MAGNETIC ROTATOR WINDS AND KEPLERIAN DISKS OF HOT STARS

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ABSTRACT

We set up equations and boundary conditions for magnetic rotator winds and disks in axially symmetric hot stars in a steady state. We establish a theorem stating that if a disk region has no meridional motion but its magnetic field has a normal component at a point \( Q \) on its shock boundary, the angular velocity of the disk region at \( Q \) is the same as the angular velocity of the star at the point \( P_\star \) at which the magnetic field line through \( Q \) is anchored. When there is equatorial symmetry, all points of the disk along the field line through \( Q \) will have the same angular velocity as \( P_\star \). Also, we show that for a given value of the magnetic field strength, if the rotation rate is too high or the flow velocity into the shock boundary is too low, a Keplerian disk region will not be formed. We consider the formation of disks in magnetic rotators through the processes of fill-up and diffusion into Keplerian orbits. At the end of the fill-up stage the density of the disk increases significantly and the magnetic force in the disk becomes negligible. If the meridional component \( B_m \) of the field at the surface is larger than a minimum value \( B_{m,\text{min}} \), a Keplerian disk can form. The radial extent of the Keplerian region is larger for larger values of \( B_m \) and is largest when \( B_m \) equals an optimal value \( B_{m,\text{opt}} \). The extent does not increase when \( B_m \) is larger than \( B_{m,\text{opt}} \). If \( \alpha \) is the ratio of rotational speed to the critical rotation speed at the photosphere, the inner and outer radii of the maximal quasi-steady Keplerian disk region are given by \( \alpha^{-2/3} R \) and \( 2^{4/3} \alpha^{-2/3} R \), respectively, where \( R \) is the stellar radius. For models with dipole-type fields, the values of \( B_{m,\text{min}} \) in B-type stars are of the order of \( 1\text{–}10 \) G, and in O-type stars they are about 500 G. Because the values of \( B_m \) required for disk formation in B-type stars are relatively small and the fill-up time is short, we suggest that meridional circulation may play a role in some of the time variation observed in disks of Be stars through its effect on the magnetic field near the photosphere.

Subject headings: circumstellar matter — stars: emission-line, Be — stars: magnetic fields — stars: rotation — stars: winds, outflows

1. INTRODUCTION

This paper has three main objectives. The first is to develop a framework of equations and boundary conditions with general validity for magnetic rotator winds and disks in hot stars. The second is to determine the conditions under which Keplerian disk regions may be formed. The third is to develop models to explain the formation and properties of Keplerian disk regions. In a future paper we intend to consider applications to wind-compressed disk (WCD) regions. Earlier studies of magnetic rotator winds, such as Weber & Davis (1967), focused on one-dimensional models involving equatorial winds mainly to consider magnetic braking. Mestel (1968) set up a system of equations for axially symmetric stars with dipole fields. His equations were based on the solutions obtained by Chandrasekhar (1956) for steady state magnetohydrodynamic (MHD) systems. Sakurai (1985) developed a numerical method for two-dimensional magnetic rotator winds using the Weber-Davis approach. There have been several other studies of equatorial and non-equatorial winds and flows in the context of disk formation, and a review of such work is given in Cassinelli et al. (2002), in which a magnetically torqued disk (MTD) model was proposed to explain the Be star phenomenon. An important difference between the MTD model and our models is that we consider weaker magnetic fields that are sufficient to provide the torque required to sustain higher rotation rates in the flows toward the disk region, whereas the fields discussed in the MTD paper are strong fields required to spin up the disk. Since the disk material is much denser than the preshock flow material, the fields required in the MTD model are stronger than the fields required in our models.

Over the past decade, several models have been developed to explain the formation of disks and equatorial density enhancements around rapidly rotating stars. The idea of wind-compressed disks in nonmagnetic stars was developed by Bjorkman & Cassinelli (1993), and the concept of wind-compressed zones (WCZs) was proposed by Ignace, Cassinelli, & Bjorkman (1996). In the WCD model, trajectories of supersonic flow material from a rotating star travel to the equatorial plane. After the matter enters the disk region to form a WCD it continues to flow outward. However, the angular speed of this material is far below the Keplerian orbital speed. An important similarity between the WCD model and the magnetically channeled model is that the flow passes through a compression shock, which leads to a significant enhancement of the density in the equatorial region. The WCZ model is a milder version of the WCD model and does not involve a magnetic field. Because it applies to stars that are rotating too slowly to form WCD structures, the WCZ model does not have a shock-compressed disk region and the enhancement of density is less than in the WCD model. The WCZ idea has been modified by Ignace, Cassinelli, & Bjorkman (1998) to include the effects of a weak magnetic field. It is called the WCFields model. In this model, the magnetic fields are unimportant in accelerating the flow, but an enhancement of the field near the equatorial plane occurs through compression of field.
lines. The WCFIELDS model may be relevant in discussing the flow in the equatorial region beyond and above the disk in magnetic rotator models.

Babel & Montmerle (1997a) developed a magnetically confined wind-shock (MCWS) model to explain properties of Ap and Bp stars. In this model, a strong dipole magnetic field channels the flow of wind from an early-type star to form a disk. Babel & Montmerle (1997b) and Donati et al. (2001) have applied the MCWS model to hot stars in which observational data have been interpreted to indicate the presence of dipole magnetic fields with strengths in the range from 300 to 400 \( \text{G} \) and with significant obliquity. Unfortunately, these papers neglected the contribution of rapid rotation to the dynamics of the wind. The hot stars of interest to us, especially Be-type, are known to be extremely fast rotators, and the effect of rotation on the dynamics of the wind will be more significant than that of the magnetic field. The MTD model addressed the question of disks in fast rotators with strong magnetic fields.

Several phases in the development of disks around Be stars have been revealed by long-term observational studies of these objects. Disks can form and disappear on timescales of about 30 yr. When the disks of Be stars achieve a certain mass, as estimated by the equivalent width of the \( \text{H}_\alpha \) emission line, the stars tend to develop the observational phenomenon of \( V/R \) variations. During the \( V/R \) variation phase, the star displays a modification in which the \( V \)-component periodically changes so as to appear stronger and weaker relative to the \( R \)-component. The \( V/R \) phenomenon has a typical timescale of about 7 yr, which has been interpreted by Okazaki (1997) and others as the orbital period of a one-armed spiral pattern and is typically much longer than the orbital time of matter in the disk region around the star.

In § 2 we discuss the spatial features of flow regions or wind zones in our models. In §§ 3 and 4 we set up systems of equations that can be used to study properties of flows and winds in rotating stellar models with magnetic fields. The models have axial symmetry and are in a steady state. In a general study of magnetic rotator winds it would be necessary to consider details of the magnetic field over the entire stellar surface. If the objective is restricted to studying the formation of disks, we need to consider magnetic fields that are sufficiently strong only over a smaller localized sector of the stellar surface through which flux tubes emerge from the interior.

In § 5 we set up jump conditions that must be satisfied across the shock boundary between a flow region and a disk region. We establish an important theorem in § 6 connecting the angular velocity of a disk region in which there is no meridional motion with the angular velocity at the photosphere. In § 7 we consider conditions that must be satisfied during the fill-up stage in a disk region with no meridional motion, and we derive a necessary condition for the formation of a Keplerian disk. In § 8 we set up equations that govern the structure of a disk region in a steady state. In § 9 we discuss the rotation of disk regions during the fill-up stage and, also, the extent of Keplerian regions after a quasi-steady state has been reached.

We consider two processes in the formation of Keplerian disk regions. These are a fill-up process and a process of adjustment of disk material with super-Keplerian rotation speeds into Keplerian orbits. We use the term “quasi-steady Keplerian disk” to describe a disk region that is formed after the processes of fill-up and adjustment into Keplerian orbits. We study models in which it is reasonable to consider the processes separately, and we discuss this in §§ 9, 13, and 14. Several authors (Lee, Saio, & Osaki 1991; Porter 1999; Okazaki 2001) have suggested that viscous stresses play an important role in the structure and properties of disks. We assume that viscous forces are not important during the fill-up stage.

In § 10 we discuss some constraints on the magnetic field, and in § 11 we develop a prescription for a stream function and the corresponding dipole-type magnetic field configurations that enable us to study disk formation. We discuss further details of our models and develop a method of solution in § 12. We present our results in § 13. Conclusions and possible interpretation of observations are discussed in § 14.

### 2. FLOW REGIONS AND WIND ZONES

We consider the flow of matter away from the stellar surface caused by traditional wind-driving processes. While most of the flow from higher latitudes will travel large distances away from the star in the form of a wind, material from lower latitudes can flow into equatorial disk regions or even flow back onto the star. In order to simplify terminology, we refer to any flow region surrounding the star other than a disk region as a wind zone. This includes a region in which matter flows to a disk region or back toward the stellar surface. Figure 1 shows a schematic drawing of a meridional cross section of the different wind zones and, also, the disk region during the fill-up stage. In each hemisphere of our models, we consider three wind zones. \( W_{\text{disk}} \) denotes the wind zone in which the matter flows into the disk region. \( W_{\text{outer}} \) is the wind zone in which material actually leaves the stellar environment in the form of a wind. A third zone that may be present is \( W_{\text{inner}} \), where matter flows toward the equatorial plane and then back toward the stellar surface. When disks are present, we consider models with a single contiguous disk region \( D \).

We denote the photosphere by \( \Sigma^* \). Let \( \Sigma_0 \), which we call the initial surface, be a surface close to the photosphere on which initial conditions for the flow in the wind zones are specified. We can take \( \Sigma_0 \) to be the sonic surface or some

![Fig. 1. Schematic figure of magnetic field lines, streamlines, and the equatorial disk during the fill-up stage. Material from the sector \( \Sigma_{0,\text{disk}} \) of the initial surface travels through the wind zone \( W_{\text{disk}} \) and enters the disk region \( D \). Material from \( \Sigma_{0,\text{inner}} \) flows through the wind zone \( W_{\text{outer}} \) out into space. Material from the inner sector \( \Sigma_{0,\text{int}} \) flows into the wind zone \( W_{\min} \).](image-url)
other appropriate surface. Quantities associated with the photosphere and the initial surface are denoted by subscripts "\( * \)" and "\( 0, \)" respectively. We identify different sectors of the photosphere and the initial surface that are linked to the different wind zones. At higher latitudes, we have the sector \( \Sigma_{0,\text{outer}} \) of the initial surface from which material flows through the wind zone \( W_{\text{outer}} \) along open streamlines to a large distance from the star. The meridional speed of the flow in this zone reaches the meridional Alfvén velocity at some distance from the star and exceeds this value at points beyond the Alfvén distance so that the magnetic field lines are drawn out by the flow. The sector of the photosphere that is associated with \( \Sigma_{0,\text{outer}} \) is denoted by \( \Sigma_{*,\text{outer}} \). The flow from the sector \( \Sigma_{*,\text{disk}} \) of the initial surface travels through the wind zone \( W_{\text{disk}} \) and passes through a shock surface \( \Sigma_{\text{shock}} \) to supply material to the disk region \( D \).

Let \( \Sigma_{*,\text{disk}} \) be the sector of the photosphere that is linked to the sector \( \Sigma_{0,\text{outer}} \) by the magnetic field. If a third wind zone \( W_{\text{inner}} \) exists such that initially the flow is toward the equatorial plane and then back toward the surface, we take \( \Sigma_{0,\text{inner}} \) and \( \Sigma_{*,\text{inner}} \) to be the sectors of the initial surface and photosphere that are associated with \( W_{\text{inner}} \), respectively. If the sector \( \Sigma_{*,\text{inner}} \) is present, it would be the closest to the equator. This zone does not play any significant role in our models. The meridional speed of the flow at all points in the regions \( W_{\text{inner}} \) and \( W_{\text{disk}} \) is less than the meridional Alfvén speed so that the magnetic field of the photosphere is able to influence the flow in these zones. Although we separate the flow into different zones, the basic equations that we set up in §3 and 4 are valid for all flow regions or wind zones.

An important feature in our models is the meridional streamline between the wind zones \( W_{\text{outer}} \) and \( W_{\text{disk}} \). We use the term limiting streamline to refer to this streamline, which separates the flow entering the disk region from the wind flow to large distances from the star. During the fill-up stage, the shock boundary \( \Sigma_{\text{shock}} \) between the zone \( W_{\text{disk}} \) and the disk region \( D \) does not extend outward beyond the limiting streamline. After the fill-up stage, the material in the disk region with super-Keplerian rotation speed diffuses out until a quasi-steady region is formed. The outer boundary of the quasi-steady disk region may extend beyond the limiting streamline, and if this happens, we assume that the wind material in the region \( W_{\text{outer}} \) flows outside the extended disk region.

We assume that, in addition to axial symmetry, streamlines of the flow and magnetic field lines have equatorial symmetry. Therefore, we need to set up our equations only in the northern hemisphere, which is determined by the direction of the angular momentum vector. Equatorial symmetry together with appropriate changes in sign, where necessary, will yield the values of quantities and flow patterns in the southern hemisphere. In models with surface magnetic fields that have predominantly dipole-type structure, we may assume, without loss of generality, that the field and the flow are in the same direction in the northern hemisphere and in opposite directions in the southern hemisphere.

