Conditions for entanglement purification with general two-qubit states

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(Dated: August 31, 2016)

We present the convergence study of a recurrence entanglement purification protocol using arbitrary two-qubit initial states. The protocol is based on a rank two projector in the Bell basis which serves as a two-qubit operation replacing the usual controlled-NOT gate. We show that the whole space of two-qubit density matrices is mapped onto an invariant subspace characterized by seven real parameters. By analyzing this type of density matrices we are able to find general conditions for entanglement purification in the form of two inequalities between pairs of diagonal elements and pairs of coherences. We show that purifiable initial states do not necessary require a fidelity larger than one half with respect to any maximally entangled pure state. Furthermore, we find a family of states parametrized by their concurrence that can be perfectly converted into a Bell state in just one step of the protocol with probability proportional to the square of the concurrence.

I. INTRODUCTION

Entanglement purification protocols [1,3] generate Bell states from an ensemble of noisy entangled states. They have found application in many areas of quantum information theory [4] and prominently in proposed quantum communication technologies [5]. From a mathematical perspective, the effect of these protocols can be described as a non-linear map of the parameter defining the input density matrices. The first of the so-called recurrence protocols [6] was introduced by Bennett and collaborators [1]. This protocol relies on Werner states [5] which are described by one parameter and therefore the convergence of the corresponding map [1] can be studied in a simple way. A more efficient protocol was introduced by Deutsch et al. [2] based on Bell diagonal states involving a three dimensional parameter space. The convergence analysis is more intricate in this case and was studied separately in detail by C. Macchiavello [7].

There are two essential steps in recurrence entanglement purification protocols [1], which are reconsidered in this work: First, the application of random local unitary rotations transforming any initial density matrix into a Werner [2] or a Bell-diagonal state [3]. Second, the application of bilateral controlled-NOT gates on the spatially separated quantum systems. The latter step has already been modified in our proposed purification protocol [8] based on Refs. [1, 2]. The modification was motivated by the design of a multiphoton-assisted quantum repeater and instead of controlled-NOT gates it is based on a rank two projection in the Bell basis which can be implemented in a one-atom maser setup [9]. We have studied in Ref. [8] the performance of the modified protocol using Werner and other Bell-diagonal states. A natural question arises whether this protocol can still work for arbitrary input states making the step of random local unitary rotations (“twirling”) superfluous. This problem holds great significance, because random operations can waste important entanglement and their implementation is not straightforward. Therefore, a solution to this problem could offer a less intricate purification protocol and a better compatibility with certain experimental settings.

In this paper we show that our protocol, first introduced in Ref. [8], can exploit the entanglement of arbitrary initial states without using random unitary operations. After one iteration of the protocol, any input state is mapped onto a density matrix described by seven real parameters. This form is invariant under further iterations of the protocol. We exploit this feature to analyze the convergence properties for any input state of this form. Based on this analysis, we generalize to the case of arbitrary two-qubit states and find conditions that allow purification of Bell states. We demonstrate that in contrast to previous schemes [1, 2], a purifiable state does not require an initial fidelity larger than one half with respect to any Bell state. In particular, we find a class of states with overlap less than one half with any maximally entangled pure state that can be purified in just one step of the protocol.

The paper is organized as follows. In Sec. [1] we reintroduce and explain the steps of our protocol. In Sec. [II] we study the set of density matrices characterized by seven real parameters and derive the conditions for a successful entanglement purification. In Sec. [IV] we extend the analysis to density matrices characterized by fifteen real parameters. Examples which obey the newly found conditions are presented in Sec. [V] in Appendix [A] we present the stability analysis of the fixed points for the seven dimensional parameter space.

II. ENTANGLEMENT PURIFICATION PROTOCOL

In this section we review the entanglement purification protocol introduced in Ref. [8] which is based on the protocols of Refs. [1, 2]. Let us consider as initial condition
the product state of two qubit pairs
\[ \rho = \rho^{A_1, B_1} \rho^{A_2, B_2}. \]  
(1)

Both pairs are assumed to start in the same state \( \rho \) with certain degree of entanglement and their qubit components to be in distant locations labelled by \( A \) and \( B \).

