GALERKIN METHOD OF WEAKLY DAMPED CUBIC NONLINEAR SCHRÖDINGER WITH DIRAC IMPURITY, AND ARTIFICIAL BOUNDARY CONDITION IN A HALF-LINE

Abderrazak Chrifi*, Mostafa Abounouh and Hassan Al Moatassime
Department of Mathematics
Faculty of Science and Technology, Cadi Ayyad University
B.P. 549, Av. Abdelkarim ElKhattabi, Guéliz, Marrakesh, 40000, Morocco

Abstract. We consider a weakly damped cubic nonlinear Schrödinger equation with Dirac interaction defect in a half line of $\mathbb{R}$. Endowed with artificial boundary condition at the point $x = 0$, we discuss the global existence and uniqueness of solution of this equation by using Faedo–Galerkin method.

1. Introduction. We consider in this paper the nonlinear Schrödinger equation (NLS) with Dirac interaction function

$$iu_t + \frac{1}{2}u_{xx} + q\delta_a u + |u|^2 u + i\gamma u = f, \quad \text{in } \Omega \times \mathbb{R}_+^*,$$

where the unknown $u$ maps $\Omega \times \mathbb{R}_+^*$ into $\mathbb{C}$, the parameters $q, a \in \mathbb{R}$ and $\delta_a$ is the Dirac delta distribution at the point $a \in \Omega$ defined by $\langle \delta_a, v \rangle = v(a)$ for $v \in H^1(\Omega)$. The constant $\gamma > 0$ denotes the damping parameter, $f$ is a given external time-independent forcing of $L^2(\Omega)$.

The model (1) arises in many applications, quantum mechanics, hydrodynamic, optic, see for an overview [19, 8, 16, 10, 18, 20]. In the case when $a = \gamma = 0$, $g(s) = s$ and $f = 0$, this introduced model coincides with the Gross-Pitaevskii model [14, 17].

The well-posedness of the solutions of nonlinear equation (1) has been already studied in the literature. In the case when $q = 0$, with the assumption that $\Omega = \mathbb{R}$, the global existence in $H^1(\mathbb{R})$ and in $L^2(\mathbb{R})$ were proved in [12, 7]. For bounded $\Omega \subset \mathbb{R}$ and with standard boundary conditions (Dirichlet, Neumann and periodic boundary conditions), the NLS equation (1) possesses a unique global solution in $H^1(\Omega)$, it was proved in [11]. In the case $q \neq 0$ and $\gamma = f = 0$ was studied in [13] in which it was shown that the equation (1) is well posed in $H^1(\mathbb{R})$. In the case $\Omega = \mathbb{R}$, it was proved in [15] that the well-posedness of the solutions in $H^1(\mathbb{R})$. In [2], the long-time behavior of this equation endowed the non-standard boundary condition at $x = 0$ can be described by the existence of a maximal compact attractor in the weak topology of $H^1([0, +\infty[)$.

The aim of this paper is to investigate the nonlinear Schrödinger (NLS) equation (1) in the case $\Omega = ]-\infty, 0]$ with Dirac interaction defect. In addition, we
consider artificial boundary conditions at the limit point 0 in order to avoid the perturbations on the behavior of the solutions caused by the boundary conditions

$$\partial_n u + \sqrt{2} e^{-i\pi/4} e^{-\gamma t} e^{iV(x,t)} \partial_t^{1/2} \left( e^{\gamma t} e^{-iV(x,t)} u \right) = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}, \quad (2)$$

where $\partial_n$ represents the normal derivative operator. $V$ is the phase function defined by

$$V(x, t) = \int_0^t |u(x, s)|^2 ds.$$

The operator $\partial_t^{1/2}$ represents the Riemann-Liouville fractional derivative of $\frac{1}{2}$ order defined by

$$\partial_t^{1/2}(h(t)) = \frac{1}{\sqrt{\pi}} \partial_t \left( \int_0^t \frac{h(s)}{\sqrt{t-s}} ds \right). \quad (3)$$

This issue (2) represents an artificial boundary condition for (1) in the case $f = 0$, it was treated in [3, 5, 4] in connection to the numerical behavior of the solutions. We assume that the support of the initial data $u_0 \in H^1(\Omega)$ and the external force $f \in L^2(\Omega)$ is compact in $\Omega$. Our main result that is proved by using Faedo–Galerkin method the NLS equation (1) with the boundary condition (2) has a unique solution in $H^1(\Omega)$, which prove the following theorem

**Theorem 1.1.** Let $u_0 \in H^1(\Omega)$ be the initial data with compact support in $\Omega$, $\gamma > 0$ and $f \in L^2(\Omega)$ with compact support in $\Omega$. Then, there exists a unique function $u \in C([0, +\infty[; H^1(\Omega)) \cap C^1([0, +\infty[; [H^1(\Omega)]')$ solution of the following problem

$$\begin{cases}
i u_t + \frac{1}{2} u_{xx} + q \partial_n u + |u|^2 u + i\gamma u = f, & \text{in } \Omega \times \mathbb{R}_+ \\
\partial_n u(0, t) + \sqrt{2} e^{-i\pi/4} e^{-\gamma t} e^{iV(0,t)} \partial_t^{1/2} \left( e^{\gamma t} e^{-iV(0,t)} u(0, t) \right) = 0, & t \in \mathbb{R}_+, \\
u(x, 0) = u_0(x), & x \in \Omega.
\end{cases} \quad (4)$$

The remainder of the paper is organized as follows. In Section 2, we provide some necessary key technical results. In Section 3, we demonstrate that the problem (4) has a solution in $H^1(\Omega)$ by using Faedo–Galerkin method. Finally, in Section 4, we prove the uniqueness solution of the problem (4).

