ON ADDITIVE HIGHER CHOW GROUPS OF AFFINE SCHEMES

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Abstract. We show that multivariate additive higher Chow groups of a smooth affine $k$-scheme $\text{Spec}(R)$ essentially of finite type over a perfect field $k$ of characteristic $\neq 2$ form a differential graded algebra. In the univariate case, we show that additive higher Chow groups of $\text{Spec}(R)$ form a Witt-complex over $R$.

1. Introduction

The additive higher Chow groups $\text{TCH}^q(X,n;m)$ emerged originally in [6] in part as an attempt to understand certain relative higher algebraic $K$-groups of Quillen [29]. Since then, several papers [15], [16], [17], [18], [27], [28], [30] have studied various aspects of the groups. But, lack of a suitable moving lemma for smooth affine varieties has been a hindrance in studies of their local behaviors. Its projective sibling was already known by [16]. During the period of stagnation, the subject has evolved into the notion of ‘cycles with modulus’ $\text{CH}^q(X|D,n)$ by Binda-Kerz-Saito in [2], [14] associated to pairs $(X,D)$ of a scheme and an effective divisor $D$, setting a more flexible ground, while this desired moving lemma for the affine case was obtained by W. Kai [13]. (See Theorem 4.1)

This moving lemma now propels the authors to continue their journeys. This article is the first of some papers that relate the additive higher Chow groups to the big de Rham-Witt complexes $\mathbb{W}_m\Omega^*_R$ of [10] and to the crystalline cohomology theory studied by [1], [3], and [11]. This gives a motivic description of the latter two objects.

While the general notion of cycles with modulus for $(X,D)$ provides a wider picture, the additive higher Chow groups still have a nontrivial operation not shared by the general case. One such is an analogue of the Pontryagin product on homology groups of Lie groups, which turns the additive higher Chow groups into a differential graded algebra (DGA). This structure was envisioned by the earliest papers on the subject by Bloch-Esnault [6] and Rülling [30] for 0-cycles. The Pontryagin product on higher dimensional additive higher Chow cycles was given in [18] for smooth projective varieties. In Section 5 of this paper, we extend the product structure in two directions: (1) toward multivariate additive higher Chow groups and (2) on smooth affine varieties. In doing so, we generalize some of the necessary tools, such as the normalization theorem (Theorem 3.2). Here is the summary of the results we obtain:

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Theorem 1.1. When $X$ is a smooth quasi-projective scheme essentially of finite type over any field $k$, and $D$ is an effective Cartier divisor on $X$, each cycle class in $\text{CH}^q(X|D, n)$ has a representative, all of whose codimension 1 faces are trivial.

Theorem 1.2. For a smooth $k$-scheme either affine essentially of finite type or projective over a perfect field $k$ of characteristic $\neq 2$, the multivariate additive higher Chow groups with modulus $m$ form a differential graded algebra. Furthermore, each direct summand is a module over the big Witt ring $\mathbb{W}_m(k)$ for $m \gg 0$.

The last results of this paper, discussed in detail in Theorems 6.1, 6.9, 6.13, and 6.16 are the following:

Theorem 1.3. Let $X = \text{Spec} (R)$ be an equidimensional smooth $k$-scheme essentially of finite type over a perfect field $k$ of characteristic $\neq 2$.

1. The additive higher Chow groups $\{\text{TCH}^n(X, n; m)\}_{n,m \in \mathbb{N}}$ in the Milnor range have a functorial structure of a restricted Witt-complex over $k$.
2. Moreover, $\{\text{TCH}^n(X, n; m)\}_{n,m \in \mathbb{N}}$ is a restricted Witt-complex over $R$, and there are natural maps $\tau^R_{n,m} : \mathbb{W}_m \Omega^{-1}_R \to \text{TCH}^n(R, n; m)$.
3. If $R$ is a smooth UFD, then $\tau^R_{1,m} : \mathbb{W}_m(R) \to \text{TCH}^1(R, 1; m)$.
4. If $f : X \to Y$ is finite in $\text{SmAff}^\text{ess}_k$, then $f_* : \{\text{TCH}^n(X, n; m)\}_{n,m \in \mathbb{N}} \to \{\text{TCH}^n(Y, n; m)\}_{n,m \in \mathbb{N}}$ is a morphism of restricted Witt-complexes over $k$.

Necessary definitions are recalled in Section 2. In the following up paper [21], we prove that the above map $\tau^R_{n,m}$ is an isomorphism, and we show how one deduces crystalline cohomology of Berthelot [1] from additive higher Chow groups.

Conventions In this paper, a $k$-scheme is a separated scheme of finite type over $k$, unless we specifically say otherwise. A $k$-variety is a reduced $k$-scheme. The product $X \times Y$ means usually $X \times_k Y$, unless we specify otherwise. We let $\text{Sch}_k$ be the category of $k$-schemes, $\text{Sm}_k$ of smooth $k$-schemes, and $\text{SmAff}_k$ of smooth affine $k$-schemes. A scheme essentially of finite type is a scheme obtained by localizing at a finite subset (including $\emptyset$) of a finite type $k$-scheme. For $C = \text{Sch}_k, \text{Sm}_k, \text{SmAff}_k$, we let $C^\text{ess}$ be the extension of category $C$ obtained by localizing at a finite subset (including $\emptyset$) of objects in $C$. We let $\text{SmLoc}_k$ be the category of smooth semi-local $k$-schemes essentially of finite type over $k$. So, $\text{SmAff}^\text{ess}_k = \text{SmAff}_k \cup \text{SmLoc}_k$ for the objects. When we say a semi-local $k$-scheme, we always mean one that is essentially of finite type over $k$. Let $\text{SmProj}_k$ be the category of smooth projective $k$-schemes.

2. Recollection of some basic definitions

For $\mathbb{P}^1 = \text{Proj}_k(k[s_0, s_1])$, we let $y = s_1/s_0$ its coordinate. Let $\square := \mathbb{P}^1 \setminus \{1\}$. For $n \geq 1$, let $(y_1, \ldots, y_n) \in \square^n$ be the coordinates. A face $F \subset \square^n$ means a closed subscheme defined by the set of equations of the form $\{y_i = \epsilon_1, \ldots, y_s = \epsilon_s\}$ for an increasing sequence $\{i_j\}_{1 \leq j \leq s} \subset \{1, \ldots, n\}$ and $\epsilon_j \in \{0, \infty\}$. We allow $s = 0$, in which case $F = \square^n$. Let $\square := \mathbb{P}^1$. A face of $\square^n$ is the closure of a face in $\square^n$. For $1 \leq i \leq n$, let $F^1_{n,i} \subset \square^n$ be the closed subscheme given by $(y_i = 1)$. Let $F^1_n := \sum_{i=1}^n F^1_{n,i}$, which is the cycle associated to the closed subscheme $\square^n \setminus \square^n$. Let $\square^0 := \square^0 := \text{Spec} (k)$. Let $\iota_{n,i,\epsilon} : \square^{n-1} \hookrightarrow \square^n$ be the inclusion $(y_1, \ldots, y_{n-1}) \mapsto (y_1, \ldots, y_{i-1}, \epsilon, y_i, \ldots, y_{n-1})$. 

2.1. Cycles with modulus. Let \( X \in \text{Sch}^{\text{ess}}_k \). For effective Cartier divisors \( D_1 \) and \( D_2 \) on \( X \), we say \( D_1 \leq D_2 \) if \( D_1 + D = D_2 \) for some effective Cartier divisor \( D \) on \( X \). A scheme with an effective divisor (sed) is a pair \((X, D)\), where \( X \in \text{Sch}^{\text{ess}}_k \) and \( D \) an effective Cartier divisor. A morphism \( f : (Y, E) \to (X, D) \) of sed is a morphism \( f : Y \to X \) in \( \text{Sch}^{\text{ess}}_k \) such that \( f^*(D) \) is defined as a Cartier divisor on \( Y \) and \( f^*(D) \leq E \). In particular, \( f^{-1}(D) \subset E \). If \( f : Y \to X \) is a morphism of \( k \)-schemes, and \((X, D)\) is a sed such that \( f^{-1}(D) = \emptyset \), then \( f : (Y, \emptyset) \to (X, D) \) is a morphism of sed.

**Definition 2.1** ([2], [14]). Let \((X, D)\) and \((Y, E)\) be schemes with effective divisors. Let \( Y = Y \setminus E \). Let \( V \subset X \times Y \) be an integral closed subscheme with closure \( V \subset X \times \overline{Y} \). We say \( V \) is a has modulus \( D \) (relative to \( E \)) if \( \nu_V(D \times \overline{Y}) \leq \nu_V(D \times E) \) on \( \overline{Y} \), where \( \nu_V : \overline{Y} \to X \times Y \) is the normalization followed by the closed immersion.

In case \( Y = \overline{Y} = \text{Spec}(k) \), that \( V \) has modulus \( D \) on \( X \times Y \) is equivalent to \( V \cap D = \emptyset \). Recall (see [16] Proposition 2.4) or [20] Proposition 2.4):

**Proposition 2.2** (Containment lemma). Let \((X, D)\) and \((Y, E)\) be schemes with effective divisors and \( Y = Y \setminus E \). If \( V \subset X \times Y \) is a closed subscheme with modulus \( D \) relative to \( E \), then any closed subscheme \( W \subset V \) has modulus \( D \) relative to \( E \), too.

**Definition 2.3** ([2], [14]). Let \((X, D)\) be a scheme with an effective divisor. For \( r \in \mathbb{Z} \) and \( n \geq 0 \), let \( z_r(X|D, n) \) be the free abelian group on integral closed subschemes \( V \subset X \times \square^n \) of dimension \( r + n \) satisfying the following conditions:

1. (Face condition) for each face \( F \subset \square^n \), \( V \) intersects \( X \times F \) properly.
2. (Modulus condition) \( V \) has modulus \( D \) relative to \( F^n \) on \( X \times \square^n \).

We usually drop the phrase “relative to \( F^n \)” for simplicity. A cycle in \( z_r(X|D, n) \) is called an admissible cycle with modulus \( D \). The groups form a complex with the boundary map \( \partial = \sum_{i=1}^n (-1)^i(\partial_i \ominus \partial_i) \), where \( \partial_i = t_{r,n,i,\epsilon}^* \). One checks that \( n \mapsto z_r(X|D, n) \) is a cubical abelian group.

**Definition 2.4** ([2], [14]). The complex \((z_r(X|D, \bullet), \partial)\) is the nondegenerate complex associated to \((n \mapsto z_r(X|D, n))\), i.e., \( z_r(X|D, n) := z_r(X|D, n)/z_r(X|D, n)_{\text{degn}} \). The homology \( \text{CH}_r(X|D, n) := H_n(z_r(X|D, \bullet)) \) for \( n \geq 0 \) is called higher Chow group of \( X \) with modulus \( D \). If \( X \) is equidimensional of dimension \( d \), for \( q \geq 0 \), we write \( \text{CH}^q(X|D, n) = \text{CH}_{d-q}(X|D, n) \).

Here is a special case:

**Definition 2.5.** Let \( X \in \text{Sch}^{\text{ess}}_k \). For \( r \geq 1 \), let \( X[r] := X \times \mathbb{A}^r \). When \( (t_1, \cdots, t_r) \in \mathbb{A}^r \) are the coordinates, and \( m_1, \cdots, m_r \geq 1 \) are integers, let \( D_{\overline{m}} \) be the divisor on \( X[r] \) given by the equation \( t_1^{m_1} \cdots t_r^{m_r} = 0 \). The groups \( \text{CH}^q(X[r]|D_{\overline{m}}, n) \) are called multivariate additive higher Chow groups of \( X \). For simplicity, we often say “a cycle with modulus \( \overline{m} \)” for “a cycle with modulus \( D_{\overline{m}} \).” When \( r = 1 \), we obtain additive higher Chow groups, and as in [13], we use the old notations \( Tz^q(X, n + 1; m - 1) \) for \( z^q(X[1]|D_m, n) \) and \( \text{TCH}^q(X, n + 1; m - 1) \) for \( \text{CH}^q(X[1]|D_m, n) \).
Definition 2.6. Let $\mathcal{W}$ be a finite set of locally closed subsets of $X$ and let $e : \mathcal{W} \to \mathbb{Z}_{\geq 0}$ be a set function. Let $z_{W,e}^q(X|D,n)$ be the subgroup generated by integral cycles $Z \in z^q(X|D,n)$ such that for each $W \in \mathcal{W}$ and each face $F \subset \mathbb{A}^n$, we have $\text{codim}_{W \times F} Z \cap (W \times F) \geq q - e(W)$. They form a subcomplex $z_{\mathcal{W},e}^q(X|D,\bullet)$ of $z^q(X|D,\bullet)$. Modding out by degenerate cycles, we obtain the subcomplex $z_{\mathcal{W},e}^q(X|D,\bullet) \subset z^q(X|D,\bullet)$. We write $z_{W}^q(X|D,\bullet) := z_{W,0}^q(X|D,\bullet)$. For additive higher Chow cycles, we write $\mathcal{T}^q_{W}(X,n;m)$ for $z_{W[n]}^q(X[1]|D_{m+1},n - 1)$, where $W[1] = \{W[1] \mid W \in \mathcal{W}\}$.

Here are some basic lemmas we use often in the paper:

**Lemma 2.7** ([16, Lemma 2.1]). Let $X$ be a normal scheme and let $D_1$ and $D_2$ be effective Cartier divisors on $X$ such that $D_1 \geq D_2$ as Weil divisors. Let $Y \subset X$ be a closed subset which intersects $D_1$ and $D_2$ properly. Let $f : Y^N \to X$ be the composite of the inclusion and the normalization of $Y$. Then, $f^*(D_1) \geq f^*(D_2)$.

**Lemma 2.8** ([20, Lemma 2.2]). Let $f : Y \to X$ be a dominant map of normal integral $k$-schemes. Let $D$ be a Cartier divisor on $X$ such that the generic points of $\text{Supp}(D)$ are contained in $f(Y)$. Suppose that $f^*(D) \geq 0$ on $Y$. Then $D \geq 0$ on $X$.

**Lemma 2.9** ([20, Lemma 2.9]). Let $f : Y \to X$ be a proper morphism of quasi-projective $k$-varieties. Let $D \subset X$ be an effective Cartier divisor such that $f(X) \not\subset D$. Let $Z \in z^q(Y|f^*(D),n)$ be an irreducible cycle. Let $W = f(Z)$ on $X \times \mathbb{A}^n$. Then, $W \in z^s(X|D,n)$, where $s = \text{codim}_X X \times \mathbb{A}^n(W)$.

**Lemma 2.10.** Let $X$ be a $k$-scheme, and let $\{U_i\}_{i \in I}$ be an open cover of $X$. Let $Z \in z^q(X \times \mathbb{A}^n)$ and let $Z_{U_i}$ be its restriction to $U_i$, i.e. the flat pull-back to $U_i \times \mathbb{A}^n$. Then, $Z \in z^q(X|D,n)$ if and only if for each $i \in I$, we have $Z_{U_i} \in z^q(U_i|D_{U_i},n)$, where $D_{U_i}$ is the restriction of $D$ on $U_i$.

**Proof.** We may assume $Z$ is irreducible. The direction ($\Rightarrow$) is obvious since flat pull-backs respect admissibility of cycles with modulus by [20, Proposition 2.12]. For the direction ($\Leftarrow$), note that $Z$ satisfies the proper intersection condition with all faces, by regarding $Z_{U_i}$ as higher Chow cycles in $z^q(U_i,n)$. For the modulus condition, let $\overline{Z}$ be the Zariski closure of $Z$ in $X \times (\mathbb{P}^1)^n$ and let $\nu : \overline{Z}^N \to \overline{Z} \hookrightarrow X \times (\mathbb{P}^1)^n$ be the normalization composed with the closed immersion. Then, the Zariski closure $\overline{Z}_{U_i}$ of $Z_{U_i}$ in $U_i \times (\mathbb{P}^1)^n$ is just the pull-back of $\overline{Z}$ to $U_i$, while the normalization of $\overline{Z}_{U_i}$ is just the pull-back of $\nu$ to $U_i$.

To check the modulus condition for $Z$, we need to check whether $\nu^*(F_{n}^{1} - D \times (\mathbb{P}^1)^n) \geq 0$. This means, for each prime Weil divisor $p$ of $\overline{Z}^N$, we have $\text{ord}_p \nu^*(F_{n}^{1} - D \times (\mathbb{P}^1)^n) \geq 0$. But, each prime Weil divisor $p$ corresponds to a point of height 1, and since $\{U_i\}$ is a cover of $X$ each $p$ belongs to some $U_i$. But, the value $\text{ord}_p$ is independent of whether we work over $X$ or over $U_i$, because $k(X) = k(U_i)$. Hence, the modulus condition for $Z$ is implied by the modulus conditions for $Z_{U_i}$ over all $i \in I$. \qed

2.2. De Rham-Witt complexes.
2.2.1. Rings of big Witt-vectors. Let $R$ be a commutative ring with unity. We recall the definition of the ring of big Witt-vectors of $R$ (cf. [30, Appendix A]). A truncation set $S \subset \mathbb{N}$ is a nonempty subset such that if $s \in S$ and $t|s$, then $t \in S$. As a set, let $\mathbb{W}_S(R) := R^S$ and define the map $w : \mathbb{W}_S(R) \to R^S$ by sending $a = (a_s)_{s \in S}$ to $w(a) = (w(a)_s)_{s \in S}$, where $w(a)_s := \sum_{t|s} t a_t^{s/t}$. When $R^S$ on the target of $w$ is given the component-wise ring structure, it is known that there is a unique functorial ring structure on $\mathbb{W}_S(R)$ such that $w$ is a ring homomorphism. When $S = \{1, \ldots, m\}$, we write $\mathbb{W}_m(R) := \mathbb{W}_S(R)$.

There is another description. Let $\mathbb{W}(R) := \mathbb{W}_N(R)$. Consider the multiplicative group $(1 + tR[[t]])^\times$, where $t$ is an indeterminate. Then, there is a natural bijection $\mathbb{W}(R) \simeq (1 + tR[[t]])^\times$, where the addition in $\mathbb{W}(R)$ corresponds to the multiplication of formal power series. For a truncation set $S$, we can describe $\mathbb{W}_S(R)$ as the quotient of $(1 + tR[[t]])^\times$ by a suitable subgroup $I_S$. See [30, A.7] for details. In case $S = \{1, \ldots, m\}$, we can write $\mathbb{W}_m(R) = (1 + tR[[t]])^\times/(1 + t^{m+1}R[[t]])^\times$.

For $a \in R$, the Teichmüller lift $[a] \in \mathbb{W}_S(R)$ corresponds to the image of $1 - at \in (1 + tR[[t]])^\times$.

2.2.2. De Rham-Witt complexes. Let $R$ be a $\mathbb{Z}_{(p)}$-algebra for a prime $p$. For each truncation set $S$, there is a differential graded algebra $\mathbb{W}_S\Omega^*_R$. They define a functor on the category of truncation sets. This $\mathbb{W}_S\Omega^*_R$ is called the big de Rham-Witt complex over $R$. This is an initial object in the category of $V$-complex and in the category of Witt-complexes. For details, see [10] and [30, §1]. When $S$ is a finite truncation set, we have $\mathbb{W}_S\Omega^*_R = \Omega^*_0 S(R)/\mathfrak{N}_S$, where $\mathfrak{N}_S$ is the differential graded ideal given by some generators ([30, Proposition 1.2]). In case $S = \{1, 2, \ldots, m\}$, we write $\mathbb{W}_m\Omega^*_R$ for this object.

Here is a more relevant object for this paper from [10, Definition 1.1.1]; a restricted Witt-complex over $R$ is a pro-system of differential graded $\mathbb{Z}$-algebras $((E_m)_{m \in \mathbb{N}}, \mathfrak{F} : E_{m+1} \to E_m)$, with homomorphisms of graded rings ($F_r : E_{rm+r-1} \to E_m$) called the Frobenius maps, and homomorphisms of graded groups ($V_r : E_m \to E_{rm+r-1}$) called the Verschiebung maps, satisfying the following relations for all $n, r \in \mathbb{N}$:

(i) $\mathfrak{F}F_r = F_r\mathfrak{F}$, $\mathfrak{F}V_r = V_r\mathfrak{F}$, $F_1 = V_1 = \text{Id}$, $F_rF_s = F_{rs}$, $V_rV_s = V_{rs}$;

(ii) $F_rV_r = v$. When $(r, s) = 1$, $F_rV_s = V_sF_r$ on $E_{rm+r-1}$;

(iii) $V_r(F_r(x)y) = xV_r(y)$ for all $x \in E_{rm+r-1}$ and $y \in E_m$; (projection formula)

(iv) $F_r dV_r = d$, where $d$ is the differential of the DGAs.

Furthermore, we require that there is a pro-system of ring homomorphisms ($\lambda : \mathbb{W}_m(R) \to E^0_m$) that commutes with $F_r$ and $V_r$, satisfying

(v) $F_r d\lambda([a]) = \lambda([a]^{r-1}) d\lambda([a])$ for all $a \in R$ and $r \in \mathbb{N}$.

The pro-system $\{\mathbb{W}_m\Omega^*_R\}_{m \geq 1}$ is an initial object in the category of restricted Witt-complexes over $R$. See [30, Proposition 1.15].

3. Normalization theorem

The aim of Section 3 is to prove Theorem 3.2, which is used in Section 5. Such results were known when $D = \emptyset$, or when $X$ is replaced by $X \times \mathbb{A}^1$ with $D = \{t^{m+1} = 0\}$ for $t \in \mathbb{A}^1$. We generalize it to higher Chow groups with modulus.
Definition 3.1. Let \((X, D)\) be a scheme with an effective divisor. Let \(z^q_Y(X|D, n)\) be the subgroup of cycles \(\alpha \in z^q(X|D, n)\) such that \(\partial_i^q(\alpha) = 0\) for all \(1 \leq i \leq n\) and \(\partial_i^\infty(\alpha) = 0\) for \(2 \leq i \leq n\). One checks that \(\partial_i^\infty \circ \partial_i^\infty = 0\). Writing \(\partial_i^\infty\) as \(\partial^N\), we obtain a subcomplex \(i : (z^q_Y(X|D, \bullet), \partial^N) \hookrightarrow (z^q(X|D, \bullet), \partial)\).

