On CSM classes via Chern-Fulton classes of f-schemes

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Abstract

The Chern-Fulton class is a generalization of Chern class to the realm of arbitrary embeddable schemes. While Chern-Fulton classes are sensitive to non-reduced scheme structure, they are not sensitive to possible singularities of the underlying support, thus at first glance are not interesting from a singularity theory viewpoint. However, we introduce a class of objects which we think of as ‘fractional schemes’, or f-schemes for short, and then show that when one broadens the domain of Chern-Fulton classes to f-schemes one may indeed recover singularity invariants of varieties and schemes. More precisely, we show that for a Zariski dense subset of the space of complete intersections in a smooth variety of fixed codimension, their Chern-Schwartz-Macpherson classes (which are important singularity invariants) may be computed via Chern-Fulton classes of f-schemes.

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1. Introduction

Fix an algebraically closed field \( K \) of characteristic zero, let \( M \) be a smooth \( K \)-variety and \( X \hookrightarrow M \) an arbitrary closed subscheme. In [4], Fulton defines a ‘canonical class’ \( c_F(X) \in A_*X \) (the Chow group of \( X \)) associated with \( X \) which we refer to as the Chern-Fulton class of \( X \). Chern-Fulton classes are intrinsic invariants of embeddable schemes (i.e., independent of an embedding into a smooth variety \( M \)), and they coincide with the usual (total homology) Chern class for \( X \) smooth. While Chern-Fulton classes are
sensitive to non-reduced scheme structure, they are not sensitive to the singularities of its underlying support, thus at first glance it seems as though Chern-Fulton classes would not be of interest for those concerned with invariants of singular varieties. However, in the main result of [2], Aluffi shows that for \( X \) a hypersurface

\[
(1.1) \quad c_{\text{SM}}(X) = c_F(X^{(-1)}),
\]

where \( c_{\text{SM}}(X) \) denotes the Chern-Schwartz-MacPherson (or simply CSM) class of \( X \), and \( X^{(-1)} \) is a formal object (associated with \( X \)) with well defined Chern-Fulton class which one may think of as \( X \) ‘negatively thickened’ along its singular scheme. Though CSM classes are not sensitive to non-reduced scheme structure (as Chern-Fulton classes are), they are in some sense the most fundamental characteristic class of a singular variety \( Y \) since for \( R = \mathbb{C} \) they are the appropriate generalization of Chern class which extends the Gauß-Bonnet theorem to the realm of singular varieties, i.e.,

\[
(1.2) \quad \int_Y c_{\text{SM}}(Y) = \chi(Y),
\]

where \( \chi(Y) \) denotes the topological Euler characteristic of \( Y \) with compact support and the integral sign denotes proper pushforward to a point. Moreover, CSM classes satisfy inclusion-exclusion, and they are the unique characteristic class with these properties such that they are functorial with respect to proper morphisms. Thus in the hypersurface case formula (1.1) shows that indeed we may recover all the nice properties of CSM classes via the (scheme-theoretic) Chern-Fulton class.

In [loc. cit., pg. 2], Aluffi states “We do not know whether our result is an essential feature for hypersurfaces, or whether a formula similar to (1.1) may compute Chern-Schwartz-MacPherson’s class for arbitrary varieties.”. We give a partial answer to this question by showing that indeed a similar formula holds for a large class of complete intersections of arbitrary codimension, and that the ‘\( X^{(-1)} \)’ appearing in the RHS of formula (1.1) is a special case of a class of objects we introduce which we refer to as a ‘fractional schemes’, or \( f \)-schemes for short. We define the group of \( f \)-schemes of \( M \), denoted \( \mathfrak{F}(M) \), as the Grothendieck group of the monoid generated by isomorphism classes of quasi-coherent ideal sheaves over \( M \) (with the monoid operation induced by the product of ideal sheaves), and we show that the formal object ‘\( X^{(-1)} \)’ has a precise interpretation as the \( f \)-scheme \( [\mathcal{I}_X] \cdot [\mathcal{I}_J]^{-1} \), where \( \mathcal{I}_J \) is the ideal sheaf of the singular scheme \( J \) of \( X \) (i.e., the ideal sheaf generated by the partial derivatives of a local equation for \( X \)).

