qKZ equation and ground state of the $O(1)$ loop model with open boundary conditions

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Abstract

We consider the qKZ equations based on the two boundaries Temperley Lieb algebra. We construct their solution in the case $s = q^{-3/2}$ using a recursion relation. At the combinatorial point $q^{1/2} = e^{-2\pi i/3}$ the solution reduces to the ground state of the dense $O(1)$ loop model on a strip with open boundary conditions. We present an alternative construction of such ground state based on the knowledge of the ground state of the same model with mixed boundary conditions and prove that the sum rule as of its components is given by the product of four symplectic characters.

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1 Introduction

The interplay between statistical mechanics and combinatorics has always been of great interest both for physicists and for mathematician. The observations of Razumov and Stroganov in their seminal papers [1,2] (see also [3]), and the body of work that followed, exemplifies an instance of such an interplay. What Razumov and Stroganov found was that the properly normalized components of the ground state of the dense $O(1)$ model enumerate classes of so called Fully Packed Loop (FPL) configurations of given topology. The boundary conditions on the $O(1)$ model are reflected on the symmetries of the FPL (see [4] for further explications). An approach to these problems, that has revealed to be particularly powerful, was initiated by Di Francesco and Zinn-Justin [5]. They used the integrable structure of the dense $O(1)$ model and generalized it by introducing spectral parameters, still preserving integrability. The advantage in dealing with a more complicated problem was at the beginning a technical one: one could exploit the richer structure coming from the polynomial nature of the ground state of the generalized model to better handle it. Later it was realized that the idea of Di Francesco and Zinn-Justin opened new perspective on the problem by relating it to the study of so called qKZ equations [6] and affine Hecke algebras [7], to algebraic geometry of certain affine varieties [6] and allowing to push forward some further intriguing conjectures (in part already proved [8]) relating the homogeneous specialization of the solution of certain qKZ equation to refined enumeration of Plane Partitions.

In the present work, we will study the qKZ equations related to the so called two boundaries Temperley Lieb algebra [10, 9, 11]. When specialized at $q^{1/2} = e^{-2\pi i/3}$ the solution of such a system of equations is the ground state of the inhomogeneous $O(1)$ loop model with so called open boundary conditions. For generic $q$ instead, the problem is related to the study of certain Laurent polynomial representations of the affine Hecke algebras $\mathcal{H}(C_N)$ of type $C_N$ and of the doubly affine Hecke algebras of type $C^\vee C_N$. The main difficulty we encounter in dealing with two open boundaries is the absence of a completely factorized component. The presence of such a component in the cases with other boundary conditions, previously studied, allowed to fix the unknown factor in the recursion relations which in turn allowed to derive the sum rule at the combinatorial point $q=1$. Here we adopt the opposite strategy, we derive first the recursion relations by a method that circumvent the full knowledge of a component and then we use such recursions to fix the simplest component from which all the other can be derived.

Still we will show that at $q^{1/2} = e^{-2\pi i/3}$ the recursion relations are not sufficient to derive the sum of the components. Therefore we must resort to a different derivation of the whole eigenstate, based on mappings to systems with a single open boundary. This way we show that the degree is preserved and we compute the sum rule, which is given
by the product of four symplectic characters, correcting some recent claims [12].

The paper is organized as follows. In Section 2 we introduce the two boundaries Temperley Lieb algebra and the representation of this algebra on extended link patterns. The $\tilde{R}$ and $K$ matrices based on baxterization of the two boundaries Temperley Lieb algebra, solution of the Yang-Baxter equation and of the boundary Yang-Baxter equation, is presented in Section 3. Then, in Section 4 we introduce the qKZ equations and explain some of the properties of their solution. In particular in Section 4.1 we explain the relation between our qKZ equations and the representation theory of the (doubly) affine Hecke algebras of type $C$ and in Section 4.3 we derive the recursion relations. In Section 5 we concentrate on the case where the parameter $s$ of qKZ assumes the value $s = q^{-3/2}$, we use in such a case the recursion relations to construct the solution of qKZ. The specialization $q^{1/2} = e^{-2\pi i/3}$ is studied in Section 6 where we explain how to derive the full solution of the qKZ equations with open boundaries from the solution with mixed boundary conditions, by defining certain mappings of representations.

2 The two boundaries Temperley Lieb algebra

The problem we are going to consider is based on a boundary extension of the well known Temperley-Lieb algebra, called 2 boundaries Temperley Lieb algebra [10, 4, 9]. The Temperley Lieb algebra $TL_N$ can be defined as the free algebra with generators $e_i$, for $i = 1, \ldots, N - 1$, and relations

$$e_i^2 = \tau e_i; \quad e_i e_j = e_j e_i \quad \text{for} \quad |i - j| > 1;$$

$$e_i e_{i+1} e_i = e_i \quad \text{for} \quad 2 \leq i \leq N - 2;$$

This algebra has an appealing graphical representation in terms of non crossing link patterns connecting $N$ points on the the top and $N$ points on the bottom of a finite strip, with the graphical rules
A first extension of the TL algebra is obtained by adding to the generators of $TL_N$ a so-called boundary operator $f_R$ with the following commutation rules

$$f_R^2 = \tau f_R \quad e_{N-1} f_R e_{N-1} = e_{N-1} \quad e_i f_R = f_R e_i \quad \text{for} \quad i < N - 1.$$ 

This algebra, which is sometimes called 1BTL (one boundary Temperley-Lieb algebra), appeared for the first time in the paper [13] where it was called blob algebra, then it was studied in different contexts (for example [14]). Here we call it $TL_N^{(c,o)}$ since it is naturally related to certain statistical mechanics loop models having closed boundary condition on one side and open boundary conditions on the other side. Graphically the generator $f_R$ can be represented as

$$f_R = \begin{array}{c} \cdots \hline \end{array} \begin{array}{c} N-1 \hline \end{array} \begin{array}{c} N \hline \end{array}$$

and the commutation rules correspond to the following graphical relations
The algebra obtained by adding to the generators of the Temperley-Lieb algebra a generator $f_L$ “to the left”, with commutation relation

$$f_L^2 = \tau_L f_L \quad e_1 f_L e_1 = e_1 \quad e_i f_L = f_L e_i \quad \text{for} \quad i > 1;$$

will be called $TL_N^{(o,c)}$. The graphical representation of the generator $f_L$ is similar to the one of $f_R$ and the commutation rules look as follows

$$f_L = \begin{array}{c}
\begin{array}{c}
\ldots \\
1 \\
2 \\
\ldots
\end{array}
\end{array} = \tau_L \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots
\end{array}
\end{array}$$

In the statistical mechanics model the boundary conditions are now exchanged with respect to the case with $TL_N^{(o,c)}$.

Finally we define the algebra corresponding to open boundary conditions on both sides $TL_N^{(o,o)}$ if we add to $TL_N$ both $f_R$ and $f_L$ and require them to commute among themselves. While the algebras $TL_N$, $TL_N^{(o,c)}$ and $TL_N^{(c,o)}$ are finite dimensional, this is not the case for $TL_N^{(o,o)}$. It is quite easy to understand why by looking at the graphical representation: the commutation rules given above do not allow to erase the lines connecting the two boundaries. This means that for example all the element in the following picture have to be considered as distinct.
However it has been proved \cite{[9]} that all finite dimensional irreducible representations come from a further the quotient of this algebra which consist in giving a weight $\sqrt{\tau_c}$ to the lines going from one boundary to the other. Algebraically one has to distinguish the case $N$ odd and the case $N$ even, then introduce the following elements

- $N = 2M + 1$ : $g_1 = f_L \prod_{i=1}^{M} e_{2i}$, $g_2 = f_R \prod_{i=1}^{M} e_{2i-1}$
- $N = 2M$ : $g_1 = f_L f_R \prod_{i=1}^{M-1} e_{2i}$, $g_2 = \prod_{i=1}^{M} e_{2i-1}$

and take the quotients over

$$g_1 g_2 g_1 = \tau_c g_1 \quad g_2 g_1 g_2 = \tau_c g_2.$$

In the rest of the paper we will moreover restrict to the case $\tau_R = \tau_L = 1$. The reason will be explained in Appendix A.

### 2.1 Representation of $TL_N^{(o,o)}$ on extended link patterns

The representation of $TL_N^{(o,o)}$ we will be interested in acts on the space $\mathcal{H}_N^{(oo)}$ with basis labelled by extended link patterns. This is the Hilbert space corresponding to open-open boundary conditions. We call “extended link pattern” a diagram with $N$ points on a line, numbered from left to right, and two more points, the first called $L$ is situated on the left of point 1; the second called $R$ is situated on the right of point $N$. The point 1, ..., $N$ are either connected in pairs, or they are connected to the point $L$ or $R$, by non intersecting curves. Here is an example

![Extended link pattern diagram](image)

The action of the generators $TL_N^{(o,o)}$ is almost obvious from the graphical representation, for simplicity we restrict to the case $\tau_L = \tau_R = 1$. In such a case each time we close a loop in the bulk we remove it and multiply the link pattern by $\tau$. If instead we close a loop touching one of the two boundaries we simply remove it.
These rules are supplemented by the requirement that a line joining the two boundaries can be removed at the cost of multiplying the obtained link pattern a weight. We call the weight of the removed line $\sqrt{\tau_c}$. Graphically this looks as follows

In Section 6.2 we will consider also the representations of $T L_N^{(o,c)}$ on the space of left extended link patterns that we call $\mathcal{H}_N^{(oc)}$. This space is the subspace of $\mathcal{H}_N^{(oo)}$ consisting of link patterns having no lines connected to the point $R$ and is the Hilbert space for open-closed b.c. considered in [15].

