THE BOHR COMPACTIFICATION OF AN ABELIAN GROUP
AS A QUOTIENT OF ITS STONE-ČECH COMPACTIFICATION

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Abstract. We will prove that, for any abelian group $G$, the canonical (surjective and continuous) mapping $\beta G \rightarrow bG$ from the Stone-Čech compactification $\beta G$ of $G$ to its Bohr compactification $bG$ is a homomorphism with respect to the semigroup operation on $\beta G$, extending the multiplication on $G$, and the group operation on $bG$. Moreover, the Bohr compactification $bG$ is canonically isomorphic (both in algebraic and topological sense) to the quotient of $\beta G$ with respect to the least closed congruence relation on $\beta G$ merging all the Schur ultrafilters on $G$ into the unit of $G$.

For any (discrete) semigroup $S$, its Stone-Čech compactification $\beta S$ admits a semigroup operation extending the original multiplication on $S$ and turning it into the universal compact right topological semigroup densely extending $S$. The semigroups $\beta S$ have proved their usefulness, versatility and importance in various branches of mathematics, mainly in combinatorial number theory and topological dynamics. In particular, the algebraic and topological structure of the semigroups $\beta N$ and $\beta Z$ has been spectacularly applied in proving a handful of striking combinatorial results in number theory. The reader is referred to the monograph Hindman, Strauss [6] for a more complete account.

Similarly, the Bohr compactification $bG$ of a locally compact abelian group $G$ is the universal compact abelian group densely extending $G$. It is of crucial importance in harmonic analysis, mainly as the tool enabling to treat the almost periodic functions on $G$ through their (continuous) extensions to $bG$. For more details see, e.g., Hewitt, Ross [4], [5].

In the present paper we will bring to focus the relation between the Stone-Čech and the Bohr compactification of any (discrete) abelian group $G$. Since the Stone-Čech compactification $\beta G$ is “more universal” than the Bohr compactification $bG$, the embedding $G \rightarrow bG$ induces a canonical surjective and continuous mapping $\xi: \beta G \rightarrow bG$. We will show that $\xi$ is a homomorphism with respect to the semigroup operation on $\beta G$, extending the multiplication on $G$, and the group operation on $bG$. Then the set of all pairs $(u, v) \in \beta G \times \beta G$ such that $\xi(u) = \xi(v)$ is a closed congruence relation on $\beta G$. We will explicitly describe this relation as the least closed congruence relation $\Xi(G)$ on $\beta G$ merging all the Schur ultrafilters on $G$ (a notion to be defined later on) into the unit of $G$. Thus the Bohr compactification $bG$ is canonically isomorphic (both in algebraic and topological sense) to the

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quotient $\beta G/\Xi(G)$. As an intermediate result we will show that, for any compact right topological semigroup $S$, the quotient $S/\Theta(S)$ of $S$ with respect to the least closed congruence relation $\Theta(S)$ on $S$ containing all the pairs $(eu, u)$ where $e, u \in S$ and $e$ is an idempotent, is a right topological group with certain universal property.

In [6], Chapter 21, various compactifications of semigroups $S$ were studied and characterized as some particular quotients of the compact semigroups $\beta S$. However, the corresponding equivalence relations were always described in terms of certain families of continuous functions from $\beta S$ to $\mathbb{R}$ or $\mathbb{C}$, obtained as extensions of various kinds of “almost periodic” functions defined on $S$. In our approach, the corresponding congruence relations $\Theta(\beta G)$ and $\Xi(G)$ are described in terms of the inner algebraic and topological structure of the Stone-Čech compactification $\beta S$.

Analogous questions make sense within a more general framework: at least for the class of locally compact abelian groups or even for more general classes of topological groups. The former case is intended as a subject of our further study [8].

1. Right topological semigroups and groups

The reader is assumed to be familiar with the basic notions and results of general topology. All the unexplained notions can be found, e.g., in Engelking [2]. In what follows we will tacitly assume that all the topological spaces dealt with are Hausdorff. As a consequence, passing to a quotient $X/E$ of such a topological space $X$, we will have to guarantee that the corresponding equivalence relation, regarded as a subset $E \subseteq X \times X$, is closed.

