THE YOKONUMA-HECKE ALGEBRAS AND THE HOMFLYPT POLYNOMIAL

MARIA CHLOUVERAKI AND SOFIA LAMBROPOULOU

Abstract. We compare the invariants for classical knots and links defined using the Juyumaya trace on the Yokonuma-Hecke algebras with the HOMFLYPT polynomial. We show that these invariants do not coincide with the HOMFLYPT except in a few trivial cases.

Introduction

The Yokonuma-Hecke algebras $Y_{d,n}(u)$ were introduced by Yokonuma [Yo] in the context of Chevalley groups, as generalizations of the Iwahori-Hecke algebras. The algebras $Y_{d,n}(u)$ may be also viewed as quotients of the framed braid group algebra over a quadratic relation (see (2.2)) involving the framing generators by means of certain weighted idempotents $e_i$. Thus the classical braid groups are also represented in the algebras $Y_{d,n}(u)$.

In [Ju] Juyumaya constructed a unique linear Markov trace $tr$ on the algebras $Y_{d,n}(u)$, depending on $d$ parameters, $z, x_1, \ldots, x_{d-1}$. The trace $tr$ was used subsequently in [JuLa2] for defining isotopy invariants for framed knots. As it turned out, the trace $tr$ would not re-scale directly according to the braid equivalence moves. Therefore, certain conditions had to be imposed, implying that the trace parameters $x_1, \ldots, x_{d-1}$ had to satisfy a non linear system of equations, the so-called $E$-system (see (2.13)). Gérardin proved that the solutions of the $E$-system are parametrized by the non-empty subsets $S$ of $\mathbb{Z}/d\mathbb{Z}$ (see Appendix of [JuLa2]). Given now any solution of the $E$-system, 2-variable isotopy invariants for framed, classical and singular knots were constructed respectively in [JuLa2 JuLa3 JuLa4].

For classical knots we have the well-known HOMFLYPT or 2-variable Jones polynomial $P$ [Jo], which is determined by the Ocneanu trace (with parameter $\zeta$) on the Iwahori-Hecke algebras $H_n(q)$ of type $A$. Therefore, it is natural to ask how the invariant $P$ compares with every invariant $\Delta_S$ derived from the Juyumaya trace on the algebras $Y_{d,n}(u)$, for any $d \in \mathbb{N}$ and for any non-empty subset $S$ of $\mathbb{Z}/d\mathbb{Z}$. Computational data so far do not indicate that one invariant is topologically stronger than the other (see [CJJKL]).

In order to compare the knot invariants $P$ and $\Delta_S$, we would like to be able to specialize the indeterminates $x_1, \ldots, x_{d-1}$ to a solution of the $E$-system as early as possible during the construction of $\Delta_S$. This goal is achieved in Section 3 with the construction of the linear map $\varphi$ on $Y_{d,n}(u)$. However, as we show, there is no appropriate algebra homomorphism between $Y_{d,n}(u)$ and $H_n(q)$, unless $E := tr(e_i) = 1$, and this makes a connection between the corresponding trace functions impossible. In this paper we prove that the invariants $P$ and $\Delta_S$ do not coincide except in a few trivial cases, that is, $u = 1$ or $q = 1$ or $E = 1$, given by Theorem 6.

In fact we show (Theorem 6) that these are the only cases where the one invariant is a scalar multiple of the other, with scalars in $\mathbb{C}(q, \zeta, u, z, E)$.

The paper is organized as follows: In the first two sections we present some preliminary results on Iwahori-Hecke algebras and Yokonuma-Hecke algebras. In Section 3 we introduce the specialized Juyumaya trace, where the indeterminates $x_1, \ldots, x_{d-1}$ specialize to complex numbers, and we show that it factors through the linear map $\varphi$ that we construct. We compare the specialized Juyumaya trace with the Ocneanu trace and we obtain one case (when $E = 1$) where the invariants $P$ and $\Delta_S$ coincide. In the last two sections we proceed with comparing further the invariants, in order to obtain all cases where they coincide. More precisely, in Section 4 we give some necessary conditions for $P$ and $\Delta_S$ to coincide, by evaluating the invariants on specific

2010 Mathematics Subject Classification. 57M27, 57M25, 20F36, 20C08.

Key words and phrases. Yokonuma-Hecke algebras, Markov trace, HOMFLYPT polynomial, $E$-system.

The research project is implemented within the framework of the Action “Supporting Postdoctoral Researchers” of the Operational Program “Education and Lifelong Learning” (Action’s Beneficiary: General Secretariat for Research and Technology), and is co-financed by the European Social Fund (ESF) and the Greek State. The first author would also like to thank Guillaume Pouchin for fruitful conversations and comments.
braid words. Finally, in Section 5 we prove, with the use of an elaborate induction, that these conditions are also sufficient.

1. The 2-variable Jones or HOMFLYPT Polynomial

1.1. The symmetric group. The symmetric group \( \Sigma_n \) is generated by the transpositions \( s_1, s_2, \ldots, s_{n-1} \), with \( s_i = (i, i+1) \), subject to the relations:

\[
\begin{align*}
    s_is_j &= s_js_i \quad \text{for } |i-j| > 1; \\
    s_{i+1}s_is_{i+1} &= s_is_{i+1}s_i \quad \text{for all } i; \\
    s_i^2 &= 1 \quad \text{for all } i.
\end{align*}
\]

Let \( S = \{s_1, s_2, \ldots, s_{n-1}\} \) and let \( w \in \Sigma_n \). Then \( w = s_{i_1}s_{i_2}\cdots s_{i_r} \), with \( i_j \in S \), is an expression for \( w \).

If \( r \) is minimal such that there exists an expression \( w = s_{i_1}s_{i_2}\cdots s_{i_r} \), then this expression is called reduced and \( r \) is called the length of \( w \). We denote the length of \( w \) by \( \ell(w) \).

1.2. Conjugacy classes of \( \Sigma_n \). The conjugacy classes of \( \Sigma_n \) correspond to the cycle types of permutations; that is, two elements of \( \Sigma_n \) are conjugate in \( \Sigma_n \) if and only if they consist of the same number of disjoint cycles of the same length. It is well-known that the conjugacy classes of \( \Sigma_n \) are naturally parametrized by the partitions \( \mu \) of \( n \). If \( \mu \) has non-zero parts \( \mu_1, \mu_2, \ldots \), then we take \( w_\mu := s_{i_1}\cdots s_{i_{\mu_1}}s_{i_{\mu_1+1}} \) as representative in the class labelled by \( \mu \), where \( \{i_1, i_2, \ldots, i_{\mu_1}\} \) is the set obtained from \( \{1, 2, \ldots, n\} \) by removing the integers \( \mu_1, \mu_1 + \mu_2, \mu_1 + \mu_2 + \mu_3, \ldots \). For example, if \( n = 8 \) and \( \mu = (4, 3, 1) \), then \( \mu_1 = 4, \mu_1 + \mu_2 = 7 \) and \( \mu_1 + \mu_2 + \mu_3 = 8 \), whence \( w_\mu = s_6s_5s_3s_2s_1 \). The point about choosing these representatives is that \( w_\mu \) has minimal length in its conjugacy class, that is, we have \( \ell(w_\mu) \leq \ell(w) \) for any \( w \in \Sigma_n \) which is conjugate to \( w_\mu \).

Now let

\[
    \mathcal{D} := \{s_{i_1}\cdots s_{i_{\mu_1}}s_{i_{\mu_1+1}} \mid i_1 < i_2 < \cdots < i_{\mu_1}\} \subseteq \Sigma_n.
\]

We obviously have \( w_\mu \in \mathcal{D} \) for every partition \( \mu \) of \( n \). Moreover, if \( w = s_{i_1}\cdots s_{i_{\mu_1}}s_{i_{\mu_1+1}} \in \mathcal{D} \), one can easily check that, because of its cycle type, \( w \) has minimal length in its conjugacy class.

The following result, which relates elements of minimal length in a conjugacy class, will be useful in Subsection 3.3.

**Theorem 1.** [GePf, Theorem 3.2.9] Let \( C \) be a conjugacy class of \( \Sigma_n \) and let \( w, w' \) be two elements of minimal length in \( C \). Then \( w \) and \( w' \) are “strongly conjugate” in \( \Sigma_n \), that is, there exists a finite sequence \( w_0 = w, w_1, \ldots, w_r = w' \) such that, for all \( i = 0, 1, \ldots, r - 1 \),

\[
    \ell(w_i) = \ell(w_{i+1}) \quad \text{and} \quad \ell(w_iw_j) = \ell(w_i) + \ell(w_j) \quad \text{for all } i, j.
\]

1.3. The Iwahori-Hecke algebra. Let \( q \in \mathbb{C} \setminus \{0\} \). The Iwahori-Hecke algebra \( \mathcal{H}_n(q) \) of type \( A \) is the \( \mathbb{C} \)-associative algebra with presentation on generators \( G_1, G_2, \ldots, G_{n-1} \), and relations:

\[
\begin{align*}
    (b_1) & \quad G_iG_j = G_jG_i \quad \text{for } |i-j| > 1; \\
    (b_2) & \quad G_{i+1}G_iG_{i+1} = G_iG_{i+1}G_i \quad \text{for all } i; \\
    (h) & \quad G_i^2 = (q-1)G_i + q \quad \text{for all } i.
\end{align*}
\]

The relations (b_1) and (b_2) are defining relations for the classical Artin braid group \( B_n \); hence \( \mathcal{H}_n(q) \) can be viewed as the quotient of the group algebra \( \mathbb{C}B_n \) over the quadratic relations (h). Moreover, for \( q = 1 \) the algebra \( \mathcal{H}_n(1) \) is isomorphic to the group algebra of the symmetric group \( \Sigma_n \).

If \( w \in \Sigma_n \) and \( w = s_{i_1}s_{i_2}\cdots s_{i_r} \) is a reduced expression, we set \( G_w := G_{i_1}G_{i_2}\cdots G_{i_r} \). The set

\[
    \{G_w \mid w \in \Sigma_n\}
\]

is the “standard” \( \mathbb{C} \)-basis of \( \mathcal{H}_n(q) \). Now, the following set forms another linear \( \mathbb{C} \)-basis for \( \mathcal{H}_n(q) \) [Jo, §4]:

\[
    \mathcal{S}_H = \{G_{(i_1, \ldots, i_p)}(G_{i_1+1}G_{i_1-1})\cdots(G_{i_p+1}G_{i_p-1}) \mid 1 \leq i_1 < \cdots < i_p \leq n - 1\}.
\]

Note that all generators \( G_i \) are invertible in \( \mathcal{H}_n(q) \), with

\[
    G_i^{-1} = q^{-1}G_i + (q^{-1} - 1) \quad \text{for all } i.
\]

\( ^1 \)This result also follows from [GePf, Lemma 3.1.14], since \( w \) is a Coxeter element of the parabolic subgroup \( W_J \) of \( \Sigma_n \), where \( J = \{s_{i_1}, s_{i_2}, \ldots, s_{i_k}\} \subseteq S \).
1.4. Computation formulas in the Iwahori-Hecke algebra. Let \( m \in \mathbb{N} \). It is easy to check that
\[
G_i^m = (q^{m-1} - q^{m-2} + \cdots + (-1)^{m-2}q + (-1)^{m-1}) G_i + (q^{m-1} - q^{m-2} + \cdots + (-1)^{m-2}q).
\]
Hence, if \( m \) is even, we have
\[
(1.4) \quad G_i^m = \frac{q^m - 1}{q + 1} G_i + \frac{q^m - 1}{q + 1} + 1,
\]
and if \( m \) is odd, we have
\[
(1.5) \quad G_i^m = \frac{q^m + 1}{q + 1} G_i + \frac{q^m + 1}{q + 1} - 1.
\]

1.5. The Ocneanu trace \( \tau \). The natural inclusions \( B_n \subset B_{n+1} \) of the classical braid groups give rise to the algebra inclusions:
\[
\mathbb{C}B_0 \subset \mathbb{C}B_1 \subset \mathbb{C}B_2 \subset \ldots
\]
(setting \( \mathbb{C}B_0 := \mathbb{C} \)), which in turn induce the following algebra inclusions:
\[
\mathcal{H}_0(q) \subset \mathcal{H}_1(q) \subset \mathcal{H}_2(q) \subset \ldots
\]
(setting \( \mathcal{H}_0(q) := \mathbb{C} \)). Then we have the following.

**Theorem 2.** [Jo, Theorem 5.1] Let \( \zeta \) be an indeterminate. There exists a unique linear Markov trace
\[
\tau : \bigcup_{n \geq 0} \mathcal{H}_n(q) \longrightarrow \mathbb{C}[\zeta]
\]
defined inductively on \( \mathcal{H}_n(q) \) for all \( n \), by the following rules:
\[
\begin{align*}
\tau(hh') &= \tau(h'h) \\
\tau(1) &= 1 \\
\tau(hG_n) &= \zeta \tau(h) \quad \text{(Markov property)}
\end{align*}
\]
where \( h, h' \in \mathcal{H}_n(q) \).

The trace \( \tau \) is the Ocneanu trace with parameter \( \zeta \). Diagrammatically, in the second rule, 1 corresponds to the identity braid on any number of strands. The third rule is the so-called Markov property of the trace. One can look at the left-hand illustration of Figure 1 for a topological interpretation of the Markov property.

Another characterization of the Ocneanu trace can be given as follows. Every trace function is uniquely determined by its values on the basis elements \( G_w \), where \( w \) runs over a certain set of representatives of the various conjugacy classes of \( S_n \). Following [GePf, 3.1.16], these particular representatives are the elements \( w_\mu \), defined in §1.2 for every partition \( \mu \) of \( n \). Applying the defining formula for the Ocneanu trace \( \tau \) to the element \( G_{w_\mu} \), we see that
\[
(1.7) \quad \tau(G_{w_\mu}) = \zeta^{\ell(w_\mu)}.
\]
Conversely, if \( \psi \) is any trace function on \( \mathcal{H}_n(q) \) such that \( \psi(G_{w_\mu}) = \zeta^{\ell(w_\mu)} \) for all partitions \( \mu \) of \( n \), then \( \psi = \tau \).

