NON-ABELIAN MONOPOLES, VORTICES AND
CONFINEMENT

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Three closely related issues will be discussed. Magnetic quarks having non-Abelian
color charges have been found recently to appear as the dominant infrared degrees of
freedom in some vacua of softly broken $\mathcal{N} = 2$ supersymmetric QCD with $SU(n_c)$
gauge group. Their condensation upon $\mathcal{N} = 1$ perturbation causes confinement
and dynamical symmetry breaking. We argue that these magnetic quarks can
be naturally related to the semiclassical non-Abelian monopoles of the type first
discussed by Goddard, Nuyts, Olive and E. Weinberg. We discuss also general
properties of non-Abelian vortices and discuss their relevance to the confinement
in QCD. Finally, calculation by Douglas and Shenker of the tension ratios for
vortices of different $\mathcal{N}$-alities in the softly broken $\mathcal{N} = 2$ supersymmetric $SU(N)$
Yang-Mills theory, is carried to the second order in the adjoint multiplet mass. A
correction to the ratios violating the sine formula is found, showing that the latter
is not a universal quantity.

1. Confining vacua of softly broken $\mathcal{N} = 2$ supersymmetric QCD

Recently detailed properties of confining vacua have been studied in a
class of softly broken $\mathcal{N} = 2$ supersymmetric gauge theories. Confining
vacua in $SU(n_c)$, $USp(2n_c)$ or $SO(n_c)$ gauge theories with softly broken
$\mathcal{N} = 2$ supersymmetry, with various number of flavors $n_f < 2n_c$, $2n_c + 2$, $n_c - 2$, respectively, have been found\(^1,^2\) to fall into (roughly speaking)
the following three types (see Table 1 for the phases in $SU(n_c)$ theories):

In some of the vacua (the $r = 0$ or $r = 1$ vacua of $SU(n_c)$ theories;
also confining vacua of all flavorless cases\(^3,^4,^5\)), the gauge group of the
low-energy dual theory is the maximal Abelian subgroup $U(1)^R$, where
$R$ is the rank of the original gauge group; confinement is described by ’t Hooft-Mandelstam mechanism\(^6\);
Table 1. Phases of SU\((n_c)\) gauge theory with \(n_f\) flavors, taken from [1].

| Label \(r\) | Deg. Freed. | Eff. Gauge Group | Phase | Global Symmetry |
|-------------|-------------|------------------|-------|-----------------|
| 0           | monopoles   | \(U(1)^{n_c-1}\) | Conf. | \(U(n_f)\) |
| 1           | monopoles   | \(U(1)^{n_c-1}\) | Conf. | \(U(n_f-1) \times U(1)\) |
| \(< \frac{n_f}{2}\) | dual quarks | \(SU(r) \times U(1)^{n_c-r}\) | Conf. | \(U(n_f-r) \times U(r)\) |
| \(\tilde{n}_c\) | dual quarks | \(SU(\tilde{n}_c) \times U(1)^{n_c-\tilde{n}_c}\) | Free Mag | \(U(n_f)\) |

In the general \(r\) vacua \(2 \leq r < \frac{n_f}{2}\) of the \(SU(n_c)\) theory, the effective low-energy theory is a non-Abelian \(SU(r) \times U(1)^{n-r}\) gauge theory; massless magnetic monopoles in the fundamental representation of dual \(SU(r)\) gauge group appear as the low-energy degrees of freedom. Their condensation, together that of Abelian monopoles of the \(U(1)^{n-r-1}\) factors, describes the confinement as a generalized dual Meissner effect. The vacua in the same universality classes appear in \(USp(2n_c)\) and \(SO(n_c)\) theories with nonzero bare quark masses.

In the \(r = \frac{n_f}{2}\) vacua of \(SU(n_c)\) theory, as well as in all of confining vacua of \(USp(2n_c)\) and \(SO(n_c)\) theories with massless flavor \(^a\), the low-energy degrees of freedom involve relatively non-local objects: the low-energy theory is a deformed superconformal theory, i.e., near an infrared fixed-point.

2. Non-Abelian Monopoles

We argue first that the “dual quarks” appearing in the \(r\)-vacua of the softly broken \(\mathcal{N} = 2\) \(SU(n_c)\) theories can naturally be identified with the non-Abelian magnetic monopoles of the type first discussed by Goddard, Nuyts and Olive \(^7\) and studied further by E. Weinberg \(^8\). Our argument is based on the simple observations as regards to their charges, flavor quantum numbers, and some general properties of electromagnetic duality \(^9\).

