THE NOETHER PROBLEM FOR SPINOR GROUPS OF SMALL RANK

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Abstract. Building on prior work of Bogomolov, Garibaldi, Guralnick, Igusa, Kordonskiıı, Merkurjev and others, we show that the Noether Problem for Spin\(_n\) has a positive solution for every \(n \leq 14\) over an arbitrary field of characteristic \(\neq 2\).

1. Introduction

Let \(k\) be a field, \(\overline{k}\) be an algebraic closure of \(k\), \(G\) be a linear algebraic group defined over \(k\), and \(\rho: G \hookrightarrow \text{GL}(V)\) be a \(k\)-representation. Assume that \(\rho\) is generically free; that is, the scheme-theoretic stabilizer \(\text{Stab}_G(v)\) is trivial for a point \(v \in V(\overline{k})\) in general position. The Noether Problem asks whether the field of rational \(G\)-invariants \(k(V)^G\) is a purely transcendental extension of \(k\). Equivalently, it asks whether the Rosenlicht quotient \(V/G\) is rational over \(k\). (For the definition of the Rosenlicht quotient, see Section 2.) The following variants of the Noether Problem are also of interest: Is \(V/G\) stably rational? Is \(V/G\) retract rational? Recall that a \(d\)-dimensional algebraic variety \(X\) is called rational if \(X\) is birationally equivalent to the affine space \(A^d\), stably rational if \(X \times A^r\) is birationally equivalent to \(A^{d+r}\) for some \(r \geq 0\) and retract rational if the identity morphism \(\text{id}: V/G \to V/G\), viewed as a rational map, can be factored through the affine space \(A^m\) for some \(m \geq d\).

By the no-name lemma [RV06, Lemma 2.1] the answer to the Noether Problem for stable and retract rationality depends only on the group \(G\) and not on the choice of the representation \(V\). Following A. Merkurjev [Mer17], we will say that the classifying stack \(BG\) is stably (respectively, retract) rational if \(V/G\) is stably (respectively, retract) rational for some \(r \geq 0\) and thus every (and thus every) generically free representation \(G \hookrightarrow \text{GL}(V)\). We will also say that \(BG\) and \(BH\) are stably birationally equivalent if \(V/G\) and \(W/H\) are stably birationally equivalent, where \(H \hookrightarrow \text{GL}(W)\) is a generically free representation of \(H\). This terminology is related to the fact that \(V/G\) can be thought of as an approximation to \(BG\). Note that for us “\(BG\) is stably rational” will be a convenient short-hand for “the Noether Problem for stable rationality has a positive answer for \(G\)”; we will not actually work with stacks in this paper.

In the case where \(G\) is a finite group, and \(V\) is the regular representation of \(G\), the question of rationality of \(V/G\) was posed by E. Noether in the context of her work on the Inverse

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Galois Problem [Noe17]. For a finite group $G$, $V/G$ may not be stably (or even retract) rational. The first such examples over number fields $k$ were given by R. Swan [Swa69] and V. Voskresenskii [Vos70] and over $k = \mathbb{C}$ by D. Saltman [Sal84]. For many specific finite groups $G$ the Noether Problem remains open.

In the case where $G$ is a connected split semisimple groups over $k$, no counter-examples to the Noether Problem are known. It is known that $BG$ is stably rational for some $G$ (e.g., for $G = \text{GL}_n$, $\text{SL}_n$, $\text{SO}_n$) but for many other connected split semisimple groups the Noether Problem remains open. Among these, projective linear groups $\text{PGL}_n$ and spinor groups $\text{Spin}_n$ have received the most attention.

For $G = \text{PGL}_n$ the Noether Problem arose independently in ring theory in connection with universal division algebras; see [Pro67, p. 254]. It is known that $BG$ is stably rational for every $n$ dividing 420; the remaining cases are open. See [LB91] for an overview.

In [Bog86], F. A. Bogomolov claimed that $BG$ is stably rational for every simply connected simple complex algebraic group $G$. However, there is a mistake in his argument; in particular, it breaks down for the spinor groups. V. Kordonski˘ı [Kor00] subsequently proved that $B\text{Spin}_n$ and $B\text{Spin}_{20}$ are stably rational (again, over the field of complex numbers). More recently, Merkurjev [Mer19, Section 4] showed that $B\text{Spin}_n$ is retract rational for $n \leq 14$ over any field $k$ of characteristic $\neq 2$, and conjectured that $B\text{Spin}_n$ is, in fact, not retract rational for $n \geq 15$. We strengthen Merkurjev’s result as follows.

**Theorem 1.1.** Let $k$ be a field of characteristic $\neq 2$. Then $B\text{Spin}_n$ is stably rational over $k$ for every $n \leq 14$.

The remainder of this paper will be devoted to proving Theorem 1.1. Our strategy will be as follows. For $n \leq 6$, it is well known that $\text{Spin}_n$ is special; see [Gar09, Section 16.1]. Hence, $B\text{Spin}_n$ is stably rational; see Lemma 3.1. Merkurjev [Mer19, Corollary 5.7] showed that $B\text{Spin}_{2m+1}$ is stably birationally equivalent to $B\text{Spin}_{2m+2}$ for every $m \geq 0$. Thus in order to prove Theorem 1.1 it suffices to show that $B\text{Spin}_n$ is stably rational for $n = 7, 10, 11$ and 14. We will give self-contained proofs that work over an arbitrary field $k$ of characteristic $\neq 2$ in Propositions 6.2, 6.4, 7.1 and 8.1 respectively. Along the way we will show that $B(\text{SL}_n \ltimes \mathbb{Z}/2\mathbb{Z})$ is stably rational; see Section 4. In the last section we give an alternative proof of Theorem 1.1 over $k = \mathbb{C}$ suggested to us by G. Schwarz.

**2. Rosenlicht Quotients**

For the remainder of this paper $k$ will denote a field of arbitrary characteristic and $G$ will denote a smooth linear algebraic group defined over $k$. Starting from Section 6 we will assume that $\text{char}(k) \neq 2$, but for now $k$ is arbitrary.

Let $X$ be a reduced and absolutely irreducible algebraic variety equipped with an action of $G$ over $k$. A rational map $\pi : X \dashrightarrow Y$ is called a Rosenlicht quotient map for the $G$-action on $X$ if

- $\pi \circ g = \pi$ for every $g \in G(\overline{k})$,
- $Y$ is reduced and irreducible,
- $k(Y) = k(X)^G$ and

\[1\]Over a field of characteristic 0 retract rationality of $\text{Spin}_n$ for $n \leq 12$ was proved earlier by J.-L. Colliot-Thélène and J.-J. Sansuc [CTS07].
equivariant morphism, where $G$ is readily deduced from Rosenlicht’s Theorem. For a modern proof of Rosenlicht’s theorem, see [BGR17, Section 7]. The following Lemma refers to it as “the Rosenlicht quotient” in this paper in order not to overuse the term “rational”.

