REMARK ON FALTINGS THEOREM

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Abstract. We prove Faltings Finiteness Theorem using Rieffel’s classification of the noncommutative tori.

1. Introduction

Diophantine geometry studies rational solutions of the polynomial equations in terms of geometry of the corresponding solutions in complex numbers [Hindry & Silverman 2000] [2, Introduction]. The idea that arithmetic depends on geometry is amazingly fruitful, e.g. the Grothendieck’s concept of a scheme over the Dedekind domain led to the proof of Weil’s Conjectures. Usually the arithmetic schemes corresponding to the polynomial rings with rational coefficients support morphisms defined over the algebraic closure of the underlying field. As a result, such morphisms do not preserve rational points on the algebraic varieties.

It is known since the works of J. -P. Serre and P. Gabriel, that algebraic geometry can be recast in terms of non-commutative rings $R$ [3, Chapter 13]. Unlike commutative rings, the $R$ are usually simple rings. In particular, the spectrum of the ring $R$ is a singleton and one cannot localize $R$ or define a scheme. However the simplicity of $R$ is beneficial for the diophantine geometry and the aim of our note is to show that the morphisms of $R$ preserve rational points on the algebraic variety. Such a property allows to recover subtle arithmetic invariants in terms of the invariants of the ring $R$, see Section 4.

Let us recall some definitions.

Denote by $V$ a complex projective variety and let $A_V$ be the Serre $C^*$-algebra, i.e. the norm closure of a self-adjoint representation of the twisted homogeneous coordinate ring of $V$ by the bounded linear operators on a Hilbert space [3, Section 5.3.1]. Recall that the $C^*$-algebras $A$ and $A'$ are said to be (strongly) Morita equivalent, if $A \otimes K \cong A' \otimes K$, where $K$ is the $C^*$-algebra of all compact operators on a Hilbert space and $\cong$ is the $C^*$-algebra isomorphism [Blackadar 1986] [1, 13.7.1(c)]. The Morita equivalence can be viewed as a generalization of an isomorphism meaning that the $C^*$-algebras $A$ and $A'$ have an isomorphic category of the finitely generated projective modules over $A$ and $A'$, respectively. It is easy to see, that if $A \cong A'$ are isomorphic $C^*$-algebras, then they are Morita equivalent but not vice versa. It is known, that the Morita equivalences of the $C^*$-algebra $A_V$ preserve the complex points on $V$, i.e. the projective variety $V$ is $C$-isomorphic to a projective variety $V'$ if and only if the corresponding Serre $C^*$-algebras are Morita equivalent [3, Theorem 5.3.3].

Let $k$ be a number field and let $V(k)$ be a projective variety over the field $k$. Consider a $C^*$-algebra $M_n(A_V(k))$ consisting of the $n \times n$ matrices with the entries in $A_V(k)$, see [Blackadar 1986] [1, p. 16] for the details. It follows from the K-theory

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of \( \mathcal{A}_V(k) \), that the \( C^* \)-algebra \( M_n(\mathcal{A}_V(k)) \) is simple, see lemma 3.1. Denote by \( K \) an extension of the field \( k \) obtained by adjoining the roots of all polynomials of degree \( n \) over \( k \). Our main results can be stated as follows.

**Theorem 1.1.** Projective variety \( V(k) \) is \( K \)-isomorphic to a projective variety \( V'(k) \) if and only if
\[
M_n(\mathcal{A}_V(k)) \cong M_n(\mathcal{A}_{V'}(k)).
\] (1.1)

**Remark 1.2.** Roughly speaking, theorem 1.1 says that the isomorphisms of the \( C^* \)-algebra \( M_n(\mathcal{A}_V(k)) \) preserve the \( K \)-rational points on the variety \( V(k) \).

**Corollary 1.3.** Projective variety \( V(k) \) is \( k \)-isomorphic (\( C \)-isomorphic, resp.) to a projective variety \( V'(k) \) if and only if the Serre \( C^* \)-algebra \( \mathcal{A}_V(k) \) is isomorphic (Morita equivalent, resp.) to the Serre \( C^* \)-algebra \( \mathcal{A}_{V'}(k) \).

