Relativistic, finite temperature multifluid hydrodynamics in a neutron star from a variational principle

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We develop a relativistic multifluid dynamics appropriate for describing neutron star cores at finite temperatures based on Carter’s convective variational procedure. The model includes seven fluids, accounting for both normal and superfluid/superconducting neutrons and protons, leptons (electrons and muons) and entropy. The formulation is compared to the non-variational relativistic multifluid hydrodynamics of Gusakov and collaborators and shown to be equivalent. Vortex lines and flux tubes, mutual friction, vortex pinning, heat conduction and viscosity are incorporated into the model in steps after the basic hydrodynamics is described. The multifluid system is then considered at the mesoscopic scale where the currents around individual vortex lines and flux tubes are important, and this mesoscopic theory is averaged to determine the detailed vortex line/flux tube contributions to the macroscopic “effective” theory. This matching procedure is largely successful, though obtaining full agreement between the averaged mesoscopic and macroscopic effective theory requires discarding subdominant terms. The matching procedure allow us to show that the magnetic $H$-field inside a neutron star differs from that given in previous astrophysical works, but is in agreement with condensed matter physics literature.

I. INTRODUCTION

Neutron stars are fundamentally relativistic objects, so it is necessary to have a relativistic hydrodynamic formalism to accurately model their internal dynamics. As it is believed that neutron star cores can consist of both superfluid neutrons and superconducting protons [1–6], this formalism needs to incorporate multiple separately-moving fluids. An ideal formalism will also incorporate effects such as superfluid–normal fluid phase transitions, superfluid neutron vortex lines, type-II superconducting proton flux tubes and dissipation. Vortex lines/flux tubes can affect the fluid dynamics through mutual friction due to scattering between vortex lines and normal fluid particles, most importantly leptons, and pinning between neutron and proton vortex lines. These effects may be important in determining the oscillation modes of neutron stars that could be excited during binary inspiral [7, 8], and in explaining pulsar glitches [9–12].

In this paper we develop a fully general relativistic formulation of finite temperature, multifluid hydrodynamics appropriate for neutron star cores. We consider a core consisting of four particle species: neutrons, protons, electrons and muons. The neutrons and protons exist in both superfluid/superconducting and normal phases, whose relative motions are dynamically connected through superfluid entrainment. Our approach has a few advantages compared to previous formulations of relativistic multifluid dynamics applied to neutron stars. We follow the convective variational approach originated by Taub [13] and later elaborated by Carter and collaborators [14–17], making only limited assumptions about the dependence of the master function (Langrangian) on Lorentz-invariant combinations of vectors and tensors. We thus retain the full symmetry of the variational procedure while being connected, through strategic rearrangement of terms, to relativistic formu-
the electromagnetic auxiliary field and vortex self-tension tensors, which are conjugate to the electromagnetic field tensor $F_{\mu\nu}$ and vorticity tensors respectively, by considering a simplified model of the multifluid hydrodynamics at the “mesoscopic” scale where currents arise from individual vortex lines and flux tubes are considered. The mesoscopic theory is then averaged to determine an effective macroscopic theory, with most of the details of this procedure relegated to an appendix. We are successfully able to average the mesoscopic theory, but only find an approximate match to the effective macroscopic theory. We conclude by using the results of the averaged mesoscopic-to-macroscopic matching procedure to resolve some disagreements about the interpretation of the magnetic $H$-field in a rotating superfluid–superconducting neutron star and also clarify the form of the Maxwell equations and Lorentz force acting on the charged fluids in neutron star MHD. An alternate form of the relativistic stress-energy tensor is included in an appendix. $c = G = 1$ units and the ($- , + , + , +$) metric convention are used throughout.

II. CONVECTIVE VARIATIONAL PROCEDURE

Starting with a Lagrangian density describing the finite temperature multifluid in a neutron star core, we employ the convective variational procedure [14–17] to compute the relevant equations of motion. There has been recent interest in this formulation [31, 32] for application to problems involving neutron star astrophysics, pulsar glitches and gravitational waves from binary neutron stars. Compared to other fluid variational methods [13, 33], with the convective variational procedure we can transparently include additional forces between the fluids that are not obviously incorporated directly via a variational method. An additional advantage which we exploit is the ability to include viscosity using a convective variational-type method.

In the first subsection, we describe our Lagrangian density and define the dynamical variables, adding in steps the fluid number currents, electromagnetism and vorticity. In the second subsection we introduce the Lagrangian displacement fields employed in the convective variational procedure and derive the equations of motion.

A. Lagrangian and its variation

Consider a multifluid neutron star core consisting of neutrons ($n$), protons ($p$), electrons ($e$), muons ($m$) and entropy ($s$). The neutrons and protons will have both superfluid/superconducting and normal fluid excitation components, with the former being distinguished using an overline ($\overline{n, p}$). There exists a four-current $n^\mu_x$, $x \in \{n, p, e, m, \overline{n, p}\}$, for each species/quantity. $x = s$ is the entropy four-current $s^\mu$, which will later be related to the four-currents of the entropy-carrying normal fluids. In principal each normal fluid could have its own corresponding entropy current, but as we will later constrict the normal fluids to move together, we introduce only a single entropy current here. The following Lorentz-invariant scalars can be constructed by contracting the four-currents:

$$n^2_x = - n^\mu_x n^\mu_x , \quad \alpha^2_{xy} = - n^\mu_x n^\mu_y = \alpha^2_{yx}, \quad (1)$$

where $y \in \{n, p, e, m, \overline{n, p}, s\} \neq x$. $\alpha^2_{xy}$ is equivalent to the product of the Lorentz factor for the relative motion between fluids $x$ and $y$ and the two number densities $n_x$ and $n_y$ as measured in the respective fluids’ rest frames, as will be clear from the definition of $n^\mu_x$ given in Eq. (57). $\alpha_{xy}$ with $y \neq s$ will be responsible for superfluid entrainment, while the $\alpha_{xs}$ are “entropy entrainment” terms representing heat convection by the particle currents. The $\alpha_{xs}$ will later allow for heat conduction independent of the particle currents. There will be 10 nonzero $\alpha_{xy}$ ($\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$, $\alpha_{np}$). The superfluids do not carry entropy, so $\alpha_{\overline{n}, p} = \alpha_{p, \overline{n}} = 0$. The exclusion of entropy entrainment results in instabilities and causality violation [34, 35], and we discuss the effects of heat conduction on the entropy current in Section IV A.

The Lagrangian density will be a function of dynamical variables $n^\mu_x$, the electromagnetic field tensor $A_{\mu\nu}$, and vorticity tensors $\omega_{\mu\nu}$ associated with the vortex line/flux tube arrays for each superfluid species. We can split the Lagrangian density into a master function $\Lambda$, interaction terms and spacetime curvature terms. $\Lambda$ includes the thermodynamic internal energy density of the fluid, the electromagnetic field energy and the vortex line/flux tube energy, and is a function of Lorentz invariant scalars. To begin, we consider only the dependence of this master function on the number currents and the metric:

$$\Lambda = \Lambda (n^2_x, \alpha^2_{xy}, g_{\mu\nu}), \quad (2)$$

where all $x$ and distinct combinations of $x$ and $y$ are implicitly included. Varying this $\Lambda$ gives

$$\delta \Lambda = \frac{\partial \Lambda}{\partial n^2_x} \delta n^2_x + \frac{\partial \Lambda}{\partial \alpha^2_{xy}} \delta \alpha^2_{xy} + \frac{\partial \Lambda}{\partial g_{\mu\nu}} \delta g_{\mu\nu}. \quad (3)$$

The variations with respect to the four-currents can be rewritten in terms of the number and entropy four-currents using

$$\frac{\partial \Lambda}{\partial n^2_x} \delta n^2_x = \left( - 2 \frac{\partial \Lambda}{\partial n^2_x} n^\mu_x \right) \delta n^\mu_x , \quad (4)$$

$$\frac{\partial \Lambda}{\partial \alpha^2_{xy}} \delta \alpha^2_{xy} = \left( - \frac{\partial \Lambda}{\partial \alpha^2_{xy}} n^\mu_x \right) \delta n^\mu_x + \left( - \frac{\partial \Lambda}{\partial \alpha^2_{xy}} n^\mu_y \right) \delta n^\mu_y , \quad (5)$$

where we adopt the convention of [14, 27, 36], among others, in defining

$$B^x = - 2 \frac{\partial \Lambda}{\partial n^2_x} , \quad A^{zy} = A^{yz} = - \frac{\partial \Lambda}{\partial \alpha^2_{zy}}. \quad (6)$$
There will be 7 $B^x$, one for each current particle current plus the entropy current, and 10 nonzero $A^{xy}$ corresponding to each nonzero $\alpha_{xy}$. Using Eq. (6) and noting which $A^{xy}$ are zero, we can define the conjugate dynamical momenta or generalized chemical potential four-vectors

$$\mu_\mu^x = B^x n_\mu^x + \sum_{y \neq x} A^{xy} n_\mu^y + A^{xx} s_\mu,$$  \hspace{1cm} (7)

where $x, y \in \{n, p, e, m, \pi, \bar{\pi}, \}$, and a conjugate “thermal momentum”

$$\Theta_\mu = \mu_\mu^x = B^x s_\mu + \sum_{x} A^{xx} n_\mu^x,$$  \hspace{1cm} (8)

where $x \in \{n, p, e, m\}$. To determine $\partial \Lambda / \partial g_{\mu\nu} = \partial \Lambda / \partial g_{\mu\nu}$, following Carter [15], the variations in Eq. (3) are specified by their Lie derivative $\xi^\mu$ with respect to a single infinitesimal displacement field $\xi^\mu$ which acts on the background manifold. This displacement field is not the same as the displacement fields which specify the motion of the individual fluids and which are introduced in Section II B. For the purposes of determining $\partial \Lambda / \partial g_{\mu\nu}$, we use

$$\delta \Lambda = \mathcal{L} \xi^\mu = \xi^\mu \nabla_\mu \Lambda,$$  \hspace{1cm} (9a)

$$\delta n_\mu^x = \mathcal{L} n_\mu^x = \xi^\mu \nabla_\mu n_\mu^x - n_\mu^x \nabla_\mu \xi^\mu,$$  \hspace{1cm} (9b)

$$\delta g_{\mu\nu} = \mathcal{L} g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 2 \nabla_{(\mu} \xi_{\nu)},$$  \hspace{1cm} (9c)

which, inserted into Eq. (3), give the following relation

$$\left( \sum_{x} \mu_\mu^x \nabla_\mu n_\mu^x - \nabla_\mu \Lambda \right) \xi^\mu = \left( \sum_{x} \mu_\mu^x n_\mu^x - \frac{1}{2} \frac{\partial \Lambda}{\partial g_{\mu\nu}} \right) \nabla_\mu \xi_\nu.$$  \hspace{1cm} (10)

Since this must be true for arbitrary $\xi^\mu$ and $\nabla_\mu \xi_\nu$, both sides of this must be zero independently, giving

$$\nabla_\mu \Lambda = \sum_{x} \mu_\mu^x \nabla_\mu n_\mu^x,$$  \hspace{1cm} (11)

$$\frac{\partial \Lambda}{\partial g_{\mu\nu}} = \frac{1}{2} \sum_{x} \mu_\mu^x n_\mu^x.$$  \hspace{1cm} (12)

Inserting the second of these into Eq. (3) and using the definitions of the conjugate momenta, $\delta \Lambda$ becomes

$$\delta \Lambda = \sum_{x} \mu_\mu^x \delta n_\mu^x + \frac{1}{2} \sum_{x} n_\mu^x \mu_\mu^x \delta g_{\mu\nu}.$$  \hspace{1cm} (13)

As written, the extremization of the action with respect to each current density would require the conjugate momentum to be zero. This is of course too restrictive, and the correct variation of the current densities in terms of Lagrangian displacement fields is introduced in Section II B.

To include electromagnetism, we allow the master function to depend on the electromagnetic field tensor $F_{\mu\nu} = 2 \nabla_{(\mu} A_{\nu)}$ through a contraction with another antisymmetric rank-two tensor. The electromagnetic four-potential $A_{\mu}$ is minimally coupled to the total charge current

$$\mathcal{L}_{\mathrm{EM\ coupl.}} = j_\mu^\mu A_{\mu},$$  \hspace{1cm} (14)

where the charge current is

$$j_\mu^\mu = e (n_\mu^\mu + n_\mu^\pi - n_\mu^e - n_\mu^m).$$  \hspace{1cm} (15)

The variation of the action thus contains additional terms

$$\delta (\Lambda_{\mathrm{EM}} + \mathcal{L}_{\mathrm{EM\ coupl.}}) = - \frac{1}{8 \pi} K^{\mu\nu} \delta F_{\mu\nu} + j_\mu^\mu \delta A_{\mu} + A_{\mu} \delta j_\mu^\mu,$$  \hspace{1cm} (16)

where we have defined the (antisymmetric) electromagnetic auxiliary tensor

$$K^{\mu\nu} = -8 \pi \frac{\partial \Lambda}{\partial F_{\mu\nu}} \bigg|_{n_\mu^\pi, w_\mu^\pi}.$$  \hspace{1cm} (17)

This tensor has been defined as the electromagnetic displacement tensor $\mathcal{H}^{\mu\nu}$ in previous works [17], but for reasons explained in Section V, we reserve this notation and nomenclature for a different quantity. We have explicitly denoted that all number currents $n_\mu^\pi$ and vorticity tensors $w_\mu^\pi$ are held fixed during this variation.

In a rotating superfluid-superconducting neutron star, there will be quantized neutron vortex lines. If the proton superconductivity is type-II in some or all regions of the core, there will also be quantized flux tubes in those regions. These are incorporated into the variational formalism by adding terms coupling the superfluid currents to the vorticity and allowing $\Lambda$ to depend on the vorticity tensors $w_\mu^\pi, \pi \in \{\pi, \bar{\pi}\}$. This method was developed in [16, 17, 37], though we take a somewhat different approach.

We first rewrite the vorticity tensor in terms of a lattice field $X_\mu$

$$w_\mu^\pi = 2 \nabla_{(\mu} A_{\pi)}.$$  \hspace{1cm} (18)

$X_\mu$ will be dynamically identified with the canonical momentum four-vector $\pi_\pi$. $w_\mu^\pi$ can also be expressed in terms of two lattice scalars $X_a, a \in \{1, 2\}$

$$w_\mu^\pi = 2 \nabla_{(\mu} X_{\pi a} \nabla_{\pi a} X_{\pi}.$$  \hspace{1cm} (19)

The gradients of $X_a$ define a plane that is locally orthogonal to the vortex lines. We choose for a current-vorticity coupling

$$\mathcal{L}_{\nu \ coupl.} = - \sum_{\pi} \nabla_{\pi} X_{\pi}^a.$$  \hspace{1cm} (20)

The variation of the action will thus contain the additional vorticity terms

$$\delta (\Lambda_{\nu} + \mathcal{L}_{\nu \ coupl.}) = - \sum_{\pi} \left[ \frac{1}{2} \lambda_{\mu\nu} \delta w_{\mu\pi} + \lambda_{\mu} \delta n_{\mu\pi} + n_{\mu\pi} \delta \lambda_{\mu} \right].$$  \hspace{1cm} (21)
where we have defined the vortex line self-tension \cite{17} tensor
\[
\lambda_{\mu
u} \equiv -2 \frac{\partial \Lambda}{\partial g_{\mu\nu}} \bigg|_{n_\mu^x,F_{\mu\nu}}.
\] (22)

The generalization of Eq. (10) to incorporate electromagnetism and vorticity modifies Eq. (11)–(12) into

\[
\nabla_{\mu} \Lambda = \sum_{\alpha} \mu_{\alpha} \nabla_{\mu} n_{\alpha} - \frac{1}{8\pi} K_{\mu
u} \nabla_{\mu} F_{\nu}\rho - \frac{1}{2} \sum_{\pi} \lambda_{\mu\rho} \nabla_{\mu} w_{\rho\pi},
\] (23)

\[
\frac{\partial \Lambda}{\partial g_{\mu\nu}} = \frac{1}{2} \left( \sum_{\alpha} \mu_{\alpha} n_{\alpha} + \frac{1}{4\pi} K_{\mu\rho} F_{\nu\rho} + \sum_{\pi} \lambda_{\mu\rho} w_{\rho\pi} \right).
\] (24)

This was found using as the variation for rank two tensors
\[
\delta Y_{\mu\nu} = \delta \nabla_{\mu} n_{\nu} + \frac{1}{4\pi} \delta K_{\mu\nu} F_{\rho\sigma} + \sum_{\pi} \delta \lambda_{\mu\rho} w_{\rho\pi}.
\] (25)

The minimal coupling terms between the currents and both electromagnetic field and vorticity are not part of \( \Lambda \) and hence do not contribute to Eq. (23–24).

We incorporate general relativity by including the Einstein–Hilbert term in the action, which corresponds to adding the following term to the Lagrangian
\[
\mathcal{L}_{\text{EH}} = \frac{1}{16\pi} R,
\] (26)

for Ricci scalar \( R \), which adds the expected additional terms to the variation of the action
\[
\delta \mathcal{L}_{\text{EH}} = -\frac{1}{16\pi} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \delta g_{\mu\nu},
\] (27)

where \( R_{\mu\nu} \) is the Ricci tensor. To account for the Jacobian in the action
\[
\delta S = \int d^{4} x \sqrt{-g} \Lambda,
\] (28)

for metric determinant \( g \), we add a term \( -\frac{1}{2} \Lambda g^{\mu\nu} \delta g_{\mu\nu} \) to the variation of \( \mathcal{L} \). We thus end up with
\[
\delta \mathcal{L} = \delta \Lambda + \delta \mathcal{L}_{\text{v coup.}} + \delta \mathcal{L}_{\text{EM coup.}} + \delta \mathcal{L}_{\text{EH}} + \frac{1}{2} \Lambda g^{\mu\nu} \delta g_{\mu\nu},
\] (29)

where \( \Lambda \) includes \( \Lambda_{\text{EM}} \) and \( \Lambda_{\nu} \).

