Elements of the Kopula (eventological copula) theory

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Abstract. New in the probability theory and eventology theory, the concept of Kopula (eventological copula) is introduced. The theorem on the characterization of the sets of events by Kopula is proved, which serves as the eventological pre-image of the well-known Sklar’s theorem on copulas (1959). The Kopulas of doublets and triplets of events are given, as well as of some \( N \)-sets of events.

Keywords. Eventology, probability, Kolmogorov event, event, set of events, Kopula (eventological copula), Kopula characterizing a set of events.

1 Introduction

Long time ago and little by little, the incentive for this work materialized in the theory of sets of events, eventology [1], where the need to locate the classes of event-probability distributions (e.p.d.) of the sets of events (s.e.), which were so arbitrary and spacious to be able without let or hindrance to deal with the relationships between pairs, triples, quadruplets, etc., of events, in other words, to understand the structure of statistical dependencies and generalities between events from some s.e. A similar need is perhaps the only one that has always fueled the development of the probability theory and statistics, which in one way or another are theories of studying and evaluating the structures of statistical dependencies and generalities in the distributions of sets of events.

The classical copula theory [2, 3, 4, 5], existing since the 50s of the last century, allows us to construct classes of joint distribution functions that have given marginal distribution functions. In eventology, the theory of sets of events, proposed in the paper the theory of Kopula (eventological copula) allows us to solve a similar problem — to build classes of e.p.d.’s of sets of events whose events happen with given probabilities of marginal events.

1.1 General statement of the problem of the Kopula theory

We formulate the general statement of the problem of the \( N \)-Kopula theory for \( N \)-sets of events. If in the classical theory the copula is the tool for selecting some family of joint d.f.’s of a set of random variables from the set of all d.f.’s with given marginal d.f.’s, then in the eventological theory the Kopula is the tool for selecting a family of e.p.d.’s of the 1st kind of the set events from the set of all e.p.d.’s with the given probabilities of marginal events.

However, unlike the classical d.f.’s the functions of e.p.d.’s of the 1st kind of the \( N \)-s.e. \( \mathcal{X} \) are functions that are defined as the sets

\[
\{ p(X/\mathcal{X}), X \subseteq \mathcal{X} \}
\]

of all its \( 2^N \) values, probabilities of the 1st kind \( p(X/\mathcal{X}) \), on the set of all subsets of this \( N \)-s.e.

So, let’s clarify, Kopula is the tool for selecting a family of sets of the form (1.1) from the set of all sets with given probabilities of marginal events.

To specify a family of sets of \( 2^N \) probabilities of the 1st kind (1.1), it is necessary and sufficient to specify a family of sets from the \( 2^N - N - 1 \) parameters, since all the probabilities in each set must be non-negative and give in the sum of one. And to specify a family of sets of \( 2^N \) probability values of the 1st kind (1.1) with given probabilities of marginal events that form the \( \mathcal{X} \)-set

\[
\tilde{p} = \{ p_x, x \in \mathcal{X} \},
\]

it is necessary and sufficient to specify a family of sets from the \( 2^N - N - 1 \) parameters, since for each collection there must be another \( N \) constraints for events \( x \in \mathcal{X} \):

\[
\sum_{x \in \mathcal{X} \subseteq \mathcal{X}} p(X/\mathcal{X}) = p_x.
\]

Therefore, “to define the family of functions of e.p.d.’s of the 1st kind with given probabilities of marginal events” means “to define a family of sets from the \( 2^N - N - 1 \) parameters” as sets of functions of these probabilities. Eventological theory should solve this problem with the help of a convenient tool, the Kopula, which allows us to define a family of sets from the \( 2^N - N - 1 \) parameters as sets.
of functions from marginal probabilities, which in turn can be made dependent on a number of auxiliary parameters.

In general, the $N$-Kopula in the eventological theory is an instrument for defining the family of probability distributions of the $1$st kind of the $N$-s.e., in the form of a family of $2^N$-set of functions of the probabilities of their $N$ marginal events.

### 1.2 Prolegomena of the Kopula theory

The main results of this paper are presented in a rather rigorous mathematical manner. And although the definitions, statements and proofs are provided with examples and illustrations, in order to visualize the ideas underlying the Kopula theory, in my opinion, a number of preliminary explanations in a less strict context, which are collected in several prolegomena, may be necessary.

If in some set of events $\mathcal{X}$, some events from the subset $\mathcal{X} \subseteq \mathcal{X}$ are replaced by their complements, then we get a new set of events $\mathcal{X}^{(1)} = \mathcal{X} + (\mathcal{X} - \mathcal{X})^{(2)}$, which is called the $X$-phenomenon of s.e. $\mathcal{X}$. The set of all such $X$-phenomena for $\mathcal{X} \subseteq \mathcal{X}$ is called the $2^X$-phenomenon-dom of s.e. $\mathcal{X}$. In [6] a rather distinct theory of set-phenomena and the phenomenon-dom of some s.e.

Similarly, the theory of set-phenomena [6] defines the phenomena and phenomenon-dom of the set $\mathcal{P} = \{ p_x, x \in \mathcal{X} \}$ of the probabilities of marginal events $x \in \mathcal{X}$: the $X$-phenomenon $\mathcal{P}^{(1)} = \{ p_x, x \in \mathcal{X} \} + \{ 1 - p_x, x \in \mathcal{X} - \mathcal{X} \}$ of the set of marginal probabilities $\mathcal{P}$ is obtained by replacing the marginal probabilities $p_x$ by complementary marginal probabilities $1 - p_x$ when $x \in \mathcal{X} - \mathcal{X}$.

**Prolegomenon 1 (set-phenomenon of a set of events and a set of probabilities of events).** The main conclusion of the above theory is obvious: probability distribution of the s.e. $\mathcal{X}$ characterizes the probability distribution and the set of marginal probabilities each set-phenomenon from its $2^X$-phenomenon-dom.

**Prolegomenon 2 (set-phenomenal transformation).** 1) For each pair $\mathcal{X}^{(1)}(X), \mathcal{X}^{(c)}(Y)$ set-phenomena of the set of events $\mathcal{X}$ their probability distributions are related to each other set-phenomenal transformations. 2) In each pair $\mathcal{P}^{(1)}(X), \mathcal{P}^{(c)}(Y)$ set-phenomena of the set of marginal probabilities $\mathcal{P}$ events from $\mathcal{X}$ are also interconnected by set-phenomenal transformations.

The event $x \in \mathcal{X}$ is called half-rare [6] if the probability $p_x = P(x)$ with which it happens is not more than half: $p_x \leq 1/2$. If all events from the s.e. $\mathcal{X}$ are half-rare, we speak of a set of half-rare events, or a half-rare s.e.

**Prolegomenon 3 (sets of half-rare events and its Kopulas).** 1) It is not difficult to guess that for any s.e. $\mathcal{X}$, in the $2^X$ phenomenon-dom of the sets of events $\mathcal{X}^{(1)}(X), X \subseteq \mathcal{X}$, and in the $\mathcal{X}$-phenomenon-dom of the sets of its marginal probabilities $\tilde{p}^{(1)}(X), X \subseteq \mathcal{X}$, there is always a half-rare set-phenomenon. If, in addition, there are no events in $\mathcal{X}$ happening with probability $1/2$, then such a half-rare set-phenomenon is unique. 2) A Kopula of some family of half-rare $N$-s.e.’s is generated by $2^N$ functions from half-rare variables defined on the half-hypercube $[0,1/2]^N$ with values from $[0,1]$ that are continued by the set-phenomenal transformations of the half-rare variables to the corresponding half-hypercubes, all together completely filling the unit hypercubes.

**Prolegomenon 4 (an invariance of the copula with respect to the order of half-rare events).** Our task is to construct a Kopula of a family of arbitrary (unordered) sets of half-rare events, i.e. a 1-function, the arguments of which form an unordered set of probabilities of their marginal events. Therefore, it is natural to require such a function to be invariant with respect to the order of its arguments; with respect to the order of events in these sets. In other words, it is natural to consider this 1-function as a function of a set of arguments, rather than a vector of arguments with ordered components, as it is usually assumed.

**Prolegomenon 5 (insertable sets of half-rare events and a frame half-rare event).** Two insertable sets of half-rare events for a given set of half-rare events $\mathcal{X} = \{ x_0 \} + \mathcal{X}$ with the frame half-rare event $x_0 \in \mathcal{X}$, happening with the highest probability among all events from $\mathcal{X}$, are two sets of half-rare events $\mathcal{X}' = \{ x_0 \} (\cap) \mathcal{X}$ and $\mathcal{X}'' = \{ x_0 \} (\cap) \mathcal{X}''$ that partition the set of other events $\mathcal{X} = \mathcal{X} - \{ x_0 \}$ into two: $\mathcal{X} = \mathcal{X}'(+)\mathcal{X}''$. The events of one of them, $\mathcal{X}'$, are contained in the frame half-rare event $x_0$, and the events of the other, $\mathcal{X}''$, are contained in its complement $x_0' = \Omega - x_0$.

**Prolegomenon 6 (the insertable sets of events and conditional e.p.d.’s of a set of events with respect to the frame event and its complement).** Conditional e.p.d.’s of the 1st kind of one s.e. $\mathcal{X}$ with respect to the other s.e. $\mathcal{Y}$ are defined in the traditional way [1]. However, until now attempts to define such a “conditional” s.e., which would have given a conditional e.p.d. of the 1st kind, turned out to be completely impractical [7]. The concept of two insertable s.e.’s in a frame event is a well-defined “ersatz” of such “conditional” s.e.’s. The e.p.d.’s of this “ersatz”, although they do not coincide with two conditional e.p.d.’s of the 1st kind with respect to the frame event and its complement, but they are fully characterized by them. The converse is also true: the e.p.d.’s of two frame s.e.’s characterize the corresponding two conditioned e.p.d.’s of the 1st kind.

**Prolegomenon 7 (a frame method of constructing a Kopula of an arbitrary set of half-rare events).** A frame method constructs a Kopula of a family of arbitrary sets of half-rare events on the basis of a conditional scheme by means of a recurrence formula via conditional e.p.d.’s concerning the frame event and its complement. A recurrent
formula associates this Kopula with two Kopulas of families of their insertable sets of half-rare events of smaller dimension, which are characterized by the corresponding conditional e.p.d.’s of the 1st kind (see Prolegomenon 6).

Prolegomenon 8 (a set-phenomenal transformation of a half-rare Copula to an arbitrary one). To construct the Kopula of a family of arbitrary s.e.’s it is enough to construct the Kopula of the family of their half-rare set-phenomena and apply a set-phenomenal transformation to this Kopula.

Prolegomenon 9 (Cartesian representation of the N-Kopula in \( \mathbb{R}^N \)). It follows from the Prolegomenon 4 that the Cartesian representation of the N-Kopula in \( \mathbb{R}^N \) should be a symmetric function of \( N \) ordered variables, marginal probabilities of events from N-s.e. \( \mathcal{X} \), which is defined on the N-dimensional unit hypercube \([0, 1]^N\). The Cartesian representation of the N-Kopula is based on the fact that its symmetric image takes the same values on all permutations of its arguments, that is, is defined by the permutation of \( N \) events group. Moreover, the value of such a symmetric function on an arbitrary N-vector \( w = \{w_1, ..., w_N\} \in [0, 1]^N \) is equal to the value of the N-Kopula on an ordered \( X \)-set of marginal probabilities of half-rare events \( \bar{p} = \{p_x, x \in \mathcal{X}\} \), the ordered half-rare projection of the N-vector \( w \), the order of the variables in which is defined by an \( N \)-permutation \( \pi_w \). This has the components of the half-rare projection \( w^* \) in decreasing order, where

\[
  w^*_n = \begin{cases} 
    w_n, & w_n \leq 1/2, \\
    1 - w_n, & w_n > 1/2 
  \end{cases} \quad (1.4)
\]

are components of the N-vector \( \bar{w}^* \) of the half-rare projection of the N-vector \( \bar{w} \), \( n = 1, ..., N \). As a result, the ordered half-rare \( \bar{X} \) is the set of marginal probabilities \( \bar{p} = \bar{p}(\bar{w}) \), on which the N-Kopula takes the same value as a symmetric function on \( w \), is given by the formula

\[
  \bar{p}(\bar{w}) = \pi_w(\bar{w}^*), \quad (1.5)
\]

which defines the Cartesian representation of the N-Kopula in \( \mathbb{R}^N \) for each \( \bar{w} \in [0, 1]^N \).

### 2 The Kopula: definition, theorem and the simplest Kopulas

We consider the general probability space of Kolmogorov events \((\Omega, \mathcal{A}, P)\), some particular probability space of events\((\Omega, \mathcal{A}, P)\) and the N-set of events (N-s.e.) \( \mathcal{X} \subset \mathcal{A} \) with the event-probability distribution (e.p.d.\(^2\)) of the 1st kind

\[
  p(\mathcal{X}) = \{p(X/\mathcal{X}) : X \subseteq \mathcal{X}\},
\]

and of the second kind

\[
  P_X = \{p_X/\mathcal{X} : X \subseteq \mathcal{X}\},
\]

which, recall, are related to each other by the Mobius inversion formulas:

\[
  p_X/\mathcal{X} = \sum_{X \subseteq Y} p(Y/\mathcal{X}),
\]

\[
  p(X/\mathcal{X}) = \sum_{X \subseteq Y} (-1)^{|Y| - |X|} p_Y/\mathcal{X}.
\]

Definition 1 (set-phenomen of a s.e. and its phenomenon-dom). Every N-s.e. \( \mathcal{X} \subset \mathcal{A} \) generates its own 2\( N \)-phenomen-dom, defined as a 2\( N \)-family

\[
  2(\mathcal{X}) = \{X(\mathcal{X}), X \subseteq \mathcal{X}\}, \quad (2.1)
\]

composed of N-s.e. in the form

\[
  X(\mathcal{X}) = X + (X - X)\subseteq \mathcal{X},
\]

which for each \( X \subseteq \mathcal{X} \) is called its set-phenomen [6], more precisely, \( X \)-phenomen, where

\[
  X(\mathcal{X}) = \{x^c : x \in X\}
\]

is an -complement of the s.e. \( X \subseteq \mathcal{X} \).

We also recall that probabilities of the second kind

\[
  p_x = p(x) = P\left(\bigcap_{x \in \{x\} \subseteq \mathcal{X}} x\right) = P(x)
\]

are probabilities of marginal events from \( \{x\} \subseteq \mathcal{X} \) (marginal probabilities), probabilities of the second kind

\[
  p_{xy} = p(x, y) = P\left(\bigcap_{x \in \{x, y\} \subseteq \mathcal{X}} x\right) = P(x \cap y)
\]

are probabilities of double intersections of events from \( \{x, y\} \subseteq \mathcal{X} \), and probabilities of the second kind

\[
  p_{Z_n} = P\left(\bigcap_{x \in \{x\} \subseteq \mathcal{X}} x\right)
\]

are probabilities of \( n \)-intersections of events from \( Z_n \subseteq \mathcal{X} \), where \( |Z_n| = n; \)

Definition 2 (set-phenomen of the set of probabilities of events from a s.e. and its phenomenon-dom). The N-set of probabilities of events from \( \mathcal{X} \)

\[
  \bar{p} = \{p_x : x \in \mathcal{X}\}
\]

also generates its 2\( \mathcal{X} \)-phenomen-dom, the 2\( N \)-totality

\[
  2(\bar{p}) = \{\bar{p}(X/\mathcal{X}) : X \subseteq \mathcal{X}\}, \quad (2.2)
\]
composed of $N$-sets in the form

$$\hat{p}(c|X/x) = \left\{ p_z : z \in X(c|X) \right\},$$

and defined for $X \subseteq X$ as the $N$-set of probabilities of events from $X$-phenomenon $X(c|X)$ of the s.e. $X$ where for $p_z \in \hat{p}(c|X/x)$

$$p_z = \begin{cases} p_x, & z = x \in X, \\ 1 - p_x, & z = x^c \in X(c). \end{cases}$$

In particular, for $X = X$

$$\hat{p}(c|X/x) = \left\{ p_x : x \in X \right\} = \hat{p}.$$ We denote by

$$\psi : \bigotimes_{x \in X} [0, 1]^x \to \mathbb{R}^+_0 \quad (2.3)$$
a nonnegative bounded numerical function defined on the set-product \cite{8-9}, $X$-hypercube

$$[0, 1]^{\otimes X} = \bigotimes_{x \in X} [0, 1]^x.$$ Arguments of $\psi$ form the $N$-set

$$\hat{w} = \left\{ w_x : x \in X \right\} \in [0, 1]^{\otimes X}$$

which generates its own $2^X$-phenomenon-dom, the $2^N$-totality

$$2(c|\hat{w}) = \left\{ \hat{w}(c|X/x), X \subseteq X \right\}$$

of $N$-sets of arguments:

$$\hat{w}(c|X/x) = \left\{ w_z : z \in X(c|X) \right\}$$

where for $w_z \in \hat{w}(c|X/x)$

$$w_z = \begin{cases} w_x, & z = x \in X, \\ 1 - w_x, & z = x^c \in X(c). \end{cases}$$

Let

$$\Psi_X = \left\{ \psi : [0, 1]^{\otimes X} \to \mathbb{R}^+_0 \right\} \quad (2.6)$$

be the family of all the nonnegative bounded numerical functions on the $X$-hypercube.

**Definition 3 (normalized function on the $X$-hypercube).** A function $\psi \in \Psi_X$ is called normalized on the $X$-hypercube if for each $\hat{w} \in [0, 1]^{\otimes X}$

$$\sum_{X \subseteq X} \psi \left( \hat{w}(c|X/x) \right) = 1, \quad (2.7)$$
i.e., the sum of its values on all the $N$-sets of arguments from $2^X$-phenomenon-dom $2(c|\hat{w})$ is one.

**Definition 4 (a $1$-function on the $X$-hypercube).** A function $\psi \in \Psi_X$ is called a $1$-function on the $X$-hypercube if for all $\hat{w} \in [0, 1]^{\otimes X}$ $x$-marginal equalities are satisfied for all $x \in X$:

$$\sum_{x \in X \subseteq X} \psi \left( \hat{w}(c|X/x) \right) = w_x, \quad (2.8)$$
i.e., the sum of its values on $x$-halves of $N$-sets of arguments from the $2^X$-phenomenon-dom $2(c|\hat{w})$ is $w_x$.

Denote by

$$\Psi^0_X = \left\{ \psi \in \Psi_X : \sum_{x \in X \subseteq X} \psi \left( \hat{w}(c|X/x) \right) = 1; \hat{w} \in [0, 1]^{\otimes X} \right\}$$
the family of functions, normalized on the $X$-hypercube; and by

$$\Psi^1_X = \left\{ \psi \in \Psi_X : \sum_{x \in X \subseteq X} \psi \left( \hat{w}(c|X/x) \right) = w_x; \hat{w} \in [0, 1]^{\otimes X} \right\}$$
the family of $1$-functions on the $X$-hypercube.

1 (properties of $1$-functions on the $\{x, y\}$-square). A strict inclusion is fair:

$$\Psi^1_{\{x,y\}} \subset \Psi^0_{\{x,y\}}.$$ Proof. In other words, the lemma states: 1) if $\psi \in \Psi^1_{\{x,y\}}$ is a $1$-function on the $\{x, y\}$-square then $\psi \in \Psi^0_{\{x,y\}}$ is a normalized function on the $\{x, y\}$-square; 2) among the normalized functions from $\Psi^0_{\{x,y\}}$ there is one which is not a $1$-function. But this is obvious, as it is confirmed by the following simple examples.

First, indeed, for the doublet of events $X = \{x, y\}$ by the definition of a $1$-function, we have

$$\psi(w_x, w_y) + \psi(w_x, 1 - w_y) = w_x, \quad (2.9)$$

$$\psi(w_x, w_y) + \psi(1 - w_x, w_y) = w_y, \quad (2.10)$$

$$\psi(1 - w_x, 1 - w_y) + \psi(w_x, 1 - w_y) = 1 - w_y, \quad (2.11)$$

$$\psi(1 - w_x, 1 - w_y) + \psi(1 - w_x, w_y) = 1 - w_x. \quad (2.12)$$

The sums (2.9) and (2.12) as well as the sums (2.10) and (2.11) as a result give

$$\psi(w_x, w_y) + \psi(w_x, 1 - w_y) + \psi(1 - w_x, w_y) + \psi(1 - w_x, 1 - w_y) = 1,$$

i.e., $\psi \in \Psi^0_{\{x,y\}}$ is a normalized function on the $\{x, y\}$-square.

Second, the function (see its graph in Fig. 13)

$$\psi(w_x, w_y) = (w_x + w_y)/4,$$

3In this figure and others, which illustrate the doublets of events, the map of this function on a unit square is shown under the graph in conditional colors where the white color corresponds to the level 1/4.
is normalized on the \(\{x, y\}\)-square, since
\[
\begin{align*}
\psi(w_x, w_y) + \psi(w_x, 1 - w_y) + \\
\psi(1 - w_x, w_y) + \psi(1 - w_x, 1 - w_y) = \\
= (w_x + w_y)/4 + (w_x + 1 - w_y)/4 + \\
(1 - w_x + w_y)/4 + (1 - w_x + 1 - w_y)/4 = 1.
\end{align*}
\]

However, it is not a 1-function, since
\[
\begin{align*}
\psi(w_x, w_y) + \psi(w_x, 1 - w_y) = (w_x + w_y)/4 + \\
(w_x + 1 - w_y)/4 = w_x/2 + 1/4 \neq w_x,
\end{align*}
\]
\[
\begin{align*}
\psi(w_x, w_y) + \psi(1 - w_x, w_y) = (w_x + w_y)/4 + \\
(1 - w_x + w_y)/4 = w_y/2 + 1/4 \neq w_y.
\end{align*}
\]

The lemma is proved.

Of course, in the general case, for an arbitrary s.e. \(\mathfrak{x}\) the same lemma is fulfilled.

**Lemma 2 (properties of 1-functions on the \(x\)-hypercube). A strict inclusion is fair:**
\[
\Psi^1_x \subset \Psi^0_x.
\]

**Proof is similar.**

**Note 1 (a representation of a 1-function on the \(x\)-hypercube in the form of \(2^{[x]}\)-set of functions).** Any 1-function \(\psi \in \Psi^1_x\) on the \(x\)-hypercube \([0, 1]^{\otimes x}\) for each \(\hat{w} \in [0, 1]^{\otimes x}\) is represented in the form of \(2^{[x]}\)-set of the following functions:
\[
\psi(\hat{w}) = \left\{ \psi_{x}(\hat{w}), X \subseteq \mathfrak{x} \right\} = \\
= \left\{ \psi \left( \hat{w}(c|X/X) \right), X \subseteq \mathfrak{x} \right\}.
\]

**Definition 5 (Kopula).** The 1-functions \(\mathcal{K} \subseteq \Psi^1_x \subseteq \Psi_x\) is called \(\mathcal{K}\)-Kopulas\(^4\) of the s.e. \(\mathfrak{x}\).

As well as every 1-function (2.14), any \(\mathcal{K}\)-Kopula of the s.e. \(\mathfrak{x}\) can be represented for \(\hat{w} \in [0, 1]^{\otimes x}\) in the form of \(2^{[x]}\)-set of the following functions:
\[
\mathcal{K}(\hat{w}) = \left\{ \mathcal{K}_{x}(\hat{w}), X \subseteq \mathfrak{x} \right\} = \\
= \left\{ \mathcal{K} \left( \hat{w}(c|X/X) \right), X \subseteq \mathfrak{x} \right\}.
\]

**Note 2 (characteristic properties of Kopula).** Each Kopula \(\mathcal{K}\) has two characteristic properties
\begin{enumerate}
\item Kopula is nonnegative:
\[
\mathcal{K} \left( \hat{w}(c|X/X) \right) \geq 0
\]
for \(X \subseteq \mathfrak{x}\), since by definition \(\mathcal{K} \in \Psi^1_x\);
\item Kopula is satisfied \(x\)-marginal equalities:
\[
\sum_{x \in X \subseteq \mathfrak{x}} \mathcal{K} \left( \hat{w}(c|X/X) \right) = w_x
\]
for \(x \in \mathfrak{x}\), since by definition \(\mathcal{K} \in \Psi^1_x\);
\end{enumerate}

From (2.17) by Lemma 2 a probabilistic normalization of the Kopula follows:
\[
\sum_{X \subseteq \mathfrak{x}} \mathcal{K} \left( \hat{w}(c|X/X) \right) = 1.
\]

From (2.16) and (2.18) terrace-by-terrace probabilistic normalization of the Kopula follows:
\[
0 \leq \mathcal{K} \left( \hat{w}(c|X/X) \right) \leq 1
\]
for \(X \subseteq \mathfrak{x}\).

**2.1 Characterization of a set of events by Kopula**

The eventological analogue and the preimage of the well-known Škalar theorem on copulas [2] is the following theorem.

