ON MULTIPLIERS OF PAIRS OF LIE SUPERALGEBRAS

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ABSTRACT. We study the notion of the Schur multiplier $\mathcal{M}(N, L)$ of a pair $(N, L)$ of Lie superalgebras and obtain some upper bounds concerning dimensions. Moreover, we characterize the pairs of finite dimensional (nilpotent) Lie superalgebras for which

$$\dim \mathcal{M}(N, L) = \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N) - t,$$

for $t = 0, 1$, where $\dim N = (m|n)$.

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1. Introduction. The story of Lie superalgebras begins in the late 1960s. The Russian physicist Stavraki [30] is by far the first one who used the term supersymmetry, at which time it attracted little attention. Then V.G. Kac [15] started working on this new algebraic structure and published his first results on the classification of Lie superalgebras in 1971. The notion of Lie superalgebras also appeared almost simultaneously in a paper of Berezin and G.I. Kac (also written Kats) [4] in 1970. Developing the properties of Lie superalgebras has been of some interest in mathematics and theoretical physics for the last 40 years (see [2, 9, 10, 16] for more information).

In 1996, Batten et al. [3] discussed and studied the concept of the Schur multiplier of a Lie algebra, which is analogous to the Schur multiplier of a group introduced by Schur [29] in 1904. Moneyhun [18] proved for an $m$-dimensional Lie algebra $L$ that $\dim \mathcal{M}(L) \leq \frac{1}{2}m(m - 1)$, where $\mathcal{M}(L)$ denotes the Schur multiplier of $L$. Also in [22], Saeedi et al. generalized the Moneyhun’s result to a pair of Lie algebras and proved that if $(N, L)$ is a pair of Lie algebras in which $N$ admits a complement in $L$ and $\dim N = m$, then $\dim \mathcal{M}(N, L) \leq \frac{1}{2}m(m + 2 \dim(L/N) - 1)$. Recently in [17, 19], the notion of the Schur multiplier has been extended to Lie superalgebras. In [19], Nayak shows that if $L$ is a Lie superalgebra of dimension $(m|n)$, then $\dim \mathcal{M}(L) \leq \frac{1}{2}((m + n)^2 + (n - m))$.

In the present paper, following [17, 19, 22] we study the Schur multiplier of a pair of Lie superalgebras and provide an upper bound for its dimension, which simultaneously extends all above bounds. We also characterize the pairs $(N, L)$ of finite dimensional (nilpotent) Lie superalgebras whose Schur multipliers can
reach a maximum dimension (Theorem 3.7) or one less than that (Theorem 3.12). As a consequence, we show that the Heisenberg Lie algebra of dimension 3 is the only \((m|n)\)-dimensional nilpotent Lie superalgebra \(L\) for which \(\dim M(L) = \frac{1}{2}(m+n)^2 + (n-m) - 1\) (see also [17]).

2. Preliminaries. Throughout this paper, all (super)algebras are considered over an algebraically closed field \(F\) of characteristic \(\neq 2, 3\). We first discuss some terminologies on Lie superalgebras from [9, 16, 19].

Set \(\mathbb{Z}_2 = \{0, 1\}\). A \(\mathbb{Z}_2\)-graded vector space (or superspace) \(V\) is a direct sum of vector spaces \(V_0\) and \(V_1\) whose elements are called even and odd, respectively. Non-zero elements of \(V_0 \cup V_1\) are said to be homogeneous. For a homogeneous element \(v \in V_a\) with \(a \in \mathbb{Z}_2\), \(|v| = a\) is the degree of \(v\). In the sequel, when the notation \(|v|\) appears, it means that \(v\) is a homogeneous element. A vector subspace \(U\) of \(V\) is called \(\mathbb{Z}_2\)-graded vector subspace (or sub-superspace), if \(U = U_0 \oplus U_1\) where \(U_0 = U \cap V_0\) and \(U_1 = U \cap V_1\).

**Definition 2.1.** A Lie superalgebra is a superspace \(L = L_0 \oplus L_1\) equipped with a bilinear mapping \([-,-]: L \times L \to L\), usually called the graded bracket of \(L\), satisfying the following conditions:

1. \([L_a, L_b] \subseteq L_{a+b}\), for every \(a, b \in \mathbb{Z}_2\),
2. \([x, y] = -(-1)^{|x||y|}[y, x]\),
3. \((-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|[z, [x, y]]] = 0\),

for every \(x, y, z \in L\). The identities (ii) and (iii) are called graded antisymmetric property and graded Jacobi identity, respectively.

