COUNTING POINTS ON ABELIAN SURFACES OVER FINITE FIELDS WITH ELKIES’S METHOD

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Abstract. We generalize Elkies’s method, an essential ingredient in the SEA algorithm to count points on elliptic curves over finite fields of large characteristic, to the setting of p.p. abelian surfaces. Under reasonable assumptions related to the distribution of Elkies primes, we obtain improvements over Schoof’s method in two cases. If the abelian surface \( A \) over \( \mathbb{F}_q \) has RM by a fixed quadratic field \( F \), we reach the same asymptotic complexity \( \tilde{O}(\log^5 q) \) as the SEA algorithm up to constant factors depending on \( F \). If \( A \) is defined over a number field, we count points on \( A \) modulo sufficiently many primes in \( \tilde{O}(\log^6 q) \) binary operations on average. Numerical experiments demonstrate the practical usability of our methods.

1. Introduction

In this paper, we consider the problem of point counting for principally polarized (p.p.) abelian varieties over finite fields: given a p.p. abelian variety \( A \) over \( \mathbb{F}_q \), we aim to compute the characteristic polynomial of Frobenius \( \chi(A) \in \mathbb{Z}[X] \). If \( A \) is an elliptic curve, this is equivalent to computing \( \#A(\mathbb{F}_q) \).

The motivation behind this challenge comes from different directions. Counting points is a prerequisite for elliptic and hyperelliptic-curve cryptography [33, 34]. More recently, the hardness of the point counting problem itself was proposed as a source of cryptographic protocols [10]. From a more mathematical point of view, if \( A \) is defined over a number field, then counting points on \( A \) modulo primes of good reduction determines the Euler factors of the \( L \)-function attached to \( A \).

To this date, Schoof’s polynomial-time algorithm [42, 40] remains the central approach to point counting for abelian varieties of dimension 2 or more over finite fields of large characteristic, and much work has been devoted to making this algorithm practical [18, 19, 20, 1]. In the case of abelian surfaces over \( \mathbb{F}_q \), the complexity of Schoof’s method is \( \tilde{O}(\log^5 q) \) binary operations in general, and \( \tilde{O}(\log^3 q) \) binary operations if the abelian surfaces have explicit real multiplication (RM) by a fixed quadratic field. Note however that Schoof’s approach is in competition with cohomological algorithms, surveyed in [28], when the base field has small characteristic; in the context of computing \( L \)-functions, it is in competition with average polynomial time algorithms based on Hasse-Witt matrices [23, 24, 50].

Schoof’s approach is multi-modular: for a series of small primes \( \ell \neq p \), the reduction of \( \chi(A) \) mod \( \ell \) is computed as the characteristic polynomial of Frobenius on the \( \ell \)-torsion subgroup \( A[\ell] \), the latter being defined by explicit polynomial equations. The algorithm stops when sufficient information is collected to reconstruct \( \chi(A) \) using the Weil bounds and the Chinese remainder theorem.

In the case of elliptic curves, Elkies [12] showed how to accelerate Schoof’s algorithm by replacing \( A[\ell] \), the kernel of the endomorphism \( [\ell] \), by the kernel of
an isogeny $\varphi: A \to A'$ of degree $\ell$. Such an isogeny will exist as soon as $\chi(A)$ splits in linear factors modulo $\ell$; we say that $\ell$ is Elkies in this case. Heuristically, about half of the small primes are Elkies for a given $A$; this heuristic is true on average, either for all elliptic curves over a given finite field [45], or, assuming GRH, for all reductions of a given elliptic curve over a number field modulo primes of good reduction [46]. Then the resulting point counting algorithm will run in $O(\log^4 q)$ binary operations, instead of $\tilde{O}(\log^5 q)$. Elkies’s method is an essential part of the SEA algorithm [43], implemented in both Pari/GP [51] and Magma [3].

The central player in Elkies’s method is the classical modular polynomial $\Phi_\ell$ of level $\ell$, an explicit polynomial equation cutting out the moduli space of pairs of $\ell$-isogenous elliptic curves. More precisely, Elkies’s method relies on three main ingredients: first, upper bounds on the degree and height of $\Phi_\ell$ [9, 7]; second, an evaluation algorithm, to compute $\Phi_\ell(j, Y)$ as well as its derivative $\partial_Y \Phi_\ell(j, X)$ for a given value of $j \in \mathbb{F}_q$ [14, 6, 49]; and third, an isogeny algorithm to recover $\varphi$ as an explicit rational map from this data [12, 4].

The recent series of papers [31, 30, 32] extend all these three ingredients to the context of p.p. abelian surfaces. In this setting, the classical modular polynomials are replaced by modular equations for abelian surfaces, as described in [5, 37, 35, 38]. Here we reap the benefits of these works and describe their consequences on the point counting problem under heuristics related to the distribution of Elkies primes. We separate two cases depending on the moduli space of abelian surfaces we wish to consider. In the Siegel case, we assume nothing a priori on our abelian surfaces; in the Hilbert case, we fix a real quadratic field $F$ and we only consider p.p. abelian surfaces with RM by its ring of integers $\mathbb{Z}_F$ (but the action of $\mathbb{Z}_F$ is not assumed to be explicitly computable). In the Hilbert case, we reach the same asymptotic complexity as the SEA algorithm up to constant factors depending on $F$.

**Theorem 1.1.** Let $F$ be a real quadratic field, and let $\varepsilon > 0$. Then there exists an algorithm which, given a prime power $q = p^r$ with $r = o(\log p)$, and given the Igusa invariants of a p.p. abelian surface $A$ over $\mathbb{F}_q$ with real multiplication by $\mathbb{Z}_F$ for which a proportion $\varepsilon$ of primes are Elkies (see Definition 4.5), computes $\chi(A) \in \mathbb{Z}[X]$ in $\mathcal{O}_{F, \varepsilon}(\log^4 q)$ binary operations.

In the Siegel case, it turns out that Elkies’s method brings no complexity improvement (except perhaps for logarithmic factors) over Schoof’s method for a general abelian surface $A$ over $\mathbb{F}_q$. However, it does bring an improvement when the invariants of $A$ admit lifts in characteristic zero of small heights. The exponent in the complexity estimate is further decreased if we wish to count points modulo sufficiently many primes at once.

**Theorem 1.2.** Let $K$ be a number field, and let $\varepsilon > 0$. Then:

1. There exists an algorithm which, given $H \geq 0$, a p.p. abelian surface $A$ over $K$ whose Igusa invariants are well-defined and have height at most $H$, and given a prime ideal $\mathfrak{p}$ of $K$ of norm $q$ such that $A$ has good reduction at $\mathfrak{p}$ and a proportion $\varepsilon$ of primes are Elkies for its reduction $A_\mathfrak{p}$ (see Definition 3.7), computes $\chi(A_\mathfrak{p}) \in \mathbb{Z}[X]$ in $\mathcal{O}_{K, \varepsilon}(H \log^7 q)$ binary operations.

2. There exists an algorithm which, given $H \geq 0$ and $q \geq 1$, a p.p. abelian surface $A$ over $K$, and given $\Theta(H \log q)$ many distinct primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of $K$ such that $\log N(\mathfrak{p}_i) = O(\log q)$, such that $A$ has good reduction at all
primes $p_i$, and such that a proportion $\varepsilon$ of primes are Elkies for each of its reductions $A_{p_i}$, computes all characteristic polynomials $\chi(A_{p_i}) \in \mathbb{Z}[X]$ using $O_{K,\varepsilon}(\log^b q)$ binary operations on average for each $i$.

We have released an implementation of the key step of the above algorithms in terms of running time, namely the evaluation of modular equations $\chi(F)$, building on the C libraries Flint [22] and Arb [27]. This allows us to roughly estimate the total cost of the above point-counting algorithms in practice.

As remarked in [10, §4.2.1], dimension 2 is the largest dimension where Elkies’s method can be superior to Schoof’s algorithm for generic abelian varieties, at least in the asymptotic sense. However, Elkies’s method still seems promising in the context of counting points on p.p. abelian varieties with fixed RM in any dimension.

