THE BI-DIMENSIONAL EULER EQUATIONS IN YUDOVICH TYPE SPACE AND bmo-TYPE SPACE

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1. Introduction

The bi-dimensional incompressible Euler equations are governed by

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \nabla \Pi &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\text{div } u &= 0, \\
|t=0| &= u_0.
\end{aligned}
\]

(E)

Here, \( u = (u_1, u_2)(t, x_1, x_2) \) denotes the velocity vector-field. The scalar function \( \Pi \) stands for the pressure which can be recovered from \( u \) via Calderón-Zygmund operators, namely,

\[
\Pi = \sum_{i,j=1}^{2} \frac{\partial_{x_i} \partial_{x_j}}{-\Delta} (u_i u_j).
\]

(1.1)

There have been numerous studies on the bi-dimensional incompressible Euler equations by many physicists and mathematicians due to its physical importance. At first, some results on the local well-posedness of smooth solution for the large initial data were obtained in different type function spaces such as \( H^s(\mathbb{R}^2) \), \( B_{p,q}^s(\mathbb{R}^2) \) and \( F_{p,q}^s(\mathbb{R}^2) \) etc. Since the vorticity \( \omega \) satisfies the free transport equation

\[
\partial_t \omega + (u \cdot \nabla) \omega = 0, \quad \text{div } u = 0, \quad \omega|_{t=0} = \omega_0,
\]

(1.2)

the incompressible condition guarantees that the quantity \( \|\omega(t)\|_{L^\infty(\mathbb{R}^2)} \) is conserved. This together with the Beal-Kato-Majda (abbr. B-K-M) criterion established in [9] and the logarithmic Sobolev inequality entails the global existence and uniqueness of smooth solutions for the general initial data in subcritical functional spaces such as \( B_{p,q}^s(\mathbb{R}^2) \) with \( s > \frac{2}{p} + 1 \), see for example [13, 16, 15]. However, the B-K-M criterion does not work in the critical framework because the logarithmic Sobolev inequality is not available. In order to overcome this difficulty, M. Vishik [7] established the following logarithmic estimate for the vorticity equation

\[
\|\omega(t)\|_{B_{\infty,1}^0(\mathbb{R}^2)} \leq C \left( 1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} \, d\tau \right) \|\omega_0\|_{B_{\infty,1}^0(\mathbb{R}^2)},
\]

(1.3)
which also was proved in [13]. As a consequence, the global well-posedness was obtained in the critical Besov space $B^{\frac{2}{p}+1}_{p,1}(\mathbb{R}^2)$. In addition, by making good use of the structure of equations, P. Serfati [22] gave a notable existence and uniqueness result without any integrable conditions. We easily find that the above theory on the bi-dimensional Euler equations be do in the context of the Lipschitzian vector field. This ensures three fundamental properties of solution: global existence, uniqueness and regularity persistence, which are based on the Cauchy-Lipschitz theorem. And the question naturally arises, “do these properties hold for the non-Lipschitzian vector field?”

To answer this question, let us start with the global existence which has been extensively studied in the past years. It is well-known that an existence result of the weak solution for $(E)$ for the initial data $(u_0, \omega_0) \in L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ was obtained in [11] with $p \in [2, \infty]$, where the weak solution is defined in the following way:

**Definition 1.1** (Weak solutions). Any function $u \in L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ is a weak solution of $(E)$ if the following holds

(i) for $\forall \phi = (\phi_1, \phi_2) \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ and $\text{div} \phi = 0$,

$$\int_0^\infty \int_{\mathbb{R}^2} (\phi_t \cdot u + \nabla \phi : (u \otimes u)) \, dx \, dt = 0;$$

(ii) $\int_0^\infty \int_{\mathbb{R}^2} \nabla \psi \cdot u \, dx \, dt = 0$, $\forall \psi \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$,

where $u \otimes u = (u_i u_j)$, $\nabla \phi = (\partial_{x_j} \phi_i)$ and $A : B = \sum_{i,j=1}^2 A_{ij} B_{ij}$.

Solutions defined in this way are often called weak solutions in the literature and we also use this terminology. Later on, Y. Giga, T. Miyakawa and H. Osada [12] still established the global existence of the weak solution under $\omega_0 \in L^p$ with $p \in [2, \infty]$. Moreover, D. Chae [5] showed a global existence result for $\omega_0 \in L \log^+ L$ with compact support which can be viewed as a variation of $L^1(\mathbb{R}^2)$. The papers [10] and [15] were concerned with measure-valued solutions to the bi-dimensional Euler equations. Recently, Y. Taniuchi [21] also proved a global existence result for $(u_0, \omega_0) \in L^\infty \times bmo$ by establishing the local uniformly $L^p(\mathbb{R}^2)$ estimate for vorticity and the continuity argument.

Regularity persistence and uniqueness are also hot topics in the study of the bi-dimensional Euler equations with non-Lipschitzian vector field. V. Yudovich [25] proved the uniqueness of solution under the assumption that $\omega(t, x) \in Y^\Theta$ with $\Theta \in A_2$ (see Definition 1.6). The other interesting results on uniqueness can be found in [8] [22]. As for the problem of propagation of regularity, M. Vishik [8] only showed that $\omega(t, x)$ belongs to the class $B_{E_2}$ with $E_2(n) = n E_1(n)$ under the assumption $\omega_0 \in B_{E_1}$, where $B_{E_i}$ $(i = 1, 2)$ be defined by

$$||f||_{B_{E_i}} := \sup_{n \geq 2} \frac{1}{\Gamma_i(n)} \sum_{k=1}^n ||\Delta_k f||_{L^\infty} < \infty.$$ 

This implies that the losing of regularity of $\omega(t, x)$ occurs as time developing in the borderline space $B_{E_1}$. Therefore, whether the regularity of $\omega(t, x)$ is preserved or not in the borderline space which does not belong to Lipschitz class is a challenging issue.

The study of the global well-posedness to the bi-dimensional incompressible Euler equations with non-Lipschitzian vector field was initiated by V. Yudovich. In his pioneering work [25], a result of global well-posedness for essentially bounded vorticity was obtained. Subsequently, many works have been dedicated to the extension of this result to the more general spaces such as P. Serfati’s work [23] on the global well-posedness of $(E)$ for the initial data $(u_0, \omega_0) \in L^\infty(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$. More recently, F. Bernicot and S. Keraani [2] considered the equivalent form...
of (E), that is, the vorticity-stream equations:
\begin{align*}
\left\{ \begin{array}{l}
\partial_t \omega + (u \cdot \nabla) \omega = 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
u = K \ast \omega, \quad \text{with} \quad K(x) = \frac{x^+}{2 \pi |x|^2}, \\
\omega|_{t=0} = \omega_0.
\end{array} \right.
\end{align*}

They investigated the global well-posedness of (1.4) for the unbounded vorticity \( \omega_0 \in L^p \cap L^0_{bmo} \) with \( p \in [1, 2] \). Their proof strongly relies on the preserving measure of flow and Whitney covering theorem. Based on this, F. Bernicot and T. Hmidi [3] further generalized this result and established the global well-posedness of (1.4) for \( \omega_0 \) belonging to \( L^p \cap L^0_{bmo} \) with \( p \in [1, 2] \) and \( \alpha \in [0, 1] \). One can refer to Section 2 for more details of spaces \( L^0_{bmo} \).

The object of our paper is devoted to the study of the global existence and uniqueness of the weak solution for (E) in \( bmo \)-type spaces. It is necessary to point out that we are not only concerned with regularity persistence, but also the losing regularity for (1.2) in \( L^0_{bmo} \) space not \( LL_0 \) bmo space which continuously embeds \( L^0_{bmo} \).

We begin by establishing the uniformly local \( L^p \) estimate for \( \omega \) as in [21]. Our strategy is to establish a logarithmic estimate for \( u \), so as to obtain the global estimate of \( \|u(t)\|_{L^\infty} \). Unfortunately, it seems impossible to get the similar result with that of [21] if we borrow the algorithm used in [22, 23] directly. In order to overcome this difficulty, we exploit a new estimate for convection term. In combination with a new observation, we get that for \( \alpha \in [0, 1] \)
\begin{align*}
\|u(t)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} \leq C(t) \Phi \left( 1 + \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau \right) \cdot \left( 1 + \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau \right),
\end{align*}
where, \( \Phi(\cdot) = (T\Theta)(\cdot) \) (see Definition 1.4). The above estimate yields the global bound for the quantities \( \|u(t)\|_{L^\infty} \) and \( \|\omega(t)\|_{Y^\alpha} \) with \( \Theta \in \mathcal{A}_1 \). This enables us to show the global existence and uniqueness of weak solution to (E) in a large class involving unbounded and non-decaying vorticity. From this, we further obtain a global well-posedness result of problem (E) in a Spanne space. Next, we intend to investigate the preservation of the regularity of \( \omega \) in the borderline space \( L^0_{bmo} \) with \( \alpha \in [0, 1] \). As we know, F. Bernicot and S. Keraani [1] recently give an optimal estimate
\begin{align*}
\|\omega(t)\|_{bmo} \leq \left( 1 + \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right) ||\omega_0||_{bmo},
\end{align*}
which makes it impossible to preserve the regularity of \( \omega \) in \( bmo \). Inspired by [8], we proceed to care about the evolving property of the regularity of \( \omega(t, x) \) in space \( L^0_{bmo} \) with \( \alpha \in [0, 1] \). By developing some tools in classical harmonic analysis such as John-Nirenberg inequality, we obtain the following estimates with a logarithmic loss of regularity
\begin{align*}
\|\omega(t)\|_{L^{\alpha-1}_{bmo}} \leq C(t) ||\omega_0||_{L^\alpha_{bmo}}, \quad \text{for} \quad \alpha \in [0, 1],
\end{align*}
and
\begin{align*}
\|\omega(t)\|_{L^{\log}_{bmo}} \leq C(t) ||\omega_0||_{L^{\log}_{bmo}}.
\end{align*}
Based on this, we also establish the following estimates
\begin{align*}
\|\omega(t)\|_{L^\alpha_{bmo}} \leq C(t) \left\{ \begin{array}{l}
\left( 1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) ||\omega_0||_{L^\alpha_{bmo}}, \quad \text{for} \quad \alpha \in [0, 1]; \\
\left( 1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) \log \left( 1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) ||\omega_0||_{L^\alpha_{bmo}}, \quad \text{for} \quad \alpha = 1; \\
\left( 1 + \int_0^t \|\nabla u\|_{L^\infty} d\tau \right) \alpha ||\omega_0||_{L^\alpha_{bmo}}, \quad \text{for} \quad \alpha > 1.
\end{array} \right.
\end{align*}
This generalizes the sharp estimate (1.4), and we give a simple proof of (1.6) via a new observation. As a corollary, we obtain the global well-posedness of (E) in \( L^0_{bmo} \) with \( \alpha > 1 \).

Before presenting results, let us give some useful notations and definitions for clarity.
Notation: We will use the following notations. Let $m(\Omega)$ denote the Lebesgue measure of the set $\Omega$. $B_r(x_0) := \{x \in \mathbb{R}^d | |x - x_0| < r\}$ and $\lambda B_r(x_0) := \{x \in \mathbb{R}^d | |x - x_0| < \lambda r\}$ for any positive number $\lambda$. $\text{Avg}_\omega(f)$ stands for $\frac{1}{m(\Omega)} \int_\Omega f(y) \, dy$. And we agree that

$$
\| f \|_{p, \lambda} := \sup_{x \in \mathbb{R}^d} \left( \| f \|_{L^p(B(x, \lambda))} \right) = \sup_{x \in \mathbb{R}^d} \left( \int_{|x-y|<\lambda} |f(y)|^p \, dy \right)^{\frac{1}{p}},
$$

$L^p_{ul}(\mathbb{R}^d) := \{ f \in L^1_{loc}(\mathbb{R}^d); \| f \|_{p, 1} < \infty \}$, $\| f \|_{L^p_{ul}(\mathbb{R}^d)} := \| f \|_{p, 1}$.

**Definition 1.2.** Let $\alpha \in [0, 1]$. The space $LLog^\alpha$ ($LogLip^\alpha$) consist of those bounded functions $f$ such that

$$
\| f \|_{LLog^\alpha(\mathbb{R}^d)} := \sup_{x, x' \in \mathbb{R}^d, \ 0<|x-x'|<1} \frac{|f(x) - f(x')|}{|x-x'| \left( e - \log |x-x'| \right)^\alpha} < \infty,
$$

$$
\left( \| f \|_{LogLip^\alpha(\mathbb{R}^d)} := \sup_{x, x' \in \mathbb{R}^d, \ 0<|x-x'|<1} \frac{|f(x) - f(x')|}{|x-x'| \left( e - \log |x-x'| \right) \log^\alpha \left( \left( e - \log |x-x'| \right) \right)} < \infty \right).
$$

Let us remark that the spaces $LLog^0$ (abbr. Lip) and $LLog^1$ (abbr. LLog) correspond to Lipschitz and logLipschitz, respectively. In addition, space $LogLip^0$ corresponds to logLipschitz and we denote by LogLog the space $LLog^1$ for the sake of simplicity.

**Definition 1.3** ([23]). A modulus of continuity is any nondecreasing nonzero continuous function $\phi$ on $[0, \infty]$ such that $\phi(0) = 0$, and having that $\phi(\cdot)$ does not vary too rapidly, that is, the so-called $\Delta_2$-condition: $\phi(2h) \leq C_{\phi} \cdot \phi(h)$, for any $h \in [0, \infty[$.

**Definition 1.4** ([25]). Let the modulus of continuity $\phi(p) \geq 1$ be a positive function on $[p_0, \infty[$ with $p_0 \geq 1$, and the function $\Phi(a) := (T\phi)(a)$ on the positive ray $[0, \infty)$ is defined by

$$
\Phi(a) = \begin{cases} 
\inf_{p_0 \leq p < \infty} \{ a^p \phi(p) \}, & \text{for } a \geq 1; \\
\inf_{p_0 \leq p < \infty} \{ \phi(p) \}, & \text{for } a < 1.
\end{cases}
$$

**Remark 1.1.** The definition of the function $\Phi(\cdot)$ does not depend on the choice of the index $p_0$ in the sense of the germs at infinity, one can refer to [25] for more explanation.

**Definition 1.5.** Let $\phi$ be a modulus of continuity. Then we say that

(1) $\phi$ belongs to the class $A_1$ if the function $\Phi = T(\phi(p))$ satisfies the following admissible condition

$$
\int_a^\infty \frac{1}{x \Phi(x)} \, dx = \infty, \quad \text{for some positive } a \in [0, \infty[. \quad (1.7)
$$

(2) $\phi$ belongs to the class $A_2$ if the function $\phi$ satisfies $\int_a^\infty \frac{1}{x \phi(x)} \, dx = \infty$ for some positive $a \in [0, \infty[$.

**Examples:** The function $\phi : q \mapsto \phi(q)$ belongs to the class $A_1$, likes the functions

$$
q \mapsto q^\beta, \quad q \mapsto q \log(1 + q) \left( \log \left( 1 + \log(1 + q) \right) \right)^\beta \quad \text{and} \quad q \mapsto q \log(1 + q) \left( \log \left( 1 + \log(1 + q) \right) \right)^\beta \quad \text{if } \beta \in [0, 1].
$$

**Definition 1.6** (Yudovich). Let $\Theta(p) \geq 1$ be a non-decaying function on $[1, \infty[$.

(i) $Y^{\Theta}(\mathbb{R}^d) := \{ f \in \cap_{1 \leq p < \infty} L^p(\mathbb{R}^d); \| f \|_{Y^{\Theta}(\mathbb{R}^d)} < \infty \}$, where $\| f \|_{Y^{\Theta}(\mathbb{R}^d)} := \sup_{p \geq 1} \frac{\| f \|_{L^p(\mathbb{R}^d)}}{\Theta(p)}$.

(ii) $Y_{ul}^{\Theta}(\mathbb{R}^d) := \{ f \in \cap_{1 \leq p < \infty} L^p_{ul}(\mathbb{R}^d); \| f \|_{Y_{ul}^{\Theta}(\mathbb{R}^d)} < \infty \}$, where $\| f \|_{Y_{ul}^{\Theta}(\mathbb{R}^d)} := \sup_{p \geq 1} \frac{\| f \|_{L^p_{ul}(\mathbb{R}^d)}}{\Theta(p)}$. 


A direct calculation allows us to conclude that $\log \varphi$ with $\Theta \in A_1$. We easily find that $u(t, x) \in C([0, 1]; L^\infty(\mathbb{R}^2))$. Moreover, we further generalize the uniqueness results in [23, 25].

Thanks to the Biot-Savart law, one infers that

$$\partial_2 u_{1,0} - \partial_1 u_{2,0} = g^2(x) + 2 \sin(x_1) \sin(x_2).$$

We easily find that $\omega_0(x)$ is an unbounded and non-decaying function, and belongs to $Y^{\Theta}_{ul}(\mathbb{R}^2)$ with $\Theta(p) = \log p \in A_2$. This implies that we can get the global existence and uniqueness of weak solution to (E) in $Y^{\Theta}_{ul}(\mathbb{R}^2)$ involving unbounded and non-decaying vorticity. As byproduct we recover the existence result in [21], where the method is based on the continuity argument. Moreover, we further generalize the uniqueness results in [23, 25].

**Theorem 1.2.** Let $\alpha \in [0, 1/2]$ and $u_0 \in L^\infty(\mathbb{R}^2)$ and $\omega_0 \in Y^{\Theta}_{ul}(\mathbb{R}^2) \cap M_\varphi(\mathbb{R}^2)$ with $\Theta \in A_1$ and $\varphi(r) = \log^\alpha(e - r)$ with $r \in [0, 1/2]$. Then (E) admits a unique global solution $u$ such that

$$u(t, x) \in C\left([0, 1]; L^\infty(\mathbb{R}^2)\right) \quad \text{and} \quad \omega(t, x) \in L^\infty_{loc}\left([0, 1]; Y^{\Theta}_{ul}(\mathbb{R}^2) \cap M_\varphi(\mathbb{R}^2)\right).$$

Here, the Spanne space $M_\varphi(\mathbb{R}^2)$ is defined in Section 2.

**Remark 1.3.** A simple calculation yields that $g^\alpha(x)$ defined in (1.8) lies in Spanne space $M_\varphi(\mathbb{R}^2)$ with $\varphi(r) = \log^\alpha(e - r)$ for $\alpha \in [0, 1]$.

**Remark 1.4.** We can generalize Theorem 1.2 to the more general Spanne space $M_\varphi(\mathbb{R}^2)$ with

$$\varphi(r) = \log^\alpha(e - r) \log_2(e - r) \cdots \log_m(e - r)$$

for any natural $m$, where $\log_m$ stands for the $m$-th iteration of logarithm.

Next, we shall state both results concerning ill-regularity of the bi-dimensional Euler equations in bmo-type space. Specifically:
Theorem 1.3. Let \( u_0 \in L^\infty(\mathbb{R}^2) \) and \( \omega_0 \in L^\alpha \text{bmo}(\mathbb{R}^2) \) with \( \alpha \in [0, \infty[. \) Then (\ref{E}) admits at least a global solution \( u \) such that

(i) When \( 0 \leq \alpha \leq 1 \), \( u(t, x) \in C([1, \infty) ; L^\infty(\mathbb{R}^2)) \) and \( \omega(t, x) \in L^\infty_{\text{loc}}(\mathbb{R}^+ ; \text{bmo}(\mathbb{R}^2)) \).

