EQUILIBRIUM MEASURES OF MEROMORPHIC SELF-MAPS ON NON-KÄHLER MANIFOLDS

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Abstract. Let $X$ be a compact complex non-Kähler manifold and $f$ a dominant meromorphic self-map of $X$. Examples of such maps are self-maps of Hopf manifolds, Calabi-Eckmann manifolds, non-tori nilmanifolds and their blowups. We prove that if $f$ has a dominant topological degree, then $f$ possesses an equilibrium measure $\mu$ satisfying well-known properties as in the Kähler case. The key ingredients are the notion of weakly d.s.h. functions substituting d.s.h. functions in the Kähler case and the use of suitable test functions in Sobolev spaces. A large enough class of holomorphic self-maps with dominant topological degree on Hopf manifolds is also given.

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1. Introduction

Let $X$ be a compact complex manifold of dimension $k$. Let $f$ be a dominant meromorphic self-map of $X$. Let $\omega$ be a strictly positive Hermitian $(1,1)$-form on $X$. For $0 \leq q \leq k$, put

$$d_q(f) := \limsup_{n \to \infty} \left( \int_X (f^n)^* \omega^q \wedge \omega^{k-q} \right)^{1/n}.$$ 

We will write $d_q$ for $d_q(f)$ if no confusion arises. We can see easily that $d_q$ is independent of the choice of $\omega$. The number $d_0$ is always 1 and $d_k$ is the topological degree of $f$. When $f$ is holomorphic, $d_q$ is finite because the differential of $f$ is of $L^\infty$-norm uniformly bounded on $X$. We call $d_q$ the $q^{th}$ dynamical degree of $f$ for $0 \leq q \leq k$.

When $X$ is Kähler, the numbers $d_q$ are crucial finite bi-meromorphic invariants of $f$: see [13, 17, 11]. We don’t know whether $d_q$ for $1 \leq q \leq k-1$ is finite for general $X$. In what follows, we will study the dynamics of $f$ with $d_k(f) > d_{k-1}(f)$. In the Kähler case, such a map is said to have a dominant topological degree and its dynamics has been thoroughly investigated, see [18] and references therein for information. We emphasize that in our context, it is not clear whether the assumption $d_k > d_{k-1}$ implies $d_k > d_q$ for $1 \leq q \leq k-1$ as in the Kähler case.

A quasi-p.s.h. function on $X$ is a function from $X$ to $[-\infty, \infty)$ which is locally the sum of a plurisubharmonic function and a smooth one. For a given continuous $(1,1)$-form $\eta$, denote by $\operatorname{PSH}_0(\eta)$ the set of quasi-p.s.h. functions $\varphi$ such that $\dd \varphi + \eta \geq 0$ and $\sup_X \varphi = 0$. Equip $\operatorname{PSH}_0(\eta)$ with the induced distance from $L^1(X)$ by using the natural inclusion $\operatorname{PSH}_0(\eta) \subset L^1(X)$.

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Recall from [15] that a complex measure $\mu$ on $X$ is said to be PC if every quasi-p.s.h. function is $\mu$-integrable and for every sequence $(\varphi_n)_{n \in \mathbb{N}}$ of quasi-p.s.h. functions converging to $\varphi$ in $L^1$ such that $dd^c \varphi_n + \eta \geq 0$ for some smooth form $\eta$ independent of $n$, we have $\langle \mu, \varphi_n \rangle \to \langle \mu, \varphi \rangle$.

A pluripolar set in $X$ is a subset of $X$ contained in $\{ \varphi = -\infty \}$ for some quasi-p.s.h. function $\varphi$. By [39], every locally pluripolar set in $X$ is pluripolar. This result implies in particular that there exist abundantly singular quasi-p.s.h. functions on $X$. Observe that every PC measure has no mass on pluripolar sets. Here is our first main result.

**Theorem 1.1.** Let $X$ be a compact complex manifold of dimension $k$ and $f$ a dominant meromorphic self-map of $X$ with $d_k > d_{k-1}$. Let $\nu$ be a complex measure with $L^{k+1}$ density on $X$ so that $\nu(X) = 1$. Then $d_k^n (f^n)^* \nu$ converges weakly to a PC probability measure $\mu_f$ of entropy $\geq \log d_k$ independent of $\nu$ as $n \to \infty$ such that $d_k^{-1} f^* \mu = \mu$ and if $f$ is holomorphic then for every Hermitian metric $\omega$ on $X$, $\mu_f$ is Hölder continuous on $\text{PSh}_0(\omega)$.

The above measure $\mu_f$ is called the equilibrium measure of $f$. We emphasize that unlike the Kähler case, it is not clear to us whether the entropy of $\mu_f$ is equal to $\log d_k$. The Hölder continuity of $\mu_f$ on $\text{PSh}_0(\omega)$ for $f$ holomorphic implies that $\mu_f$ is moderate in the sense that there exist constants $\epsilon, M > 0$ such that for every $\varphi \in \text{PSh}_0(\omega)$, we have

$$\int_X e^{-\epsilon \varphi} d\mu_f \leq M,$$

see [8] for a proof. A large class of holomorphic endomorphisms of Hopf manifolds having dominant topological degree is given in Lemma 4.1 in Section 4 and the comment following it.

The existence of $\mu_f$ is proved by Fornaess-Sibony and Russakovskii-Shiffman [36, 22, 21, 35] for $X = \mathbb{P}^k$, Guedj [27] for $X$ projective and he also shows that quasi-p.s.h. functions are $\mu_f$-integrable, see also [10] for the case of polynomial-like maps. The stronger fact that for $X$ Kähler, $\mu_f$ is PC is proved by Dinh-Sibony [15] by using a key property that the space of d.s.h. functions (differences of two quasi-p.s.h. functions) is preserved by meromorphic maps. However, it seems that this property no longer holds in non-Kähler case. We refer to [18, 4, 31, 23] and references therein for more information on the Kähler case.

In order to prove that $\mu_f$ is PC in Theorem 1.1, we introduce a new class of functions called weakly d.s.h. functions which, to some extent, replace the role of d.s.h. functions (differences of two quasi-p.s.h. functions) in Kähler case. These functions enjoy a compactness property similar to that of d.s.h. functions and the pull-back of d.s.h. functions by meromorphic maps are weakly d.s.h.. We also obtain the exponentially mixing property of $\mu_f$ generalizing similar results in the Kähler case by Dinh-Sibony in [14, The. 1.1] and [15, The. 1.3].

**Theorem 1.2.** Let $X, f, d_{k-1}, d_k, \mu_f$ be as in Theorem 1.1. Then $\mu_f$ is exponentially mixing in the sense that for every constant $\epsilon > 0$ with $d_k > d_{k-1} + \epsilon$ and $0 < \alpha \leq 1$, there exists a constant $c_{\epsilon, \alpha}$ such that

$$\left| \langle \mu_f, (\psi \circ f^n) \varphi \rangle - \langle \mu_f, \psi \rangle \langle \mu, \varphi \rangle \right| \leq c_{\epsilon, \alpha} ||\psi||_\infty \langle \mu, \varphi \rangle \leq c_{\alpha} d_k^{-\alpha/2} (d_{k-1} + \epsilon)^{\alpha/2} d_k^{-\alpha/2}$$

for every $n \geq 0$, every $\psi \in L^\infty(X)$ and every Hölder continuous function $\varphi$ of order $\alpha$. In particular, $\mu_f$ is K-mixing.
If a real-valued Hölder continuous function $\varphi$ is not a coboundary, i.e., there doesn’t exist $\psi \in L^2(X)$ with $\varphi = \psi \circ f - \psi$, and satisfies $\langle \mu, \varphi \rangle = 0$, then $\mu_f$ satisfies the central limit theorem, that means there is a constant $\sigma > 0$ such that for every interval $I \subset \mathbb{R}$, we have

$$\lim_{n \to \infty} \mu_f \left\{ \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \varphi \circ f^j \in I \right\} = \frac{1}{\sqrt{2\pi\sigma}} \int_I e^{-x^2/(2\sigma^2)} \, dx.$$  

In the above statement, only the decay of correlation requires new arguments. The K-mixing property and the central limit theorem are deduced by using similar arguments from [14, 18]. Due to the same reason with the pull-back of d.s.h. functions presented above, the arguments in [15] couldn’t be applied directly to obtain the expected decay of correlation. Our approach is based on ideas from [14]: use a suitable class of functions in the Sobolev space $W^{1,2}$ as test functions.

The above results for meromorphic maps still hold for meromorphic correspondences. But in order to keep the presentation as simple as possible, we don’t elaborate it here. In the next section, we prove Theorem 1.1. A proof of Theorem 1.2 is given in Section 3. Examples of dynamical systems on non-Kähler manifolds are given in Section 4.

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2. Maps with Dominant topological degrees

In this section, we study meromorphic self-maps of a $k$-dimensional compact complex manifold having a dominant topological degree. For every analytic subset $V$ of a complex manifold $X$, we denote by $[V], \{V\}$ the current of integration along $V$ and its cohomology class respectively. The following result is crucial for us later.

**Lemma 2.1.** Let $X$ be a $k$-dimensional compact Kähler manifold and $W \subset W'$ relatively compact open subsets of $X$. Let $R$ be a $d$-exact current of order 0 on $W'$. Assume that there exists a current $Q$ of order 0 for which $R = dQ$ on $W'$. Then there exists a form $U_R$ on $W$ with $L^{1+1/k}(W)$-coefficients such that

$$R = dd^c U_R$$

on $W$ and

$$\|U_R\|_{L^{1+1/k}(W)} \leq c\|R\|_{W'},$$

for some constant $c$ independent of $R$. Moreover if $R_n = dQ_n$ is a sequence of $d$-exact currents of order 0 such that $R_n, Q_n$ are of uniformly bounded mass on $W'$ and $Q_n \to Q$ on $W'$, then $U_{R_n} \to U_R$ in $L^{1+1/k}(W)$.

Note that if $W'$ is a local chart of $X$ biholomorphic to a star-shaped open subset of $\mathbb{C}^k$, then by the homotopy formula, there always exists a current $Q$ of order 0 for which $R = dQ$ on $W'$, see for example [6, Th. 2.24]. In this context, for every sequence of $d$-exact currents $S_n$ of uniformly bounded mass converging to $S$ on $W'$ we can find currents $Q_n$ of uniformly bounded mass on any fixed open set $W'' \subset W'$ with $R_n = dQ_n$ and $Q_n \to Q$ on $W''$. We will use Lemma 2.1 for the case where $X = \mathbb{P}^k$ and $R$ is an $(1,1)$-current.
Proof. We follow the usual construction of $dd^c$-kernel as in [9, 19, 12, 8]. The new problem is that $R$ is only defined on an open subset of $X$.