Clearly, (i) means that \(|[x, y]| = |x| + |y|\) (modulo 2). Moreover, since \(\frac{1}{2}, \frac{1}{4} \in F\), (ii) implies that \([x, x] = 0\) for all \(x \in L_0\), and (iii) implies that \([x, [x, x]] = 0\) for all \(x \in L\). Hence the even part \(L_0\) of a Lie superalgebra \(L\) is actually a Lie algebra which means that if \(L_1 = 0\), then \(L\) becomes a usual Lie algebra. Also, the odd part \(L_1\) is an \(L_0\)-module. If \(L_0 = 0\), then \([x, y] = 0\) for all \(x, y \in L\) and hence \(L\) is an abelian Lie superalgebra. Also, one can easily see that the graded Jacobi identity may be written as

\([x, [y, z]] = [x, [y, z]] - (-1)^{|x||y|}[y, [x, z]]\).

A sub-superspace \(I\) of a Lie superalgebra \(L\) is said to be a sub-superalgebra (resp. graded ideal), if \([I, I] \subseteq I\) (resp. \([I, L] \subseteq I\)). Now, let \(N\) be a graded ideal of a Lie superalgebra \(L\). Then \((N, L)\) is called a pair of Lie superalgebras. The center and commutator of the pair \((N, L)\) are defined as \(Z(N, L) = \{n \in N | [n, x] = 0, \forall x \in L\}\) and \([N, L] = \langle [n, x] | n \in N, x \in L \rangle\), respectively, which are graded ideals of \(L\) contained in \(N\). Clearly if \(N = L\), then the above graded ideals coincide with the usual center and commutator of \(L\). Also, a Lie superalgebra \(L\) is said to be nilpotent, if \(L^c = 0\) for some \(c \geq 1\) where \(L^1 = L\) and \(L^{i+1} = [L^i, L]\), \(i \geq 1\).

Let \(L\) and \(K\) be two Lie superalgebras. A linear map \(f: L \to K\) is called a homomorphism of Lie superalgebras, if \(f(L_a) \subseteq K_a\) for every \(a \in \mathbb{Z}_2\), and
Let \( f(x, y) = [f(x), f(y)] \) for every \( x, y \in L \) (see [19, 23] for more details). Throughout this paper when a Lie superalgebra \( L = L_0 \oplus L_1 \) is of dimension \( m + n \), in which \( \dim L_0 = m \) and \( \dim L_1 = n \), we write \( \dim L = (m|n) \).

In the context of Lie algebras, the Schur multiplier of a pair \((N, L)\), denoted by \( \mathcal{M}(N, L) \), appears in the following natural exact sequence of Lie algebras

\[
H_3(L) \rightarrow H_3(L/N) \rightarrow \mathcal{M}(N, L) \rightarrow \mathcal{M}(L) \rightarrow \mathcal{M}(L/N)
\]

\[
\rightarrow L/[N, L] \rightarrow L/L^2 \rightarrow L/(L^2 + N) \rightarrow 0,
\]

where \( H_3(-) \) denotes the third homology of a Lie algebra (see [22]). In fact, this is similar to the definition of the Schur multiplier of a pair of groups given by Ellis [7]. Also it is easy to see that if the ideal \( N \) admits a complement in \( L \), then \( \mathcal{M}(L) \cong \mathcal{M}(N, L) \oplus \mathcal{M}(L/N) \). So in this paper, we consider pairs \((N, L)\) of Lie superalgebras in which \( N \) possesses a complement in \( L \).

Now, let \( 0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0 \) be a free presentation of a Lie superalgebra \( L \) and \( N \cong S/R \) such that \( S \) is a graded ideal of \( F \). We define the Schur multiplier of the pair \((N, L)\) as

\[
\mathcal{M}(N, L) = \frac{R \cap [S, F]}{[R, F]},
\]

which is an abelian Lie superalgebra, independent of the choice of the free presentation of \( L \). Clearly if \( N = L \), then \( \mathcal{M}(L, L) = \mathcal{M}(L) \) is the Schur multiplier of a Lie superalgebra \( L \) given in [17, 19]. In fact, there is an isomorphism of supermodules \( \mathcal{M}(L) \cong H_2(L) \), where \( H_2(L) \) is the second homology of \( L \) (see [9, Corollary 6.5]). This notion also extends the Schur multiplier of a pair of Lie algebras given in [22, 26, 27]. Moreover, the Schur multiplier and the Schur Lie-multiplier of a pair of Leibniz algebras are studied in [6, 14] and [24], respectively.

