Studies of an Adaptive Kaczmarz Method for Electrical Impedance Imaging

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Abstract. We present an adaptive Kaczmarz method for solving the inverse problem in electrical impedance tomography and determining the conductivity distribution inside an object from electrical measurements made on the surface. To best characterize an unknown conductivity distribution and avoid inverting the Jacobian-related term which could be expensive in terms of memory storage in large scale problems, we propose to solve the inverse problem by adaptively updating both the optimal current pattern with improved distinguishability and the conductivity estimate at each iteration. With a novel subset scheme, the memory-efficient reconstruction algorithm which appropriately combines the optimal current pattern generation and the Kaczmarz method can produce accurate and stable solutions adaptively compared to traditional Kaczmarz and Gauss-Newton type methods. Several reconstruction image metrics are used to quantitatively evaluate the performance of the simulation results.

1. Introduction

Different reconstruction methods proposed to solve the electrical impedance tomography (EIT) inversion problem can be categorized as direct methods and iterative methods. Direct methods are considered to be fast but less accurate. In contrast, iterative methods tend to search for more accurate solutions by iteration. However, these methods, such as the Gauss Newton method, can be expensive in terms of memory storage in large scale problems [3]. The choice of current patterns will also have a great impact on determining the system’s distinguishability. A system with limited measurement precision but high distinguishability is more likely to detect small conductivity changes in an object.

We propose the adaptive Kaczmarz method, which solves the inverse problem by applying the optimal current patterns for distinguishing the actual conductivity from the conductivity estimate between each iteration of the block Kaczmarz method. With a novel ordered subset scheme, this reconstruction method appropriately combines the optimal current pattern generation with the block Kaczmarz method and produces accurate solutions memory-efficiently.

2. Method

In this paper the inverse problem of EIT is a least-squares problem with the objective functional:

$$ E(\rho) = \|V - U(\rho)\|_2^2$$

(1)

where $\rho$ is the piecewise constant conductivity distribution, $U(\rho)$ are the voltages predicted by a forward model, and $V$ are the measured voltages on the electrodes.
2.1. Ordered Subset Scheme

The block Kaczmarz method requires that the complete set of both current patterns and measurements should be appropriately divided into subsets. One subset scheme, known as the sequential subset scheme, partitions the complete set in the order that the current patterns are applied and sequentially pick the block at each iteration. Another subset scheme, known as the randomized subset scheme, randomly picks the block from a pre-partitioned complete set at each iteration. We propose a novel subset scheme so that we can partition and order the subsets of current patterns and measurements by their ability to characterize the conductivity distribution. Specifically, we partition the current patterns \( I = \{I_{1\text{sub}}, I_{2\text{sub}}, ..., I_{n\text{sub}}\}\) and measurements \( V = \{V_{1\text{sub}}, V_{2\text{sub}}, ..., V_{n\text{sub}}\}\) so that the pair \( (I_{1\text{sub}}, V_{1\text{sub}})\) is the best subset to distinguish the actual conductivity from the conductivity estimate; the pair \( (I_{2\text{sub}}, V_{2\text{sub}})\) is the second best subset and so on. In this way, the best “blockwise” conductivity estimate from the current subset will be obtained and passed down as the starting point to the next subset. To obtain the ordered subset of the current patterns, the complete set of the optimal current patterns, which best distinguishes the actual conductivity distribution from a conductivity estimate [2], should be generated and partitioned.

2.2. Adaptive Kaczmarz

An adaptive Kaczmarz method with ordered subsets is proposed based on the block Kaczmarz method [4] and optimal current pattern generation. It begins with an initial guess for the current patterns \( I^{(0)} \) and an initial guess for the conductivity distribution \( \rho^{(0)} \) and proceeds to generate the optimal current patterns \( I^{(1)} \) to best distinguish the actual conductivity from the guess \( \rho^{(0)} \). Then \( I^{(1)} \) with the ordered subset scheme is used to solve for the conductivity distribution with one iteration of the block Kaczmarz method to obtain \( \rho^{(1)} \). The more accurate conductivity estimate \( \rho^{(1)} \) is used for the next iteration to find the optimal current patterns \( I^{(2)} \) to best distinguish the actual conductivity distribution from the more accurate estimate \( \rho^{(1)} \), which are then used to solve for \( \rho^{(2)} \). Therefore, a new conductivity estimate and a set of optimal current patterns are obtained adaptively at each iteration. The iteration continues until a reasonable conductivity estimate is obtained.

