QUANTUM DIMENSIONS AND FUSION PRODUCTS FOR IRREDUCIBLE $V_Q^\sigma$-MODULES WITH $\sigma^2 = 1$

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Abstract. Every isometry $\sigma$ of a positive-definite even lattice $Q$ can be lifted to an automorphism of the lattice vertex algebra $V_Q$. An important problem in vertex algebra theory and conformal field theory is to classify the representations of the $\sigma$-invariant subalgebra $V_Q^\sigma$ of $V_Q$, known as an orbifold. In the case when $\sigma$ is an isometry of $Q$ of order two, we have classified the irreducible modules of the orbifold vertex algebra $V_Q^\sigma$ and identified them as submodules of twisted or untwisted $V_Q$-modules in [BE]. Here we calculate their quantum dimensions and fusion products.

The examples where $Q$ is the orthogonal direct sum of two copies of the $A_2$ root lattice and $\sigma$ is the 2-cycle permutation as well as where $Q$ is the $A_n$ root lattice and $\sigma$ is a Dynkin diagram automorphism are presented in detail.

1. Introduction

The notion of a vertex algebra has been a powerful tool for studying representations of infinite-dimensional Lie algebras. The theory of vertex algebras was introduced by Borcherds [B] and has since been developed in a number of works (see e.g. [FLM, K, FB, LL, KRR]).

Let $\sigma$ be an automorphism of a vertex algebra $V$. An orbifold is the subalgebra of $\sigma$-invariants in $V$, denoted $V^\sigma$ (see [DVVV, KT, DLM]). It will follow from definitions below that every $\sigma$-twisted representation of $V$ becomes untwisted when restricted to $V^\sigma$.

It has been a long-standing conjecture that all irreducible $V^\sigma$-modules are obtained by restriction from twisted or untwisted $V$-modules. This conjecture has recently been proved in a series of works by M. Miyamoto (see [M1, M2, M3, M4]) under suitable assumptions. In particular, it is shown in [M4] that if $V$ is a simple regular vertex operator algebra of CFT type, then the fixed point subalgebra $V^\sigma$ for a finite automorphism $\sigma$ of $V$ is also regular.

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In this paper, we are concerned with the case when $Q$ is a positive-definite even lattice and $\sigma$ is an isometry of $Q$ of order two. In [BE], we have classified and explicitly constructed all irreducible modules of the orbifold vertex algebra $V^\sigma_Q$, and have realized them as submodules of twisted or untwisted $V_Q$-modules. This paper is a continuation, where we give explicitly the quantum dimensions and fusion products for the irreducible $V^\sigma_Q$-modules.

Our general approach is to restrict from $V^\sigma_Q$ to $V^\sigma_L$, where $L$ is the sublattice of $Q$ spanned by eigenvectors of $\sigma$. The subalgebra $V^\sigma_L$ then factors as a tensor product $V^\pm_L \otimes V^\sigma_L$, where $L^\pm = \{ \alpha \in Q \mid \sigma\alpha = \pm\alpha \}$, and $V^\sigma_L$ is rational by results of [DLM2, ABD, A2, DJL, FHL]. We then use the known irreducible $V^\sigma_L$-modules (see [DN, AD]) and their intertwining operators (see [A1, ADL]) to determine the quantum dimensions and fusion products for the irreducible $V^\sigma_Q$-modules given in [BE].

The paper is organized as follows. In Section 2, we briefly review lattice vertex algebras, their twisted modules, and the results for $\sigma = -1$ that we need. We also review the notions of quantum dimension and fusion product including results from [DJX] that we need. In Section 3, we review the irreducible orbifold modules given in [BE]. Our main results are presented in Sections 4 and 5. Examples where $Q$ is the orthogonal direct sum of two copies of the $A_2$ root lattice and $\sigma$ is the 2-cycle permutation as well as where $Q$ is the $A_n$ root lattice and $\sigma$ is a Dynkin diagram automorphism are presented in detail in Section 6.

2. Lattice Vertex algebras and their twisted modules

In this section, we briefly review lattice vertex algebras and their twisted modules, and recall the results in the case when $\sigma = -1$. These results will aid in the general description. General references on vertex algebras are e.g. [FLM, K, FB, LL, KRR], among many other works.

2.1. Vertex algebras and twisted modules. A vertex algebra is a vector space $V$ with a vacuum vector $1 \in V$ together with a linear map

\begin{equation}
Y(\cdot, z) : V \otimes V \to V[[z]][z^{-1}]
\end{equation}

satisfying the axioms listed below. For $v \in V$, we have

\begin{equation}
Y(v, z) = \sum_{n \in \mathbb{Z}} v^{(n)} z^{-n-1}, \quad v^{(n)} \in \text{End } V.
\end{equation}
The vacuum vector is the identity in the sense that
\[ a_{(-1)}1 = 1_{(-1)}a = a, \quad a_{(n)}1 = 0, \quad n \geq 0. \]
In particular,
\[ \lim_{z \to 0} Y(a, z)1 = a. \]
For every \( a \in V \), we call the image \( Y(a, z) \) a field, meaning it can be viewed as a formal power series from \( (\text{End} V)[[z, z^{-1}]] \) involving only finitely many negative powers of \( z \) when applied to any vector.

The main axiom for a vertex algebra is the **Borcherds identity** (also called Jacobi identity in [FLM]) satisfied by the fields:

\[
(2.3) \quad y^{-1} \delta \left( \frac{z_1 - z_2}{y} \right) Y(u, z_1)Y(v, z_2) - y^{-1} \delta \left( \frac{z_2 - z_1}{y} \right) Y(v, z_2)Y(u, z_1) = z_2^{-1} \delta \left( \frac{z_1 - y}{z_2} \right) Y(Y(u, y)v, z_2),
\]
where \( u, v \in V \) and \( \delta \) is the delta function \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \). It is important to note that the commutator \([Y(u, z_1), Y(v, z_2)]\) follows from Borcherds identity by taking residue of both sides with respect to \( y \).

A **V-module** for a vertex algebra \( V \) is a vector space \( M \) endowed with a linear map \( Y^M(\cdot, z) : V \otimes M \to M[[z, z^{-1}]] \) (cf. (2.1)) such that the Borcherds identity (2.3) holds for all \( u, v \in V \) which act as operators on \( M \) (see [FB, LL, KRR]). Let \( \sigma \) be an automorphism of \( V \) of finite order \( r \). Then \( \sigma \) is diagonalizable. In the definition of a **\( \sigma \)-twisted V-module** \( M \) [FFR, D2], the image of \( Y^M \) is allowed to have nonintegral rational powers of \( z \):

\[
Y(u, z) = \sum_{n \in \mathbb{Z}} u_{(n)} z^{n-1}, \quad \text{if} \quad \sigma u = e^{-2\pi ip} u, \quad p \in \frac{1}{r} \mathbb{Z},
\]
where \( u_{(n)} \in \text{End} M \). The Borcherds identity in the twisted case is similar to (2.3), and it requires that \( u \) be an eigenvector of \( \sigma \) (see [FLM, FFR, D2]).

Recall from [FHL] that if \( V_1 \) and \( V_2 \) are vertex algebras, their tensor product is again a vertex algebra via

\[
Y(v_1 \otimes v_2, z) = Y(v_1, z) \otimes Y(v_2, z), \quad v_i \in V_i.
\]
Furthermore, if \( M_i \) is a \( V_i \)-module, then the above formula defines the structure of a \( (V_1 \otimes V_2) \)-module on \( M_1 \otimes M_2 \) (see [FHL]). A similar statement is also true for twisted modules (see [BE, Lemma 2.2]).
2.2. Lattice vertex algebras. Consider an integral lattice, i.e., a free abelian group $Q$ of finite rank together with a symmetric nondegenerate bilinear form $(\cdot, \cdot) : Q \times Q \rightarrow \mathbb{Z}$. We assume that $Q$ is even, i.e., $|\alpha|^2 = (\alpha|\alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$. The corresponding complex vector space $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} Q$ is considered as an abelian Lie algebra with a bilinear form extended from $Q$.

The Heisenberg algebra $\hat{\mathfrak{h}} = \mathfrak{h}[t, t^{-1}] \oplus \mathbb{C} K$ is the Lie algebra with brackets $[K, \mathfrak{h}] = 0$, and

\begin{equation}
[a_m, b_n] = m \delta_{m-n} (a|b) K, \quad a_m = at^m.
\end{equation}

Its induced irreducible highest-weight representation

\begin{equation}
M(1) = \text{Ind}_{\hat{\mathfrak{h}}[t] \oplus \mathbb{C} K}^\mathfrak{h} \mathbb{C} \cong S(\mathfrak{h}[t^{-1}]t^{-1})
\end{equation}

on which $K = 1$ is known as the (bosonic) Fock space.