**Remark 1.4.** An independent proof of corollary 1.3 using the Galois cohomology can be found in [4]. Our current proof is more elegant and follows from (1.1) and an analog of the Zariski's Lemma for the algebra \( \mathcal{A}_V \) (lemma 3.2).

**Corollary 1.5.** (Faltings Theorem) If \( \mathcal{C}(k) \) is a curve of genus \( g \geq 2 \), then the set of its \( k \)-points is finite.

The paper is organized as follows. The preliminary facts can be found in Section 2. Theorem 1.1 and corollary 1.3 are proved in Section 3. Section 4 contains proof of corollary 1.5.

## 2. Preliminaries

We briefly review noncommutative algebraic geometry, operator algebras and Serre \( C^* \)-algebras. For a detailed exposition we refer the reader to [Stafford & van den Bergh 2001] [8], [Blackadar 1986] [1, Chapter II] and [3, Section 5.3.1], respectively.

### 2.1. Noncommutative algebraic geometry

Let \( V \) be a complex projective variety. For an automorphism \( \sigma : V \to V \) and an invertible sheaf \( \mathcal{L} \) of the linear forms on \( V \), one can construct a twisted homogeneous coordinate ring \( B(V, \mathcal{L}, \sigma) \) of the variety \( V \), i.e. a non-commutative ring such that:
\[
Mod(B(V, \mathcal{L}, \sigma)) / \text{Tors} \cong \text{Coh}(V),
\] (2.1)
where \( Mod \) is the category of graded left modules over the graded ring \( B(V, \mathcal{L}, \sigma) \), \( \text{Tors} \) the full subcategory of \( Mod \) of the torsion modules and \( \text{Coh} \) the category of quasi-coherent sheaves on the variety \( V \) [Stafford & van den Bergh 2001] [8, p.180]. The correspondence \( V \mapsto B(V, \mathcal{L}, \sigma) \) is a functor which maps \( C \)-isomorphic projective varieties to the Morita equivalent rings \( B(V, \mathcal{L}, \sigma) \).

**Example 2.1.** Let \( k \) be a field and \( U_\infty(k) \) the algebra of polynomials over \( k \) generated by two non-commuting variables \( x_1 \) and \( x_2 \) satisfying the relation \( x_1x_2 - x_2x_1 - x_1^2 = 0 \). Let \( \mathbb{P}^1(k) \) be the projective line over \( k \). Then \( B(V, \mathcal{L}, \sigma) \cong U_\infty(k) \) and \( V \cong \mathbb{P}^1(k) \) satisfy (2.1). Notice that the \( B(V, \mathcal{L}, \sigma) \) is far from being a commutative ring.

**Example 2.2.** Let \( S(\alpha, \beta, \gamma) \) be the Sklyanin algebra, i.e. a free \( \mathbb{C} \)-algebra on four generators \( \{x_1, \ldots, x_4\} \) satisfying six quadratic relations: \( x_1x_2 - x_2x_1 = \alpha(x_3x_4 + x_4x_3), \ x_1x_2 + x_2x_1 = x_3x_4 - x_4x_3, \ x_1x_3 - x_3x_1 = \beta(x_4x_2 + x_2x_4), \ x_1x_3 + x_3x_1 = 
\)
Example 2.4. (Rieffel 1990) The Grothendieck group of the abelian semi-group $K$ is a twisted homogenous coordinate ring of the elliptic curve (2.2).

