LINES INDUCED BY BICHROMATIC POINT SETS

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Abstract. An important theorem of Beck says that any point set in the Euclidean plane is either “nearly general position” or “nearly collinear”: there is a constant $C > 0$ such that, given $n$ points in $\mathbb{E}^2$ with at most $r$ of them collinear, the number of lines induced by the points is at least $Cr(n - r)$.

Recent work of Gutkin-Rams on billiards orbits requires the following elaboration of Beck’s Theorem to bichromatic point sets: there is a constant $C > 0$ such that, given $n$ red points and $n$ blue points in $\mathbb{E}^2$ with at most $r$ of them collinear, the number of lines spanning at least one point of each color is at least $Cr(2n - r)$.

1. Introduction

Let $p$ be a set of $n$ points in the Euclidean plane $\mathbb{E}^2$ and let $\mathcal{L}(p)$ be the set of lines induced by $p$. A line $\ell \in \mathcal{L}(p)$ is $k$-rich if it is incident on at least $k$ points of $p$. A well-known theorem of Beck relates the size of $\mathcal{L}(p)$ and the maximum richness.

**Theorem 1 (Beck’s Induced Lines Theorem [1]).** Let $p$ be a set of $n$ points in $\mathbb{E}^2$, and let $r$ be the maximum richness of any line in $\mathcal{L}(p)$. Then $|\mathcal{L}(p)| \gg r(n - r)$.

Here $f(n) \gg g(n)$ means that $f(n) \geq Cg(n)$, for an absolute constant $C > 0$.

In this note, we give an elaboration (using pretty much the same arguments) of Beck’s Theorem to bichromatic point sets, which arises in relation to the work of Gutkin and Rams [2] on the dynamics of billiard orbits. Let $p$ be a set of $n$ red points and let $q$ be a set of $n$ blue points with all points distinct (for a total of $2n$). We define $p \cup q$ to be the bichromatic point set $(p, q)$ and define the set of bichromatic induced lines $\mathcal{B}(p, q)$ to be the subset of $\mathcal{L}(p, q)$ that is incident on at least one point of each color.

**Theorem 2 (Beck-type theorem for bichromatic point sets).** Let $(p, q)$ be a bichromatic point set with $n$ points in each color class (for a total of $2n$). If the maximum richness of any line in $\mathcal{L}(p, q)$ is $r$, then $|\mathcal{B}(p, q)| \gg n(2n - r)$.

In the particular case where $r = n$, which is required by Gutkin and Rams, this shows that $|\mathcal{B}(p, q)| \gg n^2$.

Beck’s Theorem 1, and the present Theorem 2, may be deduced from the famous Szemeredi-Trotter Theorem on point-line incidences (Beck himself uses a weaker, but similar, statement as his key lemma). The following form is what we require in the sequel.

**Theorem 3 (Szemeredi-Trotter Theorem [4]).** Let $p$ be a set of $n$ points in $\mathbb{E}^2$ and let $\mathcal{L}$ be a finite set of lines in $\mathbb{E}^2$. Then the number $r$ of $k$-rich lines in $\mathcal{L}$ is $r \ll n^2/k^3 + n/k$.

Notations. We use $p = (p_i)^n$ and $q = (q_i)^n$ for point sets in $\mathbb{E}^2$. The notation $f(n) \gg Cg(n)$ means there is an absolute constant $C > 0$ such that $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. 
2. Proofs

The proof of the main theorem follows a similar line to Beck’s original proof. For a pair of points \((p, q)\), we define the richness of the pair to be the richness of the line \(p.q\).

**Lemma 4.** Let \((p, q)\) be a bichromatic point set in \(\mathbb{E}^2\). Then there is an absolute constant \(K_1 > 0\) such that the number of bichromatic point pairs that are either at most \(1/K_1\)-rich or at least \(K_1n\)-rich is at least \(n^2/2\).

**Proof.** There are exactly \(n^2\) pairs of points \((p_i, q_i)\), and each of these induces a line in \(\mathcal{B}(p, q)\). Define the subset \(\mathcal{B}_j(p, q) \subset \mathcal{B}(p, q)\) to be the set of bichromatic lines with richness between \(2^{j-1}\) and \(2^j\).

By the Szemeredi-Trotter Theorem with \(k = 2^j\),

\[
|\mathcal{B}_j(p, q)| \leq C(n^2/2^{3j} + n/2^j) \tag{1}
\]

The number of bichromatic pairs inducing any line \(\ell \in \mathcal{B}_j(p, q)\) is maximized when there are \(2^j\) red points and \(2^j\) blue ones on \(\ell\), for a total of \(2^{2j}\) bichromatic pairs. Multiplying by the estimate of (1), the number of bichromatic pairs inducing lines in \(\mathcal{B}_j(p, q)\) is at most

\[
C(n^2/2^j + n2^j) \tag{2}
\]

for a large absolute constant \(C\) coming from the Szemeredi-Trotter Theorem.

