Remark on the N-barrier method for a class of autonomous elliptic systems

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Dedicated to my grandmother in memoriam

Abstract

In this note, we aim to extend the previous work on an N-barrier maximum
principle ([1], 2) to a more general class of systems of two equations. Moreover, an
N-barrier maximum principle for systems of three equations is established.

1 N-barrier maximum principle for two equations

We are concerned with the autonomous elliptic system in \( \mathbb{R} \):

\[
\begin{align*}
\begin{cases}
    d_1 u_{xx} + \theta u_x + u f(u, v) = 0, & x \in \mathbb{R}, \\
    d_2 v_{xx} + \theta v_x + v g(u, v) = 0, & x \in \mathbb{R},
\end{cases}
\end{align*}
\]

(1.1)

which arises from the study of traveling waves in the following reaction-diffusion system
([3]):

\[
\begin{align*}
\begin{cases}
    u_t = d_1 u_{yy} + u f(u, v), & y \in \mathbb{R}, \quad t > 0, \\
    v_t = d_2 v_{yy} + v g(u, v), & y \in \mathbb{R}, \quad t > 0.
\end{cases}
\end{align*}
\]

(1.2)

A positive solution \((u(x), v(x)) = (u(y, t), v(y, t)), x = y - \theta t\) of (1.1) stands for a
traveling wave solution of (1.2). Here \(d_1\) and \(d_2\) represent the diffusion rates and \(\theta\) is the

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propagation speed of the traveling wave. Throughout, we assume, unless otherwise stated, that the following hypotheses on $f(u,v) \in C^0(\mathbb{R}^+ \times \mathbb{R}^+)$ and $g(u,v) \in C^0(\mathbb{R}^+ \times \mathbb{R}^+)$ are satisfied:

[H1] the unique solution of $f(u,v) = g(u,v) = 0$ and $u,v > 0$ is $(u,v) = (u^*, v^*)$;

[H2] the unique solution of $f(u,0) = 0$ and $u > 0$ is $u = u_1 > 0$; the unique solution of $f(0,v) = 0, v > 0$ is $v = v_1 > 0$; the unique solution of $g(u,0) = 0, u > 0$ is $u = u_2 > 0$; the unique solution of $g(0,v) = 0, v > 0$ is $v = v_2 > 0$;

[H3] as $u,v > 0$ are sufficiently large, $f(u,v), g(u,v) < 0$; as $u,v > 0$ are sufficiently small, $f(u,v), g(u,v) > 0$;

[H4] the two curves

$$C_f = \{(u,v) \mid f(u,v) = 0, \ u, v \geq 0\}$$

(1.3)

and

$$C_g = \{(u,v) \mid g(u,v) = 0, \ u, v \geq 0\}$$

(1.4)

lie completely within the region $\mathcal{R}$, which is enclosed by the $u$-axis, the $v$-axis, the line $\mathcal{L}$ given by

$$\mathcal{L} = \{(u,v) \mid \frac{u}{u} + \frac{v}{v} = 1, \ u, v \geq 0\}$$

(1.5)

and the line $\mathcal{L}$ given by

$$\mathcal{L} = \{(u,v) \mid \frac{u}{u} + \frac{v}{v} = 1, \ u, v \geq 0\},$$

(1.6)

for some $\bar{u} > y > 0$, $\bar{v} > y > 0$.

In this note, the following boundary value problem for (1.10) is studied:

\[
\begin{aligned}
&
\begin{cases}
    d_1 u_{xx} + \theta u_x + u f(u,v) = 0, & x \in \mathbb{R}, \\
    d_2 v_{xx} + \theta v_x + v g(u,v) = 0, & x \in \mathbb{R}, \\
    (u,v)(-\infty) = e_-, & (u,v)(+\infty) = e_+,
\end{cases}
\end{aligned}
\]

(1.7)

where $e_-, e_+ = (0,0), (u_1,0), (0,v_2)$, or $(u^*, v^*)$. We call a solution $(u(x),v(x))$ of (1.10) an $(e_-, e_+)$-wave. As in [1, 2], adding the two equations in (1.10) leads to an equation involving $p(x) = \alpha u(x) + \beta v(x)$ and $q(x) = d_1 \alpha u(x) + d_2 \beta v(x)$, i.e.