### 3. EQUATIONS FOR FLOWS AND WINDS

In this section, we set up equations that are valid for the material in all flow regions, or wind zones. We assume that the flow is steady, which is reasonable when all other phenomena of interest in the wind zones have timescales that are longer than the characteristic time of the flow. We consider a rotating hot star with a magnetic field. The star has mass \( M \), luminosity \( L \), equatorial radius \( R \), and effective temperature \( T_{\text{eff}} \). For simplicity, we assume that the star is approximately spherical and that the system is axially and equatorially symmetric. We use a stationary, inertial frame of reference with spherical polar coordinates \((r, \theta, \phi)\). Let \( e_r, e_\theta \), and \( e_\phi \) be unit vectors in the \( r, \theta, \) and \( \phi \)-directions, respectively. Subscripts \( m \) and \( \phi \) denote meridional and azimuthal components of vectors. The velocity and magnetic field can be written in the form \( \mathbf{v} = v_m e_\phi + v_\phi e_\phi \) and \( \mathbf{B} = B_m e_\phi + B_\phi e_\phi \), respectively. Let \( v_m \) and \( B_m \) denote the magnitudes of \( v_m \) and \( B_m \), respectively. The quantities \( v_m \) and \( B_m \) are always positive because they are the magnitudes of vectors. However, \( v_\phi \) and \( B_\phi \) can be positive or negative because they are the \( \phi \)-components of the vectors.

#### 3.1. Kinematic Conditions for Steady Flow

The density \( \rho \) and the velocity \( \mathbf{v} \) of the flow material satisfy the continuity condition

\[
\text{div} \, \rho \mathbf{v} = 0 .
\]  

The divergence-free condition for the magnetic field is

\[
\text{div} \mathbf{B} = 0 ,
\]  

and the steady state MHD induction equation for material with extremely high electrical conductivity is

\[
\text{curl} (\mathbf{v} \times \mathbf{B}) = 0 .
\]  

Chandrasekhar (1956) and Mestel (1968) have obtained useful integrals of these differential equations. In an axially symmetric system, the azimuthal component of equation (3) requires that

\[
\mathbf{v}_m = \chi \mathbf{B}_m ,
\]  

where \( \chi = (v_m \cdot \mathbf{B}_m) / B_m^2 \) is a scalar function of position. Equation (4) shows that the meridional field lines coincide with the meridional streamlines. Also, equations (3) and (4) give us

\[
v_\phi = \chi B_\phi + \omega r \sin \theta ,
\]  

where \( \omega \) is a constant along a meridional streamline line. We interpret \( \omega \) to be the constant angular velocity of the magnetic field line passing through the point \( P_0 \) at the initial surface. Thus, if we picture the magnetic field line as being anchored at the point \( P_\ast \) of the photosphere, then \( \omega \) is the angular velocity of the star at \( P_\ast \). Equations (4) and (5) can be combined to give

\[
\mathbf{v} = \chi \mathbf{B} + \omega r \sin \theta \, e_\phi .
\]  

Equations (1), (2), and (4) require that the quantity defined by \( \xi = \rho \chi \) must be constant along a meridional streamline in a hemisphere. This constant for a meridional streamline can be written

\[
\xi = \rho \chi = \frac{\nu_m \cdot \mathbf{B}_m}{B_m^2} .
\]  

Note that \( \chi > 0 \) when \( \mathbf{v}_m \) and \( \mathbf{B}_m \) are in the same direction and that \( \chi < 0 \) when they are in opposite directions. In §4.2 we discuss properties of \( v_\phi \) and \( B_\phi \) and show that the quantity \( \chi B_\phi \) is negative in both hemispheres. Therefore,
Equation (5) implies that, relative to a stationary frame of reference, the rotational velocity \( v_\phi \) of a flow particle moving along a field line is less than the corotation velocity \( \omega r \sin \theta \). If we consider points on a streamline through the point \( P_0(r_0, \theta_0, 0) \) on the initial surface, equation (7) gives

\[
\frac{\rho v_m}{B_m} = \frac{\rho_0 v_{m,0}}{B_{m,0}},
\]

where the subscript " 0 " refers to values at \( P_0 \).

Equation (4) has important consequences for modeling stellar winds and disks. First, consider a star in which the dominant component of the surface magnetic field is an odd spherical harmonic, such as a dipole-type field. In both hemispheres, material flows outward from the stellar surface, while the magnetic field is directed outward in one hemisphere and inward in the other. Thus, \( \chi \) must have opposite signs in opposite hemispheres. This raises the question of continuity of field lines across the equatorial plane. We resolve this by noting that there are two possible scenarios for the flow of material close to the equatorial plane. In one of these, opposing flows from the different hemispheres will cross a shock surface and form a disk in which the radial velocity is zero or negligible. Here shock conditions will account for the discontinuity in \( \chi \). In this paper, we consider models in which this condition prevails. In the other scenario with a dipole-type field at the surface, the flow near the equatorial plane is deflected parallel to the plane, and we do not have to consider the continuity of \( \chi \) across the plane.

If the flow crosses a shock surface before moving parallel to the equatorial plane, we expect a WCD region to be formed. In both scenarios, the flow near the equatorial plane in the region \( W_{\text{outer}} \), outside the limiting streamline, will be almost parallel to the plane so that corresponding streamlines in opposite hemispheres are not connected. Part of this flow may form a WCZ. In the case of a star with a magnetic field in which the dominant component at the initial surface is an even spherical harmonic, such as that due to a quadrupole field, the field lines near the equatorial plane in the different hemispheres will be parallel to the plane and the question of the continuity of \( \chi \) does not arise.

3.2. Dynamical Equations

The general form of the steady state equation of motion for the flow of matter is, in the usual notation,

\[
\rho(\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p - \rho \left( \frac{1 - \Gamma}{\rho^2} \right) \mathbf{e}_r + \frac{\text{curl}(\mathbf{B} \times \mathbf{B})}{4\pi} + \mathbf{F}_{\text{line}},
\]

where \( \Gamma = \sigma_e L / 4\pi G M \) is the ratio of the stellar luminosity to the Eddington luminosity and \( \sigma_e \) is the electron-scattering opacity. The term involving \( \Gamma \) gives the force due to continuum radiation written as a fraction of local gravity. The force due to line radiation is \( \mathbf{F}_{\text{line}} = \rho \theta L \mathbf{e}_r \), where \( \theta L \) is given by the usual CAK expression (Castor, Abbott, & Klein 1975). We assume that the pressure is given by the isothermal equation of state

\[
p = \rho c_s^2,
\]

which is a reasonable approximation for the wind zones and the disk. Here \( c_s \) is the isothermal speed of sound. If we apply the continuity condition (eq. [1]) to the left-hand side of the \( \phi \)-component of equation (9) and then multiply both sides by \( r \sin \theta \), we obtain

\[
\text{div}(\rho \mathbf{v} \sin \theta \mathbf{e}_\phi) = \frac{r \sin \theta}{4\pi} \left[ \text{curl}(\mathbf{B} \times \mathbf{B}) \right]_{\mathbf{e}_\phi}.
\]

The right-hand side of this equation represents the torque exerted by the magnetic force on the flow material. This equation gives the integral

\[
r \sin \theta v_\phi - \frac{1}{4\pi \xi} r \sin \theta B_\phi = \mathcal{L},
\]

where \( \mathcal{L} \) is a constant along a meridional streamline. We interpret \( \mathcal{L} \) to be the total angular momentum per unit mass at points along the streamline.

In general, the line radiation force \( \mathbf{F}_{\text{line}} \) given by the CAK expression is not conservative. In models in which this force is important, in order to obtain an energy equation we would have to find a suitably defined potential function whose gradient will be approximately equal to \( \mathbf{F}_{\text{line}} \). Since our primary interest is in understanding the effects of magnetic fields, we consider models in which the line radiation force is negligible, and we set \( \mathbf{F}_{\text{line}} = 0 \). This is a valid approximation in the case of late Be stars (Cassinelli et al. 2002). We can write equation (9) in the form

\[
\rho \left[ \frac{1}{2} \mathbf{v}^2 + c_s^2 \ln \rho - \left( \frac{1 - \Gamma}{r} \right) G M \right] = \rho \mathbf{v} \times (\mathbf{v} \times \mathbf{v}) + \frac{\text{curl}(\mathbf{B} \times \mathbf{B})}{4\pi}.
\]

We take the scalar product of this equation with \( \mathbf{v} \) and use equations (6) and (11) to obtain the Bernoulli equation,

\[
\frac{\mathbf{v}^2}{2} + c_s^2 \ln \rho - \left( \frac{1 - \Gamma}{r} \right) G M - \omega r \sin \theta v_\phi = \mathcal{E},
\]

where \( \mathcal{E} \) is a constant along a meridional streamline and has a dimension of energy per unit mass. The term \( \omega r \sin \theta v_\phi \) expresses the work done by the magnetic torque on the flow material. If we move this term to the right-hand side of the equation and compare the new equation with the corresponding equation for a nonmagnetic system, we see that the energy of the flow material is increased by the work done by the magnetic field. Only the azimuthal component of the magnetic force does work on the material and supplies energy to it. This is because the flow velocity, given by equation (6), is the sum of a vector parallel to \( \mathbf{B} \) and another in the azimuthal direction, while the magnetic force is perpendicular to \( \mathbf{B} \).

Our equations give us the set \{\( \xi, \omega, \mathcal{L}, \mathcal{E} \)\} of streamline constants for the flow of material in the wind zones. Here \( \xi \) is the ratio of the meridional mass flux density to the meridional magnetic flux density and \( \omega \) is the angular velocity of a flux tube associated with a streamline. Also, \( \mathcal{L} \) is the total angular momentum per unit mass and \( \mathcal{E} \) represents an energy per unit mass and is not the total energy per unit mass of the flow material.

4. ALFVÉN VELOCITIES AND AZIMUTHAL COMPONENTS

4.1. Alfvén Velocities

Since the magnetic field plays an important role in determining the nature of the flow, it is useful to compare the flow
velocity with the Alfvén velocity of the magnetic field, which is given in vector form by $\mathbf{B}/(4\pi \rho)^{1/2}$. We introduce the quantities

$$\mathcal{A}_m = \frac{B_m}{\sqrt{4\pi \rho m}} \quad \text{and} \quad \mathcal{A}_\phi = \frac{B_\phi}{\sqrt{4\pi \rho \phi}}.$$  \hspace{1cm} (15)

Note that $\mathcal{A}_m$ is always positive but $\mathcal{A}_\phi$ can be positive or negative depending on the sign of $B_\phi$. Also, $\mathcal{A}_m$ and $|\mathcal{A}_\phi|$ are, respectively, the reciprocals of the meridional and azimuthal Alfvén Mach numbers.

Streamlines flowing through the wind zone $W_{\text{outer}}$ are associated with the material flowing away to large distances and are characterized by the presence of an Alfvén point, $P_{\text{Alf}}$, which satisfies the condition that $\mathcal{A}_m > 1$ is satisfied at all points between $P_0$ and $P_{\text{Alf}}$, and $\mathcal{A}_m < 1$ for all points beyond $P_{\text{Alf}}$. The limiting streamline that separates the wind zones $W_{\text{outer}}$ and $W_{\text{disk}}$ is the first meridional streamline above the equatorial plane. For streamlines in all wind zones, we define two quantities, $\rho_\text{c}$ and $\ell$, which are constants for a given streamline but can be different for different streamlines. The first is given by the equation

$$\rho_\text{c} = \frac{\rho_0}{\mathcal{A}_{m,0}^2} = \frac{\rho}{\mathcal{A}_m^2},$$  \hspace{1cm} (18)

This may be interpreted as the characteristic density associated with a streamline. The second quantity is defined by

$$\omega \ell^2 = \mathcal{A}_\phi.$$  \hspace{1cm} (19)

We may interpret $\ell$ to be a characteristic distance from the rotation axis associated with a given streamline. On an open streamline in the region $W_{\text{outer}}$, we have $\rho_\text{c} = \rho_{\text{Alf}}$, and $\ell$ is the distance of $P_{\text{Alf}}$ from the rotation axis. For streamlines in the region $W_{\text{disk}}$ or $W_{\text{inner}}$, we consider $\rho_0$ and $\ell$ to be streamline parameters that enable us to conveniently express properties of the flow. The value of $\ell$ for any streamline in the northern hemisphere is given by equation (A1) in the Appendix.

4.2. Azimuthal Components

If $\omega$ is the angular velocity at the point $P_*$ of the photosphere, we have $v_{\phi,*} = \omega R \sin \theta_*$. If $\theta_* = \theta_0$, so that the points $P_*$ at the same latitude, equation (5) gives the relation between the rotational velocities at the photosphere and the initial surface as

$$\frac{v_{\phi,*}}{R} = \frac{v_{\phi,0}}{r_0} \left(1 - \frac{B_{\phi,0} v_{\phi,0}}{B_{m,0} v_{\phi,0}}\right).$$  \hspace{1cm} (20)

Equations (5), (7), (12), and (19) give us

$$v_{\phi} = \omega R \sin \theta \left\{1 - (\rho_\text{c}/\rho) \left[\ell^2 / (r^2 \sin^2 \theta)\right]\right\} \left(1 - (\rho_\text{c}/\rho)\right)$$  \hspace{1cm} (21)

and

$$B_{\phi} = 4\pi \rho \omega R \sin \theta \left\{\frac{1 - [\ell^2 / (r^2 \sin^2 \theta)]}{1 - (\rho_\text{c}/\rho)}\right\}.$$  \hspace{1cm} (22)

Because $\rho > \rho_\text{c}$ when $r \sin \theta < \ell$ and vice versa, equation (22) requires that $B_{\phi}$ and $\chi$ must have opposite signs when $\omega > 0$. That is, $\chi B_{\phi}$ is always negative. In our models, $\chi$ is positive in the northern hemisphere. Hence, $B_{\phi}$ and $\mathcal{A}_\phi$ are negative in that hemisphere. Expressions for azimuthal components and Alfvén velocities that are useful in applications to models are given in the Appendix.