The aim of an entanglement purification protocol is to trade two pairs for one pair with larger degree of entanglement using only local operations in laboratories \( A \) and \( B \) and classical communication between them. This can be achieved by increasing the fidelity with respect to any of the Bell states
\[ |\Psi^\pm\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle), \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}} (|00\rangle \pm |11\rangle). \]  
(2)

The purification protocol that we consider in this work consists of the following steps:

(I) The two-qubit quantum operation \( M \) is applied locally in locations \( A \) and \( B \), where
\[ M = |\Psi^\rangle \langle \Psi^-| + |\Phi^\rangle \langle \Phi^-|. \]  
(3)

After a successfully applied quantum operation the four qubit system attains the state
\[ \rho' = \frac{\Pi M^\dagger \Pi}{\text{Tr}(\Pi M \Pi \rho)}, \quad \Pi = M^{A_1,A_2} M^{B_1,B_2}. \]  
(4)

(II) One of the pairs is then locally measured in the computational basis, say pair \((A_2, B_2)\). The choice of the measured pair is unimportant as the operation is symmetric and the initial qubit pairs are identical. There are four possible states in which one can find pair \((A_2, B_2)\): \(|jk\rangle^{A_2,B_2} \equiv |j\rangle^{A_2} |k\rangle^{B_2} \) with \( j, k \in \{0, 1\} \). A successful measurement of one of the states \(|jk\rangle^{A_2,B_2}\) results in the two qubit state
\[ \tilde{\rho}^{A_1,B_1} = \text{Tr}_{A_2,B_2} \left\{ |jk\rangle \langle jk|^{A_2,B_2} \rho' \right\}. \]  
(5)

(III) Depending on the measurement results, the quantum gate \( V_j^{A_1} V_{k+1}^{B_1} \) is applied to the remaining qubit pair, where
\[ V_j = (|1\rangle \langle 1| + i|0\rangle \langle 0|) \sigma_x^j \]  
(6)

with the Pauli operator \( \sigma_x = |1\rangle \langle 0| + |0\rangle \langle 1| \). The final two-qubit state is then given by
\[ \rho'^{A_1,B_1} = \left( V_j^{A_1} V_{k+1}^{B_1} \right) \tilde{\rho}^{A_1,B_1} \left( V_j^{A_1} V_{k+1}^{B_1} \right)^\dagger. \]  
(7)

We remark again that there is a free choice of the qubit pair to be measured as our quantum operation \( M \) is symmetric in contrast to the controlled-NOT gate used in the seminal protocols \([1, 2]\). As the entangled pairs are assumed to start in the same state \( \rho \), in what follows we will omit the labels and study the two qubit map \( \rho \rightarrow \rho' \).

Now that we have explained the key elements of the protocol, we will study the resulting map between initial and output states of the qubit pairs. Consider a general two-qubit density matrix that can be written as
\[ \rho = \sum_{ij=1}^4 r_{ij} |i\rangle \langle j|, \quad r_j \equiv r_{jj} \]  
(8)

where we have chosen to work in the Bell basis labeled in the following way
\[ |1\rangle \equiv |\Psi^-\rangle, \quad |2\rangle \equiv |\Phi^-\rangle, \quad |3\rangle \equiv |\Phi^+\rangle, \quad |4\rangle \equiv |\Psi^+\rangle. \]  
(9)

Due to the conditions \( \text{Tr}(\rho) = 1 \) and \( \rho^\dagger = \rho \) we have the following relations:
\[ r_1 + r_2 + r_3 + r_4 = 1, \quad r_{ij} = (r_{ji})^*. \]  

The probabilistic part of this protocol takes place in step (I), i.e., in the application of the bilateral two-qubit quantum operation \( M \). The success probability of the quantum operation is \( N/2 \), with
\[ N = (r_1 + r_2)^2 + (r_3 + r_4)^2 - (r_{12} + r_{21})^2 - (r_{34} + r_{43})^2. \]  
(10)

