There is no strongly regular graph with parameters 
\((460, 153, 32, 60)\)

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Abstract. We prove that there is no strongly regular graph \((\text{SRG})\) with parameters \((460, 153, 32, 60)\). The proof is based on a recent lower bound on the number of 4-cliques in a SRG and some applications of Euclidean representation of SRGs.

1. Introduction

A finite, undirected, simple graph \(G = (V, E)\) with vertices \(V\) and edges \(E\) is called strongly regular with parameters \((v, k, \lambda, \mu)\) if \(G\) is \(k\)-regular on \(v\) vertices, and, in addition, any two adjacent vertices of \(G\) have exactly \(\lambda\) common neighbors, while any two non-adjacent vertices of \(G\) have exactly \(\mu\) common neighbors.

The parameters \((v, k, \lambda, \mu)\) of a SRG must satisfy certain known conditions (see [BvL84]), but in general it is an open question to determine parameters \((v, k, \lambda, \mu)\) for which strongly regular graphs (SRGs) exist, and, in case when they do exist, to classify such graphs. A list of known results for \(v \leq 1300\) is maintained at [Bro].

As an application of a recently established lower bound on the number of 4-cliques in a SRG (see [BPR17] and [BPR]) and of Euclidean representation of SRGs, we obtain the following non-existence result in a very special case.

Theorem 1. There is no strongly regular graph with parameters \((460, 153, 32, 60)\).

Some general background on SRGs can be found in [BH12, Chapter 9] and [Cam04], while [BH12, Chapter 8] and [BvL84] contain details on Euclidean representation of SRGs. The argument with the Gram matrix used here has been extensively applied in [BPR17].

2. Proof of Theorem 1

Assume that a SRG \(G = (V, E)\) with parameters \((v, k, \lambda, \mu) = (460, 153, 32, 60)\) exists. Then adjacency matrix \(A\) of \(G\) satisfies the equations \(AJ = 153J\) and \(A^2 + 28A - 93I = 60J\), where \(I\) is the identity matrix and \(J\) is the matrix having all entries equal to 1. Consequently, \(A\) has the

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following spectrum: $153^1 3^{144} (-31)^{45}$. The Euclidean representation of $G$ defines a mapping $V \ni u \mapsto x_u \in \mathbb{R}^{45}$ such that all $x_u$ are unit vectors and the dot products between these vectors depend only on the adjacency between the corresponding vertices of $G$. More precisely, for two different vertices $u, w \in V$, we have

$$\langle x_u, x_w \rangle = \begin{cases} p, & \text{if } u \text{ is adjacent to } w, \\ q, & \text{if } u \text{ is not adjacent to } w, \end{cases}$$

where $p = \frac{-31}{153}$, $q = \frac{5}{51}$, and $\langle x, y \rangle$ is the Euclidean dot product in $\mathbb{R}^{45}$.

The first step is to show that $G$ has at least $228111$ complete subgraphs of size $4$. This follows from [BPR17] (see also the bounds on the number of 4-cliques in [BPR]). Let us give a brief outline of the proof. For each edge $e = \{u, w\} \in E$ we consider the unit vector $y_e = \frac{x_u + x_w}{\|x_u + x_w\|}$. A simple calculation shows that the distribution of dot products between $\{x_u\}_{u \in V}$ and $\{y_e\}_{e \in E}$ depends only on the SRG parameters and the number of 4-cliques. Since the Gegenbauer polynomials $C_t^{(d-2)/2}(x)$ are positive definite on $S^{d-1}$, by applying them to the Gram matrix of $\{x_u\}_{u \in V} \cup \{y_e\}_{e \in E}$ we get a positive definite matrix (here $d = 45$, $t = 4$). To get an inequality for the number of 4-cliques we simply compute the value of the corresponding quadratic form on the vector that takes value $1$ on $x_u$’s and $a$ on $y_e$’s and optimize the parameter $a$.

Next, for any two adjacent vertices $u, w \in V$ (i.e., $\{u, w\} \in E$), let $V_{u,w}$ be the set of vertices $t \in V$ adjacent to both $u$ and $w$. Note that any pair of adjacent vertices in $V_{u,w}$ forms a 4-clique together with $u$ and $w$. Now choose $u, w$ so that the number of edges in the subgraph of $G$ induced by $V_{u,w}$ is largest possible (among all possible edges $\{u, w\} \in E$). Let $V := V_{u,w}$, $\widetilde{G}$ be the subgraph of $G$ induced by $V$, and let $m$ be the number of edges in $\widetilde{G}$. Since $G$ has $35190$ edges, we get the following inequality on $m$ from the lower bound on the number of 4-cliques: $m \geq \frac{6 \cdot 228111}{35190}$, so $m \geq 39$. For an upper bound on $m$, we will make use of the above Euclidean representation. Define $X_1 := \sum_{t \in V} x_t$ and $X_2 := x_u + x_w$. The Gram matrix $M := (\langle X_i, X_j \rangle)_{i,j=1}^{2}$ is positive semi-definite, therefore, $\det M \geq 0$. Explicitly, in terms of $m$ and graph parameters we have

