A SUFFICIENT CONDITION FOR A POLYHEDRON TO BE RIGID

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ABSTRACT. We study oriented connected closed polyhedral surfaces with non-degenerate triangular faces in three-dimensional Euclidean space, calling them polyhedra for short. A polyhedron is called flexible if its spatial shape can be changed continuously by changing its dihedral angles only. We prove that the polyhedron is not flexible if for each of its edges the following holds true: the length of this edge is not a linear combination with rational coefficients of the lengths of the remaining edges. We prove also that if a polyhedron is flexible, then some linear combinations of its dihedral angles remain constant during the flex. In this case, the coefficients of such a linear combination do not alter during the flex, are integers, and do not equal to zero simultaneously.

Keywords: flexible polyhedron, dihedral angle, Dehn invariant, Hamel basis, Bricard octahedron.

Mathematics subject classification (2010): Primary 52C25, Secondary 52B70, 51M20.

1. Introduction

We study oriented connected closed polyhedral surfaces with non-degenerate triangular faces in $\mathbb{R}^3$. For brevity, we call them polyhedra, as is customary in the theory of flexible polyhedra. A polyhedron is called flexible, if its spatial shape can be changed continuously by changing its dihedral angles only, i.e., due to such a continuous (in particular, continuous with respect to a parameter describing the process of the flex) deformation of the polyhedron, in which each of its faces remains congruent to itself in the process of deformation. A continuous family of such deformations is called the flex of the polyhedron.

We know quite a lot about flexible polyhedra, namely:

(a) they do exist (see [6], [7]; see also [26], [24], [8]; moreover, they can be of any genus and can even be non-orientable, see [35];

(b) during the flex, they necessarily keep unaltered the so-called total mean curvature, see [1];

(c) during the flex, they necessarily keep unaltered the volume of the domain they bound (see [29], [30], [31]; see also [9], [34]);

(d) during the flex, they necessarily keep unaltered the Dehn invariants, see [21];

(e) the notion of a flexible polyhedron can be introduced in all spaces of constant curvature of dimension 3 and above (see [36], [37], [16], [17], [19], as well as in pseudo-Euclidean spaces of dimension 3 and above (see [3]); moreover, it is known that in
many of these spaces flexible polyhedra do exist and possess properties similar to properties (a)–(d) (see [24], [14], [15], [18], [20]).

However, still very little is known about one of the central problems of the theory of flexible polyhedra: how do we know if a given polyhedron is flexible or not? Roughly, we can say that to answer this problem we have only the classical results that a closed convex polyhedron is necessarily nonflexible (i.e., rigid) and that a polyhedron that does not allow nontrivial infinitesimal deformations is also rigid. A large number of publications is devoted to these results, of which we will indicate only a few, in our opinion, the most significant ones, namely [2], [5], [11], [23], [38], [28], [33]. Among recent results in this direction, we mention the following one by I. Kh. Sabitov [32]: given a polyhedron, he builds a certain set of polynomials and proves that if each of these polynomials has at least one non-zero coefficient, then the polyhedron is rigid.

In this article, we give a fundamentally new sufficient condition for a polyhedron to be rigid. Namely, we prove that the condition “the length of each edge is not a Q-linear combination of the lengths of the remaining edges,” implies that the polyhedron is rigid, see Theorem 2 in Section 3.

In addition, we prove that if a polyhedron is flexible, some linear combinations of its dihedral angles remain constant during the flex, see Theorem 3 in Section 3. In this case, the coefficients of such a linear combination do not change during the flex, are integers, and not all are equal to zero.

Our arguments essentially use one auxiliary statement, proved by A. A. Gaifullin and L. S. Ignashchenko in [21], see Lemma 1 in Section 2.

2. Definitions, notation, and an auxiliary statement

**Definition 1.** A non-empty finite set \( K \) of simplices in Euclidean space is called a finite strongly connected two-dimensional simplicial complex if the following conditions (i)–(v) are satisfied:

(i) each face of a simplex in \( K \) is itself a simplex in \( K \);
(ii) the intersection of any two simplices of \( K \) is a face for each of these simplices;
(iii) each simplex of \( K \) is contained in a 2-dimensional simplex of \( K \);
(iv) every 1-dimensional simplex of the complex \( K \) is contained in exactly two 2-dimensional simplices of \( K \);
(v) for any two 2-dimensional simplices \( \sigma, \tau \in K \) there is a sequence of 2-dimensional simplices \( \sigma = \sigma_1, \sigma_2, \ldots, \sigma_n = \tau \) of the complex \( K \) such that the intersection of \( \sigma_j \cap \sigma_{j+1} \) is a 1-dimensional simplex of \( K \) for all \( j = 1, \ldots, n - 1 \).

