accumulated in state $|\downarrow, n\rangle$, where a blockade of the propagation of population through the chain of Fock states occurs, if $f_1(n) = 0$, as the transfer rate between states $|\downarrow, n\rangle$ and $|\uparrow, n + 1\rangle$ vanishes, $\Gamma_{n,n+1} = 0$. We point out that the condition $f_1(n) = 0$ can always be achieved by tuning the LD parameter to a suitable value, i.e. for every Fock state $|n\rangle$, where $n > 0$ there exists a value of the LD parameter $\eta$ for which $f_1(n, \eta) = 0$. As an example, we choose the LD parameter $\eta = 0.4518$, for which $f_1(17) = 0$, and simulate our protocol using the master equation

$$\dot{\rho} = -i[H_{\text{noJC}}, \rho] + \Gamma_m L(\sigma^-)\rho,$$

where $\Gamma_m = 2g$ is the decay rate of the internal state, and the Lindblad superoperator acts on $\rho$ as $L(X)\rho = (2X\rho X^\dagger - X^\dagger X\rho - \rho X^\dagger X)/2$.

### III. FOCK STATE GENERATION WITH DISSIPATIVE NONLINEAR ANTI-JCM

In this section we study the possibility of using the dynamics of the nonlinear anti-Jaynes-Cummings model introduced in the previous section to, along with depolarizing noise, generate high-number Fock states in a dissipative manner. In particular, the depolarizing noise that we consider corresponds to the spontaneous relaxation of the internal two-level system of the ion. Such a dissipative process, combined with the dynamics of the JCM in the LD regime (linear JCM), is routinely exploited in trapped-ion setups for the implementation of sideband cooling techniques. It is noteworthy to mention that the effect of nonlinearity on sideband cooling protocols, which arise when outside the LD regime, have also been a matter of study [7, 8, 9].

Our method works as follows: we start in the ground state of both the motional and the internal degrees of freedom $|\downarrow, 0\rangle$ as we will show later, our protocol works as well when we are outside the motional ground state, as long as the population of Fock states higher than the target Fock state is negligible. Acting with the nonlinear anti-JC Hamiltonian we induce a population transfer from state $|\downarrow, 0\rangle$ to state $|\uparrow, 1\rangle$, while at the same time, the depolarizing noise transfers population from $|\uparrow, 1\rangle$ to $|\downarrow, 1\rangle$. The simultaneous action of both processes will “heat” the motional state, progressively transferring the population of the system from one Fock state to the next one. Eventually, all the population will be accumulated in state $|\downarrow, n\rangle$, where a blockade of the propagation of population through the chain of Fock states occurs, if $f_1(n) = 0$, as the transfer rate between states $|\downarrow, n\rangle$ and $|\uparrow, n + 1\rangle$ vanishes, $\Gamma_{n,n+1} = 0$. We point out that the condition $f_1(n) = 0$ can always be achieved by tuning the LD parameter to a suitable value, i.e. for every Fock state $|n\rangle$, where $n > 0$ there exists a value of the LD parameter $\eta$ for which $f_1(n, \eta) = 0$. As an example, we choose the LD parameter $\eta = 0.4518$, for which $f_1(17) = 0$, and simulate our protocol using the master equation

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where $\Gamma_m = 2g$ is the decay rate of the internal state, and the Lindblad superoperator acts on $\rho$ as $L(X)\rho = (2X\rho X^\dagger - X^\dagger X\rho - \rho X^\dagger X)/2$.

In Fig. 3 we numerically show how our protocol is able to generate the motional Fock state $|17\rangle$, starting from a thermal state $\rho_T = \sum_{k=0}^{\infty} \frac{\langle n \rangle^k}{\langle n \rangle^{n+1}} |k\rangle \langle k|$, with $\langle n \rangle = 1$. In other words, one can obtain large final Fock states.
starting from an imperfectly cooled motional state, by a suitable tuning of the LD parameter. As an advantage of our method compared to previous approaches [57], we do not need a fine control over the Rabi frequencies or pulse durations, given that the whole wavefunction, for an arbitrary initial state with motional components smaller than \( n \), will converge to the target Fock state \( |n\rangle \). We want to point out that this protocol relies only on the precision to which the LD parameter can be set, which in turn depends on the precision to which the wave number \( k \) and the trap frequency \( \nu \) can be controlled. These parameters enjoy a great stability in trapped-ion setups [58], and therefore we deem the generation of high-number Fock states as a promising application of the nonlinear anti-JCM dynamics.