2.1. Charges of non-Abelian monopoles

Consider \(^7\) a broken gauge theory,

\[
G \xrightarrow{\phi \neq 0} H
\]

\(^a\)There are exceptions to this rule for small values of \(n_f\) and \(n_c\), e.g., \(USp(2) = SU(2)\) case. See the footnote 18 of \(^1\).
Table 2. Some examples of dual pairs of groups

| Group                              | Dual Group                   |
|------------------------------------|------------------------------|
| SU(N)/ZN                           | SU(N)                        |
| SO(2N)                             | SO(2N)                       |
| SO(2N+1)                           | USp(2N)                      |

where the unbroken group $H$ is in general non-Abelian. In order to have a finite mass, the scalar fields must behave asymptotically as

$$D\phi \stackrel{r \to \infty}{\to} 0 \Rightarrow \phi \sim U \cdot \langle \phi \rangle \cdot U^{-1}, \quad A_a^i \sim U \cdot \partial_i U^\dagger \Rightarrow \epsilon_{aij} r_j G(r), \quad (2.1)$$

with $DG = 0$, representing nontrivial elements of $\Pi_2(G/H) = \Pi_1(H)$. The function $G(r)$ can be chosen as

$$G(r) = \beta_i T_i, \quad T_i \in \text{Cartan Subalgebra of } H. \quad (2.2)$$

Topological quantization leads to the result that the “charges” $\beta_i$ take values which are weight vectors of the group $\tilde{H} = \text{dual of } H$. The dual of a group (whose roots vectors are $\alpha$’s) is by definition has the root vectors which span dual lattice, i.e., $\tilde{\alpha} = \alpha/\alpha^2$. Examples of pairs of the duals are given in the Table 2

As an example, consider an $SU(3)$ theory broken as

$$SU(3) \langle \phi \rangle \rightarrow SU(2) \times U(1), \quad \langle \phi \rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}, \quad (\% \quad (2.3)$$

Take a subgroup $SU_U(2) \subset SU(3)$

$$t^i = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad t^5 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}; \quad \frac{t^3 + \sqrt{3} t^8}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad (2.4)$$

then

$$SU_U(2) \langle \phi \rangle \rightarrow SU_U(1). \quad (\ast) \quad (2.5)$$

Embedding the ’t Hooft-Polyakov monopole solution $\phi(r), A(r)$ for $(\ast)$ one gets a $SU(3)$ solution (Sol. 1) :

$$\phi = \left( -\frac{1}{2} v \begin{pmatrix} 0 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \right) + \frac{3}{2} v \left( t^4, t^5, \frac{t^3}{2} + \frac{\sqrt{3} t^8}{2} \right) \cdot \hat{r} \phi(r), \quad (2.6)$$

$$\vec{A} = \left( t^4, t^5, \frac{t^3}{2} + \frac{\sqrt{3} t^8}{2} \right) \wedge \hat{r} A(r). \quad (2.7)$$

Together with another solution (Sol.2) with $SU_V(2) \subset SU(3)$

$$t^6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad t^7 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad \frac{-t^3 + \sqrt{3} t^8}{2} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad (2.8)$$
they yield a degenerate doublet of monopoles with charges

| monopoles | $\tilde{SU}(2)$ | $\tilde{U}(1)$ |
|-----------|-----------------|----------------|
| $\tilde{q}$ | 2               | 1              |

This construction can be generalized to cases with gauge symmetry breaking

$$SU(n) \rightarrow SU(r) \times U^{n-r}(1), \quad \langle \phi \rangle = \begin{pmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & 0 & \cdots \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & v_{n-r+1} \end{pmatrix}. \ (2.8)$$

By considering various $SU_i(2)$ subgroups $(i = 1, 2, \ldots, r)$ living in $[i, r + 1]$ subspace we find
(i) a degenerate $r$-plet of stable monopoles ($q$), gauge (Weyl-) transformed to each other by $SU(r) \subset SU(n)$;
(ii) Abelian monopoles ($e_i$), $(i = 1, 2, \ldots, n - r - 1)$ of $U^{n-r}(1)$ (non-degenerate).

The charges of these stable monopoles are identical to those found in the $r$-vacua of the softly broken $N = 2$ SQCD (Table.3)! In particular, as will be shown in the next subsection these non-Abelian monopoles can acquire flavor quantum numbers through the (generalized) Jackiw-Rebbi mechanism \textsuperscript{11}.

2.2. Fermion Zero modes in non-Abelian monopole Background

We now couple fermions in the fundamental representation of the gauge group. To be concrete consider the case of a $SU(3)$ theory. The fundamental multiplet,

$$\psi_L = \psi_{L(2)} \oplus \psi_{L(0)}, \quad \psi_R = \psi_{R(2)} \oplus \psi_{R(0)} \quad (2.9)$$

satisfies the Dirac equation $\gamma_i D_i \psi = 0$. More explicitly,

$$-\hat{\sigma} \cdot \hat{p} \psi_{L(2)} - e\hat{\sigma} \cdot (\hat{\ell} \wedge \hat{r}) A(r) \psi_{L(2)} - \frac{1}{2} v \psi_{R(2)} + 3v \hat{\ell} \cdot \hat{r} \psi_{R(2)} \phi(r) = 0,$$

$$-\hat{\sigma} \cdot \hat{p} \psi_{L(0)} + v \psi_{R(0)} = 0,$$

$$\hat{\sigma} \cdot \hat{p} \psi_{R(2)} + e\hat{\sigma} \cdot (\hat{\ell} \wedge \hat{r}) A(r) \psi_{R(2)} - \frac{1}{2} v \psi_{L(2)} + 3v \hat{\ell} \cdot \hat{r} \psi_{L(2)} \phi(r) = 0,$$
Table 3. The effective degrees of freedom and their quantum numbers at a confining \(r\)-vacua [2, 1].

