Anyons and Fractional Quantum Hall Effect in Fractal Dimensions

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The fractional quantum Hall effect is a paradigm of topological order and has been studied thoroughly in two dimensions. Here, we construct a new type of fractional quantum Hall system, which has the special property that it lives in fractal dimensions. We provide analytical wave functions and exact few-body parent Hamiltonians, and we show numerically for several different Hausdorff dimensions between 1 and 2 that the systems host Laughlin type anyons. We also find examples of fractional quantum Hall physics in fractals with Hausdorff dimension 1 and ln(4)/ln(5).

Our results suggest that the local structure of the investigated fractals is more important than the Hausdorff dimension to determine whether the systems are in the desired topological phase.

Elementary particles that exist in three dimensions are all either bosons or fermions. Nevertheless, Leinaas and Myrheim argued that in two dimensions it is allowed mathematically to have particles that are neither bosons, nor fermions, but anyons [1]. Anyons have unusual exchange properties and can have fractional charge. It was later discovered that anyons are realized physically as quasiparticles in the fractional quantum Hall effect [2, 3].

The investigations of anyons have led to more important new insights, including a large development within the description of quantum phases and phase transitions [4], and ideas to use anyons to store and process quantum information in a topologically protected way [5].

Among the most important developments in the field are the discoveries of new types of fractional quantum Hall effects and new types of systems where anyons can be realized. This includes the observation of fractional quantum Hall physics in graphene [6, 7], fractional quantum Hall physics in lattice systems [8], and generalizations of anyons and the fractional quantum Hall effect to three- or four-dimensional systems [9–11]. These developments are important, since each new type of system has its own properties: Graphene gave a relativistic version of the fractional quantum Hall effect, the introduction of lattices eliminated the need for a physical magnetic field, and generalized anyons in three or four dimensions are quite different from anyons in two dimensions.

The possibility of changing the dimension is particularly interesting, since the properties of a system in general depend strongly on the dimension of the system. Introducing fractal structures, it is even possible to consider non-integer dimensions. In the past, classical and single-particle quantum models have been studied on fractal lattices [12, 13], and renewed interest in the topic, in the form of investigations of noninteracting quantum models, has appeared recently [15–20]. This interest is, in part, motivated by experimental developments to prepare fractal structures in molecules and nanomaterials [21, 22]. Ultracold atoms in optical lattices also provide an interesting direction for realizing fractal models, in particular given the current efforts on creating innovative lattice potentials [23].

The question about topological quantum models on fractals has been taken up recently [17, 18, 20], where noninteracting Chern insulator models have been investigated. The much harder question of whether interacting topological phases, fractional quantum Hall physics, and anyons can exist in fractal space is, however, still unanswered.

In this Letter, we answer this important question in the affirmative by constructing a new type of fractional quantum Hall models, which have the special property that they live in fractal space. We provide analytical wavefunctions and corresponding few-body parent Hamiltonians. The operation needed to go from one generation to the next is shown on the left. Here, we consider a lattice model on the fractal, in which a magnetic flux goes through the center of each triangle (blue cross and circle).

The numerical resources needed to study strongly-correlated quantum many-body systems generally grow...
exponentially with the system size. Two-dimensional systems are therefore difficult to handle. One-dimensional systems are easier, but the physics is often quite different, e.g. because there are strong restrictions on the possible directions a particle can move. Systems with dimensions between 1 and 2 constitute an intriguing intermediate regime, where interesting physics is likely to happen. The present work is a first example in this direction.

**Fractal lattices.**—Spaces having fractal dimension are realized in fractals. A famous example is the Sierpinski gasket (Fig. 1) with dimension \( D = \ln(3)/\ln(2) \approx 1.5850 \). A Sierpinski gasket of generation 0 is a single triangle, and the Sierpinski gasket of generation \( n + 1 \) is obtained from the Sierpinski gasket of generation \( n \) by applying the operation shown on the left in Fig. 1 to all triangles in the gasket. The full fractal is obtained in the limit of infinite generation.

