Towards the PAC Learnability of Nash Equilibrium

Zhijian Duan 1 Dinghuai Zhang 2 Wenhan Huang 3 Yali Du 4 Jun Wang 5 Yaodong Yang 1 Xiaotie Deng 1

Abstract

Nash equilibrium (NE) is one of the most important solution concepts in game theory and has broad applications in machine learning research. While tremendous empirical success has been reported on learning approximate NE, whether NE is Probably Approximately Correct (PAC) learnable has not been addressed yet. In this paper, we make the first effort to study the PAC learnability of NE in normal-form games. We prove that NE is agnostic PAC learnable if the hypothesis class for Nash predictors has bounded covering numbers. Theoretically, we derive such a result by constructing a self-supervised PAC learning framework based on a commonly-used objective for Nash approximation. Empirically, we provide numerical evidence showing that a Nash predictor trained under our PAC learning framework, even with the most straightforward neural architecture, can efficiently learn the NE for games sampled from the same unknown distribution. Our results justify the feasibility of approximating NE through purely data-driven approaches, which benefits both game theorists and machine learning practitioners.

1. Introduction

Nash equilibrium (NE) Nash et al. (1950), in which each player’s strategy is optimal given the strategies of all other players, is one of the most important solution concepts in game theory. Recently, there has been increasing interest in NE due to its broad applications in machine learning, including Generative Adversarial Networks (GAN) Goodfellow et al. (2014) and Multi-Agent Reinforcement Learning (MARL) Yang & Wang (2020). Given that computing a two-player NE is PPAD-hard (Chen et al., 2009), machine learning practitioners have focused on approximating NE through learning-based methods (Fudenberg et al., 1998; Cesa-Bianchi & Lugosi, 2006). However, the fundamental problem of whether a NE is learnable, even in normal-form games, has not yet been addressed.

In this paper, we make the first effort to study the Probably Approximately Correct (PAC) learnability of NE. Specifically, we consider the agnostic PAC learnability of predicting NE in normal-form games where games are independently generated from an identical distribution. The main challenge is that in many cases, there are multiple NEs (See Table 1 for illustration), which brings difficulty for naively adopting supervised learning techniques in learning the NE.

To tackle the above challenge, we set up a novel self-supervised PAC learning framework based on a commonly-used loss function for approximating NE (which we name Nash approximation loss hereafter). Computing such loss only takes joint strategy and game utility as inputs. We prove that the Nash approximation loss is Lipschitz continuous with respect to both inputs, which leads to our main result that NE is agnostic PAC learnable by the hypothesis class that has bounded covering numbers. To the best of our knowledge, this is the first result that addresses the learnability of NE. Our result suggests that, with a large training set, one can learn a generalizable Nash predictor that maps from game utilities to approximate NEs for games sampled independent and identically (i.i.d.) from some distribution.

To further show the practicality of our PAC learnable result, we conduct numerical experiments by training Nash predictors under our PAC learning framework. The results suggest that Nash predictors, even constructed by standard fully-connected neural networks, can efficiently predict approximate NEs for games in the test set. On top of the prediction performance, we also show the effectiveness of using our Nash predictions as the pre-solving initialization for other Nash solvers, such as Tsaknakis & Spirakis (2007) and replicator dynamics (Schuster & Sigmund, 1983). In both cases, we report faster convergence.

In summary, our paper makes the following contributions:

• We propose to study the PAC learnability of NE in n-player normal-form games, and we prove that NE is agnostic PAC learnable with respect to Nash approximation loss. To the best of our knowledge, we are the first one to characterize the learnability of NE.
Towards the PAC Learnability of Nash Equilibrium

- We conduct extensive experiments to demonstrate the practicality of our PAC learnable result. Experimental results justify the effectiveness of the predicted NEs obtained from our PAC learning framework.

Our paper is organized as follows: In Section 2 we describe related works; In Section 3 we introduce the preliminary of game theory; In Section 4 we set up the PAC learning framework for NE learning; We present our PAC results in Section 5 and conduct numerical experiments in Section 6; We draw the conclusion in Section 7.

2. Related Work

For 2-player games, there are algorithms with a theoretical guarantee for maximum Nash approximation loss (See definition in Equation (2)). Kontogiannis et al. (2006) and Daskalakis et al. (2009) introduced simple polynomial-time algorithms based on searching small supports to reach an approximation loss of 3/4 and 1/2, respectively. Daskalakis et al. (2007) provided an algorithm of approximation loss 0.38 by enumerating arbitrarily large supports, and this approximation loss is also achieved by Czumaj et al. (2019) with a different approach. Bosse et al. (2007) proposed an algorithm based on Kontogiannis & Spirakis (2007) to reach an approximation loss of 0.36. Finally, the TS algorithm (Tsaknakis & Spirakis, 2007) achieves the current best-known approximation loss of 0.3393, and Chen et al. (2021) proved that the bound is tight. However, computing approximate NE for even arbitrary constant approximation is PPAD-hard (Daskalakis, 2013).

Learning is another paradigm to compute approximate NE by repeatedly proposing temporal strategies and updating them with the feedback rewards. Fictitious play (Monderer & Shapley, 1996) is the most well-known learning-based algorithm to approximate NE, and Conitzer (2009) proves that it reaches an approximate loss of 1/2 when given constant rounds. Double Oracle methods (McMahan et al., 2003; Dinh et al., 2021) and PSRO methods (Lanctot et al., 2017; Perez-Nieves et al., 2021), though effective, target at solving zero-sum games only. Online learning methods, including regret matching (Hart & Mas-Colell, 2000), Hedge (Auer et al., 1995) and Multiplicative weight update (Aorora et al., 2012), are proved to converge to (approximate) correlated equilibrium (Cesa-Bianchi & Lugosi, 2006). These algorithms focus on solving (approximate) NE for one specific game, while we consider games under any distributions.

Recently, several works have proposed to predict NE through data-driven approaches. Such works make use of the historical game-playing data and learn the game utility functions (Bertsimas et al., 2015; Zhang & Paschalidis, 2017; Allen et al., 2021) or game gradients (Ling et al., 2018; 2019; Heaton et al., 2021) from the observed (approximate) NE. By doing so, they can predict approximate Nash solutions for a class of games (e.g., contextual games (Heaton et al., 2021)). In our paper, we predict NE by taking game utility as the input, and we study the learnability of NE.

As for learnability analysis in games, Viqueira et al. (2019); Marchesi et al. (2020) provide the PAC analysis of learning the game utility in simulation-based games, in which the utility is obtained by query and would potentially be disturbed by noise. A Nash Oracle, which can output the exact NE for arbitrary games directly, is assumed in their papers. Similarly, Fele & Margellos (2020) considers games with noisy utilities and studies the learnability of NE given the strong assumption of Nash Oracle. As a comparison, in our paper, the real game utilities are provided, and most importantly, we do not assume any Nash Oracles. Moreover, while Jin et al. (2021b); Bai et al. (2020) propose PAC learnable algorithm to approximate NE in a zero-sum Markov game, and to approximate Coarse Correlated Equilibria (CCE) or Correlated Equilibria (CE) in a general-sum Markov game, we must highlight the difference that we consider the PAC analysis of NE in general-sum games sampled from a same arbitrary distribution, instead of approximating NE for one specific game instance.

