Energy and pressure densities of a hot quark-gluon plasma

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Abstract

We calculate the energy and hydrostatic pressure densities of a hot quark-gluon plasma in thermal equilibrium through diagrammatic analyses of the statistical average, $\langle \Theta_{\mu\nu} \rangle$, of the energy-momentum-tensor operator $\Theta_{\mu\nu}$. To leading order at high temperature, the energy density of the long wave length modes is consistently extracted by applying the hard-thermal-loop resummation scheme to the operator-inserted no-leg thermal amplitudes $\langle \Theta_{\mu\nu} \rangle$. We find that, for the long wave length gluons, the energy density, being positive, is tremendously enhanced as compared to the noninteracting case, while, for the quarks, no noticeable deviation from the noninteracting case is found.

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1 Introduction

For the long wavelength $\lambda \leq \mathcal{O}(gT)^{-1}$ or soft modes in a hot quark-gluon plasma, the contributions of “hard thermal loops” to their amplitudes are as important as tree amplitudes [1, 2, 3, 4]. Numerous applications have been made [5] of the hard thermal loop (HTL) resummation scheme [1, 2, 3, 4], the scheme which enables consistent evaluations of any thermal amplitudes to leading order in the coupling constant $g$.

Due to the HTL resummations, the energy and pressure densities of the soft modes in the quark-gluon plasma deviates from the form in the noninteracting case. Weldon [6] was the first who treat this issue. Subsequently, several work [7, 8, 9] have been devoted to the issue. The strategy for extracting the energy-momentum-tensor operator for soft modes in [6, 7, 8] and the energy functional in [9] is to use the effective action [10, 11], in which all the contributions coming from HTL resummations are taken in. A brief account is given in [9] of the energy-momentum tensors derived in [6, 7, 8].

The purpose of the present paper is to evaluate the energy density $E(p)$ and the pressure density $P(p)$ of the soft modes in the hot quark-gluon plasma from purely “diagrammatic point of view”. By “the soft mode” we mean the mode whose momentum $p$ is in the range

$$g^2 T << p << T.$$  

(1)

For analyzing the hyper-soft modes $p \leq \mathcal{O}(g^2 T)$, a treatment beyond the HTL resummation scheme is required. Our procedure is as follows: 1) We start with the energy-momentum-tensor operator, $\Theta_{\mu\nu}$, which are extracted from the QCD Lagrangian. 2) We evaluate the energy and pressure densities through calculating the statistical averages $\langle \Theta_{00} \rangle$ and $\langle \Theta_{11} \rangle$, respectively. 3) In evaluating these densities for soft modes, to leading order at high temperature, we apply the HTL resummation scheme to the operator inserted no-leg thermal amplitudes, $\langle \Theta_{00} \rangle$ and $\langle \Theta_{11} \rangle$. Through the HTL resummations, in addition to (formally) the lowest-order diagrams, formally $\mathcal{O}(g^2)$ diagrams yield dominant contributions.

It is worth mentioning that, in the approach [6, 7, 8, 9], the HTL resummations are performed first to construct the effective action and then the energy-momentum-tensor operator (for soft modes) is extracted. On the contrary, our approach starts with the energy-momentum-tensor operator $\Theta_{\mu\nu}$ extracted from the Lagrangian and...
then $\langle \Theta_{00} \rangle$ and $\langle \Theta_{11} \rangle$ are evaluated on the basis of the HTL resummation scheme. It
turns out that the resultant $\langle \Theta_{\mu\nu} \rangle$ corresponds to the thermal average of the energy-
momentum-tensor operator $T_{\mu\nu}$ found in [7]. Relationships between various expressions for $T_{\mu\nu}$, found in [3, 7, 8, 9], are not immediately obvious [9]. This is also the
case for the relations of our $\langle \Theta_{\mu\nu} \rangle$ to $T_{\mu\nu}$’s in [3, 8, 9].

The plan of the paper is as follows. In Sec. II, some preliminaries, which include
the expression for $\Theta_{\mu\nu}$ extracted from the Lagrangian, are given. The contribution to
$\langle \Theta_{\mu\nu} \rangle$ from formally lowest-order diagrams are evaluated in Sec. III. In Sec. IV, on the
basis of the HTL-resummation scheme, we calculate the contributions to $\langle \Theta_{\mu\nu} \rangle$ from
formally $O(g^2)$ diagrams. In Sec. V, the energy of the soft modes is calculated and,
in Sec. VI, the hydrostatic pressure is calculated. Sec. VII is devoted to summary
and discussion.

2 Preliminary

We consider an $SU(N)$ gauge theory with $N_f$ flavors of massless quarks in the funda-
mental representation. As is well known (see, e.g., [12]), various forms of the energy-momentum-tensor operator are available. Although they lead to different
stress-energy-momentum densities, the same total or integrated stress energy mo-
momentum results. Since we are interested in the stress-energy-momentum density of
the quark-gluon plasma, it is of vital importance which form of the energy-momentum
tensor is the physically sensible one. A natural candidate is the gravitational energy-
momentum tensor $\Theta_{\mu\nu}$, since its generalization to the case of curved space-time appears in the Einstein equation. As in all other known cases, $\Theta_{\mu\nu}$ coincides [12] with the Belinfante tensor:

\[
\Theta_{\mu\nu} \equiv \Theta_{\mu\nu}^{(g)} + \Theta_{\mu\nu}^{(q)},
\]

\[
\Theta_{\mu\nu}^{(g)} = -F_{\mu\rho}^{a} F_{\nu}^{a\rho} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta}^{a} F_{\alpha\beta}^{a}
+ \left\{ - (\partial_{\mu} \omega^{a})(D_{\nu} \omega)^{a} + \lambda A_{\mu}^{a} \partial_{\nu} \partial \cdot A^{a} \right\} + \{ \mu \leftrightarrow \nu \}
+ g_{\mu\nu} \left[ (\partial^{\rho} \omega^{a})(D_{\rho} \omega)^{a}
- \frac{\lambda}{2} (\partial \cdot A^{a})^{2} - \lambda A_{\rho}^{a} \partial^{\rho} \partial \cdot A^{a} \right],
\]
\[ \Theta^{(q)}_{\mu \nu} = \frac{i}{2} \bar{\psi} \left( \gamma_\mu \hat{D}_\nu + \gamma_\nu \hat{D}_\mu \right) \psi - ig_{\mu \nu} \bar{\psi} \gamma^5 \psi, \quad (4) \]

where

\[ (D_\mu \omega)^a \equiv \partial_\mu \omega^a + igf^{abc} A^b_\mu \omega^c, \]
\[ \hat{D}_\mu \equiv \frac{1}{2} \partial_\mu - gA^a T_a. \]

Here \( \{f^{abc}\} \) are the structure constants of \( su(N) \) and \( \{T^a\} \) are an anti-Hermitian basis of the fundamental representation. The temperature-dependent part of the energy-momentum \( P_\mu \) of a quark-gluon plasma is obtained through

\[ \langle \Theta_{\mu \nu} \rangle \equiv \frac{Tr \left( e^{-H/T} \Theta_{\mu \nu} \right)}{Tr \left( e^{-H/T} \right)} \quad (5) \]

as

\[ P_\mu \equiv \langle \Theta_{0 \mu} \rangle - \langle 0 | \Theta_{0 \mu} | 0 \rangle, \quad (6) \]

where

\[ \langle 0 | \Theta_{0 \mu} | 0 \rangle = \lim_{T \to 0} \langle \Theta_{\mu \nu} \rangle. \]

It is to be noted that, in (6), the ultraviolet (UV) divergence involved in \( \langle \Theta_{0 \mu} \rangle \) and that in \( \langle 0 | \Theta_{0 \mu} | 0 \rangle \) cancel out. Similarly the temperature-dependent part of the hydrostatic pressure \( P \) is defined as

\[ P \equiv \langle \Theta_{11} \rangle - \langle 0 | \Theta_{11} | 0 \rangle. \quad (7) \]

We employ the imaginary-time formalism \cite{2} of thermal field theory. Nevertheless, all the formulae are displayed in the Minkowski metric. Following \cite{2}, we introduce an index “\( r \)”, \( r = + \) for bosons and \( r = - \) for fermions:

\[ \frac{1}{P^2} \equiv \Delta^r (P) = \begin{cases} \Delta^+(P), & \text{for bosons,} \\ \Delta^-(P), & \text{for fermions.} \end{cases} \quad (8) \]

Here, for \( r = + \) \( \{-\} \), \( p_0 \) takes the values \( p_0 = 2n i \pi T \ (2n + 1)i \pi T \) with \( n = 0, \pm 1, \pm 2, \ldots \). Capital letters represent four-momenta, lower-case letters their components: \( P_\mu = (p_0, \mathbf{p}) \) with \( \mathbf{p} = p \hat{\mathbf{p}} \), and \( P^2 = p_0^2 - \mathbf{p}^2 \). For a loop momentum \( P \), we introduce the symbol \( \text{Tr}_P \) defined as

\[ \text{Tr}_P F(p_0, p) \equiv T \sum_{n=-\infty}^{+\infty} \int \frac{d^3 p}{(2\pi)^3} F(p_0, p). \quad (9) \]
For bosonic \{fermionic\} $P$, $p_0 = 2ni\pi T \{(2n + 1)i\pi T\}$. \[\text{Tr}_{\text{soft } P}\] is defined as in (8), where the integration over $p$ is carried out over the soft-$p$ region, Eq. (1).

In this paper, we use Feynman gauge ($\lambda = 1$) throughout.

### 3 Contributions from tadpole diagrams to $\langle \Theta_{\mu\nu} \rangle$

To the lowest order, (8) is diagrammed in Fig. 1: Figs. 1(a), (b), and (c) represent the contributions from soft gluons, FP-ghosts, and quarks, respectively. Incidentally, for hard gluons [quarks], the HTL-resummed effective gluon [quark] propagator — the line with a blob — in Fig. 1(a) [(c)] can be replaced by the bare counterpart. The form of their contributions are well known, and we do not reproduce them here.

