On the boundedness of certain generalized Hilbert operators in $\ell^p$

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May 2022

Abstract
The Hilbert matrix $H_{n,m} = (n + m + 1)^{-1}$ has been extensively studied in previous literature. In this paper we look at generalized Hilbert operators arising from measures on the interval $[0, 1]$, such that the Hilbert matrix is obtained by the Lebesgue measure. We provide a necessary and sufficient condition for these operators to be bounded in $\ell^p$ and calculate their norm.

1 Introduction

1.1 Motivation and history
The study of the classical Hilbert operator has its origins on the classical inequality of the same name.

Theorem 1.1. (Hilbert’s inequality) Suppose that $(a_m)_{m=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of non-negative terms and let $p, q > 1$ be Hölder conjugates, i.e. such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $(a_m) \in \ell^p$ and $(b_n) \in \ell^q$. Then there holds

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m + n} \leq \frac{\pi}{\sin \left( \frac{\pi}{2} \right)} \left( \sum_{m=1}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=1}^{\infty} b_n^q \right)^{\frac{1}{q}}.$$  

It was Hilbert who first proved the above for the special case $p = q = 2$ in one of his lectures on integral equations. His proof was published by Weyl (1908). The first proof for general $p, q > 1$ was given by Hardy and Riesz [5]. Perhaps the best reference for the proof is [4]. A sharp version of Hilbert’s inequality is the following: For $(a_m)_{m=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ both sequences of non-negative numbers such that $(a_m) \in \ell^p$ and $(b_n) \in \ell^q$, there holds

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{a_m b_n}{m + n + 1} \leq \frac{\pi}{\sin \left( \frac{\pi}{2} \right)} \left( \sum_{m=0}^{\infty} a_m^p \right)^{\frac{1}{p}} \left( \sum_{n=0}^{\infty} b_n^q \right)^{\frac{1}{q}}. \quad (1.1)$$

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The Hilbert operator and its associated Hilbert matrix are now introduced as follows: Consider the operator $H$ on $\ell^p$ given by

$$H : (a_n)_{n=0}^\infty \mapsto \left( \sum_{m=0}^\infty \frac{a_m}{m+n+1} \right)_{n=0}^\infty.$$ 

The fact that this is a bounded operator on $\ell^p$ for $p > 1$ follows from Hilbert’s inequality as follows.

We recall that the norm of $(a_n) \in \ell^p$ is $\| (a_n) \|_{\ell^p} = \left( \sum_{n=0}^\infty |a_n|^p \right)^{\frac{1}{p}}$. Moreover there must hold

$$\| (a_n) \|_{\ell^p} = \sup_{h \in (\ell^q)^* : \|h\| \leq 1} h((a_n)),$$

where $h$ is any element of the dual space $\ell^q$ with norm 1. Therefore

$$\| (a_n) \|_{\ell^p} = \sup_{(b_n) \in \ell^q : \| (b_n) \|_{\ell^q} \leq 1} \left| \sum_{n=0}^\infty a_n b_n \right|.$$ 

By fixing an arbitrary number $k$ and taking the supremum over all sequences $(b_n) \in \ell^q$ such that $\| (b_n) \|_{\ell^q} = k$, we arrive at the desired result. The associated matrix

$$H = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \cdots \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

acting on column vectors whose entries are terms of a sequence in $\ell^p$ is another way to look at the operator (we shall not be making a notational distinction between $H$ and its associated matrix). Further historical notes and properties of the Hilbert matrix can be found in [1].

Notice that each entry in $H$ can be written as

$$H_{n,k} = \int_0^1 g_{n,k}(t) \, dt,$$

where $g_{n,k}(t) := \binom{n+k}{k} (1-t)^n t^k$. Motivated by this integral representation, in the current paper we shall be looking at a class of generalized Hilbert operators, arising from replacing $dt$ with finite, positive Borel measures. The definition is as follows.

