An interesting modular structure associated to Landau levels

S. Twareque Ali
Department of Mathematics and Statistics, Concordia University, Montréal, Québec, CANADA H3G 1M8
E-mail: stali@mathstat.concordia.ca

Abstract. We analyze here, from a von Neumann algebraic point of view, the well known problem of a charged particle, moving on the infinite two-dimensional plane under the influence of a constant magnetic field, perpendicular to this plane. The energy levels of this system, known in the literature as Landau levels, are those of the harmonic oscillator, with each level being infinitely degenerate. There is an interesting algebraic property associated with this problem, that we discuss here. It arises from the fact that there are two mutually commuting von Neumann algebras associated to this system, which correspond to the two possible directions of the constant magnetic field. This commuting pair has a modular structure, in the sense of the Tomita-Takesaki theory, leading to the existence of KMS states for the system with physically interesting properties. An example of such a state is the Gibbs state related to the interaction part of the Landau Hamiltonian, which in this case is invariant under the modular automorphism group, generated by this Hamiltonian. Also associated to the problem is a family of orthogonal polynomials, the complex Hermite polynomials, which form a basis of the underlying Hilbert space.

1. Introduction

The quantum mechanical Hamiltonian of an electron moving in a constant electromagnetic field has eigenvalues identical to those of the harmonic oscillator, except that each level is infinitely degenerate. These are the well-known Landau levels [1], their degeneracy being explained by the fact that quantum mechanically, the centre of the (classical circular) orbit of the electron cannot be precisely determined. If the direction of the magnetic field is reversed, one obtains a second Hamiltonian, similar to the first, but commuting with it. Being oscillator type Hamiltonians, these can be written in terms of two sets of creation and annihilation operators, which mutually commute. In fact, the von Neumann algebras generated by this commuting pair of sets are mutual commutants, leading to the existence of a modular structure, in the sense of the Tomita-Takesaki theory [12]. The KMS state predicted by the theory is given essentially by the Gibbs state for a harmonic oscillator. The KMS state is then invariant under the action of the time evolution generated by the associated modular operator and obtainable in terms of the interaction Hamiltonian. The modular conjugation operator simply interchanges the two possible orientations of the magnetic field and mathematically, in an equivalent complex representation, amounts to a complex conjugation of the relevant quantities. We also look at some related complex analytic properties of the system, touching on complex Hermite polynomials which generate a basis for the underlying Hilbert space of the problem, in this
particular complex analytic representation.

Some earlier results along this direction were reported in [3, 4, 5], so that much of this paper
is of a review nature. The general mathematical theory of the modular structure associated to
the Hilbert space of Hilbert Schmidt operators has also been described in detail in [6] (Chapter
8) and [7] (Section II). Indeed, a structure similar to that reviewed here appears is a large
number of quantum optical systems (see [7] and references to the earlier literature therein). The
modular structure associated to the canonical commutation relations on $L^2(R^2)$ was already
observed in [8] and further elaborated in [5]. In [9] it was shown how a similar structure is
present in the set of discrete representations of a locally compact group. To emphasize again,
in this report we draw attention the existence of a modular structure in a rather well known
physical problem, viz that of the Landau levels. As just indicated, the interesting point here
is that the two commuting algebras correspond to the two possible directions of the magnetic
field, giving them a specific physical interpretation. Furthermore, the modular automorphism
group is mediated by the interaction Hamiltonian.

2. The physical problem

Consider a single electron of unit charge, placed in the $xy$-plane and subjected to a constant
magnetic field, pointing along the positive $z$-direction. Choosing the magnetic vector potential
to be $\vec{A}^\uparrow := \vec{A} = \frac{1}{2}(-y, x, 0)$ (so that the magnetic field, $\vec{B} = \nabla \times \vec{A}^\uparrow = (0, 0, 1)$), the classical
Hamiltonian of the system, in some rationalized units, can be written as

$$ H_{\text{elec}} = \frac{1}{2} \left( p_x + \frac{y}{2} \right)^2 + \frac{1}{2} \left( p_y - \frac{x}{2} \right)^2 ,$$

(1)

On the Hilbert space $\tilde{\mathcal{H}} = L^2(R^2, dx dy)$ of our problem, we make the replacements

$$ p_x + \frac{y}{2} \longrightarrow Q_- = -i \frac{\partial}{\partial x} + \frac{y}{2} , \quad p_y - \frac{x}{2} \longrightarrow P_- = -i \frac{\partial}{\partial y} - \frac{x}{2} .$$

(2)

Then, $[Q_-, P_-] = iI_{\tilde{\mathcal{H}}}$ and the quantum Hamiltonian corresponding to $H_{\text{elec}}$ becomes

$$ H^\uparrow = \frac{1}{2} \left( P_-^2 + Q_-^2 \right) .$$

(3)

The eigenvalues of this Hamiltonian, the so-called *Landau levels*, are $E_\ell = (\ell + \frac{1}{2})$, $\ell = 0, 1, 2, \ldots, \infty$, each level being infinitely degenerate, and we will denote the corresponding
normalized eigenvectors by $\Psi_{n\ell}$, with $\ell = 0, 1, 2, \ldots, \infty$, indexing the energy level and $n = 0, 1, 2, \ldots, \infty$, the degeneracy at each level.