IV. NONLINEAR QUANTUM RABI MODEL

Here we propose to implement the nonlinear quantum Rabi model (NQRM) in all its parameter regimes via the use of the Hamiltonian in Eq. (6). We consider off-resonant first-order red- and blue-sideband drivings with the same coupling \( \Omega \) and corresponding detunings \( \delta_r \), \( \delta_b \). The interaction Hamiltonian after the optical RWA reads [30],

\[
H_{\text{int}} = \sum_{\omega = r,b} \frac{\Omega}{2} e^{-i\omega t} e^{-i(\delta_r t + \phi_n)} + H.c., \quad (8)
\]

where \( \omega_r = \omega_0 - \nu + \delta_r \) and \( \omega_b = \omega_0 + \nu + \delta_b \), with \( \delta_r, \delta_b \ll \nu \ll \omega_0 \) and \( \Omega \ll \nu \). We consider the system beyond the Lamb-Dicke regime and set the laser field phases to \( \phi_{r,b} = 0 \). If we invoke the vibrational RWA, i.e. neglect terms that rotate with frequencies in the order of \( \nu \), the remaining terms read

\[
H_{\text{int}} = i \sigma^z (f_1 e^{-i\delta_b t} + a^d f_1 e^{-i\delta_r t}) + H.c., \quad (9)
\]

where \( g = \eta \Omega/2 \) and \( f_1 \equiv f_1(\hat{n}) \) was introduced in Eq. (5). The latter corresponds to an interaction picture Hamiltonian of the NQR with respect to the free Hamiltonian \( H_0 = \frac{1}{2} (\sigma_r + \sigma_b) \sigma_z + \frac{1}{2} (\delta_r - \delta_b) a^d a \). Therefore, undoing the interaction picture transformation, we have

\[
H_{\text{NQRM}} = \frac{\omega_0^R}{2} \sigma_z + \omega_0^R a^d a + ig(\sigma^+ - \sigma^-) (f_1 a + a^d f_1), \quad (10)
\]

where \( \omega_0^R = -\frac{1}{2} (\delta_r + \delta_b) \) and \( \omega_0^R = \frac{1}{2} (\delta_r - \delta_b) \). Equation (10) represents the general form of the NQRM, where \( \omega_0^R \) is the level splitting of the simulated two level system, \( \omega_0^R \) is the frequency of the simulated bosonic mode and \( g \) is the coupling strength between them, which in turn will be modulated by the nonlinear function \( f_1(\hat{n}, \eta) \). The different regimes of the NQRM will be characterized by the relation among these four parameters. First, in the LD regime or \( \eta/\sqrt{(a + a^d)^2} \ll 1 \), Eq. (10) can be approximated to the linear QRM [30].

Beyond the LD regime, in a parameter regime where \( |\omega_0^R - \omega_0^R| \ll g \ll |\omega_0^R + \omega_0^R| \), the RWA can be applied. This would imply neglecting terms that rotate at frequency \( \omega_0^R + \omega_0^R \) in an interaction picture with respect to \( H_0 \), leading to the nonlinear JCM studied previously in this article. On the other hand, the nonlinear anti-JCM would be recovered in a regime where \( \omega_0^R + \omega_0^R \ll g \ll |\omega_0^R - \omega_0^R| \). It is worth mentioning that the latter is only possible if the frequency of the two-level system and the frequency of the mode have opposite sign. The USC and DSC regimes are defined as \( 0.1 \lesssim g/\omega_0^R \lesssim 1 \) and \( g/\omega_0^R \gtrsim 1 \) respectively, and in these regimes the RWA does not hold anymore.