**Definition 1.2.** Let $\mu$ be a finite, positive Borel measure on $[0,1]$. We define the generalized Hilbert matrix $C^\mu$ by

$$C^\mu_{n,k} := \binom{n+k}{k} \int_0^1 (1-t)^n t^k \, d\mu(t),$$

for $n, k$ integers ranging from 0 to $\infty$. 

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Let us, finally, note that the corresponding question to the one addressed in this paper for the case of generalized Cesaro operators was tackled in [3] and throughout this work, our general approach significantly borrows from the ideas presented in it.

1.2 Preliminaries

In this subsection we will be introducing a series of lemmata which will become relevant later on.

Lemma 1.3. There holds

$$\sum_{n=0}^{\infty} \left(\begin{array}{c} n+m \\ m \end{array}\right) s^n = (1-s)^{-m-1},$$

for all $s \in (0, 1)$.

Proof. That the left-hand side converges (absolutely) is a direct consequence of the ratio test for series. One can use induction on $m$. For $m = 0$, the result holds. Assume the result holds for $m = k$. Then we have

$$(1-s)^{-m-2} = \frac{1}{1-s}(1-s)^{-m-1} = (1 + s + s^2 + \ldots) \sum_{n=0}^{\infty} \left(\begin{array}{c} n+m \\ m \end{array}\right) s^n.$$

Since both series above converge absolutely, their product is given by the series

$$\sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \left(\begin{array}{c} m+l \\ m \end{array}\right) \right) s^n,$$

which is convergent by Mertens’ theorem on the Cauchy product of series. Therefore, all one has to show is that

$$\sum_{l=0}^{n} \left(\begin{array}{c} m+l \\ m \end{array}\right) = \left(\begin{array}{c} m+n+1 \\ m+1 \end{array}\right),$$

which itself can be shown by a straightforward induction, using the fact that

$$\left(\begin{array}{c} m+n+1 \\ m+1 \end{array}\right) + \left(\begin{array}{c} m+n+1 \\ m \end{array}\right) = \left(\begin{array}{c} m+n+2 \\ m+1 \end{array}\right).$$

□
Lemma 1.4. There holds
\[ \frac{1 - t}{1 - te^{-x}} \geq e^{-\frac{1}{1 + x}}, \]
for \( x \) non-negative and \( t \in [0, 1) \).

Proof. One can consider, for a fixed \( t \), the function
\[ f_t(x) = (1 - t e^{-x}) e^{-\frac{1}{1 + x}}. \]
Differentiating with respect to \( x \) we see that the only critical point is at \( x = 0 \), whence the result follows. \( \square \)

Lemma 1.5. There holds
\[ \frac{1}{\Gamma(\omega)} \int_0^\infty e^{-a x} x^{w-1} dx = a^{-w}, \]
for any \( a > 0 \) and \( w > 0 \).

Proof. As long as \( a \) is positive, the integral is well-defined and convergent. Recall that
\[ \Gamma(\omega) = \int_0^\infty e^{-x} x^{w-1} dx. \]
Setting \( u = a x \), we have \( dx = \frac{du}{a} \) and the integral becomes
\[ \int_0^\infty e^{-u} u^{w-1} a^{-w} du = a^{-w} \Gamma(\omega). \]
The result follows. \( \square \)

2 Boundedness of \( C^\mu \) on \( \ell^p \)

In the current section we shall prove two theorems regarding the boundedness of \( C^\mu \) as an operator on \( \ell^p \) spaces. The first concerns the case where there are no Dirac masses at 0 or 1, while the latter allows for the existence of such masses at the endpoints of \([0, 1]\).