2. Preliminaries. In our forthcoming analysis we will use a slightly modified lemma of Lemma 2.1 in [6].

**Lemma 2.1.** Let $\phi \in H^{1/4}(0, T)$ be a function extended by zero for any time $s > T$, and $\gamma \geq 0$. Then, we have the properties :

$$\begin{align*}
(i) \quad & \text{Re} \left( e^{i\pi/4} \int_0^t e^{-\gamma t} e^{iV(t)} \partial_t^{1/2} \phi dt \right) \geq 0 \\
(ii) \quad & \text{Re} \left( e^{-i\pi/4} \int_0^t e^{-\gamma t} e^{iV(t)} \partial_t^{1/2} \phi dt \right) \geq 0 \\
(iii) \quad & \text{If } \phi(0) = 0 \text{ then } \text{Re} \left( e^{-i\pi/4} \int_0^t e^{-\gamma t} e^{iV(t)} \partial_t^{1/2} \phi dt \right) \geq 0
\end{align*} \quad (5)$$
Proof. We apply the Plancherel identity for the Laplace transform to $e^{-\frac{\gamma}{2} \phi}$ and $e^{-\frac{\gamma}{2} \partial_t^{1/2} \phi}$ using

$$\mathcal{L} \left( e^{-\frac{\gamma}{2} \phi} \right) (s) = \mathcal{L} (\phi)(s + \frac{\gamma}{2}),$$

and

$$\mathcal{L} \left( \partial_t^{1/2} \phi \right) (s) = \sqrt{s \mathcal{L} (\phi)}(s).$$

We have

$$\int_{0}^{\infty} e^{-\gamma t} \phi \partial_{t}^{1/2} \phi \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L} \left( e^{-\frac{\gamma}{2} \phi} \right) (iv) \mathcal{L} \left( e^{-\frac{\gamma}{2} \partial_t^{1/2} \phi} \right) (iv) \, dv$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{\gamma}{2} + iv} \left| \mathcal{L} (\phi) \left( iv + \frac{\gamma}{2} \right) \right|^2 \, dv.$$

As $\gamma \geq 0$ and $v \in \mathbb{R}$, we have

$$e^{\pi/4} \sqrt{\frac{\gamma}{2} + iv} = \sqrt{\frac{\gamma^2}{4} + v^2 e^{i\left( \frac{\pi}{2} + \frac{\pi}{2} \right)}}, \quad \text{avec } \frac{\theta(v)}{2} + \frac{\pi}{4} \in [0; \frac{\pi}{2}], \quad (8)$$

and

$$e^{-\pi/4} \sqrt{\frac{\gamma}{2} + iv} = \sqrt{\frac{\gamma^2}{4} + v^2 e^{i\left( \frac{\pi}{2} + \frac{\pi}{2} \right)}}, \quad \text{avec } \frac{\theta(v)}{2} - \frac{\pi}{4} \in [-\frac{\pi}{2}; 0], \quad (9)$$

hence, we obtain (i) and (ii).

Proof of (iii):

Since $\phi(0) = 0$, we have

$$\int_{0}^{\infty} e^{-\gamma t} \phi \partial_{t}^{1/2} \phi \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L} \left( e^{-\frac{\gamma}{2} \phi} \right) (iv) \mathcal{L} \left( e^{-\frac{\gamma}{2} \partial_t^{1/2} \phi} \right) (iv) \, dv$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{L} (\phi_t)(iv + \frac{\gamma}{2}) \mathcal{L} \left( \partial_t^{1/2} \phi \right) (iv + \frac{\gamma}{2}) \, dv$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\gamma}{2} - iv \right) \sqrt{\frac{\gamma}{2} + iv} \left| \mathcal{L} (\phi) \left( iv + \frac{\gamma}{2} \right) \right|^2 \, dv.$$

Since $\gamma \geq 0$ and $v \in \mathbb{R}$, we have

$$\frac{\gamma}{2} + iv = \sqrt{\frac{\gamma^2}{4} + v^2 e^{i\theta(v)}}, \quad \text{with } \theta(v) \in [-\frac{\pi}{2}; \frac{\pi}{2}],$$

and

$$\frac{\gamma}{2} - iv = \sqrt{\frac{\gamma^2}{4} + v^2 e^{-i\theta(v)}}, \quad \text{avec } -\theta(v) \in [-\frac{\pi}{2}; \frac{\pi}{2}].$$

Then

$$e^{-\pi/4} \left( \frac{\gamma}{2} - iv \right) \sqrt{\frac{\gamma}{2} + iv} = \sqrt{\frac{\gamma^2}{4} + v^2 e^{-i\left( \frac{\pi}{2} + \frac{\pi}{2} \right)}}, \quad \text{with } -\frac{\theta(v)}{2} - \frac{\pi}{4} \in [-\frac{\pi}{2}; 0].$$

Which give

$$\text{Re} \left( e^{-\pi/4} \left( \frac{\gamma}{2} - iv \right) \sqrt{\frac{\gamma}{2} + iv} \right) \geq 0.$$

Since

$$e^{-\pi/4} \int_{0}^{\infty} e^{-\gamma t} \phi \partial_{t}^{1/2} \phi \, dt = \frac{e^{-\pi/4}}{2\pi} \int_{-\infty}^{\infty} \left( \frac{\gamma}{2} - iv \right) \sqrt{\frac{\gamma}{2} + iv} \left| \mathcal{L} (\phi) \left( iv + \frac{\gamma}{2} \right) \right|^2 \, dv.$$

Hence, we deduce (iii).

$\square$
Next, we present our second technical result [1, 9].

**Lemma 2.2.** Consider the initial value problem in $\mathbb{C}^m$

$$
\begin{aligned}
H'(t) &= M \theta_1^{1/2} H(t) + P(H(t)), \\
H(0) &= H_0,
\end{aligned}
$$

(10)

where $M$ is a square matrix of order $m$, $P$ represents a polynomial function of $\mathbb{C}^m$, $H_0 \in \mathbb{C}^m$ is a constant vector. Then, the problem (10) has a unique local solution $H \in L^\infty(0,T;\mathbb{C}^m)$.

We introduce the technical result.

