ON THE POLYNOMIAL CONVEXITY OF THE UNION OF MORE THAN TWO TOTALLY-REAL PLANES IN $\mathbb{C}^2$

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Abstract. In this paper we shall discuss local polynomial convexity at the origin of the union of finitely many totally-real planes through $0 \in \mathbb{C}^2$. The planes, say $P_0, \ldots, P_N$, satisfy a mild transversality condition that enables us to view them in Weinstock normal form, i.e. $P_0 = \mathbb{R}^2$ and $P_j = M(A_j) := (A_j + iI)\mathbb{R}^2$, $j = 1, \ldots, N$, where each $A_j$ is a $2 \times 2$ matrix with real entries. Weinstock has solved the problem completely for $N = 1$ (in fact, for pairs of transverse, maximally totally-real subspaces in $\mathbb{C}^n \forall n \geq 2$). Using a characterization of simultaneous triangularizability of $2 \times 2$ matrices over the reals, given by Florentino, we deduce a sufficient condition for local polynomial convexity of the union of the above planes at $0 \in \mathbb{C}^2$. Weinstock's theorem for $\mathbb{C}^2$ occurs as a special case of our result. The picture is much clearer when $N = 2$. For three totally-real planes, we shall provide an open condition for local polynomial convexity of the union. We shall also argue the optimality (in an appropriate sense) of the conditions in this case.

1. Introduction and statement of results

Let $K$ be a compact subset of $\mathbb{C}^n$. The polynomially convex hull of $K$ is defined by $\hat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p|, p \in \mathbb{C}[z_1, \ldots, z_n]\}$. $K$ is said to be polynomially convex if $\hat{K} = K$. We say that a closed subset $E$ of $\mathbb{C}^n$ is locally polynomially convex at $p \in E$ if $E \cap B(p, r)$ is polynomially convex for some $r > 0$ (here, $B(p, r)$ denotes the open ball in $\mathbb{C}^n$ with centre $p$ and radius $r$). In general, it is very difficult to determine whether a given compact subset of $\mathbb{C}^n$, $n > 1$, is polynomially convex. Therefore, researchers have usually restricted their attention to specific subclasses of geometric objects. In this paper we consider the union of finitely many totally-real planes in $\mathbb{C}^2$ intersecting at $0 \in \mathbb{C}^2$, with a mild transversality condition. In this setting we shall discuss the following:

- A sufficient condition for the union to be locally polynomially convex at $0 \in \mathbb{C}^2$ that generalizes a theorem (Result 2.2 below) by Weinstock in $\mathbb{C}^2$.
- An open condition that is sufficient for the union of totally-real planes to be locally polynomially convex at $0 \in \mathbb{C}^2$ when the number of planes is three.
- Optimality, in an appropriate sense defined below, of the above open condition for three totally-real planes.

We shall see a couple of motivations for focusing attention on the above setting. However, let us first make a brief survey of known results in this direction and make the above setting a bit more formal.

It is easy to show that if $M$ is a totally-real subspace of $\mathbb{C}^n$, then any compact subset of $M$ is polynomially convex. Hence, let us now consider $P_0 \cup P_1$, where $P_0$ and $P_1$ are two transverse totally-real $n$-dimensional subspaces of $\mathbb{C}^n$. Applying a $\mathbb{C}$-linear change of coordinate, we can assume that $P_0 = \mathbb{R}^n$. A careful look at the

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second subspace under the same change of coordinate gives us $P_1 = (A + iI)\mathbb{R}^n$ for some $A \in \mathbb{R}^{n \times n}$ (see [9] for details). We shall call this form for the pair of totally-real subspaces as Weinstock’s normal form. Weinstock [9] found a way of giving complete characterization for the polynomial convexity of $P_0 \cup P_1$ at $0 \in \mathbb{C}^n$ in terms of the eigenvalues of $A$ (see Result 2.2 in Section 2).

No analogue of Weinstock’s theorem is known for more than two totally-real subspaces. Even in $\mathbb{C}^2$, the problem of generalizing Weinstock’s characterization does not seem any simpler. The works of Pascal Thomas [7, 8] give us some sense of the difficulties involved. In [7] Thomas gave an example of a one-parameter family of triples $(P^x_0, P^x_1, P^x_2)$ of totally-real planes in $\mathbb{C}^2$, intersecting at $0 \in \mathbb{C}^2$, showing that polynomial convexity of each pairwise union at the origin does not imply the polynomial convexity of the union at the origin (see Results 2.6). In fact, he showed that for the above triples $(P^x_0, P^x_1, P^x_2)$, the polynomial hull of $(\cup_{j=1}^3 P^x_j) \cap \overline{B}(0; r)$ contains an open set in $\mathbb{C}^2$. The explicit expression, given by Thomas, of this one-parameter family of triples will be used to show optimality, i.e., to prove the last assertion of Theorem 1.4. On the other hand, Thomas also found in [8] examples of triples whose union is locally polynomially convex at the origin.

In this paper, we will be far less interested in specific examples of polynomial convexity (or the failure thereof) of a finite union of totally-real planes passing through $0 \in \mathbb{C}^2$. It turns out that many of Weinstock’s ideas in [9] are the “correct” ones to follow when one considers the union of more than two totally-real subspaces containing $0 \in \mathbb{C}^2$. One of the motivations of this paper is to demonstrate the utility of those ideas. Here, we shall focus closely on how the algebraic properties of Weinstock’s normal form of a collection of totally-real planes in $\mathbb{C}^2$ influence polynomial convexity. This suggests that there is a notion of a Weinstock-type normal form for the union of more than two totally-real planes containing $0 \in \mathbb{C}^2$. Consider a finite collection of maximally totally-real subspaces $P_0, P_1, \ldots, P_N$ of $\mathbb{C}^n$, satisfying $P_0 \cap P_j = \{0\}$, $j = 1, \ldots, N$. By exactly the same arguments as in [9], we can find a $\mathcal{C}$-linear change of coordinate relative to which:

\begin{align}
P_0 &: \mathbb{R}^n \\
P_j &: M(A_j) = (A_j + iI)\mathbb{R}^n, \quad j = 1, \ldots, N,
\end{align}

where $A_j \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, N$. (Note that in this paper we shall refer to a $\mathcal{C}$-linear operator and its matrix representation relative to the standard basis of $\mathbb{C}^n$ interchangably.) We shall call \[(1.1)\] Weinstock’s normal form for $\{P_0, P_1, \ldots, P_N\}$. Note that the collection $\{P_0, P_1, \ldots, P_N\}$ above need not be pairwise transverse at the origin.

When $n = 2$, one quickly intuits that Weinstock’s methods would work if the $A_j$’s can be simultaneously conjugated over $\mathbb{R}$ to certain canonical forms. Hence if $\{A_1, \ldots, A_N\}$ is pairwise commutative then conclusions analogous to Weinstock’s can be demonstrated. This idea is the basis of our first theorem — except that the “commutation condition” above can be weakened. Note also, that:

- We recover Weinstock’s theorem for $\mathbb{C}^2$ when we take $N = 1$ below.
- We do not require the planes $P_0, \ldots, P_N$ to be pairwise transverse at $0 \in \mathbb{C}^2$.

In order to state our first theorem we need the following definition.

**Definition 1.1.** A matrix sequence $A = \{A_1, A_2, \ldots, A_N\}$, $A_j \in \mathbb{R}^{n \times n}$, is said to be reduced if there are no commuting pairs among its terms, i.e. $A_j A_k = A_k A_j \neq 0$ for
all $1 \leq j < k \leq N$. A subsequence $\mathcal{B} \subset \mathcal{A}$ is called a reduction of $\mathcal{A}$ if $\mathcal{B}$ is reduced and is obtained from $\mathcal{A}$ by deleting some of its terms. The reduced length of $\mathcal{A}$ is the greatest $k \in \mathbb{Z}_+$ such that there exists a reduction $\mathcal{B} \subset \mathcal{A}$ of cardinality $k$.

**Theorem 1.2.** Let $P_0, \ldots, P_N$ be distinct totally-real planes in $\mathbb{C}^2$ containing the origin. Assume

1. $P_0 \cap P_j = \{(0,0)\}$ for all $j = 1, 2, \ldots, N$.

