Scattering of Open and Closed Strings in 1+1 Dimensions

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The ground ring structure of 1+1 dimensional string theory leads to an infinite set of non linear recursion relations among the ‘bulk’ scattering amplitudes of open and closed tachyons on the disk, which fix them uniquely. The relations are generated by the action of the ring on the tachyon modules; associativity of this action determines all structure constants. This algebraic structure may allow one to relate the continuum picture to a matrix model.
1. Introduction.

An outstanding problem in two dimensional string theory (see e.g. [1], [2], [3] for reviews) is to reconstruct the matrix model, which is a very powerful and natural description of the physics of the theory directly from the continuum approach. It has been suggested [4] (see also [5], [6]) that certain spin zero, ghost number zero states which exist in this model and generate the so called ‘ground ring’, are important in this context. In this interpretation, the ground ring is identified with the ring of functions on the phase space of the matrix model. The physical excitations of the theory, the massless ‘tachyons’, correspond to infinitesimal perturbations of the Fermi surface, a certain curve on phase space. $W_\infty$ type symmetries of phase space are generated by certain dimension $(1,0), (0,1)$ currents. In fact, the whole structure, some aspects of which we will review below, is superficially very similar to the matrix model picture developed in [7], [8], [9], but so far the program of deriving the detailed structure of [7], [8], [9] from the continuum has not been completed.

While it may seem that the issue is of academic interest only since an exact solution exists, the real interest in this subject lies in understanding other vacua of two dimensional string theory for which the continuum picture is much more developed (either because a matrix model description is lacking or because the matrix model has so far proven too hard to solve). Examples include the black hole [10], two dimensional fermionic string theory [3], [11] and superstring theory [12], and open plus closed 2d string theory [13] (see also [14]). In these situations, the analogs of the ground ring exist, and it would be nice to be able to use them to work up to an exact solution and/or to an underlying matrix model.

In this paper we are going to investigate open plus closed 2d string theory on the disk along the lines of the above comments. In a previous paper [13] we have found the bulk [3] amplitudes for scattering of open string ‘tachyons’ (again massless) on the disk by explicitly evaluating certain Veneziano integrals. Here, we will start by showing that the result follows quite elegantly from the ring relations on the tachyon modules [6], [15]. We will then go on to consider the case of arbitrary disk scattering amplitudes with any number of closed and open string tachyons, which are difficult to evaluate directly, and find that they satisfy an infinite number of non-linear relations which again follow from the action of the ring on the modules. Compatibility of these relations determines all structure constants involved and all correlation functions. In fact the situation is infinitely over constrained, so it’s not guaranteed apriori that any solutions exist (nevertheless, this is of course the case).
The action of the ring on the modules is again reminiscent of the phase space structure of a matrix model. We conclude the paper by some observations regarding this relation, postponing further discussion to future work. But first, in the next section we start, to set the stage, by briefly describing some general properties of bulk amplitudes in $D = d + 1$ dimensional open + closed string theory, and review the two dimensional (closed) case on the sphere.

2. Some preliminaries.

2.1. Veneziano amplitudes in $d + 1$ dimensions.

Open string states first come into play on world sheets with the topology of a disk (which we will denote by $\mathcal{M}$). The appropriate action of world sheet 2$d$ gravity is (see e.g. [16]):

$$S = \frac{1}{2\pi} \int_{\mathcal{M}} d^2 \xi \sqrt{g} \left[ g^{ab} G_{\mu\nu}(X) \partial_a X^\mu \partial_b X^\nu - R^{(2)}(X) + \mathcal{T}(X) \right]$$

$$+ \frac{1}{\pi} \int_{\partial \mathcal{M}} d\xi \left[ A_\mu(X) \partial_\xi X^\mu - K \Phi(X) + g^{\frac{d}{4}} V(X) \right]$$

(2.1)

where $g_{ab}$ is the world sheet metric, $R^{(2)}$ its scalar curvature, $g = \text{det} g_{ab}$, and $K$ the extrinsic curvature on the boundary $\partial \mathcal{M}$. $X^\mu$, $\mu = 0, 1, \cdots, d$ parametrize the space-time manifold, and $G_{\mu\nu}$, $A_\mu$, $\Phi$, $\mathcal{T}$, $V$ correspond to general metric, gauge field, dilaton and closed and open string tachyon condensates, respectively.

We will be mainly discussing here the “linear dilaton” vacuum of (2.1):

$$G_{\mu\nu} = \eta_{\mu\nu}; A_\mu = 0; \Phi(X) = Q X^0; Q^2 = \frac{25 - d}{3}$$

(2.2)

As is by now standard (see e.g. [17], [3], [11]) in a vacuum such as (2.2), where the string coupling $g_{st} \simeq e^\Phi$ depends on $X^\mu$, we have to supply some sort of “wall” to keep the system away from strong coupling, e.g. by turning on appropriate potentials $\mathcal{T}(X)$, $V(X)$. However, physics in the ‘bulk’ of space-time (not to be confused with the bulk of the world sheet) is insensitive to such boundary effects, and can be studied by putting $\mathcal{T}(X) = V(X) = 0$. This reduces the problem to free field theory on the world sheet with anomalous conservation of $X^0$.