**Lemma 2.3.** Let $w \in H^1(\Omega)$ be a function defined from $\Omega = ]-\infty,0[ \to \mathbb{C}$. Then, we have the following inequality

$$
|w(0)|^2 \leq 2 \|w\|_{L^2(\Omega)} \|w_x\|_{L^2(\Omega)}. 
$$

(11)

**Proof.** Let $y \in \Omega$, we have

$$
\int_y^0 \frac{d}{dx}[w(x)]^2 dx = w^2(0) - w^2(y),
$$

which gives

$$
w^2(0) = w^2(y) + 2 \int_y^0 w_x(x) w(x) dx.
$$

Since $w \in H^1(\Omega)$, we have $\lim_{y \to -\infty} w(y) = 0$. Then, we have when $y \to -\infty$

$$
|w(0)|^2 \leq 2 \int_{-\infty}^0 |w_x(x)| |w(x)| dx.
$$

By using Cauchy-Schwarz inequality, we get (11). □

Finally, we introduce our last technical result.

**Lemma 2.4.** Assume that the sequence $(\lambda_m)_m$ of $H^1(\Omega)$ is such that $\|\lambda_m\|_{H^1(\Omega)} \leq C$ and $\lambda_m \to 0$ in $L^2(\Omega)$ when $m \to +\infty$. Then $\lambda_m(0) \to 0$ when $m \to +\infty$.

**Proof.** Let $y \in ]-\infty,0[$, we have

$$
\lambda_m^2(0) - \lambda_m^2(y) = 2 \int_y^0 \partial_x \lambda_m(x) \lambda_m(x) dx,
$$

Since $\lambda_m \in H^1(\Omega)$, we have $\lim_{y \to -\infty} \lambda_m(y) = 0$. Then, we have when $y \to -\infty$

$$
|\lambda_m(0)|^2 \leq 2 \int_{-\infty}^0 |\partial_x \lambda_m(x)| \lambda_m(x) dx
$$

$$
\leq 2 \|\lambda_m\|_{L^2(\Omega)} \|\lambda_m\|_{L^2(\Omega)}
$$

$$
\leq 2C \|\lambda_m\|_{L^2(\Omega)}.
$$

Hence the result. □
3. Existence of the solution. We set

\[ v(x, t) = \exp(-iV(0, t))u(x, t), \]

we have \(|v(x, t)| = |u(x, t)|\). Then, the problem (4) becomes

\[
\begin{cases}
iv_t + \frac{1}{2}iv_{xx} + q \delta_a v + (|v(x, t)|^2 - |v(0, t)|^2) v + iv\gamma v = \exp(-iV(0, t))f, & \text{in } \Omega \times \mathbb{R}_+ \\
\partial_n v(0, t) + \sqrt{2}e^{-i\pi/4}e^{-\gamma t}\partial_s^{1/2} (e^{\gamma t}v(0, t)) = 0, & t \in \mathbb{R}_+, \\
v(x, 0) = u_0(x), & x \in \Omega.
\end{cases}
\]

We use Faedo–Galerkin method to prove that there exists a solution \(v \in C([0, +\infty]; H^1(\Omega)) \cap C^1([0, +\infty]; H^1(\Omega))\)' of the problem (13). As a result, there exists \(u \in C([0, +\infty]; H^1(\Omega)) \cap C^1([0, +\infty]; H^1(\Omega))\) solution of (4) such that

\[ u(x, t) = \exp \left( i \int_0^t |v(0, s)| ds \right) v(x, t). \]

3.1. First step: Approximate problem. Let \((\varphi_k)_k\) be an orthogonal basis of functions in \(H^1(\Omega)\). For \(m \geq 1\), we set \(H_m = \text{Span}(\varphi_1, \ldots, \varphi_m)\) and we define the orthogonal projection operator \(P_m\) by

\[ P_m : H^1(\Omega) \rightarrow H_m : v \mapsto P_m(v) = \sum_{k=1}^m \langle v, \varphi_k \rangle_{H^1(\Omega)} \varphi_k, \]

where \(\langle \cdot, \cdot \rangle_{H^1(\Omega)}\) is the scalar product in \(H^1(\Omega)\). For \(m \geq 1\), we shall approximate \(v\) in (13) by

\[ v_m(t) = \sum_{k=1}^m h_{km}(t) \varphi_k, \]

which satisfies, \(\forall k = 1, 2, \ldots, m\)

\[
\begin{cases}
\frac{d}{dt} \langle iv_m, \varphi_k \rangle_{L^2(\Omega)} - \frac{1}{2} \langle v_{mx}, \varphi_k \rangle_{L^2(\Omega)} - \frac{1}{4} \langle v_{mm}, \varphi_k \rangle_{L^2(\Omega)} - \frac{\sqrt{2}e^{-i\pi/4}}{2} \varphi_k(0) + q \langle \delta_a v_m, \varphi_k \rangle_{H^1(\Omega)}, \quad (15)
\end{cases}
\]

where \(\langle \cdot, \cdot \rangle_{H^1(\Omega)}\) is the scalar product in \(H^1(\Omega)\).

\[ + |v_m|^2 \langle v_m, \varphi_k \rangle_{L^2(\Omega)} - |v_m(0, t)|^2 \langle v_m, \varphi_k \rangle_{L^2(\Omega)}, \quad (17)
\]

\[ + i \gamma \langle v_m, \varphi_k \rangle_{L^2(\Omega)} = \exp(-i\gamma) \langle f, \varphi_k \rangle_{L^2(\Omega)}, \quad v_m(0) = P_m(u_0), \]

where

\[ V_m(0, t) = \int_0^t |v_m(0, s)|^2 ds \]

We get the system

\[ iA H_m(t) - \frac{\sqrt{2}e^{-i\pi/4}}{2} B \partial_s^{1/2} H_m(t) = F(H_m(t)), \]

where \(F\) is a polynomial function, \(A\) and \(B\) are squares matrix of order \(m\) defined as

\[ A = \left( \langle \varphi_j, \varphi_k \rangle_{L^2(\Omega)} \right)_{k,j}, \quad B = \left( \varphi_j(0) \varphi_k(0) \right)_{k,j}, \]

and

\[ H_m(t) = \begin{pmatrix} h_{1m}(t) \\ \vdots \\ h_{mm}(t) \end{pmatrix} \in \mathbb{C}^m, \]
such that at $t = 0$ the components of $H_m(0)$ are
\[
h_{mk}(0) = k^{th} \text{ element of } u_{m0}.
\] (22)

