Lipschitz continuity of solutions of a free boundary problem involving the $p$-Laplacian

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0. Introduction

Let $\Omega$ be an open bounded domain of $\mathbb{R}^n$, $n \geq 2$. We consider the following problem

\[ (P) \quad \begin{cases} 
\text{Find } (u, \chi) \in W^{1,p}(\Omega) \times L^\infty(\Omega) \text{ such that:} \\
(i) \quad 0 \leq u \leq M, \quad 0 \leq \chi \leq 1, \quad u(1 - \chi) = 0 \text{ a.e. in } \Omega, \\
(ii) \quad \Delta_p u = -\text{div}(\chi H(x)) \text{ in } W^{-1,p}(\Omega), 
\end{cases} \]

where $\chi = (\chi_1, \ldots, \chi_n) \in \mathbb{R}^n$, $M$ and $p$ are positive constants with $p > 1$ and $p'$ is its conjugate. $H(x)$ is a vector function satisfying for some positive constant $\bar{h}$

\[ |H|_\infty \leq \bar{h}, \]  
\[ |\text{div}(H)|_\infty \leq \bar{h}. \]  

Since $\chi H \in L^\infty(\Omega)$, it is well known that $u \in C^{0,\alpha}_{\text{loc}}(\Omega)$ for all $\alpha \in (0, 1)$ (see for example [3] for a nonnegative Radon measure as a right-hand side, but the proof can also be extended to a signed Radon measure). It is our objective in this paper to show that we have in fact $u \in C^{0,1}_{\text{loc}}(\Omega)$ which is the optimal regularity for this problem because of the jump condition along the free boundary $(\partial[u > 0]) \cap \Omega$. This regularity result was proved in [2] for a constant vector $H(x) = (\delta, 0)$, and the main idea is based on the estimate $u(x_0) \leq C r$, whenever $B(x_0, r)$ is a maximal open ball satisfying $B(x_0, r) \subset [u > 0]$ and $B(x_0, r) \subset \Omega$. This estimate is in turn obtained by comparing the function $u$ with the $p$-Harmonic function, for $n = 2$,

\[ v(x) = a|x - x_0|^{n-2} + b. \]  

This function is not adapted to the present situation since we allow $\text{div}(H)$ to be a signed measure. To overcome this difficulty, we use the function $k(e^{u \delta^2} - e^{-\mu(u+\delta)^2})$, which was used in [1] and [4] to deal with linear and elliptic operators of the form $\text{div}(a(x)\nabla u)$. Even that this function works also for the $p$-Laplace operator, the calculations and estimates are obtained in different ways.

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We are also interested with the regularity up to the boundary. More precisely we assume that a solution \( u \) satisfies the Dirichlet condition \( u = 0 \) on a nonempty subset \( T \) of \( \partial \Omega \). Then we prove that \( u \in C^{0,1}_{\text{loc}}(\Omega \cup T) \). Let us mention that this question was not considered neither in [1] nor in [2], but only in [4]. The idea of the proof in [4] is to reduce the problem to a flat boundary by using a local change of variables, and then establish the estimate \( u(x) \leq C|x - x_0| \) in the half ball \( B^+(x_0, R) \), where \( x_0 \) is an interior point of the boundary where the Dirichlet condition is satisfied. This is obtained by comparing \( u \) to the test function \( \psi(|x - x_0 + Rv| - R) \), where \( v \) is the outward unit normal vector to \( T \) at \( x_0 \) and \( \psi(t) = \int_0^1 (1 + \frac{\delta}{n-1}) e^{\frac{\delta}{n-1}(D-t)} - \frac{\delta}{n-1} \frac{1}{r} ds \).

1. Interior Lipschitz continuity of \( u \)

The main result of this section is the following theorem.

**Theorem 1.1.** Let \((u, \chi)\) be a solution of \((P)\). Then \( u \in C^{0,1}_{\text{loc}}(\Omega) \).

To prove Theorem 1.1, we need two lemmas.