Provided both implementations of \( M \) are successful, the steps (II) and (III) can be given for granted as they involve unitary gates and none of the qubit measurements is discarded. The resulting two-qubit density matrix \( \rho' \) in the same basis as \([5]\) has the non-zero entries
\[ r_1' = \frac{r_1^2 + r_2^2 - r_{12}^2 - r_{21}^2}{N}, \quad r_2' = \frac{2r_{13}r_4 - |r_{34}|^2}{N}, \]  
\[ r_3' = \frac{2r_{12}r_3 - |r_{12}|^2}{N}, \quad r_4' = \frac{r_3^2 + r_4^2 - r_{34}^2 - r_{43}^2}{N}, \]  
\[ r_{14}' = \frac{r_{14}^2 + r_{23}^2 - r_{13}^2 - r_{24}^2}{N}, \quad r_{23}' = \frac{2r_{23}r_{14} - r_{13}^2}{N}. \]  
(11)

Together with \( r_{14}' = (r_{14}')^* \) and \( r_{32}' = (r_{32}')^* \), these are the only non-vanishing elements of \( \rho' \) after a single iteration of the protocol. The constant \( N \) plays here the role of a normalization factor and the resulting density matrix has eight vanishing entries. Therefore, only seven parameters characterize these type of states, in contrast to the fifteen parameters needed to describe a general two-qubit state. This simplification motivates the study of the purification protocol for density matrices described by the seven primed parameters of Eq. (11).

It is worth to note that the protocol in Ref. \([2]\) is also able to reduce the fifteen dimensional parameter space to a seven dimensional one. This happens technically in a rather different way as in our protocol. After measuring the target pair during the procedure, one keeps the control pair only if the target pair was found in the state \([11]\). The result can be obtained by a straightforward application of the protocol in Ref. \([2]\) on general two-qubit density matrices and up to our knowledge it has never been reported.
III. ENTANGLEMENT PURIFICATION OF X STATES IN THE BELL BASIS

In this section we concentrate our study on initial two-qubit states described by seven free parameters, which have been delineated in Eq. (11). In the Bell basis they have the matrix representation

\[ \rho = \begin{pmatrix} r_1 & 0 & 0 & r_{14} \\ 0 & r_2 & r_{23} & 0 \\ 0 & r_{32} & r_3 & 0 \\ r_{41} & 0 & 0 & r_4 \end{pmatrix}, \]

(12)

and therefore we refer to them them as \( X \) states. The diagonal elements \( r_1, r_2, r_3, r_4 \in [0, 1] \) and the off-diagonal elements \( r_{23}, r_{14}, r_{14} \in \mathbb{C} \) fulfill the conditions \( r_1 + r_2 + r_3 + r_4 = 1 \), \( r_{32} = r_{23} \), and \( r_{41} = r_{14} \). The four eigenvalues can be represented as the two pairs \( \lambda_{j}^{\pm} \), which can be obtained from the quadratic formula

\[ \lambda_{jk}^{\pm} = \frac{r_j + r_k \pm \sqrt{(r_j - r_k)^2 + 4|r_{jk}|^2}}{2}. \]

(13)

As the eigenvalues of \( \rho \) have to be real non-negative numbers, the radical in Eq. (13) has to be positive, and this imposes the following restriction to the absolute value of the coherences

\[ |r_{23}| \leq \sqrt{r_2 r_3}, \quad |r_{14}| \leq \sqrt{r_1 r_4}. \]

(14)

In the case of input states in the form of Eq. (12), the output state \( \rho' \) remains in the same form and according to (11) with the new coefficients:

\[ r_1' = \frac{r_1^2 + r_2^2}{N}, \quad r_2' = \frac{2 r_3 r_4}{N}, \quad r_{14}' = \frac{r_{14}^2 + r_{23}^2}{N}, \]

\[ r_4' = \frac{r_3^2 + r_4^2}{N}, \quad r_3' = \frac{2 r_1 r_2}{N}, \quad r_{23}' = \frac{2 r_{23} r_{14}^*}{N}, \]

