RANDOM ATTRACTOR AND RANDOM EXPONENTIAL ATTRACTION FOR STOCHASTIC NON-AUTONOMOUS DAMPED CUBIC WAVE EQUATION WITH LINEAR MULTIPLICATIVE WHITE NOISE

ZHAOJUAN WANG
School of Mathematical Science
Huaiyin Normal University
Huaian 223300, China

SHENGFAN ZHOU∗
Department of Mathematics
Zhejiang Normal University
Jinhua 321004, China

(Communicated by Nikolay Tzvetkov)

Abstract. In this paper, we first establish some sufficient conditions for the existence and construction of a random exponential attractor for a continuous cocycle on a separable Banach space. Then we mainly consider the random attractor and random exponential attractor for stochastic non-autonomous damped wave equation driven by linear multiplicative white noise with small coefficient when the nonlinearity is cubic. First step, we prove the existence of a random attractor for the cocycle associated with the considered system by carefully decomposing the solutions of system in two different modes and estimating the bounds of solutions. Second step, we consider an upper semi-continuity of random attractors as the coefficient of random term tends zero. Third step, we show the regularity of random attractor in a higher regular space through a recurrence method. Fourth step, we prove the existence of a random exponential attractor for the considered system, which implies the finiteness of fractal dimension of random attractor. Finally we remark that the stochastic non-autonomous damped cubic wave equation driven by additive white noise also has a random exponential attractor.

1. Introduction. It is known that the existence of attractor and the estimate of its dimension are two main topics in studying the asymptotic behavior of infinite-dimensional dynamical systems [2, 3, 9, 26, 32, 39, 41, 43, 50]. The finite fractal dimension of attractor implies that the attractor can be mapped onto a compact subset of a finite dimensional Euclidean space and hence the random attractor can be described by finite independent parameters [25]. But just knowing the

2010 Mathematics Subject Classification. Primary: 37L55; Secondary: 35B41, 35B40.
Key words and phrases. Stochastic damped wave equation, random attractor, random exponential attractor, multiplicative white noise, upper semicontinuity, fractal dimension, regularity.

This work is supported by the National Natural Science Foundation of China (under Grant Nos. 11471290, 11326114, 11401244) and Natural Science Research Project of Ordinary Universities in Jiangsu Province (grant No. 14KJB110003).
∗Corresponding author: Shengfan Zhou.
boundedness of Hausdorff dimension for a set, we can not have such an available finite parametrization (see [34, 47]).

A system in reality is usually of uncertainty due to some external “noise” (i.e., random factors). The random effects are considered as rather essential phenomena [1, 10]. The random attractor, was first studied by Ruelle [45, 46], is one of most important concepts to describe long-term behavior of solutions for a given random system to capture the essential dynamics with possibly extremely wide fluctuations. Later, Crauel, Debussche, Flandoli, Imkeller, Langa, Schmalfuss, Robinson, Bates, Lu, Caraballo, Kloeden, Wang etc., developed some general theories of random attractors (mainly on existence, semi-continuity and bound of Hausdorff/fractal dimensions) and applications to stochastic evolution equations (such as Navier–Stokes equation, reaction–diffusion equations, wave equations and lattice systems driven by random perturbation or noises), see [4, 6, 10, 11, 12, 18, 19, 28, 29, 30, 31, 33, 34, 44, 46, 49, 53, 54, 65, 66] and the references wherein.

However, there is an intrinsic drawback that random attractor sometimes is infinite dimensional and attracts orbits at a relatively slow rate so that it takes an unexpected long time to reach it. Moreover, the attractor is possible sensitive to perturbations which makes it unobservable in experiments and numerical simulations. To overcome these drawback, Shirikyan and Zelik in [49] introduced the concept of random exponential attractor and present some sufficient conditions for constructing a random exponential attractor for an autonomous random dynamical system, and gave an application to parabolic partial differential equations with random noise. By definition, a random exponential attractor for a random dynamical system is a compact random set, which has finite fractal dimension and attracts exponentially any trajectory and is positive invariant, then it contains random attractor and becomes an appropriate alternative to study the asymptotic behavior of random dynamical systems. Moreover, the existence of a random exponential attractor implies the existence of a random attractor with finite fractal dimension.

We notice that the evolution mode of states in a random system is, in some sense, similar to the deterministic non-autonomous one and there were several construction methods to obtain a pullback exponential attractor for a (deterministic) process, see [14, 15, 16, 17, 20, 35, 61]. We also notice that a trajectory of a random system is often unbounded in time along the path of sample point with probability 1 which is different from deterministic one [14, 15, 16, 17, 20, 35, 61]. Thus, in general, a simple straightforward extension from deterministic system to random system does not work. Fortunately, some time averages of those quantities bounding the trajectories of a random system in large times can be finite and possibly controlled, which provides a useful way for constructing an exponential attractor for a random system. It is observable that the existing only conditions given in [49] for constructing a random exponential attractor of random systems are not easy to be verified for some stochastic partial differential equations driven by white noises, including our considered system (1) below.

In this article, motivated by the ideas of [20, 49, 61], we first establish a new criterion (some sufficient conditions) for the existence and construction of a random exponential attractor for a continuous cocycle on a separable Banach space. It is worth mentioning that our conditions just need to check the boundedness of some random variables in the mean and can be easily verified for some stochastic evolution equations.
Then we mainly consider the following non-autonomous stochastic damped wave equation with linear multiplicative white noise and initial-boundary conditions:

\[
\begin{cases}
\alpha u_t + \alpha u + (-\Delta u + f(u, x))dt = g(x, t)dt + au \circ dW(t), x \in U, t > \tau, \\
u(x, t)|_{x \in \partial U} = 0, t \geq \tau, \tau \in \mathbb{R}, \\
u(x, \tau) = \nu_\tau(x), \quad u_\tau(x, \tau) = u_{1\tau}(x), \quad x \in U,
\end{cases}
\]

where \(\alpha > 0, a \in \mathbb{R}, \tau \in \mathbb{R}, U\) is an open bounded set of \(\mathbb{R}^3\) with a smooth boundary \(\partial U, u = u(x, t)\) is a real-valued function on \(U \times [\tau, +\infty)\), \(g(x, \cdot) \in C_0(\mathbb{R}, H_0^1(U))\), \(W(t)\) is a one-dimensional two-sided Wiener process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}\), the Borel \(\sigma\)-algebra \(\mathcal{F}\) on \(\Omega\) being generated by the compact open topology, and \(\mathbb{P}\) being the Wiener measure on \(\mathcal{F}\); the random term \("au \circ dW(t)"\) is in the Stratonovich sense.

Assume that the initial data \(u_\tau(x), u_{1\tau}(x)\) are independent of \(\omega\) and the nonlinear function \(f(u, x)\) satisfies the following conditions:

(A1): \(f(u, x) = f_1(u, x) + f_2(u, x)\) and there exist non-negative constants \(c_i, i = 0, 1, \ldots, 5\), and functions \(\beta_j \in L^1(U), j = 1, 2\), such that for \(x \in U, u \in \mathbb{R}\),

\[
\begin{cases}
0 \leq G_1(u, x) \leq c_0u f_1(u, x) \leq c_1 G_1(u, x), & G_1(u, x) = \int_0^u f_1(r, x)dr, \\
f_1(\cdot, x) \in C^2(\mathbb{R}, \mathbb{R}), & f_1'(0, x) = 0, |f_1''(u, x)| \leq c_2|u|,
\end{cases}
\]

\[
\begin{cases}
f_2(\cdot, x) \in C^1(\mathbb{R}, \mathbb{R}), & f_2'(0, \cdot) \in C(\mathbb{R}), \quad 1 \leq p < 2, \\
|f_2''(u_1, x) - f_2''(u_2, x)| \leq c_3(1 + |u_1|^{p-1} + |u_2|^{p-1}) |u_1 - u_2|,
\end{cases}
\]

and

\[
c_4u^4 - \beta_3(x) \leq G(u, x) \leq c_5 uf(u, x) + c_6 u^2 + \beta_2(x), \quad G(u, x) = \int_0^u f(r, x)dr,
\]

for some number \(c_5 \in [0, \frac{c_4}{\lambda_1}]\), where \(\lambda_1\) is the first eigenvalue of operator \(A = -\Delta, D(A) = H^2(U) \cap H_0^1(U)\).

(A2): For any fixed \(u \in \mathbb{R}, f(u, \cdot) \in C^1(U; \mathbb{R})\) and there exists a constant \(c_7 \geq 0\) such that for \(x \in U, u_1, u_2 \in \mathbb{R}\),

\[
|f_2''(u_1, x) - f_2''(u_2, x)| \leq c_7(1 + |u_1|^\tilde{p} + |u_2|^\tilde{p}) |u_1 - u_2|, \quad 0 \leq \tilde{p} \leq 2.
\]

The global attractors, pullback attractors (or kernel sections), exponential attractors and the bounds of their Hausdorff and fractal dimensions for the deterministic autonomous and non-autonomous damped wave equations (1) without noise term (i.e., \(a = 0\)) have been studied widely, see [2, 5, 9, 26, 27, 24, 32, 40, 50, 58, 59, 60, 62].

For the stochastic non-autonomous evolution equations with a time-dependent external term \(g\), Wang established an useful theory about the existence and upper semi-continuity of random attractors for the corresponding cocycle by introducing two parametric spaces, see [52, 53].

The random attractor and the bounds of its Hausdorff and fractal dimensions for the stochastic wave equations with additive noise (i.e., the random term in (1) is \("adW(t)"\) independent of \(u\)) have been studied by many authors, see [12, 13, 8, 18, 21, 38, 54, 57, 63, 65].

For the stochastic system (1) with linear multiplicative noise \("au \circ dW(t)"\) (depending on the state variable \(u\) and sufficient small coefficient \(|a|\) of random term, when the nonlinear function \(f\) has a subcubic growth exponent (i.e., \(f_1 \equiv 0\) in (A1)), the existence and the boundedness of fractal dimension of random attractor were studied, see [22, 36, 52, 66], of those, Zhou and Zhao in [66] gave some sufficient conditions to bound the fractal dimension of a random invariant set for a
cocycle and applied these conditions to get an upper bound of fractal dimension of the random attractor of system (1).

But as we know, when $f$ has a cubic growth exponent (i.e., $f_1 \neq 0$ in (A1)), there is no results about the existence and dimension of random attractor in both cases of stochastic autonomous and non-autonomous wave equations (1). In this case, there are two main essential difficulties. The first difficulty arises in showing the asymptotic compactness of system that is the key step to prove the existence of a random attractor, which is caused by the cubic growth condition (2) of $f$ and can not be overcome by decomposing the solutions of system just one time like in the deterministic case [27, 58, 62] or the subcubic growth exponential case [36, 52, 65, 66]. The second difficulty occurs in the possible unboundedness of the expectation of bound of random attractor in a “higher regularity” space here, which is a basic requiring condition in known existing methods to show the boundedness of the fractal dimension of a random attractor [34, 56, 65, 66]. To solve these problems, we have to use some new different technique.

We will do the following objects for the random attractor of system (1).

(I) Study the existence of random attractor for (1) when $f$ satisfies (A1) and the coefficient $|a|$ of random term is sufficient small. To prove the asymptotic pullback compactness of the cocycle $\Phi$ associated with (1), we decompose carefully the solutions of system and estimate the bounds of solutions through two different modes, which is different from the known publications [38, 57, 58]. Then we construct a compact measurable tempered attracting set and prove the existence of a random attractor for $\Phi$ in phase space $H_0^1(U) \times L^2(U)$.

(II) Establish the upper semicontinuity of the random attractors for (1) with respect to the coefficient $a$ of white noise term as $a \to 0$. We will show that the random attractors of (1) tends to the pullback attractor of deterministic non-autonomous system (1)$_{a=0}$ as $a \to 0$ in the sense of Hausdorff semi-distance between two subsets of phase space.

(III) Study the regularity of random attractor by constructing a compact measurable tempered attracting set through a recurrence method and hence prove the boundedness of random attractor in higher regular space $[H^2(U) \cap H_0^1(U)] \times H_0^1(U)$ for the cocycle $\Phi$.

(IV) Prove the existence of a random exponential attractor for a continuous cocycle $\Phi$ in $H_0^1(U) \times L^2(U)$ when $f$ satisfies (A1)-(A2) by applying our new criterion, which implies the finiteness of fractal dimension of random attractor for (1).

This paper is organized as follows. In section 2, we establish some sufficient conditions for the existence of a random exponential attractor for a continuous cocycle on a separable Banach space. In section 3, we first transfer the stochastic differential system (1) into an equivalent random differential system, then we show that the mapping of solutions for this random system generates a continuous cocycle, finally we prove the ultimately pullback boundedness of solutions of random equation. In section 4, we carefully decompose the solutions of random differential equation into a sum of two components in two ways: one component decays exponentially and another component is bounded in a higher regular space. In section 5, we construct a compact pullback attracting set and obtain the existence of random attractor. In section 6, we consider an upper semicontinuity of random attractors as $a \to 0$. In section 7, we prove the regularity of random attractor basing on a recurrence mode. In section 8, we prove the existence of a random exponential attractor for (1) when
f satisfies (A1)-(A2). In section 9, we consider a special case of (1) where \( f(u, x) \) is a 3 order polynomial of \( u \) with a positive leading coefficient. At last, we point out that the following non-autonomous stochastic damped wave equation with additive white noise and initial-boundary conditions:

\[
\begin{align*}
&u_{tt} + \alpha u_t - \Delta u + f(u, x) = g(x, t) + h(x)\dot{W}(t), \quad t > \tau, \quad x \in U, \quad \tau \in \mathbb{R}, \\
&u(x, t) |_{x \in \partial U} = 0, \quad t \geq \tau, \quad \tau \in \mathbb{R}, \\
&u(x, \tau) = u_\tau(x), \quad u_t(x, \tau) = u_{t, \tau}(x), \quad x \in U, \quad \tau \in \mathbb{R},
\end{align*}
\]

has a random exponential attractor under conditions (A1)-(A2).

2. Random exponential attractor for continuous cocycles. In this section, we establish some sufficient conditions for the existence and construction of a random exponential attractor for a continuous cocycle on a separable Banach space.

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and let \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\) be an ergodic metric dynamical system, that is, \(\{\theta_t : \Omega \to \Omega, t \in \mathbb{R}\}\) is a family of measure preserving transformations on \((\Omega, \mathcal{F}, \mathbb{P})\) such that \((t, \omega) \mapsto \theta_t \omega\) is \((\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})\)-measurable, \(\theta_0\) is the identity on \(\Omega\), \(\theta_{s+t} = \theta_s \theta_t\) for all \(s, t \in \mathbb{R}\), and for any \(F \in \mathcal{F}\), provided \(\mathbb{P}(\theta^{-1}_t F \Delta F) = 0\), it holds that \(\mathbb{P}(F) = 0\) or 1 for all \(t \in \mathbb{R}\).

Let \(X\) be a separable Banach space with Borel \(\sigma\)-algebra \(\mathcal{B}(X)\). Let mapping \(\Psi : \mathbb{R}^+ \times \mathcal{F} \times \Omega \times X \to X\) be a continuous cocycle on \(X\) over \(\mathbb{R}\) and \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\) with properties that (i) \(\Psi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \mathcal{F} \times E \to E\) is \((\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(E))\)-measurable; (ii) \(\Psi(0, \tau, \omega, \cdot)\) is the identity on \(X\); (iii) \(\Psi(t + s, \tau, \omega, \cdot) = \Psi(t, \tau + s, \theta_s \omega, \Psi(s, \tau, \omega, \cdot))\) for \(t, s \geq 0\); (iv) \(\Psi(t, \tau, \omega, \cdot) : X \to X\) is continuous, see [53].

A random variable \(\xi_{\omega}\) is said to be tempered with respective to \(\{\theta_t\}_{t \in \mathbb{R}}\) if for every \(\gamma > 0\) and almost every (a.e.) \(\omega \in \Omega\), \(\lim_{t \to +\infty} e^{-\gamma|t|}||\xi_{\theta_t \omega}|| = 0\) [10]. A family \(B = \{B(\tau, \omega) \subset X : \tau \in \mathbb{R}, \omega \in \Omega\}\) of nonempty bounded subsets of \(X\) is said to be tempered with respective to \(\{\theta_t\}_{t \in \mathbb{R}}\) if for every \(\gamma > 0\), \(\tau \in \mathbb{R}\) and a.e. \(\omega \in \Omega\), \(\lim_{t \to +\infty} e^{-\gamma|t|}||B(\tau + t, \theta_t \omega)||_H = 0\) for \(B(\tau, \omega) \in B\), where \(||B(\tau, \omega)||_E = \sup_{x \in E} ||B(\tau, \omega)||_E \) [53]. Denote \(D(X)\) the collection of all tempered families of nonempty bounded subsets of \(X\). Recall that the distance between a point \(u \in X\) and a subset \(F \subset X\), and the Hausdorff and symmetric distances between two subsets are defined by, respectively,

\[
d(u, F) = \inf_{v \in F} ||u - v||_X, \quad d_b(F_1, F_2) = \sup_{u \in F_1} d(u, F_2),
\]

\[
d_s(F_1, F_2) = \max\{d_b(F_1, F_2), d_b(F_2, F_1)\}, \quad \forall F_1, F_2 \subset X.
\]

**Definition 2.1.** A family \(\{\mathcal{E}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}\) of subsets of \(X\) is called a random exponential attractor in \(D(X)\) for the continuous cocycle \(\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}\) over \(\mathbb{R}\) and \((\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}})\) if there is a set of full measure \(\Omega \in F\) such that for every fixed \(\tau \in \mathbb{R}\) and \(\omega \in \Omega\), it holds that

(i) Compactness and measurability: \(\mathcal{E}(\tau, \omega)\) is measurable in \(\omega\) and compact in \(X\).

(ii) Finite-dimensionality: there exists a random variable \(\zeta_\omega < \infty\) such that \(\sup_{\tau \in \mathbb{R}} \dim \mathcal{E}(\tau, \omega) \leq \zeta_\omega < \infty\), where \(\dim \mathcal{E}(\tau, \omega) = \lim_{\varepsilon \to 0^+} \frac{\ln N_\varepsilon(\mathcal{E}(\tau, \omega))}{-\ln \varepsilon}\) is the fractal dimension of \(\mathcal{E}(\tau, \omega)\) and \(N_\varepsilon(\mathcal{E}(\tau, \omega))\) is the minimal number of balls with radius \(\varepsilon\) covering \(\mathcal{E}(\tau, \omega)\) in \(X\).

(iii) Positive invariance: \(\Psi(t, \tau - t, \theta_{-t} \omega) \mathcal{E}(\tau - t, \theta_{-t} \omega) \subseteq \mathcal{E}(\tau, \omega)\) for all \(t \geq 0\).
(iv) Exponential attraction: there exists a constant $\tilde{a} > 0$ such that for any $B \in D(X)$, there exist random variables $t_B(\tau, \omega) \geq 0$, $Q(\tau, \omega, ||B||_X) > 0$ satisfying
\[
d_h(\Psi(t, \tau - t, \theta_{-t}(\omega))B(\tau - t, \theta_{-t}(\omega), \mathcal{E}(\tau, \omega)) \leq Q(\tau, \omega, ||B||_X)e^{-\tilde{a}t}, \quad t \geq t_B(\tau, \omega).
\]

**Remark 2.1.** By Definition 2.1 and definition of random attractor (see section 5), it is easy to see that the existence of a random exponential attractor \{(\mathcal{E}(\tau, \omega))_{\tau} \in \mathbb{R} \} for \{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau, \omega, \Omega}$ with $\sup \dim_f \mathcal{E}(\tau, \omega) \leq \bar{m}$ (positive constant) implies the existence of a random attractor \{A(\tau, \omega)\}_{\tau} \in \mathbb{R} \} for \{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau, \omega, \Omega}$ with $A(\tau, \omega) \subseteq \mathcal{E}(\tau, \omega)$ and $\sup \dim_f A(\tau, \omega) \leq \sup \dim_f \mathcal{E}(\tau, \omega) \leq \bar{m} < \infty$ for every $\tau \in \mathbb{R}$ and $\omega \in \tilde{\Omega}$. Here "$\omega \in \tilde{\Omega}$" means "a.e. $\omega \in \Omega$".

In the following of this article, for simplicity, we identify "a.e. $\omega \in \Omega$" and "$\omega \in \Omega$". In fact, the property holds for a.e. $\omega \in \Omega$ throughout the article.

We make the following assumptions on the cocycle \{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau, \omega, \Omega}:

- (H1) there exists a family of tempered closed random subsets \{\hat{\chi}(\tau, \omega)\}_{\tau} \in \mathbb{R} \} of $X$ such that for any $\tau \in \mathbb{R}$ and $\omega \in \Omega$,
  - (h1) the diameter $||\hat{\chi}(\tau, \omega)||_X$ of $\hat{\chi}(\tau, \omega)$ is bounded by a tempered random variable $R_\omega$ (independent of $\tau$), i.e., $\sup \tau \in \mathbb{R} \sup u, v \in \hat{\chi}(\tau, \omega)||u - v||_X \leq R_\omega < \infty$, where $R_\omega$ is continuous in $t$ for all $t \in \mathbb{R}$;
  - (h12) $\hat{\chi}(\tau, \omega)$ is positively invariant with respect to $\{\theta_t\}_{t \in \mathbb{R}}$ in the sense that $\Phi(t, \tau - t, \theta_{-t}(\omega))\hat{\chi}(\tau - t, \theta_{-t}(\omega)) \subseteq \hat{\chi}(\tau, \omega)$ for all $t \geq 0$;

- (H2) there exist positive numbers $\lambda$, $\delta$, $t_0$, random variables $\hat{C}_0(\omega)$, $\hat{C}_1(\omega)$, $\hat{C}_2(\omega) \geq 0$ and $N$-dimensional projector $P_N: X \rightarrow P_N X$ (dim$(P_N X) = N \in \mathbb{N}$) such that for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and any $u, v \in \hat{\chi}(\tau, \omega)$,

\[
||\Psi(t, \tau, \omega)u - \Psi(t, \tau, \omega)v||_X \leq e^{\hat{C}_1(\theta_{t_0}(\omega)) + \int_{t_0}^t \hat{C}_0(\theta_{s}(\omega))ds}||u - v||_X, \quad \forall t \in [0, t_0) \tag{7}
\]

and

\[
|| (I - P_N)(\Psi(t_0, \tau, \omega)u - \Psi(t_0, \tau, \omega)v) ||_X \\
\leq (\frac{\lambda}{\bar{m}} + \int_{t_0}^t \hat{C}_1(\theta_{s}(\omega))ds + \frac{\delta}{2} e^{\hat{C}_1(\theta_{t_0}(\omega)) + \int_{t_0}^t \hat{C}_0(\theta_{s}(\omega))ds})||u - v||_X, \tag{8}
\]

where $\lambda$, $t_0$, $N$ are independent of $\tau$ and $\omega$;

- (H3) $\hat{C}_0(\omega)$, $\hat{C}_1(\omega)$, $\hat{C}_2(\omega)$, $\lambda$, $t_0$, $\delta$ satisfy:

\[
\begin{cases}
0 \leq E[\hat{C}_2(\omega)] \leq \frac{\lambda}{16}, \\
\frac{\lambda}{16} > t_0 > 0, \\
0 \leq E[\hat{C}_1^2(\omega)], E[\hat{C}_0(\omega)] < \infty, \quad i = 0, 1, \\
0 < \delta \leq \min \left\{ \frac{1}{16}, e^{-\frac{\lambda}{2} \left[ 2E[\hat{C}_1^2(\omega)] + 2\hat{C}_0^2(\omega) \right] + \lambda t_0 E[\hat{C}_1(\theta_{t_0}(\omega))] + \lambda t_0^2 E[\hat{C}_0(\omega))] \right\};
\end{cases}
\]

where "$E$" denotes the expectation;

- (H4) $\forall \tau \in \mathbb{R}$, $\omega \in \Omega$, \left\{ \begin{align*}
\lim_{t \downarrow 0} \sup_{h \in \hat{\chi}(\tau, \omega)} ||\Psi(t, \tau, \omega)u - u||_X = 0, \\
\lim_{t \downarrow 0} \sup_{h \in \hat{\chi}(\tau - t, \theta_{-t}(\omega))} ||\Psi(0, \tau - t, \theta_{-t}(\omega))u - u||_X = 0.
\end{align*} \right\}
Theorem 2.1. Assume that conditions (H1)-(H4) are satisfied. Then there exists a random exponential attractor $\{\mathcal{E}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ for the continuous cocycle $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau, \omega \in \Omega}$ with the following properties: for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, 

(i) $\mathcal{E}(\tau, \omega) (\subseteq \bar{\chi}(\tau, \omega))$ is measurable in $\omega$ and compact in $X$; 

(ii) $\Psi(t, \tau - t, \theta_{-t}\omega)\mathcal{E}(\tau - t, \theta_{-t}\omega) \subseteq \mathcal{E}(\tau, \omega)$ for all $t \geq 0$; 

(iii) $\dim_f \mathcal{E}(\tau, \omega) \leq \frac{2N\ln(\frac{2N+1}{\ln 4})}{2\ln 4} < \infty$; 

(iv) there exist a random variable $\hat{T}_\omega \geq 0$ and a tempered random variable $\hat{b}_\omega > 0$ such that 

$$d_h(\Psi(t, \tau, \omega)\bar{\chi}(\tau, \omega), \mathcal{E}(t + \tau, \theta_t\omega)) \leq \hat{b}_\omega e^{-\frac{\ln 4}{\ln 4}t}, \quad t \geq \hat{T}_\omega.$$  

(v) for any $\tau \in \mathbb{R}$, $\omega \in \Omega$, $	ext{lim}_{t \to 0} d_h(\mathcal{E}(t + \tau, \theta_t\omega), \mathcal{E}(\tau, \omega)) = 0$.

Proof. The proof consists of the following three steps:

Step 1. Construction of a random exponential attractor for the discrete cocycle $\{\Psi(n \hat{t}_0, \tau + n \hat{t}_0, \theta_{n \hat{t}_0}\omega)\}_{n \in \mathbb{N}, \hat{t}_0 \in \mathbb{R}, \omega \in \Omega}$.

Step 2. Construction of a random exponential attractor for the continuous cocycle $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau, \omega \in \Omega}$.

Step 3. Simi-continuity (v) of random exponential attractor.

Step 1. Construction of a random exponential attractor for discrete cocycle $\{\Psi(n \hat{t}_0, \tau + n \hat{t}_0, \theta_{n \hat{t}_0}\omega)\}_{n \in \mathbb{N}, \hat{t}_0 \in \mathbb{R}, \omega \in \Omega}$ in $X$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$.

