Certifying Temporal Correlations

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Self-testing has been established as a major approach for quantum device certification based on experimental statistics with minimal assumptions. However, despite more than 20 years of research effort, most of the self-testing protocols are restricted to spatial scenarios (Bell scenarios), without many temporal generalizations known. Under the scenario of sequential measurements performed on a single quantum system, we build on previous works using semi-definite programming (SDP) techniques to bound sequential measurement inequalities. For such SDPs, we show that the optimizer matrix is unique and, moreover, this uniqueness is robust to small deviations from the quantum bound. Furthermore, we consider a generalized scenario in the presence of quantum channels and draw analogies to the structure of Bell and sequential inequalities using the pseudo-density matrix formalism. These analogies allow us to show a practical use of maximal violations of sequential inequalities in the form of certification of quantum channels up to isometries.

Introduction – Newton, in his Principia Mathematica, defined time to be absolute and uniform throughout the universe. This notion was altered in 1905 by Einstein in his Special Theory of Relativity when he showed that inertial reference frames have their own relative clocks, thus placing time and space on equal footing. More recently, such analogies in space-time structure have also been explored in quantum time crystals [1] where the atoms in a crystal are arranged periodically in both space and time. From the perspective of quantum correlations, similar ideas have been explored with respect to spatial and temporal scenarios [2–5]. These examples also illuminate a symmetry in the role played by space and time in our understanding of the universe. The goal of this work is to explore the similar structure for the temporal analogue of self-testing, which emerges from temporal quantum correlations.

Since its inception in [6], self-testing [7] has become a prominent technique to certify quantum states in the device independent scenario [8–12], with extensions in literature to semi-device independent [13–17] and single party contextuality based scenarios [18–20]. Broadly speaking, self testing is a technique to infer the hidden quantum state and measurements that lead to extremum quantum correlations in a black-box scenario. In the prominent scenario of spatially separated parts of an entangled state, self testing is made possible due to properties of Bell inequalities [21] i.e. quantum realizations that achieve maximal violations of Bell inequalities are unique up to local isometries. Applications of self testing in such scenarios have been explored for device-independent (DI) randomness generation [22–27], DI quantum cryptography via quantum key distribution [28–31] and delegated quantum computing [32–34] among others. All these applications inherently share the idea of quantum state certification without explicitly using self-testing protocols making them essential for the progress of quantum technologies. Also, self testing has recently garnered interest as a theoretical tool with applications in quantum complexity theory [35].

However, most of the applications of self testing are limited to scenarios with entanglement shared between spatially separated observers without any temporal analogue. A scenario involving sequential measurements on a single quantum system has already been explored from the perspective of self-testing in [18] where non-contextuality inequalities [36] supplant the role of Bell inequalities. However, sequential measurements for this purpose need to form a context or, in other words, they need to commute. This constraint on the sequential measurements is lifted in [37] by considering sequential correlation inequalities, such as the Leggett-Garg inequality [38], where the measurements performed in the sequence do not necessarily commute and the classical bound is applicable for all theories that obey macro-realism. In this unconstrained scenario, a semi-definite programming (SDP) based method can be used to find quantum bounds of sequential correlation-based inequalities. An extension of sequential measurements to Bell scenarios is considered in [39] where an infinite hierarchy of SDPs is shown to bound the set of sequential correlations. The sequential measurement scenario also allows for the interpretation of such correlations as temporal correlations in analogy with spatial correlations since measurements are done on the same state at different times. Temporal correlations, as obtained in the sequential measurement scenarios,
have yet to find applications in quantum technologies save in few cases such as [40] where authors show that genuine temporal correlations can be used to certify the minimum dimension of the underlying quantum state. In [41] authors consider the prepare-transform-measure scenario to certify a single qubit preparation (state) and intermediate measurements up to global unitary degree of freedom based on sequential correlation based witnesses. More recently, in [42], a sequential measurement scenario has been considered where maximal violations of temporal inequalities were used to certify arbitrary outcome measurements under certain assumptions regarding the underlying state. A radical approach is adopted in [43] where the authors show how time evolution of a qubit can be viewed as temporal teleportation using Pseudo-density matrix (PDM) formalism [44] in analogy to the spatial teleportation protocol using entangled states [45]. In [5], authors study the geometry of two sequential Pauli measurements on a qubit in a more general scenario where the effect of an arbitrary quantum channel between the two sequential measurements is considered.

In this work, we study correlations arising from sequential measurements on a single party system and its generalizations to scenarios with quantum channels. Since such correlations cannot be used to certify the underlying state or measurements due to the single system isometries involved, we identify the correlation matrix itself as the candidate for certification. Furthermore, we consider the generalized scenario where quantum channels act between the sequence of measurements and use the PDM formalism to put spatial (Bell) and temporal (sequential) inequality violations on the same footing. The PDM based approach helps to identify local isometries in the temporal setting and also helps to formulate statements about the underlying channels that lead to maximal violations of sequential inequalities.

I. BOUNDING TEMPORAL CORRELATIONS VIA SDP

Consider the scenario of sequential measurements performed on a single qudit state. In order to compute quantum bounds on the inequalities constructed from such correlations between sequential measurements of binary valued ($\pm 1$) observables $\{A_i\}$ a semi-definite programming (SDP) based strategy was proposed in [37]. Optimization is performed for temporal inequalities constructed from correlations between sequential measurements on a single quantum state $\rho$. For an inequality $C = \sum_{ij} \lambda_{ij} X_{ij}$ where $\lambda_{ij}$ are the coefficients corresponding to the term $X_{ij} = \langle A_i A_j \rangle_{seq}$ with

$$\langle A_i A_j \rangle_{seq} = \frac{1}{2} \left( Tr (\rho A_i A_j) + Tr (\rho A_j A_i) \right)$$

the semi-definite program is given by:

maximize : $\sum_{ij} \lambda_{ij} X_{ij}$,  
subject to : $X = X^T \succeq 0$ and $X_{ii} = 1 \forall i$ (3)

The constraint $X \succeq 0$ follows from the fact that $X$ is the real part of $Y = Tr [\rho (A_i A_j)]$ and $v^T Y v \geq 0$ for any real vector $v$. Consider a special case of such an inequality, called an N-cycle inequality (4), given by

$$S_N \equiv \sum_{i=1}^{N-1} \langle A_i A_{i+1} \rangle_{seq} - \langle A_N A_1 \rangle_{seq} \leq N - 2 \quad (4)$$

with $N \geq 3$ and the classical bound $N - 2$ obeyed by all macro-realistic theories. Invoking the strong duality for the corresponding SDP gives the quantum bound $S_N \leq N \cos \left( \frac{\pi}{N} \right)$.