This paper is organized as follows. In Section 2, we quickly review previous results on Schoof’s method for abelian surfaces. In Sections 3 and 4, we describe Elkies’s method for p.p. abelian surfaces in the Siegel and Hilbert case respectively. Experimental results appear in Section 5. Finally, Section 6 presents possible directions to further reduce the cost of point counting for abelian surfaces in practice.

Acknowledgements. I am deeply indebted to his former advisors Damien Robert and Aurel Page for suggesting the thesis project that led to this work. I also thank Noam Elkies, John Voight and Andrew Sutherland for their insightful comments on this work. Finally, I thank the LMFDB team for allowing me to access their computational resources.

2. Background on point-counting algorithms

2.1. The characteristic polynomial of Frobenius. Let $A$ be a p.p. abelian surface over $\mathbb{F}_q$, and denote its Frobenius endomorphism by $\pi_A$. The characteristic polynomial $\chi(A)$ of $\pi_A$ takes the form

\begin{equation}
\chi(A) = X^4 - s_1 X^3 + (s_2 + 2q) X^2 - qs_1 X + q^2,
\end{equation}

where $s_1, s_2$ are integers satisfying the following inequalities [53], [41, Lem. 3.1]:

\begin{equation}
|s_1| \leq 4\sqrt{q}, \quad |s_2| \leq 4q, \quad s_1^2 - 4s_2 \geq 0, \quad s_2 + 4q \geq 2|s_1|.
\end{equation}

Denote the Rosati involution on $\text{End}(A)$ induced by the principal polarization of $A$ by $\dagger$. Then the real Frobenius $\psi_A = \pi_A + \pi_A^\dagger$ is an element of the subgroup $\text{End}^\dagger(A)$ of real endomorphisms of $A$. Its characteristic polynomial is

\begin{equation}
\xi(A) = X^2 - s_1 X + s_2.
\end{equation}

If $\ell \neq q$ is a prime, then $A[\ell] \subset A$ is a finite étale group scheme isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^4$, and we identify it with its set of points over an algebraic closure of $\mathbb{F}_q$. The reduction of $\chi(A)$ modulo $\ell$ is the characteristic polynomial of $\pi_A$ acting on $A[\ell]$. Recall that $A[\ell]$ is endowed with the Weil pairing, an alternating and nondegenerate bilinear form induced by the principal polarization of $A$; we denote it by $(x, y) \mapsto \langle x, y \rangle \in \mathbb{Z}/\ell\mathbb{Z}$. For all $x, y \in A[\ell]$, we have

\begin{equation}
\langle \pi_A(x), \pi_A(y) \rangle = q \langle x, y \rangle.
\end{equation}

The Rosati involution is equal to adjunction with respect to the Weil pairing, so (4) translates to the equality $\pi_A \pi_A^\dagger = q$.

Assume now that $A$ has RM by $\mathbb{Z}_F$, where $F$ is a real quadratic field; this means that $A$ is equipped with an embedding $\mathbb{Z}_F \hookrightarrow \text{End}^\dagger(A)$. Let $\ell \in \mathbb{Z}$ be a prime which...
splits in $F$ in a product of two principal ideals, generated by $\beta, \overline{\beta} \in \mathbb{Z}_F$. We then have an orthogonal decomposition
\begin{equation}
A[\ell] = A[\beta] \oplus A[\overline{\beta}],
\end{equation}
and both $A[\beta]$ and $A[\overline{\beta}]$ are stable under $\pi_A$, $\pi_A^\dagger$, and $\psi_A$. Since $A[\beta]$ and $A[\overline{\beta}]$ are not isotropic, the determinant of $\pi_A$ on both of these subspaces is $q$.

2.2. Schoof’s method in dimension 2. Recall that any p.p. abelian surface $A$ over $k = \mathbb{F}_q$ is either a product of two elliptic curves or the Jacobian of a hyper-elliptic genus 2 curve $C$ defined over $k$. In the point-counting context, we only have to consider this second case. The fundamental building block of Schoof’s method [18, 20] is to be able to work with the torsion subgroups $A[\ell]$ in a computationally efficient way. Since Elkies’s method ultimately involves computations with subgroups of Jacobians as well, the computational techniques developed for Schoof’s method will still apply in our context.

The first step is to choose birational coordinates on $A$. A popular choice is to consider Mumford coordinates. Let $y^2 = P(x)$ be an equation of $C$, and let $K_C$ be the canonical divisor of $C$. A generic point of $A$ is linearly equivalent to $D - K_C$ for a unique degree-two divisor $D$ on $C$; in turn, a generic divisor of degree two can be written as the zero locus of polynomials of the form $x^2 + u_1 x + u_0$ and $y - v_1 x - v_0$ in a unique way. This defines the Mumford coordinates $(u_0, u_1, v_0, v_1)$ as a rational map from $A$ to the affine space $\mathbb{A}^4_k$, and $A$ is birational to its image. Denote the coordinate ring of $\mathbb{A}^4_k$ by $k[U_0, U_1, V_0, V_1]$.

If $S$ is any finite subgroup of $A$, then $S$ is stable by $D \mapsto i(D)$ where $i(D)$ denotes the hyperelliptic involution, and hence by change of sign of $\nu$-coordinates. Assume that $S$ is generic in the sense that Mumford coordinates are well-defined at all points of $S$ and that all pairs $\{D, i(D)\}$ in $S$ have distinct $u_1$-coordinates. Then the Gröbner basis cutting out $S$ in terms of Mumford coordinates, in the monomial ordering $U_1 < U_0 < V_1 < V_0$, will take the convenient form
\begin{equation}
\begin{aligned}
V_0 - V_1 S_0(U_1) &= 0, \\
V_1^2 - S_1(U_1) &= 0, \\
U_0 - R_0(U_1) &= 0, \\
R_1(U_1) &= 0
\end{aligned}
\end{equation}
for some univariate polynomials $R_1, R_0, S_1, S_0 \in k[U_1]$. The methods of [20, §3] describe how to compute this Gröbner basis when $S = A[\ell]$, assuming that the equation of $C$ is given by a polynomial $P$ of degree five, in other words that $C$ admits a rational Weierstrass point $\infty$. The input of their algorithm is given by Cantor’s division polynomials, which provide an explicit description of the following composition as a rational map:
\[
C \xrightarrow{p \to [p] - \infty} A \xrightarrow{[\ell]} A \xrightarrow{(u_0, u_1, v_0, v_1)} \mathbb{A}^4_k.
\]
More generally, if $A'$ is the Jacobian of another hyperelliptic genus 2 curve $C'$ over $k$, is $f : A \to A'$ is any isogeny, and if $p_0 \in C(k)$ is any rational point, then the same methods will compute a Gröbner basis describing $\ker(f) \subset A$ given the explicit expression of the composed map
\begin{equation}
C \xrightarrow{p \to [p] - p_0} A \xrightarrow{f} A' \xrightarrow{(u_0, u_1, v_0, v_1)} \mathbb{A}^4_k.
\end{equation}
If the degrees of these rational fractions is bounded above by \(d\), then the whole Gröbner basis computation takes \(O(d^3)\) operations in \(k\), hence \(O(d^3 \log q)\) binary operations. Note that a complexity \(O(d^{3-1/\omega} \log q)\), where \(\omega\) denotes the exponent of matrix multiplication, could probably be achieved by computing bivariate resultants using an algorithm of Villard [52] instead of the more classical evaluation-interpolation method. The resulting polynomials in (6) have degree \(O(d^2)\).

Once the Gröbner basis (6) is known, computing the Frobenius endomorphism on \(S\) is simply a matter of computing \((U_0^q, U_1^q, V_0^q, V_1^q)\) using a square-and-multiply algorithm, reducing the result modulo the defining ideal of \(S\) at each step, for a total cost of \(O(d^2 \log^2 q)\) binary operations.

When running Schoof’s method in the generic case, one takes \(S = A[\ell]\) and \(d = O(\ell^2)\). It only remains to find the correct values of \(s_1\) and \(s_2\) in \(\mathbb{Z}/\ell \mathbb{Z}\) such that Frobenius characteristic equation (1) holds on \(A[\ell]\) [20, Alg. 1]. The dominant step in the whole method is the Gröbner basis computation, which accounts for the final complexity of \(O(\log^8 q)\) binary operations. In the RM case, one can take \(S = A[\beta]\) instead provided that \(\ell\) splits correctly in \(\mathbb{Z}_F\) [19]. Then one has \(d = O(\ell)\), giving a total point-counting complexity of \(O(\log^5 q)\) binary operations; both the Gröbner step and the Frobenius computation are asymptotically dominant. In [19], the real multiplication action of \(\mathbb{Z}_F\) is assumed to be explicitly computable; in this case the Chinese remainder theorem can be used to recover \(\psi_A \in \mathbb{Z}_F\) directly.

