NAKED SINGULARITIES FORMATION IN THE GRAVITATIONAL COLLAPSE OF BAROTROPIC SPHERICAL FLUIDS

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ABSTRACT. The gravitational collapse of spherical, barotropic perfect fluids is analyzed here. For the first time, the final state of these systems is studied without resorting to simplifying assumptions - such as self-similarity - using a new approach based on non-linear o.d.e. techniques, and formation of naked singularities is shown to occur for solutions such that the mass function is analytic in a neighborhood of the spacetime singularity.

1. INTRODUCTION

The final state of gravitational collapse is an open problem of classical gravity. It is, in fact, commonly believed that a collapsing star that is unable to radiate away - via e.g. supernova explosion - a sufficient amount of mass to fall below the neutron star limit, will certainly and inevitably form a black hole, so that the singularity corresponding to diverging values of energy and stresses will be safely hidden - at least to faraway observers - by an event horizon. However, this is nothing more than a conjecture - what Roger Penrose first called a "Cosmic Censorship" conjecture [34] - and has never been proved. Actually, it is easy to see that one just cannot prove the conjecture as a statement on the mathematical evolution of any collapsing system via Einstein field equations, because in this case what is conjectured is baldly false: it is indeed an easy exercise producing counterexamples using e.g. negative energy densities or "ad hoc" field configurations. Thus, to go beyond the conjecture what is needed is a set of hypotheses, possibly based on sound physical requirements, which would allow the proof of a mathematically rigorous theorem. However, what turned out to be the truth in the last twenty years of research is that such a theorem (and, in fact, even the hypotheses of the theorem) is/are extremely difficult to be stated (see e.g. [23]).

In the meanwhile, many examples of spherically symmetric solutions exhibiting naked singularities and satisfying the principles of physical reasonableness have been discovered.

Spherically symmetric naked singularities can be divided into two groups: those occurring in scalar fields models [6, 8] and those occurring in astrophysical sources modeled with continuous media, which are of exclusive interest here (see [22] for a recent review). The first (shell focusing) examples of naked singularities where discovered in dust models, numerically by Eardley and Smarr [10] and analytically by Christodoulou [5]. Today, the gravitational collapse of dust is known in full details [25].

The dust models can, of course, be strongly criticized from the physical point of view. In fact, they have the obvious drawback that stresses are expected to develop during the collapse, possibly influencing its dynamics. In particular, such
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models are an unsound description of astrophysical sources in the late stage of the collapse even if the latter does not form a singularity: one can, for instance, regard a white dwarf or a neutron star as being an extremely compact planet, composed by a solid crust and a liquid (super)fluid core: such objects are sustained by enormous amounts of (generally anisotropic) stresses. It is, therefore, urgent to understand models of gravitational collapse with stresses.

Recently, several new results have been obtained in this direction by considering systems sustained by anisotropic stresses (see e.g. [16, 18, 20, 21]). Besides the details of the physics of the collapse of such systems, the general pattern arising from all such examples is that existence of naked singularities persists in presence of stresses: actually, we have recently shown that the mechanism responsible for the formation or whatsoever of a naked singularity is the same in all such cases [13].

In spite of the aforementioned physical relevance of anisotropic systems, it is beyond any doubt of exceeding interest the case of isotropic stresses, i.e. the gravitational collapse of perfect fluids. In fact, for instance, the perfect fluid model is (in part for historical reasons) the preferred model used in most approximations of stellar matter of astrophysical interest. Unfortunately, although local existence ad uniqueness for the solution of the Einstein field equations has been proved [28, 35], very few sound analytical models of gravitational collapse of perfect fluids are known and, as a consequence, the problem of the final state of gravitational collapse of perfect fluids in General Relativity is still essentially open. Exceptions are the solutions describing shear-free fluids (see e.g. [26, 27]) and those obtained by matching of shock waves [37]; in both cases, however, the collapse is synchronous (i.e. the singularity is of the Friedmann-Robertson-Walker type) and therefore such solutions say little about Cosmic Censorship [3, 24].

There is a unique perfect fluid class of solutions which has been investigated in full details. This is the case of self-similar fluids, which has been treated by many authors since the pioneering work by Ori and Piran [33] (for a recent review see [4]). Self-similarity is compatible with the field equations if the equation of state is of the form $p = \alpha \epsilon$ (where $p$ is the pressure, $\epsilon$ the energy density, and $\alpha$ a constant). In this case the field equations reduce to ordinary differential equations and therefore can be analyzed with the powerful techniques of dynamical systems. Ori and Piran found that self similar perfect fluids generically form naked singularities; more precisely, they showed numerically that for any $\alpha$ in a certain range there are solutions with naked singularities. Recently, Harada added some numerical examples which remove the similarity hypotheses [17].

These results clearly go in the direction of disproving any kind of censorship at least in spherical symmetry, since they show that naked singularities have to be expected in perfect fluids with physically sound equations of state. However, although being extremely relevant as a "laboratory", the self-similar ansatz is a oversimplifying assumption, and the general case of perfect fluid collapse remained untractable up today, essentially due to the lack of exact solutions.

In the present paper we present the first (as far as we are aware) analytical study on the endstates of barotropic spherical fluids which circumvents this problem. To do this we use a combination of two new ingredients. The first is the fact that, in a suitable system of coordinates (the so-called area-radius coordinates) we are able to
reduce the field equations to a single, quasi linear, second order partial differential equation. As a consequence, the metric for a barotropic spherical fluid can be written, in full generality, in terms of only one unknown function. In this way the behavior of the null radial geodesics near the singular point can be analyzed in terms of the Taylor expansion of such a function. The second ingredient is a new framework for doing this analysis based on techniques for singular non linear ordinary differential equations [13][14]. Our results here show the existence of naked singularities in barotropic perfect fluids solutions for which the mass function is analytic in a neighborhood of the center.