Theorem 3.2. Let \(X\) be a smooth quasi-projective \(k\)-scheme essentially of finite type over \(k\), and let \(D \subset X\) be an effective Cartier divisor. Then, \(i : z^q_Y(X|D, \bullet) \to z^q(X|D, \bullet)\) is a quasi-isomorphism. In particular, every cycle class in \(\text{CH}^q(X|D, n)\) can be represented by a cycle \(\alpha\) such that \(\partial_i^\infty(\alpha) = 0\) for all \(1 \leq i \leq n\) and \(\epsilon = 0, \infty\).

Its proof follows the outline for additive higher Chow groups in [13] Appendix A with suitable modifications. The essential point is to show that \((y_2 \mapsto z^q(X|D; n))\) is an “extended cubical object” in the sense of [24] §1. In Section 3 we suppose \((X, D)\) is as in Theorem 3.2.

Let \(q_1 : \square \to \square\) be the morphism \((y_1, y_2) \mapsto y_1 + y_2 - y_1 y_2\) if \(y_1, y_2 \neq \infty\), and \((y_1, y_2) \mapsto \infty\) if \(y_1\) or \(y_2 = \infty\). Under the identification \(\psi : \square \simeq \mathbb{A}^1\) given by \(y \mapsto 1/(1 - y)\) (which sends \(\{\infty, 0\}\) to \(\{0, 1\}\)), this map \(q_1\) is equivalent to \(q_{1,\psi} : \mathbb{A}^2 \to \mathbb{A}^1\) given by \((y_1, y_2) \mapsto y_1 y_2\). For our convenience, we use this \(\square_\psi := (\mathbb{A}^1, \{0, 1\})\) and cycles on \(X \times \square_\psi^n\). The boundary operator is \(\partial = \sum_{i=1}^n (-1)^i (\partial^q_i - \partial^\infty_i)\), and we replace \(F^q_{i,n}\) by \(F^\infty_{i,n} := \{y_i = \infty\}\). We write \(F^\infty_n = \sum_{i=1}^n F^\infty_{i,n}\). We write \(\square_\psi = (\mathbb{P}^1, \{0, 1\})\). The group of admissible cycles is \(z^q_\psi(X|D, n)\). Consider \(q_{n,\psi} : X \times \square_\psi^{n+1} \to X \times \square_\psi^n\) given by \((x, y_1, \ldots, y_{n+1}) \mapsto (x, y_1, \ldots, y_{n-1}, y_n y_{n+1})\).

Proposition 3.3. For \(Z \in z^q_\psi(X|D, n)\), we have \(q_{n,\psi*} \in z^q_\psi(X|D, n+1)\).

The delicacy of its proof lies in the fact that the product map \(q_{1,\psi} : \mathbb{A}^2 \to \mathbb{A}^1\) does not extend to a morphism \((\mathbb{P}^1)^2 \to \mathbb{P}^1\) of varieties. Instead we take a different path. For \(n \geq 1\), let \(i_n : W_n \hookrightarrow X \times \square_\psi^{n+1} \times \square_\psi^1\) be the closed subscheme defined by the equation \(u_0 y_n y_{n+1} = u_1\), where \((y_1, \ldots, y_{n+1}) \in \square_\psi^{n+1}\) and \((u_0; u_1) \in \square_\psi^1\) are the coordinates. Let \(y := u_1/u_0\). Its Zariski closure \(\overline{W}_n \hookrightarrow X \times \square_\psi^{n+1} \times \square_\psi^1\) is given by the equation \(u_0 u_{n,1} u_{n+1,1} = u_1 u_n 0 u_{n+1,0}\), where \((u_{1,0}, u_{1,1}), \ldots, (u_{n+1,0}, u_{n+1,1})\) are the homogeneous coordinates of \(\square_\psi^{n+1}\) with \(y_i = u_{i,1}/u_{i,0}\).

Consider \(\theta_n : X \times \square_\psi^{n+1} \times \square_\psi^1 \to X \times \square_\psi^n\) given by \((x, y_1, \ldots, y_{n+1}, (u_0; u_1)) \mapsto (x, y_1, \ldots, y_{n-1}, y_n y_{n+1})\), and let \(\pi_n := \theta_n|_{\overline{W}_n}\). To extend this \(\pi_n\) to a morphism \(\overline{\pi}_n\) on \(\overline{W}_n\), we use the projection \(\overline{\theta}_n : X \times \square_\psi^{n+1} \times \square_\psi^1 \to X \times \square_\psi^{n-1} \times \square_\psi^1\), that drops the coordinates \((u_0; u_{n,1})\) and \((u_{n+1,0}; u_{n+1,1})\), and the projection \(p_n : X \times \square_\psi^{n+1} \times \square_\psi^1 \to X \times \square_\psi^{n+1}\), that drops the last coordinate \((u_0; u_1)\).

Lemma 3.4. (1) \(W_n \cap \{u_0 = 0\} = \emptyset\), so that \(W_n \subset X \times \square_\psi^{n+1} \times \square_\psi^1\). (2) \(\overline{\theta}_n|_{\overline{W}_n} = \pi_n\). Thus, we define \(\overline{\pi}_n := \overline{\theta}_n|_{\overline{W}_n}\), which extends \(\pi_n\). (3) The varieties \(W_n\) and \(\overline{W}_n\) are smooth. (4) Both \(\pi_n\) and \(\overline{\pi}_n\) are surjective flat morphisms of relative dimension 1.

Proof. Its proof is almost identical to that of [13] Lemma A.5. Part (1) follows from the defining equation of \(W_n\), and (2) holds by definition. Let \(\rho_n := p_n|_{W_n} : W_n \to X \times \square_\psi^{n+1}\). Since \(X\) is smooth, using Jacobian criterion we check that \(W_n\) is smooth. Furthermore, \(\rho_n\) is an isomorphism with the obvious inverse. Under this identification, the morphism \(\pi_n\) can also be regarded as the projection \((x, y_1, \ldots, y_n, y) \mapsto (x, y_1, \ldots, y_{n-1}, y)\) that drops \(y_n\). In particular, \(\pi_n\) is a smooth
and surjective of relative dimension 1. To check that $\overline{W}_n$ is smooth, one can check it locally on each open set where $u_{n,i}, u_{n+1,i}, u_i$ is nonzero for $i = 0, 1$. In each such open set, the equation for $\overline{W}_n$ takes the same form as for $W_n$, so that it is smooth again by Jacobian criterion. Similarly as for $\pi_n$, one sees $\overline{\pi}_n$ is of relative dimension 1. Since $\overline{\theta}_n$ is projective and $\pi_n$ is surjective, the morphism $\overline{\pi}_n$ is projective and surjective. So, since $\overline{W}_n$ is smooth, the map $\overline{\pi}_n$ is flat by [9 Exercise III-10.9]. Thus, we have (3) and (4).

**Lemma 3.5.** Let $n \geq 1$. Let $Z \subset X \times \square_n^\psi$ be a closed subscheme with modulus $D$. Then, $Z' := (i_n)_*(\pi_n^*(Z))$ also has modulus $D$.

**Proof.** Let $\overline{Z}$ and $\overline{Z}'$ be the Zariski closures of $Z$ and $Z'$ in $X \times \square_n^\psi$ and $X \times \square_{n+1}^\psi$, respectively. By Lemma 3.4 and the projectivity of $\overline{\theta}_n$, we see that $\overline{\theta}_n(\overline{Z}') = \overline{Z}$. Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Z}'^N & \xrightarrow{\nu_{Z'}} & \mathcal{Z}\times\mathcal{W}_n^+ \times \square_{n+1}^\psi \\
\downarrow f & & \downarrow \overline{\pi}_n \\
\mathcal{Z}^N & \xrightarrow{\nu_Z} & \mathcal{Z}\times\mathcal{W}_n^+ \times \square_{n}^\psi,
\end{array}
\]

where $f$ is induced by the surjection $\overline{\theta}_n|_{\mathcal{Z}'} : \mathcal{Z}' \to \mathcal{Z}$, the maps $g$ and $\nu_Z$ are normalizations of $\mathcal{Z}'$ and $\mathcal{Z}$ composed with the closed immersions, and $\nu_{Z'} = \overline{g} \circ g$.

By the definition of $\overline{\theta}_n$, we have $\overline{\theta}_n(D \times \square^i) = D \times \square_n^\psi \times \square_1^\psi$, $\overline{\theta}_n(F_{n,n}) = F_{n+2,n+2}$, while $\overline{\theta}_n(F_{n,n,i}) = F_{n+2,i}$ for $1 \leq i \leq n - 1$. By the defining equation of $\overline{W}_n$, note we have $\overline{\pi}_n|_{\mathcal{Z}'} = \overline{\tau}_n|_{\mathcal{Z}'} F_{n,n+2} + \overline{\tau}_n^0(n(u_{n,0} = 0) \leq \overline{\tau}_n^0(\{u_{n,0} = 0\} + \{u_{n,1,0} = 0\}) = \overline{\tau}_n^0(F_{n+2} + F_{n+2,n+2}).$

First suppose $n \geq 2$. Then, $\nu_{Z'}^* \overline{\theta}_n^* \sum_{i=1}^{n+1} F_{n,i} = \sum_{i=1}^{n-1} \nu_{Z'}^* \overline{\tau}_n^0 F_{n+2,i}^0 + g^* \overline{\tau}_n^0 F_{n,n} \leq \sum_{i=1}^{n+1} \nu_{Z'}^* \overline{\tau}_n^0 F_{n+2,i}^0 + g^* \overline{\tau}_n^0 (F_{n+2} + F_{n+2,n+1}) = \sum_{i=1}^{n+1} \nu_{Z'}^* F_{n+2,i}^0 \leq \sum_{i=1}^{n+1} \nu_{Z'}^* F_{n+2,i}^0.$ In case $n = 1$, we just ignore $\sum_{i=1}^{n-1}$ in the above.

That $Z$ has modulus $D$ means $\nu_{Z'}^*(D \times \square^1) \leq \sum_{i=1}^{n+1} \nu_{Z'}^* F_{n,i}$. Applying $f^*$ and using the commutativity of the above diagram, we have $\nu_{Z'}^* \overline{\theta}_n^* (D \times \square^1) \leq \nu_{Z'}^* \overline{\tau}_n^0 \sum_{i=1}^{n-1} F_{n,i}$, which is bounded by $\sum_{i=1}^{n+1} \nu_{Z'}^* F_{n,i}$ as we saw in the above paragraph. This means $Z'$ has modulus $D$.

**Definition 3.6.** For any closed subscheme $Z \subset X \times \square_n^\psi$, we define $W_n(Z) := p_{n,i}^* i_n^* \pi_n^*(Z)$, which is closed in $X \times \square_{n+1}^\psi$.

**Lemma 3.7.** Let $n \geq 1$. If a closed subscheme $Z \subset X \times \square_n^\psi$ intersects all faces properly, then $W_n(Z)$ intersects all faces of $X \times \square_{n+1}^\psi$ properly.

**Proof.** Our $W_n$ is equal to $\tau^* \tau_n^* \tau_{n+1}^* W_X$, where $W_X$ is that of [22 Lemma 4.1], and $\tau, \tau_n, \tau_{n+1}$ are the involutions for $y, y_n, y_{n+1}$, respectively. So, the lemma is a special case of *loc.cit.*
Proof of Proposition 3.3. Consider the commutative diagram

\[
\begin{array}{ccc}
W_n & \xrightarrow{i_n} & X \times \square_{n+1}^1 \times \square_{\psi}^1 \\
\pi_n & \downarrow & \rho_n = p_n|_{W_n} \\
X \times \square_n^1 & \xleftarrow{q_n,\psi} & X \times \square_{n+1}^1.
\end{array}
\]

Recall by Lemma 3.4, \(\rho_n\) is an isomorphism so that \(\rho_{n+1}^*p_n^* = Id\). Hence, \(q_n^*(\psi) = \rho_{n+1}^*\pi_n^*(Z) = p_{n+1}^*\pi_n^*(Z) = W_n(Z)\), where \(\delta, \delta^\perp\) are due to commutativity. So, we are reduced to show that \(W_n(Z) \in Z_{\psi}^q(X|D, n + 1)\). But, by Lemmas 3.5, 3.7 we have \(i_n^*\pi_n^*(Z) \in Z_{\psi}^q(X \times \mathbb{P}^1|D \times \mathbb{P}^1, n + 1)\). Now, for the projection \(p_n\), by Lemma 2.9, we have \(p_{n+1}^*\pi_n^*(Z) = W_n(Z) \in z_{\psi}^q(X|D, n + 1)\). This proves Proposition 3.3. □

Proof of Theorem 3.3. By \[24\] Lemma 1.6, it is enough to check that \((n \mapsto z^q(X|D, n))\) is an extended cubical abelian group, i.e., it remains to check that for the morphism \(q_1 : \square^2 \to \square^1\) previously defined, and \(Z \in z^q(X|D, 1)\), we have \(q_1^*(Z) \in z^q(X|D, 2)\). Under the isomorphism \(\psi : \square \cong A^1, y \mapsto y/(1 - y)\), this is equivalent to show that \(q_1^*\psi\) sends admissible cycles to admissible cycles, which we know by Proposition 3.3. □

4. On moving lemmas

In this section, we recall some of moving lemmas on algebraic cycles with modulus conditions. By saying ‘moving lemma’, we ask whether the inclusion \(z^q_{\psi}(Y|D, \bullet) \subset z^q(Y|D, \bullet)\) in Definition 2.6 is a quasi-isomorphism. It is known when \(Y\) is smooth quasi-projective and \(D = 0\) (by \[5\]), when \(Y = X \times A^1\), with \(X\) smooth projective, \(D = X \times \{t^{m+1} = 0\}\), and \(W\) consists of \(W \times A^1\) for finitely many locally closed subsets \(W \subset Y\) (by \[13\]). Recently, W. Kai \[13\] proved it when \(Y\) is smooth affine with a suitable condition on \(D\). Kai’s case includes the above case of \(Y = X \times A^1\), where \(X\) is this time smooth affine.

In Section 4.1, we sketch the argument of Kai in the case of multivariate additive higher Chow groups of smooth affine \(k\)-variety. In Section 4.2 we generalize the moving lemma of \[16\] in the case of pairs \((X \times S, X \times D)\) where \(X\) is smooth projective. In Sections 4.3 and 4.4 we discuss the standard pull-back property and its consequences. In Section 4.5, we discuss a moving lemma for additive higher Chow groups smooth semi-local \(k\)-schemes essentially of finite type.

4.1. Kai’s affine method for multivariate additive higher Chow groups.

The moving lemma of W. Kai \[13\] is the first moving result that applies to cycles groups with a nonzero modulus over a smooth affine scheme. We sketch the proof of the following special case on multivariate additive higher Chow groups that we need, which is a bit simpler than the general case proven by him. Following Definition 2.5, we write \(X[r] := X \times A^r\), for instance.

Theorem 4.1 (W. Kai). Let \(X\) be a smooth affine variety over any field \(k\). Let \(W\) be a finite set of locally closed subsets of \(X\). Let \(W[r] := \{W[r] | W \in W\}\). Let \(m_1, \ldots, m_r \geq 1\) be integers. Then, the inclusion \(z^q_{W[r]}(X[r]|D_{m}, \bullet) \hookrightarrow z^q(X[r]|D_{m}, \bullet)\) is a quasi-isomorphism.
First recall some preparatory results:

**Lemma 4.2** ([16 Lemma 4.5]). Let \( f : X \rightarrow Y \) be a dominant morphism of normal varieties. Suppose that \( Y \) is integral and let \( \eta \) be the generic point of \( Y \). Let \( X_\eta \) be the fiber over \( \eta \) and let \( j_\eta : X_\eta \rightarrow X \) be the inclusion.

Let \( D \) be a Weil divisor on \( X \) such that \( j_\eta^*(D) \geq 0 \). Then, there exists a nonempty open subset \( U \subset Y \) such that \( j_U^*(D) \geq 0 \), where \( j_U : f^{-1}(U) \rightarrow X \) is the inclusion.

The following generalizes [16 Proposition 4.7]:

**Proposition 4.3** (Spreading lemma). Let \( k \subset K \) be a purely transcendental extension. Let \((X,D)\) be a smooth quasi-projective \(k\)-scheme with an effective Cartier divisor, and let \( \mathcal{W} \) be a finite collection of locally closed algebraic subsets of \( X \) disjoint from \( D \). Let \((X_K,D_K)\) and \( \mathcal{W}_K \) be the base change via \( \text{Spec}(K) \rightarrow \text{Spec}(k) \). Let \( p_K : X_K \rightarrow X_k \) be the base change map. Then, the pull-back map

\[
p_K^* : z^q(X|D, \bullet) \rightarrow z^q(X_K|D_K, \bullet)
\]

is injective on homology.

**Proof.** Most arguments of [16 Proposition 4.7] work without change, so we sketch the proof. If \( k \) is finite, then we can use the the pro-\( \ell \)-extension argument as in loc.cit. so that we reduce the proof of Proposition 4.3 to the case when \( k \) is infinite, which we assume from now. We may also assume that \( \text{tr.deg}_kK < \infty \) and furthermore that \( \text{tr.deg}_kK = 1 \), by induction. So, we have \( K = k(\mathbb{A}^1_k) \).

Suppose \( Z \in z^q(X|D,n) \) satisfies \( \partial Z \in z^0_\mathcal{W}(X|D,n-1) \) and \( Z_K = \partial(B_K) + V_K \) for some \( B_K \in z^q(X_K|D_K,n+1) \) and \( V_K \in z^0_{\mathcal{W}_K}(X_K|D_K,n) \). We may assume \( B_K \) and \( V_K \neq 0 \) for otherwise the proposition is trivial. Consider the inclusion \( z^q(X_K|D_K, \bullet) \rightarrow z^q(X, \bullet) \). Then, there is a nonempty open \( U' \subset \mathbb{A}^1_k \) such that \( B_K = B_{U'K}, V_K = V_{U'K} \), \( Z \subset U' = \partial(B_{U'}) + V_{U'} \) for some \( B_{U'} \in z^q(X \times U', n+1) \), \( V_{U'} \in z^0_{\mathcal{W}_XU'}(X \times U', n) \), where \( \eta \) is the generic point of \( U' \). Let \( j_\eta : X \times \eta \rightarrow X \times U' \) be the inclusion, which is flat.

Since \( B_K, V_K \) satisfy the modulus condition, we have \( j_\eta^*(X \times U' \times F_{n+1}^1 - D \times U' \times \mathbb{A}^{n+1}) \geq 0 \) on \( B_K^N \), and similarly for \( V_K^N \). Furthermore \( B_{U'}^N \rightarrow U', V_{U'}^N \rightarrow U' \) are dominant. Thus, by Lemma 4.2 there is a nonempty open \( U \subset U' \) such that \( j_{U}^*(X \times U' \times F_{n+1}^1 - D \times U' \times \mathbb{A}^{n+1}) \geq 0 \) on \( B_U^N \) and similarly for \( V_U^N \) for \( j_U : X \times U \rightarrow X \times U' \), which proves that \( B_U \) and \( V_U \) have modulus \( D \subset U \). Hence, \( B_U \in z^q(X \times U|D \times U, n+1) \) and \( V_U \in z^0_{\mathcal{W}_XU}(X \times U|D \times U, n) \), with \( Z \subset U = \partial(B_U) + V_U \). Since \( k \) is infinite, there exists \( u \in U(k) \). Consider \( i_u : X \times \{u\} \rightarrow X \times U \). Here, we have \( i^*_u(B_U) \in z^q(X, n+1) \) and \( i^*_u(V_U) \in z^0_{\mathcal{W}_X}(X, n) \) with \( Z = \partial(i^*_u(B_U)) + i^*_u(V_U) \). The pull-backs \( i^*_u \) exists for \( X \) and \( X \times U \) are smooth. Finally, by the containment lemma (Proposition 2.2), \( i^*_u(B_U) \) and \( i^*_u(V_U) \) have modulus \( D \). Hence, \( i^*_u(B_U) \in z^q(X|D, n+1) \) and \( i^*_u(V_U) \in z^0_{\mathcal{W}}(X|D, n) \). This finishes the proof. 

**Sketch of the proof of Theorem 4.4.** Step 1. We first show it when \( X = \mathbb{A}^d \). Let \( K = k(\mathbb{A}^d_k) \) and let \( \eta \in X \) be the generic point. For simplicity, using a suitable automorphism of \( \mathbb{P}^1 \), we may assume \( \Box = \mathbb{A}^1 \), with \{0,1\} as faces. For any \( g \in \mathbb{A}^d \) and an integer \( s > 0 \), define \( \phi_{g,s} : \mathbb{A}^d[r] \times \mathbb{A}^1 \rightarrow \mathbb{A}^d[r] \) by \( \phi_{g,s}(z,t,y) := (z + yt^{m_1}, \ldots, t^{m_r}g, t) \). (N.B. In terms of W. Kai’s homotopy, our \((g,0,\ldots,0) \in \mathbb{A}^d[r] \)
is a special case of his $v \in \mathbb{A}^{d+r}$. For any cycle $V \in z^q(X[r]|D_{m,n})$, we define $H^*_{g,s}(V) := (\phi_{g,s} \times \text{id}_{\mathbb{A}^d})^*(V)$.

When $g = \eta$, using [4, Lemma 1.2], one checks that $H^*_{\eta,s}(V)$ intersects with all $W[r] \times F$ properly, where $W \in \mathcal{W}$ and $F \subset \mathbb{A}^n$ is a face. (This is slightly different from that of [13, Lemma 3.2], but the same argument works.)

On the other hand, for each irreducible $V \in z^q(X[r]|D_{m,n})$, there is an integer $s(V)$ (let’s call it the threshold of $V$ for simplicity), such that for any $g \in \mathbb{A}^d$, the cycle $H^*_{g,s}(V)$ has modulus $D_m$ whenever $s > s(V)$. ([13, Proposition 3.4]; This is the most important contribution of W. Kai’s moving lemma.)