(15)

and the normalization factor

\[ N = (r_1 + r_2)^2 + (1 - r_1 - r_2)^2. \]

(16)

In order to find the convergence properties of this map, we will first study the behavior of the off-diagonal elements after one iteration by considering the norm of the pair \((r_{14}, r_{23})\). It turns out that this quantity does not increase, but typically decreases after iteration of the map. This can be noted by analyzing the ratio

\[ \frac{|r_{14}'| + |r_{23}'|}{|r_{14}| + |r_{23}|} \leq \frac{|r_{14} + |r_{23}|}{N} \leq 1, \]

(17)

where we have used the normalization factor in Eq. (16). The first inequality can be obtained by direct evaluation of \( r_{14}' \) and \( r_{23}' \) using Eq. (15), and by taking into account the triangle inequality \(|r_{14} + r_{23}^2| \leq |r_{14}|^2 + |r_{23}|^2|\). The second inequality in Eq. (17) follows from considering the ratio between the maximum value of \(|r_{14}| + |r_{23}|\) and the minimum value of \( N \), both of which are \( 1/2 \). To show this, first note that according to Eq. (16), \( N \) attains its minimum value \( 1/2 \) when \( r_1 + r_2 = 1/2 \). Secondly, we take into account the conditions in (14) that combined with the normalization condition for \( \rho \) and the fact that for any pair of real numbers \( 2xy \leq x^2 + y^2 \), implies the relation \(|r_{14}| + |r_{23}| \leq 1/2 \) between the coherences. Taking the equality case in (14) one can note that the equality in Eq. (17) is achieved only with the initial conditions

\[ |r_{14}| = r_1 = r_4 = 0.5, \quad |r_{23}| = r_2 = r_3 = 0, \quad |r_{14}| = r_1 = r_4 = 0, \quad |r_{23}| = r_2 = r_3 = 0.5, \quad |r_{14}| = |r_{23}| = r_1 = r_2 = r_3 = r_4 = 0.25. \]

(18)

Only in these three cases the norm of the coherences remains constant after repeated iteration of the map in (15). For all other values of the coherences, the ratio in Eq. (17) is strictly less than one and therefore \( r_{14} \) and \( r_{23} \) tend to zero by repeated iterations of the map (15). We remark that in all these three cases, all the diagonal values are less than one-half.

Now we turn our attention to the diagonal elements. The aim is to find the initial conditions that after iteration of the map tend to one of the Bell states. The protocol we have presented here is not the same as the protocol in Ref. [2], however the map (15) for the diagonal elements \( r_1, r_2, r_3, r_4 \) has a similar structure. Therefore, we follow the idea of the proof in Ref. [7], where Bell diagonal density matrices were studied. The same treatment is justified in our case as the diagonal elements in Eq. (15) do not depend on the off-diagonal ones.

We start this discussion by noting that \( r_1' \geq r_3' \) and \( r_4' \geq r_2' \) which indicates that neither \( r_2 \) or \( r_3 \) can reach unit value, actually not even above \( 1/2 \). With this fact we recognize that only \( r_1 \) or \( r_4 \) can potentially increase their value to unity after repeated iterations of the map (15). Noting the symmetry of the map under interchange of \( r_1 \leftrightarrow r_4 \) and \( r_2 \leftrightarrow r_3 \), allows us to consider only the maximization of one, say \( r_1 \). Indeed the case \( r_1 = 1 \) and all other parameters equal to zero is a fixed point of the map (15). Now, let us rewrite the value of \( r_1' \) as

\[ r_1' = \frac{r_1^2 + r_2^2}{2(r_1^2 + r_2^2) - F(r_1, r_2)}, \]

where we have introduced the quadratic form

\[ F(r_1, r_2) = (2r_1 - 1)(2 - 2r_2). \]

(19)