In a semigroup $(S, \cdot)$, the left shifts $L_a : S \to S$ and the right shifts $R_a : S \to S$ are defined by $L_a(x) = ax$, $R_a(x) = xa$, respectively, for $a, x \in S$. A semigroup $(S, \cdot)$ endowed with a topology $\tau$ is called a right topological semigroup provided all the right shifts $R_a : S \to S$ are continuous. Using left shifts, the concept of a left topological semigroup can be defined analogously. A right topological semigroup which is (algebraically) a group is called a right topological group, and similarly for left topological groups. If $(S, \cdot, \tau)$ is both a left and right topological semigroup then we say that it is a semitopological semigroup. A topological semigroup $(S, \cdot, \tau)$ is a semigroup such that the multiplication $\cdot : S \times S \to S$ is (jointly) continuous. A topological group $(G, \cdot, \tau)$ is a group which is a topological semigroup such that the inverse map $J : G \to G$, where $J(x) = x^{-1}$, is continuous, as well (cf. Hindman, Strauss [6]).

Depending on context, we will omit the multiplication sign $\cdot$ or the symbol of the topology $\tau$ or both in the notation of a (left or right) topological semigroup $(S, \cdot, \tau)$.

In what follows we will heavily depend on the following two results due to Ellis (see, e.g., [6], Theorem 2.5 and Corollary 2.39, respectively).

**Proposition 1.1.** Let $(S, \cdot, \tau)$ be a compact right topological semigroup. Then $S$ contains at least one idempotent element.

**Proposition 1.2.** Let $(G, \cdot, \tau)$ be both a semitopological group and a locally compact topological space. Then $(G, \cdot, \tau)$ is a topological group.
just in the case of semigroups. An equivalence relation $E$ on a semigroup $S$ is called a \textit{congruence relation} on $S$ if it is preserved by the semigroup operation, i.e., if for any $(x_1, y_1), (x_2, y_2) \in E$ we have $(x_1x_2, y_1y_2) \in E$, as well. Denoting by $[x]_E$ the coset of the element $x \in S$ with respect to $E$, the quotient $S/E$ can be turned into a semigroup defining the operation on $S/E$ by

$$[x]_E \cdot [y]_E = [xy]_E,$$

for $x, y \in S$. If $S$ is a group then any congruence relation $E$ on $S$ is uniquely determined by the coset $[1]_E$ (which is a normal subgroup of $S$) and $E$ is preserved by the inverse map, as well, i.e., $(x^{-1}, y^{-1}) \in E$ for any $(x, y) \in E$, thus $S/E$ becomes a group with the inverse map given by $[x]_E^{-1} = [x^{-1}]_E$. However, for a semigroup this is not the case in general. If $\tau$ is a topology on $S$ and $E$ is \textit{closed} as subset of the product $S \times S$ then $S/E$ endowed with the quotient topology of $\tau/E$, i.e., the finest topology making the canonical projection $x \mapsto [x]_E: S \to S/E$ continuous, becomes a hausdorff topological space. If $(S, \cdot, \tau)$ is a right topological semigroup and $E$ is a closed congruence relation on it, then the quotient $(S/E, \cdot, \tau/E)$ is a right topological semigroup, as well. The properties “being a left topological semigroup”, “being a topological group”, etc., are preserved under the quotients with respect to closed congruence relations in an analogous way.

If $(S, \cdot, \tau)$ is a right topological semigroup, then it is clear that the full relation $S \times S$ is a closed congruence relation on $S$ and the intersection of any family $(E_i)_{i \in I}$ of closed congruence relations on $S$ is a closed congruence relation on $S$. As a consequence, for every subset $D \subseteq S \times S$, there exists the least closed congruence relation $\Phi$ on $S$ such that $D \subseteq \Phi$, namely

$$\Phi = \bigcap \{E \mid E \text{ is closed congruence relation on } S \text{ and } D \subseteq E\}.$$ 

In the discrete case, the description in purely algebraic terms of the least congruence containing a given set $D \subseteq S \times S$ can be found in \cite{1} or \cite{3}.

