ON A CYLINDER FREELY FLOATING IN OBLIQUE WAVES

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We investigate the coupled motion of a mechanical system consisting of water and a body freely floating in it. The water occupies either a half-space or a layer of constant depth into which an infinitely long surface-piercing cylinder is immersed, thus allowing us to study the so-called oblique waves. Under the assumption that the motion is of small amplitude near equilibrium and describes time-harmonic oscillations, the linear setting of the phenomenon reduces to a spectral problem with the radian frequency as the spectral parameter. If the radiation condition is fulfilled, then the total energy is finite and the equipartition of kinetic and potential energy takes place for the whole system. On this basis, it is proved that no wave modes are trapped under some restrictions on their frequencies. In the case where a symmetric cylinder has two immersed parts, restrictions are imposed on the type of mode as well. Bibliography: 12 titles. Illustrations: 1 figure.

1 Introduction

This paper continues the author’s studies dealing with the motion of a mechanical system consisting of a water layer of constant depth and a rigid body freely floating in it. The initial publication [1] was written more than ten years ago and several papers on this topic appeared since then (see, for example, [2]–[4]). It was John [5] who proposed the linear problem describing the coupled motion of water bounded from above by the atmosphere and a partially immersed body. The latter floats freely according to Archimedes’ law being unaffected by all external forces (for example, due to constraints on its motion) except for gravity. The motion of water (its viscosity is neglected as well as the surface tension on its surface) is assumed to be irrotational, whereas the motion of the whole system is supposed to be of small amplitude near equilibrium; this allows us to use a linear mathematical model.

Within the framework of the linear theory of water waves, two- and three-dimensional formulations are possible. The problem considered here is two-dimensional which is essential for formulating the geometric and other restrictions that must be imposed to guarantee the uniqueness. The latter is of paramount importance (in the classical survey [6], it is placed at the top of the list of important open problems) because there are examples of nonuniqueness (see,
for example, [1]). Therefore, some restrictions on the frequency range and on the body shape are required. It should be noted that such restrictions are unnecessary for similar problems in acoustics (see, for example, the monograph [7]).

The original John’s formulation of the floating body problem is rather cumbersome because he did not use the matrix form of his equations of the body motion (such a form of these equations described below demonstrates their simple structure; see also [2] and [3]). For this reason, the problem was neglected by researches during 60 years after publication of the article [8] in which the uniqueness theorem was proved for the three-dimensional problem under the assumption that the body satisfies the so-called John condition (it is described below for two-dimensional geometries), whereas the frequency of oscillations is sufficiently large.

On the other hand, the problem of time-harmonic oscillations of water in the presence of fixed rigid bodies attracted much attention in the second half of the 20th century. Initially, the case of a single body had been investigated in the article [8] mentioned above. Numerous results about this problem were presented in detail in the summarizing monograph [9] which also contains the extensive literature. In particular, the first nonuniqueness example due to McIver [10] was generalized in [9]; namely, nontrivial solutions were constructed for the two-dimensional homogeneous problem with an arbitrary finite number of surface-piercing bodies (only two bodies were considered in [10]); the so-called inverse procedure was applied for this purpose. The presence of multiple bodies violates the John condition according to which only a single surface-piercing body is admissible in the two-dimensional problem. From the hydrodynamic viewpoint, these nontrivial solutions describe trapped modes, i.e., free oscillations of water having final energy. These results were developed further in the article [11] in which the uniqueness theorem due to John and McIver’s nonuniqueness example was extended to the case of infinitely long, surface-piercing cylinders in oblique waves. The problem for fixed cylinders considered in [11] is a particular case of that formulated in Section 2. It is worth mentioning that another approach to the uniqueness question in the problem with fixed cylinders was proposed in [12].

