We obtain an explicit representation for quasiradial $\gamma$-harmonic functions, which shows that these functions have essentially algebraic nature. In particular, we give a complete description of all $\gamma$ which admit algebraic quasiradial solutions. Unlike the cases $\gamma = \infty$ and $\gamma = 1$, only finitely many algebraic solutions is shown to exist for any fixed $|\gamma| > 1$. Moreover, there is a special extremal series of $\gamma$ which exactly corresponds to the well-known ideal $m$-atomic gas adiabatic constant $\gamma = \frac{2m + 3}{2m + 1}$.

1. Introduction

We study specific solutions to the following quasilinear equation

$$u_{xx} \left( (\gamma + 1)u_x^2 + (\gamma - 1)u_y^2 \right) + 4u_{xy}u_xu_y + u_{yy}((\gamma + 1)u_y^2 + (\gamma - 1)u_x^2) = 0$$

where $|\gamma| > 1$ or $\gamma = 1$. Let $L_\gamma[u]$ denote the left-hand side of this equation. A solution of the form

$$u(x, y) = \rho^k f(\theta), \quad k \geq 1,$$

where $\rho$ and $\theta$ are the polar coordinates in the $(x, y)$-plane, is said to be a quasiradial. The origin of this study goes back to the well-known $p$-Laplace equation

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) = 0,$$

which is the divergence form (1) with $p = \frac{2\gamma}{\gamma - 1}$. We call solutions to (1) $\gamma$-harmonic functions.

The existence and integral representations for quasiradial $\gamma$-harmonic functions were previously established by G. Aronsson in [1]–[5]. In particular, it was shown in [2] (see also [15] and [3]) that single-valued quasiradial $\gamma$-harmonic functions do exist only for those exponents $k$ in (2) which satisfy the characteristic equation

$$\begin{align*}
(2N - 1)(\gamma + 1)k^2 - 2(N^2\gamma + 2N - 1)k + N^2(1 + \gamma) &= 0, \\
n &\in \mathbb{N}.
\end{align*}$$

We refer to the corresponding solution as the $N$-solution to (1). One can easily see that the 0-solutions are constants, and the 1-solutions are linear
functions. In what follows, such solutions are said to be the *trivial* solutions to (1).

The limit case $\gamma = \infty$ reduces to the standard Laplace equation $\Delta u = 0$, and the corresponding $N$-solutions are harmonic polynomials of degree $N$. Note that harmonic polynomials are *algebraic* functions, i.e. they satisfy some (actually, trivial one) polynomial identity $P(x, y, u) \equiv 0$. We show that this property is still valid for $N$-solutions of the Aronsson equation (i.e. $\gamma = 1$)

\[
(4) \quad u_{xx} u_x^2 + 2u_{xy} u_x u_y + u_{yy} u_y^2 = 0,
\]

It is the aim of this paper to study this phenomena for general $\gamma$’s. To make this point more explicit, we have to note that for $\gamma \neq \infty$ a (weak) solution of (1) is normally in the class $C^{1, \alpha}$. In particular, quasiradial solutions have a Hölder singularity near the origin, and one should consider them as ‘singular solutions’ (the terminology is borrowed from [4]). This non-regular character is a consequence of the general situation for $\gamma$-harmonic functions near their singular points (i.e. at the points at which $|\nabla u| = 0$), see [17], [8], [11].

Our key result is the following explicit parametric representations for $N$-solutions:

\[
(5) \quad x + iy = e^{i\phi}(\mu |\zeta|^{2(N-1)} + \bar{\zeta}^{2N-1}),
\]

\[
u_N = C|\zeta|^{k(2N-1)} \cdot \text{Re} \zeta^N.
\]

Here $\zeta \in \mathbb{C}$ is the parametrization variable, $\phi$ is an arbitrary constant, $k = k(N, \gamma)$ is the biggest root of (3), and $\mu$ is defined by (41) below. We show also that (5) represents an entire graph over the $(x, y)$-plane when $\zeta$ runs the complex plane $\mathbb{C}$.

In particular, it immediately follows from (3) and (5), that $u_N$ is an *algebraic* function whenever $k(N, \gamma)$ is a *rational* number. This makes more explicit the mentioned above Hölderian behaviour of quasiradial solutions at their singular points.

In contrast with (4), we show that for any rational $|\gamma| > 1$, the class of algebraic $N$-solutions is necessary finite in the sense that the following upper estimate holds $N \leq \left\lfloor \frac{q^2(p^2+2-q^2)}{2p^2} \right\rfloor$, where $\gamma = p/q$, with $q$ and $p$ to be co-prime numbers, and $\lfloor x \rfloor$ denotes the integer part of $x$. In particular, this yields the absence of algebraic solutions for all integer numbers $\gamma$, $|\gamma| \geq 2$.

Denote by $\mathcal{A}$ the set of all $\gamma \in \mathbb{Q}$, $|\gamma| > 1$, such that (1) admits *nontrivial* algebraic $N$-solutions. Then $\gamma \in \mathcal{A}$ iff $-\gamma \in \mathcal{A}$ (see Section 4 below). In Section 5 we show that for all rational $\gamma = p/q \in \mathcal{A}$, $\gamma > 1$, the following bilateral estimate holds

\[2 + q \leq p \leq q^2 - 2.\]
The last inequalities are sharp. Moreover, in Section 6 we prove that equality \( q = p + 2 \) holds iff \( q \) is an odd number, \( q \geq 3 \). This yields the so-called minimal series

\[
\gamma = \frac{2N + 3}{2N + 1}, \quad N = 2, 3, 4, \ldots,
\]

and \( N \) is the index of the corresponding (a unique for the given \( \gamma \)) algebraic quasiradial solution.

We end this introduction with one possible physical interpretation of (6), which gives a motivation of our choice of \( \gamma \) instead of \( p \). Namely, observe that (1) is a homogeneous form of the gas dynamics equation

\[
L_\gamma[\phi] = 2\Delta \phi,
\]

for the potential of the gas velocity \( \phi \) [7, p. 9]. The parameter \( \gamma \) in (7) is the so-called adiabatic gas constant, that is the ratio of the gas’ specific heats at constant volume and constant pressure (see, e.g. [14]). Note that, for all known gas models due to a specific combinatoric nature of the adiabatic constant \( \gamma \), it can be evaluated in terms of the freedom degrees of the corresponding gas. In particular, it follows that \( \gamma \) a rational number.

Moreover, the most important for applications is the simple gas which consists of \( m \) atoms, \( \gamma \) is given by the following ratio

\[
\gamma_m = \frac{2m + 3}{2m + 1}, \quad m = 1, 2, \ldots,
\]

(e.g., \( m = 2, \) or \( \gamma_2 = 7/5, \) describes the standard Earth’s atmosphere).

