WAVELET ANALYSIS OF THE BESOV REGULARITY OF LÉVY WHITE NOISES

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In this paper, we characterize the local smoothness and the asymptotic growth rate of Lévy white noises. We do so by identifying the weighted Besov spaces in which they are localized. We extend known results in two ways. First, we obtain new bounds for the local smoothness via the Blumenthal-Getoor indices of the Lévy white noise. We deduce the critical local smoothness when the two indices coincide, which is true for symmetric-\(\alpha\)-stable, compound Poisson and symmetric-gamma white noises. Second, we express the critical asymptotic growth rate in terms of the moments properties of the Lévy white noise. Previous analysis only provided lower bounds for both the local smoothness and the asymptotic growth rate. Showing the sharpness of these bounds requires to determine in which Besov spaces a given Lévy white noise is (almost surely) not. Our methods are based on the wavelet-domain characterization of Besov spaces and precise moment estimates for the wavelet coefficients of the noise.

1. Introduction and Main Results. The main topic of this paper is the study of the Besov regularity of Lévy white noises. We are especially interested in identifying the critical local smoothness and the critical asymptotic growth rate of those random processes for any integrability parameter. In a nutshell, our contributions are as follows:

- **Wavelet methods for Lévy white noises.** First appearing in the eighties, especially in the works of Y. Meyer [29] and I. Daubechies [7], wavelet techniques became established as primary tools in functional analysis. As such, they can naturally be used for the study of random processes, as it was done, for instance, for fractional Brownian motion [30], \(\alpha\)-stable processes [32], and for the study of stochastic partial differential equations [19, 20]. In this paper, we demonstrate that such wavelet techniques are also adapted to the analysis of Lévy white noises. In particular, all our results are derived using the wavelet characterization of weighted Besov spaces.

- **New positive results.** We determine in which weighted Besov spaces is a given Lévy white noise \(w\) is. We are able to improve known results; in particular, for the growth properties of the noise. This requires the identification of a new index associated to \(w\), characterized by moments properties.

- **Negative regularity results.** We also determine in which Besov spaces a Lévy white noise is not. This requires a more evolved analysis as compared to positive results and had, to the best of our knowledge, only received limited attention of researchers in the past.

- **Critical local smoothness and asymptotic rate.** The combination of positive and negative results allows determining the critical Besov exponents of the Lévy white noises, both for the local smoothness and the asymptotic behavior. The results are summarized in Theorem 1. Two consequences are the characterization of the Sobolev and Hölder-Zigmund regularities of Lévy white noises in Corollary 1.

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1.1. Local Smoothness and Asymptotic Rate of Tempered Generalized Functions. We construct random processes as random elements in the space $S'({\mathbb R}^d)$ of tempered generalized functions from $\mathbb R^d$ to $\mathbb R$ (see Section 2.1). We will therefore describe their local and asymptotic properties as we would do for a (deterministic) tempered generalized function. To do so, we rely on the family of weighted Besov spaces, which are embedded in $S'({\mathbb R}^d)$ and allows to jointly study the local smoothness and the asymptotic behavior of a generalized function.

Besov spaces are denoted by $B^p_{\tau,\rho}({\mathbb R}^d)$, where $\tau \in \mathbb R$ is the smoothness, $p \in (0,\infty]$ the integrability, and $\rho \in (0,\infty]$ a secondary parameter. In this paper, we focus on the case $p = \rho$. We use the simplified notation $B^\tau_p({\mathbb R}^d) = B^\tau_p({\mathbb R}^d)$ for these spaces which are sometimes refer to as Sobolev-Sobodeckij spaces. We say that $f$ is in the weighted Besov space $B^\tau_p(\mathbb R^d;\rho)$ with weight $\rho \in \mathbb R$ if $\langle \cdot, \cdot \rangle^\rho \times f$ is in the classical Besov space $B^\tau_p(\mathbb R^d)$, with the notation $\langle x \rangle = (1 + \|x\|^2)^{1/2}$. We precisely define weighted Besov spaces in Section 2.3 in terms of wavelet expansions. For the time being, it is sufficient to remember that the space of tempered generalized functions satisfies [24, Proposition 1]

\begin{equation}
S'({\mathbb R}^d) = \bigcup_{\tau,\rho \in \mathbb R} B^\tau_p(\mathbb R^d;\rho).
\end{equation}

Ideally, we aim at identifying in which weighted Besov space a given $f \in S'({\mathbb R}^d)$ is. The relation (1.1) implies that for any $p$, there exists some $\tau, \rho$ for which this is true. For $p$ fixed, Besov spaces are continuously embedded in the sense that, for $\tau_1, \tau_2$ and $\rho, \rho_1, \rho_2$ such that $\tau_1 \leq \tau_2$ and $\rho_1 \leq \rho_2$, we have

\[B^\tau_p(\mathbb R^d;\rho) \subseteq B^{\tau_1}_p(\mathbb R^d;\rho) \quad \text{and} \quad B^\tau_p(\mathbb R^d;\rho_2) \subseteq B^\tau_p(\mathbb R^d;\rho_1).\]

To characterize the properties of $f \in S'({\mathbb R}^d)$, the key is then to determine the two critical exponents $\tau_p(f) \in (-\infty,\infty]$ and $\rho_p(f) \in (-\infty,\infty]$ such that:

- if $\tau < \tau_p(f)$ and $\rho < \rho_p(f)$, then $f \in B^\tau_p(\mathbb R^d;\rho)$; while
- if $\tau > \tau_p(f)$ or $\rho > \rho_p(f)$, then $f \notin B^\tau_p(\mathbb R^d;\rho)$.

The case $\tau_p(f) = \infty$ corresponds to smooth functions, and $\rho_p(f) = \infty$ means that $f$ is rapidly decaying. The quantity $\tau_p(f)$ measures the local smoothness and $\rho_p(f)$ the asymptotic rate of $f$ for the integrability $p$. When $\rho_p(f) < 0$ (which will be the case for Lévy white noise), we talk about the asymptotic growth rate of $f$.

1.2. Local Smoothness and Asymptotic Rate of Lévy White Noises. The complete family of Lévy white noises, defined as random elements in the space of generalized functions, was introduced by I.M. Gel’fand and N.Y. Vilenkin [18]. We briefly recap the concepts required to state our main results. A more complete exposition is given in Section 2.

Our main contributions concern the localization of a Lévy white noise $w$ in weighted Besov spaces. It includes positive ($w$ is almost surely in a given Besov space) and negative ($w$ is almost surely not in a given Besov space) results. In order to characterize the local smoothness $\tau_p(w)$ and the asymptotic rate $\rho_p(w)$, we now introduce some notations. Let $X = \langle w, 1_{[0,1]^d} \rangle$ be the random variable corresponding to the integration of the Lévy white noise $w$ over the domain $[0,1]^d$. The Lévy exponent $\Psi$ of $w$ is the logarithm of the characteristic function of $x$; that is, for every $\xi \in \mathbb R$,

\begin{equation}
\Psi(\xi) = \log E \left[ e^{i\xi(w,1_{[0,1]^d})} \right] = \log E \left[ e^{i\xi X} \right].
\end{equation}
We associate to a Lévy white noises its Blumenthal-Getoor indices, defined as

\begin{align}
\beta_\infty &= \inf \left\{ p > 0 \mid \lim_{|\xi| \to \infty} \frac{\Psi(\xi)}{|\xi|^p} = 0 \right\}, \\
\bar{\beta}_\infty &= \inf \left\{ p > 0 \mid \liminf_{|\xi| \to \infty} \frac{\Psi(\xi)}{|\xi|^p} = 0 \right\}.
\end{align}

The distinction is that \( \beta_\infty \) considers the limit, while \( \bar{\beta}_\infty \) deals with the inferior limit. In general, one has that \( 0 \leq \bar{\beta}_\infty \leq \beta_\infty \leq 2 \). The Blumenthal-Getoor indices are linked to the local behavior of Lévy processes and Lévy white noises (see Section 2.2 for more details and references to the literature).

In addition, we introduce the moment index of \( w \)

\begin{equation}
p_{\text{max}} = \sup \left\{ p > 0 \mid \mathbb{E}[|\langle w, 1_{[0,1]^d} \rangle|^p] < \infty \right\},
\end{equation}

white noise which is closely related—but not identical—to the Pruitt index in general (see Section 2.2). As we shall see, \( p_{\text{max}} \) fully characterizes the asymptotic rate of the Lévy.

The class of Lévy white noise is very general. Specific examples are the Gaussian and compound Poisson white noises, which will receive a special treatment thereafter. We summarize the results of this paper in Theorem 1.

**Theorem 1.** Consider a Lévy white noise \( w \) with Blumenthal-Getoor indices \( 0 \leq \beta_\infty \leq \beta_\infty \leq 2 \) and moment index \( 0 < p_{\text{max}} \leq \infty \). We fix \( 0 < p \leq \infty \).

- If \( w \) is Gaussian, then, almost surely,

\begin{equation}
\tau_p(w) = -\frac{d}{2} \quad \text{and} \quad \rho_p(w) = -\frac{d}{p}.
\end{equation}

- If \( w \) is compound Poisson, then, almost surely,

\begin{equation}
\tau_p(w) = \frac{d}{p} - d \quad \text{and} \quad \rho_p(w) = -\frac{d}{\min(p, p_{\text{max}})}.
\end{equation}

- If \( w \) is non-Gaussian, then, almost surely, if \( p \in (0,2) \), \( p \) is an even integer, or \( p = \infty \),

\begin{equation}
\frac{d}{\max(p, \beta_\infty)} - d \leq \tau_p(w) \leq \frac{d}{\max(p, \bar{\beta}_\infty)} - d \quad \text{and} \quad \rho_p(w) = -\frac{d}{\min(p, p_{\text{max}})}.
\end{equation}

Theorem 1 provides the full answer to the non-Gaussian scenario with \( 0 < p \leq 2 \) when \( \beta_\infty = \beta_\infty \). This covers the case for most of the Lévy white noises used in practice, including symmetric-\( \alpha \)-stable or symmetric-gamma. For \( p > 2 \), we have full results for even integers. Note that the smoothness is \( \tau_p(w) = d/p - d \) in this case (because \( \beta_\infty \leq 2 \)). In Section 6, we refine Theorem 1 by showing that the relation \( \tau_p(w) \leq \frac{d}{\max(p, \beta_\infty)} - d \) and \( \rho_p(w) = -\frac{d}{\min(p, p_{\text{max}})} \) are valid for any \( p > 0 \). We conjecture that (1.8) is actually true without restriction on \( p \). Two direct consequences are the identification of the Sobolev \( (p = 2) \) and Hölder-Zigmund \( (p = \infty) \) regularity of Lévy white noises.

**Corollary 1.** Let \( w \) be a Lévy white noise in \( \mathcal{S}'(\mathbb{R}^d) \) with indices \( p_{\text{max}}, \beta_\infty, \bar{\beta}_\infty \). Then, the Sobolev local smoothness and asymptotic growth rate \( (p = 2) \) are

\begin{equation}
\tau_2(w) = -\frac{d}{2} \quad \text{and} \quad \rho_2(w) = -\frac{d}{\min(2, p_{\text{max}})}.
\end{equation}
Moreover, the Hölder-Zigmund local smoothness and asymptotic growth rate \((p = \infty)\) are

\[
\begin{align*}
\tau_\infty(w) &= -\frac{d}{2} \quad \text{and} \quad \rho_\infty(w) = 0 \quad \text{if } w \text{ is Gaussian, and} \\
\tau_\infty(w) &= -d \quad \text{and} \quad \rho_\infty(w) = -\frac{d}{\rho_{\max}} \quad \text{otherwise.}
\end{align*}
\]

**Proof.** The case \(p = 2\) is directly deduced from Theorem 1 and the relation \(\beta_\infty \leq 2\). For \(p = \infty\), we use again Theorem 1 with \(p = 2m\), together with the embedding relations (Proposition 2) from which we know that \(\tau_w(\infty) = \lim_{m \to \infty} \tau_w(2m)\) and \(\rho_w(\infty) = \lim_{m \to \infty} \rho_w(2m)\).

Theorem 1 also allows deducing the following results on the local smoothness of Lévy processes.

**Corollary 2.** Let \(X\) be a Lévy process with Blumenthal-Getoor indices \(0 \leq \bar{\beta}_\infty \leq \beta_\infty \leq 2\). Then, we have almost surely that, for any \(0 < p \leq \infty\),

\[
\begin{align*}
\tau_p(X) &= \frac{1}{2} \quad \text{if } X \text{ is Gaussian, and} \\
\tau_p(X) &= \frac{1}{p} \quad \text{if } X \text{ is compound Poisson.}
\end{align*}
\]

In the general case, we have almost surely that, for \(p \in (0, 2)\), \(p\) is an even integer, or \(p = \infty\),

\[
\frac{1}{\max(p, \beta_\infty)} \leq \tau_p(X) \leq \frac{1}{\max(p, \bar{\beta}_\infty)}.
\]

**Proof.** A Lévy white noise \(w\) is the weak derivative of the corresponding Lévy process \(X\) with identical Lévy exponent. This well-known fact has been rigorously shown in the sense of generalized random processes in [5]. A direct consequence is that \(\tau_p(X) = \tau_p(w) + 1\), where \(w = X'\). Then, Corollary 2 is a reformulation of the local smoothness results of Theorem 1 with \(d = 1\).