Theorem 2.1. Fix \( 1 \leq p \leq \infty \). If \( \mu \) has no Dirac masses at 0 or 1, then the matrix \( C^\mu \) is bounded as an operator from \( \ell^p \) to itself if and only if
\[ N^\mu_p := \int_0^1 \frac{1}{(1 - t)^{1 - \frac{1}{p}} t^{\frac{1}{p}}} \, d\mu(t) < \infty. \]
Moreover, if that is the case, the operator norm equals \( N^\mu_p \) (here we understand that when \( p = \infty \), the integral becomes \( \int_0^1 \frac{1}{1 - t} \, d\mu(t) \)).
Proof. Suppose that \(\int_0^1 \frac{1}{(1-t)^{p-1}} \, t^p \, d\mu(t)\) is finite and there are no Dirac masses at 0 or 1. We will, in the first instance, show that \(C^\mu\) is bounded from \(\ell^p\) to \(\ell^p\) for \(1 \leq p < \infty\) and we will calculate its norm. We will then treat the case \(p = \infty\).

We begin by defining, for a given sequence \((a_n)_{n=0}^\infty \in \ell^p\), the function \(e_n : [0, 1] \to \mathbb{R}\) given by

\[
e_n(t) = \sum_{m=0}^\infty \binom{n+m}{m} t^m (1-t)^n a_m.
\]  

(2.1)

We claim that \(e_n\) is well-defined on \([0, 1]\) and that there holds, for \(t \in (0, 1)\),

\[
\sum_{n=0}^\infty e_n^p \leq \frac{1}{(1-t)^{p-1} t} \sum_{n=0}^\infty a_n^p.
\]  

(2.2)

To check this, we first look at the case \(p > 1\). Here, Hölder’s inequality gives

\[
e_n(t) \leq \left( \sum_{m=0}^\infty \binom{n+m}{m} t^m (1-t)^n \right)^{\frac{1}{q}} \left( \sum_{m=0}^\infty a_m^p \right)^{\frac{1}{p}} := B_m
\]  

(2.3)

where \(q = \frac{p}{p-1}\) is the Hölder conjugate of \(p\), i.e. \(\frac{1}{q} + \frac{1}{p} = 1\). To check that the first term on the right-hand side in (2.3) is convergent, the ratio test for series gives

\[
\frac{B_{m+1}}{B_m} = \left( \frac{(n+m+1)}{m+1} t^{m+1} (1-t)^n \right)^q = \left( \frac{m+n+1}{m+1} t \right)^q \to t^q,
\]  

(2.4)

as \(m \to \infty\). Hence, for \(t \in [0, 1]\), we have

\[
\lim_{m \to \infty} \frac{B_{m+1}}{B_m} = t^q < t < 1
\]

and therefore the series is (absolutely) convergent. Finally, for \(t = 1\), the partial sums \(\sum_{m=0}^K B_m = 0\) and hence the series equals zero. This shows that for all \(t \in [0, 1]\), \(e_n(t)\) is well-defined.

For \(p = 1\), it is enough to show that, for a fixed \(t \in [0, 1]\) and for any fixed \(n\), the sequence

\[
c_m(n; t) := \binom{n+m}{m} t^m (1-t)^n
\]

is in \(\ell^\infty\). Consider a biased coin with probability of heads \(t\) and probability of tails \(1-t\). The probability of landing \(m\) heads in \((n+m)\) throws for this coin
is precisely \( c_m(n; t) \). Hence each term is bounded by 1 and the result follows. This proves that \( e_n(t) \) is well-defined when \((a_n) \in \ell^1\).

By Hölder’s inequality again, we have

\[
e_p^n(t) \leq \left( \sum_{m=0}^\infty \binom{n+m}{m} (1-t)^n t^m a_m^p \right) \left( \sum_{m=0}^\infty \binom{n+m}{m} (1-t)^n t^m \right)^{p-1} \\
\leq \frac{1}{(1-t)^{p-1}} \sum_{m=0}^\infty \binom{n+m}{m} (1-t)^n t^m a_m^p.
\]

(2.5)