3 The boundary scattering matrix

Let us consider now the following $\check{R}$-matrix
where $I$ is the identity operator and $\tau = -(q + q^{-1})$ (recall that $e^2 = \tau e$). The matrix $\hat{R}_i$ satisfies the Yang-Baxter equation

$$
\hat{R}_{i+1}(w, z) \hat{R}_i(x, z) \hat{R}_{i+1}(x, w) = \hat{R}_i(x, w) \hat{R}_{i+1}(x, z) \hat{R}_i(w, z).
$$

In a model with boundaries, integrability is assured by the presence of boundary scattering matrix which satisfies the so called Boundary Yang Baxter Equation [16, 17]

$$
K_R(w) \hat{R}_{N-1}(1/z, w) K_R(z) \hat{R}_{N-1}(w, z) = \hat{R}_{N-1}(1/z, 1/w) K_R(z) \hat{R}_{N-1}(1/w, z) K_R(w).
$$

In the present case the nontrivial boundary conditions on the right are given by a right boundary scattering matrix $K_R$ of a Baxterized form $K_R(z) = a(z) I + b(z) f_R$. The nontrivial solutions of the right boundary Yang Baxter Equation form a one parameter family, which in the case of generic $\tau_R$ reads

$$
K_R(z) = \frac{(z - \zeta_R/q)(z - kq/\zeta_R)I + \frac{(q^2-1)}{(q^2+\tau_R)}(z^2 - 1)f_R}{(z - q/\zeta_R)(zk - \zeta_R/q)}.
$$

(2)
with
\[ k = \frac{q + q^2\tau_R}{q + \tau_R}; \quad \tau_R = q \frac{k - 1}{q^2 - k}. \]

In the case \( \tau_R = 1 \), which turns out to be the most interesting one for us and to which we restrict from now on, we have \( k = q \) and the boundary scattering matrix reduces to
\[ K_R(\zeta_R|z) = \frac{(z - q^2/\zeta_R)(z - \zeta_R/q) + (q - 1)(z^2 - 1)f_R}{(qz - \zeta_R/q)(z - q/\zeta_R)}. \]  

We consider left boundary scattering matrices as well but, in order to formulate the qKZ equations, we require them to satisfy a modified boundary Yang Baxter equation in which we have a new parameter \( s \)
\[ \hat{R}_1(s/w, s/z)K_L(z)\hat{R}_1(z, s/w)K_L(w) = K_L(w)\hat{R}_1(w, s/z)K_L(z)\hat{R}_1(z, w). \]

The explicit form of the left scattering matrices when \( \tau_L = 1 \) is
\[ K_L(\zeta_L|z) = \frac{(qz - s\zeta_L/q)(z - q/\zeta_L) + (q - 1)(s - z^2)f_L}{(z - s\zeta_L/q)(z - q^2/\zeta_L)}. \]

4 The qKZ equations and basic properties of their solutions

Given the representation of \( TL_N^{(o,o)} \) on the space of extended link patterns, let us consider a function \( \Psi_N(\zeta_L; z_1, \ldots, z_N; \zeta_R) \) from \( \mathbb{C}^{N+2} \) to this space. The boundary qKZ equations are the following set of equations for \( \Psi(\zeta_L; z_1, \ldots, z_N; \zeta_R) \)
\[ \hat{R}_i(z_{i+1}, z_i) \circ \Psi(\zeta_L; z_1, \ldots, z_i, z_{i+1}, \ldots, z_N; \zeta_R) = \Psi(\zeta_L; z_1, \ldots, z_{i+1}, z_i, \ldots, z_N; \zeta_R); \quad (5) \]
\[ K_R(\zeta_R|z_N) \circ \Psi(\zeta_L; z_1, \ldots, z_N; \zeta_R) = \Psi(\zeta_L; z_1, \ldots, 1/z_N; \zeta_R); \quad (6) \]
\[ K_L(\zeta_L|z_1) \circ \Psi(\zeta_L; z_1, \ldots, z_N\zeta_R) = \Psi(\zeta_L; s/z_1, \ldots, z_N\zeta_R); \quad (7) \]

We expand now these equations on the natural basis of extended link patterns. Therefore we write
\[ \Psi_N(\zeta_L; z_1, \ldots, z_N; \zeta_R) = \sum_{\pi} \psi^N_\pi(\zeta_L; z_1, \ldots, z_N; \zeta_R)|\pi\rangle, \]
where the sum runs over all the extended link patterns. Since given an extended link pattern, the notion of an arc opening or closing at a point is well defined, we will parameterize it by the location of open and closing arcs and use the following notation \(|coo \ldots oc\rangle\), each \( o \) or \( c \) standing for “opening” or “closing”. A concrete example of our notation is
It is useful to introduce the following operators on the space of Laurent polynomials of $N$ variables

$$t_i \circ \phi(\ldots, z_i, z_{i+1}, \ldots) = \phi(\ldots, z_{i+1}, z_i, \ldots),$$  \hspace{1cm} (8)

$$t_R \circ \phi(\ldots, z_N) = \phi(\ldots, 1/z_N),$$ \hspace{1cm} (9)

$$t_L \circ \phi(z_1, \ldots) = \phi(s/z_1, \ldots).$$ \hspace{1cm} (10)

These operators allows us to rewrite the qKZ equations for the components of $\Psi_N(\zeta_L; z_1, \ldots, z_N; \zeta_R)$ in the extended link patterns basis as follows.

- From eq. (5) it follows that if $|\pi > \notin e_i \circ \mathcal{H}_N^{(oo)}$ then we have
  $$ (qz_{i+1} - z_i/q)\psi^N_\pi = t_i \circ (qz_{i+1} - z_i/q)\psi^N_\pi $$ \hspace{1cm} (11)

  Otherwise, if $|\pi > \in e_i \circ \mathcal{H}_N^{(oo)}$ then
  $$ \partial_i \psi^N_\pi := (qz_i - z_{i+1}/q) \frac{1 - t_i}{z_i - z_{i+1}} \psi^N_\pi = \sum_{\pi \propto e_i \circ \pi'} c^i_{\pi, \pi'} \psi^N_{\pi'}, $$ \hspace{1cm} (12)

  where the coefficients $c^i_{\pi, \pi'}$ are defined by $e_i \pi' = c^i_{i, \pi, \pi'} \pi$ and can be either $\sqrt{t_c}$ if in applying $e_i$ to $\pi'$ we form a line joining the two boundaries or 1 in case we do not form a line joining the two boundaries.

- From eq. (6) it follows that if $|\pi > \notin f_R \circ \mathcal{H}_N^{(oo)}$ then
  $$ (z_N - q^2/\zeta_R)(z_N - \zeta_R/q)\psi^N_\pi = (qz_N - \zeta_R/q)(z_N - q/\zeta_R)t_R \circ \psi^N_\pi.$$ \hspace{1cm} (13)

  If $|\pi > \in f_R \circ \mathcal{H}_N^{(oo)}$ there is only one preimage under $f_R$ different from $\pi$ itself, i.e. there is only one $\pi' \neq \pi$ such that $\pi \propto f_R \circ \pi'$ and the components of this preimage is given by
  $$ c^R_{\pi, \pi'} \psi^N_{\pi'} = \partial_R \psi^N_{\pi} := (qz_N - \zeta_R/q)(z_N - q/\zeta_R) \frac{1 - t_R}{(q - 1)(z_N^2 - 1)} \circ \psi^N_{\pi} $$ \hspace{1cm} (14)

  The coefficient $c^R_{\pi, \pi'}$ now is either $\sqrt{t_c}$ if by applying $f_R$ to $\pi'$ we form a line joining the two boundaries, or otherwise it is equal to 1.

- Finally from eq. (7) it follows that if $|\pi > \notin f_L \circ \mathcal{H}_N^{(oo)}$ then
  $$ (qz_1 - s\zeta_L/q)(z_1 - q/\zeta_L)\psi^N_\pi = (z_1 - s\zeta_L/q)(z_1 - q^2/\zeta_L)t_L \circ \psi^N_\pi.$$ \hspace{1cm} (15)
If $|\pi > \in f_L \circ H_N^{(oo)}$ its preimage under $f_L$, different from $\pi$ itself is unique

$$c_{L,\pi,\pi'}^L \psi^N = \partial_L \psi^N := (z_1 - s\zeta_L/q)(z_1 - q^2/\zeta_L) \frac{1-t_L}{(q-1)(s-z_N^2)} \circ \psi^N$$

(16)

The coefficient $c_{L,\pi,\pi'}^L$ now is either $\sqrt{c}$ if by applying $f_L$ to $\pi'$ we form a line joining the two boundaries, or otherwise it is equal to 1.