B. Deriving the equations of motion

We review the convective variational procedure of Carter \cite{14}, which is further developed and expounded in later papers \cite{15, 17, 36}. The main result of interest is the variation of the number four-current \( n_{\mu}^{x} \), given by
\[
\delta n_{\mu}^{x} = \xi_{\mu}^{x} \nabla_{\sigma} n_{\mu}^{x} - n_{\mu}^{x} \nabla_{\sigma} \xi_{\mu}^{x} + n_{\mu}^{x} \nabla_{\sigma} \xi_{\mu}^{x} - \frac{1}{2} n_{\mu}^{x} g^{\sigma\pi} \delta g_{\sigma\pi},
\] (30)

where \( \xi_{\mu}^{x} \) is the Lagrangian infinitesimal displacement field specifying the variation of the four-current of species \( x \). This expression differs from the Lie derivative of \( n_{\mu}^{x} \) by the inclusion of the effects of gravitational perturbations. We use the sign convention of Carter and Langlois \cite{17}, which differs by \( \xi_{\mu}^{x} \to -\xi_{\mu}^{x} \) compared to the expected nonrelativistic limit and other references such as Anderson and Comer \cite{36}. This is derived by first starting with a dual to the number current (omitting species labels)
\[
^{*}n_{\mu\nu\sigma} = \varepsilon_{\mu\nu\sigma\rho} \Pi^{\rho}
\] (31)

where \( \varepsilon_{\mu\nu\sigma\rho} \) is the Levi-Civita tensor. This three-form can be specified by the derivatives of three scalars \( N_{1}^{1}, N_{2}^{2}, N_{3}^{3} \), which label the coordinates of a particular fluid element in “matter space” and which are the same for all time. These coordinates can be pushed forward to give the coordinates of the fluid element at any time slice. So \( ^{*}n_{\mu\nu\sigma} \) can be written as
\[
^{*}n_{\mu\nu\sigma} = -f(N_{1}^{1}, N_{2}^{2}, N_{3}^{3})_{123} \nabla_{\mu} N_{1}^{1} \nabla_{\nu} N_{2}^{2} \nabla_{\sigma} N_{3}^{3},
\] (32)

where \( f(N_{1}^{1}, N_{2}^{2}, N_{3}^{3})_{123} \) is antisymmetric in the scalar indices 1, 2, 3. The variations of the scalars can be expressed in terms of an infinitesimal displacement field
\[
\delta N^{a} = \varepsilon^{a}_{\nu} \nabla_{\rho} N^{a},
\] (33)

and so Eq. (30) can be found by taking the variation of Eq. (32) and using Eq. (31) and
\[
\delta \varepsilon_{\mu\nu\sigma\rho} = \frac{1}{2} \varepsilon_{\mu\nu\sigma\rho} g^{\lambda\eta} \delta g_{\lambda\eta}.
\] (34)

Note that the form of \( ^{*}n_{\mu\nu\sigma} \) as given by Eq. (32) is closed
\[
\nabla_{\lambda} (^{*}n_{\mu\nu\sigma}) = 0,
\] (35)

which automatically means that \( n_{\mu}^{x} \) is conserved through Eq. (31). We thus assume separate conservation of each current \( \nabla_{\mu} n_{\mu}^{x} = 0 \) in the rest of this paper for those currents where the variation Eq. (30) is used. Implicit in this is the assumption that the rate of interconversion between particle species is much slower than the dynamical timescales of interest, which is certainly true for weak interactions in cold neutron stars, but not necessarily true for the formation or breaking of Cooper pairs of neutron or protons.

The vorticity tensor \( w^{\mu} \) can be specified in a similar way to the dual number current \( ^{*}n_{\mu\nu\sigma} \), except now only with two lattice scalars \( \chi^{a}_{\mu} \) and \( \chi^{2}_{\mu} \). The variation for these scalars is simply their Lie derivative with respect to \( \xi_{\mu}^{x} \):
\[
\delta \chi^{a}_{\mu} = \xi^{a}_{\mu} \nabla_{\rho} \chi^{a}_{\rho}, \quad a \in \{1, 2\},
\] (36)

where \( \xi^{a}_{\mu} \) are Lagrangian displacement fields describing the spacetime motion of the vortex line/flux tube array associated with the superfluid of species \( \pi \). The vortex line/flux tube arrays do not in general move along with the relevant superfluid species. We have assumed here
that the same infinitesimal displacement field describes the variations of both $\chi^{\mu}_{\nu}$ and $\chi_{\mu}^{\nu}$. Since it is these lattice scalars that are the freely-varying quantities relating to the vorticity [37], we must write the variations of the vorticity tensor and lattice field in terms of $\delta \chi^{\mu}_{\nu}$ and hence $\xi^{\mu}_{\nu}$. Eq. (36) gives the variation of the vorticity tensor to be

$$\delta w_{\mu\nu} = -2\nabla_{[\mu}(w_{\nu]} - \delta \chi^{\mu}_{\nu}).$$

(37)

However, the perturbation of Eq. (18) gives

$$\delta w_{\mu\nu} = 2\nabla_{[\mu} \delta \lambda^{\mu}_{\nu]},$$

(38)

so by comparison to Eq. (37) we have

$$\delta \lambda^{\mu}_{\nu} = -w_{\mu\nu} \xi^{\mu}_{\nu} + \nabla_{\nu} \phi_{\nu}$$

(39)

for a scalar $\phi_{\nu}$. If we postulate the form $\lambda^{\mu}_{\nu} = \lambda^{\mu}_{\nu} \nabla_{\mu} \chi^{\mu}_{\nu}$ based on Eq. (37–38), then we find $\phi_{\nu} = \xi^{\mu}_{\nu} \lambda^{\mu}_{\nu}$. Note that $\lambda^{\mu}_{\nu}$ itself is not determined uniquely, but only up to a physical unimportant gradient of a scalar which we set to zero here.

Combining these Lagrangian variations plus $\delta F_{\mu\nu} = 2\nabla_{[\mu} \delta A_{\nu]}$ and inserting into Eq. (29), we obtain

$$\delta \mathcal{L} = \sum_{x \neq \nu} \pi^{\mu}_{\nu} \delta n_{\mu}^{\nu} + \left( j^{\mu}_{\nu} - \frac{1}{4\pi} \nabla_{\mu} K^{\mu\nu} \right) \delta A_{\mu}$$

$$- \sum_{\nu} \left[ \frac{1}{2} \lambda^{\mu\nu}_{\nu} \delta w_{\mu\nu} + \left( \lambda^{\mu\nu} - \pi^{\mu}_{\nu} \right) \delta n_{\mu\nu}^{\nu} + n_{\mu\nu} \delta \lambda^{\mu}_{\nu} \right]$$

$$+ \frac{1}{2} \sum_{x} n_{\mu\nu}^{\nu} \mu_{\nu} g^{\sigma\nu} + \frac{1}{4\pi} K^{\mu\rho} F_{\mu\rho} + \sum_{\nu} \lambda^{\mu\nu}_{\nu} w_{\mu\nu}^{\nu} - \frac{1}{8\pi} R^{\mu\nu} + \left( \Lambda + \frac{1}{16\pi} R \right) g^{\mu\nu} \right] \delta g_{\mu\nu}$$

(40)

where we have defined the gauge-dependent canonical momentum covectors

$$\pi^{\mu}_{\nu} \equiv \mu^{\mu}_{\nu} + q_{\nu} A_{\mu}$$

(41)

where $q_{\nu} = q_{\nu} \xi = e, q_{\nu} = q_{\nu} = -e, q_{\nu} = q_{\nu} = q_{\nu} = 0$.

For the normal fluids, we use Eq. (30) to constrain the current variations, implying that the first term in Eq. (40) becomes

$$\pi^{\mu\nu}_{\mu\nu} \nabla_{\sigma} n_{\mu\nu}^{\nu} - \pi^{\mu\nu}_{\mu\nu} \nabla_{\sigma} \xi^{\mu}_{\nu} + \pi^{\mu\nu}_{\mu\nu} \nabla_{\sigma} \xi^{\nu}_{\mu} - \frac{1}{2} \pi^{\mu\nu}_{\mu\nu} \nabla_{\sigma} g^{\rho\nu} \delta g_{\sigma\rho}$$

$$= 2\xi^{\mu}_{\nu} n_{\mu\nu}^{\nu} \nabla_{\nu} \pi^{\mu}_{\nu} + \xi^{\mu}_{\nu} \pi^{\mu}_{\nu} \nabla_{\sigma} n_{\mu\nu}^{\nu} - \frac{1}{2} \pi^{\mu\nu}_{\mu\nu} \nabla_{\sigma} g^{\rho\nu} \delta g_{\sigma\rho}$$

(42)

where we integrated by parts and dropped total derivative terms. Use of Eq. (30) is unnecessary for the superfluid components, since the variation of the superfluid number currents is already constrained. However, this means we must enforce conservation of the separate superfluid current densities in a different manner. A simple way to do this is by adding a Schutz-type [33] term for each superfluid to the Lagrangian:

$$\mathcal{L}_{S} = -\sum_{\nu} n_{\mu\nu}^{\nu} \nabla_{\nu} \phi_{\nu},$$

(43)

where $\phi_{\nu}$ is a scalar phase. Taking the variation of this and setting the coefficient of $\delta \phi_{\nu}$ equal to zero gives, after an integration by parts, $\nabla_{\nu} n_{\mu\nu}^{\nu} = 0$. The variation with respect to $n_{\mu\nu}^{\nu}$ adds the additional term

$$-\sum_{\nu} \delta n_{\mu\nu}^{\nu} \nabla_{\nu} \phi_{\nu}$$

(44)

to Eq. (40). Setting the coefficient of $\delta n_{\mu\nu}^{\nu}$ equal to zero now gives

$$\pi^{\mu}_{\nu} = \nabla_{\nu} \phi_{\nu} + \lambda^{\mu}_{\nu}$$

(45)

which correctly gives us as the vorticity tensor the covariant curl of the canonical momentum covector. $\nabla_{\nu} \phi_{\nu}$ will not contribute to the vorticity, and can thus safely be set to zero. Microscopically, $\pi^{\mu}_{\nu}$ is the gradient of a potential for superfluid neutrons and the gradient of a potential plus $A_{\mu}$ for superconducting protons. This equation represents a macroscopic average. Using Eq. (37) and (39), the third and fifth terms in Eq. (40) become

$$-\frac{1}{2} \lambda^{\mu\nu}_{\nu} \delta w_{\mu\nu} + \delta \lambda^{\mu\nu}_{\nu}$$

$$= \lambda^{\mu\nu}_{\nu} \nabla_{[\mu}(w_{\nu]} - \delta \chi^{\mu}_{\nu}) + n_{\mu\nu}^{\nu} \delta w_{\mu\nu} + n_{\mu\nu}^{\nu} \delta \lambda^{\mu}_{\nu}$$

$$= \xi^{\mu}_{\nu} \left[ w_{\mu\nu}^{\nu} + \nabla_{\nu} \lambda^{\mu\nu} \right] + \pi^{\mu}_{\nu} \nabla_{\nu} n_{\mu\nu}^{\nu},$$

(46)

where we integrated by parts, dropping total derivative terms, and used Eq. (45) in the last line.

Returning to Eq. (40), $\delta \mathcal{L} / \delta A_{\mu} = 0$ gives the sourced Maxwell equations in a continuous medium

$$\nabla_{\nu} K^{\mu\nu} = 4\pi j^{\nu}_{\mu},$$

(47)

which also guarantees charge conservation $\nabla_{\nu} j^{\nu}_{\mu} = 0$ due to the asymmetry of $K^{\mu\nu}$. Both $F_{\mu\nu}$ and $w_{\mu\nu}$ are closed

$$\nabla_{\nu} F_{\mu\nu} = 0, \quad \nabla_{\nu} w_{\mu\nu} = 0,$$

(48)

this being the source-free Maxwell equations for the former. The remainder of Eq. (40) becomes, using Eqs. (42–46)

$$\delta \mathcal{L} = \sum_{\nu} \xi^{\mu}_{\nu} \left[ 2n_{\mu\nu} \nabla_{\nu} \pi_{\mu\nu}^{x} + \pi_{\mu\nu}^{x} \nabla_{\nu} n_{\mu\nu}^{x} \right]$$

$$+ \frac{1}{2} \sum_{\nu} \xi^{\mu}_{\nu} \left[ w_{\mu\nu}^{\nu} + \nabla_{\nu} \lambda^{\mu\nu}_{\nu} \right] + \pi_{\mu\nu}^{x} \nabla_{\nu} n_{\mu\nu}^{x}$$

$$+ \frac{1}{8\pi} \left[ \sum_{\nu} n_{\mu\nu}^{\nu} \mu_{\nu}^{\nu} g^{\rho\sigma} + \frac{1}{4\pi} K^{\mu\rho} F_{\mu\rho} + \sum_{\nu} \lambda^{\mu\nu}_{\nu} w_{\mu\nu}^{\nu}$$

$$+ \Psi^{\mu\nu} \right] \delta g_{\mu\nu}$$

$$= \sum_{\nu} \xi^{\mu}_{\nu} f^{\mu}_{\nu} + \frac{1}{2} \left( T^{\mu\nu} - \frac{1}{8\pi} \left( R^{\mu\nu} - \frac{1}{2} R_{\mu\nu} \right) \right) \delta g_{\mu\nu},$$

(49)
where \( f^\mu_\nu \) is the generalized force (density) acting on fluid \( x \), the generalized pressure \( \Psi \) is defined as

\[
\Psi = \Lambda - \sum_x \mu^x_\mu n^x_\nu,
\]

and the stress-energy tensor is

\[
T^{\mu\nu} = \sum_x n^x_\mu u^x_\nu + \frac{1}{4\pi} \kappa^{\mu\nu} F^\rho_\rho + \sum_x \Lambda^{\mu\nu} w^x_\nu + \Psi g^{\mu\nu}.
\]

\( T^{\mu\nu} \), of course, satisfies the Einstein field equations

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}.
\]

This stress-energy tensor does not appear to be explicitly symmetric. We will discuss in Section V why it is symmetric regardless by a comparison between this “macroscopic” stress-energy tensor and an average “mesoscopic” stress-energy tensor which accounts for small-scale motion around vortex lines and flux tubes. Using the forms of the four currents and conjugate four-momenta introduced in Section III, \( T^{\mu\nu} \) is expanded in Appendix A in a more explicitly symmetric form which is compared to the stress-energy tensor for a single perfect fluid.

The identification of the generalized forces in Eq. (49) gives the equations of motion for the normal fluids and vortex line/flux tube arrays associated to each superfluid.

\[
\begin{align*}
\dot{f}^x_\mu &= 2n^x_\mu \nabla_\nu [\pi^x_\nu] + \pi^x_\mu \nabla_\nu n^x_\nu, \\
\dot{f}^x_\nu &= u^x_\mu (n^x_\mu + \nabla_\mu \lambda^x_\mu) + \pi^x_\mu \nabla_\mu \pi^x_\nu.
\end{align*}
\]

Note that, because the variational procedure assumes conserved currents (or imposes it via Lagrange multipliers for the superfluids), that the final terms on the right-hand side of both of these equations is zero. We will, however, allow for non-conservation of the entropy current \( x = s \), which allows us to discuss entropy generation in Section IV. In Section IV B, we relate the equation of motion of the vortex line/flux tube arrays to those of the associated superfluid. If there are no forces between the constituents other than those mediated by the electromagnetic field, then the \( f^x_\mu \) must each individually equal zero.

Finally, summing Eqs. (53–54) for each fluid and using Eq. (23,48,50,51), we can show that the stress-energy tensor is conserved up to external forces acting on the fluids:

\[
\nabla_\nu T^{\mu\nu} = \sum_x f^x_\mu.
\]

The right-hand side of this equation should equal zero if energy and momentum are conserved in this system, so in that case the \( f^x_\mu \) must sum to zero. This can be accomplished if they are all zero individually, or if they cancel each other, which corresponds to forces which act between the fluid constituents. We can also add forces to the right-hand side here as long as they act on multiple fluid constituents and hence mutually cancel. This will allow us to insert forces that we are unable to derive from a variational principle.

### III. Relation to Physical Parameters

Our discussion so far has focused on a somewhat abstract variational principle and the resulting equations of motion and stress-energy tensor. To proceed, we need to relate the variables in the previous section to physical quantities. First, we introduce the four-velocities of the fluids. Because of short collisional coupling times [38–40], it is expected that all four normal fluid components \( x = n, p, e, m \) will comove and have common four-velocity \( u^\mu \), normalized in the standard manner \( u^\mu u_\mu = -1 \). Their currents are defined as

\[
n^x_\mu = n_x u^\mu, \quad x \in \{n, p, e, m\},
\]

where \( n_x \) is the number density of species \( x \) defined in the normal fluid rest frame.

The superfluids do not have to comove with the normal fluid, and we specify their four-currents by

\[
n^x_\mu = n^{\bullet}_x u^\mu = n^{\bullet}_x \gamma(v^x_\mu) (u^\mu + v^x_\mu), \quad x \in \{\pi, \rho\},
\]

where \( n^{\bullet}_x \) is a spacelike relative four-velocity between the normal fluid and the superfluid \( x \). \( n^{\bullet}_x \) is the number density of species \( x \) in its own rest frame, equal to twice the density of Cooper pairs. We will use a subscript \( \gamma^{(x)} \) to indicate a quantity measured in the normal fluid rest frame, so the superfluid density in this frame is

\[
n^{\gamma^{(x)}} = n^{\bullet}_x \gamma(v^x_\mu) (n_x u^\mu + v^x_\mu),
\]

where \( v^x_\mu = v^{\bullet}_x u^\mu \) and \( v^{\bullet}_x u^\mu = 0 \). The \( v^{\bullet}_x \) are defined in this way so that they are normalized in the same way as the normal fluid four-velocity. Strong electrostatic coupling between the normal fluid leptons and the superconducting protons means that the latter will also likely move collisionlessly with the normal fluid, but for now we permit the superconducting protons to move independently of the normal fluid.

