COMMUTATIVE HOPF ALGEBRAS
OF PERMUTATIONS AND TREES

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ABSTRACT. We propose several constructions of commutative or cocommutative Hopf algebras based on various combinatorial structures, and investigate the relations between them. A commutative Hopf algebra of permutations is obtained by a general construction based on graphs, and its non-commutative dual is realized in three different ways, in particular as the Grossman-Larson algebra of heap ordered trees. Extensions to endofunctions, parking functions, set partitions, planar binary trees and rooted forests are discussed. Finally, we introduce one-parameter families interpolating between different structures constructed on the same combinatorial objects.

1. INTRODUCTION

Many examples of Hopf algebras based on combinatorial structures are known. Among these, algebras based on permutations and planar binary trees play a prominent role, and arise in seemingly unrelated contexts [18, 16, 7, 4]. As Hopf algebras, both are noncommutative and non cocommutative, and in fact self-dual.

More recently, cocommutative Hopf algebras of binary trees and permutations have been constructed [19, 2]. In [19], binary trees arise as sums over rearrangements classes in an algebra of parking functions, while in [2], cocommutative Hopf algebras are obtained as the graded coalgebras associated with coradical filtrations.

In [21], a general method for constructing commutative Hopf algebras based on various kind of graphs has been presented. The aim of this note is to investigate Hopf algebras based on permutations and trees constructed by the method developed in [21]. These commutative algebras are, by definition, realized in terms of polynomials in an infinite set of doubly indexed indeterminates. The dual Hopf algebras are then realized by means of non commutative polynomials in variables $a_{ij}$. We show that these first resulting algebras are isomorphic (in a non trivial way) to the duals of those of [2], and study some generalizations such as endofunctions, parking functions, set partitions, trees, forests, and so on.

The possibility to obtain in an almost systematic way commutative, and in general non cocommutative, versions of the usual combinatorial Hopf algebras leads us to conjecture that these standard versions should be considered as some kind of quantum groups, i.e., can be incorporated into one-parameter families, containing an enveloping algebra and its dual for special values of the parameter. A few results supporting this point of view are presented in the final section.

In all the paper, $K$ will denote a field of characteristic zero. All the notations used here is as in [8, 21].
2. A commutative Hopf algebra of endofunctions

Permutations can be regarded in an obvious way as labelled and oriented graphs whose connected components are cycles. Actually, arbitrary endofunctions (functions from $[n] := \{1, \ldots, n\}$ to itself) can be regarded as labelled graphs, connecting $i$ with $f(i)$ for all $i$ so as to fit in the framework of [21], where a general process for building Hopf algebras of graphs is described.

In the sequel, we identify an endofunction $f$ of $[n]$ with the word

\[ w_f = f(1)f(2) \cdots f(n) \in [n]^n. \]

Let $\{x_{ij} \mid i, j \geq 1\}$ be an infinite set of commuting indeterminates, and let $\mathcal{J}$ be the ideal of $R = K[x_{ij} \mid i, j \geq 1]$ generated by the relations

\[ x_{ij}x_{ik} = 0 \quad \text{for all } i, j, k. \]

For an endofunction $f : [n] \to [n]$, define

\[ M_f := \sum_{i_1 < \cdots < i_n} x_{i_1 f(1)} \cdots x_{i_n f(n)}, \]

in $R/\mathcal{J}$.

It follows from [21], Section 4, that

\section*{Theorem 2.1.}

The $M_f$ span a subalgebra $\text{EQSym}$ of the commutative algebra $R/\mathcal{J}$. More precisely, there exist non-negative integers $C_{f,g}^h$ such that

\[ M_f M_g = \sum_h C_{f,g}^h M_h. \]

\section*{Example 2.2.}

\begin{align*}
(5) \quad M_1 M_{22} &= M_{133} + M_{323} + M_{223}. \\
(6) \quad M_1 M_{331} &= M_{1442} + M_{4241} + M_{4431} + M_{3314}. \\
(7) \quad M_{12} M_{21} &= M_{1243} + M_{1432} + M_{4231} + M_{1324} + M_{3214} + M_{2134}. \\
(8) \quad M_{12} M_{22} &= M_{1244} + M_{1443} + M_{4234} + M_{1324} + M_{3234} + M_{2234}. 
\end{align*}

The \textit{shifted concatenation} of two endofunctions $f : [n] \to [n]$ and $g : [m] \to [m]$ is the endofunction $h := f \bullet g$ of $[n+m]$ such that $w_h := w_f \bullet w_g$, that is

\[ h(i) = f(i) \text{ if } i \leq n, \quad h(i) = n + g(i - n) \text{ if } i > n. \]
We can now give a combinatorial interpretation of the coefficient $C_{f,g}^h$: if $f : [n] \to [n]$ and $g : [m] \to [m]$, this coefficient is the number of permutations $\tau$ in the shuffle product $(1 \ldots p) \shuffle (p + 1 \ldots p + n)$ such that

$$h = \tau^{-1} \circ (f \bullet g) \circ \tau.$$ 

For example, with $f = 12$ and $g = 22$, one finds the set (see Equation (8))

$$\{1244, 1434, 4234, 1334, 3234, 2234\}.$$ 

Now, still following [21], define a coproduct by

$$\Delta_h : \sum_{(f,g) : f \bullet g = h} M_f \otimes M_g.$$ 

This endows $EQSym$ with a (commutative, non cocommutative) Hopf algebra structure.

**Example 2.3.**

(13) $\Delta M_{626124} = M_{626124} \otimes 1 + 1 \otimes M_{626124}$.

(14) $\Delta M_{4232277} = M_{4232277} \otimes 1 + M_{42322} \otimes M_{22} + 1 \otimes M_{4232277}$.

Define a connected endofunction as a function that cannot be obtained by non trivial shifted concatenation. Then, the definition of the coproduct of the $M_f$ implies

**Proposition 2.4.** If $(S^f)$ denotes the dual basis of $(M_f)$, the graded dual $ESym := EQSym^*$ is free over the set

$$\{S^f | f \text{ connected}\}.$$ 

Indeed, Equation (12) is equivalent to

(16) $S^f S^g = S^{f \bullet g}$.

