NOISE EFFECTS ON THE STOCHASTIC EULER–POINCARÉ EQUATIONS

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Abstract. In this paper, we first establish the existence, uniqueness and the blow-up criterion of the pathwise strong solution to the periodic boundary value problem of the stochastic Euler–Poincaré equation with nonlinear multiplicative noise. Then we consider the noise effects with respect to the continuity of the solution map and the wave breaking phenomenon. Even though the noise has some already known regularization effects, almost nothing is clear to the problem whether the noise can improve the continuity/stability of the solution map, neither for general SPDEs nor for special examples. As a new setting to analyze initial data dependence, we introduce the concept of the stability of the exiting time (See Definition 1.4 below) and construct an example to show that for the stochastic Euler–Poincaré equations, the multiplicative noise (Itô sense) cannot improve the stability of the exiting time and improve the continuity of the dependence on initial data simultaneously. Then we consider the noise effect on the wave breaking phenomenon in the particular 1-D case, namely the stochastic Camassa–Holm equation. We show that under certain condition on the initial data, wave breaking happens with positive probability and we provide a lower bound of such probability. We also characterize the breaking rate of breaking solution.

Contents

1. Introduction 2
1.1. Stochastic EP equations 3
1.2. Noise effects 3
1.3. Notations, hypotheses and definitions 4
1.4. Main results and remarks 7
2. Preliminaries 9
3. Blow-up criterion 11
4. Regular pathwise solutions 13
4.1. Approximation scheme 13
4.2. Uniform estimates 13
4.3. Martingale solution to the cut-off problem 16
4.4. Pathwise uniqueness 19
4.5. Regular pathwise solution to the cut-off problem 21
4.6. Final proof for Theorem 4.1 22
5. Proof for Theorem 1.1 22
6. Noise effect on the dependence on initial data 27
6.1. Estimates on the approximation solutions 28
6.2. Construction of actual solutions 30
6.3. Estimates on the error 30
6.4. Proof for Theorem 1.3 32
7. Wave breaking and breaking rate 33
7.1. Blow-up scenario 34
7.2. Wave breaking 36
Acknowledgement 38
References 38

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1. Introduction

Consider the following Euler-Poincaré (EP) equations,

\[
\begin{aligned}
\partial_t m + (u \cdot \nabla) m &= - (\nabla u)^T m - (\text{div}u)u \\
m &= (1 - \alpha \Delta) u.
\end{aligned}
\] (1.1)

In (1.1), \( u = (u_j)_{1 \leq j \leq d} \) and \( m = (m_j)_{1 \leq j \leq d} \) with \( u_j = u_j(t,x) \) and \( m_j = (1 - \alpha \Delta) u_j(t,x) \) represent the velocity and momentum, respectively. \((\nabla u)^T\) denotes the transpose of \(\nabla u\) and \(\alpha\) corresponds to the square of the length scale. The EP equations (1.1) were first studied by Holm et al. [42, 43] as a framework for modeling and analyzing fluid dynamics, particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling, see also [1, 44]. There are a variety of mathematical interpretations of the (1.1), and each of them can be a point of departure for further investigation. The well-posedness of (1.1) have been studied by many researchers, and we will not attempt to survey all of them here. Here we only mention the following results. When \( d \geq 2 \), Chae and Liu [10] established the well-posedness results for both weak and strong solutions. More precisely, for given \( u_0 \in W^{2,p}, \ p > d \), Chae and Liu proved the local existence of the weak solution belonging to \( L^\infty([0,T_u); W^{2,p}(\mathbb{R}^d)) \). For \( u_0 \in H^m, \ m > d/2 + 3 \), they proved local existence and uniqueness of a strong solution belonging to \( C([0,T_u); H^m) \). They also obtained blow-up criterion and the finite time blow-up of the classical solution for the case \( \alpha = 0 \). For the case \( \alpha > 0 \), the blow-up and global existence of the solutions to (1.1) were studied in [53]. For the local solution in Besov spaces, we refer to [64].

For convenience, we assume \( \alpha = 1 \) in (1.1). Then we can rewrite (1.1) into the general form of transport equations as follows [64, 65]:

\[ u_t + (u \cdot \nabla) u + F(u) = 0, \]

where

\[ F(u) = (I - \Delta)^{-1} \text{div} F_1(u) + (I - \Delta)^{-1} F_2(u), \] (1.2)

and

\[
\begin{align*}
F_1(u) &= \nabla u (\nabla u + \nabla u^T) - \nabla u^T \nabla u - \nabla u (\text{div}u) + \frac{1}{2} I |\nabla u|^2, \\
F_2(u) &= u (\text{div}u) + u \cdot \nabla u^T.
\end{align*}
\]

In the above, \( f = (I - \Delta)^{-1} g \) means \( g = G * f \) with the Green function \( G \) for the Helmholtz operator \( I - \Delta \).

Especially, when \( d = 1, \alpha = 1 \), (1.1) becomes the Camassa–Holm (CH) equation [29, 9],

\[ m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \]

which is equivalent to

\[ u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \] (1.3)

or

\[ u_t + uu_x + q(u) = 0, \]

where

\[ q(u) = q_1(u) + q_2(u), \] (1.4)

and

\[ q_1(u) = (1 - \partial_x^2)^{-1} \partial_x (u^2), \quad q_2(u) = \frac{1}{2} (1 - \partial_x^2)^{-1} \partial_x (u_x^2). \]

In 1-D case, (1.3) has been studied by many mathematicians and physicists and (1.3) exhibits both phenomena of (peaked) soliton interaction and wave breaking. Constantin, Escher and McKeen [14, 12, 15, 54] studied the wave breaking of the CH equation. Bressan and Constantin developed a new approach to the analysis of the CH equation, and proved the existence of the global conservative and dissipative solutions in [6, 5]. Later, Holden and Raynaud [40, 41] also obtained the global conservative and dissipative solutions from a Lagrangian point of view. As pointed out in [13, 17, 18], the occurrence of the traveling waves with a peak at their crest, exactly like the waves of the greatest height solutions to the governing equations for water waves.
1.1. Stochastic EP equations. When we consider a physical model in the real world and we need to account for the influence of internal and external noise, and the background for the model may contain an inherent element of randomness and therefore becomes difficult to describe deterministically. Moreover, it is also worthwhile noting that the randomness of the background movement is one of the prevailing hypotheses on the onset of turbulence in fluid models [7, 55]. To be more appropriate to capture the reality, we are motivated to consider the following stochastic EP equations,
\[ du + [(u \cdot \nabla) u + F(u)] dt = B(t, u) dW, \]
where \( dW \) is a cylindrical Wiener process which will be specified in next section and \( B(t, u) dW \) may account for the random energy exchange. Notice that (1.5) is the type of stochastic transport type with nonlocal nonlinearities.

With the stochastic EP equation (1.5) in mind, the first target is the following

**Target 1:** Establish existence, uniqueness and blow-up criterion of pathwise solution to the following periodic boundary value problem of the stochastic EP equations (1.5):
\[
\begin{align*}
&du + [(u \cdot \nabla) u + F(u)] dt = B(t, u) dW, \\
&u(\omega, 0, x) = u_0(\omega, x), \quad x \in \mathbb{T}^d,
\end{align*}
\]
where \( F(u) \) is given in (1.2). The relevant results are stated in Theorems 1.1 and 1.2.

1.2. Noise effects. For SPDEs, noise effect is one of the probabilistically important questions worthwhile to study and many regularization effects have been observed. For example, it is known that the well-posedness of linear stochastic transport equation with noise can be established under weaker hypotheses than its deterministic counterpart (cf. [25, 27]). For stochastic scalar conservation laws, noise on flux may bring some regularization effects [31] and noisy source may trigger the discrete entropy dissipation in the numerical schemes for conservation laws so that the schemes enjoy some stability properties not present in the deterministic case [52]. For stochastic Euler equations, certain noise may prevent the coalescence of vortices (singularity) in two-dimensional space [28]. Besides, linear multiplicative noises can be used to regularize singularities caused by nonlinear effects in some PDEs, see [34, 51, 57, 58].

In this paper, we will consider this noise effect on (1.5) associated with the dependence on the initial data and the phenomenon of wave breaking.

1.2.1. Dependence on the initial data. For deterministic PDEs, the classical notion of well-posedness of an abstract Cauchy problem due to Hadamard requires the existence of a unique solution which depends continuously on initial data. For some specific problems, the solution map \( u_0 \mapsto u \) can be shown to be more than continuous (the solution map is uniformly continuous, Lipschitz or even differentiable) with suitably chosen topologies, see e.g., [3, 36, 50]. For stochastic evolution equations, the property of dependence on initial conditions turns out to be a much more complicated problem since the existence time of the solution to a stochastic evolution equation is generally a random variable and in general we do not have lifespan estimates, cf. [34]. However, in terms of numerics, continuous dependence on initial data is essential for the design of reliable simulation methods. Therefore it is interesting to study the dependence on the initial data in stochastic case. Moreover, it becomes very interesting by noticing that:

- The “regularization by noise” may formally be related to the regularization produced by an additional Laplacian;
- If we can indeed add a Laplacian to the governing equations in some cases, then by using some semilinear parabolic techniques, the dependence on initial data may be improved to Lipschitz. For example, for the deterministic Euler equations, the dependence on initial data cannot be better than continuous [38], but for the deterministic Navier-Stokes equations, it is at least Lipschitz, see pp. 79–81 in [36].

Therefore it is reasonable to ask the following question:

\[ \text{Whether the noise can improve the dependence on initial data?} \quad (1.7) \]

We notice that the previous work mainly focused on the effects of the noise on the existence and uniqueness. So far, almost nothing has been known to (1.7), neither for the general case nor on the special examples. Therefore the second goal of this paper is the following

**Target 2:** Consider the problem (1.7) for the periodic stochastic EP equations, namely
\[
\begin{align*}
&du + [(u \cdot \nabla) u + F(u)] dt = Q(t, u) dW, \\
&u(\omega, 0, x) = u_0(\omega, x), \quad x \in \mathbb{T}^d,
\end{align*}
\]
where \(Q(t, \cdot)\) satisfies some conditions. We first introduce the definition on the stability on the exiting time (see Definition 1.4 blow). After this, we give a partial answer to (1.7). More specific, we construct an example to show that the multiplicative noise (in Itô sense) cannot improve the stability of the exiting time, and simultaneously improve the continuity of the dependence on initial data. The statement is listed in Theorem 1.3.

1.2.2. On the wave breaking. As emphasized by Whitham [63], the wave breaking phenomenon is one of the most intriguing long standing problems of water wave theory. Particularly, for CH equation, we recall the following results:

- For the deterministic CH equation, the wave breaking phenomenon has been well studied and it is known that the only way singularities can occur in solutions is in the form of breaking waves, see [14, 15, 54] for example;
- With random noise, as far as we know, we can only find the work [20]. In [20] the authors proved that temporal randomness (in the sense of Stratonovich) in the diffeomorphic flow map for stochastic Camassa–Holm equation does not prevent the wave breaking process. However, their result only shows that with positive probability, wave breaking occurs.

Comparing the deterministic case and the stochastic case, it is very natural to ask the following question:

*What is the probability of wave breaking and what is its breaking rate?* (1.9)

Since it is difficult to determine whether the pathwise solutions to (1.6) is globally defined or blow-up in finite time for general nonlinear multiplicative noise \(B(t, u)\,dW\), we mainly focus on the case of non-autonomous linear multiplicative noise, namely \(B(t, u)\,dW = b(t)\,dW\), where \(W\) is a standard 1-D Brownian motion. Then the third goal in this paper is the following

**Target 3:** Study the wave breaking phenomenon of the solutions to the 1-D stochastic CH equation with particular non-autonomous linear multiplicative noise, namely

\[
\begin{align*}
    du + [u_0\,d\sigma + q(u)]\,dt &= b(t)\,dW, \quad x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad t \in \mathbb{R}^+, \\
    u(\omega, 0, x) &= u_0(\omega, x), \quad x \in \mathbb{T}.
\end{align*}
\]

(1.10)

where \(W\) is a standard 1-D Brownian motion and \(q\) is defined in (1.4). We notice that if \(u \in H^s\) with \(s > 3\), (1.10) can be reformulated as

\[
u_t - u_{xxx} + 3uu_x = 2uu_xuxx + uu_{xxx} + b(t)(u - u_{xx})\dot{W}.
\]

(1.11)

The detailed results on wave breaking of (1.10) is stated in Theorems 1.4 and 1.5.

1.3. Notations, hypotheses and definitions. Subsequently, we list some of the most frequently used notations, assumptions, and precise the notions of the solution in this paper.

1.3.1. Notations. Let \(L^p(\mathbb{T}^d; \mathbb{R}^d)\) with \(d \geq 1, d \in \mathbb{Z}^+\) and \(1 \leq p < \infty\) be the standard Lebesgue space of measurable \(p\)-integrable \(\mathbb{R}^d\)-valued functions with domain \(\mathbb{T}^d\) and let \(L^\infty(\mathbb{T}^d; \mathbb{R}^d)\) be the space of essentially bounded functions. Particularly, \(L^2(\mathbb{T}^d; \mathbb{R}^d)\) has an inner product \((f, g)_{L^2} = \int_{\mathbb{T}^d} f \cdot \bar{g} dx\), where \(\bar{g}\) denotes the complex conjugate of \(g\). The Fourier transform and inverse Fourier transform of \(f(x) \in L^2(\mathbb{T}^d; \mathbb{R}^d)\) are defined by \(\hat{f}(\xi) = \int_{\mathbb{T}^d} f(x)e^{-ix\cdot\xi}dx\) and \(f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \hat{f}(k)e^{ix\cdot k}\) \((k \in \mathbb{Z}^d)\), respectively. For any real number \(s\), the operator \(D^s = (I - \Delta)^{s/2}\) is defined by \(\hat{D}^s f(\xi) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi)\). Then the Sobolev spaces \(H^s\) on \(\mathbb{T}^d\) with values in \(\mathbb{R}^d\) can be defined as

\[H^s(\mathbb{T}^d; \mathbb{R}^d) := \left\{ f \in L^2(\mathbb{T}^d; \mathbb{R}^d) : \|f\|_{H^s(\mathbb{T}^d; \mathbb{R}^d)}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{f}(k)|^2 < +\infty \right\}\]

with inner product \((f, g)_{H^s} = (D^s f, D^s g)_{L^2}\). When the function spaces are defined on \(\mathbb{T}^d\) and take values in \(\mathbb{R}^d\), for the sake of simplicity, we omit the parentheses in the above notations from now on if there is no ambiguity. For linear operators \(A\) and \(B\), we denote by the Lie Bracket \([A, B] = AB - BA\). We will use \(\lesssim\) to denote estimates that hold up to some universal deterministic constant which may change from line to line but whose meaning is clear from the context.

\((\Omega, \mathcal{F}, \mathbb{P})\), where \(\mathbb{P}\) is a probability measure on \(\Omega\) and \(\mathcal{F}\) is a \(\sigma\)-algebra, denotes a complete probability space. Let \(t > 0\) and \(\tau \in [0, t]\). \(\sigma\{x_1(\tau), \cdots, x_n(\tau)\}_{\tau \in [0, t]}\) stands for the completion of the union \(\sigma\)-algebra generated by \((x_1(\tau), \cdots, x_n(\tau))\). All stochastic integrals are defined in Itô sense and \(\mathbb{E}x\) is the mathematical expectation of \(x\) with respect to \(\mathbb{P}\). Let \(X\) be a separable Banach space. \(\mathcal{B}(X)\) denotes the
Borel sets of $X$ and $\mathcal{P}(X)$ stands for the collection of Borel probability measures on $X$. For $E \subseteq X$, $1_E$ is the indicator function on $E$, i.e., it is equal to 1 when $x \in E$, and zero otherwise.

We call $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{W})$ a stochastic basis. Here $(\Omega, \mathcal{F}, \mathbb{P})$ is a underlying probability space, $\{\mathcal{F}_t\}_{t \geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\{\mathcal{F}_0\}$ contains all the $\mathbb{P}$-negligible subsets and $\mathbb{W}(t) = \mathbb{W}(\omega, t), \omega \in \Omega$ is a cylindrical Wiener process. More precisely, we consider a separable Hilbert space $U$ as well as a larger one $U_0$ such that the canonical injections $U \hookrightarrow U_0$ is Hilbert-Schmidt. Therefore for any $T > 0$, we have, cf. [21, 30, 46],

$$\mathbb{W} = \sum_{k=1}^{\infty} e_k W_k \in C([0, T], U_0) \ \mathbb{P} - \text{a.s.,}$$

where $\{e_k\}$ is a complete orthonormal basis of the $U$ and $\{W_k\}_{k \geq 1}$ is a sequence of mutually independent standard one-dimensional Brownian motions. To define the Itô stochastic integral

$$\int_0^T Zd\mathbb{W} = \sum_{k=1}^{\infty} \int_0^T Z e_k dW_k \quad (1.12)$$

on some separable Hilbert space $X$, it is required (see [21, 56] for example) for predictable stochastic process $Z$ to take values in the space of Hilbert-Schmidt operators from $U$ to $X$, denoted by $\mathcal{L}_2(U, X)$. Remember that

$$Z \in \mathcal{L}_2(U, X) \Rightarrow \|Z\|_{\mathcal{L}_2(U, X)}^2 = \sum_{k=1}^{\infty} \|Ze_k\|^2_X < \infty.$$

As in [21, 56], we see that for a predictable $X$-valued process $Z$ such that $Z \in \mathcal{L}_2(U, X)$, (1.12) is a well-defined continuous square integrable martingale such that for all stopping times $\tau$ and $v \in X$,

$$(\int_0^\tau Zd\mathbb{W}, v)_X = \sum_{k=1}^{\infty} \int_0^\tau (Ze_k, v)_X dW_k.$$

Here we remark that the stochastic integral (1.12) does not depend on the choice of the space $U_0$, cf. [21, 56]. For example, $U_0$ can be defined as

$$U_0 = \left\{ v = \sum_{k=1}^{\infty} a_k e_k : \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty, \|v\|_{U_0} = \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} \right\}.$$

Most notably for the analysis here, the Burkholder-Davis-Gundy (BDG) inequality holds which in the present context takes the following form

$$\mathbb{E} \left( \sup_{t \in [0, T]} \left\| \int_0^t Zd\mathbb{W} \right\|_X^p \right) \leq C \mathbb{E} \left( \int_0^T \|Z\|_{\mathcal{L}_2(U, X)}^2 dt \right)^{\frac{p}{2}}, \quad p \geq 1,$$

or in terms of the coefficients,

$$\mathbb{E} \left( \sup_{t \in [0, T]} \left\| \sum_{k=1}^{\infty} \int_0^t Ze_k dW_k \right\|_X^p \right) \leq C \mathbb{E} \left( \int_0^T \sum_{k=1}^{\infty} \|Ze_k\|^2_X dt \right)^{\frac{p}{2}}, \quad p \geq 1.$$

1.3.2. Hypotheses. We make the following hypotheses in this paper.

**Hypothesis I.** Throughout this paper, we assume that $B : [0, \infty) \times H^s \ni (t, u) \mapsto B(t, u) \in \mathcal{L}_2(U, H^s)$ for $u \in H^s$ with $s > \frac{d}{2}$ such that if $u : \Omega \times [0, T] \rightarrow H^s$ is predictable, then $B(t, u)$ is also predictable. Furthermore, we assume the following:

1. There are non-decreasing locally bounded functions $f(\cdot), h_1(\cdot) \in C([0, +\infty); [0, +\infty))$ with $f(0) = 0$ such that for all $s > \frac{d}{2}$,

$$\|B(t, u)\|_{\mathcal{L}_2(U, H^s)} \leq h_1(t)f(\|u\|_{W^{1,\infty}})(1 + \|u\|_{H^s}).$$

2. There are locally bounded non-decreasing functions $g(\cdot), h_2(\cdot) \in C([0, +\infty); [0, +\infty))$ such that for any $s > \frac{d}{2}$,

$$\|B(t, u) - B(t, v)\|_{\mathcal{L}_2(U, H^s)} \leq h_2(t)g(\|u\|_{H^s} + \|v\|_{H^s})\|u - v\|_{H^s}.$$
Hypothesis II. When we consider (1.8) in Section 6, we need a modified assumptions on \( Q(t, \cdot) \). For \( s \geq 0 \), we assume that \( Q : [0, \infty) \times H^s \ni (t, u) \mapsto \sigma(t, u) \in \mathbb{L}_2^2(U, H^s) \) for \( u \in H^s \) such that if \( u : \Omega \times [0, T] \to H^s \) is predictable, then \( Q(t, u) \) is also predictable. Moreover, we assume that \( Q(t, \cdot) \) satisfies Hypothesis I and when \( s > \frac{d}{2} \),

\[
\|Q(t, u)\|_{\mathbb{L}_2^2(U, H^s)} \leq \|F(u)\|_{H^s}, \quad \|Q(t, u) - Q(t, v)\|_{\mathbb{L}_2^2(U, H^s)} \leq \|F(u) - F(v)\|_{H^s}.
\] (1.13)

Hypothesis III. When considering (1.10) with non-autonomous linear noise \( b(t)u dW_t \), we assume that \( b(t) \in C([0, \infty); [0, \infty)) \) and there is a \( b^* > 0 \) such that \( b^2(t) \leq b^* \) for all \( t \geq 0 \).

1.3.3. Definitions. We now define the martingale and pathwise solutions to the problem (1.6).

**Definition 1.1** (Martingale solutions). Let \( s > d/2 + 1 \) with \( d \geq 2 \) and \( \mu_0 \in \mathcal{P}(H^s) \). A triple \((\mathcal{S}, u, \tau)\) is said to be a martingale solution to (1.6) if

1. \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W) \) is a stochastic basis and \( \tau \) is a stopping time relative to \( \mathcal{F}_t \);
2. \( u : \Omega \times [0, \infty) \to H^s \) is an \( \mathcal{F}_t \) predictable \( H^s \)-valued process such that \( \mu_0(Y) = \mathbb{P}\{u(0) \in Y\} \), \( \forall Y \in \mathcal{B}(H^s) \) and \( u(\cdot \wedge \tau) \in C([0, \infty); H^s) \), \( \mathbb{P} - a.s. \)\); \( \mathbb{P} - a.s. \)
3. For every \( t > 0 \),
\[
\|u(t \wedge \tau) - u_0 + \int_0^{t \wedge \tau} ([u \cdot \nabla] v + F(u)) \, dt'\| \leq \int_0^{t \wedge \tau} \|B(t', u) dW_t\| \quad \mathbb{P} - a.s.
\] (1.15)
4. If \( \tau = \infty \), \( \mathbb{P} - a.s. \), then we say the martingale solution is global.

**Definition 1.2** (Pathwise solutions). Let \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W) \) be a fixed stochastic basis. Let \( s > d/2 + 1 \) with \( d \geq 2 \) and \( u_0 \) be an \( H^s \)-valued \( \mathcal{F}_0 \)-measurable random variable (relative to \( \mathcal{S} \)). A local pathwise solution to (1.6) is a pair \((u, \tau)\), where \( \tau \) is a stopping time satisfying \( \mathbb{P}\{\tau > 0\} = 1 \) and \( u : \Omega \times [0, \tau) \to H^s \) is an \( \mathcal{F}_t \) predictable \( H^s \)-valued process satisfying (1.14) and (1.15). Additionally, \((u, \tau^*)\) is called a maximal pathwise solution to (1.6) if \( \tau^* > 0 \) almost surely and if there is an increasing sequence \( \tau_n \to \tau^* \) such that for any \( n \in \mathbb{N} \), \((u, \tau_n)\) is a pathwise solution and

\[
\sup_{\tau \in [0, \tau_n]} \|u\|_{H^s} \geq n \quad a.e. \text{ on } \{\tau^* < \infty\}.
\]

If \( \tau^* = \infty \) almost surely, then such a solution is called global.

