Learning-based Symbolic Abstractions for Nonlinear Control Systems

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Abstract

Symbolic models or abstractions are known to be powerful tools for the control design of cyber-physical systems (CPSs) with logic specifications. In this paper, we investigate a novel learning-based approach to the construction of symbolic models for nonlinear control systems. In particular, the symbolic model is constructed based on learning the un-modeled part of the dynamics from training data based on state-space exploration, and the concept of an alternating simulation relation that represents behavioral relationships with respect to the original control system. Moreover, we aim at achieving safe exploration, meaning that the trajectory of the system is guaranteed to be in a safe region for all times while collecting the training data. In addition, we provide some techniques to reduce the computational load, in terms of memory and computation time, of constructing the symbolic models and the safety controller synthesis, so as to make our approach practical. Finally, a numerical simulation illustrates the effectiveness of the proposed approach.

Key words: Symbolic models, uncertain systems, safety controller synthesis, Gaussian Processes

1 Introduction

In cyber-physical systems (CPSs), computational devices are tightly integrated with physical processes. Embedded computers monitor the behavior of the physical processes through sensors, and usually control them through actuators using feedback loops. Nowadays, CPSs are ubiquitous in modern control engineering, including automobiles, aircraft, building control systems, chemical plants, transportation systems, and so on. Many CPSs are safety critical or mission critical: it must ensure that the system operates correctly meeting the satisfaction of safety or some desired specifications. Formal methods are known to provide essential tools for the design of CPSs, as they give theoretical or rigorous mathematical proofs that the system works correctly meeting the desired specification [1]. While the formal methods have been originally developed in software engineering that aims at finding bugs or security vulnerabilities in the software, the methodologies have been recently recognized to be useful in other applications, including the control design of CPSs. In particular, one of the most successful methods that interface the formal methods and the control design of CPSs is the so-called symbolic control, see, e.g., [2]. The main objective of the symbolic control is to design controllers for CPSs with logic specifications (as detailed below). In such approaches, symbolic models or abstractions are constructed based on the original control systems. Roughly speaking, while the original control system is represented in a continu-
ous state (and input) space, the symbolic model is represented in a discrete state (and input) space, while preserving the behavior of the original control system. As such, controllers can be designed based on several algorithmic techniques from supervisory control of discrete event systems, such as a safety/controllability game [3].

The symbolic approach is known to be a powerful tool for the control design of CPSs in the following three ways. First, it allows us to synthesize controllers for general nonlinear dynamical systems with state and input constraints. Second, by constructing the symbolic model, we can take into account the constraints that are imposed on the cyber part with regard to the digital platform, such as a quantization effect. Third, it allows us to synthesize controllers under various control specifications, including safety, reachability, or more complex ones such as those expressed by linear temporal logic (LTL) formulas or automata on infinite strings. As previously mentioned, symbolic models or abstractions are constructed such that they are represented in a discrete state space while preserving the behavior of the original control system. More formally, behavioural relationships such as the concept of approximate (bi-)simulation relation, see, e.g., [3,4,5,6,7,8,9,10,11,12,13], are used to relate the behaviours of the original control system and its symbolic model. For example, [5] employs an approximate bisimulation relation to construct the symbolic model for nonlinear, incrementally asymptotically stable systems. [8] employs an approximate alternating simulation relation, so that the symbolic models can be constructed for general (incrementally forward complete) nonlinear systems without any assumption on stability. Moreover, [11,12] characterize the notion of robustness for input-output dynamical systems based on the concept of a contractive approximate simulation relation.

In this paper, we focus on investigating the construction of symbolic models for nonlinear control systems. In particular, we consider the case where the dynamics of the plant includes state-dependent, un-modeled dynamics. In contrast to the aforementioned abstraction schemes, we propose a learning-based solution to this problem, in which the symbolic model is constructed based on learning the un-modeled dynamics from training data. Moreover, we aim at achieving safe-exploration, meaning that the trajectory of the system stays inside a safe set for all times while collecting the training data. Achieving safe exploration is particularly useful for safety critical CPSs, see, e.g., [14]. More technically, as a starting point of our approach, we employ the Gaussian process (GP) regression [15] in order to estimate the un-modeled dynamics from training data. As we will see later, it is shown that, under some smoothness assumption on the un-modeled dynamics, an error bound on the un-modeled dynamics can be derived based on the result from [16]. Note that, in contrast to previous approaches of learning-based controller synthesis with the GP regression (e.g., [17,18]) that make use of an error (or regret) bound that involves an information gain, here we will make use of a deterministic error bound that does not involve the information gain, which has been also derived in [16] (for details, see Lemma 2 and Remark 2 in this paper). Based on this error bound and the concept of an approximate alternating simulation relation [8], we then provide an approach to construct the symbolic model. To achieve the safe exploration, we also provide a safety controller synthesis via a safety game [3]. Finally, we provide an overall algorithm that collects the training data from scratch and constructs the symbolic model. Along with this algorithm, we provide several techniques to reduce the computational load of constructing the symbolic model and the safety controller synthesis. In particular, we provide a lazy abstraction scheme, in which the transitions of the symbolic model are updated only around the region where the training data is collected.

( Related works): The approach presented in this paper is related to previous literature in terms of symbolic control (or temporal logics) and controller synthesis for dynamical systems learned by training data. In what follows, we discuss how our approach differs from previous works and highlight our main contributions.

As previously mentioned, there have been a wide variety of symbolic control techniques for dynamical systems, e.g., [4,5,6,7,8,9,10,11,12,13]; however, most of the previous approaches typically assume that the dynamics of the plant is completely known or they consider uniform disturbance that is not learned from training data. The learning-based approach is advantageous over the uniform disturbance-based approach in the following sense. In the uniform disturbance-based approach, every transition of the symbolic model is defined by taking the worst case effect of the un-modeled function, since the un-modeled function will not be learned from data. On the other hand, in the learning-based approach, the un-modeled function will be learned and thus its uncertainty will decrease as the state exploration progresses. Hence, the symbolic model will have fewer redundant transitions than the uniform disturbance-based approach, and this leads to obtaining a larger region that guarantees safety (i.e., controlled invariant set). To the best of our knowledge, there are only few works of symbolic or temporal logic control for a dynamical system that is partially unknown and is learned by training data (e.g., by the GP regression) [21,22]. In [21], the authors provided a way to obtain a finite abstraction using interval Markov decision processes (IMDPs) with the unknown dynamics learned by the GP regression. The abstraction has been then utilized for safety verification. Our approach is different from this previous work in the following sense: first, while the proposed approach in [21] makes it possible to provide probabilistic guarantees, in this paper, we are able to provide deterministic guarantees. Second, while in [21] the symbolic model is constructed to deal with only safety specifications, in this paper, we are constructing the symbolic abstraction in the more general
sense of alternating simulation relations, i.e., we can refine a controller for the symbolic model into a controller for the original system for any specification, and not just safety. Finally, for the particular class of safety specifications, the authors in [21] provided a way of constructing a symbolic model with given initial training data, and they did not provide an approach to update the symbolic model when new training data are collected online, and which is the main issue in learning-based control, since the objective is to exploit the new training data collected online. In contrast, our approach provides a new computationally efficient approach to collect new training data while reducing the computation load to update the symbolic model and to synthesize safety controllers. In [22], the authors provided a way to detect faults using signal temporal logic (STL) for partially unknown dynamical systems and these are learned by the GP regression. However, the problem setup considered in [22] is different from the one considered in this paper. Specifically, while [22] considered a monitoring scheme in which they monitor behaviors of the system without control inputs and check if a given STL formula is satisfied, this paper considers a synthesis scheme in which we find a controller to satisfy certain specifications (expressed by, e.g., temporal logic formulas). The proposed approach is also different; while our approach aims at constructing symbolic models from training data while guaranteeing a given specification, the approach in [22] provided a monitoring scheme by employing a robustness degree of STL formulas.

Apart from the use of symbolic control, various learning-based controller synthesis techniques with the GP regression have been proposed. Most of the previous works aim at synthesizing controllers to achieve stability/tracking [23,24,25,26], or to guarantee safety [17,18,27,28,29,30,31,32,33]. Since we here construct a safety controller to achieve a safe exploration, our approach is particularly related to the second category, i.e., [17,18,27,28,29,30,31,32,33]. For example, the authors in [17] (resp. [18]) proposed an approach to learn a region of attraction (ROA) using safety controllers for continuous-time systems: \( x(t) = f(x(t), u(t)) + g(x(t), u(t)) \) (resp. discrete-time systems: \( x(k+1) = h(x(k), u(k)) + g(x(k), u(k)) \)), where the function \( g(\cdot) \) is unknown and it is learned by the GP regression. The assumptions on the unknown function \( g(\cdot) \) that are made in [17] and [18] are the same as the ones we are using in this paper, namely the fact that the unknown function lies in the reproducing kernel Hilbert space (RKHS). However, the authors in [17,18] assumed the existence of a known Lyapunov function for the nominal system \( \dot{x}(t) = f(x(t), u(t)) + g(x(k+1), u(k+1)) \), for which the computation may be difficult for general nonlinear systems. The approach presented in this paper allows us to deal with more general complex specifications (including safety) without requiring the existence of a Lyapunov function. The approaches presented in [30,31,32] used a control barrier function and [29] used a Hamilton-Jacobi-Issac (HJI) equation to synthesize safety controllers with the GP regression. The proposed approach presented in this paper is significantly different from [30,31,32,29] in the following sense. First, note that while the goal of the previous work is to derive a safety controller, our main goal is to construct a symbolic model. Constructing the symbolic model is beneficial since it allows not only to compute a safety controller, but also controllers from more general complex specifications, such as those expressed by temporal logic specifications and automata on infinite strings. Moreover, while in [30,31] (resp. [32]), the use of barrier functions makes it only possible to deal with the class of polynomial dynamical systems (resp. input affine systems), the proposed approach in this paper makes it possible to deal with general nonlinear systems, and this is achieved by employing the symbolic models. Besides, while solving the HJI equation generally requires a heavy computational load, [29] did not provide a way of speeding up the computation of solving the HJI equation when a new set of training data is obtained. On the other hand, we here propose a way of reducing the computation load to update the symbolic model as well as synthesize safety controllers even if a new set of training data is obtained online.

Due to page limitation, some detailed proofs, remarks, and simulation results are given in the extended version of this paper [34].

