A field-theoretic model for Hodge theory

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Abstract: We demonstrate that the four (3 + 1)-dimensional (4D) free Abelian 2-form gauge theory presents a tractable field theoretical model for the Hodge theory where the well-defined symmetry transformations correspond to the de Rham cohomological operators of differential geometry. The conserved charges, corresponding to the above continuous symmetry transformations, obey an algebra that is reminiscent of the algebra obeyed by the cohomological operators. The discrete symmetry transformation of the theory represents the realization of the Hodge duality operation that exists in the relationship between the exterior and co-exterior derivatives of differential geometry. Thus, we provide the realizations of all the mathematical quantities, associated with the de Rham cohomological operators, in the language of the symmetries of the present 4D free Abelian 2-form gauge theory.

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1 Introduction

The non-Abelian 1-form (i.e. $A^{(1)} = dx^\mu A_\mu$) gauge theories, endowed with the first-class constraints in the language of Dirac’s prescription for the classification scheme [1, 2], are at the heart of the standard model of elementary particle physics where there is a stunning degree of agreement between the theory and experiment. The above cited first-class constraints of the 1-form gauge theories appear very naturally in the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism when one demands that the true physical states (of the total quantum Hilbert space) are those that are annihilated (i.e. $Q_b|\text{phys} >= 0$) by the nilpotent (i.e. $Q_b^2 = 0$) and conserved (i.e. $\dot{Q}_b = 0$) BRST charge operator $Q_b$. In fact, as it turns out, it is the operator form of the first-class constraints that annihilate the physical states of the theory due to the physicality condition $Q_b|\text{phys} >= 0$. This condition is consistent with the Dirac’s prescription for the quantization of systems with constraints.

The observations, made above, are true for any arbitrary $p$-form (with $p = 1, 2, 3...$) gauge theory. The nilpotency ($Q_b^2 = 0$) of the BRST charge and the physicality criteria ($Q_b|\text{phys} >= 0$) are the two essential ingredients that provide a thread of connection that runs through the cohomological aspects of BRST charge and the exterior derivative $d$ (with $d = dx^\mu \partial_\mu$, $d^2 = 0$) of differential geometry [3-6]. For instance, two BRST closed physical states (i.e. $Q_b|\text{phys} >= 0, Q'_b|\text{phys} >' = 0$) are said to belong to the same cohomology class with respect to the BRST charge $Q_b$ if they differ by a BRST exact state (i.e. $|\text{phys} >' = |\text{phys} > + Q_b|\chi >$ where $|\chi >$ is a non-null state). Exactly, in a similar fashion, two closed forms (i.e. $df = 0, df' = 0$) are said to belong to the same cohomology class w.r.t. $d$ if they differ by an exact form (i.e. $f' = f + dg$ for $g$ to be a non-trivial form). Thus, the nilpotent BRST charge $Q_b$, which is the generator of the well-defined nilpotent BRST symmetry transformations, is a physical realization of $d$.

It has been a long-standing problem to find out the physical realizations of the other cohomological operators $\delta = \pm * d*$ (with $\delta^2 = 0$) and the Laplacian operator $\Delta = d\delta + \delta d$ constitute the set of de Rham cohomological operators of differential geometry on a compact manifold without a boundary. These operators follow the algebra: $d^2 = \delta^2 = 0, \Delta = (d + \delta)^2 \equiv \{d, \delta\}, [d, \Delta] = 0, [\delta, \Delta] = 0$. The operator $*$ is the Hodge duality operation on a given manifold.
and $\Delta$ for the following field theoretical models, namely:

(i) two $(1+1)$-dimensional (2D) free (non-)Abelian 1-form gauge theories (without any interaction with the matter fields) [11-13],

(ii) 2D interacting U(1) Abelian gauge theory with Dirac fields [14], and

(iii) 4D free Abelian 2-form gauge theory [15-17].

In the context of the 4D Abelian 2-form gauge theory, the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations turn out to be anticommuting only up to a U(1) vector gauge transformation. The absolute anticommutativity is found to be absent in [15-17]. On the contrary, our recent work on the superfield approach to 4D free Abelian 2-form gauge theory [18], enforces the (anti-)BRST symmetry transformations of the theory to be absolutely anticommuting due to the presence of a Curci-Ferrari type restriction (cf. equation (12) below) on the theory. Taking the help of this restriction, we have been able to obtain the Lagrangian densities of the Abelian 2-form gauge theories that respect absolutely anticommuting (anti-)BRST symmetry transformations [19,20]. The connection of the above type of restriction with the concept of gerbes has also been established in [19].

In our earlier works on the BRST approach to free 4D Abelian 2-form gauge theory [19,20], the above Curci-Ferrari type restriction has been incorporated in the Lagrangian densities of the theory through a Lorentz vector auxiliary field. As a consequence of this restriction, however, the massless scalar field of the theory is constrained to possess a kinetic term with a negative signature. In our very recent works [21,22], we have shown that the BRST invariant Lagrangian densities of the 2-form gauge theory can be defined that invoke no Lagrange multiplier (auxiliary) field. For such kind of Lagrangian densities, the Curci-Ferrari type of restriction emerges from the equations of motion and the scalar field of the theory is not enforced to have a negative kinetic term. These Lagrangian densities have been further generalized [22] so as to respect the nilpotent (anti-)BRST as well as (anti-)co-BRST symmetry transformations together. These transformations are found to be absolutely anticommuting due to the Curci-Ferrari type restrictions.

The absolute anticommutativity of the off-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations is very sacrosanct because it demonstrates the linear independence of (i) the BRST versus the anti-BRST, and (ii) the co-BRST vis-à-vis the anti-co-BRST symmetry transformations (cf. (39) below). This statement becomes crystal clear within the framework of superfield approach to BRST formalism (see, e.g. [18]). The purpose of our present investigation is to obtain the well-defined physical realizations of the abstract de Rham cohomological operators of differential geometry in the language of the symmetry transformations (and the corresponding generators) in the context of the Abelian 2-form gauge theory. This exercise establishes
that the Abelian 2-form gauge theory is a field theoretic model for the Hodge theory where the absolute anticommutativity between the BRST and anti-BRST as well as co-BRST and anti-co-BRST symmetry transformations are guaranteed due to the presence of the Curci-Ferrari type restrictions (cf. (12) below). These restrictions emerge as the equations of motion and they are not imposed from outside. The absolute anticommutativity is a decisive feature of our present investigation that was absent in our earlier works [15-17].

The following central factors have motivated us to pursue our present investigation. First and foremost, it is always very important to have a gauge theory where all the de Rham cohomological operators find their physical meaning. Our free 4D Abelian 2-form gauge theory provides the same and, hence, it is a field theoretic model for the Hodge theory. Second, the nilpotent (anti-)BRST and (anti-)co-BRST symmetry transformations were not found to be absolutely anticommuting for the present model in [15-17]. It was challenging to achieve this goal so that there could be consistency between the superfield approach to Abelian 2-form gauge theory [18] and the ordinary field theoretical approach to BRST formalism for the same theory. We have accomplished this goal in our present endeavour. Finally, our present study is a modest step towards our main goal of studying the interacting non-Abelian 2-form gauge theories within the framework of BRST formalism.

The outline of our present paper is as follows. In Sec. 2, we generalize the gauge-fixed Kalb-Ramond Lagrangian density by incorporating two auxiliary vector fields and a pair of massless scalar fields. Our Sec. 3 deals with the nilpotent and anticommuting (anti-)BRST symmetry transformations and corresponding conserved charges. Our Sec. 4 is devoted to the discussion of the anticommuting (anti-)co-BRST symmetry transformations and the derivation of their generators. In Sec. 5, we derive a bosonic symmetry transformation that is the analogue of the Laplacian operator of differential geometry. Our Sec. 6 deals with the ghost and discrete symmetry transformations of the theory. We derive the extended BRST algebra in Sec. 7 where its relationship with the cohomological differential operators is established at the algebraic level. Finally, in our Sec. 8, we make some concluding remarks and point out a few future directions for further investigations.

In Appendices A, B and C, we mention some key points for specific proofs.

2 Preliminaries: gauge-fixed Lagrangian densities

We begin with the Kalb-Ramond Lagrangian density [23-26]

\[
\mathcal{L}^{(0)} = \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa}, \quad H_{\mu\nu\kappa} = \partial_{\mu} B_{\nu\kappa} + \partial_{\nu} B_{\kappa\mu} + \partial_{\kappa} B_{\mu\nu},
\]  

(1)
where the totally antisymmetric curvature tensor \( H_{\mu\nu\kappa} \) is derived from the 3-form \( H^{(3)} = dB^{(2)} = [(dx^\mu \wedge dx^\nu \wedge dx^\kappa)/(3!)] H_{\mu\nu\kappa} \). In the above, the 2-form \( B^{(2)} = [(dx^\mu \wedge dx^\nu)/(2!)]B_{\mu\nu} \) defines the antisymmetric tensor gauge potential \( B_{\mu\nu} \) and \( d = dx^\mu \partial_\mu \) (with \( d^2 = 0 \)) is the exterior derivative of differential geometry. It can be checked that \( \partial_\mu H^{\mu\nu\kappa} = 0 \) (due to the Euler-Lagrange equation of motion derived from the above Lagrangian density).