\[
0 = \alpha \left( d_1 u_{xx} + \theta u_x + u f(u,v) \right) + \beta \left( d_2 v_{xx} + \theta v_x + v g(u,v) \right)
\]

\[
=q''(x) + \theta p'(x) + \alpha u f(u,v) + \beta v g(u,v)
\]

\[
=q''(x) + \theta p'(x) + F(u,v),
\]

(1.8)

where $\alpha, \beta > 0$ are arbitrary constants and $F(u,v) = \alpha u (\sigma_1 - c_{11} u - c_{12} v) + \beta v (\sigma_2 - c_{21} u - c_{22} v)$. To show the main result, we begin with a useful lemma.
Lemma 1.1. Under $[H1 \sim H4]$, the curve $F(u, v) = 0$ in the first quadrant of the uv-plane, i.e.
\[ C_F = \{(u, v) \mid F(u, v) = 0, \ u, v \geq 0\} \] (1.9)
lies completely within the region $\mathcal{R}$.

Proof. The proof is elementary and is thus omitted here. \hfill \Box

We are now in the position to prove

Theorem 1.2 (N-Barrier Maximum Principle for Systems of Two Equations). Assume $[H1 \sim H4]$ hold. Suppose that there exists a pair of positive solutions $(u(x), v(x))$ of the boundary value problem
\[
\begin{cases}
  d_1 u_{xx} + \theta u_x + u f(u, v) = 0, & x \in \mathbb{R}, \\
  d_2 v_{xx} + \theta v_x + v g(u, v) = 0, & x \in \mathbb{R}, \\
  (u, v)(-\infty) = e_- , & (u, v)(+\infty) = e_+ ,
\end{cases}
\] (1.10)
where $e_-, e_+ = (0, 0), (u_1, 0), (0, v_2), \ (u^*, v^*)$. For any $\alpha, \beta > 0$, let $p(x) = \alpha u(x) + \beta v(x)$ and $q(x) = \alpha d_1 u(x) + \beta d_2 v(x)$. We have
\[
p \leq p(x) \leq \bar{p}, \ x \in \mathbb{R},
\] (1.11)
where
\[
\bar{p} = \max (\alpha \bar{u}, \beta \bar{v}) \frac{\max(d_1, d_2)}{\min(d_1, d_2)}
\] (1.12)
and
\[
p = \min (\alpha u, \beta v) \frac{\min(d_1, d_2)}{\max(d_1, d_2)} \chi,
\] (1.13)
where
\[
\chi = \begin{cases}
  1, & \text{if } e_+ \neq (0, 0), \\
  0, & \text{if } e_- = (0, 0).
\end{cases}
\] (1.14)

Proof. We employ the N-barrier method developed in [1, 2] to show (1.11). It is readily seen that once appropriate N-barriers are constructed, the upper and lower bounds in (1.11) are given in exactly the same way as in (1.2).

To prove the lower bound, we can construct an N-barrier which starts either from the point $(\bar{u}, 0)$ or $(0, \bar{v})$. Due to Lemma 1.1, the N-barrier is away from the region $\mathcal{R}$. Then the lower bound can be proved by contradiction as in [1, 2].

On the other hand, the upper bound can be shown in the same manner by constructing an N-barrier starting either from the point $(\bar{u}, 0)$ or $(0, \bar{v})$. In addition, it is clear that when the boundary conditions $e_+ = (0, 0)$ or $e_- = (0, 0)$, a trivial lower bound of $p(x)$, i.e. $p(x) \geq 0$ can only be given. This completes the proof. \hfill \Box
2 N-barrier maximum principle for three equations

In this section, the following assumptions on \( f(u,v,w) \in C^0(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+), g(u,v,w) \in C^0(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+), \) and \( h(u,v,w) \in C^0(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+) \) are satisfied:

[A1] (Unique coexistence state) \((u,v,w) = (u^*, v^*, w^*)\) is the unique solution of

\[
\begin{align*}
  f(u,v,w) &= 0, \quad u,v,w > 0, \\
  g(u,v,w) &= 0, \quad u,v,w > 0, \\
  h(u,v,w) &= 0, \quad u,v,w > 0.
\end{align*}
\]  

(A.1)