In any physical system, there is a limit to how small the smallest scales of a fractal can be and a limit to how large the total fractal can be, so the generation of a physical fractal is always finite. As long as we are investigating the fractal at a length scale, which is large compared to the smallest structures of the fractal and small compared to the total size of the fractal, it does, however, not make a difference whether the generation of the fractal is finite or infinite, and the system will effectively be in a space with the dimension of the fractal.

Here, we consider a lattice model on the fractal, where there is one lattice site at the center of each of the smallest triangles. In the limit of large enough generation, it does not make a significant difference, whether we treat each triangle as a triangle or a single point, since the triangles are much smaller than the length scales of interest.

**Quantum states.**—We start our search for anyons in fractal space by constructing fractional quantum Hall states on fractal lattices. The standard fractional quantum Hall effect is realized in a two-dimensional electron gas, and an important ingredient is a strong magnetic field perpendicular to the plane. Just taking a fractional quantum Hall state and restricting the possible particle positions to be on the fractal lattice is not enough to obtain the desired states. The main trick is that we also need to restrict the magnetic flux to only go through the lattice sites as shown in Fig. 1. This means that the Gaussian factor, present, e.g., in the Laughlin state, is modified. We can find the appropriate modification by using the conformal field theory approach \(^{24, 25}\) to construct the states.

We here consider the Laughlin state with \( q \) fluxes per particle, where \( q \) is an integer. We associate a vertex operator \( V_{n_i}(z_j) = : e^{i(\eta n_j - \eta) \phi(z_j)/\sqrt{\eta}} : \) to each of the lattice sites. Here, \( n_j \in \{0, 1\} \) is the number of particles on the site, \( z_j \) is the position of the site in the plane written as a complex number, \(-\eta\) is the magnetic flux through the site, \( \phi(z_j) \) is the chiral part of a free massless boson, and \( : \ldots : \) means normal ordering. We take \(-\eta < 0\), since the magnetic field points inwards (as in Fig. 1). In the two-dimensional case, we can insert anyons with charge \( p_k/q \), where \( p_k \) is an integer, into the states at the positions \( w_k \) by including a vertex operator \( H(w_k) = : e^{ip_k \phi(w_k)/\sqrt{\eta}} : \) for each of the anyons. We do the same here to investigate whether these operators also produce anyons on the fractal lattice. The Laughlin state on a fractal lattice with \( N \) sites, \( M \) particles, and \( K \) “anyons” is then defined as

\[
|\psi_{q,K,M}\rangle \propto \sum_{n_1,\ldots,n_N} \prod_{k=1}^K H(w_k) \prod_{j=1}^N V_{n_j}(z_j)|0\rangle|n_1,\ldots,n_N\rangle,
\]

where \( |0\rangle \) is the vacuum state. This evaluates to

\[
|\psi_{q,K,M}\rangle \propto \sum_{n_1,\ldots,n_N} \delta_n \prod_{i=1}^N e^{i\phi n_j} \prod_{i=1}^K \prod_{j=1}^N (w_i - z_j)^{p_i n_j} \times \prod_{i<j}(z_i - z_j)^{q n_i n_j - n_i n_j - n_i n_j}|n_1,\ldots,n_N\rangle.
\]  

(1)

Here, \( \phi_j \) are undetermined phases, which do not influence the results below, and \( \delta_n \) is one if \( M = \sum n_j = (\eta N - \sum_k p_k)/q \) and zero otherwise. Note that the wavefunction has a well-defined limit for \( w_k \to z_j \), so the anyon coordinates are allowed to be on the lattice sites.
are, indeed, screened and have the expected charge 0.

Topological properties.—Although the analytical expression
for the state (1) is similar to the Laughlin state,
there is no guarantee that the state has the correct topological
properties to qualify for being a Laughlin type state. We now
show numerically that it has. We do
this by showing that the anyons are screened and have
the same charge and braiding properties as the normal
Laughlin anyons.