Let $\sigma : \hat{X} \times \hat{X} \to X \times X$ be the blowup of $X \times X$ along the diagonal $\Delta$. Put $\hat{\Delta} := \sigma^{-1}(\Delta)$. By [38], there exists a closed $(k-1, k-1)$-form $\hat{\eta}$ on $\hat{X} \times \hat{X}$ for which $\sigma_*(\hat{\eta} \wedge [\hat{\Delta}]) = [\Delta]$ and $\hat{\eta} \wedge [\hat{\Delta}]$ belongs the cohomology class $\sigma^*\{\Delta\}$, where $\{\Delta\}$ denotes the cohomology class of $\Delta$. By Künneth’s theorem, $\{\Delta\}$ can be represented by a sum of forms $\pi_1^*\phi_1 \wedge \pi_2^*\phi_2$ where $\phi_1, \phi_2$ are closed forms on $X$. Let $\gamma$ be such a representative of $\{\Delta\}$.

Write $[\hat{\Delta}] = \hat{\beta} + dd^c\hat{u}$, where $u$ is a quasi-p.s.h. function on $\hat{X} \times \hat{X}$. This equation implies that $u$ is smooth outside $\hat{\Delta}$ and $u - \log \text{dist}(\cdot, \hat{\Delta})$ is a bounded function near $\hat{\Delta}$. Since $\hat{\beta} \wedge \hat{\eta}$ is cohomologous to $\sigma^*\gamma$, there is a real smooth form $\hat{\beta}'$ on $\hat{X} \times \hat{X}$ such that

$$
(2.1) \quad dd^c\hat{\beta}' = \hat{\beta} \wedge \hat{\eta} - \sigma^*\gamma.
$$

Denote by $\pi_1, \pi_2$ the natural projections from $X \times X$ to the first and second components respectively. Let $\Pi_j := \pi_j \circ \sigma$ for $j = 1, 2$. Let $\chi$ be a smooth function on $X^2$ such that $\chi \equiv 1$ on $W' \cap W''$. Put $\chi_{\sigma} := \chi \circ \sigma$.

Let $\tau$ be a smooth function on $X$ such that $\text{supp}\tau \subset W'$ and $\tau \equiv 1$ on $W$. Since $d(\tau Q) = d\tau \wedge Q + \tau R$ is of order 0 and equal to $R$ on $W$, without loss of generality, we can assume that $R = d(\tau Q)$ which is supported on $W'$. This is the only place where we need the hypothesis on $Q$.

By using suitable local coordinates near $\hat{\Delta}$, we can check that $\hat{u}\Pi^*_1R$ is a well-defined current and for every sequence $R_n$ converging weakly to $R$ we have $\hat{u}\Pi^*_1R_n \to \hat{u}\Pi^*_1R$, see [38, Le. 3.2] for similar computations. This allows us to define

$$
U_R := (\Pi_1)_*\left[\chi_{\sigma}(\hat{u} \hat{\eta} + \hat{\beta}') \wedge \Pi_2^*(R)\right].
$$

Suppose for the moment $R$ is smooth. Since $\chi \equiv 1$ on $W' \cap W''$, the partial derivatives of $\chi$ are 0 on $W' \cap W''$. Using this and the fact that $\text{supp}\tau \subset W'$, on $W'$ we have

$$
\begin{align*}
dd^cU_R &= (\Pi_1)_*\left[\chi_{\sigma} dd^c\hat{u} \wedge \hat{\eta} \wedge \Pi_2^*(R)\right] + (\Pi_1)_*\left[\chi_{\sigma} dd^c\hat{\beta}' \wedge \Pi_2^*(R)\right] \\
&= (\Pi_1)_*\left[\chi_{\sigma}(\hat{\beta} \wedge \hat{\eta} - \hat{\beta}' \wedge \hat{\eta}) \wedge \Pi_2^*(R)\right] \\
&\quad + (\Pi_1)_*\left[\chi_{\sigma}(\hat{\beta} \wedge \hat{\eta} - \sigma^*\gamma) \wedge \Pi_2^*(R)\right] \\
&= (\Pi_1)_*\left[\chi_{\sigma}(\hat{\beta} \wedge \hat{\eta} - \sigma^*\gamma) \wedge \Pi_2^*(R)\right] \\
&= (\Pi_1)_*\left[\chi(\hat{\beta} \wedge \hat{\eta} - \sigma^*\gamma) \wedge \Pi_2^*(R)\right].
\end{align*}
$$

The first term in the right-hand side of the last equality is equal to $R$ on $W'$ because $\chi \equiv 1$ on $\Delta \cap (W' \times W'')$. The second term is zero on $W'$ by using Fubini’s theorem, the special form of $\gamma$ and the fact that $R$ is d-exact. Consequently, $dd^cU_R = R$ on $W'$ if $R$ is smooth. In general, one only needs to regularize $R$ by using de Rham’s regularisation theorem to obtain that $dd^cU_R = R$ on $W'$ for every $R$.

As above we can suppose that $R_n = d(\tau Q_n)$ and $R = d(\tau Q)$. By writing down explicitly the kernel defining $U_R$ locally on $X \times X$, we see that

$$
U_R(x_1) = \int_{\{x_2 \in X\}} K(x_1, x_2) \wedge R(x_2),
$$

where

$$
K(x_1, x_2) = \|x_1 - x_2\|^{-2k+2}K_1(x_1, x_2)
$$

for some $K_1(x_1, x_2)$.
for some bounded measurable form $K_1(x_1, x_2)$ on $W' \times W'$ which is smooth outside $\Delta$. Thus the coefficients of $U_R$ are in $L^{1+1/k}(W)$.

In order to prove the second desired assertion, we use classical techniques in [30, Th. 3.2.13, Th. 4.1.8]. For every small constant $\epsilon > 0$, let

$$K_\epsilon(x_1, x_2) := \max\{\|x_1 - x_2\|, \epsilon\}^{-2k+2}K_1(x_1, x_2)$$

which is a continuous form. As $\epsilon \to 0$, we have $K_\epsilon(\cdot, x_2) \to K(\cdot, x_2)$ in $L^{1+1/k}(W)$ uniformly in $x_2 \in W$. Thus as $n \to \infty$, 

$$\int_{\{x_2 \in X\}} (K_\epsilon(x_1, x_2) - K(x_1, x_2)) \land (R_n(x_2) - R(x_2)) \to 0$$

in $L^{1+1/k}(W)$ because the mass of $R_n$ is uniformly bounded. On the other hand,

$$\int_{\{x_2 \in X\}} K_\epsilon(x_1, x_2) \land (R_n(x_2) - R(x_2))$$

converges uniformly to 0 as $\epsilon$ fixed because $K_\epsilon$ is continuous. We deduce that $U_{R_n} \to U_R$ in $L^{1+1/k}(W)$. The proof is finished. \hfill \Box

Let $X$ be a complex manifold. A function from $X$ to $[\infty, \infty)$ is said to be quasi-p.s.h. function if it can be written locally the sum of a plurisubharmonic (p.s.h.) function and a smooth one. For every continuous $(1,1)$-form $\eta$, a quasi-p.s.h. function $\varphi$ is $\eta$-p.s.h. if $\ddc \varphi + \eta \geq 0$. By partition of unity, every quasi-p.s.h. function is $\eta$-p.s.h. for some smooth form $\eta$. For a given form $\eta$, denote by $\text{PSH}(\eta)$ the set of quasi-p.s.h. functions $\varphi$ for which $\ddc \varphi + \eta \geq 0$.

**Definition 2.2.** A locally integrable function $\varphi$ on $X$ is said to be weakly d.s.h. if $\ddc \varphi$ is a current of order $0$ on $X$. Let $\mathcal{W}$ be the complex vector space of all weakly d.s.h. functions on $X$.

Clearly, every quasi-p.s.h is weakly d.s.h. A subset of $X$ is a pluripolar set if it is contained in $\{\varphi = -\infty\}$ for some quasi-p.s.h. function $\varphi$. If $X$ is compact, every locally pluripolar set is pluripolar by [39]. In our proofs, we only use a particular case of this result that every proper analytic subset of a compact manifold $X$ is pluripolar, see Lemma 2.8 below.

Consider now $X$ is compact. Let $\mu_0$ be a smooth probability measure on $X$. We use this measure to define $L^p$ norms on $X$. For $\varphi \in \mathcal{W}$, put

$$\|\varphi\|_{\mathcal{W}} := \int_X \varphi d\mu_0 + \|\ddc \varphi\|_X,$$

where $\|\cdot\|_X$ is the mass of a current on $X$. We will write from now on $\|\cdot\|$ instead of $\|\cdot\|_X$ if no confusion arises. The function $\|\cdot\|_{\mathcal{W}}$ is a norm on $\mathcal{W}$ because if $\ddc \varphi = 0$ then $\varphi$ must be a constant. The norm $\|\cdot\|_{\mathcal{W}}$ is similar to the norm of the space of d.s.h. functions on the Kähler case introduced by Dinh-Sibony [15]. However, we don’t know if these two norms are equivalent in this case.

As in the Kähler case, we introduce the topology on $\mathcal{W}$ as follows: we say that $\varphi_n \in \mathcal{W}$ converges to $\varphi \in \mathcal{W}$ as $n \to \infty$ if $\varphi_n \to \varphi$ as currents and $\|\varphi_n\|_{\mathcal{W}}$ is uniformly bounded. We have the following crucial compactness result.
Lemma 2.3. Let $X$ be a compact complex manifold. There exists a constant $c$ so that for every weakly d.s.h. function $\varphi$ on $X$ with $\int_X \varphi d\mu_0 = 0$, we have
\begin{equation}
\|\varphi\|_{L^{1+1/k}(X)} \leq c\|dd^c \varphi\|_X.
\end{equation}
Consequently, given a positive constant $A$, the set $\mathcal{W}_0$ of weakly quasi-p.s.h. functions $\varphi$ with $\int_X \varphi d\mu_0 = 0$ such that $\|dd^c \varphi\| \leq A$ is compact in $L^{1+1/k}(X)$.

A direct consequence of Lemma 2.3 is that if $\varphi_n \to \varphi$ in $\mathcal{W}$ then $\varphi_n \to \varphi$ in $L^{1+1/k}$. In Kähler case, a related version of the inequality (2.4) for d.s.h. functions with $L^p$-norm in place of $L^{1+1/k}$-norm and $\|\cdot\|_*$ in place of $\|\cdot\|_X$ was proved in [15] using cohomological tools for d.s.h. functions. Their proof uses cohomological arguments which are not applicable to prove (2.4) for weakly quasi-p.s.h. functions.

Proof. Consider a weakly quasi-p.s.h. function $\varphi$ with $\|dd^c \varphi\| \leq A$. Let $(W_j)$ be (finite) open covering of $X$ where $W_j$ are local charts of $X$ biholomorphic to the unit ball of $\mathbb{C}^k$. Since $\|dd^c \varphi\| \leq A$, by Lemma 2.1, we have $\tau_j \in L^{1+1/k}(W_j)$ for which $dd^c \tau_j = dd^c \varphi$ on $W_j$ and
\begin{equation}
\|\tau_j\|_{L^{1+1/k}(W_j)} \lesssim A.
\end{equation}
Hence, $\varphi - \tau_j$ can be represented by a pluriharmonic function on $W_j$. For simplicity, we identify this function with $(\varphi - \tau_j)$. We deduce that $\varphi \in L^{1+1/k}(X)$.

