A NEARLY QUATERNIONIC STRUCTURE ON SU(3)

ÓSCAR MACIÁ

Abstract. It is shown that the compact Lie group SU(3) admits an Sp(2)Sp(1) structure whose distinguished 2-forms $\omega_1, \omega_2, \omega_3$ span a differential ideal. This is achieved by first reducing the structure further to a subgroup isomorphic to SO(3).

1. Introduction

An almost quaternionic Hermitian (AQH) manifold is a Riemannian $4n$-manifold $\{M, g\}$ admitting a Sp($n$)Sp(1)-structure, that is a reduction of its frame bundle to a subbundle whose structure group is the subgroup Sp($n$)Sp(1) of SO($4n$). This means that $\{M, g\}$ is equipped locally with a triple of almost complex structures $\{I_1, I_2, I_3\}$ that behave like the imaginary quaternions $i, j, k$, and are compatible with the metric. The almost complex structures $I_i$ generate a subbundle $\mathcal{I}$ of endomorphisms of $T M$.

Following the method initiated by Gray & Hervella for the study of almost Hermitian manifolds [12], the space

$$\mathbb{R}^{4n} \otimes (\text{sp}(n) \oplus \text{sp}(1)) \perp$$

of intrinsic torsion tensors decomposes into irreducible modules under the action of Sp($n$)Sp(1), giving rise to a natural classification of AQH manifolds. The intrinsic torsion can be identified with the Levi-Civita derivative $\nabla \Omega$, where

$$\Omega = \sum_{j=1}^{3} \omega_j \wedge \omega_j,$$

is the fundamental 4-form, defined locally in terms of the 2-forms given by $\omega_i(X,Y) = g(I_i X, Y)$. Conditions describing the intrinsic torsion classes can be studied accordingly. In the general case, for $n > 2$, there exist six irreducible components of intrinsic torsion. But for $n = 2$, only four components arise, giving $2^4 = 16$ classes of AQH 8-manifolds, with a closer analogy to the almost Hermitian complex case.

If the intrinsic torsion vanishes the AQH manifold is said to be quaternionic Kähler, the holonomy reduces to Sp($n$)Sp(1) and the manifold is Einstein. In 1989, Swann [18] proved

\[\text{2000 Mathematics Subject Classification.} \text{ Primary 53C15 ; Secondary 53C26.}\]

\[\text{Key words and phrases.} \text{ Almost quaternionic Hermitian, G-structure, intrinsic torsion.}\]
Theorem 1.1. Let \( \{ M, g, \mathcal{I} \} \) be an AQH \( 4n \)-manifold, \( n > 2 \), with fundamental 4-form \( \Omega \). Then, it is quaternionic Kähler if and only if \( \Omega \) is closed. For \( n = 2 \), \( \{ M, g, \mathcal{I} \} \) is quaternionic Kähler if and only if

1. The fundamental 4-form is closed, \( d\Omega = 0 \).
2. The set \( \{ \omega_1, \omega_2, \omega_3 \} \) of 2-forms generates a differential ideal.

Condition (2) means that there exists a \( 3 \times 3 \) matrix \( (\beta^j_i) \) of 1-forms such that

\[
d\omega_i = \sum_{j=1}^{3} \beta^j_i \wedge \omega_j,
\]

(1.1)

The condition itself is easily seen to be dependent only on \( \mathcal{I} \) and not the choice of basis (see Section 3).

In the case \( n = 2 \), this result left open the existence question for manifolds satisfying one of the conditions but not the other. One affirmative answer was provided in 2001 by Salamon \[10\] with an example of an AQH 8-manifold with closed, but non-parallel, fundamental 4-form. This ‘almost parallel’ manifold is a product of a 3-torus with a 5-dimensional nilmanifold, and similar examples were found by Giovannini \[11\].

In the present paper we deal with the complementary case, namely 8-manifolds with \( \text{Sp}(2)\text{Sp}(1) \)-structure for which the \( \mathcal{I} \) generates a differential ideal, but for which the 4-form \( \Omega \) is not closed.

The paper is organized as follows. In Section 2, we discuss the four components of intrinsic torsion and their relationship with the differential ideal condition. Other properties of this condition are discussed in Section 3, which enables us to re-formulate the quaternionic Kähler condition. A one-parameter family of quaternionic structures is defined in Section 4 on \( \text{SU}(3) \), endowed with a compatible deformation \( g_\lambda \) of its bi-invariant metric. In Section 5, it is shown that for specific choices of the parameter \( \lambda \), \( \{ \text{SU}(3), g_\lambda, \mathcal{I} \} \) is AQH and satisfies condition (2), but not (1) in Theorem 1.1.

Some of the computations in Sections 4 and 5 were effectively carried out using Mathematica and the differential forms package scalarEDC \[2\].

