Integration in terms of polylogarithm

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Abstract

This paper provides a Liouville principle for integration in terms of dilogarithm and partial result for polylogarithm.

1 Introduction

Indefinite integration is classical task studied from beginning of calculus: given \( f \) we seek \( g \) such that \( f = g' \). The first step in deeper study of integration is to delimit possible form of integrals. In case of elementary integration classical Liouville-Ostrowski theorem says that only new transcendentals that can appear in \( g \) are logarithms. More precisely, when \( f \in L \) where \( L \) is a differential field with algebraically closed constant field and \( f \) has integral elementary over \( L \), then

\[
f = v_0' + \sum c_i \frac{v_i'}{v_i}
\]

where \( v_i \in L \) and \( c_i \in L \) are constants.

However, there are many elementary function which do not have elementary integrals and to integrate them we introduce new special functions in the integral. In this paper we study integration in terms of polylogarithms. Polylogarithms appear during iterated integration of rational functions [9], so they are very natural extension of elementary functions. Integration in terms of polylogarithms was studied by Baddoura [1], [2], but he only handled integrals in transcendental extensions. Also, he gave proofs only in case of dilogarithm. In recent article Y. Kaur and V. R. Srinivasan [8] give Liouville type principle for larger class of functions. They use different arguments, but for dilogarithm their results are essentially equivalent to that of Baddoura. When seeking integral we allow also algebraic extensions. We give partial result for polylogarithm (Theorem 3.1). It seems that we give first proof
of nontrivial symbolic integration result for polylogarithms of arbitrary integer order (Baddoura in \cite{3} gives a useful result, but leaves main difficulty unresolved).

In case of primitive extensions our result are based on abstract version of dilogarithm identity given by Baddoura. In exponential case we base our proof on a lemma about independence of logarithmic forms (Lemma \cite{4,2}) which may be of independent interest.

2 Setup and preliminaries

Classical polylogarithm $\text{Li}_s$ is defined by series

$$\text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

which is convergent for $|z| < 1$ and extended by analytic continuation to multivalued function. For purpose of symbolic integration of particular interest are polylogarithms of integer order. Differentiating the series we get the recurrence relation

$$z \partial_z \text{Li}_s(z) = \text{Li}_{s-1}(z)$$

so

$$\text{Li}_s(z) = \int_0^z \frac{\text{Li}_{s-1}(t)}{t} dt.$$ 

We have

$$\text{Li}_1(z) = -\log(1 - z)$$

so for positive integer $n$ polylogarithm $\text{Li}_n(z)$ is a Liouvillian function. For $n > 2$ multiple integration implied by the recurrence relation is inconvenient, so we introduce functions $I_m$ by the formula

$$I_m(z) = \int_0^z \log(t)^{m-1} \frac{dt}{1 - t}.$$ 

For integer $m \geq 1$ we have

$$\text{Li}_m(z) = \frac{(-1)^{m-1}}{(m - 1)!} I_m - \sum_{k=1}^{m-1} \frac{(-1)^k}{k!} \text{Li}_{m-k}(z) \log(z)^k.$$ 

Namely, applying $z \partial_z$ to both sides, by direct calculation we get equality, so difference between both sides is a constant. By computing limit when $z$ tends to 0 we see that the constant is zero. Consequently, $\text{Li}_m(z)$ can be expressed
in terms of $\log(z)$ and $I_k(z)$, with $k = 1, \ldots, m$. Similarly $I_m(z)$ can be expressed in terms of $\log(z)$ and $\text{Li}_k(z)$, with $k = 1, \ldots, m$. This means that integration in terms of polylogarithms is equivalent to integration in terms of $I_m(x)$.

In the sequel we assume standard machinery of differential fields (see for example [10]). We will denote derivative in differential fields by $D$, except for cases when we will need more than one derivative. When $u$ is element of a differential field $K$ we have $DI_k(u) = (D(u-1)/(u-1)) \log(u)^k = (D \log(u-1)) \log(u)^k$.

We say that a differential field $L$ is a dilogarithmic extension of $F$ iff there exists $\theta_1, \ldots, \theta_n \in L$ such that $L = F(\theta_1, \ldots, \theta_2)$ and for each $i$, $1 \leq i \leq n$, one of the following holds

1. $\theta_i$ is algebraic over $F_i$
2. $\frac{D\theta_i}{u} = Du$ for some $u \in F_i$
3. $D\theta_i = \frac{Du}{u}$ for some $u \in F_i$
4. $D\theta_i = \frac{D(u-1)}{u-1}v^k$ for some $u, v \in F_i$ such that $Du = (Dv)u$ and integer $k > 0$

where $F_i = F(\theta_1, \ldots, \theta_{i-1})$. When only first three cases appear we say that $L$ is an elementary extension of $F$. Intuitively clauses 2 to 4 above mean $\theta_i = \exp(u)$, $\theta_i = \log(u)$, $\theta_i = I_k(u)$ respectively. However, logarithm and $I_k$ is only determined up to additive constants by equations above. Similarly, exponential is only determined up to multiplicative constants. Our results about integrability do not depend on specific choice of exponentials and logarithms: different choice only changes elementary part of the integral, but does not affect integrability. We take advantage of this freedom and in the proofs assume that for nonzero $x$ and $y$ we have

$$\log(xy) = \log(x) + \log(y),$$

$$\log(-x) = \log(x).$$

Note that the second formula implies that $\log(-1) = 0$, which does not agree with definition in calculus, but still satisfies $D\log(-1) = D(-1)/(-1) = 0$. In fact, we assume that logarithms of roots of unity are all 0. Note that torsion subgroup of multiplicative group $F_*$ of a field $F$ consists of roots of unity. So $F_*$ divided by roots of unity is a torsion free group. Any finitely generated subgroup of an abelian torsion free group is a free subgroup.
So given any finite subset $S$ of $F_*$ we consider multiplicative subgroup $G$ generated by $S$ modulo roots of unity. Then on generators $g$ of $G$ we can choose logarithms in any way consistent with equation $D \log(g) = (Dg)/g$ and extend by linearity to whole $G$. This ensures that (1) and (2) hold on subgroup of $F_*$ generated by $S$. Since we will simultaneously use only finite number of elements, we can assume that (1) and (2) hold for all elements that we will use. Also, in various places when we consider logarithms we assume that they will not add new constants. Namely, adding transcendental log($a$) to a base field adds new constant iff $a$ already has logarithm in the base field. So by adding transcendental logarithms only when we can not find logarithm in base field we ensure that there will be no new constants.

It is well-known that when a field $K$ is an extension of transcendental degree 1 of $F$, which is finitely generated over $F$, then $K$ can be treated as a function field on an algebraic curve defined over $F$ ([10]). Standard tool in this situation is use of Puiseaux expansions.

