Polar varieties and bipolar surfaces of minimal surfaces in the $n$-sphere

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Received: 26 April 2021 / Accepted: 4 July 2021 / Published online: 6 August 2021
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Abstract

For a given minimal surface in the $n$-sphere, two ways to construct a minimal surface in the $m$-sphere are given. One way constructs a minimal immersion. The other way constructs a minimal immersion which may have branch points. The branch points occur exactly at each point where the original minimal surface is geodesic. If a minimal surface in the 3-sphere is given, then these ways construct Lawson’s polar variety and bipolar surface.

Keywords Conformal map · Harmonic map · Minimal surface · Clifford algebra

1 Introduction

Minimal surfaces in the unit $(r-1)$-sphere $S^{r-1}$ $(r \geq 4)$ are a classical research subject common to different research fields. In the theory of surfaces, they are surfaces with vanishing mean curvature vector. In the theory of harmonic maps, parametrized minimal surfaces in $S^{r-1}$ are conformal harmonic maps. In the theory of spectral geometry, parametrized minimal spheres in $S^{r-1}$ embedded in the Euclidean $r$-space $\mathbb{E}^r$ are eigenmaps of the Laplace-Beltrami operator.

Giving parametrizations $f: \Sigma \rightarrow S^{r-1}$ of minimal surfaces in $S^{r-1}$ explicitly is one simple but important problem. We consider this problem by restricting ourselves in the case where $\Sigma$ is oriented and compact without boundary in this paper.

Parametrizations have been given for each minimal sphere. A minimal sphere is a minimal surface of constant positive Gaussian curvature with genus zero. Minimal spheres in $S^3$ are totally geodesic spheres. Borůvka [4–6] constructed linearly full immersions from $S^2$ into $S^{2r-2}$ for each $r \geq 2$. Calabi [7] showed that Borůvka’s spheres were the only minimal spheres in $S^{r-1}$ for each $r \geq 3$.

For minimal surfaces which are not of constant Gaussian curvature or not genus zero, parametrizations were given under some conditions. Bobenko [2] gave a parametrization for each minimal torus in $S^3$. Sharipov [14] gave a parametrization for each complex normal minimal torus in $S^5$.

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Since it is difficult to completely solve the problem at present, it is meaningful to define transforms of minimal surfaces. Lawson’s polar varieties and bipolar surfaces [11] are transforms of minimal surfaces in $S^3$. Polar varieties are minimal surfaces in $S^3$ and bipolar surfaces are minimal surfaces in $S^5$. Bolton and Vrancken [3] generalized polar varieties and defined two transforms between minimal surfaces in $S^5$ with non-circular ellipse of curvature. Antić and Vrancken [1] generalized Bolton and Vrancken’s transforms and defined transforms between minimal surfaces in $S^{2r-1}$ for each $r \geq 2$.

In this paper, we will define two transforms for a given conformal immersion $f : \Sigma \to S^{r-1}$ ($r \geq 4$) which is not necessarily minimal. If a given conformal immersion is minimal into $S^3$, then our polar varieties and bipolar surfaces are Lawson’s polar varieties and bipolar surfaces, respectively. Our polar varieties are neither Bolton and Vrancken’s transforms nor Antić and Vrancken’s transforms.

Lawson used the exterior algebra in order to define a polar variety and a bipolar surface. For a minimal immersion $f : \Sigma \to S^3 \subset \mathbb{E}^4$ with local holomorphic coordinate $z$ and the metric $2F|dz|^2$ on $\Sigma$ induced by $f$, the polar variety $f^* : \Sigma \to S^3$ is the Hodge dual of the map

$$\frac{1}{iF}(f \wedge df \wedge \bar{df})$$

and the bipolar surface is the map

$$f \wedge f^* : \Sigma \to S^5.$$

We use the Clifford algebra $\mathbb{C}\ell(\mathbb{E}')$ instead to define our polar variety and bipolar surface (Definition 1). Let $\binom{m}{n}$ be the binomial coefficient. For a conformal immersion $f : \Sigma \to S^{r-1} \subset \mathbb{E}' \subset \mathbb{C}\ell(\mathbb{E}')$ which is not necessarily minimal, our polar variety and bipolar surface are maps

$$f^{\xi}_{df(T\Sigma)} : \Sigma \to S^{(r-1)} \subset \mathbb{C}\ell(\mathbb{E}'), \quad \xi_{df(T\Sigma)} : \Sigma \to S^{(r-1)} \subset \mathbb{C}\ell(\mathbb{E}')$$

respectively. The map $\xi_{df(T\Sigma)}$ is the oriented volume form of $df(T\Sigma)$. It is a counterpart of $(df \wedge df)/iF$ and considered as the generalized Gauss map of $f$. We show that the differential map of a polar variety vanishes at the points where the metric induced by a given conformal immersion is equal to that of its bipolar surface (Theorem 1).

Lawson showed that polar varieties and bipolar surfaces of minimal surfaces in $S^3$ are minimal. We show that if a given conformal immersion from $\Sigma$ to $S^{r-1}$ is minimal, then our polar variety and bipolar surface are minimal (Theorems 2, 3). For the proof, we write the equation for a map being a conformal map and that for a map being a harmonic map in terms of the Clifford algebra. These are variants of equations written in terms of quaternions [9, 13].

Lawson showed that the singularities of a polar variety of a minimal surface in $S^3$ occur precisely at the points where the Gaussian curvature of a given minimal surface in $S^3$ is equal to one. These points are geodesic points of the given minimal surface. We show that a geodesic point of a given minimal immersion into $S^{r-1}$ corresponds to a branch point of our polar variety (Theorem 4).
2 The Clifford algebra

We review the Clifford algebra [12] which plays an important role in this paper.

For a real vector space $W$, we denote the dimension of $W$ by $\dim W$.