If $\pi: X \to Y$ is a Rosenlicht quotient map, then by a theorem of Rosenlicht [Ros56, Theorem 2] there exist a $G$-invariant open $k$-subvariety $X_0 \subset X$ and an open $k$-subvariety $Y_0 \subset Y$ such that $\pi$ restricts to a regular map $\pi_0: X_0 \to Y_0$ and

$$(2.1) \quad \text{for any } x \in X_0(k), \text{ the fiber } \pi_0^{-1}(\pi_0(x)) \text{ equals the } G(k)\text{-orbit of } x.$$  

For a modern proof of Rosenlicht’s theorem, see [BGR17, Section 7]. The following Lemma is readily deduced from Rosenlicht’s Theorem.

**Lemma 2.1.** Consider an action of $G$ on $X$ as above, and let $\pi: X \to Z$ be a $G$-equivariant morphism, where $G$ acts trivially on $Z$. Suppose that

(i) there exists a dense open subvariety $Z_0 \subset Z$ such that for any $z \in Z_0(k)$ the fiber $\pi^{-1}(z) = G \cdot x$ for some $x \in X(k)$, and

(ii) $\pi$ is generically smooth (which is automatic in characteristic 0).

Then $\pi$ is a Rosenlicht quotient for the $G$-action on $X$. In particular, $\pi$ induces an isomorphism between $k(Z)$ and $k(X)^G$.

**Proof.** Let $\mu: X \to Y$ be a Rosenlicht quotient map. Since $\pi$ is $G$-equivariant, it factors through $\mu$ as follows:

$$\begin{array}{c}
\xymatrix{ X \\
\mu \ar@{|->}[r] & \pi \\
Y \ar@{|->}[ur] & Z. }
\end{array}$$

(This is called the universal property of Rosenlicht quotients.) It now suffices to show that $\alpha$ is a birational isomorphism. Choose open subvarieties $X_0 \subset X$ and $Y_0 \subset Y$ as in Rosenlicht’s theorem (2.1) and such that $\alpha$ is regular on $Y_0$. Then (i) tells us that $\alpha$ induces a bijection between $Y_0(k)$ and $Z_0(k)$ for some dense open subvariety $Z_0 \subset Z$. In characteristic 0 this implies that $\alpha$ is a birational isomorphism, and we are done.

In characteristic $0$ this implies that $\alpha$ induces a bijection between $Y_0(k)$ and $Z_0(k)$ only tells us that the field extension $k(Y)/k(Z)$ induced by $\alpha$ is purely inseparable, so we need to use condition (ii) to finish the proof. By (ii), $\pi$ is generically smooth and hence, so is $\alpha$. This implies that $\alpha$ is a birational isomorphism, as desired. \hfill \Box

**Remark 2.2.** Consider a generically free action of $G$ on a variety $X$ defined over $k$. By [BF03, Theorem 4.7], there exists a dense $G$-invariant open subvariety $X_0 \subset X$ which is the total space of a $G$-torsor $\pi: X_0 \to Y$ over some $k$-variety $Y$. Conditions (i) and (ii) of Lemma 2.1 are satisfied, because $G$ is smooth and $\pi$ is a $G$-torsor. It follows that $\pi: X_0 \to Y$ (viewed as a rational map $X \to Y$) is a Rosenlicht quotient for the $G$-action on $X$. 

\begin{itemize}
  \item $\pi$ is induced by the inclusion of fields $k(X)^G \hookrightarrow k(X)$.
\end{itemize}
3. Preliminaries on the Noether Problem

In this section we collect several known results on the Noether Problem for future use. Given two \( k \)-varieties, \( X \) and \( Y \), we will write \( X \sim Y \) if \( X \) and \( Y \) are birationally isomorphic over \( k \).

Recall that a smooth linear algebraic group \( G \) is called special if \( H^1(K, G) = \{1\} \) for every field \( K \) containing \( k \). Special groups were introduced by Serre [Ser58]; over an algebraically closed field of characteristic 0 they were classified by Grothendieck [Gro58].

**Lemma 3.1.** If \( G \) is special and stably rational, then \( BG \) is stably rational.

*Proof.* See [CTS07, Proposition 4.7] or [FR18, Remark 3.2]. \( \square \)

**Example 3.2.** It is known that the groups \( G = \text{GL}_n, \text{SL}_n, \text{Sp}_{2n} \) are special for every \( n \geq 1 \), and so are the groups \( G = \text{Spin}_n \) for \( n \leq 6 \). Thus \( BG \) is stably rational for these \( G \).

Let \( P \to S_n \) be a permutation representation of a linear algebraic group \( P \). If \( G \) is another linear algebraic group, this representation gives rise to the wreath product \( G \wr P \), which is defined as the semidirect product \( G^n \rtimes P \) via the permutation action of \( P \) on \( G^n \).

**Lemma 3.3.** Let \( G, G_1, H, \) and \( P \) be linear algebraic groups over \( k \), and \( P \to S_n \) be a permutation representation.

(a) If \( BG \) and \( BH \) are stably birational, then \( B(G \wr P) \) and \( B(H \wr P) \) are stably birational.

(b) If \( BG \) is stably rational, then \( B(G \wr P) \) is stably birational to \( BP \).

(c) If \( BG \) is stably rational, then \( B(G \times G_1) \) is stably birational to \( B(G_1) \).

*Proof.* (a) Let \( V \) be a generically free \( G \)-representation, and \( W \) be a generically free \( H \)-representation. By our assumption the Rosenlicht quotients \( V/G \) and \( W/H \) are stably birationally equivalent, say \( V/G \times \mathbb{A}^r \sim W/H \times \mathbb{A}^s \). After replacing \( V \) by \( V \oplus \mathbb{A}^r \) and \( W \) by \( W \oplus \mathbb{A}^s \), where \( G \) acts trivially on \( \mathbb{A}^r \) and \( H \) acts trivially on \( \mathbb{A}^s \), we may assume that \( V/G \sim W/H \).

The product actions of \( G^n \) on \( V^n \) and of \( H^n \) on \( W^n \) naturally extend to linear representations

\[
G \wr P \to \text{GL}(V^n) \quad \text{and} \quad H \wr P \to \text{GL}(W^n),
\]
respectively, where \( P \) acts on \( V^n \) and \( W^n \) by permuting the factors. Now let \( P \to \text{GL}(Z) \) be some generically free linear representation of \( P \). Then the representations \( G^n \rtimes P \) on \( V^n \times Z \) and of \( H^n \rtimes P \) on \( W^n \times Z \) are generically free. Comparing the Rosenlicht quotients \( (V^n \times Z)/(G \wr P) \) and \( (W^n \times Z)/(H \wr P) \), we obtain

\[
(V^n \times Z)/(G \wr P) \sim ((V/G)^n \times Z)/P \sim ((W/H)^n \times Z)/P \sim (W^n \times Z)/(H \wr P),
\]
as desired.

(b) Letting \( H \) be the trivial group in part (a), we deduce that \( B(G^n \rtimes P) \) is stably birational to \( BP \).

(c) is a special case of (b) with \( P = G_1 \), equipped with the trivial permutation representation \( P \to S_1 \). \( \square \)
Lemma 3.4. Let $G 	o \text{GL}(V)$ be a finite-dimensional representation defined over $k$. Suppose there exists a $k$-point $v_0 \in V$ such that the scheme-theoretic stabilizer $H$ of $v_0$ in $G$ is smooth, and the $G$-orbit of $v_0$ is dense in $V$. Then $BG$ and $BH$ are stably birationally equivalent.