**Lemma 1.1.** Let \( x_0 = (x_0, \ldots, x_0) \) and \( r > 0 \) such that \( B_r(x_0) \subset \{u > 0\} \), \( \overline{B_r(x_0)} \subset \Omega \) and \( \partial B_r(x_0) \cap \partial \{u > 0\} \neq \emptyset \). Then there exists a positive constant \( C \) depending only on \( n, p, \bar{h} \) and \( \delta(\Omega) \) (the diameter of \( \Omega \)) such that

\[
\min_{B_{r/2}(x_0)} u \leq C. 
\]

**Proof.** Let \( \epsilon \in (0, r) \) such that \( B_{r+\epsilon}(x_0) \subset \Omega \), and let \( v \) be defined in \( D \) by

\[
v(x) = k(e^{-\alpha\rho^2} - e^{-\alpha(r+\epsilon)^2})
\]

where

\[
D = B_{r+\epsilon}(x_0) \setminus \overline{B_{r/2}(x_0)},
\]

\[
\rho^2 = |x - x_0|^2 = \sum_{i=1}^{i=n} (x_i - x_0)^2, \quad k = \frac{m}{e^{-\alpha\rho^2} - e^{-\alpha(r+\epsilon)^2}},
\]

\[
m = \min_{B_{r/2}(x_0)} u, \quad \alpha = \frac{k}{r^2} \quad \text{with} \quad \kappa = \max\left(2, \frac{2(p + n - 2)}{p - 1} + 1\right).
\]

Then one can easily verify that \( v \) satisfies

\[
\begin{cases}
\nabla v = -2\alpha k e^{-\alpha\rho^2} (x - x_0) & \text{in } D, \\
\Delta_p v = -(2k^\alpha)^{p-1} \rho^{p-2} e^{-\alpha(p-1)\rho^2} \left[(n + p - 2) - 2\alpha(p - 1)\rho^2\right] & \text{in } D, \\
v = m & \text{on } \partial B_{r/2}(x_0), \\
v = 0 & \text{on } \partial B_{r+\epsilon}(x_0).
\end{cases}
\]

Now we claim that

\[
\Delta_p v + \text{div}(H) \geq 0 \quad \text{in } D. \tag{1.1}
\]

Indeed we have by (0.2)

\[
\Delta_p v + \text{div}(H) \geq (2k^\alpha)^{p-1} \rho^{p-2} e^{-\alpha(p-1)\rho^2} \left[2\alpha(p - 1)\rho^2 - (n + p - 2)\right] - \bar{h}
\]

\[
= 2^{p-1}k^{p-1} \left[ \left( \frac{e^{-\alpha\rho^2}}{e^{-\alpha\rho^2} - e^{-\alpha(r+\epsilon)^2}} \right)^{\frac{p-1}{2}} \left[ 2\alpha^2 (p - 1)\rho^2 - (n + p - 2) \right] - \bar{h}. \right.
\]

Note that by the choice of \( \kappa \), we have

\[
2 \frac{\kappa}{r^2} (p - 1)\rho^2 - (n + p - 2) \geq 2 \frac{\kappa}{r^2} (p - 1)(r/2)^2 - (n + p - 2) = \frac{\kappa}{2} (p - 1) - (n + p - 2) = \beta > 0.
\]
Then
\[ \Delta_p v + \text{div}(H) \geq 2^{p-1}k^{p-1}m^{p-1} \frac{(r/2)^{p-1}}{r^{2(p-1)}(r+\epsilon)} \left( \frac{\left( e^{-\alpha(r+\epsilon)^2} \right)}{e^{-\alpha(r+\epsilon)^2} - e^{-\alpha(r+\epsilon)^2}} \right) \beta - \tilde{h} \]
\[ = k^{p-1} \left( \frac{m}{r} \right)^{p-1} \frac{1}{(r+\epsilon)} \left( \frac{e^{-\alpha(r+\epsilon)^2}}{e^{-\alpha r^2}/4 - e^{-\alpha(r+\epsilon)^2}} \right)^{p-1} \beta - \tilde{h} \]
\[ = k^{p-1} \left( \frac{m}{r} \right)^{p-1} \frac{1}{(r+\epsilon)} \left( \frac{1}{(e^{(1+\epsilon)/r^2}-1)} \right)^{p-1} \beta - \tilde{h} \]
\[ = \theta(r). \quad (1.2) \]

