STRONG SUBMEASURES AND SEVERAL APPLICATIONS

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To my daughter on her birthday occasion

ABSTRACT. A strong submeasure on a compact metric space $X$ is a sub-linear and bounded operator on the space of continuous functions on $X$. A strong submeasure is positive if it is non-decreasing. By Hahn-Banach theorem, a positive strong submeasure is the supremum of a non-empty collection of measures whose masses are uniformly bounded from above.

We give several applications of strong submeasures in various diverse topics, thus illustrate the usefulness of this classical but largely overlooked notion. The applications include: 1) Pullback and pushforward measures by dominant meromorphic maps between compact Kähler manifolds of the same dimensions; 2) Wedge intersection of positive closed $(1,1)$ currents; 3) Applications to dynamics of meromorphic maps on compact Kähler manifolds; 4) Variational principle in continuous dynamics. For instance, we show that positive strong submeasures are preserved under pullback and pushforward by meromorphic maps. In another result, we show that for any dominant meromorphic selfmap $f$, there is always a non-zero positive strong submeasure $\mu$ so that $f^*(\mu) = \mu$. These results are valid for dynamics on non-smooth compact complex varieties, and some of them are valid more generally for meromorphic correspondences.

1. INTRODUCTION

In this paper, we give several applications of strong submeasures, a classical but largely overlook notion, in various different topics, ranging from variational principle in continuous dynamics, to wedge intersection of positive closed currents and dynamics of meromorphic maps on compact Kähler manifolds. The fact that strong submeasures appear naturally in these many diverse topics illustrates their usefulness and worth of further investigation.

In dynamical systems and ergodic theory, measures play a crucial role. (To clarify the notation, in the remaining of this paper, by a measure we mean a positive measure, while if we mention a signed measure we will clearly indicate so.) If $f : X \to X$ is a continuous map of a compact metric space $X$, one of the main questions is to find a measure $\mu$ on $X$ which is invariant, that is $f^*(\mu) = \mu$ and of maximal entropy (that is the measure theoretic entropy and the topological entropy are the same: $h_{\mu}(f) = h_{\text{top}}(f)$). This tradition has been started at least since H. Poincaré. For later use, we recall that for a compact subset $A \subset X$, we have ([29]):

$$\mu(A) = \inf_{\phi \in C^0(X, \geq 1_A)} \mu(\phi).$$
Here $1_A : X \to \{0, 1\}$ is the characteristic function of $A$, that is $1_A(x) = 1$ if $x \in A$ and $= 0$ otherwise, $C^0(X)$ is the space of continuous functions from $X$ into $\mathbb{R}$, and for any bounded function $g : X \to \mathbb{R}$ we use the notations

$$C^0(X, \geq g) = \{ \phi \in C^0(X) : \phi \geq g \}.$$  

Moreover, for any open set $B \subset X$ we have (29)

$$\mu(B) = \sup_{A \text{ compact} \subset B} \mu(A).$$  

While pushforward of a measure by a proper continuous map is classical, there is currently no definition of pullback of a measure by a continuous function, so that the resulting is a measure with good compatible properties, even for quite nice maps such as blowup maps in Example 1 below. (Besides being interest on its own e.g. from the categorical viewpoint, pullback of a measure is actually needed when we want to pushforward a measure by a meromorphic map, see the next paragraph, Theorem 1.10 and the discussion before it for more detail.) However, as seen there, we can pullback the measure if we allow the resulting to be a positive strong submeasure (see definition below). More generally, let $f : X \to Y$ be a continuous function between compact metric spaces, and assume that there is an open dense set $U \subset Y$ so that $f : f^{-1}(U) \to U$ is a covering map of finite degree. Then the proof of Theorem 1.10 shows that we can pullback positive strong submeasures on $Y$ to positive strong submeasures on $X$.

In complex dynamics, it happens often that while it may be hard to construct dynamically interesting holomorphic maps $f : X \to X$ of a compact Kähler manifold $X$, it is usually easy to construct many dynamically interesting meromorphic maps $f : X \to X$. So it is important to extend notions from smooth dynamical systems to this more general case. However, as of now, there is only a definition of pushing forward measures having no mass on the indeterminate set $I(f)$ of $f$. As far as we know, there is no guarantee for the existence of a non-zero invariant probabilistic measure (with no mass on $I(f)$) for a general meromorphic map $f$. On the other hand, in Theorem 1.10 we show that a non-zero invariant positive strong submeasure always exists. Moreover, for the common practice of first finding canonical Green currents and then wedge them together in order to obtain a dynamically interesting measure, there is no guarantee that the wedge intersection can be defined as a measure. There are such cases where again strong submeasures can be applied, see Theorem 1.13 and the discussion before it.

In this paper, we use generalisations of measure and signed measures, called positive strong submeasures and (signed) strong submeasures, to show that in the situation in the above paragraph, $f_\ast(\mu)$ can be canonically defined for any measure $\mu$ on $X$ as a positive strong submeasure. More generally, we can pullback and push forward any positive strong submeasure and the resulting is again a positive strong submeasure. Strong submeasures are a generalisation of signed measures, and the corresponding generalisation of measures are so-called positive strong submeasures. We will also show that wedge intersection of positive closed $(1, 1)$ currents on $X$ can also be defined in terms of strong submeasures. These questions have been open for some decades.

Remarks. An example given by A. B. Taylor and B. Shiffman states that there are positive closed $(1, 1)$ currents $T$ in a small open set $U \subset \mathbb{C}^2$, being smooth outside a non-empty proper closed analytic subset $V \subset U$ and $T|_{U \setminus V} \land T|_{U \setminus V}$ have infinite mass near $V$. 
Therefore, on compact Kähler manifolds, for various previous definitions of intersection of positive closed (1, 1) currents (see Section 3 for more details), which are local in nature, self-wedge intersection of positive closed currents cannot be both well-defined and preserving cohomology intersections. On the other hand, submeasures allow the flexibility to do that in the most general setting, as seen in the main results of our paper.

Denote by \( \varphi \in C^0(X) \) the sup-norm \( ||\varphi||_{L^\infty} = \sup_{x \in X} |\varphi(x)| \). We recall that by Riesz representation theorem (see [35]), on a compact metric space \( X \) a measure of bounded mass \( \mu \) is the same as a linear operator \( \mu : C^0(X) \to \mathbb{R} \), that is \( \mu(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1 \mu(\varphi_1) + \lambda_2 \mu(\varphi_2) \) for all \( \varphi_1, \varphi_2 \in C^0(X) \) and constants \( \lambda_1, \lambda_2 \); which is bounded: \( |\mu(\varphi)| \leq C||\varphi||_{L^\infty} \) for a constant \( C \) independent of \( \varphi \in C^0(X) \), and positive: \( \mu(\varphi) \geq 0 \) whenever \( \varphi \in C^0(X) \) is non-negative. Note that we can then choose \( C = \mu(1) \) (the mass of the measure \( \mu \)) and by linearity the positivity is the same as having

\[
(1.4) \quad \mu(\varphi_1) \geq \mu(\varphi_2)
\]

for all \( \varphi_1, \varphi_2 \in C^0(X) \) satisfying \( \varphi_1 \geq \varphi_2 \).

Inspired by this, we now define strong submeasures and positive strong submeasures. In fact, these notions are classical in functional analysis (even though they were not named so in the literature), only that before they had not been applied in the above questions or even to smooth dynamical systems (in which case, since measures behave well under the pushforward map, there is no need to use generalisations of measures). Here, we show however, that positive strong submeasures apply naturally even to smooth dynamical systems. By the Variational Principle [27, 28], the topological entropy of a continuous map \( f \) on a compact Hausdorff space \( X \) is the supremum of the measure-theoretic entropy all over regular Borel measures which is invariant by \( f \) and has mass \( \leq 1 \). However, in general, even on compact metric spaces, the supremum may not be achieved by any such measure. One of main results in this paper (Theorem 1.5), is that the supremum in Variational Principle is attained at an invariant positive strong submeasure, under an appropriate notion of entropy for such submeasures.

We recall that a functional \( \mu : C^0(X) \to \mathbb{R} \) is sub-linear if \( \mu(\varphi_1 + \varphi_2) \leq \mu(\varphi_1) + \mu(\varphi_2) \) and \( \mu(\lambda \varphi) = \lambda \mu(\varphi) \) for \( \varphi_1, \varphi_2, \varphi \in C^0(X) \) and a non-negative constant \( \lambda \). A strong submeasure is then simply a sub-linear functional \( \mu : C^0(X) \to \mathbb{R} \) which is also bounded, that is there is a constant \( C > 0 \) so that for all \( \varphi \in C^0(X) \) we have \( |\mu(\varphi)| \leq C||\varphi||_{L^\infty} \). The least such constant \( C \) is called the norm of \( \mu \) and is denoted by \( ||\mu|| \). A strong submeasure \( \mu \) is positive if it is non-decreasing, that is for all \( \varphi_1 \geq \varphi_2 \) we have \( \mu(\varphi_1) \geq \mu(\varphi_2) \). It is easy to check that a strong submeasure is Lipschitz continuous \( |\mu(\varphi_1) - \mu(\varphi_2)| \leq ||\mu|| \times ||\varphi_1 - \varphi_2||_{L^\infty} \), and convex \( \mu(t_1 \varphi_1 + t_2 \varphi_2) \leq t_1 \mu(\varphi_1) + t_2 \mu(\varphi_2) \) for \( t_1, t_2 \geq 0 \). We denote by \( SM(X) \) the set of all strong submeasures on \( X \), and by \( SM^+(X) \) the set of all positive strong submeasures on \( X \).

By Hahn-Banach’s extension theorem (see [34]) and Riesz representation theorem (see [35]), we have the following characterisation of strong submeasures and positive strong submeasures.

**Theorem 1.1.** Let \( X \) be a compact metric space, and \( \mu : C^0(X) \to \mathbb{R} \) an operator.

1) \( \mu \) is a strong submeasure if and only if there is a non-empty collection \( \mathcal{G} \) of signed measures \( \chi = \chi^+ - \chi^- \) where \( \chi^\pm \) are measures on \( X \) so that \( \sup_{\chi = \chi^+ - \chi^- \in \mathcal{G}} \chi^\pm(1) < \infty \),
\( \mu(\varphi) = \sup_{\chi \in \mathcal{G}} \chi(\varphi). \) 

2) \( \mu \) is a positive strong submeasure if and only if there is a non-empty collection \( \mathcal{G} \) of measures on a compact metric space \( X \) so that \( \sup_{\chi \in \mathcal{G}} \chi(1) < \infty \), and:

\( \mu(\varphi) = \sup_{\chi \in \mathcal{G}} \chi(\varphi). \)

**Proof.**

1) Let \( \varphi_0 \in C^0(X) \). By Hahn-Banach's extension theorem, there is a linear operator \( \Lambda : C^0(X) \to \mathbb{R} \) so that \( \Lambda(\varphi_0) = \mu(\varphi_0) \) and

\[ -\mu(-\varphi) \leq \Lambda(\varphi) \leq \mu(\varphi), \]

for all \( \varphi \in C^0(X) \). Therefore, \( |\Lambda(\varphi)| \leq ||\mu|| \times ||\varphi||_{L^\infty} \). By Riesz representation theorem, \( \Lambda \) is a signed measure on \( X \). We take \( \mathcal{G} \) to be the collection of all such \( \Lambda \)'s, then \( \mu = \sup_{\chi \in \mathcal{G}} \chi \).

The converse is obvious.

2) Let \( \Lambda \) be as in 1) and \( \varphi \geq 0 \). Since \( \mu \) is positive, it follows that \( \mu(-\varphi) \leq \mu(0) = 0 \), and hence \( \Lambda(\varphi) \geq 0 \). It follows from Riesz representation theorem that \( \Lambda \) is a positive measure on \( X \). We can then proceed as in 1). \( \square \)

**Remark.**

1) We remark that unlike measures, for strong submeasures the condition that \( \mu(\varphi) \geq 0 \) for all \( 0 \leq \varphi \in C^0(X) \) is not enough for \( \mu \) to be positive. For example, let \( X \) be the space consisting of two points \( 0 \) and \( 1 \). Then an element \( \varphi \in C^0(X) \) is characterised by a tuple \( (a, b) \in \mathbb{R}^2 \), where \( a = \varphi(0) \) and \( b = \varphi(1) \). It is easy to check that the function \( \mu : \mathbb{R}^2 \to \mathbb{R} \) given by \( \mu(a, b) = \max\{a - b, 0\} \) is a strong submeasure on \( X \) and also that \( \mu(a, b) \geq 0 \) for all \( (a, b) \). However, \( \mu \) is not positive with respect to the above definition, since \( \varphi(1, 0) = 1 > \varphi(1, 1) = 0 \), and the function \( \varphi_1 = (1, 1) \) is bigger than the function \( \varphi_2 = (1, 0) \).

2) On the other hand, the proof of part 2) of Theorem 1.1 shows that for \( \mu \) to be positive, it suffices to have \( \mu(-\varphi) \leq 0 \leq \mu(\varphi) \) for all \( \varphi \geq 0 \).

**Definition 1.2.** We say that a sequence \( \mu_1, \mu_2, \ldots \in SM(X) \) weakly converges to \( \mu \in SM(X) \) if \( \sup_n ||\mu_n|| < \infty \) and

\[ \lim_{n \to \infty} \mu_n(\varphi) = \mu(\varphi) \]

for all \( \varphi \in C^0(X) \). We use the notation \( \mu_n \rightharpoonup \mu \) to denote that \( \mu_n \) weakly converges to \( \mu \).

If \( \mu_1, \mu_2 : C^0(X) \to \mathbb{R} \), we define \( \max\{\mu_1, \mu_2\} : C^0(X) \to \mathbb{R} \) by the formula \( \max\{\mu_1, \mu_2\}(\varphi) = \max\{\mu_1(\varphi), \mu_2(\varphi)\} \).

**Theorem 1.3.** Let \( X \) be a compact metric space.

1) **Weak-compactness.** Let \( \mu_1, \mu_2, \ldots \) be a sequence in \( SM(X) \) such that \( \sup_n ||\mu_n|| < \infty \). Then there is a subsequence \( \{\mu_{n(k)}\}_{k=1,2,\ldots} \) which weakly converges to some \( \mu \in SM(X) \). If moreover \( \mu_n \in SM^+(X) \), then so is \( \mu \).

2) If \( \mu \in SM^+(X) \), then \( ||\mu|| = \max\{||\mu(1)||, ||\mu(-1)||\} \).

3) If \( \mu_1, \mu_2 \in SM(X) \) then \( \max\{\mu_1, \mu_2\} \) and \( \mu_1 + \mu_2 \) are also in \( SM(X) \). If \( \mu_1, \mu_2 \in SM^+(X) \) then \( \max\{\mu_1, \mu_2\} \) and \( \mu_1 + \mu_2 \) are also in \( SM^+(X) \).
Like measures, **positive** strong submeasures give rise naturally to **set functions**. On a compact metric space \( X \), recall that a function \( g : X \to \mathbb{R} \) is upper-semicontinuous if for every \( x \in X \)

\[
\limsup_{y \to x} g(y) \leq g(x).
\]

For example, the characteristic function of a closed subset \( A \subset X \) is upper-semicontinuous. By Baire’s theorem \([5]\), if \( g \) is a bounded upper-semicontinuous function on \( X \) then the set \( C^0(X, \geq g) \) is non-empty and moreover

\[
g = \inf_{\varphi \in C^0(X, \geq g)} \varphi.
\]

More precisely, there is a sequence of continuous functions \( g_n \) on \( X \) decreasing to \( g \). Hence, if \( \mu \) is a **measure**, we have by Lebesgue and Levi’s monotone convergence theorem in the integration theory that

\[
\mu(g) = \lim_{n \to \infty} \mu(g_n) = \inf_{\varphi \in C^0(X, \geq g)} \mu(\varphi).
\]

Inspired by this and \([1, 2]\), if \( \mu \) is an **arbitrary** strong submeasure, we define for any upper-semicontinuous function \( g \) on \( X \) the value

\[(1.9) \quad E(\mu)(g) := \inf_{\varphi \in C^0(X, \geq g)} \mu(\varphi) \in [-\infty, \infty). \]

Then for a closed set \( A \subset X \), we define \( \mu(A) := E(\mu)(1_A) \) where \( 1_A \) is the characteristic function of \( A \). If \( \mu \) is positive, we always have \( \mu(A) \geq 0 \). Then, for an open subset \( B \subset X \), following \([1, 3]\) we define \( \mu(B) := \sup\{\mu(A) : A \text{ compact} \subset B\} \). Denote by \( BUS(X) \) the set of all bounded upper-semicontinuous functions on \( X \). We have the following result.

