A Type-Directed, Dictionary-Passing Translation of Featherweight Generic Go

MARTIN SULZMANN  
Karlsruhe University of Applied Sciences, Germany  
e-mail: martin.sulzmann@h-ka.de

STEFAN WEHR  
Offenburg University of Applied Sciences, Germany  
e-mail: stefan.wehr@hs-offenburg.de

Abstract

Featherweight Generic Go (FGG) is a minimal core calculus modeling the essential features of the programming language Go. It includes support for overloaded methods, interface types, structural subtyping and generics. The most straightforward semantic description of the dynamic behavior of FGG programs is to resolve method calls based on runtime type information of the receiver.

This article shows a different approach by defining a type-directed translation from FGG to an untyped lambda-calculus. The translation of an FGG program provides evidence for the availability of methods as additional dictionary parameters, similar to the dictionary-passing approach known from Haskell type classes. Then, method calls can be resolved by a simple lookup of the method definition in the dictionary.

Every program in the image of the translation has the same dynamic semantics as its source FGG program. The proof of this result is based on a syntactic, step-indexed logical relation. The step-index ensures a well-founded definition of the relation in the presence of recursive interface types and recursive methods.

1 Introduction

Go (2022) is a statically typed programming language introduced by Google in 2009. It supports method overloading by allowing multiple declarations of the same method signature for different receivers. Receivers are structs, similar to structs in C. The language also supports interfaces; as in many object-oriented languages, an interface consists of a set of method signatures. But unlike in many object-oriented languages, subtyping in Go is structural not nominal.

Earlier work by Griesemer et al. (2020) introduces Featherweight Go (FG), a minimal core calculus that covers method overloading, structs, interfaces and structural subtyping. Their work specifies static typing rules and a dynamic semantics for FG based on runtime method lookup. However, the actual Go implementation appears to employ a different dynamic semantics. Quoting Griesemer and co-workers: “Go is designed to enable efficient implementation. Structures are laid out in memory as a sequence of fields, while an
interface is a pair of a pointer to an underlying structure and a pointer to a dictionary of methods.”

In our own prior work (Sulzmann and Wehr, 2021, 2022), we formalize a type-directed dictionary-passing translation for FG and establish its semantic equivalence with FG’s dynamic semantics. Griesemer and coworkers also introduce Featherweight Generic Go (FGG), an extension of FG with generics. In this work, we extend our translation approach to FGG. Our contributions are as follows:

• We specify the translation of source FGG programs without type assertions to an untyped $\lambda$-calculus with recursive let-bindings, constructors and pattern matching. We employ a dictionary-passing translation scheme à la type classes (Hall et al., 1996) to statically resolve overloaded method calls. The translation is guided by the typing of the FGG program.
• We establish the semantic correctness of the dictionary-passing translation. The result relies on a syntactic, step-indexed logical relation to ensure well-foundedness of definitions in the presence of recursive interface types and recursive methods.
• We explain how to extend the translation to support type assertions.
• We present an implementation of the translation, including support for type assertions.

The upcoming Section 2 gives an overview of our translation by example. Section 3 gives a recap of the source language FGG, whereas Section 4 defines the target language and the translation itself. Next, Section 5 establishes the formal properties of the translation, rigorous proofs of our results can be found in the Appendix. Section 6 presents the implementation and explains the handling of type assertions. Section 7 covers related work. Finally, Section 8 summarizes this work and points out directions for future work.

2 Overview

This section introduces Featherweight Generic Go (FGG, Griesemer et al., 2020) and our type-directed dictionary-passing translation through a series of examples. FGG is a tiny model of Go that includes essential typing features such as method overloading, structs, interfaces, structural subtyping, and the extension with generics.

An FGG program consists of declarations for structs, interfaces, methods, and a main function. Function and method bodies only contain a single return statement, all expression are free from side effects. For the examples in this section, we extend FGG with primitive types for integers and strings, with an operator + for string concatenation and a builtin function intToString, with definitions of local variables, and with function definitions.

We will first consider FGG without generics to highlight the idea behind our type-directed dictionary-passing translation scheme. We refer to FGG without generics as Featherweight Go (FG). Then, we show how the translation scheme for FG can be adapted to deal with the addition of generics. All examples have been checked against our implementation of the translation.

1 https://github.com/skogsbaer/fgg-translate
2.1 Featherweight Go

The upper part of Figure 1 shows an FG program for formatting values as strings. The code does not use generics yet.

Structs in Go are similar to structs in C, a syntactic difference is the Go convention that field or variable names precede their types. Here, struct Num has a single field val of type int, so it simply acts as a wrapper for integers.

Methods and functions are introduced via the keyword func. A method can be distinguished from a function as the receiver argument always precedes the method name. Methods can be overloaded on the receiver type. In lines 5 and 6, we find methods format and pretty, respectively, for receiver type Num. Lines 8 and 10 defines two functions.

Interfaces in Go declare sets of method signatures sharing the same receiver where method names must be distinct and the receiver is left implicit. Interfaces are types and describe all receivers that implement the methods declared by the interface. In our example, interface Format declares a method format for rendering its receiver as a string. The second interface Pretty also declares format, but adds a second method pretty with the intention to produce a visually more attractive output.

```
type Num struct { val int }
type Format interface { format() string }
type Pretty interface { format() string; pretty() string }

func (this Num) format() string { return intToString(this.val) }
func (this Num) pretty() string { return this.format() }

func formatSome(x Format) string { return x.format() }

func main() {
    var s1 string = formatSome(Num{1})
    var pr Pretty = Num{2}
    var s2 string = formatSome(pr)
}
```

Fig. 1. String-formatting and its translation
Interfaces and method definitions imply structural subtype relations. Interface `Format` contains a subset of the methods declared by interface `Pretty`. Hence, `Pretty` is a structural subtype of `Format`, written (1) `Pretty <: Format`. Line 5 defines a method `format` for receiver type `Num`, where the method body assumes a built-in function `intToString` for converting integers to strings. We say that `Num` implements method `format`. Hence, `Num` is a structural subtype of `Format`, written (2) `Num <: Format`. Receiver `Num` also implements the `pretty` method, see line 6. Hence, we also find that (3) `Num <: Pretty`. Structural subtype relations play a crucial role when typechecking programs.

For example, consider the function call `formatSome(Num{1})` in line 11. From above we find that (2) `Num <: Format`. That is, `Num` implements the `Format` interface and therefore the function call typechecks. Consider the variable declaration and assignment in line 12. Value `Num{2}` is assigned to a variable of interface type `Pretty`. Based on the subtype relation (3) `Num <: Pretty` the assignment typechecks. Consider the function call `formatSome(pr)` in line 13. Based on the subtype relation (1) `Pretty <: Format` the function call typechecks.

In Griesemer et al. (2020), the dynamic behavior of programs is explained via runtime lookup of methods, where based on the receiver’s runtime type the appropriate method definition is selected. The Go conditions demand that for each method name and receiver type there can be at most one definition. This guarantees that method calls can be resolved unambiguously.

### 2.2 Type-Directed Translation

We explain the meaning of Go programs by means of translation to an untyped \(\lambda\)-calculus with recursive top-level definitions, let-bindings, pattern matching, integers, strings, an operator `++` for string concatenation, and a built-in function `intToString`. We will use a Haskell-style notation.

Method definitions belonging to an interface are grouped together in a dictionary of methods. Thus, method calls can be turned into primitive function calls by simply looking up the appropriate method in the dictionary. Structural subtype relations are turned into coercion functions that transform, for example, a struct value into an interface value to make sure that the appropriate dictionaries are available. Where to insert dictionaries and coercions in the program is guided by the typechecking rules. Hence, the translation is type-directed.

Our translation strategy can be summarized as follows:

**Struct.** An FGG value at the type of a struct with \(n\) fields is represented by an \(n\)-tuple holding the values of the fields. We call such an \(n\)-tuple a *struct value*.

**Interface.** An FGG value at the type of an interface is represented as a pair \((V, D)\), where \(V\) is a struct value and \(D\) is a *method dictionary*. Such a method dictionary is a tuple holding implementations of all interface methods for \(V\), in order of declaration in the interface. We call the pair \((V, D)\) an *interface value*.

**Coercion.** A structural subtype relation \(\tau <: \sigma\) implies a *coercion function* to transform the target representation of an FGG value of type \(\tau\) into a representation at type \(\sigma\).
The lower part of Figure 1 gives the translation of our running example. In this overview section, we identify a 1-tuple with the single value it holds.

For each field name, we assume a helper function to access the field component, see line 16. Method calls on interface values lookup the respective method definition in the dictionary and apply it to the struct value embedded inside the interface value. See lines 19-21. Method definitions translate to plain functions, see lines 24-25. Recall that for each method name and receiver type there can be at most one definition. Hence, the generated function names are all distinct.

Structural subtype relations translate to coercions, see lines 28-30. For example, (2) Num :<: Format translates to the toFormatNum coercion. Input parameter x represents a target representation of a Num value. The output (x, formatNum) is an interface value holding the receiver and the corresponding method definition. Coercion toPrettyNum corresponds to (3) Num :<: Pretty and coercion toFormatPretty to (1) Pretty :<: Format.

The translation of the main function, starting at line 35, is guided by the typechecking of the source program. Each application of a structural subtype relation leads to the insertion of the corresponding coercion function in the target program. For example, the function call formatSome(Num{1}) translates to formatSome (toFormatNum 1) because typing of the source requires (2) Num :<: Format. The other coercions arise for similar reasons.

2.3 Adding Generics

We extend our running example by including pairs, see Figure 2. The struct type Pair[T Any, U Any] is generic in the type of the pair components, T and U are type variables. When introducing type variables we must also specify an upper type bound to constrain the set of concrete types that will replace type variables. The bounded type parameter T Any can therefore be interpreted as \( \forall T. T <: Any \). Upper bounds are always interface types. The upper bound Any is satisfied by any type because the set of methods that need to be implemented is empty.

To format pairs, we need to format the left and right component that are of generic type T and U. Hence, the method definition in line 4 states the type bound Format for type variables T and U. In general, bounds of type parameters for the receiver struct of a method declaration must be in a covariant subtype relation relative to the bounds in the struct declaration. This is guaranteed in our case as we find Format <: Any. Importantly, the type bounds in line 4 imply the subtype relations (4) T <: Format and (5) U <: Format. Thus, we can show that the method body typechecks. For example, expression this.left is of type T. Based on (4), this expression is also of type Format and therefore the method call in line 5. this.left.format() typechecks.

We consider typechecking the main function. Instances for generic type variables must always be explicitly supplied. Hence, when constructing a pair that holds number values, see line 9, we find Pair[Num, Num].

Consider the method call p.format() in line 10. The receiver struct Pair[T Format, U Format] of the method definition in line 4 matches p’s type Pair[Num, Num] by replacing T and U by Num. The type bounds in the receiver type are satisfied as we know from above that (2) Num :<: Format. Hence, the method call typechecks.
By generalizing the above argument we find that

\[ \{ T <: \text{Format}, U <: \text{Format} \} \vdash \text{Pair}[T, U] <: \text{Format}. \]

That is, under the assumptions \( T <: \text{Format} \) and \( U <: \text{Format} \) we can derive that \( \text{Pair}[T, U] <: \text{Format} \). In particular, we find that \( \text{Pair}[\text{Num}, \text{Num}] <: \text{Format} \). Hence, the function call \( \text{formatSome}(p) \) in line 11 typechecks.

Extending our type-directed translation scheme to deal with generics turns out to be fairly straightforward.

**Bounded type parameter.** A bounded type parameter \( T \text{Ifce} \) where \( T \) is a type variable and \( \text{Ifce} \) is an interface type becomes a coercion parameter \( \text{toIfce}_{T} \). At instantiation sites, coercions need to be inserted.

The lower part of Figure 2 shows the translated program. Starting at line 18 we find the translation of the definition of method \( \text{format} \) for pairs. Each bounded type parameter \( T \text{Format} \) and \( U \text{Format} \) is turned into a coercion parameter \( \text{toFormat}_{T} \) and \( \text{toFormat}_{U} \). In the target, we use a curried function definition where coercion parameters are collected in a tuple.

A method call of \( \text{format} \) needs to supply concrete instances for these coercion parameters. See line 27 which is the translation of calling \( \text{format} \) on receiver type \( \text{Pair}[\text{Num}, \text{Num}] \). Hence, we must pass as the first argument the tuple of coercions \( (\text{toFormat}_{\text{Num}}, \text{toFormat}_{\text{Num}}) \) to \( \text{formatPair} \).
Subtype relation (6) implies the (parameterized) coercion toFormat\textsubscript{Pair} in line 22. Given coercions toFormat\textsubscript{T} and toFormat\textsubscript{U} we can transform a pair p into an interface value for Format, where the method dictionary consists of the partially applied translated method definition format\textsubscript{Pair}.

We make use of toFormat\textsubscript{Pair} in the translation of the function call formatSome(p), see line 28. Based on the specific coercion toFormat\textsubscript{Num}, the call toFormat\textsubscript{Pair} transforms the pair value p into the interface value \( (p, \text{format\textsubscript{Pair}}(\text{toFormat\textsubscript{Num}}, \text{toFormat\textsubscript{Num}})) \). Then, we call formatSome on this interface value.

### 2.4 Bounded type parameters of structs and interfaces

We could strengthen the type bound in the definition for pairs by replacing Any with Format.

```go
type PPair[T Format, U Format] struct { left T; right U }
```

Such stronger type bounds only serve a purpose to rule out more programs (statically). For example, the following program will not typecheck because Int does not implement the Format interface.

```go
type Int struct { iVal int }
func main2b() {
    // Will not typecheck because Int <: Format does not hold
    var p PPair[Int, Int] = PPair[Int, Int]{ Int{1}, Int{2} }
}
```

In general, type bounds in the definition of structs have no meaning at runtime. Hence, in our translation scheme such type bounds can effectively be ignored. The same applies to type bounds that arise in the definition of generic interfaces. We will see an example of a generic interface in the later Section 4.3.

### 2.5 Bounded type parameters of methods

There may be bounded type parameters local to methods. Consider Figure 3 where we further extend our running example. Starting at line 1 we find a definition of method formatSep for pairs. This method takes an argument s that acts as a separator when formatting pairs. Argument s is of the generic type S constrained by the type bound S Format. Type parameter S is local to the method and not connected to the receiver struct. Type arguments for S must also be explicitly specified in the program text, see method calls in lines 8 and 13.

In the translation, bounded type parameters of methods simply become additional coercion parameters. Consider the translation of formatSep defined on pairs starting at line 21. The translated method definition first expects the coercion parameters (toFormat\textsubscript{T}, toFormat\textsubscript{U}) that result from the bounded type parameters T Format and U Format of the receiver. Then, we find the receiver argument this followed by the coercion parameter toFormat\textsubscript{S} resulting from S Format, and finally the method argument s. The translation of the method body follows the scheme we have seen so far, see lines 22-24. When calling method formatSep on a pair we need to provide the appropriate coercions, see line 37.
From the method definition of `formatSep` for pairs and from the definition of interface `FormatSep`, we find that the following subtype relation holds:

\[
\{ T <: \text{Format}, U <: \text{Format} \} \vdash \text{Pair}[T, U] <: \text{FormatSep}.
\]

Subtype relation (7) implies the coercion `toFormatSepPair` in line 28. Thus, the function call of `formatSepSome` from line 14 translates to the target code starting in line 38.

The point to note is that a coercion parameter corresponding to a bounded type parameter of a method is not part of the dictionary; it is only supplied at the call site of the method. Consider the call `x.formatSep[Format](s)` in line 8. In the translation (line 33), we first partially apply the respective dictionary entry on the receiver. This is done via the target expression `(formatSep[Format] x)`. Type `Format` is a valid instantiation for type parameter `S` of `formatSep` because `Format <: Format` in FGG. In the translation, this corresponds to the (identify) coercion `toFormatFormat`. Hence, we supply the remaining arguments `toFormatFormat` and `s`.

```go
func (this Pair[T Format, U Format])(s S) string {
    return this.left.format() + s.format() + this.right.format()
}

type FormatSep interface { formatSep[S Format](s S) string }

func formatSepSome(x FormatSep, s Format) string {
    return x.formatSep[Format](s)
}

func main3 () {
    var p Pair[Num, Num] = Pair[Num, Num]{ Num{1}, Num{2} }
    var s1 string = p.formatSep[Num](Num{3}) // result: 132
    var s2 string = formatSepSome(p,Num{4}) // result: 142
}

-- Method call on interface
-- call formatSep on receiver of type FormatSep
formatSepFormatSep (x, f) = f x

-- Method definition (line 1 in the FGG code)
formatSepPair (toFormatT, toFormatU) this toFormatS s =
    formatFormat (toFormatT (left this)) ++
    formatFormat (toFormatS s) ++
    formatFormat (toFormatU (right this))

-- Coercions
toFormatFormat x = x -- Format <: Format
toFormatSepPair (toFormatT, toFormatU) p =
    -- Pair[T, U] <: FormatSep (given T <: Format and U <: Format)
    (p, formatSepPair (toFormatT, toFormatU))

-- Function definitions (lines 7 and 11 in the FGG code)
formatSepSome (x, s) = (formatSepFormatSep x) toFormatFormat s

main3 =
    let p = (1,2)
        s1 = formatSepPair (toFormatNum, toFormatNum) p toFormatNum 3
        s2 = formatSepSome
            (toFormatSepPair (toFormatNum, toFormatNum) p,
             toFormatNum 4)
```

Fig. 3. Bounded type parameters of methods (extending code from Figure 2)
2.6 Outlook

Next, we formalize FGG following the description by Griesemer et al. (2020) (Section 3). Then, we give the details of our type-directed translation scheme (Section 4) and establish that the meaning of FGG programs is preserved (Section 5).

3 Featherweight Generic Go

Featherweight Go (FG, Griesemer et al., 2020) is a small subset of the full Go language (2022) supporting only essential features such as structs, interfaces, method overloading and structural subtyping. In the same article, the authors add generics to FG with the goal to scale the design to full Go. The resulting calculus is called Featherweight Generic Go (FGG). Since version 1.18, full Go includes generics as well, but with limited expressivity compared to the FGG proposal (see Section 7.1). For the translation presented in this article, we stick to the original FGG language with minor differences in presentation but excluding dynamic type assertions. Section 6 sketches how to extend the translation with dynamic type assertions.

The next two subsections introduce the syntax and the dynamic semantics of FGG. We defer the definition of its static semantics until Section 4.2, where we specify it as part of the type-directed dictionary-passing translation.

3.1 Syntax

Figure 4 introduces the syntax of FGG without dynamic type assertions. We assume several countably infinite, pairwise disjoint sets for names, ranged over by \( \mathcal{N} \) with some subscript (upper part of the figure). Meta variables \( t_s \) and \( u_s \) denote struct names, \( t_I \) and \( u_I \) interface names, \( \alpha \) and \( \beta \) type variables, \( f \) field names, \( m \) method names, and \( x, y \) denote names for variables in expressions. Overbar notation \( \overline{s}^n \) is a shorthand for the sequence \( s_1 \ldots s_n \), where \( s \) is some syntactic construct. In some places, commas separate the sequence items.

| Struct name                               | \( t_S, u_S \in \mathcal{N}_{struct} \) |
| Interface name                            | \( t_I, u_I \in \mathcal{N}_{iface} \) |
| Type variable name                        | \( \alpha, \beta \in \mathcal{N}_{tyvar} \) |
| Type name                                 | \( t, u ::= t_S \mid t_I \) |
| Type                                      | \( \tau, \sigma ::= \alpha \mid t[\overline{\tau}] \) |
| Expression                                | \( e, g ::= x \mid e.m[t](\overline{\tau}) \mid \tau.S(\overline{\tau}) \mid e.f \) |
| Method signature                          | \( R ::= m[\overline{\alpha\overline{\tau}}](\overline{x})(\overline{\tau}) \) |
| Declaration                               | \( D ::= \text{type } t_S[\alpha t_I] \mid \text{struct } \{ f \}\mid \text{type } t_I[\alpha t_I] \mid \text{interface } \{ R \} \mid \text{func } (x t_S[\overline{\alpha\overline{t_I}}]) R \{ \text{return } e \} \) |
| Program                                   | \( P ::= D \mid \text{func } \text{main}() \{ = e \} \) |

Fig. 4. Syntax of Featherweight Generic Go (FGG) without type assertions
If irrelevant, we omit the \( n \) and simply write \( \bar{s} \). Using the index variable \( i \) under an over-bar marks the parts that vary from sequence item to sequence item; for example, \( \bar{s} s^i_i \) abbreviates \( s' s_1 \ldots s' s_n \) and \( \bar{s}^j_j \) abbreviates \( s_j s_{j1} \ldots s_j s_q \).

The middle part of Figure 4 shows the syntax of types in FGG. A type name \( t, u \) is either a struct or interface name. Types \( \tau, \sigma \) include types variables \( \alpha \) and instantiated types \( t[\bar{\tau}] \). For non-generic structs or interfaces, we often write just \( t \) instead of \( t[] \). Struct types \( \tau_S \), \( \sigma_S \) and interface types \( \tau_I \), \( \sigma_I \) denote syntactic subsets of the full type syntax.

The lower part of Figure 4 defines the syntax of FGG expressions, declarations, and programs. Expressions, ranged over by \( e \) and \( g \), include variables \( x \), method calls, struct literals, and field selections. A method call \( e.m[\bar{\tau}](\bar{x}) \) invokes method \( m \) on receiver \( e \) with type arguments \( \bar{\tau} \) and arguments \( \bar{x} \). If \( m \) does not take type arguments, we often write just \( e.m(\bar{x}) \). A struct literals \( \tau_S(\bar{\sigma}^0) \) creates an instance of a struct with \( n \) fields, the arguments \( \bar{\sigma}^0 \) become the values of the fields in order of appearance in the struct definition. A field selection \( e.f \) projects the value of some struct field \( f \) from expression \( e \).