So, as in [13, Section 3.1.2], if we consider the subgroup $z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n}) \leq s \subset z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n})$, consisting of cycles all of whose irreducible components $V$ have threshold $s(V) \leq s$, then,

$$
\frac{z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n}) \leq s}{z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n}) \leq s} = \lim_{\to \infty} \frac{z^q_{\mathcal{W}[r]r,e}(X|D_{m,n}) \leq s}{z^q_{\mathcal{W}[r]r,e}(X|D_{m,n}) \leq s},
$$

Then, one has the map

$$
H^*_{\eta,s} : \frac{z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n}) \leq s}{z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n}) \leq s} \to \frac{z^q_{\mathcal{W}[r]r,e}(X_{K[r]|D_{m,n}}+1)}{z^q_{\mathcal{W}[r]r,e}(X_{K[r]|D_{m,n}}+1)},
$$

which gives a homotopy between the base change $p^*_{K/k}$ and $H^*_{\eta,s}|_{y_1=1}$. However, $H^*_{\eta,s}|_{y_1=1}$ is zero on the quotient, while $p^*_{K/k}$ is injective on homology by Proposition [13] after taking $s \to \infty$, so that the map $p^*_{K/k}$ is in fact zero on homology. This means, the quotient $z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n})/z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n})$ is acyclic, proving the moving lemma for $X = \mathbb{A}^d$.

**Step 2.** Now, suppose $X$ is a general smooth affine $k$-variety. We use a standard generic projection trick. Choose a closed immersion $X \hookrightarrow \mathbb{A}^N$ for some $N > d$. Let $\overline{X}$ be the closure of $X$ in $\mathbb{P}^N \supset \mathbb{A}^N$. Let $P := \mathbb{P}^N \setminus \mathbb{A}^N$ and let $\overline{X} \cap P$. For any linear closed subscheme $L \subset \mathbb{P}^N$ of dimension $N - d - 1$, we have the projection $\pi_L : \mathbb{P}^N \setminus L \to \mathbb{P}^d$. In case $L \subset P$, the projection $\pi_L$ gives the affine linear map $\pi_L^0 : \mathbb{A}^N \setminus \mathbb{A}^d$. The restriction $\pi_{L,X} := \pi_L^0|_X : X \to \mathbb{A}^d$ is finite if and only if $L \cap \overline{X} = \emptyset$. There is a nonempty open subset $U_X$ of the Grassmannian $\text{Gr}(N - d - 1,N - 1) = \text{Gr}(N - d - 1,P)$ that parameterizes all $L$ of dimension $N - d - 1$ in the projective space $P$ of dimension $N - 1$, such that $L \cap \overline{X} = \emptyset$. For such $L \in U_X$, the morphism $\pi_{L,X}$ is also flat by [1] Exercise III-10.9, p.276]. Hence both $\pi_{L,X}$ and $\pi_{L,X}^*$ are defined. For any closed irreducible admissible cycle $Z$ in $X[r] \times \mathbb{A}^n$, define the residual cycle by $L^*(Z) := \pi_{L,X}^*(\pi_{L,X}^*([Z])) - [Z]$. Extending this map linearly, we get a morphism of complexes $L^* : z^q(\pi_{L,X}^*|D_{m,n}) \to z^q(X[r]|D_{m,n})$. But, by Chow’s moving lemma technique (cf. [15, Lemma 1.3, p.84] [16, Lemma 6.3], [23, Lemma 3.5.4]), for the generic point $L_{gen} \in G_X$, the map $L_{gen}^* \circ \pi_{L,X}^*$ maps $z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n})$ into $z^q_{\mathcal{W}[r]r,e-1}(X_{K[r]|D_{m,n}})$ for each set function $e : \mathcal{W}[r] \to \mathbb{Z}_{\geq 0}$, where $(e - 1)(W[r]) := \max\{e(W[r]) - 1, 0\}$. Hence, the chain map $L_{gen}^* \circ \pi_{L,X}^* - p_{K/k}^* : z^q_{\mathcal{W}[r]r,e}(X[r]|D_{m,n})/z^q_{\mathcal{W}[r]r,e-1}(X[r]|D_{m,n}) \to z^q_{\mathcal{W}[r]r,e}(X_{K[r]|D_{m,n}})/z^q_{\mathcal{W}[r]r,e-1}(X_{K[r]|D_{m,n}})$ is zero, i.e. $L_{gen}^* \circ \pi_{L,X}^*$ factors through the quotient complex $z^q_{\mathcal{W}[r]r,e}(X_{K[r]|D_{m,n}})/z^q_{\mathcal{W}[r]r,e-1}(X_{K[r]|D_{m,n}})$ for some finite set $\mathcal{W}'$ of algebraic
4.2. Projective method for multivariate additive higher Chow groups.

Theorem 4.4. Let \((S, D)\) be a smooth quasi-projective \(k\)-variety with an effective Cartier divisor. Let \(X \subseteq k\) be a smooth projective \(k\)-variety. Let \(W \in \text{finite collection of locally closed subsets of } X\). Then, the inclusion \(z^q(W \times S | X \times D, \bullet) \hookrightarrow z^q(X \times S | X \times D, \bullet)\) is a quasi-isomorphism. In particular, with \((S, D) = (X, D_\text{m})\), the moving lemma holds for multivariate additive higher Chow groups of smooth projective varieties over \(k\).

Proof. Step 1. We first prove the moving lemma when \(X = \mathbb{P}^d\). The algebraic group \(SL_{d+1,k}\) acts on \(\mathbb{P}^d\). Let \(K = k(SL_{d+1,k})\). Then, there is a \(K\)-morphism \(\phi: \square_k \to SL_{d+1,k}\) such that \(\phi(0) = \text{Id}\), and \(\phi(\infty) = \eta\), where \(\eta\) is the generic point. See \([16, \text{Lemma 5}]\). For such \(\phi\), consider the composition \(H_n\) of morphisms

\[
\mathbb{P}^d \times S \times \square_k^{n+1} \xrightarrow{\mu_\phi} \mathbb{P}^d \times S \times \square_k^n \xrightarrow{pr'_K} \mathbb{P}^d \times S \times \square_K^n \xrightarrow{\phi_n} \mathbb{P}^d \times S \times \square_k^n,
\]

where \(\mu_\phi(x, s, y_1, \ldots, y_{n+1}) = (\phi(y_1) x, s, y_1, \ldots, y_{n+1})\), and \(pr'_K\) is the projection dropping \(y_1\), and \(\phi_n\) is the base change. We claim that \(H_n\) carries \(z^q_{W \times S_k}(\mathbb{P}^d \times S | P \times D, n)\) to \(z^q_{W \times S_k}(\mathbb{P}^d \times S | P \times D, n+1)\), i.e. for an irreducible cycle \(Z \in z^q_{W \times S_k}(\mathbb{P}^d \times S | P \times D, n)\), we show that \(Z' := H_n(Z) \in z^q_{W \times S_k}(\mathbb{P}^d \times S | P \times D, n+1)\).

To do so, we first claim that \(Z'\) intersects with \(W \times S \times F_k\) properly for each \(W \in \mathcal{W}\) and each face \(F \subseteq \square_k^n\).

1. In case \(F = \{0\} \times F'\) for some face \(F' \subseteq \square^n\), because \(\phi(0) = \text{Id}\), we have \(Z' \cap (W \times S \times F_k) \simeq Z_k \cap (W \times S \times F_k)\). Note that \(\dim(W \times S \times F_k) = \dim(W \times S \times F_k)\). Hence, \(\text{codim}_{W \times S \times F_k}(Z' \cap (W \times S \times F_k)) = \dim(W \times S \times F_k)\) and \(\text{codim}_{W \times S \times F_k}(Z \cap (W \times S \times F_k)) = \dim(W \times S \times F_k)\). Since \(\dim(W \times S \times F_k) = 0\), we get \(\text{codim}_{W \times S \times F_k}(Z' \cap (W \times S \times F_k)) = 0\).

2. In case \(F = \{\infty\} \times F'\) for some face \(F' \subseteq \square^n\), \(\dim(W \times S \times F_k) = \dim(W \times S \times F_k)\) and \(Z' \cap (W \times S \times F_k) \simeq Z_k \cap (W \times S \times F_k)\), where \(\eta\) is the generic point of \(SL_{d+1,k}\), and \(\text{codim}_{W \times S \times F_k}(Z' \cap (W \times S \times F_k)) = \dim(W \times S \times F_k)\).

3. In case \(F = \square \times F'\) for some face \(F' \subseteq \square^n\), the projection \(Z' \cap (W \times S \times \square \times F_k) \to \square_k\) is flat, being a dominant map to a curve, so \(\dim(Z' \cap (W \times S \times \square \times F_k)) = 0\).

Thus, for all \(F \subseteq \square_k^n\), \(\text{codim}_{W \times S \times F_k}(Z' \cap (W \times S \times F_k)) \geq 0\) and \(\text{codim}_{W \times S \times F_k}(Z \cap (W \times S \times F_k)) = 0\). Hence, \(\text{codim}_{W \times S \times F_k}(Z' \cap (W \times S \times F_k)) \geq 0\).
Now we show that $Z'$ has modulus $\mathbb{P}^d \times D$. We drop automorphisms $\tau : \mathbb{P}^d \times S \times \mathbb{P}^{n+1} \rightarrow \mathbb{P}^d \times S \times \mathbb{P}^{n+1}$ that exchange the factors. For $p : \mathbb{P}^d \rightarrow \text{Spec}(k)$, we take $V = p(Z)$ on $S \times \Delta^n$. Because $Z \subset p^{-1}(p(Z)) = \mathbb{P}^r \times V$, we have $Z' = \mu_p^*(\mathbb{P}^1 \times Z) \subset \mu_p^*(\mathbb{P}^d \times \mathbb{P}^1 \times V) = \mathbb{P}^d \times \mathbb{P}^1 \times V := Z_1$. By Lemma 2.9, $V$ is admissible on $S \times \Delta^n$. So, $p^*[V] = \mathbb{P}^d \times V$ is admissible on $S \times \Delta^n$. In particular, $\mathbb{P}^d \times V$ has modulus $\mathbb{P}^d \times D$. Hence, $Z_1 = \mathbb{P}^d \times \mathbb{P}^1 \times V$ also has modulus $\mathbb{P}^d \times D$. Now, $Z' \subset Z_1$ shows that $Z'$ has modulus $\mathbb{P}^d \times D$ by Proposition 2.2.

Thus, we proved $Z' \in z^q_{\mathcal{W} \times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, n + 1)$.

Going back to the proof, one checks that $H^q_{\bullet} : z^q(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot) \rightarrow z^q(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot + 1)$ is a chain homotopy satisfying $\partial H^* + H^* \partial(Z) = Z_K - \eta \cdot (Z_K)$, and the same holds for $z_{\mathcal{W} \times S}$ by a straightforward computation (cf. [16, Lemma 5.6]). Furthermore, for each admissible $Z$, we have $\eta \cdot Z_K \in z^q_{\mathcal{W} \times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, n)$, by the above proof of proper intersection of $Z'$ with $W \times S \times F_K$, where $F = \{\infty\} \times F'$ for a face $F' \subset \Delta^n$. Hence, the base change $p^*_{K/k} : z^q(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot) \rightarrow z^q(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot) / z^q_{\mathcal{W} \times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot)$ is homotopic to $\eta \cdot p^*_{K/k}$, which is zero on the quotient. That is, $p^*_{K/k}$ on the above quotient complex is zero on homology. However, by the spreading argument (Proposition 4.3), $p^*_{K/k}$ is injective on homology. Hence, the quotient complex $z^q(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot) / z^q_{\mathcal{W} \times S}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot)$ is acyclic, i.e., the moving lemma holds for $(\mathbb{P}^d \times S|\mathbb{P}^d \times D)$, finishing Step 1.

**Step 2.** Now let $X$ be a general smooth projective variety of dimension $d$. We use a standard generic projection argument again. Let $X \hookrightarrow \mathbb{P}^N$ be an embedding for some $N > d$. For each linear closed subscheme $L \subset \mathbb{P}^N$ of dimension $N - d - 1$, there is a linear projection $\pi_L : \mathbb{P}^N \setminus L \rightarrow \mathbb{P}^d$. Whenever $L \cap X = \emptyset$, it induces a finite morphism $\pi_{L,X} := \pi_L|X : X \rightarrow \mathbb{P}^d$. These $L$ form a nonempty open subset $G_X \subset \text{Gr}(N - d - 1, N)$ of the Grassmannian, and $\pi_{L,X}$ are actually flat as well because $X$ is smooth ([11, Exercise III-10.9]). In particular, $\pi_{L,X,*}$ and $\pi_{L,X}$ are both defined. For any admissible cycle $Z \in z^q(X \times S|X \times D, n)$ and $L \in G_X$, define $\tilde{L}(Z) := \pi_{L,X,*}([Z]) - [Z]$. It gives a chain map $\tilde{L} : z^q(X \times S|X \times D, \cdot) \rightarrow z^q(X \times S|X \times D, \cdot)$. By Chow’s moving lemma technique (cf. [16, Lemma 6.3], [15, Lemma 1.13, p.84]), for the generic point $L_{\text{gen}} \in G_X$, the map $\tilde{L}_{\text{gen}}$ maps $z^q_{\mathcal{W} \times S,e}(X \times S|X \times D, \cdot)$ into $z^q_{\mathcal{W} \times S,e^{-1}}(X_K \times S|X \times D, \cdot)$ for each set function $e : \mathcal{W} \times S \rightarrow \mathbb{Z}_{\geq 0}$, where $(e - 1)(\mathcal{W} \times S) := \max(e(W \times S) - 1, 0)$. Hence, the chain map $\tilde{L}_{\text{gen}} = \pi_{L_{\text{gen}}}^* \circ \pi_{L_{\text{gen}},*} - p^*_{K/k} : z^q_{\mathcal{W} \times S,e}(X \times S|X \times D, \cdot) / z^q_{\mathcal{W} \times S,e^{-1}}(X \times S|X \times D, \cdot) \rightarrow z^q_{\mathcal{W} \times S,e^{-1}}(X_K \times S|X \times D, \cdot) / z^q_{\mathcal{W} \times S,e^{-1}}(X_K \times S|X \times D, \cdot)$ is zero, i.e., $\pi_{L_{\text{gen}}} \circ \pi_{L_{\text{gen}},*}$ is just the base change $p^*_{K/k}$. However, $\pi_{L_{\text{gen}}} \circ \pi_{L_{\text{gen}},*}$ factors through the quotient complex $z^q_{\mathcal{W} \times S,e^{-1}}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot) / z^q_{\mathcal{W} \times S,e^{-1}}(\mathbb{P}^d \times S|\mathbb{P}^d \times D, \cdot)$ for some finite set $\mathcal{W}'$ of algebraic subsets of $\mathbb{P}^d$ and a set function $e' : \mathcal{W}' \times S \rightarrow \mathbb{Z}_{\geq 0}$ (cf. [16, 6C]). By Step 1, this quotient complex is acyclic. So, $\pi_{L_{\text{gen}}} \circ \pi_{L_{\text{gen}},*}$ is zero on homology, and so is $p^*_{K/k}$. Thus, $p^*_{K/k} : z^q(X \times S|X \times D, \cdot) / z^q_{\mathcal{W} \times S}(X \times S|X \times D, \cdot)$
is also zero on homology. But, by Proposition 4.3, $p^*_{k/k}$ is injective on homology, so the quotient $z^q(X \times S|X \times D,\bullet)/z^q_{W \times S}(X \times S|X \times D,\bullet)$ is acyclic. Thus, the moving lemma holds for $(X \times S, X \times D)$, as desired. □

4.3. Contravariant functoriality.

**Theorem 4.5.** Let $f : X \to Y$ be a morphism of $k$-varieties, with $Y$ smooth affine or smooth projective. Let $r, m_1, \cdots, m_r \geq 1$ be integers. Then, there exists a pull-back $f^* : CH^q(Y[r]|D_{m_r}, n) \to CH^q(X[r]|D_{m_r}, n)$.

**Proof.** This is a standard application of the moving lemma, so we sketch the proof. There is a finite set $W$ of locally closed subsets of $Y$ such that $f^* : z^q_{W[r]}(Y[r]|D_{m_r}, \bullet) \to z^q(X[r]|D_{m_r}, \bullet)$ is given by taking the inverse images. This gives a zig-zag of chain maps $z^q(Y[r]|D_{m_r}, \bullet) \xleftarrow{q_{iso}} z^q_{W[r]}(Y[r]|D_{m_r}, \bullet) f^* \to z^q(X[r]|D_{m_r}, \bullet)$, which defines a morphism $f^* : z^q(Y[r]|D_{m_r}, \bullet) \to z^q(X[r]|D_{m_r}, \bullet)$ in $D(\text{Ab})$. This gives the pull-back map by taking homology. □

**Remark 4.6.** As a special case, when $r = 1$, we have the pull-back map $f^* : TCH^q(Y, n;m) \to TCH^q(X, n;m)$.

4.4. The presheaf $TCH$. For the rest of the section, we concentrate on additive higher Chow groups. By Theorem 4.5, we see that $T^q_{n,m} := TCH^q(\textminus, n;m)$ is a presheaf of abelian groups on the category $\text{SmAff}_k$, but we do not know if it is a presheaf on the categories $\text{Sm}_k$ or $\text{Sch}_k$. However, we can exploit Theorem 4.5 further to define a new presheaf on $\text{Sm}_k$ and $\text{Sch}_k$. The idea of using this detour occurred to the authors while they were working on [19].

Let $X \in \text{Sch}_k$. Let $(X \downarrow \text{SmAff}_k)$ be the category whose objects are the k-morphisms $X \to A$, with $A \in \text{SmAff}_k$, and a morphism from $h_1 : X \to A$ to $h_2 : X \to B$, with $A, B \in \text{SmAff}_k$ is given by a k-morphism $g : A \to B$ such that $g \circ h_1 = h_2$. One immediately checks that $(X \downarrow \text{SmAff}_k)$ is cofiltered.

The functor $T^q_{n,m} : \text{SmAff}_k^{\text{op}} \to (\text{Ab})$ induces the functor $(X \downarrow \text{SmAff}_k)^{\text{op}} \to (\text{Ab})$, $(X \to A) \mapsto T^q_{n,m}(A)$, also denoted by $T^q_{n,m}$. Here $(\text{Ab})$ is the category of abelian groups.

**Definition 4.7.** For $X \in \text{Sch}_k$, define

$$TCH^q(X, n;m) := \colim_{(X \downarrow \text{SmAff}_k)^{\text{op}}} T^q_{n,m},$$

which is a filtered colimit.

**Proposition 4.8.** Let $X, Y \in \text{Sch}_k$, and let $f : X \to Y$ be a k-morphism. The association $X \mapsto TCH^q(X, n;m)$ satisfies the following properties:

1. There is a canonical homomorphism $\alpha_X : TCH^q(X, n;m) \to TCH^q(X, n;m)$.
2. If $X \in \text{SmAff}_k$, then $\alpha_X$ is an isomorphism.
3. There is a canonical pull-back $f^* : TCH^q(Y, n;m) \to TCH^q(X, n;m)$. If $g : Y \to Z$ is another morphism in $\text{Sch}_k$, then we have $(g \circ f)^* = f^* \circ g^*$.

In particular, $TCH^q(\textminus, n;m)$ is a presheaf of abelian groups on $\text{Sch}_k$.

**Proof.** (1) Let $(h : X \to A) \in (X \downarrow \text{SmAff}_k)^{\text{op}}$. By Theorem 4.5, we have the pull-back $h^* : TCH^q(A, n;m) \to TCH^q(X, n;m)$. Regarding $TCH^q(X, n;m)$ as a constant functor on $(X \downarrow \text{SmAff}_k)^{\text{op}}$, this gives a morphism of functors $T^q_{n,m} \to$
Taking the colimits over all \( h \), we obtain \( \mathcal{TCH}^q(X,n;m) \rightarrow \mathcal{TCH}^q(X,n;m) \), where \( \alpha_X = \text{colim}_h h^* \).

(2) When \( X \in \text{SmAff}_k \), the category \((X \downarrow \text{SmAff}_k)^{\text{op}}\) has the terminal object \( \text{Id}_X : X \rightarrow X \). Hence, the colimit \( \mathcal{TCH}^q(X,n;m) \) is just \( \mathcal{TCH}^q(X,n;m) \).

(3) The morphism \( f : X \rightarrow Y \) defines a functor \( f^* : (Y \downarrow \text{SmAff}_k)^{\text{op}} \rightarrow (X \downarrow \text{SmAff}_k)^{\text{op}} \) given by \( (Y \xrightarrow{h} A) \mapsto (X \xrightarrow{f} Y \xrightarrow{h} A) \). Thus, taking the colimits of the functors induced by \( T^*_m \), we obtain \( f^* : \mathcal{TCH}^q(Y,n;m) \rightarrow \mathcal{TCH}^q(X,n;m) \). For another morphism \( g : Y \rightarrow Z \), that \( (g \circ f)^* = f^* \circ g^* \) can be checked easily using the universal property of colimits.

\( \square \)

Remark 4.9. Since additive higher Chow groups have pull-backs for flat maps ([16, Theorem 3.1(2)]), for any \( X \in \text{Sm}_k \), \( \mathcal{TCH}^q(-,n;m) \) defines a presheaf on the small Zariski site \( \text{Open}(X) \) of \( X \). But, for each affine open \( U \in \text{Open}(X) \), by Proposition [13, 4.12], we have \( \mathcal{TCH}^q(U,n;m) \simeq \mathcal{TCH}^q(U,n;m) \) so that their Zariski sheafifications on \( \text{Open}(X) \) will be identical.

4.5. Moving lemma for smooth semi-local schemes. One remaining objective in Section 4 is to prove the following semi-local variation of Theorem 4.1:

**Theorem 4.10.** Let \( Y \in \text{SmLoc}_k \). Let \( \mathcal{W} \) be a finite set of locally closed subsets of \( Y \). Then, the inclusion \( T^*_m(Y,\bullet;m) \hookrightarrow T^*_m(Y,\bullet;m) \) is a quasi-isomorphism.

4.5.1. Semi-local schemes. We begin with some basic results related to cycles over semi-local schemes. These are also used in [21]. Recall that when \( A \) is a ring and \( \Sigma = \{ p_1, \ldots, p_N \} \) is a finite subset of \( \text{Spec}(A) \), the localization at \( \Sigma \) is the localization \( A \rightarrow S^{-1}A \), where \( S = \bigcap_{i=1}^{N}(A \setminus p_i) \). For a quasi-projective \( k \)-scheme \( X \) and a finite subset \( \Sigma \) of (not necessarily closed) points of \( X \), we can reduce it to the case when \( X \) is affine. Indeed, recall from [8 §2.2] the notion of “FA-schemes”. We say that a scheme \( X \) is FA, if for any given finite subset \( \Sigma \subset X \), there exists an affine open subset \( V \subset X \) such that \( \Sigma \subset V \).