* If \( \theta(r) \leq 0 \), then
\[ \left( \frac{m}{r} \right)^{p-1} \leq \frac{\tilde{h}}{k^{p-1} \beta} (r+\epsilon)e^{(1+\epsilon)/r^2}-1)\beta - \tilde{h}. \]

Letting \( \epsilon \to 0 \), we get
\[ m^{p-1} \leq \frac{\tilde{h}}{k^{p-1} \beta} (e^{2\gamma} - 1)\beta - \tilde{h}. \]

Hence
\[ m \leq \left( \frac{\tilde{h} \delta(\Omega)}{2\beta} \right)^{1/p} \left( \frac{e^{2\gamma} - 1}{\kappa} \right)^{1/p} \beta - \tilde{h} \quad (n, p, \tilde{h}, \delta(\Omega)). \]

* If \( \theta(r) > 0 \), we get (1.1) from (1.2).

Since \( v \leq u \) on \( \partial D \), \( \zeta = (v-u)^+ \in W_0^{1,p}(D) \), and \( \pm \zeta \) - after being extended by zero outside \( D \) - are test functions for (P). So we have
\[ \int_D (|\nabla u|^{p-2} \nabla u + \chi H(x)) \nabla (v-u)^+ = 0. \quad (1.3) \]

By (1.1) we have
\[ \int_D (|\nabla v|^{p-2} \nabla v + H(x)) \nabla (v-u)^+ \leq 0. \quad (1.4) \]

Subtracting (1.3) from (1.4), we get
\[ \int_D (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u) \nabla (v-u)^+ dx \leq \int_D (\chi - 1) H(x) \nabla (v-u)^+ dx \]

which can be written by using (0.1) and taking into account that \( \chi = 1 \) a.e. in \( [u > 0] \)
\[ \int_{D \cap [u > 0]} (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u) \nabla (v-u)^+ dx \leq \int_{D \cap [u > 0]} |\nabla v| (\tilde{h} - |\nabla v|^{p-1}) dx. \]

If \( \int_{D \cap [u = 0]} |\nabla v| (\tilde{h} - |\nabla v|^{p-1}) dx \leq 0 \), then
\[ \int_{B_r(x_0)} (|\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u) \nabla (v-u)^+ dx \leq 0 \]

from which we deduce that \( \nabla (v-u)^+ = 0 \) in \( B_r(x_0) \). Since \( (v-u)^+ = 0 \) in \( B_r(x_0) \setminus B_{r/2}(x_0) \), we get \( v \leq u \) in \( B_r(x_0) \setminus B_{r/2}(x_0) \).

By continuity one has \( v \leq u \) on \( \partial B_r(x_0) \). But \( v > 0 \) on \( \partial B_r(x_0) \) and \( \partial B_r(x_0) \cap \partial [u > 0] \neq \emptyset \). So we get a contradiction. Hence
\[ \int_{D \cap [u = 0]} |\nabla v| (\tilde{h} - |\nabla v|^{p-1}) dx > 0. \quad (1.5) \]