We now suppose on the contrary that (2.4) doesn’t hold, that means that there exists a sequence of non identically zero weakly quasi-p.s.h. functions $\varphi_n$ with $\int_X \varphi_n d\mu_0 = 0$ and
\begin{equation}
\infty > \|\varphi_n\|_{L^{1+1/k}(X)} \geq n\|dd^c \varphi_n\|_X.
\end{equation}
By multiplying $\varphi_n$ by a positive constant and the $L^{1+1/k}$-integrability of $\varphi_n$, we can assume that
\begin{equation}
\|\varphi_n\|_{L^{1+1/k}(X)} = 1.
\end{equation}
Thus, we get
\begin{equation}
\|dd^c \varphi_n\| \leq 1/n.
\end{equation}
Note that we still have $\int_X \varphi_n d\mu_0 = 0$. Let $\tau_j^n$ be the function $\tau_j$ for $\varphi_n$ in place of $\varphi$. Put $T_n := dd^c \varphi_n$. These currents of order 0 are of uniformly bounded mass and converges to 0 by (2.7). Lemma 2.1 and the remark after it tell us that $\tau_j^n$ converges to 0 in $L^{1+1/k}(W_j')$, for every $W_j' \subset W_j$. We can also arrange that $(W_j')$ is still a covering of $X$. For simplicity, we can assume that $W_j' = W_j$ for every $j$.

Recall now that $\varphi_n - \tau_j^n$ is pluriharmonic on $W_j$. The last function is of $L^{1+1/k}$-norm bounded on $W_j$ because of (2.5) and (2.6). The mean equality for pluriharmonic functions implies that $(\varphi_n - \tau_j^n)$ is of $\mathcal{E}'$-norm uniformly bounded on compact subsets of $W_j$ in $n \in \mathbb{N}$ for every $l \in \mathbb{N}$. We deduce that by extracting a subsequence, we can suppose that $\varphi_n - \tau_j^n$ converging uniformly to a pluriharmonic function $\tau_j^\infty$ on compact subsets of $W_j$ as $n \to \infty$. Since $\|\tau_j^n\|_{L^{1+1/k}(W_j)} \to 0$, we obtain that
\begin{equation}
\varphi_n \to \tau_j^\infty \quad \text{in } L^{1+1/k}(W_j).
\end{equation}
This yields that function $\tau^\infty := \tau_j^\infty$ on $W_j$ for every $j$ is a well-defined pluriharmonic function on $X$. Since $X$ is compact, $\tau^\infty$ is a constant. This combined with $\int_X \varphi_n d\mu_0 = 0$
gives $\tau^\infty = 0$. We then have proved that $\varphi_n \to 0$ in $L^{1+1/k}(X)$, hence $\|\varphi_n\|_{L^{1+1/k}} \to 0$, a contradiction. Thus, (2.4) holds.

In order to prove the second desired assertion, we use again the function $\tau_j$ above. We have $\varphi - \tau_j$ is pluriharmonic on $W_j$ and by (2.4), the $L^{1+1/k}$-norm of $\varphi$ is also $\lesssim A$. Thus, the $L^{1+1/k}$-norm of the pluriharmonic function $(\varphi - \tau_j)$ is $\lesssim A$. It follows that its $C^1$-norm is $\lesssim A$ as well. Hence, we can extract a convergent subsequence of $(\varphi - \tau_j)$ for $\varphi \in W'$ in $C^1$. This combined with the $L^{1+1/k}$ continuity of $\tau_j$ in $T$ implies the desired assertion. The proof is finished.

Notice that the proof of Lemma 2.3 also tells us that if a $(0,0)$-current $\varphi'$ in an arbitrary complex manifold $X'$ is so that $\dd^c \varphi'$ is of order 0 then $\varphi'$ is in $L^{1+1/k}(X')$.

We equip the vector space $W$ of Borel measurable functions on $X$ with the pointwise convergence topology: $h_n \to h$ if $h_n$ converges pointwise to $h$ almost everywhere (with respect to Lebesgue’s measure). Let $P$ be a continuous linear endomorphism of the last vector space. Define $W_P$ to be the set of all $\varphi \in W$ for which $P\varphi \in W$.

**Lemma 2.4.** There exists a constant $c$ such that

\begin{equation}
\|P\varphi\|_{L^{1+1/k}} \leq c\left(\|\varphi\|_W + \|\dd^c(P\varphi)\|\right),
\end{equation}

for any $\varphi \in W_P$. In particular, there is a constant $c'$ so that

\begin{equation}
\|P\varphi\|_{L^{1+1/k}} \leq c'\left(\|\dd^c\varphi\| + \|\dd^c(P\varphi)\|\right)
\end{equation}

for every $\varphi \in W_P \cap W_0$. Moreover, if $\varphi_n \in W_P \cap W_0 \to \varphi$ as currents as $n \to \infty$ so that $\left(\|\dd^c\varphi_n\| + \|\dd^c(P\varphi_n)\|\right)$ are uniformly bounded, then $P\varphi_n \to P\varphi$ in $L^{1+1/k}$.

**Proof.** The inequality (2.9) is a direct consequence of (2.8) and Lemma 2.3. Now suppose that there is a sequence $(\varphi_n) \subset W_P$ for which

\begin{equation}
\|P\varphi_n\|_{L^{1+1/k}} = 1, \quad \|\varphi\|_W + \|\dd^c(P\varphi_n)\| \leq 1/n.
\end{equation}

Applying the compactness property in Lemma 2.3 to the sequence $(P\varphi_n)_{n \in \mathbb{N}}$, we see that by extracting a subsequence of $\varphi_n$ if necessary, the sequence $P\varphi_n$ converges in $L^{1+1/k}$ to a weakly d.s.h. function $\varphi^\infty$. Consequently,

\begin{equation}
\|\varphi^\infty\|_{L^{1+1/k}} = 1, \quad \|\dd^c\varphi^\infty\| = 0.
\end{equation}

Hence $\varphi^\infty$ is a constant. Since the convergence in $L^1$ implies the almost everywhere convergence of a subsequence, we can suppose also that $P\varphi_n$ converges almost everywhere to $\varphi^\infty$.

On the other hand, the inequality of (2.10) allows us to use the compactness property in Lemma 2.3 again for $(\varphi_n)$. Hence, we can extract a subsequence of $(\varphi_n)$ converging to $\varphi^\infty := 0$ in $L^{1+1/k}$ and almost everywhere. Thus $P\varphi_n$ converges almost everywhere to $P\varphi^\infty$ because of the continuity of $P$. It follows that $\varphi^\prime = P\varphi^\infty = 0$, note here $P(0) = 0$ by the linearity of $P$. This is a contraction because of (2.11). Thus (2.8) follows. The last desired assertion follows directly from above arguments. The proof is finished.

Let $a \in \mathbb{C}^*$, $r$ a constant in $(0, |a|)$ and $\delta > 0$ a constant. Assume that $P(1) = a$, where 1 is the constant function equal to 1 on $X$. Define $W^\infty_{P,r,\delta}$ to be the set of all $\varphi \in B$ such that $P^n\varphi \in W$ for every $n \geq 0$ and

$$\|\dd^c(P^n\varphi)\| \leq \delta r^n$$
for every $n \geq 0$, here $P^0$ denote the identity map. By the linearity of $P$, every constant function belongs to $\mathcal{W}_{r,\delta}^\infty$. We equip $\mathcal{W}_{P,\delta}^\infty$ with the induced topology from that on $\mathcal{W}$. Observe that $\mathcal{W}_{P,r,\delta}^\infty$ is closed in $\mathcal{W}$ and

$$r^{-m}P^m(\mathcal{W}_{P,r,\delta}^\infty) \subset \mathcal{W}_{P,r,\delta}^\infty$$

for every positive integer $m$. Hence $\mathcal{W}_{P,r,\delta}^\infty \cap \mathcal{W}_0^\infty$ is compact and $P^m(\mathcal{W}_{P,r,\delta}^\infty)$ is contained in the complex vector subspace $\mathcal{W}_{P,r,\delta}^\infty$ of $\mathcal{W}$ generated by $\mathcal{W}_{P,r,\delta}^\infty$.

**Proposition 2.5.** There exists a continuous linear functional $\mu_P : \mathcal{W}_{P,r,\delta}^\infty \to \mathbb{C}$ such that for every complex measure $\nu$ with $L^{k+1}$ density on $X$, $\nu(X) = 1$ and for every $\varphi \in \mathcal{W}_{P,r,\delta}^\infty$, we have

$$\langle a^{-n}(P^n)\nu, \varphi \rangle \to \langle \mu_P, \varphi \rangle. \quad (2.12)$$

Here for $Q : \mathcal{B} \to \mathcal{B}$, by definition, $\langle Q\nu, \varphi \rangle := \langle \nu, Q\varphi \rangle$ for $\varphi \in \mathcal{B}$ such that $Q\varphi$ is $\nu$-integrable.

**Proof.** Recall that $\mu_0$ is a smooth probability volume form on $X$. We only need to construct $\mu_P$ on $\mathcal{W}_{P,r,\delta}^\infty$ and prove (2.12) for $\varphi \in \mathcal{W}_{P,r,\delta}^\infty$. The extension of $\mu_P$ to $\mathcal{W}_{P,r,\delta}^\infty$ is automatically done by using the linearity of $(P^n)\nu$ and (2.12).