| \(n_f \times q\) | \(SU(r)\) | \(U(1)_0\) | \(U(1)_1\) | \(\ldots\) | \(U(1)_{n_c-r-1}\) |
|----------------|--------|--------|--------|--------|----------------|
| \(e_1\) | \(r\) | \(1\) | \(0\) | \(\ldots\) | \(0\) |
| \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) | \(\vdots\) |
| \(e_{n_c-r-1}\) | \(1\) | \(0\) | \(0\) | \(\ldots\) | \(1\) |

\[
\vec{\sigma} \cdot \vec{p} \psi_{R(0)} + v \psi_{L(0)} = 0. \tag{2.10}
\]

Through the Yukawa coupling, the fermion acquired a mass, \(m = \frac{v}{\sqrt{2}}\). Generalizing the Jackiw-Rebbi analysis to the massive fermions, it can be shown that a normalizable zero mode exists if \(3v > v\) which is obviously satisfied. Each fermion gets one zero mode; quantum mechanically, the monopoles become flavor multiplets.

An analogous construction in the case of the breaking \(SU(n_c) \to SU(r) \times U(1)^{n_c-r}\), the above condition is replaced by

\[
\left| \frac{v_0 - v_{r+1}}{2} \right| > \left| \frac{v_0 + v_{r+1}}{2} \right|. \tag{2.11}
\]

Note that for the breaking \(SU(n) \to SU(n-1) \times U(1)\) such a condition is always satisfied; otherwise, only the monopoles with VEVS satisfying the above condition will give rise to fermion zero modes.

This mechanism “explains” the low-energy degrees of freedom in the \(r\) vacua of softly broken \(N = 2\) SQCD, with \(G = SU(n_c)\), with \(n_f\) quarks:

### 2.3. Duality

It is also significant that, in the softly broken \(N = 2\) \(SU(n_c)\) theory, the \(r\) vacua with a magnetic \(SU(r)\) gauge group occur only for \(r \leq \frac{2n_c}{3}\). This is a manifestation of the fact that the quantum behavior of non-Abelian monopoles depends crucially on the massless matter fermion degrees of freedom in the fundamental theory. Indeed, the magnetic \(SU(r) \times U(1)^{n_c-r}\) theory with these matter multiplets is infrared-free (i.e., non-asymptotic free). This is the correct behavior as it should be dual to the original asymptotic free \(SU(n_c)\) gauge theory. Note that the gauge coupling constant evolution, which appears as due to the perturbative loops of magnetic monopoles, is actually the result of, and equivalent to, the infinite sum of instanton contributions in the original \(SU(n_c)\) theory.

This is perfectly analogous to the observation [12] about how the old paradox related to the Dirac quantization condition and renormalization
group \(^{13}\) :

\[ g_c(\mu) \cdot g_m(\mu) = 2\pi n, \quad \forall \mu, \quad (2.12) \]
is solved within the SU(2) Seiberg-Witten theory.

Note that this also explains why in the pure \(\mathcal{N} = 2\) SU\((n_c)\) theory or on a generic point of the Coulomb branch of the \(\mathcal{N} = 2\) SQCD, the low-energy effective theory is an Abelian gauge theory \(^{3,5}\). Massless fermion flavors are needed in order for non-Abelian monopoles to get dressed, via a generalized Jackiw-Rebbi mechanism discussed above with a non trivial SU\((n_f)\) flavor quantum numbers and, as a result, to render the dual gauge interactions infrared-free. When this is not possible, non-Abelian monopoles are strongly coupled and do not manifest themselves as identifiable low-energy degrees of freedom.

In this respect, it is very interesting that the boundary case \(r = \frac{n_f}{2}\) also occurs (confining vacua of type (iii) discussed in Introduction) within the class of supersymmetric theories considered in \(^1\). In these vacua, non-Abelian monopoles and dyons are strongly coupled, but still describe the low-energy dynamics, albeit via non-local effective interactions.

Non-Abelian monopoles are actually quite elusive objects. Though their presence may be detected in a semi-classical approximation, their true nature depends on the long distance physics. If the “unbroken” gauge group is dynamically broken further in the infrared such multiplets of states simply represent an approximately degenerate set of magnetic monopoles. Only if there is no further dynamical breaking do the non-Abelian monopoles transforming as nontrivial multiplets of the unbroken, dual gauge group, appear in the theory.

There are strong indications that this occurs in the \(r\)-vacua (with an effective SU\((r) \times U(1)^{n_c-r}\) gauge symmetry) of the softly broken \(\mathcal{N} = 2\), SU\((n_c)\) supersymmetric QCD \(^1\). If our idea is correct, this is perhaps the first physical system known in which Goddard-Nuyts-Olive-Weinberg monopoles manifest themselves as infrared degrees of freedom, playing an essential dynamical role. For more about the subtle nature of non-Abelian monopoles, see \(^9\).

3. Non-Abelian Vortices

A closely related issue is that of non-Abelian vortices \(^{14,19}\). If confinement is to be described as a sort of non-Abelian dual Meissner effect, the magnetic monopoles of the type discussed above condense and break the dual gauge group. As a result, the system develops vortex configurations
which serve as confining strings.

