The spanning number and the independence number of a subset of an abelian group

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Abstract

Let \( A = \{a_1, a_2, \ldots, a_m\} \) be a subset of a finite abelian group \( G \). We call \( A \) \( t \)-independent in \( G \), if whenever
\[
\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m = 0
\]
for some integers \( \lambda_1, \lambda_2, \ldots, \lambda_m \) with
\[
|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \leq t,
\]
we have \( \lambda_1 = \lambda_2 = \cdots = \lambda_m = 0 \), and we say that \( A \) is \( s \)-spanning in \( G \), if every element \( g \) of \( G \) can be written as
\[
g = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m
\]
for some integers \( \lambda_1, \lambda_2, \ldots, \lambda_m \) with
\[
|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \leq s.
\]

In this paper we give an upper bound for the size of a \( t \)-independent set and a lower bound for the size of an \( s \)-spanning set in \( G \), and determine some cases when this extremal size occurs. We also discuss an interesting connection to spherical combinatorics.

1 Introduction

We illuminate our concepts by the following examples.

**Example 1** Consider the set \( A = \{1, 4, 6, 9, 11\} \) in the cyclic group \( G = \mathbb{Z}_{25} \). We are interested in the degree to which this set is independent in \( G \). We find, for example, that \( 1 + 4 + 4 - 9 = 0 \) and \( 11 + 11 + 9 - 6 = 0 \), but that such an equation with only three terms from \( A \) cannot be found. We therefore say that \( A \) is 3-independent in \( G \) and write \( \text{ind}(A) = 3 \). It can be shown that \( A \) is optimal in each of the following regards:
• no subset of \( G \) of size \( m > 5 \) is 3-independent in \( G \) (furthermore, \( A \) is essentially the unique 3-independent set in \( G \) of size 5);

• no subset of \( G \) of size 5 is \( t \)-independent for \( t > 3 \) (that is, for \( t > 3 \), there will always be \( t \), not necessarily distinct, elements with a signed sum of 0); and

• \( n = 25 \) is the smallest odd number for which a 3-independent set of size 5 in \( \mathbb{Z}_n \) exists. (In fact, it can be shown that \( \mathbb{Z}_n \) has a 3-independent set of size 5, if and only if, \( n = 20, 22, 24, 25, 26, \) or \( n \geq 28. \))

The fact that \( G = \mathbb{Z}_{25} \) has this relatively large 3-independent subset is due, as explained later, to the fact that 25 has a prime divisor which is congruent to 5 mod 6.

Example 2 How can one place a finite number of points on the \( d \)-dimensional sphere \( S^d \subset \mathbb{R}^{d+1} \) with the highest momentum balance? For the circle \( S^1 \), the answer is given by the vertices of a regular polygon, but the issue is far more difficult for \( d > 1 \). For a positive integer \( n \) and a set of integers \( A = \{a_1, a_2, \ldots, a_m\} \), define the set of \( n \) points \( X(A) = \{x_1, x_2, \ldots, x_n\} \) with

\[
x_i = \frac{1}{\sqrt{m}} \left( \cos\left(\frac{2\pi a_1}{n}\right), \sin\left(\frac{2\pi a_1}{n}\right), \ldots, \cos\left(\frac{2\pi a_m}{n}\right), \sin\left(\frac{2\pi a_m}{n}\right) \right)
\]

\((i = 1, 2, \ldots, n)\); thus, for example, for \( n = 25 \) and \( A = \{1, 4, 6, 9, 11\} \), \( X(A) \) is a set of 25 points on the unit sphere \( S^9 \). It can be shown that this \( X(A) \) is a spherical 3-design, that is, for every polynomial \( f : S^9 \to \mathbb{R} \) of total degree at most 3, the average value of \( f \) on \( S^9 \) equals the arithmetic average of \( f \) on \( X(A) \). We can also verify that \( X(A) \) is optimal in that

• no set of 25 points is a \( t \)-design on \( S^9 \) for \( t > 3 \);

• no set of 25 points is a 3-design on \( S^d \) for \( d > 9 \);

• \( n = 25 \) is the minimum odd size for which a 3-design on \( S^9 \) exists. (It was recently proved that an \( n \)-point 3-design on \( S^9 \) exists, if and only if, \( n = 20, 22, 24, \) or \( n \geq 25. \))

Example 3 Finally, consider \( A = \{3, 4\} \) in \( G = \mathbb{Z}_{25} \). Note that every element of \( G \) can be generated by a signed sum of at most three terms of \( A \): \( 1 = 4 - 3, 2 = 3 + 3 - 4, \ldots, 24 = 3 - 4. \) We therefore call \( A = \{3, 4\} \) a 3-spanning set in \( G = \mathbb{Z}_{25} \), and write span(\( A \)) = 3. Again, our example is extremal; it can be shown that

• no subset of \( G \) of size \( m < 2 \) is 3-spanning in \( G \);

• no subset of \( G \) of size 2 is \( s \)-spanning for \( s < 3 \); and

• \( n = 25 \) is the largest number for which a 3-spanning set of size 2 in \( \mathbb{Z}_n \) exists. (Furthermore, as we will see, \( \mathbb{Z}_n \) has a 3-spanning set of size at most 2 for every \( n \leq 25. \))

In fact, this example has an even more distinguished property: every element of \( G \) can be written uniquely as a signed sum of at most 3 elements of \( A \); we call such a set perfect. As a consequence of being a perfect 3-spanning set, \( A \) is also a maximum size 6-independent set in \( G \).
The fact that $G = \mathbb{Z}_{25}$ has a perfect spanning subset of size 2 is due to the fact that 25 is the sum of two consecutive squares, as explained later.