For fixed $\tau \in \mathbb{R}, \omega \in \Omega$ and any $m \in \mathbb{Z}, n, j \in \mathbb{N}$, write

$$\Psi(n, \tau, m - j, \omega) = \Psi(n \hat{t}_0, \tau + (m - j)\hat{t}_0, \theta_{(m-j)\hat{t}_0}\omega),$$

$$\bar{\chi}(\tau, m - j, \omega) = \bar{\chi}(\tau + (m - j)\hat{t}_0, \theta_{(m-j)\hat{t}_0}\omega).$$

From (H2), it follows that for any $u, v \in \bar{\chi}(\tau, \omega)$,

$$||P_N \Psi(t_0, \tau, \omega) u - P_N \Psi(t_0, \tau, \omega) v||_X \leq e^{\hat{C}_1(\hat{t}_0\omega) + \int_{t_0}^0 \hat{C}_0(\theta_t\omega)dt} ||u - v||_X. \quad (11)$$

1) **Covering of** $\Psi(n, \tau, m - n, \omega)\bar{\chi}(\tau, m - n, \omega)$. First we claim that $\Psi(n, \tau, m - n, \omega)\bar{\chi}(\tau, m - n, \omega)$ has a covering of closed balls of $X$ centered at points in itself by induction on $n$:

$$\left\{ \begin{array}{l}
\Psi(n, \tau, m - n, \omega)\bar{\chi}(\tau, m - n, \omega) \subset \bigcup_{i=1}^{\hat{N}^n} \hat{B}(u_{n,m,i}; r_{1-n,m,m,i}), \\
u_{n,m,i} \in \Psi(n, \tau, m - n, \omega)\bar{\chi}(\tau, m - n, \omega), \quad 1 \leq i \leq \hat{N}^n, \\
r_{1-n,m,m,i} = a_{\theta_{(m-1)\hat{t}_0}\omega}a_{\theta_{(m-2)\hat{t}_0}\omega} \cdots a_{\theta_{(m-n)\hat{t}_0}\omega} R_{\theta_{(m-n)\hat{t}_0}\omega},
\end{array} \right. \quad (12)$$

where $\hat{B}(X; r)$ denotes the closed ball of $X$ centered at $u$ with radius $r$ and $\hat{N} = \left(\frac{2\sqrt{N}}{\delta} + 1\right)^N$.

$$a_{\theta_{(m-l)\hat{t}_0}\omega} = 2e^{f(m-l)\hat{t}_0} \hat{C}_2(\theta_{l\omega}) + 2\delta e^{\hat{C}_1(\theta_{l\omega})} + f(m-l)\hat{t}_0 \hat{C}_0(\theta_{l\omega})}, \quad (13)$$

for $l = 1 \ldots n$. In fact, for $n = 0$, by $\Psi(0, \tau, m, \omega)\bar{\chi}(\tau, m, \omega) = \bar{\chi}(\tau, m, \omega)$ or (h12), we can take $u_{0,m,1} \in \bar{\chi}(\tau, m, \omega) \subset \hat{B}(u_{0,m,1}; R_{\theta_{m\hat{t}_0}\omega})$ arbitrarily by condition (h11).

Consider $n = 1$, then by the cocycle property of $\Psi$, we have

$$\Psi(1, \tau, m - 1, \omega)\bar{\chi}(\tau, m - 1, \omega) \subseteq \Psi(1, \tau, m - 1, \omega)\hat{B}(u_{-1, m-1, 1}; R_{\theta_{(m-1)\hat{t}_0}\omega}) \cap \bar{\chi}(\tau, m - 1, \omega). \quad (14)$$
For any \(u \in \bar{B}(u_{-1,m,1}; R_{\theta(m-1)\ell_0} \omega) \cap \tilde{\chi}(\tau, m - 1, \omega)\), by (11), we have
\[
\|P_N \Psi(1, \tau, m - 1, \omega)u - P_N \Psi(1, \tau, m - 1, \omega)u_{-1,m,1}\|_X \leq \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds \|u - u_{-1,m,1}\|_X \leq \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega.
\]

By Lemma 1.2 in [18], there exist \(\tilde{V}_{-1,m,1,1}, \ldots, \tilde{V}_{-1,m,1,N} \in X\) such that
\[
P_N \Psi(1, \tau, m - 1, \omega)[\bar{B}(u_{-1,m,1}; R_{\theta(m-1)\ell_0} \omega) \cap \tilde{\chi}(\tau, m - 1, \omega)] \subseteq \bar{B}_{P_N X}(P_N \Psi(1, \tau, m - 1, \omega)u_{-1,m,1,1}, \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega)
\]
\[
\subseteq \bigcup_{j=1}^N \bar{B}_{P_N X}(\tilde{V}_{-1,m,1,1,j}, \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega) \tag{15}
\]
and
\[
P_N \Psi(1, \tau, m - 1, \omega)[\bar{B}(u_{-1,m,1,1}; R_{\theta(m-1)\ell_0} \omega) \cap \tilde{\chi}(\tau, m - 1, \omega)] \cap \bar{B}_{P_N X}(\tilde{V}_{-1,m,1,1,j}, \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega) \neq \emptyset,
\]
for any \(j \in \{1, \ldots, N\}\). For each \(j \in \{1, \ldots, N\}\), take a point \(V_{-1,m,1,1,j}\) such that
\[
\begin{cases}
V_{-1,m,1,1,j} \in \bar{B}(u_{-1,m,1,1}; R_{\theta(m-1)\ell_0} \omega) \cap \tilde{\chi}(\tau, m - 1, \omega), \\
P_N \Psi(1, \tau, m - 1, \omega)V_{-1,m,1,1,j} \in \bar{B}_{P_N X}(\tilde{V}_{-1,m,1,1,j}, \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega). 
\end{cases} \tag{16}
\]
By (15),
\[
P_N \Psi(1, \tau, m - 1, \omega)[\bar{B}(u_{-1,m,1,1}; R_{\theta(m-1)\ell_0} \omega) \cap \tilde{\chi}(\tau, m - 1, \omega)] \subseteq \bigcup_{j=1}^N \bar{B}_{P_N X}(P_N \Psi(1, \tau, m - 1, \omega)V_{-1,m,1,1,j},
\]
\[
\epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega).
\]
For any \(u \in \bar{B}(u_{-1,m,1}; R_{\theta(m-1)\ell_0} \omega)\), \(\tilde{\chi}(\tau, m - 1, \omega)\), there is a \(j \in \{1, \ldots, N\}\) such that
\[
P_N \Psi(1, \tau, m - 1, \omega)u \in \bar{B}_{P_N X}(P_N \Psi(1, \tau, m - 1, \omega)V_{-1,m,1,1,j}, \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega),
\]
that is,
\[
\|P_N \Psi(1, \tau, m - 1, \omega)u - P_N \Psi(1, \tau, m - 1, \omega)V_{-1,m,1,1,j}\|_X \leq \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds R_{\theta(m-1)\ell_0} \omega. \tag{17}
\]
It then follows from (8), (16) and (17) that
\[
\|(I - P_N) \Psi(1, \tau, m - 1, \omega)u - (I - P_N) \Psi(1, \tau, m - 1, \omega)V_{-1,m,1,1,j}\|_X \leq (2e^{-\lambda_0 + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_2(\theta_{i_0}) ds} + \epsilon \hat{C}_1(\theta_{m_0}\omega) + \int_{(m_0)_{i_0}}^{m_0} \hat{C}_0(\theta_{i_0}) ds) R_{\theta(m-1)\ell_0} \omega.
\]
Applying the recursion to the inclusion (19), it follows that (12) holds.

\[ \Psi(1, \tau, m - 1, \omega) = \Psi(1, \tau, m - 1, \omega) \leq \theta \Psi(1, \tau, m - 1, \omega) + \delta \theta \int_0^t \dot{C}_2(\theta_s \omega) ds \]

Hence, by (14) and (18),

\[ \Psi(1, \tau, m - 1, \omega) \chi(\tau, m - 1, \omega) \]

is

\[ \bigcup_{j=1}^N \bar{B}(u_{-1,m,j}) \]

\[ (2e^{-\lambda_0 t} + \int_{m-1}^0 \dot{C}_2(\theta_s \omega) ds + 2\delta e \dot{C}_1(\theta_m \omega) + \int_{m-1}^0 \dot{C}_0(\theta_s \omega) ds) R_{\theta_m} \]

where

\[ u_{-1,m,j} = \Psi(1, \tau, m - 1, \omega) V_{-1,m-1,1,j} \subset \Psi(1, \tau, m - 1, \omega) \chi(\tau, m - 1, \omega), 1 \leq j \leq N. \]

Applying the recursion to the inclusion (19), it follows that (12) holds.

2) Exponentially decay of radius \( r_{1-n,m,\omega} \).

Set

\[ J = \left\{ \omega \in \Omega : \dot{C}_1(\theta_{m-1} \omega) + \int_0^t [\dot{C}_0(\theta_{m-1} \omega) - \dot{C}_2(\theta_{m-1} \omega)] ds + \lambda_0 > \ln \frac{1}{\delta} \right\}. \]

(a) If \( \theta_{m-1+t} \omega \in J \), then we have

\[ \dot{C}_1(\theta_{m-1+t} \omega) + \int_{m-1}^t \dot{C}_0(\theta_s \omega) ds + \int_{m-1}^t \dot{C}_2(\theta_s \omega) ds > e^{-\lambda_0 t} + \int_{m-1}^t \dot{C}_2(\theta_s \omega) ds \]

that is,

\[ \dot{\delta} e \dot{C}_1(\theta_{m-1+t} \omega) + \int_{m-1}^t \dot{C}_0(\theta_s \omega) ds > e^{-\lambda_0 t} + \int_{m-1}^t \dot{C}_2(\theta_s \omega) ds \]

thus, by (9), (13) and (22), we have

\[ a_{\theta_{m-1+t} \omega} = 2e^{-\lambda_0 t} + \int_{m-1}^t \dot{C}_2(\theta_s \omega) ds \]

(b) If \( \theta_{m-1+t} \omega \notin J \), then

\[ \dot{\delta} e \dot{C}_1(\theta_{m-1+t} \omega) + \int_{m-1}^t \dot{C}_0(\theta_s \omega) ds \leq e^{-\lambda_0 t} + \int_{m-1}^t \dot{C}_2(\theta_s \omega) ds \]

Write

\[ \sum_{l=1}^n \theta_{l} \theta_{m-1+t} \omega = k_1, \sum_{l=1}^n [1 - \theta_{l} \theta_{m-1+t} \omega] = k_2, n = k_1 + k_2. \]
Then
\[
\prod_{l=1}^{n} d_{\theta_l} \tilde{\omega}_l \\
\leq \left( \frac{1}{4} \right)^{k_1} e^{\sum_{l=1}^{n} \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0)} \left( \tilde{C}_1 (\theta_{(m-l+1)} \tilde{\omega}_0) + f_{(m-l)}^{(m-l+1)} \tilde{C}_0 (\theta_s \omega) ds \right) \\
\times 4^{k_2} e^{\sum_{l=1}^{n} [1 - \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0)]} \left( -\tilde{\lambda}_0 + \int_{(m-l)}^{(m-l+1)} \tilde{C}_2 (\theta \omega) ds \right) \\
\leq \frac{1}{2k_1} \left( 2e^{-\frac{\tilde{\lambda}_0}{2}} \right)^{k_1} e^{\sum_{l=1}^{n} \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0)} \left( \tilde{C}_1 (\theta_{(m-l+1)} \tilde{\omega}_0) + f_{(m-l)}^{(m-l+1)} \tilde{C}_0 (\theta_s \omega) ds \right) \\
\times 2^{k_2} \left( 2e^{-\frac{\tilde{\lambda}_0}{2}} \right)^{k_2} e^{-\frac{\tilde{\lambda}_0}{2} + \sum_{l=1}^{n} [1 - \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0)]} \left( -\tilde{\lambda}_0 + \int_{(m-l)}^{(m-l+1)} \tilde{C}_2 (\theta \omega) ds \right) \\
\leq \frac{1}{2} e^{\sum_{l=1}^{n} \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0)} \left( \tilde{C}_1 (\theta_{(m-l+1)} \tilde{\omega}_0) + f_{(m-l)}^{(m-l+1)} \tilde{C}_0 (\theta_s \omega) ds \right) \\
\times 2^n e^{-\frac{\tilde{\lambda}_0}{2} + \sum_{l=1}^{n} f_{(m-l)}^{(m-l+1)} \tilde{C}_2 (\theta \omega) ds} ,
\]

where \( \vartheta_l (\omega) = \begin{cases} 1, & \omega \in J_l \\ 0, & \omega \notin J_l \end{cases} \). By Birkhoff ergodic Theorem [51], we have that
\[
\mathbb{E}[\tilde{C}^2_1 (\theta_s \omega) = \mathbb{E}[\tilde{C}^2_2 (\omega)] \text{ for } s \in \mathbb{R}, i = 0, 1, 2 \text{ and }
\]
\[
\frac{1}{n} \sum_{l=1}^{n} \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0) \cdot \left( \tilde{C}_1 (\theta_{(m-l+1)} \tilde{\omega}_0) + \int_{(m-l)}^{(m-l+1)} \tilde{C}_0 (\theta_s \omega) ds \right) \\
\overset{n \to +\infty}{\longrightarrow} \mathbb{E} \left( \vartheta (\omega) \left( \tilde{C}_1 (\theta_{t_0} \omega) + \int_{0}^{t_0} \tilde{C}_0 (\theta_s \omega) ds \right) \right) ,
\]
\[
\frac{1}{n} \sum_{l=1}^{n} \int_{(m-l)}^{(m-l+1)} \tilde{C}_2 (\theta_s \omega) ds \\
\overset{n \to +\infty}{\longrightarrow} \mathbb{E} \left( \int_{(m-l)}^{(m-l+1)} \tilde{C}_2 (\theta_s \omega) ds \right) = t_0 \mathbb{E}[\tilde{C}_2 (\omega)] = \frac{\tilde{\lambda}_0}{16}.
\]

Then for any \( \omega \in \Omega \), there exists a large integer \( n_1 (\omega) \in \mathbb{N} \) such that for \( n \geq n_1 (\omega) \),
\[
\sum_{l=1}^{n} \vartheta_l (\theta_{(m-l)} \tilde{\omega}_0) \cdot \left( \tilde{C}_1 (\theta_{(m-l+1)} \tilde{\omega}_0) + \int_{(m-l)}^{(m-l+1)} \tilde{C}_0 (\theta_s \omega) ds \right) \\
\leq 2n \mathbb{E} \left( \vartheta_l (\omega) \cdot \left( \tilde{C}_1 (\theta_{t_0} \omega) + \int_{0}^{t_0} \tilde{C}_0 (\theta_s \omega) ds \right) \right) \\
\text{and}
\sum_{l=1}^{n} \int_{(m-l)}^{(m-l+1)} \tilde{C}_2 (\theta_s \omega) ds \leq \frac{\tilde{\lambda}_0}{8} n. \]
By (9), (15) and Hölder inequality, we have that for \( n \geq n_1(\omega) \),

\[
E \left( \theta_f(\omega) \cdot \left( \hat{C}_1(\theta_{t_0}, \omega) + \int_0^{t_0} \hat{C}_0(\theta_s, \omega) ds \right) \right) 
\]

\[
\leq \frac{1}{\ln \frac{1}{\delta}} E \left( 2\hat{C}_1^2(\theta_{t_0}, \omega) + 2\hat{\lambda}_l \int_0^{t_0} \hat{C}_0(\theta_s, \omega) ds + \hat{\lambda}_l \hat{C}_1(\theta_{t_0}, \omega) + \hat{\lambda} \int_0^{t_0} \hat{C}_0(\theta_s, \omega) ds \right) 
\]

\[
\leq \frac{1}{\ln \frac{1}{\delta}} \left( 2E[\hat{C}_1^2(\omega)] + 2\hat{\lambda}_l^2 E[\hat{C}_0^2(\omega)] + \hat{\lambda} \hat{C}_1(\theta_{t_0}, \omega) + \hat{\lambda} \hat{C}_0(\hat{\theta}_2(\omega)) \right),
\]

and

\[
2^n e^{-\frac{\hat{\lambda}_l}{\hat{\lambda} n} + \sum_{i=0}^{n-1} \frac{m - i - 1}{\hat{\lambda} n} \hat{C}_1(\theta_{t_0}, \omega) ds} \leq 2^n e^{-\frac{\hat{\lambda}_l}{\hat{\lambda} n} + \frac{1}{4}} = \left( 2e^{-\frac{\hat{\lambda}_l}{\hat{\lambda} n}} \right)^n = 1.
\]

Then by (23),

\[
\prod_{l=1}^{n} a_{\theta_{(m - l) l_0}} \omega \leq \left( \frac{1}{2} \right)^n e^{\sum_{l=1}^{n} \theta_f(\theta_{(m - l) l_0}) \cdot \left( \hat{C}_1(\theta_{(m - l) l_0}, \omega) + \int_{(m - l) l_0}^{(m - l + 1) l_0} \hat{C}_0(\theta_s, \omega) ds \right)} 
\]

\[
\leq \left( \frac{1}{2} e^{-\frac{\hat{\lambda}_l}{\hat{\lambda} n}} \left( 2E[\hat{C}_1^2(\omega)] + 2\hat{\lambda}_l^2 E[\hat{C}_0^2(\omega)] + \hat{\lambda} \hat{C}_1(\theta_{t_0}, \omega) + \hat{\lambda} \hat{C}_0(\hat{\theta}_2(\omega)) \right) \right)^n 
\]

\[
= \left( \frac{3}{4} \right)^n, \quad n \geq n_1(\omega).
\]

By (H1) and (1), there exists a tempered random variable \( b_{\omega,m} (> 0) \) such that

\[
R_{\theta_{(m - n) l_0}} \omega \leq b_{\omega,m} e^{\frac{\hat{\lambda}_l}{\hat{\lambda} n} + \frac{1}{4}} \quad \text{for } n \in \mathbb{N}.
\]

By (24), for every \( \omega \in \Omega \) and \( n \geq n_1(\omega) \), we have

\[
0 < r_{\omega,m,n} = \prod_{l=1}^{n} a_{\theta_{(m - l) l_0}} \omega R_{\theta_{(m - n) l_0}} \omega 
\]

\[
\leq \left( \frac{3}{4} \right)^n b_{\omega,m} e^{-\frac{\hat{\lambda}_l}{\hat{\lambda} n} + \frac{1}{4}} = \left( \frac{3}{4} \right)^n b_{\omega,m} \xrightarrow{n \to \infty} 0.
\]

3) Construction of a random exponential attractor.

For fixed \( \tau \in \mathbb{R}, \omega \in \Omega, m \in \mathbb{Z} \) and any \( n \in \mathbb{N} \), put

\[
E_{-n}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) = \{ u_{-n,m,i} : 1 \leq i \leq N^n \}.
\]

Then

\[
E_{-n}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) \subseteq \Psi(n, \tau, m - n, \omega) \hat{\chi}(\tau, m - n, \omega) \subseteq \hat{\chi}(\tau, m, \omega)
\]

and

\[
\Psi(p, \tau, m, \omega)E_{-n}(\tau + \lambda t_0 - p \lambda t_0, \theta_{\lambda t_0 - p \lambda t_0} \omega) \subseteq E_{-n-p}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega), \quad \forall p \in \mathbb{N},
\]

which implies that the number of element of \( E_{-n}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) \) satisfies:

\[
\#E_{-n_1}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) \leq \#E_{-n_2}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) \quad \text{for } n_1 \leq n_2.
\]

Set

\[
E(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) = \bigcup_{n=0}^{\infty} E_{-n}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) \subseteq \hat{\chi}(\tau + \lambda t_0, \theta_{\lambda t_0} \omega).
\]

Then \( \{ E(\tau + \lambda t_0, \theta_{\lambda t_0} \omega) \}_{\tau \in \mathbb{R}, \omega \in \Omega} \) is a random exponential attractor for discrete cocycle \( \{ \Psi(n, \tau, m, \omega) \}_{m \in \mathbb{Z}, n \in \mathbb{N}, \tau \in \mathbb{R}, \omega \in \Omega} \).
(1) Compactness and measurability. Let $0 < \varepsilon < 1$ be a given number. By (25), there exists an integer $n_B = n_B(\varepsilon) \in \mathbb{N}$ such that $r_{1-\varepsilon, m, \omega} \leq \varepsilon < r_{1-(n_B-1), m, \omega}$. By (27), for all $n \geq n_B$,

\[
\mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0}) \\
\subseteq \Psi(n, \tau, m-n, \omega)\chi(\tau, m-n, \omega) \\
\subseteq \Psi(n, \tau, m-n, \omega)\Psi(n-n_B, \tau, m-n, \omega)\chi(\tau, m-n, \omega) \\
\subseteq \bigcup_{i=1}^{N^*_n} B(u_{n-n_B, m, i}; r_{1-n_B, m, \omega}) \\
\subseteq \bigcup_{i=1}^{N^*_n} B(u_{n-n_B, m, i}; \varepsilon).
\]

Thus,

\[
\bigcup_{n=n_B}^{+\infty} \mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0}) \subseteq \bigcup_{i=1}^{N^*_n} B(u_{n-n_B, m, i}; \varepsilon),
\]

that is, $\bigcup_{n=n_B}^{+\infty} \mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0})$ is covered by $N^*_n$ closed balls with radius $\varepsilon$. On the other hand, $\bigcup_{n=1}^{N^n} \mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0}) = \bigcup_{n=1}^{N^n} \{u_{n, m, i} : 1 \leq i \leq N^n\}$ is a finite set. So, by definition (30),

\[
\mathcal{E}(\tau + m\ell_0, \theta_{m\ell_0}) = \left( \bigcup_{n=0}^{n_B-1} \mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0}) \right) \cup \bigcup_{n=n_B}^{+\infty} \mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0})
\]

is a compact set of $X$.

By (26), $\mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0})$ is a finite measurable set for all $n \in \mathbb{N}$. Thus, by (30) and Proposition 1.3.1 (ii)(v) in [10], it follows that $\mathcal{E}(\tau + m\ell_0, \theta_{m\ell_0})$ is measurable.

(2) Finite fractal dimension. By (25), we have $n_B \to +\infty$ as $\varepsilon \to 0$ and $\ln \varepsilon \leq \ln r_{1-(n_B-1), m, \omega}$, that is,

\[
1 - \ln \varepsilon \leq - \ln r_{1-(n_B-1), m, \omega} \leq \frac{1}{2} - \frac{2\varepsilon-1}{2} \ln \frac{2}{\varepsilon} - \ln b_{\omega,m}.
\]

(33)

Taking $\varepsilon$ sufficient small such that $n_B - 1 \geq n_4(\omega)$, then by (29) and (31),

\[
N(\mathcal{E}(\tau + m\ell_0, \theta_{m\ell_0})) \leq \#\mathcal{E}_{n_B}(\tau + m\ell_0, \theta_{m\ell_0}) + \sum_{n=0}^{n_B-1} \#\mathcal{E}_n(\tau + m\ell_0, \theta_{m\ell_0}) \\
\leq (n_B + 1)\tilde{N}^{n_B}.
\]

Thus,

\[
\ln N(\mathcal{E}(\tau + m\ell_0, \theta_{m\ell_0})) \leq \ln(n_B + 1) + n_B \ln \tilde{N}.
\]

(34)

It follows from (33) and (34) that for $\tau \in \mathbb{R}$, $\omega \in \Omega$, $m \in \mathbb{Z}$, $\dim_f \mathcal{E}(\tau + m\ell_0, \theta_{m\ell_0})$ has an upper bound (constant):

\[
\dim_f \mathcal{E}(\tau + m\ell_0, \theta_{m\ell_0}) \leq \lim_{n_B \to +\infty} \sup \frac{\ln(n_B + 1) + n_B \ln \tilde{N}}{\ln \frac{2}{\varepsilon} - \ln b_{\omega,m}} = \frac{2\ln \tilde{N}}{\ln \frac{2}{\varepsilon}} < \infty.
\]
(3) Positive invariance. For $\tau \in \mathbb{R}$, $\omega \in \Omega$, $m \in \mathbb{Z}$ and $p \in \mathbb{N}$, by (28),
\[
\Psi(p\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) \subseteq \bigcup_{n=0}^{\infty} \Psi(p\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) E^{-n}(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)
\]
\[
\subseteq \bigcup_{n=0}^{\infty} E^{-n}(\tau + m\hat{t}_0 + p\hat{t}_0, \theta_{m\hat{t}_0 + p\hat{t}_0}\omega)
\]
\[
= E(\tau + m\hat{t}_0 + p\hat{t}_0, \theta_{m\hat{t}_0 + p\hat{t}_0}\omega), \quad p \in \mathbb{N}.
\]

(4) Exponential attraction. Since $u_{-n,m,i} \in E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) \cap \Psi(n, \tau, m - n, \omega) \chi(\tau, m - n, \omega)$, $\forall 1 \leq i \leq N^n$, by (12) and (25), for $n \geq n_1(\omega)$,
\[
d_n(\Psi(n, \tau, m - n, \omega) \chi(\tau, m - n, \omega), E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega))
\]
\[
\leq r_{1-n,m,\omega} \leq \left(\frac{3}{4}\right)^n b_{\omega,m} = b_{\omega,m} e^{-\frac{\ln 4}{4} n}. \tag{37}
\]

Step 2. Construction of a random exponential attractor for continuous cocycle $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$.

For any given $\tau$, $s \in \mathbb{R}$ and $\omega \in \Omega$, let $m \in \mathbb{Z}$ be the (fixed) integer such that $m\hat{t}_0 \leq s < (m + 1)\hat{t}_0$, set
\[
E(\tau + s, \theta_{s}\omega) = \Psi(s - m\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega), \quad 0 \leq s - m\hat{t}_0 < \hat{t}_0. \tag{38}
\]
Then $\{E(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ is a random exponential attractor for continuous cocycle $\{\Psi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega}$. Let us show this fact as follows.

(1) Compactness, measurability and finite fractal dimension. By (H4), $\Psi(s - m\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)$ is Lipschitz continuous from $\chi(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)$ into $\chi(\tau + s, \theta_{s}\omega)$, and by the compactness of $E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)$, it follows that $E(\tau + s, \theta_{s}\omega)$ is compact and
\[
\dim_f E(\tau + s, \theta_{s}\omega) \leq \dim_f E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) \leq \frac{2N \ln \left(\frac{2N}{3} + 1\right)}{\ln \frac{4}{3}}. \tag{39}
\]

By (H12), $E(\tau + s, \theta_{s}\omega) \subseteq \chi(\tau + s, \theta_{s}\omega)$. By the continuity of $\Psi(t, \tau, \omega)$ and measurability in $\omega$ of $\Psi(t, \tau, \omega)$, it deduces form (30) and the measurability of $E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)$ that $E(\tau + s, \theta_{s}\omega)$ is measurable.

