Degenerate soliton solutions and their dynamics in the nonlocal Manakov system: I Symmetry preserving and symmetry breaking solutions

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Abstract In this paper, we construct degenerate soliton solutions (which preserve $\mathcal{PT}$-symmetry/break $\mathcal{PT}$-symmetry) to the nonlocal Manakov system through a non-standard bilinear procedure. Here by degenerate we mean the solitons that are present in both the modes which propagate with same velocity. The degenerate nonlocal soliton solution is constructed after briefly indicating the form of nondegenerate one-soliton solution. To derive these soliton solutions, we simultaneously solve the nonlocal Manakov equation and a pair of coupled equations that arise from the zero curvature condition. The later consideration yields general soliton solution which agrees with the solutions that are already reported in the literature under certain specific parametric choice. We also discuss the salient features associated with the obtained degenerate soliton solutions.

Keywords nonlocal Manakov equation · Hirota’s bilinear method · Soliton solutions

1 Introduction

In the context of $\mathcal{PT}$-symmetric classical optics [12], recently a nonlocal nonlinear Schrödinger (NNLS) equation, namely

$$iq_t(x,t) + iq_{xx}(x,t) + 2\sigma q(x,t)q^*(-x,t)q(x,t) = 0, \quad \sigma = \pm 1.$$ (1)
has been introduced in [3]. It has been shown that Eq. (1) is completely integrable [3, 4, 5], since it admits a Lax pair, infinite number of conservation laws and is solvable by inverse scattering transform (IST) technique. Eq. (1) has a self induced-potential $V(x, t) = 2\sigma q(x, t)q^*(−x, t)$ which obeys the $PT$-symmetry condition $V^*(−x, t) = V(x, t)$ [6]. In Eq. (1), if we replace the nonlocal $q^*(−x, t)$ term by $q^*(x, t)$, it becomes standard NLS equation. The nonlocal term in (1) implies that the field $q^*(−x, t)$ is either independent or dependent with respect to the field $q(x, t)$. In the independent case, the field $q^*(−x, t)$ is a parity conjugate of $q(x, t)$ in which the solution exhibits $PT$-symmetry while in the independent case the function $q^*(−x, t)$ is not parity transformed ($x → −x$) complex conjugate function of $q(x, t)$ which corresponds to $PT$-symmetry broken case. The NNLS equation is gauge equivalent to an unconventional system of coupled Landau-Lifshitz equation [8]. In contradiction to this the standard NLS equation has been shown to be $L$-equivalent to Heisenberg spin chain equation in the continuum limit [9]. Various recent studies have shown that the analysis of NNLS equation and its variants have become one of the active areas of research both from physical and mathematical perspectives [10]-[34].

The non-trivial generalization of Eq. (1) is the vector nonlocal NLS equation or coupled NNLS equation, namely

$$i q_j, t(x, t) + q_j, x(x, t) + 2 \sum_{l=1}^{2} \sigma_l q_l(x, t)q_l^*(-x, t)q_j(x, t) = 0, \quad j = 1, 2. \quad (2)$$

In Eq. (2), each $q_j(x, t)$ is a complex valued wave envelope and $q_j, t$ and $q_j, x$ represent the derivatives of $q_j$ with respect to $t$ and $x$, respectively. In the above equation, $q_l^*(-x, t)$ are the nonlocal fields, and the local CNLS equation can be obtained by replacing it by local fields $q_l^*(x, t)$. In Eq. (2), the nonlocal version of self phase modulation and cross phase modulation constitute the nonlocal nonlinearity. Analogues to the local CNLS equations, Eq. (2) comprises of three different equations, namely focusing, defocusing and mixed type depending on the signs of the nonlinearity coefficients $\sigma_l$'s. If $\sigma_l = +1$, $l = 1, 2$, Eq. (2) becomes the focusing CNNLS equation or the nonlocal version of the celebrated local Manakov equation. The local Manakov equation is shown to possesses several interesting properties [35-36], including shape changing property of solitons under collision. When $\sigma_l = −1$, $l = 1, 2$, Eq. (2) becomes the defocusing coupled NNLS equation. Its local counterpart is the defocusing coupled NLS equation which admits dark-dark and bright-dark soliton solutions [37-38]. Defocusing coupled NLS equation does not admit any shape changing property [38]. If $\sigma_1 = +1$, $\sigma_2 = +1$ and vice-versa), Eq. (2) becomes the coupled NNLS equation with mixed focusing-defocusing nonlinearity. The local version of it admits bright-bright, bright-dark and dark-dark type soliton solutions [39-41]. The above facts emphasize that to study the collision between solitons in the underlying nonlocal system, it is essential to derive multi-soliton solutions.

In Ref. [43] the authors have obtained a two parameter family of breathing finite time blowup one soliton solution for the Eq. (2) with $\sigma_1 = +1$. Very recently, soliton solutions have been constructed for the various coupled nonlocal field models by applying non-vanishing boundary conditions. However, to the best of our knowledge,
for the first time, we report in this paper bright one and two soliton solutions of the nonlocal Manakov equation, that is Eq. (2) with $\sigma_l = +1$, $l = 1, 2$. For convenience, we divide our investigation into two parts. In the present first part, we focus our attention only on the derivation of soliton solutions to the nonlocal Manakov equation. In the second subsequent part, we investigate the collision dynamics between the degenerate two-solitons in detail by using the obtained two soliton solution.

To explore general soliton solutions, we adopt the non-standard bilinearization procedure developed for the scalar NNLS equation [14]. Using this procedure, we bilinearize both the nonlocal Manakov equation and the following a pair of coupled equations that arise in the zero curvature condition [43], that is

\[ i q_j^{*,t}(-x,t) - q_j^{*,xx}(-x,t) - 2 \sum_{l=1}^2 q_l^{t}(-x,t) q_l(x,t) q_j^{*}(-x,t) = 0, \quad j = 1, 2. \tag{3} \]

The reason behind the inclusion of the above equations in the solution construction process is that to introduce more number of complex parameters in the soliton solutions since the number of distinct eigenvalues arise in pair in one and higher order soliton solutions and the possibility of locating eigenvalues anywhere in the complex plane leads to new eigenvalue configuration in the nonlocal family of equations while solving the left/right Riemann-Hilbert problem. Due to the above reasons we treat the functions $q_j(x,t)$ and $q_j^{*}(-x,t)$, $j = 1, 2$ as independent entities. As we pointed out earlier, in the general case, the functions $q_j^{*}(-x,t)$ need not always the parity transformed complex conjugate of $q_j(x,t)$.