Note that eq. (7) is of non-degenerate elliptic type for all adiabatic exponents \( \gamma > 1 \). In the recent paper of I. Zorina and the author [16], it is proved that for any integer \( N \geq 2 \) there exists a solution of (7) with non-trivial polynomial growth \( k_N > 1 \) (where \( k_N \) is defined by (3)) which is a real analytic function in the whole \( \mathbb{R}^2 \).

Now, the quasiradial \( N \)-solutions to (1) can be naturally regarded as the cones (after a suitable scale renormalization) over the corresponding \( N \)-solutions to (7). In other words, (1) represents a microscopy level of the gas flow. In this connection, the coincidence of the minimal series (6) and the natural adiabatic constants (8) implicitly underlines the essence of algebraic character of the corresponding \( N \)-solutions. We observe also, that in this case we have \( N = m \), i.e. the atomic number is equal to the index of the corresponding \( N \)-solution.

2. The separation equation

In this section we study the basic properties of the wave function \( f(\theta) \). For technical reasons, we assume that \( |\gamma| > 1 \). The case \( \gamma = 1 \) requires a few more care because of degenerate character of the separate equation (13).
However, all the formulated below results are still valid in this limit case if we suppose that $k > 1$.

The separation of variables in (2) yields the following ordinary differential equation

\[(9) \quad f''((\gamma - 1)k^2 f^2 + (\gamma + 1)f'^2) + \\
+ \left[f'^2(k(\gamma + 3) - 2) + ((1 + \gamma)k - 2)k^2 f^2\right]f k = 0,
\]

where the prime denotes the derivative with respect to $\theta$. Letting $W = f'^2(\theta)$, $Z = f^2(\theta)$, we can rewrite (9) as

\[
\frac{dW}{dZ} = -k((\gamma + 3)k - 2)W - k^3((\gamma + 1)k - 2)Z
\]

which splits into the following linear system

\[(10) \quad W'(\xi) = -k((\gamma + 3)k - 2)W - k^3((\gamma + 1)k - 2)Z
Z'(\xi) = (\gamma + 1)W + (\gamma - 1)k^2 Z.
\]

One can easily verify that

\[
W(\xi) = C_1 k^2 e^{-2k^2 \xi} + C_2 k((\gamma + 1)k - 2)e^{2(k-k^2)\xi}
Z(\xi) = -C_1 e^{-2k^2 \xi} - C_2 (\gamma + 1)e^{2(k-k^2)\xi},
\]

is the general solution of (10), where $C_1$ and $C_2$ are arbitrary constants.

Then

\[(11) \quad W + k^2 Z = -k\eta k^{-1}C_2 \quad W + \lambda^2 Z = \frac{2k}{\gamma + 1}\eta^k C_1,
\]

where $\eta = e^{-2k^2}$ and $\lambda^2 = k^2 - \frac{2k}{\gamma + 1}$. The right-hand side of the latter identity is positive for all $|\gamma| > 1$ and $k \geq 1$, so we can define

\[(12) \quad \lambda := \sqrt{k^2 - \frac{2k}{\gamma + 1}}.
\]

Then elimination of $\eta$ in (11) yields $(W + k^2 Z)^k = C_3(W + \lambda^2 Z)^{k-1}$. Since (9) is a homogeneous equation, it suffices to study only the case $C_3 = 1$.

Thus, we have the following first order differential equation

\[(13) \quad (f'^2(\theta) + k^2 f^2(\theta))^k = (f'^2(\theta) + \lambda^2 f^2(\theta))^{k-1}.
\]

We introduce new phase variables $z = f(\theta)$, $w = f'(\theta)$, and define the set

\[(14) \quad \Gamma = \{(z, w) \in \mathbb{R}^2 : (w^2 + k^2 z^2)^k = (w^2 + \lambda^2 z^2)^{k-1}, \ w^2 + z^2 \neq 0\}.
\]

Observe that the intersection $\Gamma$ with the $Oz$-axis consists of exactly two points (the *apexes*): $A^{\pm} = (\pm z_0, 0)$, where

\[(15) \quad z_0 = \lambda^{k-1} k^{-k}.
\]
Lemma 2.1. Let $|\gamma| > 1$ and $k > 1$. Then $\Gamma$ a real analytic closed Jordan curve. Moreover, $\Gamma \setminus \{A^+, A^-\}$ splits in two mutually symmetric graphs (which have no common points with $Oz$).

Proof. The first statement easily follows from the representation of $\Gamma$ in the polar coordinates $z = r \cos \alpha$, $w = r \sin \alpha$:

$$r = \frac{(\sin^2 \alpha + \lambda^2 \cos^2 \alpha)^{(k-1)/2}}{(\sin^2 \alpha + k^2 \cos^2 \alpha)^{k/2}}.$$

Now, consider $\Gamma$ as the 0-level set of the function

$$F(z; w) = (w^2 + k^2 z^2)^k - (w^2 + \lambda^2 z^2)^{k-1}.$$

We claim that $F'_w \neq 0$ on $\Gamma \setminus \{A^+, A^-\}$. Indeed, suppose $F'_w(z_1, w_1) = 0$ and $w_1 \neq 0$. Then we have

$$\frac{1}{2w_1} F'_w(z_1, w_1) = k(w_1^2 + k^2 z_1^2)^{k-1} - (k-1)(w_1^2 + \lambda^2 z_1^2)^{k-2} = 0,$$

which together with $F(z_1, w_1) = 0$ and $|\gamma| > 1$ implies

$$w_1^2 = \frac{1 - \gamma^2}{(1 + \gamma)^2 k^2 z_1^2} \leq 0.$$

It follows that $w_1 = z_1 = 0$, which contradicts the definition of $\Gamma$ and implies our claim. Thus, $\Gamma \setminus \{A^+, A^-\}$ splits into union of two graphs with respect to the $Oz$-axis and the lemma is proved. □ □

Now, we construct a special solution $f(\theta)$ of (13), satisfying the initial condition $f(0) = z_0$. With the above notation we have

$$\frac{dz}{w} = \frac{f'(\theta)d\theta}{f'(\theta)} = d\theta.$$

Define

$$\Theta(\xi) = \int_{A^+}^{\xi} \frac{dz}{w},$$

where $\xi \in \Gamma$, and the integral is taken clockwise along the arc $(A^+, \xi)$ of $\Gamma$. The last integrand a priori has singular behavior when $w$ vanishes (i.e. for $\xi = A^\pm$). But, it can be shown that these singularities are removable. Indeed, using again the representation of $\Gamma$ as the 0-level set of function (16) we find

$$\frac{dz}{w} = -\frac{F'_w dw}{F'_w} = \frac{k(w^2 + k^2 z^2)^{k-1} - (k-1)(w^2 + \lambda^2 z^2)^{k-2} dw}{z}.$$

Now, to show that (18) has no singularity it suffices only to verify that the denominator of the right-hand ratio in (19) is non-zero in a neighborhood of the apexes. The corresponding values at $A^\pm$ are equal $\lambda^{2(k-1)} z_0^{2(k-2)} \neq 0$. Therefore, the integral in (18) is well defined, and it follows that $\Theta(\xi)$ is an
analytic function of $\xi$ (in the sense that $\Theta(\xi(\tau))$ is analytic for any analytic parametrization $\xi(\tau)$).

