Compositional Thinking in Cyberphysical Systems Theory

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Abstract—Engineering safe and secure cyber-physical systems requires system engineers to develop and maintain a number of model views, both dynamic and static, which can be seen as algebras. We posit that verifying the composition of requirement, behavioral, and architectural models using category theory gives rise to a strictly compositional interpretation of cyber-physical systems theory, which can assist in the modeling and analysis of safety-critical cyber-physical systems.

**APPLIED COMPOSITIONAL THINKING**

Lee [2], among others, recognized early in the development of the field of cyber-physical systems that there is a need for developing competing methods to hybrid systems and process algebras. While this is true, an important observation is that both these formalisms form algebras. In fact, the design of cyber-physical systems involves the study of different algebras (Figure 1).

There is significant research in developing these individual algebras and implementing composition within a particular algebra. However, there is still an open problem about how to relate those paradigms that in practice represent individual models and to examine the behavior of the system as a whole must be composed too. Compositional cyber-physical systems theory [3], [4] uses category theory to transform data from one algebra to another and to ultimately relate them formally, such that we can compose across domains. This provides one solution to the open problem of composition between formal methods and their corresponding model views in cyber-physical system design [5], [6].

The study of compositionality is not new and certainly not only possible with categories. For example, compositionality is formalized and addressed with tools from control theory [7], contract-based design [8], or monotone co-design [9]. This is to say that we recognize that the cyber-physical systems field has had a long tradition in formalization for the study of correctness and the management of design complexity. But the idea of category theory in system science and formal methods is that those approaches may be seen as operating implicitly within a category and making that explicit can be of use in developing scalable and general purpose modeling tools that can operate across categories.

Recently all these three areas of control [10], contracts [4], and co-design [11, chapter 4] have been described, generalized, and unified with fun-
result is obtained entirely by a different methodalogies, and mainly by formalizing the notion of functoriality – structure preserving maps between categories that can translate a general syntax to the particular semantics of the application domain. This means that we need not reconstruct the progress that has occurred in each individual field or model view because we can use the algebras already in existence, but with the added benefit that we can compose between categories and, therefore, between model views.

Applied compositional thinking brings forward the practical dissemination of categorical results in the field of engineering in general. We will particularly concern ourselves with how research directions in applied category theory can be leveraged in system science and formal methods. Composition can mean different things both depending on the engineering field as well as the particular context in which composition is studied. In cyber-physical systems the term composition can be understood as either horizontal or vertical.

Horizontal composition is relatively well studied and refers to things like the block diagram al-
gebra widely used in the control community. Vertical composition, in our definition, instead spans different domains; a clear example in the application of systems modeling and formal methods is how components compose and are abstracted or refined between requirements, behaviors, and architectures. In order to do this right we need to formally relate and verify the composition of different types of algebras present in the design and analysis of cyber-physical systems (the leaf nodes of Figure 1). At the moment, engineering researchers and practitioners attempt to address this issue by developing naming spaces and conventions. The need to capture the relationships between heterogeneous system representations in contract-based design has led to the notion of vertical contracts [12]. The congruence between our framework and contract-based design is that contracts in both senses are extensions of behavior types. Across paradigms, as long as types agree, composition is possible.

Each of the individual formalisms or algebras used in the design of cyber-physical systems have had whole fields dedicated to them with steady research progress before and after the term cyber-physical systems came to existence. However, the relationship between those algebras at the moment is either ad-hoc or quasi-formal. Relating algebras formally can lead to new insights, methods, and tools for the design of cyber-physical systems that operate as we expect.

Specifically, category theory is one mathematical tool that can verify the composition of differing views and therefore lead to another dynamical computation systems theory founded on the recognition that addressing composition is important in understand the (mis-)behavior of the system as a whole. Category theory is the study of mathematical structures from the perspective that, to better determine an object’s purpose and behavior, we ought to study its relationship with other objects instead of examining the object only in itself. Categories are intuitively congruent with engineering cyber-physical systems because of the existence of both dynamics and computation in those systems, which are modeled through a multitude of perspectives that need to be related.

Formal methods have taken an increasingly important role in the design of systems both in academic [13] and industry [14] settings. In system modeling, the SysML V2 standard is a glimpse of the possible future of formal and compositional methods in a practice where the gap between requirement, behavioral, and structural formalisms is bridged. However, it is not clear how this relationship will be formally defined. This shows an increasing need for formal and compositional methods that verify the composition and scale system models.