The set of all \( i \)-dimensional simplices of the complex \( K \) is denoted by \( K_i \), \( i = 0, 1, 2 \).

A complex \( K \) is called oriented if each of its 2-dimensional simplices is equipped with an orientation and these orientations are consistent with each other. The latter means that if \( \sigma \in K_1 \) is contained both in \( \tau_1 \in K_2 \) and \( \tau_2 \in K_2 \), then \( \tau_1 \) and \( \tau_2 \) generate opposite orientations on \( \sigma \).
Remark 1. The conditions (i) and (ii) mean that $K$ is a simplicial complex; the condition (iii), in particular, means that $K$ is a two-dimensional complex; the condition (iv) means that $K$ has no neither boundary nor branch points; the condition (v) means that $K$ is strongly connected.

Definition 2. Let $K$ be a finite strongly connected two-dimensional simplicial complex. A mapping $P : K \to \mathbb{R}^3$, whose restriction to each simplex of $K$ is affine, is called a polyhedron of the combinatorial type $K$. For brevity, the image $P(K)$ of $K$ under the action of $P$ is sometimes called a polyhedron too. If $P : K \to \mathbb{R}^3$ and $\sigma \in K_i$ then $P(\sigma)$ is called a vertex of $P$, if $i = 0$; is called an edge of $P$, if $i = 1$; and is called a face of $P$, if $i = 2$. We say that a polyhedron $P : K \to \mathbb{R}^3$ has nondegenerate faces (or, equivalently, nondegenerate triangular faces), if, for every $\sigma \in K_2$, the face $P(\sigma)$ is a nondegenerate 2-dimensional simplex in $\mathbb{R}^3$. If $K$ is an oriented complex, then a polytope $P : K \to \mathbb{R}^3$ is called oriented.

Remark 2. A polyhedron $P : K \to \mathbb{R}^3$ may have self-intersections, i.e., in Definition 2 it may occur that there are simplices $\sigma, \tau \in K$ such that $\sigma \cap \tau = \emptyset$ and $P(\sigma) \cap P(\tau) \neq \emptyset$.

Definition 3. A continuous family $\{P_t\}_{t \in [0, 1]}$ of polyhedra $P_t : K \to \mathbb{R}^3$ is called a flex of a polyhedron $P : K \to \mathbb{R}^3$ if $P = P_0$ and, for every $\sigma \in K_2$ and $t \in (0, 1]$, the face $P_t(\sigma)$ of the polyhedron $P_t$ is congruent to the face $P_0(\sigma)$ of the polyhedron $P_0$. A flex $\{P_t\}_{t \in [0, 1]}$ is called trivial if $P_t$ is obtained from $P_0$ by means of an isometry of $\mathbb{R}^3$, i.e., if for every $t \in (0, 1]$ there is an isometry $A_t : \mathbb{R}^3 \to \mathbb{R}^3$ such that $P_t = A_t \circ P_0$. Otherwise, the flex is called nontrivial. A polyhedron $P$ is called flexible if there is nontrivial flex of $P$ and is called rigid if every flex of $P$ is trivial.

Remark 3. Definitions 2 and 3 concern only simplicial polytopes, but this does not reduce the generality of the results obtained in Section 3. After all, even if we would use some more general definition of a flexible polyhedron, which allows us to consider flexible polyhedra with non-triangular faces, we would additionally triangulate the faces of the flexible polyhedron and obtain a flexible polyhedron in the sense of Definition 3. Note also that, by analogy with Definitions 1 and 3 the concept of a flexible polyhedron can be defined in any spatial form (i.e., in a connected complete Riemannian manifold of constant curvature) of dimension 3 and above; see, e.g., [24], [37], [16], [17], [18], [20].