Following [FK, B], let $\varepsilon : Q \times Q \rightarrow \{\pm 1\}$ be a bimultiplicative 2-cocycle such that

\begin{equation}
\varepsilon(\alpha, \alpha) = (-1)^{|\alpha|^2/2}, \quad \alpha \in Q.
\end{equation}

Then we have the associative algebra $\mathbb{C}_\varepsilon[Q]$ with basis $\{e^\alpha\}_{\alpha \in Q}$ and multiplication

\begin{equation}
e^{\alpha} e^{\beta} = \varepsilon(\alpha, \beta) e^{\alpha + \beta}.
\end{equation}

Such a 2-cocycle $\varepsilon$ is unique up to equivalence. In addition,

\begin{equation}
\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{|\alpha||\beta|}, \quad \alpha, \beta \in Q.
\end{equation}

The lattice vertex algebra associated to $Q$ is defined as $V_Q = M(1) \otimes \mathbb{C}_\varepsilon[Q]$, where the vacuum vector is $1 = 1 \otimes e^0$. An action of the Heisenberg algebra on $V_Q$ is given by

\begin{equation}
a_n e^{\beta} = \delta_{n,0} (a|\beta) e^{\beta}, \quad n \geq 0, \quad a \in \mathfrak{h}.
\end{equation}

The map $Y$ on $V_Q$ is uniquely determined by the generating fields:

\begin{equation}
Y(a_{-1} 1, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}, \quad a \in \mathfrak{h},
\end{equation}

\begin{equation}
Y(e^\alpha, z) = e^\alpha z^{\alpha_0} \exp\left(\sum_{n>0} \alpha_n z^{-n} \right) \exp\left(\sum_{n>0} \alpha_n z^{-n} \right),
\end{equation}

where $z^{\alpha_0} e^{\beta} = z^{(\alpha|\beta)} e^{\beta}$. Since the map $\mathfrak{h} \rightarrow M(1)$ given by $a \mapsto a_{-1} 1$ is injective, we can identify $a \in \mathfrak{h}$ with $a_{-1} 1 \in M(1)$, i.e., $a_{(n)} = a_n$ for all $n \in \mathbb{Z}$.
2.3. **Twisted representations of lattice vertex algebras.** Suppose \( \sigma \) is an automorphism of \( \mathfrak{h} \) with finite order \( r \) which preserves the bilinear form \( (\cdot | \cdot) \). Then \( \sigma \) is naturally extended to both \( \hat{\mathfrak{h}} \) and \( M(1) \), again denoted \( \sigma \). The \( \sigma \)-twisted Heisenberg algebra \( \hat{\mathfrak{h}}_\sigma \) is spanned over \( \mathbb{C} \) by \( K \) and the elements \( a_m = at^m \) such that \( \sigma a = e^{-2\pi i m} a \). It becomes a Lie algebra with bracket \( (2.4) \) for \( m, n \in \frac{1}{r} \mathbb{Z} \).

Let \( \hat{\mathfrak{h}}^\geq_\sigma \) (respectively, \( \hat{\mathfrak{h}}^<_\sigma \)) be the abelian subalgebra of \( \hat{\mathfrak{h}}_\sigma \) spanned by all elements \( a_m \) with \( m \geq 0 \) (respectively, \( m < 0 \)). We let \( \hat{\mathfrak{h}}^\geq_\sigma \) act on \( \mathbb{C} \) trivially and \( K \) act as the identity operator. The \( \sigma \)-twisted Fock space is defined as the induced module
\[
M(1)_\sigma = \text{Ind}_{\hat{\mathfrak{h}}^\leq_\sigma \otimes \mathbb{C} K}^{\hat{\mathfrak{h}}_\sigma} \mathbb{C} \cong S(\hat{\mathfrak{h}}^<_\sigma),
\]
and is an irreducible highest-weight representation of \( \hat{\mathfrak{h}}_\sigma \) which has the structure of a \( \sigma \)-twisted representation of the vertex algebra \( M(1) \) (see \[FLM, KRR\]). For the map \( Y \), we let \( Y(1, z) \) be the identity operator,
\[
Y(a, z) = \sum_{n \in p + \mathbb{Z}} a_n z^{-n-1}, \quad a \in \mathfrak{h}, \quad \sigma a = e^{-2\pi ip} a, \quad p \in \frac{1}{r} \mathbb{Z},
\]
and extend linearly to all of \( \mathfrak{h} \).

Now consider \( \sigma \) as an automorphism of \( Q \). Since \( \sigma \) preserves the bilinear form, the uniqueness of the cocycle \( \varepsilon \) and \( (2.7) \) imply that
\[
(2.11) \quad \eta(\alpha + \beta)\varepsilon(\sigma \alpha, \sigma \beta) = \eta(\alpha)\eta(\beta)\varepsilon(\alpha, \beta)
\]
for some function \( \eta: Q \to \{\pm 1\} \). It is shown in \[BE, Lemma 2.3\] that we can set \( \eta = 1 \) on any sublattice whose elements satisfy \( \varepsilon(\sigma \alpha, \sigma \beta) = \varepsilon(\alpha, \beta) \). In particular, \( \eta \) can be chosen such that
\[
(2.12) \quad \eta(\alpha) = 1, \quad \alpha \in Q \cap \mathfrak{h}^\sigma,
\]
where \( \mathfrak{h}^\sigma \subset \mathfrak{h} \) denotes the subspace spanned by vectors fixed under \( \sigma \). Then \( \sigma \) can be lifted to an automorphism of the lattice vertex algebra \( V_Q \) by setting
\[
(2.13) \quad \sigma(a_n) = \sigma(a)_n, \quad \sigma(e^\alpha) = \eta(\alpha) e^{\sigma \alpha}, \quad a \in \mathfrak{h}, \quad \alpha \in Q.
\]
Note the order of \( \sigma \) on \( V_Q \) may double.

It is shown in \[BK\] that irreducible \( \sigma \)-twisted \( V_Q \)-modules are parameterized by the set \( (Q^* / Q)^\sigma \) of \( \sigma \)-invariants in \( Q^* / Q \) and every \( \sigma \)-twisted \( V_Q \)-module is a direct sum of irreducible ones (see \[BK, Theorem 4.2\]). In the special case when \( \sigma = 1 \), we get Dong’s Theorem that the irreducible \( V_Q \)-modules are classified by \( Q^* / Q \) (see \[D1\]). Explicitly, they are given by:
\[
V_{\lambda + Q} = \mathcal{F} \otimes \mathbb{C}_\varepsilon[Q] e^\lambda, \quad \lambda \in Q^*.
\]
When the lattice $Q$ is written as an orthogonal direct sum of sublattices, $Q = L_1 \oplus L_2$, we have a natural isomorphism $V_Q \cong V_{L_1} \otimes V_{L_2}$. It can be shown that if $L_1$ and $L_2$ are $\sigma$-invariant, there is a correspondence of irreducible twisted modules in the sense that every irreducible $\sigma$-twisted $V_Q$-module $M$ is a tensor product, $M \cong M_1 \otimes M_2$, where $M_i$ is an irreducible $\sigma|_{L_i}$-twisted $V_{L_i}$-module (cf. [FHL, BE]).

2.4. The case $\sigma = -1$. Now we review what is known in the case when $\sigma = -1$, which will be used in our treatment of the general case. In this subsection, we denote the even integral lattice by $L$ instead of $Q$ so that $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is the corresponding complex vector space.

Let $\hat{L}$ be the central extension of $L$ by the 2-group $\langle -1 \rangle$: 

$$1 \to \langle -1 \rangle \to \hat{L} \to L \to 1$$

with the group structure in $\hat{L}$ written multiplicatively. Then $\epsilon$ is the associated 2-cocycle (cf. (2.6)). Let

$$K = \{\sigma(a)a^{-1} | a \in \hat{L}\}$$

and $T$ be any $\hat{L}/K$-module with the natural action of $-1$. Then these irreducible $\hat{L}/K$-modules are determined by central characters $\chi$ of $\hat{L}/K$ such that $\chi(-1) = -1$. When convenient, we denote the corresponding module by $T_\chi$. Define the vector space $V_L^T = M(1)_\sigma \otimes T$ (cf. (2.10)). Then every irreducible $\sigma$-twisted $V_L$-module is isomorphic to $V_L^T$ for some irreducible $\hat{L}/K$-module $T_\chi$ (see [FLM, D2]).

An action of $\sigma$ on $V_L^T$ can be defined by

$$\sigma(h_{(n_1)}^i \cdots h_{(n_k)}^k t) = (-1)^k h_{(-n_1)}^i \cdots h_{(-n_k)}^k t$$

for $h^i \in \mathfrak{h}$, $n_i \in \frac{1}{2} + \mathbb{Z}_{\geq 0}$ and $t \in T$. Then $\sigma$ is an automorphism of $V_L^\sigma$-modules. The eigenspaces for $\sigma$ are denoted $V_L^{T, \pm}$ and are both $V_L^\sigma$-modules. Similarly, an action of $\sigma$ on the untwisted $V_L$-modules $V_{\lambda + L}$ can be defined by

$$\sigma(h_{(-n_1)}^1 \cdots h_{(-n_k)}^k e^{\lambda + \alpha}) = (-1)^k h_{(-n_1)}^1 \cdots h_{(-n_k)}^k e^{-\lambda - \alpha}$$

for $h^i \in \mathfrak{h}$, $n_i \in \mathbb{Z}_{\geq 0}$ and $\alpha \in L$. Clearly $\sigma V_{\lambda + L} \subseteq V_{-\lambda + L}$ which implies the eigenspaces $V_L^{\pm}$ are $V_L^\sigma$-modules for $\lambda \in L^*$ with $2\lambda \in L$ and $V_{\lambda + L} \cong V_{-\lambda + L}$ as $V_L^{\sigma}$-modules if $2\lambda \not\in L$. The following theorem is the classification of irreducible $V_L^{T, \pm}$-modules.

