Analysis of the influence of gear tooth friction on dynamic force in a spur gear

Ł Jedliński
Lublin University of Technology, Mechanical Engineering Faculty, Department of Machine Design and Mechatronics, ul. Nadbystrzycka 36, 20-618 Lublin, Poland
ljedlinski@pollub.pl

Abstract. The main objective of this paper is to investigate the influence of tooth friction on the dynamic force of the gearbox. This is one of the most insignificantly investigated aspects of the gearbox work, despite both experimental and theoretical research conducted. For this reason, it is not entirely possible to establish the friction force determining the friction coefficient due to the complexity of the process. Currently, various approaches are taken in calculating dynamic models, among others excluding friction, Coulomb models, models in which the value of the friction coefficient depends on the sliding velocity, and the application of complex elastohydrodynamic models. Due to such different approaches, it was decided to examine the influence of tooth friction on the dynamic force for the stationary work conditions of the gearbox. Moreover, a new gear tooth friction model was developed, based on the Coulomb model of friction and rolling friction. The results were obtained for four friction models and, in the fifth case, without including it. Furthermore, the influence of torsional vibration on the friction lever arm was analysed.

1. Introduction
Research on the technical equipment is conducted to gain a better understanding of the phenomena occurring during the exploitation of the equipment. One aims to optimise their behaviour in terms of efficiency, weight, and environment protection [1], durability, reliability [2, 3], or reduction of the undesirable leftover processes, such as vibration and noise [4].

Gearboxes frequently occur in the drive systems of various machines and devices. Despite numerous benefits, the cyclical meshing of the subsequent pairs of teeth causes time-varying forces to occur. This phenomenon has a negative influence due to durability, vibrations, and noise, among others. It is vital to investigate how these variable dynamic forces occur, and especially the influence of each factor. The influence of friction, one of those factors, is most visible in the wear of cooperating surfaces [5].

Authors of the gearboxes’ dynamic models present various approaches to the gear tooth friction. It was omitted in older publications [6, 7] due to the fact that its value amounts to less than 1 percent of the normal force. Even currently it is not always considered in the studies modelling the functioning of the gearboxes [8-11].

The simplest friction model used both in the past and currently, is sliding friction (Coulomb). The friction factor is usually constant. Force of the gear tooth friction changes its direction when reaching the pitch point. This trait has been considered by numerous researchers by changing the sign of the friction indicator [12]. A rapid change of the friction force value may be problematic due to the accuracy of the numerical solution according to Rincon et al. [13]. For this reason, the authors introduced
a dependency that “smoothes out” the sliding friction indicator. A more accurate rendering of the behaviour of the friction force was conducted by the means of developing empirical and theoretical models of the sliding friction indicator, based to a huge extent on the sliding velocity [14].

Due to the fact that the cooperation of gears occurs usually with gear oil, the designed models take this influence into account. Among those, especially popular are elastohydrodynamic lubrication (EHL) models [15, 16], created as a result of experiments. In order to conduct calculations, detailed information on the lubricant is required, e.g. viscosity, inlet temperature, and other data concerning gears, that is tooth roughness, Hertz pressure, Young module, Poisson ratio, and speed. Moreover, certain coefficients in the equations are determined on the basis of stand research on samples, e.g. disc-shaped [17, 18]. Considering the information above, it is to be stated that the obtained results are correct for the given parameter range [17, 19].

Another common defect of the discussed models is the fact that the friction coefficient in the pitch point is often equal to 0 [14, 20]. For this reason, both conducting a tribological analysis and explaining the tooth wear are not possible. The value of the friction factor in this point decreases, because of the occurrence of rolling friction, significantly smaller than sliding friction. This type of friction is also predominant in the areas near this point.

Due to the limitations of the models, a new friction model (marked as model 4) was designed. Its basis is solid since it considers the occurring rolling and sliding friction. The friction force does never equal zero. On the other hand, it is much less complex than the EHL models, which allows one to use it at the design stage without the full data on the gearbox for the majority of the working conditions and doing additional experimental testing to select the right parameters. The accuracy of the proposed model was increased by considering the change to the length of the lever arm of friction as a result of vibration. For comparison purposes, the calculations were made also for three models of Coulomb friction with varied levels of detail and not considering the gear tooth friction.