Let \( L \) and \( M \) be two Lie superalgebras. By an action of \( L \) on \( M \), we mean an \( F \)-bilinear map \( L \times M \rightarrow M \) given by \((l, m) \mapsto \mathfrak{t}^l m \) satisfying

(i) \( \mathfrak{t}^l m \in M_{a+b} \), for every \( l \in L_a \) and \( m \in M_b \), \( a, b \in \mathbb{Z}_2 \) (even grading),

(ii) \( \mathfrak{t}^{[l,l']} m = \mathfrak{t}^{l'} m - (-1)^{|l||l'|} \mathfrak{t}^l \mathfrak{t}^{l'} m \),

(iii) \( \mathfrak{t}^{[m,m']} = \mathfrak{t}^m [m,m'] + (-1)^{|m||m'|} [m,m'] \mathfrak{t}^m \),

for all \( l, l' \in L \) and \( m, m' \in M \). Clearly, if \( L \) is a sub-superalgebra of some Lie superalgebra \( P \) and \( M \) is a graded ideal of \( P \), then the Lie multiplication of \( P \) induces an action of \( L \) on \( M \) by \( \mathfrak{t}^l m = [l, m] \). We also define the semidirect product of \( M \) by \( L \), denoted by \( M \rtimes L \), with underlying supermodule \( M \oplus L \) and the graded bracket

\[
[(m,l), (m',l')] = ([m, m'] + \mathfrak{t}^{m'} m - (-1)^{|m||m'|} \mathfrak{t}^l m, [l, l']),
\]

for \( m, m' \in M \) and \( l, l' \in L \) (see [9] for more information).

Now we define the concept of a cover for a pair of Lie superalgebras. The usual notion of central extensions and covers of a Lie superalgebra is already given in [19, 17].
DEFINITION 2.2. Let \((N, L)\) be a pair of Lie superalgebras. A relative central extension of \((N, L)\) is a homomorphism of Lie superalgebras \(\sigma : M \rightarrow L\) together with an action of \(L\) on \(M\) satisfying the following conditions:

(i) \(\sigma(M) = N\).

(ii) \(\sigma(lm) = [l, \sigma(m)]\), for all \(l \in L, m \in M\).

(iii) \(\sigma(m)m' = [m, m']\), for all \(m, m' \in M\).

(iv) \(\ker \sigma \subseteq Z(M, L)\), in which \(Z(M, L) = \{m \in M \mid \forall l \in L\}\).

In addition, the relative central extension \(\sigma : M \rightarrow L\) is said to be a cover of \((N, L)\), if \(\mathcal{M}(N, L) \cong \ker \sigma \subseteq [M, L]\), where \([M, L] = \{lm \mid m \in M, l \in L\}\).

Recall from [9, Definition 2.5] that a crossed module of Lie superalgebras is a homomorphism of Lie superalgebras \(\partial : M \rightarrow P\) with an action of \(P\) on \(M\) satisfying

(i) \(\partial(pm) = [p, \partial(m)]\),

(ii) \(\partial(m)m' = [m, m']\),

for all \(p \in P\) and \(m, m' \in M\). Therefore, a relative central extension of \((N, L)\) is a crossed module \(\sigma : M \rightarrow L\) with \(\sigma(M) = N\) and \(\ker \sigma \subseteq Z(M, L)\).

The following result is a generalization of [22, Proposition 2.5].

PROPOSITION 2.3. Every pair of Lie superalgebras has at least one cover.

Proof. Let \((N, L)\) be a pair of Lie superalgebras and \(0 \rightarrow R \rightarrow F \xrightarrow{\pi} L \rightarrow 0\) be a free presentation of \(L\) such that \(N \cong S/R\) for some graded ideal \(S\) of \(F\). Consider \(T/[R, F]\) as a complement of \(\mathcal{M}(N, L)\) in the abelian Lie superalgebra \(R/[R, F]\). It is easy to see that \(l(s + T) = [f, s] + T\) where \(\pi(f) = l\), is an action of \(L\) on \(S/T\).