The matrix $A$ is Hermitian and Positive-definite, then it is invertible. Therefore

\[
H'_m(t) + \frac{\sqrt{2}e^{i\pi/4}}{2} A^{-1} B \partial_t^{1/2} H_m(t) = -iA^{-1}F(H_m(t)),
\] (23)

we set
\[
M = -\frac{\sqrt{2}e^{i\pi/4}}{2} A^{-1} B,
\]

and
\[
P(H_m(t)) = -iA^{-1}F(H_m(t)).
\]

The system (23) becomes instead
\[
\begin{cases}
    H'_m(t) = M \partial_t^{1/2} H_m(t) + P(H_m(t)), \\
    H_m(0) = H_{m0},
\end{cases}
\] (24)

where $P$ is a polynomial function of $C_m$. Using the Lemma 2.2, we get that the system (24) has a unique local solution. Then, we obtain a unique function $H_m = (h_1^m, ..., h_m^m)$ in $[0,T_m]$ solution of (19) with the initial condition (22).

As a result, the approximate problem (17) has a unique solution $v_m$ such that $v_m : [0,T_m] \rightarrow H_m$. The existence of a maximal solution $v_m$ (defined on $[0,T_{max}]$) is obtained by iterating $m$, where $T_{max}$ is the maximum time of existence such that

\[
v_m : [0,T_{max}] \rightarrow H_m.
\] (25)

Then we have $T_{max} < +\infty$ and \( \lim_{t \to T_{max}} |v_m| = +\infty \), or $T_{max} = +\infty$ and \( \lim_{t \to T_{max}} |v_m| < +\infty \).

### 3.2. Second step : A priori estimates.

**Estimate in $L^2(\Omega)$**. We multiply the equation (13) by $-iv$, we integrate in space domain $\Omega$, and we take the real part of the resulting equation and we use integration by parts for the second term to obtain

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{L^2(\Omega)}^2 + \gamma \|v(t)\|_{L^2(\Omega)}^2 = \frac{1}{2} \Re \left( i v(0,t) \partial_n v(0,t) \right) \\
+ i \Im \left( \exp(-iV(0,t)) \int_{-\infty}^{0} f v dx \right).
\] (26)

Using Cauchy–Schwarz, Young inequalities, and Gronwall lemma, we obtain

\[
\|v(t)\|_{L^2(\Omega)}^2 \leq e^{-\gamma t} \|v_0\|_{L^2(\Omega)}^2 + \frac{1 - e^{-\gamma t}}{\gamma^2} \|f\|_{L^2(\Omega)}^2 + e^{-\gamma t} \int_{0}^{t} e^{\gamma s} \Re \left( i v(0,s) \partial_n v(0,s) \right) ds.
\] (27)

We use the boundary condition for the problem (13), we have

\[
\frac{e^{-\gamma t}}{2} \int_{0}^{t} e^{\gamma s} \Re \left( i v(0,s) \partial_n v(0,s) \right) ds \\
= -\frac{\sqrt{2}e^{-\gamma t}}{2} \Re \left( e^{i\pi/4} \int_{0}^{t} e^{\gamma s} v(0,t) \partial_t^{1/2} (e^{\gamma s} v(0,s)) ds \right).
\] (28)
The Lemma 2.1 shows that the term in (28) is negative. Hence
\[ \|v(t)\|^2_{L^2(\Omega)} \leq e^{-\gamma t}\|u_0\|^2_{L^2(\Omega)} + \frac{1 - e^{-\gamma t}}{\gamma^2}\|f\|^2_{L^2(\Omega)}. \] (29)

Then, \(\forall t \geq 0\), we have
\[ \|v(t)\|^2_{L^2(\Omega)} \leq M_0^2 = \|u_0\|^2_{L^2(\Omega)} + \frac{1}{\gamma^2}\|f\|^2_{L^2(\Omega)}, \] (30)

**Estimate in \(H^1(\Omega)\).** In the following, we denote by \(M\) any positive constant depending only on \(q, \gamma, \|f\|_{L^2(\Omega)}\) and \(\|u_0\|_{H^1(\Omega)}\).

We multiply the first equality of (13) by \(\bar{v}_i + \gamma \bar{v}\) and we integrate in space domain \(\Omega\). By considering the real part, we get
\[
\frac{d}{dt} \Psi(v(t)) + 2\gamma \Psi(v(t)) = 2\mathcal{R}e \left( v_i(0,t) + \gamma v_i(0,t)\partial_n v(0,t) \right) - 2|v(0,t)|^2 \frac{d}{dt}\|v\|^2_{L^2(\Omega)}
- 4\gamma|v(0,t)|^2\|v\|^2_{L^2(\Omega)} + 2\gamma\|v\|^4_{L^4(\Omega)} + 4\gamma\mathcal{R}e \left( \exp(-i\mathcal{V}(0,t)) \int_{-\infty}^{0} f\tau dx \right)
- 4\mathcal{R}e \left( i|v(0,t)|^2 \exp(-i\mathcal{V}(0,t)) \int_{-\infty}^{0} f\tau dx \right),
\] (31)
where
\[
\Psi(v(t)) = \|v_x\|^2_{L^2(\Omega)} - 2q|v(a,t)|^2 - \|v\|^4_{L^4(\Omega)} + 4\mathcal{R}e \left( \exp(-i\mathcal{V}(0,t)) \int_{-\infty}^{0} f\tau dx \right).
\] (32)

We first need to upper-bound \(\|v_x\|^2_{L^2(\Omega)}\) in terms of \(\Psi(v(t))\). Indeed, we have by (32)
\[
\|v_x\|^2_{L^2(\Omega)} = \Psi(v(t)) + 2q|v(a,t)|^2 + \|v\|^4_{L^4(\Omega)}
- 4\mathcal{R}e \left( \exp(-i\mathcal{V}(0,t)) \int_{-\infty}^{0} f\tau dx \right).
\] (33)