Summing over \( n \), we get (for \( 0 < t < 1 \)),

\[
\sum_{n=0}^\infty e_p^n \leq \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{1}{(1-t)^{p-1}} \binom{n+m}{m} (1-t)^n t^m a_m^p \\
\leq \sum_{m=0}^\infty \frac{1}{(1-t)^{p-1}} t^m a_m^p \sum_{n=0}^\infty \binom{n+m}{m} (1-t)^n \\
\leq \sum_{m=0}^\infty \frac{1}{(1-t)^{p-1}} t^p a_m^p,
\]

(2.6)

where we have used Lemma 1.3. We now have

\[
b_n = \sum_{m=0}^\infty \int_0^1 \binom{n+m}{m} t^m (1-t)^n a_m \, d\mu(t) \\
= \int_0^1 \sum_{m=0}^\infty \binom{n+m}{m} t^m (1-t)^n a_m \, d\mu(t) \quad \text{(by the D.C.T.)} \\
= \int_0^1 e_n(t) \, d\mu(t).
\]

By Minkowski’s inequality, we have

\[
\left( \sum_{n=0}^\infty b_n^p \right)^{\frac{1}{p}} \leq \int_0^1 \left( \sum_{n=0}^\infty e_n^p \right)^{\frac{1}{p}} \, d\mu(t) \leq \int_0^1 \frac{1}{t^{\frac{p}{1+\frac{1}{p}}} (1-t)^{\frac{p}{1+\frac{1}{p}}}} \left( \sum_{n=0}^\infty a_n^p \right)^{\frac{1}{p}} \, d\mu(t),
\]

(2.7)

or equivalently

\[
\|(b_n)\|_{\ell^p} \leq \left( \int_0^1 \frac{1}{t^{\frac{p}{1+\frac{1}{p}}} (1-t)^{\frac{p}{1+\frac{1}{p}}}} \, d\mu(t) \right) \|(a_n)\|_{\ell^p}.
\]

(2.8)

To show that \( N_p^\mu \) is the optimal constant (and hence the norm of the corresponding operator, when it is finite) we work as follows. Fix any \( \eta > 0 \) small.
There exists a $\delta > 0$ such that
\[
\int_{\delta}^{1-\delta} \frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{p-1}{p}}} \, d\mu(t) > (1-\eta)N_p^\mu. \tag{2.9}
\]

That such a $\delta$ exists follows from the monotone convergence theorem, when applied for example to the functions $f_n(t) = \frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{p-1}{p}}} I_{[\frac{1}{n}, \frac{n+1}{n}]}$. We choose $c, N, \varepsilon$ (in this order) such that

- There holds $(1 + \frac{1}{c})^{-\frac{2}{p}} > 1 - \eta$.
- There holds $N \geq \lceil \frac{c}{\varepsilon} \rceil$, so that
  \[
  \int_{n}^{1-\delta} \frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{p-1}{p}}} \, d\mu(t) > \int_{\delta}^{1-\delta} \frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{p-1}{p}}} \, d\mu(t) > (1-\eta)N_p^\mu, \tag{2.10}
  \]
  for $n \geq N$.
- There holds $0 < \varepsilon < \frac{1}{p}$. Moreover, denoting $w = \frac{1}{p} + \varepsilon$, we must choose $\varepsilon$ so that the following two conditions also hold:
  \[
  \int_{\delta}^{1-\delta} t^{-w} (1-t)^{w-1} \, d\mu(t) > (1-\eta) \int_{\delta}^{1-\delta} \frac{1}{t^{\frac{1}{p}}(1-t)^{\frac{p-1}{p}}} \, d\mu(t) \tag{2.10}
  \]
  and such that
  \[
  \sum_{n=N}^{\infty} (n+1)^{-p} w > (1-\eta) \sum_{n=0}^{\infty} (n+1)^{-p} w. \tag{2.11}
  \]

The latter condition can be met for $\varepsilon$ sufficiently small, since the $p-$series diverges for $p = 1$ and converges for $p > 1$.