Now, define \(\sigma : S/T \rightarrow L\) by \(s + T \mapsto \pi(s)\). We show that \(\sigma\) is a cover of \((N, L)\). Clearly \(\sigma(M) = N\). Also, \(\sigma(l(s + T)) = \sigma([f, s] + T) = \pi([f, s]) = [l, \pi(s)] = [l, \sigma(s + T)]\) and \(\sigma(s' + T)(s + T) = \pi(s')(s + T) = [s' + T, s + T]\). Moreover, one may easily check that \(\mathcal{M}(N, L) \cong \ker \sigma \subseteq Z(M, L)\). Finally

\[
\ker \sigma = \frac{R}{T} \cong \frac{R \cap [S, F]}{[R, F]} \subseteq \frac{[S, F]}{[R, F]} = \frac{[S, F]}{[S, F] \cap T} \cong \frac{[S, F] + T}{T} = \langle [f, s] + T \mid f \in F, s \in S \rangle = \langle \pi(f)(s + T) \mid f \in F, s \in S \rangle = [S/T, L],
\]

which completes the proof.

\(\square\)

3. Main results. The Lie algebra analogue of Schur’s theorem [29], proved by Moneyhun [18], states that if \(L\) is a Lie algebra such that \(\dim(L/Z(L)) = m\), then \(\dim L^2 \leq \frac{1}{2}m(m - 1)\). This upper bound has been recently generalized to Lie superalgebras. In [19], Nayak shows that if \(L\) is a Lie superalgebra with \(\dim(L/Z(L)) = (m|n)\), then \(\dim L^2 \leq \frac{1}{2}((m + n)^2 + (n - m))\). In the following theorem, we extend this result to a pair of Lie superalgebras.
THEOREM 3.1. Let \( (N, L) \) be a pair of Lie superalgebras such that \( L/N \) is finite dimensional and \( \dim(N/Z(N, L)) = (m|n) \). Then
\[
\dim[N, L] \leq \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N).
\]

Proof. Let \( \{\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n\} \) be a basis of \( \bar{N} = N/Z(N, L) \) in which \( \bar{x}_i \in \bar{N}_0 \) (1 \( \leq \) \( i \) \( \leq \) \( m \)) and \( \bar{y}_i \in \bar{N}_1 \) (1 \( \leq \) \( i \) \( \leq \) \( n \)). Also let \( \dim(L/N) = (p|q) \). Since \( \dim(L/Z(N, L)) = (m + p|n + q) \), one may extend the above basis to
\[
\{\bar{x}_1, \ldots, \bar{x}_m, \bar{x}_{m+1}, \ldots, \bar{x}_{m+p}, \bar{y}_1, \ldots, \bar{y}_n, \bar{y}_{n+1}, \ldots, \bar{y}_{n+q}\}
\]
for \( \bar{L} = L/Z(N, L) \) where \( \bar{x}_j \in \bar{L}_0 \) (1 \( \leq \) \( j \) \( \leq \) \( m + p \)) and \( \bar{y}_j \in \bar{L}_1 \) (1 \( \leq \) \( j \) \( \leq \) \( n + q \)). Then
\[
\dim[N_0, L_0] \leq \#\{[x_i, x_j] : 1 \leq i \leq m \text{ and } i < j \leq m + p\} \leq \binom{m}{2} + mp,
\]
\[
\dim[N_1, L_1] \leq \#\{[y_i, y_j] : 1 \leq i \leq n \text{ and } i \leq j \leq n + q\} \leq \binom{n}{2} + nq,
\]
\[
\dim \left( [N_0, N_1] + [N_0, L_1 - N_1] + [N_1, L_0 - N_0] \right) \leq mn + mq + np.
\]
Therefore,
\[
\dim[N, L] \leq \binom{m}{2} + mp + \binom{n}{2} + nq + mn + mq + np
\]
\[
= \frac{1}{2}((m + n)^2 + (n - m)) + (m + n)(p + q).
\]
\[\Box\]

REMARK 3.2. Clearly if \( N = L \), then our upper bound coincides with the Nayak’s one. Also if \( (N, L) \) is a pair of Lie algebras, i.e. \( \dim(N/Z(N, L)) = (m|0) \), then the above theorem implies that \( \dim[N, L] \leq \frac{1}{2}m(m + 2 \dim(L/N) - 1) \) which is given in [22].

In the next result, we provide an upper bound on the dimension of the Schur multiplier of a pair of Lie superalgebras.

