Multisequent Gentzen Deduction Systems for $B_2^2$-valued first-order logic

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ABSTRACT

For the four-element Boolean algebra $B_2^2$, a multisequent $\Gamma,\Delta,\Sigma,\Pi$ is a generalization of sequent $\Gamma \Rightarrow \Delta$ in traditional $B_2^2$-valued first-order logic. By defining the truth-values of quantified formulas, a Gentzen deduction system $G_2^2$ for $B_2^2$-valued first-order logic will be built and its soundness and completeness theorems will be proved.

Key Words: Multisequent, Hypersequent, $B_2^2$-valued semantics, Soundness theorem, Completeness theorem

1. INTRODUCTION

In traditional propositional logic,\cite{1} the negation connective $\neg$ is eliminated via the following rules in a Gentzen deduction system:

\[
\begin{align*}
(\neg L) & \quad \Gamma, \neg A \Rightarrow \Delta \\
(\neg R) & \quad \Gamma, A \Rightarrow \Delta
\end{align*}
\]

The truth-value of a sequent $\Gamma \Rightarrow \Delta$ under an assignment $v$ is defined by a condition of ($\exists \vee \exists$)-form, that is, either some formula $A$ in $\Gamma$ is false, or some formula $B$ in $\Delta$ is true.

In $B_2^2$-valued first-order logic, where $B_2^2 = \{\top, \bot, \neg, \exists\}$ is the four-element Boolean algebra, a multisequent is a quadruple $(\Gamma, \Delta, \Sigma, \Pi)$, where $\Gamma, \Delta, \Sigma, \Pi$ are sets of formulas. Multisequent $(\Gamma, \Delta, \Sigma, \Pi)$ is true in a model $M$ and an assignment $v$ if either

- some formula $A$ in $\Gamma$ has truth-value $t$, or
- some formula $B$ in $\Delta$ has truth-value $\top$, or
- some formula $C$ in $\Sigma$ has truth-value $\bot$, or
- some formula $D$ in $\Pi$ has truth-value $\bot$.

By the semantics, multisequents are different from hypersequents.\cite{2,3} A sequent $\Gamma \Rightarrow \Delta$ is taken as a multisequent $\Delta | \Gamma$.

Here, negation $\neg$ commutes $t$ with $\bot$ and $\top$ with $\bot$. Traditional deduction rules $(\neg L)$ and $(\neg R)$ do not work here.

For quantifiers, in traditional first-order logic, $\forall x A(x)$ is true if for each element $a, A(x/a)$ is true; and $\forall x A(x)$ is false if for some element $a, A(x/a)$ is false. In $B_2^2$-valued first-order logic, we define that $\forall x A(x)$ has truth-value

- $t$ if for each element $a, A(x/a)$ has truth-value $t$;
- $\top$ if for some element $b, A(x/b)$ has truth-value $\bot$; and for each element $a, A(x/a)$ has truth-value either $t$ or $\top$;
- $\bot$ if for some element $b, A(x/b)$ has truth-value $\bot$, and for...
The paper is organized as follows: the next section gives basic definitions of first-order logic, that is, for any multisequent \( \Gamma \Delta \Sigma \Pi \),

- Soundness theorem: if \( \Gamma \Delta \Sigma \Pi \) is provable in \( G_2 \), then \( \Gamma \Delta \Sigma \Pi \) is valid;

- Completeness theorem: if \( \Gamma \Delta \Sigma \Pi \) is valid then \( \Gamma \Delta \Sigma \Pi \) is provable in \( G_2 \).

The paper is organized as follows: the next section gives basic definitions in \( B_2 \)-valued first-order logic; the third section gives basic definitions of \( B_2 \)-valued first-order logic; the fourth section gives Gentzen deduction system \( G_2 \) for \( B_2 \)-valued first-order logic and prove soundness and completeness theorems; the fifth section discusses the different constructions of trees in the proof of completeness theorem, and the last section concluded the paper.

2. MULTISEQUENT DEDUCTION SYSTEM FOR \( B_2 \)-VALUED FIRST-ORDER LOGIC

Let \( L \) be a logical language of first-order logic which contains the following symbols:

- constant symbols: \( c_0, c_1, \ldots \);
- variable symbols: \( x_0, x_1, \ldots \);
- function symbols: \( f_0, f_1, \ldots \);
- predicate symbol: \( p_0, p_1, \ldots \); and
- logical connectives and quantifiers: \( \neg, \land, \lor, \forall, \exists \).

A term \( t \) is a string of the following forms:

\[
  t ::= c[x]f(t_1, \ldots, t_n),
\]

where \( f \) is an \( n \)-ary function symbol.