Definition 4. Let $K$ be a finite strongly connected oriented two-dimensional simplicial complex and $P : K \to \mathbb{R}^3$ be an oriented polyhedron with nondegenerate faces. Suppose $\sigma \in K_1$, $\tau_1, \tau_2 \in K_2$ and $\sigma = \tau_1 \cap \tau_2$. Let us denote by $n(\tau_1)$ and $n(\tau_2)$ the unit positively oriented (according to the orientation of $K$) normals to the faces $P(\tau_1)$ and $P(\tau_2)$ respectively. Consider first the case when $n(\tau_1) + n(\tau_2) \neq 0$. Let $x$ be an arbitrary point of the edge $P(\sigma)$, which is not a vertex of $P(K)$, and let $B(x)$ be a ball in $\mathbb{R}^3$ which is centered at $x$ and does not intersect any edge of the faces $P(\tau_1)$ and $P(\tau_2)$, except the edge $P(\sigma)$. The set $P(\tau_1) \cup P(\tau_2)$ divides the ball $B(x)$...
into two bodies (slices). Let us denote by $B_+(x)$ the one whose interior contains the
points $x + \frac{1}{2} \varepsilon (n(\tau_1) + n(\tau_2))$, where a positive number $\varepsilon$ is less than the radius of the
ball $B(x)$. We call the number $\varphi_\sigma = 2\pi \text{Vol} (B_+(x))/\text{Vol} (B(x))$, the principal value of
the dihedral angle at the edge $P(\sigma)$ of the polyhedron $P(K)$ at the edge $P(\sigma)$. Here $\text{Vol} (\cdot)$ stands
for the 3-dimensional volume of the corresponding body. If $n(\tau_1) + n(\tau_2) = 0$, we put by definition $\varphi_\sigma = 0$. We call the set $\Phi_\sigma = \{ \varphi_\sigma + 2\pi m \mid m \in \mathbb{Z} \}$ the oriented
dihedral angle at the edge $P(\sigma)$ of the polyhedron $P(K)$. Here $\varphi_\sigma$ stands for the
principal value of the dihedral angle at the edge $P(\sigma)$.

Remark 4. It can be shown that $\varphi_\sigma$ is in the range from 0 to $2\pi$ and does not depend
on the choice of the point $x$ on the edge $P(\sigma)$. It can also be shown that $\varphi_\sigma$ is not
a continuous function on the set of all oriented polytopes of a given combinatorial
type $K$ with non-degenerate faces. The fact is that if $\varphi_\sigma = 0$ at the edge $P(\sigma)$ of
$P : K \to \mathbb{R}^3$, then in any neighborhood of $P$ there exist polyhedra of combinatorial
type $K$ with non-degenerate faces, for which the principal value of the dihedral angle
at the corresponding edge is close to 0 and there are polyhedra for which it is close
to $2\pi$. On the other hand, it can be shown that the many-valued function $\Phi_\sigma$ is
continuous on the set of all oriented polytopes of a given combinatorial type $K$ with
non-degenerate faces. Therefore, it is possible to find continuous selection functions
for it, see [25]. In Definition 5 we use the existence of a suitable continuous selection
function.

Definition 5. Let $K$ be a finite strongly connected oriented two-dimensional simplicial complex and let $\sigma \in K_1$. Let $P : K \to \mathbb{R}^3$ be an oriented polyhedron with nondegenerate faces and let $\{P_t\}_{t \in [0,1]}$ be a flex of $P$. We call a continuous selection function $\tilde{\varphi}_\sigma(t)$ of the multi-valued mapping $t \mapsto \Phi_\sigma(t)$ such that $\tilde{\varphi}_\sigma(0) = \varphi_\sigma$, where $\varphi_\sigma$ is a principal value of the dihedral angle of $P$ at the edge $P(\sigma)$, a dihedral angle
at the edge $P(\sigma)$ during the flex $\{P_t\}_{t \in [0,1]}$.

The following lemma is a very special case of Statement 3.4, proved by A. A. Gain-
fullin and L. S. Ignashchenko in their article [21].

Lemma 1. Let $K$ be a finite strongly connected oriented two-dimensional simplicial complex, $P : K \to \mathbb{R}^3$ be an oriented polyhedron with nondegenerate faces, and $\{P_t\}_{t \in [0,1]}$ be a flex of $P$. Then, for every $\mathbb{Q}$-linear function, $f : \mathbb{R} \to \mathbb{R}$, the expression $\sum_\sigma f(\ell_\sigma) \tilde{\varphi}_\sigma(t)$ is independent of $t$, i.e. it remains constant during the
flex. Here, summation is carried out over all 1-dimensional simplices $\sigma \in K$, while $\ell_\sigma$ and $\tilde{\varphi}_\sigma(t)$ denote the length of the edge $P(\sigma)$ and the dihedral angle of the polyhedron $P$ at the edge $P(\sigma)$ during the flex $\{P_t\}_{t \in [0,1]}$, respectively. \hfill \Box

We will use Lemma 1 in Section 3.