The gauge-fixing term for the above Lagrangian density is derived by the application of the co-exterior derivative (\( \delta = - \ast d \ast \), \( \delta^2 = 0 \)) on the 2-form (i.e. \( \delta B^{(2)} = - \ast d \ast B^{(2)} = (\partial^\nu B_{\nu\mu})dx^\mu \)) which leads to the expression for the 1-form \( G^{(1)} = (\partial^\nu B_{\nu\mu})dx^\mu \). In the above, the \( \ast \) is the Hodge duality operation on the 4D spacetime manifold. In fact, the explicit expression for the gauge-fixed Lagrangian density of the present 2-form gauge theory is

\[
\mathcal{L}^{(1)} = \frac{1}{12} H^{\mu\nu\kappa} H_{\mu\nu\kappa} + \frac{1}{2} (\partial^\nu B_{\nu\mu})(\partial_\kappa B^{\kappa\mu}).
\]

The ensuing equation of motion, derived from the above gauge-fixed Lagrangian density, is now \( \Box B_{\mu\nu} = 0 \) where \( \Box = \partial_\mu^2 - \partial_i^2 \) is the d’Alembertian operator. This equation of motion has its origin in the Laplacian operator \( \Delta = (d + \delta)^2 = d\delta + \delta d \) when one demands the validity of the Laplace equation \( \Delta B^{(2)} = 0 \). It is interesting to note that \( \mathcal{L}^{(1)} \) remains invariant under the discrete symmetry transformation \( B_{\mu\nu} \rightarrow \pm \frac{\iota}{2} \varepsilon_{\mu\nu\rho\kappa} B^{\rho\kappa} \).

One can linearize the kinetic and gauge fixing parts of the Lagrangian density (2) by introducing the Nakanishi-Lautrup type of Lorentz vector auxiliary fields \( B^{(1)}_\mu \) and \( B^{(1)}_\mu \) as given below:

\[
\mathcal{L}^{(2)} = \frac{1}{2} B^{(1)}_\mu B^{(1)}_\mu - B^{(1)}_\mu \left( \frac{1}{3!} \varepsilon_{\mu\nu\rho\kappa} H^{\nu\rho\kappa} \right) + B^{(1)}_\mu \left( \partial^\nu B_{\nu\mu} \right) - \frac{1}{2} B^{(1)}_\mu B^{(1)}_\mu.
\]

It is worthwhile to mention that the Hodge dual of 3-form \( H^{(3)} \) (i.e. \( \ast H^{(3)} = \frac{1}{3!}(dx^\mu)\varepsilon_{\mu\nu\rho\kappa} H^{\nu\rho\kappa} \)), which happens to be a 1-form, has been exploited in the linearization of the kinetic term. The above Lagrangian density in (3) respects the following discrete symmetry transformations:

\[
B_{\mu\nu} \rightarrow \pm \frac{i}{2} \varepsilon_{\mu\nu\rho\kappa} B^{\rho\kappa}, \quad B^{(1)}_\mu \rightarrow \pm i B^{(1)}_\mu, \quad B^{(1)}_\mu \rightarrow \mp i B^{(1)}_\mu.
\]

\( ^2 \)We choose here the flat metric (for the 4D Minkowski spacetime manifold) with signatures \((+1, -1, -1, -1)\). The choice of the totally antisymmetric 4D Levi-Civita tensor \( (\varepsilon_{\mu\nu\rho\kappa}) \) is such that \( \varepsilon_{0123} = +1 = -\varepsilon_{0123} \), \( \varepsilon_{0ijk} = \varepsilon_{ijk} \), \( \varepsilon_{\mu\nu\kappa\zeta} \varepsilon^{\mu\nu\kappa\zeta} = -4! \), \( \varepsilon_{\mu\nu\kappa\zeta} \varepsilon^{\mu\nu\kappa\eta} = -3! \delta^\eta_\zeta \), etc. Here \( \varepsilon_{ijk} \) is the 3D Levi-Civita tensor. In the whole body of our text, we shall be taking the Greek indices \( \mu, \nu, \eta, \kappa \ldots = 0, 1, 2, 3 \) and the Latin indices \( i, j, k \ldots = 1, 2, 3 \).
The equations of motion that emerge from the above Lagrangian density are

\[
(\partial \cdot B^{(1)}) = 0, \quad B^{(1)}_\mu = \partial^\nu B_{\nu\mu}, \quad (\partial \cdot B^{(1)}) = 0, \quad \Box B^{(1)}_\mu = 0,
\]

\[
\varepsilon^{\mu\nu\kappa\eta} \partial^\kappa B^{(1)}_\eta + (\partial^\mu B^{(1)}_\nu - \partial^\nu B^{(1)}_\mu) = 0, \quad B^{(1)}_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\kappa} H^{\nu\kappa},
\]

\[
\varepsilon^{\mu\nu\kappa\eta} \partial^\kappa B^{(1)}_\eta - (\partial^\mu B^{(1)}_\nu - \partial^\nu B^{(1)}_\mu) = 0, \quad \Box B^{(1)}_\mu = 0, \quad \Box B_{\mu\nu} = 0. \quad (5)
\]

The Lagrangian density (3) has room for further generalizations.

We have the freedom to add/subtract the 1-forms (i.e. \( F_{\mu}^{(1)} = dx^\mu \partial_\mu \phi_1 \) and \( F_{\mu}^{(2)} = dx^\mu \partial_\mu \phi_2 \)) to the gauge fixing term as well as the Hodge dual of the 3-form. These 1-forms are constructed with the massless scalar fields \( \phi_1 \) and \( \phi_2 \). The above statements can be materialized in following two different ways, namely:

\[
\mathcal{L}^{(3)} = \frac{1}{2} B^\mu B_\mu - B^\mu \left( \frac{1}{3!} \varepsilon_{\mu\nu\kappa} H^{\nu\kappa} + \frac{1}{2} \partial^\mu \phi_2 \right) + B^\mu \left( \partial^\nu B_{\nu\mu} + \frac{1}{2} \partial^\mu \phi_1 \right) - \frac{1}{2} B^\mu B_\mu, \quad (6)
\]

\[
\mathcal{L}^{(4)} = \frac{1}{2} \bar{B}^\mu \bar{B}_\mu - B^\mu \left( \frac{1}{3!} \varepsilon_{\mu\nu\kappa} H^{\nu\kappa} - \frac{1}{2} \partial^\mu \phi_2 \right) + B^\mu \left( \partial^\nu B_{\nu\mu} - \frac{1}{2} \partial^\mu \phi_1 \right) - \frac{1}{2} \bar{B}^\mu \bar{B}_\mu, \quad (7)
\]

where

\[
B_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\kappa} H^{\nu\kappa} + \frac{1}{2} \partial^\mu \phi_2, \quad B_\mu = \partial^\nu B_{\nu\mu} + \frac{1}{2} \partial^\mu \phi_1,
\]

\[
\bar{B}_\mu = \frac{1}{3!} \varepsilon_{\mu\nu\kappa} H^{\nu\kappa} - \frac{1}{2} \partial^\mu \phi_2, \quad \bar{B}_\mu = \partial^\nu B_{\nu\mu} - \frac{1}{2} \partial^\mu \phi_1. \quad (8)
\]

It should be noted that a factor of half has been taken with the scalar fields \( \phi_1 \) and \( \phi_2 \) for the algebraic convenience. The Lagrangian densities (6) and (7) are endowed with the following discrete symmetry transformations:

\[
B_{\mu\nu} \rightarrow \mp \frac{i}{2} \varepsilon_{\mu\nu\kappa} B^{\kappa}, \quad \phi_1 \rightarrow \pm i \phi_2, \quad \phi_2 \rightarrow \mp i \phi_1,
\]

\[
B_\mu \rightarrow \pm i B_\mu, \quad B_\mu \rightarrow \mp i B_\mu, \quad (9)
\]

\[
\bar{B}_{\mu\nu} \rightarrow \mp \frac{i}{2} \varepsilon_{\mu\nu\kappa} B^{\kappa}, \quad \phi_1 \rightarrow \pm i \phi_2, \quad \phi_2 \rightarrow \mp i \phi_1,
\]

\[
\bar{B}_\mu \rightarrow \pm i \bar{B}_\mu, \quad \bar{B}_\mu \rightarrow \mp i \bar{B}_\mu. \quad (10)
\]
The equations of motion (with \((\partial \cdot B) = \partial_\mu B^\mu\), etc.), that are obeyed by the fields \(B_{\mu \nu}, \phi_1, \phi_2, B_\mu, \bar{B}_\mu, \bar{B}_\mu\) of the above Lagrangian densities, are:

\[
\Box B_{\mu \nu} = 0, \quad \Box \phi_1 = 0, \quad \Box \phi_2 = 0,
\]
\[
\Box B_\mu = 0, \quad \Box \bar{B}_\mu = 0, \quad \Box \bar{B}_\mu = 0,
\]
\[
(\partial \cdot B) = 0, \quad (\partial \cdot \bar{B}) = 0, \quad (\partial \cdot \bar{B}) = 0. \tag{11}
\]

It has been shown, in our earlier work, that the gauge-fixed Lagrangian density (2) is endowed with a set of (dual-)gauge symmetry transformations when the (dual-)gauge parameters of the above transformations are restricted to obey exactly similar kind of conditions (see, e.g. [17] for details).

