EMBEDDINGS OF Ree Unitals in a Projective Plane Over a Field

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Abstract. We show that the Ree unital \( R(q) \) has an embedding in a projective plane over a field \( F \) if and only if \( q = 3 \) and \( F^3 \) is a subfield of \( F \). In this case, the embedding is unique up to projective linear transformations. Beside elementary calculations, our proof uses the classification of the maximal subgroups of the simple Ree groups.

1. Introduction

A \( t-(n,k,\lambda) \) design, or equivalently a Steiner system \( S_\lambda(t,k,n) \), is a finite simple incidence structure consisting of \( n \) points and a number of blocks, such that every block is incident with \( k \) points and every \( t \)-subset of points is incident with exactly \( \lambda \) blocks. Let \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, I) \) be a design and \( \mathcal{\Pi} = (\mathcal{P}', \mathcal{B}', I') \) a projective plane. The map \( \varrho : \mathcal{P} \cup \mathcal{B} \to \mathcal{P}' \cup \mathcal{B}' \) is an embedding of \( \mathcal{U} \), provided it is injective, \( \varrho(\mathcal{P}) \subseteq \mathcal{P}' \), \( \varrho(\mathcal{B}) \subseteq \mathcal{B}' \), and
\[
\forall P \in \mathcal{P}, \forall B \in \mathcal{B} : \quad PIB \iff \varrho(P)I'\varrho(B).
\]
The embedding \( \varrho \) is admissible, if for any automorphism \( \alpha \) of \( \mathcal{U} \), there is a collineation \( \beta \) of \( \mathcal{\Pi} \) such that \( \varrho(\mathcal{P}^\alpha) = \varrho(\mathcal{P})^\beta \) holds for all \( P \in \mathcal{P} \).

An abstract unital or a unital design of order \( n \) is a \( 2-(n^3 + 1, n + 1, 1) \) design. The problem of the embeddings of abstract unitals in projective planes is a classical one with many old unsolved questions, see [3, 4, 5, 10, 11, 12, 13, 15]. The classical hermitian unital \( H(q) \) of order \( q \) is constructed from the set of absolute points and non-absolute lines of the desarguesian plane \( PG(2,q^2) \). The abstract hermitian unital \( H(q) \) has a natural embedding in \( PG(2,q^3) \), which is unique up to projective equivalence, see [11, 14]. Moreover, this embedding is admissible.

Another class of abstract unitals of order \( q = 3^{2n+1} \) was discovered by Lüneburg [13]. Let \( \text{Ree}(q) = 2G_2(q) \) be the Ree group of order \( (q^3 + 1)q^3(q - 1) \), \( q = 3^{2n+1} \), see [1, 9]. Then \( \text{Ree}(q) \) has a 2-transitive action on \( q^3 + 1 \) points, namely by conjugation on the set of all Sylow 3-subgroups. The pointwise stabilizer of two points \( P, Q \) is cyclic of order \( q - 1 \) and thus contains a unique involution \( t \). It follows that \( \text{Ree}(q) \) has a unique conjugacy class of involutions, and any involution \( t \) fixes exactly \( q + 1 \) points. The blocks of the Ree unital \( \mathcal{R}(q) \) are the sets of fixed points of the involutions of \( \text{Ree}(q) \). \( \mathcal{R}(q) \) admits the \( \text{Ree}(q) \) as a 2-transitive automorphism group; the full automorphism group is larger, for \( n \geq 1 \), admitting also the field automorphism, see [3]. The smallest Ree unital \( \mathcal{R}(3) \) and the smallest Ree group \( \text{Ree}(3) \cong \text{PFL}(2,8) \cong \text{PSL}(2,8) \rtimes C_3 \) are a little different from the general case, see [2, 6]. For \( q > 3 \), \( \text{Ree}(q) \) is simple.

2010 Mathematics Subject Classification. 51E20, 05B05.

Key words and phrases. Ree unital, Ree group, projective embedding, admissible embedding.

Support provided from the National Research, Development and Innovation Fund of Hungary, NKFIH-OTKA Grants 119687, 115288 and SNN 132625.
Lüneburg showed that the Ree unital $\mathcal{R}(q)$ has no admissible embeddings in projective planes of order $q^2$ (desarguesian or not). For $q = 3$, Grüning proved that the smallest a Ree unital $\mathcal{R}(3)$ has no admissible embedding in any projective plane of order 9. Montinaro extended these results by showing that for $q \neq 3$ and $n \leq q^4$, $\mathcal{R}(q)$ has no admissible embedding in a projective plane of order $n$. Moreover, if $\mathcal{R}(3)$ is embedded in a projective plane $\Pi$ of order $n \leq 3^4$ in an admissible way, then either $\Pi \cong \text{PG}(2, 8)$, or $n = 2^6$, see [15, Theorem 5].