**Theorem 1.4.** Let \( X \) be a compact metric space and \( \mu \in SM(X) \). Let \( E(\mu) : BUS(X) \to [-\infty, \infty) \) be defined as in \([1, 2]\). Assume that \( E(\mu)(0) \) is finite. We have:

1) For all \( \varphi \in BUS(X) \), the value \( E(\mu)(\varphi) \) is finite. Moreover, \( E(\mu)(0) = 0 \) and \( E(\mu)(-1) = -\mu(1) \).

2) **Extension.** If \( \mu \) is positive, then for all \( \varphi \in C^0(X) \) we have \( E(\mu)(\varphi) = \mu(\varphi) \).

3) **Moreover**, \( E(\mu) \) satisfies the following properties

i) **Sub-linearity.** \( E(\mu)(\varphi_1 + \varphi_2) \leq E(\mu)(\varphi_1) + E(\mu)(\varphi_2) \) and \( E(\mu)(\lambda \varphi) = \lambda E(\mu)(\varphi) \) for \( \varphi_1, \varphi_2, \varphi \in BUS(X) \) and a non-negative constant \( \lambda \).

ii) **Positivity.** \( E(\mu)(\varphi_1) \geq E(\mu)(\varphi_2) \) for all \( \varphi_1, \varphi_2 \in BUS(X) \) satisfying \( \varphi_1 \geq \varphi_2 \).

iii) **Boundedness.** There is a constant \( C > 0 \) so that for all \( \varphi \in BUS(X) \) we have \( |E(\mu)(\varphi)| \leq C||\varphi||_{L^\infty} \). The least such constant \( C \) is in fact \( ||\mu|| \).

4) If \( A_1, A_2 \) are closed subsets of \( X \) then \( \mu(A_1 \cup A_2) \leq \mu(A_1) + \mu(A_2) \). Likewise, if \( B_1, B_2 \) are open subsets of \( X \) then \( \mu(B_1 \cup B_2) \leq \mu(B_1) + \mu(B_2) \).

**Remark.** Note that for every \( \mu \in SM(X) \), either \( E(\mu) \) is identically the function \( -\infty \) or for every \( \varphi \in C^0(X) \) the number \( E(\mu)(\varphi) \) is finite. In fact, if \( E(\mu) \) is not finite for some \( \varphi \), then by 1) of the theorem, we have \( E(\mu)(0) = -\infty \). Therefore, there is a function \( 0 \leq \varphi_0 \in C^0(X) \) so that \( \mu(\varphi_0) < 0 \). It then follows from the definition that for all \( \varphi \in C^0(X) \):

\[
E(\mu)(\varphi) \leq \inf_{n \in \mathbb{N}} \mu(\varphi + n\varphi_0) \leq \inf_{n \in \mathbb{N}} (\mu(\varphi) + n\mu(\varphi_0)) = -\infty.
\]
Here, in the second inequality we used the sub-linearity of $\mu$.

If we have a positive strong submeasure $\mu$, and define for any Borel set $A \subset X$ the number $\tilde{\mu}(A) = \inf \{ \mu(B) : B \text{ open}, A \subset B \}$, then we see easily from part 4) of Theorem 1.4 that: i) $\tilde{\mu}(\emptyset) = 0$, ii) $\tilde{\mu}(A_1) \leq \tilde{\mu}(A_2)$ for all Borel sets $A_1 \subset A_2$ and iii) $\tilde{\mu}(A_1 \cup A_2) \leq \tilde{\mu}(A_1) + \tilde{\mu}(A_2)$.

Such $\tilde{\mu}$ are known in the literature as submeasures (see e.g. [37]), and hence it is justified to call our objects $\mu$ positive strong submeasures.

Next we discuss some applications of strong submeasures. The first one is about the Variational Principle mentioned above. Given $X$ a compact metric space (or more generally a compact Hausdorff space) and $f : X \rightarrow X$ a continuous map. In Section 4 we will define a notion $h_\mu(f)$ of entropy for positive strong submeasures $\mu$ invariant by $f$, and prove the following Variational Principle.

**Theorem 1.5.** Let $X$ be a compact metric space (or more generally a compact Hausdorff space) and $f : X \rightarrow X$ a continuous map. Then

$$h_{top}(f) = \max \{ h_\mu(f) : \mu \text{ is an } f\text{-invariant positive strong submeasure of mass } \leq 1 \}.$$  

This result is relevant to part 5) of Theorem 1.10 where we show that for any compact Kähler manifold $X$ and meromorphic map $f : X \rightarrow X$, there is always a non-zero positive strong submeasure $\mu$ invariant by $f$. We will pursue a generalisation of this theorem to dynamics of meromorphic maps in a future work.

The second application is to discuss the pullback and pushforward of measures by meromorphic maps. Given $f : X \rightarrow Y$ a dominant meromorphic map of compact Kähler manifolds of the same dimension $k$. If $\mu$ is a measure having no mass on the indeterminate set $I(f)$, then the push forward $f_*(\mu)$ can be defined as a measure of the same mass as follows: For any Borel set $A \subset Y$ we define $f_*(\mu)(A) := \mu(f^{-1}(A \setminus I(f)))$. However, if $A$ has mass on $I(f)$, there is not yet a definition of what measure should $f_*(\mu)$ be. Here is an example. Let $\pi : Y \rightarrow X$ be the blowup of $X$ at a smooth point, and $f : X \rightarrow Y$ be the inverse of $\pi$. Then $f$ is bimeromorphic, but is not holomorphic. If $V$ is the exceptional divisor of the blowup $\pi$, then under $f$ the point $p$ blowups to $V$. Let $\delta_p$ be the Dirac mass at $p$. If we were able to define $f_*(\delta_p)$ as a measure, then it is natural to see that $f_*(\delta_p)$ must have support in $V$. But since on $V$, which is isomorphic to the projective space $\mathbb{P}^{k-1}$ and hence has a big automorphism group, we have many different measures it is not clear what measure should be the correct one. On the other hand, we will next show that there is a canonical way to assign $f_*(\mu)$ as a positive strong submeasure, with many good compatible properties. Under this new definition, the push forward of $f_*(\delta_p)$ can be described very simply, as will be seen in Example 1 below. (Remark: We were informed by Lucas Kaufmann Sacchato and Elizabeth Wulcan that in Example 1 they have a different definition of $f_*(\mu)$ as a measure, and this can be extended to some larger classes of measures. However, it is not yet clear to us what good properties that definition has.) In fact, the definition can be extended to all $SM^+(X)$, as follows.

Let $\Gamma_f \subset X \times Y$ be the graph of $f : X \rightarrow Y$, $\pi_X, \pi_Y : X \times Y \rightarrow X, Y$ be the projections, and $\pi_{X,f}, \pi_{Y,f}$ the restrictions of these maps to $\Gamma_f$. While $\Gamma_f$ is not smooth in general, it is a compact metric space under the induced metric from $X \times Y$, and the holomorphic map $\pi_{X,f} : \Gamma_f \rightarrow X$ is bimeromorphic. If we can define the following operators: $\pi_{X,f}^* : SM^+(X) \rightarrow SM^+(\Gamma_f)$ and $(\pi_{Y,f})_* : SM^+(\Gamma_f) \rightarrow SM^+(Y)$, then we can (as in the usual case of push forwarding a smooth form) obtain the desired push forward by the formula
Using Proposition 1.6, we define \((\pi^*_{X,f})(\mu)\). The push forward \((\pi^*_{Y,f})\) is straightforward. If \(\varphi \in C^0(Y)\), then the function \((\pi^*_{Y,f})\varphi\) is in \(C^0(\Gamma_f)\). Hence we can define for \(\chi \in SM(\Gamma_f)\) and \(\varphi \in C^0(Y)\):
\[
(\pi^*_{Y,f})(\chi)(\varphi) := \chi((\pi^*_{Y,f})(\varphi)).
\]
Hence, it remains to define \(\pi^*_{X,f}(\mu)\) for \(\mu \in SM(X)\). We would like to define, as in the case of pulling back a smooth form, for each function \(\varphi \in C^0(\Gamma_f)\) the following
\[
\pi^*_{X,f}(\mu)(\varphi) := \mu((\pi^*_{X,f})(\varphi)).
\]
The problem is that \((\pi^*_{X,f})\varphi\) can only be defined on the set \(U = X \setminus I(f)\) which is isomorphic to an open dense set of \(\Gamma_f\), and which in general only a proper subset of \(X\).

More precisely, if \(x \in U\), we define \((\pi^*_{U,f})(\varphi)(x) = \varphi((\pi^{-1}_{X,Y})(x))\), using that \(\pi^*_{U,f} : \pi^{-1}_{X,Y}(U) \subset \Gamma_f \rightarrow U\) is an isomorphism. It is easy to check that \((\pi^*_{U,f})(\varphi)(x)\) is a bounded continuous function on \(U\). In general, there is no way to extend this as a continuous function on the whole \(X\), but there is a canonical extension of \((\pi^*_{U,f})(\varphi)(x)\) to the whole \(X\) as an upper-semicontinuous function by the following result. (The next proposition is probably well known, but because of the lack of a reference, we include the statement here together with its proof.)

**Proposition 1.6.** Let \(X\) be a compact metric space, \(U \subset X\) an open dense set, and \(g : U \rightarrow \mathbb{R}\) a bounded upper semicontinuous function. Define \(E(g) : X \rightarrow \mathbb{R}\) as follows: If \(x \in U\) then \(E(g)(x) := g(x)\), and if \(x \in X \setminus U\) then
\[
E(g)(x) := \limsup_{y \in U, y \rightarrow x} g(y).
\]

Then

1) \(E(g)\) is a bounded upper-semicontinuous function, and \(E(g)|_U = g\). In other words, \(E(g)\) is a bounded upper-semicontinuous extension of \(g\).

2) If \(g\) is continuous on \(U\), \(U_1 \subset U\) is another open dense set of \(X\) and \(g_1 = g|_{U_1}\), then \(E(g_1) = E(g)\).

3) Moreover, \(E(g_1 + g_2) \leq E(g_1) + E(g_2)\) for any \(g_1, g_2 : U \rightarrow \mathbb{R}\) bounded upper-semicontinuous functions.

**Remark.** On the other hand, if \(g\) is not continuous on \(U\) then it is easy to construct examples for which the conclusion of part 2) in the proposition does not hold.

**Definition 1.7.** Using Proposition 1.6, we define \((\pi^*_{X,f})(\varphi)\) to be the upper-semicontinuous function \(E((\pi^*_{U,f})(\varphi))\) on \(X\). We emphasise that it is globally defined on the whole of \(X\), and is not changed if we replace \(U\) by one open dense subset of it. Similarly, if \(W \subset Y\) is a Zariski dense open subset so that \(\pi^*_{W,f} : \pi^{-1}_{Y,f}(W) \subset \Gamma_f \rightarrow W\) is a finite covering map, then for any continuous function \(\varphi\) on \(\Gamma_f\) we easily have that \((\pi^*_{W,f})(\varphi)\) is upper-semicontinuous on \(Y\) and is independent of the choice of \(W\) (see Section 2).

Using Proposition 1.6 and the above upper-semicontinuous pushforward, we can finally define the following pullback operator
\[
(1.11) \quad \pi^*_{X,f}(\mu)(\varphi) := \inf_{\psi \in C^0(X, \geq (\pi^*_{X,f})(\varphi))} \mu(\psi).
\]
Unlike (1.10), in (1.11) we need the condition that the strong submeasure \(\mu\) is positive, in order to have good properties (see the comments after Theorem 1.4).
**Definition 1.8.** For convenience, we write here the final formula for pushforwarding a strong submeasure by a meromorphic map $f$ with generically finite fibres:

$$(1.12) \quad f_*(\mu)(\varphi) := \inf_{\psi \in C^0(X, \geq (\pi_{X,f})_*(\pi^*_{Y,f}(\varphi)))} \mu(\psi).$$

Here $\pi^*_{Y,f}(\varphi) \in C^0(\Gamma_f)$ and $(\pi_{X,f})_*$ is the upper-semicontinuous pushforward of functions in $C^0(\Gamma_f)$.

Similarly, we can define the pullback of a strong submeasure by a meromorphic map $f$ with generically finite fibres by the formula:

$$(1.13) \quad f^*(\mu)(\varphi) := \inf_{\psi \in C^0(Y, \geq (\pi_{Y,f})_*(\pi^*_{X,f}(\varphi)))} \mu(\psi).$$

For convenience, we give also the following definition.

**Definition 1.9.** Let $\varphi \in C^0(X)$ and $\psi \in C^0(Y)$. We denote by $f_*(\varphi)$ the above upper-semicontinuous function $(\pi_{Y,f})_*(\pi^*_{X,f}(\varphi))$, and by $f^*(\psi)$ the above upper-semicontinuous function $(\pi_{X,f})_*(\pi^*_{Y,f}(\psi))$.

**Remark.** In the above definitions of pullback and pushforward (of positive strong submeasures) or upper-semicontinuous pullback and pushforward (of continuous functions) by meromorphic maps, if we replace the graph $\Gamma_f$ by its resolutions of singularities, we will obtain the same result. This will be proven in Section 2.

We then use (1.9) to extend (1.12) and (1.13) to all bounded upper-semicontinuous functions. Before stating the main result concerning these pullback and push forward operators, let us compute them for the blowup example above.

**Example 1.** Let $\pi : Y \to X$ be the blowup of $X$ at a point $p$, and $V \subset Y$ the exceptional divisor. Let $\delta_p$ be the Dirac measure at $p$. Then for any continuous function $\varphi$ on $Y$, we have

$$\pi^*\delta_p(\varphi) = \max_{y \in V} \varphi(y).$$

Therefore, $\pi^*\delta_p$ is not a measure. In particular, if $A \subset Y$ is a closed set then $\pi^*\delta_p(A) = \inf_{x \in C^0(X, \geq 1_A)} \pi^*\delta_p(\varphi)$ is $\delta_p(\pi(A \cap Y))$.

**Proof of Example 1.** By definition

$$\pi^*\delta_p(\varphi) = \inf_{\psi \in C^0(Y, \geq \pi_*\varphi)} \delta_p(\psi) = \inf_{\psi \in C^0(Y, \geq \pi_*\varphi)} \psi(p).$$

Since $\pi : Y \setminus V \to X \setminus \{p\}$ is an isomorphism, it is easy to check that $\pi_*(\varphi)(p) = \max_{y \in V} \varphi(y)$. Therefore, for any $\delta \in C^0(Y, \geq \pi_*\varphi)$, we have $\psi(p) \geq \max_{y \in V} \varphi(y)$. Hence by definition $\pi^*\delta_p(\varphi) \geq \max_{y \in V} \varphi(y)$. On the other hand, for any $\epsilon > 0$, choose a small neighborhood $U_\epsilon$ of $p$ so that

$$\sup_{y \in \pi^{-1}(U_\epsilon)} \varphi(y) \leq \epsilon + \max_{y \in V} \varphi(y).$$

It follows that $\sup_{U_\epsilon} \pi_*\varphi \leq \epsilon + \max_{y \in V} \varphi(y)$. Since $\pi_*(\varphi)$ is continuous on $X \setminus \{p\}$, it follows by elementary set theoretic topology that we can find a continuous function $\varphi$ on $X$ so that $\psi \geq \pi_*\varphi$ and $\sup_{U_\epsilon} \psi \leq \epsilon + \max_{y \in V} \varphi(y)$. It follows that $\pi^*\delta_p(\varphi) \leq \epsilon + \max_{y \in V} \varphi(y)$. Since $\epsilon$ is an arbitrary positive number, we conclude from the above discussion
that $\pi^*(\delta_p)(\varphi) = \max_{y \in V} \varphi(y)$. Similarly, we can show that $\pi^*(\delta_p)(A) = \delta_p(\pi(A \cap Y))$. (Q.E.D.)

**Theorem 1.10.** Let $f : X \to Y$ be a dominant meromorphic map of compact Kähler manifolds of the same dimension.

1) We have $f^*(SM^+(Y)) \subset SM^+(X)$ and $f_*(SM^+(X)) \subset SM^+(Y)$. Moreover, if $\mu \in SM^+(X)$ and $\nu \in SM^+(Y)$, then $f_*(\mu) = f^*(\nu) = \mu(\pm 1)$ and $f^*(\nu)(\pm 1) = \deg(f)\nu(\pm 1)$. Here $\deg(f)$ is the topological degree of $f$, that is the number of inverse images by $f$ of a generic point in $Y$. In particular, $||f_*(\mu)|| = ||\mu||$ and $||f^*(\nu)|| = \deg(f)||\nu||$.

