The Physical Process First Law for Bifurcate Killing Horizons

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(Dated: August 2007)

Abstract

The physical process version of the first law for black holes states that the passage of energy and angular momentum through the horizon results in a change in area $\frac{k}{8\pi}\Delta A = \Delta E - \Omega\Delta J$, so long as this passage is quasi-stationary. A similar physical process first law can be derived for any bifurcate Killing horizon in any spacetime dimension $d \geq 3$ using much the same argument. However, to make this law non-trivial, one must show that sufficiently quasi-stationary processes do in fact occur. In particular, one must show that processes exist for which the shear and expansion remain small, and in which no new generators are added to the horizon. Thorne, MacDonald, and Price considered related issues when an object falls across a $d = 4$ black hole horizon. By generalizing their argument to arbitrary $d \geq 3$ and to any bifurcate Killing horizon, we derive a condition under which these effects are controlled and the first law applies. In particular, by providing a non-trivial first law for Rindler horizons, our work completes the parallel between the mechanics of such horizons and those of black holes for $d \geq 3$. We also comment on the situation for $d = 2$. 

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I. INTRODUCTION

The analogy between the laws of black hole mechanics and thermodynamics has been deeply ingrained in theoretical physics for some time. In their contribution to Bekenstein’s festschrift, Jacobson and Parentani [1] emphasized that these laws also hold for much more general horizons, and in particular for what they call asymptotic Rindler horizons. Such horizons are the boundary of the past of an event at future null infinity ($I^+$) in an asymptotically flat spacetime. In many cases such horizons result from small perturbations of a Rindler horizon in flat spacetime.

Although [1] emphasizes the generality of horizon thermodynamics, in their discussion the “physical process version” of the first law appears to be an exception. This law describes the dynamical change in horizon area in response to a flux of stress energy through the generators. As was first demonstrated by Hartle and Hawking [2] (see also [3] for a review), for black holes this response can be written in the form

$$\Delta E = \frac{\kappa}{8\pi} \Delta A + \Omega \Delta J.$$  \hspace{1cm} (1.1)

The form of this expression motivated [4] to dub this result the “physical process version of the first law.”
Despite the issues raised in [1], this note clarifies the situation and demonstrates that the physical process first law holds non-trivially for a general bifurcate Killing horizon in \( d \geq 3 \) spacetime dimensions. In particular, it holds non-trivially for \( d \geq 3 \) Rindler horizons.

As suggested by [5], the first step in the argument is a straightforward generalization of the one for black holes. For sufficiently quasi-stationary processes, one may follow [3, 4] in using the Raychaudhuri equation for null geodesic congruences to derive (1.1). By ‘sufficiently quasi-stationary,’ we mean that i) the expansion and shear along each generator are weak enough to neglect second order terms and ii) no new generators are added to the horizon. We give this argument in section II below, elaborating on subtle points.

The main issue raised in [1] was whether sufficiently quasi-stationary processes exist in the context of Rindler horizons. Of particular concern was a result of Thorne, MacDonald, and Price [6] which states that, in \( d = 4 \) spacetime dimensions, the absorption of an object of mass \( m \) and radius \( r \) by a black hole of mass \( M \) will result in the formation of caustics when \( r \ll \sqrt{mM} \) in units with \( G = 1 \) (while caustics fail to form when \( r \gg \sqrt{mM} \)). Considering a Rindler horizon as the \( M \rightarrow \infty \) limit of a black hole would therefore seem to indicate that caustics always form when an object with any \( m, r \) passes through a Rindler horizon. But the formation of caustics causes two problems: i) in the region near caustics the expansion becomes large and ii) caustics generally signal the addition of generators to the horizon. Thus, the above argument for the physical process first law would not apply if one was forced to consider horizon generators in a region with caustics.

We clarify the issues surrounding caustic formation in section III for the case \( d \geq 4 \). There are two subtleties. First, we emphasize that the derivation of the physical process first law traces generators of the horizon only back to the unperturbed past horizon. Thus, it is only to the future of the past horizon that we need to avoid caustics. In the \( d = 4 \) black hole context, it is precisely for this regime that the threshold \( r \sim \sqrt{mM} \) of [6] determines whether caustics are formed. However, this expression for the threshold is only valid when the object can be thought of as having begun its fall from rest an infinite distance away from the black hole – a condition which does not admit a smooth limit to the case of Rindler horizons. In section III we show that (for \( d = 4 \)) a more local characterization of when caustics form is given by the condition \( r \sim \sqrt{E_\chi/\kappa} \). Here \( E_\chi \) and \( \kappa \) are respectively the Killing energy of the incident object and the surface gravity defined by the Killing field \( \chi \). As usual, the ratio \( E_\chi/\kappa \) is independent of the choice of normalization for the Killing
field. We show that this condition applies to the passage of a weakly self-gravitating object through any bifurcate Killing horizon so long as \( r \) is much smaller than the curvature scale of the unperturbed horizon geometry. (Strongly self-gravitating objects, such as black holes, generically lead to the formation of caustics when they pass through a null congruence.) We also show that the generalization of the above condition to arbitrary spacetime dimension \( d \geq 4 \) is

\[
\rho \sim \left( \frac{E_{\chi}}{\kappa} \right)^{\frac{1}{d-2}}.
\]

(1.2)

In section IV we show that this conclusion also holds for \( d = 3 \), but argue that there is no general analogue for \( d = 2 \). In fact, for \( d = 2 \) Rindler horizons in translation-invariant theories, we argue that either the first law fails to hold or that it holds only vacuously, in the sense that quasi-stationary processes do not arise even in a limiting sense from physical processes. This last point seems to resolve a certain tension [7, 8] associated with particular viewpoints on gravitational entropy. Finally, section V provides some physical interpretation for condition (1.2).