By Agmon and Young inequalities, we have
\[
2q|v(a,t)|^2 \leq 2q|\mathcal{M}_0|\|v_x\|^2_{L^2(\Omega)} \leq \frac{1}{4}\|v_x\|^2_{L^2(\Omega)} + M.
\] (34)

Using Gagliardo-Nirenberg and Young inequalities, we obtain
\[
\|v\|^4_{L^4(\Omega)} \leq \frac{1}{4}\|v_x\|^2_{L^2(\Omega)} + M.
\] (35)

By Cauchy–Schwarz, we have
\[
- 4\mathcal{R}e \left( \exp(-i\mathcal{V}(0,t)) \int_{-\infty}^{0} f\tau dx \right) \leq 4\|f\|_{L^2(\Omega)} M_0 \leq M.
\] (36)

Hence, by using the above inequalities (34), (35) and (36), we obtain
\[
\|v_x\|^2_{L^2(\Omega)} \leq 2\Psi(v(t)) + M.
\] (37)

Next, we upper-bound the second member of equality (31).

We use Gagliardo-Nirenberg and Young inequalities combined with (37) to obtain
\[
2\gamma\|v\|^4_{L^4(\Omega)} \leq \frac{\gamma}{4}\|v_x\|^2_{L^2(\Omega)} + M
\leq \frac{\gamma}{2}\Psi(v(t)) + M,
\] (38)
We use Young inequality and Lemma 2.3 combined with (30) and (37) to obtain
\[ -4\Re \left( i|v(0,t)|^2 \exp(-iV(0,t)) \int_{-\infty}^{0} f \pi dx \right) \leq M|v(0,t)|^2 \]
\[ \leq 2M\|v\|_{L^2(\Omega)} \|v_x\|_{L^2(\Omega)} \]
\[ \leq M\|v_x\|_{L^2(\Omega)} \]
\[ \leq \frac{\gamma}{4}\|v_x\|_{L^2(\Omega)}^2 + M \]
\[ \leq \frac{\gamma}{2}\Psi(v(t)) + M. \quad (39) \]

We have
\[ \left| -|v(0,t)|^2 \frac{d}{dt} \|v\|^2_{L^2(\Omega)} \right| = |v(0,t)|^2 \frac{d}{dt} \left( \lambda(t)\|v\|^2_{L^2(\Omega)} \right), \quad (40) \]
where
\[ \lambda(t) = \text{sign} \left( \frac{d}{dt} \|v\|^2_{L^2(\Omega)} \right). \]

Applying the Lemma 2.3 and (30), we get
\[ \left| -|v(0,t)|^2 \frac{d}{dt} \|v\|^2_{L^2(\Omega)} \right| \leq 2\|v\|_{L^2(\Omega)} \|v_x\|_{L^2(\Omega)} \frac{d}{dt} \left( \lambda(t)\|v\|^2_{L^2(\Omega)} \right) \]
\[ \leq M\|v_x\|_{L^2(\Omega)} \frac{d}{dt} \left( \lambda(t)\|v\|^2_{L^2(\Omega)} \right). \quad (41) \]

Using (31), (36), (38), (39) and (41), we obtain
\[ \frac{d}{dt} \Psi(v(t)) + \gamma\Psi(v(t)) \leq M + 2\Re \left( \bar{v_l}(0,t) + \gamma v(0,t) \partial_n v(0,t) \right) \]
\[ + M\|v_x\|_{L^2(\Omega)} \frac{d}{dt} \left( \lambda(t)\|v\|^2_{L^2(\Omega)} \right). \]

We multiply this inequality by \( e^{\gamma t} \), we get
\[ \frac{d}{dt} \left( e^{\gamma t} \Psi(v(t)) \right) \leq Me^{\gamma t} + 2\Re \left( e^{\gamma t}v(0,t) \partial_n v(0,t) \right) \]
\[ + Me^{\frac{1}{2}\gamma t} \sup_{t \in [0,T]} \left[ e^{\gamma t}\|v_x\|^2_{L^2(\Omega)} \right]^{\frac{1}{2}} \frac{d}{dt} \left( \lambda(t)\|v\|^2_{L^2(\Omega)} \right), \]
where \( T > 0 \). Using the fact that
\[ Me^{\frac{1}{2}\gamma t} \sup_{t \in [0,T]} \left[ e^{\gamma t}\|v_x\|^2_{L^2(\Omega)} \right]^{\frac{1}{2}} \frac{d}{dt} \left( \lambda(t)\|v\|^2_{L^2(\Omega)} \right) \]
\[ = \frac{d}{dt} \left( Me^{\frac{1}{2}\gamma t} \sup_{t \in [0,T]} \left[ e^{\gamma t}\|v_x\|^2_{L^2(\Omega)} \right]^{\frac{1}{2}} \lambda(t)\|v\|^2_{L^2(\Omega)} \right) \]
\[ - \frac{\gamma}{2} Me^{\frac{1}{2}\gamma t} \sup_{t \in [0,T]} \left[ e^{\gamma t}\|v_x\|^2_{L^2(\Omega)} \right]^{\frac{1}{2}} \lambda(t)\|v\|^2_{L^2(\Omega)}, \]
to obtain
\[
\frac{d}{dt} \left( e^{\gamma t} \Psi(v(t)) - Me^{\frac{1}{2} \gamma t} \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \lambda(t)\|v\|_{L^2(\Omega)}^2
\]
\[
\leq Me^{\gamma t} + 2Re \left( (e^{\gamma t}v(0,t))_x \partial_n v(0,t) \right)
\]
\[
- \frac{\gamma}{2} Me^{\frac{1}{2} \gamma t} \sup_{t \in [0,T]} \left[ e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right]^{\frac{1}{2}} \lambda(t)\|v\|_{L^2(\Omega)}^2
\]
\[
\leq Me^{\gamma t} + 2Re \left( (e^{\gamma t}v(0,t))_x \partial_n v(0,t) \right) + Me^{\frac{1}{2} \gamma t} \sup_{t \in [0,T]} \left[ e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right]^{\frac{1}{2}}.
\]