Let $\varphi \in \mathcal{W}_{P,r,\delta}^\infty$. Put $b_0 := \int_X \varphi d\mu_0$ and $\varphi_0 := \varphi - b_0$. We define two sequences $\varphi_n, b_n$ as follows. Put

$$b_n = b_n(\varphi) := \int_X (P\varphi_{n-1}) d\mu_0, \quad \varphi_n := P\varphi_{n-1} - b_n$$

for $n \geq 1$. We have $r^{-n}\varphi_n \in \mathcal{W}_0^\infty \cap \mathcal{W}_{P,r,\delta}^\infty$ and $d\varphi(P^n\varphi_n) = d\varphi(P^{n+1}\varphi_n)$ for every $n, m$. By Lemma 2.4 we have

$$\|\varphi_n\|_{L^{1+1/k}} \leq c(\|d\varphi(P\varphi_{n-1})\| + \|d\varphi\varphi_{n-1}\|), \quad |b_n| \leq c(\|d\varphi(P\varphi_{n-1})\| + \|d\varphi\varphi_{n-1}\|),$$

for some constant $c$ independent of $n, \varphi$. It follows that

$$\|\varphi_n\|_{L^{1+1/k}} \leq c(\|d\varphi(P\varphi_{n-1})\| + \|d\varphi(P^{n-1}\varphi_n)\|) \leq c\delta(r + 1)r^{-n-1}, \quad |b_n| \leq c\delta(r + 1)r^{-n-1}$$

for $n \geq 1$. Since $P(1) = a$ we have $P(b_n) = ab_n$ for every $n$. Using this gives

$$a^{-n}P^n\varphi = b_0 + a^{-n}P^n\varphi_0 = b_0 + a^{-n}P^{n-1}(P\varphi_0) = b_0 + a^{-1}b_1 + a^{-n}P^{n-1}\varphi_1 = \cdots = b_0 + a^{-1}b_1 + \cdots a^{-n}b_n + a^{-n}\varphi_n.$$  

Put $b'_n = b'_n(\varphi) := b_0 + a^{-1}b_1 + \cdots a^{-n}b_n$ which converges to a number $b'_\infty$ (depending on $\varphi$) by (2.13) and the fact that $|a| > r$. We deduce from (2.14) that

$$|a^{-n}P^n\varphi - b'_n| \leq |a|^{-n}|\varphi_n|.$$  

This combined with the first inequality of (2.13) implies that $a^{-n}P\varphi$ converges to $b'_\infty$ in $L^{1+1/k}$. Precisely, we have

$$\|a^{-n}P^n\varphi - b'_n\|_{L^{1+1/k}} \leq \delta|a|^{-n}r^n. \quad (2.15)$$

Since $\nu(X) = 1$, we get

$$\langle a^{-n}(P^n)\nu, \varphi \rangle - b'_n = \langle \nu, a^{-n}P^n\varphi - b'_n \rangle.$$
Using this, (2.15) and Hölder’s inequality implies that $\langle a^{-n}(P^n)_*, \nu, \varphi \rangle$ converges to $b'_\infty = b'_{\infty}(\varphi)$ because $\nu$ has $L^{k+1}$ density. Define $\langle \mu_P, \varphi \rangle := b'_{\infty}(\varphi)$ which is independent of $\nu$. We then obtain the desired convergence toward $\mu_P$.

Consider a sequence $\tilde{\varphi}_m \to \varphi$ in $\mathcal{W}_{P,r,\delta}$. Let $\tilde{b}_{nm}, \tilde{\varphi}_{nm}$ be respectively the $b_n$ and $\varphi_n$ for $\tilde{\varphi}_m$ in place of $\varphi$. By the last assertion of Lemma 2.4 $\tilde{b}_{nm} \to b_n$ as $m \to \infty$ for every $n$ and (2.13) still holds for $\tilde{b}_{nm}, \tilde{\varphi}_{nm}$ in place of $b_n, \varphi_n$. We infer that $\tilde{b}'_{nm} \to b'_n$ and

$$a^{-n}\tilde{\varphi}_{nm} \to 0$$

in $L^{1+1/k}$ as $m \to \infty$. Thus, $\langle \mu_P, \tilde{\varphi}_m \rangle \to \langle \mu_P, \varphi \rangle$ as $m \to \infty$. In other words, $\mu_P$ is continuous. The proof is finished. \hfill \Box

Let $X$ be a compact complex manifold and $f$ a meromorphic self-map on $X$. Denote by $\Gamma$ the graph of $f$ on $X \times X$ and $\pi_1, \pi_2$ the restrictions to $\Gamma$ of the natural projections from $X \times X$ to the first and second components respectively.

Let $\Phi$ be a form with measurable coefficients on $X$. We say that $\Phi \in L^1$ if its coefficients are $L^1$ functions (with respect to Lebesgue’s measure on $X$). If $\Omega$ is an open Zariski dense subset of $X$ such that $\pi_2$ is an unramified covering over $\Omega$, the form $f_*\Phi|_\Omega := (\pi_2|_\Omega^{-1})_*(\pi_1^\ast \Phi)$ is a measurable form on $\Omega$. Hence $f_*\Phi$ is a measurable form on $X$ independent of $\Omega$. We can check that $f_* : \mathcal{B} \to \mathcal{B}$ is continuous. Consequently, $f_*$ is an example of the map $\pi$. We infer that $f_* \Phi \in L^1(\Omega)$ for some $\Omega$ as above (hence for every such $\Omega$). In this case, we can define $f_*\Phi$ to be a current of order $0$ induced by $f_*\Phi|_\Omega$ on $X$ by extending trivially through $X\setminus\Omega$. This definition is independent of the choice of $\Omega$. Note that the pull-back by $f$ of smooth functions or smooth forms is always in $L^1$. The following is similar to results in [32, 16].

**Lemma 2.6.** For every quasi-p.s.h. function $\varphi$ on $X$, we have $f_*\varphi \in L^1$ and if $\ddc^c \varphi + \eta \geq 0$ for some continuous $(1,1)$-form $\eta > 0$, then $\ddc^c (f_* \varphi) + f_* \eta \geq 0$. In particular,

$$\langle f^n \varphi \rangle \in \mathcal{W}_f \cap \mathcal{W}.$$  \hfill (2.16)

The inclusion (2.16) explains the crucial roles of $\mathcal{W}_f, \mathcal{W}$ in our study.

**Proof.** Let $\sigma : \Gamma' \to \Gamma$ be a desingularisation of $\Gamma$. Let $\Omega$ be as above. Put $\pi_j := \pi_j \circ \sigma$ for $j = 1, 2$. Since $\varphi$ is quasi-p.s.h., $\varphi \circ \pi_1$ is so. Thus, $\varphi \circ \pi_1 = \sigma_\ast (\varphi \circ \pi_1')$ is in $\mathcal{L}^1(\Gamma')$. Since

$$\|f_* \varphi\|_{\mathcal{L}^1(\Omega)} = \|\pi_2 \ast (\varphi \circ \pi_1)\|_{\mathcal{L}^1(\Gamma)} \lesssim \|\varphi \circ \pi_1\|_{\mathcal{L}^1(\Gamma)},$$

we get the first desired assertion.

By [2] and the fact that $\eta > 0$, there exists a decreasing sequence of smooth quasi-p.s.h functions $\varphi_n$ converging pointwise to $\varphi$ locally such that $\ddc^c \varphi_n + \eta \geq 0$ for every $n$. We can have a global convergence in the last sentence. But this is not necessary for our arguments. By Lebesgue’s dominated convergence theorem, the sequence $\varphi_n \circ \pi_1'$ converges in $L^1$ to $\varphi \circ \pi_1'$. It follows that the sequence of positive smooth forms $\ddc^c (\varphi_n \circ \pi_1') + \pi_1'^\ast \eta$ converges weakly to $\ddc^c (\varphi \circ \pi_1') + \pi_1'^\ast \eta$. Thus, the last current is also positive. Now observe that

$$\pi_2^\ast(d\ddc^c (\varphi \circ \pi_1') + \pi_1'^\ast \eta) = \ddc^c ((\pi_2^\ast \varphi) + \pi_1'^\ast \eta) = \ddc^c ((\pi_2^\ast \pi_1' \varphi) + \pi_1'^\ast \eta)$$

because $\pi_1' \varphi$ and $\pi_1'^\ast \eta$ have no mass on sets of Lebesgue measure zero. Thus $\ddc^c (f_* \varphi) + f_* \eta \geq 0$.

Note that $f_* \varphi$ has finite mass on $X$. We infer that $f_* \varphi \in \mathcal{W}$. In other words, $\varphi \in \mathcal{W}_f \cap \mathcal{W}$. Applying this to $f^n$ instead of $f$ and using the formula that $(f^n)_\ast \varphi = f_\ast (f^{n-1})_\ast \varphi$ as
functions on some suitable open dense subset of $X$, we obtain (2.16). This finishes the proof.

**Lemma 2.7.** Let $X$ be a compact complex manifold of dimension $k$ and $f$ a meromorphic self-map on $X$. Let $\varphi$ be a quasi-p.s.h. function on $X$ with $\dd^c \varphi + \eta \geq 0$ for some continuous $(1,1)$-form $\eta$. Then given every positive constant $\epsilon$, there exists a constant $c_\epsilon$ independent of $\varphi, \eta$ for which

$$\|\dd^c (f^n)_* \varphi\| \leq c_\epsilon (d_{k-1}(f) + \epsilon)^n \|\eta\|_{L^\infty}$$

for every $n \geq 1$.

**Proof.** By replacing $\eta$ by a strictly positive smooth form dominating it, we can assume that $\eta > 0$. Let $\omega$ be a Gauduchon metric on $X$, that means that $\omega$ is a Hermitian metric and $\dd^c \omega^{k-1} = 0$, see [24]. Let $\Gamma_n$ be the graph of $f^n$ and $\pi_{1,n}, \pi_{2,n}$ the natural maps from $\Gamma_n$ to the first and second components of $X \times X$. By Lemma 2.6 the current $\dd^c(f^n)_* \varphi + (f^n)_* \eta$ is positive. Thus, using $\dd^c \omega^{k-1} = 0$ gives

$$\|\dd^c (f^n)_* \varphi + (f^n)_* \eta\| \leq \|\dd^c (f^n)_* \varphi + (f^n)_* \eta, \omega^{k-1}\| \leq \|\dd^c (f^n)_* \varphi + (f^n)_* \eta, \omega^{k-1}\|.$$

This combined with the definition of $d_{k-1}(f)$ gives

$$\|\dd^c (f^n)_* \varphi + (f^n)_* \eta\| \leq c_\epsilon (d_{k-1}(f) + \epsilon)^n \|\eta\|_{L^\infty}.$$ 

The desired inequality then follows immediately. The proof is finished.

We now come to the end of the proof of the first main result.

**End of Proof of Theorem [1.1]** Fix a positive constant $\epsilon$ for which $d_k > d_{k-1} + \epsilon$. Put

$$P := f_*, \ a := d_k, \ r := (d_{k-1} + \epsilon), \ \delta := c_\epsilon,$$

where $c_\epsilon$ is the constant in Lemma 2.7. Let $\varphi$ be a quasi-p.s.h. with $\dd^c \varphi + \eta \geq 0$ for some continuous $(1,1)$-form $\eta > 0$ so that $\|\eta\|_{L^\infty} \leq 1$. We have $P(1) = a$ and $\varphi \in \mathcal{H}_{P,r,\delta}$ by Lemma 2.7. Every quasi-p.s.h. function is in $\mathcal{H}_{P,r,\delta}$. Since $\nu$ has no mass on proper analytic subsets of $X$, observe that

$$\langle (f^n)^* \nu, \varphi \rangle = \langle \nu, (f^n)_* \varphi \rangle = \langle \nu, P^n \varphi \rangle$$

because we only need to consider integrals on an open Zariski dense subset of $X$. Applying Proposition 2.5 to $P$, we obtain a continuous functional $\mu_P$ on $\mathcal{H}_{P,r,\delta}$ such that

$$\langle d_k^{-n}(f^n)^* \nu, \varphi \rangle \to \langle \mu_P, \varphi \rangle,$$

for every $\varphi \in \mathcal{H}_{P,r,\delta}$. By choosing $\nu \geq 0$, we see that $\langle \mu_P, \varphi \rangle \geq 0$ if $\varphi \geq 0$. Let $\mu_f$ be the probability measure on $X$ defined by $\langle \mu_f, \varphi \rangle := \langle \mu_P, \varphi \rangle$ for every smooth function $\varphi$. Recall here that smooth functions are quasi-p.s.h. on $X$. We will prove that $\mu_f = \mu_P$ for every quasi-p.s.h. function $\varphi$.