**Definition 1.3** (Pathwise uniqueness). The local martingale (pathwise) solutions are said to be pathwise unique, if for any given two pairs of local martingale (pathwise) solutions \((\mathcal{S}, u_1, \tau_1)\) and \((\mathcal{S}, u_2, \tau_2)\) with the same basis \( \mathcal{S} \) and \( \mathbb{P}\{u_1(0) = u_2(0)\} = 1 \), we have

\[
\mathbb{P}\{u_1(t, x) = u_2(t, x), \forall (t, x) \in [0, \tau_1 \wedge \tau_2] \times \mathbb{R}^d\} = 1.
\]

We also introduce the following notions on the stability of exiting time.

**Definition 1.4** (Stability of exiting time). Let \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W) \) be a fixed stochastic basis and \( s > d/2 + 1 \) with \( d \geq 2 \). Let \( u_0 \) be an \( H^s \)-valued \( \mathcal{F}_0 \)-measurable random variable such that \( \mathbb{E}\|u_0\|_{H^s}^2 < \infty \). Assume that \( \{u_{0,n}\} \) is an arbitrary sequence of \( H^s \)-valued \( \mathcal{F}_0 \)-measurable random variables satisfying \( \mathbb{E}\|u_{0,n}\|_{H^s}^2 < \infty \). For each \( n \), let \( u \) and \( u_{n} \) be the unique solutions to (1.6) with initial value \( u_0 \) and \( u_{0,n} \), respectively. For any \( R > 0 \) and \( n \in \mathbb{N} \), define the \( R \)-exiting time as

\[
\tau_n^R := \inf \{t \geq 0 : \|u_{n}(t)\|_{H^s} > R\}, \quad \tau^R := \inf \{t \geq 0 : \|u(t)\|_{H^s} > R\},
\]

where \( \inf \emptyset = \infty \).

1. Let \( R > 0 \). If \( u_{0,n} \to u_0 \) in \( H^s \) almost surely implies

\[
\lim_{n \to \infty} \tau_n^R = \tau^R \quad \mathbb{P} - a.s.,
\] (1.16)

then the \( R \)-exiting time is said to be stable at \( u \).

2. Let \( R > 0 \). If \( u_{0,n} \to u_0 \) in \( H^{s'} \) for all \( s' < s \) almost surely also implies (1.16), then the \( R \)-exiting time is said to be strongly stable at \( u \).
1.4. Main results and remarks. Now we formulate our main results. For the problem (1.6), we have the following two results.

**Theorem 1.1** (Existence and uniqueness). Let $S = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ be a fixed stochastic basis and $s > d/2 + 1$ with $d \geq 2$. If $u_0$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable such that $\mathbb{E} \|u_0\|_{H^s}^2 < \infty$ and if Hypothesis I is verified, then (1.6) admits a unique pathwise solution $(u, \tau)$ in the sense of Definitions 1.2–1.3. Moreover, $u$ satisfies

$$u(\cdot \wedge \tau) \in L^2(\Omega; C([0, \infty); H^s)),$$

and it can be extended to a maximal solution $(u, \tau^*)$ in the sense of Definition 1.2. Moreover, for a.e. $\omega \in \Omega$, either $\tau^* = \infty$ or $\tau^* < \infty$ with $\limsup_{t \to \tau^*} \|u(t)\|_{H^s} = \infty$.

**Theorem 1.2** (Blow-up criterion). Let $u$ be the solution with maximal existence time $\tau^*$ to (1.6) obtained in Theorem 1.1. Then $u(t)$, as a $W^{1, \infty}$-valued process, is also $\mathcal{F}_t$ adapted for $t < \tau^*$ and

$$I_{\limsup_{t \to \tau^*} \|u(t)\|_{H^s} = \infty} = I_{\limsup_{t \to \tau^*} \|u(t)\|_{W^{1, \infty}} = \infty} \ P - a.s.$$

**Remark 1.1.** The proof for Theorem 1.1 is divided into the following subsections. We are highly motivated by the recent papers [34, 57, 22]. However, there are some differences between this work and the previous ones.

- The Faedo-Galerkin method used in [34, 22] is hard to be used here directly since we do not have the additional incompressible condition, which guarantees the global existence of the approximation solution (see, e.g., [26, 34]). In our case, we need to find a positive lower bound for the existence time $\tau_*$ of the approximation solution $u_\varepsilon$, which is not clear due to the lack of life span estimate in the stochastic setting. This difficulty can be overcome by constructing a suitable approximation scheme and establishing a uniform blow-up criterion such that it is not only available for $u$, but also for $u_\varepsilon$. We borrow the idea from the recent work [19] to achieve such blow-up criterion.

- There are many ways to identify the limit of the $u_\varepsilon$. One can pass to the limit directly with using some technical convergence results in [2, 35] and the recent paper [22]. Another approach is based on the martingale representation result. Namely, one can show that the limit process is a martingale, identify its quadratic variation, and apply the martingale representation, see [21, 30, 45] for example. To avoid the use of further difficult results, we identify both the quadratic variation of the corresponding martingale and its cross variation with the limit Wiener process obtained through compactness. This approach follows a rather general and elementary method introduced in [8], which has been generalized to different settings, see [39] for example.

For **Target 2** with respect to the question (1.7), we have

**Theorem 1.3.** Let $S = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ be a fixed stochastic basis and let $s > d/2 + 1$ with $d \geq 2$. If $Q$ satisfies Hypothesis II, where $F(\cdot)$ is given by (1.2), then there is at least one of the following properties holding true for the problem (1.8),

1. For any $R \gg 1$, the $R$-exit time is not strongly stable at the zero solution in the sense of Definition 1.4.

2. The solution map $u_0 \mapsto u$ defined by (1.8) is not uniformly continuous, as a map from $L^2(\Omega; H^s)$ into $L^2(\Omega; C([0, T]; H^s))$ for any $T > 0$. More precisely, there exist two sequences of solutions $u_{1,n}(t)$ and $u_{2,n}(t)$, and two sequences of stopping times $\tau_{1,n}$ and $\tau_{2,n}$, such that
   - $P\{\tau_{i,n} > 0\} = 1$ for each $n > 1$ and $i = 1, 2$. Besides,
     $$\lim_{n \to \infty} \tau_{1,n} = \lim_{n \to \infty} \tau_{2,n} = \infty \ P - a.s. \quad (1.18)$$
   - For $i = 1, 2$, $u_{i,n} \in C([0, \tau_{i,n}], H^s) \ P - a.s.$, and
     $$\mathbb{E} \left( \sup_{t \in [0, \tau_{1,n}]} \|u_{1,n}(t)\|_{H^s}^2 + \sup_{t \in [0, \tau_{2,n}]} \|u_{2,n}(t)\|_{H^s}^2 \right) \lesssim 1. \quad (1.19)$$
   - At time $t = 0$,
     $$\lim_{n \to \infty} \mathbb{E}\|u_{1,n}(0) - u_{2,n}(0)\|_{H^s}^2 = 0. \quad (1.20)$$
   - For any $T > 0$, we have
     $$\liminf_{n \to \infty} \mathbb{E} \sup_{t \in [0, T \wedge \tau_{1,n} \wedge \tau_{2,n}]} \|u_{1,n}(t) - u_{2,n}(t)\|_{H^s}^2 \gtrsim \left( \sup_{t \in [0, T]} |\sin t| \right)^2. \quad (1.21)$$
Remark 1.2. We give the following remarks concerning Theorem 1.3.

- In the deterministic case, the question on the optimal dependence of solutions on the data has been proposed in [24]. And Kato [47] proved that the solution map for the (inviscid) Burgers equation is not Hölder continuous in the $H^s(\mathbb{T})$ norm with $s > 3/2$ regardless of the Hölder exponent. Since then other methods have been developed and successfully applied to various nonlinear PDEs, see [50, 38, 59, 60] and the references therein.

- To prove Theorem 1.3, we assume that for some $R_0 > 1$, the $R_0$-exciting time of the zero solution is strongly stable. Then we will construct an example to show that the solution map $u_0 \mapsto u$ defined by (1.8) is not uniformly continuous. This example involves the construction (for each $s > d/2 + 1$) of two sequences of solutions which are converging at time zero but remain far apart at any later time. Actually, we will first construct two sequences of approximation solutions $u^{l,n}(l \in \{-1, 1\})$ such that the actual solutions $u_{l,n}(l \in \{-1, 1\})$ starting from $u_{l,n}(0) = u^{l,n}(0)$ satisfy that as $n \to \infty$,

$$
\lim_{n \to \infty} \mathbb{E} \sup_{[0, \tau_{l,n}]} \|u_{l,n} - u^{l,n}\|_{H^s}^2 = 0,
$$

(1.22)

where $u_{l,n}$ exists at least on $[0, \tau_{l,n}]$. Due to the lack of life span estimate in stochastic setting, in order to obtain (1.22), we first connect the property $\inf_n \tau_{l,n} > 0$ with the stability property of the exiting time of the zero solution. In deterministic case, we have uniform lower bounds for the existence times of a sequence of solutions (see (4.7)–(4.8) in [61] and (3.8)–(3.9) in [62] for example). If (1.22) holds true, then we can estimate the approximation solutions instead of the actual solutions and obtain (1.21) by showing that the error in $H^{2s-\sigma}$ behaves like $n^{s-\sigma}$, but the error in $H^s$ is $O(1/n^r)$, where $d/2 < \sigma < s - 1$ and $-r_s + s - \sigma < 0$. These two estimates and interpolation give (1.22).

- Theorem 1.3 implies that for the issue of the dependence on initial data, we cannot expect that the multiplicative noise (in Itô sense) to improve the stability of the exiting time of the zero solution, and simultaneously improve the continuity of the dependence on initial data. Formally speaking, the “regularization by (Itô sense) noise” actually preserves the hyperbolic structure of the equations. As for the noise in the sense of Stratonovich, whether it can improve the dependence on initial data is our future work.

- Theorem 1.3 is proved for $d \geq 2$. However, the proof holds true also for $d = 1$, namely the stochastic CH equation case (see Remark 6.1).

Now we consider the problem (1.10) with respect to the question (1.9). We first give the following result:

Theorem 1.4 ( Blow-up scenario). Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)$ be a fixed stochastic basis. Assume that $s > 3$, $b(t)$ satisfies Hypothesis III and $u_0(\omega, x)$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. Let $(u, \tau^*)$ be the corresponding unique maximal solution to (1.11). Then we have

$$
I_{\lim \sup_{t<\tau^*} \|u\|_{H^s} = \infty} = I_{\lim \inf_{t \to \tau^*} \min_{x \in \mathbb{T}} u(x, t) = -\infty},
$$

(1.23)

which means that if singularities arise, they can arise only in the breaking form. Moreover, we have

$$
\lim_{t \to \tau^*} \min_{x \in \mathbb{T}} [u(x, t)] \int_t^{\tau^*} \beta(t') dt' = -2\beta(\tau^*) \text{ a.e. on } \{\tau^* < \infty\},
$$

(1.24)

where $\beta(\omega, t) = \mathbb{E}^{\omega} \int_0^1 b(t') dW_{t'} - \int_0^1 \frac{\partial b}{\partial x} \omega \, dt'$

Remark 1.3. If $b(t) \equiv 0$ in (1.11), then everything is deterministic and $\beta \equiv 1$. We see that the blow-up rate estimate turns out to be

$$
\lim_{t \to \tau^*} \min_{x \in \mathbb{T}} [u(x, t)] \int_t^{\tau^*} 1 \, dt' = \lim_{t \to \tau^*} \left( \min_{x \in \mathbb{T}} [u(x, t)] (\tau^* - t) \right) = -2,
$$

which covers the deterministic case in [46].

Now we are in the position to give an answer to the question (1.9).

Theorem 1.5 ( Wave breaking). Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)$ be a fixed stochastic basis. Let $b(t)$ satisfy Hypothesis III, $s > 3$ and $u_0 \in H^s$ be an $H^s$-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. Let $0 < c < 1$. If almost surely we have

$$
\min_{x \in \mathbb{T}} \partial_x u_0(x) < -\frac{1}{2} \sqrt{\frac{(b^*)^2}{c^2} + 4\lambda \|u_0\|_{H^s}^2} - \frac{b^*}{2c},
$$

then other methods have been developed and successfully applied to various nonlinear PDEs, see [50, 38, 59, 60] and the references therein.
where $b^*$ is given in Hypothesis III and $\lambda$ is given in Lemma 2.4, then the corresponding maximal solution $(u, \tau^*)$ to (1.10) (or to (1.11) equivalently) satisfies

$$
P \{ \tau^* < \infty \} = P \left\{ \liminf_{t \to \tau^*} \left[ \min_{x \in \mathbb{Z}^d} u_x(t, x) \right] = -\infty \right\} \geq P \left\{ e_{\lambda}^J b(t' dW_t' > c \; \; \forall t \right\} > 0.
$$

On the other hand,

$$
P \left\{ \|u(t)\|_{L^\infty} \leq \sup_{t \geq 0} e_{\lambda}^J b(t' dW_t' - \int_0^t \frac{\varepsilon^2}{4} \|u_k\|_{H^1} dt \right\} < \infty, \; t \in [0, \tau^*) \} = 1.
$$

That is to say, $P \{ u \text{ breaks in finite time} \} \geq P \left\{ e_{\lambda}^J b(t' dW_t' > c, \; t > 0 \right\} > 0.$ And the wave breaking rate is given in Theorem 1.4.

The paper is organized as follows: In the next section, some relevant preliminaries are briefly recalled. Then we will first prove Theorem 1.2 in Section 3 and postpone the proof for existence to later sections. We establish the existence of the unique pathwise solution in $H^s$ with $s > d/2 + 3$ in Section 4 and then extend the range of the Sobolev exponent $s$ to $s > d/2 + 1$ in Section 5, which gives Theorem 1.1. Then we give a partial answer to the question (1.7) and prove Theorem 1.3 in Section 6. In Section 7, we prove Theorems 1.4 and 1.5.

2. Preliminaries

Now we briefly recall some relevant preliminaries, which will be used later. For $\varepsilon \in (0, 1)$, $J_\varepsilon$ is the Friedrichs mollifier defined by $J_\varepsilon f(x) = j_\varepsilon \ast f(x)$, where $\ast$ stands for the convolution. And $j_\varepsilon(x)$ can be constructed by first considering a Schwartz function $j(x)$ such that $0 \leq j(\xi) \leq 1$ for all the $\xi \in \mathbb{R}^d$ and $\hat{j}(\xi) = 1$ for any $\xi \in [-1, 1]^d$, and then letting $j_\varepsilon(x) = \frac{1}{2\varepsilon} \sum_{k \in \mathbb{Z}^d} \hat{j}(\varepsilon k)e^{i\varepsilon k \cdot x}$. It is obvious that $\hat{j}_\varepsilon(\xi) = \hat{j}(\varepsilon \xi)$.

And for any $u \in H^s$, \begin{equation}
\|u - J_\varepsilon u\|_{H^r} = o(\varepsilon^{s-r}), \; \varepsilon \leq 1.
\end{equation}

In fact, since $\hat{j}_\varepsilon(\xi) = 1$ for any $\xi \in [-1, 1]^d$ and $0 \leq \hat{j}(\xi) \leq 1$, we have

$$
\varepsilon^{2r-2s}\|u - J_\varepsilon u\|^2_{H^r} = \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^s \frac{\varepsilon^{2r-2s}}{(1 + |k|^2)^{s+r}} \left|1 - \hat{j}(\varepsilon k)\right|^2 |\hat{u}(k)|^2
\leq \sum_{k \in \mathbb{Z}^d; |k| \geq 1} \left(1 + |k|^2\right)^s \frac{\varepsilon^{2r-2s}}{(1 + |k|^2)^{r-s}} \left|1 - \hat{j}(\varepsilon k)\right|^2 |\hat{u}(k)|^2
\leq \|u - J_\varepsilon u\|^2_{H^r} \sim o(1).
\end{equation}

$J_\varepsilon$ also admits that for $u \in H^s$ and $r \geq s$, \begin{equation}
\|J_\varepsilon u\|_{H^r} \leq O(\varepsilon^{s-r}) \|u\|_{H^r}.
\end{equation}

To see this, for $\varepsilon \in (0, 1)$ and $r \geq s$, we consider

$$
\|J_\varepsilon u\|_{H^r} = \sum_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^r |\hat{j}(\varepsilon k)|^2 |\hat{u}(k)|^2 \leq \|u\|^2_{H^r} \left( \sup_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^r |\hat{j}(\varepsilon k)|^2 \right).
\end{equation}

By the construction of the $j_\varepsilon(x)$, there holds the following estimate \begin{equation}
\sup_{k \in \mathbb{Z}^d} \left(1 + |k|^2\right)^r |\hat{j}(\varepsilon k)|^2 = \varepsilon^{2s-2r} \sup_{m \in \mathbb{R}^d} \left(\varepsilon^2 + |m|^2\right)^{r-s} |\hat{j}(m)|^2.
\end{equation}

Since $(\varepsilon^2 + |m|^2)^{r-s} |\hat{j}(m)|^2$ is bounded uniformly in $\varepsilon \in (0, 1)$, we obtain (2.2). In addition, it has that \begin{equation}
D^s J_\varepsilon = J_\varepsilon D^s,
\end{equation}
\begin{equation}
(J_\varepsilon f \ast g)_{L^2} = (f, J_\varepsilon g)_{L^2},
\end{equation}
\begin{equation}
\|J_\varepsilon u\|_{H^r} \leq \|u\|_{H^r}.
\end{equation}

We will also need to consider $J_\varepsilon B(t, u)$, which is understood as $(J_\varepsilon B(t, u))f = J_\varepsilon (B(t, u)f)$ for any $f \in U$. Therefore we have

\begin{equation}
\left\{ \begin{array}{l}
\|J_\varepsilon B(t, u)\|_{L^2(U, H^r)} \leq \|B(t, u)\|_{L^2(U, H^r)} \\
\left( \int_0^T J_\varepsilon B(t, u)dW_t \right)_{H^r} = \sum_{k=1}^\infty \int_0^T (B(t, u)e_k, J_\varepsilon v)_{H^r} dW_k.
\end{array} \right.
\end{equation}
Moreover, \( \mu \) (Prokhorov Theorem, Lemma 2.7, Skorokhod Theorem, weakly to a probability measure \( J \)).

Since Lemma 2.2, let \( \mu \) is uniformly integrable.

The following Lemma has been established for the whole line case in [14]. When \( x \in \mathbb{T} \), using the periodic property of \( v \) in the proof as in Theorem 2.1 in [14], one can also obtain the same result as follows (cf. [12]):

**Lemma 2.3** ([14]). Let \( T > 0 \) and \( v \in C^1([0,T); H^2(\mathbb{T})) \). Then given any \( t \in [0,T) \), there is at least one point \( z(t) \) with

\[
M(t) \triangleq \min_{x \in \mathbb{T}} v_z(t, x) = v_z(t, z(t)).
\]

Moreover, \( M(t) \) is almost everywhere differentiable on \( [0,T) \) with

\[
\frac{d}{dt} M(t) = v_x(t, z(t)) \text{ a.e. on } (0,T).
\]

**Lemma 2.4** ([12]). For any \( f \in H^1(\mathbb{T}) \), there is a \( \lambda > 0 \) such that

\[
\max_{x \in \mathbb{T}} f^2(x) \leq \lambda \| f \|_{H^1}^2.
\]

**Lemma 2.5** ([60, 65]). Let \( \sigma, \alpha \in \mathbb{R} \). If \( n \in \mathbb{Z}^+ \) and \( n \gg 1 \), then

\[
\| \sin(nx_i - \alpha) \|_{H^2} = \| \cos(nx_i - \alpha) \|_{H^2} \approx n^\sigma, \quad i = 1, 2, \ldots, d, \\
\| \cos(nx_i - \alpha) \sin(nx_i - \alpha) \|_{H^2} \approx n^\sigma, \quad i, j = 1, 2, \ldots, d, \quad i \neq j.
\]

**Lemma 2.6** ([64, 65]). Let \( s > d/2 \) with \( d \geq 2 \). For any \( v_1, v_2 \) in \( H^s \), the \( F(\cdot) \) defined in (1.2) satisfies

\[
\| F(v) \|_{H^s} \leq \| v \|_{W^{1,n}} \| v \|_{H^s}, \quad s > d/2 + 1,
\]

\[
\| F(v_1) - F(v_2) \|_{H^s} \leq (\| v_1 \|_{H^{s+1}} + \| v_2 \|_{H^{s+1}}) \| v_1 - v_2 \|_{H^s}, \quad d/2 + 1 > s > d/2,
\]

\[
\| F(v_1) - F(v_2) \|_{H^s} \leq (\| v_1 \|_{H^s} + \| v_2 \|_{H^s}) \| v_1 - v_2 \|_{H^s}, \quad s > d/2 + 1.
\]

**Lemma 2.7** (Prokhorov Theorem,[21]). Let \( X \) be a complete and separable metric space. A sequence of measures \( \{ \mu_n \} \subset \mathcal{P}(X) \) is tight if and only if it is relatively compact, i.e., there is a subsequence \( \{ \mu_{n_k} \} \) converging to a probability measure \( \mu \) weakly.

**Lemma 2.8** (Skorokhod Theorem,[21]). Let \( X \) be a complete and separable metric space. For an arbitrary sequence \( \{ \mu_n \} \subset \mathcal{P}(X) \) such that \( \{ \mu_n \} \) is tight on \( (X, \mathcal{B}(X)) \), there exists a subsequence \( \{ \mu_{n_k} \} \) converging weakly to a probability measure \( \mu \), and a probability space \((\Omega, F, \mathbb{P})\) with \( X \) valued Borel measurable random variables \( x_n \) and \( x \), such that \( \mu_n \) is the distribution of \( x_n \), \( \mu \) is the distribution of \( x \), and \( x_n \to x \) \( \mathbb{P} \)-a.s.

**Lemma 2.9** (Vitali’s Convergence Theorem, [11]). Let \( p \in [1, \infty) \), \( X_n \in L^p \) and \( X_n \) converges to \( X \) in probability. Then the following are equivalent:

1. \( \lim_{n \to \infty} X_n = X \) in \( L^p \);
2. \( |X_n|^p \) is uniformly integrable;
(3) \( \lim_{n \to \infty} \mathbb{E}[[X_n]^p] = \mathbb{E}[[X]^p] \).

Particularly, if \( \sup_n \mathbb{E}[[X_n]^q] < \infty \) for some \( p < q < \infty \), or if there exists a \( Y \in L^p \) such that \( |X_n| < Y \) for all \( n \), then the above properties hold true.

**Lemma 2.10 (Gyöngy-Krylov Lemma, [35]):** Let \( X \) be a Polish space equipped with the Borel sigma-algebra \( \mathcal{B}(X) \). Let \( \{Y_j\}_{j \geq 0} \) be a sequence of \( X \) valued random variables. Let \( \mu_{j,n}(\cdot) := \mathbb{P}(Y_j \times Y_i \in \cdot) \forall \in \mathcal{B}(X \times X) \).

Then \( \{Y_j\}_{j \geq 0} \) converges in probability if and only if for every subsequence of \( \{\mu_{j,n}\}_{k \geq 0} \), there exists a further subsequence which weakly converges to some \( \mu \in \mathcal{P}(X \times X) \) satisfying

\[ \mu \left( \{(u,v) \in X \times X, u = v\} \right) = 1. \]

### 3. Blow-up criterion

Let us postpone the proof for existence and uniqueness to Sections 4 and 5. Here we will prove Theorem 1.2 first, since some similar estimates will be used later.

In the following lemma we present the relationship between the explosion time of \( \|u(t)\|_{H^s} \) and the explosion time of \( \|u(t)\|_{W^{1,\infty}} \), which is the key step in the proof for Theorem 1.2, and it is related to some ideas from the recent work for the 3D stochastic Euler equation [19].