Now, $ESym$ being a graded connected cocommutative Hopf algebra, it follows from the Milnor-Moore theorem that

$$ESym = U(L),$$

where $L$ is the Lie algebra of its primitive elements. Let us now prove

**Theorem 2.5.** As a graded Lie algebra, the primitive Lie algebra $L$ of $ESym$ is free over a set indexed by connected endofunctions.

**Proof.** Assume it is the case. By standard arguments on generating series, one finds that the number of generators of $L$ in degree $n$ is equal to the number of algebraic generators of $ESym$ in degree $n$, parametrized for example by connected endofunctions (series beginning by $(1, 3, 20, 197, 2511, 38924, \ldots)$). We will now show that $L$ has at least this number of generators and that those generators are algebraically independent, determining completely the dimensions of the homogeneous components $L_n$. 

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of $L$ (series beginning by $(1, 3, 23, 223, 2800, 42576, \ldots)$). Following Reutenauer [22] p. 58, denote by $\pi_1$ the Eulerian idempotent, that is, the endomorphism of $\text{ESym}$ defined by $\pi_1 = \log^*(Id)$. It is obvious, thanks to the definition of $S^f$ that

\begin{equation}
\pi_1(S^f) = S^f + \cdots,
\end{equation}

where the dots stand for terms $S^g$ such that $g$ is not connected. Since the $S^f$ associated with connected endofunctions are independent, the dimension of $L_n$ is at least equal to the number of connected endofunctions of size $n$. So $L_n$ is free over a set of primitive elements parametrized by connected endofunctions.

There are many Hopf subalgebras of $\text{EQSym}$ which can be defined by imposing natural restrictions to maps: being bijective (see Section 3), idempotent ($f^2 = f$), involutive ($f^2 = \text{id}$), or more generally the Burnside classes ($f^p = f^q$), and so on. We shall start with the Hopf algebra of permutations.

3. A commutative Hopf algebra of permutations

3.1. The Hopf algebra of bijective endofunctions. Let us define $\mathcal{G} \text{QSym}$ as the subalgebra of $\text{EQSym}$ spanned by the

\begin{equation}
M_\sigma = \sum_{i_1 < \cdots < i_n} x_{i_{\sigma(1)}} \cdots x_{i_{\sigma(n)}},
\end{equation}

where $\sigma$ runs over bijective endofunctions, i.e., permutations. Note that $\mathcal{G} \text{QSym}$ is also isomorphic to the image of $\text{EQSym}$ in the quotient of $R/J$ by the relations

\begin{equation}
x_{i_k}x_{j_k} = 0 \text{ for all } i, j, k.
\end{equation}

By the usual argument, it follows that

**Proposition 3.1.** The $M_\sigma$ span a Hopf subalgebra $\mathcal{G} \text{QSym}$ of the commutative Hopf algebra $\text{EQSym}$. 

As already mentioned, there exist non-negative integers $C_{\alpha,\beta}^\gamma$ such that

\begin{equation}
M_\alpha M_\beta = \sum_{\gamma} C_{\alpha,\beta}^\gamma M_\gamma.
\end{equation}

The combinatorial interpretation of the coefficients $C_{\alpha,\beta}^\gamma$ seen in Section 2 can be reformulated in the special case of permutations. Write $\alpha$ and $\beta$ as a union of disjoint cycles. Split the set $[n + m]$ into a set $A$ of $n$ elements, and its complement $B$, in all possible ways. For each splitting, apply to $\alpha$ (resp. $\beta$) in $A$ (resp. $B$) the unique increasing morphism of alphabets from $[n]$ to $A$ (resp. from $[m]$ to $B$) and return the list of permutations with the resulting cycles. On the example $\alpha = 12$ and $\beta = 321$, this yields

\begin{equation}
(1)(2)(53)(4), (1)(3)(52)(4), (1)(4)(52)(3), (1)(5)(42)(3), (2)(3)(51)(4),
(2)(4)(51)(3), (2)(5)(41)(3), (3)(4)(51)(2), (3)(5)(41)(2), (4)(5)(31)(2).
\end{equation}

This set corresponds to the permutations and multiplicities of Equation 27.
A third interpretation of this product comes from the dual coproduct point of view: $C^\gamma_{\alpha,\beta}$ is the number of ways of getting $(\alpha, \beta)$ as the standardized words of pairs $(a, b)$ of two complementary subsets of cycles of $\gamma$. For example, with $\alpha = 12$, $\beta = 321$, and $\gamma = 52341$, one has three solutions for the pair $(a, b)$, namely

$$((2)(3), (4)(51)), ((2)(4), (3)(51)), ((3)(4), (2)(51)),$$

which is coherent with Equations (22) and (27).

**Example 3.2.**

\begin{align*}
(24) & \quad M_{12 \cdots n} M_{12 \cdots p} = \binom{n + p}{n} M_{12 \cdots (n+p)}.
\end{align*}

\begin{align*}
(25) & \quad M_1 M_{21} = M_{132} + M_{213} + M_{321}.
\end{align*}

\begin{align*}
(26) & \quad M_{12} M_{21} = M_{1243} + M_{1324} + M_{1432} + M_{2134} + M_{3214} + M_{4231}.
\end{align*}

\begin{align*}
(27) & \quad M_{12} M_{321} = M_{12543} + M_{14325} + 2M_{15342} + M_{32145} + 2M_{42315} + 3M_{52341}.
\end{align*}

\begin{align*}
(28) & \quad M_{21} M_{123} = M_{12354} + M_{12435} + M_{12543} + M_{13245} + M_{14325}
+ M_{15342} + M_{21345} + M_{32145} + M_{42315} + M_{52341}.
\end{align*}

\begin{align*}
(29) & \quad M_{21} M_{231} = M_{21453} + M_{23154} + M_{24513} + M_{25431} + M_{34152}
+ M_{34521} + M_{35412} + M_{43251} + M_{43512} + M_{53421}.
\end{align*}

**3.2. Duality.** Recall that the coproduct is given by

\begin{equation}
\Delta M_{\sigma} := \sum_{(\alpha, \beta) : \alpha \bullet \beta = \sigma} M_\alpha \otimes M_\beta.
\end{equation}

As in Section 2, this implies

**Proposition 3.3.** If $(S^\sigma)$ denotes the dual basis of $(M_\sigma)$, the graded dual $S\text{Sym} := S QSym^*$ is free over the set

\begin{equation}
\{ S^\alpha \mid \alpha \text{ connected} \}.
\end{equation}

Indeed, Equation (30) is equivalent to

\begin{equation}
S^\alpha S^\beta = S^{\alpha \bullet \beta}.
\end{equation}

Note that $S\text{Sym}$ is both a subalgebra and a quotient of $E\text{Sym}$, since $S QSym$ is both a quotient and a subalgebra of $E QSym$. 
Now, as before, \( \mathcal{S}\text{Sym} \) being a graded connected cocommutative Hopf algebra, it follows from the Milnor-Moore theorem that

\begin{equation}
\mathcal{S}\text{Sym} = U(L),
\end{equation}

where \( L \) is the Lie algebra of its primitive elements.