**Lemma 4.11.** Let \( X \) be a quasi-projective \( k \)-scheme. Given any finite subset \( \Sigma \subset X \) and an open subset \( U \subset X \) containing \( \Sigma \), there exists an affine open subset \( V \subset U \) containing \( \Sigma \).

**Proof.** By [25, Proposition 3.3.36], \( X \) is FA. Furthermore, any open subset of an FA-scheme is also FA (see [8 §2.2-(2)]). In particular, \( U \subset X \) is FA. \( \square \)

So, for a quasi-projective \( k \)-scheme \( X \) with a finite subset \( \Sigma \subset X \), we may replace \( X \) by an affine open subset \( \text{Spec}(A) \) that contains \( \Sigma \), and we define the semi-local scheme \( \text{Spec}(\mathcal{O}_{X,\Sigma}) \) to be the localization of \( A \) at \( \Sigma \).

When \( X \) is a \( k \)-scheme, for each open immersion \( j^U_U : U \hookrightarrow V \) of open subsets of \( X \), we have the flat pull-back \( (j^V_U)^* : T^*_m(V,n;m) \rightarrow T^*_m(U,n;m) \). In particular, for each \( x \in X \), the open neighborhoods of \( x \) forms a directed system. Then, we have:

**Lemma 4.12.** Let \( X \) be a \( k \)-scheme and let \( x \in X \) be a scheme point. Let \( Y = \text{Spec}(\mathcal{O}_{X,x}) \). Then, we have \( T^*_m(Y,n;m) = \text{colim}_{x \in U} T^*_m(U,n;m) \), where the colimit is taken over all open neighborhoods \( U \) of \( x \).

**Proof.** By Lemma [4.11] we may assume that \( X \) is affine, so, write \( X = \text{Spec}(A) \). Let \( p_x \subset A \) be the prime ideal that corresponds to the point \( x \), and let \( S := A \setminus p_x \),
so that \( Y = \text{Spec} (\mathbb{S}^{-1} A) \). To facilitate our proof, using a suitable automorphism of \( \mathbb{P}^1 \), we identify \( \square \) with \( \mathbb{A}^1 \), and take \( \{0,1\} \subset \mathbb{A}^1 \) as the faces. So, \( X \times B_n = X \times \mathbb{A}^1 \times \mathbb{A}^{n-1} = \text{Spec} (A[t, y_1, \ldots, y_{n-1}]) \).

Let \( \alpha \in \mathcal{T}^q(Y; n; m) \). We need to find an open subset \( U \subset X \) containing \( x \), such that the closure \( \overline{U} \) in \( U \times \mathbb{A}^1 \times \mathbb{A}^{n-1} \) is admissible. For this, we may assume \( \alpha \) is irreducible, i.e. it is the closed irreducible subscheme \( Z \subset Y \times \mathbb{A}^1 \times \mathbb{A}^{n-1} \). Let \( \overline{Z} \) be the Zariski closure in \( X \times \mathbb{A}^1 \times \mathbb{A}^{n-1} \). Let \( p \) be the prime ideal of \( B := A[t, y_1, \ldots, y_{n-1}] \) such that \( V(p) = \overline{Z} \).

For the proper intersection with faces, let \( q \subset B \) be the prime ideal \( (y_1 - \epsilon_1, \ldots, y_s - \epsilon_s) \), where \( 1 \leq i_1 < \cdots < i_s \leq n - 1 \) and \( \epsilon_j \in \{0,1\} \). Let \( \mathfrak{P} \) be a minimal prime of \( p + q \). One checks immediately from the behavior of prime ideals under localizations that there is \( a \in S \) such that either \( \mathfrak{P}B[a^{-1}] = B[a^{-1}] \) or \( \text{ht}(\mathfrak{P}B[a^{-1}]) \geq q + s \). This means, over \( U_1 = \text{Spec} (A[a^{-1}]) \), either the intersection of \( \overline{Z}_{U_1} \) with \( V(q) \) is empty, or has codimension \( \geq q + s \). Thus, \( \overline{Z}_{U_1} \) intersects all faces properly.

For the modulus condition, let \( \nu : W \to \overline{Z} \leftarrow X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \) be the normalization composed with the closed immersion. Let \( F_n = \sum_{i=1}^{n-1} (y_i = \infty) \) be the divisor at infinity. The modulus condition of \( Z \) means \((m + 1)[j^*\nu^*\{t = 00\}] \leq [j^*\nu^*(F_n^\infty)] \) on \( W \). Note that there exists only finitely many prime Weil divisors on \( P_1, \ldots, P_N \) such that \( \text{ord}_{P_i}(\nu^*(F_n^\infty) - (m + 1)\nu^*(t = 0)) < 0 \). Their images \( Q_i \) under the normalization map \( W \to \overline{Z} \) are still irreducible proper closed subsets of \( \overline{Z} \), thus of \( X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \). Take the open complement \( Q^c_i \) in the ambient space, and take \( U_2 := \cap_{i=1}^N \pi(Q^c_i) \), where \( \pi : X \times \mathbb{A}^1 \times (\mathbb{P}^1)^{n-1} \to X \) is the projection, which is open. Thus, \( U_2 \subset X \) is a open dense subset, containing \( x \), and the cycle \( \overline{Z}_{U_2} \) satisfies the modulus condition by construction. So, taking an affine open subset \( U \subset U_1 \cap U_2 \) that contains \( x \), we have \( Z_U \in \mathcal{T}^q(U; n; m) \). That \((Z_U)_Y = Z \) is obvious.

We can extend this colimit description to semi-local schemes:

**Lemma 4.13.** Let \( Y \) be a semi-local \( k \)-scheme obtained by localizing at a finite set \( \Sigma \) of scheme points of a quasi-projective \( k \)-variety \( X \). Let \( Z \subset Y \times B_n \) be a closed subscheme and let \( \overline{Z} \) be its Zariski closure in \( X \times B_n \).

Then, \( Z \in \mathcal{T}^q(Y; n; m) \) if and only if there exists an affine open subset \( U \subset X \) containing \( \Sigma \), such that \( \overline{Z}_U \in \mathcal{T}^q(U; n; m) \), where \( \overline{Z}_U \) is the pull-back of \( \overline{Z} \) via the inclusion \( U \to X \).

**Proof.** The direction \((\Leftarrow)\) is obvious by taking the pull-back via the flat morphism \( Y \to U \). For the direction \((\Rightarrow)\), For each \( x \in \Sigma \), by Lemma 4.12, we have an affine open neighborhood \( U_x \subset X \) of \( x \) such that \( \overline{Z}_{U_x} \in \mathcal{T}^q(U_x; n; m) \). Take \( V = \sqcup_i U_i \). This is an open subset of \( X \) containing \( \Sigma \). By Lemma 2.10 we have \( \overline{Z}_V \in \mathcal{T}^q(V; n; m) \). On the other hand, by Lemma 4.11 there exists an affine open subset \( U \subset V \) containing \( \Sigma \). By taking the flat pull-back via the open immersion \( U \to V \), we get \( \overline{Z}_U \in \mathcal{T}^q(U; n; m) \).

**Lemma 4.14.** Let \( Y \) be a semi-local integral \( k \)-scheme obtained by localizing at a finite set \( \Sigma \) of scheme points of an integral \( k \)-scheme \( X \). Let \( Z \in \mathcal{T}^q(Y; n; m) \), \( W \in \mathcal{T}^q(Y, n + 1; m) \), and let \( \overline{Z}, \overline{W} \) be their Zariski closures in \( X \times B_n \) and \( X \times B_{n+1} \), respectively. For every open subset \( V \subset X \), the subscript \( V \) means the restriction to \( V \). Then, we have the following:
(1) Suppose \(0 = \partial Z\). Then, there exists an affine open subset \(U \subset X\) containing \(\Sigma\) such that \(\overline{Z}_U \in Tz^q(U, n; m)\) and \(0 = \partial \overline{Z}_U\).

(2) Suppose \(Z = \partial W\). Then, there exists an affine open subset \(U \subset X\) containing \(\Sigma\) such that \(\overline{Z}_U \in Tz^q(U, n; m)\), \(\overline{W}_U \in Tz^q(U, n+1; m)\), and \(\overline{Z}_U = \partial \overline{W}_U\).

Proof. Note that (1) is a special case of (2), so we prove (2) only. Let \(Z' := Z - \partial W \in z^q(X \times B_n)\). If \(Z'\) is 0 as a cycle, then we may simply take \(U_0 = X\). If not, let \(Z'_1, \ldots, Z'_s\) be the irreducible components of \(Z'\). Since \(Z = \partial W\), each component \(Z'_i\) has empty intersection with \(Y \times B_n\). So, each \(\pi((Z'_i)^c)\) is a nonempty open subset of \(X\) containing \(\Sigma\), where \(\pi : X \times B_n \to X\) is the projection, which is open. Let \(U_0 = \bigcap_{i=1}^s \pi((Z'_i)^c)\).

On the other hand, by Lemma 4.13 there exist open sets \(U_1, U_2 \subset X\) containing \(\Sigma\) such that \(\overline{Z}_{U_1} \in Tz^q(U_1, n; m)\) and \(\overline{W}_{U_2} \in Tz^q(U_2, n+1; m)\). Choose an affine open subset \(U \subset U_0 \cap U_1 \cap U_2\) containing \(\Sigma\), using Lemma 4.11. Then part (2) holds by construction.

4.5.2. Proof and applications. We now give the proof of the moving lemma for smooth semi-local \(k\)-schemes essentially of finite type:

Proof of Theorem 4.10. We show that the chain map \(Tz^q_W(Y, \bullet; m) \hookrightarrow Tz^q(Y, \bullet; m)\) induces isomorphisms on the homology groups. Let \(X\) be a smooth affine \(k\)-variety with a finite subset \(\Sigma \subset X\) such that \(Y := \text{Spec}(O_{Y, \Sigma})\).

For surjectivity on homology, let \(Z \in Tz^q(Y, n; m)\) such that \(\partial Z = 0\). Let \(\overline{Z}\) be the Zariski closure of \(Z\) in \(X \times B_n\). Here, \(\partial \overline{Z}\) may not be zero, but by Lemma 4.14 there exists an affine open subset \(U \subset X\) containing \(\Sigma\) such that we have \(\partial \overline{Z}_U = 0\), where \(\overline{Z}_U\) is the pull-back of \(\overline{Z}\) over \(U\). Let \(W_U = \{W_U \mid W \in W\}\), where \(W_U\) is the Zariski closure of \(W\) in \(U\). Then, the quasi-isomorphism \(Tz^q_{W_U}(U, \bullet; m) \hookrightarrow Tz^q(U, \bullet; m)\) by Theorem 4.1 shows that there is some \(C \in Tz^q(U, n+1; m)\) and \(Z_U' \in Tz^q_{W_U}(U, n; m)\) such that \(\partial C = \overline{Z}_U - Z_U'\). Let \(\iota : Y \hookrightarrow U\) be the inclusion. So, by applying the flat pull-back \(\iota^*\) (which is equivariant with respect to taking faces), we obtain \(\partial(\iota^* C) = Z - \iota^* Z_U'\), and here \(\iota^* Z_U' \in Tz^q_W(Y, n; m)\), i.e., \(Z\) is equivalent to a member in \(Tz^q_W(Y, n; m)\).

For injectivity on homology, suppose that \(Z \in Tz^q_W(Y, n; m)\) be such that \(Z = \partial Z'\) for some \(Z' \in Tz^q(Y, n+1; m)\). Let \(\overline{Z}\) and \(\overline{Z}'\) be the Zariski closures of \(Z\) and \(Z'\) on \(X \times B_n\) and \(X \times B_{n+1}\), respectively. Then, by Lemma 4.14 there exists a nonempty open affine subset \(U \subset X\) containing \(\Sigma\) such that \(\overline{Z}_U = \partial \overline{Z}_U\), where the subscript \(U\) means restrictions over \(U\). Then, the quasi-isomorphism \(Tz^q_{W_U}(U, \bullet; m) \hookrightarrow Tz^q(U, \bullet; m)\) by Theorem 4.1 shows that there exists \(Z'' \in Tz^q_{W_U}(U, n+1; m)\) such that \(\overline{Z}_U = \partial Z''\). Pulling back via \(\iota : Y \hookrightarrow U\) then shows \(Z = \partial(\iota^* Z'')\), with \(\iota^* Z'' \in Tz^q_W(Y, n+1; m)\).

Corollary 4.15. Let \(f : Y_1 \to Y_2\) be a morphism of semi-local \(k\)-schemes essentially of finite type, where \(Y_2\) is smooth over \(k\). Then, there is a natural pull-back \(f^* : TCH^q(Y_2, n; m) \to TCH^q(Y_1, n; m)\).

Proof. The proof is given by the identical argument as in Theorem 4.5 (see also Theorem 7.1)], except one should use Theorem 4.10 instead of Theorem 4.1. □
ON ADDITIVE HIGHER CHOW GROUPS OF AFFINE SCHEMES

5. DGA-structure on multivariate additive higher Chow groups

In Section 5 we show that the multivariate additive higher Chow groups have an extra product structure that resembles the Pontryagin product, and a differential operator, that turns the groups into differential graded algebras (DGA), when \( X \) is in \( \text{SmAff}^\text{ess}_k \) or \( \text{SmProj}_k \). This generalizes the DGA-structure on additive higher Chow groups of smooth projective varieties in [18].

5.1. Some cycle computations. We generalize some of [18] §3.2.1, 3.2.2, 3.3. Let \( k \) be a field and let \((X, D)\) be a \( k \)-scheme with an effective divisor.

Let \( n \geq 1 \). Consider the finite morphism \( \chi_{r,n} : X \times \Box^n \to X \times \Box^n \) given by \((x, y_1, \ldots, y_n) \mapsto (x, y_1', y_2, \ldots, y_n)\). Given an irreducible cycle \( Z \subset X \times \Box^n \), define \( Z\{\} := (\chi_{n,r})_*([Z]) = [k(Z) : k(\chi_{n,r}(Z))] \cdot [\chi_{n,r}(Z)] \). We extend it \( \mathbb{Z} \)-linearly.

**Lemma 5.1.** If \( Z \) is an admissible cycle with modulus \( D \), then so is \( Z\{\} \).

**Proof.** Its proof is identical to that of [18] Lemma 3.11 in verbatim, except that the divisor \((m+1)\{t=0\}\) should be replaced by \( D \times \Box^n \).

Let \( n, i \geq 1 \). Suppose \( X \) is smooth quasi-projective essentially of finite type over \( k \). Let \((x, y_1, \ldots, y_n, \lambda)\) be the coordinates of \( X \times \Box^{n+2} \). Consider the closed subschemes \( V^i_X \) on \( X \times \Box^{n+2} \) given by the equation \((1-y)(1-\lambda) = 1 - y_i \) if \( i = 1 \), and \((1-y)(1-\lambda) = (1-y_i)(1+y_1+\cdots+y_i^{-1} - \lambda(1+y_1+\cdots+y_i^{-1})) \) if \( i \geq 2 \). Let \( W^i_X \) be the Zariski closure of \( V^i_X \) in \( X \times \Box^{n+2} \). Let \( \pi_1 : X \times \Box^{n+2} \to X \times \Box^{n+1} \) be the projection that drops \( y_i \), and let \( \pi'_1 := \pi_1|_{W^i_X} \). As in [18] Lemma 3.12, one sees that \( \pi'_1 \) is proper surjective. For an irreducible cycle \( Z \subset X \times \Box^n \), define (cf. [18] Definition 3.13) \( \gamma^i_Z := \pi'_1_* (V^i_X \cdot (Z \times \Box^2)) \) as an abstract algebraic cycle. We extend it \( \mathbb{Z} \)-linearly.

**Lemma 5.2.** Let \( Z \in z^q(X|D, n) \). Then, \( \gamma^i_Z \in z^q(X|D, n+1) \).

**Proof.** Once we have Lemma 5.1, the proof of Lemma 5.2 follows that of [18] Lemma 3.15 in verbatim, except that we replace \((m+1)\{t=0\}\) by \( D \times \Box^n \).

A permutation \( \sigma \in \mathfrak{S}_n \) acts naturally on \( \Box^n \) via \( \sigma(y_1, \ldots, y_n) := (y_{\sigma(1)}, \ldots, y_{\sigma(n)}) \). This action extends to cycles on \( X \times \Box^n \).

**Lemma 5.3.** Let \( n \geq 2 \) and let \( Z \in z^q(X|D, n) \) such that \( \partial^i_1(Z) = 0 \) for all \( 1 \leq i \leq n \) and \( \varepsilon \in \{0, \infty\} \). Let \( \sigma \in \mathfrak{S}_n \). Then, there exists \( \gamma^2_Z \in z^q(X|D, n+1) \) such that \( Z = (\text{sgn}(\sigma))(\sigma \cdot Z) + \partial(\gamma^2_Z) \).

**Proof.** Its proof is identical to that of [18] Lemma 3.16, except that, instead of [18] Lemma 3.15 we use Lemma 5.2.

5.2. Pontryagin product. We discuss a product structure on the multivariate additive higher Chow groups, generalizing [18] §3. In [5.3] and [5.4], we define a differential operator and prove the Leibniz rule, generalizing [18] §4.

**Theorem 5.4.** Let \( k \) be a perfect field. Let \( m = (m_1, \ldots, m_r) \) with \( m_i \geq 1 \). Let \( X \) be in \( \text{SmAff}^\text{ess}_k \) or in \( \text{SmProj}_k \). Then, \( \text{CH}(X[r]|D_m) = \bigoplus_{q,n \geq 0} \text{CH}^q(X[r]|D_m, n) \) form a graded commutative algebra.

Its proof requires some discussions, that generalize [18] §3. The proof is over after Lemma 5.16. Let \( X_1, X_2 \in \text{Sch}^\text{ess}_k \) and suppose \( k \) is perfect.
Lemma 5.5. For \( i = 1, 2 \), let \( V_i \) be cycles on \( X_i \times \mathbb{A}^r \times \square^{n_i} \) with modulus \( m_i = (m_{i1}, \ldots, m_{ir}) \), respectively. Then, \( V_1 \times V_2 \), regarded as a cycle on \( X_1 \times X_2 \times \mathbb{A}^r \times \mathbb{A}^r \times \square^{n_1+n_2} \) after a suitable exchange of factors, has modulus \( (m_{11}, m_{22}) \).

Proof. Its proof is similar to that of [13] Lemma 2.11, with some changes. We may assume that \( V_1 \) and \( V_2 \) are irreducible. It is enough to show that each irreducible component \( W \subset V_1 \times V_2 \) has modulus \( (m_{11}, m_{22}) \). Let \( \iota_1 : V_1 \hookrightarrow X_1 \times \mathbb{A}^r \times \square^{n_1} \) be the Zariski closures of \( V_1 \) and \( V_2 \), respectively. Let \( \nu_{V_1} : V_1^N \to V_1 \) be the normalizations. By [13] Lemma 3.1, the product of two reduced normal \( \kappa \)-schemes is again normal over perfect fields. Thus, the morphism \( \nu := \nu_{V_1} \times \nu_{V_2} : V_1^N \times V_2^N \to V_1 \times V_2 = \overline{V_1 \times V_2} \) is the normalization. Hence, the composite \( \overline{W}^N \overset{\nu_N}{\twoheadrightarrow} \overline{V_1} \times \overline{V_2} \), where \( \overline{W} \) is the Zariski closure of \( W \) and \( \nu_W \) is the normalization of \( \overline{W} \), factors into \( \overline{W}^N \overset{i_N}{\twoheadrightarrow} \overline{V_1} \times \overline{V_2} \overset{\nu}{\to} V_1 \times V_2 \), where \( i_N \) is the natural inclusion.

Let \( (t_1, \ldots, t_r, t'_1, \ldots, t'_r, y_1, \ldots, y_{n_1+n_2}) \in \mathbb{A}^r \times \mathbb{A}^r \times \square^{n_1+n_2} \) be the coordinates. Here, consider two divisors \( D_1 := \sum_{i=1}^{n_1} y_i = 1 - \sum_{j=1}^{r_1} m_{1j} t_j = 0 \), \( D_2 := \sum_{i=1}^{n_1+n_2} y_i = 1 - \sum_{j=1}^{r_2} m_{2j} t'_j = 0 \). By the modulus conditions satisfied by \( V_1 \) and \( V_2 \), we have \((\iota_1 \times 1)(\nu_{V_1} \times 1))^*D_1 \geq 0\) and \((1 \times \iota_2)(1 \times \nu_{V_2}))^*D_2 \geq 0 \). Thus, we have \( \nu_1 \times \iota_2 \geq (D_1 + D_2) \geq 0 \) on \( \overline{V_1} \times \overline{V_2} \), so that \( \nu_W \geq (D_1 + D_2) \geq 0 \) on \( \overline{W} \). Since \( i \circ \nu_W = \nu \circ i_N \), this is equivalent to \( \nu_W \geq (D_1 + D_2) \geq 0 \), which shows \( W \) has modulus \( (m_{11}, m_{22}) \).

For \( 1 \leq i \leq r \), define \( \mu : X_1 \times \mathbb{A}^r \times \square^{n_1} \to X_1 \times X_2 \times \mathbb{A}^r \times \square^{n_1+n_2} \) by \((x_1, \{ t_i \}, \{ y_j \}) \times (x_2, \{ t'_i \}, \{ y'_j \}) \to (x_1, x_2, \{ t_i t'_i \}, \{ y_j y'_j \}) \). The map \( \mu \) is flat, but not proper. However, we show that for two cycles \( V_1 \) and \( V_2 \) with modulus conditions, \( \mu(V_1 \times V_2) \) is closed:

Lemma 5.6. For \( i = 1, 2 \), let \( V_i \subset X_i \times \mathbb{A}^r \times \square^{n_i} \) be closed subschemes with modulus \( m_i \). Then, \( \mu|_{V_1 \times V_2} \) is projective.