Let $V_r$ be an oriented real vector space with $\dim V_r = r$. We fix a positively-oriented basis $e_1, \ldots, e_r$ of $V_r$. Let $Q$ be a positive definite quadratic form on $V_r$ such that $e_1, \ldots, e_r$ is an orthonormal basis. A positive definite inner product $B$ of $V_r$ is associated with $Q$ by

$$B(v_1, v_2) := \frac{1}{2}(Q(v_1 + v_2) - Q(v_1) - Q(v_2)) \quad (v_1, v_2 \in V_r).$$

The vector space $V_r$ with inner product $B$ is the $r$-dimensional Euclidean space $\mathbb{E}^r$. We denote the unit hypersphere in $\mathbb{E}^r$ centered at the origin by $S^{r-1}$:

$$S^{r-1} = \{ v \in \mathbb{E}^r : Q(v) = 1 \}.$$

The Clifford algebra $\mathcal{C}l(V_r)$ is the algebra generated by $e_1, \ldots, e_r$ subject to the relations

$$e_i e_j + e_j e_i = -2\delta_{ij} \quad (i, j = 1, \ldots, r).$$

Then, $\mathcal{C}l(V_r)$ is a $2^r$-dimensional real vector space. A different choice of a positively-oriented basis $\tilde{e}_1, \ldots, \tilde{e}_r$ generates an algebra being isomorphic to $\mathcal{C}l(V_r)$. Hence,

$$\tilde{e}_i \tilde{e}_j + \tilde{e}_j \tilde{e}_i = -2\delta_{ij} \quad (i, j = 1, \ldots, r).$$

Let $\mathcal{C}l(V_r)^\infty$ be the group of all invertible elements of $\mathcal{C}l(V_r)$ and $F_r$ be the finite subgroup of $\mathcal{C}l(V_r)^\infty$ generated by $e_1, \ldots, e_r$. The set $F_r$ forms a basis of $\mathcal{C}l(V_r)$. We denote by $\mathcal{C}l(V_r)^{(i)}$ the linear subspace of $\mathcal{C}l(V_r)$ of dimension $\binom{r}{i}$ spanned by

$$e_{i_1} e_{i_2} \cdots e_{i_j} \quad (1 \leq i_1 < i_2 < \cdots < i_j \leq r).$$

Then, $\mathcal{C}l(V_r)^{(0)} = \mathbb{R}$ and $\mathcal{C}l(V_r)^{(1)} = V_r$. The Clifford algebra $\mathcal{C}l(V_r)$ has a direct decomposition

$$\mathcal{C}l(V_r) = \bigoplus_{i=0}^r \mathcal{C}l(V_r)^{(i)}.$$

For an element $\phi \in \mathcal{C}l(V_r)$, we denote by $\phi^{(i)}$ the $\mathcal{C}l(V_r)^{(i)}$-part of $\phi$. We see that $\mathcal{C}l(V_r)^{(i)} \setminus \{0\} \subset \mathcal{C}l(V_r)^\infty$ for each $i$. The adjoint representation of $\mathcal{C}l(V_r)^\infty$ is

$$\text{Ad}_j(\phi) = \lambda \phi \lambda^{-1} \quad (\lambda \in \mathcal{C}l(V_r)^\infty, \phi \in \mathcal{C}l(V_r)).$$

An involutive automorphism $\alpha$ of $\mathcal{C}l(V_r)$ is defined by extending the vector space isomorphism $\alpha : V_r \rightarrow V_r$ with $\alpha(v) = -v$ for any $v \in V_r$. An anti-automorphism $(\cdot)^\prime : \mathcal{C}l(V_r) \rightarrow \mathcal{C}l(V_r)$ is defined by extending the map

$$(v_1 \cdots v_j)^\prime = v_j \cdots v_1 \quad (v_1, \ldots, v_j \in V_r).$$

Let $\mathfrak{M}(V_r)$ be a subgroup of $\mathcal{C}l(V_r)^\infty$ that consists of all products

$$v_1 v_2 \cdots v_{2l} \quad (v_1, \ldots, v_{2l} \in V_r).$$

The spin group $\text{Spin}(V_r)$ of $V_r$ is the subgroup of $\mathfrak{M}(V_r)$ of dimension $r(r-1)/2$ defined by
Spin($V_r$) = {$\hat{v} \in \mathcal{M}(V_r) : \alpha(\hat{v})^t\hat{v} = 1$}.

The set \{Ad$_{\hat{v}} : \hat{v} \in \text{Spin}(V_r)\} forms the special orthogonal group of $V_r$.

The element

$$\hat{e}_{V_r} = e_1 \cdots e_r$$

is called the oriented volume form of $V_r$. The oriented volume form of $V_r$ is independent of the choice of orthonormal positively-oriented basis of $V_r$.

3 Orthogonal complex structures

We use a complex structure of a vector space in order to interpret conformality of maps in terms of the Clifford algebra. In this section, we extend the quadratic form $Q$ on $V_r$ to $\mathcal{C}(V_r)$ and explain an orthogonal complex structure of $\mathcal{C}(V_r)$ by the Clifford algebra.

We use $Q$ to denote the positive definite quadratic form

$$Q(\phi) = ((\alpha(\phi))^t\phi)^{(0)} = (\phi(\alpha(\phi))^t)^{(0)} \quad (\phi \in \mathcal{C}(V_r))$$

on $\mathcal{C}(V_r)$. The restriction of $Q$ to $V_r$ is the positive definite quadratic form in the definition of $\mathcal{C}(V_r)$. We use $B$ for the symmetric bilinear form associated with $Q$. Since

$$Q(e_i \cdots e_j) = 1 \quad (1 \leq i_1 < \cdots < i_j \leq r),$$

$$B(e_i \cdots e_j, e_i \cdots e_k) = 0 \quad (e_i \cdots e_j \neq e_i \cdots e_k),$$

the $F_r$ forms an orthonormal basis of $\mathcal{C}(V_r)$.