**Lemma 3.1.** Let \( u \) be the pathwise solution to (1.6) obtained in Theorem 1.1. Then the real valued stochastic process \( \|u\|_{W^{1,\infty}} \) is also \( \mathcal{F}_t \) adapted. Besides, for any \( m, n \in \mathbb{Z}^+ \), define

\[ \tau_{1,m} = \inf \{t \geq 0 : \|u(t)\|_{H^s} \geq m\}, \quad \tau_{2,n} = \inf \{t \geq 0 : \|u(t)\|_{W^{1,\infty}} \geq n\}, \]

where \( \inf \emptyset = \infty \). Denote \( \tau_1 = \lim_{m \to \infty} \tau_{1,m} \) and \( \tau_2 = \lim_{n \to \infty} \tau_{2,n} \). Then

\[ \tau_1 = \tau_2 \quad \mathbb{P} - a.s. \quad (3.1) \]

**Proof.** Firstly, \( u(\cdot \wedge \tau) \in C([0, \infty); H^s) \) implies that for any \( t \in [0, \tau] \),

\[ [u(t)]^{-1}(Y) = [u(t)]^{-1}(H^s \cap Y), \quad \forall Y \in \mathcal{B}(W^{1,\infty}). \]

Therefore \( u(t) \), as a \( W^{1,\infty} \)-valued process, is also \( \mathcal{F}_t \) adapted. We then infer from the embedding \( H^s \hookrightarrow W^{1,\infty} \) for \( s > d/2 + 1 \) that for some \( M > 0 \),

\[ \sup_{t \in [0, \tau_{1,m}]} \|u(t)\|_{W^{1,\infty}} \leq M \sup_{t \in [0, \tau_{1,m}]} \|u(t)\|_{H^s} \leq (\lfloor M \rfloor + 1)m, \]

where \( \lfloor M \rfloor \) means the integer part of \( M \) and therefore \( \tau_{1,m} \leq \tau_{2,([M]+1)m} \leq \tau_2 \mathbb{P} - a.s. \), which means that

\[ \tau_1 \leq \tau_2 \mathbb{P} - a.s. \quad (3.2) \]

Now we prove the converse inequality. We first notice that for all \( n, k \in \mathbb{Z}^+ \),

\[ \left\{ \sup_{t \in [0, \tau_{2,n} \wedge k]} \|u(t)\|_{H^s} < \infty \right\} = \bigcup_{m \in \mathbb{Z}^+} \left\{ \sup_{t \in [0, \tau_{2,n} \wedge k]} \|u(t)\|_{H^s} < m \right\} \subset \bigcup_{m \in \mathbb{Z}^+} \left\{ \tau_{2,n} \wedge k \leq \tau_{1,m} \right\}. \]

Since

\[ \bigcup_{m \in \mathbb{Z}^+} \left\{ \tau_{2,n} \wedge k \leq \tau_{1,m} \right\} \subset \left\{ \tau_{2,n} \wedge k \leq \tau_1 \right\}, \]

we see that if we can show

\[ \mathbb{P} \left\{ \sup_{t \in [0, \tau_{2,n} \wedge k]} \|u(t)\|_{H^s} < \infty \right\} = 1 \quad \forall n, k \in \mathbb{Z}^+, \quad (3.3) \]

then for all \( n, k \in \mathbb{Z}^+ \), \( \mathbb{P} \left\{ \tau_{2,n} \wedge k \leq \tau_1 \right\} = 1 \) and

\[ \mathbb{P} \left\{ \tau_2 \leq \tau_1 \right\} = \mathbb{P} \left( \bigcap_{n \in \mathbb{Z}^+} \left\{ \tau_{2,n} \leq \tau_1 \right\} \right) = \mathbb{P} \left( \bigcap_{n,k \in \mathbb{Z}^+} \left\{ \tau_{2,n} \wedge k \leq \tau_1 \right\} \right) = 1. \quad (3.4) \]

Notice that (3.2) and (3.4) imply (3.1). Since (3.4) requires the assumption (3.3), we only need to prove (3.3). However, if \( u \) is a pathwise solution, we can not directly apply the Itô formula for \( \|u\|_{H^s}^2 \) to get control of \( \mathbb{E}[\|u(t)\|_{H^s}^2] \) since \((u \cdot \nabla)u\) is only an \( H^{s-1} \)-value process and the inner (or dual) products
\[ (D^s(u \cdot \nabla) u, D^s u)_{L^2} \] does not make sense. We will use \( J_\varepsilon \) to overcome this obstacle. Indeed, applying \( J_\varepsilon \) to (1.6) and using the Itô formula for \( \| J_\varepsilon u \|_{L^2}^2 \), with noticing (2.6), we have that for any \( t > 0 \),
\[
d\| J_\varepsilon u(t) \|_{L^2}^2 = (J_\varepsilon B(t, u) dW, J_\varepsilon u)_{H^s} - 2 (D^s J_\varepsilon [(u \cdot \nabla) u], D^s J_\varepsilon u)_{L^2} dt
\]
\[
- 2 (D^s J_\varepsilon F(u), D^s J_\varepsilon u)_{L^2} dt + \| J_\varepsilon B(t, u) \|_{L^2(U, H^s)}^2 dt.
\]
Therefore we have
\[
\| J_\varepsilon u(t) \|_{L^2}^2 - \| J_\varepsilon u(0) \|_{L^2}^2 = 2 \left( \int_0^t J_\varepsilon B(t', u) dW, J_\varepsilon u \right)_{H^s} - 2 \int_0^t (D^s [J_\varepsilon (u \cdot \nabla) u], D^s J_\varepsilon u)_{L^2} dt'
\]
\[
- 2 \int_0^t (D^s J_\varepsilon F(u), D^s J_\varepsilon u)_{L^2} dt' + \int_0^t \| D^s J_\varepsilon B(t', u) \|_{L^2(U, L^2)}^2 dt' = L_{1, \varepsilon} + \sum_{j=2}^4 \int_0^t L_{j, \varepsilon} dt' \tag{3.5}
\]
We can first use (2.4), BDG inequality, Hypothesis I and then use stochastic Fubini theorem [30, 21] to find that
\[
E \left( \sup_{t \in [0, \tau_2, n \wedge k]} |L_{1, \varepsilon}(t)|^2 \right) \leq \frac{1}{2} E \sup_{t \in [0, \tau_2, n \wedge k]} \| J_\varepsilon u(t) \|_{L^2}^2 + C E \int_0^{\tau_2, n \wedge k} h_1^2(t) f^2(\| u \|_{W^{1, \infty}}) (1 + \| u \|_{H^s}^2) dt.
\]
For \( L_{2, \varepsilon} \), using (2.3) and (2.4), we have
\[
L_{2, \varepsilon} = 2 \left( [D^s, (u \cdot \nabla)] u, D^s J_\varepsilon u \right)_{L^2} - 2 \left( (u \cdot \nabla) D^s u, D^s J_\varepsilon u \right)_{L^2}.
\]
Using Lemma 2.1 and (2.2), we find that
\[
(D^s, (u \cdot \nabla)] u, D^s J_\varepsilon u \right)_{L^2} \lesssim \| u \|_{W^{1, \infty}} \| u \|_{H^s}^2.
\]
On account of Lemma 2.2, (2.3), (2.4) and integration by parts, we arrive at
\[
\| (u \cdot \nabla) D^s u, D^s J_\varepsilon u \|_{L^2} = \| J_\varepsilon, (u \cdot \nabla)] D^s u, D^s J_\varepsilon u \|_{L^2} + \| (u \cdot \nabla) D^s J_\varepsilon u, D^s J_\varepsilon u \|_{L^2} \lesssim \| \nabla u \|_{L^\infty} \| u \|_{H^s}^2.
\]
There we have
\[
E \int_0^{\tau_2, n \wedge k} |L_{3, \varepsilon}| dt \leq E \int_0^{\tau_2, n \wedge k} \sup_{t' \in [0, \tau_2, n \wedge \varepsilon]} |L_{3, \varepsilon}| dt \leq C n \int_0^k E \sup_{t' \in [0, \tau_2, n \wedge k]} \| u \|_{H^s}^2 dt'.
\]
Similarly, it follows from Hypothesis I, Lemma 2.6 and (2.6) that there is a locally bounded non-decreasing function \( \Psi(t) = h_1^2(t) + 1 \) such that
\[
E \int_0^{\tau_2, n \wedge k} |L_3| + |L_4| dt \leq C(n) \int_0^k \Psi(t) \left( 1 + E \sup_{t' \in [0, t \wedge \tau_2, n]} \| u(t') \|_{H^s}^2 \right) dt.
\]
Therefore we combine the above estimates to have
\[
E \sup_{t \in [0, \tau_2, n \wedge k]} \| J_\varepsilon u(t) \|_{H^s}^2 \leq 2 E \| u_0 \|_{H^s}^2 + C(n) \int_0^k \Psi(t) \left( 1 + E \sup_{t' \in [0, t \wedge \tau_2, n]} \| u(t') \|_{H^s}^2 \right) dt.
\]
Notice that the right hand side of the above estimate does not depend on \( \varepsilon \). And for any \( T > 0 \), \( J_\varepsilon u \) tends to \( u \) in \( C([0, T], H^s) \) almost surely as \( \varepsilon \rightarrow 0 \), we can send \( \varepsilon \rightarrow 0 \) to find that
\[
E \sup_{t \in [0, \tau_2, n \wedge k]} \| u(t) \|_{H^s}^2 \leq 2 E \| u_0 \|_{H^s}^2 + C(n) \int_0^k \Psi(t) \left( 1 + E \sup_{t' \in [0, t \wedge \tau_2, n]} \| u(t') \|_{H^s}^2 \right) dt. \tag{3.6}
\]
Then the Grönwall’s inequality shows that for each \( n, k \in \mathbb{Z}^+ \), there is a \( C(n, k, u_0) > 0 \) such that
\[
E \sup_{t \in [0, \tau_2, n \wedge k]} \| u(t) \|_{H^s}^2 \leq (2 E \| u_0 \|_{H^s}^2 + 1) \exp \left( C(n) \int_0^k \Psi(t) dt \right) < C(n, k, u_0),
\]
which gives (3.3).

\[ \square \]

**Proof for Theorem 1.2.** By continuity of \( \| u(t) \|_{H^s} \) and the uniqueness of \( u \), it is easy to check that \( \tau_1 \) is actually the maximal existence time \( \tau^* \) of \( u \) in the sense of Definition 1.2. It follows from Lemma 3.1 that \( \tau_1 = \tau_2 \) almost surely, which implies Theorem 1.2. \[ \square \]
4. Regular pathwise solutions

We will prove the following result in this section.

**Theorem 4.1** (Pathwise solution in $H^s$ with $s > d/2 + 3$). Let $S = (\Omega, F_t, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ be a fixed stochastic basis. Suppose that $B(t, \cdot)$ satisfies Hypothesis I. Let $s > d/2 + 3$ with $d \geq 2$ and $u_0$ be an $H^s$-valued $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. Then there is a unique maximal pathwise solution $(u, \tau^*)$ to (1.6) in the sense of Definition 1.2.

#### 4.1. Approximation scheme. We will first construct the approximation scheme as follows.

**Cut-off.** For any $R > 1$, we let $\chi_R(x) : [0, \infty) \to [0, 1]$ be a $C^\infty$ function such that $\chi_R(x) = 1$ for $x \in [0, R]$ and $\chi_R(x) = 0$ for $x > 2R$. Then we consider the following problem by cutting the nonlinearities in (1.6),

\[
\begin{aligned}
\frac{du + \chi_R(\|u\|_{W^{1, \infty}})(u \cdot \nabla)u + F(u)}{dt} &= \chi_R(\|u\|_{W^{1, \infty}})B(t, u)dW, \quad x \in \mathbb{T}^d, \quad t > 0, \\
u(\omega, 0, x) &= u_0(\omega, x) \in H^s, \quad x \in \mathbb{T}^d.
\end{aligned}
\]

**Mollifying.** In order to apply the theory of SDE in Hilbert space to (4.1), we will have to mollify the transport term $(u \cdot \nabla)u$ since the product $(u \cdot \nabla)u$ loses one regularity. Therefore we mollify (4.1) and consider

\[
\begin{aligned}
\frac{du + H_{1,\varepsilon}(u)}{dt} &= H_2(t, u)dW, \quad x \in \mathbb{T}^d, \quad t > 0, \\
H_{1,\varepsilon}(u) &= \chi_R(\|u\|_{W^{1, \infty}})\{J_{\varepsilon}(J_{\varepsilon}u \cdot \nabla)J_{\varepsilon}u + F(u)\}, \\
H_2(t, u) &= \chi_R(\|u\|_{W^{1, \infty}})B(t, u), \\
u(\omega, 0, x) &= u_0(\omega, x) \in H^s,
\end{aligned}
\]

where $J_{\varepsilon}$ is the Friedrichs mollifier defined in the previous section. Then it follows from Hypothesis I, Lemma 2.6 and (2.2) that for any $T > 0$ and $R > 1$, there is an $l_1 = l_1(R, \varepsilon)$ and $l_2 = l_2(R)$ such that for all $u \in C([0, T]; H^s), \rho > d/2 + 1$, $H_{1,\varepsilon}(\cdot)$ and $H_2(\cdot, \cdot)$ satisfy

\[
\|H_{1,\varepsilon}(u)\|_{H^s} \leq l_1(1 + \|u\|_{H^s}), \quad \|H_2(t, u)\|_{L^2(\mathbb{R}; H^r)} \leq l_2 h_1(t)(1 + \|u\|_{H^s}), \quad \forall t \in [0, T].
\]

For any $R > 1, \varepsilon \in (0, 1)$, the system (4.2) may be viewed as an SDE in $H^s$. Fix a stochastic basis $S = (\Omega, F_t, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ in advance and let $u_0 \in L^2(\Omega; H^s)$ with $s > d/2 + 3$ with $d \geq 2$. It is easy to check that $H_{1,\varepsilon}(u)$ and $H_2$ are locally Lipschitz and satisfy (4.3), the theory of SDE in Hilbert space (see [45, 56]) can be applied here to show that (4.2) admits a unique solution $u_\varepsilon \in C([0, T], H^s) \mathbb{P}$-a.s. Using the same way as we prove Theorem 1.2, we see that for each fixed $\varepsilon$, if $T_\varepsilon < \infty$, then

\[
\limsup_{t \to T_\varepsilon} \|u_{\varepsilon}(t)\|_{W^{1, \infty}} = \infty \quad \mathbb{P} - \text{a.s.}
\]

Due to the cut-off in (4.2), $\|u_{\varepsilon}\|_{W^{1, \infty}}$ is always bounded and hence $u_\varepsilon$ is actually a global in time solution, that is, $u_\varepsilon \in C((0, \infty), H^s) \mathbb{P}$-a.s.

#### 4.2. Uniform estimates. Since now we establish some estimates (for 4.2) uniformly in $\varepsilon$.

**Proposition 4.1.** Let $S = (\Omega, F_t, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ be a fixed stochastic basis. Let $s > d/2 + 3$ with $d \geq 2$, $r > 4$, $R > 1$ and $\varepsilon \in (0, 1)$. Assume $\sigma(t, \cdot)$ satisfies Hypothesis I and $u_0 \in L^r(\Omega; H^s)$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable. Let $u_\varepsilon \in C((0, \infty); H^s)$ solve (4.2) $\mathbb{P}$-a.s., then for $0 < \alpha < \frac{1}{2} - \frac{1}{r}$ and for any $T > 0$, it holds that

\[
\{u_\varepsilon\}_{\varepsilon \in (0, 1)} \subset L^r(\Omega; C([0, T]; H^s) \cap C^\alpha([0, T]; H^{s-1}))
\]

and $\{u_\varepsilon\}_{\varepsilon \in (0, 1)}$ is bounded uniformly in $\varepsilon$. Furthermore, there are $C_1 = C_1(R, T, u_0, r) > 0$ and $C_2 = C_2(T, u_0, r, \alpha) > 0$ such that

\[
\sup_{\varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s}^r \leq C_1,
\]

and

\[
\sup_{\varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)^r\|_{C^\alpha([0, T]; H^{s-1})}^r \leq C_2.
\]
Proof. Using the Itô formula enables us to see that for $D^s u_z$,
\[
\begin{align*}
\frac{d}{dt} \|u_z\|^2_{H^s} &= 2\chi_R(\|u_z\|_{W^{1,\infty}})(B(t, u_z) dW, u_z)_{H^s} \\
&\quad - 2\chi_R(\|u_z\|_{W^{1,\infty}})(D^s J_z [\langle J_z u_z, \nabla \rangle J_z u_z], D^s u_z)_{L^2} \, dt \\
&\quad - 2\chi_R(\|u_z\|_{W^{1,\infty}})(D^s F(u_z), D^s u_z)_{L^2} \, dt \\
&\quad + \chi_R^2(\|u_z\|_{W^{1,\infty}}) \|B(t, u_z)\|^2_{L_2(U; H^s)} \, dt \\
&= J_1 + \sum_{i=2}^{4} J_i dt.
\end{align*}
\]

Integrating the above equation, taking a supremum for $t \in [0, T]$ and using the BDG inequality yield
\[
\begin{align*}
\mathbb{E} \sup_{t \in [0, T]} \|u_z(t)\|^2_{H^s} \leq &\mathbb{E} \|u_0\|^2_{H^s} + \sum_{k=2}^{4} \int_0^T \mathbb{E} \|J_k\|^2 dt \\
&+ C \mathbb{E} \left( \int_0^T \|u_z\|^2_{H^s} \chi_R^2(\|u_z\|_{W^{1,\infty}}) \|B(t, u_z)\|^2_{L_2(U; H^s)} dt \right)^{\frac{1}{2}},
\end{align*}
\]
and in the above equation,
\[
\begin{align*}
\mathbb{E} \left( \int_0^T \|u_z\|^2_{H^s} \chi_R^2(\|u_z\|_{W^{1,\infty}}) \|B(t, u_z)\|^2_{L_2(U; H^s)} dt \right)^{\frac{1}{2}} \leq &\frac{1}{2} \mathbb{E} \sup_{t \in [0, T]} \|u_z\|^2_{H^s} + C f^2(2R) \int_0^T h_1^2(t) \left( 1 + \mathbb{E} \|u_z\|^2_{H^s} \right) dt.
\end{align*}
\]

By first commuting $J_z$ and then commuting the operator $D^s$ with $J_z u_z$, then applying the Cauchy–Schwarz inequality, Lemma 2.1 and integration by parts, we see that
\[
\begin{align*}
\mathbb{E} \int_0^T \|J_2\|^2 dt &\leq 4R \int_0^T \mathbb{E} \|u_z\|^2_{H^s} dt', \\
\end{align*}
\]
for $J_3$ and $J_4$, we simply use the Cauchy–Schwarz inequality, Hypothesis I and Lemma 2.6 to deduce that
\[
\begin{align*}
\mathbb{E} \int_0^T |J_3| + |J_4| dt' \leq &\left( 4R + 2f^2(2R) \right) \int_0^T \Psi(t') \left( 1 + \mathbb{E} \|u_z\|^2_{H^s} \right) dt', \tag{4.8}
\end{align*}
\]
where $\Psi(t) = h_1^2(t) + 1$. Combining (4.6)–(4.8), we see that $u_z$ satisfies
\[
\begin{align*}
\mathbb{E} \sup_{t \in [0, T]} \|u_z(t)\|^2_{H^s} \leq &\mathbb{E} \|u_0\|^2_{H^s} + C_{R} \int_0^T \Psi(t) \left( 1 + \sup_{t' \in [0, t]} \|u_z(t')\|^2_{H^s} \right) dt.
\end{align*}
\]
Via the Grönwall’s inequality, we find that for any $t \in [0, T]$, \{$u_z$\} $\subset L^2(\Omega; C([0, T]; H^s))$ is bounded uniformly in $\varepsilon$. Now we notice that $\|u_z\|^2_{H^s}$ can be actually expressed as
\[
\begin{align*}
\frac{d}{dt} \|u_z\|^2_{H^s} &= \sum_{k=1}^{\infty} J_{1,k} dW_k + \sum_{i=2}^{4} J_i dt, \quad J_{1,k} = 2\chi_R(\|u_z\|_{W^{1,\infty}}) (B(t, u_z) e_k, u_z)_{H^s},
\end{align*}
\]
where \{$e_k$\} is a complete orthonormal basis of $U$ and \{$W_k$\}$_{k \geq 1}$ is a sequence of mutually independent real valued Brownian motions. Given $r > 4$, then $d\|u_z\|^2_{H^s} = d(\|u_z\|^2_{H^s})^r$, we have
\[
\begin{align*}
\frac{d}{dt} \|u_z\|^2_{H^s} &= \frac{r}{2} \|u_z\|^2_{H^s} \left( \sum_{k=1}^{\infty} J_{1,k} dW_k + \sum_{i=2}^{4} J_i dt \right)^{r-2} + \frac{r(r-2)}{8} \|u_z\|^2_{H^s} J_{1,k}^2 dt,
\end{align*}
\]
which together with BDG inequality yields that for any $T > 0$ and $t \in [0, T]$,
\[
\begin{align*}
\mathbb{E} \sup_{t \in [0, T]} \|u_z(t)\|^2_{H^s} \leq &\mathbb{E} \|u_0\|^2_{H^s} + C \mathbb{E} \left( \sup_{t \in [0, T]} \|u_z\|^2_{H^s} \int_0^T \chi_R^2(\|u_z\|_{W^{1,\infty}}) f^2(\|u_z\|_{W^{1,\infty}}) (1 + \|u_z\|^2_{H^s}) dt \right)^{\frac{1}{2}} \\
&+ C \int_0^T \mathbb{E} \left( \sum_{k=1}^{\infty} \|u_z\|^2_{H^s} J_{1,k}^2 \right) dt. \tag{4.9}
\end{align*}
\]
Similarly, from Hypothesis I, we have
\[
\mathbb{E}\left( \sum_{k=1}^{\infty} \|u_{e_k}\|_{H}^{-4} |J_{1,k}|^2 \right) \leq C \mathbb{E}\left( \chi_R^2(\|u_{e}\|_{W^{1,\infty}}) \sum_{k=1}^{\infty} \|B(t, u_{e_k})\|^2_{L^\infty} \|u_{e_k}\|^2_{H}\right) 
\leq C_{r, R} (1 + \mathbb{E}\|u_{e}\|^r_{H})
\]

Using estimates analogous to those in (4.7)–(4.8), we have
\[
\sum_{i=2}^{4} \mathbb{E}\|u_{e}\|_{H}^{-2}|J_i| \leq C_{r, R} \Psi(t) (1 + \mathbb{E}\|u_{e}\|^r_{H})
\]

Combining the above estimates, we identify that for any \(T > 0\),
\[
\mathbb{E}\sup_{t \in [0, T]} \|u_{e}(t)\|_{H}^r \leq 2\mathbb{E}\|u_0\|_{H}^r + C_{r, R} \int_0^T \Psi(t) \left( 1 + \mathbb{E}\sup_{t' \in [0, t]} \|u_{e}(t')\|^r_{H} \right) dt.
\]

From the above estimate and the Grönewall inequality, we obtain (4.4). Now we prove that for \(0 < \alpha < \frac{1}{2} - \frac{1}{r}\), \(\{u_{e}\} \subset L^r(\Omega; \mathcal{C}^\alpha([0, T]; H^{s-1}))\) is also bounded uniformly in \(n\). For any \([t', t] \subset [0, T]\) with \(|t - t'| \leq 1\), we first notice that from (4.2),
\[
\|u_{e}(t) - u_{e}(t')\|_{H}^{-1} \leq \left\| \int_{t'}^t H_{1, e}(u_{e}) \, dt \right\|_{H}^{-1} + \left\| \int_{t'}^t H_{2}(\tau, u_{e}) \, dW \right\|_{H}^{-1}.
\]