Consider a sequence of smooth quasi-p.s.h. functions $\varphi'_n$ with $\dd^c \varphi'_n + \eta \geq 0$ decreasing to $\varphi$, we get $\langle \mu_f, \varphi'_n \rangle = \langle \mu_P, \varphi'_n \rangle$ and $\langle \mu_f, \varphi'_n \rangle \to \langle \mu_f, \varphi \rangle$ by Lebesgue’s monotone convergence theorem. This combined with the continuity of $\mu_P$ gives $\langle \mu_f, \varphi \rangle = \langle \mu_P, \varphi \rangle$. Thus we get

$$\lim_{n \to \infty} (d_k^n(f^n)^* \nu - \mu_f, \varphi) = 0$$

for every quasi-p.s.h. function $\varphi$ on $X$.  

Since quasi-p.s.h functions are $\mu_f$-integrable, $\mu_f$ has no mass on pluripolar sets. By Lemma 2.8 below, proper analytic subsets of $X$ are pluripolar. This implies that $\mu_f$ has no mass on proper analytic subsets of $X$. We deduce that the pull-back $f^*\mu_f$ is well-defined. Here we only take the pull-back of $\mu_f$ on an open Zariski subset $\Omega$ of $X$ where $\pi_2$ is an unramified covering. One can check that this definition is independent of the choice of $\Omega$ and if $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence of positive measures having no mass on proper analytic subsets of $X$ and converging to $\mu_f$, then $f^*\Phi_n$ converges to $f^*\mu_f$ because the mass of $f^*\Phi_n$ converges to that of $f^*\mu_f$, see for example [33, Le. 3.6]. The equality

(2.20) \[ d_k^{-1} f^* \mu_f = \mu_f \]

is obtained by applying the pull-back $f^*$ to the convergence $d_k^{-n}(f^n)^* \nu \to \mu_f$, where $\nu$ is a smooth probability measure. Since we have $f_n f^* = d_k$ on Borel measurable functions, we get $f_n \mu_f = \mu_f$, in other words, $\mu_f$ is invariant by $f$.

Let $I_f$ be the indeterminacy set of $f$. Put $Z := \cup_{n \in \mathbb{Z}} f^n(I_f)$. The measure $\mu_f$ has no mass on $Z$. The entropy of $\mu_f$ is by definition that of $1_{X \setminus Z} \mu_f$ with respect to $f|_{X \setminus Z}$. By an inequality of Parry [33, 18], using $f^* \mu_f = d_k \mu_f$, we deduce that the entropy of $\mu_f$ is at least $\log d_k$.

Assume now $f$ is holomorphic. To prove that $\mu_f$ is Hölder continuous on PSH$(\omega)$, we use a known idea from [18]. Without loss of generality, we can assume that $\|\omega\|_{L^\infty} \leq 1$. Let $\varphi, \psi$ be two quasi-p.s.h. functions in PSH$(\omega)$. Recall that they are in $\mathcal{P}_{p, \delta}^\infty$.

Let $b_n(\varphi), b_n(\psi)$ be as in the proof of Proposition 2.5. Let $J_f$ be the Jacobian of $f$. We have

\[ \| f_n \varphi - f_n \psi \|_{L^1} = \sup_{\| h \|_{L^\infty} \leq 1} | \langle f_n \varphi - f_n \psi, h \mu_0 \rangle | = \sup_{\| h \|_{L^\infty} \leq 1} | \langle \varphi - \psi, (h \circ f) f^* \mu_0 \rangle | \]

which is

\[ \leq \| J_f \|_{L^\infty} \| \varphi - \psi \|_{L^1}. \]

Applying the last inequality for $f^n$ in place of $f$ gives

\[ |b_n(\varphi) - b_n(\psi)| \leq 2^n \| J_f \|_{L^\infty} \| \varphi - \psi \|_{L^1}. \]

Put

\[ A_1 := \sum_{n=0}^{M+1} d_k^{-n} [b_n(\varphi) - b_n(\psi)], \quad A_2 := \sum_{n=M+1}^{\infty} d_k^{-n} [b_n(\varphi) - b_n(\psi)]. \]

Using (2.14) gives

\[ \langle \mu_f, \varphi - \psi \rangle = A_1 + A_2, \quad |A_1| \leq \sum_{n=0}^{M} d_k^{-n} 2^n \| J_f \|_{L^\infty} \| \varphi - \psi \|_{L^1}, \quad |A_2| \lesssim (d_k^{-1} + \epsilon)^M d_k^{-M}. \]

Consider the case where $2\| J_f \|_{L^\infty} \leq d_k$. We have $|A_1| \leq M \| \varphi - \psi \|_{L^1}$. By choosing $M$ to be smallest integer for which $M \geq -\log \| \varphi - \psi \|_{L^1}/\log \tau$, where $\tau := d_k/(d_k^{-1} + \epsilon)$, for every constant $\epsilon > 0$, we obtain that

\[ |\langle \mu_f, \varphi - \psi \rangle| \leq |A_1| + |A_2| \lesssim \| \varphi - \psi \|_{L^1}^{-t}, \]

which implies that $\mu_f$ is Hölder continuous in this case. It remains to treat the case $2\| J_f \|_{L^\infty} \geq d_k$. We have

\[ |A_1| \leq M^2 d_k^{-M} \| J_f \|_{L^\infty}^M \| \varphi - \psi \|_{L^1} + \tau^{-M}. \]
Choose $M := -\log \|\varphi - \psi\|_{L^1} / \log (2d_k^{-1}\tau \|J_f\|_{L^\infty})$. We see that

$$|A_1| + |A_2| \lesssim -\log \|\varphi - \psi\|_{L^1} \|\varphi - \psi\|^\log \tau / \log (2d_k^{-1}\tau \|J_f\|_{L^\infty}).$$

Hence, $\mu_f$ is also Hölder continuous in this case. This finishes the proof. \hfill $\square$

Now we would like to say some words about Theorem 1.2. If one tries to mimic the arguments in the proof of [15, The. 1.3] to prove Theorem 1.2, we are led to estimating $|\langle \varphi, \varphi_0 \rangle|$. The measure $\mu_f$ still satisfies the property that for every $\omega$-p.s.h. function $\varphi$ with $\sup_\chi \varphi = 0$ is of $L^1(\mu_f)$-norm uniformly bounded, see [15, Pro. 2.3]. But unlike the Kähler case, we don’t know whether $\varphi_n$ is the difference of two $\omega$-p.s.h functions. So this explains why we cannot apply directly the approach in [15] to get a decay of correlation for $\mu_f$.

**Lemma 2.8.** Every proper analytic subset $V$ of a compact complex manifold $X$ is a pluripolar set on $X$.

**Proof.** We use here the idea in [15] where the authors prove the same result when $X$ is Kähler. Suppose now that $V$ is smooth and $\text{codim} V \geq 2$ (since otherwise the problem is trivial). Let $\sigma : \tilde{X} \to X$ be the blowup of $X$ along $V$. Denote by $\tilde{V}$ the exceptional hypersurface.

Let $\omega$ be a positive definite Hermitian form on $X$. Let $\widehat{\omega}_h$ be a Chern form of $\mathcal{O}(\tilde{V})$ whose restriction to each fiber of $\tilde{V} \approx \mathbb{P}(E)$ is strictly positive. By scaling $\omega$ if necessary, we can assume that $\widehat{\omega} := \sigma^* \omega + \widehat{\omega}_h > 0$. Since $\sigma_* \widehat{\omega}_h = \sigma_* \widehat{\omega} - \omega$, the closed current $\sigma_* \widehat{\omega}_h$ is quasi-positive. Thus there exists a quasi-p.s.h. function $\varphi$ on $\tilde{X}$ such that

$$(2.21) \quad \sigma_* \widehat{\omega}_h = dd^c \varphi + \eta$$

for some smooth closed form $\eta$. By multiplying $\widehat{\omega}_h$ by a strictly positive constant, we have $\sigma^* \sigma_* \widehat{\omega}_h = \widehat{\omega}_h + [\tilde{V}]$. Thus $|\varphi \circ \sigma(\tilde{x}) - \log \text{dist}(\tilde{x}, \tilde{V})|$ is a bounded function on $\tilde{X}$. As a consequence,

$$(2.22) \quad |\varphi(x) - \log \text{dist}(x,V)| \lesssim 1$$

on compact subsets of $X$. Consequently, $V$ is contained in $\{ \varphi = -\infty \}$. Hence $V$ is pluripolar in this case.

By the above construction, we can construct a Hermitian metric on the blowup $\tilde{X}$ of $X$ along $V$ as the sum of a pull-back of a Hermitian one on $X$ and a suitable Chern form of $\mathcal{O}(-\tilde{V})$. Hence, if $\sigma' : \tilde{X}' \to X$ is a composition of blowups along smooth submanifolds, then there are a smooth closed $(1,1)$-form $\eta'$ on $\tilde{X}'$ and a Hermitian metric $\omega$ on $X$ such that $\omega' = \sigma^* \omega + \eta'$ is a Hermitian metric on $\tilde{X}'$.

Consider now the general situation where $V$ is an analytic subset of $X$. Since a finite union of pluripolar sets is again pluripolar, it is enough to prove that the regular part $\text{Reg} V$ of $V$ is a pluripolar set because we can write $V$ as a finite union of the regular parts of suitable analytic subsets of $X$. By Hironaka’s desingularisation, there is a composition $\sigma' : \tilde{X}' \to X$ of blowups along smooth submanifolds which don’t intersect $\text{Reg} V$ (or its inverse images) such that the strict transform $\tilde{V}'$ of $V$ is smooth.

Let $\tilde{\omega}', \omega, \eta$ be as above. By the above arguments, $\tilde{V}' \subset \{ \tilde{\varphi}' = -\infty \}$ for some quasi-p.s.h. function $\tilde{\varphi}'$ on $\tilde{X}'$ and $dd^c \tilde{\varphi}' + \tilde{\omega}' \geq 0$. Put $S := \sigma'_* (dd^c \tilde{\varphi}' + \eta')$ which is a closed
(1, 1)-current on $X$ and $S + \omega \geq 0$. We can write
\[ S = dd^c \varphi_S + \eta_S, \quad \sigma_\ast \eta' = dd^c \psi + \eta \]
for some smooth closed forms $\eta_S, \eta$. We have
\[ dd^c \varphi_S + \eta_S + \omega \geq 0, \quad dd^c \psi + \eta + \omega \geq 0. \]
Thus $\varphi_S, \psi$ are quasi-p.s.h. functions on $X$. Moreover, we also have
\[ \varphi_S = \sigma'_\ast(\widehat{\tau'}) + \psi + \text{a smooth function} \]
on an open neighborhood of $\text{Reg}V$ on which $\sigma'$ is biholomorphic. Consequently, $\text{Reg}V \subset \{ \varphi'_S = -\infty \}$. This finishes the proof. \qed

3. The set $W^{1,2}_{*, f}$

In this section, we prove Theorem 1.2. Our idea is to consider suitable test functions in the Sobolev space $W^{1,2}$. This approach is inspired by [14].

