Entropy Currents for Reversible Processes in a System of Differential Equations. – The Case of Latticized Classical Field Theory –

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Abstract

We consider a very complicated system of some latticized differential equations that is considered as equations of motion for a field theory. We define macro state restrictions for such a system analogous to thermodynamical states of a system in statistical mechanics. For the case in which we have assumed adiabaticity in a generalized way which is equivalent to reversible processes. It is shown that we can define various entropy currents, not only one. It is indeed surprising that, for a two dimensional example of lattice field theory, we get three different entropy currents, all conserved under the adiabaticity condition.

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1 Introduction

In classical mechanics one can define entropy by specifying macro states, $A$, as Boltzmann’s constant times the logarithm of the phase space volume of the space of micro states corresponding to such a macro state:

$$S(A) = k \log(\text{PS vol}(A)). \quad (1.1)$$

Here $A \subseteq \text{Phase Space}$, (PS for short), is the set of all the micro states $(\vec{q}, \vec{p})$ that agree with the macro state $A$;

$$A = \{ (\vec{q}, \vec{p}) \mid (\vec{q}, \vec{p}) \in A \}. \quad (1.2)$$

We suppose that we can distinguish various “macro states” in the sense that collections (subset, roughly speaking) of states of the system at the fundamental level correspond to what we collect under one state in macroscopic thermodynamics. If we can assign these macro states entropies, we can define them as logarithms of the number of states in the macro state, according to usual definition [1][2][3]. This “number of states” in the macro state may be able to be defined in quantum mechanics, but in a classical description we either have to replace the number of states by a volume of the corresponding phase space or we have to discretize the phase space.

When the system is described macroscopically and behaves adiabatically, we shall seek to define a kind of currents very generally as below. The point $(\vec{q}, \vec{p})$ is phase-space point = micro state.

Usually one specifies a reversible development of the macro state by taking it to be very slow. This means that one can invert the rate of development of the macro state in time. If there is no such fact, it may cause significant influence on the micro physics and micro development to the second order in the development rate of the macro state $\dot{A}^2$. Combining this approximation with the second order effect of $\dot{S}(A(t))$ in the second law of thermodynamics $\dot{S}(A(t)) \geq 0$, requires

$$\frac{\partial S}{\partial A} = 0 \quad (1.3)$$

which is nothing but conservation of $S$, if we use a simple formal Taylor expansion

$$\frac{d}{dt} S(A(t)) = \frac{\partial S}{\partial A} \dot{A}(t) + \frac{1}{2} \frac{\partial^2 S}{\partial A^2} \dot{A}^2(t) + \cdots. \quad (1.4)$$
Here we neglect all but first term in the slow limit.

Part of the motivation of the present article is the hope to make more general
definition of reversibility criteria.

A direction in which such generalization has already been done in many articles
and textbooks. We also go and seek to construct entropy distribution in space and
entropy current, so to speak answer where the entropy is.

We would contemplate to specify in which field variables would sit the entropy
in a latticized field theory. In such a latticized field theory one has collections of
variables and equations of motion. One could hope to obtain a general description
as to how the entropy flows under certain restrictions of macro state.

In the long run one might seek to study entropy flow in a Euclidean field theory
(latticized or not). In studying this it is most elegant and easy to work in the case
of reversible processes because the entropy gets conserved. In such processes $\dot{S} = 0$
holds so that the second law of thermodynamics is fulfilled in a trivial manner and
furthermore time reversal invariance too.

Therefore this second law is not very relevant if we restrict ourself to reversible
processes as is actually the case in the present article. This touches on a motivation
of ours for present work: We study entropy flow without necessarily assuming second
law at the outset.

This paper is the first attempt in a series of papers on this entropy flow subject.
We then expect that the entropy will be conserved. In fact in cases where we have
latticized field theories we would expect that we could find a conserved current $j^\mu_s(x)$
of entropy

$$\partial_\mu j^\mu_s(x) = 0. \quad (1.5)$$

It is the main purpose in the present article to discuss and construct such an
entropy current under an abstract and general set up. However we investigate in
detail only in a couple of examples the most important of which is a certain triangular
lattice in two space time dimensions.

We shall meet in this article a few surprising results in connection with defining
such entropy currents for the latticized field theory models: It turns out that in two
dimensional space time we actually have to define three different entropy currents
$j^\mu_A(x), j^\mu_B(x)$ and $j^\mu_C(x)$ rather than just one as one might have a priori expected.

When one talks about entropy it is normal to think about the second law of
thermodynamics tells that entropy \[4\] will always increase or stay constant, but
never fall down. If we, however, as in this article, restrict ourselves to the adiabatic
process the entropy gets constant and the second law is not so relevant anymore.
In fact it is realized trivially. We would rather like in the present work to think of
it under the condition that we have imposed an abstractly defined adiabaticity \[5\]
condition. Once this is assumed to be valid for our model we will be able to find
the conserved entropy current without having to make any use of the second law of
thermodynamics. We therefore would like to think of our calculations as performed
in a world without having any second law of thermodynamics at all. In other words
we should rather think of the present article as a work relevant for working prior
to the second law of thermodynamics. Thus we should be able to make use of
our considerations in an attempt to derive the second law from physics assumed at
a more fundamental level. We must though admit that in the present article we
assume adiabaticity: It is defined though in a so abstract manner that we do not
need second law for that either. Thus our considerations could be especially relevant
for models with compact space time for which non-trivial relevance of second law of
thermodynamics may not be possible.

Since we only consider trivial realization of the second law of thermodynamics,
\(\dot{S} = 0\), the time orientation associated with this second law becomes quite relevant.
Thus we also do not really use the metric tensor for space time and through most
of the present article it is not specified whether we use Minkowskian or Euclidean
metric. However the lattice which we use in the example is a two dimensional
triangular one and it has a 120° degree rotation symmetry. This symmetry suggests
a Euclidean metric since it is a subgroup of the Euclidean rotation.

The lack of specification of a time axis orientation immediately calls attention
to the fact that it needs a convention of sign in order to define an entropy density
\(j^0(x)\) which is part of an entropy current \(j^\mu_s(x)\).

The present paper is organized as follows: All through the article we shall make
use of the concept of macro states or rather macro restriction in the sense of a subset
of all the micro states. There are fundamental field configurations, corresponding to
a class of such states conceived as not distinguished in the macroscopic description.

In section 2 we introduce our two dimensional classical lattice field theory. In
section 3 we describe the restriction on the field \(\varphi\). In section 4 argue a reversability
(or adiabaticity) requirement as a local principle of no loss of micro solutions by local
interplay. In section 5 we give a one dimensional example that illustrates our formalism. In section 6 we then go to our main example, the two dimensional latticized theory. In section 7 we then define entropy currents $j_A^\mu$, $j_B^\mu$ and $j_C^\mu$ associated with each “class of half curves on the lattice”. In section 8 we connect these quantities with our entropy flow currents by some equations. In section 9 we emphasize that we obtained three entropy currents. In section 10 we point out two alternative way of constructing our three entropy currents. In section 11 we then actually construct one of them, $j_A^\mu$ using conservation law in our formalism. In section 12 the continuum limit is considered for this current $j_A^\mu$. In section 13 the formalism is described analogously for the rest two currents $j_B^\mu$ and $j_C^\mu$. In section 14 a relation between the three entropy currents is presented. In section 15 we seek to understand three entropy currents by relating them to numbers of solution. Section 16 is devoted to present conclusion and outlook.

2 Introduction of lattice

We want to study reversibility hypothesis tells us for a field theory with space so that we can study how entropy may flow.

For simplicity we shall exercise by a two dimensional space-time world and even discretize it into a lattice.

We may at first set up the lattice theory quite naively by assuming that we have it as a system of first order differential equations in which the derivatives are discretized and made into differences. We may think of the discretized derivatives to be one-sided; for instance we may choose the forward difference equation to the discretized differential one

$$\frac{\partial}{\partial x} f(x) \longrightarrow \frac{1}{a} \{f(x+a) - f(x)\}, \quad (2.1)$$

where $a$ is a lattice constant that can be set to one and $f(x)$ a function. We shall end by having the differential equations turned into an equation set, one per site. This equation will involve the fields on three different sites, namely

1) on the selected site

2) on the site following the above first one in the x-direction

3) on the site following the second one in the y-direction.
It is the latter two sites that are needed to make the discretized derivatives. In d-dimensional space there will be analogously a need for involving at least $d + 1$ lattice sites in the equations. It is important for having the number of equations correctly that we have just one such $d + 1$ ($= 3$ in $d = 2$ case) equations for each site. We may draw symbolically as Fig. 1 the involvement of the sites in the equations by encircling the sites involved in an equation. Then the just described set up comes to look as Fig. 1.

![Figure 1: Field equations on a lattice](image)

There is indeed the same number of sites ●'s representing fields per unit area as the number of equations of motion as depicted in Fig. 1. The encircling lines ○ represent the field equations. So if we had a compact two dimensional space - a torus say - covered by this pattern there would be equally many variables as equations. The ○-symbols represent equally many equations as the field components. Thus we expect generically that we get an equation system with just a discrete set of solutions. At first you might expect this equation system to have just of order of unity solutions. As we already know there is much more than of order of unity solutions for systems behaving chaotical behavior.

We might draw more elegantly the lattice by deforming it so as to draw as Fig. 2 in a hexagonally symmetric way:
3 Formulation of equation of motion as a relation between field values on lattice.

