ÉTALE DESCENT OF DERIVATIONS

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Abstract. We study étale descent of derivations of algebras with values in a module. The algebras under consideration are twisted forms of algebras over rings, and apply to all classes of algebras, notably associative and Lie algebras, such as the multiloop algebras that appear in the construction of extended affine Lie algebras. The main result is Theorem 2.7.

Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra over an algebraically closed field $k$ of characteristic 0. The celebrated affine Kac-Moody Lie algebras are of the form

$$\mathcal{E} = \mathcal{L} \oplus kc \oplus kd$$

where $\mathcal{L}$ is a (twisted) loop algebra of the form $L(\mathfrak{g}, \pi)$ for some diagram automorphism $\pi$ of $\mathfrak{g}$. The element $c$ is central and $d$ is a degree derivation for a natural grading of $\mathcal{L}$.

It is thus natural to study the derivations of loop, or more generally multiloop, algebras. This is a problem with a long history going back to [BM], [Bl]. In some sense there is a prescient aspect to [BM], which seems to sense the existence of Lie algebras that would have multiloop algebras play the same role that the loop algebras play in the affine case. These algebras would come into being a decade and a half later in the form of extended affine Lie algebras (EALAs) and Lie tori (see [AABGP], [N2], [N3]).

The first author has shown that any EALA is built from a Lie torus at the “bottom” in a way reminiscent of the affine construction. Loosely speaking an
EALA is always of the form
\[ \mathcal{E} = \mathcal{L} \oplus \mathcal{C} \oplus \mathcal{D} \]
where \( \mathcal{L} \) is a Lie torus, \( \mathcal{C} \) is central in \( \mathcal{L} \oplus \mathcal{C} \) and \( \mathcal{D} \) is a space of derivations of the bottom Lie torus \( \mathcal{L} \). It is known, save for perfectly understood exceptions in absolute type \( A \), that Lie tori are always multiloop algebras [ABFP] (but not conversely, except for nullity 1 as shown in [P2]). We begin to see the central importance that the understanding of the Lie algebra of \( k \)-linear derivations of multiloop algebras has for EALA theory. They are also important for the structure of universal central extensions [N1].

Exploiting the fact that a centreless Lie torus \( \mathcal{L} \) which is finitely generated over its centroid \( R \) is étale (even Galois) locally isomorphic to the \( R \)-algebra \( \mathfrak{g} \otimes_k R \), there is a very transparent way of understanding the derivations of \( \mathcal{L} \). The main idea (see [P1] for details) is disarmingly simple and we outline it here since it will serve as the blueprint for our work: Let \( S/R \) be an étale covering that trivializes \( \mathfrak{g} \otimes_k R \).

(a) As shown in [BM], upstairs, namely for \( \mathfrak{g} \otimes_k S \), the picture is perfectly understood: Besides the inner derivations we have \( \text{Der}_k(S) \) acting naturally as derivations of \( \mathfrak{g} \otimes_k S \).

(b) One can recover \( S \) from \( \mathfrak{g} \otimes_k S \) as its centroid. Thus (a) says that the derivations of the centroid of \( \mathfrak{g} \otimes_k S \) extend naturally to \( \mathfrak{g} \otimes_k S \). More precisely, \( \text{Der}_k(\mathfrak{g} \otimes_k S) \simeq \text{IDer}(\mathfrak{g} \otimes_k S) \times \text{Der}_k(S) \) with \( \text{IDer}(\mathfrak{g} \otimes_k S) \simeq \text{Der}_k(\mathfrak{g}) \otimes_k S \).

(c) By descent considerations ([GP, Lemma 4.6]) the centroid of \( \mathcal{L} \) is \( R \). Every derivation of \( \mathcal{L} \) naturally induces a derivation of \( R \), but it is not obvious that the derivations of \( R \) can be lifted to \( \mathcal{L} \) (as they are in the trivial case (a) above).

(d) Because \( S/R \) is étale, every \( k \)-linear derivation of \( R \) extends uniquely to a \( k \)-linear derivation of \( S \), hence to a derivation of \( \mathfrak{g} \otimes_k S \) by (a).

(e) The derivation of \( \mathfrak{g} \otimes_k S \) defined in (d) descends to \( \mathcal{L} \). The resulting picture is thus completely analogous to the one of the trivial case:

\[ \text{Der}_k(\mathcal{L}) \simeq \text{IDer}_k(\mathcal{L}) \times \text{Der}_k(R). \]

A close inspection shows that more important than the structure of \( \text{Der}_k(\mathcal{L}) \) to the theory of EALAs of central extensions is the structure of \( \text{Der}_k(\mathcal{L}, N) \) where \( N \) is the graded dual of \( \mathcal{L} \). Rather than studying this particular case, we set out to see if the descent formalism will shed information about the structure of \( \text{Der}_k(\mathcal{L}, N) \) for an arbitrary \( \mathcal{L} \)-module \( N \) and \( S/R \)-form \( \mathcal{L} \). The answer is a resounding yes. There are, however, subtle technical and philosophical difficulties to overcome before one can even state a structure result. This is already quite evident in (b) and (c) above. While a derivation of \( \mathcal{L} \) induces a derivation of its centroid, what does a derivation in \( \text{Der}_k(\mathcal{L}, N) \) induce, and on what? As we shall see, proceeding in a natural way will guide us — as it always seems to do — towards the correct answer: Theorem 2.7.

Although in this introduction we have emphasized Lie algebras, the main results of this paper will be established for arbitrary algebras. This substantially broadens the applications of our work. For example, in 3.3 we use our main theorem to derive a new characterization of the first Hochschild cohomology group of separable