1.3. **Related Works on Lévy Processes and Lévy White Noises.** In this section, for comparison purposes, we reinterpret all the results in terms of the critical smoothness and asymptotic rate of the considered random processes.

**Lévy processes.** Most of the attention has been so far devoted to classical Lévy processes \(s : \mathbb{R} \to \mathbb{R}\). The Brownian motion was studied in [42, 35, 4]. The work of [4] also contains results on the regularity of fractional Brownian motions and SαS processes. By exploiting the self-similarity of the stable processes, Ciesielski et al. obtained the following results the Gaussian [4, Theorem IV.3] and stable non-Gaussian [4, Theorem VI.1] scenarios:

\[
\begin{align*}
\tau_p(X_{\text{Gauss}}) &\leq \frac{1}{\min(2, p)} \\
\tau_p(X_{\alpha}) &\leq \frac{1}{\min(p, \alpha)}
\end{align*}
\]

for any \(1 \leq p < \alpha\), where \(X_{\text{Gauss}}\) is the Brownian motion and \(X_{\alpha}\) is the SαS process with parameter \(1 < \alpha < 2\).

The complete family of Lévy processes—and more generally of Lévy-type processes—has been considered by R. Schilling in a series of papers [38, 39, 40] synthesized in [3, Chapter V] and by V.
in the simplified scenario of

We then have that
\[
Gaussian white noise. As a corollary of \[(1.15)\], one then deduces that \(\tau_p(w) = -\frac{d}{p} - \frac{1}{\min(p_{\max}, 2)}\).
\]

Lévy white noises. Veraar extensively studied the local Besov regularity of the \(d\)-dimensional Gaussian white noise. As a corollary of \([47, Theorem 3.4]\), one then deduces that \(\tau_p(w_{Gauss}) = -d/2\). This work is based on the Fourier series expansion of the process, which is specific to the Gaussian case.

In our own works, we have investigated the question for general Lévy white noises in dimension \(d\) in the periodic \([16]\) and global settings \([12]\). We had obtained the lower bounds
\[
\frac{d}{\max(p, \beta_\infty)} - d \leq \tau_p(w) \quad \text{and} \quad -\frac{d}{\min(p, p_{\max}, 2)} \leq \rho_p(w).
\]

These estimates are improved by Theorem 1, which now provides an upper bound for \(\tau_p(w)\) and shows that \((1.19)\) is sharp when \(\beta_\infty = \beta_0\). It is also worth noticing that the lower bound of \((1.20)\) is sharp if and only if \(p_{\max} \leq 2\); in particular, when the Lévy white noise has an infinite variance (\(p_{\max} < 2\)).

1.4. Sketch of Proof and the Role of Wavelet Methods. Our techniques are based on the wavelet characterization of Besov spaces as presented by H. Triebel in \([44]\). We shall see that the wavelets are especially relevant to the analysis of Lévy white noises.

We briefly present the strategy of the proof of Theorem 1 in the simplified scenario of \(d = 1\). The more general case, \(d \geq 1\), is analogous (up to some normalization factors) and will be comprehensively introduced in Section 2. Let \((\psi_M, \psi_F)\) be the mother and father Daubechies wavelet of a fixed order (the choice of the order has no influence on the results), respectively. For \(j \geq 0\) and \(k \in \mathbb{Z}\), we define the rescaled and shifted functions \(\psi_{F,k} = \psi_F(\cdot - k)\) and \(\psi_{j,M,k} = 2^{j/2} \psi_M(2^j \cdot - k)\). Then, the family \((\psi_{F,k})_{k \in \mathbb{Z}} \cup (\psi_{j,M,k})_{j \geq 0, k \in \mathbb{Z}}\) forms an orthonormal basis of \(L_2(\mathbb{R}^d)\). For a given one-dimensional Lévy white noise \(w\), one considers the family of random variables
\[
((w, \psi_{F,k}))_{k \in \mathbb{Z}} \cup ((w, \psi_{j,M,k}))_{j \geq 0, k \in \mathbb{Z}}.
\]

We then have that \(w = \sum_{k \in \mathbb{Z}} (w, \psi_{F,k}) \psi_{F,k} + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} (w, \psi_{j,M,k}) \psi_{j,M,k}\), for which the convergence is almost sure in \(S'(\mathbb{R})\).
Then, for $0 < p \leq \infty$ and $\tau, \rho \in \mathbb{R}$, the random variable

\[(1.22) \quad ||w||_{B^p_\rho(\mathbb{R};\rho)} = \left(\sum_{k \in \mathbb{Z}} \langle k \rangle^p \langle w, \psi_{F,k} \rangle^p + \sum_{j \geq 0} 2^{j(\tau p - 1 + \frac{\rho}{2})} \sum_{k \in \mathbb{Z}} \langle 2^{-j} k \rangle^p \langle w, \psi_{j,M,k} \rangle^p \right)^{1/p} \]

is well-defined, and takes values in $[0, \infty]$. We adapt (1.22) in the usual manner when $p = \infty$. We recognize the Besov (quasi-)norm that characterizes the Besov localization of the Lévy white noise. This means that $w$ is a.s. (almost surely) in $B^p_\rho(\mathbb{R};\rho)$ if and only if $||w||_{B^p_\rho(\mathbb{R};\rho)} < \infty$ a.s., and a.s. not in $B^p_\rho(\mathbb{R};\rho)$ if and only if $||w||_{B^p_\rho(\mathbb{R};\rho)} = \infty$ a.s.

We then fix $0 < p \leq \infty$. We assume that we have guessed the values $\tau_p(w)$ and $\rho_p(w)$ introduced in Section 1. Here are the main steps leading to the proof that these values are the correct ones.

- For $\tau < \tau_p(w)$ and $\rho < \rho_p(w)$, we show that $||w||_{B^p_\rho(\mathbb{R};\rho)} < \infty$ a.s. For $p < \rho_{\max}$ (see (1.5)), we establish the strongest result $\mathbb{E}[||w||^p_{B^p_\rho(\mathbb{R};\rho)}] < \infty$. This requires moment estimates for the wavelet coefficients of Lévy white noise; that is, a precise estimation of the behavior of $\mathbb{E}[|\langle w, \psi_{j,G,k} \rangle|^p]$ as $j$ goes to infinity. When $p = \rho_{\max}$, the random variables $\langle w, \psi_{j,G,k} \rangle$ have an infinite $p$th moment and the present method is not applicable. In that case, we actually deduce the result using embeddings relations between Besov spaces. It appears that this approach is sufficient to obtain sharp results.

- For $\tau > \tau_p(w)$, we show that $||w||_{B^p_\rho(\mathbb{R};\rho)} = \infty$ a.s. To do so, we only consider the mother wavelet and truncate the sum over $k$ to yield the lower bound

\[(1.23) \quad ||w||^p_{B^p_\rho(\mathbb{R};\rho)} \geq C \sum_{j \geq 0} 2^{j(\tau p - \min d, \frac{\rho}{2})} \sum_{0 \leq k_1, \ldots, k_d < 2^j} ||\langle w, \psi_{j,M,k} \rangle||^p \]

for some constant $C$ such that $(2^{-j} k)^{\rho_p} \geq C$ for every $j \geq 0$ and $0 \leq k < 2^j$. We then need to show that the wavelet coefficients $\langle w, \psi_{j,G,k} \rangle$ cannot be too small altogether using Borel-Cantelli-type arguments. Typically, this requires to control the evolution of quantities such as $\mathbb{P}(|\langle w, \psi_{j,G,k} \rangle| > x)$ with respect to $j$ and is again based on moment estimates.

- For $p > \rho_p(w)$, we show again that $||w||_{B^p_\rho(\mathbb{R};\rho)} = \infty$ a.s. This time, we only consider the father wavelet in (1.22) and use the lower bound

\[(1.24) \quad ||w||^p_{B^p_\rho(\mathbb{R};\rho)} \geq \sum_{k \in \mathbb{Z}} \langle k \rangle^{\rho_p} ||\langle w, \psi_{0,F,k} \rangle||^p. \]

A Borel-Cantelli-type argument is again used to show that the $||\langle w, \psi_{0,F,k} \rangle||$ cannot be too small altogether, and that the Besov norm is a.s. infinite.

The rest of the paper is dedicated to the proof of Theorem 1. The required mathematical concepts—Lévy white noises as generalized random processes and weighted Besov spaces—are laid out in Section 2. In Sections 3, 4, and 5, we consider the case of Gaussian noises, compound Poisson noises, and finite moments Lévy white noises, respectively. The general case for any Lévy white noise is deduced in Section 6. Finally, we discuss our results and give important examples in Section 7.

2. Preliminaries: Lévy White Noises and Weighted Besov Spaces.

2.1. Lévy White Noises as Generalized Random Processes. Let $\mathcal{S}(\mathbb{R}^d)$ be the space of rapidly decaying smooth functions from $\mathbb{R}^d$ to $\mathbb{R}$. It is endowed with its natural Fréchet nuclear topology [43]. Its topological dual is the space of tempered generalized functions $\mathcal{S}'(\mathbb{R}^d)$. We shall define
random processes as random elements of the space $S'(\mathbb{R}^d)$. This allows for a proper definition of Lévy white noises even if they do not have a pointwise interpretation (they can only be described by their effect into test functions).

The space $S'(\mathbb{R}^d)$ is endowed with the strong topology and $\mathcal{B}(S'(\mathbb{R}^d))$ denotes the Borelian $\sigma$-field for this topology. Throughout the paper, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

**Definition 1.** A generalized random process $s$ is a measurable function from $(\Omega, \mathcal{F})$ to $(S'(\mathbb{R}^d), \mathcal{B}(S'(\mathbb{R}^d)))$. Its probability law is the probability measure on $S'(\mathbb{R}^d)$ defined for $B \in \mathcal{B}(S'(\mathbb{R}^d))$ by

$$\mathcal{P}_s(B) = \mathbb{P}\{\omega \in \Omega, s(\omega) \in B\}. \tag{2.1}$$

The characteristic functional of $s$ is the functional $\widehat{\mathcal{P}}_s : S(\mathbb{R}^d) \to \mathbb{C}$ such that

$$\widehat{\mathcal{P}}_s(\varphi) = \int_{S'(\mathbb{R}^d)} e^{i(u, \varphi)} d\mathcal{P}_s(u). \tag{2.2}$$

The characteristic functional is continuous, positive-definite over $S(\mathbb{R}^d)$, and normalized such that $\widehat{\mathcal{P}}_s(0) = 1$. The converse of this result is also true: if $\widehat{\mathcal{P}}$ is a continuous and positive-definite functional over $S(\mathbb{R}^d)$ such that $\widehat{\mathcal{P}}(0) = 1$, it is the characteristic functional of a generalized random process in $S'(\mathbb{R}^d)$. This is known as the Minlos-Bochner theorem [18, 31]. It means in particular that one can define generalized random processes via the specification of their characteristic functional. Following Gelfand and Vilenkin, we use this principle to introduce Lévy white noises.

We consider functionals of the form $\widehat{\mathcal{P}}(\varphi) = \exp\left(\int_{\mathbb{R}^d} s(\varphi(x))dx\right)$. It is known [18] that $\widehat{\mathcal{P}}$ is a characteristic functional over the space $D(\mathbb{R}^d)$ of compactly supported smooth functions, if and only if the function $\Psi : \mathbb{R} \to \mathbb{C}$ is continuous, conditionally positive-definite, with $\Psi(0) = 0$ [18]. A function $\Psi$ satisfying these conditions is called a Lévy exponent and can be decomposed according to the Lévy-Khintchine theorem [37] as

$$\Psi(\xi) = i\mu\xi - \frac{\sigma^2\xi^2}{2} + \int_{\mathbb{R}} (e^{i\xi t} - 1 - i\xi t \mathbb{1}_{|t| \leq 1}) d\nu(t) \tag{2.3}$$

where $\mu \in \mathbb{R}$, $\sigma^2 \geq 0$, and $\nu$ is a Lévy measure; that is, a positive measure on $\mathbb{R}$ such that $\nu\{0\} = 0$ and $\int_{\mathbb{R}} \inf(1, t^2) d\nu(t) < \infty$. The triplet $(\mu, \sigma^2, \nu)$ is unique and called the Lévy triplet of $\Psi$.

In our case, we are only interested by defining random processes, especially Lévy white noises, over $S'(\mathbb{R}^d)$. This requires an adaptation of the theory of Gelfand and Vilenkin. We say that the Lévy exponent $\Psi$ satisfies the $\epsilon$-condition if there exists some $\epsilon > 0$ such that $\int_{\mathbb{R}} \inf(|t|, t^2) d\nu(t) < \infty$, with $\nu$ the Lévy measure of $\Psi$. This condition is fulfilled by all the Lévy measures encountered in practice. Then, the functional $\widehat{\mathcal{P}}(\varphi) = \exp\left(\int_{\mathbb{R}^d} s(\varphi(x))dx\right)$ is a characteristic functional over $S(\mathbb{R}^d)$ if and only if $\Psi$ is a Lévy exponent satisfying the $\epsilon$-condition. The sufficiency is proved in [11] and the necessity in [5].