In what follows, we define Chern-Fulton classes of arbitrary \( f \)-schemes, reformulate the RHS of formula (1.1) as a precise statement about Chern-Fulton classes of \( f \)-schemes, and then prove the analogue of formula (1.1) holds for a Zariski dense subset of the space of complete intersections in \( M \) for arbitrary fixed codimension. Of course, we highly suspect that such a formula holds true for all embeddable schemes. Moreover, it would be interesting if not just the CSM class, but other notions of ‘Chern class’ for singular varieties and schemes could be incorporated within this framework, i.e., as the Chern-Fulton class of an associated \( f \)-scheme.
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2. Segre classes of $\mathfrak{f}$-schemes

Now let $S \hookrightarrow M$ be a closed subscheme, where $M$ (here and throughout) denotes a smooth $\mathfrak{K}$-variety. We first recall the following

Definition 2.1. The Segre class of $S$ (relative to $M$) is denoted $s(S,M)$, and is defined as

\[
s(S,M) := \begin{cases} 
    c(N_{S|M})^{-1} \cap [S] \in A_*S & \text{for } S \text{ regularly embedded} \\
    f|_{E^*} s(E, \widetilde{M}) \in A_*S & \text{otherwise},
\end{cases}
\]

where $c(N_{S|M})$ denotes the Chern class of the normal bundle to $S$ in $M$ (in the case that $S$ is regularly embedded), $f : \widetilde{M} \to M$ is the blowup of $M$ along $S$ with exceptional divisor $E$ and $f|_{E^*}$ denotes the proper pushforward of $f$ restricted to $E$ (note that since $E$ is always regularly embedded this is enough to define $s(S,M)$ in any case). The Chern-Fulton class of $S$ is then denoted $c_F(S)$, and is defined as

\[
c_F(S) := c(TM) \cap s(S,M) \in A_*S.
\]

While $s(S,M)$ depends on the embedding of $S$ in $M$, $c_F(S)$ does not (as shown in Example 4.2.6 of [4]), thus justifying the absence of $M$ in its notation. Useful for us will be the fact that if $p : N \to M$ is a proper birational map then

\[
p_* s(p^{-1}(S), N) = s(S,M).
\]

Remark 2.1. In the 1960s Deligne and Grothendieck conjectured that for $\mathfrak{K} = \mathbb{C}$ there exists a unique natural transformation

\[
c_* : F \to A_*,
\]

where $F$ and $A_*$ are the (covariant) constructible function and Chow functors respectively, such that for $Y$ smooth

\[
c_*(1_Y) = c(TY) \cap [Y].
\]

Then in the 1970s, an affirmative answer to this conjecture was first proven in the seminal work of MacPherson [3], where he explicitly constructs such a $c_*$. As $c_*(1_Y)$ is defined for arbitrary $Y$ and reduces to the usual total homology Chern class of $Y$ in the smooth case, $c_*(1_Y)$ is a natural generalization of Chern class to the realm of possibly singular varieties. It was then shown later on that Mac-Pherson's Chern classes were the Alexander dual of classes constructed in the 1960s by Marie-Hélène Schwartz, thus these classes eventually came to be known as ‘Chern-Schwartz-MacPherson classes’, or CSM classes for short. MacPherson's construction was later generalized to algebraically closed fields of characteristic zero in the work of Kennedy [7].

For $Y \hookrightarrow M$ a regular embedding its CSM class is closely related to its Chern-Fulton class. In particular, we have

\[
c_{\text{SM}}(S) = c_F(S) + \mathcal{M}(S),
\]
where $\mathcal{M}(S)$ is the Milnor class of $S$, a characteristic class supported on the singular locus of $S$ which is a generalization of global Milnor number. More precisely, if $S$ has only isolated singularities then the degree-zero term of $\mathcal{M}(S)$ is the sum of all the local Milnor numbers of $S$. As mentioned in [1] while Chern-Fulton classes are sensitive to non-reduced scheme structure, they are not sensitive to the singularities of the underlying variety, thus equation 2.2 says that $\mathcal{M}(S)$ captures all the singularity information contained in the CSM class that the Chern-Fulton class does not see.

Our objectives here are in the spirit of unification, as we show that for a large class of regular embeddings Chern-Fulton classes may indeed recover the information of $\mathcal{M}(S)$ (and so $c_{\text{SM}}(S)$), once their domain of definition is extended to a class of objects we call ‘f-schemes’.