To bilinearize Eqs. (2) and (3), we introduce two auxiliary functions in the bilinear process in order to obtain the bilinear forms of them. By solving the obtained bilinear equations systematically, we derive degenerate one and two bright soliton solutions for the nonlocal Manakov equation. From the obtained one soliton solution, we match the solutions that already exist in the literature under certain parametric choice. Besides deriving the one and two soliton solutions, we also discuss the salient features of the obtained soliton solutions.

The outline of the paper is as follows. In section 2, we describe the bilinearization of Eqs. (2) and (3) using the nonstandard bilinear procedure. In Sec. 3, to begin with, we construct non-degenerate one soliton solution from which we extract the degenerate one soliton solution under specific restriction on the wavenumbers and discuss the salient features associated with it. In Sec. 4, we derive the degenerate two soliton solutions of Eq. (2). We also show that the obtained two-soliton solution can be reduced to a simple form. We present our conclusions in Sec. 5.

2 Nonstandard bilinearization procedure

The nonlocal Manakov equation (2) is integrable and solvable by IST method [43]. In Ref. [43], the authors have derived a two parameter family of breathing one soliton solution for Eq. (2) with $\sigma_l = +1$, $l = 1, 2$ through IST. However, to the best of our knowledge, explicit form of two soliton solution or higher order soliton solutions for this equation has not been reported so far. To capture the known solutions we have to
modify the procedure appropriately. Interestingly, the modified procedure generates more general solutions for this equation.

As we pointed out earlier, in the bilinear process, we also incorporate Eq. (3) along with the Eq. (2). This augmentation is very much necessary to construct general soliton solutions. To bilinearize Eqs. (2) and (3) (with $\sigma_l = +1$, $l = 1, 2$) simultaneously we consider the following transformations, namely

$$q_j(x, t) = \frac{g^{(j)}(x, t)}{f(x, t)}, \quad q^*_j(-x, t) = \frac{g^{(j)*}(-x, t)}{f^*(-x, t)}, \quad j = 1, 2, \quad \text{(4)}$$

where $g^{(j)}(x, t)$, $g^{(j)*}(-x, t)$, $f(x, t)$ and $f^*(-x, t)$ are all complex functions and they are all considered as distinct to start with. To obtain the bilinear forms of (2) and (3), respectively. By introducing equal number of auxiliary functions we can match the number of bilinear equations with equal number of unknown functions \[44, 45\] which in turn provides a nontrivial consistent solution to the given problem, as we see below.

Substituting the transformation given in (4) in Eqs. (2) and (3), we obtain the following bilinear equations, that is

\begin{align*}
D_1 g^{(j)}(x, t) \cdot f(x, t) = 2g^{(j)}(x, t) \cdot s^{(1)}(-x, t), & \quad \text{(5a)} \\
D_2 f(x, t) \cdot f(x, t) = 4s^{(1)}(-x, t) \cdot f(x, t), & \quad \text{(5b)} \\
D_3 g^{(j)*}(-x, t) \cdot f^*(-x, t) = -2g^{(j)*}(-x, t) \cdot s^{(2)}(-x, t), & \quad \text{(5c)} \\
D_2 f^*(-x, t) \cdot f^*(-x, t) = 4s^{(2)}(-x, t) \cdot f^*(-x, t), & \quad j = 1, 2, \quad \text{(5d)}
\end{align*}

where $D_1 \equiv (iD_t + D_x^2)$, $D_2 \equiv D_x^2$, $D_3 \equiv (iD_t - D_x^2)$ and $D_t$ and $D_x$ are the standard Hirota’s bilinear operators \[46\]. The auxiliary functions are defined by

\begin{align*}
s^{(1)}(-x, t) \cdot f^*(-x, t) = \sum_{n=1}^{2} g^{(n)}(x, t) \cdot g^{(n)*}(-x, t), & \quad \text{(6a)} \\
s^{(2)}(-x, t) \cdot f(x, t) = \sum_{n=1}^{2} g^{(n)}(x, t) \cdot g^{(n)*}(-x, t). & \quad \text{(6b)}
\end{align*}

The above set of bilinear Eqs. (5) can be solved by expanding the unknown functions $g^{(j)}(x, t)$, $g^{(j)*}(-x, t)$, $f(x, t)$, $f^*(-x, t)$, $s^{(1)}(-x, t)$ and $s^{(2)}(-x, t)$ in the following manner:

\begin{align*}
g^{(j)} = \epsilon g^{(j)}_1 + \epsilon^3 g^{(j)}_3 + \ldots, \quad g^{(j)*} = \epsilon g^{(j)*}_1 + \epsilon^3 g^{(j)*}_3 + \ldots, & \quad \text{(7a)} \\
f = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \ldots, \quad f^* = 1 + \epsilon^2 f_2 + \epsilon^4 f_4 + \ldots, & \quad \text{(7b)} \\
s^{(1)} = \epsilon^2 s^{(1)}_2 + \epsilon^4 s^{(1)}_4 + \ldots, \quad s^{(2)} = \epsilon^2 s^{(2)}_2 + \epsilon^4 s^{(2)}_4 + \ldots, & \quad j = 1, 2. \quad \text{(7c)}
\end{align*}

Here, $\epsilon$ is a small expansion parameter. We can obtain a set of linear partial differential equations (PDEs) by collecting the coefficients of same powers of $\epsilon$ after substituting the above expansions in (5a)-(5d). By solving them recursively we can obtain the explicit forms of the unknown functions appearing in (7). Substituting the relevant
The above Eqs. (8) admit the following solutions, namely degenerate solitons [47, 48]. To explore degenerate solitons in Eq. (2), we begin our analysis with the following lowest order linear PDEs, that is expressed as,

\[ g_{11t}(x, t) + g_{1xx}(x, t) = 0, \quad ig_{11t}(-x, t) - g_{1xx}(-x, t) = 0, \quad j = 1, 2. \]  (8)

The above Eqs. (8) admit the following solutions, namely

\[ g^{(j)}_1(x, t) = \alpha^{(j)}_1 e^{\xi^{(j)}_1}, \quad \xi^{(j)}_1 = i k_1^{(j)} x - i k_1^{(j)} t, \]  (9a)

\[ g^{(j)}_1(-x, t) = \beta^{(j)}_1 e^{\xi^{(j)}_2}, \quad \xi^{(j)}_2 = i k_1^{(j)} x + i k_1^{(j)} t, \quad j = 1, 2. \]  (9b)