**Definition 2.** A Lévy white noise in $S'(\mathbb{R}^d)$ (or simply Lévy white noise) is a generalized random process $w$ with characteristic functional of the form

$$\widehat{\mathcal{P}}_w(\varphi) = \exp\left(\int_{\mathbb{R}^d} s(\varphi(x))dx\right) \tag{2.4}$$

for every $\varphi \in S(\mathbb{R}^d)$, where $\Psi$ is a Lévy exponent satisfying the $\epsilon$-condition.

The Lévy triplet of $w$ is denoted by $(\mu, \sigma^2, \nu)$. Then, we say that $w$ is a Gaussian white noise if $\nu = 0$, compound Poisson white noise if $\sigma^2 = 0$ and $\nu = \lambda \mathcal{P}$, with $\lambda > 0$ and $\mathcal{P}$ a probability measure such that $\mathcal{P}\{0\} = 0$, and a finite moment white noise if $\mathbb{E}[|\langle w, \varphi \rangle|^p] < \infty$ for any $\varphi \in S(\mathbb{R}^d)$ and $p > 0$. 
Lévy white noises are stationary and independent at every point, meaning that $\langle w, \varphi_1 \rangle$ and $\langle w, \varphi_2 \rangle$ are independent as soon as $\varphi_1$ and $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$ have disjoint supports.

One can extend the space of test functions that can be applied to the Lévy white noise. This is done by approximating a test function $\varphi$ with functions in $\mathcal{S}(\mathbb{R}^d)$ and showing that the underlying sequence of random variables converges in probability to a random variable that we denote by $\langle w, \varphi \rangle$. This principle is developed with more generality in [13] by connecting the theory of generalized random process to independently scattered random measures in the sense of B.S. Rajput and J. Rosinski [34]. In particular, as soon as $\varphi \in L_2(\mathbb{R}^d)$ is compactly supported, the random variable $\langle w, \varphi \rangle$ is well-defined. Daubechies wavelets or indicator functions of measurable sets with finite Lebesgue measures are in this case. This was implicitely used in Section 1.2 when considering the random variable $\langle w, 1_{[0,1]^d} \rangle$.

2.2. Indices of Lévy White Noises. First of all, we will only consider Lévy white noises whose Lévy exponent satisfies the sector condition; that is, there exists $M > 0$ such that

$$|\Im\{\Psi\}(\xi)| \leq M |\Re\{\Psi\}(\xi)|.$$  

This condition ensures that no drift is dominating the Lévy white noise (a drift appears as purely imaginary in the Lévy exponent) and is needed for linking the indices with the Lévy measure [39, 8].

In Theorem 1, the smoothness and growth rate of Lévy white noises is characterized in terms of the indices (1.3), (1.4), and (1.5). We give here some additional insight about these quantities. The two former are classical, while the latter has never been considered to characterize the behavior of Lévy processes or noises for the best of our knowledge. For the notations of the different indices, with the exception of $p_{\text{max}}$, we follow [3].

The index $\beta_{\infty}$ was introduced by R. Blumenthal and R. Getoor [2] in order to characterize the behavior of Lévy process at the origin. This quantity appears to characterize many local properties of random processes driven by Lévy white noises, including the Hausdorff dimension of the image set [3], the spectrum of singularities [23, 9], the Besov regularity [38, 40, 12, 3], the local self-similarity [14], or the local compressibility [15]. Finally, the index $\beta_{\infty}$ plays a crucial role in the specification of negative results; that is, to identify the Besov spaces in which the Lévy white noises are not. It satisfies moreover the relation $\beta_{\infty} \leq \beta_{\infty}$.

In [33], W. Pruitt proposed the index

$$\beta_0 = \sup \left\{ p > 0 \mid \lim_{|\xi| \to 0} \frac{|\Psi(\xi)|}{|\xi|^p} = 0 \right\}$$

as the asymptotic counterpart of $\beta_{\infty}$. This quantity appears in the asymptotic growth rate of the supremum of Lévy(-type) processes [39] and the asymptotic self-similarity of random processes driven by Lévy white noises [14]. The Pruitt index differs from the index $p_{\text{max}}$ appearing in Theorem 1. Actually, the two quantities are linked by the relation $\beta_0 = \inf(p_{\text{max}}, 2)$. This is shown by linking $\beta_0$ to the Lévy measure [3] and knowing that $\beta_0 \leq 2$ (see the appendix of [8] for a short and elegant proof). This means that $\beta_0 < p_{\text{max}}$ when the Lévy white noise has finite moments of order bigger than 2, and one cannot recover $p_{\text{max}}$ from $\beta_0$ in this case. It is therefore required to introduce the index $p_{\text{max}}$ in addition to the Pruitt index in our analysis.

The first part of the next result is the famous Lévy-Itô decomposition, usually given for Lévy processes and reformulated here for Lévy white noises. We add a proof in the context of generalized random processes for the sake of completeness. The second part presents how does the indices of the corresponding noise behave.
Proposition 1. A Lévy white noise \( w \) can be decomposed as
\[
w = w_1 + w_2 + w_3
\]
with \( w_1 \) a Gaussian noise, \( w_2 \) a compound Poisson noise, and \( w_3 \) a Lévy white noise with finite moments, the three noises being independent. Moreover, we have the following relations between the indices:

- If \( w \) has no Gaussian part (\( w_1 = 0 \)), then
  \[
  \beta_\infty(w) = \beta_\infty(w_3) \geq \beta_\infty(w_2) = 0;
  \beta_\infty(w) = \beta_\infty(w_3) \geq \beta_\infty(w_2) = 0;
  p_{\max}(w) = p_{\max}(w_2) \leq p_{\max}(w_3) = \infty.
  \]

- If \( w \) has a Gaussian part, then
  \[
  \beta_\infty(w) = \beta_\infty(w) = 2;
  p_{\max}(w) = p_{\max}(w_2) \leq p_{\max}(w_1) = p_{\max}(w_3) = \infty.
  \]

Proof. Two generalized random processes \( s_1 \) and \( s_2 \) are independent if and only if one has, for every \( \varphi \in S(\mathbb{R}^d) \),
\[
\hat{\mathcal{P}}_{s_1+s_2}(\varphi) = \hat{\mathcal{P}}_{s_1}(\varphi)\hat{\mathcal{P}}_{s_2}(\varphi).
\]
We split the Lévy exponent \( \Psi \) of \( w \) with Lévy triplet \((\mu, \sigma^2, \nu)\) as
\[
\Psi(\xi) = \left( i\mu \xi + \frac{\sigma^2 \xi^2}{2} \right) + \left( \int_{\mathbb{R}} (e^{i\xi t} - 1) \mathbf{1}_{|t| > 1} d\nu(t) \right) + \left( \int_{\mathbb{R}} (e^{i\xi t} - 1 - i\xi t) \mathbf{1}_{|t| \leq 1} d\nu(t) \right)
\]
\[
= \Psi_1(\xi) + \Psi_2(\xi) + \Psi_3(\xi).
\]
Then, \( \Psi_1, \Psi_2, \) and \( \Psi_3(\xi) \) are three Lévy exponents with respective triplets \((\mu, \sigma^2, 0), (0, 0, \mathbf{1}_{|\cdot| \leq 1} \nu), \) and \((0, 0, \mathbf{1}_{|\cdot| > 1} \nu)\). We denote by \( w_1, w_2, \) and \( w_3 \) the respective underlying Lévy white noises. We easily see that
\[
\hat{\mathcal{P}}_{w}(\varphi) = \hat{\mathcal{P}}_{w_1}(\varphi)\hat{\mathcal{P}}_{w_2}(\varphi)\hat{\mathcal{P}}_{w_3}(\varphi)
\]
showing the independence of the noises. It is clear that \( w_1 \) is Gaussian. Moreover, \( \int_{\mathbb{R}} d\nu_2(t) = \int_{|t| > 1} d\nu(t) < \infty \), hence \( w_2 \) is a compound Poisson noise in the sense of Definition 2. Finally, the moments of \( w_3 \) are finite since \( \int_{|t| > 1} |t|^p d\nu_3(t) = 0 < \infty \) for every \( p > 0 \) and using [37, Theorem 25.3].

The relations between the Blumenthal-Getoor indices are easily deduced from the fact that \( \Psi(\xi) = \Psi_1(\xi) + \Psi_2(\xi) + \Psi_3(\xi) \) and \( \Psi_1(\xi) = -\sigma_0^2 \xi^2 / 2 \), with \( \sigma_0^2 \) the variance of \( w_1 \). For the moment indice, we recall that for any \( p > 0 \), there exists \( c_p > 0 \) such that \( |a + b|^p \leq c_p (|a|^p + |b|^p) \). Therefore, if \( X \) and \( Y \) are two random variables such that \( \mathbb{E}[|Y|^p] < \infty \), we have
\[
c_p^{-1}(\mathbb{E}[|X|^p] - \mathbb{E}[|Y|^p]) \leq \mathbb{E}[|X + Y|^p] \leq c_p (\mathbb{E}[|X|^p] + \mathbb{E}[|Y|^p]).
\]
Applying this to \( X = \langle w_2, \mathbf{1}_{[0,1]^d} \rangle \) and \( Y = \langle w_1 + w_3, \mathbf{1}_{[0,1]^d} \rangle \), which has finite \( p \)th moments for any \( p > 0 \), we deduce from (2.11) that
\[
\mathbb{E} \left[ \left| \langle w, \mathbf{1}_{[0,1]^d} \rangle \right|^p \right] = \mathbb{E}[|X + Y|^p] < \infty \iff \mathbb{E} \left[ \left| \langle w_2, \mathbf{1}_{[0,1]^d} \rangle \right|^p \right] = \mathbb{E}[|X|^p] < \infty.
\]
Hence, \( w \) and \( w_2 \) have the same moment index. \( \square \)
2.3. Weighted Besov Spaces. We define the family of weighted Besov spaces based on wavelet methods, as exposed in [44]. Essentially, Besov spaces are subspaces of $\mathcal{S}'(\mathbb{R}^d)$ that are characterized by weighted sequence norms of the wavelet coefficients. Following Triebel, we use Daubechies wavelets, which we introduce first.

The scale and shift parameters of the wavelets are respectively denoted by $j \geq 0$ and $\mathbf{k} \in \mathbb{Z}^d$. The letters $M$ and $F$ refer to the gender of the wavelet ($F$ for the father wavelets and $G$ for the mother wavelet). Consider two functions $\psi_M$ and $\psi_F \in L_2(\mathbb{R})$. We set $G^0 = \{M, F\}^d$ and, for $j \geq 1$, $G^j = G^0 \setminus \{F^d\}$. For $G = (G_1, \ldots, G_d) \in G^j$, called a gender, we set, for every $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $\psi_G(\mathbf{x}) = \prod_{i=1}^d \psi_{G_i}(x_i)$. For $j \geq 0$, $G \in G^j$, and $\mathbf{k} \in \mathbb{Z}^d$, we define

\begin{equation}
\psi_{j,G,\mathbf{k}} := 2^{jd/2} \psi_G(2^j \cdot -\mathbf{k}).
\end{equation}

It is known that, for any regularity parameter $r_0 \geq 1$, there exists two functions $\psi_M, \psi_F \in L_2(\mathbb{R})$ that are compactly supported, with at least $r_0$ continuous derivatives such that the family

\begin{equation}
\{\psi_{j,G,\mathbf{k}}\}_{(j,G,\mathbf{k}) \in \mathbb{N} \times G^1 \times \mathbb{Z}^d}
\end{equation}

is an orthonormal basis of $L_2(\mathbb{R}^d)$ [44]. Concretely, one can consider the family of Daubechies wavelets [6].

We now introduce the family of weighted Besov spaces $B^r_p(\mathbb{R}^d; \rho)$. Traditionally, Besov spaces also depends on the additional parameter $q \in (0, \infty]$ (see for instance [44, Definition X]). We should only consider the case $q = p$ in this paper, so that we do not refer to this parameter.

The following definition of weighted Besov spaces is based on wavelets. It is equivalent with the more usual Fourier-based definitions. This equivalence is proved in [44].

**Definition 3.** Let $\tau, \rho \in \mathbb{R}$ and $0 < p \leq \infty$. Fix $r_0 > \max(\tau, d(1/p - 1)_+ - \tau)$ and consider a family of compactly supported wavelets $\{\psi_{j,G,\mathbf{k}}\}_{(j,G,\mathbf{k}) \in \mathbb{N} \times G^1 \times \mathbb{Z}^d}$ with at least $r_0$ continuous derivatives.