Finally, we pick the sequence $a_n = (n+1)^{-w}$. Using Lemma 1.5 we have
\[
a_n = \frac{1}{\Gamma(\omega)} \int_{0}^{\infty} e^{-(n+1)x} x^{w-1} \, dx \tag{2.12}
\]
Using this and (2.1), we can get the following estimate for $e_n(t)$.

\[
e_n(t) = \frac{1}{\Gamma(\omega)} \int_{0}^{\infty} e^{-x} x^{w-1} \sum_{m=0}^{\infty} \binom{n+m}{m} t^m (1-t)^n e^{-mx} \, dx
= \frac{(1-t)^n}{\Gamma(\omega)} \int_{0}^{\infty} e^{-x} x^{w-1} (1-t e^{-x})^{-n} \, dx
= \frac{1}{1-t} \frac{1}{\Gamma(\omega)} \int_{0}^{\infty} e^{-x} x^{w-1} \left( \frac{1-t}{1-t e^{-x}} \right)^{n+1} \, dx \tag{2.13}
\geq \frac{1}{1-t} \frac{1}{\Gamma(\omega)} \int_{0}^{\infty} e^{-x} x^{w-1} e^{-(n+1)\frac{1-t}{1-t e^{-x}}} \, dx
= \frac{1}{1-t} \left( \frac{1+n t}{1-t} \right)^{-w},
\]
where the penultimate step was obtained by Lemma 1.4 and the last step from Lemma 1.5. When \( \frac{s}{n} \leq t \leq 1 - \frac{s}{n} \), we therefore have

\[
e_n(t) \geq (1 - t)^{w-1} t^{-w} \left( 1 + \frac{1}{c} \right)^{-w} (n + 1)^{-w} \geq (1 - t)^{w-1} t^{-w} (1 - \eta) a_n. \tag{2.14}
\]

Using (2.14), we have

\[
b_n = \int_0^1 e_n(t) \, d\mu(t) \geq \int_{\frac{s}{n}}^{1-\frac{s}{n}} e_n(t) \, d\mu(t) \\
\geq (1 - \eta) a_n \int_{\frac{s}{n}}^{1-\frac{s}{n}} (1 - t)^{w-1} \, d\mu(t) \\
\geq (1 - \eta) a_n \int_{\delta}^{1-\delta} t^{-w} (1 - t)^{w-1} \, d\mu(t) \\
\geq (1 - \eta)^2 a_n \int_{\delta}^{1-\delta} t^{-w} (1 - t)^{\frac{1}{p} - 1} \, d\mu(t) \\
\geq (1 - \eta)^3 a_n N_\mu^p, \text{ for } n \geq N, \tag{2.15}
\]

hence

\[
\sum_{n=0}^{\infty} b_n^p \geq \sum_{n=N}^{\infty} (1 - \eta)^3 p \left( N_\mu^p \right)^p a_n^p \geq (1 - \eta)^3 p+1 \left( N_\mu^p \right)^p \sum_{n=0}^{\infty} a_n^p. \tag{2.16}
\]

This establishes the ”if” direction. For the ”only if” direction, assume that \( C_\mu \) is bounded as an operator on \( \ell^p \) for \( 1 \leq p < \infty \). We claim that the integral \( N_\mu^p \) is finite. We shall make use of the same sequences as the ones just used. With the preceding notation, for an arbitrarily small but fixed \( \eta > 0 \), we have for the sequences described above:

\[
b_n = \int_0^1 e_n(t) \, d\mu(t) \geq (1 - \eta) a_n \int_0^1 (1 - t)^{w-1} t^{-w} \, d\mu(t) \tag{2.17}
\]

and hence

\[
\left( \sum b_n^p \right)^\frac{1}{p} \geq \left( (1 - \eta) \int_0^1 (1 - t)^{w-1} t^{-w} \, d\mu(t) \right) \left( \sum a_n^p \right)^\frac{1}{p}. \tag{2.18}
\]