#### 3.1 Gluon sector

We start with the gluon sector, Figs. 1(a) and (b). Using (8), we obtain for the contribution to $\langle \Theta_{\mu\nu} \rangle$,

$$\langle \Theta^{(g_0)}_{\mu\nu} \rangle = \langle \Theta^{(g_0)}_{\mu\nu} \rangle \Big|_{\text{Fig. 1(a)}} + \langle \Theta^{(g_0)}_{\mu\nu} \rangle \Big|_{\text{Fig. 1(b)}},$$

(10)

$$\langle \Theta^{(g_0)}_{\mu\nu} \rangle \Big|_{\text{Fig. 1(a)}} = \frac{1}{2} \left(N^2 - 1\right) \text{Tr}_{\text{soft } P} *\Delta_{\alpha\beta}(P) \theta^{\alpha\beta}_{\mu\nu}(P),$$

(11)

$$\langle \Theta^{(g_0)}_{\mu\nu} \rangle \Big|_{\text{Fig. 1(b)}} = -\left(N^2 - 1\right) \text{Tr}_{\text{soft } P} \Delta^+(P) \left[g_{\mu\nu}P^2 - 2P_\mu P_\nu\right].$$

(12)

where $*\Delta_{\alpha\beta}(P)$ is the effective gluon propagator in Feynman gauge,

$$*\Delta_{\alpha\beta}(P) = \frac{P_\alpha P_\beta}{P^4} + \sum_{\ell = T, L} \frac{Q^{(\ell)}_{\alpha\beta}(P)}{P^2 - \Pi_{\ell}(P)},$$

(13)

and, in (11),

$$\theta^{\alpha\beta}_{\mu\nu}(P) = \left\{-P_\mu P_\nu g^{\alpha\beta} - P^2 \delta^\alpha_\mu \delta^\beta_\nu + \frac{1}{2} g_{\mu\nu}P^2 g^{\alpha\beta}\right\} + \{\alpha \leftrightarrow \beta\}.$$  

(14)

In (13),

$$Q^{(T)}_{\alpha\beta} = \sum_{i,j=1}^{3} g_{\alpha i} g_{\beta j} \left(\hat{p}_i \hat{p}_j - \delta_{ij}\right), \quad (\hat{p} \equiv p/|p|),$$
\[ Q_{\alpha\beta}^{(L)} = g_{\alpha0} g_{\beta0} - \frac{P_{\alpha} P_{\beta}}{p^2} - \sum_{i,j=1}^{3} g_{\alpha i} g_{\beta j} \hat{p}_i \hat{p}_j , \]  

(15)

and \( \Pi_T (\Pi_L) \) is the the transverse (longitudinal) part of the thermal vacuum polarization tensor \( [13] \), whose explicit forms are not necessary for the purpose of this paper.

### 3.2 Quark sector

Fig. 1 (c) with (4) gives

\[
\langle \Theta^{(g0)}_{\mu\nu} \rangle = \frac{1}{2} N_f Tr_p \left\{ P_\nu tr \left[ \gamma_\mu \, ^*S(P) \right] - g_{\mu\nu} tr \left[ \hat{P} \, ^*S(P) \right] \right\} + \{\mu \leftrightarrow \nu\} , \]

(16)

where \(^*S(P)\) is the effective quark propagator \( [14] \):

\[
^*S(P) = \sum_{\sigma = \pm} \hat{P}_\sigma \, ^*S^{(\sigma)}(P) , \]

(17)

\[
\hat{P}_\sigma = (1, \sigma \hat{p}) .
\]

The explicit form of \(^*S^{(\sigma)}(P)\) in \( [17] \) is given in \( [14] \).

### 4 Hard-thermal-loop resummation for diagrams with operator \( \Theta_{\mu\nu} \) insertion

#### 4.1 General observations

Now we analyze formally \( \mathcal{O}(g^2) \) contributions to \( \langle \Theta_{\mu\nu} \rangle \). The relevant diagrams are depicted in Figs. 2 and 3. As stated in Sec. 1, our analysis goes from the “diagrammatic point of view”: We apply the HTL-resummation scheme \( [1, 2, 3, 4] \) to each diagram in Figs. 2 and 3, and extract the contributions at high-\( T \), which are of the same order as those from the lowest-order diagrams, Fig. 1.

In each diagram in Figs. 2 and 3, the part enclosed with the box is the one-loop correction to the composite vertex, which we write \( \langle P | \Theta_{\mu\nu} | P \rangle \). We shall see that, for
hard $K (\sim T)$ and soft $P (\sim gT)$ [cf. Figs. 2 and 3], some of these matrix elements are HTL’s, i.e., are of the same order as the lowest-order counterparts, and some of them lead to the same-order contributions to the energy and pressure densities as those from the lowest-order diagrams.

The HTL’s summarize the leading thermal effect and are essentially classical [15]. Then, in particular, they are free from UV divergences, so that we do not need to take the operator-mixing problem into consideration. In this relation, it is worth recalling here that the $T$-dependent parts of the above-mentioned $\langle P|\Theta_{\mu\nu}|P\rangle$ represent essentially the matrix elements in tree approximation in vacuum theory. In order to see this in the present case, we take massless scalar $\phi^3$ theory, for simplicity, and consider the composite operator $\phi^2/2$. We compute one-loop correction to the composite vertex as depicted in Fig. 4:

$$\langle p|\frac{1}{2}\phi^2(0)|p\rangle \equiv -\frac{g^2}{2}\text{Tr}_K\left\{\Delta^+(K)\right\}^2\Delta^+(K+P).$$

The manipulation of (18) is carried out in Appendix B. After continuing $p_0 (= 2\pi inT)$ to real energy $p_0 + i0^+$, we have (cf. (B.4))

$$\langle p|\frac{1}{2}\phi^2(0)|p\rangle = \frac{1}{2}g^2\int\frac{d^4K}{(2\pi)^4} \sum_{i,j=1}^{2} (-)^{i+j}D_{ii}(K)D_{jj}(K)$$

$$\times D_{ij}(K+P),$$

the formula which is written in terms of the real-time formulation of thermal field theory [12], which is formulated on the time path $-\infty \rightarrow +\infty \rightarrow -\infty \rightarrow -\infty - i/T$, in a complex time plane. In (19), $P$ and $K$ are the four-vectors in the Minkowski space, “1”, “i”, and “j” are the thermal indices, and $D_{ij}$ ($i, j = 1, 2$) is the $(i, j)$-component of the matrix propagator of a massless scalar particle, whose explicit form is given, e.g., in [12, 16].

A thermal amplitude essentially represents the sum of the corresponding matrix elements $\sum_{i,f}\langle i|S|f\rangle\langle f|S^*|i\rangle$, where $S$ is the $S$-matrix in vacuum theory, $S^*$ is the complex conjugate of the $S$-matrix, and $|i\rangle$ $|f\rangle$ is the initial [final] state, which includes all the particles in the heat bath. The details and the rules for the correspondences are presented in [17]. We can apply the rules to each term in (19). As an example, we take (19) with $i = 1$ and $j = 2$ with $k_0 > 0$ and $k_0 + p_0 > 0$:

$$\frac{g^2}{8\pi^2}\int d^4K\left[\frac{1}{K^2 + i0^+}\delta(K^2) \otimes \delta((K + P)^2)\right] n_B(k) (1 + n_B(E))$$
\[ - \left\{ 2\pi i \delta^2(K^2) \otimes \delta((K + P)^2) \right\} n_B^2(k)(1 + n_B(E)) \], \quad (20) \]

where \( E = |k + p| \). It should be remembered that, although products of singular functions appear in (20), they are cancelled in (19), leaving well-defined terms. The first and second terms in (20) represent the products of *tree amplitudes*, as depicted in Figs. 5(a) and (b), respectively. In Fig. 5, the left-side part of the final-state cut line (dot-dashed line) represents the amplitude or \( S \)-matrix element in vacuum theory, while the right-side part represents the complex conjugate of the amplitude. The state at the left end and the state at the right end are the same and they represent the initial state. The lines on top of Fig. 5 stand for the spectator particles, which are the constituents of the heat bath. The left-side part of \( \otimes \) in the first [second] curly brackets in (20) stands for the amplitude \( S \) in Fig. 5(a) [(b)] and the right-side part stands for complex conjugate of the amplitude, \( S^* \), in Fig. 5(a) [(b)]. Then, each term in (20) is simply an integration over \( k \) of the product of the tree amplitude and the complex conjugate of the tree amplitude weighted with statistical factors. Since the latter factors dump at the UV region, no UV disaster arises. This observation applies to all other portions of (19), provided that the \( T = 0 \) part has been subtracted.

The above argument applies to the \( T \)-dependent parts of all \( \langle P| \Theta_{\mu\nu} |P \rangle \)'s in Figs. 2 and 3. Since no UV divergences appear, \( \langle P| \Theta_{0\mu} |P \rangle \) [\( \langle P| \Theta_{11} |P \rangle \)] enjoys a meaning of energy-momentum [pressure].

To summarize, as far as the \( T \)-dependent part of \( \langle \Theta_{\mu\nu} \rangle \), which yields leading-order contributions at high-\( T \), is concerned, we do not need to worry about the operator-mixing problem, and we can use the operators (3) and (4) for evaluating the energy-momentum density.

Now we separately study the soft-gluon case, Fig. 2, and the soft-quark case, Fig. 3.