By the formal elements $\{m\} = x_1 x_4 - x_1 x_3 x_2$ and $x_1 x_4 + x_4 x_1 = x_2 x_3 - x_3 x_2$, where $\alpha, \beta, \gamma \in \mathbb{C}$ and $\alpha + \beta + \gamma + \alpha \beta \gamma = 0$. Let $E \subset \mathbb{C} P^3$ be an elliptic curve given in the Jacobi form, i.e., as an intersection of two quartics:

$$\begin{align*}
&u^2 + v^2 + w^2 + z^2 = 0, \\
&\frac{1}{w^2} u^2 + \frac{1}{u^2} w^2 + z^2 = 0.
\end{align*}$$

Then $B(V, \mathcal{L}, \sigma) \cong S(\alpha, \beta, \gamma)$ and $V \cong E$ satisfy (2.1). In other words, the $S(\alpha, \beta, \gamma)$ is a twisted homogenous coordinate ring of the elliptic curve (2.2) modulo a two-sided ideal generated by the central elements $\Omega_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $\Omega_2 = x_2^2 + \frac{1}{x_2} x_3^2 + \frac{1}{x_2} x_4^2$ [Smith & Stafford 1992, p. 267].

2.2. Operator algebras. The $C^*$-algebra is an algebra $\mathcal{A}$ over $\mathbb{C}$ with a norm $a \mapsto \|a\|$ and an involution $\{a \mapsto a^* \mid a \in \mathcal{A}\}$ such that $\mathcal{A}$ is complete with respect to the norm, and such that $\|ab\| \leq \|a\| \|b\|$ and $\|a^* a\| = \|a\|^2$ for every $a, b \in \mathcal{A}$. Each commutative $C^*$-algebra is isomorphic to the algebra $C_0(X)$ of continuous complex-valued functions on some locally compact Hausdorff space $X$. Any other algebra $\mathcal{A}$ can be thought of as a noncommutative topological space.

Example 2.3. An $n$-dimensional noncommutative torus is the universal $C^*$-algebra $\mathcal{A}_{\Theta_n}$ generated by $n$ unitary operators $U_1, \ldots, U_n$ satisfying the commutation relations $\{U_i U_j = \exp (2\pi i \theta_{ij}) U_i U_j \mid 1 \leq i, j \leq n\}$ which depend on a skew-symmetric matrix

$$\Theta_n = \begin{pmatrix}
0 & \theta_{12} & \cdots & \theta_{1n} \\
-\theta_{12} & 0 & \cdots & \theta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\theta_{1n} & -\theta_{2n} & \cdots & 0
\end{pmatrix}, \text{ where } \theta_{ij} \in \mathbb{R}.$$ (2.3)

By $M_\infty(\mathcal{A})$ one understands the algebraic direct limit of the $C^*$-algebras $M_n(\mathcal{A})$ under the embeddings $a \mapsto \text{diag}(a, 0)$. The direct limit $M_\infty(\mathcal{A})$ can be thought of as the $C^*$-algebra of infinite-dimensional matrices whose entries are all zero except for a finite number of the non-zero entries taken from the $C^*$-algebra $\mathcal{A}$. Two projections $p, q \in M_\infty(\mathcal{A})$ are equivalent, if there exists an element $v \in M_\infty(\mathcal{A})$, such that $p = v^* v$ and $q = v v^*$. The equivalence class of projection $p$ is denoted by $[p]$. We write $V(\mathcal{A})$ to denote all equivalence classes of projections in the $C^*$-algebra $M_\infty(\mathcal{A})$, i.e., $V(\mathcal{A}) := \{[p] : p = p^* = p^2 \in M_\infty(\mathcal{A})\}$. The set $V(\mathcal{A})$ has the natural structure of an abelian semi-group with the addition operation defined by the formula $[p] + [q] := \text{diag}(p, q) = [p' \oplus q']$, where $p' \sim p$, $q' \sim q$ and $p' \perp q'$. The identity of the semi-group $V(\mathcal{A})$ is given by $[0]$, where $0$ is the zero projection. By the $K_0$-group $K_0(\mathcal{A})$ of the unital $C^*$-algebra $\mathcal{A}$ one understands the Grothendieck group of the abelian semi-group $V(\mathcal{A})$, i.e., a completion of $V(\mathcal{A})$ by the formal elements $[p] - [q]$.