2. REDUCTION OF THE FIELD EQUATIONS TO A QUASI-LINEAR P.D.E.

Consider a spherically symmetric perfect fluid. The general line element in co-moving coordinates can be written as

\[ ds^2 = -e^{2\nu} dt^2 + e^{2\lambda} dr^2 + R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

where \( \nu, \lambda \) and \( R \) are function of \( r \) and \( t \) (we shall use a dot and a prime to denote derivatives with respect to \( t \) and \( r \) respectively). Denoting by \( \epsilon \) and \( p \) the energy density and the isotropic pressure of the fluid, Einstein field equations can be written as

\[ \Psi' = 4\pi \epsilon R^2 R', \]

\[ \tilde{\Psi} = -4\pi p R^2 \tilde{R}, \]

\[ \tilde{R}' = \tilde{R}\nu' + R'\lambda, \]

\[ p' = -(\epsilon + p)\nu', \]

where \( \Psi(r, t) \) is the Misner-Sharp mass, defined in such a way that the equation \( R = 2\Psi \) spans the boundary of the trapped region, i.e. the region in which outgoing null rays re-converge:

\[ \Psi(r, t) = \frac{R}{2} \left[ 1 - g^{\mu\nu}(\partial_{\mu}R)(\partial_{\nu}R) \right] = \frac{R}{2} \left[ 1 - (R')^2 e^{-2\lambda} + (\tilde{R})^2 e^{-2\nu} \right], \]

The curve \( t_h(r) \) describing this boundary, i.e. the function defined implicitly by

\[ R(r, t_h(r)) = 2\Psi(r, R(r, t_h(r))), \]

is called apparent horizon and will play a fundamental role in what follows.

Initial data for the field equations can be assigned on any Cauchy surface (\( t = 0 \), say). Physically, the arbitrariness on the data refers to the initial distribution of energy density and the initial velocity profile, and is therefore described by two functions of \( r \) only. Data for \( R \) do not carry physical information and we parameterize the initial surface in such a way that \( R(r, 0) = r \).

The data must be complemented with the information about the physical nature of the collapsing material. In the present paper we shall consider only barotropic perfect fluids, i.e. fluids for which the equation of state can be given in the standard thermodynamical form: the pressure \( p \) equals minus the derivative w.r. to the specific volume \( v \) of the specific energy density \( e(v) \). We are going to work however with the matter density \( \rho = 1/v \) and with the energy density \( e(\rho) = \rho e(1/\rho) \).
Therefore we are going to use in the sequel the equation of state of the fluid in the form (slightly less familiar than $p = -\frac{de}{d\nu}$):

\begin{equation}
    p = \rho \frac{de}{d\nu} - \epsilon \tag{2.5}
\end{equation}

Using the comoving description of the fields the matter density is proportional to the determinant of the 3-metric, i.e.

\begin{equation}
    \rho = \frac{e^{-\lambda}}{4\pi E R^2} \tag{2.6}
\end{equation}

where $E = E(r)$ is an arbitrary positive function.

In order to simplify reading, we are going to develop in full details in the next sections the special - although physically very relevant - case of the linear equation of state

\begin{equation}
    p = \alpha \epsilon \tag{2.7}
\end{equation}

where $\alpha$ is a constant parameter. However, in the final section, we will show how the results can be easily extended to (virtually) all the - physically valid - barotropic equations of state.

In terms of the matter density eq. (2.5) implies $\epsilon = \rho^{\alpha+1}$ up to a multiplicative constant which however can be absorbed in the definition of $E(r)$. For such fluids the field equation (2.2d) integrates to

\begin{equation}
    e^\nu = \rho^{-\alpha} \tag{2.8}
\end{equation}

up to a multiplicative function of time only which can be taken equal to one by a reparameterization of $t$.

We are now going to show that the remaining field equations simplify considerably (and actually the problem of the final state becomes tractable) if another system of coordinates, the area-radius ones, are used. The advantages of this system were first recognized by Ori [32], who used it to obtain the general exact solution for charged dust. Subsequently, the area-radius framework has been successfully applied to models of gravitational collapse and cosmic censorship (see e.g. [13, 18, 30]).

Area-radius coordinates are obtained using $R$ in place of the comoving time. Denoting by subscripts derivatives w.r. to the new coordinates, we have $\Psi' = \Psi,\nu + R' \Psi,\nu, R = R' \Psi,\nu$. Substituting in eqs (2.2a), (2.2b) we obtain $R'$ and $\rho$ in terms of the mass:

\begin{equation}
    R' = -\frac{\alpha}{\alpha + 1} \Psi,\nu \tag{2.9}
\end{equation}

\begin{equation}
    \rho = \left( -\frac{\Psi,\nu \Psi,R}{4\pi \alpha R^2} \right)^{\frac{1}{\alpha+1}} \tag{2.10}
\end{equation}

In writing the above formulae we have excluded the case $\alpha = 0$. This case corresponds to the dust (Tolman-Bondi) solutions which is already very well known and will not be considered further in the present paper (see [25] and references therein).
Equation (2.3) can be used to express the velocity \( u = |\dot{R}e^{-\nu}| \) as

\[
(2.11) \quad u^2 = \frac{2\Psi}{R} + Y^2 - 1.
\]

where we have introduced the function

\[
(2.12) \quad Y = R\, e^{-\lambda},
\]

using (2.8), (2.9) and (2.10) we have

\[
(2.13) \quad Y = \frac{E\, \psi_r}{(\alpha + 1)\rho^\alpha}.
\]

This function plays the role of an “effective potential” for the motion of the shells. Notice that \( u \) is known when \( Y \) and \( \Psi \) are; \( Y \) is known when \( E(r) \) is given and \( \Psi \) is known. Thus, in particular, the initial velocity profile \( u(r, r) \) is known when the functions

\[
(2.14) \quad \Psi_0(r) = \Psi(r, r), \quad Y_0(r) = Y(r, r)
\]

are known. It is therefore convenient to use \( Y_0 \) as the second arbitrary function, eliminating \( E \):

\[
(2.15) \quad Y(r, R) = \frac{\Psi_r(r, R)}{\Psi_r(r, r)} \left[ \frac{\Psi_r(r, r) R^2}{\Psi_r(r, R) r^2} \right]^{\frac{\alpha}{\alpha + 1}} Y_0(r),
\]

where (2.10) and (2.13) have been used.

