Magnetohydrodynamic fluid flow and heat transfer over a shrinking sheet under the influence of thermal slip

Saeed Ahmad a,*, Muhammad Yousaf a, Amir Khan b, Gul Zaman a

a Department of Mathematics, University of Malakand Chakdara, Dir (L), Pakhtunkhwa, Pakistan
b Department of Mathematics and Statistics, University of Swat, Pakhtunkhwa, Pakistan

* Corresponding author.
E-mail address: saeedahmad@uom.edu.pk (S. Ahmad).

Abstract

We study the heat transfer of a Magneto Hydrodynamic (MHD) boundary layer flow of a Newtonian fluid over a porous shrinking sheet under the influence of thermal slip. The flow allows electric current to pass through. The governing PDEs are transformed into self-similar ODEs via Lie group analysis. We study the variations in the dimensionless quantities like velocity and temperature of the flow in terms of the different parameters involved in the problem. We discuss the thickness of the boundary layers under the influence of various parameters involved in the flow. Numerical simulations are carried out to explain and support the results obtained.

Keywords: Applied mathematics

1. Introduction

Due to variety applications in manufacturing industries and technological processes, like, wire drawing, production of paper and glass-fiber, processing industries of metal and polymer etc., the flow of incompressible viscous fluids through stretching sheets has attracted a considerable attention of researchers. In closed analytical form, an exact similarity solution was found, later on, by Crane [1], who considered streamline of Newtonian fluid’s flow. McLeod and Rajagopal [2], later on proved the uniqueness
of the solution established by Crane. For the same flow, Gupta and Gupta studied the transfer of heat and mass over a stretching surface [3]. The idea of a flow due to a stretching surface was extended to three dimensions by Wang [4]. The investigation of magnetohydrodynamic (MHD) flow is very interesting due to promising magnetic field effects on the boundary layer. Pavlov took an uniform magnetic field into account and studied the MHD flow over a stretching surface to obtain the exact similarity solutions [5]. Andersson investigated the MHD flow of an incompressible viscous fluid over a stretching sheet [6]. The MHD flow over a stretching permeable surface without and with blowing were respectively studied by [7] and [8], where important contributions were made.

Because of having increasing applications to various engineering problems, the flow of an incompressible fluid due to a shrinking sheet has attracted much more attention [9]. An analytical study of MHD fluid’s flow past a shrinking surface was reported in [10]. In the presence of suction, Kandasamy and Khamis studied the effects of mass and heat transfer of MHD boundary layer flow over a shrinking sheet [11]. Fang and Zhang studied MHD flow over a shrinking surface and obtained an analytical solution [12]. In a later publication, they reported the thermal boundary layer flow and calculated an exact analytic solution [13]. For the MHD fluid’s flow over shrinking surface, Krishnendu studied the effects of heat source subject to mass suction [14].

Lie group analysis is a technique applied to non-linear differential equations to obtain similarity reductions. This particular analysis reduces the number of variables in the governing PDEs. Consequently the system of PDEs is converted into a self-similar system of ODEs. In this report, we consider a shrinking sheet and study the effects mass and heat transfer under the influence of thermal slip. We use the technique of Lie group to determine the self similar solution.

The above mentioned approach has been exploited for analyzing the process of conviction in many flow configurations arising in different branches of engineering and other applied sciences [15]. Ullah and Zaman used the approach of Lie group to obtain similarity transformations for the flow of tangent hyperbolic fluid over a stretching sheet subject to slip conditions [16]. Avramenko et al. used the analysis of Lie group to determine the symmetric characteristics of tumultuous flows of a boundary layer [17]. The authors of [18] studied hypothetically a declining plume of bio-convection in a profound apartment full of a fluid soaked permeable medium by exercising the method of the Lie group. The Lie group technique has been exploited to investigate blended convective flow taking the transmission of mass into account [19]. Hamad et al. used the Lie group analysis to explore the impact of thermic emission and convective surface transversality conditions over a flow of boundary layer [20]. Considering an inclined plate, Aziz et al. explored the production of heat and the reactive index’s variable in a magneto-hydrodynamic flow
by using the transformations of scaling group [21]. A free of convection nanofluid’s flow across a chemically susceptible horizontal plate in a penetrating medium was investigated by Rashidi et al. [27]. They used the technique of Lie group analysis to determine the solution.

