Partitions of graphs into small and large sets

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Abstract

Let $G$ be a graph on $n$ vertices. We call a subset $A$ of the vertex set $V(G)$ $k$-small if, for every vertex $v \in A$, $\deg(v) \leq n - |A| + k$. A subset $B \subseteq V(G)$ is called $k$-large if, for every vertex $u \in B$, $\deg(u) \geq |B| - k - 1$. Moreover, we denote by $\varphi_k(G)$ the minimum integer $t$ such that there is a partition of $V(G)$ into $t$ $k$-small sets, and by $\Omega_k(G)$ the minimum integer $t$ such that there is a partition of $V(G)$ into $t$ $k$-large sets.

In this paper, we will show tight connections between $k$-small sets, respectively $k$-large sets, and the $k$-independence number, the clique number and the chromatic number of a graph. We shall develop greedy algorithms to compute in linear time both $\varphi_k(G)$ and $\Omega_k(G)$ and prove various sharp inequalities concerning these parameters, which we will use to obtain refinements of the Caro-Wei Theorem, the Turán Theorem and the Hansen-Zheng Theorem among other things.

Keywords: $k$-small set, $k$-large set, $k$-independence, clique number, chromatic number

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1 Introduction and Notation

We start with the following basic definitions. Let $n$ and $m$ be two positive integers. Let $S = \{0 \leq a_1 \leq a_2 \leq \ldots \leq a_m \leq n - 1\}$ be a sequence of $m$ integers and $\overline{S} = \{0 \leq b_m \leq b_{m-1} \leq \ldots \leq b_1 \leq n - 1\}$ be the complement sequence, where $b_i = n - a_i - 1$ for $1 \leq i \leq m$. Let $k \geq 0$ be an integer. A subsequence $A$ of $S$ is called $k$-small if, for every member $x$
of $A$, $x \leq n - |A| + k$. A subsequence $B$ of $S$ is called $k$-large if, for every member $x$ of $B$, $x \geq |B| - k - 1$. In particular, for the terminology of graphs, we have the following definitions. Let $G$ be a graph on $n$ vertices. We call a set of vertices $A \subseteq V(G)$ $k$-small if, for every vertex $v \in A$, $\text{deg}(v) \leq n - |A| + k$. A subset $B \subseteq V(G)$ is called $k$-large if, for every vertex $v \in B$, $\text{deg}(v) \geq |B| - k - 1$. When $k = 0$, we say that $A$ is a small set ($\delta$-set in [15, 11]) and $B$ a large set. Let $S_k(G)$ denote the maximum cardinality of a $k$-small set and $L_k(G)$ denote the maximum cardinality of a $k$-large set in $G$. Further, given a graph $G$, let $\varphi_k(G)$ be the minimum integer $t$ such that there is a partition of $V(G)$ into $t$ $k$-small sets, and let $\Omega_k(G)$ be the minimum integer $t$ such that there is a partition of $V(G)$ into $t$ $k$-large sets. When $k = 0$, we will set $\varphi(G)$ instead of $\varphi_0(G)$ and $\Omega(G)$ instead of $\Omega_0(G)$.

Consider the following observations.

**Observation 1.1.** Let $n$ and $m$ be two positive integers. Let $S = \{0 \leq a_1 \leq a_2 \leq \ldots \leq a_m \leq n - 1\}$ be a sequence and let $G$ be a graph.

(i) $A$ is a $k$-small subsequence of $S$ if and only if $\overline{A}$ is a $k$-large subsequence of $\overline{S}$;

(ii) $A$ is a $k$-small set in $G$ if and only if $A$ is a $k$-large set in $\overline{G}$;

(iii) $S_k(G) = L_k(\overline{G})$ and $L_k(G) = S_k(\overline{G})$;

(iv) $\varphi_k(G) = \Omega_k(G)$ and $\Omega_k(G) = \varphi_k(\overline{G})$.

**Proof.** (i) $A$ is a small subsequence of $S$ if and only if $a_i \leq n - |A| + k$ for every $a_i \in A$, which is equivalent to $b_i \geq |A| - k - 1$ for each $b_i = n - a_i - 1 \in \overline{A}$, meaning that $\overline{A}$ is a $k$-large subsequence of $\overline{S}$.

(ii) $A$ is a $k$ small set of $G$ if and only if $\text{deg}_G(v) \leq n - |A| + k$ for every $v \in A$, which is equivalent to $\text{deg}(\overline{G})(v) = n - 1 - \text{deg}_G(v) \geq |A| - k - 1$ for every $v \in A$, meaning that $A$ is a $k$-large set in $\overline{G}$.

(iii) and (iv) follow directly from (ii). □

A $k$-independent set $A$ in $G$ is a subset of vertices of $G$ such that $|N(v) \cap A| \leq k$ for every $v \in A$. The maximum cardinality of a $k$-independent set is denoted by $\alpha_k(G)$. Note that a 0-independent set is precisely an independent set, so we will use the usual notation $\alpha(G)$ for the independence number instead of $\alpha_0(G)$. The well-known Caro-Wei bound [2, 17] $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\text{deg}(v) + 1}$ was generalized by Favaron [8] to $\alpha_k(G) \geq \sum_{v \in V(G)} \frac{k}{1 + k \text{deg}(v)}$. Other generalizations and improvements were given in [3, 12]. For more information on the $k$-independence number see also the survey [4].

Similarly, we call a subset $B \subseteq V(G)$ such that $|N(v) \cap B| \geq |B| - k - 1$ for every $v \in B$ a $k$-near clique and the cardinality of a maximum $k$-near clique will be denoted by $\omega_k(G)$. A 0-near clique is precisely a clique and so we will use the usual notation for the clique number $\omega(G)$ instead of $\omega_0(G)$.

The connection between $k$-independent sets and $k$-near cliques to $k$-small and $k$-large sets is given below.
Observation 1.2. In a graph $G$, every $k$-independent set is a $k$-small set and every $k$-near clique is a $k$-large set;

Proof. Let $A$ be a $k$-independent set and $B$ a $k$-near clique of $G$. Then $\deg(v) \leq n - |A| + k$ for every $v \in A$ and $\deg(v) \geq |B| - k - 1$ for every $v \in B$. Hence, $A$ is a $k$-small set and $B$ a $k$-large set. $\square$

We denote by $\deg(v) = \deg_G(v)$ the degree of the vertex $v$ in $G$ and $N_G(v)$ and $N_G[v]$ is its open and, respectively, closed neighborhood of $v$. With $d(G)$ we refer to the average degree $\frac{1}{|V(G)|} \sum_{v \in V(G)} \deg(v)$ of $G$. Given the degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ of $G$, we will denote by $v_1, v_2, \ldots, v_n$ the vertices of $G$ ordered accordingly to the degree sequence, i.e. such that $\deg(v_i) = d_i$. Moreover, $\chi(G)$ is the chromatic number and $\theta(G)$ the clique-partition number of $G$. For notation not mentioned here, we refer the reader to [18].

The paper is organized in several sections as follows:

1 Introduction and Notation
2 Bounds on $S_k(G)$ and $L_k(G)$ with applications to upper and lower bounds on $\alpha_k(G)$ and $\omega_k(G)$
3 Algorithms for $\varphi_k(G)$ and $\Omega_k(G)$
4 Bounds on $\varphi_k(G)$ and $\Omega_k(G)$
5 More applications to $\alpha(G)$ and $\omega(G)$
6 Variations of small and large sets
7 References

2 Bounds on $S_k(G)$ and $L_k(G)$ with applications to upper bounds on $\alpha_k(G)$ and $\omega_k(G)$

Since every $k$-independent set of $G$ is a $k$-small set and every $k$-near clique of $G$ is a $k$-large set, one expects that the bounds on $S_k(G)$, $L_k(G)$, $\varphi(G)$ and $\Omega(G)$ can be derived using their arithmetic definitions, and that some properties will be also useful in obtaining bounds on the much harder to compute $\alpha_k(G)$ and $\omega_k(G)$. As we shall see in the sequel this is indeed the case and several refinements of the Caro-Wei Theorem [2, 17], the Turán Theorem [16] and Hansen-Zheng Theorem [10] are easily derived from bounds using $k$-small sets and $k$-large sets as well as some relations between $L_0(G)$ and $\chi(G)$. A lower bound on $\alpha(G)$ and $\omega(G)$ in terms of $\Omega(G)$ and $\varphi(G)$, respectively, illustrates the usefulness of working with small and large sets.

