Regularity of some class of nonlinear transformations

February 1, 2008

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Abstract

In this paper we consider quadratic stochastic operators designed on finite Abelian groups. It is proved that such operators have the property of regularity.

Mathematics Subject Classification: 37A25, 37N25, 46T99, 47H60. Key words: Quadratic stochastic operators; Finite Abelian group; Regularity; Ergodicity.

1 Quadratic stochastic operators

Let

\[ S^{n-1} = \{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \ \forall i = 1, \cdots, n \} \]  

(1)

be the \((n - 1)\)-dimensional simplex in \(\mathbb{R}^n\). The transformation \(V : S^{n-1} \to S^{n-1}\) is called a quadratic stochastic operator (q.s.o.), if

\[ (Vx)_k = \sum_{i,j=1}^{n} p_{ij,k} x_i x_j \]  

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where

\[ p_{ij,k} \geq 0, \]

\[ p_{ij,k} = p_{ji,k}, \]

\[ \sum_{k=1}^{n} p_{ij,k} = 1 \] (3)

for arbitrary \( i, j, k \in \{1, \cdots, n\} \). Such operators have applications in mathematical biology, namely theory of heredity, where the coefficients \( p_{ij,k} \) are interpreted as coefficients of heredity [1-3].

Assume \( \{V^kx : k = 0, 1, \cdots\} \) is the trajectory of the initial point \( x \in S^{n-1} \), where \( V^{k+1}x = V(V^kx) \) for any \( k = 0, 1, \cdots \)

**Definition 1** A q.s.o. \( V : S^{n-1} \to S^{n-1} \) is called ergodic (respectively regular) if for any initial point \( x \in S^{n-1} \) the limit

\[ \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} V^i x \]

(respectively the limit \( \lim_{k \to \infty} V^kx \)) exists.

Evidently, any regular q.s.o. \( V \) has the ergodic property, but the converse is not necessarily true.

To determine whether some q.s.o. is ergodic or regular is rather complicated problem.

S. Ulam in [4] presupposed the assumption that any q.s.o. \( V \) is ergodic. Later M. Zakharevitch [5] showed, that this is an incorrect
hypothesis in general. More precisely, he proved that the q.s.o. \( V \), which is defined on the simplex

\[ S^2 = \{(x, y, z) : x, y, z \geq 0, \ x + y + z = 1\} \]

by the formula

\[
\begin{align*}
\hat{x} &= x^2 + 2xy; \\
\hat{y} &= y^2 + 2yz; \\
\hat{z} &= z^2 + 2xz.
\end{align*}
\]

is not an ergodic q.s.o. and respectively is not regular.

Later in [6] necessary and sufficient conditions were established for the ergodicity of the so-called Volterrian q.s.o.:

\[
\begin{align*}
\hat{x} &= x(1 + ay - bz); \\
\hat{y} &= y(1 - ax + cz); \\
\hat{z} &= z(1 + bx - cy)
\end{align*}
\]

where \( a, b, c \in [-1, 1] \). For \( a, b, c = 1 \) we have the initial example of Zakharevitch.

**Theorem 1** Any q.s.o. in the above form is non-ergodic if and only if the three parameters \( a, b, c \) have the same sign.
Below we’ll construct one class of q.s.o. and prove that all such q.s.o. are regular and respectively ergodic.

2 The design of quadratic stochastic operators

Let $G$ be a finite Abelian group and $S(G)$ be a set of all probabilistic measures on $G$. It is evident, that if $|G| = n$, then $S(G)$ coincides with $S^{n-1}(1)$.

Let further $H \subset G$ be a subgroup of $G$ and $\{g + H : g \in G\}$ be the cosets of $H$ in $G$. Assume $\mu \in S(G)$ is a fixed positive measure, that is $\mu(g) > 0$ for any $g \in G$. Then we define the coefficients $p_{fg,h}$, where $f, g, h \in G$ in the following way:

$$p_{fg,h} = \begin{cases} \frac{\mu(g)}{\mu(f+g+H)}, & \text{if } h \in f + g + H; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that for arbitrary $f, g, h \in G$ the conditions (3) are satisfied. It is also evident that if $H = \{e\}$, where $e$ is the neutral
element of group $G$, then

$$p_{fg,h} = \begin{cases} 
1 & \text{if } h = f + g; \\
0 & \text{otherwise}
\end{cases}$$

and if $H = G$, then

$$p_{fg,h} = \mu(h) \quad \forall f, g \in G$$

In the common case q.s.o. $V$ on $S(G)$ is defined as

$$(Vx)_h = \sum_{f, g \in G} p_{fg,h} x_fx_g$$

for all $h \in G$, where $x = \{x_t, \ t \in G\} \in S(G)$ and $p_{fg,h}$ as above.

