Selecting $\mu \to e$ Conversion Targets to distinguish Lepton Flavour-Changing Operators

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Abstract

The experimental sensitivity to $\mu \to e$ conversion on nuclei is set to improve by four orders of magnitude in coming years. However, various operator coefficients add coherently in the amplitude for $\mu \to e$ conversion, weighted by nucleus-dependent functions, and therefore in the event of a detection, identifying the relevant new physics scenarios could be difficult. Using a representation of the nuclear targets as vectors in coefficient space, whose components are the weighting functions, we quantify the expectation that different nuclear targets could give different constraints. With current or achievable theoretical uncertainties, three suitably chosen targets could determine all but one of the coefficients contributing to Spin-Independent conversion. Spin-Dependent coefficients on protons and neutrons are currently distinguishable, but dedicated nuclear calculations would be required to differentiate between axial, tensor and pseudoscalar operators.

1 Introduction

The observation of neutrino mixing and masses implies that flavour cannot be conserved among charged leptons. However, despite a long programme of experimental searches for various processes, charged lepton flavour violation (CLFV) at a point has yet to be observed.

For $\mu \leftrightarrow e$ flavour change, the current most stringent bound is $BR(\mu \to e\gamma) \leq 4.2 \times 10^{-13}$ from the MEG collaboration [1] at PSI. This sensitivity will improve in coming years [2], and the Mu3e experiment [3] at PSI aims to reach $BR(\mu \to e\bar{e}e) \sim 10^{-16}$. Several experiments under construction will improve the sensitivity to $\mu \to e$ conversion on nuclei: The COMET [4] at J-PARC and the Mu2e [5] at FNAL plan to reach branching ratios of $\sim 10^{-16}$ on Aluminium. The PRISM/PRIME proposal [6] aims to probe $\sim 10^{-18}$, and at the same time enables to use heavy $\mu \to e$ conversion targets with shorter lifetimes of their muonic atoms, thanks to its designed pure muon beam with no pion contamination. § This enhanced sensitivity and wider-range selection of $\mu \to e$ conversion targets motivate our interest in low-energy $\mu \leftrightarrow e$ flavour change.

In the coming years, irrespective of whether CLFV is observed or further constrained, it is important to maximise the amount of information that experiments can obtain about the New Physics responsible for CLFV. This is especially challenging for the operators involving nucleons or quarks, because in $\mu \to e$ conversion, the contributing coefficients add in the amplitude. So in this paper, we consider $\mu \to e$ conversion on nuclei, and present a recipe for selecting targets such that they constrain or measure different CLFV parameters. Reference [10] is an earlier discussion of the prospects of distinguishing models with $\mu \to e$ conversion, which was also explored in [11]. In this letter, we follow the perspective of [11].

We assume that the New Physics responsible for $\mu \to e$ conversion is heavy, and parametrise it in Effective Field Theory [12, 13, 14, 15]. Section 2 gives the $\mu \to e$ conversion rate, and the effective Lagrangian at the experimental scale ($\sim$ GeV), in terms of operators that are QED invariant, labelled by their Lorentz structure, and constructed out of electrons, muons and nucleons ($p$ and $n$). In Section 3 we divide the rate into pieces that do not interfere with each other. Section 4 is a toy model of two observables that depend on a sum of theoretical parameters, which illustrates the impact of theoretical uncertainties on the determination of operator coefficients. It is well-known, since the study of Kitano, Koike and Okada (KKO) [16], that different target nuclei have different relative sensitivity to the various operator coefficients. In Section 5, using the notion of targets as vectors in the space of

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§Another interesting observable at these experiments is the $\mu^- e^- \to e^- e^-$ in a muonic atom. This process could have not only photonic dipole but also contact interactions, and the atomic number dependence of its reaction rate makes possible to discriminate the type of relevant CLFV interactions [7, 8, 9].
2 \( \mu \to e \) conversion

\( \mu \to e \) conversion is the process where an incident \( \mu^- \) is captured by a nucleus, and tumbles down to the 1s state. The muon can then interact with the nucleus, by exchanging a photon or via a contact interaction, and turn into an electron which escapes with an energy \( \sim m_\mu \). This process has been searched for in the past with various target materials, as summarised in table 1; the best existing bound is \( BR < 7 \times 10^{-13} \) on Gold (\( Z = 79 \)) from SINDRUM-II [17].