In this paper, we completely characterize the embeddings of $\mathcal{R}(3)$ in a projective plane over a field, extending Montinaro’s result.

**Theorem 1.** Let $F$ be a field and $\varphi : \mathcal{R}(3) \rightarrow \text{PG}(2, F)$ an embedding. Then the following hold:

(i) $F_8$ is a subfield of $F$, and the image of $\varphi$ is contained in a subplane of order 8.

(ii) The embedding is unique up to $\text{Aut}(\mathcal{R}(3))$ and $\text{PGL}(3, F)$.

(iii) The embedding is admissible.

Our main result is the following:

**Theorem 2.** Let $n$ be a positive integer, and $q = 3^{2n+1}$. Suppose that $\Pi$ is a projective plane such that for any embedding $\varphi : \mathcal{R}(3) \rightarrow \Pi$, the image $\varphi(\mathcal{R}(3))$ is contained in a pappian subplane. Then the Ree unital $\mathcal{R}(q)$ has no embedding in $\Pi$. In particular, $\mathcal{R}(q)$ has no embedding in a projective plane over a field.

These results suggest that the problem of projective embeddings of the Ree unitals can be reduced to the question whether the smallest Ree unital has an embedding in a non-desarguesian projective plane. This question is surprisingly hard, even if we assume that the embedding is admissible.

The structure of the paper is as follows. In section 2, we present the embedding of $\mathcal{R}(3)$ in $\text{PG}(2, 8)$, and some technical lemmas to ease the calculations in $\text{PG}(2, 8)$. In section 3, we study sets of five points of $\text{PG}(2, 8)$, determining ten external lines of a conic. Such external pentagons correspond to super O’Nan configurations of $\mathcal{R}(3)$; their properties are listed in section 4. In sections 5 and 6, we prove Theorems 1 and 2.

2. Preliminaries

The embedding of $\mathcal{R}(3)$ in $\text{PG}(2, 8)$ deserves special attention. The construction was first given by Grüning, who attributes the idea to F.C. Piper. The embedding is slightly simpler to present in the dual setting. Let $K$ be a non-singular conic in $\text{PG}(2, 8)$, and some technical lemmas to ease the calculations in $\text{PG}(2, 8)$. In section 3, we study sets of five points of $\text{PG}(2, 8)$, determining ten external lines of a conic. Such external pentagons correspond to super O’Nan configurations of $\mathcal{R}(3)$; their properties are listed in section 4. In sections 5 and 6, we prove Theorems 1 and 2.

2. Preliminaries

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The tangents of $K$ have a common point $N$, which is called the nucleus of $K$ (see [8]). The set $\mathcal{O} = K \cup \{N\}$ is hyperoval, that is, a set of 10 points such that each line intersects it in 0 or 2 points. The point $P$ is external, if $P \notin \mathcal{O}$. The line $\ell$ is external, if $\ell \cap \mathcal{O} = \emptyset$. There are 63 external points, 28 external lines, and each external point is incident with 4 external lines. In other words, the external points and the external lines form a (dual) unital $\mathcal{U}$ of order 3. Let $G = \text{PFO}(3, 8)$ be the group of projective semilinear transformations of $\text{PG}(2, 8)$, preserving $\mathcal{O}$. $G$ is isomorphic to $\text{PFL}(2, 8)$ and acts 2-transitively on the set of external lines.

Hence, $\mathcal{U}$ has a 2-transitive automorphism group and $\mathcal{R}(3) \cong \mathcal{U}$ by [2]. We call the
isomorphism \( \varphi : \mathcal{R}(3) \to \mathcal{U} \) a dual embedding of \( \mathcal{R}(3) \) into \( \text{PG}(2, 8) \) with respect to the conic \( \mathcal{K} \).

To make the computation in \( \mathbb{F}_8 \) more transparent, we fix a root \( \gamma \in \mathbb{F}_8 \) of the polynomial \( X^3 + X + 1 = 0 \) in \( \mathbb{F}_8 \). Then
\[
\gamma^4 = \gamma^2 + \gamma, \quad \gamma^5 = \gamma^2 + \gamma + 1, \quad \gamma^6 = \gamma^2 + 1, \quad \gamma^7 = 1.
\]
The trace map of \( \mathbb{F}_8 \) over \( \mathbb{F}_2 \) is defined as
\[
\text{Tr}(x) = x + x^2 + x^4.
\]
We fix the coordinate frame \((X, Y, Z)\) in \( \text{PG}(2, 8) \) and extend the action of the Frobenius automorphism \( \Phi : x \mapsto x^2 \) to the points and lines of \( \text{PG}(2, 8) \). In this way, we obtain a projective semilinear transformation of order 3. For \( c \in \mathbb{F}_8 \), the map
\[
\tau_c : (x, y, z) \mapsto (x + cz, y + c^2 z, z)
\]
is an elation with axis \( Z = 0 \).