1.6. The HOMFLYPT polynomial. Let now \( \mathcal{L} \) be the set of isotopy classes of oriented links in \( S^3 \). We know from Jones’ construction [Jo] that, in order to obtain a link invariant according to the Markov equivalence for braids, the closed braids \( \hat{\alpha}, \hat{\alpha}\sigma_n \) and \( \hat{\alpha}\sigma_n^{-1} \) have to be assigned the same value for any braid \( \alpha \in B_n \). Therefore, in order to obtain a link invariant from the trace \( \tau \), it has to be re-scaled, so that \( \tau(hG_n^{-1}) = \tau(hG_n) \) for all \( h \in \mathcal{H}_n(q) \), and also normalized, so that the closed braids \( \hat{\alpha} \) and \( \hat{\alpha}\sigma_n \) be assigned the same value. Set now
\[
\lambda_H := \frac{\zeta + (1 - q)}{q\zeta}.
\]
**Definition 1.** [Jo Definition 6.1] We define a map $P$ on the set $L$ by defining $P$ on the closure $\hat{\alpha}$ of any braid $\alpha \in B_n$, for all $n \in \mathbb{N}$, as follows:

$$P(\hat{\alpha}) := \left( \frac{1 - \lambda H q}{\sqrt{\lambda H (1 - q)}} \right)^{n-1} (\sqrt{\lambda H})^{\epsilon(\alpha)} (\tau \circ \pi)(\alpha)$$

where $\pi : CB_n \rightarrow H_n(q)$ is the natural algebra epimorphism that maps the braid generator $\sigma_i$ to the algebra generator $G_i$, and $\epsilon(\alpha)$ is the sum of the exponents of the braid generators in the braid word $\alpha$. Equivalently, by setting

$$D_H := \frac{1 - \lambda H q}{\sqrt{\lambda H (1 - q)}} = \frac{1}{\zeta \sqrt{\lambda H}}$$

we can write:

$$P(\hat{\alpha}) = (D_H)^{n-1}(\sqrt{\lambda H})^{\epsilon(\alpha)} (\tau \circ \pi)(\alpha).$$

As it turns out, the map $P$ is well-defined on $L$ and it defines the well-known 2-variable Jones or HOMFLYPT polynomial, an isotopy invariant of classical knots and links. This map depends on the quadratic relation (1.2)(h), so an automorphism of the Iwahori-Hecke algebra $H_n(q)$ may give rise to a different map. However, one can easily check that the map $P'$ induced by the automorphism

(1.8) $G_i \mapsto -q^{-1}G_i$

is equal to $P$ (if $G_i' := -q^{-1}G_i$), then $G_i'^2 = (q^{-1} - 1)G'_i + q^{-1}$ and $\tau(G'_i) = -q^{-1} \zeta$. We will need this later.

**2. Knot invariants from the Yokonuma-Hecke algebras**

2.1. The Yokonuma-Hecke algebra. In the sequel we fix $d \in \mathbb{N}$. Let $u \in \mathbb{C}\setminus \{0\}$. The Yokonuma-Hecke algebra, denoted by $Y_{d,n}(u)$, is a $\mathbb{C}$-associative algebra generated by the elements $g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$

subject to the following relations:

$$(b_1) \quad g_i g_j = g_j g_i \quad \text{for } |i - j| > 1$$

$$(b_2) \quad g_i g_j g_i = g_j g_i g_j \quad \text{for } |i - j| = 1$$

$$(f_1) \quad t_i t_j = t_j t_i \quad \text{for all } i, j$$

$$(f_2) \quad t_j g_i = g_i t_{s(i)} \quad \text{for all } i, j$$

$$(f_3) \quad t_j^d = 1 \quad \text{for all } j$$

where $s_i$ is the transposition $(i, i+1)$, together with the extra quadratic relations:

$$g_i^2 = 1 + (u - 1) e_i + (u^{-1} - 1) e_i g_i \quad \text{for all } i$$

where

$$e_i := \frac{1}{d} \sum_{s=0}^{d-1} t_i^s t_{i+1}^s.$$  

It is easily verified that the elements $e_i$ are idempotents in $Y_{d,n}(u)$ [Ju Lemma 4]. Also, the elements $g_i$ are invertible, with

$$g_i^{-1} = g_i + (u^{-1} - 1) e_i + (u - 1) e_i g_i.$$

The relations $(b_1), (b_2), (f_1)$ and $(f_2)$ are defining relations for the classical framed braid group $\mathcal{F}_n \cong \mathbb{Z}^n \rtimes B_n$, with the $t_j$'s being interpreted as the ‘elementary framings’ (framing 1 on the $j$th strand). The relations $t_j^d = 1$ mean that the framing of each braid strand is regarded modulo $d$. Thus, the algebra $Y_{d,n}(u)$ arises naturally as a quotient of the framed braid group algebra over the modular relations $(f_3)$ and the quadratic relations (2.2) [Ju]. Moreover, relations (2.2) are defining relations for the modular framed braid group $\mathcal{F}_{d,n} \cong (\mathbb{Z} / d\mathbb{Z})^n \rtimes B_n$, so the algebra $Y_{d,n}(u)$ can be also seen as a quotient of the modular framed braid group algebra over the quadratic relations (2.2).

From the above, the algebra $Y_{d,n}(u)$ has natural topological interpretation in the context of framed braids and framed knots. However, in [JuLa3] a different topological interpretation to $Y_{d,n}(u)$ was given, in relation to classical knots and links. Namely, viewing the $t_j$’s only as formal generators and ignoring their framing interpretation, we have by relations $(b_1)$ and $(b_2)$ that the classical braid group $B_n$ is represented in $Y_{d,n}(u)$. 


The Yokonuma-Hecke algebra was originally introduced by T. Yokonuma [Yo]. For $d = 1$, the algebra $Y_{1,n}(u)$ coincides with the Iwahori-Hecke algebra $H_n(u)$ of type $A$. For more details and for further topological interpretations, see [JuLa1, JuLa2, JuLa3, JuLa4] and references therein.

2.2. Computation formulas in the Yokonuma-Hecke algebra. Let $i, k \in \{1, 2, \ldots, n\}$ and let

$$e_{i,k} := \frac{1}{d} \sum_{s=0}^{d-1} t^s_i t^d_k.$$

Clearly $e_{i,k} = e_{k,i}$ and it can be easily deduced that $e^2_{i,k} = e_{i,k}$. Note that $e_{i,i} = 1$ and that $e_{i,i+1} = e_i$.

Now, in $Y_{d,n}(u)$ the following relations hold (see [JuLa1, Lemma 4, Proposition 5]):

$$
t_j e_i = e_i t_j,
$$

$$e_{j} e_i = e_i e_{j},
$$

$$g_j e_i = e_i g_j$$

for $j \neq i, i + 1$

$$g_{i-1} e_i = e_{i-1,i+1} g_{i-1},
$$

$$g_{i} g_{i-1} = g_{i-1} g_{i-1,i+1},
$$

$$g_{i+1} e_{i} = e_{i,i+2},
$$

$$g_{i} g_{i+1} = g_{i+1} e_{i,i+2},
$$

$$e_{j} g_{i} g_{j} = g_{j} g_{i} e_{i}$$

for $|i - j| = 1$.

Note that, using (2.4), relations (2.6) are also valid if all $g_i$’s are replaced by their inverses $g_i^{-1}$. Moreover, the following relations hold in $Y_{d,n}(u)$.

**Lemma 1.** Let $i, k \in \{1, 2, \ldots, n\}$. We have

$$t_i e_{i,k} = t_k e_{i,k}.$$

In particular,

$$t_i e_i = t_{i+1} e_i.$$

**Proof.** We have

$$t_i e_{i,k} = \frac{1}{d} \sum_{s=0}^{d-1} t^{s+1}_i t^{d-s}_k = \frac{1}{d} \sum_{s=0}^{d-1} t^s_i t^{d-s+1}_k = \frac{1}{d} \left( \sum_{s=0}^{d-1} t^s_i t^{d-s+1}_k + t_k \right) = t_k \left( \frac{1}{d} \sum_{s=0}^{d-1} t^s_i t^{d-s}_k \right) = t_k e_{i,k}.$$



The following equalities are easy to check (see, for example, [JuLa2, Lemma 1]):

**Lemma 2.** Let $m \in \mathbb{N}$. Then, if $m$ is even, we have

$$g_i^m = \frac{u^m - 1}{u + 1} e_i g_i + \frac{u^m - 1}{u + 1} e_i + 1,$$

and if $m$ is odd, we have

$$g_i^m = \frac{u^m - u}{u + 1} e_i g_i + \frac{u^m - u}{u + 1} e_i + g_i.$$

2.3. The Juyumaya trace $\text{tr}$. The natural inclusions $F_n \subset F_{n+1}$ of the classical framed braid groups induce natural inclusions $F_{d,n} \subset F_{d,n+1}$ of modular framed braid groups and these give rise to the algebra inclusions:

$$\mathbb{C} F_{d,0} \subset \mathbb{C} F_{d,1} \subset \mathbb{C} F_{d,2} \subset \ldots$$

(setting $\mathbb{C} F_{d,0} := \mathbb{C}$), which in turn induce the following algebra inclusions:

$$Y_{d,0}(u) \subset Y_{d,1}(u) \subset Y_{d,2}(u) \subset \ldots$$

(setting $Y_{d,0}(u) := \mathbb{C}$). Then we have the following:
Theorem 3. \[Ju\] Theorem 12] Let \( z, x_1, \ldots, x_{d-1} \) be indeterminates. There exists a unique linear Markov trace
\[
\text{tr} : \bigcup_{n \geq 0} Y_{d,n}(u) \rightarrow \mathbb{C}[z, x_1, \ldots, x_{d-1}]
\]
defined inductively on \( Y_{d,n}(u) \) for all \( n \), by the following rules:
\[
\begin{align*}
\text{tr}(ab) &= \text{tr}(ba) \\
\text{tr}(1) &= 1 \\
\text{tr}(aq_n) &= z \text{tr}(a) \quad \text{(Markov property)} \\
\text{tr}(at_{n+1}^m) &= x_m \text{tr}(a) \quad (m = 1, \ldots, d-1)
\end{align*}
\]
where \( a, b \in Y_{d,n}(u) \).

We shall call the trace \( \text{tr} \) the \textit{Juyumaya trace} with parameters \( z, x_1, \ldots, x_{d-1} \). Diagrammatically, in the second rule, 1 corresponds to the identity braid on any number of strands with all framings zero. The following figure gives the topological interpretations of the last two rules.

\[
\text{Figure 1. Topological interpretations of the trace rules}
\]

The trace rules yield the following relations for all \( i \):
\[
(2.8) \quad \text{tr}(e_i) = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s} =: E \quad \text{and} \quad \text{tr}(e_i g_i) = \text{tr}(g_i) = z.
\]

Using \(2.8\), Lemma \(2\) implies that the following relations hold: For \( m \in \mathbb{Z}^>0 \), we have
\[
(2.9) \quad \text{tr}(g_i^m) = \frac{u^m - 1}{u + 1} z + \frac{u^m - 1}{u + 1} E + 1 \quad \text{if} \ m \text{ is even}
\]
and
\[
(2.10) \quad \text{tr}(g_i^m) = \frac{u^m + 1}{u + 1} z + \frac{u^m + 1}{u + 1} E - E \quad \text{if} \ m \text{ is odd}.
\]

2.4. \textit{An inductive basis for the Yokonuma-Hecke algebra.} The key in the construction of the trace \( \text{tr} \) is that \( Y_{d,n}(u) \) has a ‘nice’ inductive linear \( \mathbb{C} \)-basis. Namely, every element of \( Y_{d,n+1}(u) \) is a unique linear combination of words of the following types:
\[
(2.11) \quad w_n g_n g_{n-1} \cdots g_i t_i^k \quad \text{or} \quad w_n t_n^k \quad (k \in \mathbb{Z}/d\mathbb{Z})
\]
where \( w_n \in Y_{d,n}(u) \). Thus, the above words furnish an inductive basis for \( Y_{d,n+1}(u) \), every element of which involves \( g_n \) or a power of \( t_{n+1} \) at most once.

2.5. \textit{The split property for the Yokonuma-Hecke algebra.} Due to the relations \(2.1\)(f\(_1\)) and \(2.1\)(f\(_2\)), every monomial \( w \) in \( Y_{d,n}(u) \) can be written in the form
\[
w = t_1^{k_1} \cdots t_n^{k_n} \cdot \sigma
\]
where \( k_1, \ldots, k_n \in \mathbb{Z}/d\mathbb{Z} \) and \( \sigma \) is a word in \( g_1, \ldots, g_{n-1} \). That is, \( w \) splits into the ‘framing part’ \( t_1^{k_1} \cdots t_n^{k_n} \) and the ‘braiding part’ \( \sigma \). Applying further the braid relations \(2.1\)(b\(_1\)) and \(2.1\)(b\(_2\)) and the quadratic relations \(2.2\), we deduce that the following set is a \( \mathbb{C} \)-basis for \( Y_{d,n}(u) \) \[Ju JuLa1\]:
\[
S_Y = \left\{ t_1^{k_1} \cdots t_n^{k_n} (g_{i_1} \cdots g_{i_{1-r_1}})(g_{i_2} \cdots g_{i_{2-r_2}}) \cdots (g_{i_p} \cdots g_{i_{p-r_p}}) \mid k_1, \ldots, k_n \in \mathbb{Z}/d\mathbb{Z} \quad 1 \leq i_1 < \cdots < i_p \leq n-1 \right\}
\]
2.6. The E-system. Let now \( \mathcal{L} \) be, as above, the set of isotopy classes of oriented links in \( S^3 \). As mentioned in Subsection 1.6, likewise here, in order to obtain a link invariant from the trace \( \text{tr} \) according to the Markov equivalence for braids, the trace has to be: normalized, so that the closed braids \( \widehat{\alpha} \) and \( \widehat{\sigma} \) (\( \alpha \in B_n \)) be assigned the same value of the invariant, and re-scaled, so that the closed braids \( \widehat{\sigma} \) (\( \alpha \in B_n \)) be assigned the same value of the invariant.