2. **Dynamic model of the gearbox**

The analysis was conducted for a single-stage spur gear with two degrees of freedom $\phi_1$ and $\phi_2$ (Figure 1). The gears have a constant torque loads $T_1$ and $T_2$. Stiffness of meshing $k(t)$ is time-varying and constitutes the main reason for gear vibrations. Two tooth stiffness models were analysed. The first one according to the ISO 6336 -1 standard [21], whereas the second one progresses similarly to the real process according to Cai [7]. Damping $c$ in the meshing was assumed to be constant. Meshing friction was analysed for four variants of the friction model. Solid models of the gears subjected to analysis were made (Figure 2). On their basis the moments of inertia were determined, assuming steel as the material. The remaining parameters were shown in Table 1.

![Figure 1. Model of a spur gearbox with two degrees of freedom.](image-url)
Table 1. Property of the gears.

| Parameter                      | Pinion | Gear           |
|--------------------------------|--------|---------------|
| Number of teeth                | \( z_1 = 23 \) | \( z_2 = 48 \) |
| Module (mm)                    | \( m = 4 \) |               |
| Pressure angle (°)             | \( \alpha_0 = 20 \) |               |
| Working (operating) pressure angle (°) | \( \alpha_w = 17.61399 \) |               |
| Face width (mm)                | \( b = 40 \) |               |
| Modification coefficient       | \( x_1 = -0.1 \) | \( x_2 = -0.372 \) |
| Contact ratio                  | \( \epsilon = 1.78 \) |               |
| Moment of inertia (kgm\(^2\))  | \( J_1 = 0.00204234 \) | \( J_2 = 0.02423705 \) |
| Torque (Nm)                    | \( T_1 = 63.66 \) | \( T_2 = 132.85 \) |
| Mesh damping (Ns/m)            | \( c = 100 \) |               |
| Initial angular speed (rad/s)  | \( \omega_i = 157,0796 \) (\( n_1 = 1500 \) rpm) | \( 0 \) |

The equations of motion were introduced on the basis of the free body diagram as well as the Newton’s second law. The forces resulting from stiffness and damping act in the direction of a line of action (LOA). This line is tangential to the base circles \( r_b \). Since the relative linear displacement on LOA is equal to the relative angular displacement the substitution \( x = r_b \phi_1 - r_b \phi_2 \) and analogously for speed \( \dot{x} = r_b \dot{\phi}_1 - r_b \dot{\phi}_2 \) are obtained. The set of equations of motion without considering the gear tooth friction is:

\[
\begin{align*}
J_1 \ddot{\phi}_1 + c \left( r_{b1} \dot{\phi}_1 - r_{b2} \dot{\phi}_2 \right) \frac{1}{r_{b1}} + k \left( r_{b1} \phi_1 - r_{b2} \phi_2 \right) \frac{1}{r_{b1}} &= T_1 \\
J_2 \ddot{\phi}_2 - c \left( r_{b1} \dot{\phi}_1 - r_{b2} \dot{\phi}_2 \right) \frac{1}{r_{b2}} - k \left( r_{b1} \phi_1 - r_{b2} \phi_2 \right) \frac{1}{r_{b2}} &= -T_2
\end{align*}
\]  

(1)

Considering the gear tooth friction in a model with two degrees of freedom results in introducing an additional torque \( M_{\text{f}} \) which, depending on the location of the contact point on the line of action is either summed or subtracted from the outer torque. Equations of motion with friction moments were transformed into the form used in Simulink are shown below:

\[
\begin{align*}
\ddot{\phi}_1 &= -\frac{c r_{b1}^2 \dot{\phi}_1}{J_1} + \frac{c r_{b1} r_{b2} \dot{\phi}_2}{J_1} - \frac{k r_{b1}^2 \phi_1}{J_1} + \frac{k r_{b1} r_{b2}^2 \phi_2}{J_1} + \frac{T_1}{J_1} + \frac{M_{f1}}{J_1} \\
\ddot{\phi}_2 &= \frac{c r_{b1} r_{b2} \dot{\phi}_1}{J_2} - \frac{c r_{b1}^2 \phi_1}{J_2} + \frac{k r_{b1} r_{b2} \phi_2}{J_2} - \frac{k r_{b2}^2 \phi_1}{J_2} - \frac{T_2}{J_2} - \frac{M_{f2}}{J_2}
\end{align*}
\]  

(2)
2.1. Gear mesh tooth stiffness
Basic mesh stiffness was determined on the basis of ISO 6336-1 standard according to the B method. The flexibility of a pair of teeth was calculated:

\[ q' = C_1 + \frac{C_2}{z_1} + \frac{C_3}{z_2} + C_4 x_1 + \frac{C_5 x_1}{z_1} + C_6 x_2 + \frac{C_7 x_2}{z_2} + C_8 x_1^2 + C_9 x_2^2 = 0.0634 \ \mu m/N \]  