Theorem 2.1. Let $G$ be a graph. Then $\alpha(G) \geq \Omega(G)$ and $\omega(G) \geq \varphi(G)$.
Proof. Let $G_1 = G$ and let $x_1$ be a vertex of minimum degree in $G_1$. Now, for $i \geq 1$, let $x_i$ be a vertex of minimum degree in $G_i$ and define successively $G_{i+1} = G_i - N_{G_i}[x_i]$ and $V_i = N_{G_i}[x_i]$, until there are no vertices left, say until index $q$. In this way, we obtain a partition $V_1 \cup V_2 \cup \ldots \cup V_q$ of $V(G)$ into large sets, as, for every $v \in V_i$, $\deg(v) \geq \deg_{G_i}(v) \geq \deg_{G_{i+1}}(x_i) = |V_i| - 1$. Hence, $q \geq \Omega(G)$. On the other side, $\{x_1, x_2, \ldots, x_q\}$ is an independent set by construction and thus $\alpha(G) \geq q$. Therefore, $\alpha(G) \geq \Omega(G)$ and also $\omega(G) = \alpha(\overline{G}) \geq \Omega(\overline{G}) = \varphi(G)$ and we are done. \[\square\]

We mention that a more complicated proof of $\omega(G) \geq \varphi(G)$ was given in [13]. One of the strongest lower bounds for the independence number of a graph is the so called residue of the degree sequence denoted $R(G)$ (see [9, 15, 12]), which is the number of zeros left in the end of the Havel-Hakimi algorithm. As we shall see later, computing $\Omega(G)$ requires $O(|V(G)|)^t$-time while the Havel-Hakimi algorithm requires $O(|E(G)|)^t$-time. While $R(G)$ does better than all of the lower bounds given in the survey [19], here are two examples showing that in one case $R(G)$ does better and in the other $\Omega(G)$ does better. For the star $G = K_{1,n}$, $R(G) = n - 1$ while $\Omega(G) \sim \frac{n}{2}$. However, for the graph $G$ on 6 vertices with degree sequence 1, 2, 2, 3, 3, 3, $\Omega(G) = 3$ while $R(G) = 2$.

While the above theorem gives lower bounds on $\alpha(G)$ and $\omega(G)$ in terms of $\Omega(G)$ and $\varphi(G)$, the next one gives upper bounds on $\alpha_k(G)$ and $\omega_k(G)$ in terms of $S_k(G)$ and $L_k(G)$.

**Theorem 2.2.** Let $G$ be a graph on $n$ vertices and let $d_1 \leq d_2 \leq \ldots \leq d_n$ its degree sequence. Then

(i) $S_k(G) \geq \alpha_k(G)$ and $L_k(G) \geq \omega_k(G)$;

(ii) $S_k(G) \geq \frac{n}{\varphi_k(G)}$ and $L_k(G) \geq \frac{n}{\Omega_k(G)}$;

(iii) $S_k(G) = \max \{s : d_s \leq n - s + k\}$ and $\{v_1, v_2, \ldots, v_{S_k(G)}\}$ is a maximum $k$-small set of $G$;

(iv) $L_k(G) = \max \{t : n - t + 1 \leq d_{n-t+1}\}$ and $\{v_{n-L_k+1}, v_{n-L_k+2}, \ldots, v_n\}$ is a maximum $k$-large set of $G$.

**Proof.** (i) Since a $k$-independent set is a $k$-small set and a $k$-near-clique is a $k$-large set, $S_k(G) \geq \alpha_k(G)$ and $L_k(G) \geq \omega_k(G)$.

(ii) Let $V_1 \cup V_2 \cup \ldots \cup V_t$ be a partition of $V(G)$ into $t = \varphi_k(G)$ $k$-small sets. Then $S_k(G) \geq \max_{1 \leq i \leq t} |V_i| \geq \frac{n}{t} = \frac{n}{\varphi_k(G)}$. The other inequality follows from $L_k(G) = S_k(\overline{G}) \geq \frac{n}{\omega_k(G)} = \frac{1}{\Omega_k(G)}$.

(iii) Let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ ordered according to its degree sequence. Let $A$ be an arbitrary $k$-small set. Clearly, for every vertex $v \in A$, $\deg(v) \leq n - |A| + k$. Now order the degrees of the vertices of $A$ in increasing order such that $\deg(u_1) \leq \deg(u_2) \leq \ldots \leq \deg(u_{|A|}) \leq n - |A| + k$. Then $d_{|A|} \leq \deg(u_{|A|}) \leq n - |A| + k$. Hence, for every $k$-small set $A$, $d_{|A|} \leq \deg(u_{|A|}) \leq n - |A| + k$. Now let $s$ be the largest index in the degree sequence.
of $G$ such that $d_s \leq n - s + k$. Then $s \geq S_k(G)$, as this inequality holds for any $k$-small set. But observe that $\{v_1, v_2, \ldots, v_s\}$ is $k$-small by definition. Hence $S_k(G) \geq s$ and we conclude that $s = S_k(G)$.

(iv) Let $\overline{d_1} \leq \overline{d_2} \leq \ldots \leq \overline{d_n}$ be the degree sequence of $\overline{G}$ given through $\overline{d_i} = n - 1 - d_{n-i+1}$ for $1 \leq i \leq n$. Then, by item (iii) and Observation 1.1(iii) we have $L_k(G) = S_k(G) = \max\{t : \overline{d_t} \leq n - t + k\} = \max\{t : t - k - 1 \leq d_{n-t+1}\}$ and we are done. □

Note from previous theorem that if $k \geq \Delta$, then $S_k(G) = n$ and $\varphi_k(G) = 1$. Also, if $k \geq n - \delta - 1$, then $L_k(G) = n$ and $\Omega_k(G) = 1$. In this sense, the restrictions $k \leq \Delta$ or $k \geq n - \delta - 1$ needed in some of our theorems or observations are natural.

From Theorem 2.2 the following observation follows straightforward.

**Observation 2.3.** Let $G$ be a graph with minimum degree $\delta$ and maximum degree $\Delta$. Then

(i) $n - \Delta + k \leq S_k(G) \leq n - \delta + k$ for $k \leq \Delta$;

(ii) $\delta + k + 1 \leq L_k(G) \leq \Delta + k + 1$ for $k \leq n - \delta - 1$;

(iii) if $G$ is $r$-regular, then $S_k(G) = n - r + k$ when $k \leq r$ and $L_k(G) = r + k + 1$ when $k \leq n - r - 1$.

**Proof.** (i) Let $\delta = d_1 \leq d_2 \leq \ldots \leq d_n = \Delta$ be the degree sequence of $G$. For $k \leq \Delta$, $d_{n-\Delta+k} \leq n - (n - \Delta + k) + k = \Delta$. Therefore, $n - \Delta + k \in \{s : d_s \leq n - s + k\}$ and thus $n - \Delta + k \leq S_k(G)$. Moreover, according to Theorem 2.2(iii), from $S_k \in \{s : d_s \leq n - s + k\}$ it follows that $\delta \leq d_{S_k(G)} \leq n - S_k(G) + k$, that is, $S_k(G) \leq n - \delta + k$.

(ii) This follows from (i) applied to the graph $\overline{G}$.

(iii) This follows from (i) and (ii). □

Next we show a connection between $L_0(G)$ and the chromatic number $\chi(G)$ strengthening $L_0(G) \geq \omega(G)$. The analogon follows for $S_0(G)$ and the clique-partition number $\theta(G)$.

**Observation 2.4.** Let $G$ be a graph. Then

(i) $L_0(G) \geq \chi(G) \geq \omega(G)$;

(ii) $S_0(G) \geq \theta(G) \geq \alpha(G)$.

**Proof.** (i) By a result of Powell and Welsh ([14], see also [11], p. 148), $\chi(G) \leq \max\{\min\{i, d_i + 1\} : 1 \leq i \leq n\}$, where $d_1 \geq d_2 \geq \ldots \geq d_n$ is the degree sequence of $G$. This can be rewritten with the conventional order $d_1 \leq d_2 \leq \ldots \leq d_n$ as $\chi(G) \leq \max\{t : t \leq d_{n-t+1} + 1\}$. Since, by the above theorem, the last expression is equal to $L_0(G)$, we obtain, together with Theorem 2.1, the desired inequality chain. Another proof of $L_0(G) \geq \chi(G)$ can be given the following way. Let $V_1 \cup V_2 \cup \ldots \cup V_r$ be an
r-chromatic partition of $V(G)$, where $r = \chi(G)$. Suppose there is an index $i$ such that every vertex $v \in V_i$ has no neighbor in some set $V_j$, for an index $j \neq i$. Then we can distribute the vertices of $V_i$ among the other sets $V_j$, obtaining thus an $(r-1)$-chromatic coloring of $G$, which is a contradiction. Hence, for every $1 \leq i \leq r$, there is a vertex $v_i \in V_i$ such that $v_i$ has a neighbor in $V_j$ for every $1 \leq j \leq r$ and $i \neq j$. Therefore, $\deg(v_i) \geq r-1$ for $1 \leq i \leq r$ and hence $\{v_1, v_2, \ldots, v_r\}$ is a large set, yielding $\chi(G) = r \leq L_0(G)$.

(ii) This follows from (i) and $S_0(G) = L_0(G)$, $\theta(G) = \chi(G)$ and $\alpha(G) = \omega(G)$. □

We close this section with three observations about partitions of the vertex set of a graph into a $k$-small set and a $k$-large set.

**Observation 2.5.** Let $G$ be a graph. Then $V(G)$ can be partitioned into a $k$-small set $V_S$ and a $k$-large set $V_L$.