3.1. General Characterization

This time we consider a gauge theory in which the gauge group $G$ is spontaneously broken by the Higgs mechanism as

$$G \Longrightarrow \mathcal{C} \quad (3.1)$$

with $\mathcal{C}$ a discrete center of the group. The general properties of the vortex, which represents a nontrivial elements of the fundamental group,

$$\Pi_1(G/C) = \mathcal{C}, \quad (3.2)$$

are independent of the detailed form of the scalar potential or of the number of the Higgs fields present. Asymptotic form of the fields are:

$$A_i \sim \frac{i}{g} U(\phi) \partial_i U^\dagger(\phi); \quad \phi_A \sim U\phi_A^U U^\dagger, \quad U(\phi) = \exp i \sum_j \beta_j T_j \phi$$

where $T_j$’s can be taken in the Cartan subalgebra of $G$: then

$$A_\phi \sim \frac{1}{g r} \sum_j \beta_j T_j$$

The vortex flux

$$\oint dx_i A_i = \frac{2\pi}{g} \sum_j \beta_j T_j,$$

is characterized by the “charges” $\beta$. The quantization condition

$$U(2\pi) \in \mathcal{C}. \quad (3.3)$$

leads to the result that $\beta_j$’s are weight vectors of $\tilde{G}$ (dual of $G$). $\beta_j$’s are actually defined modulo Weyl transformations $\beta'$’s:

$$\beta' = \beta - \frac{2\alpha(\beta \cdot \alpha)}{\alpha \cdot \alpha},$$

where $\alpha$ is a root vector of $G$.

3.2. $SU(N)/Z_N$

The simplest system with non-Abelian vortices is $SO(3) = SU(2)/Z_2$ broken to $Z_2$. It has

- unique $Z_2$ vortex (the source charge additive mod 2);
8

- "flux"

\[ \int_S F_{ij}^3 \, d\sigma_{ij} = \oint dx_i A_i^3 = \frac{2\pi n}{g}. \]

which is conserved but not gauge invariant. \( n = 2 \) “vortex” can be
gauge-transformed away.

A more interesting system is \( \text{SU}(3)/\mathbb{Z}_3 \), i.e., \( \text{SU}(3) \) theories with all
fields in adjoint representation. The Cartan subalgebra can be taken to be
\[
T_3 = \frac{1}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad T_8 = \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\]

The quantization condition \( U(2\pi) \in \mathbb{Z}_3 \) leads to the equations
\[
\frac{\beta_3}{2\sqrt{3}} + \frac{\beta_8}{6} = -\frac{n_1}{3}, \quad -\frac{\beta_3}{2\sqrt{3}} + \frac{\beta_8}{6} = -\frac{n_2}{3}, \quad -\frac{1}{3} \beta_8 = -\frac{n_3}{3},
\]
(3.4)
to be solved with the condition, \( \sum_i n_i = 0 \). The simplest \( N \)-ality (triality)
one \( (n_i = [1 \mod 3]) \) solutions are:
\[
\beta = (-\sqrt{3}, 1), \quad (\sqrt{3}, 1), \text{ or } (0, -2) = 2Nw \quad (3.5)
\]
\( w = \) weight vector of 3. Thus the sources of the minimum vortex carry the
quantum number of the quarks. The dual of the theory we are studying,
\( \text{SU}(3)/\mathbb{Z}_3 \), is indeed \( \text{SU}(3)! \)

By adding four of (3.5) \( \rightarrow 6^{*} \), etc., and one could construct an
infinite number of triality-one solutions. However only the vortex with the
lowest tension is stable.

\( N \)-ality (triality)-two solutions are found by adding vectorially the minu-
mum solutions above. The source of these vortices correspond to the irre-
ducible representations has charges
\[
\beta = (-\sqrt{3}, -1), \quad (\sqrt{3}, -1), \text{ or } (0, 2),
\]
= weight vectors of 3* [\[. \]
\[
\beta = (-2\sqrt{3}, 2), \quad (2\sqrt{3}, 2), \quad (0, 2), \quad (-\sqrt{3}, -1), \quad (\sqrt{3}, -1), \text{ or } (0, -4),
\]
= 6 [\[. \]
Quantum mechanically, however, the vortex with the higher
tension (probably 6) decays through the gauge boson pair productions (Fig.
1). Somewhat similar problem of decay of metastable vortices was recently
discussed by Shifman and Yung 20.

Since 3* vortex and 3 vortex are equivalent, there is actually a unique
stable vortex with minimum \( \mathbb{Z}_3 \) charge in the \( \text{SU}(3) \) gauge theory.