We integrate this inequality between 0 and \( t \) with \( t \in [0, T] \), we obtain
\[
e^{\gamma t} \Psi(v(t)) - Me^{\frac{1}{2} \gamma t} \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \lambda(t)\|v\|_{L^2(\Omega)}^2
\]
\[
\leq \Psi(u_0) - M \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} \lambda(0)\|u_0\|_{L^2(\Omega)}^2
\]
\[
+ \frac{M}{\gamma} (e^{\gamma t} - 1) + 2 \int_0^t Re \left( (e^{\gamma s}v(0,s))_x \partial_n v(0,s) \right) ds
\]
\[
+ \frac{2M}{\gamma} (e^{\frac{1}{2} \gamma t} - 1) \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\]

Consequently,
\[
e^{\gamma t} \Psi(v(t)) \leq \Psi(u_0) + Me^{\gamma t} + M \left( 1 + e^{\frac{1}{2} \gamma t} \right) \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}
\]
\[
+ 2 \int_0^t Re \left( (e^{\gamma s}v(0,s))_x \partial_n v(0,s) \right) ds.
\]

(42)

Using the boundary condition (2) and Lemma 2.1, we have
\[
2 \int_0^t Re \left( (e^{\gamma s}v(0,s))_x \partial_n v(0,s) \right) ds
\]
\[
= -2\sqrt{2}Re \left( e^{-i\pi/4} \int_0^t e^{-\gamma s} \partial_s (e^{\gamma s}v(0,s)) \partial_s^{1/2} (e^{\gamma s}v(0,s)) ds \right) \leq 0.
\]

Then, the inequality (42) becomes
\[
e^{\gamma t} \Psi(v(t)) \leq \Psi(u_0) + Me^{\gamma t} + M \left( 1 + e^{\frac{1}{2} \gamma t} \right) \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}
\]

Using (37), we obtain
\[
e^{\gamma t} \|v_x(t)\|_{L^2(\Omega)}^2 \leq 2e^{\gamma t} \Psi(v(t)) + Me^{\gamma t}
\]
\[
\leq 2\Psi(u_0) + Me^{\gamma t} + M \left( 1 + e^{\frac{1}{2} \gamma t} \right) \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\]

(43)

Since \( t \in [0, T] \), we have
\[
e^{\gamma t} \|v_x(t)\|_{L^2(\Omega)}^2 \leq 2\Psi(u_0) + Me^{\gamma T} + Me^{\frac{1}{2} \gamma T} \sup_{t \in [0,T]} e^{\gamma t} \|v_x\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\]

(44)
Applying Young’s inequality, we have

\[ Me^{\frac{1}{2} T} \sup_{t \in [0, T]} \left[ e^{\gamma t} v_x(t)^2 \right] \leq \frac{1}{2} \sup_{t \in [0, T]} \left[ e^{\gamma t} v_x(t)^2 \right] + Me^{\gamma T}. \]  

Hence, the inequality (44) becomes

\[ e^{\gamma t} \| v_x(t) \|^2_{L^2(\Omega)} \leq 2\Psi(u_0) + Me^{\gamma T} + \frac{1}{2} \sup_{t \in [0, T]} \left[ e^{\gamma t} v_x(t)^2 \right]. \]

By taking the sup on the left side of this inequality, we get

\[ \sup_{t \in [0, T]} \left[ e^{\gamma t} v_x(t)^2 \right] \leq 2\Psi(u_0) + Me^{\gamma T} + \frac{1}{2} \sup_{t \in [0, T]} \left[ e^{\gamma t} v_x(t)^2 \right]. \]

Which gives

\[ \sup_{t \in [0, T]} \left[ e^{\gamma t} v_x(t)^2 \right] \leq 4\Psi(u_0) + Me^{\gamma T}. \]

For \( t = T \), this inequality is written

\[ e^{\gamma T} \| v_x(T) \|^2_{L^2(\Omega)} \leq 4\Psi(u_0) + Me^{\gamma T}. \]

Therefore, for all \( T \geq 0 \), we have

\[ \| v_x(T) \|^2_{L^2(\Omega)} \leq 4\Psi(u_0)e^{-\gamma T} + M \leq 4\Psi(u_0) + M \]

Then, \( \forall t \geq 0 \), we have

\[ \| v(t) \|_{H^1(\Omega)} \leq M. \]  

Then, the sequence \( (v_m)_m \) remains bounded in \( C_0([0, +\infty[; H^1(\Omega)) \), and

\[ (v'_m)_m \) remains bounded in \( C_0([0, +\infty[; [H^1(\Omega)]'). \]

**Estimate of \( xv(t) \) in \( L^2(\Omega) \).** We multiply the equation (13) by \( -ix^2 \sigma \), we integrate in space domain \( \Omega \) and we use integration by parts of the real part of the resulting equation to obtain

\[ \frac{1}{2} \frac{d}{dt} \| xv(t) \|^2_{L^2(\Omega)} - \frac{1}{2} \mathcal{I}m \left[ ix^2 v_x(x, t) \overline{v(x, t)} \right]_{-\infty}^{\infty} + \gamma \| xv(t) \|^2_{L^2(\Omega)} \]

\[ = \mathcal{I}m \left( \exp \left(-iV(0, t) \right) \int_{-\infty}^{\infty} x^2 f \nu dx \right) + 2\mathcal{I}m \left( \int_{\Omega} x v_x \nu \right). \]

Using the fact that \( \frac{1}{2} \mathcal{I}m \left[ ix^2 v_x(x, t) \overline{v(x, t)} \right]_{-\infty}^{\infty} = 0 \), and Cauchy-Schwarz, Young inequalities, we get

\[ \frac{d}{dt} \| xv(t) \|^2_{L^2(\Omega)} + \gamma \| xv(t) \|^2_{L^2(\Omega)} \leq M. \]

Finally, from Gronwall lemma, we get

\[ (v_m)_m \) remains bounded in \( C_0([0, +\infty[; H^1(\Omega) \cap L^2(\Omega; (1 + x^2) dx)). \]
3.3. Third step: Passing to the limit. We proved the existence of a unique function \( v_m \), checking (17), such that

the sequence \((v_m)_m\) remains bounded in \( C_b([0, +\infty[; H^1(\Omega)) \),

and

\[(v'_m)_m\) remains bounded in \( C_b([0, +\infty[; [H^1(\Omega)]') \).