The same argument as in Section 2 proves

**Theorem 3.4.** The graded Lie algebra \( L \) of primitive elements of \( \mathcal{S}\text{Sym} \) is free over a set indexed by connected permutations.

**Corollary 3.5.** \( \mathcal{S}\text{Sym} \) is isomorphic to \( H_0 \), the Grossman-Larson Hopf algebra of heap-ordered trees [10].

According to [2], \( \mathcal{S}Q\text{Sym} \) ( = \( \mathcal{S}\text{Sym}^* \)) is therefore isomorphic to the quotient of \( F\text{Q}\text{Sym} \) by its coradical filtration.

### 3.3. Cyclic tensors and \( \mathcal{S}Q\text{Sym} \)

For a vector space \( V \), let \( \Gamma^n V \) be the subspace of \( V \otimes^n \) spanned by **cyclic tensors**, i.e., sums of the form

\begin{equation}
\sum_{k=0}^{n-1} (v_1 \otimes \cdots \otimes v_k) \gamma^k,
\end{equation}

where \( \gamma \) is the cycle \( (1, 2, \ldots, n) \), the right action of permutations on tensors being as usual

\begin{equation}
(v_1 \otimes \cdots \otimes v_k) \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.
\end{equation}

Clearly, \( \Gamma^n V \) is stable under the action of \( GL(V) \), and its character is the symmetric function “cyclic character” \([24, 14]\):

\begin{equation}
l_n^{(0)} = \frac{1}{n} \sum_{d|n} \phi(d) p_{n/d}^d,
\end{equation}

where \( \phi \) is Euler’s function.

Let \( L_n^{(0)} \) be the subspace of \( \mathbb{C}S_n \) spanned by \( n \)-cycles. This is a submodule of \( \mathbb{C}S_n \) for the conjugation action \( \rho_\tau(\sigma) = \tau \sigma \tau^{-1} \) with Frobenius characteristic \( l_n^{(0)} \). Then one can define the analytic functor \( \Gamma \) \([12, 17]\):

\begin{equation}
\Gamma(V) = \bigoplus_{n \geq 1} V \otimes_{\mathbb{C}S_n} L_n^{(0)}.
\end{equation}

Let \( \overline{\Gamma}(V) = \bigoplus_{n \geq 1} \Gamma^n(V) \). Its symmetric algebra \( H(V) = S(\overline{\Gamma}(V)) \) can be endowed with a Hopf algebra structure, by declaring the elements of \( \overline{\Gamma}(V) \) primitive.

As an analytic functor, \( V \mapsto S(\overline{\Gamma}(V)) \) corresponds to the sequence of \( \mathcal{S}_n \)-modules \( M_n = \mathbb{C}S_n \) endowed with the conjugation action, that is,

\begin{equation}
S(\overline{\Gamma}(V)) = \bigoplus_{n \geq 0} V \otimes_{\mathbb{C}S_n} M_n,
\end{equation}
so that elements of $H_n(V)$ can be identified with symbols $\begin{bmatrix} w \\ \sigma \end{bmatrix}$ with $w \in V^\otimes n$ and \( \sigma \in S_n \) subject to the equivalences

(39) \[
\begin{bmatrix} w \tau^{-1} \\ \tau \sigma \tau^{-1} \end{bmatrix} \equiv \begin{bmatrix} w \\ \sigma \end{bmatrix}.
\]

Let $A = \{a_n \mid n \geq 1\}$ be an infinite linearly ordered alphabet, and $V = \mathbb{C}A$. We identify a tensor product of letters $a_{i_1} \otimes \cdots \otimes a_{i_n}$ with the corresponding word $w = a_{i_1} \ldots a_{i_n}$ and denote by $(w)$ the circular class of $w$. A basis of $H_n$ is then formed by the commutative products

(40) \[
m = (w_1) \cdots (w_p)
\]
of circular words, with $|w_1| + \cdots + |w_p| = n$.

With such a basis element, we can associate a permutation by the following standardization process. Fix a total order on circular words, for example the lexicographic order on minimal representatives. Write $m$ as a non-decreasing product

(41) \[
m = (w_1) \cdots (w_p) \text{ with } (w_1) \leq (w_2) \leq \cdots \leq (w_p),
\]
and compute the ordinary standardization $\sigma'$ of the word $w = w_1 \cdots w_p$. Then, $\sigma$ is the permutation obtained by parenthesing the word $\sigma'$ like $m$ and interpreting the factors as cycles. For example, if

\[
\begin{align*}
m &= (cba)(aba)(ac)(ba) = (aab)(ab)(ac)(acb) \\
w &= aababacacb \\
\sigma' &= 12637495108 \\
\sigma &= (126)(37)(49)(5108) \\
\sigma &= (2, 6, 7, 9, 10, 1, 3, 5, 4, 8)
\end{align*}
\]

We set $\sigma = \text{cstd}(m)$ and define it as the circular standardized of $m$.

Let $H_\sigma(V)$ be the subspace of $H_n(V)$ spanned by those $m$ such that $\text{cstd}(m) = \sigma$, and let $\pi_\sigma : H(V) \to H_\sigma(V)$ be the projector associated with the direct sum decomposition

(43) \[
H(V) = \bigoplus_{n \geq 0} \bigoplus_{\sigma \in S_n} H_\sigma(V).
\]

Computing the convolution of such projectors then yields the following

**Theorem 3.6.** The $\pi_\sigma$ span a subalgebra of the convolution algebra $\text{End}^\sigma H(V)$, isomorphic to $S \mathcal{QSym}$ via $\pi_\sigma \mapsto M_\sigma$.  