**Theorem 2.1** ([DN, AD]). Let $L$ be a positive-definite even lattice and $\sigma = -1$ on $L$. Then any irreducible admissible $V_L^\sigma$-module is isomorphic to one of the following:

$$V_{\lambda + L}^\pm \ (\lambda \in L^*, 2\lambda \in L), \ V_{\lambda + L} \ (\lambda \in L^*, 2\lambda \not\in L), \ V_L^{T, \pm},$$






where $T$ is an irreducible $\hat{L}/K$-module.

Now we describe intertwining operators between the irreducible $V^\sigma_L$-modules. For a vector space $U$, denote by

$$U\{z\} = \left\{ \sum_{n \in \mathbb{Q}} v(n)z^{-n-1} \middle| v(n) \in U \right\}$$

the space of $U$-valued formal series involving rational powers of $z$. Let $V$ be a vertex algebra, and $M_1, M_2, M_3$ be $V$-modules, which are not necessarily distinct. Recall from [FHL] that an intertwining operator of type $(M_3; M_1 M_2)$ is a linear map given by

$$\mathcal{Y}: M_1 \to \text{Hom}(M_2, M_3)\{z\}, \quad v \mapsto \mathcal{Y}(v, z) = \sum_{n \in \mathbb{Q}} v(n)z^{-n-1}, \quad v(n) \in \text{Hom}(M_2, M_3)$$

which satisfies $v(n)u = 0$, for $n \gg 0$, and the Borcherds identity (2.3) for $u \in V, v \in M_1$, where the action of each term is on $M_2$. The intertwining operators of type $(M_k; M_i M_j)$ form a vector space denoted $V^k_{ij}$ and the fusion rule associated with an algebra $V$ and its modules $M_i, M_j, M_k$ is $N^k_{i,j} = \dim V^k_{ij}$.

An admissible $\sigma$-twisted $V$-module $M$ is a $\mathbb{Z}$-graded $V$-module

$$M = \bigoplus_{n \in \mathbb{Z}} M(n)$$

for which $(v_m M)(n) \subset M(\text{wt}v - m - 1 + n)$ for homogeneous $v \in V$ and $m, n \in \mathbb{Z}$. The contragredient $M'$ is defined as the graded dual space

$$M' = \bigoplus_{n \in \mathbb{Z}} M(n)^*,$$

where $M(n)^* = \text{Hom}_\mathbb{C}(M(n), \mathbb{C})$. A $V$-module $M$ is self-dual if $M = M'$. It is shown in [FHL] that $M'$ has a vertex algebra module structure and the module $M$ is irreducible if and only if $M'$ is irreducible. It is also shown in [FHL] that the fusion rules satisfy a few symmetry properties:

**Proposition 2.2 (FHL).** For $V$-modules $W_1, W_2$ and $W_3$, we have

$$N^3_{1,2} = N^3_{2,1}, \quad N^3_{1,2} = N^2'_{1,3'},$$

where prime denotes contragredient module.
The fusion rules for $V_L^\sigma$ and its irreducible modules were calculated in [ADL] to be either zero or one. In order to present their theorem, we first introduce some additional notation. For $\lambda \in L^*$ such that $2\lambda \in L$, let
\begin{align}
\tau_{\lambda, \mu} &= (-1)^{|\lambda|^2|\mu|^2}, \quad \lambda, \mu \in L^*, \\
c_\chi(\lambda) &= (-1)^{(\lambda, 2\lambda)}\varepsilon(\lambda, 2\lambda)\chi(e^{2\lambda}).
\end{align}

For any central character $\chi$ of $\hat{L}/K$, also set $\chi^{(\lambda)}$ and $\chi'$ to be the central characters defined by
\begin{align}
\chi^{(\lambda)}(a) &= (-1)^{(\lambda, |a|)}\chi(a), \\
\chi'(a) &= (-1)^{\frac{1}{2}(\lambda, |a|)}\chi(a),
\end{align}
for $a$ in the center of $\hat{L}/K$, and set $T_{\chi}^{(\lambda)} = T_{\chi^{(\lambda)}}$. Note that $\chi' = \chi$ whenever 4 divides $(\bar{a}|a)$ for all $a \in \hat{L}$. It is shown in [ADL] that
\begin{align}
(V_{\lambda+L})' &\cong V_{\lambda+L}, \quad 2\lambda \not\in L \\
(V_{\lambda+L}^\pm)' &\cong \begin{cases} V_{\lambda+L}^\pm & 2|\lambda|^2 \in 2\mathbb{Z} \\
V_{\lambda+L}^\mp & 2|\lambda|^2 \in 2\mathbb{Z} + 1 \end{cases}, \quad 2\lambda \in L \\
(V_L^{T_x, \pm})' &\cong V_L^{T_{x'}, \pm}.
\end{align}

The following theorem is only a part of Theorem 5.1 from [ADL].

**Theorem 2.3 ([ADL]).** Let $L$ be a positive-definite even lattice, and $\varepsilon \in \{\pm\}$. Then for two irreducible $V_L^\sigma$-modules $M_2, M_3$, we have the following fusion rules:

1. If $\lambda \in L^* \cap \frac{1}{2}L$, the fusion rule of type \( \begin{pmatrix} M_3 \\ V_L^\sigma M_2 \end{pmatrix} \) is equal to 1 if and only if the pair $(M_2, M_3)$ is one of the following:
   \begin{align*}
   (V_{\mu+L}, V_{\lambda+\mu+L}), & \quad \mu \in L^*, \quad 2\mu \not\in L, \\
   (V_{\mu+L}^{\varepsilon_1}, V_{\lambda+\mu+L}^{\varepsilon_2}), & \quad \mu \in L^*, \quad 2\mu \in L, \quad \varepsilon_1 \in \{\pm\}, \quad \varepsilon_2 = \varepsilon_1 \varepsilon \tau_{\lambda, 2\mu}, \\
   (V_L^{T_x, \varepsilon_1}, V_L^{T_x^{(\lambda)}, \varepsilon_2}), & \quad \varepsilon_1 \in \{\pm\}, \quad \varepsilon_2 = c_\chi(\lambda)\varepsilon_1 \varepsilon.
   \end{align*}

2. The fusion rule of type \( \begin{pmatrix} M_3 \\ V_L^{T_x, \varepsilon} M_2 \end{pmatrix} \) is equal to 1 if and only if the pair $(M_2, M_3)$ is one of the following:
   \begin{align*}
   ((V_L^{T_x^{(\lambda)}, \pm})', V_{\lambda+L}), & \quad (V_{\lambda+L}, V_L^{T_x^{(\lambda)}, \pm}), \quad \lambda \not\in L^* \cap \frac{1}{2}L, \\
   ((V_{\lambda+L}^{T_x, \varepsilon_1})', (V_{\lambda+L}^{\varepsilon_2})), & \quad (V_{\lambda+L}^{\varepsilon_2}, V_L^{T_x^{(\lambda)}, \varepsilon_1}), \quad \lambda \in L^* \cap \frac{1}{2}L,
   \end{align*}
where \( \epsilon_1 \in \{ \pm \} \), and \( \epsilon_2 = c_\chi(\lambda)\epsilon_1\epsilon \). In all other cases, the fusion rules of types \( \left( M_3 V^\epsilon_{\lambda+L} M_2 \right) \) and \( \left( M_3 V^T_{\epsilon_2} M_2 \right) \) are zero.

2.5. Quantum dimension and fusion product. The notions and properties of quantum dimensions have been systematically studied in [DJX]. A vertex algebra is rational if the admissible module category is semisimple (cf. (2.18)). Let \( C_2(V) \subset V \) denote the subalgebra spanned by all \(-2\)-products. A vertex algebra is \( C_2 \)-cofinite if \( V/C_2(V) \) is finite dimensional. Suppose \( V = \bigoplus_{n \in \mathbb{Z}} V_n \) is a self dual vertex operator algebra of CFT type, i.e., \( V_n = 0 \) for \( n < 0 \), and \( V_0 = \mathbb{C}1 \). In the case when \( V \) is also rational and \( C_2 \)-cofinite, quantum dimensions of its irreducible modules have nice properties and turn out to be helpful in determining their fusion products.

For a vertex algebra \( V \) and a \( 1/\mathbb{Z} \)-graded \( \sigma \)-twisted \( V \)-module \( M \), the formal character of \( M \) is defined as
\[
\text{ch}_q M = q^{\lambda-c/24} \sum_{n \in 1/\mathbb{Z}} (\dim M_{\lambda+n}) q^n,
\]
where \( \lambda \) is the conformal weight of \( M \). The quantum dimension of \( M \) over \( V \) is defined as
\[
q \dim_v M = \lim_{q \to 1^-} \frac{\text{ch}_q M}{\text{ch}_q V}.
\]

Next we describe the fusion product of two irreducible \( V \)-modules, \( W_1 \) and \( W_2 \). Let \( I \in \mathcal{V}^W_{W_1,W_2} \) be an intertwining operator. A module \( (W,I) \) is a fusion product of \( W_1 \) and \( W_2 \) if for any \( V \)-module \( M \) and \( \mathcal{Y} \in \mathcal{V}^M_{W_1,W_2} \), there is a unique \( V \)-module homomorphism \( \phi : W \rightarrow M \) such that \( \mathcal{Y} = \phi \circ I \). The fusion product \( (W,I) \) is typically denoted by \( W_1 \boxtimes W_2 \) and in the case when \( V \) is rational, it is known that the fusion product exists:
\[
W_1 \boxtimes W_2 = \sum_W N^{W}_{W_1,W_2} W,
\]
where the sum runs over all equivalence classes of irreducible \( V \)-modules. If \( V \) is a simple vertex algebra, a simple \( V \)-module \( M \) is a simple current if for any irreducible \( V \)-module \( W \), the fusion product \( W \boxtimes M \) exists and is also a simple \( V \)-module.