Let $(S, \cdot)$ be a semigroup and $\tau$ be a topology on $S$. We denote by $\Theta(S)$ the least closed congruence relation on $S$ containing all the pairs $(eu, u)$ where $u \in S$ is an arbitrary element and $e \in S$ is an idempotent. If $(S, \cdot)$ has a unit element 1 then, obviously, $\Theta(S)$ coincides with the least closed congruence on $S$ containing all the pairs $(e, 1)$ where $e$ runs over all the idempotents in $S$.

Now, we can record the following easy consequence of the first of Ellis’ theorems (Proposition \ref{1.1}).

**Theorem 1.3.** Let $(S, \cdot, \tau)$ be a compact right topological semigroup. Then the quotient $S/\Theta(S)$ endowed with the quotient topology is a compact right topological group. Moreover, if $E$ is any closed congruence relation on $S$, then $S/E$ is a right topological group if and only if $\Theta(S) \subseteq E$.

**Proof.** Let us denote $\Theta = \Theta(S)$; then $S/\Theta$ with the quotient topology is a compact (hausdorff) right compact topological semigroup. It suffices to show that it is a group. Let $[u] = [u]_\Theta \in S/\Theta$ denote the coset of the element $u \in S$ with respect to $\Theta$. Obviously, for any idempotent $e \in S$, the coset $[e]$ is the left unit in $S/\Theta$. It remains to show that any coset $[v]$ has a left inverse in $S/\Theta$. Since the right side multiplication by $v$ is continuous, the set

$$Sv = R_v[S] = \{sv \mid s \in S\}$$
is a compact subsemigroup of \( S \), hence it is a compact right topological semigroup, as well. Thus, by the first Ellis’ theorem (Proposition 1.1), \( Sv \) contains an idempotent of the form \( e = uv \) for some \( u \in S \). Then \([u][v] = [e]\) is the unit in \( S/\Theta \) and \([u]\) is the left inverse of \([v]\). Thus \( S/\Theta \) is indeed a group.

If \( E \) is a closed congruence relation on \( S \) then \( S/E \) is a right topological semigroup. If it is a group, then it is clear that all the idempotents in \( S \) must be sent to the unit element of \( S/E \) by the canonical projection \( S \rightarrow S/E \). Hence

\[ [eu]_E = [e]_E \cdot [u]_E = [u]_E, \]

and \((eu, u) \in E\), for any \( u, e \in S \) whenever \( e \) is an idempotent. Thus \( \Theta \subseteq E \). If \( \Theta \subseteq E \) then \( S/E \) as a homomorphic image of the group \( S/\Theta \), is itself a group. □

The following universal property of the the canonical projection \( \vartheta \): \( S \rightarrow S/\Theta(S) \) (and of the quotient \( S/\Theta(S) \)) follows immediately from the last theorem.

**Corollary 1.4.** Let \((S, \cdot, \tau)\) be a compact right topological semigroup, \((G, \cdot, \tau')\) be a right topological group and \( \phi: S \rightarrow G \) be a continuous homomomorphism. Then there is a unique continuous homomorphism \( \phi': S/\Theta(S) \rightarrow G \) such that

\[ \phi = \phi' \circ \vartheta. \]

The following proposition is essentially just a more detailed reformulation of the second of Ellis’ theorems, quoted as Proposition 1.2 here.

**Proposition 1.5.** Let \((G, \cdot, \tau)\) be a locally compact right topological semigroup. Then the following conditions are equivalent:

(i) \((G, \cdot, \tau)\) is a topological group;

(ii) the inverse map \( J: G \rightarrow G \) is continuous;

(iii) \((G, \cdot, \tau)\) is a left topological group.