2.3. Non-generic cases. The above genericity assumption on \(S\) does not necessarily hold in general. As detailed in [1, §5], it could fail in a finite number of different ways: certain elements of \(S\) might be of the form \([p] - [\infty]\) or \(2[p] - K\) for some \(p \in \mathcal{C}(k)\), so that their Mumford coordinates are not defined; or they might take the generic form \([p_1] + [p_2] - K\), but the Mumford coordinates of \(\ell([p_1] - [\infty])\) might not be defined. Each of these possible degeneracy types can be managed by writing another polynomial system with a smaller number of variables or lower degrees than the original one; therefore, considering the generic case is sufficient from a complexity-theoretic point of view. It is also sufficient from a practical point of view: in large characteristics, \(S\) will be generic with overwhelming probability, and any particular \(\ell\) causing problems can simply be skipped.

3. Elkies’s method for abelian surfaces: the Siegel case

3.1. Polarized isogenies between abelian surfaces. Modular polynomials describing \(\ell\)-isogenies between elliptic curves play a central role in Elkies’s method in dimension 1. Similarly, explicit equations for moduli spaces of suitably isogenous abelian surfaces will play a central role in Elkies’s method for p.p. abelian surfaces. Such moduli spaces only exist for certain isogeny types that are directly related to the polarizations and endomorphism rings of the abelian varieties we consider.

Recall that \(\text{NS}(A)\) denotes the Néron–Severi group of \(A\), consisting of line bundles on \(A\) defined over an algebraic closure of the base field, up to algebraic equivalence.

**Theorem 3.1** ([39, Prop. 17.2]). Let \(A\) be a principally polarized abelian variety over a field \(k\). Then there is a natural isomorphism of abelian groups

\[
\mathcal{L}_A: \text{End}^1(A), +) \rightarrow (\text{NS}(A), \otimes).
\]

The line bundle \(\mathcal{L}_A(1)\) is the unique line bundle (up to algebraic equivalence) defining the principal polarization of \(A\); moreover, for each \(\beta \in \text{End}^1(A)\), the line bundle \(\mathcal{L}_A(\beta)\) is ample if and only if \(\beta\) is totally positive.
If $A$ and $A'$ are p.p. abelian varieties of the same dimension over $k$, and if $\beta \in \text{End}^1(A)$, we say that $f: A \to A'$ is a $\beta$-isogeny if $f^*\mathcal{L}_{A'}(1) = \mathcal{L}_A(\beta)$. In particular, $\beta$ must be totally positive. If $\beta$ is moreover prime to $\text{char}(k)$, an isogeny $f$ is a $\beta$-isogeny if and only if the following conditions are satisfied [11, Thm. 1.1]:

1. $\ker(f) \subset A[\beta]$ and $\ker(f)$ is maximal isotropic for the Weil pairing attached to this subgroup;

2. The image $A'$ is endowed with the natural principal polarization of $A/\ker(f)$ coming from the conditions (1).

Since $\text{End}^1(A)$ always contains a copy of $\mathbb{Z}$, it makes sense to talk about $\ell$-isogenies for any $\ell \in \mathbb{Z}_{>1}$. In the case of elliptic curves, this corresponds to the usual notion of cyclic isogenies of degree $\ell$; but in dimension $g$ the degree of an $\ell$-isogeny is $\ell^g$. In general, the degree of a $\beta$-isogeny is $(\deg \beta)^{1/2}$.

We say that two isogenies $f_1: A \to A'$ and $f_2: A \to A''$ are equivalent if there exists an isomorphism of p.p. abelian varieties $\eta: A' \to A''$ such that $f_2 = \eta \circ f_1$.

### 3.2. Elkies primes

The goal of Elkies’s method in the Siegel case is, given a p.p. abelian surface $A$ over $\mathbb{F}_q$ and a prime $\ell$, to obtain information on $\chi(A)$ mod $\ell$ using an $\ell$-isogeny $f$, with domain $A$, defined over $\mathbb{F}_p$. We say that $\ell$ is Elkies for $A$ if such an $f$ exists. Before explaining how to obtain such an $f$, let us describe how $\chi(A)$ can be computed from the action of $\pi_A$ on $\ker(f)$; and conversely, how the splitting behavior of $\chi(A)$ modulo $\ell$ can guarantee that $\ell$ is Elkies.

If $P$ is a monic polynomial of degree $d$ whose constant coefficient $a_0$ is invertible, we denote by $\text{Rec}_q(P)$ the monic polynomial $a_0^{-1}X^dP(q/X)$.

**Proposition 3.2.** Assume that $\ell$ is Elkies for $A$, and let $f: A \to A'$ be an $\ell$-isogeny defined over $\mathbb{F}_q$. Let $P \in \mathbb{Z}/\ell\mathbb{Z}[X]$ be the characteristic polynomial of $\pi_A$ on $\ker(f)$. Then $\chi(A) = P \text{Rec}_q(P)$ modulo $\ell$.

**Proof.** Choose a symplectic basis of $A[\ell]$ whose first two vectors generate $\ker(f)$. By (4), the matrix of $\pi_A$ in this basis is of the form

$$
\begin{pmatrix}
M & * \\
0 & qM^{-1}
\end{pmatrix}
$$

for some $M \in \text{GL}_2(\mathbb{Z}/\ell\mathbb{Z})$; here $M^{-t}$ denotes the inverse transpose of $M$. The characteristic polynomial of $qM^{-t}$ is $\text{Rec}_q(P)$.

**Proposition 3.3.** Let $\ell$ be a prime, and assume that one of the following holds:

1. $\chi(A)$ splits modulo $\ell$ as a product of the form $P \text{Rec}_q(P)$ where the polynomials $P$ and $\text{Rec}_q(P)$ are coprime;

2. $\chi(A)$ is totally split modulo $\ell$.

Then $\ell$ is Elkies for $A$.

Recall that the roots of $\chi(A)$ over $\mathbb{C}$ take the form $\lambda_1, q/\lambda_1, \lambda_2, q/\lambda_2$ where $\lambda_1, \lambda_2$ are complex numbers of modulus $\sqrt{q}$; hence assumption (1) means that $\chi(A)$ splits modulo $\ell$ in two coprime degree 2 factors whose roots are $\{\lambda_1, \lambda_2\}$ and $\{q/\lambda_1, q/\lambda_2\}$ respectively (up to a possible renaming of $q/\lambda_2 \mapsto \lambda_2$). Merely assuming that $\chi(A)$ splits in degree 2 factors is not sufficient to ensure that $\ell$ is Elkies: for instance, $A$ might be product of two elliptic curves over $\mathbb{F}_q$ for which $\ell$ is an Atkin prime.

**Proof.** In case (1), define $a = P(\pi_A)$ and $b = \text{Rec}_q(P)(\pi_A)$ as endomorphisms of $A[\ell]$. We have a decomposition of $A[\ell]$ as $\ker(a) \oplus \ker(b)$, and both subspaces
have dimension 2. Let us show that \( \ker(a) \) is isotropic: by (4), \( b \) is the adjoint of \( a \), hence
\[
(\ker(a), \ker(a)) = (\text{Im}(b), \text{Im}(b)) = \langle A[\ell], \text{Im}(ab) \rangle = 0.
\]
In case 2, if \( v \in A[\ell] \) is an eigenvector of \( \pi_A \), then \( v^\perp \subset A[\ell] \) is still \( \pi_A \)-stable. Therefore, there exists \( w \in v^\perp \) such that \( \langle v \rangle \oplus \langle w \rangle \subset A[\ell] \) is a \( \pi_A \)-stable subspace of dimension 2; it is isotropic by construction.

Given Proposition 3.3, it is reasonable to expect that a positive proportion of small primes \( \ell \) (in general, about three in eight) will be Elkies for a given \( A \). This heuristic will motivate the more precise Definition 3.7 later on.