In the subsequent sections of this paper we define and investigate the afore-mentioned concepts and statements. Topics similar to spanning numbers (e.g. $h$-bases) and independence numbers (e.g. sum-free sets, Sidon sets, and $B_h$ sequences) have been studied vigorously for a long time, see, for example, [9], [13], [15], [20], [22], [23], [27], and various sections of Guy’s book [14]. For general references on spherical designs, see [4], [8], [10], [11], [12], [17], [21], and [25].

2 Spanning numbers

Let $G$ be a finite abelian group of order $|G| = n$, written in additive notation. We are interested in the degree to which a given subset of $G$ spans $G$. More precisely, we introduce the following definition.

Definition 1 Let $s$ be a non-negative integer and $A = \{a_1, a_2, \ldots, a_m\}$. We say that $A$ is an $s$-spanning set in $G$, if every $g \in G$ can be written as

$$g = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m$$

for some integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with

$$|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \leq s.$$

We call the smallest $s$ for which $A$ is $s$-spanning the spanning number of $A$ in $G$, and denote it by $\text{span}(A)$.

Equivalently, $A$ is an $s$-spanning subset of $G$ if for every element $g \in G$, we can find non-negative integers $h$ and $k$ and elements $x$ and $y$ in $G$, so that $x$ is the sum of $h$ (not necessarily distinct) elements of $A$, $y$ is the sum of $k$ (not necessarily distinct) elements of $A$, $h + k \leq s$, and $g = x - y$.

The case $s = 0$ is trivial: the only group $G$ which has a 0-span is the one with a single element; therefore, we may assume that $s \geq 1$ and $n \geq 2$. Obviously, $A = G$ is an $s$-spanning subset of $G$ for every $s \geq 1$. Here we are interested in small $s$-spanning sets in $G$; we denote the size of a minimum $s$-spanning set of $G$ by $p(G, s)$.

For $s = 1$, it is clear that $\text{span}(A) = 1$, if and only if, for each $g \in G$, $A$ contains at least one of $g$ or $-g$; in particular, $A$ must contain every element of order 2. Let $O(G, 2)$ denote the set of order 2 elements of $G$; with this notation we have

$$p(G, 1) = |O(G, 2)| + \frac{|G \setminus O(G, 2) \setminus \{0\}|}{2} = \frac{n + |O(G, 2)| - 1}{2}. \quad (1)$$

As a special case, for the cyclic group of order $n$ we have

$$p(\mathbb{Z}_n, 1) = \lfloor n/2 \rfloor. \quad (2)$$
For \( s \geq 2 \), values of \( p(G, s) \) seem difficult to establish, even in the case of the cyclic groups. Computational data shows that

\[
p(Z_n, 2) = \begin{cases} 
0 & \text{if } n = 1; \\
1 & \text{if } n = 2, 3, 4, 5; \\
2 & \text{if } n = 6, 7, \ldots, 12, 13; \\
3 & \text{if } n = 14, 15, \ldots, 21; \\
4 & \text{if } n = 22, 23, \ldots, 33, \text{ and } n = 35; \\
5 & \text{if } n = 34, n = 36, 37, \ldots, 49, \text{ and } n = 51; 
\end{cases}
\]

and

\[
p(Z_n, 3) = \begin{cases} 
0 & \text{if } n = 1; \\
1 & \text{if } n = 2, 3, \ldots, 6, 7; \\
2 & \text{if } n = 8, 9 \ldots, 24, 25; \\
3 & \text{if } n = 26, 27, \ldots, 50, n = 52, \text{ and } n = 55; \\
4 & \text{if } n = 51, 53, 54, n = 56, 57, \ldots, 100, \text{ and } n = 104. 
\end{cases}
\]

(Values marked in bold-face will be discussed later.)

As these values indicate, \( p(Z_n, s) \) is, in general, not a monotone function of \( n \), though we believe that

\[
P(s) := \lim_{n \to \infty} \frac{p(Z_n, s)^s}{n}
\]

exists for every \( s \). The following theorem provides a lower bound for \( p(G, s) \) which is of the order \( n^{1/s} \) as \( n \) goes to infinity.

**Theorem 2** Let \( m \) and \( s \) be positive integers, and define \( a(m, s) \) recursively by \( a(m, 0) = a(0, s) = 1 \) and

\[
a(m, s) = a(m - 1, s) + a(m, s - 1) + a(m - 1, s - 1).
\]

1. We have

\[
a(m, s) = \sum_{k=0}^{s} \binom{s}{k} \binom{m}{k} 2^k.
\]

2. If \( G \) has order \( n \) and contains an \( s \)-spanning set of size \( m \), then \( n \leq a(m, s) \).

**Proof.** 1. Let us define

\[
a'(m, s) := \sum_{k=0}^{s} \binom{s}{k} \binom{m}{k} 2^k.
\]

Clearly, \( a'(m, 0) = a'(0, s) = 1 \); below we prove that \( a'(m, s) \) also satisfies the recursion.