Whenever \( F(r_1, r_2) \) is positive, \( r_1' \) attains values larger than \( 1/2 \). This actually happens whenever \( r_1 \) or \( r_2 \) are larger than \( 1/2 \) and in this case the function \( F \) increases after iteration of the map (15), as can be noted from the relation

\[ \frac{F(r_1', r_2')}{F(r_1, r_2)} = \frac{N - 4r_3 r_4}{N^2} = \frac{(1 - x)^2 + y^2}{(1 - x)^2 + x^2} \geq 1, \]

(20)

with \( x = r_3 + r_4 \) and \( y = r_4 - r_3 \). The inequality follows from the fact that \( 1 - x \geq (1 - x)^2 + x^2 \) for \( x \in [0, 1/2] \),
FIG. 1: Three dimensional space of the independent diagonal elements \((r_1, r_2, r_3)\). The other parameter \(r_4\) is obtained from the trace condition \(\text{Tr}\rho = 1\). In blue: the region defined by condition (21) whose points lead to the fixed point \(r_1 = 1\). In green: the region defined by condition (22) that lead to the fixed point \(r_4 = 1\). In red: all other conditions that do not lead to an entangled state. The black dots are the stable fixed points that correspond to Bell states, whereas the red one is the fixed point representing the totally mixed state.

which holds true if \(r_1 > 1/2\) or \(r_2 > 1/2\). The function \(F(r_1, r_2)\) increases its value after application of the map \((15)\) when the strict inequality is met. In this case, as \(F(r_1, r_2)\) is a monotonic function of \(r_1\) and \(r_2\), iteration of the map converges to the fixed point with \(r_1 = 1\) which is the only fixed point with the condition \(r_1 > 1/2\).

The equality in \((20)\) happens only in the case \(r_3 = r_4 = 0\), but after each iteration of the map \((15)\) with initial values \(r_3 = r_4 = 0\), the values of \(r_3\) and \(r_4\) cannot be simultaneously zero and thus \(F(r''_1, r''_2) > F(r'_1, r'_2)\) unless \(r_2 = 0\) but that would be the case of the fixed point with \(r_1 = 1\). This proves that the condition to purify to a Bell state \(|\Psi^-\rangle\), i.e. \(r_1 = 1\), can be written as

\[
(2r_1 - 1)(1 - 2r_2) > 0
\]  \(\text{(21)}\)

In complete analogy due to the symmetry \(r_1 \leftrightarrow r_4\) and \(r_2 \leftrightarrow r_3\), one can find the condition to purify to Bell state \(|\Psi^+\rangle\), i.e. \(r_1 = 1\), can be written as

\[
(2r_4 - 1)(1 - 2r_3) > 0
\]  \(\text{(22)}\)

These inequalities define four purifiable regions in the three dimensional space of the diagonal parameters shown in Fig. 1. These findings are similar to Ref. 7, where purifications of the states \(|\Phi^+\rangle\) and \(|\Psi^+\rangle\) have been proven. Here, our proof is extended taking into account off-diagonal elements of \(X\) states subject to our protocol. We have shown in Eqs. (17) and (18) that the absolute value of the off-diagonal parameters decreases in the regions subject to the conditions (21) and (22). Therefore the regions depicted in blue and green in Fig. 1 have one stable attractor point. In Appendix A we evaluate numerically the stability of the fixed points.

IV. CONDITIONS FOR GENERAL TWO-QUBIT DENSITY MATRICES

In this section we extend our analysis to the case of general two-qubit density matrices in Eq. (8). In this case and after one iteration one obtains the density matrix of an \(X\) state (see Eq. (12)). Therefore, the most important question is to define the conditions in which after the first iteration of the map \((11)\) one obtains any of the four \(r_i\)'s larger than 1/2. First we note that one can never achieve \(r'_i > 1/2\) or \(r'_3 > 1/2\) after one iteration. This due to the fact that \(r'_1 \geq r'_3\) and \(r'_4 \geq r'_2\), which is noted from \((11)\) by comparing their numerators and considering that: \(x^2 + y^2 \geq 2xy\) for any \(x, y \in \mathbb{R}\) and because \(z^2 + z^*2 \leq 2|x|^2\) for any \(z \in \mathbb{C}\). This means that we only have to find those conditions that leave \(r'_1 > 1/2\) or \(r'_4 > 1/2\). For symmetry reasons it suffices to focus in only one, we choose \(r_1\). Let us introduce the following abbreviations