(ii) When \( \alpha > 1 \), \( u(t, x) \in C([1, \infty) ; L^\infty(\mathbb{R}^2)) \) and \( \omega(t, x) \in L^\infty_{\text{loc}}(\mathbb{R}^+ ; \text{bmo}(\mathbb{R}^2)) \).

In particular, solution \( u \) is unique as \( \alpha \geq 1 \).

Remark 1.5. For the case \( 0 \leq \alpha \leq 1 \), we see that the vorticity \( \omega(t, x) \) has a logarithmic loss of regularity for all \( t > 0 \). Similar results concerning on loss of regularity on vorticity were shown in works \cite{3} and \cite{8}, see also Corollary \ref{L2} for \( \alpha \in (0, 1] \). From the optimal estimate \ref{1.6}, it seem impossible to get the regularity persistence of vorticity in bmo. Thus, the loss of regularity on vorticity in Theorem \ref{L3} seems inevitable. It is interesting to show whether this loss is optimal or not, and we plan to study it in our future work.

Remark 1.6. Since \( L^\alpha \text{bmo}(\mathbb{R}^2) \hookrightarrow B^{\alpha}_{\infty, 1}(\mathbb{R}^2) \) for \( \alpha > 1 \), \ref{E} can be obtained by the result in \cite{22} and estimate \ref{1.3}.

Next, we focus on the control estimates in \( L^\alpha \text{bmo} \ (\alpha \geq 0) \) of flow mapping determined by the vorticity-stream equations \ref{1.4} with a bi-Lipschitz vector field. The interesting point is how to optimize the control estimates by using the measure preserving property, a new observation and the generalized John-Nirenberg inequality.

Theorem 1.4. Let \( u_0 \in L^\infty(\mathbb{R}^2) \) and \( \omega_0 \in L^\alpha \text{bmo}(\mathbb{R}^2) \) with \( \alpha \in [0, \infty[. \) Assume that \( u \in L^1_{\text{loc}}(\mathbb{R}^+ ; \text{Lip}(\mathbb{R}^2)) \) is a smooth solution of \ref{E}. Then there exists a positive constant \( C \), dependent of the initial data and \( \alpha \), such that

\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} + \|\omega(t)\|_{L^\alpha \text{bmo}(\mathbb{R}^2)} \leq C \left\{
\begin{array}{l}
(1 + V_{\text{Lip}}(t)) \|\omega_0\|_{L^\alpha \text{bmo}(\mathbb{R}^2)}, \\
(1 + V_{\text{Lip}}(t)) \log (1 + V_{\text{Lip}}(t)) \|\omega_0\|_{L^\alpha \text{bmo}(\mathbb{R}^2)}, \\
(1 + V_{\text{Lip}}(t))^\alpha \|\omega_0\|_{L^\alpha \text{bmo}(\mathbb{R}^2)};
\end{array}
\right.
\] \hspace{1cm} (1.11)

for \( \alpha \in [0, 1] \); \( \alpha = 1 \); for \( \alpha > 1 \).

Remark 1.7. When \( \alpha = 0 \), Theorem \ref{L4} recovers the result established in \cite{1}, i.e.,

\[
\|\omega(t)\|_{\text{bmo}(\mathbb{R}^2)} \leq C (1 + V_{\text{Lip}}(t)) \|\omega_0\|_{\text{bmo}(\mathbb{R}^2)}. \hspace{1cm} (1.12)
\]

More importantly, they also show that \ref{1.12} is a sharp estimate by the property of \( K \)-quasi-conformal mapping and the Whitney covering theorem. In fact, we can provide a simple proof for \ref{1.12}. Firstly, using the evolving property of bi-Lipschitz flow, one can conclude for any \( p \geq 1 \)

\[
\|\omega(t)\|_{\text{bmo}^p} \leq C \left(e^{(1 + V_{\text{Lip}}(t))}\right)^{\frac{2}{\gamma}} \|\omega_0\|_{\text{bmo}^p}.
\]

In light of the Hölder inequality and Corollary \ref{L2} in Appendix A, we have

\[
\|\omega(t)\|_{\text{bmo}} \leq \|\omega(t)\|_{\text{bmo}^p} \leq C \left(e^{(1 + V_{\text{Lip}}(t))}\right)^{\frac{2}{\gamma}} \|\omega_0\|_{\text{bmo}^p} \leq C p \cdot \left(e^{(1 + V_{\text{Lip}}(t))}\right)^{\frac{2}{\gamma}} \|\omega_0\|_{\text{bmo}}.
\]

This together with Lemma \ref{L5} yields

\[
\|\omega(t)\|_{\text{bmo}(\mathbb{R}^2)} \leq C \inf_{1 \leq p < \infty} \left(p \cdot e^{(1 + V_{\text{Lip}}(t))}\right)^\frac{2}{\gamma} \|\omega_0\|_{\text{bmo}(\mathbb{R}^2)} \leq C (1 + V_{\text{Lip}}(t)) \|\omega_0\|_{\text{bmo}(\mathbb{R}^2)}.
\]

The paper is organized as follows. In Section \ref{2} we review some useful statements on functional spaces and basic analysis tools, and introduce several technical lemmas. The next section, we establish some losing estimates and logarithmic estimate for the transport equation with the vector field belonging to bmo-type spaces. Section \ref{4} is devoted to the proof of our main theorems. Finally, we generalize the classical John-Nirenberg inequality, and establish some product estimates and commutator estimates by using Bony’s para-product decomposition.
2. Preliminary

2.1. Littlewood-Paley Theory and the functional spaces. In this subsection, we first review the so-called Littlewood-Paley decomposition described, e.g., in [4, 6, 18]. Next, we introduce some useful functional spaces such as Morrey-Campanato space and its properties.

Let \((\chi, \varphi)\) be a couple of smooth functions with values in \([0, 1]\) such that \(\chi\) is supported in the ball \(\{\xi \in \mathbb{R}^d \mid |\xi| \leq \frac{3}{4}\}\), \(\varphi\) is supported in the ring \(\{\xi \in \mathbb{R}^d \mid \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}\) and

\[\chi(\xi) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} \xi) = 1 \quad \text{for each} \quad \xi \in \mathbb{R}^d.\]

For any \(u \in S'(\mathbb{R}^d)\), one can define the dyadic blocks as

\[\Delta_{-1}u = \chi(D)u \quad \text{and} \quad \Delta_j u := \varphi(2^{-j} D)u \quad \text{for each} \quad j \in \mathbb{N}.\]

We also define the following low-frequency cut-off:

\[S_j u := \chi(2^{-j} D)u.\]

It is easy to verify that

\[u = \sum_{j \geq -1} \Delta_j u, \quad \text{in} \quad S'(\mathbb{R}^d),\]

and this is called the inhomogeneous Littlewood-Paley decomposition. It has nice properties of quasi-orthogonality:

\[\Delta_j \Delta_{j'} u \equiv 0 \quad \text{if} \quad |j - j'| \geq 2.\]

\[\Delta_j (S_{j-1} u \Delta_{j'} v) \equiv 0 \quad \text{if} \quad |j - j'| \geq 5.\]

We shall also use the homogeneous Littlewood-Paley operators as follows:

\[\hat{S}_j u := \chi(2^{-j} D)u \quad \text{and} \quad \hat{\Delta}_j u := \varphi(2^{-j} D)u \quad \text{for each} \quad j \in \mathbb{Z}.\]

**Definition 2.1.** Let \(S'_h(\mathbb{R}^d)\) be the space of tempered distributions \(u\) such that

\[\lim_{q \to -\infty} \hat{S}_j u = 0, \quad \text{in} \quad S'(\mathbb{R}^d).\]

**Definition 2.2.** For any \(u, v \in S'_h(\mathbb{R}^d)\), the product \(u \cdot v\) has the Bony decomposition:

\[u \cdot v = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v),\]

where the paraproduct term

\[\hat{T}_u v = \sum_{j \leq k-2} \hat{\Delta}_j u \hat{\Delta}_k v = \sum_{j} \hat{S}_{j-1} u \hat{\Delta}_j v,\]

and the remainder term

\[\hat{R}(u, v) = \sum_{j} \hat{\Delta}_j u \widetilde{\Delta}_j v, \quad \widetilde{\Delta}_j := \sum_{k=-1}^{1} \hat{\Delta}_{j-k}.\]

Now we introduce the Bernstein lemma which will be useful throughout this paper.

**Lemma 2.1.** Let \(1 \leq a \leq b \leq \infty\) and \(f \in L^a(\mathbb{R}^d)\). Then there exists a positive constant \(C\) such that for \(q, k \in \mathbb{N}\),

\[\sup_{|\alpha| = k} ||\partial^\alpha S_q f||_{L^b(\mathbb{R}^d)} \leq C^k 2^q \left(2^q(\frac{d}{a} - \frac{1}{2})\right) ||S_q f||_{L^a(\mathbb{R}^d)},\]

\[C^{-k} 2^{qk} ||\Delta_q f||_{L^a(\mathbb{R}^d)} \leq \sup_{|\alpha| = k} ||\partial^\alpha \Delta_q f||_{L^a(\mathbb{R}^d)} \leq C^k 2^{qk} ||\Delta_q f||_{L^a(\mathbb{R}^d)}.\]
Definition 2.3. Let $s \in \mathbb{R}$, $(p, q) \in [1, \infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^d)$. Then we define the inhomogeneous Besov spaces as

$$B^{s}_{p, q}(\mathbb{R}^d) := \{ u \in \mathcal{S}'(\mathbb{R}^d) \mid \| u \|_{B^{s}_{p, q}(\mathbb{R}^d)} < \infty \},$$

where,

$$\| u \|_{B^{s}_{p, q}(\mathbb{R}^d)} := \left( \sum_{j \geq 1} 2^{jsq} \| \Delta_j u \|_{L^p(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} \quad \text{if} \quad q < \infty,$n

$$\| u \|_{B^{s}_{p, q}(\mathbb{R}^d)} := \sup_{j \geq 1} 2^{js} \| \Delta_j u \|_{L^p(\mathbb{R}^d)} \quad \text{if} \quad q = \infty.$$

Proof. (i) is obvious.

Next, we review statements of the weighted Morrey-Campanato space and give some useful properties.

Definition 2.4. Let $\alpha \in [0, \infty]$, $p \in [1, \infty]$ and the scalar function $f$ is a locally integrable. If

$$\| f \|_{L^\alpha BMO_p(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d, r \in ]0, 1[} \left( - \log r \right)^{\alpha \frac{1}{p}} \left( \frac{1}{m(B_r(x))} \int_{B_r(x)} | f(y) - \text{Avg}_{B_r(x)}(f) |^p \, dy \right)^{\frac{1}{p}},$$

then we say that $f \in L^\alpha BMO_p(\mathbb{R}^d)$. In usual, we denote $L^\alpha BMO_1(\mathbb{R}^d)$ by $L^\alpha BMO(\mathbb{R}^d)$.

Definition 2.5. Let $\alpha \in [-1, \infty]$, $p \in [1, \infty]$ and the scalar function $f$ is a locally integrable. If

$$\| f \|_{L^\alpha bmo_p(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d, r \in ]0, 1[} \left( - \log r \right)^{\alpha \frac{1}{p}} \left( \frac{1}{m(B_r(x))} \int_{B_r(x)} | f(y) - \text{Avg}_{B_r(x)}(f) |^p \, dy \right)^{\frac{1}{p}},$$

then we say that $f \in L^\alpha bmo_p(\mathbb{R}^d)$. In usual, we denote $L^\alpha bmo_1(\mathbb{R}^d)$ by $L^\alpha bmo(\mathbb{R}^d)$.

It is worthwhile to remark that $L^\alpha bmo(\mathbb{R}^d)$ is equivalent to $L^\alpha bmo_p$ for all $\alpha \geq 0$ and $p > 1$ by using Corollary 2.2. Next, we give some basic properties of the space $L^\alpha bmo(\mathbb{R}^d)$ and $L^\alpha bmo(\mathbb{R}^d)$ which will be used in the following sections.

Proposition 2.2. There hold that

(i) For $\alpha_1 \geq \alpha_2$, we have $L^{\alpha_1} bmo(\mathbb{R}^d) \hookrightarrow L^{\alpha_2} bmo(\mathbb{R}^d)$.

(ii) $L^\alpha bmo(\mathbb{R}^d)$ is a Banach space for any $\alpha \geq 0$.

(iii) If $\alpha \in ]0, \infty[$, then, for all $q > \frac{1}{\alpha}$, we have that $L^\alpha bmo(\mathbb{R}^d)$ continuously embeds $B^0_{\infty, q}(\mathbb{R}^d)$.

In particular, $bmo(\mathbb{R}^d) \hookrightarrow B^0_{\infty, \infty}(\mathbb{R}^d)$.

(iv) For every $f \in L^1(\mathbb{R}^d)$ and $g \in L^\alpha bmo(\mathbb{R}^d)$ with $\alpha \geq 0$, one has

$$\| f * g \|_{L^\alpha bmo(\mathbb{R}^d)} \leq \| f \|_{L^1(\mathbb{R}^d)} \| g \|_{L^\alpha bmo(\mathbb{R}^d)}.$$

Proof. (i) is obvious.

It is well-known that $bmo(\mathbb{R}^d)$ is a Banach space (see for example [11]). So we just need to show $L^\alpha bmo(\mathbb{R}^d)$ is a Banach space for any $\alpha > 0$. Let the family $\{ f_n \}_n$ be a Cauchy sequence in $L^\alpha bmo(\mathbb{R}^d)$. Since $bmo(\mathbb{R}^d)$ is complete, we know that the sequences $\{ f_n \}_n$ converge in $bmo(\mathbb{R}^d)$ and then in $L^1_{loc}(\mathbb{R}^d)$. According to the definition of space and the convergence in $L^1_{loc}(\mathbb{R}^d)$, we immediately get that the convergence holds in $L^\alpha bmo(\mathbb{R}^d)$. This shows completeness of space $L^\alpha bmo(\mathbb{R}^d)$ for all $\alpha > 0$.

(iii) For each $f \in L^\alpha bmo(\mathbb{R}^d)$, by Lemma 2.3 and [3] Proposition 1], we can conclude that

$$\| f \|_{B^{0, \alpha}_{\infty, \infty}(\mathbb{R}^d)} = \left( \sum_{k \geq 1} \| \Delta_k f \|_{L^\infty(\mathbb{R}^d)}^q \right)^{\frac{1}{q}} + \| \Delta_0 f \|_{L^\infty(\mathbb{R}^d)} + \| \Delta_{-1} f \|_{L^\infty(\mathbb{R}^d)}.$$
be defined as follows
By changing a variable, one can conclude that
Proof.
where

Since \( q \alpha > 1 \), the series \( \sum_{k \geq 1} k^{-\alpha q} \) is convergent.

Since the space \( L^0 \text{bmo}(\mathbb{R}^d) \) is a shift-invariant space, \( L^0 \text{bmo}(\mathbb{R}^d) \) is stable through convolution with functions in \( L^1(\mathbb{R}^d) \).

Next, we shall introduce the space which is an important generalization of Campanato space. This space was firstly studied by S. Spanne [19], see [26] also for more details.

**Definition 2.6.** Let \( \varphi \) be a positive non-decreasing function. We define the Spanne space \( \mathcal{M}_\varphi(\mathbb{R}^d) \) of all integrable functions \( f \) such that \( \| f \|_{\mathcal{M}_\varphi(\mathbb{R}^d)} < \infty \), where its norm \( \| \cdot \|_{\mathcal{M}_\varphi(\mathbb{R}^d)} \) be defined as follows

\[
\| f \|_{\mathcal{M}_\varphi(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d, r \in [0, \frac{1}{2}], y \in \mathbb{R}^d} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| \, dy + \sup_{x \in \mathbb{R}^d, r \geq \frac{1}{2}} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| \, dy.
\]

Let us remark that the Spanne space \( \mathcal{M}_\varphi(\mathbb{R}^d) \) coincides with \( L^\infty(\mathbb{R}^d) \) when \( \varphi \) is a constant function.

2.2. In this subsection, we give some useful technical lemmas and propositions which are the cornerstone in our analysis.

**Lemma 2.3** ([21]). (i) Let \( \varphi \in \mathcal{S}(\mathbb{R}^2) \), then there holds

\[ \| \varphi * f \|_{L^\infty(\mathbb{R}^2)} \leq C \| f \|_{1,1}, \quad \forall f \in L^1_{\text{ul}}(\mathbb{R}^2), \]

where \( C \) is a positive constant independent of \( f \).

(ii) If \( m \geq 1 \), then

\[ \| f \|_{q, m \lambda} \leq (2m^2)^{\frac{1}{q}} \| f \|_{q, \lambda}, \quad \forall f \in L^q_{\text{ul}}(\mathbb{R}^2) \quad \text{and} \quad \forall \lambda > 0. \]

**Lemma 2.4.** Let \( 1 \leq q \leq \infty \), \( j \in \mathbb{N} \) and \( \| f \|_{q, 2^{-j}} < \infty \). Then there holds

\[ \| \Delta_j f \|_{L^\infty(\mathbb{R}^2)} \leq C 2^{\frac{2j}{q}} \| f \|_{q, 2^{-j}}, \quad (2.1) \]

where \( C \) is a positive constant independent of \( q, j \) and \( f \).

**Proof.** By changing a variable, one can conclude that

\[ |\Delta_j f(x)| = 2^{2j} \int_{\mathbb{R}^2} \varphi(2^j y) f(x - y) \, dy = \int_{\mathbb{R}^2} \varphi(y) f\left( \frac{2^j x - y}{2^j} \right) \, dy. \]

Let \( f_j(x) := f(x/2^j) \), then we have

\[ |\Delta_j f(x)| = \int_{\mathbb{R}^2} \varphi(y) f_j(2^j x - y) \, dy = |\Delta_0 f_j(2^j x)|. \]

By using Lemma 2.3, we know

\[ \| \Delta_0 f_j \|_{L^\infty(\mathbb{R}^2)} \leq C \| f_j \|_{1,1} = C \| f(\cdot / 2^j) \|_{1,1} \leq C \| f(\cdot / 2^j) \|_{q,1}. \]

This implies

\[ \| \Delta_j f \|_{L^\infty(\mathbb{R}^2)} \leq C \| f(\cdot / 2^j) \|_{q,1}. \quad (2.2) \]
Clearly,
\[
\left( \int_{|x-y| \leq 1} \left| f \left( \frac{y}{2^j} \right) \right|^q \, dy \right)^{\frac{1}{q}} = \left( 2^{2j} \int_{|2^{-j}x-y| \leq 2^{-j}} \left| f(y) \right|^q \, dy \right)^{\frac{1}{q}} \leq 2^{\frac{2j}{q}} \| f \|_{q, 2^{-j}}.
\]
Inserting this into (2.2) yields the desired result. \( \square \)

Remark 2.1. From Lemma 2.3 and Lemma 2.4 it follows that
\[
\| \Delta_j f \|_{L^\infty(\mathbb{R}^2)} \leq \begin{cases} 
C \cdot 2^{\frac{2j}{q}} \| f \|_{q, 1}, & \forall j \geq 0; \\
C \cdot \| f \|_{q, 1}, & \forall j < 0.
\end{cases} \quad (2.3)
\]

Lemma 2.5. Let \( \alpha \in [0, 1] \), and we may take \( \phi(p) = p^\alpha \) in Definition 1.4. Then \( \Phi(a) \) is given by
\[
\Phi(a) = (T\phi)(a) = \begin{cases} 
\left( \frac{\alpha}{a} \right)^\alpha \cdot (\log a)^\alpha, & \text{if } a > e^{a_{po}}; \\
\frac{1}{a^{\alpha}} \cdot \| a \|_{p_0 a_{po}}, & \text{if } 1 \leq a \leq e^{a_{po}}; \\
\| a \|_{p_0}, & \text{if } 0 \leq a < 1.
\end{cases}
\]

Proof. Simple calculations enable us to conclude the required result, and we omit the details. \( \square \)

Proposition 2.6. Let \( f \in L^\alpha \text{bmo}(\mathbb{R}^d) \) with \( \alpha \geq 0 \), and \( B = B_r(x) \) be a ball in \( \mathbb{R}^d \) with \( r \in [0, \frac{1}{\lambda}] \). Then, for all \( 1 < \lambda < \frac{1}{r} \), we have
\[
| \text{Avg}_B(f) - \text{Avg}_{\lambda B}(f) | \leq 2^d (-\log \lambda r)^{-\alpha} \| f \|_{L^\alpha \text{bmo}(\mathbb{R}^d)}
\]
\[
+ C \begin{cases} 
\left( - \log \lambda r \right)^{-\alpha} \cdot \| f \|_{L^\alpha \text{bmo}(\mathbb{R}^d)}, & \text{for } \alpha \in [0, 1]; \\
\left( \log (-\log r) - \log (-\log \lambda r) \right) \| f \|_{L^\alpha \text{bmo}(\mathbb{R}^d)}, & \text{for } \alpha = 1; \\
\left( - \log \lambda r \right)^{-\alpha} \cdot \| f \|_{L^\alpha \text{bmo}(\mathbb{R}^d)}, & \text{for } \alpha > 1,
\end{cases}
\]
where the positive constant \( C \) depends on \( \alpha \), independent of \( f \).