The weighted Besov space $B^r_p(\mathbb{R}^d; \rho)$ is the collection of tempered generalized functions $f \in \mathcal{S}'(\mathbb{R}^d)$ that can be written as

\begin{equation}
f = \sum_{(j,G,\mathbf{k}) \in \mathbb{N} \times G^1 \times \mathbb{Z}^d} c_{j,G,\mathbf{k}} \psi_{j,G,\mathbf{k}},
\end{equation}

where the convergence holds unconditionally on $\mathcal{S}'(\mathbb{R}^d)$.

The parameter $r_0$ in Definition 3 is chosen such that the wavelet is regular enough to be applied to a function of $B^r_p(\mathbb{R}^d; \rho)$. When the convergence (2.15) occurs, the duality product $\langle f, \psi_{j,G,\mathbf{k}} \rangle$ is well defined and we have $c_{j,G,\mathbf{k}} = \langle f, \psi_{j,G,\mathbf{k}} \rangle$. Moreover, the quantity

\begin{equation}
\|f\|_{B^r_p(\mathbb{R}^d; \rho)} := \left( \sum_{j \geq 0} 2^{j(\tau p - d + dp \rho)} \sum_{G \in G^1} \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle 2^{-j} \mathbf{k}\rangle^{\rho p} |\langle f, \psi_{j,G,\mathbf{k}} \rangle|^p \right)^{1/p}
\end{equation}

is finite for $f \in B^r_p(\mathbb{R}^d; \rho)$ and specifies a norm (a quasi-norm, respectively) on $B^r_p(\mathbb{R}^d; \rho)$ for $p \geq 1$ ($p < 1$, respectively). The space $B^r_p(\mathbb{R}^d; \rho)$ is a Banach (a quasi-Banach, respectively) for this norm (quasi-norm, respectively).

**Proposition 2** (Embeddings between weighted Besov spaces). Let $0 < p_0 \leq p_1 \leq \infty$ and $\tau_0, \tau_1, \rho_0, \rho_1 \in \mathbb{R}$. 


Wavelet Analysis of Lévy White Noises

Fig 1: Representation of the embeddings between Besov spaces: If \( f \in B^{\tau_0}_{\rho_0}(\mathbb{R}^d; \rho_0) \), then \( f \) is in any Besov space that is in the shaded green regions. Conversely, if \( f \not\in B^{\tau_0}_{\rho_0}(\mathbb{R}^d; \rho_0) \), then \( f \) is not in any of the Besov spaces of the shaded red regions.

- \textbf{We have the embedding} \( B^{\tau_0}_{\rho_0}(\mathbb{R}^d; \rho_0) \subseteq B^{\tau_1}_{\rho_1}(\mathbb{R}^d; \rho_1) \) \textbf{as soon as}

\[
\tau_0 - \tau_1 > \frac{d}{\rho_0} - \frac{d}{\rho_1} \quad \text{and} \quad \rho_0 > \rho_1.
\]

- \textbf{We have the embedding} \( B^{\tau_1}_{\rho_1}(\mathbb{R}^d; \rho_1) \subseteq B^{\tau_0}_{\rho_0}(\mathbb{R}^d; \rho_0) \) \textbf{as soon as}

\[
\rho_0 - \rho_1 > \frac{d}{\rho_0} - \frac{d}{\rho_1} \quad \text{and} \quad \tau_0 > \tau_1.
\]

The embedding for the relation (2.17) is proved in [10, Section 4.2.3] and the one for (2.18) in [12, Section 2.2.2]. Proposition 2 is summarized in the two Triebel diagrams\(^1\) of Figure 1.

As a simple example, we obtain the Besov localization of the Dirac distribution. Of course, this result is known, and an alternative proof can be found for instance in [41]. We believe that it is interesting to give our own proof here. First, it illustrates how to use the wavelet-based characterization of Besov spaces, and second, it will be used to obtain sharp results for compound Poisson processes, justifying to include the proof for the sake of completeness.

**Proposition 3.** The Dirac impulse \( \delta \) is in \( B^\tau_p(\mathbb{R}^d; \rho) \) if and only if \( \tau < \frac{d}{p} - d \).

**Proof.** The definition of the Besov (quasi-)norm easily gives

\[
\|\delta\|_{B^\tau_p(\mathbb{R}^d; \rho)}^p = \sum_{j \geq 0} 2^j(\tau - d + dp) \sum_{G \in G_j} \sum_{k \in \mathbb{Z}^d} \langle 2^{-j} \hat{k} \rangle^{\rho p} |\psi_G(k)|^p.
\]

The common support \( K_\psi \) of the \( \psi_G \) is compact. Therefore, only finitely many \( \psi_G(k) \) are non zero, and for such \( k \) and every \( j \) we have

\[
0 < \min_{x \in K_\psi} \langle x \rangle^{\rho p} = \langle 2^{-j} \hat{k} \rangle^{\rho p} \leq \max_{x \in K_\psi} \langle x \rangle^{\rho p} < \infty.
\]

It is then easy to find \( 0 < A \leq B < \infty \) such that

\[
A \sum_{j \geq 0} 2^j(\tau - d + dp) \leq \|\delta\|_{B^\tau_p(\mathbb{R}^d; \rho)}^p \leq B \sum_{j \geq 0} 2^j(\tau - d + dp).
\]

The sum converges for \( \tau - d + dp < 0 \) and diverges otherwise, implying the result. \( \square \)

\(^1\)The representation of the smoothness properties in diagrams with axis \( (1/p, \tau) \) is inherited from the work of H. Triebel. It is very convenient, since the smoothness has often a simple formulation in terms of \( 1/p \). This is also valid for the asymptotic rate \( \rho \).
3. Gaussian White Noise. Our goal in this section is to prove the Gaussian part of Theorem 1. Without loss of generality, we will always assume that the noise has variance 1.

The Gaussian case is much simpler than the general one since the wavelet coefficients of the Gaussian noise are independent and identically distributed. We present it separately for three reasons: (i) it can be considered as an instructive toy problem, containing already some of the technicalities that will appear for the general case, (ii) it cannot be deduced from the other sections, where the results are based on a careful study of the Lévy measure of the noise (the Lévy measure of the Gaussian noise is zero), and (iii) we are not aware that the localization of the Gaussian noise in weighted Besov spaces has been addressed in the literature (for the local Besov regularity, an in-depth answer is given in [47]).

We first state a simple lemma that will be useful throughout the paper to show negative results (\(w\) not in a given weighted Besov space) for the asymptotic rate \(p\).

**Lemma 1.** Assume that \((X_k)_{k \in \mathbb{Z}^d}\) is a sequence of i.i.d. nonzero random variables. Then,

\[
\sum_{k \in \mathbb{Z}^d} \left| \frac{X_k}{k} \right|^d = +\infty \text{ a.s.}
\]

**Proof.** First of all, the result for any dimension \(d\) is easily deduced from the one-dimensional case. Moreover, \(|k|\) and \(|k|\) are equivalent asymptotically, so that it is equivalent to show that \(\sum_{k \geq 1} \frac{|X_k|}{k} = \infty\) for \(X_k\) i.i.d. For \(k \geq 1\), we set \(Z_k = \frac{1}{2^k} \sum_{l=2k-1}^{2k-1} |X_l|\), so that

\[
\sum_{k \geq 1} \left| \frac{X_k}{k} \right|^d = \sum_{k \geq 1} \frac{1}{2^k} \sum_{l=2k-1}^{2k-1} |X_l| = \sum_{k \geq 1} Z_k
\]

The \(Z_k\) are independent because the \(X_k\) are. Moreover, for all \(k\), we have \(\mathbb{E}[Z_k] = \mathbb{E}[|X_1|]\). The weak law of large numbers ensures that \(\mathbb{P}(Z_k > \mathbb{E}[|X_1|]/2)\) goes to 1, therefore \(\sum_{k \geq 1} \mathbb{P}(Z_k > \mathbb{E}[|X_1|]/2) = \infty\). Since the events \(\{Z_k > \mathbb{E}[|X_1|]/2\}\) are independent, we can apply the Borel-Cantelli lemma to deduce that infinitely many \(Z_k\) are bigger than \(\mathbb{E}[|X_1|]/2\) almost surely. Finally, this implies that \(\sum_{k \geq 1} Z_k = \infty\) almost surely and the result is proved. \(\square\)

**Proposition 4.** The Gaussian white noise \(w\) is

- almost surely in \(B^r_p(\mathbb{R}^d; \rho)\) if \(\tau < -d/2\) and \(\rho < -d/p\), and
- almost surely not in \(B^r_p(\mathbb{R}^d; \rho)\) if \(\tau \geq -d/2\) or \(\rho \geq -d/p\).

**Proof.** If \(\tau < -d/2\) and \(\rho < -d/p\). For \(p > 0\), we denote by \(C_p\) the \(p\)-th moment of a Gaussian random variable with 0 mean and variance 1. For the Gaussian noise, \(\langle w, \varphi_1 \rangle\) and \(\langle w, \varphi_2 \rangle\) are independent if and only if \(\langle \varphi_1, \varphi_2 \rangle = 0\), and \(\langle w, \varphi \rangle\) is a Gaussian random variable with variance \(\|\varphi\|^2\) [46]. The family of functions \((\psi_{j,G,k})_{j,G,k}\) being orthonormal, the random variables \(\langle w, \psi_{j,G,k} \rangle\) are i.i.d. with law \(\mathcal{N}(0, 1)\). We then have

\[
\mathbb{E}[\|w\|^p_{B^r_p(\mathbb{R}^d; \rho)}] = \sum_{j \geq 0} 2^{j(\tau p - d + \frac{d}{2})} \sum_{G \in G^j} \sum_{k \in \mathbb{Z}^d} \langle 2^{-j} k \rangle^{\rho p} \mathbb{E}[\|w, \psi_{j,G,k}\|^p] = C_p \sum_{j \geq 0} 2^{j(\tau p - d + \frac{d}{2})} \text{Card}(G^j) \sum_{k \in \mathbb{Z}^d} \langle 2^{-j} k \rangle^{\rho p}
\]

\[
\leq 2^d C_p \sum_{j \geq 0} 2^{j(\tau p - d + \frac{d}{2})} \sum_{k \in \mathbb{Z}^d} \langle 2^{-j} k \rangle^{\rho p}.
\]
The last inequality is due to Card(\(G^j\)) \(\leq 2^d\). Since \(\rho p < -d\) and \(\langle 2^{-j}\mathbf{k} \rangle \sim 2^{-j\|\mathbf{k}\|}\), we have that \(\sum_{\mathbf{k} \in \mathbb{Z}^d} (\langle 2^{-j}\mathbf{k} \rangle)^{\rho p} < \infty\). Moreover, we recognize a Riemann sum and have the convergence

\[
2^{-jd} \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle 2^{-j}\mathbf{k} \rangle^{\rho p} \longrightarrow \int_{\mathbb{R}^d} \langle \mathbf{x} \rangle^{\rho p} \, d\mathbf{x} < \infty.
\]

In particular, the series \(\sum_j 2^j(\tau p + dp) (2^{-jd} \sum_{\mathbf{k} \in \mathbb{Z}^d} (\langle 2^{-j}\mathbf{k} \rangle)^{\rho p})\) converges if and only if the series \(\sum_j 2^j(\tau p + dp)\) does; that is, if and only if \(\tau < d/2\). Finally, if \(\tau < d/2\) and \(\rho < -d/p\), we have shown that \(\mathbb{E}[\|w\|_{B^\rho_p(\mathbb{R}^d)}^p] < \infty\) and therefore \(w \in B^\rho_p(\mathbb{R}^d)\) almost surely.

If \(\tau \geq -d/2\). Then, we have \(2^j(\tau - d + dp/2) \geq 2^{-jd}\). We aim at lower bound the Besov norm of \(w\) and we restrict to the wavelet with gender \(G = M^d \in G^j\) for any \(j \geq 0\). Hence, we have, since \(\langle 2^{-j}\mathbf{k} \rangle \geq 1\),

\[
\|w\|_{B^\rho_p(\mathbb{R}^d)}^p \geq \sum_{j \geq 0} \frac{2^{-jd}}{2} \sum_{0 \leq k_1, \ldots, k_d < 2^j} |\langle w, \psi_{j,M^d,k} \rangle|^p := \sum_{j \geq 0} Z_j.
\]

The random variables \(Z_j = 2^{-jd} \sum_{0 \leq k_1, \ldots, k_d < 2^j} |\langle w, \psi_{j,M^d,k} \rangle|^p\) are independent, non-negative, and have the same average \(\mathbb{E}[Z_j] = C_p\) equals to the \(p\)-th moment of a Gaussian random variable with variance 1. By the weak law of large numbers, we know that \(\mathcal{P}(Z_j \geq C_p/2)\) converges to 1, and then \(\sum_{j \geq 0} \mathcal{P}(Z_j \geq C_p/2) = \infty\). The events \(\{Z_j \geq C_p/2\}\) being independent, the Borel-Cantelli lemma ensures that \(Z_j \geq C_p/2\) for infinitely many \(j\) almost surely. This ensures that \(\|w\|_{B^\rho_p(\mathbb{R}^d)}^p \geq \sum_{j \geq 0} Z_j = \infty\) almost surely.