**Proof.** Let $d_1 \leq d_2 \leq \ldots \leq d_n$ be the degree sequence of $G$ and let $j = S_k(G)$ be the largest index such that $d_j \leq n - j + k$. Let $v_1, v_2, \ldots, v_n$ be the vertices of $G$ ordered according to its degree sequence. Set $V_S = \{v_1, \ldots, v_j\}$ and set $V_L = V \setminus V_S$. Clearly, $|V_S| = j$ and $|V_L| = n - j$. By Theorem 2.2(iii), $V_S$ is a maximum $k$-small set. Since $j$ is the maximum index for which $d_j \leq n - j + k$, it follows that $d_{j+1} > n - (j+1) + k$ and thus $d_{j+1} \geq n - j + k$. But then, for $i \geq j + 1$, $d_i \geq d_{j+1} \geq n - j + k = |V_L| + k > |V_L| - k - 1$, and hence $V_L$ is a $k$-large set. Note that already a partition into small and large sets suffices to prove the statement since any small set is a $k$-small set for $k > 0$ and any large set is a $k$-large set for $k > 0$. □

From Observation 2.5 follows, in particular, that in every $n$-vertex graph there is either a $k$-small set on at least $n/2$ vertices or a $k$-large set on at least $n/2$ vertices.

**Observation 2.6.** $n \leq L_k(G) + S_k(G) \leq n + 1 + 2k$ and this is sharp.

**Proof.** From Observation 2.5, we obtain directly the lower bound $n \leq S_k(G) + L_k(G)$. Let now $A$ be a $k$-small set realizing $S_k(G)$ and $B$ a $k$-large set realizing $L_k(G)$. If $A \cap B = \emptyset$, then clearly $|A| + |B| \leq n$. Otherwise suppose there is a vertex $u \in A \cap B$. Then $\deg(u) \leq n - |A| + k$ and $\deg(u) \geq |B| - k - 1$. Hence $|B| - k - 1 \leq n - |A| + k$ and $|A| + |B| \leq n + 1 + 2k$.

To see the sharpness of the lower bound, let $G_1$ be a graph on $n_1 = 2q > 2(2k+2)$ vertices whose vertex set can be split into an independent set $V_S$ and a clique $V_L$ with $|V_S| = |V_L| = q$, and such that their vertices are joined by $k+1$ pairwise disjoint perfect matchings. Then, the vertices in $V_S$ have all degree $k+1$ and the vertices in $V_L$ all have degree $q+k$. Hence, for the degree sequence $d_1 \leq d_2 \leq \ldots \leq d_{2q}$ of $G_1$ we have $d_q = k + 1 \leq n_1 - q + k = q + k$ and $q + k = d_{q+1} > n_1 - (q+1) + k = q + k - 1$, from which follows that $V_S$ is a maximum $k$-small set, by Theorem 2.2. Also from $d_{q+1} \geq n_1 - q - k - 1 = q - k - 1$ and $d_q = k + 1 < n_1 - q - k - 1 = q - k - 1$, as $q > 2k+2$, it follows by the same theorem that $V_L$ is a maximum $k$-large set of $G_1$. Hence for this graph, $n_1 = S_k + L_k$ holds. Finally, for
the sharpness of the upper bound, let \( G_2 \) be a graph in which the largest \( 2k + 1 \) degrees in the degree sequence are \( k \). An easy check reveals the required equality. \( \square \)

**Observation 2.7.** Let \( G \) be a graph on \( n \) vertices and \( e(G) \) edges. Then there is partition of \( V(G) \) into a \( k \)-small set \( V_S \) and a \( k \)-large set \( V_L \) such that \( |V_L| \leq \frac{1}{2}(k + 1 + \sqrt{(k+1)^2 + 8e(G)}) \) and hence \( |V_S| \geq n - \frac{1}{2}(k + 1 + \sqrt{(k+1)^2 + 8e(G)}) \).

**Proof.** Let \( V(G) = V_S \cup V_L \) be a partition into a \( k \)-small and a \( k \)-large set and let \( p = |V_L| \). Then \( 2e(G) \geq \sum_{v \in V_L} \deg(v) \geq p(p - k - 1) \). Solving the quadratic inequality, we obtain \( p \leq \frac{1}{2}(k + 1 + \sqrt{(k+1)^2 + 8e(G)}) \). \( \square \)

## 3 Algorithms for \( \varphi_k(G) \) and \( \Omega_k(G) \)

In this section, we will present two algorithms with which we will be able to calculate \( \varphi_k(G) \) and \( \Omega_k(G) \) for a graph \( G \). For this, we consider any sequence of \( m \) integers \( A = \{0 \leq a_1 \leq \ldots \leq a_m \leq n - 1\} \) (not necessarily graphic). Now we want to break the sequence into \( k \)-small subsequences. With this aim, we apply the following algorithm.

**Algorithm 1**

**INPUT:** \( A \)

**STEP 1:** Set \( i := 0, \ R_0 := A. \)

**STEP 2:** Repeat

1. \( n_i := |R_i| \)
2. \( p_i := \min\{n_i, n - a_n + k\} \)
3. \( A_{i+1} := \{a_{n_i-p_i+1}, a_{n_i-p_i+2}, \ldots, a_{n_i}\} \)
4. \( R_{i+1} := R_i \setminus A_{i+1} \)
5. \( i := i + 1 \)

until \( R_i = \emptyset. \)

**OUTPUT:** \( s := i, \ A_1, A_2, \ldots, A_s. \)

Here, \( i \) stands for the current step number; \( R_i \) is the set of remaining elements and \( n_i \) its cardinality; \( A_{i+1} \) is the new subsequence constructed in step \( i \); and, on the output, \( s \) it is the number of constructed subsequences \( A_i. \)

**Theorem 3.1.** Let \( A = \{0 \leq a_1 \leq \ldots \leq a_m \leq n - 1\} \) be a sequence of \( m \) integers. Then Algorithm 1 under input \( A \) yields a minimum partition of \( A \) into \( s \) \( k \)-small subsequences \( A_1, A_2, \ldots, A_s. \)
Corollary 3.3.

Proof. Clearly, $A_i$ is a subsequence of $A$ for $i = 1, 2, \ldots, s$. By construction, in each step $i$, $A_{i+1} \subseteq R_i = R_{i-1} \setminus A_i$ and so the $A_i$’s are pairwise disjoint. Moreover, the last step $s$ is attained when $R_s = \emptyset$, i.e., $A_s = R_{s-1}$, meaning that $A_s$ consists of all remaining elements of $A$. Hence, $A_1, A_2, \ldots, A_s$ is a splitting of $A$ into subsequences. We now proceed to prove that, in each step $i \geq 0$, the produced subsequence $A_{i+1}$ is $k$-small. We distinguish between the two possible situations:

(a) $p_i = n_i \leq n - a_{n_i} + k$: Then, $A_{i+1} = \{a_1, a_2, \ldots, a_{n_i}\} = R_i$ and, for every $a \in A_{i+1}$, we have $a \leq a_{n_i} = n - (n - a_{n_i} + k) + k \leq n - n_i + k = n - |A_{i+1}| + k$. Thus, $A_{i+1}$ is a $k$-small subsequence.

(b) $p_i = n - a_{n_i} + k \leq n_i$: Then, $A_{i+1} := \{a_{n_i} - (n - a_{n_i} + k) + 1, a_{n_i} - (n - a_{n_i} + k) + 2, \ldots, a_{n_i}\}$ and, for every $a \in A_{i+1}$, we have $a \leq a_{n_i} = n - (n - a_{n_i} + k) + k = n - |A_{i+1}| + k$. Thus, $A_{i+1}$ is a $k$-small subsequence of $A$.

Finally we shall prove that the output $s$ given by Algorithm 1 is the minimum number of $k$-small subsequences in which $A$ can be partitioned. Let $A_1', A_2', \ldots, A_q'$ be an optimal splitting of $A$ into $k$-small sequences, i.e. such that $q$ is minimum. Then clearly $q \leq s$. Let $C_i = \max\{a : a \in A_i'\}$ and, without loss of generality, assume that $C_1 \geq C_2 \geq \cdots \geq C_q$. We will show by induction on $i$ that $a_{n_i} \leq C_i$. Since clearly $a_{n_1} = a_m = C_1$, the base case is done. Assume that $a_{n_i} \leq C_i$ for $i = 1, 2, \ldots, r$ and an $r < q$. Then, as $A_i'$ is a $k$-small set, we have

$$n - |A_i| + k = a_{n_i} \leq C_i \leq n - |A_i'| + k,$$

implying that $|A_i'| \leq |A_i|$, for $i = 1, 2, \ldots, r$. Suppose to the contrary that $a_{n_{r+1}} > C_{r+1}$. Then $a_{n_{r+1}} \in \bigcup_{i=1}^{r} A_i$. As $\sum_{i=1}^{r} |A_i'| \leq \sum_{i=1}^{r} |A_i|$ and, moreover, $a_{n_{r+1}} \notin \bigcup_{i=1}^{r} A_i$ by construction, there has to be an element $y$ which is contained in $\bigcup_{i=1}^{r} A_i$ but not in $\bigcup_{i=1}^{r} A_i'$. Hence, $y \in A_j'$ for some $j \geq r + 1$ and $y \geq a_{n_{r+1}}$. As $C_j$ is the largest element in $A_j'$, we conclude that $C_{r+1} \geq C_j \geq y \geq a_{n_{r+1}}$, contradicting the assumption. Hence $a_{n_{r+1}} \leq C_{r+1}$ and by induction it follows that $a_{n_i} \leq C_i$ for all $i = 1, 2, \ldots, q$. As above, this implies that $|A_i'| \leq |A_i|$ for all $i = 1, 2, \ldots, q$. Hence,

$$m = \sum_{i=1}^{q} |A_i'| \leq \sum_{i=1}^{q} |A_i| \leq \sum_{i=1}^{s} |A_i| = m,$$

from which we obtain $q = s$. Therefore, Algorithm 1 yields a partition of $A$ into the minimum possible number of $k$-small subsequences $A_1, A_2, \ldots, A_s$. \hfill $\Box$

Observation 3.2. Algorithm 1 can be written recursively by defining a function $f$ which will give the partition of an arbitrary sequence into $k$-small subsequences:

**Step 1:** Set $f(\emptyset) = \emptyset$.

**Step 2:**

$$f(\{0 \leq a_1 \leq \ldots \leq a_m \leq n - 1\}) = \{0 \leq a_{\min(m,n-a_m+k)} + 1, \ldots, a_m\} \cup f(\{0 \leq a_1 \leq \ldots \leq a_{\min(m,n-a_m+k)} \leq n - 1\})$$

When $m = n$ and $d_1 \leq d_2 \leq \ldots \leq d_n$ is the degree sequence of a graph $G$, we can use Algorithm 1 to find a partition of $V(G)$ into the minimum possible number of $k$-small sets.