(3)

Constants \( C \) are given in the standard. On their basis theoretical stiffness \( c_{th}' \) for one pair of teeth for a 1 mm face width was estimated:

\[ c_{th}' = \frac{1}{q'} = 15.77 \ \frac{N}{\mu m} \]  

(4)

The value of load intensity was determined:

\[ \frac{F_{Ka}}{b} = 43.37 < 100 \ \text{N/mm} \]  

(5)

It is lower than the minimum value, which is why low minimal intensity ought to be considered and the stiffness intensity for one pair of teeth ought to be determined from the following dependency:

\[ c' = c_{th}' C_M C_R C_B \cos \beta \left[ (F_t K_A / b) / 100 \right]^{0.25} = 10.46 \ \frac{N}{\mu m} \]  

(6)

\( C \) constants were taken from the standard. The average mesh stiffness equals:

\[ c_y = c' (0.75 \varepsilon + 0.25) = 16.60 \ \frac{N}{\mu m} \]  

(7)

Upon considering the width of the gear the stiffness of a single tooth pair equals:

\[ c_{b1}' = c' b = 418.723722 \ \frac{N}{\mu m} = 418 \ 723 \ 772 \ \frac{N}{m} \]  

(8)

for two pairs of teeth:

\[ c_{b2}' = 2c_{b1}' = 837 \ 447 \ 544 \ \frac{N}{m} \]  

(9)

whereas the average stiffness of meshing:

\[ c_{by} = c_y b = 664.033633 \ \frac{N}{\mu m} = 664 \ 033 \ 633 \ \frac{N}{m} \]  

(10)

Solving the differential equations using numerical methods is conducted in reference to time. For this reason, it is necessary to determine the time of tooth contact for one and two pairs of teeth. Figure 3 indicates that time of tooth contact for one pair of teeth \( t_1 \) equals:

\[ t_1 = 2t_z - t_p \]  

(11)

where:

\[ t_z = \frac{1}{f_z} (s) - \text{time between meshing the pairs of teeth}, \]
\[ f_z = \frac{n_1 z_1}{60} \text{ (Hz)} \] – meshing frequency,
\[ n_1 \text{ (rpm)} \] – pinion rotational speed,
\[ t_p = \frac{\theta_1}{\omega_1} \text{ (s)} \] – meshing time for one pair of teeth,
\[ \theta_1 = \theta \epsilon \text{ (rad)} \] – angle of rotation during which one pair of teeth remains in contact,
\[ \theta = \frac{2\pi}{z_1} \text{ (rad)} \] – the angle of meshing of the following pairs of teeth,

whereas for two pairs of teeth:

\[ t_2 = t_p - t_z \] (12)

Figure 3. Determining the time of contact for one and two pairs of teeth.

Figure 4 presents the progress of stiffness determined in compliance with the above-presented calculations for the rotational speed \( n_1 = 1500 \text{ rpm} \). The shape of the progress is rectangular.

Figure 4. Mesh stiffness calculated according to ISO 6336-1.

The shape of stiffness is simplified since the stiffness of teeth varies during the meshing. According to the publications [22-25] based on FEA, analytical methods, and experiment results stiffness of a single pair of teeth is the smallest in the beginning and the end of tooth contact, whereas the maximum value is reached near the pitch point. Such progress is obtained in the analytical model proposed by Cai [7]. Stiffness for one pair of teeth is determined on the basis of the results from ISO 6336-1. For one pair of teeth the stiffness is equal [7]:

\[ k_{s1} = c'_{b1} \left( -\frac{1.8}{(z\epsilon t_2)^2} t^2 + \frac{1.8}{z\epsilon t_2} t + 0.55 \right) \] (13)

where:
$t$ (s) – current time, $t = 0, t_e$.

Stiffness for two pairs of teeth is obtained by summing single sti\nences according to the time sequence (Figure 3). The results are presented in Figure 5.

Figure 5. Mesh stiffness compliant with the Cai model.