4.1 Affine Hecke generators

It is known, from the seminal papers [18, 19], that the qKZ equations are related to the representation theory of affine Hecke algebras. This has been rediscovered recently in the context of the Razumov Stroganov conjecture [7, 20].

Let us explain this observation in our case; this will lead us to consider the Laurent polynomial representations of affine Hecke algebras of type C introduced by Noumi [22]. We start from eq.(5), and introduce different generators of the $TL_N^{(o,o)}$ algebra $T_i = -e_i - 1/q$. By recombining the terms of eq.(5) we can rewrite it as

$$T_i \circ \Psi_N = \hat{T}_i \circ \Psi_N, \quad (17)$$

where the operator

$$\hat{T}_i = q + \frac{1}{q} \left( \frac{q^2 z_i - z_{i+1}}{z_i - z_{i+1}} \right) (t_i - 1)$$

acts on the polynomial part of $\Psi_N$. We proceed in the same way for the other two qKZ equation, by introducing two generators $T_N = (q_N + 1/q)T_R - 1/q_N$ and $T_0 = (q_0 + 1/q_0)T_L - 1/q_0$, where $q_0^2 = q_N^2 = -q$. Then we can rewrite eqs.(6,7) as

$$T_0 \circ \Psi_N = \hat{T}_0 \circ \Psi_N, \quad T_N \circ \Psi_N = \hat{T}_N \circ \Psi_N. \quad (18)$$

The operators $\hat{T}_0$ and $\hat{T}_N$, like the $\hat{T}_i$, act on Laurent Polynomials and are given by

$$\hat{T}_0 = q_0 + \frac{1}{q_0} \left( z - s\zeta_N \right) \left( \frac{z - z_N^2}{z^2 - s} \right) (t_L - 1).$$

$$\hat{T}_N = q_N + \frac{1}{q_N} \left( 1 - \frac{q^2}{\zeta_R} \right) \left( 1 - \frac{z_R}{z} \right) (t_R - 1)$$

It is now a matter of some straightforward computations to show that the operators $\{T_i, T_0, T_N\}$ and $\{\hat{T}_i, \hat{T}_0, \hat{T}_N\}$ satisfy separately the commutation relations of the generators
of the affine Hecke algebra $H(C_N)$ of type $C_N$

\begin{align}
(T_0 + 1/q_0)(T_0 - q_0) &= 0 \\
(T_i + 1/q)(T_i - q) &= 0 \\
(T_N + 1/q_N)(T_N - q_N) &= 0 \\
T_i T_{i \pm 1} T_i &= T_{i \pm 1} T_i T_{i \pm 1} \\
T_0 T_i T_0 T_i &= T_i T_0 T_i T_i \\
T_N T_{N-1} T_N T_{N-1} &= T_{N-1} T_N T_{N-1} T_N \\
T_i T_j &= T_j T_i \quad \text{for } |i - j| > 1 \\
T_0 T_j &= T_j T_0 \quad \text{for } j > 1 \\
T_N T_j &= T_j T_N \quad \text{for } j < N - 1
\end{align}

(19)

Indeed the representation of $H(C_N)$ given by \{$\hat{T}_i$, $\hat{T}_0$, $\hat{T}_N$\} is well known and goes under the name of Noumi representation (actually in the Noumi representation the parameters $q_0$, $q_N$ and $q$ are independent; see for example Proposition 2.2 of [23]). By adding also the operators $\hat{z}_i$, whose action on a polynomial is the multiplication by $z_i$ we obtain a representation of the doubly affine Hecke algebra of type $CV C_N$ [21, 23].

In the qKZ equation, we are considering the action of two copies of the $H(C_N)$, one acting on $H_{N}^{(\infty)}$, the other acting on a space of polynomial that for the moment we call $\hat{H}_{N}^{(\infty)}$. Then the vector $\Psi_N$ can be interpreted as a map from $H_{N}^{(\infty)}$ to the dual of $\hat{H}_{N}^{(\infty)}$

$\Psi_N : H_{N}^{(\infty)} \rightarrow \hat{H}_{N}^{(\infty)*}$

and the qKZ equations simply state that it intertwines between the two representation. Therefore, since the representation on $H_{N}^{(\infty)}$ is irreducible [9], we conclude that $\hat{H}_{N}^{(\infty)} = H_{N}^{(\infty)*}$. This means that solving the qKZ equations amounts first to find the irreducible representation of $H(C_N)$ on Laurent polynomials, dual to $H_{N}^{(\infty)}$, and then to find the basis dual to the extended link pattern basis of $H_{N}^{(\infty)}$.

4.2 Trivial factors and symmetries

From eq. (11) it follows that a component in which the points $i$ and $i + 1$ are not connected, will have the following form

$\psi(z_i, z_{i+1}) = (q z_i - z_{i+1}/q) \tilde{\psi}(z_i, z_{i+1})$

where $\tilde{\psi}(z_i, z_{i+1})$ is symmetric under exchange $z_i \leftrightarrow z_{i+1}$. In general if the consecutives points $i, i + 1, \ldots, i + r$ are not connected among themselves then

$\psi(z_i, z_{i+1}, \ldots, z_{i+r}) = \prod_{i \leq j < k \leq i+r} (q z_j - z_k / q) \tilde{\psi}(z_i, z_{i+1}, \ldots, z_{i+r})$
and $\tilde{\psi}(z_i, z_{i+1}, \ldots, z_{i+r})$ is symmetric in $z_i, \ldots, z_{i+r}$.

Analogously, from eq. (13), it follows that if the point $N$ is not connected to the right boundary $R$ then one has

$$\psi(z_N; \zeta_R) = (qz_N/\zeta_R - 1/q)(\zeta_R - q/z_N)\tilde{\psi}(z_N; \zeta_R)$$

where $\tilde{\psi}(z_N; \zeta_R)$ is invariant under $z_N \rightarrow 1/z_N$. An analogous statement relative to the left boundary is true, namely if the point 1 is not connected to the left boundary $L$ then the component can be written as

$$\psi(\zeta_L; z_1) = (z_1/\zeta_L - s/q)(\zeta_L - q^2/z_1)\tilde{\psi}(\zeta_L; z_1)$$

where now $\tilde{\psi}(\zeta_L; z_1)$ is invariant under $z_1 \rightarrow s/z_1$.

One can combine the previous remarks in order to extract more trivial factors. For example the components having all the bulk points connected to the right boundary $\psi_{oo...oo}$ or to the left boundary $\psi_{cc...cc}$ have the following form

$$\psi_{oo...oo}(\zeta_L; z; \zeta_R) = \prod_{j=1}^{N} \left( \frac{sqz_j}{z_j} - q^2 \right) \prod_{1 \leq i < j \leq N} \left( \frac{z_j}{q} - \frac{1}{q} \right) \phi^{(R)}_N(\zeta_L; z; \zeta_R)$$

$$\psi_{cc...cc}(\zeta_L; z; \zeta_R) = \prod_{i=1}^{N} \left( \frac{qz_i}{\zeta_R} - \frac{1}{q} \right) \prod_{1 \leq i < j \leq N} \left( \frac{qz_j}{z_i} - \frac{1}{q} \right) \phi^{(L)}_N(\zeta_L; z; \zeta_R),$$

where $\phi^{(R)}_N(\zeta_L; z; \zeta_R)$ is a function symmetric under exchange $z_i \leftrightarrow z_j$ and invariant under $z_j \rightarrow s/z_j$, while $\phi^{(L)}_N(\zeta_L; z; \zeta_R)$ is a function symmetric under exchange $z_i \leftrightarrow z_j$ and invariant under $z_j \rightarrow 1/z_j$.