2. Intrinsic torsion and reduction to \( \text{SO}(3) \)

We will use the \( E-H \) formalism described in \[15\]. Suppose that \( n \geq 2 \). Let \( E \) (respectively, \( H \)) denote the basic complex representation of \( \text{Sp}(n) \) (respectively, \( \text{Sp}(1) \)), with highest weight \( (1, 0, \ldots, 0) \) (resp. \( (1) \)), such that \( E \simeq \mathbb{C}^{2n} \) (resp. \( H \simeq \mathbb{C}^2 \)). We denote the \( \text{Sp}(n) \)-module with highest weight \( (1, \ldots, 1, 0, \ldots, 0) \), (with \( r \) 1’s and \( n-r \) 0’s) by \( \Lambda_r^0 E \). Also, \( S^r E \) (respectively, \( S^r H \)) will denote the \( \text{Sp}(n) \)-module (respectively, \( \text{Sp}(1) \)) with highest weight \( (r, 0, \ldots, 0) \) (respectively, \( (r) \)). Finally, let \( K \) be the \( \text{Sp}(n) \)-module with highest weight \( (2, 1, 0, \ldots, 0) \).

The fundamental 4-form of the \( \text{Sp}(n)\text{Sp}(1) \)-structure is the distinguished element arising the decomposition of \( \Lambda^4 T^* \) under the action of \( \text{Sp}(n)\text{Sp}(1) \),
where

\[(2.1) \quad T^* \otimes_\mathbb{C} \mathbb{C} = E \otimes H\]

represents the complexified cotangent space. The intrinsic torsion \(\xi = \nabla \Omega\) of an \(Sp(n)Sp(1)\)-structure is described by the following

**Theorem 2.1.** (Swann, [18]) The intrinsic torsion of an AQH 4n-manifold, \(n \geq 2\) can be identified with an element in the space

\[(A_0^3 E \oplus K \oplus E) \otimes (H \oplus S^3 H).\]

For \(n = 2\), the intrinsic torsion belongs to

\[(2.2) \quad ES^3 H \oplus KS^3 H \oplus KH \oplus EH.\]

Various examples of AQH 8-manifolds with different types of intrinsic torsion are known (see for example Cabrera & Swann [4]):

**Corollary 2.2.** An AQH 8-manifold \(M\) is quaternionic if and only if \(\xi \in KH \oplus EH\).

The adjective ‘quaternionic’ here means that the underlying \(GL(2, \mathbb{H})Sp(1)\) admits a torsion-free connection. As stated, this condition is characterized by the absence of the \(Sp(1)\) module \(S^3 H\), and ensures that the ‘twistor space’ associated to \(M\), defined in [1, Ch. 14], is a complex manifold.

**Corollary 2.3.** The fundamental 4-form of an AQH 8-manifold is closed, i.e., \(M\) is almost parallel if and only if \(\xi \in KS^3 H\).

**Corollary 2.4.** The 2-forms \(\{\omega_i\}\) of an AQH 8-manifold generate a differential ideal if and only if \(\xi \in ES^3 H \oplus EH\),

These two corollaries represent the conditions stated in Theorem [11] for \(n = 2\), namely (1) and (2) respectively. The author does not know of any example in the literature of a non quaternionic Kähler 8-manifold in the class described by Corollary [2.4].

One can establish a useful analogy between the 16 Gray–Hervella classes of almost Hermitian \(2n\)-manifolds and the 16 classes of AQH8-manifolds. This is best done by indicating the \(Sp(2)Sp(1)\)-modules in Equation (2.2) by the symbols \(\mathcal{W}_1, \ldots, \mathcal{W}_4\) respectively. This ensures that, in both cases, ‘integrability’ corresponds to vanishing of the \(\mathcal{W}_1 \oplus \mathcal{W}_2\) component, and a conformal change in the metric modifies in an essential way only the \(\mathcal{W}_4\) component.

On the other hand, the class of AQH 8-manifolds with intrinsic torsion belonging to \(\mathcal{W}_1 \oplus \mathcal{W}_4\) in Corollary [2.4] is the AQH analogue of the family of almost Hermitian manifolds containing nearly Kähler and locally conformal Kähler manifolds described by Butruille [3] and Cleyton & Ivanov [6]. For more details on the classification of AQH manifolds see Cabrera & Swann [4].
Consider the homomorphism
\[ \phi: \text{Sp}(1) \to \text{Sp}(2) \times \text{Sp}(1) \]
defined by \( \phi(g) = (i(g), g) \), where \( i: \text{Sp}(1) \hookrightarrow \text{Sp}(2) \) is the inclusion whereby \( \text{Sp}(1) \) acts irreducibly on \( \mathbb{C}^4 \). By definition, \( \text{Sp}(2)\text{Sp}(1) \) is a \( \mathbb{Z}_2 \) quotient of \( \text{Sp}(2) \times \text{Sp}(1) \) whose kernel is generated by \((-1, -1)\). Therefore, \( \phi \) induces an inclusion

\[
(2.3) \quad \text{SO}(3) = \text{Sp}(1)/\mathbb{Z}_2 \hookrightarrow \text{Sp}(2)\text{Sp}(1) \subset \text{SO}(8),
\]
and in this paper we shall effectively be considering such \( \text{SO}(3) \) structures on 8-manifolds.