According to this method, the maximum stiffness for the tooth contact for one pair of teeth is $k_s_{1\max} = 418,720,047$ N/m and for two pairs $k_s_{2\max} = 718,658,532$ N/m. Upon comparing these results with the ones obtained using the ISO method a difference of 0.99 ppm and 15.2% is to be observed, respectively. In the case of tooth contact for one pair of teeth, the difference is negligible, whereas for two pairs – significant. This phenomenon is caused by the arc-shaped stiffness. Upon analysing the results from specialist literature based on experimental results [26], as well as potential energy [23] and FEA [25] it is to be stated that maximum stiffness for the tooth contact for two pairs of teeth is always smaller than twice the maximum value of the tooth contact for one pair of teeth. Cai and ISO methods differ in the shape of the progress as well as the maximum value of tooth contact for two pairs of teeth.

2.2. Gear tooth friction

Sliding friction (Coulomb) occurs between the teeth moving against each other. As a result, friction force tangential to the profile of teeth and perpendicular to LOA equal:

$$F_f = \mu_k (N_k + N_c)$$

(14)

where:

$\mu_k$ – coefficient of sliding friction,
$N_k$ – normal force from mesh stiffness,
$N_c$ – normal force from mesh damping.

In the pitch point, the teeth roll over each other, which results in rolling friction. In the area close to the pitch point relative speed is insignificant and upon simplification, it can be stated that rolling friction occurs in the area close to the pitch point. In the case of this type of friction the moment of friction (rolling resistance) acting in the opposite direction to rolling teeth, equal:

$$M_{fr} = f \times (N_k + N_c)$$

(15)

where:

$f$ – coefficient of rolling friction (moment arm of the rolling friction).
Friction in the assumed gearbox model with 2 DOF can only be considered as the moment of friction. For this reason, the arm of sliding friction is to be determined. The speed of the movement of the contact point on the line of action is constant for the constant angular speed and with uninterrupted teeth contact. It stems from the rule of evenness of the projections of the speed of points of a solid body in a mutual direction. For the pinion, the minimal arm of friction is equal to the length of the section $|L_1A|$, whereas the maximum $|L_1E|$ (Figure 6). In order to determine the arm $|L_1A|$ and $|L_2E|$ line segment on the LOA depending on the known values will be specified:

$$|L_1E| = \sqrt{|O_1E|^2 - |O_1L_1|^2} = \sqrt{r_1^2 - r_{b1}^2}$$  \hspace{1cm} (16)$$

$$|L_1C| = |O_1C| \sin \alpha_w = r_{w1} \sin \alpha_w$$  \hspace{1cm} (17)$$

and similarly

$$|L_2A| = \sqrt{|O_2A|^2 - |O_2L_2|^2} = \sqrt{r_2^2 - r_{b2}^2}$$  \hspace{1cm} (18)$$

$$|L_2C| = |O_2C| \sin \alpha_w = r_{w2} \sin \alpha_w$$  \hspace{1cm} (19)$$

On the basis of this, the line segment for the pinion can be calculated

$$|L_1A| = |L_1E| - (|L_1E| - |L_1C|) - (|L_2A| - |L_2C|) = |L_1C| - |L_2A| + |L_2C|$$  \hspace{1cm} (20)$$

therefore

$$r_{f1\text{min}} = |L_1A| = r_{w1} \sin \alpha_w - \sqrt{r_{a2}^2 - r_{b2}^2} + r_{w2} \sin \alpha_w = (r_{w1} + r_{w2}) \sin \alpha_w - \sqrt{r_{a2}^2 - r_{b2}^2}$$  \hspace{1cm} (21)$$

$$r_{f1\text{max}} = |L_1E| = \sqrt{r_{a1}^2 - r_{b1}^2}$$  \hspace{1cm} (22)$$

and for the gear

$$|L_2E| = |L_2A| - (|L_2A| - |L_2C|) - (|L_1E| - |L_1C|) = |L_2C| - |L_1E| + |L_1C|$$  \hspace{1cm} (23)$$

therefore

$$r_{f2\text{min}} = |L_2E| = (r_{w1} + r_{w2}) \sin \alpha_w - \sqrt{r_{a1}^2 - r_{b1}^2}$$  \hspace{1cm} (24)$$

$$r_{f2\text{max}} = |L_2A| = \sqrt{r_{a2}^2 - r_{b2}^2}$$  \hspace{1cm} (25)$$

The change to the arm of friction is linear for involute meshing and a constant rotational speed of both wheels. According to the assumed direction of the rotation, the value increases from minimum to maximum in the case of the pinion and decreases in the case of the wheel. This phenomenon is presented in Figure 7 for a single pair of teeth. The equation expressing the progress of the change to the arm of friction is a linear function $r_f = ax + b$. Upon substituting it with the symbols from Figure 7 the equation of the arm of friction for the pinion equals:

$$r_{f1} = t \cdot \tan \left( \tan^{-1}\left( \frac{r_{f1\text{max}} - r_{f1\text{min}}}{t_p} \right) \right) + r_{f1\text{min}}$$  \hspace{1cm} (26)$$

where:

- $t$ – time (s), $t = (0, t_p)$
- $\tan, \tan^{-1}$ – arguments in degrees ($^\circ$)
and for the wheel

\[ r_{f2} = t \cdot \tan \left( 180 - \tan^{-1} \left( \frac{r_{f2 \text{max}} - r_{f2 \text{min}}}{t_p} \right) \right) + r_{f2 \text{max}} \]  

(27)

The results of the above-presented equations including the time sequence of the meshing of teeth are shown in Figure 8 (in Figures 8, 10-12 the blue colour means two pairs of teeth in contact and green one pair).

**Figure 6.** Determining the minimum and maximum arms of sliding friction for the pinion and a wheel. \( r_d \) – tip radius, \( r_w \) – working pitch radius, \( r_b \) – base radius, \( r_{f1 \text{min}} \) – min value of moment arm of sliding friction force acting on pinion, \( r_{f1 \text{max}} \) – max value of moment arm of sliding friction force acting on the pinion, \( C \) – pitch point, \( A, E \) – points where gears start and end to be in contact.

**Figure 7.** Theoretical schemes of the progress of changes to the arm of the sliding friction.

It is already possible to determine the Coulomb moment of friction \( M_{fc} \), the equation of which for the pinion and wheel is:

\[ M_{fc} = F_f r_f \]  

(28)

The direction of the force and so the moment of friction depends on the relative speed. In the pitch point \( C \), the relative sliding velocity equals 0. After crossing this point change to the sense of the friction force occurs (Figure 6). Time from entering the tooth contact to the pitch point can be determined from the following dependency:

\[ t_C = \frac{|L_1C| - |L_1A|}{\omega_1 r_{b1}} \ (s) \]  

(29)
In the model of friction considering the rolling friction, information on the value of the sliding velocity is required. According to the symbols in Figure 9 linear speed in the contact point equals the product of the angular speed and radius:

\[ v_1 = \omega_1 R_{v1} \]
\[ v_2 = \omega_2 R_{v2} \]

where:

\[ R_{v1} = \sqrt{r_{b1}^2 + r_{f1}^2} \]
\[ R_{v2} = \sqrt{r_{b2}^2 + r_{f2}^2} \]

The value of sliding velocity for the pinion is equal

\[ v_{s1} = v_{t1} - v_{t2} \]

and for the wheel

\[ v_{s2} = v_{t2} - v_{t1} \]
Figure 9. Speed components of the contact point in points A, C, and E.

The results of the speed components for the pinion and wheel are shown in Figure 10. Pinion and gear speeds are non-linear except for the normal components.

Figure 10. Schemes of the velocity of the contact point of the pinion and wheel.

2.3. Analysed models of gear tooth friction

The influence of friction on dynamic meshing force was analysed for four models of gear tooth friction and one model not considering it, for comparative purposes. The first three models consider only sliding friction and differ in the method of calculating the arm of friction force $r_f$. The fourth model proposed
by the author includes both the sliding and rolling friction. The fifth model does not consider friction (it equals 0) and is presented for comparative purposes.

Model 1
In the determined frictional moment, in accordance with the dependency (15), the moment arm $r_f$ is constant. For the pinon, the minimum value equals $r_f^1 = 3.06$ mm and the maximum value for the wheel is $r_f^2 = 39.30$ mm occur at the moment of tooth contact.

Model 2
In this case, the arm of the moment of sliding friction changes according to the theoretical position of wheels during the tooth contact. The arms are determined in accordance with the dependencies (26), (27), and the results presented in Figure 8.

Model 3
In model 2 the value of the arm of friction is calculated on the theoretical position of gears without including the change resulting from the torsional vibration of these gears. Moreover, this model considers this influence, as well as the moment arm, determined, for the pinion, form the following dependency:

$$r_{f\text{vib}}^1 = r_b^1 \tan(\beta_1 + \varphi_{\text{vib}})$$

and wheel

$$r_{f\text{vib}}^2 = r_b^2 \tan(\beta_2 + \varphi_{\text{vib}})$$

where:

$\varphi_{\text{vib}} = \varphi_1 - \varphi_2 i$ - the angle by which the gears move as a result of vibration,

$\varphi_1, \varphi_2$ - angular placement of the gears calculated from the equation (2),

$i = z_2/z_1$ - gear ratio.

