The equivariant volumes of the permutahedron

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Abstract. We consider the action of the symmetric group $S_n$ on the permutahedron $\Pi_n$. We prove that if $\sigma$ is a permutation of $S_n$ which has $m$ cycles of lengths $l_1, \ldots, l_m$, then the subset of $\Pi_n$ fixed by $\sigma$ is a polytope with normalized volume $n^{m-2} \gcd(l_1, \ldots, l_m)$. 

Resumen. Consideramos la acción del grupo simétrico $S_n$ sobre el permutaedro $\Pi_n$. Demostramos que si $\sigma$ es una permutación de $S_n$ que tiene $m$ ciclos de longitudes $l_1, \ldots, l_m$, entonces el subconjunto de $\Pi_n$ que permanece fijo bajo la acción de $\sigma$ es un politopo cuyo volumen normalizado es igual a $n^{m-2} \gcd(l_1, \ldots, l_m)$.

Keywords: permutahedron, volume, symmetric group, tree

1 Introduction

The $n$-permutahedron is the polytope in $\mathbb{R}^n$ whose vertices are the permutations of $[n]$: 

$$\Pi_n := \text{conv}\{(\pi(1), \pi(2), \ldots, \pi(n)) : \pi \in S_n\}.$$ 

The symmetric group $S_n$ acts on $\Pi_n \subset \mathbb{R}^n$ by permuting coordinates; more precisely, a permutation $\sigma \in S_n$ acts on a point $x = (x_1, x_2, \ldots, x_n) \in \Pi_n$, by 

$$\sigma \cdot x := (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(n)}).$$

Definition 1.1. The fixed polytope of the permutahedron $\Pi_n$ under a permutation $\sigma$ of $[n]$ is 

$$\Pi_n^\sigma = \{x \in \Pi_n : \sigma \cdot x = x\}.$$ 

Our main result is a generalization of the fact, due to Stanley [4], that $\text{Vol} \Pi_n = n^{n-2}$; see Theorem 3.1.

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Theorem 1.2. If $\sigma$ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the fixed polytope of $\Pi_n$ under $\sigma$ is

$$\text{Vol} \, \Pi_n^\sigma = n^{m-2} \gcd(l_1, \ldots, l_m).$$

This is the first step towards describing the equivariant Ehrhart theory of the permutahedron, a question posed by Stapledon [6].

1.1 Normalizing the volume

The permutahedron and its fixed polytopes are not full-dimensional. We normalize volumes so that every primitive parallelootope has volume 1. This is the normalization under which the volume of $\Pi_n$ equals $n^{n-2}$.

More precisely, let $P$ be a $d$-dimensional polytope on an affine $d$-plane $L \subset \mathbb{Z}^n$. Assume $L$ is integral, in the sense that $L \cap \mathbb{Z}^n$ is a lattice translate of a $d$-dimensional lattice $\Lambda$. We call a lattice $d$-parallelootope in $L$ primitive if its edges generate the lattice $\Lambda$; all primitive parallelootopes have the same volume. Then we define the volume of a $d$-polytope $P$ in $L$ to be $\text{Vol}(P) := \text{EVol}(P) / \text{EVol}(\Box)$ for any primitive parallelootope $\Box$ in $L$, where $\text{EVol}$ denotes Euclidean volume.

The definition of $\text{Vol}(P)$ makes sense even when $P$ is not an integral polytope. This is important because the fixed polytopes of the permutahedron are not necessarily integral.

1.2 Notation

We identify each permutation $\pi \in S_n$ with the point $(\pi(1), \ldots, \pi(n))$ in $\mathbb{R}^n$. When we write permutations in cycle notation, we do not use commas to separate the entries
of each cycle. For example, we identify the permutation 246513 in $S_6$ with the point $(2, 4, 6, 5, 1, 3) \in \mathbb{R}^6$, and write it as $(1245)(36)$ in cycle notation.