When $G$ is reductive and $\text{char } k = 0$, this lemma reduces to [CTS07, Proposition 3.13]. The following argument works in arbitrary characteristic.

Proof. Let $G \to \text{GL}(W)$ be a generically free representation of $G$. Denote the open orbit of $v_0$ in $V$ by $V_0$ and the Rosenlicht quotient map for the $H$-action on $W$ by $\pi: W \to W/H$. After possibly replacing $W/H$ by a dense open subvariety, we can choose an $H$-invariant dense open subvariety $W_0 \subset W$ such that $\pi$ restricts to a morphism $W_0 \to W/H$ whose fibers are exactly the $G$-orbits in $W_0$; see [2,1]. In fact, by Remark 2.2 we may assume that $\pi: W_0 \to W/H$ is an $H$-torsor.

We claim that $\varphi: V_0 \times W_0 \to W/H$ given by $(v, w) \to \pi(w)$ is a Rosenlicht quotient map for the $G$-action on $V_0 \times W_0$. If we establish this claim, then

$$k(V \times W)^G = k(V_0 \times W_0)^G = k(W/H) = k(W)^H,$$

and the lemma will follow. By Lemma 2.1 it suffices to show that

(i) $\varphi^{-1}(\pi(w))$ is a single $G$-orbit for any $w \in W_0(k)$, and

(ii) $\varphi$ is generically smooth.

To prove (i), suppose $\varphi(v_1, w_1) = \varphi(v_2, w_2)$ for some $(v_1, w_1)$ and $(v_2, w_2)$ in $V_0 \times W_0$. Our goal is to show that $(v_1, w_1)$ and $(v_2, w_2)$ lie in the same $G$-orbit. After translating these points by suitable elements of $G$, we may assume that $v_1 = v_2 = v_0$. Since $\pi$ is an $H$-torsor and

$$\pi(w) = \varphi(v_0, w_1) = \varphi(v_0, w_2) = \pi(w_2),$$

we conclude that $w_2 = h(w_1)$ for some $h \in H(k)$. Since $v_0$ is stabilized by $H$, we have $h(v_0, w_1) = (v_0, w_2)$, as desired.

To prove (ii), note that $\pi$ is the composition of the projection map $p: V_0 \times W_0 \to W_0$ and the Rosenlicht quotient map $\pi: W_0 \to W/H$. Clearly, $p$ is smooth. Moreover, $\pi$ is also smooth, because $H$ is smooth and $\pi$ is an $H$-torsor. Thus $\varphi = \pi \circ p$ is smooth, as desired. This completes the proof of (ii) and thus of Lemma 3.4. □

Example 3.5. Consider the 1-dimensional representation $V$ of $G = \mathbb{G}_m$, where $t \in \mathbb{G}_m$ acts on $V$ via scalar multiplication by $t^n$, where $n$ is not divisible by $\text{char}(k)$. Taking $v_0$ to be any non-zero vector in $V$, we see that the stabilizer of $v_0$ in $G = \mathbb{G}_m$ is $H = \mu_n$. Since $\mathbb{G}_m = \text{GL}_1$ is special, $B\mathbb{G}_m$ is stably rational over $k$. By Lemma 3.4 so is $B\mu_n$.

For more sophisticated applications of Lemma 3.4 see [CTS07, Section 4].

Remark 3.6. Representations of connected groups which admit a dense open orbit have been studied by M. Sato and T. Kimura [SK77] (over $\mathbb{C}$). They referred to such representations as prehomogeneous vector spaces.

Lemma 3.7. (cf. [Bog86, Corollary to Lemma 2.2]) Let $V$ be a linear representation of $G$, and consider $V$ as a vector group scheme over $k$. Then $B(V \times G)$ is stably birational to $BG$. 

Proof. Let $W$ be a generically free representation of $G$, and consider the $G$-representation $V_0 := \mathbb{A}^1 \oplus V \oplus W$, where $G$ acts trivially on $\mathbb{A}^1$. We let the vector group scheme $V$ act linearly on $\mathbb{A}^1 \oplus V$ by

$$v \cdot (\lambda, v') := (\lambda, \lambda v + v')$$

and trivially on $W$. This gives $V_0$ the structure of a $V \rtimes G$-representation. Since $G$ acts generically freely on $W$ and $V$ acts generically freely on $\mathbb{A}^1 \oplus V$, $V_0$ is generically free as a $V \rtimes G$-representation. It suffices to show that $V_0/(V \rtimes G)$ is stably birational to $W/G$.

The projection map $\pi : V_0 \to \mathbb{A}^1 \oplus W$ is $V \rtimes G$-equivariant. Moreover, $V$ acts trivially on $\mathbb{A}^1 \oplus W$ and simply transitively on the fibers of points in $(\mathbb{A}^1 \setminus \{0\}) \times W$. By Lemma 2.1, $\pi$ is the Rosenlicht quotient map for the $V$-action on $V_0$. Hence

$$V_0/(V \rtimes G) \sim (\mathbb{A}^1 \oplus W)/G \sim \mathbb{A}^1 \times W/G,$$

as desired. \hfill \Box

4. The Noether Problem for $\text{SL}_n \rtimes (\mathbb{Z}/2\mathbb{Z})$

Let $\mathbb{Z}/2\mathbb{Z} = \langle \tau \rangle$ be the cyclic group of order 2. In this section we will study the Noether Problem for the group $\text{SL}_n \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $n \geq 1$ and $\tau$ acts on $\text{SL}_n$ by $A \to (A^{-1})^T$. Our main result is as follows.

Proposition 4.1. $B(\text{SL}_n \rtimes (\mathbb{Z}/2\mathbb{Z}))$ is stably rational for every $n \geq 1$.

In the sequel we will write $M_{a \times b}$ for the space of rectangular matrices with $a$ rows and $b$ columns. Assume $n \geq 2$ and consider the linear representation of $\text{SL}_n \rtimes (\mathbb{Z}/2\mathbb{Z})$ on $V = M_{n \times (n-1)} \times M_{(n-1) \times n}$ given by

$$A : (X, Y) \mapsto (AX, YA^{-1}) \text{ for any } A \in \text{SL}_n \text{ and } \tau : (X, Y) \mapsto (Y^T, X^T).$$

Here $X^T$ denotes the transpose of $X$ and similarly for $Y$. This action is well defined:

$$\tau(A \cdot (X, Y)) = ((A^{-1})^T Y^T, X^T A^T) = (A^{-1})^T \cdot \tau(X, Y)$$

for every $A \in \text{SL}_n$, $X \in M_{n \times (n-1)}$, and $Y \in M_{(n-1) \times n}$. Set

$$\pi : V \to M_{(n-1) \times (n-1)},$$

where $\pi(X, Y) = YX$. Clearly

$$\pi(A \cdot (X, Y)) = \pi(X, Y) \tag{4.1}$$

for any $A \in \text{SL}_n$, $X \in M_{n \times (n-1)}$ and $Y \in M_{(n-1) \times n}$.