The influence of vibration on the length of moment arms of friction $r_{f\text{vib}}$ (Figure 11 a, b) is the greatest at the beginning of the simulation, when the vibration is unstable (Figure 11 b, d). After stabilisation as a result of damping the vibration decreases (Figure 11 d) along with the length of moment arms of friction.

Model 4
This model is the most accurate, since it includes sliding and rolling friction. Sliding friction is calculated in accordance with model 3, whereas rolling friction using the dependency (15). The change from sliding to rolling friction depends on the value of the sliding velocity $v_s$. It was assumed that rolling friction occurs when the speed is lower or equal $v_{\text{threshold}} = 0.1$ m/s. The frictional moment is calculated from the following dependency:

$$M_f = \begin{cases} M_{fc} = \mu_k (N_k + N_c) r_f & \text{for } v_s > v_{\text{threshold}} \\ M_{fr} = f (N_k + N_c) & \text{for } v_s \leq v_{\text{threshold}} \end{cases}$$

A conclusion of friction moments for the four models is shown in Figure 12. These exemplary calculations were conducted for the constant value of $N_k + N_c = 1$ N, coefficient of sliding friction $\mu_k = 0.05$ and coefficient of rolling friction $f = 0.00005$. 

Figure 11. Values of moment arms of frictional moments $r_{f_{vib1}}$ and $r_{f_{vib2}}$ considering torsional vibration of gears.

Figure 12. Frictional moment determined for 4 models for the parameters: $N_k + N_c = 1$ N, $\mu_k = 0.05$ and $f = 0.00005$ constant in time.
Model 5
Friction does not occur in this model. It is presented for comparative purposes.

3. Results of numerical tests on the basis of analytical models
The calculations were conducted in the Matlab\Simulink calculation package. Differential equations were solved using the Runge-Kutta method (4,5) with ode function with a variable time step. The first calculations aimed to select the proper duration of the simulation. In the beginning, the vibration amplitude decreases significantly (Figure 11), which is mostly influenced by the vibration damping coefficient \( c \). It was assumed that 30\% of the initial course will be removed in order to obtain a part in which the vibrations are stationary. The relative tolerance of the solver was 1e-5 and absolute 1e-7. The time of simulation was, respectively: 0.5 s, 1 s, 1.5 s, 2 s, 2.5 s, 3 s, 4 s, 5 s, 15 s. Models 4 and 5 were selected for this computing. Gear tooth dynamic force \( F_d \) is calculated from the following dependency:

\[
F_d = k(r_{b1}\varphi_1 - r_{b2}\varphi_2) + c(r_{b1}\dot{\varphi}_1 - r_{b2}\dot{\varphi}_2)
\]  

(39)

For short simulation time (model 5 – 0.5 s and 1 s) the values of dynamic force are higher than for the longer time (model 5 – 1.5 s to 3 s), which was explained earlier and results from the time required for vibration stabilization (Figure 13). However, the value increases again (model 5 – 4 s to 15 s). The accuracy of the tolerance of the solver was therefore changed to 1e-7 for relative tolerance and 1e-9 for absolute tolerance. Upon introducing those changes dynamic meshing force does not increase again, but the fluctuation in this value can be observed in model 4. For this reason, the accuracy of the solver has been increased again: relative tolerance 1e-10 and absolute tolerance 1e-12. The obtained values for a time from 1 s to 15 s are nearly identical and therefore the accuracy was deemed sufficient.

It was assumed that the simulation time for the further calculations will be 1.5 s and the rolling friction coefficient for model 4 \( f = 0.00005 \).

![Figure 13](image)

Figure 13. Determining the duration of the simulation and the accuracy of the solver.

3.1. Calculations of the dynamic force for the mesh stiffness compliant with ISO 6336-1
The simulations were performed for various values of the sliding friction coefficient \( \mu \). The change of this parameter in Figure 14 occurs from 0.01 by each 0.05 to 0.46. Upon comparing model 5 to the remaining model sit is to be stated that not considering friction causes a decrease in dynamic meshing force.
Moreover, a monotonous increase in the dynamic meshing force and the friction factor can be observed in the obtained results. Only the last value in model 1 is insignificantly lower than the previous one. With a few exceptions, the models are monotonous for the same values of the friction coefficient. The simpler the model, the lower the dynamic meshing force. The highest values were obtained for model 4. The differences between models 2 and 3 are exceedingly small, which indicates that considering the change in location of gears as a result of vibration influences the dynamic meshing force to an exceedingly small extent.