Next, observe that $dz/w > 0$ within our convention. Thus, applying Lemma 2.1, we conclude that $\Theta(\xi)$ is a strictly increasing function when $\xi$ runs $\Gamma$ in clockwise direction. Define function $f_k(\theta)$ by letting $f_k(\Theta(\xi)) = z(\xi)$, where $z(\xi)$ is the projection of $\xi$ onto the $Oz$-axis. Clearly, $f_k(\theta)$ becomes a real analytic periodic function, which is defined now in $\mathbb{R}$. By virtue of the symmetry of $\Gamma$, the one-quoter period $T$ satisfies

\begin{equation}
\frac{T}{4} := \int_{A^+}^{B^-} \frac{dz}{w}
\end{equation}

where $B^- = (-1, 0) \in \Gamma$.

**Lemma 2.2.**

\begin{equation}
T = 2\pi \left( 1 - \frac{k-1}{\lambda} \right).
\end{equation}

**Proof.** Define a new variable $t$ by letting $w = -zt$. Then, applying (14) we obtain the following parameterization of the arc $A^+B^-$

\begin{equation}
z(t) = (t^2 + \lambda^2)(k-1)/2 (t^2 + k^2)^{-k/2}
\end{equation}

\begin{equation}
w(t) = -t(t^2 + \lambda^2)(k-1)/2 (t^2 + k^2)^{-k/2},
\end{equation}

When $t$ runs between 0 and $+\infty$ the corresponding point $\xi(t) = (z(t), w(t))$ runs $A^+B^-$ in the clockwise direction. Moreover, we have

\[
dz(t) = t(t^2 + \lambda^2)^{-k/2} \left( k^2(k - 1) - k\lambda^2 - t^2 \right) dt,
\]

which yields

\begin{equation}
\frac{dz(t)}{w(t)} = \left( \frac{k}{t^2 + k^2} - \frac{k-1}{t^2 + \lambda^2} \right) dt.
\end{equation}

Integration of (23) gives

\begin{equation}
\Theta(z(t), w(t)) = \arctan \frac{t}{k} - \frac{k-1}{\lambda} \arctan \frac{t}{\lambda}.
\end{equation}

So, letting $t \to +\infty$ in (24), we obtain by virtue of (20): $\frac{T}{4} = \frac{\pi}{2} \left( 1 - \frac{k-1}{\lambda} \right)$, and (21) follows. $\square$

From the definition of $\Gamma$ we infer $f_k'(\Theta(\xi)) = w(\xi)$, where $z(\xi)$ is the projection of $\xi$ onto the $Oz$-axis. In particular,

\begin{equation}
f_k'(\theta) = 0 \iff \theta = \frac{Tn}{2}, \quad n \in \mathbb{Z}.
\end{equation}

Hence, the constructed function $f_k(\theta)$ satisfies (13) with the initial data

\[
f_k'(0) = 0, \quad f_k(0) = z_0 = \frac{\lambda^{k-1}}{k^k}.
\]
It is obvious that (by virtue of the autonomic character of (13)) a general solution of (13) has the form \( f(\theta) = Cf_k(\theta + a) \), where \( C \) and \( a \) are arbitrary constants.

**Corollary 2.3.** Let \( T \) be defined by (21). For \( \theta \in (-T/4, T/4) \) we have the following parametrization for \( f_k(\theta) \)

\[
\begin{align*}
  f_k &= (t^2 + \lambda^2)^{(k-1)/2}(t^2 + k^2)^{-k/2} \\
  \theta &= \arctan \left( \frac{t}{k} - \frac{k-1}{\lambda} \arctan \frac{t}{\lambda} \right), \quad t \in \mathbb{R},
\end{align*}
\]

and for other values of \( \theta \), \( f(\theta) \) satisfies the symmetry rules:

\[
  f_k(\theta) = -f_k\left(\frac{T}{2} - \theta\right), \quad f_k(-\theta) = f_k(\theta).
\]

Our further objective is to characterize all values of \( k > 1 \) which support the \( 2\pi \)-periodic wave functions \( f_k(\theta) \), that is \( f_k(\theta + 2\pi) = f_k(\theta) \). In fact, the latter condition is equivalent to absence of multivalued branches of the corresponding quasiradial solution of (1). The following assertion is a direct consequence of Lemma 2.2.

**Proposition 2.4.** Let \( |\gamma| > 1 \) and \( k \geq 1 \). Then \( f_k(\theta + 2\pi) = f_k(\theta) \) if and only if the following equality holds

\[
  \frac{k-1}{\lambda} = \frac{N-1}{N}, \quad N \in \mathbb{N},
\]

where \( N \) denotes the set of all positive integers. In this case \( f_k(\theta) \) is a \( 2\pi/N \)-periodic function.

The latter statement is not new. It has appeared in [2] for \( \gamma = 1 \) and for general \( \gamma \) in [3], [15]. However, our approach to this result seems to be complementary to these previous works and allows us to arrive at an explicit representation for the quasiradial solutions in the next section.

**Proposition 2.5.** Let \( |\gamma| > 1 \) and \( N \in \mathbb{N} \) be given. Then there exists a unique \( k = k(\gamma, N) \geq 1 \) such that (9) admits a \( 2\pi/N \)-periodic solution.