Trying to do that, it was shown in [JuLa2] that \( \text{tr} \) does not re-scale directly (being the only known Markov trace with this property). Indeed, for \( \alpha \in Y_{d,n}(u) \), we compute:

\[
\text{tr}(\alpha g_n^{-1}) = \text{tr}(\alpha g_n) + (u^{-1} - 1)\text{tr}(\alpha e_n) + (u^{-1} - 1)\text{tr}(\alpha e_n g_n).
\]

Now, although

\[
\text{tr}(\alpha e_n g_n) = \text{tr}(\alpha g_n) = z \text{tr}(\alpha) = \text{tr}(g_n)\text{tr}(\alpha)
\]

we have that \( \text{tr}(\alpha e_n) \) does not factor through \( \text{tr}(\alpha) \), that is,

\[
\text{tr}(\alpha e_n) \neq \text{tr}(e_n)\text{tr}(\alpha)
\]

leading to the fact that \( \text{tr}(\alpha g_n^{-1}) \) does not factor through \( \text{tr}(\alpha) \), that is,

\[
\text{tr}(\alpha g_n^{-1}) \neq \text{tr}(g_n^{-1})\text{tr}(\alpha).
\]

Forcing \( \text{tr}(\alpha e_n) = \text{tr}(e_n)\text{tr}(\alpha) \) yields that the trace parameters \( x_1, \ldots, x_{d-1} \) have to satisfy the so-called E-system [JuLa2] \( \S4 \). The E-system is a non-linear system of equations of the form:

\[
\sum_{s=0}^{d-1} x_{m+s}x_{d-s} = x_m \sum_{s=0}^{d-1} x_s x_{d-s} \quad (1 \leq m \leq d-1)
\]

where the sub-indices on the \( x_j \)'s are regarded modulo \( d \) and \( x_0 := 1 \). Equivalently, the E-system is written as

\[
E^{(m)} = x_m E
\]

where

\[
E^{(m)} := \frac{1}{d} \sum_{s=0}^{d-1} x_{m+s}x_{d-s} \quad \text{and} \quad E := E^{(0)} = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s} = \text{tr}(e_i).
\]

As it is shown in [JuLa2] (in the Appendix by Paul Gérardin), the solutions of the E-system are parametrized by the non-empty subsets of \( \mathbb{Z}/d\mathbb{Z} \). Let \( X_S = \{x_1, \ldots, x_{d-1}\} \) be a solution of the E-system parametrized by the non-empty subset \( S \) of \( \mathbb{Z}/d\mathbb{Z} \), and let us denote by \( \text{tr}_S \) the Juyumaya trace with the parameters \( x_1, \ldots, x_{d-1} \) specialized to \( x_1, \ldots, x_{d-1} \). Then, as it turns out [JuLa2],

\[
E = \text{tr}_S (e_i) = \frac{1}{|S|} \quad (1 \leq i \leq n-1).
\]

Moreover, we have [JuLa2] Theorem 7:

\[
\text{tr}_S (\alpha e_n) \equiv \text{tr}_S (\alpha) \text{tr}_S (e_n) = E \text{tr}_S (\alpha) \quad (\alpha \in Y_{d,n}(u)).
\]

**Notation.** The symbol ‘\( \equiv \)’ will stand for ‘\( = \)’ up to the E-condition, that is, with a given solution of the E-system.

2.7. The Case \( E = 1 \). The ‘trivial’ solutions of the E-system are the ones parametrized by the singleton subsets of \( \mathbb{Z}/d\mathbb{Z} \). By (2.14), if \( S \) is a singleton then \( E = \text{tr}_S (e_i) = 1 \). In this case, Gérardin has shown, in the Appendix of [JuLa2], that \( x_1 \) is a \( d \)-th root of unity and \( x_m = x_1^m \) (\( 1 \leq m \leq d-1 \)). Consequently,

\[
x_{k+l} = x_1^{k+l} = x_1^k x_1^l = x_k x_l \quad (k, l \in \mathbb{Z}/d\mathbb{Z}).
\]

These solutions are not very interesting topologically, but we prove here that they have the following interesting property (a stronger version of (2.15)):

**Proposition 1.** Let \( X_S \) be a solution of the E-system such that \( E = 1 \). Then

\[
\text{tr}_S (\beta e_j) \equiv \text{tr}_S (\beta) \text{tr}_S (e_j) = \text{tr}_S (\beta) \quad (\beta \in Y_{d,n+1}(u), \ 1 \leq j \leq n).
\]
Proof. It is enough to show that the above equality holds for the elements of the inductive basis given in (2.11). We will proceed by induction on \( n \). Let \( n = 1 \) and let \( k, l \in \mathbb{Z}/d\mathbb{Z} \). We have

\[
\text{tr}_s(t^k_1g_1t^l_1e_1) = \text{tr}_s(t^k_1g_1t^l_1) = 0. 
\]

Moreover, following Lemma 1, we obtain:

\[
\text{tr}_s(t^k_1t^l_1e_1) = \text{tr}_s(t^{k+l}_1e_1) = \text{tr}_s(t^{k+l}_1) = x_kx_l = tr_s(t^{k+l}_1).
\]

Now let \( n > 1 \) and assume that the statement of the proposition holds for smaller values of \( n \).

Let \( \beta = w_ng_n\ldots g_it^k_i \) for some \( k \in \mathbb{Z}/d\mathbb{Z} \) and some \( w_n \in Y_{d,n}(u) \). Assume first that \( j < n \). Then

\[
\text{tr}_s(w_ng_n\ldots g_it^k_i e_j) = z\text{tr}_s(w_ng_n\ldots g_it^k_i).
\]

By the induction hypothesis, the latter is equal to

\[
\text{tr}_s(w_ng_n\ldots g_it^k_i e_j) = \text{tr}_s(w_ng_n\ldots g_it^k_i) = z\text{tr}_s(w_ng_n\ldots g_it^k_i w_n e_{n-1}).
\]

so we are done. Now take \( j = n \). If \( i = n \), then, by (2.15) and (2.12):

\[
\text{tr}_s(w_ng_n t^k_n e_n) = \text{tr}_s(w_ng_n t^k_n).
\]

If \( i < n \), then by (2.1):

\[
\text{tr}_s(w_ng_n\ldots g_it^k_i e_n) = \text{tr}_s(w_ng_n\ldots g_it^k_i) = \text{tr}_s(w_ng_n\ldots g_it^k_i w_n e_{n-1}).
\]

By the induction hypothesis, the latter is equal to

\[
\text{tr}_s(w_ng_n\ldots g_it^k_i w_n),
\]

whence we deduce that

\[
\text{tr}_s(w_ng_n\ldots g_it^k_i e_n) = \text{tr}_s(w_ng_n\ldots g_it^k_i).
\]

Now let \( \beta = w_nt^k_{n+1} \) for some \( k \in \mathbb{Z}/d\mathbb{Z} \) and some \( w_n \in Y_{d,n}(u) \). Assume again first that \( j < n \). Applying the trace definition and the induction hypothesis we obtain:

\[
\text{tr}_s(w_n t^k_{n+1} e_j) = x_k\text{tr}_s(w_n e_j) = x_k\text{tr}_s(w_n) = \text{tr}_s(w_n t^k_{n+1}).
\]

Now take \( j = n \). With the use of Lemma 1 and (2.15), we obtain:

\[
\text{tr}_s(w_n t^k_n e_n) = \text{tr}_s(w_n t^k_n) = \text{tr}_s(w_n t^k_n) \text{tr}_s(e_n) = \text{tr}_s(w_n t^k_n).
\]

We need to show that, under the assumptions of the proposition:

\[
\text{tr}_s(w_n t^k_n) = \text{tr}_s(w_n t^k_{n+1}) = x_k\text{tr}_s(w_n).
\]

Again, it is enough to show this for the elements of the inductive basis of \( Y_{d,n}(u) \). We have already shown that for \( n = 1 \) and we will proceed by induction on \( n \). Let \( w_n = w_{n-1}g_{n-1}g_{n-2}\ldots g_l t^k_l \) for some \( l \in \mathbb{Z}/d\mathbb{Z} \) and some \( w_{n-1} \in Y_{d,n-1}(u) \). Then

\[
\text{tr}_s(w_n t^k_n) = \text{tr}_s(w_{n-1}g_{n-1}g_{n-2}\ldots g_l t^k_l) = \text{tr}_s(w_{n-1} t^k_{n-1}g_{n-1}g_{n-2}\ldots g_l t^k_l) = z\text{tr}_s(w_{n-1} t^k_{n-1}g_{n-1}g_{n-2}\ldots g_l t^k_l).
\]

By the induction hypothesis, the latter is equal to

\[
z x_k \text{tr}_s(w_{n-1} g_{n-2}\ldots g_l t^k_l) = x_k \text{tr}_s(w_{n-1} g_{n-2}\ldots g_l t^k_l) = x_k \text{tr}_s(w_{n-1}).
\]

Finally, let \( w_n = w_{n-1} t^k_n \) for some \( l \in \mathbb{Z}/d\mathbb{Z} \) and some \( w_{n-1} \in Y_{d,n-1}(u) \). Then

\[
\text{tr}_s(w_n t^k_n) = \text{tr}_s(w_{n-1} t^k_{n+1}) = \text{tr}_s(w_{n-1} t^k_{n+1}) = x_{k+l} \text{tr}_s(w_{n-1}) = x_k x_l \text{tr}_s(w_{n-1}) = x_k \text{tr}_s(w_{n-1}) = x_k \text{tr}_s(w_n).
\]

Note that there are also non-trivial solutions of the E-system, with more interesting topological interpretations. For a detailed analysis see [JuLa2].
Theorem 4. For a solution $X_S = \{x_1, \ldots, x_{d-1}\}$ of the E-system, we wish to define a link isotopy invariant $\Delta_S$. Let $\alpha \in B_n$. The E-condition guarantees that $\Delta_S(\alpha \sigma_n) = \Delta_S(\alpha \sigma_n^{-1})$. In order for $\Delta_S(\alpha \sigma_n) = \Delta_S(\hat{\alpha})$ to hold, we need to normalize. Thus, by setting:

$$\lambda_Y := \frac{z + (1 - u)E}{uz}$$

we define the following map on the set $\mathcal{L}$ of oriented classical link types.

**Definition 2.** [JuLa3] Definition 3] For a solution $X_S$ of the E-system parametrized by the subset $S$ of $\mathbb{Z}/d\mathbb{Z}$, we define a map $\Delta_S$ on the set $\mathcal{L}$ by defining $\Delta_S$ on the closure $\hat{\alpha}$ of any braid $\alpha \in B_n$, for all $n \in \mathbb{N}$, as follows:

$$\Delta_S(\hat{\alpha}) := \left( -\frac{1 - \lambda_Y u}{\sqrt{\lambda_Y (1 - u)E}} \right)^{n-1} (\sqrt{\lambda_Y})^{e(\alpha)} (\text{tr}_S \circ \delta)(\alpha)$$

where $\delta : CB_n \to Y_{d,n}(u)$ is the natural algebra homomorphism that maps the braid generator $g_i$ to the algebra generator $x_i$, and $e(\alpha)$ is the sum of the exponents of the braid generators in the braid word $\alpha$. Equivalently, by setting

$$D_Y := -\frac{1 - \lambda_Y u}{\sqrt{\lambda_Y (1 - u)E}} = \frac{1}{z \sqrt{\lambda_Y}}$$

we can write:

$$\Delta_S(\hat{\alpha}) = D_Y^{n-1} (\sqrt{\lambda_Y})^{e(\alpha)} (\text{tr}_S \circ \delta)(\alpha).$$

In [JuLa3] the following result was obtained:

**Theorem 4.** For a solution $X_S$ of the E-system, the map $\Delta_S$ is well-defined on the set $\mathcal{L}$, that is, it is a 2-variable isotopy invariant for oriented classical links, depending on the variables $u, z$.

Note that for every $d \in \mathbb{N}$, the above construction provides us with $2^d - 1$ isotopy invariants for knots.

As shown in [CJJKL], the invariants $\Delta_S$ (but not the trace $\text{tr}$) have the multiplicative property on connected sums of links. Further, it was shown in [JuLa3] that $\Delta_S$ satisfies a ‘closed’ cubic relation (closed in the sense of only involving braiding generators), which factors to the quadratic relation of the Iwahori–Hecke algebra $\mathcal{H}_n(u)$. Finally, it was shown in [JuLa2] that in the context of framed links the invariant $\Delta_S$ satisfies a skein relation. However, when $\Delta_S$ is seen as invariant of classical knots, this skein relation has no topological interpretation. This makes it very difficult to compare the invariants $\Delta_S$ with the HOMFLYPT polynomial using diagrammatic methods.

3. The Ocneanu trace vs the specialized Juyumaya trace

In order to compare the knot invariants $P$ and $\Delta_S$, in view of the E-condition, we would like to be able to specialize the indeterminates $x_1, \ldots, x_{d-1}$ to a solution of the E-system as early as possible during the construction.