While there are several applications of category theory in engineering, in this paper we will examine precisely this gap between requirements, behaviors, and architectures from the eye of compositional cyber-physical systems theory. This is possible because composition here fundamentally gives us modularity and interoperability within and across different types of models for free, provided we model things within the stricter paradigm of categories and algebras. Specifically, we will show how horizontal and vertical composition can be expressed as a formal method using categories.

### Compositional Cyber-Physical Systems Theory

Engineers and system designers use expertise, intuition and inertia to decide how different analyses factor into higher-level, harder questions about safety, resiliency, profitability and risk. We can formalize (some of) that expertise by reconstructing existing workflows and best practices in categorical terms. Though many individual features of cyber-physical systems are formal, the relationship between these pieces has not yet been mapped. In general, putting systems together requires two things per Kalman [15]:

1) Getting the physics right.

2) The rest is mathematics.

But that sparked Willems [15] to ask if we are using the right mathematics to describe the science of interconnection and he concludes that we need to represent systems as algebras. Category theory goes even further by composing those algebras together.

Instead of describing category theory in full, we want the reader to focus on the categorical toolkit and refer to the many formal treatments on the topic, including books such as Fong and Spivak [11]. However, we will give an intuitive
explanation of the fundamental concepts of category theory.

The most general notion of a category is that of an algebraic structure on a graph. A category contains a collection of objects and a collection of arrows (including an identity arrow) which can formally be joined by a composition rule. Examples of useful categories include the category of sets and functions, of graphs and graph transformations, and of states and transitions between them.

The notion of functoriality between syntax and semantics we talked about previously is implemented through a structure preserving map between categories. Functors are a predefined way of transforming some data and their associated interconnections to another form (figure 2). For example, a functor can be taking the powerset of a set, where we have predefined operations on how to do that between the same category, namely the category of sets. We can think of a functor in the application of cyber-physical system design as preserving semantics across the currently distinct algebras that define different interpretations of requirements, system behaviors, and system architectures, meaning that they assign a particular algebra to a specific model view.

Often we would like to model operations happening in parallel when speaking about cyber-physical systems. The constrained definition of a category is capable of modeling sequential processes but not parallel ones. A particularly useful type of category in system science is one that is monoidal. Monoidal categories extend this definition precisely to equip with a tensor product functor, which models processes and/or operations that need to happen in parallel.

As a representation of some data in a category, see below a commutative diagram involving objects $X$, $Y$, and $Z$ identity morphism on objects, $\text{id}$, as well as two morphisms $f$ and $g$ along with their composition rule $g \circ f$.

![Commutative Diagram](a commutative diagram involving objects $X$, $Y$, and $Z$ identity morphism on objects, $\text{id}$, as well as two morphisms $f$ and $g$ along with their composition rule $g \circ f$.)

The objects and morphisms can take several forms, the most familiar of which is perhaps the category of sets and functions, where objects are sets and morphisms are functions and composition is function composition. The example of a category we will use to show the utility of category theory in engineering is the labeled boxes and wiring diagrams category [16], where the objects are empty placeholders for processes, and the morphisms are functions or relations that construct input and output interfaces for each labeled box.

By having the mathematical structure of a category it is now possible to translate between model views as long as they form categories and their behavior can be represented as an algebra. This is achieved by defining a precise way, that can also be implemented algorithmically, of translating the objects from one category to another (Figure 2a) and similarly for the morphisms (Figure 2b). An example of a functorial operation
between categories of sets is transforming a set to its power set. A monoidal functor comes with comparison morphisms that involve the tensor product, allowing us to use the same idea in the setting of parallel operations. This might seem like new and confusing jargon for those unfamiliar with category theory but once it is understood it allows us to summarize a vast number of mathematical techniques as well as formal methods and system models by allowing us to assign algebras that contain parallel processes.

We use these algebraic structures to describe the syntax and semantics for cyber-physical systems compositionally. We show how a wiring diagram has an architecture; that is, how interfaces are constructed, and a behavior; that is, how the notion of functorial semantics takes the architecture – the syntax – and assigns multiple semantics, in this case of linear time-invariant system models.

In other cases we could use the same structures to assign other models of cyber-physical system behavior, such as labelled transitioned systems or Moore machines. Hierarchical decompositions are also necessary to construct the desired vertical composition structure. To achieves this, we are going to use the slice category of a wiring diagram to decompose the model to a particular system architecture – the hardware composition of the embedded system that implements the behaviors we have assigned, particularly for a model of an unmanned aerial vehicle (UAV). Finally, we will restrict the behavior of the system using a categorical description of contract-based design for behavior. In this respect, we regard contracts as representations of the system requirements. However, note that those requirements operate both as restrictions in behavioral and architectural system models.

