Near critical dimers and massive SLE

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Outline

1) Reminders about the classical dimer model

2) Near-critical weights
   - Definition, non Gaussian scaling limit
   - Equivalent representation: Temperley’s bijection
   - New results: convergence to massive SLE, universality, conformal covariance,
     - Along the way: scaling limit of LERW with drift

3) An exact discrete Girsanov theorem on the triangular lattice

4) Some open questions: Sine-Gordon, Ising etc?
1) The dimer model
The dimer model

Let $G$ be a finite, planar, bipartite graph.

A *dimer cover* (or *perfect matching*): a set of edges (=dimers), such that each vertex is incident to exactly one dimer.

The *dimer model* with edge weights $w_e$:

$$\mathbb{P}(m) = \frac{1}{Z} \prod_{e \in m} w_e.$$  

Typically $w_e \equiv 1$ (*→ critical*)
The dimer model as a random surface

Honeycomb lattice: *lozenge tiling* or a stack of cubes

Height function

Introduced by Thurston. Hence view as a random surface.

Note: depends on the choice of a reference frame.
Large scale behaviour?

The effect of boundary conditions is, however, not entirely trivial and will be discussed in more detail in a subsequent paper.

P. W. Kasteleyn, 1961
Temperleyan boundary conditions

Divide the vertices into black and white.
Divide further into $B_0 = \bullet, B_1 = \times$
(and $W_0, W_1$).

Temperleyan: all corners are $B_1 = \times$, and one corner is removed.
Scaling limit of height function

Theorem (Kenyon ’99)

Let $\mathcal{D} \subset \mathbb{C}$ bounded domain, $\mathcal{D}^\delta = \mathcal{D} \cap \delta \mathbb{Z}^2$ with Temperleyan boundary conditions. Let $h^\delta$ be the associated height function. Then,

$$h^\delta - \mathbb{E}(h^\delta) \to \frac{1}{\sqrt{\pi}} h^\text{GFF}_D$$

as $\delta \to 0$, in distribution.

Main ingredients of the proof:

▶ Kasteleyn theory (exact solvability): dimer correlations are given by determinants of inverse Kasteleyn matrix,

▶ Asymptotic computation of inverse Kasteleyn matrix (discrete holomorphic + boundary conditions)

▶ Computation of moments
2) Near-critical dimer model

Makarov–Smirnov (2009):

The key property of SLE is its conformal invariance, which is expected in 2D lattice models only at criticality, and the question naturally arises: Can SLE success be replicated for off-critical models? In most off-critical cases to obtain a non-trivial scaling limit one has to adjust some parameter [...], sending it at an appropriate speed to the critical value. Such limits lead to massive field theories...,
Biperiodic setup

Choose $s_i = 1 + c_i \delta$, where $\delta =$ mesh size. Gasesous/Liquid boundary...
Massive Laplacian

Let $K = \text{Kasteleyn matrix}$, $D = KK^*$. Then $D$ is (essentially) a **massive Laplacian**:

$$D(b, b) = - \sum_{i=1}^{4} s_i^2$$

but

$$\sum_{b'} D(b, b') = 2s_2s_4 + 2s_1s_3 < |D(b, b)|$$

by AM-GM.

Describes a **massive walk** (fixed killing probability).

**Natural guess:**

Scaling limit $= \text{Massive GFF}$?

$$\mathbb{E}[h(x)h(y)] = \int_0^{\infty} e^{-m^2t}p_t(x, y)dt$$
Unfortunately this guess is wrong.

Theorem (Chhita, 2012)

Limiting moments of height function can be computed; no Wick rule so non Gaussian!
New results for near-critical dimers

With Levi Haunschmid (2022) we prove:

- Exact connection with Makarov and Smirnov’s \textit{massive SLE}_2 (and with massive Laplacian).
- Existence and universality of \textit{scaling limit} of height function in Temperleyan domains
- Conformal \textit{covariance} of scaling limit
Temperley’s bijection

Dimers on $\mathbb{Z}^2 \cap D \leftrightarrow$ pairs of dual spanning trees on $B_0, B_1$ lattices.
Temperleyan boundary conditions: wired/free tree.