We conclude that the metric for a barotropic perfect fluid in area-radius coordinates can be written in terms of the data and of the function \( \Psi \) and its first derivatives as follows:

\[
(2.16) \quad ds^2 = -\frac{1}{u^2} \left[ dR^2 - 2R' dR \, dr + \left( \frac{R'}{Y} \right)^2 \left( 1 - \frac{2\Psi}{R} \right) dr^2 \right] + R^2 (d\theta^2 + \sin^2 \theta \, d\phi^2)
\]

where \( u, R' \) and \( Y \) are given by formulae (2.10), (2.9) and (2.15) above. By a tedious but straightforward calculation the remaining field equation can be rearranged as a second order equation for \( \Psi \). Remarkably enough, this equation is quasi-linear. In fact, the following holds true:

**Theorem 2.1.** The Einstein field equations for a spherical barotropic fluid in the coordinate system (2.16) are equivalent to the following, second order PDE:

\[
(2.17) \quad a \, \Psi_{,RR} + 2b \, \Psi_{,rR} + c \, \Psi_{,rr} = d,
\]
where $a, b, c, d$ are functions of $r, R, \Psi, \Psi_r, \Psi_R$ given by:

\[(2.18a)\quad a = \frac{1}{(\alpha + 1)\Psi_R} \left[ 1 - \alpha \left( \frac{Y}{u} \right)^2 \right],\]

\[(2.18b)\quad b = \left( \frac{Y}{u} \right)^2 \frac{1}{\Psi_r},\]

\[(2.18c)\quad c = -\frac{(\alpha + 1)\Psi_R}{\alpha \Psi_r^2} \left( \frac{Y}{u} \right)^2,\]

\[(2.18d)\quad d = \frac{1}{R} \left[ -\frac{2\alpha}{\alpha + 1} \left( 1 + \left( \frac{Y}{u} \right)^2 \right) + \frac{\alpha \Psi + R\Psi_R}{\alpha u^2 R} + \frac{(\alpha + 1)\Psi_R}{\alpha \Psi_r} \left( \frac{Y^\prime_0}{Y_0} - \frac{1}{\alpha + 1} \Psi_0^\prime - \frac{2\alpha}{(\alpha + 1)r} \left( \frac{Y}{u} \right)^2 R \right) \right].\]

Remark 2.2. Equation (2.17) must be supplemented with a set of data on the surface $R = r$. Since

\[(2.19)\quad ac - b^2 = -\frac{1}{\alpha} \left( \frac{Y}{u\Psi_r} \right)^2,\]

the character of the equation is determined by the sign of $\alpha$. In particular, the equation is hyperbolic for positive pressures and elliptic for the negative ones (recall that $\alpha = 0$ is excluded). For physical reasons, however, we consider here only the hyperbolic case (see next section). The initial data for equation (2.17) are thus given, in principle, by two functions. The value of $\Psi$ on the data corresponds to the physical freedom of assigning the initial mass distribution, while the first derivative can be calculated using eq. (2.9) evaluated on the data. On $R = r$ one has $R^\prime = 1$ and therefore:

\[(2.20)\quad \Psi_R(r, r) = -\frac{\alpha}{\alpha + 1} \Psi_r(r, r).\]

Remark 2.3. A perfect fluid solution need not form a singularity: one can have oscillating, regular spheres as well. This poses the problem of characterizing the space of initial data w.r. to the final state (regular or singular). As far as we know this problem has never been studied (of course, it raises the issue of global existence that, as known, is extremely difficult) so that results like those known in the case of Einstein-Vlasov systems, for which ‘small’ (in a precise analytical sense) data lead to globally regular solutions \([36]\) are not available here. In what follows, we are not going to address this problem. Therefore, we proceed further considering those data that lead to singularity formation with analytic mass function. It is, at present, unclear the degree of genericity of such data within the whole space of available data, and this will be the subject of future work.

Remark 2.4. Equation (2.17) becomes degenerate at the sonic point, when the relative velocity of the fluid equals the speed of sound. The behavior of the solutions at the sonic point is quite complicated, and not all the solutions can be extended. The problem of characterizing the structure of the space of the solutions is extremely
interesting. As far as the present authors are aware, such an analysis has been carried out in full details only in the self-similar case [1, 33, 11]. In the present paper, however, we are interested only in singularities which arise from the gravitational interaction.

3. Formation and nature of singularities

3.1. Physical requirements. We are going to impose here strict requirements of physical reasonableness. First of all, we impose the dominant energy condition, namely, energy density must be positive and the modulus of the pressure cannot exceed the energy density (so that $-1 \leq \alpha \leq 1$). We consider, however, only the case of positive pressure. It must, in fact, be taken into account that, while tensions are common in anisotropic materials, a perfect fluid can hardly be considered as physical in presence of a negative isotropic pressure.

Therefore, $\alpha > 0$ and (2.10) imply that

$$\Psi,_{R}(r, R) < 0, \quad \forall r > 0, \quad \forall R \in [0, r],$$

and since we want $R' > 0$ to avoid shell–crossing singularities (see below), it must also be, from (2.9),

$$\Psi,_{r}(r, R) > 0, \quad \forall r > 0, \quad \forall R \in [0, r].$$

As mentioned above, we require the existence of a regular Cauchy surface ($t = 0$, say) carrying the initial data for the fields. This requirement is fundamental, since it assures that the singularities eventually forming will be a genuine outcome of the dynamics. It is easy to show that, with the equation of state used here, it is equivalent to require the matter density to be finite and non vanishing on the data. Due to eqs. (2.10) and (2.20) we get

$$\lim_{r \to 0^+} \frac{\Psi,_{r}(r, r)}{r^2} \in (0, +\infty).$$

Since area–radius coordinates map the whole set \{(t, 0) : t \leq t_0\} into the point $R = r = 0$, one may ask whether this may give rise to some kind of contradiction, that is whether the hypersurface $\{R = r\}$ fails to be regular. However, note that the coordinate change, restricted on the initial data hypersurface, is regular up to the centre, since the generic point $(0, r)$ in comoving coordinates is mapped onto the point $(r, r)$ in area–radius coordinates. Moreover, we are going to put analyticity of the data into play. In a neighborhood of the center, this property has to be checked using a cartesian system of coordinates, since even powers of $r$ can give rise to loss of differentiability at finite order in such coordinates.