**Notation.** Let \( \mathbb{N}, \mathbb{N}_{\geq a}, \mathbb{N}_{> a}, \mathbb{N}_{a:b} \) be the sets of integers, integers larger than or equal to \( a \), integers larger than \( a \), and integers from \( a \) to \( b \) respectively. Let \( \mathbb{R}, \mathbb{R}_{\geq a}, \mathbb{R}_{> a} \) be the sets of reals, reals larger than or equal to \( a \) and reals larger than \( a \), respectively. Given \( a, b \in \mathbb{R} \) with \( a \leq b \), let \( [a, b] \) be the interval set from \( a \) to \( b \). Given \( a, b \in \mathbb{R}_{\geq a} \), we let \( [a, b] = \{ a \leq x \leq b \} \). Denote by \( \|x\|_{\infty} \) the infinity norm of a vector \( x \). Given \( x \in \mathbb{R}^n, \varepsilon \in \mathbb{R}_{\geq 0}, \) let \( B_\varepsilon(x) = \{ x \in \mathbb{R}^n, \|x\|_{\infty} \leq \varepsilon \} \). Given \( \mathcal{X} \subseteq \mathbb{R}^n \) and \( \eta > 0 \), denote by \( \mathcal{X}_{\eta} \subseteq \mathbb{R}^n \) the lattice in \( \mathcal{X} \) with the quantization parameter \( \eta \), i.e., \( \mathcal{X}_{\eta} = \{ x \in \mathcal{X} \mid x_i = a_i \eta, a_i \in \mathbb{N}, i = 1, 2, \ldots, n \} \), where \( x_i \in \mathbb{R} \) is the \( i \)-th element of \( x \). Given \( x \in \mathbb{R}^n \), \( \mathcal{X} \subseteq \mathbb{R}^n \), denote by \( \text{Nearest}_\mathcal{X}(x) \) the closest points in \( \mathcal{X} \) to \( x \), i.e., \( \text{Nearest}_\mathcal{X}(x) = \arg\min_{x' \in \mathcal{X}} \|x - x'\|_{\infty} \). Given \( \mathcal{X} \subseteq \mathbb{R}^n \), we let \( \text{Interior}_\mathcal{X}(\mathcal{X}) = \{ x \in \mathcal{X} \mid B_\varepsilon(x) \subseteq \mathcal{X} \} \).

## 2 Preliminaries

In this section we recall some basic concepts of the Gaussian Process (GP) regression [15], transition systems and approximate alternating simulation relations [8].

### 2.1 Gaussian process regression

Consider a nonlinear function \( h : \mathbb{R}^n \to \mathbb{R} \) perturbed by additive noise as \( y = h(x) + v \), where
x ∈ ℝⁿˣ is the input, y ∈ ℝ is the output, and
v ∼ N(0, σ²) is the Gaussian distributed white
noise. Given m : ℝⁿˣ → ℝ and some kernel function
k : ℝⁿˣ × ℝⁿˣ → ℝ ≥0, suppose that, for any
finite number of inputs X = [x₁, ..., xₜ] (xᵢ ∈ ℝⁿˣ,
t ∈ {1, ..., T}), the joint probability distribution of
the corresponding outputs y = [y₁, y₂, ..., yₜ]ᵀ follows
the multivariate Gaussian distribution: y ∼ N(M(K),
where M = [m(x₁), ..., m(xₜ)] and Kᵢⱼ = k(xᵢ, xⱼ),
t, t' ∈ {1, ..., T} (Kᵢⱼ denotes the (t, t')-element
of K). Then, we say that the function h follows a
Gaussian process (GP) [15], and it is denoted by
h(x) ∼ GP(m(x), k(x, x')).

In the GP regression problem, we start by assuming a GP
prior: h(x) ∼ GP(m(x), k(x, x')). Let D = {x₂, y₂}∈₁ denote a training data set. Then, using Bayes rule, the
posterior distribution of the output for the Gaussian distribution, i.e.,
Pr(y|x, D) = N(µ(x|D), σ²(x|D)). Here, the mean
µ(x; D) and the variance σ²(x; D) are given by

\[
\begin{align*}
\mu(x; D) &= m(x) + k_τ^T(x)(K + \sigma^2I)^{-1}(Y - M), \\
\sigma^2(x; D) &= k(x, x) - k_τ^T(x)(K + \sigma^2I)^{-1}k_τ(x),
\end{align*}
\]

where I is the identity matrix of appropriate dimension, and
k_τ(x) = [k(x, x₁), ..., k(x, xₜ)]ᵀ.

2.2 Transition system, alternating simulation relation

We provide the notion of a transition system, which will
be useful to describe a control system formalized later in this paper.

Definition 1 A transition system is a quadruple S =
(X, U, U, G), where:

- X is a set of states;
- x₀ ∈ X is an initial state;
- U is a set of inputs;
- G : X × U → 2^X is a transition map.

Roughly speaking, we denote by x' ∈ G(x, u) if and
only if the system evolves from x to x' by applying the
control input u. The state x' is called an u-successor of x.
Moreover, we denote by U(x) the set of all inputs u ∈ U,
for which G(x, u) ≠ ∅.

Next, we shall recall the notion of an approximate alter-
ning simulation relation[7,8], which is a well-known
concept to represent behavioral relationships on the
similarity between two transition systems.

Definition 2 (ε-ASR) Let S₀ = (X₀, x₀₀, U₀, G₀) and
S₁ = (X₁, x₀₁, U₁, G₁) be two transition systems. Given
e ∈ ℝ ≥0, a relation R(ε) ⊆ X₀ × X₁ is called an ε-
approximate Alternating Simulation Relation (or ε-ASR
for short) from S₀ to S₁, if the following conditions are satisfied:

(C.1) (x₀₀, x₁₀) ∈ R(ε);
(C.2) For every (x₀, x₁) ∈ R(ε), we have ||x₀ - x₁||∞ ≤ ε;
(C.3) For every (x₀, x₁) ∈ R(ε) and for every u ∈ U₀(x₀), there exist u ∈ U₁(x₁), such that the follow-
holds: for every x₁ ∈ G₁(x₁, u₁), there exists
x₁ ∈ G₁(x₁, u₁), such that (x₀, x₁) ∈ R(ε).

The transition system S₀ serves as the abstract
expression of S₁, in the sense that every transition of S₀ can be
approximately simulated by those of S₁ according to
(C.1)–(C.3) in Definition 2. The concept of an ε-ASR
is particularly useful to synthesize a controller for the
transition system S₀, based on the controller for S₁. That is,
we obtain S₀, S₁, etc., and then we can synthesize a controller for S₀
by refining a controller for S₁ that can be synthesized by
algorithmic techniques from discrete event systems,
see, e.g., [3].

3 Problem formulation

In this section, we describe a control system that we seek
to consider, provide the notion of a controlled invariant
set, and describe the goal of this paper.

3.1 System description

Let us consider the following nonlinear systems:

\[x_{t+1} = f(xₜ, uₜ) + d(xₜ) + vₜ, \quad (3)\]
\[x₀ = x, \quad u₀ ∈ U, \quad v₀ ∈ V, \quad (4)\]

for all t ∈ N ≥0, where x ∈ ℝⁿₓ is the state, u ∈ ℝⁿᵤ
is the control input, v ∈ ℝⁿᵥ is the additive noise, and
x ∈ ℝⁿₓ is the initial state. Moreover, U ⊆ ℝⁿᵤ and
V ⊆ ℝⁿᵥ are the set of control inputs and the additive
noise, respectively. It is assumed that U is compact and
V is given by V = \{v ∈ ℝⁿᵥ : ||v||∞ ≤ σ₀\} for a given
σ₀ > 0. Moreover, f : ℝⁿₓ × ℝⁿᵤ → ℝⁿₓ is the known
function that captures the modeled (or nominal) dynamics,
and d : ℝⁿₓ → ℝⁿₓ is the state-dependent, unknown
deterministic function that captures the un-modeled
dynamics. Regarding the function f, we assume the follow-
ing Lipschitz continuity:

Assumption 1 The function f is Lipschitz continu-
inous in x ∈ ℝⁿₓ, i.e., given L_f ∈ ℝ ≥0, ||f(xₜ, uₜ) -
f(xₜ, u₂)||∞ ≤ L_f ||xₜ - x₂||∞, ∀xₜ, x₂ ∈ ℝⁿₓ, ∀u ∈ U. □

Regarding the unknown function dₜ, i ∈ N₁,nₓ, in this paper
we provide a certain smoothness assumption (see,
e.g., [17,18]):
Assumption 2 For each $i \in \mathbb{N}_{1:n_x}$, let $k_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ be a given, continuously differentiable kernel function and $\mathcal{H}_k$ be the reproducing kernel Hilbert space (RKHS) corresponding to $k_i$ with the induced norm denoted by $\| \cdot \|_{k_i}$. Then, for each $i \in \mathbb{N}_{1:n_x}$, it is assumed that $d_i \in \mathcal{H}_k$. Moreover, an upper bound of the RKHS norm $\|d_i\|_{k_i} \leq B_i$ is available. \hfill $\square$

Assumption 2 implies that each $d_i : \mathbb{R}^{n_x} \rightarrow \mathbb{R}$ is characterized of the form $d_i(x) = \sum_{n=1}^{\infty} \alpha_n k_i(x, x_n)$, where $x_n \in \mathbb{R}^{n_x}$, $n \in \mathbb{N}_{>0}$ are the representer points and $\alpha_n \in \mathbb{R}$, $n \in \mathbb{N}_{>0}$ are the parameters that it is necessary to decay sufficiently fast as $n$ increases. The induced norm is given by $\|d_i\|_{k_i}^2 = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \alpha_n \alpha_{n'} k_i(x_n, x_{n'})$. In general, obtaining a large enough, yet not too conservative bound for $\|d_i\|_{k_i}$ is hard; nevertheless, there exist several ways to compute an upper bound of $\|d_i\|_{k_i}$ (see, e.g., [35]). The overview of how to compute an upper bound of $\|d_i\|_{k_i}$ is given in Appendix A of [34], and we refer the interested reader to [35] for a more detailed discussion.

Assumption 2 allows us to show the following result:

**Lemma 1** Suppose that Assumption 2 holds. Then, it follows that $|d_i(x_1) - d_i(x_2)| \leq L_i \sqrt{\|x_1 - x_2\|_2}$, for all $x_1, x_2 \in \mathcal{X}$, where $L_i = B_i \sqrt{2}\|\partial k_i/\partial x\|_\infty$. \hfill $\square$

For the proof, see [34].

