On the one dimensional cubic NLS in a critical space

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Abstract

In this note we study the initial value problem in a critical space for the one dimensional Schrödinger equation with a cubic non-linearity and under some smallness conditions. In particular the initial data is given by a sequence of Dirac deltas with different amplitudes but equispaced. This choice is motivated by a related geometrical problem; the one describing the flow of curves in three dimensions moving in the direction of the binormal with a velocity that is given by the curvature.

1 Introduction

Since the work of Da Rios [7], it is known the connection between the evolution of a vortex filament in an incompressible inviscid fluid in three dimensions and the so called Binormal Curvature Flow (BF). The equation for this flow reads

$$\partial_t \chi(t,s) = \partial_s \chi(t,s) \wedge \partial^2_{ss} \chi(t,s)$$

where $\chi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^3$ is a time dependent curve parametrized by the arclength $s$. Note that the tangencial vector to the curve $T(t,s) = \partial_s \chi(t,s)$ satisfies the so called Schrödinger map equation onto $S^2$:

$$\partial_t T = T \wedge \partial^2_{ss} T.$$ 

Thanks to the Hasimoto transformation, see [11], the above equation is associated with a focusing 1D cubic non-linear Schrödinger equation

$$i \partial_t u + \Delta u + \frac{1}{2}(|u|^2 - m(t))u = 0,$$

for some $m(t) \in \mathbb{R}$. The freedom of choice of $m(t)$ comes form the gauge invariance of the underlying geometric problem, and plays a crucial role when the initial datum of (1) is singular. A specific kind of singularity that has recently received attention is the one given by the presence of corners. In particular, in [1] and [2] the case of one corner and the stability of the solution is considered, and in [3] the more general situation of polygonal lines is studied. More concretely, in the latter work filaments made of, possibly infinity, segments of the same length $2\pi$ are analyzed. The corresponding initial datum for (1) translates to a, possibly infinite, sum of deltas of Dirac on $2\pi \mathbb{Z}$ as initial datum for the non-linear Schrödinger equation. The good choice of $m(t)$ in this situation turns out to be

$$m(t) = \frac{M}{2\pi t}, \quad M = \sum_k |\alpha_k|^2.$$ (2)

As a consequence one is led to study the Initial Value Problem (IVP)

$$\partial_t u = i \left( \Delta u + \frac{1}{2} \left( |u|^2 - \frac{M}{2\pi t} \right) u \right)$$

$$u(0,.) = \sum_k \alpha_k \delta_k(.,) \quad \delta_k(x) = \delta(x-k).$$ (4)
In [14], the IVP analogous to (3) but for subcritical nonlinearities, \( u|u|^{p-1}, \ p < 3, \) is studied, and well-posedness is proved among all the solutions that are written as the sum

\[
u(t, x) = \sum_k A_k(t) e^{it\Delta} \delta_k(x). \quad (5)
\]

Above \( e^{it\Delta} \) stands for the usual free propagator of the linear Schrödinger equation. As done in [3], one can use the identity

\[
e^{it\Delta} \delta_k = \frac{e^{i|x|^2/4t}}{\sqrt{t}} \mathbf{1}_{(1/t, t)}.
\]

As a consequence the corresponding non-linear potential becomes

\[
|u|^{p-1} = \frac{1}{t^{p-1}}|v|^{p-1},
\]

so that the factor \( \frac{1}{t^{p-1}} \) is locally integrable around \( t = 0 \) if and only if \( p < 3 \). In this case \( m(t) \) can be chosen identically zero while for \( p = 3 \) a modification is needed, (cf. [3]). In this particular situation the equation for \( v \) turns out to be

\[
iv_t + v_{xx} + \frac{1}{2t}(|v|^2 - M)v = 0,
\]

with \( M \) as in (2).

Notice that in the definition of \( T \) there is an inversion of the time variable and therefore the IVP for \( u \) becomes a scattering problem for \( v \). Also observe that the solutions of (6) formally preserve the \( L^2 \) norm

\[
\int |v|^2 \, dx.
\]

Due to the fact that from (5)

\[
v(t, x) = \sum_k A_k(t) e^{ikx},
\]

we immediately obtain that

\[
\sum_k |A_k(t)|^2 = \sum_k |\alpha_k|^2 = M,
\]

which justifies the choice done in (2). Also

\[
E(v)(t) := \frac{1}{2} \int |v_x(t)|^2 \, dx - \frac{1}{4t} \int (|v|^2 - |\alpha|^2)^2 \, dx
\]

satisfies

\[
\partial_t E(v)(t) = \frac{1}{4t^2} \int (|v|^2 - |\alpha|^2)^2 \, dx.
\]

Once the ansatz (5) is fixed, the IVP (3) is reduced to solve the following infinite dynamical system in the variables \( A_k(t) \)

\[
i \partial_t A_k = \frac{1}{8\pi t} \sum_{j_1, j_2, j_3 \in NR_k} e^{-i|\alpha|^2 - |j_1|^2 - |j_2|^2 - |j_3|^2} A_{j_1} A_{j_2} A_{j_3} - \frac{1}{8\pi t} |A_k|^2 A_k,
\]

where we have used the notation

\[
NR_k = \{(j_1, j_2, j_3) \in Z^3 \text{ such that } k = j_1 - j_2 - j_3 \text{ and } k^2 - j_1^2 + j_2^2 - j_3^2 \neq 0 \}.
\]

The lack of integrability at the origin of the \( 1/t \) factor in the above expression requires the use of the phase renormalization

\[
A_k(t) = e^{i|\alpha_k|^2/8\pi t} \log t A_k(t).
\]
Then for $t > 0$, the functions $\hat{A}_k$ satisfy the system
\begin{equation}
 i \partial_t \hat{A}_k = \frac{1}{8\pi} \sum_{j_1, j_2, j_3 \in \mathbb{N} R_k} e^{-i|\alpha_k|^2 - i\rho_j^2 + i\sigma_j^2 - i\lambda_j^2} e^{-(|\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2) i\log t} \hat{A}_{j_1} \hat{A}_{j_2} \hat{A}_{j_3} - \frac{1}{8\pi t} (-|\hat{A}_k|^2 - |\alpha_k|^2) \hat{A}_k ,
\end{equation}
with the initial condition
\begin{equation}
\hat{A}_k(0) = \alpha_k.
\end{equation}
If we define the spaces $l^{2,s}$ as those sequences $\{\alpha_k\}$ such that
\begin{equation}
\|\{\alpha_k\}\|_{l^{2,s}} = \left( \sum_k |(1 + k^2)^s \alpha_k|^p \right)^{1/p} < +\infty
\end{equation}
with $l^{p,0} = l^p$, then it is proved in [3] that the system (11)-(12) is local-in-time well-posed for $\{\alpha_k\} \in l^{2,s}$ with $s > 1/2$ and for $\{\alpha_k\} \in l^1$. Observe that due to (10) if the IVP (11)-(12) is well posed then the ones associated to (9) and to (3) is ill posed, cf. Theorem 1.4 in [3].

In this work we would like to consider initial data in $l^p$ for $p \in (1, \infty)$. We will perform a fixed point argument based on considering the regularity of the coefficients $A_k(t)$. Due to the fact that we are working directly with solutions to the system (11) it will be enough for our purposes to use the classical Sobolev spaces instead of the usual ones in this setting introduced by J. Bourgain in [4].

The results of this paper regarding the IVP (11)-(12) can be resumed as follows.

- For $p \in (1, +\infty)$,
  - (i) Large time well-posedness for small initial datum. More precisely for any $T > 0$, there exists $\varepsilon(T) > 0$ such that if the $l^p$ norm of the initial datum $\{\alpha_k\}$ is smaller then $\varepsilon(T)$, then there exists a unique solution of (11) in $[0, T]$ in an appropriate sense.
  - (ii) Short-in-time well-posedness for small enough initial datum in $l^\infty$. More precisely if the $l^\infty$ norm of $\{\alpha_k\}$ is small enough then there exits a small time $T(\|\alpha\|_{l^\infty}, \|\alpha\|_{l^p})$ such that a unique solution of (11) exists in $[0, T]$ in an appropriate sense.

- For $p = 2$, global in time well-posedness with small assumption in $l^\infty$ for the initial datum. As it can be expected this result follows from (ii) and the $l^2$ conservation law.

The first smallness condition is rather natural due to the criticality of the problem and that the method of proof is perturbative. Also the fact that the potential $1/t$ is not bounded for $t > 0$ makes necessary to consider $T < +\infty$. The second smallness condition is less natural. It comes from the linear terms in (12) below. This difficulty already appears in the case of the perturbation of one delta function and was addressed in [1] and [2]. In these papers the linear terms are not considered as perturbative. Nevertheless, there is a price to be paid for it, namely the possible growth of the zero Fourier mode of the perturbation (cf. section 6 in [2] and Appendix B in [1]). In this paper we consider another type of perturbation, a periodic one, and this growth can not be possible due to the conservation law (13)-(14). The use of this conservation law in the linearized system, cf. with (55) and (56) in [2] in the case of one delta, will require a delicate analysis that we postpone for the future.