**Theorem 1 (characterization of a s.e. by Kopula).** Let \(p = \{p(X/X) : X \subseteq \mathfrak{x}\}\) be the e.p.d. of the 1st kind of the s.e. \(\mathfrak{x}\) with \(\mathfrak{x}\)-set of probabilities of marginal events \(\hat{p} = \{p_x : x \in \mathfrak{x}\} \in [0, 1]^{\otimes \mathfrak{x}}\). Then there is a \(\mathfrak{x}\)-Kopula \(\mathcal{K} \in \Psi^1_x\) that defines a family of e.p.d.’s of the 1st kind of the s.e. \(\mathfrak{x}\). This family contains the e.p.d. \(p\), when Kopula’s arguments

\(^4\)see the footnote 1 on page 78.
coincide with \( \hat{p} \). In other words, the such Kopula that for all \( X \subseteq \mathcal{X} \)
\[
p(X/\mathcal{X}) = \mathcal{K}_X(\hat{p}) = \mathcal{K}(\hat{p}(c(X/\mathcal{X})) . \tag{2.20}
\]

Conversely, for any \( \mathcal{X} \)-set of probabilities of marginal events \( \hat{p} \in [0, 1]^{\otimes \mathcal{X}} \) and any \( |\mathcal{X}| \)-Kopula \( \mathcal{K} \in \Psi_{\mathcal{X}} \), the function \( p = \{p(X/\mathcal{X}) : X \subseteq \mathcal{X}\} \), defined by formulas (2.20) for \( X \subseteq \mathcal{X} \), is an e.p.d. of the 1st kind, which characterizes the s.e. \( \mathcal{X} \) with given \( \mathcal{X} \)-set of the probabilities of marginal events \( \hat{p} \).

Proof is a direct consequence of the properties of e.p.d. of the 1st kind of the s.e. \( \mathcal{X} \) and the \( |\mathcal{X}| \)-Kopula. First, if the e.p.d. of the 1st kind \( p = \{p(X/\mathcal{X}) : X \subseteq \mathcal{X}\} \) of some s.e. \( \mathcal{X} \) with the \( \mathcal{X} \)-set of marginal probabilities \( \hat{p} = \{p_x : x \in \mathcal{X}\} \) is defined, then from properties of probabilities of the 1st kind it follows that for \( x \in \mathcal{X} \)
\[
p_x = \sum_{x \subseteq \mathcal{X}} p(X/\mathcal{X}), \tag{2.21}
\]
i.e., the function \( \mathcal{K} \), defined by the e.p.d. of the 1st kind \( p \) and formulas (2.3), satisfies \( x \)-marginal equalities for \( x \in \mathcal{X} \):
\[
p_x = \sum_{x \subseteq \mathcal{X}} \mathcal{K}(\hat{p}(c(X/\mathcal{X})) \tag{2.22}
\]
(required for being a 1-function:: \( \mathcal{K} \in \Psi_{\mathcal{X}} \)) and serves as the \( |\mathcal{X}| \)-Kopula.

Second, if the function \( \mathcal{K} \) is the \( |\mathcal{X}| \)-Kopula, then by Lemma 1: \( \mathcal{K} \in \Psi_{\mathcal{X}} \subseteq \Psi_{\mathcal{X}} \), i.e., it is normalized and, consequently, by (2.20) the function \( p \) is normalized too:
\[
1 = \sum_{x \subseteq \mathcal{X}} p(X/\mathcal{X}). \tag{2.23}
\]
In addition, from (2.20) and from the fact that the \( |\mathcal{X}| \)-Kopula is a 1-function, (2.21) follows for all \( x \in \mathcal{X} \). Therefore, the function \( p \) is a e.p.d. of the 1st kind of the s.e. \( \mathcal{X} \) with the \( \mathcal{X} \)-set of marginal probabilities \( \hat{p} \). The theorem is proved.

2.2 Convex combination of Kopulas

Lemma 3 (convex combination of Kopulas). A convex combination of an arbitrary set of Kopulas of one and the same s.e. is its Kopula too.

Proof without tricks. Let \( \mathcal{X} \) be a s.e., and
\[
\mathcal{K}_1, \ldots, \mathcal{K}_n \tag{2.24}
\]
be some set of its Kopulas. Let us prove that their convex combination
\[
\mathcal{K} = \sum_{i=1}^n \alpha_i \mathcal{K}_i \tag{2.25}
\]
(where, of course, \( \alpha_1 + \ldots + \alpha_n = 1 \), \( \alpha_i \geq 0 \), \( i = 1, \ldots, n \)) is also a Kopula. For this it suffices to prove that \( \mathcal{K} \) is a 1-function. In other words, that for \( x \in \mathcal{X} \)
\[
\sum_{x \subseteq \mathcal{X}} \mathcal{K}(\hat{p}(c(X/\mathcal{X})) = p_x. \tag{2.26}
\]
Since each Kopula from the set (2.24) has a property of a 1-function, then for \( x \in \mathcal{X} \) we get what is required:
\[
\sum_{x \subseteq \mathcal{X}} \mathcal{K}(\hat{p}(c(X/\mathcal{X})) =
= \sum_{i=1}^n \alpha_i \sum_{x \subseteq \mathcal{X}} \mathcal{K}_i(\hat{p}(c(X/\mathcal{X}) = \tag{2.27}
= \sum_{i=1}^n \alpha_i p_x = p_x.
\]

Corollary 1 (convex combination of Kopulas). For every set of events \( \mathcal{X} \) the space of 1-functions \( \Psi_{\mathcal{X}} \), as well as the space of its Kopulas, is a convex manifold.

2.3 Kopula of free variables: A computational aspect

Without set-phenomenon transformations and variable transformations, analytic work on sets of half-rare events (s.h.r.e.’s) (see [6]) and sets of their marginal probabilities, is unlikely to be effective. However, in specific calculations at first, because of their unaccustomedness, these compulsory wisdoms can cause misunderstandings, leading to errors. Therefore, it is useful, in order to avoid unnecessary stumbling during calculations, to introduce separate notation for half-rare marginal probabilities events from s.h.r.e.’s. \( \mathcal{X} \) and its set-phenomena, that is, probabilities that are not greater than half, in order to distinguish them from free marginal probabilities, to the values of which there are no restrictions.

So, we will talk about half-rare variables (h-r variables) and free variables, assigning special notation to them \(^5\):

\[
\hat{p} = \{p_x, x \in \mathcal{X}\} \in [0, 1/2]^{\otimes \mathcal{X}} \quad \mathcal{X}\text{-set of half-rare variables.}
\]
\[
\hat{p}^\prime(\mathcal{X}/\mathcal{X}) \in [0, 1/2]^{\otimes \mathcal{X}} \otimes (1/2, 1]^{\otimes \mathcal{X}-\mathcal{X}} \quad \mathcal{X}\text{-renumbering} \hat{p},
\]
\[
\hat{w} = \{w_x, x \in \mathcal{X}\} \in [0, 1]^{\otimes \mathcal{X}} \quad \mathcal{X}\text{-set of free variables,}
\]
\[
\hat{w}^\prime(\mathcal{X}/\mathcal{X}) \in [0, 1]^{\otimes \mathcal{X}} \quad \mathcal{X}\text{-renumbering} \hat{w},
\]

\(^5\)Just remember [6], that the formula of \( \mathcal{X} \)-renumbering any \( \mathcal{X} \)-set of probabilities of events has the form for \( X \subseteq \mathcal{X} \):
\[
\hat{p}(\mathcal{X}/\mathcal{X}) = \{p_x, x \in \mathcal{X}\} + \{1-p_x, x \in \mathcal{X}-X\}. \]
and always interpreting them as probabilities of events. In particular, for the half-rare doublet \( \mathcal{X} = \{x, y\} \) we have:

\[ \bar{p} = \{p_x, p_y\} \in [0, 1/2]^x \otimes [0, 1/2]^y \]

- \( \mathcal{X} \)-set of half-rare variables,

\[ p_{xy}(p_x, p_y) \in [0, \min\{p_x, p_y\}] \]

- half-rare function of half-rare variables,

\[ \mathcal{W} = \{w_x, w_y\} \in [0, 1]^x \otimes [0, 1]^y \]

- \( \mathcal{X} \)-set of free variables,

\[ \mathcal{W}_{xy}(w_x, w_y) \in [0, \min\{w_x, w_y\}] \]

- free function of free variables.

Since for \( X \subseteq \mathcal{X} = \{x\} \)

\[ \mathcal{K}\left(\mathcal{W}(\mathcal{X}/\{x\})\right) = \begin{cases} \mathcal{K}(1-w_x), & X = \emptyset, \\ \mathcal{K}(w_x), & X = \{x\}, \end{cases} \]

then a marginal and global normalization of the function \( \mathcal{K} \) are written as:

\[ \mathcal{K}(w_x) = w_x, \quad \mathcal{K}(1-w_x) = 1, \]  

and the global normalization obviously follows from the marginal one, which agrees with Lemma 2; and from the marginal normalization it follows that the 1-copula \( \mathcal{K} \) of an arbitrary monoplet of events \( \mathcal{X} = \{x\} \) is defined for free variables \( \bar{w} = \{w_x\} \in [0, 1]^x \) by a one formula:

\[ \mathcal{K}(\bar{w}) = \mathcal{K}(w_x) = w_x, \]

which provides two values on each \( 2^{|\mathcal{X}|} \)-penomenon-dom by “free” formulas:

\[ \mathcal{K}\left(\mathcal{W}(\mathcal{X}/\{x\})\right) = \begin{cases} \mathcal{K}(1-w_x) = 1 - w_x, & X = \emptyset, \\ \mathcal{K}(w_x) = w_x, & X = \{x\}. \end{cases} \]

and the e.p.d. of the 1st kind of this monoplet with \( \{x\} \)-monoplet of probabilities of events \( \bar{p} = \{p_x\} \in [0, 1]^x \) are defined for half-rare variables \( \bar{p} = \{p_x\} \in [0, 1]^x \) and \( \{x\} \)-monoplet of marginal probabilities \( \{p_x\} \), where

\[ p_x = P(x) = p(x/\{x\}). \]

In other words, let’s construct a 1-function on the unit \( \mathcal{X} \)-segment, i.e., a such nonnegative bounded numerical function

\[ \mathcal{K} : [0, 1] \rightarrow [0, 1], \]

that for \( x \in \mathcal{X} \)

\[ \sum_{x \in \mathcal{X} \subseteq \mathcal{X}} \mathcal{K}\left(\mathcal{W}(\mathcal{X}/\{x\})\right) = w_x. \]

Note 3 (phenomenon replacement between half-rare and free variables). For every \( X \subseteq \mathcal{X} \) phenomenon replacement of half-rare variables \( \bar{p} = \{p_x\} \in [0, 1]^x \otimes [0, 1]^y \) by free variables \( \bar{w} = \{w_x, w_y\} \) and vise-versa is defined for \( X \subseteq \mathcal{X} \) by mutually inverse formulas of the set-phenomenon transformation of the form:

\[ \bar{p} = \bar{p}(\mathcal{X}/X) = \begin{cases} \bar{w}(\emptyset/X), & w_x > 1/2, x \in \mathcal{X}, \\ \ldots, \\ \bar{w}(\mathcal{X}/x), & w_x \leq 1/2, x \in \mathcal{X}, \\ \bar{w}(\mathcal{X}/x), & w_x > 1/2, x \in \mathcal{X} - X, \\ \ldots, \\ \bar{w}(\mathcal{X}/x), & w_x \leq 1/2, x \in \mathcal{X} \end{cases} \]

where for \( x \in \mathcal{X} \) the following agreement is always accepted (see, for example, paragraph 11.7):

\[ p_x = \begin{cases} w_x, & w_x \leq 1/2, \\ 1 - w_x, & w_x > 1/2. \end{cases} \]

2.4 Kopula for a monoplet of events

Let’s construct the 1-Kopula \( \mathcal{K} \in \Psi_1^x \) of a family of e.p.d.’s of monoplet of events \( \mathcal{X} = \{x\} \) with the e.p.d. of the 1st kind

\[ \left\{p(X/\{x\}), X \subseteq \{x\}\right\} = \left\{p(\emptyset/\{x\}), p(x/\{x\})\right\} \]

and \( \{x\}\)-monoplet of marginal probabilities \( \{p_x\} \), where

\[ p_x = P(x) = p(x/\{x\}). \]

In other words, let’s construct a 1-function on the unit \( \mathcal{X} \)-segment, i.e., a such nonnegative bounded numerical function

\[ \mathcal{K} : [0, 1] \rightarrow [0, 1], \]

that for \( x \in \mathcal{X} \)

\[ \sum_{x \in \mathcal{X} \subseteq \mathcal{X}} \mathcal{K}\left(\mathcal{W}(\mathcal{X}/\{x\})\right) = w_x. \]

2.5 Kopulas for a doublet of events

Let’s construct an example of 2-Kopulas \( \mathcal{K} \in \Psi_1^x \) of families of a doublet of events \( \mathcal{X} = \{x, y\} \), in other words, let’s construct on the unit \( \{x, y\}\)-square the such nonnegative bounded numerical functions

\[ \mathcal{K} : [0, 1]^{\otimes \{x, y\}} \rightarrow [0, 1], \]

that for all \( z \in \{x, y\} \)

\[ \sum_{z \subseteq \{x, y\}} \mathcal{K}\left(\mathcal{W}(\mathcal{Z}/\{x, y\})\right) = w_z. \]

Since each 2-set of arguments \( \bar{w} \in [0, 1]^x \otimes [0, 1]^y \) generates 2\( (x,y) \)-phenomenon-dom

\[ 2^{|\mathcal{X}|} = \{\bar{w}, \mathcal{W}(\mathcal{X}/\{x\})\}, \mathcal{W}(\mathcal{X}/\{y\}), \mathcal{W}(\emptyset)\}, \]

composed from forth its set-phenomena

\[ \bar{w} = \bar{w}(\mathcal{X}/\{x\}), \mathcal{W}(\mathcal{X}/\{y\}), \mathcal{W}(\emptyset), \]

\[ = \begin{cases} w_x, w_y, & \{x, y\}, \\ w_x, 1 - w_y, & \{x\}, \\ 1 - w_x, w_y, & \{y\}, \\ 1 - w_x, 1 - w_y. \end{cases} \]
then
\[
\mathcal{K}(\tilde{\omega}) = \mathcal{K}(w_x, w_y),
\]
\[
\mathcal{K}\left(\tilde{\omega}(\{x\} \setminus \mathcal{X}(x,y))\right) = \mathcal{K}(w_x, 1 - w_y),
\]
\[
\mathcal{K}\left(\tilde{\omega}(\{y\} \setminus \mathcal{X}(x,y))\right) = \mathcal{K}(1 - w_x, w_y)
\]
and normalizations for every \(\tilde{\omega} \in [0,1]^{\otimes\{x,y\}}\) are written as:
\[
\mathcal{K}(w_x, w_y) + \mathcal{K}(w_x, 1 - w_y) = w_x,
\]
\[
\mathcal{K}(w_x, w_y) + \mathcal{K}(1 - w_x, w_y) = w_y,
\]
\[
\mathcal{K}(w_x, w_y) + \mathcal{K}(1 - w_x, 1 - w_y) + \mathcal{K}(1 - w_x, w_y) + \mathcal{K}(1 - w_x, 1 - w_y) = 1.
\]

The e.p.d. of the 1st kind of doublet of events is defined by the 2-Kopula for \(X \subseteq \{x, y\}\) in half-rare variables by general formulas:
\[
p(X / \{x, y\}) = \mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) =
\]
\[
\begin{cases}
\mathcal{K}(1 - p_x, 1 - p_y), & X = \emptyset, \\
\mathcal{K}(p_x, 1 - p_y), & X = \{x\}, \\
\mathcal{K}(1 - p_x, p_y), & X = \{y\}, \\
\mathcal{K}(p_x, p_y), & X = \{x, y\},
\end{cases}
\]
(2.38)
where \(p_{xy}(\tilde{\omega})\) is functional parameter that has a sense of probability of double intersection.

This e.p.d. of the 1st kind of doublet of events in the free functional parameters and variables (after replacement (2.31)) has the form:
\[
p(X / \{x, y\}) = \mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) =
\]
\[
\begin{cases}
w_x + w_y - 1 + w_{xy}(1 - w_x, 1 - w_y), & w_x > 1/2, w_y > 1/2 \iff X = \emptyset, \\
w_x - w_{xy}(w_x, 1 - w_y), & w_x \leq 1/2, w_y > 1/2 \iff X = \{x\}, \\
w_y - w_{xy}(1 - w_x, w_y), & w_x > 1/2, w_y \leq 1/2 \iff X = \{y\}, \\
w_{xy}(w_x, w_y), & w_x \leq 1/2, w_y \leq 1/2 \iff X = \{x, y\}.
\end{cases}
\]
(2.39)

2.6 Kopula for a doublet of independent events

The simplest example of a 1-function on a \(\{x, y\}\)-square is the so-called independent 2-Kopula, which for free variables \(\tilde{\omega} \in [0,1]^{\otimes\{x,y\}}\) is defined by the formula:
\[
\mathcal{K}(\tilde{\omega}) = w_xw_y.
\]
(2.40)

This provides it on each \(2^{c(\tilde{\omega})}\)-phenomenon the following values:
\[
\mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) = w_xw_y,
\]
\[
\mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) = w_x(1 - w_y),
\]
\[
\mathcal{K}\left(\tilde{\omega}(\mathcal{X}(y))\right) = (1 - w_x)w_y,
\]
\[
\mathcal{K}\left(\tilde{\omega}(\emptyset)\right) = (1 - w_x)(1 - w_y).
\]
(2.41)

Indeed, the so-defined independent 2-Kopula is a 1-function because
\[
\sum_{x \in X \subseteq \{x\}} \mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) = w_xw_y + w_x(1 - w_y) = w_x,
\]
\[
\sum_{y \in X \subseteq \{y\}} \mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) = w_xw_y + (1 - w_x)w_y = w_y.
\]

The e.p.d. of the 1st kind of doublet of independent events with the \(\{x, y\}\)-set of probabilities of events \(\tilde{\omega}\) is defined by four values of the independent 2-Kopula (2.40) on its \(2^{c(\tilde{\omega})}\)-penomemon-dom by general formulas in half-rare variables (see Fig. 2), i.e., for \(X \subseteq \{x, y\}\):
\[
p(X / \{x, y\}) = \mathcal{K}\left(\tilde{\omega}(\mathcal{X}(x,y))\right) =
\]
\[
\begin{cases}
(1 - p_x)(1 - p_y), & X = \emptyset, \\
p_x(1 - p_y), & X = \{x\}, \\
(1 - p_x)p_y, & X = \{y\}, \\
p_xp_y, & X = \{x, y\}.
\end{cases}
\]
(2.42)

2.7 2-Kopula of free variables:
A computational aspect

With the phenomenal substitution (2.30) half-rare 2-Kopula as a function of the free variables takes the equivalent form:
\[
\mathcal{K}(\tilde{\omega}) =
\]
\[
\begin{cases}
w_{xy}(\tilde{\omega}(\mathcal{X}(x))) , & \tilde{\omega} \in [0,1/2]^x \otimes [0,1/2]^y, \\
w_x - w_{xy}(\tilde{\omega}(\mathcal{X}(x))) , & \tilde{\omega} \in [0,1/2]^x \otimes (1/2,1]^y, \\
w_y - w_{xy}(\tilde{\omega}(\mathcal{X}(y))) , & \tilde{\omega} \in [1/2,1]^x \otimes [0,1/2]^y, \\
w_x + w_y - 1 + w_{xy}(\tilde{\omega}(\emptyset)) , & \tilde{\omega} \in [1/2,1]^x \otimes (1/2,1]^y.
\end{cases}
\]
(2.43)

We rewrite this formula a pair of times, in order to understand the properties of the phenomenon substitution of variables and not get confused in the
Figure 2: Graphs of Cartesian representation of the 2-Kopula of a family of e.p.d.’s of an independent half-rare double of events \( \{x, y\} \); probabilities of the 1st kind are marked by different colors: \( p(x,y) \) (aqua), \( p(x) \) (lime), \( p(y) \) (yellow) \( p(\emptyset) \) (fuchsia).

Calculations:

\[
K(\tilde{w}) = \begin{cases} 
\wxy(w_x, w_y), & \tilde{w} \in [0, 1/2]^x \otimes [0, 1/2]^y, \\
wx - \wxy(w_x, 1 - w_y), & \tilde{w} \in [0, 1/2]^x \otimes (1/2, 1]^y, \\
w_y - \wxy(1 - w_x, w_y), & \tilde{w} \in [1/2, 1]^x \otimes [0, 1/2]^y, \\
w_x + w_y - 1 + \wxy(1 - w_x, 1 - w_y), & \tilde{w} \in (1/2, 1]^x \otimes (1/2, 1]^y; 
\end{cases}
\]

where

\[
0 \leq \wxy(w_x, w_y) \leq \min\{w_x, w_y\}, \\
0 \leq \wxy(w_x, 1 - w_y) \leq \min\{w_x, 1 - w_y\}, \\
0 \leq \wxy(1 - w_x, w_y) \leq \min\{1 - w_x, w_y\}, \\
0 \leq \wxy(1 - w_x, 1 - w_y) \leq \min\{1 - w_x, 1 - w_y\}
\]

(2.46)

are the Fréchet inequalities for \( \wxy \) as the half-rare probability of double intersection of half-rare events: either half-rare events \( x \) or \( y \), or their half-rare complements, when events \( x \) or \( y \) are not half-rare.

Note that the conditional formulas (2.43), (2.44) and (2.45) cannot be rewritten as four unconditional formulas, because these conditions are in the right, and not in the left. This is explained exclusively by the properties of the phenomenon replacement of half-rare variables by free ones (2.30), which, for this reason, leads to formulas that are convenient for calculations.

Note 4. Half-rare 2-Kopula of free variables of an independent double of events. With the functional parameter \( \wxy(w_x, w_y) = w_x w_y \), which corresponds to the probability of double intersection of independent events \( x \) and \( y \) happening with probabilities \( w_x \) and \( w_y \), and means, of course, that the all the following four equations are satisfied:

\[
w_{xy}(w_x, w_y) = w_x w_y, \\
w_{xy}(w_x, 1 - w_y) = w_x (1 - w_y), \\
w_{xy}(1 - w_x, w_y) = (1 - w_x) w_y, \\
w_{xy}(1 - w_x, 1 - w_y) = (1 - w_x)(1 - w_y);
\]

from (2.45) it follows that a half-rare 2-Kopula from free variables of the family of e.p.d.’s of the independent double of events \( X = \{x, y\} \) with \( X \)-sets of free marginal probabilities \( \tilde{w} = \{w_x, w_y\} \in [0, 1]^{\otimes X} \) has the same view on all \( 2^{(\leq \tilde{w})} \)-phenomenon-dom.

\[
K(\tilde{w}) = w_x w_y. \tag{2.47}
\]

2.8 Upper 2-Kopula of Fréchet

An example of a 1-function on a \( \{x, y\} \)-square is the so-called upper 2-Kopula of Fréchet, which suggests the probabilities of a double intersection to be its upper Fréchet boundary. In other words, the only functional free parameter in (2.39) is:

\[
w_{xy}(\tilde{w}) = w_{xy}^+(\tilde{w}) = \min\{w_x, w_y\}. \tag{2.48}
\]
The upper 2-Kopula of Fréchet from free variables 
\( \tilde{w} \in [0, 1]^2 \) is defined by the formulas:
\[
p(X//\{x, y\}) = \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = \\
\begin{cases} 
    w_x + w_y - 1 + \min\{1 - w_x, 1 - w_y\}, \\
    w_x > 1/2, w_y > 1/2 \iff X = \emptyset, \\
    w_x - \min\{w_x, 1 - w_y\}, \\
    w_x \leq 1/2, w_y > 1/2 \iff X = \{x\}, \\
    w_y - \min\{1 - w_x, w_y\}, \\
    w_x > 1/2, w_y \leq 1/2 \iff X = \{y\}, \\
    \min\{w_x, w_y\}, \\
    w_x \leq 1/2, w_y \leq 1/2 \iff X = \{x, y\}.
\end{cases}
\]
(2.49)

After simple transformations, these formulas provide the upper 2-Kopula of Fréchet on each \( 2(c|w) \)-phenomenon-dom the following four values of free variables:
\[
p(X//\{x, y\}) = \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = \\
\begin{cases} 
    \min\{w_x, w_y\}, \\
    w_x > 1/2, w_y > 1/2 \iff X = \emptyset, \\
    \max\{0, w_x + w_y - 1\}, \\
    w_x \leq 1/2, w_y > 1/2 \iff X = \{x\}, \\
    \max\{0, w_x + w_y - 1\}, \\
    w_x > 1/2, w_y \leq 1/2 \iff X = \{y\}, \\
    \min\{w_x, w_y\}, \\
    w_x \leq 1/2, w_y \leq 1/2 \iff X = \{x, y\}, \\
\end{cases}
\]
(2.50)

which, as can be seen, are defined by the upper and lower Fréchet boundaries of the probability of double intersection, depending on the combination of the values of the free variables.