Example 2.4. ([Rieffel 1990] [6])

$$K_0(\mathcal{A}_{\Theta_n}) \cong \mathbb{Z}^{2^n - 1}.$$ (2.4)

2.3. Serre $C^*$-algebra. Let $V$ be a complex projective variety and let $B(V, \mathcal{L}, \sigma)$ be twisted homogeneous coordinate ring of $V$ (Section 2.1). Denote by $\mathcal{H}$ a Hilbert space and let $B(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Fix a self-adjoint (i.e., $*$-invariant) representation

$$\rho : B(V, \mathcal{L}, \sigma) \longrightarrow B(\mathcal{H}).$$ (2.5)
By the Serre C*-algebra of V one understands the closure $\mathcal{A}_V$ of the *-algebra $\rho(B(V, L, \sigma))$ in the norm topology on the space $\mathcal{B}(\mathcal{H})$.

**Example 2.5.** Let $V \cong A_g$ be a $g$-dimensional abelian variety, where $g \geq 1$. The corresponding Serre C*-algebra $\mathcal{A}_V \cong \mathcal{A}_{\Theta_{2g}}$ is a $2g$-dimensional noncommutative torus, see Example 2.3. In particular for $g = 1$ and $V \cong \mathfrak{E}$ being an elliptic curve as in Example 2.2, one gets $\mathcal{A}_V \cong \mathcal{A}_0$ is an irrational rotation algebra, i.e. the universal C*-algebra on two generators $U_1$ and $U_2$ satisfying the relation $U_2U_1 = e^{2\pi i}U_1U_2$ for a constant $\theta \in \mathbb{R}$. This can be proved by comparing the generators and relations of the Sklyanin algebra $S(\alpha, \beta, \gamma)$ given in Example 2.2 with such for the algebra $\mathcal{A}_0$ [3, Section 1.3].

3. Proofs

3.1. **Proof of theorem 1.1.** We split the proof in a series of lemmas.

**Lemma 3.1.** The $M_n(\mathcal{A}_{V(k)})$ is a simple C*-algebra for each $n \geq 1$.

**Proof.** (i) Let us prove that the $\mathcal{A}_{V(k)}$ is a simple C*-algebra. Indeed, any Serre C*-algebra $\mathcal{A}_V$ is a crossed product C*-algebra [3, Theorem 5.3.4]. Denote by $\mathcal{A}$ an Approximately Finite-dimensional (AF-) C*-algebra [3, Section 3.5], such that $\mathcal{A}_V \hookrightarrow \mathcal{A}$ is an embedding and $K_0(\mathcal{A}_V) \cong K_0(\mathcal{A})$. In particular, the $\mathcal{A}_{V(k)}$ embeds into an AF-algebra $\mathcal{A}$ of stationary type [3, Section 3.5.2]. Such AF-algebras are always simple, since the corresponding dimension group $(K_0(\mathcal{A}), K^+(\mathcal{A}))$ [3, Definition 3.5.2] does not have order-ideals. Since $K^+_0(\mathcal{A}_{V(k)}) \cong K^+_0(\mathcal{A})$, we conclude that the Serre C*-algebra $\mathcal{A}_{V(k)}$ is simple.

(ii) Let us prove that the $M_n(\mathcal{A}_{V(k)})$ is a simple C*-algebra. Recall that there exists a one-to-one correspondence between the two-sided ideals of the matrix algebra $M_n(\mathcal{A}_{V(k)})$ and the two-sided of $\mathcal{A}_{V(k)}$. In particular, the $M_n(\mathcal{A}_{V(k)})$ is simple if and only if the $\mathcal{A}_{V(k)}$ is simple. It follows from item (i), that the matrix C*-algebra $M_n(\mathcal{A}_{V(k)})$ is simple for each $n \geq 1$. Lemma 3.1 is proved. □