The discussion can be generalized naturally to \( \text{SU}(N)/\mathbb{Z}_N \) theory in
Higgs phase. One finds \( N \) degenerate solutions of \( N \)-ality one
\[
\beta_j = 2Nw_j, \quad j = 1, 2, 3, \ldots, N
\]
There are also solutions representing vortices of higher \( N \)-alities. At the \( N \)-ality two, for instance, the solutions for \( \beta \) have the form,

\[
2N(w_i + w_j), \quad i,j = 1,2,\ldots,N.
\]

They fall into two gauge inequivalent sets of vortices: their sources would carry the quantum numbers of the two irreducible representations,

\[
\begin{array}{c}
\sigma, \\
\bar{\sigma}
\end{array}
\]

symmetric and antisymmetric in color, respectively.
Solutions of $N$-ality $k$ can be analogously be constructed by taking as $\beta$ the vector sum of arbitrary $k$ minimum solutions, Eq.(3.2). These vortices can be grouped into gauge invariant subsets, each of which has a source carrying quantum numbers of an irreducible representations of $SU(N)$ group,

\[
\begin{array}{ccc}
\begin{array}{c}
\kappa
\end{array} & \begin{array}{c}
\kappa+1
\end{array} & \ldots
\end{array}
\]

all having $k$ boxes.

The vortices of $N$-ality, $1 \leq k \leq N-1$, cannot be unwound by a gauge transformation. Nevertheless, this does not mean that each of the vortices (3.9) is stable against decay. A vortex of a given $N$-ality can decay through the pair production of gauge bosons into one of the same $Z_N$ quantum number but with a lower tension, via processes similar to the one in the $SU(3)$ example of Fig. 1. It is possible that the tension is smallest in the case of the antisymmetric representation $\binom{N}{k}$. If it is so, the solution for the vortex charge $\beta$ at $N$-ality $k$ is truely a unique gauge-invariant set

\[
2N \{ w_{i_1} + w_{i_2} + \ldots + w_{i_k} \mod \alpha \}, \quad i_m = 1, 2, \ldots, N, \quad (3.10)
\]

where $\alpha$’s are the root vectors of the $SU(N)$ group. These represent

\[
\Pi_1(SU(N)/Z_N) = Z_N.
\]

Which of these, apart from the smallest, $N$-ality one vortex, is stable against decay into a bundle of vortices with smaller $N$-alities, is again a dynamical question (i.e., depends on the form of the potential, values of coupling constants, quantum corrections, etc.). One would expect no universal formula for the relative tensions among vortices of different $N$-alities, on the general ground. However, there are some intriguing suggestions\(^{21}\) that the ratios among the vortex tensions for different $Z_N$ charges, found originally in the pure $\mathcal{N} = 2$ supersymmetric Yang-Mills theory (broken softly to $\mathcal{N} = 1$)\(^{22}\),

\[
\frac{T_k}{T_\ell} = \frac{\sin \frac{\pi k}{N}}{\sin \frac{\pi \ell}{N}}, \quad (3.11)
\]

might be universal. The results from lattice calculations with $SU(5)$ and $SU(6)$ Yang-Mills theories\(^{23,24}\) are consistent with the sine formula. More recent results on these ratios\(^{25,26}\) however seem to indicate the non-universality of these ratios.
The absence of vortices of $N$-ality, $N$, can be understood since the charges corresponding to an irreducible representation with $N$ boxes in the Young tableau, can always be screened by those of the dynamical fields (adjoint representation): the vortex is broken by copious production of massless gluons of the dual $SU(N)$ theory.

In an analogous fashion, one finds that sources of vortices in $USp(2N)$ theory in Higgs phase carry the weights of the $2^N$ dimensional spinor representation of the dual group, $SO(2N + 1)$; sources of vortices in $SO(2N + 1)$ theory in Higgs phase carry the weights of the $2N$ dimensional fundamental representation of $USp(2N)$, etc. For more details, see 27.

3.3. Remarks

(i) If confinement in $SU(N)$ theory is a dual Meissner effect with Olive-Montonen duality, $SU(N) \leftrightarrow SU(N)/Z_N$, then the universal $q - \bar{q}$ meson Regge trajectory will be naturally explained, in contrast to the case when the dual theory is $U(1)^{N-1}$;

(ii) Sources of the non-Abelian vortices have charge additive only mod $N$. Non-Abelian vortices are non BPS: linearized approximation is not valid in general;

(iii) Explicit construction of non-Abelian vortices 14-19 has been studied by using simple models for the adjoint scalar potential. However a systematic study of non-Abelian vortices, hence of non-Abelian superconductors, are still lacking.

(iv) What is the relation between vortex formation and XSB?

(v) Can we compute the ratios of vortex tensions for different $N$-alities (in the $SU(N)$ case)?

This last point brings us to our third issue, related to Eq.(3.11).

4. Non-Universal Corrections in the Tension Ratios in softly broken $\mathcal{N} = 2 \ SU(N)$ Yang-Mills

Derivation of formula such as Eq.(3.11) in the standard, continuous $SU(N)$ gauge theories still defies us. The first field-theoretic result on this issue was obtained by Douglas and Shenker 22, in the $\mathcal{N} = 2$ supersymmetric $SU(N)$ pure Yang Mills theory, with supersymmetry softly broken to $\mathcal{N} = 1$ by a small adjoint scalar multiplet mass $m$. They found Eq.(3.11) for the ratios of the tensions of abelian (Abrikosov-Nielsen-Olesen) 28 vortices corresponding to different $U(1)$ factors of the low-energy effective (magnetic) $U(1)^{N-1}$ theory.
The $n$-th color component of the quark has charges
\[ \delta_{n,k} - \delta_{n,k+1}, \quad (k = 1, 2, \ldots, N - 1; n = 1, 2, \ldots, N) \quad (4.1) \]
with respect to the various electric $U_k(1)$ gauge groups. The source of the $k$-th ANO string thus corresponds to the $N$-ality $k$ multiquark state, $|k\rangle = |q_1 q_2 \ldots q_k\rangle$, allowing a re-interpretation of Eq.(3.11) as referring to the ratio of the tension for different $N$-ality confining strings $29$.