The emission vertices for closed and open string states are:

$$\mathcal{T}_k(z, \bar{z}) = \exp(ik \cdot X + \beta_k X^0); \quad \frac{1}{2} k^2 - \frac{1}{2} \beta(\beta + Q) = 1$$

$$\mathcal{V}_k(\sigma) = \exp(ik \cdot X + \beta_k X^0); \quad \frac{1}{2} k^2 - \frac{1}{2} \beta(\beta + \frac{1}{2} Q) = 1/4$$

(2.3)
where we have conformally transformed the disk to the upper half plane \( z = \sigma + i\tau, \tau \geq 0 \), with the open vertices inserted at \( \tau = 0 \). The propagator on the upper half plane (with Neumann boundary conditions) is: 
\[
\langle X^\mu(z)X^\nu(w) \rangle = -\eta^{\mu\nu} \left( \log |z - w|^2 + \log |z - \bar{w}|^2 \right).
\]

Scattering of the tachyons (2.3) is described by the Veneziano integral representation (valid for \( m \geq 3 \)):
\[
\langle T_{k_1} \cdots T_{k_n} V_{p_1} \cdots V_{p_m} \rangle = \prod_{i=1}^{n} \int d^2 z_i \prod_{j=1}^{m} \int dw_j \prod_{i=1}^{n} |z_i|^{4k_i \cdot P_1} |1 - z_i|^{4k_i \cdot P_2} \prod_{1=i<j}^{n} |z_i - z_j|^{2k_i \cdot k_j} \prod_{1=i}^{n} |z_i - \bar{z}_i|^{2k_i \cdot k_i} \prod_{j=4}^{m} |w_j|^{4p_j \cdot P_1} |1 - w_j|^{4p_j \cdot P_2} (2.4)
\]

where we have used the notation \( k_1 \cdot k_2 \equiv k_1 \cdot k_2 - \beta_1 \beta_2 \), and (anomalous) energy momentum conservation: \( \sum_{i=1}^{n} k_i + \sum_{j=1}^{m} p_j = 0 \), \( \sum_{i=1}^{n} \beta_i + \sum_{j=1}^{m} \beta_j = -Q/2 \). As usual, one has to fix the ordering of the boundary operators (up to cyclic permutations). Different orderings correspond to different channels. The amplitude (2.4) is \( SL(2, R) \) invariant; therefore we have the freedom to fix three open vertices on the boundary (as we did in (2.4)), or one open vertex and one closed vertex operator.

The amplitude (2.4) exhibits poles corresponding to scattering of closed \( \rightarrow \) closed, closed \( \rightarrow \) open and open \( \rightarrow \) open strings. It is easy to check that they factorize on physical on shell states in all possible channels. For general dimension \( d \) one can not say much more about (2.4) due to the complicated dynamical structure of \( d + 1 \) dimensional string theory. In \( 1 + 1 \) dimensions one can actually evaluate these amplitudes and therefore in the next sections we will concentrate on the situation in two space-time dimensions.

2.2. Closed 2d strings on the sphere.

Before passing to the case of interest to us here, the disk, we will briefly review the structure on the sphere, in order to have at hand a collection of relevant facts, to illustrate the logic and to compare later to the structure on the disk.

The system is described by (2.1), (2.2) with \( M = \text{sphere} \) and \( Q = 2\sqrt{2} \) (see (2.2)). We will consider here the case where \( X^0, X^1 \) are both non compact. The compact case
can be treated along similar lines. The closed ‘tachyon’ emission vertex (in normalizations of \(T_k\) and \(k\) which will be convenient later) is:

\[
T_k^{(\pm)} = -\frac{\Gamma(\pm k/2)}{\Gamma(1 + k/2)} \exp \left[ i \frac{k}{2\sqrt{2}} X + (-\sqrt{2} \pm \frac{k}{2\sqrt{2}}) \phi \right]
\]  

(2.5)

where \(T_k\) has to be either multiplied by \(\bar{c}c\) (reparametrization ghosts) or integrated over \(\mathcal{M}\) for BRST invariance. \(T_k^{(+)} (T_k^{(-)})\) describe right (left) moving massless particles. In addition to the tachyons (2.5) the spectrum includes an infinite set of ‘discrete’ oscillator states at momenta \(k \in 2\mathbb{Z}\), which are related to the symmetries mentioned in the introduction; we will not need those below. The ground ring is generated by the operators

\[
a_\pm = -|cb - \frac{1}{\sqrt{2}}(\partial \phi + i \partial X)|^2 e^{\pm iX/\sqrt{2} + \phi/\sqrt{2}}
\]

(2.6)

which are BRST closed (but not exact) and have \(\Delta = \bar{\Delta} = 0\). The ring is spanned by the set of BRST invariant operators \(\{(a_+)^n(a_-)^m\}, n, m \in \mathbb{Z}_+\).

As pointed out in [15], tachyon dynamics is constrained by the ring relations on the modules (2.5). By using free field OPE, which is valid in bulk correlators [11], we find (for \(k \not\in 2\mathbb{Z}\)):

\[
a_\pm T_k^{(\pm)} = T_k^{(\pm)}
\]

\[
a_\mp T_k^{(\pm)} = 0
\]

(2.7)

Both relations in (2.7) are true modulo BRST commutators, but while the first survives in (bulk) correlation functions, the second receives non-linear modifications.

