Chapter

Brans-Dicke Solutions of Stationary, Axially Symmetric Spacetimes

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Abstract

One of the most known alternative gravitational theories is Brans-Dicke (BD) theory. The theory offers a new approach by taking a scalar field \( \phi \) instead of Newton’s gravitational constant \( G \). Solutions of the theory are under consideration and results are discussed in many papers. Stationary, axially symmetric solutions become important because gravitational field of celestial objects can be described by such solutions. Since obtaining exact solutions of BD is not an easy task, some solution-generating techniques are proposed. In this context, some solutions of Einstein general relativity, such as black hole or wormhole solutions, are discussed in BD theory. Indeed, black hole solutions in BD theory are not fully understood yet. Old and new such solutions and their analysis will be reviewed in this chapter.

Keywords: Brans-Dicke, stationary symmetric

1. Introduction

Einstein’s theory of general relativity (GR), which is undoubtedly one of the greatest theories of the last century, is still being tried to be understood. Recently, the theory is supported by the observations of the gravitational waves which are observed by LIGO and Virgo collaboration [1]. On the other hand, GR may have some problems regarding defining gravity accurately at all scales. One of the problems that GR faced was that it could not fully describe the accelerated expansion of the universe [2–4] without unknown materials, i.e., dark matter and dark energy. Although, in order to understand the theory and satisfy the scientific curiosity, GR is modified with higher-order Ricci scalar [5, 6] soon after the theory is published, this modifications were not paid attention. The pioneer of studies on scalar-tensor theory were done by Brans and Dicke [7] by changing Newton’s gravitational constant \( G \) with a scalar field \( \phi = 1/G \). In order to understand the BD theory, several experimental tests of GR are studied, and they are summarized in [8]. Additionally, it has been shown that BD theory can satisfy accelerated expansion of the universe with small and negative values of BD parameter \( \omega \) [9, 10]. But these values of BD parameter cannot satisfy the solutions of our solar system and latest CMB datas. Extended BD theories which include a potential for the BD field are allowed to construct a number of analytic approximations [11]. Although, in the
beginning, we construct our solutions with a potential, then we choose zero potential which is usually called as massless BD theory, in order to make it more simple.

Obtaining exact solutions of any theory is important in order to make comparison with observations or in order to obtain the results of the theory under consideration. Stationary, axially symmetric solutions are one of the important classes of these solutions, since the gravitational field of compact celestial objects such as stars, galaxies, and black holes can be represented by such solutions. Due to the complexity of the field equations, some solution-generating techniques are constructed. Obtaining Ernst BD equations is one of the most known of these techniques [12–14]. Also, Nayak and Tiwari [15] obtained vacuum stationary, axially symmetric BD solutions and generalized Maxwell field by Rai and Singh [16]. Their theory depends on finding out the relation between the field equations of BD and GR theories. After defining this relation, the corresponding BD solution of any known GR solution can be obtained. This method, which we call the Tiwari-Nayak-Singh-Rai (TNSR) method, is the most direct one. Instead of the one-parameter solution, which is called TNSR method, solutions with two parameters were constructed in [17]. A brief summary of [17] will be provided in the rest of the chapter.

The outline of the chapter will be as follow; in Section 2, we review the BD field equations and explain the Ernst equations and extended TNSR method. In Section 3, we study several solutions in order to understand how the extended TNSR method works. In addition, we mention the GR limit of the BD solutions.

2. Field equations of Brans-Dicke theory

In general a four-dimensional Brans-Dicke action with matter in Jordan frame is given by

$$S = \int d^4x \sqrt{-g} \left( \phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) + S_M$$  \hspace{1cm} (1)

where $8\pi G = c = 1$, $\phi$ is scalar field, $\omega(\phi)$ is BD coupling, $R$ is Ricci scalar, $V(\phi)$ is a potential BD field, and $S_M$ is action of the matter.