In the above solutions one may notice that the exponential functions which are present in both the modes are different, that is the exponential functions in \( g^{(1)}_1(x, t) \) and \( g^{(2)}_1(x, t) \) are different. Similarly the exponential functions in the fields \( g^{(1)}_1(-x, t) \) and \( g^{(2)}_1(-x, t) \) are also different. This consideration leads to the solitons which propagate with different velocities in different modes. Such type of solitons are non-degenerate solitons. For example, proceeding with the forms given in Eqs. (9a) and (9b), we find that the series expansion (7a), (7b) get truncated for non-degenerate one soliton solution at 7-th order in \( g^{(j)}(x, t) \) and \( g^{(j)*}(-x, t) \), at 8-th order in \( f(x, t) \) and \( f^*(-x, t) \) and 6-th order in \( s^{(1)}(-x, t) \) and \( s^{(2)}(-x, t) \). Using these forms and substituting them in (3), we obtain the expressions for one-soliton solution explicitly.

The factorized compact form of non-degenerate one-soliton solution can then be expressed as,

\[ q_j(x, t) = \frac{\alpha^{(j)}_1 e^{\xi^{(j)}_1} + e^{\xi^{(2)}_1} + e^{\xi^{(3)}_1} + \Delta^{(j)}_1}{1 + e^{\xi^{(1)}_1} + \xi^{(1)}_1 + \Delta_1 + e^{\xi^{(2)}_1} + \xi^{(2)}_1 + \Delta_2 + e^{\xi^{(3)}_1} + \xi^{(3)}_1 + \Delta_3}, \]  (10a)

\[ q_j^*(-x, t) = \frac{\beta^{(j)}_1 e^{\xi^{(j)}_2} + e^{\xi^{(2)}_2} + e^{\xi^{(3)}_2} + \gamma^{(j)}_1}{1 + e^{\xi^{(1)}_2} + \xi^{(1)}_2 + \Delta_1 + e^{\xi^{(2)}_2} + \xi^{(2)}_2 + \Delta_2 + e^{\xi^{(3)}_2} + \xi^{(3)}_2 + \Delta_3}, \]  (10b)

where the explicit forms of the constants appearing in the above soliton solution are given in Appendix A.

3 One-soliton solution

To begin, we demonstrate the method of constructing nondegenerate and degenerate one-soliton solutions for Eqs. (2) and (3).

3.1 Nondegenerate and Degenerate nonlocal one-soliton solution

The solitons in which both the modes propagate with the same velocity are called degenerate solitons. For example, proceeding with the forms given in Eqs. (9a) and (9b), we find that the series expansion (7a)-(7c) get truncated for non-degenerate one soliton solution at 7-th order in \( g^{(j)}(x, t) \) and \( g^{(j)*}(-x, t) \), at 8-th order in \( f(x, t) \) and \( f^*(-x, t) \) and 6-th order in \( s^{(1)}(-x, t) \) and \( s^{(2)}(-x, t) \). Using these forms and substituting them in (3), we obtain the expressions for one-soliton solution explicitly.
Similarly the exponential functions in \( g \) generate one bright soliton solution, namely \( \varepsilon \) requires analysis up to order of \( S \). S. Stalin et al. enforces the exponential functions in expressions for the functions \( g \) below even this degenerate soliton solutions reveal very interesting properties. The auxiliary functions \( s \) system of resultant linear partial differential equations, which result from the bilinear construction of degenerate soliton solutions for Eq. (2). Hence we impose a constraint on the wave numbers in the exponential functions in both the modes, that is the wave numbers are chosen to be \( k^1_1 = k^1_2 = k_1 \) and \( k^1_1 = k^1_2 = k_1 \). This restriction enforces the exponential functions in \( g^1_1(x, t) \) and \( g^1_2(x, t) \) to be one and the same. Similarly the exponential functions in \( g^2_1(-x, t) \) and \( g^2_2(-x, t) \) are same. This restriction allows us to explore degenerate solitons in Eq. (2). As we demonstrate below even this degenerate soliton solutions reveal very interesting properties.

Imposing the above said restriction on the wave numbers, we have the following expressions for the functions \( g^{(j)}_1 \) and \( g^{(j)*}_1 \), that is

\[
\begin{align*}
    g^{(j)}_1(x, t) &= \alpha^{(j)}_1 e^{\xi_1}, \quad \xi_1 = i k_1 x - i k^2_1 t, \quad (11a) \\
    g^{(j)*}_1(-x, t) &= \beta^{(j)}_1 e^{\xi_1}, \quad \xi_1 = i k_1 x + i k^2_1 t, \quad j = 1, 2. \quad (11b)
\end{align*}
\]

Now the modes differ from each other only in their (complex) amplitudes. The above restriction on the wave numbers enforces us to truncate the series expansion (7a–7c) at 3rd order in \( g^{(j)}(x, t) \) and \( g^{(j)*}(-x, t) \), at 4th order in \( f(x, t) \) and \( f^*(-x, t) \) and 4-th order in \( s^{(1)}(-x, t) \) and \( s^{(2)}(-x, t) \). Consequently solving the system of resultant linear partial differential equations, which result from the bilinear equations, using the inputs (11a–11b), we find

\[
\begin{align*}
    g^{(j)}_3(x, t) &= e^{2\xi_1 + \Delta^{(j)}}, \quad g^{(j)*}_3(-x, t) = e^{2\xi_1 + \gamma^{(j)}}, \quad e^{\alpha^{(j)}} = -\alpha^{(j)}_1 \Gamma_{11}^{(1)} \quad (12a) \\
    f^{(j)}_2(x, t) &= f^{(j)}_2(-x, t) = e^{\xi_1 + \xi_1 + \delta_1}, \quad e^{\delta^{(j)}} = -\delta^{(j)}_1 \Gamma_{11}^{(1)} \quad (12b) \\
    f^{(j)}_4(x, t) &= f^{(j)}_4(-x, t) = e^{2(\xi_1 + \xi_1 + R)}, \quad e^R = -\frac{\beta^{(j)}_1}{\kappa_{11}} \quad (12c)
\end{align*}
\]

whereas the auxiliary functions are reduced to

\[
s^{(1)}_2(-x, t) = s^{(2)}_2(-x, t) = \Gamma_{11}^{(1)} e^{\xi_1 + \xi_1}. \quad (13)
\]

In the above \( \kappa_{11} = (k_1 + \bar{k}_1)^2 \) and \( \Gamma_{11}^{(1)} = (\alpha^{(1)}_1 \beta^{(1)}_1 + \alpha^{(2)}_1 \bar{\beta}^{(2)}_1) \). One can check that the auxiliary functions \( s^{(1)}_1(-x, t) \) and \( s^{(2)}_1(-x, t) \) become zero at the order of \( e^4 \).

