Defects in the 3-dimensional toric code model form a braided monoidal 2-category

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Abstract

It was well known that there are $e$-particles and $m$-strings in the 3-dimensional (spatial dimension) toric code model, which realizes the 3-dimensional $\mathbb{Z}_2$ topological order. Recent mathematical result, however, shows that there are additional string-like topological defects in the 3-dimensional $\mathbb{Z}_2$ topological order. In this work, we construct all the missing string-like topological defects and all higher codimensional defects, and show that all topological defects form a braided monoidal semisimple 2-category satisfying a braiding non-degeneracy condition.

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1 Introduction

In recent years, we have witnessed a surging interaction between the theory of topological orders in condensed matter physics and the category theory in mathematics. This interaction is mainly due to the fact that a topological order can be described by its topological defects (or excitations) up to invertible topological orders (see for examples [Kit03, KW14]), and these topological defects form a higher category. This work further strengthens this interaction by emphasizing the appearance of a braided monoidal 2-category in a concrete lattice model: the 3d toric code model. Throughout this work, we use \( n_d \) to represent the spatial dimension and \( n_D \) to represent the spacetime dimension.

It was long postulated that the fusion and braiding properties of quasi-particles in a 2d topological order can be precisely formulated in terms of a non-degenerate braided fusion 1-category (see [Kit06, Appendix E] for a review). The demonstration of this idea through explicit lattice models came later [Kit03, LW05, KK12, LW14]. For 3d topological orders, it was also expected that topological defects should form a non-degenerate braided fusion 2-category (see for example [KW14, KWZ15, EN17, LKW18, KTZ20]). Generalizing the ideas in [KK12, LW14], in a lattice model realization of Dijkgraaf-Witten theories, Bullivant and Delcamp classified topological defects as modules over certain categorified tube algebras in [BD19, BD20]. These lattice models suffer the same problem as those 2d Levin-Wen type of models [LW05, KK12] of being very complex and far away from any physically realizable models. Moreover, the topological defects are realized mathematically as the representations as in [KK12] instead of physically in the lattice model. Therefore, it is highly desirable to demonstrate the postulated relation between the topological defects and 2-categories in a simple physical lattice model with all topological defects explicitly constructed.

In this work, in a very concrete and explicit way, we demonstrate the idea that topological defects form a non-degenerate braided monoidal 2-category in the simplest case: the 3d toric code model [HZW05], which realizes the 3d \( \mathbb{Z}_2 \) topological order. It was well known that there are \( e \)-particles and \( m \)-strings in the 3d toric code model. On the other hand, it was predicted that the topological defects in the 3d \( \mathbb{Z}_2 \) topological order can be described by the braided monoidal 2-category \( Z(2\text{Vec}_{\mathbb{Z}_2}) \), i.e. the center of the monoidal 2-category \( 2\text{Vec}_{\mathbb{Z}_2} \), see [EN17, LKW18, KTZ20]. Recent mathematical computation of \( Z(2\text{Vec}_{\mathbb{Z}_2}) \) [KTZ20, Example 3.8], however, suggests that there are additional string-like topological defects: \( 1_c \)-strings and \( m_c \)-strings in the 3d \( \mathbb{Z}_2 \) topological order. The \( 1_c \)-strings have already been constructed by Else and Nayak in [EN17, Section III] under the name of “Cheshire charged loops”. It turns out that the \( 1_c \)-strings can be obtained from condensing particles on the trivial strings, thus are condensation descendants. Similarly, the \( m_c \)-strings can be obtained from condensing particles on the \( m \)-strings.

In this work, we explain why we work with strings instead of loops (see Section 2.1) and why we should include above condensation descendants in our categorical description of the 3d \( \mathbb{Z}_2 \) topological order (see Section 4.2). Moreover, we show that all strings (including the trivial one) and 0d domain walls between strings form a braided monoidal semisimple 2-category with non-degenerate double braidings. This braided monoidal 2-category coincides
precisely with \( \mathcal{Z}(2\text{Vec}_Z) \) as predicted in \cite{EN17,LKW18,KTZ20}. We also show how to compute loops and links from our categorical description of strings (see Section 4.5).

We briefly explain the layout of this paper. In Section 2.1 we provide a general categorical description of topological defects in a 3d topological order. In Section 2.2 we show that the braiding of two strings, which is normally defined by a 1+0D defect in spacetime, can be deformed into a 0+1D defect that can be conveniently computed in the 3d toric code model. In Section 3 we briefly review the 3d toric code model. In Section 4 we explain that the topological defects in the 3d toric code model form a semisimple monoidal 2-category \( \mathcal{T}_C \). We also compute the double braidings in Section 4.4 but leave the study of the braiding structures to Section 5. In Section 4.5 we compute the topological invariants associated to three loop-like and link-like defects. The braiding structure on \( \mathcal{T}_C \) is gauge dependent. A gauge can be fixed by selecting a gapped boundary of the 3d toric code model then determining the braiding via the half-braidings of the defects on the boundary. In Section 5.1 we construct two gapped boundaries of the 3d toric code model: the smooth boundary and the rough boundary. In Section 5.2 we compute physically the half-braidings on the rough boundary and compare our result with the half-braidings computed mathematically in \cite{KTZ20}. In Appendix A we review the mathematical results of the fusion and braidings of \( \mathcal{Z}(2\text{Vec}_Z) \) in \cite{KTZ20}.

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2 3+1D topological orders and braided fusion 2-categories

2.1 Categorical description of topological defects

An \( n+1 \)D topological order (defined on an open \( n+1 \)-disk) is characterized by its topological defects up to invertible topological orders. Topological defects are distinguished by their spacetime dimensions and their types. An adiabatic move and a deformation (in spacetime) of a topological defect do not change its spacetime dimension nor its type.

In a 3+1D topological order, all types of topological defects of codimension 2 and higher form a structure called a braided monoidal 2-category \( \mathcal{C} \). A 2-category consists of objects, 1-morphisms between objects and 2-morphisms between 1-morphisms. Both 1-morphisms and 2-morphisms can be composed, and there are many coherent data and conditions. The braidings and monoidal structures are additional structures on a 2-category. We explain the physical meanings of all ingredients of \( \mathcal{C} \) in details below.

- 2-category structures:
  - The objects are the types of 2D (spacetime dimension) topological defects. A 2D defect in spacetime can be either the 1+1D world sheet of a string-like (1d spatial dimension) topological defects, or a 2+0D defect, which can be rotated in spacetime to become the 1+1D world sheet of a string-like topological defect. Therefore, we can also say that an object is a type of string-like topological defects, or a string for simplicity.

\(^2\)The physical interpretation of \( \mathcal{Z}(2\text{Vec}_Z) \) in \cite{BD20} Section 6.9 differs from that in \cite{KTZ20}, which slightly differs from that in \cite{EN17,LKW18} in their interpretations of the objects as either strings or loops (see Section 2.1).
The 1-morphisms from an object \( X \) to an object \( Y \) are 1D domain walls between two 2D defects \( X \) and \( Y \). Such a 1D domain wall can be either the world line of a particle-like (0d spatial dimension) topological defect or a 1+0D defect, which can be rotated to become the world line of a 0d domain wall between two strings. For example, consider a 0d domain wall \( f \) between two strings \( P, Q \) as illustrated in the following picture.

The 1D defect \( f \) can be viewed as either a 0+1D defect or a 1+0D defect.

The 2-morphisms are 0D (spacetime dimension) defects, which is also called instantons.

The instantons can be fused in spacetime. Thus for any two string-like topological defects \( X, Y \), the 0d domain walls between them and associated instantons form a 1-category, denoted by \( \text{Hom}_c(X,Y) \).

The 0d domain walls can be fused along string-like topological defects. Thus for any string-like topological defects \( X, Y \), both \( \text{Hom}_c(X, X) \) and \( \text{Hom}_c(Y, Y) \) are multi-fusion categories, and \( \text{Hom}_c(X, Y) \) is a \((\text{Hom}_c(Y, Y), \text{Hom}_c(X, X))\)-bimodule. This coincides with the intuition that a string-like topological defect can be viewed as a (potentially anomalous) 1d topological order.

- **Monoidal structures:** Two parallel string-like topological defects can be fused in the 3d space. This equips the 2-category of topological defects with a monoidal structure. The fusion of two strings \( X, Y \) is denoted by \( X \otimes Y \). The tensor unit is the trivial string (i.e. no string), denoted by \( 1 \).

- **Braiding structures:** The adiabatic move of one string around another parallel string in a spatial open 3-disk defines the braiding between two strings (see Figure 2).

As a consequence, the topological defects of codimension 2 or higher form a braided monoidal 2-category.