Finally, although it is not clear that the above results extend to initial data in $l^\infty$ we are able to exhibit some particular solutions that include the case of regular polygons for (1), see [13] and [8]. More precisely, if the initial condition is a constant sequence, in other words $\alpha_k = \alpha$ for any $k \in \mathbb{Z}$, then there exists an explicit solution to the problem. The result is given in Proposition 1 that is proved in collaboration with V. Banica. The uniqueness of this solution is quite likely a very challenging problem.

### 1.1 Connection with previous results

The well-posedness of 1D cubic NLS on the line and on the circle was firstly tackled respectively in [19] and [1] for $L^2$ initial data. A natural scaling for the equation is $u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x)$. Therefore the $L^2(\mathbb{R})$ setting is far from being scaling invariant, and something similar happens in the periodic case if the independence of the time of existence on the size of the period is

\footnote{In Theorem 1 the result is stated for $B_k(t/4) = \hat{A}(1/t)$.}
To present the result we rewrite the equations in a fixed point framework. We start by introducing some notation.

Well-posedness results for initial datum in $L^2$. This approach was extended later on in [9] where well-posedness is studied in the Fourier-Lebesgue spaces that we denote by $\mathcal{F}L^p$. These are spaces of distributions whose Fourier transforms are bounded in $L^p(\mathbb{R})$. Therefore, they are invariant respect to translations in the phase space. Moreover $\mathcal{F}L^\infty$ is also scaling invariant and therefore critical. Local in time well-posedness, also including periodic boundary conditions using the Fourier coefficients instead of the Fourier transform, was shown in [10] in the $\mathcal{F}L^p$ space for $2 < p < +\infty$.

At the same time plenty of effort has been done to understand well-posedness in the setting of Sobolev spaces. In this case the homogeneous space $\dot{H}^{-1/2}$ is critical. Many papers have been devoted to clarify what happens between $L^2$ and $\dot{H}^{-1/2}$. In particular, it has been shown ill-posedness, in the sense that a data to solution map which is uniformly continuous does not exist in $H^s$ with $s < 0$. Even more, an inflation phenomena concerning the growth of the Sobolev norms has been proved, see [16]-[15]-[13]-[17]. Finally in [12] it has been shown well-posedness in $H^s$ for $s > -1/2$. This result provides a weaker notion of continuity for the data to solution map. Moreover, the connection with this result and the Fourier-Lebesgue spaces has been recently done in [18].

In this paper, we are considering sums of Dirac’s deltas as initial datum. Note that a Dirac delta is critical for the $\dot{H}^{-1/2}$ and belongs to $\mathcal{F}L^\infty$ space. This means that we are considering a critical regime.

As said before, in [3] the weighted spaces $L^{2,s}$ for $s > 1/2$ are used. The main reason is that they are very convenient to obtain solutions of (1) from those of (3). Up to what extent solutions of (1) can be constructed using the ones obtained in this paper seems to us a very challenging question that we propose to address in the future.

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2 Well-posedness results for initial datum in $L^p$ with $p < +\infty$

To present the result we rewrite the equations in a fixed point framework. We start by introducing some notation.

Let us recall that $NR_k = \{(j_1, j_2, j_3) \in \mathbb{Z}^3 \text{ such that } k = j_1 - j_2 - j_3 \text{ and } k^2 - j_1^2 + j_2^2 - j_3^2 \neq 0\}$. Moreover, for any triple in $NR_k$, it holds $m := k^2 - j_1^2 + j_2^2 - j_3^2 = 2(k - j_1)(j_1 - j_2)$. In particular the map $(j_1, j_2, j_3) \mapsto (k^2 - j_1^2 + j_2^2 - j_3^2, k - j_1) := (m, z)$ is a bijection between $NR_k$ and $(m, z) \in \mathbb{Z}^2$ such that $m \not\equiv 0$ and $2z$ divide $m$. We finally denote by

$$r(m) = \{z \in \mathbb{Z} \text{ such that } 2z \text{ divide } m\}. \quad (13)$$

The system (11) can be rewritten as

$$i\partial_t \tilde{A}_k = \frac{1}{8\pi t} \sum_{m \not\equiv 0 r(m)} e^{-i\pi m} e^{-i\frac{2\pi}{\log 4} \log t} \tilde{A}_{j_1} \tilde{A}_{j_2} \tilde{A}_{j_3} - \frac{1}{8\pi t} \left(|\tilde{A}_k|^2 - |\alpha_k|^2\right) \tilde{A}_k,$$

where we denote $\Lambda_m = |\alpha_k|^2 - |\alpha_{j_1}|^2 + |\alpha_{j_2}|^2 - |\alpha_{j_3}|^2$. Let us introduce the new variable

$$B_k(t/4) = \tilde{A}_k(1/t).$$

The equations for $B_k$ are

$$i\partial_t B_k = -\frac{1}{8\pi t} \sum_{m \not\equiv 0 r(m)} e^{-i\pi m} e^{i\frac{2\pi}{\log 4} \log t} B_{j_1} B_{j_2} B_{j_3} + \frac{1}{8\pi t} \left(|B_k|^2 - |\alpha_k|^2\right) B_k \quad \text{ and } \quad \lim_{t \to +\infty} B_k = \alpha_k. \quad (14)$$

Writing $B_k = R_k + \alpha_k$ and integrating the above equations, we get

$$R_k(t) = \frac{i}{8\pi} \int_t^{+\infty} \frac{1}{m} \sum_{m \not\equiv 0 r(m)} e^{-i\pi m} e^{i\frac{2\pi}{\log 4} \log 4\tau} (R_{j_1} + \alpha_{j_1} + R_{j_2} + \alpha_{j_2} + R_{j_3} + \alpha_{j_3}) - \frac{1}{\tau} \left(|R_k + \alpha_k|^2 - |\alpha_k|^2\right) (R_k + \alpha_k) d\tau. \quad (15)$$

We show well-posedness of solutions for the system (15) via a fixed point argument. More precisely we introduce a map $T$ in (10) and we show that it is a contraction on some bounded subset of the Banach space $X^s_p$. To introduce this space let us start by defining the Banach spaces $\dot{H}^{s,p}$ and $\mathcal{H}^s_p$. 

4
At an informal level, for a bounded interval $I \subset \mathbb{R}$, the space $\mathcal{H}_p^s(I)$ corresponds to a Fourier-Lebesgue space for non periodic functions. We say that $f$ is an element of $\mathcal{H}_p^s(I)$ if it admits an extension $\hat{f}$ defined on a bigger interval $I' \supset I$ such that $\hat{f}$ in a Fourier-Lebesgue-type space $\hat{\mathcal{H}}_{s,p}^s(I')$. Then, the $\mathcal{H}_p^s$ norm is defined as the infimum of the $\hat{\mathcal{H}}_{s,p}^s$ norm over all the possible extensions.

Let us recall that for $p \in [1, +\infty)$, $s \in \mathbb{R}$ and $\Pi_4 = \mathbb{R}/4\pi\mathbb{Z}$ the torus of size 4\pi, the Fourier-Lebesgue space $\mathcal{F}\mathcal{L}^s_{p}(\Pi_4)$ is defined by the norm

$$
\|f\|_{\mathcal{F}\mathcal{L}^s_{p}(\Pi_4)} = \sum_{2k \in \mathbb{Z}} \left|\langle k \rangle^s \hat{f}_k \right|^p,
$$

where $\hat{f}_k$ denotes the $k$-th Fourier coefficient of $f$.

For $\nu \in \mathbb{N}$, let $I_{\nu} = [\nu \pi, (\nu + 1)\pi)$ and $I_{\nu}^c = [\pi(\nu - 1), \pi(\nu + 3)]$. For $s \in \mathbb{R}^+$ and $p \in [1, +\infty)$, we denote by $\hat{\mathcal{H}}_{s,p}^s(I_{\nu}^c)$ the Banach space of functions $\hat{f} : I_{\nu}^c \rightarrow \mathbb{C}$ with norm given by

$$
\|\hat{f}\|_{\hat{\mathcal{H}}_{s,p}^s(I_{\nu}^c)} = \sum_{2k \in \mathbb{Z}} \left|\langle k \rangle^s \hat{f}_k \right|^p,
$$

where $\hat{f}_k$ denotes the $k$-th Fourier coefficient of $\hat{f}$ as a $4\pi$-periodic function. Notice that the $\hat{\mathcal{H}}_{s,p}^s$ space is isomorphic to the Fourier-Lebesgue space $\mathcal{F}\mathcal{L}^s_{p}(\Pi_4)$ through the map that sends $\hat{f}$ to its $4\pi$ periodic extension. For $s = 0$, we use the notation $\hat{L}^p(I_{\nu}^c)$ instead of $\hat{\mathcal{H}}_{0,p}^s(I_{\nu}^c)$.