The general principle of applied compositional thinking is the following.

1) Wiring diagrams define how things ought to be connected and in what way.
2) Algebras assign a particular formalism or behavior to the empty processes (boxes) inhabiting the wiring diagram.

Wiring diagrams are the theory of the class

\[ \phi_{\text{in}} : \mathbb{R}^3 \times \mathbb{R}^2 \to \mathbb{R}^5, (s', c, s, e, d) \mapsto (s, e, s', d, c) \]

\[ \phi_{\text{out}} : \mathbb{R}^3 \to \mathbb{R}, (s', c, s) \mapsto s \]

Figure 3: A compositional model of an unmanned aerial vehicle (UAV) requires us to add extra data. The functions \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) create the interfaces of expected types of inputs and outputs of each box and then combine them. They are in a sense the architecture of the wiring diagram. In this case \( s' \) represents the calculated state, \( s \) the current state, \( e \) the environment, \( d \) the desired state, and \( c \) the control action. This operation does not tell us what processes actually inhabit the boxes, an additional step is needed by functorially assigning behavior algebras, in this case the algebra of linear time-invariant systems \( \mathcal{B} \) to each box, parallelizing the operation, and then completely determining the operation of the overall box UAV, as in \( \mathcal{B}(L) \times \mathcal{B}(C) \times \mathcal{B}(D) \xrightarrow{\phi_{L,C,D}} \mathcal{B}(L \otimes C \otimes D) \xrightarrow{\mathcal{B}(f)} \mathcal{B}(\text{UAV}) \).

Both the left and right pictures form wiring diagrams. The left representation is better suited for incorporating this diagrammatic reasoning within modeling tools.
of systems we want to model but only insofar as they are assigned a behavioral model through an algebra (monoidal functor). The implication of this statement is that for each wiring diagram, we can define many algebras and, therefore, can capture – in the same arrangement of components – a number of formalisms as well as translate between formalisms. Additionally, any algebra is a monoidal functor from the category of wiring diagrams to the category of sets. To deal with the disparate leaf nodes (Figure 1) that are needed for the design of a complete cyber-physical system it is useful to work in higher levels of abstraction, for example, topological spaces, graphs, vector bundles, but it is crucial to translate to a realizable design, which usually means that even if we do not do the original operations in the category of sets, the result should be translatable to the category of sets.

**Horizontal Composition**

As an example we model an unmanned aerial vehicle compositionally (Figure 3). There are two observations stemming from our UAV example:

1) The functions $\phi_{\text{in}}$ and $\phi_{\text{out}}$ create the interfaces of input-output relationships that completely determine the block UAV.

2) But by knowing the types and values expected at interfaces we do not really know how the system transforms this type of data. We, therefore, need an algebra $\mathcal{B}$ that assigns precisely the behavior of linear time-invariant systems.

A result of doing this categorically is the realization that extra information is often necessary, which is often omitted or glossed over by other composition operators and methods. In order for the system to compose nicely it is paramount that the type of system residing within all boxes in a wiring diagram are either the same, for example, Moore machines inhabit all boxes, or that we know the results of the composition of two types, for example, ordinary differential equations and Moore machines.

This means that in our example, all boxes must take the form of a linear time-invariant system of the standard form, even though in the block diagram algebra used in control describes the totality of the control system behavior. Adding this extra information is important but the solution is straightforward: reduce sensor and controller to trivial functions because they will compose to the form of the linear time-invariant system representing the dynamics we expect (Figure 3). The main benefit of implementing this algebra machinery is that it uniquely determines the description of the composite system in terms of its subsystems and their interconnections.

One way of assigning requirements to this behavior is to translate them into contracts [17], [18]. As with behavior using the algebra of contract we can first think of contracts as a relation of inputs and outputs of each box in the wiring diagram.

Then to each complex arrangement of boxes assign the following pullback, which given subsystems contracts produces a composite contract of the system model.

For a restriction in each subsystem in the composite we can represent that restriction as a set of allowable pairs that can be observed as inputs and outputs and from this formulation we will always be able to calculate the allowable input/output pairs for the composite system in terms of the specific wiring given by the architecture of the wiring diagram.