In physical literatures (see for example [WL14, JMR14, EN17, LKW18]), physicists usually consider loop-like topological defects rather than string-like topological defects. There are some reasons why we use string-like topological defects to describe a 3d topological order.

1. The string-like topological defects are more fundamental than the loop-like ones. For a loop-like, or even a knot-like topological defect, we can talk about its type. When we say a loop-like topological defects and a knot-like topological defect are of the same type, what we mean is that they are the same locally, i.e. on every open line segment. As depicted Figure 1(a), the type \( X \) of the string determines the type of the loop. This is similar to the fact that an \( n \)d topological order is a notion defined on an open \( n \)-disk in the infinite size limit [AKZ17]. The physical observables defined on any closed manifold \( M \), such as the ground state degeneracy on \( M \), can be obtained by integrating observables on open \( n \)-disks over the entire \( M \) [AKZ17]. This process of integration is mathematically defined by factorization homology (see for example [AF19] for a review).

2. Moreover, we can consider more general topological defects, an example of which is depicted in Figure 1(b): two different string-like topological defects \( X, Y \), together with
Figure 1: (a): A loop-like topological defect looks like a string locally; (b): Two different string-like topological defects \((X, Y)\) and 0d domain walls \((a, b)\) between them constitute a loop-like topological defect.

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\text{(a)}
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\text{(b)}
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two 0d domain walls \(a, b\), constitute a loop-like topological defect. Note that Figure 1 (a) is just a special case of (b) when \(X = Y = a = b\). It simply means that the string-like defects are more fundamental.

It is possible to compute the topological invariant associated to a loop-like or knot-like topological defect. This computation amounts to a dimensional reduction process of shrinking the size of the loop-like or knot-like topological defect to nearly a point so that it becomes a 0d domain walls between two trivial strings \(1\). Therefore, a loop-like or knot-like topological defect determines a topological invariant valued in the symmetric fusion 1-category \(\text{Hom}_C(1, 1)\). We illustrate this idea by some examples in Section 4.5.

\textbf{Remark 2.1.} We have not included topological defects of codimension 1 in our categorical description. The reason is that all topological defects of codimension 1 in an anomaly-free topological order are all condensation descendants \([KW14, DR18, GF19, KLWZZ20]\), and can be ignored in the 0-codimensional description of the topological order \([KLWZZ20]\). We will leave the study of topological defects of codimension 1 in the future.

\section{2.2 Braiding structure}

In this subsection, we provide an equivalent way of formulating the braiding between two strings that allows us to compute the braiding in lattice model efficiently.

Intuitively, the braiding of two strings \(X\) and \(Y\) is defined by an adiabatic move of the string \(X\) around the string \(Y\) in the 3d space, as depicted in Figure 2(a). Therefore, a braiding is a 1+0D defect between the 1+1D world sheets of \(X \otimes Y\) and \(Y \otimes X\), as depicted in Figure 2(c). Note that two world sheets of \(X\) and \(Y\) do not intersect in the 3+1D spacetime. Also Figure 2(b) depicts a double braiding process, in which the trajectory of \(X\) is a cylinder.

\begin{figure}[h]
\centering
\begin{subfigure}{0.3\textwidth}
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\includegraphics[width=\textwidth]{figure1.png}
\caption{}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
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\includegraphics[width=\textwidth]{figure2.png}
\caption{}
\end{subfigure}
\begin{subfigure}{0.3\textwidth}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{}
\end{subfigure}
\caption{the braiding of two strings: (a): moving a string \(X\) around \(Y\) in the 3d space is a braiding of \(X\) and \(Y\); (b): the trajectory of a double braiding is a cylinder; (c): the braiding is a 1+0D defect embedded in the 3+1D spacetime.}
\end{figure}
In order to compute the braiding structure explicitly in the 3d toric code model. We rotate this 1+0D defect of braiding to obtain a 0+1D defect. We denote $I := [0,1]$ and $I^n := [0,1]^n$. The world sheet of the braiding of two strings (see Figure 2(c)) is denoted by $S_1 \subset I^3$. At each time $t$, the time slice of $S_1$ is a braid $B_{1,t}$ in the 3d space, as depicted in Figure 3(a). Thus we have

$$S_1 = \{(p,t) \in I^3 \times I \mid p \in B_{1,t}\} \subset I^3 \times I = I^4,$$

where the first component $I^3$ represents the 3d space, and the second component $I$ represents the time axis.

Let us analyze the world sheet $S_1$ in $xyt$-spacetime by fixing the $z$-coordinate. Let $H_{z_0} \subset I^4$ be the hyperplane determined by $z = z_0$. Topologically $H_{z_0}$ is also a 3-disk. The section of $S_1$ at $z = z_0$ is $S_1 \cap H_{z_0} \subset H_{z_0} = I^3$, as depicted in Figure 3(b). It is clear that for all values $z_0$ the sections $S_1 \cap H_{z_0}$ are the same. Denote $B'_{1,t} := S_1 \cap H_{z_0}$, then we have

$$S_1 = B'_{1,t} \times I \subset I^3 \times I = I^4,$$

where the first component $I^3$ represents the $xyt$-spacetime (i.e. $H_{z_0}$), and the second component $I$ represents the $z$-axis.

After a rotation in $zt$-plane, the braid $B'_{1,t}$ is isotopic to $B_2$. This braid in the 3d space is a 0d domain wall between two strings $X \otimes Y$ and $Y \otimes X$, denoted by $R_{X,Y}$.

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$$S_1 = B'_{1,t} \times I \subset I^3 \times I = I^4,$$

where the first component $I^3$ represents the $xyt$-spacetime (i.e. $H_{z_0}$), and the second component $I$ represents the $z$-axis.

After a rotation in $zt$-plane, the braid $B'_{1,t}$ is isotopic to $B_2$ depicted in Figure 3(c). We denote

$$S_2 := B_2 \times I \subset I^3 \times I = I^4,$$

where the first component $I^3$ represents the 3d space, and the second component $I$ represents the time axis. Then we know that $S_1$ and $S_2$ is isotopic in the 3+1D spacetime, up to a rotation.

Since the time slices of $S_2$ are constant (the braid $B_2$), it can be viewed as a 0+1D defect. Figure 4(a) depicts the world sheet $S_2$, and its time slice is depicted in Figure 4(b). This braid in the 3d space is a 0d domain wall between two strings $X \otimes Y$ and $Y \otimes X$, denoted by $R_{X,Y}$.

![Figure 3](image3.png)

Figure 3: (a): The time slice of the world sheet $S_1$ of braiding two strings at time $t$ is $B_{1,t}$; (b): The section of $S_1$ at $z = z_0$ is a braid $B'_{1,t} := S_1 \cap H_{z_0}$; (c): After a rotation in $zt$-plane, the braid $B'_{1}$ is isotopic to $B_2$. 

![Figure 4](image4.png)

Figure 4: (a) After a rotation, the braiding of two strings gives a 0+1D defect; (b): The braiding structure is a 0d domain wall between $X \otimes Y$ and $Y \otimes X$. 

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In the categorical description of topological defects, the 0d domain wall \( R_{X,Y} \) is a 1-morphism \( R_{X',Y} : X \otimes Y \to Y \otimes X \). Since the braiding is a 0+1D defect, there are also additional 2-morphism data in the definition of a braiding. More precisely, let \( f \) be a 1D domain wall between two strings \( X \) and \( X' \).

\[
\begin{array}{c}
X' \otimes Y \xrightarrow{R_{X',Y}} Y \otimes X' \\
\downarrow f \otimes 1_Y \quad \quad \quad \quad 1_Y \otimes f \\
X \otimes Y \xrightarrow{R_{X,Y}} Y \otimes X
\end{array}
\]

This is the braiding structure of a braided fusion 2-category.

**Remark 2.2.** The braiding structure of a braided fusion 2-category also involves some 2-morphisms (invertible modifications) \( R_{A,X,Y} \) for all objects \( A, X, Y \) (see [KTZ20] and references therein). In this work we do not discuss these 2-morphisms because they are all trivial in the 3d toric code model.

## 3 3d toric code model

The 3d toric code model was introduced in [HZW05] as an exactly solvable model realizing the 3+1D \( \mathbb{Z}_2 \) gauge theory. In this section, we recall the construction of this model.

### 3.1 Hamiltonian

Similar to the 2d toric code model [Kit03], the 3d toric code model can be defined on any kind of lattice. But for simplicity, we consider only the cubic lattice as depicted in Figure 5.