We denote by $\mathcal{H}_p^s(I_{\nu})$ the Banach space of functions $f : I_{\nu} \rightarrow \mathbb{C}$ with norm given by

$$
\|f\|_{\mathcal{H}_p^s(I_{\nu})} = \inf_{f \in \hat{\mathcal{H}}_{s,p}^s(I_{\nu})} \|\hat{f}\|_{\hat{\mathcal{H}}_{s,p}^s(I_{\nu})} = \inf_{f \in \hat{\mathcal{H}}_{s,p}^s(I_{\nu})} \sum_{2k \in \mathbb{Z}} \left|\langle k \rangle^s \hat{f}_k \right|^p.
$$

Let us notice that for $s > 1/p$ the space $\hat{\mathcal{H}}_{s,p}^s(I_{\nu}^c) \subset C^0(\Pi_4)$. This implies that for elements $\hat{f}$ of $\hat{\mathcal{H}}_{s,p}^s(I_{\nu}^c)$ with $s > 1/p$, it holds $\hat{f}(\pi(\nu - 1)) = \hat{f}(\pi(\nu + 3))$. At the contrary a function in $\mathcal{H}_p^s(I_{\nu})$ for $s > 1/p$ is continuous in $I_{\nu}$, it admits an extension in $C^0(\Pi_4)$ but it does not need to be 2\pi-periodic in the sense that in general $f(\pi \nu) \neq f(\pi(\nu + 2))$.

As before we denote by $\hat{L}^p(I_{\nu})$ the space $\mathcal{H}_p^0$. Moreover the homogeneous seminorm $\hat{\mathcal{H}}_p^s$ is given by

$$
\|f\|_{\hat{\mathcal{H}}_p^s(I_{\nu})} = \inf_{f \in \hat{L}^p(I_{\nu})} \sum_{2k \in \mathbb{Z}} \left|\langle k \rangle^s \hat{f}_k \right|^p.
$$

For sequences $\{R_k\}_{k \in \mathbb{Z}}$ of functions $R_k : (0, +\infty) \rightarrow \mathbb{C}$, we introduce the norm

$$
\|\{R_k\}_{k \in \mathbb{Z}\}^p_{X^s_p} = \sup_{\nu \geq 0} \|\nu + 1\{R_k\}_{k \in \mathbb{Z}\}^p_{\mathcal{H}_p^s(I_{\nu})}\|_{p \mathbb{Z}^s} = \sup_{\nu \geq 0} \sum_{k \in \mathbb{Z}} \|\nu + 1\{R_k\}_{k \in \mathbb{Z}\}^p_{\mathcal{H}_p^s(I_{\nu})}\|
$$

and $X^s_p$ the space of sequences $\{R_k\}_{k \in \mathbb{Z}}$ for which the norm $\|\{R_k\}_{k \in \mathbb{Z}\}^p_{X^s_p}$ is bounded. In the following we use $R$ to denote a sequence $\{R_k\}_{k \in \mathbb{Z}}$ to simplify the notation and we use subindexes to denote the elements of a sequence. Finally for any $T > 0$, we define the space $X^s_p(T)$ the space of sequences $R$ of functions $R_k : (T; +\infty) \rightarrow \mathbb{C}$ for which the norm

$$
\|R\|_{X^s_p(T)} = \inf_{\tilde{R}_{\mathbb{Z}} X^s_p} \|\tilde{R}\|_{X^s_p(T)} = \inf_{\tilde{R}_{\mathbb{Z}} X^s_p} \|\tilde{R}\|_{X^s_p(T)}
$$

is bounded. Let us conclude this section with the main result.

**Theorem 1.** Let $\{\alpha_k\}_{k \in \mathbb{Z}} \in l^p$ an initial data for $p \in (1, \infty)$.

- **(Large in time well-posedness)** For any $T > 0$, there exists $\delta$ such that if $\|\alpha\|_p < \delta$, then there exists a unique solution $R$ to (15) in $X^s_p(T)$ for some $s$ with $\max\{1/p, 1 - 1/p\} < s < 1$.

- **(Short in time well-posedness)** There exists $\delta > 0$ such that if $\|\alpha\|_{\infty} < \delta$, then for a big enough time $T \gg 1$ there exists a unique solution $R$ to (15) in $X^s_p(T)$ for some $s$ with $\max\{1/p, 1 - 1/p\} < s < 1$.

Moreover in the special case $p = 2$, if $\|\alpha\|_{\infty} < \delta$ for $\delta$ small enough, then there exists a unique solution to (15) in $(0, +\infty)$, which belongs to $\bigcap_{T > 0} X^s_p(T)$ for some $1/2 < s < 1$.

We postpone the proof to Section 3.
2.1 Example of explicit solutions for special \(l^\infty\) initial datum

Although we are not able to extend the previous results to the case \(p = \infty\), there exist non-trivial solutions to the system (10) with initial datum in \(l^\infty \setminus \bigcup_{p \geq 1} l^p\). In particular we are able to exhibit an explicit solution in the case where the initial datum is a constant sequence, in other words the \(\alpha_k = \alpha \neq 0\) for any \(k \in \mathbb{Z}\).

This observation has been made in collaboration with V. Banica who has kindly allowed us to include it in this article.

For consistency let us present the results in the \(B\) variables.

**Proposition 1.** Given \(\alpha \in \mathbb{C}\), then there exists an explicit solution to (10) with initial datum \(\alpha_k = \alpha\) for any \(k \in \mathbb{Z}\). Namely

\[
B_k(t) = B(t) = \alpha e^{-i|\alpha|^2 \int_t^{+\infty} \sum_{m \neq 0} \sum_{r(m)} e^{-im\tau} \ d\tau},
\]

where \(r(m)\) is defined in (13).

We postpone the proof of this result in Section 4.

3 Proof of Theorem [11]

The proof is based on a fixed point argument in the space \(X^s_p\). Let \(\eta \in C^\infty(\mathbb{R})\) such that \(0 \leq \eta \leq 1\), \(\eta(t) = 1\) for \(t \geq \pi\). Moreover for \(N \in \mathbb{N}\) we define \(\eta_N(t) = \eta(t - \pi N)\). We consider the map \(\mathcal{T}_{\eta_N} : X^s_p \to X^s_p\) with \(k\)-th component given by

\[
(\mathcal{T}_{\eta_N}(R))(t)_k = \frac{i\eta_N(t)}{8\pi} \int_t^{+\infty} \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0 \ r(m)} e^{-im\tau} e^{i \frac{\pi}{8} \log 4 \tau (R_{j_1} + \alpha_{j_1})(R_{j_2} + \alpha_{j_2})(R_{j_3} + \alpha_{j_3})} d\tau
\]

\[
- \frac{i\eta_N(t)}{8\pi} \int_t^{+\infty} \frac{\eta_N(\tau)}{\tau} \left( |R_k + \alpha_k|^2 - |\alpha_k|^2 \right) (R_k + \alpha_k) d\tau.
\]

We start by presenting some bounds of the map \(\mathcal{T}_{\eta_N}\). To do that we rewrite

\[
(\mathcal{T}_{\eta_N}(R))_k = (\mathcal{T}_{\eta_N,0}(R))_k + (\mathcal{T}_{\eta_N,1}(R))_k + (\mathcal{T}_{\eta_N,2}(R))_k
\]

where \(\mathcal{T}_{\eta_N,0}\) is independent on \(R\), \(\mathcal{T}_{\eta_N,1}\) is linear and \(\mathcal{T}_{\eta_N,2}\) is super-linear. For example

\[
(\mathcal{T}_{\eta_N,0}(R))_k = \frac{i\eta_N(t)}{8\pi} \int_t^{+\infty} \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0 \ r(m)} e^{-im\tau} e^{i \frac{\pi}{8} \log 4 \tau \alpha_{j_1} \alpha_{j_2} \alpha_{j_3}} d\tau.
\]

The goal is to show that the map \(\mathcal{T}_{\eta_N}\) is a contraction.

**Lemma 1.** Let \(1 < p < +\infty\), let \(s > \max\{1/p, 1 - 1/p\}\) and let \(N \geq 0\) be a natural number. Then the \(\dot{L}^p\) estimates are as follows: for \(\nu \geq N\)

\[
\|\mathcal{T}_{\eta_N,0}(R)\|_{\dot{L}^p(\mathbb{Z}; \dot{L}^\nu(\mathbb{I}_L)))} \leq \frac{C}{\nu + 1} \|\alpha\|_{\dot{L}^p(\mathbb{Z})}^3
\]

\[
\|\mathcal{T}_{\eta_N,1}(R)\|_{\dot{L}^p(\mathbb{Z}; \dot{L}^\nu(\mathbb{I}_L)))} \leq \frac{C}{\nu + 1} \max \left\{ \|\alpha\|_{\dot{L}^p(\mathbb{Z})}^2, \frac{\|\alpha\|_{\dot{L}^p(\mathbb{Z})}^2}{\nu + 1} \right\} \|R\|_{X^s_p}
\]

and

\[
\|\mathcal{T}_{\eta_N,2}(R)\|_{\dot{L}^p(\mathbb{Z}; \dot{L}^\nu(\mathbb{I}_L)))} \leq \frac{C}{(\nu + 1)^2} \max \left\{ \|\alpha\|_{\dot{L}^p(\mathbb{Z})}, \frac{\|R\|_{X^s_p}}{\nu + 1} \right\} \|R\|_{X^s_p}^2.
\]