The interaction of the muon with the nucleus can be parametrised at the experimental scale in Effective Field Theory, via dipole operators and a variety of 2-nucleon operators:

\[
L_{\mu A \to e A}(\Lambda_{\text{exp}}) = -\frac{4G_F}{\sqrt{2}} \sum_{N=p,n} \left[ m_\mu \left( C_{DL} \bar{\pi} \sigma^{\alpha\beta} \mu_L F_{\alpha\beta} + C_{DR} \bar{\pi} \sigma^{\alpha\beta} \mu_R F_{\alpha\beta} \right) + \left( C_{SL} \bar{\tau} P_L \mu + C_{SR} \bar{\tau} P_R \mu \right) \bar{N} N \right] 
+ \left( C_{PL} \bar{\gamma}^{\alpha} P_L \mu + C_{PR} \bar{\gamma}^{\alpha} P_R \mu \right) \bar{N} \gamma_\alpha N
+ \left( C_{VL} \bar{\gamma}^{\alpha} P_L \mu + C_{VR} \bar{\gamma}^{\alpha} P_R \mu \right) \bar{N} \gamma_\alpha \gamma_5 N
+ \left( C_{AL} \bar{\gamma}^{\alpha} P_L \mu + C_{AR} \bar{\gamma}^{\alpha} P_R \mu \right) \bar{N} \gamma_\alpha \gamma_5 N
+ \left( C_{DL,LR} \bar{\gamma}^{\alpha} P_L \mu + C_{DL,RR} \bar{\gamma}^{\alpha} P_R \mu \right) i(\bar{N} \gamma_5 \gamma_\alpha N)
+ \left( C_{DL,LR} \bar{\gamma}^{\alpha} P_L \mu + C_{DL,RR} \bar{\gamma}^{\alpha} P_R \mu \right) \bar{N} \gamma_\alpha \gamma_5 N + h.c. \right]. \tag{1}
\]

Since the electron is relativistic, and the nucleons not, it is convenient to use a chiral basis for the lepton current, but not for the nucleons.

This basis of nucleon operators is chosen because it represents the minimal set onto which two-lepton-two-quark, and two-lepton-two-gluon operators can be matched at the leading order in \( \chi PT \). This explains the presence of the derivative operators \( \mathcal{O}_{\Delta r, X}^{(N)} \), which only contribute at finite momentum transfer, and have coefficients of dimension \( 1/\text{mass} \), but represent a minimal attempt to include pion exchange diagrams for the quark axial current at finite momentum transfer [11]. At higher order in \( \chi PT \), additional operators can appear, sometimes involving more than two nucleons [18].

Like in WIMP scattering on nuclei, the muon can interact coherently with the charge or mass distribution of the nucleus, called the “Spin Independent” (SI) process, or it can have Spin-Dependent (SD) interactions [19] with the nucleus at a rate that does not benefit from the atomic-number-squared enhancement of the SI rate. The Dipole, Scalar and Vector operators will contribute to the SI rate (with a small admixture of the Tensor, see eqn 3), and the Axial, Tensor and Pseudoscalar operators contribute to the SD rate.

