Research Article
Stochastic Stability Analysis of Coupled Viscoelastic Systems with Nonviscously Damping Driven by White Noise

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Nonviscously damped structural system has been raised in many engineering fields, in which the damping forces depend on the past time history of velocities via convolution integrals over some kernel functions. This paper investigates stochastic stability of coupled viscoelastic system with nonviscously damping driven by white noise through moment Lyapunov exponents and Lyapunov exponents. Using the coordinate transformation, the coupled Itô stochastic differential equations of the norm of the response and angles process are obtained. Then the problem of the moment Lyapunov exponent is transformed to the eigenvalue problem, and then the second-perturbation method is used to derive the moment Lyapunov exponent of coupled stochastic system. Lyapunov exponent also can be obtained according to the relationship between moment Lyapunov exponent and Lyapunov exponent. Finally, the effects of various physical quantities of stochastic coupled system on the stochastic stability are discussed in detail. These results are validated by the direct Monte Carlo simulation technique.

1. Introduction

Viscoelastic materials are widely used in aerospace, construction, textile, and other industries, because they have a series of excellent properties, including light weight, high strength, wide source, and good shock absorption. These materials’ stress depends on the past time history of stain; that is, the stress will increase correspondingly with the increase of time and the bucking will be more likely to occur. Therefore, the research of dynamical behavior of viscoelastic system has received a lot of interests in recent years [1–4].

To better understand the dynamical behavior of the viscoelastic system, many methods had been put forward. McTavish et al. [5] presented GHM method to analyze the linear multiple degree of freedom viscoelastic damping systems. Li et al. [6] studied the decay rate of energy functional of nonlinear viscoelastic full Marguerre-von Kármán shallow shell system. Later, Li [7] also showed the existence and uniqueness of weak solution of this system by applying the Galerkin finite element method. Nieto and Ahmad [8] used the generalized quasi-linearization method to obtain the explicit approximate solutions of an initial and terminal value problem for the forced Duffing equation with nonviscous damping. Recently, Li and Du [9] studied general energy decay of a degenerate viscoelastic Petrovsky-type plate equation with boundary feedback through a priori estimate and analysis of Lyapunov-like functional.

The dynamic stability as an important field of nonlinear dynamics has attached increasing attention in recent years. Guo et al. [10] derived a new simple sufficient condition of the global asymptotic stability of the integrodifferential systems with delay through constructing suitable Lyapunov functions combined with the analytical technique. And then Meng et al. [11, 12] obtained a sufficient condition of uniform stability for nonlinear integrodifferential system. However, when we consider the dynamical behavior of the system in a real environment, the stochastic excitation cannot be ruled out, such as wind gusts, earthquakes, and ocean waves [13–15]. Stochastic stability of the system under the stochastic excitation has attracted more and more attention [16–20]. The Lyapunov exponent specially has been researched by many scholars as an important indication for judging the stability of stochastic system. For example, Potapov [21] derived the sufficient condition of the almost-sure asymptotic
stability of elastic and viscoelastic systems by the Lyapunov’s direct method. He [22] also studied the stability of elastic and viscoelastic systems under the non-Gaussian excitation. For stochastic stability of high dimensional coupled system, Pavlović et al. [23] proposed the direct Lyapunov method to investigate the almost-sure stochastic stability of a viscoelastic double-beam system under parametric excitation. Then they used the same method to analyze the instability of coupled nanobeam systems subject to compressive axial loading [24]. These works only indicate that the system is almost-surely stable.

According to the theory of large deviation, which was first proposed by Baxendale and Stroock [25], the almost-sure stability of stochastic system does not mean the moment stability. Therefore, it is necessary to study the moment Lyapunov exponent of a stochastic system. The moment Lyapunov exponent can judge not only almost-sure stability but also the moment stability [26]. Nevertheless, analytic solutions of the moment Lyapunov exponents are very difficult to derive. To overcome this, many scholars developed different approximation methods to study the moment Lyapunov exponents of the stochastic system. For example, Sri Namachchivaya and Van Roessel [27] studied the moment Lyapunov exponents of two-degrees-of-freedom coupled elastic oscillators under real noise excitation by combining an asymptotic approximation for the moment Lyapunov exponents. Kozić and his associates [28–30] developed the first-order perturbation approach to obtain weak noise expansion of moment Lyapunov exponents and Lyapunov exponents for a stochastically coupled double-beam system and Timoshenko beam system, respectively. Subsequently, Stojanović and Petković [31] used a perturbation method to study the moment Lyapunov exponents and the Lyapunov exponents of the three elastically connected Euler beams. More recently, Deng [32] applied the averaging method to establish the moment Lyapunov exponent of a viscoelastic coupled system with nonviscous damping. Deng [33] investigated stochastic stability of two-degrees-of-freedom coupled viscoelastic system under white noise through moment Lyapunov exponent, but its damping term is viscous.

The purpose of this paper is to study stochastic stability of a viscoelastic coupled system with nonviscous damping subject to Gaussian white noise excitation through moment Lyapunov exponents and Lyapunov exponents, in which the nonviscous damping term is expressed by Boltzmann superposition integral with a hereditary relaxation kernel. The paper is organized as follows. Section 2 derives the governing equations of motion of a stochastic viscoelastic system with a nonviscous damping. And then solving the problem of the moment Lyapunov exponent is changed into the eigenvalue and eigenfunction problem through a suitable transformation. In Section 3, the zeroth-order and the first-order and second-order perturbation solutions of the moment Lyapunov exponents are derived, respectively, based on the second-order perturbation method. The effects of different physical quantities on the stochastic stability of the coupled system are discussed under the given parameters. Correspondingly, the results are verified by means of the direct Monte Carlo simulation in Section 4. Finally, conclusions will be drawn in Section 5.

2. Problem Formulation

Considering a coupled viscoelastic system with nonviscous damped structure driven by white noise excitation, the governing equations can be expressed as

\[
\ddot{q}_1 + \omega_1^2 q_1 + \epsilon \beta_1 \int_0^t \gamma(t - \tau) \dot{q}_1(\tau) d\tau + \epsilon H_1 (q_1 - q_2) - \sqrt{\epsilon} K_1 \xi(t) q_1 = 0,
\]

\[
\ddot{q}_2 + \omega_2^2 q_2 + \epsilon \beta_2 \int_0^t \gamma(t - \tau) \dot{q}_2(\tau) d\tau + \epsilon H_2 (q_2 - q_1) - \sqrt{\epsilon} K_2 \xi(t) q_2 = 0,
\]

where \(q_1\) and \(q_2\) are generalized displacements. \(\omega_1\) and \(\omega_2\) are natural frequencies. \(\epsilon\) is a small parameter. \(\beta_1\) and \(\beta_2\) are nonviscous damping coefficients. \(H_1\) and \(K_1\) are constants. \(\xi(t)\) is a Gaussian white noise with zero mean and noise intensity \(\sigma^2\). \(\gamma(t)\) is a relaxation function of nonviscous damping, which can be expressed by

\[
y(t) = \frac{1}{\alpha} \exp \left\{ \frac{-t}{\alpha} \right\}
\]

in which \(\alpha\) is the relaxation parameter. Note that the limit case \(\alpha \rightarrow 0\), the nonviscous damping force reduces to classical viscous damping.