**Lemma 3.** (i) If \( c \neq 0 \) then the line \( Y = mX + bZ \) is \( \tau_c \)-invariant if and only if \( m = c \).

(ii) \( \Phi \) and \( \tau_c \) (\( c \in \mathbb{F}_8 \)) preserve the conic \( \mathcal{K} : X^2 + YZ = 0 \) of \( \text{PG}(2, 8) \) and the line \( \ell_\infty : Z = 0 \) at infinity.

(iii) The line \( Y = mX + bZ \) is external to \( \mathcal{K} \) if and only if \( \text{Tr}(b/m^2) = 1 \).

(iv) Let \( \Gamma \) denote the group
\[
\Gamma = \{ \tau_c | \text{Tr}(c) = 0 \}
\]
of elations. \( \Gamma \) is elementary abelian of order 4. By conjugation, \( \Phi \) permutes the nontrivial elements of \( \Gamma \).

(v) The group \( G_0 = \langle \Gamma, \Phi \rangle \) of semilinear transformations is isomorphic to \( A_4 \).

**Proof.** \( \tau_c \) maps \( Y = mX + bZ \) to \( Y = mX + (b + c^2 + cm)Z \). This implies (i). (ii) is trivial. (iii) follows from the fact that in a finite field \( \mathbb{F}_q \) of even order, the quadratic form \( X^2 + mX + b \) is reducible if and only if \( \text{Tr}(b/m^2) = 0 \). (iv) and (v) are immediate. \( \square \)

Finally, we present a useful elementary result on groups acting on graphs:

**Lemma 4.** Let \( G \) be a group acting primitively on the set of vertices of the graph \( \Gamma \). Then \( \Gamma \) is either empty or connected.

**Proof.** The connected components of \( \Gamma \) are blocks of imprimitivity of \( G \). \( \square \)

### 3. External pentagons in \( \text{PG}(2, 8) \)

Let \( p \) be a prime, and \( q = p^e \) be a prime power. Let \( \mathcal{K} \) be a non-singular conic in \( \text{PG}(2, q) \). For any line \( \ell \), we have \(|\mathcal{K} \cap \ell| \leq 2\). We call \( \ell \) secant, tangent or external to \( \mathcal{K} \), according if \(|\mathcal{K} \cap \ell| = 2, 1, \) or 0. If \( q \) is even, then all tangents pass through a common point, the nucleus of \( \mathcal{K} \). (See [8].) The group of collineations preserving \( \mathcal{K} \) is \( \text{PGO}(3, q) \). One has the isomorphisms \( \text{PGO}(3, 4) \cong \text{PGL}(2, 4) \) and \( \text{PGO}(3, 2) \cong \text{PGL}(2, 2) \).

**Definition 5.** Let \( \mathcal{K} \) be a conic in \( \text{PG}(2, q) \). We say that the points \( P_0, \ldots, P_4 \) in general position form an external pentagon with respect to \( \mathcal{K} \), if \( P_iP_j \) are external lines of \( \mathcal{K} \), \( 0 \leq i < j \leq 4 \). The external pentagon is said to have type \( A_4 \), if there is a group \( G_0 \) of collineations preserving \( \mathcal{K} \) and \( \{P_0, \ldots, P_4\} \), such that \( G_0 \cong A_4 \).
Notice that the points of an external pentagon are not in $K$, and if $q$ is even, then they are also distinct from the nucleus of $K$.

**Lemma 6.** Let $\mathcal{P} = \{P_0, \ldots, P_4\}$ be a external pentagon of type $A_4$ in $\text{PG}(2, q)$ with respect to the conic $K$. Then $q$ is even and the following hold:

(i) $G_0$ fixes one point of $\mathcal{P}$ and acts 2-transitively on the remaining four.

(ii) If $q = 8$, then $G_0 \cong A_4$ is the full stabilizer of the sets $K$ and $\mathcal{P}$ in the collineation group of $\text{PG}(2, 8)$.