Lemma 4.2. Suppose $\pi(X, Y)$ is a non-singular $(n-1) \times (n-1)$ matrix for some $n \geq 2$, $X \in M_{n \times (n-1)}(k)$ and $Y \in M_{(n-1) \times n}(k)$. Then

(a) the $\text{SL}_n$-orbit of $(X, Y)$ in $V$ contains a point of the form $(J, Y')$, where

$$J = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{n \times (n-1)} \text{ and } Y' \in M_{(n-1) \times n}.$$
(b) The scheme-theoretic stabilizer $\text{Stab}_{\text{SL}_n}(X, Y)$ is trivial and $\pi^{-1}(\pi(X, Y))$ is a single $\text{SL}_n$-orbit.

(c) $\pi$ is a Rosenlicht quotient map for the $\text{SL}_n$-action on $V$. In particular, $\pi$ induces an isomorphism between $k(M_{(n-1) \times (n-1)})$ and $k(V)^{\text{SL}_n}$.

Proof. (a) is a consequence of the fact that $\text{SL}_n$ acts transitively on $(n-1)$-tuples of linearly independent vectors in $k^n$.

(b) In view of part (a), we may assume that $X = J$. If

$$Y = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n-1} & y_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n-1} & y_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{n-1,1} & y_{n-1,2} & \cdots & y_{n-1,n-1} & y_{n-1,n} \end{pmatrix},$$

then

$$\pi(J, Y) = YJ = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n-1} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{n-1,1} & y_{n-1,2} & \cdots & y_{n-1,n-1} \end{pmatrix}.\tag{4.3}$$

Now suppose $(X', Z)$ is another point in the fiber $\pi^{-1}(\pi(J, Y))$. We want to show that $(X', Z)$ is an $\text{SL}_n$-translate of $(J, Y)$. By part (a), we may assume without loss of generality that $X' = J$. Since we are assuming that $\pi(J, Y) = \pi(J, Z)$, this tells us that

$$Z = \begin{pmatrix} y_{1,1} & y_{1,2} & \cdots & y_{1,n-1} & z_{1,n} \\ y_{2,1} & y_{2,2} & \cdots & y_{2,n-1} & z_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{n-1,1} & y_{n-1,2} & \cdots & y_{n-1,n-1} & z_{n-1,n} \end{pmatrix}\tag{4.4}$$

for some $z_{1,n}, \ldots, z_{n-1,n} \in k$.

We claim that the locus $\Lambda$ of solutions to the system $A \cdot (J, Y) = (J, Z)$, or equivalently

$$\begin{cases} AJ = J \\
ZA = Y, \end{cases}$$

is (scheme-theoretically) a single point $A \in \text{SL}_n$. The fact that $\Lambda$ is non-empty implies that $(J, Y)$ and $(J, Z)$ lie in the same $\text{SL}_n$-orbit. The fact that $\Lambda$ is a single point tells us that the scheme-theoretic stabilizer of $(J, Y)$ is trivial (just set $Z = Y$ in the claim).

It thus remains to prove the claim. Note that $AJ = J$ if and only if

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & t_1 \\ 0 & 1 & \cdots & 0 & t_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & t_{n-1} \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}\tag{4.5}$$
for some \((t_1, \ldots, t_{n-1}) \in \mathbb{A}^{n-1}\). On the other hand, \(ZA = Y\) translates to
\[
\begin{align*}
y_{1,1}t_1 + \cdots + y_{1,n-1}t_{n-1} + z_{1,n} &= y_{1,n} \\
y_{2,1}t_1 + \cdots + y_{2,n-1}t_{n-1} + z_{2,n} &= y_{2,n} \\
\cdots \\
y_{n-1,1}t_1 + \cdots + y_{n-1,n-1}t_{n-1} + z_{n-1,n} &= y_{n-1,n}.
\end{align*}
\]

The matrix of this linear system is \((4.3)\). This matrix is non-singular by our assumption. Hence, the system has a unique solution, \((t_1, \ldots, t_{n-1})\) in \(\mathbb{A}^{n-1}\). This completes the proof of the claim and thus of part (b).

(c) In view of \((4.1)\) and part (b), it suffices to show that \(\pi\) is generically smooth; see Lemma \(\ref{lem:generic_smoothness}\). In other words, we need to check that the differential \(d\pi : T_v(V) \to T_{\pi}((M_{(n-1)\times(n-1)})\) is surjective for \(v \in V\) in general position. This is readily seen by restricting \(\pi\) to the affine subspace \(\{J\} \times M_{(n-1)\times n}\) and using the formula \((4.3)\). \(\square\)

**Proof of Proposition \(\ref{prop:group_extensions}\)** First let us settle the case, where \(n = 1\). Here \(SL_n \rtimes (\mathbb{Z}/2\mathbb{Z}) \simeq Z/2Z\). Examining the natural two-dimensional permutation representation \(W\) of \(\mathbb{Z}/2\mathbb{Z} \simeq S_2\), we readily see that \(k(W)^{Z/2Z}\) is rational over \(k\). (Here \(k\) is a field of arbitrary characteristic.) Consequently,
\[
B(\mathbb{Z}/2\mathbb{Z}) \text{ is stably rational over } k.
\]

Now suppose \(n \geq 2\). Set \(V = M_4 \rtimes (\mathbb{Z}/2\mathbb{Z})\) and consider the representation \(V \times W\) of \(SL_n \rtimes (\mathbb{Z}/2\mathbb{Z})\), where \(SL_2 \rtimes (\mathbb{Z}/2\mathbb{Z})\) acts on \(W\) via \(\mathbb{Z}/2\mathbb{Z}\). Since the \(SL_n\)-action on \(V\) is generically free (see Lemma \(\ref{lem:generic_free}\)) and the \(\mathbb{Z}/2\mathbb{Z}\)-action on \(V\) is generically free (obvious), we conclude that so is the \(SL_n \rtimes (\mathbb{Z}/2\mathbb{Z})\)-action on \(V \times W\). It remains to show that
\[
k(V \times W)^{SL_n \rtimes (\mathbb{Z}/2\mathbb{Z})} \text{ is stably rational over } k.
\]

In view of Lemma \(\ref{lem:stably_rational}\)(c), we have
\[
k(V \times W)^{SL_n \rtimes (\mathbb{Z}/2\mathbb{Z})} = k((V/SL_n) \times W)^{\mathbb{Z}/2\mathbb{Z}} = k(M_{(n-1)\times(n-1)} \times W)^{\mathbb{Z}/2\mathbb{Z}}.
\]

To see how \(\mathbb{Z}/2\mathbb{Z} = \langle \tau \rangle\) acts on \(V/SL_n = M_{(n-1)\times(n-1)}\), recall that \(\tau\) sends \(v = (X,Y) \in V\) to \((Y^T, X^T)\). Hence, the induced action of \(\tau\) on \(V/SL_n = M_{(n-1)\times(n-1)}\) takes \(\pi(X,Y) = YX\) to \(\pi(Y^T, X^T) = X^TY^T = \pi(X,Y)^T\). In other words, the induced \(\mathbb{Z}/2\mathbb{Z}\)-action on \(V/SL_n = M_{(n-1)\times(n-1)}\) is given by \(\tau : Z \mapsto Z^T\). In particular, this action is linear.