Indeed, consider the amplitude:

\[
F(z, \bar{z}) = \langle a_- (z) c\bar{c} T_{k_1}^{(+)} (0) c\bar{c} T_{k_2}^{(+)} (\infty) \prod_{i=3}^n \int T_{k_i}^{(+)} (z_i) c\bar{c} T_{p}^{(-)} (1) \rangle
\]

(2.8)

(most other amplitudes can be shown to vanish [3], [11]). One can convince oneself that \(\partial_z F = \partial_{\bar{z}} F = 0\) by using \(\partial_z a_- = \{Q_{\text{BRST}}, b_{-1} a_-\}\) and deforming the contour (commuting \(Q_{\text{BRST}}\) to the other operators in (2.8)). This is not completely trivial – one has to check vanishing of boundary terms from the \(z_i\) integrals. Given that \(F(z, \bar{z}) = F\) is constant one can derive a non-linear identity on correlation functions of \(T_k\) by comparing \(F(z \to 1)\) and \(F(z \to 0)\). In the former limit we get by the first equation in (2.7): \(\prod_{i=1}^n T_{k_i}^{(+)} T_{p-2}^{(-)}\) while
in the second one would find 0 by using (2.7) naively. In fact, there is a correction which can be schematically summarized\(^\ddagger\) as follows [6], [15], [18]:

\[ a_- T^{(+)}_{k_1} \int T^{(+)}_{k_2} = T^{(+)}_{k_1+k_2-2} \]  

(2.9)

I.e., in the presence of an integrated \( T_{k_2} \) the contribution of the region in which \( a_- \) approaches \( T_{k_1} \) is modified. It is easily checked that (2.9) is the only modification of (2.7) which is necessary. By combining (2.7), (2.9) in (2.8) we find a recursion relation:

\[
\langle \prod_{i=1}^n T^{(+)}_{k_i} T^{(-)}_{p-2} \rangle = \sum_{i=3}^n \langle T^{(+)}_{k_1+k_i-2} \prod_{i \neq j=2}^n T^{(+)}_{k_j} T^{(-)}_p \rangle
\]

(2.10)

whose solution (given \( \langle T^{(+)}_{k_1} T^{(+)}_{k_2} T^{(-)}_p \rangle = 1 \)) is:

\[
\langle \prod_{i=1}^n T^{(+)}_{k_i} T^{(-)}_p \rangle = (n-2)!
\]

(2.11)

A few comments in this simple case will prove useful later.

1) The fact that the structure constant on the r.h.s. of (2.9) is 1 was originally obtained by calculating a four point correlation function. On general grounds one expects a relation like:

\[ a_- T^{(+)}_{k_1} \int T^{(+)}_{k_2} = f(k_1, k_2) T^{(+)}_{k_1+k_2-2} \]  

(2.12)

with some apriori unknown structure function \( f(k_1, k_2) \). The point is that one can determine \( f(k_1, k_2) \) by requiring associativity of (2.12). Indeed, consider:

\[
(a_-)^2 T^{(+)}_{k_1} T^{(+)}_{k_2} T^{(+)}_{k_3} = f(k_1, k_2) f(k_1+k_2-2, k_3) T^{(+)}_k = (k_1 \leftrightarrow k_2) = (k_1 \leftrightarrow k_3)
\]

(2.13)

where \( k = k_1 + k_2 + k_3 - 4 \). Associativity (2.13) implies that \( f(k_1, k_2) f(k_1+k_2-2, k_3) \) must be invariant under permutations of \( k_1, k_2, k_3 \). By applying \( a_+ \) to (2.12) one also finds that \( f(k_1, k_2) \) should be periodic in \( k_1, k_2 \) (with period 2). The solution can be expressed in terms of an arbitrary (periodic) function \( M(k) \): \( f(k_1, k_2) = M(k_1 + k_2) / M(k_1) M(k_2) \); \( M(k) \) can be put to 1 by renormalizing \( T_k \).

Below we will see less trivial examples of such structure constants which will also be determined by associativity conditions.

2) The relations (2.7), (2.9) provide a link to the formalism of [8]. Consider first (2.7); it implies \( a_+ a_- T_k = 0 \), a fact that has been interpreted in [15] as the statement that the \( T_k \)

\(^\ddagger\) Here and in many subsequent formulae we ignore factors of \( \pi \) which can be easily restored.
describe infinitesimal excitations living on the line $a_+a_- = 0$ in the $(a_+, a_-)$ plane. This line is related to the Fermi surface of the matrix model fermions through the relations $a_\pm = p \pm q$ ($p, q$ are phase space variables). $\mathcal{T}^{(\pm)}$ live on the segments $a_\mp = 0$. Eq. (2.9) is very natural in this interpretation; indeed, it can be rewritten as:

$$a_- \mathcal{T}^{(+)}_{k_1} |_{\lambda_{k_2}} = \lambda_{k_2}^{(+)} \mathcal{T}^{(+)}_{k_1+k_2-2}$$

(2.14)

where $\lambda_k$ is the condensate of $\mathcal{T}_k$, corresponding to $S \to S + \lambda_k^{(\pm)} \int \mathcal{T}^{(\pm)}_k$ in (2.11). Combining (2.7) and (2.14) we see that perturbing by $\lambda_k$ shifts the Fermi surface to

$$a_+a_- = \lambda_k e^{ikX}$$

(2.15)

where as usual (2.13) is understood as a relation on the tachyon module $[15]$ (rather than a relation in the ring). In particular, for $k = 0$ we find (2.15) $a_+a_- = \mu [1]$. Equation (2.13) is very reminiscent of the results of [8].

The above discussion is a step towards establishing the connection with the matrix model. However the picture is still incomplete; one difficulty is that (2.7), (2.14) (and obvious generalizations) do not seem to hold in non bulk amplitudes. Nevertheless, we will next turn to obtaining the analogs of (2.7) – (2.15) for the disk with open and closed states.