We now introduce f-schemes via the following

**Definition 2.2.** Let $\mathcal{I}$ denote the monoid generated by isomorphism classes of quasicoherent ideal sheaves over $M$, with binary operation given by

$$[\mathcal{I}_{S_1}] \cdot [\mathcal{I}_{S_2}] = [\mathcal{I}_{S_1} \cdot \mathcal{I}_{S_2}],$$

where $\mathcal{I}_{S_1}$ and $\mathcal{I}_{S_2}$ are ideal sheaves corresponding to closed subschemes $S_1$, $S_2$ of $M$, brackets are used to denote their isomorphism class and the product on the RHS corresponds to the usual product of ideal sheaves. The group of f-schemes of $M$, denoted $\mathfrak{F}(M)$, is then defined as the Grothendieck group of the monoid $\mathcal{I}$. We will usually just denote $[\mathcal{I}_S]$ simply by $S$, and when we do so its inverse will be denoted $S^{-1}$.

Now let $T \in \mathfrak{F}(M)$ be arbitrary, which (after canceling factors with their inverses) may be written uniquely as

$$T = S_1 \cdot S_2^{-1},$$

where $S_1$ and $S_2$ correspond to closed subschemes of $M$, either of which (but not both) may be the empty subscheme of $M$ (i.e., the identity in $\mathfrak{F}(M)$). We now extend the notion of relative Segre class to f-schemes via the following

**Definition 2.3.** Let $T = S_1 \cdot S_2^{-1}$ be an arbitrary f-scheme and let $p : \tilde{M} \to M$ be the blowup of $M$ along $S_1 \cup S_2$. Then the Segre class of $T$ is defined as

$$s(T, M) := p_* (c(\mathcal{O}(S_1) \otimes \mathcal{O}(S_2)^\vee)^{-1} \cap ([S_1] - [S_2])) \in A_*(S_1 \cup S_2),$$

where $[S_i] = p^{-1}(S_i)$. The Chern-Fulton class of $T$ is then given by

$$c_F(T) := c(TM) \cap s(T, M)$$

Note that by the birational invariance of Segre classes (formula 2.1), $s(T, M)$ is essentially $s(S_1 \cup S_2, M)$, but with the contribution of $S_2$ ‘dualized’. 

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1Throughout this note, $S_1 \cup S_2$ will always denote the closed subscheme corresponding to the ideal sheaf $\mathcal{I}_{S_1} \cdot \mathcal{I}_{S_2}$. 

3. Aluffi’s Hypersurface Formula in the Language of \( f \)-Schemes

Our next task is to rephrase Aluffi’s result (equation 1.1) in the language of \( f \)-schemes, but before doing so we introduce some useful notations.

So let \( S \) be an algebraic scheme and denote by \( d \) the dimension of the largest component of \( S \). For \( \alpha = \sum \alpha^i \in A_\ast S \) with \( \alpha^i \in A_{(d-i)}S \) we let

\[
\alpha^\vee = \sum (-1)^i \alpha^i,
\]

and refer to it as the dual of \( \alpha \). We remark that by replacing \(-1\) by a positive integer \( n \) in formula 3.1 yields the \( n \)-th Adams of \( \alpha \), usually denoted \( \alpha^{(n)} \). Thus we may think of \( \alpha^\vee \) as the ‘\(-1\)th Adams’ of \( \alpha \). For a line bundle \( L \to S \) we let

\[
\alpha \otimes L = \sum \frac{\alpha^i}{c(L)^i},
\]

and refer to it as \( \alpha \) tensor \( L \). After identifying \( L \) with its class in the Picard group \( \text{Pic}(S) \) it is then straightforward to show the map \( \alpha \mapsto \alpha \otimes L \) defines an action of \( \text{Pic}(S) \) on \( A_\ast S \). This fact is proven in [1], along with the fact that if \( E \) is a class in the Grothendieck group of vector bundles on \( S \) then

\[
(c(E) \cap \alpha)^\vee = c(E^\vee) \cap \alpha^\vee,
\]

and

\[
(c(E) \cap \alpha) \otimes L = \frac{c(E \otimes L)}{c(L)^r} \cap (\alpha \otimes L),
\]

where \( r \in \mathbb{Z} \) denotes the rank of \( E \).