Now assume \( V \) is a rational, \( C_2 \)-cofinite, self dual vertex algebra of CFT type with finitely many irreducible admissible modules. Following [DJX], set \( M_0, \ldots, M_d \) as the inequivalent irreducible \( V \)-modules with \( M_0 \cong V \). Under these conditions, the following properties of quantum dimension are shown in [DJX].
Proposition 2.4 ([DJK]).

1. \( q \dim_V M_i \geq 1 \) for any \( 0 \leq i \leq d \).
2. If \( q \dim_V M \) exists, then \( q \dim_V M = q \dim_V M' \).
3. For any \( 0 \leq i, j \leq d \),
   \[ q \dim_V (M_i \boxtimes M_j) = q \dim_V M_i \cdot q \dim_V M_j. \]
4. \( M \) is a simple current of \( V \) if and only if \( q \dim_V M = 1 \).

We also list a few other basic properties of quantum dimension needed in the calculations.

Lemma 2.5. Let \( A, A_i, \) and \( B \) be \( V \)-modules for which \( A \subset B \) and \( i \) is in some index set. Then

\[ q \dim_V A \leq q \dim_V B, \]

\[ q \dim_V \bigoplus_i A_i = \sum_i q \dim_V A_i. \]

Lemma 2.6. For vertex algebras \( V \) and \( W \), let \( M \) be a \( V \)-module and \( N \) be a \( W \)-module. Then

\[ q \dim_{V \otimes W} M \otimes N = q \dim_V M \cdot q \dim_W N. \]

3. Irreducible \( V_Q^\sigma \)-modules

From now on, we assume \( \sigma \) has order 2. The map \( \sigma \) is extended by linearity to the complex vector space \( h = \mathbb{C} \otimes_{\mathbb{Z}} Q \) and we denote by

\[ \pi_\pm = \frac{1}{2}(1 \pm \sigma), \quad \alpha_\pm = \pi_\pm(\alpha) \]

the projections onto the eigenspaces of \( \sigma \). Introduce the important sublattices

\[ L_\pm = h_\pm \cap Q, \quad L = L_+ \oplus L_- \subseteq Q, \]

where \( h_\pm = \pi_\pm(h) \). Note that \( h = h_+ \oplus h_- \) is an orthogonal direct sum.

In [BE] we constructed and classified the irreducible \( V_Q^\sigma \)-modules and as a consequence of our result [BE, Theorem 3.7], we realized all of them as submodules of twisted or untwisted \( V_Q \)-modules [BE, Theorem 3.8]. For a positive-definite even lattice \( Q \) and an order two automorphism \( \sigma \) of \( Q \), the first step in our construction is to restrict \( Q \) to the sublattice \( \bar{Q} \) satisfying \( (\alpha | \sigma \alpha) \in 2\mathbb{Z} \) for all \( \alpha \in Q \). Under this condition, \( \sigma^2 = 1 \) on the lattice vertex algebra \( V_{\bar{Q}} \), and \( (V_Q)^{\sigma^2} = V_{\bar{Q}} \). Therefore

\[ V_Q^\sigma = (V_Q)^{\sigma^2} = V_{\bar{Q}}^\sigma \]

so that we may assume \( |\sigma| = 2 \) on \( V_Q \) and only work with the sublattice \( \bar{Q} \). For simplicity, we use \( Q \) instead of \( \bar{Q} \) for the rest of this work. It is
also shown in [BE] that $V_Q^\sigma$ decomposes as a direct sum of irreducible $V_L^\sigma$-modules:

$\text{(3.4)}$  

\[ V_Q^\sigma \cong \bigoplus_{\gamma+L \in Q/L} V_{\gamma+L+} \otimes V_{\gamma+L-}^{\eta(\gamma)}. \]

**Theorem 3.1 (BE).** Let $Q$ be a positive-definite even lattice, and $\sigma$ be an automorphism of $Q$ of order two such that $(\alpha|\sigma\alpha)$ is even for all $\alpha \in Q$. Then as a module over $V_L^\sigma \cong V_{L+} \otimes V_{L-}^*$ each irreducible $V_Q^\sigma$-module is isomorphic to one of the following:

$\text{(3.5)}$  

\[ \bigoplus_{\gamma+L \in Q/L} V_{\gamma+L+} \otimes V_{\gamma+L-}^{\eta(\gamma)} \quad (2\mu \not\in L-), \]

$\text{(3.6)}$  

\[ \bigoplus_{\gamma+L \in Q/L} V_{\gamma+L+} \otimes V_{\gamma+L-}^{\eta(\gamma)} \quad (2\mu \in L-), \]

where $\lambda \in L_+^*$, $\mu \in L_-^*$, $\epsilon \in \{\pm\}$,

$\text{(3.7)}$  

\[ \bigoplus_{\gamma+L Q/L} V_{\gamma+L+} \otimes V_{\gamma+L-}^{\eta(\gamma)} \quad \epsilon = \epsilon \eta(\gamma) \epsilon \chi(\gamma-), \]

where $\lambda \in (\pi_+Q)^*$, and $\chi$ is a central character of the group $L_-/K$ associated to $L_-$ (c.f. (2.14), (2.15)). Furthermore, every irreducible $V_Q^\sigma$-module is a submodule of a $V_Q$-module or a $\sigma$-twisted $V_Q$-module.

**Remark 3.2.** The proof is given in [BE]. The only difference is the condition on $\lambda$ in (3.7). Note that $(\pi_+Q)^*/\pi_+Q \subseteq L_+^*/L_+$ and set

$\text{(3.8)}$  

\[ v_\gamma = e^\gamma + \eta(\gamma)e^{\sigma\gamma} = e^{\gamma+} \otimes (e^{\gamma-} + \eta(\gamma)e^{-\gamma-}) \in V_{\gamma+L+} \otimes V_{\gamma+L-}^{\eta(\gamma)}. \]

Then by the fusion rules in [ADL], we have

$\text{(3.9)}$  

\[ Y(v_\gamma, z) : e^\lambda \otimes V_{L-}^{T(x-)} \rightarrow Y(e^{\gamma+}, z)e^\lambda \otimes V_{L-}^{T(x-)}^{\eta(\gamma)}, \]

where $Y(e^{\gamma+}, z)e^\lambda = z^{(\gamma+|\lambda)} \exp\left(\sum_{n<0} \alpha_n z^{-n}\right)e^{\gamma+\lambda}$. To be an untwisted module implies that $(\gamma+|\lambda) \in \mathbb{Z}$ for all $\gamma \in Q$, i.e., $\lambda \in (\pi_+Q)^*$.

Using a similar argument, we also have the restriction that $\lambda + \mu \in Q^*$ in (3.5) and (3.6).

In order to present the fusion products of irreducible $V_Q^\sigma$-modules, it is necessary to have more convenient notation. We will denote the modules (3.5) by $U_{\lambda, \mu}$ and refer to them as modules of type 1. Similarly the modules (3.6) are denoted $U_{\lambda, \mu}^{\pm}$ and we refer to them as modules of type 2. Lastly, the modules (3.7) are denoted $T_{\lambda, \chi}^{*\gamma}$, where the sign $\epsilon_0$ is the eigenvalue of $\sigma$ on the term with $\gamma = 0$, and we refer to them as modules of twisted type.
4. Quantum Dimension of Irreducible $V_Q^\sigma$-modules

The first important step is to reduce the quantum dimension over $V_Q^\sigma$ to the quantum dimension over the subalgebra $V_L^+$. 

**Proposition 4.1.** We have
\[ q \dim_{V_L^+} V_Q^\sigma = |Q/L|, \]
and the quantum dimension of any irreducible $V_Q^\sigma$-module $W$ can be determined from the quantum dimension of $W$ as a $V_L^+$-module. In particular,
\[ q \dim_{V_Q^\sigma} W = \left|\frac{Q}{L}\right|^{-1} q \dim_{V_L^+} W. \]

**Proof.** From the fusion rules of irreducible modules of $V_L^+$ and $V_L^+$, we have that $V_{\gamma+L_+} \otimes V_{\gamma+L_-}^{(\gamma)}$ is a simple current for every $\gamma \in Q$. The result (4.1) then follows since $V_Q^\sigma$ has precisely $|Q/L|$ summands (cf. (3.4)). Now let $W$ be an irreducible $V_Q^\sigma$-module. Then
\[ q \dim_{V_Q^\sigma} W = \frac{q \dim_{V_L^+} W}{q \dim_{V_L^+} V_Q^\sigma} = \left|\frac{Q}{L}\right|^{-1} q \dim_{V_L^+} W. \]

The quantum dimensions for untwisted type irreducible $V_Q^\sigma$-modules can now be determined using (4.2) and turn out to be independent of the lattice $Q$.