A method signature \( R ::= m[\bar{\tau}](\bar{x}) \tau \) consists of a name \( m \), bounded type parameters \( \alpha_i \) with interface type \( \tau_I \) as upper bounds, parameters \( x_i \) of type \( \tau_i \), and return type \( \tau \). It binds \( \bar{x} \) and \( \bar{\tau} \). The scope of a type variable \( \alpha_i \) is \( \bar{\tau} \), \( \tau \), and all upper bounds \( \bar{\tau}_I \), so FGG supports F-bounded quantification (Canning et al., 1989). For non-generic methods, we often write just \( m(\bar{x}_I) \tau \).

A declaration \( D \) comes in three forms: a struct \( \text{type } \tau_S[\bar{\alpha} \bar{\tau}_I] \text{ struct } \{ \bar{f} \ \tau \} \) with fields \( f_i \) of type \( \tau_i \); an interface \( \text{type } \bar{\tau}_I[\bar{\alpha} \bar{\tau}_I] \text{ interface } \{ \bar{R} \} \) with method signatures \( \bar{R} \); or a method \( \text{func } (x \ \bar{\tau}_S[\bar{\alpha} \bar{\tau}_I]) \ R \{ \text{return } e \} \) providing an implementation of method \( R \) for struct \( \bar{\tau}_S \). All three forms bind the type variables \( \bar{\tau}_I \), a method implementation additionally binds the receiver parameter \( x \). The scope of a type variable \( \alpha_i \) includes all upper bounds \( \bar{\tau}_I \), the body of the declaration enclosed in \{ . . . \}, and for method declarations also the signature \( R \). We omit the \( [\bar{\alpha} \bar{\tau}_I] \) part completely if \( \bar{\alpha} \bar{\tau}_I \) is empty. Finally, a program \( P \) consists of a sequence of declaration together with a main function. Method and function bodies only contain a single expression. We follow the usual convention and identify syntactic constructs up to renaming of bound variables or type variables.

The syntax of FGG as presented here differs slightly from its original form (Griesemer et al., 2020). The original article encloses type parameters in parenthesis, an additional \textbf{type} keyword starts a list of type parameters. Here, we follow the syntax of full Go and use square brackets without any keyword. Further, the original article prepends \textbf{package} \textbf{main} to each program, something we omit for succinctness. Finally, we reduce the number of syntactic meta-variables to improve readability.

### 3.2 Dynamic Semantics

Figure 5 defines a call-by-value dynamic semantics for FGG using a small-step reduction semantics with evaluation contexts. The definition is largely taken from Griesemer et al. (2020).

We use \( v, u, w \) to denote values, where a value is a struct literal with all fields being values. A call-by-value evaluation context \( E \) is an expression with a hole \( \square \) such that the hole marks the point where the next evaluation step should happen. We write \( E[e] \) to denote the replacement of the hole in \( E \) with expression \( e \). A value substitution \( \theta \) is a finite
mapping \(\langle x \mapsto v \rangle\) from variables to values, whereas a type substitution \(\eta\) is a finite mapping \(\langle \alpha \mapsto \tau \rangle\) from type variables to types. The (type) variables in the domain of a substitution must be distinct. Substitution application, written in prefix notation as \(\theta e\) or \(\eta e\) or \(\eta \tau\), is defined in the usual, capture-avoiding way. When combining two sequences, we implicitly assume that both sequences have the same length. For example, combining variables \(\overline{x}\) and values \(\overline{v}\) to a substitution \(\langle \overline{x} \mapsto \overline{v} \rangle\) implicitly assumes that there are as many variables as values.

The reduction relation \(e \longrightarrow e'\) denotes that expression \(e\) reduces to expression \(e'\). To avoid clutter, the sequence of declarations \(\overline{D}\) of the underlying program is implicitly available in the rules defining this reduction relation. Rule FG-CONTEXT applies a reduction step inside an expression. Rule FG-FIELD reduces a field selection \(\tau_S[\overline{\tau}\{\overline{\tau}\}]\).fi by extracting value \(v_i\) corresponding to field \(f_i\) from the struct literal. Rule FG-CALL reduces a method call \(\tau_S[\overline{\tau}\{\overline{\tau}\}].m[\overline{\tau}]\) to get a method definition for \(m\) and \(\tau_S\) and substitutes type arguments, receiver, and value arguments in the method body.

Reduction in FGG is deterministic (see Lemma A.1.1 in Appendix A.1 for a formal proof), assuming the following three restrictions:

**FGG-UNIQUE-STRUCTS** Each struct \(\tau_S\) is defined at most once in the program.

**FGG-DISTINCT-FIELDS** For each struct definition \(\textbf{type} \tau_S[\alpha \tau]\ \textbf{struct} \ \{f \tau\}\), field names \(\overline{f}\) are distinct.

**FGG-UNIQUE-METHOD-DEFS** Each method definition \(\textbf{func} (x \tau_S[\alpha \tau]\ m[\alpha' \tau'](x \sigma) \sigma \{\textbf{return}\ e\})\) is uniquely identified by struct name \(\tau_S\) and method name \(m\).

The first two restrictions ensures that the value for a field in rule FG-FIELD is unambiguous. The third restriction avoids multiple matching method definitions in rule FG-CALL.
4 Type-directed translation

This section defines a type-directed, dictionary-passing translation from FGG to an untyped \(\lambda\)-calculus extended with recursive let-bindings, constructors and pattern matching. We first introduce the target language, then specify the translation itself, and last not least give some examples. Formal properties of the translation are deferred until Section 5.

4.1 Target Language

Figure 6 defines the syntax and the call-by-value dynamic semantics of the target language (TL). We use uppercase letters for constructs of the target language. Variables \(X, Y\) and constructors \(K\) are drawn from countably infinite, pairwise disjoint sets \(\mathcal{V}_{\text{Var}}\) and \(\mathcal{V}_{\text{Con}}\), respectively. Expressions, ranged over by \(E\) and \(G\), include variables \(X\), constructors \(K\), function applications \(E \rightarrow E'\), \(\lambda\)-abstractions \(\lambda X. E\), and pattern matching via case-expressions \(\text{case } E \text{ of } \mathcal{P} \rightarrow E\). Patterns \(\mathcal{P}\) have the form \(K X\), they do not nest. We assume that all constructors in \(\mathcal{P}\) are distinct. To avoid some parenthesis, we use the conventions that application binds to the left and that the body of a \(\lambda\) extends to the right as far as possible.
A program \texttt{let } \(X = V\) \texttt{in } \(E\) consists of a sequence of (mutually recursive) definitions and a (main) expression, where we assume that the variables \(X\) are distinct. In the translation from FGG, the values \(V\) are always functions resulting as translations of FGG methods. We identify expressions, pattern clauses and programs up to renaming of bound variables. Variables are bound by \(\lambda\) expressions, patterns, and let-bindings of programs.

Some syntactic sugar simplifies the construction of patterns, expressions and programs. (a) We use nested patterns to abbreviate nested case-expressions. (b) We assume data constructors for tuples up to some fixed but arbitrary size. The syntax \((E^n)\) constructs an \(n\)-tuple when used as an expression, and \((\text{Pat}^n)\) deconstructs it when used in a pattern context. (c) We use patterns in \(\lambda\)-expressions; that is, the notation \(\lambda\text{Pat}.E\) stands for \(\lambda X.\text{case } X \text{ of } \text{Pat} \to E\) where \(X\) is fresh.

Target values \(V, U, W\) are either \(\lambda\)-expressions or constructors applied to values. A constructor value \(K \text{Pat}^n\) is short for \((\ldots (K \text{V}_1) \ldots) \text{V}_n\). A call-by-value evaluation context \(\mathcal{R}\) is an expression with a hole \(\Box\) such that the hole marks the point where the next evaluation step should happen. We write \(\mathcal{R}[E]\) to denote the replacement of the hole in \(\mathcal{R}\) with expression \(E\).

A substitution \(\rho, \mu\) is a finite mapping \(\langle X \mapsto V \rangle\) from variables to values. The variables \(X\) in the domain must be distinct. Substitution application, written in prefix notation \(\rho E\), is defined in the usual, capture-avoiding way. We use two different meta variables \(\mu\) and \(\rho\) for substitutions in the target language with the convention that the domain of \(\mu\) contains only top-level variables bound by \texttt{let}. As top-level variables result from translating FGG methods, we sometimes call \(\mu\) a method substitution.

The reduction semantics for the target language is defined by two relations: \(E \rightarrow_{\mu} E'\) reduces expression \(E\) to \(E'\) under method substitution \(\mu\), and \(\text{Prog} \rightarrow \text{Prog}'\) reduces \(\text{Prog}\) to \(\text{Prog}'\). The definition of the latter simply forms a method substitution \(\mu\) from the top-level bindings of \(\text{Prog}\) and then reduces the main expression of \(\text{Prog}\) under \(\mu\) (rule \texttt{TL-Prog}). We defer the substitution of top-level–bound variables because they might be recursive.

The definition of the reduction relation for expressions extends over four rules. Rule \texttt{TL-CONTEXT} uses evaluation context \(\mathcal{R}\) to reduce inside an expression, rule \texttt{TL-LAMBDA} reduces function application in the usual way. Pattern matching in rule \texttt{TL-CASE} assumes that the scrutinee is a constructor value \(K \text{Pat}^n\); the lookup of a pattern clause matching \(K\) yields at most one result as we assume that clauses have distinct constructors. During a sequence of reduction steps, a variable bound by \texttt{let} at the top-level might become a redex, as only \(\lambda\)-bound variables are substituted right away. Thus, rule \texttt{TL-METHOD} finds the value for the variable in the method substitution \(\mu\).

\subsection*{4.2 Translation}

Before we dive into the technical details, we summarize our translation strategy.

\textbf{Struct.} An FGG value at the type of a struct is represented in the TL as a \textit{struct value}; that is, a tuple \((\text{Pat}^n)\) where \(n\) is the number of fields and \(\text{V}_i\) represents the \(i\)-th field of the struct.
Interface. An FGG value at the type of an interface is represented in the TL as an interface value; that is a pair \((V, D)\), where \(V\) is a struct value realizing the interface and \(D\) is a dictionary.

Dictionary. A dictionary \(D\) for an interface with methods \(I\) is a tuple \((V)\) such that \(V_i\) is a dictionary entry for method \(R_i\).

Dictionary entry. A dictionary entry for a method with signature \(R = m[\alpha \tau_f](\bar{x} \sigma)\sigma\) is a function accepting a triple: (1) receiver, (2) tuple with coercions corresponding to the bounded type parameters \(\alpha \tau_f\) of the method, (3) tuple for parameters \(\bar{x}\).

Coercion. A structural subtype relation \(\tau <: \sigma\) implies a coercion function to transform the target representation of an FGG value at type \(\tau\) into a representation at type \(\sigma\).

Bounded type parameter. A bounded type parameter \(\alpha \tau_f\) becomes a coercion parameter transforming the type supplied for \(\alpha\) to its bound \(\tau_f\). At instantiation sites, coercions need to be inserted.

Method declaration. A method declaration \(\text{func}(x t_f[\alpha \tau_f]) m[\alpha' \tau_f'](\bar{x} \sigma)\sigma \{\text{return } e\}\) is represented as a top-level function \(X_{m,t_f}\) accepting a quadruple: (1) tuple with coercions corresponding to the bounded type parameters \(\alpha \tau_f\) of the receiver, (2) receiver \(x\), (3) tuple with coercions corresponding to bounded type parameters \(\alpha' \tau_f'\) of the method, (4) tuple for parameters \(\bar{x}\).

In essence, the above is a more detailed description of the translation scheme motivated in Section 2. The only difference is that dictionary entries and translations of methods are now represented as uncurried functions. For example, instead of the curried representation in Figure 3

\[
\text{formatWithSepPair (toFormat}_T, \text{toFormat}_U) \text{this toFormat}_S x = \ldots
\]

\[
\text{toFormatWithSepPair (toFormat}_T, \text{toFormat}_U) p =
(p, \text{formatWithSepPair (toFormat}_T, \text{toFormat}_U))
\]

our actual translation scheme uses uncurried functions, as in the following code:

\[
\text{formatWithSepPair ((toFormat}_T, \text{toFormat}_U), \text{this}, \text{toFormat}_S, x) = \ldots
\]

\[
\text{toFormatWithSepPair (toFormat}_T, \text{toFormat}_U) p =
(p, \{(\text{this,locals,arg}) ->
\text{formatWithSepPair ((toFormat}_T, \text{toFormat}_U),locals,\text{arg})\})
\]

Using an uncurried representation instead of a curried representation is just a matter of taste. As we have carried out the semantic equivalence proof initially based on the uncurried representation, we stick to the uncurried representation from now on.

4.2.1 Conventions and Notations

The translation relies on three total, injective functions with pairwise disjoint ranges for mapping FGG names to TL variables. The first function \(N_{\text{var}} : V_{\text{var}}\) translates a FGG variable \(x\) to a TL variable \(X\). To avoid clutter, we do not spell out the translation function explicitly but use the abbreviation that a lowercase \(x\) always translates into its uppercase counterpart \(X\). The second function \(N_{\text{tyvar}} : V_{\text{var}}\) translates an FGG type variable \(\alpha\) into a TL variable, abbreviated \(X_{\alpha}\). The third function \(N_{\text{method}} \times N_{\text{struct}} : V_{\text{var}}\) gives us the TL variable \(X_{m,t}\) representing the translation of a method \(m\) for struct \(t\). Here is a summary of the shorthand notations for name translation functions, where \(\text{methodName}(R)\) denotes
Instantiation of bounded type parameters

\[ \Delta \vdash_{\text{sub}} \alpha : \tau_T \mapsto \sigma : \eta \leadsto V \]

**TYPE-INST-CHECKED**

\[
\eta = \langle \alpha \mapsto \sigma' \rangle \quad \Delta \vdash_{\text{core}e} \sigma_i : \eta \tau_{1i} \leadsto V_i \quad (\forall i \in [n])
\]

\[ \Delta \vdash_{\text{sub}} \alpha : \tau_T \mapsto \sigma' : \eta \leadsto \langle \bigvee_i V_i \rangle \]

\[ (R, V) \in \text{methods}(\Delta, \tau_S) \quad \text{methods}(\tau_I) = \{ \overline{R} \} \]

**Method access**

**METHODS-STRUCT**

\[
\text{func} \ (x \, t_S[\alpha : \tau_T]) \ R \ {\{ \text{return} \ e \} \in D} \quad \Delta \vdash_{\text{sub}} \alpha : \tau_T \mapsto \sigma : \eta \leadsto V
\]

\[ \langle \eta, R, V \rangle \in \text{methods}(\Delta, t_S[\sigma]) \]

**METHODS-IFACE**

\[
\text{type} \ t_I[\alpha : \tau_T] \ \text{interface} \ {\{ \overline{R} \} \in D} \quad \eta = \langle \alpha \mapsto \sigma \rangle
\]

\[ \text{methods}(t_I[\sigma]) = \{ \eta \overline{R} \} \]

**Fig. 7. Auxiliary judgments for the translation**

The notation for translating names slightly differs from the approach used in the examples of Section 2. For instance, the coercion toFormat\_T from Figure 3 is now named \(X_T\) and formatWithSep\_Pair becomes \(X_{\text{formatWithSep,Pair}}\). The notation of the formal translation stresses that \(X_T\) and \(X_{\text{formatWithSep,Pair}}\) are variables of the target language.

An FGG type environment \(\Delta\) as a mapping \(\{\alpha : \tau_T\}\) from type variables \(\alpha_i\) to their upper bounds \(\tau_{1i}\). An FGG value environment \(\Gamma\) is a mapping \(\{\alpha : \tau\}\) from FGG variables \(x_i\) to their types \(\tau_i\). An environment may contain at most one binding for a type variable or variable. We write \(\emptyset\) for the empty environment, \(\text{dom}(\cdot)\) for the domain of an environment, and \(\bigcup\) for the disjoint union of two environments. The notation \(\text{distinct}(\overline{s})\) asserts that \(\overline{s}\) is a sequence of disjoint items. We let \([n]\) denote the set \(\{1, \ldots, n\}\).

In the following, we assume that the declarations \(D\) of the FGG program being translated are implicitly available in all rules. This avoids the need for threading the declarations through all translation rules.

### 4.2.2 Auxiliary Judgments

Figure 7 defines some auxiliary judgments. The judgment \(\Delta \vdash_{\text{sub}} \alpha : \tau_T \mapsto \sigma : \eta \leadsto V\), defined by rule **TYPE-INST-CHECKED**, constructs a type substitution \(\eta = \langle \alpha \mapsto \sigma \rangle\) and checks that the \(\sigma\) conform to their upper bounds \(\tau_T\). Thus, it needs a type environment \(\Delta\) to give meaning to free type variables and it returns a tuple \(\langle \bigvee_i V_i \rangle\) of \(\lambda\)-abstractions such that each \(V_i\) coerces the actual type argument to its upper bound. The relevant premise for checking upper bounds is \(\Delta \vdash_{\text{core}e} \sigma_i : \eta \tau_{1i} \leadsto V_i\), which asserts that \(\sigma_i\) is a structural subtype of \(\eta \tau_{1i}\) giving raise to a coercion function \(V_i\). The judgment will be defined and explained in the next subsection.
The lower part of Figure 7 defines two judgments for looking up methods defined for a struct or interface type. Judgment $\langle R, V \rangle \in \text{methods}(\Delta, \tau_S)$ states that method signature $R$ is available for struct type $\tau_S$ under type environment $\Delta$, see rule METHODS-STRUCT. The value $V$ is a tuple of coercion functions resulting from checking the bounds of the receiver’s type parameters. Judgment methods($\tau_I$) = $\{ R \}$ states that the set of method signatures available for interface type $\tau_I$ is $\{ R \}$, see rule METHODS-IFACE. In this rule, we form the substitution $\langle \alpha \mapsto \sigma \rangle$ by implicitly assuming that $\sigma$ and $\bar{\sigma}$ have the same length. From now on, we implicitly use the convention that two sequences forming a substitution are of the same length.

### 4.2.3 Translation of Structural Subtyping

**Figure 8** defines the relation $\Delta \vdash_{\text{coerce}} \tau <: \sigma \leadsto V$ for asserting that $\tau$ is a structural subtype of $\sigma$, yielding a coercion function $V$ to convert the target representations of $\tau$ to $\sigma$.

Rule COERCER-TYVAR covers the case of a type variable $\alpha$. The premise states that type bound $\langle \alpha : \sigma_I \rangle$ exists in the environment. By convention, $X_\alpha$ is the corresponding coercion function. We further find that $\Delta \vdash_{\text{coerce}} \sigma_I <: \sigma \leadsto V$. Hence, we obtain the coercion function for $\alpha <: \sigma$ by composition of coercion functions $V$ and $X_\alpha$.

Rule COERCER-STRUCT-IFACE covers structs. The premise $\langle R_i, V_i \rangle \in \text{methods}(\Delta, \tau_S)$ asserts that each method with name methodName($R_i$) of interface $t_I$ is defined for $\tau_S$. Value $V_i$ is a tuple with coercion parameters corresponding to the bounds of the receiver’s type parameters. Thus, $U_i = \lambda (\bar{Y}^1) . X_{m_i,t_S}(V_i, \bar{Y}^1)$ is the dictionary entry for the method: a function accepting receiver $Y_1$, coercion parameters $Y_2$ corresponding to bounded type parameters of the method, and the arguments $Y_3$. As written earlier, dictionary entries and top-level functions $X_{m_i,t_S}$ are uncurried. Thus, we need to deconstruct the argument triple $\langle \bar{Y}^1 \rangle$ and construct a new quadruple $\langle V, \bar{Y}^1 \rangle$ for calling $X_{m_i,t_S}$.

| Rule | Description |
|------|-------------|
| COERCER-TYVAR | $\Delta \vdash_{\text{coerce}} \tau <: \sigma \leadsto V$ |
| COERCER-STRUCT-IFACE | $\Delta \vdash_{\text{coerce}} t_I [\alpha \tau_I] \leadsto \lambda X_i : \langle \bar{X}^1 \rangle \leadsto \lambda (X_i, (\bar{U}^1))$ |

**Fig. 8. Translation of structural subtyping**
Rule `coerce-iface-iface` covers interfaces. The idea is to build a dictionary for $u_I$ from the methods in the dictionary for $t_I$. As $t_I[\overline{\tau}]$ has to be a structural subtype of $u_I[\overline{\sigma}]$, the former must declare all methods of the latter. Thus, the premise of the rule requires the total function $\pi$ to be chosen in such a way that the $i$-th method of $t_I$ is the same as the $\pi(i)$-th method of $u_I$. Then we use pattern matching to deconstruct the dictionary of $t_I$ as $\langle \overline{X'} \rangle$ and construct the wanted dictionary for $u_I$ as $\langle X_{\pi(1)}, \ldots, X_{\pi(q)} \rangle$.

### 4.2.4 Translation of Expressions

Figure 9 defines the typing and translation relation for expressions. The judgment $\langle \Delta, \Gamma \rangle \vdash_{exp} e : \tau \leadsto E$ states that under type environment $\Delta$ and value environment $\Gamma$ the FGG expression $e$ has type $\tau$ and translates to TL expression $E$.

Rule `var` retrieves the type of FGG variable $x$ from the environment and translates $x$ to its TL counterpart $X$. The context makes variable $X$ available, see Section 4.2.6. Rule `struct` typechecks and translates a struct literal $t_S[\overline{\tau}]$. Premise $\Delta \vdash_{tau} t_S[\overline{\tau}]$ checks that type $t_S[\overline{\tau}]$ is well-formed; the definition of the judgment $\Delta \vdash_{tau} \tau$ is given in Figure 10 and will be explained in the next subsection. Each argument $e_i$ translates to $E_i$, so the result is $\langle E' \rangle$. Rule `access` deals with field access $e.f_i$, where expression $e$ must have struct type
defines several well-formedness judgments. The judgments
Figure 8
and method arguments. Thus, the result of the translation is
type parameters, receiver, coercions for the bounded type
parameters local to the method, and method arguments. Hence, the translation result is
Δ
environment that result from checking the bounds of the receiver’s
type parameters, whereas
Figure 10
from checking the bounds of the type parameters local to the method.
Expressions
Δ
is well-formed if its type arguments are well-formed and if they are
Δ
T
is a tuple of coercion functions that result
from checking the bounds of the type parameters local to the method. Expressions
E
are the translation of the arguments
E
Following our translation strategy, receiver
E
is a pair where the first component is a struct value and the second component is a dictionary for the
interface. Thus, we use pattern matching to extract the struct as
Y
and the wanted method as
X
This
X
is a function accepting a triple: receiver, coercions for bounded type parameters of the method, and method arguments. Hence, the translation result is
X
j
(Y
V
, (E
).
The difference to rule call-struct is that there is no need to supply coercions for the bounded type parameters of the receiver. These coercions have already been supplied when building the dictionary, see rule coerce-struct iface of Figure 8.