**Remark 1** (On selecting $k_i$) Assumption 2 implies that the kernel function $k_i$ should be chosen apriori. Potential candidates of this kernel function are: $k_i(x, y) = e^{-\alpha \|x-y\|^2}$, $k_i(x, y) = (b + \|x-y\|^2)^{-a}$, where $a, b$ are positive constants. A useful property of employing these kernel functions is the universal approximation property, i.e., the RKHSs are dense in the space of all continuous functions over any compact set (see, e.g., [36]). Hence, if the kernel function is selected as above, any continuous function can be estimated arbitrarily well by a function that lies in the RKHS. Note that there are indeed other kernels satisfying the universal approximation property, which may also be useful to be employed (see, e.g., [36]). \hfill $\square$

### 3.2 Controlled invariant set and safety controller

A sequence $x_0, x_1, x_2, \ldots \in \mathbb{R}^{n_x}$ is called a trajectory of the system (3), if there exist $u_0, u_1, u_2, \ldots \in \mathcal{U}$, $v_0, v_1, v_2, \ldots \in \mathcal{V}$ such that $x_0 = x$, $x_{i+1} = f(x_i, u_i) + d(x_i) + v_i$, $\forall t \in \mathbb{N}_{>0}$. Moreover, a controller is defined as a set-valued mapping from each state onto the set of control inputs, i.e., $C : \mathbb{R}^{n_x} \rightarrow 2^\mathcal{U}$. Given $C$, a controlled trajectory is defined as any trajectory of the system (3), $x_0, x_1, x_2, \ldots \in \mathbb{R}^{n_x}$ with $u_t \in C(x_t)$, $\forall t \in \mathbb{N}_{>0}$.

Now, denote by $\mathcal{X} \subset \mathbb{R}^{n_x}$ a safe set, in which the trajectory of the system (3) must stay for all times. It is assumed that $\mathcal{X}$ is compact and can be either convex or non-convex, and that $\mathcal{X} \subseteq \mathcal{X}$. Based on the above, we define the notion of a controlled invariant set (see, e.g., [37]) and the safety controller as follows:

**Definition 3** A set $\mathcal{X}_S \subseteq \mathcal{X}$ is called a controlled invariant set in $\mathcal{X}$, if there exists a controller $C_S : \mathbb{R}^{n_x} \rightarrow 2^\mathcal{U}$ such that the following holds: for every $x \in \mathcal{X}_S$, there exists $u \in C_S(x)$ such that for every $v \in \mathcal{V}$, $f(x, u) + d(x) + v \in \mathcal{X}_S$. The controller $C_S$ is called a safety controller. \hfill $\square$

That is, $\mathcal{X}_S$ is called a controlled invariant set if there exists a controller $C_S$ such that every controlled trajectory induced by $C_S$ (starting from anywhere in $\mathcal{X}_S$) stays in $\mathcal{X}_S$ for all times.

### 3.3 The goal of this paper and overview of the approach

The goal of this paper is to construct a symbolic model of the control system (3), which indicates an abstract expression of (3). In particular, due to the existence of the unknown function $d$, we here propose a learning-based approach, in which the symbolic model is constructed by learning the unknown function $d$ from training data. Towards this end, we first provide an approach to construct a symbolic model for given training data (Section 4). The symbolic model is constructed based on the GP regression and the concept of an $\varepsilon$-ASR; for details, see Section 4. Based on the symbolic model, we proceed by developing an overall algorithm that aims at collecting the training data from scratch and constructing the symbolic model (Section 5). In particular, we propose a safe exploration algorithm, in which the trajectory of the system (3) must stay in $\mathcal{X}$ for all times while collecting the training data and constructing the symbolic model. As we will see later, this is achieved by iteratively updating the symbolic model, controlled invariant set and the safety controller after each step of the state-space exploration; for details, see Section 5.

### 4 Constructing symbolic models with Gaussian processes

In this section, we provide an approach to construct a symbolic model based on a given set of training data. In Section 4.1, we provide an approach to learn $d_i$ with the GP regression as well as a useful error bound on $d_i$ based on Assumption 2. In Section 4.2, we provide a way of how to construct a symbolic model from a given set of training data. In Section 4.3, we provide a safety controller synthesis, which will be useful to achieve the safe exploration provided in the next section.

#### 4.1 Learning $d$ with the GP regression

In this paper, we estimate each element of $d$, i.e., $d_i$, $i \in \mathbb{N}_{1:n_x}$ ($d = [d_1, d_2, \ldots, d_{n_x}]^T$) by the GP regression
with the kernel function \( k_i \). To this end, for each \( i \in \mathbb{N}_{1,n_x} \), let \( \mathcal{D}_{T,i} = \{X_T, Y_{T,i}\} \) be the set of input-output training data in order to estimate \( d_i \), given by \( X_T = [x_1, x_2, \ldots, x_T], Y_{T,i} = [y_{i,1}, y_{i,2}, \ldots, y_{i,T}]^T \), where \( T \in \mathbb{N}_{0} \) is the number of training data points and \( y_{i,t} = x_{t+1,i} - f_i(x_t, u_t), \forall t \in \mathbb{N}_{1,T} \) are the training outputs, with \( x_{t,i} \) and \( f_i(x_t, u_t) \) being the \( i \)-th element of \( x_t \) and \( f(x_t, u_t) \), respectively. Note that we have \( y_{i,t} = x_{t+1,i} - f_i(x_t, u_t) = d_i(x_t) + v_{t,i} \), where \( v_{t,i} \) denotes the \( i \)-th element of \( v_t \) with \( |v_{t,i}| \leq \sigma_v \). Hence, \( y_{i,t} \) represents the noisy output of \( d_i(x_t) \) with the additive noise bounded by \( \sigma_v \). As above, the realization of the additive noise sequence is uniformly bounded by \( \sigma_v \), i.e., \( |v_{t,i}| \leq \sigma_v, t \in \mathbb{N}_{0} \).

When learning the unknown function, on the other hand, it is approximated that the additive noise is drawn independently from \( N(0, \sigma_v^2) \), aiming at employing the GP regression. Moreover, it is assumed for simplicity that the mean function for the GP prior is zero (i.e., \( m(x) \equiv 0 \) for all \( x \) in Section 2.1). Thus, the mean and the variance for the GP model of \( d_i \) with an arbitrary input \( x \in \mathbb{R}^{n_x} \), denoted as \( \mu_i(x; \mathcal{D}_{T,i}) \) and \( \sigma_i^2(x; \mathcal{D}_{T,i}) \), are computed by

\[
\mu_i(x; \mathcal{D}_{T,i}) = k_{T,i}^T(x)(K_{T,i} + \sigma_i^2 I)^{-1}Y_{T,i}, \quad \sigma_i^2(x; \mathcal{D}_{T,i}) = k_i(x, x) - k_{T,i}^T(x)(K_{T,i} + \sigma_i^2 I)^{-1}k_{T,i}(x),
\]

where \( K_{T,i} \) denote the covariance matrix for the kernel function \( k_i \) and \( k_{T,i}(x) = [k_i(x, x_1), \ldots, k_i(x, x_T)]^T \).

Now, recall that the unknown function \( d_i \) lies in the RKHS corresponding to \( k_i \) (Assumption 2). Using this assumption, we can derive an error bound on \( d_i \), representing how the GP posterior mean \( \mu_i \) differs from the ground truth \( d_i \):

**Lemma 2** Suppose that Assumption 2 holds, and let \( \mathcal{D}_{T,i} = \{X_T, Y_{T,i}\} \) be the training data for \( d_i \) with \( X_T = [x_1, x_2, \ldots, x_T] \) and \( Y_{T,i} = [y_{i,1}, y_{i,2}, \ldots, y_{i,T}]^T \) for \( T \in \mathbb{N}_{0} \). Then, for all \( x \in \mathbb{R}^{n_x} \) and \( T \in \mathbb{N}_{0} \), it follows that \( d_i(x) \in Q_i(x; \mathcal{D}_{T,i}) \), where

\[
Q_i(x; \mathcal{D}_{T,i}) = [\mu_i(x; \mathcal{D}_{T,i}) \pm \beta_T \sigma_i(x; \mathcal{D}_{T,i})]
\]

with \( \beta_T = \sqrt{B_T^2 - Y_{T,i}^T(K_{T,i} + \sigma_i^2 I)^{-1}Y_{T,i} + T} \). \( \Box \)

For the proof, see [34]. Lemma 2 means that \( d_i(x) \) is shown to be in the interval set \( Q_i(x; \mathcal{D}_{T,i}) \), which can be computed based on the training data for \( d_i \).

**Remark 2** Note that the previous methods of learning-based controller synthesis with the GP regression [e.g., [17,18]] make use of the probabilistic error bound characterized by the notion of an information gain; see Theorem 3 in [16]. For example, [17] employs the following error (or regret) bound:

\[
d_i(x) \in [\mu_i(x; \mathcal{D}_{T,i}) \pm \sqrt{2||d_i||_k^2 + 300\gamma_T \log^3 (T/\delta)\sigma_i(x; \mathcal{D}_{T,i})}]
\]

which holds for all \( T \in \mathbb{N}_{0} \) with probability at least \( 1 - \delta \) (\( 0 < \delta < 1 \)), where \( \gamma_T, T \in \mathbb{N}_{0} \) denote the information gain. In contrast to this bound, in this paper we provide the error bound in the deterministic form as in Lemma 2. Note that this bound is a direct consequence from computing the upper bound of the RKHS norm of the error \( \mu_i(\cdot; \mathcal{D}_{T,i}) - d_i(\cdot) \) with respect to the kernel \( k_{T,i}(x, x') = k_i(x, x') - k_{T,i}(x)(K_{T,i} + \sigma_i^2 I)^{-1}k_{T,i}(x') \), which has been derived in the proof of Lemma 7.2 in [16] (in particular, see the first equation in the left column of page 3261 in [16]), and see also Appendix C in [34]. The probabilistic error bound (8) is more conservative than the deterministic one of Lemma 2 in the following sense. Note that (8) achieves a deterministic bound by setting \( \delta \to 0 \). However, setting \( \delta \to 0 \) in (8) implies \( d_i(x) \in [-\infty, \infty] \) for all \( T \in \mathbb{N}_{0} \), which leads to an unbounded interval \( \mathbb{R} \) (and is thus not useful for constructing a symbolic model). On the other hand, the error bound obtained in Lemma 2 is deterministic and always bounded (i.e., \( d_i(x) \in Q_i(x; \mathcal{D}_{T,i}) \subset \mathbb{R} \) holds for all \( T \in \mathbb{N}_{0} \)). Such conservativeness might arise due to the fact that, in [16] the probabilistic error bound (8) has been derived as a sufficient condition to the deterministic one given in Lemma 2 (or Lemma 7.2 in [16]). To see this, note that the error bound obtained in Lemma 7.2 of [16] was further upper bounded by using the information gain \( \gamma_T \) (see the second to the third inequality in the top of the right column of page 3261 in [16]), as well as the concentration inequalities (see the second to the third inequality in the bottom of the right column of page 3261). Hence, from the above upper boundings, the probabilistic error bound has been obtained as the sufficient condition to the deterministic one given in Lemma 2. In this paper, we will make use of the deterministic error bound in Lemma 2 instead of the probabilistic one, since it allows us to derive an error bound that can get smaller as the number of the training data increases (see below for details). Such a property is useful to show that the controlled invariant set can enlarge as the number of training data increases, and, moreover, we can provide some computationally efficient algorithms for updating the symbolic models and the safety controller synthesis (see Section 5.1). \( \Box \)