3.1. The algebra homomorphism approach. Our first approach to the above problem is to construct an algebra homomorphism $f : \bigcup_{n \geq 0} Y_{d,n}(u) \to \bigcup_{n \geq 0} Y_{d,n}(u)$ such that

$$f(t_i^m) = x_m \quad (1 \leq m \leq d - 1),$$

where $x_1, x_2, \ldots, x_{d-1} \in \mathbb{C} \setminus \{0\}$. Since $f$ is an algebra homomorphism, we must have

$$f(t_i^k t_l^l) = f(t_i^{k+l}) = x_{k+l} = x_k x_l = f(t_i^k) f(t_l^l) \quad \text{for } k, l \in \mathbb{Z}/d\mathbb{Z},$$

that is, $x_1$ is a $d$-th root of unity and $x_m = x_1^m \ (1 \leq m \leq d - 1)$. In this case, the set $X_S = \{x_1, \ldots, x_{d-1}\}$ is a solution of the E-system such that $E = 1$. Moreover, we have

$$f(e_i) = \frac{1}{d} \sum_{s=0}^{d-1} f(t_i^s) f(t_i^{-s}) = \frac{1}{d} \sum_{s=0}^{d-1} x_s x_{d-s} = \text{tr}_S(e_i) = E = 1$$

and

$$f(g_i^2) = f(g_i^2) = f(1 + (u - 1) e_i + (u - 1) e_i g_i) = 1 + (u - 1) + (u - 1) f(g_i) = u + (u - 1) f(g_i),$$

whence we can easily deduce that $f(Y_{d,n}(u))$ is isomorphic to the Iwahori–Hecke algebra $\mathcal{H}_n(u)$. 

The condition \( E = 1 \) is quite restrictive, thus making \( f \) an uninteresting mapping for our purposes. This is why in the rest of this section we will proceed with a step-by-step specialization \( t^m \mapsto x_m \), namely with the construction of a specialized Juyumaya trace, and a linear map \( \varphi \) on the Yokonuma-Hecke algebra through which the trace factors. This will allow us to conclude, in Subsection 3.4, that the invariants \( P \) and \( \Delta_S \) coincide when \( E = 1 \).

3.2. The specialized Juyumaya trace.

**Definition 3.** Let \( x_1, x_2, \ldots, x_{d-1} \in \mathbb{C} \setminus \{0\} \) and consider the ring homomorphism

\[
\theta : \mathbb{C}[z, x_1, \ldots, x_{d-1}] \rightarrow \mathbb{C}[z], \quad z \mapsto z, \quad x_m \mapsto x_m \quad (1 \leq m \leq d-1)
\]

The map \( \theta \) shall be called the *specialization map*. We will call the composition

\[
\theta \circ \text{tr} : \bigcup_{n \geq 0} Y_{d,n}(u) \rightarrow \mathbb{C}[z]
\]

the *specialized Juyumaya trace* with parameter \( z \).

In the case where \( X_S = \{x_1, \ldots, x_{d-1}\} \) is a solution of the E-system, we have \( \theta \circ \text{tr} = \text{tr}_x \). Note also that in the case \( d = 1 \), when the algebra \( Y_{1,n}(u) \) coincides with the Iwahori-Hecke algebra \( \mathcal{H}_n(u) \), \( \theta \) is simply the identity map on \( \mathbb{C}[z] \) and the specialized Juyumaya trace \( \theta \circ \text{tr} = \text{tr} \) coincides with the Ocneanu trace with parameter \( z \).

3.3. Similarities with the Ocneanu trace. In this subsection we will give another characterization of the specialized Juyumaya trace as follows. Let \( w \in \mathfrak{S}_n \) and let \( w = s_{i_1}s_{i_2}\ldots s_{i_l} \) be a reduced expression. Then we can set \( g_w := g_{i_1}g_{i_2}\ldots g_{i_l} \). If \( w, w' \in \mathfrak{S}_n \) are such that \( \ell(ww') = \ell(w) + \ell(w') \), then we have

\[
g_w g_{w'} = g_{ww'}.
\]

Let \( \mu \) be a partition of \( n \) and let \( w_\mu \) be the corresponding element of \( \mathfrak{S}_n \) defined in \([12]\). Applying the defining formula for the Juyumaya trace \( \text{tr} \) to the element \( g_{w_\mu} \), we see that \( \text{tr}(g_{w_\mu}) = z^{\ell(w_\mu)} \), whence we deduce:

\[
(\theta \circ \text{tr})(g_{w_\mu}) = z^{\ell(w_\mu)}.
\]

We will show that, as in the case of Iwahori-Hecke algebra of type \( A \), the specialized Juyumaya trace on \( Y_{d,n}(u) \) is uniquely determined by its values on the elements \( g_{w_\mu} \), where \( \mu \) runs over the partitions of \( n \). That is, if \( \psi \) is any trace function on \( Y_{d,n}(u) \) such that \( \psi(g_{w_\mu}) = z^{\ell(w_\mu)} \) for all partitions \( \mu \) of \( n \), then \( \theta \circ \psi = \theta \circ \text{tr} \). To achieve our aim, we shall first construct a linear map \( \varphi : \bigcup_{n \geq 0} Y_{d,n}(u) \rightarrow \bigcup_{n \geq 0} Y_{d,n}(u) \) with the property: \( \text{tr} \circ \varphi = \theta \circ \text{tr} \).

**Proposition 2.** Let \( \theta \) be as above and set \( x_0 := 1 \). Let \( \varphi : \bigcup_{n \geq 0} Y_{d,n}(u) \rightarrow \bigcup_{n \geq 0} Y_{d,n}(u) \) be the linear map defined inductively on \( Y_{d,n}(u) \), for all \( n \in \mathbb{N} \), by the following rules:

\[
\varphi(1) = 1, \quad \varphi(w_n g_{n-1} \ldots g_{t_k}) = g_n \varphi(w_n g_{n-1} \ldots g_{t_k}), \quad \varphi(w_n t_{k+1}) = x_k \varphi(w_n)
\]

where \( w_n \in Y_{d,n}(u) \) and \( k \in \mathbb{Z}/d\mathbb{Z} \). Then we have:

\[
(3.3) \quad \text{tr} \circ \varphi = \theta \circ \text{tr}.
\]

**Proof.** It is enough to show that \((3.3)\) holds on the elements of the inductive basis of \( Y_{d,n+1}(u) \), and we will do this by induction on \( n \). From now on, \( k \) and \( l \) are elements of \( \mathbb{Z}/d\mathbb{Z} \).

First, let \( n = 1 \). We have

\[
\text{tr} \left( \varphi(t_{1}^k g_1) \right) = \text{tr} \left( g_1 \varphi(t_{1}^k) \right) = \text{tr}(x_k g_1) = x_k z = \theta(x_k z) = \theta \left( \text{tr}(t_{1}^k g_1) \right)
\]

and

\[
\text{tr} \left( \varphi(t_{1}^l t_2^k) \right) = \text{tr} \left( x_k \varphi(t_{1}^l) \right) = \text{tr}(x_k x_l) = x_k x_l = \theta(x_k x_l) = \theta \left( \text{tr}(t_{1}^l t_2^k) \right).
\]
Now assume that (3.3) holds for smaller values of $n$. We have
\[ \text{tr} \left( \varphi(w_n g_n g_{n-1} \ldots g_i t_i^k) \right) = \text{tr} \left( g_n \varphi(w_n g_{n-1} \ldots g_i t_i^k) \right) = \text{tr} \left( \varphi(w_n g_{n-1} \ldots g_i t_i^k) \right), \]
since $\varphi(w_n g_{n-1} \ldots g_i t_i^k) \in Y_{d,n}(u)$. By the induction hypothesis, the last term is equal to
\[ z \theta \left( \text{tr}(w_n g_{n-1} \ldots g_i t_i^k) \right) = \theta \left( z \text{tr}(w_n g_{n-1} \ldots g_i t_i^k) \right) = \theta \left( \text{tr}(w_n g_{n-1} \ldots g_i t_i^k) \right). \]
Finally, we have
\[ \text{tr} \left( \varphi(w_n t_{n+1}^k) \right) = \text{tr} \left( x_k \varphi(w_n) \right) = x_k \text{tr} \left( \varphi(w_n) \right). \]
By the induction hypothesis, the last term is equal to
\[ x_k \theta \left( \text{tr}(w_n) \right) = \theta \left( x_k \text{tr}(w_n) \right) = \theta \left( \text{tr}(w_n t_{n+1}^k) \right). \]
\[ \square \]

**Remark 1.** The map $\varphi$ is an intermediate construction between the algebra and the trace map. An analogue of the map $\varphi$ can be constructed on the Iwahori-Hecke algebra $H_n(q)$.

**Remark 2.** By virtue of Proposition 2 for a solution $X_S = \{x_1, \ldots, x_{d-1}\}$ of the E-system, the invariant $\Delta_S$ is rewritten as:
\[ \Delta_S(\alpha) = D^{-1}_Y(\sqrt{\lambda}(\alpha)_V) \circ \varphi \circ \delta(\alpha). \]
Moreover, in view of the discussion in Subsection 3.1, and since $\varphi(g_i) = g_i$ and $\varphi(t_i^m) = x_m$, the map $\varphi$ of Proposition 2 provides us with the earliest possible specialization of $x_1, \ldots, x_{d-1}$ to $X_S$ during the construction of $\Delta_S$.

Proposition 2 implies that the specialized Juyumaya trace is uniquely determined by its values on the elements of the image of $\varphi$. We will now show that $\varphi(Y_{d,n}(u))$ is the subspace $W_n$ of $Y_{d,n}(u)$ spanned by the elements $\{w \in D\}$, where
\[ D = \{s_{i_1} \ldots s_{i_k} s_{i_1} | i_1 < i_2 < \cdots < i_k \} \subset G_n. \]

**Proposition 3.** Let $n \in \mathbb{N}$ and let $W_n$ be the $\mathbb{C}$-linear subspace of $Y_{d,n}(u)$ spanned by the elements $\{g_w \}_{w \in D}$. Then $\varphi(Y_{d,n}(u)) = W_n$.

**Proof.** First note that we have $W_n \subset W_{n+1}$ and $g_n W_n \subset W_{n+1}$. Note also that $\varphi(1) = 1 \in W_n$.

We will first show that $\varphi(Y_{d,n}(u)) \subset W_n$. We will proceed by induction on $n$. Let $n = 1$. Then $\varphi(t_1^k) = x_k \cdot 1 \in W_1$, for all $k \in \mathbb{Z}/d\mathbb{Z}$. Now assume that $\varphi(Y_{d,n}(u)) \subset W_n$. In order to show that $\varphi(Y_{d,n+1}(u)) \subset W_{n+1}$, it is enough to show that the images of the elements of the inductive basis of $Y_{d,n+1}(u)$ under $\varphi$ are contained in $W_{n+1}$. Let $w_n \in Y_{d,n}(u)$ and $k \in \mathbb{Z}/d\mathbb{Z}$. We have
\[ \varphi(w_n g_n g_{n-1} \ldots g_i t_i^k) = g_n \varphi(w_n g_{n-1} \ldots g_i t_i^k) \in g_n W_n \subset W_{n+1} \]
and
\[ \varphi(w_n t_{n+1}^k) = x_k \varphi(w_n) \in W_n \subset W_{n+1}. \]
We conclude that $\varphi(Y_{d,n+1}(u)) \subset W_{n+1}$.

On the other hand, let $w \in D$. Then $g_w = \varphi(g_w)$, and so $W_n \subseteq \varphi(Y_{d,n}(u))$. \[ \square \]

Now let $w \in D$. Following the discussion in (1.2) $w$ has minimal length in its conjugacy class in $G_n$. Suppose that the conjugacy class of $w$ is parametrized by the partition $\mu$ of $n$. By Theorem 1 the elements $w$ and $w_\mu$ are strongly conjugate, that is, there exists a finite sequence $w = w_0, w_1, \ldots, w_r = w_\mu$ such that, for all $i = 0, 1, \ldots, r - 1$,
\[ \ell(w_i) = \ell(w_{i+1}), \quad w_{i+1} = x_i w_i x_i^{-1} \quad \text{and} \quad \ell(x_i w_i) = \ell(x_i) + \ell(w_i) \quad \text{or} \quad \ell(w_i x_i^{-1}) = \ell(w_i) + \ell(x_i^{-1}) \]
for some elements $x_i \in G_n$. If $\ell(x_i w_i) = \ell(x_i) + \ell(w_i)$, then we have
\[ \ell(w_{i+1} x_i) = \ell(x_i w_i) = \ell(x_i) + \ell(w_i) = \ell(w_{i+1}) + \ell(x_i). \]
Following (3.1), since $w_{i+1} x_i = x_i w_i$ we deduce that
\[ g_{w_{i+1}} g_{x_i} = g_{x_i} g_{w_i}, \]
and thus,
\[ g_{w_{i+1}} = g_{x_i} g_{w_i} x_i^{-1}. \]
On the other hand, if $\ell(w_i x_i^{-1}) = \ell(w_i) + \ell(x_i^{-1})$, then we have
$$\ell(x_i^{-1} w_{i+1}) = \ell(w_i x_i^{-1}) = \ell(w_i) + \ell(x_i^{-1}) = \ell(x_i^{-1}) + \ell(w_{i+1}).$$
Following again (3.1), since $x_i^{-1} w_{i+1} = w_i x_i^{-1}$, we deduce that
$$g_{x_i^{-1}} w_{i+1} = g_w g_{x_i^{-1}},$$
and thus,
$$g_{w_{i+1}} = g_{x_i^{-1}}^{-1} g_w g_{x_i^{-1}}.$$
In any case, the elements $g_{w_{i+1}}$ and $g_{w_i}$ are conjugate in $Y_{d,n}(u)$, and so, if $\psi$ is any trace function on $Y_{d,n}(u)$, we have
$$\psi(g_{w_{i+1}}) = \psi(g_{w_i})$$
for all $i = 1, \ldots, r$. Thus, we have
$$\psi(g_w) = \psi(g_{w_0}).$$
In particular, we obtain that
$$\text{tr}(g_w) = \text{tr}(g_{w_0}).$$
From the above, we conclude the following:

**Proposition 4.** The specialized Juyumaya trace $\theta \circ \text{tr}$ on $Y_{d,n}(u)$ is uniquely determined by its values on the elements $g_{w_0}$, where $\mu$ runs over the partitions of $n$.