Let \( T \) be an orthonormal current matrix, \( \rho^{(0)} \) be an initial conductivity guess, \( m \) be the number of iterations. The algorithm is as follows:

1) Initialize \( \hat{T} = T, \rho = \rho^{(0)}; \)
2) for \( k = 1; m \) do
   a. repeat
      1. Apply \( \hat{T} \) and measure voltages \( V \);
      2. Apply \( \hat{T} \) and compute the simulated voltages \( U(\rho) \);
      3. Form the voltage difference matrix \( D = V - U(\rho) \) and \( P = D^*D \);
      4. Eigenvalue decomposition of \( P \) and form the eigenvector matrix \( E \) so that the corresponding eigenvalues are sorted in descending order;
      5. \( \hat{T} = \hat{T}E \);
      until the current pattern converges;
   b. Obtained the ordered subsets \( I = \{I_{1\text{sub}}, I_{2\text{sub}}, ..., I_{n\text{sub}}\}, V = \{V_{1\text{sub}}, V_{2\text{sub}}, ..., V_{n\text{sub}}\} \), and the row submatrix \( J = \{J_1, J_2, ..., J_n\} \), \( \rho_0 = \rho \);
   c. for \( i = 1; n \) do
      \( \rho_i = \rho_{i-1} + a_i J_i^T (J_i^T + \lambda_i I)^{-1} [V_i - U_i(\rho_{i-1})]; \)
   end
   d. \( \rho = \rho_n; \)
end

The adaptive Kaczmarz method updates adaptively both the optimal current pattern and a more accurate conductivity estimate at each iteration. It is empirically observed that an accurate and stable solution can be obtained within 5 iterations.
2.3 Model Setup
A normalized 2D circular tank with unit radius and 32 equally-spaced square electrodes is modelled. Two circular targets with radius 0.15 are placed inside the tank. The conductor is placed at (0.4, -0.35) with conductivity 2 S/m, and the insulator is placed (-0.3, 0.2) with the conductivity 0 S/m. Trigonometric current patterns are used with the maximum amplitude of 1 mA and the complete electrode model is assumed. The inverse mesh (2992 elements) is designed to be coarser that the forward mesh (70394 elements) to avoid an inverse crime. The non-adaptive Kaczmarz method, the Gauss Newton method, the adaptive Kaczmarz method, and the adaptive Gauss Newton method are compared for their reconstructed images. The adaptive Gauss Newton method combines the Gauss Newton and the optimal current pattern generation in a similar way as the adaptive Kaczmarz method does.

3. Results

3.1 Comparison of different subset schemes
Difference images are reconstructed using the Kaczmarz method with the sequential subset scheme, the randomized subset scheme, and the proposed ordered subset scheme. It is observed in Figure 1 that reconstructed images made with the randomized scheme and the sequential scheme have target deformation and artifacts on different levels, while the image made with ordered subsets more accurately reflects the actual target’s location, size, and shape.

![model](a) ordered subset (b) randomized subset with sequential partition (c) randomized subset with ordered partition (d) sequential subset

Figure 1. Comparison of reconstructed conductivity distribution using different subset schemes

3.2 Comparison of different reconstruction methods
In Figure 2, the improvement of the adaptive Kaczmarz and the adaptive Gauss Newton over their non-adaptive version can be easily observed. The images are further compared using several consensus figures of merit described in [1]. The comparison of different methods is shown in Table 1. From the error table, it is observed that the adaptive Kaczmarz has the smallest position error and blurring error, and the closest peak value to the true peak value 2. In terms of memory requirement, the Gauss Newton method with 10 iterations requires 231.4 Megabytes, while the block Kaczmarz method with 10 iterations require only 31.2 Megabytes.

4. Conclusion
We introduce the adaptive Kaczmarz method to solve the inversion in electrical impedance tomography. With a novel subset scheme, the cost-efficient method appropriately combines the block Kaczmarz method and optimal current pattern generation to produce accurate solutions. Unlike the Gauss Newton method, the adaptive Kaczmarz method avoids the expense of inverting large scale matrices and is quite memory-efficient. The simulation results show that the adaptive Kaczmarz is able to provide accurate 2D reconstruction images within a few iterations with less memory requirement.

Future efforts will involve further improvement in the efficiency of the algorithm and the generalization to the 3D case.

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