In many cases and origin of the BD theory, BD coupling $\omega(\phi)$ is chosen a constant as $\omega$. On the other hand, for different coupling function has a different modified gravity theory. One gets variations with respect to metric $g_{\mu\nu}$ and scalar field $\phi$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{T_{\mu\nu}}{\phi} - \frac{\omega(\phi)}{\phi^2} \left( \phi_{\mu\nu} \phi_{\rho\sigma} - \frac{1}{2} g_{\mu\nu} \phi^{\rho\sigma} \phi_{\rho\sigma} \right)$$
$$- \frac{1}{\phi} \left( \phi_{\mu\nu} - g_{\mu\nu} \Box \phi \right) - g_{\mu\nu} \frac{V(\phi)}{\phi}$$ \hspace{1cm} (2)

$$(2\omega + 3) \Box \phi = T - 2V + \phi \frac{dV}{d\phi} + \phi^{\rho\sigma} \frac{d\omega(\phi)}{d\phi}$$ \hspace{1cm} (3)

$T_{\mu\nu}$ represents energy-momentum tensor of the matter and $T$ is its trace and $\Box$ is d’Alembert operator with respect to full metric. The abstract index notation (i.e., $g_{\mu\nu}, T_{\mu\nu}$) is used in order to show component of what we concern with. Greek indices run over the spacetime manifold, starting with time component $t$ and space
components $r, \theta, \varphi$ in this work. For example, if we want to figure out $t, t$ component of Einstein tensor ($G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R$) from Eq. (2), we get

$$G_{tt} = \frac{T_{tt}}{\phi} - \frac{\omega(\phi)}{\phi^2} \left( \phi, \phi, t - \frac{1}{2} g_{tt} \phi^2 \phi_{,t} \right) - \frac{1}{\phi} \left( \phi_{,t} - g_{tt} \nabla^2 \phi \right) - \frac{V(\phi)}{\phi^2} \div (4)$$

where the repeated index $\lambda$ means summation, such as $\phi^\lambda, \phi, \phi_{,\lambda}$.

Conservation of the energy-momentum tensor of the matter leads to

$$T_{\mu \nu}^{\text{matter}} = 0. \quad (5)$$

Furthermore, we set BD coupling $\omega(\phi) \to \omega$ and scalar potential $V(\phi) = 0$ in the rest of the paper for simplicity and to easily obtain the field equations. Also, non-vanishing scalar potential and $\omega(\phi)$ are mostly used for cosmological solutions.

### 2.1 BD solution with electromagnetic field

A four-dimensional general stationary, axially symmetric spacetime can be represented with a metric in cylindrical coordinates in the canonical form as

$$ds^2 = -e^{2U} (dt^2 + A d\varphi)^2 + e^{2(K-U)} (d\rho^2 + dz^2) + e^{-2U} W^2 d\varphi^2 \quad (6)$$

where all the metric functions depend on the coordinates $\rho$ and $z$. We shall consider the field content described by Maxwell field such that the energy-momentum tensor is

$$T_{\mu \nu} = 2 \left( F_{\mu \lambda} F_{\nu \lambda} \right) \quad (7)$$

and we study on the potential one form which shares the symmetry of the metric (6) as

$$A = A_0(\rho, z) + A_3(\rho, z) \quad (8)$$

with Maxwell equation

$$F_{\mu \nu} ; \nu = 0. \quad (9)$$

For simplicity we define BD field Eq. (2) as

$$G_{\mu \nu} = G_{\mu \nu} - \frac{\omega}{\phi^2} \left( \phi, \phi, \phi_{,\lambda} - \frac{1}{2} g_{\mu \nu} \phi^2 \phi_{,\lambda} \right) - \frac{1}{\phi} \left( \phi_{,\mu \nu} - g_{\mu \nu} \nabla^2 \phi \right) - \frac{2T_{\mu \nu}}{\phi} = 0 \quad (10)$$

and some of the field equations of the metric (6) become

$$G_{\mu \nu}^{\rho} + G_{\mu \nu}^{\varphi} = \nabla^2 (\phi W) = 0 \quad (11)$$