(2) Positive invariance. For $t \geq 0$, let $m, n \in \mathbb{N}$ be integers such that $(m + n)\hat{t}_0 \leq s + t < (m + n + 1)\hat{t}_0$. When $n = 0$, by definition (38), we have
\[
\Psi(t, \tau + s, \theta_{s}\omega) E(\tau + s, \theta_{s}\omega)
\]
\[
= \Psi(t, \tau + s, \theta_{s}\omega) \Psi(s - m\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)
\]
\[
= \Psi(t + s - m\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)
\]
\[
= E(\tau + s + t, \theta_{s+t}\omega).
\]

When $n > 0$, then by the cocycle property of $\Psi$ and (36),
\[
\Psi(t, \tau + s, \theta_{s}\omega) E(\tau + s, \theta_{s}\omega)
\]
\[
= \Psi(s + t - (m + n)\hat{t}_0, \tau + (m + n)\hat{t}_0, \theta_{(m+n)\hat{t}_0}\omega) \Psi((m + n)\hat{t}_0 - s, \tau + s, \theta_{s}\omega)
\]
\[
\circ \Psi(s - m\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)
\]
\[
\subseteq \Psi(t + s - m\hat{t}_0, \tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega) E(\tau + m\hat{t}_0, \theta_{m\hat{t}_0}\omega)
\]
\[
= E(\tau + s + t, \theta_{s+t}\omega). \tag{40}
\]
(3) Exponential attraction. Put \( u_s \in \tilde{\chi}(\tau + s, \theta_s \omega) \), then by (h12),

\[ u_{(m+1)i_0} = \Psi((m+1)i_0 - s, \tau + s, \theta_s \omega)u_s \in \tilde{\chi}(\tau + (m+1)i_0, \theta_{(m+1)i_0} \omega). \]

Write

\[ \Psi(s + t, m + n, \tau, \omega) = \Psi(s + t - (m + n)i_0, s, \tau + (m + n)i_0, \theta_{(m+n)i_0} \omega), \]

\[ u_{s,1} = \Psi((n-1)i_0, \tau + (m + n)i_0, \theta_{(m+1)i_0} \omega)u_{(m+n)i_0}, \]

\[ u_{s,2} = \mathcal{E}(\tau + (m + n)i_0, \theta_{(m+n)i_0} \omega), \]

and by (7) and (37), for \( n \geq n_1(\omega) \),

\[ \frac{d}{dt}(\Psi(t, s, \theta_s \omega)u_s, \mathcal{E}(\tau + s + t, \theta_s \omega)) = \frac{d}{dt}(\Psi(s + t, m + n, \tau, \omega)u_{s,1}, \Psi(s + t, m + n, \tau, \omega)u_{s,2}) \leq e^{\tilde{C}_1(\theta_{(m+n+1)i_0} \omega)\int_{(m+n)i_0}^{(m+n+1)i_0} \tilde{C}_0(\theta_s \omega)ds} \cdot \frac{du_{s,1}}{ds} + \frac{du_{s,2}}{ds} \]

\[ \leq b_{\omega,m} e^{-\ln 4 (n - 1) + \tilde{C}_1(\theta_{(m+n+1)i_0} \omega)\int_{(m+n)i_0}^{(m+n+1)i_0} \tilde{C}_0(\theta_s \omega)ds}. \]

So, for \( n \geq n_1(\omega) \),

\[ d_h(\Psi(t, s + t, \theta_s \omega)\tilde{\chi}(\tau + s, \theta_s \omega), \mathcal{E}(\tau + s + t, \theta_s \omega)) \leq b_{\omega,m} e^{-\ln 4 (n - 1) + \tilde{C}_1(\theta_{(m+n+1)i_0} \omega)\int_{(m+n)i_0}^{(m+n+1)i_0} \tilde{C}_0(\theta_s \omega)ds}. \]

By \( \frac{\tilde{C}_1(\theta_{(m+n+1)i_0} \omega)\int_{(m+n)i_0}^{(m+n+1)i_0} \tilde{C}_0(\theta_s \omega)ds}{n} \xrightarrow{n \to \infty} 0 \), there exists \( n_2(\omega) \in \mathbb{N} \) such that

\[ \tilde{C}_1(\theta_{(m+n+1)i_0} \omega)\int_{(m+n)i_0}^{(m+n+1)i_0} \tilde{C}_0(\theta_s \omega)ds \leq \frac{\ln 4}{4} n, \quad \forall n \geq n_2(\omega). \]

Thus, for \( n \geq \max\{n_1(\omega), n_2(\omega)\} \),

\[ d_h(\Psi(t, s + t, \theta_s \omega)\tilde{\chi}(\tau + s, \theta_s \omega), \mathcal{E}(\tau + s + t, \theta_s \omega)) \leq b_{\omega,m} e^{-\frac{3\ln 4}{4} n}, \quad (42) \]

where

\[ b_{\omega,m} = b_{\omega,m} e^{-\frac{3\ln 4}{8} n} > 0. \]

In particular, for every \( t \in \mathbb{R}, \omega \in \Omega \) and \( t \geq t_0 \max\{n_1(\omega), n_2(\omega)\} \), it holds that

\[ d_h(\Psi(t, \tau - t, \theta_{-t} \omega)\tilde{\chi}(\tau - t, \theta_{-t} \omega), \mathcal{E}(\tau, \omega)) \leq b_{\omega} e^{-\frac{3\ln 4}{4} n}, \quad (43) \]

for some tempered random variable \( b_{\omega} \). This implies the attracting property (iv).

**Step 3.** Simi-continuity (v) of random exponential attractor.

The continuous property (v) of \( \{ \mathcal{E}(\tau, \omega) \}_{\tau \in \mathbb{R}, \omega \in \Omega} \) follows from Theorem 3.1 in [20] and condition (H4). The proof is completed. \( \square \)

3. **Mathematical setting and boundedness of solutions.** In this section, we consider the well-posedness of initial-boundary problem (1) and the boundedness of solutions.

Let \( A = -\Delta, D(A) = H^1_0(U) \cap H^2(U) \), then \( A \) is a self-adjoint positive linear sectorial operator with eigenvalues \( \{ \lambda_i \}_{i \in \mathbb{N}} \):

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots, \quad \lambda_m \xrightarrow{m \to \infty} +\infty. \]

For \( r \in \mathbb{R}, V_{2r} = D(A^r) \) is a Hilbert space with inner product \( (u, v)_{2r} = (A^r u, A^r v) \).

The injection \( V_{r_1} \hookrightarrow V_{r_2} \) is compact if \( r_1 > r_2 \).
For any $t \in \mathbb{R}$, define $\theta_t$ on $\Omega$: $\theta_t \omega(\cdot) = \omega(t + \cdot) - \omega(t)$, then $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system [1, 10]. Write $z(\theta_t \omega) = -\alpha \int_{-\infty}^{t} e^{\alpha(s)} \theta_t \omega(s)ds$ for $t \in \mathbb{R}$, $\omega \in \Omega$, which is an Ornstein-Uhlenbeck stationary process solving the Itô equation $dz(\theta_t \omega) + \sigma(z(\theta_t \omega))dt = dW(t)$. From [1, 7, 23, 66], it follows that $|z(\omega)|$ is tempered and for $\omega \in \Omega$, $t \mapsto z(\theta_t \omega)$ is continuous in $t$, moreover,

$$E[|z(\theta_t \omega)|^r] = \frac{\Gamma\left(\frac{1+r}{2}\right)}{\sqrt{\pi} \alpha^r}, \quad \forall r > 0, s \in \mathbb{R}, \quad (44)$$

and

$$\begin{align*}
\left\{ \begin{array}{l}
E\left[e^{\varepsilon \int_{t_1}^{t_2} |z(\theta_t \omega)|^2 ds}\right] \leq e^{\varepsilon t}, \quad \text{for } \alpha^2 \geq 2\varepsilon \geq 0, \tau \in \mathbb{R}, t \geq 0; \\
E\left[e^{\frac{r}{\alpha^2} \int_{t_1}^{t_2} |z(\theta_t \omega)|^2 ds}\right] \leq e^{\frac{r}{\alpha^2} t}, \quad \text{for } \alpha^3 \geq \varepsilon^2 \geq 0, \tau \in \mathbb{R}, t \geq 0;
\end{array} \right. \quad (45)
\end{align*}$$

where $\Gamma(\cdot)$ is the Gamma function.

Write $E^\tau = D(A^{\frac{3}{2}}) \times D(A)$ and $E = H_0^1(U) \times L^2(U)$. Now we transfer (1) into a random system without noise terms. Let

$$v = u_t + \varepsilon u - au z(\theta_t \omega), \quad \varepsilon = \frac{\lambda_1 \alpha}{\alpha^2 + 3\lambda_1}, \quad (46)$$

then system (1) can be written as the following equivalent random system in $E$:

$$\dot{\varphi} + A\varphi = F(\varphi, \theta_t \omega, t), \quad \varphi(\omega) = (u_t, u_1, t + \varepsilon u - au z(\theta_t \omega))^T, \quad t \geq \tau, \tau \in \mathbb{R}, \quad (47)$$

where

$$\varphi = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} \varepsilon I & -I \\ A - \varepsilon(\alpha - \varepsilon)I & (\alpha - \varepsilon)I \end{pmatrix},$$

$$F(\varphi, \theta_t \omega, t) = \begin{pmatrix} au z(\theta_t \omega) \\ (2\varepsilon - az(\theta_t \omega))az(\theta_t \omega)u - az(\theta_t \omega)v - f(u, x) + g(x, t) \end{pmatrix}. $$

In the following, we always assume that condition (A1) holds. It follows from Galerkin approximation method or [9, 42, 48, 52, 54, 64] and Lemma 3.1 below that for every fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\varphi_\tau \in E$, the problem (47) has a unique globally weak solution $\varphi(\cdot, \tau, \omega, \varphi_\tau) \in C([\tau, +\infty); E)$, which defines a continuous cocycle $\Phi: \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ on state space $E$ over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ by $\Phi(t, \tau, \omega, \varphi_\tau) = \Phi(t, \tau, \omega)\varphi_\tau(\omega) = \varphi(t + \tau, \tau, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega))$ with $\Phi(0, \tau, \omega)\varphi_\tau(\omega) = \varphi(\tau, \tau, \omega, \varphi_\tau(\theta_{-\tau} \omega))$ and $\Phi(t, \tau - t, \theta_{-\tau} \omega, \varphi_\tau(\theta_{-\tau} \omega)) = \varphi(\tau, \tau, \omega, \varphi_\tau(\theta_{-\tau} \omega))$.

Obviously, the dynamics of solution $(u, u_t)$ of (1) is same to that of cocycle $\Phi$ associated with (47) in $E$. From now, we consider the existence, upper semicontinuity, regularity of random attractor and the existence of random exponential attractor for $\Phi$ in $E$.

Let $D = D(E)$ denote the collection of all tempered families of nonempty subsets of $E$ and the inner and norm of $L^2(U)$ are denoted as $(\cdot, \cdot)$ and $|| \cdot ||$. We remark that all the numbers $c_i$ ($i \in \mathbb{N}$) below are positive constants independent of $(\omega, \tau, t)$. By (A1), we have

$$\begin{align*}
\left\{ \begin{array}{l}
|f(u, x)| \leq c_1(1 + |u|^3), \quad (f(u, x), u) \geq \frac{1}{c_1} \tilde{G}(r) - \frac{1}{2}||u||_2^2 - \frac{3}{c_1}, \\
0 \leq \int_U |u|^2 dx \leq \frac{1}{c_1} [\tilde{G}(r) + \beta_1], \quad \tilde{G}(r) \leq c_0(||u||_2^2 + ||u||_1^4) + \beta_2,
\end{array} \right. \quad (48)
\end{align*}$$

where $\tilde{G}(r) = \int_U G(u(x), x)dx$, $\bar{\beta}_i = \int_U \beta_i(x)dx$, $i = 1, 2$. Write

$$\rho = \varepsilon \min \left\{ \frac{1}{4}, \frac{1}{c_5}, \frac{1}{2c_0} \right\}, \quad \mu = \max \left\{ 4, \frac{c_1}{2c_0}, \frac{c_8}{c_4}, \frac{2}{\lambda_1} \right\}, \quad (49)$$
and
\[ c_{10} = \frac{c_8^2}{\sqrt{\lambda_1}} |U| + \frac{1}{\alpha^2} ||g||^2 + \frac{2 \varepsilon}{c_5} (\overline{\lambda}_1 + \overline{\lambda}_2), \quad |U| = \int_U dx, \quad ||g||^2 = \sup_{r \in \mathbb{R}} ||g(r, r)||^2. \]  

(50)

**Lemma 3.1.** Suppose $|a|$ is sufficient small such that
\[ |a| \leq \min \left\{ \frac{\sqrt{\pi} \rho}{4 \mu}, \frac{\alpha \rho}{2 \mu} \right\}. \]  

(51)

Then there exist a random variable $M_0(\omega)$
\[ M_0(\omega) = \left( 2c_{10} \int_{-\infty}^{0} e^{-\int_{\tau}^{0} \left( |a| \left| z(\theta - \omega) \right| + |a|^2 \left| z(\theta - \omega) \right|^2 \right) ds} \right)^{1/2} > 0, \]  

(52)

and a tempered measurable $\mathcal{D}(E)$-pullback absorbing set for $\Phi$
\[ B_0 = \{ B_0(\tau, \omega) = B_0(0, M_0(\omega)) = \{ \varphi \in E: ||\varphi||_E \leq M_0(\omega) \} | \tau \in \mathbb{R}, \omega \in \Omega \} \]  

(53)

(independent of $\tau$), that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$ and $B(\tau, \omega) \subset B \in \mathcal{D}(E)$, there exists a $T_B(\tau, \omega) \geq 0$ such that the solution $\varphi(\tau, t, \theta - \omega, \varphi_{\tau-t}(\theta - \omega)) \in E$ of (47) with $\varphi_{\tau-t}(\theta - \omega)$ satisfies:
\[ ||\varphi(\tau, t, \theta - \omega, \varphi_{\tau-t}(\theta - \omega))||_E \leq M_0(\omega), \quad \forall t \geq T_B(\tau, \omega). \]  

(54)

In particular, there exists a $T_{B\omega}(\omega) \geq 0$ (independent of $\tau$) such that
\[ \varphi(r, \tau - t, \theta - \omega, B_{\omega}(\theta - \omega)) \in B_{\omega}(\theta - \omega), \quad \forall t \geq T_{B\omega}(\omega), \quad r \geq \tau - t. \]  

(55)

**Proof.** For every fixed $\tau \in \mathbb{R}$ and $\omega \in \Omega$, let $\varphi(r) = \varphi(r, \tau - t, \theta - \omega, \varphi_{\tau-t}(\theta - \omega)) = (u(r), v(r))^T \in E \ (r \geq \tau - t, \ t \geq 0)$ be a solution of (47) with $\varphi_{\tau-t}(\theta - \omega) = (u_{\tau-t}, u_{\tau-t} + \epsilon z - a u_{\tau-t} z(\theta - \omega))^T \in E$. By [22, 63], (48) and $\varepsilon < \sqrt{\lambda_1}$, we have
\[ 2(\Lambda \varphi, \varphi) \geq \varepsilon (||u||^2 + ||v||^2), \]  

(56)

\[ 2(f(u, x), az(\theta - \omega) u) \leq \frac{c_8^2}{\sqrt{\lambda_1}} |U| + \frac{c_8^2 |a|^2 \left| z(\theta - \omega) \right|^2}{\lambda_1} ||u||^2 \]  

\[ + \frac{c_8}{c_4} |a| \left| z(\theta - \omega) \right||2 \bar{G}(r) + 2 \bar{\beta}_1|, \]  

(57)

\[ 2((2 \varepsilon - az)au - av, v) \leq \left( 2|a| \left| z(\theta - \omega) \right| + \frac{c_8^2 |a|^2 \left| z(\theta - \omega) \right|^2}{\lambda_1} \right) ||\varphi||_E^2 \]  

\[ + 2|a| \left| z(\theta - \omega) \right| ||v||^2, \]  

\[ 2(a(\theta - \omega) u, u_1) \leq 2|a| \left| z(\theta - \omega) \right||u||_E^2, \quad 2(g(r, x), v) \leq \frac{1}{\alpha} ||g||^2 + \alpha ||v||^2. \]  

Taking the inner product $(\cdot, \cdot)_E$ of (47) with $\varphi(r)$, we find that for $r \geq \tau - t$,
\[ \frac{d}{dt} \left( ||A^2 u||^2 + ||v||^2 + 2 \bar{G}(r) \right) + 2(\Lambda \varphi, \varphi)_E + 2z(f(u, x), u) \]  

\[ = 2az(\theta - \omega) u_1 + 2f(u, x), az(\theta - \omega) u \]  

\[ + 2g(r, x) + (2 \varepsilon - az(\theta - \omega))az(\theta - \omega) u - az(\theta - \omega) v, \]  

(57)

thus,
\[ \frac{d}{dt} y(r) + (\rho - \mu(|a| \left| z(\theta - \omega) \right| + |a|^2 \left| z(\theta - \omega) \right|^2)) y(r) \leq c_{10}, \quad \forall r \geq \tau - t, \]  

(58)

where
\[ y(r) = ||\varphi(r)||^2_E + 2 \bar{G}(r) + 2 \bar{\beta}_1 \geq ||\varphi(r)||^2_E. \]  

(59)
Thus, by (60), we have
\[ \| \varphi(r) \|_{\mathcal{E}}^2 \leq g(y) \int_{-\infty}^{y} e^{-\int_{-\infty}^{t} (\rho - \mu |a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt} ds + c_{10} \int_{-\infty}^{y} e^{-\int_{-\infty}^{t} (\rho - \mu |a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt} ds. \]  
(60)

By (47), (48) and (59),
\[ y(\tau - t) \leq \| \varphi_{\tau - \tau}(\theta - \tau) \|_{\mathcal{E}}^2 + 2c_9 (\| u_{\tau - \tau} \|^2 + \| u_{\tau - \tau} \|^4) + 2 (\tilde{\beta}_1 + \tilde{\beta}_2). \]  
(61)

Thus, by (60), we have
\[ \| \varphi(r) \|_{\mathcal{E}}^2 \leq g(y) \int_{-\infty}^{y} e^{-\int_{-\infty}^{t} (\rho - \mu |a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt} ds + \frac{1}{2} M_0^2(\omega). \]  
(62)

Since \( \{ \theta(t) \}_t \in \mathbb{R} \) is ergodic measure-preserving on \( (\Omega, \mathcal{F}, \mathbb{P}) \), by Birkhoff ergodic Theorem [51], it holds that for every \( \omega \in \Omega \) (in fact a.e. \( \omega \in \Omega \),
\[ \lim_{t \to +\infty} \frac{1}{t} \int_{-t}^{0} \mu(|a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt \to \mathbb{E}[|a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) = \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha}, \]  
(63)

which implies that there exists a \( T_0(\omega) > 0 \) such that
\[ \int_{-t}^{0} \mu(|a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt \leq \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right), \]  
(64)

and
\[ \int_{-\infty}^{0} e^{-\int_{-\infty}^{t} (\rho - \mu |a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt} ds \leq \frac{1}{\rho} \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right)^2 T_0(\omega) + \frac{2}{\rho} < \infty. \]

For the initial data \( \varphi_{\tau - \tau}(\theta - \tau) \in B(\tau, \theta - \tau) \in \mathcal{D}(E) \), by (61), we have
\[ y(\tau - t) \to \mathbb{E}[|a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt] \to T_0(\omega), \]  
(65)

Thus (62) and (65) imply (54). By (60) and the cocycle property of \( \varphi \), (55) holds.

For any \( \gamma > 0 \),
\[ e^{-2\gamma t} M_0^2(\theta - \tau) = 2c_{10} e^{-2\gamma t} \int_{-\infty}^{t} e^{-\int_{-\infty}^{s} (\rho - \mu |a||z(\theta, \omega)| + |a|^2 z(\theta, \omega))^2) dt} ds \to 0, \]  
(66)

that is, \( M_0(\omega) \) is a tempered random variable. The proof is completed. \( \square \)

**Remark 3.1.** By (45), (52) and (51), it follows that for
\[ |a| \leq \min \left\{ \frac{\alpha}{\sqrt{2\mu}}, \frac{\alpha \sqrt{\alpha}}{2\mu}, \frac{\sqrt{\alpha \rho}}{4\mu}, \frac{\sqrt{\alpha \rho}}{4\mu} \right\}, \]  
(66)

we have
\[ \mathbb{E}(M_0^2(\omega)) \leq c_{10} \int_{0}^{+\infty} e^{-\rho s} E \left( e^{2(|a| f_0^s |z(\theta, \omega)| dt) + e^{2|a|^2 f_0^s |z(\theta, \omega)|^2 dt} \right) ds \]
\[ \leq c_{10} \left( \frac{1}{\rho - \frac{2\mu |a|^2}{\sqrt{\alpha}}}, \frac{1}{\rho - \frac{2\mu |a|^2}{\alpha}} \right) < \infty, \]  
(67)

and for
\[ k \geq 2, \quad k \in \mathbb{N}, \quad |a| \leq \min \left\{ \frac{\alpha}{2\sqrt{k \mu}}, \frac{\alpha \sqrt{\alpha}}{2k \mu}, \frac{\sqrt{\alpha \rho}}{4\mu}, \frac{\sqrt{\alpha \rho}}{4\mu} \right\}, \]  
(68)
it holds that
\[ E(M_0^{2k}(\omega)) \]
\[ \leq 2^{(k-1)}k \int_0^{\infty} e^{-\frac{3\rho}{4}|\mu|s}ds \leq 2^{(k-1)}k \int_0^{\infty} e^{-\frac{3\rho}{4}|\mu|s}ds \approx \infty. \] (69)

4. Decomposition of solutions. In this section, we split the solutions of (47) into a sum of two components: one component decays “exponentially” and another one is ultimately pullback bounded in a “higher regular” space. To do this, we have to decompose the solutions of (47) in two modes with different initial data.

4.1. Decomposition (I). For given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), set
\[ B_1(\tau, \omega) = \bigcup_{t \geq T_{\delta_0}(\omega)} \varphi(\tau, \tau - t, \theta_{-r, \omega}, B_0(\theta_{-r, \omega})) \subseteq B_0(\omega). \] (70)

Let \( \varphi(r) = \varphi(r, \tau - t, \theta_{-r, \omega}, \varphi_{-t}(\theta_{-r, \omega})) (r \geq \tau - t, t \geq 0) \) be a solution of (47) with the initial data \( \varphi_{-t}(\theta_{-r, \omega}) \in B_1(\tau - t, \theta_{-r, \omega}) \subseteq B_0(\theta_{-r, \omega}) \). From (55), we have
\[ ||\varphi(r, \tau - t, \theta_{-r, \omega}, \varphi_{-t}(\theta_{-r, \omega}))||_E \leq M_0(\theta_{-r, \omega}), \quad \forall r \geq \tau - t. \] (71)

Let \( \varphi(r) = \varphi_1(r) + \varphi_2(r) \), where \( \varphi_1(r) = (u_1, v_1)^T \) and \( \varphi_2(r) = (u_2, v_2)^T \) satisfy, respectively,
\[ \left\{ \begin{array}{ll}
\dot{\varphi}_1 + \Lambda \varphi_1 &= F_1(\varphi_1, \theta_{-r, \omega}), \quad r > \tau - t, \\
\dot{\varphi}_1 &= F_1(\varphi_1, \theta_{-r, \omega}, \varphi_{-t}(\theta_{-r, \omega})), \quad r > \tau - t,
\end{array} \right. \] (72)
and
\[ \left\{ \begin{array}{ll}
\dot{\varphi}_2 + \Lambda \varphi_2 &= F_2(u_1, \varphi_2, \theta_{-r, \omega}, r), \quad r > \tau - t, \\
\dot{\varphi}_2 &= F_2(\varphi(r), \theta_{-r, \omega}, \varphi_{-t}(\theta_{-r, \omega})), \quad r = \tau - t,
\end{array} \right. \] (73)
where
\[ v_1 = u_i, \quad i = 1, 2, \] (74)
\[ F_1(\varphi_1, \theta_{-r, \omega}) = \begin{pmatrix} 2\varepsilon - a\theta(\theta_{-r, \omega})u_1 \\ a\theta(\theta_{-r, \omega})u_1 \end{pmatrix}, \] (75)
\[ F_2(u_1, \varphi_2, \theta_{-r, \omega}, r) = \begin{pmatrix} 2\varepsilon - a\theta(\theta_{-r, \omega})u_2 \\ a\theta(\theta_{-r, \omega})u_2 \end{pmatrix}. \] (76)

For the first component \( \varphi_1 \), we have its “exponentially” decay as follows.

Lemma 4.1. For every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( t \geq 0 \), there exists a tempered random variable \( M_1(\omega) \geq 0 \) such that the solution \( \varphi_1(r) \) of (72) satisfies
\[ ||\varphi_1(\tau, r - t, \theta_{-r, \omega}, \varphi_{-t}(\theta_{-r, \omega}))||_E \leq M_1(\theta_{-r, \omega})e^{-\int_0^t (\rho - \mu(|\cdot|z(\theta_{-r, \omega})) + |\cdot|^2z(\theta_{-r, \omega}))ds}, \] (77)
where \( \rho, \mu \) are defined by (49).

Proof. By (2) and (74), \( F_1(u_1, x, v_1) \geq \frac{d}{dt}G_1(r) + \left( \frac{\varepsilon - \theta_{\mu}}{\theta} \right)|\cdot|z(\theta_{-r, \omega})| \right) G_1(r). \]

Taking the inner product of (72) in \( E \) with \( \varphi_1 = (u_1, v_1)^T \), we find that
\[ \frac{d}{dt}y_1(r) + 2(\rho - \mu(|\cdot|z(\theta_{-r, \omega})| + |\cdot|^2z(\theta_{-r, \omega})^2))y_1(r) \leq 0, \quad \forall r \geq \tau - t, \] (78)
where

\[ \| \varphi_1(r) \|_E^2 \leq y_1(r) = \| \varphi_1(r) \|_E^2 + 2\tilde{G}_1(r) \leq \| \varphi_1(r) \|_E^2 + 2c_0c_2 \| u_1(r) \|_1^2. \]  (77)

By Gronwall’s inequality to (76), we have that for \( r \geq \tau - t, \)

\[ \| \varphi_1(r, \tau - t, \theta - \tau, \varphi_{\tau - t}(\theta - \tau, \omega)) \|_E^2 \]
\[ \leq y_1(\tau - t, \tau - t, \theta - \tau, \varphi_{\tau - t}(\theta - \tau, \omega))e^{2 \int_{\tau - t}^\tau (\rho - \mu(|a| |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2))ds} \]
\[ \leq [M_0^2(\theta - \tau, \omega) + 2c_0c_2 \| u_{\tau - t} \|_1^4]e^{2 \int_{\tau - t}^\tau (\rho - \mu(|a| |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2))ds} \]
\[ \leq c_1[1 + M_0^4(\theta - \tau, \omega)]e^{2 \int_{\tau - t}^\tau (\rho - \mu(|a| |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2))ds}. \]  (78)

Let \( M_0^2(\omega) = c_1[1 + M_0^4(\omega)] \) which is tempered. Taking \( r = \tau \) in (75), then (75) holds. The proof is completed. \( \square \)

For the second component \( \varphi_2, \) we have the following estimation.