Hence, let Weinstock’s normal form for $\{P_0, \ldots, P_N\}$ be

$$P_j : M(A_j) = (A_j + iI)\mathbb{R}^2, \quad j = 1, \ldots, N,$$

where $A_j \in \mathbb{R}^{2 \times 2}$. Let $L$ be the reduced length of $\{A_1, \ldots, A_N\}$. Assume further that

2. $\det[A_j, A_k] = 0$, $j \neq k$, $1 \leq j, k \leq N$, and, additionally, $\text{Tr}(ABC - CBA) = 0 \forall A, B, C \in \mathcal{B}$ if $L = 3$.

Under these conditions:

(a) If each $A_j$ has only real eigenvalues, then $\bigcup_{j=0}^NP_j$ is polynomially convex at the origin.

(b) If there exists a $j$, $1 \leq j \leq N$, such that $A_j$ has non-real eigenvalues, then the spectrum of $A_j$ is of the form $\{\lambda_k, \overline{\lambda_k}\}$ $\forall k = 1, \ldots, N$. Write $\lambda_k = s_k + it_k$, $V_k = (s_k^2 + t_k^2 - 1, 2is_k)$, $k = 1, \ldots, N$, and $V_0 = (1, 0)$. Then, $\bigcup_{j=0}^NP_j$ is locally polynomially convex at the origin if and only if there exists no pair $(l, m)$, $l \neq m, 0 \leq l, m \leq N$, satisfying $V_l = cV_m$ for some constant $c > 0$.

Our proof of Theorem 1.2 is strongly influenced by the methods in Weinstock’s paper — our result is already stated in terms of Weinstock’s normal form. However, in order to use these techniques (presented in Section 2), we would like the matrices $A_j$, $j = 1, \ldots, N$, in (1.1) be as simple in structure as possible. Thanks to Lemma 2.1 it suffices to work with the planes $P_0, \tilde{P}_1, \ldots, \tilde{P}_N$, where

$$\tilde{P}_j : M(B_j) = (B_j + iI)\mathbb{R}^2, \quad j = 1, \ldots, N,$$

$$B_j \sim A_j$$

such that each $B_j$ is sufficiently sparse/structured.

It turns out that in the difficult half of Theorem 1.2 the matrices $B_j$, $j = 1, \ldots, N$, just need to be upper-triangular. So, in order to exploit Lemma 2.1

- We require that $\{A_1, \ldots, A_N\}$ be simultaneously triangularizable.
- We also require the common conjugating matrix to be a matrix with real entries.

What is required, hence, is a sufficient condition for simultaneous triangularizability of real $2 \times 2$ matrices by a single conjugator with real entries. This requirement is met by a result of Florentino (see Result 2.3) and that is where Condition (2) comes from. We acknowledge that, under Condition (2), the collection $\{P_0, P_1, \ldots, P_N\}$ is non-generic in the space of $(N+1)$-tuples of totally-real 2-subspaces of $\mathbb{C}^2$. However, even under this closed condition, Theorem 1.2 has some utility. There is a close connection between the polynomial convexity at $0 \in \mathbb{C}^2$ of a union of $N$ totally-real 2-subspaces of $\mathbb{C}^2$ and the polynomial convexity of the graphs of homogeneous polynomials in $x$ and $y$, $z = x + iy$ of degree $N$, $N \geq 2$. This connection was first investigated by Forstneric-Stout and Thomas (also see Section 2 below). Bharali’s results in

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1This theorem is one of the results of the author’s doctoral thesis.
require the study of the union of more than two totally-real 2-subspaces, and a careful reading reveals that an attempt to weaken his hypothesis is obstructed by a non-generic family of N-tuples, N ≥ 3, of totally-real 2-subspaces in \( \mathbb{C}^2 \). It is hoped that Theorem 1.2 will provide some insight into this problem.

The natural question one may ask is: what happens when Condition 2 fails? We investigate the situation when the number of totally-real planes is restricted to three. The complexity of the set-up with three totally-real planes is low enough that we can replace Condition (2) by an open condition. Our next theorem uses essentially same idea as in Theorem 1.2:

**Theorem 1.3.** Let \( P_0, P_1, P_2 \) be three totally-real planes containing \( 0 \in \mathbb{C}^2 \). Assume \( P_0 \cap P_j = \{(0,0)\} \) for \( j = 1, 2 \). Hence, let Weinstock’s normal form for \( \{P_0, P_1, P_2\} \) be

\[
P_0 : \mathbb{R}^2
\]

\[
P_j : M(A_j) = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,
\]

where \( A_j \in \mathbb{R}^{2 \times 2} \). Let the pairwise unions of \( P_0, P_1, P_2 \) be locally polynomially convex at \( 0 \in \mathbb{C}^2 \). Then \( P_0 \cup P_1 \cup P_2 \) is locally polynomially convex at \( 0 \in \mathbb{C}^2 \) if any one of the following conditions holds:

(i) \( \det[A_1, A_2] > 0 \) and \( \det A_j > 0 \), \( j = 1, 2 \),

(ii) \( \det[A_1, A_2] < 0 \) and \( \det A_j < 0 \), \( j = 1, 2 \).

It turns out that the first part of Theorem 1.3 i.e., the case when Condition (i) holds, is a special case of our third result. However, we choose to present it separately because of the simplicity of its hypotheses, and because this hypothesis provides the motivation for the much more technical-looking hypothesis of Theorem 1.4 below.

It is easy to see that if \( (P_0, P_1, P_2) \) is a triple of totally-real planes with \( P_0 \cap P_j = \{0\} \), \( j = 1, 2 \) (with one of the three being designated as \( P_0 \) in case all three planes are mutually transverse), then the matrices \( A_1 \) and \( A_2 \) associated to Weinstock’s normal form for the triple \( (P_0, P_1, P_2) \) is unique. In short, every triple \( (P_0, P_1, P_2) \) of totally-real planes with \( P_0 \cap P_j = \{0\} \), \( j = 1, 2 \), is parametrized by a pair of matrices. Let us define

\[
\Omega := \{ (A_1, A_2) \in (\mathbb{R}^{2 \times 2})^2 : \sigma(A_1) \subset \mathbb{C} \setminus \mathbb{R} \text{ or } \sigma(A_1) \subset \mathbb{R} \text{ and } \#\sigma(A_1) = 2 \}.
\]

It is clear that \( (\mathbb{R}^{2 \times 2})^2 \setminus \Omega \) has Lebesgue measure zero. (In contrast, it turns out – see Section 4 – that the hypotheses of Theorem 1.3 rule out the possibility of \( \sigma(A_j) \subset \mathbb{C} \setminus \mathbb{R}, j = 1, 2 \).) In the following theorem we will study the triples of totally-real planes parametrized by \( \Omega \).

**Theorem 1.4.** Let \( P_0, P_1, P_2 \) be three totally-real planes containing \( 0 \in \mathbb{C}^2 \). Assume \( P_0 \cap P_j = \{(0,0)\} \) for \( j = 1, 2 \). Hence, let Weinstock’s normal form for \( \{P_0, P_1, P_2\} \) be

\[
P_0 : \mathbb{R}^2
\]

\[
P_j : M(A_j) = (A_j + i\mathbb{I})\mathbb{R}^2, \quad j = 1, 2,
\]

and assume \( (A_1, A_2) \) belongs to parameter domain \( \Omega \). By definition, \( \exists T \in GL(2, \mathbb{R}) \) such that

\[
T A_1 T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \text{ or } \begin{pmatrix} s & -t \\ t & s \end{pmatrix}, \quad \lambda_1 \neq \lambda_2 \in \mathbb{R}, s \in \mathbb{R}, t \in \mathbb{R} \setminus \{0\}.
\]

(1.2)
Let $A_1 := TA_1T^{-1}$. Assume further that the pairwise unions of $P_0, P_1, P_2$ be locally polynomially convex at $0 \in \mathbb{C}^2$.

(i) Suppose $\sigma(A_1) \subset \mathbb{R}$ and $\#\sigma(A_1) = 2$. Then either $\det[A_1, A_2] = 0$ or $\exists T \in GL(2, \mathbb{R})$ that satisfies (1.2) and such that $TA_2T^{-1}$ has the form

$$TA_2T^{-1} = \begin{pmatrix} s_{12} & t_2 \\ t_2 & s_{22} \end{pmatrix} \text{ or } \begin{pmatrix} s_{12} & -t_2 \\ t_2 & s_{22} \end{pmatrix} =: A_2(T), \ s_{12}, s_{22}, t_2 \in \mathbb{R}.$$ 

In the first case $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$. In the second case $P_0 \cup P_1 \cup P_2$ is locally polynomially convex if for some $T \in GL(2, \mathbb{R})$ as stated, $\det(A_j(T) + A_j(T)^T) > 0$, for $j = 1, 2$.