We have

\[
a'(m - 1, s - 1) = \sum_{k=0}^{s-1} \binom{s-1}{k} \binom{m-1}{k} 2^k
\]

\[
= \sum_{k=0}^{s-2} \binom{s-1}{k} \binom{m-1}{k} 2^k + \binom{m-1}{s-1} 2^{s-1}.
\]
and
\[
a'(m-1, s) = \sum_{k=0}^{s} \binom{s}{k} \binom{m-1}{k} 2^k \\
= \sum_{k=0}^{s-1} \binom{s}{k} \binom{m-1}{k} 2^k + \binom{m-1}{s} 2^s \\
= \sum_{k=0}^{s-1} \binom{s-1}{k-1} \binom{m-1}{k} 2^k + \sum_{k=0}^{s-2} \binom{s-1}{k} \binom{m-1}{k} 2^k + \binom{m-1}{s-1} 2^{s-1} + \binom{m-1}{s} 2^s.
\]

Next, we add \(a'(m-1, s)\) and \(a'(m-1, s-1)\). Note that
\[
\binom{m-1}{s-1} 2^{s-1} + \binom{m-1}{s} 2^{s-1} + \binom{m-1}{s} 2^s = \binom{m}{s} 2^s,
\]
and
\[
\sum_{k=0}^{s-2} \binom{s-1}{k} \binom{m-1}{k} 2^k + \sum_{k=0}^{s-2} \binom{s-1}{k} \binom{m-1}{k} 2^k = \sum_{k=0}^{s-2} \binom{s-1}{k} \binom{m-1}{k} 2^{k+1},
\]
and by replacing \(k\) by \(k-1\), this sum becomes
\[
\sum_{k=0}^{s-1} \binom{s-1}{k-1} \binom{m-1}{k} 2^k.
\]

Therefore,
\[
a'(m-1, s) + a'(m-1, s-1) = \sum_{k=0}^{s-1} \binom{s-1}{k-1} \binom{m-1}{k} 2^k + \sum_{k=0}^{s-1} \binom{s-1}{k-1} \binom{m-1}{k} 2^k + \binom{m}{s} 2^s \\
= \sum_{k=0}^{s} \binom{s-1}{k-1} \binom{m}{k} 2^k + \binom{m}{s} 2^s \\
= \sum_{k=0}^{s} \binom{s}{k} \binom{m}{k} 2^k - \sum_{k=0}^{s} \binom{s-1}{k} \binom{m}{k} 2^k \\
= a'(m, s) - a'(m, s-1).
\]

2. Assume that \(A = \{a_1, \ldots, a_m\}\) is an \(s\)-spanning set in \(G\) of size \(m\), and let
\[
\Sigma = \{\lambda_1 a_1 + \cdots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}, |\lambda_1| + \cdots + |\lambda_m| \leq s\}.
\]
We will count the elements in the index set
\[
I = \{(\lambda_1, \ldots, \lambda_m) \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}, |\lambda_1| + \cdots + |\lambda_m| \leq s\},
\]
and
as follows. For \( k = 0, 1, 2, \ldots, m \), let \( I_k \) be the set of those elements of \( I \) where exactly \( k \) of the \( m \) coördinates are non-zero. How many elements are in \( I_k \)? We can choose which \( k \) of the \( m \) coördinates are non-zero in \( \binom{m}{k} \) ways; w.l.o.g. let these coördinates be \( \lambda_1, \lambda_2, \ldots, \lambda_k \). Next, we choose the values of \( |\lambda_1|, |\lambda_2|, \ldots, |\lambda_k| \); since the sum of these \( k \) positive integers is at most \( s \), we have \( \binom{s}{k} \) choices. Finally, each of these coördinates can be positive or negative, and therefore

\[
|I_k| = \binom{s}{k} \binom{m}{k} 2^k,
\]

and

\[
|I| = \sum_{k=0}^{m} \binom{s}{k} \binom{m}{k} 2^k = \sum_{k=0}^{s} \binom{s}{k} \binom{m}{k} 2^k = a(m, s).
\]

Since \( A \) is \( s \)-spanning in \( G \), we must have \( n = |\Sigma| \leq |I| = a(m, s). \)

Theorem 2 thus provides a lower bound for the size of an \( s \)-spanning set in \( G \) which is of the order \( n^{1/s} \) as \( n \) goes to infinity.

For exact values, we establish the following results.

**Proposition 3** Let \( s \geq 1 \) be an integer.

1. If \( 2 \leq n \leq 2s + 1 \), then the set \( \{1\} \) is \( s \)-generating in \( \mathbb{Z}_n \) and \( p(\mathbb{Z}_n, s) = 1 \).
2. If \( 2s + 2 \leq n \leq 2s^2 + 2s + 1 \), then the set \( \{s, s + 1\} \) is \( s \)-generating in \( \mathbb{Z}_n \) and \( p(\mathbb{Z}_n, s) = 2 \).
3. If \( n \geq 2s^2 + 2s + 2 \), then \( p(\mathbb{Z}_n, s) \geq 3 \).

**Proof.** 1 is trivial. To prove 2, let

\[
\Sigma = \{\lambda_1 s + \lambda_2 (s + 1) \mid \lambda_1, \lambda_2 \in \mathbb{Z}, |\lambda_1| + |\lambda_2| \leq s\}.
\]

The elements of \( \Sigma \) lie in the interval \([- (s^2 + s), (s^2 + s)]\) and, since the index set

\[
I = \{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \in \mathbb{Z}, |\lambda_1| + |\lambda_2| \leq s\}
\]

contains exactly \( 2s^2 + 2s + 1 \) elements, it suffices to prove that no integer in \([- (s^2 + s), (s^2 + s)]\) can be written as an element of \( \Sigma \) in two different ways. Indeed, it is an easy exercise to show that

\[
\lambda_1 s + \lambda_2 (s + 1) = \lambda'_1 s + \lambda'_2 (s + 1) \in \Sigma
\]

implies \( \lambda_1 = \lambda'_1 \) and \( \lambda_2 = \lambda'_2 \); therefore, the set \( \{s, s + 1\} \) is \( s \)-generating in \( \mathbb{Z}_n \). As the \( s \)-span of a single element can contain at most \( 2s + 1 \) elements, for values \( n \geq 2s + 2 \) we must have \( p(\mathbb{Z}_n, s) = 2 \).