Like the superfluids, the entropy current can move independently of the normal fluids, and is specified by

\[
s^\mu = s \gamma(w^2) (u^\mu + w^\mu) = s^* (u^\mu + w^\mu),
\]

where \( w^\mu w_\mu = 0 \), \( \gamma(w^2) = (1 - w^2)^{-1/2} \) and \( w^2 = w^\mu w_\mu \). We suggestively relabel \( w^\mu \) as

\[
q^\mu = s^* T^* w^\mu,
\]

since this is a heat flux four-vector. Here \( s^* \) and \( T^* \) are the entropy density and temperature measured in the normal fluid rest frame, while \( s \) is the entropy density in the comoving frame.

Using Eqs. (56,57,59,60) in Eqs. (7–8), the conjugate
momentum covectors can be rewritten as

\begin{align}
\mu_n &= \mu_n u_\mu + A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu} + \frac{A_{\eta\mu}}{T_s} n_\eta, \quad (61a) \\
\mu_p &= \mu_p u_\mu + A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu} + \frac{A_{\eta\mu}}{T_s} n_\eta, \quad (61b) \\
\mu_\tau &= \mu_\tau u_\mu + B^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu}, \quad (61c) \\
\mu_p^\tau &= \mu_p^\tau u_\mu + B^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu}, \quad (61d) \\
\mu_m &= \mu_m u_\mu + \frac{A_{\eta\mu}}{T_s} n_\eta, \quad (61e) \\
\mu_m^* &= \mu_m^* u_\mu + \frac{A_{\eta\mu}}{T_s} n_\eta, \quad (61f) \\
\Theta_\mu &= T_\mu^* u_\mu + \frac{B_{\eta\mu}}{T_s} n_\eta, \quad (61g)
\end{align}

where we have defined the following chemical potentials/temperature measured in the normal fluid rest frame

\begin{align}
\mu_n &\equiv B^{\eta\tau} n_{\eta\mu} + A^{\mu\tau} n_{\mu\tau} + A^{\nu\tau} n_{\nu\tau} + A^{\eta\tau} s^\tau, \\
\mu_p &\equiv B^{\eta\mu} n_{\eta\tau} + A^{\nu\mu} n_{\nu\tau} + A^{\nu\tau} s^\tau, \\
\mu_\tau &\equiv B^{\eta\mu} n_{\eta\tau} + A^{\nu\tau} n_{\nu\tau} + A^{\nu\tau} s^\tau, \\
\mu_p^\tau &\equiv B^{\eta\mu} n_{\eta\tau} + A^{\nu\tau} n_{\nu\tau} + A^{\nu\tau} s^\tau, \\
\mu_c &\equiv B^{\eta\mu} n_{\eta\tau} + A^{\nu\tau} s^\tau, \\
\mu_m &\equiv B^{\eta\mu} n_{\eta\tau} + A^{\nu\tau} s^\tau, \\
T^* &\equiv B^{\mu\tau} s^\tau + A^{\eta\tau} n_{\eta\tau} + A^{\nu\tau} n_{\nu\tau} + A^{\nu\tau} s^\tau + A^{\nu\tau} n_{\nu\tau}.
\end{align}

The superfluid chemical potentials and temperature in the rest frame of the superfluids and entropy are given by

\begin{align}
\mu_\tau &= -u^{\tau\mu} \mu_\mu = \gamma (\nu^2) (A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu} + B^{\eta\tau} n_\mu) \\
&\quad + \gamma (\nu^2) (1 - u^{\nu\mu} u^{\nu\mu} A^{\eta\tau} n_\eta), \\
&\quad = (\nu^2) [\mu_\mu - \frac{2}{\nu^2} B^{\eta\tau} n_\eta - \nu^{\mu\nu} A^{\eta\tau} s^\tau], \quad (63a) \\
\mu_p^\tau &= -u^{\tau\mu} \mu_\mu = \gamma (\nu^2) (A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu} + B^{\eta\tau} n_\mu) \\
&\quad + \gamma (\nu^2) (1 - u^{\nu\mu} u^{\nu\mu} A^{\eta\tau} n_\eta), \\
&\quad = (\nu^2) [\mu_\mu - \frac{2}{\nu^2} B^{\eta\tau} n_\eta - \nu^{\mu\nu} A^{\eta\tau} s^\tau]. \\
T &= -\gamma (\nu^2) (u^{\tau\mu} + u^{\nu\mu}) \Theta_\mu \\
&= B^{\mu\tau} s^\tau + \gamma (\nu^2) (A^{\mu\tau} n_{\mu\tau} + A^{\nu\mu} n_{\nu\tau} + A^{\nu\tau} s^\tau + A^{\nu\tau} n_{\nu\tau}) \\
&= \gamma (\nu^2) [T^* + \nu^2 B^{\mu\tau} s^\tau]. \quad (63c)
\end{align}

If all of the normal fluids come over, the same Lagrangian displacement field \(\xi_\mu = \xi_\mu^0 = \xi_\mu^m = \xi_\mu^s\) must be used to describe their variations, and a single generalized force \(f_\mu^s\) acts on this combined normal fluid. This force is

\begin{align}
f_\mu^s &= 2 u^{\sigma\tau} \nabla_\sigma \Pi_\mu + \Pi_\mu \nabla_\tau u^\sigma + u^{\sigma\tau} \mu_\mu \nabla_\sigma n_{\eta\mu} \\
&\quad + u^{\sigma\tau} \pi_\sigma \nabla_\mu n_{\eta\tau} + u^{\sigma\tau} \pi_\sigma \nabla_\mu (n_{\eta\tau} + n_{\nu\tau} + n_{\nu\tau} + n_{\nu\tau} n_{\nu\tau}), \quad (64)
\end{align}

where \(\Pi_\mu = s^T u_\mu + n_{\eta\tau} \mu_\mu + n_{\nu\tau} \pi_\sigma + n_{\nu\tau} + n_{\nu\tau} n_{\nu\tau}\) is the effective momentum for the normal fluid. In the absence of heat conduction, the entropy will move with the same four-velocity \(u^\mu\) as the normal fluids since the superfluids carry no entropy. In that case, \(s = s^\tau, T = T^\tau, \, \xi_\mu = \xi_\mu^\tau\) and there is an entropy contribution to \(f_\mu^s\) [17].

The coefficients \(B^{\eta\tau}, A^{\eta\tau}\) need to be calculated using microphysics. Previously, relativistic entrainment coefficients have been computed using Landau Fermi liquid theory [41, 42], though these references employ a different formulation of the hydrodynamics and their relativistic entrainment coefficients thus differ from the \(B^{\eta\tau}, A^{\eta\tau}\) used here. We invert our definitions of the conjugate four-momenta and determine how these previously calculated entrainment coefficients could be used in the more symmetric hydrodynamics of this paper.

First, we assume \(\eta^\mu = 0\), which is implied in Gusakov et al. [41, 42]. Inverting \(\nu_\mu^0\) and \(\nu_\mu^m\) to obtain equations for the superfluid number currents and then adding to these the equations for the normal fluid current of each species gives

\begin{align}
(n_\mu^0)_{\text{total}} &= \frac{\mathbf{B}_{\eta\mu}}{\det(A)} \mu_\mu^0 - \frac{A^{\eta\tau}}{\det(A)} \mu_{\eta\mu} + n_\eta u^\mu \\
&\quad - \frac{B^{\eta\tau}(A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu})}{\det(A)} \mu_\mu^0 \\
&\quad + \frac{A^{\eta\tau}(A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu})}{\det(A)} \mu^0, \quad (65)
\end{align}

\begin{align}
(n_\mu^m)_{\text{total}} &= \frac{\mathbf{B}_{\eta\mu}}{\det(A)} \mu_\mu^m - \frac{A^{\eta\tau}}{\det(A)} \mu_{\eta\mu} + n_\eta u^\mu \\
&\quad - \frac{B^{\eta\tau}(A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu})}{\det(A)} \mu_\mu^m \\
&\quad + \frac{A^{\eta\tau}(A^{\eta\tau} n_{\eta\mu} + A^{\nu\tau} n_{\nu\mu})}{\det(A)} \mu^m, \quad (66)
\end{align}

where we have explicitly shown dependence on the \(B^{\eta\tau}, A^{\eta\tau}\), and where

\begin{align}
A &\equiv \begin{pmatrix} \mathbf{B}_{\eta\mu} & A^{\eta\tau} \end{pmatrix}. \quad (67)
\end{align}

Gusakov et al. [41, 42] use as the total (normal plus superfluid) baryon number currents

\begin{align}
(n_\mu^0)_{\text{total}} &= [(n_\mu^0)_{\text{total}} - \mu_\eta Y_{\eta\mu} - \mu_\nu Y_{\nu\mu}] u^\mu \\
&\quad + Y_{\eta\mu} Q_{\eta\mu} + Y_{\nu\mu} Q_{\nu\mu}, \\
(n_\mu^m)_{\text{total}} &= [(n_\mu^m)_{\text{total}} - \mu_\eta Y_{\eta\mu} - \mu_\nu Y_{\nu\mu}] u^\mu \\
&\quad + Y_{\eta\mu} Q_{\eta\mu} + Y_{\nu\mu} Q_{\nu\mu}, \quad (68, 69)
\end{align}

where the (symmetric, relativistic) entrainment matrix is

\begin{align}
\mathbf{Y} &= \begin{pmatrix} Y_{\eta\eta} & Y_{\eta\nu} \\
& Y_{\nu\eta} & Y_{\nu\nu} \end{pmatrix}. \quad (70)
\end{align}

and the number densities and chemical potentials are measured in the rest frame of the normal fluid. The \(Q_{\eta\mu}\) of the references are written in terms of superfluid
where recall that \( \mu_{\pi} \) is measured in the rest frame of the superfluid of species \( \pi \). Comparing Eqs. (65)–(66) and (68)–(69), it is obvious that

\[
Y_{nn} = \frac{E_{\pi}}{\det(A)}, \quad Y_{pp} = \frac{E_{\pi}}{\det(A)}, \quad Y_{np} = Y_{pn} = -A_{\pi} V_{\pi}, \tag{72}
\]

which can be inverted to give

\[
E_{\pi} = \frac{Y_{pp}}{\det(Y)}, \quad B_{\pi} = \frac{Y_{np}}{\det(Y)}, \quad A_{\pi} = -\frac{Y_{np}}{\det(Y)}. \tag{73}
\]

The total baryon currents in our notation are thus

\[
(n^\mu)_{\text{total}} = Y_{nn}\mu^\mu_{\pi} + Y_{np}\mu^\mu_{\pi} + [n_n - Y_{nn}(A^\pi n_n + A^\pi n_p)] u^\mu - Y_{np}(A^\pi n_n + A^\pi n_p)] u^\mu, \tag{74}
\]

\[
(n^\mu)_{\pi\text{total}} = Y_{pp}\mu^\mu_{\pi} + Y_{np}\mu^\mu_{\pi} + [n_p - Y_{pp}(A^\pi n_n + A^\pi n_p)] u^\mu - Y_{np}(A^\pi n_n + A^\pi n_p)] u^\mu. \tag{75}
\]

Using Eqs. (63a) and (63b), we can rewrite Eqs. (74) and (75) as

\[
(n^\mu)_{\text{total}} = [n_n + n_\pi - Y_{nn}\mu^\mu_{\pi} - Y_{np}\mu^\mu_{\pi}] u^\mu + Y_{nn}\mu^\mu_{\pi} + Y_{np}\mu^\mu_{\pi}, \tag{76}
\]

\[
(n^\mu)_{\pi\text{total}} = [n_p + n_\pi - Y_{pp}\mu^\mu_{\pi} - Y_{np}\mu^\mu_{\pi}] u^\mu + Y_{pp}\mu^\mu_{\pi} + Y_{np}\mu^\mu_{\pi}. \tag{77}
\]

which are nearly identical to Eq. (68) and (69) except for including additional relativistic corrections due to the relative motion between the normal and superfluid components of each baryon species. The difference in the species labels between normal and superfluid baryons on the chemical potentials is not a concern since \( \mu_x = \mu_{\pi} \) should be true in chemical equilibrium i.e. in equilibrium, the protons and neutrons should have no preference between the paired (superfluid) and unpaired (normal fluid) phases.

While we have so far been as general as possible with regards to the coefficients \( A^{xy}, \) simple physical arguments allow us to reduce their number. Since the \( A^{xy} \) parameterize the entrainment between the fluids, which is unaffected by the superfluid or normal fluid phase, we expect that \( A^{\pi\pi} = A^{\pi p} = A^{\pi n} = 0. \) There is no reason why the normal and superfluid species of each fluid should be entrained either, so \( A^{x\pi} = 0 = A^{x p}. \) We do not expect that \( B^\pi = B^p, \) though, unless \( A^{x x} s^x = 0, \) since this would prevent \( \mu_x = \mu_{\pi} \) in equilibrium.

### IV. DISSIPATION

#### A. Heat conduction

We begin our discussion of dissipation by determining the allowed form of the heat flux \( q^\mu \) introduced in Eq. (60). Its form is found by enforcing the positive definiteness of the entropy generation \( \Gamma_s = \nabla^\mu s^\mu \) using a standard procedure in relativistic dissipative hydrodynamics (see e.g. [15, 22, 34, 35, 43–45]). Like Olson and Hiscock [34], Priou [46] and Lopez-Monsalvo and Andersson [35], we are careful to note that the “regular” Carter formulation of relativistic finite temperature fluid dynamics, corresponding to setting the parameters \( A^{xx} = 0, \) is acasual, which is why we have included entropy entrainment.

The most general way to obtain the form of the heat flux is to start from the equation of motion for the entropy current

\[
2s^\sigma \nabla [\pi \Theta_{\pi}] + \Theta_{\pi} \nabla s^\sigma = f^s_{\mu}. \tag{78}
\]

Contraction with \( u^\mu \) and rearranging gives

\[
T^* \nabla s^\sigma = -\frac{\sigma^2}{T^*} [\nabla s^\sigma + T^* u_{\pi} + \frac{2B^s}{T^*} u^\mu \nabla [\pi q_{\sigma}]
+ \left( \frac{B^s}{T^*} - \frac{B^* T^*}{T_{\pi}} \right) q_{\pi} u_{\pi}] - u^\mu f_{\mu}. \tag{79}
\]

where \( \dot{a} = u^\mu \nabla \Sigma a. \) The easiest way to enforce that the entropy generation from heat conduction is positive definite is to make

\[
q^\mu = -\kappa \perp u_{\mu} \nabla s^\sigma + \frac{2B^s}{T^*} u^\mu \nabla [\pi q_{\sigma}]
+ \left( \frac{B^s}{T^*} - \frac{B^* T^*}{T_{\pi}} \right) q_{\pi} u_{\pi}, \tag{80}
\]

which matches Lopez-Monsalvo and Andersson [35] and gives the same entropy generation term due to heat conduction as Weinberg [43] up to the additional terms which are higher-order in \( q^\mu. \) These terms are necessary for causal heat conduction, since rearranging Eq. (80) following [35] gives a relativistic version of the Cattaneo–Vernotte equation

\[
t_h (\dot{q}^\mu + q^\nu \nabla_\nu u^\mu) + q^\mu = -\kappa \perp u_{\mu} (\nabla_s T^* + T^* u_{\mu}), \tag{81}
\]

where \( t_h \) is a heat conduction timescale and \( \kappa \) is a modified heat conductivity, which are given by

\[
t_h = \frac{B^s/ T^*}{1 + \frac{\kappa B^s/ T^*}{T_{\pi}}} \approx \frac{B^s}{T^*}, \tag{82}
\]

\[
\kappa = \frac{\kappa}{1 + \frac{B^s}{T^*}} \approx \kappa, \tag{83}
\]
where the approximate forms are valid if we drop higher-order terms in an expansion in the mean free collision time. The entropy entrainment parameters which appear in the definition of $T^\ast$ thus clearly affect $t_k$. Causal heat conductivity is absent from the treatment of dissipation in previous papers on relativistic multifluid neutron stars [22, 24], which use the treatment of dissipation in Weinberg [43].

The remaining term on the right-hand side of Eq. (79) is due to the generalized force on the entropy current $f_\mu^s$. Using conservation of energy-momentum, we can rewrite $f_\mu^s$ in terms of the generalized forces on the other fluids. The viscous contributions to entropy generation will be included in this manner by modifying the stress-energy tensor and hence the generalized forces. We next discuss the inclusion of mutual friction and vortex pinning forces which act between the fluids and vortex line/flux tube arrays, and then incorporate viscosity.

### B. Mutual friction and vortex pinning

Mutual friction is a dissipative drag force acting on vortex lines/flux tubes, and hence on their associated superfluids, due to scattering off of the normal fluid. Vortex pinning is an attractive force between the fluids and vortex line/flux tube array only differs from that for the superfluid by an irrelevant overall minus sign.