Before we close this section, we would like to point out that the equations in (8) imply the following relationships amongst the auxiliary fields, gauge fields and massless scalar fields:

\[
B_\mu - \bar{B}_\mu = \partial_\mu \phi_1, \quad B_\mu - \bar{B}_\mu = \partial_\mu \phi_2,
\]
\[
B_\mu + \bar{B}_\mu = 2 \partial_\nu B_{\nu \mu}, \quad B_\mu + \bar{B}_\mu = \varepsilon_{\mu \nu \eta \kappa} \partial_\nu B^{\eta \kappa}. \tag{12}
\]

We note that the auxiliary fields \(B_\mu, \bar{B}_\mu, \bar{B}_\mu, \bar{B}_\mu\) as well as the massless scalar fields \(\phi_1\) and \(\phi_2\) obey relationships that are reminiscent of the Curci-Ferrari restriction that exists in the realm of the non-Abelian 1-form gauge theory [27]. In the above, we have also used \(\frac{1}{3} \varepsilon_{\mu \nu \eta \kappa} H^{\eta \kappa} = \varepsilon_{\mu \nu \eta \kappa} \partial_\nu B^{\eta \kappa}\). It is worthwhile to mention that we have already obtained [18] the Curci-Ferrari type restriction \(B_\mu - \bar{B}_\mu = \partial_\mu \phi_1\) from the application of the superfield approach to BRST formalism for the free 4D Abelian 2-form gauge theory.

3 Absolutely anticommuting (anti-)BRST transformations

The (anti-)BRST invariant Lagrangian densities

\[
\mathcal{L}_{(B, B)} = \frac{1}{2} B \cdot B - B^\mu \left(\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \partial_\nu B^{\rho \sigma} + \frac{1}{2} \partial_\mu \phi_2\right)
\]
\[
+ B^\mu \left(\partial_\nu B_{\nu \mu} + \frac{1}{2} \partial_\mu \phi_1\right) - \frac{1}{2} B \cdot B + \partial_\mu \beta \partial_\mu \beta
\]
\[
+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda, \tag{13}
\]

\[
\mathcal{L}_{(\bar{B}, \bar{B})} = \frac{1}{2} \bar{B} \cdot \bar{B} - \bar{B}^\mu \left(\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \partial_\nu \bar{B}^{\rho \sigma} - \frac{1}{2} \partial_\mu \phi_2\right)
\]
\[
+ \bar{B}^\mu \left(\partial_\nu \bar{B}_{\nu \mu} - \frac{1}{2} \partial_\mu \phi_1\right) - \frac{1}{2} \bar{B} \cdot \bar{B} + \partial_\mu \beta \partial_\mu \beta
\]
\[
+ (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu)(\partial^\mu C^\nu) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda, \tag{14}
\]
are the generalization of the Lagrangian densities\(^3\) (6) and (7) which include the bosonic (anti-)ghost fields \((\beta)\beta\), the fermionic Lorentz vector (anti-)ghost fields \((\bar{C}_\mu)C_\mu\) and the auxiliary \((\lambda = +\frac{1}{2}(\partial \cdot C), \rho = -\frac{1}{2}(\partial \cdot \bar{C}))\) (anti-)ghost fields \((\rho)\lambda\) (with \(C_\mu^2 = \bar{C}_\mu^2 = 0, C_\mu \bar{C}_\nu = -\bar{C}_\nu C_\mu, \rho^2 = \lambda^2 = 0, \lambda \rho = -\rho \lambda\)).

It can be checked that the following off-shell nilpotent \((s^2_{(a)b} = 0)\) and anticommuting \((s_b s_{ab} + s_{ab} s_b = 0)\) (anti-)BRST\(^4\) transformations \(s_{(a)b} :\)

\[
\begin{align*}
 s_b B_{\mu\nu} &= -(\partial_\mu C_\nu - \partial_\nu C_\mu), \quad s_b C_\mu = -\partial_\mu \beta, \quad s_b \bar{C}_\mu = -B_\mu, \\
 s_b \phi_1 &= -2\lambda, \quad s_b \bar{\beta} = -\rho, \quad s_b [\rho, \lambda, \beta, \phi_2, B_\mu, B_\mu, H_{\mu\nu\kappa}] = 0, \quad (15)
\end{align*}
\]

\[
\begin{align*}
 s_{ab} B_{\mu\nu} &= -(\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \quad s_{ab} \bar{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_{ab} C_\mu = \bar{B}_\mu, \\
 s_{ab} \phi_1 &= -2\rho, \quad s_{ab} \bar{\beta} = -\lambda, \quad s_{ab}[\rho, \lambda, \bar{\beta}, \phi_2, \bar{B}_\mu, \bar{B}_\mu, H_{\mu\nu\kappa}] = 0, \quad (16)
\end{align*}
\]

are the symmetry transformations for the Lagrangian densities (13) and (14), respectively, because the following relationships:

\[
\begin{align*}
 s_b \mathcal{L}_{(B,B)} &= -(\partial_\mu (\partial^\nu C^\mu - \partial^\nu C^\mu) B_\nu + \rho (\partial^\mu \beta + \lambda B_\mu), \\
 s_{ab} \mathcal{L}_{(B,B)} &= -(\partial_\mu (\partial^\nu \bar{C}^\mu - \partial^\nu \bar{C}^\mu) \bar{B}_\nu - \rho \bar{B}_\mu + \lambda \partial^\mu \bar{\beta}), \quad (17)
\end{align*}
\]

ensure that (13) and (14) transform to the total spacetime derivatives.

Interestingly, the following explicit forms of (13) and (14), namely;

\[
\begin{align*}
 \mathcal{L}_{(B,B)} &= \frac{1}{2} B_\mu B^\mu - B^\mu \left(\frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} \partial^\nu B^{\rho\kappa} + \frac{1}{2} \partial_\mu \phi_2\right) \\
 &\quad + s_b \left[-\bar{C}^\mu \left((\partial^\nu B_{\nu\mu} + \frac{1}{2} \partial_\mu \phi_1) - \frac{1}{2} B_\mu\right) + \bar{\beta}(\partial \cdot C - 2\lambda)\right], \quad (18)
\end{align*}
\]

\[
\begin{align*}
 \mathcal{L}_{(B,B)} &= \frac{1}{2} \bar{B}_\mu \bar{B}^\mu - \bar{B}^\mu \left(\frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} \partial^\nu \bar{B}^{\rho\kappa} - \frac{1}{2} \partial_\mu \phi_2\right) \\
 &\quad + s_{ab} \left[+C^\mu \left((\partial^\nu B_{\nu\mu} - \frac{1}{2} \partial_\mu \phi_1) - \frac{1}{2} \bar{B}_\mu\right) + \beta(\partial \cdot \bar{C} + 2\rho)\right], \quad (19)
\end{align*}
\]

demonstrate the BRST and anti-BRST invariance of (13) and (14) in a very simple manner. This is due to the nilpotency of the (anti-)BRST transformations (i.e. \(s^2_{(a)b} = 0\) and the fact that \(s_{(a)b}[B_\mu, \bar{B}_\mu, \phi_2, \varepsilon_{\mu\nu\rho\kappa} \partial^\nu B^{\rho\kappa}] = 0\)). It
is worthwhile to point out that the (anti-)BRST symmetry transformations (15) and (16) are absolutely anticommuting as it can be seen that

\[(s_b s_{ab} + s_{ab} s_b) B_{\mu \nu} \equiv \{ s_b, s_{ab} \} B_{\mu \nu} = \partial_\mu (B_\nu - \bar{B}_\nu) - \partial_\nu (B_\mu - \bar{B}_\mu) = 0, \] (20)

is automatically satisfied because of the Curci-Ferrari type of restriction (12) (where \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1 \)). All the rest of the fields of the theory respect the anticommuting property as can be checked explicitly by exploiting the transformations (15) and (16) supplemented with the following inputs:

\[s_b \bar{B}_\mu = - \partial_\mu \lambda, \quad s_{ab} B_\mu = \partial_\mu \rho, \quad s_b \bar{B}_\mu = 0, \quad s_{ab} B_\mu = 0. \] (21)

The set of nilpotent transformations (15), (16) and (21) implies that the anticommutativity property (i.e. \( \{ s_b, s_{ab} \} = 0 \)) is clearly satisfied.