2 Statement of the Problem

Let an infinitely long surface-piercing cylinder of uniform cross-section float freely in water which is either infinitely deep or bounded from below by a horizontal rigid bottom. The Cartesian coordinate system \((x, y)\) is chosen in a plane orthogonal to the cylinder generators (directed along the \(z\)-axis), so that the \(y\)-axis is directed upwards, whereas the mean free surface of the water intersects this plane along the \(x\)-axis. Thus, the cross-section \(W\) of the water domain is a subset of \(\mathbb{R}^2_+ = \{x \in \mathbb{R}, y < 0\}\). Let \(\hat{B}\) denote the bounded two-dimensional domain whose closure is the cross-section of the cylinder equilibrium position. We suppose that \(\hat{B} \setminus \mathbb{R}^2_+\) – the part of the body located above the water surface – is a nonempty domain, whereas the immersed part \(B = \hat{B} \cap \mathbb{R}^2_+\) is the union of a finite number of domains. Thus, \(D = \hat{B} \cap \partial \mathbb{R}^2_+\) consists of the same number of nonempty intervals of the \(x\)-axis (see Figure for the case of two immersed parts, where \(D = \{x \in (-a, -b) \cup (b, a), y = 0\}\)). Note that \(W = \mathbb{R}^2_+ \setminus \overline{\mathcal{B}}\) if the water has infinite depth or \(W = \{x \in \mathbb{R}, -h < y < 0\} \setminus \overline{\mathcal{B}}, h > b_0 = \sup_{(x,y) \in \mathcal{B}} |y|\) if the water has the constant finite depth \(h\) (see Figure). The cross-section of the bottom is denoted by \(H = \{x \in \mathbb{R}, y = -h\}\) in the last case. Furthermore, \(W\) is assumed to be a Lipschitz domain, and so the unit normal \(n\) pointing to the exterior of \(W\) is defined almost everywhere on \(\partial W\). Finally, we denote by \(S = \partial \hat{B} \cap \mathbb{R}^2_+\) the wetted contour (the number of its components is equal to the number of immersed domains),
whereas $F = \partial \mathbb{R}^2 \setminus D$ is the free surface at rest.

To describe the small-amplitude coupled motion of the water/cylinder system, it is standard to apply the linear setting, in which case the first-order approximation of unknowns is used. These are the velocity potential $\Phi(x, y, z; t)$ and the vector column $q(t)$ describing the motion of the cylinder; its three components are as follows:

- $q_1$ and $q_2$ are the displacements of the center of mass in the horizontal and vertical directions respectively from its rest position $(x^{(0)}, y^{(0)})$.
- $q_3$ is the angle of rotation about the axis that goes through the center of mass orthogonally to the $(x, y)$-plane (the angle is measured from the $x$- to the $y$-axis).

Figure. A definition sketch of the cylinder cross-section $\hat{B}$ with two immersed parts, denoted by $B_-$ and $B_+$, and their wetted boundaries are $S_-$ and $S_+$ respectively. The cross-section $F$ of the free surface of the water consists of three parts; two of them lying on the $x$-axis outside $|x| > a$ are denoted by $F_\infty$ and the third one $F_0$ is between $x = -b$ and $x = +b$. Furthermore, $W = W_0 \cup W_\infty$, where $W_0$ is the part of the water domain located in the vertical strip under $F_0$ and $W_\infty$ is its complement. The equation of the horizontal bottom is $y = -h$.

We omit relations governing the time-dependent behavior (details can be found in [1]) and turn to the description of time-harmonic oscillations of the coupled water/cylinder system in the presence of oblique waves. For this purpose the following ansatz

$$ \left( \Phi(x, y, z; t), q(t) \right) = \text{Re}\left\{ \left( e^{i(kz - \omega t)} \varphi(x, y), i e^{-\omega t} z \right) \right\} $$

is applied, where $\omega > 0$ is the radian frequency of oscillations, to which the wavenumber $\nu = \omega^2 / g$ corresponds; $g > 0$ is the acceleration due to gravity that acts in the direction opposite to the $y$-axis. Furthermore, $k \in [0, \nu]$ is the prescribed wavenumber component that the wave train has parallel to the generators of the cylinder; $\varphi \in H^1_{\text{loc}}(W)$ is a complex-valued function and $z \in \mathbb{C}^3$. Thus, $k/\nu$ is the sine of the angle between the wave crests and the plane normal to the generators; waves are called oblique when $k > 0$.

To be specific, we consider the case of infinitely deep $W$ first. In the absence of incident waves, we obtain the following problem for $(\varphi, z)$:

$$ (\nabla^2 - k^2)\varphi = 0 \quad \text{in} \ W, $$

$$ \partial_y \varphi - \nu \varphi = 0 \quad \text{on} \ F, $$

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Partial derivative of \( \varphi \) with respect to \( n \) is equal to \( \omega N^T z \) on \( S \),

\[
\partial_n \varphi = \omega N^T z \quad (= \omega \sum_{j=1}^{3} N_j z_j) \quad \text{on } S, \tag{2.4}
\]

\[\nabla \varphi \rightarrow 0, \quad y \rightarrow -\infty, \tag{2.5}\]

\[
\omega^2 E z = -\omega \int_{S} \varphi N \, d s + g K z. \tag{2.6}
\]

Here, \( \nabla = (\partial_x, \partial_y) \) is the spatial gradient, whereas \( N = (N_1, N_2, N_3)^T \) (the operation \( \mathsf{T} \) transforms a vector row into a vector column and vice versa), where \( (N_1, N_2)^T = n \), \( N_3 = (x - x(0), y - y(0))^T \times n \) and \( \times \) stands for the vector product. In the equations of the body motion (2.6), the \( 3 \times 3 \) matrices are as follows:

\[
E = \begin{pmatrix} I^M & 0 & 0 \\ 0 & I^M & 0 \\ 0 & 0 & I^M \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I^D & I^D \\ 0 & I^D & I^D + I^B \end{pmatrix}. \tag{2.7}
\]