The equations that we have set up are valid for any appropriate rotation law at the stellar surface. We introduce the rotation rate parameter $\alpha = \omega / \omega_* (\theta_*)$ for the star at the point $P_*$ at the stellar photosphere by taking $\omega = \omega_\text{crit} \alpha_*$, where

$$\omega_\text{crit} = \left[\frac{GM (1 - \Gamma)}{R^3}\right]^{1/2}.$$  \hspace{1cm} (23)

Then the rotational speed at the point $P_*$ of the photosphere is given by

$$v_{\phi,*} = \omega_\text{crit} \alpha_* R \sin \theta_*.$$  \hspace{1cm} (24)

From equations (20) and (24) we obtain

$$v_{\phi,0} = \omega_\text{crit} \alpha_* \frac{\mathcal{A}_{m,0}}{\mathcal{A}_{m,0} - \mathcal{A}_{\phi,0}} r_0 \sin \theta_0$$  \hspace{1cm} (25)

for the rotational velocity of the flow material at the point $P_0$ of the initial surface. In models in which the angular velocity is constant over the sector $\Sigma_{\text{disk}}$ of the photosphere, we write $\alpha_* = \alpha$, where $\alpha$ is a constant.

5. Jump Conditions Across the Shock Boundary

The material from the sector $\Sigma_{0,\text{disk}}$ of the initial surface passes through the shock surface $\Sigma_{\text{shock}}$ before entering the disk region $D$. We consider jump conditions across $\Sigma_{\text{shock}}$ during the fill-up stage of the disk. We assume that the shock is isothermal, which is a reasonable approximation for the types of models that we consider (Cassinelli et al. 2002). In models in which the dominant component of the stellar magnetic field is an odd spherical harmonic, such as a dipole-type field, we assume that there is a region of the disk that has no meridional motion during the fill-up stage. We
refer to this pure rotational region of the disk as a pre-
Keplerian region, and we expect that this region or part of it
evolves into a quasi-steady Keplerian disk region after the
fill-up stage. On the other hand, in models in which the
dominant component of the stellar field is an even spherical
harmonic, we expect that the material in the disk region will
have an outward flow along field lines that are parallel to the
equatorial plane. For convenience, we refer to such a
region as a WCD region. Another possible situation is one
in which the inner part of the disk region is pre-
Keplerian and that in the outer part is of WCD type.
Although our main interest in this paper is in pre-Keplerian
regions, since one of our objectives is to set up equations
with general validity, we include conditions for pre-
Keplerian and WCD regions.

In each hemisphere, \( \Sigma_{\text{shock}} \) extends from the inner bound-
dary of the wind zone \( W_{\text{disk}} \) to its outer boundary. In the
northern hemisphere of the meridional plane, \( \phi = 0 \), the
inner boundary of \( W_{\text{disk}} \) coincides with the streamline from the
point \( P_{0,\text{lim}} \) on the initial surface to the point \( Q_{\text{lim}} \) on \( \Sigma_{\text{shock}} \). Let \( X_{\text{lim}} \) be the point on the equatorial plane that is
associated with \( Q_{\text{lim}} \). The outer boundary of \( W_{\text{disk}} \) is the limit-
ing streamline from \( P_{0,\text{lim}} \) on the initial surface to the point
\( Q_{\text{lim}} \) on \( \Sigma_{\text{shock}} \) and is associated with the point \( X_{\text{lim}} \) on the
equatorial plane. Let \( P_{0}(r_{0}, \theta_{0}, 0) \) be a general point on the
sector \( \Sigma_{0,\text{disk}} \) so that \( \theta_{0,\text{lim}} < \theta_{0} < \theta_{0,\text{inn}} \). The meridional streamline from \( P_{0} \) travels to the point \( Q \) on the shock
surface, which is associated with the point \( X(r_{0}, \pi/2, 0) \) on the
equatorial plane. In models in which the disk region is a
combination of a pre-Keplerian region and a WCD region,
the inner part of \( D \) from \( X_{\text{lim}} \) to the point \( X_{\text{med}} \) is pre-
Keplerian, and the outer part from \( X_{\text{med}} \) to \( X_{\text{lim}} \) is a WCD region.
The point \( X_{\text{med}}(r_{0,\text{med}}, \pi/2, 0) \) that separates the pre-
Keplerian and WCD parts of \( D \) is associated with the points
\( P_{0,\text{med}}(r_{0}, 0, \theta_{0,\text{med}}, 0) \) on \( \Sigma_{0,\text{disk}} \) and \( Q_{\text{med}} \) on \( \Sigma_{\text{shock}} \). The equa-
tions that we set up in \( \S 3 \) for the wind zones are also valid in
a WCD region, where there is a steady radial flow of
material superimposed on the rotational motion.

Let \( n \) be the unit normal vector to \( \Sigma_{\text{shock}} \) from the wind
zone into the disk. We define the unit tangent vector \( t \) in the
meridional plane as \( t = n \times e_{\phi} \). We use the subscript "n" to
denote components of vectors in the direction of \( n \) and the
notation \( \Delta[\ldots] \) to represent the change in a quantity across
the shock boundary. We use a hat to denote quantities in the
postshock disk region and the subscript "fill" to indi-
cate the values of quantities at the end of the fill-up stage.

In a region of the disk with no meridional motion, the
mass increases steadily. In such a region, the density and the
height will change with time until a steady state is reached.
The magnetic field is assumed to be steady during the fill-up
stage of the disk. Let \( h \) be the height of the shock surface at a
radial distance \( r \) along the equatorial plane. Let \( \delta \) be the
angle between the tangent line to the meridional cross sec-
tion of \( \Sigma_{\text{shock}} \) and the equatorial plane. In general, \( \delta \) changes
with \( r \) and with time, and we assume that it is a small
positive angle. Then \( h \) increases with \( r \) so that \( \partial h/\partial r = \tan \delta \geq 0 \). Although \( h \) and \( \delta \) change with time, they do so on a timescale of the order of the fill-up time \( t_{\text{fill}} \) of the
disk, which is much longer than the flow time \( t_{\text{flow}} = R/v_{m} \).

In both the pre-Keplerian and WCD regions of the disk,
conservation of magnetic flux requires that the normal
component of the magnetic field be continuous. Thus,
\[
\Delta[B_{n}] = 0 . \tag{26}
\]

Also, the tangential component of the electric field should
be continuous across \( \Sigma_{\text{shock}} \). The electric field can be written
as \( E = -v \times B/c \). Therefore, the jump condition requires
that the tangential component of \( v \times B \) should be con-
tinuous across the shock surface. The tangential component
of \( v \times B \) has contributions from its \( \phi \)-component and its \( t \)-
component. The \( \phi \)-component of \( v \times B \) is equal to \( v_{m} \times B_{m} \), which is equal to 0 in a flow region and a WCD
region because of equation (4) and in a pre-Keplerian region
because \( v_{m} = 0 \). Therefore, for both pre-Keplerian and
WCD regions, the jump condition reads
\[
\Delta[(v \times B) \cdot t] = 0 . \tag{27}
\]

The boundary conditions for mass flux are different for
pre-Keplerian and WCD regions. We consider them sepa-
rately. In a pre-Keplerian region, there is no radial motion,
and material flowing into such a region will steadily increase
the mass in that region. This gives
\[
\rho v_{n} \sec \delta = \frac{\partial (h\rho)}{\partial t} . \tag{28}
\]

Because \( \delta \) is quite small in thin disks, we can take \( \sec \delta \approx 1 \).
Also, as the disk region continues to fill up, \( \rho \delta \) becomes
much larger than \( \rho \), and the time taken by \( h \) to change sig-
nificantly becomes quite long compared with the timescale
\( t_{\text{flow}} \) for the flow of material toward the disk.

The continuity of mass across the shock surface into a
WCD region gives
\[
\Delta[\rho v_{n}] = 0 . \tag{29}
\]

Then, for points \( X \) between \( X_{\text{med}} \) and \( X_{\text{lim}} \) such that
\( \theta_{0,\text{lim}} < \theta_{0} < \theta_{0,\text{med}} \), since any inflow from the pre-Keplerian
part of the disk into the WCD part is negligible over the
timescales of interest to us, mass conservation for flow into a
WCD region requires that
\[
\int_{\theta_{0}}^{\theta_{0,\text{med}}} 2\pi r_{0} \sin \theta \rho_{0} v_{r,0} d\theta = 2\pi h r_{0} \hat{\rho} e_{r} . \tag{30}
\]

(Bjorkman & Cassinelli 1993), where \( \hat{\rho} \) is the speed of the
WCD flow parallel to the equatorial plane averaged over the
height of the disk. In models in which \( \rho_{0} \) and \( v_{r,0} \) are
constant, this condition simplifies to
\[
r_{0} \rho_{0} v_{r,0} (\cos \theta_{0} - \cos \theta_{0,\text{med}}) = h r_{0} \hat{\rho} e_{r} . \tag{31}
\]

Equations (30) and (31) give the rate of mass flow in the
northern hemisphere only. The total rate of mass flow is
twice the value of the integral.

If we substitute \( \text{curl}(B \times B) = (B \cdot \nabla)B - \nabla(B^{2})/2 \) in
equation (9), the continuity condition for momentum flux
across \( \Sigma_{\text{shock}} \) into a pre-Keplerian or WCD region yields
\[
\Delta \left[ \rho v_{n} v + \left( p + \frac{B^{2}}{8\pi} \right) n - \frac{B_{n} B}{4\pi} \right] = 0 , \tag{32}
\]
where we have assumed that the line radiation force is
negligible.

Next, we consider the conditions that must be satisfied by the
streamline constants \( \zeta, \omega, \varphi \), and \( \delta \) for flow from the
wind zone into a WCD region across the surface \( \Sigma_{\text{shock}} \). Equations (26), (29), and (7) imply that
\[
\Delta[\zeta] = 0 . \tag{33}
\]
Also, equations (6), (27), and (26), together with the fact that \( \mathbf{e}_\phi \times \mathbf{t} = \mathbf{n} \), require that
\[
\Delta[\omega] = 0 .
\]  
(34)

Equation (12) and the azimuthal component of equation (32) ensure that
\[
\Delta[\mathbf{L}] = 0 .
\]  
(35)

In general, the streamline constant \( \ell \) in the Bernoulli equation is not continuous across the shock boundary of the disk region. We do not consider jump conditions for streamline constants across the shock boundary of a pre-

### 6. A THEOREM ON DISK ROTATION

Here we consider the jump condition given by equation (27) at a point \( Q(r_Q, \theta_Q, 0) \) on the shock boundary of a disk region with no meridional motion. In the preshock flow region, equation (6) gives \( (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{t} = -\omega P_0 \sin \theta Q B_{n.Q} \). In the disk region we have \( \mathbf{v} = \mathbf{v}_Q \), and this gives \( (\mathbf{u} \times \mathbf{B}) \cdot \mathbf{t} = -\mathbf{v}_Q \cdot \mathbf{B}_{n,Q} \). Also, equation (26) states that \( B_{n,Q} = B_{n,Q} \). Therefore, equation (27) gives
\[
\mathbf{v}_Q = \omega Q \sin \theta Q ,
\]  
(36)

This equation shows that the point in the disk region that is adjacent to \( Q \) rotates with angular velocity \( \omega \). In \( \S \ 3 \) we noted that the angular velocity \( \omega \) associated with a field line is the same as that of the point \( P_\star \) of the star where the field line is anchored. Thus, we have an extremely important result for the rotation in disk regions where the meridional speed is negligible. We state this in the form of a theorem as follows:

**Theorem.**—Let \( Q \) be any point on the shock boundary between a magnetic rotator flow region and a disk region of an axially symmetric star that is in a steady state. Suppose that there is no meridional motion in the disk region and that the normal component of the magnetic field in the disk region at \( Q \) is not zero. Let \( Q \) be the point of the disk region adjacent to \( Q \) on the field line through \( Q \). Then the angular velocity of the disk region at \( Q \) is equal to the angular velocity \( \omega \) at the point \( P_\star \) of the star at which the magnetic field line through \( Q \) is anchored. Furthermore, if the system is equatorially symmetric and the magnetic field line through \( Q \) is continuous across the disk, then the angular velocity at all points of the disk region along the field line will be the same as that at \( P_\star \) for time periods over which the disk region remains steady.