From here we observe, given the boundedness of \( C_\mu \), that \( \int_0^1 (1 - t)^{w-1} t^{-w} \, d\mu(t) \) is finite and uniformly bounded above by the operator norm as \( \eta \to 0 \). Moreover, it is not hard to establish that, as \( \eta \to 0 \), the way \( \varepsilon \) is chosen means that \( \varepsilon \to 0 \) as well and hence the functions

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\[(1 - \eta)(1 - t)^{-1} t^{-w} \to \frac{1}{(1 - t)^{\frac{w-1}{p'}} t^\frac{1}{p}}\]

pointwise as \(\eta \to 0\). Fix any \(z > 0\) small. An application of the dominated convergence theorem on the interval \([z, 1-z]\) (on this fixed interval, the functions \((1-\eta)(1-t)^{-1} t^{-w}\) are uniformly bounded above) now implies that the integral

\[\int_z^{1-z} \frac{1}{(1-t)^{\frac{w-1}{p'}} t^\frac{1}{p}} \, d\mu(t)\]

is finite and bounded above by the operator norm. As \(z \to 0\), an application of the monotone convergence theorem gives the desired result.

We finally examine the case \(p = \infty\). Assume that \(\int_0^1 \frac{1}{1-t} \, d\mu(t)\) is finite and consider a sequence \((a_n)\) such that

\[b_n = \sum_{m=0}^{\infty} \binom{n+m}{m} \int_0^1 (1-t)^n t^m a_m \, d\mu(t)\]

\[\leq \left[ \int_0^1 \sum_{m=0}^{\infty} \binom{n+m}{m} (1-t)^n t^m \, d\mu(t) \right] \|a_n\|_{\ell^\infty} \quad (2.19)\]

Taking the supremum over \(n\), we have

\[\|b_n\|_{\ell^\infty} \leq \int_0^1 \frac{1}{1-t} \, d\mu(t) \|a_n\|_{\ell^\infty}. \quad (2.20)\]

By taking \((a_n)\) to be a constant sequence different than the zero sequence, we see that the constant is optimal. Conversely, if the operator is bounded on \(\ell^\infty\), picking any constant non-zero sequence \((a_n)\), we see that the integral \(\int_0^1 \frac{1}{1-t} \, d\mu(t)\) is finite and the result follows.

\[\square\]

### 2.1 The case where \(\mu\) is allowed to have Dirac masses at 0 and/or 1

To allow for the possibility of Dirac masses at 0 or 1, we need to be a bit more careful with semantics and notation.

#### 2.1.1 Preliminary measure-theoretic notes

Given a measure \(\mu\) on \([0, 1]\) and a function \(f\) defined on \([0, 1]\) which is measurable with respect to \(\mu\), we shall denote the integral \(\int_0^1 f(t) \, d\mu(t)\) by \(\mu_{[0,1]} f\). In general, for any measurable subset \(A\) of \([0, 1]\), we will use \(\mu_A(f)\) to denote the integral \(\mu_{[0,1]}(f \cdot 1_A)\), which is well-defined as \(f \cdot 1_A\) is \(\mu\)-measurable.
For a function $g$ defined only on a subset $\mathcal{P}$ of $[0, 1]$, by a slight abuse of notation, we shall also use $\mu_\mathcal{P}(g)$ to denote the integral

$$\int_\mathcal{P} g \, d\mu_\mathcal{P}(t),$$

where $\mu_\mathcal{P}$ is the restriction of the measure on $\mathcal{P}$ in the obvious way.

For a given measure $\mu$ and for any function $f$ on $[0, 1]$, the following decomposition holds:

$$\mu_{[0,1]}(f) = \mu_{(0,1)}(f) + c_0 f(0) + c_1 f(1),$$

where $c_1 := \mu\left(\{0\}\right)$ and $c_2 := \mu\left(\{1\}\right)$ are the Dirac masses of the measure at 0 and 1 respectively. This, in turn, provides the following decomposition for the matrix $C^\mu$:

$$C^\mu_{n,k} = \tilde{C}_{n,k}^\mu + \hat{C}_{n,k}^\mu,$$

(2.21)

where $\tilde{C}_{n,k}^\mu := \mu_{(0,1)}(g_{n,k}(t))$ and

$$\hat{C}^\mu := \begin{pmatrix} c_0 + c_1 & c_1 & c_1 & \cdots \\ c_0 & 0 & 0 & 0 & \cdots \\ c_0 & 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