The last rule sub is a subsumption rule allowing an expression
e
with translation
E
at type
T
to be assigned some supertype
σ.
Subsumption between a type and some of its supertypes is implicit in FGG via structural subtyping, but our translation makes such conversions explicit. Thus, premise
Δ
for
T
σ
<: \sigma \leadsto V
serves two purposes: it ensures that
σ
is a supertype of
T
and it yields a coercion function
V
from
T
σ
. The translation of
e
at type
σ
is then
V E
.

4.2.5 Well-formedness

Figure 10 defines several well-formedness judgments. The judgments
Δ
τ
and
Δ
σ
assert that a single type and multiple types, respectively, are well-formed under type environment
Δ.
A type variable is well-formed if it’s contained in
Δ
(rule ok-tvvar).
A named type
T
is well-formed if its type arguments are well-formed and if they are
subtypes of the upper bounds in the definition of
T.
The latter is checked by the premise
Δ
\alpha \tau \eta \leadsto \sigma \leadsto V
of rule ok-tynamed. As discussed earlier, the
V
is a tuple of coercion functions. We ignore these coercion functions as they do not have an operational meaning. The type substitution
\eta
is also ignored. Thus, our translation demonstrates that bounds in struct and interface declarations could be eliminated from FGG. We stick with them to stay close to the original presentation of FGG.

Judgment
Δ \alpha \tau \eta
asserts that bounded type parameters
\alpha \tau \eta
are well-formed under type environment
Δ
(rule ok-bounded-typparams). Judgment
Δ R
ensures that a method signature is well-formed (rule ok-msig). To form the combined environment
Δ \cup \{ \alpha \tau \eta \}
in the premise requires disjointness of the type variables in
Δ
and
\eta.
This can always be achieved by \alpha-renaming the type variables bound by
R.
Well-formedness of declarations

Section 3.2

Well-formedness of types

\[ \Delta \vdash \alpha \quad \Delta \vdash \tau \]

OK-TYNAME

\[ \Delta \vdash \tau \quad \text{type } t[\alpha] \ldots \in \overline{D} \]

\[ \Delta \vdash_{\text{ subtype}} \alpha \tau_i \mapsto \tau : \eta \leadsto V \]

OK-tyvar

\[ (\alpha : \tau_i) \in \Delta \]

\[ \Delta \vdash_{\text{without}} \alpha \tau_i \]

Well-formedness of type parameters and method signatures

\[ \Delta \vdash \alpha \quad \Delta \vdash \tau_i \quad \text{distinct } (\tau_i) \]

OK-MANY-TY

\[ \Delta \vdash \alpha \tau_i \quad \text{distinct } (\alpha) \]

\[ \Delta \vdash \tau_i \quad (\forall i \in [n]) \]

Well-formedness of declarations

\[ \Delta \vdash_{\text{without}} \alpha \tau_i \quad \Delta \vdash_{\text{with}} \alpha \tau_i \]

OK-BOUND-TYPARMS

\[ \text{dom}(\Delta) \cap \{\alpha\} = \emptyset \quad \text{distinct } (\alpha) \]

\[ \Delta \cup \{\alpha \vdash \tau_i\} \vdash \alpha \tau_i \]

\[ \Delta \vdash \alpha \tau_i \quad \Delta \vdash \alpha \tau_i \]

OK-STRUCT

\[ t_S \text{ defined once in } \overline{D} \quad \emptyset \vdash \alpha \tau_i \quad (\forall i \in [n]) \{\alpha \vdash \tau_i\} \vdash \alpha \tau_i \]

\[ \Delta \vdash_{\text{without}} \alpha \tau_i \quad \Delta \vdash_{\text{with}} \alpha \tau_i \]

Well-formedness

\[ \Delta \vdash_{\text{without}} \alpha \tau_i \quad \Delta \vdash_{\text{with}} \alpha \tau_i \]

OK-IFACE

\[ (\forall i \in [n]) \{\alpha \vdash \tau_i\} \vdash \alpha \tau_i \quad \text{distinct } (\tau_i) \]

\[ \text{distinct method names in } \overline{R} \]

\[ \Delta \vdash \alpha \tau_i \quad \Delta \vdash \alpha \tau_i \]

OK-INTERFACE

\[ t_I \text{ defined once in } \overline{D} \quad \emptyset \vdash \alpha \tau_i \quad (\forall i \in [n]) \{\alpha \vdash \tau_i\} \vdash \alpha \tau_i \]

\[ \Delta \vdash \alpha \tau_i \quad \Delta \vdash \alpha \tau_i \]

OK-METHOD

\[ \overline{D} \text{ contains one } \text{func-declaration for } t_S \text{ and } m \quad \emptyset \vdash \alpha \tau_i \quad \{\alpha \vdash \tau_i\} \vdash \alpha \tau_i \]

\[ (\text{type } t_S[\alpha \tau_i^m] \text{ struct } \ldots ) \in \overline{D} \quad \text{methods}(\tau_i^j) \subseteq \text{methods}(\tau_i) \quad (\forall i \in [n]) \]

\[ \Delta \vdash_{\text{without}} \alpha \tau_i \quad \Delta \vdash_{\text{with}} \alpha \tau_i \]

\[ \Delta \vdash \alpha \tau_i \quad \Delta \vdash \alpha \tau_i \]

Fig. 10. Well-formedness

Judgment \( \tau \Delta D \) validates declaration \( D \). A struct declaration is well-formed if it is defined only once (restriction \( \text{fgg-unique-structs} \) in Section 3.2), if all field names are distinct (restriction \( \text{fgg-distinct-fields} \)), and if the field types are well-formed. An interface declaration is well-formed if it is defined only once, if all its method signatures are well-formed, and if all methods have distinct names.

A method declaration for \( t_S \) and \( m \) is well-formed if there is no other declaration for \( t_S \) and \( m \) (restriction \( \text{fgg-unique-method-defs} \)), if the method signature is well-formed, and if each bound \( \tau_i \) of the method declaration is a structural subtype of the corresponding bound \( \tau_i' \) in the declaration of \( t_S \). In FGG, this boils down to checking that the methods of \( \tau_i' \) are a subset of the methods of \( \tau_i \). The well-formedness conditions for method declarations do not impose restrictions on the method body. We will deal with this in the upcoming translation rule for methods.
Translating method declarations

\[ \Gamma = \{ x : t_S[\alpha \tau], x : \overline{\sigma} \} \]
\[ \Delta = \{ \alpha \tau, \beta \sigma \} \]
\[ x \notin \{ x \} \]
\[ \langle \Delta, \Gamma \rangle \vdash e : \tau \leadsto E \]
\[ V = \lambda (\langle X_{\alpha_i} \rangle, X, \langle X_{\beta_j} \rangle, \langle \overline{X} \rangle).E \]
\[ \vdash_{\text{meth}} \text{func } (x t_S[\alpha \tau]) \ m_1[\beta \sigma \tau](x \overline{\sigma}) \ \sigma \ {\text{return } e} \leadsto X_{m,t_S} = V \]

Translating programs

\[ \overline{D} \text{ implicitly available in all subderivations} \]
\[ \langle \emptyset, \emptyset \rangle \vdash e : \tau \leadsto E \]
\[ \vdash_{\text{meth}} D \text{ for all } D \in \overline{D} \]
\[ \vdash_{\text{meth}} \text{func } \text{main()}\{e = e\} \leadsto \text{let } X_i = V_i \text{ in } E \]

Fig. 11. Translation of methods and programs

4.2.6 Translation of Methods and Programs

Figure 11 defines the translation for method declarations and programs. Rule METHOD deals with method declarations \( \text{func } (x t_S[\alpha \tau]) \ m_1[\beta \sigma \tau](x \overline{\sigma}) \ \sigma \ {\text{return } e} \). The translation of such a declaration is the binding \( X_{m,t_S} = V \). According to our translation strategy, \( V \) must be a function accepting a quadruple: coercions \( \langle X_{\alpha_i} \rangle \) for the bounded type parameters of the receiver, receiver \( X \) corresponding to \( x \), coercions \( \langle X_{\beta_j} \rangle \) for the bounded type parameters local to the method, and finally method arguments \( \overline{X} \) corresponding to \( \overline{x} \). Binding all these variables with a \( \lambda \) makes them available in the translated body \( E \).

Judgment \( \vdash_{\text{prog}} P \leadsto Prog \) denotes the translation of an FGG program \( P \) to a TL program \( Prog \). Rule PROG typechecks the main expression \( e \) under empty environments against some type \( \tau \) to get its translation \( E \). Next, the rule requires all struct or interface declarations to be well-formed. Finally, it translates each method declarations to a binding \( X_i = V_i \). The resulting TL program is then \( \text{let } X_i = V_i \text{ in } E \).

4.3 Example

We now give an example of the translation. The FGG code in the top part of Figure 12 defines equality for numbers \( \text{Num} \) and for generic boxes \( \text{Box}[^a \text{Any}] \). Interface \( \text{Any} \) defines no methods, it serves as an upper bound for otherwise unrestricted type variables. We take the liberty to assume a basic type \( \text{int} \) and an operator \( == \) for equality. Interface \( \text{Eq}[a] \) requires a method \( \text{eq} \) for comparing the receiver with a value of type \( a \). We provide implementations of \( \text{eq} \) for \( \text{Num} \) and \( \text{Box}[^a \text{]} \). Comparing the content of a box requires the F-bound \( \text{Eq}[a] \) (Canning et al., 1989). The main function compares two boxes for equality.

The middle part of the figure shows the translation of the FGG code, using abbreviations in the bottom part. Variable \( X_{\text{eq,Num}} \) holds the translation of the declaration of \( \text{eq} \) for \( \text{Num} \); it simply compares \( E_2 \) (translation of \( \text{this.val} \)) with \( E_3 \) (translation of \( \text{that.val} \)). Remember that the translation of a method declaration takes a quadruple with coercions for the bounded type parameters of the receiver, the receiver itself, coercions for the bounded type parameters of the method, and the method arguments. Here, \( () \) is a tuple of size zero,
```plaintext
type Any interface {}

``` type Num struct { val int }
``` type Box[α Any] struct { content α }
``` type Eq[α Any] interface { eq(that α) bool }
``` func (this Num) eq(that Num) bool { return this.val == that.val }
``` func (this Box[α Eq[α]]) eq(that Box[α]) bool { return this.content.eq(that.content) }
``` func main() { _ = Box[Num]{Num{1}}.eq(Box[Num]{Num{2}}) }

let Xeq.Num = λ((),This,(),(That)).E2 == E3
Xeq.Box = λ((Xα),This,(),(That)).E1
in Xeq.Box ((V3),((1)),(),((2)))

-- translated body of eq for Box
E1 = case V1, E2 of (Y, (X1)) → V1 = λY . V2 (Xα Y)
X1 ⟨Y, O, (E3)⟩ → -- identity coercion Eq[α] <: Eq[α]
-- selectors for field content of Box
V2 = λ(Y, (X)) . (Y, (X))
E2 = case This of (X1) → X1 -- coercion Num <: Eq[Num]
E3 = case That of (X1) → X1 V3 = λX . (X, (λ(Γ3) . Xeq.Num ((O, Γ3))))

Fig. 12. Example: FGG code (top) and its translation (middle) with abbreviations (bottom)