The first part of the theorem follows from equation (36) and requires that the meridional speed in the disk region be zero. To establish the second part, we use the equation for the pre-Keplerian region that corresponds to equation (5). Since \( \mathbf{v}_m = 0 \), we have \( \mathbf{v}_Q = \omega Q \sin \theta Q \), where \( \omega \) is constant along a meridional field line. Comparing this equation for \( \mathbf{v}_Q \) with equation (36), we see that \( \omega = \omega \) for the field line through \( Q \). Thus, all points of the pre-Keplerian region along a field line will have the same angular velocity as the point of the star where the field line is anchored over timescales during which the region is steady. During the fill-up stage, the flow time \( t_{\text{flow}} \) is much shorter than the fill-up time \( t_{\text{fill}} \) so that over timescales of the order of \( t_{\text{flow}} \) the second part will be true. A situation in which the second part of the theorem will be true over longer periods of time that are comparable to \( t_{\text{fill}} \) when the region \( \Sigma_{*,\text{disk}} \) is in uniform rotation. Then the corresponding pre-Keplerian region of the disk will have the same constant angular velocity as that sector of the star. The models that we consider in our applications satisfy this condition.

We emphasize that our theorem is quite different from Ferraro’s law of isorotation, although part of it may appear to be similar. We require the presence of a shock surface, and in the region between the stellar surface and the shock surface there is meridional motion as well as differential rotation of the outflow material. In models satisfying Ferraro’s law, we can easily show that there should be no meridional motion outside the stellar surface. From our discussion in \( \S \ 3 \), we know that \( \chi \) must be constant along field lines and have opposite signs in opposite hemispheres. If there is no shock surface, then the continuity of \( \chi \) across the equatorial plane requires that \( \chi = 0 \) along every field line. Therefore, \( \mathbf{v}_m \) must be zero along every field line, and there can be no meridional motion anywhere in an atmosphere where Ferraro’s law is satisfied.

### 7. CONDITIONS FOR THE FORMATION OF PRE-KEPLERIAN DISK REGIONS

In applications to models, we consider conical shock surfaces given by \( \theta = (\pi/2) - \delta \) in the northern hemisphere, where \( \delta \) is small and has the same value for all \( r \). It may increase steadily with time during the fill-up stage. In such a model, the cross section of \( \Sigma_{\text{shock}} \) is a straight line given by \( h = r \tan \delta \), and we have \( \mathbf{n} = \mathbf{e}_\theta \) and \( t = \mathbf{e}_r \). The jump conditions that we set up in \( \S \ 5 \) give us the following equations at the point \( Q \) on the shock boundary of a pre-Keplerian region of the disk, where \( v_r = 0 \) and \( v_\theta = 0 \):

\[
\hat{v}_Q = \hat{v}_Q - \frac{v_{Q0}}{B_{0Q}} B_{0Q} = \omega Q \sin \theta Q, \quad B_{0Q} = B_{0Q} \]
\[
B_{r,Q} = B_{r,Q} - 4\pi \xi v_r, \quad B_{r,Q} = B_{r,Q} \left( 1 - \frac{1}{\mathcal{A}_{m,Q}} \right), \quad B_{0,Q} = B_{0,Q} - 4\pi \xi v_0, \quad B_{0,Q} = B_{0,Q} \left( 1 - \frac{1}{\mathcal{A}_{m,Q} \mathcal{A}_{0,Q}} \right) . \]
\]

(39)

Equation (40) can be combined with equation (12) to obtain
\[
\hat{B}_{0,Q} = \frac{B_{00}}{r_0 \sin \theta_0} \left( 1 - \frac{1}{\mathcal{A}_{m0} \mathcal{A}_{00}} \right) .
\]

(41)

At the end of the fill-up stage, if \( \hat{\rho}_{\text{fill}} \) is the disk density averaged over the height of the disk at any point, we can use the jump condition (eq. [32]) for momentum flux to write
\[
\hat{\rho}_{\text{fill},Q} = \left[ \rho r_0 \left( \frac{c_0^2 + v_0^2}{c_0^2} + \frac{(B_r^2 - B_0^2)}{8\pi c_0^2} \right) \right] .
\]

(42)

Equation (39) shows that \( B_r^2 \) is slightly less than \( B_0^2 \), and equation (40) shows that \( B_0^2 \) is slightly larger than \( B_{0Q} \). In stellar models with dipole-type fields, we have \( B_{r,Q} \approx 0 \) near
the equatorial plane, and because of equation (38), we can write \( B^2 - B^2 = B_{\phi\phi} - B_{\phi\phi} \), which will be negative. Hence, in such models, the density of a Keplerian disk will be somewhat less than that in a nonmagnetic model. Also, if the magnetic field present is too strong, Keplerian disk formation may not be possible. A necessary condition for disk formation can be obtained by requiring that \( \rho_{\text{fill}} > 0 \) in equation (42). Combining this with equation (40), using the relation \( B_{\phi}^2 = 4\pi \rho v^2 \), and noting that \( v_\theta \gg c_s \) near the shock surface, we obtain a condition for Keplerian disk formation as

\[
\frac{v_\theta}{v_\phi} \geq \left( \frac{1 - 2 A_{m\phi} A_{\phi\phi}}{2 \omega^2} \right)^{1/2}.
\] (43)

That is, for a given rotational velocity of the preshock flow at a point of the shock surface, the normal component of the flow velocity at that point must be larger than a value determined by the magnetic field. Also, if \( v_{\theta\phi} \) and \( v_{\phi\phi} \) are directly related to the wind terminal velocity \( v_\infty \) and the angular velocity \( \omega \) of the star, respectively, we expect that for a given value of \( v_\infty \), there will be an upper bound on \( \omega \) beyond which a Keplerian disk will not form.

We define the fill-up time at radial distance \( r \) for a pre-Keplerian disk region to be

\[
t_{\text{fill}} = \frac{h_{\text{fill}} \rho_{\text{fill}}}{\partial (\dot{h})/\partial t}.
\] (44)

Our expression for \( t_{\text{fill}} \) is different from the one used by Cassinelli et al. (2002). Since \( \delta \) is small during the fill-up stage for thin disks, we take \( \cos \delta \approx 1 \). Then equations (28) and (44) give us \( t_{\text{fill}} = h_{\text{fill}} \rho_{\text{fill}} / \rho v_\theta \). For our conical shock surface with \( h_{\text{fill}} = r \tan \hat{\delta}_{\text{fill}} \), we have

\[
t_{\text{fill}} = \frac{\rho_{\text{fill}} r \tan \hat{\delta}_{\text{fill}}}{v_\theta},
\] (45)

where the value of \( \rho_{\text{fill}} \) is given by equation (42). After the fill-up stage, the density of a Keplerian disk region will be sufficiently large, and we assume that the disk temperature will be determined essentially by stellar radiative heating.

8. EQUATIONS FOR DISK REGIONS

In this section we consider equations for a disk region \( D \) that is in a quasi–steady state. We assume that \( v_\theta = 0 \) in \( D \), and the equations for the disk region are written for values of quantities averaged over the height of the disk. The \( r \)- and \( \theta \)-components of the momentum equation for the disk are

\[
\vec{v}_r = \frac{1}{\rho} \frac{\partial \vec{p}}{\partial r} + f_r + \vec{F}_{\text{line}} \cdot \vec{e}_r
\] (46)

and

\[
\frac{\vec{v}_r}{r} \frac{\partial (r \vec{v}_r)}{\partial r} = \mathcal{F}_r - \frac{1}{h \rho v^2} \frac{\partial}{\partial r} \left( r^2 \nu h \rho \hat{c}_s^2 \right).
\] (47)

Here

\[
f_r = \frac{\hat{v}_0^2}{r} - \frac{GM(1 - \Gamma)}{r^2} + \mathcal{F}_r,
\] (48)

\[
\mathcal{F}_r = \frac{1}{4 \pi \rho \nu} \left[ \frac{\vec{B}_\phi}{\partial \theta} \frac{\partial (r \vec{B}_\phi)}{\partial r} - \frac{\vec{B}_\theta}{\partial r} \frac{\partial (r \vec{B}_\theta)}{\partial \theta} - \frac{\vec{B}_\phi}{\partial r} \frac{\partial (r \vec{B}_\phi)}{\partial r} \right],
\] (49)

and

\[
\mathcal{F}_\phi = \frac{1}{4 \pi \rho \nu} \left[ \frac{\vec{B}_\phi}{\partial \theta} \frac{\partial (r \vec{B}_\phi)}{\partial r} + \frac{\vec{B}_\theta}{\partial r} \frac{\partial (r \vec{B}_\theta)}{\partial \theta} - \frac{\vec{B}_\phi}{\partial r} \frac{\partial (r \vec{B}_\phi)}{\partial r} \right].
\] (50)

The last term in equation (47) represents the effect of viscosity (see, e.g., Okazaki 2001), and \( \nu \) denotes the viscosity coefficient, which has a value of the order of 0.1. The forces per unit mass due to the magnetic field in the radial and azimuthal directions are given by \( \mathcal{F}_r \) and \( \mathcal{F}_\phi \), respectively. The quantity \( f_r \) is the resultant of the three main forces (per unit mass) that control the radial momentum of the disk. These equations are supplemented by a mass conservation equation that has the form \( 4\pi \rho h^2 \nu = M_{\text{disk}} \), which represents the amount of mass flowing across a cross section of the disk at radius \( r \). In WCD regions, this equation can be replaced by equation (30). In a strict Keplerian disk \( \vec{v}_r \) and \( M_{\text{disk}} \) are zero.

In a Keplerian region of \( D \) we assume that \( \vec{v}_r \) and \( \nabla \vec{p} \) are negligible. In stars where the force due to line radiation is not significant, we take \( \vec{F}_{\text{line}} = 0 \). Then equation (46) requires that \( f_r = 0 \), and equation (48) gives us an expression for the Keplerian rotation speed \( \hat{v}_{\theta, \text{Kep}} \) in the form

\[
\hat{v}_{\theta, \text{Kep}} = \left[ \frac{GM(1 - \Gamma)}{r} \right]^{1/2},
\] (51)

where \( \zeta = r^2 \mathcal{F}_r / [GM(1 - \Gamma)] \) is the ratio of the radial component of the magnetic force to effective gravity. Note that \( \zeta \) becomes negative when \( \mathcal{F}_r \) is directed inward and \( \zeta \approx 0 \) when the magnetic force in the disk is small compared to effective gravity. Also, in a Keplerian disk region equation (47) requires that \( \mathcal{F}_r - [\partial (r^2 \nu h \hat{c}_s^2) / \partial r] / (\rho h \nu^2) = 0 \). The results presented for our models in \( \xi \) 13 show that the magnetic force and the viscous force become negligible at the end of the fill-up stage. Thus, both terms on the right-hand side of equation (47) become negligible. Although we do not study the evolution of the disk region after a quasi-steady Keplerian region is established, if a viscous force becomes significant, then in order to maintain Keplerian rotation, either the magnetic torque on the disk material should offset any decrease in angular momentum due to viscosity or there should be a continuous supply of angular momentum to the disk material. In the case of a WCD region, the steady state equations are essentially the same as those that we derived in \( \xi \) 3 and 4 for the wind zones, with the additional conditions that \( \vec{v}_\theta = 0 \) and \( \vec{B}_\phi = 0 \).

9. ROTATION AND EXTENT OF DISK REGIONS

We introduce the nondimensional variable \( x = r_X / R \) to represent the radial distance of the general point \( X(r_X, \pi/2, 0) \) on the equatorial plane. We consider a pre-Keplerian disk region with shock surface given by \( \theta = (\pi/2) - \delta \) in the northern hemisphere. Suppose that \( Q[r_Q, (\pi/2) - \delta, 0] \) is the point on the shock surface such that the meridional field line through \( Q \) passes through \( X \). When the disk is thin we assume that \( Q \) is vertically above \( X \). Then we have \( r_X = r_Q \cos \delta \). This is a good approximation in the case of a thin disk with a dipole-type field because the field line through \( X \) is perpendicular to the equatorial plane at that point. We assume that all points in a narrow vertical column through \( X \) in the disk region have the same angular velocity.
9.1. Rotation of a Pre-Keplerian Region during the Fill-up Stage

The theorem that we derive in § 6 gives

\[ v_\theta = \omega r = \omega_{\text{Kep}} \alpha_* r \]  

(52)

for the rotational velocity of disk at distance \( r \) from the center of the star during the fill-up stage. The angular velocity \( \omega \) is constant along a field line and can have different values along different field lines. Thus, along the equatorial plane, \( \omega \) is a function of \( r \). To obtain this function, we must know \( \omega \) as a function of colatitude \( \theta \) at the photosphere and, also, the stream function \( \psi(r, \theta) \) for the field lines linking the points \( P_* \) on the sector \( \Sigma_{*, \text{disk}} \) to the points \( X \) in the disk region \( D \). However, even when the stream function is not known, we are able to discuss the rotation of a pre-Keplerian disk region of a star where the sector \( \Sigma_{*, \text{disk}} \) has an approximately constant angular velocity \( \omega \). In this case, the corresponding pre-Keplerian disk region will have the same constant angular velocity. Before we consider models with simple rotation laws at the photosphere, we consider the question of whether it possible for exact Keplerian rotation to be established in a pre-Keplerian region at the end of the fill-up stage. We look at two examples that show that this is very unlikely in real stars.