$$G_{\mu \nu}^{t} = 4 \left( A_0 \nabla A_0 - \nabla A_0 \nabla A_0 \right) + We^{-2U \nabla} \left( \frac{e^{4U} \phi \nabla A_0}{W} \right) = 0 \quad (12)$$
\[
G_{n_t} + G_{n_\phi} = 2 \omega W^2 \left( \overline{\nabla} \phi \right)^2 + 2 \phi W \overline{\nabla} \left( W \overline{\nabla} \phi \right) - \phi^2 e^{4U} \left( \overline{\nabla} \phi \right)^2 + 4 W^2 \phi^2 \nabla^2 K
\]
\[
- 4 \phi^2 W \overline{\nabla} \left( W \overline{\nabla} U \right) + 2 \phi W W \left[ \overline{\nabla} \left( U \overline{\nabla} \phi \right) \right]^2 = 0 \quad (13)
\]
\[
G_{n_t} - G_{n_\phi} = e^{-2U} \left( \overline{\nabla} A_0 \right)^2 - e^{2U} \left[ \overline{\nabla} \left( \overline{\nabla} A_0 \right)^2 - \left( \overline{\nabla} A_3 \right)^2 \right]
\]
\[
- \overline{\nabla} W \overline{\nabla}. \left( e^{2U} \phi \overline{\nabla} A \right) - e^{4U} \phi \left( \overline{\nabla} \phi \right)^2 + W \left[ 2 \overline{\nabla}. \left( \phi W \overline{\nabla} U \right) - \overline{\nabla}. \left( \phi W \overline{\nabla} \phi \right) \right] = 0
\]
\[
(14)
\]
where \( \overline{\nabla} f = \partial_\rho f, \partial_z f \). The field equation of (3) becomes
\[
(2 \omega + 3) \overline{\nabla}. \left( W \overline{\nabla} \phi \right) = 0. \quad (15)
\]
Using this result and Eq. (11) we obtain the last term in Eq. (14) vanishes except \( \omega = - \frac{3}{2} \). Maxwell equation of (9) becomes
\[
ME^t = \overline{\nabla}. \left[ e^{-2U} W \overline{\nabla} A_0 + \frac{\varepsilon e^{2U}}{W} \left( \overline{\nabla} A_3 - \overline{\nabla} A_0 \right) \right] = 0 \quad (16)
\]
\[
ME^\phi = \overline{\nabla}. \left[ \frac{e^{2U}}{W} \left( \overline{\nabla} A_0 - \overline{\nabla} A_3 \right) \right] = 0 \quad (17)
\]
BD field and Maxwell equations for stationary, axially spacetime are more complicated to obtain exact or approximate solutions. Even for GR, some solution-generating techniques are used because of this complexity. Firstly, we introduce Ernst equations obtained from BD field equations.

2.2 Ernst equations

The original Ernst equations in GR with the presence of Maxwell field are;
\[
\left( \varepsilon + e^* + |\Phi|^2 \right) \nabla^2 \varepsilon = 2 \left( \overline{\nabla} \varepsilon + 2 \Phi^* \overline{\nabla} \phi \right) \overline{\nabla} \varepsilon \quad (18)
\]
\[
\left( \varepsilon + e^* + |\Phi|^2 \right) \nabla^2 \Phi = 2 \left( \overline{\nabla} \varepsilon + 2 \Phi^* \overline{\nabla} \phi \right) \overline{\nabla} \Phi \quad (19)
\]
where \( \varepsilon \) and \( \Phi \) redefine potentials. Some exact solutions are obtained, integrating these equations. Our aim is to obtain BD-Maxwell field equations in this form. First, we rewrite the metric Ansatz:
\[
ds^2 = -ae^{\Omega/2} (dt + Ad \phi)^2 + ae^{-\Omega/2} d \phi^2 + \frac{e^{2 \nu}}{\sqrt{a}} (d \rho^2 + dz^2) \quad (20)
\]
which simplifies the forthcoming equations considerably. Metric (6) and metric (20) relations are given:
\[
W = \alpha, \quad U = \frac{\Omega}{4} + \frac{1}{2} \ln \alpha, \quad K = \frac{\Omega}{4} + \frac{1}{4} \ln \alpha + \nu. \quad (21)
\]
Defining an operator as $\tilde{\nabla} f = \partial_r f, \partial_z f$, BD-Maxwell field equations become