In this report, we study the impacts of thermal source on magnetohydrodynamics (MHD) flow and the transfer of heat across a shrinking sheet taking thermal slip into account. We apply Lie group analysis to transform the governing partial differential equations into self-similar ordinary differential equations.

The report is organized as follows. In Section 2, we present the mathematical formulation. We use the Lie group analysis in Section 3 to obtain similarity transformations for our mathematical formulation. Section 4 is devoted to the detail study of the variations in the dimensionless velocity and temperature profiles for different values of the Hertmann number, the mass suction parameter and the slip parameter. Numerical simulations are presented to geometrically show the impacts of different parameters on the velocity and temperature profiles. Finally, we conclude our work in Section 5.

2. Model

In this work, we consider a magnetohydrodynamic flow of a boundary layer Newtonian fluid that allows electric current to pass through in two dimensions. The fluid is taken over a permeable declining sheet with internal thermal radiation (absorption). The thermal shift will be investigated over a sheet which is coinciding with \( y = 0 \) and assume that the flow is limited to the region where \( y \) is strictly positive. The sheet is along the horizontal axis and is perpendicular to the vertical axis.

In the presence of uniform transverse magnetic field, the fundamental equations of continuity, momentum and energy for the steady two-dimensional flow with usual notations are given below [14]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_0^2}{\rho} u, \quad (2)
\]

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{K}{\rho c_p} \frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{\rho c_p} (T - T_\infty). \quad (3)
\]

In the above equations the horizontal and vertical components of the velocity are denoted respectively by \( u \) and \( v \). The viscosity and the density of the kinetic fluid are represented respectively by \( \nu \) and \( \rho \). The symbols \( \sigma \) and \( B_0 \) represent the fluid’s electrical conductivity and the magnetic field applied to the system. The temperature
of the system is denoted by $T$ while $T_\infty$ represents free stream temperature. The symbols $Q_0$, $\kappa$ and $c_p$ express the volumetric ratio of generation of heat, the fluid’s thermal conductivity and the specific heat respectively.

Denoting the shrinking constant and temperature of the sheet respectively by $c > 0$ and $T_w$, the transversality conditions in terms of components of velocity and the temperature may be expressed as

$$U_w(x) = u = -cx, \quad v_w = -v, \quad T_w = T - D_1 \frac{\partial T}{\partial y}, \text{ when } y = 0,$$

$$T \to T_\infty, \quad u \to 0, \text{ when } y \to \infty.$$  \tag{4} \tag{5}

Here, $v_w$ denotes a prescriptive sharing of wall mass suction of the sheet which is strictly positive, and $D_1(x)$ is the slip factor of the thermal measured in (length)$^{-1}$.

Now we focus our attention on the non-dimensionalization the system under consideration. This can be done by introducing the dimensionless quantities as follows

$$\tilde{x} = \frac{x U_\infty}{v}, \quad \tilde{y} = \frac{y U_\infty}{v}, \quad \tilde{u} = \frac{u}{U_\infty}, \quad \tilde{v} = \frac{v}{U_\infty}, \quad Pr = \frac{v}{\alpha},$$

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \quad M = \frac{\sigma B_0^2 v}{\rho U_\infty^2}, \quad Q = \frac{v Q_0}{U_\infty^2 \rho c_p}.$$  \tag{6}

Now, we plug the scalings given in (6) into the system given by Eqs. (1), (2) and (3) and, for the sake of simplicity, ignore the over bars. After a simple algebraic manipulation, the expressions for the continuity, the momentum and the energy can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$  \tag{7}

