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A novel fractional mathematical model of COVID-19 epidemic considering quarantine and latent time

Prashant Pandey a,b, Yu-Ming Chu c,d,*, J.F. Gómez-Aguilar e,f, Hadi Jahanshahi g, Ayman A. Aly b

a Department of Mathematical Sciences Indian Institute of Technology (BHU), Varanasi 221005, India
b Department of Mathematics Government M.G.M. P.G. College, IHarī 461111, India
c Department of Mathematics, Huazhou University, Huazhou 313000, PR China
d Hunan Provincial Key Laboratory of Mathematical Modeling and Analysis in Engineering, Changsha University of Science & Technology, Changsha 410114, PR China
e CONACyT-Tecnológico Nacional de México/CENIDET, Interior Internado Palmira S/N, Col. Palmira, C.P. 62490 Cuernavaca Morelos, Mexico
f Universidad Tecnológica de México - UNITEC MEXICO-Campus En Línea, Mexico
g Department of Mechanical Engineering, University of Manitoba, Winnipeg R3T 5V6, Canada
h Department of Mechanical Engineering, College of Engineering, Taif University, P.O.Box 11099, Taif 21944, Saudi Arabia

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ABSTRACT

In this paper, we investigate the fractional epidemic mathematical model and dynamics of COVID-19. The Wuhan city of China is considered as the origin of the corona virus. The novel corona virus is continuously spread its range of effectiveness in nearly all corners of the world. Here we analyze that under what parameters and conditions it is possible to slow the speed of spreading of corona virus. We formulate a transmission dynamical model where it is assumed that some portion of the people generates the infections, which is affected by the quarantine and latent time. We study the effect of various parameters of corona virus through the fractional mathematical model. The Laguerre collocation technique is used to deal with the concerned mathematical model numerically. In order to deal with the dynamics of the novel corona virus we collect the experimental data from 15th-21st April, 2020 of Maharashtra state, India. We analyze the effect of various parameters on the numerical solutions by graphical comparison for fractional order as well as integer order. The pictorial presentation of the variation of different parameters used in model are depicted for upper and lower solution both.

Introduction

Nowadays the mankind is continuously disobeys the basic and fundamental laws of Nature. In order to secure the future of their survival and to make a strong, positive, and respectful impact on other mankind they developed scientific research very rapidly. As a result of strong scientific research they invented and developed many scientific tools which make the life of mankind more wealthy, smarter, and easier. This new scientific way of tackling the existing and upcoming problems enhance the power of mankind so that in this race of advanced research they tackle many strong and dangerous problems. The fundamental and basic laws of nature have been made in such a way that the balance of the environment is being balanced for a longer time. Mankind has succeeded more enough in these new scientific developments and research.

During these advanced invention many laws of nature are continuously being violated and led society to face many diseases and natural disasters.

At the end of December 2019, there was a case of pneumonia in Wuhan city of China with undermined cause and for this case the available vaccines were ineffective. Later on, it was found that the cause of the case is the novel corona virus. The World Health Organization (WHO) has been declared a public health emergency and this new disease as a pandemic due to its exponential worldwide growth [1]. The COVID-19 disease is now a global health threat [12]. Till date there is no any available vaccine, medical treatments are registered to cure from this disease [10]. Many symptoms of COVID-19 were found, the majority of the cases have symptoms of fever and dry cough whereas some cases have symptoms which like joint and muscle pain, fatigue, sore
throat, shortness of the breath and headache. Due to the unavailability of an accurate treatment and vaccine, social distancing and quarantine are accepted by the world as an effective strategy to reduce the spreading of the transmission of corona virus. Many countries adopted the quarantine strategy for a large portion of the community and lockdowns of cities and states. Although there is a longtime lock-down across many countries, there is a class of doctors, sweepers, police and others who provide their essential services in this lock-down. Because of their dedication in service, the Govt. of India named these classes as Corona Warriors.

The World Health Organization has been defined four main stages of the transmission of COVID-19 disease. These stages are the areas or countries with: 1: Stage first is reached, when there are no any cases reported in the particular areas or countries. 2: Stage two is reached, when there are some sporadic cases reported. 3: Stage three is reached, when the cases are come in clusters at any places. 4: The stage four is a more sensitive stage where the cases start reported due to community transmission.

All the countries may be looking for which measures are to be adopted at different stages and analyze the situation of this pandemic regularly [15].

The first case of COVID-19 disease in India was reported on 30th January 2020. The starting few cases were reported in students who had a recent travel history from Wuhan, China. In the month of March, many cases were reported throughout India due to contact with some existing cases of COVID-19 [2]. Indian government realize the dangerous effect of the disease and takes some important decisions to prevent peoples from COVID-19. The Indian Prime Minister Shri Narendra Modi announced a lock-down for 21 days on 24th March throughout the country.

