Charges of long random states of the Heisenberg spin-1/2 chain model

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Abstract

We conjecture a formula which expresses charges of infinitely long states of the Heisenberg spin-1/2 chain model. Several arguments are provided which support the proposal.

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1 Introduction

Integrable spin chains are very lively developing realm of theoretical physics [1, 2]. They are characterized by an infinite set of conserved charges all of them encoded in the transfer matrix $T$. The Heisenberg $\text{su}(2)$ spin-$1/2$ chain is one of the simplest and most studies integrable models. The standard $T$ is a trace of the monodromy matrix for which the auxiliary space is in the fundamental representation of the symmetry group of the model: in our notation it is $j = 1$. Recently it has been shown that the models have additional conserved charges originating from transfer matrices $T_j$ for which the auxiliary space is in a higher spin representation\(^1\) of $\text{su}(2)$ [3]. Their existence requires spin chains to be infinitely long. The higher spin charges are not strictly speaking local but only so-called quasi-local operators. Soon after discovery they appeared to be necessary in description of steady-state averages after quantum quenches [4]. The proper framework in this case involves so-called Generalized

\(^1\)Here $j = 2s \in \mathbb{N}$ where $s$ is spin.
Gibbs Ensemble [5] which must include all the conserved charges and the corresponding chemical potentials.

In this paper we are going to discuss charges $X_j$ of states of the form $\Psi = (\psi) \otimes^{N/M}$, where $\psi$ has length $M$ of the infinite ($N \to \infty$) periodic Heisenberg model. The work is centered around a conjecture which, roughly speaking, says that for most of the very long $\psi$’s charges $X_j$ are well approximated in the physical strip (PS) of the complex spectral parameter $\mu$ by very simple formula

$$X_j(\mu) = \frac{1}{4\pi \mu^2} \frac{j}{\frac{1}{2}(j + 1)^2}.$$  \hspace{1cm} (1.1)

When the length of $\psi$ goes to infinity we expect that (1.1) is exact. Notice that (1.1) depends only on $j$. We shall be more specific about the precise meaning of the hypothesis in Sec.3.

We support (1.1) providing several arguments. First of all we derive analytically, under certain assumptions, the formula in the case $j = 1$. Next, we do certain large $\mu$ expansion which, in fact, coincides with (1.1) for general $j$, and do statistical analysis of vast numerical data obtained mostly for $j = 1$ and $j = 2$. Finally we formulate the conjecture and then show that it is in agreement with infinite temperature average in Gibbs ensemble.

The hypothesis claims enormous simplification of charges for some states. This simplicity is quite astonishing in view of known complexity of exact results. Sizes of expressions on $X_j$ grows rapidly with the state’s length and $j$: few examples will be given in Sec.A.4.

The paper is organized as follows. In the next section we introduce definition of charges and present explicit expressions helpful in calculations. We also discuss some features of the exact formulae on $X_j$ (Sec.2.1). Sec.3 contains main results of the paper leading to our hypothesis. Thus we first discuss a large $M$ limit of just $X_1$ for which we can do analytic calculations. Next we do large $\mu$ approximation. Finally we compare numerically (1.1) and the exact results on charges of randomly chosen and quite long (up to length 200) states. The main body of paper ends with Conclusions. Several appendices contains details on notation and technical aspects of the results.

2 Charges

In this section we recall definitions of charges of the spin chain state $\Psi$ [3]. The presentation culminates with the expression on $X_j(\mu)$ which will be used in the next sections.

Conserved quantities of the integrable su(2) Heisenberg spin chain model of the length
Here are given by the expectation value of the transfer matrix:

\[ T^{\Psi}_{(0)}(\mu) = \langle \Psi | T_{(0)}(\mu) | \Psi \rangle = \langle \Psi | \text{tr}_0(L_{0N}L_{0(N-1)} \ldots L_{0k} \ldots L_{01})(\mu) | \Psi \rangle, \]

where \( L_{0k} \) is the Lax operator, "0" denotes an auxiliary space, \( k \) the \( k \)-th node of the chain and \( \Psi \) is a given spin chain state belonging to quantum space \( \mathcal{H} = (\mathcal{V})^{\otimes N}, \dim \mathcal{V} = 2 \). Then \( X_{(0)}(\mu) \sim \partial_\mu T^{\mu}_{(0)}(\mu) \). Usually the spin-1/2 auxiliary space (in our notation it is \( j = 1 \)) is considered and then \( X_{(0)} \equiv X_{j=1} \equiv X_1 \) exist for any finite \( N \). For higher spin auxiliary spaces \( j = 2, \ldots \) the charges \( X_j \) also exist but they are independent on \( X_1 \) only in \( N \to \infty \) limit.

\[ X_j(\mu) = \lim_{N \to \infty} \frac{1}{2\pi i N} \langle \Psi | \partial_\mu \log \frac{T^+_{j}(\mu)}{T^{-}_{j}(\mu)} | \Psi \rangle \quad (2.1) \]

The factor \( T^{[j+1]}_0(\mu) \) appearing in the denominator of (2.1) shifts \( X_j \) by the state independent function. It was introduced for convenience. Strictly speaking \( X_j(\mu) \) depends on the spectral parameter \( \mu \) thus it is the generating function of the charges. In this paper we shall keep calling functions \( X_j(\mu) \) charges of \( \Psi \).