II. THE FIRST LAW FAR FROM CAUSTICS

We now state the argument for the physical process version of the first law following [3, 4]. The argument is essentially as outlined in [5], but for completeness we give the argument in its entirety. The derivation is based on the Raychaudhuri equation for null geodesic congruences, and on the corresponding equations for the shear and twist. We consider a bifurcate Killing horizon, so that the twist vanishes. We allow any such horizon in \( d \geq 3 \) spacetime dimensions, but require that the process be quasi-stationary. By this we mean specifically that, at least in the region to the future of the unperturbed past horizon, i) the expansion and shear along each generator remain weak enough to neglect second order terms and ii) no new generators are added to the horizon. Conditions under which these assumptions are justified will be discussed in section III.

It is convenient to parametrize the geodesics in terms of the Killing parameter \( v \) associated with some horizon-generating Killing field \( \chi \). With this understanding, the ‘focusing equation’ for the expansion becomes

\[
\frac{d\hat{\theta}}{dv} = \kappa \hat{\theta} - \frac{\hat{\theta}^2}{d-2} - \hat{\sigma}_{\mu\nu} \hat{\sigma}^{\mu\nu} - R_{\lambda\sigma} \chi^\lambda \chi^\sigma.
\]

(2.1)
where $\kappa$ is the surface gravity defined by $\chi^\sigma \nabla_\sigma \chi^\mu = \kappa \chi^\mu$ and the hats (\^) on the expansion and shear ($\hat{\theta}, \hat{\sigma}_{\mu\nu}$) remind the reader that these quantities have been defined using the Killing parameter $\nu$ (as opposed to the more usual affine parameter $\lambda$). Note that we have not fixed the normalization of $\chi$; our final results will be independent of this normalization. Equation (2.1) gives the standard result for $d = 4$ and is derived for the general case $d \geq 3$ in appendix A, along with the corresponding equations for shear and twist.

As stated above, we assume that the expansion and shear are weak enough that we may truncate (2.1) to linear order:

$$- \frac{d \hat{\theta}}{dv} + \kappa \hat{\theta} = S(v),$$

(2.2)

where we may use the Einstein equations to write the source as the non-gravitational energy flux through the horizon

$$S(v) = 8\pi T_{\lambda\sigma} \chi^\lambda \chi^\sigma.$$

(2.3)

As we wish to consider sources associated with brief departures from equilibrium, we shall assume that $S(v)$ vanishes rapidly as $v \to \pm\infty$, and that the expansion and the shear tend to zero in the final configuration. We may therefore solve (2.2) using an advanced Green’s function:

$$\hat{\theta}(v) = \int_v^\infty e^{\kappa(v-v')} S(v') dv'.$$

(2.4)

Now, recall that the expansion of a null congruence measures the fractional change in the area of a bundle of null generators over a finite range of Killing time,

$$\Delta A = \int_B \hat{\theta} dA dv,$$

(2.5)

where $B$ is the piece of the horizon generated by the bundle of null generators over the given range of Killing time. For weak perturbations, the fractional change in the area is simply the integral of the expansion over the Killing time. The asymptotic change in area $d(\Delta A)$ along a given generator of initial area $dA$ is then

$$\frac{d(\Delta A)}{dA} = \int_{-\infty}^\infty \hat{\theta} dv = \int_{-\infty}^\infty dv \int_v^\infty dv' e^{\kappa(v-v')} S(v').$$

(2.6)

Changing the order of integration and integrating over $v$ one finds

$$\frac{d(\Delta A)}{dA} = \frac{1}{\kappa} \int_{-\infty}^\infty dv' S(v') = \frac{8\pi}{\kappa} \int_{-\infty}^\infty dv \, T_{\mu\nu} \chi^\mu \chi^\nu.$$
Since the integral of $T_{\mu\nu}\chi^\mu\chi^\nu$ over the horizon gives the flux through the horizon of Killing energy $E_\chi$ associated with $\chi$, we have derived the first law:

$$\frac{\kappa \Delta A}{8\pi} = \Delta E_\chi.$$  (2.8)

The more general version of the first law with angular momentum flux follows immediately in the case where one uses different Killing fields $t^\mu, \phi^\mu$ to define energy $E$ and angular momentum $J$ and where $\chi^\mu = t^\mu + \Omega \phi^\mu$. In this case (2.8) becomes

$$\frac{\kappa \Delta A}{8\pi} = \Delta E - \Omega \Delta J.$$  (2.9)

Let us comment briefly on the physical interpretation of this law, and in particular on the left-hand side. Recall that we computed $\Delta A$ by integrating the expansion over $v \in (-\infty, \infty)$. Since we used advanced boundary conditions, it is clear that $v = \infty$ is the asymptotic future. On the unperturbed future horizon, $v = -\infty$ was the bifurcation surface where the future and past horizons intersect. In fact, even with the perturbation we may repeat the above derivation replacing the past limit of integration $v = -\infty$ with the surface where our generators intersect the (unperturbed) past horizon. The point is that, since we take second order terms to be small, $v = -\infty$ can differ from this surface only by at most a first-order error term. But since the expansion (i.e., the integrand) is also of first order, this means that integrating back to the past horizon changes (2.7) only by a second order term. Thus, the correction is negligible.