Now let \(K_1 > 0\) be a small constant to be selected later. We sum (2) over \(j\) such that \(1/K_1 \leq j \leq K_1n\):

\[
C \left( n^2 \sum_{1/K_1 \leq j \leq K_1n} 2^{-j} + n \sum_{1/K_1 \leq j \leq K_1n} 2^j \right) \leq C \left( n^2 2^{-1/K_1} + n \cdot 2K_1n \right)
\]

Picking \(K_1\) small enough (it depends on \(C\)) ensures that at most \(n^2/2\) of the monochromatic pairs induce lines with richness between \(1/K_1\) and \(K_1n\).

The following lemma is a bichromatic variant of Beck’s Two-Extremes Theorem.

**Lemma 5.** Let \((p, q)\) be a bichromatic point set in \(\mathbb{E}^2\). Then either

- **A** The number of bichromatic lines \(|\mathcal{B}(p, q)| \gg n^2\).
- **B** There is a line \(\ell \in \mathcal{B}(p, q)\) incident on at least \(K_2n\) red points and \(K_2n\) blue points for an absolute constant \(K_2 > 0\).

**Proof.** We partition the bichromatic pairs \((p_i, q_i)\) into three sets: \(L\) is the set of pairs with richness less than \(1/K_1\); \(M\) is the set of pairs with richness in the interval \([1/K_1, K_1n]\); \(H\) is the set of pairs with richness greater than \(K_1n\).

By Lemma 4, \(|L \cup H| \geq n^2/2\). There are now three cases:

**Case I:** (Alternative A) If \(|L| \geq n^2/4\), then we are in alternative A, since quadratically many pairs can be covered only by quadratically many lines of constant richness.

**Case II:** (Alternative B) If we are not in Case I, then, \(|H| \geq n^2/4\). In particular, since \(H\) is not empty, there are lines incident on at least one point of each color and at least \(K_1n\) points in total. If one of these lines is line incident to at least \(\frac{5}{6}K_1n\) points of each color, then we are in alternative B.

**Case III:** (Alternative A) If we are not in Case I or Case II, then every line induced by a bichromatic pair in \(H\) has at least \(\frac{5}{6}K_1n\) red points incident on it or \(\frac{5}{6}K_1n\) blue ones incident on it.
Since there are $|H| \geq n^2/4$ bichromatic point pairs incident on a very rich line, there must be at least $n/4$ different points of each color participating in some point pair in $H$.

Each line induced by a pair in $H$ generates at least $K_1 n$ incidences, so the number of these lines is at most $\frac{1}{K_1 n}$. But then if all the lines induced by $H$ span at most $\frac{1}{6} K_1 n$ blue points, the total number of blue incidences is less than $n/4$, which is a contradiction. We can make a similar argument for red points.

Thus there is a line $\ell_1$ spanning at least $\frac{5}{6} K_1 n$ blue points and a distinct line $\ell_2$ spanning at least $\frac{5}{6} K_1 n$ red points. From this configuration we get at least $(\frac{5}{6} K_1 n - 1)^2$ distinct bichromatic lines, putting us again in alternative A. □

Proof of Theorem 2. If alternative A of Lemma 5 holds, then we are already done.

If we are in alternative B, then there must be a line $\ell$ of richness $r \geq 2 K_2 n$ incident to at least $K_2 n$ points of each color. Now pick any subset $X$ of $K_2 (2n - r)$ points not incident to $\ell$. There are at least $\frac{1}{2} K_2 n (2n - r)$ bichromatic point pairs determined by one point in $X$ and one point incident to $\ell$. Thus we get at least

$$\frac{1}{2} K_2^2 n (2n - r) - \binom{K_2 (2n - r)}{2} \geq \frac{1}{2} K_2^3 n (2n - r)$$

bichromatic lines. □

3. Conclusion

We proved an extension of Beck’s Theorem [1] to bichromatic point sets using a fairly standard argument, completing the combinatorial step in Gutkin-Rams’s recent paper on billiards.

This kind of bichromatic result can, due to the general nature of the proofs, be extended to any setting where a Szemeredi-Trotter-type result is available (see, e.g., [3] for many examples). Moreover, by “forgetting” colors and repeatedly squaring the constants, Theorem 2 holds for bichromatic lines in multi-chromatic point sets. It would, however, be interesting to know whether this is the correct order of growth for the constants in a multi-chromatic version of Theorem 2.

References

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