![Figure 5: the local operators in the 3d toric code model](image)

There is a space of spin-1/2 on each edge (1-cell) of the lattice. In other words, the local degrees of freedom \( \mathcal{H}_e \) on each edge \( e \) form a two-dimensional Hilbert space \( \mathbb{C}^2 \), i.e. \( \mathcal{H}_e = \mathbb{C}^2 \). The total Hilbert space is \( \mathcal{H}_{\text{tot}} := \bigotimes_e \mathcal{H}_e \). For any vertex \( v \) and plaquette \( p \) we define a vertex operator \( A_v := \prod_i \sigma_x^i \) and a plaquette operator \( B_p := \prod_j \sigma_z^j \) acting on adjacent edges. Here \( \sigma_x^i \) means a \( \sigma_x \) operator only acting on the \( i \)-th edge. For example, the operators in Figure 5 are

\[
A_v = \sigma_x^1 \sigma_x^2 \sigma_x^3 \sigma_x^4 \sigma_x^5 \sigma_x^6, \quad B_p = \sigma_z^5 \sigma_z^6 \sigma_z^7 \sigma_z^8.
\]

The Hamiltonian of the 3d toric code model is defined by

\[
H := \sum_v (1 - A_v) + \sum_p (1 - B_p).
\]
Since all $A_v$ and $B_p$ operators mutually commute, the total Hilbert space can be decomposed as the direct sum of common eigenspaces of all $A_v$ and $B_p$ operators, whose eigenvalues are $\pm 1$. The ground state subspace has energy 0 and is the common eigenspace of all $A_v$ and $B_p$ operators with eigenvalues $+1$. The system is gapped because the first excited state has energy at least 2 no matter how large the system size is.

### 3.2 Ground state degeneracy

Let us compute the ground state degeneracy (GSD) of the 3d toric code model defined on a 3-dimensional oriented closed manifold $M$. Suppose there are $V$ vertices (0-cells), $E$ edges (1-cells), $F$ plaquettes (2-cells) and $B$ cubes (3-cells) in the lattice. The Poincaré duality implies the Euler number of $M$ is $0 = V - E + F - B$, because $\dim(M) = 3$ is odd.

The total Hilbert space has dimension $2^E$. The constraints of the ground states are given by $A_v = +1$ and $B_p = +1$ for all vertices $v$ and plaquettes $p$. However, these constraints are not independent, and their relations are listed below.

1. The product of all $A_v$ operators equals to 1 because each $\sigma_x$ on the edge has been counted twice.
2. For any cube $c$, the product of all $B_p$ operators on the faces of $c$ equals to 1 because each $\sigma_z$ on the edge of $c$ has been counted twice.
3. For any 2-dimensional closed surface $\Sigma \subset M$, the product of all $B_p$ operators on $\Sigma$ equals to 1 because each $\sigma_z$ on the edge of $\Sigma$ has been counted twice.

There are $B$ relations in (2), but only $(B - 1)$ of them are independent because

$$ \prod_{c} \left( \prod_{p \in \partial c} B_p \right) = 1. $$

Similarly, the relations in (3) are also not independent. If two surfaces $\Sigma$ and $\Sigma'$ are homologous, i.e. they encircle a 3-dimensional manifold $N$, then we have

$$ \left( \prod_{p \in \Sigma} B_p \right) \left( \prod_{p' \in \Sigma'} B_{p'} \right) = \prod_{c \in N} \left( \prod_{q \in \partial c} B_q \right). $$

Therefore, there is at most one independent relation associated to surfaces $\Sigma$ in each homology class in $H_2(M; \mathbb{Z}_2)$. We denote

$$ b_k := \dim_{\mathbb{Z}_2} H_k(M; \mathbb{Z}_2), $$
then only $b_2$ relations in (3) are independent. So there are $(V + F - 1 - B + 1 - b_2)$ independent constraints, and each one halves the total Hilbert space. Therefore, we obtain

$$ \text{GSD on } M = 2^{E - (V + F - B - b_2)} = 2^{b_2}, $$

which is a topological invariant.

**Example 3.1.** The GSD of the 3d toric code model on the 3-dimensional sphere $S^3$ is $2^0 = 1$, and the GSD on the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ is $2^3 = 8$.

### 4 Topological defects

There are topological defects in the 3d toric code model. In this section, we find all defects of codimension 2 or higher and show that they form a fusion 2-category. A 1d (or 2-codimensional) defect is string-like and is called a string for simplicity; a 0d defect is in general a domain wall between two strings.
4.1 Four types of strings and 0d defects

In the 3d toric code model, there are well-known topological excitations: the $e$-particle and the $m$-string as depicted in Figure 6.

![Figure 6: an $e$-particle and an $m$-string in the 3d toric code model](image)

Similar to the 2d case, an $e$-particle at a vertex $v_0$ corresponds to the eigenspace that all $B_p, A_v$ act on as $+1$ except $A_{v_0}$ acting as $-1$. A pair of $e$-particles can be created by a string operator $\otimes i \sigma^z_i$, where $i$'s are links along the string. An $e$-particle can be viewed as a $0d$ domain wall between two trivial strings, denoted by $\mathds{1}$.

As in the 2d case, one may also expect that $B_p = -1$ at a plaquette $p$ defines an $m$-particle. In 3d, however, the constraint that $\prod_{p \in c} B_p = 1$ for any cube $c$ implies there must be even plaquettes $p$ in a cube such that $B_p = -1$. Thus the correct topological excitation is the $m$-string (see Figure 6), which can not be broken by the above constraint. It is well-known that one can create a loop of $m$-strings (7) by a membrane operator [HZW05].

Besides the $e$-particles and $m$-strings, there are other topological defects. We slightly modify the lattice model. In Figure 7, we depict a string-like topological defect, called a $\mathds{1}_c$-string, where the subscript " $c$ " represents the "condensation descendant" (see Remark 4.2). The local Hilbert spaces on the edges of the $\mathds{1}_c$-string (the dashed line) are one-dimensional $\mathcal{H}_c = \mathbb{C}$, i.e. there is no local degree of freedom along the $\mathds{1}_c$-string. Also, for vertices on the string there are no $A_v$ operators. The $B_p$ operator adjacent to the $\mathds{1}_c$-string, for example the one in Figure 7, is defined by $B_p = \sigma^z_1 \sigma^z_2 \sigma^z_3$. The new Hamiltonian again consists of mutually commuting operators. An $m$-string can be fused with a $\mathds{1}_c$-string to give a new string-like defect denoted by $m_c$. An $m_c$-string can be microscopically realized by an $m$-string and an adjacent $\mathds{1}_c$-string. Hence we obtain four types of string-like topological defects: $1, \mathds{1}_c, m, m_c$.

**Remark 4.1.** The $\mathds{1}_c$-string was first discovered by Else and Nayak in [ENT17, Section III]. A $\mathds{1}_c$-string can also be created by adding to the original Hamiltonian (3.1) the $A_v$ terms and the projection operator onto the spin-$\uparrow$ subspace (i.e. $(1 + \sigma_z)/2$) on each 1-cell along the string.
**Remark 4.2.** Using string operators, we can move an \( e \)-particle onto a \( 1_c \)-string. It is easy to see that the \( e \)-particles condense on the \( 1_c \)-string. Moreover, one can also realize a \( 1_c \)-string physically by introducing a 1d condensation on a \( 1 \)-string defined by condensing the condensable algebra \( 1 \oplus e \) \cite{Kon14}. For this reason, \( 1_c \)-strings can be viewed as condensation descendants of the \( 1 \)-strings. Note that one can also introduce a 1d condensation on a \( 1_c \)-string to obtain a \( 1 \)-string by condensing the condensable algebra \( 1_c \oplus z \) on the \( 1_c \)-strings, where \( z \) is a 0d domain wall between two \( 1_c \)-strings (see Figure 9 (a)).

0d defects are 0d domain walls between two string-like defects. For example, the trivial particles (denoted by \( 1_{\uparrow} \)) and the \( e \)-particles can both be viewed as 0d domain walls between two trivial strings.

The ends of a \( 1_c \)-string (see Figure 8) are the 0d domain walls between a trivial string and a \( 1_c \)-string. The 0d wall from \( 1 \) to \( 1_c \) is denoted by \( x : 1 \to 1_c \), and by reversing its orientation, we get the 0d wall from \( 1_c \) to \( 1 \), denoted by \( y : 1_c \to 1 \).

![Figure 8: the 0d defects between a trivial string and a \( 1_c \)-string](image)

There are two kinds of 0d domain walls between two \( 1_c \)-strings. One is the trivial 0d wall, denoted by \( 1_{\uparrow} \); the other one is depicted in Figure 9, denoted by \( z : 1_c \to 1_c \). Microscopically, the 0d defect \( z \) is realized by four plaquettes \( B_p = -1 \) around the \( 1_c \)-string. It can be obtained by shrinking an \( m \)-loop winding around the \( 1_c \)-string. Note that if we shrink an \( m \)-loop around the trivial string, we simply get the trivial particle because the \( m \)-loop across the four plaquettes in Figure 9 (a) (with \( 1_c \)-string replaced by the trivial string) can be annihilated by a single \( \sigma \) operator.