The field of investigation of biological is now a day encountered by many researchers [8,5]. In many cases, the formulation of biological models by mathematical modeling based on ordinary differential equations i.e., on classical derivatives have some limitations and may not be able to describe the biological phenomenon accurately. To deal with these biological models very efficiently and accurately, fractional mathematical models are being in used [13,20,19]. The mathematical formulation of the dynamics of many infectious diseases has a very deep history. Many researchers have developed a successful formulation of biological models using fractional calculus theory [22,4]. The fuzzy logic approach for various epidemic mathematical models was also discussed and analyzed by many researchers [16,6,7].

In the present paper, we formulate the COVID-19 effect on society in a systemic way. This mathematical model will be able to explain the rate of transmission of disease, the impact of disease in the rate of susceptible cases, exposed cases, infectious cases with different parameters viz., average latent time, contact rate, etc. Here we perform the examination of model simulations and investigate the sensitive analysis of the concerned COVID-19 model for effective outcomes of the model. Here, We will also analyze the nature and behavior of the solution of the concerned COVID19 model in the fuzzy environment under the interaction of various parameters.

This paper is organized as: Section 2 contains some definitions and properties of fractional derivative. The fractional mathematical formulation of dynamics of COVID-19 is given in Section 3. A brief outline of the Laguerre polynomial is given in Section 4. The approximation of the real state variables present in the mathematical model is given in Section 5. In Section 6 the novel operational matrix is derived. Section 7 contains the application of the derived fractional operational matrix to formulated model. The obtained numerical results are discussed in Section 8. The effect of different parameters present in the model is analyzed in section. The overall outcomes of the research paper are given in Section 10.

### Preliminaries and Notations

#### Caputo fractional derivatives

In this section of the manuscript, we have given basic definitions and properties of fractional order derivatives. The Riemann–Liouville’s derivatives have some drawbacks in modeling the mathematical modeling some real and physical phenomenon. Fractional order derivative operator $D^\eta$ of the given order $\eta > 0$ of a function $k(t)$ in the Caputo sense is given by [14,21]

\[
D^\eta k(t) = \frac{1}{\Gamma(n-\eta)} \int_0^t (t-\rho)^{n-1-\eta} k^{(n)}(\rho) \, d\rho, \quad \text{if} \ n-1 < \eta < n.
\]

In the above expression $r$ be the integer number. Few properties of Caputo fractional derivatives are:

\[
D^\eta P = 0, \quad (2)
\]

where $P$ is an arbitrary constant.

\[
D^\eta r^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\eta+1)} r^{\alpha-\eta}, \quad \text{if} \ r \in N \cup \{0\}, \ v < [\eta],
\]

where $[\eta]$ be the ceiling function and $\{\eta\}$ be the floor function.

#### Fuzzy set theory

The idea of fuzzy sets has been introduced by L. Zadeh [23] in 1965 to tackle the uncertainty arises because of imprecision and vagueness. Consider a nonempty set $Z$ which is called as base a set and every member $z \in Z$ associated to a membership grade $\zeta(z)$. A nonempty subset of $Z \times [0,1]$ is considered as a fuzzy subset of $Z$ by L. Zadeh. The definition of a fuzzy set $B$ is given as: $B = \{ (z, \zeta(z)) : z \in Z \}$ where $\zeta$ is a function from $Z$ to $[0,1]$. The symbol $\zeta$ is commonly used for the notation of the fuzzy set $B$.

**Definition 1. (Fuzzy Numbers):** Now we are going to provide the Definition of a fuzzy number $\tilde{\omega}$. A real valued function $\tilde{\omega}$ from $\mathbb{R}$ to unit interval $[0,1]$ i.e., $\tilde{\omega} : \mathbb{R} \longrightarrow [0,1]$ is said to be a fuzzy number if it satisfies the following basic properties:

- The function $\tilde{\omega}$ should be upper semi-continuous.
- The function $\tilde{\omega}$ should satisfy the normality properties i.e., $\exists a$ real number $a$ such that $\tilde{\omega}(a) = 1$.
- The convexity property should be satisfied by the function $\tilde{\omega}$ i.e., $\forall k \in [0,1]$ and $\forall z_1, z_2 \in \mathbb{R}$, we have
  \[
  \tilde{\omega}(kz_1 + (1-k)z_2) = \min\{\tilde{\omega}(z_1), \tilde{\omega}(z_2)\}.
  \]

- The closure set of the support of function $\tilde{\omega}$ is a compact set. The support of the function $\tilde{\omega}$ is defined as $\text{supp}(\tilde{\omega}) = \{ z \in \mathbb{R} : \tilde{\omega}(z) > 0 \}$.