For convenience we give a few known lemmas:

**Lemma 2.1** Let $F$ be a field, $\bar{F}$ its algebraic closure, $K$ an extension of $F$ of transcendental degree 1, $\psi_1, \ldots, \psi_n \in K - \{0\}$ a finite family. Multiplicative group $G$ generated by $\psi_1, \ldots, \psi_n$ modulo $\bar{F}$ is a free abelian group.

**Proof**: Without loss of generality we may assume that $K$ is generated by $\psi_1, \ldots, \psi_n$ over $\bar{F}$, so $K$ is a finite extension of $\bar{F}$ since $G$ is finitely generated abelian group it is enough to show that it is torsion free. However, a nonzero element of $G$ is an algebraic function $f$ not in $\bar{F}$, that is having a zero at same place $p$. Any power of $f$ has zero at $p$, so is not in $\bar{F}$, so not a zero element of $G$. \hfill \Box

**Lemma 2.2** Let $F$ be a differential field, $v_1, \ldots, v_n \in F - \{0\}$. Assume that $F$ has the same constants as $F(\log(v_1), \ldots, \log(v_n))$. Then $\log(v_1), \ldots, \log(v_n)$ are algebraically dependent over $F$ if and only if $\log(v_i)$ are linearly dependent over constants modulo $F$.

**Proof**: We have $n$ equations:

$$\frac{Dv_i}{v_i} - D\log(v_i) = 0.$$ 

If $\log(v_1), \ldots, \log(v_n)$ are algebraically dependent, then transcendental degree of $K = F(\log(v_1), \ldots, \log(v_n))$ over $F$ is smaller than $n$. By [10] Theorem 1 differential forms

$$\frac{dv_i}{v_i} - d\log(v_i) \in \Omega_{K/F}$$

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are linearly dependent over constants, so there exists nonzero constants \( c_1, \ldots, c_n \) such that

\[
\sum c_i \frac{dv_i}{v_i} - d \left( \sum c_i \log(v_i) \right) = 0.
\]

Choose basis \( \gamma_1, \ldots, \gamma_r \) for vector space over rationals generated by \( c_1, \ldots, c_n \) so that each \( c_i \) can be written as \( c_i = \sum \alpha_{i,j} \gamma_j \) with integer \( \alpha_{i,j} \). Write \( t_j = v_1^{\alpha_{1,j}} \ldots v_n^{\alpha_{n,j}} \). Then

\[
\sum \gamma_j \frac{dt_j}{t_j} + dw = 0
\]

where \( w = \sum c_i \log(v_i) \). By [10] Proposition 4 \( w \) is algebraic over \( F \). Since

\[
Dw = \sum c_i \frac{Dv_i}{v_i}
\]

and right hand side is in \( F \) we have \( Dw \in F \). By taking trace we see that \( w \) is in \( F \). So \( \log(v_i) \) are linearly dependent over constants modulo \( F \).

\[\square\]

**Lemma 2.3** Let \( F \) be a differential field with algebraically closed constant field, \( v_1, \ldots, v_n \in F - \{0\} \). Assume that \( F \) has the same constants as \( F(\log(v_1), \ldots, \log(v_n)) \). If \( f \in F[\log(v_1), \ldots, \log(v_n)] \) has integral elementary over \( F \), then there exists \( E \in F[\log(v_1), \ldots, \log(v_n)], u_1, \ldots, u_k \in F \) and constants \( c_1, \ldots, c_k \) such that

\[
f = DE + \sum c_i \frac{Du_i}{u_i}.
\]

**Proof:** Without loss of generality we may assume that \( \log(v_1), \ldots, \log(v_k) \) are algebraically independent and \( \log(v_{k+1}), \ldots, \log(v_n) \) are algebraic over \( K = F(\log(v_1), \ldots, \log(v_k)) \). By Lemma 2.2 for \( l = k + 1, \ldots, n \) we can write \( \log(v_l) \) as a linear combination of \( \log(v_1), \ldots, \log(v_k) \) and element from \( F \). So, we can rewrite \( f \) in terms of \( \log(v_1), \ldots, \log(v_k) \), that is assume that \( f \in F[\log(v_1), \ldots, \log(v_k)] \). Now, we can use [5] Theorem 1. Namely, due to algebraic independence of \( \log(v_1), \ldots, \log(v_k) \) the ring \( R = F[\log(v_1), \ldots, \log(v_k)] \) is a polynomial ring in \( k \) variables and \( D \) is a derivation on \( R \). Since \( Dp \) has lower degree than \( p \) by [5] Proposition 1 there are no irreducible special polynomials. Since \( DF \subset F \) we can apply [5] Theorem 1. Our \( f \) has denominator 1, so by [5] Theorem 1 also \( E \) has denominator 1 and in logarithmic part we get logarithms of elements of \( F \) (here we use fact that there are no irreducible special polynomials).

\[\square\]
3 Main results

Let $K$ be a differential field and $F$ be a differential subfield of $K$. We say that $f \in K$ has integral with polylog terms defined over $F$ when there is extension $L$ of $K$ by some number of logarithms of elements of $K$ and in $L$ we have

$$f = DE + \sum d_i D(1 - h_i) \log(h_i)^{k_i}$$

where $h_i \in F$, $d_i$ are constants $k_i$ are positive integers and $E$ is elementary over $K$. We say that $f \in K$ has integral with dilog terms defined over $F$ when there is expression as above and all $k_i = 1$.

Note that applying Lemma 2.3 to equation $g = DE$ where $g$ is difference of $f$ and polylog terms we see that $E$ is is sum of polynomial in $\log(h_i)$ with coefficients in $K$ and linear combination of logarithms of elements of $K$ with constant coefficients.

**Theorem 3.1** Let $f \in F$, $f$ has integral with polylog terms defined over $K$. If $\theta$ is an exponential, $K$ is algebraic over $F(\theta)$, $F$ and $K$ have the same constants, then $f$ has integral with polylog terms defined over an algebraic extension of $F$.

**Theorem 3.2** Let $f \in K$, $f$ has integral with dilog terms defined over $K$. If $\theta$ is a primitive, $K$ is algebraic over $F(\theta)$, $F$ and $K$ have the same constants, then $f$ has integral with dilog terms defined over an algebraic extension of $F$.

**Theorem 3.3** Let $f \in F$, $f$ has integral in a dilogarithmic extension of $F$ then $f$ has integral with dilogarithmic terms defined over an algebraic extension of $F$.