Since \( |\nabla v| = 2\kappa \alpha e^{-\alpha r^2} \) and \( \kappa \geq 2 \), we have
\[ \frac{d}{d\rho} |\nabla v| = 4\kappa^2 \rho^2 e^{-\alpha r^2} + 2 \kappa \alpha e^{-\alpha r^2} = 2 \kappa \alpha e^{-\alpha r^2} \left( 1 - 2 \kappa \frac{\rho^2}{r^2} \right) \leq 2 \kappa \alpha e^{-\alpha r^2} (1 - \kappa/2) \leq 0. \]
Therefore $|\nabla v|$ is non-increasing with respect to $\rho$. We deduce from (1.5) that

$$|\nabla v|_{B_{2h}(x_0)} = 2k\alpha(r + \epsilon)e^{-\alpha(r + \epsilon)^2} < \frac{1}{h}r^{-\frac{1}{2}}$$

i.e.

$$\frac{2mk(r + \epsilon)e^{-\alpha(r + \epsilon)^2}}{r^2(e^{-\alpha r^2/4} - e^{-\alpha(r + \epsilon)^2})} < \frac{1}{h}r^{-\frac{1}{2}}.$$ 

Letting $\epsilon \to 0$, we get

$$\frac{2mk e^{-\kappa}}{r(e^{-\kappa r^2/4} - e^{-\kappa})} \leq \frac{1}{h}r^{-\frac{1}{2}}$$

which leads to

$$m \leq \frac{h^{-1/2}}{2\kappa}(e^{3\kappa/4} - 1)r = C(h, \kappa, p)r.$$

Lemma 1.2. Under the assumptions of Lemma 1.1, we have for a constant $C > 0$ depending only on $n$, $p$, $\tilde{h}$, and $\delta(\Omega)$

$$u(x_0) \leq \max_{B_{1/2}(x_0)} u \leq Cr.$$  

Proof. Applying Harnack’s inequality [5, Theorem 3.14, p. 178], we get

$$\max_{B_{1/2}(x_0)} u \leq C\left(\min_{B_{1/2}(x_0)} u + k(r)\right),$$

where $C$ is a positive constant depending only on $n$ and $p$, $k(r) = (r^2\lVert \tilde{h} \rVert_2)^{-\frac{1}{2}}$ and $\lVert \cdot \rVert$ is the norm of the Morrey space $M^{n/p, \epsilon}$, $\epsilon$ is an arbitrary number in $(0, 1)$.

We recall that the Morrey space $M^q(\Omega)$ with $1 \leq q \leq \infty$ is defined by

$$M^q(\Omega) = \left\{ f \in L^1(\Omega) \mid \exists K > 0 \text{ such that } \int_{\Omega \cap B(x,r)} |f| dy \leq Kr^{n(1-\frac{1}{q})} \forall B(x,r) \subset \mathbb{R}^n \right\}$$

and equipped with the norm

$$\lVert f \rVert_{M^q(\Omega)} = \inf \left\{ K \mid \int_{\Omega \cap B(x,r)} |f| dy \leq Kr^{n(1-\frac{1}{q})} \forall B(x,r) \subset \mathbb{R}^n \right\}.$$ 

Then one can easily show that $k(r) \leq Cr^{-\frac{p}{2}}$. Taking into account the result of Lemma 1.1, we get the result. 

Proof of Theorem 1.1. Let $\Omega_\epsilon = \{ x \in \Omega \mid d(x, \partial \Omega) > \epsilon \}$. We shall prove that for $\epsilon \in (0, \min(\frac{1}{3\delta}, \frac{1}{2(\tilde{h})^{n/p}}))$, $\nabla u$ is bounded in $\Omega_{4\delta}$ by a constant depending only on $n$, $p$, $M$, $\tilde{h}$ and $\epsilon$.

Let $x_0 \in \Omega_{4\delta}$. We distinguish two cases:

(i) $B_{2\epsilon}(x_0) \subset [u > 0]$;

Let $v$ be defined in $B_1$ by $v(y) = u(x_0 + 2\epsilon y)$.