**Lemma 2.** Let \(1 < p < +\infty\), let \(s \in (\max\{1/p, 1 - 1/p\}, 1)\) and let \(N \geq 0\) be a natural number. Then the \(\dot{H}^s_p\) seminorm can be estimated for \(\nu \geq N\) as

\[
\|\mathcal{T}_{\eta_N,0}(R)\|_{\dot{H}^s_p(\mathbb{Z}; \dot{L}^\nu(\mathbb{I}_L)))} \leq \frac{C}{\nu + 1} \|\alpha\|_{\dot{L}^p(\mathbb{Z})}^3
\]
\[ \|T_{\eta N,1}(R)\|_{l^p(Z,N^2_p(t_\nu))} \leq \frac{C}{(\nu + 1)} \max \left\{ \|\alpha\|^2_{l^\infty(Z)}, \frac{C}{(\nu + 1)^2} \right\} \|R\|_{X_p^*} \]

and

\[ \|T_{\eta N,2}(R)\|_{l^p(Z,N^2_p(\eta N^1(t_\nu))))} \leq \frac{C}{(\nu + 1)^2} \max \left\{ \|\alpha\|_{l^p(Z)}, \frac{C}{(\nu + 1)^3} \right\} \|R\|_{X_p^*}. \]

**Proof of Theorem** The difficult parts of the proof are the estimates in the above lemmas. From Lemma 1 and 2 together with the fact that \( T_{\eta N} \) is identically zero in \( I_\nu \) for \( \nu \leq N - 1 \), we deduce that

\[
\sup_{\nu \geq 0} \| (\nu + 1)T_{\eta N}(R) \|_{l^p(Z,N^2_p(t_\nu)))} \leq C \sup_{\nu \geq N} \left( \|\alpha\|^3_{l^p(Z)} + \|\alpha\|^2_{l^\infty(Z)} \|R\|_{X_p^*} + \frac{C}{(\nu + 1)} \|R\|_{X_p^*} \right) \]

which can be shortly rewritten as

\[
\|T_{\eta N}(R)\|_{X_p^*} \leq C \left( \|\alpha\|^3_{l^p(Z)} + \|\alpha\|^2_{l^\infty(Z)} \|R\|_{X_p^*} + \frac{C}{(\nu + 1)} \|R\|_{X_p^*} + \frac{1}{(\nu + 1)^2} \|R\|_{X_p^*} \right).
\]

for any natural number \( N \geq 0 \). Similarly one has

\[
\|T_{\eta N}(R) - T_{\eta N}(Q)\|_{X_p^*} \leq C \left( \|\alpha\|^3_{l^p(Z)} + \frac{C}{(\nu + 1)} \|R\|_{X_p^*} + \frac{C}{(\nu + 1)^2} \|Q\|_{X_p^*} + \frac{1}{(\nu + 1)^2} \|R\|_{X_p^*} \right) \|R - Q\|_{X_p^*}.
\]

To show well-posedness is enough to define a complete metric space where the map \( T_{\eta N} \) is a contraction.

- **(Large in time well-posedness for small datum)** To show large in time well-posedness for small initial datum we choose \( N = 0 \) and we define the space \( C^{L_t} = \{ R \in X_p^* \text{ such that } \|R\|_{X_p^*} \leq \varepsilon \} \). The functional
  \[
  T_{\eta 0} : C^{L_t} \rightarrow C^{L_t}
  \]
  is a contraction for \( \{\alpha\}_{l^p} \) and \( \varepsilon \) small enough.

- **(Short in time well-posedness)** To show short in time well-posedness we define the space \( C^{S_t} = \{ R \in X_p^* \text{ such that } \|R\|_{X_p^*} \leq 2C\|\alpha\|_{l^p} \} \). For \( \{\alpha\}_{l^\infty} \) small enough, there exists \( N \) sufficiently large such that the functional
  \[
  T_{\eta N} : C^{S_t} \rightarrow C^{St}
  \]
  is a contraction.

Note that in the large in time well-posedness result \( R \) satisfy the equation only for \( t \geq \pi \). But by choosing from the beginning \( \eta(x) = 1 \) for \( x \geq \delta > 0 \), we deduce the existence of solution in the interval \([\delta, +\infty)\) for any fixed \( \delta \), due to the trivial bound \(1/t \leq \delta^{-1} \).

Finally uniqueness follows from the fact that solutions are obtained by a Picard iteration process. In particular if \( \eta \) and \( \bar{\eta} \) are two cut-off such that \( \eta(t) = \bar{\eta}(t) = 1 \) for \( t \geq t_* \) then \( n \)-th iterate associated with \( \eta \) and \( \bar{\eta} \) are equal for time \( t \geq t_* \).

Finally let us remark that the smallness in \( l^\infty \) of the data in the short in time result is due to the term \( C\|\alpha\|^2_{l^\infty} \|R\|_{X_p^*} \).

### 3.1 Proof of Lemma 1 and 2

In this subsection we prove the two main lemmas. Let start with Lemma 1.
3.1.1 Proof of Lemma I

Proof. We divide the proof in three parts where we tackle the terms $T_0$, $T_1$ and $T_2$ separately. In particular we show all the details for the estimates of $T_0$ and $T_1$. In the last part we explain how to deduce the estimates for $T_2$ from the one of $T_1$.

Estimates of $T_0$ Recalling the definition of $T_0$ and by an integration by parts we deduce

\[
(T_0(R))_k = \frac{inN(t)}{8\pi} \int_t^{\infty} \frac{1}{\int_m} \sum_{\nu \neq 0} \sum_{\nu \neq 0} \eta_N(\tau) e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3} d\tau
\]

\[
= - \frac{inN(t)}{8\pi} \sum_{\nu \neq 0} \sum_{\nu \neq 0} e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3} + \frac{inN(t)}{8\pi} \int_t^{\infty} \sum_{\nu \neq 0} \sum_{\nu \neq 0} \eta_N(\tau) e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3} d\tau
\]

In the sequel it is useful to decompose the integral on $(t, +\infty)$ as the sum of integrals on $I_\mu$. To do that let us introduce a partition of unity $\{\psi_\mu\}$ on $[\pi, +\infty)$ defined as $\psi_\mu(t) = \eta_\mu(t) - \eta_{\mu+1}(t)$. It holds that $0 \leq \psi_\eta \leq 1$, $\psi_\mu$ is smooth and supported in $I_\mu = [\pi \mu, \pi(\mu + 1)]$. Moreover let us denote by

\[
\psi_\nu^\nu = \sum_{\nu \neq -1} \psi_{\nu - \nu} = \eta_{\nu - 1} - \eta_{\nu + 1} = \eta_{\nu - 1}(1 - \eta_{\nu + 1}).
\]

Observe that $\psi_\nu^\nu$ is smooth, with uniformly bounded derivative, supported in $I_\nu$, identically 1 in $I_\nu$ and $0 \leq \psi_\nu^\nu \leq 1$.

Let us recall that the $L^p$ norm in $I_\nu$ is defined as the infimum of the $L^p$ norm in $I_\nu^\nu$ of all the possible extensions. In particular for $\nu > N + 1$, we use the extension $(T_0(R))_k = \psi_\nu^\nu(T_0(R))_k$. We deduce that

\[
\|T_0(R)\|_{L^p(I_\nu)} = \sum_{\nu \neq -1} \left| \psi_{\nu - 1}(T_0(R))_k \right|_{L^p},
\]

where $[\cdot]_h$ denotes the $h$-th Fourier coefficient. Let us compute $[\psi_\nu^\nu(T_0(R))_k]_{L^p}$. First of all note that $\eta_N(t) = 1$ for $t \geq (N + 1)\pi$, in particular the second term of (17) is zero. Denote by

\[
\tilde{\psi}_\nu^\nu(t) = \nu + \frac{1}{8\pi} e^{\nu \alpha_1 \alpha_2 \alpha_3} \text{ which satisfies } \|\tilde{\psi}_\nu^\nu\|_{L^\infty} \leq C(1 + \|\alpha\|_{L^\infty}^2),
\]

with $C$ independent of $\nu \geq 1$ and $m$. For the first term on the right hand side of (17), we have

\[
\psi_\nu^\nu(t) = \frac{1}{8\pi} \sum_{\nu \neq 0} e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3} = \nu + \frac{1}{8\pi} \sum_{\nu \neq 0} \sum_{\nu \neq 0} e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3}
\]

\[
= \nu + \frac{1}{8\pi} \sum_{\nu \neq 0} \sum_{\nu \neq 0} e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3}
\]