Remark 1. Reinterpreting the correlation term $X_{ij}$ as an inner product of unit vectors $\{x_i\}$, we can write $X_{ij} = \langle x_i, x_j \rangle$ thus obtaining $\{x_i\}$ as the columns of the matrix $\sqrt{X}$. As a consequence of this reinterpretation, for every PSD $X$ one can find a set $\{x_i\}$ and a set of binary ($\pm 1$) outcome observables $\{A_i\}$ such that

$$\langle A_i A_j \rangle_{seq} = Tr \left[ \frac{1}{2} \rho (A_i A_j + A_j A_i) \right] = \langle x_i, x_j \rangle$$

for all quantum states $\rho$. Thus, for every quantum state $\rho$ there exist observables $\{A_i\}$ such that $X_{opt} = \langle A_i A_j \rangle_{seq} = Tr \left[ \frac{1}{2} \rho (A_i A_j + A_j A_i) \right]$ which maximally violate the inequality (4). Further, in the 2-qubit case it can be seen that taking $A_i = \vec{a}_i, \vec{a}_j$ gives $\langle A_i A_j \rangle_{seq} = \vec{a}_i \cdot \vec{a}_j$ implying that the correlations do not depend on the underlying quantum state.

In section II, we show the uniqueness of the optimizer matrix $X^{opt}$ for the N-cycle inequality (4) for all $N \geq 3$. One interesting aspect of this approach is the fact that in the temporal scenario a single SDP suffices to find a tight upper bound in the general case which is in contrast with the infinite (NPA) hierarchy of SDPs required in the general case to tightly bound Bell inequalities in the spatial correlation scenario as shown in [46, 47]. This significant increase in complexity can be attributed to the fact that the observables acting on the spatially separated halves of the entangled state are required to commute but no such restriction is placed on observables acting on a single quantum system at distinct time points in the sequential measurement scenario.
II. CERTIFICATION OF TEMPORAL CORRELATIONS USING OPTIMIZER MATRIX

For the purposes of this work, we define robust certification of temporal correlations as follows:

(\epsilon,\tau) robust certification of temporal correlations: Given 2 outcome observables \(\{A_i\}_{i=1}^n\) and physical set of sequential correlations \(\{(A_i,A_{i+1})\}_{i=1}^{\tilde{N}}\), they give an \((\epsilon,\tau)\) certificate of reference temporal correlations \(\{(\tilde{A}_i,\tilde{A}_{i+1})\}_{i=1}^{\tilde{N}}\) if the matrices \(X\) and \(\tilde{X}\) defined via \(X_{ij} = \langle A_i A_j \rangle\) (similarly for \(\tilde{X}\)) are close in Frobenius norm distance s.t.

\[
|X - \tilde{X}| \leq \mathcal{O}(\epsilon').
\]

Theorem 1. The optimizer matrix \(X^{\text{opt}}\) that maximizes the objective function \(C = \sum_{ij} \lambda_{ij} X_{ij}\) in (3) for the \(N\)-cycle inequality defined in (4) is unique for all \(N \geq 3\).

Proof. The proof has been deferred to Appendix section C1.

We mentioned earlier that the matrix \(X\) that is being optimized over is PSD and gives us the set of unit vectors \(\{x_i\}\) as the columns of \(\sqrt{X}\); however, these \(\{x_i\}\) vectors are not directly relatable to any state or measurements. Noting that \(X\) is the real part of PSD matrix \(Y\), where \(Y_{ij} = \text{Tr}(\rho A_i A_j)\), allows us to introduce the set of unit vectors \(\{y_i\}\) as columns of \(\sqrt{Y}\) matrix. In the special case when \(\rho = |\psi\rangle\langle\psi|\) is a pure state, we can write \(y_i = A_i|\psi\rangle\).

For \(X^{\text{opt}}\), we thus have

\[
X^{\text{opt}}_{ij} = \text{Tr} \left[ \frac{1}{N} \tilde{\rho} (\tilde{A}_i \tilde{A}_j + \tilde{A}_j \tilde{A}_i) \right] = \text{Re} \left\{ \text{Tr} \left[ \tilde{\rho} \tilde{A}_i \tilde{A}_j \right] \right\} = \text{Re} \left\{ \langle \tilde{\psi}| \tilde{A}_i \tilde{A}_j |\psi\rangle \right\} = \text{Re} \left\{ \langle \tilde{y}_i| \tilde{y}_j \rangle \right\}
\]

where in the third step we have assumed that \(\tilde{\rho}\) is a pure state, since \(X^{\text{opt}}\) can be obtained using any quantum state and suitable 2-outcome measurements. However, note that while defining the set \(\{\tilde{y}_i\}\), we have a global isometry such that there could be another set of unit vectors \(\{\tilde{y}_i'\}\), satisfying \(\text{Re} \left\{ \langle \tilde{y}_i'| \tilde{y}_j \rangle \right\} = \text{Re} \left\{ \langle U \tilde{y}_i'| U \tilde{y}_j \rangle \right\}\) with \(UU^\dagger = I\). This leads to the following.