### 2.1.2 Main Result

In this section we will show the following:

**Theorem 2.2.** Assume that $\mu$ contains Dirac masses $c_0$ at 0 and $c_1$ at 1, with $|c_0| + |c_1| > 0$. For $1 \leq p < \infty$, the only case when $C^\mu$ can be a bounded operator on $\ell^p$ is when $c_1 \neq 0 = c_0$ and $p = 1$. In that case, the operator is bounded if and only if

$$\int_{(0,1]} \frac{1}{t} \, d\mu(t) = \int_{(0,1)} \frac{1}{t} \, d\mu(t) + c_1$$

is finite. Moreover, this expression, when finite, equals the norm of the operator. Finally, when $p = \infty$, the operator is bounded if and only if $c_0 \neq 0 = c_1$ and the integral

$$\int_{(0,1]} \frac{1}{1-t} \, d\mu(t)$$

is finite. Moreover, the above, when finite, equals the norm of the operator on $\ell^\infty$. 

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Proof. We begin by establishing the following two lemmata:

**Lemma 2.3.** If the measure $\mu$ contains a Dirac mass at 0, then the matrix is unbounded on $\ell^p$ for $1 \leq p < \infty$.

This holds for the following reason. The matrix $\hat{C}_\mu$ for $c_0 \neq 0$ does not even map $\ell^p$ to itself, since

\[
\begin{pmatrix}
c_0 + c_1 & c_1 & c_1 & \ldots \\
c_0 & 0 & 0 & \ldots \\
c_0 & 0 & 0 & \ldots \\
c_0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
\vdots
\end{pmatrix} =
\begin{pmatrix}
(c_0 + c_1)a_0 + c_1(a_1 + a_2 + \ldots) \\
c_0 a_0 \\
c_0 a_0 \\
c_0 a_0 \\
\vdots
\end{pmatrix}
\]

(2.22)

and clearly, as the sequence on the right-hand side eventually consists of the same non-zero (as long as $a_0 \neq 0$) constant, it cannot be in $\ell^p$. Finally, since the matrix $\hat{C}_\mu$ comprises only positive entries, the sum $\hat{C}_\mu + \hat{C}_\mu$ also fails to map $\ell^p$ to $\ell^p$.

**Lemma 2.4.** If the measure $\mu$ contains a Dirac mass at 1 and no Dirac mass at 0, then the matrix $C_\mu$ is unbounded as an operator on $\ell^p$ for $1 < p < \infty$.

To establish this, notice that the first term of the sequence on the right-hand side of (2.22) is $c_1(a_0 + a_1 + \ldots)$. In general, the $\ell^p$ spaces are nested, in the sense that $\ell^p \subset \ell^q$ for $p < q$ and hence for each $p > 1$ there exists a sequence in $\ell^p$ which is not in $\ell^1$. Since $\hat{C}_\mu$ consists of only positive entries, it again follows that $C_\mu$ does not map $\ell^p$ to itself when $c_1 \neq 0 = c_0$ and $p > 1$. The only case left to examine, when the measure contains a Dirac mass, is when $c_1 \neq 0 = c_0$ and $\hat{C}_\mu$ is viewed as an operator on $\ell^1$. In this case we claim that the operator is bounded if and only if

\[
\mu([0,1]) \left(\int_{(0,1)} \frac{1}{t} \, d\mu(t) + c_1\right)
\]

is finite and that, when finite, the expression above equals the operator norm.