Suppose we have a lattice of points with some fields defined on each of these points. We may be so abstract that there may not even be the same number of components and type of fields at each lattice point, but such an extension will not play important role in this section. We can give a name to the set of field configurations $F$. Then a field development through all times of the whole system is described at the micro level by a function $\varphi$ defined on the lattice $L$ and taking values in $F$. That is to say

$$\varphi : L \to F.$$  \hspace{1cm} (3.1)

If we want to be very general and let $\varphi$ take values that are specific for the various sites $s \in L$ we can just formally put the value set $F_s$ allowed for $\varphi(s)$ into $F$ as subset and write $\varphi : L \to F$ with restrictions $\varphi(s) \in F_s \subseteq F$.

We then introduce the concept of a “macro restriction”. The idea is that it generalizes the restriction that some part of the world at some time in some era is in a special macro state in the thermodynamical sense.

The macro restriction is meant to be the restriction on the micro degrees of freedom, i.e. the field $\varphi$ at the lattice points which ensures this micro configuration to be conceived as the macro state. It is taken to define the “macro restriction” in question. By generalizing this concept of macro (state) restriction, we would like to introduce a more broad concept of restriction between field values on the lattice points. Especially it should be possible to have such restrictions between field
configurations in successive time moments and thus we could consider the equations of motion as a special case of a restriction.

The macro restriction is now defined to be a constraint on the function \( \varphi \) on a subset, \( B \) say, of the lattice \( L \). That is to say the “macro restriction” is defined by specifying a subset of functions

\[
R \subseteq \{ \chi : B \to F \text{ with } \chi(s) \in F_s \}.
\] (3.2)

We shall be mainly interested in “local macro restrictions” by which we mean that the lattice points of \( B \) lies in the neighborhood of each other.

Then we think about a possible macro description as a set of macro restrictions \( \{ R_i \mid i \in I \} \) where \( I \) denotes the set of all positive integers. Thus we require about an “allowed” map

\[
\varphi : L \to F \quad (\varphi(s) \in F_s)
\] (3.3)

that it should obey

\[
\forall i \in I [ \varphi \mid \text{restricted to } B_i \in R_i ].
\] (3.4)

The easiest to think about is really to think of the \( F_s \)’s (or \( F \)) as discrete countable or even finite sets, so that we simply can count the number of functions of the type

\[
\varphi : L \to F \quad (\varphi(s) \in F_s)
\] (3.5)
corresponding to a given “macro description” \( \{ R_i \mid i \in I \} \).

We have to keep in mind that such a macro description is a description from a macroscopic point of view of the time development of some classical field theory. The lattice \( L \) is a space time lattice so that each lattice point represents both a space point and a moment of time.

The latticized equation of motion would mean - typically for a first order differential equation - a relation between \( d+1 \) neighboring lattice point fields. Here \( d \) is the number of space time dimensions. Such restrictions \( \varphi \) of equation of motion to a subset of \( d+1 \) neighboring points \( B_i \) are formally of a completely similar form as the macro restrictions (3.4) The function subsets \( R_i \) connected with the equations of motion are of a special type. In fact in the simplest case

\[
CnB_i = d + 1
\] (3.6)

for these equations of motion restriction. Here cardinal number \( Cn \) is given by \( CnB = \) the number of elements in \( B \). That is to say we can consider the equation of
motion as a lot of macro restrictions with the $B_i \cdot S$ involving $d + 1$ lattice point i.e. $ktB_i = d + 1$, where $kt$ means cardial number of the set. The set $R_i$ for the equations of motion which represent macro restrictions are then subsets of dimension $d \cdot n$ of the full $(d + 1) \cdot n$ dimensional space on the space of all functions defined on the subset $B_i$ of the lattice. Here we denoted by $n$ the dimension of $F_s$’s - assumed to be the same for all $s \in B_i$ - so that the $n$ equations of motion just bring the dimension of the space of functions $\varphi$ restriction to $B_i : B_i \rightarrow F_s$ down from $(d + 1) \cdot n$ to $d \cdot n$.

4 Principle of no loss of micro solutions

developments by local interplay of restrictions

In this section we want to formulate the requirement of only reversible (or adiabatic) processes.

We have recently pointed out [6] that if one imposes a specific periodicity on a mechanical system, then the system is generically forced to behave reversibly. Thus we could use such a special model with imposed periodicity that the model has compact space time as a simple example of a model in which reversibility is imposed. But let us stress that this is just an example and that we in general we assume that we work with reversible processes only.

As the simplest case we want to consider a compactified space time so that there is no such thing as $t$ or the space coordinate running off to infinity.

Since in this case we expect very few solutions if any, it may sound a little strange to make statistical considerations concerning these solutions. Let us, however, at first think in this statistical way and imagine that for a given system of macro state restrictions one can ask for a probability for solutions there.

For many combinations of macro restrictions we really risk that that probability for at least one solution would be extremely low. One might assign even meaning to low probability for getting a solution because it would occur with a tiny probability measure in the parameter space.

In our article we actually introduced a way of working with generic equations of motion by taking parameters in the Hamiltonian as random numbers, so that one could in principle ask for probabilities of getting various numbers of solutions.

Naturally we should consider the case that the higher the number of allowed
functions $\varphi$ ("solutions") the more likely is a certain system of macro restrictions, what we called a macro description. So we should be most interested in macro scenarios with the largest number of allowed functions $\varphi : L \rightarrow F$. We can say in this way when we have taken $F$ or $F_i$'s discretized form so as to be able to simply count functions $\varphi$. However, we may treat quite analogously the cases in which we think of $F$ as a space with continuous coordinates on it, provided it is possible at the end to define a measure on the set of solutions $\varphi$ that obey the equation of restrictions. Then such measure may replace the counting in the totally discrete case.

Typically $F$ would be phase space with a Liouville measure on it, but for pedagogical reasons we shall at first consider the discretized case in which we can simply count the number of functions $\varphi$. If we had not made the space on which the lattice $L$ is distributed a compact one the lattice $L$ might be infinite and we may for that reason also get problems with pure counting. Since we shall, however, be most interested in small local regions we shall be satisfied with counting; we first of all look for pieces of the lattice $L$ with only a finite number of sites.

We should impose the following principle on the system of restrictions: Whenever the sets of lattice points $B$ and $D$ involved in two restrictions $(R_B, B)$ and $(R_D, D)$ have a non-trivial overlap $B \cap D$, then the restrictions here will be consistent in the sense that the imposition of one of the restrictions must not reduce the number of field configurations allowed on the overlap $B \cap D$.

We may have to keep in mind that if we have in certain macro scenario two macro restrictions defined on the subsets $A$ and $B$ of the lattice with $A \subseteq B$ then we can replace those macro restrictions by a single one $"R_A \cap R_B"$ on the bigger subsets of the ones of $L$, namely $B$. That is to say we can choose a new restriction defined on $B$:

$$"R_A \cap R_B" = \{ \varphi \mid \text{restriction to } B \} \varphi \mid \text{restriction to } B \in R_B \land \varphi \mid \text{restriction to } A \in R_A \}$$ (4.1)

We can thus reduce our considerations to macro scenarios built up from a set $\{(R_i, B_i) \mid i \in I\}$ where non of the $B_i$'s are contained in any other one.

If two such restrictions, say $(R_D, D)$ and $(R_B, B)$ have a non-empty intersection of their lattice sets $D \cap B \neq \emptyset$, then we can ask for whether the restriction of $\varphi$ to the overlap region $\varphi \mid \text{restricted to } B \cap D$ is allowed to be the same set of restricted
functions for $(R_D, D)$ as for $(R_B, B)$. If the \( \varphi \big|_{\text{restriction to } B \cap \overline{D}} : B \cap D \to F \) functions allowed by \( (R_D, D) \) and \( (R_B, B) \) are not essentially the same then the functions over \( B \) belonging to the set \( R_B \) which are also allowed by \( (R_D, D) \) will be reduced in number relative to the number of functions in \( R_B \).

Under the statistical way of which we would like to justify by some physical considerations above in section 3, there is only a fraction of the set of functions \( R_B \) that can be realized. That will at the end reduce the number of allowed \( \varphi \) by the ratio giving the fraction of \( R_B \) which get allowed by \( (R_D, D) \). By having such a lack of match on the overlap region \( B \cap D \) we have basically lost a fraction of the potentially achievable allowed functions \( \varphi \). So keep to the maximal number of allowed \( \varphi \) functions, at least not reduce the number unnecessarily locally, we should require that the restrictions of \( \varphi \) to \( B \cap D \) for any pair of macro restrictions \( \varphi \big|_{\text{restriction to } B \cap \overline{D}} \) be the same set of functions on \( B \cap D \) from both \( (R_D, D) \) and \( (R_B, B) \). Therefore we must require

\[
\{ \hat{\varphi} : B \cap D \to F \mid \hat{\varphi} \text{ extendable to belong to } R_B \text{ on } B \} = \{ \hat{\varphi} : B \cap D \to F \mid \hat{\varphi} \text{ extendable to belong to } R_D \text{ on } D \} \tag{4.2}
\]

We can say that this requirement ensures that the macro restriction \( (R_B, B) \) does not - alone at least - require any drastic selection of the possible solutions of \( \varphi \) to \( D \) which are to be in \( R_D \). If this condition (4.2) is not satisfied then e.g. most elements (ordered sets) in the set of \( R_D \) restricted to \( B \cap D \) are not extendable into functions over \( B \) that belong to the set of functions \( R_B \). But that means that the ones which can be extended into \( R_B \) functions are only a tiny subset of all the functions in \( R_D \), so that if an \( R_B \) function is realized it would seem miraculous.