Actually, by using (2.2), Lemma 2.6 and (4.4), we have that for any \([t', t] \subset [0, T]\),
\[
\mathbb{E}\left( \left\| \int_{t'}^t H_{1, e}(u_{e}) \, dt \right\|_{H}^{-1} \right)^r \leq |t - t'|^r \mathbb{E}\sup_{\tau \in [0, T]} \|H_{1, e}(u_{e})\|_{H}^{-1} \leq C|t - t'|^r \mathbb{E}\sup_{\tau \in [0, T]} (\chi_R(\|u_{e}\|_{W^{1,\infty}}) \|J_{1, u_{e}}\|_{W^{1,\infty}} \|J_{2, u_{e}}\|_{H} + \chi_R(\|u_{e}\|_{W^{1,\infty}}) \|F(u_{e})\|_{H}^r)^r.
\]

And for the stochastic integral, it follows from the BDG inequality and (4.3) that
\[
\mathbb{E}\left( \left\| \int_{t'}^t H_{2}(\tau, u_{e}) \, dW \right\|_{H}^{-1} \right)^r \leq C|t - t'|^r \mathbb{E}\sup_{\tau \in [0, T]} \left\| H_{2}(\tau, u_{e}) \right\|_{L^2(t; H^{s-1})}^2 \left\| \int_{t'}^t H_{2}(\tau, u_{e}) \, dW \right\|_{H}^{-1} \leq C|t - t'|^r \mathbb{E}\sup_{\tau \in [0, T]} \left\| H_{2}(\tau, u_{e}) \right\|_{L^2(t; H^{s-1})}^2 \leq C|t - t'|^r \bar{h}_{1}(T) \left( 1 + \mathbb{E}\sup_{\tau \in [0, T]} \|u_{e}(\tau)\|_{H}^r \right).
\]

Due to (4.4), we have
\[
\mathbb{E}\left( \left\| \int_{t'}^t H_{2}(\tau, u_{e}) \, dW \right\|_{H}^{-1} \right)^r \leq C(R, T, u_0, r, \alpha)|t - t'|^{\bar{h}_{1}(T)}.
\]

Combining (4.11) and (4.12) into (4.10), we have
\[
\mathbb{E}\|u_{e}(t) - u_{e}(t')\|_{H}^{-1} \leq C(R, T, u_0, r)|t - t'|^{\bar{h}_{1}(T)}.
\]

Then the Kolmogorov’s continuity theorem yields that for \(\alpha \in (0, \frac{1}{2} - \frac{1}{r})\), \(u_{e}\) has a \(\mathcal{C}^\alpha([0, T]; H^{s-1})\) path almost surely and
\[
\mathbb{E}\sup_{t \in [0, T]} \|u_{e}(t')\|_{\mathcal{C}^\alpha([0, T]; H^{s-1})} \leq C(R, T, u_0, r, \alpha),
\]
which implies (4.5).
4.3. Martingale solution to the cut-off problem. When we consider the martingale solutions, the stochastic basis $\mathcal{S}$ itself is an unknown part of the problem (1.6). Hence a random initial condition $u_0$ may only be regarded as an initial probability measure $\mu_0 \in \mathcal{P}(H^s)$. Therefore we assume that $\mu_0 \in \mathcal{P}(H^s)$ such that for some $r > 4$, 
\[
\int_{H^s} \|u\|^r_{H^s} \, d\mu_0(u) < \infty, \quad s > \frac{d}{2} + 1.
\] (4.13)
To start with, we choose a stochastic basis $\mathcal{S} = (\mathcal{O}, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W})$ and a random variable $u_0$ such that $u_0$ is an $\mathcal{F}_0$ measurable random variable with the distribution $\mu_0$ on $H^s$. Let $R > 1$, $0 < \varepsilon < 1$ and $T > 0$, we let $u_\varepsilon \in C([0,T]; H^s)$ be the solution to (4.2). Then we define the phase space $X^s$ as
\[
X^s = X_*^s \times X_W, \quad X_*^s = C([0,T]; H^s), \quad X_W = C([0,T]; U_0).
\] (4.14)

Lemma 4.1. Let $s > d/2 + 3$. Define $\nu_\varepsilon \in \mathcal{P}(X^s)$ as
\[
\nu_\varepsilon := \mu_\varepsilon \times \mu_W \quad \text{where} \quad \mu_\varepsilon(\cdot) = \mathbb{P}\{u_\varepsilon \in \cdot\} \quad \text{and} \quad \mu_W(\cdot) = \mathbb{P}\{W \in \cdot\}.
\]
Then $\{\nu_\varepsilon\} \subset \mathcal{P}(X_*^{s-1})$ has a weakly convergent subsequence, still denoted by $\{\nu_\varepsilon\}$, with limit measure $\nu$.

Proof. For any $M > 0$, let $B_2^M$ be the ball with radius $M$ in $C([0,T]; H^s)$ and $B_3^M$ be the ball with radius $M$ in $C^\alpha([0,T]; H_*^{s-1})$. Let
\[
A_{1,M} = \left\{ u_\varepsilon : \|u_\varepsilon\|_{C([0,T]; H^s)} < \frac{M}{2} \right\}, \quad A_{2,M} = \left\{ u_\varepsilon : \|u_\varepsilon\|_{C^\alpha([0,T]; H_*^{s-1})} < \frac{M}{2} \right\}.
\]
Via the Ascoli’s Theorem in a Banach space (cf. [23]), $A_M = A_{1,M} \cap A_{2,M}$ is pre-compact in $X_*^{s-1}$. For any $\eta > 0$, from the Chebyshev inequality, (4.4) and (4.5), we may identify that for $M = \frac{4C}{\eta}$ with some $C$ large enough,
\[
\mu_\varepsilon \left( \left( \frac{1}{M}X_*^{s-1} \right)^C \right) \leq \mathbb{P}\left\{ u_\varepsilon : \|u_\varepsilon\|_{C([0,T]; H^s)} \geq \frac{2C}{\eta} \right\} \\
+ \mathbb{P}\left\{ u_\varepsilon : \|u_\varepsilon\|_{C^\alpha([0,T]; H_*^{s-1})} \geq \frac{2C}{\eta} \right\} \\
\leq \frac{\eta}{2C} \mathbb{E}\|u_\varepsilon\|_{C([0,T]; H^s)} + \frac{\eta}{2C} \mathbb{E}\|u_\varepsilon\|_{C^\alpha([0,T]; H_*^{s-1})} \\
\leq \eta.
\]
Hence $\mu_\varepsilon$ is tight on $X_*^{s-1}$. For $\mu_W$, it is trivially tight since it stays unchanged.

Lemma 4.2. Let $s > d/2 + 3$ with $d \geq 2$. There is a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ on which there is a sequence of random variables $(\tilde{u}_\varepsilon, \tilde{W}_\varepsilon)$ and a $(\tilde{u}, \tilde{W})$ such that
\[
\tilde{\mathbb{P}}\left\{ (\tilde{u}_\varepsilon, \tilde{W}_\varepsilon) \in \cdot \right\} = \nu_\varepsilon(\cdot), \quad \tilde{\mathbb{P}}\left\{ (\tilde{u}, \tilde{W}) \in \cdot \right\} = \nu(\cdot),
\] (4.15)
and
\[
\tilde{u}_\varepsilon \rightarrow \tilde{u} \text{ in } C([0,T]; H_*^{s-1}), \quad \tilde{\mathbb{P}} \text{- a.s.}, \quad \tilde{W}_\varepsilon \rightarrow \tilde{W} \text{ in } C([0,T]; U_0), \quad \tilde{\mathbb{P}} - \text{ a.s.} \quad (4.16)
\]
Moreover, the following results hold
\begin{itemize}
\item $\tilde{W}_\varepsilon$ is a cylindrical Wiener process relative to $\tilde{\mathcal{F}}_t = \sigma \{ \tilde{u}_\varepsilon(\tau), \tilde{W}_\varepsilon(\tau) \}_{\tau \in [0,t]}$;
\item $\tilde{W}$ is a cylindrical Wiener process relative to $\tilde{\mathcal{F}}_t = \sigma \{ \tilde{u}(\tau), \tilde{W}(\tau) \}_{\tau \in [0,t]}$;
\item On $\left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \left\{ \tilde{\mathcal{F}}_t^{\varepsilon} \right\}_{t \geq 0} \right)$, almost surely we have
\[
\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(0) + \int_0^t H_{1,\varepsilon}(\tilde{u}_\varepsilon) \, dt' = \int_0^t H_2(t', \tilde{u}_\varepsilon) \, d\tilde{W}_\varepsilon,
\] (4.17)
as an equation in $H^{s-2}$, where $H_{1,\varepsilon}(\cdot)$ and $H_2(t, \cdot)$ are given in (4.2);
\item Almost surely it has that
\[
\begin{align*}
H_{1,\varepsilon}(\tilde{u}_\varepsilon) & \xrightarrow{\varepsilon \rightarrow 0} H_1(\tilde{u}) = \chi_{R}(\|\tilde{u}\|_{W^{1,\infty}}) [\tilde{u} \cdot \nabla] \tilde{u} + F(\tilde{u}) \text{ in } C([0,T]; H^{s-2}), \\
H_2(t, \tilde{u}_\varepsilon) & \xrightarrow{\varepsilon \rightarrow 0} H_2(t, \tilde{u}) = \chi_{R}(\|\tilde{u}\|_{H^{s-2}}) B(t, \tilde{u}) \text{ in } L_2(U, H^{s-1}), \quad t > 0.
\end{align*}
\] (4.18)
\end{itemize}
Proof. Since $X^*$ with the product metric is a Polish space, the existence of the sequence $(\tilde{u}_\varepsilon, \tilde{W}_\varepsilon)$ satisfying (4.16) comes from Lemmas 4.1, 2.7 and 2.8. It follows from [4, Theorem 2.1.35 and Corollary 2.1.36] that $\tilde{W}_\varepsilon$ and $\tilde{W}$ are cylindrical Wiener process relative to $\tilde{F}_t = \sigma\{\tilde{u}_\varepsilon(\tau), \tilde{W}_\varepsilon(\tau)\}_{\tau \in [0,t]}$ and $\tilde{F}_t = \sigma\{\tilde{u}(\tau), \tilde{W}(\tau)\}_{\tau \in [0,t]}$, respectively. As in [2, page 282] or [4, Theorem 2.9.1], one can find that $(\tilde{u}_\varepsilon, \tilde{W}_\varepsilon)$ relative to $\{\tilde{F}_t\}_{t \geq 0}$ satisfies (4.17) almost surely. Finally, (4.18) comes from (4.16) and Hypothesis 1.

Proposition 4.2. Let $s > d/2 + 3$ with $d \geq 2$ and $\mu_0 \in \mathcal{P}(H^s)$ with $r > 4$ satisfy (4.13). For any $R > 1$, if Hypothesis 1 is satisfied, then the limit process $(\tilde{u}, \tilde{W})$ obtained in Lemma 4.2 and the basis $\tilde{S} = (\tilde{\Omega}, \tilde{F}, \tilde{P}, \{\tilde{F}_t\}_{t \geq 0}, \tilde{W})$ with $\{\tilde{F}_t\}_{t \geq 0} = \sigma\{\tilde{u}(\tau), \tilde{W}(\tau)\}_{\tau \in [0,t]}$ satisfy (4.1) for all $t > 0$. in the sense of Definition 1.1.

Proof. Step 1: Existence. For any $\varepsilon \in (0,1)$ and $T > 0$, define

$$ \tilde{M}_\varepsilon(t) = \tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(0) + \int_0^t H_{1,\varepsilon}(\tilde{u}_\varepsilon)dt', \ t \in [0,T], $$

and

$$ \tilde{M}(t) = \tilde{u}(t) - \tilde{u}(0) + \int_0^t \chi_R(||\tilde{u}||_{W^1,\infty})[(\tilde{u} \cdot \nabla)\tilde{u} + F(\tilde{u})]dt', \ t \in [0,T]. $$

It follows from Lemma 4.2 that $\tilde{M}_\varepsilon(t)$ converges to $\tilde{M}(t)$ in $C([0,T]; H^{s-2})$, $\tilde{P} - a.s.$

As $(\tilde{u}_\varepsilon, \tilde{W}_\varepsilon)$ satisfies (4.2) relative to $\tilde{S}_\varepsilon$, we have that $\tilde{u}_\varepsilon$ is a predictable process and hence

$$ \tilde{M}_\varepsilon(t) = \int_0^t H_2(t', \tilde{u}_\varepsilon) d\tilde{W}_\varepsilon $$

is an $H^{s-2}$-valued square integrable martingale under $\tilde{P}$. Let $\tilde{W}_{\varepsilon,k}$ and $\tilde{W}_k$ be the real valued Brownian motions corresponding to $\tilde{W}_\varepsilon$ and $\tilde{W}$, respectively. In other words, let $\{e_k\}$ be a complete orthonormal basis of $U$ such that

$$ \sum_{k=1}^{\infty} \tilde{W}_{\varepsilon,k} e_k = \tilde{W}_\varepsilon, \ \sum_{k=1}^{\infty} \tilde{W}_k e_k = \tilde{W}. $$

Let $\{y_j\}$ be a complete orthonormal basis of $H^{s-2}$, then $(\tilde{M}_\varepsilon(t), y_j)_{L^2}$ is a real valued martingale, and

$$ h_{\varepsilon,j}(t) := (\tilde{M}_\varepsilon, y_j)_{H^{s-2}} = \int_0^t \sum_{k=1}^{\infty} (H_2(t, \tilde{u}_\varepsilon) e_k, y_j)_{H^{s-2}} d\tilde{W}_{\varepsilon,k}. $$

Hence for any $t' < t$ and for any bounded continuous $\tilde{F}_t$ measurable function $\varphi$ on $C([0,t'], L^2)$ such that for $\tilde{\varphi}_{\varepsilon,t'} = \varphi(\tilde{u}_\varepsilon|_{[0,t']}, \tilde{W}_\varepsilon|_{[0,t']})$, we have the following

$$ \tilde{E} \left[ (\tilde{M}_\varepsilon(t) - \tilde{M}_\varepsilon(t'), y_j)_{H^{s-2}}, \tilde{\varphi}_{\varepsilon,t'} \right] = 0 \quad (4.19) $$

and

$$ \tilde{E} \left[ (h_{\varepsilon,j}(t) - h_{\varepsilon,j}(t')) - \int_{t'}^t \left[ (H_2(t, \tilde{u}_\varepsilon)^* y_j \right] ||_{L^2}^2 dr \right] \tilde{\varphi}_{\varepsilon,t'} = 0. \quad (4.20) $$

Via the Itô product rule, we have

$$ d \left( \tilde{W}_{\varepsilon,k} h_{\varepsilon,j} \right) = (e_k, [H_2(t, \tilde{u}_\varepsilon)^* y_j]_U) dt + h_{\varepsilon,j} d\tilde{W}_{\varepsilon,k} + \sum_{i=1}^{\infty} \tilde{W}_{\varepsilon,k} (H_2(t, \tilde{u}_\varepsilon) e_k, y_j)_{H^{s-2}} d\tilde{W}_{\varepsilon,k}, $$

and therefore

$$ \tilde{E} \left[ (\tilde{W}_{\varepsilon,k} h_{\varepsilon,j} (t) - \tilde{W}_{\varepsilon,k} h_{\varepsilon,j} (t')) - \int_{t'}^t (e_k, [H_2(t, \tilde{u}_\varepsilon)^* y_j]_U) d\tau \right] \tilde{\varphi}_{\varepsilon,t'} = 0. \quad (4.21) $$
Now we notice that for any \( j \geq 1, \left\{ \|h_{\varepsilon,j}\|_2^2 \right\}_{0 < \varepsilon < 1} \) is uniformly integrable. Indeed, we first recall (4.4) and (4.15) to find that for \( r > 4, \)

\[
\sup_{\varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \|\tilde{u}_\varepsilon(t)\|_{H^s} = \sup_{\varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s} < \infty. \tag{4.22}
\]

Use (4.22), BDG inequality and (4.3) to find

\[
\sup_{\varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \left\{ \left( \tilde{M}_\varepsilon(t), y_j \right) \right\}^{\frac{2}{H-2}} \leq C \sup_{\varepsilon > 0} \mathbb{E} \left( \int_0^T \|H_2(t, \tilde{u}_\varepsilon)\|^2_{L_2(U, H^s)} \, dt \right)^\frac{2}{2} \leq C \left[ \int_0^T k_1(t) \left( 1 + \|\tilde{u}_\varepsilon\|_{H^s}^2 \right) \, dt \right]^\frac{2}{2} \leq C(R, T) \sup_{\varepsilon > 0} \mathbb{E} \sup_{t \in [0, T]} \|u_\varepsilon(t)\|_{H^s} < \infty.
\]

Similarly, for each \( k, j \geq 1, \left\{ \tilde{W}_{\varepsilon,k}h_{\varepsilon,j} \right\}_{0 < \varepsilon < 1} \) is also integrable.

By Lemmas 4.2 and 2.9, we can send \( n \to \infty \) in (4.19), (4.20) and (4.21) to identify that for \( \phi_t = \varphi \left( \tilde{u}|_{[0, \tau]}, \tilde{W} \right) \right|_{[0, \tau]} \) and for any \( j \geq 1, \)

\[
\tilde{E} \left[ \left( \tilde{M} - M(t), y_j \right) \right]_{H^{s-j}} \cdot \phi_t = 0,
\]

\[
\tilde{E} \left[ \left( \tilde{M} - M(t), y_j \right) \right]_{H^{s-j}}^2 - \tilde{E} \left[ \left( \tilde{M} - M(t), y_j \right) \right]_{H^{s-j}}^2 - \int_0^{\tau} \|H_2(\tau, \tilde{u})\|^2_{L_2(U, H^s)} \, dt \right)^\frac{2}{2} \phi_t = 0,
\]

and

\[
\tilde{E} \left[ \left( \tilde{M} - M(t), y_j \right) \right]_{H^{s-j}}^2 - \tilde{E} \left[ \left( \tilde{M} - M(t), y_j \right) \right]_{H^{s-j}}^2 - \int_0^{\tau} \|H_2(\tau, \tilde{u})\|^2_{L_2(U, H^s)} \, dt \right)^\frac{2}{2} \phi_t = 0.
\]

Therefore by applying the modified martingale representation theorem ([39], Theorem A.1) to \( \tilde{M}(t), \)

\[
\tilde{M}(t) = \int_0^T H_2(\tau, \tilde{u}) d\tilde{W}, \quad t \in [0, T], \quad \mathbb{P} - a.s.,
\]

which means that \( \tilde{u} \) and \( \tilde{S} = \left( \tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \{\tilde{\mathcal{F}}_t\}_{t \geq 0}, \tilde{W} \right) \) almost surely satisfy that for \( t \in [0, T], \)

\[
\tilde{u}(t) - \tilde{u}(0) + \int_0^t \chi_R(||\tilde{u}||_{L^\infty}) \left( ||\tilde{u}| \cdot \nabla \tilde{u} + F(\tilde{u}) \right) \, dt' = \int_0^t \chi_R(||\tilde{u}||_{H^s}) B(t', \tilde{u}) d\tilde{W}.
\]

Moreover, since \( u_\varepsilon(0) \equiv u_0 \) for all \( \varepsilon, \) it is easy to find that \( \tilde{\mathbb{P}} \{ \tilde{u}(0) = 0 \} = \mu_0(\cdot) \in \mathcal{P}(H^s). \)

**Step 2: Regularity.** Now we prove (1.14) holds true for \( \tilde{u}, \) relative to \( \tilde{S}. \) For simplicity, we just rewrite \( \tilde{u} \) as \( u \) and \( \tilde{S} \) as \( S. \) To estimate \( \mathbb{E} \|u(t)\|_{H'^s}^2, \) one can copy the proof for (3.5) to find that there is a continuous \( \Psi(t) \) such that for any \( T > 0 \)

\[
\mathbb{E} \sup_{t \in [0, T]} \|J_\varepsilon u(t)\|_{H'^s}^2 \leq C \|\tilde{u}\|_{H'^s} \sup_{t \in [0, t]} \Psi(t) \left( 1 + \mathbb{E} \sup_{t' \in [0, t]} \|u(t')\|_{H'^s} \right) \, dt.
\]

Since the right hand side of the above inequality does not depend on \( \varepsilon, \) and for any \( s > 0 \) and any \( u \in L^\infty(0, T; H^s), \) \( J_\varepsilon u \) tends to \( u \) in \( L^\infty(0, T; H^s) \) almost surely when \( \varepsilon \to 0, \) we can finally obtain

\[
\mathbb{E} \sup_{t \in [0, T]} \|u(t)\|_{H'^s}^2 \leq C \|\tilde{u}\|_{H'^s} \sup_{t \in [0, t]} \Psi(t) \left( 1 + \mathbb{E} \sup_{t' \in [0, t]} \|u(t')\|_{H'^s} \right) \, dt,
\]

which together with the Grönwall inequality means that for \( s > d/2 + 3, u \in L^2(\Omega; L^\infty(0, T; H^s)). \) With this in hand, we use similar estimates as in (4.9) to obtain that for \( r > 4, s > d/2 + 3, u(t) \in L^r(\Omega; L^\infty(0, T; H^s)). \) Furthermore, the techniques in proving (4.11) and (4.12) can be used here to obtain that

\[
\mathbb{E} \|u(t) - u(t')\|_{H'^{s-1}}^2 \leq C(R, T, u_0, r) |t - t'|^{\frac{2}{r}}, \quad |t - t'| < 1.
\]
With the help of the Kolmogorov test, the path of \( u \) can be chosen to be in \( C^\alpha ([0,T];H^{s-2}) \) with \( \alpha \in (0, \frac{1}{2}) \), almost surely. Besides,
\[
u \in L^f \left( \Omega; L^\infty (0,T;H^s) \right) \cap C^\alpha ([0,T];H^{s-1}) \right) .
\]

Now we only need to prove that \( \nu \in C ([0,T];H^s) \), \( \mathbb{P} - a.s. \). To serve this purpose, we will check that
\1. \( L^\infty (0,T;H^s) \cap C^\alpha ([0,T];H^{s-1}) \hookrightarrow C_w ([0,T];H^s) \), where \( C_w ([0,T];H^s) \) is the weakly continuous functions with values in \( H^s \);
\2. The map \( t \mapsto \| \nu \|_{H^s} \) is continuous, almost surely.

The second one can be obtained from the equation directly. We omit the details for brevity and hence conclude the proof. \( \square \)

### 4.4. Pathwise uniqueness

We first state the following result which indicates that for \( L^\infty (\Omega) \) initial values, the solution map is time locally Lipschitz in less regular spaces.