**Proof:** Let $L$ be a dilogarithmic extension such that $f$ has integral in $L$. First note that we can assume that $L$ has the same constants as $F$. Namely, $L = F(\theta_1, \ldots, \theta_n)$ where each $\theta_i$ satisfies differential equation over $F(\theta_1, \ldots, \theta_{i-1})$ (note that algebraic equation is treated as differential equation of order 0). In other words, $L = F(\theta_1, \ldots, \theta_n)$ is a dilogarithmic extension of $F$ if and only if $\theta_1, \ldots, \theta_n$ satisfy appropriate system of differential equations. $f = \gamma'$ is also a differential equation. Like in Lemma 2.1 [II] clearing denominators we convert system of differential equations to a differential ideal $I$ plus an inequality $g \neq 0$ (which is responsible for non-vanishing of denominators) and use result of Kolchin which says that differential ideal $I$ which has zero in some extension satisfying $g \neq 0$ has zero in extension having constants algebraic over constants of $F$ and satisfying $g \neq 0$. Solution clearly gives us dilogarithmic extension with constants algebraic over
constants of $F$ such that $f = \gamma'$ has solution. Adding new constants to $F$ we get algebraic extension which does not affect our claim. So we can assume that $L$ and $F$ have the same constants.

By definition of dilogarithmic extension there exists tower $F = F_0 \subset F_1 \subset \cdots \subset F_n = L$ such that $F_{k+1}$ is algebraic over $F_k(\eta_k)$, each $\eta_k$ is either primitive over $F_k$ or an exponential over $F_k$. By assumption $f$ has integral with dilogarithmic terms defined over $F_n$. Using induction and theorems 3.1 and 3.2 we see that $f$ has integral with dilogarithmic terms defined over $F$. □

4 Extension by exponential

**Lemma 4.1** Let $F$ be a differential field, $K$ be differential field algebraic over $F(\theta)$, $\theta$ be an exponential of a primitive or a primitive over $F$. Assume that $F$ is algebraically closed in $K$, $F$ and $K$ have the same constants, $\theta$ is transcendental and $\psi_i \in K$, $i = 1, \ldots, n$ are such that $\psi_i$ are multiplicatively independent modulo $F_*$. When $\theta$ is exponential of a primitive we assume that $\theta$ and $\psi_i$ are multiplicatively independent modulo $F_*$. If $a_i \in F$, $D a_i = 0$, $s \in K$,

$$\sum a_i \frac{D \psi_i}{\psi_i} + Ds \in F,$$

then $a_i = 0$ for all $i$.

**Proof:** The result follows from results of Rosenlicht [10]. Namely, when $\theta$ is a primitive we have two equations, one from assumption and the second $D \theta \in F$ from definition of $\theta$, but the transcendental degree is one, so by Theorem 1 of [10] there is differential form

$$\sum c_i \frac{d \psi_i}{\psi_i} + dv = 0$$

with constant $c_i$ and $v \in K$. Choose basis $\gamma_1, \ldots, \gamma_r$ for vector space over rationals generated by $c_1, \ldots, c_n$ so that each $c_i$ can be written as $c_i = \sum \alpha_{i,j} \gamma_j$ with integer $\alpha_{i,j}$. Write $t_j = \psi_1^{a_1,j} \cdots \psi_n^{a_n,j}$. Then

$$\sum \gamma_j \frac{d t_j}{t_j} + dv = 0.$$

By [10] Proposition 4 all $t_j$ are algebraic over $F$. Since $t_j \in K$ and $F$ is algebraically closed in $K$ we have $t_j \in F$, which means that $\psi_i$ are multiplicatively dependent modulo $F_*$. 7
When \( \theta \) is exponential of a primitive argument is similar, but we need to include \( \theta \) in the dependence. \( \square \)

**Lemma 4.2** Let \( F \) be a differential field, \( K \) be differential field algebraic over \( F(\theta) \), \( \theta \) be an exponential of a primitive or a primitive over \( F \). Assume that \( F \) is algebraically closed in \( K \), \( F \) and \( K \) have the same constants, \( \theta \) is transcendental and \( \psi_i \in K \), \( i = 1, \ldots, n \) are such that \( \psi_i \) are multiplicatively independent modulo \( F_\ast \). When \( \theta \) is exponential of a primitive we assume that \( \theta \) and \( \psi_i \) are multiplicatively independent modulo \( F_\ast \). If \( a_i \in F \), \( s \in K \),

\[
\sum a_i \frac{D\psi_i}{\psi_i} + Ds \in k,
\]

then \( a_i = 0 \) for all \( i \).

Remark: This differs from Lemma 4.1 because we dropped assumption that \( a_i \) are constants.

**Proof**: We may assume that \( K \) is finitely generated over \( F(\theta) \). When derivation on \( F \) is trivial the result is just Lemma 4.1.

To handle general derivative on \( F \) first note that if \( s \) has pole in normal place of ramification index \( r \), then \( Ds \) has order less than \( -r \), while \( \frac{D\psi_i}{\psi_i} \) has order at least \( -r \) (see [4], Lemma 1.7). This means that pole of \( Ds \) and poles of \( \frac{D\psi_i}{\psi_i} \) can not cancel. Consequently, since sum is in \( F \) we see that \( s \) has no normal poles.

\( \frac{D\psi_i}{\psi_i} \) is regular at special places (see [4], Lemma 1.8). If \( \theta \) is an exponential of a primitive, and \( s \) had pole at a special place, then also \( Ds \) would have pole at special case, which is impossible since the sum is in \( F \). In other words, when \( \theta \) is an exponential of a primitive, then \( s \) has no poles, so \( s \) is algebraic over \( F \). But \( F \) is algebraically closed in \( K \) so \( s \in F \).

Consider now mapping which maps \( \psi_i \) to vector of multiplicities of zeros and poles at normal places. This mapping extends by linearity to mapping \( \iota \) on vector space over \( \mathbb{Q} \) spanned by \( \frac{D\psi_i}{\psi_i} \). If \( a_i \in \mathbb{Z} \),

\[
\iota \left( \sum a_i \frac{D\psi_i}{\psi_i} \right) = 0,
\]

then \( \psi_i^{a_i} \) is a function with no normal poles or zeros. By replacing \( \psi_i \) by appropriate power products without loss of generality we can assume that \( \iota \left( \frac{D\psi_i}{\psi_i} \right) \) for \( i = 1, \ldots, l \) are linearly independent and \( \iota \left( \frac{D\psi_i}{\psi_i} \right) = 0 \) for \( i = l + 1, \ldots, n \). \( \iota \) takes values in \( \mathbb{Q}^A \) where \( A \) is set of normal zeros and poles of
\{ \psi_i \}$. We can extend $\iota$ by linearity to mapping from linear combinations of $D\psi_i$ with coefficients in $F$ into $F^A$. Of course $\iota(D\psi_i)$ for $i = 1, \ldots, l$ remain linearly independent over $F$. However, we can compute $\iota$ from coefficients of Puiseaux expansions of

$$\sum a_i \frac{D\psi_i}{\psi_i}$$

at places in $A$. Namely, coefficient of order $-r$, where $r$ is ramification index of the place is order of zero of $\psi_i$ times $a_i$. Moreover, since $s$ has no normal poles, $Ds$ has order bigger than $-r$, so

$$\sum a_i \frac{D\psi_i}{\psi_i} + Ds \in F$$

means that

$$\iota\left( \sum a_i \frac{D\psi_i}{\psi_i} \right) = 0$$

so $a_i = 0$ for $i = 1, \ldots, l$. In other words, to prove the lemma it remains to handle case when all $\psi_i$ have no normal zeros or poles.