To inspect this point we consider the whole set of Cauchy data for the fields. Let us choose a coordinate system on $\Sigma$ in such a way that the embedding reads

$$\Sigma(\sigma, \theta, \phi) \leftrightarrow M(r = \sigma, R = \sigma, \theta, \phi).$$
The induced metric and the extrinsic curvature (i.e. the second fundamental form) are respectively given by
\begin{align}
\text{d}s_\Sigma^2 &= \frac{1}{4\pi E(\sigma)\sigma^2 \rho(\sigma, \sigma)} \text{d}\sigma^2 + \sigma^2 \text{d}\Omega^2, \\
K_\Sigma &= -\frac{u(\sigma, \sigma)}{8\pi E(\sigma)} \left( \frac{1}{R^2 \rho} \right)_{,R} (\sigma, \sigma) \text{d}\sigma^2 - \sigma u(\sigma, \sigma) \text{d}\Omega^2.
\end{align}

It is now relatively easy to check that, if \( \Psi(r, R) \) is analytic and odd, and \( Y_0(r) = 1 + O(r^2) \) is even, using (2.10), (2.11) and (2.13) the above tensors on \( \Sigma \) are analytic and even in \( r \).

This means that all the physical quantities give rise to analytic functions in cartesian coordinates near the center.

Finally, we require regularity of the metric at the center that is, in comoving coordinates:
\begin{align}
R(0, t) &= 0, \quad e^{\lambda(0, t)} = R'(0, t),
\end{align}
for each \( t \geq 0 \) up to the time of singularity formation \( t_0 \).

The singularity forms whenever the denominator in (2.10) vanishes, that is \( R = 0 \). This kind of singularity is called a shell–focusing singularity (we have excluded here, via equations (2.9) and (3.2), the so called shell–crossing singularities at which the particle flow–lines intersect each other). In comoving coordinates \((r, t)\), the locus of the zeroes of \( R(r, t) \) defines implicitly a singularity curve \( t_s(r) \) via \( R(r, t_s(r)) = 0 \). The quantity \( t_s(r) \) represents the comoving time at which the shell labeled \( r \) becomes singular. The singularity forms if \( t_s(r) \) is finite for each shell. In physically viable cases the curve \( t_s(r) \) is strictly increasing and the center is the first point which can become singular. Regularity of the data then implies
\begin{align}
\lim_{r \to 0^+} t_s(r) = t_0 > 0.
\end{align}

In order to describe the singularity formation at the shells \( r > 0 \) by condition \( R = 0 \), from (2.10) we make the assumption
\begin{align}
\lim_{r \to 0^+} \frac{\Psi(0, R)}{R^2} &= -\infty,
\end{align}
for \( r \) sufficiently close to 0. Using the above requirements, toghether with (2.12), we can also translate relations (3.7) in area–radius coordinates asking
\begin{align}
\lim_{r \to 0^+} Y(r, \chi r) = 1, \quad \forall \chi \in (0, 1].
\end{align}

3.2. Taylor expansion of the mass. As said in Section 2.3, in the present paper we assume analyticity of the mass function at \((0, 0)\). It should be noticed that the ‘point’ \((0, 0)\) in mass–area coordinates ‘contains’ both a regular part (it contains the data \( R = r \) as \( r \) goes to zero) and a part at which the spacetime becomes singular (as \( R \) goes to zero along the singularity curve, see next section). The mass function itself however satisfies an equation which is regular at the spacetime singularity, so that the assumption made here is exactly equivalent to that usually made on the data in other models of gravitational collapse. Such data can be taken to be analytic in cartesian coordinates near the center, as in [5], or simply Taylor-expandable up to the required order as in [25]). In the present paper however we assume analiticity.
Moreover, coherently with our choice of initial data, we will assume odd–parity of the mass function.

The following holds true:

**Proposition 3.1.** The Taylor expansion of the mass function $\Psi(r, R)$ has the following structure

$$\Psi(r, R) = \frac{h}{2} \left( r^3 - \frac{\alpha}{\alpha + 1} R^3 \right) + \sum_{i+j=3+k} \Psi_{ij} r^i R^j + \ldots .$$

where $k$ is an even integer, $k \geq 2$ and $h$ is a positive constant.

**Proof.** Odd parity of $\Psi$ and regularity condition (3.3) and (2.20) imply the Taylor expansion to start from third order terms. Therefore, one certainly has

$$\Psi(r, R) = \sum_{i+j=3} \Psi_{ij} r^i R^j + \ldots .$$

For the sake of convenience we now set, for each $n \geq 0$,

$$A_n(\tau) = \sum_{i+j=3+n} i \Psi_{ij} \tau^i, \quad B_n(\tau) = \sum_{i+j=3+n} j \Psi_{ij} \tau^{j-1},$$

so that the $r^{n+2}$'s coefficients of Taylor expansions of $\Psi_r(r, r\tau)$ and $\Psi_R(r, r\tau)$ are $A_n(\tau)$ and $B_n(\tau)$ respectively. We recall that (3.3) implies $A_0(1) > 0$, and, from (2.20), $B_0(1) < 0$ follows. Using (2.15) we get

$$Y(r, r\tau) = \frac{A_0(\tau)}{A_0(1)} \left[ \frac{B_0(1) \tau^2}{B_0(\tau)} \right]^{\frac{\alpha}{\alpha + 1}} + o(1),$$

at least for each $\tau \in (0, 1]$ such that $B_0(\tau) \neq 0$ (but this polynomial can possibly vanish only for two values of $\tau$), and then (3.10) holds if

$$\frac{A_0(\tau)}{A_0(1)} \left[ \frac{B_0(1) \tau^2}{B_0(\tau)} \right]^{\frac{\alpha}{\alpha + 1}} = 1, \quad \forall \tau \in (0, 1] \text{ with } B_0(\tau) \neq 0. \quad (3.14)$$