**Lemma 4.2.** For every \( r \in \mathbb{R}, \omega \in \Omega \) and \( t \geq 0, \) there exists a random variable \( M_2(t, \omega) > 0 \) such that the solution \( \varphi_2(r) = (u_2(r), v_2(r))^T \) of (73) satisfies

\[ \| A^{\nu + \frac{1}{2}} u_2(\tau, \tau - t, \theta - \tau, \varphi_{\tau - t}(\theta - \tau, \omega)) \|_2^2 + \| A^\nu v_2(\tau, \tau - t, \theta - \tau, \varphi_{\tau - t}(\theta - \tau, \omega)) \|_2^2 \leq M_2(t, \omega), \]  (79)

where

\[ \nu = \min \left\{ \frac{1}{4} \left( 2 - p \right), \frac{1}{4}, \frac{1}{4} \right\}, \quad 0 \leq p \leq 1, \]

\[ \frac{1}{4} \leq p < 2. \]  (80)

**Proof.** Taking the inner product of (73) in \( E \) with \( A^{2\nu} \varphi_2 = (A^{2\nu} u_2, A^{2\nu} v_2)^T, \) we have that for \( r \geq \tau - t, \)

\[ \frac{1}{2} \int_E \{ (\| A^{\nu + 1/2} u_2 \|_2^2 + \| A^\nu v_2 \|_2^2) \}
\[ + (\Lambda \varphi_2, A^{2\nu} \varphi_2) + \varepsilon \int_E |f(u, x) - f_1(u_1, x)| A^{2\nu} u_2 dx \]
\[ - \int_E \{ f_1'(u, x) u_{2}, t + f_2'(u, x, u_{2}, t) u_{1}, t - f_{1, u}(u_1, x) u_{1, t} \} A^{2\nu} u_2 dx \]
\[ = (az(\theta - \tau) u_2, A^{2\nu} u_2)_1 - (az(\theta - \tau) v_2, A^{2\nu} v_2) \]
\[ + (f(u, x) - f_1(u_1, x), A^{2\nu} u_2) \]
\[ + (g(x, r) + (2\varepsilon - az(\theta - \tau, \omega)) a z(\theta - \tau, \omega) u_2, A^{2\nu} v_2). \]  (81)

For each term in (81), by Hölder’s inequality, (A1), (71), (75) and (80), we have that for \( r \geq \tau - t, \)

\[ (\Lambda \varphi_2, A^{2\nu} \varphi_2) \geq \frac{\varepsilon}{2} ||A^{\nu + 1/2} u_2||_2^2 + \frac{\varepsilon}{2} ||A^\nu v_2||_2^2 + \frac{\alpha}{2} ||A^\nu v_2||_2^2, \]

\[ (az(\theta - \tau) u_2, A^{2\nu} u_2)_1 - (az(\theta - \tau) v_2, A^{2\nu} v_2) \]
\[ \leq |a| \cdot |z(\theta - \tau, \omega)| (||A^{\nu + 1/2} u_2||_2^2 + ||A^\nu v_2||_2^2), \]

\[ (g(x, r), A^{2\nu} v_2)_1 \leq \frac{1}{2\alpha} ||g||_1^2 + \frac{\alpha}{2} ||A^\nu v_2||_2^2, \]

\[ ||g||_1^2 = \sup_{r \in \mathbb{R}} ||g(\cdot, r)||_1^2, \]

\[ (2\varepsilon - az(\theta - \tau, \omega)) a z(\theta - \tau, \omega) u_2, A^{2\nu} v_2) \]
\[ \leq \left( |a| \cdot |z(\theta - \tau, \omega)| + \frac{|a|^2 |z(\theta - \tau, \omega)|^2}{2\sqrt{\lambda_1}} \right) (||A^{\nu + 1/2} u_2||_2^2 + ||A^\nu v_2||_2^2), \]
Thus by (81), we have

\[ (f(u, x) - f_1(u_1, x), A^{2\nu} u_2(z(\theta, t))u_2) \]

\[ \leq c_{12}|u_2(z(\theta, t))| \left( \int_U (1 + |u|^6 + |u_1|^6)dx \right)^{1/2} \left( \int_U |A^{2\nu} u_2|^2 dx \right)^{\frac{1}{2}} \]

\[ \leq c_{13} \left( 1 + z^{2}(\theta, t) + M_0^8(\theta, t) \right) \]

\[ + c_{13} \left( M_1^8(\theta, t)e^{-8 \int_{\tau}^{t} (\rho - \mu(|a| + |(\theta, t)| + |a|^2|z(\theta, t)|^2))ds \right), \]

\[ \int_U f_1'(u, x)u_{2,t} \cdot A^{2\nu} u_2 dx \]

\[ \leq c_{14} \left( \int_U (1 + |u|^2)^3 dx \right)^{\frac{1}{3}} \left( \int_U |u_{2,t}|^{\frac{6}{5\nu}} dx \right)^{\frac{3 - \frac{4}{\nu}}{\frac{3 - 4}{\nu}}} \left( \int_U |A^{2\nu} u_2|^{\frac{6}{5\nu}} dx \right)^{\frac{1 + \frac{4}{\nu}}{\frac{3 - 4}{\nu}}} \]

\[ \leq \frac{1}{2} c_{15} \left( 1 + M_0^8(\theta, t) \right) \left( ||A^{p+\frac{2}{\nu}} u_2||^2 + ||A^{p} v_2||^2 \right) + \frac{\varepsilon}{8} ||A^{p+\frac{2}{\nu}} u_2||^2 \]

\[ + c_{16} \left( 1 + z^8(\theta, t) + M_0^{16}(\theta, t) \right) \]

\[ + c_{16} \left( M_1^6(\theta, t)e^{-4 \int_{\tau}^{t} (\rho - \mu(|a| + |(\theta, t)| + |a|^2|z(\theta, t)|^2))ds \right), \]

\[ \int_U f_2'(u, x)u_{1,t} \cdot A^{2\nu} u_2 dx \]

\[ \leq c_{17} \left( \int_U |u_{1,t}|^2 dx \right)^{\frac{1}{3}} \left( \int_U (1 + |u|^p)^{\frac{6}{5\nu}} dx \right)^{\frac{2 - \frac{4}{\nu}}{\frac{3 - 4}{\nu}}} \left( \int_U |A^{2\nu} u_2|^{\frac{6}{5\nu}} dx \right)^{\frac{1 + \frac{4}{\nu}}{\frac{3 - 4}{\nu}}} \]

\[ \leq c_{18} \left( 1 + z^8(\theta, t) + M_0^{16}(\theta, t) \right) + \frac{\varepsilon}{8} ||A^{p+\frac{2}{\nu}} u_2||^2 \]

\[ + c_{18} \left( M_1^6(\theta, t)e^{-4 \int_{\tau}^{t} (\rho - \mu(|a| + |(\theta, t)| + |a|^2|z(\theta, t)|^2))ds \right), \]

and

\[ \int_U [f_1'(u, x) - f_1'(u_1, x)]u_{1,t} A^{2\nu} u_2 dx \]

\[ \leq c_{19} \left( \int_U |u_1|^2 dx \right)^{\frac{1}{3}} \left( \int_U (|u_1| + |u|)^6 dx \right)^{\frac{1}{2}} \]

\[ \times \left( \int_U |u_2|^{\frac{6}{5\nu}} dx \right)^{\frac{1 - \frac{4}{\nu}}{\frac{3 - 4}{\nu}}} \left( \int_U |A^{2\nu} u_2|^{\frac{6}{5\nu}} dx \right)^{\frac{1 + \frac{4}{\nu}}{\frac{3 - 4}{\nu}}} \]

\[ \leq \left( c_{20} + M_0^6(\theta, t) \right) ||A^{p+\frac{2}{\nu}} u_2||^2 + \frac{a^2 z^2(\theta, t)}{2\sqrt{\lambda_1}} ||A^{p+\frac{2}{\nu}} u_2||^2 \]

\[ + c_{21} M_1^6(\theta, t)e^{-4 \int_{\tau}^{t} (\rho - \mu(|a| + |(\theta, t)| + |a|^2|z(\theta, t)|^2))ds ||A^{p+\frac{2}{\nu}} u_2||^2. \]

Thus by (81), we have

\[ \frac{d}{dr} y_2(r) \leq \left( -\frac{\varepsilon}{2} + m_2(\theta, t) \right) y_2(r) + q_2(\theta, t), \quad \forall r \geq \tau - t, \quad (82) \]

where

\[ y_2 = ||A^{p+1/2} u_2||^2 + ||A^{p} v_2||^2 + 2 \int_U [f(u, x) - f_1(u_1, x)] A^{2\nu} u_2 dx, \quad (83) \]

\[ m_2(\theta, t) = c_{22} + \mu(|a| + |(\theta, t)| + |a|^2|z(\theta, t)|^2) + M_0^8(\theta, t) \]

\[ + c_{23} M_1^6(\theta, t)e^{-4 \int_{\tau}^{t} (\rho - \mu(|a| + |(\theta, t)| + |a|^2|z(\theta, t)|^2))ds , \quad (84) \]
\(q_2(\theta_{r \to \infty}) = c_{24}[1 + z^8(\theta_{r \to \infty}) + M_0^{16}(\theta_{r \to \infty})]
\)
\[+ c_{25} M_1^2(\theta_{\infty}) e^{-8 \int_{-\tau}^{\tau} (\rho - \mu(|a|)|z(\theta_{r \to \infty})| + |a|^2 |z(\theta_{r \to \infty})|^2) ds},\] (85)
\[y_2(\tau - t, \tau - t, \theta_{\infty}, \varphi_{\infty}(\theta_{\infty})) = (0, 0)^T.\] (86)

Note that
\[
\int_U [f(u, x) - f_1(u_1, x)] A^{2\nu} w_2 dx
\leq c_{26}[1 + M_0^{14}(\theta_{r \to \infty}) + M_1^4(\theta_{\infty}) e^{-4 \int_{-\tau}^{\tau} (\rho - \mu(|a|)|z(\theta_{r \to \infty})| + |a|^2 |z(\theta_{r \to \infty})|^2) ds}]
\leq R_2(\theta_{r \to \infty}, \theta_{\infty}), \forall r \geq \tau - t.
\] (87)

By applying Gronwall’s inequality to (82) on \([\tau - t, r] (r \geq \tau - t)\) and by (83), (86), (87), we have that for \(r \geq \tau - t,\)
\[
||A^{\nu + \frac{1}{2}} q_2(r, \tau - t, \theta_{\infty}, \varphi_{r \to \infty}(\theta_{\infty}))||^2 + ||A^{\nu} v_2(r, \tau - t, \theta_{\infty}, \varphi_{r \to \infty}(\theta_{\infty}))||^2
\leq 2q_2(r, \tau - t, \theta_{\infty}, \varphi_{r \to \infty}(\theta_{\infty})) + 2R_2(\theta_{r \to \infty}, \theta_{\infty})
\leq 2 \int_{\tau - t}^{r} q_2(\theta_{\infty}) e^{-\frac{1}{2} (\tau - r) + \int_{\xi}^{r} M_2(\theta_{\infty})ds} d\xi + 2R_2(\theta_{r \to \infty}, \theta_{\infty}).
\] (88)

Taking
\[
M_2(t, \omega) = 2 \int_{-\tau}^{0} q_2(\theta_{\infty}) e^{\frac{1}{2} \int_{-\tau}^{t} M_2(\theta_{\infty})ds} d\xi + 2R_2(\omega, \theta_{\infty}),
\] (89)
then (88) and (89) implies (79). The proof is completed. \(\square\)

**Remark 4.1.** From (51), we can not obtain the ultimately boundedness of the second component \(\varphi_2(r)\) of (73) in a “higher regular” space \(E'' = D(A^{\nu + \frac{1}{2}}) \times D(A^{\nu})\) as in the case of subcubic growth [65, 66].

Based on Lemmas 4.1-4.2 and the idea of proof of Proposition 1.4 in [58], we have the following decomposition of solutions for (47) which plays an important role in showing the asymptotic compactness of the cocycle \(\Phi\) in \(E\).

**Lemma 4.3.** Given \(l \in \mathbb{N}\). Let
\[
|a| \leq \min \left\{ \frac{\alpha}{4\sqrt{\mu}}, \frac{\alpha\sqrt{\alpha}}{8\mu}, \frac{\sqrt{\alpha\rho}}{8\mu}, \sqrt{\frac{\alpha\rho}{8\mu}} \right\}.
\] (90)

Then there exist positive constants \(K_{0,l}\) and \(K_{l}\) such that for given \(r \in \mathbb{R}, \omega \in \Omega, t \geq 0\) and \(T > 0\), the solution \(\varphi(r)\) of (47) has a decomposition: \(\varphi(r) = \phi_1(r) + \phi_2(r)\), where \(\phi_1, \phi_2\) satisfy
\[
\left\{ \begin{array}{l}
\int_{-\tau}^{r} ||\phi_1(\xi, \tau - t, \theta_{\infty}, \varphi_{r \to \infty}(\theta_{\infty}))||_{E''}^2 d\xi \leq \frac{K_{0,l}}{\sigma_T} (\tau - r) + K_{l}, \tau - t \leq r \leq \tau, \\
||\phi_2(r, \tau - t, \theta_{\infty}, \varphi_{r \to \infty}(\theta_{\infty}))||_{E''}^2 \leq M_2(T, \omega), \quad r \geq \tau - t,
\end{array} \right.
\] (91)
and \(v\) is as in (80).

**Proof.** Let us construct functions \(\phi_1(r)\) and \(\phi_2(r)\). For given \(T > 0, s \geq 0\) and \(k \in \mathbb{N}\), consider equations (72) and (73) at the interval \([\tau - t + s + (k - 1)T, \tau - t + s + kT]\), respectively. Set
\[
\left\{ \begin{array}{l}
\phi_1 = (w_1, \tilde{w}_1)^T = \varphi_1, \quad \phi_1(\tau - t + s + (k - 1)T) = \varphi(\tau - t + s + (k - 1)T), \\
\phi_2 = (w_2, \tilde{w}_2)^T = \varphi_2, \quad \phi_2(\tau - t + s + (k - 1)T) = (0, 0)^T.
\end{array} \right.
\]
By (75), for \( r \in [\tau - t + s + (k - 1)T, \tau - t + s + kT] \), we have

\[
||\phi_1(r)||^2_E = ||w_1(r)||^2 + ||\tilde{w}_1(r)||^2
\leq M^2_1(\theta_{-t+s+(k-1)T}\omega)e^{-2f_{r-t+s+(k-1)T}(\rho - \mu(|a\omega| + |a|^2))ds}, \tag{93}
\]

then for \( l \in \mathbb{N} \),

\[
\int_{\tau-t+s+(k-1)T}^{\tau-t+s+kT} ||\phi_1(\xi)||^2_E d\xi
\leq M^2_1(\theta_{-t+s+(k-1)T}\omega)
\times \int_{r}^{r+T} e^{-2f_{r}(\rho - \mu(|a\omega| + |a|^2))ds} dr
\leq M^2_2(\theta_{-t+s+(k-1)T}\omega).
\]

By (45), (69) and (90), for \( l \in \mathbb{N} \) and \( \tau \in \mathbb{R} \),

\[
E[M^2_2(\theta_{r}\omega)] \leq \frac{1}{2} E[M^1_4(\omega)]
+ \frac{1}{2} E \left( \int_{0}^{\infty} e^{-2f_{t}(\rho - \mu(|a\omega| + |a|^2))ds} dr \right)^2
\leq \frac{1}{2} c^{27}(1 + E[M^4_6(\omega)])
+ \frac{1}{2} E \left( \int_{0}^{\infty} e^{-2f_{t}(\rho + 2\mu f_{s}(\omega) + |a|^2)ds} dr \right)^2
\leq \frac{1}{2} c^{27} \left( 1 + e^{4\mu} \left( \frac{(4l - 1)}{l\rho} \right)^{4l-1} \frac{1}{l} \left( \frac{1}{\rho - 4\mu |a|} + \frac{1}{\rho - 4\mu |a|^2} \right) \right)
\leq \frac{1}{2} c^{27} \left( \frac{1}{\rho - 2\mu |a|} + \frac{1}{\rho - 2\mu |a|^2} \right)
\leq K_{0,l} < \infty. \tag{94}
\]

By Birkhoff ergodic Theorem, we have that for any fixed \( T > 0, s \in \mathbb{R} \) and \( \omega \in \Omega \) (in fact for a.e. \( \omega \in \Omega \)), \( \frac{1}{k} \sum_{i=1}^{k} M^2_2(\theta_{s+iT}\omega) \xrightarrow{k \to \infty} E[M^2_2(\theta_{s}\omega)] = K_{0,l} \), which implies that for \( \omega \in \Omega \), there exists a large integer \( k_{0,l}(\omega) < \infty \) such that

\[
\frac{K_{0,l}}{2} \leq \frac{1}{k} \sum_{i=1}^{k} M^2_2(\theta_{s+iT}\omega) \leq \frac{3K_{0,l}}{2}, \forall k \geq k_{0,l}(\omega), \forall s \in \mathbb{R}. \tag{95}
\]

Taking the expectation to (95) with \( k = k_{0,l}(\omega) \), we have \( \frac{1}{2} E[k_{0,l}] \leq k_{0,l}(\omega) \leq \frac{3}{2} E[k_{0,l}], k_{0,l}(\omega) < \infty \), thus, the constant \( E[k_{0,l}] < \infty \). Therefore, for \( \tau - t \leq r \leq \tau \) and \( \tau - r = mT + \bar{r}, m \in \mathbb{Z}, \bar{r} \in [0, T) \), we have that

(a) when \( m \geq k_{0,l}(\omega) \), then by (95),

\[
\int_{r}^{\tau} ||\phi_1(\xi)||^2_E d\xi \leq \left( \int_{r}^{r+T} + \int_{r+T}^{r+2T} + \ldots + \int_{r+mT}^{r+(m+1)T} \right) ||\phi_1(\xi)||^2_E d\xi
\]
ϕ

second decomposition of solutions of (47) with different initial data from first one.

\[ \text{For our purpose in this section, we need make the} \]

\[ \text{where } \bar{K}(m + 1) \leq \frac{K_{0,l}}{\sigma_1 T} \rho - r + \frac{K_{0,l}}{\sigma_1}; \]

(b) when \( 0 < m < K_{0,l}(\omega) \), then

\[ \int_r^\tau ||\phi_1(s)||_{E}^2 ds \leq \left( \int_{\tau - k_0 T}^{\tau} + \ldots + \int_{\tau - 2T}^{\tau} \right) ||\phi_1(s)||_{E}^2 ds \leq \frac{K_{0,l}}{\sigma_1} k_0(\omega). \]

Thus, by (93),

\[ \int_r^\tau ||w_1(s)||_{E}^2 ds \leq \int_r^\tau ||\phi_1(s)||_{E}^2 ds \leq \frac{K_{0,l}}{\sigma_1 T} (\rho - r) + K_1, \]  

where \( K_1 = \frac{3K_{0,l}}{\sigma_1} E[k_0,l] + \frac{K_{0,l}}{\sigma_1} \).

For \( t \geq T \), considering (73) at the interval \([r - T, r]\) with \( \varphi_2(r - T) = (0, 0)^T \), by (79), we have

\[ ||\varphi_2(r, r - T, \theta - t, \varphi_2(T))||_{E}^2 \leq M_2(T, \omega), \quad \forall r \geq \tau - t. \]  

(97)

So, combining (79) and (97), we can take \( \varphi_2 \) such that for every \( t \geq 0 \) and \( r \geq \tau - t \), the second inequality of (91) holds. The proof is completed.

\[ \square \]

4.2. Decomposition (II). For our purpose in this section, we need make the second decomposition of solutions of (47) with different initial data from first one. Let \( \phi(r) = \varphi_L(r) + \varphi_N(r) \), where \( \varphi_L(r) = (u_L, v_L)^T \) and \( \varphi_N(r) = (u_N, v_N)^T \) satisfy, respectively,

\[ \begin{cases} 
\dot{\varphi}_L + \Delta \varphi_L = F_1(\varphi_L, \theta - t \omega), & r > \tau - t, \\
\varphi_L(\tau - t, \tau - t, \theta - t \omega) = \varphi_{L,\tau - t} = (u_{\tau - t}, u_{\tau - t} + \varepsilon u_{\tau - t})^T, 
\end{cases} \]

(98)

and

\[ \begin{cases} 
\dot{\varphi}_N + \Delta \varphi_N = F_2(u_L, \varphi_N, u, \theta - t \omega), & r > \tau - t, \\
\varphi_N(\tau - t, \tau - t, \theta - t \omega) = (0, -a \varepsilon \theta - t \omega) u_{\tau - t})^T, & t \geq 0, 
\end{cases} \]

(99)

where \( v_j = u_{j,t} + \varepsilon u_{j} - a \varepsilon \theta - t \omega) u_{j}) \), \( j = L, N \).

According to the assumption in section 1, the initial data \( \varphi_{L,\tau - t} = (u_{\tau - t}, u_{\tau - t} + \varepsilon u_{\tau - t})^T \) in solution \( \varphi_L(r) = \varphi_L(r, r - t, \theta - t \omega, \varphi_{L,\tau - t}) \) of (98) is independent of \( \omega \), but for \( r > \tau - t \), the solution \( \varphi_L(r) \) depends on \( \omega \). Similar to Lemma 4.1, the component \( \varphi_L \) also decays “exponentially”.

Lemma 4.4. For every \( r \in \mathbb{R}, \omega \in \Omega \) and \( t \geq 0 \), there exists a constant \( M_L > 0 \) (independent of \( (\omega, t, \tau) \)) such that the solution \( \varphi_L(r) = \varphi_L(r, \tau - t, \theta - t \omega, \varphi_{L,\tau - t}) \) of (98) satisfies

\[ ||\varphi_L(\tau, \tau - t, \theta - t \omega, \varphi_{L,\tau - t}(\theta - t \omega))||_E \leq M_L e^{-\int_0^t (\rho - \mu)(a||z(\omega, t)| + a^2|z(\omega, t)|^2) ds}. \]

(100)

**Proof.** Since the initial data \( \varphi_{L,\tau - t} = (u_{\tau - t}, u_{\tau - t} + \varepsilon u_{\tau - t})^T = \varphi_{L,\tau - t}(\theta - t \omega) + (0, a \varepsilon \theta - t \omega) u_{\tau - t})^T \in B_0(\theta - t \omega) \) is independent of \( \omega \), by replacing \( \omega \) by \( \theta - t \omega \), we have

\[ ||\varphi_{L,\tau - t}||_E^2 \leq 2 ||\varphi_{L,\tau - t}(\theta - t \omega)||_E^2 + |a|^2 z^2(\omega)||u_{\tau - t}||^2 \leq c_28[1 + |a|^4 z^2(\omega) + M_0^4(\omega)]. \]
By taking the expectation, we have

$$||\varphi_{L, \tau-t}||^2_E \leq c_{28} \left( 1 + |a|^4 \frac{\Gamma(1+\frac{\beta}{2})}{\sqrt{\pi \alpha^2}} + \frac{4c_1^2}{\rho} \left( \frac{1}{\rho - 4\mu|\alpha|^2} + \frac{1}{\rho - 4\mu|\alpha|^2} \right) \right)$$

$$\leq M_{L,0}.$$  

Similar to (78), we have

$$||\varphi_L(r, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega))||^2_E \leq M^2_L e^{-\int_{t\tau}^{t\tau^2} (\rho+\mu(|\omega|+|\varphi|)^2 + |\alpha|^2 (\theta, \omega))^2)ds}$$

for $r \geq \tau-t$, where $M_{L,0}^2 = M_{L,0} + 2c_0 c_2 M_{L,0}^2$. The proof is completed.  

Based on Lemmas 4.1-4.4, we have the following ultimately pullback boundedness of component $\varphi_N$ in a “higher regular” space $E^r$.