### 4.2 Soft-gluon sector

In this subsection we extract from formally \( \mathcal{O}(g^2) \) diagrams, Fig. 2, the contributions which are of the same order as those from the tadpole diagrams Figs. 1(a) and (b), Eqs. (10) - (12).
The contribution of Fig. 2(a) is written as

\[ \langle \Theta_{\mu\nu} \rangle_{\text{Fig. 2(a)}} = \frac{g^2}{2} N (N^2 - 1) \text{Tr}_{\text{soft P}} \left[ \text{Tr}_K \right] \left\{ \Delta^+ (K) \right\}^2 \Delta^+ (K + P) \times \Delta_{\gamma\delta} (P) \theta_{\mu\nu}^\alpha (K) \mathcal{V}_{\beta\gamma\epsilon} (K, P) \mathcal{V}_{\delta\alpha} (K + P, -P) , \] (21)

where

\[ \mathcal{V}_{\beta\gamma\epsilon} (K, P) = g_{\beta\gamma} (P - K) - g_{\gamma\epsilon} (K + 2P) + g_{\epsilon\beta} (2K + P) \gamma . \]

In (21), \( \Delta_{\gamma\delta} (P) \) is as in (13) and \( \theta_{\mu\nu}^\alpha (K) \) in (14). Carrying out the contractions with respect to the repeated indices and changing the integration variable \( K \) to \(-K - P\) at several places, (21) is led to a sum of terms, which are of the following generic forms,

\[ \text{Tr}_{\text{soft P}} \frac{1}{P^2 - \Pi_\ell (P)} F(P, K) \otimes I_{ij}^{++} , \quad (\ell = T, L) , \]

\[ \text{Tr}_{\text{soft P}} \frac{1}{P^{2n}} G(P, K) \otimes I_{ij}^{++} , \quad (n = 1, 2) , \] (23)

where

\[ F(P, K) \otimes I_{ij}^{rr'} = \text{Tr}_K F(P, K) \left\{ \Delta^r (K) \right\}^i \times \left\{ \Delta^{r'} (K + P) \right\}^j , \quad (r, r' = \pm) . \] (24)

Keeping the terms which lead to the contributions of the order under consideration, we have

\[ \langle \Theta_{\mu\nu}^{(g)} \rangle_{\text{ghon}} \approx g^2 N (N^2 - 1) \text{Tr}_{\text{soft P}} \left[ \sum_{\ell = T, L} \frac{1}{P^2 - \Pi_\ell (P)} \mathcal{V}_{\mu\nu}^{(\ell)} + \frac{1}{P^4} \mathcal{V}_{\mu\nu}^{(4)} + \frac{1}{P^2} \mathcal{V}_{\mu\nu}^{(2)} \right] , \] (25)

where

\[ \mathcal{V}_{\mu\nu}^{(\ell)} = \left( c_1^{(\ell)} g_{\mu\nu} + c_2^{(\ell)} Q_{\mu\nu}^{(\ell)} \right) I_{10}^+ + \left( c_3^{(\ell)} g_{\mu\nu} (K | Q_{\nu}^{(\ell)} | K) + c_4^{(\ell)} (K_{\mu} Q_{\nu\rho}^{(\ell)} + K_{\nu} Q_{\mu\rho}^{(\ell)}) K^\rho + c_5^{(\ell)} K_{\mu} K_{\nu} \right) \otimes I_{20}^+ \]

\[ + \left( c_6^{(\ell)} g_{\mu\nu} (K | Q_{\nu}^{(\ell)} | K) + c_7^{(\ell)} (K_{\mu} Q_{\nu\rho}^{(\ell)} + K_{\nu} Q_{\mu\rho}^{(\ell)}) K^\rho + c_8^{(\ell)} K_{\mu} K_{\nu} \right) \otimes I_{11}^{++} \]

\[ + c_9^{(\ell)} K_{\mu} K_{\nu} (K | Q_{\nu}^{(\ell)} | K) \otimes I_{21}^+ , \] (26)

\[ \mathcal{V}_{\mu\nu}^{(4)} = d_1 P_{\mu} P_{\nu} I_{10}^+ + P \cdot K \left\{ d_2 (K_{\mu} P_{\nu} + K_{\nu} P_{\mu}) + d_3 (P \cdot K) g_{\mu\nu} \right\} \otimes I_{20}^+ , \] (27)

\[ \mathcal{V}_{\mu\nu}^{(2)} = d_4 g_{\mu\nu} I_{10}^+ + d_5 K_{\mu} K_{\nu} \otimes I_{20}^+ + d_6 K_{\mu} K_{\nu} \otimes I_{11}^{++} . \] (28)
In (25) - (28), \( Q^{(\ell)}_{\mu
u} \) is as in (15) and
\[
\left( K | Q^{(\ell)} | K \right) \equiv K^\rho K^\sigma Q^{(\ell)}_{\rho\sigma},
\]
(29)
\[
c_j^{(L)} = c_j^{(T)} \quad \text{for } j = 2, 3, 4, 6, 7, 9,
\]
(30)
\[
c_j^{(L)} = \frac{1}{2} c_j^{(T)} \quad \text{for } j = 1, 5, 8.
\]
(31)
The coefficients, \( c \)'s and \( d \)'s, in (25) - (28) are listed in the first row of Table I.

In a similar manner, we can show that the contributions of Figs. 2(b) - (f) may also be written in the form (25) - (28) with (30) and (31). [The dashed-lines in Figs. 2(e) and (f) stand for the hard FP-ghost propagators.] For each contribution, the coefficients are tabulated in Table I.

Now we turn to analyze Figs. 2(g) - (i), where the dashed lines stand for the FP-ghost propagators. One ghost-gluon vertex is proportional to the soft ghost momentum. Then, this ghost-gluon vertex brings in one power of a soft momentum, instead of a hard loop momentum as in the case of Figs. 2(a) - (f). Then, the contributions of Figs. 2(g) - (i) are negligible,
\[
\langle \Theta^{(\text{ghost})} \rangle \sim 0.
\]
(32)

Finally, we analyze Figs. 2(j) and (k). It can easily be shown that, adding the contributions from Figs. 2(j) and (k), the “\( g_{\mu\nu} \) term” in (4) — the term that is proportional to \( g_{\mu\nu} \) — leads to vanishing contribution. It is worth pointing out in passing that the \( g_{\mu\nu} \) term vanishes provided the equation of motion is used. Then, what we should analyze is Figs. 2(j) and (k) with the first term on the r.h.s. of (4). It is an easy task to see that the term \( P_{\alpha} P_{\beta}/P^4 \) of the effective soft-gluon propagator \( *\Delta_{\alpha\beta}(P) \) (cf. (13)) does not yield leading contribution. Ignoring all other terms that lead to nonleading contributions, we obtain for the contribution to \( \langle \Theta^{(g)}_{\mu\nu} \rangle \),
\[
\left. \langle \Theta^{(g)}_{\mu\nu} \rangle \right|_{\text{quark}} = 2g^2 N_f \left( N^2 - 1 \right) \text{Tr}_{\text{soft}} \left[ \frac{1}{P^2 - \Pi_{\ell}(P)} \left\{ Q^{(\ell)}_{\mu\nu} \otimes I_{10}^{-} + \tilde{c}^{(T)} K^\mu K^\nu \otimes I_{20}^{-} - 2 \left( K^\nu Q^{(\ell)}_{\mu\sigma} + K^\mu Q^{(\ell)}_{\nu\sigma} \right) K^\sigma \otimes I_{11}^{-} + 4K^\mu K^\nu \left( K | Q^{(\ell)} | K \right) \otimes I_{21}^{-} \right\} \right],
\]
(33)
\[
\hat{c}^{(T)} = 2\hat{c}^{(L)} = -2.
\]
In (33), \( I_{ij}^- \) etc. are defined as in (24).
4.3 Soft-quark sector

In this subsection we extract from formally $O(g^2)$ diagrams the contributions which are of the same order as that from the tadpole diagram Fig. 1 (c), Eq. (16).

The diagrams to be analyzes are depicted in Figs. 3(a) - (c). The leading contribution from Figs. 3(a) and (b) is obtained as

$$\langle \Theta^{(g)}_{\mu \nu} \rangle_{\text{quark}} \simeq 2g^2 N_f \left( N^2 - 1 \right) \times \text{Tr}_{\text{soft } P} \sum_{\sigma = \pm} \tilde{S}^{(g)}(P) \left[ 4 K_\mu K_\nu \left( K \cdot \hat{P}_\sigma \right) \otimes I_{21}^+ ight. \\
\left. + \left( \hat{P}_\mu K_\nu + \hat{P}_\sigma K_\mu \right) \otimes I_{11}^+ \right] , \quad (34)$$

while Fig. 3(c) yields

$$\langle \Theta^{(g)}_{\mu \nu} \rangle_{\text{gluon}} \simeq -4g^2 N_f \left( N^2 - 1 \right) \text{Tr}_{\text{soft } P} \sum_{\sigma = \pm} \tilde{S}^{(g)}(P) \left[ 2 K_\mu K_\nu \left( K \cdot \hat{P}_\sigma \right) \otimes I_{21}^- \\
- \left( K_\mu \hat{P}_\nu + K_\nu \hat{P}_\mu \right) \otimes I_{11}^- \right] . \quad (35)$$

5 Energy density of soft modes

In this section we compute the energy $P_0$, Eq. (3), of soft modes.

5.1 Gluon sector

_Tadpole diagrams, Figs. 1(a) and (b)._ From (10) - (15), it is straightforward to evaluate the energy of soft modes, $P_0^{(g0)}$ (cf. (3)). Employing the spectral representation (A.1) in Appendix A for $(P^2 - \Pi_\ell(P))^{-1}$ in (13) and using (C.1) in Appendix C for $\Delta^+(P)$ in (12), we obtain

$$\langle \Theta_{00}^{(g0)} \rangle = \frac{1}{2} \left( N^2 - 1 \right) \text{Tr}_{\text{soft } P} \int_0^{1/T} d\tau \ e^{\rho_0 \tau} \int_{-\infty}^{+\infty} d\xi \ (1 + n_B(\xi)) e^{-\xi \tau} \times \\
\left[ 2 \rho_T(\xi) \left( p_0^2 + p^2 \right) + \rho_L(\xi) \left( p_0^2 - p^2 \right) \right] \\
- \frac{1}{2} \left( N^2 - 1 \right) \text{Tr}_{\text{soft } P} 1.$$

11
where
\[
n_B(\xi) = \frac{1}{e^{\xi/T} - 1},
\] (36)
and the spectral functions, \(\rho\)'s, are defined as in (A.2). Summation over \(n\) (cf. (9)) and integration over \(\tau\) lead to
\[
\langle \Theta(g_0) \rangle = \frac{1}{2} \left( N^2 - 1 \right) \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{+\infty} d\xi \ n_B(\xi) \left\{ 2(\xi^2 + p^2) \rho_T(\xi) + (\xi^2 - p^2) \rho_L(\xi) \right\}
- 2 \left( N^2 - 1 \right) \text{Tr}_{\text{soft}} \langle p \rangle. \tag{37}
\]