Statement 3 follows from Theorem 2 by noting that \( a(2, s) = 2s^2 + 2s + 1 \). \( \square \)

Let us now examine the extremal cases of Theorem 2.

**Definition 4** Suppose that \( A \) is an \( s \)-spanning set of size \( m \) in \( G \) and that \( a(m, s) \) is defined as in Theorem 2. If \(|G| = n = a(m, s)\), then we say that \( A \) is a perfect \( s \)-spanning set in \( G \).
Cases where $\mathbb{Z}_n$ has a perfect $s$-spanning set for $s = 2$ and $s = 3$ are marked with bold-face in (3) and (4). Trivially, the empty-set is a perfect $s$-spanning set in $\mathbb{Z}_1$ for every $s$. With (2) and Proposition 3, we can exhibit some other perfect spanning sets in the cyclic group.

**Proposition 5** Let $m$, $n$, and $s$ be positive integers, and let $G = \mathbb{Z}_n$.

1. If $n = 2m + 1$, then the set $\{1, 2, \ldots, m\}$ is a perfect 1-spanning set in $G$.

2. If $n = 2s + 1$, then the set $\{1\}$ is a perfect $s$-spanning set in $G$.

3. If $n = 2s^2 + 2s + 1$, then the set $\{s, s + 1\}$ is a perfect $s$-spanning set in $G$.

Note that the sets given in Proposition 5 are not unique: any element of the set in 1 can be replaced by its negative; in 2, the set $\{a\}$ is perfect for every $a$ which is relatively prime to $n$; it is not difficult to show that another example in 3 is provided by $A = \{1, 2s + 1\}$ (however, the set $\{s, s + 1\}$ in Proposition 3 cannot be replaced by $\{1, 2s + 1\}$). We could not find perfect spanning sets for $s \geq 2$ and $m \geq 3$. It might be an interesting problem to find and classify all perfect spanning sets.

### 3 Independence numbers

As in the previous section, we let $G$ be a finite abelian group of order $|G| = n$, written in additive notation, and suppose that $A$ is a subset of $G$. Here we are interested in the degree to which $A$ is independent in $G$. More precisely, we introduce the following definition.

**Definition 6** Let $t$ be a non-negative integer and $A = \{a_1, a_2, \ldots, a_m\}$. We say that $A$ is a $t$-independent set in $G$, if whenever

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_m a_m = 0$$

for some integers $\lambda_1, \lambda_2, \ldots, \lambda_m$ with

$$|\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| \leq t,$$

we have $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$. We call the largest $t$ for which $A$ is $t$-independent the independence number of $A$ in $G$, and denote it by $\text{ind}(A)$.

Equivalently, $A$ is a $t$-independent set in $G$, if for all non-negative integers $h$ and $k$ with $h + k \leq t$, the sum of $h$ (not necessarily distinct) elements of $A$ can only equal the sum of $k$ (not necessarily distinct) elements of $A$ in a trivial way, that is, $h = k$ and the two sums contain the same terms in some order.

Here we are interested in the size of a maximum $t$-independent set in $G$; we denote this by $q(G, t)$. 


Since $0 \leq \text{ind}(A) \leq n-1$ holds for every subset $A$ of $G$ (so no subset is “completely” independent), we see that $q(G, 0) = n$ and $q(G, n) = 0$. It is also clear that $\text{ind}(A) = 0$, if and only if, $0 \in A$, hence

$$q(G, 1) = n - 1.$$  \hfill (5)

For the rest of this section we assume that $t \geq 2$.

We can easily determine the value of $q(G, 2)$ as well. First, note that $A$ cannot contain any element of $\{0\} \cup \text{Ord}(G, 2)$ (the elements of order at most 2); to get a maximum 2-independent set in $G$, take exactly one of each element or its negative in $G \setminus \text{Ord}(G, 2) \setminus \{0\}$, hence we have

$$q(G, 2) = \frac{n - |\text{Ord}(G, 2)| - 1}{2}. \hfill (6)$$

As a special case, for the cyclic group of order $n$ we have

$$q(\mathbb{Z}_n, 2) = \left\lfloor \frac{n - 1}{2} \right\rfloor. \hfill (7)$$

Note that if $\text{Ord}(G, 2) \cup \{0\} = G$ then $q(G, 2) = 0$; for $n \geq 2$ this occurs only for the elementary abelian 2-group. If $\text{Ord}(G, 2) \cup \{0\} \neq G$ then, since $\text{Ord}(G, 2) \cup \{0\}$ is a subgroup of $G$, we have $1 \leq |\text{Ord}(G, 2)| + 1 \leq n/2$, and therefore we get the following.