Corollary 1. For the optimizer matrix \(X^{\text{opt}}\) which maximizes an \(N\)-cycle inequality of the kind (4), for every set \(\{y_i\}_{i=1}^N\) which satisfies \(X^{\text{opt}}_{ij} = \text{Re} \left\{ \langle y_i| y_j \rangle \right\}\), we have a global isometry \(U\) and a reference set \(\{\tilde{y}_i\}\) such that

\[
\text{Re} \left\{ \langle \tilde{y}_i| \tilde{y}_j \rangle \right\} = \text{Re} \left\{ \langle U y_i| U y_j \rangle \right\}
\]

Furthermore this implies the existence of reference measurements \(\{A_i\}_{i=1}^N\) and pure state \(|\psi\rangle\) such that \(\tilde{y}_i = A_i|\psi\rangle\).

Note that in Bell inequality based self testing scenarios uniqueness up to local isometries of \(A_i|\psi\rangle\) implies self test of the measurement \(A_i\); however, in the temporal scenario due to the single system global isometry involved we can only claim that uniqueness of \(X^{\text{opt}}\) fixes \(\text{Re} \left\{ \langle y_i| y_j \rangle \right\}\).

Theorem 2. Robustness: Consider the SDP (3), given that matrix \(X^{\text{real}}\) achieves a near-optimal value for the objective function \(C = \sum_{ij} \lambda_{ij} X_{ij}\) such that

\[
|\sum_{ij} \lambda_{ij} X_{ij}^{\text{opt}} - \sum_{ij} \lambda_{ij} X_{ij}^{\text{real}}| \leq \epsilon
\]

and \(X^{\text{opt}}\) is unique, then we can upper bound the Frobenius-norm distance between \(X^{\text{opt}}\) and \(X^{\text{real}}\) as

\[
|X^{\text{opt}} - X^{\text{real}}| \leq \mathcal{O}(\epsilon)
\]

Proof. The proof has been deferred to Appendix section C1.

The multiplicative constant going with \(\mathcal{O}(\epsilon)\) certification guarantee in Theorem 2 can be approximated by numerical SWAP based methods. Such SWAP based techniques along with SDP based methods were used to estimate constants in CHSH and CGLMP inequality based robust self tests in [48]. Also, analytical robust self testing bounds for certain Bell inequalities have been obtained in [49, 50]. The sequential measurement scenario considered here is closer in spirit to the scenario considered in [20] where numerical SWAP based techniques were adapted for robust self testing of a single quantum system.

Remark 2. It follows from Theorem 2 that near maximal violation of (4) by the set \(\{\langle A_i, A_{i+1}\rangle\}_{i=1}^{\tilde{N}}\) such that

\[
\left| \sum_{i=1}^{\tilde{N}-1} \langle A_i A_{i+1} \rangle - \langle A_N A_1 \rangle \right| - \beta_q \leq \epsilon
\]

(where \(\beta_q\) is the quantum bound for (4)) is an \((\epsilon,1)\) certificate for the reference set \(\{\langle A_i, A_j \rangle\}_{i,j}\) with \(\langle A_i, A_j \rangle_{\text{seq}} = X^{\text{opt}}_{ij}\).

III. GENERALIZED SCENARIO WITH QUANTUM CHANNELS

In this section we generalize the sequential measurement scenario considered in the previous part by introducing a quantum channel between the measurements of Alice and Bob as shown in Fig. 1. In the case of arbitrary channels \(E_{A|B}\) acting in between the time points, the sequential correlation is

\[
\langle A_m A_n \rangle_{\text{seq}} = \frac{1}{2} \text{Tr} \left[ (A_m \rho_A + \rho_A A_m) E_{A|B}(A_n) \right],
\]

as a result of which the matrix \(X_{ij} = \langle A_i A_j \rangle_{\text{seq}}\) is not generally PSD in this scenario. Hence, one cannot use the SDP method described earlier in Sec. 1 to bound sequential measurement inequalities.

The sequential measurement scenario can be viewed from the perspective of Pseudo Density Matrix (PDM) formalism (see Appendix section A for a review) by
considering the measurement events in a sequence and writing the corresponding PDM. The PDM formalism in general is more powerful since it can also take into account time evolution (via quantum channels) of the quantum state between the two sequential measurement events. Consider a 2-dimensional quantum state $\rho_A$ with sequentially measured events being performed at time points $t_A$ and $t_B$, additionally the channel $\mathcal{E}_{A|B}$ acts on $\rho_A$ mapping operators from state space $\mathcal{H}_A$ at $t_A$ to state space $\mathcal{H}_B$ at $t_B$. Then it can be shown [5] that the PDM $R_{AB}$ for this scenario is given by

$$R_{AB} = (\mathcal{I}_A \otimes \mathcal{E}_{A|B})\left\{\rho_A \otimes \frac{I}{2}, \frac{1}{2} \sum_{i=0}^{3} \sigma_i \otimes \sigma_i\right\} \quad (8)$$

where $\mathcal{I}_A$ is the identity superoperator.

**Lemma 1.** In this generalized scenario

$$\text{Tr}[(A_m \otimes A_n)R_{AB}] = (A_m A_n)_{\text{seq}}$$

**Proof.** The proof has been deferred to Appendix section C2.

The tensor structure introduced in Lemma 1 allows for certain local isometries to act in the temporal sense as

$$\text{Tr}[(A_m \otimes A_n)R_{AB}] = \text{Tr}\left((A_m \otimes A_n)(\mathcal{I}_A \otimes \mathcal{E}_{A|B})\left\{\rho_A \otimes \frac{I}{2}, \frac{1}{2} \sum_{i=0}^{3} \sigma_i \otimes \sigma_i\right\}\right)$$

$$= \text{Tr}\left((V A_m V^\dagger \otimes U A_n U^\dagger)(\mathcal{I}_A \otimes \tilde{\mathcal{E}}_{A|B})\left\{V \rho_A V^\dagger \otimes \frac{I}{2}, \frac{1}{2} \sum_{i=0}^{3} \sigma_i \otimes \sigma_i\right\}\right) \quad (9)$$

where $V$ is restricted to the set of Pauli matrices i.e. $V \in \{X,Y,Z\}$ acting locally at time point $t_A$. $U$ could be any unitary operator acting locally at $t_B$ and $\tilde{\mathcal{E}}_{A|B}$ is the channel with transformed Kraus operators $\{K_i\} \rightarrow \{UK_i V^\dagger\}$ as those of $\mathcal{E}_{A|B}$ (see Appendix section D for details). Thus, the transformation of the channel $\mathcal{E}_{A|B}$’s Kraus operators along with corresponding rotation of observables at times $t_A$ and $t_B$ does not change the correlation value. This local isometry in time is reminiscent of the local isometries present in the scenario with spatially separated entangled states and observables.