Fix a smooth volume form $\mu_0$ on $X$ and we use this form to define the norm on the space $L^2(X)$. Let $W^{1,2}$ be the space of real-valued function $\varphi \in L^2(X)$ such that $d\varphi$ has $L^2$ coefficients. Recall the following Poincaré-Sobolev inequality: for $\varphi \in W^{1,2}$ with $\int_X \varphi d\mu_0 = 0$, we have
\[ \| \varphi \|_{L^2} \leq c \| d\varphi \|_{L^2}, \tag{3.1} \]
for some constant $c$ independent of $\varphi$, see for example [28, Pro. 3.9] or [20]. Observe that the term $\| d\varphi \|_{L^2}^2$ is comparable with the mass of the positive current $i\partial \varphi \wedge \bar{\partial} \varphi$. We have the following lemma.

**Lemma 3.1.** ([14, Pro. 3.1]) Let $I$ be a compact subset of $X$ of the Hausdorff $(2k - 1)$-dimensional measure zero. Let $\varphi$ be a real-valued function $L^1_{\text{loc}}(X \setminus I)$. Assume that the coefficients of $d\varphi$ are in $L^2(X \setminus I)$. Then $\varphi \in W^{1,2}$ and there exists a compact subset $M$ of $X \setminus I$ and a constant $c > 0$ both independent of $\varphi$ such that
\[ \| \varphi \|_{L^1(X)} \leq c(\| \varphi \|_{L^1(M)} + \| d\varphi \|_{L^1(X)}). \]

The following is the central object in this section.

**Definition 3.2.** Let $W^{1,2}_{*, f}$ be the subset of $W^{1,2}$ consisting of $\varphi$ such that there exist $m_1 \in \mathbb{N}$, a continuous $(1, 1)$-form $\eta$ and an $\eta$-p.s.h. function $\psi$ satisfying
\[ i\partial \varphi \wedge \bar{\partial} \varphi \leq dd^c((f^m, \psi) + (f^m, \eta) \eta \]
as currents. A size representative of $\varphi$ is $m := (m_0, m_1)$, where $m_0$ is an upper bound of $\| \eta \|_{L^\infty}$.

If $X$ is Kähler, $W^{1,2}_{*, f}$ coincides with the space $W^{1,2}_{*}$ considered in [14] which is independent of $f$. In that context, the space $W^{1,2}_{* f}$ is studied in details in [37] and used in [7] for the study of correspondences on Riemann surfaces with two equal dynamical degrees. Let $\epsilon$ be a strictly positive constant such that $d_{k-1} + \epsilon < d_k$. We have the following observation.

**Lemma 3.3.** Let $\varphi \in W^{1,2}_{*, f}$ and $m = (m_0, m_1)$ a size representative of $\varphi$. Then we have
\[ \| d\varphi \|_{L^2} \leq c \epsilon m_0^{1/2} (d_{k-1} + \epsilon)^{m_1/2} \]
for some constant $c_\epsilon$ independent of $\varphi$. 
Proof. Let \( \eta \) be as in (3.2). Let \( \omega \) be a Hermitian metric on \( X \) with \( \text{dd}^c \omega^{k-1} = 0 \). By testing \( \text{dd}^c((f^{m_1})^*\psi) + (f^{m_1})^*\eta \) with this form, we see that the norm of \( \text{dd}^c((f^{m_1})^*\psi) + (f^{m_1})^*\eta \) is equal to \( \int_X (f^{m_1})^*\eta \wedge \omega^{k-1} \) which is bounded by \( c \cdot m_0 (d_{k-1} + \varepsilon)^{m_1} \) for some constant \( c \), independent of \( \eta, m_0, m_1 \). The desired inequality then follows. This finishes the proof. \( \square \)

Let \( \varphi \in W^{1,2}_{s,f} \). Define \( \varphi^+ := \max\{\varphi, 0\} \) and \( \varphi^- := \max\{-\varphi, 0\} \). Consider a Lipschitz function \( \chi : \mathbb{R} \to \mathbb{R} \). We have \( \partial(\chi \circ \varphi) = (\chi' \circ \varphi) \partial \varphi \). This can be seen by using a sequence of smooth functions converging to \( \varphi \) in \( W^{1,2} \). We deduce that

\[
i \partial(\chi \circ \varphi) \wedge \bar{\partial}(\chi \circ \varphi) = (\chi' \circ \varphi)^2 i \partial \varphi \wedge \bar{\partial} \varphi.
\]

Consequently, \( \chi \circ \varphi \in W^{1,2}_{s,f} \). In particular, by letting \( \chi(t) := |t|, \max\{t, 0\} \) or \( \max\{-t, 0\} \) for \( t \in \mathbb{R} \), we obtain the following crucial property.

**Lemma 3.4.** For every \( \varphi \in W^{1,2}_{s,f} \), if \( m = (m_0, m_1) \) is a size representative of \( \varphi \), then \( m \) is also a size representative of \( |\varphi|, \varphi^+ \) and \( \varphi^- \).

We already know that the pushforward of a quasi-p.s.h. function by \( f \) is a weakly d.s.h. function. The following result, which explains the role of \( W^{1,2}_{s,f} \) in our study, gives a more precise description in the case of bounded quasi-p.s.h. functions.

**Lemma 3.5.** Every bounded quasi-p.s.h. function is in \( W^{1,2}_{s,f} \) and \( f \), preserves \( W^{1,2}_{s,f} \). Moreover, for every \( \varphi \in W^{1,2}_{s,f} \), if \( m = (m_0, m_1) \) is a size representative of \( \varphi \), then \( m' := (d_km_0, m_1 + 1) \) is a size representative of \( f_* \varphi \) and

\[
\|f_* \varphi\|_{L^2} \leq c \|\varphi\|_{L^1} + \|d(f_* \varphi)\|_{L^2}
\]

for some constant \( c \), independent of \( \varphi \).

**Proof.** Let \( \varphi \) be a bounded quasi-p.s.h. function and \( f : X \to X \) a dominant meromorphic map. Using the identity

\[
2i \partial \varphi \wedge \bar{\partial} \varphi = i \partial \bar{\partial} \varphi^2 - 2 \varphi i \partial \bar{\partial} \varphi
\]

we see that there exist a continuous \((1, 1)\)-form \( \eta \) and an \( \eta \)-p.s.h. function \( \psi \) for which \( i \partial \varphi \wedge \bar{\partial} \varphi \leq \text{dd}^c \psi + \eta \). Hence \( \varphi \in W^{1,2}_{s,f} \).

Now let \( \varphi \) be an arbitrary element of \( W^{1,2}_{s,f} \). Let \( \eta \) and \( \psi \) be such that (3.2) holds. Fix an open Zariski dense subset \( \Omega \) of \( X \) on which \( f_* \varphi, (f^{m_1})^* \psi, (f^{m_1})_* \eta \) are well-defined functions or forms and \( \pi_1 \) is a unramified covering on \( f^{-1}(\Omega) \). We have \( f_* \varphi \in L^1_{\text{loc}}(\Omega) \) and

\[
\|f_* \varphi\|_{L^1(K)} \leq c \|\varphi\|_{L^1},
\]

for any compact \( K \) in \( \Omega \) and some constant \( c \), independent of \( \varphi \). Note that \( X \setminus \Omega \) is a proper analytic subset of \( X \), hence, is of Hausdorff \((2k - 1)\)-dimensional measure zero. On \( \Omega \), by the Cauchy-Schwarz inequality, we have

\[
i \partial(f_* \varphi) \wedge \bar{\partial}(f_* \varphi) \leq d_k f_* (i \partial \varphi \wedge \bar{\partial} \varphi) \leq d_k f_* \left[ \text{dd}^c((f^{m_1})^* \psi) + (f^{m_1})_* \eta \right]
\]

\[
= d_k \text{dd}^c((f^{m_1+1})^* \psi) + (f^{m_1+1})_* \eta.
\]

It follows that \( d(f_* \varphi) \in L^2(\Omega) \). By this and Lemma 3.3, we get \( f_* \varphi \in W^{1,2} \). Thus, \( i \partial(f_* \varphi) \wedge \bar{\partial}(f_* \varphi) \) has no mass on \( X \setminus \Omega \). It follows that

\[
i \partial(f_* \varphi) \wedge \bar{\partial}(f_* \varphi) \leq d_k 1_\Omega \text{dd}^c((f^{m_1+1})^* \psi) + (f^{m_1+1})_* \eta \leq d_k \text{dd}^c((f^{m_1+1})^* \psi) + (f^{m_1+1})_* \eta
\]
because the last current is positive by Lemma 2.6. Combining this with (3.1) and (3.4) gives (3.3). The desired assertion then follows. The proof is finished.

Let \( \varphi \in W_{*f}^{1,2} \) and \( m = (m_0, m_1) \) a size representative of \( \varphi \). Consider \( f_* \) acting on Borel measurable functions. Recall that \( f_* \) preserves the set of constant functions. As in the last section, let \( b_0 := \int_X \varphi d\mu_0 \), and \( \varphi_0 := \varphi - b_0 \). We define two sequences \( \varphi_n, b_n \) as follows. Put

\[
b_n = b_n(\varphi) := \int_X (f_* \varphi_{n-1})d\mu_0, \quad \varphi_n := f_* \varphi_{n-1} - b_n
\]

for \( n \geq 1 \). Note that \( \varphi_n \) differs from \((f^n)_* \varphi)\) by a constant. Lemma 3.5 yields that \( m_n := (d_k^n m_0, m_1 + n) \) is a size representative of \( \varphi_n \). This coupled with Lemma 3.4 implies that

**Lemma 3.6.** \( m_n := (d_k^n m_0, m_1 + n) \) is also a size representative of \( |\varphi_n|, \varphi_n^+ \) and \( \varphi_n^- \).