3. The disk with open external states.

3.1. Ring and modules.

Specializing the spectrum (2.3) to $D = 1+1$ we find, in addition to the closed tachyon (2.3), another massless field, the open string tachyon:

$$\mathcal{V}^{(\pm)}_k = -\Gamma(\pm k) \exp \left[ i \frac{k}{2 \sqrt{2}} X + \left( -\frac{1}{\sqrt{2}} \pm \frac{k}{2 \sqrt{2}} \right) \phi \right]$$

(3.1)

(we have again chosen a convenient normalization). The open string excitations form modules of the open string (or boundary) ring, which is generated by the operators $^2$:

$^2$ The boundary conditions on the ghosts are standard, $c = \bar{c}$, etc. We will use $c, b$ for the ghosts on the boundary; it should be evident from the context which of the two $c, b$’s is relevant in different expressions.
\[ A_\pm = -(cb - \frac{1}{2\sqrt{2}}(\partial_\phi \mp i\partial X))e^{\pm iX/2\sqrt{2}+\phi/2\sqrt{2}} \]  

(3.2)

One can readily verify that:

\[ A_\pm \mathcal{V}^{(\pm)}_k = \mathcal{V}^{(\pm)}_{k \pm 1} \]
\[ A_\mp \mathcal{V}^{(\pm)}_k = 0 \]  

(3.3)

A new feature of the open case is the importance of the ordering of the operators on the boundary of the disk. The relations (3.3) with opposite ordering are easily obtained by using the symmetry $\sigma \rightarrow -\sigma$, under which (3.2) $A_\pm \rightarrow -A_\pm$ while the tachyons are invariant. Hence for example, $\mathcal{V}^{(\pm)}_k A_\pm = -\mathcal{V}^{(\pm)}_{k \pm 1}$. We will use this symmetry below and write only the independent relations in each case.

3.2. Correlation functions.

In the next section we will consider scattering amplitudes with an arbitrary number of closed (2.5) and open (3.1) states. However, as a useful intermediate step, we will start with a discussion of open string scattering, since this will allow us to exhibit the new features arising in generalizing (2.8) – (2.15) to the disk.

A natural starting point is the amplitude:

\[ F(\sigma) = \langle \prod_{j=1}^{m} \mathcal{V}^{(-)}_{p_j} A_-(\sigma) \prod_{i=1}^{n} \mathcal{V}^{(+)}_{k_i} \rangle \]  

(3.4)

By (3.4) we mean the ordered amplitude on the disk with the specified ordering. To make sense out of (3.4) one has to fix three of the $\mathcal{V}^{(\pm)}$ in (3.4) and integrate over the rest. It would seem, from the way we wrote (3.4), that $F(\sigma)$ is independent of which three operators we fix. This is not the case. Different ways of fixing $SL(2, R)$ give rise to different $F(\sigma)$, but the constraints on tachyon amplitudes (2.4) that we will derive are of course independent of this choice (as are the original amplitudes (2.4)). We will start with a particularly convenient way of fixing $SL(2, R)$ and briefly describe other ways below (and present more details in an appendix).

One way of defining (3.4) is:

\[ F(\sigma) = \langle \prod_{j=1}^{m-1} \int \mathcal{V}^{(-)}_{p_j} c\mathcal{V}^{(-)}_{p_m}(0) A_-(\sigma) c\mathcal{V}^{(+)}_{k_1}(1) \prod_{i=2}^{n-1} \int \mathcal{V}^{(+)}_{k_i} c\mathcal{V}^{(+)}_{k_n}(\infty) \rangle \]  

(3.5)

\[ \text{3 More precisely, the } SL(2, R) \text{ invariant correlation function is } \int \partial_\sigma F(\sigma). \]
The reason why (3.5) is convenient is that \( \partial_\sigma F(\sigma) = 0 \) in this choice. Naively, this should always be the case since
\[
\partial_\sigma A_-(\sigma) = \{Q_{\text{BRST}}, b_1 A_-(\sigma)\}
\] (3.6)

But in general there are finite boundary terms in the moduli integrals in (3.4) and \( \partial_\sigma F(\sigma) \neq 0 \) (see below and appendix A). In (3.5) such boundary terms vanish (as one can verify by explicit calculation), therefore we can argue as in (2.8) – (2.10); compare \( F(\sigma \to 0) \) to \( F(\sigma \to 1) \). In the first case we find (3.3):
\[
-\langle \prod_{m=1}^{m-1} V_{p_j}^- \prod_{r=1}^m \frac{1}{\sin \pi \sum_{j=1}^r p_j} \prod_{i=1}^n \frac{1}{\sin \pi \sum_{i=1}^n k_i} \rangle
\] (3.7)

Of course, by \( SL(2, R) \) invariance, the same holds for \( A_- \int V_{k_1}^+ V_{k_2}^+ \). Since ordering is important, there is a second independent relation which is not necessary here but will be useful below:
\[
\int V_{k_1}^+ A_- V_{k_2}^+ = \frac{\sin \pi (k_1 + k_2)}{\sin \pi k_1 \sin \pi k_2} V_{k_1+k_2-1}^+ \] (3.8)

As explained above, there are two more implicit relations in (3.7), (3.8):
\[
V_{k_1}^+ \int V_{k_2}^+ A_- = -\frac{1}{\sin \pi k_2} V_{k_1+k_2-1}^+; \quad V_{k_1}^+ A_- \int V_{k_2}^+ = -\frac{\sin \pi (k_1 + k_2)}{\sin \pi k_1 \sin \pi k_2} V_{k_1+k_2-1}^+ .
\]