Proof. Since \( \lambda > 1 \), there exists a nonnegative integer \( k_0 \) such that \( 2^{k_0} \leq \lambda < 2^{k_0+1} \). By the triangle inequality, we have
\[
| \text{Avg}_B(f) - \text{Avg}_{\lambda B}(f) | \leq \sum_{k=0}^{k_0-1} | \text{Avg}_{2^k B}(f) - \text{Avg}_{2^{k+1} B}(f) | + | \text{Avg}_{2^{k_0} B}(f) - \text{Avg}_{\lambda B}(f) | . \quad (2.4)
\]
Using the doubling property of the Euclidean measure, one concludes that for \( k \in [0, k_0 - 1] \)
\[
| \text{Avg}_{2^k B}(f) - \text{Avg}_{2^{k+1} B}(f) | \leq \text{Avg}_{2^k B} \left| f - \text{Avg}_{2^{k+1} B}(f) \right| \leq \frac{|2^{k+1} B|}{|2^k B|} \cdot \frac{1}{2^{k+1} B} \int_{2^{k+1} B} \left| f(y) - \text{Avg}_{2^{k+1} B}(f) \right| \, dy \leq 2^d \left( - \log 2^{k+1} r \right)^{-\alpha} \| f \|_{L^\alpha \text{bmo}} \]
\[
= 2^d \left( - (k+1) - \log r \right)^{-\alpha} \| f \|_{L^\alpha \text{bmo}}. \quad (2.5)
\]
Similarly, we obtain
\[
| \text{Avg}_{2^{k_0} B}(f) - \text{Avg}_{\lambda B}(f) | \leq \text{Avg}_{2^{k_0} B} \left| f - \text{Avg}_{\lambda B}(f) \right| \leq \frac{|\lambda B|}{|2^{k_0} B|} \cdot \frac{1}{|\lambda B|} \int_{\lambda B} \left| f(y) - \text{Avg}_{\lambda B}(f) \right| \, dy \leq 2^d (- \log \lambda r)^{-\alpha} \| f \|_{L^\alpha \text{bmo}}. \quad (2.6)
\]
Inserting (2.5) and (2.6) into (2.4), we eventually get
\[
|\text{Avg}_B(f) - \text{Avg}_{\Lambda B}(f)| \leq \sum_{k=0}^{k_0-1} 2^d (- (k + 1) - \log r)^{-\alpha} \| f \|_{L^\alpha_{\text{bmo}}} + 2^d (- \log \lambda r)^{-\alpha} \| f \|_{L^\alpha_{\text{bmo}}}. \tag{2.7}
\]
We observe that for $\alpha \in [0, 1]$,
\[
\sum_{k=1}^{k_0-1} 2^d (- (k + 1) - \log r)^{-\alpha} \| f \|_{L^\alpha_{\text{bmo}}} = \sum_{k=1}^{k_0} 2^d (- k - \log r)^{-\alpha} \| f \|_{L^\alpha_{\text{bmo}}}
\leq \frac{2^d}{1 - \alpha} \left( ( - 1 - \log r)^{1-\alpha} - ( - k_0 - \log r)^{1-\alpha} \right) \| f \|_{L^\alpha_{\text{bmo}}}
\leq C \left( ( - \log r)^{1-\alpha} - ( - \log \lambda r)^{1-\alpha} \right) \| f \|_{L^\alpha_{\text{bmo}}}
\leq C (\log \lambda)^{1-\alpha} \| f \|_{L^\alpha_{\text{bmo}}},
\]
for $\alpha = 1$
\[
\sum_{k=1}^{k_0} 2^d (- k - \log r)^{-1} \| f \|_{L^\alpha_{\text{bmo}}} = 2^d \left( \log ( - 1 - \log r) - \log ( - k_0 - \log r) \right) \| f \|_{L^\alpha_{\text{bmo}}}
\leq C \left( \log ( - \log r) - \log ( - \log \lambda r) \right) \| f \|_{L^\alpha_{\text{bmo}}}
\leq C \log \left( 1 + \frac{\log \lambda}{- \log \lambda r} \right) \| f \|_{L^\alpha_{\text{bmo}}},
\]
and for $\alpha > 1$
\[
\sum_{k=1}^{k_0} 2^d (- k - \log r)^{-\alpha} \| f \|_{L^\alpha_{\text{bmo}}} \leq \frac{2^d}{\alpha - 1} \left( ( - k_0 - \log r)^{1-\alpha} - ( - 1 - \log r)^{1-\alpha} \right) \| f \|_{L^\alpha_{\text{bmo}}}
\leq C \left( ( - \log \lambda r)^{1-\alpha} - ( - \log r)^{1-\alpha} \right) \| f \|_{L^\alpha_{\text{bmo}}}
\leq C (\log \lambda)^{1-\alpha} \| f \|_{L^\alpha_{\text{bmo}}}
\leq C (\log \lambda)^{\alpha-1} ( - \log \lambda r)^{1-\alpha} \| f \|_{L^\alpha_{\text{bmo}}}
\leq C (\log \lambda)^{\alpha-1} ( - \log \lambda r)^{1-\alpha} \| f \|_{L^\alpha_{\text{bmo}}}.
\]
Plugging these estimates in (2.7), we eventually obtain the required result. \qed

**Remark 2.2.** From the above proof, we obviously see that
\[
|\text{Avg}_B(f) - \text{Avg}_{\Lambda B}(f)| \leq 2^d (- \log \lambda r)^{-\alpha} \| f \|_{L^\alpha_{\text{bmo}}(\mathbb{R}^d)}
\nonumber
+ C \| f \|_{L^\alpha_{\text{bmo}}(\mathbb{R}^d)} \begin{cases} (\log \lambda)^{1-\alpha}, & \text{for } \alpha \in [0, 1]; \\
\log \left( 1 + \frac{\log \lambda}{- \log \lambda r} \right), & \text{for } \alpha = 1; \\
(\log \lambda)^{\alpha-1} (\log \lambda r \cdot \log r)^{1-\alpha}, & \text{for } \alpha > 1. \end{cases}
\]

**Lemma 2.7.** Let $f \in L^\alpha_{\text{bmo}}(\mathbb{R}^d)$. There exists a positive constant $C$ independent of $f$ such that
\[
\sup_{1 \leq p < \infty} \frac{\| f \|_{L^p_{\text{bmo}}(\mathbb{R}^d)}}{p^{1-\alpha}} \leq C \| f \|_{L^\alpha_{\text{bmo}}(\mathbb{R}^d)}, \quad \text{for } \alpha \in [0, 1], \tag{2.8}
\]
and
\[
\sup_{1 \leq p < \infty} \frac{\| f \|_{L^p(\mathbb{R}^d)}}{\log(1 + p)} \leq C \| f \|_{L^\alpha_{\text{bmo}}(\mathbb{R}^d)}. \tag{2.9}
\]

**Proof.** Using the John-Nirenberg inequality, we easily find that
\[
\sup_{1 \leq p < \infty} \frac{\| f \|_{L^p_{\text{bmo}}(\mathbb{R}^d)}}{p^{1-\alpha}} \leq C \| f \|_{L^\alpha_{\text{bmo}}(\mathbb{R}^d)}.
\]

11
So we just need to show Lemma \[\text{2.7}\] for the case \(\alpha \in [0, 1]\).

For a fixed unit ball \(B_1(x) \subset \mathbb{R}^d\), performing the Vitali covering theorem, we conclude that there exists a collection \(\{B_{2^{-p}}(x_k)\}_k\) such that

1. \(B_1(x) \subset \bigcup_k B_{2^{-p}}(x_k)\);
2. The balls \(\{B_{2^{-(p+1)}}(x_k)\}_k\) are mutually disjoint;
3. For each \(k\), \(B_{2^{-(p+1)}}(x_k) \subset B_1(x)\).

Whence, we have

\[
\left( \frac{1}{m(B_1(x))} \int_{B_1(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \leq \left( \frac{1}{m(B_1(x))} \sum_k \int_{B_{2^{-p}}(x_k)} |f(y)|^p \, dy \right)^{\frac{1}{p}}
= \left( \sum_k \frac{m(B_{2^{-p}}(x_k))}{m(B_1(x))} \cdot \frac{1}{m(B_{2^{-p}}(x_k))} \int_{B_{2^{-p}}(x_k)} |f(y)|^p \, dy \right)^{\frac{1}{p}}
\leq 2^d \sup_{x \in \mathbb{R}^d} \left( \frac{1}{m(B_{2^{-p}}(x))} \int_{B_{2^{-p}}(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.
\]

We observe that

\[
\left( \frac{1}{m(B_{2^{-p}}(x))} \int_{B_{2^{-p}}(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}}
\leq \left( \frac{1}{m(B_{2^{-p}}(x))} \int_{B_{2^{-p}}(x)} |f(y) - \text{Avg}_{B_{2^{-1}}(x)}(f)|^p \, dy \right)^{\frac{1}{p}} + \frac{1}{m(B_{2^{-1}}(x))} \int_{B_{2^{-1}}(x)} |f(y)| \, dy
= I_1 + I_2.
\]

On one hand, it is clear that

\[I_2 \leq 2^d \int_{B_1(x)} |f(y)| \, dy \leq 2^d \|f\|_{bmo}.\]

On the other hand, by the triangle inequality, we can conclude that

\[I_1 \leq \left( \frac{1}{m(B_{2^{-p}}(x))} \int_{B_{2^{-p}}(x)} |f(y) - \text{Avg}_{B_{2^{-p}}(x)}(f)|^p \, dy \right)^{\frac{1}{p}}
+ \frac{1}{m(B_{2^{-1}}(x))} \int_{B_{2^{-1}}(x)} |f(y) - \text{Avg}_{B_{2^{-p}}(x)}(f)| \, dy
= I_{11} + I_{12}.
\]

According to the definition of \(L^\alpha\)bmo and Corollary \[\text{2.2}\], the term \(I_{11}\) can be controlled by

\[p^{-\alpha} \|f\|_{L^\alpha\text{bmo}} \leq Cp^{1-\alpha} \|f\|_{L^\alpha\text{bmo}}.\]

Next, performing Remark \[\text{2.2}\] with \(\lambda = 2^{p-1}\) and \(r = 2^{-p}\), we obtain

\[I_{12} \leq \frac{1}{m(B_{2^{-1}}(x))} \int_{B_{2^{-1}}(x)} |f(y) - \text{Avg}_{B_{2^{-1}}(x)}(f)| \, dy + \left| \text{Avg}_{B_{2^{-1}}(x)}(f) - \text{Avg}_{B_{2^{-p}}(x)}(f) \right|
\leq C \begin{cases} p^{1-\alpha} \|f\|_{L^\alpha\text{bmo}}, & \alpha \in [0, 1]; \\
\log(1 + p) \|f\|_{L^\text{bmo}}, & \alpha = 1. \end{cases}
\]

Collecting all these estimates, we eventually obtain

\[
\left( \int_{B_1(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} \leq \left( m(B_1(x)) \right)^{\frac{1}{p}} \left( \frac{1}{m(B_1(x))} \int_{B_1(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}}
\leq C \begin{cases} p^{1-\alpha} \|f\|_{L^\alpha\text{bmo}}, & \alpha \in [0, 1]; \\
\log(1 + p) \|f\|_{L^\text{bmo}}, & \alpha = 1. \end{cases}
\]
where, we have used the facts

\[ \psi(t, x) = x + \int_0^t u(\tau, \psi(\tau, x)) \, d\tau. \quad (2.10) \]

Moreover, if

\[ |x - y| < \min\{1, L_\alpha\}, \quad \text{where} \quad L_\alpha := \begin{cases} e^{-\exp \frac{2}{1+\alpha}(1+(1-\alpha)\log^{\alpha}(t))}, & \alpha \in [0,1], \\ e^{-\exp(e^{\log^{\alpha}(t)})}, & \alpha = 1, \end{cases} \]

and \( V_{\log^{\alpha}}(t) := \int_0^t \|u(\tau)\|_{\log^{\alpha}} \, d\tau \), then we have

\[ |\psi(t, x) - \psi(t, y)| \leq \begin{cases} e^{e^{-\exp((-1+\alpha)\log^{\alpha}(t))}} \exp((-1-\alpha)\log^{\alpha}(t)), & \alpha \in [0,1], \\ e^{e^{-\exp((-1+\alpha)\log^{\alpha}(t))}} \exp((-1-\alpha)\log^{\alpha}(t)), & \alpha = 1. \end{cases} \]

**Proposition 2.8.** Let \( u \) be a time-dependent vector field in \( L^1_{\text{loc}}(\mathbb{R}^+; \log^{\alpha}) \) with \( \alpha \in [0, 1] \). There exists a unique continuous map \( \psi \) from \( \mathbb{R}^+ \times \mathbb{R}^d \) to \( \mathbb{R}^d \) such that

\[ \psi(t, x) = x + \int_0^t u(\tau, \psi(\tau, x)) \, d\tau. \]

Moreover, if

\[ |x - y| < \min\{1, L_\alpha\}, \quad \text{where} \quad L_\alpha := \begin{cases} e^{-\exp \frac{2}{1+\alpha}(1+(1-\alpha)\log^{\alpha}(t))}, & \alpha \in [0,1], \\ e^{-\exp(e^{\log^{\alpha}(t)})}, & \alpha = 1, \end{cases} \]

and \( V_{\log^{\alpha}}(t) := \int_0^t \|u(\tau)\|_{\log^{\alpha}} \, d\tau \), then we have

\[ |\psi(t, x) - \psi(t, y)| \leq \begin{cases} e^{e^{-\exp((-1+\alpha)\log^{\alpha}(t))}} \exp((-1-\alpha)\log^{\alpha}(t)), & \alpha \in [0,1], \\ e^{e^{-\exp((-1+\alpha)\log^{\alpha}(t))}} \exp((-1-\alpha)\log^{\alpha}(t)), & \alpha = 1. \end{cases} \]

**Proof.** Let \( \delta(t) := |\psi(t, x) - \psi(t, y)| \). According to equations (2.10) and the vector field \( u \in L^1_{\text{loc}}(\mathbb{R}_+; \log^{\alpha}) \), one can conclude that

\[ \delta(t) \leq \delta(0) + \int_0^t \|u(\tau)\|_{\log^{\alpha}} \delta(\tau)(e - \log \delta(\tau))(\log(e - \log(\delta(\tau))))^\alpha \, d\tau \]

as long as \( \delta(\tau) < 1 \), for all \( \tau \in [0, t] \).

Let \( I(t) := \{ \tau \in [0, t] \mid F(\tau) \leq 1 \} \) with

\[ F(s) := \delta(0) + \int_0^s \|u(\tau)\|_{\log^{\alpha}} \delta(\tau)(e - \log \delta(\tau))(\log(e - \log(\delta(\tau))))^\alpha \, d\tau. \]

Our target is now to prove that \( I(t) = [0, t] \) when \( \delta(0) < \min\{1, L_\alpha\} \). Thanks to the continuity in time of the flow and the fact \( F(0) = \delta(0) < 1 \), we know that \( I(t) \) is a non-empty closed set. Thus, it remains for us to show that \( t_* = t \), where

\[ t_* = \max \{ \tau \in [0, t], [0, \tau] \subset I(t) \}. \]

In a similar fashion as (2.13), we infer that

\[ \delta(s) \leq \delta(0) + \int_0^s \|u(\tau)\|_{\log^{\alpha}} \delta(\tau)(e - \log \delta(\tau))(\log(e - \log(\delta(\tau))))^\alpha \, d\tau \]

for each \( s \in I_*(t) := [0, t_\ast] \).