Substituting the expressions found above in (4), we arrive at the following degenerate one bright soliton, namely

\[
q_j(x, t) = \frac{\alpha^{(j)}_1 e^{\xi_1} + e^{2\xi_1 + \Delta_j}}{1 + e^{\xi_1 + \xi_1 + R} + e^{2(\xi_1 + \xi_1) + R}} = \frac{\alpha^{(j)}_1 e^{\xi_1}}{1 + e^{\xi_1 + \xi_1 + \Delta}} \cdot e^{-\frac{\Gamma_{11}^{(1)}}{\kappa_{11}}}. \quad (14a)
\]
The fields \( q_j^*(-x, t) \) turns out to be

\[
q_j^*(-x, t) = \frac{\beta_1^{(j)} e^{\xi_1} + e^{2\xi_1 + \xi_i + \gamma_1}}{1 + e^{\xi_1 + \xi_i + \gamma_1} + e^{2(\xi_1 + \xi_i) + R}} \equiv \frac{\beta_1^{(j)} e^{\xi_1}}{1 + e^{\xi_1 + \xi_i + R}}. \tag{14b}
\]

It is a straightforward matter to verify the correctness of the solutions (14a) and (14b) by substituting them back in Eqs. (2) and (3). The one bright soliton solution given above is characterized by six complex parameters, namely \( \alpha_1^{(j)}, \beta_1^{(j)}, j = 1, 2, k_1, \) and \( \bar{k}_1 \), whereas the degenerate one bright soliton solution of local Manakov equation is characterized by only three complex parameters [35, 36, 39]. We note that the functions \( q_j^*(-x, t) \) given in (14b) are in general not parity conjugate of \( q_j(x, t) \) given in (14a). The one soliton solution (14a)-(14b) can also be rewritten as

\[
q_j(x, t) = \frac{A_j (k_1 + \bar{k}_1) e^{(\xi_1 - R \xi_1) i \chi_1 + i \gamma_1 \chi_1}}{2 i \left[ \cosh(\chi_1) \cos(\chi_2) + i \sinh(\chi_1) \sin(\chi_2) \right]}, \tag{15a}
\]

and

\[
q_j^*(-x, t) = \frac{\bar{A}_j (k_1 + \bar{k}_1) e^{-(\xi_1 - R \xi_1) i \chi_1 + i \gamma_1 \chi_1}}{2 i \left[ \cosh(\chi_1) \cos(\chi_2) + i \sinh(\chi_1) \sin(\chi_2) \right]}. \tag{15b}
\]
respectively. In the above, the complex coefficients
\[ A_j = \frac{\alpha^{(j)}_1}{\sqrt{(\alpha^{(1)}_1 \beta^{(1)}_1 + \alpha^{(2)}_1 \beta^{(2)}_1)}}, \quad \hat{A}_j = \frac{\beta^{(j)}_1}{\sqrt{(\alpha^{(1)}_1 \beta^{(1)}_1 + \alpha^{(2)}_1 \beta^{(2)}_1)}}, \quad j = 1, 2, \quad (15c) \]

and \( \chi_1 = \frac{\xi_1 + \xi_2 + \Delta_R}{2}, \quad \chi_2 = \frac{\xi_1 + \xi_2 + \Delta_R}{2}, \quad \xi_{1I} = k_{1R} x + \left( -k_{1I}^2 + k_{1R}^2 \right) t, \quad \xi_{1R} = k_{1I} x + \left( -k_{1I}^2 + k_{1R}^2 \right) t, \quad \Delta_R = \frac{1}{2} \log \left( \frac{|\alpha^{(1)}_1 \beta^{(1)}_1 + \alpha^{(2)}_1 \beta^{(2)}_1|^2}{|k_{1I} + k_{1R}|^2} \right) \) and \( \Delta_I = \frac{1}{2} \log \left( \frac{(\alpha^{(2)}_1 \beta^{(1)}_1 + \alpha^{(1)}_1 \beta^{(2)}_1)(\xi_1 + k_{1R})^2}{(\alpha^{(2)}_1 \beta^{(1)}_1 + \alpha^{(1)}_1 \beta^{(2)}_1)(\xi_1 + k_{1R})^2} \right) \). Here, \( k_{1R} \) and \( k_{1I} \), \( k_{1R} \) and \( k_{1I} \) are the real and imaginary parts of the wave numbers \( k_1 \) and \( \tilde{k}_1 \), respectively. Similarly, \( \xi_{1I} \) and \( \xi_{1R} \) are the real and imaginary parts of the wave numbers \( \xi_1 \) and \( \tilde{\xi}_1 \), respectively. To the best of our knowledge the one bright soliton solution given above is more general than the one already reported in the literature [43].

3.2 Some remarkable features of degenerate nonlocal soliton

For Eq. (2), we define the quasi-intensity (quasi-power) of solitons in both the modes as [37]
\[ I_j = A_j \hat{A}_j, \quad j = 1, 2. \quad (16) \]

In the local case the intensity of soliton is usually calculated by taking absolute squares of polarization vectors of the nonlinear Schrödinger field whereas in the non-local case the intensity is calculated by multiplying the polarization vectors nonlinear Schrödinger fields \( q_j(x, t) \) by the polarization vectors of fields \( q_j^*(-x, t) \). Here, hat in \( \hat{A}_j, j = 1, 2, \) denotes the polarization vectors present in field \( 15b \).

Using the expression (16), a conserved quantity can be brought out in terms of the polarization vectors of the solitons of both the components, that is
\[ \mathcal{I} = \int_{-\infty}^{\infty} (q_1(x, t)q_1^*(-x, t) + (q_2(x, t)q_2^*(-x, t))dx. \quad (17) \]

As far as the one-soliton solution, the above form yields
\[ A_1 \cdot \hat{A}_1 + A_2 \cdot \hat{A}_2 = 1, \quad (18) \]

where \( A_1, A_2, \hat{A}_1 \) and \( \hat{A}_2 \) are defined in Eq. \( 15c \).