If \(\rho \geq -d/p\). Keeping only the father wavelet \(\phi = \psi_{F^d}\) and the scale \(j = 0\) and exploiting the relation \(\rho p \geq -d\), we have the lower bound

\[
\|w\|_{B^\rho_p(\mathbb{R}^d)}^p \geq \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle \mathbf{k} \rangle^{\rho p} |\langle w, \phi_{\mathbf{k}} \rangle|^p \geq \sum_{\mathbf{k} \in \mathbb{Z}^d} \langle \mathbf{k} \rangle^d \frac{|\langle w, \phi_{\mathbf{k}} \rangle|^p}{\langle \mathbf{k} \rangle^d}
\]

Finally, the random variables \(\langle w, \phi_{\mathbf{k}} \rangle\) being i.i.d., Lemma 1 applies and \(\|w\|_{B^\rho_p(\mathbb{R}^d)} = \infty\) almost surely.

4. Compound Poisson Noises.

4.1. Moment estimates for compound Poisson noises. Our positive results for compound Poisson noises are based on a careful estimation of moments for compound Poisson noise.

**Lemma 2.** Let \(w\) be a compound Poisson process with moment index \(p_{\text{max}}\) and \(p < p_{\text{max}}\). Then, there exists a constant \(C\) such that

\[
\mathbb{E}[|\langle w, \psi_{j,G,k} \rangle|^p] \leq C 2^j p_{\text{max}} d - (2j)
\]

for every \(j \geq 0\), \(G \in G^j\), and \(\mathbf{k} \in \mathbb{Z}^d\).

**Proof.** We denote by \(\lambda > 0\) the sparsity parameter of \(w\). Being a compound Poisson noise, \(w\) is equal in law to

\[
w \overset{(\mathcal{L})}{=} \sum_{\mathbf{k} \in \mathbb{N}} a_{\mathbf{k}} \delta(\cdot - \mathbf{x}_{\mathbf{k}})
\]
according to [45]. For $M$ big enough, the support of the $\Psi_G$ is included in $[-M/2, M/2]^d$. Hence, the support of $\Psi_{j,G,k}$ is included in

\[(4.3) \quad I_{j,k} := \prod_{i=1}^{d} [2^{-j}(k_i - M/2), 2^{-j}(k_i + M/2)].\]

We set

\[(4.4) \quad A_{j,k} = \text{Card}\{n \in \mathbb{N}, \ x_n \in I_{j,k}\}. \]

It is a Poisson random variable with parameter $\lambda L(I_{j,k}) = \frac{\lambda M^d}{2^{j\pi}}$. Then, we have the equality in law

\[(4.5) \quad \langle w, \psi_{j,k} \rangle = \sum_{n=1}^{A_{j,k}} a'_n \psi_{j,k}(x'_n) \]

where the $a'_k$ are i.i.d. with the same law than the $a_k$. The law of the $x'_k$ can be specified explicitly but will play no role in the sequel. By conditioning on $A_{j,k}$ and using the relation $\left| \sum_{n=1}^{N} x_n \right|^p \leq N^{\max(0,p-1)} \sum_{n=1}^{N} |x_n|^p$, we deduce that

\[(4.6) \quad \mathbb{E}[\langle w, \psi_{j,G,k} \rangle^p] = \sum_{N=1}^{\infty} \mathcal{P}(A_{j,k} = N) \mathbb{E}[\left| \langle w, \psi_{j,G,k} \rangle \right|^p | A_{j,k} = N] \]

\[= \sum_{N=1}^{\infty} \mathcal{P}(A_{j,k} = N) \mathbb{E} \left[ \left| \sum_{n=1}^{N} a'_n \psi_{j,G,k}(x'_n) \right|^p \right] \]

\[\leq \sum_{N=1}^{\infty} \mathcal{P}(A_{j,k} = N) N^{\max(0,p-1)} \mathbb{E} \left[ \left| \sum_{n=1}^{N} a'_n \psi_{j,G,k}(x'_n) \right|^p \right] \]

\[\leq \|\psi_{j,G,k}\|_{\infty}^p \sum_{N=1}^{\infty} \mathcal{P}(A_{j,k} = N) N^{\max(1,p)} \mathbb{E}[|a_1|^p] \]

\[= 2^{dp/2} \|\psi_G\|_{\infty}^p \mathbb{E}[|a_1|^p] \sum_{N=1}^{\infty} N^{\max(1,p)} \mathcal{P}(A_{j,k} = N), \]

where we used at the end the obvious relation $\|\psi_{j,G,k}\|_{\infty}^p = 2^{dp/2} \|\psi_G\|_{\infty}^p$. Knowing the law of $A_{j,k}$, we then have

\[(4.7) \quad \sum_{N=1}^{\infty} N^{\max(1,p)} \mathcal{P}(A_{j,k} = N) = \sum_{N=1}^{\infty} N^{\max(1,p)} \frac{1}{N!} (M^d \lambda)^N 2^{-jdN} e^{-\lambda M^d 2^{-jd}}. \]

Then, $2^{-jdN} \leq 2^{-jd}$ for every $N \geq 1$ and $e^{-\lambda M^d 2^{-jd}} \leq 1$, hence,

\[(4.8) \quad \sum_{N=1}^{\infty} N^{\max(1,p)} \mathcal{P}(A_{j,k} = N) \leq \tilde{C} 2^{-jd} \]

where $\tilde{C} = \sum_{N=1}^{\infty} N^{\max(1,p)} \frac{1}{N!} (M^d \lambda)^N < \infty$. Finally, including (4.8) into (4.6), we deduce the result with $C = \tilde{C} \mathbb{E}[|a_1|^p] \|\psi_G\|_{\infty}^p$. \qed
4.2. The Besov regularity of compound Poisson noises. We shall prove the following result.

**Proposition 5.** Let \( w \) be a compound Poisson noise with index \( p_{\text{max}} \in (0, \infty] \). Then, \( w \) is

- almost surely in \( B^\tau_p(\mathbb{R}^d; \rho) \) if \( \tau < d/p - d \) and \( \rho < -d/\min(p, p_{\text{max}}) \), and
- almost surely not in \( B^\tau_p(\mathbb{R}^d; \rho) \) if \( \tau \geq d/p - d \) or \( \rho > -d/\min(p, p_{\text{max}}) \).

**Proof.** If \( p < p_{\text{max}}, \tau < d/p - d, \) and \( \rho < -d/p \). Under this assumption, we apply Lemma 2 to deduce that

\[
\mathbb{E}[||w||_{B^\tau_p(\mathbb{R}^d; \rho)}] = \sum_{j \geq 0} 2^{j(\tau p - d + dp)/2} \sum_{G, k} (2^{-j}k)^{\rho p} \mathbb{E}[||w, \psi_{j,G,k}||^p] \leq C 2^d \sum_{j \geq 0} 2^{j(\tau p - d + dp)} \frac{1}{2^d} \sum_{k \in \mathbb{Z}^d} (2^{-j}k)^{\rho p}.
\]

(4.9)

The sum over the gender was removed using that \( \text{Card}(\mathcal{G}^j) \leq 2^d \). Then, \( \frac{1}{2^d} \sum_{k \in \mathbb{Z}^d} (2^{-j}k)^{\rho p} \rightarrow \int_{\mathbb{R}^d} (x)^{\rho p} dx < \infty \) as soon as \( \rho < -d/p \). Assuming this condition on \( \rho \), the sum in is finite if and only if \( \sum_j 2^{j(\tau p - d + dp)} < \infty \), that is, if and only if \( \tau p - d + d/p < 0 \), as expected.

**If** \( p \geq p_{\text{max}}, \tau < d/p - d, \) and \( \rho < -d/p_{\text{max}} \). From the conditions on \( \tau \) and \( \rho \), one can find \( p_0, \rho_0, \) and \( \tau_0 \) such that

(4.10)

\[
\rho < \rho_0 < -\frac{d}{p_0} < -\frac{d}{p_{\text{max}}},
\]

(4.11)

\[
\tau + \frac{d}{p_0} - \frac{d}{p} < \tau_0 < \frac{d}{p_0} - d.
\]

Then, in particular, \( p_0 < p, \tau_0 - \tau > d/p_0 - d/p \), and \( \rho_0 > \rho \), so that \( B^\tau_{p_0}(\mathbb{R}^d; \rho_0) \subset B^\tau_p(\mathbb{R}^d; \rho) \) (according to (2.17)). Moreover, \( p_0 < p_{\text{max}}, \tau_0 < d/p_0 - d \), and \( \rho_0 < -d/p_0 \). We are therefore back to the first case, for which have shown that \( w \in B^\tau_{p_0}(\mathbb{R}^d; \rho_0) \) almost surely. In conclusion, \( w \in B^\tau_p(\mathbb{R}^d; \rho) \) almost surely.

Combining this first two cases, we obtain that \( w \in B^\tau_p(\mathbb{R}^d; \rho) \) if \( \tau < d/p - d \) and \( \rho < -d/\min(p, p_{\text{max}}) \).

**If** \( \tau \geq d/p - d \). The compound Poisson noise \( w \) is equal in law to

(4.12)

\[
\sum_{n \in \mathbb{Z}} a_n \delta(\cdot - x_n)
\]

with \( (a_n) \) i.i.d. with law of jumps \( \mathcal{P}_d \) and \( (x_n) \) the jumps location, independent of \( (a_n) \) and such that \( \text{Card}\{n, x_n \in B\} \) is a Poisson random variable of parameter \( \lambda \text{Leb}(B) \), with \( \text{Leb}(B) \) the Lebesgue measure of any Borelian set \( B \subset \mathbb{R}^d \). Assume that \( w \) is in \( B^\tau_p(\mathbb{R}^d; \rho) \) for some \( \rho \in \mathbb{R} \). Then, the product of \( w \) by any compactly supported smooth test function \( \varphi \) is also in \( B^\tau_p(\mathbb{R}^d; \rho) \). Choosing \( \varphi \) such that \( \varphi(x_0) = 1 \) and \( \varphi(x_n) = 0 \) for \( n \neq 0 \), we get

(4.13)

\[
\varphi \cdot w = a_0 \delta(\cdot - x_0) \in B^\tau_p(\mathbb{R}^d; \rho),
\]

what is absurd due to Proposition 3. This proves that \( w \notin B^\tau_p(\mathbb{R}^d; \rho) \) for any \( \rho \in \mathbb{R} \).
If \( \rho \geq -d/p \). By restricting to the scale \( j \geq 0 \), with only the father wavelet \( \phi \), and selecting \( k_0 \) such that the \( \phi_k \) have disjoint supports two by two for \( k \in k_0 \mathbb{Z}^d \), we have the lower bound

\[
(4.14) \quad \|w\|^p_{B^p_{p/q}(\mathbb{R}^d, \rho^p)} \geq \sum_{k \in k_0 \mathbb{Z}^d} \langle \phi, \phi_k \rangle \|w, \phi_k\|^p \geq \sum_{k \in k_0 \mathbb{Z}^d} \frac{\langle w, \phi_k \rangle^p}{\langle k \rangle^{d/p}}.
\]

The \( \phi_k \) having disjoint supports, the random variables \( \langle w, \phi_k \rangle \) are i.i.d. when \( k \in k_0 \mathbb{Z}^d \). Lemma 1 hence applies and \( \|w\|^p_{B^p_{p/q}(\mathbb{R}^d, \rho^p)} = \infty \), as expected.

If \( p > p_{\text{max}} \) and \( \rho > -d/p_{\text{max}} \). This means in particular that \( p_{\text{max}} < \infty \). We treat the case \( \rho < 0 \), the extension for \( \rho \geq 0 \) coming easily by embedding. We set \( q = -d/\rho \). Proceeding as for (4.14), we have that

\[
(4.15) \quad \|w\|^p_{B^p_{p/q}(\mathbb{R}^d, \rho^p)} \geq \sum_{k \in k_0 \mathbb{Z}^d} \frac{\langle w, \phi_k \rangle^p}{\langle k \rangle^{d/p}}.
\]

Consider the events \( A_k = \{\langle w, \phi_k \rangle \geq \langle k \rangle^{d/q}\} \) for \( k \in k_0 \mathbb{Z}^d \). The \( A_k \) are independent because the \( X_k = \langle w, \phi_k \rangle \) are. Moreover, the \( X_k \) have the same distribution, since \( w \) is stationary. Set \( Y = \|X_0\|^q \). Then,

\[
(4.16) \quad \sum_{k \in k_0 \mathbb{Z}^d} \mathcal{P}(A_k) = \sum_{k \in k_0 \mathbb{Z}^d} \mathcal{P}(Y \geq \langle k \rangle^d) \geq \sum_{m \geq 1} \mathcal{P}(Y \geq mk_0).
\]

Moreover, we have, exploiting that \( \mathcal{P}(Y \geq x) \) is decreasing in \( x \),

\[
(4.17) \quad \mathbb{E}[Y] = \int_0^\infty \mathcal{P}(Y \geq x) \, dx \geq \sum_{m \geq 1} \int_{mk_0}^{(m+1)k_0} \mathcal{P}(Y \geq x) \, dx \leq \sum_{m \geq 1} \mathcal{P}(Y \geq mk_0).
\]

The relation \( q = \frac{d}{\rho} > p_{\text{max}} \) implies that \( \mathbb{E}[Y] = \mathbb{E}[\|X_0\|^q] = \infty \). Hence, from (4.16) and (4.17), we deduce that \( \sum_{k \in k_0 \mathbb{Z}^d} \mathcal{P}(A_k) = \infty \). The Borel-Cantelli lemma implies that \( \|X_k\|^p \geq \langle k \rangle^{pd/q} \) for infinitely many \( k \) almost surely. Due to (4.15), this implies that \( \|w\|^p_{B^p_{p/q}(\mathbb{R}^d, \rho^p)} = \infty \) almost surely and the result is proved.