Figure 14. Dynamic meshing force for the sliding friction coefficient $\mu = 0.01 \div 0.46$.

Figure 15. Dynamic meshing force for the sliding friction coefficient $\mu = 0.008 \div 0.028$.

Figure 16 presents the results for a narrower range of variability of the friction coefficient and a smaller step. For this reason, changes to the value of dynamic meshing force occur in a narrower range. The conclusions that can be drawn are analogous to the previous ones. However, there is a difference in the progress of model 1, as it is not monotonous.

3.2. Calculations of dynamic meshing force for mesh stiffness compliant with Cai model

Similarly, to the models with stiffness compliant with ISO the values of dynamic meshing force for the models considering friction are higher than for the remaining model. Figure 16 shows a shape of the progress of the force values similar to the models 1-4, albeit not monotonous. The maximum value was reached for the friction factor $\mu = 0.21$. As a rule, the simpler model, the greater values of the determined forces.

The narrower progress of change of the friction factor with the smaller step was shown in Figure 17. A few maximum values occur in the results, so the progress is more varied.

Figure 16. Dynamic force for the sliding friction coefficient $\mu = 0.01 \div 0.46$.

Figure 17. Dynamic force for the sliding friction coefficient $\mu = 0.008 \div 0.028$. 
4. Conclusions
The results of the conducted research indicate that the assumed model of tooth stiffness has the most significant influence on the dynamic meshing force. The difference between the values is several hundred newtons. Not including the gear tooth friction in any model of tooth stiffness and this friction always causes the obtained results to be lower. In order to compare 4 friction models, it is advised to include the tooth stiffness model. Friction causes a monotonous increase in the dynamic meshing force for stiffness compliant with ISO 6336—1 and the results from Figure 14. The proposed model 4 obtained the highest values of dynamic meshing force. The simpler the representation of friction, the lower the dynamic meshing force.

It is the opposite in the case of stiffness according to Cai in Figure 16. An increase in the value of the friction coefficient does not cause a monotonous change to the value of the dynamic meshing force. In this case, the proposed model 4 obtained the lowest result. Considering the change to the location of gears as a result of vibration for stable working conditions does not increase the results significantly, which can be observed in the case of models 2. and 3. The results and conclusions were obtained and drawn for stable working conditions. For transient conditions, the influence of friction would presumably be different.

References
[1] Merkisz J, Idzior M, Lijewski P, Fuć P and Karpiuk W 2008 The analysis of the quality of fuel spraying in relation to selected rapeseed oil fuels for the common rail system Proc. of the Ninth Asia-Pacific Int. Symp. on Combustion and Energy Utilization pp 352–356 https://doi.org/10.1007/s11668-018-0573-7
[2] Muhammad I, Nordin S, Alwadie A, Awais M, Aman Sheikh M, Glowacz A and Kumar V 2019 An automated feature extraction algorithm for diagnosis of gear faults J. of Failure Analysis and Prevention 19 pp 98–105 https://doi.org/10.1007/s11668-018-0573-7
[3] Michalak A, Wodecki J, Wylomańska A and Zimroz R 2019 Application of cointegration to vibration signal for local damage detection in gearboxes Applied Acoustics 144 pp 4–105 http://dx.doi.org/10.1016/j.apacoust.2017.08.024
[4] Figlus T, Koziol M and Kuczyński Ł 2019 The effect of selected operational factors on the vibroactivity of upper gearbox housings made of composite materials Sensors 19 pp 1-18 doi:10.3390/s19194240
[5] Bor M, Borowczyk T, Idzior M, Karpiuk W and Smolec R 2017 Analysis of hypocycloid drive application in a high-pressure fuel pump VII International Congress on Combustion Engines Matec Web of Conferences 118 UNSP 00020. DOI: 10.1051/matecconf/201711800020
[6] Kuang J-H and Lin A-D 2003 Theoretical aspects of torque responses in spur gearing due to mesh stiffness variation Mechanical Systems and Signal Processing 17(2) pp 255–271 doi:10.1006/mssp.2002.151
[7] Byrtus M 2008 Qualitative analysis of nonlinear gear drive vibration caused by internal kinematic and parametric excitation Engineering MECHANICS 15 pp 471–480
[8] Yu W, Mechefske Ch K and Timusk M 2017 Influence of the addendum modification on spur gear back-side mesh stiffness and dynamics J. Sound Vib 389 pp 183–201 http://dx.doi.org/10.1016/j.jsv.2016.11.030
[9] Cui L, Yao T, Zhang Y, Gong X and Kang Ch 2017 Application of pattern recognition in gear faults based on the matching pursuit of a characteristic waveform Measurement 104 pp 212–222 http://dx.doi.org/10.1016/j.measurement.2017.03.021
[10] Yi Y, Huang K, Xiong Y and Sang M 2019 Nonlinear dynamic modelling and analysis for a spur gear system with time-varying pressure angle and gear backlash Mechanical Systems and Signal Processing 132 pp 18–34 https://doi.org/10.1016/j.ymssp.2019.06.013
[11] Margielewicz J, Gąska D and Litak G 2019 Modelling of gear backlash Nonlinear Dynamics 97 pp 355–368 https://doi.org/10.1007/s11071-019-04973-z
[12] Saxena A, Parey A and Chouksey M 2015 Effect of shaft misalignment and friction force on time-varying mesh stiffness of spur gear pair, *Engineering Failure Analysis* **49** pp 79–91 http://dx.doi.org/10.1016/j.engfailanal.2014.12.020