However, physics of the softly broken $\mathcal{N} = 2$ $SU(N)$ pure Yang-Mills theory is quite different from what is expected in QCD. Dynamical $SU(N) \to U(1)^{N-1}$ breaking introduces multiple of meson Regge trajectories with different slopes at low masses $29,30$, a feature which is neither seen in Nature nor expected in QCD. For instance, another $N$-ality $k$ state $|k\rangle' = |q_2 q_3 \ldots q_{k+1}\rangle$ acts as source of the $U_{k+1}(1)$ vortex and as the sink of the $U_2(1)$ vortex, which together bind $|k\rangle'$- anti $|k\rangle'$ states with a tension different from $T_k$. The Douglas-Shenker prediction is, so to speak, a good prediction for a wrong theory! Only in the limit of $\mathcal{N} = 1$ does one expect to find one stable vortex for each $N$-ality, corresponding to the conserved $Z_N$ charges $29$.

Within the softly broken $\mathcal{N} = 2$ $SU(N)$ theory, the two regimes can be in principle smoothly interpolated by varying the adjoint mass $m$ from zero to infinity, adjusting appropriately $\Lambda$. At small $m$ one has a good local description of the low-energy effective dual, magnetic $U(1)^{N-1}$ theory. The transition towards large $m$ regime involves both perturbative and non-perturbative effects. Perturbatively, there are higher corrections due to the $\mathcal{N} = 1$ perturbation, $m \langle \text{Tr} \Phi^2 \rangle$. Nonperturbatively - in the dual theory - there are productions of massive gauge bosons of the broken $SU(N)/U(1)^{N-1}$ generators, which mix different $U(1)^{N-1}$ vortices and eventually lead to the unique stable vortex with a given $\mathcal{N}$-ality.

Below is the result on the perturbative corrections to the tension ratios Eq.(3.11), due to the next-to-lowest contributions in $m$. We shall find a small non-universal correction to the sine formula Eq.(3.11). Our point is not that such a result is of interest in itself as a physical prediction but that it gives a strong indication for the non-universality of this formula, even though it could be an approximately a good one.

The problem of the next-to-lowest contributions in $m$ has been already analyzed in $SU(2)$ theory, by Vainshtein and Yung $30$ and by Hou $31$, although in that case there is only one $U(1)$ factor. When only up to the order $A_D$ term in the expansion
\[ m \langle \text{Tr} \Phi^2 \rangle = m U(A_D) = m \Lambda^2 \left( 1 - \frac{2t A_D}{\Lambda} - \frac{1}{4} \frac{A_D^2}{\Lambda^2} + \ldots \right) \quad (4.2) \]
is kept, the effective low energy theory turns out to be an \( \mathcal{N} = 2 \) SQED, \( A_D \) being an \( \mathcal{N} = 2 \) analogue of the Fayet-Iliopoulos term. As a result, the vortex remains BPS-saturated, and its tension is proportional to the monopole charge \(^{30,31}\). When the \( A^2_D \) term is taken into account, the vortex ceases to be BPS-saturated: the correction to the vortex tension can be calculated perturbatively, giving rise to the results that the vacuum behaves as a type I superconductor.

Our aim here is to generalize these analyses to \( SU(N) \) theory. In fact, Douglas-Shenker result Eq.(3.11) in \( SU(N) \) theory was obtained in the BPS approximation, by keeping only the linear terms in \( a_{Di} \) in the expansion

\[
U(a_{Di}) = U_0 + U_{0k} a_{Dk} + \frac{U_{0mn}}{2} a_{Dm} a_{Dn} + \ldots, \quad U_{0k} = -4i \Lambda \sin \frac{\pi k}{N}. \tag{4.3}
\]

The coefficients \( U_{0k} \) were computed by Douglas-Shenker\(^{22}\). Our first task is then to compute the coefficients of the second term \( U_{0mn} \). In principle it is a straightforward matter, as one must simply invert the Seiberg-Witten formula:

\[
b_{a_{Di}} = \oint_{\alpha_m} \lambda, \quad a_m = \oint_{\beta_m} \lambda, \quad \lambda = \frac{1}{2\pi i} \frac{x}{y} \partial P(x) \partial x dx, \tag{4.4}
\]

which is explicitly known, to second order. The only trouble is that \( a_{Dm} \) and \( a_m \) \((m = 1, 2, \ldots, N - 1)\) are given simply in terms of \( N \) dependent vacuum parameters \( \phi_i \), \( \sum_{i=1}^{N} \phi_i = 0 \). By denoting the formal derivatives with respect to \( \phi_i \) as \( \frac{\delta}{\delta \phi_i} \), one finds