Exploiting the Noether’s theorem, the following conserved currents

\[J^\mu_{(b)} \equiv (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\nu \beta - (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu - \lambda B^\mu - \rho \partial^\mu \bar{\beta} - \varepsilon_{\mu \nu \rho \kappa} (\partial_\nu C_\kappa) B_\kappa; \] (22)

\[J^\mu_{(ab)} \equiv \rho \bar{B}^\mu - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{B}_\nu - (\partial^\mu C^\nu - \partial^\nu C^\mu) \partial_\nu \bar{\beta} - \lambda \partial^\mu \bar{\beta} - \varepsilon_{\mu \nu \rho \kappa} (\partial_\nu \bar{C}_\kappa) \bar{B}_\kappa, \] (23)

are derived from the Lagrangian densities (13) and (14), respectively. Their conservation law can be proven by using the following equations of motion

\[
\begin{align*}
\lambda &= \frac{1}{2} (\partial \cdot C), \quad \rho = -\frac{1}{2} (\partial \cdot \bar{C}), \quad \Box \beta = 0, \quad \Box \bar{\beta} = 0, \quad \Box \lambda = 0, \\
\Box \bar{C}_\mu &= \frac{1}{2} \partial_\mu (\partial \cdot \bar{C}) \equiv - \partial_\mu \rho, \quad \Box C_\mu &= \frac{1}{2} \partial_\mu (\partial \cdot C) \equiv \partial_\mu \lambda, \quad \Box \rho = 0, \\
\varepsilon_{\mu \nu \eta \kappa} \partial^\eta B^\kappa + (\partial_\nu B_\rho - \partial_\rho B_\nu) \equiv \varepsilon_{\mu \nu \eta \kappa} \partial^\eta B^\kappa - (\partial_\nu B_\rho - \partial_\rho B_\nu) = 0, \\
\varepsilon_{\mu \nu \eta \kappa} \partial^\eta \bar{B}^\kappa + (\partial_\nu \bar{B}_\rho - \partial_\rho \bar{B}_\nu) \equiv \varepsilon_{\mu \nu \eta \kappa} \partial^\eta \bar{B}^\kappa - (\partial_\nu \bar{B}_\rho - \partial_\rho \bar{B}_\nu) = 0.
\end{align*}
\] (24)

\[\text{in addition to the equations enumerated in (8), (11) and (12).}\]

\[\text{Exploiting the transformations (15), (16) and (21), it can be checked that } s_{ab} L_{(B, \bar{B})} = -\partial_\mu [(\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu - \rho (B^\nu + \partial_\nu B^\mu)] + \lambda \partial^\mu \bar{\beta} + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\mu (B_\nu - \bar{B}_\nu) - (\partial^\mu \rho)(B_\mu + 2B_\mu - \frac{1}{2} \partial_\mu \phi_1) \text{ and } s_b L_{(B, \bar{B})} = -\partial_\mu [(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu + \lambda (B^\mu + \partial_\mu B^\nu) + \rho \partial^\mu \beta - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\mu (B_\nu - \bar{B}_\nu) + (\partial^\mu \lambda)(B_\mu + 2B_\mu + \frac{1}{2} \partial_\mu \phi_1)]. \] (25)

Due to the equations (12), it can be checked that the Lagrangian densities (13) and (14) remain invariant under anti-BRST and BRST transformations (16) and (15), respectively, on the constrained surface where \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1, \quad \bar{B}_\mu + 2B_\mu - \frac{1}{2} \partial_\mu \phi_1 = 3 \partial^\nu B_{\nu \mu}, \quad B_\mu + 2B_\mu + \frac{1}{2} \partial_\mu \phi_1 = 3 \partial^\nu B_{\nu \mu}. \)
The expression for the conserved (anti-)BRST charges are as follows:

\[
Q_{ab} = \int d^3x \left[ \bar{\rho} \tilde{B}^0 - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \tilde{B}_i - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \partial_i \bar{\beta} - \lambda \tilde{B}^0 - \varepsilon \delta^k \partial_i \tilde{C}_j \tilde{B}_k \right],
\]

\[\text{(25)}\]

\[
Q_b = \int d^3x \left[ (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \partial_i \beta - (\partial^0 C^i - \partial^i C^0) B_i - \lambda B^0 - \rho \partial^0 \beta - \varepsilon \delta^k \partial_i C_j \beta_k \right].
\]

\[\text{(26)}\]

The above charges generate the (anti-)BRST transformations (16) and (15) as can be checked from the following relationship

\[
s_r \Phi = -i \left[ \Phi, Q_r \right]_{[(\pm)]}, \quad r = b, ab,
\]

\[\text{(27)}\]

where the \([(\pm)]\) signs, as the subscript on the square bracket, correspond to the (anti)commutator for the generic field \(\Phi\) of the Lagrangian densities (13) and (14) being (fermionic) bosonic in nature.

The following algebraic structure

\[
s_b Q_b = -i \left\{ Q_b, Q_b \right\} = 0, \quad s_{ab} Q_{ab} = -i \left\{ Q_{ab}, Q_{ab} \right\} = 0,
\]

\[
s_b Q_{ab} = -i \left\{ Q_{ab}, Q_b \right\} = 0, \quad s_{ab} Q_{b} = -i \left\{ Q_b, Q_{ab} \right\} = 0,
\]

\[\text{(28)}\]

is derived from the transformations (15) and (16) when we exploit the expressions \(Q_{(a)b}\) from (25) and (26) and use the definition of the generator from (27). The proof of \(\left\{ Q_b, Q_{ab} \right\} = 0\), from the above equation (28), is a bit more involved. Some of the key algebraic steps are given in our Appendix A.

4 Absolutely anticommuting (anti-)co-BRST transformations

In this section, we shall show that, in addition to the transformations (15) and (16), there are other fermionic type symmetry transformations for the Lagrangian densities (13) and (14), under which, the total gauge-fixing term remains invariant. These (anti-)co-BRST symmetry transformations (i.e. \(s_{(a)d}\)) are off-shell nilpotent \((s^{2}_{(a)d} = 0)\) as can seen from the following:

\[
s_d B_{\mu \nu} = -\varepsilon_{\mu \nu \rho \kappa} \partial^\rho \tilde{C}^\kappa, \quad s_d \tilde{C}_\mu = -\partial_\mu \bar{\beta}, \quad s_d C_\mu = -\tilde{B}_\mu,
\]

\[
s_d \phi_2 = 2 \rho, \quad s_d \beta = -\lambda, \quad s_d \left[ \rho, \lambda, \beta, \bar{\phi}_1, B_\mu, \tilde{B}_\mu, \partial^\nu B_{\nu \mu} \right] = 0,
\]

\[\text{(29)}\]

\[
s_{ad} B_{\mu \nu} = -\varepsilon_{\mu \nu \rho \kappa} \partial^\rho C^\kappa, \quad s_{ad} C_\mu = \partial_\mu \beta, \quad s_{ad} \tilde{C}_\mu = \tilde{B}_\mu,
\]

\[
s_{ad} \phi_2 = 2 \lambda, \quad s_{ad} \bar{\beta} = \rho, \quad s_{ad} \left[ \rho, \lambda, \beta, \bar{\phi}_1, \tilde{B}_\mu, \tilde{B}_\mu, \partial^\nu B_{\nu \mu} \right] = 0.
\]

\[\text{(30)}\]
The above transformations are the *symmetry* transformations as is evident from the following forms of the change in the Lagrangian densities:

\[
\begin{align*}
    s_d \mathcal{L}_{(B,B)} &= \partial_\mu \left[ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu - \lambda \partial^\mu \bar{\beta} - \rho B^\mu \right], \\
    s_{ad} \mathcal{L}_{(B,B)} &= \partial_\mu \left[ (\partial^\mu C^\nu - \partial^\nu C^\mu) \bar{B}_\nu + \rho \partial^\mu \beta + \lambda \bar{B}^\mu \right].
\end{align*}
\] (31)

The above expressions show that the Lagrangian densities (13) and (14) are quasi-invariant under the transformations (29) and (30).

The Noether’s conserved currents, corresponding to the nilpotent and continuous symmetry transformations (29) and (30), are as follows:

\[
\begin{align*}
    J^\mu_{(d)} &= (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) B_\nu - (\partial^\mu C^\nu - \partial^\nu C^\mu) \partial_\nu \bar{\beta} - \rho B^\mu \\
    &\quad - \lambda \partial^\mu \bar{\beta} - \varepsilon^{\mu\nu\eta\kappa} (\partial_\nu \bar{C}_\eta) B_\kappa, \\
    J^\mu_{(ad)} &= (\partial^\mu C^\nu - \partial^\nu C^\mu) \bar{B}_\nu - (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\nu \beta + \lambda \bar{B}^\mu \\
    &\quad + \rho \partial^\mu \beta - \varepsilon^{\mu\nu\eta\kappa} (\partial_\nu C_\eta) \bar{B}_\kappa.
\end{align*}
\] (32, 33)

The conservation law (i.e. \( \partial_\mu J^\mu_{(a)d} = 0 \)) can be proven by exploiting the equations of motion (8), (11), (12) and (24).

The generators of the above nilpotent transformations (29) and (30) can be calculated from the above conserved currents as

\[
\begin{align*}
    Q_d &= \int d^3x J^0_{(d)} = \int d^3x \left[ (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) B_i - (\partial^0 C^i - \partial^i C^0) \partial_i \bar{\beta} \right. \\
    &\quad \left. - \rho B^0 - \lambda \partial^0 \bar{\beta} - \varepsilon^{ijk} (\partial_i \bar{C}_j) B_k \right], \\
    Q_{ad} &= \int d^3x J^0_{(ad)} = \int d^3x \left[ (\partial^0 C^i - \partial^i C^0) \bar{B}_i - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \beta \right. \\
    &\quad \left. + \lambda \bar{B}^0 + \rho \partial^0 \beta - \varepsilon^{ijk} (\partial_i C_j) \bar{B}_k \right].
\end{align*}
\] (34, 35)

We christen the above conserved charges as the co-BRST and anti-co-BRST.

Exploiting the canonical brackets, associated with the Lagrangian densities (13) and (14), it can be shown that \( Q^2_{(a)d} = 0 \) and \( \{Q_d, Q_{ad}\} = 0 \). However, it can be checked that the following relationships are true, namely;

\[
\begin{align*}
    s_d Q_d &= -i \{Q_d, Q_d\} = 0, \\
    s_{ad} Q_{ad} &= -i \{Q_{ad}, Q_{ad}\} = 0, \\
    s_d Q_{ad} &= -i \{Q_{ad}, Q_d\} = 0, \\
    s_{ad} Q_d &= -i \{Q_d, Q_{ad}\} = 0.
\end{align*}
\] (36)

We christen the above conserved charges as the co-BRST and anti-co-BRST.
due to the identification of (35) and (34) as the generators of the nilpotent (anti-)dual-BRST symmetry transformations (30) and (29).