**Proof.** The implication (i) \( \Rightarrow \) (ii) is trivial, (iii) \( \Rightarrow \) (i) is the second of Ellis’ theorems (Proposition 1.2). Thus it suffices to prove (ii) \( \Rightarrow \) (iii).

For any \( a, x \in S \) we have \( ax = (x^{-1}a^{-1})^{-1} \), i.e.,

\[ L_a = J \circ R_{a^{-1}} \circ J \]

which shows the continuity of \( L_a \). □

2. The Stone-Čech compactification, Schur ultrafilters, and the Bohr compactification

The Stone-Čech compactification of a set \( X \), regarded as a discrete topological space, consists of the set \( \beta X \) of all ultrafilters on \( X \) endowed with the topology with the base formed by clopen sets of the form \( \{ u \in \beta X \mid A \in u \} \) where \( A \subseteq X \). Identifying each element \( x \in X \) with the principal ultrafilter \( \{ A \subseteq X \mid x \in A \} \), \( X \) becomes embedded into \( \beta X \) as a dense subset. Moreover, \( \beta X \) has the following universal property: for any compact topological space \( K \) and any (automatically continuous) mapping \( f: X \rightarrow K \) there is a unique continuous mapping \( \hat{f}: \beta X \rightarrow K \) such that \( f(x) = \hat{f}(x) \) for any \( x \in X \), given by the u-limit

\[ \hat{f}(u) = u \lim_{x \in X} f(x) = \lim_{x \rightarrow u} f(x), \]

for \( u \in \beta X \) (which is well defined by the compactness of \( K \)). Then \( \hat{f} \) is surjective if and only if \( f[X] \) is dense in \( K \) (see Hindman, Strauss [6]).
If \((S, \cdot)\) is a (discrete) semigroup, then the semigroup operation can be extended from \(S\) to \(\beta S\) putting
\[
A \in uv \iff \{s \in S \mid s^{-1}A \in v\} \in u,
\]
for \(u, v \in \beta S, \ A \subseteq S\), where
\[
s^{-1}A = L_{-1}^{-1}[A] = \{x \in S \mid sx \in A\}.
\]
Then \((\beta S, \cdot)\) is a compact right topological semigroup; however, the left shifts \(L_u : \beta S \to \beta S\) are continuous just for the principal ultrafilters \(u\), i.e., elements of \(S\). In general, \(\beta S\) is not commutative even if \(S\) is (see [4], again).

Following Protasov [7], we call an ultrafilter \(u \in \beta S\) a Schur ultrafilter if for each \(A \in u\) there exist \(a, b \in A\) such that \(ab \in A\). It can be easily verified that every idempotent ultrafilter is a Schur one. On the other hand, not every Schur ultrafilter is idempotent. Namely it is known that there are idempotent ultrafilters \(e_1, e_2\) in the Stone-Čech compactification \(\beta \mathbb{Z}\) of the abelian group \((\mathbb{Z}, +)\) such that \(e_1 + e_2\) is not idempotent. However, as proved by Protasov [7], Lemma 5.1, if \((G, +)\) is an abelian group, then the sum \(u + v\) of any two Schur ultrafilters in \(\beta G\) is Schur again. Hence, \(e_1 + e_2\) is a Schur ultrafilter which is not idempotent.

If \((G, \cdot)\) is a group then, for each set \(A \subseteq G\), we denote \(A^{-1} = \{a^{-1} \mid a \in A\}\), as well as \(u^{-1} = \{A^{-1} \mid A \in u\}\) for any ultrafilter \(u \in \beta G\). The mapping \(u \mapsto u^{-1}\) is obviously a homeomorphism \(\beta G \to \beta G\). We will need the following result proved in [7], Lemma 5.2.

**Proposition 2.1.** Let \(G\) be a group. Then, for every ultrafilter \(u \in \beta G\), the ultrafilter \(uu^{-1}\) is Schur.