### 4.3 Recursion relations: bulk

As a consequence of the analysis in the previous section, it follows that if we set $z_{i+1} = q^2 z_i$, all the components of the solution of the qKZ equation at length $N$ that do not lie in the image of $e_i$ are zero. The subspace of $TL^{(o,o)}_N$ with an arc between points $i$ and $i + 1$ is isomorphic to the space $TL^{(o,o)}_{N-2}$. More precisely, if we call $p_i$ the map $p_i : TL^{(o,o)}_{N-2} \rightarrow TL^{(o,o)}_N$ which consists in adding an arc between $i - 1$ and $i$ (and renumbering the points), then $p_i$ is an isomorphism between $TL^{(o,o)}_N$ and $e_i \circ TL^{(o,o)}_N$. Since $\Psi_N(\ldots, z_i, z_{i+1} = q^2 z_i, \ldots) \in e_i \circ TL^{(o,o)}_N$ we can consider its preimage under $p_i$. We show now that such a preimage, $p_i^{-1} \circ \Psi_N(\ldots, z_i, z_i = q^2 z_i, \ldots)$, satisfies a set of modified qKZ equations. In order to work this out let us restrict for the moment to the case $i = N - 1$, i.e. when we add a small arc at right the end of our strip. It is easy to show that the map $p_{N-1}$ intertwines the
operators $K_L$ and $\hat{R}_{i<N-2}$ acting on $TL^{(o,o)}_{N-2}$ and the ones acting on $e_i \circ TL^{(o,o)}_N$ (by abuse of notation we do not adopt a different notation for operators acting on different spaces)

$$K_L(\zeta_L|z) \ p_{N-1} = p_{N-1} \ K_L(\zeta_L|z), \quad \hat{R}_{i<N-2}(z, w) \ p_{N-1} = p_{N-1} \ \hat{R}_{i<N-2}(z, w).$$

Then let us consider the following matrix

$$\bar{K}_R(\zeta_R|z) = \hat{R}_{N-2}(1/z, q^2z_{n-1}) \hat{R}_{N-1}(1/z, q^2z_{n-1}) K_R(\zeta_R|z) \hat{R}_{N-1}(q^2z_{n-1}, z) \hat{R}_{N-2}(z_{n-1}, z).$$

(22)

When this matrix acts on configurations in the image of $e_{N-1}$, it can be written as

$$\bar{K}_R(\zeta_R|z) \ e_{N-1} = \frac{u(1/z)}{u(z)} \left( \frac{(z - \zeta_R/q)(z - q^2/\zeta_R)(z - 1)(zq^2/\zeta_R - 1)}{(z\zeta_R/q - 1)(zq^2/\zeta_R - 1)} \right) e_{N-1},$$

(23)

where $u(z) = (q^4z_{n-1} - 1/z)(q^2z - z_{n-1})$. Moreover if we notice that $e_{N-1}f_re_{N-2}p_{N-1} = p_{N-1}f_R$

we obtain that $p_{N-1}$ intertwines between $\bar{K}_R(\zeta_R|z)$ and $u(1/z)/u(z)K_R(\zeta_R|z)$

$$\bar{K}_R(\zeta_R|z) \ p_{N-1} = \frac{u(1/z)}{u(z)} p_{N-1} \ K_R(\zeta_R|z).$$

(24)

Therefore $p_i^{-1} \circ \Psi_N(\ldots, z_{N-1}, z_N = q^2z_{N-1}, \ldots)$ satisfies a modified set of qKZ equations

where eq. (23) is substituted by the following equation

$$\frac{u(1/z_{n-2})}{u(1/z_{n-2})} K_R(\zeta_R|z_{N-2})p_{N-1}^{-1} \circ \Psi_N(\ldots, z_{N-2}, z_{N-1}, z_N = q^2z_{N-1}, \ldots)$$

$$= p_{N-1}^{-1} \circ \Psi_N(\ldots, 1/z_{N-2}, z_{N-1}, z_N = q^2z_{N-1}, \ldots)$$

(25)

Now let us suppose that we can find a function $g(z)$ which is invariant under $z \to 1/z$ and such that

$$\frac{u(z)g(z)}{u(1/z)g(1/z)} = 1$$

(26)

then we have that

$$\prod_{i=1}^{N-2} \frac{1}{u(z_i)g(z_i)} \ p_{N-2}^{-1} \circ \Psi_N(\ldots, z_{N-1}, z_N = q^2z_{N-1}, \ldots)$$
is a solution of the unmodified qKZ equations (5,6,7) of length \(N - 2\) and. Therefore, apart from a factor independent from all the \(z_i < N - 1\), it must be equal to \(\Psi_{N-2}(\ldots, z_{N-2}; \zeta_R)\)

\[
\Psi_N(\ldots, z_{N-1}, q^2 z_{N-1}, \ldots) = k(z_{N-1}, \zeta_L, \zeta_R) \left( \prod_{i=1}^{N-2} u(z_i) g(z_i) \right) p_{N-1} \circ \Psi_{N-2}(\ldots, z_{N-2}; \zeta_R)
\]

(27)

Since we are searching for Laurent polynomial solutions of the qKZ we require the function \(g(z)\), solution of eq.(26), to be also a Laurent polynomial. This fixes the possible form of \(s\) as function of \(q\). Indeed, in order to cancel the pole at \(z = q^2 s / z_{N-1}\) in eq.(26), the function \(g(z)\) must be of the form

\[
g(z) = \prod_{j=1}^{n-1} \left( 1 - \frac{q^2 s_j}{z_{N-1} z} \right) \left( s^j z - \frac{z_{N-1}}{q^2} \right) .
\]

(28)

The product must be such that the last term cancels the pole at \(z = sq^4 z_{N-1}\). This happens only if \(s = q^{-6/n}\).

In Section [3] we will argue that the lowest value of \(n\) for which it is possible to find a solution of the qKZ equations is \(n = 4\) or \(s = q^{-3/2}\). In such a case the recursion relation takes the form

\[
\Psi_N(\ldots, z_{N-1}, z_N = q^2 z_{N-1}, \ldots) = G(z_1, \ldots; z_{N-1}, \zeta_L, \zeta_R) p_{N-1} \circ \Psi_{N-2}(\ldots, \hat{z}_{N-1}, \hat{z}_N, \ldots).
\]

(29)

with

\[
G(z_1, \ldots; z_{N-1}, \zeta_L, \zeta_R) := k(z_{N-1}, \zeta_L, \zeta_R) \prod_{i=1}^{N-1} \prod_{j=1}^{4} \left( q^{3j/2} - \frac{q^2 s_j}{z_{N-1} z_i} \right) (z_i - q^{3(j-1)/2-2} z_{N-1})
\]

(30)

This is the recursion relation we were searching for. It will be very important in the next section when we will construct a solution of the qKZ equations. Actually we will need as well an analogous formula when we set \(z_2 = q^2 z_1\) for \(i = 1\). Of course we could repeat almost word by word the same derivation using this time a modified left boundary scattering matrix. However, in doing so we would not be able to keep track of the normalization choice of the wave function \(\Psi_N\), which is implicit in the function \(k(z_{N-1}, \zeta_L, \zeta_R)\). We can avoid this problem by deriving the recursion relation when \(z_{i+1} = q^2 z_i\) from the case \(i = N - 1\). The idea is simply to use eq.(5) to move the spectral lines. If we act on \(\Psi_N(\ldots, z_{N-1}, z_N = q^2 z_{N-1}, \ldots)\) with the operator \(\hat{R}_{N-1}(z_{N-1}, z_{N-2}) \hat{R}_{N-2}(z_N = q^2 z_{N-1}, z_{N-2})\) we obtain

14
\[ \Psi_N(\ldots, z_{N-1}, z_N = q^2 z_{N-1}, z_{N-2}; \zeta_R) = \]
\[ = \hat{R}_N^{-1}(z_{N-1}, z_{N-2}) \hat{R}_N^{-2}(z_N = q^2 z_{N-1}, z_{N-2}) \Psi_N(\ldots, z_{N-2}, z_{N-1}, z_N = q^2 z_{N-1}; \zeta_R) \]
\[ = G(z_1, \ldots; z_{N-1}, \zeta_L, \zeta_R) \hat{R}_N^{-1}(z_{N-1}, z_{N-2}) \hat{R}_N^{-2}(z_N = q^2 z_{N-1}, z_{N-2}) p_{N-1} \Psi_{N-2}(\ldots, z_{N-2}; \zeta_R), \]
where the second equality follows from the recursion relation proved above. Then we notice that when we act with \( \hat{R}_N^{-1}(z_{N-1}, z_{N-2}) \hat{R}_N^{-2}(z_N = q^2, z_N - 2) \) on \( e_{N-1}TL^{(o,o)}_N \) we have
\[ \hat{R}_N^{-1}(z_{N-1}, z_{N-2}) \hat{R}_N^{-2}(z_N = q^2, z_N - 2) e_{N-1} = f(z_{N-1}, z_{N-2}) e_{N-2} e_{N-1}, \] (31)
with
\[ f(z_{N-1}, z_{N-2}) = \frac{q z_{N-1} - q^3 z_{N-2}}{q^4 z_{N-1} - z_{N-2}}. \]
This property can be restated in terms of intertwinings, just by substituting \( e_{N-1} \) with \( p_{N-1} \) in the l.h.s. and \( e_{N-2} e_{N-1} \) with \( p_{N-2} \) in the r.h.s. of eq.(31). Therefore we arrive at
\[ \Psi_N(\ldots, z_{N-1}, z_N = q^2 z_{N-1}, z_{N-2}; \zeta_R) = \]
\[ = G(z_1, \ldots; z_{N-1}, \zeta_L, \zeta_R) f(z_{N-1}, z_{N-2}) p_{N-2} \Psi_{N-2}(\ldots, z_{N-2}; \zeta_R) \] (32)
We can continue this way and move the lines \( N - 1 \) and \( N \) to the position \( i \) and \( i + 1 \). By shuffling the indices we obtain therefore
\[ \Psi_N(\ldots, z_i, z_{i+1} = q^2 z_i, \ldots) = G(\ldots) \prod_{j > i+1} f(z_i, z_j) p_i \circ \Psi_{N-2}(\ldots, \hat{z}_i, \hat{z}_{i+1}, \ldots), \] (33)
where the hats in the right hand side mean that the variables are removed.