The spin-independent contribution to the branching ratio for \( \mu \to e \) conversion on the nucleus \( A \), was calculated by Kitano, Koike and Okada (KKO) [16], to be

\[
\text{BR}_{SI}(A \mu \to e A) = \frac{32G_F^2 m_\mu^5}{\Gamma_{\text{cap}}} \left[ \left| \bar{C}_{V,R}^{pp} V^{(p)} + \bar{C}_{S,L}^{pp} S^{(p)} \right|^2 + \left| \bar{C}_{S,L}^{nn} S^{(n)} \right|^2 + \left| C_{D,L} \frac{D}{4} \right|^2 + \{|L \leftrightarrow R\} \right] \tag{2}
\]

where \( \Gamma_{\text{cap}} \) is the rate for the muon to transform to a neutrino by capture on the nucleus [16, 20], \( \frac{32G_F^2 m_\mu^5}{\Gamma_{\text{cap}}} \approx 1.2 \times 10^5 \) for aluminium. The nucleus \( (A) \) and nucleon \( (N \in \{p, n\}) \)-dependent “overlap integrals” \( D_A, S_A^{(p)}, V_A^{(p)}, S_A^{(n)}, V_A^{(n)} \), correspond to the integral over the nucleus of the lepton wavefunctions and the appropriate nucleon density. These overlap integrals will play a central role in our analysis, and are given in KKO [16]. The primed scalar coefficient includes a small part of the tensor coefficient, because the tensor contributes at finite momentum transfer to the SI process [19, 11]:

\[
\bar{C}_{S,L}^{NN'} = \bar{C}_{S,L}^{NN} + \frac{2m_\mu}{m_N} \frac{\bar{C}_{S,L}^{NN} \bar{C}_{T,X}^{NN}}{m_N} . \tag{3}
\]

The SD rate depends on the distribution of spin in the nucleus, and therefore requires detailed nuclear modelling. The tensor and axial vector contributions were estimated in [19, 11] for light (\( Z \leq 20 \)) nuclei, where the muon wavefunction is wider than the radius of the nucleus, and the electron can be approximated as a plane wave. In this...
limit, where the muon wavefunction can be factored out of the nuclear spin-expectation-value, the nuclear calculation of SD WIMP scattering on the quark axial current can be used for $\mu \rightarrow e$ conversion. The branching ratio on a target of charge $Z$, with a fraction $\epsilon_I$ of isotope $I$ with spin $J_I$, can be estimated as

$$\text{BR}_{SD}(Z\mu \rightarrow Ze) = \frac{8G_F^2m_p^2(\alpha Z)^3}{\Gamma_{cap}\pi^2} \left[ \sum_I 4\epsilon_I \frac{J_I + 1}{J_I} \left| S_P^I(C_{AX}^L + 2C_{SR}^R) + S_n^I(C_{X}^L + 2C_{TR}^R) \right|^2 \frac{S_I(m_\mu)}{S_I(0)} + \{L \leftrightarrow R\} \right]$$

(4)

where $S_P^I$ is the proton spin expectation value in isotope $I$ at zero momentum transfer, and $\frac{S_I(m_\mu)}{S_I(0)}$ is a finite momentum transfer correction, which has been calculated for the axial current in some nuclei (see eqn [24] for Aluminium; this factor includes the derivative operators $\tilde{D}_{\alpha\chi}^{(N\bar{N})}$). The targets which have been used for $\mu \rightarrow e$ conversion searches are listed in table 1, with the abundances of some spin-carrying isotopes, and some results for the proton and neutron spin expectation values.

| target       | isotopes          | $J$  | $S_P^A$, $S_n^A$ | $S_I(m_\mu)/S_I(0)$ | BR (90% C.L.)         |
|--------------|-------------------|------|-----------------|----------------------|-----------------------|
| Sulfur       | Z=16,A=32 [95%]   | 0    |                 |                      | $< 7 \times 10^{-11}$ [23] |
| Titanium     | Z=22, A=48 [74%]  | 0    |                 |                      | $< 4.3 \times 10^{-12}$ [17] |
|              | Z=22, A=47 [7.5%] | 5/2  | 0.0 , 0.21 [24] | $\sim 0.12$          |                       |
|              | Z=22, A=49 [5.4%] | 7/2  | 0.0 , 0.29 [24] | $\sim 0.12$          |                       |
| Copper       | Z=29, A=63 [70%]  |      |                 |                      | $BR \leq 1.6 \times 10^{-8}$[25] |
|              | Z=29, A=65 [31%]  |      |                 |                      |                       |
| Gold         | Z=79, A=197       | 5/2  | -0.52$\rightarrow$0.30 , 0.0 | $BR < 7 \times 10^{-13}$[17] |
| Lead         | Z=82, A=206 [24%] | 0    |                 | 0.55 [26]           | $BR < 4.6 \times 10^{-11}$ [17] |
|              | Z=82, A=207 [22%] | 1/2  | 0.0 , -0.15 [24] |                       |                       |
|              | Z=82, A=208 [52%] | 0    |                 |                       |                       |
| Aluminium    | Z=13,A=27 [100%]  | 5/2  | 0.34 , 0.030 [21, 22] | 0.29 [21, 22]       | $\rightarrow 10^{-10}$ |