Let \((G, \cdot)\) be any group. We denote by \(\Xi(G)\) the least closed congruence relation on \(\beta G\) containing all the pairs \((u, 1)\) where \(u \in \beta G\) is a Schur ultrafilter. For the least closed congruence \(\Theta(\beta G)\) on \(\beta G\) merging together all the idempotents we obviously have \(\Theta(\beta G) \subseteq \Xi(G)\).

**Theorem 2.2.** Let \(G\) be a group. Then the quotient \(\beta G/\Xi(G)\) is a compact topological group.

**Proof.** Let us abbreviate \(\Theta = \Theta(\beta G), \ \Xi = \Xi(G)\), and denote by \([u] = [u]_{\Xi}\) the coset of any ultrafilter \(u \in \beta G\) with respect to \(\Xi\). Since \(\Theta \subseteq \Xi\) and \(\beta G/\Theta\) is (algebraically) a group, so is its homomorphic image \(\beta G/\Xi\). Obviously, \(\beta G/\Xi\) endowed with the quotient topology, is a compact (hausdorff) space. Thus it suffices to show that \(\beta G/\Xi\) is indeed a topological group. Clearly, \(\beta G/\Xi\) is a right topological group. From Proposition 2.1 it follows that the coset \([u^{-1}]\) is the inverse element of the coset \([u]\). At the same time, the inverse map \(u \mapsto u^{-1}\) is continuous on \(\beta G\), hence the inverse map \([u] \mapsto [u]^{-1} = [u^{-1}]\) is continuous on \(\beta G/\Xi\), as well. From Proposition 1.5 it follows that \(\beta G/\Xi\) is a topological group. \(\square\)

**Remark.** The fact that the quotient \(\beta G/\Xi(G)\) is (algebraically) a group could be proved also directly, realizing that every right shift \((\beta G)u\) contains an idempotent hence a Schur ultrafilter, similarly as in the proof of Theorem 1.3 and without using the fact that \(\beta G/\Xi(G)\) is a homomorphic image of \(\beta G/\Theta(\beta G)\). However, Protasov’s Lemma (Proposition 2.1) guarantees that this Schur ultrafilter has the particular form \(u^{-1}u\), as well as the continuity of the inverse map needed in order to allow for the application of Proposition 1.5.
Corollary 2.3. Let $G$ be an abelian group. Then the quotient $\beta G/\Xi(G)$ is a compact abelian group.

Proof. It suffices to realize that the topological group $\beta G/\Xi(G)$ contains a dense abelian subgroup $G/\Xi(G)$. □

Let $G$ be any locally compact abelian (LCA) group. Its dual group $\hat{G}$ consists of all continuous homomorphisms (characters) from $G$ to the multiplicative group $T$ of all complex units, and, endowed with the pointwise multiplication of characters and the compact-open topology, it is an LCA group, too. In particular, $G$ is discrete if and only if $\hat{G}$ is compact and vice versa. The celebrated Pontryagin-van Kampen duality theorem states that the canonical mapping $G \to \hat{\hat{G}}$, sending any element $x \in G$ to the character $\hat{x}: \hat{G} \to T$ given by $\hat{x}(\gamma) = \gamma(x)$, is an isomorphism of topological groups. Using this map, $G$ is identified with its second dual $\hat{\hat{G}}$, and the hat over $x$ is usually omitted (see, e.g., Hewitt, Ross [4]).

Using Pontryagin-van Kampen duality, the Bohr compactification $bG$ of any LCA group $G$ can be defined as the dual group $\hat{G}_d$ of $\hat{G}$, i.e., of the dual group $\hat{G}$ endowed with the discrete topology (see [4], [5]). Thus $bG$ consists of all homomorphisms $h: \hat{G} \to T$, and not just of the continuous ones. Since $\hat{G}_d$ is discrete, its dual $bG$ is compact and the canonical map $x \mapsto \hat{x}$ maps $G$ onto the dense subgroup $\hat{\hat{G}} \cong G$ of $bG$. Alternatively, the Bohr compactification $bG$ can be characterized through the following universal property: For any continuous homomorphism $\phi: G \to K$ from an LCA group $G$ to a compact topological group $K$ there exists a unique continuous homomorphism $\hat{\phi}: bG \to K$ such that $\hat{\phi}(\hat{x}) = \phi(x)$ for all $x \in G$. Additionally, $\hat{\phi}$ is surjective if and only if $\phi[G]$ is dense in $K$.