As we already mention the second term is zero for $\nu > N + 1$. For the last term in (17), we integrate by parts to obtain, for $h \neq 0$

\[
\int_{I_\nu^\nu} e^{-i \nu \tau/2} \psi_\nu^\nu(t) \int_t^{\infty} \sum_{\nu \neq 0} \sum_{\nu \neq 0} e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3} \frac{1 - \frac{\nu \tau}{8\pi}}{\nu \alpha_1 \alpha_2 \alpha_3} d\tau d\tau
\]

\[
= \int_{I_\nu^\nu} e^{-i \nu \tau/2} \psi_\nu^\nu(t) \int_t^{\infty} \sum_{\nu \neq 0} \sum_{\nu \neq 0} e^{-i \nu \tau} e^{\nu \alpha_1 \alpha_2 \alpha_3} \frac{1 - \frac{\nu \tau}{8\pi}}{\nu \alpha_1 \alpha_2 \alpha_3} d\tau d\tau
\]

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\[ + \int_{I_0} e^{-iht/2} \psi_i \sum_{m \neq 0} |r(m)| e^{-itm} \frac{1}{im}\alpha_j, \alpha_j, \alpha_j, \alpha_j, dt \]

\[ \lesssim \frac{1}{(h)(\nu + 1)} \sum_{m \neq 0} \frac{1}{m} |\alpha_j, \alpha_j, \alpha_j, (1 + \|\alpha\|_{L^p})|. \]

Then, taking $1/p + 1/q = 1$ and recalling that the number of elements of $r(m)$ are less or equal to $C_{\varepsilon}m^\varepsilon$ for any $\varepsilon > 0$, we have

\[ \sum_k \|\mathcal{T}_k^\nu(R)\|_{L^p(L_n)} \lesssim \frac{1}{(\nu + 1)^p} \sum_k \sum_h \sum_{l \leq 2^{\frac{\nu}{q}}} \frac{2^q}{l} \langle l \rangle^p |\alpha_j, \alpha_j, \alpha_j, \rangle^p \left( \sum_{m \neq 0} \frac{1}{m} |\alpha_j, \alpha_j, \alpha_j, \rangle^p \right) \left( \sum_{l \neq 0} \frac{1}{l} \right)^{\frac{q}{q-\varepsilon}} \]

where we choose $\varepsilon < q - 1$.

Let us consider the case $\nu \in \{N, N+1\}$. Notice that $\mathcal{T}_0(R)(t)$ is zero for $t \leq \pi N$. Then it is enough to multiply $\mathcal{T}_0$ by $(1 - \eta_{\nu+1})$ to extend $\mathcal{T}_0$ to a periodic function in $I_0$ for $\nu \in \{N, N+1\}$. While estimating the $L^p(L^p)$ norm of $\mathcal{T}_0$ the only new term is

\[ \eta_N(t) (1 - \eta_{\nu+1}) \int_{I_0} e^{-irm} \frac{1}{im}\alpha_j, \alpha_j, \alpha_j, \alpha_j, d\tau. \]
Notice that the 0-th Fourier coefficient is bounded by
\[ \left| \frac{1}{(\nu + 1)} \sum_{m \neq 0} \frac{1}{|m|} |\alpha_j \bar{\alpha}_j \alpha_j| \right|. \]

Let \( h \neq 0 \), then the \( h \)-th Fourier coefficient
\[
\left| \int_{I^*_h} e^{-iht/2} \eta_N(t)(1 - \eta_{\nu + 1}) \int_t^\infty \sum_{m \neq 0} \sum_{r(m)}^\infty \eta_N^*(\tau) e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau dt \right|
\]
\[
\left| \int_{I^*_h} e^{-iht/2} \eta_N(t)(1 - \eta_{\nu + 1}) \int_t^\infty \sum_{m \neq 0} \sum_{r(m)}^\infty \eta_N^*(\tau) e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau dt \right|
\]
\[
+ \int_{I^*_h} \frac{e^{-ih/2} \eta_N(t)(1 - \eta_{\nu + 1})}{\nu + 1} \sum_{m \neq 0} \sum_{r(m)}^\infty \eta_N^*(\tau) e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau dt \right|
\]
\[
\leq \frac{1}{(\nu + 1)} \sum_{m \neq 0} \sum_{r(m)} \frac{1}{|m|} |\alpha_j \bar{\alpha}_j \alpha_j| ,
\]

which can be treated as before. This concludes the proof for the term \( \mathcal{T}_0 \) because we can argue as for \( \nu > N + 1 \).

**Estimates of \( \mathcal{T}_1 \)** The linear term \( \mathcal{T}_1 \) is the sum of five terms, more precisely it reads
\[
(\mathcal{T}_1(R))_k = \frac{i \eta_N(t)}{8\pi} \int_t^\infty \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau + \frac{i \eta_N(t)}{8\pi} \int_t^\infty \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau
\]
\[
+ \frac{i \eta_N(t)}{8\pi} \int_t^\infty \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau - \frac{i \eta_N(t)}{8\pi} \int_t^\infty \frac{\eta_N(\tau)}{\tau} \alpha_k R_k d\tau - \frac{i \eta_N(t)}{8\pi} \int_t^\infty \frac{\eta_N(\tau)}{\tau} \alpha_k R_k d\tau .
\]

Let us denote the first term of the right hand side by \( (F(R))_k \). In the following we estimate only the term \( F(R) \). The estimate of the other terms follow similarly. As in the previous case, we start by considering \( \nu > N + 1 \). We have
\[
\| F(R) \|_{L^p(\mathbb{Z}; L^p(I_\nu))}^p \leq \sum_k \sum_h \left| \langle \psi \rangle_k (F(R))_k \right|^p .
\]

First of all, we note that \( \psi \) is supported in the interval \( I^*_\nu \), so we can decompose \( (F(R))_k \) as the sum of a time dependent part plus a constant as follows
\[
(\mathcal{F}(R))_k = \frac{i}{8\pi} \int_t^\infty \frac{1}{\tau} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau
\]
\[
= \frac{i}{8\pi} \int_t^\infty \sum_{\mu \geq -2} \sum_{\mu \leq \nu + 2} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau
\]
\[
= \frac{i}{8\pi} \sum_{\mu \geq -2} \sum_{\mu \leq \nu + 2} \int_t^\infty \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau
\]
\[
+ \frac{i}{8\pi} \sum_{\mu \geq \nu + 3} \int_{I^*_\nu} \frac{\psi \mu}{\tau} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} d\tau .
\]

Let us start with the constant term. Note that
\[
\frac{1}{8\pi} \int_{I^*_\nu} \frac{\psi \mu}{\tau} \sum_{m \neq 0} \sum_{r(m)}^\infty e^{-i\tau m} e^{i\Delta m \log 4^r \alpha_j \bar{\alpha}_j \alpha_j} = \frac{1}{\mu + 1} \sum_{m \neq 0} \sum_{r(m)}^\infty \left( \int_{I^*_\nu} e^{-i\tau m} \psi \mu R_j d\tau \right) \bar{\alpha}_j \alpha_j
\]
where
\[ \tilde{\psi}_\mu(t) = \frac{\mu + 1}{8 \pi t} e^{\frac{\mu}{8 \pi t} \log 4t} \psi_\mu, \]
which satisfies
\[ \| \tilde{\psi}_\mu \|_{\dot{H}^{s,p}} \leq C(1 + \| \alpha \|_{l^{\infty}}^2), \]
with \( C \) independent of \( \mu \geq 0 \) and \( m \).

By using the previous computation we have
\[
\sum_k \left| \sum_{m \neq 0} \frac{1}{(n + 1)^2} \sum_{m \neq 0} \sum_{m \neq 0} e^{-i m \tau} e^{\frac{\mu}{8 \pi t} \log 4t} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3} \right|^p \leq \sum_k \left| \sum_{m \neq 0} \frac{1}{(n + 1)^2} \sum_{m \neq 0} \sum_{m \neq 0} \| \tilde{\psi}_\mu R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3} \|_p \right|^p.
\]
where we recall that \( j_1 = k - z \) for \( z \in r(m) \). Moreover in the last inequality we use the fact that for any function \( f \in \mathcal{H}_p^s(I_\mu) \) and \( \psi \in C_c^\infty((\mu, (\mu + 2) \pi)) \) the function \( \check{f}(t) = f(t) \psi(t) \) in \( I_\mu \) and \( \check{f}(t) = 0 \) in \( I_\mu \setminus I_\mu \) satisfies
\[
\| \check{f} \|_{\dot{H}^{s,p}(I_\mu)} \leq C \| \check{f} \|_{\dot{H}^{s,p}(I_\mu)}, \quad \| \check{f} \|_{\dot{H}^{s,p}(I_\mu)} = C \| \check{f} \|_{\dot{H}^{s,p}(I_\mu)} \| \check{f} \|_{\mathcal{H}_p^s(I_\mu)}, \quad (20)
\]
for \( s > 1 - 1/p \). Inequality (20) holds true because for any \( \tilde{f} \in \dot{H}^{s,p}(I_\mu) \) such that \( \tilde{f}|_{I_\mu} = f \), it holds \( \tilde{f} \psi = f \psi \). Moreover using the fact that \( \dot{H}^{s,p} \) is an algebra for \( s > 1 - 1/p \), we have
\[
\| \psi f \|_{\dot{H}^{s,p}(I_\mu)} = \| \check{f} \|_{\dot{H}^{s,p}(I_\mu)} \leq C \| \check{f} \|_{\dot{H}^{s,p}(I_\mu)} \| \check{f} \|_{\dot{H}^{s,p}(I_\mu)}.
\]
Taking the infimum on both sides, we deduce (20).