**Proof.** The trivial case \( N = 1 \) by (27) gives \( k = 1 \). Let \( N \geq 2 \). Then (27) is equivalent to the following quadratic equation

\[
  (2N-1)(\gamma+1)k^2 - 2(N^2\gamma + 2N - 1)k + N^2(1+\gamma) = 0
\]

which has two separate roots because its discriminant is strictly positive:

\[
  D = 4(N-1)^2(N^2\gamma^2 - 2N + 1) > 4(N-1)^3 > 0.
\]

Then one can easily infer from the Viète theorem that

\[
  (k_1 - 1)(k_2 - 1) = -\frac{(N-1)^2}{2N-1} \cdot \frac{\gamma - 1}{\gamma+1} < 0,
\]
where \( k_1 \neq k_2 \) are the roots of (28). Inequality (29) implies \( k_1 < 1 < k_2 \), so that exactly one root \( k_2 > 1 \) is consistent with our constraint \( k > 1 \). □ □

3. \( N \)-solutions

From now on, we adopt a new notation \( f_N \) for the \( N \)-th wave function \( f_k \) with \( k = k(\gamma, N) \).

**Definition 3.1.** Let \( N \in \mathbb{N} \). The quasiradial solution of the form
\[
u_N(x, y) := C\rho^k f_N(\theta),
\]
where \( C \) is an arbitrary constant, is said to be a basic \( N \)-solution of (1).

Similarly, \( u = C\rho^k f_N(\theta + a) \) with an arbitrary \( a \in \mathbb{R} \) is said to be a (general) \( N \)-solution.

**Theorem 3.2.** Let \( |\gamma| > 1, N \in \mathbb{N}, \) and \( k = k(\gamma, N) \) be the biggest root of (28). Then the basic \( N \)-solution has the following representation
\[
\begin{align*}
x &= h^{2N-1}((k + \lambda) \cos \tau + (k - \lambda) \cos(2N - 1)\tau), \\
y &= h^{2N-1}((k + \lambda) \sin \tau - (k - \lambda) \sin(2N - 1)\tau), \\
u_N &= Ch^{k(2N-1)} \cos N\tau,
\end{align*}
\]
where \( \lambda \) is defined by (27), and \( \tau \in [0; 2\pi], h > 0 \) are the variables of parametrization.

**Proof.** By virtue of (26) we have the following parametrization for \( f_N(\theta) \)
\[
(30) \quad \begin{aligned}
\theta &= \arctan \frac{t}{k} - \frac{k-1}{\lambda} \arctan \frac{t}{\lambda}, \\
t &= \lambda \tan(N\tau), \quad \tau \in \left(-\frac{\pi}{2N}, \frac{\pi}{2N}\right).
\end{aligned}
\]
Define a new variable \( \tau \) by
\[
(31) \quad \begin{aligned}
t &= \lambda \tan(N\tau), \quad \tau \in \left(-\frac{\pi}{2N}, \frac{\pi}{2N}\right).
\end{aligned}
\]
Then we have from (30) and (27)
\[
(32) \quad \begin{aligned}
\theta + (N-1)\tau &= \arctan \left( \frac{\lambda}{k} \tan(N\tau) \right),
\end{aligned}
\]
which by (31) yields
\[
(33) \quad t = k \tan(\theta + (N-1)\tau).
\]
Inserting (31) and (33) into the first identity in (30) we get
\[
(34) \quad \begin{aligned}
f_N &= (t^2 + \lambda^2)^{(k-1)/2}(t^2 + k^2)^{-k/2} = \\
&= z_0 \left(1 + \tan^2(N\tau)\right)^{\frac{k-1}{2}} \left(1 + \tan^2(\theta + (N-1)\tau)\right)^{-\frac{k}{2}} = \\
&= z_0 \cos N\tau \left(\frac{\cos(\theta + (N-1)\tau)}{\cos(N\tau)}\right)^k,
\end{aligned}
\]
where $z_0$ is defined by (15). On the other hand,

\begin{equation}
\cos(\theta + (N-1)\tau) = \cos \theta \cos(N-1)\tau \cdot [1 - \tan(N-1)\tau \cdot \tan \theta].
\end{equation}

Now, we apply to (35) the addition formula

\[ 1 - \tan p \cdot \tan(\beta - p) = \frac{1}{\cos^2 p} \cdot \frac{1}{1 + \tan \beta \tan p} \]

with $p = (N-1)\tau$, $\beta = \arctan \left( \frac{\lambda}{k} \tan (N\tau) \right)$. By virtue of (32), we have $\theta = \beta - p$, therefore

\begin{equation}
1 - \tan(N-1)\tau \tan \theta = \frac{1}{\cos^2(N-1)\tau} \cdot \frac{1}{1 + \frac{\lambda}{k} \tan(N-1)\tau \tan N\tau} = \frac{k \cos N\tau}{\cos(N-1)\tau} \cdot \frac{k \cos N\tau \cos N\tau + \lambda \sin(N-1)\tau \sin N\tau}{2k \cos N\tau} = \frac{1}{\cos(N-1)\tau} \cdot \frac{(k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau}{(k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau}.
\end{equation}

Then, applying (35) and (36) to (34) we obtain

\[ f_N = z_0 \cos N\tau \left[ \frac{2k \cos \theta}{(k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau} \right]^k. \]

Taking into account that $u_N(x, y) = \rho^k f_N$ and $x = \rho \cos \theta$ we find

\[ u_N(x, y) = z_0 \cos N\tau \left[ \frac{2k \cos \theta}{(k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau} \right]^k \rho^k = (2k)^k z_0 \left[ \frac{x}{(k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau} \right]^k. \]

Setting $h^{2N-1}$ for the expression in the last brackets we arrive at

\[ x = h^{2N-1}((k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau), \]

and $u_N = C_N h^{k(2N-1)} \cos N\tau$, where $C_N = (2k)^k z_0 = 2^k \lambda^{k-1}$.

Finally, to express $y$ we eliminate the polar coordinates as follows

\[
\frac{y}{x} = \tan \theta = \frac{\lambda \tan N\tau - k \tan(N-1)\tau}{k + \lambda \tan N\tau \tan(N-1)\tau} = \frac{(k + \lambda) \sin \tau - (k - \lambda) \sin(2N-1)\tau}{(k + \lambda) \cos \tau + (k - \lambda) \cos(2N-1)\tau}.
\]

Thus we get (3.2) for all $\tau \in (-\pi/2N, \pi/2N)$. Using the analyticity of $f_N(\theta)$ we conclude that (3.2) is valid for all $\tau$. The theorem is proved completely.
In order to simplify (3.2) we make use a special intermediate parameter \( \mu \). Namely, in the above notation, put

\[
\mu = \frac{k + \lambda}{k - \lambda},
\]

so \( \lambda = k(\mu - 1)/(\mu + 1) \), and we have from (12)

\[
k = \frac{(1 + \mu)^2}{2\mu(\gamma + 1)}.
\]

Then, an easy computation shows that (27) becomes

\[
N = \frac{\mu^2 - 1}{2(\mu\gamma - 1)}.
\]