**Definition 2. (v-Level Set of Fuzzy Numbers):** Consider the collection of all fuzzy numbers defined on set of real number $\mathbb{R}$ is denoted by $\mathbb{R}_v$. Let the fuzzy number $\tilde{\omega} \in \mathbb{R}_v$ for some $v \in [0,1]$ then for every $v \in [0,1]$ the v-level set of fuzzy number $\tilde{\omega}_v$ is defined as

\[
\tilde{\omega}_v = \{ z \in \mathbb{R} : \tilde{\omega}(z) \leq v \}, \quad \forall v \in [0,1].
\]

From the above Definition we can find that the v-level set $\tilde{\omega}_v$ is a closed and bounded set. Let $\tilde{\omega} \rightarrow (v)$ and $\tilde{\omega} \leftarrow (v)$ are the end points of the v-level fuzzy interval then we can write the v-level fuzzy interval as $[\tilde{\omega}_v = \tilde{\omega}_v \rightarrow (v), \tilde{\omega}_v \leftarrow (v)]$.
\[ \omega (v) = \begin{cases} 1, & y = z, \\ 0, & y \neq z. \end{cases} \]  

(6)

**Definition 3. (Parametric Interval Form):** For any fuzzy number \( \tilde{\omega} \in \tilde{R}_x \), the parametric interval form can be given as:

\[ \tilde{\omega}[v] = [\tilde{\omega}_l(v), \tilde{\omega}_r(v)], v \in [0, 1]. \]  

(7)

The above form of the fuzzy number satisfies the following properties:

- For every \( v \in [0, 1] \) the functions \( \tilde{\omega}_l(v) \) and \( \tilde{\omega}_r(v) \) satisfy the inequality \( \tilde{\omega}_l(v) \leq \tilde{\omega}_r(v) \).
- The function \( \tilde{\omega}_l(v) \) is a non-increasing and left continuous function of \( v \).
- The function \( \tilde{\omega}_r(v) \) is a non-decreasing and left continuous function of \( v \).

The arithmetic operations i.e. vector addition and scalar multiplication of any two arbitrary fuzzy numbers \( \tilde{\omega}_1(v) \) and \( \tilde{\omega}_2(v) \) are defined for \( v \in [0, 1] \) as:

\[ \left( \tilde{\omega}_1 \oplus \tilde{\omega}_2 \right)[v] = \left[ \tilde{\omega}_{1l}(v), \tilde{\omega}_{1r}(v), \tilde{\omega}_{2l}(v), \tilde{\omega}_{2r}(v) \right], \quad (k \odot \tilde{\omega})[v] = \left[ k \tilde{\omega}_l(v), k \tilde{\omega}_r(v), k \tilde{\omega}_1(v), k \tilde{\omega}_2(v) \right], \quad k \in [0, 1]. \]  

(8)

**Definition 4. (gH-difference):** Let Mand N are two nonempty compact set then there gH-difference (generalized Hukuhara difference) as the compact set \( P \) is given as:

\[ M \cap \rho P = P \iff \{ (a)M = N + P, \quad (b)N = M - P. \} \]  

(9)

**Definition 5. (gH-derivatives):** Here we will provide the definition of fuzzy derivatives (gH-derivatives) of any arbitrary fuzzy valued function. Consider a point \( z_0 \) in \( (l, m) \) and a fuzzy valued function \( \zeta \) such that \( \zeta : (l, m) \rightarrow \tilde{R}_x \). Then the function \( \zeta \) is H-differentiable at point \( z_0 \) and is equal to a fuzzy number \( \zeta (z_0) \) if it satisfies the following equations:

(i) **Case 1:** If the H-difference for two fuzzy number \( \zeta (z_0 + h \odot \zeta (z_0)) \) and \( \zeta (z_0) \) exists then have:

\[ \zeta (z_0) = \lim_{h \to 0^+} \frac{\zeta (z_0 + h \odot \zeta (z_0)) - \zeta (z_0)}{h}. \]  

(10)

This Definition of differentiation is called as 1-differentiation of function \( \zeta \) at \((l, m)\).

(ii) **Case 2:** If the H-difference for two fuzzy number \( \zeta (z_0 + h \odot \zeta (z_0)) \) and \( \zeta (z_0) \) exists then have:

\[ \zeta (z_0) = \lim_{h \to 0^-} \frac{\zeta (z_0 + h \odot \zeta (z_0)) - \zeta (z_0)}{-h}. \]  

(11)

This Definition of differentiation is called as 2-differentiation of function \( \zeta \) at \((l, m)\).

The gH-derivative can also be given in same manner as:

\[ \zeta (z_0) = \lim_{h \to 0^+} \frac{\zeta (z_0 + h \odot \mu \zeta (z_0))}{h}. \]  

(12)

In order to establish the fractional derivatives in Caputo and Riemann–Liouville we are going to provide the Definition of Lebesgue integration of any function \( \zeta(t) \) in parametric fuzzy interval form as:

\[ \left[ \int_0^1 \zeta (z) \, dz \right]_v = \left[ \int_0^1 \zeta (z) \, dz \right]_v = \begin{cases} \int_0^1 \zeta (z) \, dz + \int_0^1 \zeta (z) \, dz, & \text{for case-1,} \\ \int_0^1 \zeta (z) \, dz + \int_0^1 \zeta (z) \, dz, & \text{for case-2.} \end{cases} \]  

(13)

**Fuzzy fractional derivatives**

Here we are going to present fuzzy fuzzy fractional derivatives of a fuzzy differential function \( \zeta(t) \). The fuzzy fractional derivatives are the generalization of classical fractional differentiation in crisp sense.