Here, we will use the PDM formalism to explore the maximal violation of sequential inequalities in the generalized scenario. Going back to $N$-cycle sequential inequality (4), using Lemma 1 we can write:

$$\text{Tr}\left[\left(\sum_{i=0}^{N-2} A_i \otimes A_{i+1} - A_{N-1} \otimes A_0\right)R_{AB}\right] \leq N - 2 \quad (10)$$

We note that (10) possesses a structure similar to the Bell inequalities under entanglement-based scenarios where the tensor product appears due to spatially separated Hilbert spaces.

**Remark –** Considering the spatial analogue of (10) where the measurement events are performed on spatially separated qubits with $N = 4$ shows that the PDMs maximally violating the inequality are the maximally entangled Bell states (which are valid density matrices).

**Case: $N = 3$** – In this case the N-cycle inequality is the Leggett-Garg inequality [38]

$$S_3 \equiv (A_1 A_2)_{\text{seq}} + (A_2 A_3)_{\text{seq}} - (A_3 A_1)_{\text{seq}}, \quad (11)$$

we will consider maximal violation of this inequality in the presence of channels. Throughout the draft we highlighted the various degrees of freedom involved in obtaining correlations from sequential measurements on a quantum system. We saw that in the simple case without channels being present, sequential correlations depend only on the angle between the observables and not on the particular direction of the observable and not even on the underlying quantum state. However, in the generalised scenario with channels other isometries also come into play thus allowing us to choose non ideal angles to obtain maximal violation of the 3-cycle sequential inequality (11). Following this discussion, we do not base our self testing statement on certification of the quantum state or of the measurements but rather on the quantum channel acting in between the
sequence of measurements. A protocol for channel certification has been previously described in [51] where the physical channel was compared to a target reference channel via the action of the channel on the maximally entangled state thus requiring the physical input state to be an entangled one. On the contrary, our channel certification scheme does not require entanglement but utilizes sequential correlations instead.

**Theorem 3.** Temporal certification statement: In a sequential measurement setting with channel $\mathcal{E}_{AB}$ acting between the sequence of 2-dim. measurements $\{A_i\}$ on 2-dim. state $\rho$, maximal violation of the 3-cycle sequential inequality (11) implies that $\mathcal{E}_{AB}$ is a 1-qubit Pauli channel with Kraus rank 1.

**Proof.** The proof has been deferred to Appendix section C 3. □

IV. DISCUSSION

Motivated by the SDP based formulations of bounding temporal correlations, we show uniqueness of $X^{opt}$ as well as robustness of this uniqueness property. Further, we show that uniqueness of $X^{opt}$ allows for existence of sets of unit vectors $\{y_i\}$ which are unique up to a global isometry. Next, we considered the generalized scenario with quantum channels and connected sequential correlations in this scenario with PDMs. This connection allows us to establish analogies between the structure of spatial and temporal inequalities along with local isometries involved. Further, we show maximal violation of sequential inequality in the generalised scenario implies unitarity of the channel. This result could prove to be useful in developing quantum process tomography protocols [52, 53] which make the assumption of working with unitary channels.

We made use of the $N = 3$ case for our channel certification result; this result could be extended to non-cyclic sequential inequalities. We believe that our work is experimentally testable. Also, robustness of the channel certification result remains to be explored which we leave for future work.

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possible negative eigenvalues. As a measure of causality, authors in \[44\] introduced a valid density matrix, however; in the presence of causal or temporal correlations.

Appendix A: Pseudo Density Matrices

The density matrix of a system contains information of all possible Pauli observables acting on the system thus the density matrix has information of all correlations between spatially separated subsystems. The Pseudo-Density Matrix (PDM) formalism was introduced in \[44\] by extending this analogy to account for causal correlations between observables acting on the same subsystem at different time points. The PDM of an \(n\)-measurement event is defined as

\[
R = \frac{1}{2^n} \sum_{i_1=0}^{3} \cdots \sum_{i_n=0}^{3} \langle \{\sigma_{i_j}\}_{j=1}^{n} \rangle \otimes \sigma_{i_j}
\]

where \(\sigma_0 = I\), \(\sigma_1 = X\), \(\sigma_2 = Y\), and \(\sigma_3 = Z\). The sub-indices \(j\) go over all measurement events which could be done in a sequence on a single qubit, on spatially separated qubits or a combination of both thus accounting for both spatial and causal correlations arising due to the \(n\) measurement events. The factor \(\langle \{\sigma_{i_j}\}_{j=1}^{n} \rangle \) denotes the correlation term corresponding to the size-\(n\) measurement events. The properties retained by the PDM \(R\) are hermiticity and unit trace. If all the measurement events are performed on distinct qubits then \(R\) is positive semidefinite (PSD) and thus a valid density matrix, however; in the presence of causal or temporal correlations \(R\) is not necessarily PSD with possible negative eigenvalues. As a measure of causality, authors in \[44\] introduced \(f_{tr}(R) = \|R\|_{tr} - 1\). Let us expound on PDMs using the following example.