\[
a = (r_1^2 + r_2^2 - r_{12}^2 - r_{21}^2),
\]

\[
b = (2r_1 - 1)(1 - 2r_2) - (r_{12} - r_{21})^2 + (r_{34} + r_{43})^2,
\]

which relate to the normalization factor as \(N = 2a - b\). This form of rewriting \(N\) allows to identify the fidelity with respect to \(|\Psi^-\rangle\) after one step of the map in Eq. (11), namely

\[
r'_1 = a/(2a - b).
\]  \(\text{(24)}\)

This expression is larger than 1/2 whenever \(b > 0\) or equivalently when the following inequality is met

\[
(2r_1 - 1)(1 - 2r_2) > -(2\text{Im}[r_{12}])^2 - (2\text{Re}[r_{34}])^2.
\]  \(\text{(25)}\)

This condition generalizes (21) as any state fulfilling it is transformed by the map into an \(X\) state of the form of (12), but with the coefficient \(r'_1 > 1/2\). In Sec. III we proved that these type of states can be purified. Because the right-hand side of Eq. (25) is always zero or negative, this inequality is always fulfilled whenever \(r_1\) or \(r_2\) are larger that 1/2 and actually allows them to be smaller than this threshold. This actually means that the purification protocol is able to convert the entanglement of other states into \(|\Psi^-\rangle\). Symmetry arguments lead to the analogue condition for purifying \(|\Psi^+\rangle\) with \(r_3\) and \(r_4\), namely

\[
(2r_4 - 1)(1 - 2r_3) > -(2\text{Im}[r_{34}])^2 - (2\text{Re}[r_{12}])^2.
\]  \(\text{(26)}\)

Eqs. (25) and (26) are the main result of this work and define the condition to purify entanglement in the presented protocol. These conditions compared with (21) and (22) relax the requirement for the diagonal parameters, because the right-hand side of the inequalities in
In order to illustrate the difference with the previous section we consider the following dependence of the coherences and the condition in Eq. (25). The examples show how our protocol is able to purify initial entangled states with low fidelity with respect to Bell states.

**Example 1.** Consider the parameter \( x \in (0.5, 1) \) determining the state

\[
\rho_1 = x |\Psi^-\rangle \langle \Psi^-| + (1 - x) |\Phi^+\rangle \langle \Phi^+| \\
|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|\psi^-\rangle + i|\psi^+\rangle).
\]  

(29)

The state \( \rho_1 \) has no fidelity larger than one half with respect to any of the four Bell states. However, it has fidelity \( x > 1/2 \) with respect to the maximally entangled state \( |\Psi^-\rangle \). Taking into account that in this example \( r_1 = r_2 = x r_3 = x/2 \) and \( r_4 = 0 \), one can find that after one iteration of the map

\[
r'_1 = \frac{x^2}{x^2 + (1 - x)^2}
\]

which is always larger than \( x \) for \( x \in (0.5, 1) \). This is an example of an initial state where the fidelities with respect to any of the four Bell states is smaller than one half and still the state \( |\Psi^-\rangle \) can be purified. Of course, here the fidelity with respect to the maximally entangled state \( |\Psi^-\rangle \) is being exploited by the protocol. In contrast, this feature would be lost with other known protocols \([1,2]\) that use random unitary gates to bring a state into a Bell diagonal form. This procedure would destroy the entanglement of \( \rho_1 \).
Example 2. Now take the parameter $c \in (0, 0.5]$ that parametrizes the state

$$\rho_2 = c|\Psi^-\rangle\langle\Psi^-| + (1 - c)|\Upsilon_{\text{sep}}\rangle\langle\Upsilon_{\text{sep}}|$$

$$|\Upsilon_{\text{sep}}\rangle = \frac{1}{\sqrt{2}} (|\Phi^+\rangle + |\Psi^+\rangle).$$ (30)