Let \(q_1 = x_1, \dot{q}_1 = \omega_1 x_2, q_2 = x_3, \dot{q}_2 = \omega_2 x_4\); system (1) can be represented as

\[
\dot{x}_1 = \omega_1 x_2,
\]

\[
\dot{x}_2 = -\omega_1 x_1 - \epsilon h_1 (x_1 - x_3) - \epsilon \gamma_1 \int_0^t \gamma(t - \tau) \dot{x}_1(\tau) d\tau + \sqrt{\epsilon} k_1 x_1 \xi(t),
\]

\[
\dot{x}_3 = \omega_2 x_4,
\]

\[
\dot{x}_4 = -\omega_2 x_3 - \epsilon h_2 (x_3 - x_1) - \epsilon \gamma_2 \int_0^t \gamma(t - \tau) \dot{x}_3(\tau) d\tau + \sqrt{\epsilon} k_2 x_3 \xi(t),
\]

where

\[
h_1 = \frac{H_1}{\omega_1},
\]

\[
h_2 = \frac{H_2}{\omega_2},
\]

\[
k_1 = \frac{K_1}{\omega_1},
\]

\[
k_2 = \frac{K_2}{\omega_2},
\]

\[
\gamma_1 = \frac{\beta_1}{\omega_1},
\]

\[
\gamma_2 = \frac{\beta_2}{\omega_2}.
\]
Using a coordinate transformation,
\begin{align*}
    x_1 &= a \cos \varphi \cos \theta_1, \\
    x_2 &= -a \cos \varphi \sin \theta_1, \\
    x_3 &= a \sin \varphi \cos \theta_2, \\
    x_4 &= -a \sin \varphi \sin \theta_2,
\end{align*}
\hspace{1cm} (5)
where $a$ represents the norm of the response. $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_2 \leq 2\pi$ are the angles process of the coupled stochastic system. $0 \leq \varphi \leq \pi/2$ denotes the coupling or exchange of energy between the coupled system. Equation (3) can be rewritten as
\begin{align*}
    \dot{\theta}_1 &= \omega_1 + \frac{\zeta}{a} \sin \varphi \sin \theta_1 \int_0^t \gamma(t - \tau) \ddot{x}_1(\tau) \ddt + \frac{\epsilon_1}{2} (1 + \cos 2\theta_1 - 2 \cos \theta_1 \cos \theta_2 \tan \varphi) \\
    &\quad - \sqrt{\kappa} \cos \theta_1 \xi(t), \\
    \dot{\theta}_2 &= \omega_2 + \frac{\zeta}{a} \cos \varphi \sin \theta_1 \int_0^t \gamma(t - \tau) \ddot{x}_2(\tau) \ddt + \frac{\epsilon_2}{2} (1 + \cos 2\theta_2 - 2 \cos \theta_1 \cos \theta_2 \cot \varphi) \\
    &\quad - \sqrt{\kappa} \cos \theta_2 \xi(t), \\
    \dot{\psi} &= \epsilon \left[ \frac{h_1}{4} \left( 2 \cos 2\varphi \sin \theta_2 \left( \cos \theta_1 + \sin \theta_1 \right) - \frac{\zeta}{a} \sin \varphi \sin \theta_1 \int_0^t \gamma(t - \tau) \ddt \\
    &\quad \cdot \dot{x}_1(\tau) \ddt + \frac{h_2}{4} \left( 2 \cos 2\varphi \sin \theta_2 \left( \cos \theta_1 + \sin \theta_1 \right) - \frac{\zeta}{a} \sin \varphi \sin \theta_1 \int_0^t \gamma(t - \tau) \ddt \\
    &\quad \cdot \dot{x}_2(\tau) \ddt \right) + \sqrt{\kappa} \left( k_1 \sin \theta_1 - k_2 \sin 2\theta_2 \right) \right], \\
    \dot{a} &= \epsilon_1 \left[ \frac{h_1}{4} \left( 2 \cos 2\varphi \sin \theta_2 \left( \cos \theta_1 + \sin \theta_1 \right) - \frac{\zeta}{a} \sin \varphi \sin \theta_1 \int_0^t \gamma(t - \tau) \ddt \\
    &\quad \cdot \dot{x}_1(\tau) \ddt + h_2 \left( 2 \cos 2\varphi \sin \theta_2 \left( \cos \theta_1 + \sin \theta_1 \right) - \frac{\zeta}{a} \sin \varphi \sin \theta_1 \int_0^t \gamma(t - \tau) \ddt \\
    &\quad \cdot \dot{x}_2(\tau) \ddt \right) - \sqrt{\kappa} \left( k_1 \sin \theta_1 - k_2 \sin 2\theta_2 \right) \right],
\end{align*}
\hspace{1cm} (6)
Therefore, the solution forms of $\theta_i$ can be expressed as $\theta_i = \omega_i (t-s) + \phi_i (t-s) = \theta_i (t) - \omega_i s$, $\dot{x}_i (t-s) = \omega_i^2 x_i (t-s) = -a\omega_i \cos \varphi \sin \theta_i (t-s)$ $\approx -a\omega_i \cos \varphi \sin (\theta_i - \omega_i s)$ $= \omega_i x_i \cos \omega_i s + \omega_i x_i \sin \omega_i s$, $\dot{x}_3 (t-s) = \omega_i x_4 (t-s) = -a\omega_i \cos \varphi \sin \theta_2 (t-s)$ $\approx -a\omega_i \cos \varphi \sin (\theta_2 - \omega_2 s)$ $= \omega_i x_4 \cos \omega_i s + \omega_i x_4 \sin \omega_i s$.

Substituting (7) into the nonviscous damping, one obtains
\begin{align*}
    \int_0^t \gamma(t - \tau) \ddt \ddt &= \frac{1}{\alpha} \int_0^t \exp \left\{ -\frac{1}{\alpha} \right\} \ddt(\ddt - s) \\
    &= M_1 x_2 + N_1 x_1, \\
\end{align*}
\hspace{1cm} (8)
Similarly, the following results also can be derived:
\begin{align*}
    \int_0^t \gamma(t - \tau) \ddt \ddt &= M_2 x_4 + N_2 x_3, \\
\end{align*}
\hspace{1cm} (9)
where
\begin{align*}
    M_1 &= \frac{\omega_1}{1 + \alpha^2 \omega_1^2}, \\
    N_1 &= \frac{\alpha \omega_1^2}{1 + \alpha^2 \omega_1^2}, \\
    M_2 &= \frac{\omega_2}{1 + \alpha^2 \omega_2^2}, \\
    N_2 &= \frac{\alpha \omega_2^2}{1 + \alpha^2 \omega_2^2}.
\end{align*}
\hspace{1cm} (10)
Substituting (8) and (9) into (6) and letting $P = \alpha^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{p/2}$ ($-\infty < p < +\infty$), (6) can be written as the following Itô stochastic differential equations:
\begin{align*}
    d\theta_1 &= m_1 (\theta_1, \theta_2, \varphi) dt + \sigma_{11} (\theta_1, \theta_2, \varphi) dW(t), \\
    d\theta_2 &= m_2 (\theta_1, \theta_2, \varphi) dt + \sigma_{21} (\theta_1, \theta_2, \varphi) dW(t), \\
    d\varphi &= m_3 (\theta_1, \theta_2, \varphi) dt + \sigma_{31} (\theta_1, \theta_2, \varphi) dW(t), \\
    dP &= Pm_4 (\theta_1, \theta_2, \varphi) dt + P\sigma_{41} (\theta_1, \theta_2, \varphi) dW(t),
\end{align*}
\hspace{1cm} (11)
where $W(t)$ is the standard Wiener process; the drift and diffusion coefficients can be rewritten as

$$m_1(\theta_1, \theta_2, \phi) = \omega_1 - \frac{\xi_1}{2} [M_1 \sin 2\theta_1 - N_1 (1 + \cos 2\theta_1)] + \frac{h_1}{2} (1 + \cos 2\theta_1 - 2 \cos \theta_1 \cos \theta_2) \cdot \tan \phi - \frac{k^2\sigma^2}{8} (2 \sin 2\theta_1 + \sin 4\theta_1),$$