Here and in the following, unless otherwise indicated, integration over \(p\) is carried out over the region (1). In deriving (37), use has been made of the spectral sum rule (A.3) in Appendix A. It is to be noted that, in \(P_0(g_0)\) (cf. (8)), the last term in (37) is cancelled by the term coming from \(\langle 0|\Theta_{00}(g_0)|0 \rangle\). We recall that we are interested in the region \(p \leq \mathcal{O}(gT)\) and note that \(\rho_T/L(\xi)\) vanishes [13] in the region \(|\xi| > p + \mathcal{O}(gT)\), where the term of \(\mathcal{O}(gT)\) comes from the pole of \((P^2 - \Pi_\ell(P))^{-1}\). Then, in the region of our interest, \(\xi/T \ll 1\), and we may approximate \(n_B(\xi)\) as
\[
n_B(\xi) \simeq \frac{T}{\xi}. \tag{38}
\]
Using the sum rules, (A.3) and (A.4) in Appendix A, we obtain
\[
\begin{align*}
P_0(g_0) & = \langle \Theta(g_0) \rangle - \langle 0|\Theta_{00}(g_0)|0 \rangle \\
& \simeq (N^2 - 1) \int \frac{d^3p}{(2\pi)^3} \left\{ 2T + \frac{3}{2} \frac{m_g^2 T}{p^2 + 3 m_g^2} \\
& \hspace{1cm} + \mathcal{O}\left(\frac{p^2}{T}\right) \right\} - p, \tag{40}
\end{align*}
\]
where \(m_g\) is the effective gluon mass induced by the quark-gluon plasma,
\[
m_g^2 = \left( N + \frac{N_f}{2} \right) \frac{g^2 T^2}{9}. \tag{41}
\]
In (40), \(-p\) in the square brackets has come from \(-\langle 0|\Theta_{00}(g_0)|0 \rangle\) in (B9). If we were to use the bare thermal gluon propagator \(g_{\alpha\beta}/P^2\) instead of \(*\Delta_{\alpha\beta}(P)\) in (11), we would obtain (11) with the second term in the square brackets being absent. It is to be noted in passing that (10) is valid for \(T >> p\). Then, it is of no surprise that (10) does not vanish in the naive limit \(T \to 0\).
Figure 2.

Figs. 2(a) - (f): The starting formulae are (25) - (31) with Table I. All the necessary formulae for evaluating $\langle \Theta_{00} \rangle$ are displayed in Appendix C. The result may be written in the form,

$$
\langle \Theta_{00}^{(g)} \rangle^{\text{gluon}} \simeq g^2 N(N^2 - 1) T \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{7}{16} \frac{T^2}{p^2} - \frac{7}{16} \frac{T^2}{p^2 + 3m_g^2} \right],
$$

$$
= \frac{21}{16} g^2 N(N^2 - 1) T \int \frac{d^3 p}{(2\pi)^3} \frac{m_g^2 T^2}{p^2(p^2 + 3m_g^2)},
$$

(42)

which comes from small parts of the integration region in (21), where $K$ is hard, $k = \mathcal{O}(T)$ [HTL]. In (42), the contribution to the coefficients 7/16 (−7/16) from Figs. 2(a) - (f) are, in respective order, as follows,

$$
\frac{7}{16} = \frac{49}{48} - \frac{11}{24} + \frac{1}{8} - \frac{3}{16} - \frac{3}{16} + \frac{1}{8},
$$

$$
-\frac{7}{16} = -\frac{1}{16} - \frac{1}{8} - \frac{1}{8} - \frac{3}{16} + \frac{1}{48} + \frac{1}{24}.
$$

Figs. 2(g) - (i): As discussed in Sec. IV, in conjunction with (32), the contributions are nonleading as compared to (40), in which the first two terms in the curly brackets are kept.

Figs. 2(j) - (k): The relevant formula is (33). Using (C.23) in Appendix C together with (C.6) - (C.21), we obtain

$$
\langle \Theta_{00}^{(g)} \rangle^{\text{quark}} \simeq -\frac{1}{4} g^2 N_f (N^2 - 1) T \int \frac{d^3 p}{(2\pi)^3} \frac{T^2}{p^2 + 3m_g^2}.
$$

(43)

Adding all the contributions (40), (42), (32), and (43), we obtain for the leading contribution to $P_0^{(g)}$,

$$
P_0^{(g)} = \langle \Theta_{00}^{(g)} \rangle - \langle 0 | \Theta_{00}^{(g)} | 0 \rangle
$$

$$
\simeq (N^2 - 1) T \int \frac{d^3 p}{(2\pi)^3} \left[ 2 + \frac{7}{16} N \left( \frac{gT}{p} \right)^2 

+ \frac{1}{2 \left( p^2 + 3m_g^2 \right)} \left\{ 3m_g^2 - \frac{1}{2} \left( \frac{7}{4} N + N_f \right) (gT)^2 \right\} \right].
$$
\[
\mathcal{E}_g(p) = \left(N^2 - 1\right) T \left[2 + \frac{63}{8} \frac{N}{2N + N_f} \left(\frac{m_g}{p}\right)^2 - 3 \frac{13N + 8N_f}{8} \frac{m_g^2}{2N + N_f} p^2 + 3 m_g^2\right]
\]
(45)

where we have used (41). It is an elementary task to show that \(\mathcal{E}_g(p)\) is positive definite for any values of \(N, N_f, T,\) and \(p\).

5.2 Quark sector

Tadpole diagram, Fig. 1(c).

The starting formulae are (16) with \(\mu = \nu = 0\) and (17). Using the spectral representation (A.5) in Appendix A, summing over \(n\) (cf. (9)), and carrying out the integration over \(\tau\), we have

\[
\langle \Theta^{(q_0)}_{00} \rangle = -4N_f \int \frac{d^3p}{(2\pi)^3} p \int d\xi \{1 - n_F(\xi)\} \times \left[\sum_{\sigma = \pm} \sigma \rho_{\sigma}(\xi)\right],
\]
(46)

where

\[
n_F(\xi) = \frac{1}{e^{\xi/T} + 1},
\]
(47)

and the spectral function \(\rho_{\sigma}\) is defined as in (A.6). Since \(\rho_{\sigma}(\xi)\) is nonzero only in the region \(|\xi| \leq p + \mathcal{O}(gT)\), we may approximate \(n_F(\xi)\) as

\[
n_F(\xi) \approx \frac{1}{2} - \frac{1}{4} \frac{\xi}{T} + \ldots
\]
(48)

Then, using the sum rules (A.7) - (A.10) in Appendix A, we obtain

\[
\langle \Theta^{(q_0)}_{00} \rangle = N_f \int_{\text{soft}} \frac{d^3p}{(2\pi)^3} \frac{p^2}{T} \left[1 + \mathcal{O}\left(\frac{p^2 + 5m_f^2/3}{T^2}\right)\right],
\]
(49)

where

\[
m_f^2 = \frac{1}{16} \frac{N^2 - 1}{N} (gT)^2.
\]
(50)

Note that the effect of the HTL resummation for the quark propagator appears at the term of \(\mathcal{O}(m_f^2/T^2) = \mathcal{O}(g^2)\) relative to 1. In fact, (16) with \(S(P)\) in place of \(\ast S(P)\)
leads to (46) in which $\rho_\sigma(\xi) = \frac{1}{2}\delta(\xi - \sigma p)$:

$$\langle \Theta_{00}^{(q0)} \rangle_{S \to S} = -2N_f \int \frac{d^3p}{(2\pi)^3} \rho \sum_{\sigma = \pm} \sigma \{1 - n_F(\sigma p)\}$$

$$\simeq N_f \int \frac{d^3p}{(2\pi)^3} \rho^2 \left[ 1 + O\left(\frac{p^2}{T^2}\right)\right].$$

(51)

Now we rewrite (54) as

$$\langle \Theta_{00}^{(q0)} \rangle_{S \to S} = 4N_f \int \frac{d^3p}{(2\pi)^3} \left\{pn_F(p) - \frac{p}{2}\right\}$$

$$= 4N_f \int \frac{d^3p}{(2\pi)^3} pn_F(p) + \langle 0|\Theta_{00}^{(q0)}|0\rangle.$$  (54)

Using (49), (52), and (54), we obtain

$$\mathbf{P}_0^{(q0)} = \langle \Theta_{00}^{(q0)} \rangle - \langle 0|\Theta_{00}^{(q0)}|0\rangle$$

$$\simeq \int_{\text{soft}} \frac{d^3p}{(2\pi)^3} \mathcal{E}_q(p),$$

$$\mathcal{E}_q(p) = N_f \left\{2p + O\left(\frac{p^2 + 5m_f^2/3}{T}\right)\right\}.$$  (57)

In contrast to the case of gluon sector, the $T$-dependent part of $\mathcal{E}_q(p)$, Eq. (57), is nonleading. The origin of this difference may be traced back to the different statistics, (38) and (48).

Here some comments are in order. The factor “4” in (53) - (54) represents four degrees of freedom, $4 = 2 \times 2$, where the first “2” comes from a quark and an antiquark and the second “2” from spin degrees of freedom. $pn_F(p)$ in (53) represents the thermal energy per one mode of momentum $p$ and $-p/2$ is the zero-point energy. In fact the zero-point energy per one fermionic mode of momentum $p$ is obtained as $\langle 0|\frac{p}{2}(a_p^\dagger a_p - a_p a_p^\dagger)|0\rangle = -\frac{p}{2}$, where $a_p$ $[a_p^\dagger]$ is the annihilation [creation] operator of an (anti)quark with $p$ and with definite spin. It is interesting to note that, for soft $p$, $n_F(p) \simeq 1/2$ (cf. (48)) and then, in (53), (the leading part of) the thermal energy and the zero-point energy cancel out to yield (52). $\mathcal{E}_q(p)$ in (57) is the thermal energy of (anti)quarks with momentum $p$, $N_f \times 4 \times p \times n_F(p) \simeq 2N_f \times p$.