We can easily verify that $v$ satisfies $\Delta_p v = -(2\epsilon)^p(\text{div}H)(x_0 + 2\epsilon y) = g$ with $|g|_{\infty} \leq (2\epsilon)^p\tilde{h} \leq 1$.

Applying Proposition 2 of [6] to $v$, we get for a positive constant $C(n, p, M)$

$$\sup_{B_1} |\nabla v| \leq C(n, p, M)$$

which leads to

$$\sup_{B_{2\epsilon}(x_0)} |\nabla u| \leq \frac{C(n, p, M)}{2\epsilon}.$$ 

(ii) $B_{2\epsilon}(x_0) \cap [u = 0] \neq \emptyset$:

Let $x \in B_\epsilon(x_0)$ such that $u(x) > 0$ and let $r(x) = \text{dist}(x, [u = 0])$ be the distance between $x$ and the set $[u = 0]$. Clearly we have $B_r(x) \subset [u > 0]$. Moreover since the distance function is Lipschitz continuous with Lipschitz constant equal to 1, we have

$$r(x) \leq |x - x_0| + r(x_0) < \epsilon + 2\epsilon = 3\epsilon.$$
which leads to
\[ \overline{B}_{r(x)}(x) \subset B_{3\epsilon}(x) \subset B_{4\epsilon}(x_0). \]
Since \( u(x) > 0 \) in \( B_{r(x)}(x) \), \( \overline{B}_{r(x)}(x) \subset B_{4\epsilon}(x_0) \subset \Omega \) and \( \partial B_{r(x)}(x) \cap \partial[u > 0] \neq \emptyset \), we deduce from Lemma 1.2 that \( u(x) \leq C_1 r(x) \) for some positive constant \( C_1 \) depending only on \( n, p, \tilde{R}, \) and \( \delta(\Omega) \).

It follows that the function defined in \( B_1 \) by
\[ v(y) = \frac{u(x + r(x)y)}{r(x)} \]

is uniformly bounded by \( C_1 \) in \( B_1 \). Moreover, it satisfies
\[ \Delta_p v = -r(x) \text{div}(H)(x + r(x)y) \quad \text{in} \ B_1, \]

Since \( |r(x) \text{div}(H)(x + r(x)y)|_\infty \leq r(x)\tilde{h} \leq 3e\tilde{h} < 1 \), we deduce (see [6]) that we have for a positive constant \( C(n, p, C_1) \)

\[ \sup_{B_{1/2}} |\nabla v| \leq C(n, p, C_1) \]

which leads to
\[ \sup_{B_{10/2}(x)} |\nabla u| \leq C(n, p, C_1). \]

In particular \( |\nabla u(x)| \leq C \).

Since \( \nabla u(x) = 0 \) a.e. in \( B_{\epsilon}(x_0) \cap \{u = 0\} \), it follows that \( \nabla u \) is uniformly bounded in \( B_\epsilon(x_0) \).

\[ \Box \]

2. Boundary Lipschitz continuity of \( u \)

In this section we assume that \( u = 0 \) on a nonempty subset \( T \) of \( \partial \Omega \) and that the uniform exterior sphere condition is satisfied locally on \( T \) i.e. for each open and connected subset \( S_0 \subset T \)

\[ \exists R_0 > 0 \quad \text{such that} \forall x \in S_0 \ \exists y \in \mathbb{R}^d \setminus \overline{S}_0 \ \overline{B}_{R_0}(y) \cap S_0 = \{x\}. \]  \hfill (2.1)

Note that we can always assume that \( R_0 < d_0/3 \) where \( d_0 = d(S_0, \partial \Omega \setminus T) > 0 \) which we will assume throughout this section.

The main result here is the following:

**Theorem 2.1.** Let \((u, \chi)\) be a solution of \((P)\). Then we have \( u \in C^{0,1}_{\text{loc}}(\Omega \cup T) \).

The proof of Theorem 2.1 is based on the following lemma.