In a general model, the condition for Keplerian rotation of the disk is obtained using equations (51), (52), and (23). It reads

\[ x^3 \alpha_*^2 = 1 - \zeta, \]  

(53)

where \( \zeta \) is function of \( r \) and \( \alpha_* \) is a function of \( \theta_* \). Suppose that we consider an idealized model in which the angular velocity distribution at points in the sector \( \Sigma_{*, \text{disk}} \) of the photosphere results in a pre-Keplerian disk region that has exact Keplerian rotation. Although such a possibility may be mainly of theoretical interest, it is quite instructive. Also, suppose that the disk is at the end of the fill-up stage when \( \zeta \) has become negligible. In such a model, equation (53) requires that \( \alpha_* = x^{-3/2} \) for exact Keplerian rotation at a point \( X \) of the disk. Suppose that all of the disk from \( X_1 \) to \( X_2 \) are in exact Keplerian rotation and let \( P_{*,1} \) and \( P_{*,2} \), respectively, be the points at the photosphere that are linked to \( X_1 \) and \( X_2 \) by the magnetic field. Let us select \( x_1 = 1.3 \) and \( x_2 = 3.5 \). Then the rotation rate \( \alpha_* \) at points in the sector \( \Sigma_{*, \text{disk}} \) of the photosphere must decrease from 0.67 at \( P_{*,1} \) to 0.15 at \( P_{*,2} \). We do not expect that the rotation rate at the photosphere in real stars will decrease so drastically for a relatively small increase in latitude. Thus, for realistic angular velocity distributions at the photosphere, we expect that if the rotational velocity of the pre-Keplerian disk has a Keplerian value at a point \( X_{\text{Kep}} \), then the rotation rate will be super-Keplerian at points between \( X_{\text{Kep}} \) and \( X_{\text{lim}} \).

Another hypothetical model with exact Keplerian rotation is one in which the magnetic field in the disk region is quite strong and has a configuration such that the magnetic force is able to offset the difference between the centrifugal force and gravity. However, for the types of models that are of interest to us, the magnetic field required will be far too strong. This is because the radial component of the magnetic force in the disk must satisfy the condition \( 1 - x^3 \alpha_*^2 \), and, for example, if a star with \( \alpha = 0.6 \) is to have a pre-Keplerian disk extending to a distance of \( x = 3.5 \), the equations that are given in § 8 together with the disk density at the end of the fill-up stage given by equation (42) show that the value of \( \zeta \) must be approximately equal to \(-14\). Such a large value of \( \zeta \) in the disk region will require an exceptionally strong magnetic field at the photosphere. Also, the topology of the field would have to be highly contrived to produce a magnetic force in the disk region that has the required magnitude and direction.

The different cases that we have considered here suggest quite strongly that the most plausible scenario for the formation of a quasi-steady Keplerian disk region is one in which a pre-Keplerian disk region is formed with material having super-Keplerian rotation speeds, and then this material diffuses into Keplerian orbits when the magnetic force becomes small.

9.2. Radial Extent of Quasi-steady Keplerian Disks

Consider a model in which the sector \( \Sigma_{*, \text{disk}} \) of the photosphere is in uniform rotation with angular velocity \( \omega \). Let the surface magnetic field be sufficiently strong for the formation of a pre-Keplerian region of the disk during the fill-up stage. If there is no meridional motion in the disk region, our theorem implies that the angular velocity of this pre-Keplerian disk region will be constant and equal to \( \omega \). Suppose that the surface field is not too strong and that at the end of the fill-up stage the disk density \( \rho_{\text{bulk}} \) becomes sufficiently large such that the magnetic force in the disk becomes negligible compared with the centrifugal force. Let \( X_{\text{Kep}} \) be the innermost point of the disk where the rotation speed is Keplerian. To simplify our discussion, we take \( X_{\text{Kep}} \) to be at the inner boundary of the pre-Keplerian region of the disk. Let \( x_{\text{Kep}} \) be the value of \( x \) at the point \( X_{\text{Kep}} \).

If \( \alpha \) is the rotation rate of the sector \( \Sigma_{*, \text{disk}} \), equations (24) and (53) give

\[ x_{\text{Kep}} = \alpha^{-2/3}. \]  

(54)

The rotational speed at points in the pre-Keplerian region between \( X_{\text{Kep}} \) and \( X_{\text{lim}} \) will exceed the local Keplerian speed. Let \( x_{\text{esc}} \) be the point at which the rotational speed of the disk reaches the escape velocity so that \( x_{\text{esc}} = 2^{1/3} \alpha^{-2/3} \). In some models, \( x_{\text{esc}} \) will be located between \( X_{\text{Kep}} \) and \( X_{\text{lim}} \), and in other models the rotational speed may not reach the escape speed between \( X_{\text{Kep}} \) and \( X_{\text{lim}} \). Let \( X_{\text{int}} \) denote the nearer of \( x_{\text{esc}} \) or \( X_{\text{lim}} \) to the point \( X_{\text{Kep}} \). Since the magnetic torque on the disk material is negligible after the fill-up stage, we expect that the super-Keplerian material in the region from \( X_{\text{Kep}} \) to \( X_{\text{int}} \) will diffuse outward into Keplerian orbits while conserving angular momentum if viscosity is negligible. The material in the region beyond \( X_{\text{int}} \) will have sufficient kinetic energy to flow away from the star.

When a quasi-steady Keplerian disk region is formed after diffusion of the material in the pre-Keplerian region, let the point \( x_{\text{end}} \) be at its outer boundary. Then, assuming that the material at \( x_{\text{end}} \) arrived from \( X_{\text{int}} \) through angular momentum conservation, we have \( x_{\text{end}} = x_{\text{int}}^4 \alpha^2 \). Thus,

\[ x_{\text{end}} = \begin{cases} 2^{4/3} \alpha^{-2/3}, & \text{if } x_{\text{lim}} \geq x_{\text{esc}}, \\ x_{\text{lim}}^{4/3} \alpha^2, & \text{if } x_{\text{lim}} < x_{\text{esc}}. \end{cases} \]  

(55)

In order to form a disk region with Keplerian rotation, we require that \( x_{\text{lim}} > x_{\text{Kep}} \). When \( x_{\text{lim}} \) is slightly larger than \( x_{\text{Kep}} \), the Keplerian region will be in the form of a ring. For larger values of \( x_{\text{lim}} \), the radial extent of the Keplerian region increases until \( x_{\text{lim}} = x_{\text{esc}} \), which we refer to as our...
optimal model. When $x_{\text{lim}}$ increases beyond $x_{\text{esc}}$, there is no increase in size of the Keplerian region. The radial extent of the largest Keplerian disk region is given by $\alpha^{-2/3} \leq x \leq 2^{4/3} \alpha^{-2/3}$. If viscosity plays a role during the process of diffusion of super-Keplerian material into Keplerian orbits after fill-up, the radial extent of the Keplerian region will be slightly different. The presence of a weak viscous force may be helpful in the process of readjustment of angular velocity.

We should point out that there are other situations in which quasi-steady Keplerian disks with different radial extents can be formed. For example, if the magnetic flux tube that assists in forming the disk region is localized over the sector $\Sigma_{*\text{disk}}$ such that the rotation speed at the innermost point $x_{\text{lim}}$ of the pre-Keplerian disk region is super-Keplerian, then the inner radius of the quasi-steady disk that is formed will be larger than $\alpha^{-2/3} R$.

10. CONSTRAINTS ON THE MAGNETIC FIELD

10.1. Magnetic Field at the Initial Surface

In this section we derive expressions for $A_{m,0,\text{lim}}$ and $A_{s,0,\text{lim}}$ that are useful for finding solutions. In our models we require that $A_{m,0} > 1$ at the points $P_0$ on the initial surface. In the wind zone $W_{\text{disk}}$, the streamline from $P_0$ travels to the point $Q$ on the shock surface and $A_{m} > 1$ at all points between $P_0$ and $Q$. In the wind zone $W_{\text{outer}}$, the streamline from $P_0$ travels into space and $A_{m} > 1$ at all points between $P_0$ and $P_{\text{All}}$, which is the Alfvén point on the streamline. In §4.2, we found that $B_0 < 0$ and $A_{s} < 0$ in the northern hemisphere in our models. Equations (A4) and (A5) show that $A_{s}$ will be negative if $A_{m} > \ell / (r_0 \sin \theta_0)$ because $\ell^2 = r^2 \sin^2 \theta$ for points between $P_0$ and $Q$, or between $P_0$ and $P_{\text{All}}$. Therefore, the field at the initial surface should satisfy the condition $A_{m,0} > \ell / (r_0 \sin \theta_0)$, which requires that $A_{m,0,\text{lim}} > y_{\text{lim}}$, where

$$y_{\text{lim}} = \frac{x_{\text{lim}}}{x_0 \sin \theta_{0,\text{lim}}}.$$  (56)

The dynamical equations governing the flow will also impose constraints on the fields at the initial surface, but we do not consider them here. When the density distribution in the wind zone is known, equation (18) gives

$$A_{m,0,\text{lim}} = \left( \frac{P_{0,\text{lim}}}{P_{X,\text{lim}}} \right)^{1/2}.$$  (57)

This is not an explicit equation for $A_{m,0,\text{lim}}$ because the value of $\rho_{X,\text{lim}}$ depends on the value of $A_{m,0,\text{lim}}$. Equation (12) gives a steady state solution of the azimuthal equation of motion, which leads to the expression for $A_{s}$ in equation (A5). When an appropriate value of $A_{m,0,\text{lim}}$ has been determined, the value of $A_{s,0,\text{lim}}$ is given by

$$A_{s,0,\text{lim}} = A_{m,0,\text{lim}} \frac{1 - \frac{x_2}{x_{\text{lim}}}}{\frac{x_2^2}{x_{\text{lim}}^2} - \frac{x_1}{x_{\text{lim}}^2}}.$$  (58)

Equations (57) and (58) are constraints that must be satisfied in a magnetic rotator wind model in a steady state.

10.2. Critical Values of the Surface Magnetic Field

Our approach to Keplerian disk formation does not depend on which theory of meridional circulation speeds is valid, and we consider circulation only as a possible cause of some of the time-dependent behavior of disks. The radial and azimuthal components of the surface magnetic fields in rapidly rotating hot stars are subject to certain bounds (Maheswaran & Cassinelli 1988, 1992), which are

$$B_{r,*,\text{circ}} = (4\pi \rho_{\text{circ}})^{1/2} v_{\text{circ}} \left( \frac{r_{\text{circ}}}{R} \right)^{1/2}.$$  (59)

and

$$B_{\phi,*,\text{circ}} = \left[ 4\pi GM (1 - \alpha^2)/(1 - \alpha^2) \rho_{\text{circ}} \right]^{1/2}.$$  (60)

The subscript “circ” denotes quantities relating to meridional circulation. The effect of rotationally driven circulation on meridional magnetic fields has been studied by Maheswaran (1969) and Roberts & Wood (1985). The critical value given in equation (59) is important for our models. If the radial component $B_{r,*}$ of the surface magnetic field is less than $B_{r,*,\text{circ}}$, then meridional circulation currents driven by rapid rotation can draw the meridional field lines beneath the stellar surface in a time of the order of $t_{\text{circ}} = 2\pi r_{\text{circ}} / v_{\text{circ}}$. In this case, the meridional magnetic field that provides the torque on the flow material will not persist over periods of time that are much longer than $t_{\text{circ}}$. However, if $t_{\text{fill}}$ is less than $t_{\text{circ}}$, a disk can form before the field lines submerge. For circulation to be involved in the time variation of disks, the meridional fields at the surface should be strong enough to assist in the formation of disks but weaker than $B_{r,*,\text{circ}}$. Also, a meridional component of the field should be able to periodically emerge from the stellar surface after being submerged by the circulation. The fields need not cover the entire photosphere but emerge at least through the sector $\Sigma_{*\text{disk}}$. The causes for the resurgence of meridional field lines could be the buoyancy of magnetic flux tubes or other MHD phenomena that occur in the stellar interior. We know that the Sun displays such phenomena, and it is reasonable to expect that other stars do the same. The critical value given by equation (60) is used for ensuring that $B_{\phi,*,\text{circ}}$ is less than $B_{\phi,*,\text{circ}}$, so that the sum of the centrifugal force and magnetic force does not exceed gravity at the photosphere. However, the magnetic fields in our models easily satisfy this condition.