Now, for every \( T \in \mathbb{N}_{0} \), it follows from Lemma 2 that \( d_i(x) \in Q_i(x; \mathcal{D}_{T,i}) \) for all \( t \in \mathbb{N}_{1,T} \). Thus, for every \( T \in \mathbb{N}_{0} \), we have \( d_i(x) \in \bigcap_{t=1}^T Q_t(x; \mathcal{D}_{T,i}) \). Therefore, for every \( T \in \mathbb{N}_{0} \), we have \( d_i(x) \in R_i(x; \mathcal{D}_{T,i}) \), where \( R_i(x; \mathcal{D}_{T,i}) = \bigcap_{t=1}^T Q_t(x; \mathcal{D}_{T,i}) \). From the definition of \( R_i(x; \mathcal{D}_{T,i}) \), it follows that \( R_i(x; \mathcal{D}_{T,i}) \supseteq R_i(x; \mathcal{D}_{T,i}) \supseteq \cdots \). Let \( \tau_i(x; \mathcal{D}_{T,i}) \in R_i(x; \mathcal{D}_{T,i}) \) be
given by
\[
\tau_i(x; D_{T,i}) = \max\{r \in \mathbb{R} | r \in \mathcal{R}_i(x; D_{T,i})\}, \quad (9)
\]
\[
\ell_i(x; D_{T,i}) = \min\{r \in \mathbb{R} | r \in \mathcal{R}_i(x; D_{T,i})\}. \quad (10)
\]
Since \( \mathcal{R}_i(x; D_{T,i}) \supseteq \mathcal{R}_i(x; D_{T,i}) \supseteq \cdots \), it follows that \( \tau_i(x; D_{T,i}) \) and \( \ell_i(x; D_{T,i}) \) are non-increasing and non-decreasing with respect to \( T \) (for fixed \( x \)), respectively. Thus, \( d_i(x) \in \mathcal{R}_i(x; D_{T,i}) \) implies that the error bound on \( d_i \) never grows or potentially gets smaller as the number of the training data increases. Note also that \( d_i(x) \in \mathcal{R}_i(x; D_{T,i}) \) implies \( |d_i(x) - \hat{d}_i(x; D_{T,i})| \leq \Delta_i(x; D_{T,i}), \)
\[
\hat{d}_i(x; D_{T,i}) = 0.5 (\tau_i(x; D_{T,i}) + \ell_i(x; D_{T,i})), \quad (11)
\]
\[
\Delta_i(x; D_{T,i}) = 0.5 (\tau_i(x; D_{T,i}) - \ell_i(x; D_{T,i})). \quad (12)
\]

4.2 Constructing symbolic models from training data

Let us now construct a symbolic model of the system (3) provided that the training data is obtained. We start by showing that the system (3) can be described within the class of a transition system (Definition 1) as follows:

**Definition 4** A transition system induced by the system (3) is a quadruple \( S = (\mathbb{R}^n, x_0, U, G) \), where:

- \( \mathbb{R}^n \) is a set of states;
- \( x_0 \in \mathbb{R}^n \) is an initial state;
- \( U \subseteq \mathbb{R}^n \) is a set of inputs;
- \( G : \mathbb{R}^n \times U \to 2^{\mathbb{R}^n} \) is a transition map, where \( x^+ \in G(x, u) \) iff there exists \( v \in U \) such that \( x^+ = f(x, u) + d(x) + v \).

Based on the transition system \( S \), a symbolic model of \( S \) is constructed by discretizing the state and the input spaces, whose transitions are defined based on the training data \( D_{T,i} \), \( i \in \mathbb{N}_{1,n} \). More specifically, the symbolic model is constructed with a tuple \( q = (D_T, \eta_x, \eta_u, \varepsilon) \), where

- \( D_T = \{D_{T,1}, \ldots, D_{T,n}\} \) is the set of training data;
- \( \eta_x \in \mathbb{R}_{>0} \) is the discretization parameter for the state space \( \mathbb{R}^n \);
- \( \eta_u \in \mathbb{R}_{>0} \) is the discretization parameter for the input space \( U \);
- \( \varepsilon \in \mathbb{R}_{>0} \) is the parameter for the precision.

The corresponding symbolic model is denoted as \( S_q \) and formally defined as follows:

**Definition 5** Let \( S = (\mathbb{R}^n, x_0, U, G) \) be the transition system induced by the system (3). Given \( q = (D_T, \eta_x, \eta_u, \varepsilon) \), a symbolic model of \( S \) is defined as a quadruple \( S_q = (\mathcal{X}_q, x_{q0}, U_q, G_q) \), where

- \( \mathcal{X}_q = [\mathbb{R}^n]_{\eta_x} \) is a set of states;
- \( x_{q0} \in \mathcal{X}_q \) is an initial state satisfying \( x_{q0} \in \text{Nearest}_q(x_0) \);
- \( U_q = \{U\}_{\eta_u} \) is a set of inputs;
- \( G_q : \mathcal{X}_q \times U_q \to 2^{\mathbb{R}^n} \) is a transition map, where \( x_q^+ \in G_q(x_q, u_q) \) iff \( x_q^+ = \hat{h}_i(x_q, u_q; D_{T,i}) \), \( \forall i \in \mathbb{N}_{1,n} \), where \( x_q^+_{T,i} \) is the \( i \)-th element of \( x_q^+ \), and

\[
\hat{h}_i(x_q, u_q; D_{T,i}) = \tau_i(x_q; D_{T,i}) + f_i(x_q, u_q) + \sigma_v \quad (13)
\]
\[
\hat{h}_i(x_q, u_q; D_{T,i}) = \ell_i(x_q; D_{T,i}) + f_i(x_q, u_q) - \sigma_v \quad (14)
\]

Recall that \( L_f \) is the Lipschitz constant for the function \( f \), and \( L_i \) is defined in Lemma 1. Moreover, \( \tau_i, \ell_i \) are defined in (9) and (10), respectively. As shown in Definition 5, the symbolic model provides an abstract expression of \( S \), in the sense that it considers the transitions only among the discretized points in the state and the input spaces. The following result indeed shows that there exists an \( \varepsilon \)-ASR from \( S_q \) to \( S \):

**Proposition 1** Suppose that Assumptions 1,2 hold, and let \( S = (\mathbb{R}^n, x_0, U, G) \). Moreover, given \( q = (D_T, \eta_x, \eta_u, \varepsilon) \), let \( S_q = (\mathcal{X}_q, x_{q0}, U_q, G_q) \) be the symbolic model of \( S \) in Definition 5. Then,

\[
R(\varepsilon) = \{ (x_q, x) \in \mathcal{X}_q \times \mathcal{X}_q | \| x_q - x \|_\infty \leq \varepsilon \} \quad (15)
\]

is an \( \varepsilon \)-ASR from \( S_q \) to \( S \).

The proof follows in the same way to [8] and is given in [34]. In addition to the above, we also have the following result:

**Lemma 3** Let \( D_{T,i} = \{X_T, Y_T, i \}, i \in \mathbb{N}_{1,n} \), \( x \in \mathbb{R}^n \) be the training data with \( X_T = \{x_1, x_2, \ldots, x_T\} \) and \( Y_T = \{y_1, y_2, \ldots, y_T\} \), \( \forall T \in \mathbb{N}_{0,n} \), and \( D_T = \{D_{T,1}, \ldots, D_{T,n}\}, T \in \mathbb{N}_{0,n} \). Moreover, for any \( T_1, T_2 \in \mathbb{N}_{0,n} \) with \( T_1 \leq T_2 \), let \( q_1 = (D_{T,1}, \eta_x, \eta_u, \varepsilon) \) and \( q_2 = (D_{T,2}, \eta_x, \eta_u, \varepsilon) \), and let \( S_{q_1} \) and \( S_{q_2} \) be the corresponding symbolic models according to Definition 5. Then, the relation

\[
R = \{ (x_q, x'_q) \in \mathcal{X}_q \times \mathcal{X}_q | x_q = x'_q \} \quad (16)
\]

is a 0-ASR from \( S_{q_1} \) to \( S_{q_2} \).

**PROOF.** Let the two symbolic models be given by \( S_{q_1} = (\mathcal{X}_{q_1}, x_{q10}, U_{q1}, G_{q1}) \), \( S_{q_2} = (\mathcal{X}_{q_2}, x_{q20}, U_{q2}, G_{q2}) \). Note that \( \mathcal{X}_{q_1} = \mathcal{X}_{q_2} = [\mathbb{R}^n]_{\eta_x}, x_{q10} = x_{q20} \) and \( U_{q1} = U_{q2} = [U]_{\eta_u} \), since we use the same discretization parameters \( \eta_x, \eta_u \) for both \( q_1 \) and \( q_2 \). Hence, the condition (C.1) in Definition 2 holds. The condition (C.2) holds from the definition of \( R \) (16). To show the condition (C.3), let us recall that for every \( x_q \in [\mathbb{R}^n]_{\eta_x}, \)
Lemma 3 implies that, since $G_q(x_q, u_q) \subseteq G_q(x_q, u_q)$ for every $x_q \in [\mathbb{R}^n]$ and $u_q \in [\mathcal{U}]$, the redundant transitions that are present in the symbolic model can be removed by increasing the number of the training data. This is due to the fact that the uncertainty (or the error bound) on the unknown function $d$ can be smaller as the training data increases (see Section 4.1).