As an example, here we investigate the NQRM in the DSC regime with initial Fock state \( |0, g\rangle \), where \( |0\rangle \) is the ground-state of the bosonic mode, and \( |g\rangle \) stands for the ground state of the effective two-level system. In Fig. 4 we study the case for \( \eta = 0.67898 \), where \( f_1(\hat{n}) = 0 \), \( g/\omega_0^R = 4 \) and \( \omega_0^R = 0 \). (b) Phonon statistics at different times for the NQRM evolved from the initial state \( |0, g\rangle \). The propagation of population through Fock states stops at \( \langle n = 7 \rangle \), with Fock states of \( n > 7 \) never getting populated.
ingly, the system dynamics never surpasses Fock state $|n\rangle$, for which $f_1(n) = 0$. Regarding the simulated regime of the nonlinear QRM, we point out that the nonlinear term also contributes to the coupling strength. Therefore, to keep the NQR of the DSC regime, the ratio $g/\omega_R$ should be larger than that for the linear QRM since $f_1(n) < 1$ always. Summarizing, our result illustrates that the Hilbert space is effectively divided into two subspaces by the NQR, namely those spanned by Fock states below and above Fock state $|n\rangle$. We denote the Fock number $n$, where $f_1(n) = 0$, as “the barrier” of the NQR. To benchmark the effect of the barrier, we also provide simulations starting from an initial coherent state with $\alpha = 1$ whose average phonon number is $\langle n \rangle = |\alpha|^2 = 1$, and make the comparison between the QRM and the NQRM in the DSC regime. For the parameter regime $g/\omega_R = 2$ and $\omega_0^R = 0$, the fidelity with respect to the initial coherent state in the linear QRM performs periodic collapses and full revivals as it can be seen in Fig. 5(a). In Fig. 5(b), we observe a round trip of the phonon-number wave packet, similarly to what was shown in Ref. 12 for the case of the linear QRM starting from a Fock state. The NQRM, on the other hand, has an associated dynamics that is aperiodic and more irregular, as shown in Fig. 6 and never crosses the motional barrier produced by the corresponding $f_1(n) = 0$. Therefore, it can be employed as a motional filter, which is determined by the location of the barrier with respect to the initial state distribution. Here, by filter we mean that the population of Fock states above a given threshold can be prevented. For the simulation we choose the LD parameter $\eta = 0.57838$ for which $f_1(10) = 0$, which is far from the center of the distribution of the initial coherent state, as well as most of its width. The simulated parameter regime corresponds to the DSC regime with $g/\omega_R = 3.7$ and $\omega_0^R = 0$. This case could also be simulated with trapped ions with detunings of $\delta_r = 2\pi \times 11.311$kHz and $\delta_0 = -2\pi \times 11.311$kHz, and a Rabi frequency of $\Omega = 2\pi \times 133.26$kHz. As for the corresponding case with initial Fock state $|0, g\rangle$, the evolution of the NQR in the coherent state case, depicted in Fig. 6 never exceeds the barrier.

V. CONCLUSIONS

We have proposed the implementation of nonlinear QRM in arbitrary coupling regimes, with trapped-ion analog quantum simulators. The nonlinear term that appears in our model is characteristic of the region beyond the Lamb-Dicke regime. This nonlinear term causes the blockade of motional propagation at $|n\rangle$, whenever $f_1(n)|n\rangle = 0$. In order to compare our models with standard linear quantum Rabi models, we have plotted the evolution of the population of the internal degrees of freedom of the ion evolving under the linear JCM and the nonlinear JCM, and observe that for the latter the collapses and revivals disappear. Also, we have proposed a method for generating large Fock states in a dissipative

FIG. 5: (color online) (a) Overlap of the instantaneous state with the initial state $P(t) = |\langle \psi_0 | \psi(t) \rangle |^2$ versus time, for a coherent initial state $|\alpha = 1, g\rangle$ evolving under the linear QRM. Collapses and revivals are observed, as expected in the DSC regime of the linear QRM. (b) Phonon statistics at different times, where we see the round trip of a phonon number wavepacket.

FIG. 6: (color online) (a) Overlap with the initial state $P(t) = |\langle \psi_0 | \psi(t) \rangle |^2$ versus time, for initial state $|\alpha = 1, g\rangle$ evolving under the NQR with LD parameter $\eta = 0.57838$, where $f_1(10) = 0$, $g/\omega_R = 3.7$ and $\omega_0^R = 0$. (b) Phonon statistics at different times for the NQRM evolved from the initial state $|\alpha = 1, g\rangle$. The Fock state $|10\rangle$ is never surpassed because $f_1(10) = 0$. 
manner, making use of the nonlinear anti-JCM and the spontaneous decay of the two-level system. Finally, we have studied the dynamics of the linear and nonlinear full QRM on the DSC regime and notice that the nonlinear case can act as a motional filter. Our work sheds light on the field of nonlinear QRMs implemented with trapped ions, and suggests plausible applications.