2) Let $g : Y \to Z$ be another dominant rational meromorphic map of compact Kähler manifolds of the same dimension. Then for all $\mu \in SM^+(X)$ we have $g_*(f_*(\mu)) \geq (g \circ f)_*(\mu)$. If $f$ and $g$ are holomorphic, then equality happens.

3) Let $g : Y \to Z$ be another dominant rational meromorphic map of compact Kähler manifolds of the same dimension. Then for all $\nu \in SM^+(Z)$ we have $(g \circ f)^*(\nu) \leq f^*g^*(\nu)$. If moreover, $f$ and $g$ are holomorphic, and $g$ is bimeromorphic, then equality happens.

4) If $\mu_n \in SM^+(X)$ weakly converges to $\mu$, and $\nu$ is a cluster point of $f_*(\mu_n)$, then $\nu \leq f_*(\mu)$. If $f$ is holomorphic, then $\lim_{n \to \infty} f_*(\mu_n) = f_*(\mu)$.

5) Let $Y = X$.

i) If $\mu_0$ is a positive strong submeasure so that $\mu_0 \geq f_*(\mu_0)$, then the weak convergence limit $\mu = \lim_{n \to \infty}(f_*)^*(\mu_0)$ exists and has the following properties: $f_*(\mu) = \mu$, $\mu \leq \mu_0$ and $||\mu|| = ||\mu_0||$.

ii) Consequently, there is always a non-zero positive strong submeasure $\mu$ so that $f_*(\mu) = \mu$.

6) If $\mu$ is a positive measure without mass on $I(f)$, then $f_*(\mu)$ is the same as the usual one.

7) For any positive strong submeasure $\mu$, we have $f_*(\mu) = \sup_{\chi \in G(\mu)} f_*(\chi)$, where $G(\mu) = \{\chi : \chi$ is a measure and $\chi \leq \mu\}$.

In Example 2 in Section 2 we will show that strict inequality can happen in parts 2), 3) and 4) in general. It also shows that, in contrast to the case of a continuous map - see Section 4, part 7) does not hold in general if we replace $\mathcal{G}(\mu)$ by a smaller set (still satisfying $\mu = \sup_{\chi \in G} \chi$). In part 5 i), note that if $f$ is a continuous map on a compact metric space and $\mu_0$ is a measure so that $\mu_0 \geq f_*(\mu_0)$, then $\mu_0$ itself is $f$-invariant. Generalising Example 1, we will give an explicit expression in Theorem 2.2 for the pushforward $f_*$ and a corresponding collection $\mathcal{G}$ in part 2) of Theorem 1.1. It is clear that the definitions of pullback and pushforward of positive strong submeasures and Theorem 1.10 are valid in the case where $X$ and $Y$ are not smooth or Kähler. We also remark that several results in Theorem 1.10 (such as parts 1, 2, 3) can be extended easily to meromorphic correspondences.

Next we discuss an application to wedge intersection of positive closed currents. Monge-Ampère operators of positive closed (1,1) currents are a very classical and active area, both on compact and non-compact manifolds. While most of the works in the subject concerns Monge-Ampère operators with no mass on pluripolar sets, there are also several works with masses on pluripolar sets, for example [32] [33] [12] [13] [31] [32] [41] [4]. For intersection of currents of higher bi-degrees, there are the methods of using super-potentials and tangent currents [22] [23] [21] with applications to complex dynamics and geometry. In all of these works, the resulting (whenever well defined) is
always a **positive measure**. However, in any of these methods, there are cases where either it is not known how to define the intersection (for example if the Lelong numbers of the given currents are positive on a common set of positive dimension) or the cohomology class of the intersection is not the intersection of the corresponding cohomology classes. (This is the case of using non-pluripolar intersection in [6] [23] [8] or using residue currents in [2] [4] [3]. For example, we can see that for the case in Example 3 in Section 3 both these two approaches give the answer 0.) The latter property is desirable when discussing invariant measures of maps. We also remark that most of these approaches are **local**, that is the intersection can be defined on small open sets and then patched together, while our approach in the below is of a global nature.

In previous work [40] [39], using regularisation of currents [14] [24], we can define a wedge intersection of positive closed currents whenever a continuity property (more precisely the existence and independence of limits of the wedge intersection of regularisations of the given positive closed currents) is satisfied. The resulting (whenever well defined) need not be positive measures, but is only a signed measure. However, not every wedge intersection of positive closed currents can be defined by that manner. Here, we use an idea different from previous works, and show that the Monge-Ampere operator of positive closed \((1,1)\) currents can be defined for **all** positive closed \((1,1)\) currents - even if their intersection number in cohomology is **negative** - if instead of signed measures we allow strong submeasures. The main idea is that in the classical case, when a wedge intersection of positive closed \((1,1)\) currents are defined, then Bedford-Taylor’s monotone convergence is satisfied. In general, we do not have Bedford-Taylor’s monotone convergence, but we can consider the set \(\mathcal{G}\) of all cluster points (which are signed measures of the form \(\chi^+ - \chi^-\) where \(\chi^\pm\) are positive measures) obtained from all monotone approximations of the given currents with some control on the masses. The masses of all signed measures in \(\mathcal{G}\) may **not be bounded** (see Example 3 for more detail), but \(\chi^+(1) - \chi^-(1)\) is a constant (indeed is the intersection number of the corresponding cohomology classes). Let \(\mathcal{G}^*\) be the closure of \(\mathcal{G}\) with respect to the weak convergence of signed measures. If we restrict to only signed measures in \(\mathcal{G}^*\) whose negative part’s mass \(\chi^-\) is smallest then the corresponding masses are **bounded**. We can then use (1.5) to obtain a strong submeasure, the so-called least negative intersection. More details will be given in Section 3 The main result concerning this subject is the following.

**Theorem 1.11.** Let \(X\) be a compact Kähler manifold of dimension \(k\), \(T_1,\ldots,T_p\) positive closed \((1,1)\) currents on \(X\) and \(R\) a positive closed \((k-p,k-p)\) current on \(X\). Then there is a strong submeasure \(\Lambda(T_1,\ldots,T_p,R)\) with the following properties:

1) **Symmetry.** \(\Lambda(T_1,\ldots,T_p,R)\) is symmetric in \(T_1,\ldots,T_p\).

2) **Compatibility with cohomology.** The mass \(\Lambda(T_1,\ldots,T_p,R)(1)\) is the intersection number of cohomology classes \(\{T_1\} \cdot \{T_2\} \cdots \{T_p\} \cdot \{R\}\).

3) **Compatibility with classical wedge intersection.** If \(U \subset X\) is an open set on which the wedge intersection \(T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U\) is classically defined (that is, Bedford-Taylor’s monotone convergence is satisfied, see Section 3 for more detail), then

\[
\Lambda(T_1,\ldots,T_p,R)|_U = T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U.
\]
Proposition 1.12. 1) Let \( X = \mathbb{P}^k \) be a projective space. Then the least negative intersection \( \Lambda(T_1, \ldots, T_p, R) \) in Theorem 1.11 is always in \( \mathcal{SM}^+(X) \).

2) Let \( X = \mathbb{P}^2, \ D \subseteq \mathbb{P}^2 \) be a line, and \([D]\) the current of integration on \( D \). Then for all \( \varphi \in \mathcal{C}^0(X)\)

\[
\Lambda([D], [D])(\varphi) = \sup_{D} \varphi.
\]

We end this section mentioning some applications to dynamics of meromorphic maps. Let \( f : X \rightarrow X \) be a dominant meromorphic map of a compact Kähler surface. The study of dynamics of such maps is very active. It is now recognised that maps which are algebraically stable (those whose pullback on cohomology group is compatible with iterates, that is \((f^n)^* = (f^*)^n\) on \( H^{1,1}(X) \) for all \( n \geq 0 \)) have good dynamical properties. An important indication of the complexity of such maps is dynamical degrees defined as follows. Let \( \lambda_1(f) \) be the spectral radius of the linear map \( f^* : H^{1,1}(X) \rightarrow H^{1,1}(X) \) and let \( \lambda_2(f) \) be the spectral radius of the linear map \( f^* : H^{2,2}(X) \rightarrow H^{2,2}(X) \). There are two large interesting classes of such maps: those with large topological degree (\( \lambda_2(f) > \lambda_1(f) \)) and those with large first dynamical degree (\( \lambda_1(f) > \lambda_2(f) \)). The dynamics of the first class is shown in our paper [20] to be as nice as expected. For the second class, the most general result so far belongs to [15, 17, 16], who showed the existence of canonical Green (1, 1) currents \( T^+ \) and \( T^- \) for \( f_1 \) and who used potential theory to prove that the dynamics is nice (in particular, the wedge intersection \( T^+ \wedge T^- \) is well-defined as a positive measure) if the so-called finite energy conditions on the Green currents are satisfied. While these conditions are satisfied for many interesting subclasses, it is known however that in general they are false [9]. On the other hand, since it is known that \( T^+ \) has no mass on curves [18], it follows from Theorem 3.11 that we have the following result.

Theorem 1.13. Let \( f : X \rightarrow X \) be a dominant meromorphic map of a compact Kähler surface which is algebraically stable and has \( \lambda_1(f) > \lambda_2(f) \). Let \( T^+ \) and \( T^- \) be the canonical Green (1, 1) currents of \( f \). Then the least negative intersection \( \Lambda(T^+, T^-) \) is in \( \mathcal{SM}^+(X) \).

At the moment we do not know whether \( f_*(\Lambda(T^+, T^-)) = \Lambda(T^+, T^-) \). By part 5) of Theorem 1.10 if we can show that \( f_* (\Lambda(T^+, T^-)) \leq \Lambda(T^+, T^-) \), then we can construct a corresponding non-zero \( f \)-invariant positive strong submeasure. Proposition 1.12 may also be applied to dynamics in \( \mathbb{P}^k \). In [39], for pseudo-automorphisms in dimension 3, we showed that besides positive closed (1, 1) currents, any positive closed (2, 2) currents can also be pulled back and pushed forward. Moreover, we showed the existence of a positive closed (1, 1) current \( T^+ \) and a positive closed (2, 2) current \( T^- \) so that \( f^*(T^+) = \lambda_1(f)T^+ \)
and \( f_\ast(T^-) = \lambda_1(f)T^- \). Hence, similarly, we can consider the least negative intersection \( \Lambda(T^+, T^-) \) and check whether it is in \( SM^+(X) \) or is invariant by \( f_\ast \).

**Remark.** We note a parallel between the least negative intersection and the tangent currents in this situation. Under the same assumptions as in Theorem 1.13, it was shown in our paper [19] that the h-dimension (defined in [21]) between \( T^+ \) and \( T^- \) is 0, the best possible.

**Organisation of the paper.** In the next section we prove several basic properties of strong submeasures, including Theorems 1.3 and 1.4. We will also prove results around the pullback and pushforward of strong submeasures by meromorphic maps, including the upper-semicontinuous Proposition 1.6 Theorem 1.10 and the observation that pullback and pushforward in (1.12) and (1.13) can also be defined using any resolution of singularities of the graph \( \Gamma_f \). In Section 3 we define least negative intersection of positive closed (1, 1) currents and prove Theorem 1.11 and Proposition 1.12 together with some other results. In Section 4 we define entropy for positive strong submeasures invariant by a continuous maps, and prove Theorem 1.5.

**Acknowledgments.** Section 3 of the current paper, inspired by a talk by Elizabeth Wulcan at the Workshop on d-bar at NTNU Trondheim in October 2017, strengthens and incorporates our previous preprint [35]. The author would like to thank Lucas Kaufmann Sacchatto for an invitation for a visit to University of Gothenburg and the useful discussions there with him, Elizabeth Wulcan and David Witt Nyström. We are grateful to Mattias Jonsson for suggesting the use of Hahn-Banach’s theorem (Theorem 1.1) as well as several other comments, which helped to greatly simplify the proofs and presentation of an earlier version of this paper. We would like also to thank Viet-Anh Nguyen for his comments and encouragement on [35].

## 2. Pullback and push-forward of positive strong submeasures by meromorphic maps

This section proves some basic properties of strong submeasures and results around pullback and pushforward by meromorphic maps. We will prove Theorems 1.3, 1.4, Proposition 1.6, Theorem 1.10 and the observation that pullback and pushforward in (1.12) and (1.13) can also be defined using any resolution of singularities of the graph \( \Gamma_f \). Some other results will also be proven, including Theorem 2.2 in which we describe in more detail the pushforward map.

First, we prove basic properties of strong submeasures.

**Proof of Theorem 1.3**

1) Since \( X \) is a compact metric space, the space \( C^0(X) \), equipped with the \( L^\infty \) norm, is separable. Therefore, there is a countable set \( \varphi_1, \varphi_2, \ldots \) which is dense in \( C^0(X) \). Because \( \sup_n ||\mu_n|| = C < \infty \), for each \( j \) the sequence \( \{\mu_n(\varphi_j)\}_{n=1,2,\ldots} \) is bounded. Therefore, using the diagonal argument, we can find a subsequence \( \{\mu_n(k)\}_{k=1,2,\ldots} \) so that for all \( j \) the following limit exists:

\[
\lim_{k \to \infty} \mu_n(\varphi_j) =: \mu(\varphi_j).
\]

As observed in the introduction, the fact that \( \mu_n \) is sublinear and bounded implies that it is Lipschitz continuous: \( ||\mu_n(\varphi) - \mu_n(\psi)|| \leq ||\mu_n|| \times ||\varphi - \psi|| \leq C||\varphi - \psi|| \) for all \( n \) and all
\[ \varphi, \psi \in C^0(X). \] Then from the fact that \( \{ \varphi_j \}_{j=1,2,...} \) is dense in \( C^0(X) \), it follows that for all \( \varphi \in C^0(X) \), the following limit exists:

\[ \lim_{k \to \infty} \mu_n(\varphi) =: \mu(\varphi). \]

It is then easy to check that \( \mu \) is also a strong submeasure, and if \( \mu_n \) are all positive then so is \( \mu \).

2) For any \( \varphi \in C^0(X) \) we have \(-\|\varphi\|_{L^\infty} \leq \varphi \leq \|\varphi\|_{L^\infty}\). Therefore, since \( \mu \) is positive, we have \( \mu(-\|\varphi\|_{L^\infty}) \leq \mu(\varphi) \leq \mu(\|\varphi\|_{L^\infty}) \). By the sub-linearity of \( \mu \), we have \( \mu(-\|\varphi\|_{L^\infty}) = \|\varphi\|_{L^\infty} \mu(-1) \) and \( \mu(\|\varphi\|_{L^\infty}) = \|\varphi\|_{L^\infty} \mu(1) \).

Therefore, \( ||\mu|| = \|\varphi(1)|, |\varphi(1)|\}. \) The reverse inequality follows from the fact that \( ||-1||_{L^\infty} = ||1||_{L^\infty} = 1 \).

3) This is obvious. \( \square \)

**Proof of Theorem 1.4**

1) We first observe that for all \( \varphi \in C^0(X, \geq 0) \) then \( \mu(\varphi) \geq 0 \). In fact, otherwise, there would be \( \varphi_0 \in C^0(X, \geq 0) \) so that \( \mu(\varphi_0) < 0 \). Then by the definition of \( E(\mu) \) and sublinearity of \( \mu \) we have

\[ E(\mu)(0) \leq \inf_{n \in N} \mu(n\varphi_0) = \inf_{n \in N} n\mu(\varphi_0) = -\infty, \]

which is a contradiction with the assumption that \( E(\mu)(0) \) is finite.

Therefore, if \( \varphi \in C^0(X, \geq 0) \), we obtain

\[ 0 \leq \inf_{\psi \in C^0(X, \geq \varphi)} \mu(\psi) \leq \mu(\varphi). \]

Therefore, for these functions \( \varphi \) we have \( E(\mu)(\varphi) \) is a finite number. In particular, \( 0 \leq E(\mu)(1) \leq \mu(1) \).

Next, we observe that if \( \varphi_1, \varphi_2 \in BUS(X) \) such that either \( E(\mu)(\varphi_1) \) or \( E(\mu)(\varphi_2) \) is finite, then the proof of 3ii) is still valid and gives \( E(\mu)(\varphi_1 + \varphi_2) \leq E(\mu)(\varphi_1) + E(\mu)(\varphi_2) \). Apply this sub-linearity to \( \varphi_1 = \varphi_2 = 0 \) we obtain \( E(\mu)(0) = E(\mu)(0 + 0) \leq 2E(\mu)(0) \), which implies \( E(\mu)(0) \geq 0 \). On the other hand, \( E(\mu)(0) \leq \mu(0) = 0 \). Therefore, \( E(\mu)(0) = 0 \).