**Definition 6. (Caputo fractional g-derivatives):** The fractional g-derivatives of any fuzzy valued measurable continuous function \( \zeta(t) \) of any arbitrary fractional order in Caputo sense at point \( t \) is given as:

\[ D_t^\alpha \zeta(t) = \lim_{h \to 0} \frac{\lambda (t+h) \odot \lambda (t)}{h}. \]

(14)

where the function \( \lambda \) is given by

\[ \lambda (t) = \frac{1}{\Gamma(1-\mu)} \int_t^\infty (t - \rho)^{-\mu} \zeta (\rho) \, dp. \]  

(15)

Let the function \( \zeta(t) \) is absolutely continuous fuzzy valued function then Caputo fractional fuzzy derivatives is defined for both previously cases as:

\[ D_t^\alpha \zeta(t) = \begin{cases} D_t^\alpha D_t^\mu \zeta(t), & \text{for case-1,} \\ D_t^\alpha D_t^\mu \zeta(t), & \text{for case-2.} \end{cases} \]

(16)

where \( D_t^\alpha D_t^\mu \zeta(t) \) and \( D_t^\alpha D_t^\mu \zeta(t) \) are given by the following equations:

\[ D_t^\alpha D_t^\mu \zeta(t) = \frac{1}{\Gamma(1-\mu)} \int_t^\infty (t - \rho)^{-\mu} \zeta (\rho) \, dp. \]  

(17)

\[ D_t^\alpha D_t^\mu \zeta(t) = \frac{1}{\Gamma(1-\mu)} \int_t^\infty (t - \rho)^{-\mu} \zeta (\rho) \, dp. \]  

(18)

**Definition 7. (Riemann Integrability):** Here we are going to provide the Definition of Riemann integrability in fuzzy approach. A fuzzy valued function \( \zeta \) such that \( \zeta : [l, m] \rightarrow \tilde{R}_x \) is said to be Riemann integrable if for any \( \rho > 0 \) there exist \( \tau > 0 \) such that for every partition \( E = \{k_1, k_2, \ldots \} \) of the given domain \([l, m] \) we have

\[ \sum_{k=1}^{n} \left( \sum_{k} (k_k - k_{k-1}) \odot \zeta (k) \right) < \rho. \]  

(19)

with norms of partition \( E \) less than \( \tau \). The symbol \( \sum_k \) denotes the summation for addition under fuzzy calculus. We can also rewrite the Riemann integrability in fuzzy sense as

\[ J = \left( \int E \right) \int_0^\infty \zeta (k) \, dk. \]  

(20)

The symbol \( \sum_k \) in Eq. (19) denotes the metric on set of all fuzzy numbers \( \tilde{R}_x \) which is defined as

\[ \sum_{k \in [l, m]} = \sup \left( \int \zeta (v) - \tilde{\omega}_1 (v) \mid \zeta (v) - \tilde{\omega}_2 (v) \right). \]

(20)

With respect to metric \( \sum_k \), the set of fuzzy numbers \( \tilde{R}_x \) forms a complete metric space.
Fractional COVID-19 model formulation

In this scientific contribution, authors have reformulated a modified mathematical model observing the threshold behavior in recovery rate and quarantined cases concerning the latent time and quarantine time with a significant difference to SEIR model [9] w.r.t to Caputo fractional derivative.