The positive entries of the mass/inertia matrix \( E \) are as follows:

\[
I^M = \rho_0^{-1} \int_B \rho(x, y) \, d x \, d y, \\
I^M_2 = \rho_0^{-1} \int_B \rho(x, y) \left[ (x - x(0))^2 + (y - y(0))^2 \right] \, d x \, d y.
\]

Here, \( \rho(x, y) \geq 0 \) is the density distribution within the body and \( \rho_0 > 0 \) is the constant density of water. On the right-hand side of the relation (2.6), we have forces and their moments: the first term is due to the hydrodynamic pressure and the second one is related to the buoyancy (see, for example, [5]). The nonzero entries of the matrix \( K \) are as follows:

\[
I^D = \int_D \, d x > 0, \quad I^D_x = \int_D (x - x(0)) \, d x, \\
I^D_{xx} = \int_D (x - x(0))^2 \, d x > 0, \quad I^B_y = \int_B (y - y(0)) \, d x \, d y.
\]

It should be noted that the matrix \( K \) is symmetric.

In (2.3), (2.4), and (2.6), \( \omega \) is a spectral parameter which is sought together with the eigenvector \( (\varphi, z) \). Since \( W \) is a Lipschitz domain and \( \varphi \in H^1_{\text{loc}}(W) \), the relations (2.2)–(2.4) are understood in the sense of the following integral identity:

\[
\int_W \nabla \varphi \nabla \psi \, d x \, d y = \nu \int_F \varphi \psi \, d x + \omega \int_S \psi N^T z \, d s. \tag{2.7}
\]

It must hold for an arbitrary smooth \( \psi \) having a compact support in \( \overline{W} \).

Together with (2.5), the following condition

\[
\int_{W \cap \{|x|=b\}} \left| \partial_{|x|} \varphi - i \ell \varphi \right|^2 \, d s = o(1), \quad b \rightarrow \infty, \quad \ell = (\nu^2 - k^2)^{1/2}, \tag{2.8}
\]
specifies the behavior of \( \varphi \) at infinity; (2.5) means that the velocity field decays with depth, whereas (2.8) yields that the potential given by formula (2.1) describes outgoing waves. This radiation condition is similar to that used in [8], where the problem, that describes water waves in two dimensions in the presence of a fixed obstacle, was considered.

The relations listed above must be augmented by the following subsidiary conditions concerning the equilibrium position:

- Archimedes’ law, \( I^M = \int_B d x \, d y \) (the mass of the displaced liquid is equal to that of the body),
- \( \int_B (x - x^{(0)}) \, d x \, d y = 0 \) (the center of buoyancy lies on the same vertical line as the center of mass),
- The matrix \( K \) is positive semi-definite; moreover, the \( 2 \times 2 \) matrix \( K' \) that stands in the lower right corner of \( K \) is positive definite (see [5]).

The last of these requirements yields the stability of the body equilibrium position which follows from the results formulated, for example, in [5, Section 2.4]. The stability is understood in the classical sense, i.e., an instantaneous infinitesimal disturbance causes the position changes which remain infinitesimal, except for purely horizontal drift, for all subsequent times.

In conclusion of this section, we note that the relations (2.5) and (2.8) must be amended in the case where \( W \) has finite depth. Namely, the no flow condition
\[
\partial_y \varphi = 0 \quad \text{on } H
\]
(2.9)
replaces (2.5), whereas \( \ell \) must be changed to \( \ell_0 \) in (2.8), where \( \ell_0 \) is the unique positive root of the equation \( \ell_0 \tanh(\ell_0 h) = \ell \).

3 Equipartition of Energy

It is known (see, for example, [9, Section 2.2.1]) that a potential, satisfying the relations (2.2), (2.3), (2.5), and (2.8) has an asymptotic representation at infinity of the same type as the Green function. Namely, if \( W \) has infinite depth, then
\[
\varphi(x, y) = A_\pm(y) e^{i \ell |x|} + r_\pm(x, y),
\]
\[
|r_\pm|^2, |\nabla r_\pm| = O([x^2 + y^2]^{-1}), \quad x^2 + y^2 \to \infty,
\]
and the following equality is valid:
\[
\ell \int_{-\infty}^{0} (|A_+(y)|^2 + |A_-(y)|^2) \, d y = -\text{Im} \int_S \varphi \nabla \varphi \cdot d s.
\]