The degeneracy of the eigenvalues can be understood in the following way. If the magnetic
field were aligned along the negative $z$-axis (with $\vec{A}^\downarrow = \frac{1}{2}(y, -x, 0)$ and $\vec{B} = \nabla \times \vec{A}^\downarrow = (0, 0, -1)$),
the corresponding quantum Hamiltonian would be

$$ H^\downarrow = \frac{1}{2} \left( P_+^2 + Q_+^2 \right) .$$

(4)

with

$$ Q_+ = -i \frac{\partial}{\partial y} + \frac{x}{2} , \quad P_+ = -i \frac{\partial}{\partial x} - \frac{y}{2} ,$$

(5)

and $[Q_+, P_+] = iI_{\tilde{\mathcal{H}}}$. The two sets of operators $\{Q_\pm, P_\pm\}$, mutually commute:

$$ [Q_+, Q_-] = [P_+, Q_-] = [Q_+, P_-] = [P_+, P_-] = 0 .$$

(6)
Thus, \([H^\dagger, H^\ddagger]\) = 0 and the eigenvectors \(\Psi_{n\ell}\) of \(H^\dagger\) can be chosen so that they are also the eigenvectors of \(H^\ddagger\) in the manner

\[
H^\dagger \Psi_{n\ell} = \left( n + \frac{1}{2} \right) \Psi_{n\ell}, \quad H^\ddagger \Psi_{n\ell} = \left( \ell + \frac{1}{2} \right) \Psi_{n\ell},
\]

(7)

so that \(H^\dagger\) lifts the degeneracy of \(H^\dagger\) and vice versa. It ought to be pointed out, however, that this lifting of the degeneracy is not just a consequence of the commutativity of the two Hamiltonians \(H^\dagger, H^\ddagger\), but more so of the existence of the two commuting algebras generated by the \(\{Q_\pm, P_\mp\}\), or equivalently, by the \(\{A_\pm, A_\pm^*\}\), in (9) below.

We also see that the quantized Hamiltonian may be split into a free part \(H_0\) and an interaction or angular momentum part, \(H_{\text{int}}^\ddagger\:

\[
\left\{
\begin{array}{l}
H^\dagger = H_0 + H_{\text{int}}^\dagger, \\
H_0 = H_{0,x} + H_{0,y} = \frac{1}{2} \left( \hat{p}_x^2 + \frac{z^2}{4} \right) + \frac{1}{2} \left( \hat{p}_y^2 + \frac{\tilde{y}^2}{4} \right), \\
H_{\text{int}} = -\frac{1}{2} (\hat{x} \hat{p}_y - \hat{y}\hat{p}_x) = -\hat{I}_z.
\end{array}
\right.
\]

(8)

with the usual definitions of \(\hat{x}, \hat{p}_x\), etc. Of course, \([\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = i\hat{I}_z\), while all the other commutators are zero.

Let us next define

\[
A_+ = \frac{1}{\sqrt{2}} (Q_+ + iP_+) \quad \quad A_+^* = \frac{1}{\sqrt{2}} (Q_+ - iP_+) \\
A_- = \frac{1}{\sqrt{2}} (iQ_- - P_-) \quad \quad A_-^* = \frac{1}{\sqrt{2}} (-iQ_- - P_-)
\]

(9)

These satisfy the commutation relations,

\[
[A_\pm, A_\pm^*] = 1,
\]

(10)

with all other commutators being zero. In terms of these, we may write the two Hamiltonians as (see (3) and (4)),

\[
H^\dagger = N_- + \frac{1}{2} \hat{I}_z, \quad H^\ddagger = N_+ + \frac{1}{2} \hat{I}_z, \quad \text{with} \quad N_\pm = A_\pm^* A_\pm.
\]

(11)

Furthermore,

\[
H_0 = \frac{1}{2} (N_+ + N_- + 1) \quad \text{and} \quad H_{\text{int}}^\dagger = -\frac{1}{2} (N_+ - N_-), \quad H_{\text{int}}^\ddagger = \frac{1}{2} (N_+ - N_-).
\]

(12)

The joint eigenstates of \(H^\dagger\) and \(H^\ddagger\) are now easily written down. Let \(\Psi_{00}\) be such that \(A_- \Psi_{00} = A_+ \Psi_{00} = 0\). Then,

\[
\Psi_{n\ell} := \frac{1}{\sqrt{n!\ell!}} (A_+^*)^n (A_-^*)^\ell \Psi_{00},
\]

(13)

where \(n, \ell = 0, 1, 2, \ldots\) and we have, \(N_+ \Psi_{n\ell} = n \Psi_{n\ell}\), \(N_- \Psi_{n\ell} = \ell \Psi_{n\ell}\), \(H_0 \Psi_{n\ell} = \frac{1}{2} (n+\ell+1) \Psi_{n\ell}\) and \(H_{\text{int}}^\dagger \Psi_{n\ell} = \frac{1}{2} (n-\ell) \Psi_{n\ell}\). A simple argument then shows that the vectors \(\Psi_{n\ell}\) span the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^2, dx \, dy)\).
We now change over to a complex representation, introducing the complex variable $z = \frac{1}{\sqrt{2}} (x - iy)$ and its associated derivative $\partial_z = \frac{1}{\sqrt{2}} (\partial_x + i \partial_y)$. In terms of these

$$A_- = \frac{1}{2} \overline{z} + \partial_z, \quad A_+ = \frac{1}{2} z + \partial_{\overline{z}} \quad \text{and} \quad A_-^* = \frac{1}{2} \overline{z} - \partial_\overline{z}, \quad A_+^* = \frac{1}{2} z - \partial_z , \quad (14)$$

In this representation, the ground state $\Psi_{00}(\overline{z}, z)$ is the solution of the equations $A_+ \Psi_{00}(\overline{z}, z) = A_- \Psi_{00}(\overline{z}, z) = 0$, so that, $\Psi_{00}(\overline{z}, z) = \sqrt{\frac{1}{2\pi}} e^{-\frac{1}{2}|z|^2}$. Thus, using (13) we get

$$\Psi_{n\ell}(\overline{z}, z) = \frac{1}{\sqrt{n! \ell!}} \left( \frac{1}{2} \overline{z} - \partial_z \right)^n \left( \frac{1}{2} z - \partial_\overline{z} \right)^\ell \Psi_{00}(z, \overline{z}) . \quad (15)$$