In order to characterize the effect of COVID-19 disease in state Maharashtra of India, here we consider the main six various parameters viz., \( S_m(t), E_m(t), I_m(t), Q_m(t), D_m(t) \) and \( R_m(t) \) at some time \( t \). Let \( N_m \) denotes the total population in Maharashtra in current time, \( S_m(t) \) denotes the total number of susceptible cases at time \( t \), \( E_m(t) \) denotes the total number of exposed cases at time \( t \) i.e. total number of infected cases which not yet be infectious, \( I_m(t) \) denotes the total number of infectious cases which is not quarantined, \( Q_m(t) \) denotes the total number of quarantined case, \( D_m(t) \) denotes the death cases and \( R_m(t) \) denotes the recovered cases. We can write \( N_m(t) = S_m(t) + E_m(t) + I_m(t) + Q_m(t) + D_m(t) + R_m(t) \) at time \( t \) for any state or place. The relation of interaction among these parameters in the epidemic fractional mathematical model of COVID-19 is demonstrated graphically in the Fig. 1. In this pictorial presentation the constant \( \alpha \) is a rate of natural mortality rate in the state, \( \beta \) is contact rate among peoples, \( 1/\delta \) denotes the average latent time, \( 1/\omega \) is average quarantined time, \( \theta(t) \) is cure rate at a time \( t \), \( \xi(t) \) is mortality rate, \( \omega \) denotes the recovery rate of infectious peoples without going through the quarantine state and \( \omega \) denotes the death rate of infectious peoples without going through the quarantine state. The term \( \alpha S_m(t) \) is the natural death rate of susceptible cases and the term \( \beta I_m(t) \) denotes the number of susceptible peoples who got infected (but not infectious yet) through the contact with infected peoples. The term \((\gamma/\beta)E_m(t)\) arises in the average latent time. In the Fig 1 it can be seen that the death rate \( \xi(t) \) and the cure rate \( \theta(t) \) is time dependent variable coefficients. In an instance by the treatment of medical team and uses of newly tested drugs it can be found that the recovery rate is increases so that death rate decreases very quickly. The term \( \omega I_m(t) \) denotes the death cases for recovered peoples by natural cause. The number of recovered peoples among the infectious class without going through the quarantined are expressed by term \( \omega I_m(t) \). Here it is assumed that the time dependent coefficients \( \xi(t) \) denotes the mortality rate and the total closed cases (mortality cases) during the quarantine state is given by the term \( \xi(t)Q_m(t) \). Assuming that the there is quarantine system of infectious persons, in that Case the quarantine Case is computed by the average quarantine time with infectious class i.e. by the term \( \delta I_m(t) \). This fractional mathematical model of the dynamics of corona virus in the Maharashtra (India) is formulated on the assumption that peoples need some average time to come into the class of infectious peoples, quarantine class, recovery class or death class. The total number of quarantined cases, death cases, infected cases and the recovered cases are available of common man by the Govt. of India.

Because of the time dependence of cure and death rate the term \( Q_m(t) \) plays an important role in the modeling of such epidemic disease. The average latent time is calculated within many days of observations.

The formulated experimental fractional model for corona virus disease in Caputo sense is given by the following system of differential equations:

\[
\begin{align*}
\frac{d^\mu S_m(t)}{dt^\mu} &= -\beta \frac{S_m(t)I_m(t)}{N_m} - \alpha S_m(t), \\
\frac{d^\mu E_m(t)}{dt^\mu} &= \beta \frac{S_m(t)I_m(t)}{N_m} - \left( \alpha + \gamma \right) E_m(t), \\
\frac{d^\mu I_m(t)}{dt^\mu} &= \gamma E_m(t) - \left( \alpha + \delta + \omega + \sigma \right) I_m(t), \\
\frac{d^\mu Q_m(t)}{dt^\mu} &= \delta I_m(t) - \left( \theta(t) + \xi(t) \right) Q_m(t), \\
\frac{d^\mu D_m(t)}{dt^\mu} &= \sigma I_m(t) + \xi(t) Q_m(t), \\
\frac{d^\mu R_m(t)}{dt^\mu} &= \theta(t) Q_m(t) + \alpha I_m(t) - \alpha R_m(t),
\end{align*}
\]

with the non-negative initial conditions

\[
S_m(0) = a, \ E_m(0) = b, \ I_m(0) = c, \ Q_m(0) = d, \ D_m(0) = e, \ R_m(0) = f.
\]

In the above fractional epidemic COVID-19 model the term \( \mu \) denotes the fractional order of Caputo sense derivatives. To find the numerical solution of this experimental model we adopt the novel numerical scheme developed in the literature [18].

Although there are wide application of fractional PDEs but the accurate mathematical model of many complex physical processes can not be found. Zadeh developed the concept of fuzzy theory to overcome this lackness of fractional PDEs and the application of fuzzy theory with fractional PDEs can be able to mathematical modeling of such complex physical process. Fuzzy DEs are very efficient tools which explain the many dynamical process accurately where the nature of dynamical process is uncertain with vague information [11]. There we consider COVID19 model (21) under fuzzy environment with \( \tilde{k}(v) = [0.9 + 0.1v, 1.1 - 0.1v] \) as
\[ c_0 \partial^\mu_s \tilde{S}_m(t) = -\frac{\tilde{S}_m(t) \tilde{E}_m(t)}{N_m} - a \tilde{S}_m(t) \partial^\mu_s \tilde{E}_m(t) = \frac{\tilde{S}_m(t) \tilde{E}_m(t)}{N_m} \]

\[ a + \alpha \tilde{E}_m(t) \partial^\mu_s \tilde{E}_m(t) = \beta \tilde{E}_m(t) = (a + \alpha + \sigma) \tilde{E}_m(t) \]

\[ \Delta t = \frac{1}{T} \int_{a}^{b} e^{-s} \mu(t) dt \]

\[ \partial \frac{d}{d \mu} L_n(t) + \delta L_n(t) = 0, t \in I, \]

\[ L_n(t) = \frac{d L_n(t)}{d \mu}, \delta L_n(t) = 0, \partial \mu \geq 0, \]

where \( \Delta t = e^{-s} \) is weight function and \( \delta_{n:m} \) is the Kronecker delta function.