Assuming that \( (\varphi, z) \) is a solution to the problem (2.2)–(2.6), (2.8), we rearrange the last formula using the coupling conditions (2.4) and (2.6). First, transposing the complex conjugate
of Equation (2.6), we obtain
\[ \omega^2 (EZ)^T = -\omega \int_S \varphi N^T d s + g (Kz)^T. \]

This relation and the condition (2.4) yield that the inner product of both sides with \( z \) can be written in the form
\[ \omega^2 z^T Ez - g z^T Kz = -\int_S \varphi \partial_n \varphi d s. \]

Second, substituting this equality into (3.2), we obtain
\[ \ell \int_{-\infty}^{0} (|A_+(y)|^2 + |A_-(y)|^2) \, dy = \text{Im} \left\{ \omega^2 z^T Ez - g z^T Kz \right\}. \]

In the same way as in [3], this yields the following assertion about the kinetic and potential energy of the water motion.

**Proposition 3.1.** Let \( (\varphi, z) \) be a solution to the problem (2.2)–(2.6), (2.8). Then
\[ \int_W (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, dx \, dy < \infty, \quad \nu \int_F |\varphi|^2 \, dx < \infty, \]
i.e., \( \varphi \in H^1(W) \). Moreover, the following equality is true:
\[ \int_W (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, dx \, dy + \omega^2 z^T Ez = \nu \int_F |\varphi|^2 \, dx + g z^T Kz. \] (3.3)

Here, the kinetic energy of the water/body system stands on the left-hand side, whereas we have the potential energy of this coupled motion on the right-hand side. Thus, the last formula generalizes the energy equipartition equality valid when a fixed body is immersed into water. Indeed, \( z \) vanishes for such a body, and (3.3) turns into the well-known equality (see, for example, formula (4.99) in [9]).

Proposition 3.1 shows that, if \( (\varphi, z) \) is a solution to the problem (2.2)–(2.6), (2.8) with complex-valued components, then its real and imaginary parts separately satisfy this problem. This allows us to consider \( (\varphi, z) \) as an element of the real product space \( H^1(W) \times \mathbb{R}^3 \) in what follows.

**Definition 3.1.** Let the subsidiary conditions concerning the equilibrium position (see Section 2) hold for the freely floating body \( \hat{B} \). A nontrivial real solution \( (\varphi, z) \in H^1(W) \times \mathbb{R}^3 \) to the problem (2.7), (2.6) is called a mode trapped by this body, whereas the corresponding value of \( \omega \) is referred to as a trapping frequency.

In the case of finite depth, the remainder in formula (3.1) has the following behavior uniformly in \( y \in [-h, 0] \):
\[ |r_\pm(x, y)|, \quad |\nabla r_\pm(x, y)| = O(|x|^{-1}), \quad |x| \to \infty, \]
whereas formula (3.2) is valid with \( \ell \) changed to \( \ell_0 \). Therefore, Proposition 3.1 is true for the problem (2.2)–(2.6), (2.8) with the condition (2.5) replaced by (2.9) and \( \ell \) changed to \( \ell_0 \) in (2.8). Definition 1 remains unchanged for the finite depth case.
4 On the Absence of Trapped Modes

In order to determine conditions on the domain $\hat{B}$ and on the frequency $\omega$ guaranteeing the absence of nontrivial solutions $(\varphi, z) \in H^1(W) \times \mathbb{R}^3$, we write (3.3) as follows:

$$z^T(\omega^2 E - gK)z = \nu \int_F |\varphi|^2 \, dx - \int_W |\nabla \varphi|^2 \, dx \, dy. \tag{4.1}$$

It is clear that the left-hand side is nonnegative for a nonzero $z$ provided that $\omega^2 \geq \lambda_0$; the latter is the largest $\lambda$ satisfying $\det(\lambda E - gK) = 0$. Let the equilibrium position be stable for the cylinder whose cross-section is $\hat{B}$ (see Section 2 for the conditions of stability), it is convenient to say that the property $\Omega$ is valid for $\omega$ if $\omega^2 \geq \lambda_0$. Thus, we arrive at the following.

**Proposition 4.1.** Let the cylinder cross-section be $\hat{B}$, and let the property $\Omega$ be valid for $\omega$. If the inequality

$$\nu \int_F |\varphi|^2 \, dx < \int_W \left(|\nabla \varphi|^2 + k^2 |\varphi|^2\right) \, dx \, dy, \tag{4.2}$$

is valid for every nontrivial $\varphi \in H^1(W)$ satisfying (2.7) and (2.6), then $\omega$ is not a trapping frequency.

Indeed, for a nontrivial $(\varphi, z)$ the inequality (4.2) is incompatible with the property $\Omega$.

**4.1. Cylinders with a single immersed part.** For the sake of simplicity we suppose that $W$ has infinite depth. We recall that the John condition requires a simply connected domain $\hat{B} \cap \mathbb{R}^2$ to belong to the strip lying between the verticals going through the points where the contour $\partial \hat{B}$ intersects the $x$-axis.