### 3.4. Subalgebras of $S \mathcal{QSym}$

For a permutation $\sigma \in S_n$, let $\text{supp}(\sigma)$ be the partition $\pi$ of the set $[n]$ whose blocks are the supports of the cycles of $\sigma$. The sums

(44) \[
U_\pi := \sum_{\text{supp}(\sigma) = \pi} M_\sigma
\]
span a Hopf subalgebra $ΠQSym$ of $S\ QSym$, which, as we shall see in the next section, is isomorphic to the dual of the Hopf algebra of symmetric functions in noncommuting variables (such as in [23, 3], not to be confused with $\mathbf{Sym}$).

One can also embed $QSym$ into $ΠQSym$: take as total ordering on finite sets of integers $\{i_1 < \cdots < i_r\}$ the lexicographic order on the words $(i_1, \ldots, i_r)$. Then, any set partition $\pi$ of $[n]$ has a canonical representative $B$ as a non-decreasing sequence (of blocks $B_1 \leq B_2 \leq \cdots \leq B_r$). Let $I = b(\pi)$ be the composition $(|B_1|, \ldots, |B_r|)$ of $n$. The sums

$$U_I := \sum_{b(\pi) = I} U_\pi = \sum_{b(\sigma) = \lambda} M_\sigma$$

where $b(\sigma)$ denotes the ordered cycle type of $\sigma$, span a Hopf subalgebra of $ΠQSym$ and $S\ QSym$, which is isomorphic to $QSym$.

Furthermore, if we denote by $\Lambda(I)$ the partition associated with a composition $I$ by sorting $I$ and by $\Lambda(\pi)$ the partition $\lambda$ whose parts are the sizes of the blocks of $\pi$, the sums

$$u_\lambda := \sum_{\Lambda(I) = \lambda} U_I = \sum_{\Lambda(\pi) = \lambda} U_\pi = \sum_{C(\sigma) = \lambda} M_\sigma$$

where $C(\sigma)$ denotes the cycle type of $\sigma$, span a Hopf subalgebra of $QSym$, $ΠQSym$, and $S\ QSym$, which is isomorphic to $Sym$ (ordinary symmetric functions).

An explicit Hopf embedding of $Sym$ into $S\ QSym$ is given by

$$j : p^*_\lambda \rightarrow u_\lambda$$

where $p^*_\lambda = \frac{p_\lambda}{z_\lambda}$ is the adjoint basis of products of power sums. The images of the usual generators of $Sym$ under this embedding have simple expressions in terms of the infinite matrix $X = (x_{ij})_{i,j \leq 1}$:

$$j(p_n) = \text{tr}(X^n)$$

which implies that $j(e_n)$ is the sum of the diagonal minors of order $n$ of $X$:

$$j(e_n) = \sum_{i_1 < \cdots < i_n} \sum_{\sigma \in S_n} \varepsilon(\sigma)x_{i_1i_{\sigma(1)}} \cdots x_{i_ni_{\sigma(n)}}$$

and $j(h_n)$ is the sum of the same minors of the permanent

$$j(h_n) = \sum_{i_1 < \cdots < i_n} \sum_{\sigma \in S_n} x_{i_1i_{\sigma(1)}} \cdots x_{i_ni_{\sigma(n)}}.$$

More generally, the sum of the diagonal immanants of type $\lambda$ gives

$$j(s_\lambda) = \sum_{i_1 < \cdots < i_n} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)x_{i_1i_{\sigma(1)}} \cdots x_{i_ni_{\sigma(n)}}.$$

Finally, one can check that the $M_\sigma$ with $\sigma$ involutive span a Hopf subalgebra of $S\ QSym$. Since the number of involutions of $S_n$ is equal to the number of standard Young tableaux of size $n$, this algebra can be regarded as a commutative version of
FSym. Notice that this version is also isomorphic to the image of $\mathcal{G} QSym$ in the quotient of $R/J$ by the relations

$$x_{ij}x_{jk} = 0, \text{ for all } i \neq k.$$ (52)

This construction generalizes to the algebras built on permutations of arbitrary given order.

4. Structure of $\mathcal{G} Sym$

4.1. A realization of $\mathcal{G} Sym$. In the previous section, we have built a commutative algebra of permutations from explicit polynomials on a set of auxiliary variables $x_{ij}$. One may ask whether its non-commutative dual admits a similar realization in terms of non-commuting variables $a_{ij}$.

We shall find such a realization, in a somewhat indirect way, by first building from scratch a Hopf algebra of permutations $\Phi Sym \subset K \langle a_{ij} \mid i, j \geq 1 \rangle$, whose operations can be described in terms of the cycle structure of permutations. Its coproduct turns out to be cocommutative, and the isomorphism with $\mathcal{G} Sym$ follows as above from the Milnor-Moore theorem.

Let $\{a_{ij}, i, j \geq 1\}$ be an infinite set of non-commuting indeterminates. We use the biword notation

$$a_{ij} \equiv \begin{bmatrix} i \\ j \end{bmatrix}, \quad \begin{bmatrix} i_1 \\ j_1 \end{bmatrix} \cdots \begin{bmatrix} i_n \\ j_n \end{bmatrix} \equiv \begin{bmatrix} i_1 \ldots i_n \\ j_1 \ldots j_n \end{bmatrix}$$ (53)

Let $\sigma \in \mathcal{S}_n$ and let $(c_1, \ldots, c_k)$ be a decomposition of $\sigma$ into disjoint cycles. With any cycle, one associates its cycle words, that is, the words obtained by reading the successive images of any element of the cycle. For example, the cycle words associated with the cycle (3142) are 1342, 2134, 3421, 4213.

We now define

$$\phi_{\sigma} := \sum_{x, a} \begin{bmatrix} x \\ a \end{bmatrix},$$ (54)

where the sum runs over all words $x$ such that $x_i = x_j$ iff $i$ and $j$ belong to the same cycle of $\sigma$, and such that the standardized word of the subword of $a$ whose letter positions belong to cycle $c_l$ is equal to the inverse of the standardized word of a cycle word of $c_l$.