**Corollary 3.3.** Let $G$ be a graph and $d_1 \leq \ldots \leq d_n$ its degree sequence. Let $A = \{0 \leq d_1 \leq \ldots \leq d_n \leq n - 1\}$ and let $V_1, V_2, \ldots, V_s$ be the sets of vertices corresponding to the degree
subsequences $A_1, A_2, \ldots, A_s$ given by Algorithm 1 under input $A$. Then $V_1 \cup V_2 \cup \ldots V_s$ is a partition of $V(G)$ into $s = \varphi_k(G)$ $k$-small sets.

By the duality between $k$-small and $k$-large sequences and since $\Omega_k(G) = \varphi_k(G)$, we can modify Algorithm 1 to an algorithm that leads us to find the exact value of $\Omega_k(G)$. Again, consider any sequence of $m$ integers $B = \{0 \leq b_m \leq b_{m-1} \leq \ldots \leq b_1 \leq n - 1\}$ (not necessarily graphic).

**Algorithm 2**

**INPUT:** $B$

**STEP 1:** Set $i := 0$, $S_0 := B$.

**STEP 2:** Repeat

1. $n_i := |S_i|$
2. $q_i := \min\{n_i, b_{n_i} + k + 1\}$
3. $B_{i+1} := \{b_{n_i}, b_{n_i-1}, \ldots, b_{n_i-q_i+1}\}$
4. $S_{i+1} := S_i \setminus B_{i+1}$
5. $i := i + 1$

until $S_i = \emptyset$.

**OUTPUT:** $t := i, B_1, B_2, \ldots, B_t$.

**Theorem 3.4.** Let $B = \{0 \leq b_m \leq b_{m-1} \leq \ldots \leq b_1 \leq n - 1\}$ be a sequence of $m$ integers. Then Algorithm 2 under input $B$ yields a minimum partition of $B$ into $s$ $k$-large subsequences $B_1, B_2, \ldots, B_t$.

**Proof.** Let $A = \overline{B} = \{0 \leq a_1 \leq a_2 \leq \ldots \leq a_m \leq n - 1\}$ be the complementary sequence to $B$, where $a_i = n - b_i - 1$. Then, from the application of Algorithm 1 under input $A$ and of Algorithm 2 under input $B$, it follows:

1. $R_0 = A = \overline{B} = \overline{S_0}$
2. $R_i = \overline{S_i}$, $|R_i| = n_i = |S_i|$
3. $q_i = \min\{n_i, b_{n_i} + k + 1\} = \min\{n_i, n - (n-1-b_{n_i}) + k\} = \min\{n_i, n - a_{n_i} + k\} = p_i$
4. $B_{i+1} = \{b_{n_i}, b_{n_i-1}, \ldots, b_{n_i-q_i+1}\} = \{n - a_{n_i} - 1, n - a_{n_i-1} - 1, \ldots, n - a_{n_i-q_i+1} - 1\} = \overline{A}_{i+1}$ and
5. $S_{i+1} = S_i \setminus B_{i+1} = \overline{R_i} \setminus A_{i+1}$.

Moreover, $S_i = \emptyset$ if and only if $R_i = \emptyset$ and thus the number of steps performed by Algorithm 1 under input $A$ is the same as the number of steps performed by Algorithm 2 under input.
$B$ and hence $s = t$. Since Algorithm 1 yields a partition of $A = \overline{B}$ into the $k$-small sets $A_1, A_2, \ldots, A_s$, the output $B_1, B_2, \ldots, B_t$ of Algorithm 2 is a partition of $B$ into $k$-large sets. □

Again, when $m = n$ and $d_n \leq d_{n-1} \leq \ldots \leq d_1$ is the degree sequence of a graph $G$, we can use Algorithm 2 to find a partition of $V(G)$ into the minimum possible number of $k$-large sets.

**Corollary 3.5.** Let $G$ be a graph and $d_n \leq \ldots \leq d_1$ its degree sequence. Let $B = \{0 \leq d'_n \leq \ldots \leq d'_{n-1} \leq d'_1 \leq n-1\}$ and let $V_1, V_2, \ldots, V_t$ be the sets of vertices corresponding to the subsequences $B_1, B_2, \ldots, B_t$ given by Algorithm 2 under input $B$. Then $V_1 \cup V_2 \cup \ldots V_t$ is a partition of $V(G)$ into $t = \Omega_k(G)$ $k$-large sets.

4** Bounds on $\varphi_k(G)$ and $\Omega_k(G)$

**Theorem 4.1.** Let $G$ be a graph on $n$ vertices and with average degree $d$. Then

(i) $\varphi_k(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v) + k} \geq \frac{n}{n - d + k}$;

(ii) $\Omega_k(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + k + 1} \geq \frac{n}{d + k + 1}$.

**Proof.** (i) Let $V_1, V_2, \ldots, V_t$ be a partition of $V(G)$ into $t = \varphi_k(G)$ $k$-small sets and set $|V_i| = n_i$, for $1 \leq i \leq t$. Then, as $\deg(v) \leq n - n_i + k$ for each $v \in V_i$, we have

$$\sum_{v \in V(G)} \frac{1}{n - \deg(v) + k} = \sum_{i=1}^{t} \sum_{v \in V_i} \frac{1}{n - \deg(v) + k} \leq \sum_{i=1}^{t} \sum_{v \in V_i} \frac{1}{n_i} = t = \varphi_k(G)$$

Now, Jensen’s inequality yields

$$\varphi_k(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v) + k} \geq \frac{n}{n - d + k}.$$ 

(ii) Since $\Omega_k(G) = \varphi_k(\overline{G})$, we obtain from (i)

$$\Omega_k(G) = \varphi_k(\overline{G}) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(\overline{G})(v) + k} \geq \frac{1}{n - d(G) + k},$$

which is equivalent to

$$\Omega_k(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + k + 1} \geq \frac{n}{d + k + 1}.$$ 

□

Theorems 2.1 and 4.1 for $k = 0$ imply the following corollary.
Corollary 4.2. Let $G$ be a graph on $n$ vertices and average degree $d$. Then

(i) $\omega(G) \geq \varphi(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v)} \geq \frac{n}{n - d};$

(ii) $\alpha(G) \geq \Omega(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \geq \frac{n}{d + 1}.$

The first explicit proof of $\alpha(G) \geq \frac{n}{d + 1}$ can be found in \[7\]. Note also that item (ii) of the previous corollary improves the Caro-Wei bound $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1}$ \[2,17\]. Moreover, the bound $\varphi(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v)}$ was given in \[13\]. From the result that $\alpha(G) \geq \Omega(G)$, one may ask if $\alpha_k(G) \geq (k + 1)\Omega_k(G)$ holds in general. However, this is in general wrong, as can be seen by the following counter example. Let $n = (k + 2)q + 1$ and let $G = K_{1,n}$ be a star with $n$ leaves. Then, clearly, $\alpha_k(G) = (k + 2)q + 1$. Moreover, $\Omega_k(G) = \left\lceil \frac{n + 1}{k+2} \right\rceil = q + 1$, since every $k$-large set containing a vertex of degree one has cardinality at most $k + 2$. Hence, in this case we have $\alpha_k(G) = (k + 2)q < (k + 1)(q + 1) = (k + 1)\Omega(G)$ for $q < k + 1$.

In view of the above counter example the following problem seems natural.

**Problem.** Let $G$ be a graph on $n$ vertices. Is it true that

$$\alpha_k(G) \geq \sum_{v \in V(G)} \frac{k + 1}{\deg(v) + k + 1} \geq \frac{n}{d(G) + k + 1}?$$

Corollary 4.3. Let $G$ be a graph on $n$ vertices and $e(G)$ edges. Then

(i) $e(G) \leq \frac{1}{2} \left( n^2 - \frac{n^2}{\varphi_k(G)} + nk \right);$ 

(ii) $e(G) \geq \frac{1}{2} \left( \frac{n^2}{\Omega_k(G)} - n(k + 1) \right).$

**Proof.** (i) From Theorem 4.1(i) and the fact that $nd = 2e(G)$, it follows $\varphi_k(G) \geq \frac{n}{n-d+k} = \frac{n^2}{n^2 - 2n(e(G)) + kn}$. Solving this inequality for $e(G)$, we obtain the desired result.