Our main goal is to find the volume of the fixed polytope $\Pi_{\sigma}^n$ for a permutation $\sigma \in S_n$. We assume that $\sigma$ has $m$ cycles of lengths $l_1 \geq \cdots \geq l_m$. In fact, for the goals of this paper, it suffices to assume

$$\sigma = (1\ 2\ \ldots\ l_1)(l_1 + 1\ l_1 + 2\ \ldots\ l_1 + l_2) \cdots (l_1 + \cdots + l_{m-1} + 1\ \ldots\ n-1\ n).$$

We let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$, and $e_S := e_{s_1} + \cdots + e_{s_k}$ for $S = \{s_1, \ldots, s_k\} \subseteq [n]$. Recall that the Minkowski sum of polytopes $P, Q \subset \mathbb{R}^n$ is the polytope $P + Q := \{p + q : p \in P, q \in Q\} \subset \mathbb{R}^n$. [3]

1.3 Organization

Section 2 presents Theorem 2.11, which describes the fixed polytope $\Pi_{\sigma}^n$ in terms of its vertices, its defining inequalities, and a Minkowski sum decomposition. Section 3 uses this to prove our main result, Theorem 1.2, on the normalized volume of $\Pi_{\sigma}^n$. This is an extended abstract; for complete statements and proofs, see [1].

2 Describing the fixed polytopes of the permutahedron

Proposition 2.1 ([7]). The permutahedron $\Pi_n$ can be described in the following three ways:

1. (Inequalities) It is the set of points $x \in \mathbb{R}^n$ satisfying
   
   $(a)$ $x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n$, and
   $(b)$ for any proper subset $\{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}$,
   $$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1 + 2 + \cdots + k.$$

2. (Vertices) It is the convex hull of the points $(\pi(1), \ldots, \pi(n))$ as $\pi$ ranges over the permutations of $[n]$.

3. (Minkowski sum) It is the Minkowski sum: $\sum_{1 \leq j < k \leq n} [e_k, e_j] + \sum_{1 \leq k \leq n} e_k$.

The $n$-permutahedron is $(n - 1)$-dimensional and every permutation of $[n]$ is indeed a vertex.

Our first goal is to prove the analogous result for the fixed polytopes of $\Pi_n$; we do so in Theorem 2.11.
2.1 Standardizing the permutation

We define the cycle type of a permutation $\sigma$ to be the partition of $n$ consisting of the lengths $l_1 \geq \cdots \geq l_m$ of the cycles of $\sigma$.

**Lemma 2.2.** The volume of $\Pi_n^\sigma$ only depends on the cycle type of $\sigma$.

We wish to measure the various fixed polytopes of $\Pi_n$, and by Lemma 2.2 we can focus our attention on the polytopes $\Pi_n^\sigma$ fixed by a permutation of the form

$$\sigma = (1 \ 2 \ \ldots \ l_1)(l_1 + 1 \ l_1 + 2 \ \ldots \ l_1 + l_2) \cdots (l_1 + \cdots + l_{m-1} + 1 \ \ldots \ n-1 \ n) \quad (2.1)$$

for a partition $l_1 \geq l_2 \geq \cdots \geq l_m$ with $l_1 + \cdots + l_m = n$. We do so from now on.

2.2 Towards the inequality description

**Lemma 2.3.** For a permutation $\sigma \in S_n$, the fixed polytope $\Pi_n^\sigma$ consists of the points $x \in \Pi_n$ satisfying $x_j = x_k$ for any $j$ and $k$ in the same cycle of $\sigma$.

**Corollary 2.4.** If a permutation $\sigma$ of $[n]$ has $m$ cycles then $\Pi_n^\sigma$ has dimension $m - 1$.

2.3 Towards a vertex description

In this section we describe a set $\text{Vert}(\sigma)$ of $m!$ points associated to a permutation $\sigma$ of $S_n$. We will show in Theorem 2.11 that this is the set of vertices of the fixed polytope $\Pi_n^\sigma$. For a point $w \in \mathbb{R}^n$, let $\overline{w}$ be the average of the $\sigma$-orbit of $w$, that is,

$$\overline{w} := \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot w, \quad (2.2)$$

where $|\sigma|$ is the order of $\sigma$ as an element of the symmetric group $S_n$.