Now \((4.6)\), tells us that \(k(M_{(n-1)\times(n-1)} \times W)^{\mathbb{Z}/2\mathbb{Z}}\) is stably rational over \(k\). This completes the proof of \((4.7)\) and thus of Proposition \(\ref{prop:group_extensions}\). \(\square\)

**Remark 4.3.** For \(n \geq 3\), the \(SL_n \rtimes (\mathbb{Z}/2\mathbb{Z})\)-action on \(V\) is generically free, so we can work directly with \(V\), rather than \(V \times W\). The extra factor of \(W\) is only needed when \(n = 2\). Note also that if \(\text{char}(k) \neq 2\), then \(\mathbb{Z}/2\mathbb{Z}\) is isomorphic to \(\mu_2\), and thus \((4.6)\) is a special case of Example \(\ref{ex:mu_2}\).

5. **Group extensions**

In the sequel we will apply Proposition \(\ref{prop:group_extensions}\) in combination with the following proposition.
**Proposition 5.1.** Let $n$ be an odd integer and let $d \geq 1$. Consider a short exact sequence

$$1 \longrightarrow \text{SL}_n \longrightarrow H \overset{\pi}{\longrightarrow} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

of algebraic groups. Then either $H \cong \text{SL}_n \times (\mathbb{Z}/2\mathbb{Z})$ or $H \cong \text{SL}_n \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the generator $\tau$ of $\mathbb{Z}/2\mathbb{Z}$ acts by $\tau : A \mapsto (A^{-1})^T$, as in Section 4.

**Proof.** Note that the fiber $\pi^{-1}(\tau)$ is an $\text{SL}_n$-torsor. Since $\text{SL}_n$ is a special group, this torsor is split. In other words, there exists an $x \in H(k)$ such that $\pi(x) = \tau$. Let $\varphi_x : \text{SL}_n \to \text{SL}_n$ denote conjugation by $x$: $\varphi_x(A) = AxA^{-1}$. Since $\varphi_x$ is a $k$-group automorphism of $\text{SL}_n$, there exists $B \in \text{SL}_n(k)$ such that either $\varphi_x(A) = BAB^{-1}$ for every $A \in \text{SL}_n$, or $\varphi_x(A) = B(A^{-1})^TB^{-1}$. After replacing $x$ by $Bx$, we may assume that either $\varphi_x = \text{Id}$ or $\varphi_x(A) = (A^{-1})^T$. It now suffices to show that there exists a $y \in H(k)$ such that $\pi(y) = \tau$ and $y^2 = 1$.

In both cases, $\varphi_x^2 = (\varphi_x)^2$ equals the identity, i.e., $x^2Ax^{-2} = A$ for every $A \in \text{SL}_n(k)$. It follows $x^2$ lies in the center of $\text{SL}_n$, i.e., $x^2 \in \mu_n(k) \subseteq \text{SL}_n(k)$ is a diagonal matrix.

Let $\langle x \rangle \subseteq H(k)$ be the subgroup generated by $x$. The restriction of $\pi$ to $\langle x \rangle$ is surjective and it sends $x^n$ to $\tau^m$. It follows that $\text{Ker}(\pi) \cap \langle x \rangle = \langle x^2 \rangle$, so that we have a short exact sequence

$$1 \longrightarrow \langle x^2 \rangle \longrightarrow \langle x \rangle \overset{\pi}{\longrightarrow} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$ 

The order of $\mu_n(k)$ divides $n$, hence $\mu_n(k)$ is cyclic of odd order. Since $\langle x^2 \rangle$ is a subgroup of $\mu_n(k)$, it is also cyclic of odd order. On the other hand, since $\langle x \rangle$ surjects onto $\mathbb{Z}/2\mathbb{Z}$, $\langle x \rangle$ is of even order. We conclude that $\langle x \rangle$ contains an element $y$ of order 2, and $\pi(y) = \tau$ as desired. 

**6. The Noether Problem for $G_2$, Spin$_7$ and Spin$_{10}$**

For the remainder of this paper we will assume that $k$ is a field of characteristic $\neq 2$. In particular, $\mu_2 \cong \mathbb{Z}/2\mathbb{Z}$ over $k$.

**Proposition 6.1.** $BG_2$ is stably rational.

**Proof.** Let $V$ be the octonion representation of $G_2$. If we let $\mathbb{G}_m$ act on $V$ by scalar, the induced $G_2 \times \mathbb{G}_m$-action on $V$ has an open orbit, and the stabilizer in general position is isomorphic to $\text{SL}_3 \rtimes \mu_2$, where $\mu_2$ acts on $\text{SL}_3$ by $A \mapsto (A^{-1})^T$; see [Jac58 Theorem 4], [ST05 Proposition 3.3] or [PT19 Lemma 3.3]. By Lemma 3.4, $BG_2$ is stably birationally equivalent to $B(\text{SL}_3 \rtimes \mu_2)$. By Proposition 4.1, $B(\text{SL}_3 \rtimes \mu_2)$ is stably rational, and hence so is $BG_2$. 

For an alternative proof of Proposition 6.1, see [ZS18 Corollary 2, p. 568].

**Proposition 6.2.** $B\text{Spin}_7$ is stably rational.

**Proof.** Let $V$ be the spin representation of $\text{Spin}_7$. Letting $\mathbb{G}_m$ act on $V$ by scalar multiplication, we obtain an action of $\text{Spin}_7 \times \mathbb{G}_m$ on $V$. If $\gamma$ is the generator of the center of $\text{Spin}_7$, then $\gamma$ acts as $-\text{Id}$ on $V$. It follows that the subgroup

$$C := \langle (\gamma, -1) \rangle \cong \mu_2$$

In [PT19] it is assumed that $\sqrt{-1} \in k$. However, this assumption can be dropped if one works with the split form of $G_2$ throughout.
of Spin$_7 \times \mathbb{G}_m$ acts trivially on $V$. The quotient $(\text{Spin}_7 \times \mathbb{G}_m)/C$ acts faithfully on $V$. This quotient group is usually called “the even Clifford group” and is denoted by $\Gamma_7^\perp$. The natural short exact sequence

$$1 \to \text{Spin}_7 \to \Gamma_7^\perp \to \mathbb{G}_m \to 1$$

shows that $B\text{Spin}_7$ and $B\Gamma_7^\perp$ are stably birational; see [Mer19, Lemma 4.3] or [Mer19, Corollary 4.4].

It remains to show that $B\Gamma_7^\perp$ is stably rational. By [Igu70] Proposition 4], there exists a quadratic form $g : V \to \mathbb{A}^1$ such that the orbits of the Spin$_7$-action on $V$ are exactly the fibers of $g$. In particular, the $\Gamma_7^\perp$-action on $V$ has an open orbit. Furthermore, if $p \in V(k), g(p) \neq 0$, and $S$ is the stabilizer of $p$ in Spin$_7$, then

$$(6.1) \quad S \simeq G_2$$

see also [GG17, Table 1]. Note that Igusa only showed that $S \simeq G_2$ at the level of points. This settles (6.1) in characteristic 0. To complete the proof of (6.1) in finite characteristic (â¬≠ 2) it remains to show that $S$ is smooth, or equivalently, that $\dim \text{Lie}(S) = \dim S$, where $\text{Lie}(S)$ denotes the Lie algebra of $S$. This is done in [SK77, pp. 115-116], where it is shown that $\text{Lie}(S) \simeq \mathfrak{g}_2$; see Remark 6.3.