3.3. Modular equations of Siegel type. Denote by \( A_g \) the Siegel moduli space of \( p.p. \) abelian varieties of dimension \( g \), considered as an algebraic variety over \( \mathbb{Q} \). Then for each prime \( \ell \), we have the following diagram of \( \mathbb{Q} \)-varieties:

\[
\begin{array}{ccc}
A_g^0(\ell) & \xleftarrow{p_1} & A_g \\
p_2 & & \\
A_g & \xrightarrow{p_2} & A_g
\end{array}
\]

where \( A_g^0(\ell) \) denotes the coarse moduli space of pairs \((A, K)\) where \( A \) is a \( p.p. \) abelian variety of dimension \( g \) and \( K \subset A[\ell] \) is the kernel of an \( \ell \)-isogeny. The morphisms \( p_1 \) and \( p_2 \) are \((A, K) \mapsto A \) and \((A, K) \mapsto A/K \) respectively. Both \( p_1 \) and \( p_2 \) are finite coverings; moreover \((p_1, p_2)\) realizes a birational isomorphism between \( A_g^0(\ell) \) and its image in \( A_g \times A_g \).

If \( g = 2 \), the graded \( \mathbb{Q} \)-algebra of Siegel modular forms is free over four generators \( I_1, I_6', I_{10}, I_{12} \) [25]. Therefore \( A_2 \) is a rational variety. The zero locus of \( I_{10} \) exactly corresponds to the locus of products of elliptic curves; moreover, it is computationally convenient to work with coordinates on \( A_2 \) which share a common, small-degree denominator, so a common choice of coordinates \( j = (j_1, j_2, j_3) \) on \( A_g \) is given by the Igusa invariants [47], which are scalar multiples of

\[
\begin{array}{ccc}
\frac{I_4I_6'}{I_{10}}, & \frac{I_4I_{12}}{I_{10}}, & \frac{I_5'}{I_{10}}
\end{array}
\]

The Igusa invariants define a local isomorphism from \( A_2 \) to \( \mathbb{A}^3 \) at every point where both \( I_4 \) and \( I_{10} \) are nonzero. Hitting this singular locus may cause problems during the point-counting algorithm. In this case, one can always consider another set of coordinates on \( A_2 \); we postpone this discussion to §3.6 below.

The Siegel modular equations of level \( \ell \) are explicit equations for the image of \( A_g^0(\ell) \) in \( A_2 \times A_2 \). They take the form of three multivariate rational fractions \( \Psi_{\ell, k} \in \mathbb{Q}(J_1, J_2, J_3)[Y] \) for \( 1 \leq k \leq 3 \). Writing \( j \circ p_1 = (j_1, j_2, j_3) \) and \( j \circ p_2 = (j'_1, j'_2, j'_3) \), the equations of \( A_g^0(\ell) \) are the following:

\[
\begin{align*}
\Psi_{\ell, 1}(j_1, j_2, j_3, j'_1) &= 0, \\
\partial_X \Psi_{\ell, 1}(j_1, j_2, j_3, j'_1) \cdot j'_k - \Psi_{\ell, k}(j_1, j_2, j_3, j'_1) &= 0 \quad \text{for } k = 2, 3.
\end{align*}
\]

Let \( d(\ell) = \ell^2 + \ell + 1 \) be the degree of \( p_1: A_g^0(\ell) \rightarrow A_2 \). Then, for any \( p.p. \) abelian surface \( A \) over \( \mathbb{C} \), there are exactly \( d(\ell) \) non-equivalent \( \ell \)-isogenies with domain \( A \). Let \( A_i' \) for \( 1 \leq i \leq d(\ell) \) be their codomains. Assume that neither \( A \) nor any of the isogenous surfaces \( A_i' \) lie in the singular locus of \( j \) as defined above; assume moreover that all coordinates \( j_1(A_i') \) are distinct. Then the denominator
of Siegel modular equations does not vanish at \( j(A) \); moreover the roots of Siegel modular equations evaluated at \( A \), by which we mean all tuples \( (j_1, j_2, j_3) \in \mathbb{C}^3 \) such that the equations (10) are satisfied with \( j_k = j_k(A) \) for \( 1 \leq k \leq 3 \), are precisely the \( d(\ell) \) tuples of the form \( j(A'_i) \) for \( 1 \leq i \leq d(\ell) \). The same properties hold if we replace \( \mathbb{C} \) by an algebraic closure of the finite field \( \mathbb{F}_p \) for any prime \( p \neq \ell \); this can be deduced either from the classical lifting theorems, or from the fact that both \( A_2 \) and \( A_3^{[\ell]}(\ell) \) are actually smooth stacks over \( \mathbb{Z}[1/\ell] \) [16, Chap. 1, §4].

### 3.4. Algorithms for Siegel modular equations

We now present the prerequisites of Elkies’s method in the Siegel case, namely an upper bound on the size of modular equations of Siegel type, as well as algorithms that allow us to evaluate modular equations at a given point and compute the associated \( \ell \)-isogenies.

In the following result, the **total degree** of a multivariate rational fraction \( F \) over \( \mathbb{Q} \) means the maximum between the total degrees of its numerator and denominator; moreover, the **height** of \( F \) is the maximum of \( \log |c| \), where \( c \in \mathbb{Z} \) runs through the coefficients of \( F \) written in irreducible form. The notion of height generalizes to arbitrary number fields.

**Theorem 3.4** ([31, Thm. 1.1 and Prop. 4.11]). The degree of \( \Psi_{\ell,k} \) in \( X \) equals \( d(\ell) \) for \( k = 1 \), and equals \( d(\ell) - 1 \) for \( k = 2, 3 \). The total degree of \( \Psi_{\ell,k} \) in \( j_1, j_2, j_3 \) is bounded above by \( 5d(\ell)/3 \) for \( k = 1 \), and \( 10d(\ell)/3 \) if \( k = 2, 3 \). The height of \( \Psi_{\ell,k} \) is \( O(\ell^3 \log \ell) \) as \( \ell \) tends to infinity.

**Theorem 3.5** ([30]). Let \( K \) be a fixed number field. There exists an algorithm which, given a prime \( \ell \), given \( H \geq 0 \), and given \( (j_1, j_2, j_3) \in K^3 \) where the denominator of Siegel modular equations of level \( \ell \) does not vanish, computes the polynomials

\[
\Psi_{\ell,k}(j_1, j_2, j_3, X) \quad \text{and} \quad \partial_{j_i} \Psi_{\ell,k}(j_1, j_2, j_3, X)
\]

for all indices \( 1 \leq i, k \leq 3 \) as elements of \( K[X] \) in quasi-linear time, in other words in \( \tilde{O}_K((^\ell H) \text{ binary operations}) \).

**Theorem 3.6** ([32, Thm. 1.1]). Let \( k \) be a field. Then there exists an algorithm which, given:

- a prime \( \ell \) such that \( \text{char}(k) > 8\ell + 7 \) if it is finite;
- the Igusa invariants of two \( \ell \)-isogenous p.p. abelian varieties \( A \) and \( A' \) defined over \( k \), lying outside the singular locus of \( j \), such that \( \text{Aut}(A) \) and \( \text{Aut}(A') \) are both equal to \( \{ \pm 1 \} \), and such that the subvariety of \( \mathbb{A}^3 \times \mathbb{A}^3 \) cut out by the equations (10) is normal at \( (j(A), j(A')) \);
- the nine values

\[
\partial_{j_i} \Psi_{\ell,k}(j_1(A), j_2(A), j_3(A), j_1(A')) \in k
\]

for \( 1 \leq i, k \leq 3 \);

computes the following data:

- a tower \( k'/k \) of at most three quadratic extensions;
- equations for two genus \( 2 \) hyperelliptic curves \( C \) and \( C' \) over \( k' \) whose Jacobians are isomorphic to \( A \) and \( A' \) respectively;
- a point \( p_0 \in C(k') \); and
- four rational fractions in \( k'(x, y) \) of total degree \( O(\ell) \) describing an \( \ell \)-isogeny \( f : A \to A' \) in the sense of (7) using \( p_0 \in C(k') \) as a base point;
for the cost of $\tilde{O}(\ell)$ elementary operations and $O(1)$ square roots in $k'$.