The same four values from the half-rare variables have the form:
\[
p(X//\{x, y\}) = \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = \\
\begin{cases} 
    \min\{1 - p_x, 1 - p_y\}, \\
    X = \emptyset, \\
    \max\{0, p_x - p_y\}, \\
    X = \{x\}, \\
    \max\{0, p_y - p_x\}, \\
    X = \{y\}, \\
    \min\{p_x, p_y\}, \\
    X = \{x, y\}.
\end{cases}
\]
(2.51)

If \( p_x \geq p_y \), this formula takes the form:
\[
p(X//\{x, y\}) = \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = \\
\begin{cases} 
    1 - p_x, \\
    X = \emptyset, \\
    p_x - p_y, \\
    X = \{x\}, \\
    0, \\
    X = \{y\}, \\
    p_y, \\
    X = \{x, y\}.
\end{cases}
\]
(2.52)

And if \( p_x < p_y \), then this formula takes the form:
\[
p(X//\{x, y\}) = \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = \\
\begin{cases} 
    1 - p_y, \\
    X = \emptyset, \\
    p_y - p_x, \\
    X = \{y\}, \\
    p_y, \\
    X = \{x, y\}.
\end{cases}
\]
(2.53)

So the definite upper 2-Kopula of Fréchet is indeed a 1-function, due to the fact that when \( w_x \geq w_y \)
\[
\sum_{x \in X \subseteq \{x, y\}} \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = w_y + (w_x - w_y) = w_x,
\]
(2.54)

and when \( w_x < w_y \)
\[
\sum_{y \in X \subseteq \{x, y\}} \mathcal{K} (\tilde{w}(c|X//\{x, y\})) = w_y + 0 = w_y,
\]
(2.55)

2.9 Lower 2-Kopula of Fréchet

A once more example of a 1-function on a \( \{x, y\} \)-square is the so-called lower 2-Kopula of Fréchet, which suggests the probabilities of a double intersection to be its upper Fréchet boundary. In other words, the only functional free parameter in (2.39) is:
\[
w_{xy}(\tilde{w}) = w_{xy}^-(\tilde{w}) = \max\{0, w_x + w_y - 1\}.
\]
(2.56)
After simple transformations, these formulas provide the lower 2-Kopula of Fréchet on each $2^{(c|\tilde{w})}$-phenomenon-dom the following four values of free variables:

$$p(X\parallel\{x,y\}) = \mathcal{K}\left(\tilde{w}(c|X\parallel\{x,y\})\right) =$$

$$= \begin{cases} 
\max\{0, w_x + w_y - 1\}, & X = \emptyset, \\
\min\{w_x, w_y\}, & X = \{x\}, \\
\max\{0, w_x + w_y - 1\}, & X = \{y\}, \\
\min\{w_x, w_y\}, & X = \{x, y\}.
\end{cases}$$

(2.58)

which, as can be seen, are also defined by the upper and lower Fréchet boundaries of the probability of double intersection only in other combinations of the values of the free variables.

The same four values from the half-rare variables have the more simple form:

$$p(X\parallel\{x,y\}) = \mathcal{K}\left(\tilde{p}(c|X\parallel\{x,y\})\right) =$$

$$= \begin{cases} 
\max\{0, 1 - p_x - p_y\}, & X = \emptyset, \\
\min\{p_x, 1 - p_y\}, & X = \{x\}, \\
\max\{0, p_x + p_y - 1\}, & X = \{y\}, \\
\min\{1 - p_x - p_y\}, & X = \{x, y\}.
\end{cases}$$

(2.59)

So the definite lower 2-Kopula of Fréchet is indeed a 1-function, due to the fact that for all half-rare variables

$$\sum_{x \in X \subseteq \{x,y\}} \mathcal{K}\left(\tilde{p}(c|X\parallel\{x,y\})\right) = p_x + 0 = p_x,$$

$$\sum_{y \in X \subseteq \{x,y\}} \mathcal{K}\left(\tilde{w}(c|X\parallel\{x,y\})\right) = p_y + 0 = p_y.$$ 

(2.60)
are the lower and upper Fréchet-boundaries of probability of double intersection.

For \( \alpha = -1 \), the probability \( w_{xy}(\hat{w}) = w_{xy}^-(\hat{w}) \) coincides with the lower Fréchet boundary of half-rare marginal probabilities; for \( \alpha = 1 \), the probability \( w_{xy}(\hat{w}) = w_{xy}^+(\hat{w}) \) coincides with the upper Fréchet boundary of marginal probabilities. Unfortunately, these are the properties of a convex combination such that for \( \alpha = 0 \) the probability of double intersection is equal to half of the sum of its lower and upper Fréchet boundaries: \( w_{xy}(\hat{w}) = (w_{xy}^-(\hat{w}) + w_{xy}^+(\hat{w}))/2 \) (see Figure 6), and not an independent 2-Kopula, no matter how much we want it. This “blunder” of the convex combination can easily be corrected by conjugation of two convex combinations, as done below.

In Fig. 5 it is a graph of this 2-Kopula for a deliberately intricate weight function with values from \([-1, 1]\):

\[
\alpha = \alpha(\hat{w}) = \sin(15(w_x - w_y)). \tag{2.63}
\]

2.10 Convex combinations of the lower, upper and independent 2-Kopulas of Fréchet

A rather general example of a 1-function on a \((x, y)\)-square is the convex combinations of the upper, lower, and independent 2-Kopulas of wise Fréchet, which propose the probabilities of a pair intersection to become a convex combination (this is allowed by the lemma 3) of its upper and lower Fréchet boundaries, as well as the probability of double intersection of independent events with some functional weighting parameter.

2.10.1 Convex combination of the lower and upper 2-Kopulas of Fréchet

A convex combination of the lower and upper 2-Kopula of Fréchet can be ensured by the unique functional free parameter \( w_{xy}(\hat{w}) \) in (2.39), in which the probability of double intersection is computed by the following formula:

\[
w_{xy}(\hat{w}) = \frac{(1 - \alpha)}{2}w_{xy}^-(\hat{w}) + \frac{(1 + \alpha)}{2}w_{xy}^+(\hat{w}) \tag{2.61}
\]

where \( \alpha = \alpha(\hat{w}) \in [-1, 1] \) is an arbitrary function on \([0, 1] \otimes [x,y] \) with values from \([-1, 1] \), and

\[
w_{xy}^-(\hat{w}) = \max\{0, w_x + w_y - 1\},
\]

\[
w_{xy}^+(\hat{w}) = \min\{\hat{w}\} \tag{2.62}
\]

We construct two convex combinations of the the independent 2-copula and the lower and upper 2-Kopulas of Fréchet. The conjugation of these two
convex combinations can be ensured by the unique functional free parameter \( w_{xy}(\bar{w}) \) in (2.39) by the following conjugation formula for two convex combinations:

\[
w_{xy}(\bar{w}) = \begin{cases} 
  \max\{\bar{w}\} (1 + \alpha), & \alpha \leq 0; \\
  \max\{\bar{w}\} (1 - \alpha) + \alpha, & \alpha > 0,
\end{cases}
\]

(2.64)

where \( \alpha = \alpha(\bar{w}) \in [-1, 1] \) is an arbitrary function on \([0, 1]^{x \times y}\) with values from \([-1, 1]\), and

\[
w_{xy}^+(\bar{w}) = \min\{\bar{w}\}
\]

(2.65)
is the upper Fréchet-boundary of probability of double intersection. For \( \alpha = 0 \), the probability \( w_{xy}(\bar{w}) = \min\{\bar{w}\} \max\{\bar{w}\} = w_x w_y \) coincides with the probability of double intersection of independent events; for \( \alpha = -1 \), the probability \( w_{xy}(\bar{w}) = 0 \) coincides with the lower Fréchet-boundary of half-rare marginal probabilities; for \( \alpha = 1 \), the probability \( w_{xy}(\bar{w}) = w_x^+ (\bar{w}) \) coincides with the upper Fréchet-boundary of marginal probabilities.

In Fig. 5 it is a graph of this 2-Kopula for the same weight function with values from \([-1, 1]\) as in the previous example.

\[
\alpha = \alpha(\bar{w}) = \sin(15(w_x - w_y)).
\]

(2.66)

### 3 The frame method of construction of Kopula

#### 3.1 Inserted sets of events and conditional e.p.d.'s

**Definition 6** (inserted s.e.'s). For each pairs of s.e.'s \( \mathcal{X} \) and \( \mathcal{Y} \) with the joint e.p.d.

\[
\{p(X + Y/\mathcal{X} + \mathcal{Y}) \}, X \subseteq \mathcal{X}, Y \subseteq \mathcal{Y}
\]

(3.1)

for every \( Y \subseteq \mathcal{Y} \) the \( Y \)-inserted s.e.'s, **generated by \( \mathcal{X} \), in the frame s.e. \( \mathcal{Y} \) are s.e.'s, which are denoted by \( \mathcal{X}'(\cap Y/\mathcal{Y}) \), and defined as the following M-intersection\(^7\):

\[
\mathcal{X}'(\cap Y/\mathcal{Y}) = \bigcap_{\mathcal{X}'(\cap Y/\mathcal{Y})} \{\text{ter}(Y/\mathcal{Y})\} = \{x \cap \text{ter}(Y/\mathcal{Y}) : x \in \mathcal{X}\}
\]

(3.2)

and have the e.p.d., which coincides with the projection of the joint e.p.d. (3.1) for fixed \( Y \subseteq \mathcal{Y} \) and every \( X \subseteq \mathcal{X} \):

\[
p(X(\cap)\{\text{ter}(Y/\mathcal{Y})\}) = p(X + Y/\mathcal{X} + \mathcal{Y})
\]

(3.3)

\(^7\)M-intersection is an intersection by Minkowski.

**Definition 7** (event-probabilistic pseudo-distribution of an inserted s.e.). For each \( Y \subseteq \mathcal{Y} \) the \( Y \)-inserted s.e.

\[
\mathcal{X}'(\cap Y/\mathcal{Y}) = \bigcap_{\mathcal{X}'(\cap Y/\mathcal{Y})} \{\text{ter}(Y/\mathcal{Y})\}
\]

(3.4)

with the e.p.d. (3.3) has the event-probabilistic \( Y \)-pseudo-distribution, which is defined as a set of probabilities of terraced events that coincide with probabilities from the e.p.d. (3.3) for all \( X \subseteq \mathcal{X} \) excepting \( X = \emptyset \):

\[
p(Y)(X + Y/\mathcal{X} + \mathcal{Y}) =
\]

\[
= \begin{cases} 
  p(X + Y/\mathcal{X} + \mathcal{Y}), & X \neq \emptyset, \\
  p(Y/\mathcal{X} + \mathcal{Y}) - 1 + p(Y/\mathcal{Y}), & X = \emptyset,
\end{cases}
\]

(3.5)

The sum of all probabilities from every \( Y \)-pseudo-distribution (3.5) is \( p(Y/\mathcal{Y}) = P(\text{ter}(Y/\mathcal{Y})) \), the probability of a terraced event, generated by the frame s.e. \( \mathcal{Y} \), in which the given s.e. \( \mathcal{X}'(\cap Y/\mathcal{Y}) \) is inserted.

Thus, the only difference of e.p.d.'s of \( Y \)-inserted s.e.'s from their event-probabilistic \( Y \)-pseudo-distributions, lies in the fact that the sums of the probabilities of the terraced events, from which the \( Y \)-pseudo-distributions are composed, are normalized not by unity, but by the probabilities of the corresponding frame terraces events \( p(Y/\mathcal{Y}) \). And the sum of the normalizing constants by \( Y \subseteq \mathcal{Y} \) is obviously equal to one.

**Note 5** (symmetry of inserted and frame s.e.'s). In Definition 6 the s.e. \( \mathcal{X} \) and \( \mathcal{Y} \) can always be swapped, i.e., to take the s.e. \( \mathcal{X} \) on a role of the frame one, and the s.e. \( \mathcal{Y} \) to take on a role of s.e., that generates \( X \)-inserted s.e.'s for every \( X \subseteq \mathcal{X} \):

\[
\mathcal{Y}'(\cap X/\mathcal{X}) = \bigcap_{\mathcal{Y}'(\cap X/\mathcal{X})} \{\text{ter}(X/\mathcal{X})\} = \{y \cap \text{ter}(X/\mathcal{X}) : y \in \mathcal{Y}\}
\]

(3.6)

**Note 6** (M-sum of the all inserted s.e.'s). The M-sum\(^8\) of the all \( Y \)-inserted s.e.'s \( \mathcal{X}'(\cap Y/\mathcal{Y}) \) for \( Y \subseteq \mathcal{Y} \) forms the given s.e. \( \mathcal{X} \):

\[
\mathcal{X} = \left( \bigcup_{Y \subseteq \mathcal{Y}} \mathcal{X}'(\cap Y/\mathcal{Y}) \right) = \mathcal{X}'(\cap \emptyset/\mathcal{Y}) + \ldots + \mathcal{X}'(\cap \mathcal{Y}/\mathcal{Y}).
\]

(3.7)

\(^8\)M-sum is a sum by Minkowski.

**Note 7** (characterization of \( Y \)-inserted s.e.'s by conditional e.p.d.'s of the 1st kind). The e.p.d. of \( Y \)-inserted s.e. \( \mathcal{X}'(\cap Y/\mathcal{Y}) \) with every \( Y \subseteq \mathcal{Y} \) has a
form for \( X \subseteq \mathcal{X} \):
\[
p\left( X^{(\cap)} \{ \text{ter}(Y/Y) \} / \mathcal{X}^{(\cap)}/Y \right) =
\begin{cases}
  p(X + Y/X + Y), & \emptyset \neq X \subseteq \mathcal{X}, \\
  p(Y/X + Y) + 1 - p(Y/Y), & X = \emptyset.
\end{cases}
\tag{3.8}
\]

where for every \( Y \subseteq \mathcal{Y} \)
\[
p(X/X | Y/Y) = \frac{p(X + Y/X + Y)}{p(Y/Y)},
\tag{3.9}
\]
i.e., the probabilities of the 1st kind, forming for \( X \subseteq \mathcal{X} \) the \( \mathcal{Y} \)-conditional e.p.d. of the 1st kind of the s.e. \( \mathcal{X} \) with respect to terraced event \( \text{ter}(Y/Y) \) generated by the s.e. \( \mathcal{Y} \).

In other words, \( \mathcal{Y} \)-inserted s.e. \( \mathcal{X}^{(\cap)}/Y \) for \( Y \subseteq \mathcal{Y} \) are characterized by formulas (3.8) and \( \mathcal{Y} \)-conditional e.p.d.'s of the 1st kind of the s.e. \( \mathcal{X} \) with respect to the terraced event \( \text{ter}(Y/Y) \), generated by the s.e. \( \mathcal{Y} \).

Note 8 (mutual characterization of conditional e.p.d.'s of the 1st kind and pseudo-distributions of inserted s.e.'s). The connection between each \( \mathcal{Y} \)-pseudo-distribution of the \( \mathcal{Y} \)-inserted s.e. with the corresponding \( \mathcal{Y} \)-conditional e.p.d. looks simpler. It is sufficient for each fixed \( Y \subseteq \mathcal{Y} \) to normalize all its probabilities of "inserted" terraced events by the probability of a terraced event \( p(Y/Y) \) in order to obtain corresponding to the \( \mathcal{Y} \)-conditional probabilities regarding the fact that the corresponding frame terraces event \( \text{ter}(Y/Y) \) happened. As a result, we have the following obvious inversion formulas:
\[
p(X/X | Y/Y) = \frac{p(Y)(X + Y/X + Y)}{p(Y/Y)},
\tag{3.10}
p(Y)(X + Y/X + Y) = p(X/X | Y/Y)p(Y/Y).
\]

Note 9 (about the appropriateness of the concept of inserted s.e.'s). It would seem, why develop a theory of inserted s.e.'s, pseudo-distributions of which are simply characterized by conditional e.p.d.'s. Is not it better to instead practice the theory of conditional e.p.d., especially since this theory has long had excellent recommendations in many areas. However, in eventology, as the theory of events, which prefers to work directly with sets of events, there is one rather serious objection. The fact is that conditional e.p.d., as any e.p.d. in eventology, there must be a set of some events, in this case, a set of well-defined "conditional events". But until now it has not been possible to give a satisfactory definition of the "conditional event", except for my impractical definition in [7]. So, the inserted s.e.'s are a completely satisfactory "surrogate" definition of the sets of "conditional events". Such that e.p.d.'s of inserted s.e.'s, although they do not coincide with the desired conditional e.p.d.'s, but are associated with them by well-defined mutual-inverse transformations. As a result, inserted s.e.'s play the role of a convenient eventological tool for working with conditional e.p.d.'s of a one set of events regarding terrace events generated by another set of events.

Example 1 (two inserted s.e.'s in a frame monoplet). Let in formulas (3.1) the s.e. \( \mathcal{X} \) is an arbitrary set, and the s.e. \( \mathcal{Y} = \{ y \} \) is a frame monoplet of events, which have the joint e.p.d. in a form:
\[
\{p(X + Y/\mathcal{X} + \{ y \}), X \subseteq \mathcal{X}, Y \subseteq \{ y \}\}.
\tag{3.11}
\]

Then there is the \( \{ y \} \)-inserted s.e. and the \( \emptyset \)-inserted s.e.:
\[
\mathcal{X}^{(\cap)}/\{ y \} = \{ x \cap y, x \in \mathcal{X}\},
\mathcal{X}^{(\cap)}/\emptyset = \{ x \cap y, x \in \mathcal{X}\}.
\tag{3.12}
\]

These inserted s.e.'s are characterized for every of two subsets of the monoplet \( Y = \{ y \} \subseteq \{ y \} \) and \( \bar{Y} = \emptyset \subseteq \{ y \} \) by formulas (3.3) and by two corresponding e.p.d.'s
\[
\{p(X/\mathcal{X} + \{ y \}), X \subseteq \mathcal{X}\},
\{p(X + \{ y \}/\mathcal{X} + \{ y \}), X \subseteq \mathcal{X}\}.
\tag{3.13}
\]

which by formulas (3.5) define two \( \mathcal{Y} \)-pseudo-distributions for \( X \subseteq \mathcal{X} \):
\[
p^{(\{ y \})}(X + \{ y \}/\mathcal{X} + \{ y \}) =
\begin{cases}
  p(X + \{ y \}/\mathcal{X} + \{ y \}), & X \neq \emptyset, \\
  p(\{ y \}/\mathcal{X} + \{ y \}) - 1 + p(\{ y \}/\mathcal{X} + \{ y \}), & X = \emptyset.
\end{cases}
\tag{3.14}
\]

\[
p(\emptyset)(X/X + \{ y \}) =
\begin{cases}
  p(X/X + \{ y \}), & X \neq \emptyset, \\
  p(\emptyset/X + \{ y \}) - 1 + p(\emptyset/X + \{ y \}), & X = \emptyset.
\end{cases}
\tag{3.15}
\]

where \( p_{y} = P(y) \) is a probability of the frame event \( y \in \{ y \} \).

First of all, note that the sum of the probabilities of terraced events from the \( \{ y \} \)-pseudo-distribution (3.14) is \( p_{y} \), and the probabilities of the \( \emptyset \)-pseudo-distribution (3.15) is \( 1 - p_{y} \); and secondly, that these two pseudo-distributions define a joint e.p.d. of the s.e. \( \mathcal{X} \) and the monoplet \( \{ y \} \), i.e., e.p.d. of the s.e. \( \mathcal{X} + \{ y \} \), which is related to them by fairly obvious formulas for \( Z \subseteq \mathcal{X} + \{ y \} \):
\[
p(Z/\mathcal{X} + \{ y \}) =
\begin{cases}
  p^{(\{ y \})}(Z/\mathcal{X} + \{ y \}), y \in Z, \\
  p(\emptyset)(Z/\mathcal{X} + \{ y \}), & y \not\in Z.
\end{cases}
\tag{3.16}
\]
The formulas (3.16) are recurrent, connecting the e.p.d. of s.e. \( X + \{ y \} \) with two pseudo-distributions of the inserted s.e. \( X' = X'(\cap(y/Y)) \) and \( X'' = X''(\cap(y/Y)) \) whose power is less by one. The inversion formulas (3.10) allow recurrence formulas (3.16) to express the e.p.d. of \( X + \{ y \} \) via the conditional e.p.d. with respect to one of its events \( y \in X + \{ y \} \) and its complements \( y^C = \Omega - y \):

\[
p(Y/X + \{ y \}) =
\begin{cases}
p(X/X + \{ y \} | \{ y \})p_y, & Z = X + \{ y \}, \\
p(X/X | \Omega \{ y \})(1 - p_y), & Z = X, \\
X \subseteq X',
\end{cases}
\]

(3.17)

where \( Z \subseteq X + \{ y \} \) Note that these formulas, like (3.16), can be used recursively to express the e.p.d. of \( X + \{ y \} \) through two conditional e.p.d.'s of the s.e. \( X' \) whose power is less by one.

### 3.2 Inserted and conditional Kopulas of a family of sets of events with respect to the set of events

**Definition 8 (inserted Kopulas).** The \( N \)-Kopulas of \( Y \)-inserted \( N \)-s.e.'s

\[
\mathcal{X}(\cap(Y/Y)) = X(\cap(Y/Y)) = \{ x \cap \text{ter}(Y/Y), x \in X \},
\]

(3.18)

which for each \( Y \subseteq Y \) are defined (see Definition 6) as intersections by Minkowski's of the s.e. \( X' \) with terraced events \( \text{ter}(Y/Y) \), generated by the s.e. \( Y \), are called the \( Y \)-inserted \( N \)-Kopulas with respect to the s.e. \( Y \). Such \( Y \)-inserted \( N \)-Kopulas characterizes e.p.d.'s of the 1st kind of \( Y \)-inserted \( N \)-s.e.'s by formulas for \( X \subseteq X ' 

\[
p(X(\cap(Y/Y))) = \mathcal{X}(X(Y))\left(\tilde{p}(c(X(Y)) \cap X(Y))\right),
\]

(3.19)

where

\[
\tilde{p}(c(X(Y)) \cap X(Y)) = \tilde{p}(Y) = \begin{cases} p_x(Y), x \in X \end{cases} = \begin{cases} P(x \cap \text{ter}(Y/Y)), x \in X \end{cases}
\]

(3.20)

is the set of probabilities of “inserted” marginal events from the \( X''(Y/Y) \), and

\[
\tilde{p}(c(X(Y)) \cap X(Y)) = \begin{cases} p_x(Y), x \in X \end{cases} + \begin{cases} p(Y/X) - p_x(Y), x \in X - X \end{cases}
\]

(3.21)

are \( X' \)-phenomena of the set of “inserted” marginal probabilities \( p(Y) \).

We also need to define an inserted \( Y \)-pseudo-Kopula with respect to the s.e. \( Y \), which characterizes the \( Y \)-pseudo-distribution of the \( Y \)-inserted s.e. \( X'(Y/Y) \), inserted into the terraces event \( \text{ter}(Y/Y) \) generated by the s.e. \( Y \). Although the \( Y \)-pseudo-Kopula is not a Kopula, i.e., is not a 1-function, it has properties very reminiscent of the Kopula properties.

**Definition 9 (inserted pseudo-Kopulas).** The \( Y \)-pseudo-Kopula of the \( Y \)-pseudo-distribution of \( Y \)-inserted s.e.

\[
\mathcal{X}(X(Y)) = X'(X(Y)) = X(\cap(Y/Y)) \text{ter}(Y/Y))
\]

(3.22)

with respect to the s.e. \( Y \) is a such function \( \mathcal{X}(Y) \) on \( X' \)-hypercubewith sides \( \{0, p(Y/Y)\} \) that 1) is non-negative:

\[
\mathcal{X}(Y) (\tilde{w}(c(X(Y)) \cap X(Y))) \leq 0
\]

(3.23)

for \( \tilde{w}(c(X(Y)) \cap X(Y)) \in [0, p(Y/Y)] \otimes X \subseteq X; \

2) satisfies the \( Y \)-marginal equalities for \( x \in X \):

\[
\sum_{x \in X} \mathcal{X}(Y) (\tilde{w}(c(X(Y)) \cap X(Y))) = \tilde{w}(c(X(Y)) \cap X(Y)), x \in X
\]

(3.24)

where

\[
\tilde{w}(c(X(Y)) \cap X(Y)) = \begin{cases} \tilde{w}(c(X(Y)) \cap X(Y)), x \in X \end{cases}
\]

(3.25)

is a \( X' \)-phomena of the \( X \)-set of marginal probabilities of the \( Y \)-pseudo-distribution of \( Y \)-inserted s.e. \( X'(Y/Y) \), i.e.,

\[
\tilde{w}(c(X(Y)) \cap X(Y)) = \begin{cases} \tilde{w}(c(X(Y)) \cap X(Y)), x \in X \end{cases}
\]

(3.26)

From (3.24) and (3.26) it follows the probabilistic \( Y \)-normalization of pseudo-Kopula:

\[
\sum_{X \subseteq X} \mathcal{X}(Y) (\tilde{w}(c(X(Y)) \cap X(Y))) = p(Y/Y).
\]

(3.27)

And from (3.23) and (3.27) it follows the terraced-by-terrace probabilistic \( Y \)-normalization of pseudo-Kopula:

\[
0 \leq \mathcal{X}(Y) (\tilde{w}(c(X(Y)) \cap X(Y))) \leq p(Y/Y)
\]

(3.28)

for \( X \subseteq X \).

Such \( Y \)-pseudo-Kopulas characterize the \( Y \)-pseudo-distribution (3.5) of \( Y \)-inserted s.e.'s \( X'(Y/Y) \) by formulas for \( X \subseteq X'

\[
p(Y/Y)(X + Y/X + Y) = \mathcal{X}(Y) (\tilde{w}(c(X(Y)) \cap X(Y))
\]

(3.29)
where
\[
\bar{p}^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) = \bar{p}^Y(Y) = \\
= \left\{ p_x^Y, x \in X \right\} = \left\{ \mathbf{P}(x \cap \text{ter}(Y/Y)), x \in X \right\} \tag{3.30}
\]
is a set of \(Y\)-marginal probabilities, coinciding with the set of marginal probabilities of \(Y\)-inserted s.e.'s \(X^{(\cap Y)}\), and
\[
\bar{p}^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) = \left\{ \frac{p_x(Y), x \in X}{p(Y/Y)} \right\} + \\
+ \left\{ p(Y/X) - p_x(Y), x \in X - X \right\} \tag{3.31}
\]
are \(X\)-phenomena of the set \(Y\)-marginal probabilities \(\bar{p}^Y\).