B.–Laslier–Ray point of view

Often easier to work with Temperleyan trees.
Keeps all the information, even in scaling limit (“Imaginary Geometry”).
Temperley’s bijection 1

Dimers on $\mathbb{Z}^2 \cap D$, Temperleyan boundary conditions.
Temperley’s bijection 2

Orient dimers black $\rightarrow$ white (just $B_0 = \bullet$ for now)
Temperley’s bijection 3

Double each oriented dimer to get spanning tree on $B_0$ lattice (wired boundary conditions).
Temperley’s bijection 4

On $B_1$ lattice, get dual (free boundary conditions) spanning tree.
Remarks

▶ The bijection is local.

▶ Temperleyan boundary conditions for dimer $\Rightarrow$ wired/free boundary conditions for trees.

▶ If $w_e \equiv 1$ then $(\mathcal{T}, \mathcal{T}^\dagger)$ uniform.

▶ More generally, in biperiodic setup,

$$\mathbb{P}(\mathcal{T} = t) \propto \prod_{e \in t} w_e/2.$$ 

Owing to biperiodic structure, (directed) edges of $\mathcal{T}$ come with weight $s_1, \ldots, s_4$. 
Extends to hexagonal lattice
Scaling limit of Temperleyan tree

Consider off-critical dimer model on square with $s_i = 1 + c_i \delta$. Let $\mathcal{T} = $ Temperleyan $B_0$-tree.

$$\mathbb{P}(\mathcal{T} = t) \propto \prod_{v \in B_0} s_v(t)$$

where $s_v(t) \in \{s_1, \ldots, s_4\}$ depending on the direction of the unique outgoing edge from $v$ in $t$.

Wilson’s algorithm

The branch connecting $z$ to $\partial D$ is LERW for the random walk on $B_0$ with jump probabilities $\left(s_i\right)_{i=1}^4$.

The random walk itself converges to BM with drift $\alpha$,

$$\alpha = \frac{1}{4}(c_1 + c_2 i + c_3 i^2 + c_4 i^3)$$

But what is the scaling limit of LERW?
Connection with massive SLE\(_2\)

Suppose

\[
c_1 + c_3 = c_2 + c_4 = 0
\]

**Theorem 1 (B.–Haunschmid)**

Let \( z \in \Omega \). Let \( \gamma^\delta = \) path in Temperleyan tree to \( \partial \Omega \), \( Y_\delta = \) endpoint. Then conditionally on \( Y_\delta = y_\delta \),

\[\gamma^\delta \to \text{mSLE}_2,\]

where mass \( m = \|\alpha\|. \)
Massive SLE$_2$

Consider random walk killed with probability $m^2 \delta^2$ at each step.

Condition to leave $\Omega$ without dying. What is scaling limit of LERW?

**Theorem (Makarov–Smirnov (2009), Chelkak-Wan (2019))**

Massive LERW converges to “massive SLE$_2$”

Described by Loewner’s equation with driving function:

$$d\xi_t = \sqrt{2} dB_t + 2\lambda_t dt;$$

with

$$\lambda_t = \frac{\partial}{\partial w} \log \left. \frac{P_{\Omega_t}^{(m)}(z, w)}{P_{\Omega_t}^{(0)}(z, w)} \right|_{w=\gamma(t)}$$

[$m = 0$: *Lawler–Schramm–Werner 2002*]
Unconditional convergence also holds, then global Radon–Nikodym derivative:

$$\frac{d\mathbb{P}}{d \text{mSLE}_2}(\gamma) = \exp(2\langle Y - z, \Delta \rangle)$$

where $Y = \text{exit point}$.

Exact same statement for hexagonal lattice $a_i = 1 + c_i \delta$, $\alpha = \frac{1}{3}(c_1 + c_2 \tau + c_3 \tau^2)$. 
Convergence of height function

**Corollary (B.–Haunschmid)**

The Temperleyan tree $\mathcal{T}_\delta$ has a scaling limit (in Schramm topology); the limit law depends only on $\Delta$ and so is the same for hexagonal and square lattice cases.

Proof: Wilson’s algorithm.

**Corollary (B.–Haunschmid)**

The height function of near-critical dimers in Temperleyan domains converge to the same scaling limit.