Let \( M \) be an 8-manifold with an \( \text{SO}(3) \)-structure compatible \( (2.3) \). Using the well-known formula
\[
S^p H \otimes S^q H \cong \bigoplus_{n=0}^{\min(p,q)} S^{p+q-2n} H,
\]
the complexified tangent space \( (2.1) \) now splits as

\[
S^3 H \otimes H \cong S^2 H \oplus S^4 H.
\]
The underlying quaternionic action is defined by a suitable inclusion of \( S^2 H \) in the space of anti-symmetric endomorphisms of the tangent space, isomorphic to
\[
\Lambda^2 T^* \cong 2S^6 H \oplus S^4 H \oplus 3S^2 H.
\]
Its image is a coefficient bundle of purely imaginary quaternions.

Relative to \( (2.3) \), we have \( E \cong S^3 H \) where \( H \) now denotes the spin representation of \( \text{Spin}(3) \). It follows that \( \Lambda^2_3 E \cong S^3 H \) and \( K \cong S^7 H \oplus S^5 H \oplus S^3 H \). The intrinsic torsion space \( (2.2) \) then decomposes as follows
\[
\begin{align*}
W_1 &= E S^3 H \cong S^6 H \oplus S^3 H \oplus S^2 H \oplus S^0 H \\
W_2 &= K S^3 H \cong S^{10} H \oplus 2S^8 H \oplus 2S^6 H \oplus 3S^4 H \oplus 2S^2 H \\
W_3 &= K H \cong S^8 H \oplus 2S^6 H \oplus S^4 H \oplus S^2 H \oplus S^0 H \\
W_4 &= E H \cong S^4 H \oplus S^2 H.
\end{align*}
\]
These isomorphisms reveal the presence of a 2-dimensional space of \( \text{SO}(3) \)-invariant tensors. Our aim is to describe an example with intrinsic torsion in the summand \( S^0 H \) in \( W_1 \). Such a ‘nearly quaternionic structure’ will be found on \( \text{SU}(3) \), although the general theory of \( \text{SO}(3) \) structures on 8-manifolds will be pursued elsewhere \[5\].

Manifolds with \( \text{SO}(3) \) structure as in \( (2.4) \) are special cases of those considered by Swann \[19\], and later Gambioli \[9\], in the context of nilpotent coadjoint orbits of a complex Lie group. The same structure arises naturally on the total space of a rank 3 vector bundle over \( \text{SU}(3)/\text{SO}(3) \) \( [7, 10] \), and can be analysed with the methods of Conti \[8\].
For the special case of SU(3), the tangent space can be identified with the Lie algebra
\begin{equation}
\mathfrak{su}(3) = \mathfrak{f} \oplus \mathfrak{p}
\end{equation}
of complex anti-Hermitian 3 \times 3 matrices. Here \( \mathfrak{f} \) is an abbreviation for the subalgebra \( \mathfrak{so}(3) \) of real anti-symmetric matrices, whereas \( \mathfrak{p} \) is the space of matrices of the form \( iS \) with \( S \) symmetric and trace-free. Observe that the decomposition (2.5) is consistent with (2.4) with \( \mathfrak{f} \cong S^2H \) and \( \mathfrak{p} \cong S^4H \).

On an 8-manifold, in view of (2.5), any reduction to SO(3) determines not only an almost quaternionic structure, but also a PSU(3)-structure in the sense of Hitchin [13].

In Section 4, we shall define endomorphisms \( I_i \) that act on (2.5) extending the adjoint representation on \( \mathfrak{f} \).

3. The Ideal Condition

We suppose throughout this section that \( \{\omega_1, \omega_2, \omega_3\} \) is a locally-defined set of 2-forms associated to a basis \( \{I_1, I_2, I_3\} \) of \( \mathcal{F} \) on an AQH 8-manifold, and that the differential ideal condition (1.1) is satisfied.

Any two bases \( \{\omega_i\}, \{\tilde{\omega}_i\} \) are related by a gauge transformation of the form
\[
\tilde{\omega}_i = \sum_{j=1}^{3} A^j_i \omega_j,
\]
with \( A = (A^j_i) \) taking values in SO(3) at each point. Then we can write
\[
d\tilde{\omega}_i = \sum_{j=1}^{3} \tilde{\beta}^j_i \wedge \tilde{\omega}_j,
\]
where
\[
\tilde{\beta}^j_i = (A^{-1})^j_i dA^l_i + (A^{-1})^j_i \beta^l_k A^k_i,
\]
with summation over repeated indices. The matrix \( \beta \) therefore transforms as
\begin{equation}
\tilde{\beta} = A^{-1}dA + \text{Ad}(A^{-1})\beta.
\end{equation}
It follows that \( \beta \) represents a connection on the rank 3 vector bundle, isomorphic to \( \mathcal{F} \), generated by the \( \{\omega_i\} \). However, this connection does not reduce to SO(3) unless \( \beta \) is anti-symmetric, a point we discuss next before a brief analysis of curvature.