$$m_2(\theta_1, \theta_2, \phi) = \omega_2 - \frac{\xi_2}{2} [M_2 \sin 2\theta_2 - N_2 (1 + \cos 2\theta_2)] + \frac{h_2}{2} (1 + \cos 2\theta_2 - 2 \cos \theta_1 \cos \theta_2) \cdot \cot \phi - \frac{k^2\sigma^2}{8} (2 \sin 2\theta_2 + \sin 4\theta_2),$$

$$m_3(\theta_1, \theta_2, \phi) = \left. e \left[ \frac{h_1}{4} \left( 2 (1 - \cos 2\phi) \sin \theta_1 \cos \theta_2 - \sin 2\phi \sin 2\theta_1 \right) + \frac{\xi_1}{4} \sin 2\phi [M_1 (1 - \cos 2\theta_1) - N_1 \sin 2\theta_1] + \frac{h_2}{2} \left( 12 (1 + \cos 2\phi) \cos \theta_1 \sin \theta_2 - \sin 2\phi \sin 2\theta_2 \right) - \frac{\xi_2}{4} \sin 2\phi \times [M_2 (1 - \cos 2\theta_2) - N_2 \sin 2\theta_2] + \frac{\sigma^2}{64} [-4k^2_1 (1 + 2 \cos 2\theta_1 + \cos 4\theta_1) \cdot \sin 2\phi + k^2_1 (1 - \cos 4\theta_1) \sin 4\phi - 4k_2 \sin 2\theta_1] \cdot \sin 2\theta_2 \sin 4\phi + 4k^2_2 (1 + 2 \cos 2\theta_2 + \cos 4\theta_2) \cdot \sin 2\phi \sin 4\phi \right],$$

$$m_4(\theta_1, \theta_2, \phi) = e \left. \frac{P}{4} \left( h_1 \left[ (1 + \cos 2\phi) \sin 2\theta_1 - 2 \sin 2\phi \cos \theta_2 \sin \theta_1 \right] - \xi_1 (1 + \cos 2\phi) \times [M_1 (1 - \cos 2\theta_1) - N_1 \sin 2\theta_1] + h_2 \left[ (1 - \cos 2\phi) \sin 2\theta_2 - 2 \sin 2\phi \cos \theta_1 \sin \theta_2 \right] - \xi_2 (1 - \cos 2\phi) [M_2 (1 - \cos 2\theta_2) - N_2 \sin 2\theta_2] + \frac{\sigma^2}{32} [10 + 3p + 16 \cos 2\theta_1 - 3 (p - 2) \cos 4\theta_1] + [8 + 4p + 16 \cos 2\theta_1 - 4 (p - 2) \cos 4\theta_1] \cos 2\phi + (p - 2) \cdot (1 - \cos 4\theta_1) \cos 4\phi + \frac{\sigma^2}{8} k^2_1 k^2_2 (p - 2) \times [10 + 3p + 16 \cos 2\theta_2 - 3 (p - 2) \cos 4\theta_2] - [8 + 4p + 16 \cos 2\theta_2 - 4 (p - 2) \cos 4\theta_2] \cos 2\phi + (p - 2) \cdot (1 - \cos 4\theta_2) \cos 4\phi \right) \right],$$

$$\sigma_{11}(\theta_1, \theta_2, \phi) = -\sqrt{\epsilon_1} \sigma \cos^2 \theta_1,$$

$$\sigma_{21}(\theta_1, \theta_2, \phi) = -\sqrt{\epsilon_2} \sigma \cos^2 \theta_2,$$

$$\sigma_{31}(\theta_1, \theta_2, \phi) = \sqrt{\epsilon} \frac{\sigma}{4} (k_1 \sin 2\theta_1 - k_2 \sin 2\theta_2) \sin 2\phi,$$

$$\sigma_{41}(\theta_1, \theta_2, \phi) = -\sqrt{\epsilon} \frac{\sigma}{4} \left[ k_1 (1 + \cos 2\phi) \sin 2\theta_1 + k_2 (1 - \cos 2\phi) \sin 2\theta_2 \right].$$

(12)

To obtain the eigenvalue problem for the $p$th moment Lyapunov exponents of a four-dimensional linear Itô stochastic system, a linear stochastic transformation [34] is introduced

$$S = T(\theta_1, \theta_2, \phi) P, \quad P = T^{-1}(\theta_1, \theta_2, \phi) S,$$

where $S$ is a new norm process depending on the transformation function $T(\theta_1, \theta_2, \phi)$. The transformation function $T(\theta_1, \theta_2, \phi)$ is defined on the independent stationary phase processes $\theta_1, \theta_2,$ and $\phi$ in the ranges $0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi, 0 \leq \phi \leq \pi/2$. Applying Itô formula, one can obtain

$$dS = \left[ \frac{1}{2} \sigma_{11}^2 T_{\theta_1}'' + \sigma_{11} \sigma_{21} T_{\theta_1}'' + \sigma_{11} \sigma_{31} T_{\theta_1}'' \right] + \left[ \sigma_{11} \sigma_{21} T_{\theta_1}'' + \sigma_{11} \sigma_{31} T_{\theta_1}'' + \frac{1}{2} \sigma_{31}^2 T_{\theta_1}'' \right] + \left[ \sigma_{11}^2 T_{\theta_2}'' + \sigma_{11} \sigma_{31} T_{\theta_2}'' \right] + \left[ \sigma_{11} \sigma_{21} T_{\theta_2}'' + \sigma_{11} \sigma_{31} T_{\theta_2}'' \right] \left[ \sigma_{31}^2 T_{\theta_2}'' + \sigma_{31} \sigma_{41} T_{\theta_2}'' \right] + \left[ \sigma_{41}^2 T_{\theta_2}'' + \sigma_{41} \sigma_{41} T_{\theta_2}'' \right] \right] dt + \left[ \sigma_{31}^2 T_{\theta_1}'' + \sigma_{31} \sigma_{31} T_{\theta_1}'' \right] dW(t).$$

(14)

If the transformation function $T(\theta_1, \theta_2, \phi)$ is bounded and nonsingular, both processes $P$ and $S$ have the same stability behavior. Therefore, transformation function $T(\theta_1, \theta_2, \phi)$ is chosen so that the drift term of the Itô differential equation (14) does not depend on the phase processes $\theta_1, \theta_2,$ and $\phi$, so one can obtain

$$dS = \Lambda(p) S dt$$

(15)

$$+ S T^{-1} (\sigma_{11} T_{\theta_1}'' + \sigma_{21} T_{\theta_2}'' + \sigma_{31} T_{\theta_2}'' + \sigma_{41} T_{\theta_2}'') dW(t).$$

Comparing (14) and (15), such transformation function $T(\theta_1, \theta_2, \phi)$ can be written by the following equation:

$$[L_0 + eL_1] T(\theta_1, \theta_2, \phi) = \Lambda(p) T(\theta_1, \theta_2, \phi).$$

(16)
where $L_0$ and $L_1$ are the following first-order and second-order differential operators:

\begin{align}
L_0 &= \omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}, \\
L_1 &= a_1 (\theta_1, \theta_2, \phi) \frac{\partial^2}{\partial \theta_1^2} + a_2 (\theta_1, \theta_2, \phi) \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \\
&+ a_3 (\theta_1, \theta_2, \phi) \frac{\partial^2}{\partial \theta_1 \partial \phi} + a_4 (\theta_1, \theta_2, \phi) \frac{\partial^2}{\partial \theta_2^2} \\
&+ a_5 (\theta_1, \theta_2, \phi) \frac{\partial^2}{\partial \theta_2 \partial \phi} + a_6 (\theta_1, \theta_2, \phi) \frac{\partial^2}{\partial \phi^2} \\
&+ b_1 (\theta_1, \theta_2, \phi) \frac{\partial}{\partial \theta_1} + b_2 (\theta_1, \theta_2, \phi) \frac{\partial}{\partial \theta_2} \\
&+ b_3 (\theta_1, \theta_2, \phi) \frac{\partial}{\partial \phi} + c (\theta_1, \theta_2, \phi),
\end{align}