$$ EE^\alpha_\nu + EE^\nu_\alpha = \nabla^2 (\phi \alpha) = 0 \quad (22) $$

$$ EE^\mu_i = \frac{1}{2} \nabla_i \left( \alpha \phi e^\Omega \nabla A_0 \right) - \frac{e^\Omega}{\mu_0} \left[ A_3 \left( \nabla A_0 \right)^2 - \nabla A_0 \cdot \nabla A_3 \right] = 0 \quad (23) $$

$$ EE^\nu_i + EE^i_\nu = \frac{1}{8} \alpha \phi \left( \nabla \Omega \right)^2 + 2 \alpha \phi \nabla^2 \nu - \frac{1}{2} \alpha \phi e^\Omega \left( \nabla A \right)^2 $$

$$ \frac{1}{2} \nabla^2 (\alpha \phi) + \alpha \left[ \frac{3}{2} \nabla^2 \phi + \frac{m}{\phi} \left( \nabla \phi \right)^2 \right] = 0 \quad (24) $$

$$ EE^i_i - EE^i_\nu = \frac{2}{\mu_0} \left[ e^\Omega / \left( \nabla A_0 \right)^2 - \left( \nabla A_3 \right)^2 - e^{-\Omega/2} \left( \nabla A_0 \right)^2 \right] $$

$$ \frac{1}{2} \nabla_i \left( \alpha \phi \nabla \Omega \right) + e^\Omega \alpha \phi \left( \nabla \phi \right)^2 + \nabla \left( e^{\Omega/2} \alpha \phi \nabla A \right) = 0 $$

$$ (3 + 2 \alpha \Omega) \nabla_i \left( \alpha \nabla \phi \right) = 0 \quad (26) $$

From the equation of (9), we can write Maxwell equations as

$$ E^i = \nabla \left[ e^{-\Omega/2} \nabla A_0 + e^{\Omega/2} \left( \nabla A_3 - \nabla A_0 \right) \right] = 0 \quad (27) $$

$$ E^\nu = \nabla \left[ e^{\Omega/2} \left( \nabla A_3 - \nabla A_0 \right) \right] = 0 \quad (28) $$

With a new potential from the last Eq. (28),

$$ \tilde{e} \times \nabla \tilde{A}_3 = e^{\Omega/2} \left( \nabla A_3 - \nabla A_0 \right), $$

$$ \tilde{e} \times \tilde{e} \times \nabla \tilde{A}_3 = e^{\Omega/2} \left( \tilde{e} \times \nabla \tilde{A}_3 - A \tilde{e} \times \nabla A \right), $$

$$ \nabla \left[ \tilde{e} \times \nabla \tilde{A}_3 \right] = \nabla \left[ A \tilde{e} \times \nabla A \right] = 0 \quad (29) $$

With the new potential, the other Maxwell Eq. (27) is written:

$$ \nabla \left[ e^{-\Omega/2} \nabla A_0 + A \tilde{e} \times \nabla \tilde{A}_3 \right] = 0. \quad (30) $$