**Lemma 3.2.** **(Zariski’s Lemma for the algebra $\mathcal{A}_V$)** If $K$ is an extension of the number field $k$ obtained by adjoining the roots of all polynomials of degree $n$ over $k$, then

$$\mathcal{A}_{V(K)} \cong M_n(\mathcal{A}_{V(k)}). \quad (3.1)$$

**Proof.** (i) Recall that the $\mathcal{A}_{V(k)}$ is the norm-closure of a self-adjoint representation of the $k$-algebra $B := B(V(k), L, \sigma)$ of non-commutative polynomials over the number field $k$, see Example 2.2 and formula (2.5). Consider the Frobenius companion matrix given by an element

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & -c_0 \\ 1 & 0 & \ldots & 0 & -c_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -c_{n-1} \end{pmatrix}, \quad c_i \in k \quad (3.2)$$

of the $K$-algebra $M_n(B)$. To determine the field of constants $K$ of the algebra $M_n(B)$, denote by $I$ the identity of $M_n(B)$ and consider the set $\{cI \mid c \in K\}$. It is easy to see, that

$$cI = C \in M_n(B) \quad (3.3)$$
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if an only if $c$ is a root of the characteristic polynomial of matrix $C$ given by (3.2), i.e.

\[
\text{Char } C = x^n + c_{n-1}x^{n-1} + \cdots + c_0.
\]  

(3.4)

We conclude from (3.3) that the field of constants $K$ of the algebra $M_n(B)$ must contain all roots of the polynomial (3.4). Since $c_i \in k$, these roots are algebraic numbers of degree $n$ over $k$.

One can run through the set of all matrices $C$ given by formula (3.2) with $c_0 \neq 0$. It is clear that in this way we get all algebraic extensions of degree $n$ over the number field $k$. Thus the field of constants $K$ of the algebra $M_n(B)$ is obtained from $k$ by adjoining all algebraic numbers of degree $n$ over $k$.

(ii) It follows from item (i) that the $M_n(B)$ is an algebra of non-commutative polynomials over the field $K$. Since the field of constants of the twisted homogeneous coordinate ring coincides with the field of definition of projective variety (see Example 2.2), we conclude that

\[
M_n(B) \cong B(V(K), \mathcal{L}, \sigma).
\]  

(3.5)

(iii) It remains to compare (3.5) with the notation $B \cong B(V(k), \mathcal{L}, \sigma)$ and to obtain a ring isomorphism:

\[
B(V(K), \mathcal{L}, \sigma) \cong M_n(B(V(k), \mathcal{L}, \sigma)).
\]  

(3.6)

Taking the norm-closure of a self-adjoint representation (2.5) of the rings in the both sides of (3.6), one gets a $C^*$-algebra isomorphism:

\[
\mathcal{A}_V(K) \cong M_n\left(\mathcal{A}_{V(k)}\right).
\]  

(3.7)

Lemma 3.2 is proved.

\[\square\]

Lemma 3.3. The varieties $V(k)$ and $V'(k)$ are $K$-isomorphic if and only if

\[
M_n\left(\mathcal{A}_{V(k)}\right) \cong M_n\left(\mathcal{A}_{V'(k)}\right).
\]  

(3.8)

Proof. (i) Let $V(k) \cong V'(k)$ be a pair of isomorphic projective varieties defined over the number field $k$. The canonical embedding

\[
\mathcal{A}_{V(k)} \hookrightarrow M_n\left(\mathcal{A}_{V(k)}\right) \cong \mathcal{A}_{V(K)}
\]  

(3.9)

gives rise to an embedding $i$ of the variety $V(k)$ into the variety $V(K)$. Consider a commutative diagram in Figure 1, where $\psi$ is an isomorphism defined over the number field $K$. Since $\varphi = i \circ \psi \circ i^{-1}$ is a composition of maps defined over the number field $K$, we conclude that the isomorphism $\varphi : V(k) \to V'(k)$ must also be defined over the field $K$.