**Figure 3.**

Figs. 3(a) and (b): The contribution to the energy density is obtained from (14) as
\[
\Theta_{00}^{(q)}(\text{quark}) \approx 4g^2 N_f (N^2 - 1) \times \text{Tr}_{\text{soft \begin{pmatrix} \sigma & 0 \end{pmatrix}}} \left\{ 2k^2 \left( K \cdot \hat{P}_\sigma \right) \otimes I_{21}^+ + (3k_0 - 2\sigma \hat{k} \cdot \hat{p}) \otimes I_{11}^- \right\}.
\]

(58)

With the help of (A.5) in Appendix A and (C.1) in Appendix C, it is straightforward to show that the \( I_{11}^- \) part in (58) leads to a nonleading contribution. As to the contribution with \( I_{21}^+ \) part in (58), using (C.1) and (C.2), the relevant HTL parts are easily evaluated:

\[
T \sum_{n=-\infty}^{+\infty} \left[ \sum_{s=\pm} S^{(s)}(P) k^3 F(\hat{k} \cdot \hat{p}) \otimes I_{21}^+ \right]
\]

\[
\approx -\frac{1}{8T} \int_{-\infty}^{+\infty} d\xi \rho_\sigma(\xi) \int \frac{d^3k}{(2\pi)^3} F(\hat{k} \cdot \hat{p})
\]

\[
\times \sum_{s=\pm} \left[ n_F(1 - n_F)(1 + 2n_B) \frac{s}{\hat{k} \cdot \hat{p} - s\xi}
\]

\[
+ n_B(1 + n_B)(1 - 2n_F) \frac{s\hat{k} \cdot \hat{p}}{(\hat{k} \cdot \hat{p} - s\xi)^2} \right]
\]

\[
= -\frac{7}{128\pi^2} \zeta(3) T^2 \int d(\hat{k} \cdot \hat{p}) F(\hat{k} \cdot \hat{p})
\]

\[
\times \int_{-\infty}^{+\infty} d\xi \rho_\sigma(\xi) \sum_{s=\pm} \left[ \frac{s}{\hat{k} \cdot \hat{p} - s\xi} + \frac{s\hat{k} \cdot \hat{p}}{(\hat{k} \cdot \hat{p} - s\xi)^2} \right],
\]

(59)

\[
T \sum_{n=-\infty}^{+\infty} \left[ \sum_{s=\pm} S^{(s)}(P) k_0 k^2 G(\hat{k} \cdot \hat{p}) \otimes I_{21}^+ \right]
\]

\[
\approx -\frac{7}{128\pi^2} \zeta(3) T^2 \int d(\hat{k} \cdot \hat{p}) G(\hat{k} \cdot \hat{p})
\]

\[
\times \int_{-\infty}^{+\infty} d\xi \rho_\sigma(\xi) \sum_{s=\pm} \left[ \frac{1}{\hat{k} \cdot \hat{p} - s\xi} + \frac{\hat{k} \cdot \hat{p}}{(\hat{k} \cdot \hat{p} - s\xi)^2} \right].
\]

(60)

We note that, in (58),

\[
K \cdot \hat{P}_\sigma = k_0 - \sigma \hat{k} \cdot \hat{p}.
\]

(61)

Eq. (58) with the \( k_0 \) part in (61) leads to (60) with \( G(\hat{k} \cdot \hat{p}) = 1 \), which vanishes because the integrand is odd with respect to \( \hat{k} \cdot \hat{p} \). The other part in (61), \(-\sigma \hat{k} \cdot \hat{p}\), leads to (59) with \( F(\hat{k} \cdot \hat{p}) = -\sigma \hat{k} \cdot \hat{p} \), which also vanishes due to the same reason as
above. Hence the contribution \((q)\) is of nonleading,
\[
\langle \Theta_{00} \rangle \big|_{\text{quark}} \simeq 0 .
\]

Fig. 3(c): The starting formula is \((33)\). The same analysis as above shows that
\[
\langle \Theta_{00} \rangle \big|_{\text{gluon}} \simeq 0 .
\]

In conclusion, to leading order at high-\(T\), \(P_0^{(q)}\), Eq. \((57)\), does not receive additional contributions from Figs. 3(a) - (c).

6 Hydrostatic pressure

In this section, we compute the hydrostatic or kinetic pressure density of soft modes. The \(T\)-dependent part of the hydrostatic pressure \(P\) is defined as in \((5)\):
\[
P \equiv \langle \Theta_{11} \rangle - \langle 0 | \Theta_{11} | 0 \rangle \\
= -\frac{1}{3} \left\{ \{ \langle \Theta_{\mu}^{\mu} \rangle - \langle 0 | \Theta_{\mu}^{\mu} | 0 \rangle \} - \{ \langle \Theta_{00} \rangle - \langle 0 | \Theta_{00} | 0 \rangle \} \right\} .
\]

(62)

We are interested in the leading contribution at high \(T\) to \((62)\) for soft modes. Then, as in Sec. V, no diverging integral appears.

We shall show that, to leading order,
\[
\langle \Theta_{\mu}^{\mu} \rangle - \langle 0 | \Theta_{\mu}^{\mu} | 0 \rangle \simeq 0 .
\]

(63)

Using this in \((62)\), we obtain
\[
P \simeq \frac{1}{3} \{ \langle \Theta_{00} \rangle - \langle 0 | \Theta_{00} | 0 \rangle \} = \frac{1}{3} P_0 ,
\]
where \(P_0\) is as in \((5)\) with \(\mu = 0\). Then, defining the “pressure density” \(P(p)\) as
\[
P \equiv \int_{\text{soft } p} \frac{d^3 p}{(2\pi)^3} P(p) ,
\]
we have, for the gluon [quark] sector, \(P_g(p) \simeq E_g(p)/3\ [P_q(p) \simeq E_q(p)/3\ ]\), where \(E_g\ [E_q]\), Eq. \((44)\) [Eq. \((57)\)], is the energy of soft gluon [quark] with momentum \(p\).
We are now in a position to confirm (63). From (3) with $\lambda = 1$ and (4), we obtain
\begin{equation}
\Theta^{(g)\mu}_{\mu} = 2 \left[ (\partial_{\mu} \bar{\omega}^a) (D^a \omega)^a - \partial_{\mu} \left( A^a_{\mu} \partial \cdot A^a \right) \right],
\end{equation}
(64)
\begin{equation}
\Theta^{(q)\mu}_{\mu} = -3i \bar{\psi} \gamma^{\mu} \gamma^5 \psi.
\end{equation}
(65)

We note that a term with a total derivative does not contribute to (63). Then, the second term in the square brackets in (64) may be ignored, and we may replace (64) by
\begin{equation}
\Theta^{(g)\mu}_{\mu} = -2 \bar{\omega}^a \partial_{\mu} (D^a \omega)^a.
\end{equation}
(66)

It is to be noted in passing that (65) and (66) vanish when the equations of motion are imposed.

The diagrams to be analyzed are Figs. 1(b), Figs. 2(e), (f), (g), and (h), with $\Theta^{(g)\mu}_{\mu}$ insertion, and Fig. 1(c), Figs. 2(j), (k), Figs. 3(a), and (b), with $\Theta^{(q)\mu}_{\mu}$ insertion.

Fig. 1(b): We obtain for the contribution to $\langle \Theta^{(q)\mu}_{\mu} \rangle$,
\begin{equation}
\left. \langle \Theta^{(q)\mu}_{\mu} \rangle \right|_{\text{Fig. 1(b)}} = -(N^2 - 1) \frac{1}{P} \cdot P^2 \\
= -(N^2 - 1) \frac{1}{P} \cdot 1.
\end{equation}
(67)

As in (67) above, (67) is cancelled by $\langle 0 | \Theta^{(q)\mu}_{\mu} | 0 \rangle$ in (63):
\begin{equation}
\left[ \langle \Theta^{(q)\mu}_{\mu} \rangle - \langle 0 | \Theta^{(q)\mu}_{\mu} | 0 \rangle \right]_{\text{Fig. 1(b)}} = 0.
\end{equation}

Fig. 1(c): The contribution to $\langle \Theta^{(q)\mu}_{\mu} \rangle$ is
\begin{equation}
\left. \langle \Theta^{(q)\mu}_{\mu} \rangle \right|_{\text{Fig. 1(c)}} = -3N_f \frac{1}{P} \cdot tr \left[ \bar{P} \gamma^5 (P) \right].
\end{equation}
(68)

Computation of (68) goes through the similar procedure as in Sec. VB. In place of (69), we have
\begin{equation}
\langle \Theta^{(q)\mu}_{\mu} \rangle = 12N_f \int d\xi \left\{ 1 - n_F(\xi) \right\} \\
\times \sum_{\sigma=\pm} \int \frac{d^3p}{(2\pi)^3} (\xi - \sigma p) \rho_\sigma(\xi).
\end{equation}

Then, using (68) and (A.7) - (A.10) in Appendix A, we obtain
\begin{equation}
\left[ \langle \Theta^{(q)\mu}_{\mu} \rangle - \langle 0 | \Theta^{(q)\mu}_{\mu} | 0 \rangle \right]_{\text{Fig. 1(k)}} = 3 \int_{\text{soft } p} \frac{d^3p}{(2\pi)^3} \theta^{(q)\mu}_{\mu}(p),
\end{equation}
(69)
\begin{equation}
\theta^{(q)\mu}_{\mu}(p) \simeq N_f \frac{m_f^2}{T},
\end{equation}
(70)
where $m_f$ is as in (50).

In the region of our interest, Eq. (1), (69) - (70) are subleading as compared to (57).

Figs. 2(e) and (f): It is straightforward to show that the contribution of Fig. 2(e) cancels the contribution of Fig. 2(f).

Figs. 2(g) and (h): The contribution is

$$-2g^2 N (N^2 - 1) \text{Tr}_{\text{soft}} \frac{P \cdot K}{P^2} I_{11}^{++}. \quad (71)$$

Comparing (71) with the “d₆ term” in (28) and noting that $K(P)$ is hard (soft), we see that (71) is subleading.

Figs. 2(j) and (k): It can easily be shown that the sum of the contributions of Fig. 2(j) and Fig. 2(k) vanishes.

Figs. 3(a) and (b): The contribution is

$$12g^2 N_F (N^2 - 1) \text{Tr}_{\text{soft}} \sum_{\sigma = \pm} S^{(\sigma)} (P) (k_0 - \sigma k \cdot \hat{p}) I_{11}^{++}.$$ 

This is of the same type as (58) with the second term in the square brackets and is nonleading as compared to (57).

This completes the proof of (63).