[13] Fernandez del Rincon, Viadero F, Sancibrian R, Garcia Fernandez P and de Juan A 2009 A dynamic model for the study of gear transmissions *Computational Methods and Experimental Measurements XIV* pp 523 – 534 doi:10.2495/CME090471

[14] Jones R G 2012 The Mathematical Modelling of Gearbox Vibration under Applied Lateral Misalignment Ph.D. thesis University of Warwick

[15] Liu H, Liu H, Zhu C and Parker R G 2020 Effects of lubrication on gear performance: A review *Mechanism and Machine Theory* **145** 103701 https://doi.org/10.1016/j.mechmachtheory.2019.103701

[16] Shi J-f, Gou X-f and Zhu L.-y 2019 Modeling and analysis of a spur gear pair considering multi-state mesh with time-varying parameters and backlash *Mechanism and Machine Theory* **134** pp 582–603 https://doi.org/10.1016/j.mechmachtheory.2019.01.018

[17] Brethee K F, Gu F and Ball A D 2016 Frictional effects on the dynamic responses of gear systems and the diagnostics of tooth breakages *Systems Science & Control Engineering* **4** pp 270 – 284 http://dx.doi.org/10.1080/21642583.2016.1241728

[18] He S, Cho S and Singh R 2008 Prediction of dynamic friction forces in spur gears using alternate sliding friction formulations *Journal of Sound and Vibration* **309** pp 843–851 doi:10.1016/j.jsv.2007.06.077

[19] Vaishya M and Singh R 2003 Strategies for modeling friction in gear dynamics *Journal of Mechanical Design* **125** pp 383 – 393 DOI: 10.1115/1.1564063

[20] Kahraman A, Lim J and Ding H 2007 A dynamic model of a spur gear pair with friction *12th IFToMM World Congress, Besancon* (France) June pp 18-21

[21] ISO 6336-1 :1996. Calculation of load capacity of spur and helical gears

[22] Chen T, Wang Y and Chen Z 2019 A novel distribution model of multiple teeth pits for evaluating time-varying mesh stiffness of external spur gears *Mechanical Systems and Signal Processing* **129** pp 479–501 https://doi.org/10.1016/j.ymssp.2019.04.029

[23] Yang Y, Cao L, Li H and Dai Y 2019 Nonlinear dynamic response of a spur gear pair based on the modeling of periodic mesh stiffness and static transmission error *Applied Mathematical Modelling* **72** pp 444–469 https://doi.org/10.1016/j.apm.2019.03.026

[24] Pedersen N L and Jørgensen M F 2014 On gear tooth stiffness evaluation *Computers and Structures* **135** pp 109–117 https://doi.org/10.1016/j.compstruc.2014.01.023

[25] Cooley Ch G, Liu Ch, Dai X and Parker R G 2016 Gear tooth mesh stiffness: A comparison of calculation approaches *Mechanism and Machine Theory* **105** pp 540–553 http://dx.doi.org/10.1016/j.mechmachtheory.2016.07.021

[26] Raghuwanshi N K and Parey A 2016 Experimental measurement of gear mesh stiffness of cracked spur gear by strain gauge technique *Measurement* **86** pp 266–275 http://dx.doi.org/10.1016/j.measurement.2016.03.001