**Proposition 7** If $G$ is isomorphic to the elementary abelian 2-group, then $q(G, 2) = 0$. Otherwise

$$\frac{1}{4} n \leq q(G, 2) \leq \frac{1}{2} n.$$  

Let us now consider $t = 3$. As before, if $G$ does not contain elements of order at least 4, then $q(G, 3) = 0$; this occurs if and only if $G$ is isomorphic to the elementary abelian $p$-group for $p = 2$ or $p = 3$. In [3] we proved the following.

**Theorem 8** ([3]) If $G$ is isomorphic to the elementary abelian $p$-group for $p = 2$ or $p = 3$, then $q(G, 3) = 0$. Otherwise

$$q(G, 3) = \left\lfloor \frac{1}{4} n \right\rfloor \leq q(G, 3) \leq \frac{1}{4} n.$$  

These bounds can be attained since $q(\mathbb{Z}_9, 3) = 1$ and $q(\mathbb{Z}_4, 3) = 1$.

For the cyclic group $\mathbb{Z}_n$, we can find explicit 3-independent sets as follows. For every $n$, the odd integers which are less than $n/3$ form a 3-independent set; if $n$ is even, we can go up to (but not including) $n/2$ as then the sum of two odd integers cannot equal $n$. We can do better in one special case when $n$ is odd; namely, when $n$ has a prime divisor $p$ which is congruent to 5 mod 6, one can show that the set

$$\left\{ pi_1 + 2i_2 + 1 \mid i_1 = 0, 1, \ldots, \frac{n}{p} - 1, \ i_2 = 0, 1, \ldots, \frac{p - 5}{6} \right\} \hfill (8)$$

is 3-independent. It is surprising that these examples cannot be improved, as we have the following exact values.
Theorem 9 \([3]\) For the cyclic group \(G = \mathbb{Z}_n\) we have

\[ q(\mathbb{Z}_n, 3) = \begin{cases} \left\lfloor \frac{n}{4} \right\rfloor & \text{if } n \text{ is even}, \\ \left(1 + \frac{1}{2}\right) \frac{n}{6} & \text{if } n \text{ is odd, has prime divisors congruent to 5 } \pmod{6}, \\ \left\lfloor \frac{n}{6} \right\rfloor & \text{and } p \text{ is the smallest such divisor}, \\ \text{otherwise}. \end{cases} \]

For \(t \geq 4\), exact results seem more difficult. With the help of a computer, we generated the following values.

\[ q(\mathbb{Z}_n, 4) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4; \\ 1 & \text{if } n = 5, 6, \ldots, 12; \\ 2 & \text{if } n = 13, 14, \ldots, 26; \\ 3 & \text{if } n = 27, 28, \ldots, 45, \text{ and } n = 47; \\ 4 & \text{if } n = 46, n = 48, 49, \ldots, 68, \text{ and } n = 72, 73; \\ 5 & \text{if } n = 69, 70, 71, \text{ and } n = 74, 75, \ldots, 102; \end{cases} \]  

(9)

\[ q(\mathbb{Z}_n, 5) = \begin{cases} 0 & \text{if } n = 1, 2, 3, 4, 5; \\ 1 & \text{if } n = 6, 7, \ldots, 17, \text{ and } n = 19, 20; \\ 2 & \text{if } n = 18, n = 21, 22, \ldots, 37, n = 39, 40, 41, n = 43, 44, 45, 47; \\ 3 & \text{if } n = 38, 42, 46, n = 48, 49, \ldots, 69, n = 71, 72, 73, 75, 76, 77, 79, 81, 83, 85, 87; \end{cases} \]  

(10)

and

\[ q(\mathbb{Z}_n, 6) = \begin{cases} 0 & \text{if } n = 1, 2, 3, \ldots, 6; \\ 1 & \text{if } n = 7, 8, 9, \ldots, 24; \\ 2 & \text{if } n = 25, 26, 27, \ldots, 69; \\ 3 & \text{if } n = 70, 71, \ldots, 151, \text{ and } n = 153, 154, 155, 158, 159, 160. \end{cases} \]  

(11)

(Values marked in bold-face will be discussed later.)

Again we see that \(q(\mathbb{Z}_n, t)\) is not, in general, a monotone function of \(n\); although for even values of \(t\) the sequence seems to possess more regularity and we conjecture that

\[ Q(t) := \lim_{n \to \infty} \frac{q(\mathbb{Z}_n, t)^{t^2}}{n} \]

exists for every even \(t\). The following theorem establishes an upper bound for \(q(G, s)\) which is of the order \(n^{1/[t/2]}\) as \(n\) goes to infinity.

Theorem 10 Let \(m\) and \(t\) be positive integers, \(t \geq 2\), and let us denote

\[ q(m, t) = \begin{cases} a(m, t/2) & \text{if } t \text{ is even}, \\ a(m, (t-1)/2) + a(m-1, (t-1)/2) & \text{if } t \text{ is odd}, \end{cases} \]

where \(a(m, t)\) is defined in Theorem \([2]\). If \(G\) has order \(n\) and contains a \(t\)-independent set of size \(m\), then \(n \geq q(m, t)\).
Proof. Assume that $A = \{a_1, \ldots, a_m\}$ is a $t$-independent set in $G$ of size $m$, and define
\[ \Sigma = \{\lambda_1 a_1 + \cdots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}, |\lambda_1| + \cdots + |\lambda_m| \leq \lfloor t/2 \rfloor \} \]
and
\[ I = \{(\lambda_1, \ldots, \lambda_m) \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}, |\lambda_1| + \cdots + |\lambda_m| \leq \lfloor t/2 \rfloor \}. \]

As in the proof of Theorem 2 we have $|I| = a(m, \lfloor t/2 \rfloor)$. Since $A$ is a $t$-independent set in $G$, the elements listed in $\Sigma$ must be all distinct, hence $n \geq |\Sigma| = |I| = a(m, \lfloor t/2 \rfloor)$. If $t$ is even, we are done.