Now let \( J \hookrightarrow X \hookrightarrow M \) be closed embeddings with \( X \) a hypersurface in \( M \) and \( J \) the singular scheme of \( X \), i.e., the closed subscheme whose ideal sheaf is locally generated by all partial derivatives of a local defining equation for \( X \). In [2], the RHS of 1.1 is computed explicitly, thus yielding Aluffi’s formula

\[
c_{SM}(X) = c(TM) \cap (s(X, M) + c(E(X))^{-1} \cap (s(J, M)^\vee \otimes \mathcal{O}(X)))
\]

A restatement of this result in the language of \( f \)-schemes is made in the following

**Claim 3.1.** \( c_{SM}(X) = c_F(X \cdot J^{-1}) \).

**Proof.** It suffices to show

\[
s(X \cdot J^{-1}, M) = s(X, M) + c(E(X))^{-1} \cap (s(J, M)^\vee \otimes \mathcal{O}(X)).
\]

The proof of this fact is similar to the proof of Proposition 9.2 in [4], though using the tensor and dual notations introduced above. Indeed, since \( X \) is a Cartier divisor, the blowup of \( M \) along \( X \cup J \) coincides with the blowup of \( M \) along \( J \). So let \( p: \tilde{M} \to M \) denote the blowup of \( M \) along \( J \), and we abuse notations by denoting the exceptional divisor simply by \( J \) and \( p^*X \) by \( X \). Note that using the tensor notation we have

\[
s(X \cdot J^{-1}, M) = p_* (([X] - [J]) \otimes \mathcal{O}(X - J)),
\]
where the tensor symbol denotes the tensor of a class by a line bundle (equation 3.2). Now we compute:

\[
p_* (\left([X] - [J]\right) \otimes \mathcal{O}(X - J)) = p_* (\left([X] - [J]\right) \otimes (\mathcal{O}(-J) \otimes \mathcal{O}(X)))
\]

\[
= p_* (\left([X] \otimes \mathcal{O}(-J) - [J] \otimes \mathcal{O}(-J)\right) \otimes \mathcal{O}(X))
\]

\[
= p_* \left(\left([X] + \frac{[-J]}{1 - J}\right) \otimes \mathcal{O}(X)\right)
\]

\[
= p_* \left(\left([X] - \frac{[X] \cdot [-J]}{1 - J} + \frac{[-J]}{1 - J}\right) \otimes \mathcal{O}(X)\right)
\]

\[
= p_* \left(\left([X] + c(\mathcal{O}(-X)) \cap \frac{[-J]}{1 - J}\right) \otimes \mathcal{O}(X)\right)
\]

\[
= p_* (s(X, M) + c(\mathcal{O}(X))^{-1} \cap (s(J, M) \vee \otimes \mathcal{O}(X)))
\]

\[
= s(X, M) + c(\mathcal{O}(X))^{-1} \cap (s(J, M) \vee \otimes \mathcal{O}(X)),
\]

where the last equality follows from the projection formula as we made no notational distinction between \(X\) and \(J\) and their pullbacks, and the second to last equality follows from the definition of Segre class and formula 3.3. This proves our claim. \(\square\)

**Remark 3.1.** The language of \(\mathcal{f}\)-schemes along with Claim 3.1 yields a simple proof in the hypersurface case that CSM classes are not sensitive to non-reduced scheme structure. Indeed, let \(X\) be a reduced hypersurface given by the equation \(F = 0\) and denote its singular scheme by \(J\). Then the scheme \(X^k\) corresponding to the ideal sheaf \(\mathcal{I}_X^k\) is given by \(F^k = 0\), and \(d(F^k) = kF^{k-1}dF\). Now since \(J\) corresponds to \(dF = 0\), the singular scheme of \(X^k\) corresponds to the ideal sheaf \(\mathcal{I}_X^{k-1} \cdot \mathcal{I}_J\). Thus

\[
c_{\text{SM}}(X^k) = c_F(X^k \cdot (X^{(1-k)} \cdot J^{-1})) = c_F(X \cdot J^{-1}) = c_{\text{SM}}(X),
\]

where the first and last equalities follow from Claim 3.1.