COROLLARY 3.3. Let \( (N, L) \) be a pair of finite dimensional Lie superalgebras such that \( \dim N = (m|n) \). Then
\[
\dim \mathcal{M}(N, L) \leq \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N) - \dim[N, L].
\]

Proof. Using the notation of Proposition 2.3, we have
\[
[N, L] \cong \frac{[S, F] + R}{R} \cong \frac{[S, F]/[R, F]}{(R \cap [S, F])/[R, F]},
\]
and hence
\[
\dim[N, L] + \dim \frac{R \cap [S, F]}{[R, F]} = \dim \frac{[S, F]}{[R, F]}.
\] (3.1)
Also, since
\[ \dim \frac{S/[R, F]}{Z(S/[R, F], F/[R, F])} \leq \dim \frac{S/[R, F]}{R/[R, F]} = (m|n) \]
and \( \frac{F/[R, F]}{S/[R, F]} \cong L/N \), by Theorem 3.1 we get
\[ \dim \frac{[S, F]}{R, F} \leq \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N). \]

Now equality (3.1) completes the proof.

Remark 3.4. One interesting problem in the theory of Lie algebras is the characterization of Lie algebras using \( t(L) = \frac{1}{2}m(m - 1) - \dim \mathcal{M}(L) \), in which \( L \) is an \( m \)-dimensional Lie algebra. The characterization of nilpotent Lie algebras for \( 0 \leq t(L) \leq 8 \) has been studied in [3, 12, 13]. Similarly, Green’s result [11] yielded a lot of interest on the classification of finite \( p \)-groups by \( t(G) \), investigated by several authors (see [5, 8, 31, 32]).

Now, let \( L \) be a Lie superalgebra. A natural question arises whether one could characterize \( (m|n) \)-dimensional Lie superalgebras by \( t(L) = \frac{1}{2}((m + n)^2 + (n - m)) - \dim \mathcal{M}(L) \). Lemma 3.6 below is the answer for \( t(L) = 0 \), and in [17] this question is answered for \( t(L) \leq 2 \) (see also [20, 21] for more details on classification of Lie superalgebras by their Schur multipliers). In [25, 28], it is discussed on \( n \)-Lie superalgebras and Leibniz \( n \)-algebras, respectively.

In what follows, we deal with a similar problem for the pair case.

Lemma 3.5. If \( L \) is an \( m \)-dimensional Lie algebra, then \( \dim \mathcal{M}(L) \leq \frac{1}{2}m(m - 1) \). Moreover the equality holds if and only if \( L \) is abelian.

Proof. See [18, Lemmas 22 and 23].

Lemma 3.6. If \( L \) is a Lie superalgebra of dimension \( (m|n) \), then
\[ \dim \mathcal{M}(L) \leq \frac{1}{2}((m + n)^2 + (n - m)). \]
Moreover the equality holds if and only if \( L \) is abelian.

Proof. See [19, Theorems 3.3 and 3.4].

Theorem 3.7. Let \((N, L)\) be a pair of finite dimensional Lie superalgebras such that \( \dim N = (m|n) \). Then
\[ \dim \mathcal{M}(N, L) \leq \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N). \]
In particular, if \( L \) is abelian then the equality holds, and if the equality holds then \( N \subseteq Z(L) \).
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Proof. Corollary 3.3 yields the above inequality. Clearly if the equality holds, then Corollary 3.3 implies that $N$ is central. Now, suppose that $L$ is abelian and put $\dim(L/N) = (p|q)$. Then by Lemma 3.6 we have $\dim \mathcal{M}(L/N) = \frac{1}{2}((p+q)^2 + (q-p))$ and $\dim \mathcal{M}(L) = \frac{1}{2}((m+p+n+q)^2 + (n+q-m-p))$. Thus the isomorphism $\mathcal{M}(L) \cong \mathcal{M}(N, L) \oplus \mathcal{M}(L/N)$ implies that

$$\dim \mathcal{M}(N, L) = \frac{1}{2}((m+n)^2 + (n-m)) + (m+n) \dim(L/N),$$

which completes the proof. \qed

COROLLARY 3.8. Let $(N, L)$ be a pair of finite dimensional Lie superalgebras such that $\dim(\frac{N + L^2}{L^2}) = (r|s)$. Then

$$\dim \mathcal{M}(N, L) \geq \frac{1}{2}((r+s)^2 + (s-r)) + (r+s) \dim(\frac{L}{N + L^2}) - \dim[N, L].$$