The vortex lines/flux tubes would move along with their associated superfluid if not for their scattering off of the normal fluid (mutual friction) or due to pinning to the vortex lines/flux tubes associated with the other superfuid (vortex pinning). We consider the mutual friction first, and represent it in Eq. (55) and Eq. (53–54) through equal but opposite contributions to $f_\mu^\nu$ and $f_\mu^\nu$, the generalized force on the combined normal fluid. To lowest order, this force should depend only on the relative velocity between the normal fluid and the vortex lines/flux tubes of species $q_\mu^\nu$, which we define analogously to Andersson et al. [48]

\[
u^\mu = \gamma(q^2) (u^\mu_L + q^\mu), \quad \gamma(q^2) = (1 - q^2)^{-1/2}, \tag{87}\]

where $u^\mu_L = 0, q^\mu = q_\mu^\nu q^\nu$. So we modify the generalized force on superfluid $\pi$ and the combined normal fluid by setting

\[
l_{\mu} = f_{\mu} = \mathcal{R}_{\mu\nu} \gamma_{\nu}, \quad f_{\mu} = f_{\mu}^{\mu\nu} = -\sum_{\pi} \mathcal{R}_{\mu\nu} \gamma_{\nu}, \tag{88}\]

where the $\mathcal{R}_{\mu\nu}$ projects out components of $q_\mu^\nu$ either along the direction tangent to the corresponding vortex line/flux tube array or along the respective vortex line/flux tube array velocity

\[
l_{\mu} = \mathcal{R}_{\mu\nu} \gamma_{\nu} \left( u_{\mu}^L - \bar{t}_{\mu} \right) \tag{89}\]

where $\bar{t}_{\mu}$ is the average spacelike tangent vector to the vortex lines/flux tube array. $\mathcal{R}_{\mu\nu}$ are dissipative coefficients parameterizing the mutual friction. Since these additional forces cancel out in the right-hand side of Eq. (55), the total stress-energy tensor is still conserved. In this case the equation of motion for a superfluid becomes

\[
\mathcal{R}_{\mu\nu} \gamma_{\nu} = n_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} + \mathcal{R}_{\mu\nu} \gamma_{\nu}. \tag{90}\]

We would like to remove references to the vortex line velocity and $q_\mu^\nu$ from Eq. (90) and rewrite it in the form of Eq. (86). Equating the two forms $u^\mu_L$ using Eq. (84,87) gives

\[
q_\mu^\nu = -\frac{1}{\gamma} \gamma_{\mu} \gamma_{\nu} - \left( \frac{1}{\gamma} - \gamma_{\nu} \right) u^\mu_L. \tag{91}\]
where \( \gamma_{\mu \nu} \equiv \gamma(\beta_{\mu \nu}^2) \), \( \gamma_{\mu} \equiv \gamma(g_{\mu}^2) \), \( \gamma_{\overline{\mu}} \equiv \gamma(v_{\overline{\mu}}^2) \). To perform the necessary manipulations, it will be convenient to rewrite the vorticity tensor in terms of the corresponding “electric” and “magnetic” four-fields in the frame co-moving with the vortex lines,

\[
W_{\mu}^{E, \gamma} = u_{L, \mu}^\nu W_{\mu}^{\gamma} = 0, \quad W_{\mu}^{\mu} = \frac{1}{2} \varepsilon_{\mu \nu \rho \lambda} u_{\nu}^\rho W_{\mu}^{\lambda} \equiv W_{\mu}^{\gamma},
\]

in terms of which we can write \( \mathbf{v}^\gamma \) as

\[
\mathbf{v}^\gamma = W_{\mu}^{\gamma} \mathbf{\tau}^\mu / \sqrt{W_{\alpha}^{\gamma} W_{\mu}^{\gamma}}.
\]

We can of course invert \( W_{\nu}^{\gamma} \) to find

\[
u_{\mu}^{\gamma} = -\varepsilon_{\mu \nu \rho} u_{\rho}^{\gamma} W_{\rho}^{\mu}.
\]

Using Eqs. (90–91,94), we solve for \( v_{\mu}^{\gamma} \),

\[
\mathbf{v}^\mu = \frac{1}{\eta^\mu} \varepsilon_{\mu \nu \rho} \mathbf{v}^{\nu} W_{\rho}^{\nu} - \frac{\gamma^\mu}{\mathbf{R}_{\mu \nu}^{\gamma \nu} \mathbf{w}^{\nu} \mathbf{v}^{\mu}} - \frac{1}{\tau^\mu \mathbf{v}^{\mu} - (\mathbf{v}^{\nu} \mathbf{v}^{\mu} \mathbf{v}^{\nu})^T},
\]

where for simplicity we have defined,

\[
\varepsilon_{\mu \nu \rho} \equiv \varepsilon_{\sigma \mu \rho \nu} \mathbf{w}^{\sigma}, \quad \eta^\mu \equiv \mathbf{R}_{\mu \nu}^{\gamma \nu} \mathbf{w}^{\nu} \mathbf{v}^{\mu}, \quad \tau^\mu \equiv \mathbf{\tilde{v}}^\mu \mathbf{v}^{\mu}.
\]

Contracting Eq. (95) with \( \mathbf{v}_{\alpha}^{\mu} W_{\alpha}^{\gamma} \) and then using the same equation to replace \( \varepsilon_{\mu \nu \rho} \mathbf{w}^{\rho} W_{\nu}^{\mu} \) gives

\[
\varepsilon_{\mu \nu \rho} W_{\nu}^{\gamma} W_{\rho}^{\delta} = \frac{1}{\mathbf{R}_{\mu \nu}^{\gamma \nu} \mathbf{w}^{\nu} \mathbf{v}^{\mu}} - \frac{\gamma^\mu}{\mathbf{R}_{\mu \nu}^{\gamma \nu} \mathbf{w}^{\nu} \mathbf{v}^{\mu}} - \frac{1}{\tau^\mu \mathbf{v}^{\mu} - (\mathbf{v}^{\nu} \mathbf{v}^{\mu} \mathbf{v}^{\nu})^T}.
\]

The two arrays to move together for small relative velocities. The pinning force \( f_{\mu}^{\text{pin}} \) acting on the neutron vortex lines due to the proton vortex lines is incorporated into the force balance equations on the two arrays as

\[
f_{\mu}^{\gamma} = f_{\mu}^{\text{pin}}, \quad f_{\mu}^{\gamma} = f_{\mu}^{\text{pin}}. \]

Since \( f_{\mu}^{\gamma} \) are force densities, the force per unit length on a vortex line/flux tube equals \( f_{\mu}^{\gamma} / \mathbf{N}_{\gamma} \), where \( \mathbf{N}_{\gamma} \) is the areal number density of vortex lines/flux tubes of species \( \gamma \) measured perpendicular to them (we give a relativistic definition of \( \mathbf{N}_{\gamma} \) in Section V A). It is reasonable to expect that the vortex pinning force should be proportional to the product of \( \mathbf{N}_{\gamma} \) and \( \mathbf{N}_{\overline{\gamma}} \), so the vortex pinning force per unit length acting on a proton flux tube will be proportional to \( \mathbf{N}_{\overline{\gamma}} \sim 2\Omega / \kappa_{\gamma} \sim 10^4 (\Omega / 10 \text{ s}^{-1}) \text{ cm}^{-2} \), where \( \Omega \) is the angular rotational frequency of the neutron star and \( \kappa_{\gamma} \) is the circulation quantum. This is much smaller than the number density of proton flux.
tubes $N_\tau \sim B/\Phi_\tau \sim 5 \times 10^{18} (B/10^{12} \text{ G}) \text{ cm}^{-2}$ where $B$ is the magnetic field strength and $\Phi_\tau$ is the flux quantum. $\kappa_\tau$ and $\Phi_\tau$ are also defined in Section V A. For this reason, the vortex pinning force acting on a single proton flux tube is negligible and often ignored. However, as we are interested in force densities, we will retain the pinning force acting on the proton flux tubes.

To lowest order, the vortex pinning force depends only on the (average) relative velocity between the two vortex line arrays contracted into an as yet undetermined rank two tensor:

$$f_\mu^{\text{pin}} \equiv R_\mu^{\text{pin}} b^\nu,$$  

where $b^\nu$ is the (spacelike, average) relative velocity of the proton vortex lines in the (average) neutron vortex line rest frame defined such that

$$u_\mu = \gamma(b^2)(u_\mu^L + b^\mu), \quad \gamma(b^2) = (1 - b^2)^{-1/2},$$

where $b^\mu u_\mu^L = 0$ and $b^2 = b^\mu b_\mu$. A reasonable nonrelativistic version of vortex pinning drag force would point in the direction defined by the cross product of the tangent vectors to both arrays, and only the component of the relative velocity between the two arrays that is in this direction will contribute to a drag force. One possible relativistic generalization of this is

$$R_\mu^{\text{pin}} = -R_\mu^{\text{pin}} \varepsilon_{\alpha\sigma\rho(\mu} \rho_\nu\beta_\eta\lambda_\nu u_\eta^L p^\sigma p^{\rho_\nu},$$

The coefficient $R_\mu^{\text{pin}}$ should be a function of $b = \sqrt{b^\mu b_\mu}$, the relative orientation between the vortex line/flux tube arrays or $\hat{\mu} \hat{\nu} / \hat{\nu} / \hat{\nu}$, and should scale linearly with both $N_\tau$ and $\mathcal{N}_\tau$ as discussed previously. The dependence on $b$ should be $b^{-3/2}$ [50, 51], as this will give the correct behaviour for the pinning force: at large $b^\mu$, the vortex pinning drag becomes insignificant compared to the mutual friction drag, while for small $b^\mu$, the vortex lines become pinned to the flux tubes [52]. The principal dissipation mechanism in the drag regime of vortex pinning is the excitation of kelvons, and in calculations like those in [50, 51], the interactions exciting the kelvons were with individual nuclei. However, in the core the pinning interaction is of course between lines of macroscopic extent, so a modification of $R_\mu^{\text{pin}}$ may be required when the finite length of the lines are considered [53–55].

It should be noted that the pinning drag force would be relevant only to a precessing neutron star with sufficiently large precession amplitude. Even in that case, the drag force estimated by Link [56] is large enough for pinning to happen on rather short timescales of days to weeks. Simple relative motion with energy stored in the Baym–Chandler kinetic energy [57] would damp away almost instantly.

The Magnus force acting on superfluid $\mathcal{T}$ can thus be written as

$$f_\mu^{\text{Magn}} = n_\mu n_\nu f_\mu^{\text{Magn}} = w_\mu^{\text{Magn}} \nabla_\nu \nabla_\mu \mathcal{T} + R_\mu^{\text{Magn}} \mathcal{T} \mathcal{T} + R_\mu^{\text{pin}} b^\nu.$$  

with $\pm$ corresponding to $\mathcal{T} = \mathcal{T}/\mathcal{T}$. It should be possible in principle to rewrite this equation in terms of only the vorticity tensor or vector, the normal fluid velocity and the superfluid relative velocities $\nu_\mu$ in a manner similar to what was done in Eq. (95–99). We do not attempt this calculation here because of the unessential complication it would add to this paper.

### C. Bulk and shear viscosity

To incorporate viscosity into this variational formalism, we follow Carter [15], the review of his work in Andersson and Comer [36] and the nonrelativistic generalization by Andersson and Comer [58], though we specify to the fluids expected in a superfluid–superconducting neutron star core. We also neglect chemical reactions that convert between fluid species as we have implicitly assumed current conservation for the separate species. Introducing the (assumed symmetric) viscosity tensor $\tau_{\mu}^{\nu}$, where the label $\Sigma$ is used to specify the origin of the viscosity. The variation of the master function to include viscosity takes the form (summing over $\Sigma$)

$$\delta \Lambda_{\text{vis}} = \frac{1}{2} \kappa_{\Sigma}^{\mu} \delta \tau_{\mu}^{\nu}, \quad \kappa_{\Sigma}^{\mu} \equiv 2 \frac{\partial \Lambda}{\partial \tau_{\mu}^{\nu}},$$

where $\kappa_{\Sigma}^{\mu}$ is a strain tensor. The new form of Eqs. (12) and (23), giving the new form of $\partial \Lambda / \partial g_{\mu\nu}$, is

$$\nabla_\nu \Lambda = \sum_x \mu_\nu \nabla_\mu n_x^\nu - \frac{1}{8 \pi} \kappa_{\Sigma}^{\mu} \nabla_\mu F_\mu - \frac{1}{2} \lambda_{\Sigma}^{\mu} \nabla_\mu w_{\nu}^{\mu}$$

$$+ \frac{1}{2} \kappa_{\Sigma}^{\mu} \nabla_\mu \tau_{\nu}^{\rho},$$

$$\frac{\partial \Lambda}{\partial g_{\mu\nu}} = \frac{1}{2} \left( \sum_x \mu_\nu \nabla_\mu n_x^\nu + \frac{1}{4 \pi} \kappa_{\Sigma}^{\mu} F_\mu + \lambda_{\Sigma}^{\mu} w_{\nu}^{\mu}$$

$$+ \kappa_{\Sigma}^{\mu\nu} \tau_{\mu}^{\nu} \right),$$

where we used

$$\delta \tau_{\Sigma}^{\mu\nu} = L_{\xi} \tau_{\Sigma}^{\mu\nu} = \xi^{\rho} \nabla_\rho \tau_{\Sigma}^{\mu\nu} - 2 \tau_{\Sigma}^{\rho\nu} \nabla_\rho \xi^{\nu}.$$  

The full variation of $\tau_{\Sigma}^{\mu\nu}$ is, from Carter [15]

$$\delta \tau_{\Sigma}^{\mu\nu} = \xi_{\Sigma}^{\rho} \nabla_\rho \tau_{\Sigma}^{\mu\nu} - 2 \tau_{\Sigma}^{\rho\nu} \nabla_\rho \xi^{\nu} - \frac{1}{2} \tau_{\Sigma}^{\rho\nu} g^{\mu\nu} \delta g_{\rho\sigma},$$

so $\delta \mathcal{L}$ becomes

$$\delta \mathcal{L} = \xi^{\mu} f_\mu^\tau + \xi^{\mu} f_\mu^\nu + \sum_\tau \xi^{\mu}_{\tau} f_\mu^\tau + \sum_\Sigma \xi^{\mu}_{\Sigma} f_\mu^\Sigma$$

$$+ \frac{1}{2} \left( T_{\mu}^{\nu\nu} - \frac{1}{8 \pi} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) \right) \delta g_{\mu\nu},$$

(111)
where $T^\mu\nu$, $\Psi$ and $f_\Sigma^\mu$ are given by

$$
T^\mu\nu = \sum_x n_x^\mu n_x^\nu + \frac{1}{4\pi} \kappa_{\mu\nu} g_{\mu\nu} + \sum_\Sigma \lambda_{\mu\nu} u_\Sigma^\mu u_\Sigma^\nu \\
+ \sum_\Sigma \kappa_\Sigma \tau_\Sigma^{\mu\nu} + \Psi g_{\mu\nu},
$$
(112)

$$
\Psi = \Lambda - \sum_x n_x^\mu n_x^\nu - \frac{1}{2} \sum_\Sigma \tau_\Sigma^{\mu\nu} \Sigma_\mu,\Sigma_\nu,
$$
(113)

$$
f_\Sigma^\mu = \kappa_\Sigma^{\mu\nu} \nabla_\nu \tau_\Sigma^{\mu\nu} + \tau_\Sigma^{\mu\nu} \left( \nabla_\nu \Sigma_\nu - \frac{1}{2} \Sigma_\mu \Sigma_\nu \right),
$$
(114)

$\xi^\mu$ is the common displacement field for the normal fluid and $f^\mu = f_1^\mu + f_2^\mu + f_3^\mu + f_4^\mu$.

We now look at the $u^\mu f^\mu$ term in Eq. (79). Conservation of energy-momentum implies

$$
f_\mu^a = -f_\mu^m - \sum_\Sigma f_\Sigma^\mu - \sum_\Sigma f_\Sigma^\mu,
$$
(115)

so contracting with $w^\mu$ and then using

$$
\begin{align*}
 u^\mu f_\mu^a &= \sum_\Sigma \gamma_{\Sigma}^2 \mathcal{R}^\mu_{\mu\nu} \tau_\Sigma^{\mu\nu} w^\mu, \\
 u_\mu^\nu f_\nu^a &= \sum_\Sigma \gamma_{\Sigma}^2 \mathcal{R}^\mu_{\mu\nu} \nabla_\nu \lambda_\Sigma^{\mu\nu}, \\
 -u_\mu^\nu f_\nu^m &= -u_\nu f_\mu^m - \frac{1}{\eta_\Sigma} f_\mu^m \nabla_\nu \lambda_\Sigma^{\mu\nu},
\end{align*}
$$
(116)

Eq. (79) becomes

$$
T^\ast \nabla_\sigma s^\ast = \frac{1}{\kappa T^*} \sum_{\mu\nu} q_\mu^\ast q_\nu^\ast + \sum_\Sigma \left[ \gamma_{\Sigma}^2 \mathcal{R}^{\mu\nu}_\Sigma \tau_\Sigma^{\mu\nu} w^\mu + \frac{1}{\eta_\Sigma} f_\mu^m \nabla_\nu \lambda_\Sigma^{\mu\nu} - u_\nu f_\mu^m \right] + \sum_\Sigma u^\mu f_\Sigma^\mu,
$$
(118)

The second law of thermodynamics requires that $\nabla_\sigma s^\ast \geq 0$, which is most easily satisfied if each term on the right hand side of Eq. (118) is individually greater than or equal to zero.