We would like to put in a remark about the very low accuracy with which we really intend to work here: Since our main goal is to obtain results about entropy flowing and conservation and entropy in an in most situations to be considered tiny Boltzmann’s constant \( k \) as unit. We will be satisfied with an accuracy in the number of \( \varphi \) function (by solving the restrictions). It is so low that deviations by a few orders of magnitude is considered quite negligible difference.

In the same spirit we also take it that two different sets of \( \varphi \) functions have typically numbers of elements which deviate by several orders of magnitude. Thus one will generically dominate the other one. In fact entropy is \( k \) times the logarithm of the number of micro states in a macro state and thus we do not need a high
accuracy. So if the number of elements in the two sets do not differ by a big factor it will not be so significant. Therefore suppose that there is indeed a significant lack of matching of the two sets of functions on \( B \cap D \) differ in number by such a big factor. Then if you find a \( \varphi \bigg|_{\text{restriction to } D} \) which matches \((R_B, B)\) by having its restriction to \( B \cap D \) extendable into an \( R_B \) function then it would seem very unlikely.

It should be recognized that our assumption (4.2) actually imply that we have only reversible processes and that the second law of thermodynamics is also fulfilled in the trivial manner \( \dot{S} = 0 \). In fact we could imagine that we attempted to have a system of macro states and macro states such that entropy increased (i.e. irreversible process). In this case we know that at later times \( t_L \) than the era of the irreversible process the possible micro developments – in our notation \( \varphi \) is a field in space time – could not be constructed. It represents all the configurations allowed by the high entropy macro states at such a later time \( t_L \). In other words there would be many states in the high entropy macro state at late time, but they could not be realized via equations of motion from a micro state in a previous low entropy macro state.

We may now argue that somewhere in between the two times, “low” and “high” entropies as discussed above, there must be some moment and somewhere, where we have some overlap between two sets of lattice points associated with equations of motion restrictions in which (4.2) is violated. We must have some moment of time from which we start to get those macro restrictions. There all the allowed micro states cannot be realized, because of the relation by equation of motion to a moment shortly before. If we denote by \((R_B, B)\) the earliest macro state restriction while the offensive restriction by \((R_D, D)\), we might reach a situation illustrated in Figure 3. In order that the equation of motion restriction symbolized by \((R_D, D)\)

![Figure 3](image)

**Figure 3**: Schematical figure of overlapping of two subsets \( B \) and \( D \) on the lattice.

or rather Fig. 3 will disturb all micro state possibilities in \((R_B, B)\), there must be a
non-empty overlap $B \cap D \neq \phi$. Since we actually selected the $(R_B, B)$ and $(R_D, D)$ such that the imposition of $\varphi$ obeying $R_D$ should diminish the development $\varphi$ that is allowed for the macro restriction $(R_B, B)$ we will not fulfill (4.2). Rather we should have

$$\left\{ \varphi \mid_{B \cap D} \varphi \in R_B \land \varphi \in R_D \right\} \subset \left\{ \varphi \mid_{B \cap D} \varphi \in R_D \right\}$$

(4.3)

Thus we see that, if we assume validity of (4.2) in all the cases of overlapping restrictions which are either macro – or equation of motion ones, then there is no place for irreversible processes.

By making the similar argument time reversed way we can also find that our assumption (4.2) also means that the entropy cannot decrease because that would imply that at some earlier than the entropy decreasing era the micro states allowed by the macro restriction could not be realized due to equations of motion.

This completes the argument that our assumption (4.2) leads to be satisfied trivially second law so that no irreversible processes occur.

5 An example in one dimension

To illustrate our formalism we may use it to a simple case, one dimensional lattice of time moments and a general mechanical system developing through a series of macro states $A(t)$ with entropies $S(A(t))$, at a site $t$.

At first this models has the following two types of macro restrictions:

1) The ones specifying just the restriction of $\varphi$ to a single discrete site $t$ by

$$\varphi \mid_{\text{restriction to } \{t\} \in R_t} = \{ \hat{\varphi} : \{t\} \to F \mid \varphi(t) \in A(t) \}$$

(5.1)

2) The ones restricting the restriction of $\varphi$ to two neighboring sites on $t$-axis i.e. to $\{A, A + a\}$ say. These are the macro restrictions implementing the equations of motion and have the form

$$R_{\{t, t+a\}} = \{ \varphi \mid_{\text{restriction to } \{t, t+a\} \in R_{\{t, t+a\}}} \}$$

$$= \left\{ \hat{\varphi} : \{t, t+a\} \to F \mid \hat{\varphi}(t+a) = \hat{\varphi}(t) + a\eta \frac{\partial H(\hat{\varphi}(t))}{\partial \hat{\varphi}} \right\}$$

(5.2)
which is nothing but the discretized Hamilton equations. Here $\eta$ is the antisymmetric matrix which takes the form

$$
\eta = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

(5.3)

with an appropriate ordering of the components of $\hat{\varphi}$ conceived of as generalized coordinates and their conjugate momenta. We argued above that one should not consider macro restrictions for subsets of lattice subsets used for other macro restrictions. Rather one should combine them into macro restrictions for the biggest of the subsets so as to end up with subsets either fully disjoint or only partly overlapping.

In the example here it means that we shall so to speak absorb the macro restrictions for $\{t\}$ and $\{t + a\}$ into that for $\{t, t + a\}$. In this way we get replacement macro restrictions for the subsets of the lattice of the form $\{t, t + a\}$. In fact we get the following simple expression,

$$
\varphi \bigg|_{\text{restriction to } \{t, t+a\} \in R^{\text{repl.}}_{\{t, t+a\}}} = \{ \hat{\varphi} : \{t, t + a\} \rightarrow F \bigg| \hat{\varphi}(t) \in A(t) \wedge \varphi(t + a) \in A(t + a) \wedge \hat{\varphi}(t + a) = \hat{\varphi}(t) + a\eta \frac{\partial H(\hat{\varphi}(t))}{\partial \hat{\varphi}} \bigg\}
$$

(5.4)

We either work with undiscretized phase space for $F$ or we discretize into points with a given density so that each point gets a cell of volume $h^N$ where $N$ is the number of degrees of freedom, $n = 2N$. It is a density proportional to phase space density. In any case we can use that the time development under the Hamilton equations is a canonical transformation and as such conserves the phase space volume. This means that the original relation $R_{\{t, t+a\}}$ from (5.2) allows just one value at $t$ for each value at $t + a$ and oppositely. If the two entropies $S(A(t))$ and $S(A(t + a))$ happen to be equally big it is at least possible that the replacing macro restriction (5.4) simply allows realization of any point in $A(t)$ as well as any point in $A(t + a)$. However if these two entropies are not equal it will be the smaller one of the two entropies $S(A(t))$ and $S(A(t + a))$ that determines the size of the set $R^{\text{repl.}}_{\{t, t+a\}}$ of restrictions of $\varphi$ to $\{t, t + a\}$ allowed by the combined macro restriction eq.(5.4). In fact

$$
\log CnR^{\text{repl.}}_{\{t, t+a\}} = \min\{S(A(t)), S(A(t + a))\}.
$$

(5.5)

If we go to consider if two neighboring ones out of these replacing macro restrictions $R_{\{t, t+a\}}$ and $R_{\{t-a, t\}}$ we ask whether the restrictions to the overlap region $\{t - a, t\} \cap \}$
\{t, t + a\} = \{t\} can match. For such a matching it would at least be needed that the cardinal number \(C_n\) for the two different sets of functions would be the same:

\[
C_n \left\{ \hat{\phi} : \{t\} \to F \middle| \hat{\phi} \text{ extendable to } R_{\{t,t+a\}}^{\text{repl.}} \right\} = C_n \left\{ \hat{\phi} : \{t\} \to F \middle| \hat{\phi} \text{ extendable to an } R_{\{t-a,t\}}^{\text{repl.}} \text{ function} \right\}. \tag{5.6}
\]

But now we quickly see that these numbers are \(\min\{S(A(t)), S(A(t + a))\}\) and \(\min\{S(A(t - a)), S(A(t))\}\) respectively. The condition for any chance of matching is therefore needed that

\[
\min\{S(A(t)), S(A(t + a))\} = \min\{S(A(t)), S(A(t - a))\}. \tag{5.7}
\]

We can also say that the only entropy numbers relevant for the replacement \(R_{\{t,t+a\}}^{\text{repl.}}\)’s, namely \(\log C_n R_{\{t,t+a\}}^{\text{repl.}}\), have to be the same i.e. \(\log C_n R_{\{t,t+a\}}^{\text{repl.}} = \log C_n R_{\{t-a,t\}}^{\text{repl.}}\). That is to say for all the information about the entropies of the \(A(t)\)’s it has left any footprint into the \(\log C_n R_{\{t,t+a\}}^{\text{repl.}}\) and actually thereby to the physics of the model these entropies \(S(A(t))\) must be constant down along the chain of time. We could simply use the Taylor expansion of a smooth \(S(A(t))\) as function of \(t\) and insert it into our condition eq.(5.7) and we would deduce

\[
\dot{S}(A(t)) = 0. \tag{5.8}
\]

In other words, our matching condition “to have no miracles locally” (in either way of time direction thinking) leads to the constancy of entropy in the one dimensional example which we just presented.