The set

$$\mathcal{S}^{2r-1} = \{\phi \in \mathcal{C}(V_r) : Q(\phi) = 1\}$$

is the unit hypersphere with center at the origin in $\mathcal{C}(V_r)$. Then, $S' \subset \mathcal{S}^{2r-1}$.

We consider linear subspaces of $V_r$. If $1 \leq n \leq r$, then we may regard $V_n$ as a subspace of $V_r$ spanned by $e_1, \ldots, e_n$. For $\hat{v} \in \text{Spin}(V_r)$, we have

$$(\text{Ad}_{\hat{v}}e_1) \cdots (\text{Ad}_{\hat{v}}e_n) = \text{Ad}_{\hat{v}}\hat{e}_{V_n} \in \mathcal{C}(V_r)^{(n)} \cap \mathcal{S}^{2r-1}.$$

Let

$$G_n(V_r) = \{\pm \text{Ad}_{\hat{v}}(\hat{e}_{V_n}) : \hat{v} \in \text{Spin}(V_r)\} \subset \mathcal{C}(V_r)^{(n)} \cap \mathcal{S}^{2r-1}.$$

We see that $G_r(V_r) = \{\hat{e}_{V_r}, -\hat{e}_{V_r}\}$. The element $-\hat{e}_{V_r}$ is the volume form of $V_r$ with the opposite orientation. We may consider $G_n(V_r)$ as the oriented Grassmannian of $n$-dimensional linear subspaces in $V_r$. We have

$$(\alpha(\xi))^t = (-1)^{i+1}\xi, \quad \xi^2 = (-1)^{i+1}$$

($$\xi \in G_{4m+2l+k}(V_r), m = 0, 1, 2, \ldots, l = 0, 1, \ldots, k = 1, 2$).

Let GL($\mathcal{C}(V_r)$) be the general linear group of the real vector space $\mathcal{C}(V_r)$ and O($\mathcal{C}(V_r)$) be the orthogonal group of $\mathcal{C}(V_r)$. Let $\mu^L : \mathcal{C}(V_r)^{\times} \rightarrow \text{GL}(\mathcal{C}(V_r))$ and
Lemma 1 Let $\lambda \in C\ell(V_r)$. If $(\alpha(\lambda))' \lambda = 1$, then $\mu^L(\lambda)$, $\mu^R(\lambda) \in O(C\ell(V_r))$. Moreover, the sets $\mu^L(G_n(V_r))$ and $\mu^R(G_n(V_r))$ are subsets of $O(C\ell(V_r))(1 \leq n \leq r)$.

Proof If $\alpha(\lambda)' \lambda = 1$, then $\lambda(\alpha(\lambda))' = 1$. Let $\phi \in C\ell(V_r)$. We have

$$Q(\lambda \phi) = ((\alpha(\lambda)\phi)' \lambda \phi)^{(0)} = ((\alpha(\phi))' \alpha(\lambda)' \lambda \phi)^{(0)} = Q(\phi),$$

$$Q(\phi \lambda) = (\phi \lambda(\alpha(\phi))')^{(0)} = (\phi \lambda \alpha(\phi)' \phi)^{(0)} = Q(\phi).$$

Then, $B(\lambda \phi, \lambda \psi) = B(\phi \lambda, \psi \lambda) = B(\phi, \psi)$. Hence, $\mu^L(\lambda)$, $\mu^R(\lambda) \in O(C\ell(V_r))$. Since $\alpha(\xi)' \xi = 1$ for each $\xi \in G_n(V_r)$, the sets $\mu^L(G_n(V_r))$ and $\mu^R(G_n(V_r))$ are subsets of $O(C\ell(V_r))(1 \leq n \leq r)$.

Set

$$\mathcal{C} = \bigcup_{m=0,1,2,...} (G_{4m+1}(V_r) \cup G_{4m+2}(V_r)).$$

Lemma 2 If $\xi \in \mathcal{C}$, then $\mu^L(\xi)$ and $\mu^R(\xi)$ are orthogonal complex structures of $C\ell(V_r)$.

Proof If $\xi \in \mathcal{C}$, then $\mu^L(\xi)$, $\mu^R(\xi) \in O(C\ell(V_r))$ by Lemma 1 and $\xi^2 = -1$ by (1). Hence $\mu^L(\xi)$ and $\mu^R(\xi)$ are orthogonal complex structures of $C\ell(V_r)$.

Let $W_2$ be a two-dimensional oriented linear subspace $V_r$. If $r = 2$, then we assume that the orientation of $W_2$ is the same as that of $V_r$. If $\xi \in \mathcal{C}$ and $\xi W_2 = W_2$, then $\mu^L(\xi)$ is an orthogonal complex structure of $W_2$. Similarly if $\xi \in \mathcal{C}$ and $W_2 \xi = W_2$, then $\mu^R(\xi)$ is an orthogonal complex structure of $W_2$. The volume form $\xi_{W_2}$ of $W_2$ is associated with $W_2$.

Lemma 3 The endomorphisms $\mu^L(\xi_{W_2})$ and $\mu^R(\xi_{W_2})$ are orthogonal complex structures of $W_2$. Moreover, $\mu^L(\xi_{W_2}) = -\mu^R(\xi_{W_2})$ on $W_2$.

Proof There exists $\hat{\psi} \in \text{Spin}(V_r)$ such that $e_1 := \text{Ad}_\psi e_1, e_2 := \text{Ad}_\psi e_2$ is an orthonormal positively-oriented basis of $W_2$. Then, $W_2 = \text{Ad}_\psi V_2$ and $\xi_{W_2} = \text{Ad}_\psi \xi_{V_2} \in \mathcal{C}$.