$$M = \begin{pmatrix} \lambda + 2mp + (\lambda^2 - \lambda - 2m)q & 2\lambda p \\ 2\lambda p & 2 + 2p \end{pmatrix} = \frac{1}{153^2} \begin{pmatrix} 19776 - 92m & -1984 \\ -1984 & 244 \end{pmatrix}.$$

The inequality $\det M \geq 0$ leads to $m \leq \frac{2416}{61}$, therefore $m \leq 39$.

Thus, $m = 39$, and the graph $\widetilde{G}$ on 32 vertices has 39 edges. Let $\mathcal{W}$ be a set of 14 vertices of $\widetilde{G}$ which have the largest degrees (in $\widetilde{G}$). We claim that the sum $\alpha$ of the degrees of vertices of $\mathcal{W}$ in $\widetilde{G}$ is at least 42. Indeed, if each such degree is at least 3, then we are clearly done.
Otherwise, the sum of the degrees of vertices not in $W$ is at most $(32 - 14)2 = 36$, which means that $\alpha \geq 2 \cdot 39 - 36 = 42$. Denote by $\beta$ the number of edges in the subgraph induced by $W$. Then we have $\alpha - 2\beta$ edges between $W$ and $V \setminus W$, and $39 + \beta - \alpha$ edges in $V \setminus W$. We take $Y_1 := \sum_{t \in V \setminus W} y_t$, $Y_2 := \sum_{t \in W} y_t$, and $Y_3 := x_u + x_w$ and apply previous considerations. For the Gram matrix $\tilde{M} := (\langle Y_i, Y_j \rangle)_{i,j=1}^3$ we clearly have $\det \tilde{M} \geq 0$. On the other hand, we compute

\[
\langle Y_1, Y_1 \rangle = 18 + 2(39 + \beta - \alpha)p + (18 \cdot 17 - 2(39 + \beta - \alpha))q,
\]

\[
\langle Y_2, Y_2 \rangle = 14 + 2\beta p + (14 \cdot 13 - 2\beta)q,
\]

\[
\langle Y_1, Y_2 \rangle = (\alpha - 2\beta)p + (18 \cdot 14 - (\alpha - 2\beta))q,
\]

\[
\langle Y_1, Y_3 \rangle = 18 \cdot 2p, \quad \langle Y_2, Y_3 \rangle = 14 \cdot 2p, \quad \langle Y_3, Y_3 \rangle = 2 + 2p,
\]

and therefore

\[
\det \tilde{M} = \left( - \frac{516304}{3581577} \alpha^2 + \frac{35785792}{3581577} \alpha \right) - \left( \frac{1252672}{3581577} \beta + \frac{198599296}{1193859} \right) =: \varphi(\alpha) - \psi(\beta).
\]

The quadratic function $\varphi$ is decreasing for $\alpha \geq \frac{35785792}{2 \cdot 516304} = \frac{2114}{61}$, in particular for $\alpha \geq 42$. The linear function $\psi$ is clearly increasing. Since $39 + \beta - \alpha \geq 0$, we have $\beta \geq 3$. Now, since

\[
0 \leq \det \tilde{M} = \varphi(\alpha) - \psi(\beta) \leq \varphi(42) - \psi(3) = \frac{-270848}{132651} < 0,
\]

we get a contradiction and hence Theorem 1 is proved.

3. Conclusion

Let us remark that the exact same reasoning from the proof above can be applied to some other strongly regular graphs. For instance, with some trivial changes we obtain non-existence of strongly regular graphs with parameters $(5929, 1482, 275, 402)$ and $(6205, 858, 47, 130)$, for which the number of 4-cliques is bounded from below by 4805 and 113 respectively. The key property that these three graphs have in common is that they have a very small (but strictly positive) value of the Krein parameter $q_{22}^2$. The proof also goes through for some strongly regular graphs that satisfy $q_{22}^2 = 0$, or equivalently

\[
(s + 1)(k + s + rs) = (k + s)(r + 1)^2.
\]

In this case the above reasoning shows that all $\lambda$-subgraphs must be regular. The smallest set of parameters that can be ruled out in this way is $(2950, 891, 204, 297)$. Alternatively, the non-existence in this case can be shown by noting that the first subconstituent must be strongly regular, but there exist no strongly regular graphs on 891 vertices of degree 204 (since there are no feasible parameters with $v = 891$ and $k = 204$).
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