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - Mu,$$  \tag{8}

$$u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^2} + Q\theta.$$  \tag{9}

The scalings defined in Eq. (6), transform the boundary conditions (4) and (5) into

$$u \ U_\infty = -cx, \quad v \ U_\infty = -v_w, \quad \theta = 1 + \frac{U_\infty D_1(x)}{v} \frac{\partial \theta}{\partial y}, \text{ when } y = 0,$$

$$u \to 0, \quad T \to T_\infty, \quad \text{when } y \to \infty.$$  \tag{10} \tag{11}

Now, for the reduction of parameters and the number of equations, the stream function $\psi$ is chosen in such a way that it satisfies the equations $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$.

We observe that the continuity of the second order partial derivatives of the stream function $\psi$ is confirmed from Eq. (7). Consequently, Eq. (7) reduces to an identity. Eqs. (8) and (9), in terms of the function $\psi$, respectively transformed into
\[
\left( \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial^2 \psi}{\partial x \partial y} \right) - \left( \frac{\partial \psi}{\partial x} \right) \left( \frac{\partial^2 \psi}{\partial y^2} \right) = \frac{\partial^3 \psi}{\partial y^3} - M \left( \frac{\partial \psi}{\partial y} \right),
\]
(12)

\[
\left( \frac{\partial \psi}{\partial y} \right) \left( \frac{\partial \theta}{\partial x} \right) - \left( \frac{\partial \psi}{\partial x} \right) \left( \frac{\partial \theta}{\partial y} \right) = \frac{1}{Pr} \left( \frac{\partial^2 \theta}{\partial y^2} \right) + Q \theta.
\]
(13)

By introducing the function \( \psi \) the boundary conditions (10) and (11) can be written as

\[
\frac{\partial \psi}{\partial y} = -\frac{c x}{U_{\infty}}, \quad \frac{\partial \psi}{\partial x} = \frac{V_{\infty}}{U_{\infty}} \theta = 1 + \frac{U_{\infty} D_1(x)}{v} \frac{\partial \theta}{\partial y} \quad \text{at} \quad y = 0,
\]
(14)

\[
\frac{\partial \psi}{\partial y} \rightarrow 0, \quad T \rightarrow T_w \quad \text{as} \quad y \rightarrow \infty.
\]
(15)

Our next step is to find the invariant solutions for the system described by (12) and (13) under a particular continuous one parametric group. This is the same as to determine the similarity solutions to the system given by (12) and (13). In this regard, we look for a group of transformations from the set of one parametric scaling transformation simplified form of the Lie group analysis.

### 3. Analysis

Our goal in this section is to apply the technique of the Lie group to determine transformations of similarity. Consequently, the system of nonlinear PDE’s will be converted into a self similar system of linear ODE’s. To do so, we need to introduce the following scaling group of transformations (see, for instance, [22, 23]),

\[
\Gamma : \quad x^* = x e^{\gamma_1}, \quad y^* = y e^{\gamma_2}, \quad \psi^* = \psi e^{\gamma_3}, \quad \theta^* = \theta e^{\gamma_4}.
\]
(16)

The letter \( \epsilon \) in (16) denotes the parameter of the group \( \Gamma \) and \( \gamma_i \), \( i = 1, 2, 3, 4 \), are arbitrary real numbers to be calculated. The introduction of the above scalings (16) transforms the ordered triplet \((x, y, \psi, \theta)\) into \((x^*, y^*, \psi^*, \theta^*)\).