5. Lévy White Noises with Finite Moments.

5.1. Moment estimates for Lévy white noise. As a preliminary result, we estimate the evolution of the even moments of the wavelet coefficients of \( w \) with the scale \( j \). This result is adapted to the case where the noise has finite moments of order bigger than 2. Many moment estimates in the literature usually deal with \( p \)-th moments where \( p \leq 2 \) [28, 8, 26, 12].

**Lemma 3.** Let \( w \) be a Lévy white noise with finite moments. Then, for every integer \( m \geq 1 \), there exists a constant \( C > 0 \) such that, for every \( j \geq 0 \), \( G \in G^j \), and \( k \in \mathbb{Z}^d \),

\[
(5.1) \quad \mathbb{E}[(\langle w, \psi_{j, G, k} \rangle)^{2m}] \leq C2^{jd(m-1)}.
\]

**Proof.** Consider a test function \( \varphi \in S(\mathbb{R}^d) \) and set \( X = \langle w, \varphi \rangle \). The characteristic function of \( X \) is

\[
(5.2) \quad \widehat{\mathcal{P}}_X(\xi) = \exp \left( \int_{\mathbb{R}^d} \Psi(\xi, \varphi) \, dx \right) = \exp(\Psi(\varphi(\xi))).
\]

The functions \( \widehat{\mathcal{P}}_X \) and \( \Psi(\varphi) \) are infinitely differentiable because all the moments of \( X \) are finite. Their Taylor expansions give respectively the moments and the cumulants of \( X \). In particular, we have
\[ \mathbb{E}[X^{2m}] = (-1)^m \hat{P}_X^{(2m)}(0). \] The \((2m)\)th derivative of \(\hat{P}_X\) is deduced for instance from the Faà di Bruno formula, and is

\[ (5.3) \quad \hat{P}_X^{(2m)}(\xi) = \left( \sum_{n_1 \ldots n_{2m}: \sum_{u} u n_u = 2m} \frac{(2m)!}{n_1! \ldots n_{2m}!} \prod_{v=1}^{2m} \left( \frac{\Psi_{\varphi_v}(\xi)}{v!} \right)^{n_v} \right) \mathbb{E}[X^\xi]. \]

Exploiting that \(\Psi_{\varphi_v}(0) = (\int_{\mathbb{R}^d} (\varphi(x))^v dx) \Psi_v(0)\) we obtain the bound, for \(\xi = 0,\)

\[ (5.4) \quad \left| \hat{P}_X^{(2m)}(0) \right| \leq C' \sum_{n_1 \ldots n_{2m}: \sum_{u} u n_u = 2m} \prod_{v=1}^{2m} \left( \int_{\mathbb{R}^d} |\varphi(x)|^v dx \right)^{n_v} \]

with \(C' > 0\) a constant.

We now apply (5.4) to \(\varphi = \psi_{j,G}\). Since we have

\[ (5.5) \quad \int_{\mathbb{R}^d} |\psi_{j,G}(x)|^v dx = 2^{jd/2} \int_{\mathbb{R}^d} |\psi_G(2^j x - k)|^v dx = 2^{jd(v/2 - 1)} \int_{\mathbb{R}^d} |\psi_G(x)|^v dx, \]

we deduce from (5.4) the new bound

\[ (5.6) \quad \mathbb{E}[(w, \psi_{j,G,k})^{2m}] = \left| \hat{P}_{(w, \psi_{j,G,k})}^{(2m)}(0) \right| \leq C'' \sum_{n_1 \ldots n_{2m}: \sum_{u} u n_u = 2m} \prod_{v=1}^{2m} 2^{jd(v/2 - 1)n_v} \]

Finally, since \(\sum_v v n_v = 2m\) and \(\sum_v n_v \geq 1\), we have \(\sum_v (n_v(v/2 - 1)) \leq m - 1\), and therefore

\[ (5.7) \quad \mathbb{E}[(w, \psi_{j,G,k})^{2m}] \leq C 2^{jd(m - 1)} \]

for an adequate \(C > 0\), as expected.

Until now, we gave moment estimates by proposing upper bounds for the quantity \(\mathbb{E}[|\langle w, \varphi \rangle|^p]\) (see Lemmas 2 and 3, but also Theorem 2 in [12]). This allows in particular to identify in which Besov space is \(w\). We now address the following problem: Can we lower bound \(\mathbb{E}[|\langle w, \varphi \rangle|^p]\) with the moments of \(\varphi\)? Lemma 4 answers positively to this question.

**Lemma 4.** Let \(w\) be a Lévy white noise whose indices satisfy \(0 < \beta_\infty\) and \(\beta_\infty < p_{\max}\). We also fix a nontrivial bounded measurable function \(\psi\) (typically a Daubechies wavelet). Consider an integrability \(0 < p < \beta_\infty\) satisfying the conditions. Then, for \(\epsilon > 0\) small enough, there exists \(A, B > 0\) such that

\[ (5.8) \quad A 2^{-j^2 \epsilon^2 p(A_{1/2-1/2}^{1/2})} \leq \mathbb{E}[|\langle w, \psi_{j,k} \rangle|^p] \leq B 2^{j^2 \epsilon^2 p(B_{1/2-1/2}^{1/2})} \]

for any \(j \geq 0\) and \(k \in \mathbb{Z}^d\).

**Proof.** As a preliminary remark, the shift parameter \(k\) in (5.8) can be omitted since \(w\) is stationary. We also remark that the upper bound of (5.8) has already been proven [12, Corollary 1], where the conditions \(p < \beta_\infty < p_{\max}\) are required. Actually, [12] does not consider the index \(p_{\max}\).
and distinguishes between the conditions $\beta_\infty < \beta_0 < 2$ and $\beta_\infty \leq \beta_0 = 2$ with finite variance. These two scenarios are covered by the condition $\beta_\infty < p_{max}$. Hence we focus on the lower bound.

Because $p < 2$, one can use the following representation of the $p$th-moment of $\langle w, \varphi \rangle$, which can be found for instance in [27, 8]:

$$
\mathbb{E}[|\langle w, \varphi \rangle|^p] = c_p \int_\mathbb{R} \frac{1 - \Re \widehat{\mathcal{F}}_{(w, \varphi)}(\xi)}{\xi^{p+1}} d\xi
$$

for some explicit constant $c_p > 0$. For a short proof, see [12, Proposition 6]. We then remark that

$$
\Re \widehat{\mathcal{F}}_{(w, \varphi)}(\xi) \leq \left| \widehat{\mathcal{F}}_{(w, \varphi)}(\xi) \right| = \left| \widehat{\mathcal{F}}_{w}(\xi, \varphi) \right| = \exp \left( \int_{\mathbb{R}^d} \psi(\xi \varphi(x)) dx \right) = \exp \left( \int_{\mathbb{R}^d} \Re \Psi(\xi \varphi(x)) dx \right).
$$

The sector condition (2.5) implies that one can find $c_\Psi > 0$ such that $-\Re \Psi(\xi) = |\Re \Psi(\xi)| \geq c_\Psi |\Psi(\xi)|$. Thus, one has that $\left| \widehat{\mathcal{F}}_{(w, \varphi)}(\xi) \right| \leq \exp \left( -c_\Psi \int_{\mathbb{R}^d} |\Psi(\xi \varphi(x))| dx \right)$, and then

$$
\mathbb{E}[|\langle w, \varphi \rangle|^p] \geq c_p \int_\mathbb{R} \frac{1 - e^{-c_\Psi \int_{\mathbb{R}^d} |\Psi(\xi \varphi(x))| dx}}{\xi^{p+1}} d\xi.
$$

Now, by definition of the index $\beta_\infty$, the function $|\Psi|$ is dominating $|\xi|^{\beta_\infty - \delta}$ at infinity for $0 < \delta < \beta_\infty$ arbitrarily small. This domination together with the continuity of the functions $|\Psi|$ and $|\cdot|^{\beta_\infty - \delta}$ imply the existence of $C > 0$ such that, for any $\xi \in \mathbb{R}$, $|\Psi(\xi)| \geq C |\xi|^{\beta_\infty - \delta}$ over $[1, \infty)$. In particular, we have that

$$
\int \left| \Psi(\xi \varphi(x)) \right| dx \geq C |\xi|^{\beta_\infty - \delta} \int_{|\varphi(x)| > 1} |\varphi(x)|^{\beta_\infty - \delta}.
$$

For simplicity, we write $\beta = \beta_\infty - \delta$ in this proof. We now set $\varphi = 2^{j/2} \psi(2^j \cdot)$, then we have by chance of variable $x = 2^j x$ that

$$
\int_{|\varphi(x)| > 1} |\varphi(x)|^{\beta} = 2^{jd\beta} \left( \frac{1}{2^\beta} \right) \int_{|\psi(y)| > 2^{j/2}|\xi|^{-1}} |\psi(y)|^{\beta} dy.
$$

Let $\gamma \geq 0$ and set $F(\gamma) = \int_{|\psi(y)| > \gamma} |\psi(y)|^{\beta} dy$. Then, $F$ is clearly decreasing, for every $0 \leq \gamma \leq \|\psi\|_{\infty}/2$ (this last quantity is finite by assumption on $\psi$), we have that $F(\gamma) \geq F(\|\psi\|_{\infty}/2)$. Applying this with $\gamma = 2^{j/2}$ to (5.13), one deduces that

$$
\int_{|\psi(y)| > 2^{j-2j/2}|\xi|^{-1}} |\psi(y)|^{\beta} dy \geq \|\psi\|_{\infty}^\beta 1_{|\psi| > \|\psi\|_{\infty}/2} \|\psi\|_{\infty}^\beta \|\psi\|_{\infty}^\beta 1_{|\xi| \geq 2^{j-2j/2}/\|\psi\|_{\infty}}.
$$

Restarting from (5.11), we then have

$$
\mathbb{E}[|\langle w, 2^{j/2} \psi(2^j \cdot) \rangle|^p] \geq c_p \int_\mathbb{R} \frac{1 - \exp \left( -c_p C |\xi|^{\beta} 2^{jd\beta} \left( \frac{1}{2^\beta} \right) \int_{|\psi(y)| > 2^{j/2}|\xi|^{-1}} |\psi(y)|^{\beta} dy \right)}{|\xi|^{p+1}} d\xi
$$

$$(i) \quad \geq c_p \int_\mathbb{R} \frac{1 - \exp \left( -c_p C |\xi|^{\beta} 2^{jd\beta} \left( \frac{1}{2^\beta} \right) \|\psi\|_{\infty}^\beta 1_{|\psi| > \|\psi\|_{\infty}/2} \|\psi\|_{\infty}^\beta \|\psi\|_{\infty}^\beta 1_{|\xi| \geq 2^{j-2j/2}/\|\psi\|_{\infty}} \right)}{|\xi|^{p+1}} d\xi
$$

$$(ii) \quad \geq c_p C^{p/\beta} \|\psi\|_{\infty}^\beta \|\psi\|_{\infty}^\beta 1_{|\psi| > \|\psi\|_{\infty}/2} \|\psi\|_{\infty}^\beta \int_\mathbb{R} \frac{1 - e^{-|u|^\beta \|\psi\|_{\infty}^\beta 1_{|u| > 2^{j-2j/2}/\|\psi\|_{\infty}}} du}{|u|^{p+1}}
$$

$$(iii) \quad \geq c_p C^{p/\beta} \|\psi\|_{\infty}^\beta \|\psi\|_{\infty}^\beta 1_{|\psi| > \|\psi\|_{\infty}/2} \|\psi\|_{\infty}^\beta 2^{jd\beta} \left( \frac{1}{2^\beta} \right) \int_\mathbb{R} \frac{1 - e^{-|u|^\beta \|\psi\|_{\infty}^\beta 1_{|u| > 2^{j-2j/2}/\|\psi\|_{\infty}}} du}{|u|^{p+1}}
$$
where we used (5.14) in (i), the change of variable $u = \xi^{1/\beta} \langle \psi \rangle_{|\psi|}^{\|\psi\|_\infty/2} \|\psi\|_\infty^{2jd(\frac{1}{2}-\frac{1}{p})}$ in (ii), and the inequality $1_{|u|>2^{-jd/2}/\|\psi\|_\infty} \geq 1_{|u|>2/\|\psi\|_\infty}$ in (iii), valid for any $j \geq 0$. Finally, we have shown that $\mathbb{E}[\langle w, 2^{jd/2}\psi(2^j \cdot) \rangle] \geq A 2^{jd(\frac{1}{2}-\frac{1}{p})}$ with $A > 0$ a constant independent on $j$.