When using this representation, we shall write our Hilbert space as $\tilde{L}^2(\mathbb{C}, \frac{d\overline{z} \wedge dz}{i})$ and then it is useful to make the further unitary transformation

$$\mathcal{V} : L^2(\mathbb{C}, \frac{d\overline{z} \wedge dz}{i}) \rightarrow L^2(\mathbb{C}, d\nu(\overline{z}, z)) \quad \text{where} \quad d\nu(\overline{z}, z) = \frac{e^{-|z|^2}}{2\pi} \frac{d\overline{z} \wedge dz}{i} , \quad (16)$$

to the more convenient Hilbert space $L^2(\mathbb{C}, d\nu(\overline{z}, z))$, using the mapping

$$(\mathcal{V} \Psi)(\overline{z}, z) = \sqrt{2\pi} e^{\frac{|z|^2}{2}} \Psi(\overline{z}, z) , \quad (17)$$

and to rewrite all the operators in question on this new space. Note that this space contains the two subspaces $\mathcal{H}_{\text{hol}}$ and $\mathcal{H}_{\text{a-hol}}$, of holomorphic and antiholomorphic functions, respectively. Both these subspaces contain the constant unit vector, $H_{00}(\overline{z}, z) = 1$, $\forall (\overline{z}, z)$. Apart from this one vector, all other vectors in the complementary subspaces of $\mathcal{H}_{\text{hol}}$ and $\mathcal{H}_{\text{a-hol}}$ are mutually orthogonal. Then,

$$A_- := \mathcal{V} A_- \mathcal{V}^{-1} = \partial_z , \quad A_+ := \mathcal{V} A_+ \mathcal{V}^{-1} = \partial_{\overline{z}} ,$$

$$A_-^* := \mathcal{V} A_-^* \mathcal{V}^{-1} = z - \partial_{\overline{z}} , \quad A_+^* := \mathcal{V} A_+^* \mathcal{V}^{-1} = \overline{z} - \partial_z , \quad (18)$$

and

$$\mathcal{N}_+ := \mathcal{V} \mathcal{N}_+ \mathcal{V}^{-1} = A_+ A_+ = -\partial_z \partial_{\overline{z}} + \overline{z} \partial_z$$

$$\mathcal{N}_- := \mathcal{V} \mathcal{N}_- \mathcal{V}^{-1} = A_- A_- = -\partial_\overline{z} \partial_z + z \partial_\overline{z} . \quad (19)$$

Writing $H_{n\ell} = \mathcal{V} H_{n\ell}$, for the transformed basis vectors (15), we have,

$$H_{n\ell}(\overline{z}, z) = \frac{1}{\sqrt{n! \ell!}} (\overline{z} - \partial_z)^n (z - \partial_\overline{z})^\ell H_{00}(\overline{z}, z) = \frac{1}{\sqrt{n! \ell!}} \left( A_+^* \right)^n \left( A_+^* \right)^\ell H_{00}(\overline{z}, z) ,$$

$$= \frac{1}{\sqrt{n! \ell!}} (\overline{z} - \partial_z)^n (z')^\ell = \frac{1}{\sqrt{n! \ell!}} (z - \partial_\overline{z})^\ell (\overline{z})^n . \quad (20)$$

Also,

$$H_{n0}(\overline{z}, z) = \frac{\overline{z}^n}{\sqrt{n!}} \quad \text{and} \quad H_{0\ell}(\overline{z}, z) = \frac{z^\ell}{\sqrt{\ell!}} , \quad (21)$$

so that $\mathcal{H}_{\text{a-hol}}$ is spanned by the vectors $H_{n0}$, $n = 0, 1, 2, \ldots$, and the space $\mathcal{H}_{\text{hol}}$ by the vectors $H_{0\ell}$, $\ell = 0, 1, 2, \ldots$. 

4
The vectors $H_{n\ell}$ are joint eigenstates of the number operators:

\[ \hat{N}_+ H_{n\ell} = \ell H_{n\ell}, \quad \hat{N}_- H_{n\ell} = n H_{n\ell}. \]  

(22)

Moreover, writing

\[ H_\uparrow = \mathcal{V} H_{\uparrow} \mathcal{V}^{-1}, \quad H_\downarrow = \mathcal{V} H_{\downarrow} \mathcal{V}^{-1}, \] 
\[ H_0 = \mathcal{V} H_0 \mathcal{V}^{-1}, \quad H^{\uparrow, \downarrow}_{\text{int}} = \mathcal{V} H^{\uparrow, \downarrow}_{\text{int}} \mathcal{V}^{-1}, \]  

(23)

for the two Hamiltonians, we clearly have,

\[ H_\uparrow H_{n\ell} = \left( \ell + \frac{1}{2} \right) H_{n\ell}, \quad H_\downarrow H_{n\ell} = \left( n + \frac{1}{2} \right) H_{n\ell}. \]  

(24)

The functions $h_{n,k}(\zeta, z) = \sqrt{n!k!} H_{nk}(\zeta, z)$ are just the complex Hermite polynomials \[10, 11\], also obtainable as:

\[ h_{n,k}(\zeta, z) = (-1)^{n+k} \zeta^{|z|^2} \partial_n x \partial_k z e^{-|z|^2}. \]  

(25)

Explicitly, the $h_{n,k}$ are given by

\[ h_{n,k}(\zeta, z) = n! k! \sum_{j=0}^{\min(n,k)} (-1)^j \frac{(\zeta)^{n-j}}{j! (n-j)!} \frac{z^{k-j}}{(k-j)!}, \]  

(26)

where $n \leq k$ denotes the smaller of the two numbers $n$ and $k$. In particular,

\[ h_{0,k}(\zeta, z) = z^k \quad \text{and} \quad h_{n,0}(\zeta, z) = \zeta^n. \]  

(27)