We define a new complex potential

$$ \Phi = A_0 + i \tilde{A}_3 \quad (31) $$

and Maxwell equations become

$$ \nabla \left[ e^{-\Omega/2} \nabla \Phi - i A \tilde{e} \times \nabla \Phi \right] = 0 \quad (32) $$

where the real part is equal to Eq. (27) and the complex part is equal to (28). Eq. (23) is written in the form

$$ \nabla \left[ e^{\Omega/2} \alpha \nabla A - 2 \tilde{e} \times Im \left( \Phi^* \nabla \Phi \right) \right] = 0 \quad (33) $$
where $1/\mu_0 = 1$. From the last equation, we can define a new potential like

$$
\vec{e}_\varphi \times \vec{\nabla} h = e^{\Omega} \phi \alpha \vec{\nabla} A - 2 \vec{e}_\varphi \times \text{Im} \left( \Phi^* \vec{\nabla} \Phi \right)
$$

$$
\vec{e}_\varphi \times \vec{e}_\varphi \times \vec{\nabla} h = e^{\Omega} \phi \alpha \vec{e}_\varphi \times \vec{\nabla} A - 2 \vec{e}_\varphi \times \vec{e}_\varphi \times \text{Im} \left( \Phi^* \vec{\nabla} \Phi \right)
$$

$$
\vec{e}_\varphi \times \vec{\nabla} A = -\frac{e^{-\Omega}}{\phi \alpha} \left( \vec{\nabla} h + 2 \text{Im} \left( \Phi^* \vec{\nabla} \Phi \right) \right)
$$

and the equation of (23) becomes

$$
\vec{\nabla} \left[ \frac{e^{-\Omega}}{\phi \alpha} \left( \vec{\nabla} h + 2 \text{Im} \left( \Phi^* \vec{\nabla} \Phi \right) \right) \right] = 0.
$$

(35)

A new function is defined

$$
f = e^{\Omega/2} \alpha \phi
$$

and (25) is obtained:

$$
\frac{f}{\alpha \phi} \vec{\nabla} \left( \alpha \phi \vec{\nabla} f \right) - \vec{\nabla} f \cdot \vec{\nabla} f - \frac{f^2}{\alpha \phi} \nabla^2 (\alpha \phi) = 2 f \vec{\nabla} \Phi \cdot \vec{\nabla} \Phi^* - \left[ \vec{\nabla} h + 2 \text{Im} \left( \Phi^* \vec{\nabla} \Phi \right) \right]^2
$$

(37)

If one introduces the complex function

$$
\epsilon = f - |\Phi|^2 + ih
$$

field equations of (23) and (25) and Maxwell Eqs. (27) and (28) can be written as

$$
\left( \epsilon + \epsilon^* + |\Phi|^2 \right) \frac{1}{\alpha \phi} \vec{\nabla} \left( \alpha \phi \vec{\nabla} \epsilon \right) = \left( \vec{\nabla} \epsilon + 2 \Phi^* \vec{\nabla} \Phi \right) \cdot \vec{\nabla} \epsilon + \text{Re}^2 \left( \epsilon + |\Phi|^2 \right) \frac{\nabla^2 (\alpha \phi)}{\alpha \phi}
$$

(39)

$$
\left( \epsilon + \epsilon^* + |\Phi|^2 \right) \frac{1}{\alpha \phi} \vec{\nabla} \left( \alpha \phi \vec{\nabla} \Phi \right) = \left( \vec{\nabla} \epsilon + 2 \Phi^* \vec{\nabla} \Phi \right) \cdot \vec{\nabla} \Phi.
$$

(40)

The last term of Eq. (39) becomes zero from the field equation of (22). Additionally, this field equation permits us to choose $\alpha \phi = \rho$ which reduces the field equations to Ernst equations of (18) and (19). Besides, BD-Maxwell Ernst equations do not include the field equations of (24) and (26). For obtaining solutions, the chosen appropriate physical potentials for $\epsilon$ and $\Phi$ which satisfy the Ernst equations are not sufficient. They must also satisfy the field equations of (24) and (26). For that reason, we introduce another method for solution of BD-Maxwell equations of stationary, axially symmetric spacetimes.