Since \( E(\mu)(1) \) is finite, applying the above sub-linearity for \( \varphi_1 = 1 \) and \( \varphi_2 = -1 \), we obtain \( 0 = E(0) = E(\mu)(1 + (-1)) \leq E(\mu)(1) + E(\mu)(-1) \). Therefore, \( E(\mu)(-1) \geq -E(\mu)(1) = -\mu(1) \).

Finally, applying the proof of part 3iii) we deduce that for all \( \varphi \in BUS(X) \), the number \( E(\mu)(\varphi) \) is finite.

2) Let \( \varphi \in C^0(X) \), and choose any \( \psi \in C^0(X, \geq \varphi) \). Since \( \mu \) is positive, we have by definition that \( \mu(\psi) \geq \mu(\varphi) \). Since \( \varphi \) is itself contained in \( C^0(X, \geq \varphi) \), it follows that

\[ E(\mu)(\varphi) = \inf_{\psi \in C^0(X, \geq \varphi)} \mu(\psi) = \mu(\varphi). \]

3) Let \( \varphi_1, \varphi_2 \in BUS(X) \).

i) If \( \psi_1 \in C^0(X, \geq \varphi_1) \) and \( \psi_2 \in C^0(X, \geq \varphi_2) \) then \( \psi_1 + \psi_2 \in C^0(X, \geq \varphi_1 + \varphi_2) \). Therefore, by sub-linearity of \( \mu \):

\[ E(\mu)(\varphi_1 + \varphi_2) = \inf_{\psi \in C^0(X, \geq \varphi_1 + \varphi_2)} \mu(\psi) \leq \mu(\psi_1 + \psi_2) \leq \mu(\psi_1) + \mu(\psi_2). \]
To this end, for each $x$ closed ball

We can choose

$\mu (\psi_1)$ is arbitrarily close to $E(\mu)(\varphi_1)$ and $\mu (\psi_2)$ is arbitrarily close to $E(\mu)(\varphi_2)$, and from that obtain the desired conclusion $E(\mu)(\varphi_1 + \varphi_2) \leq E(\mu)(\varphi_1) + E(\mu)(\varphi_2)$. The other part of i) is easy to check.

ii) If $\varphi_1 \geq \varphi_2$ then $C^0(X, \geq \varphi_1) \subset C^0(X, \geq \varphi_2)$. From this the conclusion follows.

iii) We observe that we can find $\psi \in C^0(X, \geq \varphi)$ so that $||\psi||_{L^\infty} = ||\varphi||_{L^\infty}$, simply by defining $\psi = \max\{\min\{\psi_0, ||\varphi||_{L^\infty}\}, -||\varphi||_{L^\infty}\}$ for any $\psi_0 \in C^0(X, \geq \varphi)$. Then

$$E(\mu)(\varphi) \leq \mu (\psi) \leq ||\mu|| \times ||\psi||_{L^\infty} = ||\mu|| \times ||\varphi||_{L^\infty}.$$  

By the positivity of $E(\mu)$ in ii), we have

$$E(\mu)(\varphi) \geq ||\varphi||_{L^\infty} E(\mu)(-1),$$

and hence $|E(\mu)(\varphi)| \leq \max\{|E(\mu)(-1), ||\mu||\} = ||\mu||$. In the last equality we used that 1) and positivity imply $-\mu(1) \leq E(\mu)(-1) \leq E(\mu)(0) = 0$.

4) By definition we have for closed subsets $A_1, A_2 \subset X$

$$\mu(A) = E(\mu)(1_{A_1 \cup A_2}) \leq E(\mu)(1_{A_1} + 1_{A_2}) \leq E(\mu)(1_{A_1}) + E(\mu)(1_{A_2}) = \mu(A_1) + \mu(A_2).$$

In the first inequality we used $1_{A_1 \cup A_2} \leq 1_{A_1} + 1_{A_2}$ and the positivity of $E(\mu)$. In the second inequality we used the sub-linearity of $E(\mu)$.

If $B_1, B_2$ are open subsets of $X$ and $A \subset B_1 \cup B_2$ is closed in $X$, then since $X$ is compact metric we can find closed subsets $A_1, A_2$ of $X$ so that $A_1 \subset B_1$, $A_2 \subset B_2$ and $A_1 \cup A_2 = A$. To this end, for each $x \in A$, we choose an open ball $B(x, r_x)$ (in the given metric on $X$) where $r_x > 0$ is chosen as follows: if $x \in B_1$ then the closed ball $\overline{B(x, r_x)}$ belongs to $B_1$, if $x \in B_2$ then the closed ball $\overline{B(x, r_x)}$ belongs to $B_2$, and if $x \in B_1 \cap B_2$ then the closed ball $\overline{B(x, r_x)}$ belongs to $B_1 \cap B_2$. Since $A$ is compact, there is a finite number of such balls covering $A$: $A \subset \bigcup_{i=1}^m B(x_i, r_i)$. Then the choice of $A_1 = A \cap (\bigcup_{x_i \in B_1} B(x_i, r_i))$ and $A_2 = A \cap (\bigcup_{x_i \in B_2} B(x_i, r_i))$ satisfies the requirement. Then from the above sub-linearity of $\mu$ for compact sets and the definition, we have also sub-linearity for open sets $\mu(B_1 \cup B_2) \leq \mu(B_1) + \mu(B_2)$.


Now we turn to properties of the pullback and pushforward maps.

We will first show that the upper-semicontinuous pushforwards $(\pi_{X, f})_*$ and $(\pi_{Y, f})_*$ on functions in $C^0(\Gamma_f)$ indeed produce upper-semicontinuous functions on $X$ and $Y$ as intended. Then we prove the observation that pullback and pushforward in [1.12] and [1.13] can also be defined using any resolution of singularities of the graph $\Gamma_f$. Finally, we prove Proposition 1.6 and Theorem 1.10 and give an example showing that the inequality in part 3) of Theorem 1.10 may be strict in general.

We recall the setting. We have a dominant meromorphic map $f : X \to Y$ with generically finite fibres, its graph $\Gamma_f \subset X \times Y$, and two induced projections $\pi_{X, f}, \pi_{Y, f} : \Gamma_f \to X, Y$. We let $U \subset X$ be a Zariski open dense set so that $\pi_{U, f} = \pi_{X, f}|_{\pi_{X, f}^{-1}(U)} : \pi_{X, f}^{-1}(U) \to U$ is a finite covering, and similarly $W \subset Y$ a Zariski open dense set so that $\pi_{W, f} = \pi_{Y, f}|_{\pi_{Y, f}^{-1}(W)} : \pi_{Y, f}^{-1}(W) \to U$ is a finite covering.

Let $\varphi \in C^0(\Gamma_f)$. Then obviously $(\pi_{U, f})_*(\varphi)$ is a continuous function on $U$, and hence upper-semicontinuous. Therefore, $(\pi_{X, f})_*(\varphi) = E((\pi_{U, f})_*(\varphi))$ is an upper-semicontinuous on $X$. Part 2) of Proposition 1.6 shows that $(\pi_{X, f})_*(\varphi)$ is independent of the choice of $U$. 


Now consider the upper-semicontinuous pushforward \((\pi_{Y,f})_*(\varphi)\). Recall that for \(y \in W\), we have

\[(\pi_{W,f})_*(\varphi)(y) = \sum_{z \in \pi_{W,f}^{-1}(y)} \varphi(z).\]

Even though the map \(\pi_{(W,f)}\) is not an isomorphism, it is a finite covering map of degree \(\deg(f)\). Moreover, if \(y_n \in Y \to y\), then when counted with multiplicities \(\pi_{W,f}^{-1}(y_n)\) converge to \(\pi_{W,f}^{-1}(y)\). Therefore we also have that \((\pi_{W,f})_*(\varphi)\) is continuous on \(W\), and again \((\pi_{Y,f})_*(\varphi) = E((\pi_{W,f})_*(\varphi))(y))\) is upper-semicontinuous. Again, part 2) of Proposition 1.6 shows that \((\pi_{Y,f})_*(\varphi)\) is independent of the choice of \(W\).

Now let \(\Gamma\) be any compact complex variety with surjective holomorphic maps \(\pi : \Gamma \to X\) and \(h : \Gamma \to Y\), so that \(\pi\) is bimeromorphic and \(f = h \circ \pi^{-1}\). Let \(U_1 \subset X\) be a Zariski open dense set so that \(\pi_{U_1} = \pi : \pi^{-1}(U_1) \to U_1\) is a finite covering, and \(W_1 \subset X\) be a Zariski open dense set so that \(h_{W_1} = h : h^{-1}(W_1) \to W_1\) is a finite covering. We have \((h_{W_1})_*\pi^*(\varphi)\) is continuous on \(W_1\) and \((\pi_{W,f},\pi_{X,f})^{*}\varphi)\) is continuous on \(W\), and they are the same on \(W_1 \cap W\) which is itself a Zariski open dense set. Using part 2) of Proposition 1.6 again, we have that \(E((h_{W_1})_*\pi^*(\varphi))\) and \(E((\pi_{W,f}),\pi_{X,f}^{*}\varphi)\) are the same.

The above observation shows that the value \(f_*\mu(\varphi)\) can also be defined when we use any such \(\Gamma\), for example any resolution of singularities of the graph \(\Gamma_f\). Similarly, we can also show that the value \(f^*(\mu)(\varphi)\) can also be defined when we use any such \(\Gamma\).

**Proof of Proposition 1.6** 1) Since \(g\) is bounded, it is clear that \(E(g)\) is also bounded. By definition, it is clear that \(E(g)|_U = g\).

Next we show that \(E(g)\) is upper-semicontinuous. If \(x \in U\), then there is a small ball \(B(x,r) \subset U\), and hence it can be seen that

\[\limsup_{y \in X \to x} E(g)(y) = \limsup_{y \in U \to x} E(g)(y) = \limsup_{y \in U \to x} g(y) \leq g(x),\]

since \(g\) is upper-semicontinuous on \(U\).

It remains to check that if \(x \in X \setminus U\), and \(x_n \in X \to x\) then

\[\limsup_{n \to \infty} E(g)(x_n) \leq E(g)(x).\]

To this end, choose \(y_n \in U\) so that \(d(y_n, x_n) \leq 1/n\) (here \(d(,,)\) is the metric on \(X\)) and \(|g(y_n) - E(g)(x_n)| \leq 1/n\) for all \(n\). Then \(y_n \to x\), and hence

\[\limsup_{n \to \infty} E(g)(x_n) = \limsup_{n \to \infty} g(y_n) \leq E(g)(x).\]

2) Using the assumption that \(g\) is continuous on \(U\) and \(U_1 \subset U\), we first check easily that \(E(g_1)|_U = g = E(g)|_U\), and then the equality \(E(g_1) = E(g)\) on the whole of \(X\).

3) Finally, note that if \(g_1\) and \(g_2\) are two upper-semicontinuous functions then \(g_1 + g_2\) is also upper-semicontinuous, and the inequality \(E(g_1 + g_2) \leq E(g_1) + E(g_2)\) follows from properties of limsup.

**Proof of Theorem 1.10.** 1) We will give the proof only for the pullback operator in (1.13), since the proof for the pushforward is similar. Let \(\nu \in SM^+(Y)\), we will show that \(f^*(\nu) \in SM^+(X)\). Let \(\varphi, \varphi_1, \varphi_2 \in C^0(X)\) and \(0 \leq \lambda \in \mathbb{R}\).
First, we show that $f^*(\nu)(\pm 1) = \deg(f)\nu(\pm 1)$. In fact, it follows from the definition that $(\pi_{W,f})_*\pi_{X,f}^*(\pm 1) = \pm \deg(f)$ on $W$, and hence

$$(\pi_Y,f)_*\pi_{X,f}^*(\pm 1) = E((\pi_{W,f})_*\pi_{X,f}^*(\pm 1)) = \pm \deg(f)$$
on $X$ as well. Then we have

$$f^*(\nu)(\pm 1) = \inf_{\psi \in C^0(Y,\pm \deg(f))} \nu(\psi) = \nu(\pm \deg(f)) = \deg(f)\nu(\pm 1).$$

In the first equality we used that $\nu$ is positive, and in the second equality we used that $\nu$ is sub-linear.

Second, we show the positivity of $f^*(\mu)$. If $\varphi_1 \geq \varphi_2$, then it can be seen from the definition that $(\pi_Y,f)_*\pi_{X,f}^*(\varphi_1) \geq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_2)$. Therefore $C^0(Y,\geq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_1)) \subset C^0(Y,\geq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_2))$, and hence it follows from definition that $f^*(\nu)(\varphi_1) \geq f^*(\nu)(\varphi_2)$.

Next, we show that $f^*(\nu)$ is bounded and moreover $\|f^*(\nu)\| = \deg(f)\|\nu\|$. By positivity of $f^*(\nu)$, we have $f^*(\nu)(-||\varphi||_{L^\infty}) \leq f^*(\nu) \leq f^*(\nu)(||\varphi||_{L^\infty})$. Hence $f^*(\nu)$ is bounded, and we conclude by Theorem 1.3.

Finally, we show the sub-linearity. The equality $f^*(\nu)(\lambda \varphi) = \lambda f^*(\nu)(\varphi)$, for $\lambda \geq 0$, follows from the fact that $(\pi_Y,f)_*\pi_{X,f}^*(\lambda \varphi) = \lambda (\pi_Y,f)_*\pi_{X,f}^*(\varphi)$ and properties of infimum. We now prove that $f^*(\nu)(\varphi_1 + \varphi_2) \leq f^*(\nu)(\varphi_1) + f^*(\nu)(\varphi_2)$. In fact, from Proposition 1.6 we have

$$(\pi_Y,f)_*\pi_{X,f}^*(\varphi_1 + \varphi_2) \leq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_1) + (\pi_Y,f)_*\pi_{X,f}^*(\varphi_2),$$

and hence if $\psi_1 \in C^0(Y,\geq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_1))$ and $\psi_2 \in C^0(Y,\geq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_2))$ then $\psi_1 + \psi_2 \in C^0(Y,\geq (\pi_Y,f)_*\pi_{X,f}^*(\varphi_1 + \varphi_2))$. Hence, by definition

$$f^*(\nu)(\varphi_1 + \varphi_2) \leq \nu(\psi_1 + \psi_2) \leq \nu(\psi_1) + \nu(\psi_2).$$

In the second inequality we used the sub-linearity of $\nu$. If we choose $\psi_1$ and $\psi_2$ so that $\nu(\psi_1)$ is close to $f^*(\nu)(\varphi_1)$ and $\nu(\psi_2)$ is close to $f^*(\nu)(\varphi_2)$, then we see that $f^*(\nu)(\varphi_1 + \varphi_2) \leq f^*(\nu)(\varphi_1) + f^*(\nu)(\varphi_2)$ as wanted.

2) By definition, we have

$$(g \circ f)_*(\mu)(\varphi) = \inf_{\psi \in C^0(X,\geq (g \circ f)^*(\varphi))} \mu(\psi).$$

Here we recall that $(g \circ f)^*(\varphi)$ is the upper-semicontinuous pullback of $\varphi$ by $g \circ f$.

On the other hand,

$$g_*f_*\mu(\varphi) = \inf_{\psi_1 \in C^0(Y,\geq g^*(\varphi))} f_*(\mu)(\psi_1) = \inf_{\psi_1 \in C^0(Y,\geq g^*(\varphi))} \inf_{\psi_2 \in C^0(X,\geq f^*(\psi_1))} \mu(\psi_2).$$

Then, since there is a dense Zariski open set $U \subset X$ so that all maps $f : U \to f(U)$ and $g : f(U) \to (g \circ f)(U)$ are all covering maps with finite fibres, it follows by the proof of part 2) of Proposition 1.6 that whenever $\psi_1 \in C^0(Y,\geq g^*(\varphi))$ and $\psi_2 \in C^0(Y,\geq g^*(\psi_1))$, then $\psi_2 \in C^0(X,\geq (g \circ f)^*(\varphi))$. From this, we get $g_*f_*\mu(\nu) = (g \circ f)_*(\mu)(\varphi)$.