Putting (16) in the Eqs. (12) and (13), one arrives at

\[
e^{(\gamma_1 + 2\gamma_2 - 2\gamma_3)} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial \psi^*}{\partial x^*} \frac{\partial^2 \psi^*}{\partial x^* \partial y^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial^2 \psi^*}{\partial y^* \partial y^*} \right) = e^{(3\gamma_2 - \gamma_3)} \frac{\partial^3 \psi^*}{\partial y^*^3} - M e^{(\gamma_2 - \gamma_3)} \frac{\partial \psi^*}{\partial y^*},
\]
(17)

\[
e^{(\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4)} \left( \frac{\partial \psi^*}{\partial y^*} \frac{\partial \theta}{\partial x^*} - \frac{\partial \psi^*}{\partial x^*} \frac{\partial \theta}{\partial y^*} \right) = e^{(2\gamma_2 - \gamma_3)} \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y^*^2} + e^{-\epsilon \gamma_4} Q \theta^*.
\]
(18)

The newly structured system described by Eqs. (17) and (18) will remain unchanged if their exponents are forced to satisfy the ensuing relations (see, e.g., [24, 25])

\[
\gamma_1 + 2\gamma_2 - 2\gamma_3 = 3\gamma_2 - \gamma_3 = \gamma_2 - \gamma_3,
\]
(19)

\[
\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4 = 2\gamma_2 - \gamma_4 = -\gamma_4.
\]
(20)

The Eqs. (19) and (20) can be simultaneously solved to give
$$\gamma_1 = \gamma_1, \quad \gamma_2 = \gamma_4 = 0, \quad \gamma_3 = \gamma_1.$$

By inserting (21) into the similarity (16), one may observe that the transformations given by \( \Gamma \) are changed to the subsequent mono parametric class of transformations

\[
\Gamma : \quad \psi^* = \psi e^{\gamma_1}, \quad x^* = xe^{\gamma_1}, \quad y^* = y, \quad \theta^* = \theta.
\]

Expanding mono parametric group of transformations (22) by Taylor’s series considering \( \epsilon \) very small and confining up to leading order terms in \( \epsilon \), the transformations \( \Gamma \) are converted to the elementary form

\[
\Gamma : \quad x^* - x = xe^{\gamma_1}, \quad y^* - y = 0, \quad \psi^* - \psi = \psi e^{\gamma_1}, \quad \theta^* - \theta = 0.
\]

A simple algebraic manipulation of Eq. (23) leads mono parametric group of transformations (22) to the characteristic equation

\[
\frac{d x}{x \gamma_1} = \frac{d y}{0} = \frac{d \psi}{\psi \gamma_1} = \frac{d \theta}{0}.
\]

Applying algebraic manipulation to Eq. (24), one may be able to deduce the following similarity transformations

\[
y = \eta, \quad \psi = xf(\eta), \quad \theta = \theta(\eta),
\]

where \( \eta \) is the similarity variable, and \( f, \theta \), are the dependent variables. Our next task is the similarity equations. Introducing the transformations (25) to the governing equations (12) and (13), we arrive at the following system of ordinary differential equations

\[
M f' + (f')^2 - f'' f - f''' = 0, \quad (26)
\]

\[
\theta'' + Pr(f \theta' + Q \theta) = 0, \quad (27)
\]

where the primes denote derivatives with respect to the parameter \( \eta \). Finally, solving the system presented by the relations (26) and (27) under the boundary conditions (14) and (15) leads to

\[
f = S, \quad f' = -1, \quad \theta = 1 + b \theta'(0),
\]

\[
f' \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty,
\]

where \( S = -\frac{V_w}{U_\infty} \) and \( b = \frac{U_\infty D_1(x)}{\nu} \) are the mass suction and thermal slip parameters respectively.

4. Discussion

4.1. Technique of solution

To find the solution of system consisting of the nonlinear ODE’s presented by (26) and (27) under the BC’s (28), we follow the finite difference code [26] together with
Figure 1. Profiles of the velocity (left) and temperature (right) of the flow as function of $\eta$ for several values of Hertmann number $M$.

Shooting method using the symbolic computation software MATHEMATICA 11. Taking an infinite interval into account, one can not solve the given system. Even if the interval is finite but large enough, solution of the system over it would be improbable. To verify the persistent behavior of the solution in the limit where the boundary approaches infinity, we solve the system for increasingly larger intervals. The infinity condition is taken at large but finite value of $\eta$. The desired value of $\eta$ is chosen in a way where there does not occur extensive change in velocity and temperature profiles.