**Lemma 4.5.** Assume that

$$|a| \leq \rho_0 = \min \left\{ \frac{\alpha}{4\sqrt{2\mu}}, \frac{\alpha}{8\mu}, \frac{\sqrt{\alpha \rho}}{8\mu} \right\}.$$  

Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, there exist $t_\rho(\omega) > 0$ and a tempered random variable $M_\rho(\omega) > 0$ such that $\varphi_N(\tau, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega))$ of (99) satisfies: for $t \geq t_\rho(\omega)$,

$$||A^{\rho + \frac{1}{2}} u_N(\tau, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega))||^2 + ||A^2 v_N(\tau, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega))||^2 \leq M_{\rho}^2(\omega).$$

**Proof.** Taking the inner product of (99) with $A^{2\rho} \varphi_N$ in $E$, we have

$$\frac{d}{dt} \left( ||A^{\rho+1/2} u_N||^2 + ||A^{2\rho} v_N||^2 + 2 \int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} \right)$$

$$+ \int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} u_N ||^2 dx$$

$$= \int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} v_N ||^2 dx$$

$$+ \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} u_N ||^2 dx$$

$$+ \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} v_N ||^2 dx$$

for $r \geq \tau-t$. By (A1), Lemma 4.3 and Hölder’s inequality, we have that for $r \geq \tau-t$,

$$\int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} u_N ||^2 dx \leq c_{28} \int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} v_N ||^2 dx$$

and

$$\int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} u_N ||^2 dx \leq c_{29} \int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} v_N ||^2 dx,$$

where

$$\int_U \{ f(u, \tau-t, \theta, \tau-\omega, \varphi_{\tau-t}(\theta, \omega)) \} A^{2\rho} u_N ||^2 dx$$

$$\leq c_{30} \left( e^{-2 \int_{t\tau}^{t\tau^2} (\rho+\mu(|\omega|+|\varphi|)^2 + |\alpha|^2 (\theta, \omega))^2)ds} \right) ||A^{\rho+1/2} u_N||^2$$

$$+ c_{30} \left( e^{-4 \int_{t\tau}^{t\tau^2} (\rho+\mu(|\omega|+|\varphi|)^2 + |\alpha|^2 (\theta, \omega))^2)ds} \right) ||A^{\rho+1/2} v_N||^2$$

$$+ \frac{c_2^2 \theta(\theta, \omega)}{4\sqrt{\lambda_1}} ||A^{\rho+1/2} u_N||^2.$$
\[
\int_U |u_{L,t}| |w_1| |u_N| |A^{2\nu} u_N| dx \\
\leq c_{31} \left( e^{-2 \int_{-\tau}^t (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|)^2) ds} + \frac{\sigma^2}{2 \lambda_1} \right) ||A^{\nu + \frac{1}{2}} u_N||^2 \\
+ c_{31} \left( e^{-4 \int_{-\tau}^t (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|)^2) ds} + \frac{\sigma^2}{4 \lambda_1} \right) ||A^{\nu + \frac{1}{2}} u_N||^2 \\
+ c_{32} (||w_1||^2 + ||w_1||^2) ||A^{\nu + \frac{1}{2}} u_N||^2 + \frac{\sigma^2 (\theta - \tau)^2}{4 \lambda_1} ||A^{\nu + \frac{1}{2}} u_N||^2,
\]

\[
\int_U |u_{N,t}| |w_2| |u_N| |A^{2\nu} u_N| dx \\
\leq c_{33} \left( 1 + z^{16}(\theta - \tau, \omega) + M_0^{32}(\theta - \tau, \omega) \right) \\
+ c_{33} \left( e^{-32 \int_{-\tau}^t (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|)^2) ds} + ||A^{\nu + \frac{1}{2}} u_N||^2 \right) \\
+ \frac{\varepsilon}{16} ||A^{\nu + \frac{1}{2}} u_N||^2,
\]

\[
\int_U |u_{N,t}| |w_1| |u_N| |A^{2\nu} u_N| dx \\
\leq c_{35} ||w_1||^2 \left( ||A^{\nu + \frac{1}{2}} u_N||^2 + ||A^{\nu} u_N||^2 \right) + \frac{\varepsilon}{16} ||A^{\nu + \frac{1}{2}} u_N||^2 \\
+ c_{36} \left( 1 + z^{8}(\theta - \tau, \omega) + M_0^{16}(\theta - \tau, \omega) \right) \\
+ c_{36} \left( e^{-16 \int_{-\tau}^t (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|)^2) ds} + ||w_1||^2 \right),
\]

\[
\int_U |u_{N,t}| |w_2| |u_N| |A^{2\nu} u_N| dx \\
\leq \frac{\varepsilon}{16} ||A^{\nu + \frac{1}{2}} u_N||^2 + c_{43} ||A^{\nu + \frac{1}{2}} w_2||^8 \\
+ c_{37} \left( 1 + z^{8}(\theta - \tau, \omega) + M_0^{16}(\theta - \tau, \omega) \right) \\
+ c_{37} e^{-16 \int_{-\tau}^t (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|)^2) ds},
\]

\[
\int_U f_2 u(x) u_{L,t} \cdot A^{2\nu} u_N dx \\
\leq c_{38} \left( 1 + z^{8}(\theta - \tau, \omega) + M_0^{8p}(\theta - \tau, \omega) + e^{-4 \int_{-\tau}^t (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|)^2) ds} \right) \\
+ \frac{\varepsilon}{16} ||A^{\nu + \frac{1}{2}} u_N||^2.
\]

Thus, we have

\[
\frac{d}{dt} y_N(r) + m_N(r)y_N(r) \leq q_N(\theta - \tau), \quad \forall r \geq \tau - t,
\]

where

\[
y_N = ||A^{\nu + 1/2} u_N||^2 + ||A^{\nu} u_N||^2 + 2 \int_U [f(u, x) - f_1(u, x)] A^{2\nu} u_N dx,
\]
\[ m_N(r) = \varepsilon - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2) - c_{39}(|w_1(r)|^2 + ||w_1(r)||^2) \]
\[ \quad - c_{40} e^{-2 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) ds} \]
\[ \quad - c_{40} e^{-4 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) ds}, \]
\[ q_N(\theta, \tau - \omega)) = c_{41} (1 + \varepsilon^8 (\theta, \tau - \omega) + M_0^6 (\theta, \tau - \omega)) \]
\[ + c_{41} e^{-16 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) ds} \]
\[ + c_{42} \left( ||w_1(r)||^8 + ||A^{\alpha + \frac{1}{2}} w_2(r)||^8 \right), \]  
\[ (106) \]
\[ y_N(\tau - t, \tau - t, \theta(\tau - \omega), \varphi_{\tau - t}(\theta - \omega)) \leq ||h||^2 \cdot \varepsilon^2 (\tau - \omega), \]  
\[ (107) \]
\[ \int_U [f(u, x) - f_1(u, x)] A^{2\nu} u_N dx \]
\[ \leq c_{43} (1 + M_0^4 (\theta - \omega) + e^{-4 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) ds} \]
\[ \leq R_N (\theta, \tau - \omega), \forall \tau \geq t - t, \]  
\[ (108) \]

By applying Gronwall’s inequality to (104) on \([\tau - t, r] (r \geq \tau - t)\), we have
\[ y_N(r) \leq y_N(\tau - t) e^{-\int_{\tau - t}^r m_N(s) ds} + \int_{\tau - t}^r q_N(\theta - \omega)) e^{-\int_{\tau - t}^r m_N(s) ds} d\xi. \]  
\[ (109) \]

Setting \( r = \tau, \) we have
\[ y_N(\tau) \leq y_N(\tau - t) e^{-\int_{\tau - t}^r m_N(s) ds} + \int_{\tau - t}^r q_N(\theta - \omega)) e^{-\int_{\tau - t}^r m_N(s) ds} d\xi. \]  
\[ (110) \]

Taking
\[ T = \hat{T} = \frac{4c_{39}(K_{0,1} + K_{0,2})}{\sigma_1 \varepsilon} \]  
\[ \text{in (91),} \quad K = K_1 + K_2, \]  
\[ (111) \]
then by (91) and (101), we have that for \( \tau - t \leq r \leq t, \)
\[ \int_r^\tau \left( ||w_1(s)||^2 + ||w_1(s)||^4 \right) ds \leq \frac{\varepsilon}{4c_{39}} (\tau - r) + \hat{K}, \quad ||A^{\alpha + \frac{1}{2}} w_2(r)||^2 \leq M_2(\hat{T}, \omega). \]

By (63), there exists a \( t_1(\omega) > 0 \) such that
\[ \int_{-t}^s \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2) dl \leq \left( \mu \left( \frac{|a|}{\sqrt{2\alpha}} + \frac{|a|^2}{2\alpha} \right) + \frac{\rho}{2} \right) (s + t) \]
for \( s + t \geq t_1(\omega). \) Then for \( 0 \leq t \leq t_1(\omega), \) we have
\[ \int_{-t}^0 e^{-2 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) dl} ds \leq \frac{1}{2\rho} e^2 \left( \rho + \mu \left( \frac{|a|}{\sqrt{2\alpha}} + \frac{|a|^2}{2\alpha} \right) + \frac{\rho}{2} \right) t_1(\omega) < \infty, \]
and
\[ \int_{-t}^0 e^{-4 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) dl} ds \leq \frac{1}{4\rho} e^4 \left( \rho + \mu \left( \frac{|a|}{\sqrt{2\alpha}} + \frac{|a|^2}{2\alpha} \right) + \frac{\rho}{2} \right) t_1(\omega) < \infty. \]
Thus for \( t \geq t_1(\omega), \) we have
\[ \int_{-t}^0 e^{-2 \int_{-t}^r (\rho - \mu(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)) dl} ds \leq \frac{1}{2\rho} e^2 \left( \mu \left( \frac{|a|}{\sqrt{2\alpha}} + \frac{|a|^2}{2\alpha} \right) + \frac{\rho}{2} \right) t_1(\omega) + \frac{1}{\rho} \]
\[ < \infty, \]  
\[ (111) \]
\[ \int_{-t}^{0} e^{-4 \int_{-t}^{0} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) } ds \leq \frac{1}{4\rho} e^{\frac{1}{4} \left(\frac{1}{\rho} \left(\frac{|a|}{\sqrt{|a|}} + \frac{|a|^2}{\sqrt{k}}\right) + \frac{t}{2}\right)} t_1(\omega) + \frac{1}{2\rho} < \infty. \]

So,

\[ K_1(\omega) = c_{40} \sup_{t \geq 0} \int_{-t}^{0} e^{-2 \int_{-t}^{0} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) } ds \]

\[ + c_{40} \sup_{t \geq 0} \int_{-t}^{0} e^{-4 \int_{-t}^{0} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) } ds \]

\[ \leq c_{40} \left( \frac{1}{\rho} e^{\frac{1}{4} \left(\frac{1}{\rho} \left(\frac{|a|}{\sqrt{|a|}} + \frac{|a|^2}{\sqrt{k}}\right) + \frac{t}{2}\right)} t_1(\omega) + \frac{2}{\rho} \right) < \infty. \] (112)

For \( \tau \geq \xi \geq \tau - t \), we have

\[ \int_{\xi}^{\tau} m_N(s) ds \]

\[ \geq \int_{\xi}^{\tau} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) ds - c_{39} K \]

\[ - c_{40} \int_{-t}^{0} e^{-2 \int_{-t}^{0} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) } ds \]

\[ - c_{40} \int_{-t}^{0} e^{-4 \int_{-t}^{0} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) } ds \]

\[ \geq \int_{\xi - t}^{\tau} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) ds - K_1(\omega) - c_{39} K. \] (113)

Thus,

\[ y_N(\tau - t) e^{- \int_{-t}^{0} m_N(s) ds} \]

\[ \leq \cdot 2(\theta - t) e^{- \int_{-t}^{0} (\rho - \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2)) ds + K_1(\omega) + c_{39} K \]

\[ \lim_{t \to +\infty} = 0, \] (114)

and

\[ \int_{-t}^{0} q_N(\theta \xi) e^{|\xi| + \int_{-t}^{0} \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2) ds } d\xi \]

\[ \leq \left( \int_{-t}^{0} q_N(\theta \xi) e^{\frac{|\xi|}{2} + \int_{-t}^{0} \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2) ds } d\xi \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{2} \left( \int_{-t}^{0} q_N(\theta \xi) e^{\frac{|\xi|}{2} + \int_{-t}^{0} \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2) ds } d\xi + K_2(\omega) \right), \]

where

\[ K_2(\omega) = \int_{-\infty}^{0} e^{\frac{|\xi|}{2} + \int_{-t}^{0} \mu (|a| \cdot |z(\theta,s)| + |a|^2 |z(\theta,s)|^2) ds } dr < \infty. \] (115)
Thus, similar to (111), for $t \geq 0$,

$$
\int_{-t}^{0} e^{-\frac{32}{\rho} \int_{mT}^{(m-1)T} (\rho - \mu(|\omega| + |\alpha|^2 z(\theta, \omega)^2)) ds} e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr}
\leq \left( \frac{1}{64} e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \right) K_3(\omega) < \infty,
$$

and

$$
\int_{-\infty}^{0} [z^1(\theta, \omega) + M_0^{32}(\theta, \omega)] e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \leq K_4(\omega) < \infty,
$$

$$
\int_{-\infty}^{0} M_5^{6}(T, \omega) e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \leq \sqrt{\frac{2}{\rho}} M_5^{8}(T, \omega) < \infty.
$$

By (96), we have

$$
\int_{-T}^{-mT} ||w_1(r)||_1^{16} e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \leq \int_{-T}^{-mT} M_1^{16}(\theta, mT) e^{-\frac{32}{\rho} \int_{mT}^{(m-1)T} (\rho - \mu(|\omega| + |\alpha|^2 z(\theta, \omega)^2)) ds} e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr}
\leq M_3(\theta, mT) e^{-\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr}, \quad \forall m \geq 1,
$$

where

$$
M_3(\omega) = \sqrt{\frac{2}{\varepsilon}} e^{\frac{32}{\rho} T} M_1^{16}(\omega) \left( \int_{0}^{\infty} e^{-\frac{32}{\rho} \int_{mT}^{(m-1)T} (\rho - \mu(|\omega| + |\alpha|^2 z(\theta, \omega)^2)) ds} dr \right)^{\frac{1}{4}}.
$$

Thus,

$$
\int_{-\infty}^{0} ||w_1(r)||_1^{16} e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \leq \left( \sum_{m=1}^{+\infty} M_3(\theta, mT) e^{-\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \right)
\leq \sum_{m=1}^{+\infty} M_3(\theta, mT) e^{-\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr}.
$$

Since $M_3(\omega)$ is tempered and $M_3(\theta, \omega)$ is continuous in $t$, by [1], there exists a tempered random variable $\zeta(\omega)(> 0)$ such that

$$
M_3(\theta, mT) \leq \zeta(\omega) e^{\frac{32}{\rho} mT}, \quad \forall m \geq 1,
$$

then

$$
\int_{-\infty}^{0} ||w_1(r)||_1^{16} e^{\frac{32}{\rho} \int_{mT}^{(m-1)T} \frac{\omega_1}{\sqrt{\rho}} dr} \leq \zeta(\omega) \sum_{m=1}^{+\infty} e^{\frac{32}{\rho} mT} = \frac{e^{-\frac{32}{\rho} T}}{1 - e^{-\frac{32}{\rho} T}} \zeta(\omega) = K_6(\omega) < \infty.
$$
Thus,
\[
\int_{t-r}^{t} q_N(\theta_{r-r}\omega) e^{-\int_{t-r}^{t} m_N(s) \, ds} \, dr \\
\leq e^{K_1(\omega)+c_{49}K} \int_{t-r}^{t} q_N(\theta_{r-r}\omega) e^{-\int_{t-r}^{t} (\rho-\mu(|a| \cdot |z(\theta, \omega)|+|a|^2|z(\theta, \omega)|^2)) \, ds} \, dr \\
\leq \frac{1}{2} e^{K_1(\omega)+c_{49}K} (K_7(\omega) + K_2(\omega)), 
\]
where
\[
K_7(\omega) = c_{48} \sup_{t \geq 0} \int_{-t}^{0} e^{-32 f_{-t}^{0} (\rho-\mu(|a| \cdot |z(\theta, \omega)|+|a|^2|z(\theta, \omega)|^2)) \, ds} e^{2T} dr \\
+ c_{47} \int_{-\infty}^{0} [1 + z^{16}(\theta, \omega) + M_0^{32}(\theta, \omega) + ||w_1(r)||^2 + M_0^2(T, \omega)] e^{2r} dr \\
\leq c_{48}[1 + K_3(\omega) + K_4(\omega) + K_5(\omega) + K_6(\omega)] < \infty.
\]
By (109), (107), (108) and (105), we have that for \( t \geq 0 \),
\[
||A^{1/2} u_N(\tau, \tau - t, \theta_{-t}, \omega, \varphi_{-t})||^2 + ||A^T v_N(\tau, \tau - t, \theta_{-t}, \omega, \varphi_{-t})||^2 \\
\leq 2y_N(\tau, \tau - t, \theta_{-t}, \omega, \varphi_{-t}) + 2R_N(\omega) \\
\leq 2||h||_1^2 \cdot z^2(\theta_{-t}, \omega) e^{-\int_{t}^{0} (\rho-\mu(|a| \cdot |z(\theta, \omega)|+|a|^2|z(\theta, \omega)|^2)) \, ds} + K_1(\omega) + c_{47}K \\
+ 2c_{49} \left(1 + M_0^4(\omega) + e^{-4 \int_{t}^{0} (\rho-\mu(|a| \cdot |z(\theta, \omega)|+|a|^2|z(\theta, \omega)|^2)) \, ds}\right) \\
+ \frac{1}{2} e^{K_1(\omega)+c_{49}K} (K_7(\omega) + K_2(\omega)).
\]
By (114), there exists a random variable \( t_\nu(\omega) \geq 0 \) such that for \( t \geq t_\nu(\omega) \),
\[
\begin{cases} 
0 \leq 2||h||_1^2 \cdot z^2(\theta_{-t}, \omega) e^{-\int_{t}^{0} (\rho-\mu(|a| \cdot |z(\theta, \omega)|+|a|^2|z(\theta, \omega)|^2)) \, ds} + K_1(\omega) + c_{47}K \leq \frac{1}{2}, \\
0 \leq 2c_{49} e^{-4 \int_{t}^{0} (\rho-\mu(|a| \cdot |z(\theta, \omega)|+|a|^2|z(\theta, \omega)|^2)) \, ds} \leq \frac{1}{2}.
\end{cases}
\]
Set
\[
M_\nu^2(\omega) = 1 + 2c_{49} \left(1 + M_0^4(\omega)\right) + \frac{1}{2} e^{K_1(\omega)+c_{49}K} (K_7(\omega) + K_2(\omega)),
\]
then \( M_\nu(\omega) \) is tempered and (102) holds by (124) and (125). The proof is completed.

5. **Existence of random attractor.** According to [53], we knew that a family \( K = \{K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) of nonempty subsets of \( E \) is called a measurable \( D \)-pullback attracting set for \( \Phi \) if (i) \( K \) is measurable with respect to \( F \) in \( \omega \); (ii) for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( B \in D \), \( \lim_{t \to +\infty} d_h(\Phi(t, \tau - t, \theta_{-t}, \omega, B(\tau - t, \theta_{-t}, \omega)), K(\tau, \omega)) = 0 \). A family \( A = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D \) is called a \( D \)-pullback random attractor for \( \Phi \) if (i) \( A(\tau, \omega) \) is measurable in \( \omega \) and compact in \( E \) for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \); (ii) \( A \) is invariant, i.e., for every \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( t \geq 0 \), \( \Phi(t, \tau, \omega, A(\tau, \omega)) = A(t + \tau, \theta_{t}, \omega) \); (iii) \( A \) is an attracting set in \( D \). From [53, 65], it follows that if \( \Phi \) has a compact measurable (w.r.t. \( F \)) \( D \)-pullback attracting set \( K \in D \), then \( \Phi \) has a unique \( D \)-pullback random attractor \( A \) in \( D \) given by \( A(\tau, \omega) = \bigcap_{t \geq 0} \bigcup_{t \geq r} \Phi(t, \tau - t, \theta_{-t}, K(\tau - t, \theta_{-t})) \) for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \).
Theorem 5.1. Suppose condition (A1) and (101) hold, then the cocycle Φ possesses a compact measurable $D(E)$-pullback attracting set $A_{\nu} \subset D(E)$ and a $D(E)$-pullback random attractor $A \subset D(E)$ satisfying $A(\tau,\omega) \subset A(\nu,\omega) \cap B_0(\omega)$ for every $\tau \in \mathbb{R}$, $\omega \in \Omega$.

Proof. For every $\tau \in \mathbb{R}$, $\omega \in \Omega$, let $A_{\nu}(\tau,\omega) = A_{\nu}(\omega)$ (independent of $\tau$) be the closed ball of $V_{1+2\nu} \times V_{2\nu}$ of radius $M_{\nu}(\omega)$ centered at 0, where $M_{\nu}(\omega)$ is defined by (126).

Since the embedding $V_{1+2\nu} \times V_{2\nu} \hookrightarrow H^1_0(U) \times L^2(U)$ is compact, $A_{\nu}(\tau,\omega)$ is compact in $E$. For every $B \subset D(E)$, by Lemma 3.1, there exists $T_B = T_B(\tau,\omega) \geq 0$ such that $\varphi(\tau,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)) \subset B_0(\omega)$ for $t \geq T_B$. Take $t > T_B + T_B(\tau,\omega)$ and set $t_B = t - T_B > T_B(\tau,\omega) \geq 0$. By the cocycle property of $\Phi$, we have $\varphi(\tau,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)) \subset \varphi(\tau-t_B,\theta_{-t_B}\omega, B_0(\theta_{-t_B}\omega)) \subset B_1(\tau,\omega)$. For $t > T_B + T_B(\tau,\omega)$, take any $\varphi_{\tau-t}(\theta_{-t}\omega) \in B(\tau-t,\theta_{-t}\omega)$, it follows from Lemma 4.5 that $\varphi_N(\tau-t,\theta_{-t}\omega, \varphi_{\tau-t}) = \varphi(\tau-t,\theta_{-t}\omega, \varphi_{\tau-t}) - \varphi_L(\tau-t,\theta_{-t}\omega, \varphi_{\tau-t}) \subset \Lambda_{\nu}(\tau,\omega)$. Thus, by Lemma 4.4, we have that for $t > T_B + T_B(\tau,\omega)$,
\[
\inf_{\nu \in \Lambda_{\nu}(\tau,\omega)} \| \varphi(\tau,\tau-t,\theta_{-t}\omega, \varphi_{\tau-t}(\theta_{-t}\omega)) - \psi \|_{E}^{2} \leq M_{LE} \int_{-T_B}^{\tau} (\rho - \mu |a|^2 + |a|^2 |z(\theta_{-t}\omega)|^2) ds .
\]

This implies that
\[
d_{E}(\varphi(\tau,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)), \Lambda_{\nu}(\tau,\omega)) \leq M_{LE} \int_{-T_B}^{\tau} (\rho - \mu |a|^2 + |a|^2 |z(\theta_{-t}\omega)|^2) ds \rightarrow 0 ,
\]
which is compact, $\Lambda_{\nu}(\tau,\omega)$ is attractive. So, $\Phi$ possesses a $D(E)$-pullback random attractor $A \subset D(E)$ such that $A(\tau,\omega) \subset A(\nu,\omega) \cap B_0(\omega) \subset V_{1+2\nu} \times V_{2\nu}$, The proof is completed.

6. Upper semicontinuity of random attractors. In this section, we regard the coefficient $a \in \mathbb{R}$ of random term in system (47) as a parameter. Thus by section 2 and Theorem 5.1, we can define a family of cocycles $\{\Phi^a(t,\tau,\omega)\}_{a \in \mathbb{R}}$ associated to (47) which possess a family of random attractors $\{A^a(\tau,\omega)\}_{a \in \mathbb{R}}$. Now let us consider the upper semicontinuity of random attractors $\{A^a(\tau,\omega)\}_{a \in \mathbb{R}}$ as $a \rightarrow 0$ basing on a criteria established by Wang [52, 55].

When $a = 0$, the system (47) reduces to the following deterministic non-autonomous system in $E$:
\[
\phi^0 + \Lambda \phi^0 = F_0(\phi^0, t), \quad \phi^0 = (u^0, u^0_1 + \varepsilon u^0_2)^T, \quad t \geq \tau, \quad \tau \in \mathbb{R},
\]
where
\[
\phi^0 = \begin{pmatrix} v^0 \\ w^0 \end{pmatrix}, \quad F_0(\phi^0, t) = \begin{pmatrix} 0 \\ -f(u^0, x) + g(x, t) \end{pmatrix}.
\]

It follows from sections 2-5 (or [58, 62]) that under condition (A1) and (101), the solutions $\phi^0(t, \tau, \phi^0)$ of (128) generate a continuous process $\Phi^0(t, \tau)$: $\phi^0 \rightarrow \phi^0(t, \tau, \phi^0)$, $E \rightarrow E$, $t \geq \tau$.

Theorem 6.1. For the cocycles $\{\Phi^a(t, \tau, \omega)\}_{a \leq \rho, a \neq 0}$ and the process $\{\Phi^a(t, \tau)\}_{t \geq \tau}$, we have the following results:

(i) The process $\{\Phi^0(t, \tau)\}_{t \geq \tau}$ has a closed uniform pullback absorbing set $B^0_0 = \{ \varphi \in E : \| \varphi \|_E \leq R_0 \} \subset E$, where $R_0 = 2 \varepsilon_{10}$ (independent of $\tau$) and has a pullback attractor $\{A^0(\tau)\}_{\tau \in \mathbb{R}}$ satisfying: (a) for each $\tau \in \mathbb{R}$, $A^0(\tau)$ is compact and
\[ A^0(\tau) \subseteq B^0_0; \quad \text{(b) } \Phi^0(t,\tau)A^0(\tau) = A^0(t) \text{ for } t \geq \tau; \quad \text{(c) for given } \tau \in \mathbb{R} \text{ and any bounded set } B \subseteq \mathbb{C}, \lim_{t \to +\infty} d_h(\Phi^0(\tau, t - t)B, A^0(\tau)) = 0. \]

(ii) For each \( a \in [-\rho_0, \rho_0] \setminus \{0\} \), the pullback random attractor \( \{A^a(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega} \) for \( \Phi^a(t, \tau, \omega) \) and pullback random absorbing set \( \{B^a_0(\tau, \omega) = B_0(0, M_0(\omega))\}_{\tau \in \mathbb{R}, \omega \in \Omega} \) for \( \Phi^a(t, \tau, \omega) \) satisfy:

\[
\limsup_{a \to 0} ||A^a(\tau, \omega)||_E \leq \limsup_{a \to 0} ||B^a_0(\tau, \omega)||_E \leq R^0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega. \quad (129)
\]

(iii) For every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), \( \cup_{|a| \leq \rho_0, a \neq 0} A^a(\tau, \omega) \cup A^0(\tau) \) is precompact in \( E \).

(iv) For every \( t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega \), \( a_n \in [-\rho_0, \rho_0] \setminus \{0\} \) with \( a_n \to 0 \) and \( \varphi_n, \varphi \in E \) with \( \varphi_n \to \varphi \), it holds \( \lim_{a \to \infty} \Phi^a_n(t, \tau - t, \theta_{-1}, \varphi_n) = \Phi^0(t, \tau - t, \varphi) \).

(v) For given \( \tau \in \mathbb{R}, \omega \in \Omega \),

\[
\lim_{a \to 0} d_h(A^a(\tau, \omega), A^0(\tau)) = \sup_{\varphi \in A^0(\tau), \psi \in A^0(\tau)} \inf_{\varphi \neq \psi} ||\varphi - \psi||_E = 0.
\]

**Proof.** (i) It is true from the proof of Lemmas 3.1, 4.1-4.5 and [9].

(ii) It follows from Lemma 3.1 and Theorem 5.1 that for given \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( a \in [-\rho_0, \rho_0] \setminus \{0\} \), \( A^a(\tau, \omega) \subseteq B^a_0(\tau, \omega) = B_0(0, M_0(\omega)) \subseteq E \). This shows

\[
\limsup_{a \to 0} ||A^a(\tau, \omega)||_E \leq \limsup_{a \to 0} ||B^a_0(\tau, \omega)||_E \leq \limsup_{a \to 0} ||M_0^2(\omega)||_E = R^0. \quad (130)
\]

(iii) Given \( a \in [-\rho_0, \rho_0] \setminus \{0\} \). For every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), by the proof of Theorem 5.1, \( A^a(\tau, \omega) \subseteq A^0(\omega) = \Lambda_0(\omega) \subseteq E^\nu \). Moreover, by (126), \( M_0^\nu(\omega) \) is increasing in \( |a| \), i.e., \( \Lambda_0^a(\omega) \subseteq \Lambda_0^\nu(\omega) \) for \( a \in [-\rho_0, \rho_0] \setminus \{0\} \). Thus,

\[
\cup_{|a| \leq \rho_0, a \neq 0} A^a(\tau, \omega) \cup A^0(\tau) \subseteq \Lambda_0^\nu(\omega) \cup A^0(\tau) \subseteq E.
\]

So, \( \cup_{|a| \leq \rho_0, a \neq 0} A^a(\tau, \omega) \cup A^0(\tau) \) is precompact in \( E \).