Consider now Puiseaux expansion of $\frac{D\psi_i}{\psi_i}$ at a normal place $p$ of ramification index $r$. Denoting by $\lambda$ parameter of expansion we have

$$\psi_i = c_0 + c_1 \lambda^{1/r} + c_2 \lambda^{2/r} + \ldots$$

$$D\psi_i = D(\lambda) \partial_\lambda \psi_i + D(c_0) + D(c_1) \lambda^{1/r} + \ldots$$

The second part contains only terms of nonnegative order, so $D\psi_i$ has the same terms of negative order as $D(\lambda) \partial_\lambda \psi_i$. Since $s$ has nonnegative order at $p$ the same argument shows that terms of negative order in $Ds$ are the same as terms of negative order of $D(\lambda) \partial_\lambda s$. At normal place $p$ $D(\lambda)$ has order 0 so negative part of Puiseaux expansion at $p$ of

$$\sum a_i \frac{D\psi_i}{\psi_i} + Ds$$

is the same as

$$D(\lambda) \left( \sum a_i \frac{\partial_\lambda \psi_i}{\psi_i} + \partial_\lambda s \right).$$

In particular terms of negative order vanish if and only if terms of negative order in expansion of

$$\sum a_i \frac{\partial_\lambda \psi_i}{\psi_i} + \partial_\lambda s$$
vanish. But the last expression does not depend on derivative $D$. Let $X$ be derivative on $K$ such that $X$ is zero on $F$ and $X\theta = D\theta$. Note that for $D$ and $X$ we have the same set of special places. By reasoning above,

$$\sum a_i \frac{D\psi_i}{\psi_i} + Ds \in F$$

implies that

$$\sum a_i \frac{X\psi_i}{\psi_i} + Xs$$

has nonnegative order when $p$ is normal place. For special places $\frac{X\psi_i}{\psi_i}$ has nonnegative order. When $\theta$ is exponential of a primitive we observed that $s \in k$, so $Xs = 0$ and consequently also has nonnegative order. So in all places

$$\sum a_i \frac{X\psi_i}{\psi_i} + Xs$$

has nonnegative order, so is algebraic over $F$ so in $F$, which by the Lemma 4.1 means that $a_i = 0$.

It remains to handle case when $\theta$ is a primitive. Then we have $D\theta = \eta \in F$ and we can use $\theta^{-1/r}$ as parameter in Puiseaux expansion at special places. We have

$$s = \sum c_i \theta^{-i/r}$$

$$Ds = \sum Dc_i \theta^{-i/r} - \sum \frac{i}{r} c_i \eta \theta^{-(i+r)/r}$$

$$= \sum Dc_i \theta^{-i/r} - \sum \frac{i - r}{r} c_i \eta \theta^{-i/r}$$

$Ds$ has nonnegative order at special places, so all terms above with negative $i$ vanish. When $i < 0$ is lowest order term such that $c_i \neq 0$, then $Dc_i \theta^{-i/r}$ can not cancel with other terms so $Dc_i = 0$. Now, $i < -r$ implies that

$$Dc_{i+r} - \frac{i}{r} c_i \eta = 0$$

that is

$$D\theta = \eta = D \frac{-c_{i+r}}{c_i}$$

but this is impossible since $\theta$ is transcendental and $F$ and $K$ have the same constants. So $-r \leq i < 0$. Now, similarly like $Dc_i$ we see that $Dc_j = 0$ for $j < 0$. We can do the same calculation for $Xs$ and we see that all terms of negative order in $Xs$ vanish. So, we can finish like in exponential case. □
Now we are ready to prove Theorem 3.1.

Proof: When \( \theta \) is algebraic over \( F \) there is nothing to prove, so we may assume that \( \theta \) is transcendental. Let \( \bar{F} \) be algebraic closure of \( F \). By assumption we have

\[
f = g_F + DE + \sum d_i \frac{D(1 - h_i)}{1 - h_i} \log(h_i)^k_i
\]

where \( g_F \) is sum of polylog terms with arguments in \( \bar{F} \), \( E \) denotes elementary part and \( h_i \in K - \bar{F} \) are arguments of polylogs outside \( \bar{F} \). Consider group \( G \) generated by arguments of logarithms in elementary part, \( h_i \) and \( 1 - h_i \) modulo \( \bar{F} \). By Lemma 2.1 \( G \) is a free abelian group. Let \( \alpha_j \in K \) be generators of \( G \). In particular they are multiplicatively independent over \( \bar{F} \). Let \( p \) be a place of \( K \) over 0. We may assume that each \( \alpha_j = \theta^k \beta_j \) where \( \beta_j \) has nonzero value at \( p \) and \( k \) depend on \( j \). Namely, replacing \( \theta \) by a fractional power we may assume that order of \( \theta \) at \( p \) divides orders of all \( \alpha_j \). Next, we normalize \( \beta_j \) so that each has value 1 at \( p \) and chose a multiplicatively independent family \( \psi_j \) which generate the same subgroup of \( (\bar{F}K)_* \) as \( \beta_j \) (again, we can do this due to Lemma 2.1). Note that due to Lemma 4.1 \( \log(\psi_j) \) are linearly independent over constants modulo \( \bar{F} \) so by and 2.2 they are algebraically independent over \( K \). Then each \( h_i \) can be written as

\[
h_i = u_i \prod \psi_j^{n_{i,j}}\]

where \( n_{i,j} \) are integers and \( u_i = w_i \theta^{l_j} \) with \( w_i \) algebraic over \( F \) and integer \( l_j \) so

\[
\log(h_i) = \sum n_{i,j} \log(\psi_j) + \log(u_i).
\]

Similarly

\[
\log(1 - h_i) = \sum m_{i,j} \log(\psi_j) + \log(v_i)
\]

where \( m_{i,j} \) are integers and \( v_i = r_i \theta^{o_i} \) with \( r_i \) algebraic over \( F \) and integer \( o_i \). After rewriting \( h_i, \log(1 - h_i) \) as above from polylog terms we get polynomial in \( \log(\psi_j) \), with coefficients in \( N = F(\Delta \cup \{\eta\} \cup \{w_i, \log(w_i), r_i, \log(r_i)\}) \)

where \( \Delta \) is set of logarithms needed to express \( g_F \) and \( \eta = \frac{D\theta}{\theta} \). We also rewrite logarithms in elementary part like above. By Lemma 2.3 we see that now \( E \) is a polynomial in \( \log(\psi_j), j > 0 \) with coefficients in \( M = NK \). Put \( l_\alpha = \prod \log(\psi_j)^{\alpha_j} \). We have

\[
E = \sum s_\alpha l_\alpha
\]

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with \( s_\alpha \in M \). 