The estimate for the second term of (19) follows then from the fact that \( \psi_\mu \) is bounded in \( \dot{L}^p(I_\mu) \) independently of \( \nu \).

Let us now treat the remaining terms of (19). In other words the once with \( \mu \in [\nu - 2, \nu + 2] \). Without loss of generality we consider only the case \( \mu = \nu \). Let us compute the \( h \)-th Fourier coefficient
\[
\int_{I_\mu} e^{-i \frac{\mu}{8 \pi t} \log 4t} \psi_\nu(t) \int_t^{x(\nu + 2)} \psi_\nu(\tau) \sum_{m \neq 0} e^{-i m \tau} e^{\frac{\mu}{8 \pi t} \log 4t} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3} d\tau dt.
\]
For $h \neq 0$ and by two integration by parts we have

$$
\left| \int_{I''_{\nu}} e^{-i\frac{2}{h}t\psi_\nu(t)} \int_t^{\pi(\nu+2)} e^{-i(m-l)\tau} d\tau dt \right| \lesssim \frac{1}{(\nu+1)^2} \sum_{m \neq 0} \sum_{r(m)} \sum_{l} \left| \int_{I''_{\nu}} e^{-i\frac{2}{h}t\psi_\nu(t)} \int_t^{\pi(\nu+2)} e^{-i(m-l)\tau} d\tau dt \right| \left[(\nu+1)\psi_\nu R_{j_1} \bar{\alpha}_{j_2} \alpha_{j_3} \right].
$$

We deduce that

$$
\sum_{k} \sum_{h} \left| \int_t^{\pi(\nu+2)} \frac{\psi_\nu}{8\pi \tau} \sum_{m \neq 0} \sum_{r(m)} \sum_{l} \frac{1}{(m-l) + \frac{1}{(m-l+h/2)}} \left((\nu+1)\psi_\nu R_{j_1} \bar{\alpha}_{j_2} \alpha_{j_3} \right)_{2/3} \right|_{h}^{p} \lesssim \frac{1}{(\nu+1)^2} \sum_{k} \sum_{h} \left| \frac{1}{\mu} \left(1+R_{\mu} \right) \right|_{h}^{p}.
$$

Note that in the last estimate we use Hölder’s inequality and the restriction to $q > 1$ i.e. $s > 1 - 1/p$ and [20].

Let us now consider the case $\nu \in \{N, N + 1\}$. As before, we extend $F(R)$ by $(1 - \eta_{\nu+1})F(R)$. Moreover we decompose $F_k(R)$ as the sum of a time dependent part plus a constant as follows

$$
(F(R))_k = \frac{i\eta_N(t)}{8\pi} \int_{t}^{\infty} \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0} \sum_{r(m)} e^{-i\tau m} e^{\frac{\Delta m}{\log 4 R}} R_{j_1} \bar{\alpha}_{j_2} \alpha_{j_3} d\tau
$$

$$
= \frac{i\eta_N(t)}{8\pi} \int_{t}^{\infty} \frac{\psi_\mu}{\tau} \sum_{\mu \geq N} \sum_{m \neq 0} e^{-i\tau m} e^{\frac{\Delta m}{\log 4 R}} R_{j_1} \bar{\alpha}_{j_2} \alpha_{j_3} d\tau
$$

$$
= \frac{i\eta_N(t)}{8\pi} \sum_{N \leq \mu \leq \nu+2} \int_{t}^{\infty} \frac{\psi_\mu}{\tau} \sum_{m \neq 0} e^{-i\tau m} e^{\frac{\Delta m}{\log 4 R}} R_{j_1} \bar{\alpha}_{j_2} \alpha_{j_3} d\tau
$$

$$
+ \frac{i\eta_N(t)}{8\pi} \sum_{\mu \geq \nu+3} \int_{t}^{\infty} \frac{\psi_\mu}{\tau} \sum_{m \neq 0} e^{-i\tau m} e^{\frac{\Delta m}{\log 4 R}} R_{j_1} \bar{\alpha}_{j_2} \alpha_{j_3} d\tau.
$$

From this point we can argue as for the case $\nu > N + 1$.

**Estimates of $T_2$** The functional $T_2$ is the sum of eight terms, six of them are quadratic in $R$ and two are cubic. We will just look at the most delicate one. Namely

$$
G_k(R) = \frac{i\eta_N(t)}{8\pi} \int_{t}^{\infty} \frac{\eta_N(\tau)}{\tau} \sum_{m \neq 0} \sum_{r(m)} e^{-i\tau m} e^{\frac{\Delta m}{\log 4 R}} R_{j_1} \bar{R}_{j_2} R_{j_3} d\tau.
$$

For $\nu > N + 1$ we consider the extension $\psi_\nu G(R)$ in $I''_{\nu}$. We have

$$
\|G(R)\|_{L_p(\mathbb{R},L_p(I_{\nu}))} \leq \sum_{k} \sum_{h} \left| \psi_\nu G(R) \right|_{h}^{p}
$$
First of all we note that $\psi^c_k(G(R))$ is supported in the interval $I'_\nu$, so we can decompose $G(R)$ as the sum of a time dependent part plus a constant as follows

$$(G(R))_k = \int_t^\infty \frac{1}{8\pi t} \sum_{m \neq 0 r(m)} e^{-irm} e^{\frac{\lambda m}{8\pi t} \log 4\tau} R_j \bar{R}_j \bar{R}_j \ d\tau$$

$$= \int_t^\infty \sum_{\mu \geq \nu - 2} \frac{\psi_\mu}{8\pi t} \sum_{m \neq 0 r(m)} e^{-irm} e^{\frac{\lambda m}{8\pi t} \log 4\tau} R_j \bar{R}_j \bar{R}_j \ d\tau$$

$$= \sum_{\nu - 2 \leq \mu \leq \nu + 2} \int_t^\infty \frac{\psi_\mu}{8\pi t} \sum_{m \neq 0 r(m)} e^{-irm} e^{\frac{\lambda m}{8\pi t} \log 4\tau} R_j \bar{R}_j \bar{R}_j + \sum_{\mu \geq \nu + 3} \int_t^\infty \frac{\psi_\mu}{8\pi t} \sum_{m \neq 0 r(m)} e^{-irm} e^{\frac{\lambda m}{8\pi t} \log 4\tau} R_j \bar{R}_j \bar{R}_j. \quad (21)$$

Let us start by considering the constant part. Note that

$$\mu = \eta^{\nu - 2} \sum_{m \neq 0 r(m)} \left( \int_{I_\nu} e^{-i\lambda m} \eta^{\nu - 1} \psi_\mu \ d\tau \right)$$

$$= \frac{1}{(\mu + 1)^4} \sum_{m \neq 0 r(m)} \sum_{\nu - 1} (\mu + 1)^3 \left( \int_{I_\nu} e^{-i\lambda m} \eta^{\nu - 1} \psi_\mu \ d\tau \right)$$

To estimate the above term we follow exactly the case $i = 1$. For the time dependent term w.l.o.g. we consider only the case $\mu = \nu$. Let compute the $h$-th Fourier coefficient

$$\int_{I_\nu} e^{-i\lambda t} \psi_\nu(t) \int_t^{\pi(\nu + 2)} \frac{1}{8\pi t} \sum_{m \neq 0 r(m)} e^{-i\lambda m} e^{\frac{\lambda m}{8\pi t} \log 4\tau} R_j \bar{R}_j \bar{R}_j \ d\tau \ d\tau$$

$$= \int_{I_\nu} e^{-i\lambda t} \psi_\nu(t) \int_t^{\pi(\nu + 2)} \frac{1}{(\nu + 1)^4} \sum_{m \neq 0 r(m)} \sum_{l} e^{-i(l(m - l))\tau} (\nu + 1)^3 \eta^{\nu - 1} \psi_\mu \ d\tau$$

$$= \frac{1}{(\nu + 1)^4} \sum_{m \neq 0 r(m)} \sum_{l} \left( \int_{I_\nu} e^{-i\lambda t} \psi_\nu(t) \int_t^{\pi(\nu + 2)} e^{-i(l(m - l))\tau} \ d\tau \right) (\nu + 1)^3 \eta^{\nu - 1} \psi_\mu \ d\tau$$

From this point we can proceed as in the case $i = 1$.

For $\nu = \{ N, N + 1 \}$, it is enough to extend $G(R)$ by $(1 - \eta_{\nu + 1})G(R)$ and follow the estimates for $\nu > N + 1$.

3.1.2 Proof of Lemma 2

Proof of Lemma 2 As before we divide the proof in three parts. In particular we present all the details for the terms $\mathcal{T}_0$ and $\mathcal{T}_1$. Then we explain how to deduce the estimates of $\mathcal{T}_2$ following the ideas used for $\mathcal{T}_1$.