Since

$$\xi_{W_2} W_2 = \text{Ad}_\psi (e_1 e_2 V_2) = \text{Ad}_\psi V_2 = W_2,$$

$$W_2 \xi_{W_2} = \text{Ad}_\psi (V_2 e_1 e_2) = \text{Ad}_\psi V_2 = W_2,$$

the endomorphisms $\mu^L(\xi_{W_2})$ and $\mu^R(\xi_{W_2})$ are complex structures of $W_2$. Since
\[ \xi_{W_2}e_1 = \text{Ad}(e_1 e_2 e_1) = \text{Ad}e_2 = e_2, \]
\[ \xi_{W_2}e_2 = \text{Ad}(e_1 e_2 e_2) = - \text{Ad}e_1 = -e_1, \]
\[ e_1 \xi_{W_2} = \text{Ad}(e_1 e_1 e_2) = - \text{Ad}e_2 = -e_2, \]
\[ e_2 \xi_{W_2} = \text{Ad}(e_2 e_1 e_2) = \text{Ad}e_1 = e_1, \]
we have \( \mu^L(\xi_{W_2}) = - \mu^R(\xi_{W_2}) \) on \( W_2 \). \( \square \)

### 4 Conformal immersions

We explain a conformal immersion from a two-dimensional oriented manifold to Euclidean space by the Clifford algebra.

We assume that \( r \geq 3 \) and that a linear map is not the zero map. A linear map \( \gamma : V_2 \to \mathcal{C}^\ell(V_r) \) is said to be **conformal** if there exists a positive real number \( \rho \) such that
\[
B(\gamma(v_1), \gamma(v_2)) = \rho B(v_1, v_2) \quad (v_1, v_2 \in V_2).
\]

The endomorphism \( J = \mu^L(\xi_{V_2}) = - \mu^R(\xi_{V_2}) \) on \( V_2 \) is the orthogonal complex structure of \( V_2 \) such that \( Je_1 = e_2 \) by Lemma 3. Then, \( \gamma : V_2 \to \mathcal{C}^\ell(V_r) \) is conformal if and only if there exists an orthogonal complex structure \( Y \) of \( \mathcal{C}^\ell(V_r) \) such that
\[
\gamma \circ J = Y \circ \gamma.
\]

We have the following lemma by Lemma 2 immediately.

**Lemma 4** Let \( \gamma : V_2 \to \mathcal{C}^\ell(V_r) \) be a linear map. If there exists \( \lambda \in \mathbb{C} \) with \( \gamma \circ J = \lambda \gamma \) or \( \gamma \circ J = \gamma \lambda \), then \( \gamma \) is conformal.

Assume that \( \gamma(V_2) \subset V_r \) and \( \dim \gamma(V_2) = 2 \). Then, \( \gamma(e_1), \gamma(e_2) \) is a basis of \( \gamma(V_2) \). We call the orientation of the ordered basis \( \gamma(e_1), \gamma(e_2) \) of \( \gamma(V_2) \) positive.

**Lemma 5** A linear map \( \gamma : V_2 \to V_r \) is conformal if and only if
\[
\gamma \circ J = \xi_{\gamma(V_2)} \gamma = -\gamma \xi_{\gamma(V_2)}.
\]

**Proof** By Lemma 3, the endomorphisms \( \mu^L(\xi_{\gamma(V_2)}) \) and \( \mu^R(\xi_{\gamma(V_2)}) \) are complex structures of \( \gamma(V_2) \) and \( \mu^L(\xi_{\gamma(V_2)}) = - \mu^R(\xi_{\gamma(V_2)}) \) on \( \gamma(V_2) \). If \( \gamma \) is conformal, then
\[
\xi_{\gamma(V_2)} = \frac{\gamma(e_1) \gamma(e_2)}{\sqrt{Q(\gamma(e_1))Q(\gamma(e_2))}}.
\]
Since \( Je_1 = e_2 \), we have \( \gamma \circ J = \xi_{\gamma(V_2)} \gamma = -\gamma \xi_{\gamma(V_2)} \). The converse is trivial. \( \square \)

The wedge product of \( \mathcal{C}^\ell(V_r) \) explains a relations between two conformal linear maps.

**Lemma 6** Let \( \gamma : V_2 \to V_r \) be a conformal linear map such that \( \gamma(V_2) \) is a two-dimensional subspace. Assume that \( \eta : V_2 \to \mathcal{C}^\ell(V_r) \) be a linear map.
The map $\eta$ is conformal with $\eta \circ J = \eta \xi_p(V_j)$ if and only if $\eta \wedge \gamma = 0$.

(ii) The map $\eta$ is conformal with $\eta \circ J = -\xi_{p(V_j)}\eta$ if and only if $\gamma \wedge \eta = 0$.

**Proof** (i) By Lemma 5, we have

$$(\eta \wedge \gamma)(v, Jv) = \eta(v)\gamma(Jv) - \eta(Jv)\gamma(v) = (\eta(v)\xi_p(V_j) - \eta(Jv))\gamma(v)$$

for any $v \in V_2$. Since $\gamma(v)$ is invertible for any $v \in V_2 \setminus \{0\}$, the map $\eta$ satisfies equation $\eta \circ J = \eta \xi_p(V_j)$ if and only if $\eta \wedge \gamma = 0$. If $\eta \circ J = \eta \xi_p(V_j)$, then the map $\eta$ is conformal by Lemma 4.

(ii) By Lemma 5, we have

$$(\gamma \wedge \eta)(v, Jv) = \gamma(v)\eta(Jv) - \gamma(Jv)\eta(v) = \gamma(v)(\eta(Jv) + \xi_{p(V_j)}\eta(v))$$

for any for any $v \in V_2$. Since $\gamma(v)$ is invertible for any $v \in V_2 \setminus \{0\}$, the map $\eta$ satisfies equation $\eta \circ J = -\xi_{p(V_j)}\eta$ if and only if $\gamma \wedge \eta = 0$. If $\eta \circ J = -\xi_{p(V_j)}\eta$, then the map $\eta$ is conformal by Lemma 4.

We use Lemma 4, Lemma 5 and Lemma 6 in order to explain conformal immersions from a two-dimensional Riemannian oriented manifold to the Clifford algebra.

