VARIVANCE OF SUMS IN ARITHMETIC PROGRESSIONS OF DIVISOR FUNCTIONS ASSOCIATED WITH HIGHER DEGREE L-FUNCTIONS IN $\mathbb{F}_q[t]$

CHRIS HALL, JONATHAN P. KEATING, AND EDVARODITY-GERSHON

Abstract. We compute the variances of sums in arithmetic progressions of generalised $k$-divisor functions related to certain $L$-functions in $\mathbb{F}_q[t]$, in the limit as $q \to \infty$. This is achieved by making use of recently established equidistribution results for the associated Frobenius conjugacy classes. The variances are thus expressed, when $q \to \infty$, in terms of matrix integrals, which may be evaluated. Our results extend those obtained previously in the special case corresponding to the usual $k$-divisor function, when the $L$-function in question has degree one. They illustrate the role played by the degree of the $L$-functions; in particular, we find qualitatively new behaviour when the degree exceeds one. Our calculations apply, for example, to elliptic curves defined over $\mathbb{F}_q[t]$, and we illustrate them by examining in some detail the generalised $k$-divisor functions associated with the Legendre curve.

1. Introduction

For a fixed integer $k \geq 2$ the generalised divisor function $d_k(n)$ denotes the number of ways of writing a (positive) integer as a product of $k$ factors:

$$d_k(n) = \sum_{m_1 \cdot m_2 \cdot \ldots \cdot m_k = n} 1$$

It is related to the $k$th power of the Riemann zeta-function, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

when $\text{Re} s > 1$, in that

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

for $\text{Re} s > 1$.

Define $\Delta_k(x)$ by

$$\Delta_k(x) := \sum_{n \leq x} d_k(n) - \text{Res}_{s=1} \frac{x^s \zeta^k(s)}{s} = \sum_{n \leq x} d_k(n) - xP_{k-1}(\log x)$$

where $P_{k-1}(u)$ is a certain polynomial of degree $k-1$; see, for example, [Tit86] Chapter XII.

The mean square of $\Delta_2(x)$ was computed by Cr\'{a}mer [Cra22] for $k = 2$, and by Tong [Ton56] for $k \geq 3$ (assuming the Riemann Hypothesis (RH) if $k \geq 4$), to be

$$\frac{1}{X} \int_{X}^{2X} \Delta_k(x)^2 \, dx \sim c_k X^{1-\frac{k}{2}}$$

We are pleased to acknowledge support under EPSRC Programme Grant EP/K034383/1 LMF: $L$-Functions and Modular Forms. JPK is also grateful for support from a Royal Society Wolfson Research Merit Award and ERC Advanced Grant 740900 (LogCorRM).
for a certain constant $c_k$. Heath-Brown [HB92] showed that $\Delta_k(x)/x^{1/2-\frac{1}{3k}}$ has a limiting value distribution (for $k \geq 4$ one needs to assume RH); it is non-Gaussian.

Our main focus will be on sums of divisor functions over arithmetic progressions

$$S_{d_k}(A) = S_{d_k;X;Q}(A) = \sum_{n \leq X \atop n=1 \mod Q} d_k(n)$$

For the standard divisor function ($k = 2$), it is known that if $Q < X^{2/3-\epsilon}$ then

$$S_{d_2}(A) = \frac{Xp_Q(\log X)}{\Phi(Q)} + O(X^{1/3+o(1)})$$

for some linear polynomial $p_Q$. This is due to unpublished work of Selberg. For recent work on the asymptotics of sums of $d_3$ over arithmetic progressions, see [FKM15] and the literature cited therein.

The variance $\text{Var}(S_{d_2;X;Q})$ of $S_{d_2}$ has been studied by Motohashi [Mot73], Blomer [Blo08], Lau and Zhao [LZ12], the result being [LZ12] (we assume $Q$ prime for simplicity):

(i) If $1 \leq Q < X^{1/2+\epsilon}$ then

$$\text{Var}(S_{d_2;X;Q}) \ll X^{1/2} + (\frac{X}{Q})^{2/3+\epsilon}.$$ 

(ii) For $X^{1/2} < Q < X$,

$$\text{Var}(S_{d_2;X;Q}) = \frac{X}{Q}p_3(\log \frac{Q^2}{X}) + O((\frac{X}{Q})^{5/6}(\log X)^3)$$

where $p_3$ is a polynomial of degree 3 with positive leading coefficient.

For $k \geq 3$, Kowalski and Ricotta [KR14] considered smooth analogues of the divisor sums $S_{d_k;X;Q}(A)$, and among other things computed the variance for $Q^{k-1/2+\epsilon} < X < Q^{k-\epsilon}$.

In the function field setting, the analogous problem was studied by Rodgers, Rudnick and the second and third authors [KRRGR18]. Let $\mathcal{M}_n \subset \mathbb{F}_q[t]$ be the set of monic polynomials of degree $n$ with coefficients in $\mathbb{F}_q$. For a polynomial $Q \in \mathbb{F}_q[x]$ of degree at least 2, and $A$ co-prime to $Q$, set

$$S_{d_k;A;Q}(A) := \sum_{f \in \mathcal{M}_n \atop f=1 \mod Q} d_k(f),$$

Then for $n \leq k(d_Q - 1)$ the variance of this sum is given by

$$\lim_{q \to \infty} \frac{\text{Var}_Q(S_{d_k;A;Q})}{q^n/|Q|} = I_k(n; \deg Q - 1)$$

where $I_k(n; \deg Q - 1)$ is a certain matrix integral over the unitary group $U_{\deg Q - 1}(\mathbb{C})$ — see [KRRGR18] and [L.L11] below. This integral can be evaluated in a number of ways. Some of the main results obtained in [KRRGR18] are set out in Section 5 below. It follows from these, for example that, for the classical divisor function $d = d_2$, if $\deg Q \geq 2$ and $n \leq 2(\deg Q - 1)$, then

$$\lim_{q \to \infty} \frac{\text{Var}_Q(S_{d_2;A;Q})}{q^n/|Q|} = \begin{cases} \text{Pol}_3(n), & n \leq \deg Q - 1 \\ \text{Pol}_3(2(\deg Q - 1) - n), & \deg Q \leq n \leq 2(\deg Q - 1) \end{cases}$$

where

$$\text{Pol}_3(x) = \frac{(x + 3)}{3} = (x + 1)(x + 2)(x + 3)/6.$$ 

For general $k$, the variance may be expressed in terms of a lattice-point count and is a piecewise polynomial function of degree $k^2 - 1$. 


The function-field expressions for the variance of the generalized divisor function lead immediately to conjectures in the standard number-field setting [KRRGR18]. The conjecture when $k = 3$ has recently been proved by Rodgers and Soundararajan [RS17].

Our aim here is to extend the line of research reviewed above to arithmetic functions associated with other $L$-functions in the same way that the generalized divisor function is related to the Riemann zeta-function. Our function field results (which we will soon state below; Theorem 1.1.2 and Theorem 2.0.8) can be used to motivate predictions in the number field setting. In order to illustrate these predictions, we focus now on two representative examples: elliptic curve $L$-functions and the Ramanujan $L$-function.

Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ defined over $\mathbb{Q}$. The associated $L$-function $F(s)$ will be denoted by $L(s, E)$ and is given by

$$L(s, E) = \prod_{p \mid N} (1 - a_p p^{-s})^{-1} \prod_{p \nmid N} (1 - a_p p^{-s} + p^{-2s+1})^{-1}$$

where $a_p$ is the difference between $p+1$ and the number of points on the reduced curve mod $p$

$$a_p = p + 1 - \#\tilde{E}(\mathbb{F}_p).$$

When $p \mid N$, $a_p$ is either 1, -1, or 0. In general, we have the Hasse bound on $a_p$, $|a_p| < 2\sqrt{p}$, hence the product converges and gives an analytic function for all $\Re(s) > 3/2$. The $L$ function of $E$ expands as

$$L(s, E) = \sum_{n=1}^\infty a_n n^{-s}$$

where

$$a_p^c = p^c + 1 - \#\tilde{E}(\mathbb{F}_{p^c}).$$

and for $n = \prod_{i=1}^r p_i^{e_i}$ with $p_1, \ldots, p_r$ distinct primes

$$a_n = a_{p_1^{e_1}} \cdots a_{p_r^{e_r}}.$$

For each positive integer $k$ consider the multiplicative function $d_{k, E}: \mathbb{Z} \to \mathbb{Z}$ given by

$$d_{k, E} := \prod_{n_1 \cdots n_k = n} a_{n_1} \cdots a_{n_k}.$$

Note that $d_{k, E}$ gives the coefficients in the Dirichlet series expansion

$$L(s, E)^k = \sum_{n=1}^\infty d_{k, E}(n)n^{-s}.$$

Our results in the function field setting are analogous to computing the variance of the sum of $d_{k, E}$ in arithmetic progressions Define the sum over arithmetic progressions

$$S_{x, c, E}(A) := \sum_{n<A \mod c \leq x} d_{k, E}(n).$$

Our function field result (see Theorem 2.0.8) leads us to predict that for $x^\epsilon < c$, $\epsilon > 0$, the following holds:

$$\text{Var}(S_{x, c, E}) \sim \frac{x}{\phi(c)} a_k(L(s, E)) \gamma_k \left(\frac{\log x}{2 \log c}\right)^2 \left(2 \log c\right)^{k^2-1}.$$

with

$$\gamma_k(c) = \frac{1}{k! G(1+k)^2} \int_{[0,1]^k} \delta_c(w_1 + \ldots + w_k) \prod_{i<j} (w_i - w_j)^2 d^k w,$$
with \(\delta_c(x) = \delta(x - c)\) being the delta distribution translated by \(c\), and \(G\) is the Barnes \(G\)-function, so that for positive integers \(k\), \(G(1 + k) = 1! \cdot 2! \cdot 3! \cdots (k - 1)!\). Here \(a_k(L(s, E))\) is an arithmetic factor that can be written explicitly by (2.5.9) in [CFK+05].

Note that we can detect the degree of the \(L\)-function in question as the coefficient of \(\log c\).

Another example of a degree-two \(L\)-function is the Ramanujan \(L\)-function:

\[
L(s, \tau) = \prod_{p} \left(1 - \tau(p)p^{-s} + p^{-2s+11}\right)^{-1},
\]

where \(\tau\) is the Ramanujan tau function \(\tau : \mathbb{N} \to \mathbb{Z}\) defined by the following identity:

\[
\sum_{n \geq 1} \tau(n)q^n = q \prod_{n \geq 1} \left(1 - q^n\right)^{24},
\]

where \(q = \exp(2\pi i z)\). Ramanujan conjectured (and his conjecture was proved by Deligne) that \(|\tau(p)| \leq 2p^{11/2}\) for all primes \(p\), hence the product converges and gives an analytic function for all \(\text{Re}(s) > 13/2\). The \(L\) function associated with \(\tau\) expands as

\[
L(s, \tau) = \sum_{n=1}^{\infty} \tau(n)n^{-s}
\]

For each positive integer \(k\) consider the multiplicative function \(d_{k,\tau} : \mathbb{Z} \to \mathbb{Z}\) given by

\[
d_{k,\tau} := \prod_{n_1 \cdots n_k=n} \tau(n_1) \cdots \tau(n_k).
\]

Note that \(d_{k,E}\) gives the coefficients in the Dirichlet series expansion

\[
L(s, \tau)^k = \sum_{n=1}^{\infty} d_{k,\tau}(n)n^{-s}.
\]

Again we are led to speculate that for \(x^\epsilon < c\), \(\epsilon > 0\), if

\[
S_{x,c,\tau}(A) := \sum_{n \leq x \mod c} d_{k,\tau}(n)
\]

then the following holds:

\[
\text{Var}(S_{x,c,\tau}) \sim \frac{x}{\phi(c)} a_k(L(s, \tau)) \gamma_k\left(\frac{\log x}{2 \log c}\right)(2 \log c)^{k^2 - 1}.
\]

Where \(a_k(L(s, \tau))\) is an arithmetic factor that can be written explicitly by (2.5.9) in [CFK+05].

We set out our main results below, but first, by way of illustration, we examine a specific example where the \(L\)-function in question take a relatively simple and explicit form, and can defined in a self-contained way.

1.1. **Divisors associated to Legendre curve \(L\)-function.** Let \(q\) be a power of a prime \(p\). Let \(\mathcal{M} \subset \mathbb{F}_q[t]\) be the subset of monic polynomials and \(\mathcal{I} \subset \mathcal{M}\) be the subset of irreducibles. For each \(n \geq 1\), let \(\mathcal{M}_n \subset \mathcal{M}\) and \(\mathcal{I}_n \subset \mathcal{I}\) be the respective subsets of elements of degree \(n\).

Suppose \(q\) is odd, and let \(E/\mathbb{F}_q(t)\) be the Legendre curve, that is, the elliptic curve with affine model

\[
y^2 = x(x - 1)(x - t).
\]

Its \(L\)-function is given by an Euler product

\[
L(T, E/\mathbb{F}_q(t)) = \prod_{\pi \in \mathcal{I}} L(T^{\deg(\pi)}, E/\mathbb{F}_\pi)^{-1}
\]

where \(\mathbb{F}_\pi\) is the residue field \(\mathbb{F}_q[t]/\pi \mathbb{F}_q[t]\).
Each Euler factor of \( L(T, E/F_q(t)) \) is the reciprocal of a polynomial in \( \mathbb{Q}[T] \) and satisfies
\[
T \frac{d}{dT} \log L(T, E/F_q) = \sum_{m=1}^{\infty} a_{\pi m} T^m \in \mathbb{Z}[T].
\]

We regard the coefficients \( a_{\pi m} \) as values of the multiplicative function \( f \mapsto a_f \) on \( \mathcal{M} \) given by the coefficients of the Dirichlet series expansion
\[
L(T, E/F_q(t)) = \sum_{f \in \mathcal{M}} a_f T^{\deg(f)}.
\]