In this example $r_1 = c$ and $r_2 = r_4 = r_{34} = (1 - c)/2$. After a single iteration of the protocol one finds that the fidelity with respect to $|\Psi^+\rangle$ takes the value

$$r_1' = \frac{c^2}{2c^2 + (1 - 2c) - (1 - c)^2} = 1.$$

This means that no matter the value of $c$, as long it is not zero, the state can be purified to $|\Psi^-\rangle$ in just one iteration of the protocol. The success probability is $c^2/2$ and therefore the smaller the value of $c$ the less probable is the occurrence of an event that generates $|\Psi^-\rangle$. However, the fact that perfect purification is achieved in just one step makes this protocol with this initial state an attractive candidate. Typically the overall success probability scales exponentially with the number of steps [4] and the convergence is formally achieved in an asymptotic way.

Interestingly, state $\rho_2$ has overlap smaller than one half with any maximally entangled pure state. In order to proof this statement, let us consider the general form of these type of states in the Bell basis. First, we recall that the concurrence [10] is a measure of entanglement which takes unit value for maximally entangled pure states. For a pure state $|\Psi\rangle$ it is defined as

$$C(|\Psi\rangle) = |\langle\Psi|\sigma_y \otimes \sigma_y|\Psi\rangle^*|,$$ (31)

with the Pauli matrix $\sigma_y$ and where $|\Psi\rangle^*$ is the complex conjugated vector of $|\Psi\rangle$. The Bell states are eigenstates of the operator in Eq. (31), namely $\sigma_y \otimes \sigma_y |\Psi^\pm\rangle = \pm |\Psi^\pm\rangle$ and $\sigma_y \otimes \sigma_y |\Phi^\pm\rangle = \mp |\Phi^\pm\rangle$. With this one can note that, up to a global phase, any maximally entangled pure state has the form

$$|\Psi\rangle = a_-|\Phi^-\rangle + ia_+|\Phi^+\rangle + ib_-|\Psi^-\rangle + b_+|\Psi^+\rangle,$$ (32)

with coefficients $a_\pm, b_\pm \in \mathbb{R}$ and satisfying the normalization condition $\langle\Psi|\Psi\rangle = 1$. Then, by introducing $y = (a_+^2 + b_-^2)/2$, the fidelity of $\rho_2$ with respect to any maximally entangled pure state takes the form

$$F_{\rho_2} = \langle\Psi|\rho_2|\Psi\rangle = cd^2 + (1 - c)y$$ (33)

$$\leq c(1 - y) + (1 - c)y \leq \frac{1}{2}.$$

The first inequality takes into account the normalization conditions of $|\Psi\rangle$. The last inequality is valid provided both $c$ and $y$ are positive numbers less than a half, which is true by our choice of $c$ and by the definition of $y$.

Finally, let us point out that the parameter $c$ is actually the concurrence [10] of the state $\rho_2$. Indeed, it is not hard to realize that the matrix $\rho_2 (\sigma_y \otimes \sigma_y) \rho_2^* (\sigma_y \otimes \sigma_y)$ has only one eigenvalue given by $c^2$ and therefore, according to Ref. [10], the concurrence is $c$.

VI. CONCLUSIONS

We have studied the convergence of arbitrary input two-qubit states, under the influence of the entanglement purification protocol presented in Sec. [1] and first introduced in Ref. [8]. Our protocol was introduced in the context of a multiphoton-assisted quantum repeater, where a two-qubit quantum operation can be more efficiently realized than a controlled-NOT gate.