(ii) Suppose $\sigma(A_1) \subset \mathbb{C} \setminus \mathbb{R}$. Then $\exists T \in GL(2, \mathbb{R})$ that satisfies (1.2) and such that

$$TA_2T^{-1} = \begin{pmatrix} s_{12} & -t_2 \\ t_2 & s_{22} \end{pmatrix} =: A_2(T), \ s_{12}, s_{22}, t_2 \in \mathbb{R}.$$ 

If $\det(A_j(T) + A_j(T)^T) > 0$, $j = 1, 2$, for some $T \in GL(2, \mathbb{R})$ as just stated, then $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at $0 \in \mathbb{C}^2$.

The above conditions are optimal in the sense that, writing $\Omega^* \subseteq \Omega$ to be set of pairs $(A_1, A_2) \in \Omega$ that satisfy the conditions in (i) or (ii), there is a one-parameter family of triples $(P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon)$ parametrized by $(A_1^\varepsilon, A_2^\varepsilon) \in \Omega \setminus \Omega^*$ such that

- pairwise unions of $P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon$ are locally polynomially convex at the origin;
- the union of the above planes is not locally polynomially convex at $0 \in \mathbb{C}^2$; and
- $(A_1^\varepsilon, A_2^\varepsilon) \to \partial\Omega^*$ (considered as a subset of $\Omega$) as $\varepsilon \searrow 0$.

A few words about the layout of this paper. The first half of the next section collects some useful technical results and in the second half we state and prove some useful lemmas in linear algebra. In the next three sections (Sections 3–5), we shall give the proof of the theorems. We would like the reader to realise that Part (i) of Theorem 1.3 is subsumed by Theorem 1.4. For this reason, we shall prove Theorem 1.3 in Section 5 after we prove Theorem 1.4.

2. Technical preliminaries

We shall require some preliminaries to set the stage for proving the theorems. First, we state a lemma from Weinstock’s paper [9] — whose proof is quite easy — that allows us to conjugate the matrices coming from Weinstock’s normal form by real nonsingular matrices.

Lemma 2.1. Let $T$ be an invertible linear operator on $\mathbb{C}^n$ whose matrix representation with respect to the standard basis is an $n \times n$ matrix with real entries. Then $T$ maps $M(A) \cup \mathbb{R}^n$ onto $M(TAT^{-1}) \cup \mathbb{R}^n$.

We now state the result by Weinstock [9] which was already referred repeatedly in Section 1. This theorem will play a vital role in the proof of Theorem 1.2.

Result 2.2 (Weinstock). Suppose $P_1$ and $P_2$ are two totally-real subspaces of $\mathbb{C}^n$ of maximal dimension intersecting only at $0 \in \mathbb{C}^n$. Denote the normal form for this pair as:

$$P_1 : \mathbb{R}^n,$$

$$P_2 : (A + iI)\mathbb{R}^n.$$
$P_1 \cup P_2$ is locally polynomially convex at the origin if and only if $A$ has no purely imaginary eigenvalue of modulus greater than 1.

Next, we state two lemmas from the literature which will be used repeatedly in the proofs of our theorems. The first one — due to Kallin [4] — deals with the polynomial convexity of the union of two polynomially convex sets. The second one — which is a version of a lemma from Stolzenberg's paper [5, Lemma 5] (also see Stout's book [6]) — gives a criterion for polynomial convexity of a compact set $K$ in terms of the existence of a function that belongs to the uniform algebra on $K$ generated by the polynomials and satisfies some special property.

**Lemma 2.3** (Kallin). Let $K_1$ and $K_2$ be two compact polynomially convex subsets in $\mathbb{C}^n$. Suppose $L_1$ and $L_2$ are two compact polynomially convex subsets of $\mathbb{C}$ with $L_1 \cap L_2 = \{0\}$. Suppose further that there exists a holomorphic polynomial $P$ satisfying the following conditions:

(i) $P(K_1) \subset L_1$ and $P(K_2) \subset L_2$; and

(ii) $P^{-1}\{0\} \cap (K_1 \cup K_2)$ is polynomially convex.

Then $K_1 \cup K_2$ is polynomially convex.

Given a compact $X \subset \mathbb{C}^n$, $\mathcal{P}(X)$ will denote the uniform algebra on $X$ generated by holomorphic polynomials.

**Lemma 2.4** (Stolzenberg). Let $X \subset \mathbb{C}^n$ be compact. Assume $\mathcal{P}(X)$ contains a function $f$ such that $f(X)$ has empty interior and $\mathbb{C} \setminus f(X)$ is connected. Then, $X$ is polynomially convex if and only if $f^{-1}\{w\} \cap X$ is polynomially convex for each $w \in f(X)$.

The version of Lemma 2.4 that we stated above originates in a remark following [6, Theorem 1.2.16] in Stout’s book.

The next result, due to Florentino [2], concerns the simultaneous triangularizability of a family of matrices over the field of real numbers. Though the following theorem is actually valid over any integral domain, we shall state it over the field of real numbers, which is the field relevant to our proof.

**Result 2.5** (Florentino, [2]). Let $A = (A_1, \ldots, A_n)$, $A_j \in \mathbb{R}^{2 \times 2}$, have reduced length $l \leq n$. Let $A' \subset A$ be a maximal reduction and, without loss of generality, let $A' = (A_1, \ldots, A_l)$. Then:

(i) If $l = 3$, $A$ is triangularizable if and only if each $A_k$ is triangularizable, $\det[A_j, A_k] = 0$, $j, k \leq l$, and $Tr(ABC - CBA) = 0$ for all $A, B, C \in A'$.

(ii) If $l \neq 3$, $A$ is triangularizable if and only if each $A_k$ is triangularizable and $\det[A_j, A_k] = 0$, $j, k \leq l$.

We must clarify that, in the above result, the expression “$A_k$ is triangularizable” means that $A_k$ is similar to a real upper triangular matrix by conjugation with a real invertible matrix. Likewise, the expression “$A$ is triangularizable” means that each member of $A$ is triangularizable by conjugation by the same matrix. We refer to the reader to Definition 1.11 for the definitions of reduction and reduced length.

We can now appreciate the complexity of Condition (2) in Theorem 1.2; the latter half of this condition is inherited from part (i) of Result 2.3. The case $l = 3$ is genuinely exceptional. Florentino shows in [2, Example 2.11] that the condition $tr(A_1A_2A_3 - A_3A_2A_1) = 0$ cannot, in general, be dropped.

Let us now state a result by Thomas [7] which will play the key role in our argument in the proof of the optimality-part of Theorem 1.4.
**Result 2.6** (Thomas, [7]). There exist three pairwise transversal totally-real planes $P_j$, $1 \leq j \leq 3$, in $\mathbb{C}^2$ passing through origin such that:

(i) $P_j \cup P_k$ is locally polynomially convex at $0 \in \mathbb{C}^2$ for all $j \neq k$;

(ii) $((P_1 \cup P_2 \cup P_3) \cap \mathbb{B}(0;1))$ contains an open ball in $\mathbb{C}^2$.

Note that, in the statement (ii) of the above theorem, the radius of the closed ball has no significant role. Since the set $P_1 \cup P_2 \cup P_3$ is invariant under all real dilations, (ii) would hold true with any $\mathbb{B}(0;r)$, $r > 0$, replacing the unit ball. We will see some more discussions on these planes [7] in Section 4.

We now prove some lemmas that will be used in the proofs of the theorems. All the lemmas are linear algebraic in nature. We also prove a proposition — an identity showing conditions of Theorem 1.4 are invariant under conjugation — at the end of this section.

**Lemma 2.7.** Let $A \in \mathbb{R}^{2 \times 2}$ and suppose $A$ has non-real eigenvalues $p \pm iq$. Then, there exists $S \in GL(2, \mathbb{R})$ such that

$$S^{-1}AS = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}.$$  

*Proof.* Let $v$ be an eigenvector of $A$ corresponding to the eigenvalue $p + iq$. Hence, $\overline{v}$ is an eigenvector of $A$ corresponding to the eigenvalue $p - iq$. Since $q \neq 0$, the set $\{v, \overline{v}\}$ is linearly independent over $\mathbb{C}$. Now, writing $v = v_1 + iv_2$, where $v_1, v_2 \in \mathbb{R}^2$, we have the following:

$$Av = (p + iq)v \implies Av_1 = pv_1 - qv_2 \text{ and } Av_2 = qv_1 + pv_2.$$  \hspace{1cm} (2.1)

Since $\{v, \overline{v}\}$ is linearly independent over $\mathbb{C}$, $\{v_1, v_2\}$ is also linearly independent over $\mathbb{C}$. By (2.1), the representation of $A$ with respect to the basis $\{v_1, v_2\}$ is $\begin{pmatrix} p & -q \\ q & p \end{pmatrix}$. Hence, the basis-change matrix that transform $A$ to its representation with respect to the basis $\{v_1, v_2\}$ is the desired real invertible matrix $S$.  