But

$$\frac{B_0(1) \tau^2}{B_0(\tau)} = \tau^2 \frac{\Psi_{12} + 2\Psi_{21} + 3\Psi_{03}}{\Psi_{21} + 2\Psi_{12} \tau + 3\Psi_{03} \tau^2},$$

and therefore if $\Psi_{21}$ was not vanishing, the above quantity would tend to zero as $\tau \to 0$, which is in contradiction with (3.14). Then $\Psi_{21} = 0$. A similar argument applies to $\Psi_{12}$ to show that this quantity is zero as well. Finally, relation (2.20) imposes a constraint on $A_n(1)$ and $B_n(1)$:

$$-\alpha A_n(1) = (\alpha + 1) B_n(1), \quad \forall n \geq 0. \quad (3.15)$$

Using this equation for $n = 0$ and setting $h := 2A_0(1)$ we finally get formula (3.11). □

**Remark 3.2.** A tedious but straightforward calculation shows that the Taylor expansion (3.11) is compatible with (2.17) "in the Cauchy-Kowaleski sense" at any order, that is, the equation allows the iterative calculation of all the higher order terms once the data are chosen. Of course, we stress that this is not a proof of global existence up to singularity formation but only a - fundamental - consistency check for solutions here assumed a priori as regular.
Remark 3.3. The Taylor expansion given above excludes the self-similar solutions from what follows. It can, in fact, be easily shown that analyticity in self-similar variables leads to mass functions of the form $\Psi = r \tilde{\psi}(R/r)$ where $\tilde{\psi}$ is finite at $R = 0$. One recovers here a fact which is very well known in the case of dust spacetimes, where the self-similar mass profile has a constant $\tilde{\psi}$ (linear profile) while analyticity of the data for non self-similar solutions requires $\Psi$ to start from cubic terms.

3.3. The apparent horizon. A key role in the study of the nature of a singularity is played by the apparent horizon $t_h(r)$ defined in (2.4) (see for instance [23]). The apparent horizon is the boundary of the trapped surfaces, and therefore represents the comoving time at which the shell labeled $r$ becomes trapped. In area-radius coordinates this boundary is defined by $R_{\text{hor}} = 2\Psi(r, R_{\text{hor}})$. Since $\Psi_{,R}(0,0) = 0$, implicit function theorem ensures that the curve $R_{\text{hor}}$ is defined in a right neighborhood of $r = 0$. In what follows, we shall need the behavior of this curve near $r = 0$. It is easy to check that $R_{\text{hor}}$ is strictly increasing and such that $R_{\text{hor}}(r) < r$. Moreover it is $R_{\text{hor}}(r) \approx 2\Psi(r,0)$, since from (2.4) it is

$$R_{\text{hor}} = 2\Psi(r,0) + 2R_{\text{hor}} \Psi_{,R}(r,0) + R_{\text{hor}}^2 g(r, R_{\text{hor}}),$$

where $g$ is bounded and $\Psi_{,R}(r,0)$ is infinitesimal. Therefore, due to eq. (3.11), we conclude that

$$R_{\text{h}}(r) = hr^3 + \ldots$$

Next section is devoted to the study of the nature of the central ($R = r = 0$) singularity. We restrict ourselves to this singularity since, in barotropic perfect fluid models with positive pressures, it is the only one that can be naked. This is easily seen using comoving coordinates. Indeed, a singularity cannot be naked if it occurs after the formation of the apparent horizon (i.e. it must be $t_h(r) \geq t_s(r)$). A necessary condition for this is that the singularity must be massless ($\Psi(r, t_s(r)) = 0$). But, due to equation (2.2b), in presence of a positive pressure the mass is strictly increasing in a collapsing ($\dot{R} < 0$) situation, while it is zero at the regular centre. The situation can be completely different if negative pressures are allowed: in this case non central singularities can be naked as well [9].

3.4. Nakedness of the central singularity. At the center ($R = r = 0$) the apparent horizon and the singularity form simultaneously and the necessary condition for nakedness is satisfied. The singularity will be (locally) naked if there exists a radial lightlike future pointing local solution $R_g(r)$ of the geodesic equation with initial condition $R_g(0) = 0$ "travelling before the apparent horizon", that is - in area radius coordinates - $R_g(r) > R_{\text{hor}}(r)$ for $r > 0$. We will study in full details only the existence of radial null geodesics emanating from the singularity. It can in fact be proved that, if a singularity is radially censored (that is, no radial null geodesics escape), then it is censored [31, 13].

The equation of radial null geodesics in the coordinate system $(r, R)$ is easily found from (2.16) setting $d\sigma^2 = 0$ together with $d\theta = d\phi = 0$:

$$\frac{dR}{dr} = -\frac{\alpha}{\alpha + 1} \frac{\Psi_{,R}}{\Psi} \left(1 - \frac{u}{Y}\right).$$

Our main result can be stated as follows:
Theorem 3.4. For any choice of initial data $Y_0(r), \Psi_0(r)$ for the Einstein field equations such that

1. the central singularity forms in a finite amount of comoving time, and
2. the Taylor expansion of the mass function is given by (3.11),

there exists solutions of (3.17) that extend back to the central singularity, which is therefore locally naked.

To show the result we first need the following lemma.

Lemma 3.5. Called $t_x(r)$ the curve defined by $R(r, t_x(r)) = x r^3$, there exists a $x > h$ such that

(3.18) $\lim_{r \to 0^+} t_x(r) = t_0$.

Proof. It must be shown that for some $x > h$

(3.19) $\lim_{r \to 0^+} \int_0^{x r^3} \frac{\rho^\alpha(r, \sigma)}{u(r, \sigma)} d\sigma = 0$.

With the variable change $\sigma = \tau r^3$ the integral above becomes

\[ r^3 \int_0^x \frac{\rho^\alpha(r, r^3 \tau) r^{3/2} \sqrt{\tau}}{(2\Psi(r, r^3 \tau) + \tau r^3 (Y^2(r, r^3 \tau) - 1))^{1/2}} d\tau, \]

and to prove (3.19) using Fatou's lemma it suffices to show that

(3.20) $\int_0^x \limsup_{r \to 0^+} \left( \frac{\rho^\alpha(r, r^3 \tau) \sqrt{\tau}}{(2\Psi(r, r^3 \tau) + \tau r^3 (Y^2(r, r^3 \tau) - 1))^{1/2}} - \tau \right) d\tau < +\infty$.

