Mathematical model and stability investigation of plasma equilibrium around a current-carrying conductor

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Abstract. Magnetic Galatea-traps with current-carrying conductors immersed into the plasma are permanent topics of theoretical, experimental and numerical investigations in the controlled thermonuclear fusion field. Our paper presents a mathematical model of equilibrium magnetoplasma configurations around a straight conductor non-contacting with it. Some necessary regularities in the connection between quantitative characteristics of configurations are established. Problems on their MHD stability are set up and partially solved.

1. Introduction

The present paper is connected with the series of works devoted to the specific class of the traps in which conductors with electric current creating the magnetic field are immersed into the plasma, but don’t contact with it. Attention to them was drawn by A.I. Morozov [1, 2], who called them “Galateas” and initiated the development of “Stellarator-Galatea” (SG) [3], “Galatea-Belt” [4] and “Trimix” [5].

In development and investigations of the magnetic traps, mathematical modeling and calculations [6] play a significant role in addition to the theory and experiments. The plasma parameters in the traps allow to study them in terms of magnetohydrodynamics (MHD). In the widespread cases of the traps allowing symmetry (plane, axial and helical), their plasmastatic models are reduced to two-dimensional boundary value problems with the scalar Grad-Shafranov equation [7-9] for the magnetic flux function. They are widely used in calculation (see, e.g. [10-13]).

Stability of the considered equilibrium plasma configurations is a common problem of investigations in the controlled thermonuclear fusion. The basic principles of the linear MHD stability theory are known since the 1950s [9, 14, 15]. The researches use the energy principle [16], which still successfully works in analysis of various instability manifestations of a tokamak [17].

During the numerical solution of problems with the Grad-Shafranov equation, the concept of “diffusion stability” [6, 18] was appeared.

In the present paper, the problem on models of one-dimensional equilibrium plasma configurations in a ring area of a straight cylindrical conductor and their stability is considered. Earlier in a such model,
the opportunity to isolate a conductor from plasma was demonstrated [19]. In [20, 21], the configuration stability in such areas is studied, but the reasons for instability obtained there are caused by the plasma pressure decreasing near the external boundary, where the model is not one-dimensional in real Galateas. Therefore, the recent study is focused on a part of the area near the conductor, where the plasma pressure increases with the radius. On the periphery, it is expected to be constant, and this case simulates the area where the plasma pressure is maximal and the electric current vanishes. The obtained results are related to the dependence of the configuration parameters and their stability on the problem geometry.

2. Model of cylindrical conductor ring area

The one-dimensional plasmastatic model of plasma equilibrium in the conductor ring area $1 < r < R$ is based on the three functions: the plasma pressure $p(r)$, the azimuthal component of the magnetic field intensity $H(r)$ and the axial component of the electric current density $j(r)$. They satisfy two differential equations

$$\frac{dp}{dr} = -jH; \quad j = \frac{1}{r} \frac{dHr}{dr}$$

(1)

following from the plasmastatics equations [6, 14, 15].

The problems with the equations (1) are formulated in the dimensionless form. The units of measurement of all values are composed from two given dimensional constants – the conductor radius $r_c$ and the electric current value in the conductor $J_c$ [21].

A number of the functions in the model is greater by one than a number of the equations (1), therefore one of the functions should be determined in accordance with the requirement to isolate the conductor from the plasma. We define the pressure in a form in which it vanishes at, for example,

$$p(r) = \begin{cases} p_0 \left(1 - \left(\frac{r - r_1}{r_1 - 1}\right)^2\right), & 1 \leq r < r_1; \\ p_0, & r_1 \leq r \leq R \end{cases}$$

(2)

Unlike in [20, 21], $p = \text{const}$ at $r > r_1$ is defined up to the external boundary.

The equations (1) should be added by the condition $Hr = 1$ at $r = 1$ that corresponds to the magnetic field circulation on the conductor which is equal to the electric current $J_c$ in it. This implies

$$G(r) \equiv (Hr)^2 = 1 - \frac{4p_0}{(r_1 - 1)^2} \left(\frac{r^3}{3} - \frac{r^4}{4} - 1\right); \quad H(r) = \frac{\sqrt{G(r)}}{r}; \quad j(r) = \left(\frac{2p_0(r - r_1)}{(r_1 - 1)^2} \sqrt{G(r)}\right) \quad (1 \leq r < r_1)$$

$$G(r) = G(r_1); \quad H(r) = \frac{\sqrt{G(r_1)}}{r}; \quad j(r) = 0 \quad (r_1 \leq r \leq R)$$

(3)

and the obvious restriction on the parameter $p_0 \leq p_0^\alpha = \frac{3}{2} \left(r_1^2 + 2r_1 + 3\right)$.