**Definition 10 (conditional Kopulas).** The \(N\)-Kopulas, characterizing \(Y\)-inserted e.p.d.'s of the 1st kind of the \(N\)-s.e. \(X\) with respect to the terraced event \(\text{ter}(Y/Y)\), generated by the s.e. \(Y\), i.e., e.p.d.'s of the 1st kind, defined by joint e.p.d. \(X\) and \(Y\) by formulas with fixed \(Y \subseteq Y\) for \(X \subseteq X\):
\[
p(X/X | Y/Y) = \frac{p(X + Y/X + Y)}{p(Y/Y)},
\]
are called the \(Y\)-conditional \(N\)-Kopulas of the \(N\)-s.e. \(X\) with respect to the terraced event \(\text{ter}(Y/Y)\), generated by the s.e. \(Y\).

Such \(Y\)-conditional \(N\)-Kopulas characterize the \(Y\)-conditional e.p.d. of the 1st kind (3.32) by formulas for \(X \subseteq X\):
\[
p(X/X | Y/Y) = \mathbf{K}^{(Y)} \left( \bar{p}^Y(X^{(\cap Y)} | Y/Y) \right),
\]
where
\[
\bar{p}^Y(X^{(\cap Y)} | Y/Y) = \bar{p}^Y(Y) = \left\{ \frac{p_x(Y), x \in X}{p(Y/Y)} \right\} = \\
= \left\{ \mathbf{P}(x \cap \text{ter}(Y/Y))/p(Y/Y), x \in X \right\} \tag{3.34}
\]
is a set of conditional marginal probabilities of events \(x \in X\) with respect to the terraced event \(\text{ter}(Y/Y)\), and
\[
\bar{p}^Y(X/X | Y/Y) = \\
= \left\{ \frac{p_x(Y), x \in X}{p(Y/Y)} \right\} + \left\{ 1 - \frac{p_x(Y), x \in X - X}{} \right\} \tag{3.35}
\]
are \(X\)-phenomenon of the set of conditional marginal probabilities \(\bar{p}^Y\).

**Note 10 (connection between conditional and ‘’inserted’’ marginal probabilities).** Conditional marginal probabilities are connected with “inserted” marginal probabilities for \(x \in X\) by the formula of conditional probability:
\[
p_x^Y = \frac{1}{p(Y/Y)} p_x(Y), \tag{3.36}
\]
since “inserted” marginal probabilities (3.20) are probabilities of intersections of events \(x \in X\) with the terraced event \(\text{ter}(Y/Y)\). The connection between the corresponding set of conditional “inserted” marginal probabilities we shall write in the similar way:
\[
\bar{p}_x^Y = \frac{1}{p(Y/Y)} p_x(Y), \bar{p}_x^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) = \\
= \frac{1}{p(Y/Y)} \bar{p}_x^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) \tag{3.37}
\]

**Note 11 (connection between conditional Kopulas and inserted pseudo-Kopulas).** From Definition 10 of conditional Kopula and Definition 9 of inserted pseudo-Kopula with respect to the s.e. \(Y\), and also from the formula (3.37) it follows the simple inversion formulas that connect conditional Kopulas and inserted Pseudo-Kopulas of the family of sets of events \(X\) for \(X \subseteq X\):
\[
\mathbf{K}^{Y} \left( \bar{p}^Y(X^{(\cap Y)} | Y/Y) \right) = \\
= \frac{1}{p(Y/Y)} \mathbf{K}^{(Y)} \left( p(Y/Y) \bar{p}^Y(X^{(\cap Y)} | Y/Y) \right),
\]
\[
\mathbf{K}^{Y} \left( \bar{p}^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) \right) = \\
= \frac{1}{p(Y/Y)} \mathbf{K}^{(Y)} \left( \bar{p}^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) \right).
\]

**Note 12 (two formulas of full probability for a Kopula).** The Kopula \(\mathbf{K}\) of s.e. \(X\) is expressed through \(Y\)-conditional Kopulas \(\mathbf{K}^Y\) for \(Y \subseteq Y\) by the usual formula of full probability:
\[
\mathbf{K} \left( \bar{p}^Y(X/X) \right) = \sum_{Y \subseteq Y} \mathbf{K}^Y \left( \bar{p}^Y(X^{(\cap Y)} | Y/Y) \right) p(Y/Y). \tag{3.39}
\]

From (3.39) and (3.38) we obtain an analogue of the formula of total probability — the representation of the Kopula of s.e. \(X\) in the form of sum of \(Y\)-pseudo-Kopulas by \(Y \subseteq Y\):
\[
\mathbf{K} \left( \bar{p}^Y(X/X) \right) = \sum_{Y \subseteq Y} \mathbf{K}^Y \left( \bar{p}^Y(X^{(\cap Y)} \not\in X^{(\cap Y)}) \right). \tag{3.40}
\]

**Note 13 (Kopula of a sum of sets).** A Kopula of sum \(X + Y\) of two s.e.'s \(X\) and \(Y\) characterizes their joint e.p.d. of the 1st kind and by definition has the form
\[
p(X + Y/X + Y) = \mathbf{K} \left( \bar{p}^Y(X^{(\cap Y)} + Y + Y) \right), \tag{3.41}
\]
where
\[
\bar{p}^Y(X^{(\cap Y)} + Y + Y) = \{p_x, x \in X\} + \{p_y, y \in Y\} + \\
+ \{1 - p_x, x \in X - X\} + \{1 - p_y, y \in Y - Y\} \tag{3.42}
\]
is the \((X + Y)\)-phenomenon of the set of marginal probabilities
\[
\bar{p}^Y(X^{(\cap Y)} + Y + Y) = \{p_x, x \in X\} + \{p_y, y \in Y\}. \tag{3.43}
\]
for the sum $\mathcal{X} + \mathcal{Y}$.

From previous formulas (3.32), (3.33), and (3.29) for a inserted pseudo-Kopula and conditional Kopula we obtain formulas

$$\mathcal{K}(\tilde{p}(\cdot|X+Y/X+Y)) = \mathcal{K}(Y)(\tilde{p}(\cdot|X^{(N-Y)/X}|X^{(N-Y)/Y})), \quad (3.44)$$

$$\mathcal{K}(\tilde{p}(\cdot|X+Y/X+Y)) = \mathcal{K}(Y)(\tilde{p}(\cdot|X^{(N-Y)/X}|X^{(N-Y)/Y})),$$  

that for each $Y \subseteq \mathcal{Y}$ connect the Kopula of sum $\mathcal{X} + \mathcal{Y}$ with the product of $Y$-conditional Kopula $\mathcal{X}$ with respect to $Y$ and the value of Kopula $\mathcal{Y}$ at $Y$-phenomenon; and also with the $Y$-inserted pseudo-Kopula of $\mathcal{X}$ which is inserted in the terraced eventer $(Y/Y)$, generated by $Y$.

### 3.3 Theory of the frame method for constructing Kopula

The basis of the frame method of constructing Kopula is a rather simple idea of composing an arbitrary $N$-s.e. $\mathcal{X}$ using the recurrence frame formula:

$$\mathcal{X} = \{x_0, x_1, ..., x_{N-1}\} = \{x_0\} + \mathcal{X}, \quad (3.46)$$

where $(N-1)$-s.e.’s

$$\mathcal{X} = \mathcal{X} - \{x_0\} = \{x_1, ..., x_{N-1}\} = (\mathcal{X}' (+) \mathcal{X}'') \quad (3.47)$$

are composed from two $(N-1)$-s.e.’s $\mathcal{X}'$ and $\mathcal{X}''$ by set-theoretic operation of $M$-union\(^8\) and defined as the inserted s.e.’s in the frame monoplet $\{x_0\}$ by the following formulas:

$$\mathcal{X}' = \mathcal{X}'(\cap \{x_0\})$$  
$$\mathcal{X}' = \{x_0\} \cap \{x_1, ..., x_{N-1}\}, \quad (3.48)$$

$$\mathcal{X}'' = \mathcal{X}''(\cup \{x_0\}) = \{x_0\} \cup \{x_1, ..., x_{N-1}\}.$$ 

This simple idea allows us to find the recurrent frame formulas for the $N$-Kopula of s.e. $\mathcal{X}$ as functions of the set of marginal probabilities $\tilde{p} = \{p_0, p_1, \ldots, p_{N-1}\}$.

The frame method relies on formulas (3.16) and (3.17) and also correspondingly on (3.44) and (3.45), and constructs two recurrent formulas:

$$\mathcal{K}_X(\tilde{p}) = \text{Recursion}_1(\mathcal{K}_{X'}(\tilde{p}), \mathcal{K}_{X''}(\tilde{p}), \mathcal{K}_{X'}^{(\{x_0\})}(\tilde{p}), \mathcal{K}_{X''}^{(\{x_0\})}(\tilde{p}), \mathcal{K}_{X'}^{(\{x_0\})}(\tilde{p}), \mathcal{K}_{X''}^{(\{x_0\})}(\tilde{p}), \mathcal{K}_{X'}^{(\emptyset)}(\tilde{p}), \mathcal{K}_{X''}^{(\emptyset)}(\tilde{p}), p_0), \quad (3.49)$$

$$\mathcal{K}_X(\tilde{p}) = \text{Recursion}_2(\mathcal{K}_{X'}(\tilde{p}), \mathcal{K}_{X''}(\tilde{p}), \mathcal{K}_{X'}^{(\{x_0\})}(\tilde{p}), \mathcal{K}_{X''}^{(\{x_0\})}(\tilde{p}), \mathcal{K}_{X'}^{(\emptyset)}(\tilde{p}), \mathcal{K}_{X''}^{(\emptyset)}(\tilde{p}), p_0), \quad (3.50)$$

\(^8\)Intersection and union are an intersection and union odd sets by Minkowski (see details in [11]).
half-rare, i.e., its marginal probabilities from \( \bar{p} = \{p_0, p_1, \ldots, p_{N-1}\} \) are not more than half, for example:

\[
1/2 \geq p_0 \geq p_1 \geq \cdots \geq p_{N-1},
\]

then the both inserted s.e.'s \( \mathcal{X}' = \{x'_1, \ldots, x'_{N-1}\} \) and \( \mathcal{X}^* = \{x^*_1, \ldots, x^*_{N-1}\} \), and together with them and the s.e. \( \mathcal{X} \) are also half-rare by its Definition (3.48). In other words, their marginal probabilities from \( \bar{p}' = \{p'_1, \ldots, p'_{N-1}\} \) and \( \bar{p}^* = \{p'^*_1, \ldots, p'^*_{N-1}\} \) do not exceed the corresponding marginal probabilities events from the frame s.e. \( \mathcal{X} \):

\[
p_1 \geq p'_1, \ldots, p_{N-2} \geq p'_{N-1},
\]

and marginal probabilities from \( \bar{p}_{N-1} = \{p_1, \ldots, p_{N-1}\} \) are half-rare by definition. Thus, any half-rare N-s.e. is composed by the frame method with the formula (3.47) from two inserted \( (N-1) \)-s.e.'s \( \mathcal{X}' \) and \( \mathcal{X}^* \), which are required to be half-rare.

Lemma 4 (about independent half-rare s.e.'s, constructed by the frame method from two inserted half-rare s.e.'s). That in the family of half-rare s.e. \( \mathcal{X} \) with sets of marginal probabilities \( \bar{p} \), constructed by the frame method from two inserted half-rare s.e.'s \( \mathcal{X}' \) and \( \mathcal{X}^* \), there was an independent half-rare s.e., it is necessary so that the sets of marginal probabilities are related to the marginal probabilities of the frame s.e. \( \mathcal{X} = \{x_0\} + \mathcal{X} = \{x_0\} + (\mathcal{X}' + \mathcal{X}^*) \) by the following way:

\[
\begin{align*}
\bar{p}' &= \{p'_1, \ldots, p'_{N-1}\} \\
&= \{p_1 p_0, \ldots, p_{N-1} p_0\}, \\
\bar{p}^* &= \{p'^*_1, \ldots, p'^*_{N-1}\} \\
&= \{p_1(1-p_0), \ldots, p_{N-1}(1-p_0)\};
\end{align*}
\]

and sufficient so that the e.p.d. of the 1st kind of inserted s.e.'s \( \mathcal{X}' \) and \( \mathcal{X}^* \) to be calculated from the formulas for \( X \subseteq \mathcal{X} \):

\[
\begin{align*}
p(X'//\mathcal{X}') &= \mathbf{P} \left( \bigcap_{x' \in \mathcal{X}'} x' \cap \bigcap_{x \in \mathcal{X} \setminus \mathcal{X}'} x \right) \\
&= \left\{ \begin{array}{ll}
p_0 \prod_{x \in \mathcal{X}} \prod_{x' \in \mathcal{X} \setminus \mathcal{X}'} (1 - p_{x'}), & X \neq \emptyset, \\
p_0 \prod_{x \in \mathcal{X}} (1 - p_x) + 1 - p_0, & X = \emptyset,
\end{array} \right. \\
p(X^*//\mathcal{X}^*) &= \mathbf{P} \left( \bigcap_{x^* \in \mathcal{X}^*} x^* \cap \bigcap_{x \in \mathcal{X} \setminus \mathcal{X}^*} x \right) \\
&= \left\{ \begin{array}{ll}
(1 - p_0) \prod_{x \in \mathcal{X}} \prod_{x^* \in \mathcal{X} \setminus \mathcal{X}^*} (1 - p_{x'}), & X \neq \emptyset, \\
(1 - p_0) \prod_{x \in \mathcal{X}} (1 - p_x) + p_0, & X = \emptyset,
\end{array} \right.
\end{align*}
\]

where \( \mathcal{X}' = \{x', x \in X\} = \{x_0 \cap x, x \in X\} \subseteq \mathcal{X}' \), \( \mathcal{X}^* = \{x^*, x \in X\} = \{x_0^* \cap x, x \in X\} \subseteq \mathcal{X}^* \).

Proof. The necessity is obvious, since the inserted marginal probabilities of the independent s.e. \( \mathcal{X} \) are probabilities of double intersections of independent events which have the required form for \( n = 1, \ldots, N-1 \):

\[
\begin{align*}
p'(X) &= \mathbf{P}(x_0 \cap x_n) = p_n p_0, \\
p^*(X) &= \mathbf{P}(x_0^* \cap x_n) = p_n (1 - p_0).
\end{align*}
\]

The efficiency follows from (3.57) and formulas that connect the e.p.d. of the 1st kind of frame s.e. \( \mathcal{X} \) with the e.p.d. of the 1st kind of inserted s.e.'s \( \mathcal{X}' \) and \( \mathcal{X}^* \), which have the form for \( X \subseteq \mathcal{X} \):

\[
\begin{align*}
p(X + Y//\mathcal{X} + \{x_0\}) &= \\
&= \left\{ \begin{array}{ll}
p(X'//\mathcal{X}'), & Y = \{x_0\}, X \neq \emptyset, \\
p(\emptyset//\mathcal{X}') - 1 + p_0, & Y = \{x_0\}, X = \emptyset,
\end{array} \right. \\
p(X^*//\mathcal{X}^*), & Y = \emptyset, X \neq \emptyset, \\
p(\emptyset//\mathcal{X}^*) - p_0, & Y = \emptyset, X = \emptyset.
\end{align*}
\]

Demanding (3.60) to perform sufficient conditions (3.57), we get

\[
\begin{align*}
p(X + Y//\mathcal{X} + \{x_0\}) &= \\
&= p(X + Y//\mathcal{X} + \{x_0\}) = \\
&= p(Z//\mathcal{X}) = \prod_{x \in Z} \prod_{x' \in \mathcal{X} \setminus Z} (1 - p_x),
\end{align*}
\]

as a result, for the s.e. \( \mathcal{X} \) we have the e.p.d. of the 1st kind of independent events:

\[
\begin{align*}
p(X + Y//\mathcal{X} + \{x_0\}) &= \\
&= p(Z//\mathcal{X}) = \prod_{x \in Z} \prod_{x' \in \mathcal{X} \setminus Z} (1 - p_x),
\end{align*}
\]

The lemma is proved.

4 The Kopula theory for monoplets of events

Theory of the Kopula of monoplets of events (1-Kopula) seemed to be completed by the formula (2.35). This formula defines the 1-Kopula of an arbitrary monoplet of events \( \{x\} \) with \( \{x\}\)-monoplet of marginal probabilities \( \bar{p} = \{p_x\} \in [0, 1]^x \) in the unique form:

\[
\mathbf{K}(\bar{p}) = \mathbf{K}(p_x) = p_x,
\]

10By the way, the necessary condition also follows from (3.57).
which which provides 2 values on each \(2^{|\mathcal{P}|}\)-phenomenon-don by general formulas for \(X \subseteq \{x\} \):

\[
\mathcal{K} \left( \hat{p}(X/x) \right) =
\begin{cases} 
\mathcal{K}(1 - p_x) = 1 - p_x, & X = \emptyset, \\
\mathcal{K}(p_x) = p_x, & X = \{x\}.
\end{cases}
\] (4.2)

However, the formula (4.2) can be generalized in the following simple way:

\[
\mathcal{K} \left( \hat{p}(X/x) \right) =
\begin{cases} 
\mathcal{K}^0 (1 - p_x) = 1 - \mathcal{K}^x (p_x), & X = \emptyset, \\
\mathcal{K}^x (p_x), & X = \{x\},
\end{cases}
\] (4.3)

where \(\mathcal{K}^x\) is any function such that in half-rare variables:

\[
\mathcal{K}^x : [0, 1/2] \to [0, 1/2],
\] (4.4)

and in free variables:

\[
\mathcal{K}^x : [0, 1] \to [0, 1].
\] (4.5)

In this case, the 1-Kopula (4.2) is an important special case of 1-Kopula (4.3) when \(\mathcal{K}^x (p_x) = p_x\). This case corresponds to a uniform marginal distribution function on the unit interval in the theory of the classical copula [2].

5 The Kopula theory for doublets of events

5.1 The frame method for constructing a half-rare doublet of events

In order to construct by the frame method the \(\hat{p}\)-ordered frame half-rare doublet of events

\[
\mathcal{X} = \{x, y\} = \{x\} + \mathcal{X} = \{x\} + (\mathcal{X}' + \mathcal{X}^*)
\] (5.1)

with the \(\mathcal{X}\)-set of marginal probabilities \(\hat{p} = \{p_x, p_y\}\), where

\[
1/2 \geq p_x \geq p_y \geq 0,
\] (5.2)

let's suppose that we have at our disposal two half-rare inserted monoplets of events

\[
\mathcal{X}' = \{x \cap y\} = \{s'\}, \quad \mathcal{X}^* = \{x^c \cap y\} = \{s^*\},
\] (5.3)

with known 1-Kopulas:

\[
\mathcal{K}^* \left( \hat{p}(S/x) \right) =
\begin{cases} 
\mathcal{K}^* (1 - p_{s^*}) = 1 - p_{s^*}, & S = \emptyset, \\
\mathcal{K}^* (p_{s^*}) = p_{s^*}, & S = \{s'\},
\end{cases}
\] (5.4)

By Definition of inserted monoplets (5.3) (see Fig. 7)

\[
s' = x \cap y \subseteq x,
\]

\[
s^* = x^c \cap y \subseteq x^c,
\] (5.6)

and also because of the \(\hat{p}\)-ordering assumption (5.2), we get that

\[
p_{s'} + p_{s^*} = p_y \leq p_x \leq 1/2 \leq 1 - p_x.
\] (5.7)

Consequently, the 1-Kopulas of inserted monoplets (5.4) and (5.5) are bound by the sum of their marginal probabilities:

Consequently, the 1-Kopulas of inserted monoplets (5.4) and (5.5) are bound by the sum of their marginal probabilities:

\[
p_{s'} \in [0, p_y],
\] (5.9)

and depend on only one parameter:

\[
s_{s'} \subseteq x, \quad s^* \subseteq x^c,
\] (5.10)

Figure 7: Venn diagrams of the frame half-rare doublet of events \(\mathcal{X} = \{x, y\}\), 1/2 \(\geq p_x \geq p_y\) (up), and two inserted monoplets \(\mathcal{X}' = \{s'\}\) (down) agreed with the frame doublet \(\mathcal{X}\) in the following sense \(y = s' + s^*\) \(\subseteq x, \quad s^* \subseteq x^c\).
We get the following formulas:
\[ p(xy \mid \{x, y\}) = p(s' \mid \mathcal{X}') = p_{s'}, \]
\[ p(x \mid \{x, y\}) = p(0 \mid \mathcal{X}') - 1 + p_x = p_x - p_{s'}, \]
\[ p(y \mid \{x, y\}) = p(s'' \mid \mathcal{X}'') = p_y - p_{s''}, \]
\[ p(0 \mid \{x, y\}) = p(0 \mid \mathcal{X}') - p_x = 1 - p_y - p_x + p_{s'}. \]

These formulas express the e.p.d. of the 1st kind of the \(\tilde{p}\)-ordered half-rare frame doublet of events \(\mathcal{X} = \{x, y\}\) through the e.p.d. of the 1st kind of inserted monoplets \(\mathcal{X}'\) and \(\mathcal{X}''\), and the probability of frame event \(x\), and, in the final result, through their marginal probabilities \(p_x\) and \(p_y\), and marginal probability \(p_{s'}\) of the inserted monoplet \(\mathcal{X}' = \{s'\} = \{x \cap y\}\).

The formulas (5.10) express values of the 2-Kopula of \(\tilde{p}\)-ordered doublet \(\mathcal{X} = \{x, y\}\) through 1-Kopulas of inserted monoplets \(\mathcal{X}' = \{s'\}\) and \(\mathcal{X}'' = \{s''\}\). Rewrite this in a form of an explicit recurrent formula:
\[ p(X \mid \{x, y\}) = \mathcal{K}_X \left( \frac{\tilde{p}(c \mid X \cap \{x, y\})}{\tilde{p}(c \mid \{x, y\})} \right) = \begin{cases} \mathcal{K}_X(p_{s'}), & X = \{x, y\}, \\ \mathcal{K}_X(1 - p_{s'}) - 1 + p_x, & X = \{x\}, \\ \mathcal{K}_X(p_{s''}), & X = \{y\}, \\ \mathcal{K}_X(1 - p_{s''}) - p_x, & X = \emptyset. \end{cases} \] (5.11)

Considering (2.34) and (5.9), we will continue:
\[ p(X \mid \{x, y\}) = \mathcal{K}_X \left( \frac{\tilde{p}(c \mid X \cap \{x, y\})}{\tilde{p}(c \mid \{x, y\})} \right) = \begin{cases} p_{s'}, & X = \{x, y\}, \\ p_x - p_{s'}, & X = \{x\}, \\ p_{s''}, & X = \{y\}, \\ 1 - p_x - p_{s''}, & X = \emptyset. \end{cases} \] (5.12)

\[ = \begin{cases} p_{s'}, & X = \{x, y\}, \\ p_x - p_{s'}, & X = \{x\}, \\ p_y - p_{s''}, & X = \{y\}, \\ 1 - p_x - p_y + p_{s''}, & X = \emptyset. \end{cases} \] (5.13)

We note, by the way, that the restriction (5.9) by the assumption of \(\tilde{p}\)-ordered (5.2) is a special case of Fréchet-inequalities:
\[ 0 \leq p_{s'} \leq p_{x_{zy}}^+ = \min\{p_x, p_y\} = p_y. \] (5.14)

**Note 17** (Frame method for otherwise \(\tilde{p}\)-ordered half-rare doublet of events). For otherwise \(\tilde{p}\)-ordered half-rare doublet of events
\[ \mathcal{X} = \{y, x\} = \{y\} + \mathcal{X} = \{y\} + (\mathcal{X}'(+)\mathcal{X}''), \] (5.15)
where
\[ \mathcal{X}' = \{y \cap x\} = \{s'\}, \mathcal{X}'' = \{y \cap x\} = \{s''\}, \] (5.16)

The values of the margin of simplices \(s'\) and \(s''\) are connected with cause of the assumption of \(\tilde{p}\)-ordering, we get that
\[ p_{s'}^+ + p_{s''} = p_x \leq p_y \leq 1/2 \leq 1 - p_y. \] (5.19)

Consequently, 1-copulas of inserted monoplanes are connected by a restriction on the sum of their marginal probabilities:
\[ p_{s'}^+ + p_{s''} = p_x, \] (5.20)
and depend on only one parameter:
\[ p_{s'} \in [0, p_x). \] (5.21)

By the assumptions, the following formulas are
valid:
\[ p(xy \parallel \{y,x\}) = p(s' \parallel X') = p_4, \]
\[ p(y \parallel \{y,x\}) = p(0 \parallel X') - 1 + p_y = p_y - p_4, \]
\[ p(x \parallel \{y,x\}) = p(s'' \parallel X') = p_x - p_4, \]
\[ p(0 \parallel \{y,x\}) = p(0 \parallel X') - p_y = 1 - p_x - p_y - p_4. \] (5.22)

These formulas express the e.p.d. of the 1st kind otherwise \( p \)-ordered half-rare frame doublet of events \( \mathcal{X} \) through the e.p.d. of the 1st kind of inserted monoplets \( \mathcal{X}' \) and \( \mathcal{X}'' \), and the probability of frame event \( y \), and, in the final result, through own marginal probabilities \( p_x \), \( p_y \), and the marginal probabilities of inserted monoplet \( \mathcal{X}' = \{ s' \} = \{ x \cap y \} \) (see Fig. 8).