We note here that for a specific parametric choice the one-soliton solution Eq. \( 15a-15b \) admits singularities for finite values of \( t \) at \( x = 0 \) when the following condition is satisfied:
\[ \Delta_R(k_{1R}^2 - k_{1I}^2 + k_{1R}^2 - k_{1I}^2) = 2[(2n + 1)\pi - \Delta_I](k_{1R}k_{1I} + k_{1R}\bar{k}_{1I}), \quad (19) \]

where, \( n = 0, 1, 2, \ldots, \) respectively.

The long time evolution of the degenerate one soliton solution brings out yet another interesting feature for the nonlocal case. In Fig. 1 we plot the absolute value
of the degenerate one soliton solution \( q_j(x, t) \) given in (14a) for the parametric values \( k_1 = 1 + i, \bar{k}_1 = -1.4 + i, \alpha_1^{(1)} = 1 + i, \alpha_1^{(2)} = 1.5 + i, \beta_1^{(1)} = 1 - i, \beta_1^{(2)} = 1 - i \). As one can see in Figs. 1a and 1d, the amplitudes of the soliton in both the modes decay as \( t \rightarrow +\infty \) in the \(-x\) direction. The absolute value of the fields \( q_j^+(-x, t) \) grow at \( t \rightarrow +\infty \) in the \(-x\) direction which is illustrated in Figs. 1b and 1e for the same parametric values. In other words a simultaneous loss and gain occur in the amplitudes of the solitons in the modes \( q_j^+ \) and \( q_j^- \), \( j = 1, 2 \). However, a stable propagation of soliton can be visualized in the case \( |q_j(x, t)|q_j^+(-x, t) \) which is demonstrated in Figs. 1c and 1f. The amplitudes (or energy) of the soliton are preserved as specified by the conserved quantity of Eq. (2). In view of \( \mathcal{PT}\)-symmetric classical optics, the real and imaginary parts of the \( \mathcal{PT}\)-symmetric self induced potential, \( V(x, t) = \sum_{j=1}^{2} q_j(x, t)q_j^*(x, t) \), that is present in the system (2). The stable propagation occurs due to the combined effect of loss and gain.

The complex amplitudes of the soliton in both the modes are \( A_j(k_1 + k_1^+) \), \( j = 1, 2 \) where \( A_j \)'s are unit polarization vectors which are given in (15a). The soliton in the first and second components travels with the same velocity, that is \( \frac{2(k_{1n}k_{1n}^*-k_{1t}k_{1t}^*)}{(k_{1n}+k_{1t})} \).

We call such soliton as degenerate soliton. The central position of the degenerate soliton in the two modes given by \( \frac{\tilde{\eta}_n}{(k_{1n}+k_{1t})} = \frac{1}{(k_{1n}+k_{1t})} \ln \left( \frac{\beta}{\eta_{1n}} \right) \). We recall here that in the local case, the velocity of the soliton is represented by the imaginary part of the wavenumbers [35][36][39].

3.3 Sub-cases of general soliton solution

From the one-bright soliton solution, (14a), we can also extract a two parameter family of breathing one-soliton solution which is reported in Ref. [43] by considering \( \alpha_1^{(3)} = -\sqrt{2}(\eta_1 + \bar{\eta}_1)e^{i\theta_j}, \beta_1^{(3)} = -\sqrt{2}(\eta_1 + \bar{\eta}_1)e^{i\theta_j}, j = 1, 2 \), where \( \theta_j, \theta_j, \eta_1 \) and \( \bar{\eta}_1 \) are all real parameters and by restricting the wave numbers \( k_1 \) and \( \bar{k}_1 = 2i\bar{\eta}_1 \) and \( k_1 = 2i\eta_1 \) (pure imaginary). Substituting these restrictions in Eq. (14a), we obtain

\[
q_j(x, t) = \frac{\sqrt{2}(\eta_1 + \bar{\eta}_1)e^{i\theta_j}e^{-4i\bar{\eta}_1^2x}e^{-2i\eta_1 x}}{1 + e^{i(\theta_1 + \theta_1)}e^{4i(\eta_1^2 - \bar{\eta}_1^2)x}e^{-2i(\eta_1 + \bar{\eta}_1)x}} , \quad j = 1, 2.
\]  

(20a)

The solution (20a) coincides with the one reported in Ref. [43].

We can also obtain a similar expression for fields \( q_j^+(-x, t) \) by imposing the same restrictions on Eq. (14b). Doing so, we obtain

\[
q_j^+(-x, t) = -\frac{\sqrt{2}(\eta_1 + \bar{\eta}_1)e^{i\theta_j}e^{-4i\bar{\eta}_1^2x}e^{-2i\eta_1 x}}{1 + e^{i(\theta_1 + \theta_1)}e^{4i(\eta_1^2 - \bar{\eta}_1^2)x}e^{-2i(\eta_1 + \bar{\eta}_1)x}} , \quad j = 1, 2.
\]  

(20b)

The above two parameter solutions develop a singularity in finite time which may be verified from the condition given in Eq. (19) along with \( k_{1R} = k_{1R} = 0, k_{1I} = 2\eta_1, k_{2I} = 2\bar{\eta}_1, \alpha_1^{(j)} = -\sqrt{2}(\eta_1 + \bar{\eta}_1)e^{i\theta_j}, \beta_1^{(j)} = -\sqrt{2}(\eta_1 + \bar{\eta}_1)e^{i\theta_j}, j = 1, 2 \), and \( \theta_1 + \theta_1 = \bar{\theta}_2 + \bar{\theta}_2 \). From the expressions (20a) and (20b) it is noted that \( q_j^+(-x, t) \) is parity conjugate of \( q_j(x, t) \).
We can capture the envelope soliton solution of the local Manakov equation, that is
\[ q_j(x, t) = -\sqrt{2}\eta e^{-i(\theta_j-4\eta^2 t)} \text{sech}(2\eta x), \] 
(20c)
by considering \( \eta_1 = \bar{\eta}_1 = \eta \) and \( \theta_j = -\bar{\theta}_j, j = 1, 2 \), and imposing the above restrictions in the soliton solution (14a).