### 2.3 Extended Tiwari-Nayak-Rai-Singh method

In this subsection, we try to analyze how to obtain BD-Maxwell solution from a known Einstein-Maxwell solution for stationary, axially symmetric spacetime. We start with writing a metric as

$$
ds^2 = -e^{2U} \left( dt + A_e d\varphi \right)^2 + e^{2K} \left( d\varphi^2 + dz^2 \right) + W_e e^{-2U} d\varphi^2
$$

(41)
where the subscript refers to Einstein metric functions. The first of the field equations in GR is obtained:

\[ G_{\rho}^{\rho} + G_{z}^{z} = \nabla^2 W_{\epsilon} = 0, \tag{42} \]

\[ Gd_{t}^{\phi} = W_{\epsilon} \nabla \left( \frac{e^{4U_{\epsilon}} \nabla A_{\epsilon}}{W_{\epsilon}} \right) + 2e^{2U_{\epsilon}} \left( \nabla A_{0} \nabla A_{3} - A_{\epsilon} \left( \nabla A_{0} \right)^2 \right). \tag{43} \]

When we choose the Einstein and BD field share the same Maxwell field which means \( A_{0} \) and \( A_{3} \) are the same for GR and BD, Eqs. (11) and (12) become more similar to Eqs. (42) and (43). From the first equations, we can write \( W_{\epsilon} = \phi W \), and it permit us to choose \( W = W_{e}^{k} \) and \( \phi = W^{1-k} \). In the next step, we take the metric function \( A \) of BD and \( A_{\epsilon} \) of GR are the same which satisfy the relation between \( U = U_{\epsilon} - \frac{1}{2} \ln \phi \). Finally, by using the equation \( Gdn_{t}^{t} + Gdn_{\phi}^{\phi} - \left( Gd_{t}^{t} + Gd_{\phi}^{\phi} \right) = 0 \), we obtain \( K = K_{e} + \frac{1}{4} (2\omega - 1 - k(2\omega + 3)) \ln \phi \). Previous studies took the metric functions \( K = K_{e} \) where \( k = \frac{2\omega - 1}{2\omega + 3} \) [15]. We can summarize that, if there is a GR solution in the form of metric (6), it has a corresponding BD solution with transformation of the metric function as

\[ A = A_{\epsilon}, \quad W = W_{e}^{k}, \quad \phi = W^{1-k}, \tag{44} \]

\[ U = U_{\epsilon} - \frac{1}{2} \ln \phi, \tag{45} \]

\[ K = K_{e} + \frac{1}{4} (2\omega - 1 - k(2\omega + 3)) \ln \phi. \tag{46} \]

Additionally, if the seed solution is in the form of Eq. (20), the corresponding BD solution may be obtained by the transformation as

\[ \alpha = \alpha_{e}, \quad \phi = \alpha_{e}^{1-k}, \quad \Omega = \Omega_{e} \tag{47} \]

\[ A = A_{\epsilon}, \quad \nu = \nu_{e} + \frac{2\omega - k(2\omega + 3)}{4} \ln \phi. \tag{48} \]

3. Examples of BD solutions with extended TNSR method

3.1 BD solution of Kerr-Taub-NUT metric

We also know Kerr-Taub-NUT (KTN) vacuum solution is

\[
\begin{align*}
\frac{dr^2}{\rho^2} &= \Delta - a^2 \sin^2 \theta dt^2 + 2 \Delta \alpha - a (\rho^2 + a^2 \sin^2 \theta) dt d\phi \\
&\quad + \frac{1}{\rho^2} \left( (\rho^2 + a^2 \sin^2 \theta)^2 \right) d\phi^2 + \rho^2 \left[ \frac{dr^2}{\Delta} + d\theta^2 \right]
\end{align*}
\]