To perform numerical simulations, we assign various values to the physical parameters in the system under consideration. Large values are assigned to the Hartmann number, represented by the symbol $M$ and the parameter of mass suction $S$ to assure the appearance of steady flow near the sheet.

4.2. Discussion

In the following, we explore the impacts of distinct parameters of the system upon the velocity as well as the temperature of the flow. From the practical point of view, this has great significance. We vary one of the parameters and fix the remaining taking physical relevancy into account.

First we study the impacts of the Hertmann number $M$ upon the velocity and temperature profiles of the flow. In Figure 1, we depict the variations in the profiles of the velocity $f'(\eta)$, and the temperature $\theta(\eta)$, as functions of $\eta$. From the left panel of the figure, one can observe that at a fixed value of $\eta$, the dimensionless velocity increases of the flow as the Hertmann number $M$ increases. Consequently the thickness of the momentum boundary layer decreases. The right panel of the figure shows that the temperature of the flow decreases with the increase in the Hertmann number. We have fixed the mass suction parameter $S = 2$, the slip parameter $b = 0.2$, the Prandtl number $Pr = 0.7$, and the source or sink parameter $Q = 0.5$. 
Figure 2. Profiles of the dimensionless velocity $f'(|eta|)$ (left) and the temperature $\theta(\eta)$ (right) for various values of the mass suction parameter $S$.

Figure 3. Impacts of the source/sink parameter on the dimensionless temperature of the flow.

The impacts of the mass suction parameter $S$ upon the profiles of the dimensionless velocity and temperatures are depicted in Figure 2. The Hertmann number $M = 2$, and the remaining parameters are the same as for Figure 1. For a fixed value of $\eta$, one can notice an increase in the dimensionless velocity of the flow as one increases the parameter of the mass suction $S$. Consequently, the thickness of the momentum boundary layer becomes thinner. The right panel of the same figure shows the variation of the dimensionless temperature for various values of $S$. We observe that, for a certain value of $\eta$, the temperature of the flow reduces as the mass suction parameter increases. Thus, the thickness of the thermal boundary layer can be reduces by increasing the suction parameter. Whenever the transfer of heat for a flow is given a prime importance, the effects of the parameter $Q$ upon the profile of the temperature of the flow becomes very important from the practical point of view. In Figure 3, we study the impacts of the sink parameter on the temperature profiles of the flow. It is clear from the figure that as strength of the heat source increases, the dimensionless temperature decreases. This indicates increasing the sink (source) parameter one can observe reduction (increases) in the thickness of the layer of thermal boundary.

In Figure 4 we have plotted the dimensionless temperature field $\theta(\eta)$ for different values of the Prandtl number $Pr$ and $S = 2$, $M = 2$, $Q = 0.5$ and $b = 0.2$. It may be observed from the profile that an increase in the heat conductivity of the fluid causes
Figure 4. Variations in the temperature profile under the influence of the Prandtl number $Pr$.

Figure 5. Temperature profiles $\theta(\eta)$ for various values of the thermal slip parameter $b$.

a decrease in the thickness and temperature of the boundary layer. We also note the $\theta$-independency of the momentum equation (26) and hence deduce that the Prandtl number $Pr$, has no impact on the profile of the dimensionless velocity field.

The temperature profiles in the $(\eta, \theta(\eta))$-plane for several values of the thermal slip parameter $b$ and $S = 2$, $M = 2$, $Q = 0.5$ and $Pr = 0.7$ are depicted in Figure 5. It can be observed that for a certain value of $\eta$, an increment in the parameter of the thermal slip causes a strict decrease in the temperature of the system. One may also observe the independence of the parameter of the thermal slip and equation of the momentum. Hence, there is no impact of the velocity field on this parameter.