4 Degenerate two bright soliton solution

We obtain the degenerate two-soliton solution of (2) is,
\[ q_j(x, t) = \frac{\alpha_j^{(1)} e^{\xi_j} + \alpha_j^{(2)} e^{\xi_j} + e^{\xi_j + \xi_j + \Delta_1^{(j)}} + e^{\xi_j + \xi_j + \Delta_2^{(j)}}}{D}, \] 
(21a)
\[ q_j^*(-x, t) = \frac{\beta_j^{(1)} e^{\xi_j} + \beta_j^{(2)} e^{\xi_j} + e^{\xi_j + \xi_j + \gamma_1^{(j)}} + e^{\xi_j + \xi_j + \gamma_2^{(j)}}}{D}, \] 
(21b)
\[ D = 1 + e^{\xi_j + \xi_j + \delta_1} + e^{\xi_j + \xi_j + \delta_2} + e^{\xi_j + \xi_j + \delta_3} + e^{\xi_j + \xi_j + \delta_4} + e^{\xi_j + \xi_j + \delta_5}, \]
where \( \xi_j = ik_j x - ik_j^2 t, \xi_j = ik_j x + ik_j^2 t, j = 1, 2 \) and the other constants are given in Appendix B. We note here that the above degenerate two-soliton solution is obtained from the expressions (28a) - (28c) given in Appendix C, after appropriate factorization. This is because the expression for the functions \( f(x, t) \) and \( f^*(-x, t) \) for degenerate one-soliton solution as well as two-soliton solution are equal at all orders of \( \epsilon \). Due to this fact the degenerate two-soliton solution (28a) - (28c) gets factorized into the above simple form. One can easily verified that expressions (21a) - (21b) satisfy Eqs. (2) and (3) simultaneously. The above degenerate two bright soliton solution is characterized by twelve complex parameters, namely \( \alpha_1^{(j)}, \alpha_2^{(j)}, \beta_1^{(j)}, \beta_2^{(j)}, k_j, \bar{k}_j, j = 1, 2 \). In the second part of the present work, we discuss the interaction between the degenerate two solitons in detail by carefully examining the two-soliton solution in the asymptotic regime.

5 Conclusion

In this work, we have constructed more general one and two soliton solutions for the nonlocal Manakov equation through a nonstandard bilinearization procedure. The obtained one- and two-soliton solutions are more general than the already reported ones. Besides deriving the soliton solutions, we have discussed the special features of the obtained soliton solutions. Next, we plan to investigate the collision dynamics through intensity redistribution, phase shift and relative separation distance by performing the asymptotic analysis of the two soliton solutions reported in this paper.
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Appendix

A. The constants appear in non-degenerate one-soliton solution (10a)-(10b)

The constants which appear in the non-degenerate one-soliton solution (10a)-(10b) have the explicit forms,

\[
e^{\alpha_1} = \frac{(-1)^j(k_1^{(1)} - k_1^{(2)})\alpha_1^{(1)}\beta_1^{(1)(3-j)}}{(k_1^{(1)} + k_1^{(2)})^2(k_1^{(3-j)} + k_1^{(3-j)})^2},
\]

\[
e^{\alpha_2} = \frac{(-1)^j(k_1^{(1)} - k_1^{(2)})\alpha_1^{(2)}\beta_1^{(2)(3-j)}}{(k_1^{(1)} + k_1^{(2)})^2(k_1^{(3-j)} + k_1^{(3-j)})^2},
\]

\[
e^{\delta_1} = \frac{-\alpha_1^{(1)}\beta_1^{(1)}}{(k_1^{(1)} + k_1^{(2)})^2},
\]

\[
e^{\delta_2} = \frac{-\alpha_1^{(2)}\beta_1^{(2)}}{(k_1^{(2)} + k_1^{(2)})^2},
\]

\[
e^{\delta_3} = \frac{\alpha_1^{(1)}\alpha_2^{(2)}\beta_1^{(1)(2)}(k_1^{(1)} - k_1^{(2)})(k_1^{(1)} - k_1^{(2)})}{(k_1^{(1)} + k_1^{(2)})^2(k_1^{(1)} + k_1^{(2)})^2(k_1^{(1)} + k_1^{(2)})^2}.
\]

B. The constants which appear in the reduced form of two-soliton solution (21a)-(21b)

The following constants appear in the two-soliton solution (21a)-(21b)

\[
e^{\delta_1} = \tilde{\alpha}_{12}((-1)^{3-j}k_1\beta_1^{(3-j)}\nu_1 + \tilde{k}_2\alpha_2^{(j)}\Gamma_1 - \bar{k}_1\alpha_1^{(j)}\Gamma_2)/\kappa_1\kappa_2, \quad (26a)
\]

\[
e^{\delta_2} = \tilde{\alpha}_{12}((-1)^{3-j}k_2\beta_2^{(3-j)}\nu_1 + \tilde{k}_2\alpha_2^{(j)}\Gamma_2 - \bar{k}_1\alpha_1^{(j)}\Gamma_2)/\kappa_1\kappa_2, \quad (26b)
\]

\[
e^{\gamma_1} = \tilde{\alpha}_{12}(k_2\beta_2^{(j)}\Gamma_1 - k_1\beta_1^{(j)}\Gamma_2 + (-1)^{(3-j)}k_1\alpha_1^{(j)}\nu_1)/\kappa_1\kappa_2, \quad (26c)
\]

\[
e^{\gamma_2} = \tilde{\alpha}_{12}(k_2\beta_2^{(j)}\Gamma_2 - k_1\beta_1^{(j)}\Gamma_2 + (-1)^{(3-j)}k_2\alpha_2^{(j)}\nu_1)/\kappa_1\kappa_2, \quad (26d)
\]

\[
e^{\delta_3} = \gamma_1\Gamma_1\Gamma_2, \quad \delta_4 = \gamma_2\Gamma_1\Gamma_2, \quad \delta_5 = -\gamma_3\Gamma_1\Gamma_2, \quad \delta_6 = -\gamma_3\Gamma_2\Gamma_2, \quad (26e)
\]

\[
e^{\delta_8} = \gamma_{12}\tilde{\gamma}_{12}(\gamma_1\Gamma_1\Gamma_2 - \gamma_2\nu_1\nu_2 - \gamma_3\Gamma_2\Gamma_2)/\kappa_1\kappa_2\kappa_2, \quad (26f)
\]

C. An un-factored degenerate two-soliton solution

A general un-factored degenerate two-soliton solution can be deduced by considering the following forms of seed solution for the functions \(g_1^{(j)}(x, t)\) and \(g_1^{(j)}(-x, t)\), for
The above form of seed solutions truncates the series expansions (7a)-(7c) at in

The explicit expression of all the constants that appear in two-soliton solution are

g^{(j)}(x,t) = \alpha_1^{(j)} e^{\xi_1} + \alpha_2^{(j)} e^{\xi_2}, \xi_j = ik_j x + i\beta_j^2 t,(27a)
g^{(j)*}(-x,t) = \beta_1^{(j)} e^{\xi_1} + \beta_2^{(j)} e^{\xi_2}, \xi_j = ik_j x - i\beta_j^2 t, j = 1,2. (27b)