5 Solution at \( s = q^{-3/2} \)

We search for a solution of the qKZ equations (5,6,7), in the simplest situation, i.e. when we have only two internal points \( (N = 2) \). For the moment we leave the parameters \( s \) and \( \tau_c \) free and we will see whether they will be fixed by the solution. Let us start
from the components \( \psi_{oo}(\zeta_L; z_1, z_2; \zeta_R) \) and \( \psi_{co}(\zeta_L; z_1, z_2; \zeta_R) \). We have seen in Section 4.2 that they have the form reported in equation eqs.(20, 21). Some tedious but straightforward calculations show that if we demand the Laurent polynomials \( \chi_2^{(R)}(\zeta_L; z_1, z_2; \zeta_R) \) and \( \chi_2^{(L)}(\zeta_L; z_1, z_2; \zeta_R) \) to be constant in the \( z_i \), then there are no solutions of the qKZ equation. If instead we allow the \( \chi \)'s to be of degree width 1, we see that the requirement that

\[
\psi_{oc} = \partial_R \psi_{oo} = \partial_L \psi_{cc}
\]

determines \( s = q^{-3/2} \) and fixes completely (except of course for an irrelevant global constant normalization) both \( \phi_2^{(R)} \) and \( \phi_2^{(L)} \)

\[
\phi_2^{(R)} = \left( z_1 + \frac{1}{q^{3/2} z_1} + z_2 + \frac{1}{q^{3/2} z_2} \right) - (1 + q^{-1/2}) \left( \frac{q}{\zeta_R} + \frac{\zeta_R}{q^2} \right),
\]

\[
\phi_2^{(L)} = \left( z_1 + \frac{1}{z_1} + z_2 + \frac{1}{z_2} \right) - (1 + q^{1/2}) \left( \frac{q^2}{\zeta_L} + \frac{\zeta_L}{q^{5/2}} \right).
\]

We find \( \psi_{co} \) by applying \( \partial_1 \) to \( \psi_{oc} \)

\[
\psi_{co} = \frac{1}{\sqrt{\tau_c}} \partial_1 \psi_{oc} - \psi_{oo} - \psi_{cc},
\]

then the qKZ equations close if we have

\[
\partial_R \psi_{co} = \sqrt{\tau_c} \psi_{cc} \quad \text{and} \quad \partial_L \psi_{co} = \sqrt{\tau_c} \psi_{oo}.
\]

These conditions are satisfied only if \( \tau_c = \frac{1}{q^{1/2} + 2 + q^{-1/2}} \). Armed with this simple solution we try to construct the solution at \( N = 3 \). We notice that the representation of \( TL_N^{(o,o)} \) on extended link patterns at \( \tau = -q - 1/q \) and \( \tau_c = \frac{1}{q^{1/2} + 2 + q^{-1/2}} \) is irreducible, therefore the discussion in Section 4.1 tells us that if we know a component of \( \Psi_N(\zeta_L; z; \zeta_R) \), then using the qKZ equations as in [5] we can reconstruct all the other components. It turns out that making some minimal assumptions we are able to construct \( \phi_3^{(R)} \). For this we have to come back to the bulk recursion relations of Section 4.3. In order for them to be really useful we need to determine as much as possible the unknown factor \( k(z, \zeta_L, \zeta_R) \). First of all using the qKZ equation [5] we have

\[
\psi_{o...oc} = \partial_R \psi_{o...oo}
\]

moreover if we specialize at \( z_N = q^2 z_{N-1} \) we have

\[
\psi_{o...oc}(z_N = q^2 z_{N-1}) = \frac{\left( q^3 z_{N-1} - \frac{\zeta_R}{q} \right) \left( q^2 z_{N-1} - \frac{q}{\zeta_R} \right)}{(1 - q)(q^2 z_{N-1}^2 - 1)} \psi_{o...oo}(\ldots, z_{N-1}, 1/(q^2 z_{N-1})).
\]
Applying the bulk recursion relation to the left hand side and comparing it to the right hand side (where we use eq. (20)) we find a factor of \( k(z_{N-1}, \zeta_L, \zeta_R) \)
\[
k(z, \zeta_L, \zeta_R) \propto \left( \frac{sq \zeta_L}{z} - q^2 \right) \left( \frac{z - q}{q \zeta_L} \right) \left( sq^3 z - \frac{q^2}{\zeta_L} \right) \left( \frac{\zeta_L}{q^3 z} - q \right)
\]
Repeating the same argument using \( \psi_{cc...cc} \) and the recursion relation for \( z^2 = q^2 z \) we obtain a further factor
\[
k(z, \zeta_L, \zeta_R) \propto \left( \frac{q^2 s z \zeta_R - 1}{s \zeta_R} \right) \left( \zeta_R - \frac{q z}{s} \right) \left( z \frac{q z - 1}{q^3} \zeta_R \right) \left( \frac{\zeta_R - 1}{q^3 z} \right)
\]
Then we assume that \( k(z, \zeta_L, \zeta_R) \) is only given by the product of the two factors above. With this expression for \( k(z, \zeta_L, \zeta_R) \), the recursion relation for \( \phi_{N}^{(R)} \) reads
\[
\phi_{N}^{(R)} (\zeta_L; z_j = sq^2 z_i; \zeta_R) = \tilde{G}(z_1, \ldots; z_{N-1}; \zeta_R) \phi_{N}^{(R)} (\zeta_L; \hat{z}_j, \hat{z}_i; \zeta_R) \tag{37}
\]
where
\[
\tilde{G}(z_1, \ldots; z_{N-1}; \zeta_R) = \frac{q^6 (1 - q)}{s (1 - sq^4)} \left( \frac{q^2 s}{z \zeta_R - 1} \right) \left( \zeta_R - \frac{q z_i}{s} \right) \prod_{k \neq i, j} \prod_{l=2}^3 \left( s^{-l} - \frac{q^2}{z_i z_k} \right) \left( z_k - \frac{s^{-1} - z_i}{q^2} \right)
\]
The further assumption that we make is that these recursion relations completely fix \( \phi_{N}^{(R)} \). In Section 6.3 we will show how to construct the solution of our qKZ equation in the special case \( q^{1/2} = e^{-2\pi i/3} \), using the known solution of the problem with mixed boundary conditions. The solution we will found has a degree width \( 4N - 2 \) in each variables \( z_i \), which means that the degree width of \( \phi_{N}^{(R)} \) in each variables \( z_i \) is \( 2N - 1 \). If we suppose that the same remains true for generic \( q \) at \( s = q^{-3/2} \) then the bulk recursion relations allow to fix \( \phi_{N}^{(R)} \) completely. Indeed thanks to its symmetries, once given eq. (37), we know the values of \( \phi_{N}^{(R)} \) for \( z_i = q^{\pm 2} z_{j \neq i} \). We have constructed \( \phi_{N}^{(R)} \), using Lagrange interpolation in \( z_N \), up to \( N = 6 \). It is actually quite remarkable that the interpolating formula turns out to be a Laurent polynomial in the other variables even though it doesn’t look of this form. This is also the case for all its expected symmetries. Once \( \psi_{oo...oo} \) known, we have checked up to length \( N = 6 \) that the qKZ equations hold. We believe that some representation theoretical argument should prove that the construction is valid for generic \( N \).

6 The solution at \( q^{1/2} = e^{-2\pi i/3} \) : norm and homogeneous limit

As mentioned in the introduction, when the parameter \( q^{1/2} \) assumes the value \( e^{-2\pi i/3} \), the solution of the qKZ equation is also an eigenvector of the transfer matrix of the dense
$O(1)$ loop model with open boundary conditions. Let us recall this fact here briefly. Let us consider the lattice in the following picture

![Diagram of a lattice](image)

the strip has horizontal length $N$ and vertically it extends to infinity. Each face in the bulk of the lattice is filled with the following configurations

\[ a = \begin{array}{c} \\ \end{array} \quad ; \quad b = \begin{array}{c} \\ \end{array} \]

with probabilities $a$ and $b = 1 - a$, while the boundary faces can be in the following configurations

\[ a_R = \begin{array}{c} \hat{\quad} \\ \end{array} \quad b_R = \begin{array}{c} \hat{\quad} \\ \end{array} \quad a_L = \begin{array}{c} \hat{\quad} \\ \end{array} \quad b_L = \begin{array}{c} \hat{\quad} \\ \end{array} \]

with probability $a_R$, $b_R = 1 - a_R$, $a_L$ and $b_L = 1 - a_L$. The end points of the lines on the bottom boundary are labelled from 1 to $N$. We are interested in the probability of the connectivity patterns of such boundary points: a point $i$ is connected by a line either to another point $j$ or to the left boundary or to the right boundary. Therefore these connectivity patterns are encoded by the extended link patterns defined in the Section 2.1. The standard way to deal with the question of finding the probability of a link pattern is through a transfer matrix approach. If one adds to the bottom a further double row as in the picture