Now let
\[ \Sigma' = \{\lambda_1 a_1 + \cdots + \lambda_m a_m \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}, \lambda_1 \geq 1, |\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| = \lfloor t/2 \rfloor + 1 \} \]
and
\[ I' = \{(\lambda_1, \ldots, \lambda_m) \mid \lambda_1, \ldots, \lambda_m \in \mathbb{Z}, \lambda_1 \geq 1, |\lambda_1| + |\lambda_2| + \cdots + |\lambda_m| = \lfloor t/2 \rfloor + 1 \}. \]

We will count the elements in the index set $|I'|$ as follows. For $k = 0, 1, 2, \ldots, m - 1$, let $I_k$ be the set of those elements of $I'$ where exactly $k$ of the $m - 1$ coordinates $\lambda_2, \ldots, \lambda_m$ are non-zero. An argument similar to that in the proof of Theorem 2 shows that
\[ |I_k| = \binom{m-1}{k} \binom{\lfloor t/2 \rfloor}{k} 2^k, \]

hence
\[ |I'| = \sum_{k=0}^{m-1} \binom{m-1}{k} \binom{\lfloor t/2 \rfloor}{k} 2^k = a(m - 1, \lfloor t/2 \rfloor). \]

If $t$ is odd, then the elements listed in $\Sigma'$ must be distinct from each other and from those in $\Sigma$ as well, thus $n \geq |\Sigma| + |\Sigma'| = |I| + |I'| = a(m, \lfloor t/2 \rfloor) + a(m - 1, \lfloor t/2 \rfloor)$. $\square$

Theorem 10 thus provides an upper bound for the size of a $t$-independent set in $G$ which is of the order $n^{1/\lfloor t/2 \rfloor}$ as $n$ goes to infinity.

For exact values, we establish the following results.

**Proposition 11** Let $t \geq 2$ be an integer.

1. If $1 \leq n \leq t$, then $q(\mathbb{Z}_n, t) = 0$.
2. If $t + 1 \leq n \leq \lfloor t^2/2 \rfloor + t$, then the set $\{1\}$ is $t$-independent in $\mathbb{Z}_n$ and $q(\mathbb{Z}_n, t) = 1$.
3. (a) Suppose that $t$ is even. If $n \geq t^2/2 + t + 1$, then the set $\{t/2, t/2 + 1\}$ is $t$-independent in $\mathbb{Z}_n$ and $q(\mathbb{Z}_n, t) \geq 2$.
   (b) Suppose that $t$ is odd. If $n = (t^2 - 1)/2 + t + 1$, then the set $\{1, t\}$ is $t$-independent in $\mathbb{Z}_n$ and $q(\mathbb{Z}_n, t) = 2$. 

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Proof. Let \( q(m, t) \) be defined as in Theorem [10]. Since \( q(1, t) = t + 1 \), our first claim follows from Theorem [10]. To prove 2, note that if \( n \geq t + 1 \), then \( \{1\} \) is \( t \)-independent in \( \mathbb{Z}_n \); furthermore, \( q(2, t) = \lfloor t^2/2 \rfloor + t + 1 \).

Now let \( t \) be even, and assume that \( n \geq t^2/2 + t + 1 \). We define

\[
\Sigma = \{ \lambda_1 \frac{t}{2} + \lambda_2 (\frac{t}{2} + 1) \mid \lambda_1, \lambda_2 \in \mathbb{Z}, |\lambda_1| + |\lambda_2| \leq t \}.
\]

The elements of \( \Sigma \) lie in the interval \( -(t^2/2 + t), (t^2/2 + t) \) and therefore, to prove 3 (a), it suffices to show that

\[
0 = \lambda_1 \frac{t}{2} + \lambda_2 (\frac{t}{2} + 1) \in \Sigma
\]

implies \( \lambda_1 = \lambda_2 = 0 \), which is an easy exercise. Statement 3 (b) is essentially similar. 

We now turn to the extremal cases of Theorem [10].

Definition 12 Suppose that \( A \) is a \( t \)-independent set of size \( m \) in \( G \) and that \( q(m, t) \) is defined as in Theorem [10]. If \( |G| = n = q(m, t) \), then we say that \( A \) is a tight \( t \)-independent set in \( G \).

Cases where \( \mathbb{Z}_n \) has a tight \( t \)-independent set for \( t = 4, t = 5, \) and \( t = 6 \) are marked with bold-face in (9), (10), and (11). Trivially, the empty-set is a perfect \( t \)-independent set in \( \mathbb{Z}_1 \) for every \( t \). With (5), (7), Theorem [9] Proposition [11] and one other (sporadic) example, we have the following tight \( t \)-independent sets in the cyclic group.

Proposition 13 Let \( m, n, \) and \( t \) be positive integers, and let \( G = \mathbb{Z}_n \).