```

```
corresponding to the non-existing type parameters, \((\text{That})\) denotes a tuple of size one, corresponding to the single argument \(\text{that}\).

The translation of \(eq\) for \(\text{Box}\) is more involved. Figure 13 shows its derivation. We omit “obvious” premises and some trivial details from the derivation trees. Rule \text{CALL-IFACE} translates the body of the method. It coerces the receiver to the interface type \(\text{Eq}[\alpha]\) and then extracts the method to be called via pattern matching, see \(E_1\). The construction of the coercion is done via \(\Delta \vdash_{\text{core}} \alpha <: \text{Eq}[\alpha] \leadsto V_1\), see subderivation \(\overline{1}\). Coercion \(V_1\) is slightly more complicated then necessary because the translation does not optimize the identity coercion \(V_2\). Inside of \(V_1\), we use \(X_\alpha\). This variables denotes a coercion from \(\alpha\) to the representation of \(\text{Eq}[\alpha]\); it is bound by the \(\lambda\)-expression in the definition of \(X_{eq,\text{Box}}\).

The translation of the main expression calls \(X_{eq,\text{Box}}\) with appropriate arguments, see Figure 14 for the derivation. The values \(( (1))\) and \(( (2))\) are nested tuples of size one, representing numbers wrapped in \(\text{Num}\) and \(\text{Box}\) structs. The method call of \(eq\) is translated by rule \text{CALL-STRUCT}, relying on rule \text{METHODS-STRUCT} to instantiate the type variable \(\alpha\) to \(\text{Num}\), as witnessed by the coercion \(V_3\).

![Fig. 14. Example: translation of the main function](image)

### 5 Formal Properties

In this section, we establish that the type-directed translation from Section 4.2 preserves the static and dynamic semantics of FGG programs. Detailed proofs for all lemmas and theorems are given in the appendix.

#### 5.1 Preservation of Static Semantics

It is straightforward to verify that the type system originally defined for FGG is equivalent to the type system induced by the type-directed translation presented in Section 4.2. In the following, we write \(\Delta \vdash_{\mathcal{G}} \tau <: \sigma\) for FGG’s subtyping relation, \(\Delta; \Gamma \vdash_{\mathcal{G}} e : \tau\) for its typing relation on expressions, and \(\vdash_{\mathcal{G}} P\ ok\) for the FGG typing relation on programs. These three relations were specified by Griesemer et al. (2020). The original article on
FGG also includes support for dynamic type assertions, something we do not consider for our translation. Hence, we assume that FGG expressions do not contain type assertions.

**Lemma 5.1.1 (FGG typing equivalence).** Typing in FGG is equivalent to the type system induced by the translation.

(a) If \( \Delta \vdash_G \tau <: \sigma \) then either \( \Delta \vdash_{\text{coarse}} \tau <: \sigma \leadsto V \) for some \( V \) or \( \sigma = \tau \) and \( \tau \) is not an interface type.

(b) If \( \Delta \vdash_G \tau <: \sigma \leadsto V \) then \( \Delta \vdash_{\text{coarse}} \tau <: \sigma \).

(c) If \( \Delta; \Gamma \vdash_G e : \tau \leadsto E \) then \( \langle \Delta, \Gamma \rangle \vdash_{\text{exp}} e : \tau \leadsto E \) for some \( E \).

(d) If \( \langle \Delta, \Gamma \rangle \vdash_{\text{exp}} e : \tau \leadsto E \) then \( \Delta; \Gamma \vdash_G e : \tau' \) for some \( \tau' \) and \( \Delta \vdash_G \tau' \leadsto \tau \).

(e) \( \vdash_G P \text{ ok iff } \vdash_{\text{prog}} P \leadsto \text{Prog} \).

Claims (a) and (b) state that structural subtyping in FGG is equivalent to the relation from Figure 8, except that the latter is not reflexive for type variables and struct types. Claims (c) and (d) establish that expression typing in FGG and our expression typing from Figure 9 are equivalent modulo subtyping. FGG’s expression typing rules do not have a general subsumption rule, so the type computed by the original rules for FGG might be a subtype of the type deduced by our system.

FGG enjoys type soundness (see Theorem 4.3 and 4.4 of Griesemner et al. 2020). With Lemma 5.1.1, we get the following type soundness result for our type system:

**Corollary 5.1.2.** Assume \( \langle \emptyset, \emptyset \rangle \vdash_{\text{exp}} e : \tau \leadsto E \) for some \( e, \tau, \) and \( E \). Then either \( e \) reduces to some value of type \( \tau \) or \( e \) diverges.

### 5.2 Preservation of Dynamic Semantics

This section proves that evaluating a well-typed FGG program yields the same behavior as evaluating its translation. Thereby, we consider all possible outcomes of evaluation: reduction to a value or divergence. The proof of semantic equivalence is enabled by a syntactic, step-indexed logical relation that relates an FGG expression and a TL expression at some FGG type.

We write \( e \rightarrow^k e' \) if \( e \) reduces to \( e' \) in exactly \( k \in \mathbb{N} \) steps, where \( \mathbb{N} \) denotes the natural numbers including zero. By convention, we write \( e \rightarrow^0 e' \) to denote \( e = e' \). The notation \( e \rightarrow^* e' \) states that \( e \rightarrow^k e' \) for some unknown \( k \in \mathbb{N} \). We write \( \text{dive}rge(e) \) to denote that \( e \) does not terminate; that is, for all \( k \in \mathbb{N} \) there exists some \( e' \) with \( e \rightarrow^k e' \). The same definitions apply analogously to reductions in the target language.

#### 5.2.1 The Logical Relation

The definition of the logical relation spreads over two figures 15 and 16. In these figures, we assume that the declarations \( \overline{D} \) of the FGG program being translated are implicitly available in all rules. Also, we assume that an arbitrary but fixed method substitution \( \mu \) is implicitly available to all rules.

We now explain the logical relation on expressions, see Figure 15. The relation \( e \approx E \in [\tau]_k \) denotes that FGG expression \( e \) and TL expression \( E \) are equivalent at type \( \tau \) for at most \( k \) reduction steps. We call \( k \) the *step index*. Rule `EQUIV-EXP` has two implications as
\[ e \approx E \in \llbracket \tau \rrbracket_k \]

\[ \text{Expressions} \]

\begin{align*}
\text{EQUIV-EXP} & : \\
(\forall k' < k, v . e \rightarrow^{k'} v \implies \exists V . E \rightarrow^{*}_\mu V \land v = V \in \llbracket \tau \rrbracket_{k-k'}) & \quad (\forall k' < k, e' . e \rightarrow^{k'} e' \land \text{diverge}(e') \implies \text{diverge}(E)) \\
\end{align*}

\[ v = V \in \llbracket \tau \rrbracket_k \]

\[ \text{Values} \]

\[ \text{EQUIV-STRUCT} \]

\[ \text{EQUIV-IFACE} \]

\[ \text{Method lookup} \]

\[ \text{Method dictionary entries} \]

\[ \text{Bounded type parameters} \]

\[ \text{Fig. 15. Relating FGG to TL expressions} \]

its premises. The first states that if \( e \) reduces to a value \( v \) in \( k' < k \) steps, then \( E \) reduces to some value \( V \) in an arbitrary number of steps and \( v \) is equivalent to \( V \) at type \( \tau \) for the remaining \( k - k' \) steps. The second premise is for diverging expressions: if \( e \) reduces in less than \( k \) steps to some expression \( e' \) and \( e' \) diverges, then \( E \) diverges as well.

The relation \( v = V \in \llbracket \tau \rrbracket_k \) defines equivalence of FGG value \( v \) and TL value \( V \) at type \( \tau \) with step index \( k \). Rule \text{EQUIV-STRUCT} handles the case where \( \tau \) is a struct type. Then \( v \) must be a value of this struct type and \( V \) must be a struct value such that all field values of \( v \) and \( V \) are equivalent. Rule \text{EQUIV-IFACE} deals with the case that \( \tau \) is an interface type. Hence, \( V \) must be an interface value \( \langle U, (\overline{V}) \rangle \) with two requirements. First, \( v \) and \( U \) are
equivalent for all step indices $k_1 < k$ at some struct type $\sigma_S$. Second, $\langle \overline{V} \rangle$ must be an appropriate dictionary for the methods of the interface with receiver type $\sigma_S$. To check this requirement, rule method-lookup defines the auxiliary methodLookup($m_1$, $\sigma_S$) to retrieve a quadruple $(x, \tau_S, R, e)$ from the declaration of $m_1$. This quadruple has to be equivalent to dictionary entry $V_i$ for all step indices $k_2 < k$ at the signature of the method.

Equivalence between such a quadruple and dictionary entry $U$ is written $\langle x, \tau_S, R, e \rangle \approx U \in \llbracket R \rrbracket_k$. Rule equiv-method-dict-entry defines this equivalence such that method body $e$ and $U$ take related arguments to related outputs. Thus, the premise of the rule requires for all step indices $k' \leq k$, all related type parameters $\overline{\tau}$ and $W$, all related receiver values $v$ and $V$, and all related arguments $\overline{\tau}$ and $\overline{V}$ that $e$ and $U$ yield related results when applied to the respective arguments.

The judgment $\overline{\tau} \approx V \in \llbracket \overline{\tau} \rrbracket_k$ denotes equivalence between concrete type arguments $\overline{\tau}$ and their TL realization $V$ when checking the bounds of type parameters $\overline{\tau}$. The definition in rule equiv-bounded-typarms relies on our translation strategy that bounded type parameters are represented by coercions.

Having explained all judgments from Figure 15, we verify that the recursive definitions of $e \approx E \in \llbracket \tau \rrbracket_k$ and $v \equiv V \in \llbracket \tau \rrbracket_k$ are well-founded. Often, logical relations are defined by induction on the structure of types. In our case, this approach does not work because interface types in FGG might be recursive, see our previous work (Sulzmann and Wehr, 2022) for an example. Thus, we use the step index as part of a decreasing measure $M$. Writing $|V|$ for the size of some target value $V$, we define $M(e \approx E \in \llbracket \tau \rrbracket_k) = (k, 1, 0)$ and $M(v \equiv V \in \llbracket \tau \rrbracket_k) = (k, 0, |V|)$. In equiv-exp, either $k$ decreases or stays constant but the second component of $M$ decreases. In equiv-struct, $k$ and the second component stay constant but $|V|$ decreases, and in equiv-interface together with equiv-method-dict-entry and equiv-bounded-typarms step index $k$ decreases. Note that equiv-method-dict-entry and equiv-bounded-typarms only require $k' \leq k$. This is ok because we already have $k_2 < k$ in equiv-interface.

Figure 16 extends the logical relation to whole programs. Judgment $\eta \approx \rho \in \llbracket \Delta \rrbracket_k$ denotes how a FGG type substitution $\eta$ intended to substitute the type variables from $\Delta$ is related to a TL substitution $\rho$. The definition in rule equiv-ty-subst falls back to equivalence of type parameters. Judgment $\theta \approx \rho \in \llbracket \Gamma \rrbracket_k$ similarly relates a FGG value substitution $\theta$ intended for value environment $\Gamma$ with a TL substitution $\rho$. See rule equiv-val-subst.

Judgment func $(x \; t_S[\overline{\sigma} \overline{\tau}]$) $R \{\text{return } e\} \equiv_k X$ states equivalence of a function declaration with a TL variable $X$. Rule equiv-method-decl takes an approach similar as in rule equiv-method-dict-entry: method body $e$ and variable $X$ must yield related outputs when applied to related arguments. Thus, for all related type arguments $\overline{\tau}$, $\overline{\tau}$ and $\langle \overline{W}, \overline{W}' \rangle$, all related receiver values $v$ and $V$, and all related arguments $\overline{v}$ and $\overline{V}$, the expression $e$ and variable $X$ must be related when applied to the appropriate arguments. However, different than in equiv-method-dict-entry, we only require this to hold for all $k' < k$.

Judgment $\overline{D} \equiv_k \mu$ defines equivalence between FGG declarations $\overline{D}$ and TL method substitution $\mu$. The definition in rule equiv-decls is straightforward: each method declaration for some method $m$ and struct $t_S$ must be equivalent to variable $X_{m,t_S}$.
Substitutions

\[ \eta \approx \rho \in \llbracket \Delta \rrbracket_k \quad \theta \approx \rho \in \llbracket \Gamma \rrbracket_k \]

EQUIV-TY-SUBST

\[ \eta a_i \approx (\rho X a_i) \in \llbracket \alpha \tau \rrbracket_k \]

\[ \eta \approx \rho \in \llbracket \{ \alpha \tau \} \rrbracket_k \]

EQUIV-VAL-SUBST

\[ \forall (x : \tau) \in \Gamma, \theta(x) \approx \rho(X) \in \llbracket \tau \rrbracket_k \]

\[ \theta \approx \rho \in \llbracket \Gamma \rrbracket_k \]

Method declarations

\[ \text{func} \ (x \ ts [\overline{\alpha \tau}]) \ R \ \{ \text{return} \ e \} \approx_k X \]

EQUIV-METHOD-DECL

\[ \forall k' < k, W^p, W^q, V, V' \in \llbracket \alpha \tau \rrbracket_k \land \forall i \in [n], V_i \approx \eta \sigma_i \in \llbracket \eta \sigma \rrbracket_k \]

\[ \Rightarrow \ (x \mapsto v, x' \mapsto v') \eta e \approx X \ ((W^p), V, (W^q), (V')) \in \llbracket \eta \sigma \rrbracket_k \]

\[ \text{func} \ (x \ ts [\overline{\alpha \tau}]) m[\overline{\alpha' \tau'}] (x \sigma') \sigma \ \{ \text{return} \ e \} \approx_k X \]

\[ \text{Equiv-decls} \]

\[ \text{D}, \mu \text{ are implicitly available in all subderivations} \]

\[ \forall D_i \in \text{D}, D_i = \text{func} \ (x \ ts [\overline{\alpha \tau}]) mM \ \{ \text{return} \ e \} \Rightarrow D_i \approx_k X_{m,ts} \]

\[ \text{D} \approx_k \mu \]

Fig. 16. Relating FGG to TL substitutions and declarations

5.2.2 Results

To establish the desired result of semantic equivalence, we implicitly make the following assumptions about the globally available declarations \( D \) and method substitution \( \mu \).

Assumption 5.2.1. We assume that the globally available declarations \( D \) are well-formed; that is, \( \tau_m D_i \) for all \( D_i \in D \) and \( \tau_m D_i' \sim X_i = V_i \) for some \( X_i \) and \( V_i \) and all \( D_i' = \text{func} \ldots \in D \). Further, we assume that the globally available method substitution \( \mu \) has only variables of the form \( X_{m,ts} \) in its domain.

Several basic properties hold for our logical relation. For example, monotonicity gives us that with \( e \approx E \in \llbracket \tau \rrbracket_k \) and \( k' \leq k \) we also have \( e \approx E \in \llbracket \tau \rrbracket_{k'} \). Another property is how target and source reductions preserve equivalence:

Lemma 5.2.2 (Target reductions preserve equivalence). If \( e \approx E \in \llbracket \tau \rrbracket_k \) and \( E_2 \xrightarrow{*} E \) then \( e \approx E_2 \in \llbracket \tau \rrbracket_k \).

Lemma 5.2.3 (Source reductions preserve equivalence). If \( e \approx E \in \llbracket \tau \rrbracket_k \) and \( e_2 \Rightarrow e \) then \( e_2 \approx E \in \llbracket \tau \rrbracket_{k+1} \).

The lemmas for monotonicity and several other properties are stated in Appendix A.3, together with all proofs. We can then establish that an FGG expression \( e \) is semantically equivalent to its translation \( E \).
Lemma 5.2.4 (Expression equivalence). Assume $D \approx_k \mu$ and $\eta \approx \rho \in \llbracket \Delta \rrbracket_k$ and $\theta \approx \rho \in \llbracket \eta \Gamma \rrbracket_k$. If $\langle \Delta, \Gamma \rangle \vdash_{\exp} e : \tau \leadsto E$ then $\theta \eta e \approx \rho E \in \llbracket \eta \tau \rrbracket_k$.

The proof is by induction on the derivation of $\langle \Delta, \Gamma \rangle \vdash_{\exp} e : \tau \leadsto E$, see Appendix A.3.1 for the full proof. We next establish semantic equivalence for method declarations.

Lemma 5.2.5 (Method equivalence). Let $D$ and $\mu$ such that for each $D = \text{func } (x : t_S [\tau \rightarrow \tau]) R \{ \text{return } e \} \in D$ with $m = \text{methodName}(R)$ we have $\tau_{\text{men}} D \leadsto X_{m,t_S} = V$ and $\mu(X_{m,t_S}) = V$ for some $V$. Then $D \approx_k \mu$ for any $k$.

The proof of this lemma is by induction on $k$, see Appendix A.3.2 for the full proof. Finally, the following theorem states our desired result: semantic equivalence between an FGG program and its translation.

Theorem 5.2.6 (Program equivalence). Let $\vdash_{\exp} D \text{ func main()} \{ e = e \} \leadsto\text{ let } X_i = V_i \text{ in } E$ where we assume that $e$ has type $\tau$. Let $\mu = \{ X_i \mapsto V_i \}$. Then both of the following holds:

1. If $e \rightarrow^* v$ for some value $v$ then there exists a target language value $V$ such that $E \rightarrow^* \mu V$ and $v \equiv V \in \llbracket \tau \rrbracket_k$ for any $k$.
2. If $e$ diverges then so does $E$.

Obviously, $D$ and $\mu$ meet the requirements of Assumption 5.2.1. The theorem then follows from Lemma 5.2.4 and Lemma 5.2.5. See Appendix A.3.3 for the full proof.

5.3 Getting the step index right

At some places, we require the step index in the premise to be strictly smaller than in the conclusion ($<$), other places require only less-than-or-equal ($\leq$). In \textsc{equiv-exp}, we have $<$ to keep the definition of the LR well-founded. The $<$ in rule \textsc{equiv-method-decl} is required for the inductive argument in the proof of Lemma 5.2.5. Rule \textsc{equiv-iface} also has $<$, but rule \textsc{equiv-method-dict-entry} only requires $\leq$. For well-foundedness, it is crucial that one of these rules decreases the step index. However, equally important is that the step index is not forced to decrease more than once, so we need $<$ in one rule and $\leq$ in the other. If both rules had $<$, then the proof of Lemma 5.2.4 would not go through for case \textsc{call-iface}.

Consider the following example in the context of Figure 12:

\begin{align*}
    \text{w}_1 = \text{Num}\{1\} \text{ at type } Eq[\text{Num}] &\leadsto W_1 = ( \langle 1 \rangle, \langle U \rangle ) \\
&\quad \text{where } U = \lambda (\overline{Y}); \ X_{eq,\text{Num}} (\langle \rangle, \overline{Y})
\end{align*}

\begin{align*}
    \text{w}_2 = \text{Num}\{2\} \text{ at type } \text{Num} &\leadsto W_2 = ( \langle 2 \rangle )
\end{align*}

\begin{align*}
    \text{w}_1.\text{eq}(\text{w}_2) &\leadsto E = \text{case } W_1 \text{ of } (Y, (X_1)) \rightarrow X_1 (Y, \langle \rangle, (W_2))
\end{align*}

For values $w_1$ and $w_2$, we may assume (1) $w_1 \equiv W_1 \in \llbracket Eq[\text{Num}] \rrbracket_k$ and $w_2 \approx W_2 \in \llbracket \text{Num} \rrbracket_k$ for some $k$. To verify that the translation yields related expressions, we must show

\begin{align*}
    w_1.\text{eq}(w_2) &\approx E \in \llbracket \text{bool} \rrbracket_k
\end{align*}

(2)
From (1), via inversion of rule $\text{equiv-iface}$, we can derive
\[
\text{methodLookup}(eq, \text{Num}) \approx U \in \llbracket eq(\text{that Num) bool} \rrbracket_{k-1}
\]
(3)
because the premise of the rule requires this to hold for all $k_2 < k$. Let $e$ be the body of the method declaration of $eq$ for $\text{Num}$. Inverting rule $\text{equiv-method-dict-entry}$ for (3) yields
\[
\langle this \mapsto w_1, that \mapsto w_2 \rangle e \approx U ((1), (), ((2))) \in \llbracket \text{bool} \rrbracket_{k'}
\]
(4)
for $k' = k - 1$ because rule $\text{equiv-method-dict-entry}$ has $\leq$ in its premise. Also, we have $w_1.eq(w_2) \longrightarrow^1 \langle this \mapsto w_1, that \mapsto w_2 \rangle e$ and $E \longrightarrow^* U ((1), (), ((2)))$. Thus, with (4), Lemma 5.2.2, and Lemma 5.2.3 we get $w_1.eq(w_2) \approx E \in \llbracket \text{bool} \rrbracket_{k' + 1}$. For $k' = k - 1$, this is exactly (2), as required. But if rule $\text{equiv-method-dict-entry}$ required $< \text{in its premise, then (4) would only hold for } k' = k - 2$ and we could not derive (2).

Whether we have $< \text{in equiv-iface and } \leq \text{in equiv-method-dict-entry or vice versa is a matter of taste. In our previous work at MPC (Sulzmann and Wehr, 2022), we established a dictionary-passing translation for Featherweight Go without generics. The situation is slightly different there. With generics, we need two rules with respect to methods: equiv-method-decl for method declarations and equiv-method-dict-entry for dictionary entries where the coercions for the bounds of the receiver’s type parameters have already been supplied. Without generics, there are no type parameters, so a single rule suffices (rule red-rel-method in MPC). So in the article at MPC, we use $< \text{for rule red-rel-method and } \leq \text{for rule red-rel-iface}, the pendant to rule equiv-iface of the current article.}$

## 6 Implementation and Type Assertions

We provide an implementation of the translation\(^2\) written in Haskell (2022). The target language of the translation is Racket (2022). The implementation supports the source language from Section 3, extended with type assertions, generic functions, and several base types (integers, characters, strings, and booleans). All extensions except type assertions are straightforward to support. Thus, the remainder of this section explains how to add type assertions to the translation.

Type assertions are Go’s notion for dynamic type casts. If the Go compiler sees a type assertion, written $e.(\tau)$ for some expression $e$ and type $\tau$, it treats the whole expression as having type $\tau$ and inserts a runtime conversion that fails if the type of $e$ is not a subtype of $\tau$. Type assertions are quite expressive: the asserted type $\tau$ might be a (generic) interface type or even a type variable.

We have extended our translation with support for type assertions, and we also have a rough proof sketch to verify that the extended translation preserves the dynamic semantics. However, the translation rules for type assertions are quite verbose and contain a lot of boilerplate. Further, their treatment is largely orthogonal to the main aspect of this article, namely the handling of structural subtyping through a dictionary-passing translation. Thus, we only provide an informal explanation how to extend the translation with type assertions.

\(^2\) https://github.com/skogsbaer/fgg-translate
6.1 Example for Type Assertions

To get an intuition for the treatment of type assertions, we start with a simple example and ignore generics. The later sections then fill the pieces missing. Consider the following FGG code in the context of Figure 1 and Figure 2 that converts an arbitrary value into a value of interface type Format. If the conversion fails, a runtime error occurs.

```go
func asFormat(x Any) Format {
    return x.(Format)
}
```

To translate the Go code, the representation of a struct value carries a tag specific to the struct. Each interface value carries the tag Iface to distinguish interface from struct values.

```
Num{1} at type Num  \rightarrow (K_{Num}, 1)
Num{1} at type Format  \rightarrow Iface (K_{Num}, 1) format_{Num}
```

To construct method dictionaries dynamically, we need an encoding for method signatures. This encoding consists of a constructor for the method name, a tuple with the argument types, and the result type. (We ignore generics for the moment, so we do not need to represent type parameters.) For example, the encoding of the signature `format()string` is `K_{format}() K_{string}`.

The output of the translation then contains a small “runtime system”. Function `getDictEntry` matches on the tag of a struct and the encoding of a method signature to return an appropriate dictionary entry. If no method declaration is found, an error is raised.

```
getDictEntry K_{Num} (K_{format} () K_{string}) = format_{Num}
-- other cases for getDictEntry ommitted
getDictEntry _ _ = error "type assertion failed"
```

Function `dynAssert` takes a struct value and a list of method signatures and tries to construct an interface value for the method signatures given. The map-function is defined in the usual way.

```
dynAssert (tag , fields) sigs =
    ((tag , fields), map (getDictEntry tag) sigs)
```

The translation of `asFormat` is now straightforward. We use `[...]` to denote a list of values.