The particle density difference on site \( j \), defined as

\[
p_j = \langle \psi_{q,K,M} | \sum_k p_k/q | n_j \rangle \langle n_j | \sum_k p_k/q \psi_{q,K,M} \rangle
- \langle \psi_{q,0,M} | n_j \rangle \langle n_j | \psi_{q,0,M} \rangle,
\]

gives the expectation value of the number of particles on
the site, when there are anyons in the system, relative
to the same quantity, when there are no anyons in the
system. Note that the number of particles in the two
states is chosen such that the magnetic flux \(-\eta N\) is the
same for the two states. Therefore, we have a system
with screened anyons, if \( p_j \) is only different from zero in
a small region around each anyon. The charge of the
kth anyon is defined as \( Q_k = -\sum_{j \in R_k} p_j \), where \( R_k \)
is a small region around the anyon, which is large enough
to enclose the anyon, but small enough to not enclose other
anyons.

We compute \( p_j \) numerically using the Metropolis
Monte Carlo algorithm [26]. The results for \( q = 2\),
\( M = 30 \), and 2 anyons in Fig. 3 show that the anyons
are, indeed, screened and have the expected charge 0.5.

Property that determines the relevant length scale is the
density of particles in the system, and we have chosen
the number of particles such that the typical distance
between two particles is large compared to the smallest
lattice spacing and small compared to the complete fractal.
We can also see this at the level of the anyons. The
two generations shown in the figure have the same typical
distance between the particles. The size of the anyons
do not change significantly, when going from generation
4 to generation 5. At the same time, the anyons are
small compared to the full lattice. This means that the
anyons would not be affected significantly, if we replaced
the finite generation fractal with an infinite generation
fractal.

Computing \( p_j \) for different positions of the anyons
show that the size of the anyons depends on the local
distribution of lattice sites around the anyons. It is reason-
able that the screening is affected by, at which points
close to the anyons, we allow particles to be present. In
all cases, however, the anyons are screened and have the
correct charge. We have done similar computations for
\( q = 3 \), and the conclusions are the same.

To braid the anyons, we need to move the \( w_k \) along a
continuous path. We can do this by allowing the \( w_k \) to
be at any point in the two-dimensional plane. We note
that even when \( w_k \) is between the lattice sites, the anyon
itself is still present only on the fractal lattice. This is so
because all the particles forming the quantum state are
only allowed to be on the fractal lattice. For Laughlin
states in two dimensions, it is known analytically that
the anyons have the correct braiding properties as long
as the anyons are screened and have the correct charge [27].

The derivation of the braiding properties in [27] also
holds for the fractal lattice. We hence conclude that the
constructed states, indeed, have the desired topological
properties.

Other dimensions.—We have now demonstrated that
anyons and fractional quantum Hall physics can be re-
alyzed for a particular fractal dimension. This raises the
question whether the effects can be seen in any dimen-
sion between 1 and 2, or only for this special dimension.
We therefore investigate a family of fractals that allows
us to vary the dimension. We start from a square, and
we go from one generation to the next by dividing each of
the squares present into \( L \times L \) squares of equal size
and only keeping \( U \) of the squares in a particular pattern.
The dimension of this fractal is \( \ln(U)/\ln(L) \). We study

![FIG. 3. Anyons on fractals of different dimensions. The color of each lattice site gives \( \rho_j \), and the fractals are generated as shown in the insets of Fig. 4.](image-url)
state of the states to construct a Hamiltonian, which has the possibility to use the conformal field theory properties for further studies.

Comparisons between fractional quantum Hall physics in continuous systems and in lattices have revealed important differences and new possibilities. The fractal lattices also provide new possibilities, and this strongly motivates a detailed investigation of the interplay between fractional quantum Hall physics and the structure of fractals. It seems particularly promising to study properties that have a strong dependence on the dimension of space, such as transport and entanglement.

FIG. 4. We generate fractals of different dimensions by dividing a square into 16 squares, keeping only the squares in purple (insets), and then repeating (the generation is 4 for $D < 1.20$, 3 for $1.20 < D < 1.55$, and 2 for $D > 1.55$). In all cases, $q = 2$ and $M = 40$. The main plot shows the charge of two anyons inserted into the model as a function of the dimension of the fractal (we use the same size of the local region $R_k$ for all cases). The charge is seen to be 0.5 (marked by the green line) independent of the dimension. This suggests that anyons and the fractional quantum Hall effect can exist in the whole range of dimensions from 2 to 1.

