Gapped symmetric edges of symmetry protected topological phases

Yuan-Ming Lu\textsuperscript{1,2} and Dung-Hai Lee\textsuperscript{1,2}

\textsuperscript{1}Department of Physics, University of California, Berkeley, CA 94720, USA
\textsuperscript{2}Materials Science Division, Lawrence Berkeley National Laboratories, Berkeley, CA 94720

Symmetry protected topological (SPT) phases are gapped quantum phases which host symmetry-protected gapless edge excitations. On the other hand, the edge states can be gapped by spontaneously breaking symmetry. We show that topological defects on the symmetry-broken edge cannot proliferate due to their fractional statistics. A gapped symmetric boundary, however, can be achieved between an SPT phase and certain fractionalized phases by condensing the bound state of a topological defect and an anyon. We demonstrate this by two examples in two dimensions: an exactly solvable model for the boundary between topological Ising paramagnet and double semion model, and a fermionic example about the quantum spin Hall edge. Such a hybrid structure containing both SPT phase and fractionalized phase generally support ground state degeneracy on torus.

PACS numbers: 03.65.Vf, 73.20.-r, 73.43.-f, 05.30.Pr

I. INTRODUCTION

Topological insulators\textsuperscript{13} (TIs) support gapless boundary excitations in spite of a gapped bulk spectrum. The edge states are believed to be stable against any perturbation, as long as certain symmetries are preserved. When symmetries are broken, however, TIs can be continuously tuned into a trivial atomic insulator without phase transitions. Recently it’s realized that aside from weakly-interacting electrons, such phases generally exist in interacting bosons and they are dubbed “symmetry protected topological” (SPT) phases.

When symmetries are spontaneously broken, a gap can open up in the edge spectrum of SPT phases. There are always topological defects (kinks)\textsuperscript{5} associated with spontaneous symmetry breaking, such as the domain wall excitation in an Ising ferromagnet. Usually by proliferating the defects one can restore symmetry, leading to a gapped symmetric state: e.g. the disordered phase of a transverse Ising model can be achieved by “condensing” the domain walls. Similarly can one achieve a gapped symmetric state on the edge of an SPT phase?

In this work we answer this question constructively, focusing on two spatial dimensions (2+1-D). We show that topological defects (kinks)\textsuperscript{5} on the boundary always carry fractional statistics\textsuperscript{6} or symmetry quantum numbers, hence their proliferation is either forbidden or breaks symmetry. However, on a boundary between SPT phase and certain fractionalized phase (which hosts anyon excitation\textsuperscript{5}), one can form a bosonic bound state of the kink on SPT side and anyon on fractionalized side. Proliferating this composite object will lead to a gapped symmetric boundary. This can be generalized to any spatial dimensions. Two examples are presented: 1) boundary between bosonic $Z_2$-SPT and double semion model, equipped with an exactly solvable model; 2) boundary between quantum spin Hall insulator (QSHI) and a fractionalized QSHI$^*$ phase. We show that a hybrid structure containing SPT and fractionalized phases support ground state degeneracy (GSD) on a torus (FIG. 2).

II. EDGE FIELD THEORY AND “FRACTIONAL” DEFECTS OF SPT PHASES

SPT phases\textsuperscript{7} in two spatial dimensions can be described\textsuperscript{8} by multi-component Chern-Simons theory\textsuperscript{9-10} with a symmetric unimodular matrix $K$. In particular, 2+1-D SPT phases host gapless edge excitations, described by chiral bosons $\{\phi_i\}$ with the following effective field theory\textsuperscript{11}:

$$L_{\text{edge}} = \frac{1}{4\pi} K_{I,J} \partial_I \phi_I \partial_J \phi_J - V_{I,J} \partial_I \phi_I \partial_J \phi_J$$

(1)

where $V$ is a positive-definite real symmetric matrix. Backscattering terms $\sim \cos(\sum_I \partial_I \phi_I)$ are generally not allowed by symmetry\textsuperscript{11-12} denoted by group $G_s$.