Proof: “imaginary geometry approach” by B.–Laslier–Ray (2020, 2019+).
**Conformal covariance**

**Conformal covariance:**

Image under conformal map preserved, up to power \( \beta \) of derivative of conformal map.

\((\beta = 0 \text{ means conformal invariance.})\)

This requires allowing for general vector field \( \alpha : \Omega \to \mathbb{R}^2 \equiv \mathbb{C} \).

**Generalised near-critical dimers**

At each point \( z \in B_0 \), assign weights \( s_i = 1 + c_i \delta \), with 
\[ c_1 + c_3 = 0, \quad c_2 + c_4 = 0, \]
\[ \frac{1}{4}(c_1 + c_2 i + c_3 i^2 + c_4 i^3) = \alpha \]

Any drift vector \( \alpha \) is uniquely encoded in this way.

The random walk with these weights converge in the scaling limit not to a Brownian motion, but to the solution of the SDE

\[ dX_t = dB_t + \alpha(X_t)dt. \]
Consider a smooth vector field $\alpha : \tilde{\Omega} \to \mathbb{R}^2$. Does the LERW have a scaling limit?

**Assumptions**

Suppose we have sequence of planar graphs $G^\delta$, and:

- $\alpha = \nabla \phi$ of gradient type.
- Two laws $P^\delta, Q^\delta$ such that:
  - under $P^\delta$, RW converge to BM;
  - under $Q^\delta$, RW converges to SDE.
- Uniform absolute continuity: i.e., $dQ^\delta/dP^\delta$ is Uniformly Integrable.

Holds on square lattice and triangular lattices, and conformal deformations thereof.
Theorem 3. (B.–Haunschmid)

The loop-erased random walk with local drift $\alpha$ has a scaling limit.

Described by Loewner’s equation with driving function in $\mathbb{D}$:

$$d\xi_t = \sqrt{2} dB_t + 2\lambda_t dt;$$

with

$$\lambda_t = \frac{\partial}{\partial w} \log \frac{P_{\Omega_t}^{(\alpha)}(z, w)}{P_{\Omega_t}^{(0)}(z, w)} \bigg|_{w = \gamma(t)}$$

where $\Omega_t = \mathbb{D} \setminus \gamma[0, t]$, $P_{\Omega}^{(\alpha)}(z, w) = \text{Poisson kernel}$ in $\Omega$ for the SDE.

The existence of this Poisson kernel is not trivial. (Smooth $\Omega$: Ben Arous, Kusuoka and Stroock 1984).

cf. Chelkak–Wan 2019.
Corollary

Let $\alpha = \nabla \phi$ be a smooth vector field in $\bar{\Omega}$.
Associate near-critical weights on square/hexagonal lattices.
Then height function has a scaling limit, call it $h^{(\alpha)};\Omega$. Depends just on $\alpha$.

Remark: not sure what is analogue on more general lattices.

Theorem 4. (B.–Haunschmid)

Let $F : \tilde{\Omega} \to \Omega$ be a conformal map (with bounded derivative). In law,

$$h^{(\alpha)};\Omega \circ \phi = h^{(\tilde{\alpha})};\tilde{\Omega}$$

where at a point $w \in \tilde{\Omega}$,

$$\tilde{\alpha}(w) = \overline{F'(w)} \cdot \alpha(F(w)).$$
On the imaginary geometry approach to dimers

A powerful approach to dimer models:

- Temperleyan & more, even for **balanced random environments** (B.–Laslier–Ray, 2020)

- Riemann Surfaces (B.–Laslier–Ray, 2019, 2022)

- Piecewise Temperleyan
  → **multiple SLE$_8$** (B.–Liu, 2023)
Define $\beta(v) > 0$ by

$$\exp(-\beta(v)^2) = (a/3)^{-3} \prod_{k=1}^{3} e^{\alpha_k},$$

well defined by AM-GM.
Discrete Girsanov on triangular lattice $\mathbb{T}$.