The advantage of tracing the generators back to a cross-section of the unperturbed past horizon is that, since the unperturbed horizon is at equilibrium, the area of any such cross section is just the area of the unperturbed horizon. Thus, as desired, the left-hand side of (2.9) represents the difference between the area of the perturbed horizon in the asymptotic future and the area of the unperturbed horizon.

Now, in practice, there is typically even more flexibility in choosing the past limit of integration. Note that all of the integrals in the derivation converge, and that the characteristic response time associated with the solution (2.4) is of order $\kappa^{-1}$. Thus, if the perturbation is well-localized in time, one may think of $\Delta A$ as describing the change in area between times $v_i, v_f$ which precede and follow the perturbation by any interval significantly greater than $\kappa^{-1}$. With this interpretation, it is clear that the first law in fact applies to many horizons which only approximate a bifurcate Killing horizon. For example, it applies not only to
strict Rindler horizons associated with an exact boost symmetry, but also to the asymptotic Rindler horizons of [1]. (See also [5], where they were called “partial horizons.”)

III. CHARACTERIZING THE FORMATION OF CAUSTICS

In section II above we considered the physical process first law for quasi-stationary processes. By ‘quasi-stationary’ we mean processes in which, at least to the future of the unperturbed past horizon, \( \hat{\theta}, \hat{\sigma}_{\mu\nu} \) remain weak enough to ignore all second order effects and to avoid the addition of new generators to the horizon. One expects that sufficiently weak perturbations are quasi-stationary in this sense. However, it is important to check that there do indeed exist perturbations for which this is the case.

Regarding the expansion, we see from (2.1) that the \( \hat{\theta}^2 \) term becomes relevant when \( \hat{\theta} \sim \kappa/(d - 2) \). To understand the effects of this term, let us consider the solution to (2.1) in a region where \( \hat{\sigma}^2 = 0 = S(v) \). As noted in [6], the desired solution is then

\[
\bar{\theta}(v) = \frac{1}{1 + (\theta_0^{-1} - 1) e^{\kappa(v_0 - v)}}, \tag{3.1}
\]

where \( \bar{\theta} = \frac{\hat{\theta}}{(d-2)\kappa} \) and \( \bar{\theta}(v_0) = \theta_0 \). If \( \theta_0 < 1 \), then \( \bar{\theta} \) decreases toward zero as \( v \) decreases into the past. If, however, \( \theta_0 > 1 \), then \( \bar{\theta} \) increases into the past and diverges at some finite time. Therefore, if the horizon is perturbed strongly enough to cause \( \bar{\theta} \gtrsim 1 \) (i.e., \( \hat{\theta} \gtrsim \kappa/(d - 2) \)) at any \( v = v_0 \), then the focusing equation implies that a caustic developed at some \( v < v_0 \). Thus, the requirement that non-linear terms can be ignored is essentially the requirement that no caustics form. This is also a necessary condition to avoid the addition of new generators.

Below, we generalize the discussion of [6, 9] to show that when a small, weakly self-gravitating object passes through an arbitrary bifurcate Killing horizon, the condition (1.2) sets the threshold for caustic formation to the future of the unperturbed past horizon. By “small and weakly self-gravitating,” we mean that the radius \( r \) satisfies \( m^{1/(d-3)} \ll r \ll \ell \), where \( \ell \) is the background curvature scale near the horizon. We consider here the case \( d \geq 4 \); lower dimensions will be discussed in section IV.

We noted above that the evolution of the expansion \( \hat{\theta} \) is controlled by the focusing equation (2.1) of section II. We will also require the corresponding ‘tidal-force equation’ which governs evolution of the shear \( \hat{\sigma}_{\mu\nu} \) along a congruence with vanishing twist. This equation
is derived in appendix A and takes the form

$$\frac{d\hat{\sigma}_{\mu\nu}}{dv} = \left(\kappa - \frac{2\dot{\theta}}{d-2}\right)\hat{\sigma}_{\mu\nu} - \hat{\sigma}_{\mu\rho}\hat{\sigma}^{\rho}_{\nu} + \frac{\dot{\sigma}^2}{d-2}Q_{\mu\nu} + \left(2\hat{\sigma}_{\mu\sigma} + \frac{2\dot{\theta}}{d-2}Q_{\mu\sigma}\right)\hat{\sigma}^{\sigma}_{\nu} - E_{\mu\nu}. \quad (3.2)$$

Here, for each $v$, the tensor $Q_{\mu\nu}$ is the projector onto the spacelike cut of the horizon naturally associated with constant Killing time $v$ as explained in appendix A. The final source term $E_{\mu\nu} := Q^\alpha_{\mu}Q^\beta_{\nu}C_{\alpha\lambda\beta\sigma}\chi^\lambda\chi^\sigma$ is the electric part of the Weyl tensor.

Note that if the energy flux through the horizon is of first order in a small dimensionless perturbation parameter $\epsilon$, then so is $E_{\mu\nu}$. We now distinguish between generators which intersect the matter and generators which do not. For those which do, the horizon perturbations $\dot{\theta}$ and $\dot{\sigma}_{\mu\nu}$ are again of order $\epsilon$. However, for those which do not, we see that the expansion $\dot{\theta}$ is only of second order in $\epsilon$. As a result, we will need to keep the $\dot{\sigma}^2$ term in the focusing equation (2.1) below. We will, however, drop the term $\dot{\theta}^2$.