We first notice that the quantity in square brackets at the denominator in the above expression must be positive for $r$ small. This is to ensure dynamics near the central singularity (see, e.g., (2.11)). But, using (3.11), it is

\[ \left( \frac{2\Psi(r, r^3 \tau)}{r^3} - \tau \right) = (h - \tau) + O(r^2), \]

where $O(r^2)$ is infinitesimal uniformly in $\tau$ (again, this notation means infinitesimal behaviour, uniform in $\tau$). Since $\tau$ can be greater than $h$, then $Y(r, r^3 \tau)$ cannot be infinitesimal as $r$ goes to 0. Recalling $Y = \frac{E(r) \Psi_r}{(\alpha + 1) \rho^\alpha}$, and exploiting (3.10) for $x = 1$, it is also a simple task to check that $E(r)$ behaves like $r^{-2}$,

\[ E(r) \Psi_r(r, r^3 \tau) = c_0 + O(r), \]

and so $\rho^\alpha(r, r^3 \tau)$ cannot be infinite as $r$ approaches 0. The expression for $\rho$ is given by (2.10); for simplicity we compute $\rho^{\alpha+1}$, using (3.11):

\[ \rho^{\alpha+1}(r, r^3 \tau) = -c_1 \frac{\Psi_r(r, r^3 \tau)}{r^{\alpha+1} \tau^2} = c_1 \cdot \left[ \frac{3}{2} \frac{h \alpha}{\alpha + 1} - \frac{1}{\tau^2} \left( \frac{\Psi_{41}}{r^2} + \Psi_{61} + O(r) \right) - \frac{2}{\tau} (\Psi_{32} + O(r)) + O(r) \right], \]
where \( c_1 \) is the positive constant \((4\pi\alpha)^{-1}\). As said above, this cannot be infinite and therefore \( \Psi_{41} \) vanishes, giving for some constant \( c_2 \)

\[
\rho^a(r, r^3\tau) = \frac{c_2}{\tau^{\frac{2\alpha}{\alpha+1}}} (b(\tau) + O(\tau))\alpha+1, 
\]

where \( b(\tau) \) is a regular function. This yields, passing to the limit \( r \to 0^+ \), the following expression for the integral in (3.20):

\[
\int_0^x \frac{c_2b(\tau)}{\tau^{\frac{2\alpha}{\alpha+1}}} \sqrt{\tau} \left[ (h - \tau)b(\tau)\alpha+1 + \tau c_0^2 c_2^2 \tau^{\frac{2\alpha}{\alpha+1}} \right]^{1/2} d\tau.
\]

The term in square brackets at the denominator is bounded away from zero for \( \tau \leq h \) and so is for \( x \) greater than but sufficiently near to \( h \). Recalling the bound \( \alpha < 1 \), the above integral is therefore finite, and the lemma is proved.

**Remark 3.6.** Let us observe that we have incidentally shown here that

\[
\Psi_{,R}(r, xr^3) = -a(x)r^6 + \ldots, \quad \left(1 - \frac{h}{Y}\right)(r, xr^3) = d(x) + \ldots,
\]

where \( a(x) \) and \( d(x) \) are some positive functions.

Also observe that the same argument of the above lemma can be used to show that also \( t_{\text{hor}}(r) \) tends to \( t_0 \) as \( r \to 0^+ \), that is the centre gets trapped at the same comoving time it becomes singular.

**Proof of theorem 3.4.** To show the existence of singular geodesics we use a simple technique developed earlier [13]. First of all, we recall that a function \( y_0(r) \) is called a subsolution (respectively supersolution) of an ordinary differential equation of the kind \( y' = f(r, y) \) if it satisfies \( y_0' \leq f(r, y_0) \) (respectively \( \geq \)). Now, it can be shown [12] that the apparent horizon \( R_h(r) \) is a supersolution of the geodesic equation (3.17). The singularity is certainly naked if it is possible to find a subsolution \( R_{+}(r) \) of the same equation which stays over the horizon. In fact, choose a point \((r_0, R_0)\) in the region \( S = \{(r, R) : r > 0, R_{\text{hor}}(r) < R < R_{+}(r)\} \). At this point the (regular) Cauchy problem with datum \( R(r_0) = R_0 \) admits a unique local solution \( R_g(r) \). Now the extension of this solution in the past cannot escape from \( S \) since either it would cross the supersolution from above or it would cross the subsolution from below. Thus it must extend back to the singularity with \( \lim_{r \to 0^+} R_g(r) = 0 \).

We now proceed to show that a subsolution always exist. For this aim, it suffices to consider a curve \( R_x(r) = xr^3 \), with \( x > h \). Indeed, computing the righthand side of (3.17) for \( R_x(r) \), using (3.22), we get that \( R_x(r) \) is certainly a subsolution of (3.17) if

\[
x < \frac{\alpha}{\alpha + 1} \frac{h}{2a(x)r^4} d(x),
\]

that is always satisfied, independently of \( x \), for \( r \) sufficiently small.

Therefore, if we consider the curve \( R_x(r) \) for \( x > h \) sufficiently near to \( h \), then Lemma 3.5 ensures that – re–translated in comoving coordinates – it emanates from the central singularity, and so the theorem is proved.

We stress that the theorem holds for any solution satisfying (3.8) and (3.11).
4. Extension to the general barotropic case

We are going to show in the present section that our main result, namely the existence of naked singularities, actually hold for the general (i.e. not necessarily linear) barotropic equation of state $\epsilon = \epsilon(\rho)$ provided that a set of (physically motivated) requirements are satisfied by the state function:

**Assumption 4.1.** We assume $\epsilon = \epsilon(\rho)$ to be a $C^1$ function in $[\bar{\rho}, +\infty)$ (where $\bar{\rho} \geq 0$), such that $\epsilon(\rho) \geq 0$ ($= 0$ iff $\rho = \bar{\rho}$). Recalling (2.5), that is $p(\rho) = \rho \frac{d\epsilon}{d\rho} - \epsilon$, we also assume $p(\rho)$ is a strictly positive $C^1$ function with $\frac{dp}{d\rho} > 0$, except at most for a bounded interval $[\bar{\rho}, \rho_1]$, possibly coinciding with a single point, where $p(\rho)$ can vanish.