Expanding formula for \( f \) in terms of \( l_\alpha \) we get system of equations

\[
\sum c_{\alpha,j} \frac{D(\psi_j)}{\psi_j} + Ds_\alpha + c_{\alpha,0} = 0
\]

where \( c_{\alpha,j} \) are in \( M \). Now, \( M \) is algebraic over \( N(\theta) \). We claim that \( c_{\alpha,j} \in \bar{N} \) and \( s_\alpha \in \bar{N} \) where \( \bar{N} \) is algebraic closure of \( N \) in \( M \). We prove this inductively. The claim is vacuously true if length of \( \alpha \) is big enough so that \( c_{\alpha,i} = 0 \) and \( s_\alpha = 0 \). So we may assume that our claim is true for all multiindices of higher length. Note that for \( j > 0 \) \( c_{\alpha,j} \) is sum of \( s_{\alpha+e_j} \) and terms coming from polylogs. \( c_{\alpha,0} \) is sum of terms coming from polylogs and for \( \alpha = 0 \) also includes \( g_F - f \). Of course terms coming from polylogs and \( g_F - f \) are in \( N \). Consequently by the inductive assumption \( c_{\alpha,j} \in \bar{N} \). Hence, by the lemma 4.2 for \( j > 0 \) we have \( c_{\alpha,j} = 0 \), so \( s_\alpha \) has derivative in \( \bar{N} \). Consequently, since \( \theta \) is an exponential we have \( s_\alpha \in \bar{N} \).

Now, we look at equality with \( \alpha = 0 \). From derivative of \( E \) we get

\[
Ds_0 + \sum_{j>0} s_{e_j} \frac{D(\psi_j)}{\psi_j}
\]

We proved above that \( c_{0,j} = 0 \) for \( j > 0 \), so this simplifies to

\[
f = g_F + \sum d_i \frac{D(v_i)}{v_i} \log(u_i)^{k_i} + Ds_0.
\]

We look at \( u_i \) and \( v_i \). First, if \( u_i \) or \( v_i \) have a pole at \( p \), then \( v_i = -u_i \) and we chose logarithm in such a way that \( \log(v_i) = \log(u_i) \), so we get

\[
\frac{D(u_i)}{u_i} \log(u_i)^{k_i} = \frac{1}{k_i+1} D(\log(u_i)^{k_i+1})
\]

and we can move such terms to elementary part. If \( u_i = 1 \) or \( v_i = 1 \), then the \( \frac{D(u_i)}{u_i} \log(v_i)^{k_i} \) term vanishes. In particular this happens when \( u_i \) or \( v_i \) have zero at \( p \). Otherwise \( u_i = 1 - v_i \) and we get polylog with argument algebraic over \( F \). \( \square \)
5 Extension by primitive

We would like to investigate equalities for $I_k$. To motivate our approach consider a vector space $V \subset K$ and its $k$-th tensor power $V^{\otimes (k)}$. On $V^{\otimes (k)}$ we consider linear map $\Psi$ on simple tensor given by

$$g \otimes f_1 \cdots \otimes f_{k-1} \mapsto Dg \prod_{l=1}^{k-1} f_l.$$ 

When $g = \log(1 - u)$ and $f_l = \log(u)$ we have

$$\Psi(g \otimes f_1 \cdots \otimes f_{k-1}) = DI_k.$$ 

When tensor $s$ is symmetric then $\Psi(s)$ is a derivative of element of $K$, so is negligible from the point of view of integration. So we are lead to study identities in $V^{\otimes (k)}$ modulo symmetric tensors. Below we consider only case when $k = 2$.

Lemma 5.1 Assume $V$ is a vector space, $u, v, w_i, w_{i,j} \in V$, $k_i, l_i \in \mathbb{Z}$, $k_i = l_i$ when $k_i < 0$ or $l_i < 0$, $u = \sum_i k_i w_{i,j}$ for $l_j > 0$, $v = \sum_i l_i w_{i,j}$ for $k_j > 0$, $v - \sum_i l_i w_{i,j} = u - \sum_i k_i w_{i,j}$ when $k_j < 0$. Put

$$M_{i,j} = w_i \otimes w_j + w_i \otimes w_{j,i} + w_{i,j} \otimes w_j.$$ 

Then

$$\left( \sum_i k_i w_i + u \right) \otimes \left( \sum_j l_j w_j + v \right) - \sum_{i,j} k_i l_j M_{i,j} - u \otimes v$$

is a symmetric tensor.

Proof: We have

$$\left( \sum_i k_i w_i + u \right) \otimes \left( \sum_j l_j w_j + v \right) = \left( \sum_i k_i w_i \right) \otimes \left( \sum_j l_j w_j \right)$$

$$+ u \otimes \left( \sum_j l_j w_j \right) + \left( \sum_i k_i w_i \right) \otimes v + u \otimes v.$$ 

From $M_{i,j}$ we get

$$\sum k_i l_j M_{i,j} = \left( \sum_i k_i w_i \right) \otimes \left( \sum_j l_j w_j \right) + \sum_i k_i w_i \otimes \left( \sum_j l_j w_{j,i} \right)$$

$$+ \sum_j \left( \sum_i k_i w_{i,j} \right) \otimes l_j w_j.$$
So, sum in our claim is
\[ S_1 = \sum_j (u - \sum_i k_i w_{i,j}) \otimes l_j w_j + \sum_i k_i w_i \otimes (v - \sum_j l_j w_{j,i}) \]

It remains to show that \( S_1 \) is symmetric.