One also has the useful series expansion,

\[ \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{v^n}{n!} \frac{v^k}{k!} H_{nk}(\zeta, z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{v^n}{n!} \frac{v^k}{k!} H_{nk}(\zeta, z) = e^{\zeta z + \zeta - \pi v}. \]  

(28)

Furthermore,

\[ h_{n,k}(\zeta, z) = h_{k,n}(z, \zeta) \quad \text{and} \quad h_{n,k}(\zeta, z) = (A^{\dagger}_+)^n h_{0,k}(\zeta, z). \]  

(29)

They also satisfy the recursion relations,

\[ h_{n+1,k}(\zeta, z) = \zeta h_{n,k}(\zeta, z) - k h_{n,k-1}(\zeta, z), \quad h_{n,k+1}(\zeta, z) = z h_{n,k}(\zeta, z) - n h_{n-1,k}(\zeta, z). \]  

(30)

If we formally take $z$ to be real in (25), the complex Hermite polynomials $h_{n,k}(\zeta, z)$ become the well-known real Hermite polynomials $h_{n+k}(x)$:

\[ h_n(x) = (-1)^n e^{x^2} \partial^n_x e^{-x^2}. \]

The above complex representation of the problem is
3. Summary of the Tomita-Takesaki theory

In the next two sections we shall make a connection between the Landau levels obtained above and the Tomita-Takesaki theory of modular Hilbert algebras [12]. But first, we give in this section a highly condensed summary of this theory.

Let $\mathfrak{A}$ be a von Neumann algebra on a Hilbert space $\mathfrak{H}$ and $\mathfrak{A}'$ its commutant. Let $\Phi \in \mathfrak{H}$ be a unit vector which is cyclic and separating for $\mathfrak{A}$. Then the corresponding state $\phi$ on the algebra, $\langle \phi : A \rangle = \langle \Phi | A \Phi \rangle$, $A \in \mathfrak{A}$, is faithful and normal. Consider the antilinear map,

$$S : \mathfrak{H} \mapsto \mathfrak{H}, \quad SA = A^* \Phi, \quad \forall A \in \mathfrak{A}. \quad (31)$$

Since $\Phi$ is cyclic, this map is densely defined and in fact it can be shown that it is closable. We denote its closure again by $S$ and write its polar decomposition as

$$S = J\Delta^{1/2} = \Delta^{-1/2}J, \quad \text{with} \quad \Delta = S^*S. \quad (32)$$

The operator $\Delta$, called the modular operator, is positive and self-adjoint. The operator $J$, called the modular conjugation operator, is antiunitary and satisfies $J = J^*$, $J^2 = I_\mathfrak{H}$. The antiunitarity of $J$ is to be understood as:

$$\langle J\phi | J\psi \rangle = \langle \psi | \phi \rangle, \quad \forall \phi, \psi \in \mathfrak{H}. \quad (33)$$

Since $\Delta$ is self-adjoint, using its spectral representation, we see that for $t \in \mathbb{R}$, the family of operators $\Delta^{-it\beta}$, for some fixed $\beta > 0$, defines a unitary family of automorphisms of the algebra $\mathfrak{A}$. Denoting these automorphisms by $\alpha_\phi(t)$, we may write,

$$\alpha_\phi(t)[A] = \Delta^{-it\beta} A \Delta^{it\beta}, \quad \forall A \in \mathfrak{A}. \quad (34)$$

Thus, they constitute a strongly continuous one-parameter group of automorphisms, called the modular automorphism group. Denoting the generator of this one-parameter group by $H_\phi$, we get

$$\Delta^{-it\beta} = e^{itH_\phi} \quad \text{and} \quad \Delta = e^{-\beta H_\phi}. \quad (35)$$

It can then be shown that the state $\phi$ is invariant under this automorphism group:

$$e^{-\beta H_\phi} \Phi = \Phi, \quad \Delta^{it\beta} \mathfrak{A} \Delta^{-it\beta} = \mathfrak{A}, \quad (36)$$

and the antilinear map $J$ interchanges $\mathfrak{A}$ with its commutant $\mathfrak{A}'$:

$$J\mathfrak{A}J = \mathfrak{A}'. \quad (37)$$

Finally, the state $\phi$ can be shown to satisfy the KMS (Kubo-Martin-Schwinger) condition (see for example [13]), with respect to the automorphism group $\alpha_\phi(t)$, $t \in \mathbb{R}$, in the following sense: for any two $A, B \in \mathfrak{A}$, the function

$$F_{A,B}(t) = \langle \phi : A \alpha_\phi(t)[B] \rangle, \quad (38)$$

has an extension to the strip $\{z = t + iy \mid t \in \mathbb{R}, \; y \in [0, \beta]\} \subset \mathbb{C}$ such that $F_{A,B}(z)$ is analytic in the open strip $(0, \beta)$ and continuous on its boundaries. Moreover, it also satisfies the boundary condition (at inverse temperature $\beta$),

$$\langle \phi : A \alpha_\phi(t + i\beta)[B] \rangle = \langle \phi : \alpha_\phi(t)[B] A \rangle, \quad t \in \mathbb{R}. \quad (39)$$
4. The case of Hilbert-Schmidt operators

We now proceed to demonstrate how a simple example of the Tomita-Takesaki theory is provided by two naturally commuting von-Neumann algebras carried by a Hilbert space of Hilbert-Schmidt operators. (A much fuller discussion is given, e.g., in [6, 7]). In the next section we shall transfer this structure to the Hilbert space of the Landau problem and the two commuting von-Neumann algebras generated by the operators \( A_\pm, A_\pm^* \) and the identity operator, using a specific unitary map.