Before we close this section, a few side remarks are in order. First, the anticommutativity of the transformations $s_{(a)d}$ on the gauge field

$$\{s_d, s_{ad}\} B_{\mu\nu} = \varepsilon_{\mu\nu\rho\kappa} \partial^\rho (B^\kappa - \bar{B}^\kappa) = 0,$$  \hspace{1cm} (37)

is satisfied on the constrained surface defined by the field equation $B_\mu - \bar{B}_\mu = \partial_\mu \phi_2$ given in (12). Second, it can be checked explicitly that the transformations (29) and (30) are anticommuting on the rest of the fields of the theory if we include the following transformations on the auxiliary fields:

$$s_d \bar{B}_\mu = \partial_\mu \rho, \quad s_{ad} B_\mu = -\partial_\mu \lambda, \quad s_{ad} B_\mu = 0, \quad s_d \bar{B}_\mu = 0,$$  \hspace{1cm} (38)

in addition to (29) and (30). Finally, in the proof of $s_d Q_{ad} = -i \{Q_{ad}, Q_d\} = 0$ as well as $s_{ad} Q_d = -i \{Q_d, Q_{ad}\} = 0$, one has to exploit both the constrained field equations $B_\mu - \bar{B}_\mu = \partial_\mu \phi_2$ as well as $B_\mu - \bar{B}_\mu = \partial_\mu \phi_2$ when we compute the explicit expressions $s_d Q_{ad}$ and $s_{ad} Q_d$ by using (29), (30), (34) and (35). One such computation, in a concise manner, is illustrated in our Appendix B.

5 Bosonic symmetries: analogue of the Laplacian operator

It is clear, from the previous sections, that we have four nilpotent symmetry transformations (i.e. $s_{(a)b}, s_{(a)d}$) in our present theory. We have also shown that the following anticommutators are true, namely;

$$\{s_b, s_{ab}\} = 0, \quad \{s_d, s_{ad}\} = 0, \quad \{s_b, s_{ad}\} = 0, \quad \{s_d, s_{ab}\} = 0,$$  \hspace{1cm} (39)

on the constrained surface defined by the field equations in (12).

At this juncture, it is very natural to expect that the remaining non-zero pair of the anticommutators, as listed below:

$$s_\omega = \{s_b, s_d\}, \quad s_\varpi = \{s_{ab}, s_{ad}\},$$  \hspace{1cm} (40)

would correspond to the bosonic symmetry transformations in the theory because $s_{(b)d}$ and $s_{(b)b}$ are individually nilpotent (fermionic) symmetry transformations. A close look at (17) and (31) certify the above assertions.

---

Footnote 6: Using the nilpotent transformations (29), (30) and (38), it can be readily verified that $s_{ad} \mathcal{L}_{(B,B)} = \partial_\mu[\left(\partial^\mu C^\nu - \partial^\nu C^\mu\right)\bar{B}_\mu + \rho \partial^\mu \beta + \lambda (B^\mu + \frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} \partial_\rho B_\kappa)] - (\partial^\mu C^\nu - \partial^\nu C^\mu) \partial_\mu (B_\mu - \bar{B}_\mu) - (\partial^\mu \lambda) (\bar{B}_\mu + 2B_\mu - \frac{1}{2} \partial_\nu \phi_2)$ and $s_d \mathcal{L}_{(B,B)} = \partial_\mu[\left(\partial^\mu C^\nu - \partial^\nu C^\mu\right)\bar{B}_\mu + \rho \partial^\mu \beta + \lambda (B^\mu + \frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} \partial_\rho B_\kappa) - \lambda \partial^\mu \beta] + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\mu (B_\mu - \bar{B}_\mu) + (\partial^\mu \rho)(B_\mu + 2\bar{B}_\mu + \frac{1}{2} \partial_\nu \phi_2)$. Due to the relations (12), it can be verified that the Lagrangian densities (13) and (14) remain invariant under the transformations (30) and (29), respectively, on the constrained surface where $B_\mu - \bar{B}_\mu = \partial_\mu \phi_2$, $\bar{B}_\mu + 2B_\mu - \frac{1}{2} \partial_\nu \phi_2 = \frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} \partial^\rho B_\kappa$, $B_\mu + 2\bar{B}_\mu + \frac{1}{2} \partial_\nu \phi_2 = \frac{1}{2} \varepsilon_{\mu\nu\rho\kappa} \partial^\rho \bar{B}^\kappa$. 

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The following bosonic transformations \( s_\omega = \{s_b, s_d\} \):

\[
\begin{align*}
    s_\omega B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu + \varepsilon_{\mu\nu\kappa} \partial^\kappa B^\lambda, \\
    s_\omega C_\mu &= \partial_\mu \lambda, \\
    s_\omega \bar{C}_\mu &= \partial_\mu \rho, \\
    s_\omega \left[ \phi_1, \phi_2, \beta, \bar{\beta}, \lambda, \rho, B_\mu, \bar{B}_\mu, \mathcal{B}_\mu, \bar{\mathcal{B}}_\mu \right] &= 0,
\end{align*}
\]  

are the symmetry transformations of the Lagrangian density (13) because:

\[
\begin{align*}
    s_\omega \mathcal{L}_{(B, B)} &= \partial_\mu \left[ \mathcal{B}^\mu (\partial \cdot \mathcal{B}) - \bar{B}^\mu (\partial \cdot \bar{B}) + B^\nu \partial^\mu \bar{B}_\nu - \bar{B}^\nu \partial^\mu B_\nu \\
    &\quad + (\partial^\mu \lambda) \rho - \lambda (\partial^\mu \rho) \right] + \lambda (\partial^\mu \rho) - (\partial^\mu \lambda) \rho \right],
\end{align*}
\]

shows that the Lagrangian density \( \mathcal{L}_{(B, B)} \) remains quasi-invariant under (41).

Exactly, in the above manner, it can be checked that following bosonic infinitesimal transformations \( s_\varphi = \{s_{ab}, s_{ad}\} \):

\[
\begin{align*}
    s_\varphi B_{\mu\nu} &= -(\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu + \varepsilon_{\mu\nu\kappa} \partial^\kappa \bar{B}^\lambda), \\
    s_\varphi C_\mu &= -\partial_\mu \lambda, \\
    s_\varphi \bar{C}_\mu &= -\partial_\mu \rho, \\
    s_\varphi \left[ \phi_1, \phi_2, \beta, \bar{\beta}, \lambda, \rho, B_\mu, \bar{B}_\mu, \mathcal{B}_\mu, \bar{\mathcal{B}}_\mu \right] &= 0,
\end{align*}
\]

leave the Lagrangian density (14) (i.e. \( \mathcal{L}_{(B, B)} \)) quasi-invariant as the latter transforms in the following fashion:

\[
\begin{align*}
    s_\varphi \mathcal{L}_{(B, B)} &= \partial_\mu \left[ \bar{B}^\mu (\partial \cdot \bar{B}) - \bar{B}^\mu (\partial \cdot \bar{B}) + \bar{B}^\nu \partial^\mu \bar{B}_\nu - \bar{B}^\nu \partial^\mu B_\nu \\
    &\quad + \lambda (\partial^\mu \rho) - (\partial^\mu \lambda) \rho \right] + \lambda (\partial^\mu \rho) - (\partial^\mu \lambda) \rho \right].
\end{align*}
\]

Thus, equations (42) and (44) imply that \( s_\omega \) and \( s_\varphi \) are the symmetry transformations for the Lagrangian densities (13) and (14), respectively. These symmetry transformations owe their origin to the four basic fermionic (anti-)BRST and (anti-)co-BRST symmetry transformations of the theory.

On their face value, the transformations (41) and (43) look completely independent. However, a close observation of (41) and (43), using the constrained field equations (12), reveal that they differ only by a sign factor. To be specific, using \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1, B_\mu - \bar{B}_\mu = \partial_\mu \phi_2 \), it can be seen that \( s_\omega + s_\varphi = 0 \). Thus, there is nothing profound in the observation that \([s_\omega, s_\varphi] \Phi = 0\) for the generic field \( \Phi \) of the Lagrangian densities (13) and (14). It is a sheer coincidence that in (41) and (43), we observe that \( s_\omega^2 = 0 \) and \( s_\varphi^2 = 0 \). Strictly speaking, however, these transformations are bosonic.

The following Noether conserved current emerges when we exploit the continuous symmetry transformations (41):

\[
\begin{align*}
    J^\mu_\omega &= \varepsilon^{\mu\nu\kappa} \left\{ (\partial_\kappa B_\eta) \mathcal{B}_\nu + (\partial_\nu B_\eta) \bar{B}_\kappa \right\} + \partial_\nu \left[ B^\mu \mathcal{B}^\nu - \bar{B}^\mu \bar{B}^\nu \right] \\
    &\quad + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \partial_\nu \lambda - \partial^\mu (\partial^\nu \bar{C}^\mu - \partial^\nu C^\mu) \partial_\nu \rho.
\end{align*}
\]
The conservation law (i.e. $\partial_\mu J^\mu_\omega = 0$) of the above current can be proven by exploiting the equations of motion (8), (11), (12) and (24).