Consider the decomposition
\begin{equation}
\beta = \alpha + \sigma,
\end{equation}
where \( \alpha^j_i = \frac{1}{2}(\beta^j_i - \beta^i_j) \) and \( \sigma^j_i = \frac{1}{2}(\beta^j_i + \beta^i_j) \) are the anti-symmetric and symmetric parts. The fact that \( A \in \text{SO}(3) \), implies that \( A^{-1}dA \) lies in the Lie algebra \( \mathfrak{so}(3) \) of anti-symmetric matrices. Given that Ad preserves
the decomposition (3.2), we see that the symmetric part $\sigma$ transforms as a tensor:

$$\tilde{\sigma} = \text{Ad}(A^{-1})\sigma = A^{-1}\sigma A,$$

in contrast to $\beta$.

The tensor represented by $\sigma$ can be identified with the remaining non-zero components

$$\mathcal{W}_1 \oplus \mathcal{W}_4 \simeq ES^3H \oplus EH$$

of the intrinsic torsion, or equivalently $d\Omega$. Indeed,

$$(3.3) \quad d\Omega = 2 \sum_{i=1}^{3} d\omega_i \wedge \omega_i = 2 \sum_{i,j=1}^{3} \beta_i^j \wedge \omega_i \wedge \omega_j = 2 \sum_{i,j=1}^{3} \sigma_i^j \wedge \omega_i \wedge \omega_j.$$ We can easily identify the component in $\mathcal{W}_1$:

**Lemma 3.1.** If an Sp(2)Sp(1)-structure satisfies (1.1) then its intrinsic torsion belongs to $\mathcal{W}_1$ if and only if $\text{tr}(\beta) = \beta_1^1 + \beta_2^2 + \beta_3^3$ vanishes.

**Proof.** The $\mathcal{W}_4$ part of the torsion is represented by the component of $d\Omega$ in the submodule $EH$ of $\Lambda^5T^*$. But (3.3) belongs to

$$EH \otimes S^2(S^2H) \simeq ES^5H \oplus ES^3H \oplus EH,$$

and its $EH$ component can only be obtained by taking the trace over each term $\omega_i \wedge \omega_j$, leaving us with $2\text{tr}(\beta) = 2\text{tr}(\sigma)$.

We can also use (3.3) to give an alternative characterization of quaternionic Kähler 8-manifolds. Theorem 1.1 implies the

**Corollary 3.2.** Let $\{M, g, \mathcal{J}\}$ be an AQH 8-manifold. It is quaternionic Kähler if and only if $\mathcal{J}$ generates a differential ideal with $\sigma = 0$, so that (1.1) applies with $\beta_i^j = -\beta_i^j$.

Returning to (1.1), we may consider the matrix $B = (\beta_i^j)$ of curvature 2-forms associated to the connection we have considered. These 2-forms arise in the computation

$$(3.4) \quad 0 = d^2\omega_i = \sum_j (d\beta_i^j - \sum_k \beta_i^k \wedge \beta_j^k) \wedge \omega_j = \sum_j B_i^j \wedge \omega_j,$$

which also provides a constraint on them. In particular, they have no $S^2E$ component, because $\Lambda^4T^*$ contains the module $S^2ES^2H$ [15]; thus

$$B_i^j \in S^2H \oplus \Lambda_0^2ES^2H \subset \Lambda^2T^*.$$ But in contrast to the quaternionic Kähler case, there will in general be a component of $B_i^j$ in $\Lambda_0^2ES^2H$. This will be treated in a forthcoming paper.
4. Quaternionic Endomorphisms of $\mathfrak{su}(3)$

Let $E_{ij}$ denote a $3 \times 3$ matrix with a 1 in the $ij$ position, and 0’s elsewhere. We adopt the following basis of the Lie algebra (2.5) of anti-Hermitian matrices:

\[
\begin{align*}
 e_1 &= i(E_{11} - E_{33}) \\
 e_2 &= \frac{1}{\sqrt{3}} i(-E_{11} + 2E_{22} - E_{33}) \\
 e_3 &= i(E_{21} + E_{12}) \\
 e_4 &= i(E_{31} + E_{13}) \\
 e_5 &= i(E_{32} + E_{23}) \\
 e_6 &= (E_{21} - E_{12}) \\
 e_7 &= (E_{31} - E_{13}) \\
 e_8 &= (E_{32} - E_{23})
\end{align*}
\]

(4.1)

The choice of basis has been taken such that it is conformal relative to minus the Killing form:

\[ \text{tr}(e_ie_j) = -2\delta_{ij}. \]

Observe that $\{e_6, e_7, e_8\}$ is a basis of the subalgebra $\mathfrak{so}(3)$ of real anti-symmetric matrices.