Denote by $H$ the stabilizer of $p$ in Spin$_7 \times \mathbb{G}_m$. If $(h, t) \in H(\overline{k})$, then

$$g(p) = g((h, \lambda)p) = g(\lambda hp) = \lambda^2 g(hp) = \lambda^2 g(p).$$

Since $g(p) \neq 0$, we deduce that $\lambda^2 = 1$. Thus the projection Spin$_7 \times \mathbb{G}_m \to \mathbb{G}_m$ to the second coordinate gives rise to a short exact sequence

$$1 \to S \to H \xrightarrow{\pi} \mu_2 \to 1.$$ 

Since $S \simeq G_2$ has trivial center, $C$ intersects $S$ trivially. The inclusion $S \hookrightarrow H$ now induces an isomorphism between $S$ and $H/C$, which is the stabilizer of $p$ in $\Gamma_7$.

$\Box$

Remark 6.3. The base field in [SK77] is assumed to be $\mathbb{C}$. However, the Lie algebra calculation of [SK77, pp. 115-116] remains valid over any field $k$ of characteristic $\neq 2$.

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Proposition 6.4. $B\text{Spin}_{10}$ is stably rational.

Proof. Let $V$ be the half-spin representation of Spin$_{10}$. By [Igu70] Proposition 2] there are two non-zero orbits in $V$. Let $H$ be the stabilizer of a $k$-point in the open orbit. The subgroup $H$ is explicitly described in [Igu70, Lemma 3] (with $n = 5$): we have $H = W \rtimes G_0$, where $W$ has the structure of an 8-dimensional vector space and $G_0$ acts linearly on $W$. By [Igu70, Proposition 1], $G_0 \simeq \text{Spin}_7$. Note that $H$ is smooth by the Lie algebra calculation of [SK77] which will be used in the proofs of Propositions 6.4 and 7.1.

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algebra computation in [SK77, p. 121]; once again, see Remark 6.3. Hence $H \cong W \rtimes \text{Spin}_7$ as group schemes (and not just at the level of points).

By Lemma 3.4, $B\text{Spin}_10$ is stably birational to $BH = B(W \rtimes \text{Spin}_7)$. By Lemma 3.7, $B(W \rtimes \text{Spin}_7)$ is stably birational to $B\text{Spin}_7$, which is stably rational by Proposition 6.2. We conclude that $B\text{Spin}_10$ is stably rational.

7. The Noether Problem for Spin$_{11}$

**Proposition 7.1.** $B\text{Spin}_{11}$ is stably rational.

Our proof will follow the same pattern as the proof of stable rationality of $B\text{Spin}_7$ in Proposition 6.2, except that the second half of the argument will be more involved.

**Proof of Proposition 7.1.** Let $V$ be the spin representation of Spin$_{11}$. Our starting point is the following result of J. Igusa [Igu70, Proposition 6].

(a) There exists a non-zero Spin$_{11}$-invariant homogeneous form $J : V \to \mathbb{A}^1$ of degree 4, such that the Spin$_{11}$-orbits in $V \setminus \{0\}$ are the $J^{-1}(\lambda)$, $\lambda \in \mathbb{A}^1 \setminus \{0\}$, together with four orbits inside $J^{-1}(0)$.

(b) If $-\lambda \in \mathbb{A}^1(k)$ is a non-zero square, the orbit $J^{-1}(\lambda)$ contains a rational point whose stabilizer is $k$-isomorphic to SL$_5$. This is an isomorphism of group schemes; see the proof of [SK77, Proposition 39] or [GG17, Table 1].

Letting $G_m$ act on $V$ by scalar multiplication, we obtain an action of Spin$_{11} \times G_m$ on $V$. If $\gamma$ is the generator of the center of Spin$_{11}$, then $\gamma$ acts as $-\text{Id}$ on $V$. It follows that the subgroup $C := \langle (\gamma, -1) \rangle \cong \mu_2$ of Spin$_{11} \times G_m$ acts trivially on $V$. The quotient $(\text{Spin}_{11} \times G_m)/C$ acts faithfully on $V$. This quotient is usually called the even Clifford group and is denoted by $G_m^+$.

Applying [Mer19, Lemma 4.3] to the short exact sequence

$$1 \longrightarrow \text{Spin}_{11} \longrightarrow G_{11}^+ \overset{pr_2}{\longrightarrow} G_m \longrightarrow 1$$

we see that $B\text{Spin}_{11}$ and $B\Gamma_{11}^+$ are stably birational; cf. [Mer19, Corollary 4.4]. It remains to show that $B\Gamma_{11}^+$ is stably birational.

By (a), Spin$_{11} \times G_m$ acts transitively on $U := V \setminus J^{-1}(0)$. Let $p \in U(k)$, $H$ be the stabilizer of $p$ in Spin$_{11} \times G_m$, and $\overline{H}$ be the stabilizer of $p$ in $\Gamma_{11}^+ = (\text{Spin}_{11} \times G_m)/C$. Since $C$ acts trivially on $V$, $C$ is a central subgroup of $H$ and $\overline{H} = H/C$. Since $p$ belongs to the open orbit of the $\Gamma_{11}^+$-action, Lemma 3.3 tells us that $B\Gamma_{11}^+$ is stably birationally equivalent to $B\overline{H}$. Thus it suffices to show that $B\overline{H}$ is stably rational.

If $(h, \lambda) \in H(\overline{k})$, then

$$J(p) = g((h, \lambda)p) = J(\lambda hp) = \lambda^4 J(hp) = \lambda^4 J(p).$$

Since $J(p) \neq 0$, we deduce that $\lambda^4 = 1$. This shows that $H \subseteq \text{Spin}_{11} \times \mu_4$. The kernel of the projection $\pi : H \to \mu_4$ is the stabilizer of $p$ in Spin$_{11} \times \{1\}$. By (b), this stabilizer is isomorphic to SL$_5$. We thus obtain a short exact sequence

$$1 \longrightarrow \text{SL}_5 \overset{\pi}{\longrightarrow} \overline{H} \overset{\pi}{\longrightarrow} \mu \longrightarrow 1,$$
where $\mu := \text{Im}(\pi)$. Note that $C \cap \text{Spin}_{11} = 1$ and thus $C \cap \text{SL}_5 = 1$. Modding out by $C$, we obtain a short exact sequence

$$1 \to \text{SL}_5 \to \pi(C) \to \mu/\pi(C) \to 1.$$ 

Since $C \cap \text{SL}_5 = 1$, we have $\pi(C) \simeq \mu_2$. If $\mu = \pi(C)$, then $\pi(C) \simeq \text{SL}_5$ is special. In this case $B\pi(C)$ is stably rational by Lemma 3.1. Hence we may assume that $\mu = \mu_4$. In this case $C \cap \text{SL}_5 = 1$ and our exact sequence reduces to

$$1 \to \text{SL}_5 \to \pi(C) \to \mu_2 \to 1.$$ 

By Proposition 5.1 either (i) $H = \text{SL}_5 \times \mu_2$ or (ii) $H = \text{SL}_5 \rtimes \mu_2$, where $\mu_2$ acts on $\text{SL}_5$ by $A \mapsto (A^{-1})^T$.