![Figure 9: (a): The microscopic realization of the nontrivial 0d defect between two \( 1_c \)-strings; (b): This 0d defect can be obtained by shrinking an \( m \)-loop winding around a \( 1_c \)-string.](image)

There is no 0d wall between a trivial string and an \( m \)-string due to the constraint that an \( m \)-string can not be broken. Similarly, there is no 0d walls between any one of \( \{1, 1_c\} \) and any one of \( \{m, m_c\} \). By fusing with an \( m \)-string, we can see that the 0d domain walls between the two strings of type either \( m \) or \( m_c \) can be obtained from those between the two strings of type either \( 1 \) or \( 1_c \).

**Remark 4.3.** Similar to the relation between \( 1 \) and \( 1_c \), the \( m_c \)-strings can also be viewed as condensation descendants of the \( m \)-strings.

\( \diamond \)
4.2 2-category $\text{TC}_3$

In this subsection, we show that the topological defects in the 3d toric code model form a semisimple 2-category denoted by $\text{TC}_3$.

- **Objects = 1d defects (i.e. strings):** There are four simple objects in the 2-category $\text{TC}_3$ given by the four types of simple strings $\mathbb{1}_c$, $\mathbb{1}_m$, $c$, $m_c$ in the model. General objects are direct sum of simple strings, and are called composite strings. When we discuss the fusion rule of $\mathbb{1}_c \otimes \mathbb{1}_m$ in the next subsection, we will see that the direct sums of simple strings are absolutely necessary.

- **1-morphisms = 0d defects; 2-morphisms = instantons:** 0d domain walls between any two strings $i$ and $j$ and instantons form a 1-category, denoted by $\text{Hom}_{\text{TC}_3}(i, j)$. We explain the ingredients of these 1-categories below for $i, j = \mathbb{1}_c, m, m_c$.

  - $\text{Hom}_{\text{TC}_3}(\mathbb{1}, \mathbb{1})$: There are two simple particles: the trivial particle $\mathbb{1}_t$ and the $e$-particle in $\text{Hom}_{\text{TC}_3}(\mathbb{1}, \mathbb{1})$. General particles are the direct sum of simple particles. For example, the 0d defect $\mathbb{1}_t \oplus e$ located at vertex $v_0$ can be obtained by throwing away the term $(1 - A_{v_0})$ from the Hamiltonian.

    As in the 2d case, two $e$-particles fuse into the trivial particle, i.e. $e \otimes e = \mathbb{1}_t$. Thus we have $\text{Hom}_{\text{TC}_3}(\mathbb{1}, \mathbb{1}) \cong \text{Rep}(\mathbb{Z}_2)$ as fusion 1-categories, where $\text{Rep}(\mathbb{Z}_2)$ denotes the category of (finite-dimensional) representations of the $\mathbb{Z}_2$-group.

  - $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1}_c)$: There are also two simple 0d defects $\mathbb{1}_c$ and $\mathbb{1}_s$ in $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1}_c)$. Figure 10 shows that two $z$-particles fuse into $\mathbb{1}_s$ because the operator $a_t^1 a_z^2 a_z^0 a_z^4$ annihilates two $m$-strings, i.e. $\mathbb{1}_s \otimes \mathbb{1}_s = \mathbb{1}_s$. Thus we have $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1}_c) \cong \text{Vec}_{\mathbb{Z}_2}$ as fusion 1-categories, where $\text{Vec}_{\mathbb{Z}_2}$ denotes the category of $\mathbb{Z}_2$-graded vector spaces. Note that $\text{Rep}(\mathbb{Z}_2) \cong \text{Vec}_{\mathbb{Z}_2}$ as fusion 1-categories, but we

    \begin{figure}[h]
    \centering
    \includegraphics[width=\textwidth]{fig10.png}
    \caption{The fusion of two $z$-particles is $\mathbb{1}_s$.}
    \end{figure}

  - $\text{Hom}_{\text{TC}_3}(\mathbb{1}_m, \mathbb{1}_m)$: There is only one simple 0d defect $x$ in $\text{Hom}_{\text{TC}_3}(\mathbb{1}_m, \mathbb{1}_m)$. General 0d defects in $\text{Hom}_{\text{TC}_3}(\mathbb{1}_m, \mathbb{1}_m)$ are the direct sums of $x$. Therefore, we have $\text{Hom}_{\text{TC}_3}(\mathbb{1}_m, \mathbb{1}_m) \cong \text{Vec}$ as 1-categories, where $\text{Vec}$ denotes the category of finite-dimensional vector spaces.

  - $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1})$: There is only one simple 0d defect $\mathbb{1}_c$ in $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1})$. General 0d defects in $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1})$ are the direct sums of $\mathbb{1}_c$. Again we have $\text{Hom}_{\text{TC}_3}(\mathbb{1}_c, \mathbb{1}) \cong \text{Vec}$ as 1-categories.

  - The fusion of 0d defects $y \otimes x$ gives a particle in $\text{Hom}_{\text{TC}_3}(1, \mathbb{1})$. This particle $y \otimes x$ can be represented by a $z_e$-string of length zero, or equivalently, by deleting the $A_v$ operator at a vertex $v$ from the Hamiltonian. This particle is precisely $\mathbb{1}_z \otimes e$ because it corresponds to the eigenspaces with $A_v = \pm 1$ in the total Hilbert space. Thus $y \circ x = \mathbb{1}_1 \oplus e$. 

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The fusion of $x \otimes y$ gives a 0d defect in $\text{Hom}_{TC_3}(1_c, 1_c)$. To compute it, we let $x$ and $y$ be close enough as depicted in Figure 11. The spin on the edge between $x$ and $y$ can take the value $\uparrow$ or $\downarrow$. If the spin is $\uparrow$, we get a $1_c$-string (recall the convention in Remark 4.1); if the spin is $\downarrow$, then four $B_p$ operators adjacent to this edge have to take the value $-1$, that is, we get a $z$-particle. Hence we have $x \circ y = 1_c \oplus z$.

$1_c x 1_c y = 1_c x 1_c y \uparrow \oplus 1_c x 1_c y \downarrow = 1_c \oplus z$.

Figure 11: The fusion of an $x$-particle and a $y$-particle is $x \circ y = 1_c \oplus z$.

- $\text{Hom}_{TC_3}(i, j) = 0$ for $i = 1, 1_c$ and $j = m, m_c$: This is because $m$-strings can not be broken.
- $\text{Hom}_{TC_3}(m, m) \simeq \text{Hom}_{TC_3}(1_c, 1_c) \simeq \text{Rep}(Z_2)$: This is because when $e$-particles are moved close to an $m$-string, no condensation happens. We denote the trivial 0d domain wall on the $m$-string by $1_m$, and denote the $e$-particle on the $m$-string by $e_m := e \otimes 1_m$.
- $\text{Hom}_{TC_3}(m_c, m_c) \simeq \text{Hom}_{TC_3}(1_c, 1_c) \simeq \text{Vec}_{Z_2}$. We denote the trivial 0d defect on the $m_c$-string by $1_{m_c}$. Since $m_c = 1_c \otimes m$, we denote the non-trivial 0d defect on the $m_c$-string by $z_m := z \otimes 1_m$.
- $\text{Hom}_{TC_3}(m, m_c) \simeq \text{Hom}_{TC_3}(1_c, 1_c) \simeq \text{Vec}$;
- $\text{Hom}_{TC_3}(m_c, m) \simeq \text{Hom}_{TC_3}(1_c, 1_c) \simeq \text{Vec}$.

- According to the mathematical definition of the semisimpleness and idempotent completeness of a 2-category [DR18], $TC_3$ is a semisimple 2-category.