One can differentiate log producing

\[ X_j(\mu) = \lim_{N \to \infty} \frac{1}{2\pi i N} \langle \Psi | \partial_\mu \log \frac{T^+_{j}(\mu)}{T^{-}_{j}(\mu)} | \Psi \rangle \quad (2.2) \]

with the help of so-called inversion formula, which says that \( T^{-}_{j}|T^+_{j} = T^{-}_{j-1}|T^+_{j+1} \) in the \( N \to \infty \) limit [3, 6, 7, 8].

Taking \( N \to \infty \) is always a delicate matter. One must carefully define the whole procedure. Here we consider certain family of states \( \Psi \) of the form \( \Psi = (\psi)^{\otimes N/M} \), where the substate \( \psi \) has length \( M \). Following [9, 10] we define composite two-channel Lax operator (see App.A.1 for the notation)

\[ \mathbb{L}_j(\mu, x) = n(\mu, x) L_j^{-}(\mu) L_j^{+}(x) \quad (2.3) \]

where \( L_j \) denote the Lax operator in the representation \( j \) and \( n \) is a normalization factor originating from \( T^{-}_{0}|T^+_{0} \) in (2.2). We define a monodromy operator as

\[ \mathbb{M}_j^{\Psi}(\mu, x) = \langle \psi | \mathbb{L}_j^{(M)} \ldots \mathbb{L}_j^{(1)}(\mu, x) | \psi \rangle \quad (2.4) \]

Then

\[ X_j^{\Psi}(\mu) = \lim_{N \to \infty} \frac{1}{2\pi i N} \text{tr}_{V_j \otimes V_j} [\partial_x (\mathbb{M}_j^{\Psi}(\mu, x))^{N/M}]|_{x=\mu} \quad (2.5) \]

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2Here \( f^{\pm}(\mu) = f(\mu \pm \frac{i}{2}) \) and we shall also use \( f^{[\pm k]}(\mu) = f(\mu \pm k \frac{i}{2}) \). We shall often suppress any decoration of \( M, \mathbb{L}, v, w \) if from the context it will be clear what are \( \Psi, \psi \) and \( j \).
The operator $M_j^\psi(\mu, x)$ has generically one eigenvalue $\lambda$ which tends to 1 when $x \to \mu$ \cite{10}.

\[ M_j^\psi(\mu, x)v_j(\mu) = (1 + (x - \mu)\delta_j)\nu_j(\mu) + O((x - \mu)^2) \tag{2.6} \]

The unit eigenstate and its eigenvalue dominate the trace $\text{tr}$ in (2.5)

\[ X_j^\psi(\mu) = \lim_{2\pi i N} \frac{1}{(\partial_x (1 + (x - \mu)\delta_j)^N/M_{x=\mu})} = \frac{1}{2\pi i M}\delta_j \tag{2.7} \]

It follows from (2.6) that we can find left and right unit eigenvalues of $M_j^\psi = M_j^\psi(\mu, \mu)$

\[ M_j^\psi v_j = v_j, \quad w_j^\dagger M_j^\psi = w_j^\dagger \tag{2.8} \]

By standard quantum mechanical perturbative calculations one obtains

\[ \delta_j = \frac{w_j^\dagger \partial M_j^\psi v_j}{w_j^\dagger v_j}_{x=\mu} \tag{2.9} \]

where $\partial M_j^\psi = \partial_x M_j^\psi(\mu, x)|_{x=\mu}$. The above expression is equivalent to what was derived in \cite{10}.

It appears to be very handy for various types of calculations presented in some details in Appendices. It is specially useful for efficient numerical calculations when $\psi$’s are, what we shall call, simple substates. In that case $\psi$ is just single sequence of spins up and down represented by 1 and 2. The reason for this simplification follows from triviality of the right unit eigenvector $w$. For more details we send the reader to App.A.1.

### 2.1 Charges $X_j$’s and their analytic structure

The charges $X_j(\mu)$ have very rich analytic structure on complex $\mu$-plane. For simple $\psi$ they are rational functions with poles and zeros which number grows rapidly with their length and the representation index $j$. Higher spin charges of long $\psi$ have mammoth sizes, thus they are completely impractical. To give the reader a flavor how the exact expressions may look like we display an example ($M = 7, \psi = \{1, 1, 1, 1, 2, 1, 2\}$) which still fit in the paper: see App.A.4. For larger length $\psi$ the formulae would occupy several pages e.g. for $M=40$ and typical $\psi$ the charge $X_3(\mu)$ denominator is a polynomial of the degree 234 with integer coefficients containing 80 digits.

It is much easier to see structure of charges displaying their poles on the complex $\mu \geq 0$ half-plane\(^4\). The examples are shown of Fig.1. In spite of complexity $X_j(\mu)$’s possess several general properties which can be spelled out.