A simple example is topological paramagnet protected by $Z_n$ symmetry\textsuperscript{2,13} where $G_s = Z_n \equiv \{g, g^n, \cdots, g^n = e\}$. Its edge structure is characterized by $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in effective theory\textsuperscript{7}, where under $Z_n$ symmetry operation the chiral bosons transform as:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} + \frac{2\pi}{n} \frac{1}{k} \cdot 1 \leq k \leq n - 1.$$

(2)

Here backscattering terms $H_{bs} \sim \cos(\phi_{1,2} - \alpha_{1,2})$ are forbidden by the above $Z_n$ symmetry. Once symmetry is broken, edge states in $\phi_1, \phi_2$ can be gapped out by $H_{bs}$.

Spontaneous symmetry breaking on the edge will pin chiral boson fields $\phi_{1,2}$ at certain classical values. Distinct classical values $\langle \phi_{1,2} \rangle$ correspond to different ways to break discrete $Z_n$ symmetry, and there are topological defects – domain walls (or kinks) which spatially separates these different “vacua”. Different kinks are classified\textsuperscript{8} by homotopy group $\pi_0(Z_n) = Z_n$, i.e. there are $n$ distinct types of domain walls, including the trivial one – no domain wall. All stable kinks can be generated by a fundamental domain wall, which in our case $G_s = Z_n$ can be written in terms of chiral bosons\textsuperscript{14}:

$$\tilde{D}_{Z_n,k}(x) = \exp \left[ i \frac{k \phi_1(x) + \phi_2(x)}{n} \right]$$

(3)
As implied by \([7]\) chiral bosons \(\phi_{1,2}\) obey commutation relation \(\{\phi_1(x), \phi_2(y)\} = 2\pi i \cdot \theta(y-x)\). It’s straightforward to see the classical values \(\langle \phi_{1,2}\rangle\) on two sides of domain wall \([3]\) are related by the symmetry transformation \([2]\), e.g. \(\tilde{D}_{Z_n|k}(x)\phi_1(y)\tilde{D}_{Z_n|k}^{-1}(x) = \phi_1(y) + \frac{2\pi}{n} \theta(x-y)\).

A natural question is: can we restore symmetry \((G_s = Z_n\) here) simply by proliferating topological defects \((\text{kinks})\) and obtain a gapped symmetric edge of SPT phases? Remarkably the kink \([3]\) is neither a boson nor a fermion: generally it obeys fractional statistics\([31,32]\):

\[
\tilde{D}_{Z_n|k}(x)\tilde{D}_{Z_n|k}(y) = \tilde{D}_{Z_n|k}(y)\tilde{D}_{Z_n|k}(x)e^{\frac{2\pi i}{n}\text{sgn}(x-y)}
\]

with statistical angle \(\theta_{Z_n|k} = 2\pi \frac{k}{n}\). Moreover it has fractional mutual statistics with bosonic excitations \(\{\phi_{1,2}\}\):

\[
\begin{align*}
\tilde{D}_{Z_n|k}(x)e^{i\phi_1(y)} &= e^{i\phi_1(y)}\tilde{D}_{Z_n|k}(x)e^{i\frac{2\pi i}{n}\theta(x-y)}, \\
\tilde{D}_{Z_n|k}(x)e^{i\phi_2(y)} &= e^{i\phi_2(y)}\tilde{D}_{Z_n|k}(x)e^{-i\frac{2\pi i}{n}\theta(y-x)}.
\end{align*}
\]

Its fractional statistics lead to destructive interference in the path integral when domain walls proliferate, therefore suppressing instanton events which create/annihilate domain walls. As a result in \(Z_n\)-SPT phases, it is impossible to disorder the symmetry-broken edges simply by proliferating solitons. More generally on the boundaries of \(d+1\)-D SPT phases, there are similar obstructions to proliferate symmetry-breaking topological defects\([33]\).

### III. GAPPED SYMMETRIC BOUNDARY BETWEEN A SPT PHASE AND A FRACTIONALIZED PHASE

On the other hand, on the boundary between a SPT phase \(S\) and a fractionalized phase \(\mathcal{F}\) (i.e. intrinsic topological order\([13]\) which supports anyon excitations in the bulk), the bound state of a kink from SPT side and an anyon from fractionalized side can have bosonic statistics. This object in principle can proliferate and gap out all boundary excitations, restoring symmetry \(G_s\). Notice that fractionalized phase \(\mathcal{F}\) should also respect symmetry \(G_s\), in other words it is a symmetry enriched topological (SET) phase\([17,21]\).