**Remark 4.2.** Observe that:

1. The assumptions made imply that
   \begin{equation}
   \frac{d\epsilon}{d\rho}(\rho) > 0 \text{ if } \rho > \bar{\rho}.
   \end{equation}
   and therefore $\epsilon(\rho)$ is a strictly increasing positive function.

2. Differentiating (2.5) we have, where it makes sense,
   \begin{equation}
   \frac{dp}{d\rho} = \rho \frac{d^2\epsilon}{d\rho^2},
   \end{equation}
   then $\epsilon(\rho)$ is strictly convex for $\rho$ sufficiently large, and so
   \begin{equation}
   \lim_{\rho \to +\infty} \frac{d\epsilon}{d\rho}(\rho) = +\infty.
   \end{equation}

3. The assumptions made imply the existence of $\lim_{\rho \to +\infty} p(\rho)$. In addition, if the limit would be finite, say $l$, then we should have $\frac{d\epsilon}{d\rho}(\rho) < \frac{1}{\rho}(l + \epsilon(\rho))$, and then $\epsilon(\rho) < \rho + l$ by a simple comparison argument in o.d.e., which is in contradiction with (4.3). Thus
   \begin{equation}
   \lim_{\rho \to +\infty} p(\rho) = +\infty.
   \end{equation}

**Remark 4.3.** We stress that the above mentioned hypotheses are quite natural from the physical point of view. Besides obviously including the $p = \alpha\epsilon$ equation of state considered so far, they include, for instance, the equation of state of the perfect gas $p(\rho) = K_2 \rho$ for which $\epsilon(\rho) = K_1 \rho + K_2 \rho \log \rho$ where $K_1$ and $K_2$ are positive constants (in this case one obviously has $\rho_1 = \bar{\rho} = e^{-K_1/K_2}$).

Einstein’s equation (2.2b) reads
\begin{equation}
   p = -\frac{\Psi_{,R}}{4\pi R^2}.
\end{equation}
Using it, together with (2.2a), (2.5), (2.6) and the coordinate change formulae $\Psi' = \Psi_x + R' \Psi_{,R}$ and $\dot{\Psi} = \dot{R} \Psi_{,R}$, we obtain the general counterparts for equations (2.9) and (2.13), namely
\begin{equation}
   R' = \frac{\Psi_x}{\Psi_{,R}} \frac{p}{\epsilon + p} = \frac{\Psi_x}{4\pi R^2(\epsilon + p)}.
\end{equation}
Actually, to the notation used in Section 3.2 we get

\[ Y(r, R) = \frac{E(r)\Psi_x(r, R)}{(de/d\rho)(\rho(r, R))} \]

Using these formulae, we can again express the metric in the form (2.16). The crucial point is now that we can express \( \frac{4\rho}{d\rho} \) as a function of \( p(\rho) \) and, as a consequence, the dynamics of the system is, also in the general case, expressed in terms of the mass function and its derivatives only. To this end, consider the parameterized curve in \( \mathbb{R}^2 \)

\[ \mathbb{R}^+ \ni \rho \mapsto \left( \xi(\rho) = p(\rho), \zeta(\rho) = \frac{d\rho}{d\rho}(\rho) \right). \]

This curve is globally the graphic of a function \( \zeta = \zeta(\xi) \), recalling that \( \xi \) is a non-decreasing function of \( \rho \) (by the assumptions on \( p(\rho) \)), and that, by (4.2),

\[ \frac{d\zeta}{d\rho} = \xi(\rho) \neq 0. \]

Since also \( \zeta \) is increasing by (1) of Remark 4.2, there exists \( \lim_{\rho \to \rho^+} \zeta(\rho) = \zeta_0 \) finite. Denoting by \( \zeta_0 \) the finite number \( \lim_{\rho \to \rho^+} \zeta(\rho) \), the function may be prolonged up to the point \( (\xi_0, \zeta_0) \). Let also observe that this function is \( C^1 \), for each \( \xi > \xi_0 \), where indeed \( \frac{d\zeta}{d\rho} \) is strictly positive by the assumptions made on \( p(\rho) \).

Using this result, and recalling (2.5), one finds that \( Y \) in (4.7) (and then \( u \) in (2.11)) can be expressed as functions of the data and of the mass function \( \Psi(r, R) \) and its derivatives. Then, again, with some calculations one obtains a second order PDE that must be satisfied by \( \Psi \). As in the case treated so far, we consider only analytic solutions of this equation, and proceed to analyze the structure of the lower order terms of the mass profile.

First of all, since \( R' \equiv 1 \) on the data surface \( R = r \), the expression for the initial energy is:

\[ \epsilon(r, r) = \frac{\Psi_x(r, r) + \Psi_R(r, r)}{4\pi r^2}. \]

Imposing the regularity condition \( \lim_{r \to 0^+} \epsilon(r, r) \in (0, +\infty) \) and making reference to the notation used in Section 3.2 we get

\[ A_0(1) + B_0(1) > 0. \]

Actually, \( A_0(1) \neq 0 \). Otherwise, \( \epsilon(r, r) + p(r, r) \to 0 \) as \( r \to 0^+ \), since \( \epsilon(r, r) + p(r, r) = \frac{\Psi_x(r, r)}{4\pi r^2} = \frac{A_0(1)}{\pi} + o(1) \). But \( (\epsilon + p)(\rho) \) is a strictly increasing and nonnegative function of \( \rho \), then it would be \( \rho(r, r) \to \bar{\rho} \), which would imply \( \epsilon(r, r) \to 0 \), that is a contradiction.

As in section 3.1 for physical reasonableness we suppose the initial energy \( \epsilon(r, r) \) (and therefore \( p(r, r) \)) to be a non increasing function of \( r \). This implies that we can consider, without loss of generality, the case in which also \( B_0(1) \neq 0 \). In fact, if \( B_0(1) \) vanishes by (2.5) it has to be \( p(r, r) \to 0 \) as \( r \to 0^+ \). This fact, recalling the assumptions made on the pressure, shows that \( p(r, r) \) (that is a non increasing function of \( r \)) must be identically zero. But \( \rho \) (and therefore \( p \)) must diverge at the spacetime singularity, and therefore there exists an hypersurface, such that \( p \) is non zero but the energy \( \epsilon \) is still regular, where we can re-assign the initial data on. On this hypersurface, the pressure must converge to a finite non-zero value
as \( r \to 0^+ \). Then we will suppose \( B_0(1) \neq 0 \). Finally, we note that positivity of pressure on the data further implies that \( A_0(1) > 0 \) and \( B_0(1) < 0 \).