Thus \( a_1 = 1 \), and if \( f = \prod_{i=1}^{r} \pi_i^{e_i} \) with \( \pi_1, \ldots, \pi_r \) distinct elements of \( \mathcal{I} \), then
\[
a_f = a_{\pi_1^{e_1}} \cdots a_{\pi_r^{e_r}}.
\]

For each positive integer \( k \), consider the multiplicative function \( d_{k, \text{Leg}} : \mathcal{M} \to \mathbb{Z} \) given by
\[
d_{k, \text{Leg}}(f) = \prod_{f_1, \ldots, f_k \in \mathcal{M}} a_{f_1} \cdots a_{f_k}.
\]

Equivalently, \( d_{k, \text{Leg}} \) gives the coefficients in the Dirichlet series expansion
\[
L(T, E/F_q(t))^k = \sum_{f \in \mathcal{M}} d_{k, \text{Leg}}(f) T^{\deg(f)}.
\]

It is easy to see that this is a generalization of the \( k \)th divisor function: replace \( L(T, E/F_q(t)) \) by the zeta function \( Z(T, \mathbb{A}^1/F_q) \) so that \( f \mapsto a_f \) becomes the constant function \( f \mapsto 1 \).

Let \( c \in \mathcal{M} \) be square free and \( \Gamma_q(c) = (\mathbb{F}_q[t]/c\mathbb{F}_q[t])^\times \). For each \( n \geq 1 \) and \( A \in \Gamma_q(c) \), consider the sum
\[
S_{k,n,c}(A) := \sum_{f \in \mathcal{M}_n} d_{k, \text{Leg}}(f).
\]

Let \( A \) vary uniformly over \( \Gamma(c) \), and consider the expected value
\[
\mathbb{E}_A[S_{k,n,c}(A)] := \frac{1}{|\Gamma_q(c)|} \sum_{A \in \Gamma_q(c)} S_{k,n,c}(A)
\]
and the variance
\[
\text{Var}_A[S_{k,n,c}(A)] := \frac{1}{|\Gamma_q(c)|} \sum_{A \in \Gamma_q(c)} |S_{k,n,c}(A) - \mathbb{E}_A[S_{k,n,c}(A)]|^2.
\]

These moments depend on \( q \), so one can ask how they behave when we replace \( F_q \) by a finite extension, that is, let \( q \to \infty \). Our main result gives the variance \( \text{Var}_A[S_{k,n,c}(A)] \), in the limit \( q \to \infty \), in terms of a matrix integral. Let \( U \) be an \( R \times R \) matrix. Let \( \text{std} : U_R(\mathbb{C}) \to \text{GL}_R(\mathbb{C}) \) be the representation given by the inclusion \( U_R(\mathbb{C}) \subset \text{GL}_R(\mathbb{C}) \) and
\[
\wedge^3 \text{std} : U_R(\mathbb{C}) \to \text{GL}_{R_j}(\wedge^3 \mathbb{C})
\]
be its \( j \)th exterior power where \( R_j = \binom{R}{j} \); we define \( (\wedge^3 \text{std})(g) = 0 \) unless \( 0 \leq j \leq R \). Define the matrix integrals with respect to Haar measure \( d\theta \) on \( U_R(\mathbb{C}) \) the group of \( R \times R \) unitary matrices
\[
I_k(n; R) = \int_{U_R(\mathbb{C})} \left| \sum_{n_1 + \cdots + n_k = n} \text{Tr} \left( (\otimes_{i=1}^{k} (\wedge^{n_i} \text{std})) (\theta) \right) \right|^2 d\theta.
\]

Then, as a special case of a general theorem we present in this paper, one can prove the following theorem:
Theorem 1.1.2. If \( \gcd(c, t(t - 1)) = t \) and if \( \deg(c) \gg 1 \), then

\[
\lim_{q \to \infty} \frac{|\Gamma_q(c)|}{q^{2n}} \text{Var}_A[S_{k, n, c}(A)] = I_k(n; 2 \deg(c) - 1)
\]

See Remark 2.0.9 for a brief explanation of how this follows from our general theorem. The factor 2 multiplying \( \deg(c) \) corresponds to the degree of the Legendre curve L-function.

Corollary 1.1.3. In the case of \( k = 2 \) we have for \( n \leq 2 \deg(c) - 1 \)

\[
\lim_{q \to \infty} \frac{|\Gamma_q(c)|}{q^{2n}} \text{Var}_A[S_{2, n, c}(A)] = \left(\frac{n + 3}{3}\right)
\]

and for \( 2 \deg(c) - 1 < n < 4 \deg(c) - 2 \)

\[
\lim_{q \to \infty} \frac{|\Gamma_q(c)|}{q^{2n}} \text{Var}_A[S_{2, n, c}(A)] = \left(\frac{4 \deg(c) - n + 1}{3}\right).
\]

The Corollary will follow from evaluating the matrix integral in \( \S5 \).

2. Statement of Main Theorem

Let \( \bar{K} \) be an algebraic closure of \( K = \mathbb{F}_q(t) \) and \( G_K \) be the Galois group \( \text{Gal}(\bar{K} / K) \). We regard \( \mathcal{P} = \mathcal{I} \cup \{\infty\} \) as the set of places of \( K \), and for each \( v \in \mathcal{P} \), we write \( \mathbb{F}_v \) for the residue field and \( \text{deg}(v) \) for its degree over \( \mathbb{F}_q \). We also fix inertia and decomposition groups \( I(v) \subset D(v) \subset G_K \) respectively and write \( G_v \) for the quotient group \( D(v)/I(v) \) and \( \text{Frob}_v \in G_v \) for the Geometric Frobenius element.

For each finite subset \( \mathcal{S} \subset \mathcal{P} \), let \( \bar{K}_S \subseteq \bar{K} \) be the maximal subfield which is unramified over \( \mathcal{P} \setminus \mathcal{S} \) and \( G_{K,S} \) be the Galois group \( \text{Gal}(\bar{K}_S/K) \). We abuse notation and write \( I(v) \subset D(v) \subset G_K \) for the images of \( I(v) \subset D(v) \subset G_K \) in \( G_{K,S} \) via the canonical quotient \( G_K \to G_{K,S} \). If \( v \in \mathcal{S} \), then \( I(v) \subseteq G_{K,S} \) is isomorphic to \( I(v) \subseteq G_K \), and otherwise \( I(v) \subseteq G_{K,S} \) is the trivial group.