**Proof.** (i) The Sylow 2-subgroup $T = G_0'$ of $G_0$ is normal in $G_0$, and $T \leq \text{PGO}(3, q)$. Hence, $T$ acts faithfully on $\mathcal{P}$, with a unique fixed point, say, $P_0$. As $T$ is normal in $G_0$, $G_0$ fixes $P_0$. Moreover, $T$ acts regularly and $G_0$ acts 2-transitively on $\{P_1, \ldots, P_4\}$. Let $H$ be the stabilizer of $P_0$ in $\text{PGO}(3, q)$. If $q$ is odd then $H \cong D_{2(q^2+1)} \rtimes C_q$. The dihedral subgroup $D_{2(q^2+1)}$ contains a unique central involution $\alpha$, which is contained in each elementary abelian subgroup of order 4. Hence, no subgroup of $H$ of order 12 can be isomorphic to $A_4$, a contradiction.

(ii) Assume $q = 8$. Then $H \cong F_8^+ \rtimes C_3$ has order 24 and we have to show that $H$ does not leave $\mathcal{P}$ invariant. Let $t_0$ be the tangent line through $P_0$ to $K$. The elementary abelian part $A$ of $H$ consists of elations with respect to $t_0$, that is, for any 2-element $\alpha \in A$, the set of fixed point of $\alpha$ is $t_0$. As for $i \geq 1$, $P_i \not\in t_0$, and the $A$-orbit of $P_i$ has length 8. \qed

**Definition 7.** The fundamental pentagon $\mathcal{F}$ of $\text{PG}(2, 8)$ is the set of points $A(1, 1, 0)$, $C_1(0, 1, 1)$ and

$$C_2 = (\gamma, \gamma^6, 1), \quad C_3 = (\gamma^2, \gamma^5, 1), \quad C_4 = (\gamma^4, \gamma^3, 1).$$

**Lemma 8.** The fundamental pentagon $\mathcal{F}$ is an external pentagon with respect to the conic $K : X^2 + YZ = 0$. Moreover, $\mathcal{F}$ is of $A_4$ type with collineation group $G_0 = \langle \Gamma, \Phi \rangle$.

**Proof.** The following facts can be checked by calculations: $A$ is fixed by $G_0$. $\Phi$ fixes $C_1$ and permutes $C_1, C_2, C_3$. The $\Gamma$-orbit of $C_1$ is $\{C_1, \ldots, C_4\}$. The lines $AC_1 : Y = X + Z$ and $C_1C_2 : Y = \gamma X + Z$ are external. \qed

**Lemma 9.** Let $K$ be a conic in $\text{PG}(2, 8)$ and $\mathcal{P}$ an external pentagon of type $A_4$ with respect to $K$. The projective coordinate frame can be chosen such that $K$ has equation $X^2 + YZ = 0$ and $\mathcal{P}$ is the fundamental pentagon.

**Proof.** We can assume the equation $K : X^2 + YZ = 0$ and that $A(1, 1, 0)$ is the point of $\mathcal{P}$ which is fixed by $G_0$. Then $G_0$ is a subgroup of

$$H = \{\tau_c \mid c \in F_8\} \rtimes \langle \Phi \rangle,$$

which is the stabilizer of $K$ and $A$. The 2-subgroup $\{\tau_c \mid c \in F_8\}$ has two $\Phi$-invariant, irreducible proper subgroups: $Z(H) = \langle \tau_1 \rangle$ and $\Gamma$. Hence, $\langle \Gamma, \Phi \rangle$ is the unique subgroup $H$ which is isomorphic to $A_4$. $G_0 = \langle \Gamma, \Phi \rangle$ follows. Let $C_1$ denote the point of $\mathcal{P} \setminus \{A\}$ that is fixed by $\Phi$. As $AC_1$ is an external line, $C_1$ must have coordinates $(x, x + b, 1)$ with $x, b \in F_2$ and $\text{Tr}(b) = 1$. This means that either $C_1 = (0, 1, 1)$ or $C_1 = (1, 0, 1)$. Applying $\tau_1$ to $\mathcal{P}$, we can assume $C_1 = (0, 1, 1)$. Straightforward computation shows that $\{C_1, \ldots, C_4\}$ is the $\Gamma$-orbit of $C_1$, and $\{A, C_1, \ldots, C_4\}$ is indeed the fundamental pentagon. \qed

**Remark 10.** Lemma 8(ii) and Lemma 9 imply that with fixed conic $K$ of $\text{PG}(2, 8)$, the number of $A_4$-type external pentagons is $|\text{PGO}(3, 8) : G_0| = 126$. 

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For the rest of this section, we use the notation of Lemma 3 for $\mathcal{K}$, $\Gamma$, $G_0$, and $\tau_c$.