Table 1: Current experimental bounds on $\mu \rightarrow e$ conversion, and the future sensitivity on Aluminium. The isotope abundances are from [27]. The estimate for $S_P^A$ is based on the Odd Group Model of [24], assuming $J=1/2$.

3 To determine or constrain how many coefficients?

The Lagrangian of eqn (1) contains twenty-two unknown operator coefficients. These coefficients contribute to various observables, so can be constrained, or measured, in different ways:

1. we neglect the two dipole coefficients, because the upcoming MEG-II and Mu3e experiments, respectively searching for $\mu \rightarrow e\gamma$ and $\mu \rightarrow ee$, have a slightly better sensitivity: if MEG-II and Mu3e set bounds $BR(\mu \rightarrow e\gamma) < 6 \times 10^{-14}$ and $BR(\mu \rightarrow ee) < 10^{-16}$, this would translate to $|C_{D,X}| \lesssim 2.0 \times 10^{-9}$. Whereas a SI branching ratio of $10^{-16}$ on Aluminium can be sensitive to $|C_{D,X}| \gtrsim 3.1 \times 10^{-9}$.

2. the remaining 20 coefficients involving nucleons can be divided into two classes, labelled by the chirality/helicity of the outgoing (relativistic) electron. The interference between these classes is usually neglected (suppressed by $m_\gamma^2/m_n^2$), so an experimental upper bound on the rate simultaneously sets bounds on the coefficients of both chiralities. If a $\mu \rightarrow e$ conversion signal is observed, it could be possible to identify the chirality of the operator by measuring the polarisation of the muon or electron. For simplicity, we will in the following only discuss the ten operators that create an $e_L$.

Notice that the conventions of eqn (1) label operator coefficients with the chirality of the muon, which is opposite to the electron for dipole, scalar, pseudoscalar and tensor operators.

3. Finally, the operators can also be divided into those that mediate SI or SD conversion. In the body of the paper, we will discuss the SI rate, to which contribute the dipole that we neglect, and the vector and scalar on the neutron and proton. These appear in the amplitude weighted by overlap integrals (see after eqn 2), which are nucleus-dependent. This suggests that to constrain the four operator coefficients, one just needs to search for $\mu \rightarrow e$ conversion on four sufficiently different targets. (In order to measure the SI coefficients independently from SD ones, the targets could/should be chosen without SD contributions.)

In the Appendix A, we make some remarks on the SD rate, which can be sensitive to six coefficients. However, quantitative calculations would require nuclear matrix elements that we did not find in the literature.
4 Targets as vectors, and the problem of theoretical uncertainties

In a previous publication[11], a representation of targets as vectors in coefficient space was introduced. The targets are labelled by $Z$, and for SI transitions, the elements of the vector are the overlap integrals of KKO [16]:

$$\vec{v}_Z = \left( \frac{D_Z}{4}, V_Z^{(p)}, S_Z^{(p)}, V_Z^{(n)}, S_Z^{(n)} \right)$$  \hspace{1cm} \text{(5)}