```
asFormat x =
    case x of
        Iface y () -> dynAssert y [K_{format} () () K_{string}]`
```

6.2 Runtime-representation of Types

We have already seen to we need to encode FGG’s types and method signatures at the expression-level of the target language. For a struct or interface types \(\tau\), the encoding is a unique constructor \(K_{\tau}\) applied to the encodings of \(\tau\). The encodings of a method signature includes the method name, the bounds of its type parameters, the argument types, and the result type. Type parameters local to a method are represented as de Bruijn indices, so the encoding of method signatures is invariable under renaming of type variables.
The encoding of type variables not local to a method signature or appearing outside of a method signature are passed as additional parameters. In fact, a type variable instantiated with an interface type requires two encodings. The first is the encoding of the interface type, and the second is a list with the encodings of all method signatures of the interface. Thus, a type variable now gives raise to three additional parameters: two encodings and the coercion to its upper bound.

### 6.3 Enforcing Type Assertions

The translation of a type assertion $e.\tau$ is a $\lambda$-expression that converts the translation of $e$ to a representation of the asserted type $\tau$. Clearly, such a conversion might fail at runtime. To realize type assertions, we assume that we can distinguish struct and interface values in the target, and that each struct value carries its type encoding.

In the following, we describe how a type assertion with asserted type $\tau$ is enforced on some value $V$ of the target language. Three different cases may arise:

1. Asserted type $\tau$ is a **struct type**. Such a type assertion only succeeds if $V$ is a value of the same struct. Thus, we dynamically check if $V$ is a struct value and if the type encoding carried by $V$ is equal to the encoding of $\tau$. If yes, the type assertion yields $V$. Otherwise, a runtime-error arises.

2. Asserted type $\tau$ is an **interface type**. Thus, we need to construct a new interface value by dynamically creating a dictionary for $\tau$. To support dynamic dictionary creation, the output of the translation provides a global lookup table. For each method declaration, the table maps the encoding of the receiver type and the encoding of the method signature to a dictionary entry for the method. In the example of Section 6.1, the global lookup table has been realized as the `getDictEntry` function. For the type assertion to succeed, $V$ can be either a struct value or an interface value with an embedded struct value. In any case, we have a type encoding of a struct value. Then, for each method signature $R$ of $\tau$, we use the type encoding and the encoding of $R$ to extract a dictionary entry for the method. If any of these lookup operations fails, the whole type assertions fails. Otherwise, we are able to construct an interface value for $\tau$.

3. Asserted type $\tau$ is a **type variable**. We proceed as in the two preceding cases, depending on whether the type variable has been instantiated with a struct or interface type.

The lookup in the global table performs an equality comparison on encodings of method signatures. This simple approach works because a struct implements a method of some interface only if the method signatures in the method declaration and in the interface match exactly (apart from names for type and argument variables). For example, it is not allowed to provide a more general signature in the method declaration.
7 Related Work

The related work section covers generics in Go, type classes in Haskell, logical relations, and a summary of our own prior work. At the end, we give an overview of the existing translations with source language Featherweight Generic Go.

7.1 Generics in Go

The results of this work rest on the definition of Featherweight Generic Go (FGG) provided by Griesemer and colleagues (2020). FGG is a minimal core calculus modeling the essential features of the programming language Go (2022). It includes support for overloaded methods, interface types, structural subtyping, generics, and type assertions. Our formalization of FGG ignores dynamic type assertions but otherwise sticks to the original definition of FGG, apart from some minor cosmetic changes in presentation. We prove that the type system implied by our translation is equivalent to the original type system of FGG, and that translated programs behave the same way as under the original dynamic semantics.

The original dynamic semantics of FGG uses runtime method lookup, in the same way as we did in Section 3. The authors define an alternative semantics via monomorphisation; that is, they specialize generic code for all type arguments appearing in the program. This alternative semantics is equivalent to the one based on runtime method lookup, but there exist type-correct FGG programs that cannot be monomorphized. Further, monomorphization often leads to a blowup in code size. In contrast, our translation handles all type-correct FGG programs, and instantiations of generic code with different type arguments do not increase the code size. However, we expect that monomorphized code will offer better performance than code generated by our dictionary-passing translation, because method dictionaries imply several indirections not present in monomorphized programs.

The current implementation of generics (Taylor and Griesemer, 2021) in Go versions 1.18 and 1.19 (2022) differs significantly from the formalization in FGG. For example, full Go requires a method declaration for a generic struct to have exactly the same type bounds as the struct. In FGG, bounds of the receiver struct in a method declaration might be stricter than the bounds in the corresponding struct declaration. In Figure 12, we used this feature to implement equality on the fully generic Box type, provided the type parameter can be compared for equality. Go cannot express this scenario without falling back to dynamic type assertions.

Ellis et al. (2022) formalize a dictionary-passing translating from a restricted subset of FGG to FG. The restriction for FGG is the same as previously explained for full Go: a method declaration must have the same type bounds as its receiver struct. The translation utilizes this restriction to translate an FGG struct together with all its methods into a single FG struct (dictionary). This approach would scale to full Go even with separate compilation because a struct and all its methods must be part of the same package. Further, the translation of Ellis and coworkers replaces all types in method signatures with the top-type Any, relying on dynamic type assertions to enable type checking of the resulting FG program. The authors provide a working implementation and a benchmark suite to
compare their translation against several other approaches, including the current implementation of generics in full Go. Our translation targets an extended \( \lambda \)-calculus and does not restrict the type bounds of the receiver struct in a method declaration. We also provide an implementation but no evaluation of its performance.

Method dictionaries bear some resemblance to virtual method tables (vtables) used to implement virtual method dispatch in object-oriented languages (Driesen and Hölzle, 1996). The main difference between vtables and dictionaries is that there is a fixed connection between an object and its vtable (via the class of the object), whereas the connection between a value and a dictionary may change at runtime, depending on the type the value is used at. Dictionaries allow access to a method at a fixed offset, whereas vtables in the presence of multiple inheritance require a more sophisticated lookup algorithm (Alpern et al., 2001).

A possible optimization to the dictionary-passing translation is selective code specialization (Dean et al., 1995). With this approach, the dictionary-passing translation generates code that runs for all type arguments. In addition, specialized code is generated for frequently used combinations of type arguments. This approach allows to trade code size against runtime performance. The GHC compiler for Haskell supports a SPECIALIZE pragma (GHC User’s Guide, 2022, § 6.20.11.) that allows developers to specialize a polymorphic function to a particular type. The specialization also supports type-class dictionaries.

### 7.2 Type Classes in Haskell

The dictionary-passing translation is well-studied in the context of Haskell type classes (Wadler and Blott, 1989; Hall et al., 1996). A type-class constraint translates to an extra function parameter, constraint resolution provides a dictionary with the methods of the type class for this parameter. In FGG, structural subtyping relations imply coercions and bounded type parameters translate to coercion parameters. An interface value pairs a struct value with a dictionary for the methods of the interface. Thus, interface values can be viewed as representations of existential types (Mitchell and Plotkin, 1988; Läufer, 1996; Thiemann and Wehr, 2008).

Another important property in the type class context is coherence. Bottu and coworkers (2019) make use of logical relations to state equivalence among distinct target terms resulting from the same source type class program. Thanks to our main result (Theorem 5.2.6), we get coherence for free. We believe it is worthwhile to establish a property similar to this theorem for type classes. We could employ a simple denotational semantics for source type class programs similar as Thatte (1994) or Morris (2014), which would then be related to target programs obtained via the dictionary-passing translation.

---

3 Several points in the following discussion were already included in own prior work (Sulzmann and Wehr, 2021, 2022).
7.3 Logical Relations

Logical relations have a long tradition of proving properties of typed programming languages. Such properties include termination (Tait, 1967; Statman, 1985), type safety (Skorstengaard, 2019), and program equivalence (Pierce, 2004, Chapters 6, 7). A logical relation (LR) is often defined inductively, indexed by type. If its definition is based on an operational semantics, the LR is called syntactic (Pitts, 1998; Crary and Harper, 2007). With recursive types, a step-indexed LR (Appel and McAllester, 2001; Ahmed, 2006) provides a decreasing measure to keep the definition well-founded. See Mitchell (1996, Chapter 8) and Skorstengaard (2019) for introductions to the topic.

LRs are often used to relate two terms of the same language. For our translation, the two terms are from different languages, related at a type from the source language. Benton and Hur (2009) prove correctness of compiler transformations. They use a step-indexed LR to relate a denotational semantics of the \( \lambda \)-calculus with recursion to configurations of a SECD-machine. Hur and Dreyer (2011) build on this idea to show equivalence between an expressive source language (polymorphic \( \lambda \)-calculus with references, existentials, and recursive types) and assembly language. Their biorthogonal, step-indexed Kripke LR does not directly relate the two languages but relies on abstract language specifications.

Our setting is different in that we consider a source language with support for overloading. Besides structured data and functions, we need to cover recursive interface values. This leads to some challenges to get the step index right (Sulzmann and Wehr, 2022).

Simulation or bisimulation (see e.g. Sumii and Pierce 2007) is another common technique for showing program equivalences. In our setting, using this technique amounts to proving that reduction and translation commutes: if source term \( e \) reduces to \( e' \) and translates to target term \( E \), then \( e' \) translates to \( E' \) such that \( E \) reduces to \( E'' \) (potentially in several steps) with \( E' = E'' \). One challenge is that the two target terms \( E' \) and \( E'' \) are not necessarily syntactically equal but only semantically. In our setting, this might be the case if \( E' \) and \( E'' \) contain coercions for structural subtyping. Even if such coercions behave the same, their syntax might be different. With LR, we abstract away certain details of single step reductions, as we only compare values, not intermediate results. A downside of the LR is that getting the step index right is sometimes not trivial.

Paraskevopoulou and Grover (2021) combine simulation and an untyped, step-indexed LR (Acar et al., 2008) to relate the translation of a reduced expression (the \( E' \) from the preceding paragraph) with the reduction result of the translated expression (the \( E'' \)). They use this technique to prove correctness of CPS transformations using small-step and big-step operational semantics. Resource invariants connect the number of steps a term and its translation might take, allowing them to prove that divergence and asymptotic runtime is preserved by the transformation. Our LR does not support resource invariants but includes a case for divergence directly.

7.4 Prior Work

Our own work published at APLAS (Sulzmann and Wehr, 2021) and MPC (Sulzmann and Wehr, 2022) laid the foundations for the dictionary-passing translation and its correctness.
proof of the present article. For the APLAS paper, we defined a dictionary-passing translation for Featherweight Go (FG, Griesemer et al., 2020), the non-generic variant of FGG. That translation is similar in spirit to the translation presented here, it supports type assertions but not generics. The APLAS paper includes a proof for the semantic equivalence between the source FG program and its translation. The result is, however, somewhat limited as semantic equivalence only holds for terminating programs whose translation is also known to terminate.

In the MPC paper, we addressed the aforementioned limitation by extending the proof of semantic equivalence to all possible outcomes of an FG program: termination, panic (failure of a dynamic type assertion), and divergence. The proof uses a logical relation similar to the one used here, but without support for generics. We have already shown more differences in Section 5.3.

### 7.5 Summary of Translations

The diagram in Figure 17 summarizes the existing translations by Griesemer et al. (OOPSLA 2020), by Ellis et al. (OOPSLA 2022), from our MPC 2022 paper (Sulzmann and Wehr, 2022), and from the article at hand. The three resulting target language programs $P_{TL}$, $P'_{TL}$, and $P''_{TL}$ are semantically equivalent because all translations preserve the dynamic semantics.\(^4\) Each translation with $P_{FGG}$ as its source has different restrictions. OOPSLA 2022 requires the receiver struct of some method declaration to have exactly the same type bounds as the struct declaration itself. OOPSLA 2020 requires $P_{FGG}$ to be monomorphizable, checked by a simple syntactic condition. The formal translation of this article does not support type assertions, but we informally explained in Section 6 how to extend the translation in this direction.

---

\(^4\) Strictly speaking, the MPC 2022 paper uses a target language slightly different from the one presented in this article. The differences, however, are straightforward to level out.
8 Conclusion and Future Work

This article defined a type-directed dictionary-passing translation from Featherweight Generic Go (FGG) to an extension of the untyped \( \lambda \)-calculus. The translation represents a value at the type of an FGG interface as an existential combining a concrete struct value with a dictionary for all methods of the interface. Bounded type parameters in FGG become extra function arguments in the target. These extra arguments are coercions from the instantiation of a type variable to its upper bound. The formal translation covers all features of FGG except type assertions, which we treated only informally.

Every program in the image of the translation has the same dynamic semantics as its source FGG program. The proof of this result is based on a syntactic, step-indexed logical relation. The step-index ensures a well-founded definition of the relation in the presence of recursive interface types and recursive methods. We also reported on an implementation of the translation.

In this article, we relied on FGG as defined by Griesemer and coworkers (2020), without reconsidering design decisions. But our translation raises several questions with respect to the design of generics in Go. For example, the translation clearly shows that type bounds in structs and interfaces have no operational meaning. Should we eliminate these type bounds? Or should we give them a meaning inspired by Haskell’s type class mechanism? Further, a method declaration in full Go must reuse the type bounds of its struct and must be defined in the same package as the struct. Clearly, this limits extensibility and flexibility. Can we provide a more flexible design to solve the expression problem (Wadler, 1998) in Go, without resorting to unsafe type assertions? We would like to use the insights gained through this article to answer these and similar questions in future work.

A somewhat related point is performance. As explained earlier, generics in Go are compiled by monomorphization. This gives the best possible performance because the resulting code is specialized for each type argument. However, not all programs can be monomorphized and the increase in code size is often considered problematic. This raises another interesting question for future work. Could selective monomorphization or specialization offer a viable trade-off between performance, code size, and the ability to compile Go programs which are not monomorphizable?

A statically-typed target language typically offers more room for compiler optimization. Thus, another interesting direction for future work is a translation to a typed backend, for example System F\(_C\) (Sulzmann et al., 2007).

References

Acar, U. A., Ahmed, A. & Blume, M. (2008) Imperative self-adjusting computation. Proc. of POPL 2008. ACM.

Ahmed, A. (2006) Step-indexed syntactic logical relations for recursive and quantified types. Proc. of ESOP 2006. Springer-Verlag.

Alpern, B., Cocchi, A., Fink, S. J., Grove, D. & Lieber, D. (2001) Efficient implementation of Java interfaces: Invokeinterface considered harmless. Proc. of OOPSLA 2001. ACM.

Appel, A. W. & McAllester, D. A. (2001) An indexed model of recursive types for foundational proof-carrying code. *ACM Trans. Program. Lang. Syst.*, 23(5).

Benton, N. & Hur, C. (2009) Biorthogonality, step-indexing and compiler correctness. Proc. of ICFP
2009. ACM.
Bottu, G.-J., Xie, N., Marntirosian, K. & Schrijvers, T. (2019) Coherence of type class resolution. Proc. ACM Program. Lang. 3(ICFP).
Canning, P., Cook, W., Hill, W., Olthoff, W. & Mitchell, J. C. (1989) F-bounded polymorphism for object-oriented programming. Proc. of FPCA 1989. ACM.
Crary, K. & Harper, R. (2007) Syntactic logical relations for polymorphic and recursive types. Electron. Notes Theor. Comput. Sci. 172.
Dean, J., Chambers, C. & Grove, D. (1995) Selective specialization for object-oriented languages. Proc. of PLDI 1995. ACM.
Driesen, K. & Hölzle, U. (1996) The direct cost of virtual function calls in C++. Proc. of OOPSLA 1996. ACM.
Ellis, S., Zhu, S., Yoshida, N. & Song, L. (2022) Generic Go to Go: Dictionary-passing, monomorphisation, and hybrid. To appear at OOPSLA 2022. https://arxiv.org/abs/2208.06810.
GHC User’s Guide. (2022) https://downloads.haskell.org/ghc/9.4.1/docs/users_guide/index.html.
Go Programming Language. (2022) https://golang.org.
Griesemer, R., Hu, R., Kokke, W., Lange, J., Taylor, I. L., Toninho, B., Wadler, P. & Yoshida, N. (2020) Featherweight Go. Proc. ACM Program. Lang. 4(OOPSLA).
Hall, C. V., Hammond, K., Peyton Jones, S. L. & Wadler, P. L. (1996) Type classes in Haskell. ACM Trans. Program. Lang. Syst. 18(2), 109–138.
Haskell Programming Language. (2022) https://www.haskell.org.
Hur, C. & Dreyer, D. (2011) A Kripke logical relation between ML and Assembly. Proc. of POPL 2011. ACM.