4. CSM classes via Chern-Fulton classes of \(\mathcal{f}\)-schemes

We note that while Aluffi never actually defines the object \(X^{(-1)}\) appearing in equation 1.1, it has a precise interpretation as the \(\mathcal{f}\)-scheme \(X \cdot J^{-1}\), which we now explain.\(^2\) To arrive at equation 1.1, what he first does is compute the Chern-Fulton class of the closed subscheme \(X \cup J^k \hookrightarrow M\) corresponding to the ideal sheaf \(\mathcal{I}_X \cdot \mathcal{I}_J^k\), with \(k\) an arbitrary non-negative integer (so that \(X \cup J^k\) is an honest scheme). He then shows that \(c_F(X \cup J^k)\) is a polynomial in \(k\) with coefficients in \(A_*X\), and then proves the remarkable fact that evaluating the polynomial \(c_F(X \cup J^k)\) at \(k = -1\) yields precisely \(c_{\text{SM}}(X)\). Thus \(X^{(-1)}\) is some formal geometric object corresponding to the formal ideal sheaf \(\mathcal{I}_X \cdot \mathcal{I}_J^{-1}\).

We were led to the formulation of \(\mathcal{f}\)-schemes in our efforts to make this business more concrete, so that \(X^{(-1)}\) now has a precise definition as the \(\mathcal{f}\)-scheme \(X \cdot J^{-1}\). In a heuristic manner we think of \(X \cdot J^{-1}\) as \(X\) ‘negatively thickened’ along its singular scheme \(J\), but other than this seductive picture we have yet to uncover any deeper geometric significance.

\(^2\)This formal object \(X^{(-1)}\) is not to be confused with the \(\mathcal{f}\)-scheme \(X^{-1}\).
underlying $X \cdot J^{-1}$. In any case, we strongly suspect that with any closed subscheme of $M$ there exists an associated $f$-scheme such that the Chern-Fulton class of its associated $f$-scheme recovers its CSM class. This is further evidenced by the following

**Theorem 4.1.** Let $Y \hookrightarrow M$ be a closed subscheme which is a complete intersection of hypersurfaces $X_1, \ldots, X_m$ such that $Z = X_1 \cap \cdots \cap X_{m-1}$ is smooth. Then

$$c_{\text{SM}}(Y) = c_F(Y \cdot K^{-1}),$$

where $\mathcal{I}_K$ is the ideal sheaf which is locally generated by the $m \times m$ minors of the matrix $k_{ij} = \partial_j F_i$, with $F_i = 0$ a local defining equation for $X_i$.

Before proving the theorem we note that $K$ generalizes the notion of singular scheme of a hypersurface, and is well defined by the theory of Fitting ideals. We also suspect the assumption that the open condition $Z = X_1 \cap \cdots \cap X_{m-1}$ be smooth is unnecessary, though at the moment we don’t see a way around it. CSM classes of many examples of complete intersections satisfying this assumption were computed in [5]. We remark that if an analogue of Theorem 4.1 indeed holds for arbitrary embeddable schemes, it would give a remarkably simple characterization of CSM classes, without any reference to MacPherson’s original construction or resolution of singularities. We now proceed with the

**Proof.** (We make no notational distinction between classes and their pushforwards/pullbacks associated with inclusions, and the same goes for line bundles and their restrictions.) By Theorem 1.1 of [3]

$$c_{\text{SM}}(Y) = c(TM) \cap \left( s(Y, M) + (-1)^{m-1} \frac{c(\mathcal{E}^\vee \otimes \mathcal{O}(X_m))}{c(\mathcal{E})} \cap (s(K, M)^\vee \otimes \mathcal{O}(X_m)) \right),$$

where $\mathcal{E} = \bigoplus_{i=1}^m \mathcal{O}(X_i)$. Thus it suffices to show

$$s(Y \cdot K^{-1}, M) = s(Y, M) + (-1)^{m-1} \frac{c(\mathcal{E}^\vee \otimes \mathcal{O}(X_m))}{c(\mathcal{E})} \cap (s(K, M)^\vee \otimes \mathcal{O}(X_m)).$$

Indeed, denote by $K^k$ the closed subscheme of $Z = X_1 \cap \cdots \cap X_{m-1}$ corresponding to the ideal sheaf $\mathcal{I}_K^k$. It is a general fact from the theory of Segre classes that $s(K^k, Z) = s(K, Z)^{(k)}$, i.e., the $k$-th Adams of $s(K, Z)$. Then by replacing $M$ by $Z$, $X$ by $Y$ and $-J$ by $K^k$ in the proof of Claim 3.1 yields

$$s(Y \cup K^k, Z) = s(Y, Z) + c(\mathcal{O}(Y))^{-1} \cap (s(K, Z)^{(k)} \otimes Z \mathcal{O}(Y)),\tag{4.1}$$

where we use a subscript $Z$ on the tensor to emphasize we are tensoring with respect to codimension in $Z$. Moreover, since $Z$ is smooth and regularly embedded in $M$, $Y \cup K^k \hookrightarrow M$ is a linear embedding [6], thus