For any (discrete) abelian group $G$, the existence of the canonical map $\beta G \to bG$ follows from the universal property of $bG$: There is a unique continuous map $\xi: \beta G \to bG$ such that $\xi(x) = \hat{x} = x$ for each $x \in G$. A more detailed description of this mapping uses the same universal property of $bG$ once again. Every character $\gamma \in \hat{G}$, being a continuous map $\gamma: G \to T$, extends to a continuous map $\hat{\gamma}: \beta G \to T$. This mapping sends each ultrafilter $u \in \beta G$ to the $u$-limit

$$\hat{\gamma}(u) = u\lim_{x \to u} \gamma(x) = \lim_{x \to u} \gamma(x),$$

which is well defined due to the compactness of the unit circle $T$. At the same time, since the multiplication on $T$ is continuous, we have

$$\hat{\gamma}\chi(u) = \lim_{x \to u} (\gamma\chi)(x) = \lim_{x \to u} \gamma(x) \lim_{x \to u} \chi(x) = \hat{\gamma}(u)\hat{\chi}(u),$$

for $\gamma, \chi \in \hat{G}$. That way every ultrafilter $u \in \beta G$ induces a character $\xi_u: \hat{G}_d \to T$ given by

$$\xi_u(\gamma) = \hat{\gamma}(u),$$

for $\gamma \in \hat{G}$. Obviously, for a principal ultrafilter $x \in G$, we have $\xi_x(\gamma) = \gamma(x) = \hat{x}(\gamma)$. Hence the assignment $u \mapsto \xi_u$ necessarily coincides with the canonical continuous surjective map $\xi: \beta G \to bG$ induced by the inclusion map $G \to bG$.

Because of its nice logarithmic properties, we find more convenient to use the arc metric $|\arg(x/y)|$ on the unit circle $T$ in the proofs of the next two propositions, instead of the euclidian one. Obviously, they both induce the same topology on $T$. 
Proposition 2.4. Let $G$ be an abelian group. Then the canonical map $\xi: \beta G \to bG$ is a homomorphism $(\beta G, \cdot) \to (bG, \cdot)$.

Proof. We will show that the ultralimit

$$\xi_{uv}(\gamma) = \overline{\gamma}(uv) = \lim_{x \to uv} \gamma(x)$$

equals the product $\xi_u(\gamma)\xi_v(\gamma) = \overline{\gamma}(u)\overline{\gamma}(v)$, for any $u, v \in \beta G$ and $\gamma \in \hat{G}$. Let $0 < \varepsilon < \pi/2$. There are sets $A \in u, B \in v$ such that $|\arg(\overline{\gamma}(u)/\gamma(a))| < \varepsilon$ for each $a \in A$, as well as $|\arg(\overline{\gamma}(v)/\gamma(b))| < \varepsilon$ for each $b \in B$. Then the set $C = AB$ obviously belongs to $uv$, and for any $c \in C$ we can find $a \in A, b \in B$ such that $c = ab$. Then $\gamma(c) = \gamma(a)\gamma(b)$ and

$$\left|\arg\frac{\gamma(c)}{\xi_u(\gamma)\xi_v(\gamma)}\right| = \left|\arg\left(\frac{\gamma(a)}{\overline{\gamma}(u)} \cdot \frac{\gamma(b)}{\overline{\gamma}(v)}\right)\right| \leq \left|\arg\frac{\gamma(a)}{\overline{\gamma}(u)}\right| + \left|\arg\frac{\gamma(b)}{\overline{\gamma}(v)}\right| < 2\varepsilon,$$

showing that $\xi_{uv} = \xi_u \xi_v$. \qed

Remark. A quick inspection of the proof shows that we have proved a bit more. Namely,