(17)

in which

\begin{align}
a_1 (\theta_1, \theta_2, \phi) &= \frac{1}{2} k_1^2 \sigma^2 \cos^2 \theta_1, \\
a_2 (\theta_1, \theta_2, \phi) &= \sigma^2 k_1 k_2 \cos^2 \theta_1 \cos^2 \theta_2, \\
a_3 (\theta_1, \theta_2, \phi) &= -\frac{k_1^2 \sigma^2}{4} (k_1 \sin 2\theta_1 - k_2 \sin 2\theta_2) \cos^2 \theta_1 \\
&\quad \cdot \sin 2\phi, \\
a_4 (\theta_1, \theta_2, \phi) &= \frac{1}{2} k_2^2 \sigma^2 \cos^2 \theta_2, \\
a_5 (\theta_1, \theta_2, \phi) &= -\frac{k_1^2 \sigma^2}{4} (k_1 \sin 2\theta_1 - k_2 \sin 2\theta_2) \cos^2 \theta_2 \\
&\quad \cdot \sin 2\phi, \\
a_6 (\theta_1, \theta_2, \phi) &= \frac{\sigma^2}{32} (k_1 \sin 2\theta_1 - k_2 \sin 2\theta_2)^2 \sin^2 2\phi, \\
b_1 (\theta_1, \theta_2, \phi) &= -\frac{k_1}{2} [M_1 \sin 2\theta_1 - N_1 (1 + \cos 2\theta_1)] \\
&+ \frac{1}{2} h_1 (1 + \cos 2\theta_1 - 2 \cos \theta_1 \cos \theta_2 \tan \phi) - \frac{k_1^2 \sigma^2}{8} \\
&\times (2 \sin 2\theta_1 + \sin 4\theta_1) + \frac{k_2^2 \sigma^2 p}{4} [k_1 (1 + \cos 2\phi) \\
&\quad \cdot \sin 2\theta_1 + k_2 (1 - \cos 2\phi) \sin 2\theta_2] \cos^2 \theta_1, \\
b_2 (\theta_1, \theta_2, \phi) &= -\frac{k_1}{2} [M_2 \sin 2\theta_2 - N_2 (1 + \cos 2\theta_2)] \\
&+ \frac{1}{2} h_2 (1 + \cos 2\theta_2 - 2 \cos \theta_1 \cos \theta_2 \cot \phi) - \frac{k_2^2 \sigma^2}{8} \\
&\times (2 \sin 2\theta_2 + \sin 4\theta_2) + \frac{k_1^2 \sigma^2 p}{4} [k_1 (1 + \cos 2\phi) \\
&\quad \cdot \sin 2\theta_1 + k_2 (1 - \cos 2\phi) \sin 2\theta_2] \cos^2 \theta_1,
\end{align}

(18)

\begin{align*}
b_3 (\theta_1, \theta_2, \phi) &= \frac{L}{4} [2 (1 - \cos 2\phi) \sin \theta_1 \cos \theta_2, \\
&\quad \cdot \sin 2\phi] + \frac{\kappa}{4} \sin 2\phi [M_1 (1 - \cos 2\theta_1) \\
&\quad - N_1 \sin 2\theta_1] + \frac{b_3}{4} [-2 (1 + \cos 2\phi) \cos \theta_1 \sin \theta_2 \\
&\quad + \sin 2\phi \sin 2\theta_2] - \frac{\kappa}{4} \sin 2\phi [M_2 (1 - \cos 2\theta_2) \\
&\quad - N_2 \sin 2\theta_2] + \frac{\sigma^2}{64} [-4 k_1^2 (1 + 2 \cos 2\theta_1 + \cos 4\theta_1) \\
&\quad \times \sin 2\phi + k_1^2 (1 - \cos 4\theta_1) \sin 4\phi - 4 k_1 k_2 \sin 2\theta_1 \\
&\quad \cdot \sin 2\theta_2 \sin 4\phi + 4 k_2^2 (1 + 2 \cos 2\theta_2 + \cos 4\theta_2) \\
&\quad \cdot \sin 2\phi + k_2^2 (1 - \cos 4\theta_2) \sin 4\phi] - \frac{\sigma^2 p}{16} (k_1 \sin 2\theta_1 \\
&\quad - k_2 \sin 2\theta_2) \times [k_1 (1 + \cos 2\phi) \sin 2\theta_1 + k_2 (1 \\
&\quad - \cos 2\phi) \sin 2\theta_2] \sin 2\phi, \\
c (\theta_1, \theta_2, \phi) &= \frac{P}{4} \left[ h_1 [(1 + \cos 2\phi) \sin 2\theta_1 - 2 \sin 2\phi \\
&\quad \cdot \cos \theta_2 \sin \theta_1] - \frac{\sigma^2}{32} \times [10 + 3 p + 16 \cos 2\phi \\
&\quad - 3 (p - 2) \cos 4\theta_1] + [8 + 4 p + 16 \cos 2\theta_1 \\
&\quad - 4 (p - 2) \cos 4\theta_1] \times \cos 2\phi + (p - 2) (1 \\
&\quad - \cos 4\theta_1) \cos 4\phi] + \frac{\sigma^2 k_1 k_2}{8} (p - 2) (1 - \cos 4\phi) \\
&\quad \cdot \sin 2\theta_1 \sin 2\theta_2 + \frac{\sigma^2 k_1 k_2}{32} [(10 + 3 p + 16 \cos 2\theta_2 \\
&\quad - 3 (p - 2) \cos 4\theta_2] - [8 + 4 p + 16 \cos 2\theta_2 \\
&\quad - 4 (p - 2) \cos 4\theta_2] \cos 2\phi + (p - 2) (1 - \cos 4\theta_2) \\
&\quad \cdot \cos 4\phi \right].
\end{align*}
In the following investigation, the perturbation theory will be used to obtain the solution $\Lambda(p)$ of (16).

3. Moment Lyapunov Exponents

The method of deriving the eigenvalue problem for the moment Lyapunov exponents of a two-dimensional linear Itô stochastic system was first applied by Wedig \[34\]. The eigenvalue problem for a differential operator of three independent variables $\theta_1, \theta_2,$ and $\varphi$ will be identified from (16), in which $\Lambda(p)$ is the eigenvalue and $T(\theta_1, \theta_2, \varphi)$ is the associated eigenfunction. Meanwhile, the eigenvalue $\Lambda(p)$ is seen to be the Lyapunov exponent of the $p$th moment of system (1) from (15). Applying the perturbation theory, both the moment Lyapunov exponent $\Lambda(p)$ and the eigenfunction $T(\theta_1, \theta_2, \varphi)$ are expanded in power series of $\varepsilon$; that is,

\[
\Lambda(p) = \Lambda_0(p) + \varepsilon \Lambda_1(p) + \varepsilon^2 \Lambda_2(p) + \cdots
\]

Substituting (17) into the Lyapunov exponent of the $p$th moment Lyapunov exponent was first applied by Wedig \[34\]. The moment Lyapunov exponents of a two-dimensional linear

\[
T(\theta_1, \theta_2, \varphi) = T_0(\theta_1, \theta_2, \varphi) + \varepsilon T_1(\theta_1, \theta_2, \varphi)
\]

\[
+ \varepsilon^2 T_2(\theta_1, \theta_2, \varphi) + \cdots
\]

\[
+ \varepsilon^n T_n(\theta_1, \theta_2, \varphi) + \cdots
\]

Substituting (19) into (16) and equating terms of the equal powers of $\varepsilon$ lead to the following equations:

\[
e^0: L_0 T_0(\theta_1, \theta_2, \varphi) = \Lambda_0(p) T_0(\theta_1, \theta_2, \varphi),
\]

\[
e^1: L_0 T_1(\theta_1, \theta_2, \varphi)
\]

\[
+ L_1 T_0(\theta_1, \theta_2, \varphi) = \Lambda_0(p) T_1(\theta_1, \theta_2, \varphi)
\]

\[
+ \Lambda_1(p) T_0(\theta_1, \theta_2, \varphi),
\]

\[
\cdots
\]

\[
e^n: L_0 T_n(\theta_1, \theta_2, \varphi)
\]

\[
+ L_1 T_{n-1}(\theta_1, \theta_2, \varphi) = \Lambda_0(p) T_n(\theta_1, \theta_2, \varphi)
\]

\[
+ \Lambda_1(p) T_{n-1}(\theta_1, \theta_2, \varphi) + \cdots
\]

\[
+ \Lambda_n(p) T_0(\theta_1, \theta_2, \varphi),
\]

where $T_i(\theta_1, \theta_2, \varphi)$ must be positive and periodic in the range $0 \leq \theta_1 \leq 2\pi, 0 \leq \theta_2 \leq 2\pi$.