From definition of \( F(s) \), a simple calculation yields

\[ F'(s) = \|u(s)\|_{\log^{\alpha}} \delta(s)(e - \log \delta(s))(\log(e - \log(\delta(s))))^\alpha \]

\[ \leq \|u(s)\|_{\log^{\alpha}} F(s)(e - \log F(s))(\log(e - \log(F(s))))^\alpha, \]

where we have used the facts \( s(e - \log s)(\log(e - \log s))^\alpha \) is a positive increasing function on \( [0, 1] \) and \( \delta(s) \leq F(s) \). This implies

\[ -H'_\alpha(F(s)) \leq \|u(s)\|_{\log^{\alpha}}, \]

where

\[ H_\alpha(\sigma) = \begin{cases} \frac{1}{1-\alpha}(\log(e - \log \sigma))^{1-\alpha}, & \text{if } \alpha \in [0,1], \\ \log(\log(e - \log \sigma)), & \text{if } \alpha = 1. \end{cases} \]
Accordingly, we have

$$H_\alpha(F(s)) \geq H_\alpha(F(0)) - \int_0^s \|u(\tau)\|_{\text{LogLog}^\alpha} \, d\tau = H_\alpha(\delta(0)) - \int_0^s \|u(\tau)\|_{\text{LogLog}^\alpha} \, d\tau. \quad (2.15)$$

Thanks to the representation formula of $H_\alpha$ with $\alpha \in [0, 1]$, we find that $H_1$ is bijective from $[0, 1]$ to $[0, \infty[$, and $H_\alpha$ also is bijective from $[0, 1]$ to $[\frac{1}{1-\alpha}, \infty[$ for $\alpha \in [0, 1]$. Thus, there exists a unique inverse function $H_\alpha^{-1}$ of $H_\alpha$ as follows:

$$H_\alpha^{-1}(\sigma) = \begin{cases} 
    e^{e^{-\exp((1-\alpha)\sigma)^{\frac{1}{1-\alpha}}}} & \text{if } \alpha \in [0, 1[, \\
    e^{e^{-\exp(\sigma)}} & \text{if } \alpha = 1.
\end{cases}$$

Next, we see that (2.11) means that for all $t \in [0, t_*]$

$$\int_0^t \|u(\tau)\|_{\text{LogLog}^\alpha} \, d\tau \leq H_\alpha(\delta(0)), \quad \text{for all } \alpha \in [0, 1].$$

For $\alpha \in [0, 1]$ and $0 < c < b < \infty$, we have that

$$H_\alpha^{-1}(b - c) = e^{e^{-\exp((1-\alpha)^{\frac{1}{1-\alpha}}(b-c)^{\frac{1}{1-\alpha}})}}$$

$$\leq e^{e^{-\exp(1-\alpha)^{\frac{1}{1-\alpha}}(2^{\frac{\alpha}{1-\alpha}}b^{\frac{1}{1-\alpha}}-c^{\frac{1}{1-\alpha}})}}$$

$$= e^{e^{-\exp(1-\alpha)^{\frac{1}{1-\alpha}}b^{\frac{1}{1-\alpha}}-c^{\frac{1}{1-\alpha}}}} \cdot \exp\left(-(1-\alpha)^{\frac{1}{1-\alpha}}c^{\frac{1}{1-\alpha}}\right)$$

in the second line of (2.16), we have used the following inequality

$$(b - c)^{\frac{1}{1-\alpha}} \geq 2^{\frac{\alpha}{1-\alpha}}b^{\frac{1}{1-\alpha}} - c^{\frac{1}{1-\alpha}}.$$ 

This together with (2.15) and (2.11) allows us to conclude that for all $s \in [0, t_*]$

$$\delta(s) \leq F(s) \leq e^{e^{-\exp(e^{-\log(\delta(0))})}^{\frac{1}{1-\alpha}}} \exp\left(-(1-\alpha)^{\frac{1}{1-\alpha}}f_0^s \|u(\tau)\|_{\text{LogLog}^\alpha} \, d\tau\right) < 1.$$ 

For $\alpha = 1$ and $0 < c < b < \infty$, we observe that

$$H_1^{-1}(b - c) = e^{e^{-\exp(e^{b-c})}} = e^{e^{-\exp(\exp(b-c))}} = e^{e^{-\exp(\exp(b-c))}}.$$ 

Combining this with (2.15) and (2.11), it follows that for all $s \in [0, t_*]$

$$\delta(s) \leq F(s) \leq e^{e^{-\exp(e^{-\log(\delta(0))})}^{\frac{-f_0^s \|u(\tau)\|_{\text{LogLog}^\alpha}}{d\tau}}} < 1.$$ 

Therefore, we can conclude that the desired result in term of the continuity argument.  

3. A priori estimates

This section is devoted to giving some useful a priori estimates which can be viewed as an preparation for proving our theorems.
3.1. In this subsection, our target is to establish a priori estimates for the vorticity equation in $Y_{ul}^0(\mathbb{R}^2)$. Let us begin by establishing the uniformly local $L^p$ estimate for transport equation.

**Proposition 3.1.** Let vector field $u \in L^1(\mathbb{R}^+; L^\infty(\mathbb{R}^d))$. Assume that $f(t, x)$ is a smooth solution of the following equation

$$
\begin{aligned}
\partial_t f + (u \cdot \nabla)f &= 0, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\
\operatorname{div} u &= 0, \\
f|_{t=0} &= f_0.
\end{aligned}
$$

Then there exists a positive constant $C$, independent of $p$ and $r$, such that

$$
\|f(t)\|_{p, r} \leq C \left( \frac{1}{r} \right)^{\frac{2}{d}} \left( r + \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^d)} \, d\tau \right)^{\frac{2}{d}} \|f_0\|_{p, r}.
$$

**Proof.** Let $\phi_\lambda(\cdot) = \phi(\frac{\cdot}{\lambda})$ for any positive number $\lambda$, where $\phi(\cdot)$ is a non-negative smooth function satisfying

$$
\phi(x) = \begin{cases} 
1, & y \in B_1(y) \\
0, & y \in B_2(y).
\end{cases}
$$

Obviously, we get from (3.1) that

$$
\partial_t \left( \phi_\lambda f + (u \cdot \nabla)(\phi_\lambda f) \right) = (u \cdot \nabla)\phi_\lambda f.
$$

Multiplying (3.3) by $|\phi_\lambda f|^{p-2}\phi_\lambda f$ and then integrating the resulting equation yield that

$$
\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|\phi_\lambda f(t, \cdot)|_{L^p(\mathbb{R}^d)}^p &= \int_{\mathbb{R}^d} ((u \cdot \nabla)\phi_\lambda f)|\phi_\lambda f|^{p-2}\phi_\lambda f \, dy \\
&= \int_{\mathbb{R}^d} \phi_{2\lambda}(u \cdot \nabla)\phi_\lambda f)|\phi_\lambda f|^{p-2}\phi_\lambda f \, dy \\
&\leq \|u\|_{L^\infty(\mathbb{R}^d)}\|\nabla\phi_\lambda f\|_{L^\infty(\mathbb{R}^d)} \|\phi_\lambda f(t, \cdot)|_{L^p(\mathbb{R}^d)}^{p-1} \|\phi_{2\lambda} f(t, \cdot)|_{L^p(\mathbb{R}^d)} \\
&\leq \frac{C}{\lambda^p} \|u\|_{L^\infty(\mathbb{R}^d)} \|\phi_\lambda f(t, \cdot)|_{L^p(\mathbb{R}^d)} \|\phi_{2\lambda} f(t, \cdot)|_{L^p(\mathbb{R}^d)}.
\end{aligned}
$$

From this, it follows that

$$
\frac{d}{dt} \|\phi_\lambda f(t, \cdot)|_{L^p(\mathbb{R}^d)} \leq \frac{C}{\lambda^p} \|u(t, \cdot)|_{L^\infty(\mathbb{R}^d)} \|\phi_{2\lambda} f(t, \cdot)|_{L^p(\mathbb{R}^d)}.
$$

Integrating the above inequality with respect to time $t$, we immediately obtain that

$$
\|\phi_\lambda f(t, \cdot)|_{L^p(\mathbb{R}^d)} \leq \|\phi_\lambda f_0|_{L^p(\mathbb{R}^d)} + \frac{1}{\lambda^p} \int_0^t \|u(\tau, \cdot)|_{L^\infty(\mathbb{R}^d)} \|f(\tau, \cdot)|_{L^p(\mathbb{R}^d)} \, d\tau.
$$

Taking the supremum of the above inequality over all $y \in \mathbb{R}^d$ leads to

$$
\begin{aligned}
\|f(t)|_{p, \lambda} &\leq \|f_0|_{p, 2\lambda} + \frac{1}{\lambda^p} \int_0^t \|u(\tau, \cdot)|_{L^\infty(\mathbb{R}^d)} \|f(\tau)|_{p, 4\lambda} \, d\tau \\
&\leq C \|f_0|_{p, \lambda} + C \frac{1}{\lambda^p} \int_0^t \|u(\tau, \cdot)|_{L^\infty(\mathbb{R}^d)} \|f(\tau)|_{p, \lambda} \, d\tau.
\end{aligned}
$$

By the Gronwall inequality and $\lambda \geq 1$, we have

$$
\|f(t)|_{p, r} \leq \|f(t)|_{p, \lambda} \leq \|f_0|_{p, \lambda} e^{C \frac{1}{\lambda^p} \int_0^t \|u(\tau, \cdot)|_{L^\infty(\mathbb{R}^d)} \, d\tau} \\
\leq C \lambda^\frac{2}{d} \|f_0|_{p, r} e^{C \frac{1}{\lambda^p} \int_0^t \|u(\tau, \cdot)|_{L^\infty(\mathbb{R}^d)} \, d\tau}.
$$

If, moreover, we choose a suitable $\lambda$ satisfying

$$
\lambda = \max \left\{ \frac{1}{r} \int_0^t \|u(\tau)|_{L^\infty(\mathbb{R}^d)} \, d\tau, \quad 1 \right\},
$$

then

$$
\|f(t)|_{p, r} \leq C \lambda^\frac{2}{d} \|f_0|_{p, r} e^{C \frac{1}{\lambda^p} \int_0^t \|u(\tau, \cdot)|_{L^\infty(\mathbb{R}^d)} \, d\tau}.
$$
we finally obtain
\[ \|f(t)\|_{p,r} \leq C \left( 1 + \frac{1}{r} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^d)} d\tau \right)^\frac{q}{p} \|f_0\|_{p,r}. \]

This completes the proof. \( \square \)

**Lemma 3.2.** Let \( \alpha \in [0,1] \) and \( \mathcal{R} \) is a Calderón-Zygmund operator. There holds, for any positive integer \( N \), that
\[
\| \dot{S}_{-N} \mathcal{R}((u \cdot \nabla)u) \|_{L^\infty(\mathbb{R}^2)} \leq C 2^{-N\alpha} \|u\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} \|\omega\|_{p,1}^{1-\alpha} + C 2^{-N} \|\omega\|_{p,1}^2, \quad \text{for} \ p > 2. \quad (3.6)
\]

**Proof.** According to Bony’s paraproduct decomposition, one writes
\[ (u \cdot \nabla)u = T_{u_1} \partial_1 u + T_{\partial_1 u} u_i + R(u_i, \partial_1 u). \]

For the paraproduct terms \( T_{u_1} \partial_1 u \) and \( T_{\partial_1 u} u_i \). By the Hölder inequality and the discrete Young inequality, we have
\[
\| \dot{S}_{-N} \mathcal{R}(T_{u_1} \partial_1 u) \|_{L^\infty(\mathbb{R}^2)} \leq C \sum_{k \leq -N} 2^k \| \dot{\Delta}_k (T_{u_1} u) \|_{L^\infty(\mathbb{R}^2)}
\leq C \sum_{k \leq -N} 2^k \sum_{k-q \leq 5} \| \dot{S}_{q-1} u \|_{L^\infty(\mathbb{R}^2)} \| \dot{\Delta}_q u \|_{L^\infty(\mathbb{R}^2)}
\leq C \sum_{k \leq -N} 2^{k\alpha} \sum_{k-q \leq 5} 2^{(k-q)(1-\alpha)} \| u \|_{L^\infty(\mathbb{R}^2)} 2^{-q\alpha} \| \dot{\Delta}_q \omega \|_{L^\infty(\mathbb{R}^2)}
\leq C 2^{-N\alpha} \| u \|_{L^\infty(\mathbb{R}^2)} \| \dot{S}_{-N+5} \omega \|_{\dot{B}^{0,\alpha}_{\infty,\infty}(\mathbb{R}^2)}.
\]

Resorting to the interpolation theorem and [28], we have
\[
\| \dot{S}_{-N+5} \omega \|_{\dot{B}^{0,\alpha}_{\infty,\infty}(\mathbb{R}^2)} \leq C \| \omega \|_{\dot{B}^{1,1}_{\infty,\infty}(\mathbb{R}^2)} \| \dot{S}_{-N+5} \omega \|_{L^\infty(\mathbb{R}^2)}^{1-\alpha} \| \dot{S}_{-N+5} \omega \|_{L^\infty(\mathbb{R}^2)}^{\alpha}
\leq C \| u \|_{L^\infty(\mathbb{R}^2)}^{\alpha} \| \omega \|_{L^\infty(\mathbb{R}^2)}^{1-\alpha}.
\]

Inserting (3.8) into (3.7), we get
\[
\| \dot{S}_{-N} \mathcal{R}(T_{u_1} \partial_1 u) \|_{L^\infty(\mathbb{R}^2)} \leq C 2^{-N\alpha} \| u \|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} \|\omega\|_{p,1}^{1-\alpha}.
\]

Similarly, we obtain
\[
\| \dot{S}_{-N} \mathcal{R}(T_{\partial_1 u} u_i) \|_{L^\infty(\mathbb{R}^2)} \leq C 2^{-N\alpha} \| u \|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} \|\omega\|_{p,1}^{1-\alpha}.
\]

It remains for us to deal with the remainder term \( R(u_i, \partial_1 u) \). Thanks to the low-high decomposition technique, we decompose it into two parts as follows:
\[ R(u_i, \partial_1 u) = \sum_{k < 0} \dot{\Delta}_k u_i \dot{\Delta}_k \partial_1 u + \sum_{k \geq 0} \dot{\Delta}_k u_i \dot{\Delta}_k \partial_1 u := R^5 + R^6. \]

By using the support property of \( R^6(u_i, \partial_1 u) \), we see that the quantity \( \| \dot{S}_{-N} \mathcal{R}(R^6) \|_{L^\infty(\mathbb{R}^2)} \) can be controlled by
\[
C \sum_{j \leq -N} \| \dot{\Delta}_j R^6 \|_{L^\infty(\mathbb{R}^2)} \leq C \sum_{j \leq -N} \sum_{j-5 \leq k < 0} \| \dot{\Delta}_j u_i \|_{L^\infty(\mathbb{R}^2)} \| \dot{\Delta}_k u \|_{L^\infty(\mathbb{R}^2)}
\leq C \sum_{j \leq -N} \sum_{j-5 \leq k \leq 0} 2^{(j-k)(1-\alpha)} \| u \|_{L^\infty(\mathbb{R}^2)} 2^{-k\alpha} \| \dot{\Delta}_k \omega \|_{L^\infty(\mathbb{R}^2)}
\leq C 2^{-N\alpha} \| u \|_{L^\infty(\mathbb{R}^2)} \| \dot{\Delta}_0 \omega \|_{\dot{B}^{0,\alpha}_{\infty,\infty}(\mathbb{R}^2)}. \quad (3.9)
\]
By the same argument as in proof (8.8), we can infer that
\[
\|\Delta_0 \omega\|_{\dot{B}^{-\alpha}_{\infty,\infty}(\mathbb{R}^2)} \leq C \|u\|_{L_\infty(\mathbb{R}^2)}^\alpha \|\omega\|_{p,1}^{1-\alpha}.
\] (3.10)
Plugging (3.10) in (3.9), we obtain
\[
\|\dot{S}_{-N} \mathcal{R}(R^2)\|_{L_\infty(\mathbb{R}^2)} \leq C 2^{-N} \|u\|_{L_\infty(\mathbb{R}^2)}^{1+\alpha} \|\omega\|_{p,1}^{1-\alpha}.
\]
Finally, since \( p > 2 \), the last term \( \|\dot{S}_{-N} \mathcal{R}(R^2)\|_{L_\infty(\mathbb{R}^2)} \) can be bounded by
\[
C \sum_{j \leq -N} \|\hat{\Delta}_j R^2\|_{L_\infty(\mathbb{R}^2)} \leq C \sum_{j \leq -N} 2^j \sum_{k \geq 0} \|\hat{\Delta}_k u_t\|_{L_\infty(\mathbb{R}^2)} \|\hat{\Delta}_k u\|_{L_\infty(\mathbb{R}^2)}
\leq C 2^{-N} \sum_{k \geq 0} 2^{-2k} \|\hat{\Delta}_k \omega\|_{L_\infty(\mathbb{R}^2)}^2
\leq C 2^{-N} \sum_{k \geq 0} 2^{-2k} 2^{\frac{4 \alpha}{p}} \|\omega\|_{p,1}^2 \leq C 2^{-N} \|\omega\|_{p,1}^2.
\]
Collecting all these estimates yields the desired result. □

**Proposition 3.3.** Let \( u_0 \in L_\infty(\mathbb{R}^2) \) and \( \omega_0 \in Y_u^0(\mathbb{R}^2) \) with \( \Theta \in \mathcal{A}_1 \). Assume that \( u \) is a smooth solution of (14). Then we have
\[
\|u(t)\|_{L_\infty(\mathbb{R}^2)} + \|\omega(t)\|_{Y_u^0(\mathbb{R}^2)} \leq C(t),
\]
where the positive smooth function \( C(t) \) depends on the initial data.

**Proof.** Thanks to the low-high decomposition technique, one can write
\[
\|u(t)\|_{L_\infty(\mathbb{R}^2)} \leq \|\dot{S}_{-N} u\|_{L_\infty(\mathbb{R}^2)} + \sum_{q \geq -N} \|\hat{\Delta}_q u\|_{L_\infty(\mathbb{R}^2)},
\] (3.11)
where \( N \) is a positive integer to be specified later.

Let us recall that
\[
\partial_t u + \mathcal{P}((u \cdot \nabla) u) = 0.
\]
Performing the low frequency cut-off operator \( \dot{S}_{-N} \) to the above equality, we get
\[
\partial_t \dot{S}_{-N} u + \dot{S}_{-N} \mathcal{P}((u \cdot \nabla) u) = 0.
\] (3.12)
Integrating (3.12) in time \( t \) and using Lemma 3.2, one has that for \( p > 2 \)
\[
\|S_{-N} u(t)\|_{L_\infty(\mathbb{R}^2)} \leq \|S_{-N} u_0\|_{L_\infty(\mathbb{R}^2)} + \int_0^t \|\dot{S}_{-N} \mathcal{P}((u \cdot \nabla) u)(\tau)\|_{L_\infty(\mathbb{R}^2)} d\tau
\leq C + C 2^{-N} \int_0^t \left( \|u(\tau)\|_{L_\infty(\mathbb{R}^2)}^{1+\alpha} \|\omega(\tau)\|_{p,1}^{1-\alpha} + \|\omega(\tau)\|_{p,1}^2 \right) d\tau.
\]
For the high frequency part, by resorting to (2.3), the interpolation theorem and the Hölder inequality, we easily find that for \( \alpha \in [0,1] \) and \( p > 2 \)
\[
\sum_{q \geq -N} \|\hat{\Delta}_q u\|_{L_\infty(\mathbb{R}^2)} \leq \sum_{-N \leq q \leq -1} \|\hat{\Delta}_q u\|_{L_\infty(\mathbb{R}^2)} + \sum_{q \geq 0} \|\hat{\Delta}_q u\|_{L_\infty(\mathbb{R}^2)}
\leq C \sum_{-N \leq q \leq -1} 2^{-q \alpha} \|\hat{\Delta}_q u\|_{L_\infty(\mathbb{R}^2)}^{1-\alpha} \|\hat{\Delta}_q \omega\|_{L_\infty(\mathbb{R}^2)}^\alpha + C \sum_{q \geq 0} 2^{-q (1-\frac{2}{p})} \|\omega\|_{p,1}
\leq C 2^{-N} \|u(t)\|_{L_\infty(\mathbb{R}^2)}^{1-\alpha} \|\omega(t)\|_{p,1}^{1-\alpha} + C \|\omega(t)\|_{p,1}
\leq \frac{1}{2} \|u(t)\|_{L_\infty(\mathbb{R}^2)} + C 2^{-N} \|\omega(t)\|_{p,1}.
\]
Combining these estimates and then plugging the resulting estimate in (3.11), we immediately obtain that

\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C 2^{-N\alpha} \int_0^t \left( \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} \|\omega(\tau)\|_{L^p(\mathbb{R}^2)}^{1-\alpha} + \|\omega(\tau)\|_{L^p(\mathbb{R}^2)}^2 \right) d\tau + C 2^N \|\omega\|_{p,1} + C
\]

\[
\leq C 2^{-N\alpha} \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau + Ct 2^{-N\alpha} \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1}^2 + C 2^N \|\omega(t)\|_{p,1} + C.
\]

Taking a suitable integer \( N \) such that

\[
2^N \sim \left( \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau \right)^{\frac{1}{1+\alpha}} \left( 1 + \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1} \right)^{\frac{1}{p}} + 1.
\]