3.4. **Consequences on the case $E = 1$.** Suppose now that $X = \{x_1, \ldots, x_{d-1}\}$ is a solution of the E-system such that $E = 1$, that is, $x_1$ is a $d$-th root of unity and $x_m = x_1^m$ ($1 \leq m \leq d - 1$). In this case, we can define the algebra epimorphism
$$\gamma : Y_{d,n}(u) \rightarrow \mathcal{H}_n(u)$$
by
$$g_i \mapsto G_i, \quad t_i^m \mapsto x_m \quad (1 \leq m \leq d - 1).$$
This is indeed an algebra homomorphism, since it respects all the defining relations of the algebra $Y_{d,n}(u)$. In particular:
$$\gamma(g_i^2) = \gamma(1 + (u - 1) e_i + (u - 1) e_i g_i) = u + (u - 1) G_i = G_i^2 = \gamma(g_i)^2$$
and
$$\gamma(t_i^k t_i^l) = \gamma(t_i^{k+l}) = x_{k+l} = x_i = \gamma(t_i^k) \gamma(t_i^l) \quad \text{for } k, l \in \mathbb{Z}/d\mathbb{Z}.$$ 

The map $\gamma$ is also clearly surjective.

**Remark 3.** Note that the map $\gamma$ is not an algebra homomorphism if $E \neq 1$, because neither of the above equalities holds.

Now consider the Ocneanu trace on $\mathcal{H}_n(u)$ with parameter $\zeta = z$. The composition $\tau \circ \gamma$ is a Markov trace on $Y_{d,n}(u)$ which takes the same values as the specialized Juyumaya trace $\text{tr}_s$ on the elements $g_{w_0}$, where $\mu$ runs over the partitions of $n$. By Proposition 4, we obtain
$$\tau \circ \gamma = \text{tr}_s.$$ 

The following result is a consequence of (3.4).

**Proposition 5.** Let $X_S$ be a solution of the E-system such that $E = 1$. Let $\text{tr}_s$ be the corresponding specialized Juyumaya trace on $Y_{d,n}(u)$ with parameter $z$, and let $\tau$ be the Ocneanu trace on $\mathcal{H}_n(u)$ with parameter $\zeta$. If we take $u = q$ and $z = \zeta$, then
$$(\tau \circ \pi)(\alpha) = (\text{tr}_s \circ \delta)(\alpha) \quad (\alpha \in B_n)$$
for all $n \in \mathbb{N}$.

**Proof.** Let $\alpha \in B_n$. By definition of the map $\gamma$, we have $(\gamma \circ \delta)(\alpha) = \pi(\alpha)$. Now Equation (5.4) yields the desired result. \qed

Under the assumptions of Proposition 5, we automatically obtain $\lambda_H = \lambda_Y$. We conclude the following.
Corollary 1. Let $X_S$ be a solution of the E-system such that $E = 1$. Let $\operatorname{tr}_S$ be the corresponding specialized Juyumaya trace on $Y_{d,n}(u)$ with parameter $z$, and let $\tau$ be the Ocneanu trace on $H_n(q)$ with parameter $\zeta$. If we take $q = u$ and $\zeta = z$, then
\[ P(\alpha) = \Delta_S(\alpha) \quad (\alpha \in B_n) \]
for all $n \in \mathbb{N}$.

Since the map $P$ is invariant under the Hecke algebra automorphism (1.8), we also obtain the following.

**Corollary 2.** Let $X_S$ be a solution of the E-system such that $E = 1$. Let $\operatorname{tr}_S$ be the corresponding specialized Juyumaya trace on $Y_{d,n}(u)$ with parameter $z$, and let $\tau$ be the Ocneanu trace on $H_n(q)$ with parameter $\zeta$. If we take $q = 1/u$ and $\zeta = -z/u$, then
\[ P(\alpha) = \Delta_S(\alpha) \quad (\alpha \in B_n) \]
for all $n \in \mathbb{N}$.

In the next sections we will explore the remaining cases where the maps $P$ and $\Delta_S$ coincide, and show that they are all trivial, that is, either $u = 1$ or $q = 1$ or $E = 1$.

4. Comparing $P$ and $\Delta_S$

From now on, let $X_S = \{x_1, \ldots, x_{d-1}\}$ be a solution of the E-system parametrized by a subset $S$ of $\mathbb{Z}/d\mathbb{Z}$. Let $\operatorname{tr}_S$ be the corresponding specialized Juyumaya trace on $Y_{d,n}(u)$ with parameter $z$, and let $E = \operatorname{tr}_S(e_i) = 1/|S|$. Let $\tau$ be the Ocneanu trace on $H_n(q)$ with parameter $\zeta$. In this section, we will assume that the maps $P$ and $\Delta_S$ coincide, and we will see what restrictions this assumption imposes on the values that $q$, $z$, $u$, $z$ and $E$ can take.

4.1. Some equalities. First of all, if $P$ and $\Delta_S$ coincide, they should take the same value on the closure of any braid in any $B_n$. In particular for the identity braid $1$ in each $B_n$ we have:
\[ P(1) = D_{n-1}^n = D_Y^1 = \Delta_S(1) \]
for all $n \in \mathbb{N}$, whence we deduce that
\[ (4.1) \quad D_H = D_Y. \]

From (4.1) we obtain the following equality:
\[ (4.2) \quad (u_\zeta + z^2 + E_z)q = u_\zeta(\zeta + 1). \]
If $\zeta = -1$, then we must have $u_\zeta + z^2 - E_z + E_z = 0$. If $\zeta \neq -1$, then (4.2) yields the following equality for $q$:
\[ (4.3) \quad q = \frac{u_\zeta^2 + u_\zeta}{u_\zeta + z^2 - E_z + E_z}. \]

Now if $P = \Delta_S$ and $D_H = D_Y$, we must have
\[ (4.4) \quad \frac{\tau(\pi(\alpha))}{\operatorname{tr}_S(\delta(\alpha))} = \left( \frac{\lambda_Y}{\lambda_H} \right)^{\epsilon(\alpha)} = \left( \frac{\zeta}{z} \right)^{\epsilon(\alpha)} \]
for all $\alpha \in B_n$ and for all $n \in \mathbb{N}$. Taking $\alpha = \sigma^2 \in B_n$ and $\alpha = \sigma^2 \in B_n$, for $n \geq 2$, we obtain respectively:
\[ (4.5) \quad (z^2 + z^2)q = \zeta(b_\zeta + z^2), \]
where
\[ b := \operatorname{tr}_S(g^2) = 1 + (u - 1)E + (u - 1)z, \]
and
\[ (4.6) \quad (bz_\zeta + z^3)q = \zeta(c_\zeta + bz), \]
where
\[ c := \operatorname{tr}_S(g^3) = (u^2 - u)E + (u^2 - u + 1)z. \]
Note that Equation (4.5) implies that $\zeta = -1$ if and only if $b_\zeta + z^2 = 0$. 


From now on, let us assume that $\zeta \neq -1$. Then (4.5) and (4.6) yield respectively the following equalities for $q$:

\[(4.7) \quad q = \frac{b\zeta^2 + z^2\zeta}{z^2\zeta + z^2} \]

and

\[(4.8) \quad q = \frac{c\zeta^2 + bz\zeta}{bz\zeta + z^3} \]

Suppose first that $z^2 \neq b$. Combining (4.3) and (4.7) yields:

\[(4.9) \quad \zeta^2 = \frac{bz^2 - ubEz + bEz - uz^2}{u^2 - ub} \cdot \zeta + \frac{z^4 - uE^2z^3 + E^3 - uz^2}{u^2 - ub}. \]

Suppose now that $bz \neq c$. Combining (4.3) and (4.8) yields:

\[(4.10) \quad \zeta^2 = \frac{cz^2 - ucEz + cEz - uz^2}{uz - uc} \cdot \zeta + \frac{b^2 - ubEz^2 + bEz^2 - uz^3}{uz - uc}. \]

Combining (4.9) and (4.10) yields $\zeta = -1$, which contradicts our assumption, unless

\[u = 1 \text{ or } E = 1 \text{ or } z = \frac{1 - E + u + uE}{2}. \]

We conclude that the only cases where the invariants $P$ and $\Delta_S$ may coincide are the following:

1. $\zeta = -1$;
2. $\zeta \neq -1, z^2 = b$;
3. $\zeta \neq -1, bz = c$;
4. $\zeta \neq -1, u = 1$;
5. $\zeta \neq -1, E = 1$;
6. $\zeta \neq -1, z = (1 - E + u + uE)/2$.

4.2. The case $\zeta = -1$. If $\zeta = -1$, then Equations (4.2) and (4.3) imply that

\[z^2 = uEz - Ez + u \text{ and } z^2 = b = 1 + (u - 1)E + (u - 1)z. \]

Combining the two equalities above yields

\[(u - 1)(E + z) = (u - 1)(Ez + 1) \]

which is true only if $u = 1$ or $E = 1$ or $z = 1$.

If $u = 1$, then $z = -1$ or $z = 1$. If $E = 1$, then $z = -1$ or $z = u$. If $z = 1$ and $u \neq 1$, then $E = -1$ which is absurd.

4.3. The case $\zeta \neq -1, z^2 = b$. If $z^2 = b = 1 + (u - 1)E + (u - 1)z$, then Equation (4.7) yields $q = \zeta$. Replacing $q = \zeta$ in (4.3), we obtain that $z^2 = uEz - Ez + u$. As in the previous subsection, we conclude that $u = 1$ or $E = 1$ or $z = 1$.

If $u = 1$, then $z = -1$ or $z = 1$. If $E = 1$, then $z = -1$ or $z = u$. If $z = 1$ and $u \neq 1$, then $E = -1$ which is absurd.

4.4. The case $\zeta \neq -1, bz = c$. We have

\[bz = z + uEz - Ez + uz^2 - z^2 = u^2z - uz + z + u^2E - uE = c \]

which yields

\[z(u - 1)(E + z) = u(u - 1)(E + z). \]

In the next two subsections, we will see what happens when $u = 1$ and $E = 1$. Thus, for the moment we may assume that $u \neq 1$ and $E \neq 1$.

If $z = -E$, then $b = 1$, and combining Equations (4.3) and (4.7) yields $\zeta = \pm E$ and $q = 1$. If $z \neq -E$, then $z = u$, and combining Equations (4.3) and (4.8) yields a contradiction.
4.5. *The case* $\zeta \neq -1$, $u = 1$. If $\zeta \neq -1$ and $u = 1$, Equation 4.3 becomes

\[ q = \frac{\zeta^2 + \zeta}{\zeta + z^2}. \]  

Moreover, we have $b = 1$, so Equation 4.7 becomes

\[ q = \frac{\zeta^2 + z^2\zeta}{z^2\zeta + z^2}. \]  

Combining Equations 4.11 and 4.12 gives us

\[ \zeta^2(z^2 - 1) = z^2(z^2 - 1), \]

which holds only if $\zeta = z$ or $\zeta = -z$ or $z = 1$ or $z = -1$.

If $\zeta = \pm z$, then $q = 1$. If $z = \pm 1$, then $q = \zeta$.

4.6. *The case* $\zeta \neq -1$, $E = 1$. Suppose that $z^2 \neq b$ (the case $z^2 = b$ has been completely covered in 4.3). Then Equation 4.9 becomes:

\[ \zeta^2 = \frac{zu - z^2}{u} \zeta + \frac{z^2}{u}. \]

The above quadratic equation has two solutions: $\zeta = z$ or $\zeta = -z/u$. If $\zeta = z$, then Equation 4.3 yields $q = u$. If $\zeta = -z/u$, then Equation 4.3 yields $q = 1/u$.

4.7. *The case* $\zeta \neq -1$, $z = (1 - E + u + uE)/2$. As in the previous subsection, we may assume that $z^2 \neq b$. Then Equation 4.9 becomes:

\[ \zeta^2 = \frac{-1 + E - 2u - Eu^2 - u^2}{2u} \zeta + \frac{-1 + 2E - 2u - 2Eu^2 - E^2 - u^2 + 2E^2u - E^2u^2}{4u}. \]

The above quadratic equation has two solutions: $\zeta = -z$ or $\zeta = -z/u$. If $\zeta = -z$, then Equation 4.3 yields $q = -u$. If $\zeta = -z/u$, then Equation 4.3 yields $q = -1/u$.