3. The main results

Theorem 1. Let $K$ be a finite strongly connected oriented two-dimensional simplicial complex, $P : K \to \mathbb{R}^3$ be an oriented polyhedron with nondegenerate triangular
faces, and \( \{P_t\}_{t \in [0,1]} \) be a flex of \( P \). Suppose that a 1-dimensional simplex \( \sigma_* \in K \) is such that the length of the edge \( P(\sigma_*) \) is not a \( \mathbb{Q} \)-linear combination of the lengths of the remaining edges of \( P \). Then the dihedral angle at the edge \( P(\sigma_*) \) during the flex \( \{P_t\}_{t \in [0,1]} \) remains unaltered, i.e., \( \tilde{\varphi}_\sigma(t) = \text{const.} \)

**Proof.** As usual, denote by \( K_1 \) the set of all 1-dimensional simplices of \( K \) and put \( K_1^* = K_1 \setminus \{\sigma_*\} \). Let us denote by \( \Lambda_* \) the \( \mathbb{Q} \)-linear span of the set \( \{\ell_\sigma\}_{\sigma \in K_1^*} \subset \mathbb{R}^3 \), where \( \ell_\sigma \) stands for the length of the edge \( P(\sigma) \) of \( P \). Let \( \{\lambda_1, \ldots, \lambda_m\} \) denote an arbitrary \( \mathbb{Q} \)-basis in \( \Lambda_* \).

According to the conditions of Theorem 1, the set \( \{\ell_\sigma, \lambda_1, \ldots, \lambda_m\} \) is \( \mathbb{Q} \)-linearly independent. Viewing \( \mathbb{R} \) as a vector space over the field \( \mathbb{Q} \), expand \( \{\ell_\sigma, \lambda_1, \ldots, \lambda_m\} \) to a Hamel basis in this space. Define a \( \mathbb{Q} \)-linear function \( f_\sigma : \mathbb{R} \to \mathbb{R} \) by putting for every element \( \lambda \) of the above constructed Hamel base in \( \mathbb{R} \)

\[
f_\sigma(\lambda) = \begin{cases} 1, & \text{if } \lambda = \ell_\sigma; \\ 0, & \text{if } \lambda \neq \ell_\sigma. \end{cases}
\]

Since \( \{\lambda_1, \ldots, \lambda_m\} \) is a basis in \( \Lambda_* \), for every \( \sigma \in K_1^* \), there are rational numbers \( \alpha_{\sigma_j}, j = 1, \ldots, m \) such that \( \ell_\sigma = \sum_{j=1}^m \alpha_{\sigma_j} \lambda_j \). Therefore, from the definition of the function \( f_\sigma \), we obtain for every \( \sigma \in K_1^* \)

\[
f_\sigma(\ell_\sigma) = f_\sigma \left( \sum_{j=1}^m \alpha_{\sigma_j} \lambda_j \right) = \sum_{j=1}^m \alpha_{\sigma_j} f_\sigma(\lambda_j) = 0.
\]

Hence,

\[
\sum_{\sigma \in K_1}(\|\sigma\| \tilde{\varphi}_\sigma(t)) = f_\sigma(\|\sigma\| \tilde{\varphi}_\sigma(t)) + \sum_{\sigma \in K_1^*}(\|\sigma\| \tilde{\varphi}_\sigma(t)) = \tilde{\varphi}_\sigma(t).
\]

However, according to Lemma 1, the left-hand side of the last equality is independent of \( t \). So, its right-hand side does not depend on \( t \) too, i.e., \( \tilde{\varphi}_\sigma(t) = \text{const.} \)

**Theorem 2.** Let \( K \) be a finite strongly connected oriented two-dimensional simplicial complex and \( P : K \to \mathbb{R}^3 \) be an oriented polyhedron with nondegenerate triangular faces. Suppose that the set of the lengths of the edges of \( P \) is \( \mathbb{Q} \)-linearly independent. Then \( P \) is rigid.