From this, it follows that

\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C \left( \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1} \right)^{\frac{1}{1+\alpha}} \left( \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau \right)^{\frac{1}{1+\alpha}} + Ct 2^{-N\alpha} \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1}^2 + C.
\]

Furthermore, we have

\[
\|u(t)\|_{L^{1+\alpha}(\mathbb{R}^2)} \leq C \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau + Ct^{1+\alpha} \left( \sup_{\tau \in [0,t]} \|\omega(\tau)\|_{p,1} \right)^{2(1+\alpha)} + C.
\]

(3.13)

Next, applying Proposition 3.1 to the vorticity equation, we can conclude that for any \( p \geq 1 \)

\[
\|\omega(t)\|_{p,1} \leq C \|\omega_0\|_{p,1} \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)} \right)^{\frac{2}{p}}.
\]

(3.14)

Inserting (3.14) into (3.13) leads to

\[
\|u(t)\|_{L^{1+\alpha}(\mathbb{R}^2)} \leq C \|\omega_0\|_{p,1} \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)} \right)^{\frac{2}{p}} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau
\]

\[
+ C t^{1+\alpha} \|\omega_0\|_{p,1} \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)} \right)^{\frac{2(1+\alpha)}{p}} + C
\]

\[
\leq C \|\omega_0\|_{p,1} \left( 1 + t^{2(1+\alpha)p} \right) \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)}^{2(1+\alpha)} \right)^{\frac{2}{p}} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau
\]

\[
+ C t^{1+\alpha} \|\omega_0\|_{p,1} \left( 1 + t^\frac{4\alpha}{p} \right) \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)}^{4(1+\alpha)} \right)^{\frac{1}{p}} + C
\]

\[
\leq C (1 + t^2) \|\omega_0\|_{p,1} \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)} \right)^{\frac{2}{p}} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)}^{1+\alpha} d\tau
\]

\[
+ C (1 + t^4) \|\omega_0\|_{p,1} \left( 1 + \|u\|_{L^1_t L^\infty(\mathbb{R}^2)} \right)^{\frac{2(1+\alpha)}{p}} + C.
\]
Thus, the quantity $\|u(t)\|_{L^{2(1+\alpha)}(\mathbb{R}^2)}$ can be bounded by
\[
C(1 + t^2) \sup_{2 < q < \infty} \frac{\|\omega(t)\|_{q,1}}{\Theta(q)} \cdot \Theta(p) \left(1 + \|u\|_{L^{1+\alpha}_t L^\infty(\mathbb{R}^2)}\right)^{\frac{2}{p}} \int_0^t \|u(\tau)\|_{L^{2(1+\alpha)}(\mathbb{R}^2)} \, d\tau \\
+ C(1 + t^4) \left( \sup_{2 < q < \infty} \frac{\|\omega(t)\|_{q,1}}{\Theta(q)} \cdot \left( \frac{p}{2} \right)^{2(1+\alpha)} \cdot \Theta(p) \right) \left(1 + \|u\|_{L^{1+\alpha}_t L^\infty(\mathbb{R}^2)}\right)^{\frac{4(1+\alpha)}{p}} + C
\]
\[
\leq C(1 + t^2) \sup_{2 < q < \infty} \frac{\|\omega(t)\|_{q,1}}{\Theta(q)} \cdot \left( \frac{p}{2} \right)^{2(1+\alpha)} \cdot \Theta(p) \left(1 + \|u\|_{L^{1+\alpha}_t L^\infty(\mathbb{R}^2)}\right)^{\frac{4(1+\alpha)}{p}} + C
\]
(3.15)
where we have used the relation that $\Theta(p) \leq C \Theta \left( \frac{p}{2} \right)$ for all $p > 2$ because $\Theta(\cdot)$ satisfies the $\Delta_2$ condition.

Taking the infimum of (3.15) over all $p \in [4, \infty[$, one gets
\[
\|u(t)\|_{L^{2(1+\alpha)}(\mathbb{R}^2)} \leq C(1 + t^2) \|\omega(t)\|_{Y_0^\Theta(\mathbb{R}^2)} \cdot \Phi \left( 1 + \int_0^t \|u(\tau)\|_{L^{1+\alpha}(\mathbb{R}^2)} \, d\tau \right) \cdot \int_0^t \|u(\tau)\|_{L^{2(1+\alpha)}(\mathbb{R}^2)} \, d\tau \\
+ C(1 + t^4) \|\omega(t)\|_{Y_0^\Theta(\mathbb{R}^2)} \cdot \Phi(2(1+\alpha)) \left(1 + \int_0^t \|u(\tau)\|_{L^{1+\alpha}(\mathbb{R}^2)} \, d\tau \right) + C.
\]

Since $\Theta \in A_1$, the admissible condition guarantees that $\int_1^\infty \frac{1}{s \Phi(s)} \, ds = \infty$. By the Hölder inequality, we have
\[
\int_1^\infty \frac{1}{s \Phi(t)} \, dt \leq \left( \int_1^\infty t^{-\frac{2(1+\alpha)}{p}} \, dt \right)^{\frac{1}{2(1+\alpha)}} \left( \int_1^\infty \Phi^{-2(1+\alpha)}(t) \, dt \right)^{\frac{1}{2(1+\alpha)}},
\]
which implies that $\int_1^\infty \Phi^{-2(1+\alpha)}(t) \, dt = \infty$. Moreover, by taking advantage of the Osgood theorem, we end up with
\[
\int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)} \, d\tau \leq C(t).
\]
(3.16)
Plugging (3.16) in (3.14) enables us to infer that $\|\omega(t)\|_{Y_0^\Theta(\mathbb{R}^2)} \leq C(t)$. Taking $\alpha = \frac{1}{2}$ in (3.13), we easily find that
\[
\|u\|_{L^2_t L^{\infty}(\mathbb{R}^2)} \leq C(t) \|u\|_{L^2_t L^{\infty}(\mathbb{R}^2)} \int_0^t \|u(\tau)\|_{L^\infty(\mathbb{R}^2)} \, d\tau + C(t) \leq \frac{1}{2} \|u\|_{L^\infty_t L^{\infty}(\mathbb{R}^2)}^2 + C(t).
\]
This implies that $\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C(t)$ and then the proof is achieved.

Now, we turn to study the regularity of vorticity in the Spanne space $\mathcal{M}_\varphi$. More precisely:

**Proposition 3.4.** Let $\alpha \in [0, 1/2]$ and $u_0 \in L^\infty(\mathbb{R}^2)$ and $\omega_0 \in Y_0^\Theta(\mathbb{R}^2) \cap \mathcal{M}_\varphi(\mathbb{R}^2)$ with $\Theta \in A_1$ and $\varphi(r) = \log^\alpha(e + \log r)$. Assume that $u$ is a smooth solution of (3.14). Then we have
\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} + \|\omega(t)\|_{Y_0^\Theta(\mathbb{R}^2)} \leq C(t),
\]
where the positive smooth function $C(t)$ depends on the initial data and $\alpha$.

Before proving it, we first review some properties of flow maps. Assume that $\psi \in \mathcal{L}$ the group of all bi-Lipschitz homeomorphism of $\mathbb{R}^d$ is measure preserving, we know that $\psi(B_r(x))$ is a bounded open set and $m(\psi(B_r(x))) = m(B_r(x))$. By using the Whitney covering Theorem, one can conclude that there exists a bounded collection $\{O_k\}_k$
(A) \(\{2O_k\}_k\) is a bounded covering:

\[
\psi(B_r(x)) \subset \bigcup_k 2O_k; \tag{3.18}
\]

(B) The balls \(O_k\) are pairwise disjoint and for each \(k\), \(O_k \subset \psi(B_r(x))\);

(C) The Whitney property is verified:

\[
r_{O_k} \approx d(O_k, \psi(B_r(x))^c). \tag{3.19}
\]

Clearly, the measure preserving property ensures that \(m(O_k) \le m(B_r(x))\) for all \(k\), which implies that \(r_{O_k} \le r_B\) for all \(k\). Moreover, it entails the following useful lemma.

**Lemma 3.5.** Let \(\alpha \in [0,1]\), \(\Theta_k := \sum e^{-(k+1)r} \le r \le e^{-k} \ m(O_j)\) for any \(k \ge 1\) and

\[
N_\alpha := \begin{cases} 
\exp 2^{r\alpha} \left( 1 + \left( (1 - \alpha)V_{\log \log} (t) \right)^{1/\alpha} \right) - e + \log r, & \alpha \in [0,1], \\
\exp \left( e^{V_{\log \log} (t)} \right) - e + \log r, & \alpha = 1.
\end{cases}
\]

Then there exists a universal constant \(C > 0\) such that for each \(k \ge \max \{1, N_\alpha\}\)

\[
\Theta_k \le \begin{cases} 
Cr e^{-(e+k+\log r)^2} \exp \left( (1-\alpha)^{1/\alpha} \left( V_{\log \log} (t) \right)^{1/\alpha} \right), & \alpha \in [0,1], \\
Cr e^{-(e+k+\log r)} \exp \left( -V_{\log \log} (t) \right), & \alpha = 1.
\end{cases} \tag{3.20}
\]

**Proof.** Here we just give the proof of estimate \((3.20)\) for \(\alpha \in [0,1]\), because the proof for \(\alpha = 1\) is similar. Thanks to the preservation of Lebesgue measure by \(\psi(x,t)\), we find

\[
m \left( \{y \in \psi(B) : d(y, \psi(B)^c) \le Ce^{-kr} \} \right) = m \left( \{x \in B : d(\psi(x), \psi(B)^c) \le Ce^{-kr} \} \right),
\]

This together with the fact \(\psi(B)^c = \psi(B^c)\) ensures

\[
\Theta_k \le m \left( \{x \in B : d(\psi(x), \psi(B^c)) \le Ce^{-kr} \} \right) := D_k.
\]

Since \(\psi(\partial B)\) is the frontier of \(\psi(B)\) and \(d(\psi(x), \psi(B^c)) = d(\psi(x), \partial \psi(B))\), then we have

\[
D_k \subset \{x \in B : \exists y \in \partial B \text{ with } |\psi(x) - \psi(y)| \le Ce^{-kr}\}.
\]

The condition on \(k\) allows us to use Proposition \(2.8\) to get

\[
D_k \subset \{x \in B : d(x, \partial B) \le C e^{-(e+k+\log r)^2} \exp \left( (1-\alpha)^{1/\alpha} \left( V_{\log \log} (t) \right)^{1/\alpha} \right) \},
\]

which implies the desired estimate \((3.20)\) for \(\alpha \in [0,1]\).

**Proof of Proposition 3.4.** From Proposition 3.3, we know the following estimate

\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} + \|\omega(t)\|_{Y^q_{ul}(\mathbb{R}^2)} \le C(t) \quad \text{for each } t > 0. \tag{3.21}
\]

This together with Lemma 2.3 allows us to conclude that for \(r > 1\)

\[
\frac{1}{m(B_r(x))} \int_{B_r(x)} |\omega(y,t)| \, dy \le C \frac{1}{m(B_1(x))} \|\omega(t)\|_{1,1} \le C \|\omega(t)\|_{Y^q_{ul}(\mathbb{R}^2)} \le C(t). \tag{3.22}
\]

So, we just show the case where \(r \in [0,1/e]\). Since \(\text{div } u = 0\), we have

\[
\frac{1}{m(B_r(x))} \int_{B_r(x)} |\omega_0(\psi(y))| \, dy = \frac{1}{m(\psi(B_r(x)))} \int_{\psi(B_r(x))} |\omega_0| \, dy.
\]
Moreover, applying the Whitney covering theorem, we find that
\[
\frac{1}{m(\psi(B_r(x)))} \int_{\psi(B_r(x))} |\omega_0| \, dy \leq \frac{1}{m(\psi(B_r(x)))} \sum_k m(2O_k) \frac{1}{m(2O_k)} \int_{2O_k} |\omega_0(y)| \, dy
\]
\[
\leq C \frac{1}{m(\psi(B_r(x)))} \sum_k \Theta_k \frac{1}{(e^{-kr})^2} \omega_0 \|_{1, e^{-kr}} \tag{3.23}
\]
\[
\leq C \frac{1}{m(\psi(B_r(x)))} \sum_k \Theta_k \log^\alpha (e + k - \log r) \omega_0 \|_{M_\varphi}.
\]
We split the series into two parts as follows:
\[
\frac{1}{m(\psi(B_r(x)))} \sum_k \Theta_k \log^\alpha (e + k - \log r) = \frac{1}{m(\psi(B_r(x)))} \sum_{k=0}^N \Theta_k \log^\alpha (e + k - \log r)
\]
\[
+ \frac{1}{m(\psi(B_r(x)))} \sum_{k>N} \Theta_k \log^\alpha (e + k - \log r),
\]
where the positive integer \(N\) to be fixed later.

**Step 1:** We first consider the case where \(r = r_\varphi := e^{-\exp 2\frac{\mu}{\alpha}} \left(1 + ((1 - \alpha)V_{\log^\alpha(t)}(1 - \alpha)^{\frac{1}{\alpha}})\right) \leq \log r. \tag{3.24}
\]
Denote by \(N \geq N_\alpha\) a undetermined constant. For \(k \leq N\), a simple calculation yields
\[
\frac{1}{\log^\alpha(e + k - \log r)} \frac{1}{m(\psi(B_r(x)))} \sum_{k=0}^N \Theta_k \log^\alpha (e + k - \log r)
\]
\[
\leq \frac{\log^\alpha (e + N - \log r)}{\log^\alpha(e - \log r)} \frac{1}{m(\psi(B_r(x)))} \sum_{k=0}^N \Theta_k \leq \frac{\log^\alpha \log^\alpha (e + N - \log r)}{\log^\alpha(e - \log r)}.
\tag{3.25}
\]
As for \(k > N\), Lemma 3.5 and inequality 3.24 allow us to obtain that
\[
\frac{1}{\log^\alpha(e - \log r)} \frac{1}{m(\psi(B_r(x)))} \sum_{k>N} \Theta_k \log^\alpha (e + k - \log r)
\]
\[
\leq C \frac{1}{r} \sum_{k>N} e^{-(e + k - \log r)^2 \frac{\alpha}{1 - \alpha}} \exp \left(- (1 - \alpha)^{\frac{1}{1 - \alpha}} \left(f_0^i \|u(\tau)\|_{\log^\alpha} \, d\tau \right)^{\frac{1}{1 - \alpha}} \right) \log^\alpha (e + k - \log r)
\]
\[
\leq C \frac{1}{r} e^{-(e + N - \log r)^2 \frac{\alpha}{1 - \alpha}} \exp \left(- (1 - \alpha)^{\frac{1}{1 - \alpha}} \left(f_0^i \|u(\tau)\|_{\log^\alpha} \, d\tau \right)^{\frac{1}{1 - \alpha}} \right) \log^\alpha (e + N - \log r)
\]
This together with 3.25 enables us to conclude that
\[
\frac{1}{\log^\alpha(e - \log r)} \frac{1}{m(\psi(B_r(x)))} \int_{\psi(B_r(x))} |\omega_0| \, dy
\]
\[
\leq C \left( 1 + \frac{1}{r} e^{-(e + N - \log r)^2 \frac{\alpha}{1 - \alpha}} \exp \left(- (1 - \alpha)^{\frac{1}{1 - \alpha}} \left(f_0^i \|u(\tau)\|_{\log^\alpha} \, d\tau \right)^{\frac{1}{1 - \alpha}} \right) \right) \log^\alpha (e + N - \log r)
\]
\[
\leq C \left( 1 + \frac{1}{r} e^{-(e + N - \log r)^2 \frac{\alpha}{1 - \alpha}} \exp \left(- (1 - \alpha)^{\frac{1}{1 - \alpha}} \left(f_0^i \|u(\tau)\|_{\log^\alpha} \, d\tau \right)^{\frac{1}{1 - \alpha}} \right) \right) \log^\alpha (e + N - \log r)
\]
Taking \(N = (e - \log r)^{\frac{1}{1 - \alpha}} \exp \left(2 \frac{\alpha}{1 - \alpha} \left((1 - \alpha)V_{\log^\alpha(t)}(1 - \alpha)^{\frac{1}{\alpha}}\right)\right) - e + \log r\), we easily find that
\[
\frac{1}{r} e^{-(e + N - \log r)^2 \frac{\alpha}{1 - \alpha}} \exp \left(- (1 - \alpha)^{\frac{1}{1 - \alpha}} \left(f_0^i \|u(\tau)\|_{\log^\alpha} \, d\tau \right)^{\frac{1}{1 - \alpha}} \right) \leq C.
\]
Consequently, we have by (3.24)
\[
\frac{1}{\log^\alpha(e - \log r)} \frac{1}{m(\psi(B_r(x)))} \int_{\psi(B_r(x))} |\omega_0| \, dy \leq C. \tag{3.26}
\]

**Step 2:** We are now in a position to show the case where \( r_\varphi \leq r < 1 \). By Lemma 2.3 and estimate (3.26), we get
\[
\frac{1}{\log^\alpha(e - \log r)} \frac{1}{m(\psi(B_r(x)))} \int_{\psi(B_r(x))} |\omega_0| \, dy \leq C \left( \frac{1}{\log^\alpha(e - \log r)} \frac{1}{m(B_{r_\varphi}(x))} \right) \|\omega_0 \circ \psi\|_{1, r_\varphi}
\leq C \frac{\log^\alpha(e - \log r)}{\log^\alpha(e - \log r)}
\leq C (1 + V_{\log^\alpha}(t))^{\frac{\alpha}{r}}.
\tag{3.27}
\]

Since \( \alpha \in ]0, 1/2] \), we finally get that for \( \varphi(r) = \log^\alpha(e - \log r) \)
\[
\|\omega(t)\|_{L^\varphi(\mathbb{R}^2)} \leq C (1 + V_{\log^\alpha}(t))^{\frac{\alpha}{r}} \leq C \left( 1 + \int_0^t \|u(\tau)\|_{L^{\varphi}(t)} \, d\tau \right).
\]

Thus, our main task is now to show that
\[
\|u\|_{L^{\omega(t)}} \leq C \|u\|_{L^\varphi(\mathbb{R}^2)} + C \|\omega(t)\|_{L^\varphi(\mathbb{R}^2)} \quad \text{with} \quad \varphi(r) = \log^\alpha(e - \log r). \tag{3.28}
\]

Recall from [4, Proposition 2.111] that
\[
\frac{1}{C} \|u\|_{L^{\omega(t)}} \leq \sup_{j \geq 1} \frac{||S_j \nabla u||_{L^\varphi(\mathbb{R}^2)}}{j \log^\alpha(1 + j)} \leq C \|u\|_{L^{\omega(t)}}. \tag{3.29}
\]

By the Bernstein inequality and Lemma 2.4, we see that for \( \varphi(r) = \log^\alpha(e - \log r) \)
\[
\|S_j \nabla u\|_{L^\varphi(\mathbb{R}^2)} \leq \|S_1 \nabla u\|_{L^\varphi(\mathbb{R}^2)} + \sum_{1 \leq k < j} \|\Delta_k \nabla u\|_{L^\varphi(\mathbb{R}^2)}
\leq C \|u\|_{L^\varphi(\mathbb{R}^2)} + C \sum_{1 \leq k < j} 2^j \|\omega\|_{L^\varphi(\mathbb{R}^2)}
\leq C \|u\|_{L^\varphi(\mathbb{R}^2)} + C \sum_{1 \leq k < j} 2^{2j} \|\omega\|_{L^\varphi(\mathbb{R}^2)}
\leq C \|u\|_{L^\varphi(\mathbb{R}^2)} + C \int_0^t \|u(\tau)\|_{L^{\varphi(\mathbb{R}^2)}} \, d\tau.
\]

This implies claim (3.28) and then we finish the proof of Proposition 3.4. \( \square \)

**3.2.** The target of this subsection is to show an estimate with a logarithmic loss of regularity in the borderline space \( L^\alpha bmo \) by developing the classical analysis tools such as John-Nirenberg inequality.