4.8. *Summarizing*. The cases below are the only cases where $D_H = D_Y$ and

\[ \frac{\tau(G^m)}{\text{tr}_S(g^m)} = \left( \frac{\zeta}{z} \right)^m \text{ for } m \in \{2, 3\}. \]

One can easily check, using (1.4), (1.5), (2.9) and (2.10), that the above equality holds for all $m \in \mathbb{N}$.

| Case | $q$ | $\zeta$ | $u$ | $z$ | $E$ |
|------|-----|---------|-----|-----|-----|
| 1    | 1   | $z$     | 1   | $C^*$| any |
| 2    | 1   | $-z$    | 1   | $C^*$| any |
| 3    | $C^*$| $q$     | 1   | 1   | any |
| 4    | $C^*$| $q$     | 1   | $-1$| any |
| 5    | $C^*$| $-1$    | 1   | 1   | any |
| 6    | $C^*$| $-1$    | 1   | $-1$| any |
| 7    | 1   | $E$     | $C^*$| $-E$| any |
| 8    | 1   | $-E$    | $C^*$| $-E$| any |
| 9    | $C^*$| $q$     | $C^*$| $-1$| 1   |
| 10   | $C^*$| $q$     | $C^*$| $u$ | 1   |
| 11   | $C^*$| $-1$    | $C^*$| $-1$| 1   |
| 12   | $C^*$| $-1$    | $C^*$| $u$ | 1   |
| 13   | $u$ | $z$     | $C^*$| $C^*$| 1   |
| 14   | $1/u$| $-z/u$  | $C^*$| $C^*$| 1   |
| 15   | $-u$| $-z$    | $C^*$| $(1 - E + u + uE)/2$| any |
| 16   | $-1/u$| $-z/u$ | $C^*$| $(1 - E + u + uE)/2$| any |
4.9. **Dismissing two cases.** Let us now take \( \alpha = \sigma_1 \sigma_2^2 \sigma_1 \sigma_2^2 \) in \( B_n \) for \( n \geq 3 \). We have
\[
\tau(\pi(\alpha)) = \tau(G_1 G_2^2 G_1 G_2^2) = (q^2 \zeta - 2q \zeta + \zeta)(q^2 \zeta - q \zeta + \zeta + q^2 - q) + (2q^2 \zeta - 2q \zeta + q^2)(q \zeta - \zeta + q).
\]
and
\[
\text{tr}_s(\delta(\alpha)) = \text{tr}_s(g_1 g_2^2 g_1 g_2^2) = \text{tr}_s(g_1^2) + (u - 1) \text{tr}_s(g_1^2 e_2) + 2(u - 1) \text{tr}_s(g_1^2 g_2 e_2) - (u - 1)^2 [\text{tr}_s(g_1 e_2 g_1 e_2) + 2\text{tr}_s(g_1 e_2 g_1 g_2 e_2) + \text{tr}_s(g_1 g_2 e_2 g_1 e_2)].
\]
Here, the fact that \( X_S = \{x_1, \ldots, x_{d-1}\} \) is a solution of the \( E \)-system simplifies calculations a lot. For example, we automatically deduce that \( \text{tr}_s(g_1^2 e_2) = E \text{tr}_s(g_1^2) \). Now let us see what happens when we try to calculate \( \text{tr}_s(g_2 e_2 g_1 e_2) \). This is always equal to:
\[
\frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} \text{tr}_s(g_1 t_2^l t_3^{-k} g_1 t_2^m t_3^{-m}) = \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} \text{tr}_s(g_1^2 t_2^l t_3^{-k} t_2 t_3^{-m}) = \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} x_{d-k-m} \text{tr}_s(g_1^2 t_2^l t_3^m)
\]
We have
\[
\text{tr}_s(g_1^2 t_2^l t_3^m) = \text{tr}_s(t_2^l t_2^m) + (u - 1) \text{tr}_s(e_1 t_2^l t_2^m) + (u - 1) \text{tr}_s(g_1 e_1 t_2^l t_2^m)
\]
\[
= x_k x_m + (u - 1) \frac{1}{d} \sum_{l=0}^{d-1} x_{k+l} x_{m-l} + (u - 1) z x_{k+m}
\]
\[
= x_k x_m + (u - 1) E x_{k+m} + (u - 1) z x_{k+m}
\]
Thus,
\[
\text{tr}_s(g_1 e_2 g_1 e_2) = \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} x_{d-k-m} x_k x_m + [(u - 1)(E + z)] \frac{1}{d^2} \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} x_{d-k-m} x_k x_m
\]
\[
= \frac{1}{d} \sum_{k=0}^{d-1} x_{d-k} x_k E + [(u - 1)(E + z)] \frac{1}{d} \sum_{k=0}^{d-1} E
\]
\[
= E^2 + [(u - 1)(E + z)] E = E(uE + uz - z) = E \text{tr}_s(e_1 g_1^2)
\]
We finally obtain that
\[
\text{tr}_s(g_1 g_2^2 g_1 g_2^2) = b(2b - 1) + (u - 1)^2(E + uz + z)(uE + uz - z) + u(u - 1)^2 z^2.
\]
In Cases 1–14, we obtain:
\[
\frac{\tau(\pi(\alpha))}{\text{tr}_s(\delta(\alpha))} = \frac{\tau(G_1 G_2^2 G_1 G_2^2)}{\text{tr}_s(g_1^2 g_2^2)} \equiv \left(\frac{\zeta}{z}\right)^6 = \left(\frac{\zeta}{z}\right)^{\tau(\alpha)}.
\]
Cases 15 and 16 collapse, unless \( E = 1 \) or \( u = \pm 1 \). However:
- Case 15, \( E = 1 \) is covered by Case 10;
- Case 15, \( u = 1 \) is covered by Case 3;
- Case 15, \( u = -1 \) is covered by Case 7;
- Case 16, \( E = 1 \) is covered by Case 12;
- Case 16, \( u = 1 \) is covered by Case 3;
- Case 16, \( u = -1 \) is covered by Case 8.

5. **The only cases where \( P \) and \( \Delta_S \) coincide**

We will show that the cases where the invariants \( P \) and \( \Delta_S \) coincide are precisely the Cases 1–14 in the table of \( \Pi \). We have already shown that if \( P \) and \( \Delta_S \) coincide, then we must be in one of these cases. We will now show that in all these cases, \( P \) and \( \Delta_S \) do coincide. Note that these results hold for any \( d \in \mathbb{N} \) and for any non-empty subset \( S \) of \( \mathbb{Z}/d\mathbb{Z} \).
5.1. General methodology. We already know (see [4,8]) that, for \( \alpha = \sigma_i^n \in B_n \) \((1 \leq i \leq n - 1)\), in Cases 1–14 we have

\[
D_H = D_V \quad \text{and} \quad \frac{\tau_i(\pi(\sigma_i^n))}{\text{tr}_H(\delta(\sigma_i^n))} = \left( \frac{\zeta}{z} \right)^m \quad \text{for all } m \in \mathbb{N}.
\]

In particular, following (1.4), (1.5), (2.9) and (2.10), in Cases 3–6 and 9–12 we have

\[
\tau_i(\pi(\sigma_i^n)) = \zeta^m \quad \text{and} \quad \text{tr}_H(\delta(\sigma_i^n)) = z^m \quad \text{for all } m \in \mathbb{N}.
\]

We will proceed by induction on \( n \). The cases (i) and (ii) will be handled separately. We will only need to prove (5.3) (respectively (5.4) and (5.5)) for Cases 3–6 and 9–12. We will even show that (5.5) holds in Cases 7–8 and 13–14 (respectively (5.4) and (5.5)) in Cases 3–6 and 9–12.

To do this we will use induction on the non-negative number \( \nu(\alpha) := | \text{sum of all negative exponents of the braid generators in } \alpha | \).

We will prove the above by induction on the following non-negative number:

\[
\nu(\alpha) := | \text{sum of all negative exponents of the braid generators in } \alpha |.
\]

Note that \( \nu(\alpha) \) depends on the expression of \( \alpha \) in terms of braid generators.

For the inductive step, we will show (5.3) in Cases 7–8 and 13–14 (respectively (5.4) and (5.5) in Cases 3–6 and 9–12) for \( \alpha \sigma_i^{-1} \) \((\alpha \in B_n, 1 \leq i \leq n - 1)\) using the induction hypothesis. The formulas for \( \pi(\sigma_i^{-1}) = G_i^{-1} \) and \( \delta(\sigma_i^{-1}) = g_i^{-1} \) are given respectively by Equations 1.3 and 2.4.

For the first step \( \nu(\alpha) = 0 \), we will proceed by induction on \( n \). Set

\[
B^+_n := \{ \alpha \in B_n \mid \nu(\alpha) = 0 \}.
\]

The cases \( n = 1 \) and \( n = 2 \) are taken care of by (5.1) for Cases 7–8 and 13–14 (respectively (5.2) for Cases 3–6 and 9–12). We will only need to prove (5.3) (respectively (5.4) and (5.5)) for \( B^+_n \) assuming that it holds for \( B^+_n \). To do this we will use a second induction on the non-negative number \( \epsilon(\beta) + \epsilon_n(\beta) \), where \( \epsilon_n(\beta) \) denotes the sum of the exponents of the braid generator \( \sigma_n \) in the braid word \( \beta \in B^+_n \). Note that \( \epsilon(\beta) \) is uniquely defined for \( \beta \), whereas \( \epsilon_n(\beta) \) depends on the expression of \( \beta \) in terms of braid generators. However, we always have \( \epsilon(\beta) \geq \epsilon_n(\beta) \).

If \( \epsilon(\beta) + \epsilon_n(\beta) = 0 \), then \( \beta = 1 \) and there is nothing to prove. If \( \epsilon(\beta) + \epsilon_n(\beta) = 1 \), then \( \epsilon(\beta) = 1 \) and \( \epsilon_n(\beta) = 0 \). Hence, \( \beta = \sigma_i \) for some \( 1 \leq i \leq n - 1 \) and the desired result holds. Now assume that \( \epsilon(\beta) + \epsilon_n(\beta) > 1 \) and that the result holds for smaller values of \( \epsilon + \epsilon_n \). We will distinguish three cases:

- If \( \epsilon_n(\beta) = 0 \), then \( \beta = 1 \), and the induction hypothesis on \( n \) yields the desired result.
- If \( \epsilon_n(\beta) = 1 \), then there exist \( \alpha_1, \alpha_2 \in B^+_n \) such that \( \beta = \sigma_i \sigma_n \alpha_2 \). We have
  \[
  \tau(\pi(\beta)) = \zeta \tau(\pi(\alpha_1 \alpha_2)) \quad \text{and} \quad \text{tr}_H(\delta(\beta)) = z \text{tr}_H(\delta(\alpha_1 \alpha_2)).
  \]
  The induction hypothesis on \( n \) yields the rest.
- If \( \epsilon_n(\beta) \geq 2 \), then there exist \( \alpha \in B^+_n \) and \( \beta_1, \beta_2 \in B^+_n \) such that \( \beta = \beta_1 \sigma_n \alpha \sigma_n \beta_2 \). We will need the following lemma:

Lemma 3. Let \( \alpha \in B^+_n \). Then one of the following hold:

(i) \( \sigma_n \sigma_n = \alpha_1 \sigma_n \alpha_2 \), for some \( \alpha_1, \alpha_2 \in B^+_n \), or
(ii) \( \sigma_n \sigma_n = \beta_1 \sigma_n \beta_2 \), for some \( \beta_1, \beta_2 \in B^+_n \) and \( 1 \leq j \leq n \).

Proof. We will proceed by induction on \( n \). If \( n = 1 \), then \( \alpha = 1 \) and we are in Case (ii). Assume that the above holds for \( 1, 2, \ldots, n - 1 \). We will show that it holds for \( n \). We will proceed by induction on the number \( \epsilon_n(\alpha) \), that is, the sum of the exponents of the braid generator \( \sigma_n \) in the braid word \( \alpha \):

\[2\] This set is known as the braid monoid, see, for example, [20, Chapter 4].
• If $\epsilon_{n-1}(\alpha) = 0$, then $\sigma_n$ commutes with $\alpha$, and
  \[ \sigma_n \alpha \sigma_n = \alpha \sigma_n^2. \]

• If $\epsilon_{n-1}(\alpha) = 1$, then there exist $b_1, b_2 \in B_{n-1}^+$ such that $\alpha = b_1 \sigma_{n-1} b_2$. We have:
  \[ \sigma_n \alpha \sigma_n = b_1 \sigma_n \sigma_{n-1} \sigma_n b_2 = b_1 \sigma_{n-1} \sigma_n b_2 = \alpha_1 \sigma_n \alpha_2, \]
  where $\alpha_1 = b_1 \sigma_{n-1}$ and $\alpha_2 = \sigma_{n-1} b_2$.

• If $\epsilon_{n-1}(\alpha) \geq 2$, then there exist $b \in B_{n-1}^+$ and $\alpha_1, \alpha_2 \in B_{n-1}^+$ such that $\alpha = \alpha_1 b \sigma_{n-1} \alpha_2$. Then, by the induction hypothesis on $n$, one of the following hold:
  (i) $\sigma_{n-1} b \sigma_{n-1} = b_1 \sigma_{n-1} b_2$, for some $b_1, b_2 \in B_{n-1}^+$, or
  (ii) $\sigma_{n-1} b \sigma_{n-1} = \alpha_1 \sigma_2^2 \alpha_2$, for some $\alpha_1, \alpha_2 \in B_{n+1}^+$ and $1 \leq j \leq n - 1$.

In Case (i), the induction hypothesis on $\epsilon_{n-1}(\alpha)$ yields the desired result. In Case (ii), we obtain:
  \[ \sigma_n \alpha \sigma_n = \sigma_n \alpha_1 \sigma_2^2 \alpha_2 \sigma_n = \beta_1 \sigma_2^2 \beta_2, \]
  where $\beta_1 = \sigma_n \alpha_1 \beta_1', \beta_2 = \alpha_2' \sigma_2 \sigma_n \in B_{n+1}^+$.

Applying now the above lemma to the word $\beta = \beta_1 \sigma_n \sigma_n \beta_2$, where $\alpha \in B_n^+$ and $\beta_1, \beta_2 \in B_{n+1}^+$, we obtain that one of the following hold:

(i) $\beta = \beta_1 \alpha_1 \sigma_2 \beta_2$, for some $\alpha_1, \alpha_2 \in B_{n+1}^+$, or

(ii) $\beta = \beta_1 \alpha_1 \sigma_2 \beta_2$, for some $\beta_1', \beta_2' \in B_{n+1}^+$ and $1 \leq j \leq n$.

The induction hypothesis on $\epsilon(\beta) \geq 2$ covers Case (i), since $\epsilon_n(\beta_1 \alpha_1 \sigma_2 \beta_2) < \epsilon_n(\beta_1 \sigma_n \sigma_n \beta_2)$.

To summarize: In order to prove Equality \[5.3\] in Cases 7–8 and 13–14, and Equalities \[5.4\] and \[5.5\] in Cases 3–6 and 9–12, we will show that these equalities hold on all words of the form:

\[ \beta \sigma_2^2 \quad (\beta \in B_{n+1}^+, 1 \leq j \leq n). \]

Assuming the induction hypotheses on $n$ and on $\epsilon + \epsilon_n$, and all words of the form

\[ \alpha \sigma_i^{-1} \quad (\alpha \in B_n, 1 \leq i \leq n - 1), \]

assuming the induction hypothesis on $\nu$.