\[
\sum_{i=1}^{N} \frac{\delta a_{Dm}}{\delta \phi_i} \frac{\partial \phi_i}{\partial a_{Dn}} = \delta_{mn}, \quad \sum_{m=1}^{N-1} \frac{\partial \phi_k}{\partial a_{Dm}} \frac{\delta a_{Dm}}{\delta \phi_j} = \delta_{ij} - \frac{1}{N}, \tag{4.5}
\]

which follow easily by using the constraint, \( \sum_{i=1}^{N} \phi_i = 0 \). In terms of \( B_{mi} = -i \frac{\delta a_{Dm}}{\delta \phi_i} \), \( A_{mi} = -i \frac{\delta a_m}{\delta \phi_i} \), which are explicitly given at the \( N \) confining vacua in\(^{22}\), one then finds

\[
\frac{\partial \phi_i}{\partial a_{Dm}} = -i B_{mi}; \quad \sum_{i=1}^{N} B_{mi} B_{ni} = \delta_{mn}; \quad \sum_{m=1}^{N-1} B_{mi} B_{mj} = \delta_{ij} - \frac{1}{N}. \tag{4.6}
\]

The explicit values of \( B_{mi} \) are (see\(^{22}\)):

\[
B_{mi} = \frac{1}{N} \frac{\sin \hat{\theta}_n}{\cos \hat{\theta}_i - \cos \hat{\theta}_m}; \quad \hat{\theta}_n = \frac{\pi n}{N}; \quad \theta_n = \frac{\pi (n - 1/2)}{N}. \tag{4.7}
\]

\(^{b}\)We follow the notation of \(^{22}\), with \( y^2 = P(x)^2 - \Lambda^2 \); \( P(x) = \frac{1}{2} \prod_{i=1}^{N} (x - \phi_i) \)
The definition of \( u(a_{Di}) \) is the following:

\[
u(a_{Di}) = \sum_i \phi_i^2. \tag{4.8}
\]

Then the desired coefficients can be found by the following expression, computed at \( a_{Di} = 0 \):

\[
U_{0mn} = \frac{\partial^2 u}{\partial a_{Dm} \partial a_{Dn}} = 2 \sum_k \frac{\partial \phi_k}{\partial a_{Dm}} \frac{\partial \phi_k}{\partial a_{Dn}} + 2 \phi_k \frac{\partial^2 \phi_k}{\partial a_{Dm} \partial a_{Dn}}. \tag{4.9}
\]

The first part of Eq.(4.9) becomes:

\[
2 \sum_k \phi_k \frac{\partial^2 \phi_k}{\partial a_{Dm} \partial a_{Dn}} = -2 \sum_k B_{km} B_{kn} = -2 \sum_{k,s} \frac{2}{N} \sin \left[ \frac{\pi ms}{N} \right] \sin \left[ \frac{\pi nk}{N} \right] \delta_{ks} = -2 \delta_{mn}. \tag{4.10}
\]

The evaluation of the second term is a little tricky \(^{26}\). The result is however simple:

\[
2 \sum_k \phi_k \frac{\partial^2 \phi_k}{\partial a_{Dm} \partial a_{Dn}} = \left( 2 - \frac{1}{N} \right) \delta_{mn}, \tag{4.11}
\]

thus

\[
U_{0mn} = (- \frac{1}{N}) \delta_{mn}. \tag{4.12}
\]

We now use this result to calculate the corrections to the tension ratios (3.11) found in the lowest order. The effective Lagrangean near one of the \( N \) confining \( \mathcal{N} = 1 \) vacua is

\[
\mathcal{L} = \sum_{i=1}^{N-1} Im \left[ \frac{i}{e_{Di}^2} \left( \int d^4 \theta A_{Di} A_{Di}^+ + \int d^2 \theta (W_{Di})^2 \right) \right] +
\quad + Re \left[ \int d^4 \theta (M_i^+ e^{V_{Di}} M_i + \tilde{M}_i^+ e^{-V_{Di}} \tilde{M}_i) \right] +
\quad + 2Re \left[ \sqrt{2} \int d^2 \theta A_{Di} M_i \tilde{M}_i + m U[A_{Di}] \right]. \tag{4.13}
\]

The coupling constant \( e_{Di}^2 \) is formally vanishing, as

\[
\frac{4 \pi}{e_{Di}^2} \simeq \frac{1}{2 \pi} \ln \frac{\Lambda \sin(\hat{\theta}_k)}{a_{Di} N}
\]

where \( \hat{\theta}_m \equiv \frac{2\pi m}{N} \) and \( a_{Di} = 0 \) at the minimum. Physically, the monopole loop integrals are in fact cut off by masses caused by the \( \mathcal{N} = 1 \) perturbation. The monopole becomes massive when \( m \neq 0 \), and \( \sqrt{2} a_{Di} \) should be
replaced by the physical monopole mass \((m \Lambda \sin(\hat{\theta}_k))^{1/2}\) which acts as the infrared cutoff for the coupling constant evolution. Thus

\[
edm^2 \sim \frac{16\pi^2}{\ln(\frac{\Lambda \sin(\hat{\theta}_m)}{mN^2})},
\]

(4.14)