**Approximation of the function \( \zeta(x) \)**

The collection of Laguerre polynomial form a basis set for \( L^2(I) \) so any function \( \zeta(t) \in L^2(I) \) can be expanded as:

\[ \zeta(t) = \sum_{m=0}^{\infty} b_m L_m(t), \quad m = 0, 1, 2, \ldots; \]

where the unknown constants \( b_m \) can be calculated as:

\[ b_m = \int_{a}^{b} \frac{\partial^\mu_s L_m(t)}{\partial \mu} \partial^\mu_s L_m(t) d \mu. \]

In the terms of first \( r + 1 \)-Laguerre polynomials the above approximation reduced to

\[ \tilde{\zeta}(t) = \sum_{k=0}^{r} b_k L_k(t) = \tilde{B}^\mu \sigma(t), \]

where \( B^\mu = [b_0, b_1, \ldots, b_r] \) is the matrix of unknown constants and \( \sigma(t) = [L_0(t), L_1(t), \ldots, L_r(t)]^T \) is the Laguerre vector.

**Approximation of fuzzy valued function \( \tilde{\zeta}(t) \)**

Here we use the orthogonal Laguerre polynomial in order to approximate a fuzzy valued measurable and continuous function \( \tilde{\zeta}(t) \). The approximation of the fuzzy valued function in terms of first \( r + 1 \)-Laguerre polynomials is given as:

\[ \tilde{\zeta}(t) = \sum_{k=0}^{r} \tilde{b}_k \circ L_k(t) = \tilde{B}^\mu \circ \sigma(t), \]

where \( \sigma(t) \) is the Laguerre vector and the unknown fuzzy coefficients are given by the following equation:

\[ \tilde{b}_k = \int_{a}^{b} \frac{\partial^\mu_s L_k(t)}{\partial \mu} \circ \sigma(t) d \mu. \]

Here the summation is taken in accordance with fuzzy algebraic addition \( \circ \) and \( \circ \) denotes the fuzzy scalar multiplication. The unknown fuzzy coefficients matrix \( \tilde{B} = [\tilde{b}_k] \) can be calculated with the help of Eq. (33) where all the operation will be taken as fuzzy set algebra.

**Fuzzy operational matrix for fractional derivative**

In order to derive the novel fuzzy operational matrix we have to provide following lemma.

**Lemma 1.** Consider the fractional fuzzy derivative operator \( D^\mu_s \) of fractional order \( \mu \) in Caputo sense. Let \( \sigma(t) \) be the Laguerre vector defined as in Eq. (31) then we have following results:

\[ c_0 \partial^\mu_s L_n(t) = 0, r = 0, 1, \ldots, [\mu] - 1, 0 < \mu \leq 1, \]

**Proof:** This result can be easily proved by using the basic properties of fractional Caputo derivative in the Eq. (26).

We are going to derive the operational matrix of orthogonal Laguerre polynomial [18]. On solving the Eqs. (29,30) and Eq. (26) together we get the following expressions

\[ c_0 \partial^\mu_s L_n(t) = \sum_{e=0}^{r} \frac{(-1)^e \Gamma e^\mu}{(l-g)!(g)!(g+1)^e} \int_{a}^{b} L_m(t) \partial^\mu_s L_m(t) d \mu \]

As set of Laguerre polynomials is a basis so we can write the term \( e^\mu \) as

\[ e^\mu = \sum_{e=0}^{r} b_e L_e(t). \]

where the constant coefficient \( b_e \) is given by

\[ b_e = \sum_{e=0}^{r} \frac{(-1)^e \Gamma e^\mu}{(l-g)!(g)!(g+1)^e}. \]

Using the equations (40)-(41), into the Eq. (35), we have

\[ c_0 \partial^\mu_s L_n(t) = \sum_{e=0}^{r} b_e L_e(t). \]
In the above equation the term $P_m(l,h)$ is given by the following equation

$$P_m(l,h) = \sum_{k=0}^{l} \frac{(-1)^k \Gamma(k-\mu+\alpha+1)}{l^k \Gamma(\mu-\alpha+1)!} \Gamma(g-\mu+1)! (l-h)^k. \quad (39)$$

The vector for of the equation (38) is written as

$$C^0D^\mu L_m(t) = [P_m(0,0), P_m(1,0), \ldots, P_m(r,0)] \sigma(t), l = [\mu], \ldots, r. \quad (40)$$

In the view of Lemma 1, we have

$$C^0D^\mu L_m(t) = [0, 0, \ldots, 0] \sigma(x), \quad l = 0, 1, \ldots, [\mu] - 1. \quad (41)$$

Using the equations (40), (41), we can write in matrix form as

$$C^0D^\mu \sigma(t) = H^\mu \sigma(t), \quad (42)$$

where $H^\mu$ is an operational matrix fractional derivatives of $r+1 \times r+1$ order and is given by the following equation:

$$H^\mu = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
P_m([\mu],0) & P_m([\mu],1) & P_m([\mu],2) & \cdots & P_m([\mu],r) \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
P_m(0,0) & P_m(0,1) & P_m(0,2) & \cdots & P_m(0,r) \\
P_m(r,0) & P_m(r,1) & P_m(r,2) & \cdots & P_m(r,r)
\end{bmatrix} \quad (43)$$

Implementation of Laguerre operational matrix on COVID-19 model

Here we are going to apply the Laguerre operational matrix and spectral collocation techniques to our concerned non-linear fractional COVID-19 model.