**Lemma 2.1.** Let \( S_0 \) be an open connected subset of \( T \) such that \( S_0 \subset T \). Then there exists a positive constant \( C \) depending only on \( n, p, M, \tilde{R}, \delta(\Omega) \) and \( R_0 \) such that

\[ u(x) \leq C|x - x_0| \quad \forall x \in \Omega \ \forall x_0 \in S_0. \]

**Proof.** Let \( x_0 \in S_0 \) and \( x_1 = x_0 + R_0 v \) with \( v \) the outward unit normal vector to \( \partial \Omega \) at \( x_0 \) \( (B_{R_0}(x_1) \cap \partial \Omega = \{x_0\}). \)

We consider the function \( v(x) = \psi(d(x)) \) where \( d \) and \( \psi \) are defined by:

\[ d(x) = |x - x_1| - R_0, \quad D = \delta(\Omega) \quad \text{and} \quad \psi(t) = \int_0^t \left( \left( 1 + \frac{\tilde{R}_0}{n-1} \right) e^{\frac{n-1}{n} (D-t)} - \frac{\tilde{R}_0}{n-1} \right)^{\frac{1}{n-1}} ds. \]

It is easy then to verify the following properties of \( \psi \).

\[ \psi(0) = 0, \quad \psi'(t) = \left( \left( 1 + \frac{\tilde{R}_0}{n-1} \right) e^{\frac{n-1}{n} (D-t)} - \frac{\tilde{R}_0}{n-1} \right)^{\frac{1}{n-1}} > 0 \quad \forall t \in [0, D], \]

\[ \psi'(D) = 1 \leq \psi'(t) \leq \psi'(0) = \left( \left( 1 + \frac{\tilde{R}_0}{n-1} \right) e^{\frac{n-1}{n} D} - \frac{\tilde{R}_0}{n-1} \right)^{\frac{1}{n-1}} \quad \forall t \in [0, D], \]

\[ (p - 1) \left( \psi'(t) \right)^{p-2} \psi''(t) + \frac{n-1}{R_0} \left( \psi'(t) \right)^{p-1} + \tilde{h} = 0 \quad \forall t \in [0, D]. \]
We claim that
\[ \Delta_p v + \text{div}(H) \leq 0 \quad \text{in } \Omega. \] (2.2)

Indeed we first have
\[
\frac{\partial v}{\partial x_i} = \psi'(d(x)) \frac{\partial d}{\partial x_i} = \psi'(d(x)) \frac{x_i - x_{i_t}}{|x - x_{i_t}|}.
\]

Indeed let
\[
\frac{\partial^2 d}{\partial x_i^2} = \frac{1}{|x - x_{i_t}|} - \frac{(x_i - x_{i_t})^2}{|x - x_{i_t}|^3}.
\]

Now using the test function \( \psi \), we have
\[
\int_{\Omega} |\nabla v|^{p-2} v \nabla \psi = (p-1) \psi'(d(x)) |\nabla v|^{p-2} \psi''(d(x)) \left( \frac{\partial d}{\partial x_i} \right)^2 + (\psi'(d(x)))^{p-2} \frac{\partial^2 d}{\partial x_i^2}.
\]

from which we deduce that
\[ \Delta_p v = (p-1) |\nabla v|^{p-2} \psi''(d(x)) \psi'(d(x)) \left( \frac{\partial d}{\partial x_i} \right)^2 + (\psi'(d(x)))^{p-2} \frac{\partial^2 d}{\partial x_i^2}. \] (2.3)

Note that \( |x - x_{i_t}| > R_0 \) \( \forall x \in \Omega \). Then using the above properties of \( \psi \) and (2.3) we get (2.2).