In the following, we assume that all manifolds and all maps are smooth. Let $M$ be a manifold. For a vector bundle $E$ over $M$, we denote a fiber of $E$ at $p$ by $E_p$. We denote the tangent bundle of $M$ by $TM$ and the cotangent bundle by $T^*M$.

Let $\Sigma$ be a two-dimensional Riemannian oriented manifold. The metric on $\Sigma$ and the orientation of $\Sigma$ determines an orthogonal complex structure $J_\Sigma$ on $\Sigma$ such that the orientation of the ordered frame field $X$, $J_\Sigma X$ is positive for any local nowhere-vanishing section $X$ of $T\Sigma$. Define an operator $*$ on one-forms on $\Sigma$ by $* \omega := \omega J_\Sigma$.

An immersion $f : \Sigma \rightarrow C\ell(V_r)$ is conformal if the differential map

$$df_p : (T\Sigma)_p \rightarrow (T\ell(V_r))_{f(p)}$$

is conformal at each point $p \in \Sigma$. Applying Lemmas 4, 5 and 6 for $df$, we have the following Lemmas 7, 8 and 9 immediately.

**Lemma 7** Let $f : \Sigma \rightarrow C\ell(V_r)$ be an immersion. If there exists a map $\lambda : \Sigma \rightarrow \mathbb{C}$ with $* df = \lambda df$ or $* df = df \lambda$, then $f$ is conformal.

Let $e_1, e_2$ be an orthonormal positively-oriented basis of $(T\Sigma)_p$. Then $df_p(e_1), df_p(e_2)$ is a basis of $df_p((T\Sigma)_p)$ at each $p \in \Sigma$. We call the orientation of the ordered basis $df_p(e_1), df_p(e_2)$ positive at any point $p \in \Sigma$. Define a map $\xi_{df(T\Sigma)} : \Sigma \rightarrow G_2(V_r) \subset \mathbb{C}$ by $\xi_{df(T\Sigma)}(p) := \xi_{df((T\Sigma)_p)}$. The map $\xi_{df(T\Sigma)}$ is considered as the generalized Gauss map of $f$.

**Lemma 8** Let $f : \Sigma \rightarrow V_r$ be an immersion. The differential map $df$ satisfies equation

$$* df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}$$

if and only if $f$ is conformal.

**Lemma 9** Let $f : \Sigma \rightarrow V_r$ be a conformal immersion with $* df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}$. Assume that $g : \Sigma \rightarrow C\ell(V_r)$ is an immersion.
1. The immersion $g$ is conformal with $\ast dg = dg \xi_{df(T\Sigma)}$ if and only if $dg \wedge df = 0$.

2. The immersion $g$ is conformal with $\ast dg = -\xi_{df(T\Sigma)}^2 df$ if and only if $df \wedge dg = 0$.

We omit the proofs of these lemmas.

5 Polar varieties and bipolar surfaces

Lawson [11] defined a polar variety and a bipolar surface of a minimal immersion into $S^3$. We define a polar variety and a bipolar surface of a conformal immersion which is not necessarily minimal into $S^{r-1}$ ($r \geq 4$).

We review Lawson’s polar varieties and bipolar surfaces in terms of the Clifford algebra. Let $f : \Sigma \to S^3 \subset V_4$ be a minimal immersion. Since $f$ is conformal, we have $\ast df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}$ by Lemma 8. The five-dimensional sphere $\mathcal{C}(V_4)^{(2)} \cap \bar{S}^{15}$ is a codomain of $\xi_{df(T\Sigma)}$. The map $f$ is orthogonal to $df(T\Sigma)$. Let $e_1, e_2$ be an orthonormal positively-oriented basis of $(T\Sigma)^\perp$ and $U_p$ be a three-dimensional linear subspace of $V_4$ spanned by $df_p(e_1), df_p(e_2), f(p)$. We call the orientation of the ordered basis $df_p(e_1), df_p(e_2), f(p)$ of $U_p$ positive. Then, a map $\xi_{U_p} : \Sigma \to G_3(V_4)$ is defined by $\xi_{U_p}(p) := \xi_{df_p}^1(p \in \Sigma)$. The three-dimensional sphere $\mathcal{C}(V_4)^{(3)} \cap \bar{S}^{15}$ is a codomain of $\xi_{U_p}$. Assume that $f$ is minimal. The map $g : \Sigma \to S^3$ such that $\xi_{U_p}g = \xi_{V_4}$ is called the polar variety of $f$. The map $f^{\ast}_{\xi_{U_p}} = -\xi_{df(T\Sigma)} : \Sigma \to \mathcal{C}(V_4)^{(2)} \cap \bar{S}^{2r-1}$ is called the bipolar surface of $f$.

Lawson defined a polar variety and a bipolar surface without using the minimality of $f$. We define analogues of a polar variety and a bipolar surface for a conformal immersion as follows.

Definition 1 Let $f : \Sigma \to S^{r-1}$ ($r \geq 4$) be a conformal immersion with $\ast df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}$. The polar variety of $f$ is the map $\xi_{U_p} : \Sigma \to \mathcal{C}(V_4)^{(3)} \cap \bar{S}^{2r-1}$. The bipolar surface of $f$ is the map $\xi_{df(T\Sigma)} : \Sigma \to \mathcal{C}(V_4)^{(2)} \cap \bar{S}^{2r-1}$.

We see that $\mathcal{C}(V_4)^{(3)} \cap \bar{S}^{2r-1}$ is a sphere with dimension $\binom{r}{3} - 1$ and $\mathcal{C}(V_4)^{(2)} \cap \bar{S}^{2r-1}$ is a sphere with dimension $\binom{r}{2} - 1$. Since $f(\Sigma) \subset G_1(V_4)$, $\xi_{df(T\Sigma)}(\Sigma) \subset G_2(V_4)$ and $\xi_{U_p}(\Sigma) \subset G_3(V_4)$, we have

\[ f^2 = -1, \quad \xi_{df(T\Sigma)}^2 = -1, \quad \xi_{U_p}^2 = 1 \]  \hspace{1cm} (3)

by (1).