To conclude, we remark that, for $\varepsilon > 0$ fixed, one can find $\delta > 0$ small enough such that $2^{-\varepsilon}. 2^{jd(\frac{1}{2}-\frac{1}{p})} \leq 2^j(d-\frac{1}{p})$ for any $j \geq 0$, giving the lower bound in (5.8).

5.2. The Besov regularity of finite moment white noises. We have all the tools to deduce the following results on the Besov regularity of finite moment white noises.

**Proposition 6.** Let $w$ be a finite moments Lévy white noise with Blumenthal-Getoor indices $0 \leq \beta_\infty \leq \beta_\infty \leq 2$. Then, $w$ is

- almost surely in $B^p_p(\mathbb{R}^d; \rho)$ if $\tau < d/p$, $\rho < -d/p$, for $0 < p \leq 2$, $p$ an even integer, or $p = \infty$ and
- almost surely not in $B^p_p(\mathbb{R}^d; \rho)$ if $\tau > d/p$, $\rho$ an integer. Then, $p = 2m$, $m \rightarrow \infty$ for positive results and $p \rightarrow \infty$ for negative results.

**Proof.** We only treat the case $p < \infty$. For $p = \infty$, the result is obtained using embeddings (with $p = 2m$, $m \rightarrow \infty$ for positive results and $p \rightarrow \infty$ for negative results).

If $\tau < d/p$, max$(p, \beta_\infty) = d$ and $\rho < -d/p$. We first remark that for $p \leq 2$, we have $\rho < -\frac{d}{\min(p, 2m_{\max})} = -\frac{d}{\max(p, 2m_{\max})}$, and the result is a consequence of our previous work [12, Theorem 3]. We can therefore assume that $p = 2m$ with $m \geq 1$ an integer. Then, $p = 2m \geq 2 \geq \beta_\infty$, and the conditions on $\tau$ and $\rho$ become $\tau < \frac{d}{2m} - d$ and $\rho < -\frac{d}{2m}$. Thanks to Lemma 3, we have

$$
\mathbb{E}\left[\|w\|_{B^p_p(\mathbb{R}^d; \rho)}^{2m}\right] = \sum_{j \geq 0} 2^{jd(2m\tau-d+dm)} \sum_{G, k} \langle 2^{-j} k \rangle^{2mp} \mathbb{E}\left[\langle w, \psi_j, G, k \rangle^{2m}\right]
$$

(5.16)

Then, $\frac{1}{2d} \sum_{k \in \mathbb{Z}^d} \langle 2^{-j} k \rangle^{2mp} \rightarrow \int_{\mathbb{R}^d} f(x)^{2mp} \, dx < \infty$ since $2mp < -d$, and the series in (5.16) is finite if and only if $2m\tau - d + 2dm < 0$, as we assumed. Finally, we have shown that $\mathbb{E}\left[\|w\|_{B^p_p(\mathbb{R}^d; \rho)}^{2m}\right] < \infty$, so that $w \in B^p_p(\mathbb{R}^d; \rho)$ almost surely.

If $\tau > d/p$, this part of the proof is actually valid for any Lévy white noise. It uses the decomposition $w = w_1 + w_2$ with $w_1$ a non-trivial compound Poisson noise and $w_2$ a white noise with finite moments (see Proposition 1). The main idea is that the jumps of the compound Poisson part are enough to make explode the Besov norm.

One writes $w_1 = \sum_{n \in \mathbb{Z}} a_n \delta(x_n)$ as in (4.12). Then, almost surely, all the $a_n$ are nonzero. Let $m > 0$ be such that $\mathcal{A}(\{a_0 \geq m\}) > 0$. One can assume without loss of generality that $m = 1$ (otherwise, consider the white noise $w/m$ that has the same Besov regularity than $w$). Then, there exists almost surely $n \in \mathbb{Z}$ such that $|a_n| \geq 1$ (Borel-Cantelli argument using that the $a_n$ are independent). For simplicity, one reordered the jumps such that this is achieved for $n = 0$; that is, $|a_0| \geq 1$.

Fix $I^d = \{a, a + 1\}^d$ a set such that the Daubechies wavelet with gender $M^d$ satisfies $|\psi_{M^d}(x)| \geq C > 0$ for some constant $C > 0$ and every $x \in I^d$. Due to the size of $I^d$, there exists for any $j \geq 0$ a unique $k_j \in \mathbb{Z}^d$ such that $2^j x_0 - k_j \in I^d$. Moreover, there exists almost surely a (random) integer
$j_0 \geq 0$ such that for any $j \geq j_0$, $\psi_{M^d}(2^j x_n - k_j) = 0$ for any $n \neq 0$. This is due to the fact that there are (almost surely) finitely many $x_n$ in $I^d$. From these preparatory considerations, one has that, for $j \geq j_0$,

$$
(5.17) \quad \left| \langle w_1, \psi_{M^d}(2^j \cdot -k_j) \rangle \right| = \left| \sum_{n \in \mathbb{Z}} a_n \psi_{M^d}(2^j x_n - k_j) \right| = \left| a_0 \psi_{M^d}(2^j x_0 - k_j) \right| \geq C
$$

almost surely.

Since $w_2$ has a finite variance, we have, using the Markov inequality

$$
(5.18) \quad \mathcal{P} \left( \left| \langle w_2, \psi_{M^d}(2^j \cdot -k_j) \rangle \right| \geq \frac{C}{2} \right) \leq \frac{4\mathbb{E} \left( \left| \langle w_2, \psi_{M^d}(2^j \cdot -k_j) \rangle \right|^2 \right)}{C^2} = \frac{4\sigma_0^2 \| \psi_{M^d} \|^2_2 2^{-jd}}{C^2},
$$

where $\sigma_0^2$ is the variance of $w_2$ such that $\mathbb{E}[(w_2, \varphi)^2] = \sigma_0^2 \| \varphi \|^2_2$. In particular,

$$
\sum_{j \geq 0} \mathcal{P} \left( \left| \langle w_2, \psi_{M^d}(2^j \cdot -k_j) \rangle \right| \geq \frac{C}{2} \right) < \infty.
$$

From a new Borel-Cantelli argument, we readily deduce the existence of integers $j$ arbitrarily large such that $\left| \langle w_2, \psi_{M^d}(2^j \cdot -k_j) \rangle \right| < \frac{C}{2}$. In particular, for $j \geq j_1$, we have, due to (5.17), $\left| \langle w, \psi_{M^d}(2^j \cdot -k_j) \rangle \right| \geq \frac{C}{2}$.

Putting the pieces together, we can now lower bound the Besov norm of $w$ by keeping only the mother wavelet $\psi_{M^d}$, the scale $j \geq j_1$ and the corresponding shift parameter $k_j$. Then, we obtain the almost sure lower bound

$$
(5.19) \quad \| w \|_{B_p^\tau_d(\mathbb{R}^d; \rho)}^p \geq 2^{j(\tau p - d + dp)} \left| \langle w, \psi_{M^d}(2^j \cdot -k_j) \rangle \right|^p \geq 2^{j(\tau p - d + dp)} (C/2)^p.
$$

Finally, this is valid for any $j \geq j_1$ and because $\tau p - d + dp > 0$, one concludes that $\| w \|_{B_p^\tau_d(\mathbb{R}^d; \rho)}^p = \infty$ almost surely.

**If** $0 < p < \beta_\infty$ and $d/\beta_\infty - d < \tau < d/p - d$. Together with the case $\tau > d/p - d$ above and by embeddings, we then deduce the result for $\tau < d/\max(\beta_\infty, p) - d$. Note that the case $\beta_\infty = 0$ is already covered by the previous case. Moreover, as soon as $f \notin B_p^\tau(\mathbb{R}^d; \rho)$ for some $p > 0$, we also have that $f \notin B_p^{\tau + \epsilon}(\mathbb{R}^d; \rho)$ for any $q > p$ and $\epsilon > 0$ (see Figure 1). A crucial consequence for us is that it suffices to work with arbitrarily small $p$ in order to obtain the negative result we expect. We assume here that

$$
(5.20) \quad p < \beta_\infty/2, \quad p < \frac{\beta_\infty \beta_\infty}{2(\beta_\infty - \beta_\infty)}
$$

Note that the latter condition is simply $p < \infty$ (that is, no restriction) when $\beta_\infty = \beta_\infty$.

Let $k_0 \geq 1$ be such that the families of random variables $(\langle w, \psi_{j,G,k} \rangle)_{k \in k_0 \mathbb{Z}^d}$ are independent when $j \geq 0$ and $G \in G^j$ are fixed. This is possible because the wavelets are compactly supported and because $\langle w, \varphi_1 \rangle$ and $\langle w, \varphi_2 \rangle$ are independent as soon as $\varphi_1$ and $\varphi_2$ have disjoint support. By restricting the range of $k$ and the gender to $G = M^d$, we have that

$$
(5.21) \quad \| w \|_{B_p^\tau_d(\mathbb{R}^d; \rho)}^p \geq C \sum_{j \geq 0} 2^{j(\tau p - d + dp)/2} \sum_{k \in k_0 \mathbb{Z}^d, 0 \leq k_i < k_0 2^j} \left| \langle w, \psi_{j,M^d,k} \rangle \right|^p,
$$

and because
with $C = \inf \|x\|_{\infty} \leq k_0 \langle x \rangle^p > 0$ a constant such that $\langle 2^{-j} k \rangle \geq C$. We set $X_{j,k} = 2^{-jd} \left( \frac{\varphi_{j(k)}}{\varphi_{j(k)}} \right) \langle w, \psi_{j,M^d,k} \rangle$ and

$$M_{j,p} := 2^{-jd} \sum_{k \in k_0 \mathbb{Z}^d, 0 \leq k_i < k_0 2^j} |X_{j,k}|^p,$$

which is an average among $2^{jd}$ random variables.

Recall that $p < p_{\text{max}}/2, \beta_{\infty}/2$. Moreover, since all the moments are finite, $p_{\text{max}} = \infty > \beta_{\infty}$. Hence, one can apply Lemma 4 with integrability $q = p$ or $q = 2p$. There exists $\epsilon > 0$ that can be chosen arbitrarily small and constants $m_q, M_q$ such that

$$m_q 2^{-j\epsilon} 2^{jd} \left( \frac{1}{\mathcal{L}} - \frac{1}{\mathcal{K}} \right) \leq \mathbb{E} \left[ \left| \langle w, \psi_{j,M^d,k} \rangle \right|^q \right] \leq M_q 2^{j\epsilon} 2^{jd} \left( \frac{1}{\mathcal{L}} - \frac{1}{\mathcal{K}} \right)
$$

for any $j \geq 0, k \in \mathbb{Z}^d$. In particular, with our notations, we have that

$$m_q 2^{-j\epsilon} \leq \mathbb{E}[M_{j,q}] \leq M_q 2^{j\epsilon} 2^{jd} \left( \frac{1}{\mathcal{L}} - \frac{1}{\mathcal{K}} \right)
$$

for any $j, k$ and for $q = p$ or $2p$.

Then, we control the variance of $M_{j,q}$ as follows (here, the sum over $k$ are as in (5.21); that is, over $2^{jd}$ terms):

$$\text{Var}(M_{j,p}) = \mathbb{E}[(M_{j,p} - \mathbb{E}[M_{j,p}])^2] \overset{(i)}{=} 2^{-jd} \mathbb{E} \left[ 2^{-jd} \left( \sum_k |X_{j,k}|^p - \mathbb{E}[|X_{j,k}|^p] \right)^2 \right] \overset{(ii)}{=} 2^{-jd} 2^{-jd} \sum_k \left( |X_{j,k}|^p - \mathbb{E}[|X_{j,k}|^p] \right)^2 \overset{(iii)}{=} 2^{-jd} \mathbb{E} \left[ 2^{-jd} \sum_k |X_{j,k}|^{2p} \right] \overset{(iv)}{=} 2^{-jd} \mathbb{E}[M_{j,2p}],$$

where we used that $\mathbb{E}[M_{j,p}] = \mathbb{E}[|X_{j,k}|^p]$ for every $k$ in (i), the independence of the $X_{j,k}$ in (ii), and the relation $\text{Var}(X) \leq \mathbb{E}[X^2]$ in (iii).