Given a \(2+1\)-D SPT phase \(S\), only certain \(2+1\)-D fractionalized phases can have a gapped symmetric boundary with \(S\). For example edge states of \(\mathcal{F}\) and \(S\) must have the same number of chiral edge modes, i.e. same chiral central charge \(c_\perp\). Meanwhile \(\mathcal{F}\) must support anyon excitations with the same statistics and symmetry quantum numbers as topological defects (kinks) on the edge of \(S\). We propose the following conjecture, which provides a way to look for fractionalized phase \(\mathcal{F}\) sharing a gapped symmetric boundary with SPT phase \(S\):

An SPT phase \(S\) in any spatial dimensions always possesses a gapped symmetric boundary with a fractionalized (SET) phase \(\mathcal{F}\), where \(\mathcal{F}\) is obtained by gauging\([22]\) an Abelian discrete symmetry in \(S\).

In the following we give a proof of this conjecture in 2+1-D based on Chern-Simons approach.

An Abelian discrete symmetry group always has the form of a direct product of cyclic groups:

\[
G_s = \prod_{n \geq 2} (Z_n)^{\alpha_n}, \quad \alpha_n = 0, 1, 2, \cdots.
\]

Here we explicitly prove the conjecture for the case of a generic \(Z_n\) symmetry (cyclic group of order \(n\)).

Consider an arbitrary \(2+1\)-D SPT phase \(S\) with \(Z_n\) symmetry. Since it’s an Abelian \(2+1\)-D phase its bulk effective theory is a multi-component Chern-Simons theory\([10]\):

\[
\mathcal{L}_{\text{bulk}} = \frac{\epsilon^{\mu\nu\rho}}{4\pi} \sum I,J \, a_I^J K_{I,J} \partial_\mu a_I^{J\mu} - \sum I \, a_I^J \rho_I^\mu
\]

where \(\rho_I^\mu\) are the quasiparticle currents. In the long-wavelength limit its edge excitations are described by\([19]\):

\[
\mathcal{L}_{\text{edge}} = \frac{1}{4\pi} \sum I,J \, K_{I,J} \partial_I \phi_I \partial_J \phi_I - V_{I,J} \partial_I \phi_I \partial_J \phi_J
\]

where \(K\) is a unimodular matrix and \(V\) is a real positive-definite matrix. Denoting the generator of \(Z_n\) symmetry by \(g\) \((g^n = e)\), the edge chiral bosons \(\{\phi_I\}\) transform\([17,20]\):

\[
\phi_I \xrightarrow{g} \phi_I + \delta \phi_I^g; \quad nK\delta \phi^g = 0 \mod 2\pi.
\]

First of all, what is the fractionalized (SET) phase \(\mathcal{F}\) after gauging \(Z_n\) symmetry? A well-defined way to gauge the symmetry in a lattice model\([13]\) is to couple the local degrees of freedom (which lives on lattice sites and transforms under \(Z_n\) symmetry) to a dynamical \(Z_n\) gauge field (which lives on links). In an effective field theory, the effect of gauging a symmetry is captured by deforming the symmetry twist (or symmetry flux)\([13,20,22]\) in the original SPT phase. A symmetry twist has the following property: when a particle carrying symmetry quantum numbers goes around this symmetry twist once, it transforms under a symmetry operation. In the language of gauge field theory\([6]\), particles are integer gauge charges of fields \(\{a_I^J\}\): they are labeled by an integer vector \(l\) and represented by operators \(e^{i \sum I \phi_I} \) on the edge. Notice that particle \(e^{i \phi_I}\) pick up a phase \(e^{i \delta \phi_I^g}\) under \(Z_n\) symmetry operation \(g\), therefore symmetry twists are nothing but gauge fluxes in \([6]\). In a Chern-Simons theory, gauge fluxes are also gauge charges, which becomes transparent in the following equations of motion

\[
\frac{\delta \mathcal{L}_{\text{bulk}}}{\delta a_I^J} = 0 \implies \rho_I^\mu = \frac{\epsilon^{\mu\nu\rho}}{2\pi} \sum J \, K_{I,J} \partial_J a_I^{J\rho}.
\]