**Definition 2.5.** Given $\sigma \in S_n$, we say a permutation $v = (v_1, \ldots, v_n)$ of $[n]$ is $\sigma$-standard if it satisfies the following property: for each cycle $(j_1 \ j_2 \ \cdots \ j_r)$ of $\sigma$, $(v_{j_1}, v_{j_2}, \ldots, v_{j_r})$ is a sequence of consecutive integers in increasing order. We define the set of $\sigma$-vertices to be

$$\text{Vert}(\sigma) := \{ \overline{w} : w \text{ is a } \sigma\text{-standard permutation of } [n] \}.$$

These points should not be confused with the vertices of the ambient permutahedron $\Pi_n$. Let us illustrate this definition in an example and prove some preliminary results.
Example 2.6. For \( \sigma = (1234)(567)(89) \), the \( \sigma \)-standard permutations in \( S_9 \) are

\[
\begin{align*}
(1,2,3,4, 5,6,7, 8,9), & \quad (1,2,3,4, 7,8,9, 5,6), \\
(4,5,6,7, 1,2,3, 8,9), & \quad (3,4,5,6, 7,8,9, 1,2), \\
(6,7,8,9, 1,2,3, 4,5), & \quad (6,7,8,9, 3,4,5, 1,2),
\end{align*}
\]

and the corresponding \( \sigma \)-vertices are

\[
\begin{align*}
\frac{1+2+3+4}{4} e_{1234} + \frac{5+6+7}{3} e_{567} + \frac{8+9}{2} e_{89}, & \quad \frac{1+2+3+4}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{5+6}{2} e_{89}, \\
\frac{4+5+6+7}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{8+9}{2} e_{89}, & \quad \frac{3+4+5+6}{4} e_{1234} + \frac{7+8+9}{3} e_{567} + \frac{1+2}{2} e_{89}, \\
\frac{6+7+8+9}{4} e_{1234} + \frac{1+2+3}{3} e_{567} + \frac{4+5}{2} e_{89}, & \quad \frac{6+7+8+9}{4} e_{1234} + \frac{3+4+5}{3} e_{567} + \frac{1+2}{2} e_{89}.
\end{align*}
\]

Let us give a more explicit description of \( \overline{w} \) in general, and of the \( \sigma \)-vertices in particular, which will be important in the proof of Theorem 2.11.

Lemma 2.7. For any \( w \in \mathbb{R}^n \), the average of the \( \sigma \)-orbit of \( w \) is

\[
\overline{w} = \sum_{k=1}^{m} \frac{\sum_{j \in \sigma_k} w_j}{l_k} e_{\sigma_k}.
\]

Notice that the entries of \( \overline{w} \) within each cycle \( \sigma_k \) are constant, bearing witness to the fact that \( \overline{w} \), being the average of a \( \sigma \)-orbit, must be in the fixed polytope \( \Pi_{n}^{\sigma} \).

Corollary 2.8. The set \( \text{Vert}(\sigma) \) of \( \sigma \)-vertices consists of the \( m! \) points

\[
\overline{\sigma} \prec := \sum_{k=1}^{m} \left( \frac{l_k + 1}{2} + \sum_{j : \sigma_j \prec \sigma_k} l_j \right) e_{\sigma_k}
\]

as \( \prec \) ranges over the \( m! \) possible linear orderings of \( \sigma_1, \sigma_2, \ldots, \sigma_m \).

2.4 Towards a zonotope description

We will show in Theorem 2.11 that the fixed polytope \( \Pi_{n}^{\sigma} \) is the zonotope given by the following Minkowski sum.

Definition 2.9. Let \( M_{\sigma} \) denote the Minkowski sum

\[
M_{\sigma} := \sum_{1 \leq j < k \leq m} [l_j e_{\sigma_k}, l_k e_{\sigma_j}] + \sum_{k=1}^{m} \frac{l_k + 1}{2} e_{\sigma_k} = \sum_{1 \leq j < k \leq m} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] + \sum_{k=1}^{m} \left( \frac{l_k + 1}{2} + \sum_{j < k} l_j \right) e_{\sigma_k}.
\]

Proposition 2.10. The zonotope \( M_{\sigma} \) is combinatorially equivalent to the standard permutahedron \( \Pi_{m} \), where \( m \) is the number of cycles of \( \sigma \).
2.5 Three descriptions of the fixed polytope of the permutahedron