Läufer, K. (1996) Type classes with existential types. Journal of Functional Programming. 6(3).
Mitchell, J. C. (1996) Foundations for programming languages. Foundation of computing series. MIT Press.
Mitchell, J. C. & Plotkin, G. D. (1988) Abstract types have existential type. ACM Trans. Program. Lang. Syst. 10(3).
Morris, J. G. (2014) A simple semantics for Haskell overloading. Proc. of Haskell 2014. ACM.
Paraskevopoulou, Z. & Grover, A. (2021) Compiling with continuations, correctly. Proc. ACM Program. Lang. 5(OOPSLA).
Pierce, B. (2004) Advanced Topics in Types and Programming Languages. The MIT Press.
Pitts, A. M. (1998) Existential types: Logical relations and operational equivalence. Proc. of ICALP 1998. Springer.
Racket Programming Language. (2022) https://racket-lang.org.
Skorstengaard, L. (2019) An introduction to logical relations. http://arxiv.org/abs/1907.11133.
Statman, R. (1985) Logical relations and the typed lambda-calculus. Inf. Control. 65(2/3).
Sulzmann, M., Chakravarty, M. M. T., Jones, S. L. P. & Donnelly, K. (2007) System F with type equality coercions. Proc. of TLDI 2007. ACM. pp. 53–66.
Sulzmann, M. & Wehr, S. (2021) A dictionary-passing translation of Featherweight Go. Proc. of APLAS 2021. Springer.
Sulzmann, M. & Wehr, S. (2022) Semantic preservation for a type directed translation scheme of Featherweight Go. Proc. of MPC 2022. Springer.
Sumii, E. & Pierce, B. C. (2007) A bisimulation for type abstraction and recursion. J. ACM. 54(5).
Tait, W. W. (1967) Intensional interpretations of functionals of finite type I. J. Symb. Log. 32(2), 198–212.
Taylor, I. L. & Griesemer, R. (2021) Type parameters proposal. https://go.googlesource.com/proposal/+/refs/heads/master/design/43651-type-parameters.md.
Thatte, S. R. (1994) Semantics of type classes revisited. Proc. of LISP 1994. ACM.
Thiemann, P. & Wehr, S. (2008) Interface types for Haskell. Proc. of APLAS 2008. Springer.
Wadler, P. (1998) The expression problem. Posted on the Java Genericty mailing list. http://homepages.inf.ed.ac.uk/wadler/papers/expression/expression.txt.
A Proofs

A.1 Deterministic Evaluation in FGG and TL

Lemma A.1.1 (Deterministic evaluation in FGG). If \( e \rightarrow e' \) and \( e \rightarrow e'' \) then \( e' = e'' \). If \( E \rightarrow E' \) and \( E \rightarrow E'' \) then \( E' = E'' \).

Proof. We first state and prove three sublemmas:

(a) If \( e = E_1[E_2[e']] \) then there exists \( E_3 \) with \( e = E_3[e'] \). The proof is by induction on \( E_1 \).

(b) If \( e \rightarrow e' \) then there exists a derivation of \( e \rightarrow e' \) that ends with at most one consecutive application of rule \( \text{FG-CONTEXT} \). The proof is by induction on the derivation of \( e \rightarrow e' \). From the IH, we know that this derivation ends with at most two consecutive applications of rule \( \text{FG-CONTEXT} \). If there are two such consecutive applications, (a) allow us to merge the two evaluation contexts involved, so that we need only one consecutive application of \( \text{FG-CONTEXT} \).

(c) We call an FGG expression directly reducible if it reduces but not by rule \( \text{FG-CONTEXT} \). If \( e_1 \) and \( e_2 \) are now directly reducible and \( E_1[e_1] = E_2[e_2] \) then \( E_1 = E_2 \) and \( e_1 = e_2 \). For the proof, we first note that \( E_1 = \square \) iff \( E_2 = \square \). This holds because directly reducible expressions have no inner redexes. The rest of the proof is then a straightforward induction on \( E_1 \).

Now assume \( e \rightarrow e' \) and \( e \rightarrow e'' \). By (b) we may assume that both derivations end with at most one consecutive application of rule \( \text{FG-CONTEXT} \). It is easy to see (as values do not reduce) that both derivations must end with the same rule. If this rule is \( \text{FG-FIELD} \), then \( e' = e'' \) by restrictions \( \text{FG-UNIQUE-STRUCTS} \) and \( \text{FG-DISTINCT-FIELDS} \). If this rule is \( \text{FG-CALL} \), then \( e' = e'' \) by \( \text{FG-UNIQUE-METHOD-DEFS} \). If the rule is \( \text{FG-CONTEXT} \), we have the following situation with \( R_1 \neq \text{FG-CONTEXT} \) and \( R_2 \neq \text{FG-CONTEXT} \):

\[
\begin{align*}
\text{FG-CONTEXT} & \\
R_1 & : g_1 \rightarrow g_1' \quad R_2 : g_2 \rightarrow g_2'\\
E_1[g_1] & \rightarrow E_1'[g_1'] \quad E_2[g_2] \rightarrow E_2'[g_2']
\end{align*}
\]

As neither \( R_1 \) nor \( R_2 \) are \( \text{FG-CONTEXT} \), we know that \( g_1 \) and \( g_2 \) are directly reducible. Thus, with \( E_1[g_1] = E_2[g_2] \) and (c) we get \( E_1 = E_2 \) and \( g_1 = g_2 \). With \( R_1 \) and \( R_2 \) not being \( \text{FG-CONTEXT} \), we have \( g_1' = g_2' \), so \( e' = e'' \) as required.

Lemma A.1.2 (Deterministic evaluation in TL). If \( E \rightarrow\mu E' \) and \( E \rightarrow\mu E'' \) then \( E' = E'' \). Further, if \( E \rightarrow E' \) and \( E \rightarrow E'' \) then \( E' = E'' \).

Proof. We first prove the first implication of the lemma

\[
\forall E, E', E'', \mu : E \rightarrow\mu E' \land E \rightarrow\mu E'' \Rightarrow E' = E''
\]
There are three sublemmas, analogously to the proof of Lemma A.1.1.
(a) If \( E = R_1[R_2[E']] \) then there exists \( R_3 \) with \( E = R_3[E'] \).
(b) If \( E \rightarrow_{\mu} E' \) then there exists a derivation of \( E \rightarrow_{\mu} E' \) that ends with at most one consecutive application of rule TL-CONTEXT.
(c) We call a target-language expression directly reducible if it reduces but not by rule TL-CONTEXT. If \( E_1 \) and \( E_2 \) are now directly reducible and \( R_1[E_1] = R_2[E_2] \) then \( R_1 = R_2 \) and \( E_1 = E_2 \).

The proofs of these lemmas are similar to the proofs of the sublemmas in Lemma A.1.1. Then (1) follows with reasoning similar to the proof of Lemma A.1.1. If the derivations of \( E \rightarrow \) are both end with rule TL-CASE, then our assumption that the constructors of a case-expression are distinct ensures determinacy.

The second claim of the lemma \( (E \rightarrow E' \) and \( E \rightarrow E'' \) imply \( E' = E'' \)) then follows directly from (1). Our assumption that the variables of a top-level let-binding are distinct ensures that the substitution \( \mu \) built from the top-level let-bindings is well-defined. ■

A.2 Preservation of Static Semantics

Proof of Lemma 5.1.1. We prove (a) and (b) by case distinctions on the last rule of the given derivations; (c) and (d) follow by induction on the derivations, using (a) and (b). Claim (e) then follows by examining the typing rules, using (c) and (d). ■

A.3 Preservation of Dynamic Semantics

Convention A.3.1. We omit \( \mu \) from reductions in the target language, writing \( E \rightarrow E' \) instead of \( E \rightarrow_{\mu} E' \).

Definition A.3.2. We make use of some extra metavariables and notations.
• \( \Phi, \Psi \) denote formal type parameters \( \overline{\tau} \).
• \( \hat{\Phi} \) denotes the type variables of \( \Phi \); that is, if \( \Phi = \overline{\tau} \) then \( \hat{\Phi} = \overline{\tau} \).
• \( \phi, \psi \) denote actual arguments \( \tau \).
• \( M := [(\Phi)(\xi \tau)] \) denotes the type-part of a method signature \( R \).
• \( L \) denotes the type literal struct \{ \( \overline{\tau} \) \} or interface \{ \( \overline{R} \) \}.
• \( \Phi \implies \phi : \eta \) create a type substitution \( \eta \) form type parameters \( \Phi \) and arguments \( \phi \). It is defined like this: \( \overline{\tau} ^* \implies \overline{\eta} ^* : (\overline{a} _i \mapsto \overline{\alpha} _i ^*) \)

Lemma A.3.3 (Monotonicity for expressions). Assume \( k' \leq k \). If \( e \equiv E \in \| \tau \|_k \) then \( e \equiv E \in \| \tau \|_{k'} \). If \( v \equiv V \in \| \tau \|_k \) then \( v \equiv V \in \| \tau \|_{k'} \).

Proof. We proceed by induction on \( (k, s) \) where \( s \) is the combined size of \( v, V \).
Case distinction on the last rule used in the two derivations.
• Case rule EQUIV-EXP: We label the two implications in the premise of the rule as (a) and (b).
Lemma A.3.6 (Monotonicity for method declarations). If \( \langle x, \tau_S, R, e \rangle \approx V \in \llbracket R' \rrbracket_k \) and \( k' \leq k \), then \( \langle x, \tau_S, R, e \rangle \approx V \in \llbracket R' \rrbracket_{k'} \).

Proof. Obvious. ■

Lemma A.3.5 (Monotonicity for type parameters). If \( \phi \approx V \in \llbracket \Phi \rrbracket_k \) and \( k' \leq k \), then \( \phi \approx V \in \llbracket \Phi \rrbracket_{k'} \).

Proof. Obvious. ■

Lemma A.3.6 (Monotonicity for method declarations). Assume declaration \( D = \text{func} \ (x \ t_S \ [\Phi]) \ R \ \{\text{return} \ e\} \) and \( k' \leq k \). If \( D \approx_k X \) then \( D \approx_{k'} X \).

Proof. Obvious. ■

Lemma A.3.7 (Monotonicity for programs). If \( \overline{D} \approx_k \mu \) and \( k' \leq k \), then \( \overline{D} \approx_{k'} \mu \).

Proof. Follows from Lemma A.3.6. ■

Proof of Lemma 5.2.2. Straightforward.

Proof of Lemma 5.2.3. We label the two implications in the premise of rule \textsc{equiv-exp} with (a) and (b).

(a) Assume \( k'' < k' \) and \( e \rightarrow^{k''} u \) for some value \( u \). From \( e \approx E \in \llbracket \tau \rrbracket_k \)

\[
\exists U . \ E \rightarrow^* U \\
u \equiv U \in \llbracket \tau \rrbracket_{k-k''} \\
\]

If \( k = k' \) then \( u \equiv U \in \llbracket \tau \rrbracket_{k'-k''} \). Otherwise, \( k' - k'' < k - k'' \), so the IH (induction hypothesis) applied to (1) also yields \( u \equiv U \in \llbracket \tau \rrbracket_{k'-k''} \). This proves implication (a).

(b) Assume \( k'' < k' \) and \( e \rightarrow^{k''} e' \) and \( \text{diverge}(e') \). Then we get with \( e \approx E \in \llbracket \tau \rrbracket_k \) and \( k' \leq k \) that \( \text{diverge}(E) \).

- Case rule \textsc{equiv-struct}. Follows from IH.
- Case rule \textsc{equiv-iface}. Obvious.

End case distinction. ■
Lemma A.3.8 (Expression equivalence implies value equivalence). If $k \geq 1$ and $\nu \approx V \in \llbracket \tau \rrbracket_k$ then $\nu \equiv V \in \llbracket \tau \rrbracket_k$.

Proof. From the first implication of rule `equiv-exp` we get for $k' = 0 < k$ and with $\nu \rarr_0 V$ that $\exists V'. V' \rarr V \land \nu \equiv V' \in \llbracket \tau \rrbracket_k$. But $V$ is already a value, so $V' = V$. ■

Lemma A.3.9 (Value equivalence implies expression equivalence). If $\nu \equiv V \in \llbracket \tau \rrbracket_k$ then $\nu \approx V \in \llbracket \tau \rrbracket_k$ for any $k$.

Proof. We have $\nu \rarr_0 V$, so we get the first implication of rule `equiv-exp` by setting $E = V$ and by assumption $\nu \equiv V \in \llbracket \tau \rrbracket_k$. The second implication holds vacuously because values do not diverge. ■

Lemma A.3.10. Assume `func (x t_S[Φ]) mM {return e} ≈_k X`. Then the following holds:

$$\forall k' < k, \phi, W. \phi \approx W \in \llbracket \Phi \rrbracket_{k'} \implies \Phi \rarr \phi : \eta \land$$

$$\langle x, t_S[\phi], \eta m M, \eta e \rangle \approx \lambda (Y_1, Y_2, Y_3). X (W, Y_1, Y_2, Y_3) \in \llbracket \eta m M \rrbracket_{k'}$$

Proof. Let $M = [\Phi'](\bar{x}_1 \bar{t}_1) \tau$ and assume for any $k', \phi, W$

$$k' < k$$
$$\phi \approx W \in \llbracket \Phi \rrbracket_{k'}$$

Obviously

$$\Phi \rarr \phi : \eta$$

To show that

$$\langle x, t_S[\phi], \eta m [\Phi'](\bar{x}_1 \bar{t}_1) \tau, \eta e \rangle \approx \lambda (Y_1, Y_2, Y_3). X (W, Y_1, Y_2, Y_3) \in \llbracket \eta m M \rrbracket_{k'}$$

holds, we assume the left-hand side of the implication in the premise of rule `equiv-method-dict-entry` for some $k'', \phi', W', u, U, V', \bar{V'}$:

$$k'' \leq k'$$
$$\eta \Phi' \rarr \phi' : \eta'$$
$$\phi' \approx W' \in \llbracket \eta \Phi' \rrbracket_{k''}$$
$$u \approx U \in \llbracket t_S[\phi] \rrbracket_{k''}$$

$$(\forall i \in [n]). v_i \approx V_i \in \llbracket \eta' \eta t_i \rrbracket_{k''}$$

We then need to prove (9) to show the overall goal.

$$\theta \eta' \eta e \approx V (U, W', (\bar{V'})) \in \llbracket \eta' \eta \tau \rrbracket_{k''}$$
$$\theta = \langle x \rarr u, x_i \rarr v_i' \rangle$$
$$V = \lambda (\bar{Y}^3). X (W, Y_1, Y_2, Y_3)$$
Let $\Phi = \alpha \sigma^p$, $\Phi' = \beta \sigma'^i$, $\phi = \sigma^m$, $\phi' = \sigma'^m$. Then by (3) and (5)

$$\eta = \langle \alpha_i \mapsto \sigma'_i \rangle$$  \hspace{1cm} (11)
$$\eta' = \langle \beta_i \mapsto \sigma'^{m_i} \rangle$$  \hspace{1cm} (12)

Define

$$\Psi := \alpha_i \sigma^p_i \beta_i \sigma'^{m_i}$$
$$\phi'' := \sigma'^p \sigma'^m$$
$$\eta'' := \langle \alpha_i \mapsto \sigma'^p_i \beta_i \mapsto \sigma'^m_i \rangle$$  \hspace{1cm} (13)

Then

$$\Psi \mapsto \phi'' : \eta''$$  \hspace{1cm} (14)

The $\beta^j$ are sufficiently fresh, so ftv($\sigma^m$) ∩ $\beta^j$ = 0. Hence by (11), (12), (13)

$$\eta'' \sigma^p_i = \eta'' \sigma'^{m_i}$$
$$\eta'' \sigma_i^{m_i} = \eta'' \sigma'_i$$
$$\eta'' \tau = \eta'' \tau$$
$$\eta'' \epsilon = \eta'' \epsilon$$  \hspace{1cm} (15)

We have from (2) and (6)

$$W = \langle W^p \rangle$$  \hspace{1cm} (16)
$$W' = \langle W'^{m_i} \rangle$$  \hspace{1cm} (17)

We now prove

$$\phi'' \approx \langle W^p, W'^{m_i} \rangle \in \llbracket \Psi \rrbracket$$  \hspace{1cm} (18)

by verifying the implication in the premise of rule $\text{EQUIV-BOUND-TYPARAMS}$. We consider two cases for every $\ell \leq k''$.

Case distinction whether $i$ in $[p]$ or in $[q]$.

- Case $i \in [p]$: We need to prove $\forall u, U . u \approx U \in \llbracket \sigma''_i \rrbracket \ell \implies u \approx W_i U \in \llbracket \eta'' \sigma_i \rrbracket \ell$.

  From (2) we get with $u \approx U \in \llbracket \sigma''_i \rrbracket \ell$ and $\ell \leq k'' \leq k'$ that $u \approx W_i U \in \llbracket \eta \sigma_i \rrbracket \ell$. By assumption 5.2.1 ftv($\sigma_i$) ⊆ $\alpha^p$, so $\eta'' \sigma_i = \eta \sigma_i$ by (11) and (13).

- Case $i \in [q]$: We need to prove $\forall u, U . u \approx U \in \llbracket \sigma''''_i \rrbracket \ell \implies u \approx W'_i U \in \llbracket \eta'' \sigma'_i \rrbracket \ell$.

  From (6) we get with $u \approx U \in \llbracket \sigma''''_i \rrbracket \ell$ that $u \approx W'_i U \in \llbracket \eta \sigma'_i \rrbracket \ell$. Also, $\eta'' \sigma'_i = \eta'' \sigma'_i$ by (15).

End case distinction. This finishes the proof of (18).

From (7) and (13) we have

$$u \approx U \in \llbracket \eta'' \iota_S[\alpha^p] \rrbracket$$  \hspace{1cm} (19)

From (8) and (15)

$$v_i \approx V_i \in \llbracket \eta'' \tau_i \rrbracket$$  \hspace{1cm} (20)

From the assumption $\text{func} (x \ i_S[\Phi] \ m M \ {\text{return} \ e}) \approx_k X$, we can invert rule $\text{EQUIV-METHOD-DECL}$. Noting that $k'' \leq k' < k$ and that (14), (18), (19), (20) give us the left-hand side of the implication in the premise of the rule, we get by the right-hand side
of the implication

\[ \theta \eta'' e \approx X (\overline{W^0}, U, (\overline{W'})^i, (\overline{V^i})) \in \llbracket \eta'' \tau \rrbracket_{k''} \]

With (15), (16), (17)

\[ \theta \eta' \eta e \approx X (W, U, W', (\overline{V^i})) \in \llbracket \eta' \eta \tau \rrbracket_{k''} \quad (21) \]

We have by (10)

\[ V (U, W', (\overline{V^i})) = (\lambda (\overline{V^i}). X (W, Y_1, Y_2, Y_3)) (U, W', (\overline{V^i})) \]

so

\[ V (U, W', (\overline{V^i})) \leadsto^* X (W, U, W', (\overline{V^i})) \]

Thus, with (21) and Lemma 5.2.2

\[ \theta \eta' \eta e \approx V (U, W', (\overline{V^i})) \in \llbracket \eta' \eta \tau \rrbracket_{k''} \]

as required to prove (9).

\[ \square \]

**Definition A.3.11** (Domain). We write \( \text{dom}(\cdot) \) for the domain of a substitution \( \eta, \theta, \rho \) or \( \mu \), of a type environment \( \Delta \), of a value environment \( \Gamma \), or some type parameters \( \Phi \).

**Definition A.3.12** (Free variables). We write \( \text{fv}(\cdot) \) for the set of free term variables, and \( \text{ftv}(\cdot) \) for the set of free type variables.

**Lemma A.3.13** (Subtyping preserves equivalence). Let \( \overline{D} \approx_k \mu \). Assume \( \Delta_{\text{corea}} \tau <: \sigma \leadsto V \) and \( \eta \approx \rho \in \llbracket \Delta \rrbracket_k \) and \( e \approx E \in \llbracket \eta \tau \rrbracket_k \). Then \( e \approx (\rho V) E \in \llbracket \eta \sigma \rrbracket_k \).

We prove Lemma A.3.13 together with the following two lemmas.

**Lemma A.3.14**. Assume \( \overline{D} \approx_k \mu \) and \( \eta \approx \rho \in \llbracket \Delta \rrbracket_k \). Let \( \langle R, V \rangle \in \text{methods}(\Delta, t_S[\phi]) \) and define \( U = \lambda (\overline{Y^i}). X_{m, t_S} (\rho V, Y_1, Y_2, Y_3) \). Then we have for all \( k' < k \) that \( \text{methodLookup}(m, \eta t_S[\phi]) \approx U \in \llbracket \eta R \rrbracket_{k'} \).

**Lemma A.3.15** (Substitution preserves equivalence). Assume \( \overline{D} \approx_k \mu \) and \( \eta \approx \rho \in \llbracket \Delta \rrbracket_k \). If \( \Delta_{\text{subd}} \Phi \leftrightarrow \phi : \eta' \leadsto V \) then \( \eta \phi \approx \rho V \in \llbracket \eta \Phi \rrbracket_k \).

**Proof of Lemmas A.3.13, A.3.14, and A.3.15**. We show the three lemmas by induction on the combined height of the derivations for \( \Delta_{\text{corea}} \tau <: \sigma \leadsto V \) and \( \langle R, V \rangle \in \text{methods}(\Delta, t_S[\phi]) \) and \( \Delta_{\text{subd}} \Phi \leftrightarrow \phi : \eta' \leadsto V \).

We start with the proof for Lemma A.3.13. We have from the assumptions

\[ e \approx E \in \llbracket \eta \tau \rrbracket_k \quad (1) \]

Assume \( k' < k \) and \( e \leadsto^{k'} e' \). The second implication in the premise of rule \text{EQUIV-EXP} holds obviously, because with \( \text{diverge}(e') \) we get from (1) \( \text{diverge}(E) \), so also \( \text{diverge}((\rho V) E) \).
Thus, we only need to prove the first implication. Assume that \( e' = v \) for some value \( v \). Then via (1) for some \( U \)

\[
E \rightarrow^* U
\]

\[
v \equiv U \in \llbracket \eta \tau \rrbracket_{k-k'} \quad (2)
\]

We then need to verify that \((\rho V) \ U \rightarrow^* U'\) for some \( U' \) with \( v \equiv U' \in \llbracket \eta \sigma \rrbracket_{k-k'}\). In fact, \( k' < k \), so with Lemma A.3.8 it suffices to show that \( v \approx U' \in \llbracket \eta \sigma \rrbracket_{k-k'}\).

**Case distinction** on the last rule in the derivation of \( \Delta \vdash_{\text{coerce}} \tau < \sigma \leadsto V \).

- **Case** \text{coerce-tyvar}:

\[
\begin{array}{c}
X \text{ fresh} \quad (a : \tau_I) \in \Delta \\
\Delta \vdash_{\text{coerce}} \tau_I < \sigma \leadsto \lambda X. W (X_a X) \\
\end{array}
\]

\[
\Delta \vdash_{\text{coerce}} \alpha < \sigma \leadsto W
\]

Our goal to show is

\[
e \approx (\rho X_\alpha) \ U \in \llbracket \eta \tau_I \rrbracket_k \quad (4)
\]

With (4) and the IH for Lemma A.3.13 we then get

\[
e \approx (\rho W) ((\rho X_\alpha) \ U) \in \llbracket \eta \sigma \rrbracket_k
\]

Then, with \( e \rightarrow^{k'} v \), we get \((\rho V) \ U \rightarrow (\rho W) ((\rho X_\alpha) \ U) \rightarrow^* U'\) for some \( U' \) with \( v \equiv U' \in \llbracket \eta \sigma \rrbracket_{k-k'}\).

We now prove (4). From the assumption \( \eta \approx \rho \in \llbracket \Delta \rrbracket_k \) we have

\[
\begin{align*}
\Delta &= \alpha_i : \tau_i^n \\
\eta &= (\alpha_i \mapsto \sigma_i^n) \\
\rho &= (X_\alpha_i \mapsto V_i^n) \\
\sigma_i^n &\approx V_i^n \in \llbracket \alpha_i \tau_i^n \rrbracket_k
\end{align*}
\]

such that \( \alpha = \alpha_j \) and \( \tau_I = \tau_j \) for some \( j \in [n] \). Inverting rule \text{equiv-bounded-typarams} on (5) yields

\[
\forall k'' \leq k, w, W'. w \approx W' \in \llbracket \sigma_j \rrbracket_{k''} \implies w \approx V_j W' \in \llbracket \eta \tau_j \rrbracket_{k''} \quad (6)
\]

From (3) by \( \eta \tau = \sigma_j \) then \( v \equiv U \in \llbracket \sigma_j \rrbracket_{k-k'} \). Thus with (6) and Lemma A.3.9

\[
v \approx V_j U \in \llbracket \eta \tau_j \rrbracket_{k-k'}
\]

With Lemma 5.2.3 and \( e \rightarrow^{k'} v \) then

\[
e \approx V_j U \in \llbracket \eta \tau_j \rrbracket_k
\]

But \( \tau_j = \tau_I \) and \( V_j = \rho X_\alpha \), so this proves (4).
• **Case** COERC-STRUCT-IFACE:

\[
X, Y_1, Y_2, Y_3 \text{ fresh}
\]

\[