In view of Eq. (29), the function $\zeta(t)$ can be expressed in terms of initial $r+1$ Laguerre polynomials as:

$$\zeta_l(t) = \sum_{\mu=0}^{l} b_{\mu} L_{\mu}(t) = B^l H^\mu \sigma(t), \quad (44)$$

where $b_{\mu}$ is the unknown coefficients to be determined and $\sigma(t)$ is a Laguerre vector.

We operate the fractional derivatives in the Eq. (44) with respect to time of fractional order $\mu$ and using the Eq. (43), we get

$$\frac{d^\mu \zeta(t)}{dt^\mu} = H^\mu \sigma(t). \quad (45)$$

Now in view of Eq. (44), the initial conditions (22) can be rewritten as:

$$B^l \sigma(0) = a, \quad B^l \sigma(0) = b, \quad B^l \sigma(0) = c, \quad B^l \sigma(0) = d, \quad B^l \sigma(0) = e, \quad B^l \sigma(0) = f. \quad (46)$$

In view of the Eq. (45) the concerned model’s variables can be approximated as:

$$C^0D^\mu S_m(t) = B^l H^\mu \sigma(t), \quad (47)$$

$C^0D^\mu E_m(t) = B^l H^\mu \sigma(t), \quad C^0D^\mu I_m(t) = B^l H^\mu \sigma(t), \quad C^0D^\mu R_m(t) = B^l H^\mu \sigma(t).$

Definition 8. (Fuzzy system of linear equation:) Consider a system of linear equations as

$$b_{11} \zeta_1 + b_{12} \zeta_2 + \cdots + b_{1m} \zeta_m = d_1,$$
$$b_{21} \zeta_1 + b_{22} \zeta_2 + \cdots + b_{2m} \zeta_m = d_2,$$
$$\vdots \vdots $$
$$b_{m1} \zeta_1 + b_{m2} \zeta_2 + \cdots + b_{mm} \zeta_m = d_n. \quad (48)$$

We can write the above system of equations in the matrix form as

$$BZ = D. \quad (49)$$
In the above equation the constant numbers $d_j's$ are fuzzy numbers and $C_{n \times n}$ is crisp matrix of order $n \times n$. This system of linear equations can be solved by the techniques given in the literature [3].

Now collocating Eq. (21) with the help of Eq. (46) at points $t_a=\frac{a}{r}$ for $a = 0, 1, 2, \ldots, r$. After collocating, we get a system of equations. Further by simplifying this algebraic system of equations and finding the unknown matrix coefficients $B_i's$, we obtain numerical solution of our given problem by substituting $B_i's$ in Eq. (44).

Now we are going to investigate the proposed Covid-19 model into fuzzy environment.

**Numerical results**

In this section of the article the adopted numerical scheme is applied to our concerned nonlinear fractional corona virus model (21) with the non-negative initial conditions (22) which is estimated and collected for the total population of Maharashtra $N_m(0) = 12486220$. The total susceptible Case for is $S_m(0) = 12113755$, exposed cases $E_m(0) = 300000$, infectious cases $I_m(0) = 2916$, death cases $D_m(0) = 186$, quarantine cases $Q_m(0) = 69068$ and the recovered cases $R_m(0) = 295$.

The values of other coefficients parameter is given by

Natural Mortality Rate $\left( \alpha \right) = \frac{1}{69.73 \times 365} = 3.93056 \times 10^{-3}$,

Contact Rate $\left( \beta \right) = 0.05$,

Average Latent Time $\left( 1/\gamma \right) = 0.00042914$,

Average Quarantine Time $\left( 1/\delta \right) = 0.005$,

Recovery Rate of $I_m(\omega) = 0.09648$,

Death Rate of $I_m(\sigma) = 0.913012$,

Cure Rate $\gamma(\iota) = 1.4$,

Death Rate $\xi(\iota) = 0.6$.  

Fig. 3. Plot of number of recovered Case between reported Case vs. experimental result of COVID-19 in Maharashtra.

Fig. 4. Comparison between numerical fuzzy solution and normal solution for susceptible cases at different values of fractional order $\mu$. 