**Proof.** Let \( \{P_t\}_{t \in [0,1]} \) be a flex of the polyhedron \( P = P_0 : K \to \mathbb{R}^3 \). Since, for every \( \sigma \in K_1 \), the length of the edge \( P(\sigma) \) of \( P \) cannot be represented as a \( \mathbb{Q} \)-linear combination of the lengths of remaining edges of \( P \) then, by Theorem 1, the dihedral angle at the edge \( P(\sigma) \) during the flex \( \{P_t\}_{t \in [0,1]} \) remains constant, i.e., \( \tilde{\varphi}_\sigma(t) = \text{const.} \)

Let us fix \( t \in (0, 1] \) and a 2-dimensional simplex \( \tau \in K \). Since \( \{P_t\}_{t \in [0,1]} \) is a flex, there is an isometry \( A_t : \mathbb{R}^3 \to \mathbb{R}^3 \), which matches the faces \( P_0(\tau) \subset P_0(K) \) and \( P_t(\tau) \subset P_t(K) \) together with their orientation, i.e., such that \( P_t|_\tau = (A_t \circ P_0)|_\tau \).

Since, for every \( \sigma \in K_1 \) (including the case \( \sigma \subset \tau \)), we have \( \tilde{\varphi}_\sigma(t) = \tilde{\varphi}_\sigma(0) \), the
equality $P_t = A_t \circ P_0$ holds true on the union of the simplex $\tau$ and all three 2-dimensional simplices of $K$, adjacent to $\tau$. Continuing in the same way and using the strong connectivity of the complex $K$, we make sure that the equality $P_t = A_t \circ P_0$ holds true on $K$. Hence, the flex $\{P_t\}_{t \in [0,1]}$ is trivial. Thus, we see that the polyhedron $P = P_0 : K \to \mathbb{R}^3$ allows only trivial flexes, i.e., it is rigid.

**Remark 5.** For the first time, Theorem 2 was published without proof in [21] as Corollary 1.3.

**Remark 6.** It would be very interesting to determine to what extent Theorem 2 holds for non-orientable polyhedra. The above arguments are not suitable for this, since they rely on Lemma 1, which is applicable to oriented polyhedra only.

**Remark 7.** Theorem 2 gives a sufficient condition for a polyhedron to be rigid, which is fundamentally different from the two classical sufficient conditions for a polyhedron to be rigid mentioned in Section 1 (we mean the convexity and the absence of nontrivial infinitesimal deformations), and from a sufficient condition for rigidity of polyhedra found by I.Kh. Sabitov in [32]. Recall that, in [32] Theorem 2], for each small diagonal of a polyhedron $P : K \to \mathbb{R}^3$, a polynomial $q(x)$ in one real variable $x$ is constructed such that:

- the coefficients of $q(x)$ are real numbers and depend only on $K$ and the edge lengths of $P$;
- the length of this small diagonal is a root of $q(x)$.

Obviously, if $P$ is flexible then at least one of its small diagonals is nonconstant and, therefore, takes infinitely many values. Hence, all the coefficients of the corresponding polynomial $q(x)$ are zero. This consideration led I. Kh. Sabitov to the following sufficient condition for a polyhedron to be rigid: If for each small diagonal at least one coefficient of the corresponding polynomial $q(x)$ is not equal to zero then the polyhedron $P$ is rigid, see [32, Corollary 2]. However, the application of Sabitov’s condition is complicated by the fact that the problem of finding the polynomial $q(x)$ in itself is very difficult.

**Remark 8.** Theorem 2 in a new way expresses the well-known fact that “almost all closed polytopes are inflexible.” The novelty is that the condition interpreted as “almost all” is imposed on the edge lengths, while in previously known theorems of this type a similar condition was imposed on the coordinates of the vertices. To be more precise, we recall that there is a natural one-to-one mapping from the set of all polyhedra $P : K \to \mathbb{R}^3$ of a given combinatorial type $K$ onto $\mathbb{R}^{3v}$, where $v$ is the number of 0-dimensional simplices of $K$. This mapping maps $P$ to the vector in $\mathbb{R}^{3v}$ such that the set of its coordinates is the union of the sets of coordinates of all vertices of $P$ ordered according to a preassumed order on the set of 0-dimensional simplices of $K$. The following properties of the set $\mathcal{F}$ consisting of the points of $\mathbb{R}^{3v}$ corresponding to flexible polyhedra $P : K \to \mathbb{R}^3$ are known:

(A) $\mathcal{F}$ is contained in the complement to an open everywhere dense subset of $\mathbb{R}^{3v}$, see, e.g., [5], [32], [22];
(B) $\mathcal{F}$ is contained in the complement to a set which consists of the points $x \in \mathbb{R}^{3w}$ whose coordinates are algebraically independent over $\mathbb{Q}$ (this means that, given $x$, there is no nonzero polyhedron with rational coefficients for which $x$ is a root), see, e.g., [13], [12]. For completeness, we note that properties (A) and (B) are also valid in more general situations, namely, for various and very wide classes of frameworks in $\mathbb{R}^n$, $n \geq 2$, which, for $n \geq 3$, include 1-skeletons of $(n-1)$-dimensional simplicial polyhedra, see, e.g., [27], [10] and literature mentioned there.

The following theorem is a generalization of Theorem 1:

**Theorem 3.** Let $K$ be a finite strongly connected oriented two-dimensional simplicial complex, $K_1$ be the set of all 1-dimensional simplices of $K$, $P : K \to \mathbb{R}^3$ be an oriented polyhedron with nondegenerate triangular faces, $L = \bigcup_{\sigma \in K_1} \{\ell_\sigma\}$ be the set of the lengths $\ell_\sigma$ of all edges $P(\sigma)$ of $P$, and let $\{\lambda_1, \ldots, \lambda_m\}$ be a $\mathbb{Q}$-basis in the $\mathbb{Q}$-linear span of $L$, i.e., the reals $\lambda_1, \ldots, \lambda_m$ are $\mathbb{Q}$-linearly independent and, for every $\sigma \in K_1$, there are $\alpha_{\sigma j} \in \mathbb{Q}$ such that

$$\ell_\sigma = \sum_{j=1}^m \alpha_{\sigma j} \lambda_j.$$ 

Then, for every flex $\{P_t\}_{t \in [0,1]}$ of $P$ and every $j = 1, \ldots, m$, the expression

$$\sum_{\sigma \in K_1} \alpha_{\sigma j} \varphi_\sigma(t)$$

is independent of $t$. In (1), the summation is taken over all 1-dimensional simplices $\sigma$ of $K$ and $\varphi_\sigma(t)$ stands for the dihedral angle at the edge $P(\sigma)$ during the flex $\{P_t\}_{t \in [0,1]}$.

**Proof.** Treating $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$, expand the set $\{\lambda_1, \ldots, \lambda_m\}$ to a Hamel basis in this space. For every $j = 1, \ldots, m$, define a $\mathbb{Q}$-linear function $f_j : \mathbb{R} \to \mathbb{R}$ by putting for an element $\lambda$ of the above constructed Hamel base in $\mathbb{R}$

$$f_j(\lambda) = \begin{cases} 1, & \text{if } \lambda = \lambda_j, \\ 0, & \text{if } \lambda \neq \lambda_j. \end{cases}$$

Then, using (1) and the definition of $f_j$, we get

$$\sum_{\sigma \in K_1} f_j(\ell_\sigma) \varphi_\sigma(t) = \sum_{\sigma \in K_1} f_j \left( \sum_{k=1}^m \alpha_{\sigma k} \lambda_k \right) \varphi_\sigma(t) = \sum_{\sigma \in K_1} \sum_{k=1}^m \alpha_{\sigma k} f_j(\lambda_k) \varphi_\sigma(t) = \sum_{\sigma \in K_1} \alpha_{\sigma j} \varphi_\sigma(t).$$

However, according to Lemma 1, the left-hand side of the last equality does not depend on $t$. Hence, its right-hand side does not depend on $t$. \qed
Remark 9. Reducing rational numbers $\alpha_{ij}$ in (1) to a common denominator, we find that some linear combination of its dihedral angles remains constant during the flex of a polyhedron. Moreover, the coefficients of this linear combination do not change during the flex, are integers, and not all are equal to zero. Thus, Theorem 3 clarifies the origin of linear relations with integer coefficients between dihedral angles of Bricard octahedra, found in [4] (to be more precise, in [4], linear relations for dihedral angles of Bricard octahedra of types I, II, and III were obtained and used in the proofs of Theorems 2, 4, and 6, respectively).

4. Acknowledgement

The author is grateful to Alexander A. Gaifullin and Idzhad Kh. Sabitov for their comments on a preliminary version of this article.

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December 26, 2018