(49)

where

\[
\Delta = r^2 - 2Mr + a^2 - n^2, \quad \rho^2 = r^2 + (n + a \cos \theta)^2 \]

\[
\alpha = a \sin^2 \theta - 2n \cos \theta
\]

and \( n \) is NUT parameter, \( a \) is rotational parameter, and \( M \) is the mass. By doing a coordinate transformation like \( r = e^R + M + \left( \frac{M^2 - a^2 + n^2}{4} \right) e^{-R} \), Kerr-Taub-NUT metric becomes
\[ ds^2 = -\frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta) dt^2 + \frac{2}{\rho^2} [\Delta \alpha - a (\rho^2 + a\alpha) \sin^2 \theta] dtd\phi + \frac{1}{\rho^2} (\rho^2 + a^2 \sin^2 \theta) d\phi^2 + \rho^2 [dR^2 + d\theta^2]. \]  

(50)

which is similar to metric (6) and the functions are

\[ \Delta = L^2 - 2ML + a^2 - n^2, \quad \rho^2 = L^2 + (n + a \cos \theta)^2 \]

where \( L = e^R + M + \left(\frac{M^2 - a^2 + n^2}{4}\right)e^{-R} \). From this metric we can easily find that

\[ e^{2U_e} = \frac{1}{\rho^2} (\Delta - a^2 \sin^2 \theta) \]
\[ A_e = \frac{a(\rho^2 + a\alpha)^2 \sin^2 \theta - \Delta \alpha}{\Delta - a^2 \sin^2 \theta} \]
\[ e^{2K_e} = \Delta - a^2 \sin^2 \theta \]
\[ W_e = \sqrt{\Delta} \sin \theta \]

From Eqs. (44) to (46) and by doing a coordinate transformation again, the solution looks

\[ ds^2 = \left[\sqrt{\Delta} \sin \theta\right]^{k-1} \left[-\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - \frac{2}{\rho^2} [a(\rho^2 + a\alpha) \sin^2 \theta - \Delta \alpha] dtd\phi + (\rho^2 + a\alpha)^2 \sin^2 \theta - \Delta \alpha^2] d\phi^2\right] + \left[\sqrt{\Delta} \sin \theta\right]^{(1-k)(1+2n-k(2\alpha+3))} \left(\frac{\rho^2 dR^2}{\Delta} + \rho^2 d\theta^2\right) \]

(51)

3.2 BD solution of Kerr-Newman-Taub-NUT metric

The solution for Kerr-Newman-Taub-NUT (KNTN) metric is the same as Eq. (49), but the functions are

\[ \Delta = r^2 - 2Mr + a^2 - n^2 + Q^2, \quad \rho^2 = r^2 + (n + a \cos \theta)^2 \]
\[ \alpha = a \sin^2 \theta - 2n \cos \theta \]

which correspond to mass of KTN solution which is \( M = M - Q^2/2r \). By doing the same procedure for KTN solution, we can easily obtain KNTN solution, but there are some differences such as for metric transformation, \( r = e^R + M + \left(\frac{M^2 - a^2 + n^2}{4}\right)e^{-R} \) is used. Brans-Dicke solution of KNTN becomes

\[ ds^2 = \left[\sqrt{\Delta} \sin \theta\right]^{k-1} \left[-\frac{\Delta - a^2 \sin^2 \theta}{\rho^2} dt^2 - \frac{2}{\rho^2} [a(\rho^2 + a\alpha) \sin^2 \theta - \Delta \alpha] dtd\phi + (\rho^2 + a\alpha)^2 \sin^2 \theta - \Delta \alpha^2] d\phi^2\right] + \left[\sqrt{\Delta} \sin \theta\right]^{(1-k)(1+2n-k(2\alpha+3))} \left(\frac{\rho^2 dR^2}{\Delta} + \rho^2 d\theta^2\right). \]

(52)
This solution was obtained by a sigma-model theory in [18] where the parameters have a relation like \( \alpha = \frac{1}{C_0 k(\omega + 3)/4} \). Moreover, analyses of the charged particle geodesics around this spacetime were discussed in [17] analytically.