Let \( \mathcal{H} \) be a (complex, separable) Hilbert space of dimension \( N \) (finite or infinite) and \( \{ \zeta_i \}_{i=1}^N \) an orthonormal basis of it (\( \langle \zeta_i \mid \zeta_j \rangle = \delta_{ij} \)). We denote by \( B_2(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H} \) the space of all Hilbert-Schmidt operators on \( \mathcal{H} \). This is a Hilbert space with scalar product

\[
\langle X \mid Y \rangle_2 = \text{Tr}[X^*Y] .
\]

The vectors,

\[
\{X_{ij} = |\zeta_i\rangle\langle \zeta_j| \mid i,j = 1,2,\ldots,N \}, \tag{39}
\]

form an orthonormal basis of \( B_2(\mathcal{H}) \),

\[
\langle X_{ij} \mid X_{k\ell} \rangle_2 = \delta_{ik}\delta_{\ell j} .
\]

In particular, the vectors,

\[
P_i = X_{ii} = |\zeta_i\rangle\langle \zeta_i| , \tag{40}
\]

are one dimensional projection operators on \( \mathcal{H} \). In what follows \( I \) will denote the identity operator on \( \mathcal{H} \) and \( I_2 \) that on \( B_2(\mathcal{H}) \).

We identify a special class of linear operators on \( B_2(\mathcal{H}) \), denoted by \( A \lor B, A,B \in \mathcal{L}(\mathcal{H}) \), which act on a vector \( X \in B_2(\mathcal{H}) \) in the manner:

\[
(A \lor B)(X) = AXB^* .
\]

It is easy to see that

\[
(A \lor B)^* = A^* \lor B^* \quad \text{and} \quad (A_1 \lor B_1)(A_2 \lor B_2) = (A_1A_2) \lor (B_1B_2) . \tag{41}
\]

Next we specify the two special von Neumann algebras,

\[
\mathfrak{A}_\ell = \{A_\ell = A \lor I \mid A \in \mathcal{L}(\mathcal{H})\} , \quad \mathfrak{A}_r = \{A_r = I \lor A \mid A \in \mathcal{L}(\mathcal{H})\} , \tag{42}
\]

which turn out to be both factors and mutual commutants:

\[
(\mathfrak{A}_\ell)' = \mathfrak{A}_r , \quad (\mathfrak{A}_r)' = \mathfrak{A}_\ell , \quad \mathfrak{A}_\ell \cap \mathfrak{A}_r = \mathbb{C}I_2 . \tag{43}
\]

The operator \( J : B_2(\mathcal{H}) \longrightarrow B_2(\mathcal{H}) \), acting on the basis vectors \( X_{ij} \) in (39) as

\[
JX_{ij} = X_{ji} \implies J^2 = I_2 \quad \text{and} \quad J(|\phi\rangle \langle \psi|) = |\psi\rangle \langle \phi| , \quad \forall \phi, \psi \in \mathcal{H} , \tag{44}
\]

is antiunitary, and in fact,

\[
J\mathfrak{A}_\ell J = \mathfrak{A}_r . \tag{45}
\]
4.1. A KMS state

We take \( N = \infty \) and for some \( \beta > 0 \), consider the sequence of positive numbers

\[
\alpha_n = (1 - e^{-\beta})e^{-(n-1)\beta}, \quad \sum_{n=1}^{\infty} \alpha_n = 1 .
\]

Let

\[
\Phi = \sum_{n=1}^{\infty} \alpha_n^\frac{1}{2} \varphi_n = \sum_{n=1}^{\infty} \alpha_n^\frac{1}{2} X_{nn} \in \mathcal{B}_2(\mathcal{H}) .
\]

We note the following properties of \( \Phi \):

- \( \Phi \) defines a vector state \( \varphi \) on the von Neumann algebra \( \mathfrak{A}_\ell \). This follows from the fact that for any \( A \vee I \in \mathfrak{A}_\ell \), we may define the state \( \varphi \) on \( \mathfrak{A}_\ell \) by

\[
\langle \varphi ; A \vee I \rangle = \langle \Phi | (A \vee I)(\Phi) \rangle_2 = \text{Tr}[\Phi^* A \Phi] = \text{Tr}[\rho_\varphi A] , \quad \text{with} \quad \rho_\varphi = \sum_{i=1}^{\infty} \alpha_i \mathbb{P}_i . \quad (48)
\]

Explicitly, \( \Phi \) represents the mixed state \( \rho_\varphi \), defined on the algebra \( \mathcal{L}(\mathcal{H}) \), as a vector state defined on an algebra which is isomorphic to the initial algebra, but acting on the enlarged Hilbert space of Hilbert-Schmidt operators on \( \mathcal{H} \) (this is a concrete and explicit example of the well known GNS construction).

- The state \( \varphi \) is faithful and normal. Normality follows from the last equality in (48) and the fact that \( \rho_\varphi \) is a density matrix. To check for faithfulness, note that for any \( A \vee I \in \mathfrak{A}_\ell \),

\[
\langle \varphi ; (A \vee I)^*(A \vee I) \rangle = \text{Tr}[\rho_\varphi A^* A] = \sum_{i=1}^{\infty} \alpha_i \| A \zeta_i \|^2 ,
\]

from which it follows that \( \langle \varphi ; (A \vee I)^*(A \vee I) \rangle = 0 \) if and only if \( A = 0 \) (since the \( \zeta_i \) are an orthonormal basis set and the \( \alpha_i > 0 \)), hence if and only if \( A \vee I = 0 \).