We are now ready to investigate lower order terms in the mass function. Recall that regularity of pressure along the initial data implies that \( \Psi \) cannot contain first order terms (see (4.5)). Then, as in (3.12) we set

\[ \Psi(r, R) = \sum_{i+j=3} \Psi_{ij} r^i R^j + \ldots. \]

We now denote by \( \epsilon_0 > 0 \) the limit \( \lim_{r \to 0^+} \epsilon(r, r) \). By the Remark 4.2 there exists a unique \( \rho_0 > 0 \) such that \( \epsilon(\rho_0) = \epsilon_0 \), and clearly \( \rho_0 = \lim_{r \to 0^+} \rho(r, r) \). We also denote by \( \beta_0 \) the positive number \( \frac{d\rho}{d\epsilon}(\rho_0) \). Using (3.10) at \( \tau = 1 \) we have

\[ E(r) = \frac{\beta_0}{A_0(1)} \frac{1}{r^2} + \ldots, \]

plus higher order terms. Observe now that, for a fixed \( \tau \),

\[ p(r, r\tau) = \frac{B_0(\tau)}{4\pi r^2} + \ldots = -\frac{1}{4\pi} \left( \frac{\Psi_{21}}{r^2} + 2 \frac{\Psi_{12}}{r} + 3 \Psi_{03} \right) = p_0(\tau). \]

If \( (\Psi_{21}, \Psi_{12}) \neq (0, 0) \), then \( p_0(\tau) \to \infty \) as \( \tau \to 0 \), and so \( \rho(r, r\tau) \) (and therefore \( \frac{d\rho}{d\epsilon}(\rho(r, r\tau)) \)) is sufficiently large, for \( \tau \) near to 0. This leads to a contradiction, since using (4.8) in (4.7) shows that \( Y(r, r\tau) \approx \frac{\mu}{(d/\epsilon)(r, r\tau)} \), for some non–zero constant \( \mu \) independent of \( \tau \), but (3.10) must hold. Then, again, \( B_0(\tau) = B_0(1)\tau^2 \), and \( p_0(\tau) = p_0(1) \equiv p_0 > 0 \).

Then the above argument shows that the lower order terms of the mass have the structure, analogue to (3.11),

\[ \Psi(r, R) = \frac{h}{2} \left( r^3 - \frac{p_0}{\epsilon_0 + p_0} R^3 \right) + \ldots. \]

We now proceed analyzing the nature of the singularity forming at the center. With arguments similar to Lemma 3.5 opportunely modified, it can be checked that some of the curves \( R_+ = x r^3 \), for \( x > h \) sufficiently near, are emanating from the central singularity (if seen in comoving coordinates). Indeed, we first observe that, in the case of a barotropic equation of state, (2.24) yields \( -d\nu = \frac{dp}{\epsilon + p} \leq \frac{dp}{\nu} \), where the inequality is given by dominant energy condition \( \epsilon - p \geq 0 \). This implies \( e^{-\nu} \leq \sqrt{p} \), and so the counterpart for the integral in (3.20) in this case has the following upper bound

\[
\int_0^x \lim_{r \to 0^+} \sup_{r \to 0^+} \left( \frac{e^{-\nu}(r, r^3\tau)\sqrt{\tau}}{\left( \frac{2\Psi(r, r^3\tau)}{r^3} - \tau + \tau Y^2(r, r^3\tau) \right)^{1/2}} \right) d\tau \leq \\
\leq \int_0^x \lim_{r \to 0^+} \sup_{r \to 0^+} \left( \frac{\sqrt{p(r, r^3\tau)\tau}}{\left( \frac{2\Psi(r, r^3\tau)}{r^3} - \tau + \tau Y^2(r, r^3\tau) \right)^{1/2}} \right) d\tau.
\]
Taking into account (3.22) (that still holds) in (4.5) to evaluate \( p(r, r^3\tau) \), one can see that the integral above takes a similar form to (3.21):
\[
\int_0^x \frac{c b_1(\tau)}{\sqrt{\tau [(h - \tau)b_1(\tau) + \tau b_2(\tau)]^{1/2}}} d\tau.
\]
and so it is finite, as in Lemma 3.5.

Now, using (4.6), one can compute both sides of the null radial geodesic equation
\[
(4.10) \quad \frac{dR}{dr} = R' \left(1 - \frac{u}{Y}\right)
\]
for \( R = x r^3 \), obtaining a similar expression to (3.23). We only remark that in this case, since \( Y(r, r^3\tau) \) cannot be infinitesimal as \( r \) goes to zero, then \( \frac{d}{dr}(\rho(r, r^3\tau)) \) is finite (see (4.7)) and so is \( \rho(r, r^3\tau) \). We can therefore conclude this section with the analogue of Theorem 3.4 that is:

**Theorem 4.4.** Under the hypotheses made on the equation of state in the assumption 4.1, for any choice of initial data for the Einstein field equations such that

1. the central singularity forms in a finite amount of comoving time, and
2. the Taylor expansion of the mass function is given by (4.9),

there exists solutions of (4.10) that extend back to the central singularity, which is therefore locally naked.

5. DISCUSSION AND CONCLUSIONS

Up today all analytical studies on naked singularities formation in collapsing matter of astrophysical interest (i.e. fluids) have assumed simplifying hypotheses such as dust or self-similarity.

We have shown here for the first time that among non self-similar barotropic perfect fluid solutions, all those describing complete collapse for which the mass function is regular in a neighborhood of the regular center up to singularity formation form naked singularities. Besides of spherical symmetry, this result is independent on any simplifying assumption.

The problem of the classification of the data which leads to such singularities remains for future work. In particular, it is unclear if the set generating naked singularities is really of non-zero measure in the space of the data or not.

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