Horizontal composition is defined as the interconnection of operations in one model type. The two step process is to use the theory of interconnection to define the architecture of the base wiring diagram and then inhabit the boxes of this arrangement with a particular behavioral model [4]. This brings our compositional model into the same realm as behavioral diagrams in SysML but with the possibility of simulated behavior as in MATLAB Simulink. The main benefit of this approach is that we can assign a number of algebras by describing models in categories. In this case, we assigned part of our
Figure 4: The architecture of the system resides within the slice category of our initial system behavior diagram (Figure 3) producing a formal trace between the two, thereby allowing us to perform more analyses formally. While this decomposition focuses on the one possible implementation of the embedded system aspects, this is only because it was modeled by a computer engineer. An aerospace engineer could augment this model with further decompositions of the airframe or motors as long as they got the physics right.

Vertical Decomposition

There are cases where the behavioral interpretation of the system is not sufficient to examine certain properties about a system. One such case is security, where one approach to finding vulnerabilities is to traverse a graph of the expected or implemented system architecture [19]. However, those approaches are static and often are behaviorally unaware, meaning that there is no way to know what behavior is being affected by a particular successful exploit. This is not necessarily an issue in information technology (IT) systems but in the case of cyber-physical systems, where the dynamics are richer it is necessary to know what physical behaviors the exploit might affect.

One way of understanding what a slice category is in a given wiring diagram is to think about it as splitting. We split or open up morphisms (the arrows) within the wiring diagram category in order to allow a higher degree of refinement. We can think of this as working with hyper graphs but with the extra structure of a category. Visually this morphism split is opening up the model of system behavioral (Figure 3) to a model of system architecture (Figure 4). The splitting is a form of refinement, but by using the slice category it forms a formal trace between behaviors and architectures.

Vertical decomposition not only refers to decomposing system models, but at the same time it can mean the relation of different algebras, in this case, for example, the algebra of behavior with the algebra of contracts. A similar argument could be made for applying contracts to the eventual architecture and its associated algebra (with its own corresponding architecture and behavior in the wiring diagram representation).

To understand the disjointedness of the current modeling process we can think about how we might mathematically assign behaviors and restrictions to an empty process. To the same wiring diagram (a collection of empty processes and wired between them) we can assign different formalisms. This is the reason for studying empty processes, namely that it gives us a way of decoupling syntax from semantics.
This represents the way a system designers might approach developing contracts over an already known behavior. We would instead prefer a more formal relationship of these two algebras. In the categorical case we have been explicit in assigning behavior and contracts in the form of algebras. This allows us to use a natural transformation — a map between functors — to reflect changes from the behavioral paradigm to the contracts paradigm, which can be seen as the relationship between (distinct) algebraic representations, precisely as the behavior algebra \((B)\) and contracts algebra \((C)\) in our work.

Assume that we assign to an empty process a particular formalism of behavior and contracts in the form of algebras. But, then, as it so often happens within design a change must occur in the behavior, a range of outputs from one box has changed. By having a clearly defined natural transformation — that moreover behaves well with the parallel operation — it is possible to reflect what that means in the contracts we have defined for the same system model. By using the above relationship between functors; that is, the natural transformation, we are doing precisely that.

The same argument has also implications for scalability. Knowing when a change in one modeling paradigm causes something else to be changed, augmented, or rethought within a modeling tool could reduce errors and increase efficiency in the system modeling process. However, while we can make a clear mathematical argument for verified composition, scalability requires that we build the tools and test our assumptions with practitioners in the field.

The promise of category theory is not that it will change the mathematics of a particular application field but, rather, that it will organize the information such that there might be new insights and techniques associated with translating between data (verified composition) and managing complexity (scalability).

**Category Theory for the Engineer**

Compositionality is one of the open problems in cyber-physical system design, as is argued in the NIST cyber-physical system framework [5, pages 11-12]. In general, we perceive that categorical models in engineering are currently rather unfledged and category theory is a ripe field with developed mathematical tools that can be transitioned to formal methods in the design of cyber-physical systems. The main benefits of applied compositional thinking is verified composition and scalability of different system models.

As practitioners of formal methods, we would like to know if our contracts or linear temporal specifications agree with our linear time-invariant system or labelled transition state space models. Therefore, the problem in cyber-physical systems is not only contained on which algebra to use to examine or assure a particular property of the system but also how to relate those algebras. The importance of this perspective is that algebras, even implicitly, exist at any given level of system design and their composition gives rise to well-defined traces in models.

Further, as we move throughout the lifecycle, system architecture models must be developed and tested for behavioral bisimilarity with the
models we construct in the early concept design phase, for example, to assure that our safety analysis holds. Additionally, we might at that point want to check qualities that heavily depend upon the composition of the system in hardware and software, such as security [20]. Additionally, even later in the lifecycle we want to test our models with data harvested from the designed and deployed system. Here too, the same algebraic structures can be extended by this data gathered in the field.