\text{type } t_1[\Phi] \text{ interface } \{\overline{mM'}\} \in \overline{D}
\] (7)

\[
\Phi \mapsto \phi : \eta'
\] (8)

\[
\langle \eta'(m_iM_i), V_i \rangle \in \text{methods}(\Delta, t_S[\psi])
\] (9)

\[
V_i' = \lambda(Y_1,Y_2,Y_3).X_{m_i,t_S}(V_i,Y_1,Y_2,Y_3) \quad (\forall i \in [n])
\] (10)

\[
\Delta_{\text{corer}} t_S[\psi] \ll: t_1[\phi] \leadsto \lambda X. (X, (V_i'))
\]

Hence \((\rho V) U \rightarrow (U, \rho(V''))\) and \(U' := (U, \rho(V'''))\) is a value. We now want to show \(v \approx U' \in \ll[\eta \sigma]_{k-k'}\) via rule **EQUIV-IFACE**. Define the \(\sigma_S\) in the premise of **EQUIV-IFACE** as \(\eta \tau = t_S[\eta \psi]\).

The first premise of **EQUIV-IFACE**

\[
\forall k_1 < k - k' \cdot v \equiv U \in \ll[\eta \tau]_{k_1}
\] (11)

follows from (3) and **Lemma A.3.3**. From (7) and (8) we get with \(\sigma = t_1[\phi]\) the second premise as

\[
\text{methods}(\eta \sigma) = \eta' \overline{mM'}
\] (12)

We next prove the third premise of **EQUIV-IFACE**. Pick some \(j \in [n]\) and \(k_2 < k - k'\).

With the assumptions \(\overline{D} \approx_k \mu\) and \(\eta \approx \rho \in \ll[\Delta]_{k}\) and with (9), (10), and the IH for **Lemma A.3.14** we get

\[
\text{methodLookup}(m_j, \eta t_S[\psi]) \approx \rho V_j' \in \ll[\eta \sigma]_{k-k'}
\] via rule **EQUIV-IFACE**.

• **Case** COERC-IFACE-

\[
Y, \overline{X}^i \text{ fresh} \quad \pi : [q] \rightarrow [n] \text{ total}
\]

\[
\text{type } t_1[\Phi_1] \text{ interface } \{\overline{R}^i\} \in \overline{D}
\] (14)

\[
\text{type } u_1[\Phi_2] \text{ interface } \{\overline{R''}^i\} \in \overline{D}
\] (15)

\[
\Phi_1 \mapsto \phi_1 : \eta_1 \quad \Phi_2 \mapsto \phi_2 : \eta_2
\]

\[
\eta_2 R'_i = \eta_1 R_{\pi(i)} \quad (\forall i \in [q])
\] (16)

\[
\Delta_{\text{corer}} t_1[\phi_1] \ll: u_1[\phi_2] \leadsto \lambda (Y, (\overline{X}^i)) : (Y, (X_{\pi(1)}, \ldots, X_{\pi(q)}))
\]

As \(\eta \tau = t_S[\phi_1]\) is an interface type, we get from (3) by inverting rule **EQUIV-IFACE** for some \(W, \sigma_S, \overline{W}^i\) that

\[
\forall k_1 < k - k' \cdot v \approx W \in \ll[\sigma_S]_{k_1}
\] (18)

\[
\text{methods}(\eta \tau) = \eta \eta_1 \overline{R}^i = \overline{mM'}
\] (19)

\[
\forall i \in [n], k_2 < k - k' \cdot \text{methodLookup}(m_i, \sigma_S) \approx W_i \in \ll[m_iM_i]_{k_2}
\] (20)

\[
U = (W, (\overline{W}^i))
\] (21)
Our goal is to show \((\rho V) U \xrightarrow{\ast} U'\) for some \(U'\) with \(v \approx U' \in \llbracket \eta \sigma \rrbracket_{k-k'}\). Via (17) and (21)

\[
(\rho V) U = V U \xrightarrow{\ast} (W, (W_{\pi(1)}, \ldots, W_{\pi(q)})) =: U'
\]

From (14), (15), (16), and (19) we have

\[
m'M'^l = \eta_1 R_{\pi(1)}, \ldots, \eta_1 R_{\pi(q)} = m_{\pi(1)} M_{\pi(1)}, \ldots, m_{\pi(q)} M_{\pi(q)}
\]

\[
\text{methods}(\eta \sigma) = \{m'M'^l\}
\]

Pick \(j \in [q]\). Then via (23)

\[
\text{methodLookup}(m'_j, \sigma_S) = \text{methodLookup}(m_{\pi(j)}, \sigma_S)
\]

Hence with (20) and (23)

\[
\forall j \in [q], k_2 < k - k'. \text{methodLookup}(m'_j, \sigma_S) \approx W_{\pi(j)} \in \llbracket m'_j M'_j \rrbracket_{k_2}
\]

With (18), (25), (24) and the definition of \(U'\) in (22) we then get by applying rule \textsc{equiv-face} \(V \approx U' \in \llbracket \eta \sigma \rrbracket_{k-k'}\) and with (22) also \((\rho V) U \xrightarrow{\ast} U'\).

End case distinction on the last rule in the derivation of \(\Delta \vdash \text{core} \tau < : \sigma \sim V\).

This finishes the proof of Lemma A.3.13.

We next prove Lemma A.3.14. By inverting rule \textsc{methods-struct} for the assumption \(\langle R, V \rangle \in \text{methods}(\Delta, t_S[\phi])\) we get

\[
\text{func} (x \ t_S[\Phi]) \ mM \ \{\text{return } e\} \in \overline{D}
\]

\[
\Delta \vdash \text{sub} \Phi \mapsto \phi : \eta' \sim V
\]

\[
R = \eta'mM
\]

Inverting (27) yields

\[
\Phi = \overline{\alpha \tau^l}
\]

\[
\eta' = \langle \overline{\alpha_i \rightarrow \sigma_i^{l_i}} \rangle
\]

\[
\phi = \overline{\sigma^l}
\]

\[
\Delta \vdash \text{core} \ \sigma_i < : \eta' \tau_i \sim V_i \quad (\forall i \in [n])
\]

\[
V = \langle \overline{V'^l} \rangle
\]

Define \(\eta'' = \langle \overline{\alpha_i \rightarrow \eta \sigma_i^{l_i''}} \rangle\). Then by rule \textsc{method-lookup} and (26)

\[
\text{methodLookup}(m, \eta t_S[\phi]) = \langle x, \eta t_S[\Phi], \eta''mM, \eta''e \rangle
\]

By assumption 5.2.1, the \(\overline{\alpha_i}^{l_i''}\) can be assumed to be fresh, \(\text{ftv}(\Phi) \subseteq \overline{\alpha_i}^{l_i''}\), and \(\eta \Phi = \Phi\).

Applying the IH for Lemma A.3.15 on (27) yields \(\eta \phi \approx \rho V \in \llbracket \eta \Phi \rrbracket_k\).

\[
\eta \phi \approx \rho V \in \llbracket \Phi \rrbracket_k
\]

From the assumption \(\overline{D} \approx_k \mu\) we get with (26)

\[
\text{func} (x \ t_S[\Phi]) \ mM \ \{\text{return } e\} \approx_k X_{m,t_S}
\]
Then for any \( k' < k \) by Lemma A.3.10, where (30) and Lemma A.3.5 give the left-hand side of the implication

\[
\Phi \mapsto \eta \phi : \eta'' \quad \langle x, \eta t_S[\phi], \eta''mM, \eta''e \rangle \approx \\
\lambda (\vec{T}^3). X_{m,t_S} (\rho V, Y_1, Y_2, Y_3) \in \llbracket \eta''mM \rrbracket_{k'} = U
\]

(31)

We have \( \eta R = (28) \eta \eta' mM = \eta'' mM, \) where the last equality holds because \( \text{ftv}(mM) \subseteq \overline{\alpha}'' \) and \( \overline{\alpha}'' \) fresh by assumption 5.2.1. Hence (29) and (31) give the desired claim.

Finally, we prove Lemma A.3.15. By inverting rule TYPE-INST-CHECKED for the assumption \( \Delta \gamma_{\text{subst}}, \Phi \mapsto \phi : \eta' \leadsto V \) we get

\[
\Phi = \overline{\alpha} \tau^m \\
\phi = \overline{\alpha} \tau^m \\
\eta' = \langle \alpha_i \mapsto \overline{\sigma}_i'' \rangle \\
\Delta \gamma_{\text{coerce}} \sigma_i' <: \eta' \tau_i \leadsto V_i \quad (\forall i \in [n]) \\
V = (\overline{V}')
\]

(32)

Define \( \eta'' = \langle \alpha_i \mapsto \overline{\eta} \overline{\sigma}_i'' \rangle. \) To prove \( \eta \phi \approx \rho V \in \llbracket \eta \Phi \rrbracket_k \) we need to show the implication \( \forall j \in [n], \ k' \leq k \ . \ u \approx U \in \llbracket \eta \sigma_j \rrbracket_{k'} \implies u \approx (\rho V) U \in \llbracket \eta'' \eta \sigma_j \rrbracket_{k'} \) from the premise of rule EQUIV-BOUNDED-TYPARMS. Assume \( j \in [n], \ k' \leq k, \) and \( u \approx U \in \llbracket \eta \sigma_j \rrbracket_{k'}. \) Applying the IH for Lemma A.3.13 on (32) yields together with Lemma A.3.5 and Lemma A.3.7 that

\[
u \approx (\rho V_j) U \in \llbracket \eta \eta' \tau_j \rrbracket_{k'}
\]

(33)

As the \( \overline{\alpha} \) are bound in \( \Phi, \) we may assume that \( \text{dom}(\eta) \cap \overline{\alpha} = \emptyset = \text{ftv}(\eta) \cap \overline{\alpha}. \) We now argue that

\[
\eta \eta' \tau_j = \eta'' \eta \tau_j
\]

(34)

by induction on the structure of \( \tau_j. \) The interesting case is were \( \tau_j \) is a type variable (otherwise the claim follows by the IH). If \( \tau_j \in \overline{\alpha} \) then

\[
\eta \eta' \tau_j \overset{\text{def. of } \eta''}{=} \eta'' \tau_j \overset{\text{dom}(\eta) \cap \overline{\alpha} = \emptyset}{=} \eta'' \eta \tau_j
\]

If \( \tau_j \in \text{dom}(\eta) \) then

\[
\eta \eta' \tau_j \overset{\text{dom}(\eta) \cap \overline{\alpha} = \emptyset}{=} \eta \tau_j \overset{\text{ftv}(\eta) \cap \overline{\alpha} = \emptyset}{=} \eta'' \eta \tau_j
\]

If \( \tau_j \) is some other type variable, (34) holds obviously. With (33) and (34) we get \( u \approx (\rho V_j) U \in \llbracket \eta'' \eta \tau_j \rrbracket_{k'} \) as required.

Lemma A.3.16 (Free variables of coercion values). If \( \Delta \gamma_{\text{coerce}} \tau <: \sigma \leadsto V \) then \( \text{ftv}(V) \subseteq \{ X_{\alpha} \mid \alpha \in \text{dom}(\Delta) \} \cup X \) where \( X = \{ X_{m,t_S} \mid m \ \text{method name}, \ t_S \ \text{struct name} \}. \)

Proof. By straightforward induction on the derivation of \( \Delta \gamma_{\text{coerce}} \tau <: \sigma \leadsto V. \)
A.3.1 Proof of Lemma 5.2.4

By induction on the derivation of \( \langle \Delta, \Gamma \rangle \vdash_{\text{exp}} e : \tau \leadsto E \).

**Case distinction** on the last rule in the derivation.

- **Case** \( \text{var} \):

\[
\frac{(x : \tau) \in \Gamma}{\langle \Delta, \Gamma \rangle \vdash_{\text{exp}} x : \tau \leadsto X}
\]

with \( \theta e = \theta x \) and \( \rho E = \rho X \). From the assumption \( \theta \equiv \rho \in \llbracket \eta \Gamma \rrbracket_k \) we get \( \theta x \equiv \rho X \in \llbracket \eta \tau \rrbracket_k \) as required.

- **Case** \( \text{struct} \):

\[
\Delta \vdash_{\text{as}} t_S[\phi] \\
\textbf{type}\ t_S[\Phi] \textbf{ struct} \{ \bar{\tau}' \} \in \overline{D} \tag{1}
\]

\[
\Phi \mapsto \phi : \eta' \tag{2}
\]

\[
\frac{\langle \Delta, \Gamma \rangle \vdash_{\text{exp}} e_i : \eta' \tau_i \leadsto E_i \quad (\forall i \in [n])}{\langle \Delta, \Gamma \rangle \vdash_{\text{exp}} t_S[\phi] \{ \bar{\tau}' \} : t_S[\phi] \leadsto \langle E' \rangle} \tag{3}
\]

Applying the IH to (3) yields

\[
\theta e_i \equiv \rho E_i \in \llbracket \eta \eta' \tau_i \rrbracket_k \quad (\forall i \in [n]) \tag{5}
\]

We now consider the two implications in the premise of rule \( \text{EQUIV-EXP} \)

(a) Assume \( k' < k \) and \( \theta e \mapsto_{\text{v}}^{k'} v \) for some value \( v \). The goal is to show that there exists some value \( V \) with \( \rho E \mapsto^* V \) and \( v \equiv V \in \llbracket \eta \tau \rrbracket_{k-k'} \).

With \( \theta e \mapsto_{\text{v}}^{k'} v \) there must exist values \( \bar{\tau}' \) such that

\[
\theta e_i \mapsto_{\text{v}}^{k_i} V_i \quad (\forall i \in [n]) \tag{6}
\]

\[
k_i \leq k' \quad (\forall i \in [n]) \tag{7}
\]

Via (5) and \( k_i \leq k' < k \) then for all \( i \in [n] \)

\[
\rho E_i \mapsto^* V_i \text{ for some } V_i \quad (\forall i \in [n]) \tag{8}
\]

\[
v_i \equiv V_i \in \llbracket \eta \eta' \tau_i \rrbracket_{k-k_i} \quad (\forall i \in [n]) \tag{9}
\]

We have \( k - k' \leq k - k_i \) for all \( i \in [n] \) by (6). Thus with (9) and Lemma A.3.3

\[
v_i \equiv V_i \in \llbracket \eta \eta' \tau_i \rrbracket_{k-k'} \quad (\forall i \in [n]) \tag{10}
\]

We also have with (8) and the definition of \( E \) in (4)

\[
\rho E \mapsto^* \langle \bar{\tau}' \rangle \tag{11}
\]

Assume \( \Phi = \bar{\alpha}' \) and \( \phi = \bar{\sigma}' \). Then by (2) \( \eta' = \langle \bar{\alpha}_i \mapsto \bar{\sigma}_i \rangle \) and for \( \eta'' = \langle \bar{\alpha}_i \mapsto \bar{\sigma}_i \rangle \) we have

\[
\Phi \mapsto \eta \phi : \eta'' \tag{12}
\]
By assumption 5.2.1 we have ftv(π′) ⊆ {α}, so

\[ η^nτ_i = η''τ_i \quad (\forall i ∈ [n]) \]  \hspace{1cm} (13)

With (1), (10), (7), (12), (13), and rule equiv-struct then

\[ v ≡ (\overline{V''}) ∈ \llbracket t_5[ηφ] \rrbracket_{k-k'} \]

Together with (11), this finishes subcase (a) for \( V = (\overline{V''}) \).

(b) Assume \( k' < k \) and \( θηe \rightarrow k' e' \) and \( \text{diverge}(e') \). Then \( \text{diverge}(e_j) \) for some \( j ∈ [n] \), so with (5) and (6) also \( \text{diverge}(ρE_j) \). Thus, by definition of \( E \) in (4),
\( \text{diverge}(ρE) \) as required.

• Case access:

\[ \langle Δ, Γ \rangle τ_{eq} e' : t_5[φ] \leadsto E' \]  \hspace{1cm} (14)

\[ \begin{align*}
\text{type} & \quad t_5[Φ] \text{ struct } \{ \overline{f \tau^i} \} ∈ D \\
\Phi & \mapsto \phi : η'' \\
\langle Δ, Γ \rangle τ_{eq} e'.f_j : η''τ_j \leadsto \text{ case } E' \text{ of } (\overline{X''}) \rightarrow X_i \\
= & \hspace{1cm} = e \\
= & \hspace{1cm} = τ \\
= & \hspace{1cm} = E
\end{align*} \]

Applying the IH to (14) yields

\[ θηe' ≈ ρE' ∈ \llbracket t_5[ηφ] \rrbracket_{k} \]  \hspace{1cm} (16)

We now consider the two implications in the premise of rule equiv-exp

(a) Assume \( k' < k \) and \( θηe \rightarrow k' v \) for some value \( v \). The goal is to show that there exists some value \( V \) with \( ρE \rightarrow^* V \) and \( v ≡ V ∈ \llbracket ητ \rrbracket_{k-k'} \). With \( θηe \rightarrow k' v \) then \( θηe' \rightarrow k'' v' \) for some \( v' \) and \( k'' < k' \). With (16) then for some \( V' \)

\[ ρE' \rightarrow^* V' \]  \hspace{1cm} (17)

\[ v' ≡ V' ∈ \llbracket t_5[ηφ] \rrbracket_{k-k''} \]  \hspace{1cm} (18)

Inverting rule equiv-struct on (18) yields

\[ v' = t_5[ηφ] \{ π'' \} \]  \hspace{1cm} (19)

\[ V' = (\overline{V''}) \quad \text{ for some } \overline{V''} \]

\[ v_i ≡ V_i ∈ \llbracket η''τ_i \rrbracket_{k-k''} \quad (∀i ∈ [n]) \]  \hspace{1cm} (20)

where \( η'' = (α_i ↦ ησ_i)^p \), assuming \( Φ = α̂p \) and \( φ = σ̂p \). By assumption 5.2.1 we have ftv(τ_j) ⊆ {α}. Thus, \( η''τ_j = η''τ_j \) and \( k'' < k' \). Hence with (20) and Lemma A.3.3

\[ v_j ≡ V_j ∈ \llbracket ητ \rrbracket_{k-k'} \]  \hspace{1cm} (21)

With (17) and the definition of \( E \) in (15) we get

\[ ρE \rightarrow^* V_j \]  \hspace{1cm} (22)

With \( θηe \rightarrow k' v \) and \( θηe' \rightarrow k'' v' \) and the form of \( v' \) in (19), we get \( θηe \rightarrow k' v_j \) and \( v = v_j \) by rule fg-field. Define \( V = V_j \) and we are done with subcase (a) by (21) and (22).
(b) Assume \( k' < k \) and \( \theta \eta e \rightarrow^{k'} e'' \) and \( \text{diverge}(e'') \). Then we must have that \( \theta \eta e' \rightarrow^{k''} e''' \) for some \( k'' < k' \) and some \( e''' \). Thus, \( \text{diverge}(e'') \) by the definition of \( e \) in (15) and the evaluation rules for FGG. With (16) then \( \text{diverge}(\rho E') \). By definition of \( E \) in (15) then \( \text{diverge}(\rho E) \) as required.

- **Case** \texttt{call-struct}:

\[
\begin{align*}
\langle \Delta, \Gamma \rangle \xrightarrow{r_{\text{sup}}} g : t_S [\phi] \leadsto G \quad (23) \\
\langle m[\Psi](x \sigma'^i) \sigma, W \rangle \in \text{methods}(\Delta, t_S [\phi]) \quad (24) \\
\Delta \xrightarrow{r_{\text{sub}}} \Psi \Rightarrow \psi : \eta_1 \leadsto W' \quad (25) \\
\langle \Delta, \Gamma \rangle \xrightarrow{r_{\text{sup}}} e_i : \eta_1 \sigma_i \leadsto E_i \quad (\forall i \in [n]) \quad (26)
\end{align*}
\]

From the IH applied to (23) and (26)

\[
\begin{align*}
\theta \eta g \approx \rho G \in \llbracket \eta t_S [\phi] \rrbracket_k \\
\theta \eta e_i \approx \rho E_i \in \llbracket \eta_1 \sigma_i \rrbracket_k \quad (\forall i \in [n]) \quad (29)
\end{align*}
\]

Assume \( \theta \eta e \rightarrow^{k'} e' \) for some \( k' < k \). We first consider the following situation for some values \( u, \overline{v} \):

\[
\begin{align*}
\theta \eta g \rightarrow^{k''} u \\
\theta \eta e_i \rightarrow^{k_i} v_i \\
\theta \eta e \rightarrow^{k'' + \Sigma k_i} u.m[\Psi](\overline{v}^i) \rightarrow^{k' - k'' - \Sigma k_i} e'
\end{align*}
\]

with \( k'' + \Sigma k_i \leq k' \). (30), (31), and (32) are intermediate assumptions, which become true when we later prove the two implications of rule \texttt{equiv-exp}.

We have from (28), (30), (29), and (31)

\[
\begin{align*}
\rho G \rightarrow^* U \text{ for some } U \text{ with } u \equiv U \in \llbracket \eta t_S [\phi] \rrbracket_{k - k''} \quad (33) \\
(\forall i \in [n]) \rho E_i \rightarrow^* V_i \text{ for some } V_i \text{ with } v_i \equiv V_i \in \llbracket \eta_1 \sigma_i \rrbracket_{k - k_i} \quad (34)
\end{align*}
\]

From (24) we get by inverting rule \texttt{methods-struct}:

\[
\begin{align*}
\textbf{func} \ (x \ t_S [\Phi]) \ m[\Psi](x \sigma'^i) \sigma' \ {\text{return} \ g'} \in \overline{D} \quad (35) \\
\Delta \xrightarrow{r_{\text{sub}}} \Phi \Rightarrow \phi : \eta_2 \leadsto W \\
m[\Psi](x \sigma'^i) \sigma = \eta_2 (m[\Psi'](x \sigma'^i) \sigma') \quad (36)
\end{align*}
\]

From the assumption \( \overline{D} \approx_k \mu \) and (35)

\[
\begin{align*}
\textbf{func} \ (x \ t_S [\Phi]) \ m[\Psi'](x \sigma'^i) \sigma' \ {\text{return} \ g'} \approx_k X_{m, t_S} \quad (38)
\end{align*}
\]

Define

\[
k''' := \min(k - k'', k - \Sigma k_i) - 1 \quad (39)
\]
We have $k' - k'' - \Sigma k_i < k''' + 1$ by the following reasoning:

$$
\begin{align*}
k''' + 1 & \overset{(39)}{=} \min(k - k'', k - \Sigma k_i) \\
& = k - \max(k'', \Sigma k_i) \\
& \geq k - k'' - \Sigma k_i \\
k' < k & \quad \implies k' - k'' - \Sigma k_i > 0
\end{align*}
$$

(40)

With (38) and $k''' < k$, we now want to use the implication from the premise of rule `equiv-method-decl`. We instantiate the universally quantified variables of the implication as follows: $k' = k'''$, $\phi = \eta(\phi, \psi)$, $\overline{W}^p = \rho W$, $\overline{W}' = \rho W'$, $\nu = u$, $V = U$, $\overline{\nu} = \overline{v}$, $\overline{V} = \overline{V}'$. Next, we prove the left-hand side of the implication. But first assume (see (36), (25), (37))

$$
\begin{align*}
\Phi = \overline{\alpha \tau^p} & \quad \phi = \overline{\tau^p} \\
\Psi' = \overline{\beta \tau'^q} & \quad \psi = \overline{\tau'^q} \\
W = \overline{W}^p & \quad W' = \overline{W}'^q
\end{align*}
$$

(41)

(42)

and define

$$
\eta_3 = \langle \alpha_i \mapsto \eta \tau_i^p, \beta_i \mapsto \eta \tau_i'^q \rangle
$$

(43)

– We start by showing the first two conjuncts of the implication’s left-hand side.

$$
\Phi, \Psi' \mapsto \eta(\phi, \psi) : \eta_3 \land \eta(\phi, \psi) \approx \rho \langle \overline{W}, \overline{W}' \rangle \in \llbracket \Phi, \Psi' \rrbracket_{k''}
$$

(44)

The left part of the conjunction follows from (43). We then show $\eta(\phi, \psi) \approx \rho \langle \overline{W}, \overline{W}' \rangle \in \llbracket \Phi, \Psi' \rrbracket_k$ by proving the two implications required to fulfill the premise of rule `equiv-bounded-typars`. The right part of the conjunction in (44) then follows via Lemma A.3.5.

* First implication: $u_j \approx U_j \in \llbracket \eta \tau_j^p \rrbracket_k \implies u_j \approx (\rho W_j) U_j \in \llbracket \eta_3 \tau_j \rrbracket_k$ for all $j \in [\rho]$ and all $u_j, U_j$.

From (36) and Lemma A.3.15 we have $\eta \phi \approx \rho W \in \llbracket \eta \Phi \rrbracket_k$. Hence, with $u_j \approx U_j \in \llbracket \eta \tau_j^p \rrbracket_k$ and the implication in the premise of rule `equiv-bounded-typars`, we have $u_j \approx (\rho W_j) U_j \in \llbracket \langle \alpha_i \mapsto \eta \tau_i^p \rangle \eta_3 \tau_j \rrbracket_k$.

From assumption 5.2.1, (35), and (41), we know that \text{ftv}(\tau_j) \subseteq \{\tau\}$ and $\overline{\tau}$ fresh, so $\langle \alpha_i \mapsto \eta \tau_i^p \rangle \eta_3 \tau_j = \eta_3 \tau_j$. Thus $u_j \approx (\rho W_j) U_j \in \llbracket \eta_3 \tau_j \rrbracket_k$ as required.

* Second implication: $u_j \approx U_j \in \llbracket \eta \tau_j'^q \rrbracket_k \implies u_j \approx (\rho W_j) U_j \in \llbracket \eta_3 \tau_j'' \rrbracket_k$ for all $i \in [\rho]$ and all $u_j, U_j$.

From (25) and Lemma A.3.15 we have $\eta \psi \approx \rho W' \in \llbracket \eta \Psi \rrbracket_k$.

Hence, with $u_j \approx U_j \in \llbracket \eta \tau_j'' \rrbracket_k$, the implication in the premise of rule `equiv-bounded-typars`, and (37) then $u_j \approx (\rho W_j') U_j \in \llbracket \langle \beta_i \mapsto \eta \tau_i'' \rangle \eta_2 \tau_j'' \rrbracket_k$. We have with (36) and (41) that $\eta_2 = \langle \alpha_i \mapsto \tau_i' \rangle$.

Because of assumption 5.2.1, (35), and (42), we know that \text{ftv}(\tau_j') \subseteq \{\overline{\tau}, \overline{\beta}\}$ and $\overline{\alpha}, \overline{\beta}$ fresh. Hence, $\langle \beta_i \mapsto \eta \tau_i'' \rangle \eta_2 \tau_j'' = \langle \beta_i \mapsto \eta \tau_i'' \rangle \langle \alpha_i \mapsto \eta \tau_i' \rangle \tau_j'' \approx (43)$.

Thus $u_j \approx (\rho W_j') U_j \in \llbracket \eta_3 \tau_j'' \rrbracket_k$ as required.
This finishes the proof of (44).

- We next show the third conjunct of the implication’s left-hand side.

\[ u \approx U \in \llbracket t_S[\eta_3\sigma'] \rrbracket_{k'''} \quad (45) \]

We have \( t_S[\eta_3\sigma'] = t_S[\eta\phi] \) by (43) and (41). Hence, with (33), Lemma A.3.9, and Lemma A.3.3, it suffices to show that \( k''' \leq k - k'' \). But this follows from construction of \( k''' \) in (39).

- Finally, we show the fourth conjunct:

\[ v_i \approx V_i \in \llbracket \eta_3^i\sigma'^i \rrbracket_{k''}(\forall i \in [n]) \quad (46) \]

By (25), (37), (42) we have \( \eta_1 = \{\beta_i \mapsto \tau_i^{\eta_2}\} \). By (36) and (41) we have \( \eta_2 = \{\alpha_i \mapsto \tau'^{\eta_1}_i\} \). Thus,