Let \( T > 0 \). Since the sequence \((v_m)_m\) remains bounded in \( C_b([0, +\infty[; H^1(\Omega)) \). Then, \((v_m)_m\) is bounded in \( L^\infty(0, T; H^1(\Omega)) \).

By Banach-Alaoghi Theorem, we deduce that \((v_m)_m\) admits a sub-sequence still denoted \((v_m)_m\) such that

\[ v_m \xrightarrow{\ast} v \text{ weakly } \ast \text{ in } L^\infty(0, T; H^1(\Omega)), \]

i.e. \( \forall \psi \in D([0, T]; [H^1(\Omega)]') \) we have

\[
\int_0^T \langle v_m(t), \psi(t) \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt \longrightarrow \int_0^T \langle v(t), \psi(t) \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt. \tag{51}
\]

By uniqueness of the limit in \( D'((0, T) \times \Omega) \), we obtain

\[
\frac{dv_m}{dt} \xrightarrow{\ast} \frac{dv}{dt} \text{ weakly } \ast \text{ in } L^\infty(0, T; [H^1(\Omega)]'). \tag{52}
\]

We consider \( \omega \in D([0, T]) \) such that \( \omega(T) = 0 \). We pass to the limit in each term in the equation

\[
\int_0^T \langle iv_m' + \frac{1}{2} v_{mxx} + q\delta v_m + i\gamma v_m + [v_m^2 - |v_m(0, t)|^2] v_m - \exp(-i\mathcal{V}m(0, t))f, \omega(t)\varphi_k \rangle dt = 0,
\]

where \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{[H^1(\Omega)]', H^1(\Omega)} \).

Passing to the limit:

By Integration by parts with respect to time, we get

\[
I_m = \int_0^T \langle iv_m', \omega(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt = -\langle iv_m(0), \omega(0)\varphi_k \rangle_{L^2(\Omega), L^2(\Omega)} - \int_0^T \langle iv_m, \omega'(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt.
\]

Then

\[
\lim_{m \to +\infty} I_m = -\langle iu_0, \omega(0)\varphi_k \rangle_{L^2(\Omega), L^2(\Omega)} - \int_0^T \langle iv, \omega'(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt, \tag{53}
\]

and

\[
\lim_{m \to +\infty} \int_0^T \langle i\gamma v_m, \omega(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt = \int_0^T \langle i\gamma v, \omega(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt. \tag{54}
\]

By using Lemma 2.4, we have

\[
\lim_{m \to +\infty} \int_0^T \langle \exp(-i\mathcal{V}m(0, t))f, \omega(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt
\]

\[
= \int_0^T \langle \exp(-i\mathcal{V}(0, t))f, \omega(t)\varphi_k \rangle_{H^1(\Omega), [H^1(\Omega)]'} dt.
\]

We note by

\[
J_m = \int_0^T \langle v_{mxx}, \omega(t)\varphi_k \rangle_{[H^1(\Omega)]', H^1(\Omega)} dt, \tag{55}
\]
$$K_m = \int_0^T \langle |v_m|^2 v_m, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt,$$

(56)

$$L_m = \int_0^T \langle |v_m(0, t)|^2 v_m, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt,$$

(57)

and

$$A_m = \int_0^T \langle \delta_n v_m, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt.$$

(58)

For the term $J_m$, by applying Green formula, we have

$$J_m = \int_0^T \partial_n v_m(0, t) \overline{\varphi_k(0)} \omega(t) dt - \int_0^T \langle v_{mx}, \omega(t) \varphi_{kx} \rangle_{L^2(\Omega), L^2(\Omega)} dt$$

By using the fact that

$$\int_0^T \partial_n (v_m(0, t) - v(0, t)) \overline{\varphi_k(0)} \omega(t) dt$$

$$= - \sqrt{2} e^{-i \pi/4} \varphi_k(0) \int_0^T \partial_t^{1/2} (v_m(0, t) - v(0, t)) \omega(t) dt$$

$$= - \sqrt{2} e^{-i \pi/4} \varphi_k(0) \int_0^T \partial_t \left( \int_0^t \frac{v_m(0, s) - v(0, s)}{\sqrt{t-s}} ds \right) \omega(t) dt$$

$$= \sqrt{2} e^{-i \pi/4} \varphi_k(0) \int_0^T \omega(t) \left( \int_0^t \frac{v_m(0, s) - v(0, s)}{\sqrt{t-s}} ds \right) dt,$$

and by applying Lemma 2.4, we get

$$\lim_{m \to +\infty} J_m = \int_0^T \partial_n v(0, t) \overline{\varphi_k(0)} \omega(t) dt - \int_0^T \langle v_{x}, \omega(t) \varphi_{kx} \rangle_{L^2(\Omega), L^2(\Omega)} dt$$

$$= \int_0^T \langle v_{xx}, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt.$$

We use (48) and the fact that the injection

$$H^1(\Omega) \cap L^2(\Omega; (1 + x^2)dx) \hookrightarrow L^2(\Omega)$$

is compact to get

$$\lim_{m \to +\infty} K_m = \int_0^T \langle |v|^2 v, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt.$$

(59)

By applying Lemma 2.4, we get

$$\lim_{m \to +\infty} L_m = \int_0^T \langle |v(0, t)|^2 v, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt.$$  

(60)

Since for all $\varepsilon > 0$, the sequence $(\delta_n v_{m}(t))_m$ is bounded in $[H^{1+\varepsilon}(\Omega)]'$, we get

$$\lim_{m \to +\infty} A_m = \int_0^T \langle \delta_n v, \omega(t) \varphi_k \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} dt.$$  

(61)

We deduce, by continuity and density, that for all $\varphi \in H^1(\Omega)$, for $\omega \in \mathcal{D}(0, T)$, we see that $v$ checks

$$\int_0^T \langle iv' + \frac{1}{2} v_{xx} + q \delta_n v + |v|^2 - |v(0, t)|^2 |v + i \gamma v - \exp(-i\sqrt{v(0, t)}) f, \varphi \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} \omega(t) dt = 0.$$ 