Proof. For the morphism \( \mu : \mathbb{A}^r \times \mathbb{A}^r \to \mathbb{A}^r \) given by \((t_1, \ldots, t_r) \times (t'_1, \ldots, t'_r) \mapsto (t_1 t'_1, \ldots, t_r t'_r) \), let \( \Gamma \subset \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r \) be its graph, and let \( \overline{\Gamma} \subset (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \) be its Zariski closure. Since the projection \( \text{pr}_3 : \overline{\Gamma} \to (\mathbb{P}^1)^r \) to the third \( (\mathbb{P}^1)^r \) is projective, so is its base change \( \overline{\Gamma} := \overline{\Gamma} \times (\mathbb{P}^1)^r \to \mathbb{A}^r \) via the open inclusion \( \mathbb{A}^r \hookrightarrow (\mathbb{P}^1)^r \). Set-theoretically, we check by inspection that \( \overline{\Gamma} = \Gamma \cup \bigcup_{i=1}^r (E_i \cup F_i) \) (the union is taken in \( (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \)), where \( E_i = A_{i1} \times \cdots \times A_{ir} \times C_{i1} \times \cdots \times C_{ir} \times \mathbb{A}^r \) and \( F_i = C_{i1} \times \cdots \times C_{ir} \times A_{i1} \times \cdots \times A_{ir} \times \mathbb{A}^r \), with \( A_i \ell = C_i \ell := A_i^1 \) if \( i \neq \ell \), and \( A_{ii} = \{0\}, C_{ii} = \{\infty\} \).

We let \( V \subset X_1 \times X_2 \times \Gamma \times \square^{n_1+n_2} \) be the closed subscheme isomorphic to \( V_1 \times V_2 \) via the isomorphism \( \Gamma \to \mathbb{A}^r \times \mathbb{A}^r \) (up to an exchange of factors). Let \( \overline{V} \) be the Zariski closure of \( V \) in \( X_1 \times X_2 \times \square^{n_1+n_2} \). The map \( \overline{V} \to X_1 \times X_2 \times (\mathbb{P}^1)^r \times \square^{n_1+n_2} \) induced by \( \text{pr}_3 \) is projective, so that its pull-back

\begin{equation}
\overline{V} \cap (X_1 \times X_2 \times \overline{\Gamma} \times \square^{n_1+n_2}) \to X_1 \times X_2 \times \mathbb{A}^r \times \square^{n_1+n_2},
\end{equation}

via \( \mathbb{A}^r \hookrightarrow (\mathbb{P}^1)^r \), is also projective.

Claim: \( \overline{V} \cap (X_1 \times X_2 \times E_i^j \times \square^{n_1+n_2}) = \emptyset \) for \( 1 \leq i \leq r \) and \( j = 1, 2 \).

To prove it, consider the composition \( \pi_1 : X_1 \times X_2 \times \overline{\Gamma} \times \square^{n_1+n_2} \to X_1 \times \overline{\Gamma} \times \square^{n_1+n_2} \overset{\text{pr}_3}{\to} X_1 \times (\mathbb{P}^1)^r \times \square^{n_1} \), where the first arrow is the obvious projection and \( \text{pr}_3 : \overline{\Gamma} \to (\mathbb{P}^1)^r \)}
is the restriction of the projection \((\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \to (\mathbb{P}^1)^r\). Let \(V_1\) be the closure of \(V_1\) in \(X_1 \times (\mathbb{P}^1)^r \times \square^{n_1}\). Then, \(\pi^{-1}_i(V_1)\) is closed in \(X_1 \times X_2 \times \Gamma \times \square^{n_1+n_2}\) and contains \(V\), thus contains \(V_1\) as well. But, the modulus condition on \(V_1\) implies that \(\pi^{-1}_i(V_1) \cap (X_1 \times \mathbb{A}^{i-1} \times \{0\} \times \mathbb{A}^{r-i} \times \square^{n_1}) = \emptyset\) for \(1 \leq i \leq r\). Hence, \(\pi^{-1}_i(V_1) \cap (X_1 \times X_2 \times E_i^1 \times \square^{n_1+n_2}) \subset \pi^{-1}_i(V_1) \cap (X_1 \times \mathbb{A}^{i-1} \times \{0\} \times \mathbb{A}^{r-i} \times \square^{n_1}) = \emptyset\). In particular, since \(V \subset \pi^{-1}_i(V_1)\), this implies \(V \cap (X_1 \times X_2 \times E_{i}^1 \times \square^{n_1+n_2}) = \emptyset\) for \(1 \leq i \leq r\). By symmetry, using \(\pi_2: X_1 \times X_2 \times \Gamma \times \square^{n_1+n_2} \to X_2 \times (\mathbb{P}^1)^r \times \square^{n_2}\) and \(V_2\), we deduce \(V \cap (X_1 \times X_2 \times E_{i}^2 \times \square^{n_1+n_2}) = \emptyset\) for \(1 \leq i \leq r\) as well, so we have proved the claim.

Now, by \([5,7]\), \(\sqrt{V} \cap (X_1 \times X_2 \times \Gamma \times \square^{n_1+n_2}) = V \cap (X_1 \times X_2 \times \mathbb{A}_r \times \square^{n_1+n_2}) \simeq V_1 \times V_2\) is projective over \(X_1 \times X_2 \times \mathbb{A}_r \times \square^{n_1+n_2}\) via \(\mu\). This means, \(\mu|_{V_1 \times V_2}\) is projective.

**Lemma 5.7.** Let \(W\) be a \(k\)-variety and consider \(\mu: \mathbb{A}_r \times \mathbb{A}_r \times W \to \mathbb{A}_r \times W\) given by \((t_1, \cdots, t_r, t'_1, \cdots, t'_r, w) \mapsto (t_1t'_1, \cdots, t_rt'_r, w)\). Let \(V \subset \mathbb{A}_r \times \mathbb{A}_r \times W\) be a closed subscheme disjoint from the divisor \(\{t_1, \cdots, t_r, t'_1, \cdots, t'_r = 0\} \subset \mathbb{A}_r \times \mathbb{A}_r \times W\) such that \(\mu|_V\) is projective. Then, \(\mu|_V\) is quasi-finite.

**Proof.** By base change to \(\bar{k}\), we may assume that \(k\) is algebraically closed.

**Step 1.** We first consider the case when \(r = 1\). Let \(p = (a, b) \in \mu(V)\) be a closed point, with \(a \in (\mathbb{A}_1 \setminus \{0\})(k)\) and \(b \in W(k)\). Since \(\mu|_V: V \to \mu(V)\) is projective, so is its base change \((\mu|_V)^{-1}(p) \to \{p\}\) via \(\{p\} \to \mu(V)\). Here, \((\mu|_V)^{-1}(p) = (C_a \times \{b\}) \cap V\), for the affine curve \(C_a = \{(t_1, t'_1) \in \mathbb{A}_2 \mid t_1t'_1 = a\}\). We show \((\mu|_V)^{-1}(p)\) is finite. Toward contradiction, suppose not. Then, \((C_a \times \{b\}) \cap V\) is an infinite closed subscheme of the curve \(C_a \times \{b\} \simeq C_a\), in which case \((C_a \times \{b\}) \cap V\) is all of \(C_a \times \{b\}\). Thus, \(C_a \times \{b\} \to \{(a, b)\}\) is projective, i.e. \(C_a \to \{a\}\) is projective, which is absurd because \(C_a\) is not projective. Thus, \((\mu|_V)^{-1}(p)\) is finite.

**Step 2.** Now, suppose \(r > 1\). We regard \(\mu\) as the composition \(\mu_r \circ \cdots \circ \mu_1\), where \(\mu_i\) sends \((t_i, t'_i)\) to \(t_it'_i\), while it leaves all other coordinates unchanged. More precisely, ignoring all exchanges of factors, for \(1 \leq i \leq r\), we write \(\mu_i: \mathbb{A}^{r-1} \times \mathbb{A}^{r-1} \times \mathbb{A}^1 \times W \to \mathbb{A}^{r-1} \times \mathbb{A}^{r-1} \times \mathbb{A}^1 \times W\). Let \(V \to \mathbb{A}^r \times \mathbb{A}^r \times W\) replaced by \(\mathbb{A}^{r-1} \times \mathbb{A}^{r-1} \times \mathbb{A}^1 \times W\) and \(V\) replaced by \(\mu_{i-1} \circ \cdots \mu_1(V)\) in Step 1. Being a finite composition of quasi-finite morphisms, \(\mu|_V\) is quasi-finite.

**Proposition 5.8.** For \(i = 1, 2\), let \(V_i \subset X_i \times \mathbb{A}_r \times \square^{n_i}\) be a cycle with modulus \(\mathbb{m}_i\). Then, \(\mu|_{V_1 \times V_2}\) is finite.

**Proof.** By Lemma \([5.6]\), \(\mu|_{V_1 \times V_2}\) is projective. This implies that \(\mu|_{V_1 \times V_2}\) is quasi-finite as well, by Lemma \([5.7]\). Hence, \(\mu|_{V_1 \times V_2}\) is finite.

**Definition 5.9.** For any irreducible closed subscheme \(V \subset X \times \mathbb{A}_r \times \mathbb{A}_r \times \square^n\) such that \(\mu|_V: V \to \mu(V)\) is finite, define \(\mu_*(V)\) as the push-forward, i.e. \(\mu_*(V) = \text{deg}(V/\mu(V)) \cdot [\mu(V)]\). We extend it \(\mathbb{Z}\)-linearly.
If $V_i$ are cycles on $X_i \times \mathbb{A}^r \times \mathbb{A}^r$ for $i = 1, 2$ such that $\mu|_{V_i \times V_2}$ is finite, we define the external product $V_1 \times \mu V_2 := \mu_*(V_1 \times V_2)$. If $p_i = \dim V_i$, then $\dim(V_1 \times \mu V_2) = p_1 + p_2$. If $q_i$ is the codimension of $V_i$, then $V_1 \times \mu V_2$ has codimension $q_1 + q_2 - r$.

**Lemma 5.10.** Let $X \in \text{Sch}^r_k$, and let $V$ be a cycle on $X \times \mathbb{A}^r \times \mathbb{A}^r$ with modulus $m = (m_{11}, \ldots, m_{1r}, m_{21}, \ldots, m_{2r})$. Suppose $\mu|V$ is finite. Then, the cycle $\mu_*(V)$ on $X \times \mathbb{A}^r \times \mathbb{A}^r$ has modulus $m' = (\min(m_{11}, m_{21}), \ldots, \min(m_{1r}, m_{2r})).$

**Proof.** The proof is similar to that of [18, Proposition 3.8] that we follow. Let $c_i := \min(m_{1i}, m_{2i})$. We may assume that $V$ is irreducible. It is enough to show that $\mu(V)$ has modulus $m'$. Let $\overline{V}$ and $\mu(V)$ be the Zariski closures of $V$ and $\mu(V)$ in $X \times \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r$ and $X \times \mathbb{A}^r \times \mathbb{A}^r$, respectively. Let $\nu_V : \overline{V} \to V$ and $\nu_{\mu(V)} : \mu(V) \to \mu(V)$ be the normalizations. Then, we have the diagram

$$
\begin{array}{ccc}
\overline{V} & \xrightarrow{\nu_V} & V \\
\mu(\overline{V}) & \xrightarrow{\nu_{\mu(V)}} & \mu(V) \\
& \xrightarrow{\mu} & V \times \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r,
\end{array}
$$

where $\nu_V$ and $\nu_{\mu(V)}$ are closed immersions, and $\overline{V}$ is induced by the universal property of normalization.

Let $(w_1, \ldots, w_r, y_1, \ldots, y_n) \in \mathbb{A}^r \times \mathbb{A}^r$ and $(t_1, \ldots, t_r, t'_1, \ldots, t'_r, y_1, \ldots, y_n) \in \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r$ be the coordinates, and consider the divisor $D := F_n - \sum_j (c_j t_j = 0)$ on $X \times \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r$. Since $V$ has modulus $m$, we have $\nu_V^* \nu_V^* (D) \geq 0$. On the other hand, for the divisor $D' := F_n - \sum_j (c_j t_j = 0)$ on $X \times \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r$, by pulling back via $\mu$ we obtain $\mu^* D' = F_n - \sum_j (c_j t_j = 0)$ on $X \times \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r$. Hence, by pulling back via $\nu_V \circ \nu_V$, we obtain $\nu_V^* \nu_V^* \mu^* D' \geq \nu_V^* \nu_V^* D \geq 0$. By the commutativity, this is equivalent to $(\overline{V})^* (\nu_V^* \nu_V^* \mu^* (V)) D' \geq 0$. Now, by Lemma 2.8, we obtain $\nu_V^* \nu_V^* \mu^* (V) D' \geq 0$ on $\mu(V)^N$. Thus $\mu(V)$ has modulus $m'$.

**Lemma 5.11.** For $i = 1, 2$, let $V_i \in z^e(X_i[r]|D_{m_i}, n_i)$. Then, $V_1 \times \mu V_2$ intersect all faces of $X_1 \times X_2 \times \mathbb{A}^r \times \mathbb{A}^r$ properly.

**Proof.** Its proof is similar to that of [18, Lemma 3.7]. We may assume that $V_1$ and $V_2$ are irreducible. Let $F \subset \mathbb{A}^r \times \mathbb{A}^r$ be a face of codimension $c$. For $Z = V_1 \times \mu V_2$, we need to prove that $\dim(Z \cap (X_1 \times X_2 \times F)) \leq \dim Z - c$. Since $\partial^e(V_i \times \mu V_2) = \partial^e(V_i) \times \mu V_2$ if $1 \leq i \leq n_1$ and $V_i \times \mu \partial^e(V_2)$ if $n_1 + 1 \leq i \leq n_1 + n_2$, by induction on the codimension of $F$, $\dim(Z \cap (X_1 \times X_2 \times F)) \leq \dim(V_1) - 1 + \dim(V_2) - 1$, where $\dagger$ and $\dagger$ hold by Lemmas 5.5 and 5.7, while the inequality holds by the assumption that $V_i$ intersects with the codimension 1 faces properly. The other case is similar. This completes the proof.

**Proposition 5.12.** For $i = 1, 2$, let $V_i \in z^e(X_i[r]|D_{m_i}, n_i)$ with $m_i = (m_{1i}, \ldots, m_{ri})$. Then, $V_1 \times \mu V_2 \in z^{e+q_2-e}(X_1 \times X_2) [\mu V_2]$ with $m = (c_1, \ldots, c_r)$ and $c_i = \min(m_{1i}, m_{2i})$. This product $\times \mu$ is associative.
Proof. By Proposition 5.8, $\mu_\ast (V_1 \times V_2) = V_1 \times_\mu V_2$ has modulus $m$. By Lemma 5.11, $V_1 \times_\mu V_2$ intersects with all faces properly. Associativity follows from the associativity of the Cartesian product $\times$ and the product $\mu : A^1 \times A^1 \to A^1$.

Corollary 5.13. Let $m = (m_1, \ldots, m_r)$. Then, the operations $\times_\mu : z^q (X_1[r] | D_{m_1}, n_1) \otimes z^{q_2} (X_2[r] | D_{m_2}, n_2) \to z^{q_1 + q_2 - r} ((X_1 \times X_2)[r] | D_{m_1 + n_1}, n_2)$ satisfy the relation $\partial (\xi \times_\mu \eta) = \partial (\xi) \times_\mu \eta + (-1)^{n_1} \xi \times_\mu \partial (\eta)$. In particular, they induce operations $\times_\mu : CH^{q_1} (X_1[r]| D_{m_1}, n_1) \otimes CH^{q_2} (X_2[r]| D_{m_2}, n_2) \to CH^{q_1 + q_2 - r} ((X_1 \times X_2)[r] | D_{m_1 + n_1}, n_2)$ that is associative.

Proof. By definition, one checks $\partial (\xi \times \eta) = \partial (\xi) \times \eta + (-1)^{n_1} \xi \times \partial (\eta)$. So, by applying $\mu_\ast$, we get the relation. That $\times_\mu$ descends to the level of $CH$ follows.

Remark 5.14. The product $\times_\mu$ is quite delicate. Following the notations of Corollary 5.13 consider the special case when $X_1 = X_2 = Spec (k)$, $r = 1$, $n_1 = n_2 = 0$, $q_1 = q_2 = 1$, dim$V_1 = $ dim$V_2 = 0$, $m_1 = m_2 = m \geq 1$, so that one considers the 0-cycles. In this case, Lemma 5.3 and Proposition 5.8 obviously hold. Then, the perceptive reader who knows the vanishing theorem of [20] and the product structure $\times_\mu$ of this paper and [30] coexist peacefully.

The answer is that, the vanishing theorem does not imply the vanishing of $V_2 \times_\mu V_2$. The reason is that, while the process of taking $(V_1, V_2) \mapsto V_1 \times V_2 \mapsto V_1 \times_\mu V_2$ does induce a homomorphism $\times_\mu : CH^1 (pt[1]| D_m, 0) \otimes CH^1 (pt[1]| D_m, 0) \to CH^1 (pt[1]| D_m, 0)$ as seen in Corollary 5.13, this process does not factor through the homology group $CH^2 (pt[2]| D_{m,m}, 0)$, and in particular the operation $\mu_\ast$ does not induce a homomorphism from $CH^2 (pt[2]| D_{m,m}, 0)$ to $CH^1 (pt[1]| D_m, 0)$, which maps $V_1 \times V_2$ to the class of $V_1 \times_\mu V_2$. So, the vanishing theorem of [20] and the product structure $\times_\mu$ of this paper and [30] coexist peacefully.

Definition 5.15. Let $m = (m_1, \ldots, m_r)$. Let $X$ be in $SmAff_{\text{proj}}$ or in $SmProj$. For two cycle classes $\alpha_i \in CH^n (X[r]| D_{m_i}, n_i)$ for $i = 1, 2$, form the product $\alpha_1 \times_\mu \alpha_2$ in $CH^{n_1 + q_2 - r} ((X \times X)[r] | D_{m_1 + n_1}, n_2)$, and define the internal product $\alpha_1 \wedge X \alpha_2$ to be $\Delta_\ast^x (\alpha_1 \times_\mu \alpha_2)$ via the diagonal pull-back $\Delta_\ast^x : CH^{n_1 + q_2 - r} ((X \times X)[r] | D_{m_1, n_1 + n_2}) \to CH^{n_1 + q_2 - r} (X[r] | D_{m_1}, n_1 + n_2)$. This pull-back exists by Theorem 4.15 and Corollary 4.15.

Lemma 5.16. $\wedge_X$ is associative and graded-commutative on $CH (X[r] | D_{m_1})$.

Proof. The associativity holds by Proposition 5.12. For the graded-commutativity, first note by Theorem 5.2 we can find representatives $\alpha_1$ and $\alpha_2$ of the given cycle classes whose codimension 1 faces are all trivial. For such $\alpha_1$ and $\alpha_2$, we let $\sigma$ be the permutation that sends $(1, \ldots, n_1, n_1 + 1, \ldots, n_1 + n_2)$ to $(n_1 + 1, \ldots, n_1 + n_2, 1, \ldots, n_1)$. Here $\text{sgn}(\sigma) = (-1)^{n_1}$. So, by Lemma 5.5, $\alpha_1 \wedge_X \alpha_2 = (-1)^{n_1} \alpha_2 \wedge_X \alpha_1 + \partial (W)$ for some admissible cycle $W$, as desired.

Combining the above discussion, now we have proved Theorem 5.4.

Remark 5.17. One can generalize the above a bit. Suppose $r_1 \leq r_2$. Let $m_1 = (m_1, \ldots, m_{r_1})$, and let $m_2 = (m_1, m_{r_1+1}, \ldots, m_{r_2})$. For $\alpha_1 \in z^{n_1} (X_1[r_1]| D_{m_1}, n_1)$
and $\alpha_2 \in \mathcal{W}(X_2)[r_2]|D_{m_2}|n_2)$, we define $\alpha_1 \times_\mu \alpha_2$ as follows. Consider the closed immersion $\iota : \mathbb{A}^{r_1} \hookrightarrow \mathbb{A}^{r_2}$ given by $(t_1, \ldots, t_{r_1}) \mapsto (t_1, \ldots, t_{r_1}, 1, \ldots, 1)$. This gives a proper morphism $\iota : (X_1[r_1], D_{m_1}) \hookrightarrow (X_1[r_2], D_{m_2})$, and thus $\iota_*(\alpha_1) \in \mathcal{W}(X_1)[r_2]|D_{m_2}|n_1)$. Define $\alpha_1 \times_\mu \alpha_2$ to be $\iota_*(\alpha_1) \times_\mu \alpha_2$ as in Definition 5.9.

In case $X_1 = X_2 = X$ is in $\text{SmAff}^{\text{ess}}$ or in $\text{SmProj}_k$, then define $\alpha_1 \times_X \alpha_2$ to be $\iota_*(\alpha_1) \times_X \alpha_2$ as in Definition 5.15. Note that in the above embedding $\iota$, we may permute the coordinates of $\mathbb{A}^{r_2}$ as well.

**Corollary 5.18.** Let $X \in \text{Sch}_k^{\text{ess}}$ over a perfect field $k$ of characteristic $\neq 2$. Let $1 \leq r' \leq r$. Let $\overline{m} = (m_1, \ldots, m_r)$. Let $\overline{m}' = (m'_1, \ldots, m'_r)$ be such that for some increasing sequence $1 \leq i_1 < \cdots < i_{r'} \leq r$, we have $m_j' \geq m_j$ for $1 \leq j \leq r'$. Let $pt = \text{Spec}(k)$. Then, we have a natural product $\text{CH}(pt[r]|D_{\overline{m}'}, n') \otimes \text{CH}(X[r]|D_{\overline{m}}, n) \rightarrow \text{CH}^{\mu + \nu - r'}(X[r]|D_{\overline{m}}, n + n')$. In particular, when $q' = 1$, $\text{CH}(X[r]|D_{\overline{m}}, n)$ has a $\mathcal{W}_m(k)$-module structure for any $m > m_i$ for some $1 \leq i \leq r$. When $\text{char}(k) = 0$, $\text{CH}(X[r]|D_{\overline{m}}, n)$ is a $k$-vector space.