**Proposition 3.6.** Let \( u_0 \in L^\infty(\mathbb{R}^2) \) and its vorticity \( \omega_0 \in L^\alpha bmo(\mathbb{R}^2) \) with \( \alpha \in [0, 1] \). Assume that \( u \) is a smooth solution of (1). Then we have
\[
\|u(t)\|_{L^\varphi(\mathbb{R}^2)} + \|\omega(t)\|_{L^\varphi bmo(\mathbb{R}^2)} \leq C_2(t) \left( 1 + \|\omega_0\|_{L^\varphi bmo(\mathbb{R}^2)} \right) \quad \text{for} \quad \alpha \in [0, 1], \tag{3.30}
\]
and
\[
\|u(t)\|_{L^\varphi(\mathbb{R}^2)} + \|\omega(t)\|_{L^\varphi bmo(\mathbb{R}^2)} \leq C_2(t) \left( 1 + \|\omega_0\|_{bmo(\mathbb{R}^2)} \right). \tag{3.31}
\]

Here, \( C_1 \) and \( C_2 \) are two positive functions dependent of the initial data.

**Proof.** According to Lemma 2.4, we know that \( L^\alpha bmo(\mathbb{R}^2) \) continuously embeds \( Y_0^{\Theta}(\mathbb{R}^2) \) with
\[
\Theta(p) = \begin{cases} p^{1-\alpha}, & \alpha \in [0, 1]; \\ \log(1 + p), & \alpha = 1. \end{cases}
\]
From this, it is easy to verify that $\Theta(p)$ belongs to the class $A_1$. Thus, we immediately obtain by using Proposition 3.3 that
\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} + \|\omega(t)\|_{Y^\infty_{A_1}(\mathbb{R}^2)} \leq C(t). \tag{3.32}
\]
For any $r \in ]0,1[ \times x \in \mathbb{R}^2$, using the Hölder inequality, we can deduce that for $\alpha \in [0,1[$
\[
\frac{1}{m(B_r(x))} \int_{B_r(x)} |\omega(t,y)| \, dy \leq \left(\frac{1}{m(B_r(x))}\right)^{\frac{1}{\alpha}} \|\omega(t)\|_{p,1}^{\frac{1}{\alpha}} \leq p^{1-\alpha} \left(\frac{1}{m(B_r(x))}\right)^{\frac{1}{\alpha}} \sup_{1 \leq p < \infty} \|\omega\|_{p,1}^{\frac{1}{p-1}}.
\]
According to the arbitrariness of $p$, we can conclude from the estimate (3.14) and Lemma 2.5 that
\[
(-\log r)^{\alpha-1} \frac{1}{m(B_r(x))} \int_{B_r(x)} |\omega(t,y)| \, dy \leq C(t) \sup_{1 \leq p < \infty} \frac{\|\omega_0\|_{p,1}}{p^{1-\alpha}}. \tag{3.33}
\]
On the other hand, using Lemma 2.7 again, we observe that $\sup_{1 \leq p < \infty} \frac{\|\omega_0\|_{p,1}}{p^{1-\alpha}} \leq C\|\omega_0\|_{bmo,\mathbb{R}^2}$ for $\alpha \in [0,1]$. Inserting this into (3.32) and then taking the supremum over all $r \in ]0,1[$ entails
\[
\|\omega(t)\|_{L^{\alpha-1}\text{bmo}(\mathbb{R}^2)} \leq C(t)\|\omega_0\|_{L^\infty\text{bmo}(\mathbb{R}^2)}.
\]
Now, we are in a position to show (3.31). Firstly, we see that (one may take $p = \log a$)
\[
\inf_{p \geq 1} \log (1+p) \cdot a^p \leq e \log (1+\log a), \quad \text{for} \quad a > e.
\]
Thus, using the same argument as above, we can infer that
\[
\left(\log(1 - \log r)\right)^{-1} \frac{1}{m(B_r(x))} \int_{B_r(x)} |\omega(t,y)| \, dy \leq C(t) \sup_{1 \leq p < \infty} \frac{\|\omega_0\|_{p,1}}{p^{1-\alpha}} \leq C(t)\|\omega_0\|_{bmo(\mathbb{R}^2)}.
\]
This completes the proof. \hfill \Box

3.3. In the following part, we mainly focus on the proof of Theorem 1.4.

Proof of Theorem 1.4. By Lemma 2.7 and Proposition 3.3 one can conclude that
\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} + \sup_{1 \leq p < \infty} \frac{\|\omega(t)\|_{p,1}}{p} \leq C(t). \tag{3.34}
\]
Let us fixed a ball $B_r(x) \subset \mathbb{R}^2$. Our task is now to bound the following quantity
\[
\frac{1}{m(B_r(x))} \int_{B_r(x)} \left|\omega(y) - \text{Avg}_{B_r(x)}(\omega)\right| \, dy.
\]
In order to do this, we split it into two cases.

Case 1: $r \leq e^{-4L_{bip}(t)}$.

Simple calculations lead to
\[
\left(\frac{1}{m(B_r(x))} \int_{B_r(x)} \left|\omega(y) - \text{Avg}_{B_r(x)}(\omega)\right|^p \, dy\right)^{\frac{1}{p}} = \left(\frac{1}{m(B_r(x))} \int_{\psi(B_r(x))} \left|\omega_0(y) - \text{Avg}_{\psi(B_r(x))}(\omega_0)\right|^p \, dy\right)^{\frac{1}{p}}
\]
\[
\leq 2 \left( \frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \omega_0(y) - \Avg_{B_{r}\psi(x)}(\omega_0) \right|^p \, dy \right)^{\frac{1}{p}}.
\]

If we take \(r_\psi := re^{V_{\text{Lip}}(t)}\), it is easy to verify that

\[
\log r \sim \log r_\psi.
\]

This enables us to conclude that

\[
(- \log r)^\alpha \left( \frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \omega(y) - \Avg_{B_{r}\psi}(\omega) \right|^p \, dy \right)^{\frac{1}{p}} \\
\leq 2 (- \log r)^\alpha \left( \frac{1}{m(B_r(x))} \int_{B_{r}\psi(x)} \left| \omega_0(y) - \Avg_{B_{r}\psi}(\omega_0) \right|^p \, dy \right)^{\frac{1}{p}} \\
\leq C (- \log r_\psi)^\alpha \left( \frac{m(B_{r}\psi(x))}{m(B_r(x))} \right)^{\frac{1}{p}} \left( \int_{B_{r}\psi(x)} \left| \omega_0(y) - \Avg_{B_{r}\psi(x)}(\omega_0) \right|^p \, dy \right)^{\frac{1}{p}} \\
\leq C \left( e^{V_{\text{Lip}}(t)} \right)^{\frac{1}{\alpha} \| \omega_0 \|_{L^\omega\text{bmo}}}.
\]

Moreover, by the Hölder inequality and Corollary 2.2, we immediately obtain that

\[
\| \omega(t) \|_{L^\omega\text{bmo}} \leq \| \omega(t) \|_{L^\omega\text{bmo}} \leq C \left( e^{V_{\text{Lip}}(t)} \right)^{\frac{1}{\alpha} \| \omega_0 \|_{L^\omega\text{bmo}}} \\
\leq C p \left( e^{V_{\text{Lip}}(t)} \right)^{\frac{1}{\alpha} \| \omega_0 \|_{L^\omega\text{bmo}}}.
\]

Combining this with Lemma 2.5 leads to

\[
\| \omega(t) \|_{L^\omega\text{bmo}} \leq C \| \omega_0 \|_{L^\omega\text{bmo}} \cdot \inf_{1 \leq p < \infty} p \left\| e^{V_{\text{Lip}}(t)} \right\|^{\frac{1}{\alpha} \| \omega_0 \|_{L^\omega\text{bmo}}} \\
\leq C \left( 1 + V_{\text{Lip}}(t) \right) \| \omega_0 \|_{L^\omega\text{bmo}}.
\]

Case 2: \(e^{-4V_{\text{Lip}}(t)} < r < 1\).

First of all, we notice that \(e^{-4V_{\text{Lip}}(t)} \leq r \leq \frac{1}{2}\) implies

\[- \log r \leq 4V_{\text{Lip}}(t)\]

When \(\alpha \in [0, 1]\). By Proposition 3.6, we have

\[
(- \log r)^\alpha \frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \omega(y) - \Avg_{B_{r}\psi}(\omega) \right| \, dy \leq (- \log r) \| \omega(t) \|_{L^{\alpha-1}\text{bmo}(\mathbb{R}^d)} \\
\leq C \left( 1 + V_{\text{Lip}}(t) \right) \| \omega_0 \|_{L^\omega\text{bmo}(\mathbb{R}^d)}.
\]

Similarly, one can infer that

\[
(- \log r) \frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \omega(y) - \Avg_{B_{r}\psi}(\omega) \right| \, dy \\
\leq C \left( 1 + V_{\text{Lip}}(t) \right) \log \left( 1 + V_{\text{Lip}}(t) \right) \| \omega_0 \|_{L^{\text{bmo}}(\mathbb{R}^d)}.
\]

When \(\alpha > 1\). Using Proposition 2.3, we know that \(L^\omega\text{bmo}(\mathbb{R}^d)\) continuously embeds \(B_{\infty,1}^0(\mathbb{R}^d)\). This together with the well-known fact \(\| \omega(t) \|_{L^\infty(\mathbb{R}^d)} \leq \| \omega_0 \|_{L^\infty(\mathbb{R}^d)}\) yields

\[
(- \log r)^\alpha \frac{1}{m(B_r(x))} \int_{B_r(x)} \left| \omega(y) - \Avg_{B_{r}\psi}(\omega) \right| \, dy \leq 2 (- \log r)^\alpha \frac{1}{m(B_r(x))} \int_{B_r(x)} |\omega(y)| \, dy \\
\leq 2 (- \log r)^\alpha \| \omega(t) \|_{L^\infty(\mathbb{R}^d)} \\
\leq C \left( 1 + V_{\text{Lip}}(t) \right) \| \omega_0 \|_{L^\omega\text{bmo}(\mathbb{R}^d)}.
\]

Collecting all these estimates entails the desired result. \(\square\)
From Theorem 1.4 it follows from the inclusion relation \( L^\alpha \text{bmo}(\mathbb{R}^2) \hookrightarrow B_{\infty,1}^0(\mathbb{R}^2) \) that (1.11) is closed for \( \alpha > 1 \). Specifically:

**Corollary 3.7.** Let \( u_0 \in L^\infty(\mathbb{R}^2) \) and \( \omega_0 \in L^\alpha \text{bmo}(\mathbb{R}^2) \) with \( \alpha > 1 \). Assume that \( u \) is a smooth solution of (12). Then there exist a positive smooth function \( C(t) \), dependent of the initial data and \( \alpha \), such that

\[
\|u(t)\|_{L^\infty(\mathbb{R}^2)} + \|\omega(t)\|_{L^\alpha \text{bmo}(\mathbb{R}^2)} \leq C(t).
\]

**Proof.** From [7], we already know that

\[
\|\omega(t)\|_{B_{\infty,1}^0(\mathbb{R}^2)} \leq C(1 + V_{\text{Lip}}(t)) \|\omega_0\|_{B_{\infty,1}^0(\mathbb{R}^2)}.
\]

Combining this with the inclusion relation \( L^\alpha \text{bmo}(\mathbb{R}^2) \hookrightarrow B_{\infty,1}^0(\mathbb{R}^2) \) entails \( V_{\text{Lip}}(t) \leq C(t) \). Inserting this inequality into (1.11) with \( \alpha > 1 \), we eventually get the required result. \( \square \)

### 4. Proof of the main theorems

This section is devoted to proving Theorem 1.1, Corollary 1.2, and Theorem 1.3 in Section 1. We first restrict our attention to the existence statement. Here we just need to give the proof of Theorem 1.3 for the case \( \alpha > 1 \). On the other hand, it is easy to check that

\[
\|u^n\|_{L^\infty(\mathbb{R}^2)} \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty.
\]

Next, we turn to show the time continuity that

\[
\|u(t_1) - u(t_2)\|_{L^\infty(\mathbb{R}^2)} \leq \int_{t_2}^{t_1} \|\partial_t u(\tau)\|_{L^\infty(\mathbb{R}^2)} \, d\tau \leq C|t_1 - t_2|,
\]

which implies \( u \in C(\mathbb{R}^+; B^{-\epsilon}_{\infty,1}(\mathbb{R}^2)) \). This together with the fact that \( \omega \in L^\infty(\mathbb{R}^2; L^\alpha \text{bmo}(\mathbb{R}^2)) \) yields that \( u \in C(\mathbb{R}^+; B^{1-\epsilon}_{\infty,1}(\mathbb{R}^2)) \). Mimicking the above proof, we can show the existence of a weak solution to Theorem 1.1 and Corollary 1.2.

Next, we focus on the uniqueness statement. Let \( (u, \Pi) \) and \( (\tilde{u}, \tilde{\Pi}) \) be two solutions of (12) with the same initial data, then the differences \( (\delta u, \delta \Pi) := (u - \tilde{u}, \Pi - \tilde{\Pi}) \) satisfies

\[
\left\{ \begin{array}{l}
\partial_t \delta u + (u \cdot \nabla) \delta u + \nabla \delta \Pi = -\delta (u \cdot \nabla) \tilde{u}, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\
\text{div} \delta u = 0, \\
\delta u|_{t=0} = 0.
\end{array} \right.
\]

Here and in what follows, we define

\[
\|\cdot\|_{Y_{\text{Lip}}^\alpha(\mathbb{R}^d)} := \|\cdot\|_{L^\infty(\mathbb{R}^d)} + \|\cdot\|_{Y_{\text{Lip}}^\alpha(\mathbb{R}^d)}.
\]

In order to prove the uniqueness of solution, it suffices to show the following proposition.
Proposition 4.1. Let \( u \) and \( \tilde{u} \) belong to \( L^\infty_T \( \Theta(\cdot) \) with \( \Theta(x) \) satisfies \( \int_1^\infty \frac{1}{\Theta(x)} \, dx = \infty \). Assume that \((u, \Pi)\) and \((\tilde{u}, \Pi)\) are two solutions of \((E)\) with the same initial data. Then \( u \equiv \tilde{u} \) on interval \([0, T]\).

Proof. The incompressible condition implies that \( \nabla \Pi = B(u, u) \) with \( B(u, v) := \nabla \text{div} |D|^{-2}((u \cdot \nabla) v) \). Thus, we get from \( (1.2) \) that
\[
\partial_t \delta u + (u \cdot \nabla) \delta u = B(\delta u, \tilde{u}) + B(u, \delta u) - \delta (u \cdot \nabla) \tilde{u}.
\] Applying the operator \( \Delta_q \) to the above equality with \( q \geq -1 \) yields that
\[
\partial_t \Delta_q \delta u + S_{q+1}(u \cdot \nabla) \delta u = \Delta_q B(\delta u, \tilde{u}) + \Delta_q B(u, \delta u) - \Delta_q \delta (u \cdot \nabla) \tilde{u} + R_q(u, \delta u),
\]
where \( R_q = S_{q+1}(u \cdot \nabla) \delta u - \Delta_q ((u \cdot \nabla) \delta u) \).

It follows that
\[
\| \Delta_q \delta u \|_{L^\infty} \leq \int_0^t \bigg| \Delta_q B(\delta u, \tilde{u}) + \Delta_q B(u, \delta u) - \Delta_q \delta (u \cdot \nabla) \tilde{u} + R_q(u, \delta u) \bigg| \, d\tau.
\]
By using Lemma \((E.3)\), Lemma \((E.4)\) and Lemma \((E.5)\) we readily get that for all \( \varepsilon \in ]0, 1[ \),
\[
\| \Delta_q \delta u(t) \|_{L^\infty} \leq C \Theta(q + 2) 2^{q\varepsilon} \int_0^t \left( \| u(\tau) \|_{L^\Theta_t Y_t} + \| \tilde{u}(\tau) \|_{L^\Theta_t Y_t} \right) \| \delta u(\tau) \|_{B^{-\varepsilon}_t, \infty} \, d\tau.
\] (4.4)

By resorting to the low-high frequency decomposition technique, we know
\[
\| \delta u \|_{B^{-\varepsilon}_{t, \infty}} \leq \sup_{q < N} 2^{-q\varepsilon} \| \Delta_q \delta u(t) \|_{L^\infty} + \sup_{q \geq N} 2^{-q\varepsilon} \| \Delta_q \delta u(t) \|_{L^\infty},
\] (4.5)

where \( N \) is a positive integer to be specified later.

For the high frequency part, by the Bernstein inequality, we obtain
\[
\sup_{q > N} 2^{-q\varepsilon} \| \Delta_q \delta u(t) \|_{L^\infty} \leq C \sup_{q > N} 2^{-q(1+\varepsilon)} \left( \| \Delta_q \nabla u(t) \|_{L^\infty} + \| \Delta_q \nabla \tilde{u}(t) \|_{L^\infty} \right)
\]
\[
\leq C \sup_{q > N} \Theta(q) 2^{-q(1+\varepsilon)} \left( \| u(t) \|_{L^\Theta_t Y_t} + \| \tilde{u}(t) \|_{L^\Theta_t Y_t} \right)
\]
\[
\leq C \sup_{q > N} 2^{-q\varepsilon} \left( \| u \|_{L^\Theta_t Y_t} + \| \tilde{u} \|_{L^\Theta_t Y_t} \right) \leq C 2^{-N\varepsilon}.
\] (4.6)

Next, we deal with the low frequency part. We observe that \((4.4)\), the properties of \( \Theta(\cdot) \) and the Hölder inequality allow us to get
\[
\sup_{q \leq N} 2^{-q\varepsilon} \| \Delta_q \delta u(t) \|_{L^\infty} \leq C \Theta(N) \int_0^t \left( \| u(\tau) \|_{L^\Theta_t Y_t} + \| \tilde{u}(\tau) \|_{L^\Theta_t Y_t} \right) \| \delta u(\tau) \|_{B^{-\varepsilon}_t, \infty} \, d\tau
\]
\[
\leq C \left( \| u \|_{L^\Theta_t Y_t(\mathbb{R}^2)} + \| \tilde{u} \|_{L^\Theta_t Y_t(\mathbb{R}^2)} \right) \Theta(N) \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_t, \infty} \, d\tau.
\] (4.7)

Plugging \((1.6)\) and \((1.7)\) in \((1.5)\), we readily obtain
\[
\| \delta u \|_{B^{-\varepsilon}_{t, \infty}} \leq C \Theta(N) \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_{t, \infty}} \, d\tau + C 2^{-N\varepsilon}.
\] (4.8)

According to the continuity of \( \| (u, \tilde{u})(t) \|_{B^{-\varepsilon}_{t, \infty}} \), we conclude that there exists \( T_0 \in ]0, \min\{T, 1\} \) such that \( \sup_{t \in [0, T_0]} \| \delta u(t) \|_{B^{-\varepsilon}_{t, \infty}} \leq \frac{1}{2} \). Moreover, we may take \( N \) satisfying
\[
2^{-N\varepsilon} \sim \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_{t, \infty}} \, d\tau,
\]
that is,
\[
N \sim N_0 = -\varepsilon \log \left( \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_{t, \infty}} \, d\tau \right).
\]
Thus, \( (4.3) \) becomes that for \( t \in [0, T_0] \)
\[
\| \delta u(t) \|_{B^{-\varepsilon}_{\infty, \infty}} \leq C \Theta \left( -\frac{1}{\varepsilon} \log \left( \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_{\infty, \infty}} \ d\tau \right) \right) \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_{\infty, \infty}} \ d\tau
\]
\[+ C \int_0^t \| \delta u(\tau) \|_{B^{-\varepsilon}_{\infty, \infty}} \ d\tau.\]
We observe that \( \Theta(\cdot) \) fulfills
\[
\int_0^e \frac{1}{s \Theta(\log 1/s)} \ ds = \int_1^\infty \frac{1}{\Theta(s)} \ ds = \infty. \tag{4.9}
\]
By using Osgood’s Theorem, we obtain that \( \delta u(t) \equiv 0 \) on the interval \( [0, \min\{T_0, 1\}] \). Since \( u \) and \( \bar{u} \) are in \( L^\infty([0, T]; \overline{Y^\Theta_{bmo}(\mathbb{R}^2)}) \), we eventually conclude that \( u \equiv \bar{u} \) on the whole interval \([0, T]\) via a standard connectivity argument.