The formulas (5.22) express values of the 2-Kopula of \( p \)-ordered doublet \( \mathcal{X} = \{ y, x \} \) through 1-Kopulas of inserted monoplets \( \mathcal{X}' = \{ s' \} \) and \( \mathcal{X}'' = \{ s'' \} \). Rewrite this in the form of explicit recurrent formula:
\[ p(X \parallel \{y,x\}) = \mathcal{K}_X \left( \tilde{p}^c(X \parallel \{y,x\}) \right) = \begin{cases} \mathcal{K}_X(p_4), & X = \{y, x\}, \\ \mathcal{K}_X(1 - p_4) - 1 + p_y, & X = \{y\}, \\ \mathcal{K}_X(p_4), & X = \{x\}, \\ \mathcal{K}_X(1 - p_4) - p_y, & X = \emptyset. \end{cases} \] (5.23)

Continue:
\[ p(X \parallel \{x\}, y) = \mathcal{K}_X \left( \tilde{p}^c(X \parallel \{x\}, y) \right) = \begin{cases} p_4, & X = \{y \cap x\}, \\ p_y - p_4, & X = \{y\}, \\ p_4, & X = \{x\}, \\ 1 - p_x - p_y, & X = \emptyset. \end{cases} \] (5.24)

We note, as above, that the restriction (5.21) by the assumption of another \( p \)-ordering is a special case of Fréchet inequalities:
\[ 0 \leq p_4 \leq p_4^+ = \min\{p_x, p_y\} = p_x. \] (5.26)

### 5.2 The frame method: recurrent formulas for a half-rare doublet of events

The formulas (5.11), as well as formulas (5.13), can be rewrite in the form of special cases of recurrent formulas (3.52) and (3.53) from Note 15 for the doublet \( \mathcal{X} = \{x, y\} = \{x\} + \{y\} = \{x\} + \mathcal{X}'\)
\[ \mathcal{K} \left( \tilde{p}^c(X \parallel Y \parallel X + \{x\}) \right) = \begin{cases} \mathcal{K}(x) \left( \tilde{p}^c(X \parallel \{x\} \parallel \{x\}) \parallel X' \right), & Y = \{x\}, \\ \mathcal{K}(0) \left( \tilde{p}^c(X \parallel \{y\} \parallel X') \parallel X' \right), & Y = \emptyset. \end{cases} \] (5.27)

In the formulas (5.27) the pseudo-Kopulas \( \mathcal{K}(x) \) and \( \mathcal{K}(0) \) of inserted monoplets \( X' \) and \( X'' \), correspondingly, are defined by the first and the second pairs of probabilities from (5.13) correspondingly, i.e., by formulas:
\[ \mathcal{K}(x) \left( \tilde{p}^c(X \parallel \{x\} \parallel \{x\}) \parallel X' \right) = \begin{cases} p_4, & X = \{x, y\}, \\ p_y - p_4, & X = \{y\}, \\ 1 - p_x - p_y + p_4, & X = \emptyset, \end{cases} \] (5.28)

where, for example, for \( X = \{y\} \)
\[ \{y\} \parallel \{x\} \parallel \{x\} = \{y \cap x\} \subseteq X' = \{s'\}, \]
\[ \{y\} \parallel \{y\} \parallel \{x\} = \{y \cap x\} \subseteq X'' = \{y - s'\}, \] (5.31)

and the corresponding sets of marginal probabilities of inserted monoplets \( \mathcal{X}' \) and \( \mathcal{X}'' \) have the form
\[ \tilde{p}^c(X \parallel \{x\} \parallel \{x\}) = \{p_4, p_4\}, \]
\[ \tilde{p}^c(X \parallel \{y\} \parallel \{x\}) = \{p_y - p_4\}. \] (5.32)

In the formulas (5.28) the conditional Kopulas \( \mathcal{K}(x) \) and \( \mathcal{K}(0) \) are defined by the first and the second pairs of probabilities from (5.13), normalized by \( p_x \) and by \( 1 - p_x \) correspondingly, i.e., by the formulas:
\[ \mathcal{K}(x) \left( \tilde{p}^c(X \parallel \{x\} \parallel \{x\}) \parallel \{x\} \right) = \begin{cases} \frac{1}{p_x} p_4, & X = \{x, y\}, \\ \frac{1}{p_x} (1 - p_x - p_y + p_4), & X = \emptyset, \end{cases} \] (5.33)

\[ \mathcal{K}(0) \left( \tilde{p}^c(X \parallel \{y\} \parallel \{x\}) \parallel \{x\} \right) = \begin{cases} \frac{1}{1 - p_x} (p_y - p_4), & X = \{y\}, \\ \frac{1}{1 - p_x} (1 - p_x - p_y + p_4), & X = \emptyset. \end{cases} \] (5.34)

The corresponding sets of marginal conditional probabilities of events \( y \in \mathcal{X} \) with respect to the frame terraced events \( \text{ter}(\{x\} \parallel \{x\}) = x \)
\[ \text{ter}(\emptyset \parallel \{x\}) = x^c \] correspondingly have the form:
\[ \tilde{p}^c(X \parallel \{x\} \parallel \{x\}) = \begin{cases} p_4, & X = \{x, y\}, \\ \frac{p_y - p_4}{1 - p_x}, & X = \emptyset. \end{cases} \] (5.35)
Remind, that Fréchet restrictions on the functional parameter \( p_x' = p_x(p_x, p_y) \) for \( \tilde{p} \)-ordered half-rare doublet of events \( \mathcal{X} = \{x, y\} \) have the form:

\[
0 \leq p_x' \leq p_{xy}^+ = \min\{p_x, p_y\} = p_y, \tag{5.36}
\]

and for otherwise \( \tilde{p} \)-ordered half-rare doublet of events \( \mathcal{X} = \{y, x\} \) have the form:

\[
0 \leq p_x' \leq p_{xy}^+ = \min\{p_x, p_y\} = p_x. \tag{5.37}
\]

6 The Kopula theory for triplets of events

6.1 Independent 3-Kopula

First, without the frame method, which is not required here, consider the simplest example of a \( 3\)-Kopula \( \mathcal{K} \in \Psi_3^k \) of the \((N - 1)\)-set of events \( \mathcal{X} = \{x, y, z\} \), i.e., a 1-function on the unit \( \mathcal{X} \)-cube. In other words, construct such a nonnegative bounded numerical function

\[
\mathcal{K} : [0, 1]^{\mathcal{X}} \rightarrow [0, 1],
\]

that for all \( z \in \mathcal{X} \)

\[
\sum_{x \in X \subseteq \mathcal{X}} \mathcal{K}\left(\tilde{w}(e_X|X)\right) = w_z.
\]

Such a simple example of a 1-function on \( \mathcal{X} \)-cube is so-called \( \text{independent } (N - 1)\)-Kopula, which for all free variables \( \tilde{w} = \{w_x, w_y, w_z\} \in [0, 1]^x \otimes [0, 1]^y \otimes [0, 1]^z = [0, 1]^{\mathcal{X}} \) is defined by the formula:

\[
\mathcal{K}(\tilde{w}) = w_xw_yw_z, \tag{6.1}
\]

that provides it on each \( 2^{(c|\tilde{w})} \)-phenomenon-dom the following 2\(|X|\) values:

\[
\mathcal{K}\left(\tilde{w}(e_X|X)\right) = \prod_{x \in X} w_x \prod_{x \in X - X} (1 - w_x) \tag{6.2}
\]

for \( X \subseteq \mathcal{X} \). Indeed, as in the case of the doublet of events this function is a 1-function, since for all \( x \in \mathcal{X} \)

\[
\sum_{x \in X \subseteq \mathcal{X}} \left( \prod_{x \in X} w_x \prod_{x \in X - X} (1 - w_x) \right) = w_x.
\]

The e.p.d. of the 1st kind of independent triplet of events \( \mathcal{X} \) with the \( \mathcal{X} \)-set of probabilities of events \( \tilde{p} \) is defined by \( 2^{3^2} \) values of the independent 3-Kopula (6.1) on \( 2^{(c|\tilde{p})} \)-phenomenon-dom by the general formulas of half-rare variables, i.e., for \( X \subseteq \{x, y\} \):

\[
p(X/\mathcal{X}) = \mathcal{K}\left(\tilde{p}(e_X|X)\right) = \prod_{x \in X} p_x \prod_{x \in X - X} (1 - p_x). \tag{6.3}
\]

6.2 Three-dimensional maps of the independent 3-Kopula

In Fig. 9 it is shown the results of visualization of the three-dimensional graph of independent 3-Kopula (8.1) of the triplet \( \mathcal{X} = \{x, y, z\} \), defined on the cube \([0, 1]^3\), in projections on planes, which are orthogonal to the axis \( p_y \).

Figure 9: The visualization of projections of the same three-dimensional map of Cartesian representation of independent 3-Kopula of the triplet \( \mathcal{X} = \{x, y, z\} \) on the unit cube in conditional colors with values of marginal probability \( p_x = 0, 1, \ldots, 0.9, 1.0 \), where the white color corresponds to points in which probabilities of all terraced events are 1/8. The orientation of axes: \((p_x, p_z)\) = (horizontal, vertical).

6.3 The frame method for constructing a half-rare triplet of events

In order to construct by the frame method the \( \tilde{p} \)-ordered frame half-rare triplet of events \( \mathcal{X} = \{x, y, z\} \) with the \( \mathcal{X} \)-set of marginal probabilities \( \tilde{p} = \{p_x, p_y, p_z\} \), where

\[
1/2 \geq p_x \geq p_y \geq p_z \geq 0, \tag{6.4}
\]

let’s suppose that

\[
\mathcal{X} = \{x\} + \{y, z\} = \{x\} + (\mathcal{X}'+(+\mathcal{X}'')) \tag{6.5}
\]

and in our disposal we have two inserted half-rare doublets of events

\[
\mathcal{X}' = \{s', t'\}, \quad \mathcal{X}'' = \{s'', t''\},
\]

with the known 2-Kopulas (see Fig. 10) which by the definition satisfy the following inclusions:

\[
s' = x \cap y \subseteq x, \quad t' = x \cap z \subseteq x, \quad s' \cup t' \subseteq x,
\]

\[
s'' = x \cap y \subseteq x'' \cap x' \subseteq x'' \cap z \subseteq x'', \quad s'' \cup t'' \subseteq x''. \tag{6.6}
\]

In view of the assumptions made (see Fig. 10)

\[
\text{ter}(xyz/\mathcal{X}) = xy \cap z = s' \cap t'' = \text{ter}(s't''/\mathcal{X}''),
\]

\[
\text{ter}(xy/\mathcal{X}) = x \cap y \cap z = s' \cap t' = \text{ter}(s't'/\mathcal{X}''),
\]

\[
\text{ter}(xz/\mathcal{X}) = x \cap z \cap y = s'' \cap t'' = \text{ter}(s''t''/\mathcal{X}''),
\]

\[
\text{ter}(yz/\mathcal{X}) = y \cap z \cap x = s'' \cap t' = \text{ter}(s''t'/\mathcal{X}''),
\]

\[
\text{ter}(z/\mathcal{X}) = x \cap y \cap z = s'' \cap t'' = \text{ter}(t''/\mathcal{X}''), \tag{6.7}
\]

these 6 terraced events are defined. All of them are generated by the frame half-rare triplet \( \mathcal{X} \), with the
In view of this, we obtain the formulas:

\[
\begin{align*}
\text{ter}(x/y/z) &= \text{ter}(0/x/y) - x, \\
\text{ter}(0/y/z) &= \text{ter}(0/x/y) - x.
\end{align*}
\]

or, equivalently,

\[
\begin{align*}
\text{ter}(x/y/z) &= \text{ter}(0/x/y) - x, \\
\text{ter}(0/y/z) &= \text{ter}(0/x/y) - x.
\end{align*}
\]

that are defined by the formulas.

In addition, the third pair of inclusions under the assumptions (6.6) means that the probability of the 1st kind of inserted half-square doubles \(x'\) and \(x''\) and the kind of inserted half-square doubles \(x'\) and \(x''\) in the language of e.p.d. of the 1st kind assumptions (6.9) mean that
(6.16):

\[
p(xyz/x) = p_{x'y'},
p(xy/x) = p_{y'} - p_{x'y'},
p(xz/x) = p_{y} - p_{x'y'},
p(yz/x) = p_{x'} - p_{x'y'},
p(z/x) = p_{x'} - p_{x'z} - p_{x'y'} + p_{x'z} + p_{x'y'},
p(\emptyset/x) = 1 - p_{x} - p_{y} - p_{z} + p_{x'} + p_{y'} + p_{x'y'}.
\]

(6.18)

6.4 The frame method: recurrent formulas for a half-rare triplet of events

The formulas (6.18) as well as the formulas (6.11) can be written in the form of special cases of recurrence formulas (3.52) and (3.53) from Note 15 for the triplet \( X = \{x, y, z\} = \{x\} + \{y, z\} = \{x\} + X' \):

\[
\mathcal{K}\left(\hat{p}(c/X \cup Y/X + \{x\})\right) = \begin{cases} 
\mathcal{K}(\{x\}) \left(\hat{p}(c(X^{\cap(x) \cup z}) \cap X')\right), & Y = \{x\}, \\
\mathcal{K}(\emptyset) \left(\hat{p}(c(X^{\cap(x) \cup z}) \cap X')\right), & Y = \emptyset.
\end{cases}
\]  

(6.19)

\[
\mathcal{K}\left(\hat{p}(c/X \cup Y/X + \{x\})\right) = \begin{cases} 
\mathcal{K}(\{x\}) \left(\hat{p}(c(X^{\cap(x) \cup z}) \cap X')\right) p_x, & Y = \{x\}, \\
\mathcal{K}(\emptyset) \left(\hat{p}(c(X^{\cap(x) \cup z}) \cap X')\right) (1 - p_x), & Y = \emptyset.
\end{cases}
\]  

(6.20)

In the formulas (6.19) the inserted pseudo-Kopulas \( \mathcal{K}(\{x\}) \) and \( \mathcal{K}(\emptyset) \) are defined by the first and the second four probabilities from (6.18) correspondingly, i.e., by the formulas:

\[
\mathcal{K}(\{x\}) \left(\hat{p}(c(X^{\cap(x) \cup z}) \cap X')\right) = \begin{cases} 
p_{x'y'}, & X = \{y, z\}, 
p_{y'} - p_{x'y'}, & X = \{y\}, 
p_{y} - p_{x'y'}, & X = \{z\}, 
p_{x'} - p_{x'y'} - p_{x'z} - p_{x'y'}, & X = \emptyset.
\end{cases}
\]  

(6.21)

\[
\mathcal{K}(\emptyset) \left(\hat{p}(c(X^{\cap(x) \cup z}) \cap X')\right) = \begin{cases} 
p_{x'y'}, & X = \{y, z\}, 
p_{y} - p_{x'y'} - p_{x'z} + p_{x'y'}, & X = \{y\}, 
p_{y} - p_{x'y'} - p_{x'z}, & X = \{z\}, 
1 - p_{x} - p_{y} - p_{z} + p_{x'} - p_{y'} + p_{x'y'}, & X = \emptyset.
\end{cases}
\]  

(6.22)

The corresponding sets of marginal conditional probabilities of events \( y, z \in \mathcal{X}' \) with respect to the frame terraced events \( \text{ter}(\{x\}/\{x\}) = x \) ter(\(\emptyset/\{x\}\)) = x, correspondingly have the from:

\[
\hat{p}(c/X \cup Y/X + \{x\}) = \begin{cases} 
p_{x'y'}, & X = \{y, z\}, 
p_{y} - p_{x'y'} - p_{x'z} + p_{x'y'}, & X = \{y\}, 
p_{y} - p_{x'y'} - p_{x'z}, & X = \{z\}, 
1 - p_{x} - p_{y} - p_{z} + p_{x'} - p_{y'} + p_{x'y'}, & X = \emptyset.
\end{cases}
\]  

(6.23)

Remind, that the four functional parameters \( p_{x'y'}, p_{y'}, p_{x'z}, \) and \( p_{x'y'} \) in the recurrent formulas (6.19), (6.20), and also in the formulas for pseudo-Kopulas (6.21), (6.22), and the conditional Kopulas (6.25), (6.26), obey the Fréchet-constraints (6.16).

7 The Kopula theory for quadruplets of events

7.1 The frame method for constructing a half-rare quadruplet of events

In order by the frame method to construct the ordered frame half-rare quadruplet of events \( \mathcal{X} = \{x, y, z, v\} \) with the \( X \)-set of marginal probabilities \( \tilde{p} = \{p_x, p_y, p_z, p_v\} \), where

\[
1/2 \geq p_x \geq p_y \geq p_z \geq p_v \geq 0,
\]  

(7.1)

let’s suppose that

\[
\mathcal{X} = \{x\} + \{y, z, v\} = \{x\} + \{x'\} \cup \{x'\}
\]  

(7.2)
and we have two inserted half-rare triplets of events 
\[ X' = \{s', t', u'\} \quad X^s = \{s^s, t^s, u^s\}, \]
with the known 3-Kopulas, which by definition satisfy the following inclusions (see Fig. 11):
\[ s' = x \cap y \subseteq x, \quad t' = x \cap z \subseteq x, \quad u' = x \cap v \subseteq x, \]
\[ s' \cup t' \cup u' \subseteq x, \quad (7.3) \]
\[ s^s = x^c \cap y \subseteq x^c, \quad t^s = x^c \cap z \subseteq x^c, \quad u^s = x^c \cap v \subseteq x^c, \]
\[ s^s \cup t^s \cup u^s \subseteq x^c. \]

The recurrent formulas, which express the e.p.d. of the 1st kind of \( \beta \)-ordered half-rare quadruplet of events \( X \) through the e.p.d. of the 1st kind of two inserted triplets \( X' \) and \( X^s \), follow from the general recurrent formulas (3.52) and (3.53) in Note 15 as well as in cases of a doublet and a triplet of events. And therefore, and because of the cumber-someness, these formulas are not represented here, but are only illustrated by Venn diagrams (see Fig. 11).

### 7.2 Recurrent Fréchet-restrictions in the frame method

Let us dwell in more detail on Fréchet-restrictions for \( 11 = 2^4 - 4 - 1 \) functional parameters of a Kopula of quadruplet of events, to derive the recurrent sequence of such Fréchet-restrictions, which begins with Fréchet-restrictions for a doublet of events (5.36), continues with Fréchet-restrictions for a triplet of events (6.16), and should be supported by Fréchet-restrictions for parameters of a Kopula of quadruplet of events \( X = \{x, y, z, v\} \) and so on.

To this end, we first recall Fréchet-restrictions for parameters of Kopulas of a doublet and a triplet of events.

#### 7.2.1 Fréchet-restrictions for a doublet of events

For a Kopula of doublet of events, the Fréchet-restrictions of a \( 1 = 2^2 - 2 - 1 \) parameter of inserted monoplets \( X' \) and \( X^s \) have the form:
\[ 0 \leq p_{s'} \leq p_y, \]
\[ (7.4) \]

#### 7.2.2 Fréchet-restrictions for a triplet events

For a Kopula of doublet of events, the Fréchet-restrictions of \( 4 = 2^3 - 3 - 1 \) parameters of inserted doubles \( X' \) and \( X^s \) have the form:
\[ 0 \leq p_{s'} \leq p_y, \]
\[ 0 \leq p_{t'} \leq 1, \]
\[ 0 \leq p_{u'} \leq 1, \]
\[ 0 \leq p_{s't'} \leq p_{s't'}^+, \]
\[ p_{s't'}^+ \leq p_{s't'} \leq p_{s't'}^-, \]
\[ (7.5) \]

where
\[ p_{s't'}^- = \max\{0, p_x + p_y + p_z - 1 - p_{s't'}\}, \]
\[ p_{s't'}^+ = \min\{p_y - p_{s't'}, p_z - p_{s't'}\}, \]
\[ (7.6) \]
are the lower and upper Fréchet-boundaries probabilities of double intersections of events from inserted doubles \( X' \) and \( X^s \) with respect to the frame monoplet \( \{x\} \).

The case of triplet of events gives a new level of Fréchet-restrictions (the two last Fréchet-boundaries in (7.6)), when probabilities of double intersections of events from inserted doubles have Fréchet-boundaries that depend not only on marginal probabilities of the triplet, but and on inserted marginal probabilities on which, in turn, the usual Fréchet-restrictions mentioned above are imposed.

#### 7.2.3 Fréchet-restrictions for a quadruplet of events

For a Kopula of quadruplet of events the Fréchet-restrictions of \( 11 = 2^4 - 4 - 1 \) parameters of the in-
sented triplet $\lambda'\,$ and $\lambda''\,$ have the form:

\[
\begin{align*}
0 &\leq p_{s't'} \leq p_y, \\
0 &\leq p_{tv} \leq p_z, \\
0 &\leq p_{u't'} \leq p_v, \\
\bar{p}_{s't'} &\leq p_{s't'} \leq \bar{p}_{s't'}, \\
\bar{p}_{s'u'} &\leq p_{s'u'} \leq \bar{p}_{s'u'}, \\
p_{tv} &\leq p_{tv} \leq \bar{p}_{tv}, \\
\bar{p}_{s'tu'} &\leq p_{s'tu'} \leq \bar{p}_{s'tu'}, \\
\bar{p}_{s'u't'} &\leq p_{s'u't'} \leq \bar{p}_{s'u't'}, \\
p_{tv} &\leq p_{tv} \leq \bar{p}_{tv}, \\
\bar{p}_{s'tu'u'} &\leq p_{s'tu'u'} \leq \bar{p}_{s'tu'u'},
\end{align*}
\]

(7.7)

where

\[
\begin{align*}
p_{s't'} &\equiv \max\{0, p_{s't'} + p_{tv} - p_x\}, \\
p_{s't'} &\equiv \min\{p_y - p_{s't'}, p_{tv} - p_v\}, \\
p_{s'u'} &\equiv \max\{0, p_{s'u'} + p_{tv} - p_{u't'}\}, \\
p_{s'u'} &\equiv \min\{p_y - p_{s'u'}, p_{tv} - p_{u't'}\}, \\
p_{s'tu'} &\equiv \max\{0, p_{s'tu'} + p_{tv} - p_{s'tu'}\}, \\
p_{s'tu'} &\equiv \min\{p_y - p_{s'tu'}, p_{tv} - p_{s'tu'}\}, \\
p_{s'tu'u'} &\equiv \max\{0, p_{s'tu'u'} + p_{tv} - p_{s'tu'u'}\}, \\
p_{s'tu'u'} &\equiv \min\{p_y - p_{s'tu'u'}, p_{tv} - p_{s'tu'u'}\}, \\
p_{tv} &\equiv \max\{0, p_{tv} + p_{u't'} - p_{tv}\}, \\
p_{tv} &\equiv \min\{p_y - p_{tv}, p_{tv} - p_{u't'}\}, \\
p_{s'tu} &\equiv \max\{0, p_{s'tu} + p_{tv} - p_{s'tu}\}, \\
p_{s'tu} &\equiv \min\{p_y - p_{s'tu}, p_{tv} - p_{s'tu}\}, \\
p_{s'tu'} &\equiv \max\{0, p_{s'tu'} + p_{tv} - p_{s'tu'}\}, \\
p_{s'tu'} &\equiv \min\{p_y - p_{s'tu'}, p_{tv} - p_{s'tu'}\}, \\
p_{s'tu'u'} &\equiv \max\{0, p_{s'tu'u'} + p_{tv} - p_{s'tu'u'}\}, \\
p_{s'tu'u'} &\equiv \min\{p_y - p_{s'tu'u'}, p_{tv} - p_{s'tu'u'}\},
\end{align*}
\]

(7.8)

are the lower and upper Fréchet-borders of probabilities of double and triple intersections of events from the inserted triplets $\lambda'\,$ and $\lambda''\,$ with respect to the frame monoplet \{x\}.

The case of a quadruplet of events gives the following level of Fréchet-restrictions (the four last Fréchet-borders in (7.8)), when probabilities of triple intersections of events from inserted triplets have Fréchet-borders that depend directly not so much on marginal probabilities as on inserted probabilities of double intersections, on which, in turn, Fréchet-restrictions of the previous level, mentioned above, are imposed.

The all Fréchet-restrictions in the considered frame methods for a doublet, a triplet and a quadruplet of events differ from the usual Fréchet-restrictions, which are functions of only corresponding marginal probabilities. They differ in that they have a recurrent structure. When, as the power of intersections of inserted events increases, the Fréchet-borders of their probabilities are functions of Fréchet-borders for probabilities of intersections of lower power.

The such Fréchet-restrictions and Fréchet-borders for a doublet (7.4), a triplet (7.5,7.6), a quadruplet (7.7,7.8) of events and so on, will call the recurrent Fréchet-restrictions and recurrent Fréchet-borders.