One way to partially combat the increasing but necessary complexity of those systems we need to make sure that all these algebras are traceable. By being traceable we are therefore able to apply several lightweight formal methods within one overarching model. The recent development of algebraic software based on category theory can assist us in developing modeling tools. By developing compositional modeling tools we can manage and make formal all diverse views for design, thereby bringing to practice all this theoretical work.

REFERENCES

1. P. Asare, G. Bakirtzis, R. Bernard, D. Broman, E. Lee, G. Prinsloo, M. Torngren, and S. Sunder, “Cyber-physical systems – a concept map,” URL https://cyberphysicalsystems.org or https://perma.cc/NQ6T-AVH5, 2020.

2. E. A. Lee, “Cyber-physical systems – Are computing foundations adequate,” Position paper for NSF workshop on cyber-physical systems: Research motivation, techniques and roadmap, 2006.

3. G. Bakirtzis, C. Vasilakopoulou, and C. H. Fleming, “Compositional cyber-physical systems modeling,” in Proceedings of the 2020 Applied Category Theory Conference (ACT 2020), ser. Electronic Proceedings in Theoretical Computer Science. Open Publishing Association, 2020.

4. G. Bakirtzis, C. H. Fleming, and C. Vasilakopoulou, “Categorical semantics of cyber-physical systems theory,” ACM Transactions on Cyber-Physical Systems, 2021.

5. E. R. Griffon, C. Greer, D. A. Wollman, and M. J. Burns, “Framework for cyber-physical systems: Volume 1, overview,” NIST, Special Publication (NIST SP) - 1500-201, 2017.

6. J. B. Michael, G. W. Dinolt, and D. Drusinsky, “Open questions in formal methods,” Computer, 2020.

7. R. Alur, S. Moarref, and U. Topcu, “Compositional and symbolic synthesis of reactive controllers for multi-agent systems,” Information and Computation, 2018.

8. P. Nuzzo, A. Sangiovanni-Vincentelli, D. Bresolin, L. Geretti, and T. Villa, “A platform-based design methodology with contracts and related tools for the design of cyber-physical systems,” Proceedings of the IEEE, vol. 103, no. 11, Nov. 2015.

9. A. Censi, “Uncertainty in monotone codeign problems,” IEEE Robotics and Automation Letters, 2017.

10. J. Culbertson, P. Gustafson, D. E. Koditschek, and P. F. Stiller, “Formal composition of hybrid systems,” Theory and Applications of Categories, 2019.

11. B. Fong and D. I. Spivak, An invitation to applied category theory: seven sketches in compositionality. Cambridge University Press, 2019.

12. P. Nuzzo and A. L. Sangiovanni-Vincentelli, “Hierarchical system design with vertical contracts,” in Principles of Modeling. Springer, 2018.

13. M. Sirjani, E. A. Lee, and E. Khamespanah, “Verification of cyberphysical systems,” Mathematics, 2020.

14. C. Newcombe, T. Rath, F. Zhang, B. Munteanu, M. Brooker, and M. Deardeuff, “Use of formal methods at Amazon web services,” 2014. [Online]. Available: http://research.microsoft.com/en-us/um/people/la.mp/formal-methods-amazon.pdf

15. J. C. Willems, “The behavioral approach to open and interconnected systems,” IEEE Control Systems Magazine, 2007.

16. P. Schultz, D. I. Spivak, and C. Vasilakopoulou, “Dynamical systems and sheaves,” Applied Categorical Structures, 2020.

17. A. Benveniste, B. Caillaud, D. Nickovic, R. Passerone, J. Raclet, P. Reinkemeier, A. L. Sangiovanni-Vincentelli, W. Damm, T. A. Henzinger, and K. G. Larsen, “Contracts for system design,” Foundations and Trends in Electronic Design Automation, 2018.

18. I. Filippidis and R. M. Murray, “Layering assume-guarantee contracts for hierarchical system design,” Proceedings of the IEEE, 2018.

19. G. Bakirtzis, B. J. Simon, A. G. Collins, C. H. Fleming, and C. R. Elks, “Data-driven vulnerability exploration for design phase system analysis,” IEEE Systems Journal, 2020.

20. G. Bakirtzis, F. Genovese, and C. H. Fleming, “Yoneda hacking: The algebra of attacker actions,” 2021.