4.3 Synthesizing a safety controller

Given $q = (D_T, \eta_T, \eta_u, \varepsilon)$, suppose that the symbolic model $S_q$ is obtained according to Definition 5. Based on the symbolic model, we can find a controlled invariant set $X_S$ in $X$ and the corresponding safety controller $C_S$ by employing a safety game, see, e.g., [3]. The algorithm of the safety game is illustrated in Algorithm 1. In the algorithm, the operator $\text{Pre}_{S_q} : 2^{Q_x} \rightarrow 2^{Q_x}$ is called a predecessor operator and is defined by

$$\text{Pre}_{S_q}(Q) = \{x_q \in Q \mid \exists u_q \in U_q : G_q(x_q, u_q) \subseteq Q\},$$

for a given $Q \subseteq X$. That is, $\text{Pre}_{S_q}(Q)$ is the set of all states in $Q$, for which there exists a control input in $U_q$ such that all the corresponding successors are inside $Q$. The controlled invariant set $X_S$ is computed based on the fixed point set of $Q$ (i.e., $X_{S,q}$). In particular, if $X_{S,q}$ is non-empty, the controlled invariant set is computed based on the $\varepsilon$-ASR $R(\varepsilon)$ (line 9). On the other hand, if $X_{S,q}$ is empty, it indicates that the controlled invariant set is not found (and so we set $X_S \leftarrow \emptyset$ as shown in line 13). Roughly speaking, $C_{S,q}$ (line 10) serves as a safety controller for the symbolic model $S_q$, and the safety controller $C_S$ for $S$ is refined based on the $\varepsilon$-ASR $R(\varepsilon)$ (line 11). Note that Algorithm 1 is guaranteed to terminate after a finite number of iteration, since $[\text{Interior}(X)]$ and $[\mathcal{U}]$ are both finite. The following result is an immediate consequence from the fact that $R(\varepsilon)$ is the $\varepsilon$-ASR from $S_q$ to $S$ and thus the proof is omitted (see, e.g., [3]).

Lemma 4 Suppose that for given $S_q$ and $X$. Algorithm 1 is implemented and $X_S \neq \emptyset$. Then, $X_S$ is a controlled invariant set in $X$, and $C_S$ is the corresponding safety controller.

Algorithm 1 SafeCon($S_q, X$) (safety controller synthesis).

```
Input: $S_q$ (symbolic model of $S$), $X$ (safe set);
Output: $X_S$ (if $\emptyset \neq X_S$, it yields a controlled invariant set in $X$), $C_S$ (if $\emptyset \neq X_S$, it yields a safety controller);
1: $\ell \leftarrow 0$;
2: $Q_0 \leftarrow [\text{Interior}(X)]$;
3: repeat
4: $\ell \leftarrow \ell + 1$;
5: $Q_{\ell} \leftarrow \text{Pre}_{S_q}(Q_{\ell-1})$;
6: until $Q_{\ell-1} = Q_{\ell}$
7: $X_{S,q} \leftarrow Q_\ell$;
8: if $\emptyset \neq X_{S,q}$ then
9: $X_S \leftarrow \{x \in X \mid \exists x_q \in X_{S,q} \mid x = R(\varepsilon)\}$;
10: $C_{S,q}(x_q) \leftarrow \{u_q \in U_q : G_q(x_q, u_q) \subseteq X_{S,q}\}, \forall x_q \in X_{S,q}$;
11: $C_S(x) \leftarrow \{C_{S,q}(x_q) \mid (x_q, x) \in R(\varepsilon)\}, \forall x \in X$;
12: else
13: $X_S \leftarrow \emptyset$, $C_S(x) \leftarrow \emptyset, \forall x \in X$ (which indicates that the controlled invariant set and safety controller are not found);
14: end if
```

In addition to the above, we also have the following result:

Lemma 5 Let $D_{T,i} = \{X_T, Y_{T,i}\}, i \in \mathbb{N}_{>0}$, be the training data with $X_T = \{x_1, x_2, \ldots, x_T\}$ and $Y_{T,i} = \{y_{T,i}, y_{2T,i}, \ldots, y_{i, T}\}$ for all $T \in \mathbb{N}_{>0}$, and let $D_T = \{D_{T,1}, \ldots, D_{T,n}\}, T \in \mathbb{N}_{>0}$. Moreover, for any $T_1, T_2 \in \mathbb{N}_{>0}$ with $T_1 \leq T_2$, let $q_1 = (D_{T_1}, \eta_T, \eta_u, \varepsilon)$ and $q_2 = (D_{T_2}, \eta_T, \eta_u, \varepsilon)$ and let $S_q$ and $S_{q_2}$ be the corresponding symbolic models according to Definition 5. In addition, let $X_{S_{q_2}}, X_{S_q}$ be the resulting controlled invariant sets by executing SafeCon($S_{q_2}, X$) and SafeCon($S_q, X$), respectively. Then, $X_{S_{q_2}} \subseteq X_{S_q}$.

In essence, Lemma 5 means that the controlled invariant set does not shrink or can be enlarged by increasing the number of training data. As previously mentioned, this is due to the fact that the symbolic model becomes more and more accurate (i.e., the redundant transitions are removed) as the training data increases, since the error bound on $d$ can be smaller as the training data increases.

While Lemma 5 might trivially follow from the existence of a 0-ASR from $S_{q_2}$ to $S_q$ (see Lemma 3), we here provide a detailed proof below, since the proof procedure will be useful to provide an approach to reduce the computational load for the safety controller synthesis (for details, see Section 5.1).

PROOF. Let the two symbolic models be given by $S_{q_1} = (X_{q_1}, x_{q_1, 0}, U_{q_1}, G_{q_1})$, $S_{q_2} = (X_{q_2}, x_{q_2, 0}, U_{q_2}, G_{q_2})$. Then, from the proof of Lemma 3, it follows that $G_{q_2}(x_{q_2, 0}) \subseteq G_{q_1}(x_{q_1, 0})$ for every $x_{q_2, 0} \in [\mathbb{R}^n]_{\eta_u}$ and $u_{q_2} \in [\mathcal{U}]$. Now, let $Q_1, \ldots, Q_{2,\ell}$, $\ell = 0, 1, \ldots$ be the sets of $Q_\ell$ obtained by executing SafeCon($S_{q_1}, X$) and SafeCon($S_{q_2}, X$), respectively. Note that $Q_{1,0} = Q_{2,0}$.
Algorithm 2 Learning-based symbolic abstractions with safe exploration (overall, main algorithm).

**Input:** $x_0$ (initial state), $C_{S,\text{init}}$ (initial safety controller), $\eta_x, \eta_u, \varepsilon$ (some parameters for the symbolic model), $T_{\text{exp}} \in \mathbb{N}$ (number of training data collected for each iteration of safe exploration).

**Output:** $X_{S,N}$ (symbolic model);
1: $D_{0,i} \leftarrow \emptyset, \forall i \in \mathbb{N}_{1,n_i}$;
2: $D_0 \leftarrow \{D_{0,1}, \ldots, D_{0,n_u}\}$;
3: $T_1 \leftarrow T_{\text{exp}}$;
4: $\{x_{T_1}, D_{T_1}\} \leftarrow $ SafeExp($x_0, T_{\text{exp}}, C_{S,\text{init}}, D_0$);
5: $q_1 \leftarrow \{D_{T_1}, \eta_x, \eta_u, \varepsilon\}$;
6: $S_0 \leftarrow (x_0, x_0, U_{x_0}, G_{q_1})$ (Definition 5);
7: $\{X_{S,1}, X_{S,2}\} \leftarrow $ SafeCon($S_1, \mathcal{X}$) (Algorithm 1);
8: if $\emptyset \neq X_{S,1}$ then
9: repeat
10: $T_{N+1} \leftarrow T_N + T_{\text{exp}}$;
11: $\{x_{T_{N+1}}, D_{T_{N+1}}\} \leftarrow $ SafeExp($x_{T_N}, T_{\text{exp}}, C_{S,N}, D_{T_N}$);
12: $N \leftarrow N + 1$;
13: $q_N \leftarrow \{D_{T_{N+1}}, \eta_x, \varepsilon\}$;
14: $S_N \leftarrow (x_{T_{N+1}}, x_{T_{N+1}}, U_{x_{T_{N+1}}}, G_{q_N})$ (Definition 5);
15: $\{X_{S,N}, X_{S,N}\} \leftarrow $ SafeCon($S_N, \mathcal{X}$) (Algorithm 1);
16: until $X_{S,N-1} = X_{S,N}$
17: end if

Hence, it follows that
\[
G_{q_1}(x_0, u_0) \subseteq Q_{1,0} \Rightarrow G_{q_2}(x_i, u_i) \subseteq Q_{2,0}. \tag{19}
\]

Thus, we obtain $\text{Pres}_{S_1}(Q_1,0) \subseteq \text{Pres}_{S_2}(Q_2,0)$. Hence, from the fact that $Q_{1,0} = Q_{2,0}$ and line 5 in Algorithm 1, it follows that $Q_{1,1} \subseteq Q_{2,1}$. By recursively applying the same reasoning as above, it then follows that $Q_{1,t} \subseteq Q_{2,t}, \; t = 0, 1, \ldots$. In other words, we have $X_{S_1,t} \subseteq X_{S_2,t}$ and namely, $X_{S_1} \subseteq X_{S_2}$.

5 Learning-based safe symbolic abstractions

In this section we present an overall algorithm that aims at collecting the training data from scratch and constructing the symbolic model while achieving the safe exploration. Before providing the algorithm, we need to make the following assumption:

**Assumption 3** There exists a known safety controller $C_{S,\text{init}} : \mathcal{X} \rightarrow 2^d$ such that any trajectory induced by $C_{S,\text{init}}$ stays in the safety set $\mathcal{X}$ for all times. \hfill \Box

Assumption 3 implies the existence of an initial safety controller, so that the training data can be collected at the initial phase. The initial safety controller $C_{S,\text{init}}$ may be obtained by employing an expert or heuristically based on the nominal model $f(x_k, u_k)$; for details, see Remark 5 in [34].