**Theorem 4.2.** Let $W$ be an irreducible $V_Q^\sigma$-module of untwisted type. Then we have the following quantum dimensions:
\[ q \dim_{V_Q^\sigma} W = 2, \quad \text{if } W \text{ is of type 1}, \]
\[ q \dim_{V_Q^\sigma} W = 1, \quad \text{if } W \text{ is of type 2}. \]

In particular, the irreducible $V_Q^\sigma$-modules of type 2 are simple currents.

**Proof.** Let $W = \bigoplus_{\gamma \in Q/L} V_{\gamma+L_+} \otimes V_{\gamma+L_-}$ be a $V_Q^\sigma$-module of type 1 (cf. (3.5)). Using (2.32) and (2.33), we have
\[ q \dim_{V_L^+} W = q \dim_{V_L^+} \bigoplus_{\gamma \in Q/L} V_{\gamma+L_+} \otimes V_{\gamma+L_-}, \]
\[ = \sum_{\gamma \in Q/L} q \dim_{V_L^+ \otimes V_L^+} V_{\gamma+L_+} \otimes V_{\gamma+L_-}, \]
\[ = \sum_{\gamma \in Q/L} q \dim_{V_L^+} V_{\gamma+L_+} \cdot q \dim_{V_L^+} V_{\gamma+L_-}. \]
By the fusion rules of irreducible modules of $V_{L^+}$, the simple module $V_{\gamma+\lambda+L^+}$ is a simple current for $V_{L^+}$ so that
\begin{equation}
q \dim V_{L^+} V_{\gamma+\lambda+L^+} = 1.
\end{equation}
Similarly, $q \dim V_{L^-} V_{\gamma^-+\mu+L^-} = 1$. By the fusion rules of irreducible modules of $V_{L^-}$, both simple modules $V_{L^-}^\pm$ are simple currents for $V_{L^-}$ so that
\begin{equation}
q \dim V_{L^-}^- = q \dim V_{L^-}^+ V_{L^-}^+ + q \dim V_{L^-}^+ V_{L^-}^- = 2.
\end{equation}
Since $V_{L^-}$ is a $V_{L^-}^+$-module, we can write
\begin{equation}
q \dim V_{L^-} V_{\gamma^-+\mu+L^-} = q \dim V_{L^-} V_{\gamma^-+\mu+L^-} \cdot q \dim V_{L^-}^- = 2.
\end{equation}
and therefore
\begin{equation}
q \dim V_{L^-}^- = q \dim V_{L^-} V_{\gamma^-+\mu+L^-} \cdot q \dim V_{L^-}^- = 2.
\end{equation}
Hence $q \dim W = 2|Q/L|$ and $q \dim V_{Q/L} W = 2$ using (4.2).

Now let $W = \bigoplus_{\gamma+L\in Q/L} V_{\gamma+\lambda+L^+} \otimes V_{\gamma^-+\mu+L^-}$ be a $V_{Q/L}^\pm$-module of type 2, where each $\epsilon_\gamma \in \{\pm\}$ (cf. 3.6). By fusion rules of irreducible modules of $V_{L^-}^\pm$, both simple modules $V_{L^-}^\pm$ are simple currents. Then similarly by (2.32), (2.33),
\begin{equation}
q \dim V_{L^-}^- = \sum_{\gamma+L\in Q/L} q \dim V_{L^+} V_{\gamma+\lambda+L^+} \cdot q \dim V_{L^-}^- = 1 = |Q/L|.
\end{equation}
Hence $q \dim W = 1$ by (4.2).

Quantum dimensions for irreducible $V_{Q/L}^\pm$-modules of twisted type can also be realized in terms of quantum dimension of twisted type $V_{L^-}^\pm$-modules (cf. Proposition 4.1). In order to write this relationship explicitly, we require a closer look at the fusion rules of twisted type modules in [ADL], from which we get the following fusion product:
\begin{equation}
V_{L^-}^{T_\chi,\pm} \boxtimes (V_{L^-}^{T_\chi,\pm})' = \sum_{\gamma \in Q/L} V_{\mu+L^-} + \sum_{\gamma \in Q/L} V_{\epsilon_\mu L^-}^{\chi},
\end{equation}
where each $\epsilon_\mu \in \{\pm\}$. We first show that the quantum dimension corresponding to $\chi$ and $\chi^{(\gamma-)}$ are the same for all $\gamma \in Q/L$. In order to use (4.2), we must investigate the solutions to the character equation $\chi^{(\mu)} = \chi$. The condition for the summations in (4.9) for $\chi^{(\gamma-)}$
is \( (\chi^\gamma)^{\mu} = \chi^\gamma \), and this is equivalent to the character equation
\[
\chi^{(\gamma+\mu)} = \chi^\gamma.
\]
It follows that there is a one-to-one correspondence between solutions to the character equations
\[
\chi^{(\gamma+\mu)} = \chi^\gamma
\]
and
\[
\chi^{\mu} = \chi.
\]
Using this correspondence, we then obtain:

**Lemma 4.3.** The quantum dimensions for the irreducible twisted type
\( V^+_L \)-modules \( V^{T\chi^{\gamma}}_L \pm \) are constant over \( \gamma \in Q/L \).

**Proposition 4.4.** Let \( W \) be an irreducible \( V^\sigma_Q \)-module of twisted type. Then
\[
q \dim_{V^\sigma_Q} W = q \dim_{V^+_L} V^{T\chi^{\gamma}}_L \pm.
\]

**Proof.** Let \( W = \bigoplus_{\gamma+L \in Q/L} V^{\gamma+\lambda+L+} \otimes V^{T\chi^{\gamma}}_L \pm \) be a \( V^\sigma_Q \)-module of twisted type, where each \( \epsilon_\gamma \in \{\pm\} \) (cf. (3.7)). Then using (2.32), (2.33), (4.5), (4.2), and Lemma 4.3 we have
\[
q \dim_{V^\sigma_Q} W = |Q/L|^{-1} q \dim_{V^+_L} W
\]
\[
= |Q/L|^{-1} \sum_{\gamma+L \in Q/L} q \dim_{V^+_L} V^{\gamma+\lambda+L+} \cdot q \dim_{V^+_L} V^{T\chi^{\gamma}}_L \pm \epsilon_\gamma
\]
\[
= |Q/L|^{-1} \sum_{\gamma+L \in Q/L} q \dim_{V^+_L} V^{T\chi^{\gamma}}_L \pm \epsilon_\gamma
\]
\[
= q \dim_{V^+_L} V^{T\chi^{\gamma}}_L \pm \epsilon_0.
\]

Since \( q \dim_{V^+_L} V^{T\chi^{\gamma}}_L \pm = q \dim_{V^+_L} (V^{T\chi^{\gamma}}_L \pm)^t \), the quantum dimensions of twisted type irreducible \( V^\sigma_Q \)-modules can be calculated using (4.9), Proposition 4.4, and Theorem 4.2. In view of this, we set
\[
M = L^*_+ \cap \frac{1}{2} L_-,
\]
so that the number of type 2 irreducible \( V^\sigma_L \)-modules is \( |M/L_-| \).

It is clear for any central character \( \chi \) of \( \hat{L}/K \) that \( \chi^{(\mu)} = \chi \) if and only if \( \mu \in 2L^*_+ \). It follows from (4.9) and Proposition 4.4 that
$q \dim_{V^\sigma_Q} W$ depends on those elements in $2L^*$ which correspond to distinct irreducible $V_{L^-}^\sigma$-modules of untwisted type. We denote this finite subset of $2L^*$ by $R_\sigma$. Then the number of summands in (4.9) is $|R_\sigma|$ and the number of elements in $R_\sigma$ corresponding to type 2 irreducible $V_{L^-}^\sigma$-modules which appear in (4.9) is $|R_\sigma \cap M|$. We can now write $q \dim_{V^\sigma_Q} W$ in terms of $R_\sigma$ and $M$:

**Theorem 4.5.** Let $W$ be an irreducible $V_Q^\sigma$-module of twisted type and $R_\sigma$, $M$ be as defined above. Then the square of the quantum dimension of $W$ over $V_Q^\sigma$ is given by

\[
(q \dim_{V^\sigma_Q} W)^2 = 2|R_\sigma| - |R_\sigma \cap M|.
\]

In particular, $(q \dim_{V^\sigma_Q} W)^2 \in \mathbb{Z}$, and $(q \dim_{V^\sigma_Q} W)^2 = |M|$ if $R_\sigma = M$.

**Proof.** By Proposition 4.4, it is sufficient to find $q \dim_{V_{L^-}^\sigma} V_{L^-}^T \chi, \pm$. We do this using (4.9) and counting the number of irreducible $V_{L^-}^\sigma$-modules of untwisted type. There are precisely $|R_\sigma| - |R_\sigma \cap M|$ distinct irreducible modules of type 1 in the first sum in (4.9) and $|R_\sigma \cap M|$ distinct irreducible modules of type 2 in the second sum in (4.9). The result now follows from (4.9), Proposition 4.4, and Theorem 4.2.

**Remark 4.6.** A special case of (4.12) has been proved recently in [DXY2] where the lattice $Q = L \oplus L$ is the direct sum of two copies of a positive definite even lattice $L$ of general rank and $\sigma$ is the transposition of the two summands (see also [DXY1] for rank $L = 1$). In this case they find $(q \dim_{V^\sigma_Q} W)^2 = |L^*/L|$.