Let \( \ell \) be a prime distinct from \( p \) and \( V \) be a finite-dimensional \( \bar{Q}_\ell \)-vector space. Consider a Galois representation

\[
\rho: G_{K,S} \to \text{GL}(V),
\]

that is, a continuous group homomorphism. For each \( v \in \mathcal{P} \), let \( V_v = V^\rho(I(v)) \). We say that \( s \) is the conductor of \( \rho \) if \( s \) is a square-free monic polynomial divisible by prime polynomials \( v \) for which \( V_v \) is strictly smaller than \( V \). Let

\[
\rho_v: G_v \to \text{GL}(V_v)
\]

be the composition of the restriction of \( \rho \) to \( D(v) \) and the quotient \( D(v) \	o G_v \), and let

\[
L(T, \rho_v) := \det(1 - T \rho(\text{Frob}_v) | V_v).
\]

We attach several L-functions to \( \rho \). One is the complete L-function and is given by the Euler product

\[
L(T, \rho) := \prod_{v \in \mathcal{P}} L(T^{\text{deg}(v)}, \rho_v)^{-1}.
\]

Another is a partial L-function given by the Euler product

\[
L_{\text{fin}}(T, \rho) := \prod_{\pi \in \mathcal{I}} L(T^{\text{deg}(\pi)}, \rho_\pi)^{-1}
\]

where now the Euler product is taken over the ‘finite’ places of \( K \). If \( L(T, \rho_\infty) \) is trivial, that is, if \( V_\infty = 0 \), then these L-functions coincide, but in general

\[
L_{\text{fin}}(T, \rho) = L(T, \rho) \cdot L(T, \rho_\infty).
\]
One reason for considering the partial $L$-function $L_{\text{fin}}(T, \rho)$ is that we can expand its Euler product as a Dirichlet series

$$L_{\text{fin}}(T, \rho) = \sum_{f \in \mathcal{M}} a_f T^{\deg(f)}$$

where $f \mapsto a_f$ is the multiplicative function $\mathcal{M} \to \bar{\mathbb{Q}}_\ell$ given on prime powers by writing

$$T \frac{d}{dT} \log(L(T, \rho_\pi)^{-1}) = \sum_{m=1}^{\infty} a_{\pi^m} T^m.$$

Equivalently, for each $\pi \in \mathcal{I}$ and $m \geq 1$, we have

$$a_{\pi^m} = \text{Tr}(\rho_\pi(\text{Frob}_\pi)^m | V_\pi);$$

and if $f = \prod_{i=1}^{r} \pi_i^{e_i}$ is a prime factorization in $\mathbb{F}_q[t]$ with $\pi_i \neq \pi_j$ for $i \neq j$, then

$$a_f = a_{\pi_1^{e_1}} \cdots a_{\pi_r^{e_r}}$$

where $a_1 = 1$.

For each positive integer $k$, we define the $k$th divisor function of $\rho$ as follows: it is the multiplicative function $d_{k, \rho}: \mathcal{M} \to \bar{\mathbb{Q}}_\ell$ given by

$$d_{k, \rho}(f) = \prod_{f_1, \ldots, f_k \in \mathcal{M}, f_1 \cdots f_k = f} a_{f_1} \cdots a_{f_k}.$$

When $\rho$ is the trivial representation $\mathbf{1}$ (and thus $\dim(V) = 1$), then this is the usual $k$ divisor function on $\mathcal{M}$:

$$d_{k, \mathbf{1}}(f) = |\{ f_1, \ldots, f_k \in \mathcal{M} : f_1 \cdots f_k = f \}|$$

In general, we have the identity

$$L_{\text{fin}}(T, \rho)^k = \sum_{f \in \mathcal{M}} d_{k, \rho}(f) T^{\deg(f)} = \sum_{n=0}^{\infty} \left( \sum_{f \in \mathcal{M}_n} d_{k, \rho}(f) \right) T^n.$$

Let $\mathcal{C} \subset \mathcal{P}$ be a finite subset containing $\infty$ and

$$L_{\mathcal{C}}(T, \rho) := \prod_{\pi \not\in \mathcal{C}} L(T^{\deg(\pi)}, \rho_\pi)^{-1}$$

where $\pi$ runs over $\mathcal{I}$ and let

$$R = \deg(L_{\mathcal{C}}(T, \rho)).$$

Thus $L_{\text{fin}}(T, \rho) = L_{\{\infty\}}(T, \rho)$, and in general,

$$L_{\mathcal{C}}(T, \rho)^k = \sum_{n=0}^{\infty} \left( \sum_{f \in \mathcal{M}_n, \gcd(f,c) = 1} d_{k, \rho}(f) \right) T^n$$

where $c \in \mathcal{M}$ is the product of all primes $\pi \in \mathcal{I} \cap \mathcal{C}$.

Let $\Gamma_q(c)$ be the finite abelian group $(\mathbb{F}_q[t]/c \mathbb{F}_q[t])^\times$, and for each $A \in \Gamma_q(c)$, let

$$S_{n,c,q}(A) := \sum_{f \in \mathcal{M}_n(A)} d_{k, \rho}(f)$$

where

$$\mathcal{M}_n(A) := \{ f \in \mathcal{M}_n : f \equiv A \text{ mod } c \}.$$
We regard $A$ as a random variable uniformly distributed over $\Gamma_q(c)$ and define its expected value
\[
\mathbb{E}_A[S_{n,c,q}(A)] := \frac{1}{|\Gamma_q(c)|} \sum_{A \in \Gamma_q(c)} S_{n,c,q}(A)
\]
and variance
\[
\text{Var}_A[S_{n,c,q}(A)] := \frac{1}{|\Gamma_q(c)|} \sum_{A \in \Gamma_q(c)} |S_{n,c,q}(A) - \mathbb{E}_A[S_{n,c,q}(A)]|^2
\]
accordingly.

It follows easily the definition and (2.0.4) that
\[
L_C(T, \rho) = |\Gamma_q(c)| \cdot \sum_{n=0}^{\infty} \mathbb{E}_A[S_{n,c,q}(A)] T^n,
\]
so each expected value is the coefficient of an $L$-series. The main goal of this paper is to analyze the asymptotic behavior of $|\text{Var}_A[S_{n,c,q}(A)]|^2$ as $q \to \infty$, that is, as we replace $q$ by a power $q^r$ and take $r \to \infty$. To do so, we must impose some hypotheses on $\rho$, e.g., we suppose that $\rho$ is punctually pure of weight $w$ (see section 6 in [HKRG17]). We also impose hypotheses on the Mellin transform of $\rho$, but before doing so we need some additional notation and terminology.

Let $\Phi_q(c)$ be the finite abelian group $\text{Hom}(\Gamma_q(c), \bar{\mathbb{Q}}_\ell^*)$. For each $\varphi \in \Phi_q(c)$, there is a corresponding Dirichlet character
\[
\varphi: G_{K,C} \to \text{GL}(\bar{\mathbb{Q}}_\ell)
\]
which we regard as a representation of $G_{K,R}$ for $R = C \cup S$ by composing $\varphi$ with the quotient $G_{K,R} \to G_{K,C}$. We also regard $\rho$ as a representation of $G_{K,R}$ via the quotient $G_{K,R} \to G_{K,S}$, and we define the tensor-product representation
\[
\rho \otimes \varphi: G_{K,R} \to \text{GL}(V_\varphi)
\]
where $V_\varphi = V$ and $f \mapsto \rho(f) \varphi(f)$.