(ii) Similar as in (i), from Theorem 4.1(ii) and the fact that $nd = 2e(G)$, it follows that $\Omega_k(G) \geq \frac{n^2}{2e(G) + k(n+1)}$. Solving the obtained inequality for $e(G)$, the result follows. \[\square\]

In the special case $k = 0$, Corollary 4.3 yields $e(G) \leq \frac{n^2(\varphi(G) - 1)}{2\varphi(G)}$. This bound is better than the bound $e(G) \leq \frac{n^2(\omega(G) - 1)}{2\omega(G)}$ from classical Turán’s Theorem, because $\omega(G) \geq \varphi(G)$. To illustrate this by an example, let $G$ be the graph obtained from the graph $2K_n$ by adding $n$ new independent edges between the two copies of $K_n$. Then $\varphi(G) = 2$ and $\omega(G) = n$. From Turán’s Theorem we have $e(G) \leq 2n(n - 1)$ and from Corollary 4.3(ii) follows that $e(G) \leq n^2$. The last inequality gives us the exact value of $e(G)$. 

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Theorem 4.4. Let $G$ be a graph on $n$ vertices with minimum degree $\delta$, maximum degree $\Delta$ and average degree $n$. Then:

(i) $\left\lfloor \frac{n}{n-d+1} \right\rfloor \leq \varphi_k(G) \leq \left\lceil \frac{n}{n-k-\Delta} \right\rceil$;

(ii) $\left\lfloor \frac{n}{d+k+1} \right\rfloor \leq \Omega_k(G) \leq \left\lceil \frac{n}{d+k+1} \right\rceil$;

(iii) If $\frac{r}{r-1} n + k < d \leq \Delta \leq \frac{r-1}{r} n + k$, then $\varphi_k(G) = r$;

(iv) If $\frac{n}{r} - k - 1 \leq \delta \leq d < \frac{n}{r} - k - 1$, then $\Omega_k(G) = r$;

(v) If $G$ is $r$-regular, then $\varphi_k(G) = \left\lfloor \frac{n}{n-k-r} \right\rfloor$ and $\Omega_k(G) = \left\lceil \frac{n}{r+k+1} \right\rceil$.

Proof. (i) From Theorem 4.1(i), it follows directly

$$\varphi_k(G) \geq \left\lfloor \frac{n}{n-d+k} \right\rfloor.$$ 

Let now $G$ be a graph on $n$ vertices and with maximum degree $\Delta$. If $k > \Delta$, then $\varphi_k(G) = 1$ and the right inequality side is obvious. So let $k \leq \Delta$ and let $A \subseteq V(G)$ be a set of cardinality $n - \Delta + k$. Then, for any $v \in A$, $\deg(v) \leq \Delta = n - (n - \Delta + k) + k = n - |A| + k$ and hence $A$ is a $k$-small set. Now we will partition $V(G) \setminus A$ into $k$-small sets. Note that $|V(G) \setminus A| = \Delta - k$. So take a partition $V_1, V_2, \ldots, V_t$ of $V(G) \setminus A$ into $t = \left\lceil \frac{\Delta - k}{n - \Delta + k} \right\rceil$ sets such that $|V_i| = n - \Delta + k$ for $i = 1, 2, \ldots, t - 1$ and $|V_t| \leq n - \Delta + k$. Since, for every vertex $v \in V_i$, $\deg(v) \leq \Delta = n - (n - \Delta + k) + k \leq n - |V_i| + k$, $V_i$ is a $k$-small set, for $1 \leq i \leq t$. Hence $A \cup V_1 \cup V_2 \cup \ldots \cup V_t$ is a partition of $V(G)$ into $1 + t = 1 + \left\lfloor \frac{\Delta - k}{n - \Delta + k} \right\rfloor = \left\lfloor \frac{n}{n - \Delta + k} \right\rfloor$ $k$-small sets, and thus

$$\varphi_k(G) \leq \left\lfloor \frac{n}{n - \Delta + k} \right\rfloor.$$ 

(ii) Theorem 4.1(ii) yields

$$\Omega_k(G) \geq \left\lfloor \frac{n}{d+k+1} \right\rfloor.$$ 

The other inequality side is obtained from (i) through $\Omega_k(G) = \varphi_k(G) \leq \left\lceil \frac{n}{n+k-\Delta} \right\rceil = \left\lceil \frac{n}{\delta+k+1} \right\rceil$.

(iii) If $\frac{r}{r-1} n + k < d \leq \Delta \leq \frac{r-1}{r} n + k$, we obtain from (i)

$$r - 1 = \left\lfloor \frac{n}{n-k-r} \right\rfloor < \left\lfloor \frac{n}{n-k-d} \right\rfloor \leq \varphi_k(G) \leq \left\lfloor \frac{n}{n+k-\Delta} \right\rfloor \leq \left\lfloor \frac{n}{n-r-1} \right\rfloor = r$$

and thus $\varphi_k(G) = r$.

(iv) If $\frac{n}{r} - k - 1 \leq \delta \leq d < \frac{n}{r} - k - 1$, we obtain from (ii)

$$r - 1 = \left\lfloor \frac{n}{r} \right\rfloor < \left\lfloor \frac{n}{d+k+1} \right\rfloor \leq \Omega_k(G) \leq \left\lfloor \frac{n}{\delta+k+1} \right\rfloor \leq \left\lfloor \frac{n}{r} \right\rfloor = r.$$
and thus $\Omega_k(G) = r$.

(v) Recall from (i) that $\left\lceil \frac{n}{n+k-d} \right\rceil \leq \varphi_k(G) \leq \left\lceil \frac{n}{n+k-\Delta} \right\rceil$ and thus, if $d = \Delta = r$, we have $\varphi_k(G) = \left\lceil \frac{n}{n+k-r} \right\rceil$. Analogously, item (ii) yields $\Omega_k(G) = \left\lceil \frac{n}{r+k+1} \right\rceil$. $\blacksquare$

5 More applications to $\alpha(G)$ and $\omega(G)$

**Theorem 5.1.** Let $G$ be a graph on $n$ vertices and with minimum degree $\delta$ and maximum degree $\Delta$. Then

(i) $\alpha_k(G) \leq S_k(G) \leq \frac{n-\Delta+k}{2} + \sqrt{\frac{(n-\Delta+k)^2}{4} + n\Delta - 2e(G)}$;

(ii) $\omega_k(G) \leq L_k(G) \leq \frac{\delta+k+1}{2} + \sqrt{\frac{(\delta+k+1)^2}{4} - n\delta + 2e(G)}$.

Moreover, all bounds are sharp for regular graphs.

**Proof.** (i) Let $A$ be a maximum $k$-small set and let $\Delta$ be the maximum degree of $G$. Then $\deg(v) \leq n - |A| + k$ for all $v \in A$. Then

$$2e(G) = \sum_{v \in V(G)} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in V(G) \setminus A} \deg(v) \leq (n - |A| + k)|A| + \Delta(n - |A|) = -|A|^2 + (n - \Delta + k)|A| + n\Delta,$$

which implies that $|A|^2 - (n - \Delta + k)|A| - n\Delta + 2e(G) \leq 0$. Solving the quadratic inequality, we obtain the desired bound

$$\alpha_k(G) \leq S_k(G) \leq \frac{n-\Delta+k}{2} + \sqrt{\frac{(n-\Delta+k)^2}{4} + n\Delta - 2e(G)}.$$ 

Finally, if $G$ is $r$-regular, by Observation 2.3(iii), all inequalities become equalities. (ii) This follows from $\omega_k(G) = \alpha_k(\overline{G})$ and item (i). $\blacksquare$

The following corollary is straightforward from previous theorem and Observation 2.4.

**Corollary 5.2.** Let $G$ be a graph on $n$ vertices, with maximum degree $\Delta$ and minimum degree $\delta$. Then

(i) $\alpha(G) \leq \theta(G) \leq S_0(G) \leq \left\lfloor \frac{n-\Delta}{2} + \sqrt{\frac{(n-\Delta)^2}{4} + n\Delta - 2e(G)} \right\rfloor \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor$;

(ii) $\omega(G) \leq \chi(G) \leq L_0(G) \leq \left\lfloor \frac{\delta+1}{2} + \sqrt{\frac{(\delta+1)^2}{4} - n\delta + 2e(G)} \right\rfloor \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2e(G)} \right\rfloor$. 

Proof. (i) This follows from Observation 2.4(i) and Theorem 5.1(i) setting $k = 0$. The last inequality follows because the expression is monotone increasing with $\Delta$ and $\Delta \leq n - 1$.

(ii) This follows from (i), Observation 2.4(ii) and $L_0(G) = S_0(G)$. $\square$

Note that item (i) of Corollary 5.2 is a refinement of the Hansen-Zheng bound [10] which states that $\alpha(G) \leq \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right]$. The inequality $\chi(G) \leq \left[ \frac{1}{2} + \sqrt{\frac{1}{4} + 2e(G)} \right]$ also is well known (cf. Proposition 5.2.1 in [5]).

We will need the following notation. For a set $A$ of vertices of a graph $G$, let $d_r(A) = \sqrt{\frac{1}{|A|} \sum_{v \in A} \deg^r(v)}$. When $r = 1$, we will set $d(A)$ for $d_1(A)$ and when $A = V(G)$, we will set $d_r(G)$ instead of $d_r(V(G))$. Note that $d(G)$ is the average degree of $G$. In the following, we will show that the inequality $\varphi(G) \geq \frac{n}{n - d(G)}$ given in Corollary 4.2 can be improved when $d(G)$ is substituted by $d_3(G)$. However, we will also show that, for $r \geq 4$, $d(G)$ will not be able to be replaced by $d_r(G)$ that easily. First, we need to prove the following lemma.