Example: Consider a qubit system on which 2-measurement events \(E_1\) and \(E_2\) are made. Next, on calculating \(\langle \{\sigma_{i_1}, \sigma_{i_2}\} \rangle\) associated with the 2 events, we observe that only the following terms are non-zero,

\[
\langle \{X, X\} \rangle = 1, \quad \langle \{Y, Y\} \rangle = 1, \quad \langle \{Z, Z\} \rangle = 1, \\
\langle \{I, I\} \rangle = 1, \quad \langle \{Z, I\} \rangle = 1, \quad \langle \{I, Z\} \rangle = 1.
\]
Subsequently, the PDM $R_{ex}$ for this sequence of 2 events on a single qubit system is

$$R_{ex} = \sum_{i_1} \sum_{i_2} \langle \{\sigma_{i_1}, \sigma_{i_2}\} \rangle (\sigma_{i_1} \otimes \sigma_{i_2})$$

$$= \begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 0 & 1/2 & 0 \\
    0 & 1/2 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}$$

with eigenvalues 0, $1, \frac{1}{2}, -\frac{1}{2}$. We see that $R_{ex}$ has a negative eigenvalue since the 2 measurement events $E_1$ and $E_2$ are causally related by virtue of acting on the same qubit system.

**Appendix B: Semidefinite programming basics**

An $n \times n$ matrix $X$ is said to be positive semidefinite (PSD) if and only if $v^T X v \geq 0$ for any $v \in \mathbb{R}^n$, denote PSD matrix $X$ by $X \succeq 0$. It can be easily checked that the set of PSD matrices forms a convex cone. Then, a semidefinite program (SDP) is an optimization problem of the form

$$\begin{align*}
\text{maximize : } & \quad \text{Tr}(CX) \\
\text{subject to : } & \quad \text{Tr}(A_i X) = b_i, i = 1, 2, \ldots, m \\
& \quad X \succeq 0,
\end{align*}$$

where we note that the objective function $\text{Tr}(CX)$ is linear in $X$ and the $m$ linear equation constraints are given by $m$ matrices $A_1, \ldots, A_m$ and the $m$-vector $b$. Referring to the problem above as the primal (P) SDP, we can define the dual of it as given below.

**Semidefinite programming duality:** The dual (D) problem of the above SDP (SDD) is defined to be

$$\begin{align*}
\text{minimize : } & \quad \sum_{i=1}^m y_i b_i \\
\text{subject to : } & \quad \sum_{i=1}^m y_i A_i + S = C \\
& \quad S \succeq 0,
\end{align*}$$

The following theorem summarizes relevant results from SDP duality theory.

**Theorem 4.** Consider a pair of primal (P) and dual (D) SDPs. The following holds:

i (Complementary slackness) Let $X, (y, Z)$ be a pair of primal-dual feasible solutions for (P) and (D), respectively. Assuming that $p^* = d^*$ we have that $X, (y, Z)$ are primal-dual optimal if and only if $(X, Z) = 0$.

ii (Strong duality) Assume that $d^* > -\infty$ (resp. $p^* < +\infty$) and that (D) (resp. (P)) is strictly feasible. Then $p^* = d^*$ and furthermore, the primal (resp. dual) optimal value is attained.

**Theorem 5.** [54] Let $Z^*$ be a dual optimal and nondegenerate solution of a semidefinite program. Then, there exists a unique primal optimal solution for that SDP.

The notion of dual nondegeneracy is given by the following definition.

**Definition 1.** (Dual nondegeneracy) Let $Z^*$ be an optimal dual solution and let $M$ be any symmetric matrix. If the homogeneous linear system

$$MZ^* = 0,$$
\[ \text{Tr}(MA_i) = 0 \quad (\forall i \in [m]), \]  

(B4)

only admits the trivial solution \( M = 0 \), then \( Z^* \) is said to be dual nondegenerate.

**Theorem 6.** [18] Consider a pair of primal/dual SDPs \((P)\) and \((D)\), where the primal/dual values are equal and both are attained. Furthermore, assume that the set of feasible solutions of \((P)\) is contained in some compact subset \( U \subseteq \mathbb{S}^n \). Let \( \mathcal{P} \) be the set of primal optimal solutions and \( d \) be the singularity degree of \((P)\) defined (in [55]) as the least number of facial reduction steps required to make \((P)\) strictly feasible, we have that

\[ \text{dist}(\tilde{X}, \mathcal{P}) \leq O(\epsilon^{2^{-d}}), \]

for any primal feasible solution \( \tilde{X} \) with \( p^* - \epsilon \leq \langle C, \tilde{X} \rangle \).

**Appendix C: Proofs**

1. **Proofs of Theorems 1 and 2 from Main Text (MT)**

   a. **Proof of Theorem 1**

   In order to prove Theorem 1, we will first establish a couple of Lemmas.

   **Lemma 2.** The dual optimal solution for the SDP in (3) is given by

   \[ W_N = \cos \left( \frac{\pi}{N} \right) I_N + 0.5 \left( e_1 e_N^T + e_N e_1^T \right) - \sum_{i=1}^{N-1} 0.5 \left( e_i e_{i+1}^T + e_{i+1} e_i^T \right), \]  

   (C1)

   where \( I_N \) is an \( N \times N \) identity matrix and \( e_i \) is an \( N \times 1 \) column vector with 1 at the \( i \)-th place and 0 elsewhere.

   **Proof.** To prove that the matrix in \( C1 \) is dual-optimal, we need to prove that

   1. \( W_N \) is dual-feasible, and
   2. Corresponds to the optimal value, i.e., \( N \cos \left( \frac{\pi}{N} \right) \).