We are now ready to return to (3.5). Using (3.7) in \( F(\sigma \to 1) \) and comparing to \( F(\sigma \to 0) \) we find the following recursion relation:
\[
\langle \prod_{j=1}^{m-1} V_{p_j}^- \prod_{r=1}^m \frac{1}{\sin \pi \sum_{j=1}^r p_j} \prod_{i=1}^n \frac{1}{\sin \pi \sum_{i=1}^n k_i} \rangle = -\frac{1}{\sin \pi k_1} \langle \prod_{j=1}^m V_{p_j}^- V_{k_1+k_2-1}^+ \prod_{i=1}^n V_{k_i}^+ \rangle
\] (3.9)

which is easily solved:
\[
\langle \prod_{j=1}^m V_{p_j}^- \prod_{i=1}^n V_{k_i}^+ \rangle = (-1)^n \frac{n(n-1)}{2} \prod_{r=1}^{m-1} \frac{1}{\sin \pi \sum_{j=1}^r p_j} \prod_{i=1}^{n-1} \frac{1}{\sin \pi \sum_{i=1}^n k_i} .
\] (3.10)

in agreement with the answer obtained in [13]. Similarly one can check that as claimed in [13] the amplitudes for more complicated orderings of \( V_+, V_- \) vanish. We see that as on the sphere, the correlation functions are determined by the ring action on the tachyon modules (3.3), (3.7).
3.3. Other ways of fixing SL(2,R).

The reader may have noticed a peculiar feature of the derivation of (3.9). Unlike the closed case, since ordering is important here, the result (3.9) seems to depend strongly on the fact that one of the vertices \( V_{k_1}^{(+)} \), \( V_{k_2}^{(+)} \) in (3.4) is fixed, while the other one is integrated (allowing us to use one of the two versions of (3.7)). Naively one would get a different answer if they were both fixed or both integrated. The resolution of this ‘paradox’ is that in those cases, as in other important cases below, \( \partial_\sigma F(\sigma) \neq 0 \). As noted above, (3.6), \( \partial_\sigma F(\sigma) \) is a correlation function with an insertion of a BRST commutator, and naively it should vanish. However, one can show that precisely in the cases just mentioned there are finite contributions from the boundaries of moduli space which precisely complete (3.9). The reader should consult appendix A for a more detailed discussion of this phenomenon.

We would like to emphasize, that the fact that the BRST commutator (3.6) does not decouple in general (\( \partial_\sigma F(\sigma) \neq 0 \)) does not imply breakdown of gauge invariance of the theory. The amplitude (3.4), (3.5) is not the full scattering amplitude – one has to sum over different orderings of the vertices. After doing the sum, one can show that BRST invariance is restored. For our purposes it is more convenient to consider the individual ‘channels’.

3.4. The algebraic structure.

The action of the ground ring on the tachyon modules is, as in the closed case, highly constrained. Indeed, replace (3.7), (3.8) by:

\[
A_- V_{k_1}^{(+)} \int V_{k_2}^{(+)} = f(k_1, k_2) V_{k_1+k_2-1}^{(+)} \\
\int V_{k_1}^{(+)} A_- V_{k_2}^{(+)} = g(k_1, k_2) V_{k_1+k_2-1}^{(+)}
\]

(3.11)

Consider the operator \( \Theta = \int V_{k_1}^{(+)} A_- V_{k_2}^{(+)} \int V_{k_3}^{(+)} A_- \). We can calculate \( \Theta \) in two different ways by using (3.11). Comparing the coefficients of \( V_{k_1+k_2+k_3-2} \) we obtain a consistency relation on \( f, g \):

\[
f(k_3, k_2) g(k_1, k_2 + k_3 - 1) = f(k_3, k_2 + k_1 - 1) g(k_1, k_2) + f(k_2, k_3) f(k_2 + k_3 - 1, k_1)
\]

(3.12)

The functions \( f, g \) are periodic in \( k_1, k_2 \) with period 1 (for the same reasons as in (2.12), (2.13)). One can also show that the condition (3.12) together with periodicity determine \( f, g \) essentially uniquely. It is a non trivial fact that \( f(k_1, k_2) = \frac{1}{\sin \pi k_1} \) and
$g(k_1, k_2) = \frac{\sin \pi (k_1 + k_2)}{\sin \pi k_1 \sin \pi k_2}$ indeed satisfy \(3.12\) as well as an infinite number of more complicated consistency conditions implied by \(3.11\). This gives rise to some quite non trivial trigonometric identities.

Another way of probing the consistency of the structure we found \(3.3\), \(3.7\), \(3.8\) is to consider the action of the ring in more complicated correlation functions (than \(3.4\)). Consider for example:

$$G(\sigma) = \langle \cdots \mathcal{V}^{(+)}_{k_1}(0) \int \mathcal{V}^{(+)}_{k_2} A_{-}(\sigma) \mathcal{V}^{(+)}_{k_3}(1) \int \mathcal{V}^{(+)}_{k_4} \cdots \rangle$$

where the \(\cdots\) stand for other operators which may be present. One can show that $\partial_{\sigma} G(\sigma) = 0$, and therefore, $G(\sigma \to 0) = G(\sigma \to 1)$. Using the ring relations \(3.11\) one gets an equation relating three amplitudes which can be viewed as a consistency condition on $f, g$. That condition is equivalent to \(3.12\). Similar manipulations with higher powers of $A_{-}$ give rise to the infinite number of associativity conditions mentioned above.

