Numerical Simulations of Thermal Convection in Rapidly Rotating Spherical Shell

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ABSTRACT. We present a novel numerical model used to simulate convection in the atmospheres of the Gas Giant planets Jupiter and Saturn. Nonlinear, three-dimensional, time-dependent solutions of the anelastic hydrodynamic equations are presented for a stratified, rotating spherical fluid shell heated from below. This new model is specified in terms of a grid-point based methodology which employs a hierarchy of tessellations of the regular icosahedron onto the sphere through the process of recurrent dyadic refinements of the spherical surface.

We describe discretizations of the governing equations in which all calculations are performed in Cartesian coordinates in the local neighborhoods of the almost uniform icosahedral grid, a methodology which avoids the potential mathematical and numerical difficulties associated with the pole problem in spherical geometry. Using this methodology we have built our model in primitive equations formulation, whereas the three-dimensional vector velocity field and temperature are directly advanced in time.

We show results of thermal convection in rapidly rotating spherical shell which leads to the formation of well pronounced prograde zonal jets at the equator, results which previous experiments with two-dimensional models in the limit of freely evolving turbulence were not able to achieve.

Key words: computational fluid dynamics, multigrid methods, icosahedral mesh, convection, anelastic model, zonal jets PACS: 02.60.Cb, 02.70.Dh, 92.60.Bh, 92.60.Hk, 96.15.Hy

1. Introduction

Understanding how large-scale zonal flows are generated on the Gas Giant planets is a longstanding problem in planetary and geophysical fluid dynamics (Busse [1], Williams [2], Cho and Polvani [3], Peltier and Stuhne [4], Aurnou and Heimpel [5], Glatzmaier et al. [6]). Strong eastward equatorial jets are the dominant dynamical feature for both Jupiter and Saturn, and there is a growing body of evidence that such features are also predominant on the other two Gas Giant planets Uranus and Neptune, although the direction of the equatorial jets are in the opposite direction.

We study the emergence and evolution of large-scale zonal flows, as observed on the Gas Giant planets, using a newly developed 3D GCM in spherical shell geometry. The physical basis of the model is the anelastic approximation of the hydrodynamic equations of motion, continuity, and classical thermodynamics. We describe a representative investigation of the convective response of a layer of Boussinesq fluid in rapidly rotating spherical shell subject to isothermal temperature boundary conditions. The physical scaling is governed by the 3 non-dimensional parameters: Taylor, Prandtl and Rayleigh numbers, while the depth of the shell is a variable parameter.

The model is specified in terms of a grid-point based methodology which employs a hierarchy of tessellations derived from the successive dyadic refinements of the spherical icosahedron. One major
advantage of this multi-grid methodology is that it allows for nearly linear growth of complexity in operation count as opposed to the spectral transform models, which by their nature are at least quadratic in computational complexity. Another potential advantage is the absence of pole problems, and therefore the ability of the code to capture important features of the dynamics in the polar regions. An added bonus of this new methodology is the possibility for greater local control over the computational mesh.

The spatial operators of the model are constructed around a multitude of quasi-regular icosahedral grids at each spherical level for the surface part, and second or higher order finite differences in the radial direction. They use new, performance optimized data structures which enhances the stability of the code in long-term integrations. Another novelty is the introduction of multigrid finite element methods for the solution of a fully 3D generalized Elliptic equation that arises as a consequence of the anelastic approximation. These finite element methods are explicitly defined in terms of the icosahedral grid structure and takes into account the curvature of the spherical shell in the form of an asymptotic series expansion.

This work is direct continuation of previous work of Stuhne and Peltier (1996, 1999 [7, 8]), who reported results with this novel grid-point based methodology within the framework of one-layer barotropic and shallow water models.

2. The model and its grid structure

For the purposes of this study, our 3D numerical simulations of thermal convection in rotating spherical shell is assumed to be governed by the Boussinesq approximation of the anelastic system. The dynamical equations in this approximation can be written in the following reference independent form:

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla^2 \mathbf{u} - \left( \frac{Ra}{Pr} \right) \nabla p' + \left( \frac{Ra}{Pr} \right) \tilde{g}(r) T \hat{r} - Ta^{1/2} \left( e_z \times \mathbf{u} \right)
\]  

(1)

\[
\nabla \cdot \mathbf{u} = 0
\]  

(2)

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \left( \frac{1}{Pr} \right) \nabla^2 T
\]  

(3)

under the assumption of constant density profile. In the above system \( \mathbf{u} \) is the 3D velocity field, \( T \) is the temperature field, and \( \tilde{g}(r) \) is the radial profile of acceleration due to gravity. The principle three non-dimensional scaling parameters are: Rayleigh \( Ra \), Taylor \( Ta \) and Prandtl \( Pr \) numbers.