The overall learning algorithm is shown in Algorithm 2 and the details are described as follows. The algorithm starts by initializing the training data by applying the initial safety controller and then updating the controlled invariant set and the safety controller (line 1–line 7). Then, if $\emptyset \neq X_{S,1}$ (i.e., $X_{S,1}$ is a controlled invariant set in $\mathcal{X}$), we move on to the iteration (lines 9–16). In the iteration, we first update $T_{N+1}$ (line 10). Roughly speaking, $T_N, \; N \in \mathbb{N}_{0}$ indicate the number of training data that has been collected until the $N$-th iteration of Algorithm 2. The algorithm proceeds by executing a safe exploration algorithm SafeExp (line 11), which aims at collecting the new training data while guaranteeing safety. In detail, the safe exploration algorithm is shown in Algorithm 3. In the algorithm, the control input $u_t$ (line 4) is computed as follows:

\[
\begin{align*}
\sigma_t &= \left\{ \begin{array}{ll}
\text{select arbitrarily from } C_{S,N}(x_t), & (N = 0), \\
\arg \max_{u \in C_{S,N}(x_t)} \sum_{i=1}^{n_u} \sigma_i^2(\tilde{x}_i; D_{T,\max}), & (N > 0),
\end{array} \right. \\
\end{align*}
\tag{21}
\]

where $\tilde{x}_i = f(x_t, u) + \tilde{d}(x_t; D_{T,\max})$ with $\tilde{d}(x_t; D_{T,n}) = [d_1(x_t; D_{T,1}), \ldots, d_n(x_t; D_{T,n})]^T$. Recall that $\sigma_i^2$ and $d_i$ are defined in (6) and (11), respectively. That is, we select the control input randomly from $C_{S,N}$ for the initial exploration, and, otherwise, select from $C_{S,N}$ such that the corresponding (predictive) next state has the largest variance on $d$. By doing so, the system actively explores the state-space so as to reduce the uncertainty on $d$ and enlarge the controlled invariant set while guaranteeing safety. The exploration is given until it collects the number of training data $T_{\text{exp}}$, training data, and it outputs the new training data $D_{T,N+1}$ and the current state $x_{T,N+1}$ after the exploration. Afterwards, the symbolic model $S_{Q}$ is updated with the new training data according to Definition 5 (line 13, line 14 in Algorithm 2), and the controlled invariant set $X_{S,N}$ and the safety controller $C_{S,N}$ are updated by SafeCon (line 15 in Algorithm 2). The above procedure is
iterated until the controlled invariant set converges, i.e., 
\( X_{S,N-1} = X_{S,N} \) (see line 12 in Algorithm 2). Note that 
\( X_{S,N} \) is computed by refining the set of discretized states, i.e., 
\( X_{S,N} = \{ x \in X \mid \exists x_q \in X_{S,qN}, (x_q,x) \in R(\varepsilon) \} \) (see line 8 in Algorithm 1). Hence, we have 
\( X_{S,N-1} = X_{S,N} \) if and only if \( X_{S,qN} = X_{S,qN-1} \). Since \( X_{S,qN-1} \) and \( X_{S,qN} \) are both finite and \( X_{S,qN-1} \subseteq X_{S,qN} \) (for details, see 
the proof of Theorem 1 below), the condition \( X_{S,qN} = X_{S,qN-1} \) can be checked in a finite time (i.e., check if every 
\( x_q \in X_{S,qN} \) is contained in \( X_{S,qN-1} \)).

Regarding the overall algorithm, we can conclude the following result:

**Theorem 1** Suppose that Assumptions 1–3 hold and 
Algorithm 2 is implemented. Then, Algorithm 2 terminates 
after a finite number of iteration. Moreover, the relation 
\( R(\varepsilon) = \{ (x_q,x) \in X_q \times \mathbb{R}^n \mid \| x_q - x \|_\infty \leq \varepsilon \} \) 
is an \( \varepsilon \)-ASR from \( S_{qN} \) to \( S \) for all \( N \in \mathbb{N}_{\geq 1} \), where \( e_{GD,qN} \) is the transition map that 
is (newly) constructed by applying Algorithm 4. The 
\( 10 \)

\( u = R \)

after a finite number of iteration. Given 
achieved, i.e., during the implementation of Algorithm 3, 
Algorithm 2 terminates. In addition, the 
safe exploration 
is achieved, i.e., during the implementation of Algorithm 3, 
it is shown that the trajectory of the system (3) stays in 
the safe set \( X \) all times.

**Proof.** Let us first show that Algorithm 2 terminates 
after a finite number of iteration. Given \( N \in \mathbb{N}_{\geq 0} \), let 
\( X_{S,qN} \) be the set of \( X_{S,q} \) in Algorithm 1 (line 7) computed 
by executing SafeCon\((S_{qN}, X)\). Since \( T_{N+1} \geq T_N \), and 
from the proof of Lemma 5, we obtain \( X_{S,qN} \subseteq X_{S,qN+1} \). 
In general, it follows that \( X_{S,q0} \subseteq X_{S,q1} \subseteq X_{S,q2} \subseteq \cdots \). 
Note that \( X_{S,qN} \subseteq \text{Interior}_\varepsilon(X)^{\mu}_N \) for all \( N \in \mathbb{N}_{\geq 0} \) 
and that \( \text{Interior}_\varepsilon(X)^{\mu}_N \) is finite. Hence, there exists an 
\( N^* \in \mathbb{N}_{\geq 0} \) such that \( X_{S,qN^*} = X_{S,qN^*-1} \). This in turn 
implies that \( X_{S,N^*} = X_{S,N^*-1} \), and, therefore, Algorithm 2 
terminates after a finite number of iteration. The fact 
that \( R(\varepsilon) \) is an \( \varepsilon \)-ASR from \( S_{qN} \) to \( S \) for all \( N \in \mathbb{N}_{\geq 0} \) (until 
Algorithm 2 terminates) trivially holds from Proposition 1. Moreover, we can achieve the safe exploration, 
since control inputs are always chosen from the safety controller \( C_{SN} \).

Note that, in order to make the implementation of 
Algorithm 2 tractable, it is only necessary to construct the 
symbolic model within the space-state \( [X]_{\mu} \). Specifically, 
the update of the symbolic model (line 14 in Algorithm 2) is replaced by defining a new symbolic model 
\( S_{D,qN} \):

\[
S_{D,qN} \leftarrow (X_{D,qN} \times D_{qN}, U_{D,qN} \times D_{qN}), \quad (23)
\]

where \( X_{D,qN} = [X]_{\mu}, \quad D_{qN} = 0 = x_0, \quad U_{D,qN} = U_N, \quad \) and 
\( G_{D,qN} : X_{D,qN} \times U_{D,qN} \rightarrow 2^{X_{D,qN}}, \) with \( x^+ \in \) 
\( G_{D,qN}(x,u) \) if and only if \( x^+ \in G_{qN}(x,u) \) and 
\( G_{qN}(x,u) \subseteq [X]_{\mu} \) (i.e., if \( G_{D,qN}(x,u) \subseteq [X]_{\mu} \) then 
\( u \in U_{D,qN}(x) \)). It can be easily shown that the relation 
\( R = \{ (x_q,x'_q) \in [X]_{\mu} \times [X]_{\mu} \mid x_q = x'_q \} \) is a 0-ASR
from \( S_{D,qN} \) to \( S_{qN} \). From this and the fact that the relation 
\( R(\varepsilon) = \{ (x_q,x) \in X_q \times \mathbb{R}^n \mid \| x_q - x \|_\infty \leq \varepsilon \} \) 
is an \( \varepsilon \)-ASR from \( S_{qN} \) to \( S \), it is shown that the relation 
\( R_{D,N} = \{ (x_q,x) \in [X]_{\mu} \times \mathbb{R}^n \mid \| x_q - x \|_\infty \leq \varepsilon \} \) 
is an \( \varepsilon \)-ASR from \( S_{D,qN} \) to \( S \) (see, e.g., [8] for a detailed discussion). Hence, any controller synthesized for 
the symbolic model \( S_{D,qN} \) can be refined to a controller 
for the original system \( S \) satisfying the same specification.

5.1 Some approaches to efficient computation

Since the symbolic model needs to be updated for every \( N \), the whole re-computation of this abstraction (as 
well as the safety controller synthesis) for every iteration 
clearly leads to a heavy computational load. Therefore, 
in this section we provide some techniques to reduce the 
computational load so as to make our approach more 
practical. Specifically, we propose the following two 
approaches to speed up the abstraction and controller 
synthesis procedures:

- **(Lazy abstraction):** It should be expected that, the 
  transitions are necessary to be updated only for the 
  states where the uncertainty (or the variance) on \( d \) 
is sufficiently reduced by collecting the new training 
data. Hence, we propose a lazy abstraction scheme, in 
which, starting from the initial abstraction, transitions 
from states in \( [X]_{\mu} \) are then updated only when 
the reduction of the variance on \( d \) is large enough. The 
update of the transitions allows to reduce the redundant 
transitions and hence and the abstraction becomes 
less conservative. In the proposed procedure, we do not have to recompute the abstraction for the 
whole states in \( [X]_{\mu} \), but only for the states on which 
new training data is collected.

- **(Speeding up the computation of predecessors):** It 
  should be expected that the main source of the heavy 
  computation for the safety controller synthesis is the 
  predecessor operator \( Pre_{S}(Q_{\ell}) \) (see (18)); clearly, 
  checking for every state in \( Q_{\ell} \) if there exists a control 
  input such that all the corresponding successors are in 
  \( Q_{\ell} \) requires a heavy computation, as this operation 
  needs to be done for every \( \ell \) and \( N \). Therefore, 
  we propose an approach to reduce the computational 
  load of computing this predecessor operator, in order 
  to speed up the safety controller synthesis. In particular, 
  we eliminate redundant computations of the predecessor 
  operator by making use of the earlier computed predecessors.

Regarding the first approach in the above, the update of the 
symbolic model (line 14 in Algorithm 2) is replaced by defining a new symbolic model \( S_{D,qN} \):

\[
S_{D,qN} \leftarrow (X_{D,qN} \times D_{qN}, U_{D,qN} \times D_{qN}), \quad (24)
\]

for all \( N \in \mathbb{N}_{\geq 1} \), where \( G_{D,qN} \) is the transition map that 
is (newly) constructed by applying Algorithm 4. The
Algorithm 4 Derivation of $\hat{G}_{D,q^N}$ for all $N \in \mathbb{N}_{\geq 1}$ (lazy abstraction).