Let $L(T, \rho \otimes \varphi)$ and $L_C(T, \rho \otimes \varphi)$ be the $L$-functions respectively defined by the Euler products in (2.0.1) and (2.0.2) with $\rho \otimes \varphi$ in lieu of $\varphi$. A priori each of these is a power series with coefficients in $\bar{\mathbb{Q}}_\ell$, but Grothendieck showed both are rational functions in $\bar{\mathbb{Q}}_\ell(T)$, compare (1.4.7) of [Del80]. We say that $\varphi \in \Phi_q(c)$ is good for $\rho$ if it lies in the set
\[
(2.0.5) \quad \Phi_q(c)_\rho \text{good} := \{ \varphi \in \Phi_q(c) : L(T, \rho \otimes \varphi) = L_C(T, \rho \otimes \varphi) \in \bar{\mathbb{Q}}_\ell[T] \},
\]
and otherwise we say that $\varphi$ is bad for $\rho$.

The hypothesis that $\rho$ is punctually pure implies that both $L$-functions lie in $\bar{\mathbb{Q}}(T)$, and $\varphi$ is good for $\rho$ if every zero $\alpha \in \bar{\mathbb{Q}}$ of $L_C(T, \rho \otimes \varphi)$ satisfies $|\iota(\alpha)|^2 = 1/q^{1+w}$ for every field embedding $\iota: \bar{\mathbb{Q}} \to \mathbb{C}$. Equivalently, $\varphi$ is good for $\rho$ iff the ‘unitarized’ $L$-function
\[
L_C^*(T, \rho \otimes \varphi) := L_C(T/(\sqrt{q})^{1+w}, \rho \otimes \varphi)
\]
is the characteristic polynomial of a unitary conjugacy class $\theta_{\rho,q,\varphi} \subset U_R(\mathbb{C})$ for $R = \text{deg}(L_C(T, \rho))$.

We further distinguish bad characters by saying that $\rho \in \Phi_q(c)$ is mixed for $\rho$ if it lies in the set
\[
(2.0.6) \quad \Phi_q(c)_\rho \text{mixed} := \{ \varphi \in \Phi_q(c) \setminus \Phi_q(c)_\rho \text{good} : L_C(T, \rho \otimes \varphi) \in \bar{\mathbb{Q}}_\ell[T] \},
\]
and otherwise we say that the elements of
\[
(2.0.7) \quad \Phi_q(c)_\rho \text{heavy} := \Phi_q(c) \setminus (\Phi_q(c)_\rho \text{good} \cup \Phi_q(c)_\rho \text{mixed})
\]
are heavy for $\rho$. The mixed characters are those for which $L_C^*(T, \rho \otimes \varphi)$ is not the characteristic polynomial of a unitary matrix, and the heavy characters are those for which $L_C^*(T, \rho \otimes \varphi)$ is not even a polynomial.

We are now in a position to state our main theorem:
Lemma 3.0.1. If the Mellin transform of \( \rho \) has big monodromy (see section \( \S \)), and that \( \Phi_q(c)_{\rho_{\text{heavy}}} \subseteq \{1\} \) for all \( q \). Then
\[
\lim_{q \to \infty} \frac{|\Phi_q(c)|}{q^{n(1+w)}} \cdot \text{Var}_A[S_{k,n,c}(A)] = I_k(n; R)
\]
for each \( n \geq 1 \).

Remark 2.0.9. If \( \rho \) is the representation associated to the \( \ell \)-adic Tate module of the Legendre curve, then the hypotheses on \( c \) in Theorem 1.1.2 imply that the Mellin transform of \( \rho \) has big monodromy (see [HKRG17, \S 8]). One can also show that \( R = 2 \deg(e) - 1 \) (cf. loc. cit.). Thus Theorem 1.1.2 follows from Theorem 2.0.8.

Corollary 2.0.10. In the case of \( k = 2 \) we have for \( n \leq R \)
\[
\lim_{q \to \infty} \frac{|\Gamma_q(c)|}{q^{2n}} \text{Var}_A[S_{2,n,c}(A)] = \left(\frac{n + 3}{3}\right)
\]
and for \( R < n < 2R \)
\[
\lim_{q \to \infty} \frac{|\Gamma_q(c)|}{q^{2n}} \text{Var}_A[S_{2,n,c}(A)] = \left(\frac{2R - n + 3}{3}\right)
\]

3. Big Monodromy and Equidistribution

In [HKRG17, \S 10], we defined a subgroup \( \Phi_q(u)^\nu \subseteq \Phi_q(c) \), and for each coset \( \varphi \Phi_q(u)^\nu \), we defined a monodromy group
\[
\mathcal{G}_{\text{geom}}(\varphi \Phi_q(u)^\nu, \rho \Phi_q(u)^\nu) \subseteq \text{GL}_R(\bar{\mathbb{Q}}_\ell)
\]
generated by Frobenii corresponding to the good characters in \( \varphi \Phi_q(u)^\nu \), where \( R := \deg(L_C(T, \rho)) = \deg(L_C(T, \rho \otimes \varphi)) \) (see [HKRG17, Prop. 4.3.1]). More precisely, the Frobenius attached to a character \( \varphi \alpha^\nu \in \varphi \Phi_q(u)^\nu \) is an element of \( \text{GL}_R(\bar{\mathbb{Q}}_\ell) \) with characteristic polynomial \( L_C(T, \rho \otimes \chi\alpha^\nu) \), and \( \mathcal{G}_{\text{geom}}(\varphi \Phi_q(u)^\nu, \rho \Phi_q(u)^\nu) \) is the Zariski closure of all such elements when one takes \( q \to \infty \).

We say that \( \varphi \in \Phi_q(c) \) is big for \( \rho \) iff it lies in the set
\[
\Phi_q(c)_{\rho_{\text{big}}} := \{ \varphi \in \Phi_q(c) : \mathcal{G}_{\text{geom}}(\varphi \Phi_q(u)^\nu, \rho \Phi_q(u)^\nu) = \text{GL}_R(\bar{\mathbb{Q}}_\ell) \},
\]
and we say that the Mellin transform of \( \rho \) has big monodromy iff
\[
|\Phi_q(c)| \sim |\Phi_q(c)_{\rho_{\text{big}}}| \text{ as } q \to \infty.
\]

Lemma 3.0.1. If the Mellin transform of \( \rho \) has big monodromy, then
\[
|\Phi_q(c)| \sim |\Phi_q(c)_{\rho_{\text{good}}}| \sim |\Phi_q(c)_{\rho_{\text{big}}}| \text{ as } q \to \infty
\]
\[\]

Proof. See [HKRG17, Corollary 10.4.3]. \( \square \)

If the Mellin transform of \( \rho \) has big monodromy it implies that \( \theta_{\rho, q, \varphi} \) become equidistributed in \( U_R(\mathbb{C}) \) (for reference see Theorem 10.0.4 combined with remark 8.2.4 in [HKRG17]):

Theorem 3.0.2. Suppose that \( \rho \) is punctually pure of weight \( w \), that its Mellin transform has big monodromy, and that \( \Phi_q(c)_{\rho_{\text{heavy}}} \subseteq \{1\} \) for all \( q \). Then for any continuous function \( F : U_R(\mathbb{C}) \to \mathbb{C} \)
\[
\lim_{q \to \infty} \frac{1}{|\Phi_q(c)_{\rho_{\text{good}}}|} \sum_{\varphi \in \Phi_q(c)_{\rho_{\text{good}}}} F(\theta_{\rho, q, \varphi}) = \int_{U_R(\mathbb{C})} F(\theta) d\theta
\]
with respect to Haar measure \( d\theta \) on \( U_R(\mathbb{C}) \).