By Lemma 3.3 we get

\[
\|d\varphi_n\|_{L^2} \leq c_n m_0^{1/2} d_k^{n/2} (d_k-1 + \epsilon)^{(n+m_1)/2}
\]

Using (3.5), (3.1) and (3.3) gives

\[
\|\varphi_n\|_{L^2} \leq c_n m_0^{1/2} d_k^{n/2} (d_k-1 + \epsilon)^{(n+m_1)/2}, \quad |b_n| \leq c_n m_0^{1/2} d_k^{n/2} (d_k-1 + \epsilon)^{(n+m_1)/2}
\]

for \( n \geq 1 \) and some possible different constant \( c_n \). We are now in a situation very similar to that in the last section. Using similar arguments as in the last section, we can show that \( \lim_{n \to \infty} \langle d_k^{-n} (f^n)_* \omega^k, \varphi \rangle \) exists and denote by \( b'_\infty(\varphi) \) this limit. Actually, we have

\[
b'_\infty = \sum_{j=0}^{\infty} d_k^{-j} b_j.
\]

It follows that

\[
|b'_\infty(\varphi)| \leq \|\varphi\|_{L^1} + c_n m_0^{1/2} (d_k-1 + \epsilon)^{m_1/2}
\]

for some constant \( c_n \) independent of \( \varphi \). Clearly, if \( \varphi \) is a bounded quasi-p.s.h. function, \( b'_\infty \) is equal to the same number defined in the last section. Hence we have

\[
\langle \mu_f, \varphi \rangle = b'_\infty(\varphi)
\]

for bounded quasi-p.s.h. function \( \varphi \). Let \( W_{*f}^{1,2} \) be the subset of \( W_{*f}^{1,2} \) consisting of functions which is continuous outside a closed pluripolar set. Observe that \( f_* \) preserves \( W_{*f}^{1,2} \) because \( f \) is a covering outside an analytic subset of \( X \). We now claim that

**Lemma 3.7.** For \( \varphi \in W_{*f}^{1,2} \), we have \( \langle \mu_f, \varphi \rangle = b'_\infty(\varphi) \).

**Proof.** The proof is similar to that of [14, Le. 5.5]. We recall here for the readers’ convenience. We prove first that \( \varphi \) is \( \mu_f \)-integrable. We assume for the moment that \( \varphi \geq 0 \). Let \( V \) be a closed pluripolar set such that \( \varphi \) is continuous outside \( V \). Recall that \( \mu_f \) has no mass on pluripolar sets, hence, on \( V \). Since \( d_k^{-n} (f^n)_* \omega^k \) converges to \( \mu_f \) as positive measures and \( X \setminus V \) is open, we get

\[
\langle \mu_f, \varphi \rangle \leq \liminf_{n \to \infty} \langle d_k^{-n} (f^n)_* \omega^k, \varphi \rangle = \lim_{n \to \infty} \sum_{j=0}^{n} d_k^{-j} b_j + \liminf_{n \to \infty} \langle \omega^k, d_k^{-n} \varphi_n \rangle
\]
which is equal to \( b'_\infty(\varphi) \). Hence \( \varphi \) is \( \mu_f \)-integrable if \( \varphi \geq 0 \). In general, write \( \varphi = \varphi^+ - \varphi^- \) and applying the last property shows that \( \varphi \) is \( \mu_f \)-integrable. If \( m = (m_0, m_1) \) is a size representative of \( \varphi \), then we also obtain that

\[
|\langle \mu_f, \varphi \rangle| \leq |b'_{\infty}(\varphi^+)| + |b'_{\infty}(\varphi^-)| \leq c_1 (\|\varphi\|_{L^1} + m_0^{1/2} (d_{k-1} + \epsilon)^{m_1/2}),
\]

for some constant \( c_1 \) independent of \( \varphi \). Now using \( f^* \mu_f = d_k \mu_f \) gives

\[
|\langle \mu_f, \varphi \rangle - b'_\infty(\varphi)| = |\langle \mu_f, d_k^n(f^n)_* \varphi - b'_\infty(\varphi) \rangle| \leq c_n + |\langle \mu_f, d_k^{-n} \varphi_n \rangle|,
\]

where \( c_n := -\sum_{j>n+1} d_k^{-j} |b_j| \). Observe that the first term in the right-hand side of the last inequality tends to 0 because of (3.6). On the other hand, by (3.8) and Lemma 3.6 the second term is bounded by

\[
c_c d_k^{n}(\|\varphi_n\|_{L^1} + m_0^{1/2} d_k^{n/2} (d_{k-1} + \epsilon)^{(m_1+n)/2})
\]

which tends to 0 as \( n \to \infty \). This yields the desired equality. The proof is finished. \( \square \)

**Theorem 3.8.** Let \( X, f, d_k, d_k-1, \epsilon \) be as above with \( d_k > d_k-1 + \epsilon \). Then there exists a constant \( c_c \) such that

\[
I_n(\psi, \varphi) := |\langle \mu_f, (\psi \circ f^n) \varphi \rangle - \langle \mu_f, \psi \rangle \langle \mu_f, \varphi \rangle| \leq c_c \|\psi\|_{L^1} A_n(\varphi),
\]

where

\[
A_n(\varphi) := \left[ \|\varphi\|_{L^1} + m_0^{1/2} (d_{k-1} + \epsilon)^{m_1/2} \right] d_k^{n/2} (d_{k-1} + \epsilon)^{n/2},
\]

for every \( \psi \in L^\infty(\mu_f) \), \( \varphi \in W^{1,2}_{\mathcal{A},f} \) and \( (m_0, m_1) \) a size representative of \( \varphi \).

Note that if \( \varphi \) is a bounded \( \eta \)-p.s.h. function for some continuous \((1, 1)\)-form \( \eta \) of \( L^\infty \)-norm \( \leq 1 \), then there exists a constant \( \tilde{m}_0 \) independent of \( \varphi \) such that \((\tilde{m}_0, 1)\) is a size representative of \( \varphi \). Hence the above theorem gives a decay of correlation uniformly for every such \( \varphi \).

**Proof.** Let the notations be as above. Since \( I_n(\psi, \varphi + c) = I_n(\psi, \varphi) \) for every constant \( c \) because of the invariance of \( \mu_f \). We can assume that \( \langle \mu_f, \varphi \rangle = 0 \). By Lemma 3.7 we get \( b'_{\infty}(\varphi) = 0 \). Hence, \( d_k^{-n}(f^n)_* \varphi = c_n + d_k^{-n} \varphi_n \). Using \( f^* \mu_f = d_k \mu_f \) gives

\[
I_n(\psi, \varphi) = d_k^{-n} |\langle \mu_f, \psi(f^n)_* \varphi \rangle| = |\langle \mu_f, \psi(c_n + d_k^{-n} \varphi_n) \rangle| \leq |c_n| + d_k^{-n} |\langle \mu_f, \varphi_n \rangle|.
\]

Note that as before we have

\[
|c_n| \leq c_c A_n(\varphi)
\]

for some constant \( c_c \) independent of \( \varphi \). On the other hand, \( f_* \) preserves \( W^{1,2}_{\mathcal{A},f} \), hence \( \varphi_n \in W^{1,2}_{\mathcal{A},f} \) and so is \( |\varphi_n| \). By Lemma 3.6 \((d_k^{n} m_0, m_1 + n)\) is a size representative of \( |\varphi_n| \) if \((m_0, m_1)\) is a size representative of \( \varphi \). Arguing as in the proof of Lemma 3.7 gives that

\[
d_k^{-n} |\langle \mu_f, |\varphi_n| \rangle| \leq c_c A_n(\varphi)
\]

for some constant \( c_c \) independent of \( \varphi \). Hence the desired inequality follows. This finishes the proof. \( \square \)

**End of the proof of Theorem 1.2.** The central limit theorem for \( \mu_f \) is a direct consequence of its decay of correlation as shown in (14). Hence it remains to prove the decay of correlation property. By Theorem 3.8 for every \( C^1 \) function \( \varphi \) on \( X \), we have

\[
I(\psi, \varphi) \leq c_c \|\psi\|_{L^1} \|\varphi\|_{C^1} d_k^{-n/2} (d_{k-1} + \epsilon)^{n/2}.
\]
This combined with the interpolation inequality for functionals on the Banach spaces $C^1, C^0$ gives the desired decay of correlation for $\mu_f$, see [14] p. 765.

Recall that $\mu_f$ is K-mixing if for every $\varphi \in L^2(\mu_f)$, we have

$$\sup_{\psi \in L^2(\mu_f)} I_n(\psi, \varphi) \to 0.$$  

(3.10)

Note that the operator $d_k^{-1}f_\ast$ can be extended to a continuous linear operator on $L^2(\mu_f)$ because $|f\ast\varphi|^2 \leq d_k f_\ast(|\varphi|^2)$. As above, in order to prove (3.10), we can assume that $\langle \mu_f, \varphi \rangle = 0$. Using (3.9) gives

$$I(\psi, \varphi) \leq \|d_k^{-n}(f^n)\ast\varphi\|_{L^2(\mu_f)}.$$  

(3.11)

Consider now $\varphi$ to be a bounded function in $W^{1,2}_{\ast\ast,f}$. The set of these functions is dense in $L^2(\mu_f)$. We have

$$\|d_k^{-n}(f^n)\ast\varphi\|_{L^2(\mu_f)} \leq \|\varphi\|_{\infty} \|d_k^{-n}(f^n)\ast\varphi\|_{L^1(\mu_f)}$$

which tends to 0 by the proof of Theorem 3.8. This combined with (3.11) gives (3.10). The proof is finished.

□

Remark 3.9. By the inequality (3.6), we see that for every complex measure $\nu$ with $L^2$ density and $\nu(X) = 1$, $d_k^n(f^n)\ast\nu$ converges to $\mu_f$. This is a slight improvement of Theorem I.1

4. Examples

In this section, we present examples of holomorphic dynamical systems on non-Kähler manifolds. Some of them were already considered by Gromov [26].

4.1. Hopf manifolds. Let $H$ be the standard Hopf manifold $(\mathbb{C}^k\setminus\{0\})/\{z \sim \lambda z\}$, for $\lambda \in \mathbb{C}^\ast$ and $k \geq 2$. Recall that the natural map $p_H : H \to \mathbb{P}^{k-1}$ defined by $(z_1, \ldots, z_k) \mapsto [z_1 : \cdots : z_k]$ is a fiber bundle whose fibers are $\mathbb{C}^\ast/\{t \sim \lambda t\}$ which are compact Riemann surfaces of genus 2. Every holomorphic endomorphism $f$ of $H$ is induced by an endomorphism $F$ of $\mathbb{C}^k\setminus\{0\}$ whose components are homogeneous polynomials of the same degree $\tilde{d}$. Hence $f$ is open and of finite fibers. Clearly $F$ induces naturally a holomorphic endomorphism $f'$ of $\mathbb{P}^{k-1}$ and

$$p_H \circ f = f' \circ p_H.$$  

(4.1)

Using this we get

$$d_k(f) = \tilde{d}^{k+1}, \quad P_n = \tilde{d}^{k+1}$$

where $P_n$ is the set of periodic points of period $n$ of $f$. By [26], the topological entropy $h_t(f)$ is equal to $\log d_k(f)$.