The aim was to give a geometric, intuitive measure of different targets ability to distinguish coefficients. If the operator coefficients are lined up in a pair of vectors labelled by the chirality of the outgoing electron, such that for $\epsilon_L$:

$$\vec{C}_L = (\vec{C}_{D,R}, \vec{C}_{V,L}^{up}, \vec{C}_{S,R}^{up}, \vec{C}_{V,L}^{nn}, \vec{C}_{S,R}^{nn})$$  \hspace{1cm} \text{(6)}

(and similarly for $\vec{C}_R$), then the Spin Independent Branching Ratio on target $Z$ can be written

$$BR \propto |\vec{v}_Z \cdot \vec{C}_L|^2 + |\vec{v}_Z \cdot \vec{C}_R|^2.$$  \hspace{1cm} \text{(7)}

If two target vectors are parallell, they probe the same combination of couplings, and if they are misaligned, they could allow to distinguish among the coefficients.

To quantify how “misaligned” targets need to be, in order to differentiate among coefficients, we should take into account the theoretical uncertainties. These are a significant complication, because they make uncertain which combination of coefficients is constrained by which target. To illustrate the problem, we suppose coefficient space is two-dimensional. This allows to draw pictures.

If a first observable $T_1$, can be computed with negligible theoretical uncertainty to depend on $|C_2|^2$, and a second observable $T_2$, similarly can be computed to depend on $|C_1|^2$, then the values of the coefficients respectively allowed by null results in the two experiments are inside the thick lines of the top left plot in figure 1. The central striped (dark) region is allowed when the two experimental results are combined. In reality, the allowed region should be more the shape of a circle, since the experimental uncertainties are (in part) statistical. However, we neglect this detail because it is not the subject of our discussion.

Suppose now that there is some theoretical uncertainty $\epsilon$ in the calculations, such that $T_1$ depends on $|C_1(1 \pm \epsilon) \pm \epsilon C_2|^2$, and $T_2$ depends on $|C_2(1 \pm \epsilon) \pm \epsilon C_1|^2$. Then provided $\epsilon \ll 1$ ($\epsilon \approx \pi/32 \approx .1$ in the figure), the regions respectively allowed by the two experiments are the bowties within the thin lines of the upper left plot in figure 1. The region allowed by the combined experiments is essentially unchanged (still the central square).

Consider next a situation more relevant to $\mu \rightarrow e$ conversion, illustrated in the upper right plot of figure 1. The second observable $T_2$ again depends on $|C_2(1 \pm \epsilon) \pm \epsilon C_1|^2$, but $T_1$ depends on $|\cos \theta C_2 - \sin \theta C_1|$, where $\theta \approx \pi/8 \pm \epsilon$. Neglecting theoretical uncertainties, the allowed regions for the two experiments are respectively between the thick blue lines, and thick black lines. The striped diamond is the parameter range consistent with both experiments. But if the theory uncertainty is taken into account, the allowed regions of the two experiments are respectively enclosed by the thin blue and black lines. The region allowed by the combined observations is the grey diamond, which includes the striped one. So we see that the theoretical uncertainty changes the allowed region by factors of $O(1)$.

Finally, in the lower two plots of figure 1, $T_1$ depends on $|\cos \theta C_2 - \sin \theta C_1|$, where $\theta \approx 2\epsilon \pm \epsilon$. If the theory uncertainty is neglected, as illustrated in the lower left plot, the region allowed by the two experiments corresponds to the striped diamond. However, when the angle uncertainty is taken into account, both bars can be rotated towards each other, such that they point in the same direction, and any value of $C_1$ is allowed. This is illustrated in the lower right plot, where the allowed region is grey, and gives no constraint on $C_1$. The allowed range for $C_1$ would be finite for

$$\theta > 2\epsilon.$$  \hspace{1cm} \text{(8)}

which we take as the condition that two observables constrain independent directions in coefficient space. (Recall that $\theta$ is the angle between the two observables, represented as vectors in coefficient space, and $\epsilon$ is the theoretical uncertainty on the calculation of $\theta$).