5. **Fusion Products**

Recall the notation $U_{\lambda,\mu}$, $U_{\lambda,\mu}^\pm$, and $T_{\lambda,\chi}^{\sigma_0}$ in Section 3 for the irreducible $V_Q^\sigma$-modules. The following is the other main result about fusion products of irreducible $V_Q^\sigma$-modules:
Theorem 5.1. For any $\lambda \in L^*_+$ and $\mu \in L^*$, the following are fusion products for irreducible $V_Q^\sigma$-modules given in Theorem 3.1:

\begin{align}
(5.1) & \quad U_{\lambda,\mu} \boxtimes T_{\gamma,\lambda}^{\epsilon_0} = T_{\lambda+\gamma,\lambda}^{\epsilon_0} + T_{\lambda+\gamma,\lambda}^{\epsilon_0} \\
(5.2) & \quad U_{\lambda,\mu}^{\epsilon} \boxtimes T_{\gamma,\lambda}^{\epsilon_0} = T_{\lambda+\gamma,\lambda}^{\epsilon_0}, \quad \epsilon_\mu = \epsilon_0 c_\chi(\mu) \\
(5.3) & \quad U_{\lambda,\mu} \otimes U_{\chi',\mu'} = U_{\lambda+\chi',\mu+\mu'} \\
(5.4) & \quad U_{\lambda,\mu}^{\epsilon} \otimes U_{\chi',\mu'}^{\epsilon'} = U_{\lambda+\chi',\mu+\mu'}^{\epsilon\epsilon'} \\
(5.5) & \quad U_{\lambda,\mu} \otimes U_{\chi',\mu'} = \begin{cases} 
\sum \epsilon(U_{\lambda+\chi',\mu+\mu'} + U_{\lambda+\chi',\mu'-\mu'}) \\
U_{\lambda+\chi',\mu+\mu'} + U_{\lambda+\chi',\mu'-\mu'} \\
U_{\lambda+\chi',\mu'+\mu'} + U_{\lambda+\chi',\mu'-\mu'} + U_{\lambda+\chi',\mu-\mu'} 
\end{cases} \\
(5.6) & \quad T_{\lambda,\lambda}^{\epsilon_0} \otimes T_{\chi',\psi}^{\epsilon_0} = \sum_{\chi^{(\mu)} = \psi'} U_{\lambda+\chi',\mu} + \sum_{\chi^{(\mu)-\psi'} = \mu} U_{\lambda+\chi',\mu}^{\epsilon_\mu}
\end{align}

where $\epsilon_\mu = \epsilon_0 c_\chi(\mu)(-1)^{2|\mu^2}$ in (5.6), the first case in (5.5) is when $2(\mu \pm \mu') \in L_-$ the second case is when $2(\mu \pm \mu') \notin L_-$, and the third case is when $2(\mu + \mu') \notin L_-$ and $2(\mu - \mu') \in L_-$.

Proof. We first prove (5.1). By the fusion rules for irreducible $V_{L^*_+}$-modules in [DL] and irreducible $V_{L^*_+}$-modules in [ADL], the fusion rules of type $\left(\frac{T_{\lambda,\mu}^{\epsilon_0}}{U_{\lambda,\mu} \otimes T_{\gamma,\lambda}^{\epsilon_0}}\right)$ are nonzero and this yields the sum on the right hand side of (5.1). Using (4.12) and Theorem 4.2 to count quantum dimensions of each side of (5.1), we see that there can be no other nonzero fusion rules used in the fusion product.

Similarly, the fusion rules in [DL ADL] yield nonzero intertwining operators of types $\left(\frac{T_{\lambda,\mu}^{\epsilon_0}}{U_{\lambda,\mu} \otimes T_{\gamma,\lambda}^{\epsilon_0}}\right)$, $\left(\frac{U_{\lambda+\chi',\mu+\mu'}^{\epsilon_\mu}}{U_{\lambda,\mu} \otimes U_{\chi',\mu'}}\right)$, $\left(\frac{U_{\lambda+\chi',\mu+\mu'}^{\epsilon_\mu}}{U_{\lambda,\mu} \otimes U_{\chi',\mu'}}\right)$ and a similar argument as for (5.1) proves the fusion products (5.2)-(5.4). For (5.3), note that $2\mu \notin L_-$ and $2\mu' \in L_-$ implies $U_{\lambda+\chi',\mu+\mu'} \cong U_{\lambda+\chi',\mu'-\mu'}$. Also, since $q \dim(U_{\lambda,\mu} \otimes U_{\chi',\mu'}) = 1$ by (4.4) and Proposition 2.4, the fusion product $U_{\lambda,\mu}^{\epsilon} \boxtimes U_{\chi',\mu'}^{\epsilon'}$ is a simple current, and therefore must equal one irreducible module of type 2 (cf. (4.12), Theorem 4.2). We then obtain (5.4), where the given rule of signs follows because $2\mu \in L_-$ implies $\pi_{2,\mu} = 1$ (cf. (2.21)).

We now prove (5.5) and (5.6). By the fusion rules in [DL ADL], the intertwining operators of type $\left(\frac{U_{\lambda+\chi',\mu+\mu'}^{\epsilon_\mu}}{U_{\lambda,\mu} \otimes U_{\chi',\mu'}}\right)$ are nonzero only when
\[ \mu + \mu' \notin 2L_+ \] and the intertwining operators of type \( \left( U_{X_+X',\mu+\mu'}^+, U_{X_+X',\mu+\mu'}^- \right) \) are nonzero only when \( \mu + \mu' \in 2L_- \). By counting quantum dimensions of both sides of (5.5) we see that there can be no other nonzero fusion rules used in the fusion product for either case. This proves (5.5). Again by fusion rules in [DL, ADL], the intertwining operators of type \( \left( U_{\lambda+\chi,\mu}^+, U_{\lambda+\chi,\mu'}^+, U_{\lambda+\chi,\mu}^- \right) \) are nonzero only when \( \chi (\mu) = \psi' \) and \( 2\mu \notin L_- \), and the intertwining operators of type \( \left( U_{\lambda+\chi,\mu}^+, U_{\lambda+\chi,\mu'}^+, T_{X,\psi}^\epsilon \right) \) are nonzero only when \( \chi (\mu) = \psi' \) and \( 2\mu \notin L_- \) (cf. (2.24)). In order to count the quantum dimension of the right-hand side of (5.6), we count the number of solutions \( \mu + L_- \) to the character equation \( \chi (\mu) = \psi' \). Let \( R_{X,\psi}^\chi \) be the solution set, where each element is distinct under \( \sigma \). Clearly then for any \( \mu + L_- \in R_{X,\psi}^\chi \), we have \( \mu + L_- + R_{\sigma} \subset R_{X,\psi}^\chi \) (cf. (??)). We show these two sets are equal, i.e., that \( \mu, \mu' \in R_{X,\psi}^\chi \) implies \( \mu - \mu' + L_- \in R_{\sigma} \):

\[
\chi (\mu - \mu') (U_\alpha) = (-1)^{\langle \mu - \mu' \alpha \rangle} \chi (U_\alpha) = (-1)^{\langle \mu \alpha \rangle + \langle \mu' \alpha \rangle + \langle \alpha \alpha \rangle} \chi (U_\alpha) = (-1)^{\langle \mu \alpha \rangle + \frac{1}{2} \langle \alpha \alpha \rangle \langle -1 \rangle} \chi (U_\alpha) = (-1)^{\langle \mu \alpha \rangle + \frac{1}{2} \langle \alpha \alpha \rangle} \psi (U_\alpha) = \chi (U_\alpha),
\]

where the last two steps use \( \chi (\mu) = \psi' \). Therefore \( |R_{X,\psi}^\chi| = |R_{\sigma}| \). By counting quantum dimensions of both sides of (5.6) we now see there can be no other nonzero fusion rules used in the fusion product. Finally, the rule for the sign \( \epsilon_\mu \) in (5.6) follows from the fusion rules in Theorem 2.3 and (2.26). This proves (5.6).

\[ \square \]

6. Examples

Here we show explicit details of quantum dimensions and fusion products of twisted type orbifold modules in the cases of a 2-permutation orbifold and for the \( A_n \) root lattice with a Dynkin diagram automorphism.

6.1. \( (V_K \otimes V_K)^{Z_2} \) orbifolds. The case when \( Q \) is an orthogonal direct sum of two copies of an arbitrary rank positive definite even lattice \( K \) with the 2-cycle permutation has recently been studied in [DXY2]. Here we present how some of our results can simplify in these cases.
and show the explicit results in the special case of the $A_2$ root lattice. In this section, we use the symbol $K$ to avoid confusion with (3.2).

Let $Q = K \oplus K$ be an orthogonal direct sum, where $K = \langle \alpha_1, \ldots, \alpha_n \rangle$ is a positive definite even lattice with rank $n$. Set $(\alpha_i | \alpha_i) = 2k_i$ and for $\gamma \in K$, set $\gamma^1 = (\gamma, 0)$ and $\gamma^2 = (0, \gamma)$. Then a 2-cocycle $\varepsilon$ on $K$ can be extended to a 2-cocycle on $Q$ for which we may take $\eta = 1$ on $Q$ (cf. (2.13)). Let $\sigma$ be the 2-cycle permutation $\sigma(a, b) = (b, a)$. Then the vectors

$$\alpha^i = \alpha_1^i + \alpha_2^i = (\alpha_i, \alpha_i), \tag{6.1}$$

$$\beta^i = \alpha_1^i - \alpha_2^i = (\alpha_i, -\alpha_i), \tag{6.2}$$

for $i = 1, \ldots, n$, are eigenvectors of $\sigma$ with eigenvalues $\pm 1$ and

$$(\alpha^i | \alpha^i) = 4k_i = (\beta^i | \beta^i), \tag{6.3}$$

$$(\alpha^i | \alpha^j) = 2(\alpha_i | \alpha_j) = (\beta^i | \beta^j). \tag{6.4}$$

These relations then imply $Q = \bar{Q}$ (cf. Section 3).