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[1] I. I. Rabi, “On the process of space quantization”, Phys. Rev. 49, 324 (1936).
[2] D. Braak, Q.-H. Chen, M. T. Batchelor, and E. Solano, “Semi-classical and quantum Rabi models: in celebration of 80 years”, J. Phys. A: Math. Theor. 49, 300301 (2016).
[3] D. Braak, “Integrability of the Rabi Model”, Phys. Rev. Lett. 107, 100401 (2011).
[4] E. T. Jaynes and F. W. Cummings, “Comparison of quantum and semi-classical radiation theories with application to beam maser”, Proc. IEEE 51, 80 (1963).
[5] W. P. Schleich, “Quantum Optics in Phase Space”, Wiley (2001).
[6] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, “Quantum Dynamics of Single Trapped Ions”, Rev. Mod. Phys. 75, 281 (2003).
[7] A. A. Houck, H. E. Türeci, and J. Koch, “On-chip Quantum Simulation with Superconducting Circuits”, Nat. Phys. 8, 292 (2012).
[8] J. M. Raimond, M. Brune, and S. Haroche, “Manipulating quantum entanglement with atoms and photons in a cavity”, Rev. Mod. Phys. 73, 565 (2001).
[9] A. Moroz, “A hidden analytic structure of the Rabi model”, Ann. Phys. (N.Y.) 340, 252 (2014).
[10] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. García-Ripoll, D. Zueco, T. Hümmer, E. Solano, A. Marx, and R. Gross, “Circuit quantum electrodynamics in the ultrastrong-coupling regime”, Nat. Phys. 6, 772 (2010).
[11] P. Forn-Díaz, J. Lisienfeld, D. Marcos, J. J. García-Ripoll, E. Solano, C. J. P. M. Harmans, and J. E. Mooij, “Observation of the Bloch-Siegert Shift in a Qubit-Oscillator System in the Ultrastrong Coupling Regime”, Phys. Rev. Lett. 105, 237001 (2010).
[12] J. Casanova, G. Romero, I. Lizuain, J. J. García-Ripoll, and E. Solano, “Deep Strong Coupling Regime of the Jaynes-Cummings Model”, Phys. Rev. Lett. 105, 263603 (2010).
[13] S. De Liberato, “Light-Matter Decoiling in the Deep Strong Coupling Regime: The Breakdown of the Purcell Effect”, Phys. Rev. Lett. 112, 016401 (2014).
[14] D. Z. Rossatto, C. J. Villas-Bôas, M. Sanz, and E. Solano, “Spectral classification of coupling regimes in the quantum Rabi model”, Phys. Rev. A 96, 013849 (2017).
[15] M.-J. Hwang, R. Puebla, and M. B. Plenio, “Quantum Phase Transition and Universal Dynamics in the Rabi Model”, Phys. Rev. Lett. 115, 180404 (2015).
[16] Z.-J. Ying, M.-X. Liu, H.-G. Luo, H.-Q. Lin, and J. Q. You, “Ground-state phase diagram of the quantum Rabi model”, Phys. Rev. A 92, 053823 (2015).
[17] R. Puebla, M.-J. Hwang, and M. B. Plenio, “Excited-state quantum phase transition in the Rabi model”, Phys. Rev. A 94, 023835 (2016).
[18] L. Henriet, Z. Ristivojevic, P. P. Orth, and K. Le Hur, “Quantum dynamics of the driven and dissipative Rabi model”, Phys. Rev. A 90, 023820 (2014).
[19] L.-H. Du, X.-F. Zhou, Z.-X. Zhou, X.-X. Zhou, and G.-C. Guo, “Generalized Rabi model in quantum-information processing including the $\tilde A^2$ term”, Phys. Rev. A 86, 014303 (2012).
[20] Q.-T. Xie, S. Cui, J.-P. Cao, L. Amico, and H. Fan, “Anisotropic Rabi model”, Phys. Rev. X 4, 021046 (2014).
[21] A. Moroz, “Generalized Rabi models: diagonalization in the spin subspace and differential operators of Dunkl type”, Europhys. Lett. 113, 50004 (2016).
[22] J. Casanova, R. Puebla, H. Moya-Cessa, and M. B. Plenio, “Equivalence Among Generalized nth Order Quantum Rabi Models”, arXiv:1709.02714.
[23] S. A. Chilingaryan and B. M. Rodriguez-Lara, “The quantum Rabi model for two qubits”, J. Phys. A: Math. Theor. 48, 335301 (2013).
[24] L. Lamata, “Digital-analog quantum simulation of generalized Dicke models with superconducting circuits”, Sci. Rep. 7, 43768 (2017).
[25] L. Garbe, I. L. Egusquiza, E. Solano, C. Ciuti, T. Courdreaux, P. Milman, and S. Felicetti, “Superradiant phase transition in the ultrastrong coupling regime of the two-photon Dicke model”, Phys. Rev. A 95, 053854 (2017).
[26] D. Barberena, L. Lamata, and E. Solano, “Dispersive Regimes of the Dicke Model”, Sci. Rep. 7, 8774 (2017).
[27] F. Altintas, A. U. C. Hardal, and Ö. E. Müstecaplıoğlu, “Rabi model as a quantum coherent heat engine: From quantum biology to superconducting circuits”, Phys. Rev. A 91, 023816 (2015).
[28] S. Felicetti, E. Rico, C. Sabin, T. Ockenfels, J. Koch, M. Leder, C. Grossert, M. Weitz, and E. Solano, “Quantum Rabi model in the Brillouin zone with ultracold atoms”, Phys. Rev. A 95, 013827 (2017).
[29] A. Mezzacapo, U. Las Heras, J. S. Pedernales, L. DiCarlo, E. Solano, and L. Lamata, “Digital Quantum Rabi and Dicke Models in Superconducting Circuits”, Sci. Rep. 4, 7482 (2014).
[30] J. S. Pedernales, I. Lizuain, S. Felicetti, G. Romero, L. Lamata, and E. Solano, “Quantum Rabi Model with Trapped Ions”, Sci. Rep. 5, 15472 (2015).
[31] D.-S. Lv, S.-M. An, Z.-Y. Liu, J.-N. Zhang, J. S. Pedernales, L. Lamata, E. Solano, and K. Kim, “Quantum simulation of the quantum Rabi model in a trapped ion”, arXiv:1711.00582.
[32] C. Huerta Alderete and B. M. Rodríguez-Lara, “Cross-cavity quantum Rabi model”, J. Phys. A: Math. Theor. 49, 414001 (2016).
[33] R. Puebla, J. Casanova, and M. B. Plenio, “A robust
scheme for the implementation of the quantum Rabi model in trapped ions”, New J. Phys. 18, 113039 (2016).