When $f$ and $g$ are both holomorphic and $\varphi$ is continuous, then $g^*(\varphi)$, $f^*(g^*(\varphi))$ and $(g \circ f)^*(\varphi)$ are all continuous functions. Then using the positivity of $\mu$, we can easily see that

$$(g \circ f)_*(\mu)(\varphi) = \mu(f^*g^*(\varphi)) = f_*(\mu)(g^*(\varphi)) = g_*f_*(\mu)(\varphi).$$
3) In fact, the same proof as in 2) implies that \( f^*g^*(\mu) \geq (g \circ f)^*(\mu) \). Now we prove the converse, under the assumption that both \( f \) and \( g \) are holomorphic, and \( g \) is bimeromorphic.

By definition, for all \( \varphi \in C^0(X_2) \)

\[
f^*g^*(\mu)(\varphi) = \inf_{\psi \in C^0(X_1, g_\mu)} g^*(\mu)(\psi).
\]

Choose \( \psi_0 \in C^0(X, (g \circ f)_\mu(\varphi)) \) so that \( \mu(\psi_0) \) is close to \( (g \circ f)^*(\mu)(\varphi) \). Then, since on a Zariski dense open set of \( X_1 \) where \( f_\mu(\varphi) \) is continuous we have \( g^*((g \circ f)_\mu(\varphi)) = f_\mu(\varphi) \), and \( g^*(\psi_0) \) is continuous, it follows that \( g^*(\psi_0) \geq f_\mu(\varphi) \). Therefore, we have

\[
f^*g^*(\mu)(\varphi) \leq g^*(\mu)(g^*(\psi_0)).
\]

Note that since \( g \) is a bimeromorphic holomorphism and \( \psi_0 \) is continuous, the upper-semicontinuous pushforward \( g_\mu(g^*(\psi_0)) \) is equal to \( \psi_0 \). Hence, from definition and the positivity of \( \mu \) we have

\[
g^*(\mu)(g^*(\psi_0)) = \mu(\psi_0).
\]

Combining the above inequalities, we get that \( f^*g^*(\mu)(\varphi) \leq (g \circ f)^*(\mu)(\varphi) \). Hence, we get the desired conclusion \( f^*g^*(\mu)(\varphi) = (g \circ f)^*(\mu)(\varphi) \).

4) It is enough to show the following: for all \( \varphi \in C^0(Y) \) then

\[
\lim_{n \to \infty} \sup_{\psi \in C^0(X, f^*(\varphi))} \mu_n(\psi) \leq \inf_{\psi \in C^0(X, f^*(\varphi))} \mu(\psi).
\]

If we choose \( \psi_0 \in C^0(X, f^*(\varphi)) \) so that \( \mu(\psi_0) \) is close to \( f_\mu(\varphi) \), then from \( \mu_n(\psi_0) \to \mu(\psi_0) \) we obtain the conclusion.

If \( f \) is holomorphic, then \( f^*(\varphi) \) is itself a continuous function. Then it is easy to see that \( \lim_{n \to \infty} f_\mu(\varphi)(\psi) = f_\mu(\varphi) \).

5)

i) Since \( \mu_0 \geq f_\mu(\mu_0) \), it follows that \( (f_\mu)^n(\mu_0) \geq (f_\mu)^{n+1}(\mu_0) \) for all \( n \geq 0 \). Therefore, the limit \( \mu = \lim_{n \to \infty} (f_\mu)^n(\mu_0) \) exists. The claim that \( ||\mu|| = ||\mu_0|| \) follows from part 1). Since \( (f_\mu)^n(\mu_0) \) is a decreasing sequence, it follows that \( \mu \leq \mu_0 \). Next we show that \( f_\mu(\mu) = \mu \).

Define \( \mu_n = (f_\mu)^n(\mu_0) \), then both \( \{\mu_n\}_n \) and \( \{f_\mu(\mu_n)\}_n \) weakly converge to \( \mu \). Therefore, by part 4) we have \( f_\mu(\mu) \geq \mu \).

On the other hand, since \( \mu \leq (f_\mu)^n(\mu_0) \) for all \( n \), it follows that \( f_\mu(\mu) \leq (f_\mu)^n(\mu_0) = (f_\mu)^{n+1}(\mu_0) \) for all \( n \). Since \( (f_\mu)^n(\mu_0) \) weakly converges to \( \mu \), it follows that \( f_\mu(\mu) \leq \mu \).

Hence, we obtain \( f_\mu(\mu) = \mu \) as wanted.

ii) Choose \( \mu_0 = \sup_{x \in X} \delta_x \), where \( \delta_x \) is the Dirac measure at \( x \). Then by part 1), \( f_\mu(\mu_0) \) has the same mass as \( \mu_0 \), which is 1. Hence for all \( \varphi \in C^0(X) \) we obtain

\[
f_\mu(\mu_0)(\varphi) \leq ||\varphi||_{L^\infty} = \mu_0(\varphi).
\]

This means that \( \mu_0 \geq f_\mu(\mu_0) \). Hence, we can apply part i) to obtain an \( f \)-invariant non-zero positive strong submeasure \( \mu \).

More generally, if \( A \subset X \) is any closed or open set so that \( f(A) \subset A \), we can start with \( \mu_0 = \sup_{x \in A} \delta_x \) and produce as above a non-zero positive strong submeasure \( \mu \) satisfying: \( f_\mu(\mu) = \mu \) and \( \mu \leq \mu_0 \).

6) The upper-semicontinuous pullback \( f^*(\varphi) \) of a function \( \varphi \in C^0(Y) \) is continuous on the open set \( U = X \backslash f(I(f)) \). Therefore, by choosing a small open neighborhood \( U_1 \) of \( I(f) \) and a partition of unity subordinate to \( U \) and \( U_1 \), it is easy to find for any \( U_2 \subset U \) a
By Hironaka’s resolution of singularities, we can find a smooth compact Kähler manifold $Z$, together with two surjective holomorphic maps $\pi : Z \to X$ and $h : Z \to Y$, so that $\pi$ is a finite composition of blowups at smooth centres and $f = h \circ \pi^{-1}$. Then we have for any positive strong submeasure $\mu$ that $f_\ast(\mu) = h_\ast(\pi^\ast(\mu))$. It is easy to check that the conclusion holds for $h$, and hence to prove the result it suffices to prove that $\pi^\ast(\mu) = \sup_{\chi \in \mathcal{G}(\mu)} \pi^\ast(\chi)$. By part 3), we only need to prove the conclusion in the case $\pi : Z \to X$ is the blowup at a smooth centre.

Since $\pi^\ast(\mu) \geq \pi^\ast(\chi)$ for all $\chi \in \mathcal{G}(\mu)$, it follows that $\pi^\ast(\mu) \geq \sup_{\chi \in \mathcal{G}(\mu)} \pi^\ast(\chi)$. Now we will show the reverse inequality. To this end, it suffices to show that for any measure $\chi' \leq \pi^\ast(\mu)$, there is a measure $\chi \leq \mu$ so that $\chi' \leq \pi^\ast(\chi)$.

We first show that $\pi_\ast \pi^\ast(\mu) = \mu$. In fact, if $\varphi \in C^0(X)$ then $\pi^\ast(\varphi) \in C^0(Z)$ and $\varphi = \pi_\ast \pi^\ast(\varphi)$. Hence, by definition
\[
\pi_\ast \pi^\ast(\mu)(\varphi) = \pi^\ast(\mu)(\pi^\ast(\varphi)) = \mu(\pi_\ast \pi^\ast(\varphi)) = \mu(\varphi).
\]
Hence $\pi_\ast \pi^\ast(\mu) = \mu$ as wanted.

Now if $\chi'$ is any measure on $Z$, then $\chi = \pi_\ast (\chi')$ is a measure on $X$. If moreover, $\chi' \leq \pi^\ast(\mu)$, then
\[
\chi = \pi_\ast (\chi') \leq \pi_\ast \pi^\ast(\mu) = \mu.
\]
To conclude the proof, we will show that $\pi^\ast(\chi) \geq \chi'$. To this end, let $\varphi \in C^0(Z)$, we will show that $\pi^\ast(\chi)(\varphi) \geq \chi'(\varphi)$.

By definition, the value of the positive strong submeasure $\pi^\ast(\chi)$ at $\varphi$ is defined as: $\pi^\ast(\chi)(\varphi) = \inf_{\psi \in C^0(X, \geq \pi^\ast(\varphi))} \chi(\psi)$, and since $\chi = \pi_\ast (\chi')$ the RHS is equal to $\inf_{\psi \in C^0(X, \geq \pi^\ast(\varphi))} \chi'(\psi) \geq \chi'(\varphi)$. The latter follows from the fact that $\chi'$ is a positive strong submeasure and that for all $\psi \in C^0(X, \geq \pi^\ast(\varphi))$ we have $\pi^\ast(\psi) \geq \varphi$.

Example 2. Let $J : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the standard Cremona map given by $J[x_0 : x_1 : x_2] = [1/x_0 : 1/x_1 : 1/x_2]$. It is a birational map and is an involution: $J^2 = \text{id}$. Let $e_0 = [1 : 0 : 0]$, $e_1 = [0 : 1 : 0]$ and $e_2 = [0 : 0 : 1]$, and $\Sigma_i = \{x_i = 0\}$ ($i = 0, 1, 2$). Let $\pi : X \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at $e_0, e_1$ and $e_2$, and let $E_0, E_1$ and $E_2$ be the corresponding exceptional divisors. Let $h = f \circ \pi : X \to \mathbb{P}^2$, then $h$ is a holomorphic map. Moreover, $\pi^{-1}(e_0) = E_0$ and $h(E_0) = \Sigma_0$. More precisely, we have $h^{-1}(\Sigma_0 \setminus \{e_1, e_2\}) \subset E_0$. From this, we can compute, as in Example 1 in the introduction and proof of part 2) of Proposition 1.4 that for all $\varphi \in C^0(X)$ and for $\delta_{e_0}$ the Dirac measure at $e_0$
\[
J_\ast(\delta_{e_0})(\varphi) = \sup_{\Sigma_0} \varphi.
\]
It follows that $J_\ast(\delta_{e_0}) \geq \max\{\delta_{e_1}, \delta_{e_2}\}$, where $\delta_{e_1}$ is the Dirac measure at $e_1$ and $\delta_{e_2}$ is the Dirac measure at $e_2$. Therefore, by the positivity of $J_\ast$ we obtain:
\[
J_\ast J_\ast(\delta_{e_0})(\varphi) \geq J_\ast(\max\{\delta_{e_1}, \delta_{e_2}\})(\varphi) \geq \max\{J_\ast(\delta_{e_1}(\varphi)), J_\ast(\delta_{e_2}(\varphi))\} = \max\{\sup \varphi, \sup \varphi\}.
\]

On the other hand, $J \circ J$ is the identity map, and hence $(J \circ J)_\ast(\delta_{e_0}) = \delta_{e_0}$. Hence the inequality in part 2) of Theorem 1.10 is strict in this case. Since $J = J^{-1}$, this example also shows that the inequality in part 3) of Theorem 1.10 is strict in general.
This example also shows that the inequality in part 4) of Theorem 1.10 is strict in general. In fact, let \( \{p_n\} \subset X \setminus I(f) \) be a sequence converging to a point \( p = e_0 \) and \( \{J(p_n)\} \) converges to a point \( q \in \Sigma_0 \). Let \( \mu_n = \max\{\delta_{p_1}, \ldots, \delta_{p_n}\} \). It can be checked easily that \( \mu_n \) is an increasing sequence of positive strong submeasures, with \( J_*(\mu_n) = \max\{\delta_{J(p_1)}, \ldots, \delta_{J(p_n)}\} \) for all \( n \). Then the weak convergence limit \( \mu = \lim_{n \to \infty} \mu_n = \sup_n \delta_{p_n} \) exists. In particular, \( \mu \geq \delta_{e_0} \), and hence from the above calculation we find \( J_*(\mu) \geq \sup_{x \in \Sigma_0} \delta_x \). On the other hand, \( \nu = \lim_{n \to \infty} J_*(\mu_n) = \sup_n \delta_{J(p_n)} \). It is clear that if \( x \in \Sigma_0 \setminus \{q\} \), then \( \nu \) cannot be compared with \( \delta_x \). Therefore, we have the strict inequality \( J_*(\mu) > \nu \) in this case.

If we choose a sequence of points \( \{p_n\}_{n=1,2,\ldots} \subset X \setminus \{e_0, e_1, e_2\} \) converging to \( e_0 \) and such that \( q_n = J(p_n) \) converges to a point \( q_0 \in \Sigma_0 \), then it can be seen that for \( \mu = \sup_n \delta_{p_n} \) we have \( J_*(\mu) \geq \sup_n J_*(\delta_{p_n}) \). Hence part 7) of Theorem 1.10 does not hold in general if we replace \( G(\mu) \) by the smaller set \( G = \{\delta_{p_n}\}_n \).

We end this section describing in detail the pushforward map on positive strong submeasures. The next result about a good choice of \( \psi \in C^0(X, \geq \varphi) \) for some special bounded upper-semicontinuous functions will be needed for that purpose.

**Lemma 2.1.** Let \( X \) be a compact metric space, \( A \subset X \) a closed set and \( U = X \setminus A \). Let \( \varphi \) be a bounded upper-semicontinuous function on \( X \) so that \( \varphi \mid U \) is continuous on \( U \) and \( \gamma = \varphi \mid A \) is continuous on \( A \). For any \( U' \subset U \) an open set and \( \epsilon > 0 \), there is a function \( \psi \in C^0(X, \geq \varphi) \) so that:

i) \( \psi \mid U' = \chi \); ii) \( \sup_A |\psi| - \gamma| \leq \epsilon \); and iii) \( \sup_X |\psi| \leq \sup_X |\varphi| + \epsilon \).

**Proof.** Let \( \epsilon_1 > 0 \) be a small number to be determined later. Since \( \varphi \) is upper-semicontinuous, for each \( x \in A \), there is \( r_x > 0 \), which we choose so small that \( \overline{U'} \cap \overline{B(x, r_x)} = \emptyset \), so that

\[
\sup_{y \in U \cap \overline{B(x, r_x)}} \chi(y) \leq \gamma(x) + \epsilon_1.
\]

Since \( \gamma \) is continuous on \( A \), by shrinking \( r_x \) if necessary, we can assume that

\[
\sup_{x' \in A \cap \overline{B(x, r_x)}} |\gamma(x') - \gamma(x)| \leq \epsilon_1.
\]

Hence we obtain

\[
\sup_{y \in U \cap \overline{B(x, r_x)}} \chi(y) \leq \inf_{x' \in A \cap \overline{B(x, r_x)}} \gamma(x') + 2\epsilon_1.
\]

The function \( \gamma \mid _{A \cap \overline{B(x, r_x)}} \) can be extended to a continuous function \( \gamma_x \) on \( \overline{B(x, r_x)} \). We can assume, by shrinking \( r_x \) for example, that

\[
\sup_{x', x'' \in \overline{B(x, r_x)}} |\gamma_x(x') - \gamma_x(x'')| \leq \epsilon_1.
\]

Now, since \( A \) is compact, we can find a finite such balls, say \( B(x_1, r_1), \ldots, B(x_m, r_m) \), which cover \( A \). We choose \( U'' \) another open subset of \( X \) so that \( U' \subset U'' \subset U \) and so that \( U'', B(x_1, r_1), \ldots, B(x_m, r_m) \) is a finite open covering of \( X \). Let \( \tau, \tau_1, \ldots, \tau_m \) be a partition of unity subordinate to this open covering. Then the function

\[
\psi(x) = \tau(x)\gamma(x) + \sum_{i=1}^m \tau_i(x)[\gamma_{x_i}(x) + 4\epsilon_1],
\]
with $4\epsilon_1 < \epsilon$, satisfies the conclusion of the lemma. \qed

Given $f : X \to Y$ a dominant meromorphic map between compact Kähler manifolds of the same dimension and $\mu$ a positive strong submeasure on $X$. By part 7) of Theorem 1.10 and its proof, to describe $f_*(\mu)$ it suffices to describe $\pi^*(\mu)$ where $\pi : Z \to X$ is a blowup at a smooth centre and $\mu$ is a measure. The following result addresses this question.

**Theorem 2.2.** Let $\pi : Z \to X$ be the blowup of $X$ at an irreducible smooth subvariety $A \subset X$. Let $\varphi \in C^0(Z)$. Let $\mu$ be a positive measure on $X$, and decompose $\mu = \mu_1 + \mu_2$ where $\mu_1$ has no mass on $A$ and $\mu_2$ has support on $A$. Then $\pi^*(\mu_1)$ is a positive measure on $Z$, $\pi_*(\varphi)|_A$ is continuous, and we have

$$\pi^*(\mu)(\varphi) = \pi^*(\mu_1)(\varphi) + \mu_2(\pi_*(\varphi)|_A).$$

Moreover, an explicit choice of the collection $\mathcal{G}$ in part 2) of Theorem 1.1 for $\pi^*(\mu)$ will be explicitly described in the proof.