5.2. The Cases 1 and 2. In the first two cases, although our general methodology applies, we prefer to use the following, simpler approach. Since $q = 1$, the quadratic relation \[1.2\] (h) in the Iwahori-Hecke algebra $H_2(1) \cong \mathbb{G}_2 = 1$. Similarly, since $u = 1$, the quadratic relation \[2.2\] in the Yokonuma-Hecke algebra $Y_{d, n}(1)$ becomes $g_2^2 = 1$. Therefore, there exist two natural isomorphisms $\iota^+$ and $\iota^-$ between $\pi(\mathbb{G}_2)$ and $\delta(\mathbb{G}_2)$ given by $\iota^{\pm}(G_1) = g_1$ and $\iota^{\pm}(G_1) = -g_1$, respectively. Now, if we take $\zeta = z$, then $\tau \circ \iota^+$ is a Markov trace on $H_n(1)$ that satisfies all three rules of Theorem \[2\]. The uniqueness of the Ocneanu trace yields $\tau \circ \iota^+ = \zeta$. So in Case 1 we have:

\[ \frac{\tau(\pi(\alpha))}{\tau(\delta(\alpha))} = \frac{\tau\iota^+(\pi(\alpha))}{\tau\iota^+(\delta(\alpha))} = \frac{\tau\iota^-(\delta(\alpha))}{\tau\iota^-(\delta(\alpha))} = 1 = \frac{z}{z} \epsilon(\alpha) = \frac{\zeta}{z} \epsilon(\alpha) \quad \text{for all } \alpha \in B_n. \]

Similarly, if we take $\zeta = -z$, then $\tau \circ \iota^-$ is a Markov trace that satisfies all three rules of Theorem \[2\]. Therefore, we obtain $\tau \circ \iota^- = \tau$. So in Case 2 we have:

\[ \frac{\tau(\pi(\alpha))}{\tau(\delta(\alpha))} = \frac{\tau\iota^-(\pi(\alpha))}{\tau\iota^-(\delta(\alpha))} = \frac{(-1)^\epsilon(\alpha)\tau\iota^-(\delta(\alpha))}{\tau\iota^-(\delta(\alpha))} = (-1)^\epsilon(\alpha) = \frac{-z}{z} \epsilon(\alpha) = \frac{-z}{z} \epsilon(\alpha) \quad \text{for all } \alpha \in B_n. \]
5.3. The Cases 3–6. Following our general methodology, let first $\beta \in B_{i+1}$ and $1 \leq j \leq n$. If $\zeta = q$, we have

$$\tau (\pi(\beta^j)) = (q-1) \tau (\pi(\beta^j)) + q \tau (\pi(\beta)) \quad \text{ind. hyp.} \quad (q-1) q^{\epsilon(\beta)+1} + q \cdot q^{\epsilon(\beta)} = q^{\epsilon(\beta)+2} = \zeta^{(\beta^j)}.$$ 

If $\zeta = -1$, we have

$$\tau (\pi(\beta^j)) = (q-1) \tau (\pi(\beta^j)) + q \tau (\pi(\beta)) \quad \text{ind. hyp.} \quad (q-1) (-\epsilon(\beta)+1) + q (-\epsilon(\beta)) = (-\epsilon(\beta)+2) = \zeta^{(\beta^j)}.$$ 

If $u = 1$ and $z = 1$, we have

$$\operatorname{tr}_S (\delta(\beta^j)) = \operatorname{tr}_S (\delta(\beta)) \quad \text{ind. hyp.} \quad 1 = \zeta^{(\beta^j)}.$$ 

If $u = 1$ and $z = -1$, we have

$$\operatorname{tr}_S (\delta(\beta^j)) = \operatorname{tr}_S (\delta(\beta)) \quad \text{ind. hyp.} \quad (-\epsilon(\beta)) = (-\epsilon(\beta)+2) = \zeta^{(\beta^j)}.$$ 

Now let $\alpha \in B_n$ and $1 \leq i \leq n - 1$. If $\zeta = q$, we have

$$\tau (\pi(\alpha \sigma_i^{-1})) = q^{-1} \tau (\pi(\alpha \sigma_i)) + (q^{-1} - 1) \tau (\pi(\alpha)) \quad \text{ind. hyp.} \quad q^{-1} q^{\epsilon(\alpha)+1} + (q^{-1} - 1) q^{\epsilon(\alpha)} = q^{\epsilon(\alpha)+1} = \zeta^{(\alpha \sigma_i^{-1})}.$$ 

If $\zeta = -1$, we have

$$\tau (\pi(\alpha \sigma_i^{-1})) = q^{-1} \tau (\pi(\alpha \sigma_i)) + (q^{-1} - 1) \tau (\pi(\alpha)) \quad \text{ind. hyp.} \quad q^{-1} (-\epsilon(\alpha)+1) + (q^{-1} - 1) (-\epsilon(\alpha)) = (-\epsilon(\alpha)+1) = \zeta^{(\alpha \sigma_i^{-1})}.$$ 

If $u = 1$ and $z = 1$, we have

$$\operatorname{tr}_S (\delta(\alpha \sigma_i^{-1})) = \operatorname{tr}_S (\delta(\alpha \sigma_i)) \quad \text{ind. hyp.} \quad 1 = \zeta^{(\alpha \sigma_i^{-1})}.$$ 

If $u = 1$ and $z = -1$, we have

$$\operatorname{tr}_S (\delta(\alpha \sigma_i^{-1})) = \operatorname{tr}_S (\delta(\alpha \sigma_i)) \quad \text{ind. hyp.} \quad (-\epsilon(\alpha)+1) = (-\epsilon(\alpha)+1) = \zeta^{(\alpha \sigma_i^{-1})}.$$ 

Thus, we conclude that (5.3) and (5.4) hold, whence we deduce (5.5).

5.4. The Cases 7 and 8. In order to show (5.5) in Cases 7 and 8, we will first show that

$$\tau (hG^2_j) = \tau (h) \quad (h \in H_n(1), \ 1 \leq j \leq n - 1)$$

and

$$\operatorname{tr}_S (yg^j) = \operatorname{tr}_S (y) \quad (y \in Y_{d,n}(u), \ 1 \leq j \leq n - 1).$$

Note that (5.7) is equivalent to

$$\operatorname{tr}_S (ye_j) = \operatorname{tr}_S (yg_j e_j) \quad (y \in Y_{d,n}(u), \ 1 \leq j \leq n - 1).$$

Equation 5.6 is straightforward, since $G^2_j = 1$, for all $j = 1, \ldots, n - 1$, in $H_n(1)$. To prove (5.8) we will proceed by induction on $n$. Recall that $z = -E$. It is enough to show that (5.8) holds on the elements of the inductive basis of $Y_{d,n}(u)$.

Let $n = 2$. Let $y = t_1^k g_1 t_1^l$ for some $k, l \in Z/dZ$. Then, by (2.6) and (2.12), we have

$$\operatorname{tr}_S (ye_1) = \operatorname{tr}_S (t_1^k g_1 t_1^l e_1) = \operatorname{tr}_S (t_1^{k+l} g_1 e_1) = -E \operatorname{tr}_S (t_1^{k+l}) = -E x_{k+l},$$

and, by (2.1)(f2), Lemma 1 and (2.15), we have

$$\operatorname{tr}_S (yg_1 e_1) = \operatorname{tr}_S (t_1^k g_1 t_1^l g_1 e_1) = \operatorname{tr}_S (t_1^{k+l} g_1 e_1 + (u-1) \operatorname{tr}_S (t_1^{k+l} e_1) + (u-1) \operatorname{tr}_S (t_1^{k+l} g_1 e_1) = E x_{k+l}.$$ 

So (5.8) holds. Now let $y = t_1^k t_2^l$ for some $k, l \in Z/dZ$. Then, by (2.1)(f2), (2.6) and Lemma 1 we have

$$\operatorname{tr}_S (ye_1) = \operatorname{tr}_S (t_1^{k+l} e_1) = \operatorname{tr}_S (t_1^k g_1 e_1) = E x_{k+l},$$

and

$$\operatorname{tr}_S (yg_1 e_1) = \operatorname{tr}_S (t_1^k t_2^l g_1 e_1) = \operatorname{tr}_S (t_1^k g_1 e_1) = -E x_{k+l}.$$ 

So (5.8) holds.

Now assume that (5.8) holds for $n$. We will prove it for $n+1$. Let $y = w_n g_n g_{n-1} \cdots g_{i} t_1^k$ for some $k \in Z/dZ$ and some $w_n \in Y_{d,n}(u)$. If $j < n$, then, following the definition of the trace and the induction hypothesis, we have

$$\operatorname{tr}_S (ye_j) = \operatorname{tr}_S (w_n g_n g_{n-1} \cdots g_{i} t_1^k e_j) = -E \operatorname{tr}_S (w_n g_n g_{n-1} \cdots g_{i} t_1^k g_j e_j) = E \operatorname{tr}_S (w_n g_n g_{n-1} \cdots g_{i} t_1^k g_j e_j) = -E \operatorname{tr}_S (yg_j e_j).$$
If \( j = n \), then, we have to distinguish two cases: If \( i = n \), then we have, by (2.4),
\[
\text{tr}_S(ye_n) = \text{tr}_S(w_n g_n t_k e_n) = \text{tr}_S(t_k w_n g_n e_n) = -E \text{tr}_S(t_k w_n),
\]
and, by (2.1)(f), (2.6) and Lemma 1
\[
\text{tr}_S(g y e_n) = \text{tr}_S(w_n g_n t_k g_n e_n) = \text{tr}_S(t_k w_n g_n^2 e_n) = u \text{tr}_S(t_k w_n e_n) + (u - 1) \text{tr}_S(t_k w_n g_n e_n) = E \text{tr}_S(t_k w_n).
\]
If \( i < n \), then
\[
\text{tr}_S(y e_n) = \text{tr}_S(w_n g_n g_{n-1} \ldots g_{i+1} e_n) = \text{tr}_S(w_n e_{n-1} g_{n-1} \ldots g_{i+1} e_n) = -E \text{tr}_S(g_{n-1} \ldots g_{i+1} w_n e_n).
\]
Following the induction hypothesis, the latter is equal to
\[
E \text{tr}_S(g_{n-1} \ldots g_{i+1} w_n e_{n-1}) = -\text{tr}_S(w_n g_{n-1} g_{n-2} \ldots g_{i+1} e_n) = -\text{tr}_S(g y e_n).
\]
Finally, let \( y = w_n t_k^{n+1} \) for some \( k \in \mathbb{Z}/d\mathbb{Z} \) and some \( w_n \in Y_{d,n}(u) \). If \( j < n \), then, following the definition of the trace and the induction hypothesis, we have
\[
\text{tr}_S(y e_j) = \text{tr}_S(w_n t_k^{n+1} e_j) = x_k \text{tr}_S(w_n e_j) = -x_k \text{tr}_S(w_n g_j e_j) = -\text{tr}_S(y g_j e_j).
\]
If \( j = n \), then, with repeated use of Lemma 1 we obtain:
\[
\text{tr}_S(y g_j^{-1}) = \text{tr}_S(y g_j) \quad (y \in Y_{d,n}(u), \ 1 \leq j \leq n - 1).
\]
We are now ready to prove (5.3). Let \( \beta \in B_{n+1}^+ \) and \( 1 \leq j \leq n \). Following Equations 5.6 and 5.7 we obtain
\[
\frac{\text{tr}_S(y g_j^{-1})}{\text{tr}_S(y g_j)} = \left( \frac{\zeta}{z} \right) \left( \frac{\zeta}{z} \right) = \left( \frac{\zeta}{z} \right),
\]
since
\[
\frac{\zeta}{z} = -1 \text{ in Case 7 and } \frac{\zeta}{z} = 1 \text{ in Case 8}.
\]
Now let \( \alpha \in B_n \) and \( 1 \leq i \leq n - 1 \). Following Equations 5.9 and 5.10 we obtain
\[
\frac{\text{tr}_S(\delta(\beta \sigma_i^2))}{\text{tr}_S(\delta(\beta \sigma_i^2)^{-1})} = \text{tr}_S(\delta(\beta \sigma_i^2)) = \text{tr}_S(\delta(\beta)) \quad \text{ind. hyp.} \quad \left( \frac{\zeta}{z} \right)^{\epsilon(\alpha)} \left( \frac{\zeta}{z} \right) = \left( \frac{\zeta}{z} \right)^{\epsilon(\alpha)}
\]
Again because of (5.11). Therefore, in both Cases 7 and 8, our general methodology yields (5.3).

5.5. The Cases 9–12. We will show that, in these cases, (5.4) and (5.5) hold again for all \( \alpha \in B_n \). On the Hecke algebra side, we have already shown in (5.3) that (5.4) holds when \( \zeta = q \) or \( \zeta = -1 \). Thus, it remains to show (5.5) for Cases 9–12.