![Diagram of a lattice](image)

the probabilities of the different link patterns must be preserved. This means that the vector $\Psi_N \in \mathcal{H}_N^{(oo)}$ of probabilities is an eigenvector of the double row transfer matrix.
defined above, with eigenvalue equal to 1. One can show the integrability of this model by constructing a family of commuting transfer matrices $T(t)$, depending on a “horizontal” spectral parameter. For this one needs to introduce the $R$ and $K$ matrices, which parametrise the probabilities by means of the spectral parameters, as in done for example in [5]. The existence of a family of commuting transfer matrices is a consequence of the Yang-Baxter and Boundary Yang-Baxter equations. This remains true even if we consider different bulk and boundary spectral parameters on each vertical spectral line [5] and the double row transfer matrix depends on such parameters: $T(t|\zeta_L,z,\ldots,\zeta_R)$. When we restrict to the homogeneous problem (all the vertical spectral parameters equal to 1) it is not difficult to show that the following Hamiltonian

$$H^{{(\alpha_L,\alpha_R)}}_N = \sum_{i=1}^{N-1} (e_i - 1) + \alpha_R(f_R - 1) + \alpha_L(f_L - 1),$$

where $\alpha_L = \frac{1}{\zeta_L+\zeta_i^{-1}+1}$ and $\alpha_R = \frac{1}{\zeta_R+\zeta_i^{-1}+1}$, commutes with the transfer matrix. This is a consequence of the fact that apart from a constant term one has

$$H^{{(\alpha_L,\alpha_R)}}_N = T^{-1}(0) \frac{d}{dt}T(t)|_{t=0}.$$

The vector of probabilities is an eigenvector of $H^{{(\alpha_L,\alpha_R)}}_N$ with eigenvalue equal to zero, i.e. the stationary measure of the stochastic matrix $H^{{(\alpha_L,\alpha_R)}}_N$. Closed boundary conditions, for example to the right, are obtained by sending the boundary spectral parameter $\zeta_R \to \infty$. If we consider systems with mixed boundary conditions (closed-open or open closed) or with closed BCs on both sides we can perform also only a partial homogeneous specialization. For example if we have open BCs on the right and closed on the left we can take all the spectral parameters equal to 1 except the boundary one and the last bulk one. The corresponding Hamiltonian (i.e. the logarithmic derivative of the transfer matrix in $t = 1$) assumes again a simple form

$$H^{{(c,o)}}_N(\alpha_1,\alpha_R) = \sum_{i=2}^{N-1} (e_i - 1) + \alpha_1(e_1 - 1) + \alpha_R(f_R - 1)$$

where $\alpha_1 = \frac{1}{z_1+z_1^{-1}+1}$. This remark will be important in the following when we will prove certain mappings between the vectors $\Psi_N$ with different boundary conditions and different size.

The general problem of finding the stationary measure in presence of different spectral parameter is strictly related to the qKZ equations (see [5] for an extended discussion). Indeed the eigenvector of the transfer matrix $\Psi_N(\zeta_L,z,\zeta_R)$ can be normalized in such a way it is a polynomial in the spectral parameters. Moreover as a simple consequence of the YBE and of the BYBE, at $q^{1/2} = e^{-2\pi i/3}$ we have

$$T(t|\zeta_L,\ldots,z_i,z_{i+1},\ldots,\zeta_R) \hat{R}_i(z_i,z_{i+1}) = \hat{R}_i(z_i,z_{i+1})T(t|\zeta_L,\ldots,z_{i+1},z_i,\ldots,\zeta_R)$$

(38)
\[ T(t|\zeta_L, \ldots, 1/z_N, \zeta_R)K_R(\zeta_R, z_N) = K_R(\zeta_R, z_N)T(t|\zeta_L, \ldots, z_N, \zeta_R) \]  
\[ T(t|\zeta_L, z_1, \ldots, \zeta_R)K_L(\zeta_L, z_1) = K_L(\zeta_L, z_1)T(t|\zeta_L, z_1, \ldots, \zeta_R). \]

This means that (by slightly changing the normalization, which makes it a Laurent polynomial) \( \Psi_N(\zeta_L, \bar{z}, \zeta_R) \) satisfies the qKZ equations. Notice that at \( q^{1/2} = e^{-2\pi i/3} \) all the loops have weight 1.

### 6.1 Norm of \( \Psi_N(\zeta_L, \bar{z}, \zeta_R) \)

At the special point \( q^{1/2} = e^{-2\pi i/3} \) we can compute the sum rule for the components in the basis of extended link patterns. At this special point each operator \( e_i, f_R \) and \( f_L \) has an eigenvector with eigenvalue 1 in the dual representation on \( \mathcal{H}_N^{(aa)*} \), given by \( \langle v| = \sum_{\pi} \langle \pi| \), where the sum extends to all the the link patterns. This means that we have

\[ \langle v| \tilde{R}_i(z_{i+1}, z_i) = \langle v|, \quad \langle v| K_L(\zeta_L, z_1) = \langle v| \quad \langle v| K_R(\zeta_R, z_N) = \langle v|. \]

The sum of the components of \( \Psi_N \) is given by

\[ \text{Sum}_N(\zeta_L; \bar{z}; \zeta_R) := \sum_\pi \Psi_{N,\pi}(\zeta_L; \bar{z}; \zeta_R) = \langle v| \Psi_N(\zeta_L; \bar{z}; \zeta_R) \]

As a consequence of the qKZ equation and of eq. (41) we have

\[ \text{Sum}_N(\zeta_L; \ldots, z_i, z_{i+1}, \ldots; \zeta_R) = \langle v| \tilde{R}_i(z_{i+1}, z_i)|\Psi_N(\zeta_L; \ldots, z_i, z_{i+1}, \ldots; \zeta_R) \rangle 
= \langle v| \Psi_N(\zeta_L; \ldots, z_{i+1}, z_i, \ldots; \zeta_R) \rangle = \text{Sum}_N(\zeta_L; \ldots, z_{i+1}, z_i, \ldots; \zeta_R) \]

and analogously

\[ \text{Sum}_N(\zeta_L; z_1, \ldots, z_N; \zeta_R) = \text{Sum}_N(\zeta_L; 1/z_1, \ldots, z_N; \zeta_R) = \text{Sum}_N(\zeta_L; z_1, \ldots, 1/z_N; \zeta_R) \]

Therefore \( \text{Sum}_N(\zeta_L; \bar{z}; \zeta_R) \) is a symmetric Laurent polynomial invariant under \( z_i \rightarrow 1/z_i \) of degree width \( 4N - 2 \) in each variable, hence it determined by the values at \( 4N - 1 \) distinct values of say \( z_N \). The bulk recursion relations (combined with the symmetry of the polynomial under exchange \( z_i \leftrightarrow z_{i+1} \)) provide us with the value of \( \text{Sum}_N(\zeta_L; z_1, \ldots, z_N; \zeta_R) \) at \( z_N = q^{\pm 2\pi i/3} z_{i+1}^{\pm 1} \), which means only at \((4N - 1)\) points. But this is not enough to construct \( \text{Sum}_N(\zeta_L; \bar{z}; \zeta_R) \) by Lagrange interpolation. Moreover it is also possible that at \( q^{1/2} = e^{-2\pi i/3} \) there could be some accidental simplifications leading to the appearance of a common factor in the solution of qKZ for \( q \) generic. Such a common factor should be canceled if one consider the lowest degree solution. The aim of the following sections is to explain how to determine \( \text{Sum}_N(\zeta_L; \bar{z}; \zeta_R) \) following a different path and will be the consequence of a stronger result. The idea will be to reconstruct the full vector \( \Psi_N(\zeta_L; \bar{z}; \zeta_R) \) from the knowledge of the solution of the qKZ equations with a closed boundary conditions on the left and open on the right. For this we need to construct certain intertwining maps which is done in the next subsection.
6.2 Quotients

From its definition one can think for example at the algebra $TL_{N}^{(o,c)}$ as a subalgebra of $TL_{N}^{(o,o)}$. However in the case $\tau = \tau_{L} = \tau_{R} = \tau_{c} = 1$ we can think at it as a quotient of $TL_{N}^{(o,o)}$ or of $TL_{N-1}^{(o,o)}$. Indeed one obtains $TL_{N}^{(o,c)}$ from $TL_{N}^{(o,o)}$ by quotienting the relation $f_{R} = 1$, or from $TL_{N-1}^{(o,o)}$ by relabelling $e_{N-1} = f_{R}$ and quotienting over the missing relation i.e. $e_{N-1}e_{N-2}e_{N-1} = e_{N-1}$. In the present section we want to implement this idea at the level of representations. We construct a map: $\Upsilon_{N}: \mathcal{H}_{N}^{(oo)} \rightarrow \mathcal{H}_{N}^{(oc)}$, which intertwines between the representation of $TL_{N}^{(o,c)}$ and the one of $TL_{N}^{(o,o)}$ with

$$\Upsilon_{N} \epsilon_{i<N} = \epsilon_{i<N} \Upsilon_{N}, \quad \Upsilon_{N} f_{L} = f_{L} \Upsilon_{N}, \quad \Upsilon_{N} f_{R} = \Upsilon_{N}.$$  \hspace{1cm} (45)