**Theorem 2.11.** Let \( \sigma \) be a permutation of \([n]\) whose cycles \( \sigma_1, \ldots, \sigma_m \) have respective lengths \( l_1, \ldots, l_m \). The fixed polytope \( \Pi^\sigma_n \) can be described in the following ways:

0. It is the set of points \( x \) in the permutahedron \( \Pi_n \) such that \( \sigma \cdot x = x \).

1. It is the set of points \( x \in \mathbb{R}^n \) satisfying

   \[(a) \quad x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n,\]

   \[(b) \quad \text{for any proper subset } \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, n\}, \]

   \[x_{i_1} + x_{i_2} + \cdots + x_{i_k} \leq 1 + 2 + \cdots + k, \text{ and} \]

   \[(c) \quad \text{for any } i \text{ and } j \text{ which are in the same cycle of } \sigma, \ x_i = x_j.\]

2. It is the convex hull of the set \( \text{Vert}(\sigma) \) of \( \sigma \)-vertices, as described in Corollary 2.8.

3. It is the Minkowski sum \( M_\sigma \) of Definition 2.9

Consequently, the fixed polytope \( \Pi^\sigma_n \) is a zonotope that is combinatorially isomorphic to the permutahedron \( \Pi_m \). It is \((m - 1)\)-dimensional and every \( \sigma \)-vertex is indeed a vertex of \( \Pi^\sigma_n \).

**Proof.** Description 0. is the definition of the fixed polytope \( \Pi^\sigma_n \), and we already observed in Lemma 2.3 that description 1. is accurate. Recall that we denoted the polytopes described in 2. and 3. by \( \text{conv}(\text{Vert}(\sigma)) \) and \( M_\sigma \), respectively. It remains to prove that

\[ \Pi^\sigma_n = \text{conv}(\text{Vert}(\sigma)) = M_\sigma. \]

We proceed in three steps as follows:

**A.** \( \text{conv}(\text{Vert}(\sigma)) \subseteq \Pi^\sigma_n \)  \quad **B.** \( M_\sigma \subseteq \text{conv}(\text{Vert}(\sigma)) \)  \quad **C.** \( \Pi^\sigma_n \subseteq M_\sigma \)

**A.** \( \text{conv}(\text{Vert}(\sigma)) \subseteq \Pi^\sigma_n \): It suffices to show that \( \Pi^\sigma_n \) contains any point in \( \text{Vert}(\sigma) \), say

\[ \overline{v}_\prec = \frac{1}{|\sigma|} \sum_{i=1}^{|\sigma|} \sigma^i \cdot v_\prec, \]

where \( \prec \) is a total order of \( \sigma_1, \ldots, \sigma_m \) and \( v_\prec \) is the associated \( \sigma \)-standard permutation. Since \( v_\prec \) is a vertex of \( \Pi_n \), we conclude that \( \sigma^i \cdot v_\prec \) is a vertex of \( \Pi_n \) for all \( i \), and hence their average \( \overline{v}_\prec \) is in \( \Pi_n \). Also, since \( \sigma^{[\sigma]} = 1 \), we have that \( \sigma \cdot \overline{v}_\prec = \overline{v}_\prec \). Therefore, \( \overline{v}_\prec \) is in \( \Pi^\sigma_n \) by 0., as desired.

**B.** \( M_\sigma \subseteq \text{conv}(\text{Vert}(\sigma)) \): It suffices to show that any vertex of \( M_\sigma \) is in \( \text{Vert}(\sigma) \).
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For a polytope $P \subset \mathbb{R}^n$ and a linear functional $c \in (\mathbb{R}^n)^*$, we let $P_c$ denote the face of $P$ where $c$ is maximized. In particular, for any given vertex $v$ of $M_\sigma$, consider a linear functional $c = (c_1, c_2, \ldots, c_n) \in (\mathbb{R}^n)^*$ such that $v = (M_\sigma)_c$ is the unique point in $M_\sigma$ maximizing $c$. For $k = 1, \ldots, m$, let $c_{\sigma_k} := \frac{1}{k} \sum_{i \in \sigma_k} c_i$. One can verify that

(a) $c_{\sigma_j} \neq c_{\sigma_k}$ for $j \neq k$, and

(b) $v = \overrightarrow{w}$ for the linear order $\prec$ on $\sigma_1, \sigma_2, \ldots, \sigma_m$ where $\sigma_j \prec \sigma_k$ if and only if $c_{\sigma_j} < c_{\sigma_k}$. This shows that every vertex of $M_\sigma$ is a $\sigma$-vertex, as desired.