If $H = \{e\}$ then the q.s.o. $V$ is defined as

$$(Vx)_h = \sum_{f, g \in G, f+g=h} x_fx_g$$

and if $H = G$, the q.s.o. is defined as

$$(Vx)_h = \mu(h)$$

for arbitrary $h \in G$.

Let us fix a positive measure $\mu \in S(G)$ and subgroup $H$ of group $G$. Assume $\mu_H$ is the factor-measure on factorgroup $G/H$, that is

$$\mu_H(g + H) = \sum_{h \in H} \mu(g + h)$$
for any \( g \in G \) and \( V_H \) is a q.s.o. on \( S(G/H) \), which is defined by measure \( \mu_H \). It is easy to show, that the trajectorial behaviour of \( V \) and \( V_H \) are similar, and so it is enough to study the q.s.o. generated by the trivial subgroup.

Below we consider q.s.o. constructed by the trivial subgroup \( H = \{e\} \).

3 The main result

Let \( \nu \in S(G) \) be a Haar measure on \( G \). As \( G \) is a finite Abelian group, the Haar measure \( \nu \) on \( G \) is a uniform distribution on \( G \), that is it is a centre of the simplex \( S(G) \). We prove the following

**Theorem 2** Almost all orbits tend to the center of the simplex.

**Proof:** The following lemma is a key ingredient:

**Lemma 1** \( ||Vx||_\infty \leq ||x||_\infty \).

**Proof of lemma:** Let’s define the following function

\[
 f(p) = f_n(p) = \max_{\sum_{i=1}^{n} x_i = 1, x_i \in [0,p]} \sum_{0}^{n} x_i^2.
\]

Notice that there is no dependence on \( n \). Moreover, \( f \) has an explicit form:

\[
 f(p) = kp^2 + (1 - kp)^2 \text{ if } p \in [\frac{1}{k+1}, \frac{1}{k}].
\]
We can prove this formula by induction.

The sum $\sum x_i^2$ cannot have a maximum in the interior of the domain (by the Lagrange method). So, we can look for the maximum at the set $x_n = p$. It follows that

$$f_n(p) = p^2 + \max_{x_i \in [0, p]} \frac{1}{\sum_{0}^{n-1} x_i = 1 - p} \sum_{0}^{n-1} x_i^2$$

Let $x_i = (1 - p)y_i$. Then $\sum_{0}^{n-1} y_i = 1$ and $y_i \in [0, \frac{p}{1-p}]$. So, the following recurrent formula is true:

$$f_n(p) = p^2 + (1 - p)^2 f_{n-1}(\frac{p}{1-p})$$

This allows an easy check of the expected formula. Now it is easy to see that

$$f(p) - p = kp^2 - p + (1-kp)^2 = (1-kp)(1-kp-p) = (1-kp)(1-(k+1)p) \leq 0,$$

that is

$$f(p) \leq p.$$  

From Cauchy-Bunyakowski inequality:

$$(Vx)_i \leq \sqrt{\sum_{j \in G} x_j^2} \sqrt{\sum_{j \in G} x_{i-j}^2} = \sum_{j \in G} x_j^2$$

If $\max x_i \leq p$ then, $f(p) \leq p$ implies $(Vx)_i \leq p$ or $\max(Vx)_i \leq p$.

So, the norm $||x||_\infty$ decreases on the orbits. Lemma is thus proven.
Actually, the set \( \{ x : ||x||_\infty \leq p \} \) is mapped to the set \( \{ x : ||x||_\infty \leq f(p) \} \). We can easily check, that the iterations of the function \( f \) tend to its fixed points.