   The dual of the SDP in \( (3) \) is given by

   \[ \min \sum_i y_i \]  

   (C2)

   such that

   \[ \sum_{i=1}^N y_i e_i e_i^T - \Lambda \succeq 0, \]  

   (C3)

   where \( \Lambda \) is the matrix \(-0.5 \left( e_1 e_N^T + e_N e_1^T \right) + \sum_{i=1}^{N-1} 0.5 \left( e_i e_{i+1}^T + e_{i+1} e_i^T \right) \). The matrix in \( C3 \) is equal to the matrix in \( C1 \) for \( y_i = \cos \left( \frac{\pi}{N} \right) \forall i \in \{1, 2, \cdots, N\} \). Since \( \sum_i y_i \) is equal to \( N \cos \left( \frac{\pi}{N} \right) \), it remains to be shown that \( W_N \) is positive semidefinite for all \( N \geq 3 \). We proceed by rewriting \( W_N \) as

   \[ W_N = c_N I_N + 0.5 \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 \ -1 & 0 & -1 & \cdots & 0 \ 0 & -1 & 0 & \cdots & 0 \ \vdots & \vdots & \vdots & \ddots & \vdots \ \vdots & \vdots & \vdots & \ddots & \vdots \ 1 & 0 & \cdots & \cdots & -1 \end{bmatrix} \]

   (C4)

   call this matrix \( T_N \)
where \( c_N = \cos \frac{\pi}{N} \). Note that \( W_N \) is PSD if all eigenvalues of \( T_N \) are greater than or equal to \(-2c_N\) which is what we show next. In order to find the eigenvalues of \( T_N \), consider the determinant

\[
|T_N - \lambda I_N| =
\begin{vmatrix}
-\lambda & -1 & 0 & \cdots & \cdots & 1 \\
-1 & -\lambda & -1 & \cdots & \cdots & 0 \\
0 & -1 & -\lambda & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & \cdots & -1 & -\lambda \\
\end{vmatrix}
\tag{C5}
\]

The matrix above is an ordinary tridiagonal matrix and the determinant can be evaluated using the following formula involving multiplication of \( 2 \times 2 \) matrices

\[
|a_1 b_1 \cdots c_0 | = (-1)^{n+1}(b_n \cdots b_1 + c_{n-1} \cdots c_0) + \text{Tr}
\left[
\begin{pmatrix}
a_n & -b_{n-1}c_{n-1} \\
1 & 0 \\
\end{pmatrix}
\cdots
\begin{pmatrix}
a_2 & -b_1c_1 \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
a_1 & -b_nc_0 \\
1 & 0 \\
\end{pmatrix}
\right]
\tag{C6}
\]

Plugging the values of matrix elements for our case gives

\[
|T_N - \lambda I_N| = (-1)^{N+1}(2(-1)^{N-1}) + \text{Tr}
\left[
\begin{pmatrix}
-\lambda & -1 \\
1 & 0 \\
\end{pmatrix}
\right]^N
\tag{C7}
\]

Note that the matrix \( P \equiv \begin{pmatrix}
-\lambda & -1 \\
1 & 0 \\
\end{pmatrix} \) has eigenvalues \( \mu = \frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} \) and \( \frac{1}{\mu} \). Also, \( P \) is not diagonalizable for \( \lambda = 2 \) which is an eigenvalue of \( T_N \) for odd \( N \). However, \( W_N \) remains PSD in this case.

For \( \lambda \neq 2 \), we have

\[
\text{Tr}
\left[
\begin{pmatrix}
-\lambda & -1 \\
1 & 0 \\
\end{pmatrix}
\right]^N = \mu^N + \frac{1}{\mu^N}
\tag{C8}
\]

giving

\[
|T_N - \lambda I_N| = 2 + \mu^N + \frac{1}{\mu^N}
\tag{C9}
\]

The expression above vanishes if \((\mu^N + 1)^2 = 0\). The roots of \( \mu^N + 1 = 0 \) are \( \mu = e^{(2m+1)i\pi/N} \) with \( m \in \mathbb{Z} \). Thus,

\[
\frac{-\lambda + \sqrt{\lambda^2 - 4}}{2} = e^{(2m+1)i\pi/N}
\]

\[
\sqrt{\lambda^2 - 4} = \lambda + 2e^{i(2m+1)\pi/N}
\]

\[
\lambda^2 - 4 = \lambda^2 + 4\lambda e^{i(2m+1)\pi/N} + 4\lambda e^{i(2m+1)\pi/N}
\]

\[
\lambda = -2 \cos \frac{(2m+1)\pi}{N} \geq -2 \cos \frac{\pi}{N} \quad \forall m \text{ and all } N \geq 3
\tag{C10}
\]

Therefore, \( W_N \) is PSD for all \( N \geq 3 \).

Lemma 3. The only solution to the system of linear equations

\[
X_NW_N = 0
\]

is \( X_N = 0 \), where \( X_N \) is a symmetric \( N \times N \) matrix with diagonal elements equal to zero and \( W_N \) is the dual optimal for the SDP in (3).
Proof. The proof is trivial and follows from simple linear algebra. There are $N^2$ linear equations where the maximum number of variables in a equation is three. Let us consider the equations with variables corresponding to the first row of $X_N$. There are $N$ such linear equations. Furthermore, there are three equations with number of variables equal to two. These three equations fix the value of the variables involved either equal to each other (one such equation) or a constant times the other variable. Here, the constant is $2 \cos \frac{\pi}{N}$. Substituting these constraints in the equations with three variables, we get new equations with two variables. Following this approach, we get the value of all the variables corresponding row 1 of $X_N$ equal to zero as the only self-consistent solution. Same argument applies for the variables in the other rows. This completes the proof.

Proof of Theorem 1 follows as;

Proof. Lemmas 2 and 3 imply that the dual optimal $W_N$ is non-degenerate. Together with Theorem 5, this implies that the primal optimal for the SDP in (3) is unique. This completes the proof of Theorem 1.

b. Proof of Theorem 2

Proof. Since the identity matrix belongs to the strictly feasible set of the SDP in (3), the singularity degree is 0. Once we substitute $d = 0$ in the statement of Theorem 6, we recover our robustness statement. This completes the proof of Theorem 2.