The explicit expression of all the constants that appear in two-soliton solution are
g^{(j)}(x,t) = \alpha_1^{(j)} e^{\xi_1} + \alpha_2^{(j)} e^{\xi_2} + e^{\xi_1+2\xi_2+\Delta^{(j)}}, e^{\xi_2+2\xi_1+\Delta^{(j)}} + e^{\xi_1+2\xi_2+\Delta^{(j)}} (28a)

f(x,t) = 1 + e^{\xi_1+\xi_2+\Delta^{(j)}} + e^{\xi_1+\xi_2+\Delta^{(j)}} + e^{\xi_1+\xi_2+\Delta^{(j)}} (28b)

The explicit expression of all the constants that appear in two-soliton solution are
e^{\xi_1} = -2\Gamma_{11}/\kappa_{11}, e^{\xi_2} = -2\Gamma_{12}/\kappa_{12}, e^{\xi_3} = -2\Gamma_{21}/\kappa_{21}, e^{\xi_4} = -2\Gamma_{22}/\kappa_{22},

\Gamma_{11} = (\alpha_1^{(1)}\beta_1^{(1)} + \alpha_1^{(2)}\beta_1^{(1)}), \Gamma_{12} = (\alpha_1^{(1)}\beta_1^{(2)} + \alpha_1^{(2)}\beta_1^{(2)}),

\Gamma_{21} = (\alpha_2^{(1)}\beta_2^{(1)} + \alpha_2^{(2)}\beta_2^{(1)}), \Gamma_{22} = (\alpha_2^{(1)}\beta_2^{(2)} + \alpha_2^{(2)}\beta_2^{(2)}), \kappa_{lm} = (k_l + k_m)^2, l, m = 1,2,

\Gamma_{11} = (\alpha_1^{(1)}\beta_1^{(1)} + \alpha_1^{(2)}\beta_1^{(1)}), \Gamma_{12} = (\alpha_1^{(1)}\beta_1^{(2)} + \alpha_1^{(2)}\beta_1^{(2)}),

\Gamma_{21} = (\alpha_2^{(1)}\beta_2^{(1)} + \alpha_2^{(2)}\beta_2^{(1)}), \Gamma_{22} = (\alpha_2^{(1)}\beta_2^{(2)} + \alpha_2^{(2)}\beta_2^{(2)}), \kappa_{lm} = (k_l + k_m)^2, l, m = 1,2,

\e^{\Delta^{(j)}} = -\alpha_1^{(j)}\Gamma_{11}/\kappa_{11}, \e^{\Delta^{(j)}} = -\alpha_1^{(j)}\Gamma_{12}/\kappa_{12}, \e^{\Delta^{(j)}} = -\alpha_1^{(j)}\Gamma_{21}/\kappa_{21}, \e^{\Delta^{(j)}} = -\alpha_1^{(j)}\Gamma_{22}/\kappa_{22},
\[ e^{\Delta^{(i)}} = -\left( \alpha^{(j)}_1 \Gamma_{21}(k_1 + k_2)(k_1 + k_2 - k_1) + \alpha^{(j)}_2 \Gamma_{11}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{11} \kappa_{12}, \]
\[ e^{\Delta^{(k)}} = -\left( \alpha^{(j)}_1 \Gamma_{22}(k_2 + k_1)(k_2 + k_1 - k_1) + \alpha^{(j)}_2 \Gamma_{12}(k_2 + k_1)(k_2 + k_1 - k_1) \right) / \kappa_{21} \kappa_{22}, \]
\[ e^{\beta^{(i)}} = -\left( \beta^{(j)}_1 \Gamma_{11}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{12}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{11} \kappa_{21}, \]
\[ e^{\beta^{(k)}} = -\left( \beta^{(j)}_1 \Gamma_{21}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{22}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{21} \kappa_{22}, \]
\[ \varepsilon^{\Delta^{(i)}} = -\left( \beta^{(j)}_1 \Gamma_{11}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{12}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{11} \kappa_{21}, \]
\[ \varepsilon^{\Delta^{(k)}} = -\left( \beta^{(j)}_1 \Gamma_{21}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{22}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{21} \kappa_{22}, \]
\[ \varepsilon^{\beta^{(i)}} = -\left( \beta^{(j)}_1 \Gamma_{11}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{12}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{11} \kappa_{21}, \]
\[ \varepsilon^{\beta^{(k)}} = -\left( \beta^{(j)}_1 \Gamma_{21}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{22}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{21} \kappa_{22}, \]
\[ \varepsilon^{\beta^{(m)}} = -\left( \beta^{(j)}_1 \Gamma_{21}(k_1 + k_2)(k_1 + k_2 - k_1) + \beta^{(j)}_2 \Gamma_{22}(k_1 + k_2)(k_1 + k_2 - k_1) \right) / \kappa_{21} \kappa_{22}, \]

\[ A_3 = (k_1(2k_1 + k_2 - k_2) + 2k_2k_2 + k_1(k_2 - k_2))(\alpha^{(1)}_1 \beta^{(1)}_1 \alpha^{(2)}_2 \beta^{(2)}_2 + \alpha^{(1)}_1 \beta^{(2)}_2 \alpha^{(2)}_1 \beta^{(1)}_2), \]
\[ A_4 = (-k_2k_2 + k_1(2k_2 + k_2) + k_1(2k_2 - k_1 + k_2))(\alpha^{(1)}_1 \beta^{(1)}_1 \alpha^{(1)}_2 \beta^{(1)}_2 + \alpha^{(1)}_1 \beta^{(2)}_2 \alpha^{(2)}_1 \beta^{(1)}_2), \]
\[ A_5 = (k_2^2 + k_2k_2 - k_2 + k_1^2 + k_2k_2 + k_2^2 + k_1(k_2 - k_2) + k_2k_2 + k_1k_2 \alpha^{(2)}_2 \beta^{(1)}_2 + \alpha^{(1)}_1 \beta^{(1)}_1 \alpha^{(2)}_1 \beta^{(1)}_2) \]