Now consider a point \( x \). We consider the case, when a point in orbit does not tend to a centre. It’s \( \omega \)-limit set lies at a set \( \{ x : g(x) = \frac{1}{k} \} \).

This \( \omega \)-limit set is finite. So, it is nothing else, then a periodic orbit.

We consider the periodic orbits on the set \( ||x||_\infty = \frac{1}{k} \).

Since \( ||Vx||_\infty = \frac{1}{k} \), there exists \( k \) coordinates of \( x \) equal to \( \frac{1}{k} \). All other coordinates are zeros. This is also true for \( Vx \). So, there exists sets \( A = \{ i : x_i = \frac{1}{k} \} \) and \( B = \{ j : (Vx)_j = \frac{1}{k} \} \), such that \( |A| = |B| = k \)

and

1. \( \forall i \in B \ (i - A) \cap A| = k \rightarrow i - A = A, \)
2. \( \forall i \notin B \ (i - A) \cap A| = 0 \rightarrow (i - A) \cap A = \emptyset. \)

From the second item:

\[ \forall i \notin B \ \forall x, y \in A : \ i - x \neq y; \]

\[ \forall i \notin B \ \forall j \in A + A, \ i \neq j; \]

\[ A + A \subset B. \]
From the first item

\[ \forall i \in B \ \exists j, k \in A : \ i = j + k; \]

\[ B \subset A + A. \]

So, there exists a set \( A \), such that \( |A| = |A + A| = k, \)

\[ \forall i \in A + A : \ i - A = A. \]

**Claim:** This is equivalent to the following property:

\[ \forall i, j, k \in A : \ i + j - k \in A. \]

**Proof:** \( \rightarrow \forall x, y \in A \ x + y - A = A. \) \( \forall x, y, z \in A \ x + y - z \in A. \)

\( \leftarrow \forall x, y \in A \ x + y - A \subset A. \) \( \forall z \in A + A \ z - A \subset A. \) Since \( |z - A| = |A|, \) \( z - A = A. \)

\[ \forall a, i, j \in A \ a + i - j \in A. \) \( \forall i, j \in A \ A + i - j \subset A. \)

\[ \forall i, j \in A \ A + i \subset A + j. \]

So, \( \forall i, j \in A \ A + i = A + j, \) and hence ,

\[ A + A = \bigcup_{j \in A} (A + j) = A + i. \]

So, \( |A + A| = k. \) The claim is proved.

This property is equivalent to the following : there exists a point \( p \in A \) and a subgroup \( H, \) such that \( A = p + H. \) So, the points
$V_1, V^2_2, V^3_3, \cdots$ correspond to the sets $2p + H, 4p + H, 8p + H, \cdots$.

Actually, such sequences are pre-periodic.

We now prove the instability of such periodic orbits. Let $l$ be its period and $V^l = T$ be the corresponding first return map. Then $T$ has an instability direction (an eigenvector, whose eigenvalue is greater than 1). It is easy to check, that vector $e$ with coordinates

$$e_s = \begin{cases} 1, & \text{if } s \in i + H; \\ -1, & \text{if } s \in j + H; \\ 0, & \text{other cases} \end{cases}$$

belongs to the plane \(\{x : \sum x_i = 0\}\) (a tangent plane for the simplex) and realizes this instability direction.

There is no such directions for the centre by the coincidense of each two classes.

So, the basin of attraction for such orbit consists of subvarieties of strictly positive codimension. We conclude that almost all orbits tend to the centre of the simplex.

**Remark**

We can also consider an infinite-dimensional case. All preceding arguments remain true. But there is no natural measure on a sim-
plex, and so we cannot use the term ”almost all”.

Another disappointment is that ”most” of the orbits diverge in the $l_1$ topology. It follows from the fact, that for ”most” orbits the sequence $\|V^nx\|_\infty$ tends to zero. All possible $l_1$ limit points are $l_\infty$ limit points. Since the only $l_\infty$ limit point is zero, then the subsequence of the orbit $l_1$ tends to a point, which does not belong to a simplex. This contradiction implies the absence of $l_1$ limit points.

**Acknowledgements**

This research was supported in part by the Uz.R. grant F-2.1.56.

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