**Input:** $D_{T_1,N}$ (training data), $\rho \in \mathbb{R}_{>0}$ (threshold to update transitions in $\hat{G}_{D,q^N}$), $\hat{G}_{D,q^N-1}$ (transition map of $\hat{S}_{D,q^N-1}$ (if $N > 1$));

**Output:** $\hat{G}_{D,q^N}$ (transition map of $\hat{S}_{D,q^N}$);

1: if $N = 1$ (initial execution of Algorithm 4) then
2: $\mathcal{X}^*_q = \emptyset$;
3: end if
4: for each $x_q \in \mathcal{X}^*_q$ do
5: for each $u_q \in [\mathcal{U}]_{\eta_q}$ do
6: $\hat{G}_{D,q^N}(x_q, u_q) \leftarrow \hat{G}_{D,q^N-1}(x_q, u_q)$;
7: end for
8: end for
9: for each $x_q \in [\mathcal{X}]_{\eta_q} \setminus \mathcal{X}^*_q$ do
10: for each $u_q \in [\mathcal{U}]_{\eta_q}$ do
11: \(\mathcal{X}^*_q \leftarrow \{x_q \in [\mathbb{R}^{n_q}]_{\eta_q} \mid x_q \in (\hat{h}_x(x_q, u_q; D_{T_1,i}) \setminus \hat{T}_q(x_q, u_q; D_{T_1,i}), \forall i \in \mathbb{N}_{1,n_x})\};\)
12: if $\mathcal{X}^*_q \subseteq [\mathcal{X}]_{\eta_q}$ then
13: $\hat{G}_{D,q^N}(x_q, u_q) \leftarrow \mathcal{X}^*_q$;
14: \(\Delta_q(x_q; D_{T_1,i}) \leq \rho \) for all $i \in \mathbb{N}_{1,n_x}$ then
15: $\mathcal{X}^*_q \leftarrow \mathcal{X}^*_q \cup \{x_q\}$;
16: end if
17: end if
18: end for
19: end for

The following result shows that the existence of an $\varepsilon$-ASR is still guaranteed from $\hat{S}_{D,q^N}$ to $S$.

**Theorem 2** Suppose that Assumptions 1–3 hold and Algorithm 2 is implemented, in which the symbolic model is given by (24) whose transition map $\hat{G}_{D,q^N}$ is computed by executing Algorithm 4 for all $N \in \mathbb{N}_1$. Then, for every $N \in \mathbb{N}_{\geq 1}$, the relation $R_D(\varepsilon) = \{(x_q, x) \in [\mathcal{X}]_{\eta_q} \times [\mathbb{R}^{n_x}] \mid \|x_q - x\|_{\infty} \leq \varepsilon\}$ is an $\varepsilon$-ASR from $\hat{S}_{D,q^N}$ to $S$.

**PROOF.** The result follows by induction. For $N = 1$, $R_D$ is the $\varepsilon$-ASR from $\hat{S}_{D,q^N}$ to $S$, since $\hat{S}_{D,q^N} = S_{D,q^N}$.
and $R_D$ is the $\varepsilon$-ASR from $S_{D,q_N}$ to $S$ (see the discussion after Theorem 1). For a given $N \in \mathbb{N}_{>1}$, assume that $R_D$ is the $\varepsilon$-ASR from $S_{D,q_N}$ to $S$, and suppose that, at the next iteration $N+1$, $G_{D,q_N+1}$ is given by Algorithm 4. In what follows, it is shown that there exists a 0-ASR from $S_{D,q_N}$ to $S_{D,q_N+1}$. From the derivation of $G_{D,q_N+1}$ in Algorithm 4, for every $x_q \in [X]_{n_x}$, $u_q \in \mathcal{U}_{n_u}$ with $\tilde{G}_{D,q_N}(x_q,u_q) \neq \varnothing$, it follows either $\tilde{G}_{D,q_N+1}(x_q,u_q) = \tilde{G}_{D,q_N}(x_q,u_q)$, or $\tilde{G}_{D,q_N+1}(x_q,u_q) = \{x^+_q \in [X]_{n_x} \mid x^+_q \in \tilde{h}((x_q,u_q;D_{T_N+i}), \tilde{t}_i(x_q,u_q;D_{T_N+i}),\forall i \in N_1_{n_x}\}$.

Note that for the latter case, we have

$$\tilde{h}((x_q,u_q;D_{T_N+i}), \tilde{t}_i(x_q,u_q;D_{T_N+i}),\forall i \in N_1_{n_x}\}$$

for all $N^i \leq N$, since $T_N^i \leq T_N$ (see the proof of Lemma 3). Hence, for every $x_q \in [X]_{n_x}$, $u_q \in \mathcal{U}_{n_u}$ with $\tilde{G}_{D,q_N}(x_q,u_q) \neq \varnothing$, it follows that $\tilde{G}_{D,q_N+1}(x_q,u_q) \subseteq \tilde{G}_{D,q_N}(x_q,u_q)$. This implies that the relation $R = \{(x_q,x'_q) \in [X]_{n_x} \times [X]_{n_x} \mid x_q = x'_q\}$ is a 0-ASR from $S_{D,q_N}$ to $S_{D,q_N+1}$. Thus, from the assumption that $R_D$ is the $\varepsilon$-ASR from $S_{D,q_N}$ to $S$, it is shown that $R_D$ is the $\varepsilon$-ASR from $S_{D,q_N}$ to $S$ for all $N \in \mathbb{N}_{>0}$.

Hence, any controller synthesized for the symbolic model $S_{D,q_N}$ can be refined to a controller for the original system $S$ satisfying the same specification.

**Remark 3** $S_{D,q_N}$ can have more (redundant) transitions than $S_{D,q_N}$, since the transitions of $S_{D,q_N}$ are updated only for some states, while in $S_{D,q_N}$ these are updated for all states in $[X]_{n_x}$. From Lemma 5, this implies that using $S_{D,q_N}$ may result in a larger controlled invariant set than using $S_{D,q_N}$, which may be a drawback of using $S_{D,q_N}$. Nevertheless, as will be illustrated in the numerical example in the next section (Section 6), constructing $S_{D,q_N}$ should be more practical and useful than constructing $S_{D,q_N}$, since it achieves a significant reduction of the computational load.

**Remark 4** Let us mention that the use of lazy approaches has been previously used in the symbolic control literature (see the approaches proposed in [38,39,40] and a review of the lazy techniques in [41]). In these approaches, the refinement of the abstraction is done for the regions that are not able to achieve the safety specification (either by using finer discretizations or lower inputs). In this paper, the criteria of the refinement are different, since we are refining on the regions where we are able to collect new data. Moreover, the method of refinement is also different since we conserve the same discretizations, the same input, but we benefit from the

**Algorithm 5** Derivation of $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$ for all $N \in N_{>0}$, $\ell \in N_{>0}$ (speeding up the computation of the predecessors).

**Input:** $S_{D,q_N}$, $Q_{N,\ell}$, and $Q_{N-1,\ell+1}$ (available if $N > 1$);

**Output:** $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$;

1: if $N = 1$ then
2: $Q \leftarrow \emptyset$, $Q_{N-1,\ell+1} \leftarrow \emptyset$;
3: else
4: $Q \leftarrow Q_{N-1,\ell+1}$;
5: end if
6: for each $x_q \in Q_{N,\ell} \setminus Q_{N-1,\ell+1}$ do
7: for each $u_q \in \mathcal{U}_{n_u}$ do
8: if $\tilde{G}_{D,q_N}(x_q,u_q) \subseteq Q_{N,\ell}$ then
9: $Q \leftarrow Q \cup \{x_q\}$;
10: end if
11: end for
12: end for
13: $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell}) \leftarrow Q$;

supplementary knowledge on the un-modeled dynamics to reduce the redundant transitions. □

Let us now proceed by reducing the computational load of the predecessor operator $\text{Pre}_{S_{D,q_N}}$ in order to speed up the safety controller synthesis. To this end, let $Q_{N,\ell}$, $\ell = 0,1, \ldots$ denote the sequence of sets $Q_{\ell}$, $\ell = 0,1, \ldots$ in Algorithm 1 by executing SafeCon($\tilde{S}_{q_N}, X$). Then, it follows from $T_N \geq T_{N-1}$ for all $N \in N_{>0}$ that $Q_{N,\ell+1} \subseteq Q_{N-1,\ell+1} = \text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$, for all $N \in N_{>0}$ and $\ell \in N_{>0}$ (see the proof of Lemma 5). Hence, when we aim at computing $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$, it is known that $Q_{N-1,\ell+1}$ is the subset of $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$. This implies that all states in $Q_{N-1,\ell+1}$ can be directly added to the predecessors for $Q_{N,\ell}$ without checking the existence of a control input such that all successors are in $Q_{N,\ell}$ according to (18).

Based on the above observation, we propose Algorithm 5 so as to speed up the computation of $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$.

In the algorithm, $Q$ represents the set of predecessors for $Q_{N,\ell}$ which are mainly updated according to line 6–line 12. For $N = 1$, $Q$ is initialized by the empty set (line 2). In other words, all states in $Q_{N,\ell}$ (since $Q_{N,\ell} \subseteq Q_{N-1,\ell+1}$) are evaluated to check the existence of a control input such that all the successors are in $Q_{N,\ell}$ according to line 6–12. For $N > 1$, on the other hand, $Q$ is initialized by $Q_{N-1,\ell+1}$ (line 4). This is due to the fact that it is already known that $Q_{N-1,\ell+1}$ is a subset of the predecessors for $Q_{N,\ell}$ (i.e., $Q_{N-1,\ell+1} \subseteq \text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$). Hence, only the states in $Q_{N,\ell} \setminus Q_{N-1,\ell+1}$ (instead of $Q_{N,\ell}$) are necessary to be evaluated to check the existence of a control input such that all the successors are in $Q_{N,\ell}$ according to line 6–12.

In summary, during execution of SafeCon($\tilde{S}_{q_N}, X$) (line 15 in Algorithm 2) for all $N \in N_{>1}$, $\text{Pre}_{S_{D,q_N}}(Q_{N,\ell})$,
\[ \ell = 0, 1, \ldots \] are computed by Algorithm 5. The computational complexity of computing predecessors according to Algorithm 5 for \( N = 1 \) is \( O(|Q_{N,\ell}| \cdot |U|) \). For \( N > 1 \), we have \( O(|Q_{N,\ell}| \cdot |Q_{N-1,\ell+1}| \cdot |U|) \). Hence, the computation of the predecessors becomes faster as the cardinality of \( Q_{N,\ell} \) \( Q_{N-1,\ell+1} \) becomes smaller, or in other words, \( Q_{N,\ell} \) is closer to \( Q_{N-1,\ell+1} \), i.e., \( Q_{N,\ell} \approx Q_{N-1,\ell+1} \). Note that we have \( Q_{N,\ell} = Q_{N-1,\ell+1} \) if the controlled invariant set converges \( X_{S,N} = X_{S,N} \) (i.e., \( Q_{N,\ell} = Q_{N-1,\ell} \), \( \forall \ell \in \mathbb{N}_0 \) and \( Q_{N,\ell} = Q_{N,\ell+1} \) (i.e., \( Q_{N,\ell} \) converges to a fixed point). Hence, it is expected that Algorithm 5 becomes faster as both the controlled invariant set and \( Q_{N,\ell} \) get closer to their fixed points. The memory requirement is \( O(|\text{Interior}_t(X)|) \), since it needs to store the set \( Q_{N,\ell} \subseteq |\text{Interior}_t(X)| \).