C. $\Pi_n^\sigma \subseteq M_\sigma$: Any point $p \in \Pi_n^\sigma$ can be written as a convex combination $p = \sum_{\tau \in S_n} \lambda_\tau \tau$ of the $n!$ permutations of $[n]$, where $\lambda_\tau \geq 0$ for all $\tau$ and $\sum_{\tau \in S_n} \lambda_\tau = 1$. Recall from (2.2) that $\overrightarrow{w}$ represents the average of the $\sigma$-orbit of $w \in \mathbb{R}^n$. Since $p$ is fixed by $\sigma$, we have

$$p = \overrightarrow{p} = \sum_{\tau \in S_n} \lambda_\tau \overrightarrow{\tau}.$$ 

It follows that $\Pi_n^\sigma \subseteq \text{conv}\{\overrightarrow{\tau} : \tau \in S_n\}$. Therefore, to show that $\Pi_n^\sigma \subseteq M_\sigma$, it suffices to show that $\tau \in M_\sigma$ for all permutations $\tau$. To do so, let us first derive an alternative expression for $\overrightarrow{\tau}$.

Let us begin with the vertex $\text{id} = (1, 2, \ldots, n)$ of $\Pi_n$ corresponding to the identity permutation. As described in Corollary 2.8, this is the $\sigma$-standard permutation corresponding to the order $\sigma_1 \prec \sigma_2 \prec \cdots \prec \sigma_m$, so

$$\overrightarrow{\text{id}} = \sum_{k=1}^{m} \left( \frac{l_k + 1}{2} + \sum_{j<k} l_j \right) e_{\sigma_k}. \quad (2.4)$$

Notice that this is the translation vector for the Minkowski sum of (2.3).

Now, let us compute $\overrightarrow{\tau}$ for any permutation $\tau$. Let

$$l = \text{inv}(\tau) = |\{(a,b) : 1 \leq a < b \leq n, \ \tau(a) > \tau(b)\}|$$

be the number of inversions of $\tau$. Consider a minimal sequence $\text{id} = \tau_0, \tau_1, \ldots, \tau_l = \tau$ of permutations such that $\tau_{i+1}$ is obtained from $\tau_i$ by exchanging the positions of numbers $p$ and $p + 1$, thus introducing a single new inversion without affecting any existing inversions. Such a sequence corresponds to a minimal factorization of $\tau$ as a product of simple transpositions $(p \ p + 1)$ for $1 \leq p \leq n - 1$. We have $\text{inv}(\tau_i) = i$ for $1 \leq i \leq l$.

Now we compute $\overrightarrow{\tau}$ by analyzing how $\overrightarrow{\tau_i}$ changes as we introduce new inversions, using that

$$\overrightarrow{\tau} - \overrightarrow{\text{id}} = (\overrightarrow{\tau_l} - \overrightarrow{\tau_{l-1}}) + \cdots + (\overrightarrow{\tau_1} - \overrightarrow{\text{id}}). \quad (2.5)$$

If $a < b$ are the positions of the numbers $p$ and $p + 1$ that we switch as we go from $\tau_i$ to $\tau_{i+1}$, then regarding $\tau_i$ and $\tau_{i+1}$ as vectors in $\mathbb{R}^n$ we have

$$\tau_{i+1} - \tau_i = e_a - e_b.$$
If $\sigma_j$ and $\sigma_k$ are the cycles of $\sigma$ containing $a$ and $b$, respectively, we have

$$
\tau_{i+1} - \tau_i = \bar{e}_a - \bar{e}_b = \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} = \frac{1}{l_j l_k} (l_k e_{\sigma_j} - l_j e_{\sigma_k})
$$

in light of Lemma 2.7. This is the local contribution to (2.5) that we obtain when we introduce a new inversion between a position $a$ in cycle $\sigma_j$ and a position $b$ in cycle $\sigma_k$ in our permutation. Notice that this contribution is 0 when $j = k$. Also notice that we will still have an inversion between positions $a$ and $b$ in all subsequent permutations, due to the minimality of the sequence. We conclude that

$$
\tau - \text{id} = \sum_{j < k} \text{inv}_{j,k}(\tau) \frac{1}{l_j l_k} (l_k e_{\sigma_j} - l_j e_{\sigma_k})
$$