A formula \( A \) is a string of the following forms:

\[
  A ::= p(t_1, \ldots, t_n) | \neg A_1 | A_1 \land A_2 | A_1 \lor A_2 | \forall x A_1(x),
\]

where \( p \) is an \( n \)-ary predicate symbol.

Let \( B_2 = (\{\top, \bot\}, \neg, \lor, \land) \) be the least Boolean algebra, where

\[
  \begin{array}{c|cccc}
  & \top & \bot & \land & \lor \\
  \top & \top & \bot & \top & \top \\
  \bot & \bot & \top & \bot & \bot \\
  \land & \bot & \bot & \bot & \top \\
  \lor & \top & \top & \top & \top \\
  \end{array}
\]

A model \( M \) is a pair \((U, I)\), where \( U \) is a universe and \( I \) is an interpretation such that for any constant symbol \( c, I(c) \in U \); for any \( n \)-ary function symbol \( f, I(f) : U^n \rightarrow U \) is a function; and for any \( n \)-ary predicate symbol \( p, I(p) : U^n \rightarrow B_2 \) is a relation on \( U \).

An assignment \( v \) is a function from variables to \( U \). The interpretation \( t^{I,v} \) of \( t \) in \((M, v)\) is

\[
  t^{I,v} = \begin{cases} 
  I(c) & \text{if } t = c \\
  v(x) & \text{if } t = x \\
  I(f)(t_1^{I,v}, \ldots, t_n^{I,v}) & \text{if } t = f(t_1, \ldots, t_n).
  \end{cases}
\]

Given a formula \( A \), define

\[
  v(A) = \begin{cases} 
  I(p)(t_1^{I,v}, \ldots, t_n^{I,v}) & \text{if } A = p(t_1, \ldots, t_2) \\
  \neg(A_1^{I,v}) & \text{if } A = \neg A_1 \\
  v(A_1) \land v(A_2) & \text{if } A = A_1 \land A_2 \\
  v(A_1) \lor v(A_2) & \text{if } A = A_1 \lor A_2 \\
  \min\{v_{x/a}(A_1(x)) : a \in U\} & \text{if } A = \forall x A_1(x)
  \end{cases}
\]

where for any variable \( y \),

\[
  v_{x/a}(y) = \begin{cases} 
  v(y) & \text{if } y \neq x \\
  a & \text{otherwise.}
  \end{cases}
\]

Hence,

\[
  v(\forall x A_1(x)) = \top \text{ iff } Aa \in U(v_{x/a}(A_1(x)) = \top) \\
  v(\forall x A_1(x)) = \bot \text{ iff } Aa \notin U(v_{x/a}(A_1(x)) = \bot),
\]

where in syntax, we use \( \neg, \land, \lor, \forall, \exists \) to denote logical connectives and quantifiers; and in semantics we use \( \neg, \land, \lor, \exists, \forall \) to denote the corresponding connectives and quantifiers.

A formula \( A \) is satisfied in \((I, v)\), denoted by \( I, v \vDash A \), if \( v(A) = \top \); \( A \) is valid in \( I \), denoted by \( I \vDash A \), if for any assignment \( v, I, v \vDash A \); and \( A \) is valid, denoted by \( \vDash A \), if for any interpretation \( I, I \vDash A \).

Let \( \Delta, \Gamma \) be sets of formulas. A multisequent \( \delta \) is of form \( \Gamma \Delta \). We say that \( \delta \) is satisfied in an interpretation \( I \) and an assignment \( v \), denoted by \( I, v \vDash \Gamma \Delta \), if either \( I, v \vDash \Gamma \) or \( I, v \vDash \Delta \), where \( I, \vDash \Delta \) if \( v(A) = \top \) for some \( A \in \Delta \); and \( I, v \vDash \Gamma \) if \( v(B) = \bot \) for some \( B \in \Gamma \).
δ is satisfied in an interpretation I, denoted by I ⊧ δ if I, v ⊧ δ for any assignment v; and δ is valid, denoted by |= δ, if δ is satisfied in any interpretation I. Gentzen deduction system G_2 contains the following axioms and deduction rules:

- **Axioms:**
  \[ \Gamma \cap \Delta \neq \emptyset \quad \Delta \models \Gamma \] (A),

  where Δ, Γ are sets of atomic formulas.

- **Deduction rules:**
  \[ \Delta \models - \Delta | \Gamma \quad \Delta | \neg \Delta | \Gamma \]
  \[ \Delta, \neg A | \Gamma \quad \Delta, A | \neg A | \Gamma \]
  \[ \Delta, A_1 | \Delta, A_2 | \Gamma \]
  \[ \Delta, A_1 \wedge A_2 | \Gamma \]
  \[ \Delta, x A(x) | \Gamma \]
  \[ \Delta, \forall x A(x) | \Gamma \]
  \[ \Delta, \forall x A(x) | \Gamma \]

  where *x* is a new variable not occurring free in \( \forall x A(x) \), and \( t \) is a term.