Proof. Similar to [22, Proposition 2.6], one may easily show that

$$\dim \mathcal{M}(\frac{N + L^2}{L^2}, \frac{L}{L^2}) \leq \dim \mathcal{M}(N, L) + \dim[N, L].$$

Now since $L/L^2$ is abelian, by Theorem 3.7 we have

$$\dim \mathcal{M}(\frac{N + L^2}{L^2}, \frac{L}{L^2}) = \frac{1}{2}((r+s)^2 + (s-r)) + (r+s) \dim(\frac{L}{N + L^2}),$$

which completes the proof. \qed

Recall from [2, 3] that a finite dimensional Lie (super)algebra $L$ is called Heisenberg, if $L^2 = Z(L)$ and $\dim L^2 = 1$. A Heisenberg Lie algebra, denoted by $H(m)$, has dimension $2m + 1$ with a basis $\{x_1, \ldots, x_{2m}, z\}$ and non-zero multiplications $[x_i, x_{m+i}] = z$, for $1 \leq i \leq m$. Heisenberg superalgebras consist of two types according to the parity of the center:

(1) Heisenberg superalgebra of even center and dimension $(2m + 1|n)$:

$$\mathcal{H}(m, n) = \langle x_1, \ldots, x_{2m}, z \rangle \oplus \langle y_1, \ldots, y_n \rangle \quad (m + n \geq 1)$$

with non-zero multiplications $[x_i, x_{m+i}] = z = [y_j, y_j]$, for $1 \leq i \leq m$ and $1 \leq j \leq n$.

(2) Heisenberg superalgebra of odd center and dimension $(n|n+1)$:

$$\mathcal{H}(n) = \langle x_1, \ldots, x_n \rangle \oplus \langle y_1, \ldots, y_n, z \rangle \quad (n \geq 1),$$

with non-zero multiplications $[x_i, y_i] = z$, for $1 \leq i \leq n$.

The following lemmas are needed to prove the next result.

LEMMA 3.9. ([3, Theorem 3]) Let $L$ be an $m$-dimensional nilpotent Lie algebra. Then $\dim \mathcal{M}(L) = \frac{1}{2}m(m-1) - 1$ if and only if $L \cong H(1)$. 
Lemma 3.10. ([19, Theorem 4.3]) For $\mathcal{H}(m, n)$ we have
\[
\dim \mathcal{M}(\mathcal{H}(m, n)) = \begin{cases} 
2m^2 - m - 1 + 2mn + \frac{1}{2}n(n + 1) & \text{if } m + n \geq 2, \\
0 & \text{if } m = 0, n = 1, \\
2 & \text{if } m = 1, n = 0.
\end{cases}
\]

Lemma 3.11. ([17, Proposition 4.5]) For $\mathcal{H}(n)$ we have
\[
\dim \mathcal{M}(\mathcal{H}(n)) = \begin{cases} 
2n^2 - 1 & \text{if } n \geq 2, \\
2 & \text{if } n = 1.
\end{cases}
\]

Now, we are ready to prove the following main theorem which generalizes [3, Theorem 3] and [22, Theorem B (v)].

Theorem 3.12. Let $(N, L)$ be a pair of finite dimensional nilpotent Lie superalgebras with $\dim N = (m|n)$. If
\[
\dim \mathcal{M}(N, L) = \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N) - 1,
\]
then $L$ is a non-abelian Lie superalgebra for which one of the following holds:

(i) $[N, L] = 0$,

(ii) $\dim[N, L] = 1$ and $[N, L] = Z(N, L)$.

Proof. Clearly Theorem 3.7 and equality (3.2) imply that $L$ is non-abelian. Also Corollary 3.3 and equality (3.2) imply that $\dim[N, L] \leq 1$. Suppose that $[N, L] \neq 0$. Since $L$ is nilpotent, we have $\dim[[N, L], L] \leq \dim[N, L] = 1$ and hence $[N, L] \subseteq Z(N, L)$. We only need to show that $\dim Z(N, L) = 1$.

Considering Proposition 2.3, let $\sigma : M \to L$ be a cover of the pair $(N, L)$. Put $P = M \rtimes \frac{L}{N}$, in which the action of $L/N$ on $M$ is induced by the one of $L$ on $M$. By construction we have $P/M \cong L/N$, $Z(M, P) = Z(M, L)$, $[M, L] = [M, P]$, $P/Z(M, P) \cong L$ and $M/Z(M, P) \cong N$. Therefore,
\[
\dim \frac{M}{Z(M, P)} = \dim \frac{M}{Z(M, L)} \leq \dim \frac{M}{\ker \sigma} = \dim N = (m|n),
\]
and by applying Theorem 3.1 on the pair $(M, P)$ we get
\[
\dim \mathcal{M}(N, L) = \dim \ker \sigma \leq \dim[M, L] = \dim[M, P] \leq \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N). \tag{3.3}
\]
One may consider the following two cases:

Case I. $\ker \sigma \subsetneq Z(M, L)$.