3. Preliminary of Game Theory

Normal-form Games. We study the PAC learnability of Nash equilibrium in finite n-player normal-form games. We define such a n-player game with jointly utility function u as \( \Gamma_u = (N, A, u) \), in which

- \( N = \{1, 2, \ldots, n\} \) is the set of all the n players. Each player is represented by the index \( i \in N \).
- \( A = A_1 \times A_2 \times \ldots \times A_n \) is the combinatorial action space of all players, in which \( A_i \) is the action space for player \( i \). For player \( i \in N \), let \( a_i \in A_i \) be a specific action and \( |A_i| \) be the number of actions. An action is also referred to as a pure strategy. An action profile \( a = (a_1, a_2, \ldots, a_n) \in A \) represents one play of the game in which the player \( i \) taking her corresponding action \( a_i \in A_i \). The action space \( A \) is a Cartesian product that contains all possible action profiles. Therefore, we have \( |A| = \prod_{i \in N} |A_i| \).
- \( u = (u_1, \ldots, u_n) \in U \) is the game utility, or jointly utility/payoff function, in which \( u_i : A \rightarrow \mathbb{R} \) is the utility function for player \( i \). \( u_i \) describes the utility of the player \( i \) on each possible action profile \( a = (a_1, a_2, \ldots, a_n) \in A \), thus we have \( |u_i| = |A| \) and \( |u| = n|A| \). In this paper, without loss of generality, we make the assumption that each utility is in the range of \([0, 1]\).
Mixed strategy. A mixed strategy of player $i$, denoted by $\sigma_i$, is a distribution over her action set $A_i$. Specifically, $\sigma_i(a_i)$ represents the probability that player $i$ chooses action $a_i$. Under such definition, we have $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$.

Let $\Delta A_i$ be the set of all the possible mixed strategies for player $i$, that is, the set of all the possible distribution over set $A_i$. We have $\sigma_i \in \Delta A_i$. A mixed strategy profile $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a jointly mixed strategy for all the players. On top of the mixed strategy profile $\sigma$, the probability of action profile $a = (a_1, a_2, \ldots, a_n)$ being played is denoted as $\sigma(a) := \prod_{i \in N} \sigma_i(a_i)$. Notice that an action profile $a$ (i.e., a pure strategy profile) can be also seen as a mixed strategy profile $\sigma$ with $\sigma(a_i) = 1$ for all $i \in N$.

We can calculate player $i$’s expected utility under mixed strategy profile $\sigma$ as:

$$u_i(\sigma) = \mathbb{E}_{a \sim \sigma}[u_i(a)] = \sum_{a \in A} \sigma(a) u_i(a) \quad (1)$$

Besides, on behalf of player $i$, the other players’ strategy profile is denoted as $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$ for convenience.

**Nash equilibrium (NE).** Nash equilibrium is one of the most important solution concepts in game theory. A (mixed) strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ is called a Nash equilibrium if and only if for each player $i \in N$, her strategy is the best response given the strategies $\sigma^*_{-i}$ of all the other players. Formally, in $n$-player normal-form games, a Nash equilibrium $\sigma^*$ satisfies:

$$u_i(\sigma_i^*, \sigma^*_{-i}) \geq u_i(\sigma_i, \sigma^*_{-i}), \quad \text{(NE)}$$

for all player $i \in N$ and all the other mixed strategies $\sigma_i^* \in \Delta A_i$ for player $i$. No one can benefit more by deviating from her current strategy in a Nash equilibrium. However, computing Nash equilibrium for even general 2-player games is PPAD-hard (Chen et al., 2009).

**$\epsilon$-approximate Nash equilibrium ($\epsilon$-NE).** Given the hardness of computing Nash equilibrium, many works focus on finding the approximate solutions. Given $\epsilon > 0$, we say a strategy profile $\tilde{\sigma}$ is an $\epsilon$-approximate Nash equilibrium if no one can achieve more than $\epsilon$ utility gain by deviating from her current strategy. Formally, in $n$-player normal-form games, an $\epsilon$-approximate Nash equilibrium $\tilde{\sigma}$ satisfies:

$$u_i(\sigma_i^*, \sigma^*_{-i}) \leq u_i(\tilde{\sigma}_i, \sigma^*_{-i}) + \epsilon, \quad \text{($\epsilon$-NE)}$$

for all player $i \in N$ and all the other mixed strategies $\sigma_i^* \in \Delta A_i$. The definition of $\epsilon$-approximate Nash equilibrium reflects the idea that players might not be willing to deviate their strategies when the amount of utility that they could gain by doing so is tiny (not more than $\epsilon$).

Towards the PAC Learnability of Nash Equilibrium

4. PAC Learning Framework Setup

In this section, we set up the PAC learning framework to learn a parameterized Nash predictor that maps a game utility to an approximate NE. In general, the learner has access to the following basic components: a domain set $U$; a game-generation distribution $D$, a hypothesis class $\mathcal{H}$ and training set $S$ of size $m$. Afterward, based on the Nash approximation loss (NASHAPR, defined on Equation (2)), we adopt an empirical risk minimization algorithm to get the learned Nash predictor.

4.1. Basic Component

**Domain set $U$.** This is the set of all the possible input samples, i.e., the input games. In our learning framework, we set each input sample as a game utility $u = (u_1, u_2, \ldots, u_n)$, so that our domain set is $U$, the set of all the possible jointly utility functions. Notice that we consider games with a fixed number of players and actions so that for arbitrary two games in $U$ they only differ in their game utilities. Furthermore, no side information (e.g. exact NE or $\epsilon$-NE) is provided in our learning framework.

**Game-generation distribution $D$.** Each game instance in $U$ is sampled i.i.d. from a game-generation distribution $D$. Under such distribution, the generated/sampled games may belong to a specific game class (e.g., zero-sum games). The learner does not know the exact form of $D$, but she can access the generated games. Besides, we make no assumption about $D$.

**Hypothesis class $\mathcal{H}$.** The learner should choose in advance (before seeing the data) a class of functions $\mathcal{H}$, where each function $h : U \rightarrow \Delta A_1 \times \Delta A_2 \times \cdots \times \Delta A_n$ in $\mathcal{H}$ is a Nash predictor that maps game utility to approximate NE. We call such function class $\mathcal{H}$ the hypothesis class. For finite $\mathcal{H}$, it is trivial to provide a PAC learnable result (Shalev-Shwartz & Ben-David, 2014), therefore in our paper, we consider a hypothesis class with infinite size. We will describe the way how we measure the capacity of $\mathcal{H}$ in Section 5.

**Training set $S$.** During learning, a training set $S$ of size $m$ is provided to the learner. $S = \{u^{(1)}, u^{(2)}, \ldots, u^{(m)}\}$ contains $m$ game utilities drawn i.i.d. from domain set $U$ according to game-generation distribution $D$.

4.2. Nash Approximation Loss

The main challenge for Nash predictor learning is the non-unique issue of exact NE. That is, there may be multiple NEs for a game instance (See Table 1 for an illustration). Given such an issue, existing works of predicting NE from data have made some strong assumptions to ensure the uniqueness of NE (Bertsimas et al., 2015; Zhang & Paschalidis, 2017; Li et al., 2020; Heaton et al., 2021), to make the
Table 1: An example illustrating the non-unique issue of exact NE, in which $A_1 = \{L, R\}$ and $A_2 = \{U, D\}$. Each element $(x, y)$ in the table represents $u_1 = x$ and $u_2 = y$ for the corresponding joint action profile. There are two pure strategy Nash equilibria (bolded) and one mixed strategy NE in the example.

| $P_1$ | $P_2$ | $U$ | $D$ |
|-------|-------|-----|-----|
| $L$   |       | $(0,0)$ | $(1,0.5)$ |
| $R$   |       | $(0.5,1)$ | $(0,0)$ |

learning problem well-posed. However, such uniqueness assumption does not hold in $n$-play normal-form games.