- The vector \( \Phi \) is cyclic and separating for \( \mathfrak{A}_\ell \). Indeed, cyclicity follows from the fact that if \( X \in \mathcal{B}_2(\mathcal{H}) \) is orthogonal to all \( (A \vee I)\Phi \), \( A \in \mathcal{L}(\mathcal{H}) \), then

\[
\text{Tr}[X^* A \Phi] = \sum_{i=1}^{\infty} \alpha_i^\frac{1}{2} \langle \zeta_i | X^* A \zeta_i \rangle = 0, \quad \forall A \in \mathcal{L}(\mathcal{H}) .
\]

Taking \( A = X_{k\ell} \), we easily get from the above equality, \( \langle \zeta_k | X^* \zeta_\ell \rangle = 0 \) and since this holds for all \( k, \ell \), we get \( X = 0 \). In the same way, \( \Phi \) is also cyclic for \( \mathfrak{A}_\ell \), hence separating for \( \mathfrak{A}_\ell \), i.e., \( (A \vee I)\Phi = (B \vee I)\Phi \iff A \vee I = B \vee I \).

4.2. Time evolution and modular automorphism

It will turn out (see below) that the state \( \varphi \) constructed above is indeed KMS for the time evolution \( \alpha_\varphi(t) \), \( t \in \mathbb{R} \), that we now define on the algebra \( \mathfrak{A}_\ell \). With respect to this evolution, this state will be shown to satisfy,

\[
\langle \varphi ; A_\ell \alpha_\varphi(t + i\beta)[B_\ell] \rangle = \langle \varphi ; \alpha_\varphi(t)[B_\ell] A_\ell \rangle , \quad \forall A_\ell, B_\ell \in \mathfrak{A}_\ell . \quad (49)
\]

Furthermore, the function,

\[
F_{A_\ell, B_\ell}(z) = \langle \varphi ; A_\ell \alpha_\varphi(z)[B_\ell] \rangle , \quad (50)
\]
is analytic in the strip \( \{ \Im(z) \in (0, \beta) \} \) and continuous on its boundaries.

Define the operators,
\[
P_{ij} = \mathbb{P}_i \lor \mathbb{P}_j , \quad i, j = 1, 2, \ldots, \infty
\]
the \( \mathbb{P}_i \) being the projection operators on \( \mathcal{F} \) defined in (40). Then, the \( P_{ij} \) are projection operators on the Hilbert space \( B_2(\mathcal{F}) \). Next, using \( \rho_\varphi \) in (48), define the Hamiltonian \( H_\varphi \) as:
\[
\rho_\varphi = e^{-\beta H_\varphi} \implies H_\varphi = -\frac{1}{\beta} \sum_{i=1}^{\infty} (\ln \alpha_i) \mathbb{P}_i ,
\]
and using it define the operators:
\[
H^t_\varphi = H_\varphi \lor I , \quad H^t_\varphi = I \lor H_\varphi , \quad H_\varphi = H^t_\varphi - H^t_\varphi .
\]
A straightforward computation then establishes the spectral form of \( H_\varphi \):
\[
H_\varphi = -\frac{1}{\beta} \sum_{i,j=1}^{\infty} \ln \left[ \frac{\alpha_i}{\alpha_j} \right] P_{ij} = \sum_{i,j=1}^{\infty} (i - j) P_{ij} .
\]
Writing
\[
\Delta_\varphi := \sum_{i,j=1}^{\infty} \left[ \frac{\alpha_i}{\alpha_j} \right] P_{ij} = e^{-\beta H_\varphi} ,
\]
we define a time evolution operator on \( B_2(\mathcal{F}) \):
\[
e^{iH_\varphi t} = [\Delta_\varphi]^{-\frac{i}{\pi}} . \quad t \in \mathbb{R} ,
\]
and we note that, for any \( X \in B_2(\mathcal{F}) \),
\[
e^{iH_\varphi t}(X) = \sum_{i,j=1}^{\infty} \left[ \frac{\alpha_i}{\alpha_j} \right]^{-\frac{i}{\pi}} P_{ij}(X) = \left[ \sum_{i=1}^{\infty} (\alpha_i)^{-\frac{i}{\pi}} \mathbb{P}_i \right] \lor \left[ \sum_{j=1}^{\infty} (\alpha_j)^{-\frac{j}{\pi}} \mathbb{P}_j \right] (X)
\]
so that
\[
e^{iH_\varphi t} = e^{iH_\varphi t} \lor e^{-iH_\varphi t} ,
\]
where \( H_\varphi \) is the operator defined in (52). From the definition of the vector \( \Phi \) in (47), it is clear that it commutes with \( H_\varphi \) and hence that it is invariant under this time evolution:
\[
e^{iH_\varphi t}(\Phi) = e^{iH_\varphi t} \Phi = e^{-iH_\varphi t} = \Phi .
\]
Finally, using \( e^{iH_\varphi t} \) we define the time evolution \( \alpha_\varphi \) on the algebra \( \mathfrak{A}_\ell \), in the manner (see (33)):
\[
\alpha_\varphi(t)[A_\ell] = e^{iH_\varphi t} A_\ell e^{-iH_\varphi t} \quad \forall A_\ell \in \mathfrak{A}_\ell .
\]
Writing \( A_\ell = A \lor I \), \( A \in \mathcal{L}(\mathcal{F}) \), and using the composition law (41), we see that
\[
e^{iH_\varphi t} A_\ell e^{-iH_\varphi t} = [e^{iH_\varphi t} A e^{-iH_\varphi t}] \lor I ,
\]
so that by virtue of (48),
\[
\langle \varphi ; \alpha_\varphi(t)[A_\ell] \rangle = \text{Tr} [\rho_\varphi \ e^{iH_\varphi t} A \ e^{-iH_\varphi t}] = \langle \varphi ; A_\ell \rangle ,
\]
since $\rho_\varphi$ and $H_\varphi$ commute. Thus, the state $\varphi$ is invariant under the time evolution $\alpha_\varphi$.