For all the considered cases, we find that the anyons are screened and have a charge of 0.5. These results suggest that it is, indeed, possible to have anyons and fractional quantum Hall physics in all dimensions between 1 and 2.

If we put the model on a one-dimensional line (Fig. 5(a)), we find that the anyons are not screened. This is expected, given that this model is critical [25]. Interestingly, however, it is possible to have screened anyons and fractional quantum Hall physics also in one dimension, if we instead choose to keep 4 squares as shown in the last inset in Fig. 4. For both cases, there are $N = 4^J$ sites and $M = 40$ particles in the system, $q = 2$, and the color shows $\rho_j$.

FIG. 5. (a) If we put our model on a one-dimensional lattice on a line, the anyons are not screened. (Note that the circles representing the lattice points overlap each other.) (b) It is, however, possible to have screened anyons and fractional quantum Hall physics in one dimension, if we instead choose to keep 4 squares as shown in the last inset in Fig. 4. For both cases, there are $N = 4^J$ sites and $M = 40$ particles in the system, $q = 2$, and the color shows $\rho_j$.

We find numerically that the ground state of $H$ is unique, when the number of particles in the system is fixed to $M$.

**Discussion.**—We have constructed a new type of fractional quantum Hall models that are defined on fractals, and we have shown that anyons can exist in dimensions between 1 and 2. We have also shown that fractional quantum Hall physics can appear in systems with Hausdorff dimension less than 1. These results are an important first step that opens up more interesting directions for further studies:

More examples with $L = 4$ as shown in Figs. 3 and 4. For all the considered cases, we find that the anyons are screened and have a charge of 0.5. These results suggest that it is, indeed, possible to have anyons and fractional quantum Hall physics in all dimensions between 1 and 2.

**Exact Hamiltonian.**—So far, we have shown that anyons can exist in fractal dimensions, and we have constructed fractional quantum Hall states on fractal lattices hosting anyons. As long as $\eta < 1 + q/N + \sum_k p_k/N$, it is possible to use the conformal field theory properties of the states to construct a Hamiltonian, which has the state $|\psi_{q,K,M}\rangle$, defined on a general lattice, as ground state [27]. This results in the Hamiltonian

$$H = \sum_i \sum_{k(i \neq j)} \sum_{j(i \neq k)} \frac{1}{z_i - z_k} \frac{1}{z_i - z_j} [\bar{T}_k^{-1} T_j^{-1} b_i^\dagger b_j - T_k^{-1} T_i^{-1} b_i^\dagger b_j (q n_j - 1) - T_i^{-1} T_j^{-1} (q n_k - 1) b_i^\dagger b_j + |T_i|^{-2} n_i (q n_k - 1) (q n_j - 1)].$$  \hspace{1cm} (3)

Here, $b_j$ is the operator that annihilates a hardcore boson (fermion) on site $j$, when $q$ is even (odd), $n_j = b_j^\dagger b_j$ is the number operator, and

$$T_k = e^{i d_k} e^{-i \pi (k - 1)} \prod_i (w_i - z_k) \prod_j (z_j - z_k)^{1-\eta}. \hspace{1cm} (4)$$

The $\bar{T}_k$ are the quantum Hall states.

We find numerically that the ground state of $H$ is unique, when the number of particles in the system is fixed to $M$.
The idea presented in this work of restricting the magnetic field to a fractal lattice may give some hints on how to construct fractional Chern insulator type Hamiltonians on fractal lattices. A natural choice would be a model with interactions and complex hopping terms that realize the magnetic field. Such models could pave the way for implementations in ultracold atoms in optical lattices in the future.

Tools to detect topological order in quantum systems often rely on defining the investigated models on closed surfaces. This is, however, not possible for fractals, and the work hence motivates the development of additional methods to test for topology.

Finally, as discussed above, the study of strongly-correlated quantum systems on fractals constitute an interesting intermediate regime between one and two dimensions, where little is currently known.

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