On top of the non-unique issue, it is preferable to design a self-supervised loss function that can be computed without any NEs. Inspired by the definition of $\epsilon$-approximate NE in Equation (\epsilon-NE), we define Nash approximation loss as follows:

**Definition 4.1 (Nash approximation loss, NASHAPR).** For a finite $n$-player normal-form game $\Gamma = (N, A, u)$, the Nash approximation loss of strategy profile $\sigma$ with respect to game utility $u$ is the maximum utility gain each player can obtain by deviating from her own strategy. Formally,

$$
\text{NASHAPR}(\sigma, u) := \max_{i \in N} \max_{\sigma' \in \Delta A_i} [u_i(\sigma'_i, \sigma_{-i}) - u_i(\sigma)],
$$

$$
= \max_{i \in N} \max_{a_i \in A_i} [u_i(a_i, \sigma_{-i}) - u_i(\sigma)].
$$

(2)

Although it may be called differently, Nash approximation loss is widely used in the literature of NE computation (Kontogiannis et al., 2006; Tsaknakis & Spirakis, 2007). The computation of NASHAPR($\sigma, u)$ only involves $\sigma$ and $u$. By applying such loss, we bypass the non-unique issue of NE, since we don’t need any NE or side information. Besides, as we will discuss in Section 5, the Nash approximation loss is Lipschitz continuous with respect to both inputs ($\sigma$ and $u$), which leads to our PAC result.

4.3. Learning Algorithm

Based on the aforementioned basic components and Nash approximation loss, we apply empirical risk minimization (ERM) with inductive bias to learn a Nash predictor $h$ from hypothesis class $H$. Specifically, let $L_S(h)$ be the empirical average loss of $h$ on training set $S$, i.e.,

$$
L_S(h) = \frac{1}{|S|} \sum_{u \in S} \text{NASHAPR}(h(u), u),
$$

(5.1)

and then we select a predictor $\text{ERM}_H(S)$ from $H$ that minimize $L_S(h)$:

$$
\text{ERM}_H(S) \in \arg \min_{h \in H} L_S(h).
$$

(ERM)

In practice, it is usually intractable to implement the ERM algorithm we described above. Following the approaches in deep learning community (Goodfellow et al., 2016), we can approximate ERM by minibatch Stochastic Gradient Descent (minibatch SGD). Specifically, we first parameterize the Nash predictor as $h^w$ with $d$-dimensional parameter variable $w \in \mathbb{R}^d$ (e.g., the weights of a neural network). We optimize $w$ by the standard minibatch SGD algorithm (See Algorithm 1).

**Algorithm 1 Nash Predictor Learning via minibatch SGD**

1: **Input:** Training set $S$ of size $m$
2: **Parameters:** Number of iterations $T > 0$, batch size $B > 0$, learning rate $\eta > 0$, initial parameters $w_0 \in \mathbb{R}^d$ of the Nash predictor model.
3: for $t = 0$ to $T$ do
4: Receive minibatch $S_t = \{u^{(1)}, \ldots, u^{(B)}\} \subset S$
5: Compute the empirical average loss of $S_t$:
6: $L_S(h^w) \leftarrow \frac{1}{B} \sum_{i=1}^{B} \text{NASHAPR}(h^u(u^{(i)}), u^{(i)})$
7: Update model parameters:
8: $w_{t+1} \leftarrow w_t - \eta \nabla_w L_S(h^w)$
9: end for

5. Theoretical Learnability Result

In this section, we present our main conclusion: Nash equilibrium is agnostic PAC learnable with respect to Nash approximation loss, assuming that the hypothesis class $H$ has bounded covering numbers. We derive such a result through the Lipschitz property of Nash approximation loss.

5.1. Main Assumption

Before we analyze the learnability of our PAC learning framework, we make our main assumption to measure the capacity of hypothesis class $H$ through (external) covering numbers (Shalev-Shwartz & Ben-David, 2014). Afterward, we discuss the realizability of our assumption.

To define the external covering numbers of $H$, we need some notations of the whole predictor class and a definition of the distance between two predictors. Let $F$ be the set
of all predictors \( f : U \rightarrow \Delta A_1 \times \Delta A_2 \times \cdots \times \Delta A_n \) from game utility to mixed strategy profile. We have \( \mathcal{H} \subset \mathcal{F} \) by definition. For two predictors \( f_1, f_2 \in \mathcal{F} \), we measure the distance between them via the \( \ell_{\infty,1} \)-distance, defined as follows:

**Definition 5.1** (\( \ell_{\infty,1} \)-distance). The \( \ell_{\infty,1} \)-distance between two predictors \( f_1, f_2 \in \mathcal{F} \) is defined as

\[
\| f_1 - f_2 \|_{\ell_{\infty,1}} := \max_{u \in U} \| f_1(u) - f_2(u) \|_1,
\]

in which

\[
\| \sigma - \sigma' \|_1 := \sum_{i \in N} \sum_{a_i \in A_i} |\sigma_i(a_i) - \sigma'_i(a_i)|
\]

is the \( \ell_1 \)-distance between two mixed strategy profiles \( \sigma, \sigma' \in \Delta A_1 \times \Delta A_2 \times \cdots \times \Delta A_n \).

On top of \( \mathcal{F} \) and \( \ell_{\infty,1} \)-distance, we define the \( r \)-cover and the \( r \)-covering number of a predictor class (hypothesis class) \( \mathcal{H} \subset \mathcal{F} \), which will be used to make our main assumption.

**Definition 5.2** (\( r \)-cover). For two predictor classes \( \mathcal{H}, \mathcal{H}' \subset \mathcal{F} \), we say \( \mathcal{H}' \) \( r \)-covers \( \mathcal{H} \) (w.r.t. \( \ell_{\infty,1} \)-distance) if for all predictor \( h \in \mathcal{H} \), there exists a predictor \( h' \in \mathcal{H}' \) such that \( \| h - h' \|_{\ell_{\infty,1}} \leq r \).

**Definition 5.3** (\( r \)-covering number). We define the \( r \)-covering number of a predictor class \( \mathcal{H} \subset \mathcal{F} \), denoted by \( \mathcal{N}_r(\mathcal{H}, r) \), as the cardinality of the smallest \( \mathcal{H}' \subset \mathcal{F} \) that \( r \)-covers \( \mathcal{H} \) with respect to \( \ell_{\infty,1} \)-distance.

Based on the \( r \)-covering number, we can measure the capacity (or complexity, equivalently) of a hypothesis class \( \mathcal{H} \): \( \mathcal{N}_r(\mathcal{H}, r) \) represents the minimum number of predictors to \( r \)-cover \( \mathcal{H} \). The larger \( \mathcal{N}_r(\mathcal{H}, r) \) a hypothesis class \( \mathcal{H} \) has, the more complex \( \mathcal{H} \) is (and the more representative the parameterized model \( h^w \) with \( w \in \mathbb{R}^d \) is).