**Lemma 2.8.** Let $A_1, \ldots, A_N \in \mathbb{R}^{2 \times 2}$ and assume that $\det[A_j, A_k] = 0 \forall j \neq k$. Assume also that $A_1$ has non-real eigenvalues. Then, there exists $S \in GL(2, \mathbb{R})$ such that

$$S^{-1}A_jS = \begin{pmatrix} s_j & -t_j \\ t_j & s_j \end{pmatrix}, \quad j = 1, \ldots, N,$$

where $s_j, t_j \in \mathbb{R}$, $j = 1, \ldots, N$.

*Proof.* Since $A_1$ has non-real eigenvalues, by Lemma 2.7, there exists $S \in GL(2, \mathbb{R})$ such that

$$S^{-1}A_1S = \begin{pmatrix} p & -q \\ q & p \end{pmatrix},$$

where $p \pm iq$ ($q \neq 0$) are the eigenvalues of $A_1$. Suppose

$$S^{-1}A_jS = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \text{ for some } j : 1 \leq j \leq N.$$  

Since the determinant remains invariant under conjugation by an invertible matrix, by hypothesis:

$$\det[S^{-1}A_1S, S^{-1}A_jS] = 0.$$  \hspace{1cm} (2.2)

A simple calculation gives us

$$[S^{-1}A_1S, S^{-1}A_jS] = \begin{pmatrix} -q(b + c) & q(a - d) \\ q(a - d) & q(b + c) \end{pmatrix}.$$
Hence, from (2.2) and the fact that \( q \neq 0 \), we infer that \( b = -c, \ a = d \). Thus, under the conjugation by \( S \), we have

\[
S^{-1}A_jS = \begin{pmatrix} s_j & -t_j \\ t_j & s_j \end{pmatrix},
\]

where \( s_j, t_j \in \mathbb{R}, \ j = 1, \ldots, N. \)

\( \square \)

**Lemma 2.9.** Let \( A_1, A_2 \in \mathbb{R}^{2 \times 2} \) be two matrices such that \( \det[A_1, A_2] = 0 \) and \( A_1 - A_2 \) is invertible. Suppose \( A_1 \) has non-real eigenvalues. Then

- \( A_2 \) either has non-real eigenvalues or is a scalar matrix; and
- \( B := (A_1A_2 + I)(A_1 - A_2)^{-1} \) has complex conjugate eigenvalues.

**Proof.** Since \( A_1 \) has non-real eigenvalues and \( \det[A_1, A_2] = 0 \), appealing Lemma 2.8 we see that there exists \( S \in GL(2, \mathbb{R}) \) such that

\[
S^{-1}A_jS = \begin{pmatrix} s_j & -t_j \\ t_j & s_j \end{pmatrix}, \ j = 1, 2.
\]

This shows that \( A_2 \) either has non-real eigenvalues or is a scalar matrix. Note that conjugating by the matrix \( \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \), we see that

\[
A_j \sim \begin{pmatrix} \overline{\lambda_j} & 0 \\ 0 & \lambda_j \end{pmatrix}, \ j = 1, 2.
\]

Hence, \( (A_1 - A_2) \) and \( (A_1A_2 + I) \) can be conjugated by \( S \) to diagonal matrices with diagonal entries \( (\lambda_1 - \lambda_2, \lambda_1 - \lambda_2) \) and \( (\lambda_1\lambda_2 + 1, \lambda_1\lambda_2 + 1) \) respectively. Hence, by examining \( S^{-1}BS \), we see that the matrix \( B = (A_1A_2 + I)(A_1 - A_2)^{-1} \) has complex conjugate eigenvalues. \( \square \)

**Lemma 2.10.** Let \( A_1, A_2 \in \mathbb{R}^{2 \times 2} \). Suppose \( A_1 \) has two distinct eigenvalues. Then \( \exists T \in GL(2, \mathbb{R}) \) such that:

(i) If \( A_1 \) has real eigenvalues and \( \det[A_1, A_2] \neq 0 \), then

\[
TA_1T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad TA_2T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}
\]

for \( \lambda_j, s_{2j}, t_2 \in \mathbb{R}, \ j = 1, 2, \)

(ii) If \( A_1 \) has non-real eigenvalues, then

\[
TA_1T^{-1} = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix} \quad \text{and} \quad TA_2T^{-1} = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}
\]

for \( s_j, s_{2j}, t_j \in \mathbb{R}, \ j = 1, 2. \)

**Proof.** (i) Since \( A_1 \) has two distinct real eigenvalues, \( A_1 \) is diagonalizable over \( \mathbb{R} \), i.e. there exists a \( S \in GL(2, \mathbb{R}) \) such that

\[
SA_1S^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \ \lambda_1 \neq \lambda_2 \in \mathbb{R}.
\]

Hence, without loss of generality, we can assume that \( A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \). Suppose \( A_2 = \begin{pmatrix} s_{21} & t_1 \\ t_2 & s_{22} \end{pmatrix}, \ t_j, s_{2j} \in \mathbb{R} \) for \( j = 1, 2. \) Observe that, in view of Result 2.5
$t_1 t_2 = 0 \iff \det [A_1, A_2] = 0$. Hence neither $t_1$ nor $t_2$ is zero. We have, since $A_1$ commutes with all diagonal matrices, that

$$GA_1 G^{-1} = A_1 \text{ for } G = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, \text{ where } g_1 g_2 \neq 0.$$ 

We also have, after conjugating $A_2$ by $G$, that

$$GA_2 G^{-1} = \begin{pmatrix} s_{21} & t_1 g_1 / g_2 \\ t_2 g_2 / g_1 & s_{22} \end{pmatrix}. \quad (2.3)$$

Now observe that if $t_1$ and $t_2$ are of same sign, then there exist $g_1, g_2 \in \mathbb{R} \setminus \{0\}$ such that

$$t_1 g_1^2 = t_2 g_2^2.$$ 

Therefore, in this case, $\tilde{\ell}_2 := t_1 g_1 / g_2 = t_2 g_2 / g_1$, and we conclude from (2.3) that

$$GA_2 G^{-1} = \begin{pmatrix} s_{21} & \tilde{\ell}_2 \\ \tilde{\ell}_2 & s_{22} \end{pmatrix}.$$ 

We also observe that, if $t_1$ and $t_2$ are of different sign, then there exist $g_1, g_2 \in \mathbb{R} \setminus \{0\}$ such that

$$t_1 g_1^2 + t_2 g_2^2 = 0.$$ 

Therefore, in this case, $\tilde{\ell}_2 := -t_1 g_1 / g_2 = t_2 g_2 / g_1$, and we conclude from (2.3) that

$$GA_2 G^{-1} = \begin{pmatrix} s_{21} & -\tilde{\ell}_2 \\ \tilde{\ell}_2 & s_{22} \end{pmatrix}.$$ 

(ii) Since $A_1$ has non-real eigenvalues, say $s_1 \pm i t_1$, by Lemma 2.7 there exists $S \in GL(2, \mathbb{R})$ such that $S A_1 S^{-1} = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix}, s_1, t_1 \in \mathbb{R}$. Hence, without loss of generality, we may assume that $A_1 = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix}$. Let

$$A_2 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}, m_j \in \mathbb{R}, j = 1, 2, 3, 4,$n

with $m_2 + m_3 \neq 0$; otherwise, there is nothing to prove.