Meridional circulation velocities in rotating stars have been discussed by several authors, e.g., Sweet (1950), Baker & Kippenhahn (1959), Maheswaran (1968), Pavlov & Yakovlev (1978), and Tassoul & Tassoul (1995). Unfortunately, no circulation models are available for rapidly rotating hot stars with winds. Sweet’s first-order expansion method gave relatively modest values for circulation speeds near the stellar surface. Maheswaran showed that the first-order expansion method was inadequate when the angular velocity is taken to be constant because the term in $1/\rho$ does not appear in the first-order term for the circulation speed. In the case of uniform rotation, the $1/\rho$ term appears in the second-order and higher terms so that the circulation speeds are much larger than the values estimated by Sweet. For nonuniform angular velocity distributions, the $1/\rho$ factor in the circulation speed appears in the first-order term (Baker & Kippenhahn 1959; Pavlov & Yakovlev 1978). Tassoul & Tassoul (1995) find that when eddy viscosity due to turbulence is included, the $1/\rho$ factor disappears and circulation speeds are similar to those obtained by Sweet (1950).
However, the Tassoul & Tassoul model is not appropriate for hot stars with winds because it imposes the boundary condition that $\mathbf{v} \cdot \mathbf{n} = 0$, which is essentially the same as enforcing $v_r, \pi = 0$ at the stellar surface. The fact that they obtain values of the circulation speed that are smaller than the velocities for mass-loss rates near the surface implies that their boundary condition is not appropriate for hot stars with winds. Also, they have not treated the surface region in detail, so that it is not clear whether the choking of circulation by eddy viscosity is a consequence of the stringent boundary condition that requires the flow to become transverse at the surface. The boundary condition used by Maheswaran (1968) involves only the conservation of mass so that $v_r, \pi$ does not have to be zero near the photosphere. In the absence of models with more appropriate boundary conditions, we use the circulation velocity that is consistent with the results of Maheswaran (1968) and Pavlov & Yakovlev (1978), which is

$$v_{\text{circ}} = \frac{L R^2}{G M (1 - \Gamma) \alpha^2} \left\{ 1 + \frac{\rho}{\rho_{\text{circ}}} \left[ (1 - \Gamma) \alpha^2 + \frac{\Delta \Omega}{\Omega} \right] \right\}.$$  

(61)

Here $\Omega$ is the angular velocity at the equator and $\Delta \Omega$ is the change in angular velocity from equator to pole; $\rho = 3M/(4\pi R^3)$ is the mean density of the star. In the case of stars that are in almost uniform rotation, we can write the circulation velocity in the outer part of the radiative diffusion zone in the form

$$v_{\text{circ}} \approx \frac{3L(1 - \Gamma)^2 \alpha^4}{4\pi \rho_{\text{circ}} G M R}.$$  

(62)

We should point out that even if the Tassoul & Tassoul (1995) approach to meridional circulation is appropriate, the theory that we propose for Keplerian disk formation is not affected. The only difference is that meridional components of surface fields with the strengths that we require will be able to survive for much longer periods of time because the circulation speeds are very small and the turnover time is long. In this case, rotationally driven circulation will not be a factor in the time variation of disks.

10.3. Timescale for Ohmic Diffusion in the Disk

The azimuthal component $B_\phi$ of the magnetic field in the disk region will be in opposite directions in opposite hemispheres, and we wish to determine whether there could be early decay of the field through Ohmic diffusion because the height of the disk is small. The diffusion timescale of the field is given by $t_{\text{diff}} = 4\pi \ell^2 / \eta \ell$, where $\ell$ is the length scale of the field and $\eta$ is the magnetic diffusivity. Using the expression given by Spitzer (1962) for diffusivity transverse to the field, we obtain $\eta = 1.42 \times 10^{14} T^{-3/2}$ cm$^2$ s$^{-1}$ for disk regions. Then equation (45) for the disk fill-up time, together with $\ell = h_{\text{fill}}$, gives

$$t_{\text{fill}} = \frac{t_{\text{diff}}}{r \rho v_n T^{3/2} \tan \delta_{\text{fill}}} = \frac{1.13 \times 10^{13} \rho_{\text{fill}}}{r \rho v_n T^{3/2} \tan \delta_{\text{fill}}}.$$  

(63)

We find that the diffusion times for the fields in the disk regions of our models are very much longer than the fill-up times, so that decay due to Ohmic diffusion will not be a factor.

11. DIPOLE-TYPE MAGNETIC FIELDS

IN WIND ZONES

Since we wish to study the formation and properties of Keplerian disks, we consider models in which meridional fields have a dipole-type structure. In particular, we look for solutions with stream functions that have the form given by

$$\psi(r, \theta) \equiv \frac{\sin^2 \theta}{r^3} = \frac{\sin^2 \theta_0}{r_0^3},$$  

(64)

where $\lambda$ is a constant. The field lines that coincide with streamlines given by this equation correspond to a dipole-type field with

$$B_r = \frac{2C}{r^{2+\lambda}} \cos \theta \quad \text{and} \quad B_\theta = \frac{\lambda C}{r^{2+\lambda}} \sin \theta,$$  

(65)

where $C$ is a constant. When $\lambda = 1$ this gives a regular dipole field. However, even if the magnetic field at the stellar surface is that of a regular dipole, the field lines in the wind zones may be slightly modified by the flow. In realistic models it is likely that the values of $\lambda$ will be smaller than 1. When $\lambda$ approaches 0 the transverse component $B_\theta$ approaches 0, and when $\lambda = 0$ the field becomes purely radial. We consider models in which $0 < \lambda \leq 1$.

In our applications, the disk has a shock boundary given by $\theta = (\pi/2) - \Delta$ in the northern hemisphere. If the meridional streamline from the point $P_\theta(r_0, \theta_0, 0)$ on the initial surface arrives at the point $Q(\theta, (\pi/2) - \Delta, 0)$ on the shock boundary, we have

$$r_Q = r_0 \left( \frac{\cos \Delta}{\sin \theta_0} \right)^{2/\lambda}.$$  

(66)

As described in § 9, we take $X(r_X, \pi/2, 0)$ to be the point on the equatorial plane that is associated with the point $Q$ such that $r_X = r_Q \cos \Delta$. Equation (66) gives the relationship between $\theta_0$ and $x$ in the form

$$\sin^2 \theta_0 = \left( \frac{x_0}{x} \right)^{\lambda} \cos^2 + \lambda \delta.$$  

(67)

The initial surface in our models is close enough to the stellar surface so that we can take $r_0 \approx R$. Also, the disk is sufficiently thin such that $\cos \Delta \approx 1$. Then, equation (67) can be written as

$$\sin \theta_0 = \frac{1}{\lambda^{\lambda/2}}.$$  

(68)

We can construct models by prescribing the values of any two of the quantities $x_{\text{lim}}, \theta_{0, \text{lim}}$, and $\lambda$. Then equation (68) fixes the value of the third quantity. In models in which $\theta_{0, \text{lim}}$ and $x_{\text{lim}}$ are known, we obtain

$$\lambda = -\frac{2 \ln \left( \sin \theta_{0, \text{lim}} \right)}{\ln x_{\text{lim}}}.$$  

(69)

Equations (65) and (67) yield

$$B_{m, Q} = B_{m, 0} \left( \frac{\lambda^2}{\lambda^2 + \lambda [4(\lambda^2 - 1) + \lambda]} \right)^{1/2}.$$  

(70)

We assume that over the sector $\Sigma_{0, \text{disk}}$ of the initial surface, the density $\rho_0$ and meridional speed of the flow $v_{m, 0}$ are
constant. Then, using \(B_{m,0} = (4\pi\rho_0)^{1/2} v_{m,0} \mathcal{A}_{m,0}\), we obtain

\[
\mathcal{A}_{m,0} = \mathcal{A}_{m,0,\lim} \left( \frac{x_{\lim}^4}{x_{\lim}^4} \right)^{1/2}.
\] (71)

For our dipole-type fields, we obtain \(x_{\lim}^{(2+\lambda)/2}\) for the quantity defined in equation (56). This expression can be substituted in equation (58) to compute the value of \(\mathcal{A}_{\phi,0,\lim}\) in terms of \(x_{\lim}\) and \(\mathcal{A}_{m,0,\lim}\).

### 12. MODELS AND APPLICATIONS

If we know the density, velocity, and magnetic field strength at the stellar surface, we can use our system of equations and boundary conditions to compute solutions for the wind zones and the disk region. Because the flow streamlines and the magnetic field lines coincide in a meridional plane, a convenient approach is to introduce a meridional stream function \(\psi(r, \theta)\) for the wind zone and obtain solutions for it in terms of known values at the initial surface. In this paper we do not attempt to completely solve our system of equations. For the present, we focus on the formation and properties of Keplerian disks. We derive two results that help us develop a method to pursue this goal. For given values of \(\theta_0\) and \(\mathcal{A}_{\phi,0}\), equation (A1) gives \(d\ell/d\mathcal{A}_{m,0} > 0\), which means that when \(B_{m,0}\) increases so does \(x_{\lim}\). Also, for given values of \(\theta_0\) and \(\mathcal{A}_{m,0}\), we obtain \(d\ell/d\mathcal{A}_{\phi,0} < 0\). Since \(\mathcal{A}_{\phi,0} < 0\), when \(|B_{m,0}|\) increases \(x_{\lim}\) also increases.

We consider models in which \(\psi(r, \theta)\) has the form given by equation (64) and compute solutions for a large number of different cases. First, we select a value for the uniform rotation rate \(\alpha\) of the sector \(\Sigma^{*, \text{disk}}\). Then we select a value for \(\theta_{0,\lim}\) from a range of plausible values. This specifies the location of the point \(P_{0,\lim}\) which is at the upper boundary of the sector \(\Sigma^{*, \text{disk}}\) on the initial surface. Next, we choose an appropriate value of \(x_{\lim}\) which fixes the magnitudes of \(B_{m,0,\lim}\) and \(B_{\phi,0,\lim}\) at \(P_{0,\lim}\). In our models, the minimum value of \(x_{\lim}\) that is required for the formation of a quasi-steady Keplerian disk is given by \(x_{\lim} = x_{\text{Kepl}} = \alpha^{-2/3}\). Let \(B_{m,0,\text{min}}\) denote the value of \(B_{m,0,\lim}\) in this case. Then \(B_{m,0,\text{min}}\) is the minimum meridional magnetic field that is required at the initial surface for Keplerian disk formation. When we increase the value of \(x_{\lim}\), the pre-Keplerian region between \(X_{\text{Kepl}}\) and \(x_{\lim}\) increases in size, as does the radial extent of the quasi-steady Keplerian disk region into which it evolves. This continues until the value of \(x_{\lim}\) reaches the value \(x_{\text{esc}}\).

Let \(B_{m,0,\text{opt}}\) be the value of \(B_{m,0,\lim}\) when \(x_{\lim} = x_{\text{esc}} = 1/\alpha^{1/3} \alpha^{-2/3}\). For values of \(x_{\lim}\) larger than \(x_{\text{esc}}\), although the pre-Keplerian region is larger, the radial extent of the corresponding quasi-steady Keplerian disk region does not increase in size because only the portion of the pre-Keplerian region between \(x_{\lim}\) and \(x_{\text{esc}}\) evolves into the quasi-steady Keplerian disk region. Thus, \(B_{m,0,\text{opt}}\) is smallest meridional field for which the quasi-steady Keplerian disk has the largest radial extent. The model with \(B_{m,0,\text{lim}} = B_{m,0,\text{opt}}\) is our optimal model. We denote the corresponding azimuthal field by \(B_{\phi,0,\text{opt}}\). For values of \(B_{m,0,\text{lim}} > B_{m,0,\text{opt}}\), the radial extent of the quasi-steady Keplerian region is the same as that of the optimal model. Let \(P_{0,\text{Kepl}}(\theta_{0,\text{Kepl}}, 0)\) be the point on the initial surface that corresponds to \(X_{\text{Kepl}}\). Then the sector \(\Sigma^{*, \text{disk}}\) is given by \(\theta_{0,\text{Kepl}} < \theta_0 < \theta_{0,\lim}\).

In applications to models, we take the initial surface \(\Sigma_0\) to be the sonic surface. Then \(v_{m,0} = c_s\), where \(c_s^2 = \alpha R T / \mu\). Here \(\alpha\) is the gas constant and \(\mu\) is the mean molecular weight. Since this surface is close enough to the stellar surface, we take \(r_0 \approx R\). We consider a thin disk region with conical shock boundaries given by \(\theta = (\pi/2) - \delta\) and \(\theta = (\pi/2) + \delta\) during the fill-up stage, where \(\delta\) is the same for all points on the shock surface. At the end of the fill-up stage \(\delta = \delta_{\text{fill}}\). Fill-up occurs on a time of the order of \(\delta_{\text{fill}}\), which is much longer than the flow time \(t_{\text{flow}} = R/c_s\) near the shock surface. The magnetic field emerging from the sector \(\Sigma_0^{*, \text{disk}}\) has a dipole-type structure given by equation (65). Since the values of \(\theta_{0,\text{lim}}\) and \(x_{\lim}\) have been chosen, equation (69) fixes the value of \(\lambda\). We assume that the initial density \(\rho_0\) is constant over \(\Sigma_0\) and we use the mass-loss rate \(M\) to obtain

\[
\rho_0 = \frac{M}{4\pi \rho_0 c_s}.
\] (72)

Using the definition of Alfvén velocity and the values of azimuthal quantities given in §§ 4.1 and 4.2, we can write

\[
B_{m,0} = \mathcal{A}_{m,0} \left( \frac{R T}{\mu} \right)^{1/4} \frac{M^{1/2}}{r_0}
\] (73)

and

\[
B_{\phi,0} = \mathcal{A}_{\phi,0} \mathcal{A}_{m,0} - \mathcal{A}_{\phi,0} \left( \frac{\mu}{R T} \right)^{1/4} \frac{M^{1/2}}{R} v_{\phi,*},
\] (74)

where \(v_{\phi,*}\) is the rotational speed at the point \(P_*\) of the photosphere and is given by equations (24) and (23) with \(\alpha_* = \alpha\) and \(\theta_* = \theta_0\). In general, \(\mathcal{A}_{m,0}\) and \(\mathcal{A}_{\phi,0}\) are functions of colatitude \(\theta_0\).