7.3 The frame method: recurrent formulas for a half-rare quadruplet of events

The recurrent formulas for Kopula of a quadruplet of events immediately can be written in the form of special cases of recurrence formulas (3.52) and (3.53) from Note 15 for the quadruplet $\mathcal{X} = \{x, y, z, v\} = \{x\} + \mathcal{X}':$

\[
\mathcal{K}\left(\tilde{\rho}(c)\,|\,X \cap X + \{x\}\right) =
\begin{cases}
\mathcal{K}(x), & Y = \{x\}, \\
\mathcal{K}(0), & Y = \emptyset.
\end{cases}
\]

(7.9)

\[
\mathcal{K}\left(\tilde{\rho}(c)\,|\,X \cap X + \{x\}\right) =
\begin{cases}
\mathcal{K}(x), & Y = \{x\}, \\
\mathcal{K}(0), & Y = \emptyset.
\end{cases}
\]

(7.10)

In the formulas (7.9) the inserted pseudo-Kopulas $\mathcal{K}(x)$ and $\mathcal{K}(0)$ are defined by the octuples of probabilities, i.e., by the formulas:

\[
\mathcal{K}(x)\left(\tilde{\rho}(c)\,|\,X \cap X + \{x\}\right) =
\begin{cases}
p_{s't'u'}, & X = \{y, z, v\}, \\
p_{s't'u'} - p_{s't'u'}, & X = \{y, z\}, \\
p_{s't'u'} - p_{s't'u'}, & X = \{y, v\}, \\
\tilde{p}_{tv} &\leq p_{tv} \leq \tilde{p}_{tv}, \quad X = \{y, z\},
\end{cases}
\]

(7.11)
The corresponding sets of marginal conditional probabilities of events \( y, z \in \mathcal{X} \) with respect to the frame terraced events \( \text{ter}(\{ x \} \mid \{ x \}) = x \) and \( \text{ter}(\emptyset \mid \{ x \}) = x' \) correspondingly have the form:

\[
\tilde{p}(c\mid X\mid X_0\mid \{ x \}) = \begin{cases} 
\frac{1}{p_x}p_{s't'uv'}, & X = \{ y, z, v \}, \\
\frac{1}{p_x}(p_{s't'v'} - p_{s't'v'}), & X = \{ y, z \}, \\
\frac{1}{p_x}(p_{st'v'} - p_{st'v'}), & X = \{ y, v \}, \\
\frac{1}{p_x}(p_{ut'v'} - p_{ut'v'}), & X = \{ z, v \}, \\
\frac{1}{p_x}(p_{st'u'v'} - p_{st'u'v'}), & X = \{ z, v \}, \\
\frac{1}{p_x}(p_{st'u'} - p_{st'u'}) & X = \emptyset,
\end{cases}
\]

(7.17)

Recall that 11 functional parameters

\[
\begin{align*}
& p_{s't'v'}, p_{st'v'}, p_{st'u'}, \\
& p_{s't'uv'}, p_{s't'u'v'}, p_{st'u'v'}, \\
& p_{s't'u'v'}
\end{align*}
\]

in the recurrent formulas (7.10), (7.12) and (7.15), (7.16) obey the Fréchet-restrictions (7.7) and Fréchet-boundaries (7.8).

8 The Kopula theory for a set of events

8.1 Independent N-Kopula

First, without the frame method, which is not required here, let's consider the simplest example of the N-Kopula \( \mathcal{K} \in \Psi_1 \) of an \( N \)-set of events \( \mathcal{X} \), i.e., a 1-function on the unit \( \mathcal{X} \) hypercube. In other words, we construct a nonnegative bounded numerical function

\[
\mathcal{K} : [0, 1]^{\otimes \mathcal{X}} \rightarrow [0, 1],
\]

that for all \( z \in \mathcal{X} \)

\[
\sum_{x \in \mathcal{X} \subseteq \mathcal{X}} \mathcal{K}(\tilde{w}(c\mid X\mid \mathcal{X})) = w_z.
\]

A such simplest example of a 1-function on the unit \( \mathcal{X} \) hypercube is the so-called independent N-Kopula
which for all free variables \( \tilde{w} \in [0,1]^{\mathcal{X}} \) is defined by the formula:

\[
\mathcal{K}(\tilde{w}) = \prod_{x \in \mathcal{X}} w_x, \tag{8.1}
\]

that provides it on each \( g(\tilde{w}) \)-phenomen-dom the following \( 2^N \) values:

\[
\mathcal{K}
\left(\tilde{w}(c|X/\mathcal{X})\right) = \prod_{x \in \mathcal{X}} w_x \prod_{x \in \mathcal{X} - X} (1 - w_x) \tag{8.2}
\]
for \( X \subseteq \mathcal{X} \). Indeed\(^{11}\) as in the case of doublet of events this function is a 1-function, since for all \( x \in \mathcal{X} \)

\[
\sum_{x \in X \subseteq \mathcal{X}} \left( \prod_{x \in X} w_x \prod_{x \in \mathcal{X} - X} (1 - w_x) \right) = w_x.
\]

The e.p.d. of the 1st kind of ondependent \( N \)-s.e. \( \mathcal{X} \) with the \( \mathcal{X} \)-set of probabilities of events \( \tilde{p} \) is defined by \( 2^N \) values of the independent \( N \)-Kopula (8.1) on the \( g(\tilde{p}) \)-phenomen-dom by the general formulas of half-rare variables, i.e., for \( X \subseteq \{x, y\} \):

\[
p(X/\mathcal{X}) = \mathcal{K}
\left(\tilde{p}(c|X/\mathcal{X})\right) = \prod_{x \in \mathcal{X}} p_x \prod_{x \in \mathcal{X} - X} (1 - p_x). \tag{8.3}
\]

### 8.2 The frame method for constructing a half-rare set of events

The general recurrent formulas (3.52, 3.53) of the frame method for constructing a Kopula of a set of events are derived in Note 15. Recall these formulas:

\[
\mathcal{K}
\left(\tilde{p}(c|X + Y/\mathcal{X} + \{x_0\})\right) = \left\{ \begin{align*}
\mathcal{K}(\{x_0\}) \left(\tilde{p}(c|X|\{x_0\})/\mathcal{X}^{\prime}\right) Y = \{x_0\}, \\
\mathcal{K}(\emptyset) \left(\tilde{p}(c|X^\emptyset|\{x_0\})/\mathcal{X}^\emptyset\right) Y = \emptyset,
\end{align*} \right. \tag{8.4}
\]

\[
\mathcal{K}
\left(\tilde{p}(c|X + Y/\mathcal{X} + \{x_0\})\right) = \left\{ \begin{align*}
\mathcal{K}(\{x_0\}) \left(\tilde{p}(c|X + Y|\{x_0\})/\mathcal{X}^{\prime}\right) p_0, Y = \{x_0\}, \\
\mathcal{K}(\emptyset) \left(\tilde{p}(c|X + Y|\{x_0\})/\mathcal{X}^\emptyset\right) (1 - p_0), Y = \emptyset,
\end{align*} \right. \tag{8.5}
\]

which express the \( N \)-Kopula of \( N \)-s.e. \( \mathcal{X} = \mathcal{X} + \{x_0\} \), where \( \mathcal{X} = \mathcal{X}^{\prime}(+)\mathcal{X}^\emptyset \), through the known probability \( p_0 \) of the event \( x_0 \) and together with it either two known inserted pseudo-(\( N \)-1)-Kopulas (see Definition 9), i.e., pseudo-(\( N \)-1)-Kopulas of inserted \( (N - 1) \)-s.e.’s \( \mathcal{X}^{\prime} \) and \( \mathcal{X}^\emptyset \) in the frame monoplet \( \{x_0\} \), or through two known conditional (\( N \)-1)-Kopulas (see Definition 10) with respect to the frame monoplet \( \{x_0\} \) for the same \( \mathcal{X}^{\prime} \) and \( \mathcal{X}^\emptyset \).

\(^{11}\)Perhaps this statement deserves to be called a lemma, which, incidentally, is not difficult to prove.

We write out more detailed formulas for corresponding pseudo-(\( N \)-1)-Kopulas:

\[
\mathcal{K}((x_0)) \left(\tilde{p}(c|X^{\prime}/\mathcal{X}^{\prime})/\mathcal{X}^{\prime}\right) = \mathcal{K}(\{x_0\}) \left(\tilde{p}(c|X^{\prime}/\mathcal{X}^{\prime})\right) = \left\{ \begin{align*}
p(X^{\prime}/\mathcal{X}^{\prime}), & \quad \mathcal{X}^{\prime} \neq \emptyset, \\
p(\emptyset/\mathcal{X}^{\prime}) - 1 + p_0, & \quad \mathcal{X}^{\prime} = \emptyset,
\end{align*} \right. \tag{8.6}
\]

\[
\mathcal{K}(\emptyset) \left(\tilde{p}(c|X^\emptyset/\mathcal{X}^\emptyset)/\mathcal{X}^\emptyset\right) = \mathcal{K}(\emptyset) \left(\tilde{p}(c|X^\emptyset/\mathcal{X}^\emptyset)\right) = \left\{ \begin{align*}
p(X^\emptyset/\mathcal{X}^\emptyset), & \quad \mathcal{X}^\emptyset \neq \emptyset, \\
p(\emptyset/\mathcal{X}^\emptyset) - p_0, & \quad \mathcal{X}^\emptyset = \emptyset,
\end{align*} \right. \tag{8.6}
\]

and for conditional (\( N \)-1)-Kopulas:

\[
\mathcal{K}(\{x_0\}) \left(\tilde{p}(c|X/\mathcal{X}|\{x_0\})/\mathcal{X}\right) = \left\{ \begin{align*}
\frac{1}{p_0} p(X^{\prime}/\mathcal{X}^{\prime}), & \quad \mathcal{X}^{\prime} \neq \emptyset, \\
\frac{1}{p_0} (p(\emptyset/\mathcal{X}^\emptyset) - 1 + p_0), & \quad \mathcal{X}^\emptyset \neq \emptyset,
\end{align*} \right. \tag{8.7}
\]

\[
\mathcal{K}(\emptyset) \left(\tilde{p}(c|X^\emptyset/\mathcal{X}^\emptyset)/\mathcal{X}^\emptyset\right) = \left\{ \begin{align*}
\frac{1}{p_0} p(X^\emptyset/\mathcal{X}^\emptyset), & \quad \mathcal{X}^\emptyset \neq \emptyset, \\
\frac{1}{p_0} (p(\emptyset/\mathcal{X}^\emptyset) - p_0), & \quad \mathcal{X}^\emptyset = \emptyset,
\end{align*} \right. \tag{8.7}
\]

where \( X^{\prime} = X^{\prime}(c|X^{\prime}/\mathcal{X}^{\prime}) = X^{\prime}(\{x_0\}) = \{x \cap x_0, x \in X\} \), \( X^\emptyset = X^{\emptyset}(c|X^{\emptyset}/\mathcal{X}^{\emptyset}) = X^{\emptyset}(\{x_0\}) = \{x \cap x_0, x \in X\} \), \( X \subseteq \mathcal{X} \).

### 8.3 Recurrent formulas for Fréchet-boundaries and Fréchet-restrictions

Now let us consider recurrent formulas for Fréchet-boundaries Fréchet-restrictions and for the \( 2^N - N - 1 \) functional parameters of an \( N \)-Kopula of \( N \)-set of events

\[
\mathcal{X} = \{x_0, x_1, \ldots, x_{N-1}\} = \{x_0\} + \{x_1, \ldots, x_{N-1}\} = \{x_0\} + \mathcal{X} = \{x_0\} + (\mathcal{X}^{\prime}(+)\mathcal{X}^\emptyset), \tag{8.8}
\]

where

\[
\mathcal{X}^{\prime} = \{x_0 \cap x_1, \ldots, x_0 \cap x_{N-1}\}, \\
\mathcal{X}^\emptyset = \{x_0^\emptyset \cap x_1, \ldots, x_0^\emptyset \cap x_{N-1}\} \tag{8.9}
\]

are inserted (\( N \)-1)-s.e.’s, and

\[
\tilde{p}(c|\mathcal{X}/\mathcal{X}) = \{p_0, p_1, \ldots, p_{N-1}\} \tag{8.10}
\]

is the \( \mathcal{X} \)-set of probabilities of marginal events from \( \mathcal{X} \), i.e., \( p_n = P(x_n), n = 0, 1, \ldots, N - 1 \).

Judging by the form of Fréchet-boundaries Fréchet-restrictions for a doublet, a triplet and
a quadruplet of events, collected in paragraph 7.2, these Fréchet-restrictions consists of two groups, such that one of them, which refers to the parameters of the inserted \((N-1)\)-s.e. \(X'\), consists of \(2^{N-1} - 1\) Fréchet-restrictions, and the other, which refers to the parameters of the inserted \((N-1)\)-s.e. \(X''\), consists of \(2^{N-1} - (N-1) - 1\) Fréchet-restrictions. And, as it should:

\[
2^N - N - 1 = (2^{N-1} - 1) + (2^{N-1} - (N-1) - 1). \tag{8.11}
\]

The first group, related to the inserted \((N-1)\)-s.e. \(X'\), contains Fréchet-restrictions for probabilities of the second kind

\[
p_{X'\mid X'} = P\left( \bigcap_{x' \in X'} x' \right), \tag{8.12}
\]

that are numbered by nonempty subsets \(X' \neq \emptyset\) of inserted \((N-1)\)-s.e. \(X'\) (the number of such subsets: \(2^{N-1} - 1\)); the second group, related to the inserted \((N-1)\)-s.e. \(X''\), contains Fréchet-restrictions for the such probabilities of the second kind:

\[
p_{X''\mid X''} = P\left( \bigcap_{x'' \in X''} x'' \right), \tag{8.13}
\]

that are numbered by subsets \(X'' \subseteq X''\) with the power \(|X''| \geq 2\) (number of such subsets: \(2^{N-1} - (N-1) - 1\)).

**Note 18 (denotations for subsets of fixed power).** To more conveniently represent the recurrent Fréchet-restrictions, agree to denote

\[
X'_n \subseteq X' \iff |X'_n| = n, \quad X''_n \subseteq X'' \iff |X''_n| = n. \tag{8.14}
\]

the subsets consisting of \(n\) events. In this notation, for example, the \(X\)-set of marginal probabilities \(p^{\text{marg}}(X\mid x)\) is written as the \(X\)-set probabilities of the second kind that are numbered by monoplets of events \(X_1 = \{x\}, x \in X\):

\[
\{p_0, p_1, \ldots, p_{N-1}\} = \{p_{X_1\mid X}, X_1 \subseteq X\}. \tag{8.15}
\]

The set of probabilities of double intersection of events \(x \in X\), i.e., the set of probabilities of the second kind that are numbered by doublets, has the form:

\[
\{p_{(x,y)\mid X}, \{x,y\} \subseteq X\} = \{p_{X_2\mid X}, X_2 \subseteq X\}. \tag{8.16}
\]

And the set of probabilities of triple intersections of events \(x \in X\), i.e., the set of probabilities of the second kind that are numbered by triplets, has the form:

\[
\{p_{(x,y,z)\mid X}, \{x,y,z\} \subseteq X\} = \{p_{X_3\mid X}, X_3 \subseteq X\} \tag{8.17}
\]

and so on.

**Note 19 (recurrent formulas for Fréchet-boundaries and Fréchet-restrictions).** Probabilities of \(n\)-intersections \((n = 2, \ldots, N-1)\) of events from the inserted s.e.’s \(X'\) and \(X''\) have the recurrent Fréchet-restrictions (see paragraph 7.2) that are written by denotations from Note 18 by the following way with respect to \(X''\):

\[
p_{X''\mid X''} \leq p_{X'_n\mid X'} \leq p_{X''_n\mid X''}, \tag{8.18}
\]

where

\[
\bar{p}_{X''_n\mid X''} = \max \left\{ 0, p_x - \sum_{X'_n \subseteq X'_n} \left( p_x - p_{X'_n\mid X'} \right) \right\}, \tag{8.19}
\]

are recurrent the lower and upper Fréchet-boundaries. And the lower Fréchet-boundary we can write somewhat differently after simple transformations:

\[
\bar{p}_{X''_n\mid X''} = \max \left\{ 0, \sum_{X'_n \subseteq X'_n} p_{X'_n\mid X'} - (n-1)p_x \right\}. \tag{8.20}
\]

Similar look the recurrent Fréchet-restrictions with respect to the inserted s.e. \(X'\):

\[
p_{X''_n\mid X''} \leq p_{X'_n\mid X'} \leq p_{X''_n\mid X''}, \tag{8.21}
\]

where

\[
\bar{p}_{X''_n\mid X''} = \max \left\{ 0, 1 - p_x - \sum_{X'_n \subseteq X'_n} (1 - p_x - p_{X'_n\mid X'}) \right\}, \tag{8.22}
\]

are recurrent the lower and upper Fréchet-boundaries. And the lower Fréchet-boundary we can write somewhat differently after simple transformations:

\[
\bar{p}_{X''_n\mid X''} = \max \left\{ 0, \sum_{X'_n \subseteq X'_n} p_{X'_n\mid X'} - (n-1)(1-p_x) \right\}. \tag{8.23}
\]

It remains to write out more \(N-1\) recurrent Fréchet-restrictions on probabilities of marginal events from the inserted s.e. \(X'\), i.e., on probabilities of the second kind that are numbered by monoplets \(X'_1 \subseteq X'\):

\[
0 \leq p_{X'_1\mid X'} \leq p_{X'_1\mid X'}, \tag{8.24}
\]

which are restricted by marginal probabilities of events from the \((N-1)\)-s.e. \(X'\) and which together with the recurrent Fréchet-restrictions (8.18,
8.21) form the all totality of \textit{recurrent Fréchet-restrictions}. This totality consists of $2^N-N-1$ restrictions. And \textit{recurrent the lower and upper Fréchet-boundaries} in these restrictions are defined by recurrent formulas (8.19, 8.22).

9 Parametrization of functional parameters of Kopula by Fréchet-correlations of inserted events

Let’s consider parametrization on an example of functional parameters $p_{x'}, p_{y'}, p_{z'}, p_{x'y'}, p_{x'z'}, p_{y'z'}$ of 3-Kopula of the $j$-ordered half-rare triplet $\mathcal{X} = \{x, y, z\}$, which in the frame method is constructed from two inserted pseudo-2-Kopulas.

9.1 Parametrization of functional parameters $p_{x'}$ and $p_{y'}$

The Fréchet-restriction of the functional parameter $p_{x'} = p_{x'}(p_x, p_y, p_z)$

$$0 \leq p_{x'} \leq p_y,$$  

(9.1)

that in the frame method has a sense of probability of double intersection of events $x$ and $y$:

$$p_{x'} = p_{xy} \cap x = P(x \cap y),$$

(9.2)

is based on the notion of Fréchet-correlation [1]

$$\text{Kor}_{xy} = \begin{cases} \frac{\text{Kov}_{xy}}{\text{Kov}_{xy}^0}, & \text{Kov}_{xy} < 0, \\ \frac{\text{Kov}_{xy}}{\text{Kov}_{xy}^0}, & \text{Kov}_{xy} \geq 0, \end{cases}$$

(9.3)

where

$$\text{Kov}_{xy} = P(x \cap y) - P(x)P(y)$$

(9.4)

is a covariance of events $x$ and $y$, and

$$\text{Kov}_{xy}^0 = \max\{0, p_x + p_y - 1\} - p_x p_y = -p_x p_y,$$

$$\text{Kov}_{xy}^+ = \min\{p_x, p_y\} - p_x p_y = p_y - p_x p_y$$

(9.5)

are its the lower and upper Fréchet-boundaries.

From Definition (9.3) we get the parametrization of functional parameter $p_{x'}$ by the double Fréchet-correlation on the following form:

$$p_{x'}(p_x, p_y, p_z) =$$

$$= \begin{cases} p_x p_y - \text{Kor}_{xy} \text{Kov}_{xy}, & \text{Kor}_{xy} < 0, \\ p_x p_y + \text{Kor}_{xy} \text{Kov}_{xy}, & \text{Kor}_{xy} \geq 0 \end{cases}$$

(9.6)

are the lower and upper Fréchet-boundaries of probabilities of double intersections of events from inserted doubles $x'$ and $x''$ with respect to the frame monoplet $\{x\}$, which should serve inserted the lower and upper Fréchet-boundaries of probabilities of triple intersections (9.20) of events from the triplet $\mathcal{X} = \{x\} + (x')^+(x'')$. However, as might be expected, these Fréchet-boundaries are not always ready to serve as the lower and upper Fréchet-boundaries for probabilities of triple intersections.

The parametrization of functional parameter $p_{y'}$ by the double Fréchet-correlation is similar:

$$p_{y'}(p_x, p_y, p_z) =$$

$$= \begin{cases} p_x p_y - \text{Kor}_{xy} \text{Kov}_{xy}, & \text{Kor}_{xy} < 0, \\ p_x p_y + \text{Kor}_{xy} \text{Kov}_{xy}, & \text{Kor}_{xy} \geq 0 \end{cases}$$

(9.7)

9.2 Inserted triple covariances and Fréchet-correlations

We recall first that an \textit{absolute triple Fréchet-correlation} [1] of three events $x, y$ and $z$ is defined similarly to the double one:

$$\text{Kor}_{xyz} = \begin{cases} \frac{\text{Kov}_{xyz}}{\text{Kov}_{xyz}^0}, & \text{Kov}_{xyz} < 0, \\ \frac{\text{Kov}_{xyz}}{\text{Kov}_{xyz}^0}, & \text{Kov}_{xyz} \geq 0, \end{cases}$$

(9.8)

where

$$\text{Kov}_{xyz} = P(x \cap y \cap z) - P(x)P(y)P(z)$$

(9.9)

is the triple covariance of events $x, y$ and $z$, and

$$\text{Kov}_{xyz}^0 = \max\{0, p_x + p_y + p_z - 2\} - p_x p_y p_z =$$

$$= -p_x p_y p_z,$$

$$\text{Kov}_{xyz}^+ = \min\{p_x, p_y, p_z\} - p_x p_y p_z =$$

$$= p_z - p_x p_y p_z$$

(9.10)

are its \textit{absolute} the lower and upper Fréchet-boundaries.

The definition of the \textit{inserted} triple Fréchet-correlation differs of the definition of \textit{absolute} one (9.8) in that its the lower and upper Fréchet-boundaries must depend on the e.p.d. of the inserted doublets $X'$ and $X''$. So they differ from \textit{absolute} Fréchet-boundaries (9.10) and have the form (9.19), where

$$p_{x'y'} = \max\{0, p_{x'} + p_{y'} - p_x\},$$

$$p_{x'y'}^+ = \min\{p_{x'}, p_{y'}\},$$

$$p_{x'y''} = \max\{0, p_{x'} + p_{y'} + p_z - 1 - p_{x'} - p_{y'}\},$$

$$p_{x'y''}^+ = \min\{p_{x'} - p_{x''}, p_{y'} - p_{y''}\}$$

(9.11)

are the lower and upper Fréchet-boundaries of probabilities of double intersections of events from inserted doubles $X'$ and $X''$ with respect to the frame monoplet $\{x\}$, which should serve inserted the lower and upper Fréchet-boundaries of probabilities of triple intersections (9.20) of events from the triplet $\mathcal{X} = \{x\} + (X'(+)X'')$. However, as might be expected, these Fréchet-boundaries are not always ready to serve as the lower and upper Fréchet-boundaries for probabilities of triple intersections.
For this reason, it is necessary to modify the definitions of two inserted triple covariances and, respectively, — inserted the lower and upper Fréchet-boundaries of these covariances.