We now apply the Chebyshev’s inequality $\mathcal{P}(|X - \mathbb{E}[X]| \geq x) \leq \text{Var}(X)/x^2$ to $X = M_{j,p}$ and $x = 2^{-j\epsilon} m_p/2$ to get

$$\mathcal{P}(|M_{j,p} - \mathbb{E}[M_{j,p}]| \geq 2^{-j\epsilon} m_p/2) \leq \frac{4 \text{Var}(M_{j,p}) 2^{2j\epsilon}}{m_p^2} \leq \frac{4}{m_p^2} 2^{-jd} \mathbb{E}[M_{j,2p}] 2^{2j\epsilon} \leq \frac{4 M_{2p} 2j}{m_p^2} (d(2p(1/\beta_{\infty} - 1/\beta_{\infty}) - 1) + 3\epsilon)
$$

(5.25)

where we used (5.24) and (5.23) (applied to $q = 2p$). Due to the second relation in (5.20), the exponent in (5.25) is strictly negative for $\epsilon$ small enough (what we assume from now). Hence, $\sum_j \mathcal{P}(|M_{j,p} - \mathbb{E}[M_{j,p}]| \geq 2^{-j\epsilon} m_p/2) < \infty$. From the lemma of Borel-Cantelli, only a finite number of the $j$ can therefore satisfy the relation $|M_{j,p} - \mathbb{E}[M_{j,p}]| \geq 2^{-j\epsilon} m_p/2$. A direct consequence is then that there exists almost surely $J \geq 0$ such that for every $j \geq J$,

$$M_{j,p} \geq \mathbb{E}[M_{j,p}] - |M_{j,p} - \mathbb{E}[M_{j,p}]| \geq m_p 2^{-j\epsilon} - \frac{m_p}{2} 2^{-j\epsilon} = \frac{m_p}{2} 2^{-j\epsilon}. \quad (5.26)$$
Using the latter inequality, we deduce that
$$
\|w\|_{B_p^r(\mathbb{R}^d; \rho)}^p \geq C \sum_{j \geq 1} 2^{jp(\tau - d + d/\beta_\infty)} M_{j,p} \geq \frac{C m_p}{2} \sum_{j \geq J} 2^{jp(\tau - d + d/\beta_\infty - \epsilon)}.
$$
For $\epsilon$ small enough, we have that $\tau - d + d/\beta_\infty - \epsilon > 0$ and therefore $\|w\|_{B_p^r(\mathbb{R}^d; \rho)} = \infty$ almost surely.

If $\rho \geq -d/p$. The proof is identical to the one of compound Poisson noises where we also have to treat the case $\rho \geq -d/p$. \hfill \Box

6. Non-Gaussian Lévy White Noise. This section is the occasion to synthesized the results and to deduce the general case from the previous ones. We consider non-Gaussian Lévy white noises, the Gaussian white noise having already been treated in Section 3. We conclude with the proof of Theorem 1.

**Proposition 7.** Consider a non-Gaussian Lévy white noise $w$ with indices $0 \leq \beta_\infty \leq \beta_\infty \leq 2$ and $0 < p_{\text{max}} \leq \infty$. Then, $w$ is

- almost surely in $B_p^r(\mathbb{R}^d; \rho)$ if $\tau < d/\max(p, \beta_\infty) - d$ and $\rho < -d/\min(p, p_{\text{max}})$, for $0 < p \leq 2$, $p$ an even integer, or $p = \infty$; and
- almost surely not in $B_p^r(\mathbb{R}^d; \rho)$ if $\tau > d/\max(p, \beta_\infty) - d$ or $\rho > -d/\min(p, p_{\text{max}})$ for every $0 < p \leq \infty$.

**Proof.** According to Proposition 1, $w = w_1 + w_2$ with $w_1$ a compound Poisson noise, $w_2$ a Lévy white noise with finite moments (that can include a Gaussian part). Moreover, we have

$$
\beta_\infty = \beta_\infty(w_2) \geq \beta_\infty(w_1) = 0,
$$
$$
p_{\text{max}} = p_{\text{max}}(w_1) \leq p_{\text{max}}(w_2) = \infty.
$$

If $\tau < \left(\frac{d}{\max(p, \beta_\infty)} - d\right)$ and $\rho < -\frac{d}{\min(p, p_{\text{max}})}$. Then, $\tau < \left(\frac{d}{p} - d\right)$ and $\rho < -\frac{d}{\min(p, p_{\text{max}}(w_1))}$ so that $w_1 \in B_p^r(\mathbb{R}^d; \rho)$ due to Proposition 5. Similarly, $w_2 \in B_p^r(\mathbb{R}^d; \rho)$ thanks to Proposition 6. Finally, $w = w_1 + w_2 \in B_p^r(\mathbb{R}^d; \rho)$.

If $\tau > \left(\frac{d}{\max(p, \beta_\infty)} - d\right)$. The arguments of Proposition 6 for this case are still valid for $w$.

If $\rho > -\frac{d}{\min(p, p_{\text{max}})}$, if $\rho \geq -d/p$, then the same proof as for the compound Poisson case implies that $w \notin B_p^r(\mathbb{R}^d; \rho)$. We already know that $w \notin B_p^r(\mathbb{R}^d; \rho)$ if $\tau \geq d/\max(p, \beta_\infty) - d$ and if $\rho \geq -d/p$. The only remaining case is when $p < p_{\text{max}}$, $\tau < d/\max(p, \beta_\infty) - d$, and $-d/p_{\text{max}} < \rho < -d/p$. In this case, $w_2 \in B_p^r(\mathbb{R}^d; \rho)$ from Proposition 6, while $w_1 \notin B_p^r(\mathbb{R}^d; \rho)$ with Proposition 5 due to the condition $\rho > -d/p_{\text{max}}$. Finally, $w \notin B_p^r(\mathbb{R}^d; \rho)$ as the sum of a element of $B_p^r(\mathbb{R}^d; \rho)$ and a function not in $B_p^r(\mathbb{R}^d; \rho)$.

Finally, we can translate our results in terms of the local smoothness $\tau_p(w)$ and the asymptotic growth rate $\rho_p(w)$ of Lévy white noises.

**Proof of Theorem 1.** The values of $\tau_p(w)$ and $\rho_p(w)$ are directly deduced from Propositions 4, 5, and 7. Positive results ($w \in B_p^r(\mathbb{R}^d; \rho)$) directly give lower bounds for $\tau_p(w)$ and $\rho_p(w)$, while negative results ($w \notin B_p^r(\mathbb{R}^d; \rho)$) provide upper bounds. For the Gaussian and compound Poisson cases, studied separately, there are no restrictions on $0 < p \leq \infty$. The case of a general non-Gaussian Lévy white noise is deduced from Proposition 7. \hfill \Box
7. Discussion and Examples.

7.1. Application to some Subfamilies of Lévy White Noises. We apply Theorem 1 to deduce the local smoothness and asymptotic growth rate of notorious Lévy white noise. This includes Gaussian, symmetric-α-stable, symmetric Gamma (including Laplace), compound Poisson, inverse Gaussian and layered stable white Lévy noises. All the underlying laws are known to be infinitely divisible [37, 22]. In Table 1, we define the different families in terms of the characteristic functional of $X = \langle w, 1_{[0,1]} \rangle$ and give adequate references. Most of these families together with the convention we are following in this paper are introduced with more details in [14, Section 5.1]. This is especially relevant for the layered stable white noises, which have the particularity of describing the complete spectrum of possible couples $(\alpha_1, \alpha_2) = (\beta_\infty, p_{\text{max}}) \in (0, 2)^2$, and for which the Lévy exponent is

$$
\Psi_{\alpha_1, \alpha_2}(\xi) = \int_{\mathbb{R}} (\cos(t\xi) - 1) \left( 1_{|t| \leq 1} |t|^{-(\alpha_1+1)} + 1_{|t| > 1} |t|^{-(\alpha_2+1)} \right) dt. \tag{7.1}
$$

We also provide a visualization of our results in terms of Triebel diagrams. In Figures 2 to 5, we plot the local smoothness $\frac{1}{p} \mapsto \tau_{p}(w)$ and asymptotic growth rate $\frac{1}{p} \mapsto \rho_{p}(w)$ for different Lévy white noises (with the exception of $\tau_{p}(w)$ that is not fully determined for the general case in Figure 5; here, we represent the lower and upper bounds of (1.8)). A given noise is almost surely in a Besov space $B^{\tau}_{p}(\mathbb{R}^d; \rho)$ is the points $(1/p, \tau)$ and $(1/p, \rho)$ are in the green regions. A contrario, the noise is almost surely not in $B^{\tau}_{p}(\mathbb{R}^d; \rho)$ if $(1/p, \tau)$ or $(1/p, \rho)$ are in the red region. In Figure 5, the white region corresponds to the case where we do not now if the Lévy white noise is or not in the corresponding Besov spaces.

7.2. Lévy White Noises with Distinct Blumenthal-Getoor Indices. In Theorem 1, we obtained lower and upper bounds for the local regularity of a Lévy white noise that are equal for every $p$ if and only if $\beta_\infty = \beta_\infty$. This equality is valid for all the examples presented in Section 7.1. It is however possible to find counterexamples, and it is even possible to define Lévy exponents with indices taking any values $(\beta_\infty, \beta_\infty) \in [0, 2]^2$ with the obvious constraint that $\beta_\infty \leq \beta_\infty$. This is done

| White noise                | Parameters | $\hat{\mathcal{F}}_{\xi}(\xi)$ | $\beta_\infty = \beta_\infty$ | $p_{\text{max}}$ |
|---------------------------|------------|---------------------------------|-------------------------------|------------------|
| Gaussian                  | $\sigma^2 > 0$ | $e^{-\sigma^2 \omega^2/2}$ | 2                             | $\infty$         |
| Cauchy [36]               | $\gamma > 0$  | $e^{-\gamma|\xi|}$              | 1                             | 1                |
| SoS [36]                  | $0 < \alpha < 2$ | $e^{-|\xi|^\alpha}$          | $\alpha$                      | $\alpha$         |
| sum of SoS                | $0 < \alpha_1, \alpha_2 \leq 2$ | $e^{-|\xi|^{\alpha_1} - |\xi|^{\alpha_2}}$ | $\max(\alpha_1, \alpha_2)$ | $\min(\alpha_1, \alpha_2)$ |
| Laplace [25]              | $\sigma^2 > 0$ | $(1 + \sigma^2 \xi^2/2)^{-1}$ | 0                             | $\infty$         |
| symmetric Gamma [25]      | $\sigma^2, \lambda > 0$ | $(1 + \sigma^2 \xi^2/2)^{-\lambda}$ | 0                             | $\infty$         |
| finite-moments            | $\lambda > 0$  | $e^{\lambda(\xi^2 - 1)}$     | $\alpha_1$                    | $\alpha_2$       |
| compound Poisson          | $\beta$      | $\beta(\xi - 1)$             | $\alpha_1$                    | $\alpha_2$       |
| layered stable [22]       | $0 < \alpha_1, \alpha_2 < 2$ | see (7.1)                     | $1/2$                         | $\infty$         |
| inverse Gaussian [1]      | -           | $\left( 1 - (1 - 2\xi^2)^{1/2} \right)$ | $1/2$                         | $\infty$         |
in [17, Examples 1.1.14, 1.1.15], where the authors introduce

\[
\Psi_{\beta_2,M}(\xi) = \sum_{k \geq 1} 2^{\beta_2 M^k - k} (\cos(2^{-M^k} \xi) - 1),
\]

(7.2)

\[
\Psi_{\beta_1,\beta_2,M}(\xi) = \int_{|t| \leq 1} \frac{\cos(t\xi) - 1}{|t|^{|\beta_1| + 1}} dt + \Psi_{\beta_2,M}(\xi).
\]

(7.3)

with $0 < \beta_1 \leq \beta_2 < 2$ and $M > 2/(2 - \beta_2)$, and show that $\Psi_{\beta_2,M}$ (resp. $\Psi_{\beta_1,\beta_2,M}$) is a Lévy exponent with Blumenthal-Getoor indices $\beta_\infty = 0$ and $\beta_\infty = \beta_2$ ($\beta_\infty = \beta_1$ and $\beta_\infty = \beta_2$, respectively).

7.3. Conclusive Remarks and Open Questions. We have obtained new results on the localization of Lévy white noises in weighted Besov spaces. This includes the identification of the local smoothness in many cases, and lower and upper bounds for the general case, together with the identification of the asymptotic growth rate.

Some open questions remain for a complete answer on the Besov regularity of Lévy white noise.

- Our general results are valid for integrability $0 < p < 2$, $p = 2m$ with $m \geq 1$ integer, or $p = \infty$. We conjecture that the derived formulas remain true for any $0 < p \leq \infty$.  

Fig 2: Gaussian noise

Fig 3: SαS noise with $\alpha = 2/3$

Fig 4: Compound Poisson noise with $p_{\text{max}} = 2$
More importantly, we have distinct lower and upper bound for the local smoothness when $\beta_\infty \neq \beta_\infty$ in Theorem 1 for $\beta_\infty \leq p \leq \beta_\infty$. For worst-case examples of Section 7.2, this results in the least sharp bound $d/\max(p, 2) - d \leq \tau_p(w) \leq d/p - d$. The identification of $\tau_p(w)$ for these cases is unknown at this stage, and it is not sure that it can be expressed in terms of the indices considered so far.

Finally, we did not investigate the localization of a general Lévy white noise for the critical value $\tau = \tau_p(w)$ or $\rho = \rho_p(w)$. However, partial answers have been given for compound Poisson white noises (Proposition 5) and finite moments white noises (Proposition 6), and a complete characterization was given in the Gaussian case (Proposition 4). For the general case, we conjecture that $w \notin B^\tau_{p}(\mathbb{R}^d; \rho)$ as soon as $\tau = \tau_p(w)$ or $\rho = \rho_p(w)$, in accordance with the known results.

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