Define four-vectors

$$
u_{v}^\mu = \gamma(v_{v}^\mu)(u^\mu + v_{v}^\mu), \quad \gamma(v_{v}^\mu) = (1 - v_{v}^2)^{-1/2},
$$
(119)

where $u^\mu v^\mu = 0$ and $v_{v}^\mu = v_{v}^\mu u^\mu$, such that

$$
\tau_\Sigma^{\mu\nu} u_\nu^\Sigma = 0, \quad \kappa_\Sigma^{\mu\nu} u_\Sigma^\nu = 0.
$$
(120)

That is, the viscosity tensor and the strain tensor are both purely spacelike in the frame moving with $u_{v}^\mu$, and the viscous and strain tensors have been constrained to have only six independent components. The entropy generation equation can be rewritten as

$$
T^\ast \nabla_\sigma s^\ast = \frac{1}{\kappa T^*} \sum_{\mu\nu} q_\mu^\ast q_\nu^\ast + \sum_\Sigma \left[ \gamma_{\Sigma}^2 \mathcal{R}^{\mu\nu}_\Sigma \tau_\Sigma^{\mu\nu} w^\mu + \frac{1}{\eta_\Sigma} f_\mu^m \nabla_\nu \lambda_\Sigma^{\mu\nu} - u_\nu f_\mu^m \right] + \sum_{\Sigma} v_{v}^\mu f_\mu^m + \frac{1}{2\gamma(v_{v}^\mu)} \tau_\Sigma^{\mu\nu} L_{\Sigma} \kappa_\Sigma^{\mu\nu}.
$$
(121)

where $v_{v}^\mu = w^\mu$ and

$$
\tau_\Sigma^{\mu\nu} L_{\Sigma} \kappa_\Sigma^{\mu\nu} = -2\Sigma^{\mu\nu} f_\Sigma^{\mu\nu},
$$
(122)

Analogously to Carter [15], introduce linear combinations of the $u_{v}^\mu$ and $v_{v}^\mu = w^\mu$ such that

$$
\sum_{\alpha = \Sigma, a} \zeta_{\alpha}^a v_{v}^\mu = u_{v}^\mu, \quad \sum_{\alpha} \zeta_{\alpha}^a = 1,
$$
(123)

so the terms depending on the forces can be combined using

$$
\tilde{f}_{\mu}^a = f_{\mu}^a + \sum_{\Sigma} \zeta_{\alpha}^a f_{\Sigma} = -\sum_{b=\Sigma, a} \mathcal{R}_{\mu\nu}^{ab} v_{v}^b,
$$
(124)

where $\mathcal{R}_{\mu\nu}^{ab}$ is a positive-definite symmetric resistivity tensor. This tensor must be symmetric by the Onsager reciprocal relations. This procedure assumes that there are no other dynamical velocities in the problem than $u_{v}^\mu$, $v_{v}^\mu$ and $w^\mu$. There will also be a contribution to the viscosity from the normal fluid

$$
\sum_{\alpha = \Sigma, a} \zeta_{\alpha}^a v_{v}^\mu = 0,
$$
(125)

corresponding to $u_{v}^\mu = w^\mu$. To make the viscosity term look more like a standard entropy generation equation, we use

$$
\kappa_{\mu\nu}^{\Sigma} = \kappa_{\mu\nu} = g_{\mu\nu} + u_{\mu}^{\Sigma} u_{\nu}^{\Sigma},
$$
(126a)

$$
f_{\mu}^{\Sigma} = \nabla_{\mu} t_{\Sigma}^{\Sigma},
$$
(126b)

$$
\tau_{\mu\nu}^{\Sigma} = -\eta_{\mu\nu}^{\rho\sigma} L_{\Sigma} \kappa_{\rho\sigma}^{\Sigma},
$$
(126c)

$$
\eta_{\Sigma}^{\mu\nu\rho\sigma} = \eta_{\Sigma} \perp_{\Sigma}^{\rho} \perp_{\Sigma}^{\sigma} + \left( \frac{\zeta_{\Sigma}}{2} - \frac{\zeta_{\Sigma}}{3} \right) \perp_{\Sigma}^{\mu} \perp_{\Sigma}^{\nu},
$$
(126d)

where $\eta_{\Sigma}$ and $\zeta_{\Sigma}$ are (dynamic) shear and bulk viscosity coefficients, respectively. This form ensures that the entropy generation is positive definite. We do not include the higher-order corrections discussed in Carter [15] and hence assume that the we only have viscosity linear in the fluid velocities. The viscous tensor can also be rewritten as

$$
\begin{align*}
\tau_{\mu\nu}^{\Sigma} &= -2\eta_{\Sigma} \left[ \nabla_{\mu} t_{\Sigma}^{\Sigma} + u_{\mu}^{\Sigma} u_{\nu}^{\Sigma} - \frac{1}{3} \perp_{\Sigma}^{\mu} \nabla_{\nu} u_{\Sigma}^{\Sigma} \right] \\
&\quad - \zeta_{\Sigma} \perp_{\Sigma}^{\mu} \nabla_{\nu} u_{\Sigma}^{\Sigma} \\
&\quad = -\eta_{\Sigma} \perp_{\Sigma}^{\mu} \nabla_{\nu} u_{\Sigma}^{\Sigma} - \Sigma_{\nu}^{\beta} W_{\Sigma}^{\mu} - \zeta_{\Sigma} \perp_{\Sigma}^{\mu} \nabla_{\nu} u_{\Sigma}^{\Sigma},
\end{align*}
$$
(127)

where $u_{v}^\mu = \nabla_{\mu} u_{\Sigma}^{\Sigma}$ and $W_{\nu}^{\mu} = \nabla_{\mu} u_{\Sigma}^{\Sigma} + \nabla_{\nu} u_{\Sigma}^{\Sigma} - 2/3 g_{\mu\nu} \nabla_{\rho} u_{\Sigma}^{\Sigma}$ is the shear tensor. Eq. (121) becomes
In the case of the shear and bulk viscosity of the normal fluid, \( v_0^2 = u^\mu \), \( \gamma(v_0^2) = 1 \). We also expect a bulk viscosity term from the superfluids [19, 20, 22]. The most general form of the viscosity contribution to the entropy generation (the second line on the right-hand side of Eq. (128)) should thus be of the form

\[
[T^v \nabla s]_{\text{visc}} = \frac{\eta_r}{2} \left( \nabla_\mu u_\nu + \nabla_\nu u_\mu \right) \left( \nabla_\sigma u^\mu + \nabla_\sigma u^\mu \right) - \frac{4}{3} \left( \nabla_\sigma u^\sigma \right)^2 + \eta_r \dot{u}_\mu \dot{u}^\mu + \zeta_r \left( \nabla_\sigma u^\sigma \right)^2
+ \frac{1}{\gamma(v_0^2)} \zeta(\nabla_\sigma u_\sigma)^2 + \frac{1}{\gamma(v_0^2)} \zeta_r(\nabla_\sigma u^\sigma)^2 + \frac{1}{\gamma(v_0^2)} \zeta_{rr}(\nabla_\sigma \left[ \gamma(v_0^2)(u^\sigma + v_r^2/2 + v_r^2/2) \right])^2, \tag{129}
\]

where the subscript \( r \) is used to specify the viscosity coefficients for the normal fluid. We included all reasonable combinations of \( v_0^\mu \) and \( v_r^\mu \) in \( v_0^2 \) i.e. \( v_0^2 \in \{ v_0^\mu, v_r^\mu, v_r^\mu = 1/2(v_0^\mu + v_r^\mu) \} \). The choice of a \( v_0^2 \) which includes contributions from both relative superfluid velocities is required to make our formulation agree with previous versions of superfluid hydrodynamics viscosity as discussed below.

Comparing Eq. (129) to the relativistic version of the Landau–Khalatnikov superfluid viscosity [19, 20], our formulation has only four distinct bulk viscosity coefficients compared to six in Guskov [22]. However, the last four terms of Eq. (129) can be expanded into six terms which are all proportional to the product of the divergences of two of \( u_\mu, v_0^\mu \) or \( v_r^\mu \). We can then identify the coefficients of these six terms, themselves sums of \( \zeta_r, \zeta_r, \zeta_r \) and \( \zeta_r \), as six bulk viscosity coefficients, analogously but not identical to those of Guskov [22] due to the different representation of the fluid degrees of freedom. In a realistic neutron star core, with the superconducting protons conoving with the normal fluid due to electrostatic attraction, \( v_r^\mu = 0 \) and the coefficients \( \zeta_r \) and \( \zeta_r \) do not contribute to entropy generation, leaving only relevant two bulk viscosity coefficients \( \zeta_r \) and \( \zeta_r \).

The different viscosity coefficients are, in principle, possible to calculate from microphysics. The shear viscosity will have contributions from lepton–lepton, lepton–proton [39] and proton-mediated lepton–neutron scattering [40]. The bulk viscosity in both the normal fluid and neutron superfluid is due to modified and direct Urca processes [22, 59, 60]. Superfluidity generally increases the shear viscosity of the normal fluid and lowers the bulk viscosity.
pare the resulting electromagnetism to previous studies of superconducting neutron star cores with flux tubes and magnetized vortex lines. Finally, we discuss how to compute the magnetic field in a superconducting neutron star core given an electric current density, and the form of the Lorentz force in the total equation of motion for the charged fluids.

A. Mesoscopic stress-energy tensor, averaging procedure and effective theory

We postulate the following Lorentz-invariant splitting of the (macroscopic) master function \( \Lambda \) as a function of the contractions of \( F_{\mu\nu} \) and \( u_{\mu\nu}^\pi \):

\[
\Lambda = \Lambda_0 + \Lambda_{\text{EM}+V}(X_F, X_\pi, X_\tau, Y_\tau, Y_\pi, Z) \tag{130}
\]

where the scalars of which \( \Lambda_{\text{EM}+V} \) is a function are defined as

\[
X_F = \frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \quad X_\pi = \frac{1}{4} u_{\mu\nu}^\pi u_{\mu\nu}^\pi,
\]

\[
X_\tau = \frac{1}{4} u_{\mu\nu}^\tau u_{\mu\nu}^\tau, \quad Y_\tau = \frac{1}{2} F_{\mu\nu} u_{\mu\nu}^\tau,
\]

\[
Y_\pi = \frac{1}{2} F_{\mu\nu} u_{\mu\nu}^\pi, \quad Z = \frac{1}{2} u_{\mu\nu}^\pi u_{\mu\nu}^\pi.
\]

\( \Lambda_0 \) is the contribution to the master function from the four-currents alone, while \( \Lambda_{\text{EM}+V} \) contains all contributions from flux tubes/vortex lines and electromagnetic fields. \( \Lambda_{\text{EM}+V} \) will also contain functional dependence on contractions of the superfluid/superconducting four-currents, since the flux tube/vortex line energies will depend on density numbers through dependence on the London length \( \Lambda_s \) and coherence lengths \( \xi \), but we have assumed that there are no terms involving contractions between the number currents and the tensors \( F_{\mu\nu} \) and \( u_{\mu\nu}^\pi \). According to Eq. (17,22) \( K_{\mu\nu} \) and \( \Lambda_{\mu\nu}^\pi \) will then take the forms

\[
K_{\mu\nu} \equiv -4\pi \left( \frac{\partial \Lambda_{\text{EM}+V}}{\partial X_F} F_{\mu\nu} + \frac{\partial \Lambda_{\text{EM}+V}}{\partial Y_\tau} u_{\mu\nu}^\tau \right)
\]

\[
\Lambda_{\mu\nu}^\pi \equiv -\frac{\partial \Lambda_{\text{EM}+V}}{\partial X_\pi} u_{\mu\nu}^\pi - \frac{\partial \Lambda_{\text{EM}+V}}{\partial Y_\pi} u_{\mu\nu}^\pi - \frac{\partial \Lambda_{\text{EM}+V}}{\partial Y_\tau} F_{\mu\nu}, \tag{131}
\]

\[
\Lambda_{\mu\nu}^\tau \equiv -\frac{\partial \Lambda_{\text{EM}+V}}{\partial X_\tau} u_{\mu\nu}^\tau - \frac{\partial \Lambda_{\text{EM}+V}}{\partial Z} u_{\mu\nu}^\tau - \frac{\partial \Lambda_{\text{EM}+V}}{\partial Y_\tau} F_{\mu\nu}. \tag{132}
\]

The goal of the mesoscopic averaging procedure is to determine what \( \Lambda_{\text{EM}+V} \) and its partial derivatives are.

We define the mesoscopic scale \( \ell \) such that there are many vortex lines and flux tubes within an area \( \ell^2 \). \( \ell \) obeys the following hierarchy of length scales:

\[
\ell \gg \ell \gg d_n \gg d_p > \Lambda_s > \xi_p, \xi_n. \tag{134}
\]
to replace the sum over vortex lines/flux tubes with multiplication by the relevant areal number density $N_{\tau}$. This averaged mesoscopic stress-energy tensor is then compared to the macroscopic stress-energy tensor to determine the macroscopic effective master function $\Lambda_{EM+V}$ and its partial derivatives.

We relegateing most of the details of the calculation to Appendix B, but discuss the averaging procedure for the electromagnetic field and vorticity here. The canonical four momenta for the superfluid neutrons and superconducting protons, and hence the vorticity tensors, are quantized

$$\oint \tau^{\mu}_\nu dx^\mu = \int w_{\mu}^\tau dS^\mu = hN_{\tau},$$  \hspace{1cm} (138)

where $N_{\tau} \in \mathbb{Z}$ and the generalized Stokes’ theorem [62] was used. Recall that in Section IV B we defined a vorticity vector

$$W^\alpha_{\tau} \equiv \frac{1}{2} \epsilon^{\beta\gamma\delta\mu} u^\beta_{\tau} \phi^\delta_{\tau} F_{\mu\gamma},$$  \hspace{1cm} (139)

where $u^\beta_{\tau}$ is the average four-velocity of the vortex lines/flux tubes of species $\tau$. Since we are ignoring mutual friction and vortex pinning in the averaging calculation, to lowest order the vortex lines/flux tubes co-move with their corresponding fluids i.e. $u^\beta_{\tau} = w^\beta_{\mu}$ and $u^\mu_{\tau} = w^\mu_{\mu}$. This assumption is equivalent to assuming that the vortex lines are straight and uniformly distributed, since we are ignoring the vortex line self-tension force in Eq. (54). Since we will find a general expression for the vortex vector that does not necessarily correspond to zero vortex line self-tension force, this force can be considered a first-order correction to the equations of motion. Over length scales $\ell$ much larger than the separation of the vortex lines/flux tubes, the quantization condition allows us to write $W^\alpha_{\tau}$ as

$$W^\alpha_{\tau} = N_{\tau} \kappa_{\tau}^\alpha \phi^\beta_{\tau} F_{\mu\gamma} = N_{\tau} \Phi_{\tau} e^\alpha_{\tau} = \frac{1}{\kappa_{\tau}} e^\alpha_{\tau},$$  \hspace{1cm} (140)

where $\kappa_{\tau} = h/(2\mu^\tau)$ is the relativistic generalization of the quantum of circulation with a factor of 2 because the superfluid neutrons/superconducting protons will form Cooper pairs, $\phi_{\tau}^\beta = h/(2e)$ is the flux quantum associated with a proton flux tube, $\phi_{\tau}^\beta$ is the average spatial tangent vector to the vortex lines/flux tubes defined in Eq. (93), and $N_{\tau}$ is the areal number density of vortex lines/flux tubes in the spatial plane perpendicular to $\phi_{\tau}$. $N_{\tau}$ is Lorentz-invariant and defined by

$$N_{\tau} \equiv \frac{1}{\kappa_{\tau}} \sqrt{W_{\mu}^\tau W_{\mu}^\tau} = \frac{1}{\kappa_{\tau}} \left( \frac{1}{2} \epsilon^{\beta\gamma\delta\mu} w^\beta_{\tau} w^\gamma_{\mu} \right).$$  \hspace{1cm} (141)

We expect contributions to $\Lambda_{EM+V}$ that will be proportional to the $N_{\tau}$ times an energy per unit length. The electromagnetic field contributions due to flux tubes/magnetized vortex lines should also be linearly proportional to $N_{\tau}$. The separations between proton flux tubes/neutron vortex lines $d_p/d_n$ are defined by $N_{\tau}/N_{\pi}$ through

$$N_{\pi} = \frac{2}{\sqrt{3d_z}},$$  \hspace{1cm} (142)

assuming equilateral triangular lattices.

Because of entrainment, the neutron vortex lines will become magnetized. This is made apparent by combining the vorticity tensors $w_{\mu}^\tau$ and $w_{\mu}^\pi$ in a way to eliminate the superfluid neutron current, which itself does not source a magnetic field. In our formulation, this corresponds to eliminating $u_{\mu}^\pi$. We thus add the two tensors in such a way as to give

$$F_{\mu\nu} = F_{\mu\nu}^L + \frac{1}{e} w_{\mu\nu}^\pi + \frac{Y_{np}}{eY_{pp}} u_{\mu\nu}^\tau,$$  \hspace{1cm} (143)

where $F_{\mu\nu}^L$ is the London electromagnetic field tensor. If we can ignore derivatives of $\mu_x$, $n_x$ and the coefficients $B^\alpha$, $A^xy$, it takes the form

$$F_{\mu\nu}^L \approx -\frac{2}{e} \left( \mu_{\tau}^\gamma + \frac{Y_{np}}{Y_{pp}} \mu_{\tau}^\delta \phi_{\tau}^\delta \right) \partial_{[\mu} u_{\nu]} - \frac{2n_{\tau}^\gamma}{eY_{pp}} \partial_{[\mu} u_{\nu]}^\tau,$$  \hspace{1cm} (144)

where $u_{\mu}^\tau$ has canceled out as expected. In the general case where the gradients of $\mu_x$, $n_x$, $B^\alpha$, $A^xy$ cannot be ignored, this will not be true.

Based on Eq. (143), we split the mesoscopic electromagnetic field tensor $F_{\mu\nu}$ into

$$F_{\mu\nu} = F_{\mu\nu}^L + F_{\mu\nu}^\tau + F_{\mu\nu}^\pi,$$  \hspace{1cm} (145)

with the right-hand side terms corresponding to the London field, proton flux tube field and magnetized neutron vortex line field, respectively. $F_{\mu\nu}^L$ is a large-scale quantity and is the same when averaged i.e. $\langle F_{\mu\nu}^L \rangle = F_{\mu\nu}^L$. The average of the second and third terms on the right-hand side of Eq. (145) can be identified with the second and third terms on the right-hand side of Eq. (143) i.e. $\langle F_{\mu\nu}^\tau \rangle = \langle F_{\mu\nu}^\pi \rangle = \langle F_{\mu\nu}^\tau \rangle = \langle F_{\mu\nu}^\pi \rangle = 0$ when the entrainment is zero ($Y_{np} = 0$) as required. Finally, since $\langle F_{\mu\nu}^\pi \rangle = \langle w_{\mu\nu}^\pi \rangle$, we enforce

$$u_{\mu}^{L,\tau} F_{\mu\nu}^\pi = 0,$$  \hspace{1cm} (146)

that is, there is no electric field due to the flux tube/magnetized vortex lines in their respective rest frame.