\(\text{3One can easily show that } v_j w_j^\dagger \sim \text{Adj}(M - 1). \)

\(\text{4For simple } \psi \text{'s charges } X_j \text{ are even and real functions of } \mu.\)
Figure 1: Positions of poles of $X_1(\mu), X_4(\mu)$ for $\psi = \{1, 1, 1, 2, 1, 2\}$ on the complex half-plane ($\mu, \text{Im}(\mu) \geq 0$). The explicit expression for $X_4$ would occupy too much space here, thus we present $X_1$ only: $X_1(\mu) = \frac{1}{7\pi} \times \frac{10\mu^{10}+25\mu^8+34\mu^6+26\mu^4+11\mu^2+2}{7\mu^8+21\mu^6+35\mu^4+21\mu^2+7\mu+1}$.

1. All poles in $X_j(\mu)$ appears beyond the so-called Physical Strip (PS) which is: $PS = \{\mu \in \mathbb{C}; |\text{Im}(\mu)| < 1/2\}$. The poles we interpret as bound states of auxiliary spins in the background of fixed $\Psi$ (see also [11]).

On the technical level poles originates from zeros of $\mathbf{w} \mathbf{v}$ and corresponds to those $\mu$’s for which $M_{\psi}^j$ contains nontrivial 2-dim Jordan block with eigenvalue 1. The proof of this statement is given in App.A.2 for the simplest case of $X_1$ only. For the higher charges we checked that fact numerically only.

2. Poles of $X_1$ lay on the hyperbola $\text{Re}(\mu^2) = -1/2$ which represents the relativistic dependence of the real and imaginary part on $\mu$ (App.A.2). Poles of the higher charges seems to align certain curves too, but their nature is more complicated (see Fig.(6) in App.A.4).

3. Most of the poles and zeros of $X_j$’s are very close to each other what means that their contribution to charges inside PS is very small. This suggest big redundancy of information contained in exact expressions.

3 The conjecture

From the analysis presented in the previous sections it is clear that the exact structure of the charges is very complicated. For long chains one may doubt if exact expressions on $X_j$
(if known) would be of any practical use. Thus a formula that would well approximate charges in PS might be very useful. In this section we shall make proposal which seems to do the job for very long random and simple $\psi$. First we present arguments which will justify our final statement of Sec.3.4.

3.1 $X_1$ in $M \to \infty$ limit

One may wonder what is the distribution of poles thus the density of charges on the complex $\mu$-plane in large $M$ limit. We are not going to consider here the most general case of arbitrary $X_j$. Quick look at the distribution of poles suggests that the problem might be very hard. But for $X_1$ thanks to the results of App.A.2 we can present calculations which lead to conceivable picture of the limit. The procedure we propose is a direct analog of the thermodynamic limit [12, 13].

First of all we decompose the formula on $X_1$ as

$$X_1 = \frac{1}{2\pi} \sum_n \frac{c_n}{\mu - \mu_n}$$  \hspace{1cm} (3.1)

where $\mu_n$ are positions of poles and $c_n \in \mathbb{C}$. From App.A.2 we have

$$\mu_n^2 = iy_n - 1/2, \quad \frac{i y_n - 1/2}{i y_n + 1/2} = e^{2i\pi k_n/M}, \quad k_n = 1, \ldots M - 1, \quad y_n \in \mathbb{R}$$

One must remember that for non-generic substates not all $k_n$ correspond to poles: there are holes in the distribution $k_n$ i.e. there are less poles then $M - 1$. At large $M$ and generic random $\psi$ we expect that there are no holes i.e. there is one-to-one $k_n \leftrightarrow n$ linear relation thus we shall set $k_n = n$. Fig.2 shows poles for $M = 40$ state. Notice that density of a charge is not given uniquely by distribution of poles. This would hold only if all residues $c_n$ were equal what we are going to assume from now on. Thus we set $c_n = c$. Denoting the continuous variable as $\alpha$ i.e. $n/M \to \alpha$ in $M \to \infty$ limit we rewrite (3.1) as

$$X_1^{ThR}(\mu) = \frac{c}{2\pi} \int d\alpha \frac{1}{\mu - \mu(\alpha)}$$  \hspace{1cm} (3.2)

where

$$\mu^2(\alpha) = iy(\alpha) - 1/2, \quad \frac{i y(\alpha) - 1/2}{i y(\alpha) + 1/2} = e^{2i\pi \alpha}, \quad y \in \mathbb{R}$$  \hspace{1cm} (3.3)

Changing integration variable to $a = \text{Re}(\mu(\alpha))$ we get:

$$X_1^{ThR}(\mu) = \frac{c}{2\pi^2} \int_{-\infty}^{\infty} \frac{da}{4a^2 + 1} \frac{1}{a^2 + (\mu - a)^2 + 1/2} = \frac{c}{4\pi(\mu^2 + 1)}$$  \hspace{1cm} (3.4)

\footnote{We leave aside experimental problems related to the ability to control initial condition of spins in chain i.e. the state $\psi$.}
The above constant $c$ has been redefined to include numerical factors appearing in the course of calculations. The representation (3.4) is properly defined for $\mu \in \mathbb{R}$ but can be analytically extended to the whole complex plane. The final value of $c$ can be fixed by comparing with large $\mu$ result of Sec.3.2 and App.A.3 yielding $c = 1$.

From (3.4) the density of charge in the variable $a$ can be read to be $1/(4a^2 + 1)$. The letter is included in Fig.2 and it nicely fits the density of poles on the hyperbola.