Now we observe that for \( \gamma > 1 \) we have \( k > \lambda > 0 \), so that \( \mu > 1 \). Similarly, \( \gamma < -1 \) implies \( \mu < -1 \). On the other hand, considering (39) as a quadratic equation for \( \mu \)

\[
F(\mu) := \mu^2 - 2\gamma N\mu + (2N - 1) = 0,
\]

we find \( F(1) = 2N(1 - \gamma) \) and \( F(-1) = 2N(1 + \gamma) \). If \( \gamma > 1 \) then \( F(1) < 0 \), i.e. exactly one root of (40)

\[
\mu^+ = N\gamma + \sqrt{N^2\gamma^2 - 2N + 1}
\]

agrees the above constrain \( \mu > 1 \). Similarly, for \( \gamma < -1 \) we have

\[
\mu^- = N\gamma - \sqrt{N^2\gamma^2 - 2N + 1}.
\]

Define

\[
\mu \equiv \mu(\gamma, N) = \begin{cases} 
N\gamma + \sqrt{N^2\gamma^2 - 2N + 1}, & \text{for } \gamma > 1; \\
N\gamma - \sqrt{N^2\gamma^2 - 2N + 1}, & \text{for } \gamma < -1.
\end{cases}
\]

For the further purpose, notice that

\[
\mu(-\gamma, N) = -\mu(\gamma, N).
\]

Now, by using the homogeneity character of (43), one can rewrite it as follows

\[
x = h^{2N-1}(\mu \cos \tau + \cos(2N - 1)\tau),
\]

\[
y = h^{2N-1}(\mu \sin \tau - \sin(2N - 1)\tau),
\]

\[
u_N = Ch^{(2N-1)k} \cos N\tau.
\]

A general \( N \)-solution can be obtained from a certain basic \( N \)-solution by a suitable rotation of in the \((x, y)\) plane. Let

\[
x' = x \cos \psi + y \sin \psi, \quad y' = -x \sin \psi + y \cos \psi,
\]
be such a rotation. Then (43) implies
\begin{align*}
x &= h^{2N-1}(\mu \cos \tau + \cos((2N - 1)\tau + 2N\psi)), \\
y &= h^{2N-1}(\mu \sin \tau - \sin((2N - 1)\tau + 2N\psi)), \\
u &\equiv u_{N,\psi} = Ch^{(2N-1)k} \cos N(\tau + \psi),
\end{align*}
In particular, $u_{N,0} = u_N$. Thus, (44) gives the representation for general $N$-solutions to (1).

By putting in (44) $X = h \cos \tau$ and $Y = h \sin \tau$, we obtain an algebraic representation of a basic $N$-solution $u(x, y)$ (see also an equivalent complex form (5) given in the Introduction)
\begin{align*}
x &= \mu X(X^2 + Y^2)^{N-1} + \text{Re}(X + iY)^{2N-1}, \\
y &= \mu Y(X^2 + Y^2)^{N-1} - \text{Im}(X + iY)^{2N-1}, \\
u_N &= C(X^2 + Y^2)^{(2N-1)-N} \text{Re}(X + iY)^N.
\end{align*}

**Corollary 3.3.** All $N$-solutions are quasialgebraic functions in the sense that $u_N^\alpha$ is an algebraic function, where $\alpha = \frac{2}{k(2N-1)-N}$.

To illustrate the last property, we briefly mention the following well-known example. Note (see, also [2] and [10] for further examples) that $u_2 = x^{4/3} - y^{4/3}$ is a basic 2-solution of (4). Then it is easily verified that $u \equiv u_2(x, y)$ satisfies the following polynomial identity
\[27x^4y^4u^3 = (x^4 - y^4 - u^3)^3.\]

**4. Conjugate solutions**

We recall that the main equation (1) can be represented as a $p$-Laplace equation for $p = \frac{2\gamma}{\gamma - 1}$. It is well known fact (cf. [5], [6]) that in two-dimensional case there is the canonical correspondence between $p$-harmonic and $p'$-harmonic functions for
\[\frac{1}{p} + \frac{1}{p'} = 1.
\]
More precisely, given a solution $u$ of equation $L_\gamma[u] = 0$ we define the conjugate function $U$ by
\begin{align*}
U_x &= |\nabla u|^{2/(\gamma - 1)}u_y, \\
U_y &= -|\nabla u|^{2/(\gamma - 1)}u_x, \\
U(0) &= 0.
\end{align*}
Note that $U(x, y)$ is not necessarily a single valued function. But at least locally, $U$ is a quasiradial solution of the conjugated equation
\begin{equation}
L_{-\gamma}[U] = 0.
\end{equation}
It turns out that there is a simple relation between the conjugate quasiradial solutions. We define an adjoint (to basic one) $N$-solution $u_N^*$ as follows
\begin{align}
    x &= h^{2N-1} (\mu \cos \tau - \cos((2N - 1)\tau)), \\
y &= h^{2N-1} (\mu \sin \tau + \sin((2N - 1)\tau)), \\
u_N^* &= Ch^{(2N-1)k} \sin N\tau.
\end{align}

Applying (44), we obtain an equivalent definition:
\[
u_N^* = u_{N,-\pi/2N}.
\]
We have also $u_{N}^{**} = -u_N$. The functions $u_N$ and $u_N^*$ form a conjugate pair, analogous to conjugate harmonic functions. More precisely, if $\gamma = 1$ one can easily derive from the above representation that $u_N(x, y) = \text{Re}(x + iy)^N$ and $u_N^*(x, y) = \text{Im}(x + iy)^N$.

**Theorem 4.1.** Let $u_N$ be a basic $N$-solution of (1) and $U_N^*$ be an adjoint $N$-solution of (47). Then there is a constant $c$ such that $cu_N$ and $U_N^*$ form the conjugate pair in the sense (46).

**Proof.** Without loss of generality, we can assume $N \geq 2$. Let $u_N$ be an $N$-solution of (1) and $U$ be the corresponding conjugate function defined by (46), normalized by $U(0) = 0$. It follows from homogeneity of $u_N$ that $U$ is a homogeneous function as well. Hence, there is a real $\beta$ such that
\[
    U = r^\beta G(\theta).
\]
Here $G(\theta)$ is a priori a multivalued function. We will prove that $U = U_N^*$ with a suitable constant $C$ in (48).