### 6 The two dimensional example

Using the same procedure in our two dimensional lattice example we take it that the biggest subsets of the lattice for which we have macro restrictions are the three element ones associated with the equation of motion. Basically we should then absorb the other macro restrictions into these three point ones, which should then replace all the others. After this replacement - a kind of absorption - there will be no longer any tracks left of macro restrictions associated with subsets \(B\) of the lattice contained in the surviving three-point subsets. Suppose e.g. that the macro states imposed on the three sites in a subset \(D\) of the lattice associated with the equation
of motion are called $G$, $J$, $Q$. That is to say we have in the lattice depicted in Fig. 4:

![Diagram of lattice with points G, J, Q]

Figure 4: Equation of motion for three points on the lattice

Here $s_G$, $s_J$ and $s_Q$ are the names of the sites. Whatever these macro restrictions to the macro states $G$, $J$, and $Q$ originally would have been the information surviving into the replacement $R_{D}^{\text{repl.}}$. macro restriction from these macro states would only be some macro states which could then be called $G_D$, $J_D$ and $Q_D$. They will be defined from the $R_{D}^{\text{repl.}}$ as

$$G_D \doteq \{ \hat{\phi}(s_G) \mid \hat{\phi} : \{s_G, s_J, s_Q\} \to F \land \hat{\phi} \in R_{D}^{\text{repl.}} \},$$

$$J_D \doteq \{ \hat{\phi}(s_J) \mid \hat{\phi} : D \to F \land \hat{\phi} \in R_{D}^{\text{repl.}} \},$$

$$Q_D \doteq \{ \hat{\phi}(s_Q) \mid \hat{\phi} : D \to F \land \hat{\phi} \in R_{D}^{\text{repl.}} \}.$$

where $\{s_G, s_J, s_Q\} = D$. \hspace{1cm} (6.1)

Defined from another set of three points $E$ say which also contains say $s_G$ we get analogously

$$G_E \doteq \{ \hat{\phi}(s_G) \mid \hat{\phi} : E \to F \land \hat{\phi} \in R_{E}^{\text{repl.}} \}. \hspace{1cm} (6.2)$$

The condition to avoid locally miraculous restrictions - our principle - now comes to say e.g.

$$G_D = G_E \hspace{1cm} (6.3)$$

for a situation like:
This relation implies $S(G_D) = S(G_E)$ for the entropies.

In the spirit of only the information surviving into the $R^{\text{repl}}_E$, $R^{\text{repl}}_D$, etc accessible after all at all we could also redefine correlation of entropies between neighboring points - whenever present in one of our three - point $B$-sets - by putting say

$$S(G_D) + S(J_D) + (M_{GJ})_D = \log C n\{\hat{\varphi} : \{s_G, s_D\} \to F \mid \hat{\varphi} : D \to F \land \hat{\varphi} \in R^{\text{repl}}_D\}.$$  \hfill (6.4)

### 7 Defining three different entropy currents $j^\mu_A$, $j^\mu_B$ and $j^\mu_C$

We have found above a sort of conservation rules associated with parallelogram-like figures. We should have in mind that there are in our lattice three such types of parallelogram-like structure orientations. It is therefore natural that we shall construct three different entropy currents, which we may in the continuum limit denote $j^\mu_A$, $j^\mu_B$ and $j^\mu_C$ corresponding to the three different orientations of parallelograms to be associated with the conservation.

Before writing down complete form of the continuum limit currents $j^\mu_A$, $j^\mu_B$ and $j^\mu_C$ by means of the lattice quantities as it is our goal to do soon, we should look a bit more on how it comes that we have these three different concepts of entropy flow.

For this purpose we want to point out corresponding to one of the orientations of our parallelogram-like structures there is a restricted class of curves on the lattice.
Such a class of curves is by definition a curve of the type allowed for either the full drawn or the broken lines forming what we could call the two sides of the “parallelogram-structures” as is depicted in Fig. 6. These curves may be described by thinking of them as oriented meaning with an arrow along them and then they are allowed to be composed from only two types of links. For instance we define corresponding to a parallelogram-like structure of the orientation

![Figure 6: Two sides of parallelogram structure](image)

have that in the “positive” direction along the curves allowed in the class which we call here class \(A\), there are only links corresponding to space time vectors \((\sqrt{\frac{3}{4}}a, \frac{1}{2}a)\) and \((\sqrt{\frac{3}{4}}a, -\frac{1}{2}a)\). Here \(a\) is the hexagonal lattice constant, i.e. the length of the sides of the triangles. Starting from a lattice site \(x^\mu\) say we can by composing these two vectors in succession construct an infinite number of half-curves extending from this point \(x^\mu\) as in figure 7:

![Figure 7: Half-curve with only the positive direction](image)

We will call such a family of half-curves extending from \(x^\mu\) the \(A\)-class of half curves extending from \(x^\mu\). It will be later of interest that with the equations of motion associated to every other of the triangles in the lattice as already described, we can use these equations of motion to predict the restriction of a solution to the
equations of motion to one of the half-curves in class $A$ extending from $x^\mu$ say $\alpha$ to the restriction of an other half curve in the same class $A$ also extending from $x^\mu$, say $\beta$.

Figure 8: Two half-curves in class $A$, both of which start from $x^\mu$. Two points of the black triangle uniquely specifies the 3rd point and its field values which give a restriction to solutions of equations of motion.

In fact one may easily convince oneself by looking at figure 8 that supposing a solution on the half curve $\alpha$ - here $\alpha$ is written with the line - is given, then the values of that solution on the $\beta$ - half-curve can be calculated without further input information. In fact one shall just use the rule that when at two points in a black triangle the solution values are known then the value on the third corner of this triangle is uniquely given by equation of motion. Basically the uniqueness is argued for it relatively from curve $\alpha$ to curve $\beta$. From this unique predictability from one half-curve $\alpha$ to another one $\beta$ in the class we may think of a speculation of $\varphi$ along any curve in class $A$ from $x^\mu$ to tell the same information about $\varphi$. That is to say that by choosing say the class $A$ one has specified in detail what could be called the “information” about a solution $\varphi$ to the equations lying to the $A$-side of a space-time point $x^\mu$. We will say that the information about solutions $\varphi$ contained to the $A$-side of $x^\mu$ is the information contained in the restriction of $\varphi$ to one of the half-curves, $\beta$ say, extending from $x^\mu$ and belonging to class $A$, which again means composed successively from $(\frac{1}{2}a, \sqrt{\frac{3}{4}}a)$ and $(-\frac{1}{2}a, \sqrt{\frac{3}{4}}a)$. 

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As stressed it does not matter which of these half-curves we use, they all give the same information about solutions. This question of information about solutions really gets precise by talking about that solutions $\varphi$ can be classified into classes inside which the solutions have the same restriction to all the half-curves of $A$-type extending from $x^\mu$. But again we have to have in mind it is enough to require two solutions $\varphi_1$ and $\varphi_2$ to have the same restriction

$$\varphi_1|_\alpha = \varphi_2|_\alpha$$

(7.1)

to one half-curve in the class of curves then it will be true for all the half-curves of type $A$ extending from $x^\mu$.

If you therefore want to count the number of allowed solutions corresponding to a macro scenario and to define how the information needed to specify a solution lies to one side or to the other side of the space time point $x^\mu$, we may make the concept of these “sides” meaningful by saying that the information on the $A$-side is the one contained in the restriction of $\varphi$ to one of the half-curves, $\alpha$, i.e. the information contained in $\varphi|_\alpha$.

Ignoring the problems of infrared divergences we could say that, if we consider the set of all solutions $\{\varphi\}$ of a given macro scenario and ask for how many different restrictions $\varphi|_\alpha$ to a half-curve $\alpha$ in the $A$-class extending from $x^\mu$, the logarithm of this number represents the entropy to the $A$-side of $x^\mu$. The counting just by numbers is only possible if we have discretized the value space for solutions $\varphi$ so that they become countable, but then one could instead imagine using a method of the type of measuring phase space volume. Even if infrared divergence problems makes it ill-defined what the full amount of entropy to the $A$-side of $x^\mu$ is, we can still get quite meaningful convergent results for the entropy on the $A$-side of $x^\mu$ subtracted from that to the $A$-side of another space time point $y^\mu$, because we can now find $A$-type half courses extending from $x^\mu$ going to follow or coincide with an $A$-type half-curve extending from $y^\mu$ from some point on as depicted in Fig. 9:
Figure 9: The $A$-type half-curve of $x^\mu$ coincides with another $A$-type half-curve extending from $y^\mu$.

Then we can use alone the restrictions of the solutions to the pieces of the half-curve which are not coinciding, and these pieces will be finite.

It should be kept in mind that we want to identify the entropy on the $A$-side of $x^\mu$ with the number of solutions in the given macro scenario which is characterized by each class having its own restriction to the type A half-curves extending from $x^\mu$.

It should thus be understood that to the extend that we can identify this newly defined concept to the $A$-side of a space time point $x^\mu$ from a more usual concept of one side of a point in a one-space dimension world we may use such a concept to make a meaning where entropy is placed and thus entropy density and furthermore entropy flow.