Based on this, we turn to prove the uniqueness of solution one by one.

- **Uniqueness of Theorem 1.1**
  We see that
  \[
  \sup_{2 \leq j < \infty} \frac{\| S_{j+1} \nabla u(t) \|_{L^\infty(\mathbb{R}^2)}}{j \Theta(j)} \leq C \| u(t) \|_{L^\infty(\mathbb{R}^2)} + C \sup_{2 \leq j < \infty} \frac{\| \omega(t) \|_{j, \frac{1}{j}}}{\Theta(j)}. \tag{4.10}
  \]
  Indeed, by using (2.3), we can infer that
  \[
  \| S_{j+1} \nabla u \|_{L^\infty(\mathbb{R}^2)} \leq \| \Delta_0 u \|_{L^\infty(\mathbb{R}^2)} + \sum_{1 \leq k \leq j} \| \Delta_k \nabla u(t) \|_{L^\infty(\mathbb{R}^2)}
  \]
  \[
  \leq C \| u(t) \|_{L^\infty(\mathbb{R}^2)} + C \sum_{1 \leq k \leq j} \| \Delta_k \omega(t) \|_{L^\infty(\mathbb{R}^2)}
  \]
  \[
  \leq C \| u(t) \|_{L^\infty(\mathbb{R}^2)} + C \sum_{1 \leq k \leq j} 2^{2k} \| \omega(t) \|_{j, \frac{1}{j}}
  \]
  \[
  \leq C \| u(t) \|_{L^\infty(\mathbb{R}^2)} + C j \| \omega(t) \|_{j, \frac{1}{j}}.
  \]
  Since \( \omega \in Y^\Theta_{bmo}(\mathbb{R}^2) \) with \( \Theta \in \mathcal{A}_2 \), then we have \( u \in Y^\Theta_{bmo}(\mathbb{R}^2) \) with \( \Theta \) satisfies \( \int_1^\infty \frac{1}{\Theta(x)} \ dx = \infty \). Moreover, applying Proposition 1.1, we can conclude that the uniqueness of solution.

- **Uniqueness of Theorem 1.2**
  We can apply (3.24) to Proposition 4.1 to get the uniqueness of solution.

- **Uniqueness of Theorem 1.3**
  By Proposition 2.2, we know that
  \[
  \sup_{j \geq 2} \frac{\| S_j \nabla u \|_{L^\infty(\mathbb{R}^2)}}{j \log(1 + j)} \leq C \| u \|_{L^\infty(\mathbb{R}^2)} + \| \omega \|_{L_{\log bmo}(\mathbb{R}^2)}.
  \]
  It is obvious that \( \Theta(p) = p \log(1 + p) \) satisfies \( \int_1^\infty \frac{1}{\Theta(p)} \ dx = \infty \). Whence, it follow the uniqueness of solution from Proposition 2.2.

Now, the proof of our results is achieved completely.

**Appendix**

In this section, we first show the generalized John-Nirenberg inequality and its corollary. Next, we further generalize the estimates for convection term which play an important role in proving the uniqueness, in the spirit of [4, 8].

27
Clearly, decomposition of $Q$

Suppose that $\zeta$

Now, let us define $\lambda > 0$

Thus, for any $f \in L^p(Q)$, we get

Moreover, by the Calderón-Zygmund decomposition theorem, we get

According to construction of $E.1$, we have

Theorem E.1 (The generalized John-Nirenberg inequality). Let $f(x)$ belongs to $L^\alpha BMO(\mathbb{R}^d)$ and $\alpha \in [0, \infty[$. There exist constants $B$ and $b$ dependent of $d$ such that for all cube $Q \subset \mathbb{R}^d$ with $r_Q \in ]0, 1]$ and $\beta > 0$

\[
\mu_Q(\beta) \leq B \exp \left( -\frac{b\beta(-\log r_Q)^\alpha}{\|f\|_{L^\alpha BMO}} \right) \cdot m(Q),
\]

where $\mu_Q(\beta)$ be defined by

\[
\mu_Q(\beta) := m \left( \{ x \in Q : |f(x) - \text{Avg}_Q(f)| > \beta \} \right).
\]

Let us remark that when $\alpha = 0$, Theorem E.1 comes back to the classical John-Nirenberg inequality, see for instance [17].

Proof. When $\beta(-\log r_Q)^\alpha \leq \|f\|_{L^\alpha BMO}$, one can conclude (5.1) by taking $B = e$ and $b = 1$.

When $\beta(-\log r_Q)^\alpha > \|f\|_{L^\alpha BMO}$. For a fixed cube $Q_0$ with $r_Q < 1$ and we assume $\text{Avg}_{Q_0}(f) = 0$. Otherwise, $f(x)$ may be instead by $g(x) = f(x) - \text{Avg}_{Q_0}(f)$ which fulfills $\text{Avg}_{Q_0}(g) = 0$ and $\|g\|_{L^\alpha BMO} = \|f\|_{L^\alpha BMO}$. Also, we may assume that $\|f\|_{L^\alpha BMO} = 1$ without loss of generality.

Now, let us define

\[
\mu_Q(\beta) = m \left( \{ x \in Q_0 : |f(x)| > \beta \} \right) := m(E_\beta).
\]

Thus, for any $\lambda > \|f\|_{L^\alpha BMO}(-\log r_Q_0)^{-\alpha} = (-\log r_Q_0)^{-\alpha}$, it is obvious that

\[
\frac{1}{m(Q_0)} \int_{Q_0} |f(x)| \, dx < \lambda.
\]

Moreover, by the Calderón-Zygmund decomposition theorem, we get

\[Q_0 = F^\lambda \bigcup \left( \bigcup_{k=1}^\infty Q_k^\lambda \right),\]

where, the cubes $Q_k^\lambda$ are mutually disjoint and satisfies

\[|f(x)| \leq \lambda, \quad \text{for a.e } x \in F^\lambda,\]

and

\[\lambda < \frac{1}{m(Q_k^\lambda)} \int_{Q_k^\lambda} |f(x)| \, dx \leq 2^d \lambda.
\]

According to construction of $Q_k^\lambda$, there exists a mother cube $Q_k^\lambda$ such that $Q_k^\lambda$ is one of $2^d$ equal children cubes which $Q_k^\lambda$ satisfying

\[
\frac{1}{m(Q_k^\lambda)} \int_{Q_k^\lambda} |f(x)| \, dx \leq \lambda.
\]

Thus, it follows that

\[
\frac{1}{m(Q_k^\lambda)} \int_{Q_k^\lambda} |f(x)| \, dx \leq \frac{1}{m(Q_k^\lambda)} \int_{Q_k^\lambda} |f(x) - \text{Avg}_{Q_k^\lambda}(f)| \, dx + |\text{Avg}_{Q_k^\lambda}(f)|
\]

\[\leq 2^d (-\log r_{Q_k})^{-\alpha} \|f\|_{L^\alpha BMO} + \lambda
\]

\[\leq 2^d (-\log r_{Q_k})^{-\alpha} + \lambda.
\]

Suppose that $\zeta \geq \lambda$, in the same way as above, it is easy to construct Calderón-Zygmund decomposition of $Q_0$ as follows:

\[Q_0 = F^\zeta \bigcup \left( \bigcup_{j=1}^\infty Q_j^\zeta \right).
\]

Clearly,

\[\bigcup_{j=1}^\infty Q_j^\zeta \subset \bigcup_{k=1}^\infty Q_k^\lambda.
\]
Whence, for each cube $Q_j^\zeta$, there exists a cube $Q_k^\lambda$ such that $Q_j^\zeta \subset Q_k^\lambda$. Now let us take $\zeta = \lambda + 2^{d+1}(-\log r_{Q_k})^{-\alpha}$ and denote

$$Q_{j,k}^{\zeta,\lambda} := \bigcup_{j, Q_j^\zeta \subset Q_k^\lambda} Q_j^\zeta.$$

Thus, we easily find that

$$\zeta = \lambda + 2^{d+1}(-\log r_{Q_k})^{-\alpha}$$

$$\leq \frac{1}{m(Q_{j,k}^{\zeta,\lambda})} \int_{Q_{j,k}^{\zeta,\lambda}} |f(x)| \, dx$$

$$\leq \frac{1}{m(Q_{j,k}^{\zeta,\lambda})} \int_{Q_{j,k}^{\zeta,\lambda}} |f(x) - \text{Avg}_{Q_k^\lambda}(f)| \, dx + |\text{Avg}_{Q_k^\lambda}(f)|$$

$$\leq \frac{m(Q_k^\lambda)}{m(Q_{j,k}^{\zeta,\lambda})} (-\log r_{Q_k})^{-\alpha} \|f\|_{L^\alpha \text{-BMO}} + 2^d(-\log r_{Q_k})^{-\alpha} + \lambda,$$

from which, it follow that

$$m(Q_{j,k}^{\zeta,\lambda}) \leq 2^{-d} m(Q_k^\lambda).$$

Consequently,

$$\sum_j m(Q_j^\zeta) \leq 2^{-d} \sum_k m(Q_k^\lambda),$$

and

$$\zeta - \lambda = 2^{d+1}(-\log r_{Q_k})^{-\alpha}.$$

We observe that $\beta > \|f\|_{L^\alpha \text{-BMO}}(-\log r_{Q_0})^{-\alpha} = (-\log r_{Q_0})^{-\alpha}$. If, moreover, we take $r = \lfloor \frac{\beta(-\log r_{Q_0})^{-\alpha}}{2(d+1)\log r_{Q_0}^{-\alpha}} \rfloor$ and $\zeta = (-\log r_{Q_0})^{-\alpha} + 2^{d+1}(-\log r_{2Q_k})^{-\alpha}$, then we have $(-\log r_{Q_0})^{-\alpha} \leq \zeta \leq \beta$ which implies $E_{\beta} \subset E_{\zeta}$. Since

$$f(x) \leq \zeta, \quad \text{for a.e } x \in F^\zeta,$$

we get $E_{\zeta} = \bigcup_j Q_j^\zeta$. This together with (5.3) enables us to infer that

$$\sum_j m(Q_j^\zeta) \leq 2^{-rd} \sum_k m(Q_k^1).$$

So,

$$m(E_{\beta}) \leq m(E_{\zeta}) \leq \sum_j m(Q_j^\zeta) \leq 2^{-rd} \sum_k m(Q_k^1).$$

This entails

$$\mu_{Q_0}(\beta) \leq 2^{-rd} m(Q_0).$$

Taking $B = 2^{d+1}2^{-d-1}$ and $b = \log(d2^{-d-1})$ and using (5.4), we eventually obtain that

$$Bm(Q_0) \exp(-b(-\log r_{Q_0})^{-\alpha}) = 2^d m(Q_0) \cdot 2^{d(1-(\log r_{Q_0})^{-\alpha} - 2^{-d-1})} \geq 2^d m(Q_0) \cdot 2^{d(r+1)} \geq \mu_{Q_0}.$$

This completes the proof. \hfill \Box

**Corollary E.2.** Suppose that $1 \leq q < \infty$ and $\alpha \in [0, \infty]$. Then there holds that $L^\alpha \text{-BMO}_q = L^\alpha \text{-BMO}$. 

29
Proof. We just need to show that \(L^\alpha BMO_q = L^\alpha BMO\) for \(1 < q < \infty\). For an arbitrary cubic \(Q_r(x) \subset \mathbb{R}^d\) with \(r \in [0, 1]\), by the Hölder inequality, we have

\[
\frac{1}{m(Q_r(x))} \int_{Q_r(x)} |f(x) - \text{Avg}_{Q_r(x)}(f)| \, dx \leq \left( \frac{1}{m(Q_r(x))} \int_{Q_r(x)} |f(x) - \text{Avg}_{Q_r(x)}(f)|^q \, dx \right)^{rac{1}{q}} \leq (-\log r)^\alpha \|f\|_{L^\alpha BMO_q}.
\]

On the other hand, the generalized John-Nirenberg inequality ensures that for \(r \in [0, 1]\)

\[
\frac{(-\log r)^\alpha}{m(B_r(x))} \int_Q |f(x) - \text{Avg}_Q(f)|^q \, dx = \frac{q(-\log r)^\alpha}{m(B_r(x))} \int_0^\infty \xi^{q-1} \mu_Q(\xi) \, d\xi 
\leq Bq(-\log r)^\alpha \int_0^\infty \xi^{q-1} \exp \left( -\frac{b(-\log r)^\alpha}{\|f\|_{L^\alpha BMO}} \right) \, d\xi 
= Bq \int_0^\infty \xi^{q-1} \exp \left( -\frac{b\xi}{\|f\|_{L^\alpha BMO}} \right) \, d\xi 
= Bq b^{-q} \Gamma(q) \|f\|_{L^\alpha BMO}^q. \tag{5.5}
\]

This implies \(\|f\|_{L^\alpha BMO_q} \leq Cq \|f\|_{L^\alpha BMO}\). \(\square\)

In the following part, we always assume that \(\Theta\) is a modulus of continuity for the convenience of presentation.

Lemma E.3. Let \(1 \leq p \leq \infty\). Then there exist a positive constant \(C\) such that

\[
\|\Delta_q((u \cdot \nabla)v)\|_{L^p} \leq C\Theta(q + 2)2^{q\varepsilon}(\|v\|_{L^p} + \|v\|_{Y^{\Theta}_\varepsilon})\|u\|_{B^{p,\varepsilon}}. \quad \forall \varepsilon \in [0, 1]. \tag{5.6}
\]

Proof. Thanks to the Bony para-product decomposition, one can write

\[
(u \cdot \nabla)v = T_{u_j} \partial_j v_i + T_{\partial_j v_i} u_j + \partial_j \mathcal{R}(u_j, v_i).
\]

For the first term \(T_{u_j} \partial_j v_i\),

\[
\|\Delta_q T_{u_j} \partial_j v_i\|_{L^p} \leq C \sum_{|k-q| \leq 5} \|\Delta_q(S_{k-1} u_j \Delta_k \partial_j v_i)\|_{L^p} 
\leq C \sum_{|k-q| \leq 5} \|S_{k-1} u_j\|_{L^p} \|\Delta_k \partial_j v_i\|_{L^\infty} 
\leq C \sum_{|k-q| \leq 5} \|S_{k-1} u_j\|_{L^p} \|S_{q+5} \partial_j v_i\|_{L^\infty} 
\leq C\Theta(q)2^{q\varepsilon}\|u_j\|_{B^{p,\varepsilon}} \|v\|_{Y^{\Theta}_\varepsilon}.
\]

In a similar fashion as above, we have

\[
\|\Delta_q T_{\partial_j v_i} u_j\|_{L^p} \leq C \sum_{|k-q| \leq 5} \|\Delta_q(S_{k-1} \partial_j v_i \Delta_k u_j)\|_{L^p} 
\leq C \sum_{|k-q| \leq 5} \|S_{k-1} \partial_j v_i\|_{L^\infty} \|\Delta_k u_j\|_{L^p} 
\leq C\Theta(q)2^{q\varepsilon}\|u_j\|_{B^{p,\varepsilon}} \|v\|_{Y^{\Theta}_\varepsilon}
\]

Finally, the remainder term can be bounded as follows:

\[
\|\Delta_q \partial_j \mathcal{R}(u_j, v_i)\|_{L^p} \leq 2^{q\varepsilon}\|\Delta_q \mathcal{R}(u_j, v_i)\|_{L^p} 
\leq C2^{q\varepsilon} \sum_{k \geq q-2} \|\Delta_k u_j \Delta_k v_i\|_{L^p}
\]

30
\[ \leq C 2^{qe} \sum_{k \geq q-2} 2^{(q-k)(1-\epsilon)} 2^{-k\epsilon} \| \tilde{\Delta}^k u_j \|_{L^p} 2^k \| \tilde{\Delta}^k v_i \|_{L^\infty} \]
\[ \leq C 2^{qe} \left( \| \Delta_{-1} v \|_{L^\infty} + \| v \|_{Y_{1,i}^0} \right) \sum_{k \geq q-2} 2^{(q-k)(1-\epsilon)} 2^{-k\epsilon} \Theta(k+2) \| \tilde{\Delta}^k u_j \|_{L^p}. \tag{5.7} \]

Since \( \Theta \) is a modulus of continuity, we know that \( \Theta(2^h) \leq C \Theta(h) \) for all \( h \geq 1 \). Thus, we have
\[ \sum_{k \geq q-2} 2^{(q-k)(1-\epsilon)} \Theta(k+2) 2^{-k\epsilon} \| \tilde{\Delta}^k u_j \|_{L^p} \leq C \Theta(q+2) \sum_{k \geq q-2} 2^{(q-k)(1-\epsilon)} \left( 1 + \frac{k-q}{q} \right) \log_2 C \Theta(k+2) 2^{-k\epsilon} \| \tilde{\Delta}^k u_j \|_{L^p} \]
\[ \leq C \Theta(q+2) \| u \|_{B_{p,\infty}^\epsilon}. \tag{5.8} \]

In the second line of (5.8), we have used that for \( k \geq q \leq 1 \),
\[ \Theta(k) \leq C \Theta \left( \frac{k}{2q} \right) \leq C \Theta \left( \frac{k}{2^m q} \right) \leq \left( \frac{k}{q} \right)^{\log_2 C} \Theta(q), \tag{5.9} \]
with \( a = \log_2 \frac{k}{q} \).

Inserting (5.8) in (5.7) leads to
\[ \| \Delta_q \partial_j R(u_j, v_i) \|_{L^p} \leq C \Theta(q+2) 2^{qe} \left( \| v \|_{L^p} + \| v \|_{Y_{1,i}^0} \right) \| u \|_{B_{p,\infty}^\epsilon}. \]

Collecting all these estimates yields the desired result.