For $\mu \rightarrow e$ conversion, the theoretical uncertainties in the calculation of the rate and the overlap integrals were discussed in [11]. The current uncertainties were estimated as $\epsilon \lesssim 10\%$. This is based on KKO’s estimate of the uncertainties in their overlap integrals, which is $\lesssim 5\%$ for light nuclei, and $\lesssim 10\%$ for heavier nuclei, and on NLO effects in $\chi$PT. These are parametrically 10\%, and could, for instance change the form of eqn (7), as occurs in WIMPs scattering [28], making it impossible to parametrise targets as vectors. In the following section, we take the current uncertainties to be $\epsilon \sim 10\%$, or possibly $\epsilon \sim 5\%$ for light targets, implying that two targets can give independent constraints if they are misaligned by $\theta \gtrsim .2$ (or possibly $\theta \gtrsim .1$ for light targets).
Figure 1: Illustration of the impact of theoretical uncertainties on the determination of operator coefficients, when combining results from two experiments. The allowed region neglecting theory uncertainties is stripped; the larger grey areas are allowed when theoretical uncertainties are included. The upper left plots is for two experiments that measure orthogonal parameters, the upper right plot is for two experiments who measure correlated parameters but with manageable uncertainty, and the lower two plots represent the case where the two experiments do not give independent information when the theory uncertainty is included.

5 Comparing current bounds

In section 3, it was suggested that the four scalar and vector coefficients could be constrained or measured by searching for $\mu \rightarrow e$ conversion in four “sufficiently different” targets. And we see from table 1 that there is data for Sulfur, Titanium, Copper, Gold and Lead. However, as estimated in the previous section, “sufficiently different” means misaligned by 10-20%, so in this section, we calculate the misalignment between the targets for which there is data.

Recall that targets are described by vectors (see eqn 5), that live in the same space as the operator coefficients. However, the components of the target vectors are all positive, meaning the misalignment angle between any two target vectors is $< \pi/2$, or equivalently, that the target vectors point all in the first quadrant.

The angle between target $T$ and target $Z$ can be estimated from the normalised inner product

$$\frac{\vec{v}_Z \cdot \vec{v}_T}{|\vec{v}_Z||\vec{v}_T|} \simeq \cos \theta \simeq 1 - \frac{\theta^2}{2}$$

In figure 2 are plotted the misalignment angles* between the targets of table 1, and the other possibilities given by KKO, labelled by $Z$. The thin blue line in figure 2 (the line with largest $\theta$ at high $Z$) is the misalignment angle with respect to Sulfur, and the thin green line (the solid line with the second largest $\theta$ at high $Z$) is the misalignment angle with respect to Titanium. So the blue line at $Z=22$ (Titanium) is equal to the green line at 16 (Sulfur), and both give $\theta \sim 0.08$ between Sulfur and Titanium, suggesting that these constrain the same combination of coefficients. On the other hand, Gold probes different coefficients from the light targets (as anticipated by KKO [16]), but Gold and

*Since the current MEG bound on the dipole coefficients constrains them to be below the sensitivity of the current $\mu \rightarrow e$ conversion bounds, the dipole overlap integral was set to zero in obtaining this figure.
Figure 2: Angle $\theta$ between a target vector (e.g., dashed red = Aluminium) and other targets labelled by $Z$. The angle is obtained as in eqn (9), with all the dipole coefficients set to zero. The solid lines represent the targets for which there is currently data (see table 1). From smallest to largest value of $\theta$ at large $Z$, they are: thick green = Lead, thick blue = Gold, black = Copper, thin green = Titanium, dashed red = Aluminium, and thin blue is Sulfur. We assume that two targets can probe different coefficients if their misalignment angle is $\theta > 0.2$ radians (or 0.1).

Lead cannot distinguish coefficients. Also Copper and Titanium do not give independent constraints. So the current experimental bounds on $\mu \rightarrow e$ conversion constrain two combinations of the four coefficients present in $\vec{C}_L$ (similarly, two combinations in $\vec{C}_R$). Thus, the current experimental bounds can be taken as the SINDRUM-II constraints from Titanium and Gold.