(ii) Let $\psi$ be the $K$-isomorphism between the varieties $V(K)$ and $V'(K)$ shown on the diagram in Figure 1. The $\psi$ gives rise to an isomorphism $\mathcal{A}_{V(K)} \cong \mathcal{A}_{V'(K)}$ of the corresponding Serre $C^*$-algebras.

In view of lemma 3.2, one gets the isomorphisms $\mathcal{A}_{V(K)} \cong M_n\left(\mathcal{A}_{V(k)}\right)$ and $\mathcal{A}_{V'(K)} \cong M_n\left(\mathcal{A}_{V'(k)}\right)$. Comparing the latter with the results of item (i), we conclude that the varieties $V(k) \cong V'(k)$ are $K$-isomorphic if and only if

\[
M_n\left(\mathcal{A}_{V(k)}\right) \cong M_n\left(\mathcal{A}_{V'(k)}\right).
\]  

(3.10)

Lemma 3.3 is proved.

\[\square\]
Theorem 1.1 follows from lemma 3.3.

3.2. Proof of corollary 1.3. (i) Let us prove that the projective variety $V(k)$ is $k$-isomorphic to a projective variety $V'(k)$ if and only if the Serre $C^*$-algebra $\mathcal{A}_{V(k)}$ is isomorphic to the Serre $C^*$-algebra $\mathcal{A}_{V'(k)}$. Indeed, suppose that in theorem 1.1 we have $n = 1$. In this case the field $K$ is an algebraic extension of the field $k$ of degree 1. In other words, the fields $k \cong K$ are isomorphic.

On the other hand, theorem 1.1 says that the variety $V(k)$ is $k$-isomorphic to the variety $V'(k)$ if and only if $\mathcal{A}_{V(k)} \cong \mathcal{A}_{V'(k)}$ by formula (1.1) with $n = 1$. Item (i) is proved.

(ii) Let us prove that the projective variety $V(k)$ is $C$-isomorphic to a projective variety $V'(k)$ if and only if the Serre $C^*$-algebra $\mathcal{A}_{V(k)}$ is Morita equivalent to the Serre $C^*$-algebra $\mathcal{A}_{V'(k)}$.

We shall denote by $\bar{k}$ the algebraic closure of the number field $k$ obtained by adjoining the roots of all polynomials of finite degree over $k$. It is easy to see, that this case of theorem 1.1 corresponds to $n = \infty$ and $K \cong \bar{k}$.

Recall that the $C^*$-algebra $M_n(\mathcal{A}_{V(k)})$ can be written in the form of a tensor product, i.e.

$$M_n(\mathcal{A}_{V(k)}) \cong \mathcal{A}_{V(k)} \otimes M_n(C). \quad (3.11)$$

On the other hand, the canonical inclusions of the $C^*$-algebras $C \subset M_2(C) \subset M_3(C) \subset \ldots$ are known to converge to an inductive limit:

$$\lim_{n \to \infty} M_n(C) \cong K, \quad (3.12)$$

where $K$ is the $C^*$-algebra of all compact operators on a Hilbert space. Thus in view of (3.11), the isomorphism (1.1) for $n = \infty$ can be written in the form:

$$\mathcal{A}_{V(k)} \otimes K \cong \mathcal{A}_{V'(k)} \otimes K. \quad (3.13)$$

By definition, the isomorphism (3.13) says that the Serre $C^*$-algebras $\mathcal{A}_{V(k)}$ and $\mathcal{A}_{V'(k)}$ are Morita equivalent.

It remains to apply the Lefschetz Principle to the number field $\bar{k}$. Namely, one can pass from the algebraically closed field $\bar{k}$ of characteristic zero to the field of complex numbers $C$. This remark finishes the proof of item (ii).