The hierarchy of grids obtained through the process of recursive dyadic refinement is best illustrated graphically in Figure 1 which shows the mesh structures of grid levels 0 through 5. This sub-devision of the computational domain leads to quasi-regular coverage of the surface of the sphere that has no pole problem.

The connectivity of the nearest and next-to-nearest grid points on the surface of the sphere, as well as the full three-dimensionality of the computational mesh forming the so called computational molecule is shown in Figure 2.

The spatial differential operators of the model are constructed using affine basis states on the surface of the sphere and finite differences with the possibility of variable resolution in the vertical.

Other important details concerning the anelastic model are the following:

- dynamical core consists of a dry adiabatic atmosphere in a non-hydrostatic formulation
- non-dimensionalized Cartesian form of the dynamical equations
- 4D Cartesian embedding of spherically constrained 3D velocity field
- vertical component of the Coriolis force and the centrifugal force taken into account
- instantaneous pressure adjustment of velocity field carried out by a generalized 3D elliptic solver
- freely adjustable spatial and temporal resolution
- isothermal shell boundaries with linearly varying gravity with radius
FIG. 1. Dyadic refinement of the computational mesh structure on the surface of the sphere for refinement levels \( l = 0 \) up to level \( l = 5 \) of the basic icosahedron.

FIG. 2. Nearest and next-to-nearest neighbours grid points on the surface of the sphere (a) and nearest neighbours grid points belonging to the 3D computational molecule (b).
- the initial state is a thermally perturbed hydrostatic equilibrium
- the end state is zonal geostrophic flow - invariant along the axis of rotation, with alternating prograde/retrograde surface jets.

3. Thick-thin zonal flow formation
We apply our icosahedral model to the problem of zonal jet formation in the atmospheres of the Gas Giant planets. We investigate the formation and evolution of these zonal jets, which are driven by vigorous convection and strong Coriolis force. Such a test case is of particular relevance to the outer layers of the gas giant planets where a number of open questions associated with the formation and evolution of coherent structures await solution.

For this problem, the model parameters we have employed were selected to reflect (within the technical limitations of a fully 3D model) our current understanding of the Jovian dynamics. The Rayleigh and Taylor numbers were chosen to be as high as is numerically feasible although they are many orders of magnitude smaller than those assumed for Jupiter and Saturn. Regardless of this limitation, our ultimate goal is to demonstrate that the anelastic icosahedral model can simulate successfully the formation and evolution of zonal flows in spherical shell geometry. These zonal flows are spontaneously generated out of purely random thermal initial conditions under the joint action of strong rotation and vigorous convection.

![Zonal Velocity - Surface Projection (R=0)](image)

**FIG. 3.** Surface projections of zonal velocity at the outer shell boundary for free-rigid velocity boundary condition: (a) thick shell depth 0.65%; (b) thin shell depth 0.25%.

As one example of successful numerical simulation we present results from a global flow comparison study for thick-thin shell experiment in the Boussinesq approximation. The important point here is that the anelastic model has been run at more extreme values of the control parameters ($Ra = 10^7$, $E = 10^{-5}$ or $Ta = 4 \times 10^{10}$). The intent was to test the code under strong forcing of both convection and rotation where the characteristic dynamical time scale is very short in comparison to the diffusion time scale. The model was run with free-rigid velocity boundary conditions. The effect of the shell-depth on the width of the equatorial zonal flow is represented in Figure 3. The two panels show surface projections of the mean zonal velocity field at the free outer boundary level of the shell. For the thick shell (shell depth 65%) the prograde equatorial jet is much wider in comparison to the equatorial jet for the thin shell (shell depth 25%). Consequently, the strength of the broader zonal flow is presumably diminished in comparison to that for the shell with shallower depth. This figure serves primarily an illustrative purpose since the actual shell-depth at which more realistic numerical experiments for Jupiter and Saturn should be run would be considerably smaller (shell depth ~10%). It should also be noted that this shell geometry
would be computationally much more demanding in a 3D numerical experiment with the full anelastic icosahedral formulation because of the increased complexity of the dynamical system.

4. Conclusions and future work

Numerical simulations of global convection in rapidly rotating spherical shells remain one of the outstanding challenges for computational fluid dynamics. To more truthfully simulate the global zonal flows as observed on Jupiter and Saturn, for instance, including the broad prograde equatorial jets and multiple alternating jets at higher latitudes, the new anelastic icosahedral model in spherical geometry should be run at considerably higher spatial resolution (presumably multigrid level \( l = 7, 8 \) on a spherical surface, and number of radial levels in the vertical \( R > 100 \)) to ensure that the small scale structure of the flow is adequately represented. In addition, the simulations should be performed at shell depths \( < 0.1 \) in order to capture the development and evolution of multiple jets not only in the equatorial region, but also the complex jet structure in the polar regions as recently observed by the Cassini spacecraft. The hope is that the new code will be very well suited for this because it does not have a "pole problem" and the resolution on the surface of the sphere is almost regular, including the polar regions. At the same time the principle scaling parameters, the Rayleigh number \( Ra \) and the Taylor number \( Ta \), should be boosted up in accord with the changed shell geometry. A rough estimate could be: \( Ra > 10^8 \div 10^{10} \), and \( Ta > 10^{10} \div 10^{12} \). These parameter values are still many orders of magnitude below the physical values for Jupiter and Saturn, but will allow to test the performance of the code at more extreme regimes. Even higher Rayleigh numbers will be needed to simulate the Ice Giants - Uranus and Neptune, where it is believed that strong convection dominates over the Coriolis force and vigorous angular momentum mixing forces their zonal winds in the retrograde direction. All this undoubtedly will put a premium on computer resources in terms of memory and speed and ultimately will require more powerful computer architectures.