We want to stress that $c_n \to c(\alpha) = c = const$ can not hold for general $\psi$ e.g. for $\psi$'s which are of the form $\psi = (\psi')^{M/M'}$, where the length of $\psi'$ is $M'$. Then $X_1^{\psi} = X_1^{\psi'}$ thus it does not depend on $M$ at all. In the extreme case $\psi' = \{1, 2\}$ (Néel state) the whole density $c(\alpha)$ is localized at two points $\mu = \pm i/\sqrt{2}$. On the other hand we expect that for most of the long random $\psi$'s, $c(\alpha) = c = const$ is good approximation. This point will be under thorough scrutiny in Sec.3.3.

### 3.2 Large $\mu$ expansion

In this section we shall discuss certain large $\mu$ approximation of the exact expression on $X_j$. The procedure we propose is a hybrid: we do large $\mu$ expansion of the Lax operators and the vector $v$ but keep intact the normalization factor $n$. This is well motivated by the
previous derivation of Eq.(3.4) where \( n \) appears naturally from continuous distribution of the charge density localized along a hyperbola. Detailed derivation is presented in App.A.3. The obtained result is

\[
\tilde{X}_j(\mu) \approx \frac{1}{2\pi \mu^2} \frac{1}{\frac{1}{4}(j+1)^2} \left( \frac{j}{2} - \frac{j(1-r)(r^{j+1}+1) - 2r(1-r^j)}{2(r+1)(1-r^{j+1})} \right)
\]

(3.5)

where \( r = n_2/n_1 \) and \( n_1, n_2 \) denote numbers of spins up and down in \( \psi \), respectively. The formula is \( r \rightarrow 1/r \) invariant. Few remarks are necessary at this point. The singularities at \( \mu = \pm i^{j+1}/2 \) come from normalization factor \( n(\mu) \). In the approximation made the denominator of 2.9 i.e. \( w^+v \) is \( \mu \) independent contrary to exact results on charges. Recall that zeros of \( w^+v \) give spectra of bound states. These we do not expect to appear in \( \mu \rightarrow \infty \) limit, at least at the leading order of the expansion. Thus \( w^+v = const \) is physically well motivated.

The r.h.s. of (3.5) has the following expansion for small \( \epsilon \equiv 1 - r \):

\[
\tilde{X}_j(\mu) \approx \frac{1}{4\pi \mu^2} \frac{j}{(j+1)^2} \left( 1 + \frac{1}{12} \epsilon^2 \right) + O(\epsilon^3).
\]

(3.6)

For random states of the length \( M \) the average deviation of \( \left| \frac{n_1-n_2}{n_1+n_2} \right| = \left| (1-r)/(1+r) \right| \sim \sqrt{M}/M \rightarrow 0 \) when \( M \rightarrow \infty \) thus \( \epsilon = 0 \) for most of the random long \( \psi \)'s.

Several pictures comparing \( X_j \) and \( \tilde{X}_j \) are given in App.A.4 (Fig.7). From there we see that (3.5) works quite well even for relatively short \( \psi \)'s. Moreover the higher representation \( j \) the better are approximations. But we need more quantitative checks. The next section is devoted to a simple statistical analysis of estimates provided by (3.5).

### 3.3 Statistics

It is interesting to check how well (3.5) estimates the exact expression. Previously given arguments for \( X_1 \) gives hope that the proposed formula is, in a sense, exact in the \( M \rightarrow \infty \) limit. Thus (3.5) for \( j = 1 \) should be good approximation even for large but finite \( M \). The situation is less clear for \( \tilde{X}_j \) \( (j > 1) \) where we do not have similar analytical arguments for higher charges thus we are forced to rely on statistical analysis only. Moreover due to length of exact formulae we have been unable to go too far with value of \( M \) and \( j \).

Hereafter we shall compare values of \( X \)'s and \( \tilde{X} \)'s on the real line \( \mu \in \mathbb{R} \). As a measure of deviation between \( X_j \) and \( \tilde{X}_j \) we have chosen

\[
\text{dis}_j = \sup_{\mu \in [-10,10]} \left| \frac{X_j(\mu) - \tilde{X}_j(\mu)}{X_j(\mu)} \right|
\]

(3.7)

which will be calculated for the following cases:
Figure 3: Histogram of deviation $\text{dis}_j$ for $j = 1, 3, 5, 7$ for random chains $M = 20$.

(a) fixed 100 random $\psi$'s of the length $M = 20$ for different representation index $j = 1, 3, 5, 7$

(b) $\text{dis}_1$ calculated for 100 random $\psi$'s of the lengths $M = 20, 50, 100, 200$.

(c) $\text{dis}_2$ calculated for 100 random $\psi$'s of the lengths $M = 20, 50, 100$.

The obtained data were plotted on histograms Fig.3 for the case (a) and Fig.4 for the case (b)$^6$. For the case (c) i.e. $X_2$, the histogram appears to be very similar to (b) hence it is not displayed here.