8 Equations for $S(x)$ and $\log CnR(x)$

We want to compute what the conservation laws $\partial_\mu j^\mu_A = \partial_\mu j^\mu_B = \partial_\mu j^\mu_C = 0$ for the three currents mean for the two scalar fields $S(x)$ and $\log CnR(x)$ in terms of which we managed to write them. Since $S(x)$ always come into the expressions for the currents via the difference $S(x) - \log CnR(x)$ we shall give this quantity the name $D(x)$

$$D(x) \equiv S(x) - \log CnR(x).$$

(8.1)

Let us also define for the three types half-curve systems $A$, $B$, and $C$ the unit vectors orthogonal to the average direction of these half-curves

$$\theta^\mu_A = (0, 1)^\mu, \quad \theta^\mu_B = (-\sqrt{\frac{3}{4}}, -\frac{1}{2})^\mu, \quad \text{and} \quad \theta^\mu_C = (-\sqrt{\frac{3}{4}}, -\frac{1}{2})^\mu$$

(8.2)

as well as the vectors in these average directions

$$H^\mu_A = (1, 0)^\mu, \quad H^\mu_B = (-\frac{1}{2}, \sqrt{\frac{3}{4}})^\mu, \quad \text{and} \quad H^\mu_C = (-\frac{1}{2}, -\sqrt{\frac{3}{4}})^\mu.$$

(8.3)

Notice that you obtain $\theta^\mu_A$ by rotating $H^\mu_A$ by $90^\circ$ anticlockwise and analogously $\theta^\mu_B$ by rotating $90^\circ$ $H^\mu_B$ and so on. Using this notation we can write expressions for the three entropy currents become

$$j^\mu_A(x) = \theta^\mu_A \left \{ D(x) + \frac{a}{\sqrt{3}} H_A^\mu \frac{\partial}{\partial x^\mu} \log CnR(x) \right \} + \frac{3a}{4} \varepsilon^\mu_{\nu\rho} \frac{\partial}{\partial x^\rho} \log CnR(x)$$
\[ j^\mu_B(x) = \theta^\mu_B \left\{ D(x) + \frac{a}{\sqrt{3}} H^\rho_B \frac{\partial}{\partial x^\rho} \log CnR(x) \right\} + \frac{\sqrt{3} a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x) \]

\[ j^\mu_C(x) = \theta^\mu_C \left\{ D(x) + \frac{a}{\sqrt{3}} H^\rho_C \frac{\partial}{\partial x^\rho} \log CnR(x) \right\} + \frac{\sqrt{3} a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x). \] (8.4)

Since the topological current term \( \frac{\sqrt{3} a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x) \) is trivially conserved the nontrivial information from the conservation of the three currents comes only from the conservation of the first parts and that tells

\[ \theta^\mu_A \partial_\mu D + \frac{a}{\sqrt{3}} \theta^\mu_A H^\rho_A \partial_\mu \partial_\rho \log CnR(x) = 0 \]

\[ \theta^\mu_B \partial_\mu D + \frac{a}{\sqrt{3}} \theta^\mu_B H^\rho_B \partial_\mu \partial_\rho \log CnR(x) = 0 \]

\[ \theta^\mu_C \partial_\mu D + \frac{a}{\sqrt{3}} \theta^\mu_C H^\rho_C \partial_\mu \partial_\rho \log CnR(x) = 0. \] (8.5)

It is easy to see - as we essentially already did - that the sum of these three equations is trivially zero just from \( \partial_\rho \partial_\mu = \partial_\mu \partial_\rho \). So there is in reality only two independent equations, just enough to determine \( D(x) \) up to an additive constant. However, one may wonder whether there exists any \( D(x) \) function at all satisfying these equations. Indeed the condition for that to be the case is that the partial derivatives of second order for \( D(x) \) derived from these equations are consistent with commutativity of the partial derivatives. For that we can consider the condition that

\[ (\theta^\mu_A \theta^\nu_B - \theta^\mu_B \theta^\nu_A) \partial_\mu \partial_\nu D = 0 \] (8.6)

The idea to obtain eq. (8.6) is to differentiate the first of the three equations in the \( \theta^\mu_B \)-direction, i.e. act on it with \( \theta^\nu_B \partial_\nu \), and then compare with what we get from the second equation by acting with \( \theta^\nu_A \partial_\nu \). It is easily seen the consistency condition for the two first equations manipulated this way becomes

\[ (H^\mu_A - H^\mu_B) \theta^\rho_B \partial_\rho \partial_\nu \log CnR(x) = 0. \] (8.7)

Since indeed \( H^\mu_A - H^\mu_B \propto \theta^\mu_C \) and we can symmetrize in the three indices \( \mu, \nu, \rho \) because \( \partial_\mu \partial_\nu \partial_\rho \log CnR(x) \) is a symmetric third rank tensor we may write this condition

\[ \theta_{\mu A}^\mu \theta_{\nu B}^\nu \theta_{\rho C}^\rho \partial_\mu \partial_\nu \partial_\rho \log CnR(x) = 0. \] (8.8)

We can consider the symmetric third rank tensor

\[ \hat{\xi}^{\mu\nu\rho} = \theta_{\mu A}^{\mu} \theta_{\nu B}^\nu \theta_{\rho C}^\rho \] (8.9)
connected with our lattice and the selection of triangles to be associated with equations of motion.

It is easy to see that if indeed \( \log CnR(x) \) obey the third order homogeneous differential equation then we can from \( \log CnR(x) \) integrate up to an additive constant.

9 The stressing of the three types of entropy current

Now, however, it is crucial for our discussion that there is on our lattice three types of entropy currents under rotation by 120° in the space-time plane choices of half-curve classes or equivalently parallelogram-like structure of orientations. We denote them by A, B, and C and we have already discussed that the curves in class A are made by successively compassing steps \( (\sqrt{\frac{3}{4}}a, \frac{1}{2}a) \) and \( (\sqrt{\frac{3}{4}}a, -\frac{1}{2}a) \). Analogously we let the B-type of half-curves be compared successively from steps \( (0, a) \) and \( (-\sqrt{\frac{3}{4}}a, \frac{1}{2}a) \). And finally we have also a class of half-curves defined by the steps \( (0, -a) \), and \( (-\sqrt{\frac{3}{4}}a, -\frac{1}{2}a) \). It should be seen that apart from the sign with which the links are allowed into the three defined types of half-curves which are thought of as the one oriented away from the starting point \( x^\mu \), there are only three directions of links in the lattice that can be used as steps for the curves. Thus for example \( (\sqrt{\frac{3}{4}}a, -\frac{1}{2}a) \) used in the A-type half-curves is re-used with opposite sign, \( (-\sqrt{\frac{3}{4}}a, \frac{1}{2}a) \) as a step for type B.

Let us also have in mind that there is one type of what we called parallelogram-like structures that can be called type B, one that can be called type A and so on.

10 Construction of the three entropy currents

To construct continuum limit entropy flows we may choose one of the following routes:

1) We may take as starting point of the link-entropy conservation rules which really says:

If we form a closed curve from two finite pieces of curves of say type \( \alpha \), and \( \beta \), -
so that they form together one of the parallelogram-like structures of orientation $A$ - then the sum of the link-entropies is

$$K_{GH} = \frac{1}{2}(S_G + S_H) - \log Cn(R_{GHL})$$

(10.1)

where along $\alpha$ and along $\beta$ are equal

$$\sum_{\rightarrow \in \alpha} K_{\rightarrow} = \sum_{\rightarrow \in \beta} K_{\rightarrow}$$

(10.2)

2) We may take as the starting point the entropy definition by means of number of different restriction to a half-curve $\alpha$ say, i.e. $\varphi |_\alpha$, in the solution set for the given macro scenario.

Note that in both cases we have to select one of the types of curves $A$, $B$ or $C$. We must therefore a priori expect that can and must define three different entropy currents, each of which have to be marked by this choice $A$, $B$ or $C$. That is to say we shall define in continuum limit at first three different entropy currents $j_A^\mu(x)$, $j_B^\mu(x)$ and $j_C^\mu(x)$, all being conserved in the adiabatic case

$$\partial_\mu j_A^\mu(x) = \partial_\mu j_B^\mu(x) = \partial_\mu j_C^\mu(x) = 0.$$  

(10.3)

It would be unexpected to find so many entropy currents being truly different, and we would therefore expect - and indeed shall find below - that these three entropy currents $j_A^\mu(x)$, $j_B^\mu(x)$ and $j_C^\mu(x)$ are indeed related, so that they only deviate in a rather trivial way by constants.

### 11 Construction starting from the parallelogram-like structure conservation rule

Let us now contemplate how to construct a continuum limit entropy current density say $j_A^\mu(x)$ so that its conservation is related to the sum of the link entropy rule using curve pieces $\alpha$, $\beta$ of the $A$-type. From the links of the relevance in the $A$-case, the ones going in direction $(\sqrt{3}a, \frac{1}{2}a)$ or $(\sqrt{3}a, -\frac{1}{2}a)$, we can use the $K_{GH} = \frac{1}{2}(S_G + S_H) - \log Cn(R_{GHL})$ link entropies to construct the current density $j_A^\mu(x)$ in the region around the link(s) in question.

From the continuum limit approximation we will take it that when the lattice constant $a$ is small the value $K_{\rightarrow}$ of the link entropy varies slowly from one link to
the neighboring ones of the same direction. We do not know however a priori any good reason for that links with different direction should have approximately the same $K$ even in close to each other in space-time.

A priori we would set up an expression for $j^\mu_A(x)$ of the form

$$j^\mu_A(x) = \text{average around } (b^\mu K_\times + c^\mu K_\cdot)$$

(11.1)

where $K_\times$ and $K_\cdot$ symbolize the two different directions of links relevant for the curves of type $A$, and $b^\mu$ and $c^\mu$ are some constant space-time vectors (2-vectors) to be chosen so as to make $j^\mu_A(x)$ conserved and to satisfy possible other wishes. The average around $x$ means that we strictly speaking extract $j^\mu_A(x)$ over a region so large that the lattice structure is no longer felt.

In the continuum limit we can think of the two link entropies relevant for case $A$ as two functions $K_\times(x)$ and $K_\cdot(x)$ of the space-time point $x$. We may also introduce for instance the unit vectors along these link directions $e^\mu_\times$ and $e^\mu_\cdot$. Then the conservation law expressed by the parallelograms can be written in the continuum limit

$$e^\mu_\times \partial_\mu K_\times - e^\mu_\cdot \partial_\mu K_\cdot = 0$$

(11.2)

This would mean that if we defined a current

$$j^\mu_A(x) = b^\mu K_\times(x) + c^\mu K_\cdot(x)$$

(11.3)

with

$$b^\mu = e^\mu_\times$$
$$c^\mu = -e^\mu_\cdot$$

then the conservation rule with the parallelogram would lead to the conservation of this current.