The impacts of the Prandtl number $Pr$ upon the temperature gradient at sheet against $\eta$, for $M = 2$, $S = 2$ and $b = 0.2$, are depicted in Figure 6. We note that $\theta'(0)$ is negative sign definite in for all values of the Prandtl number $Pr$. This implies that there is no heat absorption in the sheet and there is a heat transfer from the sheet. The increase in the rate of heat transfer is related with the increase in the Prandtl number $Pr$. This is of particular interest when one evaluates the rate of heat transfer from the sheet.

In Figure 7 we depict skin friction in terms of the magnetic field for several values of the mass suction parameter $S$. One may observe that for a certain value of $M$, skin friction increases with an increase in the value of parameter of mass suction $S$. From
Figure 6. Profiles of the temperature gradient $\theta'(0)$ versus the source/sink parameter $Q$, for various values of the Prandtl number $Pr$.

Figure 7. Plot of skin friction $f''(0)$ in terms of the magnetic field, for various values of the mass suction parameter $S$.

the momentum equation one may also observe that the skin friction independent of $\theta$, so that there is no impact of the temperature on the skin friction.

5. Conclusion

We investigated a flow of a boundary layer of a magnetohydrodynamic (MHD) Newtonian fluid over a diminishing sheet having internal thermal source. The fluid allows an electric current to pass through. We applied the approach of Lie group to the system to obtain the similarity transformations in the form of ODE’s.

The impacts of the Hertmann number upon the profiles of the dimensionless velocity and temperature of the flow has been discussed. We have demonstrated that the temperature (velocity) decreases (increases) as we bring increase in the Hertmann number.

The variations in the velocity and temperature of the flow have been discussed in terms the mass suction parameter. We have shown that the dimensionless temperature of the flow decreases when one increases the Prandtl number, the parameter of the thermal slip and source/sink parameters. By increasing the Prandtl number and the heat source parameter, the transfer of heat can be reinforced. To
improve the quality of the final product, this result can play a vital role in production engineering. We also demonstrated that by including the thermal slip parameter in the flow, the heat absorption from the sheet can be vanished. It has predicted that the thickness of the thermal boundary layer can be altered by varying the sink (source) parameter.

Declarations

Author contribution statement

Saeed Ahmad, Muhammad Yousaf, Amir Khan, Gul Zaman: Conceived and designed the analysis; Analyzed and interpreted the data; Contributed analysis tools or data; Wrote the paper.

Funding statement

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Competing interest statement

The authors declare no conflict of interest.

Additional information

No additional information is available for this paper.

References

[1] L.J. Crane, Flow past a stretching plate, Z. Angew. Math. Phys. 21 (4) (1970) 645–647.
[2] J.B. McLeod, K.R. Rajagopal, On the uniqueness of flow of a Navier–Stokes fluid due to a stretching boundary, in: Analysis and Continuum Mechanics, Springer, Berlin, Heidelberg, 1989, pp. 565–573.
[3] P.S. Gupta, A.S. Gupta, Heat and mass transfer on a stretching sheet with suction or blowing, Can. J. Chem. Eng. 55 (6) (1977) 744–746.
[4] C.Y. Wang, The three-dimensional flow due to a stretching flat surface, Phys. Fluids 27 (8) (1984) 1915–1917.
[5] K.B. Pavlov, Magnetohydrodynamic flow of an incompressible viscous fluid caused by deformation of a plane surface, Magn. Gidrodin. 4 (1) (1974) 146–147.

[6] H.I. Andersson, MHD flow of a viscoelastic fluid past a stretching surface, Acta Mech. 95 (1) (1992) 227–230.

[7] Ioan Pop, Tsung-Yen Na, A note on MHD flow over a stretching permeable surface, Mech. Res. Commun. 25 (3) (1998) 263–269.

[8] K. Bhattacharyya, G.C. Layek, Chemically reactive solute distribution in MHD boundary layer flow over a permeable stretching sheet with suction or blowing, Chem. Eng. Commun. 197 (12) (2010) 1527–1540.