**Proof.** Let $B \subset Z$ be the exceptional divisor of the blowup. Then $\pi : B \to A$ is a smooth holomorphic fibration, whose fibres are isomorphic to $\mathbb{P}^{r-1}$ where $r =$ the codimension of $A$. As in Example 1, it can be computed that for $x \in A$ then $\pi_*(\varphi)(x) = \sup_{y \in \pi^{-1}(x)} \varphi$. Therefore, from what was said about the map $\pi : B \to A$, it follows that $\pi_*(\varphi)|_A$ is continuous. Then it is easy to see that the upper-semicontinuous function $\pi_*(\varphi)$ satisfies the conditions of Lemma 2.1. It is easy to check that $\pi^*(\mu_1)$ is a positive measure on $Z$. Hence, by the conclusion of Lemma 2.1, it is easy to see that

$$\pi^*(\mu)(\varphi) = \pi^*(\mu_1)(\varphi) + \mu_2(\pi_*(\varphi)|_A).$$

Now we provide an explicit choice of the collection $\mathcal{G}$ associated to $\pi^*(\mu)$ in part 2) of Theorem 1.1. From Equation (2.1), it suffices to provide such a $\mathcal{G}$ for $\mu_2$, since then the corresponding collection for $\mu$ will be $\pi^*\mu_1 + \mathcal{G}$. Therefore, in the remaining of the proof, we will assume that $\mu = \mu_2$ has support on $A$.

Define $\psi(\varphi) = \pi_*(\varphi)|_A$, it then follows that $\psi(\varphi) \in C^0(A)$, and

$$\pi^*(\mu)(\varphi) = \mu(\psi(\varphi)).$$

We now present an explicit collection $\mathcal{G}$ of positive measures on $X$ so that

$$\mu(\psi(\varphi)) = \sup_{\chi \in \mathcal{G}} \chi(\varphi).$$

To this end, let us consider for each finite open cover $\{U_i\}_{i \in I}$ of $A$, a partition of unity $\{\tau_i\}$ subordinate to the finite open cover $\{U_i\}$ of $A$, and local continuous sections $\gamma_i : U_i \to \pi^{-1}(U_i)$, the following assignment on $B$:

$$\chi(\{U_i\}, \{\tau_i\}, \gamma_i)(\varphi) = \mu(H(\varphi)),$$

where $H(\varphi) \in C^0(A)$ is the following function

$$H(\varphi)(x) = \sum_{i \in I} \tau_i(x)\varphi(\gamma_i(x)).$$

Since $H(\varphi)$ is linear and non-decreasing in $\varphi$, it is easy to see that $\chi$ is indeed a measure. Moreover, since $|H(x)| \leq \max_B |\varphi|$, it follows that $||\chi|| \leq ||\mu||$. 

We let $G$ be the collection of such positive measures. We now claim that for all $\varphi \in C^0(Z)$
\[ \mu(\psi(\varphi)) = \sup_{\chi \in G} \chi(\varphi). \]

We show first $\mu(\psi(\varphi)) \geq \chi(\varphi)$ for all $\chi \in G$. In fact, since $\gamma_i(x) \in \pi^{-1}(x)$ for all $x \in A$, it follows by definition that
\[ H(\varphi)(x) \leq \sup_{\pi^{-1}(x)} \varphi = \psi(\varphi)(x), \]
for all $x \in A$. Hence $\mu(\psi(\varphi)) \geq \chi(\varphi)$.

Now we show the converse. Let $\varphi$ be any continuous function on $Z$. Then for any $\epsilon > 0$ we can always find a finite open covering $\{U_i\}_{i \in I}$ of $X$, depending on $\varphi$ and $\epsilon$, so that for all $x \in U_i$ we have
\[ |\varphi(\gamma_i(x)) - \sup_{\pi^{-1}(x)} \varphi| \leq \epsilon. \]

It then follows that correspondingly $|H(x) - \psi(\varphi)(x)| \leq \epsilon$ for all $x \in A$. Therefore, for this choice of $\chi \in G$
\[ |\mu(\psi(\varphi)) - \mu(H)| \leq \epsilon, \]
and hence letting $\epsilon \to 0$ concludes the proof $\square$

3. Least-negative intersection of almost positive closed $(1,1)$ currents

In this section we define the least negative intersection of almost positive closed $(1,1)$ currents, more precisely of $p$ almost positive closed $(1,1)$ currents and a positive closed $(k-p,k-p)$ current on a compact Kähler manifold of dimension $k$. Then we prove Theorem 1.11, Proposition 1.12 and some other results. The content in this section is an improvement of our previous preprint [38].

For background about positive closed currents and intersection theory the readers may consult the book [13]. Let $(X,\omega)$ be a Kähler manifold. We recall that an upper-semicontinuous function $u : X \to [-\infty, \infty)$ is quasi-plurisubharmonic if there is a constant $A > 0$ so that $A\omega + \ddc u$ is a positive closed $(1,1)$ current. If $\Omega$ is any smooth closed $(1,1)$ form on $X$, the current $T = \Omega + \ddc u$ is almost positive. We then say that $u$ is a quasi-potential of $T$. For a quasi-psh function $u$, its Lelong number at $x \in X$ is defined as follows:
\[ \nu(u,x) = \liminf_{z \to x} \frac{u(z)}{\log |z-x|}. \]

We then define $\nu(T,x) = \nu(u,x)$.

From the seminal work of Bedford and Taylor [7, 6] on intersection of positive closed $(1,1)$ currents whose quasi-potentials are locally bounded, to later developments with important contributions from Demailly [14], Fornaess and Sibony [26], Kolodziej [30], Cegrell [11, 10], Guedj and Zeriahi [25], Boucksom, Eyssidieux, Guedj and Zeriahi [8] and many others, the following monotone convergence is the cornerstone.

**Definition 3.1. Monotone convergence.** Let $(X,\omega)$ be a Kähler manifold of dimension $k$, and $u_1, \ldots, u_p$ be quasi-plurisubharmonic functions and $R$ a positive closed $(k-p,k-p)$. If $\{u_i^{(n)}\}_n$ ($i = 1, 2, 3, \ldots$) is a sequence of smooth quasi-psh functions decreasing to $u_i$,
and there is a constant $B > 0$ such that $dd^c u_i^{(n)} \geq -B \omega$ for all $i$ and $n$, then for any $1 \leq i_1 < i_2 < \ldots < i_q \leq p$

$$\lim_{n \to \infty} dd^c u_{i_1}^{(n)} \wedge dd^c u_{i_2}^{(n)} \wedge \ldots \wedge dd^c u_{i_q}^{(n)} \wedge R$$

exists, and the limit does not depend on the choice of the regularisation $u_i^{(n)}$. We denote the limit by $dd^c u_{i_1} \wedge \ldots \wedge dd^c u_{i_q} \wedge R$.

Note that the existence of the regularisations $\{u_i^{(n)}\}_n$ in the above monotone convergence condition is classical in the local setting, and for a compact Kähler manifold the existence is proven by Demailly [14]. We recall that a quasi-psh function $u$ has analytic singularities if locally it can be written as

$$u = \gamma + c \log(|f_1|^2 + \ldots + |f_m|^2),$$

where $\gamma$ is a smooth function, $c > 0$ is a positive constant, and $f_1, \ldots, f_m$ are holomorphic functions. We will need the following theorem, see [14].

**Theorem 3.2.** Let $(X, \omega)$ be a compact Kähler manifold. Let $T = \Omega + dd^c u$ be a positive closed $(1, 1)$ current on $X$.

1) There are a sequence of smooth quasi-psh functions $u_n$ decreasing to $u$, and a constant $A > 0$ (depending on $T$) so that $dd^c u_n \geq -A \omega$ for all $n$.

2) There are a sequence of quasi-psh functions $u_n$ with analytic singularities decreasing to $u$, and a sequence of positive numbers $\epsilon_n$ decreasing to 0, so that the following 2 conditions are satisfied: i) $\Omega + dd^c u_n \geq -\epsilon_n \omega$ for all $n$; and ii) $\nu(u_n, x)$ increases to $\nu(u, x)$ uniformly with respect to $x \in X$.

**Definition 3.3. Classical wedge intersection.** Let $(X, \omega)$ be a Kähler manifold and $u_1, \ldots, u_p$ quasi-psh functions on $X$. Assume that for every open subset $U \subset X$, the restrictions $u_i|_U$ ($i = 1, \ldots, p$) and $R|_U$ on the Kähler manifold $(U, \omega|_U = \omega|_U)$ satisfy the monotone convergence property. Then for every smooth closed $(1, 1)$ forms $\Omega_1, \Omega_2, \ldots, \Omega_p$ on $X$ and any open set $U \subset X$, the wedge intersection $(\Omega_1 + dd^c u_1)|_U \wedge (\Omega_2 + dd^c u_2)|_U \wedge \ldots \wedge (\Omega_p + dd^c u_p)|_U \wedge R|_U$ is well-defined by a corresponding monotone convergence. We say that in this case the intersection $(\Omega_1 + dd^c u_1) \wedge (\Omega_2 + dd^c u_2) \wedge \ldots \wedge (\Omega_p + dd^c u_p) \wedge R$ is classically defined.

**Remark 3.4.** 1) By the seminal work [7] classical wedge intersection is satisfied when $u_1, \ldots, u_p$ are locally bounded. It is also true when the singularities of $u_1, \ldots, u_p$ are "small" in a certain sense, see [14] [26].

2) If the above monotone convergence holds (for smooth regularisations $u_i^{(n)}$), then it also holds when we use more generally bounded regularisations $u_i^{(n)}$. See Lemma 3.7 for more details.

3) In the approach using residue currents [2] [4] [3], for positive closed $(1, 1)$ currents of analytic singularities, it was shown that while the monotone convergence does not hold for all bounded regularisations, it does hold for special bounded regularisations such as $u_i^{(n)} = \max\{u_i, -n\}$. We note that this special class of regularisations is used in the non-pluripolar approach [6] [25] [8]. However, both these two approaches does not preserve cohomology classes, since they give the answer 0 to Example 3 below.
The following notion will be frequently used in the remaining of the paper.

**Definition 3.5. Good monotone approximation.** Let \((X, \omega)\) be a Kähler manifold, and \(u\) a quasi-psh function on \(X\). A sequence \(\{u_n\}_{n=1,2,...}\) of quasi-psh functions on \(X\) is a good monotone approximation of \(u\) if it satisfies the following two properties:

i) There exists \(A > 0\) so that \(dd^c u_n \geq -A\omega\) for all \(n\), and

ii) \(u_n\) decreases to \(u\).

**Lemma 3.6.** Let \(u_1, \ldots, u_p\) be quasi-psh functions on a Kähler manifold \((X, \omega)\), and \(R\) a positive closed \((k-p,k-p)\) current on \(X\). For each \(i\), let \(\{u_i^{(n)}\}\) be a good monotone approximation for \(u_i\). Assume that for each \(n\), the intersection \(dd^c u_1^{(n)} \wedge \ldots \wedge dd^c u_p^{(n)} \wedge R\) is classically defined. Then we can write \(dd^c u_1^{(n)} \wedge \ldots \wedge dd^c u_p^{(n)} \wedge R = \mu_n^+ - \mu_n^-\), where \(\mu_n^\pm\) are positive measures on \(X\) so that for any compact set \(K \subset X\):

\[
\sup_n ||\mu_n^\pm||_K < \infty.
\]

**Proof.** For simplicity, we prove only for the case \(p = 1\). The case where \(p > 1\) is similar.

By definition, there is \(A > 0\) so that \(A\omega + dd^c u_1^{(n)}\) is a positive closed current on \(X\) and \(\mu_p^+ = (A\omega + dd^c u_1^{(n)}) \wedge R\) is a positive measure. Hence \(dd^c u_1^{(n)} \wedge R = \mu_n^+ - \mu_n^-\) where \(\mu_n^- = A\omega \wedge R\) is also a positive measure on \(X\). Note that \(\mu_n^\pm\) is independent of \(n\), and hence for any compact \(K \subset X\): \(\sup_n ||\mu_n^\pm||_K < \infty\). The claim for \(||\mu_n^\pm||_K\) follows from the assumption that \((A\omega + dd^c u_1^{(n)}) \wedge R\) is classically defined. \(\square\)

Let \((X, \omega)\) be a compact Kähler manifold, and \(R\) a positive closed \((k-p,k-p)\) current on \(X\). We denote by \(\mathcal{E}(R)\) the set of all tuples \((u_1, \ldots, u_p)\) where \(u_i\)'s \((i = 1, \ldots, p)\) are quasi-psh functions so that the intersection \(dd^c u_1 \wedge \ldots \wedge dd^c u_p \wedge R\) is classically defined. Thus \(\mathcal{E}(R)\) is the largest class of tuples where Bedford-Taylor monotone convergence is still valid.

Assume that \((u_1, \ldots, u_p) \in \mathcal{E}(R)\). Then by definition, for any smooth closed \((1,1)\) forms \(\Omega_1, \ldots, \Omega_p\) on \(X\), any open set \(U \subset X\) and sequences of smooth functions \(\{u_i^{(n)}\}\) defined on \(U\) which are good monotone approximations of \(u_i|_U\), we have

\[
\lim_{n \to \infty} (\Omega_1|_U + dd^c u_1^{(n)}) \wedge (\Omega_2|_U + dd^c u_2^{(n)}) \wedge \ldots \wedge (\Omega_p|_U + dd^c u_p^{(n)}) \wedge R|_U = (\Omega_1|_U + dd^c u_1|_U) \wedge (\Omega_2|_U + dd^c u_2|_U) \wedge \ldots \wedge (\Omega_p|_U + dd^c u_p|_U) \wedge R|_U.
\]

If in the above, we choose more generally \((u_1^{(n)}, \ldots, u_p^{(n)})\) not smooth, but only in \(\mathcal{E}(R)\) for each \(n\), then each term in the LHS of the above equality is still defined and the same equality occurs. That is, using more general good monotone approximations does not expand the set \(\mathcal{E}(R)\).

**Lemma 3.7.** Let \((X, \omega)\) be a compact Kähler manifold and \(R\) a positive closed \((k-p,k-p)\) current on \(X\). Let \(u_1, \ldots, u_p\) be quasi-psh functions on \(X\) and \(\Omega_1, \ldots, \Omega_p\) smooth closed \((1,1)\) forms on \(X\). For each \(i\), let \(\{u_i^{(n)}\}_n\) be a good monotone approximation (not necessarily smooth) of \(u_i\). Assume that for each \(n\), the tuple \((u_1^{(n)}, \ldots, u_p^{(n)})\) is in \(\mathcal{E}(R)\), and
the limit (in the weak convergence of signed measures) \( \mu = \lim_{n \to \infty} (\Omega_1 + dd^c u_1^{(n)}) \land \ldots \land (\Omega_p + dd^c u_p^{(n)}) \land R \) exists. Then, for each \( i \) there is a smooth good monotone approximation \( \{v_i^{(n)}\}_n \) of \( u_i \) and so that

\[
\lim_{n \to \infty} (\Omega_1 + dd^c v_1^{(n)}) \land \ldots \land (\Omega_p + dd^c v_p^{(n)}) \land R = \mu.
\]

In particular, if \((u_1, \ldots, u_p) \in \mathcal{E}(R)\) then for all open set \( U \subset X \)

\[
\lim_{n \to \infty} (\Omega_1|_U + dd^c u_1^{(n)}|_U) \land \ldots \land (\Omega_p|_U + dd^c u_p^{(n)}|_U) \land R|_U = (\Omega_1|_U + dd^c u_1|_U) \land \ldots \land (\Omega_p|_U + dd^c u_p|_U) \land R|_U.
\]

Proof. For simplicity, we may assume that \( \Omega_1 = \ldots = \Omega_p = 0 \).

By Theorem 3.2 for each \( i \) and \( n \), there is a good monotone approximation \( \{\Phi_m(u_i^{(n)})\}_m \) of \( u_i^{(n)} \).

Since \( X \) is a compact metric space, the space \( C^0(X) \) is separable. Therefore, there is a dense countable set \( \mathcal{F} \subset C^0(X) \). We enumerate the elements in \( \mathcal{F} \) as \( \varphi_1, \varphi_2, \ldots \).