[A2] (Competitively exclusive states) For some \(u_i, v_i, w_i > 0\) \((i = 1, 2, 3)\) with \((\Lambda_1 - \Lambda_2)^2 + (\Lambda_1 - \Lambda_3)^2 + (\Lambda_2 - \Lambda_3)^2 \neq 0\) \((\Lambda = u, v, w),\)

\[
\begin{align*}
  f(u_1,0,0) &= f(0,v_1,0) = f(0,0,w_1) = 0, \\
  g(u_2,0,0) &= g(0,v_2,0) = g(0,0,w_2) = 0, \\
  h(u_3,0,0) &= h(0,v_3,0) = h(0,0,w_3) = 0.
\end{align*}
\]  

(A.2)

[A3] (Logistic-growth nonlinearity) For \(u, v, w > 0\), as \(u, v, w\) are sufficiently large

\[
\begin{align*}
  f(u,v,w), g(u,v,w), h(u,v,w) &< 0, \\
  f(u,v,w), g(u,v,w), h(u,v,w) &> 0,
\end{align*}
\]  

(A.3)

and when \(u, v, w\) are sufficiently small.

[A4] (Covering pentahedron) the three surfaces

\[
S_f = \{(u,v,w) \mid f(u,v,w) = 0, \ u,v,w \geq 0\},
\]  

(A.4)

\[
S_g = \{(u,v,w) \mid g(u,v,w) = 0, \ u,v,w \geq 0\},
\]  

(A.5)

\[
S_h = \{(u,v,w) \mid h(u,v,w) = 0, \ u,v,w \geq 0\},
\]  

(A.6)

lie completely within the pentahedron \((a polyhedron with five faces) PH, which is enclosed by the uv-plane, the uw-plane, and the vw-plane as well as the planes

\[
\mathcal{P} = \{(u,v,w) \mid \frac{u}{u} + \frac{v}{v} + \frac{w}{w} = 1, \ u,v,w \geq 0\}
\]  

(A.7)

and

\[
\bar{\mathcal{P}} = \{(u,v,w) \mid \frac{u}{\bar{u}} + \frac{v}{\bar{v}} + \frac{w}{\bar{w}} = 1, \ u,v,w \geq 0\}
\]  

(A.8)

for some \(\bar{u} > u > 0, \bar{v} > v > 0, \bar{w} > w > 0.\)
We can prove in a similar manner that an N-barrier maximum principle remains true for systems of three equations.

**Theorem 2.1 (N-Barrier Maximum Principle for Systems of Three Equations).** Assume \([A1] \sim [A4]\) are satisfied. Suppose that there exists a pair of positive solutions \((u(x), v(x), w(x))\) of the boundary value problem

\[
\begin{align*}
&d_1 \left( u_{xx} + \theta u_x + u f(u, v, w) \right) = 0, \quad x \in \mathbb{R}, \\
&d_2 \left( v_{xx} + \theta v_x + v g(u, v, w) \right) = 0, \quad x \in \mathbb{R}, \\
&d_3 \left( w_{xx} + \theta w_x + w h(u, v, w) \right) = 0, \quad x \in \mathbb{R}, \\
&(u, v, w)(-\infty) = e_-, \quad (u, v, w)(+\infty) = e_+,
\end{align*}
\]

(2.12)

where \(e_-, e_+ = (0, 0, 0), \; (u_1, 0, 0), \; (0, v_2, 0), \; (0, 0, w_3), \; (u^*, v^*, w^*)\). For any \(\alpha, \beta, \gamma > 0\), let \(p(x) = \alpha u(x) + \beta v(x) + \gamma w(x)\). We have

\[
\min \left( \alpha u, \beta v, \gamma w \right) \frac{\min(d_1, d_2, d_3)}{\max(d_1, d_2, d_3)} \chi \leq p(x) \leq \max \left( \alpha \tilde{u}, \beta \tilde{v}, \gamma \tilde{w} \right) \frac{\max(d_1, d_2, d_3)}{\min(d_1, d_2, d_3)},
\]

(2.13)

for \(x \in \mathbb{R}\), where

\[
\chi = \begin{cases} 
1, & \text{if } e_\pm \neq (0, 0, 0), \\
0, & \text{if } e_\pm = (0, 0, 0).
\end{cases}
\]

(2.14)

**References**

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