First, notice that $U$ is a quasiradial solution of (47). Denote by $k$ the growth exponent of $u_N$. Then $k > 1$ and the components of the gradient $\nabla u_N$ are homogeneous functions of order $(k - 1)$. Moreover,
\[
    |\nabla u_N|^2 = r^{2k-2} (k^2 f_N^2(\theta) + f_N^2(\theta)).
\]
In particular, $\nabla u_N \neq 0$ for $r \neq 0$. Applying (46) gives
\[
    \nabla U = \begin{bmatrix} U_x' \\ U_y' \end{bmatrix} = |\nabla u_N|^{2/\gamma - 1} \begin{bmatrix} u_{N,y} \\ -u_{N,x} \end{bmatrix} = r^{(k-1)\frac{1}{\gamma - 1} + 1} \begin{bmatrix} G_1(\theta) \\ G_2(\theta) \end{bmatrix},
\]
where $G_i(\theta)$ are certain $2\pi$-periodic functions of $\theta$. Since $|\gamma| > 1$ and $k > 1$,
\[
    \beta = (k - 1)\frac{\gamma + 1}{\gamma - 1} + 1 > 1.
\]

On the other hand, let $k^*$ be the growth exponent of $U_N^*$. Let $\mu$ and $\mu^*$ be the corresponding auxiliary parameters defined by (37) for $k$ and $k^*$ respectively. Then it follows that from (42) that $\mu^* = -\mu$, and from (38) we have
\[
k = \frac{(1 + \mu)^2}{2(1 + \gamma)\mu}.
and
\[ k^* = \frac{(1 + \mu^*)^2}{2(1 + \gamma^*)\mu^*} = \frac{(1 - \mu)^2}{2(\gamma - 1)}, \]
where \( \gamma^* = -\gamma \). Hence
\[ (50) \quad k(1 + \gamma) + k^*(1 - \gamma) \equiv k(1 + \gamma) + k^*(1 + \gamma^*) = 2. \]
Thus, by (50) we have
\[ \beta = (k - 1)\frac{\gamma + 1}{\gamma - 1} + 1 = \frac{k(\gamma + 1)}{\gamma - 1} - \frac{\gamma + 1}{\gamma - 1} + 1 = \frac{2 - k^*(-\gamma + 1)}{\gamma - 1} - \frac{2}{\gamma - 1} = k^*. \]
The latter means that \( U \) is an \( N \)-solution of (47) with \( \beta = k^* \) (in particular, the function \( G \) in (49) is a single valued function).

It remains only to show that \( U \) is an \( \text{adjoint} \) \( N \)-solution. Since \( U \) is an \( N \)-solution, there is \( \psi \) such that \( U = U_{N,\psi} \) in representation (44). Thus, we have to prove that \( N\psi = \pm \pi/2 \mod \pi \).

Choosing \( \tau_0 = -\psi + \pi/2N, \ h = 1 \) in representation (44) for \( U = U_{N,\psi} \), we obtain the corresponding Cartesian coordinates \( (x_0, y_0) \neq 0 \). Then \( U(x_0, y_0) = 0 \) and applying the Euler theorem on homogeneous functions yields
\[ (51) \quad x_0 \cdot U'_x(x_0, y_0) + y_0 \cdot \frac{\partial U}{\partial y}(x_0, y_0) = k^*U(x_0, y_0) = 0. \]
Hence, the gradient \( \nabla U(x_0, y_0) \) is orthogonal to the radius vector \( \nabla r(x_0, y_0) \) of the point \( (x_0, y_0) \). On the other hand, one can readily see from (44) that
\[ x_0 = (\mu^* - 1) \cos \left( \frac{\pi}{2N} - \psi \right), \]
\[ y_0 = (\mu^* - 1) \sin \left( \frac{\pi}{2N} - \psi \right), \]
and
\[ (52) \quad \theta_0 = \frac{\pi}{2N} - \psi \mod \pi. \]
By (46), the gradients of \( u_N \) and \( U \) are mutually orthogonal. Then from (51) we infer that vectors \( \nabla u_N(x_0, y_0) \) and \( \nabla r(x_0, y_0) \) are collinear. Since
\[ \nabla u_N = f_N(\theta) r^{k-1} \nabla r + r^k f'_N(\theta) \nabla \theta, \]
we conclude that \( f'_N(\theta_0) = 0 \). Hence, by (25) there is \( n \in \mathbb{Z} \) such that
\[ \theta_0 = \frac{Tn}{2} = \frac{\pi n}{N}, \]
which by (52) yields \( N\psi \equiv \frac{\pi}{2} \mod \pi \), and the theorem follows. \( \Box \)
5. Algebraic $N$-solutions

In this section we settle the following question: For which rational numbers $\gamma \in \mathbb{Q}$ such that $|\gamma| > 1$, equation (1) does admit nontrivial (i.e. $N \geq 2$) algebraic solutions? Clearly, algebraicity of a general $N$-solution is equivalent to that property for basic $N$-solutions. Therefore, in what follows we confine ourselves by only basic $N$-solutions.

**Lemma 5.1.** Let $\gamma \in \mathbb{Q}$, $|\gamma| > 1$ and $N \in \mathbb{N}$. Then inclusions $k(\gamma, N) \in \mathbb{Q}$, $\lambda(\gamma, N) \in \mathbb{Q}$ and $\mu(\gamma, N) \in \mathbb{Q}$ are pairwise equivalent.

**Proof.** It immediately follows from (27) that inclusions $k \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}$ are equivalent. By (38), $\mu \in \mathbb{Q}$ implies both $k \in \mathbb{Q}$ and $\lambda \in \mathbb{Q}$. On the other hand, if $k \in \mathbb{Q}$ then $\lambda \in \mathbb{Q}$, so by (37) $\mu \in \mathbb{Q}$ and the lemma is proved. □ □

For the further convenience, we put $\gamma \in \mathcal{A}$, if there exists $N \geq 2$, $N \in \mathbb{N}$, such that $u_N(x, y)$ is an algebraic function. In this case we also define $N(\gamma) = \{N \in \mathbb{N} : u_N \text{ is an algebraic function}\}$.

**Lemma 5.2.** Let $\gamma \in \mathcal{A}$, $|\gamma| > 1$ and $N \geq 2$. Then $N \in N(\gamma)$ iff the corresponding exponent $k(\gamma, N) \in \mathbb{Q}$.