**Proposition 11.** Let $\mathcal{F} = \{A, C_1, \ldots, C_4\}$ be the fundamental pentagon in $\text{PG}(2, 8)$. For any even permutation $ijk \ell$ of $\{1, 2, 3, 4\}$, let $d_{ijk \ell}$ denote the line connecting the points $AC_i \cap C_j C_\ell$ and $AC_j \cap C_i C_\ell$. The following hold:

(i) $G_0$ permutes the lines $d_{ijk \ell}$ regularly. In particular, the lines $d_{ijk \ell}$ are distinct.

(ii) The lines $d_{ijk \ell}$ are external to $\mathcal{K}$.

(iii) For any coset $\Gamma g$ of $\Gamma$, the four lines $d_\pi$, $\pi \in \Gamma g$, share a common point at infinity $Z = 0$.

(iv) The lines $AC_4$, $d_{1234}$ and $d_{3241}$ are concurrent.

**Proof.** Obviously, $G_0$ acts on the lines $d_{ijk \ell}$. We have

$$AC_1 \cap C_2 C_4 = (\gamma^6, \gamma^2, 1), \quad AC_2 \cap C_3 C_4 = (\gamma, \gamma^4, 1),$$

with connecting line $d_{1234} : Y = \gamma^6 X + \gamma^3 Z$. The intersection $d_{1234} \cap \ell_\infty = (1, \gamma^6, 0)$ is not fixed by any element of order $3$ of $G$. Thus, the stabilizer of $d_{1234}$ in $G$ is contained in $\Gamma$. Since $\text{Tr}(\gamma^6) = 1$, the stabilizer of $d_{1234}$ in $\Gamma$ is trivial by Lemma 3(i).

This proves (i), and also (ii), since $d_{1234}$ is an external line. (iii) follows from the fact that $\Gamma$ fixes the points at infinity. Computing the equations and the determinant

$$\det \begin{bmatrix} 1 & 1 & \gamma^6 \\ \gamma^6 & 1 & \gamma^3 \\ \gamma^3 & 1 & \gamma^4 \end{bmatrix} = 0,$$

we obtain (iv). \qed

4. **Super O’Nan configurations in $\mathcal{R}(3)$**

In a 2-design, an O’Nan (or Pasch) configuration consists of four pairwise intersection blocks, no three of which pass through the same point. Brouwer [2] observed that in $\mathcal{R}(3)$, each O’Nan configuration is contained in a super O’Nan configuration, that is, in a set of five pairwise intersecting blocks in general position. In this section, we collect some facts on super O’Nan configurations of $\mathcal{R}(3)$.

**Lemma 12 ([2]).** The number of super O’Nan configurations in $\mathcal{R}(3)$ is $126$. \qed

For the rest of this section, we fix a dual embedding $\varphi^*$ of $\mathcal{R}(3)$ in $\text{PG}(2, 8)$ with respect to the conic $\mathcal{K} : X^2 + YZ = 0$. Notice that the blocks $a, b$ of $\mathcal{R}(3)$ intersect if and only if the points $\varphi^*(a), \varphi^*(b)$ determine an external line of $\mathcal{K}$. This implies that $\{b_0, \ldots, b_4\}$ is a super O’Nan configuration of $\mathcal{R}(3)$ if and only if $\{\varphi^*(b_0), \ldots, \varphi^*(b_4)\}$ is an external pentagon with respect to $\mathcal{K}$.

**Lemma 13.** Ree(3) acts transitively on the set of super O’Nan configurations of $\mathcal{R}(3)$. The stabilizer of a super O’Nan configuration is isomorphic to $A_4$. It fixes one of the blocks $b_i$ and acts 2-transitively on the remaining four.

**Proof.** Remark 10 and Lemma 12 imply that all external pentagons of $\text{PG}(2, 8)$ are of $A_4$ type. Our claim follows from Lemma 9. \qed

**Proposition 14.** Let $\mathcal{B}$ be a super O’Nan configuration of $\mathcal{R}(3)$ with stabilizer subgroup $S \cong A_4$. We can label the blocks of $\mathcal{B}$ by $a, c_1, \ldots, c_4$ such that for any even permutation $ijk \ell$, the blocks $(a \cap c_i)(c_j \cap c_\ell)$ and $(a \cap c_j)(c_k \cap c_\ell)$ have a unique intersection $D_{ijk \ell}$. Moreover, the following hold:
(i) The points \(D_{ijk}\) are distinct.