At a graphical level the map $\Upsilon_{N}$ is obtained as follows: starting from the right remove the first two lines connected to $R$ and connect the unmatched points with an arc. If the initial number of lines emanating from $R$ was even, repeat this procedure with the remaining lines joining $R$ until the lines from $R$ are exhausted. If the number of lines from $R$ was odd repeat the procedure until there is a single line connected to $R$ then remove this line and connect the unmatched bulk point with the left boundary $L$. Here is an example

The map $\Upsilon_{N}$ is easier to visualize if we use a different diagrammatic representation of the components, in terms of doubly-blobbed arcs $[\sqcup \uparrow \sqcup \downarrow]$, then $\Upsilon_{N}$ simply consists in removing the right blobs. The other map we want to define is $\Theta_{N}: \mathcal{H}_{N}^{(oo)} \rightarrow \mathcal{H}_{N+1}^{(oc)}$, which intertwines $f_{R}$ and $e_{N-1}$

$$\Theta_{N} \epsilon_{i<N} = \epsilon_{i<N} \Theta_{N}, \quad \Theta_{N} f_{L} = f_{L} \Theta_{N}, \quad \Theta_{N} f_{R} = e_{N} \Theta_{N}.$$  \hspace{1cm} (46)

At the diagrammatic level the map $\Theta_{N}$ goes as follows: change the name of the the point the right boundary point $R \rightarrow N + 1$. If there are no lines connected to $N + 1$ draw a line from it to $L$. If there are more than two lines connected to $N + 1$ remove the second and the third one starting from the right and draw an arc connecting the unmatched bulk points. Repeat this procedure until there are either one or two remaining lines. In the former case do nothing, while in the latter case remove the left most line and connect the unmatched bulk point with the left boundary $L$. Here is an example
The proof of eqs. (45, 46) is a tedious but straightforward graphical case by case analysis.

6.3 Reconstruction of $\Psi_N(\zeta_L; \bar{z}; \zeta_R)$

Now we use the maps $\Upsilon_N$ and $\Theta_N$ to reconstruct $\Psi_N(\zeta_L; \bar{z}; \zeta_R)$ from the known form of the solution of qKZ with mixed boundary conditions \cite{15} and gain information on the sum rule. First of all we notice that (with a slight abuse of notation)

$$\Upsilon_N K_L(\zeta_L|z) = K_L(\zeta_L|z)\Upsilon_N, \quad \Upsilon_N \hat{R}_i(z, w) = R_i(z, w)\Upsilon_N, \quad \Upsilon_N K_R(\zeta_R|z) = \Upsilon_N, \quad (47)$$

therefore we find that $\Upsilon |\Psi_N(\zeta_L, \bar{z}, \zeta_R))$ satisfies the qKZ equations with mixed closed-open boundary conditions. Therefore it must be

$$\Upsilon |\Psi_N(\zeta_L, \bar{z}, \zeta_R)) = f_1(\zeta_L, \bar{z}, \zeta_R)|\Psi_N^{(o,c)}(\zeta_L, \bar{z})\rangle \quad (48)$$

where $f_1(\zeta_L, \bar{z}, \zeta_R)$ is a symmetric polynomials in the $z_i$s invariant under $z_i \leftrightarrow 1/z_i$ and $|\Psi_N^{(o,c)}(\zeta_L, \bar{z})\rangle$ is the known solution of the qKZ equations with mixed B.C. \cite{15}. Moreover since we have $\langle v | = \langle v_N^{(o,c)} | \Upsilon$, where $\langle v_N^{(o,c)} | = \sum_{\pi(o,c)} \langle \pi^{(o,c)} |$ and the sum runs over all right extended link patterns, we obtain for the sum rule

$$\langle v |\Psi_N(\zeta_L, \bar{z}, \zeta_R)) = f_1(\zeta_L, \bar{z}, \zeta_R)\langle v^{(o,c)} |\Psi_N^{(o,c)}(\zeta_L, \bar{z}, \zeta_R))\rangle.$$  

In \cite{15} it is proven that $\langle v_N^{(o,c)} |\Psi_N^{(o,c)}(\zeta_L, \bar{z})\rangle = \chi_{N-1}(\bar{z})\chi_N(\bar{z}, \zeta_L)$, this means that we have found two factors of the sum rule

$$\text{Sum}_N(\zeta_L; \bar{z}; \zeta_R) \propto \chi_{N-1}(\bar{z})\chi_N(\bar{z}, \zeta_L), \quad (49)$$

where $\chi_n(z_1, \ldots, z_n)$ is the symplectic character

$$\chi_n(z_1, \ldots, z_n) = \frac{\det(z_i^{j+[j/2]-1} - z_i^{j-[j/2]+1})_{1\leq i,j\leq n}}{\det(z_i^j - z_i^{-j})_{1\leq i,j\leq n}}. \quad (50)$$

In order to find the other factors, we use the map $\Theta_N$. As done above with $\Upsilon$ one can prove that

$$\Theta_N |\Psi_N(\zeta_L, \bar{z}, \zeta_R)) = f_2(\zeta_L, \bar{z}, \zeta_R)|\Psi_N^{(o,c)}(\zeta_L, \bar{z}, \zeta_R))\rangle \quad (51)$$
where now $|\Psi_{N+1}^{(a,c)}(\zeta_L; \vec{z}, \zeta_R)\rangle$ is solution of the qKZ equations with mixed boundary conditions at size $N + 1$ and the parameter $\zeta_R$ plays the role of a bulk spectral parameter. We have also $\langle \nu \rangle = \langle \nu_{N+1}^{(a,c)} | \Theta_N \rangle$, and using again the results of [15] we obtain for the sum rule

$$\sum_N(\zeta_L; \vec{z}; \zeta_R) \propto \chi_N(\vec{z}, \zeta_R)\chi_{N+1}(\vec{z}, \zeta_L, \zeta_R). \quad (52)$$

To conclude the derivation we notice that these four factors exhaust the degree of $\sum_N(\zeta_L; \vec{z}; \zeta_R)$ (and do not have common factors), therefore

$$\sum_N(\zeta_L; \vec{z}; \zeta_R) = \chi_N(\vec{z})\chi_{N+1}(\zeta_L, \vec{z})\chi_{N+2}(\zeta_R, \zeta_L, \vec{z}). \quad (53)$$

Let us notice that eqs. (49,52) provides us also with a way of finding all the components of $|\Psi_N(\zeta_L; \vec{z}, \zeta_R)\rangle$ without the need of solving the qKZ equations for generic $q$. Let us show this in a concrete example at $N = 3$. Component wise eq. (49,52) read

$$\psi_{ccc} + \psi_{coco} = f_1\psi_{ccc(a,c)}; \quad \psi_{occc} + \psi_{occo} = f_1\psi_{occc}^{(a,c)}$$

$$\psi_{coc} + \psi_{coo} + \psi_{oooc} + f_1\psi_{coc}^{(a,c)} = f_2\psi_{coc}, \quad \psi_{ococ} + f_2\psi_{ococ} = f_2\psi_{ococ}^{(a,c)},$$

$$\psi_{ccoc} = f_2\psi_{ccc}, \quad \psi_{coco} = f_2\psi_{coco}, \quad \psi_{occc} = f_2\psi_{occc}, \quad \psi_{ocoo} = f_2\psi_{ocoo}.$$  

This system of equations is immediately solved

$$\psi_{ccc} = f_2\psi_{ccc}^{(a,c)}; \quad \psi_{coco} = f_2\psi_{coco}^{(a,c)}; \quad \psi_{occc} = f_2\psi_{occc}^{(a,c)}; \quad \psi_{ocoo} = f_2\psi_{ocoo}^{(a,c)}$$

$$\psi_{ccoc} = f_1\psi_{ccc} - f_2\psi_{ccc}^{(a,c)}; \quad \psi_{ocoo} = f_1\psi_{occc} - f_2\psi_{occc}^{(a,c)};$$

$$\psi_{coo} = f_2(\psi_{ccc}^{(a,c)} + \psi_{coco}^{(a,c)}) - f_1\psi_{ccc}^{(a,c)}; \quad \psi_{ccc} = f_2(\psi_{occc}^{(a,c)} + \psi_{ocoo}^{(a,c)}) - f_1\psi_{ocoo}^{(a,c)}.$$

giving an alternative way of computing the components of the ground state of the dense $O(1)$ model with open boundary conditions. We do not have a proof that this method allows to reconstruct the full ground state for a generic system size $N$, but we have checked it up to $N = 6$, checking as well that the result coincide with the ones found by solving the qKZ equation and then taking the limit $q^{1/2} \rightarrow e^{-2\pi i/3}$.