In a first approach, we have studied in detail the convergence in a subset of density matrices which is left invariant by the protocol. This subset is characterized by seven real parameters, three of which are diagonal elements of the density matrix in the Bell basis. These diagonal elements do not depend on the coherences after the iteration of the protocol. Therefore, we have based our proof on the work of C. Macchiavello [2]. In order to show that the off-diagonal elements monotonically decrease, we have introduced a quadratic form which decreases under each iteration of the map. Combining these results we were able to define those conditions for the initial states which lead to the purification of two Bell states. We have shown that these conditions can be generalized to arbitrary density matrices with 15 real parameters. This contrasts with the two seminal entanglement purification protocols [1] [2] which consider density matrices characterized by one or three real parameters. Random unitary rotations are required in those cases to bring any state into a Bell diagonal state. This step in the original protocols may destroy useful entanglement. The fact that our protocol does not require such step is the reason why we obtain less constrained conditions for purifiable initial states.

Exploiting these new findings, we have shown that our protocol allows purification of entangled states having an overlap less than one half with any Bell state. Some states might even have fidelity less than one half with respect to any maximally entangled pure state. We have shown a class of these type of states that can be converted into a Bell state in just one step of the purification protocol. The states are parametrized by their concurrence $c$ and the probability of success is given by $c^2/2$. This feature is interesting, as the resources of entanglement purification scale exponentially with the number of states. Therefore, our result could initiate the study of other type of local two-qubit operations which might lead to an optimal entanglement purification for specific type of states. Provided that the source of the entangled pairs is well under control, the engineering of an efficient local two-qubit operation might be more resourceful than the application of a controlled-NOT gate.

Acknowledgments

This work is supported by the BMBF project Q.com.
Appendix A: Fixed point analysis of X states

In order to show that there are no other stable fixed points in regions defined by the conditions (21) and (22), we have numerically investigated the stability of the fixed points of the map (15). This is accomplished by studying the Jacobian matrix with entries $J_{ij}$, where the entries can be defined as $J_{ij} = \frac{\partial f_i}{\partial x_j}$, where each $x_i$ is a component of the eight dimensional vector $v = (r_1, r_2, r_3, r_4, \text{Re}[r_{14}], \text{Im}[r_{14}], \text{Re}[r_{23}], \text{Im}[r_{23}])$. The introduced function fulfills the relation $v' = f(v)$ where the primed variables are defined in the non-linear map (15). Fixed points are obtained by solving the equation $v_\ast = f(v_\ast)$ and their stability is specified with the eigenvalues of $J|v\ast=v_\ast$. If all eigenvalues have absolute value smaller than one, the point is stable. If the absolute value of any eigenvalue is larger than one, the fixed point is unstable.

Table I shows that there are three stable fixed points, of which two represent the cases when we purify the states $|\Psi^\ast\rangle$ and $|\Psi^+\rangle$ and the other when we converge towards the maximally mixed state. We have found that when considering initial density matrices with one of the parameters $r_i = 0$ and the rest $r_j, r_k, r_l \in [0, 0.5]$ the iteration of the protocol can lead to fixed points of at least second order, meaning that they are fixed points of the second iteration of the function $f$, namely $v_\ast\ast = f(f(v_\ast\ast))$. It also can happen that with right initial conditions the map stays or cycles around unstable one- or multi-period fixed points.

| Fixed points $(r_1, r_2, r_3, r_4, r_{14}, r_{23})$ | Stability |
|------------------------------------------------|--|
| (1, 0, 0, 0, 0, 0) | stable |
| (0, 0, 0, 1, 0, 0) | stable |
| (0.1409, 0.2344, 0.1245, 0.5, 0, 0) | unstable |
| (0.5, 0.1245, 0.2344, 0.1409, 0, 0) | unstable |
| (0.25, 0.25, 0.25, 0.25, 0.25) | stable |
| (0.25, 0.25, 0.25, 0.25, 0.25, −0.25) | unstable |
| (0.25, 0.25, 0.25, 0.25, 0.25) | unstable |
| (0.5, 0, 0, 0.5, 0, 0) | unstable |
| (0.5, 0, 0, 0.5, 0, 0) | unstable |

**TABLE I:** List of fixed points of the map (15) with their respective stability. Two fixed points are given with 4-digit precision.

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