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The plot of the infected cases between the experimental infected cases and the estimated, collected data fitting for the Maharashtra between the 15th and 21st April 2020 is shown through the Fig. 2 for $\mu = 1$. This figure ensure that the model fitting curve quite agree with the experimental outcomes. Fig. 3 is the plot for the recovered cases between the data fitting and experimental recovered cases for $\mu = 1$. A good agreement between the fitting curve and experimental outcomes can be easily noticed from the Fig. 3. The effectiveness and variation of the degree of approximation $r$ is discussed in the literature [17]. One can easily understand that the order of convergence of the numerical techniques provided in this article [17] is increases as we increase the degree of approximation and the errors in computing the numerical solution is decreases as we increase the degree of approximation. In the present article we consider the degree approximation $r = 5$ for each numerical computation and pictorial presentation.

The comparison between the variation of fuzzy solution and normal solution for susceptible cases at different values of fractional order $\mu = 0.6, 0.8, 1$ for the obtained experimental results are shown through the Fig. 4. In Fig. 5 the comparison between the variation for number of exposed cases are depicted for the different value of the contact rate $\beta$ at fractional order $\mu = 0.8$ for both fuzzy solution as well as normal solution. The comparison between the variation of number of infected cases at fractional order $\mu = 0.8$ for different contact rate is shown through the Fig. 6 for both the solution. For the different fractional order $\mu = 0.6, 0.8, 1$ the variation of quarantined cases is also depicted through the Fig. 7. The pictorial representation of susceptible cases for upper and lower solutions are depicted through Fig. 8 and Fig. 9 respectively at different values of fractional order $\mu = 0.6, 0.8, 1$. The variation of exposed cases for upper and lower solutions are depicted through Fig. 10 and Fig. 11 respectively at different values of contact rate $\beta = 0.05, 15, 30$. The variation of infected cases for upper and lower solutions are depicted through Fig. 12 and Fig. 13 respectively at different values of
contact rate $\beta = 0.05, 15, 30$.

**Results and discussions**

Here we are going to analyze the sensitiveness and effect of the different parameters present in the experimental COVID-19 model on the numerical outcomes for different variables. From the above plots between the reported cases and experimental cases for different unknown variables, we see that our proposed method is effective and valid. The plot for the susceptible cases for fractional order advances towards the plot for integer order as we move fractional order system to integer order system which justifies the modeling of the dynamics of concerned experimental model from the Fig. 4. The enhancement of the exposed cases can be easily calculated for the different contact rate $\beta$. We can notice that the exposed cases increases as we increase the contact rate this justifies the fact of social distancing to prevent our self from this epidemic through Fig. 5. Through the Fig. 6 the variation in infected cases can be visualized for different contact rate. The theory behind the lock-down is well justified from the variation in infected cases for different contact rate as infected cases increases as we increase the contact rate. The graph for quarantined cases is shown through the Fig. 7 for different fractional order system. Figs. 8 and 9 are the plots between susceptible cases and time for the Case of upper and lower solutions of concerned fuzzy fractional COVID19 model. Figs. 10 and 11 are the plots between exposed cases and time for the Case of upper and lower solutions. Figs. 12 and 13 are the plots between infected cases and time for the Case of upper and lower solutions.

**Conclusion**

This scientific work achieve the many novel consequences. The first one is the accurate modeling of the dynamics of COVID-19 by fractional

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*Fig. 7. Comparison between numerical fuzzy solution and normal solution for quarantined cases at different values of fractional order $\mu$.*

*Fig. 8. Variation of susceptible cases for upper numerical solution at different values of fractional order $\mu$.***
order system of differential equations and by fuzzy fractional order systems for the state Maharashtra of India. The Laguerre operational matrix is derived to tackle the concerned fractional mathematical model numerically. Here we shows the effect of different parameters for the concerned system. The numerical outcomes of the concerned model has good agreement with the available data for the both cases of concerned model i.e. in fractional differential environment as well as in fractional fuzzy environment. The whole calculations and simulations are done by using the fractional derivatives in Caputo sense. The model fitting curve and the numerical outcomes have been plotted for the value $\mu = 1$. In order to explore the concerned model for complete factor of epidemic few more important parameters will be added in future after a justification with available data sources.

The authors of this scientific work are optimistic to use the concerned fractional COVID19 model with more data sets for India and other countries in their future work. They are also interested to deal financial and economic factors in the concerned model with adaptive fuzzy logic approach. In addition, authors are planning to use neural network schemes to analyze various predictions with different factors.

**Availability of data and materials**

Not applicable.

**Authors contributions**

All authors have equally contributed to the manuscript, and read and approved it.

![Graph showing variation of susceptible cases for lower numerical solution at different values of fractional order $\mu$.](image1)

![Graph showing variation of exposed cases for upper numerical solution at different values of contact rate $\beta$.](image2)
Fig. 11. Variation of exposed cases for lower numerical solution at different values of contact rate $\beta$.

Fig. 12. Variation of infected cases for upper numerical solution at different values of contact rate $\beta$. 
Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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