Let \( \Gamma, \Delta, \Sigma, \Pi \) be sets of formulas. A multisequent \( \delta \) is of form \( \Gamma | \Delta | \Sigma | \Pi \). We say that \( \delta \) is satisfied in \( (I, v) \), denoted by \( I, v \models \Gamma | \Delta | \Sigma | \Pi \), if either \( I, v \models \Gamma \), \( I, v \models \Delta \), \( I, v \models \Sigma \), or \( I, v \models \Pi \), where

\[ I, v \models \Delta \text{ if for some formula } A \in \Delta, v(A) = t; \]
\[ I, v \models \Theta \text{ if for some formula } B \in \Theta, v(B) = T; \]
\[ I, v \models \Gamma \text{ if for some formula } C \in \Theta, v(C) = \bot; \text{ and } \]
\[ I, v \models \Pi \text{ if for some formula } D \in \Pi, v(D) = \#. \]

A multisequent \( \Gamma | \Delta | \Sigma | \Pi \) is valid, denoted by \( |= \Gamma | \Delta | \Sigma | \Pi \), if for any interpretation \( I \) and assignment \( v, I, v \models \Gamma | \Delta | \Sigma | \Pi \).

**Theorem 2.1** (Soundness theorem). For any multisequent \( \Delta \models \Gamma \), if \( \models \Delta | \Gamma \) then \( \models \Delta | \Gamma \).

**Theorem 2.2** (Completeness theorem). For any multisequent \( \Delta | \Gamma \models \Delta | \Gamma \) only if \( |= \Delta | \Gamma \).

3. **B^2_3-valued first-order logic**

Let \( B^2_3 \) be a Boolean algebra \( \{ \top, \top, \perp, \perp \} \), where

\[
\begin{array}{cccccccc}
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\top & \top & \top & \top & \top & \top & \top & \top \\
\end{array}
\]

A model \( M \) is a pair \((U, I)\), where \( U \) is a universe and \( I \) is an interpretation such that for any constant symbol \( c, I(c) \in U \); for any \( n \)-ary function symbol \( f, I(f) : U^n \to U \) is a function; and for any \( n \)-ary predicate symbol \( p, I(p) : U^n \to B^2_3 \) is a relation on \( U \).

Given a formula \( A \), define

\[ v(A) = \begin{cases} I(p)(t_1, \ldots, t_n) & \text{if } A = p(t_1, \ldots, t_2) \\ \neg(A^1^\top) & \text{if } A = \neg A_1 \\ v(A_1) \wedge v(A_2) & \text{if } A = A_1 \wedge A_2 \\ v(A_1) \cup v(A_2) & \text{if } A = A_1 \vee A_2 \\ \text{defined below} & \text{if } A = \forall x A_1(x) \end{cases} \]

**Proposition 3.1** Let \( \Gamma, \Delta, \Sigma, \Pi \) be sets of atomic formulas. Then, \( \Gamma | \Delta | \Sigma | \Pi \) is valid if and only if \( \models \Gamma \cap \Delta \cap \Sigma \cap \Pi \neq \emptyset \).

**Proof.** Assume that \( p \in \Gamma \cap \Delta \cap \Sigma \cap \Pi \neq \emptyset \). For any interpretation \( I \) and assignment \( v, v(p) = t \) then \( I, v \models \Gamma \); \( v(p) = \bot \) then \( I, v \models \Delta \); \( v(p) = T \) then \( I, v \models \Sigma \); otherwise, \( v(p) = \# \), \( I, v \models \Pi \). Hence, \( I, v \models \Gamma | \Delta | \Sigma | \Pi \).

Conversely, assume that \( \Gamma \cap \Delta \cap \Sigma \cap \Pi = \emptyset \). Let \( U \) be the set of all the constants occurring in \( \Gamma, \Delta, \Sigma \) or \( \Pi \). Define \( I \) and \( v \) such that for any atomic formula \( l \),

\[ v(l) = \begin{cases} *4 & \text{if } l = \Gamma_s \cap \Gamma_{s_2} \cap \Gamma_{s_3} \\ *3 & \text{otherwise, if } l \in \Gamma_s \cap \Gamma_{s_2} - \bigcup_{i=1,2,3} \Gamma_{s_i} \cap \Gamma_{s_2} \cap \Gamma_{s_3} \\ f (...) & \text{otherwise, if } l \in \Gamma_{s_2} \cap \Gamma_{s_3} \end{cases} \]
where \( \{*, \ast_2, \ast_3, \ast_4\} = \{t, \top, \bot, \mathsf{f}\} \), and \( \Gamma_1 = \Gamma, \Gamma = \Delta, \Gamma_\bot = \Sigma, \Gamma_\Pi = \Pi; \ast_1 \neq \ast_2 \neq \ast_3 \neq \ast_1; \ast_1 \neq \ast_2 \in \mathcal{B}_2 \), and \( \# \in \mathcal{B}_2 \). Then, \( I, v \not\models \Gamma|\Delta|\Sigma|\Pi \).