We can summarize all the ingredients of the 2-category $TC_3$ heuristically by the following diagram:

\[
\begin{array}{ccc}
\text{Rep}(Z_2) & \rightarrow & \text{Vec}_{Z_2} \\
1_c & \downarrow & \downarrow m_c \\
\text{Vec} & \rightarrow & \text{Vec} \\
\end{array}
\]

Mathematically, the full 2-subcategory represented by the following subdiagram

\[
\begin{array}{ccc}
\text{Rep}(Z_2) & \rightarrow & \text{Vec}_{Z_2} \\
1_c & \downarrow & \downarrow m_c \\
\text{Vec} & \rightarrow & \text{Vec} \\
\end{array}
\]
is precisely the well-known 2-category $\mathcal{Z}(2\text{Vec}_{Z_2})$, which is the 2-category of finite semisimple module categories over $\text{Rep}(Z_2)$ (or $\text{Vec}_{Z_2}$ equivalently). As a consequence, the complete 2-category $\mathcal{T}C_3$ of topological defects of the 3d toric code model is equivalent to the direct sum $\mathcal{Z}(2\text{Vec}_{Z_2}) \cong 2\text{Rep}(Z_2)$. This is precisely the 2-category $\mathcal{Z}(2\text{Vec}_{Z_2})$ of topological excitations in the 3d $Z_2$ topological order predicted in [KTZ20, Example 3.8], where $\mathcal{Z}(2\text{Vec}_{Z_2})$ denotes the Drinfeld center of the monoidal 2-category $2\text{Vec}_{Z_2}$ of $Z_2$-graded 2-vector spaces.

We denote the full 2-subcategory consisting of 1 and $m$ by $\mathcal{T}C_0^0$. According to [DR18], the 2-category $\mathcal{T}C_3$ is semisimple (i.e. idempotent complete) but $\mathcal{T}C_0^0$ is not semisimple. According to [KW14], $\mathcal{T}C_0^0$ is the core of $\mathcal{T}C_3$. By definition, the core is a subcategory consisting of elementary defects such that all the rest of defects can be obtained from the elementary ones by condensations. The non-elementary defects are called condensed excitations or condensation descendants (of the elementary ones). It is clear that the $1_c$-string and the $m_c$-string are both condensation descendants. The process of adding more defects to $\mathcal{T}C_0^0$ to obtain $\mathcal{T}C_3$ is called “condensation completion” [GJF19] [KLWZZ20], which is also called an idempotent completion [DR18] or a Karoubi completion [GJF19].

In the end of this subsection, we try to answer an obvious question: why should we include the condensation descendants: the $1_c$-strings and the $m_c$-strings in our categorical description of the 3d $Z_2$ topological order? We provide a few reasons below.

1. The $1_c$-strings and the $m_c$-strings are physically natural. We cannot ignore them for physical processes involving condensations of particles. It is also unnatural to ignore them when we consider dimensional reduction processes. For example, by a dimensional reduction from 3d to 2d, both an $m$-string and an $m_c$-string can be viewed as two gapped 1d boundaries of the same 2d topological order.

2. The $1_c$-strings and the $m_c$-strings are mathematically natural. Without them, $\mathcal{T}C_0^0$ is not idempotent complete (or semisimple) as shown in [DR18].

3. We construct the rough boundary of the 3d toric code model in Section 5.1. Although there is no condensation descendants on this boundary (with only the trivial strings and the $m$-strings), as shown in [KTZ20], the $1_c$-strings and the $m_c$-strings naturally pop up in the 3d bulk as they are demanded by the boundary-bulk relation [KWZ17].

4. A gapped boundary of a topological order can be viewed as a consequence of a condensation of the topological excitations in the bulk. It turns out that the rough boundary of the 3d toric code model is a consequence of condensing the $1_c$-strings, which should be viewed as a condensable algebra in $\mathcal{T}C_3$ (see Remark 5.2). Using $\mathcal{T}C_0^0$, there is no way to develop a condensation theory.

### 4.3 Monoidal structure on $\mathcal{T}C_3$

The monoidal structure on $\mathcal{T}C_3$ is given by the fusion of string-like topological defects. The fusion of two strings $X, Y$ is denoted by $X \otimes Y$.

By definition, a $1_c$-string and an $m$-string fuse into an $m_c$-string, i.e. $m_c = 1_c \otimes m$.

It is not hard to see that two $m$-strings fuse into the trivial string, i.e. $m \otimes m = 1$. Figure 10 shows two $m$-loops can be annihilated by several $\sigma_x$ operators, and a similar argument applies to the fusion of two $m$-strings.

Let us consider the fusion of two $1_c$-strings. As depicted in Figure 12, if two $1_c$-strings are closed enough, all the spins on the edges between them have to be $\uparrow$ or $\downarrow$, due to the constrains that $B_p = +1$ for all plaquettes $p$ between two strings. If all of these spins take the value $\uparrow$, we get a single “thick” $1_c$-string. If all of these spins take the value $\downarrow$, we need to flip them.
(recall our convention in Remark 4.1) by applying $\sigma_x$ operators on these edges. The result is a “thick” $1_c$-string and two $m$-strings, and clearly these strings fuse into a single $1_c$-string. Hence we obtain $1_c \otimes 1_c = 1_c \oplus 1_c$, where two direct summands $1_c$ correspond to spin $\uparrow$ and spin $\downarrow$ respectively. It also implies that $m_c \otimes m_c = 1_c \oplus 1_c$, $1_c \otimes m_c = m_c \oplus m_c$, and $m_c \otimes 1_c = m_c \oplus m_c$.

**Remark 4.4.** Using the fact that $e$-particles condense on a $1_c$-string and $1_c \otimes 1_c = 1_c \oplus 1_c$, it is easy to see that a $1_c$-string shrink to a particle-like excitation $1_c \oplus e$ (see also Section 4.5).

Comparing with the fusion rule of $Z(2\text{Vec}_{Z_2})$ in [KTZ20, Example 3.8], we see that $\text{TC}_3 \cong Z(2\text{Vec}_{Z_2})$ as monoidal semisimple 2-categories [DR18].

**Remark 4.5.** Note that the full 2-subcategory $\text{TC}_0^3$ is a monoidal 2-subcategory of $\text{TC}_3$.

### 4.4 Double braidings in $\text{TC}_3$

It is obvious that the fusion product is commutative, i.e. $i \otimes j = j \otimes i$ for $i, j = 1_c, m, m_c$. From the 3d toric model, one can compute the double braidings by moving one string, together with all possible particles (or 0d defects) on it, around another string along a path of a full circle.

1. The double braiding between two $1_c$-strings (including the particles on them) is obviously trivial. This is compatible with the physical fact that $e$ is a boson, and with the mathematical fact that Rep($Z_2$) is a symmetric fusion 1-category.

2. The double braiding between a $1_c$-string and a $m_c$-string (including all 0d defects on them) is obviously trivial. Note that this is compatible with the mathematical fact that 2Rep($Z_2$) is a symmetric fusion 2-category.

3. The double braiding between a $1_c$-string (including particles $1_c$ or $e$) and an $m$-string is non-trivial because the double braiding between an $e$-particle and an $m$-string is $-1$ as illustrated in Figure 13.
Figure 13: The double braiding of the $e$-particle and the $m$-string is $-1$.

Figure 14: The double braiding of the $1_c$-string and the $m$-string is $z \otimes 1_m$.

4. The double braiding between a $1_c$-string and an $m_c$-string is non-trivial and is given by $z_m = z \otimes 1_m$ as illustrated in Figure 14.

5. The double braiding between two $m$-strings is trivial unless we attach an $e$-particle on one of the $m$-strings, in which case the double braiding gives $-1$.

6. All the rest double braidings can be obtained from above double braidings using the fusion properties.

These double braidings coincide with those of $\mathcal{Z}(2\text{Vec}_{\mathbb{Z}_2})$, which was implicitly given in [KTZ20]. See more detail in Appendix A.

The double braidings are physical and independent of any gauge choices. But the braiding structure is gauge dependent. Namely, different choices of the braiding can be braided equivalent. One way to fix the gauge is to select a gapped boundary of the 3d toric code model, then determine the braidings via the half-braidings of the defects on the boundary, as we do in Section 5.2.

4.5 Loops and links

In this subsection we give some examples of loop-like topological defects in the 3d toric code model. By a dimensional reduction process, they shrink to particle-like topological defects, i.e. 0d domain walls between two trivial strings $1$.

The first example is an $m$-loop. As depicted in Figure 15, an $m$-loop shrinks to a trivial particle $1_1$ because it can be annihilated by a membrane operator $\sigma_1^1 \sigma_2^2 \sigma_3^3 \sigma_4^4$.

The second example is a $1_c$-loop. As depicted in Figure 16, the dimensional reduction process can be done in two steps: first we horizontally squeeze the $1_c$-loop to get a $1_c \otimes 1_c$-string with two ends; then we vertically squeeze the string to get a particle. In other words, the dimensional reduction of a $1_c$-loop is equal to the composition of two 0d domain walls along a $1_c \otimes 1_c$-string.