**Proof.** Let $N \in N(\gamma)$. Then it follows from (44) and $|\mu| > 2N - 1 \geq 3$ that

$$(x^2 + y^2) = h^{4N - 2}\left[\mu^2 + 1 + 2\mu \cos(2N\tau + 2\phi)\right] \sim h^{4N - 2}, \quad h \to \infty$$

while

$$u_n(x, y) = h^{(2N-1)k} \cos(2N\tau + \phi),$$

where $k = k(\gamma, N)$. Thus, the growth exponent of $u$ is equal to $k$ and it follows that $k \in \mathbb{Q}$. Now, let $k(\gamma, N) \in \mathbb{Q}$. Then (45) gives a rational parametrization of $u_N^{2d}$, where $d$ is the denominator of $k$. Hence, $u_N(x, y)$ is an algebraic function. □ □

**Corollary 5.3.** For $\gamma = 1$ all the $N$-solutions are algebraic functions, i.e. $N(1) = \mathbb{N}$.

The next assertion is an easy corollary of (50)

**Lemma 5.4.** Let $\gamma \in \mathbb{Q}$. Then $k_N \in \mathbb{Q} \iff k_N^* \in \mathbb{Q}$. In particular,

$$N(\gamma) = N(-\gamma).$$

By Lemma 5.4, we can assume without loss of generality that $\gamma > 1$. In what follows, we suppose that $p$ and $q$ have no common divisors. Then by Lemma 5.2, $\gamma \in \mathcal{A}$ if and only if there is an integer $N \geq 2$ such that $k(\gamma, N) \in \mathbb{Q}$. By virtue of Lemma 5.1, this is equivalent to existence of a rational solution $\mu = \frac{A}{B} > 1$ of (39):

$$N = \frac{q(A^2 - B^2)}{2B(Ap - Bq)}, \quad A > B.$$
Thus, we arrive at the following diophantine equation
\begin{equation}
A^2q - 2ABpN + q(2N - 1)B^2 = 0. \tag{53}
\end{equation}

**Theorem 5.5.** The following assertions are equivalent

(i) \( \gamma = p/q \in A \) with \( N \in N(\gamma) \), \( N \geq 2 \);

(ii) equation (53) has an integer solution \((A, B)\): \( A, B \in \mathbb{Z} \), and \( A > B \geq 1 \);

(iii) the discriminant
\begin{equation}
D = N^2p^2 - q^2(2N - 1) \tag{54}
\end{equation}
is an squared integer.

Moreover, if \( \gamma \in A, \gamma \neq 1 \), then the set \( N(\gamma) \) is finite and the following upper bounds holds
\begin{equation}
N < \frac{q^2(p^2 + 2 - q^2)}{2p^2}. \tag{55}
\end{equation}

In particular,
\begin{equation}
q \geq 3. \tag{56}
\end{equation}

and
\begin{equation}
q + 1 \leq p \leq q^2 - 1; \tag{57}
\end{equation}

**Proof.** Clearly, we have only to establish the equivalence (ii) and (iii). In turn, the only nontrivial implication is (iii) \( \Rightarrow \) (ii).

Let (iii) be true. Then \( p = q\gamma > q \) and \( D = d^2 \) for some \( d \in \mathbb{N} \). Since
\[
D = N^2(p^2 - q^2) + q^2(N - 1)^2 > 0,
\]
we have \( d > 0 \). Let \( V = A/B \) and consider the following associated with (53) quadratic equation
\begin{equation}
F(V) := V^2q - 2pNV + q(2N - 1) = 0. \tag{58}
\end{equation}
Then (58) has two distinct rational solutions \( v_1 \) and \( v_2, v_1 < v_2 \). Since
\[
F(2N - 1) = -2(N - 1)(2N - 1)(p - q) < 0,
\]
we have \( v_2 > 2N - 1 \). Let \( v_2 = A/B \) where \( A \) and \( B \) have no common divisors, and \( B > 0 \). Hence, \( A > (2N - 1)B \) and \((A, B)\) is a desired solution of (ii).

In order to establish the finiteness of \( N(\gamma) \), we make use (iii). We have
\begin{equation}
D = \left( Np - \frac{q^2}{p} \right)^2 + \frac{q^2(p^2 - q^2)}{p^2}. \tag{59}
\end{equation}
Let \( D = d^2 \), with \( d \in \mathbb{N} \). Then (59) yields
\begin{equation}
d > Np - \frac{q^2}{p}. \tag{60}
\end{equation}
while the Bernoulli inequality and (59) imply the upper following bound
\[
(61) \quad d < \left( Np - \frac{q^2}{p} \right) + \frac{q^2(p^2 - q^2)}{2p(Np^2 - q^2)}.
\]

Since \( p \) and \( q \) have no common divisors, we can write
\[ \frac{q^2}{p} = M + \frac{m}{p} \]
where \( M > 0 \) is an integer, and \( m \in \{1, 2, \ldots, p-1\} \) is the non-zero remainder of the latter ratio. On the other hand, since \( m = q^2 - Mp \), it follows that \( m \) and \( p \) have no common divisors. Thus, by virtue of \( Np \in \mathbb{N} \), and strict inequalities (60) and (61) we obtain
\[
\frac{q^2(p^2 - q^2)}{2p(Np^2 - q^2)} > \frac{1}{p},
\]
which easily implies (55).

To verify (56) we notice that the cases \( q = 1 \) and \( q = 2 \) together with (55) easily yield the contradiction: \( N < 2 \).

The left inequality in (57) immediately follows from \( p > q \). Finally, to prove the right hand side inequality we put as above \( D = d^2, d \geq 1 \). Then \( D < N^2p^2 \), whence \( d < Np \). Taking into account integrity of \( d \) we obtain \( d \leq Np - 1 \), or what is the same
\[
0 \leq (Np - 1)^2 - D = 2N(q^2 - p) + 1 - q^2,
\]

hence
\[
(62) \quad q^2 - p \geq (q^2 - 1)/2N \geq 0.
\]

Therefore, \( p < q^2 \) and by virtue of integrity of \( q \) and \( p \) we arrive at a stronger inequality
\[
q + 1 \leq p \leq q^2 - 1
\]
and the theorem is proved. \( \square \) \( \square \)

**Corollary 5.6.** If \( \gamma, |\gamma| > 1 \), is an integer number then (1) has no (nontrivial) algebraic \( N \)-solutions.

**Proposition 5.7.** The following representation holds
\[
(63) \quad A = \left\{ \frac{2N - 1 + s^2}{2sN} : s \in \mathbb{Q} \cap (0, 1), N \in \mathbb{N} \right\}.
\]

**Proof.** We notice that the discriminant \( D \) in (54) will be the a squared integer if and only if a positive integer valued solution \((x, y)\) to the following Pell type equation
\[
(64) \quad N^2x^2 - (2N - 1)y^2 = 1,
\]
does exist such that \( x > y \). Here \( x = p/d, y = q/d \) and \( \gamma \) is uniquely defined by \( \gamma = x/y \).
Then (64) can be resolved by the standard rationalization technique. Really, let \( x = \frac{(1 + sy)}{N} \), whence \( s \) should be a rational number. Substituting the last expression in (64) gives

\[
y = \frac{2s}{2N - 1 - s^2}, \quad x = \frac{2N - 1 + s^2}{N(2N - 1 - s^2)}.
\]

Thus, we have

\[
\gamma = \frac{2N - 1 + s^2}{2sN}.
\]

To define the admissible values of \( s > 0 \) which provide \( \gamma > 1 \) we notice that

\[
\gamma - 1 = \frac{(s - 2N + 1)(s - 1)}{2sN},
\]

whence \( s \in (0; 1) \cup (2N - 1; +\infty) \). We can reduce our representation to the interval \((0; 1)\) only since the invariance property of (63) with respect to involution \( s \rightarrow (2N - 1)/s \).

\[\square\]

\[\square\]

6. Maximal and minimal series

Now we study the set \( A \) in dependence of the fractional decomposition \( \gamma = p/q \). We suppose as before that \( p \) and \( q \) have no common divisors.