A conformal immersion $f$, the polar variety $\xi_{U_p}$ of $f$, the bipolar surface $\xi_{df(T\Sigma)}$ of $f$ and their differential maps are related as follows.

Lemma 10 Equations

\[ f\xi_{df(T\Sigma)} = \xi_{df(T\Sigma)} f = \xi_{U_p}, \]  \hspace{1cm} (4)

\[ \xi_{df(T\Sigma)} d\xi_{df(T\Sigma)} + \xi_{U_p} d\xi_{U_p} - fdf = 0. \]  \hspace{1cm} (5)

hold.
Proof
Let \( e_1, e_2 \) be a local orthonormal positively-oriented basis of \((T \Sigma)_p\). Since \( r \geq 4 \), there exists \( \hat{v} \in \text{Spin}(V_r) \) such that

\[
\begin{align*}
df_p(e_1) &= (Q(df_p(e_1)))^{1/2} \text{Ad}_{\hat{v}}(e_1), \\
df_p(e_2) &= (Q(df_p(e_2)))^{1/2} \text{Ad}_{\hat{v}}(e_2), \\
f(p) &= \text{Ad}_{\hat{v}}(e_3).
\end{align*}
\]

Then,

\[
\begin{align*}
\xi_{U \ell_p} &= \text{Ad}_{\hat{v}}(e_1 e_2 e_3) = \text{Ad}_{\hat{v}}(e_1) \text{Ad}_{\hat{v}}(e_2) \text{Ad}_{\hat{v}}(e_3) = \xi_{df(T \Sigma)} f(p), \\
\xi_{U \ell_p} &= \text{Ad}_{\hat{v}}(e_3) = \text{Ad}_{\hat{v}}(e_1 e_2) = \text{Ad}_{\hat{v}}(e_1) \text{Ad}_{\hat{v}}(e_2) = f(p) \xi_{df(T \Sigma)}.
\end{align*}
\]

Hence, Eq. (4) holds.

Differentiating (4), we have

\[
d\xi_{U \ell} = df \, \xi_{df(T \Sigma)} + f \, d\xi_{df(T \Sigma)}.
\]

Then,

\[
\begin{align*}
\xi_{U \ell} \, d\xi_{U \ell} &= f \, \xi_{df(T \Sigma)} (df \, \xi_{df(T \Sigma)} + f \, d\xi_{df(T \Sigma)}) \\
&= f \, \xi_{df(T \Sigma)} df \, \xi_{df(T \Sigma)} + f \, \xi_{df(T \Sigma)} f \, d\xi_{df(T \Sigma)} \\
&= f \, df - \xi_{df(T \Sigma)} d\xi_{df(T \Sigma)}
\end{align*}
\]

by (4). Hence, Eq. (5) holds. \( \square \)

The metrics on \( \Sigma \) induced by \( f \), \( \xi_{df(T \Sigma)} \) and \( \xi_{U \ell} \) are related as follows.

Lemma 11
The metrics \( f^* B \), \((\xi_{df(T \Sigma)})^* B \) and \((\xi_{U \ell})^* B \) on \( \Sigma \) satisfy equation

\[
f^* B + (\xi_{U \ell})^* B - (\xi_{df(T \Sigma)})^* B = 0.
\]

Proof
The differential of \( \xi_{df(T \Sigma)} \) is

\[
d\xi_{df(T \Sigma)} = -\xi_{df(T \Sigma)} (df - \xi_{U \ell} \, d\xi_{U \ell}) = f (- \ast df - d\xi_{U \ell}),
\]

by (3) and Lemma 10. Since \( \ast df_p((T \Sigma)_p) \subset V_r \) and \((d\xi_{U \ell})_p((T \Sigma)_p) \subset \mathcal{C}_c(V_r)\) at each \( p \in \Sigma \), the linear subspace \( \ast df_p((T \Sigma)_p) \) is orthogonal to \((d\xi_{U \ell})_p((T \Sigma)_p) \) at each \( p \in \Sigma \).

Since \( G_1(V_r) \) is a codomain of \( f \) and \( G_2(V_r) \) is a codomain of \( \xi_{df(T \Sigma)} \), endomorphisms \( \mu^L(f(p)) \) and \( \mu^L(\xi_{df_p((T \Sigma)_p)}) \) are orthogonal transformation of \( \mathcal{C}_c(V_r) \) at each \( p \in \Sigma \) by Lemma 1. Then,

\[
B(d\xi_{df(T \Sigma)}(X), d\xi_{df(T \Sigma)}(Y)) = B(- \ast df(X), - \ast df(Y)) + B(-d\xi_{U \ell}(X), -d\xi_{U \ell}(Y)) = B(\xi_{df(T \Sigma)} df(X), \xi_{df(T \Sigma)} df(Y)) + B(d\xi_{U \ell}(X), d\xi_{U \ell}(Y)) = B(df(X), df(Y)) + B(d\xi_{U \ell}(X), d\xi_{U \ell}(Y))
\]

for any local sections \( X \) and \( Y \) of \( T \Sigma \). Hence, the lemma holds. \( \square \)
Lawson’s bipolar surface is an immersion and his polar variety is an immersion admitting singularities. The same is true for our bipolar surface and polar variety.

**Theorem 1** Let \( f : \Sigma \to S^{r-1} \) be a conformal immersion. Then, the bipolar surface of \( f \) is an immersion. The differential map of the polar variety of \( f \) vanishes at \( p \in \Sigma \) if and only if the metric on \( \Sigma \) induced by \( f \) is equal to the metric on \( \Sigma \) induced by the bipolar surface of \( f \) at \( p \).