Let \( \beta \in B_{n+1}^+ \) and \( 1 \leq j \leq n \). We have
\[
\text{tr}_S(\delta(\beta \sigma_i^2)) = \text{tr}_S(\delta(\beta)) \quad \text{ind. hyp.} \quad \text{tr}_S(\delta(\beta)) = \text{tr}_S(\delta(\beta)) + (u - 1) \text{tr}_S(\delta(\beta)e_j) + (u - 1) \text{tr}_S(\delta(\beta)e_jg_j).
\]
By Proposition 1 if \( E = 1 \), the latter is equal to
\[
\text{tr}_S(\delta(\beta)) + (u - 1) \text{tr}_S(\delta(\beta)) + (u - 1) \text{tr}_S(\delta(\beta)g_j).
\]
On the one hand, if \( z = u \), we deduce, using the induction hypothesis, that
\[
\text{tr}_S(\delta(\beta \sigma_i^2)) = u^{\epsilon(\beta)} + (u - 1) u^{\epsilon(\beta)} + (u - 1)^2 u^{\epsilon(\beta) + 1} = u^{\epsilon(\beta) + 2} = \zeta^{\epsilon(\beta \sigma_i^2)}.
\]
On the other hand, if \( z = -1 \), we deduce, using the induction hypothesis, that
\[
\text{tr}_S(\delta(\beta \sigma_i^2)) = (-1)^{\epsilon(\beta)} + (u - 1) (-1)^{\epsilon(\beta)} + (u - 1) (-1)^{\epsilon(\beta) + 1} = (-1)^{\epsilon(\beta) + 2} = \zeta^{\epsilon(\beta \sigma_i^2)}.
\]
Now let $\alpha \in B_n$ and $1 \leq i \leq n-1$. Using formula (2.4), we obtain
\[ \text{tr}_s(\delta(\alpha \sigma_i^{-1})) = \text{tr}_s(\delta(\alpha)g_i^{-1}) = \text{tr}_s(\delta(\alpha)g_i) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)e_i) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)g_i). \]
By Proposition 11 if $E = 1$, the latter is equal to
\[ \text{tr}_s(\delta(\alpha)g_i) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)g_i). \]
On the one hand, if $z = u$, we deduce, using the induction hypothesis, that
\[ \text{tr}_s(\delta(\alpha \sigma_i^{-1})) = u^{\epsilon(\alpha) + 1} + (u^{-1} - 1) u^{\epsilon(\alpha)} + (u^{-1} - 1) u^{\epsilon(\alpha) + 1} = u^{\epsilon(\alpha) + 1} = z^{\epsilon(\alpha \sigma_i^{-1})}. \]
On the other hand, if $z = -1$, we deduce, using the induction hypothesis, that
\[ \text{tr}_s(\delta(\alpha \sigma_i^{-1})) = (-1)^{\epsilon(\alpha) + 1} + (u^{-1} - 1)(-1)^{\epsilon(\alpha)} + (u^{-1} - 1)(-1)^{\epsilon(\alpha) + 1} = (-1)^{\epsilon(\alpha) - 1} = z^{\epsilon(\alpha \sigma_i^{-1})}. \]
Thus, we conclude that (5.3) holds. Since (5.4) also holds, we deduce that (5.5) holds.

5.6. The Cases 13 and 14. The Cases 13 and 14 have been covered by Corollaries 11 and 2 respectively. Nevertheless, we will see here how our general methodology applies also to these cases.
Let $\beta \in B_{n+1}^1$ and $1 \leq j \leq n$. We have
\[ \tau(\pi(\beta \sigma_j^2)) = \tau(\pi(\beta)G_j^2) = (q - 1) \tau(\pi(\beta)G_j) + q \tau(\pi(\beta)) = (q - 1) \tau(\pi(\beta \sigma_j)) + q \tau(\pi(\beta)) \]
and
\[ \text{tr}_s(\delta(\beta \sigma_j^2)) = \text{tr}_s(\delta(\beta)g_j^2) = \text{tr}_s(\delta(\beta)) + (u - 1) \text{tr}_s(\delta(\beta)e_j) + (u - 1) \text{tr}_s(\delta(\beta)e_jg_j). \]
Since $E = 1$, by Proposition 11, the last equation becomes:
\[ \text{tr}_s(\delta(\beta \sigma_j^2)) = \text{tr}_s(\delta(\beta)) + (u - 1) \text{tr}_s(\delta(\beta)) + (u - 1) \text{tr}_s(\delta(\beta)g_j) = (u - 1) \text{tr}_s(\delta(\beta)g_j) + u \text{tr}_s(\delta(\beta)). \]
If $q = u$ and $\zeta = z$, the induction hypothesis on (5.3) yields:
\[ \tau(\pi(\beta \sigma_j^2)) = (u - 1) \cdot 1^{\epsilon(\beta) + 1} \cdot \text{tr}_s(\delta(\beta \sigma_j)) + u \cdot 1^{\epsilon(\beta)} \cdot \text{tr}_s(\delta(\beta)) = \text{tr}_s(\delta(\beta \sigma_j^2)), \]
as desired. If $q = 1/u$ and $\zeta = -z/u$, the induction hypothesis on (5.3) yields:
\[ \tau(\pi(\beta \sigma_j^2)) = \left(\frac{1}{u} - 1\right) \left(\frac{-1}{u}\right)^{\epsilon(\beta) + 1} \text{tr}_s(\delta(\beta \sigma_j)) + \frac{1}{u} \left(\frac{-1}{u}\right)^{\epsilon(\beta)} \text{tr}_s(\delta(\beta)) = \left(\frac{-1}{u}\right)^{\epsilon(\beta) + 2} \text{tr}_s(\delta(\beta \sigma_j^2)), \]
as desired.
Now let $\alpha \in B_n$ and $1 \leq i \leq n-1$. We have
\[ \tau(\pi(\alpha \sigma_i^{-1})) = q^{-1} \tau(\pi(\alpha \sigma_i)) + (q^{-1} - 1) \tau(\pi(\alpha)) \]
and
\[ \text{tr}_s(\delta(\alpha \sigma_i^{-1})) = \text{tr}_s(\delta(\alpha)g_i^{-1}) = \text{tr}_s(\delta(\alpha)g_i) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)e_i) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)g_i). \]
Since $E = 1$, by Proposition 11, the last equation becomes:
\[ \text{tr}_s(\delta(\alpha \sigma_i^{-1})) = \text{tr}_s(\delta(\alpha)g_i) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)g_i) = u^{-1} \text{tr}_s(\delta(\alpha \sigma_i)) + (u^{-1} - 1) \text{tr}_s(\delta(\alpha)). \]
If $q = u$ and $\zeta = z$, the induction hypothesis on (5.3) yields:
\[ \tau(\pi(\alpha \sigma_i^{-1})) = u^{-1} \cdot 1^{\epsilon(\alpha) + 1} \cdot \text{tr}_s(\delta(\alpha \sigma_i)) + (u^{-1} - 1) \cdot 1^{\epsilon(\alpha)} \cdot \text{tr}_s(\delta(\alpha)) = \text{tr}_s(\delta(\alpha \sigma_i^{-1})), \]
as desired. If $q = 1/u$ and $\zeta = -z/u$, the induction hypothesis on (5.3) yields:
\[ \tau(\pi(\alpha \sigma_i^{-1})) = u \left(\frac{-1}{u}\right)^{\epsilon(\alpha) + 1} \text{tr}_s(\delta(\alpha \sigma_i)) + (u - 1) \left(\frac{-1}{u}\right)^{\epsilon(\alpha)} \text{tr}_s(\delta(\alpha)) = \left(\frac{-1}{u}\right)^{\epsilon(\alpha) - 1} \text{tr}_s(\delta(\alpha \sigma_i^{-1})), \]
as desired. Following our general methodology, (5.3) holds also in Cases 13 and 14.
5.7. Conclusion. The following result, proved in Subsections 5.1–5.6, is the main result of this paper.

**Theorem 5.** Let \( X \) be a solution of the E-system. Let \( \text{tr}_S \) be the corresponding specialized Juyumaya trace on \( Y_{d,n}(u) \) with parameter \( z \), and let \( \tau \) be the Ocneanu trace on \( H_n(q) \) with parameter \( \zeta \). Let \( E = \text{tr}_S(e_i) \) for all \( i = 1, \ldots, n - 1 \). Then \( P = \Delta_S \) if and only if we are in one of the cases portrayed in the following table:

| Case | \( q \) | \( \zeta \) | \( u \) | \( z \) | \( E \) |
|------|--------|--------|------|------|------|
| 1    | 1      | \( z \) | 1    | \( \mathbb{C}^* \) | any  |
| 2    | 1      | \(- z\) | 1    | \( \mathbb{C}^* \) | any  |
| 3    | \( \mathbb{C}^* \) | \( q \) | 1    | 1    | any  |
| 4    | \( \mathbb{C}^* \) | \( q \) | 1    | \(- 1\) | any  |
| 5    | \( \mathbb{C}^* \) | \(- 1\) | 1    | \(- 1\) | any  |
| 6    | 1      | \( E \) | \( \mathbb{C}^* \) | \(- E\) | any  |
| 7    | 1      | \(- E\) | \( \mathbb{C}^* \) | \(- E\) | any  |
| 8    | \( \mathbb{C}^* \) | \( q \) | \( \mathbb{C}^* \) | \(- 1\) | 1    |
| 9    | \( \mathbb{C}^* \) | \( q \) | \( \mathbb{C}^* \) | 1    | 1    |
| 10   | \( \mathbb{C}^* \) | \(- 1\) | \( \mathbb{C}^* \) | 1    | 1    |
| 11   | \( \mathbb{C}^* \) | \(- 1\) | \( \mathbb{C}^* \) | 1    | 1    |
| 12   | \( u \) | \( z \) | \( \mathbb{C}^* \) | \( \mathbb{C}^* \) | 1    |
| 13   | \( 1/u \) | \(- z/u\) | \( \mathbb{C}^* \) | \( \mathbb{C}^* \) | 1    |
| 14   | \( u \) | \( z \) | \( \mathbb{C}^* \) | \( \mathbb{C}^* \) | 1    |

5.8. Comparing further \( P \) and \( \Delta_S \). In Theorem 5 we give a necessary and sufficient condition for the invariants \( P \) and \( \Delta_S \) to coincide. However, as we mentioned in the introduction, computational data do not indicate that one invariant is topologically stronger than the other. A simple explanation would be that \( P \) is a scalar multiple of \( \Delta_S \), that is, there exist \( (c_n)_{n \in \mathbb{N}} \) in \( \mathbb{C}(q, \zeta, u, z, E) \) such that

\[
(5.12) \quad P(\hat{\alpha}) = c_n \Delta_S(\hat{\alpha}) \quad (\alpha \in B_n)
\]

for all \( n \in \mathbb{N} \). Then for the identity braid \( 1 \) in each \( B_n \) we have:

\[
P(\hat{1}) = D_{H}^{n-1} = c_n D_{Y}^{n-1} = c_n \Delta_S(\hat{1}).
\]

We deduce that

\[
c_n = \frac{D_{H}^{n-1}}{D_{Y}^{n-1}} = \left( \frac{D_{H}}{D_{Y}} \right)^{n-1}.
\]

Thus, if (5.12) holds, we must have

\[
(5.13) \quad \frac{\tau(\pi(\alpha))}{\text{tr}_S(\delta(\alpha))} = \left( \frac{\zeta}{z} \right)^{c(\alpha)}
\]

for all \( \alpha \in B_n \) and for all \( n \in \mathbb{N} \). Taking \( \alpha = \sigma_1^{-1} = B_n \), for \( n \geq 2 \), we obtain

\[
(5.14) \quad (u\zeta + z^2 - uEz + Ez)q = u\zeta(\zeta + 1),
\]

which in turn yields (see §4.1)

\[
D_{H} = D_{Y}.
\]

We conclude that \( c_n = 1 \) for all \( n \in \mathbb{N} \). Combining this with Theorem 5, we obtain the following result:

**Theorem 6.** Let \( X \) be a solution of the E-system. Let \( \text{tr}_S \) be the corresponding specialized Juyumaya trace on \( Y_{d,n}(u) \) with parameter \( z \), and let \( \tau \) be the Ocneanu trace on \( H_n(q) \) with parameter \( \zeta \). Let \( E = \text{tr}_S(e_i) \) for all \( i = 1, \ldots, n - 1 \). Then there exist \( (c_n)_{n \in \mathbb{N}} \) in \( \mathbb{C}(q, \zeta, u, z, E) \) such that, for all \( n \in \mathbb{N} \),

\[
P(\hat{\alpha}) = c_n \Delta_S(\hat{\alpha}) \quad (\alpha \in B_n)
\]

if and only if \( P = \Delta_S \), that is, if and only if we are in one of the cases portrayed in the table of Theorem 5.
References

[CJJKL] S. Chmutov, S. Jablan, J. Juyumaya, K. Karvounis, S. Lambropoulou, Computations on the Yokonuma–Hecke algebras, work in progress. See http://www.math.ntua.gr/~sofia/yokonuma/index.html.

[GePf] M. Geck, G. Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Math. Soc. Monographs, New Series 21, Oxford University Press, New York, 2000.

[Jo] V. F. R. Jones, Hecke algebra representations of braid groups and link polynomials, Annals of Math. 126 (1987), no. 2, 335–388.

[Ju] J. Juyumaya, Markov trace on the Yokonuma–Hecke algebra, J. Knot Theory and its Ramifications 13 (2004) 25–39.

[JuLa1] J. Juyumaya, S. Lambropoulou, p-adic framed braids, Topology and its Applications 154 (2007) 1804–1826.

[JuLa2] J. Juyumaya, S. Lambropoulou, p-adic framed braids II, to appear in Advances in Mathematics. See \[\text{http://arxiv.org/abs/0905.3626v2}\].

[JuLa3] J. Juyumaya, S. Lambropoulou, An adelic extension of the Jones polynomial, M. Banagl, D. Vogel (eds.) The mathematics of knots, Contributions in the Mathematical and Computational Sciences, Vol. 1, Springer.

[JuLa4] J. Juyumaya, S. Lambropoulou, An invariant for singular knots, J. Knot Theory and its Ramifications, 18(6) (2009) 825–840.

[Yo] T. Yokonuma, Sur la structure des anneaux de Hecke d’un groupe de Chevalley fini, C.R. Acad. Sc. Paris, 264, 344–347 (1967).

Laboratoire de Mathématiques UVSQ, Bâtiment Fermat, 45 avenue des États-Unis, 78035 Versailles cedex, France.

E-mail address: maria.chlouveraki@uvsq.fr

Department of Mathematics, National Technical University of Athens, Zografou campus, GR-15780 Athens, Greece.

E-mail address: sofia@math.ntua.gr