Truncating the tidal force equation to $O(\epsilon)$ and the focusing equation to $O(\epsilon^2)$ yields

$$-\frac{d\hat{\sigma}_{\mu\nu}}{dv} + \kappa\hat{\sigma}_{\mu\nu} = E_{\mu\nu}, \quad (3.3)$$

and

$$-\frac{d\dot{\theta}}{dv} + \kappa\dot{\theta} = S(v) + \dot{\sigma}^2. \quad (3.4)$$

As in section II, the desired solutions follow by integrating the sources against an advanced Green’s function:

$$\hat{\sigma}_{\mu\nu}(v) = \int E_{\mu\nu}(v') \, e^{\kappa(v-v')}\Theta(v' - v)dv' \quad (3.5)$$

$$\dot{\theta}(v) = \int \left(S(v') + \dot{\sigma}^2(v')\right) \, e^{\kappa(v-v')}\Theta(v' - v)dv'. \quad (3.6)$$

We wish to apply this analysis to the situation in which an object of mass $m$ falls freely through the horizon. Following [6], our strategy will be to describe this process as the passage of the mass through a Rindler horizon in flat spacetime. One might think that this requires the curvature of the spacetime to be small. Indeed, as stated above, we require the curvature scale of the spacetime near the horizon to satisfy $\ell \gg r$. However, we need make no further restrictions on $\ell$. To see this, note first that since $\chi$ generates an isometry the scale $\ell$ is invariant under the diffeomorphism generated by $\chi$. This diffeomorphism acts like a flat-space boost near the bifurcation surface. Thus, we may approximate any region near the bifurcation surface as being in flat spacetime, so long as there is some reference frame
in which the size of this region is much less than $\ell$ as measured by the corresponding locally inertial coordinates.

One might ask if we can choose such a frame so that this small region includes the event where the center of our mass falls across the future horizon. But since this event is null-separated from the bifurcation surface, it is clear that such a choice is always possible! We need only choose a sufficiently "boosted" frame in which the coordinate separation of this point from the bifurcation surface is small. Furthermore, since we assumed $r \ll \ell$ above, it follows that every event where the horizon intersects our object is in fact contained in the desired region.

Note that, having determined that a flat space analysis is valid in this particular frame, we are free to apply an additional boost to transform this flat-space description to any other convenient frame. Below, we will choose the frame in which the object is at rest, say at $Z = z_0$, $x_i = 0$ in terms of the usual Minkowski coordinates $(T, Z, x_i)$.

Let us first consider those geodesics through which the matter energy flux is negligible. In other words, we consider geodesics that pass outside the object itself. Since we assumed $r \gg m^{1/(d-3)}$, we may describe the object by a linearized solution to the Einstein equations. For simplicity, we take the object to be spherically symmetric; the general case follows by linear superposition. The corresponding linearized metric takes the form

$$ds^2 = -\left(1 - \frac{cm}{(\sqrt{\rho^2 + (Z - z_0)^2})^{d-3}}\right) dT^2 + \left(1 + \frac{1}{d-3} \frac{cm}{(\sqrt{\rho^2 + (Z - z_0)^2})^{d-3}}\right) (dZ^2 + d\rho^2 + \rho^2 d\Omega_{d-3}^2),$$

(3.7)

where $\rho^2 = x_1^2 + \ldots + x_{d-2}^2$ and the line element on the $S^{d-3}$ is parametrized by angles $\phi_j$, with $j = 1, \ldots, d - 3$. The constant $c$ is

$$c = \frac{16\pi}{(d-2)\Omega_{d-2}},$$

(3.8)

where $\Omega_n$ is the volume of $S^n$. The diagonal transverse components of the electric part of the Weyl tensor for this metric are

$$\mathcal{E}_{\rho\rho} = -\frac{d-3}{g_{\phi_j\phi_j}} \mathcal{E}_{\phi_j\phi_j} = -\frac{(d-3)(d-1)c m \kappa^2}{2} \frac{\rho^2 T^2}{(\sqrt{\rho^2 + (Z - z_0)^2})^{d+1}},$$

(3.9)
while all the components with mixed indices vanish, i.e., $E_{ρφj} = 0 = E_{φjφk}(j ≠ k)$.

We wish to evaluate (3.9) along the horizon $T - Z = 0$ and to express the result in terms of the advanced Killing time

$$v = \kappa^{-1} \ln \left( \frac{\kappa}{2}(T + Z) \right)$$

(3.10)

used in (3.5, 3.6). It is useful to work in terms of the shifted coordinate $\bar{v} = v - v_0$, where $v_0$ is the value of $v$ where the object crosses the horizon. Note that along the horizon we have

$$T = z_0 e^{κ\bar{v}}.$$  

(3.11)

Using this relation in (3.9) and expanding the exponentials for $κ\bar{v} ≪ 1$ we have

$$E_{ρρ} = \frac{-(d - 3)(d - 1) cmκ^2}{2} \frac{ρ^2}{\sqrt{ρ^2 + (z_0 κ \bar{v})^2}} \left. \right|_{d+1}. $$

(3.12)

Our analysis will now simplify considerably if we approximate the time dependence in this result by a delta function [9],

$$E_{ρρ} = \frac{- (d - 3) 8πmκz_0}{Ω_d - 3} \frac{ρ^2}{ρ^{d-2}} \delta(\bar{v})$$

(3.13)

which has the same time integral as (3.12). Note that the tidal forces then depend on $m$ and the object’s trajectory only through the quantity $mκz_0$, which is just the Killing energy $E_χ = -p_μχ^μ$, where $p^μ$ is the four-momentum. The replacement of (3.12) by (3.13) is a good approximation when $ρ ≪ z_0 = E_χ/(mκ)$. Let us assume for the moment that $r ≪ z_0 = E_χ/(mκ)$, so that $ρ ≪ z_0 = E_χ/(mκ)$ is the region of greatest interest. We will return to the more general case shortly.