(ii) Let \(T\) be the Sylow 2-subgroup of \(S\). For any coset \(Tg\) of \(S\), the four points \(D_x, \pi \in Tg\), form a block.

(iii) The points \(a \cap c_4, D_{1234}\) and \(D_{3241}\) are contained in a block.

Proof. We use Lemma 13 and Proposition 11 and the dual embedding \(\varphi^*\) of \(\mathcal{R}(3)\) in \(\text{PG}(2, 8)\). □

5. Embeddings of \(\mathcal{R}(3)\)

Proof of Theorem 5. Let \(\{a, c_1, \ldots, c_4\}\) be a super O’Nan configuration of \(\mathcal{R}(3)\), as given in Proposition 14. Let us choose the projective coordinate frame of \(\text{PG}(2, F)\) such that \(\varphi(c_1) : X + Y + Z = 0, \varphi(c_2) : X = 0, \varphi(c_3) : Y = 0, \varphi(c_4) : Z = 0\). There are elements \(u, v \in F \setminus \{0, 1\}, u \neq v\), such that \(\varphi(a) : uX + vY + Z = 0\). We have

\[
\varphi(D_{1234}) = (v^2 - v, v - u, v^2 - uv), \quad \varphi(D_{2143}) = (u - uv, u - 1, v - u),
\]

\[
\varphi(D_{3412}) = (v, uv - u, -uv), \quad \varphi(D_{4321}) = (v - u, u^2 - u, u - uv).
\]

By Proposition 14(ii), these are collinear points, thus,

\[
\det \begin{bmatrix}
v^2 - v & v - u & v^2 - uv \\
u - uv & u - 1 & v - u \\
v & uv - u & -uv
\end{bmatrix} = u(u - 1)v(v - 1)(v^2 - uv + 2u - 3v) = 0,
\]

and

\[
\det \begin{bmatrix}
v^2 - v & v - u & v^2 - uv \\
u - uv & u - 1 & v - u \\
v - u & u^2 - u & u - uv
\end{bmatrix} = u(u - 1)v(v - 1)(uv - u^2 + 2u - 3v + 1) = 0.
\]

The difference of these two determinants is

\[
u(u - 1)v(v - 1)((u - v)^2 - 1) = 0,
\]

which implies \(u = v \pm 1\). Substituting back, we obtain either \(2v - 2 = 0\) or \(2 = 0\), which are not possible unless \(\text{char}(F) = 2\). In this case, all equations so far reduce to \(u + v + 1 = 0\). Computing \(\varphi(a \cap c_4) = (v, v + 1, 0)\) and

\[
\varphi(D_{3241}) = (v + 1, 1, v^2),
\]

we have

\[
\det \begin{bmatrix}
v & v + 1 & 0 \\
v^2 + v & 1 & v \\
v + 1 & 1 & v^2
\end{bmatrix} = v(v + 1)(v^3 + v^2 + 1) = 0.
\]

This shows that \(u, v \in \mathbb{F}_8\), and for any even permutation \(ijkl\), \(\varphi(D_{ijkl})\) is contained in the subplane \(\text{PG}(2, 8)\). Hence, at least 22 points of \(\varphi(\mathcal{R}(3))\) is contained in \(\text{PG}(2, 8)\). If \(Q\) is one of the remaining 6 points, then there are at least two blocks through \(Q\) with equation over \(\mathbb{F}_8\), and therefore \(Q\) is in \(\text{PG}(2, 8)\) as well. The computation shows that up to the action of the Frobenius map \(\Phi\), the embedding \(\varphi\) is uniquely determined by the images of the blocks \(c_1, \ldots, c_4\). In particular, \(\varphi\) must be an embedding with respect to a dual conic \(\mathcal{K}^*\). As all dual conics of subplanes of order 8 are projectively equivalent, we obtain (ii). Montinaro’s 13, Theorem 5 implies (iii). □
Corollary 15. Let $F$ be a field and $\varphi : \mathcal{R}(3) \to \text{PG}(2, F)$ an embedding. Let $S = \{1, a_1, \ldots, a_7\}$ be a Sylow 2-subgroup of $\text{Ree}(3)$. Then the lines $\varphi(a_1), \ldots, \varphi(a_7)$ are concurrent.