4. Scattering of Open and Closed Strings.

4.1. Algebraic structure and Recursion relations.

We now turn to the scattering amplitudes with an arbitrary number of open and closed states. Following the logic of section 3 we will obtain recursion relations by studying the ring relations for the amplitude:

$$C(\sigma) = \langle \prod_{j=1}^{m} \mathcal{V}^{(-)}_{p_j} A_{-}(\sigma) \prod_{i=1}^{n} \mathcal{V}^{(+)}_{k_i} \prod_{a=1}^{M} \mathcal{T}^{(-)}_{r_a} \prod_{s=1}^{N} \mathcal{T}^{(+)}_{q_s} \rangle$$

In the previous section (and appendix A) we have discussed the case $N = M = 0$, and saw that the ring relations on the (open) tachyon modules \(3.3\), \(3.7\), \(3.8\) determine the correlation functions \(4.1\) uniquely. In the general case $(N, M \neq 0)$ the same conclusion holds. The only new element is the action of $A_{\pm}$ on the closed tachyons $\mathcal{T}^{(\pm)}$. The relevant calculations are presented in appendix B, where we show that $\partial_{\sigma} C(\sigma) \neq 0$ (for any choice of $SL(2, R)$ fixing); for the particular choice used in section 3 \(3.5\) the relevant boundary terms are due to closed tachyons $(\mathcal{T}^{(+)}_{q_s})$ approaching $A_{-}$. They can be summarized by the following action of the boundary ground ring on the closed tachyons:

$$A_{-} \mathcal{T}^{(+)}_{q} = \sin \left( \frac{\pi q}{2} \right) \mathcal{V}^{(+)}_{q-1}$$

$$A_{-} \mathcal{T}^{(-)}_{q} = 0$$

\(4.2\)
The structure constant in (4.2) is again determined by consistency with the other relations (3.3), (3.7), (3.8) as in section 3.4. The details of how (4.2) emerges from (4.1) appear in appendix B, but combining it with the previous relations one can immediately write a recursion relation for the general amplitude (4.1):

\[
\langle \prod_{j=1}^{m-1} V_{p_j}^{(-)} \prod_{i=1}^{n} V_{k_i}^{(+)} \prod_{a=1}^{M} T_{r_a}^{(-)} \prod_{s=1}^{N} T_{q_s}^{(+)} \rangle = \]

\[-\frac{1}{\sin \pi k_1} \langle \prod_{j=1}^{m} V_{p_j}^{(-)} V_{k_1 + k_2 - 1}^{(+)} \prod_{i=3}^{n} V_{k_i}^{(+)} \prod_{a=1}^{M} T_{r_a}^{(-)} \prod_{s=1}^{N} T_{q_s}^{(+)} \rangle \]

\[-\sum_{i=1}^{N} \sin \frac{\pi q_i}{2} \langle \prod_{j=1}^{m} V_{p_j}^{(-)} V_{q_1 - 1}^{(+)} \prod_{i=1}^{n} V_{k_i}^{(+)} \prod_{a=1}^{M} T_{r_a}^{(-)} \prod_{i \neq s=1} T_{q_s}^{(+)} \rangle \]  (4.3)

The first term on the r.h.s. of (4.3) vanishes for \( n < 2 \), while the second vanishes for \( N = 0 \). Eq. (4.3) generalizes (3.9) by expressing a general correlation function of open and closed states (l.h.s.) in terms of one with one fewer open string state (the first term on the r.h.s.) and another with one fewer closed string state (second term on r.h.s.).

A few comments about (4.3) are in order:

1) It clearly determines all correlation functions uniquely. By iterating it enough times, an arbitrary correlation function (4.1) is related to the (known) three point function \( \langle VVV \rangle \).

2) The dynamics splits into that of left and right moving particles in space time; the solution to (4.3) has the form:

\[
\langle \prod_{j=1}^{m} V_{p_j}^{(-)} \prod_{i=1}^{n} V_{k_i}^{(+)} \prod_{a=1}^{M} T_{r_a}^{(-)} \prod_{s=1}^{N} T_{q_s}^{(+)} \rangle = W_{n,N}(k_1 | q_s) W_{m,M}(p_j | r_a) \]  (4.4)

where \( W_{n,N} \) is a certain function of momenta determined by the recursion relation (4.3) and the only coupling between left and right being through the zero mode sum rules:

\[
\sum_{i=1}^{n} k_i + \sum_{s=1}^{N} q_s = - \sum_{j=1}^{m} p_j - \sum_{a=1}^{M} r_a = n + m + 2(N + M) - 2
\]

This generalizes a similar structure observed in [13] for the case \( N = M = 0 \) in (4.4).

3) The solution to (4.3) clearly has the periodicity \( k_i \rightarrow k_i + 1; q_s \rightarrow q_s + 2 \) (and similarly for the left movers), again generalizing [13].