The conserved charge, corresponding to the above conserved current, is

$$W = \int d^3 x J^0_\omega \equiv \int d^3 x \left[ \epsilon^{ijk} \left( (\partial_i B_j) B_k + (\partial_i B_j) B_k + (\partial^0 C^i - \partial^i C^0) \partial_\lambda \right. \right.$$ 

$$ \left. - (\partial^i C^i - \partial^i C^0) \partial_\rho \right].$$

(46)

There are other ways to compute this conserved charge. For instance, it can be checked that $s_b Q_d = -i\{Q_d, Q_b\}, s_d Q_b = -i\{Q_b, Q_d\}$ can be used to deduce the expression for $W$. Similarly the expressions $s_{ab} Q_{ad} = -i\{Q_{ad}, Q_{ab}\}$ and $s_{ad} Q_{ab} = -i\{Q_{ab}, Q_{ad}\}$ lead to the derivation of $W$. Some subtlety of these computations are discussed briefly in our Appendix C.

6 Ghost and discrete symmetry transformations: the ghost charge

It will be noted that the ghost part of the Lagrangian densities (13) and (14)

$$\mathcal{L}_g = \partial_\mu \bar{\beta} \partial^\mu \beta + (\partial_\mu C_{\nu} - \partial_\nu C_{\mu}) (\partial^\mu C^{\nu}) + (\partial \cdot C - \lambda) \rho + (\partial \cdot \bar{C} + \rho) \lambda,$$

(47)

respects the following continuous global ($\Sigma \neq \Sigma(x)$) scale symmetry transformations for the ghost fields of the theory, namely;

$$C_\mu \rightarrow e^{+\Sigma} C_\mu, \quad \bar{C}_\mu \rightarrow e^{-\Sigma} \bar{C}_\mu, \quad \beta \rightarrow e^{+2\Sigma} \beta, \quad \bar{\beta} \rightarrow e^{-2\Sigma} \bar{\beta}, \quad \rho \rightarrow e^{-\Sigma} \rho, \quad \lambda \rightarrow e^{+\Sigma} \lambda,$$

(48)

where numbers ($\pm 1$) and ($\pm 2$), in the exponentials, stand for the ghost numbers of the corresponding (anti-)ghost fields. It is evident that $\lambda$ and $\rho$ have the ghost number (+1) and (-1), respectively, because of the fact that $\lambda = +\frac{1}{2}(\partial \cdot C), \rho = -\frac{1}{2}(\partial \cdot \bar{C})$. Furthermore, the ghost number for the rest of the fields of the theory (i.e. $B_{\mu \nu}, B_{\mu}, \bar{B}_{\mu}, B_{\mu}, \bar{B}_{\mu}, \phi_1, \phi_2$) is zero. Thus, under ghost symmetry transformations: $B_{\mu \nu} \rightarrow B_{\mu \nu}, B_{\mu} \rightarrow B_{\mu}, B_{\mu}, \bar{B}_{\mu} \rightarrow B_{\mu}, \bar{B}_{\mu}, \phi_1 \rightarrow \phi_1, \phi_2 \rightarrow \phi_2$.

The infinitesimal version (i.e. $\Sigma \rightarrow 0$) of the above global scale transformations (i.e. $s_g$) is as given below:

$$s_g C_\mu = +\Sigma C_\mu, \quad s_g \bar{C}_\mu = -\Sigma \bar{C}_\mu, \quad s_g \rho = -\Sigma \rho,$$

$$s_g \lambda = +\Sigma \lambda, \quad s_g \beta = +2\Sigma \beta, \quad s_g \bar{\beta} = -2\Sigma \bar{\beta}.$$ 

(49)

The above symmetry transformations lead to the derivation of the conserved Noether current (i.e. the ghost current) as:

$$J^\mu_\omega = 2\beta \partial^\mu \bar{\beta} - 2\bar{\beta} \partial^\mu \beta + (\partial^\mu C^{\nu} - \partial^\nu C^\mu) \bar{C}_\nu$$

$$+ (\partial^\mu \bar{C}^{\nu} - \partial^\nu \bar{C}^\mu) C_\nu + C^\mu \rho - \bar{C}^\mu \lambda.$$

(50)
The conservation law \( \partial_{\mu} J_{(g)}^{\mu} = 0 \) can be readily proven by exploiting the equations of motion for the (anti-)ghost fields from (24).

The generator of the infinitesimal transformations (49) is the conserved (i.e. \( \dot{Q}_{(g)} = 0 \)) ghost charge \( Q_g \) defined by the following expression:

\[
Q_g = \int d^3 x J_{(g)}^0 = \int d^3 x \left[ 2\beta \partial^0 \bar{\beta} - 2\bar{\beta} \partial^0 \beta + (\partial^0 C^i - \partial^i C^0) \bar{C}_i + (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) C_i + C^0 \rho - \bar{C}^0 \lambda \right].
\]

Exploiting the infinitesimal transformations (15), (16), (29), (30), (41) and (49) (with \( \Sigma = 1 \)), the following algebraic structure can be deduced:

\[
s^2_{(a)d} = 0, \quad s^2_{(a)b} = 0, \quad \{s_b, s_{ab}\} = 0, \quad \{s_d, s_{ad}\} = 0, \quad \{s_d, s_b\} = s_\omega,
\]

\[
\{s_{ad}, s_{ab}\} = s_\omega \equiv -s_\omega, \quad [s_\omega, s_r] = 0, \quad r = b, ab, d, ad, g,
\]

\[
[s_g, s_b] = +s_b, \quad [s_g, s_d] = -s_d, \quad [s_g, s_{ab}] = -s_{ab}, \quad [s_g, s_{ad}] = +s_{ad}.
\]

All the rest of the (anti)commutators of the above infinitesimal transformations (e.g. \([s_g, s_d] = 0\), etc.) are found to be trivially zero.

In addition to the continuous symmetry transformations (49), the ghost part of the Lagrangian density (i.e. the equation (47)) respects the following discrete symmetry transformations:

\[
C_\mu \rightarrow \pm i\bar{C}_\mu, \quad \bar{C}_\mu \rightarrow \pm iC_\mu, \quad \beta \rightarrow \pm i\bar{\beta},
\]

\[
\bar{\beta} \rightarrow \mp i\beta, \quad \rho \rightarrow \mp i\lambda, \quad \lambda \rightarrow \mp i\rho.
\]

Thus, we note that, under the discrete symmetry transformations (9), (10) and (53), the total Lagrangian densities (13) and (14) remain invariant.

The discrete symmetry transformations (9), (10) and (53), combined together, correspond to the Hodge duality \( \ast \) operation of differential geometry. To corroborate this assertion, it is essential to note that a pair of above discrete transformations, on the bosonic (B) and fermionic (F) fields of the theory, lead to the following expressions [28]:

\[
*(*B) = +B, \quad B = B_{\mu\nu}, B_\mu, \bar{B}_\mu, B_\mu, \bar{B}_\mu, \phi_1, \phi_2, \beta, \bar{\beta},
\]

\[
*(*F) = -F, \quad F = C_\mu, \bar{C}_\mu, \rho, \lambda.
\]

The above signs are important for our purpose because it can be seen that, in the following relationships [28] (see, e.g. [28] for details)

\[
\quad s_{(a)d} \Phi = \pm * s_{(a)b} * \Phi, \quad \Phi = B, F,
\]

\[
\text{It can be checked that the relation } s_{(a)b} \Phi = \mp * s_{(a)d} * \Phi \text{ (with } \Phi = B, F \text{) is also true. Here } B \text{ and } F \text{ are defined in (54). The sign-flip on the r.h.s. is due to the dimensionality of the spacetime manifold on which the fields of the theory are defined (see, e.g. [28]).}
\]

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the (+)- signs are dictated by the signs in (54). The above relationship is the analogue of the relationship \( \delta = \pm \ast d\ast \) that exists between the exterior and co-exterior derivatives \( d \) and \( \delta \) of differential geometry.

It is worthwhile to point out that, in the realm of differential geometry on a compact manifold without a boundary, the signature in the relationship \( \delta = \pm \ast d\ast \) is decided by the dimension of the manifold and the degree of the differential forms that are involved in the inner product. For instance, for the even dimensional compact manifold \( \delta = - \ast d\ast \) is always true (see, e.g. [3,4] for details). In the realm of BRST formalism for the 4D free Abelian 2-form gauge theory, the signature in (55) is dictated by (54).

Before we close this section, it is interesting to note that the application of the discrete symmetry transformations (9), (10) and (53) (i.e. the analogue of the \( \ast \) operation) on the conserved charges is

\[
\ast Q_b = Q_d, \quad \ast Q_d = -Q_b, \quad \ast W = -W, \\
\ast Q_{ad} = -Q_{ab}, \quad \ast Q_{ab} = Q_{ad}, \quad \ast Q_g = -Q_g.
\] (56)

The above equation shows that, under the discrete symmetry transformations of the theory, \( Q_b \to Q_d, Q_d \to -Q_b \) (and \( Q_{ab} \to Q_{ad}, Q_{ad} \to -Q_{ab} \)) which are exactly like the electromagnetic duality (i.e. \( \vec{E} \to \vec{B}, \vec{B} \to -\vec{E} \)) that exists for the Maxwell’s source free field equations.