After performing the averaging procedure on the mesoscopic stress-energy tensor, we obtain the following aver-
The resulting mesoscopic stress-energy tensor is given by

\[
\langle T^{\mu\nu} \rangle = \left( \sum_{x=e,m,s} n_x \nabla_x + \sum_{x=n,p} B^x n_x^2 + 2 A^T n_{n,p} \right) u^\mu u^\nu + \sum_{x} B^x n_x^2 u_x^\mu u_x^\nu + 2 A^T n_{n,p} u_{n,p} (u_x^\mu u_x^\nu) + 2 A^T \left( n_{n,p} u_{n,p} + n_{h} u_{h} \right)
\]

\[+ \left( \tilde{\lambda}_0 + \sum_{x} n_x \nabla_x \right) g^{\mu\nu} + \frac{1}{4\pi} \left( F_{L\mu} F_{L\nu} - \frac{1}{4} F_{L\rho} F_{L\sigma} g^{\mu\nu} \right) + \sum_{x} \frac{\Phi_x}{2\pi e} \left( u_x^{(\mu} u_x^{\nu)} - \frac{1}{4} u_x^{\rho} u_x^{\lambda} g^{\rho\lambda} \right) + \sum_{x} \frac{\pi e \pi_x}{N_x} \left( u_x^{(\mu} u_x^{\nu)} - \frac{1}{2} u_x^{\rho} u_x^{\lambda} g^{\rho\lambda} \right) + \sum_{x} \frac{1}{32\pi^3 N_x \xi_x^2} u_x^{\mu\nu}, \tag{147}
\]

where \(\tilde{\lambda}_0\) only includes dependence on the large-scale current densities, and we have defined the energy per unit length per flux tube/vortex line

\[
E_{\nu,\pi} \equiv \frac{\Phi_x}{16\pi^2 \Lambda_x^2} \ln \left( \frac{1.12 \Lambda_x}{\xi_x} \right) \tag{148}
\]

\[
E_{\nu,\pi} = \frac{\pi h^2}{8B^x} \ln \left( \frac{0.0712}{N_x \xi_x^2} \right) + \frac{\Phi_x}{16\pi^2 \Lambda_x^2} \ln \left( \frac{1.12 \Lambda_x}{\xi_x} \right). \tag{149}
\]

We ignore condensation energy in the \(E_{\nu,\pi}\), which is much smaller than the other contributions. Since \(u_x^{\mu\nu} \propto N_x\), the final two terms in Eq. (147) are proportional to the areal density of vortex lines/flux tubes as expected.

Eq. (147) is then matched to Eq. (51). The averaged mesoscopic-macroscopic stress energy tensor matching procedure is described in full detail in Appendix B.3. The resulting \(\Lambda_{EM+V}\) is

\[
\Lambda_{EM+V} = -\frac{F_{L\mu} F_{L\nu}}{16\pi} - \sum_{x} \left( N_x E_{\nu,\pi} + \frac{\Phi_x}{8\pi e \Phi_x} F_{L\mu} F_{L\nu} u_x^{\mu\nu} \right), \tag{150}
\]

or in terms of the scalars \(X_F, X_\pi, Y_F, Y_\pi, Z\) and using

\[
F_{L\mu} F_{L\nu} = 4X_F + \frac{4Y_F}{e^2} X_\pi + \frac{4Y^2_{\pi}}{e^4 Y_{\pi}^2} X_\pi - \frac{4Y_{\pi}}{e^2} Y_{\pi} + 4Y_{\pi} Y_{\pi} Z, \tag{151}
\]

plus Eq. (141,143), we can write

\[
\Lambda_{EM+V} = -\frac{1}{4\pi} X_F + \frac{1}{4\pi e^2} X_\pi + \frac{Y_{\pi}}{4\pi e^2 Y_{\pi}} X_\pi + \frac{\sqrt{2} X_F}{\Phi_\pi} E_{\nu,\pi} + \sum_{x} \frac{\sqrt{2} X_F}{\Phi_\pi} E_{\nu,\pi}. \tag{152}
\]

In the matching procedure, we dropped the final term in Eq. (147). This is certainly legitimate in the strong type-II limit where the kinetic energy associated with flux tubes \(\approx \varepsilon_{\nu,\pi}\) is much larger than the flux tube/vortex line magnetic field energy per unit length \(\Phi_\pi^2/(32\pi^2 \Lambda_x^2)\). If this term is not removed, there would be an inconsistency between (a) the \(\Lambda_{EM+V}\) found by comparing the terms proportional to the metric in Eq. (147) to those in Eq. (51), and (b) the partial derivatives of \(\Lambda_{EM+V}\) found by comparing the rest of the terms in Eq. (147) and Eq. (51). For consistency we also must ignore the derivative of \(\varepsilon_{\nu,\pi}\) with respect to \(X_\pi \propto N_x \Phi_\pi\), which is justified since

\[
|N_x \frac{\partial \varepsilon_{\nu,\pi}}{\partial \frac{\Phi_\pi}{\Phi_\pi^2}}| = \frac{\pi h^2}{8 B^x} \ll \varepsilon_{\nu,\pi} \sim \frac{\pi h^2}{4 B^x} \ln \left( \frac{d_n}{\xi_n} \right). \tag{153}
\]

since \(d_n \gg \xi_n\). That some terms in either the averaged mesoscopic stress-energy tensor or the partial derivatives of \(\Lambda_{EM+V}\) must be ignored to obtain a consistent \(\Lambda_{EM+V}\) is not unexpected, as there was no guarantee that an exact macroscopic effective action could be found to reproduce the averaged mesoscopic action and stress-energy tensor. That this procedure works so well suggests that we could simply use the averaging method as a motivation for an effective theory, for which we would use Eq. (150) as the macroscopic master function, and then use this to derive the macroscopic stress-energy tensor.

We note that Eq. (150) differs from the expression for the vortex line-flux tube-electromagnetic energy density obtained in [29, 30] by its inclusion of terms coupling the London field and electromagnetic fields associated with the flux tubes/vortex lines. Such terms were eliminated in the references by the “rotation energy cancellation lemma”. We do not remove these terms for two reasons. First, the rotation energy cancellation lemma requires integrating by parts and dropping a boundary term that we are sure actually vanishes. This boundary term is identically zero in the zero entrainment or fully pinned \(u_x^{\mu\nu} = u_{L\mu}^{\nu} = u_{\nu}^{\mu}\) limits as long as Eq. (146) is true, but is not necessarily zero otherwise. Secondly, while the rotation energy cancellation lemma can be applied to those terms in the mesoscopic stress-energy tensor in which the indices on \(F_{L\mu\nu}\) and the flux tube/vortex line electromagnetic field tensor \(F_{\nu\mu}\) are fully contracted, it does not work on terms like \(F_{L\mu\nu} F_{L\mu\nu}\). As the energy density is frame dependent, the rotation energy cancellation lemma is not guaranteed to work in all frames, in particular in the case of multiple fluids with different velocities.

As the final result of this section, we calculate \(K^{\mu\nu}\) and the \(\lambda^{\mu\nu}\). Using Eq. (152) and the argument in Eq. (153),
Eq. (131–133) gives
\[ K_{\mu\nu} = F_{\mu\nu}, \]
(154)
\[ \lambda_{\mu\nu} = \frac{E_{\mu\nu}}{N_p} - \frac{1}{4\pi e^2} \left( \frac{u_{\mu\nu}}{\lambda_{\mu\nu}} + \frac{Y_{n\mu}}{Y_{pp}} \lambda_{\mu\nu} \right), \]
(155)
\[ \lambda_{\mu\nu} = \frac{E_{\mu\nu}}{N_p} - \frac{Y_{np}}{4\pi e^2} \left( \frac{u_{\mu\nu}}{\lambda_{\mu\nu}} + \frac{Y_{n\mu}}{Y_{pp}} \lambda_{\mu\nu} \right). \]
(156)

B. Magnetic H-field and Maxwell equations in a neutron star

There is disagreement in the literature about what the electromagnetic displacement field tensor \( \mathcal{H}^{\mu\nu} \), or equivalently the magnetic H-field (and electric displacement field \( D \) if we were concerned about electric fields) is inside a superconducting neutron star core. One of the early studies of neutron star MHD by Mendell [63] found \( \mathbf{H} = \mathbf{B} \). This result was rejected by later studies, the first of which appears to be Carter and Langlois [17], who argued that \( \mathbf{H} = \mathbf{B}_L \). This result has been the standard since then [24, 28, 29]. However, \( \mathbf{H} = \mathbf{B}_L \) disagrees with the accepted value for a type–II superconductor in the condensed matter literature: in the nonrotating case it standard electronic superconductivity result is [64] \( \mathbf{H} = H^c_\perp \) where \( H^c_\perp \) is the first critical field for proton superconductivity. We clarify this disagreement below, and further discuss its implications for the Maxwell equations inside a neutron star.

According to [65], the thermodynamic definition of the magnetic H-field is
\[ \mathbf{H} = 4\pi \frac{\partial u}{\partial \mathbf{B}} \bigg|_{n_s, n_t}, \]
(157)
for internal energy density \( u \), average magnetic field \( B \), entropy density \( s \), and number density \( n_t \). In our formulation, the analog to the internal energy density is the master function \( \Lambda \), and the analog to the entropy and number densities are the currents \( n^e_s \), including the entropy current \( s^e \). This means that the electromagnetic displacement tensor \( \mathcal{H}^{\mu\nu} \), whose components in the fluid rest frame are the electric displacement field \( D \) and magnetic H-field, is not equal to the electromagnetic auxiliary tensor defined in Eq. (17), but is instead defined via the variation
\[ \mathcal{H}^{\mu\nu} = -8\pi \frac{\partial \Lambda}{\partial F_{\mu\nu}} \bigg|_{n^e_s} = -8\pi \left[ \frac{\partial \Lambda}{\partial F_{\mu\nu}} \bigg|_{n^e_s, w_{\mu\nu}} + \sum \frac{\partial \Lambda}{\partial w_{\mu\nu}} \bigg|_{n^e_s, F_{\mu\nu}} \frac{\partial w_{\mu\nu}}{\partial F_{\mu\nu}} \bigg|_{n^e_s} \right] = \mathcal{K}^{\mu\nu} + 4\pi e \lambda^{\mu\nu} . \]
(158)
\( \mathcal{H}^{\mu\nu} \) can then be related to \( F_{\mu\nu} \) by defining a magnetization-polarization tensor \( \mathcal{M}_{\mu
u} \) and writing
\[ F_{\mu\nu} = \mathcal{H}^{\mu\nu} + 4\pi \mathcal{M}^{\mu\nu}. \]
(159)
Thus the magnetization-polarization tensor is \( \mathcal{M}^{\mu\nu} = e \lambda^{\mu\nu} \). We believe the subtle distinction between \( K^{\mu\nu} \) (which has often been called \( \mathcal{H}^{\mu\nu} \) and \( \mathcal{H}^{\mu\nu} \) as defined in Eq. (158), to be the source of disagreement between neutron star MHD and condensed matter superconductivity literature regarding the magnetic H-field in a type–II proton superconducting neutron star. Based on Eq. (154–155), \( \mathcal{H}^{\mu\nu} \) is
\[ \mathcal{H}^{\mu\nu} = H^c_\perp \frac{\partial u_{\mu\nu}}{\partial F_{\mu\nu}} + F_{\mu\nu}. \]
(160)
where the first critical field for proton superconductivity is
\[ H^c_\perp = \frac{4\pi e E_{\mu\nu}}{\Phi_\perp}, \]
(161)
and \( \frac{\partial u_{\mu\nu}}{\partial F_{\mu\nu}} \equiv u_{\mu\nu} / (\mathcal{N}_p \Phi_\perp) \). Eq. (160) agrees with the standard condensed matter result which ignores rotation and hence does not include a London field.

The distinction between \( K^{\mu\nu} \) and \( \mathcal{H}^{\mu\nu} \) as we define them here has implications on the interpretation of the Maxwell equations. The variation of the Lagrangian with respect to \( A_\mu \) in Section II B gives Eq. (47), which by Eq. (154) gives as the sourced Maxwell equations
\[ \nabla_\nu F^{\mu\nu} = 4\pi j^e_\nu. \]
(162)
This has the same form as the microscopic sourced Maxwell equations, and appears to suggest that \( \mathbf{H} = \mathbf{B} \) is correct, though \( K^{\mu\nu} \neq \mathcal{H}^{\mu\nu} \) as already discussed. For straight vortex lines/flux tube arrays of uniform density, \( \nabla_\nu u_{\mu\nu} = 0 \), so ignoring corrections due to curvature of the lines and variations in \( \mathcal{N}_p \) Eq. (162) becomes
\[ \nabla_\nu F^{\mu\nu}_L \approx 4\pi j^e_\nu . \]
(163)
This appears (technically incorrectly) to suggest that \( \mathbf{H} = \mathbf{B}_L \) and agrees with the Maxwell equations in [17, 28]. Using Eq. (144) and working in the zero temperature approximation such that \( n^e_p = Y_{pp} u^e_p + Y_{np} \mu^e_p \) [41], Eq. (163) then implies
\[ n^e_m + n^e_p - n^e_p \approx n^e_p + \Lambda^2 \partial_x (\partial^\mu n^e_p - \partial^\mu n^e_p) \approx n^e_p (1 + L^{-2} \Lambda^2), \]
(164)
where \( L \) is the hydrodynamic length scale \( \sim 10^5 \) cm. Since \( \Lambda \sim 10^{-12} \) cm, the right-hand side of Eq. (163) is very close to \( n^{e}_p \). So to a very good approximation, \( j^e_\nu = 0 \) inside a neutron star core, a conclusion drawn by Jones [66] and which is a consequence of the proton superconductivity. Whatever approximation is made on the left-hand side of Eq. (162), the equation tells us how to compute the gradient of \( F^{\mu\nu} \) given an electric current density.
We conclude our discussion on electromagnetism in the presence of vortex lines and flux tubes by finding the Lorentz force acting on the charged fluids. Using $\mathcal{T}_{\mu\nu} = (\mathcal{H}_{\mu\nu} - F_{\mu\nu})/(4\pi)$ and Eq. (53–54), the combined force acting on the charged fluids $x \in \{p, \bar{p}, e, m\}$ is

$$
\sum_{x=p,\bar{p},e,m} f_x^\mu = \sum_{x=p,\bar{p},e,m} 2n_x^\mu \nabla_{[\mu} H_{\rho]}^x + 2\nabla_{[\mu \mathcal{H}_{\rho]}^x} \nabla_{[\rho} \lambda^\rho_{\mu]} + \frac{1}{4\pi} F_{\mu\nu} \nabla_\rho \mathcal{H}^{\rho\nu}.
$$

(165)

This can be clearly separated into three parts: the sum of the relativistic Euler equation for each fluid, the flux tube self-tension force where the electromagnetic field contribution is subtracted from the vorticity tensor, and the Lorentz force. The last of these has the standard relativistic form in a magnetizable medium and reduces to $(\nabla \times \mathbf{H}) \times \mathbf{B}/4\pi$ nonrelativistically.

VI. CONCLUSION

This article has extended the elegant convective variational principle first developed by Carter to a finite temperature, multifluid system including neutron superfluidity and proton superconductivity that is appropriate for use in studying the fluid dynamics of neutron star cores. The hydrodynamics includes the proton flux tubes and magnetized neutron vortex lines, with mutual friction and vortex pinning incorporated covariantly. Viscosity and heat conduction are also included in the equations of the motion to further extend the scope of the hydrodynamics in a way that agrees with different formulations based on Landau–Khalatnikov hydrodynamics. This formulation has a few practical advantages compared to other versions of neutron star hydrodynamics, notably being fully relativistic and using the distinct fluid species as degrees of freedom. One advantage of this is that it allows sources of buoyancy among the different fluids to emerge naturally.

The averaging procedure used to determine the form of the macroscopic action from the mesoscopic theory allowed us to find an approximate effective macroscopic theory, but not an exact term-by-term match. In particular, we were forced to ignore certain terms in the averaged mesoscopic stress-energy tensor, and to drop subdominant terms in partial derivatives of the electromagnetic-vorticity master function $\mathcal{A}_{\text{EM}+\text{V}}$, to obtain a consistent macroscopic effective theory. This may not have been necessary had we gave more careful consideration to, for example, magnetic interactions between flux tubes or between vortex lines, which could provide additional pressure terms orthogonal to the flux tube/vortex line arrays and help to remove the terms in the last line of Eq. (147). Unlike previous attempts at obtaining vortex energy contributions starting from a mesoscopic theory, we did not make use of the “rotation energy cancellation lemma”, finding that this was inappropriate for certain terms in the mesoscopic stress-energy tensor, and thus its use would have led to further inconsistencies between the averaged mesoscopic stress-energy tensor $\langle T_{\mu\nu} \rangle$ and $A_{\text{EM}+\text{V}}$, whose partial derivatives are used to construct $\langle T_{\mu\nu} \rangle$. We find that frame dependence of the energy density invalidates the lemma when entrainment is included unless the vortex line and flux tube arrays are forced to move together.

Based on the effective macroscopic theory found by averaging the mesoscopic theory, we have also clarified an interpretational issue regarding the magnetic field in a type-II superconducting neutron star core, arguing that the field interpreted as $\mathbf{H}$ in most previous works is not in fact the physical $\mathbf{H}$ field defined thermodynamically by Eq. (157). Instead, when ignoring rotation the correct $\mathbf{H}$-field is that found in the condensed matter literature; that is, the first critical field $H_{\text{c1}}$ for proton superconductivity in the low flux tube density limit. The Maxwell equations found using the effective macroscopic theory are the same as the microscopic Maxwell equations, though to a good approximation the sourced equation reduces to depending only on the London field. This provides some justification for why previous authors found $\mathbf{H} = \mathbf{B}$ and $\mathbf{H} = \mathbf{B}_0$ in neutron star core MHD. We are also able to combine the charged fluid equations of motions into a single equation and show that the Lorentz force is the relativistic analog of $(\nabla \times \mathbf{H}) \times \mathbf{B}/4\pi$. These modifications to neutron star MHD could have implications on our understanding of the neutron star magnetic field and phenomena such as pulsar glitches.