Estimates for $\mathcal{T}_0$ We start by recalling some useful tools that has been already introduced in the proof of Lemma 1. Let $\{ \psi_\mu \}$ a partition of unity defined as $\psi_\mu(t) = \eta_\mu(t) - \eta_{\mu + 1}(t)$. It holds $0 \leq \psi_\mu \leq 1$, $\psi_\mu$ is smooth and supported in $I_\mu = [\pi \mu, \pi (\mu + 2)]$. Moreover take

$$\psi^c_\mu = \sum_{d = -1}^1 \psi_{\nu + d} = \eta_{\nu - 1} - \eta_{\nu + 1} = \eta_{\nu - 1}(1 - \eta_{\nu + 1})$$

which is a set of smooth cut-off functions supported in the interior of $I^c_\nu$, identically 1 in $I_\nu$, with uniformly bounded derivative and $0 \leq \psi^c_\mu \leq 1$.

For $\nu > N + 1$, we estimate the $L^p(\hat{H}_p^\nu(I_\nu))$ seminorm of $\mathcal{T}_0(R)$ by the seminorm of the extension $\hat{\mathcal{T}}_0(R) = \psi^c_\nu \mathcal{T}_0(R)$. We have

$$\| \mathcal{T}_0(R) \|^p_{L^p(\hat{H}_p^\nu(I_\nu))} \leq \sum_k \sum_k \| h \|^p_{\mathcal{H}} \left( |\psi^c_\nu(\mathcal{T}_0(R))_k| \right)^p = \sum_k \sum_k \frac{1}{|h|^{1 - s}} \left( |\partial_t(\psi^c_\nu(\mathcal{T}_0(R))_k)| \right)^p.$$
Let us compute the Fourier coefficients. First of all note that
\[ \partial_t (\psi^\nu_\nu R(R))_k = (\psi^\nu_\nu)' R(R))_k + \psi^\nu_\nu (T(R))'_k. \]

For the term \((\psi^\nu_\nu)' R(R))_k\), we use (17) and the proof of the estimate follows as for the analogous case in Lemma 1. Let now rewrite
\[ \psi^\nu_\nu (T(R))'_k = - \frac{\psi^\nu_\nu(t)}{8\pi t} \sum_{m \neq 0} \sum_{r(m)} e^{-itm} e^{i \frac{44\log 4}{\alpha_j \bar{\alpha}_{j_3}}} \]
\[ = - \frac{1}{\nu + 1} \sum_{m \neq 0} \sum_{l \in \mathbb{Z}} \sum_{l+h \neq 0} e^{-it(m-1/2)} [\tilde{\psi}^\nu_\nu]_l \alpha_j \bar{\alpha}_{j_3} \]
Using the above equality and some Hölder inequalities, we deduce
\[ \sum_{k} \sum_{k \neq 0} \frac{1}{|h|^{1/2} p} \left| [\psi^\nu_\nu (T(R))'_k]_h \right|^p = \frac{1}{(\nu + 1)^p} \sum_{k} \sum_{h \neq 0} \frac{1}{|h|^{1/2} p} \left( \sum_{l \in \mathbb{Z}} \sum_{l+h \neq 0} \frac{1}{l} \left| \tilde{\psi}^\nu_\nu l \right| \alpha_j \bar{\alpha}_{j_3} \right)^p \]
\[ \leq \frac{1}{(\nu + 1)^p} \sum_{k} \sum_{h \neq 0} \frac{1}{|h|^{1/2} p} \left( \sum_{l \in \mathbb{Z}} \sum_{l+h \neq 0} r(l/2) \left| \tilde{\psi}^\nu_\nu l \right| \alpha_j \bar{\alpha}_{j_3} \right)^p \]
\[ \lesssim \frac{1}{(\nu + 1)^p} \sum_{k} \sum_{h \neq 0} \frac{h^\varepsilon}{|h|^{1/2} p} \sum_{l \in \mathbb{Z}} \sum_{l+h \neq 0} r(l/2) \left| \tilde{\psi}^\nu_\nu l \right| \alpha_j \bar{\alpha}_{j_3} \]
\[ \lesssim \frac{1}{(\nu + 1)^p} \left\| \alpha \right\|_p^3 \sup_{h \neq 0} \frac{1}{|h|^{1/2} p - \varepsilon} \]
\[ \lesssim \frac{1}{(\nu + 1)^p} \left\| \alpha \right\|_p^3. \]

Note that the last inequality holds true only if we consider \( s < 1 \).

For \( \nu \in \{N, N+1\} \), as in Lemma 1 we extend \( T_0 \) by \((1 - \eta_{\nu+1}) T_0\). For these terms the estimates follow easily as for \( \nu > N + 1 \).

**Estimates \( T_1 \)** As before, we restrict only to the term \( F \) which is the first element of the right hand side of (18). For \( \nu > N + 1 \), to estimate the \( l^p(\mathcal{H}_p^?) \) seminorm we use the extension \( \tilde{F}(R) = \psi^\nu_\nu F(R) \). As before
\[ \left\| F(R) \right\|_{l^p(\mathcal{H}_p^?)} \leq \sum_{h \neq 0} \left| [\tilde{\psi}^\nu_\nu (F(R))_k]_h \right|^p = \sum_{k} \sum_{h \neq 0} \frac{1}{|h|^{1/2} p} \left| \partial_t (\psi^\nu_\nu (F(R))_k) \right|_h \]

Let us compute the Fourier coefficients. We use Leibnitz’s rule as before and we use (19) instead of (17). Let now rewrite
\[ \psi^\nu_\nu (F(R))'_k = - \frac{1}{8\pi t} \sum_{d = -1} \sum_{m \neq 0} e^{-itm} e^{i \frac{44\log 4}{\alpha_j \bar{\alpha}_{j_3}}} R_j. \]
inequality, we deduce

\[ -\sum_{d=-1} \sum_{h \neq 0} \sum_{m \neq 0} \sum_{r(m)} e^{-it(m-l/2)} (\nu + 1 + d) [\tilde{\psi}_{\nu+d} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3}] \]

\[ = -\sum_{d=-1} \sum_{h \neq 0} \sum_{m \neq 0} e^{-i\theta/2} \left( \sum_{l \in \mathbb{Z}} \sum_{r((h+l)/2)} (\nu + 1 + d) [\tilde{\psi}_{\nu+d} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3}] \right) \]

\[ = -\sum_{h \neq 0} \sum_{d=-1} (\nu + 1 + d)^2 \left( \sum_{l \in \mathbb{Z}} \sum_{r((h+l)/2)} (\nu + 1 + d) [\tilde{\psi}_{\nu+d} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3}] \right). \]

For the \( L^p(\mathcal{H}_p^{l-s}) \) seminorm of \( \psi^e_p(F(R))_k \), we get

\[ \sum_k \sum_{h \neq 0} \frac{1}{|h|^{(1-s)p}} \left| [\psi^e_p(F(R))_k]_h \right|^p = \sum_k \sum_{h \neq 0} \frac{1}{|h|^{(1-s)p}} \left| \sum_{d=-1} (\nu + 1 + d)^2 \sum_{l \in \mathbb{Z}} \sum_{r((h+l)/2)} (\nu + 1 + d) [\tilde{\psi}_{\nu+d} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3}] \right|^p. \]

Note that the sum in \( d \) is of three elements so it is enough to consider for example the case \( d = 0 \). After applying Hölder’s inequality, we deduce

\[ \frac{1}{(\nu + 1)^{2p}} \sum_k \sum_{h \neq 0} \frac{1}{|h|^{(1-s)p}} \sum_{l \in \mathbb{Z}} \sum_{r((h+l)/2)} (\nu + 1) |\tilde{\psi}_{\nu} R_{j_1} \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3}|^p \]

\[ \lesssim \left( \frac{1}{(\nu + 1)^{2p}} \right)^p \left( \frac{1}{|h|^{(1-s)p}} \right)^\varepsilon \sum_k \sum_{h \neq 0} \sum_{l \in \mathbb{Z}} \sum_{r((h+l)/2)} (\nu + 1) \left| \tilde{\alpha}_{j_2} \tilde{\alpha}_{j_3} \right|^p \]

\[ \lesssim \frac{\|a\|^{2p}}{(\nu + 1)^{2p}} \|\nu + 1\|^{\varepsilon} \sup_{h \neq 0} \frac{1}{|h|^{(1-s)p-\varepsilon}}, \]

the desired estimates hold for \( s \in (0,1) \).

For \( \nu \in \{N, N+1\} \), as in Lemma \( \text{II} \) we extend \( F \) by \((1 - \eta_{\nu+1})T_0 \) and the estimates follow easily as for \( \nu > N + 1 \).