Using (3.13) in our expression for the shear (3.5), we have

$$\dot{σ}_{ρρ} = \frac{-(d - 3)}{g_{φjφj}} \sigma_{φjφj} = -(d - 3) \frac{8πE_χ e^{κ\bar{v}}Θ(−\bar{v})}{Ω_d - 3} \frac{ρ^2}{ρ^{d-2}}. $$

(3.14)

We then substitute this result into (3.6) to find the expansion,

$$\dot{θ} = \frac{d - 2}{κ(d - 3)} \left( \frac{8π(d - 3)E_χ}{Ω_d - 3ρ^{d-2}} \right)^2 e^{κ\bar{v}}(1 - e^{-κ\bar{v}})Θ(−\bar{v})$$

(3.15)

The arguments given above then imply that caustics will form along our geodesic if

$$\left( \frac{\dot{θ}}{(d - 2)κ} \right)_{max} ≥ 1,$$

(3.16)
that is, if
\[ r \lesssim \left( \frac{4\pi \sqrt{d-3} E_{\chi}}{\Omega_{d-3} \kappa} \right)^{\frac{1}{d-2}}, \]  
(3.17)
where we have set \( \rho = r \), the radius of the object. On the other hand, no caustics will form along our geodesic if
\[ r \gg \left( \frac{4\pi \sqrt{d-3} E_{\chi}}{\Omega_{d-3} \kappa} \right)^{\frac{1}{d-2}}. \]  
(3.18)
For \( d = 4 \) and \( \kappa^{-1} = 4M \), this reduces exactly to the result stated in [6].

The above results were derived using the approximation \( r \ll z_0 = E_{\chi}/(m\kappa) \) to replace (3.12) by (3.13). However, it turns out that this need not be taken as an independent assumption. Let us first consider the case when (3.17) holds. Recall that we take the object to be weakly gravitating, so that \( r^{d-3} \gg m \). Combining this statement with (3.17), we arrive at \( r \ll z_0 = E_{\chi}/(m\kappa) \), so that the replacement of (3.12) by (3.13) is justified.

On the other hand, let us consider the case when (3.18) holds. For \( r \gtrsim z_0 \), the approximation of (3.12) by (3.13) makes an error. However, one notes that the solution (3.6) has a tendency to “forget” about the source at \( v' > v \). As a result, compressing the source to a delta-function can only increase the effect on the expansion. Thus, the maximum expansion resulting from (3.12) is strictly less than the maximum resulting from (3.13). It follows that condition (3.18) forbids the formation of caustics along our geodesics without further qualification.

We now consider those geodesics which do pass through the object. For simplicity, we consider a homogeneous object of constant Killing-energy density \( E_{\chi}/r^{d-1} \). In this case, we see from (3.13) that the electric part of the Weyl tensor is smaller inside the object than just outside. But the Weyl tensor is the only source for the shear. Thus, when (3.18) holds, the shear also remains small along all geodesics which intersect the matter.

It remains only to analyze the expansion, which is a linear functional of \( S(v) + \hat{\theta}^2 \). Let us first note that for the above source the contribution from \( S(v) \) to \( \hat{\theta}/\kappa \) is bounded by a term of order \( E_{\chi}/(\kappa r^{d-2}) \). On the other hand, we saw above that the shear term contributes a term of order \( (E_{\chi}/(\kappa r^{d-2}))^2 \). Adding the two such terms makes it clear that conditions (3.17), (3.18) again determine whether or not caustics form in the region of interest.
IV. LOWER DIMENSIONS: \( d = 2, 3 \)

It is interesting to discuss the remaining cases of lower spacetime dimension \( d = 2, 3 \). (There are no horizons for \( d < 2 \).) The analysis for \( d = 3 \) is quite similar to that above. The arguments of section III already hold for \( d = 3 \), and the main difference in section III is that both the Weyl tensor and the shear vanish identically. For those generators which pass through the object, we have already seen that ignoring the shear changes the threshold only by a coefficient of order one, so for \( d = 3 \) we again obtain \( r \sim (E_\chi/\kappa)^{1/(d-2)} = E_\chi/\kappa \). For generators which pass outside the object, the expansion simply remains zero. Locally, this part of the congruence is non-singular. However, the left and right sides of the congruence will nevertheless cross due to global effects. So long as \( r \) is much smaller than any curvature scale in the unperturbed background, one may use the conical deficit angle \( \delta = 4m \) generated by a point mass to check that the threshold for this to occur is once again set by \( r \sim E_\chi/\kappa = (E_\chi/\kappa)^{1/(d-2)} \). Thus, our results extend to the case \( d = 3 \).

Let us now consider the case \( d = 2 \). Here the main complication is that the Einstein-Hilbert action becomes trivial. Nevertheless, one may define a non-trivial theory either by studying a scalar gravity theory (dilaton gravity) or by considering compactifications of a higher-dimensional theory. The latter is, in a sense, a special case of the former. We proceed by considering various examples.

Let us first suppose that one simply compactifies \( n \)-dimensional Einstein gravity on an \( n-2 \) torus. One may again then argue as in section III that a first law holds for any quasi-stationary process. However, since there is only one uncompactified spatial direction, the gravitational field induced by any perturbation now tends to grow linearly with distance and can even change the asymptotics of the spacetime. There is thus no analogue of the arguments in section III. In particular, it seems likely that the passage of such an object would destroy any asymptotic Rindler horizon.