**Lemma E.4.** Let \( \epsilon \in ] -1, 1 [ \) and \( 1 \leq p \leq \infty \). Assume that \( u \) be a divergence free vector field over \( \mathbb{R}^d \). There exists a positive constant \( C \) such that for all \( q \geq -1 \)
\[ \| R_q(u, v) \|_{L^p} \leq C \Theta(q+2) 2^{-q\epsilon} \| u \|_{Y_{1,i}^0} \| v \|_{B_{p,\infty}^\epsilon}, \tag{5.10} \]
where \( R_q(u, v) := S_{q+1} (u \cdot \nabla) \Delta_q v - \Delta_q ((u \cdot \nabla)v). \)

**Proof.** We first decompose \( R_q(u, v) \) as follows:
\[ R_q(u, v) = S_{q+1} (u \cdot \nabla) \Delta_q v - \Delta_q (S_{q+1} (u \cdot \nabla)v) - \Delta_q ((I_d - S_{q+1})(u \cdot \nabla)v) \]
\[ = - [\Delta_q, S_{q+1} \tilde{u}] \cdot \nabla v - [\Delta_q, S_{q+1} u] \cdot \nabla v - \Delta_q ((I_d - S_{q+1})(u \cdot \nabla)v), \tag{5.11} \]
where \( \tilde{u} = (I_d - S_1)u \).

Note that
\[ [\Delta_q, S_{q+1} \tilde{u}] \cdot \nabla v = [\Delta_q, S_{q+1} T_{\partial u}] \partial_i v + \Delta_q \left( T_{\partial u} S_{q+1} \tilde{u}_i \right) + \Delta_q \left( R(S_{q+1} \tilde{u}_i, \partial_i v) \right) \]
\[ - T_{\Delta q \partial u} S_{q+1} \tilde{u}_i - R(S_{q+1} \tilde{u}_i, \Delta_q \partial_i v) \]
\[ := R^1_q(u, v) + R^2_q(u, v) + R^3_q(u, v) + R^4_q(u, v) + R^5_q(u, v), \]
and
\[ \Delta_q ((I_d - S_{q+1})(u \cdot \nabla)v) = \Delta_q (T(I_d - S_{q+1})u, \partial_i v) + \Delta_q (T(\partial u)(I_d - S_{q+1})u_i) + \Delta_q R((I_d - S_{q+1})u_i, \partial_i v) \]
\[ := R^6_q(u, v) + R^7_q(u, v) + R^8_q(u, v). \]

First of all, we observe that
\[ [S_{q-1} S_{q+1} \tilde{u}_i, \Delta_q] \partial_i \Delta_q v \]
\[ = 2^d \int_{\mathbb{R}^d} \left( S_{q-1} S_{q+1} \tilde{u}_i(x) - S_{q-1} S_{q+1} \tilde{u}_i(x - y) \right) \varphi(2^{q-1} (x - y)) \partial_i \Delta_q v(y) \, dy \]
\[ = -2^d \int_{\mathbb{R}^d} \int_0^1 \partial_k S_{q-1} S_{q+1} \tilde{u}_i (x + (1 - \tau)(x - y)) \, d\tau \, (x_k - y_k) \varphi(2^{q-1} (x - y)) \partial_i \Delta_q v(y) \, dy \]
\[-2^{q(d-1)} \int_{\mathbb{R}^d} \int_0^1 \partial_k S_{q'-1} S_{q+1} \bar{u}_i (x + (1 - \tau) (x - y)) \, d\tau \, 2^{q'} (x_k - y_k) \varphi (2^{q'} (x - y)) \partial_i \Delta_{q'} v (y) \, dy,\]

where used the relation \( \Delta_{q'} f = 2^{q' d} \int_{\mathbb{R}^d} \varphi (2^{q'} (x - y)) f (y) \, dy. \)

Therefore, we immediately get that
\[
\| R_1^q (u, v) \|_{L^p} \leq C \sum_{|q'-q| \leq 4} 2^{-q'} \| \partial_k S_{q'-1} \bar{u}_i \|_{L^\infty} \| \partial_i \Delta_{q'} v \|_{L^p} \int_{\mathbb{R}^d} |x| \varphi (x) \, dx
\]
\[
\leq C \sum_{|q'-q| \leq 4} \| \partial_k S_{q'-1} u_i \|_{L^\infty} \| \Delta_{q'} v \|_{L^p}
\]
\[
\leq C \Theta (q + 2) \| u \|_{Y_{Lip}^q} \sum_{|q'-q| \leq 2} \| \Delta_{q'} v \|_{L^p}.
\]

In a similar fashion as in proof of \( R_1^q (u, v) \), we can bounded \( [\Delta_{q'}, S_1 u] \cdot \nabla v \) as follows:
\[
\| [\Delta_{q'}, S_1 u] \cdot \nabla v \|_{L^p} \leq C \sum_{|q'-q| \leq 2} \| \nabla S_1 u \|_{L^\infty} \| \Delta_{q'} v \|_{L^p}
\]
\[
\leq C \| u \|_{Y_{Lip}^q} \sum_{|q'-q| \leq 2} \| \Delta_{q'} v \|_{L^p}.
\]

For the second term \( R_2^q (u, v) \), by the same way as in proving Lemma \([E.3]\), we infer that
\[
\| R_2^q (u, v) \|_{L^p} \leq C \sum_{|q'-q| \leq 4} \| \Delta_{q'} \bar{u}_i \|_{L^\infty} \| S_{q'-1} \partial_i v \|_{L^p}
\]
\[
\leq C \sum_{|q'-q| \leq 4} 2^{q'} \| \Delta_{q'} \nabla u_i \|_{L^\infty} \sum_{-1 \leq k \leq q'-2} 2^{k-q} \| \Delta_k v \|_{L^p}
\]
\[
\leq C \Theta (q + 2) \sum_{|q'-q| \leq 4} \frac{\Theta (q' + 2)}{\Theta (q + 2)} \sum_{-1 \leq k \leq q+2} 2^{k-q} \| \Delta_k v \|_{L^p}
\]
\[
\leq C \Theta (q + 2) \| u \|_{Y_{Lip}^q} \sum_{-1 \leq k \leq q+2} 2^{k-q} \| \Delta_k v \|_{L^p}.
\]

Similarly, we can conclude that
\[
\| R_3^q (u, v) \|_{L^p} \leq C \Theta (q + 2) \| u \|_{Y_{Lip}^q} \sum_{-1 \leq k \leq q+2} 2^{k-q} \| \Delta_k v \|_{L^p}.
\]

Since \( \epsilon \in ]0, 1] \), the reminder term \( R_4^q (u, v) \) can be bounded by
\[
\| \partial_i \Delta_q (R (S_{q+1} \bar{u}_i, v)) \|_{L^p} \leq C \sum_{q' \geq q-3} 2^{q'} \| \Delta_{q'} v \|_{L^p} \| \Delta_{q'} S_{q+1} \bar{u}_i \|_{L^\infty}
\]
\[
\leq C \sum_{q' \geq q-3} 2^{q'} \| \Delta_{q'} v \|_{L^p} \| S_{q+1} \nabla \bar{u}_i \|_{L^\infty}
\]
\[
\leq C \Theta (q + 2) 2^{-q} \| u \|_{Y_{Lip}^q} \| v \|_{B_{p, \infty}^\epsilon}.
\]

Similarly, we can conclude that
\[
\| R_5^q (u, v) \|_{L^p} \leq C \Theta (q + 2) 2^{-q} \| u \|_{Y_{Lip}^q} \| v \|_{B_{p, \infty}^\epsilon}.
\]

It remain for us to bound the last three terms \( R_6^q (u, v) \), \( R_7^q (u, v) \), and \( R_8^q (u, v) \). Thanks to the property of support and the Hölder inequality, one has
\[
\| R_6^q (u, v) \|_{L^p} \leq C \sum_{|q'-q| \leq 5} \| S_{q'-1} (I_d - S_{q+1}) \bar{u}_i \|_{L^\infty} \| \Delta_{q'} \partial_i v \|_{L^p}
\]

32
\[ \leq C \sum_{|q' - q| \leq 5} 2^{-q'} \| S_{q' - 1} (I_d - S_{q+1}) \nabla u_i \|_{L^\infty} \| \Delta_q \partial_i v \|_{L^p} \]
\[ \leq C \sum_{|q' - q| \leq 5} 2^{q' - q} \| S_{q' - 1} \nabla u_i \|_{L^\infty} \| \Delta_q v \|_{L^p} \]
\[ \leq C \Theta(q + 2) \| u \|_{Y^{q_0}_{Lip}} \sum_{|q' - q| \leq 5} 2^{q' - q} \| \Delta_q v \|_{L^p}. \]

For the term \( R^7_q(u,v) \), by the Hölder inequality, we obtain
\[
\| R^7_q(u,v) \|_{L^p} \leq C \sum_{|q' - q| \leq 5} \| S_{q' - 1} \partial_i v \|_{L^p} \| \Delta_q (I_d - S_{q+1}) u_i \|_{L^\infty}
\leq C \sum_{|q' - q| \leq 5} 2^{q' - q} \sum_{-1 \leq k \leq q'} 2^{k - q'} \| \Delta_k v \|_{L^p} \| \Delta_q \nabla u_i \|_{L^\infty}
\leq C \Theta(q + 2) \| u \|_{Y^{q_0}_{Lip}} \sum_{-1 \leq k \leq q + 3} 2^{k - q} \| \Delta_k v \|_{L^p}.
\]

As for the last term \( R^8_q(u,v) \), by the Hölder inequality and (5.11), we obtain
\[
\| R^8_q(u,v) \|_{L^p} \leq C \sum_{q' \geq q - 3} 2^{q'} \| \Delta_q v \|_{L^p} \| \Delta_q (I_d - S_{q+1}) u_i \|_{L^\infty}
\leq C \sum_{q' \geq q - 3} 2^{q'} \| \Delta_q v \|_{L^p} \| \Delta_q u_i \|_{L^\infty}
\leq C \sum_{q' \geq q - 3} 2^{q' - q} \| \Delta_q v \|_{L^p} \| \Delta_q \nabla u_i \|_{L^\infty}
\leq C \Theta(q + 2) 2^{-q} \| u \|_{Y^{q_0}_{Lip}} \| v \|_{B^p_{\infty,\infty}}.
\]

Combining all these bounds yields the desired result. \( \square \)

**Lemma E.5.** Let \( B(u,v) = \nabla \text{div} |D|^{-2} ((u \cdot \nabla) v) \) and \( \varepsilon \in [0,1] \). Then there holds that for \( q \geq -1 \):
\[
\| \Delta_q B(u,v) \|_{L^\infty} \leq C \Theta(q + 2) 2^{q_0} \min \left\{ \| u \|_{B^{-\infty,q}_{\infty,\infty}}, \| v \|_{B^{-\infty,q}_{\infty,\infty}}, \| u \|_{Y^{q_0}_{Lip}} \right\}. \tag{5.12}
\]

**Proof.** Thanks to the Bony-paraproduct decomposition, one can decompose the \( \nabla \Pi = B(u,u) \) as follows:
\[
B(u,v) := B_1(u,v) + B_2(u,v) + B_3(u,v) + B_4(u,v) + B_5(u,v).
\]
Denoting \( \theta \in \mathcal{D}(B(0,2)) \) is a smooth function with value 1 on the ball \( B(0,1) \), we have
\[
B_1(u,v) := \nabla (-\Delta)^{-1} \mathcal{T}_{\partial_i u_j} \partial_j v_i,
B_2(u,v) := \nabla (-\Delta)^{-1} \mathcal{T}_{\partial_j v_i} \partial_j u_i,
B_3(u,v) := \partial_i \partial_j \nabla (-\Delta)^{-1} (I_d - \Delta_{-1}) \mathcal{R}(u_j, v_i),
B_4(u,v) := \theta E_d \ast \nabla \partial_i \partial_j \Delta_{-1} \mathcal{R}(u_j, v_i), \quad \text{with} \quad E_d = c_d \frac{1}{|x-y|^{d-2}},
B_5(u,v) := \tilde{E}_d \ast \nabla \partial_i \partial_j \Delta_{-1} \mathcal{R}(u_j, v_i) \quad \text{with} \quad \tilde{E}_d := (1 - \theta) E_d.
\]
First, we tackle with the para-product terms. By using the Hölder inequality, we obtain
\[
\| \Delta_q B_1(u,v) \|_{L^\infty} \leq \| \Delta_q \nabla (-\Delta)^{-1} (\mathcal{T}_{\partial_i u_j} \partial_j v_i) \|_{L^\infty}
\leq C \sum_{|k-q| \leq 5} \| \Delta_q \nabla (-\Delta)^{-1} (S_{k-1} \partial_i u_j \Delta_k \partial_j v_i) \|_{L^\infty}
\]

Combining all these bounds yields the desired result. \( \square \)
For the remainder term $B$

Inserting (5.14) and (5.15) in (5.13), one obtains

\[
\leq C \sum_{|k-q| \leq 5} 2^{-q} \| S_{k-1} \partial u_j \|_{L^\infty} \| \Delta_k \partial_j v_i \|_{L^\infty}. \tag{5.13}
\]

Furthermore, the fact

\[
\| \Delta_k \partial_j v_i \|_{L^\infty} = \| S_{k+2} \Delta_k \partial_j v_i \|_{L^\infty} \leq C \| S_{k+2} \Delta_k \partial_j v_i \|_{L^\infty} \leq C \Theta(k+2) \| v \|_{Y_{\text{Lip}}}^\infty
\]

yields

\[
\leq C \Theta(q+2) 2^{q \epsilon} \sum_{|k-q| \leq 5} 2^{-(q-k)(1+\epsilon)} \frac{\Theta(k+2)}{\Theta(q+2)} 2^{-k \epsilon} \| S_{k-1} u_j \|_{L^\infty} \frac{1}{\Theta(k+2)} \| \Delta_k \partial_j v_i \|_{L^\infty} \leq C \Theta(q+2) 2^{q \epsilon} \| v \|_{Y_{\text{Lip}}}^\epsilon \| u \|_{B_{\infty,\infty}^{-\epsilon}}. \tag{5.14}
\]

On the other hand, we see that

\[
\sum_{|k-q| \leq 5} 2^{-q} \| S_{k-1} \partial u_j \|_{L^\infty} \| \Delta_k \partial_j v_i \|_{L^\infty} \leq C \sum_{|k-q| \leq 5} 2^{-(q-k)(1+\epsilon)} \| S_{k-1} \partial u_j \|_{L^\infty} 2^{-k \epsilon} \| \Delta_k v_i \|_{L^\infty} \leq C \Theta(q+2) 2^{q \epsilon} \| u \|_{Y_{\text{Lip}}}^\epsilon \| v \|_{B_{\infty,\infty}^{-\epsilon}}. \tag{5.15}
\]

Inserting (5.14) and (5.15) in (5.13), one obtains

\[
\| \Delta_q B_1(u, v) \|_{L^\infty} \leq C \Theta(q+2) 2^{q \epsilon} \min \left\{ \| u \|_{B_{\infty,\infty}^{-\epsilon}}, \| v \|_{Y_{\text{Lip}}}^\epsilon, \| v \|_{B_{\infty,\infty}^{-\epsilon}} \right\}. \]

Similarly, we have

\[
\| \Delta_q B_2(u, v) \|_{L^\infty} \leq \| \Delta_q \nabla (-\Delta)^{-1}(T_{\partial_j u_j} \partial_j u_j) \|_{L^\infty} \leq C \sum_{|k-q| \leq 5} \| \Delta_q \nabla (-\Delta)^{-1}(S_{k-1} \partial_j v_i \Delta_k \partial u_j) \|_{L^\infty} \leq C \sum_{|k-q| \leq 5} 2^{-q} \| S_{k-1} \partial u_j \|_{L^\infty} \| \Delta_k \partial_j v_i \|_{L^\infty} \leq C \Theta(q+2) 2^{q \epsilon} \min \left\{ \| u \|_{B_{\infty,\infty}^{-\epsilon}}, \| v \|_{Y_{\text{Lip}}}^\epsilon, \| v \|_{B_{\infty,\infty}^{-\epsilon}} \right\}. \]

For the remainder term $B_3(u, v)$, it can be bounded by

\[
\| \Delta_q \partial_j \nabla (-\Delta)^{-1}(I_{d - \Delta - 1} R(u_j, v_i)) \|_{L^\infty} \leq \| \Delta_q \partial_j \nabla (-\Delta)^{-1} R(u_j, v_i) \|_{L^\infty} \leq C \sum_{k \geq q-2} 2^q \| \Delta_k u_j \|_{L^\infty} \| \Delta_k v_i \|_{L^\infty} \leq C 2^{q \epsilon} \sum_{k \geq q-2} 2^{(q-k)(1-\epsilon)} 2^{k q} 2^{-k \epsilon} \| \Delta_k u_j \|_{L^\infty} \| \Delta_k v_i \|_{L^\infty} \leq C \Theta(q+2) 2^{q \epsilon} \sum_{k \geq q-2} \frac{\Theta(k+2)}{\Theta(q+2)} 2^{(q-k)(1-\epsilon)} \min \left\{ \| u \|_{B_{\infty,\infty}^{-\epsilon}}, \| v \|_{Y_{\text{Lip}}}^\epsilon, \| v \|_{B_{\infty,\infty}^{-\epsilon}} \right\}. \tag{5.19}
\]

By using (5.19), we can deduce that

\[
\| \Delta_q B_3(u, v) \|_{L^\infty} \leq C \Theta(q+2) 2^{q \epsilon} \min \left\{ \| u \|_{B_{\infty,\infty}^\infty}, \| v \|_{Y_{\text{Lip}}}^\infty, \| v \|_{B_{\infty,\infty}^{-\epsilon}} \right\}. \]
Since $\theta E_d \in L^1(\mathbb{R}^d)$, then we get

$$\|\Delta q B(\theta u, v)\|_{L^\infty} \leq \|\Delta q (\theta E_d \ast \nabla \partial_i \partial_j \Delta_{-1} \mathcal{R}(u_j, v_i))\|_{L^\infty} \leq C \|\theta E_d\|_{L^1} \|\nabla \partial_i \partial_j \Delta_{-1} \mathcal{R}(u_j, v_i)\|_{L^\infty} \leq C \|\theta E_d\|_{L^1} \sum_{k \geq 1} \|\Delta_k u_j\|_{L^\infty} \|\Delta_k v_i\|_{L^\infty} \leq C \Theta(q + 2) 2^q \min \left\{ \|u\|_{B^{\infty, \infty}^\infty \ast \|v\|_{Y^{\infty}_\mathrm{Lip}}, \|v\|_{B^{\infty, \infty}^\infty \ast \|u\|_{Y^{\infty}_\mathrm{Lip}}} \right\}.$$

Finally, the fact that $\nabla \partial_i \partial_j \tilde{E}_d \in L^1$ enables us to conclude that

$$\|\Delta q B(\theta u, v)\|_{L^\infty} \leq \|\Delta q (\nabla \partial_i \partial_j \tilde{E}_d \ast \Delta_{-1} \mathcal{R}(u_j, v_i))\|_{L^\infty} \leq C \|\nabla \partial_i \partial_j \tilde{E}_d\|_{L^1} \sum_{k \geq 1} \|\Delta_k u_j\|_{L^\infty} \|\Delta_k v_i\|_{L^\infty} \leq C \Theta(q + 2) 2^q \min \left\{ \|u\|_{B^{\infty, \infty}^\infty \ast \|v\|_{Y^{\infty}_\mathrm{Lip}}, \|v\|_{B^{\infty, \infty}^\infty \ast \|u\|_{Y^{\infty}_\mathrm{Lip}}} \right\}.$$

This ends the proof. \(\square\)

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