**Lemma 4.3.** Let \( s > d/2 + 1 \) with \( d \geq 2 \), \( S = (\Omega, \mathcal{F}, \mathbb{P}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{W}) \) be a fixed stochastic basis and Hypothesis 1 be verified. Let \( u_0 \) and \( v_0 \) be two \( H^2 \)-valued \( \mathcal{F}_0 \)-measurable random variables (relative to \( S \)) satisfying \( \| u_0 \|_{H^s}, \| v_0 \|_{H^s} < M \) almost surely for some deterministic \( M > 0 \). Let \((S,u,\tau_1),(S,v,\tau_2)\) be two local pathwise solutions to (1.6) such that \( u(0) = u_0 \), \( v(0) = v_0 \) almost surely. For any \( T > 0 \), we denote
\[
\tau^T_{u,v} := \inf \{ t \geq 0 : \| u(t) \|_{H^s} > M + 2 \} \land T, \quad \tau^T_{v} := \inf \{ t \geq 0 : \| v(t) \|_{H^s} > M + 2 \} \land T,
\]
and \( \tau^T_{u,v} = \tau^T_{u} \land \tau^T_{v} \). Then we have that for \( s' \in \left( \frac{d}{2}, \min \left\{ s - 1, \frac{d}{2} + 1 \right\} \right) \),
\[
\mathbb{E} \sup_{t \in [0, \tau^T_{u,v}]} \| u(t) - v(t) \|_{H^{s'}}^2 \leq C(M,T) \mathbb{E} \| u_0 - v_0 \|_{H^{s'}}^2 .
\]

**Proof.** Let \( w(t) = u(t) - v(t) \) for \( t \in [0, \tau_1 \land \tau_2] \). We have
\[
dw + \left( \| \nabla \| u_1 + (u_2 \cdot \nabla)w \right) dt + [F(u_1) - F(u_2)] dt = \left[ B(t, u_1) - B(t, u_2) \right] d\mathbb{W}.
\]

Then we use the Itô formula for \( \| w \|_{H^{s'}}^2 \) with \( s' \in \left( \frac{d}{2}, \min \left\{ s - 1, \frac{d}{2} + 1 \right\} \right) \) to find that
\[
\text{d} \| w \|_{H^{s'}}^2 = 2 \left[ \| B(t, u_1) - B(t, u_2) \| d\mathbb{W}, w \right]_{H^{s'}}^2$
\]
\[
= 2 \left[ (w \cdot \nabla)u_1, w \right]_{H^{s'}} dt + 2 \left[ (u_2 \cdot \nabla)w, w \right]_{H^{s'}} dt
\]
\[
+ 2 \left[ F(u_1) - F(u_2), w \right]_{H^{s'}} dt + \| B(t, u_1) - B(t, u_2) \|_{L_2(U,H^{s-1})}^2 dt
\]
\[
= J_1 + \sum_{k=2}^5 J_k dt.
\]

Taking a supremum over \( t \in [0, \tau^T_{U,V}] \) and using the BDG inequality, (4.23) and the Cauchy–Schwarz inequality yield that there is a \( C > 0 \) such that
\[
\mathbb{E} \sup_{t \in [0, \tau^T_{U,V}]} \| w(t) \|_{H^{s'}}^2 - \mathbb{E} \| w(0) \|_{H^{s'}}^2 \leq 2 \left[ \| B(t, u_1) - B(t, u_2) \|_{L_2(U,H^{s-1})} \| w \|_{H^{s'}} \right] dt
\]
\[
\leq C \mathbb{E} \left( \int_0^{\tau^T_{U,V}} \| B(t, u_1) - B(t, u_2) \|_{L_2(U,H^{s-1})} \| w \|_{H^{s'}} \right] dt + \sum_{k=2}^5 \mathbb{E} \int_0^{\tau^T_{U,V}} |J_k| dt
\]
\[
\leq C g_0(2M + 4) \mathbb{E} \left( \sup_{t \in [0, \tau^T_{U,V}]} \| w \|_{H^{s'}} \right) \int_0^{\tau^T_{U,V}} h_2(t) \| w \|_{H^{s'}} \right] dt + \sum_{k=2}^5 \mathbb{E} \int_0^{\tau^T_{U,V}} |J_k| dt
\]
\[
\leq \frac{1}{2} \mathbb{E} \sup_{t \in [0, \tau^T_{U,V}]} \| w \|_{H^{s'}} + C(M,T) \int_0^{\tau^T_{U,V}} h_2(t) \| w \|_{H^{s'}} dt + \sum_{k=2}^5 \mathbb{E} \int_0^{\tau^T_{U,V}} |J_k| dt .
\]

Using the fact \( H^{s'} \) is an algebra, we have
\[
|J_2| \leq \| w \|_{H^{s'}} \| u_1 \|_{H^{s'}} .
\]

When \( d \geq 3 \), it follows from Lemma 2.1 and integration by parts that
\[
|J_3| \leq \| \nabla u_1 \|_{L_2(U)} \| w \|_{H^{s'}} + \| \nabla u_2 \|_{L_2(U)} \| w \|_{H^{s'}}
\]
\[
\leq \| \nabla u_1 \|_{L_2(U)} \| w \|_{L^2(U)} \| w \|_{H^{s'}} + \| \nabla u_2 \|_{L_2(U)} \| w \|_{H^{s'}} .
\]
Using the facts $H^s \hookrightarrow W^{s-1, \frac{2^d}{s}} \hookrightarrow W^{s', \frac{2^d}{s'}}$, $H^{s'} \hookrightarrow W^{1,d}$ and $H^s \hookrightarrow W^{1,\infty}$, we find
\[ |J_3| \leq ||u_2||_{H^s} ||w||^2_{H^{s'}} \quad d \geq 3. \] (4.28)

When $d = 2$, since $1 < s' < \min\{2, s - 1\}$, we let $p > 2$ such that $s' - 1 = 1 - \frac{2}{p}$ and then find $q \in (2, \infty)$ by solving $\frac{1}{q} = \frac{1}{s} + \frac{1}{p}$. Then we have $H^s \hookrightarrow W^{s-1+\frac{2}{q}, q} \hookrightarrow W^{s', q}$, $H^{s'} \hookrightarrow W^{1,p}$ and $H^s \hookrightarrow W^{1,\infty}$. As a result,
\[ |J_3| \lesssim \|D^{s'} u_2\|_{L^q} \|\nabla u\|_{L^p} \|w\|_{H^{s'}}^2 + \|\nabla u_2\|_{L^\infty} \|w\|_{H^{s'}}^2 \lesssim ||u_2||_{H^s} ||w||^2_{H^{s'}}, \quad d = 2. \] (4.29)

From Hypothesis I and Lemma 2.6, we have that for some locally bounded function $\Phi(t)$ satisfying $\Phi(t) > 1 + h_2(t)$,
\[ |J_4| + |J_5| \lesssim \Phi(t) \left[ (||u_1||_{H^s} + ||u_2||_{H^{s'}}) + \gamma^2 (||u_1||_{H^s} + ||u_2||_{H^{s'}}) \right] ||w||^2_{H^{s'}.} \] (4.30)

Therefore we combine (4.26)-(4.30) and (4.23) to arrive at
\[ \sum_{i=2}^5 \mathbb{E} \int_0^T \! |J_i| \! dt \leq C \int_0^T \! \Phi(t) \mathbb{E} \sup_{t' \in [0, \tau_{K,2}^w]} \|w(t')\|_{H^{s'}}^2 dt, \quad C = C(M). \]

As a result, we find that for some $C = C(M, T)$,
\[ \mathbb{E} \sup_{t \in [0, \tau_{K,2}^w]} \|w(t)\|_{H^{s'}}^2 \leq 2 \mathbb{E} \|w(0)\|_{H^{s'}}^2 + C \int_0^T \! \Phi(t) \mathbb{E} \sup_{t' \in [0, \tau_{K,2}^w]} \|w(t')\|_{H^{s'}}^2 \! dt. \]

It follows from the Grönwall’s inequality that
\[ \mathbb{E} \sup_{t \in [0, \tau_{K,2}^w]} \|w(t)\|_{H^{s'}}^2 \leq C(M, T) \mathbb{E} \|w(0)\|_{H^{s'}}^2, \]
which is (4.24). \hfill \Box

Now we establish the pathwise uniqueness for the original problem (1.6).

**Lemma 4.4.** Let $s > d/2 + 1$ with $d \geq 2$, $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{W})$ be a fixed stochastic basis and Hypothesis I be verified. Let $u_0$ be an $H^s$-valued $\mathcal{F}_0$-measurable random variable (relative to $S$) satisfying $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$. If $(S, u_1, \tau_1)$ and $(S, u_2, \tau_2)$ are two local pathwise solutions to (1.6) satisfying $u_i(\cdot \wedge \tau_i) \in L^2(\Omega; C([0, \infty); H^s))$ for $i = 1, 2$ and $\mathbb{P}(u_1(0) = u_2(0) = u_0(x)) = 1$, then
\[ \mathbb{P} \left\{ u_1(t, x) = u_2(t, x), \quad \forall (t, x) \in [0, \tau_1 \wedge \tau_2) \times T^d \right\} = 1. \]

**Proof.** We first assume that $\|u_0||_{H^s} < M$, $P$-a.s. for some deterministic $M > 0$. Let $K > 2M$, $T > 0$ and define
\[ \tau_K^T := \inf \left\{ t \geq 0 : \|u_1(t)||_{H^s} + \|u_2(t)||_{H^s} > K \right\} \wedge T. \] (4.31)

Then one can repeat the proof for (4.24) by using $\tau_K^T$ instead of $\tau_{K,w}^T$ to find that for any $K > 2M$ and any $T > 0$,
\[ \mathbb{E} \sup_{t \in [0, \tau_K^T]} \|u_1(t) - u_2(t)||_{H^{s'}}^2 \leq C(K,T) \mathbb{E} \|u_1(0) - u_2(0)||_{H^{s'}}^2 = 0. \]

Hence
\[ \mathbb{E} \sup_{t \in [0, \tau_1 \wedge \tau_2]} \|u_1(t) - u_2(t)||_{H^{s'}}^2 = 0. \] It is easy to see that
\[ \mathbb{P} \left\{ \lim_{K \to \infty} \inf_{t \in [0, \tau_1 \wedge \tau_2]} \tau_K > \tau_1 \wedge \tau_2 \right\} = 1. \] (4.32)

Sending $K \to \infty$ and using the monotone convergence theorem and (4.32) yield the desired result. Now we remove the restriction that $u_0$ is almost surely bounded. Motivated by [33, 34], for general $H^s$-valued $\mathcal{F}_0$-measurable initial value such that $\mathbb{E}\|u_0\|_{H^s}^2 < \infty$, we consider the decomposition $\bigcup_{k \geq 1} \Omega_k = \Omega$, where $\Omega_k = \{ k - 1 \leq \|u_0||_{H^s} < k \}$, $k \in \mathbb{N}$, $k \geq 1$. Then we see that
\[ u_0(\omega, x) = \sum_{k \geq 1} u_0,k(\omega, x) \triangleq \sum_{k \geq 1} u_0(\omega, x) 1_{\Omega_k}. \]

Moreover, $u_1 = u_1 \times 1 = u_1 \times \left( \sum_{k \geq 1} 1_{\Omega_k} \right) = \sum_{k \geq 1} u_1 1_{\Omega_k}$ and similarly $\tau_1 = \sum_{k \geq 1} \tau_1 1_{\Omega_k}$. Let $(u(k), \tau(k))$ be the solution to (1.6) with initial data $u_0,k$. Since $\Omega_k \cap \Omega_{k'} = \emptyset$ for $k \neq k'$, we can infer from the assumption $B(t, 0) = 0, F(0) = 0$ that $(u_1 1_{\Omega_k}, \tau_1 1_{\Omega_k})$ is also a solution with initial data $u_0,k$. Hence the previous step means that $u_1 1_{\Omega_k} = u(k) = u(k) 1_{\Omega_k}$ on $[0, \tau_1 1_{\Omega_k} \wedge \tau(k) 1_{\Omega_k}]$ almost surely. Therefore we can
assume that \( \tau(k) \Omega_k \geq \tau_1 \Omega_k \), \( \mathbb{P} - a.s. \), otherwise we can extend \( u(k) \Omega_k \) on \([0, \tau_2 \Omega_k]\) almost surely, then

\[
P \{ u_1 = u_2, \ t \in [0, \tau_1 \wedge \tau_2] \} \geq \mathbb{P} \left\{ \bigcup_{k \in \mathbb{N}, \ k \geq 1} \Omega_k \right\} = \mathbb{P} \{ \Omega \} = 1,
\]

which completes the proof. \( \square \)

**Remark 4.1.** We remark that if we only focus on uniqueness, the estimate \( \mathbb{E} \sup_{t \in [0, \tau_k^+]} \| w(t) \|^2_{L^2} = 0 \) is already enough. However, the estimate (4.24) in \( H^s \) with \( s' \in \left( \frac{d}{2}, \min \left\{ s - 1, \frac{d}{2} + 1 \right\} \right) \) will be needed in Section 5 to extend the range of \( s \).

Similarly, for the cut-off problem (4.1), we also have the pathwise uniqueness.

**Lemma 4.5.** Let \( s > d/2 + 3 \) with \( d \geq 2 \) and Hypothesis I be satisfied. Assume that \((\mathcal{S}, u_1, \infty)\) and \((\mathcal{S}, u_2, \infty)\) are two solutions, on the same basis \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W}) \), to (4.1) such that \( \mathbb{P} \{ u_1(0) = u_2(0) = u_0(x) \} = 1 \), then

\[
P \{ u_1(t, x) = u_2(t, x), \ \forall (t, x) \in [0, \infty) \times \mathbb{R}^d \} = 1. \tag{4.33}
\]

**Proof.** In this case, we will prove \( \mathbb{E} \sup_{t \in [0, \tau_k^+]} \| u_1(t) - u_2(t) \|^2_{H^{s-1}} = 0 \). Actually, \( (u_2 \cdot \nabla) w, w \) can be handled more easily since \( H^{s-2} \hookrightarrow W^{1, \infty} \), i.e.,

\[
(|(u_2 \cdot \nabla) w, w|)_{H^{s-1}} \lesssim \|D^{s-1} u_2 \|_{L^2} \| \nabla w \|_{L^\infty} + \| \nabla u_2 \|_{L^\infty} \| w \|_{H^{s-1}}^2 \lesssim \| u_2 \|_{H^s} \| w \|_{H^{s-1}}^2.
\]

For the additional terms coming from the cut-off function \( \chi_R(\cdot) \), one can apply the mean value theorem to obtain

\[
|\chi_R(\| u_1 \|_{W^{1, \infty}}) - \chi_R(\| u_2 \|_{W^{1, \infty}})| \leq C \| u_1 - u_2 \|_{W^{1, \infty}} \leq C \| u_1 - u_2 \|_{H^{s-1}}.
\]

Then, one can modify the proof for Lemma 4.3 to get \( \mathbb{E} \sup_{t \in [0, \tau_k^+]} \| u_1(t) - u_2(t) \|^2_{H^{s-1}} = 0 \) and then proceed along the same lines as in Lemma 4.4 to obtain the desired result. \( \square \)

### 4.5. Regular pathwise solution to the cut-off problem

Then we can prove the existence and uniqueness of a smooth pathwise solution to (4.1). To be more precise, we are going to prove the following result.

**Proposition 4.3.** Let \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W}) \) be a fixed stochastic basis. Suppose that \( F(\cdot) \) and \( B(t, \cdot) \) satisfy Hypothesis I. Let \( s > d/2 + 3 \) with \( d \geq 2 \), \( r > 4 \) and \( u_0 \) be an \( H^s \)-valued \( F_0 \)-measurable random variable such that \( \mathbb{E} \| u_0 \|^2_{H^s} < \infty \). Then (4.1) has a unique global pathwise solution in the sense of Definitions 1.2–1.3.

**Proof.** Let \( \mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{W}) \) be given and let \( u_\varepsilon \) be the global pathwise solution to (4.2). We define sequences of measures \( \nu_{\varepsilon_1, \varepsilon_2} \) and \( \mu_{\varepsilon_1, \varepsilon_2} \) as

\[
\nu_{\varepsilon_1, \varepsilon_2}(\cdot) = \mathbb{P} \left\{ (u_{\varepsilon_1, \varepsilon_2}) \in \cdot \right\} \text{ on } X_u^s \times X_u^s,
\]

\[
\mu_{\varepsilon_1, \varepsilon_2}(\cdot) = \mathbb{P} \left\{ (u_{\varepsilon_1, \varepsilon_2}, W) \in \cdot \right\} \text{ on } X_u^s \times X_u^s \times X_W,
\]

where \( X_u^s \) and \( X_W \) are given in (4.14). Let \( \left\{ \nu_{\varepsilon_1, \varepsilon_2}^k \right\}_{k \in \mathbb{N}} \) be an arbitrary subsequence of \( \left\{ \nu_{\varepsilon_1, \varepsilon_2} \right\} \) such that \( \varepsilon_k \to 0 \) as \( k \to \infty \). With minor modifications in the proof for Lemma 4.1, the tightness of \( \left\{ \nu_{\varepsilon_1, \varepsilon_2}^k \right\}_{k \in \mathbb{N}} \) can be obtained. Then by Lemma 2.8, one can find a probability space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \) on which there is a sequence of random variables \( \left\{ u_{\varepsilon_1, \varepsilon_2}^k, W_k \right\} \) and a random variable \( \left\{ \mu_{\varepsilon_1, \varepsilon_2}, W \right\} \) such that

\[
\lim_{k \to \infty} \left( u_{\varepsilon_1, \varepsilon_2}^k, W_k \right) = \left( \mu_{\varepsilon_1, \varepsilon_2}, W \right) \text{ in } X_{u_\varepsilon}^{s-1} \times X_{u_\varepsilon}^{s-1} \times X_W \quad \mathbb{P} - a.s.
\]

Notice that \( \nu_{\varepsilon_1, \varepsilon_2}^k \) also converges weakly to a measure \( \nu \) on \( X_{u_\varepsilon}^{s-2} \times X_{u_\varepsilon}^{s-2} \) defined by \( \nu(\cdot) = \hat{\mathbb{P}} \left\{ (\mu_{\varepsilon_1, \varepsilon_2}, \nu) \in \cdot \right\} \). Similar to Proposition 4.2, we see that both \( \left( \hat{S}, u_\varepsilon, \infty \right) \) and \( \left( \hat{S}, \nu, \infty \right) \) are martingale solutions to (4.1). Moreover, since \( u_\varepsilon(0) \equiv u_0 \) for all \( \varepsilon \), it is easy to obtain that \( \mu(0) = \nu(0) \) almost surely in \( \hat{\Omega} \) (cf. [4, page 210] for example). Then we use Lemma 4.5 to see

\[
\nu \left\{ (\mu, \nu) \in X_{u_\varepsilon}^{s-2} \times X_{u_\varepsilon}^{s-2} \right\} = 1.
\]
Then Lemma 2.10 can be used to show that the original sequence $u_\varepsilon$ defined on the initial probability space $(\Omega, \mathcal{F}, P)$ has a subsequence converging almost surely to a random variable $u$ in $X_u^{s-2}$. Repeating the procedure in Proposition 4.2 again, we obtain the unique global pathwise solution to (4.1). \hfill \Box

4.6. Final proof for Theorem 4.1. According to Proposition 4.3, to prove Theorem 4.1, we need to remove the cut-off function and the $r$-th order moment restriction of $u_0$. The method used here is inspired by the works [34, 33].

Proof for Theorem 4.1. Let $u_0(\omega, x) \in L^2(\Omega; H^s)$ and

\[ \Omega_k = \{ k - 1 \leq \| u_0 \|_{H^r} < k \}, \quad k \in \mathbb{N}, \quad k \geq 1. \]

Since $\mathbb{E}[\| u_0 \|_{H^s}^2] < \infty$, we have $1 = \sum_{k \geq 1}^\infty \Omega_k \mathbb{P} - \text{a.s.}$, which means that

\[ u_0(\omega, x) = \sum_{k \geq 1}^\infty u_{0,k}(\omega, x) = \sum_{k \geq 1}^\infty u_{0,k}(\omega) 1_{k-1 \leq \| u_0 \|_{H^r} < k} \mathbb{P} - \text{a.s.} \]

On account of Proposition 4.3, we let $u_{k,R}$ be the pathwise unique global solution to the cut-off problem (4.1) with initial value $u_{0,k}$ and cut-off function $\chi_R(\cdot)$. Define

\[ \tau_{k,R} = \inf \left\{ t > 0 : \sup_{\omega \in [0,t]} \| u_{k,R}(\omega') \|_{H^s}^2 > \| u_{0,k} \|_{H^s}^2 + 2 \right\}. \tag{4.34} \]

Then for any $R > 0$, we have $\mathbb{P}\{\tau_{k,R} > 0, \forall k \geq 1\} = 1$. Now we let $R = R_k$ be discrete and then denote $(u_k, \tau_k) = (u_{k,R_k}, \tau_{k,R_k})$. If $R_k^2 > \| u_{0,k} \|_{H^s}^2 + 2$, then $\mathbb{P}\{\tau_k > 0, \forall k \geq 1\} = 1$ and

\[ \mathbb{P}\{ \| u_k \|_{H^{s-2}}^2 \leq \| u_k \|_{H^s}^2 \leq \| u_{0,k} \|_{H^s}^2 + 2 < R_k^2, \forall t \in [0, \tau_k], \forall k \geq 1\} = 1, \]

which means

\[ \mathbb{P}\{ \chi_{R_k}(\| u_k \|_{H^{s-2}}) = 1, \forall t \in [0, \tau_k], \forall k \geq 1\} = 1. \]

Therefore $(u_k, \tau_k)$ is the unique pathwise solution to (1.6) with initial value $u_{0,k}$. Since $\bigcup_{k \in \mathbb{N}, \ k \geq 1} \Omega_k$ is a set of full measure, $F(0) = 0$ (cf. (1.2)), $B(t, 0) = 0$ (cf. Hypothesis 1) and $\Omega_k \cap \Omega_{k'} = \emptyset$ with $k \neq k'$, direct computation shows that

\[ (u = \sum_{k \geq 1}^\infty u_{0,k} 1_{k-1 \leq \| u_0 \|_{H^r} < k}, \quad \tau = \sum_{k \geq 1}^\infty \tau_k 1_{k-1 \leq \| u_0 \|_{H^r} < k}) \]

is the unique pathwise solution to (1.6) corresponding to the initial condition $u_0$. Besides, using (4.34), we have

\[ \sup_{t \in [0,\tau]} \| u \|_{H^r}^2 = \sum_{k \geq 1}^\infty 1_{k-1 \leq \| u_0 \|_{H^r} < k} \sup_{t \in [0,\tau_k]} \| u_k \|_{H^s}^2 \leq \sum_{k \geq 1}^\infty 1_{k-1 \leq \| u_0 \|_{H^r} < k} \left( \| u_{0,k} \|_{H^s}^2 + 2 \right) \leq \| u_0 \|_{H^s}^2 + 1. \]

Taking expectation gives rise to (1.17). Finally, the passage, from $(u, \tau)$ to a maximal pathwise solution in the sense of Definition 1.2, may be carried out as in [19, 34, 33, 57]. We omit the details here for brevity. \hfill \Box

5. Proof for Theorem 1.1

Now we are in the position to prove Theorem 1.1. With Theorem 4.1 in hand, when $s > d/2 + 1$ with $d \geq 2$, we first consider the following problem

\[ \begin{cases} du + [(u \cdot \nabla)u + F(u)] dt = B(t, u) dW, \quad x \in \mathbb{T}^d, \quad t > 0, \\ u(\omega, 0, x) = J_0 u_0(\omega, x) \in H^\infty, \quad x \in \mathbb{T}^d. \end{cases} \tag{5.1} \]

where $u_0$ is an $H^s$-valued initial process such that $\| u_0 \|_{H^s} < M$ for some $M > 0$. Let $\varepsilon = \frac{1}{k}$ with $k \in \mathbb{N}$. Theorem 4.1 shows that for a given stochastic basis $S = (\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0}, W)$, (5.1) admits a unique solution $(u_k, \tau_k^*)$ such that for any $\eta > 3$, $u_k \in C([0, \tau_k^*), H^\eta) \mathbb{P} - \text{a.s.}$ Moreover, (2.5) implies that

\[ \sup_{k \in \mathbb{N}} \| J_0 u_0 \|_{H^r} \leq M. \tag{5.2} \]

Motivated by [34, 32], we are going to show that $u_k$ is a Cauchy sequence, as $k \to \infty$, in $C([0, \tau], H^s)$ for some almost surely positive stopping time $\tau$ and $s > d/2 + 1$ with $d \geq 2$. To this end, we will first prove the following results.
Lemma 5.1. For any $T > 0$, $s > d/2 + 1$ with $d ≥ 2$ and $s' ∈ (\frac{d}{2}, \min\{s - 1, \frac{d}{2} + 1\})$, we let

$$τ^T_k := \inf \left\{ t ≥ 0 : \|u_k\|_{H^s} ≥ \|J^\perp_k u_0\|_{H^s} + 2 \right\} ∧ T, \tag{5.3}$$

and for $k, m > 1$, we define

$$τ^T_{k,m} = τ^T_k ∧ τ^T_m. \tag{5.4}$$

Then $w_{m,k} = u_m - u_k$ with $m, k > 1$ satisfies

$$E \sup_{t \in [0,τ^T_{k,m}]} \|w_{m,k}(t)\|_{H^s}^2 ≤ E \left\{ \|w_{m,k}(0)\|_{H^s}^2 + \|w_{m,k}(0)\|_{H^{s+1}}^2 \right\} + E \sup_{t \in [0,τ^T_{k,m}]} \|w_{m,k}(t)\|_{H^{s+1}}^2. \tag{5.5}$$