As a result, in the fractionalized phase \(\mathcal{F}\) obtained by gauging \(Z_n\) symmetry, quasiparticles corresponding to (fractional gauge charge) vector

\[
l^g = K \delta \phi^g / 2\pi,
\]
are new excitations in SET phase $\mathcal{F}$. One can immediately see from (8) that $n\mathbf{g}$ must be an integer vector. Moreover in a $\mathbb{Z}_n$-SPT the symmetry transformations form a faithful representation of $\mathbb{Z}_n$ group, meaning that at least one component of integer vector $\mathbf{g}$ is 1/n. Without loss of generality, we assume that $\mathbf{g}_{p+1} = 1/n$ where \( \dim \mathbf{K} = p + 1 \). In other words we have

$$
\mathbf{g} = (v^T, 1)^T/n, \quad v \in \mathbb{Z}^p.
$$

(10)

As discussed in Ref. 20, the fractionalized SET phase $\mathcal{F}$ obtained by gauging $\mathbb{Z}_n$ symmetry is described by Chern-Simons theory with matrix

$$
\mathbf{K} = \mathbf{M}^{-1}\mathbf{K}(\mathbf{M}^{-1})^T, \quad \mathbf{M} = \left(\begin{array}{cc} 1 & \mathbf{v}/n \\ 0 & 1/n \end{array}\right).
$$

It’s easy to see

$$
\mathbf{M}^{-1} = \left(\begin{array}{cc} 1_{p \times p} & -\mathbf{v} \\ 0_{1 \times p} & n \end{array}\right), \quad \det \mathbf{K} = n^2 \det \mathbf{K}.
$$

(11)

We label chiral bosons on the edge of SET phase $\mathcal{F}$ by \( \{\phi_I | 1 \leq I \leq p+1\} \). They have the following correspondence with the edge chiral bosons \( \{\tilde{\phi}_I\} \) in SPT phase $\mathcal{S}$:

$$
\tilde{\phi}_I \leftrightarrow \sum_j \mathbf{M}_{j,I} \phi_I
$$

(12)

Similar relations hold for their symmetry transformations \( \{\delta \phi_I\} \) and \( \{\delta \tilde{\phi}_I\} \).

In SPT phase $\mathcal{S}$, mutual statistics between particle I and the new (fractional) particle $\mathbf{g}$ is given by

$$
\theta_{\mathbf{g}} = 2\pi I^T \mathbf{K}^{-1} \mathbf{g} = I^T \delta \phi_0.
$$

(13)

Generally $\delta \phi_0/2\pi = p_I/q_I$ where \( (p_I, q_I) \) are two mutually prime integers. It’s not hard to check that the following set of backscattering terms between the SPT edge and SET edge gives rise to a gapped symmetric boundary between them:

$$
\mathcal{L}_1 = \sum_{I=1}^p \cos \big[q_I(\tilde{\phi}_I - \phi_I)\big] + \cos \big[n(\tilde{\phi}_{p+1} - \sum_I \mathbf{g} \phi_I)\big].
$$

In the case of gauging a continuous symmetry, the conjecture is not valid anymore. An obvious example is $U(1)$-SPT in 2+1-D, i.e. bosonic integer quantum Hall effect. The chiral central charge is always $c_- = 0$ for such a $U(1)$-SPT phase $\mathcal{S}$. After gauging $U(1)$ symmetry, we obtain a fractionalized phase $\mathcal{F}$ described by $U(1)$ level-$\sigma_{xy}$ Chern-Simons term, which has chiral central charge $c_- = \text{Sgn} \left(\sigma_{xy}\right)$. Apparently there cannot be a gapped edge between $\mathcal{S}$ and $\mathcal{F}$.

### IV. EXAMPLES

In this section we’ll demonstrate the conjecture in two examples.

#### A. Topological Ising paramagnet ($G_s = Z_2$)

The simplest example of 2+1-D bosonic SPT phases is the topological Ising paramagnet with $G_s = Z_2$ symmetry. In its edge effective theory with $\mathbf{K} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$ and Ising symmetry acts on chiral bosons $\phi_{1,2}$ as $\left(\begin{array}{c} \phi_1 \\ \phi_2 \end{array}\right)$ with $n = 2, k = 1$. The domain wall operator $\hat{D}_{\mathbb{Z}_2} |1\rangle$ in (4) is a semion, with fractional statistics $\theta = \pi/2$. In order to acquire a gapped symmetric edge, the corresponding fractionalized (SET) phase must support bulk semion excitations and non-chiral edge states. The simplest choice is the double semion state with $\mathbf{K} = \left(\begin{array}{cc} 2 & 0 \\ 0 & -2 \end{array}\right)$ in (7).