Another sort of \( d = 2 \) compactification arises when some method has been used to ‘stabilize the moduli’ at particular values (see e.g. [11, 12] for reviews). In practice, this means incorporating various quantum and/or stringy effects to create a potential for the size of the compactified directions. For a \( d = 2 \) compactification, this effectively creates a mass term for the gravitational degrees of freedom and removes the linear growth of gravitational fields described above. It thus stabilizes the boundary conditions. However, it also removes
us from the regime where the Einstein equations alone can be used to study the response of the horizon. In particular, since the volume of the compactified dimensions tends to remain constant, the area of the horizon is not changed by the passage of any object. In fact, if all moduli are stabilized at particular values, the horizon after the passage of the object will be indistinguishable from the original horizon. Thus, no analogue of the physical process first law can hold in this context. For the same reason, it is also clear that no stationary comparison version of the first law can hold. We therefore see no reason to assign a finite entropy to asymptotic horizons in this context.

As a third example of $d = 2$ gravity, we consider dilaton gravity associated with linear dilaton vacua. In such theories the dilaton runs from infinitely weak coupling on one side (say, at the right infinity) to infinitely strong coupling at the other (left) infinity. Unlike the previous two examples, these theories can admit black hole solutions. In fact, all of the known 1+1 black holes [13, 14] arise in this context.

Such linear dilaton boundary conditions turn out to be stable. However, in contrast to the case where the dilaton modulus is ‘stabilized’ at a particular value, here the value of the dilaton tends to change monotonically along horizons. As a result, in this context, there can be a first law for both black hole and asymptotic Rindler horizons. However, since our methods do not apply directly to this case, the first law must be verified by another method (see e.g. [14]).

The above examples suggest that 1+1 gravity systems enjoy a first law of horizon mechanics only when the boundary conditions break asymptotic spatial translation symmetry. This observation is interesting in the context of [7], which predicted that there would be no Poincaré-invariant 1+1 compactifications of consistent quantum gravity theories (and of string theory in particular). To arrive at this claim, the authors assumed that, since $d = 2$ Rindler horizons have finite area, one can assign them finite entropy$^1$. But our observation

$^1$ More precisely, the authors of [7] supposed that $i)$ compactifications of higher dimensional gravity with stabilized moduli would lead to 1+1 horizons with finite entropy and $ii)$ the entropy of such horizons should agree with the von-Neumann entropy of $\exp(-\beta H)$, where $\beta, H$ are respectively the inverse temperature and the Killing energy operator associated with the horizon. Ref. [7] showed that $(i)$ and $(ii)$ conflict with Poincaré symmetry, which led to the prediction stated above. We note that, so long as one adds the assumption that $iii)$ the theory contains localized excitations, this conclusion continues to hold if assumption $(ii)$ above is replaced with the somewhat different assumption that $ii'$) the entropy of such horizons counts the total number of quantum states associated with the system behind the horizon. In
above suggests that horizons in such theories do not enjoy a first law. Thus, we see no reason to assign them a finite entropy. From this viewpoint, it is no surprise that \cite{8} did in fact construct 1+1 Poincaré-invariant compactifications of string theory which are free of massless moduli.

V. DISCUSSION

We have argued that the physical process version of the first law holds non-trivially for any bifurcate Killing horizon in spacetime dimensions \( d \geq 3 \). We have also seen that it holds for \( d \geq 3 \) approximate Killing horizons such as general asymptotic Rindler horizons.

In addition to giving a straightforward derivation of the first law for quasi-stationary processes, we generalized the arguments of \cite{6, 9} to processes where a homogeneous weakly-gravitating object passes through any bifurcate Killing horizon in any spacetime dimension \( d \geq 3 \). In particular, we showed that for \( d \geq 3 \) the condition

\[
r_{d-2} \sim \frac{E_\chi}{\kappa}
\]

sets the threshold for the formation of caustics to the future of the past horizon. When \( r_{d-2} \gg \frac{E_\chi}{\kappa} \), no caustics form in the region to the future of the unperturbed past horizon and the process is indeed quasi-stationary. Thus, the first law applies. In particular, for \( d \geq 3 \) our work completes the analogy between black hole and asymptotic Rindler horizons outlined in \cite{1, 5}.

The condition \((5.1)\) should not be a surprise. If the first law holds, this threshold is

\[
r_{d-2} \sim \Delta A.
\]

In other words, the first law is valid when the horizon area (entropy) through which the object passes is much larger than the change in area (entropy) induced by the object itself. From the thermodynamic perspective, this is a natural definition of a quasi-stationary process\(^2\).

One may use a related perspective to briefly summarize the arguments of section III without going through the technical details. The point is that one expects caustics to form

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footnote{1} We thank Ted Jacobson for suggesting this interpretation of (1.2).

footnote{2} We thank Ted Jacobson for suggesting this interpretation of (1.2).

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first along those geodesics which pass through the edge of the matter distribution, where both the matter-energy density and the electric part of the Weyl tensor act as sources for (2.1) and (3.2). But as discussed in section II, dropping non-linear terms in (2.1), (3.2) implies that an energy $\delta E_\chi$ passing through a bundle of generators with area $\delta A$ causes a change in the area of that bundle given by $\Delta(\delta A) = \delta E_\chi / \kappa$. So long as this is much less than $\delta A$ itself one must be far from caustics, as caustics arise when the area of a bundle of null geodesics shrinks to zero. Thus a first-order treatment should be valid and the physical process first law should hold. Integrating this condition over the matter distribution gives precisely (5.1).