**Theorem 4.1.** Let $W$ have infinite depth, and let the domain $\hat{B} \cap \mathbb{R}^2$ be simply connected and satisfy the John condition. Let also the subsidiary conditions concerning the equilibrium position (see Section 2) be fulfilled. Then the inequality (4.2) is valid for a nontrivial solution $(\varphi, z) \in H^1(W) \times \mathbb{R}^3$ of the problem (2.7), (2.6).

**Proof.** Since $\hat{B} \cap \mathbb{R}^2$ is simply connected, $D$ consists of a single interval, and so the intersection of the free surface and the $(x, y)$-plane is $F = F_\infty$ (see Figure); moreover, $F = F_+ \cup F_-$, where $F_+$ ($F_-$) is the ray lying on the $x$-axis to the right (left) of $D$. Let us prove the inequality

$$\nu \int_{F_\pm} |\varphi|^2 \, dx < \int_{W_\pm} \left(|\nabla \varphi|^2 + k^2 |\varphi|^2\right) \, dx \, dy, \tag{4.3}$$

where $W_\pm \subset W$ is the subdomain lying strictly under $F_\pm$. In view of the John condition, these subdomains are well defined.

Following John, let us define

$$a^{(\pm)}(x) = \int_{-\infty}^{0} \varphi(x, y) e^{\nu y} \, dy \quad \text{on} \quad F_\pm. \tag{4.4}$$

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Differentiating this function twice and using Equation (2.2), we obtain

$$a_{xx}^{(\pm)} = k^2 a^{(\pm)} - \int_{-\infty}^{0} \varphi_{yy}(x, y) e^{\nu y} \, dy.$$  

Integrating by parts twice and taking into account the conditions (2.3) and (2.5), we conclude that

$$a_{xx}^{(\pm)} + \ell^2 a^{(\pm)} = 0 \quad \text{on } F_{\pm}$$

because $k^2 - \nu^2 = -\ell^2$.

Since $\varphi \in H^1(W)$, we have $\lim_{|x| \to \infty} a^{(\pm)}(x) = 0$, and so $a^{(\pm)} \equiv 0$ on $F_{\pm}$. Now, integrating by parts in (4.4), we see that

$$\varphi(x, 0) = \int_{-\infty}^{0} \varphi_y(x, y) e^{\nu y} \, dy, \quad (x, 0) \in F_{\pm}.$$  

Squaring both sides, applying the Schwarz inequality to the integral and integrating over $F_{\pm}$, we obtain

$$\nu \int_{F_{\pm}} |\varphi^{(\pm)}(x, 0)|^2 \, dx \leq \frac{1}{2} \int_{W_{\pm}} |\varphi_y|^2 \, dx \, dy < \int_{W_{\pm}} |\nabla \varphi|^2 \, dx \, dy;$$

here, the coefficient $1/2$ results from integration of $e^{2\nu y}$. The last inequality is stronger than (4.3), and so it implies (4.2) because $F = F_+ \cup F_-$. \hfill \square

It is clear that Propositions 3.1 and 4.1, and Theorem 4.1 yield.

**Corollary 4.1.** Let $W$ have infinite depth, and let the domain $\hat{B} \cap \mathbb{R}^2_-$ be simply connected and satisfy the John condition. Let also the subsidiary conditions concerning the equilibrium position (see Section 2) be fulfilled. If the property $\Omega$ is valid for $\omega$, then the problem (2.2)–(2.6), (2.8) has only a trivial solution for this $\omega$.

An analogous assertion is true when the water domain has a finite depth.

**4.2. Cylinders with two immersed parts.** In the case of a cylinder with two immersed parts (see Figure), we are going to apply the trick based on the John condition which was used in Subsection 4.1. This is possible if the cylinder cross-section is symmetric about a vertical axis (the $y$-axis in Figure), the density distribution $\rho$ is also symmetric within the cylinder, and each of the domains $B_+$ and $B_-$ is simply connected and satisfies the John condition, i.e., $\beta \geq \pi/2$ (see Figure). Let also the subsidiary conditions concerning the equilibrium position (see Section 2) be fulfilled. Moreover, we restrict the cylinder admissible motions to sway, heave, and roll and consider separately the classes of velocity potentials consisting of odd and even functions of $x$.