Note that gauging Ising symmetry in $Z_2$-SPT leads to nothing but double semion model. On its edge there are two branches of chiral bosons $\phi_{s,\bar{s}}$, where $\exp(i \phi_s)$ creates a semion and $\exp(i \phi_{\bar{s}})$ creates an anti-semion. Now the following tunneling terms between the SPT edge $(\phi_{1,2})$ and double semion edge $(\phi_{s,\bar{s}})$

$$
\mathcal{H}_t = T_1 \cos(\phi_1 + \phi_2 - 2\phi_s) + T_2 \cos(\phi_1 - \phi_2 - 2\phi_{\bar{s}})
$$

(14)

can open up a gap on their boundary without breaking Ising symmetry. Here spins in double semion model doesn’t transform under Ising symmetry. In contrast, on the “pure” boundary between this $Z_2$-SPT phase and the vacuum, there is no way to get rid of gapless excitations without breaking symmetry.

In the following we present an exactly solvable model for such a boundary between bosonic $Z_2$-SPT and double semion model. Its Hamiltonian consists of commuting local projectors:

$$
\mathcal{H} = \mathcal{H}_{SPT} \{\sigma_a\} + \mathcal{H}_{SET} \{\tau_i\} + \mathcal{H}_{BDY},
$$

(15)

On $Z_2$-SPT side

$$
\mathcal{H}_{SPT} = \sum_a \frac{\sigma_a^x}{(a, b, c)} \bigg( \prod_{(a, b, c)} -i \frac{1 - \sigma_a^x}{2} \bigg)
$$

where $(a, b, c)$ runs over all six (nearest neighbor) triangles containing $a$. On the side of double semion mode

$$
\mathcal{H}_{SET} = -\sum_{I \text{ legs of } I} \tau_i + \sum_{r \text{ edges of } r} \frac{1 - \tau_r}{2} + \sum_{R \text{ legs of } r} \frac{1 - \tau_r}{2}
$$

$I$ denotes vertices and $r$ denotes hexagonal plaquette center. The boundary Hamiltonian connects the two sides symmetrically

$$
\mathcal{H}_{BDY} = \sum_{r \text{ boundary}} -\tau_r + \sigma_r^x \sigma_r^z \bigg( \prod_{(r, b, c)} -i \frac{1 - \tau_r}{2} \bigg) + \sigma_r^x \bigg( \prod_{(r, b, c)} -i \frac{1 - \sigma_r^x}{2} \bigg) \bigg( \prod_{r \text{ edges of } r} \frac{1 - \tau_r}{2} \bigg) \bigg( \prod_{R \text{ legs of } r} \frac{1 - \tau_r}{2} \bigg)
$$

as shown in FIG. 1. It’s easy to verify that all terms commute with each other. Physical the 1st term in
$H_{BDY}$ guarantees a domain wall from $Z_2$-SPT side is always bound to a string from double semion side, while the 2nd term provides kinetic energy which allows this bound state to hop on the boundary. Therefore they can “condense” on the boundary and gap out the edge states without breaking Ising symmetry. The low-energy effective theory for boundary Hamiltonian $H_{BDY}$ is nothing but \[ (14). \]

The Ising symmetry in this model is implemented by $g = \prod \sigma^z$, thus $\sigma^z$ spins in double semion model is invariant under Ising spin flip. Now let’s consider model \[ (15) \] on a torus (see FIG. 2), where a half of the torus hosts double semion model and the other half hosts $Z_2$-SPT phase. Both shared boundaries between $Z_2$-SPT and double semion model are gapped by Hamiltonian \[ (16). \] However there is a 2-fold ground state degeneracy (GSD) in such a hybrid structure on torus. This can be easily verified by comparing the number of independent stabilizers \[ (23) \] (local commuting projectors in $H$) and the total number of spins, since there is a global constraint for the local projectors in model \[ (15) \] on torus:

$$
\prod_{\text{legs of } I} (\prod_{r \in \text{boundary}} \tau^x_i) \cdot \prod_{\text{edges of } S} \prod_{\text{R-legs}} (1 + \tau^z_j) \cdot \prod_{\text{L-vertices}} (-1)^{1 - \tau^x_i} = 1
$$

These two degenerate ground states can be labeled by eigenvalues of string “order parameter” \[ (23) \]

$$
\hat{O}_S = \prod_{\text{edges of } S} \prod_{\text{R-legs}} \prod_{\text{L-vertices}} (-1)^{1 - \tau^x_i}
$$

The closed string $S$ winds around a non-contractible loop of torus once, parallel to the boundary between $Z_2$-SPT and double semion model, as illustrated by the horizontal green loop in FIG. 1. It’s easy to verify $(\hat{O}_S)^2 = 1$, hence it has eigenvalues $\pm 1$.