Now we have
\[ u(x) = 0 \leq v(x) \quad \text{for all } x \in T. \] (2.4)

Moreover we claim that
\[ v(x) \geq \psi(R_0) \quad \text{for all } x \in \partial \Omega \setminus T. \] (2.5)

This is due to the fact that \( d(x) = |x - x_{i_t}| - R_0 \) and
\[ |x - x_{i_t}| \geq 2R_0 \quad \forall x \in \partial \Omega \setminus T. \]

Indeed let \( x \in \partial \Omega \setminus T \). We have
\[ |x - x_0| \leq |x - x_{i_t}| + |x_{i_t} - x_0| = |x - x_{i_t}| + R_0 \]
which leads to
\[ |x - x_{i_t}| \geq |x - x_0| - R_0 \geq d(S_0, \partial \Omega \setminus T) - R_0 = d_0 - R_0 > 3R_0 - R_0 = 2R_0. \]

Now we have two cases:

* If \( \psi(R_0) \geq M \), then by (2.5), we have
\[ v \geq u \quad \text{on } \partial \Omega \setminus T. \]

* If \( \psi(R_0) < M \), we take \( w = \frac{M}{\psi(R_0)} v \). Then
\[ w \geq \frac{M}{\psi(R_0)} \psi(R_0) = M \geq u \quad \text{on } \partial \Omega \setminus T. \]

Moreover the function \( w \) satisfies
\[ \Delta_p w + \text{div}(H) = \left( \frac{M}{\psi(R_0)} \right)^{p-1} \Delta_p v + \text{div}(H) \leq 0 \quad \text{in } \Omega \]
since \( \frac{M}{\psi(R_0)} > 1 \) and \( \Delta_p v \leq -\tilde{h} < 0 \) in \( \Omega \).

We deduce that in both cases, we have a function \( v \) that satisfies \( \Delta_p v + \text{div}(H) \leq 0 \) in \( \Omega \) and \( v \geq u \) on \( \partial \Omega \).

Now using the test function \( (u - v)^+ \) for the problem (P) and for (2.2), we obtain
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u. \nabla (u - v)^+ \, dx = - \int_{\Omega} \chi H(x). \nabla (u - v)^+ \, dx.
\]
\[
- \int_{\Omega} |\nabla v|^{p-2} \nabla v. \nabla (u - v)^+ \, dx \leq \int_{\Omega} H(x). \nabla (u - v)^+ \, dx.
\] (2.6)
Taking into account that \( \chi = 1 \) a.e. in \([u > 0]\) and adding (2.6) and (2.7), we obtain
\[
\int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v). \nabla (u - v) \, dx \leq 0
\]
which leads to \((u - v)^+\) constant in \(\Omega\). Since \(u < v\) on \(\partial \Omega\), we get \(u \leq v\) in \(\Omega\).

We conclude that for all \(x \in \Omega\) and all \(x_0 \in S_0\), we have
\[
u(x) \leq v(x) = |v(x) - v(x_0)| \leq \sup_{x \in \Omega} |\nabla v(x)| |x - x_0| \leq \left( \sup_{t \in [0, \epsilon]} |\psi'|(t) \right) |x - x_0| = \psi'(0)|x - x_0| = C(n, p, M, \bar{h}, D, R_0)|x - x_0|. \quad \square
\]

**Proof of Theorem 2.1.** Let \(d_0 > 0\) be small enough such that
\[
S(d_0) = \{ x \in T \mid d(x, \partial \Omega \setminus T) > d_0 \} \neq \emptyset.
\]