Example 4.1.

$$\phi_{12} = \sum_{x \neq y} \begin{bmatrix} x \\ y \\ a \\ b \end{bmatrix}. $$ (55)

$$\phi_{41352} = \sum_{x \neq y; \text{Std}(abde)^{-1} = 1342, 3421, 4213, \text{or } 2134} \begin{bmatrix} x \\ x \\ y \\ x \end{bmatrix}. $$ (56)

Theorem 4.2. The $\phi_{\sigma}$ span a subalgebra $\Phi Sym$ of $K \langle a_{ij} \mid i, j \geq 1 \rangle$. More precisely, there exist non-negative integers $g_{\alpha, \beta}^\sigma$ (0 or 1) such that $\phi_{\alpha} \phi_{\beta} = \sum g_{\alpha, \beta}^\sigma \phi_{\sigma}$. 
ΦSym is free over the set
\[ \{ \phi_\alpha | \alpha \text{ connected} \}. \] (57)

To give the precise expression of the product \( \phi_\alpha \phi_\beta \), we first need to define two operations on cycles.

The first operation is just the circular shuffle on disjoint cycles: if \( c'_1 \) and \( c''_1 \) are two disjoint cycles, their cyclic shuffle \( c'_1 \triangleright c''_1 \) is the set of cycles \( c_1 \) such that their cycle words are obtained by applying the usual shuffle on the cycle words of \( c'_1 \) and \( c''_1 \). This definition makes sense because a shuffle of cycle words associated with two words on disjoint alphabets splits as a union of cyclic classes.

For example, the cyclic shuffle \( (132) \triangleright (45) \) gives the set of cycles:
\[ \{(13245), (13425), (14325), (14352), (14532), (13254), (13524), (15324), (15342), (15432)\}. \] (58)

These cycles correspond to the following list of permutations which are those appearing in Equation (65), except for the first one which will be found later:
\[ \{ 34251, 35421, 31452, 45231, 41532, 41253, 35214, 34512, 31524, 54213, 51423, 51234 \}. \] (59)

Let us now define an operation on two sets \( C_1 \) and \( C_2 \) of disjoint cycles. We call matching a list of all those cycles, some of the cycles being paired, always one of \( C_1 \) with one of \( C_2 \). The cycles remaining alone are considered to be associated with the empty cycle. With all matchings associate the set of sets of cycles obtained by the product \( \triangleright \) of any pair of cycles. The union of those sets of cycles is denoted by \( C_1 \triangleright C_2 \).

For example, the matchings corresponding to \( C_1 = \{(1), (2)\} \) and \( C_2 = \{(3), (4)\} \) are:
\[ \{(1) \triangleright (2)\} \triangleright \{(3) \triangleright (4)\}, \{(1) \triangleright (2), (3)\} \triangleright \{(4)\}, \{(1) \triangleright (2), (4)\} \triangleright \{(3)\}, \{(1) \triangleright (3), (2)\} \triangleright \{(4)\}, \{(1) \triangleright (3), (2), (4)\} \triangleright \{(3)\}, \{(1), (4)\} \triangleright \{(2), (3)\}, \]
and the product \( C_1 \sim C_2 \) is then
\[ \{(1), (2), (3), (4)\}, \{(1), (23), (4)\}, \{(1), (24), (3)\}, \{(13), (2), (4)\}, \{(13), (24)\}, \{(14), (2), (3)\}, \{(14), (23)\}. \] (61)

Remark that this calculation is identical with the Wick formula in quantum field theory (see \[ \] for an explanation of this coincidence).

We are now in a position to describe the product \( \phi_\sigma \phi_\tau \): let \( C_1 \) be the cycle decomposition of \( \sigma \) and \( C_2 \) be the cycle decomposition of \( \tau \), shifted by the size of \( \sigma \). Then the permutations indexing the elements appearing in the product \( \phi_\sigma \phi_\tau \) are the permutations whose cycle decompositions belong to \( C_1 \sim C_2 \).

For example, with \( \sigma = \tau = 12 \), one finds that \( C_1 = \{(1), (2)\} \) and \( C_2 = \{(3), (4)\} \). It is then easy to check that one goes from Equation (61) to Equation (63) by computing the corresponding permutations.
Example 4.3.

(62) \[ \phi_{12}\phi_{21} = \phi_{1243} + \phi_{1342} + \phi_{1423} + \phi_{3241} + \phi_{4213}. \]

(63) \[ \phi_{12}\phi_{12} = \phi_{1234} + \phi_{1324} + \phi_{1432} + \phi_{3412} + \phi_{4321} + \phi_{4312}. \]

(64) \[ \phi_1\phi_{4312} = \phi_{15423} + \phi_{25413} + \phi_{35421} + \phi_{45123} + \phi_{51423} + \phi_{54213}. \]

(65) \[ \phi_{312}\phi_{21} = \phi_{31254} + \phi_{31452} + \phi_{34152} + \phi_{43124} + \phi_{34521} + \phi_{35214} + \phi_{35421} + \phi_{41253} + \phi_{41532} + \phi_{45231} + \phi_{51234} + \phi_{51423} + \phi_{54213}. \]

Let us recall a rather general recipe to obtain the coproduct of a combinatorial Hopf algebra from a realization in terms of words on an ordered alphabet \( X \). Assume that \( X \) is the ordered sum of two mutually commuting alphabets \( X' \) and \( X'' \). Then define the coproduct as \( \Delta(F) = F(X' \dot{+} X'') \), identifying \( F' \otimes F'' \) with \( F'(X')F''(X'') \) [7, 20].

There are many different ways to define a coproduct on \( \PhiSym \) compatible with the realization since there are many ways to order an alphabet of biletters: order the letters of the first alphabet, order the letters of the second alphabet, or order lexicographically with respect to one alphabet and then to the second.

In the sequel, we only consider the coproduct obtained by ordering the biletters with respect to the first alphabet. Thanks to the definition of the \( \phi \), it is easy to see that it corresponds to the unshuffling of the cycles of a permutation:

(66) \[ \Delta \phi_\sigma := \sum_{(\alpha, \beta)} \phi_\alpha \otimes \phi_\beta, \]

where the sum is taken over all pairs of permutations \( (\alpha, \beta) \) such that \( \alpha \) is obtained by standardizing any subset of cycles of \( \sigma \), and \( \beta \) by standardizing the complementary subset of cycles.

Example 4.4.