3.1. The Zeroth-Order Perturbation. Substituting (17) into (20), the zeroth-order perturbation equation is

\[
\omega_1 \frac{\partial T_0(\theta_1, \theta_2, \varphi)}{\partial \theta_1} + \omega_2 \frac{\partial T_0(\theta_1, \theta_2, \varphi)}{\partial \theta_2} = \Lambda_0(p) T_0(\theta_1, \theta_2, \varphi).
\]

Through $\Lambda(0) = \Lambda_0(0) + \varepsilon \Lambda_1(0) + \varepsilon^2 \Lambda_2(0) + \cdots + \varepsilon^n \Lambda_n(0) = 0$, thus, $\Lambda_n(0) = 0, n = 1, 2 \ldots$ will be obtained. Then (21) does not contain $p$, and the eigenvalue $\Lambda_0(p)$ is independent of $p$. Therefore, $\Lambda_0(0) = 0$ will lead to $\Lambda_n(p) = 0$ for arbitrary $p$.

At this point, (21) will be reduced to

\[
\omega_1 \frac{\partial T_0(\theta_1, \theta_2, \varphi)}{\partial \theta_1} + \omega_2 \frac{\partial T_0(\theta_1, \theta_2, \varphi)}{\partial \theta_2} = 0.
\]

Using the method of separation of variables, so we can express $T_0(\theta_1, \theta_2, \varphi)$ as

\[
T_0(\theta_1, \theta_2, \varphi) = \Theta_1(\theta_1) \Theta_2(\theta_2) \psi(\varphi),
\]

where $\Theta_1(\theta_1), \Theta_2(\theta_2),$ and $\psi(\varphi)$ are functions about $\theta_1, \theta_2$, and $\varphi$, respectively.

Substituting (23) into (22), (22) will become

\[
\omega_1 \frac{\partial \Theta_1}{\partial \theta_1} + \omega_2 \frac{\partial \Theta_2}{\partial \theta_2} = 0.
\]

Solving the above differential equation (24), one can easily obtain

\[
\Theta_1(\theta_1) = C_1 e^\omega_1 \theta_1,
\]

\[
\Theta_2(\theta_2) = C_2 e^\omega_2 \theta_2.
\]

In addition, since function $T_0(\theta_1, \theta_2, \varphi)$ is periodic about $\theta_1$ and $\theta_2$, the following boundary conditions can be obtained:

\[
T_0(\theta_1 + 2\pi, \theta_2, \varphi) = T_0(\theta_1, \theta_2 + 2\pi, \varphi) = T_0(\theta_1, \theta_2, \varphi).
\]

The adjoint equation of (22) is

\[
-\omega_1 \frac{\partial T_0^*(\theta_1, \theta_2, \varphi)}{\partial \theta_1} - \omega_2 \frac{\partial T_0^*(\theta_1, \theta_2, \varphi)}{\partial \theta_2} = 0.
\]

Applying the method of separation of variables to solve (27), one obtains

\[
T_0^* = \frac{F(\varphi)}{4\pi^2}, \quad \varphi \in (0, 2\pi),
\]

where $F(\varphi)$ is an arbitrary function.

3.2. The First-Order Perturbation. From (20), the first-order perturbation equation is

\[
L_0 T_1(\theta_1, \theta_2, \varphi) + L_1 T_0(\theta_1, \theta_2, \varphi) = \Lambda_0(p) T_1(\theta_1, \theta_2, \varphi) + \Lambda_1(p) T_0(\theta_1, \theta_2, \varphi),
\]

Based on the analysis results in Section 3.1, one obtains

\[
L_0 T_1(\theta_1, \theta_2, \varphi) = \Lambda_1(p) T_0(\varphi) - L_1 T_0(\theta_1, \theta_2, \varphi).
\]
Then the solvability condition of (30) is
\[
\langle \Lambda_1 (p) T_0 (\varphi) - L_1 T_0 (\varphi), T_0' \rangle = \frac{1}{4\pi^2} \int_0^{\pi/2} F(\varphi) \cdot \int_0^{2\pi} \int_0^{2\pi} \left[ \Lambda_1 (p) T_0 (\varphi) - L_1 T_0 (\varphi) \right] d\theta_1 d\theta_2 d\varphi = 0,
\]
(31)

Then the boundary conditions of (34) are determined by
\[
L_1 T_0 (\varphi) = c T_0 (\varphi) + b_3 \frac{\partial T_0 (\varphi)}{\partial \varphi} + a_6 \frac{\partial^2 T_0 (\varphi)}{\partial \varphi^2}.
\]

Applying the sense of the arbitrary of function \(F(\varphi)\), (31) will become
\[
\int_0^{\pi/2} \left[ (\Lambda_1 (p) - p Q(\varphi)) T_0 (\varphi) - (\mu (\varphi) + p \bar{\mu} (\varphi)) \frac{\partial T_0 (\varphi)}{\partial \varphi} - \frac{1}{2} R^2 (\varphi) \frac{\partial^2 T_0 (\varphi)}{\partial \varphi^2} \right] d\varphi = 0,
\]
(32)

where
\[
Q(\varphi) = \frac{1}{32} \left\{ \frac{1}{2} (6 + p) \left( k_1^2 + k_2^2 \right) \sigma^2 - 8 (M_1 \zeta_1 + M_2 \zeta_2) + \left[ (p + 2) \left( k_1^2 - k_2^2 \right) \sigma^2 - 8 (M_1 \zeta_1 - M_2 \zeta_2) \right] \right. \\
\left. \times \cos 2\varphi + \frac{1}{2} (-2 + p) \left( k_1^2 + k_2^2 \right) \sigma^2 \cos 2\varphi \right\},
\][33]

\[
\mu (\varphi) = -\frac{1}{32} \left\{ 2 \left( k_1^2 - k_2^2 \right) \sigma^2 - 8 (M_1 \zeta_1 - M_2 \zeta_2) \right\} \sin 2\varphi - \left( k_1^2 + k_2^2 \right) \sigma^2 \sin 2\varphi \cos 2\varphi, \]

\[
\bar{\mu} (\varphi) = -\frac{1}{32} \left\{ \left( k_1^2 - k_2^2 \right) \sigma^2 \sin 2\varphi - \left( k_1^2 + k_2^2 \right) \sigma^2 \sin 2\varphi \cos 2\varphi \right\},
\][33]

\[
R^2 (\varphi) = \frac{1}{64} \left( k_1^2 + k_2^2 \right) \sigma^2 \sin^2 2\varphi;
\]

then (32) is simplified as the following equation:
\[
L (p) T_0 (\varphi) = \Lambda_1 (p) T_0 (\varphi),
\]
\[
L (p) = \frac{1}{2} R^2 (\varphi) \frac{\partial}{\partial \varphi} + [\mu (\varphi) + p \bar{\mu} (\varphi)] \frac{\partial}{\partial \varphi} + p Q(\varphi).
\]
(34)