Let us compute the 0d domain walls at the two ends of the $1_c \otimes 1_c$-string in Figure 16. The upper one, which is viewed as a 1-morphism from $1$ to $1_c \otimes 1_c$, is depicted in Figure 17. All the spins on the edges between two $1_c$-strings should take the value $\uparrow$ or $\downarrow$. But at the end of the
Figure 15: An $m$-loop can be created/annihilated by a membrane operator.

\[ 1_c \bigcirc = 1_c \otimes 1_c. \]

Figure 16: A $1_c$-loop is equal to the composition of two 0d domain walls along a $1_c \otimes 1_c$-string. The $1_c \otimes 1_c$-string, the spins have to take the value $\uparrow$ because the $B_p$ operator at the end consists of a single $\sigma_z$ operator. Therefore, the result is a single $1_c$-string, which corresponds to the first direct summand of $1_c \otimes 1_c = 1_c \oplus 1_c$. Recall that the end of this $1_c$-string, i.e. the 0d domain wall between the trivial string $1$ and this $1_c$-string is nothing but $x : 1 \rightarrow 1_c$. Hence, the upper 0d domain wall in Figure 16 is equal to

\[
1 \xrightarrow{x} 1_c \oplus 1_c = 1_c \otimes 1_c.
\]  

(4.4)

Similarly, the lower 0d domain wall in Figure 16 viewed as a 1-morphism from $1_c \otimes 1_c$ to $1$, is equal to

\[
1_c \otimes 1_c = 1_c \oplus 1_c \xrightarrow{y \ 0} 1.
\]

Then we have

\[
\left( 1 \xrightarrow{x} 1_c \oplus 1_c \xrightarrow{y \ 0} 1 \right) = \left( y \ 0 \right) \left( x \ 0 \right) = y \circ x = 1_1 \oplus e.
\]

(4.6)
Hence, the dimensional reduction of a $1_c$-loop to a point gives a composite particle: $1 \oplus c$.

The third example is a link consisting of an $m$-loop and a $1_c$-loop. As depicted in Figure 18, we first shrink the $m$-loop to get a $z$-particle on the $1_c$-loop, then horizontally squeeze the $1_c$-loop. Thus the link equals to the composition of three 0d domain walls, where the upper one and the lower one is known and the middle one is equal to $z \otimes 1_c : 1_c \otimes 1_c \to 1_c \otimes 1_c$.

![Diagram](https://example.com/diagram.png)

Figure 18: An $m$-$1_c$-link is equal to the composition of three 0d domain walls.

Let us compute the 0d domain wall $z \otimes 1_c$, as depicted in Figure 19. Due to the middle blue plaquette with $B_p = -1$, there are two possibilities of the spins between two $1_c$-strings: the spins above the middle plaquette take the value $\uparrow$ and the spins below the middle plaquette take the value $\downarrow$, or the spins above the middle plaquette take the value $\downarrow$ and the spins below the middle plaquette take the value $\uparrow$. Therefore, the 0d domain wall $z \otimes 1_c$ exchanges two direct summands of $1_c \otimes 1_c = 1_c \oplus 1_c$, and these two possibilities correspond to two direct summands of the 0d domain wall:

$$z \otimes 1_c = \left( \begin{array}{cc} 0 & 1_c \\ 1_c & 0 \end{array} \right) \rightarrow 1_c \oplus 1_c = 1_c \otimes 1_c.$$  \hspace{1cm} (4.7)

![Diagram](https://example.com/diagram.png)

Figure 19: The 0d domain wall $z \otimes 1_c$ is equal to (4.7).
Then we have

\[
(1 \rightarrow 1_c \oplus 1_c (0 \rightarrow 1_{1_c} \oplus 1_c \rightarrow 1_c \oplus 1_c (y \rightarrow 0)) = 0. \tag{4.8}
\]

Hence, the dimensional reduction of an \(m\)-\(1_c\)-link to a point gives 0. This result means the configuration of an \(m\)-\(1_c\)-link is physically forbidden. One can see this fact explicitly in lattice model by reading Figure 17 and Figure 19.

**Remark 4.6.** The fact that an \(m\)-\(1_c\)-link is physically forbidden was first pointed out by Else and Nayak in [EN17, Section III].

5 Gapped boundaries and braidings

In this section, we first construct two gapped boundaries of 3d toric code model, then use the boundary-bulk relation to realize the braiding structure in \(TC_3\).

5.1 Smooth and rough gapped boundaries

Similar to the 2d case, there are two obvious gapped boundaries of the 3d toric code model: the smooth and rough boundaries.

The smooth boundary is depicted in Figure 20. The \(A_v\) operators on the smooth boundary is the product of five \(\sigma_x\) operators, and the other operators are the same as the original 3d toric code model. Thus the Hamiltonian is still the sum of mutually commuting operators.

![Figure 20: the topological defects on the smooth boundary](image)

The 1d and 0d topological defects on the smooth boundary are also depicted in Figure 20. Figure 21 shows that the \(m\)-strings are condensed on the smooth boundary, because they can be created/annihilated by local operators on the boundary (for example the operator \(\sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x \sigma_5^x\) in the figure). The \(e\)-particle and \(1_c\)-string survive on the smooth boundary, and the \(m\)-string condenses to \(1_c\) on the smooth boundary. Hence the topological defects on the smooth boundary form a fusion 2-category \(2\text{Rep}(\mathbb{Z}_2)\).

![Figure 21: An \(m\)-string can be created/annihilated by local operators on the smooth boundary.](image)
Remark 5.1. The smooth boundary can also be understood as condensing \( A = \mathbb{1} \oplus m \), which should be viewed as a condensable algebra in \( \text{TC}_3 \). From this point of view, the fusion 2-category \( \text{2Rep}(\mathbb{Z}_2) \) of boundary defects can also be viewed as the 2-category \( \text{RMod}_A(\text{TC}_3) \) of right \( A \)-modules in \( \text{TC}_3 \). The 2-category \( \text{RMod}_A(\text{TC}_3) \) has two simple objects \( \mathbb{1} \oplus m \) and \( \mathbb{1} \oplus m_c \). It is easy to check that \( \mathbb{1} \oplus m_c \) is not a local \( A \)-module. Therefore, we obtain that the 2-category of local \( A \)-modules in \( \text{TC}_3 \) is 2Vec and describes the trivial 3d topological order. This story is completely parallel to the condensation theory in 2d topological order [Kon14].

The rough boundary is depicted in Figure 22. There is no spin on the dashed edges on the boundary, and the \( B_p \) operators near the boundary is the product of three \( \sigma_z \) operators. On the rough boundary, the \( e \)-particles are condensed. For example, the \( e \)-particle Figure 22 can be created/annihilated by \( \sigma^1_x \). Also, Figure 23 shows that \( \mathbb{1}_c \)-strings are condensed on the rough boundary. Thus there are only two simple 1d topological defects on the rough boundary: the trivial string \( \mathbb{1} \) and the \( m \)-string, and there is no nontrivial 0d topological defects. Hence the topological defects on the rough boundary form a fusion 2-category 2Vec\( \mathbb{Z}_2 \), which is equivalent to 2Vec \( \boxtimes \) 2Vec as 2-categories.

![Figure 22](image)

Figure 22: The only nontrivial topological defects on the rough boundary are the \( m \)-strings. The \( e \)-particles are condensed.

Remark 5.2. The rough boundary is an example of the so-called string-only boundary of a 3d topological order [KW18]. It can be obtained by condensing all particles in the bulk. Note that \( \mathbb{1}_c \)-strings also condense because they are condensation descendants of the particles. Mathematically, the rough boundary can be obtained by condensing \( \mathbb{1}_c \), which should be viewed as a condensable algebra in \( \text{TC}_3 \) with the multiplication defined by the projection onto the first direct summand of \( \mathbb{1}_c \otimes \mathbb{1}_c \). From this point of view, the fusion 2-category 2Vec\( \mathbb{Z}_2 \) of boundary defects can also be realized by the 2-category \( \text{RMod}_e(\text{TC}_3) \) of right \( \mathbb{1}_c \)-modules. It has two simple objects: the trivial \( \mathbb{1}_c \)-module \( \mathbb{1}_c \) and a non-local \( \mathbb{1}_c \)-module \( m_c \). Therefore, the 2-category of local \( \mathbb{1}_c \)-modules is 2Vec and describes the trivial 3d topological order. This story is again parallel to the condensation theory in 2d topological order [Kon14]. We will develop the condensation theory for 3d topological orders elsewhere.