If we choose the $b^\mu$ and $c^\mu$ in an other ratio or in other directions the current $j^\mu_A(x)$ will not be conserved.
12 Continuum limit for the expressions for $K_{GH}$ in terms of $S(x)$ and $\log Cn(R(x))$

In the continuum limit we should notice that in the expression for the link entropy

$$K_{GH} = \frac{1}{2}(S_G + S_H) - \log Cn(R_{GHL})$$

(12.1)

the center of the triangle $\triangle GHL$ associated with an equation of motion element is not quite at the same positions as lattice sites associated with the “entropies at sites” $S_G$ and $S_H$. Although in the very crudest approximation we would just put $K(x) = S(x) - \log Cn(R(x))$, this is therefore not quite true. We should rather say if we want to $K_{GH} = K(x)$ the $x$ is middle of the link $GH$ then clearly $\frac{1}{2}(S_G + S_H)$ will even including linear terms in a Taylor expansion, i.e. of order of the lattice constant $a$, be equal to $S(x)$. But the center of the triangle is displaced and we must rather take

$$\log Cn(R_{GHL}) = \log Cn(R(x)) + \chi^\mu \frac{\partial}{\partial x^\mu} \log Cn(R(x))$$

(12.2)

where $\chi^\mu$ is defined by

$$\chi^\mu = x^\mu_{center\triangle GHL} - x^\mu_{centerGH}.$$ 

(12.3)

The $\chi^\mu$ only depends on the lattice orientation and the link direction and on which of the triangles that are associated with the equation of motion. This latter dependence is in fact crucial for the sign of the 2-vector $\chi^\mu$ because a link $GH$ lies in the lattice between two triangles, but it is only one of them that is associated with an equation of motion element. If one chooses the wrong triangle one would get the opposite sign for $\chi^\mu$, but $R_{GHL}$ is associated with a triangle in correspondence with an element of equation of the motion.

In the $A$-type half-curve system for constructing the $A$-type entropy current $j_A^\mu(x)$ we use the link directions $(\sqrt{\frac{3}{2}}a, \frac{1}{2}a)$ and $(\sqrt{\frac{3}{2}}a, -\frac{1}{2}a)$ and we imagine we have chosen the lattice so that the triangles associated with equations of motion have one tip pointing just in the negative $x^1$-axis direction. Tip here means an angle in the triangle and that it points in the negative $x^1$-direction is supposed to that this “tip” has a coordinate only deviating from that of the center of triangle by having a smaller $x^1$-coordinate. Taking into account this chosen orientation of our lattice
and of which triangles are equation of motion associated we easily calculate that:

1) for the links of direction \((\sqrt{\frac{3}{4}}a, \frac{1}{2}a)\) the \(\chi^\mu\) pointing from the middle of the link to the center of the neighboring triangle associated with the equation of motion is

\[
\chi^\mu = (\frac{1}{4\sqrt{3}}a, -\frac{1}{4}a)
\]  

(12.4)

2) for the links of the direction \((\sqrt{\frac{3}{4}}a, -\frac{1}{2}a)\) we get

\[
\chi^\mu = (\frac{1}{4\sqrt{3}}a, \frac{1}{4}a).
\]

(12.5)

Let us insert

\[
K_\bullet(x) = S(x) - \log Cn(R(x)) + (\frac{1}{4}\sqrt{3}a, -\frac{1}{4}a)^\mu \frac{\partial}{\partial x^\mu} \log Cn(R(x))
\]

(12.6)

and

\[
K_\triangle(x) = S(x) - \log Cn(R(x)) + (\frac{1}{4}\sqrt{3}a, \frac{1}{4}a)^\mu \frac{\partial}{\partial x^\mu} \log Cn(R(x))
\]

(12.7)

into

\[
\mathcal{J}_A^\mu(x) = - (\sqrt{\frac{3}{4}}, -\frac{1}{2})^\mu K_\bullet(x) + (\sqrt{\frac{3}{4}}, \frac{1}{2})^\mu K_\triangle(x)
\]

(12.8)

We obtain

\[
\mathcal{J}_A^\mu(x) = (\sqrt{\frac{3}{4}} \cdot \frac{1}{2} a \frac{\partial}{\partial x^2} \log Cn \{R(x)\}, S(x) - \log Cn \{R(x)\})
\]

\[
+ \frac{a}{4\sqrt{3} \frac{\partial}{\partial x^1} \log Cn \{R(x)\})^\mu}
\]

\[
= (0, 1)^\mu \left\{ S(x) - \log CnR(x) + (\frac{a}{4\sqrt{3}} + \frac{\sqrt{3}a}{4}) \frac{\partial}{\partial x^1} \log CnR(x) \right\} +
\]

\[
+ \frac{\sqrt{3}}{4} a\epsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x)
\]

\[
= (0, 1)^\mu \left\{ S(x) - \log CnR(x) + \frac{a}{\sqrt{3} \frac{\partial}{\partial x^1} \log CnR(x) \right\} +
\]

\[
+ \frac{\sqrt{3}}{16} a\epsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x)
\]

(12.9)

We may interpret the quantity

\[
\log CnR(x) - \frac{a}{\sqrt{3} \frac{\partial}{\partial x^1} \log CnR(x)}
\]

(12.10)

as is the extrapolated value of \(\log CnR(x)\) to the corner of triangle at which the two link directions associated with the \(A\)-choice meet

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Figure 10: The corner of the black triangle at which the two link directions meet, associated with the first term of eq.\((12.11)\)

So if instead of the center of the triangles associated with equations of motion we had decided to let the representative point of the triangle be the corner natural in the \(A\)-case we would have gotten rid of the extra term \(-\frac{a}{\sqrt{3} \partial x^1} \log CR(x)\) from eq.\((12.11)\). But it should be noted that when we go to the cases \(B\) and \(C\) another corner would have to be chosen to get rid of the corresponding terms.

13 Constructing analogously \(j_B^\mu(x)\) and \(j_C^\mu(x)\)

For the construction of the three different entropy currents \(j_A^\mu(x)\), \(j_B^\mu(x)\), and \(j_C^\mu(x)\) the normalization is not given just by ensuring conservation, but even if we normalize all three numerically in an analogous way there is a sign to be chosen which is fundamentally arbitrary. If you think of one direction in space-time as the positive time axis you may say that we shall make the entropy density corresponding to that have the \(K\)’s come in with positive coefficients. But ignoring second law of thermodynamics as we seek to do in this article there is no physical difference between positive and negative time directions. So such a sign choice for our currents \(j_A^\mu(x)\), \(j_B^\mu(x)\), and \(j_C^\mu(x)\) is basically arbitrary. It might even be pedagogical to think of both signs and say that we are defining six currents \(\pm j_A^\mu(x)\), \(\pm j_B^\mu(x)\), and \(\pm j_C^\mu(x)\), rather than only three.

Then analogous construction for the \(B\)-half-curves associated \(B\)-entropy current \(j_B^\mu\) goes, since the \(B\)-choice applies for the half-curves the links counted with orientation

\[
(0, a), ( -\sqrt{\frac{3}{4}} a, \frac{1}{2} a )
\]

\[(13.1)\]
by using
\[ j_B^\mu(x) = (-\sqrt{\frac{3}{4}} \cdot \frac{1}{2}) K_{\mu}(x) + (0, -1) K_{\nu}(x) \] (13.2)
where we have chosen a sign corresponding to \( j_B^\mu \) as \( j_A^\mu \) rotated by 120° in the space time plane. Then we insert
\[ K_{\mu}(x) = S(x) - \log CnR(x) + \left( -\frac{a}{2\sqrt{3}}, \frac{1}{2} \right)^\mu \frac{\partial}{\partial x^\mu} \log CnR(x) \] (13.3)
and
\[ K_{\nu}(x) = S(x) - \log CnR(x) + \left( \frac{a}{4\sqrt{3}}, -\frac{1}{2} a \right)^\mu \frac{\partial}{\partial x^\mu} \log CnR(x) \] (13.4)
and we obtain
\[
j_B^\mu = (-\sqrt{\frac{3}{4}} \cdot \frac{1}{2})(S(x) - \log CnR(x)) - (\frac{1}{2\sqrt{3}}, -\frac{1}{2} a)^\rho \frac{\partial}{\partial x^\rho} \log CnR(x)) + \\
\left\{ (+\sqrt{\frac{3}{4}} \cdot \frac{a}{2\sqrt{3}} - \frac{a}{2\sqrt{3}}) \partial_1 \log CnR(x) + \frac{1}{2} \sqrt{\frac{3}{4}} a \partial_2 \log CnR(x), \\
(-\frac{a}{2\sqrt{3}}, -\frac{1}{2} a)^\rho \frac{\partial}{\partial x^\rho} \log CnR(x)) \right\} \\
= (-\sqrt{\frac{3}{4}} \cdot \frac{1}{2}) \left\{ S(x) - \log CnR(x) - (\frac{a}{2\sqrt{3}}, -\frac{1}{2} a)^\rho \frac{\partial}{\partial x^\rho} \log CnR(x) \right\} \\
+ \frac{\sqrt{3} a}{2} \cdot \frac{1}{2} \varepsilon^{\mu
u} \frac{\partial}{\partial x^\nu} \log CnR(x) \] (13.5)
Again we can interpret \( \log CnR(x) + (\frac{a}{2\sqrt{3}}, -\frac{a}{2})^\rho \frac{\partial}{\partial x^\rho} \log CnR(x) \) as the value of \( \log CnR(x) \) if we instead of identifying the position of the triangle with its center identified it as the corner selected now by the B-choice.