6 Selecting future targets

The upcoming COMET and Mu2e experiments plan to use an Aluminium target, illustrated as a red dashed line in figure 2. Unfortunately, it is only misaligned with respect to Titanium and Sulfur by a few percent, so with current theoretical uncertainties, Aluminium probes the same combination of Spin-Independent coefficients as Titanium (and Sulfur).

It is therefore interesting to explore which targets could measure a different combination of coefficients from Aluminium. As noted by KKO, the scalar and vector overlap integrals grow differently with $Z$, and using targets with different $n : p$ ratios could allow to differentiate coefficients on protons from those on neutrons. To quantify these differences, we introduce four orthonormal vectors in the space of nucleon overlap integrals:

\[
\hat{e}_1 = \frac{1}{2}(1,1,1,1) \\
\hat{e}_2 = \frac{1}{2}(-1,-1,1,1) \\
\hat{e}_3 = \frac{1}{2}(1,-1,1,-1) \\
\hat{e}_4 = \frac{1}{2}(-1,1,1,-1)
\]  

(10)

Dotted into the coefficients, $\hat{e}_1$ measures the sum of coefficients, $\hat{e}_2$ is the difference between coefficients on protons and neutrons, $\hat{e}_3$ is the vector - scalar difference, and $\hat{e}_4$ is the remaining direction. All the targets are mostly aligned on $\hat{e}_1$; this is expected as the overlap integrals are of comparable size, and all positive $^1$. Indeed, for Aluminium,

$^1$One way to see this, is to project the target vectors onto the basis of eqn 10. We find that $\vec{v}_Z \cdot \hat{e}_1 \geq 0.93|\vec{v}_Z|$ for all $Z$, so we do not plot the projection onto $\hat{e}_1$. It decreases with $Z$.  

6
the target vector $\mathbf{v}_{13}$ and $\mathbf{e}_1$ are almost parallel: $\mathbf{v}_{13} \cdot \mathbf{e}_1 \geq 0.997|\mathbf{v}_{13}|$. So we suppose that this sum of coefficients is measured on Aluminium, and plot in figure 3 the projection of the target vectors onto $\mathbf{e}_2$ (thick, continuous line), $\mathbf{e}_3$ (dashed) and $\mathbf{e}_4$ (thin).

Figure 3 shows, as noted by KKO, that comparing heavy to light targets can distinguish scalar vs vector coefficients (or constrain both in the absence of a signal). The neutron to proton ratio also increases with atomic number, but perhaps a more promising target for making this difference would be Lithium, with four neutrons and three protons: the theoretical uncertainties could be more manageable, and the scalar-vector difference is suppressed. In addition, being light, it has a long lifetime, making it appropriate for the COMET and Mu2e experiments.

Unfortunately, it seems that $\mu \rightarrow e$ conversion targets have little sensitivity to $\mathbf{e}_4$, which measures some curious simultaneous difference between scalars and vectors, and neutrons and protons.

7 Summary

This letter studies the selection of targets for $\mu \rightarrow e$ conversion, with the aim that they probe independent combinations of $\mu \rightarrow e$ flavour-changing parameters, while including the theoretical uncertainties of the calculation. The rate is parametrised via the operators given in eqn (1), and the theoretical uncertainties are reviewed in section 4. We take the current uncertainties to be $\sim 10\%$, and anticipate that this could be reduced to $5\%$ in the future.

Using a parametrisation of targets as vectors in the space of operator coefficients, we reproduce the observation of Kitano, Koike and Okada (KKO) [16], that comparing light to heavy targets allows to distinguish scalar from vector operator coefficients. We also observe that comparing light targets with very different neutron to proton ratios could allow to distinguish operators involving neutrons from those involving protons. A reduction in the theory uncertainty would help to make this distinction. Lithium is the most promising target in the list for which KKO computed overlap integrals, however other light isotopes with higher $n/p$ ratios, such as Beryllium10, could be interesting to consider.