Then the boundary conditions of (34) are determined by considering the adjoint equation for the case of \(p = 0\),
\[
L^* \bar{m} (\varphi) = 0, \quad L^* = \frac{1}{2} \frac{\partial}{\partial \varphi} R^2 (\varphi) - \frac{\partial}{\partial \varphi} \mu (\varphi),
\]
(35)

where \(L^*\) is the Fokker-Planck operator and \(\varphi = 0\) and \(\varphi = \pi/2\) are the entrance boundaries [27]. The eigenfunction \(T_0 (\varphi)\) satisfies zero Neumann boundary condition, and \(\Lambda_1 (p)\) is the largest eigenvalue of (34) with zeros Neumann boundary. Therefore, the solution of (34) can be calculated from an orthogonal expansion [34]. Under zeros Neumann boundary conditions, \(T_0 (\varphi)\) can be expressed by a Fourier cosine series [27]; that is,
\[
T_0 (\varphi) = \sum_{n=0}^{\infty} z_n \cos 2n\varphi.
\]
(36)

Substituting (36) into (34), multiplying by \(\cos 2n\varphi\) in both sides and integrating for \(\varphi\), we can obtain
\[
\sum_{m=0}^{\infty} a_{mn} z_m = 2 \Lambda_1 (p) z_0,
\]
(37)

\[
\sum_{m=0}^{\infty} a_{mn} z_m = \Lambda_1 (p) z_n, \quad n = 1, 2, \ldots,
\]

where
\[
a_{mn} = \frac{4}{\pi} \int_0^{\pi/2} L (p) \left( \cos 2m\varphi \right) \cos 2n\varphi d\varphi,
\]
(38)

In order to guarantee the existence of the nontrivial solution for each \(z_n\), the coefficient matrix \(A = (a_{mn})\) must equal zero. Therefore, the problem of calculating \(\Lambda_1 (p)\) is translated into calculating the leading eigenvalue of \(A\). The sequence of approximations will be constructed by the eigenvalues of a sequence of the following submatrices:
\[
\begin{bmatrix}
\frac{1}{2} a_{00} \\
\frac{1}{2} a_{01} \\
\frac{1}{2} a_{10} \\
\frac{1}{2} a_{11} \\
\end{bmatrix},
\]
\[
\begin{bmatrix}
\frac{1}{2} a_{00} & a_{01} & \cdots \\
\frac{1}{2} a_{10} & a_{11} & \cdots \\
\frac{1}{2} a_{20} & a_{21} & a_{22} & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]
(39)

Obviously, the set of approximate eigenvalues obtained by this method converges to the associated true eigenvalues as \(n \to \infty\). In general, we can obtain the approximate...
eigenvalues by truncating $n$, which can be written as $\Lambda_1(p) = a_{00}/2$ and

$$\Lambda(p) = \epsilon \Lambda_1(p) + O(\epsilon^2) = \frac{1}{128} \epsilon p \left[ (10 + 3 \rho) (k_1^2 + k_2^2) \sigma^2 - 32 (M_1 \zeta_1 + M_2 \zeta_2) \right] + O(\epsilon^2).$$

The comparison results of approximate analytical moment Lyapunov exponents obtained by truncating $n$ and the direct Monte Carlo simulation results are shown in Figure 1. It is easily found that the approximate results agree well with the simulation results. Therefore, the first-order perturbation will be reduced to

$$\omega_1 \frac{\partial T_1}{\partial \theta_1} + \omega_2 \frac{\partial T_1}{\partial \theta_2} + c(\theta_1, \theta_2, \varphi) = \Lambda_1(p).$$

Without loss of generality, there is a relationship between two frequencies of the form $m_1 \omega_1 = m_2 \omega_2$, in which $m_1$ and $m_2$ are integers. The second frequency can be rewritten as $\omega_2 = k_1 \omega_1$. Then the function $c(\theta_1, \theta_2, \varphi)$ in (41) can be rewritten as

$$c(\theta_1, \theta_2, \varphi) = \Lambda_1(p) + f_0(\theta_1, \theta_2) + f_1(\theta_1, \theta_2) \cos 2\varphi + f_2(\theta_1, \theta_2) \cos 4\varphi + f_3(\theta_1, \theta_2) \sin 2\varphi,$$

where function $f_i(\theta_1, \theta_2)$ is the periodic function on $\theta_1, \theta_2$ and given as

$$f_0(\theta_1, \theta_2) = \frac{p}{4} \left( h_1 + \zeta_1 N_1 \right) \sin 2\theta_1 + \frac{p}{8} \left( 2 \zeta_1 M_1 + k_1^2 \sigma^2 \right) \cos 2\theta_1,$$

$$f_1(\theta_1, \theta_2) = \frac{p}{4} \left( h_2 + \zeta_2 N_2 \right) \sin 2\theta_2 + \frac{p}{8} \left( 2 \zeta_2 M_2 + k_2^2 \sigma^2 \right) \cos 2\theta_2,$$

$$f_2(\theta_1, \theta_2) = \frac{3k_1^2 \sigma^2}{128} p (2 - p) \cos 4\theta_1 + \frac{3k_2^2 \sigma^2}{128} p (2 - p) \cos 4\theta_2.$$

Note that the general solution $T_1(\theta_1, \theta_2)$ of (41) can not be obtained except in some special cases. Considering the nature
of the coefficients of (41), the series expansion of function $T_1(\theta_1, \theta_2, \varphi)$ can be presented in the following form:

$$T_1(\theta_1, \theta_2, \varphi) = T_{10}(\theta_1, \theta_2) + T_{11}(\theta_1, \theta_2) \cos 2\varphi + T_{12}(\theta_1, \theta_2) \cos 4\varphi + T_{13}(\theta_1, \theta_2) \sin 2\varphi$$

(44)

Substituting (44) into (41) and equating terms of the equal trigonometry function to give a set of partial differential equations, one obtains

$$\frac{\partial T_{1r}(\theta_1, \theta_2)}{\partial \theta_1} + \frac{\partial T_{1r}(\theta_1, \theta_2)}{\partial \theta_2} + f_r(\theta_1, \theta_2) = 0,$$

(45)

where the function $T_{1r}(\theta_1, \theta_2)$ can be written as

$$T_{10}(\theta_1, \theta_2) = -\frac{p(2\zeta_1 M_1 + k_1^2 \sigma^2)}{16\omega_1} \sin 2\theta_1 + \frac{p(h_1 + \zeta_1 N_1)}{8\omega_1} \cos 2\theta_1$$

$$- \frac{p(2\zeta_2 M_2 + k_2^2 \sigma^2)}{16\omega_1} \sin 2\theta_2 + \frac{p(h_2 + \zeta_2 N_2)}{8\omega_1} \cos 2\theta_2$$

$$- \frac{3k_1^2 p(2 - p) \sigma^2}{512\omega_1} \sin 4\theta_1$$

$$- \frac{3k_2^2 p(2 - p) \sigma^2}{512\omega_1} \sin 4\theta_2$$

$$+ \frac{kk_1 k_2 p(2 - p) \sigma^2}{64(1 - k^2) \omega_1} \sin 2\theta_1 \cos 2\theta_2$$

$$- \frac{k_1 k_2 p(2 - p) \sigma^2}{64(1 - k^2) \omega_1} \sin 2\theta_2 \cos 2\theta_1$$

$$+ C_0(\theta_2 - k\theta_1),$$

$$T_{11}(\theta_1, \theta_2) = -\frac{p \left[8(-\zeta_1 M_1 + \zeta_2 M_2) + (k_1^2 - k_2^2)(2 + p) \sigma^2\right]}{32\omega_1} \theta_1$$

$$- \frac{p(2\zeta_1 M_1 + k_1^2 \sigma^2)}{16\omega_1} \sin 2\theta_1$$

$$+ \frac{p(h_1 + \zeta_1 N_1)}{8\omega_1} \cos 2\theta_1$$

$$+ \frac{p(2\zeta_2 M_2 + k_2^2 \sigma^2)}{16\omega_1} \sin 2\theta_2$$

in which $C_r(\theta_2 - k\theta_1)$ are arbitrary functions of two variables, and we make assumptions as follows:

$$C_r(\theta_2 - k\theta_1) = A_{1r} + B_{1r} \sin(2\theta_2 - 2k\theta_1) + C_{1r} \sin(4\theta_2 - 4k\theta_1),$$