$$\xi_u(\gamma)\xi_v(\gamma) = \overline{\gamma}(u)\overline{\gamma}(v) = (u \circ v) \cdot \lim_{x \in G} \gamma(x),$$

where $u \circ v$ denotes the filter on $\beta G$ generated by all the sets of the form $C = AB$, for $A \in u, B \in v$. As it is clear that $u \circ v = v \circ u \subseteq uv \cap vu$, we have

$$\xi_{uv}(\gamma) = (u \circ v) \cdot \lim_{x \in G} \gamma(x) = \xi_{vu}(\gamma).$$

This has the neat consequence that, in spite of that the ultrafilters $uv$ and $vu$ may differ, they still determine the same character of $\hat{G}_d$ (which, of course, follows directly from the commutativity of the group $bG$ and homomorphy of the canonical map $\beta G \to bG$, as well).

We will describe the closed congruence relation

$$\text{Eq}(\xi) = \{(u, v) \in \beta G \times \beta G \mid \xi_u = \xi_v\} = \bigcap_{\gamma \in \hat{G}} \text{Eq}(\overline{\gamma})$$

on $\beta G$, induced by the continuous surjective homomorphism $\xi$.

Proposition 2.5. Let $G$ be an abelian group. Then

$$\text{Eq}(\xi) = \Xi(G).$$

Proof. First we show that $\xi_u(\gamma) = \overline{\gamma}(u) = 1$ for every Schur ultrafilter $u \in \beta G$ and any $\gamma \in \hat{G}$; then $\xi_u$ is the unit element in $bG$ and the inclusion $\Xi(G) \subseteq \text{Eq}(\xi)$ follows immediately. So let $u \in \beta G$ be Schur and $\gamma \in \hat{G}$. Let $0 < \varepsilon < \pi/3$. Then there is an $A \in u$ such that

$$\left|\arg\frac{\overline{\gamma}(u)}{\gamma(c)}\right| < \varepsilon$$

for each $c \in A$. Let $a, b \in A$ be such that $ab \in A$, as well. Then

$$|\arg\xi_u(\gamma)| = \left|\arg\left(\frac{\overline{\gamma}(u)}{\gamma(a)} \cdot \frac{\overline{\gamma}(u)}{\gamma(b)} \cdot \frac{\gamma(ab)}{\gamma(u)}\right)\right| \leq \left|\arg\frac{\overline{\gamma}(u)}{\gamma(a)}\right| + \left|\arg\frac{\overline{\gamma}(u)}{\gamma(b)}\right| + \left|\arg\frac{\gamma(ab)}{\gamma(u)}\right| < 3\varepsilon.$$
Hence $|\arg \xi_u(\gamma)| = 0$, i.e., $\xi_u(\gamma) = \tilde{\gamma}(u) = 1$.

Since $bG$ is a universal compactification of the group $G$, the reversed inclusion $\text{Eq}(\xi) \subseteq \Xi(G)$ follows from the fact that the quotient $\beta G/\Xi(G)$ is a compact topological group, established in Theorem 2.2. □

As a consequence of Propositions 2.4 and 2.5, the mapping $\beta G/\Xi(G) \to bG$, induced by the canonical mapping $\xi: \beta G \to bG$, is a bijective continuous homomorphism of topological groups. Since $\beta G$ is compact, it is a homeomorphism, too (see, e.g., Engelking [2], Theorem 3.1.13). That way we finally obtain

**Theorem 2.6.** Let $G$ be an abelian group and $\Xi(G)$ be the least closed congruence relation on $\beta G$ merging all the Schur ultrafilters $u \in \beta G$ into the unit of $G$. Then the mapping $\beta G/\Xi(G) \to bG$, induced by the canonical mapping $\xi: \beta G \to bG$ given by $\xi_u(\gamma) = \tilde{\gamma}(u)$ for $u \in \beta G$, $\gamma \in \hat{G}$, is an isomorphism of topological groups.

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