(67) \[ \Delta \phi_{12} = \phi_{12} \otimes 1 + 2\phi_1 \otimes \phi_1 + 1 \otimes \phi_{12}. \]

(68) \[ \Delta \phi_{312} = \phi_{312} \otimes 1 + 1 \otimes \phi_{312}. \]

(69) \[ \Delta \phi_{4231} = \phi_{4231} \otimes 1 + 2\phi_{321} \otimes \phi_1 + \phi_{21} \otimes \phi_{12} + \phi_{12} \otimes \phi_{21} + 2\phi_1 \otimes \phi_{321} + 1 \otimes \phi_{4231}. \]

The next theorem can be easily proved on the realization.

Theorem 4.5. \( \Delta \) is an algebra morphism, so that \( \PhiSym \) is a graded bialgebra (for the grading \( \deg \phi_\sigma = n \) if \( \sigma \in S_n \)). Moreover, \( \Delta \) is cocommutative.

The same reasoning as in Section 3 shows that
Theorem 4.6. \( \mathfrak{S} \text{Sym} \) and \( \Phi \text{Sym} \) are isomorphic as Hopf algebras.

To get the explicit change of basis going from \( \phi \) to \( S \), let us first recall that a connected permutation is a permutation \( \sigma \) such that \( \sigma([1,k]) \neq [1,k] \) for any \( k \in [1,n-1] \). Any permutation \( \sigma \) has a unique maximal factorization \( \sigma = \sigma_1 \cdot \cdots \cdot \sigma_r \) into connected permutations. We then define

\[
S'_\sigma := \phi_{\sigma_1} \cdots \phi_{\sigma_r}.
\]

First remark that the \( S' \) form a basis of \( \Phi \text{Sym} \). It is easy to check that the \( S' \) is a multiplicative basis with product given by shifted concatenation of permutations, so that they multiply as the \( S \) do. Moreover, the coproduct of \( S'_\sigma \) is the same as for \( \phi_\sigma \), so the same as for \( S^\sigma \). So both bases \( S \) and \( S' \) have same product and same coproduct.

This proves that the linear map \( S^\sigma \mapsto S'_\sigma \) realizes the Hopf isomorphism between \( \mathfrak{S} \text{Sym} \) and \( \Phi \text{Sym} \). There is another natural isomorphism: define

\[
S''_\sigma := \sum_{x,a} \left[ \begin{array}{c} x \\ a \end{array} \right],
\]

where the sum runs over all words \( x \) such that \( x_i = x_j \) if (but not only if) \( i \) and \( j \) belong the same cycle of \( \sigma \) and such that the standardized word of the subword of \( a \) consisting of the indices of cycle \( c_l \) is equal to the inverse of the standardized word of a cycle word of \( c_l \).

The fact that both bases \( S' \) and \( S'' \) have same product and coproduct simply comes from the fact that if \( (c_1) \cdots (c_p) \) is the cycle decomposition of \( \sigma \),

\[
S''_\sigma = \sum_{(c_1) \cdots (c_p) \sim (c_2) \cdots \cdots \sim (c_p)} \phi(c).
\]

For example,

\[
S''_{2431} = S''_{(124)(3)} = \phi_{(124)(3)} + \phi_{(1423)} + \phi_{(1234)} + \phi_{(1324)}
\]

\[
= \phi_{2431} + \phi_{4312} + \phi_{2341} + \phi_{3421}.
\]

4.2. Quotients of \( \Phi \text{Sym} \). Let \( I \) be the ideal of \( \Phi \text{Sym} \) generated by the differences

\[
\phi_\sigma - \phi_\tau
\]

where \( \sigma \) and \( \tau \) have the same cycle type.

The definitions of its product and coproduct directly imply that \( I \) is a Hopf ideal. Since the cycle types are parametrized by integer partitions, the quotient \( \Phi \text{Sym}/I \) has a basis \( Y_\lambda \), corresponding to the class of \( \phi_\sigma \), where \( \sigma \) has \( \lambda \) as cycle type.

From Equations (62)-(65), one finds:

Example 4.7.

\[
Y_{11}Y_2 = Y_{211} + 4Y_{31}, \quad Y_{11}^2 = Y_{1111} + 2Y_{22} + 4Y_{211}.
\]
\begin{align*}
Y_1Y_4 &= Y_{41} + 4Y_5, \quad Y_3Y_2 = Y_{32} + 12Y_5. \\
\text{Theorem 4.8.} \Phi\text{Sym}/I \text{ is isomorphic to Sym, the Hopf algebra of ordinary symmetric functions,}
\end{align*}

If one writes \( \lambda = (\lambda_1, \ldots, \lambda_p) = (1^{m_1}, \ldots, k^{m_k}) \), an explicit isomorphism is given by
\begin{equation}
Y_\lambda \mapsto \frac{\prod_{i=1}^km_i!}{\prod_{j=1}^p(\lambda_j - 1)!}m_\lambda.
\end{equation}

5. Parking functions and trees

5.1. A commutative algebra of parking functions. It is also possible to build a commutative pendant of the Hopf algebra of parking functions introduced in [19]: let PF\(_n\) be the set of parking functions of length \( n \). For \( a \in \text{PF}_n \), set, as before
\begin{equation}
M_a := \sum_{i_1 < \cdots < i_n} x_{i_1}^a i_{a(1)} \cdots x_{i_n}^a i_{a(n)}.
\end{equation}

Then, once more, the \( M_a \) form a linear basis of a \( \mathbb{Z} \)-subalgebra \( \text{PQSym} \) of \( \text{EQSym} \), which is also a sub-coalgebra if one defines the coproduct in the usual way, that is, from special cuts in graphs (see [21] for more details).

\textbf{Example 5.1.}
\begin{align*}
M_1M_{11} &= M_{122} + M_{121} + M_{113}. \\
M_1M_{221} &= M_{1332} + M_{3231} + M_{2231} + M_{2214}. \\
M_{12}M_{21} &= M_{1243} + M_{1432} + M_{4231} + M_{1324} + M_{3214} + M_{2134}. \\
\Delta M_{525124} &= M_{525124} \otimes 1 + 1 \otimes M_{525124}. \\
\Delta M_{4131166} &= M_{4131166} \otimes 1 + M_{41311} \otimes M_{11} + 1 \otimes M_{4131166}.
\end{align*}

The main interest of the non-commutative and non-cocommutative Hopf algebra of parking functions defined in [19] was that it naturally led to two algebras of trees. We obtained a cocommutative Hopf algebra of planar binary trees by summing over the distinct permutations of parking functions, and an algebra of planar trees by summing over hypoplactic classes.