(2.7)

where

$$
\text{inv}_{j,k}(\tau) = |\{(a, b) : 1 \leq a < b \leq n, \ a \in \sigma_j, \ b \in \sigma_k \ and \ \tau(a) > \tau(b)\}|
$$

is the number of inversions in $\tau$ between a position in $\sigma_j$ and a position in $\sigma_k$ for $j < k$.

Equations (2.4) and (2.7) give us an alternative description for $\tau$. This description makes it apparent that $\tau \in M_\sigma$. Notice that $|\sigma_j| = l_j$ and $|\sigma_k| = l_k$ imply that $0 \leq \text{inv}_{j,k}(\tau) \leq l_j l_k$, so

$$
\tau - \text{id} \in \sum_{1 \leq j < k \leq n} [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}] \cup [0, l_k e_{\sigma_j} - l_j e_{\sigma_k}]
$$

combining this with (2.3) and (2.4) gives the desired result. $\square$

**Figure 2:** (a) A minimal sequence of permutations $\text{id} = \tau_0, \tau_1, \ldots, \tau_9 = 461352$ adding one inversion at a time and (b) the corresponding path from $\text{id}$ to $\tau$ in the zonotope $M_\sigma$.

**Example 2.12.** Figure 2 illustrates part C of the proof above for $n = 6$, $\sigma = (123)(45)(6)$, and the permutation $\tau = 461352$. This permutation has inv($\tau$) = 9 inversions, and the columns of the left panel show a minimal sequence of permutations $\text{id} = \tau_0, \tau_1, \ldots, \tau_9 =$
where each \( \tau_{i+1} \) is obtained from \( \tau_i \) by swapping two consecutive numbers, thus introducing a single new inversion.

The rows of the diagram are split into three groups 1, 2, and 3, corresponding to the support of the cycles of \( \sigma \). Out of the \( \text{inv}(\tau) = 9 \) inversions of \( \tau \), there are \( \text{inv}_{1,2}(\tau) = 3 \) involving groups 1 and 2, \( \text{inv}_{1,3}(\tau) = 2 \) involve groups 1 and 3, and \( \text{inv}_{2,3}(\tau) = 2 \) involving groups 2 and 3.

This sequence of permutations gives rise to a walk from \( \text{id} \), which is the top right vertex of the zonotope \( M_\sigma \), to \( \tau \). In the rightmost triangle, which is not drawn to scale, vertex \( i \) represents the point \( \frac{e_{\sigma_i}}{l_i} \) for \( 1 \leq i \leq 3 \). Whenever two numbers in groups \( j < k \) are swapped in the left panel, to get from permutation \( \tau_i \) to \( \tau_{i+1} \), we take a step in direction \( e_{\sigma_j}/l_j - e_{\sigma_k}/l_k \) in the right panel, to get from point \( \tau_i \) to \( \tau_{i+1} \). This is the direction of edge \( jk \) in the triangle, and its length is \( \frac{1}{l_j l_k} \) of the length of the generator \( l_k e_{\sigma_j} - l_j e_{\sigma_k} \) of the zonotope. Then

\[
\tau - \text{id} = \frac{3}{l_1 l_2} (l_2 e_{\sigma_1} - l_1 e_{\sigma_2}) + \frac{2}{l_1 l_3} (l_3 e_{\sigma_1} - l_1 e_{\sigma_3}) + \frac{2}{l_2 l_3} (l_3 e_{\sigma_2} - l_2 e_{\sigma_3}).
\]

Since \( 3 = \text{inv}_{1,2}(\tau) \leq l_1 l_2 = 6 \), \( 2 = \text{inv}_{1,3}(\tau) \leq l_1 l_3 = 3 \) and \( 2 = \text{inv}_{2,3}(\tau) \leq l_2 l_3 = 2 \), the resulting point \( \tau \) is in the zonotope \( M_\sigma \).