Given those definitions, we assume that the logarithm of \( \mathcal{N}_r(\mathcal{H}, r) \) can be bounded by a polynomial of \( \frac{1}{r} \), say, Poly\( (\frac{1}{r}) \). Formally,

**Assumption 5.4** (Bounded covering numbers). For a hypothesis class \( \mathcal{H} \), the logarithm of its \( r \)-covering number satisfies

\[
\ln \mathcal{N}_r(\mathcal{H}, r) \leq \text{Poly} \left( \frac{1}{r} \right).
\]

That is, we only consider \( \mathcal{H} \) with limited representativeness, as what is usually done in learning theory with infinite hypothesis class (Shalev-Shwartz & Ben-David, 2014).

Assumption 5.4 holds for many widely used machine learning models, including classical linear model (Zhang, 2002) and kernel method (Zhou, 2002). Moreover, as we are going to show, Assumption 5.4 also holds for Lipschitz hypothesis class defined as follows, which includes neural networks with parameters of bounded values (Szegedy et al., 2014; Scaman & Virmaux, 2018).

**Definition 5.5** (Lipschitz hypothesis class). We say \( \mathcal{H} \) is a Lipschitz hypothesis class if there is a constant \( L_\mathcal{H} > 0 \) such that for each function (predictor) \( h \in \mathcal{H} \) and for all game utility \( u, v \in U \), we have

\[
\| h(u) - h(v) \|_1 \leq L_\mathcal{H} \| u - v \|_{\max},
\]

in which

\[
\| u - v \|_{\max} := \max_{u \in U} \max_{a \in A} |u(a) - v(a)|
\]

is the \( \ell_{\max} \)-distance between two game utilities \( u \) and \( v \).

**Lemma 5.6**. For Lipschitz hypothesis class \( \mathcal{H} \), Assumption 5.4 holds since we have

\[
\mathcal{N}_r(\mathcal{H}, r) \leq O \left( \left( \frac{L_\mathcal{H}}{r} \right)^n |A| \ln \frac{1}{r} \right)
\]

5.2. Important Lemmas

Next, we will present some critical intermediate results with respect to the smoothness of Nash approximation loss, which will be helpful to derive our PAC result.

We start with deriving the Lipschitz continuity of \( \text{NASHAPR}(\sigma, u) \) with respect to its first input: the joint strategy profile \( \sigma \), and we get the following lemma:

**Lemma 5.7**. For normal-form game \( \Gamma_u = (N, A, u) \), for arbitrary strategy profile \( \sigma \) and strategy profile \( \sigma' \), we have

\[
|\text{NASHAPR}(\sigma, u) - \text{NASHAPR}(\sigma', u)| \leq 2 \| \sigma - \sigma' \|_1,
\]

where \( \| \sigma - \sigma' \|_1 \) is defined in Equation (4).

It turns out that, \( \text{NASHAPR}(\sigma, u) \) is 2-Lipschitz continuous with respect to \( \sigma \) under \( \ell_1 \)-distance. It provides the smoothness of Nash approximation loss with respect to different approximate NEs.

In addition, we also analyze the Lipschitz property of \( \text{NASHAPR}(\sigma, u) \) with respect to the second input: the game utility \( u \), and we get the following result:

**Lemma 5.8**. For strategy profile \( \sigma \), for arbitrary normal form games \( \Gamma_u = (N, A, u) \) and \( \Gamma_v = (N, A, v) \) where \( u, v \in U \), we have

\[
|\text{NASHAPR}(\sigma, u) - \text{NASHAPR}(\sigma, v)| \leq 2 \| u - v \|_{\max},
\]

where \( \| u - v \|_{\max} \) is defined on Equation (7).

**Remark 5.9**. While minor changes in game utility may cause different NEs (See Table 2 for illustration of such non-smooth issue), the Nash approximation loss is 2-Lipschitz continuous with respect to the game utility, as we can see in Lemma 5.8. Such continuity result plays a critical role in the theoretical analysis of game utility learning in simulation-based games (Viqueira et al., 2019), in which the goal is to recover the actual game utility through the noisy query data and to compute an approximate NE for the underlying game.
Towards the PAC Learnability of Nash Equilibrium

Table 2: An example illustrating the non-smooth issue of exact NE, in which $A_1 = \{L, R\}$ and $A_2 = \{U, D\}$. Each element $(x, y)$ in the table represents $u_1 = x$ and $u_2 = y$ for the corresponding joint action profile. Minor changes in the utility of game $\Gamma_u$ (into game $\Gamma_v$) can cause different NE point. The only difference between $\Gamma_u$ and $\Gamma_v$ is the utility $u_1(R, U)$, which only differ by arbitrary small $2\epsilon$.

(a): Game $\Gamma_u$. There is only one NE: $(L, U)$. (b): Game $\Gamma_v$. There is only one NE: $(R, U)$.

\[
\begin{array}{c|c|c|c}
 & P_2 & U & D \\
\hline
L & (0.5, 0.5) & (1, 0) \\
R & (0.5 - \epsilon, 1) & (0, 0) \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 & P_2 & U & D \\
\hline
L & (0.5, 0.5) & (1, 0) \\
R & (0.5 + \epsilon, 1) & (0, 0) \\
\end{array}
\]

5.3. Main Results

Based on Assumption 5.4, Lemma 5.7 and Lemma 5.8, we get our first result: the uniform convergence of hypothesis class $\mathcal{H}$ with respect to Nash approximation loss. It characterizes the sample complexity (i.e., the size of the training set) to probably obtain an $\epsilon$-representative training set $S$. That is, for arbitrary predictors $h \in \mathcal{H}$, let $L_D(h)$ be the expected loss of $h$ under distribution $\mathcal{D}$, i.e.,

\[
L_D(h) := \mathbb{E}_{u \sim \mathcal{D}}[\text{NASHAPR}(h(u), u)],
\]

the empirical loss $L_S(h)$ on the $\epsilon$-representative training set $S$ is close to $L_D(h)$ up to $\epsilon$. To represent the sample complexity of obtaining such $\epsilon$-representative set, we use the covering numbers of $\mathcal{H}$:

**Theorem 5.10** (Uniform Convergence). For all hypothesis class $\mathcal{H}$ of Nash predictors, for all $\epsilon, \delta \in (0, 1)$ and arbitrary game-generation distribution $\mathcal{D}$, with probability at least $1 - \delta$ over draw of the training set $S$ with

\[
m \geq m^{UC}_{\mathcal{H}}(\epsilon, \delta) := \frac{9}{2\epsilon^2} \left( \ln \frac{2}{\delta} + \ln \mathcal{N}_F(\mathcal{H}, \epsilon/6) \right)
\]

games from $\mathcal{D}$, we have

\[
\forall h \in \mathcal{H}, \left| L_S(h) - L_D(h) \right| \leq \epsilon,
\]

where $L_S(h)$ (defined on Equation (L_S)) is the empirical loss of $h$ on training set $S$, and $L_D(h)$ (defined on Equation (L_D)) is the expected loss of $h$ under distribution $\mathcal{D}$.

Theorem 5.10, the uniform convergence of $\mathcal{H}$, is the sufficient condition for *agnostic PAC learnable*. By taking an $\frac{\epsilon}{2}$-representative set $S$ as the training set, and running ERM with respect to $L_S(h)$, we can learn a predictor $\text{ERM}_{\mathcal{H}}(S) \in \mathcal{H}$ with performance near to those of optimal predictors $h^* \in \min_{h \in \mathcal{H}} L_D(h)$ in the hypothesis class.