To obtain the KMS condition (49), combining (59) and (60), we first note that, with $A_\ell = A \lor I$, $B_\ell = B \lor I$,

$$A_\ell \alpha_\varphi(t) [B_\ell] = [A e^{i H_\varphi t} B e^{-i H_\varphi t}] \lor I.$$ 

Hence, again using (48),

$$F_{A_\ell, B_\ell}(t) = \langle \varphi ; A_\ell \alpha_\varphi(t) [B_\ell] \rangle = \text{Tr} \left[ \rho_\varphi A e^{i H_\varphi t} B e^{-i H_\varphi t} \right] = \text{Tr} \left[ \rho_\varphi e^{-i H_\varphi t} A e^{i H_\varphi t} B \right],$$

the last equality following from the commutativity of $\rho_\varphi$ and $H_\varphi$. Thus, since $\rho_\varphi = e^{-\beta H_\varphi}$,

$$F_{A_\ell, B_\ell}(t + i \beta) = \text{Tr} \left[ \rho_\varphi e^{-i H_\varphi t} e^{i H_\varphi t} A e^{i H_\varphi t} e^{-i H_\varphi t} B \right] = \text{Tr} \left[ e^{-i H_\varphi t} A e^{i H_\varphi t} \rho_\varphi B \right],$$

so that

$$\langle \varphi ; A_\ell \alpha_\varphi(t + i \beta) [B_\ell] \rangle = \text{Tr} \left[ \rho_\varphi e^{i H_\varphi t} B e^{-i H_\varphi t} A \right] = \langle \varphi ; \alpha_\varphi(t) [B_\ell] A_\ell \rangle,$$

which is the KMS condition.

Finally note that from the definition in (52), it follows that $H_\varphi$ is the quantum Hamiltonian (up to an additive constant) of a one-dimensional harmonic oscillator, so that the state $\rho_\varphi$ is just the Gibbs state for the harmonic oscillator.

### 4.3. The antilinear operator $S_\varphi$

We now analyze the antilinear operator $S_\varphi : \mathcal{B}_2(\mathfrak{H}) \rightarrow \mathcal{B}_2(\mathfrak{H})$, which acts as (see (31))

$$S_\varphi(A_\ell \Phi) = A^*_\ell \Phi, \quad \forall A_\ell \in \mathcal{A}_\ell.$$  \hspace{1cm} (62)

Taking $A_\ell = A \lor I$,

$$S_\varphi(A_\ell \Phi) = A^*_\ell \Phi, \quad \forall A_\ell \in \mathcal{A}_\ell \iff S_\varphi(A \Phi) = A^* \Phi, \quad \forall A \in \mathcal{L}(\mathfrak{H}).$$

Using (47) we may write,

$$S_\varphi(A \Phi) = A^* \Phi \quad \Rightarrow \quad \sum_{i=1}^{\infty} \alpha_i^{\frac{1}{2}} S_\varphi(A \mathbb{P}_i) = \sum_{i=1}^{\infty} \alpha_i^{\frac{1}{2}} A^* \mathbb{P}_i.$$

Taking $A = X_{kl}$ (see (39)) and using $X_{kl} \mathbb{P}_i = \delta_{ki} X_{ki}$, we then get

$$\alpha_k^{\frac{1}{2}} S_\varphi(X_{kl}) = \alpha_k^{\frac{1}{2}} \mathcal{S} X_{lk} \quad \Rightarrow \quad S_\varphi(X_{kl}) = \left[ \frac{\alpha_k}{\alpha_l} \right]^\frac{1}{2} X_{lk}. \hspace{1cm} (63)$$

Since any $A \in \mathcal{L}(\mathfrak{H})$ can be written as $A = \sum_{i,j=1}^{\infty} a_{ij} X_{ij}$, where $a_{ij} = \langle \zeta_i | A \zeta_j \rangle$, and furthermore, since $\mathbb{P}_{ij}(X_{kl}) = X_{ij} \delta_{ik} \delta_{j\ell}$, we obtain using (44) and (55),

$$S_\varphi = J[\Delta_\varphi]^{\frac{1}{2}}, \hspace{1cm} (64)$$

which in fact, also gives the polar decomposition of $S_\varphi$.

In this way, within the framework of the Hilbert space of Hilbert-Schmidt operators, we have obtained all the ingredients of a modular structure, as outlined in Section 3. We now proceed to transfer this entire structure to the Hilbert space $L^2(\mathbb{C}, d\nu)$ (see (16)) incorporating the Landau levels.

10
5. Modular structure of the Landau level problem

Going back to the Hilbert space $\mathcal{H}$ (this time taken to be infinite-dimensional) with orthonormal basis $\{\zeta_n\}_{n=0}^{\infty}$, we consider the one-dimensional oscillator Hamiltonian,

$$H_{\text{osc}} = \sum_{n=0}^{\infty} \left( n + \frac{1}{2} \right) p_n = \frac{1}{2}(p^2 + q^2), \quad \rho_n = |\zeta_n\rangle \langle \zeta_n|,$$

(65)

$Q, P$ being the usual position and momentum operators satisfying $[Q, P] = i\hbar$.

On $\mathcal{H}$ we construct the operator

$$V(z, \overline{z}) = 2\pi e^{|z|^2} e^{-i(z a^\dagger + \pi a)}, \quad \text{with} \quad a = \frac{1}{\sqrt{2}} (Q + iP), \quad a^\dagger = \frac{1}{\sqrt{2}} (Q - iP).$$

(66)

for each $z \in \mathbb{C}$. Consider now the map $U : \mathcal{B}_2(\mathcal{H}) \rightarrow L^2(\mathbb{C}, d\nu)$, with

$$(UX)(x, y) = \frac{1}{(2\pi)^2} \text{Tr}[V(z, \overline{z})^* X].$$

(67)

This map is known to be unitary (see, for example, [6]). Furthermore, as shown in [5] it induces the remarkable transformations,

$$U \begin{pmatrix} a \vee I_\delta \\ a^\dagger \vee I_\delta \end{pmatrix} U^{-1} = \begin{pmatrix} A_+ \\ A_+^\dagger \end{pmatrix}, \quad U \begin{pmatrix} I_\delta \vee a \\ I_\delta \vee a^\dagger \end{pmatrix} U^{-1} = \begin{pmatrix} A_- \\ A_-^\dagger \end{pmatrix},$$