4. GÉNTEZ DEDUCTION SYSTEM
Gentzen deduction system \( G_2 \) contains the following axioms and deduction rules:

- **Axioms:**
  
  \[
  (A) \quad \frac{\Pi \cap \Sigma \cap \Delta \cap \Gamma \neq \emptyset}{\Gamma \models \Sigma, \Pi}
  \]

- **Deduction rules for binary logical connective \( \land \):**
  
  \[
  (\land^A) \quad \frac{\Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma, \Delta \models \Sigma, \Pi}
  \]
  \[
  (\land^B) \quad \frac{\Gamma \models \Delta \models \Sigma|\Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta \models \Sigma, \Pi}
  \]
  \[
  (\land^C) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]
  \[
  (\land^D) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]

- **Deduction rules for binary logical connective \( \lor \):**
  
  \[
  (\lor^A) \quad \frac{\Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma, \Delta \models \Sigma, \Pi}
  \]
  \[
  (\lor^B) \quad \frac{\Gamma \models \Delta \models \Sigma|\Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta \models \Sigma, \Pi}
  \]
  \[
  (\lor^C) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]
  \[
  (\lor^D) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]

- **Deduction rules for quantifier \( \forall \):**
  
  \[
  (\forall^A) \quad \frac{\Gamma, \Delta \models \Sigma|\Pi \quad \Gamma, \Delta \models \Sigma, \Pi}{\Gamma \models \Delta \models \Sigma, \Pi}
  \]
  \[
  (\forall^B) \quad \frac{\Gamma \models \Delta \models \Sigma|\Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta \models \Sigma, \Pi}
  \]
  \[
  (\forall^C) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]
  \[
  (\forall^D) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]

- **Deduction rules for unary logical connective \( \neg \):**
  
  \[
  (\neg^A) \quad \frac{\Gamma \models \Delta \models \Sigma|\Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta \models \Sigma, \Pi}
  \]
  \[
  (\neg^B) \quad \frac{\Gamma \models \Delta \models \Sigma|\Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta \models \Sigma, \Pi}
  \]
  \[
  (\neg^C) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]
  \[
  (\neg^D) \quad \frac{\Gamma \models \Delta \models \Sigma, \Pi \quad \Gamma \models \Delta, \Sigma|\Pi}{\Gamma \models \Delta, \Sigma \models \Pi}
  \]
where \( t \) is a term and \( x \) is a new variable not occurring free in \( \Gamma, \Delta, \Sigma \) and \( \Pi \).

**Definition 4.1** \( \vdash \Gamma|\Delta|\Sigma|\Pi \) if there is a sequence \( \{ \Delta_1|\theta_1|\Pi_1, \ldots, \Delta_n|\theta_n|\Pi_n \} \) of multisequents such that \( \Delta_n|\theta_n|\Pi_n = \Gamma|\Delta|\Sigma|\Pi \), and for each \( 1 \leq i \leq n, \Delta_i|\theta_i|\Pi_i \) is deduced from the previous multisequents by one of the deduction rules in \( G^2 \).

**Theorem 4.2** For any multisequent \( \Gamma|\Delta|\Sigma|\Pi \), if \( \models \Gamma|\Delta|\Sigma|\Pi \) then \( \vdash \Gamma|\Delta|\Sigma|\Pi \).

**Proof.** We prove that axioms and validation rules preserve validity. Fix an interpretation \( I \).

To verify the validity of the axiom, assume that \( \Gamma \cap \Delta \cap \Sigma \cap \Pi \neq \emptyset \). Then, there is an atomic formula \( l \in \Gamma \cap \Delta \cap \Sigma \cap \Pi \), and for any assignment \( v \), if \( v(l) = t \) then \( I, v \models \Gamma \); if \( v(l) = \top \) then \( I, v \models \Delta \), if \( v(l) = \bot \) then \( I, v \models \Sigma \), otherwise, \( I, v \models \Pi \), and each of which implies \( I, v \models \Gamma|\Delta|\Sigma|\Pi \).