$$s(Y \cup K^k, Z) = c(N_Z M) \cap s(Y \cup K^k, M).$$

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$^3$A factor of $(-1)^{m-1}$ was left out of the statement of Theorem 1.1 of [3], since in the context of that paper they were only concerned with the difference $c_{\text{SM}}(Y) - c_F(Y)$ up to sign. Thus we have added it as the whole class $c_{\text{SM}}(Y)$ concerns us here.
Then evaluating this equation at \( k = 0 \) yields
\[
(4.2) \quad s(Y, Z) = c(N_Z M) \cap s(Y, M),
\]
and then at \( k = -1 \) yields an equation for the Segre class of the \( f \)-scheme \( Y \cdot K^{-1} \), i.e.,
\[
(4.3) \quad s(Y \cdot K^{-1}, Z) = c(N_Z M) \cap s(Y \cdot K^{-1}, M).
\]
Furthermore, the same reasoning yields
\[
(4.4) \quad s(K, Z) = c(N_Z M) \cap s(K, M),
\]
and by \ref{3.3} we have
\[
(4.4) \quad s(K, Z) = c(N_Z M^\vee) \cap s(K, M)^\vee.
\]
Putting things together yields
\[
\begin{align*}
s(Y \cdot K^{-1}, M) &= c(N_Z M)^{-1} \cap s(Y \cdot K^{-1}, Z) \\
&= c(N_Z M)^{-1} \cap \left( s(Y, Z) + c(O(Y))^{-1} \cap (s(K, Z)^\vee \otimes_{Z} O(Y)) \right) \\
&= s(Y, M) + c(O)^{-1} \cap ((c(N_Z M^\vee) \cap s(K, M)^\vee) \otimes_{Z} O(Y)).
\end{align*}
\]
Now let
\[
\mathcal{M}(Y) := c(O)^{-1} \cap ((c(N_Z M^\vee) \cap s(K, M)^\vee) \otimes_{Z} O(Y)).
\]
Our theorem is thus proved if we show
\[
\mathcal{M}(Y) = (-1)^{m-1} \frac{c(O\vee \otimes O(X_m))}{c(O)} \cap (s(K, M)^\vee \otimes O(X_m)).
\]
For this we need to recall Lemma 2.1 of [3], which states that if \( X \) is a variety endowed with a line bundle \( \mathcal{L} \to X \), and \( X' \to X \) is a regular embedding of codimension \( d \), then for every \( \alpha \in A_* X' \) we have
\[
(4.5) \quad \alpha \otimes X' \mathcal{L} = c(\mathcal{L})^d \cap (\alpha \otimes X \mathcal{L}).
\]
Then indeed,
\[
\begin{align*}
\mathcal{M}(Y) &= c(O)^{-1} \cap ((c(N_Z M^\vee) \cap s(K, M)^\vee) \otimes_{Z} O(Y)) \\
&= (-1)^{m-1} \frac{c(O(X_m))^{m-1}}{c(O)} \cap ((c(N_Z M^\vee) \cap s(K, M)^\vee) \otimes_{M} O(X_m)) \\
&= (-1)^{m-1} \frac{c(O(X_m))^{m-1}}{c(O)} \cap \left( \frac{c(N_Z M^\vee \otimes O(X_m))}{c(O(X_m))^{m-1}} \cap (s(K, M)^\vee \otimes_{M} O(X_m)) \right) \\
&= (-1)^{m-1} \frac{c(O\vee \otimes O(X_m))}{c(O)} \cap (s(K, M)^\vee \otimes_{M} O(X_m)),
\end{align*}
\]
where the factor of \((-1)^{m-1}\) appears in the second equation due to the fact that we are taking duals in \( M \) rather than in \( Z \). We also used the fact that \( O(Y) = O(X_m)|_Z \), and
in the last equality we used that since $\mathcal{E} = N_Z M \oplus \mathcal{O}(X_m)$, $\mathcal{E}^\vee \otimes \mathcal{O}(X_m) = (N_Z M^\vee \otimes \mathcal{O}(X_m)) \oplus \mathcal{O}$, thus

$$c(\mathcal{E}^\vee \otimes \mathcal{O}(X_m)) = c(N_Z M^\vee \otimes \mathcal{O}(X_m)),$$

which concludes the proof. □

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