**4.2.1. Sway motion.** In this case, $z = (z_1, 0, 0)^T$, $z_1 \in \mathbb{R}$, and so the boundary condition (2.4) takes the form

$$\partial_n \varphi = \omega N_1 z_1 \quad \text{on } S$$
and the system (2.6) splits. Its first equation is as follows:

\[ \omega I^M z_1 = - \int_S \varphi N_1 \, ds, \]  

(4.5)

whereas the second and third equations turn into the orthogonality conditions:

\[ \int_S \varphi N_j \, ds = 0, \quad j = 2, 3. \]

Thus, we have to determine frequency intervals for which the equality

\[ I^M(\omega z_1)^2 = \nu \int_F |\varphi|^2 \, dx - \int_W (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, dx \, dy \]  

(4.6)

obtained from (4.1) cannot be valid for a nontrivial \( \varphi \).

Let us assume that the domains \( B_+ \) and \( B_- \) are simply connected and each satisfies the John condition, i.e., \( \beta \geq \pi/2 \) (see Figure). Repeating literally considerations used in Section 4.1, we arrive at the inequality:

\[ \nu \int_{F_\infty} |\varphi|^2 \, dx < \int_{W_\infty} (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, dx \, dy. \]  

(4.7)

It remains to estimate

\[ \nu \int_{F_0} |\varphi|^2 \, dx, \]

and for this purpose we consider the function

\[ a(x) = \int_{-\infty}^{0} \varphi(x, y) e^{\nu y} \, dy, \]  

(4.8)

which, in view of the John condition, is well defined for \( x \in F_0 \). In the same way as in Subsection 4.1, it satisfies the equation \( a_{xx} + \ell^2 a = 0 \) on \( F_0 \).

Assuming that \( \varphi(x, y) \) is odd in \( x \), we get \( a(x) = C \sin \ell x \), which we substitute into (4.8) and differentiate. After squaring both obtained equalities, we apply the Schwarz inequality to the integrals. In this way, we find that for every \( (x, 0) \in F_0 \)

\[ 2\nu C^2 \sin^2 \ell x \leq \int_{-\infty}^{0} |\varphi(x, y)|^2 \, dy, \]  

(4.9)

\[ 2\nu \ell^2 C^2 \cos^2 \ell x \leq \int_{-\infty}^{0} |\varphi_x(x, y)|^2 \, dy. \]  

(4.10)

Furthermore, integration by parts in (4.8) yields

\[ \varphi(x, 0) = -\nu C \sin \ell x + \int_{-\infty}^{0} \varphi_y(x, y) e^{\nu y} \, dy \]
for every \((x, 0) \in F_0\), and so

\[
\nu|\varphi(x, 0)|^2 \leq 2\nu^3 C^2 \sin^2 \ell x + \int_{-\infty}^{0} |\varphi_y(x, y)|^2 \, dy.
\]

Integrating this over \(F_0\) and using (4.9), we obtain

\[
\nu \int_{F_0} |\varphi(x, 0)|^2 \, dx \leq 2\nu(k^2 + \ell^2)C^2 \int_{F_0} \sin^2 \ell x \, dx + \int_{W_0} |\varphi_y|^2 \, dx \, dy \\
\leq k^2 \int_{W_0} |\varphi|^2 \, dx \, dy + \int_{W_0} |\varphi_y|^2 \, dx \, dy + 2\nu \ell^2 C^2 \int_{F_0} \sin^2 \ell x \, dx. \quad (4.11)
\]

Let us say that the property \(\Omega_\omega\) is valid for \(\omega\) if the inequalities

\[
\pi m \leq \ell b \leq \pi (2m + 1)/2
\]

are valid for some \(m = 0, 1, \ldots\); we recall that \(\ell = (\nu^2 - k^2)^{1/2}\), \(\nu = \omega^2/g\) and \(2b\) is the spacing between \(B_+\) and \(B_-\).

Since the property \(\Omega_\omega\) is equivalent to the inequality

\[
\int_{0}^{b} \sin^2 \ell x \, dx \leq \int_{0}^{b} \cos^2 \ell x \, dx,
\]

we can estimate the last term in (4.11) with the help of (4.10) provided that this property is fulfilled. In this way, we arrive at the estimate

\[
\nu \int_{F_0} |\varphi|^2 \, dx \leq \int_{W_0} (|\nabla \varphi|^2 + k^2|\varphi|^2) \, dx \, dy,
\]

which combined with (4.7) leads to a contradiction with the equality (4.6), unless \((\varphi, z)\) is trivial. This completes the proof of the following.

**Proposition 4.2.** Let \(W\) have infinite depth, and let the domain \(\hat{B}\) be symmetric about the \(y\)-axis and such that \(\hat{B} \cap \mathbb{R}^2_+\) is the union of two simply connected domains each satisfying the John condition. Let also the subsidiary conditions concerning the equilibrium position (see Section 2) be fulfilled.

If the property \(\Omega_\omega\) is valid for \(\omega\), then the problem (2.2)–(2.6), (2.8) has only a trivial solution \((\varphi, z)\) for this \(\omega\) provided that \(\varphi\) is odd in \(x\) and \(z = (z_1, 0, 0)^T\).