By assumption, when \( l_j > 0 \) we have \( u = \sum_i k_i w_{i,j} \) so
\[ (u - \sum_i k_i w_{i,j}) \otimes l_j w_j = 0 \]
and
\[ \sum_{l_j > 0} (u - \sum_i k_i w_{i,j}) \otimes l_j w_j = 0. \]

Similarly
\[ \sum_{k_i > 0} k_i w_i \otimes (v - \sum_j l_j w_{j,i}) = 0. \]

By assumption, when \( l_j < 0 \) (so also \( k_j < 0 \)) there are \( t_j \) such that
\[ u - \sum_i k_i w_{i,j} = v - \sum_i l_i w_{i,j} = t_j. \]

Using the equalities above we get
\[ S_1 = \sum_j (u - \sum_i k_i w_{i,j}) \otimes l_j w_j + \sum_i k_i w_i \otimes (v - \sum_j l_j w_{j,i}) \]
\[ = \sum_{l_j < 0} (u - \sum_i k_i w_{i,j}) \otimes l_j w_j + \sum_{k_i < 0} k_i w_i \otimes (v - \sum_j l_j w_{j,i}) \]
\[ = \sum_{l_j < 0} t_j \otimes l_j w_j + \sum_{k_i < 0} k_i w_i \otimes t_i = \sum_{l_j < 0} (t_j \otimes l_j w_j + l_j w_j \otimes t_j). \]

where the last equality follows since negative \( k_j \) are the same as negative \( l_j \).
Since the result is symmetric this ends the proof. \( \square \)

**Lemma 5.2** Assume that \( F \) is a differential field, \( K \) is a differential field algebraic over \( F(\theta) \), \( F \) and \( K \) have the same constants, \( \theta \) is a primitive over \( F \). Let \( h_i \in K \) be a finite family. Let \( V \) be vector space over constants spanned by logarithms with arguments algebraic over \( F \). Let \( W \) be vector space over constants spanned by logarithms of \( h_i \) and \( 1 - h_i \). Let \( A \) be set of
zeros and poles of \( h_i \) and \( 1 - h_i \) in algebraic closure of \( F \). There exist vector space over constants \( X \), embedding \( \iota \) from \( V + W \) into \( X \oplus V \), elements \( \delta_a \in X \), embedding \( \iota \) from \( V \) into \( X \oplus V \), elements \( u_i, v_i \) algebraic over \( F \), elements \( \beta_{a,b} \in V \) such that

\[
\iota(\log(h_i)) = \log(u_i) + \sum_{a \in A} \text{ord}(h_i, a) \beta_{a,b}
\]

\[
\iota(\log(1 - h_i)) = \log(v_i) + \sum_{a \in A} \text{ord}(1 - h_i, a) \beta_{a,b}
\]

when \( \text{ord}(1 - h_i, b) > 0 \) we have

\[
\log(u_i) = \sum_{a \in A} \text{ord}(h_i, a) \beta_{a,b}
\]

when \( \text{ord}(h_i, b) > 0 \) we have

\[
\log(v_i) = \sum_{a \in A} \text{ord}(1 - h_i, b) \beta_{a,b}
\]

when \( \text{ord}(h_i, b) < 0 \) we have

\[
\log(u_i) + \sum_{a \in A} \text{ord}(h_i, a) \beta_{a,b} = \log(v_i) + \sum_{a \in A} \text{ord}(1 - h_i, a) \beta_{a,b}.
\]

Moreover, for each \( i \) either \( v_i = 1 - u_i \) or \( v_i = -u_i \) or \( v_i = 1 \) or \( u_i = 1 \).

**Proof:** Let \( \bar{F} \) be algebraic closure of \( F \). Consider multiplicative group \( G \) generated by \( h_i, 1 - h_i \) and \( \bar{F}_* \). Choose place \( p \) of \( \bar{F}K \). Let \( \psi_j \) be generators of \( G \) modulo \( \bar{F}_* \), normalized so that leading coefficient of Puiseaux expansion at \( p \) is 1. By Lemma 2.1 \( G \) modulo \( \bar{F}_* \) is a free abelian group. As generators of free abelian group \( \psi_j \) are multiplicatively independent modulo \( \bar{F}_* \). Consequently, by lemmas 4.1 and 2.2 \( \log(\psi_j) \) are algebraically independent over \( F \).

There are integers \( m_{i,j} \) and \( n_{i,j} \) and elements \( u_i, v_i \in \bar{F} \) such that

\[
h_i = u_i \prod_j \psi_j^{m_{i,j}},
\]

\[
(1 - h_i) = v_i \prod_j \psi_j^{n_{i,j}}.
\]

Note that when both \( h_i \) and \( 1 - h_i \) have order 0 at \( p \), then \( v_i = 1 - u_i \). When one of \( h_i \) and \( 1 - h_i \) have negative order at \( p \), then also the second have negative order and \( v_i = u_i \). When \( h_i \) have positive order at \( p \), then
\(v_i = 1\), When \(1 - h_i\) have positive order at \(p\), then \(u_i = 1\). This covers all cases, so the last claim holds.

Let \(W_0\) be vector space over constants spanned by \(\{\log(\psi_j)\}\) and \(\bar{W}\) be vector space over constants spanned by \(W_0\) and \(V\). Let \(\tilde{A}\) be set of zeros and poles of \(\psi_j\)-s over \(\bar{F}\). We take as \(X\) vector space over constants with basis \(\delta_a, a \in \tilde{A}\) (actually, for final result we only need \(a \in A\), but for the proof we need larger \(X\)).

We define \(\iota\) on \(\bar{W}\) by the formula

\[
\iota(\log(\psi_j)) = \sum_{a \in \tilde{A}} \text{ord}(\psi_j, a) \delta_a,
\]
on \(W_0\) and as identity on \(V\). Clearly on \(V\) \(\iota\) is well-defined and is injective. Since \(\psi_j\) are multiplicatively independent modulo algebraic closure of \(F\) \(\log(\psi_j)\) are linearly independent modulo \(V\) so \(\bar{W}\) is a direct sum of \(W_0\) and \(V\). In particular it follows that \(\iota\) is well defined. Also, element in kernel of \(\iota\) has form

\[
\sum c_j \log(\psi_j)
\]
where \(c_j\) are constants such that for each \(a \in \tilde{A}\) we have

\[
\sum c_j \text{ord}(\psi_j, a) = 0.
\]

Let \(e_l\) be basis of vector space over rational numbers spanned by \(c_j\)-s. We can take \(e_l\) such that all \(c_j\) have integer coordinates. That is

\[
c_j = \sum_l q_{j,l} e_l.
\]
Then, for each \(a \in \tilde{A}\)

\[
\sum_l e_l(\sum_j q_{j,l} \text{ord}(\psi_j, a)) = 0.
\]
Since \(e_l\) are linearly independent over rationals that means that for each \(l\) and \(a\)

\[
\sum_j q_{j,l} \text{ord}(\psi_j, a) = 0.
\]
Since function without zeros and poles is in \(\bar{F}_*\) and \(\psi_j\) are multiplicatively independent over \(\bar{F}_*\) equality above means that \(q_{j,l} = 0\) so also \(c_j = 0\). This means that kernel of \(\iota\) is trivial, that is \(\iota\) is injective.