The Spin-Dependent (SD) conversion rate is mentioned in the Appendix. We reiterate that the neutron vs proton operators can be distinguished by searching for SD conversion on nuclei with an odd neutron and with an odd proton. Comparing the SD rate on light vs heavy nuclei could allow to distinguish axial from tensor coefficients, but dedicated nuclear calculations would be required to confirm this.
A Appendix: the SD contribution

In this Appendix, we discuss how different targets could distinguish among the many operators that contribute to SD conversion.

We first recall the operators that contribute to the SD rate. For a fixed electron chirality, the pseudoscalar, axial vector and tensor nucleon currents become, in the non-relativistic limit \[29\]

\[
\begin{align*}
\pi_N(p_f)\gamma_5 u_N(p_i) & \rightarrow \bar{q}\cdot \bar{S}_N/m_N \\
\pi_N(p_f)\gamma^\alpha \gamma_5 u_N(p_i) & \rightarrow (\bar{P}\cdot \bar{S}_N, 2S_N)/m_N \\
\pi_N(p_f)\sigma^{jk} u_N(p_i) & \rightarrow 2\epsilon^{ijkl} S_l \\
\pi_N(p_f)\sigma_{\mu\nu} u_N(p_i) & \rightarrow (iq^\nu - 2P^\nu \epsilon^{ijkl})/2m_N
\end{align*}
\]

(11)

where \( q = p_i - p_f \), and \( P = p_f + p_i \). So they all connect the lepton current to the spin of the nucleon, and at zero-momentum transfer, the pseudoscalar nucleon current vanishes and the tensor current is twice the axial current. (We neglect the derivative operator given in eqn (1), because it is induced by the quark axial current at finite momentum transfer, the pseudoscalar nucleon current vanishes and the tensor current is twice the axial current. (We neglect the derivative operator given in eqn (1), because it is induced by the quark axial current at finite momentum transfer, and the quark axial current also induces the nucleon axial current.)

The SD coefficients on neutrons, can be distinguished from those on protons, by comparing targets with an odd number of protons or neutrons [19]. This can be seen from table 1, where the spin of a nucleus is largely due to the spin of the one unpaired nucleon. For instance, searching for \( \mu \rightarrow e + e + e \) conversion on Aluminium, and on a Titanium target containing a sufficient abundance of spin-carrying isotope, would give independent constraints on SD coefficients on the proton and neutron.

It is possible that comparing heavy and light targets could distinguish axial from tensor operators. The estimate for the SD Branching ratio given in eqn (4) assumes light nuclear targets (where the muon wavefunction is broader than the atom), and exhibits a degeneracy between the tensor and axial coefficients. If the same light-nucleus approximation is used to compute the SI rate, then the scalar and vector overlap integrals would be the same. Indeed, as pointed out by KKO, the scalar overlap integral becomes different from the vector in heavy nuclei, because the negative energy component of the electron wavefunction becomes relevant, and has opposite sign for vector and scalar (see KKO, eqns 20-23). There is a similar sign difference between tensor and axial operators, so one could hope to distinguish tensor from axial operators by comparing the SD rate in light and heavy nuclei. For instance, table 1 suggests that \( \mu \rightarrow e + e \) conversion on gold and lead could distinguish the axial from tensor operators respectively for protons and neutrons. However, one difficulty is that the SD rate is relatively suppressed with respect to the SI rate by a factor \( \sim 1/A^2 \), which becomes more significant for heavier nuclei. The second difficulty is that dedicated nuclear calculations of the expectation value in the nucleus of the various SD operators, weighted by the lepton wavefunctions, would be required. These calculations currently do not exist.

Finally, in order to be sensitive to the Pseudoscalar operator, and to obtain reliable predictions for the SD rates, the finite momentum transfer should be taken into account. However, then in squaring the matrix element, the spin sums do not factorise from the sum of operator coefficients (as occurs for the SI rate, see eqn 2). This suggests that the nuclear calculation would need to be performed in the presence of the A,P and T operators in order to explore the prospects of distinguishing the pseudoscalar.

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