Let $r$ be a strictly positive number. Let $\mathcal{D}(r, \tilde{d})$ be the set of holomorphic maps $F : \mathbb{C}^k\setminus\{0\} \to \mathbb{C}^k\setminus\{0\}$ whose components are homogeneous polynomials of degree $\tilde{d}$ for which

$$\|DF(z)\| \leq r\tilde{d}\|F(z)\|/\|z\|,$$  

(4.2)

for $z \in \mathbb{C}^k\setminus\{0\}$, where the norms are the Euclidean norms. Using

$$\|DF^n(x)\| \leq \|DF(F^{n-1}(x))\| \cdots \|DF(F(x))\| \cdot \|DF(x)\|$$
and (4.2), we infer that

\[
\|DF^n(z)\| \leq r^n \tilde{d}^n\|F^n(z)\|/\|z\|, \tag{4.3}
\]

for \(z \in \mathbb{C}^k \setminus \{0\}\) and \(n \in \mathbb{N}^*\). We can see that the map \(F_0(z) := (z_1^d, \ldots, z_k^d)\) belongs to \(\mathcal{D}(2k, \tilde{d})\) because

\[
\|DF_0(z)\| = \tilde{d} \left( \sum_{j=1}^{k} |z_j|^{2(d-1)} \right)^{1/2} \leq 2kd \left( \sum_{j=1}^{k} |z_j|^{2d} \right)^{1/2} \left( \sum_{j=1}^{k} |z_j|^2 \right)^{-1/2} = 2kd \|F_0(z)\|/\|z\|.
\]

We can construct easily some other examples.

**Lemma 4.1.** Let \(\tilde{d}\) be a positive integer \(\geq 2\) and \(r\) a positive real number \(\geq 1\). Let \(f : H \to H\) be the holomorphic map induced by a map \(F \in \mathcal{D}(r, \tilde{d})\). Then we have \(d_q(f) \leq r^2 \tilde{d}^{q+1}\) for every \(0 \leq q \leq k - 1\). In particular, if \(\tilde{d} > r^2\), then \(f\) has a dominant topological degree, i.e., \(d_k > d_q\) for \(0 \leq q \leq k - 1\).

Arguing as in [18, Pro. 2.7], we can see that as in the case of polynomial-like maps, the property of having a dominant topological degree is preserved under small perturbations. Hence the above lemma provides us a rich class of self-maps with a dominant topological degree.

**Proof.** Let the notation be as above. We already know that \(d_k = \tilde{d}^{k+1}\). Hence, if we can prove the desired inequality, then \(f\) has a dominant topological degree provided that \(\tilde{d} > (r^2 + 2)\). Recall that \(d_0(f) = 1\).

Let \(1 \leq q \leq k - 1\). Let \(\omega_{FS}\) be the Fubini-Study form on \(\mathbb{P}^{k-1}\) and \(p_{\mathbb{C}^k}\) the natural projection from \(\mathbb{C}^k \setminus \{0\}\) to \(\mathbb{P}^{k-1}\). The pull-back \(p_{\mathbb{C}^k}^* \omega_{FS}\) on \(\mathbb{C}^k \setminus \{0\}\) is given by

\[
2p_{\mathbb{C}^k}^* \omega_{FS} = \ddbar c \log(|z_1|^2 + \cdots + |z_k|^2).
\]

Recall here \(\ddbar c = i\partial \bar{\partial} / \pi\). Define

\[
\omega := (2\pi)^{-1}\|z\|^{-2} \sum_{j=1}^{k} dz_j \wedge d\bar{z}_j
\]

and

\[
\eta := \sum_{j=1}^{k} z_j dz_j, \quad \omega' := (2\pi)^{-1}\|z\|^{-4}\eta \wedge \bar{\eta} \geq 0.
\]

Both \(\omega, \omega'\) induce well-defined smooth forms on \(H\) which are denoted by the same notations \(\omega, \omega'\) respectively for simplicity. Observe that \(\omega\) is a Hermitian metric on \(H\). Direct computations show that

\[
\omega = p_{\mathbb{C}^k}^* \omega_{FS} + \omega', \quad \omega' \wedge \omega' = 0. \tag{4.4}
\]

By (4.4), on \(H\), we have

\[
\omega^q = p_{H}^* \omega_{FS}^q + p_{H}^* \omega_{FS}^{q-1} \wedge \omega'
\]

for \(1 \leq q \leq k - 1\). Using the last equality and (4.1) gives

\[
f^* \omega^q = p_{H}^* f^* \omega_{FS}^q + p_{H}^* f^* \omega_{FS}^{q-1} \wedge f^* \omega'.
\]
It follows that
\[(4.5)\]
\[
\int_H (f^n)^* \omega^q \wedge \omega^{k-q} = \int_H p^*_H((f^m)^* \omega^q_{FS} \wedge \omega^{k-q-1}_{FS}) \wedge \omega' + \int_H p^*_H((f^m)^* \omega^{q-1}_{FS} \wedge \omega^{k-q}_{FS}) \wedge (f^n)^* \omega'.
\]
\[
+ \int_H p^*_H((f^n)^* \omega^q_{FS} \wedge \omega^{k-q-1}_{FS}) \wedge (f^n)^* \omega' \wedge \omega' .
\]
Denote by $I_1, I_2, I_3$ the first, second and third integrals in the right-hand side of the last equality. Using Fubini's theorem gives
\[
I_1 = \int_{[z] \in \mathbb{P}k-1} (f^m)^* \omega^q_{FS} \wedge \omega^{k-q-1}_{FS}) \int_{P^*_H([z])} \omega'.
\]
and
\[
I_2 = \int_{[z] \in \mathbb{P}k-1} (f^m)^* \omega^q_{FS} \wedge \omega^{k-q}_{FS}) \int_{P^*_H([z])} (f^m)^* \omega' = \tilde{d}^2n \int_{[z] \in \mathbb{P}k-1} (f^m)^* \omega^q_{FS} \wedge \omega^{k-q}_{FS}) \int_{P^*_H([z])} \omega'.
\]
because the topological degree of $f_{[z]}$ is equal to $\tilde{d}^2$. Thus we get
\[(4.6)\]
\[
\lim_{n \to \infty} I_1^{1/n} = d_q(f') = \tilde{d}^q, \quad \lim_{n \to \infty} I_2^{1/n} = \tilde{d}^2 d_{q-1}(f') = \tilde{d}^{q+1},
\]
where recall that the dynamical degree $d_q(f')$ of $f'$ is equal to $\tilde{d}^q$ for $1 \leq q \leq k - 1$.

It remains to estimate $I_3$. By $(4.4)$,
\[(f^n)^* \omega' \wedge \omega' = (f^n)^* \omega \wedge \omega' - p^*_H(f^n)^* \omega_{FS} \wedge \omega'.
\]
Direct computations show that
\[(F^n)^* \omega = (2\pi)^{-1} \| F^n \|^{-2} \sum_{j=1}^k dF^n_j \wedge d\Gamma^n_j \leq (2\pi)^{-1} k(r\tilde{d})^{2n} \| z \|^{-2} \sum_{j=1}^k dz_j \wedge d\bar{z}_j = k(r\tilde{d})^{2n} \omega
\]
by $(4.3)$. It follows that
\[(f^n)^* \omega' \wedge \omega' \leq k(r\tilde{d})^{2n} \omega \wedge \omega' + p^*_H(f^n)^* \omega_{FS} \wedge \omega' \leq k[(r\tilde{d})^{2n} + 1]p^*_H \omega_{FS} \wedge \omega'
\]
which implies
\[I_3 \leq k[(r\tilde{d})^{2n} + 1] \int_H p^*_H((f^m)^* \omega^{q-1}_{FS} \wedge \omega^{k-q}_{FS}) \wedge \omega' .
\]
Taking the power $1/n$ in the last inequality and letting $n \to \infty$ give
\[(4.7)\]
\[
\limsup_{n \to \infty} I_3^{1/n} \leq r^2 \tilde{d}^2 d_{q-1}(f') = r^2 \tilde{d}^{q+1}
\]
for $1 \leq q \leq k - 1$. Combining $(4.7)$, $(4.6)$ and $(4.5)$ yields
\[d_q(f) \leq \limsup_{n \to \infty} (I_1 + I_2 + I_3)^{1/n} \leq r^2 \tilde{d}^{q+1}.
\]
The proof is finished.
4.2. Calabi-Eckmann manifolds. Let \( \alpha \in \mathbb{C} \setminus \mathbb{R} \). Let \( k, l \) be integers \( \geq 2 \). The Calabi-Eckmann manifold \( X \) is defined by
\[
X := (\mathbb{C}^k \setminus \{0\}) \times (\mathbb{C}^l \setminus \{0\}) / \sim,
\]
where the equivalence relation \( \sim \) is given by \((z, w) \sim (e^t z, e^{i\theta} w)\) for every \( t \in \mathbb{C} \). Recall that \( X \) is diffeomorphic to \( S^{2k-1} \times S^{2l-1} \) and non-Kähler because \( H^2(X) = \{0\} \). For every homogeneous polynomials \( F(z), G(w) \) of the same degree, the self-map of \((\mathbb{C}^k \setminus \{0\}) \times (\mathbb{C}^l \setminus \{0\})\) given by \((z, w) \mapsto (F(z), G(w))\) can descend to a self-map of \( X \). It is likely that we can obtain a good class of self-maps of \( X \) with dominant topological degree as above.

4.3. Nilmanifolds. Consider \( G \) a complex Lie group and \( \Gamma \) a closed complex Lie subgroup of \( G \) such that \( X := G/\Gamma \) is a compact non-Kähler manifold of dimension \( k \). By [1], nilmanifolds which are not tori are examples of such manifolds. For such \( X \), every \( g \in G \) and every \( A \in \text{Aut}(G) \) preserving \( \Gamma \), the affine transformation \( gA \) induces a holomorphic automorphism on \( X \). In the real setting, the dynamical systems associated to such maps possess interesting properies and has been studied extensively. We refer to [34, 25, 5] for informations.

4.4. Blowups. Let \( X \) be a compact complex manifold. Let \( \tilde{X} \) be a compact manifold bimeromorphic to \( X \) via a map \( \sigma : \tilde{X} \to X \). Then given a meromorphic self-correspondence \( f \) on \( X \), \( f_\sigma := \sigma^{-1} \circ f \circ \sigma \) is a self-correspondence on \( \tilde{X} \). We can take, for example, \( X \) to be a nilmanifold, a Hopf manifold or a Calabi-Eckmann manifold and \( \tilde{X} \) to be the blowup of \( X \) along a smooth submanifold \( V \) of \( X \) (a point for example). By a well-known example of Hironaka [29], there exist a compact Kähler manifold \( X \) and a non-Kähler manifold \( \tilde{X} \) bimeromorphic to \( X \). Such manifolds \( \tilde{X} \) are in the class \( \mathcal{C} \) of Fujiki. Due to the lack of a Kähler form, we don’t know whether the dynamical degrees of \( f \) and \( f_\sigma \) are the same even for \( X \) Kähler.

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