Proof. Consider the dual embedding $\varphi^* : \mathcal{R}(3) \to \text{PG}(2, F)$. By Theorem 11, $\varphi^*(\mathcal{R}(3))$ is contained in a subplane $\text{PG}(2, 8)$. As before, we identify the blocks of $\mathcal{R}(3)$ and the involutions of $\text{Ree}(3)$. For a block $a$, $\varphi^*(a)$ is an external point of the conic $\mathcal{K}$. Moreover, $a$ determines a unique collineation $\hat{a}$ of order 2. The following are equivalent:

1. Two involutions $a_1, a_2 \in \text{Ree}(3)$ commute.
2. The involutions $\hat{a}_1, \hat{a}_2 \in \text{PGL}(3, F)$ commute.
3. The line $\varphi^*(a_1)\varphi^*(a_2)$ is tangent to $\mathcal{K}$.
4. $\hat{a}_1, \hat{a}_2$ fix the same point of $\mathcal{K}$.

This implies that $\varphi^*(a_1), \ldots, \varphi^*(a_7)$ are contained in a tangent of $\mathcal{K}$, which is our claim in the dual setting. \hfill \Box

6. THE NONEXISTENCE OF EMBEDDINGS OF $\mathcal{R}(q)$, $q \geq 27$

In this section, we write $q = 3^{2n+1}$, and $G = \text{Ree}(q)$. We have $2n + 1 = |\text{Out}(\text{Ree}(q))|$ and for any divisor $\alpha$ of $2n + 1$ there is an outer automorphism $\psi_\alpha$ of $\text{Ree}(q)$ of order $\alpha$. (See [9, Lemma 6.2].) Write $q_0 = q^{1/\alpha} = 3^{2n_0+1}$ and $G_0 = C_{\text{Ree}(q)}(\psi_\alpha)$. We have $G_0 \cong \text{Ree}(q_0)$.

In order to be self-contained, we resume Kleidman’s classification [9, Theorem C] of the maximal subgroups of $G$, see also [11]. If $q \geq 27$ and $H$ is a maximal subgroup of $G$, then one of the following cases occur:

(M1) $H$ is a 1-point stabilizer, isomorphic to the semidirect product of a group of order $q^3$ with the cyclic group of order $q - 1$.
(M2) $H \cong \text{Ree}(q_0)$, where $q_0 = q^{1/\alpha}$, $\alpha$ prime.
(M3) $H \cong C_2 \times \text{PSL}(2, q)$ is the centralizer of an involution.
(M4) $H \cong (C_2^3 \times D_{(q+1)/2}) \rtimes C_3$ is the normalizer of a subgroup of order 4.
(M5) $H \cong C_{q^2 - q + 1} \times C_6$.
(M6) $H \cong C_{q^2 - q + 1} \rtimes C_6$.

If $q = 3$ then (M2) and (M6) does not occur. Moreover, $H \cong \text{PSL}(2, 8)$, or $H \cong (C_2^3 \times C_7) \rtimes C_3$ is the normalizer of a Sylow 2-subgroup, that contains the subgroups (M3) and (M4).

In $\text{Ree}(q)$, the stabilizer of two points is cyclic of order $q - 1$. Hence, the intersection of two Sylow 3-subgroups is trivial. This implies that any Sylow 3-subgroup $S_0$ of $G_0$ is contained in a unique Sylow 3-subgroup $S$ of $G$, and $S$ is left invariant by $\psi_\alpha$. Conversely, let $S$ be a $\psi_\alpha$-invariant Sylow 3-subgroup of $G$. The normalizer $N_G(S)$ is a parabolic subgroup of $G$, isomorphic to the semidirect product of a group of order $q^3$ with the cyclic group of order $q - 1$. The centralizer of the field automorphism in $N_G(S)$ has order $q_0^3(q_0 - 1)$. This shows that $S_0 = S \cap G_0$ is a Sylow 3-subgroup in $G_0$.

Proposition 16. Let $q_0 = 3^{2n_0+1}$ such that $F_q$ is an extension of $F_{q_0}$. Then $\mathcal{R}(q)$ has a subdesign $D \cong \mathcal{R}(q_0)$. Moreover, the stabilizer of $D$ in $\text{Ree}(q)$ is isomorphic to $\text{Ree}(q_0)$. In particular, $\mathcal{R}(3)$ is a subdesign of $\mathcal{R}(q)$ with stabilizer subgroup $\text{Ree}(3)$.