The newest addition to this model is that the radial direction can be treated also in a fully spectral manner using Chebyshev polynomials which are the polynomials with the smallest error bounds for the radial components of the gradient and Laplacian operators. This new option allows for a hybrid gridpoint-spectral formulation of the model which can considerably enhance the flexibility of the model to operate in different regimes.

The non-tensor nature of the nearly uniform icosahedral grid in 3D (see Doyd, 2001 [9] and references therein) still poses a huge research challenge. Apart from programming complexities, the accuracy of the spatial operators is not optimal since the affine basis states on the surface of the sphere are not orthogonal to each other. Current gridpoint methods however lack explicit conservation of momentum and thermal fluxes, as well as kinetic energy conservation. In long term integrations and strongly forced thermal convection the icosahedral gridpoint methodology can suffer from non-linear aliasing instabilities, in addition to problems with insufficient spatial resolution near the boundaries. To alleviate such shortcomings we started experimenting with non-uniform Chebyshev grids in the radial direction that are able to resolve well thermal boundary layers at the edges of the spherical shell, although this approach can lead to severe constraints on explicit and semi-implicit time stepping schemes. Regardless of these principle limitations, our recent numerical experiments show that the hybrid formulation of the spatial operators of the model with spectral Chebyshev representation on stretched radial mesh is superior to the fully gridpoint formulation.

A natural evolution of the code that can dramatically improve the conservation properties of the model is the conversion of the code into fully 3D finite volume formulation. This will undoubtedly improve flux balancing of the differential operators, thereby allowing long integration runs at more extreme values of the scaling parameters that are needed in order to reach statistically equilibrated final states of the dynamical system.

Currently the model is being run very efficiently on a single node of the IBM Power6 machine at SciNet with OpenMP 3.0 multithread option. Future upgrades of the model require cross node parallelization with the MPI 3.0 library that will improve the efficiency of the code at high resolution.
that is absolutely necessary in the strong convective regime.

In conclusion, we believe that the general framework for globally convecting systems in spherical geometry based on the icosahedral methodology, is robust and capable of resolving some of the more complicated processes that arise in nonlinear geophysical and astrophysical hydrodynamics. With continuing refinement and improvement of the code, we are encouraged to believe that the methodology we have based this research on may ultimately rival the currently prevalent spectral transform methods in this field. One very significant ingredient in achieving this goal is to parallelize the code using the newest Message Passing Interface (MPI) standard that ultimately will allow the model to be run efficiently on massively parallel computer architectures.

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References:
[1] Busse, F.H., "A simple model of convection in the Jovian atmosphere", Icarus 20 (1976) 255–260.
[2] Williams, G.P., "Jupiter’s atmospheric circulation", Nature 257 (1975) 778.
[3] Cho, J.Y.K., Polvani, L.M., "The morphogenesis of bands and zonal winds in the atmospheres on the giant outer planets", Science 273 (1996) 335–337.
[4] Peltier, W.R., Stuhne, G.R., "The upscale turbulent cascade — shear layers, cyclones and Gas Giant bands", in: Pearce, R.P. (Ed.), Meteorology at the Millennium, Academic Press, New York, 2002, p. 43.
[5] Heimpel, M.H., Aurnou, J.M., "Turbulent convection in rapidly rotating spherical shells: A model for equatorial and high-latitude jets on Jupiter and Saturn", Icarus 187 (2007) 540–557.
[6] Glatzmaier, G.A., Evonuk, M., Rogers, T.M., "Differential rotation in giant planets maintained by density-stratified turbulent convection", Geophys. Astrophys. Fluid Dynamics 103 (2009) 31–51.
[7] Stuhne, G.R., Peltier, W.R., "Vortex erosion and amalgamation in a new model of large scale flow on the sphere", J. Comp. Phys. 128 (1996) 58–81.
[8] Stuhne, G.R., Peltier, W.R., "New icosahedral grid-point discretizations of the shallow water equations on the sphere", J. Comp. Phys. 148 (1999) 23–58.
[9] Boyd, J.P., "Chebyshev and Fourier Spectral Methods", Dover Publications, 2001.