7 Summary and discussion

Applying the HTL resummation scheme to the one-loop correction to the composite vertex $\langle P | \Theta_{\mu \nu} | P \rangle$, we have deduced the energy density and the pressure density, in the high-$T$ limit, for soft gluons and soft quarks in a hot quark-gluon plasma. The resultant energy density for soft gluons, $E_g(p)$, and for soft quarks, $E_q(p)$, are given in (55) and (57), respectively. In contrast to the soft-gluon case, to leading order at high-$T$, the energy density for soft quarks, $E_q(p)$, does not receive contributions from HTL’s. This difference originates in different statistics. To leading order, the hydrostatic pressure density $P(p)$ is related to the energy density $E(p)$ through $P(p) \simeq E(p)/3$, for both soft-gluon and soft-quark sectors. Thus the relation that is characteristic of free-massless-particle systems still holds approximately.

Now, as in Sec. IV, we discuss the physical content of $\langle \Theta_{00} \rangle$ by decomposing it into the sum of matrix elements of $S \otimes S^*$. We take up (11) and (21), and expands
\[ \Delta_{\alpha\beta}(P) = \Delta^{(F)}_{\alpha\beta}(P) + \sum_{n=0}^{\infty} \Delta_{\alpha\beta}^{(n)}(P), \]

\[ \Delta^{(F)}_{\alpha\beta}(P) \equiv P_{\alpha}P_{\beta} \left\{ \Delta^+(P) \right\}^2, \]

\[ \Delta_{\alpha\beta}^{(n)}(P) \equiv \sum_{\ell=T,L} Q_{\alpha\beta}^{(\ell)} \left\{ \Delta^+(P) \right\}^{n+1} \{ \Pi_{\ell}(P) \}^n. \]

Inserting (72) into (11) and (21), we have, with obvious notations,

\[ \langle \Theta_{00}^{(g0)} \rangle_{\text{Fig. 1(a)}} = \langle \Theta_{00}^{(g0)} \rangle_F + \sum_{n=0}^{\infty} \langle \Theta_{00}^{(g0)} \rangle_n \]

and

\[ \langle \Theta_{00} \rangle_{\text{Fig. 2(a)}} = \langle \Theta_{00} \rangle_F + \sum_{n=0}^{\infty} \langle \Theta_{00} \rangle_n, \]

respectively. According to the general arguments in [18] (see also [19]), through analytic continuations to the Minkowski space, (73) and (74) go to the expressions written in the real-time formulation:

\[ \langle \Theta_{00}^{(g0)} \rangle_F = -\frac{i}{2} \left( N^2 - 1 \right) \int \frac{d^4P}{(2\pi)^4} \theta^{\alpha\beta}_{00}(P) P_{\alpha}P_{\beta} \frac{\partial}{\partial \lambda^2} D_{11}(P; \lambda^2) \bigg|_{\lambda = 0}, \]

\[ \langle \Theta_{00}^{(g0)} \rangle_n = -\frac{i}{2} \left( N^2 - 1 \right) \int \frac{d^4P}{(2\pi)^4} \theta^{\alpha\beta}_{00}(P) \sum_{\ell=T,L} Q_{\alpha\beta}^{(\ell)} \]

\[ \times \sum_{j=1}^{2} (-)^{j-1} D_{\ell j}(P) \left\{ \left[ \Pi_{\ell}(P) \tau_3 D(P) \right]^n \right\}_{J_1} , \]

\[ \langle \Theta_{00} \rangle_F = -\frac{g^2}{2} N \left( N^2 - 1 \right) \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4K}{(2\pi)^4} \theta^{\alpha\beta}_{00}(K) V_{\beta\gamma\epsilon}(K, P) V^{\gamma\epsilon}_{\delta\alpha}(K + P, -P) \]

\[ \times \sum_{i,j=1}^{2} D_{i1}(K) D_{ij}(K) D_{ji}(K + P) P_i P_j \frac{\partial}{\partial \lambda^2} D_{\ell j}(P; \lambda^2) \bigg|_{\lambda = 0}, \]

\[ \langle \Theta_{00} \rangle_n = -\frac{g^2}{2} N \left( N^2 - 1 \right) \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4K}{(2\pi)^4} \theta^{\alpha\beta}_{00}(K) V_{\beta\gamma\epsilon}(K, P) V^{\gamma\epsilon}_{\delta\alpha}(K + P, -P) \]

\[ \times \sum_{i,j=1}^{2} D_{i1}(K) D_{ij}(K) D_{ji}(K + P) \sum_{\ell=T,L} Q_{\gamma\delta}^{(\ell)} \]

\[ \times \sum_{j'=1}^{2} (-)^{j'-1} D_{j'\ell}(P) \left\{ \left[ \Pi_{\ell}(P) \tau_3 D(P) \right]^n \right\}_{J_{1j'}}, \]
Here, as in (19), $P$ and $K$ are the four-vectors in the Minkowski space, $D$ is the scalar $2 \times 2$ matrix propagator, $\hat{\Pi}_\ell(P)$ is the scalar $2 \times 2$ vacuum-polarization matrix, and $\tau_3$ is the third Pauli matrix.

Applying the general rules [17] to each term in (75) - (78), we can identify the corresponding set of matrix elements of $S \otimes S^*$. As an example, we take up $\langle \Theta^{(g_0)}_{00} \rangle_3$ in (76) and (78), and find the corresponding matrix elements. Thermal amplitudes representing $\langle \Theta_0^{(g_0)} \rangle_3$ and $\langle \Theta_0 \rangle_3$ are depicted, respectively, in Figs. 6(a) and (b). In Fig. 6, $P$ is soft, $K$'s are hard, and “1”, “… i_1 “… i_6 ” are the thermal indices. We consider the case where $i_1 = i_2 = i_3 = i_6 = 1$, $i_4 = i_5 = 2$, and all $k_0$’s are positive. The rules [17] tell us that the thermal amplitude, Fig. 6(a) [(b)], under consideration represents $S \otimes S^*$ as depicted in Fig. 7(a) [(b)].

In $S \otimes S^*$ in Fig. 7(a), the “probe” $\Theta_{00}$ is put on the soft line $P$ in the $S$-matrix element side, while in Fig. 7(b), $\Theta_{00}$ is on the hard line $K$. From the very definition (5), $\langle \Theta_{00} \rangle$ is the (thermal) expectation value of $\Theta_{00}$ in the initial states — the states at the left end and the right end of Fig. 7, which consist of hard active gluons only. Here “active gluons” means those gluons which directly participate in the thermal reaction. We recall here that, as has been stressed by Weldon [20] in a different context, intermediate states or virtual particles are a mathematical figment of Feynman-Dyson perturbation theory. Then, Fig. 7(a) [(b)] represents the contribution through the figment, intermediate soft [hard] gluon with $P$ [K], to the energy density of the quark-gluon plasma, which is composed of interacting gluons and quarks and is in thermal equilibrium. Thus Fig. 7(a) [(b)] has nothing to do with the energy of the soft [hard] gluon. As a matter of course, Fig. 6 or the original Figs. 1, 2, and 3 include the diagrams, $S \otimes S^*$'s, in which $\Theta_{00}$ is put on the gluon line in the initial state.

In this relation, we may view Figs. 2 and 3 as giving the correction to the energy-momentum density of hard gluons and quarks. To the lowest order, the energy density of hard gluons is obtained from (37) by setting $\rho_T(\xi) = \rho_L(\xi) = \epsilon(\xi) \delta(\xi^2 - p^2),

\left. P_{\text{hard gluon}}^{(g_0)} \right|_{\text{hard gluon}} = 2(N^2 - 1) \int \frac{d^3k}{(2\pi)^3} k n_B(k). \tag{79}

The contributions from Figs. 2(a) - (f) to the energy density are of the following two
types:

\[
\begin{align*}
\int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + (3m_0^2)} & \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k} n_B(k), \\
\int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^2 + (3m_0^2)} & \int \frac{d^3 k}{(2\pi)^3} n_B(k) \{1 + n_B(k)\}.
\end{align*}
\]

Carrying out the integration over \( k \) in each contribution, and summing over the contributions from all other diagrams, we obtain (44) and (45). In order to obtain the correction to \( P_{0}^{(g0)} \) \( \big|_{\text{hard gluon}} \), Eq. (79), from Figs. 2(a) - (f), we carry out the integrations over soft-\( p \) in (80) and (81):

\[
\begin{align*}
\text{Eq. (80)} & \propto g^2 p^* T \int \frac{d^3 k}{(2\pi)^3} \frac{1}{k} n_B(k), \\
\text{Eq. (81)} & \propto g^2 p^* \int \frac{d^3 k}{(2\pi)^3} n_B(k) \{1 + n_B(k)\}.
\end{align*}
\]

Here \( p^* \) denotes the boundary between the soft- and hard-\( P \) regions, \( gT << p^* << T \). Eqs. (82) and (83) is \( O(p^*/T) \) smaller than (79) in the hard \( k \) \( \{= O(T)\} \) region, which gives the leading contributions to the integrals in (79), (82), and (83). Thus Figs. 2(a) - (f) yield \( O(p^*/T) \) smaller corrections to the hard-gluon energy density. This is also the case for Figs. 2(g) - (k). The diagrams Figs. 3(a) - (c) yield higher-order corrections to \( P_{0}^{(g0)} \) \( \big|_{\text{hard quark}} \).

**Appendix A Effective propagators**

In this Appendix, we display the spectral representations for the effective propagators.

*Effective gluon propagator (cf. (13))*

\[
\frac{1}{P^2 - \Pi_{T/L}(P)} = - \int_0^{1/T} d\tau e^{p_0\tau} \int_{-\infty}^{+\infty} d\xi \rho_{T/L}(\xi) (1 + n_B(\xi)) e^{-\xi\tau},
\]
\( \rho_{T/L}(\xi) \equiv -\frac{1}{\pi} \left. \frac{1}{P^2 - \Pi_{T/L}(P)} \right|_{p_0 = \xi + i\sigma^+} \), \hspace{1cm} \text{(A.1)}

where \( n_B(\xi) \) is as in \( \text{[13]} \). Explicit expression of \( \Pi_{T/L}(P) \) is given in \( \text{[13]} \). The spectral function, \( \rho_{T/L}(\xi) \), defined above satisfy various sum rules, among which we use the following ones in the text:

\[
\int_{-\infty}^{+\infty} d\xi \xi^n \rho_{T/L}(\xi) = n, \quad (n = 0, 1), \hspace{1cm} \text{(A.3)}
\]

\[
\int_{-\infty}^{+\infty} \frac{d\xi}{\xi} \rho_\ell(\xi) = \frac{1}{p^2 + 3 \delta_{\ell L} m_g^2}, \quad (\ell = T, L). \hspace{1cm} \text{(A.4)}
\]

In \( \text{[A.4]} \), \( m_g \) is as in \( \text{[11]} \).