### 3 The volumes of the fixed polytopes of the permutahedron

To compute the volume of \( \Pi_{\Pi_\nu} ^\sigma \) we use its description as a zonotope, recalling that a zonotope can be tiled by parallelotopes as follows. If \( A \) is a set of vectors, then \( B \subseteq A \) is called a basis for \( A \) if \( B \) is linearly independent and \( \text{rank}(B) = \text{rank}(A) \). We define the parallelotope \( \square B \) to be the Minkowski sum of the segments in \( B \), that is,

\[
\square B := \left\{ \sum_{b \in B} \lambda_b b : 0 \leq \lambda_b \leq 1 \text{ for each } b \in B \right\}.
\]

**Theorem 3.1** ([2, 4, 7]). Let \( A \subset \mathbb{Z}^n \) be a set of lattice vectors of rank \( d \).

1. The zonotope \( Z(A) \) can be tiled using one translate of the parallelotope \( \square B \) for each basis \( B \) of \( A \). Therefore, the volume of the \( d \)-dimensional zonotope \( Z(A) \) is

\[
\text{Vol} (Z(A)) = \sum_{B \subseteq A \text{ basis}} \text{Vol}(\square B).
\]

2. For each \( B \subset \mathbb{Z}^n \) of rank \( d \), \( \text{Vol}(\square B) \) equals the index of \( \mathbb{Z}B \) as a sublattice of \( (\text{span } B) \cap \mathbb{Z}^n \). Using the vectors in \( B \) as the columns of an \( n \times d \) matrix, \( \text{Vol}(B) \) is the greatest common divisor of the minors of rank \( d \).
By Theorem 2.11, the fixed polytope $\Pi^\sigma_n$ is a translate of the zonotope generated by the set

$$F_\sigma = \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : 1 \leq j < k \leq m \right\}.$$

This set of vectors has a nice combinatorial structure, allowing us to describe the bases $B$ and the volumes $\text{Vol}(\square B)$ combinatorially. We do this in the next two lemmas. For a tree $T$ whose vertex set is $[m]$, let

$$F_T = \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : j < k \text{ and } jk \text{ is an edge of } T \right\},$$

$$E_T = \left\{ \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} : j < k \text{ and } jk \text{ is an edge of } T \right\}.$$

**Lemma 3.2 ([4]).** The vector configuration

$$F_\sigma := \left\{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : 1 \leq j < k \leq m \right\}$$

has exactly $m^{m-2}$ bases: they are the sets $F_T$ as $T$ ranges over the spanning trees on $[m]$.

**Lemma 3.3.** For any tree $T$ on $[m]$ we have

1. $\text{Vol}(\square F_T) = \prod_{i=1}^{m} l_i^{\deg_T(i)} \text{Vol}(E_T)$,
2. $\text{Vol}(\square E_T) = \frac{\gcd(l_1, \ldots, l_m)}{l_1 \cdots l_m},$

where $\deg_T(i)$ is the number of edges containing vertex $i$ in $T$.

**Proof.** 1. Since $l_k e_{\sigma_j} - l_j e_{\sigma_k} = l_j l_k \left( \frac{e_{\sigma_j}}{l_j} - \frac{e_{\sigma_k}}{l_k} \right)$ for each edge $jk$ of $T$, and volumes scale linearly with respect to each edge length of a parallelotope, we have

$$\text{Vol}(\square F_T) = \left( \prod_{jk \text{ edge of } T} l_j l_k \right) \text{Vol}(\square E_T) = \prod_{i=1}^{m} l_i^{\deg_T(i)} \text{Vol}(\square E_T).$$

2. The parallelotopes $\square E_T$ are the images of the parallelotopes $\square A_T$ under the linear bijective map

$$\phi : \mathbb{R}^m \to (\mathbb{R}^n)^\sigma$$

$$f_i \mapsto \frac{e_{\sigma_i}}{l_i},$$

where

$$A_T := \{ f_j - f_k : j < k, jk \text{ is an edge of } T \}.$$
Since the vector configuration \( \{ f_j - f_k : 1 \leq j < k \leq m \} \) is unimodular, all parallelotopes \( \Box A_T \) have unit volume. Therefore, the parallelotopes \( \Box E_T = \phi(\Box A_T) \) have the same normalized volume, so \( \text{Vol}(E_T) \) is independent of \( T \).