To verify that \( \neg B \) preserves validity, assume that for any assignment \( v, I, v \models \Gamma|\Delta|\Sigma|\Pi \). If \( I, v \models \Gamma|\Delta|\Sigma|\Pi \) then \( I, v \models \Gamma|\Delta|, \neg A|\Sigma|\Pi \); otherwise, \( v(A) = \bot \), and by the definition of \( \neg A \), \( v(\neg A) = \top, I, v \models \Delta, \neg A \), and hence, \( I, v \models \Gamma|\Delta|, \neg A|\Sigma|\Pi \).

To verify that \( \land^D \) preserves validity, assume that for any assignment \( v \),
\[
\begin{align*}
& I, v \models \Gamma|\Delta|D_1|\Sigma|\Pi, \\
& I, v \models \Gamma|\Delta|D_2|\Sigma|\Pi.
\end{align*}
\]

For any assignment \( v \), if \( v \models \Gamma|\Delta|\Sigma|\Pi \) then \( v \models \Gamma|\Delta|\Sigma|D_1 \land D_2|\Pi \); otherwise, \( v(D_1) = \top, v(D_2) = \bot \), and by the definition of \( \land \), \( v(D_1 \land D_2) = \bot \), \( v \models D_1 \land D_2, \Pi \), and hence, \( v \models \Gamma|\Delta|\Sigma|B_1 \land B_2|\Pi \).

To verify that \( \forall^A \) preserves validity, assume that for any assignment \( v \),
\[
\begin{align*}
& I, v \models \Gamma|\Delta, A_1|\Sigma|\Pi, \\
& I, v \models \Gamma|\Delta|A_2|\Sigma|\Pi.
\end{align*}
\]

For any assignment \( v \), if \( v \models \Gamma|\Delta|\Sigma|\Pi \) then \( v \models \Gamma, A_1 \land A_2|\Delta|\Sigma|\Pi \); otherwise, \( v(A_1) = \top, v(A_2) = \bot \), and by the definition of \( \land \), \( v(A_1 \land A_2) = \top \), \( v \models A_1 \land A_2, \Gamma \), and hence, \( v \models \Gamma, A_1 \land A_2|\Delta|\Sigma|\Pi \).

To verify that \( \forall^B \) preserves the validity, assume that for any assignment \( v, I, v \models \Gamma|\Delta, B(t)|\Sigma|\Pi \) and \( I, v \models \Gamma, B(x)|\Delta|\Sigma|\Pi \) then \( I, v \models \Gamma, \forall x B(x)|\Sigma|\Pi \); otherwise, by induction assumption, \( v(B(t)) = \top \) or \( v(B(x)) = \bot \), i.e., for any \( a \in U \), either \( v(B(t)) = \top \) or \( v_{x/a}(B(x)) = \bot \), i.e., \( v(\forall x B(x)) = \top \).

To verify that \( \forall^D \) preserves the validity, assume that for any assignment \( v, I, v \models \Gamma|\Delta|\Sigma|\Pi \) and \( I, v \models \Gamma, \forall x B(x)|\Sigma|\Pi \) then \( I, v \models \Gamma, \forall x B(x)|\Sigma|\Pi \); otherwise, by induction assumption, \( v(B(t)) = \top \) or \( v(B(x)) = \bot \), i.e., for any \( a \in U \), either \( v(B(t)) = \top \) or \( v_{x/a}(B(x)) = \bot \), i.e., \( v(\forall x B(x)) = \top \).

Similar for other deduction rules.

**Theorem 4.3** For any multisequent \( \Gamma|\Delta|\Sigma|\Pi \), if \( \models \Gamma|\Delta|\Sigma|\Pi \) then \( \vdash \Gamma|\Delta|\Sigma|\Pi \).

**Proof.** Given a multisequent \( \Gamma|\Delta|\Sigma|\Pi \), we construct a tree \( T \) such that either

(i) for each branch \( \xi \) of \( T \), each multisequent \( \Gamma'|\Delta'|\Sigma'|\Pi' \) at the leaf of \( \xi \) is an axiom, or

(ii) there is an assignment \( v \) such that \( v \not\models \Gamma|\Delta|\Sigma|\Pi \).

\( T \) is constructed as follows:

* the root of \( T \) is \( \Gamma|\Delta|\Sigma|\Pi \);
* for a node \( \xi \), if for each sequent \( \Gamma'|\Delta'|\Sigma'|\Pi' \) at \( \xi, \Gamma' \cup \Delta' \cup \Sigma' \cup \Pi' \) is a set of atomic formulas then the node is a leaf;
* otherwise, \( \xi \) has the direct child node containing the following multisequents:

\[
\begin{align*}
& \Gamma_1|\Delta_1|\Sigma_1|A|\Pi_1 \quad \text{if } \Gamma_1, \neg A|\Delta_1|\Sigma_1|\Pi_1 \in \xi, \\
& \Gamma_1|\Delta_1|\Sigma_1|B|\Pi_1 \quad \text{if } \Gamma_1, \neg B|\Delta_1|\Sigma_1|\Pi_1 \in \xi, \\
& \Gamma_1|\Delta_1|\Sigma_1|C|\Pi_1 \quad \text{if } \Gamma_1, \neg C|\Delta_1|\Sigma_1|\Pi_1 \in \xi, \\
& \Gamma_1, D|\Delta_1|\Sigma_1|\Pi_1 \quad \text{if } \Gamma_1, \neg D|\Delta_1|\Sigma_1|\Pi_1 \in \xi.
\end{align*}
\]

and
\[
\begin{align*}
\Gamma_1, A_1 | \Delta_1 | \Sigma_1 | \Pi_1 & \quad \text{if } \Gamma_1, A_1 \land A_2 | \Delta_1 | \Sigma_1 | \Pi_1 \in \xi \\
\Gamma_1, A_2 | \Delta_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, B_1 | \Sigma_1 | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1, B_1 \land B_2 | \Sigma_1 | \Pi_1 \in \xi \\
\Gamma_1, B_1 | \Delta_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1, B_2 | \Delta_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1, C_1, C_2 | \Delta_1 | \Sigma_1 | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1, C_1 \land C_2 | \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1, C_1, C_2 | \Pi_1 & \\
\Gamma_1, C_1 | \Delta_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1, C_1 | \Delta_1 | \Sigma_1, D_1, | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1 | D_1 \land D_2, \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_1, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1, D_1 | \Sigma_1, D_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_1, | B_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_2, | B_2 & \\
\Gamma_1 | \Delta_1, D_1, D_2, | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1 | D_1 \land D_2, \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1 | \Sigma_1, C_1, D_1, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, C_2, D_1, | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1, C_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, C_1, C_2, | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1, C_1 \land C_2 | \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1 | \Sigma_1, C_1, D_1, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, C_2, D_1, | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1, C_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, C_1, C_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_1, | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1 | D_1 \land D_2, \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1, D_1, D_2, | \Pi_1 & \\
\end{align*}
\]

and

\[
\begin{align*}
\Gamma_1, A_1 | \Delta_1 | \Sigma_1 | \Pi_1 & \quad \text{if } \Gamma_1, A_1 \lor A_2 | \Delta_1 | \Sigma_1 | \Pi_1 \in \xi \\
\Gamma_1, A_2 | \Delta_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, A_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1, A_1 | \Delta_1, A_2 | \Pi_1 & \\
\Gamma_1 | \Delta_1, A_1 | \Sigma_1, A_2 | \Pi_1 & \\
\Gamma_1, A_1 | \Delta_1, A_2 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, B_1, B_2 | \Sigma_1 | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1, B_1 \lor B_2 | \Sigma_1 | \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1, B_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, B_2 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1 | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1 | C_1 | \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1, C_1, C_2 | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1, C_2 | \Sigma_1 | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1, C_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1, C_1 | \Sigma_1, D_1, | \Pi_1 & \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_1, | \Pi_1 & \quad \text{if } \Gamma_1 | \Delta_1 | \Sigma_1 | D_1 \lor D_2, \Pi_1 \in \xi \\
\Gamma_1 | \Delta_1 | \Sigma_1, D_2, | \Pi_1 & \\
\Gamma_1 | \Delta_1, D_1, D_2, | \Pi_1 & \\
\end{align*}
\]

and
Proof. By the definition of $T$, $T$ is a proof tree of $\Gamma|\Delta|\Sigma|\Pi$.

**Lemma 4.4** If there is a branch $\xi \subseteq T$ such that each multisequent $\Gamma'|\Delta'|\Sigma'|\Pi'$ is an axiom in $G_2$ then $\xi$ is a proof of $\Gamma|\Delta|\Sigma|\Pi$.

**Proof.** Let $\gamma$ be the set of all the atomic multisequents in $T$ which is not an axiom.

$$
\begin{align*}
&\Gamma_1, A_1(c)|\Delta_1|\Sigma_1|\Pi_1 \\
&\text{if } \Gamma_1, \forall x A_1(x)|\Delta_1|\Sigma_1|\Pi_1 \in \xi
\end{align*}
$$