Then $\dim(M/Z(M, P)) \leq \dim(M/\ker \sigma) = (m|n)$, and we have two cases again:

Subcase I-1. If $\dim(M/Z(M, P)) \leq (m|n - 1)$, then Theorem 3.1 and equality (3.2) imply that
\[
\frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim(L/N) - 1 \leq \dim[M, P] \leq \frac{1}{2}((m + n - 1)^2 + (n - 1 - m)) + (m + n - 1) \dim(L/N).
\]
This yields \( \dim L \leq 1 \), and in both cases \( \dim L = (1|0) \) and \((0|1)\), \( L \) is abelian which is a contradiction.

**Subcase I-2.** If \( \dim(M/Z(M, P)) \leq (m - 1|n) \), then by a similar manner one can show that \( \dim L \leq 2 \). If \( \dim L = (2|0) \), then \( L \) is a Lie algebra. Since it is non-abelian, by Lemma 3.5, \( \dim M(L) = 0 \). Hence Lemma 3.9 implies that \( \dim L = 3 \), which is a contradiction. If \( \dim L = (0|2) \), then \( L \) is abelian which is impossible. Thus \( \dim L = (1|1) \). Let \( L = L_0 \oplus L_1 \) where \( L_0 = \langle x \rangle \) and \( L_1 = \langle y \rangle \). Then Definition 2.1 implies that there exist only two non-abelian Lie superalgebras. The only non-zero multiplication of the first one is \([x, y] = y\), but this Lie superalgebra is not nilpotent since \( Z(L) = 0 \). In the second one, we must have \([y, y] = x\) as the only non-zero multiplication. In this case \( L^2 = Z(L) = \langle x \rangle \) and \( L \cong \mathcal{H}(0, 1) \). Thus by Lemma 3.10, we get \( \dim M(L) = 0 \). Now, if \( \dim N = (1|0) \), then \( N = Z(L) \) which is impossible since \([N, L] \neq 0\). Also the multiplication \([y, y] = x\) shows that \( \dim N \) cannot be \((0|1)\). Hence we must have \( N = L \). But equality (3.2) implies that \( \dim M(L) = \dim M(L, L) = 1 \), which is impossible. Hence Case I leads to a contradiction.

**Case II.** \( \ker \sigma = Z(M, L) \).

By (3.2) and (3.3), we have two cases:

**Subcase II-1.** \( \ker \sigma = Z(M, P) = [M, P] \). Then we have \( N \cong \frac{M}{Z(M, P)} \subseteq Z(\frac{P}{Z(M, P)}) \cong Z(L) \), which is impossible.

**Subcase II-2.** \( \ker \sigma = Z(M, P) \not\subseteq [M, P] \) and hence
\[
\dim [M, P] = \frac{1}{2}((m + n)^2 + (n - m)) + (m + n) \dim (L/N). \tag{3.4}
\]

In this subcase, we show that \( \dim Z(\frac{M}{Z(M, P)}, \frac{P}{Z(M, P)}) = \dim Z(N, L) = 1 \). As mentioned above, this completes the proof. Suppose on the contrary that \( Z(\frac{M}{Z(M, P)}, \frac{P}{Z(M, P)}) \) has a basis containing at least two elements \( x + Z(M, P) \) and \( y + Z(M, P) \). Considering the adjoint map \( \text{ad}_x : P \to P^2 \), we have \( P/C_P(x) \cong [x, P] \), where \( C_P(x) \) is the centralizer of \( x \) in \( P \). Since \( x + Z(M, P) \) is non-zero, \( Z(M, P) \not\subseteq C_P(x) \) and hence
\[
\dim [x, P] = \dim (P/C_P(x)) \not\subseteq \dim (P/Z(M, P)) = \dim L. \tag{3.5}
\]