**Proof.** Let $\iota : \mathbb{A}^{r'} \hookrightarrow \mathbb{A}^r$ be given by $(t'_1, \ldots, t'_{r'}) \mapsto (s_1, \ldots, s_r)$, where $s_i = 1$ if $i \notin \{i_1, \ldots, i_{r'}\}$, while $s_i = t'_{i_j}$ if $i = i_j$. This induces a proper morphism $\iota : (pt[r']|D_{\overline{m}'}) \rightarrow (pt[r]|D_{\overline{m}})$ of schemes with effective divisors. Thus, we have $\iota_!: \text{CH}^{\mu + \nu - (r'-r)}(pt[r]|D_{\overline{m}}, n') \otimes \text{CH}(X[r]|D_{\overline{m}}, n) \rightarrow \text{CH}^{\mu + \nu - r'}(X[r]|D_{\overline{m}}, n + n')$. So, we obtain the desired product given by the composition $\times_\mu \circ (\iota_! \otimes \text{Id})$.

When $r' = q' = 1$, $n' = 0$, $\overline{m}' = 1 + m$, by [30] we have $\text{CH}(1|D_{1+m}, 0) \cong \mathcal{W}_m(k)$, the ring of big Witt vectors of $k$. Thus, $\text{CH}(X[r]|D_{\overline{m}}, n)$ is a $\mathcal{W}_m(k)$-module. When $\text{char}(k) = 0$, the ring $\mathcal{W}_m(k)$ is a $k$-vector space (in a non-canonical way), so that $\text{CH}(X[r]|D_{\overline{m}}, n)$ is also a $k$-vector space.

### 5.3. Differential

Let $k$ be a field of characteristic $\neq 2$. Let $X$ be a smooth quasi-projective scheme essentially of finite type over $k$. Let $r \geq 1$ and let $\overline{m} = (m_1, \ldots, m_r)$ for integers $m_i \geq 1$. Let $(G^r_m)^{\times} := G^r_m \setminus \{(t_1, \ldots, t_r) \in G^r_m | t_1 \cdots t_r = 1\}$. Consider the morphism $\delta_n : (G^r_m)^{\times} \times \square^n \rightarrow G^r_m \times \square^{n+1}$, $(t_1, \ldots, t_r, y_1, \ldots, y_n) \mapsto (t_1, \ldots, t_r, \frac{1}{t_1 \cdots t_r}, y_1, \ldots, y_n)$. It induces $\delta_n : X \times (G^r_m)^{\times} \times \square^n \rightarrow X \times G^r_m \times \square^{n+1}$.

Recall a closed subscheme $Z \subset X \times A^r \times \square^n$ with modulus $\overline{m}$ does not intersect the divisor $\{t_1 \cdots t_r = 0\}$. So, it is closed in $X \times G^r_m \times \square^n$. For such $Z$, we define $Z^{\times} := Z|_{X \times (G^r_m)^{\times} \times \square^n}$. We can extend it $\mathbb{Z}$-linearly.

**Lemma 5.19.** For a closed subscheme $Z \subset X \times A^r \times \square^n$ with modulus $\overline{m}$, the image $\delta_n(Z^{\times})$ is closed in $X \times G^r_m \times \square^{n+1}$.

**Proof.** It is enough to show that $\delta_n : X \times (G^r_m)^{\times} \times \square^n \rightarrow X \times G^r_m \times \square^{n+1}$ is a closed immersion. It reduces to show that the map $(G^r_m)^{\times} \rightarrow G^r_m \times (\mathbb{P}^n \setminus \{1\})$ given by $(t_1, \ldots, t_r) \mapsto (t_1, \ldots, t_r, 1/(t_1 \cdots t_r))$ is a closed immersion. This is obvious because the image coincides with the closed subscheme given by the equation $t_1 \cdots t_r y = 1$, where $(t_1, \ldots, t_r, y) \in G^r_m \times \square$ are the coordinates.

**Definition 5.20.** For a closed subscheme $Z \subset X \times A^r \times \square^n$ with modulus $\overline{m}$, for simplicity we write $\delta_n(Z) := \delta_n(Z^{\times})$. If $Z$ is a cycle, we write $\delta(Z)$ by extending it $\mathbb{Z}$-linearly. We may often write $\delta(Z)$ if no confusion arises.
Lemma 5.21. Let $Z$ be a cycle on $X \times \mathbb{A}^r \times \square^n$ with modulus $m$. Then, $\delta_n(Z)$ is a cycle on $X \times \mathbb{A}^r \times \square^{n+1}$ with modulus $m$.

Proof. We may suppose that $Z$ is irreducible. Let $V = \delta_n(Z)$, which is a priori closed in $X \times \mathbb{G}^r_m \times \square^{n+1}$. If the closure $V'$ of $V$ in $X \times \mathbb{A}^r \times \square^{n+1}$ has modulus $m$, then it does not intersect the divisor $\{t_1 \cdots t_r = 0\}$ of $X \times \mathbb{A}^r \times \square^{n+1}$, so $V = V'$, and $V$ is closed in $X \times \mathbb{A}^r \times \square^{n+1}$ with modulus $m$. So, we reduce to show $V'$ has modulus $m$.

Let $\overline{Z}$ and $\overline{V}$ be the Zariski closures of $Z$ and $V'$ in $X \times \mathbb{A}^r \times \square^n$ and $X \times \mathbb{A}^r \times \square^{n+1}$, respectively. Observe that $\delta_n$ extends to $\overline{\delta}_n : X \times \mathbb{A}^r \times \square^i \to X \times \mathbb{A}^r \times \square^{i+1}$, which is induced from $\mathbb{A}^r \xrightarrow{\Gamma} \mathbb{A}^r \times \square \xrightarrow{\text{Id} \times \sigma} \mathbb{A}^r \times \square$, where $\Gamma$ is the graph morphism of the composite $\mathbb{A}^r \to \mathbb{A}^1 \hookrightarrow \square$ of the product map followed by the open inclusion, $(t_1, \cdots, t_r) \mapsto (t_1 \cdots t_r) \mapsto (t_1 \cdots t_r) : 1)$, while $\sigma : \square \to \square$ is the antipodal automorphism $(a; b) \mapsto (b; a)$ where $(a; b) \in \square = \mathbb{P}^1$ are the homogeneous coordinates. Since $\Gamma$ is a closed immersion and $\text{Id} \times \sigma$ is an isomorphism, the morphism $\overline{\delta}_n$ is projective. Hence, the dominant map $\delta_n|_{X^\times} : Z^\times \to V$ induces $\overline{\delta}_n|_{\overline{\mathcal{X}}} : \overline{Z} \to \overline{V}$. Hence, we have a commutative diagram

\[
\begin{array}{ccc}
\overline{Z}^N & \xrightarrow{\nu_Z} & \overline{V}^N \\
\delta_n \downarrow & & \delta_n \downarrow \\
X \times \mathbb{A}^r \times \square^n & \xrightarrow{\nu} & X \times \mathbb{A}^r \times \square^{n+1},
\end{array}
\]

where $\nu_Z, \nu_V$ are the closed immersions, $\nu_Z, \nu_V$ are normalizations, and $\overline{\delta}_n$ is given by the universal property of normalization.

By definition, $\overline{\delta}_n \{t_j = 0\} = \{t_j = 0\}$ for $1 \leq j \leq r$. First consider the case $n \geq 1$. Then, $\overline{\delta}_n F_{n+1,i}^1 = F_{n+1,i}^1$ for $2 \leq i \leq n + 1$. Now, $\overline{\delta}_n \nu_V^r \tau_V^r \{ \sum_{i=1}^{r+1} F_{n+1,i}^1 - \sum_{j=1}^r m_j \{t_1 = 0\} \} = \nu_Z^r \tau_Z^r \{ \sum_{i=1}^{r+1} F_{n+1,i}^1 - \sum_{j=1}^r m_j \{t_1 = 0\} \} \geq \nu_Z^r \tau_Z^r \{ \sum_{i=1}^{r+1} F_{n+1,i}^1 - \sum_{j=1}^r m_j \{t_1 = 0\} \} \geq 0$, where $\tau$ holds by the commutativity of (5.2) and $\delta$ holds for $Z$ has modulus $m$. Then, by Lemma 2.28, we can drop $\overline{\delta}_n^\ast$, i.e. $V'$ has modulus $m$.

When $n = 0$, we have for $1 \leq j \leq r$, $\nu_Z^r \nu_V^r \nu_V^r \{t_j = 0\} = \nu_Z^r \nu_V^r \nu_V^r \{t_j = 0\} = \nu_Z^r \nu_V^r \nu_V^r \{t_j = 0\} = \nu_Z^r \nu_V^r \nu_V^r \{t_j = 0\}$, which is 0 because $\overline{Z} \cap \{t_j = 0\} = \emptyset$. Hence, $\delta_n^\ast \nu_V^r \nu_V^r (F_{1,1}^1 - \sum_{j=1}^r m_j \{t_j = 0\}) = \delta_n^\ast \nu_V^r \nu_V^r F_{1,1}^1 \geq 0$. By Lemma 2.28, we drop $\overline{\delta}_n^\ast$, i.e. $V'$ has modulus $m$.

\[
\begin{array}{c}
\text{Proposition 5.22. Let } Z \in z^q(X[r]|D_m, n) \text{. Then, } \delta(Z) \in z^{q+1}(X[r]|D_m, n+1) \text{. Furthermore, } \delta \text{ and } \partial \text{ satisfy the equality } \partial \delta + \delta \partial = 0 \text{.}
\end{array}
\]

Proof. We may assume that $Z$ is an irreducible cycle. Let $\partial_{n,i}$ be the boundary given by the face $F_{n,i}^\epsilon$ on $X \times \mathbb{A}^r \times \square^n$, for $1 \leq i \leq n$ and $\epsilon = 0, \infty$.

Claim: For $\epsilon = 0, \infty$, (i) $\partial_{n+1,i} \circ \delta_n = 0$, (ii) $\partial_{n+1,i} \circ \delta_n = \delta_{n-1} \circ \partial_{n,i-1}$ for $2 \leq i \leq n + 1$.

For (i), we show that $\delta_n(Z) \cap \{y_1 = \epsilon\} = \emptyset$ for $\epsilon = 0, \infty$. Since $\delta_n(Z) \subset V(t_1 \cdots t_r y_1 = 1)$, we have $\delta_n(Z) \cap \{y_1 = 0\} = \emptyset$. On the other hand, if $\delta_n(Z)$ intersects $\{y_1 = \infty\}$, then one of $t_i$ must be zero on $Z$, i.e. $Z$ intersects $\{t_i = 0\}$
for some $1 \leq i \leq r$. However, since $Z$ has modulus $m$, this cannot happen. Thus, $\delta_n(Z) \cap \{y_1 = \infty\} = \emptyset$. This shows (i). For (ii), by the definition of $\delta_n$, the diagram

\[
\begin{array}{ccc}
(G_m^r)^{\times} \times \square^{n-1} & \xrightarrow{\iota_{i-1}} & (G_m^r)^{\times} \times \square^n \\
\delta_{n-1} \downarrow & & \downarrow \delta_n \\
G_m^r \times \square^n & \xrightarrow{\iota_i} & G_m^r \times \square^{n+1}
\end{array}
\]

is Cartesian. Thus, $\delta_{n-1}((\iota_i^*\alpha(Z)) = (\iota_i^*)^*(\delta_n(Z))$ by [7 Proposition 1.7], i.e. (ii) holds. This proves the claim.

By Lemma \ref{lemma:5.26} we know $\delta_n(Z)$ has modulus $m$. Since $Z$ intersects all faces properly, so does $\delta_n(Z)$ by applying (i) and (ii) of the above Claim repeatedly. For $\partial \delta + \delta \partial = 0$, note that $\partial \delta_n(Z) = \sum_{i=1}^{n-1}(\delta_{n+1,i}(\partial_{n+1,i}(Z) - \partial_{n+1,i}(\delta_n(Z))) \uparrow \sum_{i=2}^{n-1}(\delta_{n-1}^\partial\delta_{n-1,i}(Z) - \delta_{n-1}^\partial\delta_{n-1,i}(\delta_n(Z))) = -\delta_{n-1}^\partial\delta_{n-1,i}(\partial_{n-1,i}(Z)) = -\delta_{n-1}^\partial\delta_{n-1,i}(Z), \therefore \overline{\partial \delta} = \delta_{n-1}^\partial \partial(Z)$, where $\uparrow$ holds by the Claim. \hfill \Box

**Lemma 5.23.** Let $Z \in z^t(X[r]|D_m, n)$ be such that $\partial_i(Z) = 0$ for $1 \leq i \leq n$ and $\epsilon = 0, \infty$. Then, $2\delta^2(Z)$ is the boundary of an admissible cycle with modulus $m$.

**Proof.** Note that $\delta^2(Z)$ is an admissible cycle on $X \times \mathbb{A}^r \times \square^{n+2}$ with modulus $m$, by Proposition \ref{proposition:5.22}. For the transposition $\tau = (1, 2)$ on the set $\{1, \ldots, n+2\}$, by the definition of $\delta$ we have $\tau \cdot \delta^2(Z) = \delta^2(Z)$. On the other hand, by Lemma \ref{lemma:5.3} we have $\tau \cdot \delta^2(Z) = -\delta^2(Z) + \partial(\gamma)$ for some admissible cycle $\gamma$. Hence, we have $-\delta^2(Z) + \partial(\gamma) = \delta^2(Z)$, i.e. $2\delta^2(Z) = \partial(\gamma)$, as desired. \hfill \Box

**Corollary 5.24.** Let $k$ be a perfect field of characteristic $\neq 2$ and let $X$ be a smooth quasi-projective scheme essentially of finite type over $k$. Then, $\delta^2 = 0$ on $\text{CH}^q(X[r]|D_m, n)$.

**Proof.** By Proposition \ref{proposition:5.22} the map $\delta$ is defined on $\text{CH}^q(X[r]|D_m, n)$. By Corollary \ref{corollary:5.18} the groups $\text{CH}^q(X[r]|D_m, n)$ are $\mathbb{W}_m(k)$-modules for some sufficiently large $m > 0$. By Lemma \ref{lemma:5.23} we saw that $2\delta^2 = 0$ as operators on the groups, but since $\text{char}(k) \neq 2$, this implies $\delta^2 = 0$. \hfill \Box

5.4. **Leibniz rule.** Let $k$ be a perfect field of characteristic $\neq 2$. Let $X$ be a smooth quasi-projective scheme essentially of finite type over $k$. Let

\[(x, t_1, \ldots, t_r, t'_1, \ldots, t'_r, y_1, \ldots, y_n, y, \lambda) \in X \times \mathbb{A}^{2r} \times \square^{n+2}\]

be the coordinates. Let $T \subset X \times \mathbb{A}^{2r} \times \square^{n+2}$ be the closed subscheme defined by the equation $t_1 \cdot \cdot \cdot t_r y = \lambda(t_1 \cdot \cdot \cdot t_r t'_1 \cdot \cdot \cdot t'_r y - 1)$.

**Definition 5.25.** Let $Z \subset X \times \mathbb{A}^{2r} \times \square^n$ be a closed subscheme. Define $C_Z := T \cdot (Z \times \square^2)$ on $X \times \mathbb{A}^{2r} \times \square^{n+2}$. We extend it $Z$-linearly to cycles.

**Lemma 5.26.** Let $Z$ be a cycle on $X \times \mathbb{A}^{2r} \times \square^n$ with modulus $m$ = $(m_1, \ldots, m_{2r})$. Then, $C_Z$ has modulus $m$ on $X \times \mathbb{A}^{2r} \times \square^{n+2}$.

**Proof.** We may assume $Z$ is irreducible. We show that each irreducible component $V \subset C_Z$ has modulus $m$. Let $\overline{Z}$ and $\overline{V}$ be the Zariski closures of $Z$ and $V$ in $X \times \mathbb{A}^{2r} \times \square^n$ and $X \times \mathbb{A}^{2r} \times \square^{n+2}$, respectively. The projection $pr : X \times \mathbb{A}^{2r} \times \square^{n+2} \rightarrow$
$X \times \mathbb{A}^{2r} \times □^t$ that ignores the last two $□^2$ is projective, while its restriction to $X \times \mathbb{A}^{2r} \times □^{n+2}$ maps $V$ into $Z$. So, $\text{pr}$ maps $V$ to $Z$, giving a commutative diagram

\[
\begin{array}{ccc}
\nabla^N & \overset{\nu_V}{\longrightarrow} & \nabla \longrightarrow X \times \mathbb{A}^{2r} \times □^{n+2} \\
\downarrow \text{pr}^N & & \downarrow \text{pr} \downarrow \\
\nabla^N & \overset{\nu_Z}{\longrightarrow} & \nabla \longrightarrow X \times \mathbb{A}^{2r} \times □^n,
\end{array}
\]

where $\nu_V$ and $\nu_Z$ are the closed immersions, $\nu_V$ and $\nu_Z$ are normalizations, and $\text{pr}^N$ is induced by the universal property of normalization. Since $Z$ has modulus $m$, we have $\nu_Z^* \nu_Z^* \left( \sum_{i=1}^n F_{n,i} - \sum_{j=1}^{2r} m_j \{ t_j = 0 \} \right) \geq 0$. After pulling back via $\text{pr}^N$, by (5.3), we have $\nu_V^* \nu_V^* \text{pr}^N(\sum_{i=1}^n F_{n,i} - \sum_{j=1}^{2r} m_j \{ t_j = 0 \}) \geq 0$. But, $\text{pr}^N(\{ t_j = 0 \}) = \{ t_j = 0 \}$ for $1 \leq j \leq 2r$ and $\text{pr}^N F_{n,i} = F_{n+2,i}$ for $1 \leq i \leq n$. Hence, we have $\sum_{j=1}^{2r} m_j \nu_V^* \nu_V^* \{ t_j = 0 \} \leq \sum_{i=1}^n \nu_V^* \nu_V^* F_{n+2,i} \leq \sum_{i=1}^{n+2} \nu_V^* \nu_V^* F_{n+2,i},$ i.e. $V$ has modulus $m$.

\[
\square
\]

Remark 5.27. [15] Corollary 4.11], which considers the case when $r = 1$, is unfortunately not quite correct. Corollary 5.28 below corrects and supersedes it:

**Corollary 5.28.** Let $r \geq 1$. For $i = 1, 2$, let $X_i$ be smooth quasi-projective schemes essentially of finite type over $k$, and let $V_i \subset z^n(X_i)[r]D_{m_i}, n_i)$ with $m_i = (m_{i1}, \cdots, m_{ir})$. Under the exchange of factors $X_1 \times \mathbb{A}^r \times □^m \times X_2 \times \mathbb{A}^r \times □^n \simeq X_1 \times X_2 \times \mathbb{A}^{2r} \times □^n$, where $n = n_1 + n_2$, consider the cycle $C_{V_1 \times V_2}$ on $X_1 \times X_2 \times \mathbb{A}^{2r} \times □^{n+2}$. The morphism $\mu : \mathbb{A}^{2r} \to \mathbb{A}^r$ given by $(t_1, \cdots, t_r, t_1', \cdots, t_r') \mapsto (t_1, \cdots, t_r)$ induces $\mu : X_1 \times X_2 \times \mathbb{A}^{2r} \times □^{n+2} \to X_1 \times X_2 \times \mathbb{A}^r \times □^{n+2}$. Then, $\mu|_{C_{V_1 \times V_2}}$ is finite. In particular, $\mu_*(C_{V_1 \times V_2})$ as in Definition 5.9 is well-defined, and it has modulus $(\min(m_{i1}, m_{i2}), \cdots, \min(m_{ir1}, m_{ir2})).$

**Proof.** If $\mu|_{C_{V_1 \times V_2}}$ is indeed finite, then by Lemma 5.10, $\mu_*(C_{V_1 \times V_2})$ is a well-defined cycle with modulus $(\min(m_{i1}, m_{i2}), \cdots, \min(m_{ir1}, m_{ir2})).$ So, it is enough to show that $\mu|_{C_{V_1 \times V_2}}$ is finite. On the other hand, if $\mu|_{C_{V_1 \times V_2}}$ is projective, then by Lemma 5.7, it is quasi-finite, thus it is finite. So, it reduces to check that $\mu|_{C_{V_1 \times V_2}}$ is projective.

We borrow ideas from the proof of Lemma 5.6. Let $\Gamma \subset \mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r$ be the graph of $\mu$ and let $\Gamma \subset (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r \times (\mathbb{P}^1)^r$ be its Zariski closure. We had the projective morphism $\Gamma^0 := \Gamma \times (\mathbb{P}^1)^r \to \mathbb{A}^r$, and set-theoretically we had $\Gamma^0 = \Gamma \cup \bigcup_{i=1}^r (E_i \cup E_i')$ for some closed subsets $E_i \subset \mathbb{A}^{3r}$, for $1 \leq i \leq r, j = 0, 1$.

Via the isomorphism $\Gamma \to \mathbb{A}^{2r}$, we have the closed subscheme $V \subset X_1 \times X_2 \times \Gamma \times \mathbb{A}^n$ isomorphic to $V_1 \times V_2$. Similarly, let $C \subset X_1 \times X_2 \times \Gamma \times \mathbb{A}^n$ be the closed subscheme isomorphic to $C_{V_1 \times V_2}$. Let $\nabla$ and $\nabla$ be the Zariski closures of $C$ and $V$ in $X_1 \times X_2 \times \Gamma \times \mathbb{A}^n$ and $X_1 \times X_2 \times \Gamma \times \mathbb{A}^n$, respectively. Since the maps $\nabla \to X_1 \times X_2 \times (\mathbb{P}^1)^r \times □^r$ and $\nabla \to X_1 \times X_2 \times (\mathbb{P}^1)^r \times □^{n+2}$ are projective, by base change so are (5.1) and

\[
\nabla \cap (X_1 \times X_2 \times \Gamma^0 \times □^{n+2}) \to X_1 \times X_2 \times \mathbb{A}^r \times □^{n+2}.
\]

In Lemma 5.6, we saw $\nabla \cap (X_1 \times X_2 \times E_i \times □^{n+2}) = \emptyset$ for $1 \leq i \leq r, j = 1, 2$. We claim that we have $\nabla \cap (X_1 \times X_2 \times E_i \times □^{n+2}) = \emptyset$ for all $1 \leq i \leq r$ and $j = 1, 2$. 