**Effective quark propagator (cf. \( \text{[17]} \))**

\[
\tilde{S}^{(\sigma)}(P) = -\int_0^{1/T} d\tau e^{p_0 \tau} \int_{-\infty}^{+\infty} d\xi e^{-\xi \tau} \rho_{\sigma}(\xi) \times (1 - n_F(\xi)) \quad (\sigma = \pm), \hspace{1cm} \text{(A.5)}
\]

\[
\rho_{\sigma}(\xi) \equiv -\frac{1}{\pi} \left. \frac{1}{P^2 - \Pi_{T/L}(P)} \right|_{p_0 = \xi + i\sigma^+}, \hspace{1cm} \text{(A.6)}
\]

where \( n_F(\xi) \) is as in \( \text{[17]} \). Explicit expression of \( \tilde{S}^{(\sigma)}(P) \) is given in \( \text{[14]} \). The spectral function, \( \rho_{\sigma}(\xi) \), defined above satisfy the sum rules,

\[
\int_{-\infty}^{+\infty} d\xi \rho_\sigma(\xi) = \frac{1}{2}, \hspace{1cm} \text{(A.7)}
\]

\[
\int_{-\infty}^{+\infty} d\xi \xi \rho_\sigma(\xi) = \frac{1}{2} \sigma p, \hspace{1cm} \text{(A.8)}
\]

\[
\int_{-\infty}^{+\infty} d\xi \xi^2 \rho_\sigma(\xi) = \frac{1}{2} \left( p^2 + m_f^2 \right), \hspace{1cm} \text{(A.9)}
\]

\[
\int_{-\infty}^{+\infty} d\xi \xi^3 \rho_\sigma(\xi) = \frac{1}{2} \sigma p \left( p^2 + 5 \frac{2}{3} m_f^2 \right), \hspace{1cm} \text{(A.10)}
\]

where \( m_f \) is as in \( \text{[50]} \).
Appendix B Derivation of (19)

In this Appendix, we deduce (19) from (18). Using the \( \tau \)-representations (C.1) and (C.2) (in Appendix C) for \( \Delta^+(K+P) \) and \( \{\Delta^+(K)\}^2 \), after standard manipulations, we obtain

\[
\langle p | \frac{1}{2} \phi^2(0) | p \rangle = -g^2 \frac{2}{16} \text{Tr} \left\{ \Delta^+(K) \right\}^2 \Delta^+(K+P)
\]

\[
= g^2 \int_{-\infty}^{+\infty} \frac{d^3p}{(2\pi)^3} \frac{1}{E k^2} \sum_{\tau=\pm} \sum_{\sigma=\pm} \left[ \left\{ \left( 1 + n \right) \left( \frac{1+\sigma}{2} + n' \right) - n \left( \frac{1-\sigma}{2} + n' \right) \right\} \times \left\{ \frac{1}{k} \frac{1}{k + \sigma E + \tau p_0} + \frac{1}{(k + \sigma E + \tau p_0)^2} \right\} \right. \\
\left. + \frac{n(1+n)}{T} \frac{\sigma}{k + \sigma E + \tau p_0} \right] , \quad (B.1)
\]

where \( n \equiv n_B(k) \) and \( n' \equiv n_B(E) \) with \( E = |k + p| \). We note that (B.1) may be written as

\[
(B.1) = -g^2 \frac{\partial}{\partial \lambda^2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{E \lambda} \\
\times \sum_{\tau=\pm} \sum_{\sigma=\pm} \left( 1 + n_B(E_{\lambda}) \right) \left( \frac{1+\sigma}{2} + n' \right) - n_B(E_{\lambda}) \left( \frac{1-\sigma}{2} + n' \right) \bigg|_{\lambda = 0} , \quad (B.2)
\]

where \( E_{\lambda} \equiv \sqrt{k^2 + \lambda^2} \). In the form (B.1) or (B.2), we can continue \( p_0 (= 2\pi i n T) \) to real energy \( p_0 + i0^+ \). Straightforward calculation shows that

\[
(B.2) = \frac{i}{2} g^2 \int \frac{d^4K}{(2\pi)^4} \sum_{i=1}^{2} (-)^{i-1} D_{1i}(K+P) \\
\times \frac{\partial D_{1i}(K; \lambda^2)}{\partial \lambda^2} \bigg|_{\lambda = 0} . \quad (B.3)
\]

The formula (B.3) is written in terms of real-time thermal \( \phi^3 \) theory [12], which is formulated on the time path \( -\infty \rightarrow +\infty \rightarrow -\infty \rightarrow -\infty \rightarrow i/T \), in a complex time plane. \( D_{1i}(K; \lambda^2) \) is the bare thermal propagator of a boson with mass \( \lambda \). In passing, the result (B.3) is in accord with the general argument of analytic continuation of thermal amplitudes (cf. [18, 19]).
Employing the mass derivative formula \([21, 16]\) for \(\partial D_{1i}/\partial \lambda\), we obtain

\[
(B.3) = \frac{1}{2} g^2 \int \frac{d^4 K}{(2\pi)^4} \sum_{i,j=1}^{2} (-)^{i+j} D_{1i}(K) \times D_{ij}(K)D_{j1}(K + P). \tag{B.4}
\]

**Appendix C List of useful formulae**

In this Appendix we collect various formulae used in the text. For the purpose of deriving those formulae, we introduce the “\(\tau\)-representations” \([2, 22]\) for the bare propagators. As in Sec. I, following \([2]\), we use an index \(r\), \(r = +\) for bosons and \(r = -\) for fermions:

\[
n^r = \begin{cases} 
    n^+(k) = n_B(k), & \text{for bosons} \\
    n^-(k) = n_F(k), & \text{for fermions}
\end{cases}
\]

and introduce

\[
f_+^r(k) = 1 + r n^r(k), \quad f_-^r = r n^r(k),
\]

\[
g^r(k) = r n^r(k) \left(1 + r n^r(k)\right).
\]

Then, the boson and fermion propagators (cf. \([8]\)) may be written as

\[
\Delta^r(K) = -\frac{1}{2k} \int_0^{1/T} d\tau e^{k\alpha\tau} \sum_{s=\pm} f^r_s(k)e^{-sk\tau}, \tag{C.1}
\]

\[
\{\Delta^r(K)\}^2 = -\frac{1}{2k^2 K^2} \left[ \frac{1}{4k^2 T} g^r(k) \int_0^{1/T} d\tau e^{k\alpha\tau} \left[e^{-k\tau} + e^{k\tau}\right] + \frac{1}{4k^2} \int_0^{1/T} \tau d\tau e^{k\alpha\tau} \sum_{s=\pm} s f^r_s(k)e^{-sk\tau}. \right. \tag{C.2}
\]

**Formulae for evaluating Figs. 2(a) - (f)**
As an illustration, we sketch how to evaluate, under the HTL approximation,

\[
T \sum_{n=\pm \infty} \frac{k^2}{P^2 - \Pi_\ell(P)} \otimes I_{11}^{++} = T \sum_{n=\pm \infty} \frac{1}{P^2 - \Pi_\ell(P)} \text{Tr}_k k^2 \Delta^+(K) \Delta^+(K + P),
\]

(\ell = T, L).

Using (C.1) for \(\Delta^+\)'s, standard manipulations lead to

\[
\text{Tr}_k k^2 \Delta^+(K) \Delta^+(K + P) = \frac{1}{4} \int \frac{d^3k}{(2\pi)^3} k \sum_{\tau = \pm} \left[ \frac{1 + n_B(k) + n_B(E)}{k + E + \tau p_0} \right. \\
\left. + \frac{n_B(E) - n_B(k)}{k - E + \tau p_0} \right],
\]  

(C.3)

where \(E = |k + p|\). We take out from (C.3) the UV-divergent piece;

\[
\frac{1}{4} \int \frac{d^3k}{(2\pi)^3} k \sum_{\tau = \pm} \frac{1}{k + E + \tau p_0}.
\]  

(C.4)

Continuing the form (C.4) to real \(p_0\), we can easily show that

\[
(C.4) = -i \int_{-\infty}^{+\infty} \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{K^2} \frac{1}{(K + P)^2},
\]

which is nothing but the “T = 0 sector” of (C.3). Thus (C.3) is not the HTL and we ignore it.

Using (A.1) for \((P^2 - \Pi_\ell(P))^{-1}\), under the HTL approximation, we have

\[
T \sum_{n=\pm \infty} \frac{k^2}{P^2 - \Pi_\ell(P)} \otimes I_{11}^{++} \approx -\frac{1}{4} \int d\xi \rho_\ell(\xi) \int \frac{d^3k}{(2\pi)^3} \sum_{\tau = \pm} \left[ \frac{1}{k} n_B(k) n_B(\xi) \\
- \tau (1 + n_B(k)) n_B(k) - \tau (n_B(k) - n_B(E)) n_B(\xi) \right].
\]  

(C.5)

Since \(\xi, p << k\), we can employ the approximations,

\[
n_B(k) - n_B(E) \simeq n_B(k) (1 + n_B(k)) \frac{p \cdot k}{T},
\]

\[
n_B(\xi) \simeq \frac{T}{\xi},
\]

26
to get

\[(C.5) = -\frac{1}{2} \int_{-\infty}^{+\infty} \frac{d\xi}{\xi} \rho_i(\xi) \int \frac{d^3k}{(2\pi)^3} \times \left[ \frac{T}{k} n_B(k) + n_B(k)(1 + n_B(k)) \right].\]

Using the spectral sum rule \((\text{A.4})\), we finally obtain

\[T \sum_{n=-\infty}^{+\infty} \frac{k^2}{P^2 - \Pi_\ell(P)} \otimes I_{11} \simeq -\frac{1}{8} \frac{T^3}{p^2 + 3 \delta_{\ell L} m_g^2}, \quad (\ell = T, L), \]

where \(m_g\) is as in \((\text{II})\).