Proof. Remember that the points and blocks of $\mathcal{R}(q)$ can be identified with the Sylow 3-subgroups, and the involutions of $\text{Ree}(q)$, respectively. Hence, any automorphism
of \( G = \text{Ree}(q) \) induces an automorphism of \( \mathcal{R}(q) \). The involutions fixed by \( \psi_a \) and the Sylow 3-subgroups left invariant by \( \psi_a \) form a subdesign \( \mathcal{D} \) of \( \mathcal{R}(q) \). As explained above, \( \psi_a \)-invariant involutions and Sylow 3-subgroups of \( G \) correspond to involutions and Sylow 3-subgroups of \( G_0 \). Hence, \( \mathcal{D} \cong \mathcal{R}(q_0) \). Let \( T_0 \) be the stabilizer of \( \mathcal{D} \) in \( G \); clearly \( G_0 \leq T_0 \). Looking at the list of maximal subgroups of \( G \) in [9, Theorem C], we see that either \( T_0 = \text{Ree}(q) \), or \( T_0 \) is contained in a subgroup isomorphic to \( \text{Ree}(q_1) \) with \( q_1 = q^{1/\beta} \), \( \beta \) prime. Repeating this argument, we conclude that \( T_0 \) itself is isomorphic to a Ree group \( \text{Ree}(q_\ast) \), where \( \mathbb{F}_{q_\ast} \) is a subfield of \( \mathbb{F}_q \). As \( T_0 \) preserves the set of involutions of \( G_0 \), the only possibility is \( q_0 = q_\ast \).

We are now in the position to prove Theorem 2.

**Proof of Theorem 3**. Let us suppose that an embedding \( \varphi : \mathcal{R}(q) \to \Pi \) exists. Let \( I \) denote the set of involutions of \( \text{Ree}(q) \). In three steps, we show that all lines \( \varphi(a) \) (\( a \in I \)) are concurrent, a contradiction.

Claim 1: For any Sylow 2-subgroup \( S = \{1, a_1, \ldots, a_7\} \) of \( \text{Ree}(q) \), the lines \( \varphi(a_1), \ldots, \varphi(a_7) \) are concurrent.

By Proposition [13], \( \mathcal{R}(q) \) has a subdesign \( \mathcal{D} \cong \mathcal{R}(3) \), whose stabilizer is the subgroup \( \text{Ree}(3) \) of \( \text{Ree}(q) \). Both \( \text{Ree}(q) \) and \( \text{Ree}(3) \) have an elementary abelian Sylow 2-subgroup of order 8. Hence, we can assume w.l.o.g. that \( S \leq \text{Ree}(3) \). By the assumption, \( \varphi(\mathcal{D}) \) is contained in a pappian subplane \( \Pi_0 \). We apply Corollary [15] to the restriction \( \varphi : \mathcal{D} \to \Pi_0 \) to prove the claim.

Claim 2: For any \( a \in I \), there is a point \( P_a \in \varphi(a) \) of \( \text{PG}(2, F) \) such that \( P_a \in \varphi(b) \) for all \( b \in I \) with \( ab = ba \).

Fix \( a \in I \). The centralizer \( C_G(a) \) is \( \langle a \rangle \times T \), with \( T \cong \text{PSL}(2, q) \). \( T \) has a unique class \( J \) of involutions. For arbitrary \( b, c \in J \), \( \langle a, b, c \rangle \) is contained in a Sylow 2-subgroup of \( G \). By claim 1, the lines \( \varphi(b), \varphi(c) \) intersect on \( \varphi(a) \). Fix \( b \) and define \( P_a = \varphi(a) \cap \varphi(b) \). As \( T \) acts primitively on \( J \), Lemma [4] implies that for any \( c \in J \), there are elements \( b_0 = b, b_1, \ldots, b_k = c \in J \) such that \( b_i b_{i+1} = b_{i+1} b_i \) for all \( i = 0, \ldots, k - 1 \). For all indices \( i, \varphi(a) \cap \varphi(b_i) = \varphi(a) \cap \varphi(b_{i+1}) \). Hence, \( \varphi(a), \varphi(b) \) and \( \varphi(c) \) are concurrent. If \( c \) is an involution of \( C_G(a) \), not in \( J \cup \{a\} \), then \( ac \in J \) and \( \varphi(a), \varphi(b) \) and \( \varphi(ac) \) are concurrent. Also, the lines \( \varphi(a), \varphi(c) \) and \( \varphi(ac) \) are concurrent, that shows \( P_a \in \varphi(c) \).

Claim 3: All lines \( \varphi(a) \) (\( a \in I \)) of the embedding are concurrent.

If \( a, b \in I \) commute then \( P_a = P_b \). Fix arbitrary elements \( a, b \in I \). By [9, Theorem C], \( G \) acts primitively on \( I \). Hence by Lemma [4] there are elements \( a_0 = a, a_1, \ldots, a_k = b \) such that \( a_i a_{i+1} = a_{i+1} a_i \) for all \( i = 0, \ldots, k - 1 \). Then \( P_a = P_{a_0} = \ldots = P_{a_k} = P_b \), that finishes the proof.

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