(iv) Given \( a \in [-\rho_0, \rho_0] \). For every \( t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega \), let \( \varphi^a(r, \tau - t, \theta_{-1}, \varphi_{-1}^a(\theta_{-1})) \) and \( \Phi^0(r, \tau - t, \varphi_{-1}^0(\theta_{-1})) \) \( (r \geq \tau - t) \) be the solutions of (47) and (128) with initial data \( \varphi_{-1}^a(\theta_{-1}) \) and \( \varphi_{-1}^0(\theta_{-1}) \), respectively. Set \( Y = \varphi^a - \varphi^0 = (\xi, \zeta) = (u^a - u^0, v^a - v^0) \), then

\[
\dot{Y} + AY = F^n(\varphi^a, \theta_{-1}, \omega, r) - F^n(\varphi^0, r), \quad Y_{-1}(\theta_{-1}) = \varphi_{-1}^a(\theta_{-1}) - \varphi_{-1}^0(\theta_{-1}). \quad (131)
\]

for \( r \geq \tau - t \), where

\[
F^n(\varphi^a, \theta_{-1}, r) - F^n(\varphi^0, r) = \left( \begin{array}{c}
z(\theta_{-1})u^a
az(\theta_{-1})u^a - az(\theta_{-1})v^a + f(u^0, x) - f(u^a, x)
\end{array} \right).
\]

Taking the inner product of (131) with \( Y \) in \( E \), we have

\[
\frac{1}{2} \frac{d}{dt} ||Y||^2 + \frac{\xi}{2} ||\xi||^2 + ||\zeta||^2 = (az(\theta_{-1})u^a, \xi_1) + (f(u^0, x) - f(u^a, x), \zeta) \quad (132)
\]

By (60),

\[
\max\{||u^a||^2_1, ||u^0||^2_1, ||v^a||^2_1, ||v^0||^2_1\} \leq Q_1(r, \tau - t, \theta_{-1}) \leq Q_1(r, \tau - t, \theta_{-1})
\]

\[
= (||\varphi_{-1} - \varphi_{-1}(\theta_{-1})||^2_1 + 2c_0(||u_{-1}||^2_1 + ||u_{-1}||^4_1) + c_0)
\times e^{-\int_{\tau-t}^{\tau} (r - \mu ||a||^2 z(\theta_{-1})) + ||a||^2 ||z(\theta_{-1})||^2) ds}
\]

\[
+c_{10} \int_{\tau-t}^{\tau} e^{-\int_{\tau-t}^{\tau} (r - \mu ||a||^2 z(\theta_{-1})) + ||a||^2 ||z(\theta_{-1})||^2) ds}
\]
is continuous in \( r \) but independent of \( a \), and
\[
(2\varepsilon - az(\theta_{r-\tau}\omega))az(\theta_{r-\tau}\omega)u^a - az(\theta_{r-\tau}\omega)\nu^a, \zeta)
\leq |a|[(\varepsilon + \frac{1}{2})z(\theta_{r-\tau}\omega) + \rho z(\theta_{r-\tau}\omega)^2](||u^a||^2 + 3||\nu^a||^2 + 2||\nu^0||^2), \tag{133}
\]
\[
(az(\theta_{r-\tau})u^a, \xi) \leq |a| \cdot \frac{1}{2}z(\theta_{r-\tau}\omega)(||u^a||^2 + ||\xi||^2), \tag{134}
\]
\[
(f(u^0, x) - f(u^a, x), \zeta) \leq \frac{1}{2\alpha}c_{50} \int_U (1 + ||u^0||^4 + ||u^a||^4)||\zeta||^2 dx + \frac{\alpha}{2}||\zeta||^2
\leq \frac{1}{2\alpha}c_{51}(1 + ||u^0||^4 + ||u^a||^4)||\zeta||^2 + \frac{\alpha}{2}||\zeta||^2
\leq Q_2(r, \tau - t, \theta_{r-\tau}\omega)||\zeta||^2 + \frac{\alpha}{2}||\zeta||^2. \tag{135}
\]

By \text{(133)-(135)}, it follows that
 \[
\frac{d}{dt}||Y||^2_E \leq Q_3(r, \tau - t, \theta_{r-\tau}\omega)\|\tilde{\psi}\|^2_E + |a| \cdot Q_4(r, \tau - t, \theta_{r-\tau}\omega), \quad r \geq \tau - t, \tag{136}
\] where \( Q_3(r, \tau - t, \theta_{r-\tau}\omega) \), \( Q_4(r, \tau - t, \theta_{r-\tau}\omega) \) are continuous in \( r \) but independent of \( a \) and
\[
||Y(r)||^2_E = ||\varphi^a(r, \tau - t, \theta_{r-\tau}\omega, \varphi^a_{r-t}(\theta_{r-\tau}\omega)) - \varphi^0(r, \tau - t, \varphi^0_{r-t})||^2_E
= ||\xi(r)||^2 + ||\zeta(r)||^2.
\]

By applying Gronwall inequality to \text{(136)} on \([\tau - t, \tau]\), we have
\[
||\varphi^a(r, \tau - t, \theta_{r-\tau}\omega, \varphi^a_{r-t}(\theta_{r-\tau}\omega)) - \varphi^0(r, \tau - t, \varphi^0_{r-t})||^2_E
\leq ||\varphi^a_{r-t}(\theta_{r-\tau}\omega) - \varphi^0_{r-t}||^2_E + \int_{\tau - t}^{\tau} Q_3(r, \tau - t, \theta_{r-\tau}\omega) dr + |a| \int_{\tau - t}^{\tau} Q_4(r, \tau - t, \theta_{r-\tau}\omega) dr. \tag{137}
\]

From \text{(137)}, we see that for every \( \tau \in \mathbb{R}, \omega \in \Omega, t \geq 0, |a_n| \to 0, \) and \( \varphi^a_{r-t}(\theta_{r-\tau}\omega), \varphi^0_{r-t} \in E \) with \( \varphi^a_{r-t}(\theta_{r-\tau}\omega) \to \varphi^0_{r-t} \), it holds that:
\[
\lim_{n \to \infty} \varphi^a_{r-t}(\tau, \tau - t, \theta_{r-\tau}\omega, \varphi^a_{r-t}(\theta_{r-\tau}\omega)) = \varphi^0(\tau, \tau - t, \varphi^0_{r-t}). \tag{138}
\]
(v) It is proved by Proposition 2.15 of [52] (or Theorem 2.2 in [64]). The proof is completed. \( \square \)

7. Regularity of random attractor. In this section, we suppose \((A1)\) and \text{(101)} hold, we will show that \( \mathcal{A}(\tau, \omega) \) is included in the bounded ball of \( E^1 \) basing on the “iteration” method. First we present the following result.

Lemma 7.1. For given \( \tau \in \mathbb{R}, \omega \in \Omega, \) assume that \( B_{\nu}(\tau, \omega) \subseteq B_1(\tau, \omega) \) and \( B_\nu(\tau, \omega) \in \mathcal{D}(E^\nu) \), where \( \nu \) is as in \text{(80)}. Then there exist \( t_{\nu}(\omega) > 0 \) and a tempered random variable \( M_\nu(\omega) > 0 \) such that the solution \( \varphi(\tau, \tau - t, \theta_{r-\tau}\omega, \varphi_{r-t}(\theta_{r-\tau}\omega)) \) of \text{(47)} with \( \varphi_{r-t}(\theta_{r-\tau}\omega) \in B_\nu(\tau - t, \theta_{r-\tau}\omega) \) satisfies
\[
||\varphi(\tau, \tau - t, \theta_{r-\tau}\omega, \varphi_{r-t}(\theta_{r-\tau}\omega))||^2_E \leq \tilde{M}_\nu^2(\omega), \quad \forall t \geq t_{1\nu}(\omega). \tag{139}
\]
Thus, by (140) we have
\[ r \]
where, for \( r \geq t - \tau \),
\[
\int_U f'(u, x) u_t A^{2\nu} u dx \leq c_{52} \int_U |u_t| (1 + |w_1|^2 + |w_2|^2) A^{2\nu} u dx,
\]
\[
\int_U |u_t| \cdot |w_1|^2 \cdot A^{2\nu} u dx \leq c_{53} \left( 1 + z^2(\theta_{r-\tau} - \omega) + M_0^4(\theta_{r-\tau} - \omega) \right),
\]
\[
\int_U |u_t| \cdot |w_2|^2 \cdot A^{2\nu} u dx \leq \frac{\varepsilon}{16} |A^{\nu + \frac{1}{2}} u|^2 + c_{54} |A^{\nu + \frac{1}{2}} w_2|^2 + c_{55} \left( 1 + z^8(\theta_{r-\tau} - \omega) + M_0^{16}(\theta_{r-\tau} - \omega) \right),
\]
\[
\int_U f(u, x) A^{2\nu} u dx \leq c_8 \int_U (1 + |u|^3) \cdot A^{2\nu} u dx \leq c_{56} [1 + M_0^4(\theta_{r-\tau} - \omega)].
\]
Thus, by (140), we have
\[
\frac{d}{dt} \tilde{g}_1(r) + \tilde{m}_1(r) \tilde{g}_1(r) \leq \tilde{q}_1(\theta_{r-\tau} - \omega), \quad \forall r \geq t - \tau,
\]
where
\[
\tilde{g}_1 = ||A^{\nu + \frac{1}{2}} u||^2 + ||A^{\nu} v||^2 + 2 \int_U f(u, x) A^{2\nu} v dx,
\]
\[
\tilde{q}_1(\theta_{r-\tau} - \omega) = c_{57} \left[ 1 + z^{8}(\theta_{r-\tau} - \omega) + M_0^{16}(\theta_{r-\tau} - \omega) + |w_1|^6 + M_2^4(T, \omega) \right],
\]
\[
\tilde{m}_1(r) = \frac{\varepsilon}{2} - \mu |a| z(\theta_{r-\tau} - \omega) + |a|^2|z(\theta_{r-\tau} - \omega)|^2 - c_{59} |w_2(r)|^2.
\]
By applying Gronwall's inequality to (115) on \([t - \tau, r] \ (r \geq t - \tau)\), we have
\[
\tilde{g}_1(r) \leq \tilde{g}_1(t - \tau) e^{- \int_{t - \tau}^{r} \tilde{m}_1(r) dr} + \int_{t - \tau}^{r} \tilde{q}_1(\theta_{r-\tau} - \omega)) e^{- \int_{\xi}^{r} \tilde{m}_1(r) dr} d\xi.
\]
Similar to (113), for \( \tau \geq \xi \geq t - \tau \), we have
\[
\int_{\xi}^{r} \tilde{m}_1(r) dr \geq \int_{t - \tau}^{0} \left( \rho - \mu |a| z(\theta_{\omega}) + |a|^2|z(\theta_{\omega})|^2 \right) ds - c_{59} K_1.
\]
By \( \varphi_{r-\tau}(\theta_{r-\tau} - \omega) \in B_0(\theta_{r-\tau} - \omega) \cap \mathcal{D}(E^\nu) \), we have
\[
\tilde{g}_1(r - t) e^{- \int_{t - \tau}^{r} \tilde{m}_1(r) dr} \leq (||A^{\nu + 1/2} u_{r-\tau - t}||^2 + ||A^{\nu} v_{r-\tau - t}||^2 + 2c_{56} [1 + M_0^4(\theta_{r-\tau} - \omega)])
\times e^{- \int_{t - \tau}^{0} \left( \rho - \mu |a| z(\theta_{\omega}) + |a|^2|z(\theta_{\omega})|^2 \right) ds + c_{59} K}
\to_{t \to +\infty} 0.
\]
Thus, there exists a random variable \( \tilde{\tau}_\nu(\omega) \geq 0 \) such that
\[
0 \leq \tilde{g}_1(r - t) e^{- \int_{t - \tau}^{r} \tilde{m}_1(r) dr} \leq 1, \quad \forall t \geq \tilde{\tau}_\nu(\omega).
\]
proof of Lemma 7.2 below should be simpler than those of Lemmas 4.1-4.5 and for \( t \geq \tau, \omega \) the solution \( \varphi(\cdot, t, \theta, \omega) \) of (99) and \( \varphi(\cdot, t, \theta, \omega) \) of (47) satisfy, respectively, (i) for \( \nu \leq \kappa \leq 1 - \nu, \)</p>

\[
\| \varphi_N(\tau, t, \theta, \omega, \varphi_{(\theta, \omega)}) \|_{\mathcal{F}^{\mathbb{R}}}^2 \leq b_2^\nu(\omega),
\]

(148) (ii) for \( \nu \leq \kappa \leq 1, \)

\[
\| \varphi_N(\tau, t, \theta, \omega, \varphi_{(\theta, \omega)}) \|_{\mathcal{F}^{\mathbb{R}}}^2 \leq b_2^\nu(\omega),
\]

(149) where \( \nu \) is as in (80).

To construct a non-empty compact measurable tempered attracting set for \( \Phi \) in \( E^1 \), we appeal to the following result given in [66].

**Lemma 7.3.** [66] Let \( (X, d) \) be a Polish space, \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \) be an ergodic metric dynamical system and \( \{\Phi(t, \tau, \omega)\}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega} \) be a continuous cocycle on \( X \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, \{\theta_t\}_{t \in \mathbb{R}}) \). Assume that for each \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( t \geq 0, \)

(i) there exist a constant \( \tilde{C}_1 > 0 \) and a random variable \( \tilde{Q}_1(\omega) \) such that

\[
d(\Phi(t, \tau, \omega)\varphi_1, \Phi(t, \tau, \omega)\varphi_2) \leq \tilde{C}_1 e^{\tilde{Q}_1(\omega) t} d(\varphi_1, \varphi_2), \quad \forall \varphi_1, \varphi_2 \in X;
\]

(150)
(ii) there exist random variables \( \hat{C}_2(t, \omega), \hat{C}_3(t, \omega), \hat{Q}_2(\omega), \hat{Q}_3(\omega) \) and three subsets \( K_1, K_2, K_3 \subset X \) such that
\[
\begin{align*}
& d_h(\Phi(t, \tau, \omega)K_1, K_2) \leq \hat{C}_2(t, \omega)e^{\int_0^t \hat{Q}_2(\theta, \omega)ds}, \\
& d_h(\Phi(t, \tau, \omega)K_2, K_3) \leq \hat{C}_3(t, \omega)e^{\int_0^t \hat{Q}_3(\theta, \omega)ds};
\end{align*}
\] (151)

(iii) the expectations of \( \hat{Q}_i(\omega) \) \( (i = 1, 2, 3) \) satisfy:
\[
|\mathbf{E}[\hat{Q}_1(\omega)]| < \infty, \quad -\infty < \mathbf{E}[\hat{Q}_2(\omega)], \mathbf{E}[\hat{Q}_3(\omega)] < 0.
\] (152)

Then for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( T_0(\omega) > 0 \) (independent of \( \tau \)) such that for positive constant \( \sigma \) with
\[
0 < \sigma \leq \frac{-\mathbf{E}[\hat{Q}_2(\omega)]}{3\mathbf{E}[\hat{Q}_1(\omega)] - \mathbf{E}[\hat{Q}_2(\omega)] - 3\mathbf{E}[\hat{Q}_3(\omega)]}
\] (153)
and \( \min\{\sigma t, (1 - \sigma)t\} \geq T_0(\omega), t > 0 \), it holds that
\[
d_h(\Phi(t, \tau, \omega)K_1, K_3) \leq \left( \hat{C}_1 \hat{C}_2(\sigma t, \omega) + \hat{C}_3((1 - \sigma)t, \omega) \right)e^{\frac{2}{3}\mathbf{E}[\hat{Q}_3(\omega)t]}.
\] (154)

From Lemma 3.1, it is sufficient to verify that the solution \( \varphi(t + \tau, \tau, \omega, \varphi_\tau(\omega)) \) of (47) satisfies Lemma 7.3 on the bounded random set \( B_1(\tau, \omega) \) for \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( t \geq 0 \).

First we consider the Lipschitz property of \( \varphi(t + \tau, \tau, \omega, \varphi_\tau(\omega)) \). Let \( \varphi_{j\tau}(\omega) = (u_{j\tau}(\omega), v_{j\tau}(\omega)) \in B_1(\tau, \omega), \varphi_{j\tau}(r) = \varphi_j(r, \tau, \theta_{-\tau}, \varphi_\tau(\theta_{-\tau})) = (u_j(r), v_j(r)), r \geq \tau, j = 1, 2 \), and
\[
\psi(r) = \varphi_1(r) - \varphi_2(r) = (u_1(r) - u_2(r), v_1(r) - v_2(r)) = (\xi(r), \eta(r)),
\] (155)
then
\[
\dot{\psi} + \Lambda \psi = F(\varphi_1, \theta_{-\tau}, \omega, r) - F(\varphi_2, \theta_{-\tau}, \omega, r), \quad r \geq \tau,
\] (156)
where
\[
\psi_\tau(\omega) = (\xi_\tau, \eta_\tau) = (u_{1\tau} - u_{2\tau}, v_{1\tau} - v_{2\tau}),
\]
\[
F(\varphi_1, \theta_{-\tau}, \omega, r) - F(\varphi_2, \theta_{-\tau}, \omega, r) = \left( \frac{az(\theta_{-\tau}, \omega)\xi}{(2\varepsilon - az)az\xi - az\eta - f(u_1, x) + f(u_2, x)} \right).
\]

By (71), we have
\[
||\varphi_1(r)||_E \leq M_0(\theta_{-\tau}, \omega), \quad ||\varphi_2(r)||_E \leq M_0(\theta_{-\tau}, \omega), \quad \forall r \geq \tau.
\] (157)

**Lemma 7.4.** There exists a tempered random variable \( C_1(\omega) > 0 \) with \( 0 < \mathbf{E}[C_1(\omega)] < \infty \) such that for every \( \tau \in \mathbb{R}, \omega \in \Omega, t \geq 0 \), it holds that
\[
||\varphi(t + \tau, \tau, \theta_{-\tau}, \omega, \varphi_\tau(\theta_{-\tau}))-\varphi(t + \tau, \tau, \theta_{-\tau}, \omega, \varphi_{2\tau}(\theta_{-\tau}))||_E \leq e^{\int_0^t C_1(\theta, \omega)ds}||\varphi_1 - \varphi_{2\tau}||_E.
\] (158)

**Proof.** By (A1), Hölder inequality and (157), we have
\[
||f(u_2(r), x) - f(u_1(r), x)||^2 \leq c_{63}(1 + M_0^4(\theta_{-\tau}, \omega))||\xi(r)||_E^2.
\]
Taking the inner product \((\cdot, \cdot)_E\) of (156) with \( \psi(r) \), we find that for \( r \geq \tau, \)
\[
\frac{d}{dt}||\psi(r)||_E^2 \leq \left( -\varepsilon + 4|a||z(\theta_{-\tau}, \omega)| + \frac{|a|^2||z(\theta_{-\tau}, \omega)||^2}{\sqrt{\lambda_1}} \right)||\psi(r)||_E^2
+ \frac{c_{63}}{\alpha}(1 + M_0^4(\theta_{-\tau}, \omega))||\psi(r)||_E^2.
\] (159)
Then by applying the Gronwall inequality to (159), we have that for \( r \geq \tau \),
\[
\|\varphi_1(r) - \varphi_2(r)\|_{L^2}^2 \leq \|\varphi_1 - \varphi_2\|_{L^2}^2 e^{\int_0^r \left( 4|a||z(\theta, \omega)| + \frac{|a|^2|z(\theta, \omega)|^2}{\sqrt{\lambda_1}} + \frac{c_{63}}{4\alpha} (1 + M_0^4(\omega)) \right) ds}.
\]
(160)

Set \( r = t + \tau \), we obtain (158), where
\[
C_1(\omega) = 2|a||z(\omega)| + \frac{|a|^2|z(\omega)|^2}{2\sqrt{\lambda_1}} + \frac{c_{63}}{2\alpha} (1 + M_0^4(\omega))
\]
(161)

and by (69),
\[
0 < \mathbb{E}(C_1(\omega)) = \frac{2|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha \sqrt{\lambda_1}} + \frac{c_{63}}{2\alpha} (1 + \mathbb{E}[M_0^4(\omega)]) < \infty.
\]
(162)

The proof is completed.

\[ \square \]

**Lemma 7.5.** For every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exist a positive number \( \delta \), a random bounded ball \( B_1(\omega) \) of \( E^1 \) with radius \( b_1(\omega) \) and a tempered random variable \( Q(\omega) > 0 \) such that the solution \( \varphi(\tau, \tau - t, \theta_{-t}, \varphi_{-t}(\theta_{-t})) \) of (47) with \( \varphi_{-t}(\theta_{-t}) \in B_1(\tau - t, \theta_{-t}) \) satisfies
\[
d_h(\varphi(\tau, t, \theta_{-t}, B_1(\tau - t, \theta_{-t})), B_0(\omega)) \leq Q(\theta_{-t}) e^{-\rho\mu(\frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha})} t.
\]
(163)

for \( t \geq T(\omega) \); moreover, \( B_1(\omega) \subseteq B_1(\tau, \omega) \subseteq B_0(\omega) \).

**Proof.** Let \( \varphi_{-t}(\theta_{-t}) \in B_1(\tau - t, \theta_{-t}) \). Let \( \Lambda(\omega) \subseteq E^\nu \subseteq E \) be as in Theorem 5.1. By Lemma 4.4, we have that for \( t \geq 0 \),
\[
d_h(\varphi(\tau, \tau - t, \theta_{-t}, B_1(\tau - t, \theta_{-t})), \Lambda(\omega)) \leq M_2 e^{-\int_0^t (\rho - \mu(\frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha})) ds}.
\]
(164)

By Lemma 4.4, Lemma 7.1 and Lemma 7.2 to \( \varphi_{-t}(\theta_{-t}) \in \Lambda(\omega) \), there exist \( t_{1\nu}(\omega) \geq 0 \) and a random ball \( A_{2\nu}(\omega) \) of \( E^{2\nu} \) with radius \( M_{2\nu}(\omega) \) (defined by (148)) such that
\[
d_h(\varphi(\tau, \tau - t, \theta_{-t}, \Lambda(\omega)), A_{2\nu}(\omega)) \leq P_{1\nu}(\theta_{-t}) e^{-\int_0^t (\rho - \mu(\frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha})) ds}, \quad t \geq t_{1\nu}(\omega),
\]
(165)

where \( P_{1\nu}(\theta_{-t}) = 2[1 + |a|^4 z(\omega)]^4 M_{1\nu}(\theta_{-t}), M_{1\nu}(\omega) = M_0^4(\omega) \) is defined by (146).

By Lemmas 7.3-7.4 and (164)-(165), there exists \( T_{1\nu}(\omega) > 0 \) (independent of \( \tau \)) such that for \( t \geq T_{1\nu}(\omega) \),
\[
d_h(\varphi(\tau, \tau - t, \theta_{-t}, B_1(\tau - t, \theta_{-t})), A_{2\nu}(\omega)) \leq P_{2\nu}(\theta_{-t}) e^{-\int_0^t (\rho - \mu(\frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha})) ds},
\]
(166)

where \( P_{2\nu}(\theta_{-t}) = M_L + P_{1\nu}(\theta_{-t-\sigma_1}) \) and
\[
0 < \sigma_1 \leq \frac{\rho - \mu \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right)}{3\mathbb{E}[C_1(\omega)] + 4\rho - 4\mu \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right)} < 1.
\]
(167)

For a fixed \( \nu > 0 \) in (80), there exists \( \tilde{k} \in \mathbb{N} \) such that \( 1 - \nu \leq (\tilde{k} - 1)\nu < 1 \). By making the above recursion at most \( \tilde{k} \left( \frac{1}{4} + 2 \right) \) steps, there must exist \( T_{k\nu}(\omega) > 0 \)
By (172) and (173), it holds that

\[
\text{d}_h(\psi, \tau - t, \theta_{\tau}, B_1(\tau - t, \theta_{\tau})), B_1(\omega)) \leq \text{P}_{k^j}(\theta_{-t} \omega) e^{\frac{-\sigma_{k^j}^j}{2}(\rho - \mu \frac{|w|}{\sqrt{\pi \alpha}} - \mu |w|^2)} \quad (168)
\]

for \( t \geq T_{k^j}(\omega), \) where

\[
0 < \sigma_j \leq \frac{\sigma_{k^j}^j}{2} \left( \rho - \mu \frac{|w|}{\sqrt{\pi \alpha}} - \mu |w|^2 \right) \quad (169)
\]

\[
P_{k^j}(\theta_{-t} \omega) = M_k + P_{(k-1)^j}(\theta_{-(1-\sigma_{k^j}^j)}t \omega) \quad \text{is tempered.}
\]

By combining Lemmas 7.5 and the proof of Theorem 5.1, we obtain that the random attractor \( A \) for \( \Phi \) is in fact included in the random ball \( \hat{B}_1(\omega) \) of \( E^1 \).

**Theorem 7.1.** Suppose (A1) and (101) hold. Then the \( \mathcal{D}(E) \)-pullback random attractor \( A \in \mathcal{D}(E) \) for \( \Phi \) satisfies:

\[
A(\tau, \omega) \subseteq \hat{B}_1(\omega), \quad ||A(\tau, \omega)||_{E^1} = \sup_{\psi \in A(\tau, \omega)} ||\psi||_{E^1} \leq b_1(\omega), \quad \forall \tau \in \mathbb{R}, \omega \in \Omega,
\]

where \( b_1(\omega) \) is the radius of \( \hat{B}_1(\omega) \) in the norm of \( E^1 \).

**Proof.** It is easy to see that \( \hat{B}_1(\omega) \) is a compact measurable \( \mathcal{D}(E) \)-pullback attracting ball for \( \Phi \) in \( E \) which implying (170). From (126) and (149), the radius \( b_1(\omega) = \sup_{\varphi \in \hat{B}_1(\omega)} ||\varphi||_{E} \) has the following form

\[
b_1^2(\omega) = c_{64} + c_{65} M_0^{k^j}(\omega) + c_{66} e^{k^j \nu_1(\omega)} \left( K^{k^j}_T(\omega) + K^k_2(\omega) \right).
\]

The proof is completed. \( \square \)

**Remark 7.1.** From the expression (171) of \( b_1(\omega) \) (the obtained bound of random attractor in the norm of \( E^1 \)), it is seen that the expectation of \( b_1^2(\omega) \) is possible unbounded, which is different from the case of subcubic growth exponent in [65, 66].

8. Existence of random exponential attractor. In this section, we assume that conditions (A1)-(A2) hold. Now let us prove the existence of a random exponential attractor for \( \Phi \) in \( E \) according to Theorem 2.1. For this aim, in the following, we will check that \( \Phi \) satisfies the conditions (H1)-(H4) in Theorem 2.1.

First, from lemma 7.2 (ii) and lemma 7.5, it follows that for given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there exists \( T_1(\omega) > 0 \) such that

\[
\psi(\tau - t, \theta_{-t} \omega, \psi_{-t}(\theta_{-t} \omega)) \subseteq \hat{B}_1(\omega), \forall \psi_{-t}(\theta_{-t} \omega) \in \hat{B}_1(\theta_{-t} \omega), t \geq T_1(\omega).
\]

Take a family of bounded random subsets \( \{\hat{\chi}(\tau, \omega)\} \in \mathbb{R}, \omega \in \Omega \) of \( E \) as

\[
\hat{\chi}(\tau, \omega) = \cup_{s \geq T_1(\omega)} \Phi(\tau, \tau - s, \theta_{-s} \omega, \hat{B}_1(\theta_{-s} \omega))
\]

(173)

By (172) and (173), it holds that

\[
\hat{\chi}(\tau, \omega) \subseteq B_1(\omega) \subseteq B_1(\tau, \omega) \subseteq B_0(\omega)
\]

and

\[
\Phi(t, \tau - t, \theta_{-t} \omega) \hat{\chi}(\tau - t, \theta_{-t} \omega) \subseteq \hat{\chi}(\tau, \omega), \quad \forall t \geq 0;
\]

(174)
thus, by the continuity and the cocycle property of $\Phi(t, \tau - t, \theta_{-t}\omega)$ for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, a family of tempered closed random subsets $\{\tilde{\chi}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies conditions (h11) and (h12) in Theorem 2.1.