(46)

Here, the unknown constants $A_{1r}, B_{1r}, C_{1r} (r = 0, 1, 2, 3)$ will be determined by using the following conditions:

$$T_{1r}(0,0) = T_{1r}(0,2\pi) = T_{1r}(2\pi,0) = T_{1r}(2\pi,2\pi) = 0,$$

$$\frac{\partial T_{1r}(0,0)}{\partial \theta_1} = \frac{\partial T_{1r}(2\pi,0)}{\partial \theta_1},$$

$$\frac{\partial T_{1r}(0,0)}{\partial \theta_2} = \frac{\partial T_{1r}(2\pi,0)}{\partial \theta_2},$$

(47)

$$r = 0, 1, 2, 3.$$
Based on the solvability condition, one obtains

\[ A_{10} = \frac{p (h_1 + \zeta_2 N_2 + h_1 k + k_1 \zeta_3 N_3)}{8 k \omega}, \]

\[ A_{11} = \frac{p (h_1 + \zeta_2 N_2 - h_1 k - k_1 \zeta_3 N_3)}{8 k \omega}, \]

\[ A_{12} = 0, \]

\[ A_{13} = \frac{p (h_1 - h_1 k)}{2 (1 - k^2) \omega}, \]

\[ B_{10} = 0, \]

\[ B_{11} = -\frac{p \left[ 8 (-\xi_1 M_1 + \xi_2 M_2) + (k_1^2 - k_2^2) (2 + p) \sigma^2 \right] \cos^2 k \pi}{4 \sin 2k \pi (5 \cos 2k \pi + \cos 6k \pi) \omega} \]

\[ B_{12} = \frac{p \left( k_1^2 + k_2^2 \right)(2 - p) \sigma^2 \cos 2k \pi}{16 \left( 3 \sin 2k \pi + \sin 6k \pi \right) \omega}, \]

\[ C_{10} = 0, \]

\[ C_{11} = 0, \]

\[ C_{12} = 0, \]

\[ C_{13} = 0, \]

\[ C_{14} = \frac{p \left[ 8 (-\xi_1 M_1 + \xi_2 M_2) + (k_1^2 - k_2^2) (2 + p) \sigma^2 \right]}{32 \sin 2k \pi (5 \cos 2k \pi + \cos 6k \pi) \omega}, \]

\[ C_{15} = -\frac{p \left( k_1^2 + k_2^2 \right)(2 - p) \sigma^2}{64 \left( 4 \sin 4k \pi + \sin 8k \pi \right) \omega}. \]

3.3. The Second-Order Perturbation. To further investigate the effects of parameter \( H_1 \) on stochastic stability of system, the second-order perturbation equation of (20) will be used; that is,

\[ L_0 T_2 (\theta_1, \theta_2, \varphi) + L_1 T_1 (\theta_1, \theta_2, \varphi) + L_2 (\varphi) T_0 (\theta_1, \theta_2, \varphi) = \Lambda (p) T_1 (\theta_1, \theta_2, \varphi) + \Lambda (2) (\theta_1, \theta_2, \varphi). \]  

(50)

Based on the solvability condition, one obtains

\[
\Lambda_2 (p) = \frac{1}{2 \pi^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^{2\pi} \left[ a_1 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \theta_1^2} + a_2 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \theta_1 \partial \theta_2} + a_3 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \varphi} + a_4 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \theta_1 \partial \varphi} + a_5 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \theta_1 \partial \varphi} + a_6 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \theta_2 \partial \varphi} + a_7 (\theta_1, \theta_2, \varphi) \frac{\partial^2 T_1 (\theta_1, \theta_2, \varphi)}{\partial \varphi^2} \right] \]  

(51)

The solution has the form:

\[
\Lambda_2 (p) = \Lambda_{21} (p) \sin 4k \pi \frac{\cos 4k \pi}{2 + \cos 4k \pi} + \Lambda_{22} (p) \cos 4k \pi \frac{\cos 4k \pi}{2 + \cos 4k \pi} + \Lambda_{23} (p) \frac{1}{2 + \cos 4k \pi},
\]

(52)

where the values \( \Lambda_{21} (p), \Lambda_{22} (p), \Lambda_{23} (p) \) are given in Appendix A.

Substituting (40) and (52) into (19), the second-order approximate solution of the moment Lyapunov exponent is obtained as

\[
\Lambda (p) = \epsilon \Lambda_1 (p) + \epsilon^2 \Lambda_2 (p) + O (\epsilon^3),
\]

(53)

The corresponding Lyapunov exponents can be expressed as

\[
\lambda = \left. \frac{d \Lambda (p)}{dp} \right|_{p=0} = \epsilon \lambda_1 + \epsilon^2 \lambda_2 + O (\epsilon^3),
\]

(54)

where the values \( \lambda_{21}, \lambda_{22}, \) and \( \lambda_{23} \) are given in Appendix B.

4. Stochastic Stability Analysis

Through the above-mentioned analysis, the analytic expressions of moment Lyapunov exponents \( \Lambda (p) \) and Lyapunov
be determined from second-order perturbation in the subsequent research.

Therefore, we only consider the $\Lambda(p)$ exponents where $\Lambda(p)$ denotes the expected value and $\mathbb{E}[\cdot]$ is a solution process of a random system. If $\Lambda(p) < 0$, the system is almost-surely stable. Meanwhile, the moment stability of the system enhances with the increase of $H_2$. 

The variations of the moment Lyapunov exponents for different $H_2$ are also analyzed. From Figure 7, we find that the coupled stochastic system may be almost-surely stable due to the value of slope $\Lambda(p)$ less than 0 at the origin. However, the moment stability of system is not assured for $p < 0$ or $p$ is sufficiently large due to the moment Lyapunov exponents greater than 0. Besides, one can easily see that the moment stability of the system enhances with the increase of $H_2$.

Finally, the effects of $K_2$ on the moment Lyapunov exponents are investigated and shown in Figure 8. From those figures, the system may be almost-surely stable in terms of the Lyapunov exponents. It is also indicated that the system has the characteristic of the moment stability. And the moment stability of the coupled system will increase with the increase of $K_2$ under given parameters. The approximate analytical results agree well with numerical simulation results by the direct Monte Carlo.

5. Conclusions
In this paper, the moment stability and almost-sure stability of coupled viscoelastic system with nonviscous damping subject to Gaussian white noise excitation are investigated. The nonviscously damped structure is assumed to follow the stochastic differentialequations by means of the method of stochastic averaging. Then the linear transformation is introduced to derive the eigenvalue problem. In order to obtain the analytical expressions of eigenvalue, the second-order perturbation method is used. Finally, the effects of Gaussian noise, the relaxation parameter, the nonviscous damping coefficient, and two physical quantities, that is $H_2$, $K_2$, on the moment stability are discussed in detail. Then the corresponding results are verified through the direct Monte Carlo simulation technique. Based on the results above, the
Figure 3: Variation of the moment Lyapunov exponent for different $\alpha$. Solid-line in the analytical results, circle in the numerical results by Monte Carlo.

Figure 4: Variation of the moment Lyapunov exponent for different $\beta_2$. Solid-line in the analytical results, circle in the numerical results by Monte Carlo.