We observe that $A_1$ commutes with all the matrices having the same structure as that of itself. Let $G := \begin{pmatrix} g_1 & -g_2 \\ g_2 & g_1 \end{pmatrix}$ with $g_1, g_2 \in \mathbb{R}, g_1^2 + g_2^2 = 1$ and $g_1 g_2 \neq 0$. Therefore,

$$GA_1 G^{-1} = A_1,$n

and

$$GA_2 G^{-1} = \begin{pmatrix} g_1 & -g_2 \\ g_2 & g_1 \end{pmatrix} \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ -g_2 & g_1 \end{pmatrix}$$

$$= \begin{pmatrix} g_1 m_1 - g_1 g_2 (m_2 + m_3) + g_2^2 m_4 & g_2 m_2 + g_1 g_2 (m_1 - m_4) - g_2^2 m_3 \\ g_2^2 m_3 + g_1 g_2 (m_1 - m_4) - g_2^2 m_2 & g_2^2 m_1 + g_1 g_2 (m_2 + m_3) + g_2^2 m_4 \end{pmatrix}$$

$$= \begin{pmatrix} f_1(g_1, g_2) & f_2(g_1, g_2) \\ f_3(g_1, g_2) & f_4(g_1, g_2) \end{pmatrix}.$$
Let us now look closely at the quadratic equation \( f_2(g_1, g_2) + f_3(g_1, g_2) = 0 \) in \( g_1, g_2 \). This gives:

\[
(m_2 + m_3)(g_1^2 - g_2^2) + 2(m_1 - m_4)g_1g_2 = 0
\]

This implies, since \( m_2 + m_3 \neq 0 \) and \( g_1g_2 \neq 0 \),

\[
\frac{g_1}{g_2} - \frac{g_2}{g_1} + 2\frac{m_1 - m_4}{m_2 + m_3} = 0
\]

Looking at the above as a quadratic in \( \frac{g_1}{g_2} =: \mu \), we have

\[
\mu^2 + 2\frac{m_1 - m_4}{m_2 + m_3}\mu - 1 = 0.
\]

The discriminant of the above quadratic is

\[
4 \left( \frac{m_1 - m_4}{m_2 + m_3} \right)^2 + 4,
\]

which is greater than zero for all \( m_j \in \mathbb{R}, j = 1, 2, 3, 4 \). Hence (2.4) has a real root, say \( \mu_1 \). Therefore, the conjugation with a common matrix \( S \in GL(2, \mathbb{R}) \).

3. The proof of Theorem 1.2

We remind the reader that, in this section, the word “triangularizable” will refer to triangularization by a real matrix.

*Proof of Theorem 1.2.* (a) Since all \( A_j, j = 1, \ldots, N \), have real eigenvalues, each \( A_j, j = 1, \ldots, N \), is triangularizable. We now appeal to Result 2.5 to get:

\[
A_j \sim \begin{pmatrix} \mu_j & a_j \\ 0 & \nu_j \end{pmatrix}, \quad \mu_j, \nu_j, a_j \in \mathbb{R}, j = 1, \ldots, N,
\]

by the conjugation with a common matrix \( S \in GL(2, \mathbb{R}) \). In view of the Lemma 2.1, we may assume that

\[
A_j = \begin{pmatrix} \mu_j & a_j \\ 0 & \nu_j \end{pmatrix}, \quad \mu_j, \nu_j, a_j \in \mathbb{R}, j = 1, \ldots, N.
\]

From this point, we will — for simplicity of notation — refer to each \( S(P_j) \) as \( P_j, j = 0, \ldots, N \). Hence, \( P_j = M(A_j) \) for the preceding choice of \( A_j, j = 1, \ldots, N \). We have

\[
M(A_j) = \{(\mu_j + i)x + a_jy, (\nu_j + i)y) : x, y \in \mathbb{R} \}, \text{ for all } j = 1, \ldots, N.
\]

Let \( K := (\cup_{j=0}^{N} P_j) \cap \mathbb{B}(0,1) \) and \( K_j := P_j \cap K, j = 0, \ldots, N \). Since, for every \( j = 0, \ldots, N \), \( K_j \) is a compact subset of a totally real plane, \( K_j \) is polynomially convex. We shall use Lemma 2.4 to show the polynomial convexity of \( K = \cup_{j=1}^{N} K_j \). Consider the polynomial

\[
F(z, w) = w.
\]

Clearly, there exists a real number \( R > 0 \) such that:

\[
F(K) \subset (\cup_{j=1}^{N}((\nu_j + i)y : y \in \mathbb{R}, |y| \leq R)) \cup \{y : y \in \mathbb{R}, |y| \leq R\}.
\]
Since each of the members in the union of the right hand side of (3.1) is a bounded real line segment in $\mathbb{C}$, $F(K)$ has no interior and $\mathbb{C} \setminus F(K)$ is connected. We shall now calculate $F^{-1}\{\zeta\} \cap K$, where $\zeta \in F(K)$. If $\zeta \neq 0$, then $\zeta \in F(K_{j,0})$ for some $j, k \leq N$. Hence, we get

$$F^{-1}\{\zeta\} \cap K = \begin{cases} \{((\mu_j + i)x/a_{j,0}\zeta/(\nu_j + i), \zeta) : x \in \mathbb{R}\} \cap K, & \text{if } 1 \leq j, k \leq N, \\ \{(x, \zeta) \in \mathbb{C}^2 : x \in \mathbb{R}\} \cap K, & \text{if } j = 0. \end{cases}$$

(3.2)

We remark that $\zeta \in F(K_{j,0})$ implies that $\zeta/(\nu_j + i) \in \mathbb{R}$. Also, $\nu_j + i \neq 0$ because, in the present case, $\nu_j \in \mathbb{R}$. If $\zeta = 0$, then we have

$$F^{-1}\{0\} \cap K = \left[\bigcup_{j=1}^{N}\{((\mu_j + i)x, 0) \in \mathbb{C}^2 : x \in \mathbb{R}\}\right] \cup \{(x, 0) \in \mathbb{C}^2 : x \in \mathbb{R}\} \cap K$$

(3.3)

In view of (3.2), we have the set $F^{-1}\{\zeta\} \cap K$ is a single line segment in $\mathbb{C}^2$ when $\zeta \neq 0$. Hence,

$$F^{-1}\{\zeta\} \cap K = F^{-1}\{\zeta\} \cap K, \text{ for } \zeta \neq 0.$$ 

(3.4)

From (3.3), we see that $F^{-1}\{0\} \cap K$ is a union of line segments in $\mathbb{C} \times \{0\}$ intersecting only at the origin. Hence,

$$F^{-1}\{0\} \cap K = F^{-1}\{0\} \cap K.$$ 

(3.5)

Thus, in view of (3.3) and (3.5), we can appeal to Lemma 2.4 to get the polynomial convexity of $K$.

(b) Without loss of generality, assume that $A_1$ has non-real complex eigenvalues. From Lemma 2.8 it follows that there exists $S \in GL(2, \mathbb{R})$ such that

$$S^{-1}A_jS = \begin{pmatrix} s_j & -t_j \\ t_j & s_j \end{pmatrix}, s_j, t_j \in \mathbb{R}, j = 1, \ldots, N.$$

Again, by Lemma 2.1 there is no loss of generality to assume

$$A_j = \begin{pmatrix} s_j & -t_j \\ t_j & s_j \end{pmatrix}, s_j, t_j \in \mathbb{R}, j = 1, \ldots, N.$$

We will relabel $S(P_j)$ as $P_j$, $j = 0, \ldots, N$, exactly as in (a). Then

$$M(A_j) = \{(s_j + i)x - t_jy, t_jx + (s_j + i)y : x, y \in \mathbb{R}\}, j = 1, \ldots, N.$$

Note that, in this case,

$$\det(A_j - A_k) = (s_j - s_k)^2 + (t_j - t_k)^2 > 0,$$

(3.6)

and consequently, $M(A_j) \cap M(A_k) = \{0\}$ for $j \neq k$.