Lemma 5.3. Let $\beta_1, \beta_2, \ldots, \beta_r \in [0, 1]$ be real numbers such that $\beta_1 + \beta_2 + \ldots + \beta_r \leq r - 1$. Then

$$\sum_{i=1}^{r} (1 - \beta_i) \beta_i^r \leq \left( \frac{r - 1}{r} \right)^r$$

and equality holds if and only if $\beta_1 = \beta_2 = \ldots = \beta_r = \frac{r - 1}{r}$.

Proof. If $r = 1$, then $\beta_1 = 0$ and the inequality is obvious. Let $r \geq 2$. We consider the function $f(x) = (1 - x)x^{r-1}$, $x \geq 0$. From $f'(x) = x^{r-2}(r - 1 - rx)$ we see that $f(x)$ attains its absolute maximum exactly when $x = \frac{r - 1}{r}$ and thus

$$f(x) \leq f\left( \frac{r - 1}{r} \right) = \frac{1}{r} \left( \frac{r - 1}{r} \right)^{r-1}.$$ 

Hence, we have

$$(1 - \beta_i) \beta_i^r = (1 - \beta_i) \beta_i^{r-1} \beta_i \leq \frac{1}{r} \left( \frac{r - 1}{r} \right)^{r-1} \beta_i, \quad i = 1, 2, \ldots, r.$$ 

Now the condition $\beta_1 + \beta_2 + \ldots + \beta_r \leq r - 1$ yields

$$\sum_{i=1}^{r} (1 - \beta_i) \beta_i^r \leq \frac{1}{r} \left( \frac{r - 1}{r} \right)^{r-1} (\beta_1 + \beta_2 + \ldots + \beta_r) \leq \left( \frac{r - 1}{r} \right)^r$$

and the desired inequality holds. Suppose now that we have equality in (11). Then we have equality in all the above given inequalities and hence

$$(1 - \beta_i) \beta_i^{r-1} = \frac{1}{r} \left( \frac{r - 1}{r} \right)^{r-1}, \quad i = 1, 2, \ldots, r,$$

implying thus $\beta_1 = \beta_2 = \ldots = \beta_r = \frac{r - 1}{r}$. $\square$
Theorem 5.4. Let $G$ be a graph on $n$ vertices. Then, the following statements hold:

(i) For every integer $r \leq \varphi(G)$, $\varphi(G) \geq \frac{n}{n - d_r(G)}$. Moreover, equality holds if and only if $G$ is an $\frac{n(\varphi(G)-1)}{\varphi(G)}$-regular graph.

(ii) $\varphi(G) \geq \frac{n}{n - d_1(G)}$. Moreover, equality holds if and only if $G$ is an $\frac{n(\varphi(G)-1)}{\varphi(G)}$-regular graph.

(iii) If $\varphi(G) \neq 2$, then $\varphi(G) \geq \frac{n}{n - d_1(G)}$. Moreover, there exists a graph $G$ for which $\varphi(G) = 2$ and $\varphi(G) < \frac{n}{n - d_1(G)}$.

Proof. (i) Since $d_{r-1}(G) \leq d_r(G)$ for all $r \leq \varphi(G)$, it is enough to prove $\varphi(G) \geq \frac{n}{n - d_\varphi(G)}$. Let $\varphi(G) = \varphi$ and let $V(G) = V_1 \cup V_2 \cup \ldots \cup V_\varphi$ be a partition of $V(G)$ into small sets and let $n_i = |V_i|$, $1 \leq i \leq \varphi$. As $\deg(v) \leq n - n_i$ for every $v \in V_i$ and $1 \leq i \leq \varphi$, we have

$$\beta_i = 1 - \frac{n_i}{n} \leq 1$$

for $1 \leq i \leq \varphi$, the inequality above can be rewritten as

$$(d_{\varphi}(G))^2 n = \sum_{v \in V(G)} \deg^\varphi(v) = \sum_{i=1}^\varphi \sum_{v \in V_i} \deg^\varphi(v) \leq \sum_{i=1}^\varphi n_i(n - n_i)^\varphi. \quad (2)$$

Setting $\beta_i = 1 - \frac{n_i}{n}$ for $1 \leq i \leq \varphi$, the inequality above can be rewritten as

$$(d_{\varphi}(G))^2 n = \sum_{v \in V(G)} \deg^\varphi(v) \leq n^{\varphi+1} \sum_{i=1}^\varphi (1 - \beta_i)^\varphi. \quad (3)$$

Since $\beta_1 + \beta_2 + \ldots + \beta_\varphi = \varphi - 1$, Lemma 5.3 yields $d_{\varphi}(G) \leq \frac{n(\varphi-1)}{\varphi}$, from which the desired inequality $\varphi(G) = \varphi \geq \frac{n}{n - d_{\varphi}(G)}$. Hence we have proved

$$\varphi \geq \frac{n}{n - d_{\varphi}} \geq \frac{n}{n - d_{\varphi-1}(G)} \geq \ldots \geq \frac{n}{n - d_r(G)}. \quad (4)$$

for any $1 \leq r \leq \varphi(G)$.

Suppose now that we have $\varphi(G) = \frac{n}{n - d_r(G)}$ for some $1 \leq r \leq \varphi = \varphi(G)$. Then, we have equality all over the inequality chain (4). In particular, $\varphi = \frac{n}{n - d_{\varphi}(G)}$, which is equivalent to

$$d_{\varphi} = \frac{n(\varphi-1)}{\varphi},$$

and hence we have equality in (2) and (3), too. From the equality in (2), it follows $\deg(v) = n - n_i$ for $v \in V_i$, $1 \leq i \leq \varphi$. From $d_r = \frac{n(\varphi-1)}{\varphi}$ and the equality in (3), we see that in (4) there is equality, too. Moreover, from Lemma 5.3 (it follows (with $r = \varphi$) that $\beta_i = \frac{r-1}{\varphi}$ for $1 \leq i \leq \varphi$ and thus $n_i = \frac{n}{\varphi}$ and $\varphi$ divides $n$. Hence, $\deg(v) = n - n_i = \frac{n(\varphi-1)}{\varphi}$ for all $v \in V_i$ and $1 \leq i \leq \varphi$, turning out that $G$ is $\frac{n(\varphi-1)}{\varphi}$-regular. Conversely, if $G$ is $\frac{n(\varphi-1)}{\varphi}$-regular, then evidently $d_{\varphi}(G) = \frac{n(\varphi-1)}{\varphi} = d_r(G)$ for every $r \leq \varphi$. Then from Theorem 4.4 we have $\varphi(G) = \frac{n}{n - d_r(G)} = \frac{n}{n - d_{\varphi}(G)}$.

(ii) If $\varphi = \varphi(G) \geq 3$, then from item (i) we have $\varphi(G) \geq \frac{n}{n - d_{\varphi}(G)}$ with equality if and only if $G$ is $\frac{(\varphi-1)n}{\varphi}$-regular. It remains to consider the cases $\varphi(G) = 1$ and $\varphi(G) = 2$. Note that $\varphi(G) = 1$ holds if and only if $G = K_n$. Hence, in this case $d_3(G) = 0$ and
\( \varphi(G) = 1 = \frac{n}{n - d_3(G)} \). So assume that \( \varphi(G) = 2 \) and let \( V(G) = V_1 \cup V_2 \) be a partition of \( V(G) \) into two small sets. Setting \( |V_1| = n_1 \) and \( |V_2| = n_2 = n - n_1 \), we have

\[
\sum_{v \in V(G)} \text{deg}^3(v) = \sum_{v \in V_1} \text{deg}^3(v) + \sum_{v \in V_2} \text{deg}^3(v) \leq n_1(n - n_1)^3 + n_2(n - n_2)^3 = n_1n_2(n^2 - 2n_1n_2).
\]

The last expression takes its maximum when \( n_1n_2 = \frac{n^2}{4} \). Hence, it follows \( \sum_{v \in V(G)} \text{deg}^3(v) \leq \frac{n^4}{8} \) and thus \( d_3(G) \leq \frac{n}{2} \), which yields \( \frac{n}{n - d_3(G)} \leq 2 = \varphi(G) \).

Now suppose that \( \varphi(G) = 2 = \frac{n}{n - d_3(G)} \). Then we have equality in the inequality given above. Hence, \( n_1n_2 = \frac{n^2}{4} \) and \( \text{deg}(v) = n - n_i \) for \( v \in V_1 \), \( i = 1, 2 \). Therefore, \( n_1 = n_2 = \frac{n}{2} = \frac{n(\varphi - 1)}{\varphi} \) and \( G \) is an \( \frac{n(\varphi - 1)}{\varphi} \)-regular graph. On the other side, if \( G \) is an \( \frac{n}{2} \)-regular graph, then \( d_3(G) = \frac{n}{2} \) and, from Theorem 4.4 (v), \( \varphi(G) = 2 \). Hence \( \varphi(G) = 2 = \frac{n}{n - d_3(G)} \).