We just justify that for all $\varphi \in H^1(\Omega)$ we have

$$\langle iv' + \frac{1}{2} v_{xx} + q \delta_n v + |v|^2 - |v(0, t)|^2 |v + i \gamma v - \exp(-i\sqrt{v(0, t)}) f, \varphi \rangle_{H^1(\Omega)^\prime, H^1(\Omega)} = 0.$$  

(62)
We get that \( v \) satisfies the initial condition \( v(0) = u_0 \). We then justify
\[
v \in C([0, +\infty [, H^1(\Omega)) \cap C^1 ([0, +\infty [; [H^1(\Omega)]') ,
\]
and satisfies the problem (13). Hence, there exists
\[
u \in C([0, +\infty [, H^1(\Omega)) \cap C^1 ([0, +\infty [; [H^1(\Omega)]')
\]
solution of (4) such that
\[
u(x, t) = \exp \left( i \int_0^t |v(0, s)| ds \right) v(x, t).
\]

4. **Uniqueness of the solution.** Let \( v(t) \) and \( \tilde{v}(t) \) be two solutions verifying the problem (13).

We set
\[
w(t) = v(t) - \tilde{v}(t),
\]
we get
\[
iw_t + \frac{1}{2} w_{xx} + q\delta_a w + |v|^2 u - |\tilde{v}|^2 \tilde{v} - |v(0, t)|^2 v + |\tilde{v}(0, t)|^2 \tilde{v}
+ i\gamma w = \left( \exp(-i\tilde{V}(0, t)) - \exp(-i\tilde{\mathcal{V}}(0, t)) \right) f,
\]
and the initial data
\[
w(0) = u_0 - u_0 = 0,
\]
with the boundary conditions
\[
\partial_n w(0, t) + \sqrt{2} e^{-i\pi/4} e^{-\gamma t} \partial_t^{1/2} \left( e^{\gamma t} w(0, t) \right) = 0, \quad t \in \mathbb{R}_+,
\]
where
\[
\tilde{\mathcal{V}}(0, t) = \int_0^t |\tilde{v}(0, s)|^2 ds.
\]

We multiply the equation (65) by \( \overline{w} \) taking the imaginary part, we get
\[
\frac{1}{2} \frac{d}{dt} \|w\|_{L^2(\Omega)}^2 + \gamma \|w\|_{L^2(\Omega)}^2 = -\mathcal{I} m \left( \partial_n w(0, t) \overline{w}(0, t) \right)
- \mathcal{I} m \left( \int_{\Omega} (|v|^2 v - |\tilde{v}|^2 \tilde{v}) \overline{w} dx \right) + \mathcal{I} m \left( \int_{\Omega} (|v(0, t)|^2 v - |\tilde{v}(0, t)|^2 \tilde{v}) \overline{w} dx \right)
+ \mathcal{I} m \left( \exp(-i\mathcal{V}(0, t)) - \exp(-i\tilde{\mathcal{V}}(0, t)) \right) \int_{\Omega} f \overline{w} dx,
\]
we have
\[
-\mathcal{I} m \left( \partial_n w(0, t) \overline{w}(0, t) \right) = -\sqrt{2} \mathcal{R} e \left( e^{i\pi/4} e^{-\gamma t} w(0, t) \partial_t^{1/2} \left( e^{\gamma t} w(0, t) \right) \right).
\]

We use Cauchy–Schwarz and Young inequalities, and we consider the fact that the injection of \( H^1(\Omega) \) in \( L^\infty(\Omega) \) is continuous and that \( v \) and \( \tilde{v} \) are in \( C_b([0, +\infty [; H^1(\Omega)) \), it comes that
\[
\frac{d}{dt} \|w(t)\|_{L^2(\Omega)}^2 + \gamma \|w(t)\|_{L^2(\Omega)}^2 \leq K \|w(t)\|_{L^2(\Omega)}^2
- \sqrt{2} \mathcal{R} e \left( e^{i\pi/4} e^{-\gamma t} w(0, t) \partial_t^{1/2} \left( e^{\gamma t} w(0, t) \right) \right).
\]

Using Gronwall’s lemma, and by Lemma 2.1, we get
\[
\|w(t)\|_{L^2(\Omega)}^2 \leq e^{(K-\gamma)t} \|w(0)\|^2 = 0,
\]
and therefore the uniqueness of the solution of (13).
Then, there exists a unique function
\[ u \in C([0, +\infty[; H^1(\Omega)) \cap C^1([0, +\infty[; [H^1(\Omega)]') \]
solution of the problem (4) such that
\[ u(x, t) = \exp \left( i \int_0^t |v(0, s)|ds \right) v(x, t). \quad (70) \]

**Conclusion.** In this paper, we have studied a weakly damped cubic nonlinear Schrödinger equation with Dirac distribution in \( \Omega = ]-\infty, 0[ \). For this purpose, a non-standard boundary condition was considered in order to demonstrate the well-posedness of the solution of:
\[
\begin{cases}
  iu_t + \frac{1}{2} u_{xx} + q\delta_a u + |u|^2 u + i\gamma u = f, & \text{in } \Omega \times \mathbb{R}_+ \\
  \partial_n u(0, t) + \sqrt{2}e^{-\pi/4}e^{-\gamma t}e^{i\varphi(0, t)}\partial_t \left( e^{\gamma t}e^{-i\varphi(0, t)}u(0, t) \right) = 0, & t \in \mathbb{R}_+,
\end{cases}
\]
\[ u(x, 0) = u_0(x), \quad x \in \Omega. \]
Then, by using the Galerkin method, we have shown that this equation can have a unique solution in \( H^1(\Omega) \). The remaining question is to investigate the weakly damped nonlinear Schrödinger equation with Dirac distribution on a bounded domain of \( \mathbb{R} \) or \( \mathbb{R}^2 \) with artificial boundary conditions.

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E-mail address: abderrazak.chrifi@gmail.com
E-mail address: m_abounouh@yahoo.fr
E-mail address: hassan.al.moatassime@gmail.com