**Proposition 6.1** (Maximal series). For \( p/q \in A \) the following sharp estimates hold

\[
p \leq q^2 - 2
\]

with equality only for odd denominator \( q = 2s + 1 \), \( p = 4(s^2 + s) - 1 \), and \( N = s(s + 1) \), where \( s \) is a positive integer. Moreover, if the denominator \( q = 2s \) is even we have a stronger inequality

\[
p \leq \frac{q^2 - 2}{2} = 2s^2 - 1
\]

with equality iff

\[
p = \frac{q^2 - 2}{2}, \quad N = \frac{q^2 - 4}{4}.
\]

**Proof.** First prove (65). In view of (57), it suffices to exclude the case \( q = p^2 - 1 \). Assuming the contrary, and let the last equality hold. Then (62) and (55) together yield

\[
q^2 \leq 2N < q^2 \frac{(p^2 + 2 - q^2)}{p^2},
\]

which easily implies \( q^2 \leq 2 \) that contradicts to the lower bound (56).

In order to analyze the equality case, assume that \( p = q^2 - 2 \). It follows then from (62) that \( 4N \geq q^2 - 1 = p + 1 \) and (54) can be rewritten as follows

\[
d^2 = N^2p^2 - (2N - 1)(p + 2) = (Np - 1)^2 - (4N - 1 - p) \leq (Np - 1)^2
\]
where $d > 0$ is an integer. Hence, $d \leq (Np - 1)$.

On the other hand,
\[
d^2 = (Np - 2)^2 + (p - 2)(2N - 1) > (Np - 2)^2
\]
which together with the preceding inequality implies $d = Np - 1$, and consequently
\[
p = q^2 - 2 = 4N - 1.
\]
In particular, $q$ must be an odd number. One can readily find that in this case $N$ is an integer. Thus, the first case of the corollary is proved.

To prove the second statement we suppose $q = 2s$. We have from (56) $s \geq 2$ and due to irreducibility of $p/q$ we notice that $p$ should be an odd number.

By Theorem 5.5 we have for the discriminant
\[
D \equiv d^2 = N^2p^2 - 4s^2(2N - 1).
\]
The last identity shows that $d$ has the same parity as $Np$ does. Thus, $d \leq pN - 2$. On the other hand,
\[
(Np - 2)^2 \geq d^2 = (Np - 2)^2 + 4(Np - 1 - s^2(2N - 1))
\]
and therefore,
\[
p - 2s^2 \leq -\frac{s^2 - 1}{N}.
\]
The last inequality and $s \geq 2$ yield
\[
p < 2s^2 = \frac{q^2}{2}.
\]
Now, the inequality (66) follows from the evenness of $q$.

To analyze the equality case in (66) we observe that in our notation $d > Np - 3$, otherwise we would have
\[
(Np - 3)^2 \leq d^2 = N^2p^2 - 2(2N - 1)(p + 1),
\]
which implies $2N(p-2) \leq 7-2p$, and contradiction follows. Thus, $d = Np-2$ and a straightforward computation shows
\[
N = \frac{p - 1}{2} = s^2 - 1
\]
which completes the proof. \qed \qed

**Proposition 6.2** (Minimal series). For $p/q \in \mathcal{A}$ the following lower bound holds
\[
p \geq q + 2
\]
with equality iff $q = 2s + 1$ is an odd number, $p = q + 2$ and $N = (q - 1)/2$. 

Proof. From (57) we have \( p \geq q + 1 \). Again, we argue by contradictory and assume that \( p = q + 1 \). Then we have the following relation for the discriminant

\[
D = N^2(q + 1)^2 - q^2(2N - 1) = d^2,
\]

and \( d \) is a positive integer. Hence

\[
d^2 = q^2(N - 1)^2 + 2qN^2 + N^2 = \left(q(N - 1) + \frac{N^2}{N - 1}\right)^2 - \frac{(2N - 1)N^2}{(N - 1)^2}.
\]

In particular,

\[d < q(N - 1) + \frac{N^2}{N - 1} = q(N - 1) + N + 1 + \frac{1}{N - 1}\]

which implies

\[d \leq q(N - 1) + N + 1 \equiv d_1 + 1.\]

On the other hand

\[d_1^2 = (q(N - 1) + N)^2 = q^2(N - 1)^2 + 2qN(N - 1) + N^2 < d^2,
\]

and it follows that \( d = d_1 + 1 \). Then by virtue of (68) we arrive at \( 2q = 2N - 1 \). Thus, the contradictions shows that \( p \geq q + 2 \) and (67) is proved.

Now assume that the equality \( p = q + 2 \) holds. Arguing as above we obtain the following inequality

\[
d < (N - 1)q + 2N + 2 + \frac{2}{N - 1} \leq (N - 1)q + 2N + 3.
\]

On the other hand,

\[
d^2 \equiv q^2(N - 1)^2 + 4qN^2 + 4N^2 > ((N - 1)q + 2N)^2,
\]

which implies by strong inequality in (69)

\[(N - 1)q + 2N + 1 \leq d \leq (N - 1)q + 2N + 2.\]

But it follows from (70) that \( d \) and \( (N - 1)q \) have the same parity. This gives exactly one choice in the last inequality

\[d = (N - 1)q + 2N + 2\]

which after comparison with the definition of \( d \) in (68) implies \( q = 2N + 1 \), and the required relation follows.

The ‘algebraic’ constants \( \gamma \in \mathcal{A} \) for small denominators \( q \leq 30 \) are displayed in Figure 1.
We wish point out that our formula (5) can also be deduced by using the general representation of \(p\)-harmonic functions in the plane near their singular points. The mentioned representation is established in [9] and [12] by using the hodograph method. Our approach, nevertheless, is more direct and use no the quasiregularity of the complex gradient of the \(N\)-solutions.

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