The Lie bracket of $\mathfrak{su}(3)$ is defined by $[A, B] = AB - BA$. For the basis above, they are given by

\[
\begin{align*}
 [e_1, e_2] &= 0 \\
 [e_1, e_3] &= e_6 \\
 [e_1, e_4] &= 2e_7 \\
 [e_1, e_5] &= e_8 \\
 [e_1, e_6] &= -e_3 \\
 [e_1, e_7] &= -2e_4 \\
 [e_1, e_8] &= -e_5 \\
 [e_2, e_3] &= -\sqrt{3}e_6 \\
 [e_2, e_4] &= 0 \\
 [e_2, e_5] &= \sqrt{3}e_8 \\
 [e_2, e_6] &= \sqrt{3}e_3 \\
 [e_2, e_7] &= 0 \\
 [e_2, e_8] &= -\sqrt{3}e_5 \\
 [e_3, e_4] &= e_8 \\
 [e_3, e_5] &= e_7 \\
 [e_3, e_6] &= e_1 - \sqrt{3}e_2 \\
 [e_3, e_7] &= -e_5 \\
 [e_3, e_8] &= -e_4 \\
 [e_4, e_5] &= e_6 \\
 [e_4, e_6] &= -e_5 \\
 [e_4, e_7] &= 2e_1 \\
 [e_4, e_8] &= e_3,
\end{align*}
\]

Together with

\[
\begin{align*}
 [e_5, e_6] &= e_4 \\
 [e_5, e_7] &= e_3 \\
 [e_5, e_8] &= e_1 + \sqrt{3}e_2 \\
 [e_6, e_7] &= e_8 \\
 [e_6, e_8] &= -e_7 \\
 [e_7, e_8] &= e_6.
\end{align*}
\]

We can regard these elements as left-invariant vector fields on the Lie group $\text{SU}(3)$.

Now let $\{e^1, \ldots, e^8\}$ be the dual basis $\mathfrak{su}(3)^*$, or equivalently left-invariant 1-forms on $\text{SU}(3)$, so that $e^i(e_j) = \delta^i_j$. Using the Cartan formula, we arrive at the exterior differential system
\[\begin{align*}
d e^1 &= -e^{36} - 2e^{47} - e^{58}, \\
de^2 &= \sqrt{3}(e^{36} - e^{58}), \\
de^3 &= e^{16} - \sqrt{3}e^{26} - e^{48} - e^{57}, \\
de^4 &= 2e^{17} + e^{38} - e^{56}, \\
de^5 &= e^{18} + \sqrt{3}e^{28} + e^{37} + e^{46}, \\
de^6 &= -e^{13} + \sqrt{3}e^{23} - e^{45} - e^{78}, \\
de^7 &= -2e^{14} - e^{35} + e^{68}, \\
de^8 &= -e^{15} - \sqrt{3}e^{25} - e^{34} - e^{67}.
\end{align*}\]

Referring to \([2.5]\), we shall use the notation \(S^2 H\) to indicate the space of quaternionic endomorphisms, and identify it with \(\mathfrak{b}\) in a natural way by setting

\[I_1 = e_8, \quad I_2 = -e_7, \quad I_3 = e_6.\]

We are going to define an \(\text{SO}(3)\)-equivariant linear mapping

\[S^2 H \otimes (\mathfrak{b} \oplus \mathfrak{p}) \to (\mathfrak{b} \oplus \mathfrak{p}),\]

by considering the associated four maps one at a time. In view of the isomorphisms

\[
S^2 H \otimes S^2 H \simeq S^4 H \oplus S^2 H \oplus S^0 H, \\
S^2 H \otimes S^4 H \simeq S^6 H \oplus S^4 H \oplus S^2 H,
\]

each of the four maps is uniquely determined up to a scalar multiple.

Any equivariant linear map \([4.3]\) must therefore be a linear combination of the following four non-zero maps:

\[
\begin{align*}
\phi_1 &: \quad S^2 H \otimes \mathfrak{b} \to \mathfrak{b} \quad (A, B) \mapsto [A, B], \\
\phi_2 &: \quad S^2 H \otimes \mathfrak{b} \to \mathfrak{p} \quad (A, B) \mapsto i\{(A, B) - \frac{2}{3} \text{tr}(AB)1\}, \\
\phi_3 &: \quad S^2 H \otimes \mathfrak{p} \to \mathfrak{b} \quad (A, C) \mapsto i\{A, C\}, \\
\phi_4 &: \quad S^2 H \otimes \mathfrak{p} \to \mathfrak{p} \quad (A, C) \mapsto [A, C].
\end{align*}
\]

Here, \(A \in S^2 H\) is identified with an element of \(\mathfrak{b}\) via \([4.2]\), \(B \in \mathfrak{b}\), and \(C \in \mathfrak{p}\). Also, \(\{A, B\} = AB + BA\) is the anti-commutator, and \(1\) denotes the \(3 \times 3\) identity matrix. Note that all the images on the right-hand side have zero trace and are anti-Hermitian, as required.

**Proposition 4.1.** There is a one-parameter family of \(\text{SO}(3)\)-invariant quaternionic actions on \(\mathfrak{su}(3)\).