Equilibrium configurations described by (2) and (3) depend on the dimensionless parameters $p_0$ and $r_1$. On the figures 1–2, the examples of some configurations at their different values are presented.

Figure 1. Distributions of basic plasma parameters in configuration with $r_1 = 2.0$. 2
Figure 2. Distributions of basic plasma parameters in configuration with $\eta_1 = 3.5$.

They show that the electric current in plasma increases with the pressure maximal value $p_0$. The critical value $p_0^{cr}$ and the current $j(r)$ decrease when the value $\eta_1$ increases, i.e. the place where the pressure reaches its maximum moves to the outer boundary.

3. Necessary condition for stability

A mathematical model of two-dimensional equilibrium plasma configurations is constructed during numerical solution of the boundary value problems with the Grad-Shafranov equation for the scalar magnetic flux function using the iterative relaxation method [6]. Convergence of the method corresponds to solution stability relative to disturbances of the same dimension. It was called “diffusion stability” and is necessary for the strictly MHD stability relative to arbitrary small three-dimensional disturbances, including dynamical ones. At the same time, it provides the MHD stability if it takes place for all helical harmonics of arbitrary disturbances [6, 18].

The model of the configurations considered above is presented by the formulas (2), (3) and doesn’t require solution of the Grad-Shafranov problems. However, their one-dimensional analogue can be of interest to analyze the diffusion stability. In this case, the boundary value problem has the form

$$\frac{1}{r} \frac{d}{dr}\left( r \frac{d\psi}{dr} \right) + \frac{dp}{d\psi} = 0; \quad \frac{d\psi}{dr}(1) = -1; \quad \psi(R) = 0$$

Here $\psi$ is the magnetic flux function connected with the magnetic field by the equation

$$\frac{d\psi}{dr} = -H; \quad \frac{dp}{d\psi} = \frac{dp}{dr} \frac{d\psi}{dr}$$

Convergence of the iterative relaxation process is provided by positivity of the linear differential operator [6]

$$\mathbf{L}[u] = -\frac{1}{r} \frac{d}{dr}\left( r \frac{du}{dr} \right) - Q(\psi) u = 0; \quad \frac{du}{dr}(1) = 0; \quad u(R) = 0,$$

where

$$Q(\psi) = \frac{d}{d\psi} \left( \frac{d\psi}{dr} \right) = \frac{1}{H} \frac{d}{dr} \Rightarrow Q(r) = \frac{2p\sigma^2}{(\eta_1 - 1)^2 G(r)} \left[ -2 + \frac{\eta_1}{r} + \frac{2p\sigma^2}{(\eta_1 - 1)^2 G(r)} \right]$$

By comparing the operator (4) with

$$\mathbf{L}_0[u] = -\frac{1}{r} \frac{d}{dr}\left( r \frac{du}{dr} \right) = 0; \quad \frac{du}{dr}(1) = 0; \quad u(R) = 0,$$

we will get the sufficient criterion of its spectrum positivity $\max Q(r) \leq \mu_1$, where $\mu_1$ is the first eigenvalue of the operator $\mathbf{L}_0$ simply calculated using cylinder functions. The expression (5) implies that the diffusion stability can be disturbed if the plasma configuration approaches the conductor ($\eta_1 \rightarrow 1$). In this case, $Q(r)$ increases indefinitely, but $\mu_1$ doesn’t depend on $\eta_1$, therefore the criterion above is violated.
4. MHD stability

Strict investigations of the equilibrium configuration MHD stability are maintained in the linear approximation. Stability are determined by behavior in time of arbitrary small three-dimensional disturbances of the considered configurations, including the medium velocity.

The MHD equations linearized on the equilibrium

\[ \nabla p = \mathbf{j} \times \mathbf{H}; \quad \mathbf{j} = \text{rot} \mathbf{H}; \quad \text{div} \mathbf{H} = 0 \]  

have the form

\[ \rho \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 = \text{rot} \mathbf{H}_1 \times \mathbf{H} + \mathbf{j} \times \mathbf{H}; \quad \frac{\partial p_1}{\partial t} + \nabla \cdot \mathbf{v}_1 + \gamma p \text{ div } \mathbf{v}_1 = 0; \quad \frac{\partial \mathbf{H}_1}{\partial t} = \text{rot} (\mathbf{v}_1 \times \mathbf{H}), \]  

where disturbances noted by the index “1” depend on time \( t \) and all three spatial coordinates regardless to the configuration symmetry.