A pure double-semion model on torus has 4-fold GSD labeled by two commuting string operators: $\hat{O}_L$ and $\hat{O}_S^\prime$ = $\prod_{\text{R-legs}} \tau^z_j$. However in the ground states of our model, the 1st term of boundary Hamiltonian \[ (16) \] fixes the eigenvalue of $\hat{O}_S^\prime$ to be 1. Therefore the original 4-fold GSD of double-semion model on torus reduces to 2-fold in the hybrid structure here. The two degenerate ground states can be alternatively labeled by another string operator

$$
\hat{O}_L = \sigma^z_r (\prod_{\text{R-legs}} \tau^z_j) \sigma^z_e, \quad [\hat{O}_L, H] = 0, \quad (\hat{O}_L)^2 = 1
$$

where the open oriented line $L$ starts on one boundary ($r_b$ is the closet $\sigma$ spin on its r.h.s.) and ends on the
other boundary (r_e on its r.h.s.) in FIG. 2. Notice that open line L crosses the closed string S only once, hence \{O_T, O_S\} = 0. Both string operators are even under Ising symmetry. This means operator \(\hat{O}_L\) tunnels between two \(\hat{O}_S\) eigenstates in the ground states’ manifold:

\[\hat{O}_L|O_S = \pm 1\rangle = |O_S = \mp 1\rangle.\]

As we shrink the SET region in FIG. 2, the length of line L also decreases and ultimately \(O_L\) will become a local operator. In this limit the two \(\hat{O}_S\) eigenstates start to mix by local interactions and the system picks up a unique ground state as a superposition of \(|O_S = \pm 1\rangle\) states. As a result we go back to the case of pure \(\mathbb{Z}_2\) ground state as a superposition of\(^{\text{(14)}}\) \(\{O_T, O_S\} = 0\). Both string operators are even under Ising symmetry. This means operator \(\hat{O}_L\) tunnels between two \(\hat{O}_S\) eigenstates in the ground states’ manifold:

\[\hat{O}_L|O_S = \pm 1\rangle = |O_S = \mp 1\rangle.\]

We consider the hybrid geometry in FIG. 2, there is a 2-fold GSD. Remarkably the two degenerate ground states are labeled by different electron number parity on each edge. Here the closed string operator becomes

\[\hat{O}_S = e^i\int_0^L dx \partial_x \phi_e(x) = e^i\int_0^L dx \partial_x \phi_e(x).\]  

When such a anyon is created from one boundary, another anyon must be created from the other boundary. The double-semion-model part of this operator (i.e. \(e^{i(T_2\phi)}\) part) from both boundary can be brought into the bulk of double-semion-model region and annihilated\(^{25}\), while the \(\mathbb{Z}_2\)-SPT part of operator \(\psi\) (i.e. \(e^{-i\phi_e^2}\) part) are left on both boundaries. Such a process of of creating (on the boundary) and annihilating (in the SET bulk) anyons is precisely realized by string operator \(\hat{O}_L\).

**B. Quantum spin Hall insulators \(\left(G_s = U(1) \times \mathbb{Z}_2^T\right)\)**

Now let’s turn to a more familiar example of SPT phases of electrons: \(\mathbb{Z}_2\) quantum spin Hall insulators\(^{23,25}\) (QSHI). Its edge theory is \(\Theta\) with \(K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\). Let’s denote the two branches of chiral bosons as \(\phi_{R/L}\), and they transform under \(U(1) \times \mathbb{Z}_2^T\) symmetry as

\[e^{i\theta N_f} \phi_{R/L} e^{-i\theta N_f} = \phi_{R/L} + \theta, \quad \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix} T_c \begin{pmatrix} -\phi_L \\ -\phi_R + \pi \end{pmatrix}.\]

Electrons on the edge are \(\psi_{R/L} \sim e^{i\phi_{R/L}}\). A gap will open up on the edge spectrum, by magnetic order \(M = M \cos(\phi_R - \phi_L + \phi_M)\) which breaks time reversal \(T_c\), or by superconductivity \(\Delta = \Delta \cos(\phi_R + \phi_L + \phi_D)\) which breaks \(U(1)\) charge conservation. Can we restore symmetry and obtain a gapped symmetric edge by proliferating defects of order parameters \(M\) and \(\Delta\)? The answer is no. For example, domain wall of magnetic order \(D_T = \exp[i(\phi_R + \phi_L)/2]\) is a bosonic object, but it carries unit charge. Therefore proliferating this object will break \(U(1)\) charge conservation!