**Proof.** Introducing a constant \(\lambda_i\) for each \(\phi_i\), \([4.3]\) must be given by

\[A \cdot X = \lambda_1 [A, X^a] + i\lambda_2 \left\{\{A, X^a\} - \frac{2}{3} \text{tr}(AX^a)1\right\} + i\lambda_3 \{A, X^s\} + \lambda_4 [A, X^s],\]

where \(A \in S^2 H\), and \(X^a = \frac{1}{2}(X - X^t)\) and \(X^s = \frac{1}{2}(X + X^t)\) are the (anti-)symmetric components of \(X \in \mathfrak{su}(3)\).
Recall the formula (4.2) to identify endomorphisms with elements of $\mathfrak{p}$. We first impose the identities
\begin{equation}
I_i \cdot (I_i \cdot X) = -X, \quad i = 1, 2, 3.
\end{equation}
These can be used to find the $\lambda_k$ by making simple choices of the matrix $X$. Calculations show that (4.4) holds fully when
\begin{equation}
\lambda_1 = \frac{1}{2} \epsilon, \quad \lambda_3 = -\frac{3}{4} (\lambda_2)^{-1}, \quad \lambda_4 = -\frac{1}{2} \epsilon,
\end{equation}
provided $\epsilon = \pm 1$. We take $\epsilon = +1$, since this makes the identity
\[ I_1 \cdot (I_2 \cdot X) = I_3 \cdot X = -I_2 \cdot (I_1 \cdot X) \]
automatically valid (whereas $\epsilon = -1$ would give us $I_2 I_1 = I_3$). We can now parametrise the quaternionic structures by $\lambda_2$, and the proof is complete. \[ \square \]

We shall denote by $\mathcal{J}_\lambda$ the quaternionic structure defined by (4.5) in terms of the parameter $\lambda := \lambda_2$. Recall that \{$e^i$\} is an orthonormal basis of $\mathfrak{su}(3)$ for a multiple of the Killing form. Next, we deform this metric by rescaling on the subspace $\mathfrak{p}$.

**Proposition 4.2.** The Riemannian metric
\begin{equation}
g_\lambda = \sum_{i=1}^{5} e^i \otimes e^i + \frac{4}{3} \lambda^2 \sum_{i=6}^{8} e^i \otimes e^i
\end{equation}
is compatible with the structure $\mathcal{J}_\lambda$.

**Proof.** If \{$I_i$\} are the endomorphisms defined in (4.2), with the quaternionic action defined in Proposition 4.1, we need to show that
\[ g_\lambda(I_i \cdot X, I_i \cdot Y) = g_\lambda(X, Y), \quad i = 1, 2, 3. \]
Since $\mathcal{J}_\lambda$ is $\text{SO}(3)$-invariant, and both subspaces $\mathfrak{p}$, $\mathfrak{p}$ are irreducible, this equation must hold for some choice of $\lambda$. The rest is a computation. \[ \square \]

5. THE MAIN RESULT

The formula for $A \cdot X$ in the proof of Proposition 4.1 can be used to compute the endomorphisms $I_i$ explicitly. For example, the action of $I_3$ on $\mathfrak{su}(3)$ is given by
\begin{equation}
\begin{cases}
e_1 & \mapsto \frac{1}{2} e_3 + \frac{3}{4} \lambda^{-1} e_6 \\
e_2 & \mapsto -\frac{1}{2} \sqrt{3} e_3 + \frac{1}{4} \sqrt{3} \lambda^{-1} e_6 \\
e_3 & \mapsto -\frac{1}{2} e_1 + \frac{1}{2} \sqrt{3} e_2 \\
e_4 & \mapsto \frac{1}{2} e_5 - \frac{3}{4} \lambda^{-1} e_8 \\
e_5 & \mapsto -\frac{1}{2} e_4 + \frac{3}{4} \lambda^{-1} e_7 \\
e_6 & \mapsto -\lambda e_1 - \frac{1}{\sqrt{3}} \lambda e_2 \\
e_7 & \mapsto -\frac{1}{2} e_8 - \lambda e_5 \\
e_8 & \mapsto \frac{1}{2} e_7 + \lambda e_4.
\end{cases}
\end{equation}
We can use these formulas, and analogous ones for $I_1, I_2$ to prove

**Proposition 5.1.** A set of 2-forms $\{\omega_i\}$ associated to the AQH manifold $\{SU(3), g_\lambda, \mathcal{F}_\lambda\}$ is given by

\[
\begin{align*}
\omega_1 &= \frac{1}{2}(e^{15} + \sqrt{3}e^{25} + e^{34}) + \lambda(\frac{1}{\sqrt{3}}e^{28} - e^{46} + e^{37} - e^{18}) - \frac{2}{3}\lambda^2 e^{67}, \\
\omega_2 &= -e^{14} - \frac{1}{2}e^{35} + \lambda(\frac{2}{\sqrt{3}}e^{27} - e^{38} - e^{56}) - \frac{2}{3}\lambda^2 e^{68}, \\
\omega_3 &= \frac{1}{2}(e^{13} - \sqrt{3}e^{23} + e^{45}) + \lambda(\frac{1}{\sqrt{3}}e^{26} - e^{48} + e^{57} + e^{16}) - \frac{2}{3}\lambda^2 e^{78}.
\end{align*}
\]