$^6$All calculations have been done by Mathematica. We have used RandomInteger[1,2,20] as generator of random $\psi$ of $M = 20$. Negative values of $\text{dis}_j$ in Figs.3 and 4 follows from interpolation done by SmoothHistogram function.
It is clear that the bigger $M$ the relative difference between $X_j$ and $\tilde{X}_j$ is smaller. For $j = 1$ and $M = 50$ the deviation $\text{dis}_1$ for random substates peaks about 0.1 while for $M = 200$ it is only 0.05. Similar tendency is seen for $j = 2$ but we had poorer statistics in this case. Moreover Fig.3 suggests that statistically the formula works better if the representation $j$ is higher although we did not do enough numerics to make any convincing claim to what extent $\tilde{X}_j$ works better for e.g. $j = 3$ compared to $j = 1$.

It is of primer necessity to increase amount of numerical data to support (3.5) and our main conjecture discussed in the next paragraph.

### 3.4 The conjecture

In this section we shall spell out our main hypothesis and clarify some of vague statements appearing in the paper. Our claims are based on arguments given in the previous subsections. Moreover we present new reasons which let us extend the conjecture to non-simple $\psi$’s.

Substates $\psi$ of the previous section have been chosen randomly. The random choice include those $\psi$’s which charges are far from being close to (3.5). These we call non-generic. For example: $\psi = (\psi')^\otimes M/M'$ ($\psi'$ has length $M'$, $M'$ is a nontrivial divisor of $M$) are non-generic: $X_j^\psi = X_j^{\psi'}$ for any $M$. The important fact (supported by numerics of the previous subsection) is that for large $M$ probability that random $\psi$ is non-generic is close to zero. In this sense the conjecture is formulated for most of simple $\psi$’s.

The space of states of the model is very reach but up to this point we have been solely working with simple $\psi$’s in the form of one sequence of spins up and down. These are rare in the space of all states. The most general $\psi$’s are of the form

$$\psi = \sum_n \alpha_n \psi_n, \quad \alpha_n \in \mathbb{C} \quad (3.8)$$

where now $\psi_n$’s are all different and simple. Hence we need to calculate

$$\langle \psi_m | L^{(M)} \ldots L^{(1)}(\mu, x) | \psi_n \rangle, \quad m \neq n \quad (3.9)$$

for all $m, n$. The claim is that if both $\psi_m$ and $\psi_n$, $m \neq n$ are random then the above expression vanish in $M \to \infty$ limit. The crucial point is that (3.9) always contains off-diagonal terms of $L$ i.e. $L^1_2$ and $L^2_2$ which number grows to infinity when $M \to \infty$. Inspection of (A.1) reveals that $L^1_2(\mu, x)$ and $L^2_2(\mu, x)$ are contracting operators i.e. $||L^1_2(\mu, x) \cdot v|| \leq p||v||$ $v \neq 0, v \in \mathbb{C}^{2j+2}$ and $q \in [0, 1)$ for any representation $j$ and $\mu, x \in \mathbb{R}\{0\}$ (the same holds for $L^2_2$). Indeed, e.g. for $j = 1$ we have

$$\frac{||L^1_2(\mu, x) \cdot v||^2}{||v||^2} = \frac{||(i + x)v_2 + (i + x)v_3||^2 + (\mu^2 + x^2)||v_4||^2}{(1 + \mu^2)(1 + x^2)} \leq 1 \quad (3.10)$$
where the equality can hold only for $\mu = x = 0$. For higher $j$ the bound $p$ is smaller then 1 e.g. for $j = 2$ it is $p = 8/9$ for all $\mu, x \in R$. Infinite product of contracting operators and bounded by 1 operators $L_1^1, L_2^2$ yields zero. Assuming that analyticity in $\mu, x \in \text{PS}$ is preserved by the limiting procedure we infer that (3.9) vanishes. Thus if the hypothesis is true for simple $\psi$ it is true for all long, random $\psi$.

Conjecture. For almost all states of the form $\Psi = (\psi)^{\otimes N/M}$ ($N$ is divisible by $M$) where $\psi$ is a random substate of the length $M$ the charges (2.2) in the limit $M \to \infty$ are given by:

$$
\lim_{M \to \infty} X_j(\mu) \equiv X_j^{ThR}(\mu) = \frac{1}{4\pi \mu^2 + \frac{1}{4}(j+1)^2}
$$

(3.11)

3.5 $T \to \infty$ average

The conjecture might be very hard to prove by direct means as it has been discussed in previous sections. But if correct it has direct consequences which can be easily checked. Here we shall calculate the average of the charges over infinite temperature Gibbs ensemble for infinitely long spin chain and show that it is equal to the r.h.s of (3.11) $^7$. This should be expected if states of charge (3.11) dominates the ensemble.