In order to proceed further, we need to determine either the meridional velocity \(v_m\) or the density \(\rho\) of the flow as it approaches the shock boundary of the disk. When one of these quantities is known the other can be computed using equation (8) in models with dipole-type fields. In a study in which detailed properties of the wind zones are required, we should convert the Bernoulli equation (14) to an equation in \(v_m\) as done by Mestel (1968) or to an equation in \(\rho\) as in Sakurai (1985). Both forms of the equation are quite complicated, and the numerical methods that must be used to find solutions are too laborious for our immediate needs. Since we require values of \(v_m\) for the flow only near the shock surface, we use the \(\beta\)-law for the velocity of stellar winds and assume that

\[
v_m = v_{\infty} \left( \frac{1 - R}{r_0} \right)^{\beta/2},
\] (75)

where \(\beta\) is a constant such that \(0.5 < \beta < 2\) (see, e.g., Lamers & Cassinelli 1999). We use equation (75) to compute the flow velocity adjacent to the point \(X_{\lim}\). Then we compute the density at that point by using equations (8) and (70). Next, we use the equations in § 10.1 to obtain the values of \(\mathcal{A}_{m,0,\lim}\) and \(\mathcal{A}_{\phi,0,\lim}\) and then use equation (71) to find the value of \(\mathcal{A}_{m,0}\). At this stage we have the value of \(\mathcal{A}_{\phi,0,\lim}\), but because we have not specified the azimuthal field along the initial surface, we do not have any conditions on how \(\mathcal{A}_{\phi,0}\) relates to \(\mathcal{A}_{\phi,0,\lim}\) or how \(B_{\phi,0}\) relates to \(B_{\phi,0,\lim}\). In our models, we choose the azimuthal field at the initial surface such that \(B_{\phi,0}\) is directly proportional to \(v_{\phi,0}\) at...
points on the sector $\Sigma_{\text{disk}}$. Then $\mathcal{A}_{\theta,0}$ is constant over this sector so that its value is equal to $\mathcal{A}_{\theta,0,\lim}$. When $\mathcal{A}_{m,0}$ and $\mathcal{A}_{\theta,0}$ are known, equations (73) and (74) give the values of $B_{m,0}$ and $B_{\theta,0}$.

13. RESULTS

We apply the method of solution described in § 12 to stellar models whose basic properties are given in Table 1. The values of the density and the flow velocity at the initial surface are also shown in this table. We consider a variety of different cases by taking different combinations of values of $\alpha$, $\theta_{\lim}$, $x_{\lim}$, and $\beta$. Table 2 gives the upper bounds on rotation rate $\alpha$ for which quasi-steady Keplerian disks can be formed when the magnetic field components have their optimal values. We see that the upper bound on $\alpha$ for Keplerian disk formation decreases when the terminal velocity $v_\infty$ of the wind decreases, and this can be explained in terms of the necessary condition (eq. [43]) that we derive in § 7, where the magnetic pressure in the disk plays a role even though the magnetic force is small compared with gravity or centrifugal force. The upper bounds on the rotation rate $\alpha$ for Keplerian disk formation will be larger for stars in which the normal component of the flow velocity into the shock surface is larger than what we have assumed in our models.

Table 3 displays the values of the critical quantities $B_{\text{m,0,min}}$, $B_{\text{m,0,opt}}$, and $B_{\text{\theta,0,opt}}$ for the magnetic field at the initial surface for models with selected values of $\alpha$ when $\theta_{\lim} = 65^\circ$ and $\theta_{\lim} = 75^\circ$. We use equation (45) to compute the fill-up times shown in the table. The table also shows the values of $B_{\text{\theta,*,L}}$, which is the minimum field strength required by the meridional component of the field at the photosphere to withstand the effects of rotationally driven circulation. We find that the values of $B_{\text{m,0,min}}$ and $B_{\text{\theta,0,opt}}$ are about 30 and 70 G, respectively, for a B0 star and about 1 and 1.6 G for a B9 star. The magnitudes of $B_{\text{\theta,0,opt}}$ are about 15 and 0.2 G for the same stars. Except in models of the B9 star, we find that the values of $B_{\text{m,0,opt}}$ are less than $B_{\text{\theta,*,L}}$. In the B9 models, they are of the same order.

In a rapidly rotating O3 star, $B_{\text{m,0,min}}$ is about 500 G and $B_{\text{\theta,0,opt}}$ is about 1000 G. In an O6.5 star the corresponding values are about 100 and 200 G, respectively. In models of O3 stars, the values of $B_{\text{m,0}}$ required to form Keplerian disks are larger than $B_{\text{\theta,*,L}}$, and in O6.5 stars they are of the same order. At the end of the fill-up stage, the densities of the disks vary from about $5 \times 10^{-13} \text{ g cm}^{-3}$ for B0 stars to about $7 \times 10^{-17} \text{ g cm}^{-3}$ for B9 stars. The values for O3 and O6.5 stars are, respectively, of the order of $10^{-11}$ and $10^{-12} \text{ g cm}^{-3}$. The values of the disk density $\rho$ depend on the values of $M$ and $v_\infty$. If the mass-loss rates near the equator for the different stellar models are higher than the values shown in Table 1, the disk densities will be larger than the values shown in Table 3. The disk fill-up times have been computed for models with $\delta_{\text{fill}} = 10^2$, and we find that these times are of the order of a few months for models of O-type stars and about a week for B9 models.

Table 4 shows the average values of the ratio of the magnetic force to centrifugal force and the ratio of the viscous force to magnetic force in the disk regions after the fill-up stage. Because of the increased density, the magnetic force components $\mathcal{F}_r$ and $\mathcal{F}_\theta$ in the disk region, given by equations (49) and (50), become small compared with the centrifugal force or gravity. The results given in Table 5 for the optimal models of the B2 star show that when $\Sigma_{\text{disk}}$ is located near the latitude of $10^\circ$, its spread in latitude is only about $2^\circ$, and when it is located near the latitude of $25^\circ$ the spread is about $5^\circ$. The spread in latitude required for the sector $\Sigma_{\text{*,disk}}$ in optimal models is smaller when it is closer to the equator. Table 6 displays values of $B_{\text{m,0,lim}}$ and $B_{\text{\theta,0,lim}}$ for different values of $x_{\lim}$ in models of the B2 star with $\alpha = 0.5$ and $\theta_{\lim} = 75^\circ$. This verifies the result we derived in § 12 that the magnitudes of $B_{\text{m,0}}$ and $B_{\text{\theta,0}}$ increase when the values of $x_{\lim}$ are increased.

14. DISCUSSION

In § 6 we establish an important theorem connecting the angular velocity at points on the photosphere with the angular velocities at corresponding points in a disk region with no meridional motion, when the two sets of points are linked by field lines of a magnetic field of sufficient strength and the system is in a steady state with axial symmetry. This theorem is applicable in astrophysical situations in which a central rotating object is linked to an equatorial disk region by a magnetic field with the required strength and structure. In this paper, we use the theorem to discuss the formation of Keplerian disk regions around magnetic rotator stars. If the sector $\Sigma_{\text{*,disk}}$ of the photosphere rotates with approximately uniform angular velocity and is linked to a pre-Keplerian disk region by a dipole-type magnetic field of sufficient strength, then during the fill-up stage the disk region will have the same uniform angular velocity. In § 7 we derive a necessary condition for the formation of a Keplerian disk. It requires that the normal component of the flow velocity near the shock boundary be sufficiently large for Keplerian disks to form when a magnetic field is present and when the rotation rate is high. This constraint results from the fact that magnetic pressure in the disk region plays a role in inhibiting Keplerian disk formation even when the magnetic field is small compared with gravity or centrifugal force.

| Spectral Type | $M$ ($M_\odot$) | $R$ ($R_\odot$) | $L$ ($L_\odot$) | $T_{\text{eff}}$ ($\times 10^4$ K) | $M$ ($\times 10^{-9} M_\odot$ yr$^{-1}$) | $v_\infty$ (km s$^{-1}$) | $\rho_\infty$ (g cm$^{-3}$) | $v_{\text{fill}}$ ($\times 10^9$ cm s$^{-1}$) | $v_{\infty,\text{eq, crit}}$ (km s$^{-1}$) |
|--------------|------------|-------------|--------------|---------------------------|-------------------|----------------|----------------|------------------|----------------|
| O3............| 55         | 14          | 1.10 $\times 10^6$ | 5.0              | 9100              | 3100           | 1.85 $\times 10^{-11}$ | 2.60            | 592             |
| O6.5..........| 29         | 10          | 2.31 $\times 10^3$ | 4.0              | 310               | 2500           | 1.38 $\times 10^{-12}$ | 2.32            | 661             |
| B0............| 15         | 6.6         | 4.12 $\times 10^4$ | 3.2              | 27                | 1300           | 3.09 $\times 10^{-13}$ | 2.08            | 634             |
| B2............| 8.3        | 4.5         | 5110          | 2.3              | 0.4               | 840            | 1.16 $\times 10^{-14}$ | 1.76            | 589             |
| B5............| 4.5        | 3.5         | 560           | 1.5              | 0.01              | 580            | 5.95 $\times 10^{-16}$ | 1.42            | 495             |
| B9............| 2.6        | 2.6         | 61            | 1.0              | 0.0013            | 460            | 1.72 $\times 10^{-16}$ | 1.16            | 437             |

Note.—Entries for the basic stellar properties are from Bjorkman & Cassinelli 1993. We use $L/L_\odot = (R/R_\odot)^2 (T_{\text{eff}}/T_{\text{eff, eff}})^4$ to compute luminosity.
The evolutionary time for all the stars except for the B9 star. Although the second process may begin during the adjustment of super-Keplerian disk material into Keplerian disks consists of a disk fill-up process and a process of transported to the surface region in a time shorter than an evolutionary time for all the stars except for the B9 star. For the B9 star, they are of the same order. In the radiative envelopes of the stars, T is much larger than T_{\text{eff}}, and the decay time will be much longer for all the stars. Recently, Charbonneau & MacGregor (2001) showed that magnetic fields may be generated through dynamo action at the core-envelope boundary of a hot star, and MacGregor & Cassinelli (2003) have found that such a field can be transported to the surface region in a time shorter than an evolutionary time of the star. In view of the observational evidence and theoretical support, it is reasonable for us to consider models with modest surface fields.

Our approach to modeling the formation of Keplerian disks consists of a disk fill-up process and a process of adjustment of super-Keplerian disk material into Keplerian orbits. Although the maximal process may begin during the later stages of the first process, it is reasonable to consider them separately because initially the density in the disk region is small and the magnetic field plays a role in the rotation of the disk material. Toward the end of the fill-up stage, the density of the disk region increases to much larger values, and the magnetic force in the disk becomes negligible when compared with centrifugal force or gravity. This allows the super-Keplerian material in the pre-Keplerian disk region to move into Keplerian orbits while conserving angular momentum when there is no viscous force. Although viscosity is negligible at the end of the fill-up stage, the onset of a weak viscous force during the process of diffusion of super-Keplerian material into Keplerian orbits will be helpful.

When the meridional field B_{m,0} at the initial surface is only slightly larger than the minimum value B_{m,0,min} that is required for disk formation, the Keplerian region will be in the shape of a ring. For increasing values of B_{m,0}, the radial extent of the Keplerian region increases, and the region reaches its largest extent when B_{m,0} equals the optimal value B_{m,0,opt}. For values of B_{m,0} larger than B_{m,0,opt}, there is no increase in the radial extent of the quasi-steady Keplerian region. When the sector \Sigma_{*,\text{disk}} of the photosphere has uniform rotation, the inner and outer radii of the maximal quasi-steady Keplerian region of the disk are given by R = \alpha^{-2/3} R and \alpha^{-2/3} R, respectively, where R is the stellar radius and \alpha is the ratio of the rotational speed to the Keplerian speed at \Sigma_{*,\text{disk}}. While the magnetic field strength at the stellar surface determines the extent of a quasi-steady Keplerian disk region, we see that both the inner and outer radii of a maximal disk are given in terms of \alpha. The inner radius is the distance at which there is corotational Keplerian rotation speed. The outer radius of the maximal disk is determined by the point where the escape velocity is reached in a pre-Keplerian region, and it is larger for smaller values of \alpha. Our result for the maximal radial extent of a quasi-steady Keplerian region of the disk is obtained without including the effect of viscous forces. As shown in Table 4, the viscous force in the disk region is negligible at the end of the fill-up stage. If a weak viscous force plays a role during the transition from a pre-Keplerian to a quasi-steady Keplerian disk region, the radial extent of the quasi-steady region will be slightly different. There are other situations

### Table 2

**Upper Bounds on the Rotation Rate \( \alpha \)**

| Spectral Type | \( \beta = 0.75 \) | \( \beta = 1.00 \) | \( \beta = 1.50 \) |
|---------------|-----------------|-----------------|-----------------|
| O3            | 0.91            | 0.86            | 0.72            |
| O6.5          | 0.89            | 0.80            | 0.65            |
| B0            | 0.76            | 0.65            | 0.50            |
| B2            | 0.65            | 0.55            | 0.41            |
| B5            | 0.59            | 0.49            | 0.36            |
| B9            | 0.55            | 0.45            | 0.33            |

Note.—Values shown are for optimal models of Keplerian disks when \( \theta_{\text{0,lim}} = 70^\circ \).