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Appendix A: Alternative form of stress-energy tensor

Starting from Eq. (112), insert the forms of $n_x^\mu$, $\mu_x^\mu$, $s^\mu$ and $\Theta_\mu$, as given in Section IV A and $\kappa_{\mu\nu}$, as given by Eq. (126a) to obtain

$$
\begin{align*}
\text{Stress-energy tensor } T_{\mu\nu} &= \langle \mathcal{H}_{\mu\nu} \rangle - \frac{1}{4\pi} \langle F_{\mu\nu} \rangle \\
&= \langle \mathcal{H}_{\mu\nu} \rangle - \frac{1}{4\pi} \{}\pi \mathcal{F}_{\mu\nu} \rangle \times \langle \mathbf{B} \rangle/4\pi.
\end{align*}
$$
\( T^{\mu\nu} = (n_{\mu\nu} + n_{p\mu p} + n_{\varepsilon\mu\varepsilon} + n_{m\mu m} + n_{p\mu p} + n_{\varepsilon\mu\varepsilon} + n_{m\mu m} + n_{p\mu p} + sT^*)u^\mu u^\nu + \Psi g^{\mu\nu} + 2u^{(\mu}q^{\nu)} + 2n_{\mu\nu}^{s}\delta^{(\mu}_{\nu} + 2n_{p\mu p}^{s}\delta^{(\mu}_{\nu} + 2n_{\varepsilon\mu\varepsilon}^{s}\delta^{(\mu}_{\nu} + 2n_{m\mu m}^{s}\delta^{(\mu}_{\nu}) + 2A^{\tau\rho}n_{\tau\rho}^{s}(^{(\mu}_{\nu}) + \frac{B^{s}}{2}q^{\mu}q^{\nu} + B^{*}(n_{\mu\nu})^{2}u^{\mu}u^{\nu} + B^{*}(n_{\mu\nu})^{2}u^{\mu}u^{\nu} + \frac{1}{4\pi}K^{\mu\nu}F_{\rho}^{\nu} + \sum_{\tau=\pi,\rho}^{\Sigma} \lambda_{\tau}^{\mu\nu}u_{\tau\rho}^{\mu} + \sum_{\tau=\Sigma}^{\Sigma} \tau_{\tau}^{\mu\nu} \)

\( = (\Psi - \Lambda)u^\mu u^\nu + \Psi g^{\mu\nu} + \frac{B^{s}}{T^*}q^{\mu}q^{\nu} + 2u^{(\mu}q^{\nu)} + \frac{1}{4\pi}K^{\mu\nu}F_{\rho}^{\nu} + \sum_{\tau=\pi,\rho}^{\Sigma} \lambda_{\tau}^{\mu\nu}u_{\tau\rho}^{\mu} + \sum_{\tau=\Sigma}^{\Sigma} \tau_{\tau}^{\mu\nu} \)

\( + \left( \left( B^{f}(n_{\mu\nu})^{2}u^{\mu}u^{\nu} + B^{*}(n_{\mu\nu})^{2}u^{\mu}u^{\nu} + 2A^{\tau\rho}n_{\tau\rho}^{s}(^{(\mu}_{\nu}) \right) \right) \quad \text{(A1)} \)

The first term proportional to \( u^\mu u^\nu \) is found by rewriting \( \Psi \), given by Eq. (113), as

\( \Psi = \Lambda + n_{\mu\nu} + n_{p\mu p} + n_{\varepsilon\mu\varepsilon} + n_{m\mu m} + n_{p\mu p} + n_{\varepsilon\mu\varepsilon} + n_{m\mu m} + n_{p\mu p} + sT^* - \frac{B^{s}}{T^*}q^{\mu}q^{\nu} - \frac{1}{2} \sum_{\tau=\Sigma}^{\Sigma} \tau_{\tau}^{\mu\nu} \).

Note that, according to Eq. (127), \( \tau_{\tau}^{\mu\nu} \) is traceless, so the final term in Eq. (A2) is zero. The similarity of this form of \( T^{\mu\nu} \) to that of a single perfect fluid [43] is now evident; this form is effectively the same as that for such a fluid, plus electromagnetism, vorticity and viscosity, with differences depending on the relative motion of heat and the superfluids separated out.

**Appendix B: Full details of mesoscopic stress-energy tensor and averaging procedure**

We continue from the main text immediately following the introduction of the mesoscopic Lagrangian and stress-energy tensor Eq. (135). On the mesoscopic scale, currents around vortex lines/flux tubes are represented within the currents \( \tilde{n}_{\mu}^{\tau} \) not by using the vorticity tensors \( u^{\mu\nu} \) as is the case in the macroscopic dynamics. We incorporate these purely “mesoscopic” currents \( \delta n_{\mu}^{\tau} \) by defining \( \tilde{n}_{\mu}^{\tau} \) as

\( \tilde{n}_{\mu}^{\tau} \equiv \tilde{n}_{\tau\gamma}(\delta v_{\gamma}^{\mu})(u_{\mu}^{\tau} + \delta u_{\mu}^{\tau}), \quad \tau = \pi, \rho, \)

where \( u_{\mu}^{\tau} \) is defined as in Eq. (57). \( \tilde{n}_{\mu}^{\tau} \) satisfies the normalization condition \( -\tilde{n}_{\mu}^{\tau}\delta v_{\tau}^{\mu} = \tilde{n}_{\mu}^{\tau} \), since \( u_{\mu}^{\tau}\delta v_{\tau}^{\mu} = 0 \) as a result of the approximation that the vortex lines/flux tubes move with their respective superfluid. We enforce that the \( \delta n_{\mu}^{\tau} \) average to zero over scales larger than the typical cross-section of a vortex line/flux tube, and that any large-scale average part of a relative velocity between the normal fluid and superfluids is included in \( u_{\mu}^{\tau} \). \( \tilde{n}_{\tau}^{\tau} \) is the number density of species \( \tau \) measured in the frame comoving with the total current of that species, and it is related to the number density \( n_{\tau} \) in the frame of the bulk flow (the frame of \( u_{\mu}^{\tau} \)) by

\( n_{\tau} = -u_{\mu}^{\mu}\tilde{n}_{\mu}^{\tau} = \tilde{n}_{\tau\gamma}(\delta v_{\gamma}^{\mu}). \quad \text{(B2)} \)

We first expand out the terms in Eq. (135), removing any dependence on the vortex line/flux tube mesoscale currents from the master function \( \Lambda \) and replace it with \( \Lambda_0 \), which represents only the internal energy of the fluid and the kinetic energy of macroscopic currents. Following [30], we write

\( \tilde{\Lambda}(\tilde{n}_{\gamma}^{2}, \tilde{\alpha}_{\gamma}^{2}) = \Lambda_0(n_{\gamma}^{2}, \alpha_{\gamma}^{2}) \)

\( + \sum_{x, y} \left( \frac{\partial \Lambda_0}{\partial n_{\gamma}^{2}}(\delta v_{x}^{2}) + \frac{\partial \Lambda_0}{\partial \alpha_{\gamma}^{2}}(\delta v_{y}^{2}) \right) \quad \text{(B3)} \)

where the “0” subscript denotes the master function with the \( \delta n_{\mu}^{\tau} \) removed, and where

\( \delta v_{x}^{2} = \tilde{n}_{x}^{2} - n_{x}^{2} = \begin{cases} -n_{x}^{2}\delta v_{x}^{2}, & x = \pi, \rho, \\ 0, & \text{otherwise} \end{cases} \)

\( \delta v_{y}^{2} = \tilde{n}_{y}^{2} - n_{y}^{2} = \begin{cases} 0, & x = \pi, \rho, y = p, \\ \frac{n_{\mu\nu}^{\pi\rho}(\delta v_{\pi}^{\mu})(\delta v_{\rho}^{\nu}), & x = \pi, y = n, \\ n_{\mu\nu}^{\pi\rho}(\delta v_{\pi}^{\mu})(\delta v_{\rho}^{\nu}), & x = \rho, y = n, \\ n_{\mu\nu}^{\pi\rho}(\delta v_{\pi}^{\mu})(\delta v_{\rho}^{\nu}), & x = \pi, y = p, \\ n_{\mu\nu}^{\pi\rho}(\delta v_{\pi}^{\mu})(\delta v_{\rho}^{\nu}), & x = \rho, y = p, \\ 0, & \text{otherwise} \end{cases} \)

where \( u_{\mu}^{\tau}\delta v_{\tau}^{\mu} = -u_{\mu}^{\tau}\delta v_{\tau}^{\mu} \) was used. For the normal fluids, \( \tilde{n}_{\tau}^{\tau} \) simply equals \( n_{\tau}^{\tau} \), since the normal fluid currents are unchanged by the inclusion of the mesoscale currents. We have kept only terms that are order \( \delta n_{\mu}^{\tau} \) in the mesoscopic-scale velocities.

The partial derivatives of \( \Lambda_0 \) in Eq. (B3) are identified with the entrainment coefficients as defined in the macroscopic theory (Eq. (6)). We use the physical arguments presented at the end of Section III to reduce the number of entrainment coefficients i.e. \( A^{\tau\rho} = 0 = A^{\tau\rho} \).
\[ \mathcal{A}^{\nu} = \mathcal{A}^{\mu} = \mathcal{A}^{\nu} = \mathcal{A}^{\mu} \] 

Hence \( \tilde{\Lambda}(\tilde{n}_x^2, \tilde{\alpha}_x^2) \) becomes

\[ \tilde{\Lambda}(\tilde{n}_x^2, \tilde{\alpha}_x^2) = \tilde{\Lambda}_0 + \frac{1}{2} B^2 \mathcal{F}_{\mu \nu} \delta \mathcal{F}^\mu_{\nu} \]

It is also convenient to define a mesoscale superfluid neutron canonical momentum vector

\[ \delta \pi_{\mu}^\text{me} \equiv \mathcal{B}^\mu n_{\mu} \delta v_{\mu}^n + \mathcal{A}^{\nu} n_{\nu} \delta v_{\mu}^p, \] 

which is simply the part of \( \mu_{\text{fl}} \) that depends on the mesoscale velocities \( \delta v_{\mu}^p \). Note that, because the neutron superfluid is not coupled to the electromagnetic field, we could also have called \( \delta \pi_{\mu}^\text{me} \) simply \( \delta \mu_{\text{fl}} \). The definition of \( \delta \pi_{\mu}^\text{me} \) will simply write some terms of the stress-energy tensor immensely by canceling terms which couple \( \delta v_{\mu}^p \) and \( \delta v_{\mu}^p \).

Combining these definitions and results with Eq. (72) and Eq. (61a–62g), the mesoscopic stress-energy tensor resulting from Eq. (B3) is

\[ \tilde{T}^{\mu \nu} = \left( \sum_{x=x,m,s} n_x \mu_x + \sum_{x=n,p} B^\mu n_{\mu}^2 + 2 \mathcal{A}^{\nu} n_{\nu} n_{\mu} \right) u^{\mu} u^{\nu} + \sum_{x=n,p} B^\mu n_{\mu}^2 u^{\mu} u^{\nu} + 2 \mathcal{A}^{\nu} n_{\nu} n_{\mu} u^{\mu} u^{\nu} \]

Next, the form of the electromagnetic tensor and \( \delta A_{\mu} \). We use the splitting Eq. (145) for \( \tilde{F}^{\mu \nu} \). Along with the split of \( \tilde{F}^{\mu \nu} \) in Eq. (145), we split the Maxwell equation Eq. (137) into large-scale and mesoscale parts, with the mesoscale proton current \( \propto \delta A_{\mu} \) sourcing the mesoscale fields and the other parts of the current sourcing the large-scale field (the London field). Using Eq. (56,57,B1) and assuming local charge neutrality

\[ \nabla_{\mu} \tilde{F}_{\mu \nu} = 4 \pi e n_{\nu} \delta \mu_{\text{fl}} \quad \nabla_{\nu} (\tilde{F}_{\mu \nu} + \tilde{F}_{\nu \mu}) = 4 \pi e n_{\mu} \delta v_{\mu}^p. \]

We can also split the four-potential \( \tilde{A}_{\mu} \) into large-scale and mesoscale contributions \( \tilde{A}_{\mu}^L \) and \( \delta A_{\mu} = \delta A_{\mu}^L + \delta A_{\mu}^M \) respectively where \( \tilde{F}_{\mu \nu} = 2 \nabla_{\nu} \delta A_{\mu} \). Then, defining a mesoscale canonical momentum vector for the protons analogously to Eq. (B5)

\[ \delta \pi_{\mu}^\text{me} \equiv B^\mu n_{\mu} \delta v_{\mu}^p + \mathcal{A}^{\nu} n_{\nu} \delta v_{\mu}^p + \epsilon \delta A_{\mu}, \]

and combining it with Eq. (B5) to eliminate \( \delta v_{\mu}^p \), we obtain

\[ \frac{1}{V_{pp}} n_{pp} \delta v_{\mu}^p + \epsilon \delta A_{\mu} = \delta \pi_{\mu}^\text{me} + \frac{Y_{pp}}{V_{pp}} \delta \pi_{\mu}^\text{me}. \]
or, eliminating \( \delta v^\rho_\tau \) with Eq. (B11) and using \( \Phi_\tau = h/(2e) \) and \( \Phi_\tau = Y_{pp} \Phi_\tau/Y_{pp} \) and \( \Lambda_\tau = (4\pi e^2 Y_{pp})^{-1/2} \),

\[
\sum_\tau \left( \nabla_\nu F^{\mu\nu} - \frac{1}{\Lambda_\tau^2} \delta A^{\mu}_\tau \right) = \frac{2}{h \Lambda_\tau^2} \sum_\tau \Phi_\tau \delta \pi^\mu_\tau. \tag{B14}
\]

Eq. (B13) is used to obtain the London equation for proton flux tubes or magnetized neutron vortex lines. We will assume that the magnetic fields due to the flux tubes or magnetized vortex lines will have negligible overlap, and so we can fix \( \tau \) to be either \( \tau \) or \( \bar{\tau} \) and drop the other contribution to Eq. (B14). As a consequence of Eq. (146) with \( u^L_\mu = u^\mu_\mu \), we have

\[
F^{\mu\nu}_\tau = -\varepsilon^{\mu\nu\alpha\beta} u^\alpha_\tau \delta B^{\beta}_\tau, \tag{B15}
\]

where \( \delta B^\mu_\tau \) is the magnetic field due to the flux tubes/vortex lines measured in their rest frame. Then contracting with \( \varepsilon^{\rho\sigma\mu\nu} u^\rho_\tau \), ignoring spatial curvature (which is a very good approximation for microscopic structures like vortex lines and flux tubes), assuming the flux tubes/vortex lines move with their respective superfluid and using Eq. (B10), we obtain

\[
\nabla^2 \delta B^\mu_\tau - \frac{\delta B^\mu_\tau}{\Lambda_\tau^2} = -\frac{\Phi_\tau}{\Lambda_\tau^2} \sum_a u^\mu_\tau \delta^2((x_\tau,a)). \tag{B16}
\]

This is the London equation in the comoving frame, whose solutions are the magnetic fields for flux tubes/magnetized vortex lines, \( \nabla^2 \) is the flat space Laplacian, and \( \Lambda_\tau \) is the London length. The right-hand side of the equation is a sum over flux tubes/magnetized vortex lines labeled by index \( a \) and represented as two-dimensional delta functions. The solutions in the comoving frame for single flux tubes/magnetized vortex lines take the familiar form [30]

\[
\delta B^\mu_{\tau,a} = \frac{\varepsilon_\tau \mu K_0((x_\tau,a)/\Lambda_\tau)}{2\pi \Lambda_\tau^2 x_\tau^0 K_1(x_\tau^0/\Lambda_\tau)} \equiv B_{\tau,a}(x_\tau,a) \varepsilon_\tau \mu_{\tau,a}, \tag{B17}
\]

where \( x_\tau^0 \equiv \xi_\tau/\Lambda_\tau \). Flux in the core of the flux tubes/vortex lines, included in e.g. [30], is ignored here.

Using Eq. (B15,B17), the mesoscale electromagnetic field tensors are

\[
F^\mu_\tau = -\sum_a \varepsilon^{\mu\nu\alpha\beta} u^\alpha_\tau \delta B^{\beta}_{\tau,a}. \tag{B18}
\]

Hence we can replace \( \delta B^\mu_\tau \) in Eq. (B6) using the mesoscale Maxwell equation Eq. (B11) with the gradient of Eq. (B18).