First we describe the orbifold $V^\sigma_Q$. We have (cf. (5.2), (5.4))

$$L_+ = \langle \alpha_1, \ldots, \alpha_n \rangle, \quad L_- = \langle \beta_1, \ldots, \beta_n \rangle, \tag{6.5}$$

$$V^\sigma_Q = \bigoplus_{(b_1, \ldots, b_n)} \left( V_{\frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_+^{\frac{1}{2} \sum b_i \beta^i + L_-} \right), \tag{6.6}$$

where each $b_i \in \{0, 1\}$ and there are $|Q/L| = 2^n$ summands. We have

$$M = \left\{ \frac{\beta^i}{2} \middle| i = 0, \ldots, n \right\}, \tag{6.7}$$

$$R_\sigma \cap M = \text{span}_{\mathbb{Z}_2} \left\{ \frac{\beta^i}{2} \middle| (\alpha_i | K) \in 2\mathbb{Z} \right\}, \tag{6.8}$$

and in this case, the Gram matrix for $K$ can determine the size of $R_\sigma \cap M$.

**Lemma 6.1.** Let $M$ be as in (6.11). Then $|R_\sigma \cap M| = 1$ if and only if for every $i$, $(\alpha_i | \alpha_j)$ is odd for at least one $j$.

**Proof.** Let $\frac{\beta^i}{2} \in M$ and $j$ be such that $(\alpha_i | \alpha_j)$ is odd. Then

$$\left( \frac{\beta^i}{2} \bigm| \beta^j \right) = (\alpha_i | \alpha_j), \quad \chi'(\frac{\beta^i}{2}) = -\chi(\beta^j), \tag{6.9}$$

so that $\frac{\beta^i}{2} \notin R_\sigma$. \qed

Due to (6.3), we have

$$\chi' = \chi. \tag{6.10}$$
for all central characters $\chi$ of $L_-$. Hence all irreducible $V_\mathcal{Q}^\sigma$-modules of twisted type are self dual.

6.1.1. *Rank 1 case.* Here we provide explicit details of Theorem 4.5 in the case when $K$ is a rank 1 even positive definite lattice. These results will correspond to the results previously done in [DXY1].

Write $K = \mathbb{Z}\gamma$, where $(\gamma|\gamma) = 2k$ for some positive integer $k$ and let $\sigma$ be the 2-cycle permutation. Then it is easy to see that

$$L_- = \mathbb{Z}\beta,$$

$$L_-^* = \frac{1}{4k}L_-,$$

We also have $|R_\sigma \cap M| = 2$ and the other elements of $L_-^*/L_-$ are paired under $\sigma$. Therefore the quantum dimension of any twisted type irreducible $V_\mathcal{Q}^\sigma$-module $W$ is given by Theorem 4.5:

$$q\dim_{V_\mathcal{Q}^\sigma} W)^2 = 2|R_\sigma| - |R_\sigma \cap M|$$

$$= 2 \left( \frac{2k - 2}{2} + 2 \right) - 2$$

$$= 2k.$$

6.1.2. $A_2 \oplus A_2$ case. Here we provide explicit details of Theorems 3.1 and 4.5 in the case when $K = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the $A_2$ root lattice.

Consider the orthogonal direct sum $Q = K \oplus K$ and let $\sigma$ be the 2-cycle permutation. Then

$$L_+ = \langle \alpha^1, \alpha^2 \rangle, \quad L_- = \langle \beta^1, \beta^2 \rangle,$$

$$|\alpha^i|^2 = 4 = |\beta^i|^2, \quad (\alpha^1|\alpha^2) = -2 = (\beta^1|\beta^2),$$

and the orbifold is given by

$$V_\mathcal{Q}^\sigma = \bigoplus_{(b_1, b_2)} \left( V^+_{\frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V^+_{\frac{1}{2} \sum b_i \beta^i + L_-} \right),$$

where $b_1, b_2 \in \{0, 1\}$. To describe the dual lattices $L_\pm^*$ we set

$$\delta^1 = \frac{\alpha^1 + 2\alpha^2}{6}, \quad \delta^2 = \frac{2\alpha^1 + \alpha^2}{6},$$

$$\rho^1 = \frac{\beta^1 + 2\beta^2}{6}, \quad \rho^2 = \frac{2\beta^1 + \beta^2}{6},$$

so that $L_+^* = \langle \delta^1, \delta^2 \rangle$ and $L_-^* = \langle \rho^1, \rho^2 \rangle$ with quotients $L_+^*/L_\pm \simeq \mathbb{Z}_2 \times \mathbb{Z}_6$. Note that $\sigma(\rho^i + L_-) = 5\rho^i + L_-, 2\rho^2 + L_- = 4\rho^1 + L_-,$
and $2\delta^2 + L_+ = 4\delta^1 + L_+$. By Theorem 3.1, the type 1 irreducible $V^\sigma_Q$-modules are

$$U_{\lambda_{ij},\mu} = \bigoplus_{(b_1, b_2)} \left( V_{\lambda_{ij} + \frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_{\mu + \frac{1}{2} \sum b_i \beta^i + L_-} \right),$$

where $\lambda_{ij} = i\delta^1 + j\delta^2$, $0 \leq i \leq 5$, $j = 0, 1$ and $\mu \in \{\rho^1, 2\rho^1, \rho^2, \rho^1 + 5\rho^2\}$. Set $\mu = k\rho^1 + l\rho^2$. Then for (6.21) to be untwisted, we also require that $i + k \in 2\mathbb{Z}$ and $j + l \in 2\mathbb{Z}$ (cf. Remark 3.2). This gives 12 distinct irreducible modules of type 1.

Now we find

$$M/L_- = \langle 3\rho^1 + L_-, 3\rho^2 + L_- \rangle \cong \mathbb{Z}_2^2,$$

(cf. (4.11)) so that the type 2 irreducible $V^\sigma_Q$-modules are given by $U_{\pm \lambda_{ij},\mu}$ with $\mu + L_- \in M/L_-$. In order for $U_{\pm \lambda_{ij},\mu}$ to be untwisted we also require that $i + k \in 2\mathbb{Z}$ and $j + l \in 2\mathbb{Z}$ (cf. Remark 3.2). This gives 24 distinct irreducible modules of type 2.

For the orbifold modules of twisted type, consider the following labelling of the central characters:

| $\chi$ | $1$ | $e^{\beta^1}$ | $e^{\beta^2}$ | $e^{\beta_1^1 + \beta^2}$ |
|--------|-----|----------------|----------------|--------------------------|
| $\chi_{00}$ | $1$ | $1$ | $1$ | $1$ |
| $\chi_{10}$ | $1$ | $1$ | $-1$ | $-1$ |
| $\chi_{01}$ | $1$ | $-1$ | $1$ | $-1$ |
| $\chi_{11}$ | $1$ | $-1$ | $-1$ | $1$ |

For $\gamma = c_1\beta^1 + c_2\beta^2$, notice that $\chi_{ij}(e^{\gamma}) = (-1)^{c_1j + c_2i}$. We can then use this labelling to show the following properties:

**Lemma 6.2.** Let $b_i, c_i \in \{0, 1\}$ and $\mu = \frac{1}{2}b_1\beta^1 + \frac{1}{2}b_2\beta^2$. Then we have

(6.23) \[ c_{\chi}(\mu) = \chi(e^{b_1\beta^1})\chi(e^{b_2\beta^2}), \]

(6.24) \[ \chi^{(\mu)}_{ij} = \chi_{i+b_1,j+b_2} \]

(6.25) \[ \chi^{(\rho^1 + k\rho^2)}(e^{\bar{\epsilon}\bar{\beta}}) = (-1)^{rc_2 + kc_1}\chi(e^{\bar{\epsilon}\bar{\beta}}), \]

where $\bar{\epsilon} \cdot \bar{\beta} = c_1\beta^1 + c_2\beta^2$ and the indices in (6.21) are taken modulo 2.