[34] R. Blatt and C. F. Roos, “Quantum Simulations with Trapped Ions”, Nat. Phys. 8, 277 (2012).

[35] K. Kim, M.-S. Chang, S. Korenblit, R. Islam, E.E. Edwards, J. K. Freericks, G.-D. Lin, L.-M. Duan, and C. Monroe, “Quantum Simulation of Frustrated Ising Spins with Trapped Ions”, Nature 465, 590 (2010).

[36] J. Casanova, L. Lamata, I. L. Egusquiza, R. Gerritsma, C. F. Roos, J. J. García-Ripoll, and E. Solano, “Quantum Simulation of Quantum Field Theories in Trapped Ions”, Phys. Rev. Lett. 107, 260501 (2011).

[37] J. Casanova, A. Mezzacapo, L. Lamata, and E. Solano, “Quantum Simulation of Interacting Fermion Lattice Models in Trapped Ions”, Phys. Rev. Lett. 108, 190502 (2012).

[38] A. Mezzacapo, J. Casanova, L. Lamata, and E. Solano, “Digital Quantum Simulation of the Holstein Model in Trapped Ions”, Phys. Rev. Lett. 109, 200501 (2012).

[39] L. Lamata, A. Mezzacapo, J. Casanova, and E. Solano, “Efficient Quantum Simulation of Fermionic and Bosonic Models in Trapped Ions”, EPJ Quantum Technology 1, 9 (2014).

[40] L. Lamata, J. León, T. Schätz, and E. Solano, “Dirac Equation and Quantum Relativistic Effects in a Single Trapped Ion”, Phys. Rev. Lett. 98, 253005 (2007).