Proof. Notice that $w_{m,k}$ satisfies

$$dw_{m,k} + [(w_{m,k} \cdot \nabla)u_m + (u_k \cdot \nabla)w_{m,k}]dt + [F(u_m) - F(u_k)]dt = (B(t, u_m) - B(t, u_k))dW$$

with $w_{m,k}(0) = J^\perp_k u_0 - J^\perp_k u_0 ∈ H^\infty$. Applying the Itô formula gives rise to

$$d\|w_{m,k}\|_{H^s}^2 = 2 \sum_{j=1}^2 \left( \int_0^T \left[ (\partial_t \|w_{m,k}\|_{H^s}) e_j, w_{m,k} \right]_H^s \, dW_j \right. \tag{5.5}$$

Remember that $s' ∈ (\frac{d}{2}, \min\{s - 1, \frac{d}{2} + 1\})$. Then one can use Lemma 2.1 to find that

$$|A_{2,s}| ≤ \|w_{m,k}\|_{H^s}^2 \|\nabla u_m\|_{H^s} + \|w_{m,k}\|_{H^{s+1}}^2 \|u_m\|_{H^{s+1}},$$

and

$$|A_{3,s}| ≤ \|u_k\|_{H^s} \|\nabla w_{m,k}\|_{L\infty} \|w_{m,k}\|_{H^s} + \|\nabla u_k\|_{L\infty} \|w_{m,k}\|_{H^s} \|w_{m,k}\|_{H^{s+1}}^2.$$
Combining the above estimates into (5.5), and using the Grönwall’s inequality, we have that for some $C = C(M, T)$,

$$
E \sup_{t \in \left[0, \tau_{k,m}^T \right]} \| w_{m,k}(t) \|^2_{H^s} \leq C E \| w_{m,k}(0) \|^2_{H^s} + C E \sup_{t \in \left[0, \tau_{k,m}^T \right]} \| w_{m,k}(t) \|^2_{H^s} \| u_m(t) \|^2_{H^{s+1}},
$$

(5.6)

where $s' \in \left( \frac{d}{2}, \min \{ s - 1, \frac{d}{2} + 1 \} \right)$. Now we estimate $E \sup_{t \in \left[0, \tau_{k,m}^T \right]} \| w_{m,k}(t) \|^2_{H^s} \| u_m(t) \|^2_{H^{s+1}}$. We first use the Itô formula to deduce that for any $\rho > 0$,

$$
d\| w_m \|^2_{H^s} = 2 \sum_{i=1}^{\infty} (B(t, u_m) c_i, u_m)_{H^s} \, dw_i - 2 \left( D^\rho [((u_m \cdot \nabla) u_m), D^\rho u_m] \right)_{L^2} \, dt
$$

$$
= \sum_{i=1}^{\infty} D_{i,\rho, t} \, dw_i + \sum_{i=2}^{4} D_{i,\rho, t} \, dt.
$$

(5.7)

As a result, by the Itô product rule for (5.5) and (5.7) with $s > d/2 + 1$ with $d \geq 2$, we have

$$
d\| w_{m,k} \|^2_{H^s} \| u_m \|^2_{H^{s+1}} = \sum_{j=1}^{\infty} \left( \| w_{m,k} \|^2_{H^s} \| D_{1,s+1,j} + \| u_m \|^2_{H^{s+1}}, A_{1,s',j} \right) \, dw_j
$$

$$
+ \sum_{j=2}^{4} \| w_{m,k} \|^2_{H^s} \| D_{1,s+1,j} \| dt + \sum_{j=2}^{5} \| u_m \|^2_{H^{s+1}, A_{1,s'}} \| dt
$$

$$
+ \sum_{j=1}^{\infty} A_{1,s',j} D_{1,s+1,j} \, dt.
$$

Therefore for any $T > 0$ and $t \in \left[0, \tau_{k,m}^T \right]$, using BDG inequality as before, we arrive at

$$
E \sup_{t \in \left[0, \tau_{k,m}^T \right]} \| w_{m,k} \|^2_{H^s} \| u_m \|^2_{H^{s+1}} - E \| w_{m,k}(0) \|^2_{H^s} \| u_m(0) \|^2_{H^{s+1}}
$$

$$
\leq C E \left( \int_0^{\tau_{k,m}^T} \| w_{m,k} \|^4_{H^s} \| B(t, u_m) \|^2_{L^2(U, H^{s+1})} \| u_m \|^2_{H^{s+1}} \, dt \right)^{\frac{1}{2}}
$$

$$
+ C E \left( \int_0^{\tau_{k,m}^T} \| u_m \|^4_{H^{s+1}} \| B(t, u_m) \|^2_{L^2(U, H^{s'})} \| w_{m,k} \|^2_{H^s} \, dt \right)^{\frac{1}{2}}
$$

$$
+ \sum_{i=2}^{4} E \int_0^{\tau_{k,m}^T} \| w_{m,k} \|^2_{H^s} \| D_{1,s+1} \| dt + \sum_{i=2}^{5} E \int_0^{\tau_{k,m}^T} \| u_m \|^2_{H^{s+1}, A_{1,s'}} \| dt
$$

$$
+ E \int_0^{\tau_{k,m}^T} \sum_{j=1}^{\infty} |A_{1,s',j} D_{1,s+1,j}| \, dt.
$$

(5.8)

After using Lemma 2.1, Hypothesis I, Lemma 2.6 and the embedding of $H^s \hookrightarrow W^{1,\infty}$ for $s > d/2 + 1$, we can then apply (5.2), (5.3) and (5.4) to the resulting inequality to obtain that for some $C = C(M)$ and some $\Psi_1(t)$,

$$
\sum_{i=2}^{4} E \int_0^{\tau_{k,m}^T} \| w_{m,k} \|^2_{H^s} \| D_{1,s+1} \| dt
$$

$$
\leq C E \int_0^{\tau_{k,m}^T} \| w_{m,k} \|^2_{H^s} \| u_m \|_{H^s} \| u_m \|^2_{H^{s+1}} \| h_1(t) f^2(\| u_m \|_{H^s})(1 + \| u_m \|^2_{H^{s+1}}) \| dt
$$

$$
\leq C \int_0^{\tau_{k,m}^T} \Psi(t) E \sup_{t' \in \left[0, \tau_{k,m}^T \right]} \| w_{m,k}(t') \|^2_{H^s} \| u_m(t') \|^2_{H^{s+1}} \, dt + C \int_0^{\tau_{k,m}^T} \Psi(t) E \sup_{t' \in \left[0, \tau_{k,m}^T \right]} \| w_{m,k}(t') \|^2_{H^s} \, dt.
$$

Repeating the estimates for (4.27), (4.28), (4.29) and (4.30), and then using (5.3) and (5.4), we have

$$
\sum_{i=2}^{5} E \int_0^{\tau_{k,m}^T} \| u_m \|^2_{H^{s+1}, A_{1,s'}} \| dt \leq C \int_0^{\tau_{k,m}^T} \Psi(t) E \sup_{t' \in \left[0, \tau_{k,m}^T \right]} \| u_m(t') \|^2_{H^{s+1}}, \| w_{m,k}(t') \|^2_{H^s} \, dt, \quad C = C(M).
$$
We can infer from Hypothesis I, (5.2), (5.3) and (5.4) that for some \( C = C(M) \),

\[
\begin{align*}
CE & \left( \int_0^T \left| \sup_{t \in [0, T]} |w_{m,k}(t)| \right|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \leq \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\leq & \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 + C \int_0^T h_1(t) \sup_{\nu \in [0, T]} \left| \sup_{t \in [0, T]} |w_{m,k}(t')| \right|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\leq & \frac{1}{4} \left( \sup_{t \in [0, T]} \|w_{m,k}\|_{H^{r+1}}^2 \|u_m\|_{H^{r+1}}^2 + C \int_0^T h_1(t) \sup_{\nu \in [0, T]} \left| \sup_{t \in [0, T]} |w_{m,k}(t')| \right|_{H^{r+1}}^2 \right) \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\end{align*}
\]

and

\[
\begin{align*}
CE & \left( \int_0^T \left| \sup_{t \in [0, T]} |w_{m,k}(t)| \right|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \leq \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\leq & \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \leq \frac{1}{4} \left( \sup_{t \in [0, T]} \|w_{m,k}\|_{H^{r+1}}^2 \|u_m\|_{H^{r+1}}^2 + C \int_0^T h_1(t) \sup_{\nu \in [0, T]} \left| \sup_{t \in [0, T]} |w_{m,k}(t')| \right|_{H^{r+1}}^2 \right) \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\end{align*}
\]

Finally, it follows from the Hölder inequality and Hypothesis I that

\[
\sum_{j=1}^{\infty} |A_{1,s', s,j} D_{1,s+1,j}| \leq \left( \sum_{j=1}^{\infty} \|B(t, u_m) - B(t, u_k)\|_{H^{r+1}}^2 \|w_{m,k}\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \leq \frac{1}{4} \left( \sup_{t \in [0, T]} \|w_{m,k}\|_{H^{r+1}}^2 \|u_m\|_{H^{r+1}}^2 + C \int_0^T h_1(t) \sup_{\nu \in [0, T]} \left| \sup_{t \in [0, T]} |w_{m,k}(t')| \right|_{H^{r+1}}^2 \right) \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\]

Consequently, we have that for some \( C = C(M) \) and some locally bounded function \( \Phi(t) \),

\[
\begin{align*}
\mathbb{E} & \left( \int_0^T \left| \sup_{t \in [0, T]} |w_{m,k}(t)| \right|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \leq \frac{1}{4} \left( \sup_{t \in [0, T]} \|w_{m,k}\|_{H^{r+1}}^2 \|u_m\|_{H^{r+1}}^2 + C \int_0^T \|w_{m,k}(t')\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\end{align*}
\]

Combining the above estimates into (5.8), we have that for some \( C = C(M, T) > 0 \),

\[
\begin{align*}
\mathbb{E} \sup_{t \in [0, T]} \|w_{m,k}\|_{H^{r+1}}^2 \|u_m\|_{H^{r+1}}^2 \leq 2 \mathbb{E} \|w_{m,k}(0)\|_{H^{r+1}}^2 \|u_m(0)\|_{H^{r+1}}^2 + C \int_0^T \mathbb{E} \|w_{m,k}(t')\|_{H^{r+1}}^2 \right) \left( \int_0^T \|w_{m,k}\|_{L^2(U; H^{r+1})}^2 \|u_m\|_{H^{r+1}}^2 \right)^{\frac{1}{2}} \\
\end{align*}
\]
Then we see that for some $C = C(M, T) > 0$,

$$E \sup_{t \in [0, \tau^T_{k,m}]} \|w_{m,k}(t)\|_{L^2} \leq CE \|w_{m,k}(0)\|_{H^{s'}} + CE \sup_{t \in [0, \tau^T_{k,m}]} \|w_{m,k}(t)\|_{H^{s'}}. \tag{5.9}$$

Combining (5.6) and (5.9), we obtain the desired result. \hfill \square

**Lemma 5.2.** Let $\tau^T_k, \tau^T_{k,m}$ be defined as in (5.3),(5.4). Then $\{u_k\}_{k \in \mathbb{N}}$ satisfies

$$\lim_{m \to \infty} \sup_{k \geq m} \mathbb{E} \sup_{t \in (0, \tau^T_{k,m})} \|u_k - u_m\|_{H^s} = 0, \tag{5.10}$$

and

$$\lim_{K \to 0} \sup_{k \geq 1} \mathbb{P} \left\{ \sup_{t \in [0, \tau^T_{k,m} \wedge K]} \|u_k\|_{H^{s'}} \geq \|J_\Phi^k u_0\|_{H^{s'}} + 1 \right\} = 0. \tag{5.11}$$

**Proof.** Recalling Lemma 5.1, we have

$$E \sup_{t \in [0, \tau^T_{k,m}]} \|w_{m,k}(t)\|_{H^{s'}} \leq E \left\{ \|w_{m,k}(0)\|_{H^{s'}}^2 + \|w_{m,k}(0)\|_{H^{s'}}^2 \|u_m(0)\|_{H^{s'}}^2 \right\} \leq E \sup_{t \in [0, \tau^T_{k,m}]} \|w_{m,k}(t)\|_{H^{s'}}^2. \tag{5.12}$$

We notice that (2.1) and Lemma 4.3 yield

$$\lim_{m \to \infty} \sup_{k \geq m} \mathbb{E} \|w_{m,k}(0)\|_{H^{s'}}^2 = 0, \tag{5.13}$$

and

$$\lim_{m \to \infty} \sup_{k \geq m} \mathbb{E} \sup_{t \in [0, \tau^T_{k,m}]} \|w_{m,k}(t)\|_{H^{s'}}^2 \leq C(M, T) \lim_{m \to \infty} \sup_{k \geq m} \mathbb{E} \|w_{m,k}(0)\|_{H^{s'}}^2 = 0. \tag{5.14}$$

Moreover, it follows from (5.2), (2.1) and (2.2) that

$$\sup_{k \geq m} \|w_{m,k}(0)\|_{H^{s'}} \|u_m(0)\|_{H^{s'}} \sim o \left( \left( \frac{1}{m} \right)^{s-s'} \right), \tag{5.15}$$

which gives

$$\lim_{m \to \infty} \sup_{k \geq m} \mathbb{E} \|w_{m,k}(0)\|_{H^{s'}}^2 \|u_m(0)\|_{H^{s'}}^2 = 0. \tag{5.16}$$

Combining (5.12)–(5.16), we obtain (5.10). As for (5.11), we recall (5.7) to obtain that for any $K > 0$,

$$\sup_{t \in [0, \tau^T_{k,m} \wedge K]} \|u_k(t)\|_{H^{s'}}^2 \leq \|J_\Phi^k u_0\|_{H^{s'}}^2 + \sup_{t \in [0, \tau^T_{k,m} \wedge K]} \left| \int_0^t \sum_{j=1}^{\infty} D_{1,s,j} \mathbb{d}W_j \right| + \frac{4}{\tau^T_{k,m} \wedge K} \int_0^{\tau^T_{k,m} \wedge K} |D_{1,s}| \mathbb{d}t. \tag{5.17}$$

As a result, we have

$$\mathbb{P} \left\{ \sup_{t \in [0, \tau^T_{k,m} \wedge K]} \|u_k(t)\|_{H^{s'}}^2 > \|J_\Phi^k u_0\|_{H^{s'}}^2 + 1 \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, \tau^T_{k,m} \wedge K]} \left| \int_0^t \sum_{j=1}^{\infty} D_{1,s,j} \mathbb{d}W_j \right| > \frac{1}{2} \right\} + \mathbb{P} \left\{ \frac{4}{\tau^T_{k,m} \wedge K} \int_0^{\tau^T_{k,m} \wedge K} |D_{1,s}| \mathbb{d}t > \frac{1}{2} \right\}. \tag{5.18}$$

Using the Chebyshev inequality, Lemma 2.1, Hypothesis 1, the embedding of $H^s \hookrightarrow W^{1,\infty}$ for $s > d/2 + 1$, (5.3) and (5.2), we have

$$\mathbb{P} \left\{ \sum_{i=2}^{\tau^T_{k,m} \wedge K} |D_{1,s}| \mathbb{d}t > \frac{1}{2} \right\} \leq C \sum_{i=2}^{\tau^T_{k,m} \wedge K} |D_{1,s}| \mathbb{d}t$$

$$\leq CE \int_0^{\tau^T_{k,m} \wedge K} \left[ \|u_k\|_{H^{s'}}^3 + h_1^2(t) \|u_k\|_{H^{s'}} (1 + \|u_k\|_{H^{s'}})^2 \right] \mathbb{d}t$$

$$\leq CE \int_0^{\tau^T_{k,m} \wedge K} C(M, T) \mathbb{d}t \leq C(M, T) K. \tag{5.19}$$
Similarly, from the Doob’s maximal inequality and the Itô isometry, we have
\[
P \left\{ \sup_{t \in [0, \tau_k^T \wedge K]} \left| \int_0^t \sum_{j=1}^\infty D_{1,a,j}dW_j \right| > \frac{1}{2} \right\} \leq 4\mathbb{E} \left( \int_0^{\tau_k^T \wedge K} \sum_{j=1}^\infty D_{1,a,j}dW_j \right)^2 \\
\leq C\mathbb{E} \int_0^{\tau_k^T \wedge K} [h^2_1(t)f^2(\|u_k\|_{W^s})](1 + \|u_k\|_{H^s})^2 \|u_k\|_{H^s} dt \\
\leq C\mathbb{E} \int_0^{\tau_k^T \wedge K} C(M,T)dt \leq C(M,T)K,
\]
Hence we have
\[
P \left\{ \sup_{t \in [0, \tau_k^T \wedge K]} \|u_k(t)\|_{H^s}^2 > \|J_\frac{1}{2}u_0\|_{H^s}^2 + 1 \right\} \leq C(M,T)K,
\]
which gives (5.11).

\[\square\]

**Lemma 5.3** ([33], Lemma 5.1). Let \(\tau_k^T, \tau^T_{k,m}\) be defined as in (5.3) and (5.4). If \(\{u_k\}_{k \in \mathbb{N}}\) satisfies (5.10) and (5.11), then there is a stopping time \(\tau\) satisfying \(P \{0 < \tau \leq T\} = 1\) and a process \(u \in C([0, \tau], H^s)\) such that for some subsequence \(k_n\),
\[
\lim_{n \to \infty} \sup_{t \in [0, \tau]} \|u_{k_n} - u\|_{H^s} = 0 \quad P - a.s.
\]
Besides,
\[
\sup_{t \in [0, \tau]} \|u\|_{H^s} \leq \|u_0\|_{H^s} + 2 \quad P - a.s.
\]

Employing the above result, we can obtain the pathwise solution under the additional assumption that the initial process is almost surely bounded.

**Proposition 5.1.** Let \(S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)\) be a fixed stochastic basis and Hypothesis I be verified. Let \(s > d/2 + 1\) with \(d \geq 2\) and let \(u_0\) be an \(H^s\)-valued \(\mathcal{F}_0\)-measurable random variable such that \(\|u_0\|_{H^s} < M\) \(P\)-a.s. for some deterministic \(M > 0\). Then (1.6) has a unique pathwise solution in the sense of Definitions 1.2.

**Proof.** Lemmas 5.2 and 5.3 yield that for the solutions \(\{u_k\}_{k \in \mathbb{N}}\) to (5.1) with \(\varepsilon = \frac{1}{2}\), there is a stopping time \(\tau\) satisfying \(P \{0 < \tau \leq T\} = 1\) and a process \(u \in C([0, \tau], H^s)\) such that for some subsequence \(k_n\),
\[
\lim_{n \to \infty} \sup_{t \in [0, \tau]} \|u_{k_n} - u\|_{H^s} = 0 \quad P - a.s.
\]
With this almost sure convergence, we can repeat the method as in Proposition 4.2 to prove that \((u, \tau)\) is a pathwise solution, in the sense of Definitions 1.2, to (1.6). Uniqueness comes from Lemma 4.4. \(\square\)

Finally, we are in the position to finish the proof for Theorem 1.1.

**Proof for Theorem 1.1.** With Proposition 5.1 in hand, one can use the cutting argument as employed in the passage from Proposition 4.3 to Theorem 4.1 (subsection 4.6) to remove the boundedness assumption on initial data and to obtain (1.17). Besides, one may pass from the case of local to maximal pathwise solutions as in [34, 32, 57]. Here the details are omitted for simplicity. \(\square\)

6. Noise effect on the dependence on initial data

In this section, we consider the periodic boundary value problem (1.8), i.e.,
\[
\begin{align*}
\frac{d u}{dt} + \left[u \cdot \nabla\right] u + F(u) dt &= Q(t, u) dW, & t > 0, & x \in \mathbb{T}^d, \\
u(\omega, 0, x) &= u_0(\omega, x), & t > 0, & x \in \mathbb{T}^d.
\end{align*}
\]

Now we proceed to prove Theorem 1.3. We assume that for some \(R_0 \gg 1\), the \(R_0\)-exiting time is strongly stable at the zero solution. Then we will show that the solution map \(u_0 \mapsto u\) defined by (1.8) is not uniformly continuous. We will firstly assume that the dimension \(d \geq 2\) is even.
6.1. Estimates on the approximation solutions. Let $l \in \{-1,1\}$. Define divergence-free vector field as

$$u^{l,n} = (ln^{-1} + n^{-s} \cos \theta_1, ln^{-1} + n^{-s} \cos \theta_2, \ldots, ln^{-1} + n^{-s} \cos \theta_d),$$  \hspace{1cm} (6.1)

where $\theta_i = nx_{d+1-i} - lt$ with $1 \leq i \leq d$ and $n \in \mathbb{Z}^+$. Substituting $u^{l,n}$ into (1.8), we see that the error $E^{l,n}(t)$ can be defined as

$$E^{l,n}(t) = u^{l,n}(t) - u^{l,n}(0) + \int_0^t \left[ (u^{l,n} \cdot \nabla)u^{l,n} + F(u^{l,n}) \right] dt' - \int_0^t Q(t', u^{l,n}) dW.$$  \hspace{1cm} (6.2)

Now we analyze the error as follows.