Let the water motion in the presence of a symmetric, freely floating cylinder (see Figure) be described by \(\varphi\) even in \(x\). Then Equation (4.5) implies that \(z_1 = 0\), which means that such a potential does not comporte with the free sway motion of a symmetric cylinder. However, there are examples of motionless symmetric cylinders with two immersed parts that trap modes at particular frequencies (see [11, Sections 5 and 6]).
Now, the equality (4.6) turns into
\[ \nu \int_{F} |\varphi|^2 \, d x = \int_{W} (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, d x \, d y, \] (4.12)
and so the uniqueness theorem proved in [11, Section 3], where the case of fixed \( B_- \) and \( B_+ \) was discussed, is applicable to the problem under consideration here. The corresponding assertion formulated below is analogous to Proposition 4.2, but involves the property \( \Omega_+ \) instead of \( \Omega_- \); namely, the property \( \Omega_+ \) is valid for \( \omega \) if the inequalities
\[ \pi(2m + 1)/2 \leq \ell b \leq \pi(m + 1), \quad \ell = (\nu^2 - k^2)^{1/2}, \quad \nu = \omega^2 / g, \]
take place for some \( m = 0, 1, \ldots \). Thus, these properties are of the same kind; either of them is valid when \( \ell b \) belongs to one interval in a sequence. These sequences for \( \Omega_+ \) and \( \Omega_- \) are complementary and their intervals have common endpoints.

**Proposition 4.3.** Let \( W \) have infinite depth, and let the domain \( \hat{B} \) satisfy the assumptions of Proposition 4.2. If the property \( \Omega_+ \) is valid for \( \omega \), then the problem (2.2)–(2.6), (2.8) has only the trivial solution \((\varphi, z)\) for this \( \omega \) provided that \( \varphi \) is even in \( x \) and \( z = (z_1, 0, 0)^T \).

Comparing Propositions 4.2 and 4.3 with the uniqueness theorem proved in [11], we see that for a cylinder with two immersed parts the frequency intervals, where the potential (either odd or even in \( x \)) is trivial, are the same irrespective whether the cylinder floats freely or is fixed and depend only on the potential parity.

**4.2.2. Heave motion.** In this case, \( z = (0, z_2, 0)^T, \ z_2 \in \mathbb{R}, \) and so the boundary condition (2.4) takes the form
\[ \partial_n \varphi = \omega N_2 z_2 \text{ on } S, \] (4.13)
and the system (2.6) splits. Its second equation is as follows:
\[ (\omega^2 I^M - g I^D) z_2 = -\omega \int_S \varphi N_2 \, d s, \] (4.14)
whereas the first and third equations turn into the orthogonality conditions
\[ \int_S \varphi N_1 \, d s = 0, \quad \int_S \varphi N_3 \, d s = 0. \]
The last one is a consequence of the equality \( I^D_x = 0 \) which follows from the symmetry about the \( y \)-axis. Indeed, \( x^{(0)} = 0 \) in this case. Furthermore, the relation (4.1) takes the form
\[ (\omega^2 I^M - g I^D) z_2^2 = \nu \int_F |\varphi|^2 \, d x - \int_W (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, d x \, d y. \] (4.15)

Assuming that \( \varphi(x, y) \) is odd in \( x \), we see that the right-hand side of (4.14) vanishes because the integrand has the same parity as \( \varphi \). Then the last equality turns into (4.12), and so we are in a position to refer to the considerations used in [11, Section 3]. In this way, we arrive at the following assertion (it is similar to Propositions 4.2 and 4.3) for the present problem.
Proposition 4.4. Let $W$ have infinite depth, and let the domain $\hat{B}$ satisfy the assumptions of Proposition 4.2. If the property $\Omega_-$ is valid for $\omega$, then the problem (2.2)–(2.6), (2.8) has only a trivial solution $(\varphi, z)$ for this $\omega$ provided that $\varphi$ is odd in $x$ and $z = (0, z_2, 0)^T$.

Proof. According to considerations in [11, Section 3], the property $\Omega_-$ guarantees that $\varphi$ (satisfying (4.12) and odd in $x$) is trivial. Then the boundary condition (4.13) yields that $z$ is also trivial.

Let us turn to the case where $\varphi(x, y)$ is even in $x$ and apply the approach used in the proof of Proposition 4.2 for determining the frequency intervals when the equality (4.15) cannot hold for a nontrivial $\varphi$. Considerations based on the property $\Omega_+$ show that the right-hand side of (4.15) is strictly negative when this property is valid. On the other hand, the left-hand side of this equality is nonnegative if $\omega^2 \geq gI^D/I^M$. This yields the following.