*Proof.* The expression of $\omega_3$ follows easily from (5.1). For example, using (4.6),

\[
\omega_3(e_1, e_6) = g_\lambda(I_3 e_1, e_6) = \frac{3}{4} \lambda^{-1} g_\lambda(e_6, e_6) = \lambda,
\]

explaining the coefficient of $e^{16}$. Minus the same coefficient is visible in the expression for $I_3 e_6$, which is consistent.

The expressions for $\omega_1, \omega_2$ follow in a similar way from the computation of $I_1, I_2$ that we omit. \(\square\)

We are now in a position to verify if and when the ideal condition (1.1) holds. Since $\beta = (\beta^j_i)$ is a matrix of 1-forms, we first impose the condition that its trace vanishes. By Lemma 3.1, this amounts to assuming that the intrinsic torsion lies in $\mathcal{W}_1$. Then $\beta$ takes values in

\[
\mathfrak{sl}(3, \mathbb{R}) = \mathfrak{b} \oplus \mathfrak{ip},
\]

where $\mathfrak{ip}$ denotes the 5-dimensional space of real symmetric trace-free matrices. The decomposition (5.2) is merely the Cartan dual of (2.5) in the theory of symmetric spaces.

Once we express the 1-forms $\beta^j_i$ in terms of the basis dual to (4.1), we can regard $\xi \mapsto \beta(\xi)$ as a linear mapping from $\mathfrak{su}(3)$ to (5.2). It is natural to suppose that the restriction of this mapping to each of $\mathfrak{b}$ and $\mathfrak{p}$ separately is a multiple of the identity. We therefore suppose that

\[
\beta: e_i \mapsto aE_i^a + sE_i^s,
\]

where $E_i$ is the matrix associated to $e_i$ (for example, $E_1 = E_{11} - E_{33}$), and (with some abuse of notation) the coefficients $a, s$ are to be determined. More explicitly,

\[
\beta = (\beta^j_i) = \begin{pmatrix}
  s(e^1 - \frac{1}{\sqrt{3}}e^2) & se^3 + ae^6 & se^4 + ae^7 \\
  se^3 - ae^6 & \frac{2}{\sqrt{3}}se^2 & se^5 + ae^8 \\
  se^4 - ae^7 & se^5 - ae^8 & -s(e^1 + \frac{1}{\sqrt{3}}e^2)
\end{pmatrix}.
\]

By construction, $\text{tr}(\beta) = 0$.

With this set-up, we can state
**Theorem 5.2.** The compact AQH manifold \( \{ SU(3), g_\lambda, J_\lambda \} \) satisfies the ideal condition (2) of Theorem 1.1 if and only if \( \lambda = \pm \frac{3}{20} \). The resulting structure is invariant by the action of \( SU(3) \) on the left and \( SO(3) \) on the right, and its intrinsic torsion is \( SO(3) \)-invariant.

**Proof.** This is a direct computation. Solving the equations (1.1) with the \( \omega_i \) as in Proposition 5.1 and the \( \beta_i^j \) as in (5.4), we first find that a necessary condition is

\[
a = 1 + \frac{16}{3} \lambda^2, \quad s = -2\lambda.
\]

Once these values are assigned, the remaining equations are satisfied by taking \( \lambda^2 = \frac{3}{20} \).

By construction, the forms \( e^i \) are left invariant, so all the structures considered in this paper are invariant by left translation. Right translation by \( g \in SU(3) \) can then be identified with action of \( \text{Ad}(g) \) on the Lie algebra \( su(3) \). In our case, we are free to take \( g \in SO(3) \), as defined in (2.3).

We already know that the intrinsic torsion belongs to \( \mathcal{W}_1 \). The fact that it belongs to the 1-dimensional subspace \( S^0 H \) follows because the intrinsic torsion is completely determined by the map (5.3) that is itself \( SO(3) \)-equivariant. \( \square \)

**Remarks.** 1. The two choices of sign for \( \lambda \) give a different quaternionic action and ideal structure for the same metric \( g_\lambda \). Since \( (\beta_i^j) \) is not anti-symmetric, the resulting structure on \( SU(3) \) is not quaternionic Kähler. Note that \( SU(3) \) cannot in any case admit an AQH structure with \( d\Omega = 0 \) since otherwise \( [\Omega] \) would be a non-zero element in cohomology, but \( b_4(SU(3)) = 0 \).