Now let us show $\{\tilde{\chi}(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ satisfies conditions (H2)-(H4) in Theorem 2.1. For $j \in \mathbb{N}$, let $e_j \in D(A)$ be the eigenvector of operator $A$ to eigenvalue $\lambda_j$ with $Ae_j = \lambda_j e_j$, then $\{e_j\}_{j \in \mathbb{N}}$ form an orthonormal base of $L^2(U)$. Let

$$L^2_n(U) = \text{span}\{e_1, e_2, \cdots, e_n\}, \quad [L^2_n(U)]^\perp = \text{span}\{e_{n+1}, e_{n+2}, \cdots\}, \quad n \in \mathbb{N},$$

then $L^2_n(U) \times L^2_n(U)$ is a $2n$-dimensional subspace of $E$. Let

$$P_n : E \to L^2_n(U) \times L^2_n(U), \quad Q_n = I - P_n : E \to [L^2_n(U)]^\perp \times [L^2_n(U)]^\perp$$

be the orthonormal projectors. For $u_{nq} \in [L^2_n(U)]^\perp$, we have

$$\lambda_{n+1}||u_{nq}||^2_E \leq ||u_{nq}||^2. \quad (175)$$

**Lemma 8.1.** Suppose (101) holds. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, there exist random variables $C_2(\omega) > 0$, $C_3(\omega) > 0$, $C_4(\omega) > 0$ and a $2n$-dimensional projector $P_n : E \to P_n E = L^2_n(U) \times L^2_n(U)$ such that for any $\phi_{j, \tau-t}(\theta_{-t}\omega) \in \tilde{\chi}(\tau - t, \theta_{-t}\omega), j = 1, 2,$

$$||\Phi(t, \tau - t, \theta_{-t}\omega)\phi_{j, \tau-t}(\theta_{-t}\omega) - \Phi(t, \tau - t, \theta_{-t}\omega)\phi_{j, \tau-t}(\theta_{-t}\omega)||_E \leq e^{\int_0^\tau C_2(\theta_{-t}\omega)ds}||\phi_{j, \tau-t}(\theta_{-t}\omega) - \phi_{2, \tau-t}(\theta_{-t}\omega)||_E \quad (176)$$

and

$$||(I - P_n)\Phi(t, \tau - t, \theta_{-t}\omega)\phi_{j, \tau-t}(\theta_{-t}\omega) - \Phi(t, \tau - t, \theta_{-t}\omega)\phi_{2, \tau-t}(\theta_{-t}\omega)||_E \leq \left(e^{-\xi(t) C_3(\theta_{-t}\omega)} + \frac{1}{\lambda_{n+1}} e^{C_4(\theta_{-t}\omega) + \int_0^\tau C_2(\theta_{-t}\omega)ds} \right) \times ||\phi_{j, \tau-t}(\theta_{-t}\omega) - \phi_{2, \tau-t}(\theta_{-t}\omega)||_E. \quad (177)$$

**Proof.** For any fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, let

$$\phi_{j, \tau-t}(\theta_{-t}\omega) = (u_{j, \tau-t}(\theta_{-t}\omega), \nu_{j, \tau-t}(\theta_{-t}\omega)) \in \tilde{\chi}(\tau - t, \theta_{-t}\omega), \quad j = 1, 2,$$

$$\phi_1(r) = \phi_2(r, \tau-t, \theta_{-t}\omega, \phi_{j, \tau-t}(\theta_{-t}\omega)) = (u_1(r), \nu_1(r)) \quad \text{and} \quad \psi(r) = \phi_1(r) - \phi_2(r) = (\xi(r), \eta(r))$$

for $r \geq \tau - t$, then $\psi(r)$ satisfies (156). By (174), it follows that for $r \geq \tau - t$, $\phi_1(r), \phi_2(r) \in \tilde{\chi}(r, \theta_{-t}\omega) \subseteq B_1(\theta_{-t}\omega) \subseteq E^1$, thus, $||\phi_1(r)||_{E^1} \leq b_1(\theta_{-t}\omega), \quad ||\phi_2(r)||_{E^1} \leq b_1(\theta_{-t}\omega)$, which implies that

$$|u_j(r)| \leq c_0 T(\Omega)||Au_j(r)||_E \leq c_0 b_1(\theta_{-t}\omega), \quad r \geq \tau - t, \quad j = 1, 2. \quad (178)$$

Taking the inner product of (156) with $\psi_{nq} = Q_n \psi$ in $E$, we have that for $r \geq \tau - t$,

$$\frac{d}{dt}||\psi_{nq}(r)||^2_E \leq -\epsilon + 4|a|^2|z(\theta_{-t}\omega)| + \frac{|a|^2|z(\theta_{-t}\omega)|^2}{\lambda_1}||\psi_{nq}(r)||^2_E - \alpha||\eta_{nq}(r)||^2 + 2(f(u_2(r), x) - f(u_1(r), x), \eta_{nq}). \quad (179)$$
Thus, by (2), (3) and (5), we have that for \( \tau \geq \tau - t \),
\[
\begin{align*}
&\| f(u_1, x) + f(u_2, x) \|_2^2 \\
&\leq \int_U |f_u'(u_2, x)\nabla u_2 - f_u'(u_1, x)\nabla u_1 + f_x'(u_2, x) - f_x'(u_1, x)|^2 \, dx \\
&\leq \int_U \left( |f_u'(u_2, x) - f_u'(u_1, x)|\nabla u_2 + |f_x'(u_1, x)|\nabla \xi + c_T(1 + |u_1|^2 + |u_2|^2)|\xi| \right)^2 \, dx \\
&\leq 2 \int_U (c_69(1 + |u_1| + |u_2|)|\nabla u_2|\|\xi\| + c_70(1 + |u_1|^2)|\nabla \xi|)^2 \, dx \\
&\quad + 2 \int_U (c_T(1 + c_68b_1(\theta_{\tau})^2)|\nabla u_2|\|\xi\| + c_71(1 + b_1(\theta_{\tau})) (|\xi|^2 + ||\xi||^2)) \, dx \\
&\leq c_72(1 + 2c_68b_1(\theta_{\tau})^2)||A u_2||^2||\xi||^2 + c_71(1 + b_1(\theta_{\tau})) (|\xi|^2 + ||\xi||^2) \\
&\leq c_73(1 + b_1(\theta_{\tau}))|\xi|^2, \quad (180)
\end{align*}
\]

thus, by (175) and (180),
\[
\begin{align*}
&2(f(u_2(r), x) - f(u_1(r), x), \eta_{nq}) \\
&\leq 2||f(u_2(r)) - f(u_1(r))||_1 \cdot ||\eta_{nq}||_1 \\
&\leq \frac{2c_{74}}{\alpha} (1 + b_1(\theta_{\tau}))|\xi(r)||^2 + \alpha ||\eta_{nq}(r)||^2. \quad (181)
\end{align*}
\]

Putting (181) into (179), we have that for \( r \geq \tau - t \),
\[
\begin{align*}
&\frac{d}{dt}||\psi_{nq}(r, \tau - t, \theta_{\tau}, \psi_{\tau-t}(\theta_{\tau}))||_E^2 \\
&\leq \left( -\varepsilon + 4|a||z(\theta_{\tau})| + \frac{|a|^2|z(\theta_{\tau})|^2}{\sqrt{\lambda_1}} \right) ||\psi_{nq}(r, \tau - t, \theta_{\tau}, \psi_{\tau-t}(\theta_{\tau}))||_E^2 \\
&\quad + c_{74} \frac{1}{\lambda_{n+1}} (1 + b_1(\theta_{\tau}))e^{\int_{r-t}^{r}2C_1(\theta_{\tau})ds}||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2. \quad (182)
\end{align*}
\]

By applying Gronwall inequality to (182) on \([\tau - t, \tau] (t \geq 0)\), we have
\[
\begin{align*}
&||\psi_{nq}(\tau, \tau - t, \theta_{\tau}, \psi_{\tau-t}(\theta_{\tau}))||_E^2 \\
&\leq \left( -\varepsilon + 4|a||z(\theta_{\tau})| + \frac{|a|^2|z(\theta_{\tau})|^2}{\sqrt{\lambda_1}} \right) \int_{r-t}^{r} e^{\int_{r-t}^{s}2C_1(\theta_{\tau})ds} \frac{1}{\lambda_{n+1}} ||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2 ds \\
&\quad + c_{74} \frac{1}{\lambda_{n+1}} (1 + b_1(\theta_{\tau}))e^{\int_{r-t}^{r}2C_1(\theta_{\tau})+4|a||z(\theta_{\tau})|+\frac{|a|^2|z(\theta_{\tau})|^2}{\sqrt{\lambda_1}}) \int_{r-t}^{r} ds ||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2 \\
&\quad + \frac{1}{\lambda_{n+1}} ||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2 e^{\int_{r-t}^{r}2C_1(\theta_{\tau})ds} \\
&\quad + \frac{1}{\lambda_{n+1}} \int_{r-t}^{r} e^{\int_{r-t}^{s}2C_1(\theta_{\tau})+4|a||z(\theta_{\tau})|+\frac{|a|^2|z(\theta_{\tau})|^2}{\sqrt{\lambda_1}}) ||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2 ds \\
&\quad \times \int_{r-t}^{r} c_{74}(1 + b_1(\theta_{\tau}))e^{\int_{r-t}^{s}2C_1(\theta_{\tau})ds} dr ||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2 \\
&\quad \times \int_{r-t}^{r} c_{74}(1 + b_1(\theta_{\tau}))e^{\int_{r-t}^{s}2C_1(\theta_{\tau})ds} dr ||\varphi_{\tau-t}(\theta_{\tau}) - \varphi_{\tau-t}(\theta_{\tau})||_E^2. \quad (183)
\end{align*}
\]

By (171),
\[
1 + b_1(\omega) = c_{75}e^{2\tilde{k}1(\omega)} \left( 1 + M_{\tilde{k}}^5(\omega) + K_{\omega}^{2k}(\omega) + K_{\omega}^{2k}(\omega) \right), \quad (184)
\]
where
\[
K_{2}^{2k}(\omega) \leq \left(\frac{2(2^k - 1)}{k}\right)^{2k-1} \int_{-\infty}^{0} e^{\frac{s}{2}+4k\int_{0}^{s} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds} dr < \infty,
\]
\[
K_{2}^{2k}(\omega) \leq c_{76} \sup_{|r| < T} \int_{0}^{T} e^{-64k\int_{0}^{s} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds} e^{\rho r} dr + c_{77} M_{2}^{16k}(T, \omega)
\]
\[+ c_{78} \int_{-\infty}^{0} [1 + z^{32k}(\theta, \omega) + M_{0}^{54k}(\theta, \omega) + \|w_{1}(r)\|_{1}^{32k}] e^{\rho r} dr,
\]
\[
M_{2}^{16k}(T, \omega) \leq c_{79} \int_{-T}^{0} q_{2}^{16k}(\theta, \omega) e^{4k\varepsilon r + 16k\int_{0}^{s} m_{2}(\theta, \omega) ds} dr + c_{80} R_{2}^{16k}(\omega, \theta, \omega),
\]
\[
R_{2}^{16k}(\omega, \theta, \omega) = c_{81} \left(1 + M_{0}^{44k}(\omega)\right) + c_{81} M_{1}^{44k}(\theta, \omega) e^{-64k\rho T + 64k\mu \int_{0}^{T} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds},
\]
\[
q_{2}^{16k}(\theta, \omega) = c_{82} [1 + z^{128k}(\theta, \omega) + M_{0}^{256k}(\theta, \omega)] + c_{83} M_{1}^{128k}(\theta, \omega) e^{-128k\int_{-T}^{0} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds},
\]
For \( t = T \) and \( r \in [0, T] \), by (84), we have
\[
e^{\mu_{T} \int_{s}^{T} |\hat{w}(r)| |\hat{z}(\theta, \omega)| + |\hat{a}|^2|\hat{z}(\theta, \omega)|^2)ds,
\]
and
\[
q_{2}^{16k}(\theta, \omega) = c_{82} [1 + z^{128k}(\theta, \omega) + M_{0}^{256k}(\theta, \omega)] + c_{83} M_{1}^{128k}(\theta, \omega) e^{-128k\int_{-T}^{0} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds},
\]
\[
\int_{-T}^{0} q_{2}^{16k}(\theta, \omega) e^{4k\varepsilon r + 16k\int_{0}^{s} m_{2}(\theta, \omega) ds} dr \leq e^{16k\max_{-T < s < 0}[22 + \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2] + M_{\mu}^{4}(\theta, \omega)]}
\times e^{\frac{4k\mu}{\rho} T + 4k\mu \int_{0}^{T} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds}
\times \left(c_{82} \int_{-T}^{0} [1 + z^{128k}(\theta, \omega) + M_{0}^{256k}(\theta, \omega)] e^{4k\varepsilon r} dr + c_{83} \int_{-T}^{0} M_{1}^{128k}(\theta, \omega) e^{-128k\int_{-T}^{0} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds} e^{16k\varepsilon r} dr\right).
\]
Thus, by (184),
\[
1 + b_{11}(\omega) \leq c_{84} e^{2kK_{1}(\omega) + 16kK_{9}(\omega) K_{10}(\omega)},
\]
where
\[
K_{9}(\omega) = \max_{-T \leq s \leq 0} \left[ c_{22} + \mu([a] \cdot |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2) + M_{\mu}^{4}(\theta, \omega)\right]
\times \frac{c_{23}}{4 \rho} M_{1}^{4}(\theta, \omega) e^{40T + 4\mu \int_{0}^{T} \mu([a] + |z(\theta, \omega)| + |a|^2|z(\theta, \omega)|^2)ds},
\]
Lemma 8.2. Assume that

\[ |a| \leq \min \left\{ \frac{\alpha}{64\sqrt{k\mu}}, \frac{\alpha\sqrt{\alpha}}{2048k\mu}, \sqrt{\alpha \rho}, \frac{\sqrt{\rho \alpha}}{16\mu}, \sqrt{\frac{\sqrt{\pi \alpha \varepsilon}}{128}}, \sqrt{\frac{\varepsilon \alpha \sqrt{\lambda_1}}{4}} \right\}. \]
then
\[
0 \leq E[C_4(\omega)] \leq \frac{\varepsilon}{32},
\]
\[
0 \leq E[C_2(\omega)], E[C_3(\omega)], E[C_2^2(\omega)], E[C_3(\omega)] < \infty.
\]

**Proof.** (i) By (44), (195) and (196), we have
\[ E[C_4(\omega)] = \frac{2|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{4 \alpha \sqrt{\lambda_1}} \leq \frac{\varepsilon}{32}.
\]

(ii) By (194),
\[
C_3^2(\theta, \omega) = 2k^2 \left( \max_{-t \leq r \leq 0} K_1^2(\theta, \omega) + 64 \max_{-t \leq r \leq 0} K_3^2(\theta, \omega) \right),
\]
where
\[
K_1^2(\theta, \omega) \leq c_{86} \sup_{t \geq 0} \left( \int_{-t}^{0} e^{-2\rho(s+t)+2} f_{-t}^{+} \mu(|a| \cdot |z(\theta_{s+r}, \omega)| + |a|^2 |z(\theta_{s+r}, \omega)|^2) \text{d}t \text{d}s \right)^2
\]
and
\[
K_3^2(\theta, \omega) \leq c_{87} + \max_{-T \leq r \leq 0} \left[ |z(\theta_{s+r}, \omega)|^2 + |z(\theta_{s+r}, \omega)|^4 + M_0^8(\theta_{s+r}, \omega) \right]
\]
\[
\quad + c_{89} M_0^{32}(\theta_{s+r}, \omega) + c_{90} e^{8T+8T} \mu f_{-t}^{+}(|a| \cdot |z(\theta_{s+r}, \omega)| + |a|^2 |z(\theta_{s+r}, \omega)|^2) \text{d}s.
\]

By (45) and (196),
\[
E \left( \int_{-t}^{0} e^{-2\rho(s+t)+2} f_{-t}^{+} \mu(|a| \cdot |z(\theta_{s+r}, \omega)| + |a|^2 |z(\theta_{s+r}, \omega)|^2) \text{d}t \text{d}s \right)
\]
\[
\leq \frac{1}{2} \left( \frac{1}{\rho - \frac{2\mu|a|}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\mu|a|^2}{\alpha}} \right) < \infty,
\]
\[
E \left( \int_{-t}^{0} e^{-4\rho(s+t)+4} f_{-t}^{+} \mu(|a| \cdot |z(\theta_{s+r}, \omega)| + |a|^2 |z(\theta_{s+r}, \omega)|^2) \text{d}t \text{d}s \right)^2
\]
\[
\leq \frac{1}{4} \left( \frac{1}{\rho - \frac{4\mu|a|}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\mu|a|^2}{\alpha}} \right) < \infty.
\]
\[
E \left( \int_{-t}^{0} e^{-2\rho(s+t)+2} f_{-t}^{+} \mu(|a| \cdot |z(\theta_{s+r}, \omega)| + |a|^2 |z(\theta_{s+r}, \omega)|^2) \text{d}t \text{d}s \right)^2
\]
\[
\leq \frac{1}{4\rho} \left( \frac{1}{\rho - \frac{4\mu|a|}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\mu|a|^2}{\alpha}} \right) \leq \rho_1,
\]
and
\[
E \left( \int_{-t}^{0} e^{-4\rho(s+t)+4} f_{-t}^{+} \mu(|a| \cdot |z(\theta_{s+r}, \omega)| + |a|^2 |z(\theta_{s+r}, \omega)|^2) \text{d}t \text{d}s \right)^2
\]
Thus, it follows that

\[
E\left[\left|z(\theta_{s+\omega})\right|^2 + \left|z(\theta_{s+\omega})\right|^4 + M^8_0(\theta_{s+\omega})\right] \leq \frac{\|a\|^2}{2\alpha} + \frac{3\|a\|^4}{4\alpha^2} + \frac{27c^4_1}{\rho^3} \left(\frac{1}{\rho - \frac{4\|a\|^2}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\|a\|^2}{\alpha}}\right) \leq \rho_1 < 0,
\]

and

\[
E\left[\left|z(\theta_{\tilde{r}+\omega})\right|^2 + \left|z(\theta_{\tilde{r}+\omega})\right|^4 + M^8_0(\theta_{\tilde{r}+\omega})\right] \leq \frac{1515c_{10}^4}{240\rho_{15}^4} \left(\frac{1}{\rho - \frac{4\|a\|^2}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\|a\|^2}{\alpha}}\right) \leq \rho_5 < \infty.
\]

Thus, \(E[C_3(\theta_i\omega)] < \infty\) and

\[
E[C^2_3(\theta_i\omega)] = E[C^2_3(\omega)] \leq 2\tilde{k}^2 (c_{86}(\rho_1 + \rho_2) + c_{87}c_{88}\rho_3 + c_{89}\rho_4 + c_{90}\rho_5) \leq \rho_6 < \infty.
\]  

(201) (iii) By (193),

\[
C^2_2(\omega) \leq c_{91} \left(1 + \left|z(\omega)\right|^2 + \left|z(\omega)\right|^4 + M^8_0(\omega) + K^4_{10}(\omega)\right),
\]  

where

\[
E[1 + \left|z(\omega)\right|^2 + \left|z(\omega)\right|^4 + M^8_0(\omega)] \leq 1 + \rho_3,
\]  

and

\[
K^4_{10}(\omega) = c_{92} \left(1 + M^{32k}_0(\omega) + K^{8k}_2(\omega) + R^{64k}_2(\omega, \theta_{\tilde{r}+\omega}) + K^{4}_{11}(\omega)\right).
\]  

(204) By computation, we have

\[
E[M^{32k}_0(\omega)] \leq \frac{216k^4c_{10}^4}{8k}\left(\frac{16k - 1}{8k\rho}\right)^{16k-1} \left(\frac{1}{\rho - \frac{4\|a\|^2}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\|a\|^2}{\alpha}}\right) \leq \rho_7,
\]

\[
E[K^{8k}_2(\omega)] \leq \left(\frac{8k - 1}{2k\rho}\right)^{8k-1} \left(\frac{1}{8k} + \frac{1}{8k - \frac{8\|a\|^2}{\sqrt{\alpha}}} + \frac{1}{8k - \frac{8\|a\|^2}{\alpha}}\right) \leq \rho_8.
\]

By (185),

\[
R^{64k}_2(\omega, \theta_{\tilde{r}+\omega}) \leq c_{93} \left(1 + M^{256k}_0(\omega) + M^{1024k}_0(\theta_{\tilde{r}}\omega)\right) + c_{93} \left(e^{-512k\rho\tilde{r}} + 512\rho\int_0^{\tilde{r}}(\|a\| + \|az\| + \|az\|^2)ds\right),
\]
where

$$E[M_0^{256\tilde{k}}(\omega)] \leq \frac{2^{128\tilde{k}}}{64\tilde{k}} \left( \frac{128\tilde{k} - 1}{64\tilde{k} \rho} \right)^{128\tilde{k} - 1} \left( \frac{1}{\rho - \frac{4\mu|a|}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\mu|a|^2}{\alpha}} \right) \leq \rho_9,$$

$$E[M_0^{1024\tilde{k}}(\theta_{-\tilde{r}}\omega)] \leq \frac{2^{512\tilde{k}}}{256\tilde{k}} \left( \frac{512\tilde{k} - 1}{256\tilde{k} \rho} \right)^{512\tilde{k} - 1} \left( \frac{1}{\rho - \frac{4\mu|a|}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{4\mu|a|^2}{\alpha}} \right) \leq \rho_{10},$$

$$E \left( e^{-512\tilde{k}\rho T + 512\tilde{k}\mu \int_{-\tilde{T}}^{0}(|a| \cdot |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2)ds} \right) \leq e^{-512\tilde{k}\rho T} \left( e^{\frac{512\tilde{k}|a|}{\sqrt{\alpha}}t} + e^{\frac{512\tilde{k}|a|^2}{\alpha}t} \right) \leq \rho_{11},$$

then

$$E[P_{21}^{(\hat{y})}(\omega, \theta_{-\tilde{r}}\omega)] \leq c_{93} (1 + \rho_9 + \rho_{10} + \rho_{11}) \leq \rho_{12}.$$ 

By (189),

$$K_{11}(\omega) \leq c_{94} \sup_{t \geq 0} \int_{-t}^{0} e^{-256\tilde{k} \int_{-r}^{\rho \cdot \mu |a| |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2} ds} dr$$

$$+ c_{95} \int_{-t}^{0} \left[ 1 + z^{128\tilde{k}}(\theta_{\tilde{r}}\omega) + M_0^{256\tilde{k}}(\theta_{-\tilde{r}}\omega) \right] \left| \left| 1^{128\tilde{k}} \right| \right| e^{\mu r} dr$$

$$+ c_{96} \int_{-t}^{0} \left[ 1 + z^{512\tilde{k}}(\theta_{\tilde{r}}\omega) + M_0^{1024\tilde{k}}(\theta_{-\tilde{r}}\omega) \right] e^{4\tilde{k}\tilde{r} r} dr$$

$$+ c_{97} \int_{-t}^{0} M_1^{512\tilde{k}}(\theta_{-\tilde{r}}\omega) e^{-512\tilde{k} \int_{-r}^{\rho \cdot \mu |a| |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2} ds} e^{4\tilde{k}\tilde{r} r} dr,$$

where

$$E \left( \int_{-t}^{0} e^{-256\tilde{k} \int_{-r}^{\rho \cdot \mu |a| |z(\theta, \omega)| + |a|^2 |z(\theta, \omega)|^2} ds} dr \right) \leq \int_{-t}^{0} e^{-256\tilde{k} \rho (r + t)} \frac{1}{2} \left( E e^{\int_{-r}^{\rho |a| \sqrt{\alpha}} M_0^{256\tilde{k}}(\theta_{-\tilde{r}}\omega) ds} + E e^{\int_{-r}^{\rho |a| \sqrt{\alpha}} M_0^{1024\tilde{k}}(\theta_{-\tilde{r}}\omega) ds} \right) dr$$

$$\leq \frac{1}{2} \int_{-t}^{0} \left( e^{-256\tilde{k} \rho (r + t)} + e^{-256\tilde{k} \rho \cdot \frac{2\mu|a|^2}{\alpha}} + e^{-256\tilde{k} \rho \cdot \frac{2\mu|a|^2}{\alpha}} \right) dr$$

$$\leq \frac{1}{512\tilde{k}} \left( \frac{1}{\rho - \frac{2\mu|a|}{\sqrt{\alpha}}} + \frac{1}{\rho - \frac{2\mu|a|^2}{\alpha}} \right) \leq \rho_{13}, \text{ (independent of } t)$$

$$E \left( \int_{-\infty}^{0} \left[ 1 + z^{128\tilde{k}}(\theta_{\tilde{r}}\omega) + M_0^{256\tilde{k}}(\theta_{-\tilde{r}}\omega) \right] e^{\mu r} dr \right) \leq \frac{1}{\rho} \left( 1 + \frac{\Gamma \left( 1 + \frac{128\tilde{k}}{2} \right)}{\sqrt{\pi} \alpha^{128\tilde{k}}} + \rho_9 \right) \leq \rho_{14}. \quad (205)$$
and

\[ E \left( \int_{-\infty}^{0} \left[ 1 + z^{512k} (\theta, r) + M_0^{1024k} (\theta, r) \right] e^{4k \xi r} \, dr \right) \leq \frac{1}{2k} \left( 1 + \frac{\Gamma(1+512k)}{\sqrt{\pi} \alpha^{512k}} + \rho_{10} \right) \lesssim \rho_{15}. \]

By (93), we have

\[ \int_{-m\hat{T}}^{-(m-1)\hat{T}} \| w_1 (r) \|^2 \, e^{\rho \xi r} \, dr \leq \int_{-m\hat{T}}^{-(m-1)\hat{T}} M_1^{128k} (\theta, -\hat{T}) e^{-128k \int_{-m\hat{T}}^{m\hat{T}} (\rho - \mu (|z| + |z|^2)) \, ds} \, e^{\rho \xi r} \, dr \]

\[ \leq \frac{1}{2\rho} e^{\rho \hat{T}} M_1^{256k} (\theta, -\hat{T}) e^{-\rho m\hat{T}} \times \left( \int_{0}^{\infty} e^{-256k \int_{0}^{s} (\rho - \mu (|z| + |z|^2)) \, ds} \, dr \right)^{\frac{1}{2}} \lesssim M_4 (\theta, -\hat{T}) e^{-\rho m\hat{T}}, \quad \forall m \geq 1, \]

where

\[ M_4 (\theta, -\hat{T}) = \frac{1}{2\rho} e^{\rho \hat{T}} M_1^{256k} (\theta, -\hat{T}) \]

\[ \times \left( \int_{0}^{\infty} e^{-256k \int_{0}^{s} (\rho - \mu (|z| + |z|^2)) \, ds} \, dr \right)^{\frac{1}{2}}. \]