In the following Theorem we present a sufficient criteria for the Mellin transform of \( \rho \) to have big monodromy (explicit example for representations meeting this criteria can be found in [HKRG17, \S 12]).
Theorem 3.0.4. Let \( s \) be the conductor of \( \rho \) and suppose that \( \gcd(s,c) = t \) and that \( \deg c \geq 3 \). Suppose moreover that \( V(0) \) has a unique unipotent block of exact multiplicity one and that \( \rho \) is geometrically simple and pointwise pure. If \( r := \dim(V) \) and \( \deg c \) satisfy

\[
\deg c > \frac{1}{r}(72(r^2 + 1)^2 - r - \deg(L(T, \rho))) + \sum_{\nu \in C} \deg \nu(r - \deg L(T, \rho_\nu))
\]

then the Mellin transform of \( \rho \) has big monodromy.

4. Proof of Theorem 2.0.8

We express the variance of the arithmetic progressions sums \( S_{k,n,c}(A) \) in terms of sums of divisor functions twisted by Dirichlet characters. Let \( k \) be a positive integer and let

\[
c_{k,\rho \otimes \varphi, n} = \sum_{\substack{f \in \mathcal{M}_n \\ \gcd(f,c) = 1}} d_{k,\rho \otimes \varphi}(f)
\]

be the coefficients in the expansion of the k-th power of the partial L-function (2.0.4)

\[
L_C(T, \rho \otimes \varphi)^k = \sum_{n=0}^{\infty} c_{k,\rho \otimes \varphi, n} T^n
\]

For each \( \varphi \in \Phi_q(c) \), we extend \( \varphi \) to a multiplicative map \( \varphi_1 : \mathcal{M} \to \bar{\mathbb{Q}}_\ell \) by defining:

\[
\varphi_1(f) = \begin{cases} 
\varphi(f + c F_q[t]) & \text{if } \gcd(f,c) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

It is multiplicative and satisfies

\[
\varphi_1(\pi) = \begin{cases} 
\varphi(\text{Frob}_v(\pi)) & \text{if } \pi \nmid c \\
0 & \text{otherwise for } \pi \in \mathcal{I}.
\end{cases}
\]

The Orthogonality relations for Dirichlet characters are:

(i) for each \( A_1, A_2 \in \Gamma_q(c) \)

\[
\sum_{\varphi \in \Phi_q(c)} \varphi_1(A_1) \varphi_2(A_2) = \begin{cases} 
1 & \text{if } A_1 = A_2 \mod c \\
0 & \text{otherwise,}
\end{cases}
\]

(ii) for every \( \varphi_1, \varphi_2 \in \Phi_q(c) \)

\[
\sum_{A \in \Gamma_q(c)} \varphi_1(A) \varphi_2(A) = \begin{cases} 
1 & \text{if } \varphi_1 = \varphi_2 \\
0 & \text{if } \varphi_1 \neq \varphi_2.
\end{cases}
\]

We use (4.0.2) and (4.0.1) to express the sum of \( d_{k,\rho}(f) \) over arithmetic progressions in terms of \( c_{k,\rho \otimes \varphi, n} \):

\[
S_{n,c,q}(A) = \sum_{f \in \mathcal{M}_n} d_{k,\rho}(f) \sum_{\varphi \in \Phi_q(c)} \varphi_1(A) \varphi_1(f) = \frac{1}{|\Phi_q(c)|} \sum_{\varphi \in \Phi_q(c)} c_{k,\rho \otimes \varphi, n} \cdot \varphi_1(A)
\]
Therefore, if we write $1 \in \Phi_q(c)$ for the trivial character and by using the second orthogonality relation (4.0.3), then expected value of $S_{n,c,q}(A)$ equals

$$
\mathbb{E}_A[S_{n,c,q}(A)] := \frac{1}{|\Phi_q(c)|} \sum_{A \in \Gamma_q(c)} S_{n,c,q}(A)
$$

(4.0.4)

$$
= \frac{1}{|\Phi_q(c)|} \sum_{\varphi \in \Phi_q(c)} c_{k,\rho \otimes \varphi,n} \sum_{A \in \Gamma_q(c)} \bar{\varphi}_1(A)
$$

$$
= \frac{1}{|\Phi_q(c)|} c_{k,\rho \otimes 1,n}
$$

In particular, we have the identity

$$
S_{n,c,q}(A) - \mathbb{E}_A[S_{n,c,q}(A)] = \frac{1}{|\Phi_q(c)|} \sum_{\varphi \in \Phi_q(c), \varphi \neq 1} c_{k,\rho \otimes \varphi,n} \cdot \bar{\varphi}(A).
$$

Now consider the variance

$$
\text{Var}_A[S_{n,c,q}(A)] = \frac{1}{|\Phi_q(c)|} \sum_{A \in \Gamma_q(c)} |S_{n,c,q}(A) - \mathbb{E}_A[S_{n,c,q}(A)]|^2.
$$

If we apply identities (4.0.3) and (4.0.5), then the right side equals

$$
\frac{1}{|\Phi_q(c)|^3} \sum_{A \in \Gamma_q(c)} \sum_{\varphi_1, \varphi_2 \in \Phi_q(c), \varphi_1, \varphi_2 \neq 1} c_{k,\rho \otimes \varphi_1,n} c_{k,\rho \otimes \varphi_2,n} : \bar{\varphi}_1(A) \varphi_2(A) = \frac{1}{|\Phi_q(c)|^2} \sum_{\varphi \in \Phi_q(c), \varphi \neq 1} |c_{k,\rho \otimes \varphi,n}|^2.
$$

In summary, the function $S_{n,c,q}(A)$ of the random variable $A$ satisfies

$$
\mathbb{E}_A[S_{n,c,q}(A)] = \frac{1}{|\Phi_q(c)|} c_{k,\rho \otimes 1,n}, \quad \text{Var}_A[S_{n,c,q}(A)] = \frac{1}{|\Phi_q(c)|^2} \sum_{\varphi \in \Phi_q(c), \varphi \neq 1} |c_{k,\rho \otimes \varphi,n}|^2.
$$

Next, we break $c_{k,\rho \otimes \varphi,n}$ into smaller pieces which we will express in terms of the associated Frobenius matrices:

**Lemma 4.0.7.**

$$
c_{k,\rho \otimes \varphi,n} = \sum_{n_1 + \cdots + n_k = n} \prod_{i=1}^k c_{1,\rho \otimes \varphi,n_i}
$$

**Proof.** This follows immediately from the definition of $c_{k,\rho \otimes \varphi,n}$ as the coefficients in the expansion of $L_C(T, \rho \otimes \varphi)^k$:

$$
L_C(T, \rho \otimes \varphi)^k = (\sum_{n=0}^\infty c_{1,\rho \otimes \varphi,n} T^n)^k = \sum_{n=0}^\infty \prod_{i=1}^k c_{1,\rho \otimes \varphi,n_i} T^n
$$

(4.0.8)

Recall that std: $U_R(\mathbb{C}) \to \text{GL}_R(\mathbb{C})$ is the representation given by the inclusion $U_R(\mathbb{C}) \subset \text{GL}_R(\mathbb{C})$ and

$\wedge^j \text{std}: U_R(\mathbb{C}) \to \text{GL}_{R_j}(\wedge^j \mathbb{C})$

is its $j$th exterior power where $R_j = {R \choose j}$; and we defined $(\wedge^j \text{std})(g) = 0$ unless $0 \leq j \leq R$.