We shall now investigate whether similar constructions can be found for the commutative version \( \text{PQSym} \).
5.2. From labelled to unlabelled parking graphs. A first construction, which can always be done for Hopf algebras of labelled graphs is to build a subalgebra by summing over labellings. Notice that this subalgebra is the same as the subalgebra obtained by summing endofunctions graphs over their labellings.

The dimension of this subalgebra in degree $n$ is equal to the number of unlabelled parking graphs $1, 1, 3, 7, 19, 47, \ldots$ For example, here are the 7 unlabelled parking graphs of size 3 (to be compared with the 16 parking functions):

![Unlabelled parking graphs of size 3](image)

The product of two such unlabelled graphs is the concatenation of graphs and the coproduct of an unlabelled graph is the unshuffle of its connected subgraphs. So this algebra is isomorphic to the polynomial algebra on generators indexed by connected parking graphs.

5.3. Binary trees and nondecreasing parking functions. One can easily check that in $PQSym$, summing over parking functions having the same reordering does not lead to a subalgebra. However, if we denote by $I$ the subspace of $PQSym$ spanned by the $M_a$ where $a$ is not equal to its nondecreasing reordering, it turns out that $I$ is an ideal and a coideal, and $CQSym := PQSym/I$ is therefore a commutative Hopf algebra with basis given by the classes $M_\pi := M_a$ labelled by nondecreasing parking functions.

Notice that $CQSym$ is also isomorphic to the image of $PQSym$ in the quotient of $R/J$ by the relations

$$x_{ij}x_{kl} = 0, \text{ for all } i < k \text{ and } j > l.$$

The dual basis of $M_\pi$ is

$$S^\pi := \sum_a S^a,$$

where the sum is taken over all permutations of $\pi$.

The dual $CQSym^*$ is free over the set $S^\pi$, where $\pi$ runs over connected nondecreasing parking functions. So if one denotes by $CQSym$ the Catalan algebra defined in [19], the usual Milnor-Moore argument then shows that

$$CQSym \sim CQSym^*, \quad CQSym \sim CQSym^*.$$
5.4. **From nondecreasing parking functions to rooted forests.** Nondecreasing parking functions correspond to parking graphs of a particular type: namely, rooted forests with a particular labelling (it corresponds to nondecreasing maps), the root being given by the loops in each connected component.

Taking sums over the allowed labellings of a given rooted forest, we get that the

\[ M_F := \sum_{\text{supp}(\pi) = F} M_\pi, \]

span a commutative Hopf algebra of rooted forests, which is likely to coincide with the quotient of the Connes-Kreimer algebra \([6]\) by its coradical filtration \([2]\).

6. **Quantum versions**

6.1. **Quantum quasi-symmetric functions.** When several Hopf algebra structures can be defined on the same class of combinatorial objects, it is tempting to try to interpolate between them.

This can be done, for example with compositions: the algebra of quantum quasi-symmetric functions \(QSym_q\) \([26, 7]\) interpolates between quasi-symmetric functions and non-commutative symmetric functions.

However, the natural structure on \(QSym_q\) is not exactly that of a Hopf algebra but rather of a twisted Hopf algebra \([15]\).

Recall that the coproduct of \(QSym(X)\) amounts to replace \(X\) by the ordered sum \(X' + X''\) of two isomorphic and mutually commuting alphabets. On the other hand, \(QSym_q\) can be realized by means of an alphabet of \(q\)-commuting letters

\[ x_j x_i = qx_i x_j, \text{ for } j > i. \]

Hence, if we define a coproduct on \(QSym_q\) by

\[ \Delta_q f(X) = f(X' + X''), \]

with \(X'\) and \(X''\) \(q\)-commuting with each other, it will be an algebra morphism

\[ QSym_q \rightarrow QSym_q(X' + X'') = QSym_q \otimes \chi QSym_q \]

for the twisted tensor product

\[ (a \otimes b) \cdot (a' \otimes b') = \chi(b, a')(aa' \otimes bb'), \]

where

\[ \chi(b, a') = q^{\text{deg}(b) - \text{deg}(a')} \]

for homogeneous elements \(b\) and \(a'\).

It is easily checked that \(\Delta_q\) is actually given by the same formula as the usual coproduct of \(QSym\), that is

\[ \Delta_q M_I = \sum_{H = K - 1} M_H \otimes M_K. \]
The dual twisted Hopf algebra, denoted by $\text{Sym}_q$, is isomorphic to $\text{Sym}$ as an algebra. If we denote by $S^I$ the dual basis of $M_I$, $S^I S^J = S^{I+J}$, and $\text{Sym}_q$ is freely generated by the $S^n = S_n$, whose coproduct is

$$\Delta_q S_n = \sum_{i+j=n} q^{ij} S_i \otimes S_j.$$  

(94)

As above, $\Delta_q$ is an algebra morphism

$$\text{Sym}_q \to \text{Sym}_q \otimes \chi \text{Sym}_q,$$

(95)

where $\chi$ is again defined by Equation (92).

6.2. Quantum free quasi-symmetric functions. The previous constructions can be lifted to $\text{FQSym}$. Recall that $\phi(F_\sigma) = q^{l(\sigma)}F_{c(\sigma)}$ is an algebra homomorphism $\text{FQSym} \to \text{QSym}_q$, which is in fact induced by the specialization $\phi(a_i) = x_i$ of the underlying free variables $a_i$ to $q$-commuting variables $x_i$.

The coproduct of $\text{FQSym}$ is also defined by

$$\Delta_F(A) = F(A' + A''),$$

(96)

where $A'$ and $A''$ are two mutually commuting copies of $A$ [7]. If instead one sets $a''a' = qa'a''$, one obtains again a twisted Hopf algebra structure $\text{FQSym}_q$ on $\text{FQSym}$, for which $\phi$ is a homomorphism.