We have

\[
\iota(\log(h_i)) = \iota(\log(u_i) + \sum m_{i,j} \log(\psi_j))
\]
\[
= \log(u_i) + \sum m_{i,j} \sum_a \text{ord}(\psi_j, a)\delta_a = \log(u_i) + \sum_a \text{ord}(h_i, a)\delta_a
\]

and similarly for \(\log(1 - h_i)\). This shows first two conditions in conclusion of the lemma.

We need to define \(\beta_{a,b}\) and show that they have required properties. We first define \(\alpha_{j,a}\) as leading coefficient of Puiseaux expansion of \(\psi_j\) at \(a\) and put \(\gamma_{j,a} = -\log(\alpha_{j,a})\).

When \(a\) is a zero of \(h_i\) at \(a\) we have \(1 - h_i = 1\) so
\[
1 = v_i \prod \alpha_{j,a}^{n_{i,j}}
\]
and
\[
\log(v_i) = \sum_j n_{i,j} \gamma_{j,a}.
\]

When \(a\) is z zero of \(1 - h_i\) at \(a\) we have \(h_i = 1 - (1 - h_i) = 1\) so
\[
1 = u_i \prod \alpha_{j,a}^{m_{i,j}}
\]
and
\[
\log(u_i) = \sum_j m_{i,j} \gamma_{j,a}.
\]

When \(a\) is a pole of \(h_i\), then \(a\) is also a pole of \(1 - h_i\) and we have
\[
u_i \prod \alpha_{j,a}^{m_{i,j}} = -w_i \prod \alpha_{j,a}^{n_{i,j}}
\]
so
\[
\log(u_i) - \sum_j m_{i,j} \gamma_{j,a} = \log(w_i) - \sum_j n_{i,j} \gamma_{j,a}.
\]

For fixed \(b\) we can view \(\gamma_{j,b}\) as values of a linear operator \(T_b\) defined on \(W_0\) by the formula:
\[
T_b(\sum_j c_j \log(\psi_j)) = \sum c_j \gamma_{j,b}.
\]

Since \(\iota\) is injective on \(W_0\) and takes values in \(X\) we can treat \(T_b\) as operator defined on a subspace of \(X\) and extend it to linear operator \(\overline{T}_b\) defined on whole \(X\). We put \(\beta_{a,b} = T_b(\delta_a)\). Since \(\iota(\log(\psi_j)) = \sum_a \text{ord}(\psi_j, a)\delta_a\) we have \(\gamma_{j,b} = \sum_a \text{ord}(\psi_j, a)\beta_{a,b}\). Next, when \(\text{ord}(1 - h_i, b) > 0\) (that is \(b\) is zero of \(1 - h_i\)) we have
\[
\log(u_i) = \sum_j m_{i,j} \gamma_{j,b} = \sum_j m_{i,j} \sum_a \text{ord}(\psi_j, a)\beta_{a,b}
\]
\[
= \sum_a (\sum_j m_{i,j} \text{ord}(\psi_j, a))\beta_{a,b} = \sum_a \text{ord}(h_i, a))\beta_{a,b}.
\]
Similarly when \( \text{ord}(h_i, b) > 0 \) we have

\[
\log(v_i) = \sum_a \text{ord}(1 - h_i, a))\beta_{a,b}
\]

and when \( \text{ord}(h_i, b) < 0 \) we have

\[
\log(u_i) - \sum_a \text{ord}(h_i, a))\beta_{a,b} = \log(v_i) - \sum_a \text{ord}(1 - h_i, a))\beta_{a,b}
\]

so \( \beta_{a,b} \) satisfy conclusion of the lemma.

Now we can prove Theorem 3.2

Proof: When \( \theta \) is algebraic over \( F \) there is nothing to prove. So we may assume that \( \theta \) is transcendental over \( F \). Let \( \bar{F} \) be algebraic closure of \( F \). We have

\[
f = g_F + DE + \sum d_i \frac{D(1 - h_i)}{1 - h_i} \log(h_i)
\]

where \( g_F \) is sum of dilog terms with arguments in \( \bar{F} \), \( E \) denotes elementary parts and \( h_i \in K - \bar{F} \). Since

\[
D(\log(1 - h_i) \log(h_i)) = \frac{D(1 - h_i)}{1 - h_i} \log(h_i) + \frac{D(h_i)}{h_i} \log(1 - h_i)
\]

by changing elementary part we may assume that dilog terms are antisymmetric, that is

\[
f = g_F + DE + \sum d_i \left( \frac{D(1 - h_i)}{1 - h_i} \log(h_i) - \frac{D(h_i)}{h_i} \log(1 - h_i) \right). \tag{3}
\]

Without loss of generality we may assume that arguments of logarithms in elementary part are either in \( F \) or appear among \( h_i \). We now use Lemma 5.2 obtaining \( u_i, v_i, \) etc. with properties stated in the Lemma. Like in the proof of the Lemma we introduce space \( \bar{W} \) containing \( \log(h_i), \log(1 - h_i), \log(u_i), \log(v_i) \). On \( \bar{W} \otimes \bar{W} \) we consider mapping \( \Psi \) given by the formula

\[
\Psi(t \otimes s) = (Dt)s.
\]

We have

\[
\sum d_i \left( \frac{D(1 - h_i)}{1 - h_i} \log(h_i) - \frac{D(h_i)}{h_i} \log(1 - h_i) \right)
\]

\[
= \sum d_i \Psi(\log(1 - h_i) \otimes \log(h_i) - \log(h_i) \otimes \log(1 - h_i)).
\]
Now, to prove the theorem it is enough to show that

\[ S_1 = \sum d_i (\log(1 - h_i) \otimes \log(h_i) - \log(h_i) \otimes \log(1 - h_i)) \]

\[ - \log(v_i) \otimes \log(u_i) + \log(u_i) \otimes \log(v_i)) \]

is a symmetric tensor. Namely, on symmetric tensor \( \Psi(t \otimes s + s \otimes t) = D(st) \) so values of \( \Psi \) on symmetric tensors are derivatives of elementary functions. So by changing elementary part we can replace

\[ \sum d_i \left( \frac{D(1 - h_i)}{1 - h_i} \log(h_i) - \frac{D(h_i)}{h_i} \log(1 - h_i) \right) \]

by

\[ \sum d_i \left( \frac{D(v_i)}{v_i} \log(u_i) - \frac{D(u_i)}{u_i} \log(v_i) \right). \]

By the last claim of Lemma 5.2 we have four possibilities for \( u_i \) and \( v_i \). When \( u_i = 1 - v_i \) term of the sum above is just dilog term with argument algebraic over \( F \). When \( v_i = -u_i \), then \( \log(v_i) = \log(u_i) \) and corresponding term is zero. Also, when \( u_i = 1 \) or \( v_i = 1 \), then corresponding term is zero. So all dilog terms have arguments algebraic over \( F \) as claimed.