\[ \eta_1^i\sigma'^i = \eta_1\eta_2^i\sigma'^i = \eta_3^i\sigma'^i \quad (43) \]

For the last equation, note that ftv(\( \sigma'^i \)) \( \subseteq \{\alpha_i, \beta_i\} \) by assumption 5.2.1 and (35), (41), (42). Hence, with (34), Lemma A.3.9, and Lemma A.3.3, it suffices to show that \( k''' \leq k - k_i \). But this follows from construction of \( k''' \) in (39).

Now (44), (45), and (46) are the left-hand side of the implication of rule equiv-method-decl, which we get from (38). The right-hand side of the implication then yields

\[ \langle x \mapsto u, \overline{x_i} \mapsto V_i \rangle \eta_3g' \approx X_{m,t_S}(\rho W, U, \rho W', \langle \overline{\eta} \rangle) \in \llbracket \eta_3\sigma' \rrbracket_{k'''} \quad (47) \]

\[ \vdash \theta' \quad (48) \]

From (33) we have \( u = t_S[\eta\phi] \) by inverting rule equiv-struct. Hence by (35), (43), and rule fg-call

\[ u.m[\eta\psi](\overline{\eta}) \rightarrow \theta'\eta_3g' \quad (48) \]

Also we have

\[ \eta\tau \quad (27) \]

\[ \eta_1\sigma' \quad (37) \]

\[ \eta_1\eta_2\sigma' = \eta_3\sigma' \]

where the last equation follows from (25), (37), (42), (36), (41) and ftv(\( \sigma' \)) \( \subseteq \{\alpha_i, \beta_i\} \) with assumption 5.2.1. Thus, with (47), (48), and Lemma 5.2.3

\[ u.m[\eta\psi](\overline{\eta}) \approx X_{m,t_S}(\rho W, U, \rho W', \langle \overline{\eta}\rangle) \in \llbracket \eta\tau \rrbracket_{k''' + 1} \quad (49) \]

By definition of \( E \) in (27) and with (33) and (34) we have

\[ \rho E \rightarrow^* \mu(X_{m,t_S})(\rho W, U, \rho W', \langle \overline{\eta}\rangle) \quad (50) \]

Also, we have by rules tl-context and tl-method

\[ X_{m,t_S}(\rho W, U, \rho W', \langle \overline{\eta}\rangle) \rightarrow \mu(X_{m,t_S})(\rho W, U, \rho W', \langle \overline{\eta}\rangle) \quad (51) \]

So far, we proved everything under the assumptions (30), (31), (32). We next consider the two implications of rule equiv-exp.

(a) Assume \( e' = v \) for some value \( v \). Our goal is to prove that there exists some value \( V \) such that \( \rho E \rightarrow^* V \) and \( v \equiv V \in \llbracket \eta\tau \rrbracket_{k-k'} \). Noting that (30), (31), (32)
hold, we have together with (48)
\[ \theta \eta e \rightarrow^{k''+\Sigma k_i} u.m[\eta \psi](\overline{\nu}) \rightarrow^{k'-k''-\Sigma k_i} v \] (52)
with \( k'' + \Sigma k_i < k' \). We have \( k' - k'' - \Sigma k_i < k''' + 1 \) by (40). Hence with (49) and (52) we know that there exists some value \( V \) with
\[ X_{m,tS}(\rho W_1 U, \rho W', (\overline{V})) \rightarrow^* V \] (53)
\[ v \equiv V \in \llbracket \eta \tau \rrbracket_{k''+1-k''+\Sigma k_i} \] (54)
We have \( k - k' \leq k''' + 1 - k' + k'' + \Sigma k_i \) by the following reasoning:
\[ k''' + 1 - k' + k'' + \Sigma k_i = \min(k - k'', k - \Sigma k_i) - k' + k'' + \Sigma k_i \]
\[ = k - \max(k'', \Sigma k_i) - k' + k'' + \Sigma k_i \]
\[ \geq k - k'' - \Sigma k_i - k' + k'' + \Sigma k_i \]
\[ = k - k' \]
With (54) and Lemma A.3.3 then \( v \equiv V \in \llbracket \eta \tau \rrbracket_{k-k'} \). And from (50), (51), (53), and Lemma A.1.2 we have that \( \rho E \rightarrow^* V \).

(b) Assume \( \text{dive}(e') \). We then have to show \( \text{dive}(\rho E) \).

Case distinction whether receiver, argument or method call diverges.
- Case receiver diverges: Then \( \theta \eta g \rightarrow^{k'} g'' \) and \( \text{dive}(g'') \). With (28) and \( k' < k \) then \( \text{dive}(\rho G) \), so by the definition of \( E \) in (27) we get \( \text{dive}(\rho E) \).
- Case \( j \)-th argument diverges: Then \( \theta \eta g \rightarrow^{k''} u \) and \( \theta \eta e_i \rightarrow^{k_i} v_i \) for all \( i < j \) and \( \theta \eta e_j \rightarrow^{k_j} e'' \) and \( \text{dive}(e'') \). With (29) and \( k_j \leq k' < k \) we get \( \text{dive}(\rho E_j) \). By definition of \( E \) in (27) then \( \text{dive}(\rho E) \).
- Case method call diverges: Then we are in the situation that (30), (31), and (32) hold. We then have
\[ u.m[\eta \psi](\overline{\nu}) \rightarrow^{k'-k''-\Sigma k_i} e' \]
Hence, with (40), (49), and the second implication in the premise of rule \textsc{equiv-exp}, we have that \( \text{dive}(X_{m,tS}(\rho W, U, \rho W', (\overline{V}))) \). With (50) and (51) and Lemma A.1.2 then also \( \text{dive}(\rho E) \) as required.

End case distinction.

This finishes the proof for rule \textsc{call-struct}.

• Case \textsc{call-iface}:

\[ \langle \Delta, \Gamma \rangle \triangleright e g : \tau_1 \leadsto G \] (55)
methods(\( \tau_j \)) = \( \overline{\tau}^q \) (56)
\[ R_j = m[\Psi](\overline{x}^{m})^{\sigma} \quad \text{(for some } j \in [q] \text{)} \] (57)
\[ \Delta \triangleright \text{val} \quad \Psi \mapsto \psi : \eta_1 \leadsto V \] (58)
\[ \langle \Delta, \Gamma \rangle \triangleright e_i : \eta_1 \sigma_i \leadsto E_i \quad (\forall i \in [n]) \] (59)
\( Y, \overline{X}' \) fresh
\[ \langle \Delta, \Gamma \rangle \triangleright e g \cdot m[\psi](\overline{\nu}^l) : \eta_1 \sigma \leadsto E \] (60)
\( =e \)
\( =\tau \)
with

\[ E = \text{case } G \text{ of } (Y, (\overline{X^i})) \rightarrow X_j (Y, V_j, (\overline{E^j})) \]  

(61)

From the IH applied to (55), (59)

\[ \theta \eta g \approx \rho G \in \llbracket \eta \tau i \rrbracket_k \]  

(62)

\[ \theta \eta e_i \approx \rho E_i \in \llbracket \eta \eta_i \sigma_i \rrbracket_k \quad (\forall i \in [n]) \]  

(63)

Assume \( \theta \eta e \rightarrow^{k'} e' \) for some \( k' < k \). We first consider the following situation for some \( u, \overline{v}^n \):

\[ \theta \eta g \rightarrow^{k''} u \]  

(64)

\[ \theta \eta e_i \rightarrow^{k_i} v_i \]  

(65)

\[ \theta \eta e \rightarrow^{k'' + \Sigma k_i} u.m[\eta \psi](\overline{v}^n) \rightarrow^{k' - k'' - \Sigma k_i} e' \]  

(66)

with \( k'' + \Sigma k_i \leq k' \). (64), (65), and (66) are intermediate assumptions, which become true when we later prove the two implications of rule \textsc{equiv-exp}.

We have from (62), (63), (64), and (65)

\[ \rho G \rightarrow^* U \text{ for some } U \text{ with } u \equiv U \in \llbracket \eta \tau i \rrbracket_{k - k''} \]  

(67)

\[ (\forall i \in [n]) \rho E_i \rightarrow^* V_i \text{ for some } V_i \text{ with } v_i \equiv V_i \in \llbracket \eta \eta_i \sigma_i \rrbracket_{k - k_i} \]  

(68)

From (67) and (56) we get by inverting rule \textsc{equiv-iface}

\[ \exists \sigma_s = t_S[\phi] \]  

(69)

\[ U = (U', (\overline{U'})) \]  

(70)

\[ \forall \ell_1 < k - k''. \ u \equiv U' \in \llbracket \sigma_S \rrbracket_{\ell_1} \]  

(71)

\[ \forall \ell_2 < k - k''. \ \text{methodLookup}(m_j, \sigma_S) \approx U_j \in \llbracket \eta R_j \rrbracket_{\ell_2} \]  

(72)

Hence we have by (69), (72), (57), and rule \textsc{method-lookup}

\[ \text{func} (x \ t_S[\Phi]) \ | \ m[\Psi]'(x \ \overline{s}^{m_i}) \ \sigma' \ {\text{return } e''} \in \overline{D} \]  

(73)

\[ = ^{R'} \phi \ \eta_2 \]  

(74)

\[ \text{methodLookup}(m_j, \sigma_S) = \langle x, t_S[\phi], \eta_2 R', \eta_2 e'' \rangle \]  

(75)

\[ \eta_2 R' = \eta R_j = \eta(m[\Psi](\overline{x_i} \ \overline{s}^{m_i})) \ \sigma \]  

(76)

Then by (72) and (75)

\[ \langle x, t_S[\phi], \eta_2 R', \eta_2 e'' \rangle \approx U_j \in \llbracket \eta R_j \rrbracket_{k - k'' - 1} \]  

(77)

Define \( k''' := \min(k - k'' - 1, k - \Sigma k_i - 1) \). Then

\[ k''' \leq k - k'' - 1 \]  

(78)

\[ k''' < k \]  

(79)

\[ k''' < k - k'' \]  

(80)

\[ k''' < k - k_i \quad (\forall i \in [n]) \]  

(81)

\[ k' - k'' - \Sigma k_i < k''' + 1 \]  

(82)
The first four of these claims are straightforward to verify. The last can be shown with the following reasoning:

\[
    k''' + 1 = k - \max(k'' + 1, \Sigma k_i + 1) + 1 \\
    > k - (k'' + 1 + \Sigma k_i + 1) + 1 \\
    = k - 1 - k'' - \Sigma k_i \\
    \geq k' - k'' - \Sigma k_i
\]

From (77) we get the implication in the premise of rule \textsc{equiv-method-dict-entry}. We now show that the left-hand side of the implication holds. The universally quantified variables of the rule’s premise are instantiated as follows: \( k' = k''', \phi = \eta \psi, W = \rho V, v = u, V = U', \overline{w} = \overline{w}' = \overline{w}'' \). The variables in the conclusion are instantiated as follows: \( x = x, \tau_S = t_S[\phi], m[\Phi](\overline{x_1} \overline{w}'') \tau = \eta_2 R', e = \eta_2 e'' \). The requirement \( k''' \leq k - k'' - 1 \) follows from (78).

We have from (58) and (76) the first conjunct:

\[
    \eta \Psi = \eta \psi : (\alpha \mapsto \eta \overline{\tau}) \quad (\text{assuming } \eta_1 = (\alpha \mapsto \overline{\tau}), \hat{\Psi} = \overline{\alpha}, \psi = \overline{\tau}) \quad (83)
\]

From (58) we get the second conjunct by \textbf{Lemma A.3.15}, (79), by the assumptions \( D \approx_k \mu \) and \( \eta \approx \rho \in \llbracket \Delta \rrbracket_k \), and by \textbf{Lemma A.3.5}:

\[
    \eta \psi \approx \rho V \in \llbracket \eta \Psi \rrbracket'''
\]

With (80), (71), (69), and \textbf{Lemma A.3.9} we get the third conjunct:

\[
    u \approx U' \in \llbracket t_S[\phi] \rrbracket'''
\]

With (81), (68), \textbf{Lemma A.3.9}, and \textbf{Lemma A.3.3} we have

\[
    \nu_i \approx V_i \in \llbracket \eta \eta_1 \sigma_i \rrbracket'''' \quad (\forall i \in [n]) \quad (84)
\]

We next prove

\[
    \eta \eta_1 \sigma_i = \eta_4 \eta \sigma_i \quad (\forall i \in [n]) \quad (85)
\]

\[
    \eta \eta_1 \sigma = \eta_4 \eta \sigma \quad (86)
\]

by induction on \( \sigma_i \) or \( \sigma \). The interesting case is where \( \sigma_i \) or \( \sigma \) is a type variable \( \alpha \in \text{dom}(\eta_1) \cup \text{dom}(\eta) \). As the \( \overline{\alpha} = \text{dom}(\eta_1) = \text{dom}(\eta_4) \) are bound in \( \Psi \) (see (83)), we may assume that \( \overline{\alpha} \cap \text{dom}(\eta) = \emptyset = \overline{\alpha} \cap \text{ftv}(\eta) \). If \( \alpha \in \text{dom}(\eta_1) \) then

\[
    \eta \eta_1 \alpha \overset{(83)}{=} \eta_4 \alpha \overset{\text{dom}(\eta_1) \cap \text{dom}(\eta) = \emptyset}{=} \eta_4 \eta \alpha
\]

If \( \alpha \in \text{dom}(\eta) \) then

\[
    \eta \eta_1 \alpha \overset{\text{dom}(\eta) \cap \text{dom}(\eta_1) = \emptyset}{=} \eta \alpha \overset{\text{dom}(\eta_1) \cap \text{ftv}(\eta) = \emptyset}{=} \eta_4 \eta \alpha
\]

We now get with (76) and (85) that \( \eta \eta_1 \sigma_i = \eta_4 \eta_2 \sigma_i' \). Hence with (84) the fourth conjunct:

\[
    (\forall i \in [n]) \quad \nu_i \approx V_i \in \llbracket \eta_4 \eta_2 \sigma_i' \rrbracket''''
\]
Now the right-hand side of the implication of rule `equiv-method-dict-entry` yields with (77)
\[
\langle x \mapsto u, \overline{v_i^m} \rangle \eta_4 \eta_2 e'' \simeq U_j (U', \rho V, (\overline{V''})) \in [\eta_4 \eta_2 \sigma']_{k'''}
\]  
(87)
Define $\eta_3$ such that
\[
\Phi, \Psi' \mapsto \phi, \eta \psi : \eta_3
\]  
(88)
Then with (74) and (83)
\[
\eta_4 \eta_2 e'' = \eta_3 e''
\]  
(89)
by induction on $e''$. The interesting case is the one for a type variable $\alpha$. By assumption 5.2.1 and (73), we know that $\alpha \in \Phi \cup \Psi'$. Further we may assume that the type variables $\Psi'$ are fresh, and we have $\text{dom}(\eta_2) = \Phi$ by (74) and $\text{dom}(\eta_4) = \Psi'$ by (83).
Thus, if $\alpha \in \Phi$ then $\eta_4 \eta_2 \alpha = \eta_2 \alpha$ because $\Psi'$ fresh, and $\eta_3 \alpha = \eta_2 \alpha$ by (74) and (88). If $\alpha \in \Psi'$ then $\eta_4 \eta_2 \alpha = \eta_4 \alpha$ because $\Psi'$ fresh, and $\eta_3 \alpha = \eta_4 \alpha$ by (88) and (83).
With (86) and (76) $\eta_4 \eta_2 \sigma' = \eta_1 \sigma$. Hence we have with (89), (87)
\[
\langle x \mapsto u, \overline{v_i^m} \rangle \eta_3 e'' \simeq U_j (U', \rho V, (\overline{V''})) \in [\eta_1 \sigma]_{k'''}
\]  
(90)
From (69) and (71) we get by inverting rule `equiv-struct` that $u = t_S [\phi] \ldots$. Hence by rule `fg-call` with (73) and (88)
\[
u.m[\eta \psi](\overline{v''}) \longrightarrow \langle x \mapsto u, \overline{v_i^m} \rangle \eta_3 e''
\]  
Then with (90) and Lemma 5.2.3
\[
\langle x \mapsto u, \overline{v_i^m} \rangle \eta_3 e'' \simeq U_j (U', \rho V, (\overline{V''})) \in [\eta_1 \sigma]_{k''' + 1}
\]  
(91)
We also have
\[
\rho E \longrightarrow^* \text{case } u \text{ of } (Y, (\overline{X}')) \longrightarrow X_j (Y, V, (\overline{E''})) 
\]  
(92)
So far, we proved everything under the assumptions (64), (65), (66). We next consider the two implications of rule `equiv-exp`.

(a) Assume $e' = v$ for some value $v$. Then (64), (65), and (66) hold. We now need to show that there exists some $W$ with $\rho E \longrightarrow^* W$ and $v \equiv W \in [\eta \tau]_{k'''}$. We have $k' - k'' - k_1 < k'''' + 1$ by (82) Also, we have with (66) that
\[
u.m[\eta \psi](\overline{v''}) \longrightarrow_{k'''}^{k''' - k''} \Sigma k_i v
\]  
Hence, (91) gives us the existence of some $W$ such that
\[
U_j (U', \rho V, (\overline{V''})) \longrightarrow^* W
\]  
(93)
We get \( k - k' \leq k''' + 1 - (k' - k'' - \Sigma k_i) \) by 
\[
k''' + 1 - (k' - k'' - \Sigma k_i) = k - \max(k'' + 1, \Sigma k_i + 1) + 1 - k' + k'' + \Sigma k_i \\
= k - \max(k'', \Sigma k_i) - k' + k'' + \Sigma k_i \\
\geq k - (k'' + \Sigma k_i) - k' + (k'' + \Sigma k_i) \\
= k - k'
\]
Hence by Lemma A.3.3

\[
v = W \in \llbracket \eta_1 \sigma \rrbracket_{k-k'}
\]
By (60) \( \eta_1 \sigma = \tau \) so \( v \approx W \in \llbracket \eta \tau \rrbracket_{k-k'} \) and with (92) and (93) \( \rho E \rightarrow^* W \).

(b) Assume \( \text{diverge}(e') \). We then have to show \( \text{diverge}(\rho E) \).

**Case distinction** whether receiver, argument or method call diverges.

- **Case** receiver diverges: Then \( \theta \eta g \rightarrow_k g' \) and \( \text{diverge}(g') \). With (62) and \( k' < k \) then \( \text{diverge}(\rho G) \), so by the definition of \( E \) in (61) we get \( \text{diverge}(\rho E) \).

- **Case** \( j \)-th argument diverges: Then \( \theta \eta g \rightarrow_k u \). By (62) and rule \text{EQUIV-IFACE} we know that \( U = (U', \overline{U'}^q) \) for some \( U', \overline{U'}^q \). Hence

\[
\rho E \rightarrow^* U_j (U', \rho V, \rho (\overline{E'}^q))
\]

Because the \( j \)-th argument diverges, we also have \( \theta \eta e_i \rightarrow_{k_i} \nu_i \) for all \( i < j \) and \( \theta \eta e_j \rightarrow_{k_j} \nu_j \) and \( \text{diverge}(e'') \). With (63) we get \( \text{diverge}(\rho E_j) \), so with (94) also \( \text{diverge}(\rho E) \).

- **Case** method call diverges: Then we are in the situation that (64), (65), and (66) hold. Thus, we get with (66), (91), (82)

\[
u.m[\eta \psi](\overline{v}^q) \approx U_j (U', \rho V, (\overline{U'}^q)) \in \llbracket \eta_1 \sigma \rrbracket_{k''+1} \]

\( k' - k'' - \Sigma k_i < k''' + 1 \)

Hence \( \text{diverge}(U_j (U', \rho V, (\overline{U'}^q))) \) by the implication in the premise of rule \text{EQUIV-EXP}. So by (92) also \( \text{diverge}(\rho E) \) as required.

**End case distinction.**

- **Case sub:**

\[
\begin{array}{c}
\langle \Delta, \Gamma \rangle \triangleright_{\text{exp}} e : \sigma \leadsto E' \\
\Delta \triangleright_{\text{conv}} \sigma \prec \tau \leadsto V
\end{array}
\]

\[
\langle \Delta, \Gamma \rangle \triangleright_{\text{exp}} e : \tau \leadsto V \quad E'
\]

From the IH then \( \theta \eta e \approx \rho E' \in \llbracket \eta \sigma \rrbracket_k \). From Lemma A.3.13 we get \( \theta \eta e \approx (\rho V) \rho E' \in \llbracket \eta \tau \rrbracket_k \) with \( \rho E = (\rho V) (\rho E') \) as required.

**End case distinction** on the last rule in the derivation of \( \langle \Delta, \Gamma \rangle \triangleright_{\text{exp}} e : \tau \leadsto E \). ❄️

A.3.2 Proof of Lemma 5.2.5

We proceed by induction on \( k \). For \( k = 0 \), we first note that \( e \approx E \in \llbracket \tau \rrbracket_0 \) holds for any \( e, E, \tau \) because the two implications in the premise of rule \text{EQUIV-EXP} hold trivially. Thus, we
get $D \approx_0 X_{m,ts}$ for all $D = \text{func}(x \ t_S[\Phi]) \ mm \ \text{return} \ e \in D$ by rule equiv-method-decl. Hence $D \approx_0 \mu$ by rule equiv-decls.

Now assume $D \approx_k \mu$ (IH) for some $k$ and prove $D \approx_{k+1} \mu$. By rule equiv-decls, we need to show $D \approx_{k+1} X_{m,ts}$ for all

$$D = \text{func}(x \ t_S[\Phi]) \ m[\Phi'](x \ t'^r) \ \tau \ \text{return} \ e \in \overline{D} \tag{95}$$

Thus, we assume the left-hand side of the implication in the premise of rule equiv-method-decl and then show the right-hand side of the implication. More specifically, let

$$\Phi = \overline{\alpha_i \sigma_i^p} \quad \Phi' = \overline{\alpha_i \sigma_i^{p'}} \quad \Psi = \Phi, \ \Phi' = \overline{\alpha_i \sigma_i^p \ \beta_i \ \sigma_i^{p'}}$$

and assume for arbitrary $k' < k + 1, \phi = \overline{\sigma^m^p}, \phi' = \overline{\sigma^m^{p'}}, W^p, W'^{p'}, u, U, \overline{v^m}, \overline{V'^p}$ the left-hand side of the implication:

$$\Psi \mapsto \phi, \ \phi': \eta \text{ with } \eta = (\overline{\alpha_i \mapsto \sigma_i^m \ \beta_i \mapsto \sigma_i^{m'}}) \tag{96}$$

$$\phi, \ \phi' \approx (W^p, W'^{p'}) \in \llbracket \Psi \rrbracket_{k'} \tag{97}$$

$$u \approx U \in \llbracket t_S[\eta \overline{\alpha^p}] \rrbracket_{k'} \tag{98}$$

$$v_i \approx V_i \in \llbracket \eta \tau_i \rrbracket_{k'} \quad (\forall i \in [n]) \tag{99}$$

From this we need to prove the following goal:

$$\langle x \mapsto u, \overline{x_i} \mapsto v''_i \rangle \ \eta e \approx X_{m,ts} \ (\langle W^p, U, (\overline{W'^p}), (\overline{V'^p}) \rangle) \in \llbracket \eta \tau \rrbracket_{k'} \tag{100}$$

Define

$$\rho = \langle X_{\overline{\alpha}} \mapsto W^p, X_{\overline{\beta}} \mapsto W'^{p'}, X \mapsto U, \overline{x_i} \mapsto \overline{v''_i} \rangle \tag{101}$$

$$\Delta = \{\overline{\alpha_i : \sigma_i^p}, \ \overline{\beta_i : \sigma_i^{p'}}\}$$

$$\Gamma = \{x : t_S[\overline{\alpha^p}], \overline{x_i : \overline{\tau^{p'}}}\}$$

Then with (96), (97), and rule equiv-ty-subst

$$\eta \approx \rho \in \llbracket \Delta \rrbracket_{k'} \tag{102}$$

And with (98), (99), the definition of \(\theta\) in (100), and rule equiv-val-subst

$$\theta \approx \rho \in \llbracket \eta \Gamma \rrbracket_{k'} \tag{103}$$

From the assumption $t_{\text{num}} D \sim X_{m,ts} = V$ we get by inverting rule method

$$\langle \Delta, \Gamma \rangle t_{\text{exp}} e : \tau \sim E \tag{104}$$

$$V = \lambda \ (\langle X_{\overline{\alpha'}}^p \rangle, X, (\overline{X_{\overline{\beta'}}^{p'}}), (\overline{X'^{p'}})) \cdot E \tag{105}$$

With $k' < k + 1$ we have $k' \leq k$. With the IH and Lemma A.3.7 then

$$\overline{D} \approx_{k'} \mu \tag{106}$$

(106), (102), (103) and (104) are the requirements of Lemma 5.2.4. The lemma then yields

$$\theta \eta e \approx \rho E \in \llbracket \eta \tau \rrbracket_{k'} \tag{107}$$
We also have

\[ X_{m,ts} \left( (\overline{W}^0), U, (\overline{W'}^1), (V^i) \right) \rightarrow V \left( (\overline{W}^0), U, (\overline{W'}^1), (V^i) \right) \rightarrow^* \rho E \]

where the first reduction follows from assumption \( \mu(X_{m,ts}) = V \) and rule \textsc{tl-method}, the remaining steps by (105) and (101). With (107) and Lemma 5.2.2 we then get (100) as required.

\[ \square \]

\textbf{A.3.3 Proof of Theorem 5.2.6}

We first prove that the assumptions of the theorem imply \( e \approx E \in \llbracket \tau \rrbracket_k \) for any \( k \). \( \overline{D} \) and \( \mu \) are the declarations and the substitution whose existence we assumed globally. Obviously, they meet the requirements of Assumption 5.2.1.

Assume \( k \in \mathbb{N} \). From Lemma 5.2.5 we get \( \overline{D} \approx_k \mu \). By the assumption \( \tau_{\text{prog}} \overline{D} \textbf{ main()\{ = e \}\leadsto let } X_i = V_i \textbf{ in } E \), by inverting rule \textsc{prog}, and by the assumption that \( e \) has type \( \tau \), we find \( \langle \emptyset, \emptyset \rangle \tau_{\text{exp}} e : \tau \leadsto E \). Lemma 5.2.4 then yields \( e \approx E \in \llbracket \tau \rrbracket_k \) as required.

From \( e \approx E \in \llbracket \tau \rrbracket_k \) for any \( k \) and the two implications in the premise of rule \textsc{equiv-exp}, we then get the two claims needed to show. \[ \blacksquare \]