2. Proof of Lemma 1 from MT

Proof. Starting with LHS,

$$\text{Tr}[(A_m \otimes A_n)R_{AB}] = \text{Tr}\left\{ (A_m \otimes A_n) \left[ \left( \rho_A \otimes \frac{1}{2} \right) E_{AB} + E_{AB} \left( \rho_A \otimes \frac{1}{2} \right) \right] \right\}$$

$$= \frac{1}{2} \left[ \text{Tr}((A_m \rho \otimes A_n) E_{AB}) + \text{Tr}((A_m \otimes A_n) E_{AB}(\rho \otimes I)) \right]$$

$$= \frac{1}{2} \left[ \sum_{ij} \text{Tr}(A_m \rho \langle i | j \rangle) \text{Tr}(A_n E_{A|B} (|j \rangle \langle i |)) + \sum_{ij} \text{Tr}(A_m |i \rangle \langle j | \rho) \text{Tr}(A_n E_{A|B} (|j \rangle \langle i |)) \right]$$

$$= \frac{1}{2} \left[ \sum_{ij} \text{Tr}(A_m \rho + \rho A_m) |i \rangle \langle i| \text{Tr}(A_n E_{A|B} (|j \rangle \langle i |)) \right]$$

$$= \frac{1}{2} \left[ \sum_{ij} \text{Tr}(A_m \rho + \rho A_m) |i \rangle \langle i| \sum_k V_k^\dagger A_n V_k |j \rangle \right]$$

$$= \frac{1}{2} \left[ \sum_{ij} \text{Tr}(A_m \rho + \rho A_m) |i \rangle \langle i| \sum_k V_k^\dagger A_n V_k |j \rangle \right]$$

$$= \frac{1}{2} \left[ \sum_{ij} \text{Tr}(A_m \rho + \rho A_m) E_{A|B} (A_n) \right]$$

$$= \langle A_m A_n \rangle_{seq}$$

where in the first line, $E_{AB} = \sum_{ij} (\mathcal{I}_A \otimes E_{A|B})(|i \rangle \langle j | A \otimes |j \rangle \langle i | B)$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B$ is Jamiolkowski isomorphic to $E_{A|B}$.

\qed
3. Proof of Theorem 3 from MT

We first prove the following useful result,

**Lemma 4.** Consider the correlation $\langle \sigma_{k} \sigma_{l} \rangle_{\text{seq}}$ in the generalized scenario with channels with $\sigma_{k/l}$ as the Pauli observables and $\rho_{A}$ being 2-dimensional. Then, we have

$$\langle \sigma_{k} \sigma_{l} \rangle_{\text{seq}} = \frac{1}{2} \left[ \langle \sigma_{k} \rangle_{\rho_{A}} \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{0})) + \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{k})) \right]$$

(C12)

**Proof.**

$$\langle \sigma_{k} \sigma_{l} \rangle_{\text{seq}} = \text{Tr}[(\sigma_{k} \otimes \sigma_{l}) R_{AB}] \quad \text{using Lemma 1}$$

$$= \text{Tr} \left[ (\sigma_{k} \otimes \sigma_{l})(\mathcal{I}_{A} \otimes \mathcal{E}_{A|B}) \left\{ \rho_{A} \otimes \frac{I}{2} + \frac{1}{2} \sum_{i=0}^{3} \sigma_{i} \otimes \sigma_{i} \right\} \right] \quad \text{using (8) from MT}$$

$$= \frac{1}{4} \text{Tr} \left[ (\sigma_{k} \otimes \sigma_{l})(\mathcal{I}_{A} \otimes \mathcal{E}_{A|B}) \sum_{i=0}^{3} \langle \rho_{A}, \sigma_{i} \rangle \otimes \sigma_{i} \right]$$

$$= \frac{1}{4} \text{Tr} \left[ \sum_{i=0}^{3} \sigma_{k} \{ \rho_{A}, \sigma_{i} \} \otimes \sigma_{i} \mathcal{E}_{A|B}(\sigma_{i}) \right]$$

$$= \frac{1}{4} \sum_{i=0}^{3} \text{Tr}(\sigma_{k} \rho_{A} \sigma_{i} + \sigma_{k} \rho_{A} \sigma_{i}) \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{i}))$$

$$= \frac{1}{4} \sum_{i=0}^{3} \text{Tr}(\rho_{A}(\sigma_{i} \sigma_{k} + \sigma_{k} \sigma_{i})) \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{i}))$$

$$= \frac{1}{2} \left[ \langle \sigma_{k} \rangle_{\rho_{A}} \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{0})) + \sum_{i=1}^{3} \delta_{ik} \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{i})) \right]$$

$$= \frac{1}{2} \left[ \langle \sigma_{k} \rangle_{\rho_{A}} \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{0})) + \text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{k})) \right].$$

$\square$

Proceeding with the proof of Theorem 3 from MT;

**Proof.** Take $\rho_{A} = |0\rangle \langle 0|$ since we expect maximal violation to be obtainable using a pure state and rotational freedom of observables allows us to take it as an eigenstate of $\sigma_{2}$. Then using Lemma 4, we get

$$\langle \sigma_{k} \sigma_{l} \rangle_{\text{seq}} = \begin{cases} 
\text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{k}))/2, & \text{for } k = 1, 2 \\
\text{Tr}(\sigma_{l} \mathcal{E}_{A|B}(\sigma_{0}) + \sigma_{l} \mathcal{E}_{A|B}(\sigma_{3}))/2, & \text{for } k = 3 
\end{cases}$$

(C13)

Furthermore, it has been established in [57] that all possible quantum channels acting on 2-dim. states correspond to the convex closure of the maps parametrized in the Pauli basis $\{\sigma_{i}\}_{i}$ using the following Kraus operators,

$$K_{+}^{l} = \left[ \cos \frac{v}{2} \sin \frac{u}{2} \right] \sigma_{0} + \left[ \sin \frac{v}{2} \sin \frac{u}{2} \right] \sigma_{3}$$

$$K_{-}^{l} = \left[ \sin \frac{v}{2} \cos \frac{u}{2} \right] \sigma_{1} - i \left[ \cos \frac{v}{2} \sin \frac{u}{2} \right] \sigma_{2}$$