**Lemma 4.0.9.** Let $\varphi \in \Phi_q(c)_{\rho, \text{good}}$, then

$$
c_{1,\rho \otimes \varphi,n} = (-1)^n \cdot (\sqrt{q})^{n(1+w)} \cdot \text{Tr}((\wedge^n \text{std})(\theta_{\rho,\varphi}))
$$

11
Proof. A Dirichlet character \( \varphi \) is good for \( \rho \) iff the ‘unitarized’ \( L \)-function
\[
L^*_C(T, \rho \otimes \varphi) := L_C(T/(\sqrt{q})^{1+w}, \rho \otimes \varphi)
\]
is the characteristic polynomial of a unitary conjugacy class \( \theta_{\rho,q,\varphi} \subset U_R(\mathbb{C}) \) for \( R = \deg(L_C(T, \rho)) \).

The coefficients of the characteristic polynomial of an \( N \times N \) matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) are the elementary symmetric functions \( \sum_{1 \leq i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r} \) which give the character of the exterior power representation. Thus we may write
\[
L^*_C(T, \rho \otimes \varphi) = \sum_{i=0}^{R} (-1)^i \cdot (\sqrt{q})^{(1+w)} \cdot \text{Tr}(\wedge^i \theta_{\rho,q,\varphi})T^i
\]

Lemma 4.0.10. If \( \varphi \in \Phi_q(c) \), then
\[
|c_{1,\rho \otimes \varphi,n}|^2 = \begin{cases} O\left(q^{n(2+w)}\right) & \text{if } \varphi \in \Phi_q(c)_\text{heavy} \\ O\left(q^{n(1+w)}\right) & \text{otherwise} \end{cases}
\]
The implied constants depend only on \( \rho \) and \( n \).

Proof. Fix a field embedding \( \iota : \overline{Q} \rightarrow \mathbb{C} \) and identify \( \overline{Q} \) with its image. Recall that, for some integer \( s \) depending on \( \varphi \) and satisfying \( s \leq \dim(V) \), we can express \( L_C(T, \rho \otimes \varphi) \) as a ratio
\[
\prod_{i=1}^{s+R}(1 - \alpha_i T) / \prod_{j=1}^{s}(1 - \beta_j T)
\]
where the \( \alpha_i \) and \( \beta_j \) lie in \( \overline{Q} \) and satisfy
\[
|\alpha_i|^2 \leq q^{1+w}, \quad |\beta_j|^2 \leq q^{2+w}.
\]
For reference see equation (3.4.2) and Theorem 6.2.1 in [HKRG17]. If \( \varphi \notin \Phi_q(c)_\text{heavy} \), then \( s = 0 \) and \( |\alpha_i|^2 \leq q^{1+w} \), and thus
\[
|c_{1,\rho \otimes \varphi,n}|^2 = \left| \sum_{i_1 < \cdots < i_n} \alpha_{i_1} \cdots \alpha_{i_n} \right|^2 \leq \sum_{i_1 < \cdots < i_n} |\alpha_{i_1} \cdots \alpha_{i_n}|^2 \leq \binom{R}{k} q^{1+w}.
\]
When \( \rho \) is heavy we get
\[
|c_{1,\rho \otimes \varphi,n}|^2 = \left| \sum_{n_1+n_2=n} (-1)^{n_1} \sum_{i_1 < \cdots < i_{n_1}} \alpha_{i_1} \cdots \alpha_{i_{n_1}} \sum_{j_1 \leq \cdots \leq j_{n_2}} \beta_{j_1} \cdots \beta_{j_{n_2}} \right|^2
\]
\[
\leq \sum_{n_1+n_2=n} \sum_{i_1 < \cdots < i_{n_1}} \sum_{j_1 \leq \cdots \leq j_{n_2}} |\alpha_{i_1} \cdots \alpha_{i_{n_1}} \beta_{j_1} \cdots \beta_{j_{n_2}}|^2
\]
\[
= O(q^{n(2+w)}).
\]

Corollary 4.0.11. If \( \varphi \in \Phi_q(c) \), then
\[
|c_{k,\rho \otimes \varphi,n}|^2 = \begin{cases} O\left(q^{n(2+w)}\right) & \text{if } \varphi \in \Phi_q(c)_\text{heavy} \\ O\left(q^{n(1+w)}\right) & \text{otherwise} \end{cases}
\]
The implied constants depend only on \( \rho \) and \( n \).

Proof. Combine Lemma 4.0.7 and Lemma 4.0.10.
Lemma 4.0.12. If the Mellin transform of \( \rho \) has big monodromy and if \( \Phi_q(c)_{\rho \text{ heavy}} \subseteq \{1\} \), then

\[
\sum_{\varphi \in \Phi_q(c) \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2 \sim \sum_{\varphi \in \Phi_q(c)_{\rho \text{ good}} \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2 \quad \text{as } q \to \infty
\]

Proof. We break the sum over all Dirichlet characters mod \( c \) to sums over "good", "heavy" and "mixed" characters (see definitions (2.0.5), (2.0.7), (2.0.6))

\[
\sum_{\varphi \in \Phi_q(c) \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2 = \sum_{\varphi \in \Phi_q(c)_{\rho \text{ good}} \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2 + \sum_{\varphi \in \Phi_q(c)_{\rho \text{ heavy}} \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2 + \sum_{\varphi \in \Phi_q(c)_{\rho \text{ mixed}} \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2
\]

which give by Corollary 4.0.11

\[
\sum_{\varphi \in \Phi_q(c)_{\rho \text{ good}} \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2 + |\Phi_q(c)_{\rho \text{ heavy}} \setminus \{1\}| \cdot O(q^{n(2+w)}) + |\Phi_q(c)_{\rho \text{ mixed}} \setminus \{1\}| \cdot O(q^{n(1+w)})
\]

Lemma 3.0.1 and having \( \Phi_q(c)_{\rho \text{ heavy}} \subseteq \{1\} \) conclude the proof. \( \square \)

Proposition 4.0.13. If the Mellin transform of \( \rho \) has big monodromy and if \( \Phi_q(c)_{\rho \text{ heavy}} \subseteq \{1\} \), then

\[
\lim_{q \to \infty} \frac{|\Phi_q(c)|}{q^{n(1+w)}} \cdot \text{Var}_A[S_{n,c,q}(A)] = \int_{U_R(\mathbb{C})} \left| \sum_{n_j + \cdots + n_k = n \atop 0 \leq n_1, \ldots, n_k \leq R} \text{Tr} \left( \bigotimes_{i=1}^k (\wedge^{n_i \text{ std}})(\theta) \right) \right|^2 d\theta.
\]