As we did for $V$ and $C$, let $S \subset X_1 \times X_2 \times \Gamma \times \Box^{n+2}$ be the closed subscheme isomorphic to $T$ via $\Gamma \xrightarrow{\sim} \mathbb{A}^2$, where $T$ is defined just above Definition 5.25. So, $C = S \cap (V \times \Box)$ by definition. Let $\overline{S}$ be the Zariski closure of $S$ in $X_1 \times X_2 \times \Gamma \times \Box^{n+2}$. Then, $\overline{C} \cap (X_1 \times X_2 \times E_i^j \times \Box^{n+2}) \subset \overline{S} \cap (V \times \Box^2) \cap (X_1 \times X_2 \times E_i^j \times \Box^{n+2}) = \overline{S} \cap (\nabla \cap (X_1 \times X_2 \times E_i^j \times \Box^n)) \times \Box^2) = \overline{S} \cap (\emptyset \times \Box^n) = \emptyset$, where $\dagger$ holds because $\nabla \cap (X_1 \times X_2 \times E_i^j \times \Box^n) = \emptyset$. This proves the claim.

Thus, $\overline{C} \cap (X_1 \times X_2 \times \Gamma \times \Box^{n+2}) = \overline{C} \cap (X_1 \times X_2 \times \Gamma \times \Box^{n+2}) = C \simeq C_{V_1 \times V_2}$, but by (5.4), the morphism from this to $X_1 \times X_2 \times \mathbb{A}^r \times \Box^{n+2}$, induced by the third projection $\mathbb{A}^r \times \mathbb{A}^r \times \mathbb{A}^r \rightarrow \mathbb{A}^r$ is projective, i.e. $\mu|_{C_{V_1 \times V_2}}$ is projective, as desired. This completes the proof.

**Definition 5.29.** For $i = 1, 2$, let $V_i \in z^{q_i}(X_i[r]|D_{m,n_i})$. Let $n = n_1 + n_2$. We let $V_1 \times_{\mu} V_2$ be the cycle $\sigma \cdot \mu_*(C_{V_1 \times V_2})$, where $\sigma = (n + 2, n + 1, \ldots, 1)^2 \in \mathcal{S}_{n+2}$.

**Remark 5.30.** Set-theoretically, $V_1 \times_{\mu} V_2$ can be described as the Zariski closure of $\mu(V_1 \times V_2 \times \Box)$, where the rational morphism $\mu : X_1 \times X_2 \times \mathbb{A}^2 \times \Box^n \times \Box \rightarrow X_1 \times X_2 \times \mathbb{A}^r \times \Box^{n+2}$ is given by $(x_1, x_2, \{t_i\}, \{t_1\}, \{y_i\}, y) \mapsto (x_1, x_2, \{t_i\}, \{t_1\}, \{y_i\}, y, \frac{1}{(t_1, \ldots, t_i, \ldots, y_i)}y - t_i, y_i, \ldots, y_n)$.

**Lemma 5.31.** For $i = 1, 2$, let $V_i \in z^{q_i}(X_i[r]|D_{m,n_i})$. Then, $V_1 \times_{\mu} V_2 \in z^{q_1+q_2-r}((X_1 \times X_2)[r]|D_{m,n_1 + n_2 + 2})$.

**Proof.** By Corollary 5.28, the cycle $\mu_*(C_{V_1 \times V_2})$ has modulus $m$, thus so does $W := V_1 \times_{\mu} V_2$. It remains to prove that $W$ intersects all faces properly. Let $s_{n_1} = (n_1 + 1, n_1, \ldots, 1) \in S_{n+1}$. Then, by direct calculations, we have

$$
\begin{align*}
\partial_i^0 W &= \sigma_{n_1}(V_1 \times_{\mu} \delta(V_2)), \\
&= 0, \\
\partial_i^2 W &= \delta(V_1 \times_{\mu} V_2), \\
\partial_i^2 W &= \delta(V_1) \times_{\mu} V_2, \\
\partial_i^2 W &= \left\{ \begin{array}{ll}
\partial_{i-1}^2(V_1) \times_{\mu} V_2, & \text{for } 3 \leq i \leq n_1 + 2, \\
V_1 \times_{\mu} \partial_{i-n_1-2}^2(V_2), & \text{for } n_1 + 3 \leq i \leq n_1 + 2.
\end{array} \right.
\end{align*}
$$

Since each $V_i$ is admissible, (5.5), Propositions 5.12 5.22 and induction on the codimension of faces, show that $W$ intersects all faces properly. □

**Proposition 5.32.** Let $X_1, X_2$ be smooth quasi-projective schemes essentially of finite type over $k$. Let $\xi \in z^{q_1}(X_1[r]|D_{m,n_1})$ and $\eta \in z^{q_2}(X_2[r]|D_{m,n_2})$. Let $n = n_1 + n_2, q = q_1 + q_2$. Suppose that all codimension 1 faces of $\xi$ and $\eta$ vanish. Then, in the group $z^{q_1+q_2-r}((X_1 \times X_2)[r]|D_{m,n+1})$, the cycle $\delta(\xi \times_{\mu} \eta) - \delta\xi \times_{\mu} \eta$ is the boundary of an admissible cycle.

**Proof.** By (5.5), for $3 \leq i \leq n_1 + 2$, we have $\partial_i^0 (\xi \times_{\mu} \eta) = \partial_{i-2}^0(\xi) \times_{\mu} \eta = 0$, while for $n_1 + 3 \leq i \leq n + 2$, we have $\partial_i^0(\xi \times_{\mu} \eta) = \xi \times_{\mu} \partial_{i-n_1-2}^0(\eta) = 0$. Hence, $\partial(\xi \times_{\mu} \eta) = \sum_{i=1}^{n+2}(1)^i(\partial^\infty - \partial^0)(\xi \times_{\mu} \eta) = \delta(\xi \times_{\mu} \eta) - \left\{ s_{n_1}\cdot(\xi \times_{\mu} \delta\eta) + \delta\xi \times_{\mu} \eta \right\}$ by (5.5) for $i = 1, 2$. Equivalently,

$$
\delta(\xi \times_{\mu} \eta) - \delta\xi \times_{\mu} \eta - \sigma_{n_1}\cdot(\xi \times_{\mu} \delta\eta) = \partial(\xi \times_{\mu} \eta).
$$

But, for $\xi \times_{\mu} \delta\eta$, notice that

$$
\partial_i^0(\xi \times_{\mu} \delta\eta) = \left\{ \begin{array}{ll}
\partial_i^0 \xi \times_{\mu} \delta\eta = 0, & \text{for } 1 \leq i \leq n_1, \\
\xi \times_{\mu} \partial_{i-n_1}^0(\delta\eta), & \text{for } n_1 + 1 \leq i \leq n + 1.
\end{array} \right.
$$

When $i = n_1 + 1$, by Claim (i) of Proposition 5.22, we have $\partial_i^0(\delta\eta) = 0$, while, when $n_1 + 2 \leq i \leq n + 1$, by Claim (ii) of Proposition 5.22, we have $\partial_{i-n_1}^0(\delta\eta) = \delta(\partial_{i-n_1-1}\eta) = \delta(0) = 0$. Hence, $\xi \times_{\mu} \delta\eta$ is a cycle with trivial codimension 1
faces, so, by Lemma 5.3 for some admissible cycle \( \gamma \), we have \( \sigma_{m_1} \cdot (\xi \times \mu \delta \eta) = sgn(\sigma_{m_1})(\xi \times \mu \delta \eta) + \partial(\gamma) = (-1)^{m_1} \xi \times \mu \delta \eta + \partial(\gamma) \). Putting this back in (5.6), we obtain \( \delta(\xi \times \mu \eta) - \delta(\xi \times \mu \eta) - (-1)^{m_1} \xi \times \mu \delta \eta = \partial(\xi \times \mu \eta) - \partial(\gamma) \), as desired. \( \square \)

**Corollary 5.33.** Let \( X \) be in \( \text{SmAff}_k \) or in \( \text{SmProj}_k \). Let \( \xi \in \text{CH}^m(X[r]_m, n_1) \) and \( \eta \in \text{CH}^m(X[r]_m, n_2) \). Then, for the internal product \( \wedge \) in Definition 5.13 the differential \( \delta \) satisfies the Leibniz rule \( \delta(\xi \wedge \eta) = \delta(\xi) \wedge \eta + (-1)^{m_1} \xi \wedge \delta(\eta) \) in \( \text{CH}^{m+q-r}(X[r]_m, n+1) \).

**Proof.** By Theorem 5.32 we may choose representatives of \( \xi \) and \( \eta \), all of whose codimension 1 faces are trivial. Then, \( \delta(\xi \times \mu \eta) - \delta(\xi \times \mu \eta) - (-1)^{m_1} \xi \times \mu \delta \eta \) is the boundary of an admissible cycle by Proposition 5.32. Now, applying the pull-back via the diagonal map \( \Delta_X^* : \text{CH}^{q-r}(X \times X)[r]_m, n+1) \to \text{CH}^{q-r}(X[r]_m, n+1) \), we deduce the Leibniz rule for \( \wedge_X \).

The above discussion summarizes as follows:

**Theorem 5.34.** Let \( X \) be in \( \text{SmAff}_k \) or in \( \text{SmProj}_k \) over a perfect field \( k \) of \( \text{char}(k) \neq 2 \). Then, \( (\text{CH}(X[r]_m, \wedge_X; \delta) \) forms a commutative differential graded algebra.

6. Witt-complex structure

**Convention:** In this section, a smooth affine \( k \)-scheme means an object in \( \text{SmAff}^\text{ess}_k \), i.e. an object of either \( \text{SmAff}_k \) or \( \text{SmLoc}_k \). We often refer the former as the finite type case, and the latter as the semi-local case, in short.

The objective of Section 6 is to prove that the additive higher Chow groups of the schemes in \( \text{SmAff}^\text{ess}_k \) have restricted Witt-complex structures considered in [18] and [30]. Since we exclusively use the case \( r = 1 \) only, we use the notations \( T_\mathbb{Z}^q(X, n; m) \) and \( \text{TCH}^q(X, n; m) \) instead of \( z^q(X[1]; D_{m+1}, n-1) \) and \( \text{CH}^q(X[1]; D_{m+1}, n-1) \).

For a field \( k \), Rülling proved in [30] that the big de Rham-Witt complex \( \mathbb{W}_m \Omega^\bullet_k \) is geometrically realized by additive higher Chow groups of \( \text{Spec}(k) \) of 0-cycles. When \( X \) is a smooth projective variety over a perfect field \( k \), it was proven in [18] that additive higher Chow groups of \( X \) form a restricted Witt-complex over \( k \).

One result of the section is Theorem 6.9 which shows that when \( \text{Spec}(R) \) is in \( \text{SmAff}^\text{ess}_k \), additive higher Chow groups of \( R \) form a restricted Witt-complex over \( R \), not just over \( k \).

6.1. Witt-complex structure over \( k \). In Section 6.1 we aim to show that additive higher Chow groups form a functorial restricted Witt-complex over \( k \) on \( \text{SmAff}^\text{ess}_k \). From now on, our ground field \( k \) is perfect of any characteristic other than 2. For \( X \in \text{Sch}^\text{ess}_k \), we let \( \text{TCH}(X; m) := \{\text{TCH}^q(X, n; m)\}_{n,q \in \mathbb{N}} \) and \( \text{TCH}^M(X; m) := \{\text{TCH}^M(X, n; m)\}_{n,m \in \mathbb{N}} \). The superscript \( M \) is for Milnor. Let \( \text{TCH}(X) := \{\text{TCH}(X; m)\}_{m \in \mathbb{N}} \) and \( \text{TCH}^M(X) := \{\text{TCH}^M(X; m)\}_{m \in \mathbb{N}} \). We similarly define \( \mathcal{TCH}(X; m), \mathcal{TCH}^M(X; m), \mathcal{TCH}(X), \) and \( \mathcal{TCH}^M(X) \) for \( X \in \text{Sch}_k \).

For \( r \geq 1 \), let \( \phi_\ast : \mathbb{G}_m \to \mathbb{G}_m \) be the morphism \( x \mapsto x^r \), which induces \( \phi_\ast : \text{Spec}(R) \times B_n \to \text{Spec}(R) \times B_n \). By [18] §5.1, 5.2, we have the Frobenius \( F_r : \text{TCH}^q(R, n; rm+r-1) \to \text{TCH}^q(R, n; m) \) and the Verschiebung \( V_r : \text{TCH}^q(R, n; m) \to \text{TCH}^q(R, n; rm+r-1) \) given by \( F_r = \phi_{rs} \) and \( V_r = \phi^\ast_r \). On the other hand, for any \( m \geq 1 \), we have a natural inclusion \( \mathcal{R} : \mathcal{TCH}(R, \bullet; m+1) \to \)
\(Tz^q(R, \bullet; m)\), which induces \(\mathcal{R} : \text{TCH}^q(R, n; m + 1) \to \text{TCH}^q(R, n; m)\), called the restriction. Finally, by Sections 5.3 and 5.4 (see also [18, §4.1]), there is a differential \(\delta : Tz^q(R, \bullet; m) \to Tz^q(R, \bullet + 1; m)\), which induces \(\delta : \text{TCH}^q(R, n; m) \to \text{TCH}^q(R, n + 1; m)\).

**Theorem 6.1.** Let \(X \in \text{SmAff}^{\mathrm{ess}}_k\) be equidimensional. Let \(m \geq 1\). Then, \(\text{TCH}(X; m)\) is a DGA and \(\text{TCH}_M^M(X; m)\) is its sub-DGA. Furthermore, \(\text{TCH}(X)\) is a restricted Witt-complex over \(k\) and \(\text{TCH}_M^M(X)\) is a restricted sub-Witt-complex over \(k\). These structures are functorial.

**Proof.** In [18, Theorem 1.1, Scholium 1.2], it was stated if the moving lemma holds for \(X\), then \(\text{TCH}(X; m)\) and \(\text{TCH}_M^M(X; m)\) are DGAs, and that \(\text{TCH}(X)\) and \(\text{TCH}_M^M(X)\) are restricted Witt-complexes over \(k\) with respect to the above \(\mathcal{R}, F_r, V_r\). Now the moving lemma now holds by Theorem 4.1 and Theorem 4.10 essentially by [13], so the theorem holds. We sketch the structure using the results of Section 5 and the results proven in [18].

That the groups form a DGA was already proven in Theorem 5.34.

We now go to the restricted Witt-complex structure. The functoriality of the restriction operator \(\mathcal{R}\) recalled in the above, was stated in [18, Corollary 5.19], which we easily check here: let \(f : X \to Y\) be a morphism in \(\text{SmAff}^{\mathrm{ess}}_k\), and consider the following commutative diagram:

\[
\begin{array}{ccc}
Tz^q_W(Y, \bullet; m + 1) & \xrightarrow{f^*} & Tz^q(X, \bullet; m + 1) \\
\downarrow & & \downarrow \\
Tz^q_W(Y, \bullet; m) & \xrightarrow{f^*} & Tz^q(X, \bullet; m),
\end{array}
\]

where \(W\) is a finite set of locally closed subsets of \(Y\), where the horizontal maps are chain maps given by inverse images, as in the proof of Theorem 4.5 and Corollary 4.15. The diagram and Theorems 4.1 and 4.10 imply that \(f^* \mathcal{R} = \mathcal{R} f^*\) because the vertical inclusions induce \(\mathcal{R}\) by definition.

For each \(r \geq 1\), the Frobenius \(F_r\) and Verschiebung \(V_r\) recalled in the above are functorial as proven in [18, Lemmas 5.4, 5.9], and that \(F_r\) is a graded ring homomorphism is proven in [18, Corollary 5.6]. The maps \(\lambda : \mathbb{W}_m(k) \to \text{TCH}^1(X, 1; m)\) are given by the pull-back (using Theorem 4.5) via the structure map \(p : X \to \text{Spec}(k)\), because the main theorem of [30] shows that \(\text{TCH}^1(X, 1; m) \cong \mathbb{W}_m(k)\).

Finally, for the properties (i), (ii), (iii), (iv), (v) in Section 2.2.2 all of them are proven in [18, Theorem 5.13], where none requires the projectivity assumption. \(\square\)

**Corollary 6.2.** Let \(m \geq 1\) be an integer. Then, \(\text{TCH}(-; m)\) and \(\text{TCH}_M^M(-; m)\) define presheaves of DGAs on \(\text{Sch}_k\), and the pro-systems \(\text{TCH}(\cdot)\) and \(\text{TCH}_M^M(\cdot)\) define presheaves of restricted Witt-complexes over \(k\) on \(\text{Sch}_k\).

**Proof.** Let \(X \in \text{Sch}_k\). By definition, \(\text{TCH}(X; m)\) is the colimit over all \((X \to A) \in (X \downarrow \text{SmAff}^{\mathrm{ess}}_k)^{\mathrm{op}}\) of \(\text{TCH}(A; m)\). But the category of DGAs is closed under filtered colimits (see [12]) so that \(\text{TCH}(X; m)\) is a DGA. For each morphism \(f : X \to Y\) in \(\text{Sch}_k\), one checks \(f^* : \text{TCH}(Y; m) \to \text{TCH}(X; m)\) is a morphism of DGAs. The other assertions follow easily using Theorem 6.1. \(\square\)
6.2. Witt-complex structure over $R$. Let $X = \text{Spec}(R) \in \text{SmAff}_k^{\text{ess}}$. In Theorem \ref{thm:6.1} we saw that $\text{TCH}(X)$ is a restricted Witt-complex over $k$. The objective of Section \ref{sec:6.2} is to show that $\text{TCH}(X)$ is not only a restricted Witt-complex over $k$, but also over $R$.

Let $m \geq 1$ be an integer. Let $R$ be an equidimensional $k$-algebra. We first define a group homomorphism $\tau_R : \mathbb{W}_m(R) \to \text{TCH}^1(R, 1; m)$. Recall that the underlying abelian group of $\mathbb{W}_m(R)$ identifies with the multiplicative group $(1 + tR[[t]])^\times$ of $(1 + tR[[t]])^\times$. Here, every $p(t) \in (1 + tR[[t]])^\times$ is uniquely written as $p(t) = \prod_{n \geq 1} (1 - a_n t^n)$ for $a_n \in R$. In particular, the group $(1 + tR[[t]])^\times/(1 + t^m + 1R[[t]])^\times$ is generated by polynomials of the form $\prod_{n=1}^m (1 - a_n t^n)$ with $a_n \in R$.

For each polynomial $1 - tf(t)$ contained in $(1 + R[[t]])^\times$, consider the closed subscheme of $\text{Spec}(R[t])$ given by the ideal $(1 - tf(t))$, and let $\Gamma_{(1-tf(t))}$ be its associated cycle. By definition, $\Gamma_{(1-tf(t))} \cap \{ t = 0 \} = \emptyset$ so that $\Gamma_{(1-tf(t))} \in \text{Tz}^1(R, 1; m)$. In case $f(t) = at^n - 1$, $n \geq 1$, let $\Gamma_{a,n} := \Gamma_{(1-at^n)}$. Observe first:

**Lemma 6.3.** Let $f(t), g(t)$ be polynomials in $R[t]$, and let $h(t) \in R[t]$ be the unique polynomial such that $(1 - tf(t))(1 - tg(t)) = 1 - th(t)$. Then, $\Gamma_{(1-th(t))} = \Gamma_{(1-tf(t))} + \Gamma_{(1-tg(t))}$ in $\text{Tz}^1(R, 1; m)$.

**Proof.** This is obvious by $(1 - tf(t))(1 - tg(t)) = 1 - th(t)$. \hfill $\square$

**Lemma 6.4.** For $n \geq m + 1$ and $a \in R$, we have $\Gamma_{a,n} \equiv 0$ in $\text{TCH}^1(R, 1; m)$.

**Proof.** Consider the closed subscheme $\Gamma \subset X \times \mathbb{G}_m \times \square$ given by $y_1 = 1 - at^n$. Let $\nu : \Gamma^N \to \overline{\Gamma} \leftarrow X \times \mathbb{A}_1 \times \mathbb{P}_1$ be the normalization of the Zariski closure $\overline{\Gamma}$ in $X \times \mathbb{A}_1 \times \mathbb{P}_1$. Since $at^n = 1 - y_1$ on $\overline{\Gamma}$, we see that $\nu^* \{ t = 0 \} \leq \nu^* \{ y_1 = 1 \}$ on $\Gamma^N$. Since $n \geq m + 1$, this shows that $\Gamma$ satisfies the modulus $m$ condition. Since $\partial \Gamma = 0$ and $\partial \nu^* \Gamma = \Gamma_{a,n}$ (which is of codimension 1), the cycle $\Gamma$ is an admissible cycle in $\text{Tz}^1(R, 2; m)$ such that $\partial \Gamma = \Gamma_{a,n}$. This shows $\Gamma_{a,n} \equiv 0$ in $\text{TCH}^1(R, 1; m)$ as desired. \hfill $\square$

**Corollary 6.5.** Let $R$ be an equidimensional $k$-algebra. Then, the map $\tau_R : \mathbb{W}_m(R) \to \text{TCH}^1(R, 1; m)$ that sends a polynomial $1 - tf(t)$ to $\Gamma_{(1-tf(t))}$ is a well-defined group homomorphism.