**Proof** Let \( f : \Sigma \to S^{r-1} \) be a conformal immersion with \( \ast df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}. \) By Lemma 11, we have

\[
(\xi_{df(T\Sigma)})^\ast B = f^\ast B + (\xi_{Uf})^\ast B.
\]

Since \( f \) is an immersion, \( \xi_{df(T\Sigma)} \) is an immersion.

At a point \( p \in \Sigma \) where \( (d\xi_{Uf})_p = 0 \), we have \( (\xi_{df(T\Sigma)})^\ast B)_p = (f^\ast B)_p \) by Lemma 11.

Conversely, if \( (\xi_{df(T\Sigma)})^\ast B)_p = (f^\ast B)_p \), then we have \( (\xi_{Uf})^\ast B)_p = 0 \) by Lemma 11. Then, \( (d\xi_{Uf})_p = 0 \).

\( \square \)

### 6 Minimal surfaces in a sphere

Lawson [11] showed that if a conformal immersion \( f : \Sigma \to S^3 \) is minimal, then the polar variety of \( f \) and the bipolar surface of \( f \) are minimal. We show that if a conformal immersion \( f : \Sigma \to S^{r-1} (r \geq 4) \) is minimal, then the polar variety of \( f \) and the bipolar surface of \( f \) are minimal.

A conformal harmonic map \( f : \Sigma \to S^{r-1} (r \geq 4) \) is a minimal immersion or a minimal branched immersion [8]. In the beginning, we write the harmonic map equation in terms of the Clifford algebra.

Let \( \Delta \) be the Laplace-Beltrami operator with respect to the metric of \( \Sigma \). Then, an immersion \( f : \Sigma \to G_n(V_r) \subset C\ell(V_r) \) is a harmonic map [8] if and only if

\[
\Delta f = 2f. \tag{7}
\]

Let \( dA \) be the area element of the metric induced by \( f \) on \( \Sigma \). The Hodge dual of Eq. (7) is

\[
d \ast df = 2f dA. \tag{8}
\]

Recall that \( G_n(V_r) \) is a subset of a sphere \( C\ell(V_r)^{(n)} \cap \tilde{S}^{2r-1} \).

**Lemma 12** An immersion

\[
f : \Sigma \to G_n(V_r) \subset C\ell(V_r)^{(n)} \cap \tilde{S}^{2r-1}
\]

is a harmonic map if and only if

\[
d(f \ast df) = 0 \tag{9}
\]

or, equivalently,

\[
d(\ast df f) = 0.
\]
Theorem 2 If a conformal immersion \( f : \Sigma \to S^{r-1} \subset V_r \) with
\[
\ast df = \xi df(\Sigma) df = -df \xi df(\Sigma),
\]
is minimal, then the polar variety \( \xi Uf : \Sigma \to C\epsilon(V_r)^{(3)} \cap S^{r-1} \) is a minimal immersion with
\[
\ast d\xi Uf = -\xi df(\Sigma) d\xi Uf = d\xi Uf \xi df(\Sigma).
\]
or a minimal branched immersion with (10).

Proof Assume that \( f : \Sigma \to S^{r-1} \subset V_r \) is a minimal immersion with
\[
\ast df = \xi df(\Sigma) df = -df \xi df(\Sigma).
\]

Since
\[
d(f \ast df) = d(f \xi df(\Sigma) df) = d(\xi Uf) df = d\xi Uf \land df = 0,
\]
\[
d(\ast df f) = d(-df \xi df(\Sigma) f) = d(-df \xi Uf) = df \land d\xi Uf = 0
\]
by Lemma 12 and (4), the polar variety \( \xi Uf \) is conformal with
\[
\ast d\xi Uf = -\xi df(\Sigma) d\xi Uf = d\xi Uf \xi df(\Sigma).
\]
by Lemma 9, except at the points where \( d\xi Uf \) vanishes.

We have
by (3) and (4). Hence, the polar variety $\xi_{U^f}$ is a harmonic map by Lemma 12.
Since $\xi_{U^f}$ is a conformal harmonic map, it is a minimal immersion or a minimal branched immersion.

Next, we show that a bipolar surface of a minimal immersion is minimal.

**Theorem 3** If a conformal immersion $f: \Sigma \to S^{r-1} \subset V_r$ with

\[
* df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}
\]

is minimal, then the bipolar surface $\xi_{df(T\Sigma)}: \Sigma \to C\ell(V_r)^{(2)} \cap S^{2r-1}$ is a minimal immersion.

**Proof** Let $f: \Sigma \to S^{r-1} \subset V_r$ be a minimal immersion with $* df = \xi_{df(T\Sigma)} df = -df \xi_{df(T\Sigma)}$.
The bipolar surface $\xi_{df(T\Sigma)}$ is an immersion by Theorem 1.
Since $\xi_{U^f}$ is conformal by Theorem 2, there exists a non-negative function $\rho$ such that $(\xi_{U^f})^*B = \rho(f^*B)$. Then,

\[
(\xi_{df(T\Sigma)})^*B = f^*B + (\xi_{U^f})^*B = (1 + \rho)(f^*B)
\]

by Lemma 11. Then, $\xi_{df(T\Sigma)}$ is conformal.
We have

\[
d(\xi_{df(T\Sigma)} \ast d\xi_{df(T\Sigma)}) = d(f \ast df) - d(\xi_{U^f} \ast d\xi_{U^f}) = 0
\]

by (5), Lemma 12 and Theorem 2. Then, the map $\xi_{df(T\Sigma)}$ is a harmonic map by Lemma 12.
Since $\xi_{df(T\Sigma)}$ is a conformal harmonic map, it is minimal.

We may think of showing that $\xi_{df(T\Sigma)}$ is conformal by the existence of a map $\gamma$ such that $* d\xi_{df(T\Sigma)} = \gamma d\xi_{df(T\Sigma)} = -d\xi_{df(T\Sigma)} \gamma$ like Lemma 8. However, we see that there does not exist such a map from the proof of Theorem 4 later.
We obtain more information about the metrics induced by $f$, $\xi_{U^f}$ and $\xi_{df(T\Sigma)}$ on $\Sigma$.