Figure 5: Variation of the moment Lyapunov exponent for different $\epsilon$. Solid-line in the analytical results, circle in the numerical results by Monte Carlo.
Figure 6: Variation of the moment Lyapunov exponent for different $\sigma$. Solid-line in the analytical results, circle in the numerical results by Monte Carlo.

Figure 7: Variation of the moment Lyapunov exponent for different $H_2$. Solid-line in the analytical results, circle in the numerical results by Monte Carlo.

Figure 8: Variation of the moment Lyapunov exponent for different $K_2$. Solid-line in the analytical results, circle in the numerical results by Monte Carlo.
appropriate parameters will help to strengthen the stability of the coupled viscoelastic system with nonviscous damping.

Appendix

A. Values of $\Lambda_{21}(p), \Lambda_{22}(p),$ and $\Lambda_{23}(p)$

$$\Lambda_{21}(p) = \frac{p(2 + p)}{262144(1 + k^2)\omega_1} \left( -4096 \left( -1 + k^2 \right) + k^2 \right) M_2 \zeta_2 (M_1 \zeta_1 - M_2 \zeta_2) - 128 \left( -1 + k^2 \right) \times \left[ 64k_1k_2 (M_1 \zeta_1 - M_2 \zeta_2) - 4k_2^2 (2 + p) M_2 \zeta_2 + k_1^2 \left[ (-10 + p) M_1 \zeta_1 + 3 (6 + p) M_2 \zeta_2 \right] \sigma^2 + 1024k_1k_2 (k_1^2 - k_2^2) (2 + p) + 16kk_1k_2 \left[ k_1^2 (-136 - 62p + p^2) + k_2^2 (120 + 66p + p^3) \right] + k_2^2 (-1 + k^2) \times \left[ k_1^2 (-312 - 130p + 15p^2) + k_2^2 (328 + 126p - 17p^2) \right] \sigma^2 \right) \right)\sigma^2$$

$$\Lambda_{22}(p) = \frac{p}{98304 (1 + k^2) \omega_1 \pi} \left( -3072 \left( -1 + k^2 \right) \cdot (2 + p) \pi (M_1 \zeta_1 - M_2 \zeta_2) \times [h_2 + N_2 \zeta_2 + k\pi (M_1 \zeta_1 - M_2 \zeta_2)] + 256k (k_1^2 + k_2^2) (2 + p) \cdot (4 + p) (h_1 - kh_2) - 768k\pi (1 + k^2) k_1^2 (10 + 3p) (h_1 + \omega_1 \zeta_1) + 384\pi (1 + k^2) (h_3 + N_2 \zeta_2) \left[ k_1^2 (2 + p)^2 - k_2^2 (4 + p) (6 + p) \right] + 768k\pi^2 (-1 + k^2) \cdot (k_1^2 (2 + p)^2 - k_2^2 (4 + p) (6 + p) \right] + 768k\pi^2 (-1 + k^2) \cdot (k_1^2 (2 + p)^2 - k_2^2 (4 + p) (6 + p) \right] + 768k\pi^2 (-1 + k^2) \cdot (2 + p) \pi \left( [M_1 \zeta_1 - M_2 \zeta_2] \times [h_2 + N_2 \zeta_2 + k\pi (M_1 \zeta_1 - M_2 \zeta_2)] + 256k (k_1^2 + k_2^2) (2 + p) \cdot (4 + p) (h_1 - kh_2) + 96k^2 \pi^2 (-1 + k^2) (h_2 + N_2 \zeta_2) \left[ 3k_1^2 (2 + p)^2 - k_2^2 (92 + 36p + 3p^2) \right] - 768(k_2^2 (10 + 3p) (h_1 + \omega_1 \zeta_1) + 768k\pi^2 (-1 + k^2) \right) \left( k_1^2 (10 + 3p) (h_1 + \omega_1 \zeta_1) + 768k\pi^2 (-1 + k^2) \right) \left( k_1^2 (10 + 3p) (h_1 + \omega_1 \zeta_1) \right) \sigma^2 \cdot \sigma^4 \right)$$

$$\Lambda_{23}(p) = \frac{p}{49152 (1 + k^2) \omega_1 \pi} \left( -768 \left( -1 + k^2 \right) \cdot (2 + p) \pi (M_1 \zeta_1 - M_2 \zeta_2) \times [3 (h_2 + N_2 \zeta_2 + 4k\pi (M_1 \zeta_1 - M_2 \zeta_2))] + \left[ 128k_1k_2 (k_1^2 + k_2^2) - 24\pi \sigma^2 \left( k_1^2 + k_2^2 \right) \sigma^2 - 24k (1 + k^2) \right] \pi^2 \left( k_1^2 - k_2^2 \right)^2 \sigma^4 \right)$$

B. Values of $\lambda_{21}, \lambda_{22},$ and $\lambda_{23}$

$$\lambda_{21} = -\frac{1}{16384 (1 + k^2) \omega_1} \left\{ -512 \left( -1 + k^2 \right) \cdot M_2 \zeta_2 (M_1 \zeta_1 - M_2 \zeta_2) - 32 \left( -1 + k^2 \right) \times \left[ 32kk_2 (M_1 \zeta_1 - M_2 \zeta_2) - 4k_1^2 M_2 \zeta_2 + k_2^2 (5M_1 \zeta_1 + 9M_2 \zeta_2) \right] \}^2 + k_2^2 (16k_1^2 (17) + 16k_2^2) - 39k_2^2 (1 - k^2) + 16kk_1k_2^2 (15 - 16k^2) + 41k_1k_2^2 \left( -1 + k^2 \right) \sigma^4 \right)$$

$$\lambda_{22} = \frac{1}{6144 (1 + k^2) \omega_1} \left\{ 384\pi \left( -1 + k^2 \right) \cdot (M_1 \zeta_1 - M_2 \zeta_2) \times [(h_2 + N_2 \zeta_2) + k\pi (M_1 \zeta_1 - M_2 \zeta_2) - 3k \left( -1 + k^2 \right) + k^2] \left( k_1^2 - k_2^2 \right) (M_1 \zeta_1 - M_2 \zeta_2) - 3k \left( -1 + k^2 \right) + k^2] \left( k_1^2 - k_2^2 \right) (M_1 \zeta_1 - M_2 \zeta_2) \} - 192k\pi^2 \left( -1 + k^2 \right) \cdot \pi^2 \left( k_1^2 + k_2^2 \right) \sigma^2 - 24k (1 + k^2) \pi^2 \left( k_1^2 - k_2^2 \right)^2 \cdot \sigma^4 \right)$$

$$\lambda_{23} = \frac{1}{3072 (1 + k^2) \omega_1 \pi} \left\{ 96\pi \left( -1 + k^2 \right) \cdot (M_1 \zeta_1 - M_2 \zeta_2) \times [3 (h_2 + N_2 \zeta_2 + 4k\pi (M_1 \zeta_1 - M_2 \zeta_2))] - \left[ 128h_2k^2 (k_1^2 + k_2^2) - 24\pi \left( -1 + k^2 \right) \right] \left( k_1^2 - k_2^2 \right) (h_2 + N_2 \zeta_2) + 32k [4h_1 (k_1^2 + k_2^2) + 15k_1^2 \pi \left( -1 + k^2 \right) (h_1 + N_1 \zeta_1)] - 192k\pi^2 \left( -1 + k^2 \right) \cdot \pi^2 \left( k_1^2 + k_2^2 \right) \sigma^2 - 24k (1 + k^2) \pi^2 \left( k_1^2 - k_2^2 \right)^2 \cdot \sigma^4 \right)$$

Data Availability

All the numerical calculated data used to support the findings of this study can be obtained by calculating the equations in the paper, and the white noise is generated randomly. The codes used in this paper are available from the corresponding author upon request.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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