In contrast, all of the above results may fail for $d = 2$. In particular, there can be no first law when higher-dimensional gravity is compactified to $d = 2$ in a manner that stabilizes all moduli, or in the string theories of [8]. Such theories effectively assign gravity a mass $m > 0$, and turn off the long-range interaction. Thus, one might say that the long-distance Newton’s constant $G$ has been renormalized to zero. This viewpoint suggests that any horizon entropy is strictly infinite, and it is no surprise that the first law becomes trivial. On the other hand, horizon entropy is often thought of as a short-distance phenomenon. From this perspective one might not expect the mass $m$ to influence the entropy, since $m$ may be much less than any fundamental scale (such as the short-distance Planck Mass $m_{pl}$). It would thus be especially interesting to understand these effects from a microscopic perspective.

Acknowledgements

D.M. thanks Ted Jacobson for many interesting discussions on these and related issues, and for detailed comments on an early draft of the manuscript. He also thanks Steve Giddings and Gary Horowitz for a discussion of 1+1 compactifications. This work was supported in part by the National Science Foundation under Grant No PHY05-55669, and by funds from the University of California.

APPENDIX A: RAYCHAUDHURI EQUATIONS

This appendix reviews the derivation of the Raychaudhuri equation for null geodesic congruences and the associated equations for the shear and twist in general spacetime dimension
\( d \geq 3 \). As in the main text, we consider a congruence associated with a bifurcate Killing horizon\(^3\).

Let \( k^\mu \) be the affinely parametrized future-pointing null normal generating the bifurcate Killing horizon \( \mathcal{K} \). On the bifurcation surface, there is a second future-pointing null normal \( l^\mu \), which one may think of as pointing in the opposite spatial direction to \( k^\mu \). We choose 
\( l^\mu k_\mu = -1 \) and \( k^\nu \nabla_\mu l^\nu = 0 \) on the bifurcation surface, and define \( l^\mu \) all along the congruence by parallel transport along the geodesics. Thus, the above inner products hold at each point in the congruence.

For our purposes, it is more convenient to parametrize the null geodesics so that their tangents are given by the Killing vector field \( \chi^\mu \), rather than \( k^\mu \). This vector satisfies
\[
\chi^\sigma \nabla_\sigma \chi^\mu = \kappa \chi^\mu, \tag{A1}
\]
where \( \kappa \) is the surface gravity of \( \mathcal{K} \). Similarly we can now define a second null vector field \( \hat{l}^\mu \) satisfying
\( \hat{l}^\mu \chi_\mu = -1 \) and \( \chi^\nu \nabla_\mu \hat{l}^\nu = -\kappa \hat{l}^\nu \). Now, following for example [3], we can define a projection tensor
\[
Q_{\mu\nu} = g_{\mu\nu} + \chi_\mu \hat{l}_\nu + \chi_\nu \hat{l}_\mu \tag{A2}
\]
that projects onto the \((d-2)\)-dimensional space spanned by the deviation vectors orthogonal to both \( \chi^\mu \) and \( \hat{l}^\mu \). This also coincides with the space tangent to the cut \( \mathcal{C} \) of the horizon obtained by Lie dragging the bifurcation surface along the affine tangent field \( k^\mu \).