It follows that we can use any tree \( T \) to compute \( \text{Vol}(E_T) \) or, equivalently, \( \text{Vol}(F_T) \). We choose the tree \( T = \text{Claw}_m \) with edges \( 1m, 2m, \ldots, (m-1)m \). Writing the \( m-1 \) vectors of

\[
F_{\text{Claw}_m} = \{ l_{i}\sigma_i - l_i\sigma_m : 1 \leq i \leq m-1 \}
\]

as the columns of an \( n \times (m-1) \) matrix, then \( \text{Vol}(F_{\text{Claw}_m}) \) is the greatest common divisor of the non-zero maximal minors of this matrix. This quantity does not change when we remove duplicate rows; the result is the \( m \times (m-1) \) matrix

\[
\begin{bmatrix}
  l_m & 0 & 0 & \cdots & 0 \\
  0 & l_m & 0 & \cdots & 0 \\
  0 & 0 & l_m & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & l_m \\
  -l_1 & -l_2 & -l_3 & \cdots & -l_{m-1}
\end{bmatrix}
\]

This matrix has \( m \) maximal minors, whose absolute values equal \( l_m^{m-2}l_1, \ldots, l_m^{m-2}l_{m-1}, l_m^{m-1} \). Therefore,

\[
\text{Vol}(\Box F_{\text{Claw}_m}) = l_m^{m-2} \text{gcd}(l_1, \ldots, l_{m-1}, l_m)
\]

and part 1 then implies that

\[
\text{Vol}(\Box E_{\text{Claw}_m}) = \frac{\text{Vol}(\Box F_{\text{Claw}_m})}{l_1 \cdots l_{m-1} l_m^{m-1}} = \frac{\text{gcd}(l_1, \ldots, l_m)}{l_1 \cdots l_m}
\]

as desired. \( \square \)

**Lemma 3.4.** For any positive integer \( m \geq 2 \) and unknowns \( x_1, \ldots, x_m \), we have

\[
\sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} x_i^{\deg_T(i)-1} = (x_1 + \cdots + x_m)^{m-2}. \]

**Sketch of proof.** This is a variant of the analogous result for rooted trees [5, Theorem 5.3.4], which states that

\[
\sum_{(T,r) \text{ rooted tree on } [m]} \prod_{i=1}^{m} x_i^{\text{children}(T,r)(i)} = (x_1 + \cdots + x_m)^{m-1}
\]

where \( \text{children}(T,r)(v) \) counts the children of \( v \). It can be proved similarly, or derived directly from it. \( \square \)
Theorem 1.2. If $\sigma$ is a permutation of $[n]$ whose cycles have lengths $l_1, \ldots, l_m$, then the normalized volume of the fixed polytope of $\Pi_n$ under $\sigma$ is

$$\text{Vol } \Pi_n^\sigma = n^{m-2} \gcd(l_1, \ldots, l_m).$$

Proof. Since $\Pi_n^\sigma$ is a translate of the zonotope for $F_\sigma := \{ l_k e_{\sigma_j} - l_j e_{\sigma_k} : 1 \leq j < k \leq m \}$, we invoke Theorem 3.1. Using Lemmas 3.2 to 3.4, it follows that

$$\text{Vol } \Pi_n^\sigma = \sum_{T \text{ tree on } [m]} \text{Vol}(\square F_T) = \sum_{T \text{ tree on } [m]} \prod_{i=1}^{m} l_i^{\text{deg}_T(i)-1} \gcd(l_1, \ldots, l_m) = (l_1 + \cdots + l_m)^{m-2} \gcd(l_1, \ldots, l_m) = n^{m-2} \gcd(l_1, \ldots, l_m).$$

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