Now since $(\pi_+ Q)^*/(\pi_+ Q) = \langle 2\delta^1 + \pi_+ Q \rangle \cong \mathbb{Z}_3$, there are $3 \cdot 4 = 12$ orbifold modules of twisted type:

(6.26) \[ T_{2i\delta^1,\chi_{mn}}^{\sigma_0} = \bigoplus_{\mu = (b_1, b_2)} V_{2i\delta^1 + \frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_{\mu + \frac{1}{2} \sum b_i \beta^i + L_-}^{T_{\chi_{mn+b_1,n+b_2}}} e^\mu, \]

where $i = 0, 1, 2$, $e_\mu = ec_{\chi}(\mu)$ (cf. (6.23)).
Now we determine the quantum dimensions for irreducible modules of twisted type:

\[(6.27) \quad R_\sigma = \{0, 2\rho^1\}, \quad |R_\sigma \cap M| = 1, \]
\[(6.28) \quad q \dim_{V_Q^\sigma} T_{\lambda, \chi_{ij}}^{\rho^0} = \sqrt{3}, \]
(c.f. Lemma 6.1) and a few of their fusion products:

\[(6.29) \quad T_{\lambda, \chi_{00}}^{\rho^0} \otimes T_{\lambda', \chi_{10}}^{\rho^0} = U_{\lambda + \lambda', \rho^1} + V_{\lambda + \lambda', \rho^3}, \]
\[(6.30) \quad T_{\lambda, \chi_{00}}^{\rho^0} \otimes T_{\lambda', \chi_{01}}^{\rho^0} = U_{\lambda + \lambda', \rho^2} + V_{\lambda + \lambda', \rho^3}, \]
\[(6.31) \quad T_{\lambda, \chi_{00}}^{\rho^0} \otimes T_{\lambda', \chi_{11}}^{\rho^0} = U_{\lambda + \lambda', \rho^3 + 5\rho^2} + V_{\lambda + \lambda', \rho^3 + \rho^2}. \]

The other fusion products are similar. Note that each fusion product \((6.32)\) and \((6.33)\) can be written in the form \(U_{\lambda, \mu} + U_{\lambda, \mu'}\) \((\lambda \in L_+, \mu, \mu' \in L_-)\), and that \(q \dim(T_{\lambda, \chi_{ij}}^{\rho^0} \otimes T_{\lambda', \chi_{kl}}^{\rho^0}) = 3\) for \(i, j, k, l \in \{0, 1\}\) (c.f. (6.28)).

6.2. The \(A_n\) root lattice. Consider the \(A_n\) Dynkin diagram with the simple roots \(\{\alpha_1, \ldots, \alpha_n\}\), the associated root lattice \(Q = \sum_{i=1}^n \mathbb{Z}\alpha_i\), and the Dynkin diagram automorphism \(\sigma : \alpha_i \leftrightarrow \alpha_{n-i+1}\). Set

\[(6.32) \quad \alpha^i = \alpha_i + \alpha_{n-i+1} \quad \text{and} \quad \beta^i = \alpha_i - \alpha_{n-i+1}. \]

The construction of all irreducible \(V_Q^\sigma\)-modules was done in [E]. Since the results depend on the parity of \(n\), we treat the cases separately.

6.2.1. Results for \(n\) odd. Throughout this section, set \(l = \frac{n-1}{2}\) and let \(i < l + 1\). Then

\[|\alpha^i|^2 = 4 = |\beta^i|^2, \quad (\alpha^i|\beta^j) = 0, \quad (\alpha^i|\alpha^{i+1}) = -2 = (\beta^i|\beta^{i+1}), \]

and \(\langle \alpha^i|\alpha^j \rangle = 0 = (\beta^i|\beta^j)\) otherwise, and

\[L_+ = \sum_{i=1}^l \mathbb{Z}\alpha^i + \mathbb{Z}\alpha_{l+1}, \quad L_- = \sum_{i=1}^l \mathbb{Z}\beta^i. \]

Therefore \(Q = \bar{Q}\) (c.f. Section 3), and the orbifold is given by

\[(6.33) \quad V_Q^\sigma \cong \bigoplus_{(b_1, \ldots, b_l)} \left( V_{\frac{l}{2} \sum b_i \alpha^i + L_+} \otimes V_{\frac{l}{2} \sum b_i \beta^i + L_-} \right), \]

where \(b_1, \ldots, b_l \in \{0, 1\}\). Now the Gram matrix for \(L_-\) is twice the Gram matrix for \(A_l\) so that we may use the dual lattice \(\{\lambda_1, \ldots, \lambda_l\}\) for \(A_l\) to describe the dual lattice for \(L_-\). A basis for \(L_+^\ast\) is then

\[\mu_i = \frac{1}{2(l+1)}((l-i+1)\beta^1 + 2(l-i+1)\beta^2 + i(l-i+1)\beta^i + i(l-i)\beta^{i+1} + \cdots + i\beta^l), \]
where \( i = 1, \ldots, l \). Then there are \( l + 4 \) irreducible \( V_Q^\sigma \)-modules of untwisted type (see \[E\] Theorem 5.5.5):

\[
U_0^\pm = \bigoplus_{(b_1, \ldots, b_l)} \left( V_{\frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_{\frac{1}{2} \sum b_i \beta^i + L_-} \right),
\]

\[
U_k^\pm = \bigoplus_{(b_1, \ldots, b_l)} \left( V_{\frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_{\frac{1}{2} \sum b_i \beta^i + L_-} \right), \quad k = 1, \ldots, l
\]

\[
U_{l+1}^\pm = \bigoplus_{(b_1, \ldots, b_l)} \left( V_{\frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_{\frac{1}{2} \sum b_i \beta^i + L_-} \right),
\]

where \( b_1, \ldots, b_l \in \{0, 1\} \) and \( \epsilon_i = 0 \) for \( i \) odd and \( \epsilon_i = 1 \) for \( i \) even.

From this it is clear that \( |R_\sigma| = \lfloor \frac{l}{2} \rfloor + 2 \) and \( |R_\sigma \cap M| = 2 \). If \( W \) is an irreducible \( V_Q^\sigma \)-module of twisted type, then from Theorem 4.5 we have

\[
(q \text{dim}_{V_Q^\sigma} W)^2 = \begin{cases} l + 2, & \text{if } l \text{ even,} \\ l + 1, & \text{if } l \text{ odd.} \end{cases}
\]

6.2.2. Results for \( n \) even. Throughout this section, set \( l = \frac{n}{2} \) and let \( i < l + 1 \). Then for \( i < l \),

\[
|\alpha^i|^2 = 4 = |\beta^i|^2, \quad (\alpha^i|\beta^i) = 0, \quad (\alpha^i|\alpha^{i+1}) = -2 = (\beta^i|\beta^{i+1}),
\]

\[
(\beta^l|\beta^l) = 6, \quad (\alpha^l|\alpha^l) = 2, \quad \text{and } (\alpha^i|\alpha^j) = 0 = (\beta^i|\beta^j) \text{ otherwise. Then}
\]

\[
\bar{Q} = \sum_{i=1}^{l-1} \mathbb{Z} \alpha_i + \mathbb{Z} \alpha^l + \mathbb{Z} \beta^l + \sum_{i=l+2}^{n} \mathbb{Z} \alpha_i,
\]

and the cosets \( Q/L \) are in correspondence with \( \{0, 1\} \)-valued \((l-1)\)-tuples (cf. Section 3). Therefore the orbifold is given by

\[
(6.34) \quad V_Q^\sigma \simeq \bigoplus_{(b_1, \ldots, b_{l-1})} \left( V_{\frac{1}{2} \sum b_i \alpha^i + L_+} \otimes V_{\frac{1}{2} \sum b_i \beta^i + L_-} \right),
\]

where \( b_1, \ldots, b_{l-1} \in \{0, 1\} \). A basis of \( L_-^\sigma \) was computed in \[E\] Proposition 5.6.2:

\[
\mu_i = \frac{1}{2(2l+1)}((2l-2i+1)\beta^1 + \cdots + i(2l-2i+1)\beta^i
\]

\[
+ i(2l-2i-1)\beta^{i+1} + \cdots + i\beta^l),
\]

where \( 1 \leq i \leq l \). Set \( \gamma = \frac{1}{2}(\beta^1 + \beta^3 + \cdots + \beta^s) \), where \( s = \begin{cases} l, & \text{if } l \text{ odd} \\ l-1, & \text{if } l \text{ even.} \end{cases} \)
Then there are $l + 4$ irreducible $V^\sigma_Q$-modules of untwisted type (see [E] Theorem 5.6.6):

$$U^\pm_0 = \bigoplus_{(b_1, \ldots, b_{l-1})} \left( V^{\pm} \sum b_i \alpha^i + L_++ \otimes V^{\pm} \frac{1}{2} \sum b_i \beta^i + L_- \right)$$

$$U^\pm_k = \bigoplus_{(b_1, \ldots, b_{l-1})} \left( V^{\pm} \sum b_i \alpha^i + L_+ \otimes V^{\pm} \frac{1}{2} \sum b_i \beta^i + L_- \right), \quad k = 1, \ldots, l - 1,$$

$$U^\pm_\gamma = \bigoplus_{(b_1, \ldots, b_{l-1})} \left( V^{\pm} \sum b_i \alpha^i + L_+ \otimes V^{\pm} \frac{1}{2} \sum b_i \beta^i + L_- \right),$$

where $b_1, \ldots, b_{l} \in \{0, 1\}$. Note that $\frac{\beta}{2} \notin R_\sigma$. From this it is clear that

$$|R_\sigma| = \begin{cases} l + 1, & \text{if } l \text{ even}, \\ l, & \text{if } l \text{ odd} \end{cases}, \quad |R_\sigma \cap M| = \begin{cases} 2, & \text{if } l \text{ even}, \\ 1, & \text{if } l \text{ odd} \end{cases}.$$  

If $W$ is an irreducible $V^\sigma_Q$-module of twisted type, then from Theorem 4.5 we have

$$(q \dim_{V^\sigma_Q} W)^2 = \begin{cases} 2l, & \text{if } l \text{ even}, \\ 2l - 1, & \text{if } l \text{ odd} \end{cases}.$$  

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