In case (i), $B\pi(C)$ is stably birational to $B\mu_2$ by Lemma 3.3(c), and $B\mu_2$ is stably rational by Example 3.5. Thus $B\pi(C)$ is stably rational. In case (ii), $B\pi(C)$ is stably rational by Proposition 4.1.

7. The Noether Problem for $\text{Spin}_{14}$

Proposition 8.1. $B\text{Spin}_{14}$ is stably rational.

Proof. Let $V$ be the half-spin representation of $\text{Spin}_{14}$; $v \in V$ be a $k$-point in general position, $S$ be the stabilizer of $v$, and $N$ be the normalizer of $S$. By [Gar09, Example 21.1],

$$N \simeq (G_2 \times G_2) \rtimes \mu_8, \quad G_2 \times G_2 \subseteq S \subseteq N,$$

and the $\text{Spin}_{14}$-orbit of $[v]$ is open in $\mathbb{P}(V)$; cf. also [GG17, §8]. The $\mu_8$-action on $G_2 \times G_2$ factors through the surjection $\mu_8 \to \mu_2$, where $\mu_2 \simeq S_2$ acts on $G_2 \times G_2$ by switching the two factors. Note that this action is well defined even if $k$ does not contain an 8th root of unity. We will write $N \simeq G_2 \rtimes \mu_8$, as in Section 3.

Letting $\mathbb{G}_m$ act on $V$ by scalar multiplication, we obtain an action of $\text{Spin}_{14} \rtimes \mathbb{G}_m$ on $V$. The orbit of $v$ under this action is open and dense in $V$. Let $H$ be the stabilizer of $v$ in $\text{Spin}_{14} \rtimes \mathbb{G}_m$. Consider the composition

$$\varphi : H \hookrightarrow \text{Spin}_{14} \times \mathbb{G}_m \xrightarrow{\text{pr}_1} \text{Spin}_{14}.$$ 

Note that $\varphi$ is injective. Indeed, its kernel, the stabilizer of $v$ in $\mathbb{G}_m$, is trivial. Thus $H \simeq \text{Im}(\varphi)$. Moreover, clearly $S \times \{1\} \subset H$, and thus

$$G_2 \times G_2 \subset S \subseteq \text{Im}(\varphi) \simeq H.$$ 

We claim that $\text{Im}(\varphi) \subseteq N$. Here the inclusion should be understood scheme-theoretically. To prove this claim, let $R$ be a $k$-algebra and $g \in \text{Spin}_{14}(R)$ be in the image of $\varphi$. Then $gv = \lambda v$ for some $\lambda \in R^\times$. For any $h \in S(R)$, we have

$$g^{-1}hg = \lambda g^{-1}hv = \lambda g^{-1}v = \lambda \lambda^{-1}v = v.$$ 

This shows that $g^{-1}hg \in S(R)$ for any $h \in S(R)$. In other words, $g \in N(R)$, as claimed.
We have thus shown that $\text{Im}(\varphi)$ is a subgroup of $N \simeq (G_2 \times G_2) \rtimes \mu_8$ containing $G_2 \times G_2$. Thus $\text{Im}(\varphi)$ is the preimage of a subgroup of $\mu_8$ under the natural projection $N \to N/(G_2 \times G_2) \simeq \mu_8$. We conclude that

$$H \simeq \text{Im}(\varphi) \simeq (G_2 \times G_2) \rtimes \mu_m \overset{\text{def}}{=} G_2 \wr \mu_m,$$

where $\mu_m$ is a subgroup of $\mu_8$, i.e., $m$ is a divisor of 8. In particular, in view of our standing assumption that $\text{char}(k) \neq 2$, this shows that $H$ is smooth. (As an aside, we remark that the semidirect product $(G_2 \times G_2) \rtimes \mu_m$ is direct if $m = 1, 2$ or 4 and not direct if $m = 8$.)

To finish the proof, observe that,

(i) by Proposition 6.1, $B G_2$ is stably rational.

(ii) By Lemma 3.3(b), $B(G_2 \wr \mu_m)$ is stably birationally equivalent to $B \mu_m$. On the other hand, $B \mu_m$ is stably rational by Example 3.5.

(iii) By Lemma 3.4 applied to the representation of Spin$^1_4 \times \mathbb{G}_m$ on $V$, $B(\text{Spin}^1_4 \times \mathbb{G}_m)$ is stably birationally equivalent to $B H$, where $H \simeq \text{Im}(\varphi) \simeq G_2 \wr \mu_m$. By (ii), $B H$ is stably rational, and hence, so is $B(\text{Spin}^1_4 \times \mathbb{G}_m)$.

(iv) By Lemma 3.3(c), $B \text{Spin}^1_4$ is stably birationally equivalent to $B(\text{Spin}^1_4 \times \mathbb{G}_m)$. Thus $B \text{Spin}^1_4$ is stably rational.

Remark 8.2. Assume that $-1$ is a square in $k$, and let $D_{2m+1}, D'_{2m+1}$ be the two non-isomorphic extraspecial 2-groups of order $2^{2m+1}$. It is shown in [BB13] that $BD_{2m+1}$ and $BD'_{2m+1}$ are stably birationally equivalent. By [Mer19, Corollary 6.2], $B \text{Spin}_n$ is stably birational to $BD_{2m+1}$. Therefore, Theorem 1.1 has the following consequence: $BD_{2m+1}$ and $BD'_{2m+1}$ are stably rational for any $m \leq 6$.

9. COREGULAR REPRESENTATIONS OF SPINOR GROUPS

In this section we present an alternative proof of Theorem 1.1 over $k = \mathbb{C}$ suggested to us by G. Schwarz.

Let $G$ be a linear algebraic group over $k$. A linear representation $\rho: G \to \text{GL}(V)$ is called coregular if $k[V]^G$ is a polynomial ring over $k$. Coregular representations of finite groups $G$, whose order is not divisible by $\text{char}(k)$, are described by the celebrated theorem of Chevalley-Shephard-Todd: $\rho$ is coregular if and only if $\rho(G)$ is generated by pseudo-reflections. Now suppose that $G$ is a simple linear algebraic groups over $k = \mathbb{C}$. In this setting irreducible coregular representations were classified by V. Kac, V. Popov and E. Vinberg in [KPV76] and arbitrary (not necessarily irreducible) coregular representations by G. Schwarz in [Sch78] and (independently) by O. Adamovich and E. Golovina [AG83]. For an overview of this area of research we refer the readers to [PV94, §8]. We will not need the full classification here; we will only use the fact that certain specific representations of Spin$_n$ ($n = 7, 10, 11, 14$) are coregular.

Coregular representations are related to the Noether Problem via the following simple observation.

Lemma 9.1. Suppose a smooth algebraic $k$-group $G$ has no non-trivial characters (over $k$). If $G$ admits a generically free coregular representation over $k$, then $BG$ is stably rational over $k$. 