Finally we may now construct the C-half-curves related C- entropy current \( j_C^\mu(x) \), where we now have the oriented links from which the half-curves are constructed as
\[ (0, -1 a), (-\sqrt{\frac{3}{4}} a, -\frac{1}{2} a). \] (13.6)
The C-entropy current is with again by 120° rotations chosen sign
\[ j_C^\mu(x) = (0, -1) K_{\mu}(x) - \left( -\sqrt{\frac{3}{4}}, -\frac{1}{2} \right) K_{\nu}(x). \] (13.7)
Herein we shall insert the already above given expressions for \( K_{\mu} \) and \( K_{\nu} \) to obtain
\[ j_C^\mu(x) = (0, -1) \left\{ S(x) - \log CnR(x) - (\frac{a}{4\sqrt{3}}, \frac{1}{4} a)^\rho \frac{\partial}{\partial x^\rho} \log CnR(x) \right\} \]
\[
\begin{align*}
&+ (\sqrt{\frac{3}{4}} + \frac{1}{2}) \left\{ S(x) - \log C_nR(x) - \left( \frac{a}{2\sqrt{3}}, 0 \right) \frac{\partial}{\partial x^\rho} \log C_nR(x) \right\} \\
&= (\sqrt{\frac{3}{4}} - \frac{1}{2}) \left\{ S(x) - \log C_nR(x) - \left( \frac{a}{2\sqrt{3}}, \frac{1}{2} \right) \frac{\partial}{\partial x^\rho} \log C_nR(x) \right\} \\
&+ \left\{ -\sqrt{\frac{3}{4}} \cdot \frac{a}{2\sqrt{3}} + \sqrt{\frac{3}{4}} \cdot \frac{a}{2\sqrt{3}} \right\} \partial_1 \log C_nR + (\sqrt{\frac{3}{4}} \cdot 0 \\
&+ \sqrt{\frac{3}{4}} \cdot \frac{1}{2}a) \partial_2 \log C_nR, (-1 \cdot (-\frac{1}{4}a) - \frac{1}{2} \cdot \frac{1}{2}a) \partial_2 \log C_nR \\
&+ (-\frac{a}{4\sqrt{3}} - \frac{a}{2\sqrt{3}} - \frac{a}{2\sqrt{3}} \cdot \partial_1 \log C_nR \right\} \\
&= \left( \sqrt{\frac{3}{4}} - \frac{1}{2} \right) \left\{ S(x) - \log C_nR(x) - \left( \frac{a}{2\sqrt{3}}, \frac{1}{2} \right) \frac{\partial}{\partial x^\rho} \log C_nR(x) \right\} \\
&+ \frac{\sqrt{3}a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log C_nR(x)
\end{align*}
\tag{13.8}
\]

14 Relation between the different entropy currents \(j^\mu_A, j^\mu_B, \text{ and } j^\mu_C\)

We may resume by writing the three entropy currents which we have constructed:

\[
\begin{align*}
\langle j^\mu_A(x) \rangle &= (0, 1)^\mu \left\{ S(x) - \log C_nR(x) + \frac{a}{\sqrt{3}} \frac{\partial}{\partial x^1} \log C_nR(x) \right\} + \frac{\sqrt{3}a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log C_nR(x); \\
\langle j^\mu_B(x) \rangle &= (-\sqrt{\frac{3}{4}} - \frac{1}{2})^\mu \left\{ S(x) - \log C_nR(x) - \left( \frac{a}{2\sqrt{3}} - \frac{a}{2} \right) \frac{\partial}{\partial x^\rho} \log C_nR(x) \right\} \\
&+ \frac{\sqrt{3}a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log C_nR(x); \\
\langle j^\mu_C(x) \rangle &= \left( \sqrt{\frac{3}{4}} + \frac{1}{2} \right)^\mu \left\{ S(x) - \log C_nR(x) - \left( -\frac{a}{2\sqrt{3}} -\frac{1}{2}a \right) \frac{\partial}{\partial x^\rho} \log C_nR(x) \right\} \\
&+ \frac{\sqrt{3}a}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log C_nR(x). 
\end{align*}
\tag{14.1}
\]

It is easily seen from these expressions that

\[
\langle j^\mu_A(x) \rangle + \langle j^\mu_B(x) \rangle + \langle j^\mu_C(x) \rangle = \frac{a\sqrt{3}}{4} \varepsilon^{\mu\nu} \frac{\partial}{\partial x^\nu} \log C_nR(x). \tag{14.2}
\]

In fact it is first easily seen that what we could call the main terms meaning the ones without any \(a\)-factor cancel

\[
\{(0, 1) + (-\sqrt{\frac{3}{4}}, -\frac{1}{2}) + (\sqrt{\frac{3}{4}}, -\frac{1}{2}) \} \cdot \{ S(x) - \log C_nR(x) \} = 0. \tag{14.3}
\]

30
Next we may write the terms connected with shift between the corners and the center of the triangle in matrix form

\[
\begin{pmatrix}
0 & \frac{a}{\sqrt{3}} & 0 \\
1 & 0 & \frac{a}{\sqrt{3}} \\
-\frac{1}{2} & -\frac{1}{2} & 0
\end{pmatrix}
\begin{pmatrix}
-\frac{\sqrt{3}}{4} & \frac{a}{2\sqrt{3}} & 0 \\
\frac{a}{2\sqrt{3}} & -\frac{a}{2} & -\frac{a}{2}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x_1} \log CnR(x) \\
\frac{\partial}{\partial x_2} \log CnR(x)
\end{pmatrix}
\]

\[= \left( 0 + \frac{a}{4} - \frac{a}{4} 0 - \frac{a\sqrt{3}}{4} - \frac{a\sqrt{3}}{4} \right) \begin{pmatrix}
0 & -\frac{a\sqrt{3}}{2} \\
\frac{a\sqrt{3}}{2} & 0
\end{pmatrix}
\begin{pmatrix}
\frac{\partial}{\partial x_1} \log CnR(x) \\
\frac{\partial}{\partial x_2} \log CnR(x)
\end{pmatrix}
\]

\[= -\frac{3a\sqrt{3}}{2} \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x) \quad (14.4)
\]

together with the three identical topological charge terms

\[\frac{3\sqrt{3}a}{4} \varepsilon_{\mu\nu} \frac{\partial}{\partial x^\nu} \log CnR(x) \quad (14.5)
\]

15 Understanding our entropy currents from counting of number of solution

In this section we should seek to describe the three above defined entropy currents \( j_A^\mu, j_B^\mu \) and \( j_C^\mu \) by counting solutions. We have already above explained that to each of the three choices \( A, B \) or \( C \) there corresponds a series of half-curves. Theses bunches of half-curves have the property that whether two solutions \( \varphi_1 \) and \( \varphi_2 \) to the equations of motion - obeying also the macro scenario restrictions imposed - have the same restriction to one of the half-curves, \( \delta \) say, is equivalent to whether they have it on any other, \( \gamma \) say, in the same class of half-curves. Imagining without going in details that we have chosen both an infrared cut off and a discretization in a field value space we can simply ask for the number of classes of solutions \( \varphi \) with the same restriction to a curve in say the \( B \)-bundle \( \delta, Cn(\{ \varphi | \delta \}) \). We can construct from this number

\[Cn\{ \varphi | \delta \} | \varphi \quad \text{a solution obeying the macro scenario}\]  

(15.1)
what we could call the entropy to the $B$-side of the point $x^\mu$, where $\delta$ is taken to extend from $x^\mu$. In fact

$$S(B - \text{side of } x^\mu) = \log Cn\{\varphi|\varphi \text{ a solution obeying the macro scenario}\}. \quad (15.2)$$

Having defined such an entropy on "one side" the current should be obtained simply as a topological charge out of this $S(B - \text{side of } x^\mu)$:

$$j_B^\mu(x) = e^{\mu\nu} \frac{\partial}{\partial x^\nu} S(B - \text{side of } x^\rho) \quad (15.3)$$

Thinking of $x^\rho$ as a site in our triangular lattice and noting that our link entropies $K_{GH}$ (entropy of link between G and H), if they are exponentiated, provide the number of variation possibilities for a solution restriction to a half-curve $\delta$, say, which comes extra by having the link $GH$ included. We can thus see that - even with respect to normalization - the entropy to the $A$-side of $x^\mu$ can be written as

$$S(A - \text{side of } x^\mu) = \Sigma_{\text{along half-curve } b K_{GH}}. \quad (15.4)$$

To avoid the infrared divergencies the easiest is to ask for the difference like

$$S(A - \text{side of } x^\mu) - S(A - \text{side of } y^\mu) = \Sigma_{\text{along a } B\text{-type curve for } x^\nu \text{ to } y^\nu K_{GH}}. \quad (15.5)$$

Now the links in the $A$-type half-curves go in the directions $(\frac{3}{4}a, \frac{1}{2}a)$ or $(\frac{3}{4}a, -\frac{1}{2}a)$ and the links of these two directions are displaced by $\chi^\mu = (\frac{1}{4\sqrt{3}}a, -\frac{1}{4}a)$ and $\chi^\mu = (\frac{1}{4\sqrt{3}}a, \frac{1}{4}a)$ respectively. So we found the continuum limit expression for link entropies for these two directions to be $(12.6)$ and $(12.7)$ respectively. If we consider a continuous curve which extends in a possible direction for an $A$-type curve and parametrize it by $y^\mu(s)$, and infinitesimal piece contribute to the sum $(15.5)$. The contribution is calculated in the following way: First expand the infinitesimal $dy^\mu(s)$ on the two directions

$$dy^\mu = (\sqrt{\frac{3}{4}a, \frac{1}{2}a})dn_\bullet + (\sqrt{\frac{3}{4}a, -\frac{1}{2}a})dn_\circ. \quad (15.6)$$