As \(U_{0mn}\) is found to be diagonal, the description of the ANO vortices in terms of effective magnetic Abelian theory description continues to be valid for each \(U(1)\) factor. In the linear approximation \(U(A_D) = m \Lambda^2 + \mu A_D\), where \(\mu \equiv |4m \Lambda \sin(\hat{\theta}_k)|\) for the \(k\)-th \(U(1)\) theory, the theory can be (for the static configurations) effectively reduced to an \(N = 4\) theory in 2+1 dimensions. In this way, Bogomolny’s equations for the BPS vortex can be easily found from the condition that the vacuum to be supersymmetric:

\[
F_{12} = \sqrt{2}(\sqrt{2} M^+ \tilde{M}^+ - \mu) \quad (D_1 + iD_2)M = 0
\]

(4.15)

\[
M = \tilde{M}^+, \quad A_D = 0.
\]

(4.16)

The solutions of these equations are similar to the one considered by Nielsen and Olesen:

\[
M = \left(\frac{\mu}{\sqrt{2}}\right)^{1/2} e^{i\alpha} f[re\sqrt{\mu}], \quad A_\phi = -2n \frac{g(r,e\sqrt{\mu})}{r}
\]

(4.17)

where

\[
f' = \frac{f}{r}(1 - 2g) \quad g' = \frac{1}{2n} r(1 - f^2)
\]

(4.18)

with boundary conditions \(f(0) = g(0) = 0, \quad f(r \rightarrow \infty) = 1, \quad g(r \rightarrow \infty) = +1/2\). The tension turns out to be independent of the coupling constant: for the minimum vortex

\[
T = \sqrt{2} \pi \mu = 4\sqrt{2} \pi |m \Lambda| \sin \frac{\pi k}{N}.
\]

(4.19)

That the absolute value of \(m\) appears in Eq.(4.19) as it should, and also in Eq.(4.22) below, is not obvious. This can actually be shown by an appropriate redefinition of the field variables, used in (4.12), which renders all equations real.

When the second order term in \(U(A_D) = \mu A_D + \frac{1}{2} \eta A_D^2\), \(\eta \equiv U_{kk}\), is taken into account, the vortex ceases to be BPS saturated. The corrections to the vortex tension due to \(\eta\) can be taking into account by perturbation theory, following (4.2). To first order, the equation for \(A_{Dk} = A_D\) is

\[
\nabla^2 A_D = -2e^2 \eta (\mu - \sqrt{2} M \tilde{M}^+) + 2e^2 A_D (M M^+ + \tilde{M} \tilde{M}^+)
\]

(4.20)
where unperturbed expressions from Eq.(4.17) can be used for \(M, \tilde{M}\). The vortex tension becomes simply

\[
T = \int d^2x \left[ (-\sqrt{2}\mu F_{12}) - 2e^2\eta A_D(\mu - \sqrt{2}M^\dagger \tilde{M}^\dagger) \right]
\]  

(4.21)

where the second term represents the correction. By restoring the \(k\) dependence, we finally get for the tension of the \(k\)-th vortex,

\[
T_k = 4\sqrt{2}\pi |m| \Lambda \sin\left(\frac{\pi k}{N}\right) - C \frac{16\pi^2|m|^2}{N^2 \ln \frac{\Lambda \sin(k\pi/N)}{|m| N}},
\]  

(4.22)

where \(C = 2\sqrt{2}\pi(0.68) = 6.04\). The correction term has a negative sign, independently of the phase of the adjoint mass. Note that the relation \(T_k = T_{N-k}\) continues to hold. Eq.(4.22) is valid for \(m \ll \Lambda\). Qualitative feature of this correction is shown in Fig.3, for \(N = 6\).

In the above consideration, we have taken into account exactly the \(m^2\) corrections in the F-term of the effective low-energy action. On the other hand, the corrections to the D-terms are subtler. Indeed, based on the physical consideration, \(a_D\) in the argument of the logarithm in the effective low energy coupling constant was replaced by the monopole mass, of \(O(\sqrt{m}\Lambda)\). This amounts to the \(m\) insertion to all orders in the loops. Such a resummation is necessitated by the infrared divergences and represents a standard procedure. Another well-known example is the chiral perturbation theory in which quark masses appear logarithmically, e.g., in the expansion of the quark condensate. This explains the non-analytic dependence on \(m\) as well as on \(\frac{1}{N}\).  

Also, there are corrections due to nondiagonal elements in the coupling constant matrix \(\tau_{ij}\), which mix the different \(U(1)\) factors, neglected in Eq.(4.13). These nondiagonal elements are suppressed by \(O(\frac{1}{\log \Lambda^2/m})\) relatively to the diagonal ones, apparently of the same order of suppression as the correction calculated above. However, these nondiagonal elements gives rise to corrections to the tension of one order higher, \(O(\frac{1}{\log^2 \Lambda^2/m})\), hence is negligible to the order considered.

We thus find a non-universal correction to the Douglas-Shenker formula, Eq.(3.11). In the process of transition towards fully non-Abelian superconductivity at large \(m\) nonperturbative effects such as the \(W\) boson productions are probably essential. Nonetheless, the presence of a calculable deviation from the sine formula is qualitatively significant and shows that such a ratio is not a universal quantity.
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