**Lemma 6.1.** Let $d \geq 2$ be even and $s > 1 + \frac{d}{2} \geq 2$. For $\frac{d}{2} < \sigma < s - 1$, we have that for any $T > 0$ and $n \gg 1$,

$$\mathbb{E} \sup_{t \in [0,T]} \| E^{l,n}(t) \|_{H^s}^2 \leq Cn^{-2r_s}, \hspace{0.5cm} C = C(T),$$  \hspace{1cm} (6.3)

where

$$r_s = \begin{cases} 2s - \sigma - 1 & \text{if } 1 + \frac{d}{2} < s \leq 3, \\ s - \sigma + 2 & \text{if } s > 3. \end{cases}$$

**Proof.** Direct computation shows that

$$(u^{l,n} \cdot \nabla)u^{l,n} = (-ln^{-s} \sin \theta_1 - n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d}.$$  \hspace{1cm} (6.4)

which means that

$$u^{l,n}(t) - u^{l,n}(0) + \int_0^t (u^{l,n} \cdot \nabla)u^{l,n} dt' = \int_0^t (-n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d} dt'.$$

Then we have

$$E^{l,n}(t) + \int_0^t \left[ (n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d} - F(u^{l,n}) \right] dt' + \int_0^t Q(t, u^{l,n}) dW = 0.$$  \hspace{1cm} (6.5)

We notice that by Lemma 2.5,

$$\| (n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i})_{1 \leq i \leq d} \|_{H^s} \leq C \sum_{i=1}^d \| n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i} \|_{H^s} \lesssim n^{-2s+1+\sigma} \lesssim n^{-r_s}. \hspace{1cm} (6.6)$$

For $F() = F_{EP,1}() + F_2()$ given by (1.2), some calculations reveal that

$$F_{EP,1}(u^{l,n}) = n^{-2s+2} \times \begin{pmatrix} a_{11} + \frac{1}{2} \sum_{i=1}^d \sin^2 \theta_i \\ a_{22} + \frac{1}{2} \sum_{i=1}^d \sin^2 \theta_i \\ \ldots \\ a_{dd} + \frac{1}{2} \sum_{i=1}^d \sin^2 \theta_i \end{pmatrix},$$

where $a_{ii} = \sin \theta_i (\sin \theta_i + \sin \theta_{d+1-i}) - \sin^2 \theta_{d+1-i}$. Therefore

$$\text{div} F_{EP,1}(u^{l,n}) = n^{-2s+3} (\sin \theta_i \cos \theta_{d+1-i} - \sin \theta_{d+1-i} \cos \theta_{d+1-i})_{1 \leq i \leq d}.$$  \hspace{1cm} (6.7)

Similarly, since $\text{div} u^{l,n} = 0$, we have

$$F_2(u^{l,n}) = (-ln^{-s} \sin \theta_{d+1-i} - n^{-2s+1} \sin \theta_{d+1-i} \cos \theta_{d+1-i})_{1 \leq i \leq d}.$$  \hspace{1cm} (6.8)

Therefore

$$F(u^{l,n}) = (-I - \Delta)^{-1} P_l)_{1 \leq l \leq d},$$

where

$$P_l = \begin{pmatrix} n^{-2s+3} \sin \theta_i \cos \theta_{d+1-i} - \frac{n^{-2s+1} + n^{-2s+3}}{2} \sin 2\theta_{d+1-i} - ln^{-s} \sin \theta_{d+1-i} \end{pmatrix}.$$
Since \(-(I - \Delta)^{-1}\) is bounded from \(H^\sigma\) to \(H^{\sigma+2}\), we can use Lemma 2.5 to derive that

\[
\|F(u, l, n)\|_{H^\sigma} \leq C \sum_{i=1}^{d} \left( \|n^{-2s+3} \sin \theta_i \cos \theta_{d+1-i}\|_{H^{\sigma+2}} + \|n^{-2s+3} \sin 2\theta_{d+1-i}\|_{H^{\sigma+2}} \right) + C \sum_{i=1}^{d} \left( \|n^{-2s+1} \sin \theta_{d+1-i}\|_{H^{\sigma+2}} + \|n^{-s} \sin \theta_{d+1-i}\|_{H^{\sigma+2}} \right) \\
\lesssim n^{-2s+3+\sigma-2} + n^{-2s+1+\sigma-2} + n^{-s+\sigma-2} \lesssim n^{-\sigma}.
\]  

(6.6)

Then we can use the Itô formula to (6.4) to find that for any \(T > 0\) and \(t \in [0, T]\),

\[
E \sup_{t \in [0, T]} \|E^{l, n}(t)\|_{H^\sigma}^2 \leq E \sup_{t \in [0, T]} \left| \left( -2 \int_0^t Q(t', u, l, n) dW, E^{l, n} \right) \right|_{H^\sigma} + \sum_{i=2}^{4} \int_0^T E|J_i|dt,
\]

where

\[
J_2 = -2 \left( D^\sigma \left( n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i} \right) \right)_{1 \leq i \leq d}, D^\sigma E^{l, n}_{L^2},
\]

\[
J_3 = 2(D^\sigma F(u, l, n), D^\sigma E^{l, n})_{L^2},
\]

\[
J_4 = \|Q(t, u, l, n)\|_{L^2(U, H^\sigma)}^2.
\]

Using (1.13) and BDG inequality, we find that

\[
E \sup_{t \in [0, T]} \left| \left( -2 \int_0^t Q(t', u, l, n) dW, E^{l, n} \right) \right|_{H^\sigma} \leq 2E \left( \sup_{t \in [0, T]} \|E^{l, n}(t)\|_{H^\sigma}^2 \right) \left( \int_0^T \|F(u, l, n)\|_{H^\sigma} dt \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} E \sup_{t \in [0, T]} \|E^{l, n}(t)\|_{H^\sigma}^2 + CTn^{-2\sigma}.
\]  

(6.7)

We use (6.5) and (6.6) to find that,

\[
\int_0^T E|J_2|dt \leq C \int_0^T E \left( \|n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i}\|_{H^{\sigma+2}} \|E^{l, n}(t)\|_{H^\sigma} \right) dt \\
\leq C \int_0^T E \left( \|n^{-2s+1} \sin \theta_i \cos \theta_{d+1-i}\|_{H^\sigma}^2 \right) dt + C \int_0^T E \|E^{l, n}(t)\|_{H^\sigma}^2 dt \\
\leq CTn^{-2\sigma} + C \int_0^T E \|E^{l, n}(t)\|_{H^\sigma}^2 dt,
\]

\[
\int_0^T E|J_3|dt \leq C \int_0^T E \left( \|F(u, l, n)\|_{H^\sigma} \|E^{l, n}(t)\|_{H^\sigma} \right) dt \\
\leq C \int_0^T E \|F(u, l, n)\|_{H^\sigma} dt + C \int_0^T E \|E^{l, n}(t)\|_{H^\sigma}^2 dt \\
\leq CTn^{-2\sigma} + C \int_0^T E \|E^{l, n}(t)\|_{H^\sigma}^2 dt,
\]

and

\[
\int_0^T E|J_4|dt \leq C \int_0^T E \|F(u, l, n)\|_{H^\sigma}^2 dt \leq CTn^{-2\sigma}.
\]

Collecting the above estimates into (6.7), we arrive at

\[
E \sup_{t \in [0, T]} \|E^{l, n}(t)\|_{H^\sigma}^2 \leq CTn^{-2\sigma} + C \int_0^T E \sup_{t' \in [0, t]} \|E^{l, n}(t')\|_{H^\sigma}^2 dt.
\]

Then it follows from the Grönwall inequality that

\[
E \sup_{t \in [0, T]} \|E^{l, n}(t)\|_{H^\sigma}^2 \leq Cn^{-2\sigma}, \quad C = C(T),
\]

which is the desired result. □
6.2. Construction of actual solutions. Now we consider the problem (1.8) with deterministic initial data \( u^{l,n}(0, x) \), i.e.,

\[
\begin{align*}
\frac{du + [u \cdot \nabla] u + F(u)] dt &= Q(t, u) dW, \\
u(0, x) &= u^{l,n}(0, x),
\end{align*}
\]

where

\[
u^{l,n}(0, x) = (ln^{-1} + n^{-s} \cos nx_{d+1-i}), 1 \leq i \leq d.
\]

Since \( Q \) satisfies Hypothesis II, Theorem 1.1 means that for each \( n \), (6.8) has a uniqueness maximal solution \((u_{l,n}, \tau_{l,n}^R)\).

6.3. Estimates on the error.

**Lemma 6.2.** Let \( d \geq 2 \) be even, \( s > 1 + \frac{d}{2} \), \( \frac{d}{2} < \sigma < s - 1 \) and \( r_s > 0 \) be given in Lemma 6.1. For \( R \gg 1 \), we define

\[
\tau_{l,n}^R := \inf \{ t \geq 0 : \|u_{l,n}\|_{H^s} > R \}; \ l \in \{-1, 1\}.
\]

Then for any \( T > 0 \) and \( n \gg 1 \), we have that for \( l \in \{-1, 1\} \),

\[
\mathbb{E}\sup_{t \in [0, T \wedge \tau_{l,n}^R]} \|u^{l,n} - u_{l,n}\|^2_{H^s} \leq Cn^{-2r_s}, \ C = C(R, T),
\]

and

\[
\mathbb{E}\sup_{t \in [0, T \wedge \tau_{l,n}^R]} \|u^{l,n} - u_{l,n}\|^2_{H^{2s-r}} \leq Cn^{2s-2r_s}, \ C = C(R, T).
\]

**Proof.** We first notice that by Lemma 2.5, for \( l \in \{-1, 1\} \),

\[
\|u^{l,n}(t)\|_{H^s} \lesssim 1, \ \text{for any} \ t > 0 \ \text{and} \ n \in \mathbb{Z}^+,
\]

which means \( \mathbb{P}\{\tau_{l,n}^R > 0\} = 1 \) for any \( n \in \mathbb{Z}^+ \) and \( l \in \{-1, 1\} \). Let \( v = v^{l,n} = u^{l,n} - u_{l,n} \). In view of (6.2), (6.4) and (6.8), we see that \( v \) satisfies

\[
v(t) + \int_0^t \left( (u^{l,n} \cdot \nabla)v + (v \cdot \nabla)u_{l,n} + (-F(u_{l,n})) \right) dt' = \int_0^t [-Q(t', u_{l,n})] dW - \int_0^t \left[ (n^{-2s+1} \sin \theta \cos \theta_{d+1-i})_{1 \leq i \leq d} \right] dt'.
\]

For any \( T > 0 \), we use the Itô formula on \([0, T \wedge \tau_{l,n}^R] \), take a supremum over \( t \in [0, T \wedge \tau_{l,n}^R] \) and use the BDG inequality to find

\[
\mathbb{E}\sup_{t \in [0, T \wedge \tau_{l,n}^R]} \|v\|^2_{H^s} \leq 2\mathbb{E}\sup_{t \in [0, T \wedge \tau_{l,n}^R]} \left( \left| \int_0^t -Q(t', u_{l,n}) dW, v \right|_{H^s} \right) + \sum_{i=2}^6 \mathbb{E} \int_0^{T \wedge \tau_{l,n}^R} |K_i| dt,
\]

where

\[
K_2 = 2 \left( D^\sigma \left(-n^{-2s+1} \sin \theta \cos \theta_{d+1-i} \right)_{1 \leq i \leq d}, D^\sigma v \right)_{L^2},
\]

\[
K_3 = -2 \left( D^\sigma [(v \cdot \nabla)u_{l,n}], D^\sigma v \right)_{L^2},
\]

\[
K_4 = -2 \left( D^\sigma [(u^{l,n} \cdot \nabla)v], D^\sigma v \right)_{L^2},
\]

\[
K_5 = (D^\sigma F(u_{l,n}), D^\sigma v)_{L^2},
\]

\[
K_6 = \|Q(t, u_{l,n})\|^2_{L^2(U, H^s)}.
\]

We can first infer from Lemma 2.6 that

\[
\|F(u_{l,n})\|_{H^s} \lesssim (\|F(u^{l,n}) - F(u_{l,n})\|_{H^s} + \|F(u^{l,n})\|_{H^s})^2 \lesssim (\|u^{l,n}\|_{H^s} + \|u_{l,n}\|_{H^s})^2 \|v\|^2_{H^s} + \|F(u^{l,n})\|^2_{H^s}.
\]
From the above estimate, (1.13), BDG inequality, (6.6), (6.9) and (6.12), we have
\[
\mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \left| -2 \int_{0}^{t} Q(t', u_{t,n}) dW, v \right|_{H^{s}} \leq 2 \mathbb{E} \left( \int_{0}^{T \wedge \tau_{t,T}^{R}} \|v\|_{H^{s}}^{2} \|F(u_{t,n})\|_{H^{s}}^{2} dt \right)^{\frac{1}{2}} = \mathbb{E} \left( \int_{0}^{T \wedge \tau_{t,T}^{R}} \|v\|_{H^{s}}^{2} \|F(u_{t,n})\|_{H^{s}}^{2} dt \right)^{\frac{1}{2}} \leq \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \|v\|_{H^{s}}^{2} \int_{0}^{T \wedge \tau_{t,T}^{R}} \|u_{t,n}\|_{H^{s}}^{2} \|v\|_{H^{s}}^{2} dt \right)^{\frac{1}{2}} + \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \|v\|_{H^{s}}^{2} \int_{0}^{T \wedge \tau_{t,T}^{R}} \|F(u_{t,n})\|_{H^{s}}^{2} dt \right)^{\frac{1}{2}} \leq \frac{1}{2} \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \|v\|_{H^{s}}^{2} + C_{R} \mathbb{E} \int_{0}^{T \wedge \tau_{t,T}^{R}} \|v(t)\|_{H^{s}}^{2} dt + \mathbb{E} \int_{0}^{T \wedge \tau_{t,T}^{R}} \|F(u_{t,n})\|_{H^{s}}^{2} dt \right) \leq \frac{1}{2} \mathbb{E} \left( \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \|v\|_{H^{s}}^{2} + C_{R} \mathbb{E} \int_{0}^{T \wedge \tau_{t,T}^{R}} \|v(t)\|_{H^{s}}^{2} dt + C T n^{-2r_{s}} \right).
\]

Applying Lemma 2.6, $H^{\sigma} \hookrightarrow L^{\infty}$, integration by parts and (6.5), we have
\[
|K_{2}| \lesssim \left\| (n^{-2s+1} \sin \theta_{i} \cos \theta_{t} + 1) \right\|_{H^{s}}^{2} \lesssim n^{-2r_{s}} + \|v\|_{H^{s}}^{2},
\]
\[
|K_{3}| \lesssim \|(|v| \cdot \nabla)u_{t,n}\|_{H^{s}}\|v\|_{H^{s}} \lesssim \|v\|_{H^{s}}^{2} n_{t,n} \|u_{t,n}\|_{H^{s}},
\]
\[
|K_{5}| \lesssim \left( \|u_{t,n}\|_{H^{s}}^{2} + \|u_{t,n}\|_{H^{s}} \right)\|v\|_{H^{s}}^{2} + \|F(u_{t,n})\|_{H^{s}}^{2} + \|v\|_{H^{s}}^{2},
\]
and
\[
|K_{6}| \lesssim \left( \|u_{t,n}\|_{H^{s}}^{2} + \|u_{t,n}\|_{H^{s}} \right)\|v\|_{H^{s}}^{2} + \|F(u_{t,n})\|_{H^{s}}^{2}.
\]

With Lemma 2.1 in hand, we consider the following two cases:
\[
|K_{4}| \lesssim \|u_{t,n}\|_{W^{2,p}} \|\nabla v\|_{L^{p}} \|v\|_{H^{s}} + \|u_{t,n}\|_{L^{\infty}} \|v\|_{H^{s}} \lesssim \|u_{t,n}\|_{H^{s}} \|v\|_{H^{s}}^{2} \text{ for even } d \geq 4,
\]
and
\[
|K_{4}| \lesssim \|u_{t,n}\|_{W^{2,p}} \|\nabla v\|_{L^{p}} \|v\|_{H^{s}} + \|u_{t,n}\|_{L^{\infty}} \|v\|_{H^{s}}^{2} \lesssim \|u_{t,n}\|_{H^{s}} \|v\|_{H^{s}}^{2} \text{ for } d = 2,
\]
where in the case $d = 2$, $p$ will be chosen such that $\sigma - \frac{d}{2} = \sigma - 1 > 1 - \frac{d}{2} > 0$ and $q$ is determined by $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$. We use $H^{s} \hookrightarrow H^{\sigma+1} \hookrightarrow W^{\sigma+2,2}, H^{\sigma} \hookrightarrow W^{1,d}$ for the case $d \geq 4$ and use $H^{\sigma} \hookrightarrow W^{\sigma+1,q} \hookrightarrow W^{\sigma,q}$ and $H^{\sigma} \hookrightarrow W^{1,p}$ for the case $d = 2$ to obtain
\[
|K_{4}| \lesssim \|u_{t,n}\|_{H^{s}} \|v\|_{H^{s}}^{2}.
\]

Therefore we can infer from Lemma 2.6, (6.6), (6.9) and (6.12) that
\[
\mathbb{E} \int_{0}^{T \wedge \tau_{t,T}^{R}} (|K_{2}| + |K_{5}| + |K_{6}|) dt \leq C T n^{-2r_{s}} + C_{R} \int_{0}^{T} \mathbb{E} \sup_{t' \in [0, T \wedge \tau_{t,T}^{R}]} \|v(t')\|_{H^{s}}^{2} dt,
\]
and
\[
\mathbb{E} \int_{0}^{T \wedge \tau_{t,T}^{R}} (|K_{3}| + |K_{4}|) dt \leq C_{R} \int_{0}^{T} \mathbb{E} \sup_{t' \in [0, T \wedge \tau_{t,T}^{R}]} \|v(t')\|_{H^{s}}^{2} dt.
\]

Over all, we arrive at
\[
\mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \|v(t)\|_{H^{s}}^{2} \leq C T n^{-2r_{s}} + C_{R} \int_{0}^{T} \mathbb{E} \sup_{t' \in [0, T \wedge \tau_{t,T}^{R}]} \|v(t')\|_{H^{s}}^{2} dt.
\]

Via the Grönwall inequality, we have
\[
\mathbb{E} \sup_{t \in [0, T \wedge \tau_{t,T}^{R}]} \|v(t)\|_{H^{s}}^{2} \leq C n^{-2r_{s}}, \quad C = C(R, T),
\]
which is (6.10). For (6.11), we first notice that $u_{t,n}$ is the unique solution to (6.8) and $2s - \sigma > 0$. For each fixed $n \in \mathbb{Z}^+$, we can repeat the proof for (3.6) with using Lemma 2.6 and (6.9) to find that,

$$
\mathbb{E} \sup_{t \in [0,T]} \|u_{t,n}(t)\|^2_{H^{2s-\sigma}} \leq 2\mathbb{E}\|u_{t,n}(0)\|^2_{H^{2s-\sigma}} + C_{R,T} \int_0^T \mathbb{E} \sup_{t' \in [0,t]} \|u(t')\|^2_{H^{2s-\sigma}} \, dt.
$$

From the above estimate, we can use the Grönwall inequality and Lemma 2.5 to infer

$$
\mathbb{E} \sup_{t \in [0,T]} \|u_{t,n}(t)\|^2_{H^{2s-\sigma}} \leq C\mathbb{E}\|u_{t,n}(0)\|^2_{H^{2s-\sigma}} \leq Cn^{2s-2\sigma}, \quad C = C(R,T).
$$

Then it follows from Lemma 2.5 that for some $C = C(R,T)$ and $l \in \{-1,1\}$,

$$
\mathbb{E} \sup_{t \in [0,T]} \|v^l_{t,n}(t)\|^2_{H^{2s-\sigma}} \leq C\mathbb{E} \sup_{t \in [0,T]} \|u_{t,n}(t)\|^2_{H^{2s-\sigma}} + C\mathbb{E} \sup_{t \in [0,T]} \|u_{t,n}(t)\|^2_{H^{2s-\sigma}} \leq Cn^{2s-2\sigma},
$$

which is (6.11). \qed

6.4. Proof for Theorem 1.3.

**Lemma 6.3.** Let $d \geq 2$ be even and $Q(t, u)$ satisfy (1.13). If for some $R_0 \gg 1$, the $R_0$-exiting time is strongly stable at the zero solution to (1.8), then for $l \in \{-1,1\}$, we have

$$
\lim_{n \to \infty} \tau_{l,n}^{R_0} = \infty \quad \mathbb{P} - a.s., \quad (6.13)
$$

where $\tau_{l,n}^{R_0}$ is given in (6.9).

**Proof.** Since $u_{t,n}$ satisfies (6.8), it follows that

$$
\lim_{n \to \infty} \|u_{t,n}(0) - 0\|_{H^s} = \lim_{n \to \infty} \|u^{l,n}(0)\|_{H^s} = 0 \quad \forall s' < s.
$$

Notice that for zero initial data, the unique solution to (1.8) is zero, and the $R_0$-exiting time at the zero solution is $\infty$. Therefore we see that if the $R_0$-exiting time is strongly stable at the zero solution to (1.8), then (6.13) holds true. \qed

With the above result at our disposal, now we can prove Theorem 1.3.

**Proof for Theorem 1.3.** Let us first consider the case $d \geq 2$ is even. Let $R_0 \gg 1$. We will show that if the $R_0$-exiting time is strongly stable at the zero solution, then $(u_{-1,n}, \tau_{-1,n})$ and $(u_{1,n}, \tau_{1,n})$ satisfy (1.18)–(1.21). For each $n > 1$, for $l \in \{-1,1\}$ and for the fixed $R_0 \gg 1$, Lemma 2.5 and (6.9) give us $\mathbb{P}\{\tau_{l,n}^{R_0} > 0\} = 1$ and Lemma 6.3 implies (1.18). Besides, Theorem 1.1 and (6.9) show that $u_{t,n} \in C([0,\tau_{l,n}]; H^s)$ $\mathbb{P} - a.s.$ and

$$
\sup_{t \in [0,\tau_{l,n}^{R_0}]} \|u_{t,n}\|_{H^s}^2 \leq R_0^2, \quad \mathbb{P} - a.s.,
$$

which gives (1.19). And (1.20) is given by

$$
\|u_{-1,n}(0) - u_{1,n}(0)\|_{H^s} = \|u^{-1,n}(0) - u^{1,n}(0)\|_{H^s} \lesssim n^{-1}.
$$

For any $T > 0$, using the interpolation inequality and Lemma 6.2, we see that for $l \in \{-1,1\}$ and $v = v^l_{t,n} = u^{l,n} - u_{t,n}$,

$$
\left( \mathbb{E} \sup_{t \in [0,T]} \|v\|_{H^s} \right)^2 \leq \mathbb{E} \sup_{t \in [0,T]} \|v\|^2_{H^s} \leq \left( \mathbb{E} \sup_{t \in [0,\tau_{l,n}^{R_0}]} \|v\|^2_{H^s} \right)^{\frac{1}{2}} \left( \mathbb{E} \sup_{t \in [0,\tau_{l,n}^{R_0}]} \|v\|^2_{H^{2s-\sigma}} \right)^{\frac{1}{2}} \lesssim n^{-r_s + (s - \sigma)}.
$$

It follows from

$$
0 > -r_s + s - \sigma = \begin{cases} 
1 - s & \text{if } 1 + \frac{d}{2} < s \leq 3, \\
-2 & \text{if } s > 3,
\end{cases}
$$

that for $l \in \{-1,1\}$,

$$
\lim_{n \to \infty} \mathbb{E} \sup_{t \in [0,\tau_{l,n}^{R_0}]} \|u^{l,n} - u_{t,n}\|_{H^s} = 0. \quad (6.14)
$$
Now we prove (1.21). For any given $T > 0$, on account of (6.14), Lemmas 2.5 and 6.3 and the fact that $\cos \alpha - \cos \beta = -2 \sin \left( \frac{\alpha + \beta}{2} \right) \sin \left( \frac{\alpha - \beta}{2} \right)$, we have

$$\liminf_{n \to \infty} \frac{1}{E} \sup_{t \in [0, T]} \|u_{1,n}(t) - u_{2,n}(t)\|_{L^2} \geq \liminf_{n \to \infty} \frac{1}{E} \sup_{t \in [0, T]} \|u_{1,n}(t) - u_{1,n}(t)\|_{L^2}$$

and the fact that

$$\liminf_{n \to \infty} \frac{1}{E} \sup_{t \in [0, T]} \|\alpha_{1,n}(t) - \alpha_{2,n}(t)\|_{L^2} \geq \liminf_{n \to \infty} \frac{1}{E} \sup_{t \in [0, T]} \|\alpha_{1,n}(t) - \alpha_{1,n}(t)\|_{L^2}$$

also hold true. Therefore we have

$$\liminf_{n \to \infty} \frac{1}{E} \sup_{t \in [0, T]} \|u_{1,n}(t) - u_{1,n}(t)\|_{L^2} \geq \|u_{1,n}(t) - u_{2,n}(t)\|_{L^2}$$

for all $t \in [0, T]$. Moreover, for the pathwise solutions $u_{t,n}$ to (6.8) with $u_{1,n}(0) = u_{1,n}(0) = (\ln^{-1} + n^{-s} \cos \theta_1, ln^{-1} + n^{-s} \cos \theta_2, \cdots, ln^{-1} + n^{-s} \cos \theta_{d-1}, 0)$, where $\theta_i = n x_{d-i} - l t$ with $1 \leq i \leq d - 1$, $n \in \mathbb{Z}^+$, $l \in \{1, -1\}$. In this case, $d - 1$ is even and we can repeat the proof for Lemma 6.1 to find that the error $E_{1,n}(t)$ also enjoys (6.3). Moreover, for the pathwise solutions $u_{t,n}$ to (6.8) with $u_{t,n}(0) = u_{t,n}(0) = (\ln^{-1} + n^{-s} \cos nx_{d-i}, 0)$, $1 \leq i \leq d - 1$, we can basically repeat the previous procedure to show that Lemmas 6.2 and 6.3 also hold true. Therefore one can establish (1.18)-(1.21) for $u_{t,n}$ similarly.

In conclusion, we see that if for some $R_0 \gg 1$, the $R_0$-exiting time is strongly stable at the zero solution, then the solution map defined by (1.8) is not uniformly continuous when $Q(t, \cdot)$ satisfies Hypothesis II. □

**Remark 6.1.** From the above proof for Theorem 1.3, it is clear that if $d = 1$, one can use $u^{I,n} = \ln^{-1} + n^{-s} \cos nx_{d-i}$, $n \geq 1$ as a sequence of approximation solutions and repeat the other part of the proof correspondingly to obtain the similar statements in $d = 1$. Therefore Theorem 1.3 also holds true for $d = 1$, namely the stochastic CH equation case.