Hence we need to play the same trick and consider the boundary between QSHI and a fractionalized SET phase. A convenient choice is the so-called QSHI* phase  obtained by coupling fermions in QSHI to a dynamical \(\mathbb{Z}_2\) gauge field\(^{23}\). As shown in Appendix \(\hat{X}\) the effective edge theory of QSHI* is described by \(J_0\) with \(K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\). If we label its edge chiral bosons as \(\phi_{s/c}\), they transform under \(U(1) \times \mathbb{Z}_2^T\) symmetry as

\[e^{i\theta N_f} \phi_{s(c)} e^{-i\theta N_f} = \begin{pmatrix} \phi_s + \theta \\ \phi_c \end{pmatrix}, \quad \begin{pmatrix} \phi_s \\ \phi_c \end{pmatrix} T_c \begin{pmatrix} \phi_s - \phi_c + \pi \\ 1 \end{pmatrix}.\]

Therefore \(e^{i\phi_c}\) is a charge-1 boson (“chargeon”), with mutual semion statistics with neutral boson \(e^{i\phi_s}\) (“spinon” since \(T^2 = -1\)). Clearly \(e^{-i\phi_c}\) can form a neutral bosonic bound state with domain wall \(D_T\), and its condensation will lead to a gapped symmetric edge between QSHI and QSHI*. More precisely the boundary tunneling term is

\[\mathcal{H}_t = T_e \cos(\phi_R + \phi_L - 2\phi_e) + T_s \cos(\phi_R - \phi_L - 2\phi_e).\]  

**V. SUMMARY**

In this paper we investigate the question of how to proliferate topological defects (domain walls or kinks)
on the symmetry-broken edge of an SPT phase, in order to achieve a gapped symmetric edge. We show that condensing these defects is either forbidden by quantum statistics or breaks symmetry. On the other hand, we can overcome these obstructions by considering a boundary between an SPT phase \( S \) and a fractionalized (SET) phase \( F \). We propose a conjecture (A1) for how to find such fractionalized phases for a given SPT, which generalizes to all spatial dimensions. Two examples in two spatial dimensions are presented with effective field theory and exactly solvable models. We found that once this hybrid structure is put on a torus as in FIG. 2, there will be ground state degeneracy accompanying the two gapped symmetric edges between \( S \) and \( F \).

Note added Upon completion of this work, we became aware of Ref. 31, where a different geometry containing both SPT and SET phases are considered. In their situation open string order parameter \( \hat{R} \) only contains operators from SET side on one boundary, therefore it breaks symmetry in a nonlocal way.

Acknowledgments

This work is supported by DOE Office of Basic Energy Sciences, Materials Sciences Division of the U.S. DOE under contract No. DE-AC02-05CH11231 (YML,DHL) and in part by the National Science Foundation under Grant No. PHYS-1066293(YML). We thank Taylor L. Hughes and Ashvin Vishwanath for helpful conversations. YML No. PHYS-1066293(YML). We thank Taylor L. Hughes and Ashvin Vishwanath for helpful conversations. YML No. PHYS-1066293(YML). We thank Taylor L. Hughes and Ashvin Vishwanath for helpful conversations. YML No. PHYS-1066293(YML). We thank Taylor L. Hughes and Ashvin Vishwanath for helpful conversations. YML No. PHYS-1066293(YML).