1. If \( n = 2 \), then the set \( \{1\} \) is a tight 1-independent set in \( G \).
2. If \( n = 2m + 1 \), then the set \( \{1, 2, \ldots, m\} \) is a tight 2-independent set in \( G \).
3. If \( n = 4m \), then the set \( \{1, 3, \ldots, 2m - 1\} \) is a tight 3-independent set in \( G \).
4. If \( n = t + 1 \), then the set \( \{1\} \) is a tight \( t \)-independent set in \( G \).
5. Let \( n = \lfloor t^2/2 \rfloor + t + 1 \). If \( t \) is even, then the set \( \{t/2, t/2 + 1\} \) is a tight \( t \)-independent set in \( G \); if \( t \) is odd, then the set \( \{1, t\} \) is a tight \( t \)-independent set in \( G \).
6. If \( n = 38 \), then the set \( \{1, 7, 11\} \) is a tight 5-independent set in \( G \).

Proposition [13] contains every tight (non-empty) \( t \)-independent set that we could find so far; in particular, we could not find tight \( t \)-independent sets for \( t \geq 4 \) and \( m \geq 3 \) other than the seemingly sporadic example listed last. The problem of finding and classifying all tight \( t \)-independent sets remains open.

As it is clear from our exposition, there is a strong relationship between \( s \)-spanning sets and \( t \)-independent sets when \( t \) is even. Namely, we have the following.
**Theorem 14** Let $s$ and $t$ positive integers, $t$ even. Let $A$ be a subset of $G$, and suppose that $\text{span}(A) = s$ and $\text{ind}(A) = t$.

1. The order $n$ of $G$ satisfies $a(m, t/2) \leq n \leq a(m, s)$.

2. We have $t \leq 2s$.

3. The following three statements are equivalent.
   
   (i) $t = 2s$; 
   (ii) $A$ is a perfect $s$-spanning set in $G$; and 
   (iii) $A$ is a tight $t$-independent set in $G$.

The analogous relationship when $t$ is odd is considerably more complicated and will be the subject of future study.

### 4 Spherical designs

Here we discuss an application of the previous section to spherical combinatorics. We are interested in placing a finite number of points on the $d$-dimensional sphere $S^d \subset \mathbb{R}^{d+1}$ with the highest momentum balance. The following definition was introduced by Delsarte, Goethals, and Seidel in 1977 [8].

**Definition 15** Let $t$ be a non-negative integer. A finite set $X$ of points on the $d$-sphere $S^d \subset \mathbb{R}^{d+1}$ is a spherical $t$-design, if for every polynomial $f$ of total degree $t$ or less, the average value of $f$ over the whole sphere is equal to the arithmetic average of its values on $X$.

In other words, $X$ is a spherical $t$-design if the Chebyshev-type quadrature formula

$$
\frac{1}{\sigma_d(S^d)} \int_{S^d} f(x) d\sigma_d(x) \approx \frac{1}{|X|} \sum_{x \in X} f(x)
$$

(12)

is exact for all polynomials $f : S^d \rightarrow \mathbb{R}$ of total degree at most $t$ ($\sigma_d$ denotes the surface measure on $S^d$).

The concept of $t$-designs on the sphere is analogous to $t-(v,k,\lambda)$ designs in combinatorics (see [21]), and has been studied in various contexts, including representation theory, spherical geometry, and approximation theory. For general references see [4], [8], [10], [11], [12], [17], [21], and [25]. The existence of spherical designs for every $t$ and $d$ and large enough $n = |X|$ was first proved by Seymour and Zaslavsky in 1984 [26].

A central question in the field is to find all integer triples $(t, d, n)$ for which a spherical $t$-design on $S^d$ exists consisting of $n$ points, and to provide explicit constructions for these parameters.
Clearly, to achieve high momentum balance on the sphere, one needs to take a large number of points. Delsarte, Goethals, and Seidel [5] provide the tight lower bound

\[ n \geq N_d^t := \left( d + \left\lfloor \frac{t}{2} \right\rfloor \right) + \left( d + \left\lfloor \frac{t-1}{2} \right\rfloor \right). \]  

We shall refer to the bound \( N_d^t \) in (13) as the DGS bound. Spherical designs of this minimum size are called tight. Bannai and Damerell [5, 6] proved that tight spherical designs for \( d \geq 2 \) exist only for \( t = 1, 2, 3, 4, 5, 7, \) or 11. All tight \( t \)-designs are known, except possibly for \( t = 4, 5, \) or 7. In particular, there is a unique 11-\( t \)-design (\( d = 23 \) and \( n = 196560 \)).

Let us now attempt to construct spherical designs. One’s intuition that the vertices of a regular polygon provide spherical designs on the circle \( S^1 \) is indeed correct; more precisely, we have the following.

**Proposition 16** Let \( t \) and \( n \) be positive integers.

1. If \( n \leq t \), then there is no \( n \)-point spherical \( t \)-design on \( S^1 \).

2. Suppose that \( n \geq t + 1 \). For a positive integer \( j \), define

\[ z_n^j := \left( \cos\left( \frac{2\pi j}{n} \right), \sin\left( \frac{2\pi j}{n} \right) \right). \]  

Then the set \( X_n := \{z_n^j | j = 1, 2, \ldots, n\} \) is a \( t \)-design on \( S^1 \).