6 Simulation results

In this section we illustrate the effectiveness of the proposed approach through a simulation of an adaptive cruise control (ACC) [42,43,44]. The simulation has been conducted on Windows 10, Intel(R) Core(TM) 2.40GHz, 8GB RAM. The state vector is given by \( x = [x_1, x_2, x_3]^T \in \mathbb{R}^3 \), where \( x_1 \) is the velocity of the leading vehicle, \( x_2 \) is the velocity of the following vehicle, and \( x_3 \) is the distance between the lead vehicle and the following vehicle. Moreover, the input vector indicates the acceleration of the following car \( u \in \mathbb{R} \). The dynamics is given by

\[
\begin{aligned}
x_{t+1} &= x_t + \Delta \begin{bmatrix} 0 \\ u_t \\ x_{1,t} - x_{2,t} \end{bmatrix} + \Delta \begin{bmatrix} a_{t} \\ 0 \\ 0 \end{bmatrix} \\
&+ \Delta \begin{bmatrix} f(x_t, u_t) \\ 0 \\ -\left((v_0 + \nu_1 x_{2,t} + \nu_2 x_{2,t}^2)/M \right) \\ 0 \end{bmatrix},
\end{aligned}
\]

where \( M \) is the weight of the following vehicles, \( \Delta \) represents the sampling time, \( a_t \), is the acceleration of the lead vehicle that is assumed to be the additive noise, and \( \nu_0, \nu_1, \nu_2 \) are the constants for the aerodynamic drag force, whose function (i.e., \( (v_0 + \nu_1 x_{2,t} + \nu_2 x_{2,t}^2)/M \)) is assumed to be unknown apriori. It is assumed that \( \Delta = 1, M = 1000, v_0 = 60, \nu_1 = 1.2, \nu_2 = 1.0 \). Moreover, we assume that the velocity of the lead vehicle fulfills \( 15 \leq x_{1,t} \leq 25 \) for all \( t \in \mathbb{N}_{20} \), and its acceleration is bounded as \( |a_t| \leq 0.2 \) for all \( t \in \mathbb{N}_{20} \). The safe set is given by \( X = Z \cdot O \), where \( Z = \{ x \in \mathbb{R}^3 \mid 15 \leq x_1 \leq 25, 15 \leq x_2 \leq 25, 30 \leq x_3 \leq 80 \} \). \( O = \{ x \in \mathbb{R}^3 \mid 2x_2 \leq x_3 \} \). The input constraint set is \( U = \{ u \in \mathbb{R} \mid |u| \leq 1.0 \} \). The initial state is given by \( x = [20, 20, 60]^T, \) and \( \eta_2 = \varepsilon = 0.2, \eta_u = 0.2, T_{\text{exp}} = 30 \). Moreover, during the implementation of Algorithm 2, we incorporate Algorithm 4 and Algorithm 5 with \( \rho = 0.01 \) so as to reduce the computational load of abstractions and the safety controller synthesis. We used a squared-exponential \( k(x_1, x_1') = \exp(-\alpha|x_1 - x_1'|) \) with \( \alpha = 1 \). The computed upper bound of the RKHS norm was \( \|d_2\| \leq 2.0 \) (for details on how to obtain this bound, see [34]).

For comparisons, we have also computed a symbolic model and a controlled invariant set by regarding \( d_2(x_{2,t}) = \nu_0 + \nu_1 x_{2,t} + \nu_2 x_{2,t}^2/M \) as the uniform disturbance (i.e., the aerodynamic drag force will not be learned from data). We assume that the uniform disturbance satisfies \( 0.30 \leq d_2(x_{2,t}) \leq 0.71 \), \( \forall t \in \mathbb{N} \), since \( \min_{x_2 \in [15,25]} d_2(x_2) = 0.30 \) and \( \max_{x_2 \in [15,25]} d_2(x_2) = 0.71 \). These lower and the upper bounds of \( d_2 \) have
been utilized to construct the symbolic model and the controlled invariant set by following the abstraction procedure given in previous work, e.g., [45].

Fig. 1 shows the trajectories of $x_1$, $x_2$ by applying the proposed approach Algorithm 2 and the phase portrait of $x_2$, $x_3$. The figure illustrates that the trajectories are always inside $\mathcal{X}$ (white region), showing the achievement of the safe exploration. The algorithm terminates at $N = 10$. The computed controlled invariant sets for $N = 1, 4, 7, 10$ are illustrated in Fig. 2. The figure shows that the volume of the controlled invariant set is enlarged by collecting the training data according to Algorithm 2.

Fig. 3 shows the controlled invariant set finally obtained by applying the proposed approach (which is equivalent to the lower right of Fig. 2) and the uniform disturbance-based approach as described above. In addition, Fig. 4 shows $|\mathcal{X}_{S,N}[\eta]|$ (i.e., the cardinality or the number of states contained in $\mathcal{X}_{S,N}[\eta]$) by applying the proposed approach against the number of iterations (blue dotted line) and the uniform disturbance-based approach (green dotted line). Note that the size of the controlled invariant set under the uniform disturbance-based approach is constant for all the iterations, since the unknown function is not learned. The figure shows that the controlled invariant set obtained by the proposed approach becomes larger than the uniform disturbance-based approach after $N = 6$. This is because, by applying the proposed algorithm, the uncertainty of the unknown function becomes smaller as the iteration progresses, which results in reducing redundant transitions of the symbolic model (and thus enlarge the controlled invariant set); on the other hand, the uniform disturbance-based approach always considers the worst case effect of the disturbance, and thus the redundant transitions will not be removed. Moreover, for further comparisons, we also implemented Algorithm 2 neither by employing Algorithm 4 (i.e., $\mathcal{S}_{D,A_N}$ in (23) is constructed for each $N$) nor by employing Algorithm 5 for the safety controller synthesis, and the results are also plotted (red dotted lines). The right figure illustrates the total execution time to implement line 14 and line 15 in Algorithm 2.

Fig. 4. The left figure indicates $|\mathcal{X}_{S,N}[\eta]|$ with Algorithm 4 and 5 (blue solid), uniform disturbance-based approach (green dotted) and without Algorithm 4 and 5 (red dotted line). The right figure illustrates the total execution time to implement line 14 and line 15 for each $N$ in Algorithm 2.

Now, using the learned symbolic model $\mathcal{S}_{D,A_N}$, we can synthesize a controller satisfying complex control specifications, such as temporal logic formulas. Following a correct-by-construction approach [42], we encode the requirements for the ACC by the linear temporal logic (LTL). First, consider two modes, called set-speed mode and time-gap mode. If the mode is in set-speed mode, the following vehicle must keep a given desired speed with some accuracy, i.e., $|x_2 - x^*_2| \leq \epsilon_1$. If the mode is in time-gap mode, the following vehicle must achieve a desired time headway $w^*$ with some accuracy, i.e., $|x_3/x_2 - w^*| \leq \epsilon_2$. Let $\text{mode}_1$, $\text{mode}_2$ be atomic propositions, such that $\text{mode}_1$ (resp. $\text{mode}_2$) is satisfied if the mode is in set-speed mode (resp. the time-gap mode). It is assumed that $\text{mode}_1$ (resp. $\text{mode}_2$) is satisfied if the state is included in the set $\mathcal{X}_1 = \{x \in \mathcal{X} \mid x_3 \leq 60\}$ (resp. $\mathcal{X}_2 = \mathcal{X} \setminus \mathcal{X}_1$). Let $\text{spec}_1$, $\text{spec}_2$ be the atomic propositions, such that $\text{spec}_1$ (resp. $\text{spec}_2$) is satisfied if $|x_2 - x^*_2| \leq \epsilon_1$ (resp. $|x_3/x_2 - w^*| \leq \epsilon_2$). Moreover, let $\text{safe}$ be the atomic proposition, such that it is satisfied if the state is included in $\mathcal{X}$. Then, we encode the control specification by the LTL formula as follows:

$$\psi = \Box \text{safe} \land \Box \land_{i=1}^2 (\text{mode}_i \implies \bigcirc \bigcirc \text{spec}_i), \quad (25)$$
where $\Box$ and $\bigcirc$ are so-called the "always" and "next" temporal operators, respectively (see, e.g., [46]). In words, the state $x$ must always stay in the safe set $\mathcal{X}$, and if the mode is in set-speed mode (resp. time-gap mode), the following vehicle must achieve the desired speed in two time steps (resp. the desired time headway in two time steps). Note that the controller for the safety specification $\Box \text{safe}$ has been already obtained after the implementation of Algorithm 2. The controller for the remaining part $\Box \land \bigwedge_{i=1}^2 (\text{mode}_i \implies \bigcirc \bigcirc \text{spec}_i)$ can be synthesized by a fixed point algorithm (see, e.g., [42]). The upper figures of Fig. 4 indicate the state trajectories by employing the synthesized controller with $x^*_2 = 22$, $\omega^* = 2.4$ and $\epsilon_1 = 0.2$, $\epsilon_2 = 0.2$. Moreover, the lower figures indicate the sequences of the error $|x_2 - x^*_2|$ and $|x_3/x_2 - \omega^*|$. It can be verified that the formula $\psi$ is satisfied by applying the synthesized controller, showing the effectiveness of the proposed approach.

7 Conclusions and future works

In this paper, we propose a learning-based approach towards symbolic abstractions for nonlinear control systems. The symbolic model is constructed by learning the un-modeled dynamics from training data, and the concept of an $\varepsilon$-approximate alternating simulation relation. Moreover, the safe exploration has been achieved by iteratively updating the controlled invariant and the safety controller, employing the safety game. In addition, we provide several techniques to alleviate the computational load to construct the symbolic models and the controlled invariant set. Finally, we illustrate the effectiveness of the proposed approach through a simulation example of an adaptive cruise control.

In our problem setup, it is of great importance to compute the upper bound of the RKHS norm $\|d_k\|_B \leq B$, since it has been utilized to construct the symbolic models. Hence, as described in Section 6, obtaining a large enough, yet not too conservative bound for $\|d_k\|_B$ should be further investigated in the future. In addition, since there exist no outliers in our problem setup, investigating how these can affect (if they exist) the estimation accuracy of the unknown function as well as how to detect them should be further pursued in future work.

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