Appendix A: Effective field theory of QSHI*

QSHI* is obtained by gauging \( Z_f^2 \) (fermion parity) symmetry (generated by \( g = (-1)^{N_f} \)) in a fermionic QSHI in 2+1-D. This means we need to couple a dynamical \( Z_2 \) gauge field to fermions in QSHI. Following discussions in the previous section, in this case we have

\[
K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \delta \phi^g = \left( \frac{\pi}{\pi} \right) \Rightarrow \lambda^g = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \quad (A1)
\]

Hence after gauging \( Z_f^2 \) symmetry, we obtained a fractionalized SET phase \( F \) with

\[
K^g = M^{-1} K (M^{-1})^T = \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \quad (A2)
\]

\[
M = \begin{pmatrix} 1/2 & 0 \\ -1/2 & 1 \end{pmatrix}.
\]

We can always make a \( X \in GL(2, Z) \) rotation to the \( K \) matrices so that

\[
K^g \simeq X^T K^g X = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \quad (A3)
\]

\[
X = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.
\]

Let’s label the chiral boson fields in QSHI (A1) as \( \phi_{R/L} \), and those in QSHI* (A3) as \( \phi_{c/s} \). They have the following correspondence:

\[
\begin{pmatrix} \phi_s \\ \phi_c \end{pmatrix} \leftrightarrow X^{-1} M^T \begin{pmatrix} \phi_R \\ \phi_L \end{pmatrix}. \quad (A4)
\]

Hence one can easily figure out the symmetry transformations of quasiparticles in QSHI* st

\[
e^{i \theta \hat{N}_f} \begin{pmatrix} \phi_s \\ \phi_c \end{pmatrix} e^{-i \theta \hat{N}_f} = \begin{pmatrix} \phi_s \\ \phi_c + \theta \end{pmatrix}, \quad (A5)
\]

\[
\begin{pmatrix} \phi_s \\ \phi_c \end{pmatrix} \rightarrow \begin{pmatrix} \phi_s \\ -\phi_c \end{pmatrix} + \frac{\theta}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

References

1 M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
2 J. E. Moore, Nature 464, 194 (2010), ISSN 0028-0836.
3 X.-L. Qi and S.-C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
4 X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013).
5 N. D. Mermin, Rev. Mod. Phys. 51, 591 (1979).
6 F. Wilczek, Fractional Statistics and Anyon Superconductivity (World Scientific Pub Co Inc, 1990).
7 Y.-M. Lu and A. Vishwanath, Phys. Rev. B 86, 125119 (2012).
8 N. Read, Phys. Rev. Lett. 65, 1502 (1990).
9 X. G. Wen and A. Zee, Phys. Rev. B 46, 2290 (1992).
10 J. Frohlich and A. Zee, Nuclear Physics B 364, 517 (1991), ISSN 0550-3213.
11 X.-G. Wen, Advances in Physics 44, 405 (1995), ISSN 0001-8732.
12 M. Levin and A. Stern, Phys. Rev. B 86, 115131 (2012).
13 M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012).
14 X. Chen, Y.-M. Lu, and A. Vishwanath, Nat Commun 5, (2014).
15 A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016 (2013).
16 X.-G. Wen, Quantum Field Theory Of Many-body Systems: From The Origin Of Sound To An Origin Of Light And Electrons (Oxford University Press, New York, 2004).
17 A. M. Essin and M. Hermele, Phys. Rev. B 87, 104406 (2013).
A. Mesaros and Y. Ran, Phys. Rev. B 87, 155115 (2013).
L.-Y. Hung and X.-G. Wen, Phys. Rev. B 87, 165107 (2013).
Y.-M. Lu and A. Vishwanath, ArXiv e-prints 1302.2634 (2013), 1302.2634.
L.-Y. Hung and Y. Wan, Phys. Rev. B 87, 195103 (2013).
L.-Y. Hung and X.-G. Wen, ArXiv e-prints 1211.2767 (2012), 1211.2767.
M. A. Levin and X.-G. Wen, Phys. Rev. B 71, 045110 (2005).
A. Y. Kitaev, Annals of Physics 303, 2 (2003), ISSN 0003-4916.
J. Wang and X.-G. Wen, ArXiv e-prints (2012), 1212.4863.
C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005).

B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006).
M. Konig, S. Wiedmann, C. Brune, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, Science 318, 766 (2007), http://www.sciencemag.org/content/318/5851/766.full.pdf.
A. Ruegg and G. A. Fiete, Phys. Rev. Lett. 108, 046401 (2012).
T. Senthil and M. P. A. Fisher, Phys. Rev. B 62, 7850 (2000).
C. Wang and M. Levin, Phys. Rev. B 88, 245136 (2013).
Strictly speaking the domain wall could also be an antiasimon, as a bound state of a semionic domain wall and a local spin flip.