**Estimates of \( T_2 \)** As before let us restrict to the estimates of the cubic term \( G \). For \( \nu > N + 1 \), to estimate the \( L^p(\mathcal{H}_p^{l-s}) \) seminorm we use the extension \( \tilde{G}(R) = \psi^e_p G(R) \). So

\[ \|G(R)\|_{p}^{\varepsilon} \leq \sum_k \sum_{h \neq 0} |h|^{\varepsilon} \left| [\psi^e_p(G(R))_k]_h \right|^p \]

Let us compute the Fourier coefficients of \( \partial_t(\psi^e_p(G(R))_k) \). First of all note that

\[ \partial_t(\psi^e_p(G(R))_k) = (\psi^e_p)'(G(R))_k + \psi^e_p(G(R))_k. \]

For the term \( (\psi^e_p)'(G(R))_k \), we use \( \text{[21]} \) and the proof of the estimate follows as for the analogous case in Lemma \( \text{II} \). Similarly

\[ \psi^e_p(G(R))_k = -\frac{1}{8\pi t} \sum_{d=-1} \sum_{m \neq 0} e^{-imt} e^{\frac{\nu + d}{8\pi t} \log 4t} R_{j_1} R_{j_2} R_{j_3}. \]
Thus we denote \( \sum \) and start by recalling equation (14) which reads

\[
\begin{align*}
&= - \sum_{d=-1}^{1} \frac{1}{(\nu+1+d)^{4}} \sum_{m \neq 0} \sum_{r(m)} \int_{l} e^{-it(m-l/2)}(\nu+1+d)^{3} |\psi_{\nu+d}R_{j_{1}}R_{j_{2}}R_{j_{3}}l| \\
&= - \sum_{d=-1}^{1} \frac{1}{(\nu+1+d)^{4}} \sum_{h} e^{-ith/2} \left( \sum_{l \in \mathbb{Z}} \sum_{r(h+l)/2} |\nu+1+d| |\psi_{\nu+d}R_{j_{1}}R_{j_{2}}R_{j_{3}}l| \right) \\
&= - \sum_{h} e^{-ith/2} \left( \sum_{d=-1}^{1} \frac{1}{(\nu+1+d)^{4}} \left( \sum_{l \in \mathbb{Z}} \sum_{r(h+l)/2} |\nu+1+d| |\psi_{\nu+d}R_{j_{1}}R_{j_{2}}R_{j_{3}}l| \right) \right).
\end{align*}
\]

The estimates then follow straightforward as in the case \( i = 1 \).

Finally for the case \( \nu \in \{N, N+1\} \). We extend \( G(R) \) by \( (1 - \eta_{\nu+1})G(R) \) and we argue as in the case \( \nu > N+1 \).

\[ \Box \]

4 Proof of Proposition 1

This section is dedicated to the proof of Proposition 1. Let start by recalling equation (14) which reads

\[
i\partial_{t}B_{k} = -\frac{1}{8\pi t} \sum_{m \neq 0} e^{-imt} e^{it \nu} \log \frac{4}{t} B_{j_{1}j_{2}j_{3}l} B_{k} + \frac{1}{8\pi t} (|B_{k}|^{2} - |\alpha_{k}|^{2}) B_{k} \quad \text{and} \quad \lim_{t \to +\infty} B_{k} = \alpha_{k}.
\]

We are looking for explicit solutions in the case the initial datum \( \alpha_{k} = \alpha \) for any \( k \in \mathbb{Z} \). First of all notice that \( \Lambda_{m} = 0 \) and let us make the ansatz that \( B_{k}(t) = B(t) \) for any \( k \in \mathbb{Z} \). The equations rewrite

\[
i\partial_{t}B = -|B|^{2}B \sum_{m \neq 0} \frac{e^{-imt}}{8\pi t} r_{m} + \frac{1}{8\pi t} (|B|^{2} - |\alpha|^{2}) B \quad \text{and} \quad \lim_{t \to +\infty} B = \alpha.
\]

where we denote \( \sum_{r(m)} = r_{m} \). Notice that by definition of \( r(m) \) we have \( r_{m} = r_{-m} \). We deduce that

\[
3 \left( \sum_{m \neq 0} e^{imr_{m}} = 0,
\right)
\]

thus

\[
d_{t}|B(t)|^{2} = 0.
\]

The equation rewrites

\[
i\partial_{t}B = -|\alpha|^{2}B \sum_{m \neq 0} \frac{e^{-imt}}{8\pi t} r_{m} \quad \text{and} \quad \lim_{t \to +\infty} B = \alpha.
\]

If we are able to show that \( \sum_{m \neq 0} e^{imr_{m}} \) is integrable in \( (T, +\infty) \) for some \( T > 0 \), then the solution is

\[
B(t) = \alpha e^{-i|\alpha|^{2} \int_{t}^{T} \sum_{m \neq 0} \sum_{r(m)} e^{imr_{m}} \, \frac{d\tau}{\pi}}
\]

We have

\[
- \int_{t}^{+\infty} \sum_{m \neq 0} \frac{e^{-im\tau}}{\tau} r_{m} \, d\tau = \sum_{m \neq 0} \frac{e^{-imt}}{imt} r_{m} + \int_{t}^{+\infty} \sum_{m \neq 0} \frac{e^{-im\tau}}{im\tau^{2}} r_{m} \, d\tau = \frac{1}{it} \sum_{m \neq 0} \frac{e^{-imt}}{m} r_{m} + \sum_{m \neq 0} r_{m} f(t, m),
\]

with

\[
|f(t, m)| \leq \frac{1}{m^{2}t^{2}}.
\]
As 
\[ r_m = 2\text{Card}\{j \in \mathbb{Z}, j|m\}, \]
we know that 
\[ r_m \lesssim \log m, \]
and the upper-estimate is optimal for \( m = 2^n \). Thus there is no issue for defining the part involving \( f(x, m) \), and the remaining part
\[ \hat{B}(t) := e^{it\sum_{m \neq 0} e^{-imt} \frac{r_m}{m}}, \]
satisfies
\[ \hat{B} \in L^\infty_T L^2(\nu, \nu + 1) \cap L^p_T \dot{H}^s(\nu, \nu + 1). \]
for all
\[ 1 \leq p \leq \infty, \quad 0 \leq s < \frac{1}{2}, \quad (1 + s)p > 1. \]

References

[1] Banica, V., Vega, L. (2012) Scattering for 1D cubic NLS and singular vortex dynamics. J. Eur. Math. Soc. (JEMS) 14, no. 1, 209-253.

[2] Banica, V., Vega, L. (2013). Stability of the self-similar dynamics of a vortex filament. Archive for Rational Mechanics and Analysis, 210(3), 673-712.

[3] Banica, V., Vega, L. (2018). Evolution of polygonal lines by the binormal flow. arXiv preprint [arXiv:1807.06948].

[4] Bourgain, J. (1993). Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. Geometric & Functional Analysis GAFA, 3(3), 209-262.

[5] Carles, R., Kappeler, T. (2017). Norm-inflation with infinite loss of regularity for periodic NLS equations in negative Sobolev spaces. Bull. Soc. Math. France, 145(4), 623-642.

[6] Christ, M., Colliander, J., Tao, T. (2003). Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations. American Journal of Mathematics, 125(6), 1235-1293.

[7] Da Rios, L. S. (1906). On the motion of an unbounded fluid with a vortex filament of any shape. Rend. Circ. Mat. Palermo, 22, 117-135.

[8] De la Hoz, F., Vega, L. (2014). Vortex filament equation for a regular polygon. Nonlinearity, 27(12), 3031.

[9] Grünrock, A. (2005). Bi-and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. International Mathematics Research Notices, 2005(41), 2525-2558.

[10] Grünrock, A., Herr, S. (2008). Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data. SIAM Journal on Mathematical Analysis, 39(6), 1890-1920.

[11] Hasimoto, H. (1972). A solution on a vortex filament. Journal of Fluid Mechanics, 51(3), 477-485.

[12] Harrop-Griffiths, B., Killip, R., Visan, M. (2020). Sharp well-posedness for the cubic NLS and mKdV in \( H^s(\mathbb{R}) \). arXiv preprint [arXiv:2003.05011].

[13] Jerrard, R. L., Smets, D. (2015). On the motion of a curve by its binormal curvature. Journal of the European Mathematical Society, 17(6), 1487–1515.

[14] Kita, N. (2006). Mode generating property of solutions to the nonlinear Schrödinger equations in one space dimension. GAKUTO Internat. Ser., Math. Sci. Appl., 26, 111-128.

[15] Kishimoto, N. (2009). Well-posedness of the Cauchy problem for the Korteweg-de Vries equation at the critical regularity. Differential and Integral Equations, 22(5/6), 447-464.
[16] Kenig, C. E., Ponce, G., Vega, L. (2001). On the ill-posedness of some canonical dispersive equations. Duke Mathematical Journal, 106(3), 617-633.

[17] Oh, T. (2017). A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces. Funkcialaj Ekvacioj, 60(2), 259-277.

[18] Oh, T., Wang, Y. (2020). Global well-posedness of the one-dimensional cubic nonlinear Schrödinger equation in almost critical spaces. J. Differential Equations 269, no. 1, 612-640

[19] Tsutsumi, Y. (1987). $L^2$-solutions for nonlinear Schrödinger equations and nonlinear groups, Funkcialaj Ekvacioj, 30, 115-125.

[20] Vargas, A., Vega, L. (2001) Global well-posedness for 1D non-linear Schrödinger equation for data with an infinite $L^2$ norm. J. Math. Pures Appl. (9) 80, no. 10, 1029-1044