2. Averaging the mesoscopic stress-energy tensor

We now average the mesoscopic stress-energy tensor, Eq. (B6). As noted before, the non-electromagnetic part of this equation consists of purely large-scale flow terms, purely mesoscale flow terms, and mixed terms. Because we specified that \( \delta \pi^\mu_\mu = 0 \) and because \( A_{\mu}^\mu \) averages to itself over volumes \( \sim \ell^2 \) i.e. \( \langle A_{\mu}^\mu \rangle = A_{\mu}^\mu \), the term averages to zero. Though \( \delta \pi^\mu_\mu \) does not average to zero, we will absorb any effect of the large scale-small scale superfluid neutron momentum term \( \propto \mu_\tau (\delta \pi^\mu_\tau) \) in \( T^{\mu\nu} \) into the purely small-scale superfluid neutron momentum terms using a cutoff length. We can thus treat \( \delta \pi^\mu_\mu \) to be either \( \mu_\tau (\delta \pi^\mu_\tau) \) and drop the \( \mu_\tau (\delta \pi^\mu_\tau) \) term vanishes upon averaging. The purely large-scale flow terms, which include \( \langle \tilde{A}_0 + \sum_a n_a \pi^\mu_\nu \rangle g^{\mu\nu} \), do not change upon averaging, and have exact matches in the macroscopic stress-energy tensor. The remaining terms--the purely mesoscale terms plus all of the electromagnetic terms--we label \( \Delta T^{\mu\nu} \):

\[
\Delta T^{\mu\nu} = \frac{1}{B^\tau} \left[ \delta \pi^\mu_\tau \delta \pi^\nu_\tau - \frac{1}{2} \delta \pi^\tau_\nu g^{\mu\nu} \right] + \frac{n_2^2}{Y_{pp}} \left[ \nu^{\mu\nu} \delta B^{\tau}_\mu \right] - \frac{1}{4\pi} \left[ F^{\mu\nu}_\tau F^\nu_\mu + F^{\mu\nu}_\tau F^\mu_\nu + F^{\mu\nu}_\tau F^\nu_\mu + F^{\mu\nu}_\tau F^\mu_\nu \right] + \frac{g^{\mu\nu}}{16\pi} \left[ F^{\alpha\nu}_\tau F^{\alpha}_\rho + F^{\beta\nu}_\tau F^{\beta}_\rho + F^{\nu}_\tau F^{\mu}_\rho + F^{\nu}_\tau F^{\mu}_\rho \right]
+ \frac{1}{B^\tau} \left[ F^{\alpha\nu}_\tau F^{\alpha}_\rho + F^{\beta\nu}_\tau F^{\beta}_\rho \right]. \tag{B19}
\]

We now integrate Eq. (B19) over a surface of size \( \sim \ell^2 \), then replace the quantities in the mesoscopic stress-energy tensor with averaged quantities. First we consider

\[
\tilde{T}^{\mu\nu}_\tau = \frac{1}{B^\tau} \left[ \delta \pi^\mu_\tau \delta \pi^\nu_\tau - \frac{1}{2} \delta \pi^\tau_\nu g^{\mu\nu} \right]. \tag{B20}
\]

\( \delta \pi^\mu_\tau \) is replaced by a sum over individual vortex lines. For this purpose, we rewrite Eq. (B8) as

\[
\delta \pi^\mu_\tau = \frac{\hbar}{2\pi_{\tau,a}} (-\sin \varphi_{\tau,a} \zeta^\mu_{\tau,a} + \cos \varphi_{\tau,a} \bar{Q}^\mu_{\tau,a}) \tag{B21}
\]

where \( \varphi_{\tau,a} \) is an azimuthal angle measured around a vortex line labeled \( a \) and \( \zeta^\mu_{\tau,a} \) and \( \bar{Q}^\mu_{\tau,a} \) are mutually orthogonal unit vectors which are also orthogonal to both \( U^\mu_{\tau,a} \) and \( \bar{U}^\mu_{\tau,a} \). When integrating over a surface area \( \sim \ell^2 \) in the plane perpendicular to the average vortex line tangent vector \( \tilde{u}^\mu_{\tau} = \langle \tilde{u}^\mu_{\tau,a} \rangle \), the sum over different vortex lines is replaced with a multiplication by the areal density of vortex lines \( N_{\tau} \). We also replace the other vectors with their average values over the area of integration \( \tilde{\zeta}_{\tau,a} = \langle \tilde{\zeta}^\mu_{\tau,a} \rangle \) and \( \tilde{\bar{Q}}_{\tau,a} = \langle \tilde{Q}^\mu_{\tau,a} \rangle \). This means we only need to consider the integral for a single vortex line, integrating radially from
the coherence length (since we’re ignoring the core) to a cutoff radius \( r_n^{\text{cut}} \):

\[
\langle \tilde{T}_{\mu\nu} \rangle = N_n \frac{\hbar^2}{8B^2} \ln \left( \frac{r_n^{\text{cut}}}{\xi_n} \right) \left( \hat{\rho}_{\mu\nu}^{-}\hat{\rho}_{\mu\nu} + \hat{\eta}_{\mu\nu}^{-}\hat{\eta}_{\mu\nu} - g^{\mu\nu} \right) .
\]

(B22)

The neutron vortex line cutoff radius \( r_n^{\text{cut}} \) accounts for the long-range nature of the vortex lines and incorporates the effect of interactions between them. \( \langle \tilde{T}_{\mu\nu} \rangle \) thus absorbs the \( \delta \tilde{H}_{\mu\nu}^{(\mu)} \) terms in Eq. (B6) that we argued earlier to zero. Based on Tkachenko [68] and Sonin [69], we expect \( r_n^{\text{cut}} \approx 0.0712(\xi_n N_n)^{-1} \). Additionally we have

\[
g^{\mu\nu} = -u_{\mu} u_{\nu} + \tilde{\rho}_{\mu\nu}^{-} + \frac{i}{2} \tilde{\eta}_{\mu\nu}^{-} - \frac{i}{2} \tilde{\eta}_{\mu\nu}^{+} + \frac{i}{2} \tilde{\eta}_{\mu\nu}^{+} .
\]

(B23)

so we can write

\[
\langle \tilde{T}_{\mu\nu} \rangle = N_n \frac{\pi \hbar^2}{8B^2} \ln \left( \frac{0.0712}{N_n \pi r_n^{\text{cut}}} \right) \left( u_{\mu} u_{\nu} - \frac{i}{2} \eta_{\mu\nu}^{+} - \frac{i}{2} \eta_{\mu\nu}^{-} \right) .
\]

(Eq. (B24) has the general form of the stress-energy tensor for a single string along \( \tilde{\eta}_{\mu} \) [70].

Next consider

\[
\tilde{T}_{\mu\nu} = \frac{\eta_{\mu\nu}}{\rho_{PP}} \left[ \delta_{\mu\nu} \hat{\rho}_{\mu\nu}^{(\mu)} - \frac{1}{2} \delta_{\mu\nu} g^{\mu\nu} \right] .
\]

(B25)

At this point, we neglect the interactions between different flux tubes/vortex lines and consider only their self-energy contributions. This allows us to simplify in our averaging integral and again integrate only over a surface locally perpendicular to the flux lines/vortex tubes, then multiply by the relevant area number density \( N_n \). Using Eq. (137,B15) and ignoring spatial curvature, we find an identical coordinate system as was used to compute \( \langle \tilde{T}_{\mu\nu} \rangle \). We use the large cutoff radius \( r_n^{\text{cut}} \to \infty \) limit and used the approximation

\[
\frac{K_0(x_0^\perp)K_2(x_0^\perp)}{K_1^2(x_0^\perp)} - 1 \approx 2 \left[ \ln \left( \frac{1}{x_0^\perp} \right) - 0.384 \right] , \quad x_0^\perp \ll 1 .
\]

(B27)

Now we consider the electromagnetic field tensor terms in the mesoscopic stress-energy tensor. As we have previously discussed, the overlap between the magnetic fields due to different flux tubes and magnetized vortex lines is negligibly small, and hence we force the \( \tilde{F}_{\mu\nu} \tilde{F}_{\mu\nu} \) terms in Eq. (B19) to vanish upon averaging. The London field is the same before and after averaging, and so the \( \tilde{F}_{\mu\nu} \) terms are unchanged by averaging other than removing the tilde. Because the London field is unchanged upon averaging, the \( \tilde{F}_{\mu\nu} \) terms become

\[
\frac{1}{2\pi} \left[ \langle F_{\mu\rho} F_{\nu\rho} \rangle - \frac{1}{4} F_{\mu\rho} F_{\alpha\sigma} g^{\mu\rho} \right] = \frac{1}{2\pi} \left[ \langle F_{\mu\rho} F_{\nu\rho} \rangle - \frac{1}{4} F_{\mu\rho} F_{\alpha\sigma} g^{\mu\rho} \right] = \frac{1}{2\pi} \left[ \langle F_{\mu\rho} F_{\nu\rho} \rangle - \frac{1}{4} F_{\mu\rho} F_{\alpha\sigma} g^{\mu\rho} \right] .
\]

where we identify \( \langle \tilde{F}_{\mu\nu} \rangle = \langle F_{\mu\nu} \rangle / \epsilon \) as argued using Eq. (143). An analogous expression is found for the \( \tilde{F}_{\mu\nu} \) terms except with additional entrainment coefficients since \( \langle \tilde{F}_{\mu\nu} \rangle = Y_{\mu\nu} / \epsilon \).

The final terms to average in Δ\( \tilde{T}_{\mu\nu} \) are

\[
\tilde{T}_{\mu\nu} = \frac{\eta_{\mu\nu}}{\rho_{PP}} \left[ \delta_{\mu\nu} \hat{\rho}_{\mu\nu}^{(\mu)} - \frac{1}{2} \delta_{\mu\nu} g^{\mu\nu} \right] .
\]

(B29)

Using Eq. (B15) and again ignoring interactions between different flux tubes and vortex lines, we can compute the average of \( \tilde{T}_{B_{\mu\nu}} \) analogously to Eq. (B22,B26), giving

\[
\langle \tilde{T}_{B_{\mu\nu}} \rangle = \sum_{\tau} \frac{\eta_{\mu\nu}}{16\pi^2 A_2} \int_0^{2\pi} d\phi_{\perp} \int_{\eta_{\perp}}^{r_n^{\text{cut}}} \frac{d\omega_{\perp} \omega_{\perp} K_2^2(\omega_{\perp}/\Lambda_* \xi_{\perp}^2)}{[x_0^\perp K_1(x_0^\perp)]^2} \times \left( u_{\mu} u_{\nu} + \frac{1}{2} g^{\mu\nu} - \frac{i}{2} \eta_{\mu\nu}^{-} \right)
\]

\[
= \sum_{\tau} \frac{\eta_{\mu\nu}}{16\pi^2 A_2} \left( u_{\mu} u_{\nu} + \frac{1}{2} g^{\mu\nu} - \frac{i}{2} \eta_{\mu\nu}^{-} \right) ,
\]

where we used the approximation \( K_2^2(x_0^\perp)/K_1^2(x_0^\perp) \approx 0 \) for \( x_0^\perp \ll 1 \).

Using Eq. (139,140,B23) we can rewrite the following contractions of tensors as

\[
u_{\mu\rho} u_{\nu\rho} = 2W_{\mu\rho} W_{\rho\nu} + 2(\nabla_{\mu} F_{\rho\nu}) = 4\pi \tau_{\mu\nu} .
\]

(B31)

\[
u_{\mu\rho} u_{\nu\rho} = (\rho^{\mu\nu} + u^\mu u^\nu) W_{\mu\rho} W_{\rho\nu} - W_{\mu\rho} W_{\rho\nu} = (\nabla_{\mu} F_{\rho\nu})^2 + (u^\mu u^\nu - \frac{i}{2} \eta_{\mu\nu}^{\mu}) .
\]

(B32)
Replacing terms in Eq. (B19) with their respective averages and using Eq. (B32), we finally obtain

\[
(\bar{T}^{\mu\nu}) = \left( \sum_{x=e,m,s} n_x \mu_x + \sum_{x=n,p} B^x n_x^2 + 2A^{\mu\nu}_x n_x n_p \right) u^\mu u^\nu \\
+ \sum_{x} B^x n_x^2 \mu_x \right) u^\mu u^\nu \\
+ \sum_{x} B^x n_x^2 \mu_x \right) u^\mu u^\nu \\
+ \left( \Lambda_0 + \sum_{x} n_x \mu_x \right) g^{\mu\nu} \\
+ \frac{1}{4\pi} \left( E^{\nu}_{\rho \sigma} F_{\rho \sigma}^{\mu} - \frac{1}{4} F_{\rho \sigma} F_{\rho \sigma}^{\mu} \right) \\
+ \sum_{x} \frac{\Phi_{x}}{2\pi e} \left( u^\mu_{x} \rho \nu \sigma \right) - \frac{1}{4} \sum_{x} \left( u^\mu_{x} \rho \nu \sigma \right) g^{\mu\nu} \\
+ \sum_{x} \frac{\mathcal{E}_{x,\pi}^{\mu\nu}}{N_{\pi}(\Phi_{x})} \left( u^\mu_{x} \rho \nu \sigma \right) - \frac{1}{2} \sum_{x} \left( u^\mu_{x} \rho \nu \sigma \right) g^{\mu\nu} \\
+ \frac{1}{32\pi^2 N_{\pi} \Lambda_{e}^{2}} \sum_{x} \mu^\rho_{x} \nu \pi^{\mu\nu}_{x},
\]

(B33)

which is Eq. (147) in the main text, and where the \( \mathcal{E}_{x,\pi}^{\mu\nu} \) are defined in Eq. (148–149).

3. Matching to the macroscopic stress-energy tensor

We now match the macroscopic stress-energy tensor as found in Eq. (51) with the averaged mesoscopic stress-energy tensor Eq. (B33). To begin, we need to expand out the \( \sum_{x} n_x^2 \mu_x \) and \( \sum_{x} n_x \mu_x \) terms using the definitions of the currents and conjugate momenta from Section III. This gives

\[
T^{\mu\nu} = \left( \sum_{x=e,m,s} n_x \mu_x + \sum_{x=n,p} B^x n_x^2 + 2A^{\mu\nu}_x n_x n_p \right) u^\mu u^\nu \\
+ \sum_{x} B^x n_x^2 \mu_x \right) u^\mu u^\nu \\
+ \left( \Lambda_0 + \sum_{x} n_x \mu_x \right) g^{\mu\nu} \\
+ \frac{1}{4\pi} \left( E^{\nu}_{\rho \sigma} F_{\rho \sigma}^{\mu} - \frac{1}{4} F_{\rho \sigma} F_{\rho \sigma}^{\mu} \right) \\
+ \sum_{x} \frac{\Phi_{x}}{2\pi e} \left( u^\mu_{x} \rho \nu \sigma \right) - \frac{1}{4} \sum_{x} \left( u^\mu_{x} \rho \nu \sigma \right) g^{\mu\nu} \\
+ \frac{1}{32\pi^2 N_{\pi} \Lambda_{e}^{2}} \sum_{x} \mu^\rho_{x} \nu \pi^{\mu\nu}_{x},
\]

(B34)

We see that the macroscopic current terms in Eq. (B34) and Eq. (B33) match, so we now focus on matching the remaining terms.

We postulate the Lorentz-invariant form of the macroscopic master function \( \Lambda \) as in Eq. (130) and identify \( \Lambda_0 = \widehat{\Lambda}_0 \). Using Eq. (130–133) in Eq. (B34) and then substituting \( F^{\mu\nu} \) using Eq. (143) gives

\[
\Delta T^{\mu\nu} = \Lambda^{\mu\nu}_{EM+V} - \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial X^F} F^{\mu\nu}_{\rho \sigma} F_{\rho \sigma}^{\mu} \\
- \left[ \frac{\kappa_{2\pi}^{\nu}}{e^2 Y_{pp}} \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial X^F} + \frac{2\pi}{e Y_{pp}} \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Y^\pi} \right] u^\mu_{\pi} u^\nu_{\pi} \\
+ \left[ \frac{1}{e} \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial X^F} + \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Y^\pi} \right] F^{\rho(\mu \nu)\rho}_{\pi} \\
+ \left[ \frac{Y_{np}}{e^2 Y_{pp}} \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial X^F} + \frac{Y_{np}}{e Y_{pp}} \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Y^\pi} \right] \frac{1}{e} \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Z} u^\mu_{\pi} u^\nu_{\pi},
\]

(B35)

where \( \Delta T^{\mu\nu} \) is defined to only include those terms in the macroscopic stress-energy tensor which do not have an exact matching term in the mesoscopic stress-energy tensor, but including all of the electromagnetic terms.

We can now match terms by comparing Eq. (B35) to \( \langle \Delta \bar{T}^{\mu\nu} \rangle \) (the last four lines of Eq. (B33)). First, matching the London magnetic field squared terms requires

\[
\frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial X^F} = -\frac{1}{4\pi}.
\]

Matching to the London field–flux tube/vortex line field cross terms and using Eq. (B36), we require

\[
\frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Y^\pi} = 0 = \frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Y^\pi}.
\]

(B37)

The flux tube/vortex line cross term in Eq. (B35) are zero as a result of our ignoring their interactions in the averaged mesoscopic theory. In accordance with Eq. (B36–B37), this requires

\[
\frac{\partial \Lambda^{\mu\nu}_{EM+V}}{\partial Z} = \frac{Y_{np}}{4\pi e^2 Y_{pp}}.
\]

(B38)
Matching to terms proportional to $u^\mu u^\nu_{\rho\sigma}$ gives

$$\frac{\partial \Lambda_{EM+V}}{\partial X_\pi} = \frac{1}{4\pi e^2} - \frac{1}{N_p(\Phi_c e)^2} \left( \xi_v \pi + \frac{\phi_\pi^2}{32\pi^2 \Lambda_\pi^2} \right),$$

(B39)

$$\frac{\partial \Lambda_{EM+V}}{\partial X_\pi} = \frac{Y_{np}}{Y_{pp}} \left[ \frac{1}{4\pi e^2} - \frac{1}{N_p(\Phi_c e)^2} \left( \xi_v \pi + \frac{\phi_\pi^2}{32\pi^2 \Lambda_\pi^2} \right) \right].$$

(B40)

Matching terms proportional to $g^{\mu\nu}$ gives the same $\Lambda_{EM+V}$ as Eq. (150). Rewriting this in terms of the scalars $X_F, X_\pi, Y_\pi, Z$ and taking the partial derivatives of $\Lambda_{EM+V}$ with respect to each scalar, we obtain the same results as in Eq. (B36–B38). However, we do not completely recover Eq. (B39–B40) and miss additional vortex line/flux tube magnetic field energy contributions $\propto \phi_\pi^2/(32\pi^2 \Lambda_\pi^2)$ (in fact, one-half the magnetic field energy per unit length). In the strong type-II superconductivity limit, the missing terms would be irrelevant and so both ways to find $\Lambda_{EM+V}$ would be consistent. We drop them regardless of the physical limit, which is equivalent to dropping the last line in Eq. (B33). We also gain an extra term in Eq. (B40) because of the $\Lambda_\pi$ dependence of the vortex line energy cutoff radius in $\xi_v \pi$. This contribution is argued to be small in Eq. (153).

That we cannot obtain a completely consistent macroscopic master function and stress-energy tensor from averaging the mesoscopic theory is not entirely surprising, as we had no reason to believe this was possible before we began. We can at least have an approximate effective macroscopic theory by using the $\Lambda_{EM+V}$ found by matching terms proportional to $g^{\mu\nu}$ between the averaged mesoscopic and macroscopic theories and then ignoring terms in the stress-energy tensor inconsistent with this—fortunately there are only two such terms, and in the strong type-II superconductivity limit and for $d_n \gg \xi_n$ both terms are negligible.

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