(iii) The case \( \varphi(G) = 1 \) is trivial. If \( \varphi(G) \geq 4 \), then the statement follows from item (i). The case \( \varphi(G) = 3 \) can be proved by straightforward calculations using Lagrange multipliers. As in the case (i), a partition of \( V(G) \) into \( \varphi(G) = 3 \) small sets \( V_1, V_2, V_3 \) with \( |V_1| = n_1 \), \( |V_2| = n_2 \) and \( |V_3| = n_3 \) leads to the inequality

\[
\left( \frac{d_4(G)}{n} \right)^4 \leq \sum_{i=1}^{3} (1 - \beta_i)\beta_i^4 = f(\beta_1, \beta_2, \beta_3),
\]

where \( \beta_i = 1 - \frac{n_i}{n} \) and clearly \( \beta_1 + \beta_2 + \beta_3 = 2 \) and \( \beta_i \in [0, 1] \), for \( i = 1, 2, 3 \). We will show that \( f(\beta_1, \beta_2, \beta_3) \leq \left( \frac{3}{4} \right)^4 \). Let

\[
F(\beta_1, \beta_2, \beta_3, \lambda) = \sum_{i=1}^{3} (1 - \beta_i)\beta_i^4 + \lambda(\beta_1 + \beta_2 + \beta_3 - 2)
\]

be the Lagrange function. The extremal points are either solutions of the system

\[
\begin{align*}
\frac{\partial F}{\partial \beta_i} &= 4\beta_i^3 - 5\beta_i^4 - \lambda = 0, & i = 1, 2, 3 \\
\frac{\partial F}{\partial \lambda} &= \beta_1 + \beta_2 + \beta_3 - 2 = 0
\end{align*}
\]

or they are points on the border. We shall prove that the system has no solution in which \( \beta_1, \beta_2, \beta_3 \) are pairwise distinct. Let us suppose the contrary. Then \( \beta_1, \beta_2, \beta_3 \) are roots of \( g(x) = 5x^4 - 4x^3 + \lambda \). As \( \beta_1 + \beta_2 + \beta_3 = 2 \) from Vieta’s formula follows that the fourth root of \( g \) is \( -\frac{\lambda}{6} \). Therefore \( \lambda = -12 \left( \frac{\beta_3}{2} \right)^2 \) and so \( g(x) \) has only two real roots, which is a contradiction. Let \( (\beta_1, \beta_2, \beta_3) \) be an extremal point which is not on the border. As \( \beta_1, \beta_2, \beta_3 \) are solutions of the system, we can suppose that \( \beta_1 = 2\beta \) and \( \beta_2 = \beta_3 = 1 - \beta \), where \( \beta \in [0, \frac{1}{2}] \). Then

\[
f(\beta_1, \beta_2, \beta_3) = f(\beta) = -30\beta^5 + 8\beta^4 + 12\beta^3 - 8\beta^2 + 2\beta
\]

and

\[
f'(\beta) = -2(3\beta - 1)(25\beta^3 + 3\beta^2 - 5\beta + 1).
\]

\( f' \) has two real roots, \( \frac{1}{3} \) and another one negative. Therefore, \( f \) attains its maximum \( \left( \frac{3}{8} \right)^4 \) in \( [0, \frac{1}{2}] \) exactly when \( \beta = \frac{1}{3} \). It is easy to see that the maximum on the border is \( \frac{1}{12} \), which
is strictly smaller than \((\frac{2}{3})^4\). Hence, we have \(\left(\frac{d_4(G)}{n}\right)^4 \leq \left(\frac{2}{3}\right)^4 = \left(\frac{\varphi(G)-1}{\varphi(G)}\right)\), implying thus that \(\varphi(G) \geq \frac{n}{n-d_4(G)}\).

Consider now the graph \(G = K_{1,9}\). It is clear that \(\varphi(G) = 2\), \(d_4(G) = \sqrt[4]{657} > 5\). Therefore \(2 = \varphi(G) < \frac{10}{10-d_4(G)}\). \(\square\)

**Corollary 5.5.** Let \(G\) be a graph on \(n\) vertices. Then, the following statements hold:

(i) For every integer \(r \leq \varphi(G)\), \(\omega(G) \geq \frac{n}{n-d_r(G)}\) and equality holds if and only if \(G\) is a complete \(\omega(G)\)-partite Turán graph \(K_{\frac{n}{\omega(G)}, \frac{n}{\omega(G)}, \ldots, \frac{n}{\omega(G)}}\).

(ii) \(\omega(G) \geq \frac{n}{n-d_3(G)}\) and equality holds if and only if \(G\) is a complete \(\omega(G)\)-partite Turán graph \(K_{\frac{n}{\omega(G)}, \frac{n}{\omega(G)}, \ldots, \frac{n}{\omega(G)}}\).

(iii) If \(\varphi(G) \neq 2\), then \(\omega(G) \geq \frac{n}{n-d_4(G)}\).

**Proof.** (i) From Theorems 2.1 and 5.4(i), we have \(\omega(G) \geq \varphi(G) \geq \frac{n}{n-d_r(G)}\). Suppose now that \(\omega(G) = \frac{n}{n-d_r(G)}\). Then we have equality in Theorem 5.4(i). Thus, setting \(\varphi(G) = \omega(G) = \omega\), \(G\) is \(\frac{n(\omega-1)}{\omega}\)-regular and \(e(G) = \frac{n^2(\omega-1)}{2\omega}\). Since \(\omega(G) = \omega\), from Turán’s Theorem it follows that \(G\) is a complete \(\omega\)-chromatic regular graph, i.e. \(G\) is a complete \(\omega\)-partite Turán graph \(K_{\frac{n}{\omega}, \frac{n}{\omega}, \ldots, \frac{n}{\omega}}\). Conversely, if \(G\) is the complete \(\omega\)-partite Turán graph \(K_{\frac{n}{\omega}, \frac{n}{\omega}, \ldots, \frac{n}{\omega}}\), then evidently \(d_r(G) = \frac{n(\omega-1)}{\omega}\) and hence \(\omega(G) = \omega = \frac{n}{n-d_r(G)}\).

(ii) From Theorems 2.1 and 5.4(ii), we have \(\omega(G) \geq \varphi(G) \geq \frac{n}{n-d_3(G)}\). Suppose now that \(\omega = \varphi(G) = \frac{n}{n-d_3(G)}\). Then \(\varphi(G) = \frac{n}{n-d_3(G)}\), i.e. we have equality in Theorem 5.4(ii).

Thus, setting \(\varphi(G) = \omega(G) = \omega\), \(G\) is \(\frac{n(\omega-1)}{\omega}\)-regular and \(e(G) = \frac{n^2(\omega-1)}{2\omega}\). Since \(\omega(G) = \omega\), from Turán’s Theorem it follows that \(G\) is a complete \(\omega\)-chromatic regular graph, i.e. \(G = K_{\frac{n}{\omega}, \frac{n}{\omega}, \ldots, \frac{n}{\omega}}\). Conversely, if \(G\) is the complete \(\omega\)-partite Turán graph \(K_{\frac{n}{\omega}, \frac{n}{\omega}, \ldots, \frac{n}{\omega}}\), then evidently \(d_3(G) = \frac{n(\omega-1)}{\omega}\) and hence \(\omega(G) = \omega = \frac{n}{n-d_3(G)}\).

(iii) This follows from Theorems 2.1 and 5.4(iii). \(\square\)

Note that Theorem 5.4(ii) improves the bound \(\varphi(G) \geq \frac{n}{n-d_2(G)}\) given in [1] and Corollary 5.5(ii) is better than the inequality \(\omega(G) \geq \frac{n}{n-d_2(G)}\), given in [6] and later in [11] where the proof was corrected.

Since \(\alpha(G) = \omega(G)\) and \(\Omega(G) = \varphi(G)\), we have the following corollaries.

**Corollary 5.6.** Let \(G\) be a graph on \(n\) vertices. Then, the following statements hold:

(i) For every integer \(r \leq \Omega(G)\), \(\Omega(G) \geq \frac{n}{n-d_r(G)}\). Moreover, equality holds if and only if \(G\) is an \((\frac{n}{\Omega(G)} - 1)\)-regular graph.

(ii) \(\Omega(G) \geq \frac{n}{n-d_3(G)}\). Moreover, equality holds if and only if \(G\) is an \((\frac{n}{\Omega(G)} - 1)\)-regular graph.
Proof. If \( \beta \) is a \( \sum (i) \), then using the definition of \( \beta \) the Caro-Wei bound.

Observation 6.1. In a graph \( G \), every small set is an \( \alpha \)-small set and every \( \alpha \)-small set is a \( \beta \)-small set.

Proof. If \( A \) is a small set of \( G \), then \( n - \deg(v) \geq |A| \) for every vertex \( v \in A \) and we have
\[
\sum_{v \in A} \frac{1}{n - \deg(v)} \leq \sum_{v \in A} \frac{1}{|A|} = 1.
\]
Hence, \( A \) is an \( \alpha \)-small set. Further, if \( A \) is an \( \alpha \)-small set of \( G \), then
\[
1 \geq \sum_{v \in A} \frac{1}{n - \deg(v)} \geq \frac{|A|}{n - d(A)}
\]
by Jensen’s inequality and hence \( d(A) \leq n - |A| \) and thus \( A \) is a \( \beta \)-small set. \( \square \)

Let now \( \varphi^\alpha(G) \) and \( \varphi^\beta(G) \) be the minimum number of \( \alpha \)-small sets and, respectively, \( \beta \)-small sets in which \( V(G) \) can be partitioned. Further, let \( CW(G) = \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \) be the Caro-Wei bound.