Define a vector $\alpha(v)$ at every vertex $v$ in the graph,

$$\alpha = \alpha_1 + \alpha_2 \tau + \alpha_3 \tau^2,$$

**Lemma**

Fix any lattice path $\gamma = (x_0, \ldots, x_n)$ on $\mathbb{T}$.

$$\frac{\mathcal{Q}}{\mathcal{P}}(\gamma) = \exp(M_n - \frac{1}{2} V_n)$$

where $M_n = \frac{2}{3} \sum_{s=0}^{n-1} \langle \alpha(x_s), dx_s \rangle$; and $V_n = \frac{2}{3} \sum_{s=0}^{n-1} \beta(x_s)^2$.

Discrete analogue of

$$\frac{d\mathcal{Q}}{d\mathcal{P}} = \exp \left( \int_0^t \Delta(X_s) \cdot dX_s - \frac{1}{2} \int_0^t \| \Delta(X_s) \|^2 ds \right).$$

**Corollary (constant drift case)**

$\mathcal{Q}_x(\cdot | x_n = y)$ is the same as a massive walk conditioned to survive up to time $n$ and $X_n = y$. 
Proof.

At each \( v \), write \( n_i = n_i(v) = \) number of times path goes in direction 1, \( \tau, \tau^2 \).

\[
Q_x(\gamma) = \prod_v \prod_{i=1}^3 \left( \frac{e^{\alpha_i}}{a} \right)^{n_i}
\]

\[
= 3^{-n} \prod_v \left[ \left( (a/3)^{-2} \prod_{i=1}^3 (e^{\alpha_i})^{\frac{n_1+n_2+n_3}{3}} \right) \prod_{i=1}^3 (e^{\alpha_i})^{n_i-\frac{n_1+n_2+n_3}{3}} \right]
\]

\[
= 3^{-n} \prod_v e^{-\beta(v)^2 \frac{n_1+n_2+n_3}{3}} \exp \left( \sum_{i=1}^3 \alpha_i (n_i - \frac{n_1+n_2+n_3}{3}) \right)
\]

\[
= 3^{-n} e^{-\frac{1}{2} \nu_n} \exp \left( \sum_v \alpha_1 \left( \frac{2n_1-n_2-n_3}{3} + \frac{2n_2-n_1-n_3}{3} + \frac{2n_3-n_1-n_2}{3} \right) \right)
\]

\[
= 3^{-n} e^{-\frac{1}{2} \nu_n} \exp \left( \frac{2}{3} \sum_v \left( \alpha_1 + \alpha_2 \tau + \alpha_3 \tau^2, n_1 + n_2 \tau + n_3 \tau^2 \right) \right)
\]

\[
= 3^{-n} e^{-\frac{1}{2} \nu_n} \exp \left( \frac{2}{3} \sum_{s=0}^{n-1} \langle \alpha(x_s), dx_s \rangle \right).
\]
Sine–Gordon

An integrable but non conformal QFT:

\[ \mathbb{P}^{SG}(dh) \propto \exp \left( z \int_D \cos(\sqrt{\beta} h(x)) dx \right) \mathbb{P}^{GFF}(dh), \]

where \( \mathbb{P}^{GFF} = \text{law of } (h/\sqrt{2\pi}), h \text{ a GFF with log correlations.} \)

Free fermion point

\( \beta = 4\pi: \textbf{Coleman correspondence}, \text{ cf. Bauerschmidt–Webb (2023).} \)

This is a massive extension of the fermion-boson correspondence.

At the free fermion point, the Sine-Gordon field is “particularly integrable”.
Conjectures

For $\alpha = \text{constant}$, we conjecture convergence of the massive dimer height function to the Sine-Gordon model at the free fermion point $\beta = 4\pi$.

Progress by S. Mason (2022) (full plane in a particular case).

More generally:

**Conjecture**

Let $h^{(\alpha)};\Omega$ denote the limiting height function of the near-critical dimer height functions associated to the drift vector field $\alpha : \Omega \rightarrow \mathbb{R}^2$. Then the law of this field is given by

$$
\mathbb{P}^{(\alpha);\Omega}(dh) \propto \exp \left( z \int_D \left\langle \alpha(x); e^{i\sqrt{\beta}h(x)} \right\rangle \, dx \right) \mathbb{P}^{\text{GFF}}(dh),
$$

as $\beta = 4\pi$. (Free fermion?)

If true, massive SLE would be “flow lines” of free fermion Sine-Gordon...