(68)

and thus,

$$U \begin{pmatrix} H_{\text{osc}} \vee I_\delta \\ I_\delta \vee H_{\text{osc}} \end{pmatrix} U^{-1} = \begin{pmatrix} H^+ \vee H^+ \\ \overline{H^+} \vee \overline{H^+} \end{pmatrix}, \quad UX_{n\ell} = H_{n-1\ell-1},$$

(69)

where the $X_{n\ell}$ are the basis vectors defined in (39) and the $H_{n\ell}$ are the normalized complex Hermite polynomials defined in (20). The two sets of operators, $\{A_+, A_+^\dagger\}$ and $\{A_-, A_-^\dagger\}$, generate the two von Neumann algebras $\mathfrak{M}_+$ and $\mathfrak{M}_-$, respectively, with $U\mathfrak{M}_+ U^{-1} = \mathfrak{M}_+$ and $U\mathfrak{M}_- U^{-1} = \mathfrak{M}_-$. Thus physically, the two commuting algebras, which are also mutual commutants and are both factors, in view of (43), correspond to the two directions of the magnetic field.

The KMS state $\mathcal{X} = U\Phi$, with $\Phi$ given by (47) is just the *Gibbs equilibrium state* for this physical system, i.e., for the *interaction Hamiltonian* $\mathcal{H}_\varphi$ (see (73) below). Note also, that the existence of the unitary map $U$ is also a consequence of the well known theorem of von Neumann on representations of the canonical commutation relations. Here we simply have an explicit example of it.

Let us consider some further aspects of the modular structure, now in the context of the physical problem of the electron in a constant magnetic field. The map $U$ transforms the modular conjugation map $J$ in (44) to $J = UJU^{-1}$ which basically acts by complex conjugation (see (18) and (29)):

$$(JH_{n\ell})(\overline{z}, z) = H_{\ell n}(z, \overline{z}) \quad \text{and} \quad J A_\pm J^{-1} = A_\mp = \overline{A}_\pm,$$

(70)

etc. (The overline indicates complex conjugation of the variables appearing in the definitions of the operators). Furthermore, with $\mathcal{H}_0, \mathcal{H}_1$, etc., as in (23),

$$JH_0J = \mathcal{H}_0, \quad JH_1J = H_1 \quad \Rightarrow \quad JH_1^\dagger J = H_1 \quad (71)$$

meaning that the modular conjugation map interchanges the two physical setups, corresponding to the two directions of the magnetic field.
The KMS state on the algebra $\mathcal{U}_+,$ which is a vector state, is given by the vector (see (47) and (46)),

$$\mathcal{X} = \mathcal{U} \Phi = (1 - e^{-\beta})^{\frac{1}{2}} \sum_{n=0}^{\infty} e^{-\frac{n^2}{2}} H_{nn} \in L^2(\mathbb{C}, d\nu(z, z)),$$

$$\mathcal{J} \mathcal{X} = \mathcal{X}.$$  \hfill (72)

The Hamiltonian

$$\mathcal{H}_\varphi = \mathcal{H}^\uparrow - \mathcal{H}^\downarrow = 2 \mathcal{H}_{\text{int}} = -2(N_+ - N_-),$$  \hfill (73)

then gives the modular operator,

$$\Delta_\varphi = \exp[-\beta \mathcal{H}_\varphi] = \sum_{n,\ell=0}^{\infty} e^{-\beta(n-\ell)}|H_{n\ell}\rangle \langle H_{n\ell}|,$$

and the one-parameter automorphism group,

$$\Delta_{\varphi}^{-\frac{it}{\beta}} = \exp[i\mathcal{H}_\varphi t] = \exp[2i\mathcal{H}_{\text{int}}^\dagger t], \quad t \in \mathbb{R},$$

$$\Delta_{\varphi}^{-\frac{it}{\beta}} \mathcal{X} = \mathcal{X},$$  \hfill (75)

which stabilizes $\mathcal{X}.$ In other words, the modular automorphism is basically the time evolution generated by the interaction Hamiltonian. One also verifies that

$$\Delta_{\varphi}^{-\frac{it}{\beta}} \mathcal{U}_+ \Delta_{\varphi}^{-\frac{it}{\beta}} = \mathcal{U}_+.$$  \hfill (76)

### 6. Discussion

As set out in the Introduction, our aim in this paper was to make a link between a somewhat arcane (at least from a physicist’s point of view) mathematical theory of modular structures in von Neumann algebras and the rather well-known physical problem of quantum mechanically describing an electron moving in a constant magnetic field and the associated Landau levels. We have shown here how the two experimental setups, corresponding to the two possible directions of the magnetic field are mathematically reflected in the appearance of two commuting von Neumann algebras. The effect of the modular conjugation operator of the algebraic theory is translated, in the experimental context, into a change of direction of the magnetic field. The equilibrium KMS state turns out to be the Gibbs state obtained from the interacting Hamiltonian and, being invariant under the modular conjugation, is independent of the choice of the direction of the magnetic field and thus reflects a symmetry of the problem. What is intriguing is that this symmetry is essentially the symmetry under complex conjugation. Let us just point out again that the complex analytic representation introduced here is unitarily equivalent to the initial representation on $L^2(\mathbb{R}^2).$ Its interest lies more in the identification of the modular conjugation operator with a simple complex conjugation. Additionally, as shown in [3], there are interesting families of coherent states that can be readily obtained from this representation. There is also the possibility that, starting from general families of coherent states on other Hilbert spaces of holomorphic functions, one could arrive at other pairs of commuting von Neumann algebras and thus, other modular structures.

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