The equations (7) can be reduced to the one vector second-order equation for the velocity

\[ \rho \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \nabla (\nabla \cdot \mathbf{v}_1 + \gamma p \text{ div } \mathbf{v}_1) + \text{rot} \text{rot} (\mathbf{v}_1 \times \mathbf{H}) \times \mathbf{H} + \mathbf{j} \times \text{rot} (\mathbf{v}_1 \times \mathbf{H}) = -\mathbf{K}[\mathbf{v}_1] \]  

In the configuration stability studies in a finite volume, it is naturally to set up the boundary condition \( \nu_n = 0 \) and include it in the definition of the linear differential operator \( \mathbf{K} \). Its coefficients don’t depend on time, therefore the solutions of the problems with the uniform equation (8) are composed from exponents

\[ \mathbf{v}_1(t, r, \varphi, z) = e^{i\omega t} \mathbf{v}_1(r, \varphi, z), \]  

and these problems transform into the eigenvalue problems

\[ \rho \omega^2 \mathbf{v}_1 = \mathbf{K}[\mathbf{v}_1] \]  

The operator \( \mathbf{K} \) is self-adjoint, therefore the eigenvalues \( \omega^2 \) are real. Non-increasing in time of the solution (9) corresponds to the stability that takes place when \( \omega^2 \geq 0 \), i.e. under positivity of the operator \( \mathbf{K} \).

These are the common concepts of the MHD stability linear theory [6, 9, 14, 15]. In the investigation of the one-dimensional configurations of the considered geometry, the equations (6) transform into (1), and the coefficients of the operator \( \mathbf{K} \) don’t depend on \( \varphi \) and \( z \) in the cylindrical coordinates. Therefore, the solutions of the problems with the equation (10) depend on them also exponentially:

\[ \mathbf{v}_1(r, \varphi, z) = e^{i\omega - ikz} \mathbf{v}_1(r) \]

In this case, the three-dimensional eigenvalue problem splits into series of one-dimensional vector problems for disturbance harmonics with integer values of \( m \) and real ones of \( k \). Each of them can be reduced to the scalar problem for the function \( u = \mathbf{v}_1, r \) with the equation

\[ -\frac{d}{dr} \left( F \frac{du}{dr} \right) + Gu = 0, \]

where the coefficients \( F \) and \( G \) depend on the eigenvalue \( \omega^2 \) and parameters \( m \) and \( k \) nonlinearly [6, 22]. The further simplification is connected with the fact that the stability problem is a qualitative one, so, instead of the eigenvalue calculations, it is sufficient to find the “stability boundary”, namely the conditions under which a minimal eigenvalue \( \omega^2 \) turns to zero, i.e. the operator \( \mathbf{K} \) ceases to be positive [6, 20]. At \( \omega^2 = 0 \), the equation (11) has the form:

\[ \frac{d}{dr} \left( \frac{H^2 du}{\eta r \frac{dr}{d\eta}} \right) + \left[ \frac{m^2 H^2}{r^3} - \frac{4\alpha^2 H^2}{\eta r} + \frac{d}{dr} \left( \frac{(2-\eta)H^2}{\eta r^2} \right) \right] u = 0, \]
where $\alpha = k/m$, $\eta = 1 + \alpha^2 r^2$. The stability boundary for the disturbance harmonics with the fixed $m$ and $k$ in the range of the problem parameters $p_0$ and $r_1$ is determined by their values under which the boundary value problem with the equation (12) has a nontrivial solution.

The equation (12) degenerates at $m = 0$, but, in this case, the stability is ensured by the inequality

$$\frac{r}{p} \frac{dp}{dr} < \frac{2\gamma H^2}{H^2 + \gamma p},$$

obtained using the energy principle [14, 16]. In the considered class of the configurations (2), it is satisfied always, because $dp/dr \geq 0$, i.e. the configurations are stable relative to the disturbances independent on $\varphi$.

The calculations of the boundary value problems with the equation (12) at $m = 1$ and $m = 2$ had demonstrated that all configurations presented on the figures 1–2 are stable with respect to disturbances with any values of the parameter $k$. It means that the requirement to isolate conductors from the plasma doesn’t produce instability in the considered one-dimensional model. More detailed investigations must deal with two-dimensional models of the traps, including their periphery where the plasma pressure decreases near the outer boundary.

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