**Theorem 5.11** (Agnostic PAC Learnable). For all hypothesis class $\mathcal{H}$ of Nash predictors, for all $\epsilon, \delta \in (0, 1)$ and arbitrary game-generation distribution $\mathcal{D}$, with probability at least $1 - \delta$ over draw of the training set $S$ with

\[
m \geq m_{\mathcal{H}}(\epsilon, \delta) := \frac{18}{\epsilon^2} \left( \ln \frac{2}{\delta} + \ln \mathcal{N}_F(\mathcal{H}, \epsilon/6) \right)
\]

games from $\mathcal{D}$, when running empirical risk minimization on Nash approximation loss, we have

\[
L_D(\text{ERM}_{\mathcal{H}}(S)) \leq \min_{h \in \mathcal{H}} L_D(h) + \epsilon,
\]

in which $L_D(h)$ and $\text{ERM}_{\mathcal{H}}(S)$ are defined in Equation (L_D) and Equation (ERM), respectively. The sample complexity $m_{\mathcal{H}}(\epsilon, \delta)$ can be bounded by a polynomial of $\frac{1}{\epsilon}$ and $\ln \frac{1}{\delta}$ when Assumption 5.4 holds.

Theorem 5.11 provides the (agnostic) PAC learnability of NE. Under Assumption 5.4, when using a training set with size larger than a polynomial of $\frac{1}{\epsilon}$ and $\ln \frac{1}{\delta}$, with probability at least $1 - \delta$, the learned Nash predictor is able to reach the near-optimal performance within $\mathcal{H}$. Such a result provides the fundamental theoretical guarantee to learn to predict NE through data-driven approaches. As we will demonstrate by experiments in Section 6, even equipped with the most simple neural architectures, the learned Nash predictor can predict Nash solutions for games under the same distribution efficiently.

**Remark 5.12**. While realizability assumption (Shalev-Shwartz & Ben-David, 2014), i.e., the assumption that $\min_{h \in \mathcal{H}} L_D(h) = 0$, is adopted in many PAC analyses (Krishnamurthy et al., 2016; Jin et al., 2021a), however, it does not feasible in our case. Due to the non-smooth issue we discussed before in Table 2, it remains an open question whether there is a hypothesis class that satisfies realizability assumption and limited capacity assumption (e.g., Assumption 5.4) at the same time. As a result, we do not assume the optimal performance in $\mathcal{H}$, and we consider the agnostic PAC learnability.

6. Numerical Experiments

We provide numerical experiments to verify the practicality of our PAC result. Specifically, we construct a parameterized model as our hypothesis class and train a Nash predictor via the PAC learning framework. We show by experiments that the learned Nash predictor is computation-efficient with low generation loss, and the output solutions can serve as...
Table 3: Experiment result of Bimatrix games of size $300 \times 300$. Each experiment is run by 10 different random seeds and the average results are presented. (a): Average Nash approximation loss (and the corresponding standard deviation) of the learned Nash predictor on each test set. (b): Average time and iterations traditional algorithms spent on each test set to reach the same performance as our trained Nash predictor. * means the method failed to reach the same performance as ours under the maximum iteration limit.

|                      | TravelersDilemma | GrabTheDollar | WarOfAttrition | BertrandOligopoly | MajorityVoting |
|----------------------|------------------|---------------|----------------|-------------------|----------------|
|                      | NASHAPR          | NASHAPR       | NASHAPR        | NASHAPR           | NASHAPR        |
| Ours                 | 1.025e-6 ± 1.18e-7 | 3.480e-5 ± 4.62e-5 | 2.976e-7 ± 2.81e-8 | 3.442e-4 ± 5.05e-5 | 1.621e-4 ± 4.84e-4 |

(b)

|                      | TravelersDilemma | GrabTheDollar | WarOfAttrition | BertrandOligopoly | MajorityVoting |
|----------------------|------------------|---------------|----------------|-------------------|----------------|
|                      | Time Iteration   | Time Iteration | Time Iteration | Time Iteration    | Time Iteration |
| FP                   | <0.5s 1.0        | <0.5s 1.0     | <0.5s 1.0      | 12.3s 1.0         | 89.9s 1.0      |
| RM                   | 5.4s 2799.4      | 4.6s 2391.9   | 2.2s 1193.3    | 2.7s 1355.7       | 9.3s 5284.2    |
| RD                   | 4.4s 3861.8      | 2.0s 1726.0   | 2.9s 2523.9    | 0.9s 800.6        | 96.4s 80604.7  |
| TS                   | 601.8s 5.4       | 302.1s 2.3    | 94.5s 3.0      | 155.8s 1.4        | 94.6s 2.8      |
| Ours                 | <0.5s 1.0        | <0.5s 1.0     | <0.5s 1.0      | <0.5s 1.0         | <0.5s 1.0      |

effective initializing points for other algorithms. We present the experiments on bimatrix games in this section (See Appendix C.1 for experiments with more players). Each experiment is run by 10 different random seeds and the average results are presented.

Experiments Setup. We use GAMUT\(^4\) (Nudelman et al., 2004), a suite of game generators designated for testing game-theoretic algorithms, to generate the game instances. We select 5 game classes as our data distribution since they are nontrivial for TS algorithm (Tsaknakis & Spirakis, 2007) to solve (Fearnley et al., 2015): TravelersDilemma, GrabTheDollar, WarOfAttrition, BertrandOligopoly and MajorityVoting (See Appendix B.1 for a detailed game classes description). We set the bimatrix game size as $300 \times 300$. For each game class, we generate $2 \times 10^4$ game instances with different random seeds\(^5\), and we randomly divided 2000 and 200 instances for validation and test.

As for the Nash predictor, we construct a fully connected neural network as the hypothesis class. We apply ReLU as the activation function and add batch normalization (without learnable parameters) before the activation function. We use 4 hidden layers with 1024 nodes of each layer in our neural network. We learn our model using Adam optimizer with a learning rate of 0.001, and we bound the parameters of our model in the range of $[0, 1]$ to meet Assumption 5.4.

Efficiency. Table 3a reports the test results of our Nash predictor. We observe that the test Nash approximation loss is sufficiently small (less than 0.0005 for all the 5 game classes). We must highlight that, unlike traditional algorithms such as fictitious play (Monderer & Shapley, 1996), our Nash predictor has never seen the test game instances during training. The approximate solution is obtained by just a simple feed-forward computation of the Nash predictor.

To further demonstrate the efficiency of learning a parameterized Nash predictor, we record the time and iterations traditional algorithms spent (on the test set) to reach the same performance as us. We set the following algorithms as baselines: Fictitious play (FP) (Monderer & Shapley, 1996): The most well-known learning algorithm to approximate Nash equilibrium; Regret matching (RM) (Hart & Mas-Colell, 2000): Representative method of no-regret learning, and it leads to correlated equilibrium. Replicator dynamics (RD) (Schuster & Sigmund, 1983): A system of differential equations that describe how a population of strategies, or replicators, evolve through time. TS Algorithm (Tsaknakis & Spirakis, 2007): It reaches the currently best known approximation ratio $\epsilon=0.3393$ for bimatrix games, and we set the approximation objective to $\delta+0.3393 = 10^{-5} + 0.3393$. During implementation, we use GPU to speed up the baselines except for TS algorithm\(^6\). For fictitious play, regret

\(^3\)All of our experiments are run on a Linux machine with 48 core Intel(R) Xeon(R) CPU (E5-2650 v4@2.20GHz) and 4 TITAN V GPU.