Suppose there does not exist any constant $c > 0$ such that

$$V_j = cV_k, \text{ for some } j \neq k, 0 \leq j, k \leq N.$$

We shall show that $K := (\bigcup_{j=0}^{N}P_j) \cap \overline{\mathbb{B}(0,1)} = \bigcup_{j=0}^{N}K_j$ is polynomially convex, where

$$K_j := P_j \cap K, 0 \leq j \leq N. \text{ Each } K_j \text{ is necessarily polynomially convex. In this case we shall use Kallin’s lemma (i.e Lemma 2.3) to show that } K \text{ is polynomially convex. Let us consider the polynomial }$$

$$F(z, w) := z^2 + w^2.$$

We shall now look at the image of $K_j$ under this map:

$$F(K_0) \subset \{z \in \mathbb{C} : z \geq 0\} = \{\alpha V_0 : \alpha \geq 0\},$$

(3.7)
\[ F((s_j + i)x - t_j y, t_j x + (s_j + i)y) = ((s_j + i)x - t_j y)^2 + (t_j x + (s_j + i)y)^2 \]
\[ = (s_j^2 + t_j^2 - 1)(x^2 + y^2) + 2i s_j (x^2 + y^2). \]
Hence,
\[ F(K_j) \subset \{ \beta V_j \in \mathbb{R}^2 : \beta \geq 0 \}, \quad j = 1, \ldots, N. \]  
(3.8)
Since there does not exist any constant \( c > 0 \) such that \( V_j = cV_k \), for some \( j \neq k \), \( 0 \leq j, k \leq N \), we have by equations (3.7) and (3.8):
\[ F(K_l) \cap F(K_m) = \{ 0 \} \text{ for } l \neq m. \]  
(3.9)
Furthermore,
\[ F^{-1}\{0\} \cap K_j = \{ 0 \} \text{ for all } j = 0, \ldots, N. \]  
(3.10)
From (3.9), (3.10), it follows that:
- for each \( j \), \( F(K_j) \) lies in different line segment of \( C \), each of which has an end at \( 0 \in \mathbb{C}^2 \); and
- \( F^{-1}\{0\} \cap K = \{ 0 \} \), which is polynomially convex.

Since each \( F(K_j) \), \( j = 0, \ldots, N \), is polynomially convex, the above shows that all the conditions of Kallin’s lemma are satisfied. Hence, \( K = \cup_{j=1}^N K_j \) is polynomially convex; i.e. \( \cup_{j=0}^N P_j \) is locally polynomially convex at the origin.

We will prove the converse in its contrapositive formulation. Let there exist two numbers \( l, m \) such that \( l \neq m \) and for some constant \( c > 0 \)
\[ V_l = cV_m. \]
This implies
\[ \text{det}(A_l + \mathbb{I}) = c \text{det}(A_m + \mathbb{I}). \]  
(3.11)
Without loss of generality, let us assume that \( l = 2 \) and \( m = 1 \). From (3.11), we get
\[ \text{det}[(A_1 - \mathbb{I})(A_2 + \mathbb{I})] = c \text{det}(A_1^2 + \mathbb{I}) > 0. \]  
(3.12)
Note that if we view \( (A_1 - \mathbb{I}) \) as a \( \mathbb{C} \)-linear transformation on \( \mathbb{C}^2 \), then
\[ (A_1 - \mathbb{I})(M(A_1)) = (A_1^2 + \mathbb{I}) \mathbb{R}^2 = \mathbb{R}^2 \]
\[ (A_1 - \mathbb{I})(M(A_2)) = [A_1 A_2 + \mathbb{I} + i(A_1 - A_2)] \mathbb{R}^2 \]
\[ = [(A_1 A_2 + \mathbb{I})(A_1 - A_2)^{-1} + i\mathbb{I}] \mathbb{R}^2 \equiv (B + \mathbb{I}) \mathbb{R}^2. \]
The first equality follows from the fact that \( (A_1^2 + \mathbb{I}) \) is invertible (because of (3.12) and the fact that \( M(A_1) \) is totally-real) and the invertibility of \( (A_1 - A_2) \) follows from (3.6). Now, from Lemma 2.9 we get that \( B \) has complex conjugate eigenvalues. We now write \( \sigma(B) = \{ \mu, \overline{\mu} \} \), and \( \mu = s + it \), \( s, t \in \mathbb{R} \). From (3.6) and (3.12), we get:
\[ \text{det}(B + \mathbb{I}) = (s^2 + t^2 - 1) + 2is > 0. \]
This implies \( s = 0 \) and \( |t| > 1 \). From an auxiliary result of Weinstock [9, Theorem 2], it follows that \( \mathbb{R}^2 \cup M(B) = (A_1 - \mathbb{I})(M(A_1) \cup M(A_2)) \) is not locally polynomially convex at the origin. Equivalently, \( \cup_{j=0}^N P_j \) is not locally polynomially convex at the origin.

Let us make the following remark.

**Remark 3.1.** We observe that under Condition (2) of the above theorem, local polynomial convexity of pairwise unions of \( P_0, \ldots, P_N \) at the origin imples local polynomial convexity of \( \cup_{j=0}^N P_j \) at \( 0 \in \mathbb{C}^2 \).
4. Proof of the Theorem 1.4

Before proceeding to the proof of Theorem 1.4 we shall state some preliminaries needed in the proof of optimality part of Theorem 1.4. Recall Result 2.6 which gives a triple of totally-real planes whose union is not locally polynomially convex at \(0 \in \mathbb{C}^2\) although each of the pairwise unions is locally polynomially convex at the origin. In the proof of Result 2.6 Thomas [7] demonstrates a family of triples \((P^0_\varepsilon, P^1_\varepsilon, P^2_\varepsilon)\), where \(\varepsilon\) is a complex number close to 0, having the above mentioned property. The planes in the above triples are graphs in \(\mathbb{C}^2\) with the following equations:

\[
P^0_\varepsilon : w = \bar{z}
\]

\[
P^1_\varepsilon : w = -\frac{\sqrt{3}(\sqrt{3} - i)}{2\varepsilon}z + \frac{1 + \sqrt{3}i}{2},
\]

\[
P^2_\varepsilon : w = -\frac{\sqrt{3}(\sqrt{3} + i)}{2\varepsilon}z - \frac{1 + \sqrt{3}i}{2}\bar{z}.
\]

In the proof of Theorem 1.4 we shall restrict our attention to the above triples when \(\varepsilon \in \mathbb{R} \setminus \{0\}\). We now apply a \(\mathbb{C}\)-linear change of coordinate \((z, w) \mapsto (z + w, i(w - z))\) from \(\mathbb{C}^2\) to \(\mathbb{C}^2\). In the new coordinate, we have

\[
P^0_\varepsilon : \mathbb{R}^2
\]

\[
P^j_\varepsilon : (A_j^\varepsilon + iI)\mathbb{R}^2, j = 1, 2,
\]

where \(A_j^\varepsilon \in \mathbb{R}^{2 \times 2}\) have the following form:

\[
A_1^\varepsilon = \begin{pmatrix}
\frac{\varepsilon}{\sqrt{3}(1+\varepsilon)} & -\frac{1+\varepsilon}{\sqrt{3}(1-\varepsilon)} \\
-\frac{1}{1-\varepsilon} & \frac{\varepsilon}{\sqrt{3}(1+\varepsilon)}
\end{pmatrix}
\]

and

\[
A_2^\varepsilon = \begin{pmatrix}
\frac{-\varepsilon}{\sqrt{3}(1-\varepsilon)} & \frac{1}{1+\varepsilon} \\
-\frac{1}{1+\varepsilon} & \frac{-\varepsilon}{\sqrt{3}(1-\varepsilon)}
\end{pmatrix}.
\]

We are now in a position to begin the proof of Theorem 1.4.

The proof of Theorem 1.4. In view of Lemma 2.10, we divide the proof of this theorem into two cases depending on the eigenvalues of \(A_1\).

Case I. When eigenvalues of \(A_1\) are real and distinct.

First, let us consider the case when \(\det[A_1, A_2] = 0\). By Lemma 2.8 \(A_1\) and \(A_2\) both have real eigenvalues, whence they are triangularizable over \(\mathbb{R}\). Hence, Theorem 1.2 applies, and from Part (a) of that theorem, we are done.

When \(\det[A_1, A_2] \neq 0\), the first assertion of (i) of our theorem follows from Part (i) of Lemma 2.10. By hypothesis, \(\det(A_1 + A_1^\top) > 0\), and there is a \(T_0 \in GL(2, \mathbb{R})\) such that \(\det(A_2(T_0) + A_2(T_0)^\top) > 0\). For simplicity of notation, for the remainder of this proof, we shall write \(A_2 := A_2(T_0)\). Hence:

\[
A_2 := T_0A_2T_0^{-1} = \begin{pmatrix}
s_{21} & t_2 \\
t_2 & s_{22}
\end{pmatrix} \text{ or } \begin{pmatrix}
s_{21} & -t_2 \\
t_2 & s_{22}
\end{pmatrix}.
\]

We shall now divide the proof into two subcases. Once again, for simplicity of notation, we shall follow the conventions of the proof of Theorem 1.2 and denote the planes \(T_0(P_j)asP_j\), \(j = 0, 1, 2\).