### 3.3 BD solution of magnetized Kerr-Newman solution

Magnetized Kerr-Newman solution was found by Gibbons and his friends [19] as

\[
ds^2 = H \left[ -f dt^2 + \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right) \right] + \frac{S \sin^2 \theta}{H \Sigma} (d\varphi - \beta dt)^2
\]

\[
A = \Phi_0 dt + \Phi_3 (d\varphi - \beta dt).
\]

where

\[
\Sigma = r^2 + a^2 \cos^2 \theta \quad \Delta = r^2 + a^2 - 2mr + p^2 + q^2
\]

\[
f = \frac{\Delta \Sigma}{S} \quad S = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta
\]

\[
\beta = \frac{a(2mr - p^2 - q^2)}{S}
\]

where \( a \) is rotational parameter, \( p \) is magnetic charge, and \( q \) is electric charge of rotating black hole. We can easily compare (53) and (6) by doing \( r = e^R + M + \frac{M^2 - a^2 p^2 - q^2}{4} e^{-R} \) transformation. GR metric functions become

\[
e^{2U_r} = \frac{H^2 f \Sigma - S \beta^2 \sin^2 \theta}{H \Sigma} \quad A_r = \frac{S \beta \sin^2 \theta}{H^2 f \Sigma - S \beta^2 \sin^2 \theta}
\]

\[
W_e = \sqrt{\Delta} \sin \theta \quad e^{2K_r} = H^2 f \Sigma - S \beta^2 \sin^2 \theta.
\]

where

\[
\Sigma = L^2 + a^2 \cos^2 \theta \quad \Delta = L^2 + a^2 - 2mL + p^2 + q^2
\]

\[
f = \frac{\Delta \Sigma}{S} \quad S = (L^2 + a^2)^2 - a^2 \Delta \sin^2 \theta
\]

\[
\beta = \frac{a(2mL - p^2 - q^2)}{S}
\]

where \( L = e^R + M + \frac{M^2 - a^2 p^2 - q^2}{4} e^{-R} \). Metric functions of BD solution of magnetized KN are obtained from Eqs. (44) to (46), and by inverse transformation of \( R \to r \), metric of BD solution of magnetized KN becomes

\[
ds^2 = \left( \sqrt{\Delta} \sin \theta \right)^{k-1} \left[ \left( -Hf + \frac{S \beta^2 \sin^2 \theta}{H \Sigma} \right) dr^2 - 2 \frac{S \beta \sin^2 \theta}{H \Sigma} dtd\varphi + \frac{S \sin^2 \theta}{H \Sigma} d\varphi^2 \right]
\]

\[
\left( \sqrt{\Delta} \sin \theta \right)^{1+(1+2\omega-k(2\omega+3))/4} H \Sigma \left( \frac{dr^2}{\Delta} + d\theta^2 \right).
\]

\[
(59)
\]
4. GR limit of the solutions

According to the common belief, since BD parameter $\omega \to \infty$, the BD solutions reduce to corresponding GR ones. Contrary to this belief, several counter examples were presented in the literature [20–24]. In our study, the GR limit of the BD solutions is out of complexity. When $k \to 1$, BD transformation equations of (44)–(46) reduce to seed GR metric functions for any finite $\omega$ since scalar field $\phi$ becomes constant. It is obvious from the given examples that, as $k \to 1$, BD metrics reduce the corresponding GR ones.

5. Conclusion

In this section, we have studied to obtain corresponding BD or BD-Maxwell solution from any known solution of the Einstein or the Einstein-Maxwell theory for stationary, axially symmetric spacetimes in Jordan frame. First we present that, although several field equations of BD are not included by Ernst equations, BD field equations can be written in the form of Ernst Eqs. BD solutions can be obtained by selecting the appropriate physical potentials or by integrating Ernst equations, but it should be remembered that the equations which are not included in the Ernst equations should be provided.

In order to obtain BD solutions, we have constructed two parameter solution-generating techniques. It was seen that, in previous works, it was studied with one parameter. From any given seed GR solution of Eqs. (6) or (20), the corresponding BD solution can be obtained by the two parameter solution-generating techniques. In order to show how this method works, we have constructed several known solutions and also some new solutions for BD theory. We have also discussed the GR limit of these solutions.

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