**Proof.** Every member of $\mathbb{W}_m(R)$ is representable by a polynomial of the form $1 - tf(t)$, where $f(t) \in R[t]$. That $\tau_R$ is well-defined follows from Lemma \ref{lem:6.4} and that this group homomorphism follows from Lemma \ref{lem:6.3} and the unique factorization of elements in $(1 + tR[[t]])^\times$ into product of the form $\prod_{n \geq 1} (1 - a_n t^n)$. \hfill $\square$

Recall from \cite{30} Appendix A that for each $r \geq 1$, we have the Frobenius $F_r : \mathbb{W}_{rm+r-1}(R) \to \mathbb{W}_m(R)$ and the Verschiebung $V_r : \mathbb{W}_m(R) \to \mathbb{W}_{rm+r+1}(R)$. It is given by $F_r(1 - at^n) = (1 - at^n)^s$, where $s = \gcd(r, n)$ and $V_r(1 - at^n) = 1 - at^rn$. On the other hand, as seen in Section \ref{sec:6.1} we have operations $F_r$ and $V_r$ on $\{ \text{TCH}^1(R, 1; m) \}_{m \in \mathbb{N}}$. Now Corollary \ref{cor:6.5} can be strengthened as follows:

**Lemma 6.6.** Let $R$ be an equidimensional $k$-algebra. Then, the map $\tau_R : \mathbb{W}_m(R) \to \text{TCH}^1(R, 1; m)$ of Corollary \ref{cor:6.5} commutes with the Frobenius $F_r$ and the Verschiebung $V_r$ operators on both sides.
Proof. That \( \tau_R V_r = V_r \tau_R \) is easy; by definition, we have \( V_r(\Gamma_{a,n}) = \Gamma_{a,r,n}, \) while 
\( V_r(1 - at^n) = 1 - at^n. \) Hence,\( \tau_R V_r(1 - at^n) = \tau_R(1 - at^n) = \Gamma_{a,r,n}, \) and \( V_r \tau_R(1 - at^n) = V_r(\Gamma_{a,n}) = \Gamma_{a,r,n}. \) This shows \( \tau_R V_r = V_r \tau_R. \)

That \( \tau_R F_r = F_r \tau_R \) is slightly more involved. Recall that \( F_r(1 - at^n) = (1 - a^r t^n)^s, \)
where \( s = \gcd(r, n). \) Write \( n = n's \) and \( r = r's, \) where \( 1 = (r', n'). \) Hence, we have \( \tau_R F_r(1 - at^n) = s \Gamma_{a,r,n} = s V_r(\Gamma_{a,r,n}) = s V_r(\Gamma_{a,r,n}) = : \heartsuit, \) while, on the other hand, \( F_r \tau_R (1 - at^n) = F_r \Gamma_{a,n} = F_r V_n(\Gamma_{a,n}) = : \diamondsuit. \)

First observe that when \( n = 1, \) we have \( s = 1, \) \( r = r', \) \( n = n', \) and we have \( \heartsuit = F_r(\Gamma_{a,1}) = \Gamma_{a,r,1} = \heartsuit, \) so that \( \tau_R F_r(1 - at) = F_r \tau_R(1 - at), \) indeed.

For a general \( n \geq 1, \) we have \( F_r V_n = F_r F_s V_{n'} = F_r(\Gamma_{a,s}) \circ (s \cdot \text{Id}) \circ V_{n'} = s F_r V_{n'} = \dagger s V_n F_r, \) where \( \dagger \) holds because \( (r', n') = 1. \) Since \( F_r(\Gamma_{a,1}) = \Gamma_{a, r,1} \) (by the first case), we have \( \heartsuit = F_r V_n(\Gamma_{a,1}) = s V_n F_r(\Gamma_{a,1}) = s V_n(\Gamma_{a, r,1}) = \heartsuit. \) This shows \( \tau_R F_r = F_r \tau_R. \)

Remark 6.7. In the proof of Lemma 6.6, we saw that for \( s = (r, n), \)
\[
(6.1) \quad F_r(\Gamma_{a,n}) = s \Gamma_{a,n}^{r/n}, \quad V_r(\Gamma_{a,n}) = \Gamma_{a, n}. \]

Proposition 6.8. Let \( \Spec(R) \in \text{SmAff}_{k}^{\text{ess}} \) be equidimensional. Then, the map \( \tau_R : \mathbb{W}_{m}(R) \to \text{TCH}^1(R; 1; m) \) is a ring homomorphism, commuting with the Frobenius \( F_r \) and Verschiebung \( V_r \) for \( r \geq 1. \)

Proof. That \( \tau_R \) is a group homomorphism commutes with \( F_r \) and \( V_r, \) was proven in Lemma 6.6. It remains to prove that \( \tau_R \) respects the products. By [3, Proposition (1.1)], it is enough to prove that for \( a, b \in R \) and \( u, v \geq 1, \)
\[
(6.2) \quad \Gamma_{a,u} \wedge \Gamma_{b,v} = w \Gamma_{a^{r/b} b^{s/v} u^w} \quad \text{in TCH}^1(R; 1; m),
\]
where \( w = \gcd(u, v) \) and \( \wedge \) is the product structure on the ring \( \text{TCH}^1(R; 1; m) \) as in Theorem 6.1 (see Step 1 below, too). (N.B. The smoothness of \( R \) is essential to have the pull-back \( \Delta^* \) in the definition of \( \wedge. \) ) We prove it in three steps.

Step 1. First, consider the case when \( u = v = 1, \) i.e. we prove \( \Gamma_{a,1} \wedge \Gamma_{b,1} = \Gamma_{ab,1}. \) Recall that \( \wedge_{\mu} \) is defined as the composition \( \Delta^* \circ \mu_{\ast} \circ \times \) in
\[
\Spec(R) \times \mathbb{G}_m \times \Spec(R) \times \mathbb{G}_m \xrightarrow{\mu} \Spec(R) \times \Spec(R) \times \mathbb{G}_m \xrightarrow{\Delta} \Spec(R) \times \mathbb{G}_m.
\]

Under the identification \( \Spec(R) \times \Spec(R) \simeq \Spec(R \otimes_k R), \) we have \( \mu_{\ast}(\Gamma_{a,1} \times \Gamma_{b,1}) = \Gamma_{(a \otimes 1)} \times \Gamma_{(1 \otimes b),1} = \Gamma_{ab,1} \) because \( \Delta : \Spec(R) \to \Spec(R \otimes_k R) \) is given by the multiplication \( R \otimes_k R \to R. \) This proves (6.2) for Step 1.

For the following remaining two steps, we use the projection formula: \( x \wedge V_s(y) = V_s(F_s(x) \wedge y), \) which we can use by Theorem 6.1

Step 2. Consider the case when \( v = 1, \) but \( u \geq 1 \) is any integer. We apply the projection formula to \( x = \Gamma_{b,1} \) and \( y = \Gamma_{a,1} \) with \( s = u. \) Since \( \text{TCH}^1(R; 1; m) \) is a commutative ring, we obtain \( V_u(\Gamma_{a,1}) \wedge \Gamma_{b,1} = V_u(\Gamma_{a,1} \wedge F_u(\Gamma_{b,1})). \) Here, the left hand side is \( \Gamma_{a,u} \wedge \Gamma_{b,1} \) by (6.1), while the right hand side is \( =^1 V_u(\Gamma_{a,1} \wedge \Gamma_{b,1}) =^2 V_u(\Gamma_{ab,1}) =^3 \Gamma_{ab,u,1}, \) where \( =^1 \) and \( =^3 \) hold by (6.1) and \( =^2 \) holds by Step 1. This proves (6.2) for Step 2.

Step 3. Finally, let \( u, v \geq 1 \) be any integers. Let \( w = \gcd(u, v). \) We again apply the projection formula to \( x = V_u(\Gamma_{a,1}) \) and \( y = \Gamma_{b,1}, \) \( s = v, \) so that \( V_u(\Gamma_{a,1}) \wedge V_v(\Gamma_{b,1}) = V_v(F_v(V_u(\Gamma_{a,1})) \wedge \Gamma_{b,1}) \). Its left hand side coincides with that of (6.2) by (6.1). Its right hand side is \( =^1 V_v(F_v(\Gamma_{a,u}) \wedge \Gamma_{b,1}) =^2 V_v(t \Gamma_{a^{r/b} b^{s/v} u^w} \wedge \Gamma_{b,1}), \) where
Corollary 6.10. Let 
\[ H \]
Hence, both hand sides of (6.3) coincide. This completes the proof. □

where 
\[ TCH \]
defines a morphism of restricted Witt-complexes over 
\[ Spec (R) \]. 
As we saw in the proof of Theorem 6.1, we already have the restriction \( \mathcal{R} \), 
the differential \( \delta \), the Frobenius \( F_r \) and the Verschiebung \( V_r \) defined by the same formulae. Furthermore, by Proposition 6.8, now we have ring homomorphisms 
\[ \lambda = \tau_R : \mathbb{W}_m(R) \to TCH^1(R, 1; m) \] for \( m \geq 1 \). The properties (i), (ii), (iii), (iv) in Section 2.2.2 are independent of the choice of the ring, so that these four properties that we checked in Theorem 6.1 still work.

What is left to be checked is the property (v) that for all \( a \in R \) and \( r \geq 1 \),
\[ F_r \delta \tau_R ([a]) = \tau_R [(a)^{r-1}] \delta \tau_R ([a]), \]
where, we have shrunk the product notation \( \wedge \). We check it in the following, 
identifying \( \mathbb{W}_m(R) \) with \( (1 + tR[[t]])^* / (1 + t^{m+1}R[[t]])^* \).

If \( a = 0 \), then \( \tau_R ([a]) = \tau_R (1 - 0 \cdot t) = \Gamma_{(1 - 0 \cdot t)} = \emptyset \), because the ideal \( (1) = (1 - 0 \cdot t) \) in \( R[t] \) is \( R \{ t \} \). So, both sides of (6.3) are zero.

If \( a = 1 \), then \( \tau_R ([a]) = \tau_R (1 - t) = \Gamma_{(1 - t)} \). But, in our definition of \( \delta \), to compute it, we should first restrict the cycle \( \Gamma_{(1 - t)} \subset Spec (R) \times \mathbb{G}_m \) onto \( Spec (R) \times (\mathbb{G}_m \setminus \{1\}) \), which becomes empty. Hence, \( \delta \tau_R ([a]) = \delta \Gamma_{(1 - t)} = \emptyset \), so again both sides of (6.3) are zero.

Let \( a \in R \setminus \{ 0, 1 \} \). Then, \( \tau_R ([a]) = \Gamma_{(1 - at)} \subset Spec (R) \times \mathbb{G}_m \), and \( \delta \tau_R ([a]) \) is given by the ideal \( (1 - at, 1 - ty_1) \) in \( R[t, y_1] \). Since \( t \) is not a zero-divisor in \( R[t, y_1] \), as ideals we have \( (1 - at, 1 - ty_1) = (1 - at, y_1 - a) \). Hence, \( F_r \delta \tau_R ([a]) \) is given by the ideal \( (1 - a^r t, y_1 - a) \) in \( R[t, y_1] \). On the other hand,
\[ \tau_R ([a]^{r-1}) \delta \tau_R ([a]) = \Gamma_{(1 - a^r \cdot t)} \wedge Spec \left( \frac{R[t, y_1]}{(1 - a^r \cdot t, y_1)} \right) = \Gamma_{\Delta^* \left( \frac{R[t, y_1]}{(1 - a^r \cdot t, y_1)} \right)} \]
where \( \Delta^* \) holds because \( \Delta \) is induced by the product homomorphism \( R \otimes_k R \to R \). Hence, both hand sides of (6.3) coincide. This completes the proof. □

Corollary 6.10. Let \( Spec (R) \in SmAff^{ess}_k \) be equidimensional. Then, for each \( n, m \geq 1 \), there is a unique homomorphism \( \tau^R_{nm} : \mathbb{W}_m \Omega^{n-1} R \to TCH^n (R, n; m) \) that defines a morphism of restricted Witt-complexes over \( R \), such that \( \tau^R_{1m} = \tau_R \).

Proof. Since \( TCH^M(R) \) is a restricted Witt-complex over \( R \) by Theorem 6.9, the universal property of \( \{ \mathbb{W}_m \Omega^{*} R \}_{m \in \mathbb{N}} \) gives unique homomorphisms \( \tau^R_{nm} \). We have \( \tau^R_{1m} = \tau_R \) because the map \( \lambda \) of Section 2.2.2 is given by \( \tau_R \) in Theorem 6.9. □

Example 6.11. Since \( \{ \tau^R_{nm} \}_{n \geq 1} \) is a morphism of DGAs, using the same sort of computations as in (6.4), one easily checks that for \( a \in R, b_1, \cdots, b_{n-1} \in R^\times \), we have \( \tau^R_{nm} (V_i ([a]) dlog [b_1] \wedge \cdots \wedge dlog [b_{n-1}]) = [Z_{b_1, \cdots, b_{n-1}}] \), where \( Z_{b_1, \cdots, b_{n-1}} := \text{Spec} \left( \frac{R[t, y_1, \cdots, y_{n-1}]}{(1 - a^r, y_1 - b_1, \cdots, y_{n-1} - b_{n-1})} \right) \subset Spec (R) \times \mathbb{A}^1 \times (\mathbb{A}^1)^{n-1} \).
Lemma 6.14. Let \( R \) be a factorial k-algebra essentially of finite type. Let \( I \subset R[t] \) be a height 1 prime ideal such that \((I, t) = (1)\). Then, \( I \) is a

Remark 6.12. Exploiting the idea we used in Section 4.4 to define the presheaf \( \mathcal{TCH} \), we may redefine \( \mathbb{W}_m \Omega^n_{(-)} \) so that it becomes a presheaf on \( \text{Sm}_k \) which maps to \( \mathcal{TCH}^n(-, n; m) \). Namely, for \( X \in \text{Sm}_k \), define \( \mathbb{W}_m \Omega^n_{X} \) to be the presheaf colim(\( X \downarrow \text{SmAff}_k \))\( \mathbb{W}_m \Omega^n_{(-)} \). By Corollary 6.10 for each \( X = \text{Spec} (R) \in \text{SmAff}_k \), we have a natural homomorphism \( \tau_{n,m}: \mathbb{W}_m \Omega^n_{R}^{-1} \to \mathcal{TCH}^n(R, n; m) \). So, by taking the colimits over all objects in the index category, and the Zariski sheafifications, we obtain the morphism of Zariski sheaves \( \tau_{n,m}: \mathbb{W}_m \Omega^n_{(\_)}^{-1} \to \mathcal{TCH}^n(-, n; m)_\text{Zar} \).

An interesting question to ask is whether this is an isomorphism. We answer it affirmatively in [21].

6.3. The case of integral k-algebras. We prove a bit more on the group homomorphism \( \tau_R: \mathbb{W}_m (R) \to \mathcal{TCH}^1(R, 1; m) \) of Corollary [6.5]. Here is the summary:

Theorem 6.13. Let \( R \) be an equidimensional k-algebra essentially of finite type.

1. If \( R \) is an integral domain, then \( \tau_R \) is an injective group homomorphism compatible with Frobenius and Verschiebung operators.
2. If \( R \) is a UFD, then \( \tau_R \) is a group isomorphism compatible with Frobenius and Verschiebung operators.
3. If \( R \) is a smooth k-algebra that is a UFD, then \( \tau_R \) is a ring isomorphism compatible with Frobenius and Verschiebung operators. In particular, the statement holds when \( R \) is a smooth semi-local k-algebra.

Proof. (1) When \( R \) is integral, let \( K := \text{Frac} (R) \), and \( \iota: R \hookrightarrow K \) be the inclusion. This induces a commutative diagram

\[
\begin{array}{ccc}
\mathbb{W}_m (R) & \xrightarrow{\mathbb{W}_m (\iota)} & \mathbb{W}_m (K) \\
\tau_R \downarrow & & \tau_K \downarrow \\
\mathcal{TCH}^1(R, 1; m) & \xrightarrow{\sim} & \mathcal{TCH}^1(K, 1; m),
\end{array}
\]

where the bottom map is the flat pull-back via \( \text{Spec} (K) \to \text{Spec} (R) \), and \( \tau_K \) is the isomorphism of [30] Theorem 1]. Recall that when \( A \to B \) is an injective (resp. surjective) ring homomorphism, then \( \mathbb{W}_m (A) \to \mathbb{W}_m (B) \) is also injective (resp. surjective) by [30] Properties A.1.(i)]. Because \( \tau_K \) and \( \mathbb{W}_m (\iota) \) are injective, \( \tau_R \) is also injective. This proves (1).

(2) Since \( R \) is a UFD, by Lemma 6.14 proven below, every closed irreducible admissible cycle in \( \text{Tz}^1(R, 1; m) \) is given by an ideal of the form \((1 - tf(t))\) for some nonzero polynomial \( f(t) \in R[t] \). In particular, the map \( \tau_R \) is surjective. Hence, combined with (1), we see that \( \tau_R \) is a group isomorphism, proving (2).

(3) Since \( R \) is smooth, by Proposition 6.8, we know that \( \tau_R \) is a ring homomorphism compatible with Frobenius and Verschiebung operators. On the other hand, \( R \) is a UFD so that (2) shows that \( \tau_R \) is bijective. Hence, the map \( \tau_R \) is a ring isomorphism. When \( R \) is a smooth semi-local k-algebra, a theorem of Auslander [26] Theorem 20.3] shows \( R \) is also a UFD. This proves (3).

In the proof of the above, we used the following lemma:

Lemma 6.14. Let \( R \) be an equidimensional factorial k-algebra essentially of finite type. Let \( I \subset R[t] \) be a height 1 prime ideal such that \((I, t) = (1)\). Then, \( I \) is a
principal ideal generated by a polynomial of the form $1 - tf(t)$ for some nonzero polynomial $f(t) \in R[t]$.

**Proof.** Recall that in a UFD, every prime ideal of height 1 is principal ([9, Proposition I.1.12A]). Since $R$ is a UFD, so is $R[t]$. Hence, $I$ is a principal ideal.

We first claim that $I$ contains an element of the form $1 - tf(t)$ with $f(t) \neq 0$. Indeed, since $(I, t) = (1)$, we have $1 = tg(t) + h(t)$ for some $g(t) \in R[t]$ and $h(t) \in I$. Express $h(t) = a_0 + a_1 t + \cdots + a_s t^s$ for some $a_0, \ldots, a_s \in R$. By plugging in $t = 0$, we have $a_0 = 1$.

If $s = 0$, then $h(t) = 1 \in I$, so that $I = R[t]$, which contradicts that $I$ is a prime ideal of $R[t]$. Hence $s > 0$. Express $h(t) = 1 - t(-a_1 - \cdots - a_s t^{s-1}) =: 1 - th_1(t)$. Since $s > 0$, we must have $h_1(t) \neq 0$, so we proved the claim.

Note that every irreducible factor of a polynomial of the form $1 - tf(t)$ is of this form again. Hence, if we choose an element $1 - tf(t) \in I$ of the form $1 - tf(t)$, where $f(t) \neq 0$, with the minimal degree, it is automatically irreducible. But, $I$ is a minimal prime ideal so that we must have $(1 - tf(t)) = I$. \qed

### 6.4. Push-forward.

In this last part of the paper, we prove that for a finite morphism $f : X \to Y$ in $\text{SmAff}_{k}^{\text{ess}}$, the push-forward $f_{*}$ on additive higher Chow groups commute with the restriction $\mathfrak{R}$, the differential $\delta$, the Frobenius $F_{r}$, and the Verschiebung $V_{r}$ operators on them.

**Proposition 6.15.** Under the above notations, for $r \geq 1$, we have

\begin{align*}
(a) \quad f_{*}\mathfrak{R} = \mathfrak{R}f_{*}; \quad (b) \quad f_{*}\delta = \delta f_{*}; \quad (c) \quad f_{*}F_{r} = F_{r}f_{*}; \quad (d) \quad f_{*}V_{r} = V_{r}f_{*}.
\end{align*}

**Proof.** The assertion (a) follows from the following commutative diagram:

\[
\begin{array}{ccc}
\text{Tz}^{q}(X, \bullet; m + 1) & \xrightarrow{f_{*}} & \text{Tz}^{q}(Y, \bullet; m + 1) \\
\downarrow & & \downarrow \\
\text{Tz}^{q}(X, \bullet; m) & \xrightarrow{f_{*}} & \text{Tz}^{q}(Y, \bullet; m).
\end{array}
\]

For the assertion (b), we use the diagram (6.5) below:

\begin{equation}
X \times \mathbb{G}_{m} \times \square^{n-1} \xrightarrow{\text{Id} \times \psi} X \times \mathbb{G}_{m} \times \square^{n-1} \xrightarrow{\text{Id} \times \delta_{n}} X \times \mathbb{G}_{m} \times \square^{n} \\
\downarrow \quad f \times \text{Id} \quad \downarrow \quad f \times \text{Id} \quad \downarrow \quad f \times \text{Id} \\
Y \times \mathbb{G}_{m} \times \square^{n-1} \xrightarrow{\text{Id} \times \psi} Y \times \mathbb{G}_{m} \times \square^{n-1} \xrightarrow{\text{Id} \times \delta_{n}} Y \times \mathbb{G}_{m} \times \square^{n}.
\end{equation}

Since $\delta = (\text{Id} \times \delta_{n})(\text{Id} \times \psi)^{*}$, it is enough to show that $f_{*}$ commutes with both $(\text{Id} \times \delta_{n})$ and $(\text{Id} \times \psi)^{*}$. From the right square, we immediately have $(\text{Id} \times \delta_{n})f_{*} = f_{*}(\text{Id} \times \delta_{n})$. On the other hand, the left square is Cartesian, where horizontal maps are flat and vertical maps are finite, so that by applying [7, Proposition 1.7], we have $f_{*}(\text{Id} \times \psi)^{*} = (\text{id} \times \psi)^{*}f_{*}$. This shows $f_{*}\delta = \delta f_{*}$ as desired.

For the assertion (c), since both $f_{*}$ and $F_{r}$ are equivariant with respect to taking faces, it suffices to check that on the level of cycles, we have $f_{*}\phi_{r*} = \phi_{r*}f_{*}$. It
reduces to check that the diagram below commutes
\[
\begin{array}{c}
\xymatrix{ X \times \mathbb{G}_m \ar[r]^{\text{Id} \times \phi_r} \ar[d]_{f \times \text{Id}} & X \times \mathbb{G}_m \ar[d]^{f \times \text{Id}} \\
Y \times \mathbb{G}_m \ar[r]_{\text{Id} \times \phi_r} & Y \times \mathbb{G}_m. }
\end{array}
\]

Indeed, one sees that both compositions are \( f \times \phi_r \). This proves the assertion \((c)\).

For the assertion \((d)\), since both \( f_* \) and \( V_r \) are equivariant with respect to taking faces, it suffices to check that on the level of cycles, we have \( f_* \phi_r^* = \phi_r^* f_* \). But, since the diagram \((6.6)\) is Cartesian, where horizontal maps are flat and vertical maps are finite, we deduce the result by \([7, \text{Proposition 1.7}]\). This proves \((d)\). \( \square \)

As a corollary, we have:

**Theorem 6.16.** Let \( f : X \to Y \) be a finite morphism in \( \text{SmAff}_{k}^{\text{ess}} \). Then, the induced map \( f_* : \{ \text{TCH}^n(X, n; m) \}_{n, m \in \mathbb{N}} \to \{ \text{TCH}^n(Y, n; m) \}_{n, m \in \mathbb{N}} \) is a morphism of restricted Witt-complexes over \( k \).

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