Since \( \mu = \lim_{n \to \infty} dd^c u_1^{(n)} \land \ldots \land dd^c u_p^{(n)} \land R \), for each \( l \) there is a number \( n_l \) so that for all \( \varphi \in \{\varphi_1, \ldots, \varphi_l\} \) and for all \( n \geq n_l \):

\[
(3.1) \quad |\mu(\varphi) - dd^c u_1^{(n)} \land \ldots \land dd^c u_p^{(n)} \land R(\varphi)| \leq 1/l.
\]

We can assume that \( n_1 < n_2 < n_3 \ldots \) Then for each \( i \), the sequence \( \{u_i^{(n)}\}_l \) is a good monotone approximation of \( u_i \). Therefore, we can assume that \(|u_i^{(n)} - u_i|_{L^1(X)} \leq 1/l^2\) for all \( l \) and \( i \).

Since for each \( l \) the tuple \((u_1^{(n_l)}, \ldots, u_p^{(n_l)}) \in \mathcal{E}(R)\), and \( \{\Phi_m(u_i^{(n_l)})\}_m \) is a smooth good monotone approximation of \( u_i^{(n_l)} \), there is \( m_l \) so that for all \( m \geq m_l \) we have for all \( \varphi \in \{\varphi_1, \ldots, \varphi_m\} \):

\[
(3.2) \quad |dd^c \Phi_m(u_1^{(n_l)}) \land \ldots \land dd^c \Phi_m(u_i^{(n_l)})(\varphi) - dd^c u_1^{(n_l)} \land \ldots \land dd^c u_p^{(n_l)} \land R(\varphi)| \leq 1/l.
\]

Now we are ready to choose the sequence \( \{v_i^{(l)}\} \). We first choose an intermediate sequence \( w_i^{(l)} \).

Choose \( w_i^{(1)} = \Phi_{m_1}(u_i^{(n_1)}) \). We can also arrange so that \(|w_i^{(1)} - u_i^{(n_1)}|_{L^1(X)} \leq 1 \).

Since \( w_i^{(1)} \) is smooth, \( w_i^{(1)} \geq u_i^{(n_1)} \geq u_i^{(n_2)} \), and \( \{\Phi_m(u_i^{(n_2)})\} \) is a good monotone approximation of \( u_i^{(n_2)} \), by Hartogs’ lemma (see [36]), for \( m \) large enough we have \( w_i^{(1)} + 1 \geq \Phi_m(u_i^{(n_2)}) \) and (3.2) is satisfied. We choose \( w_i^{(2)} = \Phi_m(u_i^{(n_2)}) \) for such \( m \).

Constructing inductively, we can find a sequence \( w_i^{(l)} = \Phi_m(u_i^{(n_l)}) \) for large enough \( m \), so that (3.2) is satisfied and: i) \(|w_i^{(l)} - u_i^{(n_l)}|_{L^1} \leq 1/l^2 \), and ii) \( w_i^{(l)} \leq w_i^{(l-1)} + 1/l^2 \).

Now we define

\[
v_i^{(l)} = w_i^{(l)} + \sum_{h \geq l+1} \frac{1}{h^2}.
\]
Then \( v_i^{(l)} - v_i^{(l-1)} = w_i^{(l)} - u_i^{(l)} - 1/l^2 \leq 0 \) by the choice of \( u_i^{(l)} \). Moreover, \( dd^c v_i^{(l)} = dd^c w_i^{(l)} \) and it is easy to check that \( v_i^{(l)} \) decreases to \( u_i \). Hence \( \{v_i^{(l)}\}_i \) is a smooth good monotone approximation of \( u_i \).

Since \( dd^c v_i^{(l)} \geq -4\omega \) for all \( i \) and \( l \), by Lemma 3.6 it follows that after working with a subsequence of \( \{v_i^{(l)}\} \) if needed, there is a signed measure \( \mu' \) so that

\[
\lim_{n \to \infty} dd^c v_1^{(n)} \wedge \ldots \wedge dd^c v_p^{(n)} \wedge R = \mu'.
\]

Then from (3.2) we have that \( \mu'(\varphi) = \mu(\varphi) \) for all \( \varphi \in \mathcal{F} \). Since \( \mathcal{F} \) is dense in \( C^0(X) \), we have the desired conclusion.

Now we turn our attention to a general tuple \((u_1, \ldots, u_p)\) of quasi-psh functions on a compact Kähler manifold \((X, \omega)\). By the very definition, if \((u_1, \ldots, u_p) \notin \mathcal{E}(R)\), then it is not guaranteed that we can assign a unique (signed) measure to the (not yet defined) intersection \((\Omega_1 + dd^c u_1) \wedge \ldots \wedge (\Omega_p + dd^c u_p) \wedge R\) using good monotone approximations. However, we can assign to the tuple \((\Omega_1 + dd^c u_1, \ldots, \Omega_p + dd^c u_p)\) a collection \(\mathcal{G}\) of signed measures in the following manner. Let \(\{u_i^{(n)}\}_n\) (for \(i = 1, \ldots, p\)) be a good monotone approximation of \(u_i\). Assume also that all \(u_i^{(n)}\) are smooth functions. Then by Lemma 3.6 we can write for each \(n\)

\[
(\Omega_1 + dd^c u_1^{(n)}) \wedge \ldots \wedge (\Omega_p + dd^c u_p^{(n)}) \wedge R = \mu_n^+ - \mu_n^-,
\]

where \(\sup_n \|\mu_n^+\| < \infty\). Therefore, there is a subsequence \(\{n(k)\}_k\) so that \(\mu_{n(k)}^\pm\) weakly converges to positive measures \(\mu^\pm\), and the signed measure \(\mu = \mu^+ - \mu^-\) is one element in the collection \(\mathcal{G}\). Below is the precise definition.

**Definition 3.8.** Let \((X, \omega)\) be a compact Kähler manifold, \(u_1, \ldots, u_p\) quasi-psh functions, \(\Omega_1, \ldots, \Omega_p\) smooth closed \((1,1)\) forms, and \(R\) a positive closed \((k - p, k - p)\) current. Let \(\mathcal{G}(u_1, \ldots, u_p, \Omega_1, \ldots, \Omega_p, R)\) be the set of signed measures \(\mu\) of the form:

\[
\mu = \lim_{n \to \infty} (\Omega_1 + dd^c u_1^{(n)}) \wedge \ldots \wedge (\Omega_p + dd^c u_p^{(n)}) \wedge R,
\]

where \(\{u_i^{(n)}\}_n\) (\(i = 1, \ldots, p\)) is a good monotone approximation of \(u_i\) by smooth quasi-psh functions.

Here are some properties of this collection.

**Proposition 3.9.** Let \((X, \omega)\) be a compact Kähler manifold. Let \(u_1, \ldots, u_p\) be quasi-psh functions, \(\Omega_1, \ldots, \Omega_p\) smooth closed \((1,1)\) forms and \(R\) a positive closed \((k - p, k - p)\) current.

1) Let \(u_1', \ldots, u_p'\) be quasi-psh functions and \(\Omega_1', \ldots, \Omega_p'\) be smooth closed \((1,1)\) forms so that \(\Omega_i + dd^c u_i = \Omega_i' + dd^c u_i'\) for all \(i = 1, \ldots, p\). Then

\[
\mathcal{G}(u_1, \ldots, u_p, \Omega_1, \ldots, \Omega_p, R) = \mathcal{G}(u_1', \ldots, u_p', \Omega_1', \ldots, \Omega_p', R).
\]

2) Let \(\{u_i^{(n)}\}_n\) (\(i = 1, \ldots, p\)) be a good monotone approximation of \(u_i\). Assume that for each \(n\), the tuple \((u_1^{(n)}, \ldots, u_p^{(n)})\) belongs to \(\mathcal{E}(R)\), and that there is a signed measure \(\mu\) so that

\[
\lim_{n \to \infty} (\Omega_1 + dd^c u_1^{(n)}) \wedge \ldots \wedge (\Omega_p + dd^c u_p^{(n)}) \wedge R = \mu.
\]
Then $\mu \in G(u_1, \ldots, u_p, \Omega_1, \ldots, \Omega_p, R)$.

Proof. 1) If we put $\phi_i = u_i - u_i'$, then $dd^c \phi_i = \Omega'_i - \Omega_i$, and hence $\phi_i$ is a smooth function. Whenever \{u_i^{(n)}\}_n is a good monotone approximation of $u_i$ by smooth quasi-psh functions, then \{u_i^{(n)}\}_n = \{u_i^{(n)} - \phi_i\}_n$ is a good monotone approximation of $u_i'$ by smooth quasi-psh functions and vice versa. Moreover, for all $n$ and $i$

$$\Omega_i + dd^c u_i^{(n)} = \Omega'_i + dd^c u'_i^{(n)}.$$ 

Hence for all $n$, we have

$$(\Omega_1 + dd^c u_1^{(n)}) \wedge \ldots \wedge (\Omega_p + dd^c u_p^{(n)}) \wedge R = (\Omega'_1 + dd^c u'_1^{(n)}) \wedge \ldots \wedge (\Omega'_p + dd^c u'_p^{(n)}) \wedge R.$$ 

From this, we obtain the conclusion.

2) The proof is similar to that of Lemma 3.7. \qed

From part 1) of Proposition 3.9, it follows that for any tuple of almost positive closed $(1,1)$ currents $(T_1, \ldots, T_p)$ and a positive closed $(k - p, k - p)$ current $R$, there is a well-defined collection $G(T_1, \ldots, T_p, R)$ of signed measures, which reduces to one element in the case when the intersection $T_1 \wedge \ldots \wedge T_p \wedge R$ is classically defined. Our main idea is to obtain one single object from this large collection, which is as close to positive measures as possible. An idea would be to define a strong submeasure using (1.5). There are some technical difficulties, since apparently this collection $G(T_1, \ldots, T_p, R)$ is a priori not bounded in mass, and hence the resulting strong submeasure might be $\infty$ identically. Also, this collection contains many signed measures of different natures, and so we need to have a criterion to say among two signed measures, which is closer to be a positive measure. We elaborate on this in the next paragraph, after an example illustrating these points.

**Example 3.** Let $X$ be a compact Kähler surface and $D \subset X$ an irreducible curve. Let $T_1 = T_2 = [D]$, where $[D]$ is the current of integration over $D$. Then, the collection $G(T_1, T_2)$ contains all signed measures of the form $\Omega \wedge [D]$, where $\Omega$ is an arbitrary smooth closed $(1,1)$ form cohomologous to $[D]$.

Proof. Let $\Omega$ be a smooth closed $(1,1)$ form cohomologous to $[D]$. Then we can write $T_1 = T_2 = \Omega + dd^c u$ for some quasi-psh function $u$. From part 2) of Proposition 3.9, we only need to construct two good monotone approximations \{u_1^{(n)}\}_n and \{u_2^{(n)}\}_n of $u$ by bounded quasi-psh functions so that

$$\lim_{n \to \infty} (\Omega + dd^c u_1^{(n)}) \wedge (\Omega + dd^c u_2^{(n)}) = \Omega \wedge [D].$$

We choose $u_1^{(n)} = \max\{u, -n\}$. Since $(\Omega + dd^c u_1^{(n)}) \wedge [D]$ is classically defined for all $n$, we can find a good monotone approximation \{u_2^{(n)}\}_n of $u$ by smooth quasi-psh functions so that

$$\lim_{n \to \infty} (\Omega + dd^c u_1^{(n)}) \wedge (\Omega + dd^c u_2^{(n)}) = \lim_{n \to \infty} (\Omega + dd^c u_1^{(n)}) \wedge [D].$$

Since $[D]$ is smooth outside of $D$, it follows that $u$ is smooth outside of $D$. Moreover, since $[D]$ has Lelong number 1 along $D$, it follows that $\lim_{x \to D} u(x) = -\infty$. Therefore, $u^{(n)}_1 = -n$ in an open neighborhood of $D$, and hence $dd^c u_1^{(n)} = 0$ in a neighborhood of $D$. Therefore, for all $n$ we have $(\Omega + dd^c u_1^{(n)}) \wedge [D] = \Omega \wedge [D]$, and this proves the conclusion. \qed
We first define an order on strong submeasures. If \( \mu \) is a (positive) measure, then we define as usual its norm: \( ||\mu|| = \mu(1) \). If \( \mu \) is a signed measure, we define its norm as follows:
\[
||\mu|| = \inf\{||\mu^+|| + ||\mu^-|| : \mu = \mu^+ - \mu^- \text{ where } \mu^\pm \text{ are positive measures}\}.
\]
We then define a negative-part norm of \( \mu \) as follows:
\[
||\mu||_{\text{neg}} = \inf\{||\mu^-|| : \mu = \mu^+ - \mu^- \text{ where } \mu^\pm \text{ are positive measures}\}.
\]
It is easy to check that \( ||\mu||_{\text{neg}} = 0 \) iff \( \mu \) is a positive measure. Hence, it is reasonable to say that a signed measure \( \mu \) with \( ||\mu|| \) fixed is closer to being a (positive) measure if \( ||\mu||_{\text{neg}} \) is smaller.

**Definition 3.10. Partial order on strong submeasures.** Let \( \mu_1 = \sup_{x_1 \in \mathcal{G}_1} x_1 \) and \( \mu_2 = \sup_{x_2 \in \mathcal{G}_2} x_2 \) be two strong submeasures, where \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be two collections of signed measures on \( X \) so that \( \sup_{x_1 \in \mathcal{G}_1} ||x_1||, \sup_{x_2 \in \mathcal{G}_2} ||x_2|| < \infty \). We write \( \mu_1 > \mu_2 \) if either
i) \( \inf_{x_1 \in \mathcal{G}_1} ||x_1||_{\text{neg}} < \inf_{x_2 \in \mathcal{G}_2} ||x_2||_{\text{neg}}, \)
or
ii) \( \inf_{x_1 \in \mathcal{G}_1} ||x_1||_{\text{neg}} = \inf_{x_2 \in \mathcal{G}_2} ||x_2||_{\text{neg}}, \) and \( \mu_1(\varphi) \geq \mu_2(\varphi) \) for all \( \varphi \in C^0(X) \).

We note that if \( \mu = \mu^+ - \mu^- \in \mathcal{G}(T_1, \ldots, T_p, R) \), where \( \mu^\pm \) are positive measures then \( \mu^+(1) - \mu^-(1) = \{T_1\} \cdots \{T_p\} \cdot \{R\} \), where the right hand side is the intersection in cohomology, is a constant. Therefore, from the above arguments, we see that the rough idea to proceed is to use the construction \( \mathcal{G}^* \) to signed measures in \( \mathcal{G}(T_1, \ldots, T_p, R) \), whose negative parts have mass of the smallest value. The only problem is that there may be no signed measure in \( \mathcal{G}(T_1, \ldots, T_p, R) \) whose negative part has the smallest value, and even so, there may be other signed measures whose negative parts have masses converging to the smallest value. Such limit signed measures should be considered as having the same roles as the signed measures in \( \mathcal{G}(T_1, T_2, \ldots, T_p, R) \) whose negative parts have masses of the smallest value. To cope with these issues, we proceed as follows.

Let the assumptions be as above. We define
\[
(3.3) \quad \kappa(T_1, \ldots, T_p, R) = \inf_{\mu \in \mathcal{G}^*(T_1, \ldots, T_p, R)} ||\mu||_{\text{neg}}.
\]

We also define \( \mathcal{G}^*(T_1, \ldots, T_p, R) \) to be the closure of \( \mathcal{G}(T_1, \ldots, T_p, R) \) with respect to the weak convergence of signed measures. More precisely,
\[
\mathcal{G}^*(T_1, \ldots, T_p, R) = \{\mu : \text{there exists } \mu_n \in \mathcal{G}(T_1, \ldots, T_p, R) \text{ weakly converging to } \mu\}.
\]
Finally, the least negative intersection \( \Lambda(T_1, \ldots, T_p, R) \) is the strong submeasure whose action on \( \varphi \in C^0(X) \) is given by
\[
\Lambda(T_1, \ldots, T_p, R)(\varphi) = \sup_{\mu \in \mathcal{G}^*(T_1, \ldots, T_p, R)} \mu(\varphi), \quad ||\mu||_{\text{neg}} = \kappa(T_1, \ldots, T_p, R).
\]

Now we give the proofs of Theorem 1.11, Proposition 1.12 and some other results.