**Proof.** 1 follows from the DGS bound as \( N_1^t = t + 1 \). To prove 2, we first note that, using spherical harmonics, one can prove (see [5]) that, in general, a finite set \( X \) is a spherical \( t \)-design, if and only if, for every integer \( 1 \leq k \leq t \) and every homogeneous harmonic polynomial \( f \) of total degree \( k \),

\[ \sum_{x \in X} f(x) = 0. \]

(A polynomial is harmonic if it is in the kernel of the Laplace operator.) The set of homogeneous harmonic polynomials of total degree \( k \) on the circle, \( \text{Harm}_k(S^1) \), is a 2-dimensional vector space over the reals and is spanned by the polynomials \( \text{Re}(z^k) \) and \( \text{Im}(z^k) \) where \( z = x + \sqrt{-1}y \) (we can think of the elements of \( X \) and \( S^1 \) as complex numbers). Therefore, we see that \( X \) is a \( t \)-design on \( S^1 \), if and only if,

\[ \sum_{z \in X} z^k = 0 \]

for \( k = 1, 2, \ldots, t \). With \( X_n \) as defined above, one finds that

\[ \sum_{j=1}^{n} (z_n^j)^k = \begin{cases} 0 & \text{if } k \not\equiv 0 \mod n, \\ n & \text{if } k \equiv 0 \mod n. \end{cases} \]

Therefore, \( X_n \) is a \( t \)-design on \( S^1 \), if and only if, \( k \not\equiv 0 \mod n \) for \( k = 1, 2, \ldots, t \) (using the terminology of our last section, if and only if, \( \{1\} \) is a \( t \)-independent set in \( \mathbb{Z}_n \)), or \( n \geq t + 1 \). \[ \square \]
A further classification of $t$-designs on the circle can be found in Hong’s paper [18]; he proved, for example, that if $n \geq 2t + 3$, then there are infinitely many $t$-designs on $S^1$ which do not come from regular polygons.

We now attempt to generalize Proposition 16 for higher dimensions. For simplicity, we assume that $d$ is odd, and let $m = (d + 1)/2$. (The case when $d$ is even can be reduced to this case by a simple technique, see [2] or [19].)

Let $a_1, a_2, \ldots, a_m$ be integers, and set $A := \{a_1, a_2, \ldots, a_m\}$. For a positive integers $n$, define

$$X_n(A) := \left\{ \frac{1}{\sqrt{m}} (z_n^j(a_1), z_n^j(a_2), \ldots, z_n^j(a_m)) \ | \ j = 1, 2, \ldots, n \right\},$$

(15)

where, like in (14),

$$z_n^j(a_i) := \left( \cos\left(\frac{2\pi j}{n}a_i\right), \sin\left(\frac{2\pi j}{n}a_i\right) \right).$$

Note that $X_n(A) \subset S^d$. In [2] we proved the following.

**Theorem 17** ([2]) Let $t, d,$ and $n$ be positive integers with $t \leq 3$, $d$ odd, and set $m = (d + 1)/2$. For integers $a_1, a_2, \ldots, a_m$, define $X_n(A)$ as in (15). If $A$ is a $t$-independent set in $\mathbb{Z}_n$, then $X_n(A)$ is a spherical $t$-design on $S^d$.

Theorem 17 yields the following results.

**Corollary 18** Let $n$ and $d$ be positive integers, $d$ odd, and set $m = (d + 1)/2$.

1. (a) If $n = 1$, then there is no $n$-point spherical 1-design on $S^d$.
   (b) If $n \geq 2$, define $a_i = 1$ for $1 \leq i \leq m$. Then the set $X_n(A)$, as defined in (15), is a spherical 1-design on $S^d$.

2. (a) If $n \leq d + 1$, then there is no $n$-point spherical 2-design on $S^d$.
   (b) If $n \geq d + 2$, define $a_i = i$ for $1 \leq i \leq m$. Then the set $X_n(A)$, as defined in (15), is a spherical 2-design on $S^d$.

3. (a) If $n \leq 2d + 1$, then there is no $n$-point spherical 3-design on $S^d$.
   (b) If $n \geq 2d + 2$ is even or if $n \geq 3d + 3$ is odd, define $a_i = 2i + 1$ for $1 \leq i \leq m$; if

$$n \geq \frac{p}{p + 1}(3d + 3)$$

where $p$ is a divisor of $n$ which is congruent to 5 mod 6, choose $A$ to be any $m$ elements of the set in (8). In each case the set $X_n(A)$, as defined in (15), is a spherical 3-design on $S^d$.
Proof. Parts (a) are from the DGS bounds $N_d^t$ for $t \leq 3$; parts (b) follow from Theorem 17 since, by (5), (7), and the paragraph before Theorem 9, the sets specified are $t$-independent for $t = 1, 2, \text{ and } 3$, respectively (note that in all cases of 2 and 3, $m = (d + 1)/2 \leq q(\mathbb{Z}_n, t)$).

Part 3 of Corollary 18 leaves the question of existence of 3-designs open for some odd values of $n$. Note that the minimum value of $\frac{p}{p + 1}(3d + 3)$ is $5(d + 1)/2$ (when $n$ is divisible by 5). In [2] we proved that a spherical 3-design on $S^d$ ($d$ odd) exists for every odd value of $n \geq 5(d + 1)/2$, and conjectured that 3-designs do not exist with $2(d + 1) < n < 5(d + 1)/2$ and $n$ odd. This conjecture is supported by the numerical evidence of Hardin and Sloane [16]. A recent result of Boumova, Boyvalenkov, and Danev [7] proves that no 3-design exists of odd size $n$ with $n \approx 2.32(d + 1)$. In particular, the case $d = 9$ of Example 2 in our Introduction is completely settled: 3-designs on $n$ points on $S^9$ exist, if and only if, $n \geq 20$ even, or $n \geq 25$ odd.

The application of $t$-independent sets to spherical $t$-designs seems more complicated when $t \geq 4$, and will be the subject of an upcoming paper.

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