The first modification of definitions (see Fig. 18, 19, 20). For brevity, we denote \( p_0^{(1)}(x) = p_x p_y p_z, p_0^{(0)} = (1 - p_x)p_y p_z \). Two inserted triple covariance are defined by the formulas:

\[
\text{Kov}_{xyz}^{(1)} = \begin{cases} 
  p_{s' t'} - p_{s}^{(1)}(x), & p_{s}^{(1)}(x) \in [p_{s' t'}, p_{s' t'}^+], \\
  p_{s' t'} - p_{s}^{(1)}(x), & p_{s}^{(1)}(x) < p_{s' t'}, \\
  p_{s' t'}^+ - p_{s}^{(1)}(x), & p_{s}^{(1)}(x) < p_{s' t'}, \\
  p_{s' t'} - p_{s}^{(1)}(x), & p_{s}^{(1)}(x) < p_{s' t'}, \\
  p_{s' t'}^+ - p_{s}^{(1)}(x), & p_{s}^{(1)}(x) < p_{s' t'}, 
\end{cases}
\]

\[
(9.12)
\]

and inserted the lower and upper Fréchet-boundaries of these covariances — by the formulas:

\[
\text{Kov}_{xyz}^{(0)} = \begin{cases} 
  p_{s' t'} - p_{s}^{(0)}(x), & p_{s}^{(0)} \in [p_{s' t'}, p_{s' t'}^+], \\
  p_{s' t'} - p_{s}^{(0)}(x), & p_{s}^{(0)} < p_{s' t'}, \\
  p_{s' t'}^+ - p_{s}^{(0)}(x), & p_{s}^{(0)} < p_{s' t'}, \\
  p_{s' t'} - p_{s}^{(0)}(x), & p_{s}^{(0)} < p_{s' t'}, \\
  p_{s' t'}^+ - p_{s}^{(0)}(x), & p_{s}^{(0)} < p_{s' t'}, 
\end{cases}
\]

\[
(9.13)
\]

Two inserted triple covariances are defined by the formulas:

\[
\text{Kov}_{xyz}^{(1)} = p_{s' t'} - p_{0}^{(1)}(x), \\
\text{Kov}_{xyz}^{(0)} = p_{s' t'} - p_{0}^{(0)}(x),
\]

and inserted the lower and upper Fréchet-boundaries of these covariances — by the formulas:

\[
\text{Kov}_{xyz}^{(1)} = p_{s' t'} - p_{0}^{(0)}(x), \\
\text{Kov}_{xyz}^{(0)} = p_{s' t'} - p_{0}^{(0)}(x),
\]

\[
(9.17)
\]

For any modification definitions of two inserted Fréchet-correlations look in the usual way:

\[
\text{Kor}_{xyz}^{(1)} = \begin{cases} 
  \text{Kov}_{xyz}^{(1)} \text{Kov}_{xyz}^{(1)}
\end{cases}
\]

\[
(9.18)
\]

\[
\text{Kor}_{xyz}^{(0)} = \begin{cases} 
  \text{Kov}_{xyz}^{(0)} \text{Kov}_{xyz}^{(0)}
\end{cases}
\]

\[
(9.19)
\]

\[
\text{Kor}_{xyz}^{(1)} = \begin{cases} 
  \text{Kov}_{xyz}^{(1)} \text{Kov}_{xyz}^{(1)}
\end{cases}
\]

\[
(9.20)
\]

\[
\text{Kor}_{xyz}^{(0)} = \begin{cases} 
  \text{Kov}_{xyz}^{(0)} \text{Kov}_{xyz}^{(0)}
\end{cases}
\]

\[
(9.21)
\]

9.3 Parametrization of functional parameters \( p_{s' t'} \) and \( p_{s' t'}^+ \)

The Fréchet-restriction of two functional parameters \( p_{s' t'} = p_{s' t'}(p_x, p_y, p_z) \) \( p_{s' t'}^+ = p_{s' t'}^+(p_x, p_y, p_z) \)

\[
p_{s' t'} \leq p_{s' t'} \leq p_{s' t'}^+,
\]

\[
p_{s' t'} \leq p_{s' t'} \leq p_{s' t'}^+,
\]

that in the frame method have a sense of probabilities of triple intersections of events:

\[
p_{s' t'} = P(x \cap y \cap z),
\]

\[
p_{s' t'} = P(x' \cap y \cap z),
\]

is based on the notion of the inserted triple Fréchet-correlation.

From definitions (9.18) and (9.11) we get the parametrization of functional parameter \( p_{s' t'} \) of the inserted triple Fréchet-correlation \( \text{Kor}_{xyz}^{(1)} \) in the following form:

\[
p_{s' t'}(p_x, p_y, p_z) = \begin{cases} 
  p_x p_y p_z - \text{Kor}_{xyz}^{(1)} \text{Kov}_{xyz}^{(1)}
\end{cases}
\]

\[
(9.21)
\]
The parametrization of functional parameter $p_{x'y'z'}$ of the inserted triple Fréchet-correlation $Kor_{xyz}^{(0)}$ follows from the same definitions (9.18) and (9.11):

$$p_{x'y'z'}(p_x, p_y, p_z) = \begin{cases} p_x p_y p_z - Kor_{xyz}^{(0)}Kov_{xyz}^{(0)}, \\ Kor_{xyz}^{(x)} < 0, \\ p_x p_y p_z + Kor_{xyz}^{(0)}Kov_{xyz}^{(0)}, \\ Kor_{xyz}^{(x)} \geq 0, \end{cases}$$

(9.22)

Note 20 (about parametrization of functional parameters of 3-Kopula by Fréchet-correlations).

The parametrization of the four functional parameters $p_x, p_y, p_{x'y'},$ and $p_{x'y'z'}$ of 3-Kopula of the $\hat{p}$-ordered half-rare triplet $X = \{x, y, z\}$ by two double Fréchet-correlations $Kor_{xyz}$ and $Kor_{x'y'}$ (9.3) and by two inserted triple Fréchet-correlations $Kor_{xyz}^{(0)}$ and $Kor_{xyz}^{(x'y')}$ (9.18) has the following advantages. Each of four Fréchet-correlations is a numerical characteristics of dependency of events with values from fixed interval $[-1, +1].$ And these values clearly indicate the proximity to Fréchet-boundaries and to independent 3-Kopula. The value “$-1$” indicates to the lower Fréchet-boundary, the value “$+1$” — to the upper Fréchet-boundary, and the value “$0$” — to independent events. For example, the equality of all these four Fréchet-correlations to zero determines a family of independent 3-Kopulas. Advantages of the proposed idea of parametrization of functional parameters of 3-Kopula are that

- an each Fréchet-correlation can take arbitrary value from $[-1, +1]$ without any connection with the values of the other three Fréchet-correlations;
- the above parametrization algorithm for functional parameters of 3-Kopula extends to the parametrization of the functional parameters of $N$-Kopulas by inserted Fréchet-correlations of higher orders.

10 Examples of Kopulas of some families of sets of events

10.1 Examples of different 2-Kopulas with a functional parameter within Frechet boundaries

Consider in Fig. 12 a number of examples of 2-Kopulas of doublets of half-rare events $X = \{x, y\}$ and its set-phenomena $\mathcal{X}^c(x) = \{x^c, y^c\},$ $\mathcal{X}^c(y) = \{x^c, y^c\},$ and $\mathcal{X}^c(xy) = \{x^c, y^c\},$ each of which is characterized by its own functional parameter $P(x \cap y) = p_{xy}(w_x, w_y),$ lying within the Fréchet-boundaries:

$$0 \leq p_{xy}(w_x, w_y) \leq \min\{w_x, w_y\}. \quad (10.1)$$

Upper 2-Kopula of Fréchet (embedded):

$$p_{xy}(w_x, w_y) = \min\{w_x, w_y\}. \quad (10.2)$$

Independent 2-Kopula of Fréchet:

$$p_{xy}(w_x, w_y) = w_xw_y. \quad (10.3)$$

Lower 2-Kopula of Fréchet (minimum-intersected):

$$p_{xy}(w_x, w_y) = \max\{0, w_x + w_y - 1\}. \quad (10.4)$$

Half-independent 2-Kopula:

$$p_{xy}(w_x, w_y) = w_xw_y/2. \quad (10.5)$$

Half-embedded 2-Kopula:

$$p_{xy}(w_x, w_y) = \min\{w_x, w_y\}/2. \quad (10.6)$$

Arbitrary-embedded 2-Kopula:

$$p_{xy}(w_x, w_y) = \min\{w_x, w_y\}(1 + \sin(15(w_x - w_y)))/2. \quad (10.7)$$

Continuously-embedded 2-Kopula:

$$p_{xy}(w_x, w_y) = w_xw_y + (\alpha(w_x, w_y) - w_xw_y)\beta(w_x, w_y), \quad (10.8)$$

where

$$\alpha(w_x, w_y) = \min\{w_x, w_y\}(1 + \sin(15(w_x - w_y)))/2, \quad (10.9)$$

$$\beta(w_x, w_y) = \sqrt{(1/2 - w_x)(1/2 - w_y)}. \quad (10.10)$$

10.2 Examples of 2-Kopulas with a functional parameter corresponding to some classical copulas

In Figs. 13, 14, 16, 17, and 15 it is shown 2-Kopulas of doublets of half-rare events $X = \{x, y\}$ and its set-phenomena $\mathcal{X}^c(x), \mathcal{X}^c(y),$ and $\mathcal{X}^c(xy),$ corresponding to some classical copulas.

2-Kopula of Ali-Mikhail-Haq, $\theta \in [-1, 1]:$

$$p_{xy}(w_x, w_y) = \frac{w_xw_y}{1 - \theta(1 - w_x)(1 - w_y)}. \quad (10.10)$$
2-Kopula of Clayton, $\theta \in [-1, \infty) - \{0\}$:

$$
\begin{align*}
    p_{xy}(w_x, w_y) &= \\
    &= \left[ \max \left\{ w_x^{-\theta} + w_y^{-\theta} - 1; 0 \right\} \right]^{-1/\theta}.
\end{align*}
$$

2-Kopula of Joe, $\theta \in [1, \infty)$:

$$
\begin{align*}
    p_{xy}(w_x, w_y) &= \\
    &= 1 - \left[ (1-w_x)^\theta + (1-w_y)^\theta - (1-w_x)^\theta (1-w_y)^\theta \right]^{1/\theta}.
\end{align*}
$$
2-Kopula of Gumbel, $\theta \in [1, \infty)$:

$$p_{xy}(w_x, w_y) = \frac{1}{\theta} \log \left[ 1 + \frac{(\exp(-\theta w_x) - 1)(\exp(-\theta w_y) - 1)}{\exp(-\theta) - 1} \right]. \quad (10.13)$$

10.3 Examples of 3-Kopulas, functional parameters of which serve Fréchet-correlations of events

In Fig.'s 13, 18, 19, and 20 it is shown 3-Kopulas of triplets of half-rare events $X = \{x, y, z\}$, functional parameters of which serve Fréchet-correlations in the first modification of definitions (see paragraph 9.2).

11 Appendix

11.1 Abbreviations in the Kopula theory

Consider the universal probability space $(\Omega, A^{(1)}, P)$ and one of its subject-name realizations, a partial probability space $(\Omega, A, P)$. The elements of the sigma-algebra $A^{(1)}$ are the universal Kolmogorov events $x^{(1)} \in A^{(1)}$, and the elements of the sigma-algebra $A \subseteq \text{events } x \in A$, which serve as names of universal Kolmogorov events $x^{(1)}$ (see in details [8]).

The notions that are relevant to a s.e. $X \subseteq A$ for which it is convenient to use the following abbreviations:

- $X = \{x : x \in X\}$ — a set of events (s.e.);
- $\bar{p} = \{p_x, x \in X\}$ — an $X$-set of probabilities of events from $X$;
- $\mathcal{X}(\cdot|X) = \mathcal{X}(\cdot|X/\mathcal{X}) = \{x : x \in X\} + \{x^c : x \in X - X\}$ — an $X$-phenomenon of $X$, $X \subseteq X$;
- $\mathcal{X}(\cdot|X) = \mathcal{X}(\cdot|X/\mathcal{X}) = \mathcal{X}$ — an $X$-phenomenon of $X$ equal to $X$;
- $\bar{p}(\cdot|X/\mathcal{X}) = \{p_x, x \in X\} + \{1-p_x, x \in X-X\}$ — an $X$-set of probabilities of events from $\mathcal{X}(\cdot|X)$, $X \subseteq \mathcal{X}$;
- $\bar{p}(\cdot|X/\mathcal{X}) = \bar{p}$ — an $X$-set of probabilities of events from $\mathcal{X}(\cdot|X)$ equal to $\bar{p}$;
- $p(X/\mathcal{X})$ — a value of e.p.d. of the 1st kind of $\mathcal{X}$ for $X \subseteq \mathcal{X}$;
- $\mathcal{K}(\bar{p}(\cdot|X/\mathcal{X}))$ — a value of the Kopula of e.p.d. of the 1st kind of $\mathcal{X}$ for $X \subseteq \mathcal{X}$;
- $p(X/\mathcal{X}) = \mathcal{K}(\bar{p}(\cdot|X/\mathcal{X}))$ — the definition of e.p.d. of the 1st kind of $\mathcal{X}$ by its Kopula, $X \subseteq \mathcal{X}$;
- $\tilde{s} = \{s_x, x \in \mathcal{X}\} \in \{0, 1/2\}^{\otimes X}$ — an $X$-set of half-rare variables;
- $\tilde{w} = \{w_x, x \in \mathcal{X}\} \in [0, 1]^{\otimes X}$ — an $X$-set of free variables.

Each figure shows maps of these 3-copulas on a cube in conditional colors, where the white color corresponds to the points at which the probabilities of terraced events are $1/8$. 

---

Figure 16: Cartesian representations of 2-Kopulas of doublets of half-rare events $X = \{x, y\}$ and its set-phenomena $X^{(1)}(\cdot)$, $X^{(1)}(\cdot)$, $X^{(1)}(\cdot)$ corresponding to Frank Kopula (from up to down): from near-upper ($\theta = 6.5$) around pinked-independent ($\theta = 0.1, -0.1$) to near-lower ($\theta = -6.5$).

Figure 17: Cartesian representations of 2-Kopulas of doublets of half-rare events $X = \{x, y\}$ and its set-phenomena $X^{(1)}(\cdot)$, $X^{(1)}(\cdot)$, $X^{(1)}(\cdot)$ corresponding to Gumbel Kopula (from up to down): from independent ($\theta = 1.0$) to near-lower ($\theta = 6.5$).
11.2 Set-phenomenon renumbering a e.p.d. of the 1st kind and its Kopulas

Lemma 5 (Set-phenomenon renumbering a e.p.d. of the 1st kind and its Kopulas). E.p.d. of the 1st kind and Kopulas of the s.e. \( X \) and of its S-phenomena \( X^{(c|S)} \) are connected by formulas of mutually inversion set-phenomenon renumbering for \( X \subseteq X, S \subseteq X \):

\[
p(X^{(c|S \cap X)} \cap X) = p((\Delta X)^c \cap X),
\]

\[
p(X \cap X) = p(((\Delta X)^c \cap (\Delta X)^c) \cap X^{(c|S)}).
\]

(11.1)

Therefore, the Kopula is obtained for \( \text{Kor} = 0 \).

\[
\mathcal{K}\left(\hat{p}^{(c|X)}(X^{(c|S \cap X)} \cap X^{(c|S)})\right) = \mathcal{K}\left(\hat{p}^{(c|((\Delta X)^c \cap (\Delta X)^c) \cap X^{(c|S)})}\right),
\]

(11.2)
events of the 1st kind of the s.e. and of its set-phenomena proved in [6].

11.3 Useful denotations for a doublet of events that are invariant relative to the $\tilde{p}$-order

The following special denotations for a doublet of events $\mathcal{X} = \{x, y\}$ and the $\mathcal{X}$-set of marginal probabilities $\tilde{p} = \{p_x, p_y\}$ that are invariant relative to the $\tilde{p}$-order, are useful.

\[ \mathcal{X} = \{x, y\} = \{x^+, x^-\}, \]
\[ \tilde{p} = \{p_x, p_y\} = \{p^+, p^-\}, \]
\[ 1/2 \geq p^+ \geq p^- , \quad 1 \geq w^+ \geq w^- , \]
\[ \{x^+\} = \max\{\mathcal{X}\} = \left\{\{x\}, p_x > p_y, \{y\}\right\} , \tag{11.4} \]
\[ \{x^-\} = \min\{\mathcal{X}\} = \left\{\{x\}, p_x < p_y, \{y\}\right\} , \]
\[ w^+ = \max\{w\} = \left\{w_x, w_x > w_y, w_y, \right\} , \tag{11.5} \]
\[ w^- = \min\{w\} = \left\{w_x, w_x \leq w_y, w_y, \right\} . \]

\[ p^+ = \max\{\tilde{p}\} = \begin{cases} p_x, p_x > p_y, \\ p_y, \end{cases} \]
\[ = \max\left\{ \min\{w_x, 1 - w_x\}, \min\{w_y, 1 - w_y\} \right\} . \tag{11.6} \]
\[ p^- = \min\{\tilde{p}\} = \begin{cases} p_x, p_x \leq p_y, \\ p_y, \end{cases} \]
\[ = \min\left\{ \min\{w_x, 1 - w_x\}, \min\{w_y, 1 - w_y\} \right\} . \]

In such invariant denotations, it is not difficult to write down the general recurrence formula for the half-rare 2-Kopula of the doublet $\mathcal{X}$, united combining both orders:

\[ \mathcal{K} \left(\tilde{p} \mid \mathcal{X} \mid X\right) = \mathcal{K} \left(\tilde{p} \mid \mathcal{X} \mid X\right) \left(\tilde{p} \mid \mathcal{X} \mid X\right) = \]
\[ = \begin{cases} \mathcal{K'} \left(p_{xy}(\tilde{p})\right), \\ \mathcal{K''} \left(p^+ - p_{xy}(\tilde{p})\right), \quad X = \{x^{+}\}, \end{cases} \quad \mathcal{X} = \{x^{+}\}, \tag{11.7} \]
\[ e = \begin{cases} \mathcal{K'} \left(1 - p_{xy}(\tilde{p})\right) - 1 + p^+, \\ \mathcal{K''} \left(1 - p^+ + p_{xy}(\tilde{p})\right) - p^+, \quad X = \emptyset \end{cases} , \]

where by Definition (11.3)

\[ p^+ = \max\left\{ \min\{w_x, 1 - w_x\}, \min\{w_y, 1 - w_y\} \right\} , \]
\[ p^- = \min\left\{ \min\{w_x, 1 - w_x\}, \min\{w_y, 1 - w_y\} \right\} . \]
The mutual set-phenomenon inversion of 2-Kopulas of half-rare \( \tilde{p} \) and free \( \tilde{w} \) marginal probabilities has the form:

\[
\mathcal{K}(\tilde{p}(c|X|X)) = \mathcal{K}(\tilde{w}(c|X|X)),
\]

\[
\mathcal{K}(\tilde{p}(c|X|X)(\tilde{p})) = \mathcal{K}(\tilde{w}(c|X|X(\tilde{w}))).
\]

For example, for \( X \subseteq \mathcal{X} = \{x, y\} \)

\[
\mathcal{K}(\tilde{p}(c|X|X)) = \mathcal{K}(\tilde{w}(c|X|X)),
\]

\[
\mathcal{K}(\tilde{p}(c|{x}|X)) = \mathcal{K}(\tilde{w}(c|{x}|X)),
\]

\[
\mathcal{K}(\tilde{p}(c|{y}|X)) = \mathcal{K}(\tilde{w}(c|{y}|X)),
\]

\[
\mathcal{K}(\tilde{p}(c|{x}{y}|X)) = \mathcal{K}(\tilde{w}(c|{x}{y}|X)).
\]

11.4 Recurrent properties of the \( \tilde{p} \)-ordering a half-rare s.e.

11.4.1 Recurrent properties of the \( \tilde{p} \)-ordering a half-rare doublets of events

Let us explain the role of \( \tilde{p} \)-ordering in the frame method using the example of constructing a 2-Kopula of the \( \tilde{p} \)-ordered half-rare events \( X = \{x, y\} \) with \( X \)-set of marginal probabilities of events \( \tilde{p} = \{p_x, p_y\} \), that is, \( 1/2 \geq p_x \geq p_y \):

\[
\mathcal{K}'(\tilde{p}(c|X|X)) = \begin{cases} p_{xy}, & X = \{x, y\}, \\
p_x - p_{xy}, & X = \{x\}, \\
p_y - p_{xy}, & X = \{y\}, \\
1 - p_x - p_y + p_{xy}, & X = \emptyset, \end{cases}
\]

(11.11)

where, when selected as a function parameter \( p_{xy} \) of the 1-Kopulas of inserted half-rare monoplates \( X' = \{s'\} = \{x \cap y\} \) and \( X'' = \{s''\} = \{x \cap z\} \), are equal, respectively:

\[
\mathcal{K}'(\tilde{p}(c|S|X')) = \begin{cases} p_{s't'}, & S = \{s', t'\}, \\
p_{st'} - p_{s't'}, & S = \{s'\}, \\
p_{st'} - p_{st'}, & S = \{s''\}, \\
1 - p_{s't'} - p_{st'} + p_{s't'}, & S = \emptyset, \end{cases}
\]

\[
\mathcal{K}'(\tilde{p}(c|S|X'')) = \begin{cases} p_{s't''}, & S = \{s', t''\}, \\
p_{s't'} - p_{st'}, & S = \{s''\}, \\
p_{st'} - p_{s't'}, & S = \{s''\}, \\
1 - p_{s't'} - p_{st'} + p_{s't'}, & S = \emptyset, \end{cases}
\]

(11.12)

under the assumption that inserted half-rare monoplates have “equally direct” \( \tilde{p} \)-orders:

\[
p_y \geq p_{s'} \geq p_{t'}, \quad p_z \geq p_{s''} \geq p_{t'}. \tag{11.13}
\]

However, nothing prevents the emergence of two more “opposite \( \tilde{p} \) orders” on the inserted half-rare monoplates:

\[
p_y \geq p_{s'} \geq p_{t'}, \quad p_z \geq p_{s''} \geq p_{t'}. \tag{11.14}
\]

\[
p_y \geq p_{s'} \geq p_{t'}, \quad p_z \geq p_{s''} \geq p_{t'}. \tag{11.15}
\]

except for the “equally inverse” \( \tilde{p} \)-order

\[
p_y \geq p_{s'} + p_{s''} \geq p_{t'} + p_{t''} = p_z. \tag{11.16}
\]

which can not be due to the consistency of the functional parameters, i.e., because

\[
p_y = p_{s'} + p_{s''} \geq p_t + p_t = p_z. \tag{11.17}
\]

11.4.2 Recurrent properties of \( \tilde{p} \)-ordering the half-rare triplets of events

Let us explain the role of \( \tilde{p} \)-ordering in the frame method using the example of constructing a 3-Kopula of the \( \tilde{p} \)-ordered half-rare events \( X = \{x, y, z\} \) with \( X \)-set of marginal probabilities of events \( \tilde{p} = \{p_x, p_y, p_z\} \), that is, \( 1/2 \geq p_x \geq p_y \geq p_z \):

\[
\mathcal{K}'(\tilde{p}(c|X|X)) = \begin{cases} \mathcal{K}'(p_{s', t'}), & X = \{x, y, z\}, \\
\mathcal{K}'(p_{s', 1-p_t}), & X = \{x, y\}, \\
\mathcal{K}'(1-p_{s'}, p_{t'}), & X = \{x, z\}, \\
\mathcal{K}'(p_{s'} - p_{s'}, 1-p_{t} - p_{z}), & X = \{y, z\}, \\
\mathcal{K}'(p_{s'} - p_{s'}, 1-p_{t} + p_{z}), & X = \{y\}, \\
\mathcal{K}'(1-p_{s'} + p_{s'}, p_{t'} - p_{z}), & X = \{z\}, \\
\mathcal{K}'(1-p_{s'} + p_{s'}, 1-p_{t} + p_{z} - p_{z}), & X = \emptyset, \end{cases}
\]

(11.18)

where, when selected as function parameters \( p_{s'}, p_{t'}, p_{st'} \) and \( p_{s't'} \) and despite the fact that \( p_{s'} = p_y - p_{s'} - p_{s't'} = p_z - p_t, \) the 2-Kopulas of inserted half-rare doublets \( X' = \{s', t'\} = \{x \cap \emptyset, x \cap z\} \) and
\( \mathcal{X}^* = \{s^*, t^*\} = \{x^\cap y, z^\cap z\} \) are equal, respectively:

\[
\mathcal{K}' \left( \hat{p}(e|S\neq \mathcal{X}') \right) = \begin{cases} 
    p_{s't'}, & S = \{s', t'\}, \\
    p_{s't'} - p_{s't'}, & S = \{s'\}, \\
    p_{st'} - p_{s't'}, & S = \{t'\}, \\
    1 - p_{s't'} - p_{s't'} + p_{s't'}, & S = \emptyset,
\end{cases}
\]

\[
\mathcal{K}' \left( \hat{p}(e|S\neq \mathcal{X}') \right) = \begin{cases} 
    p_{s't'}, & S = \{s', t'\}, \\
    p_{st'} - p_{s't'}, & S = \{s'\}, \\
    p_{s't'} - p_{s't'}, & S = \{t'\}, \\
    1 - p_{s't'} - p_{s't'} + p_{s't'}, & S = \emptyset,
\end{cases}
\]

under the assumption that the inserted half-rare doublets have “equally direct” \( \hat{p} \) orders:

\[
p_y \geq p_{s'}, \geq p_{t'}, \geq p_{s't'}, \geq p_{s's't'}.
\]

(11.19)

However, nothing prevents the emergence of two more “opposite \( \hat{p} \) orders” on the inserted half-rare doublets:

\[
p_y \geq p_{s'}, \geq p_{t'}, \geq p_{s't'}, \geq p_{s's't'}.
\]

(11.20)

except for the “equally inverse” \( \hat{p} \)-order

\[
p_y \geq p_{s'}, \geq p_{t'}, \geq p_{s't'}, \geq p_{s's't'}.
\]

(11.21)

which can not be due to the consistency of the functional parameters, i.e., because

\[
p_y = p_{s'} + p_{s'} \geq p_{t'} + p_{t'} = p_{s't'}.
\]

(11.22)

11.4.3 Extending the frame method to \( \hat{p} \)-non-ordered half-rare s.e.'s

Above we outlined the frame method for constructing \( N \)-Kopulas of \( \hat{p} \)-ordered half-rare \( N \)-s.e.'s. It remains to extend it to construct \( N \)-Kopulas of \( \hat{p} \)-disordered half-rare \( N \)-s.e.'s using the following technique, based on the obvious invariance property of permutations of events in s.e.: “as events from some s.e. do not order, the s.e. will not change”; and very useful in practical calculations.