\(^4\)http://gamut.stanford.edu/

\(^5\)The randomness comes from several aspects, including the permutation of action space. See Appendix B.2 for details.

\(^6\)It is not able to accelerate TS algorithm by GPU, so we run it
Towards the PAC Learnability of Nash Equilibrium

Figure 1: Experimental results of initializing TS algorithm with Nash predictor, where the y-axis is the total time cost for each algorithm to converge on the test set. ‘TS’ means TS algorithm starting from a random strategy, and in ‘TS+Initialization’ we set the output of our Nash predictor as the initializing point. Each experiment is run by 10 times. Average results and 95% confidence intervals are shown.

matching, and replicator dynamics, we set the maximum number of iterations to $10^5$ and terminate the algorithm once it reaches the same performance. For TS algorithm, since it terminates with probability 1, we early stop it if the same performance has already been reached.

We present the efficiency results in Table 3b. While our Nash predictor efficiently comes up with an approximate solution, however, the baseline methods spend much more time to reach the same performance, especially in the game MajorityVoting. Fictitious play even fails to converge in GrabTheDollar. Such results indicate that our Nash predictor can be served as an efficient approximate Nash solution provider. We also observe similar results in multi-player games (See Appendix C.1).

Effectiveness as Initial Solution. Although with a theoretical guarantee of Nash approximation bound, TS algorithm consumes the most time, which motivates us to provide good initializing points for TS algorithm through our Nash predictor. Figure 1 reports the experimental results. It turns out that, by taking the output strategies of Nash predictor as the starting ports, TS algorithm spends less time to terminate, especially in TravelersDilemma. Notice that TS algorithm ensures the final solution will be better than the initial solution. Thus, it would be helpful to provide a good initialization strategy for the TS algorithm.

In addition to the TS algorithm, replicator dynamics can be also initialized by the Nash predictor. See Figure 2 for the experiments in MajorityVoting (We leave the remaining experiments for other game classes in Appendix C.2). After initializing with Nash predictor, replicator dynamics starts with a lower Nash approximation loss and reaches a better approximate solution than the original version.

7. Conclusion

In this paper, we profile the learnability of Nash equilibrium in $n$-player normal-form games. Theoretically, we prove that NE is agnostic PAC learnable with respect to Nash approximation loss, a commonly-used objective for approximating NE, under a realizable assumption that the hypothesis class has bounded covering numbers. Such results provide the theoretical guarantee to learn a Nash predictor via empirical risk minimization. Empirically, we conduct numerical experiments to verify the PAC result and also demonstrate the efficiency and effectiveness of the learned Nash predictor. Experiment results show that our learned Nash predictor can efficiently approximate Nash solutions for games under the same underlying distribution even with a simple fully-connected neural network. Furthermore, the solution from our predictor can also initialize other algorithms to accelerate its execution. As for future work, we are interested in extending the PAC results beyond normal-form games. Moreover, while we set the fully connected neural networks as the hypothesis class for simplicity in our experiments, it remains a challenge to construct a more efficient Nash predictor.
Towards the PAC Learnability of Nash Equilibrium

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Towards the PAC Learnability of Nash Equilibrium

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A. Additional Theoretical Analysis

A.1. Proof of Lemma 5.7

Proof of Lemma 5.7. \( \forall i \in N, \forall \sigma, \sigma', \) we have

\[ |u_i(\sigma) - u_i(\sigma')| = |u_i(\sigma_1, \sigma_2, \ldots, \sigma_n) - u_i(\sigma'_1, \sigma'_2, \ldots, \sigma'_n)| \]

\[ = \left| \sum_{j=1}^{n} \left( u_i(\sigma_1, \ldots, \sigma_{j-1}, \sigma_j, \sigma'_j, \ldots, \sigma'_{n}) - u_i(\sigma_1, \ldots, \sigma_{j-1}, \sigma'_j, \ldots, \sigma'_{n}) \right) \right| \]

define \( y_{-j} := (\sigma_1, \ldots, \sigma_{j-1}, \sigma'_j, \ldots, \sigma'_{n}) \)

\[ = \left| \sum_{j=1}^{n} \left( u_i(\sigma_j, y_{-j}) - u_i(\sigma'_j, y_{-j}) \right) \right| \]

\[ = \left| \sum_{j=1}^{n} \sum_{a_j} (\sigma_j(a_j) - \sigma'_j(a_j)) u_i(a_j, a_{-j}) y_{-j}(a_{-j}) \right| \]

\[ \leq \sum_{j=1}^{n} \sum_{a_j} |\sigma_j(a_j) - \sigma'_j(a_j)| \sum_{a_{-j}} u_i(a_j, a_{-j}) y_{-j}(a_{-j}) \]

\[ \leq \sum_{j=1}^{n} \sum_{a_j} |\sigma_j(a_j) - \sigma'_j(a_j)| \sum_{a_{-j}} y_{-j}(a_{-j}), \quad \text{since} \quad u_i(a_j, a_{-j}) \in [0, 1] \]

\[ \leq \sum_{j=1}^{n} \sum_{a_j \in A_j} |\sigma_j(a_j) - \sigma'_j(a_j)| \]

\[ = \|\sigma - \sigma'\|_1 \]

Therefore, \( \forall a_i \in A_i, \)

\[ u_i(a_i, \sigma_{-i}) - u_i(\sigma) = u_i(a_i, \sigma_{-i}) - u_i(a_i, \sigma'_{-i}) + u_i(a_i, \sigma'_{-i}) - u_i(\sigma') + u_i(\sigma') - u_i(\sigma) \]

\[ \leq \|\sigma - \sigma'\|_1 + \text{NASHAPR}(\sigma', u) + \|\sigma - \sigma'\|_1 \]

\[ = \text{NASHAPR}(\sigma', u) + 2\|\sigma - \sigma'\|_1 \quad (16) \]

Based on this result, we get

\[ \text{NASHAPR}(\sigma, u) = \max_{i \in N, a_i \in A_i} [u_i(a_i, \sigma_{-i}) - u_i(\sigma)] \leq \text{NASHAPR}(\sigma', u) + 2\|\sigma - \sigma'\|_1 \quad (17) \]

Similarly, we can also derive that

\[ \text{NASHAPR}(\sigma', u) \leq \text{NASHAPR}(\sigma, u) + 2\|\sigma - \sigma'\|_1 \quad (18) \]

This finish the proof.

\[ \square \]

A.2. Proof of Lemma 5.8

Proof of Lemma 5.8. \( \forall u, v \in U, \forall \sigma \in \Delta A_1 \times A_2 \times \cdots \times A_n, \forall i \in N, \forall a_i \in A_i, \) we have

\[ u_i(a_i, \sigma_{-i}) - u_i(\sigma) = u_i(a_i, \sigma_{-i}) + (u_i(a_i, \sigma_{-i}) - v_i(a_i, \sigma_{-i})) \]

\[ \leq v_i(a_i, \sigma_{-i}) + \|u - v\|_{\text{max}}, \]

\[ \leq v_i(\sigma) + \text{NASHAPR}(\sigma, v) + \|u - v\|_{\text{max}}, \]

\[ \leq u_i(\sigma) + \text{NASHAPR}(\sigma, v) + 2\|u - v\|_{\text{max}} \quad (19) \]