### Table 3

**Critical Magnetic Field Strengths, Disk Fill-Up Times, and Timescales for Circulation**

| Spectral Type | \( \alpha \) | \( \theta_{0,\text{lim}} \) (deg) | \( \theta_{0,\text{Kep}} \) (deg) | \( B_{m,0,\text{min}} \) (G) | \( B_{m,0,\text{opt}} \) (G) | \( B_{0,0,\text{opt}} \) (G) | \( B_{\text{opt}} \) (G) | \( t_{\text{circ}} \) (yr) | \( t_{\text{fill}} \) (yr) | \( \rho_{\text{fill}} \) (g cm\(^{-3}\)) |
|---------------|--------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-----------------|-----------------|-----------------|
| O3            | 0.60         | 65                            | 70.7                          | 437                           | 893                           | –63                           | 139                           | 0.53            | 0.18            | 9.5 \times 10^{-11} |
| O6.5          | 0.60         | 75                            | 78.5                          | 540                           | 1100                          | –46                           |                                | 6.2 \times 10^{-11} |
| B0            | 0.60         | 75                            | 78.5                          | 101                           | 207                           | –20                           | 175                           | 0.57            | 0.12            | 6.0 \times 10^{-12} |
| B2            | 0.50         | 75                            | 78.5                          | 40                            | 83                            | –9                            |                                | 3.9 \times 10^{-12} |
| B5            | 0.45         | 65                            | 69.6                          | 7.0                            | 13                            | –2.1                          | 25                            | 1.1             | 0.02            | 8.7 \times 10^{-15} |
| B9            | 0.45         | 75                            | 77.8                          | 8.6                            | 16                            | 1.6                           |                                | 5.4 \times 10^{-15} |

Note.—We use \( \beta = 1.0 \) to compute \( B_{0,0,\text{min}}, B_{m,0,\text{opt}}, \) and \( B_{0,0,\text{opt}} \). Values of \( t_{\text{fill}} \) are computed for \( \delta_{\text{fill}} = 10^\circ \). The density of the disk at the point \( X_{\text{fill}} \) at the end of the fill-up stage is \( \rho_{\text{fill}} \).
confined to a sector that we do not discuss here in which quasi-steady Keplerian innermost point of the pre-Keplerian disk region is super-

that we do not discuss here in which quasi-steady Keplerian disks of different radial extent can be formed. For example, if the magnetic field lines from the star to the disk region are confined to a sector $\Sigma_{*,\text{disk}}$ such that the rotation speed at the innermost point of the pre-Keplerian disk region is super-Keplerian, then the inner radius will be larger than $\alpha^{-2/3} R$. Thus, if the rotation rate and the extent of a Keplerian disk region of a star can be obtained from observations, our models can be used to make some deductions about the magnetic field of the star.

The minimum and optimal meridional magnetic fields required for the formation of Keplerian disks in models of B-type stars are, respectively, of the order of 1–10 G. These fields are weaker than $B^{*,L}$, which is the lower bound derived by Maheswaran & Cassinelli (1988, 1992) for the meridional field at the stellar surface to withstand the effects of meridional circulation. The fill-up times in models of all the different stars that we consider in this paper are shorter than the corresponding circulation times. Thus, although meridional circulation may not affect the formation of disks in B-type stars, its effect on the stellar field can influence some observed properties of disks. On the other hand, the situation in regard to O-type stars is different. In models of O6.5 stars, meridional magnetic fields required for Keplerian disk formation are of the order of 100 G, which is of the same order as $B^{*,L}$, and in O3 stars they are about 1000 G, which is much larger than $B^{*,L}$. Although the fill-up times in O-type stars are shorter than the meridional circulation times, if these stars possess the meridional fields with the strengths that are required for Keplerian disk formation, circulation may not be involved in causing any observable changes in disks. Our numerical results indicate that for reasonable values of the rotation rate and surface magnetic field strengths, B-type stars are likely to possess Keplerian disks. This is consistent with the results for the distribution of Be stars obtained by Pols et al. (1991). Our models do not require that the meridional field $B_{m,0}$ be uniformly strong across the stellar surface for the formation of a Keplerian disk. The field must have the required strength on the sector $\Sigma_{0,\text{disk}}$ and the rotation should be approximately uniform on the sector $\Sigma_{*,\text{disk}}$. The entire star need not be in uniform rotation. For all reasonable locations of $\Sigma_{*,\text{disk}}$ in all our models, we find that its spread in latitude needs to be only a few degrees. We should emphasize that the assumption of axial symmetry has been made to simplify the mathematics. Our model for disk formation does not depend on an aligned dipole field or even an axially symmetric field. Our results will qualitatively apply to oblique rotator models or to stars with magnetic fields consisting of flux loops that emerge from lower latitudes and thread the disk.

The strengths of the meridional fields required for Keplerian disk formation in our models suggest that a possible scenario for the variation with time of properties of Keplerian disks in some B-type stars is the following. These stars possess magnetic fields that are mainly subphotospheric, and from time to time, because of magnetic buoyancy or other interior phenomena, meridional magnetic field lines emerge from the stellar surface. The strength of the meridional component of the field satisfies the condition $B_{m,0} < B^{*,L}$, so that it is able to assist in the formation of Keplerian disks. After the formation of a disk, the meridional field lines are drawn below the stellar surface in a timescale of the order of the circulation time, which is longer than the disk fill-up time $t_{\text{fill}}$. This shuts off the supply

### TABLE 4

| Spectral Type | $\alpha$ | $\theta_{0,\text{lim}}$ (deg) | $F_{\text{mag}}/F_{\text{cen}}$ | $F_{\text{visc}}/F_{\text{mag}}$ | $F_{2}$ |
|--------------|---------|-----------------------------|-------------------------------|-------------------------------|---------|
| O3           | 0.60    | 65                          | 0.005                         | 0.056                         |         |
| O6.5         | 0.60    | 65                          | 0.004                         | 0.045                         |         |
| B0           | 0.60    | 65                          | 0.011                         | 0.016                         |         |
| B2           | 0.50    | 65                          | 0.014                         | 0.011                         |         |
| B5           | 0.45    | 65                          | 0.017                         | 0.008                         |         |
| B9           | 0.40    | 65                          | 0.012                         | 0.010                         |         |

#### TABLE 5

| $\alpha$ | $\theta_{0,\text{lim}}$ (deg) | $x_{\text{Kep}}$ | $x_{\text{end}}$ | $B_{m,0,\text{opt}}$ (G) | $B_{x,0,\text{opt}}$ (G) | $B_{m,0,\text{min}}$ (G) |
|----------|-------------------------------|-----------------|-----------------|-------------------------|-------------------------|-------------------------|
| 0.35     | 2.01                          | 5.07            | 22.4            | --1.4                   | 13.9                    |                         |
| 0.40     | 1.84                          | 4.64            | 18.9            | --1.6                   | 11.3                    |                         |
| 0.45     | 1.70                          | 4.29            | 16.2            | --1.7                   | 9.3                     |                         |
| 0.50     | 1.59                          | 4.00            | 13.9            | --1.9                   | 7.6                     |                         |
| 0.55     | 1.49                          | 3.75            | 12.0            | --2.0                   | 6.3                     |                         |

#### TABLE 6

| $\theta_{0,\text{lim}}$ (deg) | $x_{\text{lim}}$ | $x_{\text{end}}$ | $B_{0,0,\text{lim}}$ (G) | $B_{x,0,\text{lim}}$ (G) | $\theta_{\text{Kep}}$ (deg) |
|------------------------------|-----------------|-----------------|-------------------------|-------------------------|-------------------------|
| 0.5                          | 1.62            | 1.72            | 8.1                     | --1.75                  | 70.4                    |
| 2.02                         | 4.00            | 13.9           | --1.85                  | 73.7                    |                         |
| 3.18                         | 4.00            | 32.8           | --2.26                  | 77.3                    |                         |
| 4.77                         | 4.00            | 61.4           | --2.86                  | 79.0                    |                         |

#### TABLE 4

| Spectral Type | $\alpha$ | $\theta_{0,\text{lim}}$ (deg) | $F_{\text{mag}}/F_{\text{cen}}$ | $F_{\text{visc}}/F_{\text{mag}}$ | $F_{2}$ |
|--------------|---------|-----------------------------|-------------------------------|-------------------------------|---------|

Note. — Values shown are for models of the B2 star, with $\alpha = 0.5$, $\theta_{0,\text{lim}} = 70^\circ$, and $\beta = 1.0$. In all these models, $x_{\text{Kep}} = 1.59$ and $x_{\text{visc}} = 2.00$.
of magnetically torqued material to the disk from non-equatorial latitudes. The resurgence and submersion of meridional fields across the stellar surface could explain some of the variations of the disk region over different timescales. We should emphasize that the theory that we propose for Keplerian disk formation does not depend on which theory of meridional circulation speeds is valid. We consider circulation only as a possible cause of variation of disks with time. Thus, the Be star phenomenon may be an interesting example in which the effects of rotation, circulation, and magnetic activity play a significant role. Obviously, stars in which \( B_{m,0,\min} > B_{r,s,L} \) will not exhibit this aspect of time variation. It is beyond the scope of the present work to consider details of the time-dependent behavior of the surface field caused by rotation, circulation, and outflow or the depletion of flux due to instabilities. Obviously, such a study will be greatly beneficial for a better understanding of the role of magnetic fields in the formation, properties, and variability of disks.

Some of the basic goals of this paper are to understand how magnetic fields of rotating stars can influence the formation of Keplerian disks and to explain the time variability observed in Be stars. The MTD model of Cassinelli et al. (2002) has the same objectives, and the flow regions that we consider in our models are similar to those in the MTD model. One of the main differences between our models and the MTD model is that we are able to establish a result for the angular velocity distribution during the fill-up stage of a disk. The MTD model uses an empirical formula for the rotational velocity in the disk region, which may be appropriate in WCD models. A second difference is that the meridional fields in our models can be weaker than the critical strength \( B_{r,s,L} \) and yet be strong enough to provide the torque required by the flow material to reach rotational speeds that lead to the formation of Keplerian disks. The MTD model requires stronger fields because the torque is applied to the disk, and, therefore, their strengths are larger than \( B_{r,s,L} \). As we show in this paper, a problem with stronger fields in the disk region is that they can inhibit the formation of a Keplerian region. An important aspect of our models is that for the strengths of fields that we consider, the magnetic force in the disk region becomes negligible at the end of the fill-up stage. Thus, the one-armed spiral pattern of the global disk oscillation model (Okazaki 1997) can be used to explain the \( V/R \) variability in disks of Be stars.

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APPENDIX

EQUATIONS FOR AZIMUTHAL COMPONENTS AND ALFVÉN VELOCITIES

The value of the quantity \( \ell \) for any streamline in the northern hemisphere is given by

\[
\ell = r_0 \sin \theta_0 \left[ \frac{\mathscr{A}_{m,0} (1 - \mathscr{A}_{m,0} \mathscr{A}_{\phi,0})}{\mathscr{A}_{m,0} - \mathscr{A}_{\phi,0}} \right]^{1/2} = r \sin \theta \left[ \frac{\mathscr{A}_{m} (1 - \mathscr{A}_{m} \mathscr{A}_{\phi})}{\mathscr{A}_{m} - \mathscr{A}_{\phi}} \right]^{1/2}.
\]

(A1)

We derive this by first substituting for \( \ell \) from equation (19) in equation (12) and then eliminating \( \omega \) between this new equation and equation (5). Next, we substitute for \( \xi \) and \( \chi \) from equation (7) and use equation (15) to replace \( B_m \) and \( B_\phi \) by \( \mathscr{A}_{m} \) and \( \mathscr{A}_{\phi} \).

We can write expressions for \( v_\phi \) and \( B_\phi \) in the northern hemisphere in the forms

\[
v_\phi = \omega \rho \sin \theta \frac{\mathscr{A}_{m}}{\mathscr{A}_{m} - \mathscr{A}_{\phi}}
\]

(A2)

and

\[
B_\phi = \omega \rho \sin \theta \frac{4 \pi \rho \nu \mathscr{A}_{m,0}}{B_{m,0}} \frac{\mathscr{A}_{m}^{2} \mathscr{A}_{\phi}}{\mathscr{A}_{m} - \mathscr{A}_{\phi}}.
\]

(A3)

Along an open streamline, both \( v_\phi \) and \( B_\phi \) have removable singularities at \( r \sin \theta = \ell \), where \( \rho = \rho_c \). The values of \( \rho_c \) and \( \ell \) to be inserted in these equations are given by equations (18) and (A1), respectively. Expressions for \( \mathscr{A}_{m} \) and \( \mathscr{A}_{\phi} \) in the northern hemisphere are obtained from equations (8), (18), (21), and (22). These are

\[
\mathscr{A}_{m} = \left( \frac{\rho}{\rho_c} \right)^{1/2}
\]

(A4)

and

\[
\mathscr{A}_{\phi} = \left( \frac{\rho_c}{\rho} \right)^{1/2} \left\{ \frac{1 - \left[ (\ell^2 / (r^2 \sin^2 \theta)) \right]}{1 - (\rho_c / \rho) \left[ (\ell^2 / (r^2 \sin^2 \theta)) \right]} \right\}
\]

(A5)

at points in all wind zones in the northern hemisphere.
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