There is another arguments in favour of the relation to the above $T \to \infty$. Notice that charges determine equilibrium densities through string-charge relations of $[14]$.

$$
\rho_j = X_j^+ + X_j^- - X_{j-1} - X_{j+1}
$$

(3.12)

$$
\bar{\rho}_j = \frac{1}{2\pi} \frac{4j}{j^2 + 4\mu^2} - X_j^+ - X_j^-
$$

(3.13)

For (3.11) we get:

$$
\rho_j(\mu) = \frac{1}{2\pi} \frac{8}{(4\mu^2 + j^2)(4\mu^2 + (j+2)^2)}
$$

(3.14)

$$
\bar{\rho}_j(\mu) = \frac{1}{2\pi} \frac{8j(j+2)}{(4\mu^2 + j^2)(4\mu^2 + (j+2)^2)}
$$

(3.15)

Thus the ratio of holes to particle densities is determined to be constant depending only on $j$: $\eta_j = j(j+2)$. The latter respects Y system $[15, 16, 17, 18]$.

$$
\eta_j^+ \eta_j^- = (1 + \eta_{j+1})(1 + \eta_{j-1})
$$

(3.16)

$^7$The calculations has been suggested to the author by Balázs Pozgay, Jacopo de Nardis, Enej Ilievsky and Miłosz Panfil.
which is equivalent to TBA in some cases [13, 19]. Here it is \( T \to \infty \) limit of TBA (see [13]).

The average is defined as
\[
\langle X_j \rangle = \lim_{N \to \infty} \frac{1}{2\pi i N} \text{tr}_{V_j \otimes V_j} \frac{1}{2} \left( \frac{1}{2} \text{tr}(L_j(\mu, x)) \right)^N \bigg|_{x=\mu} 
\]
(3.17)

where the inner trace is over single node quantum space. Explicitly
\[
\frac{1}{2} \text{tr}(L_j(\mu, x)) = \frac{n(\mu, x)}{4} ((2\mu - i)(2x + i) - 2C_2)
\]
(3.18)

where \( C_2 = (s^- \otimes s^+ + s^+ \otimes s^- + 2s_z \otimes s_z) \) is a Casimir acting on \( V_j \otimes V_j = \bigoplus_{r=0}^{2j} V_r \). Eigenvalues of the \( L_j \) for the \( r \)-representation \( V_r \) are:
\[
\lambda_r = \frac{(j + 1)^2 - r\left(\frac{r}{2} + 1\right) + 2i(\mu - x) + 4\mu x}{(2\mu - i(j + 1))(2x + i(j + 1))}, \quad r = 0, ... 2j.
\]
(3.19)

Only \( r = 0 \) term survives the limit \( N \to \infty \) in (3.17) yielding:
\[
\frac{1}{2\pi i N} \left. \partial_x (\lambda_0^N) \right|_{x=\mu} \to \frac{1}{\pi} \frac{j}{(j + 1)^2 + 4\mu^2}
\]
(3.20)

what is the expected result.

4 Conclusions

In this paper we conjecture a formula \( X_j^{ThR} \) expressing conserved charges of very long random states \( \Psi = (\psi) \otimes N/M \) of the Heisenberg spin chain. If the length \( M \) of the substate \( \psi \) goes to infinity the claim is that the formula is exact. Otherwise it provides a good approximation of a very complicated exact expression. In the case \( j = 1 \) we have been able to derive \( X_1^{ThR} \) in spirit of the standard thermodynamic limit. Unfortunately we do not have such arguments for bigger \( j \). The very striking feature of the formula is its simplicity. If our claim is correct this suggest existence of relatively simple analytical arguments supporting it.

We have checked numerically for \( M \) ranging up to 200 but for relatively low representations \( j = 1, 2 \) that the longer are \( \psi \)'s the conjectured formula is closer to the exact one. Due to lack of analytic proof it would be useful to increase amount of numerical data.

On the way to the main result we have also obtained leading terms of a large spectral parameter expansion of charges. It would be interesting to investigate if one can calculate next to leading terms or maybe even formulate consistent perturbative approach. The delicate point is that such an expansion should be regular for all \( \mu \in PS \).
Finally we must mention that as a consequence of the conjecture the infinite temperature limit of the average of the charges are given exactly by (3.11). This strengthen our believe that the conjecture is correct.

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A Appendices

A.1 Basic notation

Although the formula (2.9) is very explicit in practice higher spin charges are difficult to calculate for general $\psi$. Things are easier when one limits considerations to simple substates being one single chain of spins up and down e.g $\psi = \{1, 2, 1, 2, 1, 2\}$ where numbers 1,2 represent spins up and down respectively. For this state $M^{\psi}_{\psi}(u, x) = \prod_{i=1}^{M}(L_{\psi}(\mu, x))^{\psi(i)}_{\psi(i)}$, where $i$ indicates the node number and $\psi(i) = 1, 2$. Thus $(L_{\psi}(\mu, x))^{\psi(i)}_{\psi(i)}$ are

\begin{align*}
(L_{\psi}(\mu, x))^{1}_{1} &= n (\mu^- + is^z_\psi) \otimes (x^+ + is^z_\psi) - s^-_j \otimes s^+_\psi) \\
(L_{\psi}(\mu, x))^{2}_{2} &= n ((\mu^- - is^z_\psi) \otimes (x^+ - is^z_\psi) - s^+_j \otimes s^-_\psi)
\end{align*}

where $s^a_j$ respects su(2) algebra in representation $j$, $\mu^{\pm} = \mu \pm \frac{1}{2}$, $x^{\pm} = x \pm \frac{1}{2}$, $\mu$, $x \in \mathbb{C}$ and

$$n(\mu, x) = (L^{[j-1]}_0(\mu)L^{[j+1]}_0(x))^{-1} = (\mu - i\frac{j+1}{2})^{-1}(x + i\frac{j+1}{2})^{-1}. \quad (A.2)$$

is the normalization constant. We often omit arguments if $\mu = \xi$ e.g. $L_{\psi} \equiv L_{\psi}(\mu, \mu)$ All these operators act on $V_j \otimes V_j$, where $V_j$ is the module of the representation $j$ spanned by $e_k, (k = 0, ..., j)$. Useful facts are:

1. Charges are invariant under: (a) cyclic shift of nodes, (b) interchange $1 \leftrightarrow 2$.

\footnote{We follow conventions of [14].}
2. For each node: \([([\mathbb{L}_j(\mu, x))]_i, \hat{S}^z) = 0, i = 1, 2 \text{ (no sum)}, \) where \(\hat{S}^z = s^z_j \otimes I + I \otimes s^z_j.\) We decompose \(V_j \otimes V_j\) as direct sum of eigenspaces of \(S^z: V_j \otimes V_j = \bigoplus_{S^z=0} W(j, S^z).\) Thus \(M^\psi_j(\mu, x) : W(j, S^z) \rightarrow W(j, S^z).\)

3. \(v_j \in W(j, j)\)

For simple \(\psi\) one can easily obtain the left unit eigenvector \(w(2.8):\)

\[
(e_k \otimes e_{j-k}) \cdot (\mathbb{L}_j)_1^n = \frac{n}{4} \left[ ((2k + 1 - j)^2 + 4\mu^2) e_k \otimes e_{j-k} + 4k(k - j - 1)e_{k-1} \otimes e_{j-k+1} \right]
\]

\[
(e_k \otimes e_{j-k}) \cdot (\mathbb{L}_j)_2^n = \frac{n}{4} \left[ ((2k - 1 - j)^2 + 4\mu^2) e_k \otimes e_{j-k} + 4(k+1)(k-j)e_{k+1} \otimes e_{j-k-1} \right]
\]

where \(n(j) = ((1 + j)^2/4 + \mu^2)^{-1}.\) Then:

\[
w_j = \sum_{k=0}^{j} (-1)^k e_k \otimes e_{j-k}, \quad i = 1, 2, \quad (A.3)
\]

It follows that \(w_j \in W(j, j)\) and also \(v_j \in W(j, j)\) what significantly simplifies calculations of charges.

### A.2 Poles of \(X_1\)

Here we shall determine alignment of poles of \(X_1.\)

From \(w^\dagger (\mathbb{L}_j)_1^n = w^\dagger, \quad i = 1, 2 \text{ (no sum)}\) the \(2 \times 2\) matrix \(\bar{M}\) has the form

\[
\bar{M} = \begin{pmatrix}
a & b \\
(a-1) & b+1
\end{pmatrix}
\]

(A.4)

then

\[
w^\dagger v = N \frac{a + b - 1}{1 - a} = N \frac{\det(\bar{M}) - 1}{1 - a}, \quad (A.5)
\]

where \(N\) is a normalization constant. Vanishing of the numerator: \(\det(\bar{M}) - 1 = a + b - 1 = 0\) is the condition for \(x = 1\) to be double zero of

\[
det(\bar{M} - x) = 0. \quad (A.6)
\]

When additionally \(a = 1\) (i.e. also \(b = 0\)) then \(\bar{M}\) has two eigenvalues equal 1. Thus \(w^\dagger v = 0\) is the condition for the \(\bar{M}\) to have non-trivial Jordan form. From \(\det(\mathbb{L}_{1,ii}(\mu)) = \frac{\mu^2}{\mu^2 + 1}\) one gets \(\det(\bar{M}) - 1 = (\frac{\mu^2}{\mu^2 + 1})^M - 1 = 0.\) Substituting \(\mu^2 = y - 1/2\) we obtain\(^9\)

\[
\frac{y - 1/2}{y + 1/2} = e^{2i\pi k/M}, \quad k = 1, ... M - 1 \quad (A.7)
\]

\(^9\)We have excluded \(k = 0\) because it corresponds to \(y \rightarrow i\infty\) limit which is not seen for finite \(M.\)
that means that \( y \in i\mathbb{R} \). One must remember, though, that not all the solutions of \((A.7)\) are poles of \(X_1\), but certainly all these poles align the hyperbola: \( \text{Im}(\mu)^2 - \text{Re}(\mu)^2 = 1/2 \). This fact can be seen on Fig.1 and Fig.2.

### A.3 Derivation of (3.5)

We discuss derivation of (3.5) which well approximate charges in PS. We do kind of hybrid \(1/m\mu\) expansion in which the normalization factor \(n\) is kept intact.

We are looking for leading and the first subleading term of \(v\) and \(M\) (subscript \(j\) is mostly skipped here):

\[
M = \prod_{i \in \psi} (L_x^i)
\]

in \(|\mu| \to \infty\) expansion. We shall expand terms from Lax operators only. The normalization factor \(n\) will be left intact. The following observations are helpful:

- the diagonal elements of \(M\) contain the leading terms. These are \((\mu^\pm \pm i s^z) \otimes (\mu^\pm \pm i s^z)\);
- the off-diagonal terms \(s^\pm \otimes s^\mp\) are always suppressed;
- \(s^\pm \otimes s^\mp\) can be freely shifted along the chain because their commutator with \((\mu^\pm \pm i s^z) \otimes (\mu^\pm \pm i s^z)\) is \(O(\mu^0)\) i.e. suppressed by two powers of \(\mu\).