We now introduce the distortion tensor
\[
\hat{B}_{\mu\nu} = Q^{\alpha}_{\mu} Q^{\beta}_{\nu} \nabla_{\beta} \chi_{\alpha}, \tag{A3}
\]
which satisfies
\[
\chi^\sigma \nabla_\sigma \hat{B}_{\mu\nu} = \kappa \hat{B}_{\mu\nu} - \hat{B}_{\mu}^\sigma \hat{B}_{\sigma\nu} - Q^{\alpha}_{\mu} Q^{\beta}_{\nu} R_{\alpha\beta\lambda\sigma} \chi^\lambda \chi^\sigma. \tag{A4}
\]
The tensor \( \hat{B}_{\mu\nu} \) can be decomposed into expansion, shear, and twist as
\[
\hat{B}_{\mu\nu} = \frac{\hat{\theta}}{d-2} Q_{\mu\nu} + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu}, \tag{A5}
\]
\(^3\) The Raychaudhuri equations for expansion, shear, and twist in general spacetime dimension were previously derived in [10] using affine parametrization instead of Killing parametrization. It seems that the expansion equation in general spacetime dimension has been known for some time, e.g. [3].
where \( \hat{\theta} = Q^{\mu\nu} \hat{B}_{\mu\nu} \), \( \hat{\sigma}_{\mu\nu} = \hat{B}_{(\mu\nu)} - \frac{\hat{\theta}}{d-2} Q_{\mu\nu} \), and \( \hat{\omega}_{\mu\nu} = \hat{B}_{[\mu\nu]} \). Taking the trace of (A4) gives
\[
\chi^\sigma \nabla_\sigma \hat{\theta} = \kappa \hat{\theta} - \frac{\hat{\theta}^2}{d-2} - \hat{\sigma}_{\mu\nu} \hat{\sigma}^{\mu\nu} + \hat{\omega}_{\mu\nu} \hat{\omega}^{\mu\nu} - R_{\lambda\sigma} \chi^\lambda \chi^\sigma , \tag{A6}
\]
while taking the antisymmetric part gives
\[
\chi^\sigma \nabla_\sigma \hat{\omega}_{\mu\nu} = \kappa \hat{\omega}_{\mu\nu} - \frac{2}{d-2} \hat{\omega}_{\mu\nu} - 2 \hat{\sigma}_{[\mu\sigma} \hat{\omega}_{\nu]\sigma} . \tag{A7}
\]
Finally, the traceless symmetric part is
\[
\chi^\sigma \nabla_\sigma \hat{\sigma}_{\mu\nu} = \kappa \hat{\sigma}_{\mu\nu} - \frac{2\hat{\theta}}{d-2} \hat{\sigma}_{\mu\nu} - \hat{\sigma}_{\mu\sigma} \hat{\sigma}^{\sigma\nu} - \hat{\omega}_{\mu\sigma} \hat{\omega}^{\sigma\nu} + \frac{1}{d-2} (\hat{\sigma}^2 - \hat{\omega}^2) Q_{\mu\nu} - Q_{\mu\rho} Q_{\nu}^\rho C_{\alpha\beta\sigma\lambda} \chi^\lambda \chi^\sigma . \tag{A8}
\]
We now simplify these equations as in [6]. First we note that the null congruence generating a bifurcate Killing horizon is hypersurface orthogonal, i.e.,
\[
\hat{\omega}_{\mu\nu} = 0 . \tag{A9}
\]
Thus, \( \hat{B}_{\mu\nu} \) is a symmetric 2-tensor identical to the extrinsic curvature of the cut \( \mathcal{C} \) introduced above. As usual, this tensor may be written
\[
\hat{B}_{\mu\nu} = \frac{1}{2} \mathcal{L}_\chi Q_{\mu\nu} . \tag{A10}
\]
By choosing coordinates adapted to the horizon, we can also replace the Lie derivative \( \mathcal{L}_\chi \) by an ordinary derivative with respect to a Killing parameter \( v \):
\[
\hat{B}_{\mu\nu} = \frac{1}{2} \frac{dQ_{\mu\nu}}{dv} . \tag{A11}
\]
We then decompose \( \hat{B}_{\mu\nu} \) into shear and expansion as before to obtain the ‘metric evolution equation’ [6, 17, 18],
\[
\frac{1}{2} \frac{dQ_{\mu\nu}}{dv} = \hat{\sigma}_{\mu\nu} + \frac{\hat{\theta}}{d-2} Q_{\mu\nu} . \tag{A12}
\]
We would like to write (A6), (A8) as similar ordinary differential equations. Since the expansion is a scalar, we can simply replace \( \chi^\sigma \nabla_\sigma \) with \( \frac{d}{dv} \) in equation (A6). The result is
\[
\frac{d\hat{\theta}}{dv} = \kappa \hat{\theta} - \frac{\hat{\theta}^2}{d-2} - \hat{\sigma}_{\mu\nu} \hat{\sigma}^{\mu\nu} - R_{\lambda\sigma} \chi^\lambda \chi^\sigma . \tag{A13}
\]
However, when acting on a tensor quantity like the shear, the derivatives \( \mathcal{L}_\chi \) and \( \chi^\sigma \nabla_\sigma \) differ by ‘connection terms’ [17]:
\[
\mathcal{L}_\chi \hat{\sigma}_{\mu\nu} = \chi^\sigma \nabla_\sigma \hat{\sigma}_{\mu\nu} + \hat{\sigma}_{\sigma\nu} \nabla_\mu \chi^\sigma + \hat{\sigma}_{\mu\sigma} \nabla_\nu \chi^\sigma \tag{A14}
\]
\[
= \chi^\sigma \nabla_\sigma \hat{\sigma}_{\mu\nu} + \hat{\sigma}_{\sigma\nu} \hat{B}^\sigma_{\mu} + \hat{\sigma}_{\mu\sigma} \hat{B}^\sigma_{\nu} \tag{A15}
\]
\[
= \chi^\sigma \nabla_\sigma \hat{\sigma}_{\mu\nu} + 2\hat{\sigma}_{\mu}^\sigma \hat{\sigma}_{\sigma\nu} + \frac{2\hat{\theta}}{d-2} \hat{\sigma}_{\mu\nu} . \tag{A16}
\]
Thus we find
\[\frac{d\sigma_{\mu\nu}}{dv} = \left(\kappa - \frac{2\dot{\theta}}{d-2}\right)\dot{\sigma}_{\mu\nu} - \dot{\sigma}_{\mu\sigma}\sigma^\sigma\nu + \frac{\dot{\sigma}^2}{d-2}Q_{\mu\nu} + \left(2\dot{\sigma}_{\mu\sigma} + \frac{2\dot{\theta}}{d-2}Q_{\mu\sigma}\right)\hat{\sigma}^\sigma\nu - Q^\alpha_{\mu}\hat{Q}^\beta_{\nu}C_{\alpha\beta\sigma\chi}\chi^\sigma.\]  
(A17)

The ‘focusing equation’ \(\text{(A13)}\) and the ‘tidal force equation’ \(\text{(A17)}\) are the key results of this appendix. Note that, for \(d = 4\), equations \(\text{(A13)}\) and \(\text{(A17)}\) can be simplified because the indices appearing in these equations run only over two dimensions. In this case we can use the identities
\[\hat{\sigma}^\sigma\mu\nu = 0, \quad \hat{\sigma}_{\mu\sigma}\hat{\sigma}^\sigma\nu = \frac{1}{2}\dot{\sigma}^2Q_{\mu\nu}, \quad \hat{\omega}_{\mu\sigma}\hat{\sigma}^\sigma\nu = -\frac{1}{2}\dot{\omega}^2Q_{\mu\nu},\]  
(A18)
after which our results reproduce those of \([6, 9, 17, 18]\). The equations simplify even further for \(d = 3\), where the shear, twist, and Weyl tensor vanish identically.

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