We denote by

\[
\mathcal{X}^* = \{x_0^*, x_1^*, ..., x_{N-1}^*\},
\]

(11.23)

the \( \hat{p} \)-ordered half-rare \( N \)-s.e., which consists from the same events, that an “arbitrary” \( \hat{p} \)-non-ordered half-rare \( N \)-s.e.

\[
\mathcal{X} = \{x_0, x_1, ..., x_{N-1}\},
\]

(11.24)

i.e.,

\[
\mathcal{X}^* = \{x_0^*, x_1^*, ..., x_{N-1}^*\} = \{x_0, x_1, ..., x_{N-1}\} = \mathcal{X},
\]

(11.25)

but arranged in descending order of their probabilities. In other words, the \( \mathcal{X}^* \)-set of marginal probabilities

\[
\hat{p}^* = \{p_0^*, p_1^*, ..., p_{N-1}^*\},
\]

(11.26)

is such that

\[
1/2 \geq p_0^* \geq p_1^* \geq ... \geq p_{N-1}^*
\]

(11.27)

where

\[
p_0^* = \max\{p_x : x \in \mathcal{X}\},
\]

\[
p_1^* = \max\{p_x : x \in \mathcal{X} - \{x_0^*\}\},
\]

(11.28)

\[
... \]

\[
p_{n+1}^* = \max\{p_x : x \in \mathcal{X} - \{x_1^*, ..., x_n^*\}\},
\]

(11.29)

\[
... \]

\[
p_N^* = \max\{p_x : x \in \mathcal{X} - \{x_1^*, ..., x_{N-1}^*\}\}.
\]

Consequently, \( \mathcal{X}^* \)-set of marginal probabilities \( \hat{p}^* \) which consists of the same probabilities that \( \mathcal{X} \)-set of marginal probabilities \( \hat{p} \), i.e.,

\[
\hat{p}^* = \{p_0, p_1, ..., p_{N-1}\} = \hat{p}.
\]

(11.30)

but arranged in descending order.

Now, to construct the \( \mathcal{X} \)-Kopulas of the \( \hat{p} \)-disordered \( N \)-s.e \( \mathcal{X} \) by the frame method it is sufficient to construct this \( \mathcal{X} \)-Kopula of the \( \hat{p} \)-ordered \( \mathcal{X}^* \)-\( N \)-s.e \( \mathcal{X}^* = \mathcal{X} \) by this method, reasoning by (11.27) and (11.31) reasoning that

\[
N \times \mathcal{X}^* \times \mathcal{X} \times \mathcal{X}^* \mathcal{X} = \mathcal{X},
\]

(11.31)

reasoning by virtue of (11.27) and (11.31), that

\[
\mathcal{K}(\hat{p}) = \mathcal{K}(\hat{p}^*),
\]

(11.32)

i.e., for \( X \subseteq \mathcal{X} \)

\[
\mathcal{K}(\hat{p}(e|X^\neq \mathcal{X})) = \mathcal{K}(\hat{p}^*(e|X^\neq \mathcal{X}^*))
\]

(11.33)

where

\[
X^* = \{x^* : x \in X\} \subseteq \mathcal{X}^*
\]

(11.34)

are subsets of the \( \hat{p} \)-ordered \( (N-1) \)-s.e. \( \mathcal{X}^* \).

Note 21 (properties of functions of an unordered set of arguments). Equations (11.32) and (11.33) should not be regarded as a unique property of the Kopula invariance with respect to permutations of its arguments. This property is possessed by any Kopula, since it is a function of an unordered set of arguments. Therefore it is quite natural that the Kopula is invariant under permutations of the arguments, like any other such function. This property must be remembered only in practical
calculations, when we volence-nolens must introduce an arbitrary order on a disordered set in order to be able to perform calculations.

Consider the examples of Kopulas of arbitrary, i.e., \( \tilde{p} \)-disordered, s.e.'s

\[
\mathcal{X} = \{x_0, x_1, ..., x_{N-1}\}
\]

in the notation just introduced, assuming that we have available Kopulas of the \( \tilde{p} \)-ordered \( N \)-s.e's

\[
\mathcal{X}^* = \{x^*_0, x^*_1, ..., x^*_{N-1}\} = \mathcal{X}
\]

for \( N = 1, 2 \).

**Example 2** (invariant formula for the 2-Kopula of a half-rare triplet of events). Let \( \mathcal{X} = \{x_0, x_1\} \) be the \( \tilde{p} \)-non-ordered half-rare triplet of events. Then its 2-Kopula is calculated at each point \( \tilde{p}|(\mathcal{X}\{x_0, x_1\}) \in [0, 1]^{\mathcal{X}} \) by the following formulas:

\[
\mathcal{K}\left(\tilde{p}|(\mathcal{X}\{x_0, x_1\})\right) = \mathcal{K}\left(\tilde{p}|(\mathcal{X}^*\{x^*_0, x^*_1\})\right) = \begin{cases} \mathcal{K}'(p'_s), & X^* = \{x^*_0, x^*_1\}, \\ \mathcal{K}'(p_1 - p'_s), & X^* = \{x^*_1\}, \\ \mathcal{K}'(1 - p'_s - 1 + p_0), & X^* = \{x^*_0\}, \\ \mathcal{K}'(1 - p_1 + p'_s - p_0), & X^* = \emptyset, \end{cases}
\]

(11.35)

where

\[
\mathcal{X}' = \{s'\} = \{x^*_0 \cap x^*_1\}, \\
\mathcal{X}' = \{s''\} = \{(x^*_0)^c \cap x^*_1\}.
\]

(11.36)

**Example 3** (invariant formula for the 3-Kopula of a half-rare triplet of events). Let \( \mathcal{X} = \{x_0, x_1, x_2\} \) be the \( \tilde{p} \)-non-ordered half-rare triplet of events. Then its 3-Kopula is calculated at each point \( \tilde{p}|(\mathcal{X}\{x_0, x_1, x_2\}) \in [0, 1]^{\mathcal{X}} \) by the following formulas:

\[
\mathcal{K}\left(\tilde{p}|(\mathcal{X}\{x_0, x_1, x_2\})\right) = \mathcal{K}\left(\tilde{p}|(\mathcal{X}^*\{x^*_0, x^*_1, x^*_2\})\right) = \begin{cases} \mathcal{K}'(p'_s, p'_t), & X^* = \{x^*_0, x^*_1, x^*_2\}, \\ \mathcal{K}'(p_1 - p'_t), & X^* = \{x^*_1\}, \\ \mathcal{K}'(1 - p'_s - 1 + p_0), & X^* = \{x^*_0\}, \\ \mathcal{K}'(1 - p_1 + p'_s - p_0 - p_t), & X^* = \{x^*_2\}, \\ \mathcal{K}'(1 - p_1 + p'_s - p_0 - p_t), & X^* = \{x^*_2\}, \\ \mathcal{K}'(1 - p_1 + p'_s - p_0 - p_t - p_0), & X^* = \emptyset, \end{cases}
\]

(11.37)

where when selecting as function parameters \( p'_s, p'_t, p'_s t' \) and despite the fact that \( p_s = p_1 - p'_s, p_t = p_2 - p'_t, \) 2-Kopulas of the inserted half-rare doublets \( \mathcal{X}' = \{s', t'\} = \{x^*_0 \cap x^*_1, x^*_0 \cap x^*_2\} \) and \( \mathcal{X}' = \{s''\} = \{(x^*_0)^c \cap x^*_1, (x^*_0)^c \cap x^*_2\} \) are equal respectively:

\[
\mathcal{K}'\left(\tilde{p}|(\mathcal{X}'\{s', t'\})\right) = \begin{cases} \rho s', & S = \{s', t'\}, \\ \rho s' - p s' t', & S = \{s'\}, \\ \rho t - p s' t', & S = \{t'\}, \\ 1 - p s' - p t + p s' t', & S = \emptyset, \end{cases}
\]

\[
\mathcal{K}'\left(\tilde{p}|(\mathcal{X}'\{s'', t''\})\right) = \begin{cases} \rho s', & S = \{s', t''\}, \\ \rho s' - p s' t'', & S = \{s''\}, \\ \rho t' - p s' t'', & S = \{t''\}, \\ 1 - p s' - p t + p s' t'', & S = \emptyset. \end{cases}
\]

(11.38)

**11.5 Geometric interpretation of set-phenomenon renumberings**

For a subset of events \( V \subseteq \mathcal{X} \) \( V \)-phenomenon renumbering of the terrace events, generated by \( (N - 1) \)-s.h.r.e. \( \mathcal{X} \), is based on the replacement of events from the subset \( V^c = \mathcal{X} - V \) by their complements:

\[
\mathcal{X}^{(c)}(V) = V + (V^c)^{(c)} = \{x, x \in V\} + \{x^c, x \in V^c\},
\]

(11.39)

from which the mutually inverse set-phenomenon renumbering formulas follow:

\[
\text{ter}(X^{(c)}(V) \cap \mathcal{X}^{(c)}(V)) = \text{ter}((V\Delta X)^c \cap \mathcal{X}^{(c)}(V)),
\]

(11.40)

where

\[
\text{ter}(X^c \cap \mathcal{X}) = \text{ter}((V\Delta X)^c \cap \mathcal{X}^{(c)}(V)),
\]

for \( V \subseteq \mathcal{X} \) and \( X \subseteq \mathcal{X} \).

Therefore, the \( V \)-phenomenon renumbering of the terrace events, generated by \( (N - 1) \)-s.h.r.e. \( \mathcal{X} \), by the formulas (11.40) is geometrically interpreted on the \( (N - 1) \)-dimensional Venn diagram of this s.e. as a reflection of the \( \mathcal{X} \)-hypercube relative to those hyperplanes that are orthogonal to the \( x \)-axes numbered by the events \( x \in V^c \subseteq \mathcal{X} \) (see Fig. 21 for the doublet of events).

**11.6 Projection of the \( 2^N \)-simplex on the \( \mathcal{X} \)-hypercube**

Take an arbitrary \( (N - 1) \)-s.e. \( \mathcal{X} \subseteq \mathcal{A} \) with e.p.d. of the 1st kind, which, as is known [1], is defined as the \( 2^\mathcal{X} \)-set of probabilities of terrace events of the 1st kind

\[
\{p(X/\mathcal{X}), X \subseteq \mathcal{X}\}.
\]

(11.41)

Look at the \( \mathcal{X} \)-set (11.41) as a \( 2^\mathcal{X} \)-hyperpoint from a half-rare \( 2^\mathcal{X} \)-vertex simplex

\[
S_{2\mathcal{X}} = \left\{ \left\{ p(X/\mathcal{X}), X \subseteq \mathcal{X} \right\} : p(X/\mathcal{X}) \geq 0, \sum_{X \subseteq \mathcal{X}} p(X/\mathcal{X}) = 1 \right\}
\]

(11.42)
to each vertex of which the degenerate e.p.d. corresponds. In this e.p.d., as is known, only one of the 1st kind of probability, equal to one, is different from zero. Number the vertex $2^X \otimes S_{2^X}$ of the simplex $S_{2^X}$ by the subset $X \subseteq \mathfrak{X}$. The degenerate e.p.d. of the 1st kind with $p(X/\mathfrak{X})=1$ corresponds to this vertex. And associate the vertex $\mathfrak{X} \otimes \mathfrak{X}$ with the hypercube $[0,1] \otimes [0,1] \otimes \mathfrak{X}$, numbered by the $\mathfrak{X}$-set:

$$\{\mathfrak{X} \otimes \mathfrak{X}(x), x \in \mathfrak{X}\}$$

where

$$p_x = \sum_{X \subseteq \mathfrak{X}} p(X/\mathfrak{X}) \mathfrak{X} \otimes \mathfrak{X}(x) = \sum_{x \in X \subseteq \mathfrak{X}} p(X/\mathfrak{X})$$

is a convex combination of hypercube vertices, which, as known [1], is interpreted as the probability of event $x \in \mathfrak{X}$.

With projection (11.45) vertices of the $2^X$-simplex maps to vertices of the $\mathfrak{X}$-hypercube, and edges map to its edges or diagonals (see [9], [10] and Fig. 23).

Example 4 (projections of vertices of a $2^X$-simplex). For example, the vertex of the $2^X$-simplex enumerated by the subset $X_0 \subseteq \mathfrak{X}$ corresponds to the degenerate e.p.d. of the 1st kind with probabilities

$$p(X/\mathfrak{X}) = \begin{cases} 1, & X = X_0, \\ 0, & X_0 \neq X \subseteq \mathfrak{X}. \end{cases}$$

From (11.46) you obtain that

$$p_x = \mathfrak{X}_{X_0 \neq \mathfrak{X}}(x) = \begin{cases} 1, & x \in X_0, \\ 0, & x \in \mathfrak{X} - X_0. \end{cases}$$

Therefore, by (11.43)

$$\{p_x, x \in \mathfrak{X}\} = \{\mathfrak{X}_{X_0 \neq \mathfrak{X}}(x), x \in \mathfrak{X}\}$$

is a vertex of the $\mathfrak{X}$-hypercube.

In particular, the $\emptyset$-vertex of the $2^X$-simplex, i.e., the vertex numbered by the subset $X_0 = \emptyset$, is projected into the $\emptyset$-set $\{0, ..., 0\}$, consisting of the zero probabilities of marginal events, in other words, projected into the $\emptyset$-vertex of the $\mathfrak{X}$-hypercube, i.e., to the vertex located at the beginning coordinates:

$$\{0, ..., 0\} \sim p(X/\mathfrak{X}) = \begin{cases} 1, & X = \emptyset, \\ 0, & \emptyset \neq X \subseteq \mathfrak{X}; \end{cases}$$

and the $\emptyset$-vertex of the $2^X$-simplex, i.e., the vertex enumerated by the subset $X_0 = \emptyset$ is projected into the $\emptyset$-set $\{1, ..., 1\}$, consisting of the unit probabilities of marginal events, in other words, projected into the $\emptyset$-vertex of the $\mathfrak{X}$-hypercube, i.e., to the vertex opposite to the origin:

$$\{1, ..., 1\} \sim p(X/\mathfrak{X}) = \begin{cases} 1, & X = \mathfrak{X}, \\ 0, & \mathfrak{X} \neq X \subseteq \mathfrak{X}. \end{cases}$$

In general, due to the linearity of the projection (11.45), the set of such points of the $2^X$-simplex that project into the same point of the $\mathfrak{X}$-hypercube is convex and forms a sub-simplex of smaller dimension.
11.7 Half-rare events on Venn \((N-1)\)-diagram

We will figure out how a Venn \((N-1)\)-diagram of an arbitrary \((N-1)\)-s.e. is constructed on the basis of the projection (11.45), in which the role of the space of universal elementary events \(\Omega\) is played by the unit \((N-1)\)-dimensional hypercube. Such a Venn \((N-1)\)-diagram puts terraced hypercubes generated by dividing a unit hypercube in half orthogonal to each of the \(N\) axes into a one-to-one correspondence with the terraced events generated by the given \((N-1)\)-s.e.

Take first \((N-1)\)-s.h-r.e. \(x\) and represent its Venn \((N-1)\)-diagram\(^{14}\) On which \(\Omega\) is represented by a unit \((N-1)\)-dimensional hypercube that serves as ordered\(^{15}\) image of the \(x\)-hypercube

\[
[0,1]^{\otimes x} = \bigotimes_{x \in X} [0,1]^x, \tag{11.49}
\]

broken by hyperplanes orthogonal to \(x\)-axis and intersecting them at points \(1/2\) into \(2^N\) \(x\)-teraced hypercubes for \(X \subseteq x\)

\[
[0,1]^{\otimes \text{ter}(X/x)} = \bigotimes_{x \in X} [0,1/2]^x \otimes \bigotimes_{x \in x - X} (1/2,1]^x, \tag{11.50}
\]

where each marginal half-rare event \(x \in x\) is represented as a \(x\)-half of \(x\)-hypercube containing the origin:

\[
[0,1/2]^x \otimes [0,1]^{\otimes (x - \{x\})}, \tag{11.51}
\]

its complement \(x^c = \Omega - x\) is represented in the form of another \(x\)-half of \(x\)-hypercube that does not contain the origin:

\[
(1/2,1]^x \otimes [0,1]^{\otimes (x - \{x\})}, \tag{11.52}
\]

and the \(x\)-teraced hypercube \(\text{ter}(X/x)\) — as a \(x\)-teraced hypercube (11.50):

\[
\text{ter}(X/x) \sim [0,1]^{\otimes \text{ter}(X/x)}. \tag{11.53}
\]

The formula (11.53) once again points to a one-to-one correspondence between the \(2^N\)-space of terraced hypercubes (11.50) from the Venn \((N-1)\)-diagram of \((N-1)\)-s.h-r.e. \(x\) and \(2^N\)-totality of terraced events, generated by \(x\).

If the correspondence between the terraced hypercubes and the terraced events looks natural, then for the sets of half-rare events \(x\) the correspondence between the terraced hypercubes and the numbering of the vertices of the \(2^x\)-simplex projected into the corresponding vertices of the \(x\)-hypercube under the projection (11.45) is defined by the operation of the complement and requires a special

\[^{14}\text{See the Venn 2-diagram doublet of half-rare events in Fig. 23.}\]

\[^{15}\text{The role of the order of events in s.e. when working with their images in } \mathbb{R}^N \text{ is discussed in [7].}\]
Lemma 6 (on a set-phenomenon \(X\)-spectrum of normalized function). In order that the family of functions \(\{\theta_X : X \subseteq \mathfrak{X}\}\) \(\Psi_X\) is a set-phenomenon \(\mathfrak{X}\)-spectrum of some function normalized on the \(\mathfrak{X}\)-hypercube, it is necessary and sufficient that

\[
\sum_{X \subseteq \mathfrak{X}} \theta_X (\mathfrak{w}) = 1 \quad (11.58)
\]

for all \(\mathfrak{w} \in [0, 1]^\otimes \mathfrak{X}\).

Proof. 1) If the family \(\{\theta_X : X \subseteq \mathfrak{X}\}\) is a set-phenomenon \(\mathfrak{X}\)-spectrum of some normalized function, then by Definition 11 the equality (11.58) is satisfied. 2) Let now the equality (11.58) is satisfied. Construct the function \(\psi\) on the \(\mathfrak{X}\)-hypercube by the following way

\[
\psi (\mathfrak{w}) = \left\{ \begin{array}{ll}
\theta_0 (\mathfrak{w}^{\{\mathfrak{w}\}}) , & \mathfrak{w} \in [0, 1/2]^\otimes \mathfrak{X} , \\
\theta_X (\mathfrak{w}^{\{\mathfrak{w}\}}) , & \mathfrak{w} \in \mathfrak{ter}^0 (\mathfrak{X} / \mathfrak{X}) , \\
& \ldots , \\
\theta_X (\mathfrak{w}^{\{\mathfrak{w}\}}) , & \mathfrak{w} \in [2/1, 1]^\otimes \mathfrak{X} .
\end{array} \right.
\]

and show that the \(\psi\) is normalized on the \(\mathfrak{X}\)-hypercube. Indeed, noting that for an arbitrary \(X \subseteq \mathfrak{X}\) the equality \(\psi (\mathfrak{w}) = \theta_X (\mathfrak{w}^{\{\mathfrak{w}\}} (X / \mathfrak{X})\) is equivalent to the equality \(\psi (\mathfrak{w}^{\{\mathfrak{w}\}} (X / \mathfrak{X})\) = \(\theta_X (\mathfrak{w})\), we obtain the required:

\[
\sum_{X \subseteq \mathfrak{X}} \psi (\mathfrak{w}^{\{\mathfrak{w}\}} (X / \mathfrak{X})) = \sum_{X \subseteq \mathfrak{X}} \theta_X (\mathfrak{w}) = 1.
\]

Lemma 7 (on a set-phenomenon \(X\)-spectrum of the \(1\)-function). In order that the family of functions \(\{\theta_X : X \subseteq \mathfrak{X}\}\) \(\Psi_X\) is a set-phenomenon \(\mathfrak{X}\)-spectrum of some \(1\)-function on the \(\mathfrak{X}\)-hypercube, it is necessary and sufficient that for each \(x \in \mathfrak{X}\)

\[
\sum_{x \in \mathfrak{X} \subseteq \mathfrak{X}} \theta_X (\mathfrak{w}) = w_x \quad (11.59)
\]
for all \( \tilde{w} \in [0,1]^{\otimes X} \).

**Proof.** 1) If the family (1.3) is a set-phenomenon \( \mathcal{X} \)-spectrum of some 1-function, then partial sums of functions from the family at \( x \in X \subseteq \mathcal{X} \) \( w_x \):

\[
\sum_{x \in X \subseteq \mathcal{X}} \psi_X(\tilde{w}) = w_x
\]

for each \( \tilde{w} \in [0,1]^{\otimes X} \) by Definition 4. 2) Let now the equalities (1.4) are satisfied. Let's construct the function \( \psi \) on \( \mathcal{X} \)-hypercube by the following way

\[
\psi(\tilde{w}) = \begin{cases} 
\theta_{\emptyset}(\tilde{w}^{c,\emptyset}), & \tilde{w} \in [0,1/2]^{\otimes X}, \\
\ldots, & \\
\theta_X(\tilde{w}^{c,\emptyset^X}), & \tilde{w} \in \text{ter}^{X}(X/\emptyset), \\
\ldots, & \\
\theta_{\emptyset}(\tilde{w}), & \tilde{w} \in [1/2,1]^{\otimes X}
\end{cases}
\]

and show that \( \psi \) is a 1-function on the \( \mathcal{X} \)-hypercube. Indeed, noting that for an arbitrary \( X \subseteq \mathcal{X} \) the equality \( \psi(\tilde{w}) = \theta_X(\tilde{w}^{c,\emptyset^X}) \) is equivalent to the equality \( \psi(\tilde{w}^{c,\emptyset^X}) = \theta_X(\tilde{w}) \), we obtain the required:

\[
\sum_{x \in X \subseteq \mathcal{X}} \psi(\tilde{w}^{c,\emptyset^X}) = \sum_{x \in X \subseteq \mathcal{X}} \theta_X(\tilde{w}) = w_x.
\]

Figure 24: The projection of \( 2^2 \)-vertices simplex on a square, on which the scheme is superimposed, illustrating a connection of two permutations of events in a half-rare doublet with the \( 2^2 \) set-phenomena.

Figure 25: These are not geometrical projections of \( 2^3 \)-vertices simplex on a cube, but two conditional schemes of these projections, which illustrate a connection of six permutations of events in a half-rare triplet of events with its \( 2^3 \) set-phenomena. The conditional scheme of the projection on the upper half of the cube is shown at the top, on the lower half — at the bottom. In the Venn diagram of half-rare events: \( x \) is the left, \( y \) is the right, and \( z \) is the lower half of the cube.

12 Remaining behind the scenes

In the text and, in particular, in the Appendix, the value of the \( \rho \)-ordering condition of the set of events is specified, which complicates the computational implementation of the above algorithms in the frame method of constructing Kopulas as set functions of the set of marginal probabilities. The reason for this complication lies in the properties of the set-functions, i.e., functions of a set that differ from the properties of arbitrary functions of several variables. The point is that the set-function
of the set of marginal probabilities is necessarily a symmetric function of the marginal probability vector (Cartesian representation of Kopula, see Prolegomenon 9), to determine which it is sufficient to specify its values only on those vectors whose components are ordered, for example, in descending order, so that the remaining values can be determined by the appropriate permutations of the arguments. For example, the Cartesian representation of an \( N \)-Kopula in \( \mathbb{R}^N \) is sufficient to define on the \( 1/N! \) part of the unit \( N \)-hypercube so that this representation becomes definite on the whole hypercube by continuing permutations of arguments.

Although this task is purely technical, but its solution opens the way for the application of the proposed Kopula (eventological copula) theory to the construction of the eventological theory of ordinary copulas that determine the joint distribution of a given set of marginal distributions. The author encountered this when developing the program code, which calculated all the illustrations for the Kopula examples. The problem is solved programmatically, but requires a detailed description of this solution (see Fig. 24 and 25), which, of course, together with the eventological theory of copula deserves a separate publication.

In conclusion, I can not resist the temptation to quote the formulation of the tenth Prolegomenon of the Kopula theory, which reveals the content of these my next publications.

Prolegomenon 10 (Cartesian representation of the \( N \)-Kopula defines \( 2^N \) classical copulas of \( N \) marginal uniform distributions on \([0,1])\).

13 On the inevitable development of language

This first work on the theory of the eventological copula is over at the end of July 2015. It sums up the work on the eventological theory of probabilities, raising the theory of Kopula to its apex. The work is written in a mathematical language, in which the state of the eventological theory was reflected precisely at the time when the author unexpectedly, but by the way, got a brilliant example of two statisticians from sociology and ecology, who immediately forced him to postpone polishing of the Kopula theory for almost a year in order to immediately immerse themselves in the destructive creation of a new unifying eventological theory of experience and chance by the agonizing fusion of two dual theories: the eventological theory of believabilities and the eventological theory of probabilities.

Because of this, the mathematical language of this work is just a pretension to the eventological probability theory, which does not yet know that there is a very close twin that exists — the eventological theory of believabilities. Therefore, in the terminology of this work, those crucial changes in the basic concepts and notations that were invented to construct a unifying eventological theory did not find any worthy reflection. Of course, the new unifying theory suggests the development of the original mathematical language of dual Kopulas, one of which hosts the eventological probability theory, and the other — in the eventological believability theory.

★ The English version of this article was published on November 12, 2017. Therefore, my later works [11, 12, 13, 14], which expand the themes of this work, are added to the list of references. Due to the arXiv.org limitation on the volume of publication, the work is reduced by removing some illustrations. In full, the work is available at: http://www.academia.edu/35218637/.

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