## 7. Wave breaking and breaking rate

In this section, we study the blow-up of classical solutions to (1.10), and estimate the associated probabilities. We first notice that on $\mathbb{T}$, the operator $(1 - \partial_{xx}^2)^{-1}$ in $q(\cdot)$ (cf. (1.4)) has an explicit form, namely

$$[(1 - \partial_{xx}^2)^{-1} f](x) = [G_T * f](x), \quad G_T = \frac{\cosh(x - 2 \pi \frac{[x/2]}{2})}{2 \sinh(\pi)}, \quad \forall f \in L^2(\mathbb{T}),$$

where $[x]$ stands for the integer part of $x$.

Motivated by [34, 57, 58], we introduce the following Girsanov type transform

$$v = \frac{1}{\beta(\omega, t)} u, \quad \beta(\omega, t) = e^\int_0^t k(s')dW_s - \int_0^t \frac{z(s')^2}{2} ds'.$$ (7.1)
Proposition 7.1. Let $s > 3$, $b(t)$ satisfies Hypothesis III and $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)$ be fixed. If $u_0(\omega, x)$ is an $H^s$-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}\|u_0\|^2_{H^s} < \infty$ and $(u, \tau^*)$ is the corresponding unique maximal solution to (1.6) (or to (1.11), equivalently), then for $t \in [0, \tau^*)$, the process $v$ defined by (7.1) solves the following problem on $\mathbb{T}$ almost surely,

$$
\begin{align*}
v_t + \beta v u_x + \beta (1 - \beta^2 x^2)^{-1} u_x \left(v^2 + \frac{1}{2} v_x^2\right) &= 0, \\
v(\omega, 0, x) &= u_0(\omega, x).
\end{align*}
$$

Moreover, $v \in C([0, \tau^*); H^s) \cap C^1([0, \tau^*); H^{s-1}) \mathbb{P} - a.s.$, and

$$
\mathbb{P}\{\|v(t)\|_{H^s} = \|u_0\|_{H^s}\} = 1.
$$

Proof. Since $b(t)$ satisfies Hypothesis III, $B(t, u) = b(t)u$ satisfies Hypothesis I. Consequently, Theorem 1.1 implies that (1.10) has a unique maximal solution $(u, \tau^*)$. Direct computation with Itô formula yields

$$
d \frac{1}{\beta} = -b(t) \frac{1}{\beta} dW + b^2(t) \frac{1}{\beta} dt.
$$

Then we have

$$
dv = -\frac{1}{\beta} dv + u \frac{1}{\beta} + d \frac{1}{\beta} du \\
= \frac{1}{\beta} \left[ -[u \partial_x u + q(u)] dt + b(t) u dW \right] + u \left[ -b(t) \frac{1}{\beta} dW + b^2(t) \frac{1}{\beta} dt \right] - b^2(t) \frac{1}{\beta} u dt \\
= \frac{1}{\beta} \left[ -[u \partial_x u + q(u)] dt \right] \\
= \left\{ -\beta v u_x + \beta (1 - \beta^2 x^2)^{-1} u_x \left(v^2 + \frac{1}{2} v_x^2\right) \right\} dt,
$$

which is (7.2)\textsuperscript{1}. Since $v(0) = u_0(\omega, x)$, we see that $v$ satisfies (7.2). Moreover, Theorem 1.1 implies $u \in C([0, \tau^*); H^s) \mathbb{P} - a.s.$, so $v \in C([0, \tau^*); H^s) \mathbb{P} - a.s.$ Besides, from Lemma 2.6 and (7.2)\textsuperscript{1}, we see that for a.e. $\omega \in \Omega$, $v_t = -\beta v u_x - \beta(q_1(v) + q_2(v)) \in C([0, \tau^*); H^{s-1})$. Hence $v \in C^1([0, \tau^*); H^{s-1}) \mathbb{P} - a.s.$

Notice that (7.2)\textsuperscript{1} is equivalent to

$$
v_t - v_{xxx} + 3\beta v u_x = 2\beta v u_x v_x + \beta v v_{xxx}.
$$

Multiplying both sides of the above equation by $v$ and then integrating the resulting equation on $x \in \mathbb{T}$, we see that for a.e. $\omega \in \Omega$ and for all $t > 0$

$$
\frac{d}{dt} \int_{\mathbb{T}} (v^2 + v_x^2) \, dx = 0,
$$

which implies (7.3).



\section{Blow-up scenario.}

In contrast to the original SPDE (1.10), no stochastic integral appears in (7.2). Then we can give a more precise blow-up criterion and analyze the blow-up rate.

Proof for Theorem 1.4. We divided the proof into two parts.

\textbf{Step 1: Refined blow-up criterion (1.23).} By Theorem 1.2, to prove (1.23), it is sufficient to prove that

$$
1 \{\liminf_{t \to \tau^*} \min_{x \in \mathbb{T}} |u_x(t, x)| = \infty\} = 1 \{\limsup_{t \to \tau^*} \|u(t)\|_{W^{1, \infty}} = \infty\} \mathbb{P} - a.s.
$$

It is clear that if $\{\liminf_{t \to \tau^*} \min_{x \in \mathbb{T}} |u_x(t, x)| = \infty\} \subset \{\limsup_{t \to \tau^*} \|u(t)\|_{W^{1, \infty}} = \infty\}$. Now we prove $\{\liminf_{t \to \tau^*} \min_{x \in \mathbb{T}} |u_x(t, x)| = \infty\} \subset \{\limsup_{t \to \tau^*} \|u(t)\|_{W^{1, \infty}} = \infty\}^C$, where $A^C$ means the complement of $A \subset \Omega$. Notice that

$$
\left\{ \liminf_{t \to \tau^*} \min_{x \in \mathbb{T}} |u_x(t, x)| = \infty \right\}^C = \left\{ \exists K(\omega) > 0 \text{ s.t. } \min_{x \in \mathbb{T}} u_x(\omega, t, x) > -K(\omega) \forall t \text{ almost surely} \right\}.
$$

Since $v$ solves (7.2) (or equivalent (7.5)) $\mathbb{P} - a.s.$, it is easy to find that the momentum variable $V = v - v_{xx}$ satisfies

$$
V_t + \beta v V_x + 2\beta v v_x = 0 \mathbb{P} - a.s.
$$

(7.7)
Using (7.7), (7.1) and integration by parts, we find that
\[
\frac{d}{dt} \int_T V^2 dx = 2 \int_T V[-\beta v v_x - 2\beta V v_x] dx
\]
\[
= -4\beta \int_T V^2 v_x dx - 2\beta \int_T Vv v_x dx
\]
\[
= -3\beta \int_T V^2 v_x dx
\]
\[
\leq 3K \int_T V^2 dx, \quad t \in [0, \tau^*) \text{ a.e. on } \left\{ \liminf_{t \to \tau^*} u_x(t, x) = -\infty \right\}^C.
\]

It follows from the above estimate, (7.1) and the fact that \(\sup_{t > 0} \beta(t) < \infty\) almost surely that
\[
\|u(t)\|_{H^2} \lesssim \beta(t) e^{Kt} \|u(0)\|_{H^2} < \infty, \quad t \in [0, \tau^*) \text{ a.e. on } \left\{ \liminf_{t \to \tau^*} u_x(t, x) = -\infty \right\}^C.
\]

By the embedding \(H^2 \hookrightarrow W^{1, \infty}\), we see that
\[
\left\{ \liminf_{t \to \tau^*} u_x(t, x) = -\infty \right\}^C \subseteq \left\{ \limsup_{t \to \tau^*} u(t)\|_{W^{1, \infty}} = \infty \right\}^C.
\]

Hence we obtain (7.6).

**Step 2: Blow-up rate** (1.24). We notice that for the maximal solution \((u, \tau^*)\) to (1.11), Proposition 7.1 ensures that \(v(\omega, t, x)\) defined by (7.1) solves (7.4) with the same initial data \(u_0\) almost surely. Using (7.4), we identify that
\[
v_{tx} + \beta v v_{xx} = \beta v^2 - \beta \frac{1}{2} v_x^2 - \beta G_T * \left(v^2 + \frac{1}{2} v_x^2\right), \quad t \in [0, \tau^*)
\]
\[
(7.8)
\]
Let
\[
M(\omega, t) := \min_{x \in \Omega} v_x(\omega, t, x).
\]
Proposition 7.1 guarantees that \(v(\omega, t, x) \in C^1([0, \tau^*]; H^{s-1})\) with \(s > 3\). For a.e. \(\omega \in \Omega\), we let \(z(\omega, t)\) be a point where this infimum of \(v_x\) is attained. By Lemma 2.3 and \(v_{tx} \in C([0, \tau^*]; H^{s-2})\) with \(s > 3\), we have
\[
P\{ M \text{ is locally Lipschitz} \} = 1.
\]
Now we evaluate (7.8) in \((t, z(\omega, t))\) with using Lemma 2.3 to obtain that for a.e. \(\omega \in \Omega\),
\[
\frac{d}{dt} M(t) = \beta v^2(t, z(t)) - \beta \frac{1}{2} M^2(t) - \beta G_T * \left(v^2 + \frac{1}{2} v_x^2\right)(t, z(t)) \text{ a.e. on } (0, \tau^*).
\]
\[
(7.11)
\]
As \(G_T > 0\), we have
\[
-\beta \left\| G_T * \left(v^2 + \frac{1}{2} v_x^2\right) \right\|_{L^\infty} \leq \frac{d}{dt} M(t) + \beta \frac{1}{2} M^2(t) \leq \beta \|v\|_{L^\infty}^2 \text{ a.e. on } (0, \tau^*).
\]
Using \(\|G\|_{L^\infty} < \infty\) and (7.3), we have
\[
\left\| G_T * \left(v^2 + \frac{1}{2} v_x^2\right) \right\|_{L^\infty} \lesssim \left\| v^2 + \frac{1}{2} v_x^2 \right\|_{L^1} \lesssim \|v\|_{H^1}^2 = \|u_0\|_{H^1}^2.
\]
Therefore for some \(C > 0\) and \(N = C\|u_0\|_{H^1}^2\), we have
\[
-\beta N \leq \frac{d}{dt} M(t) + \beta \frac{1}{2} M^2(t) \leq \beta N \quad \text{a.e. on } (0, \tau^*).
\]
\[
(7.12)
\]
Let \(\epsilon \in (0, \frac{1}{2})\). Since \(N < \infty\) almost surely and \(\liminf_{t \to \tau^*} M(t) = -\infty\) a.e. on \(\{\tau^* < \infty\}\) (by Step 1), for a.e. \(\omega \in \{\tau^* < \infty\}\) there is a \(t_0 = t_0(\omega, \epsilon) \in (0, \tau^*)\) such that \(M(t_0) < -\sqrt{\frac{N}{\epsilon}}\).

**Claim.** For any fixed \(\epsilon \in (0, \frac{1}{2})\), we have
\[
M(t) \leq -\sqrt{\frac{N}{\epsilon}}, \quad t \in [t_0, \tau^*) \text{ a.e. on } \{\tau^* < \infty\}.
\]
\[
(7.13)
\]
To this end, we define \(\tau^* < \infty\) as
\[
\tau(\omega) := \inf \left\{ t > t_0 : M(\omega, t) > -\sqrt{\frac{N}{\epsilon}} \right\} \wedge \tau^*.
\]
Since $M(t_0) < \sqrt{\frac{N}{2}}$, $\tau(\omega) > t_0$ a.e. on $\{\tau^* < \infty\}$. Now we will show

$$\tau(\omega) = \tau^*(\omega), \text{ a.e. on } \{\tau^* < \infty\}. \quad (7.14)$$

Suppose (7.14) is not true and let $\Omega^* \subseteq \{\tau^* < \infty\}$ such that $P(\Omega^* > 0)$ and $0 < \tau(\omega^*) < \tau^*(\omega^*)$ for all $\omega^* \in \Omega^*$. In view of the continuity of the path of $M(\omega^*, t)$, we find that $M(\omega^*, \tau(\omega^*)) = -\sqrt{\frac{N}{2}}$. On the other hand, (7.12) means that on $[t_0, \tau(\omega^*))$,

$$\frac{d}{dt} M(t) \leq -\beta N - \frac{1}{2} M^2(t) \leq -\beta N - \frac{1}{2} \frac{N}{\epsilon} < 0 \text{ a.e. on } (0, \tau^*),$$

which shows that $M(\omega^*, t)$ is non-increasing for $t \in [t_0, \tau(\omega^*))$. Hence by the continuity of the path of $M(\omega^*, t)$ again, we see that $M(\tau(\omega^*)) \leq M(t_0) < -\sqrt{\frac{N}{2}}$, which is a contradiction. Hence (7.14) is true and so is (7.13).

A combination of (7.12) and (7.13) enables us to infer that a.e. on $\{\tau^* < \infty\}$,

$$\frac{\partial}{\partial t} M(t) + \beta \frac{1}{2} > -\frac{\partial}{\partial t} M(t) \geq -\beta \frac{N}{M^2} + \beta \frac{1}{2} \text{ a.e. on } (t_0, \tau^*).$$

Due to (7.10) and (7.13), $\frac{\partial}{\partial t}$ is also locally Lipschitz a.e. on $\{\tau^* < \infty\}$. Then we integrate the above estimate on $(t, \tau^*)$ with noticing $\liminf_{t \to \tau^*} M(t) = -\infty$ a.e. on $\{\tau^* < \infty\}$ to derive that

$$\left(\frac{1}{2} + \epsilon\right) \int_t^{\tau^*} \beta(t') dt' \geq -\frac{1}{M(t)} \geq \left(\frac{1}{2} - \epsilon\right) \int_t^{\tau^*} \beta(t') dt', \ t_0 \leq t < \tau^*, \text{ a.e. on } \omega \in \{\tau^* < \infty\}.$$

Then we can infer from (7.1) and (7.9) that for a.e. $\omega \in \{\tau^* < \infty\}$,

$$\frac{1}{2} + \epsilon \leq -\min_{x \in \mathbb{T}}[u_x(\omega, t, x)]\beta^{-1}(t) \int_t^{\tau^*} \beta(t') dt' \leq \frac{1}{2} - \epsilon, \ t_0 \leq t < \tau^*.$$

Since $\epsilon \in (0, \frac{1}{4})$ is arbitrary, we obtain that

$$\lim_{t \to \tau^*} \left(\min_{x \in \mathbb{T}}[u_x(t, x)] \int_t^{\tau^*} \beta(t') dt'\right) = -2\beta(\tau^*) \text{ a.e. on } \{\tau^* < \infty\},$$

which completes the proof. \hfill $\Box$

7.2. Wave breaking. Motivated by [12], we first establish the following result.

**Proposition 7.2.** Let $S = (\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0}, W)$ be a fixed stochastic basis. Let $b(t)$ satisfy Hypothesis III, $s > 3$ and $u_0 = u_0(x) \in H^s$ be an $H^s$-valued $\mathcal{F}_0$-measurable random variable with $\mathbb{E}[|u_0|^3_{H^s}] < \infty$. Let $\{u, \tau^*\}$ be the maximal solution to (1.11) (or to (1.11), equivalently) with initial random variable $u_0$. Let $\lambda$ satisfy Lemma 2.4 and $M$ be given in (7.9). Let $K = \frac{1}{2}|u_0|^2_{H^1}$ and assume that $M(0) < -\sqrt{2K}$ almost surely. Then we have

$$M(t) \leq -\sqrt{2K} \ t \in [0, \tau^*) \ \mathbb{P} - a.s. \quad (7.15)$$

**Proof.** We begin by estimating $G_\tau \ast (v^2 + \frac{1}{2} v_y^2)$. For any $v \in H^1$, we have

$$G_\tau \ast \left(v^2 + \frac{1}{2} v_y^2\right)(x) = \int_0^{2\pi} G_\tau(x - y) \left(v^2(y) + \frac{1}{2} v_y^2(y)\right) dy$$

$$= \frac{e^{x-\pi}}{4 \sinh(\pi)} \int_0^{2\pi} e^{-y} \left(v^2 + \frac{1}{2} v_y^2\right) dy + \frac{e^{-x+\pi}}{4 \sinh(\pi)} \int_0^{2\pi} e^y \left(v^2 + \frac{1}{2} v_y^2\right) dy,$$

$$+ \frac{e^{x+\pi}}{4 \sinh(\pi)} \int_0^{2\pi} e^{-y} \left(v^2 + \frac{1}{2} v_y^2\right) dy + \frac{e^{-x-\pi}}{4 \sinh(\pi)} \int_0^{2\pi} e^y \left(v^2 + \frac{1}{2} v_y^2\right) dy. \quad (7.16)$$

By using $\frac{1}{2} a^2 + \frac{1}{2} b^2 \geq \pm ab$, it is easy to obtain the following estimates,

$$\int_0^{2\pi} e^{-y} \left(v^2 + \frac{1}{2} v_y^2\right) dy \geq -\int_0^{2\pi} e^{-y} v v_y dy + \int_0^{2\pi} e^y \frac{1}{2} v_y^2 dy$$

$$= -\frac{1}{2} v^2(x) e^{-x} + \frac{1}{2} v^2(0),$$

$$\int_0^{2\pi} e^y \left(v^2 + \frac{1}{2} v_y^2\right) dy \geq -\int_0^{2\pi} e^y v v_y dy + \int_0^{2\pi} e^y \frac{1}{2} v_y^2 dy$$

$$= -\frac{1}{2} v^2(x) e^{-x} + \frac{1}{2} v^2(0).$$
\[ \int_0^\infty e^y \left( v^2 + \frac{1}{2} v_y^2 \right) \, dy \geq \int_0^\infty e^y v_y v_x \, dy + \int_0^\infty e^y \frac{1}{2} v^2 \, dy \]
\[= \frac{1}{2} v^2(x)e^x - \frac{1}{2} v^2(0), \]
\[\int_0^{2\pi} e^{-x} \left( v^2 + \frac{1}{2} v_y^2 \right) \, dy \geq \int_0^{2\pi} e^{-x} v_y v_x \, dy + \int_0^{2\pi} e^{-x} \frac{1}{2} v^2 \, dy \]
\[= \frac{1}{2} v^2(x)e^{-x} - \frac{1}{2} v^2(2\pi)e^{-2\pi}, \]
and
\[\int_0^{2\pi} e^y \left( v^2 + \frac{1}{2} v_y^2 \right) \, dy \geq \int_0^{2\pi} e^y v_y v_x \, dy + \int_0^{2\pi} e^y \frac{1}{2} v^2 \, dy \]
\[= \frac{1}{2} v^2(2\pi)e^{2\pi} - \frac{1}{2} v^2(x)e^x. \]

Now we insert the above four estimates into (7.16) to identify that
\[ G_T \ast \left( v^2 + \frac{1}{2} v_y^2 \right) (x) \geq \frac{1}{2} v^2. \]
Inserting the above estimate into (7.11) and then using Lemma 2.4 and (7.3), we see that for a.e. \( \omega \in \Omega, \)
\[ \frac{d}{dt} M(t) \leq \frac{1}{2} v^2(t, z(t)) - \frac{1}{2} c M^2(t) \]
\[\leq \frac{\beta}{2} \|v(t)\|_{L^1} - \frac{1}{2} c M^2(t) \]
\[= \beta K - \frac{1}{2} c M^2(t) \quad \text{a.e. on } (0, \tau^*). \quad (7.17) \]
Similar to the proof for (7.13), it is easy to prove the desired result and here we omit the details. \( \square \)

**Proposition 7.3.** Let all the conditions as in the statement of Proposition 7.2 hold true. Let \( 0 < c < 1 \) and
\[ \Omega^* = \{ \omega : \beta(t) \geq c e^{-\frac{\lambda t}{2}} \text{ for all } t \}. \]
If \( M(0) < \frac{1}{2} \sqrt{\frac{\langle b \rangle^2}{c^2} + 8K - \frac{b^*}{2c}} \text{ almost surely, then for a.e. } \omega \in \Omega^*, \]
\[ \tau^* (\omega) < \infty. \]

**Proof.** Now, we rewrite (7.17) as
\[ \frac{d}{dt} M(t) \leq - \frac{\beta}{2} \left( 1 - \frac{2K}{M^2(0)} \right) M^2(t) - \frac{\beta K}{M^2(0)} M^2(t) + \beta K \quad \text{a.e. on } (0, \tau^*) \quad P - a.s. \]
Since \( M(0) < -\frac{1}{2} \sqrt{\frac{\langle b \rangle^2}{c^2} + 8K - \frac{b^*}{2c}} < -\sqrt{2K}, \) it follows from Proposition 7.2 that
\[ \frac{d}{dt} M(t) \leq - \frac{\beta(t)}{2} \left( 1 - \frac{2K}{M^2(0)} \right) M^2(t) - \left( \frac{M^2(t)}{M^2(0)} - 1 \right) \beta(t) K \]
\[\leq - \frac{\beta(t)}{2} \left( 1 - \frac{2K}{M^2(0)} \right) M^2(t) \quad \text{a.e. on } (0, \tau^*) \quad P - a.s. \]
Due to (7.10) and (7.15), \( \frac{1}{M(t)} \) is also locally Lipschitz almost surely. Therefore we identify that
\[ \frac{1}{M(t)} - \frac{1}{M(0)} \geq \left( 1 - \frac{2K}{M^2(0)} \right) \int_0^t \frac{\beta(t')}{2} \, dt', \forall t \in [0, \tau^*) \quad P - a.s. \]
Therefore we use (7.15) to find that for a.e. \( \omega \in \Omega^*, \)
\[ -\frac{1}{M(0)} \geq \left( 1 - \frac{K}{M^2(0)} \right) \tau^* \beta(t') \, dt' \geq \left( 1 - \frac{K}{M^2(0)} \right) \left( \frac{2c}{b^*} - \frac{2c}{b^*} e^{-\frac{\lambda \tau^*}{2}} \right). \]
Since $M(0) < -\frac{1}{2} \sqrt{\frac{(b^*)^2}{c^2} + 8K} - \frac{b^*}{2c} < -\sqrt{2K}$ almost surely, we finally arrive at

$$\left(\frac{1}{2} - \frac{K}{M^2(0)}\right) \frac{2c}{b^*} e^{\frac{b^*}{2c} t} \leq \frac{2c}{b^*}\left(\frac{1}{2} - \frac{K}{M^2(0)}\right) + \frac{1}{M(0)} > 0,$$

which implies that $\tau^* < \infty$ a.e. on $\Omega^*$. \hfill \Box

Proof for Theorem 1.5. Recall (7.1). Since $A = A(\omega) = \sup_{t \geq 0} \beta(\omega, t) < \infty$ almost surely, we can first infer from $H^1 \hookrightarrow L^\infty$ and (7.3) that for all $t \in [0, \tau^*)$,

$$\sup_{t \in (0, \tau^*)} \|u\|_{L^\infty} \leq A \|u_0\|_{H^1} < \infty \quad \mathbb{P} - \mathbb{a.s.}$$

We can now conclude from Theorem 1.4 and Proposition 7.3 that

$$\mathbb{P} \{\tau^* < \infty\} \geq \mathbb{P} \left\{\beta(t) \geq ce^{-b^*t} \quad \forall t > 0\right\}.$$

Since $b^2(t) < b^*$ for all $t > 0$, we have

$$\left\{ e^{\int_0^t b(t')dt'}dW_t > c \quad \forall t \right\} \subseteq \left\{ \beta(t) \geq ce^{-b^*t} \quad \forall t > 0 \right\}.$$

Therefore we arrive at

$$\mathbb{P} \{\tau^* < \infty\} \geq \mathbb{P} \left\{ e^{\int_0^t b(t')dt'}dW_t > c \quad \forall t \right\} > 0,$$

which together with Theorem 1.4 gives rise to Theorem 1.5. \hfill \Box

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