(a) When \(A_1 = \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{pmatrix}\) and \(A_2 = \begin{pmatrix}
s_{21} & t_2 \\
t_2 & s_{22}
\end{pmatrix}\).
Let \( K_j = P_j \cap \overline{B(0; 1)} \), \( j = 0, 1, 2 \). Therefore, we have

\[
\begin{align*}
K_1 & \subset \{((\lambda_1 + i)x, (\lambda_2 + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}, \\
K_2 & \subset \{((s_{21} + i)x + t_{2y}, t_2x + (s_{22} + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}.
\end{align*}
\]

We shall, in view of the condition that pairwise unions are locally polynomially convex at \( 0 \in \mathbb{C}^2 \), use Kallin’s lemma to show the polynomial convexity of \( K_0 \cup K \), where \( K := K_1 \cup K_2 \). For that, consider the polynomial

\[
F(z, w) = z^2 + w^2.
\]

Clearly,

\[
F(K_0) \subset \{z \in \mathbb{C} : z \geq 0\}. \quad (4.2)
\]

For \((z, w) \in K_1\), we have

\[
F(z, w) = F((\lambda_1 + i)x, (\lambda_2 + i)y) = (\lambda_1^2 - 1)x^2 + (\lambda_2^2 - 1)y^2 + 2i(\lambda_1 x^2 + \lambda_2 y^2), \quad (4.3)
\]

and, for \((z, w) \in K_2\),

\[
F(z, w) = F((s_{21} + i)x + t_{2y}, t_2x + (s_{22} + i)y) = (s_{21}^2 + t_2^2 - 1)x^2 + (s_{22}^2 + t_2^2 - 1)y^2 + 2(s_{21} + s_{22})t_2xy + 2i(s_{21}x^2 + s_{22}y^2 + 2t_2xy). \quad (4.4)
\]

By hypothesis, we have

\[
\det(A_1 + A_1^T) > 0 \iff \lambda_1 \lambda_2 > 0, \quad (4.5)
\]

\[
\det(A_2 + A_2^T) > 0 \iff s_{21}s_{22} > t_2^2 > 0. \quad (4.6)
\]

Hence, in view of \((4.5)\) and \((4.6)\), equations \((4.3)\) and \((4.4)\) give us

\[
F(K) \subset (\mathbb{C} \setminus \mathbb{R}) \cup \{0\}.
\]

Hence, we get

\[
\overline{F(K_0)} \cap \overline{F(K)} = \{0\}, \quad (4.7)
\]

and

\[
F^{-1}\{0\} \cap (K_0 \cup K) = \{0\}. \quad (4.8)
\]

Therefore, from \((4.2)\), \((4.7)\) and \((4.8)\), all the conditions of Lemma 2.3 are satisfied. Hence, \( K_0 \cup K \) is polynomially convex.

(b) When \( A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \) and \( A_2 = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix} \).

As above, let \( K_j = P_j \cap \overline{B(0; 1)} \), \( j = 0, 1, 2 \), and \( K = K_1 \cup K_2 \). We also get \( K_1 \) to be the same as that in subcase (a) and

\[
K_2 \subset \{((s_{21} + i)x - t_{2y}, t_2x + (s_{22} + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}.
\]

Again, we consider the polynomial

\[
F(z, w) = z^2 + w^2.
\]

When \((z, w) \in K_1\), \( F(z, w) \) is as in equation \((4.3)\), and, for \((z, w) \in K_2\),

\[
F(z, w) = F((s_{21} + i)x - t_{2y}, t_2x + (s_{22} + i)y) = (s_{21}^2 + t_2^2 - 1)x^2 + (s_{22}^2 + t_2^2 - 1)y^2 + 2(s_{22} - s_{21})t_2xy + 2i(s_{21}x^2 + s_{22}y^2). \quad (4.9)
\]

The inequality in \((4.5)\) remains the same but \((4.6)\) is replaced by:

\[
det(A_2 + A_2^T) > 0 \iff s_{21}s_{22} > 0. \quad (4.10)
\]
In view of equations \((4.5)\) and \((4.10)\), the expressions in \((4.2)\), \((4.3)\) and \((4.9)\) give
\[
F(K_0) \cap F(K) = \{0\},
\]
and
\[
F^{-1}\{0\} \cap (K_0 \cup K) = \{0\}.
\]
Therefore, all the conditions of Kallin’s lemma are satisfied. Hence \(K_0 \cup K\) is polynomially convex.

**Case II. When \(A_1\) has non-real eigenvalues.**

Since \(\sigma(A_1) \subset \mathbb{C} \setminus \mathbb{R}\), the first part of (ii) of our theorem follows from Lemma 2.10 Part (ii). By hypothesis, \(\det(A_1 + A_1^T) > 0\) and \(\exists T_0 \in GL(2, \mathbb{R})\) such that \(\det(A_2(T_0) + A_2(T_0)^T) > 0\), where
\[
A_1 = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix} \quad \text{and} \quad A_2 := A_2(T_0) = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}.
\]

As in the previous case, let \(K_j = P_j \cap \mathbb{B}(0;1), \ j = 0, 1, 2\). We have
\[
K_0 \subset \{(x, y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\},
\]
\[
K_1 \subset \{((s_1 + i)x - t_1y, t_1x + (s_1 + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\},
\]
\[
K_2 \subset \{((s_{21} + i)x - t_2y, t_2x + (s_{22} + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}.
\]

We shall again use Kallin’s lemma to show the polynomial convexity of \(K_0 \cup K_1 \cup K_2\). Consider the polynomial
\[
F(z, w) = z^2 + w^2.
\]
When \((z, w) \in K_0\), \(F(z, w)\) is as in \((4.2)\). For \((z, w) \in K_1\), we have
\[
F(z, w) = F((s_1 + i)x - t_1y, t_1x + (s_1 + i)y)
= (s_1^2 + t_1^2 - 1)(x^2 + y^2) + 2is_1(x^2 + y^2),
\]
and for \((z, w) \in K_2\), \(F(z, w)\) is as in equation \((4.9)\). Let \(K = K_1 \cup K_2\). As before, from homogeneity of the totally-real planes and the hypothesis that the pairwise unions of the given totally-real planes are locally polynomially convex at the origin, \(K\) is polynomially convex. By hypotheses, we get that
\[
\det(A_1 + A_1^T) > 0 \implies s_1^2 > 0 \tag{4.12}
\]
\[
\det(A_2 + A_2^T) > 0 \implies s_{21}s_{22} > 0. \tag{4.13}
\]

Hence, in view of \((4.2)\), \((4.11)\) and \((4.9)\), we conclude that
\[
F(K_0) \subset \{z \in \mathbb{C} : z \geq 0\}; F(K) \subset (\mathbb{C} \setminus \mathbb{R}) \cup \{0\},
\]
and
\[
F^{-1}\{0\} \cap (K_0 \cup K) = \{0\}.
\]
Therefore, all the conditions of Lemma 2.10 are satisfied. Hence, \(K_0 \cup K\) is polynomially convex.

It is now time to show that our conditions are optimal. We examine the one-parameter family of triples \((P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon)\), where \(P_0^\varepsilon = \mathbb{R}^2 \forall \varepsilon\) and \(P_j^\varepsilon\) are as determined by the matrices \(A_j^\varepsilon, \ j = 1, 2\), given in \((4.1)\). From the discussion preceding \((4.1)\) we already know that pairwise unions of \(P_0^\varepsilon, P_1^\varepsilon, P_2^\varepsilon\) are locally polynomially convex at \(0 \in \mathbb{C}^2\) and that \(P_0^\varepsilon \cup P_1^\varepsilon \cup P_2^\varepsilon \forall \varepsilon \in \mathbb{R} \setminus \{0\}\), for \(\varepsilon\) sufficiently small, is not locally
By Claim 1, we have

\[ \lambda \] in the statement of Theorem 1.4 and Result 2.6, we already know that \((A^e_1, A^e_2) \) in (i) proof of Part (ii) Claim 1.

Proof of the Claim 1.

Suppose \( A^e \). Clearly, for \( \varepsilon \): 0 < \( \varepsilon \) \( \varepsilon \) we have polynomially convex at \( 0 \). Hence, \((A^e_1, A^e_2) \in \Omega \). Now, from (ii) (read in the contrapositive) in the statement of Theorem 1.4 and Result 2.6 we already know that \((A^e_1, A^e_2) \notin \Omega^* \) \( \forall \varepsilon \): 0 < \( \varepsilon \) \( \varepsilon \). Therefore, \((A^e_1, A^e_2) \in \Omega \setminus \Omega^* \forall \varepsilon \): 0 < \( \varepsilon \) \( \varepsilon \).