Let \(\epsilon \in (0, d_0/4)\) and consider the set
\[
\Omega(\epsilon, d_0) = \{ x \in \Omega \mid d(x, S(2d_0)) < \epsilon \}.
\]
We shall prove that \(u \in C^{0,1}(\Omega(\epsilon, d_0) \cup S(2d_0))\) by showing that \(\nabla u\) is uniformly bounded in \(\Omega(\epsilon, d_0)\). So let \(x \in \Omega(\epsilon, d_0)\) and let \(x_0 \in \partial \Omega\) such that \(d_0 = d(x, \partial \Omega) = |x - x_0|\). We claim that \(x_0 \in S(d_0)\). Indeed let \(x_1 \in S(2d_0)\) such that \(d(x, S(2d_0)) = |x - x_1|\) and let \(x_2 \in \partial \Omega \setminus T\). Then we have
\[
2d_0 \leq d(x_1, \partial \Omega \setminus T) \leq |x_1 - x_2|
\]
\[
= |x_1 - x| + |x - x_0| + |x_0 - x_2| < \epsilon + d(x, \partial \Omega) + |x_0 - x_2|
\]
\[
\leq \epsilon + d(x, S(2d_0)) + |x_0 - x_2| < \epsilon + \epsilon + |x_0 - x_2|
\]
\[
< d_0/4 + d_0/4 + |x_0 - x_2|.
\]

It follows that
\[
|x_0 - x_2| > 2d_0 - d_0/4 - d_0/4 = 3d_0/2 \quad \forall x_2 \in \partial \Omega \setminus T
\]
which leads to \(d(x_0, \partial \Omega \setminus T) \geq 3d_0/2 > d_0\) i.e. \(x_0 \in S(d_0)\).

Now we assume that \(u(x) > 0\). Then we claim that the function \(v\) defined in \(B_1\) by
\[
v(z) = \frac{u(x + r(x)z)}{r(x)}
\]
is uniformly bounded in \(B_1\). We distinguish two cases:

1. \(r(x) \geq \frac{d_0}{2}\):

   We have by Lemma 2.1 for all \(z \in B_1\)
\[
u(x + r(x)z) \leq C \left( |x + r(x)z - x_0| \leq C \left( |x - x_0| + r(x) \right) = C(d_x + r(x)) \leq C(4r(x) + r(x)) = 5Cr(x).\right.
\]

2. \(r(x) < \frac{d_0}{2}\):

   Since the distance function is Lipschitz continuous with Lipschitz constant equal to 1, we have for each \(z \in B_1\)
\[
r(x + r(x)z) \leq r(x) + |x + r(x)z - x| = r(x) + (r(x)|z| \leq 2r(x).
\]

In the same way we have
\[
d_x \leq d_{x + r(x)z} + |x + r(x)z - x| = d_{x + r(x)z} + r(x)|z| \leq d_{x + r(x)z} + \frac{d_x}{2}
\]
which leads to
\[
r(x + r(x)z) \leq 2r(x) < \frac{d_x}{2} \leq d_{x + r(x)z}.
\]

It follows that \(\overline{B_{r(x + r(x)z)}}(x + r(x)z) \subset B_{d_{x + r(x)z}}(x + r(x)z) \subset \Omega\). Applying Lemma 1.2, we get
\[
u(x + r(x)z) \leq Cr(x + r(x)z) \leq 2Cr(x).
\]
We have now proved that the function $v$ is uniformly bounded in $B_1$ by a constant $C$ depending only on $n$, $p$, $M$, $\tilde{h}R_0$ and $\delta(\Omega)$. Moreover, it satisfies
\[
\begin{cases}
\Delta_p v = -\text{div}(\tilde{H}) = -r(x) \text{div}(H)(x + r(x)z) & \text{in } B_1, \\
\left|\text{div}(\tilde{H})\right|_{\infty} \leq r(x)\tilde{h} \leq \delta(\Omega)\tilde{h}/2.
\end{cases}
\]
We deduce that $|\nabla v|_{\infty, B_{1/2}} \leq C(n, p, M, \tilde{h}, \delta(\Omega))$ (see [6]).

In particular we have proved that $|
abla v(0)| \leq C$, which is equivalent to $|
abla u(x)| \leq C$.

Since $\nabla u(x) = 0$ a.e. in $\Omega(\epsilon, d_0) \cap [u = 0]$, it follows that $\nabla u$ is uniformly bounded in $\Omega(\epsilon, d_0)$. □

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