and

- for each $\Gamma_2|\Delta_2, B'_1(t)|\Sigma_2|\Pi_2 \in T$ such that $\Gamma_2|\Delta_2, B'_1(t)|\Sigma_2|\Pi_2$ has not be applied to a constant $c$, and for each child node $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ of $\Gamma_2|\Delta_2, B'_1(t)|\Sigma_2|\Pi_2$, let the child node of $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ contain sequent $\Gamma_3|\Delta_3, B'_1(c)|\Sigma_3|\Pi_3$;
- for each $\Gamma_2|\Delta_2|\Sigma_2, C'_1(t)|\Pi_2 \in T$ such that $\Gamma_2|\Delta_2|\Sigma_2, C'_1(t)|\Pi_2$ has not be applied to a constant $c$, and for each child node $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ of $\Gamma_2|\Delta_2|\Sigma_2, C'_1(t)|\Pi_2$, let the child node of $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ contain sequent $\Gamma_3|\Delta_3, C'_1(c)|\Pi_3$;
- for each $\Gamma_2|\Delta_2|\Sigma_2, D'_1(t), \Pi_2 \in T$ such that $\Gamma_2|\Delta_2|\Sigma_2, D'_1(t), \Pi_2$ has not be applied to a constant $c$, and for each child node $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ of $\Gamma_2|\Delta_2|\Sigma_2, D'_1(t), \Pi_2$, let the child node of $\Gamma_3|\Delta_3|\Sigma_3|\Pi_3$ contain sequent $\Gamma_3|\Delta_3, D'_1(c), \Pi_3$,

where $\begin{align*}
\delta_1
\end{align*}$ represents that $\delta_1, \delta_2$ are at a same child node; and $\begin{align*}
\delta_2
\end{align*}$ represents that $\delta_1, \delta_2$ are at different direct children nodes.

We proved by induction on tree that each $\xi \subseteq T$ contains a multisequent $\Gamma'|\Delta'|\Sigma'|\Pi' \in \xi$ which is not satisfied by $v$.

Case 1. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'|\Pi' \notin \xi$. Then, $\xi$ has a direct child node containing $\Gamma'|\Delta'|\Sigma'|\Pi'$. By induction assumption, if $v \nmid \Gamma'|\Delta'|\Sigma'|\Pi'$ then $v \nmid \Gamma'|\Delta'|\Sigma'|\Pi'$.

Case 2. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'|\Pi' \in \xi$. Then, $\xi$ has a direct child node containing $\Gamma'|\Delta'|\Sigma'|\Pi'$ and $\Delta', A_2|\Theta|\Gamma'$. By induction assumption, if $v \nmid \Gamma'|\Delta'|\Sigma'|\Pi'$, or $v \nmid \Gamma', A_2|\Delta'|\Sigma'|\Pi'$ then $v \nmid \Gamma'|\Delta'|\Sigma'|\Pi'$.}

Case 3. $\Gamma'|\Delta'|\Sigma'|\Pi' = \Gamma'|\Delta'|\Sigma'|\Pi'$ and $\Delta', A_2|\Theta|\Gamma'$. Then, $\xi$ has three direct children node containing $\Gamma'|\Delta'|\Sigma'|\Pi'$.
respectively. By induction assumption, if

either \( v \not\models \Gamma' \vdash \Delta' \land B_1 \land \Sigma' \land \Pi' \), or \( v \not\models \Gamma' \vdash \Delta' \land B_2 \land \Sigma' \land \Pi' \);

then \( v \not\models \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Sigma' \land \Pi' \).

Case 4. \( \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Sigma' \land \Pi' \).

By induction assumption, (1) either \( v \not\models \Gamma' \vdash \Delta'' \land C_1 \land \Pi' \), or \( v \not\models \Gamma' \vdash \Delta'' \land C_2 \land \Pi' \); and (2) either \( v \not\models \Gamma' \vdash \Delta'' \land C_1 \land \Pi' \), or \( v \not\models \Gamma' \vdash \Delta'' \land C_2 \land \Pi' \).

Then \( v \not\models \Gamma' \vdash \Delta'' \land C_1 \land \Pi' \).

Case 5. \( \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Sigma' \land \Pi' \).

By induction assumption,

\( v \not\models \Gamma' \vdash \Delta'' \land B_1 \land \Pi' \),

\( v \not\models \Gamma' \vdash \Delta'' \land B_2 \land \Pi' \),

\( v \not\models \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Pi' \).

Hence, \( v \not\models \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Pi' \).

Case 6. \( \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Sigma' \land \Pi' \).

Case 7. \( \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Pi' \).

Case 8. \( \Gamma' \vdash \Delta'' \land B_1 \land B_2 \land \Pi' \).