[9] M. Miklavcic, C.Y. Wang, Viscous flow due to a shrinking sheet, Q. Appl. Math. 64 (2) (2006) 283–290.

[10] T. Hayat, Z. Abbas, M. Sajid, On the analytic solution of magnetohydrodynamic flow of a second grade fluid over a shrinking sheet, J. Appl. Mech. 74 (6) (2007) 1165–1171.

[11] R. Kandasamy, A.B. Khamis, Effects of heat and mass transfer on nonlinear MHD boundary layer flow over a shrinking sheet in the presence of suction, Appl. Math. Mech. 29 (10) (2008) 1309.

[12] T. Fang, J. Zhang, Closed-form exact solutions of MHD viscous flow over a shrinking sheet, Commun. Nonlinear Sci. Numer. Simul. 14 (7) (2009) 2853–2857.

[13] T. Fang, J. Zhang, Thermal boundary layers over a shrinking sheet: an analytical solution, Acta Mech. 209 (3) (2010) 325–343.

[14] K. Bhattacharyya, Effects of heat source/sink on MHD flow and heat transfer over a shrinking sheet with mass suction, Chem. Eng. Res. Bull. 15 (1) (2011) 12–17.

[15] G. Bluman, S. Anco, Symmetry and Integration Methods for Differential Equations, vol. 154, Springer Science & Business Media, 2008.

[16] Z. Ullah, G. Zaman, Lie group analysis of magnetohydrodynamic tangent hyperbolic fluid flow towards a stretching sheet with slip conditions, Heliyon 3 (11) (2017) e00443.

[17] A.A. Avramenko, S.G. Kobzar, I.V. Shevchuk, A.V. Kuznetsov, L.T. Iwanisov, Symmetry of turbulent boundary-layer flows: investigation of different eddy viscosity models, Acta Mech. 151 (1) (2001) 1–14.
[18] A.V. Kuznetsov, A.A. Avramenko, P. Geng, Analytical investigation of a falling plume caused by bioconvection of oxytactic bacteria in a fluid saturated porous medium, Int. J. Eng. Sci. 42 (5) (2004) 557–569.

[19] M. Jalil, S. Asghar, M. Mushtaq, Lie group analysis of mixed convection flow with mass transfer over a stretching surface with suction or injection, Math. Probl. Eng. (2010).

[20] M.A.A. Hamad, M.J. Uddin, A.M. Ismail, Radiation effects on heat and mass transfer in MHD stagnation-point flow over a permeable flat plate with thermal convective surface boundary condition, temperature dependent viscosity and thermal conductivity, Nucl. Eng. Des. 242 (2012) 194–200.

[21] A. Aziz, M. Uddin, M.A.A. Hamad, MHD flow over an inclined radiating plate with the temperature-dependent thermal conductivity, variable reactive index, and heat generation, Heat Transf. Asian Res. 41 (3) (2012) 241–259.

[22] S. Mukhopadhyay, G.C. Layek, S.A. Samad, Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity, Int. J. Heat Mass Transf. 48 (21) (2005) 4460–4466.

[23] S. Mukhopadhyay, G.C. Layek, Effects of variable fluid viscosity on flow past a heated stretching sheet embedded in a porous medium in presence of heat source/sink, Meccanica 47 (4) (2012) 863–876.

[24] A.G. Hansen, Similarity Analyses of Boundary Value Problems in Engineering, Prentice-Hall, 1964.

[25] D.Y. Shang, Theory of Heat Transfer with Forced Convection Film Flows, Springer Science & Business Media, 2010.

[26] S.G. Pinto, S.P. Rodríguez, J.M. Torcal, On the numerical solution of stiff IVPs by Lobatto IIIA Runge–Kutta methods, J. Comput. Appl. Math. 82 (1–2) (1997) 129–148.

[27] M.M. Rashidi, E. Momoniat, M. Ferdows, A. Basiriparsa, Lie group solution for free convective flow of a nanofluid past a chemically reacting horizontal plate in a porous media, Math. Probl. Eng. (2014).