(iii) Corollary 1.3 follows from items (i) and (ii).
4. Faltings Theorem

In this section we consider an application of the corollary 1.3 to an alternative proof of the Faltings Theorem about finiteness of the rational points on the higher genus curves [Hindry & Silverman 2000] [2, Theorem E.0.1]. Namely, we show that corollary 1.3 and a classification of the even-dimensional noncommutative tori up to an isomorphism (lemma 4.2) imply the Shafarevich Conjecture for the abelian varieties [Parˇsin 1970] [5, Conjecture S1]. The Faltings Theorem is known to follow from the Shafarevich Conjecture using the Parshin’s Trick. Let us pass to a detailed argument.

**Theorem 4.1. (Shafarevich Conjecture)** There exists only a finite number of the abelian varieties $A_g$ over a number field $k$, such that:

(i) $A_g$ have a polarization of degree $d \geq 1$;

(ii) $A_g$ have a good reduction outside a fixed finite set $S$ of places of the field $k$.

**Proof.** We split the proof in two lemmas having an independent interest.

**Lemma 4.2.** There exists only a finite number of pairwise distinct skew-symmetric matrices $\Theta_{2g}$ given by the formula (2.3), such that the corresponding noncommutative tori $A_{\Theta_{2g}}$ are isomorphic to each other.

**Proof.** (i) Denote by $\rho_{ij} = \exp (2\pi i \theta_{ij})$, see Example 2.3. Then the commutation relations for the noncommutative torus $A_{\Theta_{2g}}$ can be written in the form:

$$U_i U_j U_i^{-1} U_j^{-1} = \rho_{ij} \in T,$$

where $T$ is the unit circle. Since the $n$-dimensional noncommutative torus depends on $\frac{n(n-1)}{2}$ real parameters $\theta_{ij}$, one can identify the $A_{\Theta_{2g}}$ with the family of noncommutative tori

$$\left\{ A_\rho \mid \rho \in T^{g(2g-1)} \right\}.$$  \hspace{1cm} (4.2)

(ii) For a dense subalgebra $A^0_\rho \subset A_\rho$ consisting of the polynomials in the variables $\{ U_k \mid 1 \leq k \leq 2g \}$ and an element $x \in A^0_\rho$ consider a function given by the formula

$$\rho \mapsto ||x||.$$ \hspace{1cm} (4.3)

It is easy to see, that (4.3) is a (real) analytic function on $T^{g(2g-1)}$.

(iii) Lemma 4.2 can be proved by contradiction. Let $\{ \rho_k \} \in T^{g(2g-1)}$ be an infinite sequence, such that the $A_{\rho_k}$ in (4.2) are pairwise isomorphic noncommutative tori. Since $T^{g(2g-1)}$ is a compact space, one can always restrict to a Cauchy subsequence $\{ \rho'_k \}$ convergent to a point $\rho \in T^{g(2g-1)}$. Since function (4.3) is constant on $\rho'_k$ and analytic on an interval $[\rho'_k, \rho]$, we conclude that (4.3) is constant everywhere on the interval $[\rho'_k, \rho]$. This is obviously false, since the interval $[\rho'_k, \rho] \subset T^{g(2g-1)}$ contains a continuum of pairwise non-isomorphic noncommutative tori $A_\rho$ [Rieffel 1990] [6]. This contradiction finishes the proof of lemma 4.2. \hfill \Box
Example 4.3. Let \( g = 1 \) in lemma 4.2. Then matrix (2.3) can be written as:

\[
\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}, \quad \text{where} \quad \theta \in \mathbb{R}.
\] (4.4)

It is known, that the noncommutative tori \( \mathcal{A}_g \) and \( \mathcal{A}_{g'} \) are isomorphic if and only if \( \theta' = \pm \theta \mod \mathbb{Z} \) [Rieffel 1990] [6, p. 200]. In other words, there are only two distinct values \( \rho_1, \rho_2 \in \mathbb{T} \), such that \( \rho_{1,2} = \exp(\pm 2\pi i \theta) \) correspond to the isomorphic noncommutative tori.

Remark 4.4. For \( g \geq 2 \) an explicit formula for the matrices \( \Theta_{2g} \) corresponding to the pairwise isomorphic noncommutative tori is an open problem [Rieffel 1990] [6, Question 4.1]. Lemma 4.2 can be viewed as a partial answer to this problem.