Proof. Let \( \rho : G \to \text{GL}(V) \) be a representation defined over \( k \). Since \( k[V] \) is a unique factorization domain, and \( G \) has no non-trivial characters, \( k(V) \) is the fraction field of \( k[V]^G \); see [PV94, Theorem 3.3]. If \( V \) is coregular, this shows that \( k(V)^G \) is rational over \( k \). If \( V \) is also generically free, we conclude that \( BG \) is stably rational. \( \square \)

Now recall from the Introduction that in order to prove Theorem 1.1 it suffices to show that \( B \text{Spin}_n \) is stably rational for \( n = 7, 10, 11, 14 \). Over the field \( k = \mathbb{C} \) of complex numbers, Theorem 1.1 is now an immediate consequence of the following.

**Proposition 9.2.** Let \( k = \mathbb{C} \). Then \( G := \text{Spin}_n \) admits a coregular generically free representation for \( n = 7, 10, 11, 14 \).

**Proof.** Let \( V_n \) be the natural representation of \( \text{Spin}_n \) (via the projection \( \text{Spin}_n \to \text{SO}_n \)), and \( W_n \) be the spin representation. If \( n \) is even, let \( W_n^{1/2} \) be the half-spin representation. The following representations are shown to be coregular in [Sch78]:

1. \( V_7^{\mathbb{Z}_3} \oplus W_7 \), see Table 3a.9;
2. \( V_{10}^{\mathbb{Z}_5} \oplus W_{10}^{1/2} \), see Table 3a.20;
3. \( V_{11}^{\mathbb{Z}_4} \oplus W_{11} \), see Table 3a.25;
4. \( V_{14}^{\mathbb{Z}_3} \oplus W_{14}^{1/2} \), see Table 3a.31.

It thus remains to show that these representations are generically free.

(i) Let \( w \in W_7 \) be a point in general position. By [GG17, Table 1], the stabilizer of \( w \) is isomorphic to \( G_2 \). We claim that

\[
\text{(9.1) the } G_2 \text{-action on } V_7^{\mathbb{Z}_3} \text{ is generically free.}
\]

Note that \( \Lambda = \{ x \in V_7^{\mathbb{Z}_3} \oplus W_7 \mid \text{Stab}_{\text{Spin}_7}(x) = 1 \} \) is a constructible subset of \( V_7^{\mathbb{Z}_3} \oplus W_7 \). If the claim is established, then by the Fiber Dimension Theorem,

\[
\dim(\Lambda) = \dim((V_7^{\mathbb{Z}_3} \times W_7)).
\]

Thus \( \Lambda \) contains a dense open subset of \( V_7^{3} \times W_7 \), i.e., the Spin\(_7\)-action on \( V_7^{3} \times W_7 \) is generically free.

It remains to prove the claim. Since \( \dim(V_7) = 7 \), the \( G_2 \)-action on \( V_7 \) is the octonion representation of \( G_2 \). In other words, we may identify \( V_7 \) with the space of trace-zero octonions in such a way that \( G_2 \) acts via octonion automorphisms. If \( x, y, z \) generate the octonions as a \( \mathbb{C} \)-algebra, then clearly the \( G_2 \)-stabilizer of \( (x, y, z) \in V_7^{\mathbb{Z}_3} \) is trivial. The set \( U \subset V_7^{\mathbb{Z}_3} \) of generating triples is readily seen to be open. Note also that \( U \) is non-empty, because \((i, j, k) \in U \), where \( i, j \) and \( k \) are the standard generators.

(ii) If \( v := (v_1, v_2, v_3, v_4, v_5) \in V_{10}^{\mathbb{Z}_5} \) is a general point, the stabilizer of \( v \) is isomorphic to \( \text{Spin}_5 \). Arguing as in (i), it remains to show that the restriction of \( W_{10}^{1/2} \) to \( \text{Spin}_5 \) is generically free. Recall that, for every \( n \geq 1 \),

- the restriction of the spin representation \( W_{2n+1} \) to \( \text{Spin}_{2n} \) is \( W_{2n}^{1/2} \oplus W_{2n}^{1/2} \), and
- the restriction of \( W_{2n}^{1/2} \) to \( \text{Spin}_{2n-1} \) is the spin representation \( W_{2n-1} \),

see [Igu70, p. 1000] or [Bum13, Ex. 31.2, Ex. 31.3]. Restricting \( W_{10}^{1/2} \) from \( \text{Spin}_{10} \) to \( \text{Spin}_9 \), then to \( \text{Spin}_8 \), etc., we see that as a \( \text{Spin}_5 \)-representation, \( W_{10}^{1/2} \) is isomorphic to \( W_{5}^{\mathbb{Z}_4} \). Note that \( \dim(W_5) = 4 \). There is an accidental isomorphism \( \text{Spin}_5 \cong \text{Sp}_4 \), under
which \( W_5 \) corresponds to the natural 4-dimensional representation of \( \text{Sp}_4 \); see [Ada96, Proposition 5.1]. Thus we can identify the \( \text{Spin}_5 \)-action on \( W_5^4 \) with the \( \text{Sp}_4 \)-action on \( M_{4 \times 4} \) via left multiplication. The latter action is clearly generically free.

(iii) If \( v := (v_1, v_2, v_3, v_4) \in V_{11}^{\otimes 4} \) is a general point, the stabilizer of \( v \) is isomorphic to \( \text{Spin}_7 \). Once again, it suffices to show that when we restrict \( W_{11} \) from \( \text{Spin}_{11} \) to \( \text{Spin}_7 \), it becomes generically free. Arguing as in (ii) above, we see that as a \( \text{Spin}_7 \)-representation, \( W_{11} \) is isomorphic to \( W_7^{\otimes 4} \). If \( w \in W_7 \) is a general point, then the \( \text{Spin}_7 \)-stabilizer of \( w \) is isomorphic to \( G_2 \); see (6.1). Thus it suffices to show that the \( G_2 \)-action on \( W_7^{\otimes 3} \) is generically free. As a \( G_2 \)-representation, \( W_7 = \mathbb{C} \oplus V_7 \), where \( \mathbb{C} \) is the trivial 1-dimensional representation generated by the vector stabilized by \( G_2 \), and \( V_7 \) is a 7-dimensional representation, which can be identified with trace-zero octonions, as in (i). The desired conclusion now follows from (9.1).

(iv) If \( w \in W_{14}^{1/2} \) is a general point, then by [GG17, Table 1] the stabilizer of \( w \) in \( \text{Spin}_{14} \) is isomorphic to \( G_2 \times G_2 \). It thus suffices to show that \( G_2 \times G_2 \) acts generically freely on \( V_{14}^{\otimes 3} \). Since \( G_2 \times G_2 \) has trivial center, we may view it as a subgroup of \( \text{SO}(V_{14}) \) via the projection \( \text{Spin}_{14} \to \text{SO}(V_{14}) \). By [GG17, §8], \( V_{14} \cong V_7 \oplus V_7 \) as a \( G_2 \times G_2 \)-representation. Here \( V_7 \) is identified with trace-zero octonions, with the natural action of \( G_2 \), and \((g, g') \in G_2 \times G_2 \) acts on \((v, v') \in V_7 \times V_7 \) by \((v, v') \mapsto (g(v), g'(v')) \). By (9.1), \( G_2 \) acts generically freely on \( V_7^{\otimes 3} \). Hence, \( G_2 \times G_2 \) acts generically freely on \( V_{14}^{\otimes 3} \cong V_7^{\otimes 3} \oplus V_7^{\otimes 3} \), as desired.

\[ \square \]

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