3.5. Complexity bounds for Elkies’s method. Let $K$ be a fixed number field, and let $A_0$ be a p.p. abelian surface over $K$. Let $p$ be a prime of $K$, of residue field $\mathbb{F}_q$, where $A_0$ has good reduction; we denote the reduced abelian variety over $\mathbb{F}_q$ by $A$.

In order to count points on $A$, we apply Elkies’s method in the following way.

1. For a series of small primes $\ell$, we apply Theorem 3.5 to evaluate the Siegel modular equations at $j(A_0)$, and reduce the result to $\mathbb{F}_q$. This step costs $O_K(\ell^6 H)$ binary operations, and will dominate the rest of the algorithm; however, it needs to be done only once if we wish to count points on $A$ modulo $p_i$ for several primes $p_i$.

2. We then attempt to find a root of the reduced Siegel modular equations (10) over $\mathbb{F}_q$; this costs $O(\ell^3 \log^2 q)$ binary operations. If there are none, we simply skip $\ell$.

3. If we find one, then we hope that it corresponds to the Igusa invariants of a p.p. abelian surface $A'$ over $\mathbb{F}_q$ for which the genericity conditions of Theorem 3.6 hold. If they do, then $\ell$ is Elkies for $A$, and we are able to compute an explicit rational representation of such an $f$ using $O(\ell \log q)$ binary operations.

4. At this point, the methods described in §2.2 allow us to compute with formal points of $\ker(f)$, and hence recover the characteristic polynomial of Frobenius on this subgroup; the Gröbner basis and Frobenius calculations cost $\tilde{O}(\ell^3 \log q)$ and $\tilde{O}(\ell^2 \log^2 q)$ binary operations respectively. The polynomial $\chi(A) \mod \ell$ itself is finally computed using Proposition 3.2.

By the Hasse-Weil bounds, carrying out this algorithm successfully for a series of primes $\ell_i$ such that $\prod \ell_i > 8q$ is sufficient to recover $\chi(A) \in \mathbb{Z}[X]$ using the Chinese remainder theorem.

3.6. Degenerate cases. We now analyze the different failure cases of the algorithm sketched above. The following issues may arise for any Elkies prime $\ell$:

1. One or more of the p.p. abelian surfaces $\ell$-isogenous to $A$ over an algebraic closure of $\mathbb{F}_q$ may lie on the singular locus of $j$.

2. Several of these abelian surfaces may have the same $j_1$-coordinate.

3. Either $A$ or $A'$ may be the product of two elliptic curves.

4. Either $A$ or $A'$ may have extra automorphisms.

5. The subvariety of $\mathbb{A}^3 \times \mathbb{A}^3$ cut out by the Siegel modular equations of level $\ell$ may not be normal at $(j(A), j(A'))$.

Both failure cases (1) and (2) can be detected during the execution of the algorithm of Theorem 3.5; they are easily solved by taking different birational coordinates on $A_0$. For instance, we can apply a projective linear transformation $m$ with integer coefficients on the weight 20 coordinates $(I_5^5 : I_4^2 I_5^2 : I_3^2 I_6^2 : I_4 I_6 I_10 : I_6^2 I_10)$ before taking the quotients (9). When choosing $m$, we must make sure that $O(\ell^6)$ non-equalities in the algebraic closure of $\mathbb{F}_q$ are satisfied. This can always be achieved provided that we choose coefficients in $m$ of height $O(\log \ell)$. The degree and height bounds of Theorem 3.4 still hold for the modified modular equations using this new set of invariants.

In case (3), we can apply the SEA algorithm on both factors.

In case (4), we obtain a lot of new information about $A$: either $A$ is a twist of the Jacobian of the hyperelliptic curve $y^2 = x^5 - 1$ with complex multiplication
by $\mathbb{Q}(\zeta_5)$, so that $\chi(A)$ can be determined by the CM method [54]; or we can find an explicit isogeny from $A$ to the product of two elliptic curves [25, §8].

Finally, in case (5), a geometric argument shows that $(A, A')$ must be the reduction to $\mathbb{F}_q$ of a singular point in characteristic zero [32, Rem. 4.12]. Using the complex-analytic uniformization of $A_2$ then shows that such singular points either admit extraneous automorphisms, or have the property that there exist two non-equivalent $\ell$-isogenies from $A$ to $A'$. This implies that $A$ possesses a non-integral endomorphism of norm $\ell^4$. Unlike the elliptic curve case, where we would switch to the CM method straightaway, this new piece of information seems insufficient to describe $\text{End}(A)$ in a precise way in higher dimensions. Instead, we simply skip $\ell$ and carry on with Elkies’s method for other primes; we make the heuristic assumption that sufficiently many Elkies primes still exist.

**Definition 3.7.** Let $\varepsilon > 0$, and let $A$ be a p.p. abelian surface over $\mathbb{F}_q$. We say that $A$ has a proportion $\varepsilon$ of primes are Elkies for $A$ if the following holds: for every $X \geq \frac{1}{\varepsilon} \log q$, the proportion of primes $\ell \leq X$ such that $\ell$ is Elkies for $A$ and $\text{End}(A)$ admits no non-integral endomorphism of norm $\ell^4$ is at least $\varepsilon$.

If a proportion $\varepsilon$ of primes are Elkies for $A$, then we are able to collect sufficiently many Elkies primes $\ell_i$ such that $\ell_i \in O_\varepsilon(\log q)$. Thus, Theorem 1.2 directly follows from the complexity estimates given in §3.5.

**Remark 3.8.** It is known that for any fixed $\varepsilon > 0$, there exists an elliptic curve $E$ over some finite field $\mathbb{F}_q$ for which it does not hold that a proportion $\varepsilon$ of primes are Elkies for $E$ [44]. Following [49], we could relax Definition 3.7 by taking an upper bounds of the form $X = \frac{1}{2} \log q \log \log(q)^n$ for some fixed $n \geq 1$. Then we can hope that there exists a positive $\varepsilon > 0$ such that all abelian surfaces over finite fields have a proportion $\varepsilon$ of Elkies primes. This more permissive definition does not modify the complexity estimates of Theorem 1.2.

4. Elkies’s method for abelian surfaces: the Hilbert case

4.1. Elkies primes. In this section, we fix a real quadratic field $F$ of discriminant $\Delta_F$, and we consider the point counting problem for a p.p. abelian surface $A$ over $\mathbb{F}_q$ with real multiplication by $\mathbb{Z}_F$. By Theorem 3.1, real multiplication structure guarantees the presence of supplementary isogenies compared to the Siegel case: for each totally positive $\beta \in \mathbb{Z}_F$, we can look for $\beta$-isogenies $f: A \to A'$ defined over $\mathbb{F}_q$. Assume further that $\beta \in \mathbb{Z}_F$ is prime, prime to $\Delta_F$, and that $\ell = N_{F/\mathbb{Q}}(\beta) \in \mathbb{Z}$ is also prime; in other words $\ell$ is a prime that splits in $F$ in a product of two ideals $(\beta) \cdot (\mathfrak{p})$ that are trivial in the narrow class group of $\mathbb{Z}_F$. By the Čebotarev density theorem, this kind of splitting will occur for a positive proportion of primes $\ell \in \mathbb{Z}$. We say that $\beta$ is Elkies for $A$ if a $\beta$-isogeny $f$ with domain $A$ exists over $\mathbb{F}_q$. Then $\ker(f) \subset A[\ell]$ is a $\pi_A$-stable subgroup of order $\ell$; therefore we can hope to obtain information on $\chi(A) \mod \ell$ by manipulating polynomials of degree $O(\ell)$ only, as in Elkies’s original method for elliptic curves.

In the Hilbert case, the Chinese remaindering step is formulated in terms of the real Frobenius endomorphism $\psi_A = \pi_A + \pi_A^\dagger$ as an element of $\mathbb{Z}_F$. By the Weil bounds (2), we have

$$|\text{Tr}(\psi_A)| \leq 4\sqrt{q} \quad \text{and} \quad \text{Disc}(\mathbb{Z}[\psi_A]) \leq 4q.$$
Assume that \( \beta \) is Elkies, and the eigenvalue \( \lambda \in \mathbb{Z}/\ell \mathbb{Z} \) of \( \pi_A \) on \( \ker(f) \) has been computed. Then we have

\[
\psi_A = \lambda + q/\lambda \mod \beta
\]

under the canonical isomorphism \( \mathbb{Z}_F / \beta \mathbb{Z}_F \cong \mathbb{Z}/\ell \mathbb{Z} \).