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4.2. The form of the solutions

One can solve (4.3) more explicitly in terms of known functions. The dynamics of the right (left) moving particles is governed by \( W_{n,N}(k|q) \) \( (W_{m,M}(p|r)) \). In order to compute \( W_{n,N}(k|q) \) consider the partition of all \( N \) left moving closed tachyons into \( n \) sets \( (n \) is the number of left moving open tachyons): \[
P = \{q_1, q_2, ..., q_N\} = \bigoplus_{i=1}^{n} \{q_{j_1}^{(i)} ... q_{j_{v_i}}^{(i)}\}, \tag{4.5}
\]
where \( v_i \) is the number of momenta in the \( i \)-th set. Some of the \( v_i \) may be equal to zero, in which case the set is empty. Then \( W_{n,N}(k|q) \) is given by the sum over all partitions \( P \) (4.5) in terms of the basic functions \( W_{n,0}, W_{1,r} \):

\[
W_{n,N}(k|q) = \sum_{P} \left( \prod_{i=1}^{n} W_{1,v_i} \left( - \sum_{s=1}^{v_i} q_{j_s}^{(i)} | q_{j_s}^{(i)} \right) \right) W_{n,0}(k_i + \sum_{s=1}^{v_i} q_{j_s}^{(i)} | 0) \tag{4.6}
\]

\( W_{n,0} \) is known from the open case (compare (3.10), (4.4)), while \( W_{1,r} \) satisfies a simple recursion relation:

\[
W_{1,r}^{(+)}(k|q) = \sum_{j=1}^{r} \frac{\sin \pi(k + q_j)}{\sin \pi k} W_{1,r-1}^{(+)}(k + q_j|q_i, \hat{q}_j), \tag{4.7}
\]

where \( k \) and \( q_i \) are related by momentum conservation. The relation (4.7) should be supplemented with the initial condition \( W_{1,0} = 1 \). Its derivation follows from the ring relations (using \( A^2 \) instead of \( A \) in (3.4)); we will omit the details of the proof.

The simplest example of (4.3) is the scattering amplitude of one closed string tachyon and an arbitrary number of open tachyons. Taking into account that \( W_{1,1}(-q|q) = (2 \cos \frac{\pi q}{2})^{-1} \) we get

\[
\langle \prod_{j=1}^{m} \gamma_{p_j}^{(-)} \prod_{i=1}^{n} \gamma_{k_i}^{(+)} \rangle = \frac{1}{2 \cos \frac{\pi q}{2}} \sum_{l=1}^{n} \langle \prod_{j=1}^{m} \gamma_{p_j}^{(-)} \prod_{i=1}^{l-1} \gamma_{k_i}^{(+)} \gamma_{k_i+q}^{(+)} \prod_{i=l+1}^{n} \gamma_{k_i}^{(+)} \rangle. \tag{4.8}
\]

5. Summary.

In this paper we have presented the ring – module structure of open plus closed string theory on the disk. We have seen that the action of the ground ring on the modules
both linear (2.7), (3.3), and non-linear (2.9), (3.7), (3.8), (4.2) is completely determined by
consistency of the structure and furthermore determines the correlation functions uniquely
through the basic relation (4.3) which follows from the ring structure. The ring relations
are powerful and we have been able to solve for the correlation functions without doing
any integrals. This led to a verification of known results for the correlation functions of
open string tachyons (3.10) and to new results for correlation functions of open and closed
tachyons (4.3), (4.6). The algebraic structure of the disk correlation functions is quite
rich and we have certainly not exhausted it here. For higher genus Riemann surfaces one
should be able to make progress in a similar fashion by generalizing the ring relations. It
is important to study this structure (and possible generalizations) in more detail.

The action of the ring on the modules is suggestive again of a phase space interpreta-
tion with excitations living on certain curves. It is interesting that both open and closed
tachyons seem to live on the same \((A_+, A_-)\) phase space. Naively \((a_+, a_-)\) generate in-
dependent directions, but this is probably not the case – there should be relations of the
general form \(a_\pm \simeq A_\pm^2\) (in the sense of [15] – acting on the modules). One would also like
to extend the results of this paper to non-bulk amplitudes and try to solve the relevant
matrix model (see e.g. [19]) to gain further insight on the physics of these models, which
is bound to be interesting.

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**Appendix A. Open tachyon scattering.**

In the text we have extracted a recursion relation (3.9) for the scattering amplitude
of open string tachyons by considering a correlation function involving an \(A_-\) (3.4). We
have stressed that the derivation of (3.9) depended crucially on the way we defined (3.4)
in (3.5). The purpose of this appendix is to show that different ways of defining (3.4) give
rise to the same recursion relation (3.9), but in a different way. Consider:

\[
\hat{F}(\sigma) = \langle \prod_{j=1}^{n-1} \int \mathcal{V}^{(-)}_{p_j} c\mathcal{V}^{(-)}_{p_m}(0)A_-(\sigma)c\mathcal{V}^{(+)}_{k_1}(1)c\mathcal{V}^{(+)}_{k_2}(\infty) \prod_{i=3}^{n} \int \mathcal{V}^{(+)}_{k_1} \rangle \quad (A.1)
\]
\(\tilde{F}\) is not equal to \(F(\sigma)\) \((\text{3.3})\). As in the text, we are interested in computing \(\int_0^1 \partial_\sigma \tilde{F}(\sigma)\). One way to calculate it is to use the fact that it is clearly equal to \(\tilde{F}(\sigma = 1) - \tilde{F}(\sigma = 0)\). Comparing to \((\text{3.3})\) we see that the two \(\mathcal{V}(+)'s\) closest to \(A_-\) are not integrated, hence there is naively no contribution of the form \((\text{3.7})\) from \(\sigma = 1\). This is indeed correct (but still naive as we’ll see in a moment), and therefore,

\[
\int_0^1 \partial_\sigma \tilde{F}(\sigma) = -\tilde{F}(\sigma = 0) = -\langle \prod_{j=1}^{m-1} \mathcal{V}^{(-)}_{p_j} \mathcal{V}^{(-)}_{p_m-1} \prod_{i=1}^{n} \mathcal{V}^{(+)}_{k_i} \rangle
\]  