It remains to show that \( S_1 \) is a symmetric tensor. Since \( \iota \) is an embedding, it is enough to show that \( (\iota \otimes \iota)(S_1) \) is symmetric. By Lemma 5.1 putting

\[ M_{a,b} = \delta_a \otimes \delta_b + \delta_a \otimes \beta_{b,a} + \beta_{a,b} \otimes \delta_b \]

for each \( i \) we have

\[ (\iota \otimes \iota)(\log(1 - h_i) \otimes \log(h_i) - \log(v_i) \otimes \log(u_i)) \]

\[ = \sum_{a,b} \text{ord}(1 - h_i, a)\text{ord}(h_i, b)M_{a,b} \]

modulo symmetric tensors. Consequently, modulo symmetric tensors

\[ (\iota \otimes \iota)(S_1) = \sum_i d_i \sum_{a,b} \text{ord}(1 - h_i, a)\text{ord}(h_i, b)(M_{a,b} - M_{b,a}). \]

So, it is enough to show that the last sum equals 0. Consider projection \( \pi \) from \( V \oplus X \) onto space \( X \). We have

\[ (\pi \otimes \pi)(M_{a,b} - M_{b,a}) = \delta_a \otimes \delta_b - \delta_b \otimes \delta_a \]

so projections of \( M_{a,b} - M_{b,a} \) give linearly independent antisymmetric tensors. So it is enough to show that \( (\pi \otimes \pi)(\iota \otimes \iota)(S_1) \) is 0, because then coefficients
of $M_{a,b} - M_{b,a}$ must be zero. However, $S_1$ is in $(V + W) \otimes (V + W)$ so we can give more explicit formula for $(\pi \otimes \pi)(\iota \otimes \iota)(S_1)$. Let $W_0$ be space spanned by $\log(\psi_j)$ from the proof of Lemma 5.2. We showed that $V + W = V \oplus W_0$ and by formula for $\iota$ we see that $\pi \iota$ is just projection $\chi$ from $V + W$ onto $W_0$ followed by embedding from $W_0$ into $X$. So, it is enough to show that $S_2 = (\chi \otimes \chi)S_1$ equals 0. $\chi$ maps $\log(u_i)$ and $\log(v_i)$ to 0 while $\log(h_i)$ and $\log(1 - h_i)$ are mapped to linear combinations of $\log(\psi_j)$. Note that

$$\log(1 - h_i) \otimes \log(h_i) - \log(h_i) \otimes \log(1 - h_i)$$

is mapped to an antisymmetric tensor. So $S_2$ is an antisymmetric tensor. Consider $\Psi(S_2)$. We have

$$S_2 = \sum c_{k,j} \log(\psi_k) \otimes \log(\psi_j),$$

$$\Psi(S_2) = \sum c_{k,j} D(\log(\psi_k)) \log(\psi_j)$$

where $c_{k,j}$ are constants and $c_{k,j} = -c_{j,k}$. Note that $\log(h_i) - \chi(\log(h_i)) = \log(u_i)$ and similarly for $1 - h_i$, so $\Psi(S_1) = \Psi(S_2) + \Psi(R)$ where $R$ contains terms containing $\log(u_i)$ and $\log(v_i)$. Let $L$ be $F$ extended by $u_i, v_i, \log(u_i)$ and $\log(v_i)$. Now, expand equation (3) as polynomial in $\log(\psi_j)$ and consider terms linear in $\log(\psi_j)$. Elementary part is a polynomial of degree 2 with coefficients in $KL$. Since terms of second order in (3) are 0 coefficients of second order elementary terms are constants, that is

$$E = \sum b_{k,j} \log(\psi_k) \log(\psi_j) + \sum s_j \log(\psi_k) + s_0$$

where $s_j \in KL$, and $b_{k,j}$ are constant which we can assume to be symmetric, that is $b_{k,j} = b_{j,k}$. So coefficient of $\log(\psi_j)$ in (3) is

$$\sum c_{k,j} + 2b_{k,j}D(\log(\psi_k)) + D(a_j)$$

where $a_j \in KL$ is a sum of a linear combination of logarithms with arguments algebraic over $F$ coming from $\Psi(S_1)$ and $s_j$. Using Lemma [11] we see that $c_{k,j} + 2b_{k,j} = 0$. Since $c_{k,j}$ are antisymmetric and $b_{k,j}$ are symmetric it follows that $c_{k,j} = 0$. However, this means that $S_2 = 0$, which ends the proof. $\square$
6 Further remarks

Our lemmas 5.1 and 5.2 correspond to Proposition 2 in [1] and [2]. When in Lemma 5.2 field $K$ is purely transcendental over $F$ we could use infinite place only for normalization (do not include it in $A$), take $\delta_a = \log(\theta - a)$ and $\beta_{a,b} = \log(a - b)$. Baddoura instead of our $M_{i,j}$ uses Spence function evaluated at $(\theta - b)/\theta - 0)$. Both expression gave the same main part. Spence function has built in antisymmetry, we work modulo symmetric tensors. Also, our $M_{i,j}$ omits lowest order term present in Spence function. This considerably simplifies handling of lowest order terms in our proof. Baddoura uses different condition on $\beta_{a,b}$, which is equivalent to our condition in transcendental case thanks to symmetry of $\beta_{a,b}$, but in general we are unable to find symmetric $\beta_{a,b}$.

Our lemmas 5.1 and 5.2 are harder to use than Baddoura’s identity: we do not know if there is actual function associated to an element of $X$.

Tensor products seem to be standard tool used for studying polylogarithms (we learned about it from [4]) but seem to be new in context of symbolic integration.

Our Lemma 2.3 is simple and should be well-known (it could be easily extracted from proof of Liouville theorem). It replaces longish arguments used in [1] and [2].

We hope to extend Theorem 3.2 to polylogarithms of arbitrary order. For purpose of symbolic integration one would like to have more precise information. In particular, we would like to have method to find arguments of polylogarithms needed in given integral. Here, our current result have significant weakness: in principle we need arbitrary algebraic extension to find arguments of polylogarithms. It seems reasonable that we only need new algebraic constants and that we can find arguments of polylogarithms in base field extended by constants.

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