7 Algebraic structures: cohomological aspects

The algebra obeyed by the transformations \( s_r \) (with \( r = b, ab, d, ad, \omega, g \)) is replicated by the generators of these transformations. Exploiting the canonical (anti)commutators, derived from the Lagrangian densities (13) and (14), it can be shown that the following algebraic structure is true, namely;

\[
Q^2_{(a)b} = 0, \quad Q^2_{(a)d} = 0, \quad [W, Q_r] = 0, \quad (r = b, ab, d, ad, g), \\
\{Q_b, Q_{ab}\} = 0, \quad \{Q_d, Q_{ad}\} = 0, \quad \{Q_b, Q_{ad}\} = 0, \\
\{Q_d, Q_b\} = -\{Q_{ad}, Q_{ab}\} = W, \quad \{Q_d, Q_{ab}\} = 0, \\
i[Q_g, Q_b] = +Q_b, \quad i[Q_g, Q_{ab}] = -Q_{ab}, \\
i[Q_g, Q_d] = -Q_d, \quad i[Q_g, Q_{ad}] = +Q_{ad}.
\] (57)

This is the extended BRST algebra corresponding to our present 4D Abelian 2-form gauge theory which is endowed with six symmetry transformations.

We can define the ghost number of a state (in the quantum Hilbert space of states) as the eigen value of the operator \( iQ_g \). In other words, a state \( |\psi> = n|\psi> \) has the ghost number \( n \). As a result of the
served charges (corresponding to the symmetries of the 2-form theory) to the namely; 
\(Q\) cohomological operators (of differential geometry on the compact manifolds),
\(d\) respectively. As is evident from (58), the set \(Q\) and \(Q_{ad}\) have ghost
numbers \((n + 1)\) and \((n - 1)\), respectively.

The structure of the algebra in (57) and the relationship in (58) demonstrate that there are two sets of generators of transformations that correspond to the de Rham cohomological differential operators \(d, \delta, \Delta\). For instance, the first set \(Q_b, Q_d, W\) and the second set \(Q_{ad}, Q_{ab}, -W\) obey exactly the same kind of algebra as \((d, \delta, \Delta)\) which is: \(d^2 = \delta^2 = 0, \Delta = \{d, \delta\} = (d + \delta)^2, [\Delta, d] = 0, [\Delta, \delta] = 0\). Thus, the mapping is two-to-one from the conserved charges (corresponding to the symmetries of the 2-form theory) to the cohomological operators (of differential geometry on the compact manifolds), namely; \((Q_b, Q_{ad}) \rightarrow d, (Q_d, Q_{ab}) \rightarrow \delta\) and \((+W, -W) \rightarrow \Delta\).

It is well-known that the exterior derivative raises the degree of a form by one when it operates on it. On the other hand, the dual-exterior derivative lowers the degree of a form by one due to its action on the latter. These properties of \(d\) and \(\delta\) are mimicked by sets \((Q_b, Q_{ad})\) and \((Q_d, Q_{ab})\), respectively. As is evident from (58), the set \((Q_b, Q_{ad})\) raises the ghost number of a state by one and the set \((Q_d, Q_{ab})\) lowers the ghost number of the same state by one. Furthermore, it is an important point to note that \(Q_b\) and \(Q_{ab}\) are independent of each-other (i.e. \(\{Q_b, Q_{ab}\} = 0\)) as are \(Q_d\) and \(Q_{ad}\) because of \(\{Q_d, Q_{ad}\} = 0\). These observations enable us to express any arbitrary state \(|\psi >_n\) due to the Hodge decomposition theorem (HDT) \([3-6]\), as follows
\[
|\psi >_n = |\omega >_{(n)} + Q_b |\chi >_{(n-1)} + Q_d |\theta >_{(n+1)}
\equiv |\omega >_{(n)} + Q_{ad} |\chi >_{(n-1)} + Q_{ab} |\theta >_{(n+1)},
\]
(59)

where \(|\omega >_n\) is the harmonic state, \(Q_b|\chi >_{(n-1)}\) is the BRST exact state and \(Q_d|\theta >_{(n+1)}\) is the co-BRST exact state. In a similar fashion, the second line of the above equation can also be defined.

In the above, the most symmetric state is the harmonic state because it is (anti-)BRST as well as (anti-)co-BRST invariant. This is why, it is

\[\text{On a compact manifold without a boundary, any arbitrary n-form } f_n \text{ can be uniquely written as the sum of the harmonic form } h_n \text{ (with } \Delta h_n = 0, dh_n = 0, \delta h_n = 0\), an exact form \((d\epsilon_{n-1})\) and a co-exact form \((\delta\epsilon_{n+1})\). Thus, the HDT can be mathematically expressed as: } f_n = h_n + d\epsilon_{n-1} + \delta\epsilon_{n+1} \text{ where } h_n \text{ is annihilated by } d \text{ and } \delta \text{ together.}
appropriate to choose this state as the physical state of the theory. The physicality criteria (i.e. $Q_{(a)b}|\text{phys}>=0, Q_{(a)d}|\text{phys}>=0$) on the physical state $|\text{phys}>$ of the theory leads to the annihilation of the physical state by the operator form of the first-class constraints and their dual. This analysis has already been performed in our earlier works [15]. Thus, we shall not dwell on it in our present endeavour because the results are almost the same.

8 Conclusions

In our present investigation, we have demonstrated that the free 4D Abelian 2-form gauge theory is a tractable field theoretical model for the Hodge theory because all the de Rham cohomological operators of differential geometry find their physical realizations in the language of the well-defined symmetry transformations of the specific Lagrangian densities (cf. (13) and (14)) of the theory. It turns out that the total kinetic term of the gauge field, owing its origin to the exterior derivative $d = dx^\mu \partial_\mu$, remains invariant under the (anti-)BRST symmetry transformations. On the other hand, the total gauge-fixing term, owing its origin to the co-exterior derivative $\delta = \pm \ast d \ast$, is found to remain invariant under the (anti-)co-BRST symmetry transformations.

The discrete symmetry transformations (9), (10) and (53), present in our Abelian 2-form gauge theory, are found to be the realization of the Hodge duality $\ast$ operation of the differential geometry in the relationship $\delta = \pm \ast d \ast$. The interplay of the continuous symmetries and the discrete symmetries of the theory encode the above relationship in an explicit manner as is evident from our equation (55). The ($\pm$) signs of the above relationship are captured in (54) where the operation of the two successive discrete symmetry transformations on the fields of the theory, unambiguously decides it [28].

According to the Noether’s theorem, the continuous symmetry transformations lead to the conserved charges. We have six continuous symmetries in the theory which lead to six conserved charges as $Q_b, Q_{ab}, Q_d, Q_{ad}, W, Q_g$. These charges obey the algebra (57) that is reminiscent of the algebra respected by the de Rham cohomological operators of differential geometry. It turns out that, under the duality transformations (9), (10) and (53), the algebraic structure in (57) remains intact as is clear from the transformations (56). Thus, we conclude that the whole 0theory is duality invariant because (i) the Lagrangian densities (13) and (14) of the theory remain invariant under (9), (10) and (53), and (ii) the algebraic structure (57) also remains intact.

---

9Besides the role of $d$ and $\delta$, there are other specific subtleties that are also involved in our discussion of the (anti-)BRST and (anti-)co-BRST transformations (cf. Sec. 2). For instance, the massless scalar fields $\phi_2$ and $\phi_1$ also remain invariant under the (anti-)BRST and (anti-)co-BRST symmetry transformations, respectively.
invariant under the discrete symmetry transformations (9), (10) and (53).

There are significant physical implications of our present kind of studies. For instance, we have been able to demonstrate, because of the above type of studies, that the two \((1 + 1)\)-dimensional \((2D)\) free Abelian and non-Abelian gauge theories (having no interaction with matter fields) present a new type of topological field theories which capture a part of the salient features of the Witten-type of topological theories and some of the key properties of the Schwarz-type of topological theories (see, e.g. [13] for details). Furthermore, the \(2D\) interacting Abelian \(U(1)\) gauge theory (i.e. QED) presents a field theoretical model for the Hodge theory where the topological gauge field \(A_\mu\) couples with the Noether conserved current constructed with the help of Dirac fields [14]. In addition, such studies have established that the free Abelian 2-form gauge theory is a quasi-topological field theory [16].

We have established, in our very recent work [29], that the simple \(2D\) free Abelian \(U(1)\) gauge theory is a field theoretical model for the Hodge theory. In this work, we have demonstrated the usefulness of the ordinary as well as the super de Rham cohomological operators where the latter cohomological operators are defined on the \((2, 2)\)-dimensional supermanifold. We have exploited the importance of the super exterior derivative in deriving the nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations and Curci-Ferrari type restriction for the \(4D\) free Abelian gauge theory in [18]. It would be interesting venture to tap the potential of the super co-exterior derivative and super Laplacian operator, defined on the \((4, 2)\)-dimensional supermanifold, for the \(4D\) Abelian 2-form gauge theory.

It would be a challenging endeavour to capture the main features of our present investigation in the language of the Hamiltonian formalism where the constraint structure of the theory is emphasized [30-32]. The study of the topological features of the Abelian 2-form gauge theory, with the help of absolutely anticommuting (anti-)BRST as well as (anti-)co-BRST symmetry transformation, is yet another direction for further investigation. The generalization of our present results to the case of the \(4D\) non-Abelian 2-form gauge theory is a demanding problem for us. There are some interesting field theoretical models where the 2-form gauge potential appears in a compelling manner [33,34]. It would be nice to study them within the framework of the BRST formalism and look for the existence of dual-BRST type symmetry transformations. All the above issues are being investigated at the moment and our results would be reported in our future publications [35].