Lemma 4.5. The number of abelian varieties \( A_g \) over a number field \( k \) satisfying conditions (i) and (ii) of Theorem 4.1 is at most finite.

Proof. (i) Let \( A_g \) be a \( g \)-dimensional abelian variety defined over the number field \( k \). Consider a \( k \)-isomorphism \( \varphi : A_g \to A'_g \) from \( A_g \) to an abelian variety \( A'_g \). The corollary 1.3 says that the Serre \( C^* \)-algebras \( \mathcal{A}_{A_g} \cong \mathcal{A}_{\Theta_{2g}} \) and \( \mathcal{A}_{A'_g} \cong \mathcal{A}_{\Theta'_{2g}} \) must be isomorphic noncommutative tori, see Example 2.5. The corresponding commutative diagram is shown in Figure 2.

(ii) Let \((A_g, d, S)\) be the abelian variety \( A_g \) over the field \( k \) with a polarization of degree \( d \geq 1 \) and good reduction outside a finite set \( S \) of places of \( k \). Since \( \varphi \) is a \( k \)-isomorphism of \( A_g \), it must preserve the degree \( d \) of polarization and the set \( S \) of places of a bad reduction of \( A_g \). In other words,

\[
\varphi(A_g, d, S) = (A'_g, d, S). \tag{4.5}
\]

(iii) Lemma 4.2 implies that there exists at most finite number of isomorphic abelian varieties \( A_g \) satisfying (4.5). Indeed, for otherwise one gets an infinite number of isomorphic noncommutative tori \( \mathcal{A}_{\Theta_{2g}} \), see Figure 2. This contradicts lemma 4.2 and finishes the proof of lemma 4.5. \( \square \)

Theorem 4.1 follows from lemma 4.5. \( \square \)

Corollary 4.6. (Faltings Theorem) If \( C(k) \) is a curve of genus \( g \geq 2 \), then the set of its \( k \)-points is finite.
Proof. The argument based on the Shafarevich Conjecture 4.1 is well known. For the sake of completeness, we repeat it in below.

(i) One can always assume that there exists at least one \( k \)-point of \( \mathcal{C} \). The Jacobian \( \text{Jac} \mathcal{C} \) of \( \mathcal{C} \) is an abelian variety over the field \( k \). Conversely, given a principal polarization \( d = 1 \) on the Jacobian \( \text{Jac} \mathcal{C} \), one can recover a curve \( \mathcal{C} \) having the same set \( S \) as the abelian variety \( \text{Jac} \mathcal{C} \). In view of Theorem 4.1, there exists only a finite number of pairs \( (\mathcal{C}, S) \) consisting of the isomorphic curves \( \mathcal{C} \) with a fixed set \( S \).

(ii) The rest of the proof is a Parshin’s Trick described in [Parˇ sin 1970] [5, Théorème 1]. Namely, if \( P \in \mathcal{C}(k) \) is a \( k \)-point, then there exists a branched covering \( \mathcal{C}_P(K) \) of \( \mathcal{C}(k) \), such that the number field \( K \) is ramified in \( S \) over \( k \). The construction does not depend on the choice of the \( k \)-point \( P \). It is known from item (i) that there exists only a finite number of the pairs \( (\mathcal{C}_P(K), S) \) and, therefore, only a finite number of the \( k \)-points on the curve \( \mathcal{C}(k) \).

Corollary 4.6 follows from item (ii).

Remark 4.7. It follows from the proof of theorem 4.1 and corollary 4.6, that the number \( |\mathcal{C}(k)| \) of the rational points on the higher genus curve depends on the number of isomorphic \( n \)-dimensional noncommutative tori given by the distinct skew-symmetric matrices (2.3) [Rieffel 1990] [6, Question 4.1]. To the best of our knowledge, the Rieffel’s Problem is unsolved except for the special cases (see example 4.3).

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