[41] R. Gerritsma, G. Kirchmair, F. Zähringer, E. Solano, R. Blatt, and C. F. Roos, “Quantum Simulation of the Dirac Equation”, Nature 463, 68 (2010).

[42] R. Gerritsma, B. P. Lanyon, G. Kirchmair, F. Zähringer, C. Hempel, J. Casanova, J. J. García-Ripoll, E. Solano, R. Blatt, and C. F. Roos, “Quantum Simulation of the Klein Paradox with Trapped Ions”, Phys. Rev. Lett. 106, 060503 (2011).

[43] J. Casanova, C. Sabín, J. León, I. L. Egusquiza, R. Gerritsma, C. F. Roos, J. J. García-Ripoll, and E. Solano, “Quantum Simulation of the Majorana Equation and Unphysical Operations”, Phys. Rev. X 1, 021018 (2011).

[44] X. Zhang, Y. Shen, J. Zhang, J. Casanova, L. Lamata, E. Solano, M.-H. Yung, J.-N. Zhang, and K. Kim, “Time Reversal and Charge Conjugation in an Embedding Quantum Simulator”, Nat. Commun. 6, 7917 (2015).

[45] X.-H. Cheng, U. Alvarez-Rodríguez, L. Lamata, X. Chen, and E. Solano, “Time and spatial parity operations with trapped ions”, Phys. Rev. A 92, 022344 (2015).

[46] I. Arrazola, J. S. Pedernales, L. Lamata, and E. Solano, “Digital-Analog Quantum Simulation of Spin Models in Trapped Ions”, Sci. Rep. 6, 30534 (2016).

[47] X.-H. Cheng, I. Arrazola, J. S Pedernales, L. Lamata, X. Chen, and E. Solano “Switchable Particle Statistics with an Embedding Quantum Simulator”, Phys. Rev. A 95, 022305 (2017).

[48] W. Vogel and R. L. de Matos Filho, “Nonlinear Jaynes-Cummings dynamics of a trapped ion”, Phys. Rev. A 52, 4214 (1995).

[49] R. L. de Matos Filho and W. Vogel, “Nonlinear coherent states”, Phys. Rev. A 54, 4560 (1996).

[50] R. L. de Matos Filho and W. Vogel, “Quantum Nondemolition Measurement of the Motional Energy of a Trapped Atom”, Phys. Rev. Lett. 76, 4520 (1996).

[51] D. Stevens, J. Brochard, and A. M. Steane, “Simple experimental methods for trapped-ion quantum processors”, Phys. Rev. A 58, 2750 (1998).

[52] G. Morigi, J. I. Cirac, M. Lewenstein, and P. Zoller, “Ground-state laser cooling beyond the Lamb-Dicke limit”, Europhys. Lett. 39, 13 (1997).

[53] G. Morigi, J. Eschner, J. I. Cirac, and P. Zoller, “Laser cooling of two trapped ions: Sideband cooling beyond the Lamb-Dicke limit”, Phys. Rev. A 59, 3797 (1999).

[54] L. Förster, M. Karski, J.-M. Choi, A. Steffen, W. Alt, D. Meschede, A. Widera, E. Montano, J.-H. Lee, W. Rakreungdet, and P. S. Jessen, “Microwave Control of Atomic Motion in Optical Lattices”, Phys. Rev. Lett. 103, 233001 (2009).

[55] Y.-M. Hu, W.-L. Yang, Y.-Y. Xu, F. Zhou, L. Chen, K.-L Gao, M. Feng, and C. Lee, “Franck-Condon physics in a single trapped ion”, New J. Phys. 13, 053037 (2011).

[56] M. Fleischhauer and W. P. Schleich, “Revivals made simple: Poisson summation formula as a key to the revivals in the Jaynes-Cummings model”, Phys. Rev. A 47, 4258 (1993).

[57] D. M. Meekhof, C. Monroe, B. E. King, W. M. Itano, and D. J. Wineland, “Generation of Nonclassical Motional States of a Trapped Atom” Phys. Rev. Lett. 76 1796 (1996).

[58] K. G. Johnson, J. D. Wong-Campos, A. Restelli, K. A. Landsman, B. Neyenhuis, J. Mizrahi, C. Monroe, “Active stabilization of ion trap radiofrequency potentials” Rev. Sci. Instru. 87 (5), 053110 (2016).