We prove that \( \dim [x, P] = \dim L - 1 \). Since \( x + [x, P] \in Z(\frac{M}{[x, P]}, \frac{P}{[x, P]}) - \frac{Z(M, P)}{[x, P]} \), we have
\[
\dim \left( \frac{P/[x, P]}{Z(\frac{M}{[x, P]}, \frac{P}{[x, P]})} \right) \not\subseteq \dim \left( \frac{P/[x, P]}{Z(\frac{M}{[x, P]}, \frac{P}{[x, P]})} \right) = \dim (P/M) = \dim (L/N),
\]
and since
\[
\dim \left( \frac{P/[x, P]}{Z(\frac{M}{[x, P]}, \frac{P}{[x, P]})} \right) - \dim \left( \frac{M/[x, P]}{Z(\frac{M}{[x, P]}, \frac{P}{[x, P]})} \right) = \dim (P/M) = \dim (L/N),
\]
we get
\[
\dim \left( \frac{M/[x, P]}{Z(\frac{M}{[x, P]}, \frac{P}{[x, P]})} \right) \not\subseteq \dim L - \dim (L/N) = \dim N = (m|n).
Now if \( \dim \left( \frac{M}{[x,P]} \right) \leq (m|n - 1) \), then by applying Theorem 3.1 for the pair \((\frac{M}{[x,P]}, \frac{P}{[x,P]})\), and also using equality (3.4) and a similar manner to subcase I-1, we get \( \dim [x, P] \geq \dim L \), which contradicts (3.5).

But if \( \dim \left( \frac{M}{[x,P]} \right) \leq (m - 1|n) \) then by a similar computation we get \( \dim [x, P] \geq \dim L - 1 \), and using (3.5) we have \( \dim [x, P] = \dim (\frac{P}{C_P(x)}) = \dim L - 1 \). Therefore,

\[
\dim \left( \frac{C_P(x)}{Z(M, P)} \right) = \dim \left( \frac{P}{Z(M, P)} \right) - \dim \left( \frac{P}{C_P(x)} \right) = \dim L - \dim L - 1 = 1. \tag{3.6}
\]

Since \( x + Z(M, P) \in Z(\frac{M}{Z(M, P)}, \frac{P}{Z(M, P)}) \), then \( [M, P] \subseteq C_P(x) \), and since in this subcase we have \( Z(M, P) \subseteq [M, P] \), thus

\[
\dim L - 1 = \dim \left( \frac{P}{C_P(x)} \right) \leq \dim \left( \frac{P}{[M, P]} \right) = \dim \left( \frac{P/Z(M, P)}{Z(M, P)} \right) = \dim \left( \frac{L}{[N, L]} \right) = \dim L - 1,
\]

which implies that \( [M, P] = C_P(x) \). Since \( x + Z(M, P) \) is arbitrary, we also have \( [M, P] = C_P(y) \) and hence \( y \in C_P(x) \). But this contradicts (3.6). Therefore in this subcase, we must have \( \dim Z(\frac{M}{Z(M, P)}, \frac{P}{Z(M, P)}) = \dim Z(N, L) = 1 \).

The following corollary generalizes Lemma 3.9 and also shows that there is no finite dimensional nilpotent Lie superalgebra with non-trivial odd part satisfying equality (3.7) below (see also [17, Proposition 4.8]).

**Corollary 3.13.** Let \( L \) be a nilpotent Lie superalgebra of dimension \((m|n)\). Then

\[
\dim \mathcal{M}(L) = \frac{1}{2} ((m + n)^2 + (n - m)) - 1 \tag{3.7}
\]

if and only if \( L \cong \mathcal{H}(1, 0) = H(1) \).

**Proof.** If \( L \cong \mathcal{H}(1, 0) \), then Lemma 3.10 implies that \( \dim \mathcal{M}(\mathcal{H}(1, 0)) = 2 \).

Conversely, if equality (3.7) holds, then Theorem 3.12 implies that \( L^2 = Z(L) \) is one-dimensional. If \( \dim Z(L) = (1|0) \), then \( L \) is the Heisenberg Lie superalgebra \( \mathcal{H}(\frac{m-1}{2}, n) \) with even center. Using Lemma 3.10, we must have \( L \cong \mathcal{H}(1, 0) \). If \( \dim Z(L) = (0|1) \), then \( L \) is the Heisenberg Lie superalgebra \( \mathcal{H}(\frac{m+n-1}{2}) \) with odd center. In this case, equality (3.7) and Lemma 3.11 lead to a contradiction.

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ON MULTIPLIERS OF PAIRS OF LIE SUPERALGEBRAS

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