2. The matrix \( B \) of curvature 2-forms defined by (3.4) with \( \beta \) in (5.4) will reflect the overall \( SO(3) \) invariance. One finds that, if the expression for \( \omega_3 \) in Proposition 5.1 is written as \( \tau_0 + \lambda \tau_1 + \lambda^2 \tau_2 \), then

\[
B_2^1 - B_1^2 = -4(a + s^2) \tau_0 - 3a(a - 1) \tau_2.
\]

The symmetric coefficients are a bit more complicated, but the diagonal ones are given by

\[
B_1^1 = c(e^{36} + e^{47}), \quad B_2^2 = c(e^{58} - e^{36}), \quad B_3^3 = -c(e^{47} + e^{58})
\]

where \( c = 2(a - 1)s \). Assigning values to the constants as in the proof of Theorem 5.2 does not eliminate any terms.

We conclude with some observations concerning integrable quaternionic structures. The Lie group \( SU(3) \) was shown by Spindel, Servin, Troost & Van Proeyen [17] to admit a hypercomplex structure. In the treatment of Joyce [14] this hypercomplex structure arises from a 3-dimensional subalgebra \( su(2) \) inequivalent to \( \mathfrak{h} \). This provides \( SU(3) \) with AQH structures with intrinsic torsion in \( \mathcal{W}_3 \oplus \mathcal{W}_4 \), but not directly related to our construction. In our case, \( SU(3) \) cannot admit an \( SO(3) \)-invariant quaternionic structure.
Such a structure would necessarily have torsion in $W_3$ and satisfy

\begin{equation}
\label{5.5}
d\Omega \wedge \omega_i = 0, \quad i = 1, 2, 3,
\end{equation}
equations that can never be compatible with (4.6) if $\lambda \in \mathbb{R}$.

Our computations can however be performed equally for the Lie group $\text{SL}(3, \mathbb{R})$; it suffices to repeat everything with complex coefficients. For this group, the situation is reversed; it turns out that $\text{SL}(3, \mathbb{R})$ does not admit a structure of the type described in Theorem 5.2, but (5.5) can be solved when the analogue of the parameter $\lambda$ takes on the values $\pm \frac{1}{2}$. In this way, $\text{SL}(3, \mathbb{R})$ becomes a quaternionic manifold, and admits a compatible Hermitian structure with torsion in $W_3$.

Acknowledgements

The author would like to thank Simon Salamon and Simon Chiossi for enlightening discussions and useful suggestions, and the Department of Mathematics at the Politecnico di Torino for its hospitality during the preparation of this work. The latter was supported by the Spanish Ministry of Science and Education (MEC) and by the Spanish Foundation for Science and Technology (FECYT) through a post-doctoral fellowship and research contract associated with the project –2007-0857.

References

[1] A.L. Besse, Einstein Manifolds, Springer-Verlag, Berlin, 1987.
[2] S. Bonanos, Mathematica software, http://www.inp.demokritos.gr/~sbonano/
[3] J-B. Butruille, Espace de twisteurs d’une variété presque hermitiane de dimension 6, Ann. Ins. Fourier, 57 (2007), 1451–1485.
[4] F.M. Cabrera, A. Swann, The intrinsic torsion of almost quaternion-Hermitian manifolds, Ann. Inst. Fourier 58 (2008), 1455–1497.
[5] S. Chiossi, O. Macia, SO(3)-structures on 8-manifolds, in preparation.
[6] R. Cleyton, S. Ivanov, Conformal equivalence between certain geometries in dimension 6 and 7, Math. Res. Lett. 15 (2008), 631–640.
[7] D. Conti, Special holonomy and hypersurfaces, Ph.D. thesis, Scuola Normale Superiore, Pisa, 2005.
[8] D. Conti, Invariant forms, associated bundles and Calabi-Yau metrics, J. Geom. Physics 57 (2007) 2483–2508
[9] A. Gambioli, Latent quaternionic geometry, Tokyo J. Math. 31 (2008) 203–223.
[10] A. Gambioli, SU(3)-manifolds of cohomogeneity one, Annals Global Anal. Geom. 34 (2008), 77-100.
[11] D. Giovannini, Special structures and symplectic geometry, Ph.D. thesis, University of Turin, 2004.
[12] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. 123 (1980) 35–58.
[13] N.J. Hitchin: Stable forms and special metrics, Contemp. Math. 288 (2001) 70–89.
[14] D. Joyce, Compact hypercomplex and quaternionic manifolds, J. Differ. Geom. 35 (1992) 743–761.
[15] S. Salamon, Quaternionic Kähler manifolds, Invent. Math. 67 (1982) 143–171.
[16] S. Salamon, Almost parallel structures, Contemp. Math. 288 (2001), 162–181.
[17] Ph. Spindel, A. Servin, W. Troost, A. Van Proeyen, Extended supersymmetric $\sigma$-models on group manifolds, Nucl. Phys. B 308 (1988), 662–698.
[18] A.F. Swann, Aspects symplectiques de la géométrie quaternionique, C.R. Acad. Sci. Paris 308 (1989) 225-228.
[19] A.F. Swann, Hyper Kähler and quaternionic Kähler geometry, Math. Ann. 289 (1991) 421–450.

Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy
E-mail address: oscarmacia@calvino.polito.it