In the algorithm, we consider a series totally positive Elkies primes \( \beta_i \) in \( \mathbb{Z}_F \), with norms \( \ell_i \in \mathbb{Z} \). We collect the values of \( \psi_A \) modulo \( \beta_i \) as elements of \( \mathbb{Z}/\ell_i \mathbb{Z} \) as above. The Chinese remainder theorem in \( \mathbb{Z}_F \) allows us to reconstruct the value of \( \psi_A \) modulo the ideal \( \mathfrak{B} = \prod_i (\beta_i) \). The cost of this reconstruction is negligible when compared to the rest of the algorithm.

**Proposition 4.1.** Assume that \( N(\mathfrak{B}) > 16q \). Then \( \psi_A \) is uniquely determined by equations (11) and the data of \( \psi_A \mod \mathfrak{B} \).

**Proof.** Assume that \( \mathfrak{B} \) contains a nonzero \( \alpha \in \mathbb{Z}_F \) such that \( |\text{Tr}_{F/\mathbb{Q}}(\alpha)| \leq 8\sqrt{q} \) and \( \text{Disc}(\mathbb{Z}[\alpha]) \leq 16q \). Then we have

\[
N(\mathfrak{B}) \leq |N_{F/\mathbb{Q}}(\alpha)| = \frac{1}{4}|\text{Tr}(\alpha)^2 - \text{Disc}(\mathbb{Z}[\alpha])| \leq 16q.
\]

Once \( \psi_A \in \mathbb{Z}_F \) has been determined, its characteristic polynomial completely describes \( \chi(A) \), as equations (1) and (3) show. Heuristically, we expect that roughly half of the suitable primes \( \beta \) will be Elkies: indeed \( \beta \) is Elkies if and only if the characteristic polynomial of \( \pi_A \) on \( A[\beta] \), a polynomial of degree 2, splits in \( \mathbb{Z}/\ell \mathbb{Z} \).

### 4.2. Modular equations of Hilbert type.

The key fact that allows us to reach similar point-counting complexities in the RM case and the case of elliptic curves is that the associated *Hilbert modular equations* have a reasonable size.

Denote by \( \mathcal{A}_{2,F} \) the Hilbert moduli space of p.p. abelian surfaces with real multiplication by \( \mathbb{Z}_F \), seen as an algebraic variety over \( \mathbb{Q} \). For each \( \beta \) as above, we have a diagram of \( \mathbb{Q} \)-varieties

\[
\begin{array}{c c c}
\mathcal{A}_{2,F} & \mathcal{A}_{2,F}^0(\beta) & \mathcal{A}_{2,F}^0(\beta) \\
p_1 & \leftrightarrow & p_2
\end{array}
\]

where \( \mathcal{A}_{2,F}^0(\beta) \) denotes the coarse moduli space of pairs \((A,K)\) where \( A \) is a p.p. abelian surface with real multiplication by \( \mathbb{Z}_F \), and \( K \subset A[\beta] \) is the kernel of a \( \beta \)-isogeny. The Hilbert modular equations of level \( \beta \) are explicit equations for the image of \( \mathcal{A}_{2,F}^0(\beta) \) in \( \mathcal{A}_{2,F} \times \mathcal{A}_{2,F} \). To define them, we make a choice of coordinates \( j = (j_1,j_2,j_3) \) on \( \mathcal{A}_{2,F} \), related by an explicit equation of the form

\[
E(j_1,j_2,j_3) = 0.
\]

Assume further that \( j_1 \) and \( j_2 \) are algebraically independent, and write \( e = \deg_{j_3}(E) \). The Hilbert modular equations are then the data of the three multivariate rational fractions \( \Psi_{\beta,k} \in \mathbb{Q}(J_1,J_2,J_3,X) \) of degree at most \( e - 1 \) in \( J_3 \) such that the system of equations (10) holds with \( \ell \) replaced by \( \beta \).

In concrete cases, it is sometimes convenient to modify this definition and consider *symmetric* modular equations on Humbert, rather than Hilbert, surfaces. Recall that there exists a forgetful map \( \mathcal{A}_{2,F} \to \mathcal{A}_2 \) which is generically 2-1. The image \( \mathcal{H}_F \) of \( \mathcal{A}_{2,F} \), called the Humbert surface attached to \( \mathbb{Z}_F \), is often less geometrically complicated than \( \mathcal{A}_{2,F} \). Explicit coordinates on \( \mathcal{H}_F \) are also easier to
describe: for instance, the Igusa invariants (9) are always a valid choice. If the discriminant of $F$ is less than 100, then $\mathcal{H}_F$ is rational, and explicit parametrizations appear in [13]. In the case $F = \mathbb{Q}(\sqrt{5})$, the Gundlach invariants denoted by $g_1, g_2$ (see [21, Satz 6], although other normalizations are also used) are convenient coordinates on $\mathcal{H}_F$ derived from an explicit description of the associated graded ring of symmetric Hilbert modular forms.

We will denote the (symmetric) Hilbert modular equations in Igusa invariants by $\Psi^I_{\beta,k}$ for $1 \leq k \leq 3$; they are equal for the prime $\beta$ and its real conjugate $\overline{\beta}$. Similarly, we denote the Hilbert modular equations of level $\beta$ in Gundlach invariants for $F = \mathbb{Q}(\sqrt{5})$ by $\Psi^G_{\beta,k}$ for $k = 1, 2$; they are multivariate rational fractions in $\mathbb{Q}(G_1, G_2)[Y]$. Modular equations on Hilbert surfaces describe $\beta$-isogenies between abelian surfaces with RM by $\mathbb{Z}_F$, in a similar way as in §3.3 for modular equations of Siegel type. In the symmetric case, modular equations describe $\beta$- and $\overline{\beta}$-isogenies simultaneously.

**4.3. Algorithms for Hilbert modular equations.** Let $d(\beta) = \ell + 1$ be the degree of $p_1$ in diagram (12). We fix a choice of coordinates $(j_1, j_2, j_3)$ on $\mathcal{A}_{2,F}$.

**Theorem 4.2 ([31, Thm. 1.1 and Prop. 4.13]).** The degree of $\Psi^I_{\beta,k}$ in $X$ is $d(\beta)$ for $k = 1$, and $d(\beta) - 1$ for $k > 1$. The total degrees of $\Psi^I_{\beta,k}$ are $O_F(\ell)$, and their heights are $O_F(\ell \log \ell)$. In the case of $F = \mathbb{Q}(\sqrt{5})$, the total degree of $\Psi^I_{\beta,k}$ in $G_1, G_2$ is at most $10d(\beta)/3$ for $k = 1, 2$.

**Theorem 4.3 ([30, Thm. 5.3]).** Let $q = p^r$ be a prime power, and let $F = \mathbb{Q}(\sqrt{5})$. There exists an algorithm which, given $(g_1, g_2) \in \mathbb{F}_q^2$ where the denominator of $\Psi^G_{\beta,k}$ for $k = 1, 2$ does not vanish, computes the modular equations $\Psi^G_{\beta,k}(g_1, g_2, X)$ as well as their derivatives $\partial_{G_i} \Psi^G_{\beta,k}(g_1, g_2, X)$ for $1 \leq i, k \leq 2$ as elements of $\mathbb{F}_q[X]$ in $O(\ell^2r^2 \log p)$ binary operations.

This result generalizes to any other real quadratic field $F$ for which explicit generators of the graded rings of Hilbert modular forms over $\mathbb{Z}$ are known. Otherwise, the evaluation algorithm can still be run, but it involves a heuristic reconstruction of rational numbers from their complex approximations.