In this way we get

\[
M(e_k \otimes e_{j-k}) \approx [1 - \frac{1}{\mu^2} (k^2 M + k(n_1 - n_2 - jM) - jn_1)] e_k \otimes e_{j-k} \quad (A.9)
\]

\[
= -\frac{n_1}{\mu^2} (j-k)(k+1) e_{k+1} \otimes e_{j-k-1} - \frac{n_2}{\mu^2} (j-k+1) k (e_{k+1} \otimes e_{j-k-1})
\]

where \(n_1, n_2\) denotes numbers of spins up and down in \(\psi\). From the above one easily gets:

\[
v \approx \sum_{k=0}^{j} (-1)^k r^k (e_k \otimes e_{j-k}), \quad w^\dagger v \approx \frac{1 - r^{j+1}}{1 - r} \quad (A.10)
\]

where \(r = n_2/n_1\). Notice that \(w^\dagger v\) is spectral parameter \(\mu\) independent contrary to exact results on charges. Solutions to \(w^\dagger v = 0\) give spectra of the bound states which we should not expect to appear at \(\mu \to \infty\) limit, at least in the leading order. Thus \(w^\dagger v = \text{const}\) is physically well motivated.

In similar manner we calculate \(\partial M\). Derivatives \(\partial_x L\) are proportional to \((\mu^\pm \pm i s^z) \otimes 1\) which can be shifted to back of all expressions at the cost of commutators. The latter are
higher order corrections, thus irrelevant here. Hence $\partial M$ contains a sum of expressions of the form

$$\prod_{i \in \psi'} L_i \cdot (\mu^\pm \pm is^z) \otimes I$$  \hspace{1cm} (A.11)

where $\psi'$ is a subchain in which one node (where the derivative acted) was removed. Because finally we are interested in $w^\dagger \partial M v$, due to $w^\dagger L^\dagger i = w^\dagger$ the $L$’s in (A.11) can be omitted yielding

$$\partial M \approx n \left[ n_1((\mu^- + is^z) \otimes I) + n_2((\mu^- - is^z) \otimes I) \right] - \frac{M}{\mu + i(j + 1)/2}$$  \hspace{1cm} (A.12)

where the last term comes from differentiation of the normalization $n : \partial_x n(\mu, x) \big|_{x=\mu}$. Now we can use $v$ displayed in (A.10) to get our final result (3.5).

$$X_j(\mu) \approx \frac{1}{2\pi} \frac{1}{\mu^2} + \frac{1}{4(j + 1)^2} \left( \frac{j}{2} - \frac{j(1 - r)(r^{j+1} + 1) - 2r(1 - r^j)}{2(r + 1)(1 - r^{j+1})} \right)$$  \hspace{1cm} (A.13)

It is worth to notice that nontrivial denominator comes from $n$ of (2.3). The $\frac{1}{4}(j + 1)^2$ piece regularizes behaviour of $X_j(\mu)$ for small $\mu$. 
A.4 More pictures

In this section we show several additional pictures which help to understand the main paper.

Figure 5: Poles of $X_2(\mu)$ for $\psi = \{1,1,1,2,1,2\}$ displayed in the complex half-plane (Im($\mu$) $\geq$ 0) and the corresponding analytic expression below.

$$X_2(\mu) = \frac{12}{7\pi} \times$$

$$\frac{14680640\mu^{22} + 1871708160\mu^{20} + 12689080320\mu^{18} + 57839910912\mu^{16} + 189502291968\mu^{14} + 455242522624\mu^{12}}{654311424\mu^{24} + 9479127040\mu^{22} + 7148562024\mu^{20} + 30071192576\mu^{18} + 1319572668416\mu^{16} + 3603429982208\mu^{14} + 7414633218048\mu^{12}}$$

$$+ \frac{804831242240\mu^{10} + 1039513800192\mu^{8} + 958560474048\mu^{6} + 599204434384\mu^{4} + 227327105092\mu^{2} + 39573895547}{11483489935360\mu^{10} + 13232857409792\mu^{8} + 11057736083712\mu^{6} + 6304816157920\mu^{4} + 2204519902544\mu^{2} + 356177462887}$$

Figure 6: Distribution of poles of $X_2$ and $X_3$ for the substate $\psi$ of the length $M = 40$: $\psi = \{1,2,1,1,1,2,2,1,1,1,2,2,1,1,1,1,1,2,1,2,1,2,1,2,1,2,2,1,2,1,2,1,1,2,1\}$.
Figure 7: Below we present several figures comparing $X_j(\mu)$ (red dashed lines) and $\tilde{X}_j(\mu)$ (black lines) of (3.5) for real $\mu \in [-10,10]$. Separate figures on the right show the relative difference $\text{dis}_j$ given by (3.7). The displayed cases are $j = 1, 4, 8$ for $\psi = \{1, 1, 1, 2, 1, 2, 2, 1, 2, 1\}$. 

![Figure 7](image_url)
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