Now observe that:

\[
\lim_{\varepsilon \to 0} A^e_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : A^0_j, j = 1, 2.
\]

In the notation of the statement of Theorem 1.4 \( A^0_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \). Define

\[
S := \{ T \in GL(2, \mathbb{R}) : TA^0_1T^{-1} = A^0_1 \}.
\]

Clearly, \( I \in S \) and, in our notation:

\[
det(A^0_2(I) + A^0_2(I)^T) = det(A^0_2 + (A^0_2)^T) = 0.
\]

Thus, appealing to the inequalities in Part (ii) of our theorem that define \( \Omega^* \), we get \((A^0_1, A^0_2) \in \partial \Omega^* \).

5. Proof of the Theorem 1.3

Proof of Theorem 1.3 (i) First, we shall prove three claims. From these claims, the proof of Part (i) Theorem 1.3 will follow by appealing to Theorem 1.4.

Claim 1. If \( det[A_1, A_2] > 0 \), then \( A_j \) cannot have non-real eigenvalues.

Proof of the Claim 1. Suppose \( A_1 \) has non-real eigenvalues. Then, in view of Lemma 2.10 there exists a \( T \in GL(2, \mathbb{R}) \) such that

\[
TA_1T^{-1} = \begin{pmatrix} s_1 & -t_1 \\ t_1 & s_1 \end{pmatrix} \quad \text{and} \quad TA_2T^{-1} = \begin{pmatrix} s_2 & t_2 \\ -t_2 & s_2 \end{pmatrix}.
\]

Now, by a simple computation, we see that

\[
det[TA_1T^{-1}, TA_2T^{-1}] = -t_1^2(s_{22} - s_{21})^2 \leq 0,
\]

which is a contradiction to the fact that \( det[A_1, A_2] > 0 \). Hence, \( A_j \) cannot have non-real eigenvalues.

Claim 2. If \( det[A_1, A_2] > 0 \), then each \( A_j \) has distinct eigenvalues.

Proof of Claim 2. Suppose \( A_j \) does not have distinct eigenvalues. Let \( \sigma(A_j) = \{ \lambda_j \} \). By Claim 1, we have \( \lambda_1 \in \mathbb{R} \). Hence, there exists \( T \in GL(2, \mathbb{R}) \) such that

\[
TA_1T^{-1} = \begin{pmatrix} \lambda_1 & \mu \\ 0 & \lambda_1 \end{pmatrix}.
\]
Let us write $T A_2 T^{-1} = T A_2 T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, again, by a simple computation, we see that
\[ \det[T A_1 T^{-1}, T A_2 T^{-1}] = -c^2 \mu^2 \leq 0, \]
which is a contradiction.

**Claim 3.** If $\det[A_1, A_2] > 0$, then there exists a $T \in GL(2, \mathbb{R})$ such that
\[ T A_1 T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad T A_2 T^{-1} = \begin{pmatrix} s_{21} & t_1 \\ t_1 & s_{22} \end{pmatrix}. \]

**Proof of Claim 3.** In view of Claim 1 and Claim 2, we conclude that $A_1$ has distinct eigenvalues. Hence, applying Lemma 2.10, we get that there exists $T \in GL(2, \mathbb{R})$ such that
\[ T A_1 T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad T A_2 T^{-1} = \begin{pmatrix} s_{21} & t_1 \\ t_1 & s_{22} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_{21} & -t_1 \\ t_1 & s_{22} \end{pmatrix}. \]

Suppose $T A_2 T^{-1} = \begin{pmatrix} s_{21} & -t_1 \\ t_1 & s_{22} \end{pmatrix}$. Again calculating the commutator, we note that
\[ \det[T A_1 T^{-1}, T A_2 T^{-1}] = -t_2^2(\lambda_2 - \lambda_1)^2 \leq 0, \]
which is a contradiction. Hence the claim.

We now resume the proof of Theorem 1.3. In view of Claim 3, we always get a $T \in GL(2, \mathbb{R})$ such that
\[ T A_1 T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad T A_2 T^{-1} = \begin{pmatrix} s_{21} & t_1 \\ t_1 & s_{22} \end{pmatrix}. \]
We now observe that, in this case,
\[ \det(A_1 + A_1^T) = 4\det A_1, \]
\[ \det(A_2 + A_2^T) = 4\det A_2. \]
Hence, the conditions $\det A_j > 0$ imply that we can appeal Part (i) of Theorem 1.4. Therefore, $P_0 \cup P_1 \cup P_2$ is locally polynomially convex at the origin.

(ii) We shall again use Kallin’s lemma for the proof of this part. Before that, let us obtain simpler form of the matrices that will be used in the proof.

**Claim 4.** It suffices to work with the union $\mathbb{R}^2 \cup M(A_1) \cup M(A_2)$, where:
\[ A_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}. \]

**Proof of the Claim.** Since $\det A_j < 0$ for $j = 1, 2$, each $A_j$ must have real distinct eigenvalues. Hence, in view of Lemma 2.10, we can find a $T \in GL(2, \mathbb{R})$ such that
\[ T A_1 T^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad T A_2 T^{-1} = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}, \]
for $\lambda_j, s_{2j}, t_2 \in \mathbb{R}, \ j = 1, 2$. Let $A_j = T A_j T^{-1}$ for $j = 1, 2$.

Suppose $A_2 = \begin{pmatrix} s_{21} & t_2 \\ t_2 & s_{22} \end{pmatrix}$. Then, by a simple computation, we can see that
\[ \det[A_1, A_2] = \det[A_1, A_2] = (\lambda_1 - \lambda_2)^2 t_2^2 > 0. \]
This is a contradiction to the assumption that $\det[A_1, A_2] < 0$. Hence,

$$A_2 = \begin{pmatrix} s_{21} & -t_2 \\ t_2 & s_{22} \end{pmatrix}.$$ 

The claim follows from Lemma 2.1.

As before, to simplify notation, we shall denote $M(A_j)$ as $P_j$, $j = 1, 2$. As in the earlier cases, let $K_j = P_j \cap \mathbb{R}(0, 1)$, $j = 0, 1, 2$. We have

$$K_0 \subset \{(x, y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\},$$

$$K_1 \subset \{((\lambda_1 + i)x, (\lambda_2 + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\},$$

$$K_2 \subset \{((s_{21} + i)x - t_2y, t_2x + (s_{22} + i)y) \in \mathbb{C}^2 : x, y \in \mathbb{R}\}.$$ 

We now show that $K_0 \cup K$ ia polynomially convex, where $K = K_1 \cup K_2$. Consider the polynomial

$$F(z, w) = z^2 - w^2.$$ 

Clearly,

$$F(K_0) \subset \mathbb{R} \subset \mathbb{C}. \quad (5.1)$$ 

For $(z, w) \in K_1$, we have

$$F(z, w) = F((\lambda_1 + i)x, (\lambda_2 + i)y) = (\lambda_1^2 - 1)x^2 + (1 - \lambda_2^2)y^2 + 2i(\lambda_1x^2 - \lambda_2y^2), \quad (5.2)$$

and, for $(z, w) \in K_2$,

$$F(z, w) = F((s_{21} + i)x - t_2y, t_2x + (s_{22} + i)y) = (s_{21}^2 - t_2^2 - 1)x^2 + (1 - s_{22}^2 + t_2^2)y^2 - 2(s_{21} + s_{22})t_1xy + 2i(s_{21}x^2 - s_{22}y^2 - 2t_2xy). \quad (5.3)$$

We now show that $\hat{F}(K) \cap \hat{F}(K_0) = \{0\}$. We have

$$\det A_1 < 0 \iff \lambda_1 \lambda_2 < 0, \quad (5.4)$$

$$\det A_2 < 0 \iff s_{21}s_{22} < -t_2^2 < 0. \quad (5.5)$$

In view of (5.4) and (5.5), expressions (5.2) and (5.3) give us

$$F(K) \subset (\mathbb{C} \setminus \mathbb{R}) \cup \{0\}.$$ 

Hence, we have

$$\hat{F}(K) \cap \hat{F}(K_0) = \{0\} \text{ and } F^{-1}\{0\} \cap K = \{0\}. \quad (5.6)$$

We also have

$$F^{-1}\{0\} \cap K_0 = \{(x, y) \in K_0 : x = \pm y\} \quad (5.7)$$ 

Hence, $F^{-1}\{0\} \cap (K \cup K_0)$ is polynomially convex. Therefore, all the conditions of Lemma 2.3 are satisfied. Hence, $K \cup K_0$ is polynomially convex. \hfill \Box

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