7 Conclusions and perspectives

In this paper we have considered the qKZ equations related to the Baxterization of the two boundaries Temperley Lieb algebra. By deriving a recursion relation satisfied by their solution we have been able to construct it explicitly. At the combinatorial point $q^{1/2} = e^{-2\pi i/3}$ the solution of the qKZ equations is also the ground state of the dense
O(1) loop model with open boundary conditions. We have shown how to reconstruct this ground state starting from the ground state of the same model with mixed boundary conditions known from [15]. In doing that we have also been able to prove the sum rule. In Section 4.1 we have briefly discussed the relations between our problem and the theory of representation of (doubly) affine Hecke algebras of type $C \lor C_N$. It would be interesting to develop further this observation, in particular making contact with a recent work of Kasatani [23] and hopefully with the theory of non symmetric Koornwinder-MacDonald polynomials. It would be also interesting to find integral formulae as already done in the case of other boundary conditions [8]. In this paper we have only tangentially touched problems related to the combinatorics behind the homogeneous limit of the ground state of the O(1) model. In fact it is not difficult to see that, making use of the mapping between model with different boundary conditions, one can easily explain certain observation reported in Table 2 of [10]. We plan to come back to such matters and to generalization to even more exotic boundary conditions in the immediate future.

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A Recursion relations: boundary

In this appendix we want to derive a boundary recursion relation similar to the bulk one, the motivation being twofold. On one side we show that leaving the boundary weight $\tau_R$ free, the boundary recursion relation maps the solution of qKZ at size $N$ to a solution of qKZ at size $N - 1$ and $\tau_R \rightarrow 1/\tau_R$. This provides a motivation to restrict to the case $\tau_R = 1$ (and similarly $\tau_L = 1$). On the other side the recursion we find provides the most convenient way to write the Laplace interpolation formula giving the unknown factor $\phi_N^{(R)}$ of $\psi_{\omega_0...\omega_0}$. From the analysis of Section 4.2 adapted to the case $\tau_R \neq 1$, it follows that if $z_N = q/\zeta_R$ or $z_N = \zeta_R/(qk)$ then all the components in which the point $N$ is not connected to the right boundary $R$ are zero. The other components, namely the ones having an arc going from $N$ to $R$ are in one to one correspondence with the extended link patterns with one bulk point removed. One simply removes the point $N$ and the arc from it to $R$. We call this map $p_R : \mathcal{H}_N^{(oo)} \rightarrow \mathcal{H}_{N-1}^{(oo)}$. In this section we
show that \( p_R \circ \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R) \) or \( z_N = \zeta_R/(qk); \zeta_R) \) satisfy a modified qKZ equation. Let us restrict ourselves to the case \( z_N = q/\zeta_R \), the other one being completely analogous. First of all we notice that \( p_R \circ \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R) \) satisfies the qKZ eqs. (5) for \( i < N - 1 \) and eq. (7), these properties being a trivial consequence of the fact that \( \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R) \) satisfies the same equation. Then let us consider the following operator

\[
\tilde{K}_R(\zeta_R|z) = R_{N-1}(1/x, q/\zeta_R) \tilde{K}_R(\zeta_R|z) R_{N-1}(q/\zeta_R, x).
\]

We see easily that

\[
\tilde{K}_R(\zeta_R|z_{N-1}) \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R) = \Psi_N(\ldots, 1/z_{N-1}, z_N = q/\zeta_R; \zeta_R) \quad (54)
\]

As one expected, \( \tilde{K}_R(z_{N-1}) \) preserves the image of \( f_R \) (which are the components not identically zero when \( z_N = q/\zeta_R \)) therefore if we let it act only on such a subspace we see that it assumes the following form

\[
\tilde{K}_R(\zeta_R, z_{N-1}) = \frac{(z - q^4\zeta_R - q^3(\tilde{k} - \tilde{\zeta}_R))}{(\zeta_R - q^4z)(\tilde{k} - \tilde{\zeta}_R)} \left( \frac{z - \tilde{\zeta}_R}{\zeta_R - 1}(z - \tilde{k}/\tilde{\zeta}_R - 1) \right)^2
\]

where \( \tilde{f}_R = 1/\tau_R f_R e_{N-1}, \tilde{\zeta}_R = q^3/\zeta_R, \tilde{\tau}_R = 1/\tau_R \) and \( \tilde{k} \) is the same as \( k \) in which we substitute \( \tau_R \) with \( \tilde{\tau}_R \). Apart from the multiplicative factor in front of it, this has just the same form of the scattering matrix in eq. (2). This is consistent with the fact that the algebra of operators \( f_L, e_i, N-1 \) and \( \tilde{f}_R \) is a \( TL_{N-1}^{(0,0)} \) with boundary loop weight \( \tilde{\tau}_R \). From now on we will restrict ourselves to the case when \( \tau_R = \tau_L = 1 \). This implies in particular \( \tilde{k} = k = q \) and we can write

\[
\tilde{K}_R(\zeta_R|z) = \frac{(z - q\zeta_R)(z - q^2/\zeta_R)}{(1 - qz\zeta_R)(1 - zq^2/\zeta_R)} K_R(q^3/\zeta_R|z) \quad (56)
\]

We conclude therefore that \( p_R \circ \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R) \) satisfies the following set of equations

\[
\tilde{R}_{i<N-1}(z_{i+1}, z_i) p_R \circ \Psi_N(\ldots, z_i, z_{i+1}, \ldots, z_N = q/\zeta_R) = p_R \circ \Psi_N(z_{i+1}, z_i, \ldots, z_N = q/\zeta_R); \quad (57)
\]

\[
u_R(1/z_{N-1}) K_R(q^3/\zeta_R|z_{N-1}) p_R \circ \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R) = p_R \circ \Psi_N(\ldots, 1/z_{N-1}, z_N = q/\zeta_R; \zeta_R); \quad (58)
\]

where \( u_R(z) = (\frac{1}{q^2} - \zeta_R) \left( \frac{qz - \frac{8k}{q}}{q} \right) \). Now let us suppose that we can find a function \( g_R(z) \) which is invariant under \( z \to 1/z \) and such that

\[
u_R \left( \frac{z}{q^2} \right) g_R \left( \frac{z}{q^2} \right) = \frac{\left( \frac{1}{q^2} - \zeta_R \right) \left( \frac{qz - \frac{8k}{q}}{q} \right)}{\zeta_R} g_R \left( \frac{z}{q^2} \right) = 1 \]

25
then it is straightforward to see that

\[
\prod_{i=1}^{N-1} \left( \frac{1}{u_R(z_i)g_R(z_i)} \right) p_R \circ \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R)
\]

is solution of the unmodified qKZ equation with boundary parameter \( \zeta_R = q^3/\zeta_R \). If we require that the solution \( g(z) \) of eq. (60) is a Laurent polynomial in \( z \), then we find that the parameter \( s \) must be of the form \( s = q^{-3/n} \) and it is easy to check that the solution is

\[
g(z) = \prod_{j=1}^{n-1} (q^{3j/n-1}z - \zeta_R)(q^{3j/n-1}/z - \zeta_R),
\]

therefore, calling

\[
G_R(z_1, \ldots, z_{N-1}) = \prod_{i=1}^{N-1} u_R(z_i)g_R(z_i) \propto \prod_{i=1}^{N-1} \prod_{j=1}^{n}(q^{3j/n-1}z_i - \zeta_R)(q^{3(j-1)/n-1}/z_i - \zeta_R)
\]

we can rewrite eq. (61) as follows

\[
p_R \circ \Psi_N(\ldots, z_{N-1}, z_N = q/\zeta_R; \zeta_R) = k_R(\zeta_L, \zeta_R) G_R(z_1, \ldots, z_{N-1}) \Psi_{N-1}(\ldots, z_{N-1}; q^2/\zeta_R)
\]

where \( k_R(\zeta_L, \zeta_R) \) is an unknown function of \( \zeta_L \) and \( \zeta_R \), that we can fix for \( s = q^{-3/2} \) to be

\[
k_R(\zeta_L, \zeta_R) = \left( s\zeta_R - \frac{q^2}{\zeta_L} \right) \left( \frac{q}{\zeta_R} - q \right),
\]

by using minimality arguments similar to the ones employed in Section 5. Now we want to change a bit perspective on the role of the boundary parameter \( \zeta_R \); as a consequence of the bulk recursion relations, it follows that the function \( \phi_N^R(\zeta_L; \zeta_R) \) is a Laurent polynomial in \( \zeta_R \), invariant under \( \zeta_R \to q^3/\zeta_R \) and degree width \( \lfloor \frac{N-1}{2} \rfloor \). This means that we can use the boundary recursion relation in order to construct \( \phi_N^R \) by using Lagrange interpolation in \( \zeta_R \) and the symmetries of \( \phi_N^R \) under exchange of the \( z_i \)s. Forgetting a constant normalization factor that we cannot fix, the result reads

\[
\phi_N^{(R)}(\zeta_L; \zeta_R) = \sum_{i=1}^{N} \prod_{k \neq i} \left( \frac{\zeta_R + \frac{q^2}{\zeta_R} - q^2z_k - \frac{q}{z_k}}{q^2z_i + \frac{q}{z_i} - q^2z_k - \frac{q}{z_k}} \right) \phi_{N-1}^{(R)}(\zeta_L; \zeta_R; q/z_i)
\]

It is remarkable that this expression turn out not only to be a Laurent polynomial in the spectral parameters \( z_i \), but it is also endowed with all the symmetry properties expected for \( \phi_N^{(R)} \). We have checked for that for systems up to size \( N = 6 \), starting from the above expression one gets a solution of the qKZ equations.
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