Proposition 4.5. Let $W$ have infinite depth, and let the domain $\hat{B}$ satisfy the assumptions of Proposition 4.2. If the property $\Omega_+$ is valid for $\omega \geq \sqrt{gI^D/I^M}$, then the problem (2.2)–(2.6), (2.8) has only a trivial solution $(\varphi, z)$ for this $\omega$ provided that $\varphi$ is even in $x$ and $z = (0, z_2, 0)^T$.

The essential distinction between this assertion and Proposition 4.4 is as follows. Along with the property $\Omega_+$, the assumptions of Proposition 4.5 include the inequality $\omega^2 \geq gI^D/I^M$, whereas no condition other than the property $\Omega_-$ is imposed in Proposition 4.4.

4.2.3. Roll motion. In this case, $z = (0, 0, z_3)^T$, $z_3 \in \mathbb{R}$, and so the boundary condition (2.4) takes the form
\[
\partial_n \varphi = \omega N_3 z_3 \text{ on } S,
\]and the system (2.6) splits. The first and second equations turn into the orthogonality conditions
\[
\int_S \varphi N_1 \, ds = 0, \quad \int_S \varphi N_2 \, ds = 0.
\]

Again, the last one is a consequence of the equality $I_x^D = 0$ which follows from symmetry of $\hat{B}$ about the $y$-axis. The third equation is as follows:
\[
[\omega^2 I_2^M - g(I_{xx}^D + I_y^B)] z_3 = -\omega \int_S \varphi N_3 \, ds,
\]
where $N_3(x, y) = x N_2 - (y - y^{(0)}) N_1$ and $N_3(x, y) = -N_3(-x, y)$ for $(\pm x, y) \in S_\pm$ in view of symmetry. Now, the relation (4.1) takes the form:
\[
[\omega^2 I_2^M - g(I_{xx}^D + I_y^B)] z_3^2 = \nu \int_F |\varphi|^2 \, dx - \int_W (|\nabla \varphi|^2 + k^2 |\varphi|^2) \, dx \, dy.
\]

Assuming that $\varphi(x, y)$ is even in $x$, we see that the right-hand side of (4.16) vanishes (cf. Section 4.2.2), and the last equality turns into (4.12). Again we are in a position to refer to the considerations used in [11, Section 3], which lead to.

Proposition 4.6. Let $W$ have infinite depth, and let the domain $\hat{B}$ satisfy the assumptions of Proposition 4.2. If the property $\Omega_+$ is valid for $\omega$, then the problem (2.2)–(2.6), (2.8) has only a trivial solution $(\varphi, z)$ for this $\omega$ provided that $\varphi$ is even in $x$ and $z = (0, 0, z_3)^T$. 104
Comparing this assertion and Proposition 4.3, we see that for a freely floating symmetric cylinder with two immersed parts the property $\Omega_+$ guarantees uniqueness at all frequencies satisfying it provided that $\varphi$ is even in $x$ and the cylinder executes either sway or roll motion.

In the case where $\varphi(x,y)$ is odd in $x$, we again apply the approach used for the proof of Proposition 4.5. Since $I^D_{xx} = 0$ in view of symmetry of $\hat{B}$, the assumption that $K'$ is a positive definite matrix (see Section 2) yields the inequality $I^D_{xx} + I^B_y > 0$. Thus, we obtain the following assertion.

**Proposition 4.7.** Let $W$ have infinite depth, and let the domain $\hat{B}$ satisfy the assumptions of Proposition 4.2. If the property $\Omega_-$ is valid for

$$\omega \geq \sqrt{g(I^D_{xx} + I^B_y)/I^M_2},$$

then the problem (2.2)–(2.6), (2.8) has only a trivial solution $(\varphi, z)$ for this $\omega$ provided that $\varphi$ is odd in $x$ and $z = (0, 0, z_3)^T$.

As in the case of Propositions 4.4 and 4.5, the distinction between Propositions 4.6 and 4.7 is that the latter one includes the extra inequality (4.17).

In conclusion of this section, it should be mentioned that there are analogues of Propositions 4.2–4.7 for the water having a finite depth. Their formulations are similar to those above.

## 5 Discussion

First, the obtained results guarantee the uniqueness of a solution in proper classes of functions for the problem of scattering obliquely incoming plane waves by an infinitely long, freely floating cylinder. This problem is left for future research.

Second, equations (4.14) and (4.16) demonstrate that one may expect the existence of trapped modes for some particular bodies in heave and roll motion. To find these bodies and the corresponding trapped modes is another interesting problem for future research. A hint to its solution (at least for $k = 0$) can be found in the article [4].

## Declarations

**Data availability** This manuscript has no associated data.

**Ethical Conduct** Not applicable.

**Conflicts of interest** The author declares that there is no conflict of interest.

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