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Appendix A

Here we furnish some of the key steps in proving the fact that $\{Q_b, Q_{ab}\} = 0$ by exploiting the transformations (15) and the expression for $Q_{ab}$ from (25) in the computation $s_b Q_{ab} = -i \{Q_{ab}, Q_b\}$. It can be checked that

$$s_b Q_{ab} = \int d^3 x \left[ \epsilon^{ijk} (\partial_i B_j) \bar{B}_k + (\partial^0 B^i - \partial^i B^0) \bar{B}_i \right. $$

$$+ \rho \lambda - \lambda \dot{\rho} - (\partial^0 C^i - \partial^i C^0) \partial_i \lambda - (\partial^0 \bar{C}^i - \partial^i \bar{C}^{0}) \partial_i \rho \left. \right].$$

(60)

Using the constraint field equation $B_{\mu} - \bar{B}_\mu = \partial_\mu \phi_2$ from (12) and equation of motion $\epsilon^{\mu\nu\rho\sigma} \partial_\rho B_\sigma + (\partial^\mu B^\nu - \partial^\nu B^\mu) = 0$ (which implies $(\partial^0 B^i - \partial^i B^0) = -\epsilon^{ijk} \partial_j \bar{B}_k$) from (24), the first two terms of the above equation lead to

$$\int d^3 x \left[ \epsilon^{ijk} \partial_i B_j (B_k - \bar{B}_k) \right] \equiv \int d^3 x \left[ \epsilon^{ijk} \partial_i B_j (B_k - \bar{B}_k - \partial_k \phi_1) \right].$$

(61)

This expression is zero on constrained surface defined by the field equation $B_{\mu} - \bar{B}_\mu = \partial_\mu \phi_1$. The rest of the terms of (60) are as follows

$$\int d^3 x \left[ \rho \lambda - \lambda \dot{\rho} + (\partial_0 \bar{C}_i - \partial_i \bar{C}_0) \partial_i \lambda + (\partial_0 C_i - \partial_i C_0) \partial_i \rho \right].$$

(62)

Performing a partial integration and throwing away the total space derivative terms, the above equation can be recast into the following form:

$$\int d^3 x \left[ \rho \lambda - \lambda \dot{\rho} - (\partial_0 \partial_i \bar{C}_i - \partial_i \partial_i \bar{C}_0) \lambda - (\partial_0 \partial_i C_i - \partial_i \partial_i C_0) \rho \right].$$

(63)

The above equation, with the help of the equations of motion $\lambda = \frac{1}{2} (\partial \cdot \bar{C}), \rho = -\frac{1}{2} (\partial \cdot \bar{C})$ from (24), can be reduced to

$$\int d^3 x \left[ \dot{\lambda} \rho - \dot{\rho} \lambda - (\Box \bar{C}_0) \lambda - (\Box C_0) \rho \right] = 0,$$

(64)

where we have used $\Box C_0 = \dot{\lambda}$ and $\Box \bar{C}_0 = -\dot{\rho}$ from (24).

Appendix B

We very concisely provide some key inputs for the proof of $\{Q_d, Q_{ad}\} = 0$ from the computation of $s_d Q_{ad} = -i \{Q_{ad}, Q_d\}$ by exploiting the transformations (29) and expression for $Q_{ad}$ from (35). It can be seen that

$$s_d Q_{ad} = \int d^3 x \left[ \epsilon^{ijk} (\partial_i B_j) \bar{B}_k - (\partial^0 B^i - \partial^i B^0) \bar{B}_i \right. $$

$$+ \rho \lambda - \lambda \dot{\rho} - (\partial^0 C^i - \partial^i C^0) \partial_i \rho - (\partial^0 \bar{C}^i - \partial^i \bar{C}^{0}) \partial_i \lambda \left. \right].$$

(65)
The ghost part of the above expression can be easily shown to be equal to zero by exploiting the equations of motion from (24) (e.g. \( \Box C_\mu = \partial_\mu \lambda, \Box \bar{C}_\mu = -\partial_\mu \rho \) and \( \lambda = \frac{1}{2}(\partial \cdot C), \rho = -\frac{1}{2}(\partial \cdot \bar{C}) \)). Now the first two terms of (65) can be recast into the following form

\[
\int d^3x \left[ \epsilon^{ijk}(\partial_i B_j) \bar{B}_k - \epsilon^{ijk}(\partial_j B_k) \bar{B}_i \right], \quad (66)
\]

where we have exploited an appropriate equation of motion from (24) which implies that \( (\partial^0 B_i - \partial_i B^0) = \epsilon^{ijk} \partial_j B_k \). Performing a partial integration and using the constrained field equation \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1 \), the above expression can be put in the following form

\[
\int d^3x \left[ \epsilon^{ijk} (\partial_i B_j) (B_k - \bar{B}_k) \right] \equiv \int d^3x \left[ \epsilon^{ijk} (\partial_i B_j) (B_k - \bar{B}_k - \partial_k \phi_2) \right], \quad (67)
\]

which reduces to zero on the constrained surface parametrized by the field equation \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_2 \) from (12). Thus, we note that equations (12) and (24) play important roles in the proof of \( \{Q_d, Q_{ad}\} = 0 \).

**Appendix C**

We give a synopsis of the other ways to compute the conserved bosonic charge \( W \) of equation (46). Exploiting the BRST symmetry transformation (15) and (21) and applying them onto the expression of \( Q_d \) from (34), it is clear that

\[
s_b Q_d = \int d^3x \left[ \epsilon^{ijk}(\partial_i B_j) B_k - \lambda \dot{\rho} - (\partial^0 C^i - \partial^0 \bar{C}^0) \partial_i \rho \right. \\
- \left. (\partial^0 B^i - \partial^0 \bar{B}^0) \bar{B}_i \right]. \quad (68)
\]

Using the equations of motion from (24), it can be seen that \( (\partial^0 B^i - \partial^0 \bar{B}^0) = -\epsilon^{ijk}(\partial_j B_k) \) and \( -\lambda \dot{\rho} = \lambda \Box \bar{C}_0 \). As a consequence, we have

\[
s_b Q_d = \int d^3x \left[ \epsilon^{ijk}\{\partial_i (B_j B_k) + (\partial_i B_j) B_k \} \right. \\
- \left. (\partial^0 C^i - \partial^0 \bar{C}^0) \partial_i \rho + \lambda(\partial_0 \partial_0 \bar{C}_0 - \partial_i \partial_i \bar{C}_0) \right]. \quad (69)
\]

Exploiting \( \partial_0 \bar{C}_0 = -2\rho + \partial_i \bar{C}_i \) and performing a partial integration, it can be shown that the following identity is true, namely;

\[
- \int d^3x \lambda \dot{\rho} = -2 \int d^3x \lambda \dot{\rho} + \int d^3x (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \partial_i \lambda. \quad (70)
\]
As a result of the above equality, it is clear that

\begin{align*}
    s_b Q_d &= -i \{ Q_d, Q_b \} = \int d^3 x \left[ \epsilon^{ijk} \left\{ (\partial_i B_j) B_k + (\partial_i B_j) B_k \right\} \\
    &- \left( \partial^0 C^i - \partial^i C^0 \right) \partial_i \rho + \left( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 \right) \partial_i \lambda \right] \equiv W, \quad (71)
\end{align*}

where \( W \) is defined in (46). Exactly in a similar fashion, one can show that \( s_d Q_b = -i \{ Q_b, Q_d \} \) leads to the derivation of \( W \).

It can be also checked that \( s_{ab} Q_{ad} = -i \{ Q_{ad}, Q_{ab} \} = -W \). The key steps in the above computation are as follows

\begin{align*}
    s_{ab} Q_{ad} &= \int d^3 x \left[ -\epsilon^{ijk} \left\{ (\partial_i \bar{B}_j) \bar{B}_k + (\partial_i \bar{B}_j) \bar{B}_k \right\} \\
    &- \left( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 \right) \partial_i \lambda \right], \quad (72)
\end{align*}

where we have used equations (16), (21) and (35). Using the equations of motion from (24), one can express the above equation as given below

\begin{align*}
    s_{ab} Q_{ad} &= \int d^3 x \left[ -\epsilon^{ijk} \left\{ (\partial_i \bar{B}_j) \bar{B}_k + (\partial_i \bar{B}_j) \bar{B}_k \right\} \\
    &- \left( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 \right) \partial_i \lambda + \left( \partial^0 C^i - \partial^i C^0 \right) \partial_i \rho \right]. \quad (73)
\end{align*}

Exploiting the constrained field equations (i.e. \( B_\mu - \bar{B}_\mu = \partial_\mu \phi_1, B_\mu - \bar{B}_\mu = \partial_\mu \phi_2 \)), we can re-express the above equation as follows

\begin{align*}
    s_{ab} Q_{ad} &= \int d^3 x \left[ -\epsilon^{ijk} \left\{ (\partial_i B_j) B_k + (\partial_i B_j) B_k \right\} \\
    &- \left( \partial^0 C^i - \partial^i C^0 \right) \partial_i \lambda + \left( \partial^0 C^i - \partial^i C^0 \right) \partial_i \rho \right] \equiv -W. \quad (74)
\end{align*}

In the above, the expression for \( W \) is from equation (46). In a similar fashion, it can be shown that \( s_{ad} Q_{ab} = -i \{ Q_{ab}, Q_{ad} \} = -W \).

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