**Theorem 6.1.** Let $A'$ and $A''$ be $q$-commuting copies of the ordered alphabet $A$, i.e., $a''a' = qa'a''$ for $a' \in A'$ and $a'' \in A''$. Then, the coproduct

$$\Delta_q f = f(A' + A'')$$

(97)

defines a twisted Hopf algebra structure. It is explicitly given in the basis $F_\sigma$ by

$$\Delta_q F_\sigma = \sum_{\alpha \cdot \beta = \sigma} q^{\text{inv}(\alpha, \beta)} F_\alpha \otimes F_\beta$$

(98)

where $\text{inv}(\alpha, \beta)$ is the number of inversions of $\sigma$ with one element in $\alpha$ and the other in $\beta$.

More precisely, $\Delta_q$ is an algebra morphism with values in the twisted tensor product of graded algebras $\text{FQSym} \otimes \chi \text{FQSym}$ where $(a \otimes \chi b)(a' \otimes \chi b') = \chi(b, a')(aa' \otimes \chi bb')$ and $\chi(b, a') = q^{\deg(b) \cdot \deg(a')}$ for homogeneous elements $b, a'$.

The map $\phi : \text{FQSym}_q \to \text{QSym}_q$ defined by

$$\phi(F_\sigma) = q^{l(\sigma)}F_{c(\sigma)}$$

(99)

is a morphism of twisted Hopf algebras.

**Example 6.2.**

$$\Delta_q F_{2431} = F_{2431} \otimes 1 + q^2 F_{132} \otimes F_1 + q^3 F_{12} \otimes F_{21} + q F_1 \otimes F_{321} + 1 \otimes F_{2431}.$$  

(100)

$$\Delta_q F_{3421} = F_{3421} \otimes 1 + q^3 F_{231} \otimes F_1 + q^4 F_{12} \otimes F_{21} + q^2 F_1 \otimes F_{321} + 1 \otimes F_{3421}.$$  

(101)
\begin{align}
\Delta_q F_{21} &= F_{21} \otimes 1 + q F_1 \otimes F_1 + 1 \otimes F_{21}.
\end{align}

\begin{align}
(\Delta_q F_{21})(\Delta_q F_1) &= (F_{213} + F_{231} + F_{321}) \otimes 1 + (F_{21} + q^2(F_{12} + F_{21})) \otimes F_1 \\
&\quad + F_1 \otimes (q^2 F_{21} + q(F_{12} + F_{21})) + 1 \otimes (F_{213} + F_{231} + F_{321}).
\end{align}

\begin{align}
\Delta_q F_{213} &= F_{213} \otimes 1 + F_{21} \otimes F_1 + q F_1 \otimes F_{12} + 1 \otimes F_{213}.
\end{align}

\begin{align}
\Delta_q F_{231} &= F_{231} \otimes 1 + q^2 F_{12} \otimes F_1 + q F_1 \otimes F_{21} + 1 \otimes F_{231}.
\end{align}

\begin{align}
\Delta_q F_{321} &= F_{321} \otimes 1 + q^2 F_{21} \otimes F_1 + q^2 F_1 \otimes F_{21} + 1 \otimes F_{321}.
\end{align}

Finally, one can also define a one-parameter family of ordinary Hopf algebra structures on $\mathbb{F}Q\text{Sym}$, by restricting formula (98) for $\Delta_q$ to connected permutations $\sigma$, and requiring that $\Delta_q$ be an algebra homomorphism. Then, for $q = 0$, $\Delta_q$ becomes cocommutative, and it is easily shown that the resulting Hopf algebra is isomorphic to $\mathbb{S}\text{Sym}$.

However, it follows from [7] that for generic $q$, the Hopf algebras defined in this way are all isomorphic to $\mathbb{F}Q\text{Sym}$. This suggests to interpret $\mathbb{F}Q\text{Sym}$ as a kind of quantum group: it would be the generic element of a quantum deformation of the enveloping algebra $\mathbb{S}\text{Sym} = U(L)$. Similar considerations apply to various examples, in particular to the Loday-Ronco algebra $\mathbb{PBT}$, whose commutative version obtained in [19] can be quantized in the same way as $Q\text{Sym}$, by means of $q$-commuting variables [20].

There is another way to obtain $Q\text{Sym}$ from $\mathbb{F}Q\text{Sym}$: it is known [7] that $Q\text{Sym}$ is isomorphic to the image of $\mathbb{F}Q\text{Sym}(A)$ in the hypoplactic algebra $K[A^*/\equiv_H]$. One may then ask whether there exist a $q$-analogue of the hypoplactic congruence leading directly to $Q\text{Sym}_q$.

Recall that the hypoplactic congruence can be presented as the bi-sylvester congruence:

\begin{align}
ubvcaw &\equiv ubvacw, \text{ with } a < b \leq c \\
ucavbw &\equiv uacvbw, \text{ with } a \leq b < c
\end{align}

with $u, v, w \in A^*$. A natural $q$-analogue, compatible with the above $q$-commutation is

\begin{align}
ubvcaw &\equiv_qH qubvacw, \text{ with } a < b \leq c \\
ucavbw &\equiv_qH quacvbw, \text{ with } a \leq b < c
\end{align}

with $u, v, w \in A^*$. Then, we have

**Theorem 6.3.** The image of $\mathbb{F}Q\text{Sym}(A)$ under the natural projection $K\langle A \rangle \to K\langle A \rangle/\equiv_{qH}$ is isomorphic to $Q\text{Sym}_q$ as an algebra, and also as a twisted Hopf algebra for the coproduct $A \to A' + A''$, $A'$ and $A''$ being $q$-commuting alphabets.
Moreover, if one only takes the sylvester congruence
\[(109) \quad ucavbw \equiv uacvbw,\]
the quotient \(\text{FQSym}(A)\) under the natural projection \(\mathbb{K}\langle A \rangle \to \mathbb{K}\langle A \rangle / \equiv_S\) is isomorphic to the Hopf algebra of planar binary trees of Loday and Ronco \([10, 11]\). The previous construction provides natural twisted \(q\)-analogues of this Hopf algebra. Indeed, if one defines the \(q\)-sylvester congruence as
\[(110) \quad ucavbw \equiv_{qS} quacvbw, \text{ with } a \leq b < c,\]
then

**Theorem 6.4.** The image of \(\text{FQSym}(A)\) under the natural projection \(\mathbb{K}\langle A \rangle \to \mathbb{K}\langle A \rangle / \equiv_{qS}\) is a twisted Hopf algebra, with basis indexed by planar binary trees.

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