Theorem 6.2. Let \( G \) be a graph on \( n \) vertices. Then
\[
(i) \ \omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq \varphi^\beta(G) \geq \left\lceil \frac{n}{n - d(G)} \right\rceil;
\]
\[
(ii) \ \omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq CW(G) \geq \left\lceil \frac{n}{n - \omega(G)} \right\rceil.
\]

Proof. Since every small set is an \( \alpha \)-small set and every \( \alpha \)-small set is a \( \beta \)-small set and because of Theorem 2.1, we have the inequality chain \( \omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq \varphi^\beta(G) \). Now we will prove the remaining bounds.

(i) Let \( t = \varphi^\beta(G) \) and let \( V(G) = A_1 \cup A_2 \cup \ldots \cup A_t \) be a partition of \( V(G) \) into \( \beta \)-small sets. Then, using the definition of \( \beta \)-small set and Jensen’s inequality, we obtain
\[
nd(G) = 2e(G) = \sum_{v \in V(G)} \deg(v) = \sum_{i=1}^{t} \sum_{v \in A_i} \deg(v) \leq \sum_{i=1}^{t} (n - |A_i|)|A_i| \leq n \left(n - \frac{n}{t}\right).
\]

6 Variations of small and large sets

Let \( G \) be a graph on \( n \) vertices and \( A \) a subset of \( V(G) \). We call \( A \) \( \alpha \)-small if \( \sum_{v \in A} \frac{1}{n - \deg(v)} \leq 1 \) and \( \beta \)-small if \( d(A) \leq n - |A| \). Now we observe the following.

Corollary 5.7. Let \( G \) be a graph on \( n \) vertices. Then, the following statements hold:
\[
(i) \ \text{For every integer } r \leq \Omega(G), \ \alpha(G) \geq \frac{n}{n - d_i(G)} \text{ and equality holds if and only if } G \text{ is the union of } \alpha(G) \text{ copies of } K_{\frac{n}{\alpha(G)}}.
\]
\[
(ii) \ \alpha(G) \geq \frac{n}{n - d_i(G)} \text{ and equality holds if and only if } \alpha(G) \text{ copies of } K_{\frac{n}{\alpha(G)}}.
\]
\[
(iii) \ \text{If } \Omega(G) \neq 2, \ \text{then } \Omega(G) \geq \frac{n}{n - d(G)} \text{. Moreover, there exists a graph } G \text{ for which } \varphi(G) = 2 \text{ and } \Omega(G) < \frac{n}{n - d(G)}.
\]
Hence \( d(G) \leq n - \frac{n}{t} = n - \frac{n}{\varphi^\beta(G)} \), which is equivalent to \( \varphi^\beta(G) \geq \frac{n}{n - d(G)} \).

(ii) Let \( V(G) = A_1 \cup A_2 \cup \ldots \cup A_t \) be a partition of \( V(G) \) into \( t = \varphi^\alpha(G) \) \( \alpha \)-small sets. Then, Corollary \( \ref{cor1}(i) \) and the definition of \( \alpha \)-small set yield

\[
\frac{n}{n - d(G)} \leq CW(G) = \sum_{v \in V(G)} \frac{1}{n - \deg(v)} = \sum_{i=1}^{t} \sum_{v \in A_i} \frac{1}{n - \deg(v)} \leq t = \varphi^\alpha(G).
\]

\( \square \)

Let us consider an example. Let \( G \) be a graph obtained from \( 2K_n \) by joining one of the vertices of the first copy of \( K_n \) to all the vertices of the second copy of \( K_n \). Then \( \varphi(G) = 3 \), \( CW(G) = 3 - \frac{2}{n+1} \) and \( \varphi^\beta(G) = 2 \). In this case \( \varphi^\beta(G) \leq CW(G) \). We do not know if \( \varphi^\beta(G) \leq CW(G) \) is always true.

The inequality chains given in Theorem \( \ref{thm2}(i) \) and (ii) together with the fact that \( 2e(G) = nd(G) \) lead to the following corollary.

**Corollary 6.3.** Let \( G \) be a graph on \( n \) vertices. Then

(i) \( e(G) \leq \frac{(\varphi^\beta(G)-1)n^2}{2\varphi^\beta(G)} \leq \frac{(\varphi^\alpha(G)-1)n^2}{2\varphi^\alpha(G)} \leq \frac{(\varphi(G)-1)n^2}{2\varphi(G)} \leq \frac{(\omega(G)-1)n^2}{2\omega(G)} \); 

(ii) \( e(G) \leq \frac{(CW(G)-1)n^2}{2CW(G)} \leq \frac{(\varphi^\alpha(G)-1)n^2}{2\varphi^\alpha(G)} \leq \frac{(\varphi(G)-1)n^2}{2\varphi(G)} \leq \frac{(\omega(G)-1)n^2}{2\omega(G)} \).

As remarked for Corollary \( \ref{cor3} \), the above bounds on \( e(G) \) are better than the bound \( e(G) \leq \frac{n^2(\omega(G)-1)}{2\omega(G)} \) from classical Turán’s Theorem, because \( \omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq CW(G) \) and \( \varphi^\alpha(G) \geq \varphi^\beta(G) \).

Analogous to \( \alpha \)-small and \( \beta \)-small sets, we can define \( \alpha \)-large and \( \beta \)-large sets. Let \( G \) be a graph on \( n \) vertices and \( B \) a subset of \( V(G) \). \( B \) will be called \( \alpha \)-large if \( \sum_{v \in B} \frac{1}{\deg(v)+1} \leq 1 \) and \( \beta \)-large if \( d(B) \geq |B| - 1 \). As for small sets, every large set is an \( \alpha \)-large set and every \( \alpha \)-large set is a \( \beta \)-large set. We also define \( \Omega^\alpha(G) \) and \( \Omega^\beta(G) \) as the minimum number of \( \alpha \)-large sets and, respectively, \( \beta \)-large sets in which \( V(G) \) can be partitioned.

Theorem \( \ref{thm2} \) and Corollary \( \ref{cor3} \) yield, together with the known facts that \( \alpha(G) = \omega(G) \), \( \Omega(G) = \varphi(G) \), \( \Omega^\alpha(G) = \varphi^\alpha(G) \) and \( \Omega^\beta(G) = \varphi^\beta(G) \), the following corollaries.

**Corollary 6.4.** Let \( G \) be a graph on \( n \) vertices. Then

(i) \( \alpha(G) \geq \Omega(G) \geq \Omega^\alpha(G) \geq \Omega^\beta(G) \geq \frac{n}{d(G)+1} \); 

(ii) \( \alpha(G) \geq \Omega(G) \geq \Omega^\alpha(G) \geq CW(G) \geq \frac{n}{d(G)+1} \);

**Corollary 6.5.** Let \( G \) be a graph on \( n \) vertices. Then
\[(i)\ e(G) \geq \frac{n}{2} \left(\frac{n}{\Omega(G)} - 1\right) \geq \frac{n}{2} \left(\frac{n}{\Omega(G)} - 1\right) \geq \frac{n}{2} \left(\frac{n}{\Omega(G)} - 1\right);\]
\[(ii)\ e(G) \geq \frac{n}{2} \left(\frac{n}{\Omega(G)} - 1\right) \geq \frac{n}{2} \left(\frac{n}{\Omega(G)} - 1\right) \geq \frac{n}{2} \left(\frac{n}{\Omega(G)} - 1\right).\]

Let $S^\alpha(G)$ and $S^\beta(G)$ be the maximum cardinality of an $\alpha$-small set and of a $\beta$-small set of $G$, respectively. Analogously, let $L^\alpha(G)$ and $L^\beta(G)$ be the maximum cardinality of an $\alpha$-large set and of a $\beta$-large set of $G$, respectively. We finish this section with the following theorem.

**Theorem 6.6.** Let $G$ be a graph on $n$ vertices, with maximum degree $\Delta$ and minimum degree $\delta$. Then

\[(i)\ \alpha(G) \leq S_0(G) \leq S^\alpha(G) \leq S^\beta(G) \leq \left\lfloor \frac{n - \Delta}{2} + \sqrt{\frac{(n - \Delta)^2}{4} + n \Delta - 2e(G)} \right\rfloor \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor;
\]
\[(ii)\ \omega(G) \leq L_0(G) \leq L^\alpha(G) \leq L^\beta(G) \leq \left\lfloor \frac{d + 1}{2} + \sqrt{\frac{(d + 1)^2}{4} - n \delta + 2e(G)} \right\rfloor \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2e(G)} \right\rfloor.
\]

**Proof.** The inequality chains $\alpha(G) \leq S_0(G) \leq S^\alpha(G) \leq S^\beta(G)$ and $\omega(G) \leq L_0(G) \leq L^\alpha(G) \leq L^\beta(G)$ follow from Theorem 2.2(i) for $k = 0$ and Observation 6.1. The proof of the right side inequalities is analogous to the proof of the Theorem 5.1 in case $k = 0$. □

Note also that Corollary 5.2 follows from this theorem because of $S_0(G) \leq S^\alpha(G)$ and $L_0(G) \leq L^\alpha(G)$.

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