Therefore,

\[ \text{NASHAPR}(\sigma, u) = \max_{i \in N, a_i \in A_i} [u_i(a_i, \sigma_{-i}) - u_i(\sigma)] \leq \text{NASHAPR}(\sigma, v) + 2\|u - v\|_{\text{max}} \quad (20) \]

Similarly, we also have

\[ \text{NASHAPR}(\sigma, v) \leq \text{NASHAPR}(\sigma, u) + 2\|u - v\|_{\text{max}} \quad (21) \]

\[ \square \]
A.3. Proof of Theorem 5.10

Proof of Theorem 5.10. We denote \( \mathcal{H}_r \) as the minimum \( r \)-covering set of \( \mathcal{H} \), i.e., \( |\mathcal{H}_r| = N_r(\mathcal{H}, r) \). For all \( h \in \mathcal{H} \), we denote \( h_r \) as the closed function to \( h \) in \( \mathcal{H}_r \). Therefore, \( \forall u \in U \), we have \( \|h(u) - h_r(u)\|_1 \leq r \). Then,

\[
|\ell(h(u) - \ell(h_r(u))| = |N_{\text{Nash}}(h(u), u) - N_{\text{Nash}}(h_r(u), u)|
\leq 2\|h(u) - h_r(u)\|_1, \quad \text{according to Lemma 5.7 (22)}
\]

\( \forall \epsilon \in (0, 1) \), we set the covering radius \( r = \frac{\epsilon}{6} \). Then,

\[
\mathbb{P}_{\mathcal{S} \sim \mathcal{D}^m} \left[ \exists h \in \mathcal{H}, \left| L_S(h) - L_D(h) \right| > \epsilon \right]
\leq \mathbb{P}_{\mathcal{S} \sim \mathcal{D}^m} \left[ \exists h \in \mathcal{H}, \left| L_S(h) - L_S(h_r) \right| + \left| L_S(h_r) - L_D(h_r) \right| + \left| L_D(h_r) - L_D(h) \right| > \epsilon \right]
\leq \mathbb{P}_{\mathcal{S} \sim \mathcal{D}^m} \left[ \exists h \in \mathcal{H}, 2r + \left| L_S(h_r) - L_D(h_r) \right| > \epsilon \right]
\leq \mathbb{P}_{\mathcal{S} \sim \mathcal{D}^m} \left[ \exists h_r \in \mathcal{H}_r, \left| L_S(h_r) - L_D(h_r) \right| > \frac{1}{3} \epsilon, \quad r = \frac{\epsilon}{6} \right]
\leq N(\mathcal{H}, \frac{\epsilon}{6}) \mathbb{P}_{\mathcal{S} \sim \mathcal{D}^m} \left[ \left| L_S(h) - L_D(h) \right| > \frac{1}{3} \epsilon \right]
\leq 2N(\mathcal{H}, \frac{\epsilon}{6}) \exp\left(\frac{-2}{9} m \epsilon^2\right), \quad \text{according to Hoeffding Inequality (23)}
\]

When \( m \geq \frac{2N(\mathcal{H}, \frac{\epsilon}{6})}{\delta^2} \), according to Equation (23), we have

\[
\mathbb{P}_{\mathcal{S} \sim \mathcal{D}^m} \left[ \exists h \in \mathcal{H}, \left| L_S(h) - L_D(h) \right| > \epsilon \right] < \delta \tag{24}
\]

\( \square \)

A.4. Proof of Lemma 5.6

The proof is done by construction.

Construct 1: A \( \nu \)-covering set \( U_\nu \subseteq U \) for the domain of \( \mathcal{H} \) (i.e., \( U \)) with respect to \( \ell_\text{max} \) distance.

We do so by discretizing each element along \([0, 1]\) at scale \( \nu \). That is, we discretize it to \( \{0, \nu, 2\nu, \ldots, \lfloor \frac{1}{\nu} \rfloor \nu\} \). Therefore, we have

\[
|U_\nu| = \left\lceil \frac{1}{\nu} \right\rceil^{|A|} \tag{25}
\]

Construct 2: A \( \mu \)-covering set \( \Pi_\mu = \Pi_{\mu,1} \times \ldots \times \Pi_{\mu,n} \subseteq \Delta A_1 \times \cdots \times \Delta A_n \) with respect to \( \ell_1 \) distance for the range of \( \mathcal{H} \). For all \( i \in N \), \( \Pi_{\mu,i} \subseteq \Delta A_i \) is a \( \frac{\mu}{n} \)-covering set of \( \Delta A_i \) w.r.t. \( \ell_1 \) distance.

First, we define \( g(x) \) for \( x \in (0, 1) \) as the maximum value \( y \in (0, x] \) such that \( \frac{1}{y} \) is an integer. From the definition, we can easily get the following corollary:

Corollary A.1. \( g(x) \in \left( \frac{x}{\lfloor x \rfloor}, x \right] \) for \( x \in (0, 1) \).

We construct \( \Pi_{\mu,i} \) by discretizing each element along \([0, 1]\) at scale \( g(\frac{\mu}{n|A_i|}) \). By doing so, \( \Pi_{\mu,i} \) is a \( \frac{\mu}{n} \)-covering set of \( \Delta A_i \).

The cardinal number of \( \Pi_{\mu,i} \) is a combination number:

\[
|\Pi_{\mu,i}| = \left( \frac{g(\frac{\mu}{n|A_i|})}{|A_i| - 1} \right)^{|A_i|} \leq \left( \frac{\epsilon}{|A_i| - 1} \right)^{|A_i|} = O(\left( \frac{1}{\mu} \right)^{|A_i| - n}) \tag{26}
\]

So that

\[
|\Pi_{\mu}| = \prod_{i \in N} |\Pi_{\mu,i}| = \prod_{i \in N} \left( \frac{g(\frac{\mu}{n|A_i|})}{|A_i| - 1} \right)^{|A_i|} \leq \prod_{i \in N} \left( \frac{\epsilon}{|A_i| - 1} \right)^{|A_i|} = O(\left( \frac{1}{\mu} \right)^{|A| - n}) \tag{27}
\]
Towards the PAC Learnability of Nash Equilibrium

**Construct 3:** A r-covering set \( \mathcal{H}_r \subset \mathcal{F} \) of \( \mathcal{H} \) with respect to \( \ell_{\infty,1} \) distance.