Evaluation of leading-order contributions of other formulae used in the text are obtained in a similar manner:

\[T \sum_{n=-\infty}^{+\infty} \frac{1}{P^2 - \Pi_\ell(P)} I_{10}^+ \simeq \frac{1}{12} \frac{T^3}{p^2 + 3 \delta_{\ell L} m_g^2}, \quad (\ell = T, L), \quad (C.6)\]

\[T \sum_{n=-\infty}^{+\infty} \frac{k^2}{P^2 - \Pi_\ell(P)} \otimes I_{20}^+ \simeq T \sum_{n=-\infty}^{+\infty} \frac{k^2}{P^2 - \Pi_\ell(P)} \otimes I_{11}^{++} \simeq -\frac{1}{8} \frac{T^3}{p^2 + 3 \delta_{\ell L} m_g^2}, \quad (C.7)\]

\[T \sum_{n=-\infty}^{+\infty} \frac{k^4}{P^2 - \Pi_\ell(P)} \otimes I_{21}^{++} \simeq \frac{5}{32} \frac{T^3}{p^2 + 3 \delta_{\ell L} m_g^2}, \quad (C.8)\]

\[T \sum_{n=-\infty}^{+\infty} \frac{1}{P^2 - \Pi_{L}(P)} \otimes I_{10}^+ \simeq -\frac{1}{12} \frac{T^3}{p^2 + 3 m_g^2}, \quad (C.9)\]

\[T \sum_{n=-\infty}^{+\infty} \frac{k^2}{P^2 - \Pi_{L}(P)} \otimes I_{20}^+ \simeq T \sum_{n=-\infty}^{+\infty} \frac{p^2}{P^2 - \Pi_{L}(P)} \frac{k^2}{P^2} \otimes I_{11}^{++} \simeq \frac{1}{8} \frac{T^3}{p^2 + 3 m_g^2}, \quad (C.10)\]

\[T \sum_{n=-\infty}^{+\infty} \frac{1}{P^2 - \Pi_{L}(P)} \frac{P \cdot K}{P^2} \otimes I_{11}^{++} \simeq \frac{1}{24} \frac{T^3}{p^2 + 3 m_g^2}, \quad (C.11)\]

\[T \sum_{n=-\infty}^{+\infty} \frac{1}{P^2 - \Pi_{L}(P)} \frac{P \cdot K}{P^2} \otimes I_{20}^+ \simeq \frac{1}{24} \frac{T^3}{p^2 + 3 m_g^2}. \]
\[
T \sum_{n=-\infty}^{+\infty} \frac{k \cdot p}{P^2 - \Pi_L(P)} \frac{P \cdot K}{P^2} \otimes I_{11}^{++} \simeq -\frac{1}{24} \frac{T^3}{p^2 + 3 m_g^2},
\]  
(C.12)

\[
T \sum_{n=-\infty}^{+\infty} \frac{k \cdot p}{P^2 - \Pi_L(P)} \frac{P \cdot K}{P^2} \otimes I_{20}^{+} \simeq -\frac{1}{24} \frac{T^3}{p^2 + 3 m_g^2},
\]  
(C.13)

\[
T \sum_{n=-\infty}^{+\infty} \frac{k^2}{P^2 - \Pi_L(P)} \frac{(P \cdot K)^2}{P^2} \otimes I_{21}^{++} \simeq -\frac{5}{96} \frac{T^3}{p^2 + 3 m_g^2}.
\]  
(C.14)

In deriving (C.9) - (C.15), we have used (C.1) for \(1/P^2 = \Delta^+(P)\).

Using the above formulae, we obtain

\[
T \sum_{n=-\infty}^{+\infty} \left( \frac{K | Q^{(L)} | L}{P^2 - \Pi_T(P)} \right) \otimes I_{11}^{++}
\]
\[
\simeq T \sum_{n=-\infty}^{+\infty} \left( \frac{K | Q^{(T)} | K}{P^2 - \Pi_T(P)} \right) \otimes I_{20}^{+} \simeq \frac{1}{12} \frac{T^3}{p^2},
\]  
(C.16)

\[
T \sum_{n=-\infty}^{+\infty} \left( \frac{k^2 (K | Q^{(T)} | K)}{P^2 - \Pi_T(P)} \right) \otimes I_{21}^{++} \simeq -\frac{5}{48} \frac{T^3}{p^2},
\]  
(C.17)

\[
T \sum_{n=-\infty}^{+\infty} \left( \frac{K | Q^{(L)} | K}{P^2 - \Pi_L(P)} \right) \otimes I_{11}^{++}
\]
\[
\simeq T \sum_{n=-\infty}^{+\infty} \frac{k_0 K^\rho Q^{(L)}_{0\rho}}{P^2 - \Pi_L(P)} \otimes I_{11}^{++} \simeq -\frac{1}{24} \frac{T^3}{p^2 + 3 m_g^2},
\]  
(C.18)

\[
T \sum_{n=-\infty}^{+\infty} \left( \frac{K | Q^{(L)} | K}{P^2 - \Pi_L(P)} \right) \otimes I_{20}^{+}
\]
\[
\simeq T \sum_{n=-\infty}^{+\infty} \frac{k_0 K^\rho Q^{(L)}_{0\rho}}{P^2 - \Pi_L(P)} \otimes I_{20}^{+} \simeq -\frac{1}{24} \frac{T^3}{p^2 + 3 m_g^2}.
\]  
(C.19)

\[
T \sum_{n=-\infty}^{+\infty} \frac{k^2 (K | Q^{(L)} | K)}{P^2 - \Pi_L(P)} \otimes I_{21}^{++} \simeq \frac{1}{32} \frac{T^3}{p^2 + 3 m_g^2},
\]  
(C.20)

\[
T \sum_{n=-\infty}^{+\infty} Q^{(L)}_{ii} \otimes I_{10}^{+} \simeq 0.
\]  
(C.21)
Here, as in (23), \( (K|Q^{(T/L)}|K) \equiv K^\rho K^\sigma Q^{(T/L)}_{\rho\sigma} \).

For evaluating the formula of the type (23), further manipulations are not necessary. In fact, with the help of the trivial identity,

\[
T \sum_{n=-\infty}^{+\infty} \frac{1}{P^2} G(P, K) \otimes I_{ij}^{++} = \lim_{m \to 0} T \sum_{n=-\infty}^{+\infty} \frac{1}{P^2 - \Pi_L(P)} \frac{1}{P^{2(\ell-1)}} \times G(P, K) \otimes I_{ij}^{++}, \quad (\ell = 0, 1), \tag{C.22}
\]

we can evaluate the l.h.s. of (C.22) by using (C.6) - (C.21).

**Formulae for evaluating Figs. 2(j) and (k)**

When \( I_{ij}^{-} \) is substituted for \( I_{ij}^{++} \) in each of the formulae above, (C.6) - (C.21), we obtain the r.h.s. of that formula multiplied by the factor \(-\frac{1}{2}\). Namely, writing the l.h.s. of (C.6) - (C.21) in a generic form \( T \sum_{n=-\infty}^{+\infty} H(P, K) \otimes I_{ij}^{++} \), we have

\[
T \sum_{n=-\infty}^{+\infty} H(P, K) \otimes I_{ij}^{-} \simeq -\frac{1}{2} T \sum_{n=-\infty}^{+\infty} H(P, K) \otimes I_{ij}^{++}. \tag{C.23}
\]
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FIG. 1. Tadpole diagrams. The symbol $\otimes$ indicates the point where the operator $\Theta_{\mu\nu}$ is inserted. The blobs indicate the (HTL-resummed) effective propagators.

FIG. 2. Formally $\mathcal{O}(g^2)$ diagrams that contribute to the soft-gluon sector. $K$ is hard ($\sim T$) and $P$ is soft ($\leq \mathcal{O}(gT)$).

FIG. 3. Formally $\mathcal{O}(g^2)$ diagrams that contribute to the soft-quark sector. $K$ is hard ($\sim T$) and $P$ is soft ($\leq \mathcal{O}(gT)$).

FIG. 4. One-loop correction to the composite vertex $\langle P|\phi^2/2|P \rangle$.

FIG. 5. A contribution to $S \otimes S^*$, which is included in the thermal Green function Fig. 4. The left-side part of the final-state cut line (dot-dashed line) is the $S$-matrix element in vacuum theory and the right-side part is the complex conjugate of the $S$-matrix element, $S^*$. A group of particles on top of the figure stands for the constituents of the heat bath.

FIG. 6. Diagrams representing the thermal amplitude $(\Theta_{00}^{(g)})_{3}$, Eq. (76), and $(\Theta_{00})_{3}$, Eq. (78). For visual clarity, we have used solid lines for gluons. Thick lines represent hard gluons, while thin lines represent soft gluons.

FIG. 7. Matrix elements of $S \otimes S^*$, which are included in the thermal diagrams Fig. 6.
Values of the coefficients in (26) - (28) for various diagrams in Fig. 2.

| Fig. 2 | $c_1(T)$ | $c_2(T)$ | $c_3(T)$ | $c_4(T)$ | $c_5(T)$ | $c_6(T)$ | $c_7(T)$ | $c_8(T)$ | $c_9(T)$ |
|--------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| (a)    | 2        | -1       | 1        | -1       | 2        | 0        | 4        | -8       | -10      |
| (b)    | -2       | 2        | 0        | 0        | 0        | -1       | 2        | 4        | 0        |
| (c)    | 1        | -2       | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| (d)    | 0        | -1       | -1/2     | 1        | 2        | 0        | 0        | 0        | 0        |
| (e)    | 0        | 0        | 0        | 0        | 0        | -1       | 0        | 0        | 2        |
| (f)    | 0        | 0        | 0        | 0        | 0        | 1        | -1       | 0        | 0        |

| Fig. 2 | $d_1$ | $d_2$ | $d_3$ | $d_4$ | $d_5$ | $d_6$ |
|--------|-------|-------|-------|-------|-------|-------|
| (a)    | 0     | 0     | 0     | 3/2   | -3    | -5/2  |
| (b)    | 2     | 0     | 0     | -3/2  | 0     | 2     |
| (c)    | -2    | 0     | 0     | 1/2   | 0     | 0     |
| (d)    | -1    | 1     | -1/2  | 0     | 1     | 0     |
| (e)    | 1/2   | 0     | 0     | -1/2  | 0     | 1/2   |
| (f)    | -1    | 0     | 0     | 1/2   | 0     | 0     |