Thus,

\[ \int_{-\infty}^{0} \| w_1 (r) \|^2 \, e^{\rho \xi r} \, dr = \sum_{m=1}^{+\infty} \int_{-m\hat{T}}^{-(m-1)\hat{T}} \| w_1 (r) \|^2 \, e^{\rho \xi r} \, dr \leq \sum_{m=1}^{+\infty} M_4 (\theta, -\hat{T}) e^{-\rho m\hat{T}}, \]

and

\[ E[M_4 (\theta, -\hat{T})] \]

\[ \leq \frac{1}{\sqrt{2\rho}} e^{\rho \hat{T}} \left( \int_{0}^{\infty} e^{-256k \int_{0}^{s} (\rho - \mu (|z| + |z|^2)) \, ds} \, dr \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{\sqrt{2\rho}} e^{\rho \hat{T}} (c_{11} (1 + \rho_{10}) + \rho_{13}) \lesssim \rho_{16}. \]

So,

\[ E \left( \int_{-\infty}^{0} \| w_1 (r) \|^2 \, e^{\rho \xi r} \, dr \right) \leq \sum_{m=1}^{+\infty} \rho_{16} e^{-\rho m\hat{T}} \leq \frac{\rho_{16} e^{-\rho \hat{T}}}{1 - e^{-\rho \hat{T}}} \lesssim \rho_{17}. \]
By (45) and (196),
\[
E \left( M^{12k}_1 (\theta_{\tau, \omega}) e^{-512k \int_{-\tau}^{\tau} (\rho - \mu(a^2 z(\theta_{\tau, \omega})^2) \, ds \right) 
\leq 2c_{11} (1 + E[M^{2048k}_0 (\theta_{-mT, \omega})]) + 2E e^{-1024k \int_{-\tau}^{\tau} (\rho - \mu(a^2 z(\theta_{\tau, \omega})^2) \, ds 
\leq 2c_{11} \left( 1 + \frac{2102k}{512k} \frac{1024k - 1}{512k} \left( \frac{1}{\rho - 4\mu(a^2)} + \frac{1}{\rho - 4\mu(a^2)} \right) \right) 
+ e^{-1024k (\rho - 2\mu(a^2) \omega)(r+T)} + e^{-1024k (\rho - 2\mu(a^2) \omega)(r+T)} 
\leq \rho_{18} + e^{-1024k (\rho - 2\mu(a^2) \omega)(r+T)} + e^{-1024k (\rho - 2\mu(a^2) \omega)(r+T)},
\]
thus,
\[
E \left( \int_{-\tau}^{0} M^{12k}_1 (\theta_{\tau, \omega}) e^{-128k \int_{-\tau}^{\tau} (\rho - \mu(a^2 z(\theta_{\tau, \omega})^2) \, ds \right) e^{4k \epsilon r} \, dr \right) 
\leq \frac{\rho_{18}}{4k \epsilon} + \int_{-\tau}^{0} e^{-1024k (\rho - 2\mu(a^2) \omega)(r+T)} + e^{-1024k (\rho - 2\mu(a^2) \omega)(r+T)} e^{4k \epsilon r} \, dr 
\leq \frac{\rho_{18}}{4k \epsilon} + \frac{1}{1024k (\rho - 2\mu(a^2) \omega)} + \frac{1024k (\rho - 2\mu(a^2) \omega)}{1024k (\rho - 2\mu(a^2) \omega)} = \rho_{19}.
\]
By (205),
\[
E[K_{11}^{10} (\omega)] \leq c_{94} \rho_{13} + c_{95} \rho_{14} + c_{95} \rho_{15} + c_{96} \rho_{15} + c_{97} \rho_{19} \leq \rho_{20}.
\]
By (204),
\[
E[K_{10}^{10} (\omega)] \leq c_{92} (1 + \rho_{7} + \rho_{8} + \rho_{12} + \rho_{20}) \leq \rho_{21}.
\]
Therefore,
\[
E[C_{2}(\omega)] < \infty, \quad E[C_{3}^{2}(\omega)] \leq c_{91} (1 + \rho_{3} + \rho_{21}) \leq \rho_{22} < \infty. \quad (206)
\]
The proof is completed. \( \square \)

It is easy to see from (178) that \( \Phi \) satisfies condition (H4) on \( \chi(\tau, \omega) \). According to Lemmas 8.1-8.2 and Theorem 2.1, we obtain our main result in this section as follows.

**Theorem 8.1.** Suppose (A1)-(A2) and (196) hold. Then the continuous cocycle \( \{ \Phi(t, \tau, \omega) \}_{t \geq 0, \tau \in \mathbb{R}, \omega \in \Omega} \) associated with (178) has a random exponential attractor \( \{ \tilde{E}(\tau, \omega) \}_{\tau \in \mathbb{R}, \omega \in \Omega} \) with the following properties: for every \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

(i) \( \tilde{E}(\tau, \omega) \) is measurable in \( \omega \) and compact in \( E \) and \( \mathcal{A}(\tau, \omega) \subseteq \tilde{E}(\tau, \omega) \subseteq \chi(\tau, \omega) \), where \( \mathcal{A}(\tau, \omega) \) is the random attractor given in Theorem 5.1;

(ii) \( \Phi(t, \tau - t, \theta_{-t} \omega) \tilde{E}(\tau, \omega) \subseteq \tilde{E}(\tau, \omega) \) for all \( t \geq 0 \);

(iii) \( \dim_f \mathcal{A}(\tau, \omega) \leq \dim_f \tilde{E}(\tau, \omega) \leq \frac{4n_0 \ln(\sqrt{2n_0 \lambda_{n_0} + 1})}{\ln 4} \) < \( \infty \), where \( n_0 \in \mathbb{N} \) is a finite integer number;

(iv) \( \lim_{t \to 0} d_{\omega}(\tilde{E}(\tau + t, \theta_{t} \omega), \tilde{E}(\tau, \omega)) = 0 \) and \( \lim_{t \to 0} d_{\omega}(\tilde{E}(\tau - t, \theta_{-t} \omega), \tilde{E}(\tau, \omega)) = 0 \).

(v) for every \( B(\tau, \omega) \in B \in \mathcal{D}(E) \), there exist a random variable \( \tilde{T}_B(\tau, \omega) \geq 0 \) and a tempered random variable \( \tilde{b}_\omega > 0 \) such that
\[
d_{\omega}(\Phi(t, \tau - t, \theta_{-t} \omega)B(\tau - t, \theta_{-t} \omega), \tilde{E}(\tau, \omega)) \leq \tilde{b}_\omega e^{-\sigma_0 t}, \quad t \geq \tilde{T}_B(\tau, \omega), \quad (207)
\]where \( \sigma_0 > 0 \) is a constant.
Proof. It is easy to see that (196) implies (101). Taking $t = t_0 = \frac{16\ln 2}{\rho_{23}} < \infty$ in (177)-(176). By (198), we have

$$\rho_{23} = 2E[\hat{C}^2_t(\omega)] + 2\hat{\rho}_{0}^2E[C^2_0(\omega)] + \hat{\lambda}_0E[\hat{C}_1(\theta_{\omega})] + \hat{\lambda}_{0}^2E[\hat{C}_0(\omega)] < \infty.$$ 

From [37] and $U \subset \mathbb{R}^3$, it follows that $\lambda_n \geq c_{98}n^{\frac{3}{2}} \rightarrow \infty$, where $c_{98} > 0$ is a constant just depending on $|U|$ and $|\partial U|$. Thus, $0 < \frac{1}{\sqrt{\lambda_n}} \leq \frac{1}{c_{98} n^{\frac{3}{2}}} \rightarrow 0$. Then there must exists a finite integer number $n_0 \in N$ satisfying

$$n_0 + 1 \geq \max \left\{ \frac{2}{\sqrt{c_{98}^2}}, \frac{8}{\sqrt{c_{98}^3}} \right\},$$

which implying that

$$\hat{\delta} = \frac{2}{\sqrt{\lambda_{n_0} + 1}} \leq \min \left\{ \frac{1}{16}, e^{-\frac{\sqrt{\rho_{23}}}{2}} \right\}.$$ 

From Lemmas 8.1-8.2 and the continuity of $\Phi(t, \tau - t, \theta_{\omega})$ for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t \geq 0$, a family of tempered closed random subsets $\{\chi(\tau, \omega)\}$ satisfies conditions (H2) and (H4) in Theorem 2.1. According to Theorem 2.1, it follows that (i)-(iv) of Theorem 8.1 hold and there exist a random variable $\tilde{T}_\xi(\tau, \omega) \geq 0$ and a tempered random variable $\hat{b}_\omega > 0$ such that

$$d_h(\Phi(t, \tau - t, \theta_{-\omega}), \chi(\tau, \omega), \hat{E}(\tau, \omega)) \leq \hat{b}_\omega e^{-\frac{\sqrt{\rho_{23}}}{4} t}, \quad t \geq \tilde{T}_\xi(\tau, \omega). \quad (208)$$

By (172), $\varphi(\tau, t, \theta_{-\omega}, B_1(t, \theta_{-\omega})) \subset \chi(\tau, \omega)$ for $t \geq 2T_1(\omega)$. Thus, by (163), there exists $\hat{b}_\omega$ such that

$$d_h(\varphi(\tau, t, \theta_{-\omega}, B_1(t, \theta_{-\omega})), \chi(\tau, \omega)) \leq \hat{b}_\omega e^{-\hat{\sigma} t}, \quad t \geq T_\nu(\omega) + 2T_1(\omega); \quad (209)$$

where

$$\hat{\sigma} = \frac{\hat{\sigma}}{4} \left( \rho - \mu \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right) \right) > 0$$

and

$$\hat{b}_\omega = \sup_{t \in \mathbb{R}} \left\{ Q(\theta_{-\omega}) e^{-\frac{\hat{\sigma}}{2} t} \left( \rho - \mu \left( \frac{|a|}{\sqrt{\pi \alpha}} + \frac{|a|^2}{2\alpha} \right) \right) \right\} < \infty.$$ 

Again by the definition of $B_1(\tau, \omega)$ in (70), the absorbing property of $B_0(\omega)$ in Lemma 3.1, (208)-(209) and Lemma 7.3, the exponential attraction (v) of Theorem 8.1 hold. The proof is completed. \qed

9. Special example. Consider a special example of problem (1):

$$\begin{cases}
\frac{du}{dt} + \alpha u + (-\Delta u + \tilde{f}(u, x))dt = g(x, t)dt + au \circ dW(t) \text{ in } U \times (\tau, +\infty), \\
u(x, t) \big|_{t \in \partial U} = 0, \quad t \geq \tau, \quad \tau \in \mathbb{R}, \\
\nu(x, \tau) = u_\tau(x), \quad u_\tau(x, \tau) = u_\tau(x), \quad x \in U,
\end{cases} \quad (210)$$

where

$$\tilde{f}(u, x) = a_0(x) + a_1(x)u + a_2(x)u^2 + a_3(x)u^3, \quad a_3(x) > 0, \quad a_i(x) \in C^1(\bar{U}), \quad i = 0, 1, 2, 3,$$

(211)
which arising in the relativistic quantum mechanics equation \([50]\). We can check that \(\tilde{f}(u, x)\) satisfies conditions \((A1)-(A2)\). In fact, let \(\tilde{f}_1(u, x) = a_3(x)u^4\), \(\tilde{f}_2(u, x) = a_0(x) + a_1(x)u + a_2(x)u^2\), then for each \(x \in U\) and \(u \in \mathbb{R}\),
\[
0 \leq G_1(u, x) = \int_0^u a_3(x) r^3 dr = \frac{1}{4} a_3(x) u^4 \leq \tilde{c}_0 u \tilde{f}_1(u, x) \leq \tilde{c}_1 G_1(u, x),
\]
\[
\tilde{f}_1 \in C^2(\mathbb{R}, \mathbb{R}), \quad \tilde{f}_1(u, 0, x) = 0, \quad |\tilde{f}_1''(u, x)| \leq \tilde{c}_2 |u|,
\]
\[
\tilde{f}_2(\cdot, x) \in C^2(\mathbb{R}, \mathbb{R}), \quad |\tilde{f}_2''(u, x)| \leq \tilde{c}_4 + \tilde{c}_0 |u|, \quad |\tilde{f}_2''(u, x)| \leq \tilde{c}_6,
\]
\[
\tilde{c}_4 u^4 - \tilde{\beta}_1(u) \leq G(u, x) = a_0(x)u + \frac{1}{2} a_1(x)u^2 + \frac{1}{3} a_2(x)u^3 + \frac{1}{4} a_3(x)u^4 \leq \tilde{c}_5 u \tilde{f}(u, x) + \tilde{\beta}_2(x),
\]
\[
|\tilde{f}_1'(u_1, x) - \tilde{f}_1'(u_2, x)| \leq \tilde{c}_7 (1 + |u_1|^2 + |u_2|^2) |u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R},
\]
\[
|\tilde{f}(u, x)| \leq \tilde{c}_8 (1 + |u|^3),
\]
where
\[
\tilde{c}_0 = \frac{1}{4}, \quad \tilde{c}_1 = 1, \quad \tilde{c}_2 = 6 \max_{x \in U} |a_3(x)|, \quad \tilde{c}_3 = \max_{x \in U} |a_1(x)|, \quad \tilde{c}_4 = \frac{1}{8} \min_{x \in U} |a_3(x)|,
\]
\[
\tilde{c}_5 = 2, \quad \tilde{c}_6 = 2 \max_{x \in U} |a_2(x)|, \quad \tilde{c}_7 = \max_{x \in U} \{|a_1'(x)|, |a_2(x)|\}, \quad p = 1,
\]
\[
\tilde{c}_8 = \max_{x \in U} \left\{ \frac{|a_0(x)|}{3}, \frac{|a_1(x)|}{3}, \frac{|a_2(x)|}{3}, a_3(x) + 1 \right\},
\]
\[
\tilde{\beta}_1(x) = \frac{3}{4} \left( \frac{6}{a_3(x)} \right)^{\frac{3}{4}} a_0^4(x) + \frac{3}{2} a_3(x) a_2^2(x) + \frac{1}{12} \left( \frac{6}{a_3(x)} \right)^{3} a_4^2(x),
\]
\[
\tilde{\beta}_2(x) = \left( \frac{3}{4} \left( \frac{1}{a_3(x)} \right)^{\frac{3}{4}} + \frac{3}{2} \left( \frac{3}{2a_3(x)} \right)^{\frac{1}{4}} a_0^4(x) + \left( \frac{1}{4a_3(x)} + \frac{9}{2a_3(x)} \right) a_2^2(x) \right) a_3^2(x) + \left( \frac{1}{12} \left( \frac{1}{a_3(x)} \right)^{3} + \frac{1}{2} \left( \frac{9}{2a_3(x)} \right)^{3} \right) a_4^2(x).
\]

In this case,
\[
\rho = \frac{\varepsilon}{4}, \quad \mu = \max \left\{ 4, \frac{\tilde{c}_8}{\tilde{c}_4}, \frac{2}{\sqrt{4}} \right\}, \quad \nu = \frac{1}{4}, \quad k = 4,
\]
and if
\[
|a| \leq \min \left\{ \frac{\alpha}{128\sqrt{\mu}}, \frac{\alpha\sqrt{\alpha}}{8192\mu}, \sqrt{\frac{\alpha\rho}{128\mu}}, \sqrt{\frac{\alpha\rho}{128\mu}} \right\}, \quad \mu = 1,
\]
then Theorem 8.1 is valid for \((210)\).

10. **Remark.** Consider the initial boundary valued problem of damped non-autonomous wave equations with additive white noise:
\[
\begin{cases}
 u_{tt} + au_t - \Delta u + f(u, x) = g(x, t) + h(x)W(t), \ t > \tau, x \in U, \tau \in \mathbb{R}, \\
 u(x, t) = 0, \quad t \geq \tau, \quad \tau \in \mathbb{R}, \\
 u(x, \tau) = u_\tau(x), \quad u_t(x, \tau) = \tau \tau(x), \quad x \in U, \quad \tau \in \mathbb{R},
\end{cases}
\]
where \(U, \alpha, g, W\) are as in section 1, \(h \in H^3_0(U) \cap H^2(U)\). Similar to section 8, under conditions \((A1)-(A2)\), problem \((213)\) has a random exponential attractor as in Theorem 8.1.
Here we notice that for the system (1), we require that the coefficient \( a \) of the random term is small (see (196)), but for the system (213), we don’t need such a condition, this is because that the multiplicative noise term depends on the state variable \( u \) but the additive noise term is independent of \( u \).

**Acknowledgments.** The authors would like to express their sincere thanks to the anonymous referees for their time and useful suggestions and comments, which led to a much improved quality of this work.

**REFERENCES**

[1] L. Arnold, *Random Dynamical Systems*, Springer-Verlag, Berlin, 1998.

[2] A. V. Babin and M. I. Vishik, *Attractors of Evolution Equations*, North-Holland Publishing Co., Amsterdam, 1992.

[3] F. Balibrea, T. Caraballo, P. E. Kloeden and J. Valero, Recent developments in dynamical systems: Three perspectives, *Inter. J. Bifur. Chaos*, **20**(2010), 2591–2636.

[4] P. W. Bates, K. Lu and B. Wang, Random attractors for stochastic reaction-diffusion equations on unbounded domains, *J. Differential Equations*, **246**(2009), 845–869.

[5] M. D. Blair, H. F. Smith and C. D. Sogge, Strichartz estimates for the wave equation on manifolds with boundary, *Ann. Inst. H. Poincaré Anal. Non Lineaire*, **26**(2009), 1817–1829.

[6] T. Caraballo, J. A. Langa and J. C. Robinson, Stability and random attractors for a reaction-diffusion equation with multiplicative noise, *Discrete Contin. Dyn. Syst.*, **6**(2000), 875–892.

[7] T. Caraballo, P. E. Kloeden and B. Schmalfuss, Exponentially stable stationary solutions for stochastic evolution equations and their perturbation, *Appl. Math. Optim.*, **50**(2004), 183–207.

[8] R. Carmona and D. Nualart, Random nonlinear wave equations: Smoothness of the solutions, *Prob. Theory Relat. Fields*, **79**(1988), 469–508.

[9] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society, Providence, RI, 2002.

[10] I. Chueshov, *Monotone Random Systems Theory and Applications*, Springer-Verlag, New York, 2002.

[11] H. Crauel and F. Flandoli, Attractors for random dynamical systems, *Probab. Th. Re. Fields*, **100**(1994), 365–393.

[12] H. Crauel and F. Flandoli, Hausdorff dimension of invariant sets for random dynamical systems, *J. Dynam. Differential Equations*, **10**(1998), 449–474.

[13] H. Crauel, A. Debussche and F. Flandoli, Random attractors, *J. Dyn. Diff. Eqns.*, **9**(1997), 307–341.

[14] A. Carvalho and S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: theoretical results, *Comm. Pure Appl. Anal.*, **12**(2013), 3047–3071.

[15] A. Carvalho and S. Sonner, Pullback exponential attractors for evolution processes in Banach spaces: properties and applications, *Comm. Pure Appl. Anal.*, **13**(2014), 1141–1165.

[16] R. Czaja and M. Efendiev, Pullback exponential attractors for nonautonomous equations Part I: Semilinear parabolic problems, *J. Math. Anal. Appl.*, **381**(2011), 748–765.

[17] R. Czaja and M. Efendiev, Pullback exponential attractors for nonautonomous equations Part II: Applications to reaction-diffusion systems, *J. Math. Anal. Appl.*, **381**(2011), 766–780.

[18] A. Debussche, On the finite dimensionality of random attractors, *Stochastic Anal. Appl.*, **15**(1997), 473–491.

[19] A. Debussche, Hausdorff dimension of a random invariant set, *J. Math. Pures Appl.*, **77**(1998), 967–988.

[20] M. Efendiev, Y. Yamamoto and A. Yagi, Exponential attractors for non-autonomous dissipative system, *J. Math. Soc. Japan*, **63**(2011), 647–673.

[21] X. Fan, Random attractor for a damped sine-Gordon equation with white noise, *Pacific J. Math.*, **216**(2004), 63–76.

[22] X. Fan, Random attractors for damped stochastic wave equations with multiplicative noise, *Internat. J. Math.*, **19**(2008), 421–437.

[23] X. Fan, Attractors for a damped stochastic wave equation of sine-Gordon type with sublinear multiplicative noise, *Stoch. Anal. Appl.*, **24**(2006), 767–793.
[24] E. Feireisl and E. Zuazua, Global attractors for semilinear wave equations with locally distributed nonlinear damping and critical exponent, *Commun. Partial Differential Equations*, 18 (1993), 1539–1555.

[25] C. Foias and E. Olson, Finite fractal dimension and Holder-Lipschitz parametrization, *Indiana Univ. Math. J.*, 45 (1996), 603–616.

[26] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, American Mathematical Society, Providence, RI, 1988.

[27] Y. Huang, Y. Zhao and Z. Yin, On the dimension of the global attractor for a damped semilinear wave equation with critical exponent, *J. Math. Phys.*, 41 (2000), 4957–4966.

[28] P. Imkeller and B. Schmalfuss, The conjugacy of stochastic and random differential equations and the existence of global attractors, *J. Dynam. Differential Equations*, 13 (2001), 215–249.

[29] T. Jordan, M. Pollicott and K. Simon, Hausdorff dimension for randomly perturbed self affine attractors, *Commun. Math. Phys.*, 270 (2006), 519–544.

[30] Y. Kifer, Attractors via random perturbations, *Commun. Math. Phys.*, 121 (1989), 445–455.

[31] S. Kuksin and A. Shirikyan, Stochastic dissipative PDE's and Gibbs measures, *Commun. Math. Phys.*, 213 (2000), 291–330.

[32] O. A. Ladyzhenskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.

[33] J. A. Langa, Finite-dimensional limiting dynamics of random dynamical systems, *Dyn. Syst.*, 18 (2003), 57–68.

[34] J. A. Langa and J. C. Robinson, Fractal dimension of a random invariant set, *J. Math. Pures Appl.*, 85 (2006), 269–294.

[35] J. A. Langa, A. Miranville and J. Real, Pullback exponential attractors, *Discrete Contin. Dyn. Syst.*, 26 (2010), 1329–1357.

[36] H. Li, Y. You and J. Tu, Random attractors and averaging for non-autonomous stochastic wave equations with nonlinear damping, *J. Differential Equations*, 258 (2015), 148–190.

[37] P. Li and S. T. Yau, Estimate of the first eigenvalue of a compact Riemann manifold, *Proceeding of Symposition in Pure Math.*, 36 (1980), 205–239.

[38] Y. Lv and W. Wang, Limiting dynamics for stochastic wave equations, *J. Differential Equations*, 244 (2008), 1–23.

[39] J. Milnor, On the concept of attractor, *Commun. Math. Phys.*, 99 (1985), 177–195.

[40] A. Miranville, V. Pata and S. Zelik, Exponential attractors for singularly perturbed damped wave equations: A simple construction, *Asymptot. Anal.*, 53 (2007), 1–12.

[41] H. E. Nusse and J. A. Yorke, The equality of fractal dimension and uncertainty dimension for certain dynamical systems, *Commun. Math. Phys.*, 150 (1992), 1–21.

[42] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[43] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, 2001.

[44] J. C. Robinson, Stability of random attractors under perturbation and approximation, *J. Differential Equations*, 186 (2002), 652–669.

[45] D. Ruelle, Small random perturbations of dynamical systems and the definition of attractors, *Commun. Math. Phys.*, 82 (1981/82), 137–151.

[46] D. Ruelle, Characteristic exponents for a viscous fluid subjected to time dependent forces, *Commun. Math. Phys.*, 93 (1984), 285–300.

[47] T. Sauer, J. A. Yorke and M. Casdagli, Embedology, *J. Stat. Phys.*, 65 (1993), 579–616.

[48] A. Savostianov and S. Zelik, Recent progress in attractors for quintic wave equations, *Math. Bohem.*, 139 (2014), 657–665.

[49] A. Shirikyan and S. Zelik, Exponential attractors for random dynamical systems and applications, *Stoch. PDE: Anal. Comp.*, 1 (2013), 241–281.

[50] R. Temam, *Infinite-dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

[51] P. Walters, *An Introduction to Ergodic Theory*, Graduate Texts in Mathematics, 79. Springer-Verlag, New York-Berlin, 1982.

[52] B. Wang, Random attractors for non-autonomous stochastic wave equations with multiplicative noise, *Discrete Contin. Dyn. Syst.*, 34 (2014), 269–300.

[53] B. Wang, Sufficient and necessary criteria for existence of pullback attractors for non-compact random dynamical systems, *J. Differential Equations*, 253 (2012), 1544–1583.
B. Wang, Asymptotic behavior of stochastic wave equations with critical exponents on $\mathbb{R}^3$, *Trans. Amer. Math. Soc.*, 363 (2011), 3639–3663.

B. Wang, Existence and upper semicontinuity of attractors for stochastic equations with deterministic non-autonomous terms, *Stochastics and Dynamics*, 14 (2014), 1450009, 31 pp.

G. Wang and Y. Tang, Fractal dimension of a random invariant set and applications, *J. Appl. Math.*, (2013), Art. ID 415764, 5 pp.

M. Yang, J. Duan and P. Kloeden, Asymptotic behavior of solutions for random wave equations with nonlinear damping and white noise, *Nonlinear Anal. Real World Appl.*, 12 (2011), 464–478.

S. Zelik, Asymptotic regularity of solutions of a nonautonomous damped wave equation with a critical growth exponent, *Commun. Pure Appl. Anal.*, 3 (2004), 921–934.

S. Zhou, On dimension of the global attractor for damped nonlinear wave equations, *J. Math. Phys.*, 40 (1999), 1432–1438.

S. Zhou, Dimension of the global attractor for damped nonlinear wave equations, *Proc. Amer. Math. Soc.*, 127 (1999), 3623–3631.

S. Zhou and X. Han, Pullback exponential attractors for non-autonomous lattice systems, *J. Dyna. Diff Eqns.*, 24 (2012), 601–631.

S. Zhou and L. Wang, Kernel sections for damped non-autonomous wave equations with critical exponent, *Discrete Contin. Dyn. Syst.*, 9 (2003), 399–412.

S. Zhou, F. Yin and Z. Ouyang, Random attractor for damped nonlinear wave equations with white noise, *SIAM J. Appl. Dyn. Syst.*, 4 (2005), 883–903.

S. Zhou and M. Zhao, Random attractors for damped non-autonomous wave equations with memory and white noise, *Nonlinear Anal.*, 120 (2015), 202–226.

S. Zhou and M. Zhao, Fractal dimension of random invariant sets for nonautonomous random dynamical systems and random attractor for stochastic damped wave equation, *Nonlinear Anal.*, 133 (2016), 292–318.

S. Zhou and M. Zhao, Fractal dimension of random attractor for stochastic damped wave equation with multiplicative noise, *Discrete Contin. Dyn. Syst.*, 36 (2016), 2887–2914.

Received September 2016; revised June 2017.

E-mail address: wangzhaojuan2006@163.com
E-mail address: zhoushengfan@yahoo.com