5.2 Braiding structure on \( \text{TC}_3 \)

According to the boundary-bulk relation [KWZ15, KWZ17], the topological defects in the bulk should be described by the center of those on a gapped/gapless boundary. In the case of 3d toric code model, it means that we should have the following two braided monoidal equivalences \( Z(\text{2Rep}(\mathbb{Z}_2)) \simeq \text{TC}_3 \simeq Z(2\text{Vec}_{\mathbb{Z}_2}) \). In particular, the boundary-bulk relation of the rough boundary, i.e. \( \text{TC}_3 \simeq Z(2\text{Vec}_{\mathbb{Z}_2}) \), tells us how to determine the braiding structures of \( \text{TC}_3 \) via the half-braidings in the category \( 2\text{Vec}_{\mathbb{Z}_2} \), as it was done in [KTZ20]. To realize the braiding physically with respect to the rough boundary, we only need to check if the physical half-braidings of topological defects on the rough boundary coincide with the mathematical half-braidings in \( 2\text{Vec}_{\mathbb{Z}_2} \) computed in [KTZ20] (see more detail for Appendix A).
A string-like topological defect $X$ in the bulk can be moved to the boundary and becomes a boundary topological defect, denoted by $F(X)$. Since $F(X)$ comes from the bulk, it has half-braidings 1-morphism $R_{f(X),Y} : F(X) \otimes Y \to Y \otimes F(X)$ with boundary topological defects $Y$. Similar to the fact that a braiding can be realized by a 0+1D defect in spacetime as discussed in Section 2.2, this half-braiding is a 1-morphism (0d domain wall) depicted as follows:

\[
\begin{array}{c}
\xymatrix{
F(X) \otimes Y & Y \otimes F(X) \\
F(X) & Y
}
\end{array}
\]

Similarly, a 0d domain wall $f$ in the bulk between $X$ and $X'$ can be moved to the boundary and becomes a boundary topological defect $F(f)$ between $F(X)$ and $F(X')$. Then $F$ should be a functor from the category of bulk topological defects to the category of boundary topological defects, called the bulk-to-boundary map. The 0d domain wall $F(f)$ also has half-braidings $R_{f(f),Y}$ with all boundary topological defects $Y$. This half-braiding is a 2-morphism (an instanton) depicted as follows:

\[
\begin{array}{c}
\xymatrix{F(X') \otimes Y & Y \otimes F(X')
\ar[r]^-{R_{f(f),Y}} & Y \otimes F(X')}
\end{array}
\]

These half-braidings should satisfy some compatibility conditions [KTZ20] Section 2].

Let us compute the half-braidings of bulk strings with boundary topological defects. There are only two simple strings $\mathbb{1}, m$ on the boundary, and clearly the half-braiding of a bulk topological defect with the trivial string $\mathbb{1}$ on the boundary is trivial. So we only need to compute the half-braidings of bulk topological defects with the $m$-string on the boundary.

It is clear that the $\mathbb{1}$-string and $m$-string in the bulk have trivial half-braiding with the $m$-string on the boundary. More precisely, we have $F(\mathbb{1}) = \mathbb{1}$ and $F(m) = m$, and the half-braidings are

\[
R_{f(\mathbb{1}),m} = \left(F(\mathbb{1}) \otimes m = \mathbb{1} \otimes m = m \otimes \mathbb{1} = \mathbb{1} \otimes m = m \otimes F(\mathbb{1})\right),
\]

and

\[
R_{f(m),m} = \left(F(m) \otimes m = m \otimes m = \mathbb{1} \otimes \mathbb{1} = \mathbb{1} = m \otimes m = m \otimes F(m)\right).
\]

To compute the half-braidings of the $\mathbb{1}_c$-string and $m_c$-string with the $m$-string on the rough boundary, we need to compute $F(\mathbb{1}_c)$ and $F(m_c)$. Figure 23 shows that $F(\mathbb{1}_c) = 1 \oplus 1$ because all the spins on the edges between the $\mathbb{1}_c$-string and the rough boundary have to be $\uparrow$ or $\downarrow$.

**Remark 5.3.** The computation of $F(\mathbb{1}_c)$ is similar to the fusion rule of two $\mathbb{1}_c$-strings in the bulk. Indeed, $\mathbb{1}_c$ is a condensable algebra in $\text{TC}_3$, and the rough boundary can be obtained by condensing the $\mathbb{1}_c$-strings.
Figure 23: The $1_c$-strings are condensed on the rough boundary. We have $F(1_c) = 1 \oplus 1$.

Similarly we have $F(m_c) = m \oplus m$:

$$F(m_c) = m \otimes F(1_c) = F(1_c) \otimes m = m \oplus m \oplus m = m \oplus F(1_c).$$

Then we can compute the half-braiding of the $1_c$-string with the $m$-string on the boundary, as depicted in Figure 24. Similarly, all the spins on the edges between the $1_c$-string and the rough boundary have to be $\uparrow$ or $\downarrow$. Thus the 0d domain wall in Figure 24 exchanges two direct summands of $F(m_c) = m \oplus m$:

$$R_{F(1_c),m} = (F(1_c) \otimes m = m \oplus m \oplus m = m \oplus F(1_c)). \quad (5.5)$$

Similarly, the half-braiding of the $m_c$-string and the $m$-string is

$$R_{F(m_c),m} = (F(m_c) \otimes m = 1 \oplus 1 \oplus 1 = m \oplus F(m_c)). \quad (5.6)$$

This half-braiding $R_{F(m_c),m}$ is also equal to $F(z)$, as depicted in Figure 25.

Finally we compute the half-braiding of 0d domain walls in the bulk with boundary topological defects. The only nontrivial one is the half-braiding of the $e$-particle in the bulk with the $m$-string, as depicted in the following picture:
Figure 24: the half-braiding of the $1c$-string with the $m$-string on the rough boundary

Figure 25: The 0d domain wall $F(z)$ is equal to the half-braiding (5.6).
This half-braiding can be realized by the operator \( a_1^2 a_2^2 \): first an \( e \)-particle is created from the rough boundary, then moved across the \( m \)-string and annihilated to the rough boundary. This operator is nothing but the \( B_p \) operator on the middle plaquette, thus it equals to \(-1\) due to the existence of the \( m \)-string. In other words, we have

\[
R_{f(3),m} = \left( R_{f(3),m} \circ (F(e) \otimes 1_m) = 1_m \xrightarrow{-1} 1_m = (1_m \otimes F(e)) \circ R_{f(3),m} \right).
\]

### 5.3 Nondegeneracy

Now we check the nondegeneracy of the braidings of the braided fusion 2-category \( TC_3 \). Mathematically, \( TC_3 \) is nondegenerate means its sylleptic center is trivial, i.e. equivalent to \( 2Vec \). The non-degeneracy of \( TC_3 \approx Z(2Vec_{Z_2}) \) has been proved in [KTZ20]. Physically, the non-degeneracy of the braidings means:

1. If a simple string \( X \) has trivial double braiding with all strings, i.e. \( R_{Y,X} \circ R_{X,Y} = 1_{X \otimes Y} \) for all \( Y \), then \( X = 1 \) is the trivial string.

2. If a simple particle \( f : 1 \rightarrow 1 \) on the trivial string has trivial double braiding with all strings, i.e. \( R_{3,Y} \circ (f \otimes 1_Y) \xrightarrow{R_{Y}} (1_Y \otimes f) \circ R_{3,Y} \xrightarrow{R_{Y}} R_{4,Y} \circ (f \otimes 1_Y) = \text{id}_{R_{3,Y}(f \otimes 1_Y)} \) for all \( Y \), then \( f = 1_x \) is the trivial particle.

We have shown that the double braiding between an \( m \)-string and a \( 1_c \)-string is nontrivial, and the double braiding between an \( e \)-particle and an \( m \)-string is nontrivial. Therefore, we conclude that the braidings in \( TC_3 \) are non-degenerate.