For all \( u \in U \), we define \( u_\nu \in U_\nu \) as the closed utility matrix to \( u \) in \( U_\nu \) (with tie-breaking rule) so that we have \( \| u_\nu - u \|_{\max} \leq \nu \). Based on this, we construct an auxiliary function class \( \mathcal{F}_{\nu,\mu} \). It contains all the functions \( f : U \to \Pi_\nu \) that satisfies \( f(u) = f(u_\nu) \) for all \( u \in U \). By the definition of \( \mathcal{F}_{\nu,\mu} \), we have

\[
|\mathcal{F}_{\nu,\mu}| = |\Pi_\mu|^{|U_\nu|}
\]

(28)

For all \( h \in \mathcal{H} \), denote \( h_{\nu,\mu} \) as the closed function to \( h \) in \( \mathcal{F}_{\nu,\mu} \) with respect to \( \ell_{\infty,1} \) distance. We have \( \| h - h_{\nu,\mu} \| \leq \mu \). Based on this,

**Claim A.2.** \( \forall u \in U, |\ell(h, u) - \ell(h_{\nu,\mu}, u)| \leq 2L_H \nu + 2\mu \)

**Proof.**

\[
|\ell(h, u) - \ell(h_{\nu,\mu}, u)| = |\text{NASHPR}(h(u), u) - \text{NASHPR}(h_{\nu,\mu}(u), u)|
\]

\[
\leq 2\| h(u) - h_{\nu,\mu}(u) \|_1
= 2\| h(u) - h_{\nu,\mu}(u_\nu) \|_1
\leq 2\| h(u) - h(u_\nu) \|_1 + 2\| h(u_\nu) - h_{\nu,\mu}(u_\nu) \|_1
\leq 2L_H \nu + 2\mu
\]

A5. **Proof of Theorem 5.11**

**Proof of Theorem 5.11.** Denote \( h_S \in \arg\min_{h \in \mathcal{H}} L_S(h) \) and \( h^* \in \arg\min_{h \in \mathcal{H}} L_D(h) \). Define \( m_H(\epsilon, \delta) := m_H^{UC}(\epsilon, \delta) \).

when \( m \geq m_H(\epsilon, \delta) = \frac{18}{\epsilon} \ln \left( \frac{2N(\mathcal{H}, \frac{\epsilon}{\delta})}{\delta} \right) \), with probability at least \( 1 - \delta \) we have

\[
L_D(h_S) \leq L_S(h_S) + \frac{\epsilon}{2} \leq L_S(h^*) + \frac{\epsilon}{2} \leq L_D(h^*) + \frac{\epsilon}{2} + \frac{\epsilon}{2} = L_D(h^*) + \epsilon
\]

(31)

B. GAMUT

**B.1. Game Classes**

There are a total of 35 distinct game classes available in GAMUT. The 5 game classes we used are 8:

- **TravelersDilemma**: Each player simultaneously requests an amount of money and receives the lowest of the requests submitted by all players.

- **GrabTheDollar**: A price is up for grabs, and both players have to decide when to grab the price. The action of each player is the chosen times. If both players grab for it simultaneously, they will rip the price, and both will receive a low payoff. If one chooses a time earlier than the other, he will receive the high payoff, and the opposing player will receive a payoff between the high and the low.

8See GAMUT User Guide for more details: http://gamut.stanford.edu/userdoc.pdf
Towards the PAC Learnability of Nash Equilibrium

Table 4: Experiment results of 3-player and 4-player games. The game dimension is 30 × 30 × 30 for 3-player and 15 × 15 × 15 × 15 for 4-player. Each experiment is run by 10 different random seeds and the average results are presented.

(a) Average Nash approximation loss (and the corresponding standard deviation) of the learned Nash predictor on each test set; (b) Average Time and iterations traditional algorithms spent on each test set to reach the same performance as our trained Nash predictor. * represents that the method failed to reach the same performance under the iteration limit.

(a)

| Methods         | BertrandOligopoly-3 | BertrandOligopoly-4 | MajorityVoting-3 | MajorityVoting-4 |
|-----------------|---------------------|---------------------|-------------------|------------------|
| Ours            | 2.542e-6 ± 4.25e-6  | 1.468e-7 ± 3.07e-8  | 7.306e-4 ± 5.73e-4| 3.324e-4 ± 3.87e-4|

(b)

| Methods | BertrandOligopoly-3 | BertrandOligopoly-4 | MajorityVoting-3 | MajorityVoting-4 |
|---------|---------------------|---------------------|-------------------|------------------|
| Time    | 178.0s              | 304.6s              | 22.4s             | 33.0s            |
| Iteration | 100000             | 100000             | 11333.9           | 220000           |
| Fp      |                      |                      | 13.0s             | 13.0s            |
|         |                      |                      | 4317.8            | 4317.8           |
| RM      | 14.3s               | 3.3s                | 0.6s              | 0.6s             |
|         | 5487.6              | 842.0               | 221.5             | 89.7             |
|         | 19335.8             | 38927.1             | 2621.2            | 16329.1          |
| Ours    | <0.5s               | <0.5s               | <0.5s             | <0.5s            |
|         | 1.0                 | 1.0                 | 1.0               | 1.0              |

- **WarOfAttrition**: In this game, both players are competing for a single object, and each chooses a time to concede the object to the other player. If both concede at the same time, they share the object. Each player has a valuation of the object, and each player’s utility is decremented at every time step.

- **BertrandOligopoly**: All players in this game are producing the same item and are expected to set a price at which to sell the item. The player with the lowest price gets all the demand for the item and produces enough items to meet the demand so that obtain the corresponding payoff.

- **MajorityVoting**: This is an n-player symmetric game. All players vote for one of the |A_1| candidates. Players’ utilities for each candidate being declared the winner are arbitrary. If there is a tie, the winner is the candidate with the lowest number. There may be multiple Nash equilibria in this game.

B.2. Game Generation

The game generator tool GAMUT requires random seeds to generate a game. According to the source code of GAMUT, the randomness includes:

- The interpretation of each action, especially when each action corresponds to a numerical value within a range. For instance, in TravelersDilemma, each action represents a bet.

- The utility of each player when it is not unique. For example, in MajorityVoting, players’ utilities for each candidate being declared the winner are arbitrary.

- The permutation of the action space. The action space will be randomly shuffled, so as the corresponding utilities.

C. Additional Experiments

C.1. Multi-player Games

We also conduct experiments on games with more than two players. Specifically, we evaluate our Nash predictor in BertrandOligopoly and MajorityVoting with 3 and 4 players (Denote BertrandOligopoly-3 and BertrandOligopoly-4 as the 3 and 4 players version of BertrandOligopoly, so as MajorityVoting-3 and MajorityVoting-4). Due to the exponential dimension of multiple player games, we set the size of utility matrix as 30 × 30 × 30 for 3-player games and 15 × 15 × 15 × 15 for 4-player games. The data generation procedure and the hypothesis class selection are the same as in the experiments of
Towards the PAC Learnability of Nash Equilibrium

Figure 3: Experiment result of initializing replicator dynamics. ‘RD’ means replicator dynamics starting from random policy, and ‘RD+Initialization’ represents RD initialized by our Nash predictor. Each experiment is run by 5 different random seeds. The average results and the 95% confidence interval are shown.

bimatrix games. We also record the time, and iterations baseline algorithms spend to reach the same performance, except for the TS algorithm, since it is designed for bimatrix games only.

The experiment result of generalization and efficiency are shown in Table 4. As we can see in Table 4a, the learned Nash predictor still generalizes well on all the tasks, with Nash approximation value less than 0.0008. Our PAC result verifies since we build our theoretical analysis upon \(n\)-player games. In Table 4b, we can find that baseline algorithms still struggle to get the same performance as the learned Nash predictor (Fictitious play even fails in BertrandOligopoly-3 and BertrandOligopoly-4). Our Nash predictor only needs one forward pass to develop a good approximate solution.

C.2. Initialize Replicator Dynamics

We set the output solution of the trained predictor as the starting point for replicator dynamics. We present the experiment results for the remaining 4 game classes in Figure 3. It can be seen that replicator dynamics with initialization always start with a lower Nash approximation loss and converges faster than the original version.