**Theorem 4.4 ([32, Thm. 6.3]).** Let $F$ be a fixed real quadratic field. Then there exists an open subvariety $U \subset \mathcal{A}_{0, 2,F}^0(\beta)$ and an algorithm which, for any field $k$, given:

- a totally positive $\beta \in \mathbb{Z}_F$ such that $\text{char}(k) > 4 \text{Tr}_{F/\mathbb{Q}(\beta)} + 7$ if it is positive;
- the Igusa invariants of two $\beta$-isogenous p.p. abelian surfaces $A$ and $A'$ defined over $k$ with real multiplication by $\mathbb{Z}_F$, such that this $\beta$-isogeny comes from a point of $U$;
- The nine values

$$\partial_{G_i} \Psi^G_{\beta,k}(j_1(A), j_2(A), j_3(A), j_1(A')) \in k$$

for $1 \leq i, k \leq 3$;

computes the following data:

- a tower $k'/k$ of at most three quadratic extensions;
- equations for two genus 2 hyperelliptic curves $C$ and $C'$ whose Jacobians are isomorphic to $A$ and $A'$ over an algebraic closure of $k$;
- a point $p_0 \in C(k')$; and
at most four possible tuples of rational fractions in \( k'(x, y) \) of total degree \( O(\text{Tr}_{F/Q}(\beta)) \), such that one of these tuples describes a \( \beta \)-isogeny \( f: A \to A' \) in the sense of (7) using \( p_0 \in \mathcal{C}(k') \) as a base point;

using \( \tilde{O}(\text{Tr}_{F/Q}(\beta)) + O_F(1) \) elementary operations and \( O(1) \) square roots in \( k' \).

As in Theorem 3.6, the open subvariety \( U \subset \mathcal{A}_2^0(F) (\beta) \) in Theorem 4.4 can be described explicitly. It is sufficient to impose the following conditions [32, §4.2.3];

- Both \( A \) and \( A' \) have no extraneous automorphisms as \( \text{p.p.} \) abelian surfaces;
- There exists only one isogeny \( f: A \to A' \) over an algebraic closure of \( k \) whose kernel is cyclic of degree \( \ell \), up to equivalence;
- There exists only one possible real multiplication embedding of \( \mathbb{Z}_F \) inside both \( \text{End}(A) \) up to real conjugation on \( \mathbb{Z}_F \), and the same holds for \( A' \);
- Both \( A \) and \( A' \) lie outside of the singular locus of Igusa invariants.

4.4. **Complexity bounds.** Let \( q = p^r \) be a prime power, and let \( A \) be a \( \text{p.p.} \) abelian surface over \( \mathbb{F}_q \) with RM by \( \mathbb{Z}_F \). We apply Elkies’s method as follows.

1. Let \( \ell \in \mathbb{Z} \) be a prime with the correct splitting behavior in \( \mathbb{Z}_F \), and let \( \beta \in \mathbb{Z}_F \) be a totally positive prime above \( \ell \). By [19, Lem. 1], it is possible to choose \( \beta \) such that \( \text{Tr}_{F/Q}(\beta) \in O_F(\sqrt{\ell}) \). We evaluate the corresponding modular equations using Theorem 4.3, for instance in Igusa invariants. Assuming that \( r = o(\log p) \), this costs \( \tilde{O}(\ell^2 \log q) \) binary operations.
2. We then attempt to find a root of these modular equations over \( \mathbb{F}_q \); this costs \( \tilde{O}(\ell \log^2 q) \) binary operations. If there are none, we skip \( \ell \).
3. If we find one, we attempt to compute a cyclic isogeny \( f: A \to A' \) of degree \( \ell \) using Theorem 4.4. If this computation is successful, we obtain the rational representation of \( f \) as in (7) in terms of rational functions of degree \( O_F(\sqrt{\ell}) \). This costs \( \tilde{O}(\sqrt{\ell}) \) binary operations.
4. After that, computing a Gröbner basis describing \( \ker(f) \) costs \( \tilde{O}(\ell^{3/2} \log q) \) binary operations; the resulting polynomials have degree \( O(\ell) \). Computing the Frobenius eigenvalue on \( \ker(f) \) costs \( \tilde{O}(\ell \log^2 q) \) binary operations.

By Proposition 4.1, the total algorithm costs \( \tilde{O}(\log^4 q) \) binary operations provided that sufficiently many Elkies primes exist and the computations of Theorem 4.4 succeed sufficiently often.

4.5. **Degenerate cases.** The treatment of degenerate cases which may occur the algorithm above is similar to the Siegel case. The only new possible problems are the following:

1. The algorithm may involve a \( \text{p.p.} \) abelian surface \( A \) that corresponds to a point where the map from \( \mathcal{A}_2.F \) to \( \mathcal{A}_2 \) is not étale;
2. \( \text{End}(A) \) may contain an element of norm \( \ell^2 \) outside \( \mathbb{Z}_F \);
3. The isogeny algorithm of Theorem 4.4 may output several possibilities for the rational representation of \( f \).

In case (1), we can always consider coordinates on the Hilbert surface \( \mathcal{A}_2.F \) instead. As in the Siegel case, we incorporate the assumption that case (2) does not happen too often into Definition 4.5 below. Finally, in case (3), we can check which of the candidate values actually describes a group morphism between Jacobians; if more than one candidate passes this test, we are led back to case (2).
Definition 4.5. Let $\varepsilon > 0$, and let $A$ be a p.p. abelian surface over $\mathbb{F}_q$ with real multiplication by $\mathbb{Z}_F$. We say that a proportion $\varepsilon$ of primes are Elkies for $A$ if the following holds: for every $X \geq \frac{1}{\varepsilon} \log q$, the proportion of primes $\ell \leq X$ such that

- $\ell$ splits in the form $(\beta)(\overline{\beta})$ for some totally positive $\beta \in \mathbb{Z}_F$,
- one of $\beta$ or $\overline{\beta}$ is Elkies, and
- $\text{End}(A)$ admits no non-real endomorphism of norm $\ell^2$,

is at least $\varepsilon$.

Remark 4.6. Let $\ell$ be a prime that splits in $\mathbb{Z}_F$, but whose prime factors $l \cdot l$ are nontrivial in the narrow class group of $\mathbb{Z}_F$. Instead of skipping $\ell$ altogether in the point-counting algorithm, we can compute a totally positive generator $\beta$ of $l \cdot l$ where $l$ denotes a small representative of the relevant class in the narrow class group of $\mathbb{Z}_F$. Elkies’s method will also apply to the non-prime $\beta$.

5. Experimental results

We have implemented algorithms to evaluate modular equations for p.p. abelian surfaces over $\mathbb{Q}$ and prime finite fields in C [29]. The experiments presented here can be reproduced by downloading the library and running make reproduce. In practice, we expect that the evaluation of modular equations will exceed the cost of other polynomial manipulations in Elkies’s method by a large margin.

In the Siegel case, we consider the “random” tuple of Igusa invariants of small height given by $(\frac{159}{239}, -\frac{19}{28}, -\frac{193}{246})$. We record the time to evaluate Siegel modular equations at this point for prime levels $\ell \leq 17$ on a single core (AMD EPYC 7713), and compare it with the cost estimation of $0.002 \ell^6 \log(\ell)^3 \log \log(\ell)$ seconds.

| $\ell$ | 2  | 3  | 5  | 7  | 11 | 13 | 17 |
|-------|----|----|----|----|----|----|----|
| Time (s) | 1.34 | 5.12 | 96.7 | 1.23 $\cdot 10^3$ | 3.97 $\cdot 10^4$ | 1.57 $\cdot 10^5$ | 1.12 $\cdot 10^6$ |
| Estimation | -  | -  | 62  | 1.2 $\cdot 10^4$ | 4.3 $\cdot 10^4$ | 1.5 $\cdot 10^5$ | 1.1 $\cdot 10^6$ |

In light of these results, a point-counting approach based exclusively on Elkies’s method for general p.p. abelian surfaces would be unlikely to beat Schoof’s method in practical instances. However, using modular equations would still allow one to introduce several improvements inspired from the SEA algorithm (see §6).