Indeed, assume first that the expression above is finite. Proceeding as in (2.40), we get, for $0 < t < 1$:

\[
\sum_{n=0}^{\infty} e_n(t) \leq \frac{1}{t} \sum_{n=0}^{\infty} a_n(t) \quad (2.23)
\]

for all $n$. We now notice that

\[
b_0 = \int_{(0,1)} e_0(t) \, d\mu(t) + c_1 \|a_n\|_{\ell^1} \quad (2.24)
\]
and

\[ b_n = \int_{(0,1)} e_n(t) \, d\mu(t) \]  

(2.25)

for \( n \geq 1 \). Adding (2.24) and (2.25) together, we arrive at

\[ \| (b_n) \|_{\ell^1} \leq \left( \int_{(0,1)} \frac{1}{t} \, d\mu(t) + c_1 \right) \| (a_n) \|_{\ell^1} \]  

(2.26)

To show that this is the optimal constant, we again use the sequences described in the previous section. With the same notation as before, with \( a_n = (n + 1)^{-w} \), we have

\[ \sum_{n=0}^{\infty} b_n = b_0 + \sum_{n=1}^{\infty} b_n = \int_{(0,1)} e_0(t) \, d\mu(t) + c_1 \| (a_n) \|_{\ell^1} + \sum_{n=1}^{\infty} \int_{(0,1)} e_n(t) \, d\mu(t) \]

\[ \geq \left( 1 - \eta \right)^4 \int_{(0,1)} \frac{1}{t} \, d\mu(t) + c_1 \right) \| (a_n) \|_{\ell^1}. \]  

(2.27)

Letting \( \eta \to 0 \), we obtain the desired result.

Finally, let us suppose that the operator \( C^\mu \) is bounded on \( \ell^1 \). The same argument as in the previous case shows the finiteness of \( \int_{(0,1]} \frac{1}{t} \, d\mu(t) \).

We finally examine the case \( p = \infty \). The proof is similar in spirit to the previous results, although a bit simpler. Indeed, if \( c_1 \neq 0 \) then it is clear from (2.21) that \( C^\mu \) does not map \( \ell^\infty \) to itself.

Assume that \( \int_{(0,1]} \frac{1}{t} \, d\mu(t) \) is finite and consider a sequence \( (a_n) \in \ell^\infty \). We have

\[ b_n = \sum_{m=0}^{\infty} \binom{n + m}{m} \int_{(0,1]} (1 - t)^n t^m a_m \, d\mu(t) + c_0 a_n \leq \left[ \int_{(0,1)} \sum_{m=0}^{\infty} \binom{n + m}{m} (1 - t)^n t^m \, d\mu(t) \right] \| (a_n) \|_{\ell^\infty} + c_0 a_n \]  

(2.28)

Taking the supremum of (2.28) over \( n \), we have

\[ \| (b_n) \|_{\ell^\infty} \leq \int_{(0,1]} \frac{1}{1 - t} \, d\mu(t) \| (a_n) \|_{\ell^\infty}. \]  

(2.29)

By taking \( (a_n) \) to be a constant sequence different than the zero sequence, we see that the constant is optimal.
Conversely, assuming that the operator is bounded from \( \ell^{\infty} \) to itself, again picking a non-zero constant sequence, we see that

\[
b_n = \sum_{m=0}^{\infty} \binom{n+m}{m} \int_{(0,1)} (1-t)^n t^m a_m \, d\mu(t) + c_0 a_n
\]

\[
= \left[ \int_{(0,1)} \frac{1}{1-t} \, d\mu(t) \right] a_n.
\]

(2.30)

From this, the finiteness of the operator norm implies the finiteness of

\[
\int_{(0,1)} \frac{1}{1-t} \, d\mu(t)
\]

and the result follows.

\[\Box\]

Acknowledgements

I would like to thank Aristomenis Siskakis for introducing me to this problem and for various beneficial suggestions regarding the format of the paper. I would also like to thank him, as well as Vasilis Daskalogiannis, Petros Galanopoulos and Giorgos Stylogiannis for several fruitful discussions on the topic. Finally, I thank Vasilis for useful comments and suggestions on the introduction.

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