Then the contribution to the sum is

$$d\Sigma' K_{GH} = K_\bullet(x)dn_\bullet + K_\circ(x)dn_\circ. \quad (15.7)$$

$$= \left\{ S(x) - \log CnR(x) + (\frac{1}{4\sqrt{3}}a, -\frac{1}{4}a)\rho \frac{\partial}{\partial x^\rho} \log CnR(x) \right\}dn_\bullet$$

$$+ \left\{ S(x) - \log CnR(x) + (\frac{1}{4\sqrt{3}}a, \frac{1}{4}a)\rho \frac{\partial}{\partial x^\rho} \log CnR(x) \right\}dn_\circ. \quad (15.8)$$
Clearly
\[ dn_\bullet = \frac{1}{a} (\frac{1}{\sqrt{3}}, 1) \mu dy^\rho \] (15.9)
and
\[ dn_\circ = \frac{1}{a} (\frac{1}{\sqrt{3}}, -1) \rho dy^\rho \] (15.10)
and so

\[ d\Sigma_{K_{GH}} = \{ S - \log CnR(x) + \frac{1}{4\sqrt{3}} a \cdot \frac{\partial}{\partial x^1} \log CnR(x) \}(dn_\bullet + dn_\circ) \] (15.11)

\[ + \{- \frac{1}{4} a \frac{\partial}{\partial x^2} \log CnR(x)\} \cdot (dn_\bullet + dn_\circ) \]

\[ = (S - \log CnR(x) + \frac{1}{4\sqrt{3}} a \frac{\partial}{\partial x^1} \log CnR(x) \cdot \frac{2}{a} \sqrt{3} dy^1) \]

\[ + \{- \frac{1}{4} a \frac{\partial}{\partial x^2} \log CnR(x)\} \cdot \frac{2}{a} dy^2 \]

\[ = \frac{\sqrt{33}4}{A} \cdot dn_\bullet \]

\[ dn_\bullet = \frac{4}{\sqrt{3a}} dy^\mu (\frac{1}{2}, \sqrt{3})_\mu \] (15.12)

\[ = dy^\mu (\frac{1}{a} \frac{2}{\sqrt{3}}, 2)_\mu \]

\[ = \frac{2}{\sqrt{3a}} (S(x) - \log CnR(x)) dy^1 + \frac{1}{6} dy^1 \frac{\partial}{\partial x^1} \log CnR(x) \]

\[ - \frac{1}{2} dy^2 \frac{\partial}{\partial x^2} \log CnR(x) \]

We know from the fact that the integral of this contribution

\[ \int d\Sigma_{K_{GH}} \] (15.13)

should integrate up to only depend on the end points - but not on the way - that the integrability conditions should be satisfied. If it were not integrable this way for the curves with an expansion (15.6) having \( dn_\bullet \) and \( dn_\circ \) both being positive it would mean that different curves of this type - but with common end points would not give rise to the same number of classes of different solution \( \varphi \). Since the classes of solutions defined from one of these curves are in a one-to-one correspondence with the ones from the other one there must be equally many and thus the one-form (15.6) must be integrable.

We can rewrite

\[ d \sum K_{GH} (S(x) - \log CnR(x) + \frac{1}{4\sqrt{3}} a \frac{\partial}{\partial x^1} \log CnR(x)) \frac{2}{a} \sqrt{3} dy^1 \]
\[ (+\frac{a}{4}\frac{\partial}{\partial x^2}\log CnR(x))\frac{2}{a}dy^2 = \frac{1}{a}\cdot\frac{2}{\sqrt{3}}(S(x) - \log CnR(x) + \frac{a}{\sqrt{3}}\frac{\partial}{\partial x^2}\log CnR(x))dy^1 \]

\[ - \frac{1}{2}d\log CnR(x) \]  

(15.14)

In this form the condition for \( d\sum K_{GH} \) being a total differential is very easy to write because the last term \( -\frac{1}{2}d\log CnR(x) \) itself is one and can be ignored for that purpose. Thus we simply get that

\[ \frac{\partial}{\partial x^1}(\frac{1}{a}\cdot\frac{2}{\sqrt{3}}(S(x) - \log CnR(x) + \frac{a}{\sqrt{3}}\frac{\partial}{\partial x^2}\log CnR(x)) = 0 \]  

(15.15)

or that

\[ f(x^2) = S(x) - \log CnR(x) + \frac{a}{\sqrt{3}}\frac{\partial}{\partial x^2}\log CnR(x) \]  

(15.16)

only function of \( x^2 \) but not of \( x^1 \).

We recognize this expression (15.16) from the expression occurring in the form (12.10) or (14.1) for \( j^\mu_A(x) \). We should remember though that in the derivation leading up to (12.10) or (14.1) we only used the conservation but did not have the normalization. Rather we should use our expression for \( d\sum K_{GH} \) which has been physically normalized to define the correctly normalized \( j^\mu_A(x) \) current by

\[ d\sum K_{GH} = j^2_{Anorm}(x)dx - j^1_{Anorm}(x)dx^2 \]  

(15.17)

so that we see

\[ j^\mu_{Anorm}(x) = \frac{2}{a\sqrt{3}}j^\mu_A(x). \]  

(15.18)

One can similarly construct the analogous \( d\sum K_{GH} \) differential for the \( B \)- and the \( C \)-types of curves and see that the condition for them being integrable is just equivalent to the current conservation. Again we could use this interpretation of the entropy by means of solution class counting to give the correct normalization of the current densities,

\[ j^\mu_{Anorm}(x) = \frac{2}{a\sqrt{3}}j^\mu_A(x) \]

\[ j^\mu_{Bnorm}(x) = \frac{2}{a\sqrt{3}}j^\mu_B(x) \]

\[ j^\mu_{Cnorm}(x) = \frac{2}{a\sqrt{3}}j^\mu_C(x) \]  

(15.19)

From these considerations it should be understood that our \( A \)-type entropy current density \( j^2_{Anorm}(x) \), really tells the logarithm of how many classes of solutions
belonging to the macro scenario are distinguishable by their restriction to one unit length in the $x^1$-direction. This is means infinitesimally that an $dx^1$ is the classification number $\exp(j_{A_{\text{norm}}}(x)dx^1)$. This is indeed the number of states of the fields in the $dx^1$-interval and thus our entropy current does indeed deserve to be called an entropy current.

16 Conclusion and outlook

We have considered a “classical” lattice field theory on a two dimensional regular triangular lattice of a very general type. That is to say we did never write what the supposedly complicated equations would be. Rather we just in a extremely abstract manner assumed that we looked for a big class of solutions obeying all over the lattice some macro constraints which were meant to be the features to what a macroscopic observer would notice. We assumed also in a general way that the macro scenario imposed corresponding to an adiabatic development formulated by the assumptions we called reversibility in either way.

Our assumption of reversibility or adiabaticity “either way” was the assumption (4.2) in section 4. It means that entropy would be constant. Thus it becomes a setting for studying conserved entropy currents. It should be understood that such entropy flows are given in a macroscopic picture which means a system of macro restrictions, consistent with our reversibility.

Now our main study was for such macroscopic pictures or macro scenarios with our (slightly generalized) adiabaticity assumption to find and define conserved entropy currents because of the adiabaticity.

The surprise is, however, that instead of finding only one - as one would presumably have expected - we found three conserved entropy currents, $j^A_A(x)$, $j^A_B(x)$, and $j^C_C(x)$ as we denoted them in the continuum limit (lattice constant $a \to 0$). We can say that we found in our rather general latticized field theory three different kinds of entropy!

It turned out that with our special simplification of the lattice model having only restrictions by the field values that are related inside a certain system of triangles the three different currents of entropy could be described - in the continuum limit - in terms of two scalar fields $S(x)$ and $\log CNR(x)$ describing the prescribed macro scenario. From the conservation laws of these three entropies we got a certain
homogeneous third order differential equation

\[ \xi^{\mu\nu\rho} \partial_\mu \partial_\nu \partial_\rho \log CnR(x) = 0 \]  

(16.1)

where the third order tensor \( \xi^{\mu\nu\rho} \) is related to the lattice orientation.

What we did by means of the lattice non rotational invariance was to let the lattice select some direction in which to say that the entropy flows. We have these different crude directions \( A, B, \) and \( C \) “half-curves”. In Minkowskian relativistic theories one must use the distinction between forward in time going and backward going (half) curves. To keep the precise analogy we should though rather think of right going and left going half-curves. But we managed under some more general condition to obtain even several entropy current definitions.

We do probably best by admitting that we do not fully yet understand what is going on with these the somewhat mysterious three entropies. It is rather obvious that as it comes out these entropies are seemingly lattice artifacts in as far as the bunches of “half” curves extending from some point \( x^\mu \) which we used for the definition of the three entropies were clearly defined by means of our lattice only. So from the very definition we had our three entropies attached to the lattice.

It is our next important subjects of works to study further detailed investigations of the properties of three entropy currents in two dimensional example and to generalize our method of constructing entropy currents into any dimensions. Furthermore most importantly we will have to clarify the question whether all the three entropy currents are relevant ones in actual physical situation.

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