**Proof of Theorem 1.11.** To show that \( \Lambda(T_1, \ldots, T_p, R) \) is a strong submeasure, by part 3) of Theorem 1.3 it suffices to show that
\[
\sup_{\mu \in \mathcal{G}^*(T_1, \ldots, T_p, R)}, ||\mu||_{\text{neg}} = \kappa(T_1, \ldots, T_p, R) ||\mu|| < \infty.
\]
But this is clear, since if we write \( \mu = \mu^+ - \mu^- \) with \( ||\mu^-|| = \mu^-(1) = \kappa(T_1, \ldots, T_p, R), \) then from
\[
||\mu^+|| = \mu^+(1) = \mu(1) + \mu^-(1) = \{T_1\} \cdot \{T_p\} \cdot \{R\} + \kappa(T_1, \ldots, T_p, R),
\]
we have
\[ ||\mu|| \leq ||\mu^+|| + ||\mu^-|| = \{T_1\} \cdot \{T_p\} \cdot \{R\} + 2\kappa(T_1, \ldots, T_p, R) \]
for all such \( \mu \).

1) The symmetry between \( T_1, \ldots, T_p \) is clear since in the definition of \( \Lambda(T_1, \ldots, T_p, R) \) we use all possible good monotone approximations by smooth quasi-psh functions of the quasi-potentials of all \( T_1, \ldots, T_p \).

2) If \( \mu \in G(T_1, \ldots, T_p, R) \), then it is a weak convergence limit
\[ \mu = \lim_{n \to \infty} (\Omega_1 + dd^c u_1^{(n)}) \land \ldots \land (\Omega_p + dd^c u_p^{(n)}) \land R. \]
Therefore, since \( (\Omega_1 + dd^c u_1^{(n)}) \land \ldots \land (\Omega_p + dd^c u_p^{(n)}) \land R(1) \) is the cohomology intersection \( \{\Omega_1 + dd^c u_1^{(n)}\} \cdots \{\Omega_p + dd^c u_p^{(n)}\} \cdot \{R\} \), which is the same as \( \{T_1\} \cdots \{T_p\} \cdot \{R\} \) for all \( n \), we obtain the conclusion.

3) This follows from the definition of classically defined wedge intersections.

4) This follows from the fact that for any \( \mu \in G(T_1, \ldots, T_p, R) \), a function \( \varphi \in C^0(X) \) and a constant \( B \), we have
\[ \mu(\varphi + B) = \mu(\varphi) + B\mu(1) = \mu(\varphi) + B\{T_1\} \cdots \{T_p\} \cdot \{R\}. \]

5) This is obvious. \( \square \)

Proof of Proposition 1.12. 1) It is known that if \( T_i = \Omega_i + dd^c u_i \) is a positive closed \((1, 1)\) current in \( \mathbb{P}^k \), then there is a good monotone approximation \( \{u_i^{(n)}\}_n \) of \( u_i \) by smooth quasi-psh functions so that \( \Omega_i + dd^c u_i^{(n)} \) is a positive form for all \( n \). Therefore, for all \( n \), the wedge intersection \( (\Omega_i + dd^c u_i^{(n)}) \land \ldots \land (\Omega_p + dd^c u_p^{(n)}) \land R \) is a positive measure. Any subsequence limit of this sequence will be also a positive measure, and hence the number \( \kappa(T_1, \ldots, T_p, R) \) is 0. From this and definition, we see that \( \Lambda(T_1, \ldots, T_p, R) \) is in \( SM^+(X) \).

2) Since the wedge intersection \([D] \land [D]\) is classically defined outside of \( D \), and there the resulting is 0, it follows that \( \Lambda([D], [D]) \) has support in \( D \). By 1), \( \Lambda([D], [D]) \) is in \( SM^+(X) \) and its mass is \( \Lambda([D], [D])(1) = \{D\} \cdot \{D\} = 1 \). Therefore, for all \( \varphi \in C^0(X) \), by the positivity of \( \Lambda([D], [D]) \) and the fact that it has support in \( D \) we have
\[ \Lambda([D], [D])(\varphi) \leq \Lambda([D], [D])(\sup D \varphi) = \sup_D \varphi. \]
Now we prove the reverse inequality. Fix a smooth closed \((1, 1)\) form \( \Omega \) having the same cohomology class as that of \([D]\). From Example 3, we see that \( (\Omega + dd^c v) \land [D] \) belongs to \( G([D], [D]) \) for all smooth function \( v \) on \( X \). For any point \( p \in D \), we choose an other line \( D_1 \subset X = \mathbb{P}^2 \) so that \( D \cap D_1 = \{p\} \). Then \([D_1]\) has the same cohomology class as \([D]\), and hence we can write \([D_1] = \Omega + dd^c u_1\) for some quasi-psh function \( u_1 \). By the argument in the proof of 1), we can find a good monotone approximation \( \{v_n\} \) of \( u_1 \) by smooth quasi-psh functions and so that \( \Omega + dd^c v_n \geq 0 \) for all \( n \). Note that the intersection \([D_1] \land [D]\) is classically defined, and hence we have
\[ \lim_{n \to \infty} (\Omega + dd^c v_n) \land [D] = \delta_p, \]
the Dirac measure at \( p \). Therefore, \( \delta_p \in \mathcal{G}(\{D\}, \{D\}) \) with \( \|\delta_p\|_{\text{neg}} = 0 = \kappa(\{D\}, \{D\}) \). Hence, by definition of \( \Lambda(\{D\}, \{D\}) \) we have

\[
\Lambda(\{D\}, \{D\})(\varphi) \geq \delta_p(\varphi) = \varphi(p),
\]

for all \( p \in D \). Thus

\[
\Lambda(\{D\}, \{D\})(\varphi) \geq \sup_{D} \varphi,
\]

as wanted. \( \square \)

The above arguments and results in Chapter 3 in [13] yield the following result.

**Theorem 3.11.** Let \( X \) be a compact Kähler manifold, \( T_1, \ldots, T_p \) positive closed \((1,1)\) currents and \( R \) a positive closed \((k-p,k-p)\) current.

1) If \( \kappa(T_1, \ldots, T_p, R) = 0 \), then the least negative intersection \( \Lambda(T_1, \ldots, T_p, R) \) is in \( SM^+(X) \).

2) Let \( E_i = \{ x \in X : \nu(T_i, x) > 0 \} \). Assume that for any \( 1 \leq i_1 < i_2 < \ldots < i_q \leq p \), the \( 2p-2q+1 \) Hausdorff dimension of \( E_{i_1} \cap \ldots \cap E_{i_q} \cap \sup(R) = 0 \). Then \( \kappa(T_1, \ldots, T_p, R) = 0 \).

3) Assume that \( R = [W] \) is the current of integration on a subvariety \( W \subset X \). Assume that for any \( 1 < i_1 < i_2 < \ldots < i_q \leq p \), the intersection \( E_{i_1} \cap \ldots \cap E_{i_q} \cap W \) do not contain any variety of dimension \( > p-q \). Then \( \kappa(T_1, \ldots, T_p, R) = 0 \).

4) Assume that \( \kappa(T_1, \ldots, T_p, R) = 0 \). Assume also that there is a Zariski open set \( U \subset X \) on which the intersection \( T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U \) is classically defined, and whose mass is exactly \( \{T_1\} \cdot \{T_p\} \cdot \{R\} \). Then \( \Lambda(T_1, \ldots, T_p, R) \) is the extension by \( 0 \) of the positive measure \( T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U \).

**Proof.** We prove for example 3) and 4).

3) By part 2) of Theorem 3.2, we have good monotone approximations \( \{T_i^{(n)}\} \) of \( T_i \) by currents with analytic singularities with the following properties: i) \( T_i^{(n)} + \epsilon_n \omega \geq 0 \) for all \( n \) where \( \epsilon_n \to 0 \), and ii) \( \{ x \in X : \nu(T_i^{(n)}, x) > 0 \} \subset E_i \) for all \( n \) and \( i \). Then from Chapter 3 in [13], it follows from i) that for each \( n \) the intersection \( T_i^{(n)} \wedge \ldots \wedge T_p^{(n)} \wedge R \) is classically defined, and it follows from ii) that any cluster point of \( T_i^{(n)} \wedge \ldots \wedge T_p^{(n)} \wedge R \) is a positive measure. By part 2) of Proposition 3.9, any such limit measure is in \( \mathcal{G}(T_1, \ldots, T_p, R) \). Thus \( \kappa(T_1, \ldots, T_p, R) = 0 \).

4) Let \( \mu \) be any signed measure in \( \mathcal{G}(T_1, \ldots, T_p, R) \). From the definition, we have that \( \mu|_U = T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_{|U} \). It follows that the same is true also for \( \mu \in \mathcal{G}^*(T_1, \ldots, T_p, R) \). Since \( \kappa(T_1, \ldots, T_p, R) = 0 \), it follows that

\[
\Lambda(T_1, \ldots, T_p, R)(\varphi) = \sup_{\mu \text{ positive measure } \in \mathcal{G}^*(T_1, \ldots, T_p, R)} \mu(\varphi).
\]

Hence the claim follows if we can show that \( \mathcal{G}^*(T_1, \ldots, T_p, R) \) has only one measure. In fact, let \( \mu \in \mathcal{G}^*(T_1, \ldots, T_p, R) \) be a positive measure. Then \( \mu \) has mass \( \{T_1\} \cdot \{T_p\} \cdot \{R\} \), which is the same as that of \( T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U \), and by the above arguments \( \mu|_U = T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U \). Therefore, \( \mu \) must be the extension by \( 0 \) of \( T_1|_U \wedge \ldots \wedge T_p|_U \wedge R|_U \). \( \square \)
4. Positive strong submeasures and the Variational Principle in continuous dynamics

In this subsection we define a notion of entropy for positive strong submeasures invariant by a continuous map \( f \) on a compact Hausdorff space \( X \), and show that the topological entropy of \( f \) is obtained at an invariant positive strong submeasure.

We first recall relevant definitions about entropy of an invariant measure and the Variational Principle, see [28, 27]. Let \( X \) be a compact Hausdorff space and \( f : X \to X \) a continuous map. Let \( \mu \) be a probability Borel-measure on \( X \) which is invariant by \( f \), that is \( f_\ast(\mu) = \mu \). We next define the entropy \( h_{\mu}(f) \) of \( f \) with respect to \( \mu \). We say that a finite collection of Borel sets \( \alpha \) is a \( \mu \)-partition if \( \mu(\bigcup_{A \in \alpha} A) = 1 \) and \( \mu(A \cap B) = 0 \) whenever \( A, B \in \alpha \) and \( A \neq B \). Given a \( \mu \)-partition \( \alpha \), we define

\[
H_{\mu}(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A).
\]

If \( \alpha \) and \( \beta \) are \( \mu \)-partitions, then \( \alpha \vee \beta := \{ A \cap B : A \in \alpha \text{ and } B \in \beta \} \) is also a \( \mu \)-partition. Similarly, for all \( n \) the collection \( V_{i=0}^{n-1} f^{-i}(\alpha) := \{ A_0 \cap A_1 \cap \ldots \cap A_{n-1} : A_0 \in \alpha, A_1 \in f^{-1}(\alpha), \ldots, A_{n-1} \in f^{-(n-1)}(\alpha) \} \) is also a \( \mu \)-partition. We define:

\[
(4.1) \quad h_{\mu}(f, \alpha) = \lim_{n \to \infty} \frac{1}{n} H_{\mu}(V_{i=0}^{n-1} f^{-i}(\alpha)),
\]

(the above limit always exists) and

\[
h_{\mu}(f) := \sup\{ h_{\mu}(\alpha) : \alpha \text{ runs all over } \mu \text{-partitions} \}.
\]

Recall (see [25]) that a Borel-measure \( \mu \) of finite mass is regular if for every Borel set \( E \):

i) \( \mu(E) = \inf\{ \mu(V) : V \text{ open, } E \subset V \} \), and ii) \( \mu(E) = \sup\{ \mu(K) : K \text{ compact, } K \subset E \} \).

Note that if \( X \) is a compact metric space, then any Borel measure of finite mass is regular.

The Variational Principle, an important result on dynamics of continuous maps, is as follows [28, 27].

**Theorem 4.1.** Let \( X \) be a compact Hausdorff space and \( f : X \to X \) a continuous map. Then

\[
h_{\text{top}}(f) = \sup\{ h_{\mu}(f) : \mu \text{ runs all over probability regular Borel measures } \mu \text{ invariant by } f \}.
\]

It is known, however, that the supremum in the theorem may not be attained by any such invariant measure \( \mu \), even if \( X \) is a compact metric space. In contrast, here, we show that with an appropriate definition, positive strong submeasures also fit naturally with the Variational Principle and provide the desired maximum.

First, we note that even though we defined in previous sections, strong submeasures only for compact metric spaces, this definition extends easily to the case of compact Hausdorff spaces. We still have the Hahn-Banach theorem on compact Hausdorff spaces, and hence we can define a strong submeasure \( \mu \) by one of the following two equivalent definitions: i) \( \mu \) is a bounded and sublinear operator on \( C^0(X) \), or ii) \( \mu = \sup_{\nu \in \mathcal{G}} \nu \), where \( \mathcal{G} \) is a non-empty collection of signed regular Borel-measures on \( X \) whose norms are uniformly bounded from above. Such a strong submeasure \( \mu \) is positive if moreover \( \mathcal{G} \) in ii) can be chosen to consist of only positive measures.
As in the previous sections, we can then define the pushforward of $\mu$ by a continuous map $f : X \to X$. We have the following property, which is stronger than 7) of Theorem [1.10. The proof of the result is simple, and similar to that of Theorem 1.10, and hence is omitted.

**Lemma 4.2.** Let $X$ be a compact Hausdorff space and $f : X \to X$ a continuous map. Let $\mu$ be a positive strong submeasure on $X$, and assume that $\mathcal{G}$ is any non-empty collection of positive measures on $X$ such that $\mu = \sup_{\nu \in \mathcal{G}} \nu$. Then

$$f_* \mu = \sup_{\nu \in \mathcal{G}} f_* \nu.$$

Now we define an appropriate notion of entropy for a positive strong submeasure $\mu$ which is invariant by $f$. A first try would be to adapt (4.1) to the more general case of positive strong submeasures, and then proceed as before. However, this is not appropriate, as the readers can readily check with the simplest case of identity maps on spaces with infinitely many points. In this case, there are many positive strong submeasures with mass 1 and invariant by $f$, whose entropy, according to the above definition, can be as large as possible and even can be infinity. On the other hand, recall that the topological entropy of the identity map is 0.

We instead proceed as follows. Given $\mu$ a positive strong submeasure invariant by $f$ and any regular measure $\nu \leq \mu$, there is a regular measure $\nu' \leq \mu$ so that $\nu'$ is invariant by $f$, $\nu' \leq \mu$ and $\nu'$ has the same mass as $\nu$. Such a measure $\nu'$ can be constructed as any cluster point of the sequence

$$\frac{1}{n} \sum_{i=0}^{n-1} f^i_*(\nu).$$

We define $\mathcal{G}(f, \mu) = \{\nu : \nu$ is a regular Borel-measure invariant by $f$, and $\nu \leq \mu\}$. Finally, we define the desired entropy as follows:

\begin{equation}
(4.2) \quad h_\mu(f) := \sup_{\nu \in \mathcal{G}(f, \mu)} h_\nu(f).
\end{equation}

We now can prove the Variational Principle for positive strong submeasures.

**Proof of Theorem 1.5.** We first show that if $\mu$ is any positive strong submeasure of mass 1 and invariant by $f$, then $h_\mu(f) \leq h_{top}(f)$. To this end, we need only to observe that for any $\nu \in \mathcal{G}(f, \mu)$, then the mass of $\nu$ is $\leq 1$, and hence $h_\nu(f) \leq h_{top}(f)$ by Theorem 4.1.

We finish the proof by showing that there is a positive strong submeasure $\mu$ of mass $\leq 1$ and invariant by $f$ so that $h_\mu(f) = h_{top}(f)$. To this end, we let $\mathcal{G} = \{\nu : \nu$ is a regular measure of mass $\leq 1$ and is invariant by $f\}$. This set is non-empty (it contains at least the measure zero.) We define $\mu = \sup_{\nu \in \mathcal{G}} \nu$. By Lemma 4.2 we have that

$$f_* \mu = \sup_{\nu \in \mathcal{G}} f_* \nu = \sup_{\nu} \nu = \mu.$$

Hence, $\mu$ is invariant by $f$. Moreover, from the definition of $\mathcal{G}$, it follows that $\mu$ has mass $\leq 1$. Finally, by (4.2) and Theorem 4.1 we have that $h_\mu(f) = h_{top}(f)$. □
**Final remark.** Besides fitting naturally with the Variational Principle, definition (4.2) is also compatible with the philosophy that a property of a positive strong submeasure $\mu$ should be related to the supremum of the same property of measures $\nu \leq \mu$. We see some instances of this philosophy above: the definition of $\mu$ itself is such a supremum, and Lemma 4.2.

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