In the Hilbert case for $F = \mathbb{Q}(\sqrt{5})$, we consider the pair of “random” Gundlach invariants of small height given by $(-\frac{117}{64}, -\frac{199}{172})$. We evaluate Hilbert modular equations of level $\beta$, where $\beta \in \mathbb{Z}_F$ is totally positive of prime norm $\ell \leq 500$ at that point, and compare it with the estimation of $2 \cdot 10^{-4} \ell^2 \log(\ell)^3 \log \log \ell$ seconds.

| $\ell$ | 11 | 19 | 29 | 31 | 41 | 59 | 61 | 71 |
|-------|----|----|----|----|----|----|----|----|
| Time (s) | 2.45 | 4.14 | 9.66 | 11.1 | 25.6 | 59.6 | 64.0 | 107 |
| Estimation | -  | 2.0 | 7.8 | 9.6 | 23 | 66 | 73 | 113 |

For larger primes, counting in core-hours is perhaps more readable.
Consider the problem of counting points on a p.p. abelian surface $A$ over a prime field $F_q$ with $\log q \simeq 512$ given by these Gundlach invariants. Assuming that half of the primes $\beta$ are Elkies, a strategy based purely on Elkies’s method would involve all primes $\ell$ up to roughly 800 with the correct splitting behavior in $\mathbb{Z}_F$. We can roughly estimate a total cost of a few core-weeks for this computation, thus indicating that Schoof’s method can perhaps be beaten in this context [19, §5.2].

Finally, we report on the time (in seconds) to evaluate modular equations for Hilbert type in Igusa invariants for $\ell \leq 50$, at the point given by the parameters $(-239/152, 224/103)$ in the different parametrizations of Humbert surfaces found in [13], for all real quadratic fields of discriminants up to 100.

| $\Delta, \ell$ | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 | 47 |
|---------------|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 5             |   |   |   |   | 9.76 | 9.76 | 30.2 | 78.0 | 91.1 | 205 |   |   |   |   |
| 8             |   |   |   |   | 2.67 | 2.67 | 26.9 | 48.2 | 95.1 | 208 | 208 |   |   |   |   |
| 12            |   |   |   |   | 40.9 | 40.9 | 126.0 | 370.0 | 735 | 825 |   |   |   |   |   |
| 13            |   |   |   |   | 2.02 | 2.02 | 25.6 | 51.1 | 83.9 | 165 |   |   |   |   |   |
| 17            | 4.34 |   |   |   | 57.2 | 57.2 | 126 |   |   |   |   |   |   |   |   |
| 21            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 24            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 28            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 29            |   |   |   |   | 4.75 | 5.76 | 15.0 |   |   |   |   |   |   |   |   |
| 33            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 37            |   |   |   |   | 3.04 | 3.04 | 29.5 |   |   |   |   |   |   |   |   |
| 40            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 41            | 5.27 | 10.5 |   |   | 326 | 326 | 625 | 1050 | 1450 |   |   |   |   |   |   |
| 44            |   | 4.77 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 53            |   |   |   |   | 24.7 | 49.6 | 59.7 | 107 |   | 281 |   |   |   |   |   |
| 56            |   |   |   |   |   |   | 27 |   |   |   |   |   |   |   |   |
| 57            |   |   |   |   | 30.0 |   |   |   |   |   |   |   |   |   |   |
| 60            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 61            |   |   |   |   | 8.34 | 13.9 | 49.1 | 108 |   |   |   |   |   | 580 | 772 |
| 65            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 69            |   |   |   |   |   | 41.8 |   |   |   |   |   |   |   |   |   |
| 73            | 6.40 | 14.9 |   |   | 241 | 342 | 1000 | 1370 |   |   |   |   |   |   |   |
| 76            |   | 6.9 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 77            |   |   |   |   | 55.8 | 55.8 | 133 |   |   |   |   |   |   |   |   |
| 85            |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 88            |   | 18.1 |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 89            | 3.55 | 15.4 | 57.3 | 147 |   |   |   |   |   |   |   |   |   |   |   |
| 92            |   |   |   |   | 104 |   |   |   |   |   |   | 567 |   |   |   |
| 93            |   |   |   |   | 16.5 |   |   |   |   |   | 95 |   |   |   |   |
| 97            | 7.01 | 8.48 |   | 98.1 |   |   |   |   |   | 1370 |   |   |   |   |   |

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On the first line, we observe that Hilbert modular equations in Igusa invariants for $F = \mathbb{Q}(\sqrt{5})$ are indeed more expensive to evaluate than their counterparts in Gundlach invariants.

6. Perspectives

In this final section, we sketch possible improvements to Elkies’s method for abelian surfaces as described above, following existing works in the dimension 1 case. They would either reduce the constant factors hidden in complexity estimates by large amounts, or introduce exponential-time gains.

6.1. Smaller modular equations. The modular equations of Siegel and Hilbert type presented above are higher-dimensional analogues of the classical modular polynomials $\Phi_\ell$ in dimension 1. It is well-known that other kinds of modular polynomials provide explicit equations for essentially the same modular curve which are much smaller, despite sharing the same $O(\ell^3 \log \ell)$ size asymptotic: see for instance [15, §3] and the data available at [48]. In the dimension 2 case, modular equations written in terms of theta constants are considerably smaller than Siegel or Hilbert modular equations as defined above [36]. One can ask whether this choice of coordinates is the optimal one.

More generally, it could well be that systems of equations of the form (10) inherently force modular equations to have large coefficients; other ways of describing the diagram (8) might lead to smaller polynomials – for instance, a more intrinsic equation for $A_2^0(\ell)$ along with the Atkin–Lehner involution exchanging $p_1$ and $p_2$. Such equations do not even have to be defined by a formula valid for each $\ell$; all we need is an algorithm to compute such equations when $\ell$ is given, perhaps by computing a basis of Siegel modular forms of level $\Gamma_0(\ell)$, or Hilbert modular forms of level $\Gamma_0(\beta)$, on the fly.

6.2. Other SEA strategies. In the case of elliptic curves, there is more to the SEA algorithm than applying Elkies’s method to a series of distinct primes. We list some of the possible improvements below. Due to the larger implied constants in complexity estimates about modular equations, we expect these improvements to have an even larger impact on practical running times in higher dimensions.

(1) Isogeny chains. In favorable situations, modular polynomials of level $\ell$ can be used to compute not only an $\ell$-isogeny $E \to E'$, but a chain of $\ell$-isogenies $E \to E_1 \to \cdots \to E_r$ whose composition is an $\ell^r$-isogeny, for some $r \geq 2$ [17]. This yields the value of $\chi(E)$ modulo $\ell^r$. In order to remain within the same complexity bound, one should take $r$ no greater than $\log \log(q) / \log(\ell) + O(1)$. The existence of an isogeny chain of the desired length over $\mathbb{F}_q$ depends on the shape of the connected component of the $\ell$-isogeny graph on which $E$ lies. Note that a chain of length $r = 2$ can be constructed by evaluating modular polynomials only once, at $E_1$. In dimension 2, this strategy seems easier to apply in the Hilbert case, since isogeny graphs are still volcanoes in this case [26] and the composition of a non-backtracking chain of $\beta$-isogenies will always be a $\beta^r$-isogeny. This property does not hold for $\ell$-isogenies in the Siegel case, and the shape of the associated isogeny graphs is also more complicated [8].

(2) Atkin’s method. It is known that studying the factorization patterns of modular equation of level $\ell$ over $\mathbb{F}_q$, even in the absence of rational roots,
restrict the possible Frobenius eigenvalues modulo \( \ell \) \([43, \S 6], [2] \). This allows one to take advantage of Atkin (i.e. non-Elkies) primes as well. This information can be used at the end of the point-counting algorithm in an exponential-time sieve, whose practical effect is to reduce the number of necessary Elkies primes. In general, if we can compute \( n \geq 2 \) possible values of \( \chi(A) \) modulo \( \ell \), the “value” of \( \ell \) as an Atkin prime is \( \log(\ell)/\log(n) \), and one should only keep the highest-valued primes for the final sieve. Thus, once a few Atkin primes have been collected, it only makes sense to look for low-degree factors of modular equations; this is cheaper than computing the full factorization.

(3) Schoof’s method. When \( \ell = O(\sqrt{\log q}) \) is not Elkies, it is usually more interesting to apply Schoof’s original method to compute \( \chi(A) \mod \ell \) than attempting to keep \( \ell \) as part of the Atkin data; this makes space for larger primes in the final sieve. If \( \ell \) is very small (for instance \( \ell = 2 \)), then Schoof’s method can also yield \( \chi(A) \mod \ell \) for a suitable power of \( \ell \) \([20, \S 4] \).

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