(A.2)

On the other hand, we may compute \(\partial_\sigma \tilde{F}(\sigma)\) by using \(\partial_\sigma A_-(\sigma) = \{Q_{\text{BRST}}, b_{-1}A_-\} \) \((\text{3.6})\); commuting \(Q_{\text{BRST}}\) through the rest of the operators in \((\text{A.1})\) we pick up potential boundary contributions from each of the integrated \(\mathcal{V}_i\): \(\int \partial(c\mathcal{V}_i)\). The main question is whether there are finite boundary terms. One can convince oneself that the situation is the following: for all the \(\mathcal{V}^{(-)}_{p_j} j = 1, ..., m - 1\) and \(\mathcal{V}^{(+)}_{k_i} i = 4, ..., n\) there are finite and equal boundary contributions from the upper and lower limits of integration, which therefore cancel each other. For \(\mathcal{V}^{(+)}_{k_3}(z_3)\) the analysis is slightly modified: the upper limit of integration \((z_3 \to \infty)\) gives a vanishing boundary term, whereas the contribution from \(z_3 \to 0\) is finite. The value of the boundary term from \(z_3 \to 0\) is given by a product of correlation functions:

\[
\langle \int b_{-1}A_- \mathcal{V}^{(+)}_{k_1} \mathcal{V}^{(+)}_{k_2} \mathcal{V}^{(+)}_{k_3} \rangle \langle \prod_{j=1}^{m} \mathcal{V}^{(-)}_{p_j} \mathcal{V}^{(-)}_{k_m-1} \prod_{i=3}^{n} \mathcal{V}^{(+)}_{k_i} \rangle
\]

which precisely supplies the missing term to turn \((\text{A.2})\) into \((\text{3.9})\). The relation between the way \((\text{3.9})\) arises from \((\text{3.3})\) and \((\text{A.1})\) is through a conformal transformation; in \((\text{3.3})\) we allow \(A_-\), \(\mathcal{V}^{(+)}_{k_1}\) and \(\mathcal{V}^{(+)}_{k_2}\) to approach each other, and get a finite contribution from the region of degeneration; in \((\text{A.1})\) we keep the distance between \(\mathcal{V}^{(+)}_{k_1}\) and \(\mathcal{V}^{(+)}_{k_2}\) fixed and the degeneration region is the region where all other operators approach each other (while being well separated from \(\mathcal{V}^{(+)}_{k_1}, \mathcal{V}^{(+)}_{k_2}\)). We see that the result of this calculation can be summarized by the additional relation:

\[
A_-(\mathcal{T}^{(+)}_{k_1} \mathcal{T}^{(+)}_{k_2}) = \frac{1}{\sin \pi k_1} \mathcal{T}^{(+)}_{k_1 + k_2 - 1}
\]

and appropriately for \((\text{3.8})\).
Appendix B. Open and closed tachyon scattering.

Here we supply some details of the derivation of eq. (4.2), (4.3) which determine the open plus closed string correlation functions. Consider the function:

\[
C(\sigma) = \langle \prod_{j=1}^{m-1} \int \mathcal{V}_{p_j}(-) c V_{p_m}(-)(0) A_-(\sigma) c V_{k_1}^{(+)}(1) \prod_{i=2}^{n-1} \int \mathcal{V}_{k_i}^{(+)} c V_{k_n}^{(+)}(\infty) \prod_{a=1}^{M} \int \mathcal{T}_{r_a}^{(-)} \prod_{s=1}^{N} \int \mathcal{T}_{q_s}^{(+)} \rangle,
\]

(B.1)

where the \( \mathcal{V} \) are integrated over the real line in the specified order, while the \( \mathcal{T} \) are integrated over the whole upper half plane. In the by now standard fashion we consider \( \int_0^1 \partial_\sigma C(\sigma) \) and compute it in two different ways. The first, as \( C(\sigma \to 1) - C(\sigma \to 0) \) gives rise using (3.3), (3.7) to the standard terms in (4.3): the l.h.s. and the first term on the r.h.s.. The second, obtained again by writing \( \partial_\sigma A_- \) as a BRST commutator (3.6) reduces to a sum of total derivative terms from each of the integrated vertices in (B.1). Many of these terms vanish. In particular, there are no boundary contributions from \( \mathcal{V}_{p_j}^{(-)} \), \( \mathcal{V}_{k_i}^{(+)} \), \( \mathcal{T}_{r_a}^{(-)} \). Each of the \( \mathcal{T}_{q_s}^{(+)} \) gives rise to a boundary contribution from the region near \( \sigma \):

\[
\int \partial_z (c T_{q_s}^{(+)} b_{-1} A_-) = \langle T_{q_s}^{(+)} b_{-1} A_- \mathcal{V}_{k}^{(+)} \rangle \langle \mathcal{V}_{k}^{(+)} \cdot \cdot \cdot \rangle
\]

(B.2)

This is the origin of the second term on the r.h.s. of (4.3). To actually calculate the structure function \( (\sin \frac{\pi q_s}{2}) \) in (4.2), (4.3) one can either explicitly evaluate the correlator (B.2), or apply various consistency requirements, as mentioned above.
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