**Remark 5.4.** The three-loop braidings are important data to distinguish 4-cocyles in 3+1D Dijkgraaf-Witten theories [WL14], and have a deep relation with 3d modular data [JMR14, WW13], and are perhaps also related to the non-degeneracy of braidings. It is an interesting project to compute the three-loop braidings explicitly. We leave it to the future. \( \diamond \)

### Appendix

#### A Braided monoidal 2-category \( Z(2Vec_{Z_2}) \)

We recall the braiding monoidal structures of the 2-category \( Z(2Vec_{Z_2}) \) from [KTZ20]. By construction, \( 1,m = \text{Vec} \) and \( 1_c,m_c = \text{Vec}_{Z_2}, \text{Vec}_{Z_2} \)-modules. The 2-category structure of \( Z(2Vec_{Z_2}) \) can be conveniently represented by the following quiver.

\[
\begin{align*}
\begin{array}{c}
\text{Rep}(Z_3) \\
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Vec}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Vec}_{Z_2}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Vec}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Rep}(Z_3)
\end{array}
\end{array}
\end{align*}
\end{equation}
\]

(A.1)

The non-trivial fusion rules are given by

\[
m \otimes m \simeq 1, \quad 1_c \otimes m \simeq m_c, \quad 1_c \otimes 1_c \simeq m_c \otimes m_c \simeq 1_c \oplus 1_c; \quad 1_c \otimes m_c \simeq m_c \oplus m_c. \quad (A.2)
\]

We focus on the braiding structure below. For \( X,Y = 1,1_c,m,m_c \) and \( Z_2 = \{1,s\} \), we have

\[
R_{X,Y} : X \otimes Y \rightarrow Y \otimes X \\
(x,y) \mapsto (y, \rho_s(x)),
\]

23
where \( g = 1 \) for \( Y = 1, \mathbb{1} \), and \( g = s \) for \( Y = m, m_c \) and \( \rho_g : X \to X \) is the action of \( \delta_g \in \text{Vec}_{\mathbb{Z}_2} \) on \( X \), where \( \delta_g \) is a simple object of \( \text{Vec}_{\mathbb{Z}_2} \) with the \( g \)-grading. More explicitly,

\[
R_{XY} = \begin{cases} 
(x, y) \mapsto (y, \rho_s(x)) & \text{if } X = 1, m, \text{ and } Y = m, m_c; \\
(x, y) \mapsto (y, x) & \text{if otherwise}. 
\end{cases} \tag{A.3}
\]

It follows that there are seven non-trivial double-braiding 1-isomorphisms:

\[
X \otimes Y \xrightarrow{\rho_{XY}} X \otimes Y = \begin{cases} 
(x, y) \mapsto (\rho_s(x), y) & \text{for } X \otimes Y = 1, m, 1_c \otimes m, m \otimes m; \\
(x, y) \mapsto (x, \rho_s(y)) & \text{for } X \otimes Y = m \otimes 1_c, m \otimes m_c, m_c \otimes 1_c; \\
(x, y) \mapsto (\rho_s(x), \rho_s(y)) & \text{for } X \otimes Y = m_c \otimes m. 
\end{cases} \tag{A.4}
\]

Using the compatibility between the fusions and braidings, all of the seven can be reduced to only two cases: \( X \otimes Y = 1_c \otimes m, m \otimes 1_c \). Namely, it is enough to only check the double braiding between \( 1_c \) and \( m \).

When the braiding involves non-trivial 1-morphisms \( f : X \to X' \), we also have the following 2-isomorphisms:

\[
\begin{array}{c}
X \otimes Y \xrightarrow{R_{XY}} Y \otimes X \\
\downarrow f \circ 1 \quad \downarrow 1 \circ f \\
X' \otimes Y \xrightarrow{R_{XY}} Y \otimes X'
\end{array}
\quad \quad
\begin{array}{c}
X \otimes Y \xrightarrow{R_{XY}} Y \otimes X \\
\downarrow 1 \circ (R_{XY})^{-1} \quad \downarrow 1 \circ f \\
X \otimes Y' \xrightarrow{R_{XY}} Y' \otimes X
\end{array}
\]

When \( X = X' \), we have

\[
R_{f,Y} = \begin{cases} 
1 & \text{if } X = 1, m \text{ and } f = e \text{ and } Y = 1, 1_c; \\
-1 & \text{if } X = 1, m \text{ and } f = e \text{ and } Y = m, m_c; \\
1 & \text{if } X = 1_c, m_c \text{ and } f = z \text{ and } Y = 1, 1_c; \\
1 & \text{if } X = 1_c, m_c \text{ and } f = z \text{ and } Y = m, m_c. 
\end{cases} \tag{A.5}
\]

It follows that, there are only four non-trivial double-braiding 2-isomorphisms:

\[
R_{Y,f} \circ R_{f,Y} = -1 \quad \text{for } X \otimes Y = 1 \otimes m, 1 \otimes m_c, m \otimes m, m \otimes m, \text{ and } f = e. \tag{A.6}
\]

Using the compatibility between the fusions and braidings, all of the four can be reduced to only one case \( X \otimes Y = 1 \otimes m \) and \( f = e \). When \( X \neq X' \), the 2-isomorphism \( R_{f,Y} \) is always the identity, so is the associated double braiding.

### References

[AF19] David Ayala and John Francis. *A factorization homology primer*. arXiv preprint, 2019, [arXiv:1903.10961](https://arxiv.org/abs/1903.10961).

[AKZ17] Yinghua Ai, Liang Kong, and Hao Zheng. *Topological orders and factorization homology*. Advances in Theoretical and Mathematical Physics, 21(8):1845–1894, 2017, [arXiv:1607.08422](https://arxiv.org/abs/1607.08422).

[BD19] Alex Bullivant and Clement Delcamp. *Tube algebras, excitations statistics and compactification in gauge models of topological phases*. Journal of High Energy Physics, 2019(10), 2019, [arXiv:1905.08673](https://arxiv.org/abs/1905.08673).
Alex Bullivant and Clement Delcamp. *Gapped boundaries and string-like excitations in (3+1)d gauge models of topological phases*. arXiv preprint, 2020, arXiv:2006.06536.

Christopher L. Douglas and David J. Reutter. *Fusion 2-categories and a state-sum invariant for 4-manifolds*. arXiv preprint, 2018, arXiv:1812.11933.

Dominic V. Else and Chetan Nayak. *Cheshire charge in (3+1)-dimensional topological phases*. Physical Review B, 96(4), 2017, arXiv:1702.02148.

Davide Gaiotto and Theo Johnson-Freyd. *Condensations in higher categories*. arXiv preprint, 2019, arXiv:1905.09566.

Alioscia Hamma, Paolo Zanardi, and Xiao-Gang Wen. *String and membrane condensation on three-dimensional lattices*. Physical Review B, 72(3), 2005, arXiv:cond-mat/0411752.

Shenghan Jiang, Andrej Mesaros, and Ying Ran. *Generalized modular transformations in (3+1)D topologically ordered phases and triple linking invariant of loop braiding*. Physical Review X, 4(3), 2014, arXiv:1404.1062.

Alexei Kitaev. *Fault-tolerant quantum computation by anyons*. Annals of Physics, 303(1):2–30, 2003, arXiv:quant-ph/9707021.

Alexei Kitaev. *Anyons in an exactly solved model and beyond*. Annals of Physics, 321(1):2–111, 2006, arXiv:cond-mat/0506438.

Alexei Kitaev and Liang Kong. *Models for gapped boundaries and domain walls*. Communications in Mathematical Physics, 313(2):351–373, 2012, arXiv:1104.5047.

Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, and Hao Zheng. *Classification of topological phases with finite internal symmetries in all dimensions*. arXiv preprint, 2020, arXiv:2003.08898.

Liang Kong. *Anyon condensation and tensor categories*. Nuclear Physics B, 886:436–482, 2014, arXiv:1307.8244.

Liang Kong, Yin Tian, and Shan Zhou. *The center of monoidal 2-categories in 3+1d Dijkgraaf-Witten theory*. Advances in Mathematics, 360:106928, 2020, arXiv:1905.04644.

Liang Kong and Xiao-Gang Wen. *Braided fusion categories, gravitational anomalies, and the mathematical framework for topological orders in any dimensions*. arXiv preprint, 2014, arXiv:1405.5858.

Liang Kong, Xiao-Gang Wen, and Hao Zheng. *Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers*. arXiv preprint, 2015, arXiv:1502.01690.

Liang Kong, Xiao-Gang Wen, and Hao Zheng. *Boundary-bulk relation in topological orders*. Nuclear Physics B, 922:62–76, 2017, arXiv:1702.00673.

Tian Lan, Liang Kong, and Xiao-Gang Wen. *Classification of 3+1D bosonic topological orders: The case when pointlike excitations are all bosons*. Physical Review X, 8(2), 2018, arXiv:1704.04221.
[LW05] Michael A. Levin and Xiao-Gang Wen. *String-net condensation: A physical mechanism for topological phases*. Physical Review B, 71(4), 2005, arXiv:cond-mat/0404617

[LW14] Tian Lan and Xiao-Gang Wen. *Topological quasiparticles and the holographic bulk-edge relation in (2+1)-dimensional string-net models*. Physical Review B, 90(11), 2014, arXiv:1311.1784

[WL14] Chenjie Wang and Michael Levin. *Braiding statistics of loop excitations in three dimensions*. Physical Review Letters, 113(8), 2014, arXiv:1403.7437

[WW15] Juven C. Wang and Xiao-Gang Wen. *Non-abelian string and particle braiding in topological order: Modular SL(3, Z) representation and (3+1)-dimensional twisted gauge theory*. Physical Review B, 91(3), 2015, arXiv:1404.7854