For a vector space $U$, we denote by $U$ the trivial vector bundle over $\Sigma$ with fiber $U$.
The vector bundle $C\ell(V_r)$ is a real trivial vector bundle of rank $2r$ over $\Sigma$. The real vector bundle $V_r$ is a subbundle of $C\ell(V_r)$. A map $f: \Sigma \to C\ell(V_r)$ is regarded as a section of $C\ell(V_r)$.

Let $e_1, \ldots, e_r$ be a local orthonormal frame field of $V_r$. Assume that $e_1 e_2 = \xi_{df(T\Sigma)}$ and $e_r = f$. Then, there exist one-forms $\theta_1$ and $\theta_2$ on $\Sigma$ such that

\[
df = \theta_1 e_1 + \theta_2 e_2.
\]

The metric on $\Sigma$ induced by $f$ is

\[
l = \sum_{i=1}^{2} \theta_i \otimes \theta_i.
\]

Set
The structure equations are

\[ d\epsilon_i = \sum_{j=1}^{2} \omega_{ij} e_j + \sum_{a=3}^{r-1} \omega_{ia} e_a - \theta_{ij} \quad (i = 1, 2), \]

\[ d\epsilon_a = \sum_{i=1}^{2} \omega_{ai} e_i + \sum_{\beta=3}^{r-1} \omega_{a\beta} e_{\beta} \quad (a = 3, \ldots, r - 1), \]

\[ \omega_{AB} = -\omega_{BA} \quad (A, B = 1, \ldots, r - 1). \]

Then, the Gaussian curvature is

\[ K = \sum_{a=3}^{r-1} (h_{a11} h_{a22} - h_{a12} h_{a21}) + 1 \]

The second fundamental form of \( f \) is

\[ II = \sum_{i,j=1}^{2} \sum_{a=1}^{r-1} h_{aij} e_a \otimes \theta_i \otimes \theta_j. \]

The mean curvature vector of \( f \) is

\[ H = \sum_{a=3}^{r-1} \left( \sum_{i=1}^{r-1} h_{aii} \right) e_a. \]

If \( f \) is minimal, then \( h_{a11} + h_{a22} = 0 \quad (a = 3, \ldots, r - 1) \). Then, the Gaussian curvature is
\[ K = - \sum_{a=3}^{r-1} (h_{a11}^2 + h_{a12}^2) + 1. \]

Hence, the Gaussian curvature is equal to one at \( p \in \Sigma \) if and only if \( f \) is geodesic at \( p \).

**Theorem 4** Let \( f : \Sigma \rightarrow S^{r-1} \) is a minimal immersion. The metric induced by \( \xi_{df(\Sigma)} \) is

\[
\left( \sum_{a=3}^{r-1} (h_{a11}^2 + h_{a12}^2) + 1 \right) I.
\]

The metric induced by \( \xi_{Uf} \) is

\[
\left( \sum_{a=3}^{r-1} (h_{a11}^2 + h_{a12}^2) \right) I.
\]

The metric induced by \( \xi_{df(\Sigma)} \) is equal to that induced by \( f \) at a point \( p \in \Sigma \) if and only if \( f \) is geodesic at \( p \). The metric induced by \( \xi_{Uf} \) vanishes at a point \( p \in \Sigma \) if and only if \( f \) is geodesic at \( p \).

**Proof** The differential of \( \xi_{df(\Sigma)} = e_1 e_2 \) is

\[
d(e_1 e_2) = de_1 e_2 + e_1 de_2
\]

\[
= \left( \omega_{12} e_2 + \sum_{a=1}^{r-1} \omega_{1a} e_a - \theta_1 f \right) e_2 + e_1 \left( \omega_{21} e_1 + \sum_{a=3}^{r-1} \omega_{2a} e_a - \theta_2 f \right)
\]

\[
= \sum_{a=3}^{r-1} \omega_{2a} e_1 e_a - \theta_2 e_1 f - \sum_{a=1}^{r-1} \omega_{1a} e_2 e_a + \theta_1 e_2 f
\]

\[
= \sum_{a=3}^{r-1} \sum_{i=1}^{2} h_{a2i} \theta_1 e_1 e_a - \theta_2 e_1 f - \sum_{a=1}^{r-1} \sum_{i=1}^{2} h_{a1i} \theta_2 e_2 e_a + \theta_1 e_2 f
\]

\[
= \theta_1 \left( \sum_{a=3}^{r-1} h_{a21} e_1 e_a - \sum_{a=1}^{r-1} h_{a11} e_2 e_a + e_2 f \right)
\]

\[
+ \theta_2 \left( \sum_{a=3}^{r-1} h_{a22} e_1 e_a - \sum_{a=1}^{r-1} h_{a12} e_2 e_a - e_1 f \right)
\]

\[
= \theta_1 E_1 + \theta_2 E_2,
\]

\[
E_1 = \sum_{a=3}^{r-1} h_{a21} e_1 e_a - \sum_{a=1}^{r-1} h_{a11} e_2 e_a + e_2 f,
\]

\[
E_2 = \sum_{a=3}^{r-1} h_{a22} e_1 e_a - \sum_{a=1}^{r-1} h_{a12} e_2 e_a - e_1 f.
\]

Since \( h_{a22} = -h_{a11}, h_{a1j} = h_{a2j} (a = 3, \ldots, r-1, i, j = 1,2), \) we have
Then, the metric induced by \( \xi_{df(T\Sigma)} \) is (11). By Lemma 11, the metric induced by \( \xi_{Uf} \) is (12). The rest is clear. \( \square \)

Acknowledgements  This work was supported by the Japan Society for the Promotion of Science KAKENHI Grant Number 17K05217.

Funding  This work was supported by JSPS KAKENHI Grant Number 17K05217.

Date availability  Not applicable.

Declarations

Conflict of interest  Not applicable.

Code availability  Not applicable.

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