5. DISCUSSION

In the proof of completeness theorem, given a sequent \( \Gamma \Rightarrow \Delta \) to be proved and a deduction rule of form

\[
\begin{array}{c}
\quad P \quad Q \\
\quad S \\
\end{array}
\]

where \( P, Q, S \) are sequents, we decompose a node containing \( S \) into two children nodes containing \( P \) and \( Q \), respectively:

Given a deduction rule of form

\[
\begin{array}{c}
P \\
Q \\
S \\
\end{array}
\]

we merge sequents \( P \) and \( Q \) into one sequent \( P, Q \):

In the end, we get a tree \( T \) such that each sequent at the leaf node of \( T \) is atomic. If each leaf has an axiom then \( T \) is a proof tree; otherwise, there is a branch \( \gamma \) of \( T \) such that the leaf node of \( \gamma \) contains no axiom. Then, we define an assignment \( v \) in which the sequent \( \Gamma \Rightarrow \Delta \) is not satisfied.
For multisequents, a node containing \( S \) which has three deduction rules

\[
\begin{align*}
P_1 & \quad P_2 \\
\frac{Q_1 \quad Q_2}{S} \\
\frac{R_1 \quad R_2}{S}
\end{align*}
\]

has eight children nodes:

In another way, given a deduction rule

\[
\begin{array}{c}
P \\ \hline \\ Q \\
\end{array}
\]

we merge sequents \( P \) and \( Q \) into one sequent \( P, Q \):

\[
\begin{array}{c}
P, Q \\ \hline \\ S
\end{array}
\]

Given a deduction rule:

\[
\begin{align*}
P & \quad S \\
\frac{Q}{S}
\end{align*}
\]

has three children nodes:

In the end, we get a tree \( T \) such that each sequent at the leaf node of \( T \) is atomic. If there is a branch \( \gamma \) such that each sequent at the leaf node of \( \gamma \) is an axiom then \( \gamma \) is a proof of \( \Gamma \Rightarrow \Delta \); otherwise, for each leaf node \( \xi \) of \( T \), there is a sequent at \( \xi \) is not an axiom. Then, we define an assignment \( v \) in which the sequent \( \Gamma \Rightarrow \Delta \) is not satisfied.

For multisequents, a node containing \( S \) which has three deduction rules

\[
\begin{align*}
P_1 & \quad P_2 \\
\frac{Q_1 \quad Q_2}{S} \\
\frac{R_1 \quad R_2}{S}
\end{align*}
\]

has three children nodes:

Dually, for existential quantifier \( \exists \) we have the following definition of truth-value:

\[
v(\exists x A_1(x)) = \begin{cases} 
  \top & \text{if } E_a \in U(v_{x/a}(A_1(x)) = \top) \\
  \top & \text{if } A_a \in U(v_{x/a}(A_1(x)) \in \{\top, \bot\}) \& E_a \in U(v_{x/a}(A_1(x)) = \top) \\
  \bot & \text{if } A_a \in U(v_{x/a}(A_1(x)) \in \{\top, \bot\}) \& E_a \in U(v_{x/a}(A_1(x)) = \bot) \\
  \bot & \text{if } A_a \in U(v_{x/a}(A_1(x)) = \bot).
\end{cases}
\]

and deduction rules:

\[
\begin{align*} 
(\exists^A) & \quad \Gamma, A(t)|\Delta|\Sigma|\Pi \\
& \quad \Gamma, \exists x A(x)|\Delta|\Sigma|\Pi \\
(\exists^B) & \quad \Gamma|\Delta|\Sigma, C(t)|\Pi \\
& \quad \Gamma|\Delta|\Sigma, C(x)|\Pi \\
(\exists^C) & \quad \Gamma|\Delta|\Sigma, C(x)|\Pi \\
& \quad \Gamma|\Delta|\Sigma, \exists x C(x)|\Pi \\
(\exists^D) & \quad \Gamma|\Delta|\Sigma|D(x), \Pi \\
& \quad \Gamma|\Delta|\Sigma|\exists x D(x), \Pi \\
(\exists^E) & \quad \Gamma|\Delta|\Sigma|B(x), \Pi \\
& \quad \Gamma|\Delta|\Sigma|B(x)|\Pi \\
\end{align*}
\]
where \( t \) is a term and \( x \) is a new variable not occurring free in \( \Gamma, \Delta, \Sigma \) and \( \Pi \).

6. CONCLUSION

A Gentzen deduction system for \( B_3^2 \)-valued first-order logic is given and soundness and completeness theorems are proved.

A future work will consider different choices for defining the truth-values of quantified formulas. One choice is as follows:

- \( \forall x A(x) \) has truth-value \( t \) if for each element \( a, A(x/a) \) has truth-value \( t \);
- \( \forall x A(x) \) has truth-value \( \top \) if for each element \( a, A(x/a) \) has truth-value \( \top \);
- \( \forall x A(x) \) has truth-value \( \bot \) if for each element \( a, A(x/a) \) has truth-value \( \bot \);
- \( \forall x A(x) \) has truth-value \( f \) if for each element \( a, A(x/a) \) has truth-value \( f \).

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