(C14)

with $v \in [0, \pi]$, $u \in [0, 2\pi]$. Action of the channel $\mathcal{E}_{A|B}(\sigma_{l})$ can be written using these Kraus operators as,

$$K_{+} \sigma_{0} K_{+}^{l} + K_{-} \sigma_{0} K_{-}^{l} = \sigma_{0} + \sin(u) \sin(v) \sigma_{3}$$

$$K_{+} \sigma_{1} K_{+}^{l} + K_{-} \sigma_{1} K_{-}^{l} = \cos(u) \sigma_{1}$$

$$K_{+} \sigma_{2} K_{+}^{l} + K_{-} \sigma_{2} K_{-}^{l} = \cos(v) \sigma_{2}$$

$$K_{+} \sigma_{3} K_{+}^{l} + K_{-} \sigma_{3} K_{-}^{l} = \cos(u) \cos(v) \sigma_{3}.$$  

(C15)

Substituting (C15) into (C13) gives

$$\langle \sigma_{k} \sigma_{l} \rangle_{\text{seq}} = \delta_{kl}(\delta_{1k} \cos u + \delta_{2k} \cos v + \delta_{3k} \cos(u - v))$$

(C16)
Using (C16) for $S_3$ along with the fact that for 2-dimensional observables $A_{m/n} = \tilde{a}_{m/n}$, we can write $(A_m A_n)_{\text{seq}} = \sum_{k,l} a_{mk} a_{nl} (\sigma_k \sigma_l)_{\text{seq}}$ gives,

$$S_3 = \sum_{k,l=1}^{3} [(a_{1k}a_{2l} + a_{2k}a_{3l} - a_{3k}a_{1l})\delta_{kl}(\delta_{1k} \cos u + \delta_{2k} \cos v + \delta_{3k} \cos (u - v))]$$

$$= \sum_{k=1}^{3} [(a_{1k}a_{2k} + a_{2k}a_{3k} - a_{3k}a_{1k})(\delta_{1k} \cos u + \delta_{2k} \cos v + \delta_{3k} \cos (u - v))]$$

(C17)

From (C17), it can be seen that the maximal value achieved is $3/2$ in the cases with $\{u = v = 0\}$, $\{u = v = \pi\}$, $\{u = 0, v = \pi\}$ and $\{u = \pi, v = 0\}$ which correspond to special cases of the Pauli channel. However, note that the angle between the measurement choices $\{A_i\}$ would depend on the particular case of the channel $E_{A|B}$. Hence, taking a convex combination of channels corresponding to the cases above will not give maximal violation for a particular choice of 3 measurements $\{A_i\}_{i=1,2,3}$. This result again highlights the unitary degree of freedom in choosing the channel as described in (9). Interestingly, the 4 channels corresponding to the cases above also correspond to the 4 “pseudo-Bell states”

$$R_{AB}^{(1)} = \frac{1}{4} (I \otimes X + X \otimes Y + Y \otimes Z + Z \otimes Z)$$

$$R_{AB}^{(2)} = \frac{1}{4} (I \otimes X + X \otimes Y - Y \otimes Z + Z \otimes Z)$$

$$R_{AB}^{(3)} = \frac{1}{4} (I \otimes X - X \otimes Y - Y \otimes Z + Z \otimes Z)$$

$$R_{AB}^{(4)} = \frac{1}{4} (I \otimes X - X \otimes Y + Y \otimes Z + Z \otimes Z)$$

which were used for showing temporal evolution as an analogue of spatial teleportation in [43].

\[\square\]

**Appendix D: Local isometry in time**

Justification for (9) in MT,

$$\text{Tr} [(A_m \otimes A_n) R_{AB}] = \text{Tr} \left[ (A_m \otimes A_n) (I_A \otimes E_{A|B}) \left\{ \rho_A \otimes I_2, \frac{1}{2} \sum_{i=0}^{3} \sigma_i \otimes \sigma_i \right\} \right]$$

$$= \text{Tr} \left[ (A_m \otimes A_n) \frac{1}{2} \sum_{i=0}^{3} \left\{ \rho_A, \sigma_i \right\} \otimes E_{A|B}(\sigma_i) \right]$$

$$= \frac{1}{2} \sum_{i=0}^{3} \text{Tr} \left[ A_m (\rho_A \sigma_i + \sigma_i \rho_A) \right] \text{Tr} \left[ A_n E_{A|B}(\sigma_i) \right]$$

$$= \frac{1}{2} \sum_{i=0}^{3} \text{Tr} \left[ V A_m V^\dagger (V \rho_A V^\dagger V \sigma_i V^\dagger + V \sigma_i V^\dagger V \rho_A V^\dagger) \right] \text{Tr} \left[ U A_n U^\dagger \tilde{E}_{A|B}(V \sigma_i V^\dagger) \right]$$

$$= \frac{1}{2} \sum_{i=0}^{3} \text{Tr} \left[ \tilde{A}_m (\tilde{\rho}_A \sigma_i + \sigma_i \tilde{\rho}_A) \right] \text{Tr} \left[ \tilde{A}_n \tilde{E}_{A|B}(\sigma_i) \right]$$

where $\tilde{E}_{A|B}$ has transformed Kraus operators as $\{K_i\} \rightarrow \{UK_i V^\dagger\}$, $U$ is a unitary operator acting at time point $t_B$ and $V$, acting at $t_A$, is restricted to $V \in \{X,Y,Z\}$ such that applying the transformation $V \sigma_i V^\dagger$ conserves the elements of the set $\sigma_i \in \{I,X,Y,Z\}$. Note that we choose this particular choice for $V$ just to highlight the local isometry in time, there exist other choices for $V$ such as the Clifford group that would also conserve the set of Pauli matrices.