Proof. In [4.0.6] we found the following expression for the variance

\[
\text{Var}_A[S_{n,c,q}(A)] = \left| \frac{1}{\Phi_q(c)} \right|^2 \sum_{\varphi \in \Phi_q(c) \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2.
\]

In the limit of \( q \to \infty \) we have from Lemma 4.0.12

\[
\text{Var}_A[S_{n,c,q}(A)] \sim \left| \frac{1}{\Phi_q(c)} \right|^2 \sum_{\varphi \in \Phi_q(c)_{\rho \text{ good}} \setminus \{1\}} |c_{k,\rho \otimes \varphi, n}|^2
\]

which Lemma 4.0.7 and Lemma 4.0.9 equals to

\[
\left| \frac{1}{\Phi_q(c)} \right|^2 \sum_{\varphi \in \Phi_q(c)_{\rho \text{ good}} \setminus \{1\}} \left| \sum_{n_j + \cdots + n_k = n \atop 0 \leq n_1, \ldots, n_k \leq R} \text{Tr} \left( \bigotimes_{i=1}^k (\wedge^{n_i \text{ std}})(\theta_{\rho,\varphi}) \right) \right|^2.
\]

Now note that \( |\Phi_q(c)| \sim |\Phi_q(c)_{\rho \text{ good}}| \) (Lemma 3.0.1) and apply the equidistribution result Theorem 3.0.2 to conclude the proof. \( \square \)

5. Matrix Integral

In this section we state a few results evaluating the matrix integral in certain ranges and its asymptotic behaviour. Proofs can be found in [KRRGR18].

In certain ranges the matrix integral evaluates to a very simple expression. For \( n < R \) or \((k-1)R < n < kR\), we obtain the following formulae:

Theorem 5.0.1. Let \( I_k(n; R) \) be the matrix integral defined in (1.1.1). Then
(i) for \((k - 1)R < n < kR\),
\[
I_k(n; R) = \left( \frac{kR - n + k^2 - 1}{k^2 - 1} \right).
\]

(ii) for \(n < R\)
\[
I_k(n; R) = \left( \frac{n + k^2 - 1}{k^2 - 1} \right).
\]

There is also a closed-form formula for the matrix integral for any range of the parameters, in terms of a lattice point count:

**Theorem 5.0.2.** \(I_k(m; N)\) is equal to the count of lattice points \(x = (x_i^{(j)}) \in \mathbb{Z}^k\) satisfying each of the following relations:

(i) \(0 \leq x_i^{(j)} \leq N\) for all \(1 \leq i, j \leq k\);

(ii) \(x_1^{(k)} + x_2^{(k-1)} + \cdots + x_k^{(1)} = kN - m\), and

(iii) \(x\) is a \(k \times k\) matrix whose entries satisfy the following system of inequalities,

\[
\begin{align*}
x_1^{(1)} &\leq x_1^{(2)} \leq \cdots \leq x_1^{(k)} \\
\vdots &\leq \vdots \leq \cdots \leq \vdots \\
x_k^{(1)} &\leq x_k^{(2)} \leq \cdots \leq x_k^{(k)}
\end{align*}
\]

The asymptotic behavior of \(I_k(n; R)\) for \(n \approx R\) is given in the following theorem:

**Theorem 5.0.3.** Let \(c := n/R\). Then for \(c \in [0, k]\),
\[
I_k(n; R) = \gamma_k(c)R^{k^2-1} + O_k(R^{k^2-2}),
\]
with
\[
\gamma_k(c) = \frac{1}{k!G(1+k)^2} \int_{[0,1]^k} \delta_c(w_1 + \ldots + w_k) \prod_{i<j}(w_i - w_j)^2 \, dw,
\]
with \(\delta_c(x) = \delta(x - c)\) being the delta distribution translated by \(c\), and \(G\) is the Barnes \(G\)-function, so that for positive integers \(k\), \(G(1+k) = 1! \cdot 2! \cdot 3! \cdots (k-1)!\).

**References**

[Blö08] V. Blomer, *The average value of divisor sums in arithmetic progressions*, Q. J. Math. 59 (2008), no. 3, 275–286. MR2444061

[CFK+05] J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Integral moments of \(L\)-functions*, Proc. London Math. Soc. (3) 91 (2005), no. 1, 33–104. MR2149530

[CRA22] Harald Cramér, *über zwei Sätze des Herrn G. H. Hardy*, Math. Z. 15 (1922), no. 1, 201–210. MR1544568

[Del80] Pierre Deligne, *La conjecture de Weil. II*, Inst. Hautes Études Sci. Publ. Math. 52 (1980), 137–252. MR601520

[FKM15] Étienne Fouvry, Emmanuel Kowalski, and Philippe Michel, *On the exponent of distribution of the ternary divisor function*, Mathematika 61 (2015), no. 1, 121–144. MR3333965

[HB92] D. R. Heath-Brown, *The distribution and moments of the error term in the Dirichlet divisor problem*, Acta Arith. 60 (1992), no. 4, 389–415. MR1159354
Chris Hall, Jonathan P. Keating, and Edva Roditty-Gershon, *Variance of sums in arithmetic progressions of arithmetic functions associated with higher degree $l$-functions in $\mathbb{F}_q[t]$,* arXiv:1703.09190v1 (2017).

Emmanuel Kowalski and Guillaume Ricotta, *Fourier coefficients of $GL(N)$ automorphic forms in arithmetic progressions,* Geom. Funct. Anal. 24 (2014), no. 4, 1229–1297. MR3248485

J. P. Keating, B. Rodgers, E. Roditty-Gershon, and Z. Rudnick, *Sums of divisor functions in $\mathbb{F}_q[t]$ and matrix integrals,* Math. Z. 288 (2018), no. 1-2, 167–198. MR3774409

Yuk-Kam Lau and Lilu Zhao, *On a variance of Hecke eigenvalues in arithmetic progressions,* J. Number Theory 132 (2012), no. 5, 869–887. MR2890517

Yoichi Motohashi, *On the distribution of the divisor function in arithmetic progressions,* Acta Arith. 22 (1973), 175–199. MR0340196

Brad Rodgers and Kannan Soundararajan, *The variance of divisor sums in arithmetic progressions,* arXiv:1610.06900v2 (2017).

E. C. Titchmarsh, *The theory of the Riemann zeta-function,* Second, The Clarendon Press, Oxford University Press, New York, 1986. Edited and with a preface by D. R. Heath-Brown. MR882550

Kwang-Chang Tong, *On divisor problems. II, III,* Acta Math. Sinica 6 (1956), 139–152, 515–541. MR0098718

DEPARTMENT OF MATHEMATICS, WESTERN UNIVERSITY, LONDON, ON, CANADA, N6A 5B7

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL BS8 1TW, UK

SCHOOL OF MATHEMATICS, UNIVERSITY OF BRISTOL, BRISTOL BS8 1TW, UK