\[ \varepsilon^{\Delta^{(i)}} = \tilde{\theta}_{12} \Gamma_{11} \left( -\frac{1}{2} k_1 \beta^{(i)}_{11} \nu_1 - k_2 \beta^{(i)}_{12} \right) / \kappa_{11}^{2/2} \]
\[ \varepsilon^{\Delta^{(k)}} = \tilde{\theta}_{12} \Gamma_{21} \left( -\frac{1}{2} k_1 \beta^{(k)}_{11} \nu_1 - k_2 \beta^{(k)}_{12} \right) / \kappa_{11}^{2/2} \]
\[ \varepsilon^{\beta^{(i)}} = \tilde{\theta}_{12} \Gamma_{12} \left( -\frac{1}{2} k_1 \beta^{(i)}_{11} \nu_1 - k_2 \beta^{(i)}_{12} \right) / \kappa_{11}^{2/2} \]
\[ \varepsilon^{\beta^{(k)}} = \tilde{\theta}_{12} \Gamma_{22} \left( -\frac{1}{2} k_1 \beta^{(k)}_{11} \nu_1 - k_2 \beta^{(k)}_{12} \right) / \kappa_{11}^{2/2} \]
\[ \varepsilon^{\beta^{(m)}} = \tilde{\theta}_{12} \Gamma_{21} \left( -\frac{1}{2} k_1 \beta^{(m)}_{11} \nu_1 - k_2 \beta^{(m)}_{12} \right) / \kappa_{11}^{2/2} \]
\[ e^{a_2} = \frac{\bar{\theta}_{12} A_6}{\kappa_{11} \kappa_{12} \kappa_{21} \kappa_{22}}, \quad e^{\alpha_3} = \frac{\bar{\theta}_{12} A_7}{\kappa_{11} \kappa_{12} \kappa_{21} \kappa_{22}}, \quad e^{\alpha_4} = \frac{\bar{\theta}_{12} A_8}{\kappa_{11} \kappa_{12} \kappa_{21} \kappa_{22}}. \]

\[ A_6 = \left( -2k_2^2 \alpha_2 (\Gamma_{11} \Gamma_{22} + 2k_2^2 \alpha_1 (\Gamma_{21} \Gamma_{22} + k_1^2 [-2k_2^2 \alpha_2 (\Gamma_{21} \Gamma_{22} + k_1 (\alpha_2 + \bar{\kappa})]) + k_2 \nu_2 (-\alpha_2 \nu_2 + \beta_2^{(3-j)} (-1)^j \Gamma_{21}) \right) + k_1 k_2 [k_2 \Gamma_{21} (\alpha_1 \Gamma_{22} + \alpha_2 \Gamma_{12}) + k_2 \nu_2 (-\alpha_1 \nu_2 + \beta_2^{(3-j)} (-1)^j \Gamma_{12}) \right] + k_1 [k_2 \alpha_1 (\beta_2^{(3-j)} \Gamma_{11} \Gamma_{22} + \alpha_2^{(3-j)} \Gamma_{11} \Gamma_{12})] + k_1 [k_2 \alpha_3 (\beta_2^{(3-j)} \Gamma_{11} \Gamma_{22} + \alpha_2^{(3-j)} \Gamma_{11} \Gamma_{12})], \]

\[ A_7 = \left( -k_2 \Gamma_{11} (2k_2 \alpha_2 \Gamma_{12} + k_2 \bar{k} (2 \alpha_2 \Gamma_{12} + (-1)^j \beta_2^{(3-j)} \nu_1) + k_2^2 (\alpha_1 \Gamma_{22} + \alpha_2 \Gamma_{12}) \right) + k_2 \nu_2 (-\alpha_2 \nu_2 + \beta_2^{(3-j)} \Gamma_{11}) \right] + k_1 k_2 [k_2 \alpha_1 (\beta_2^{(3-j)} \Gamma_{11} \Gamma_{22} + \alpha_2^{(3-j)} \Gamma_{11} \Gamma_{12}) \right] + k_1 [k_2 \alpha_3 (\beta_2^{(3-j)} \Gamma_{11} \Gamma_{22} + \alpha_2^{(3-j)} \Gamma_{11} \Gamma_{12})], \]

\[ A_8 = \left( -k_2 \Gamma_{11} (2k_2 \alpha_2 \Gamma_{12} + k_2 \bar{k} (2 \alpha_2 \Gamma_{12} + (-1)^j \beta_2^{(3-j)} \nu_1) + k_2^2 (\alpha_1 \Gamma_{22} + \alpha_2 \Gamma_{12}) \right) + k_2 \nu_2 (-\alpha_2 \nu_2 + \beta_2^{(3-j)} \Gamma_{11}) \right] + k_1 k_2 [k_2 \alpha_1 (\beta_2^{(3-j)} \Gamma_{11} \Gamma_{22} + \alpha_2^{(3-j)} \Gamma_{11} \Gamma_{12})]. \]
\[ A_9 = -2k_2 \bar{k}_2 \Gamma_{11} \Gamma_{22} \beta_2^{(j)} + 2k_1^3 \beta_1^{(j)} \Gamma_{12} \Gamma_{22} + \bar{k}_2^2 (-1)^j \left[ (-1)^{(3-j)} 2k_2 \Gamma_{21} \beta_2^{(j)} + (-1)^{(3-j)} \right] \]
\[ + \bar{k}_2 v_2 (\nu_1 \beta_2^{(j)} + (-1)^{(3-j)} \alpha_2^{(3-j)} \Gamma_{12}) ] + k_2^2 [ \bar{k}_2 \Gamma_{22} (2 \beta_1^{(j)} \Gamma_{12} + (-1)^{(3-j)} \alpha_2^{(3-j)} v_2) + \bar{k}_2 \Gamma_{12} (2 \beta_1^{(j)} \Gamma_{12} + (-1)^{(3-j)} \alpha_2^{(3-j)} v_2) ] + \bar{k}_2 [ (-1)^{(3-j)} ] \]
\[ + \nu_2 (\alpha_1^{(3-j)} \Gamma_{22} + (-1)^{(3-j)} \beta_2^{(j)} \nu_1) + k_2 \beta_2^{(j)} (\alpha_1^{(1)} \beta_1^{(1)} \Gamma_{21} + \beta_1^{(1)} \Gamma_{22}) + \alpha_1^{(2)} (\beta_1^{(2)} \Gamma_{22}) \]
\[ + \beta_2^{(2)} \Gamma_{21} )) ] + \bar{k}_1 \bar{k}_2 \Gamma_{22} (2 \beta_2^{(j)} \Gamma_{11} + \beta_1^{(1)} \Gamma_{12}) + k_2 (2 \beta_1^{(1)} \Gamma_{22} \beta_1^{(1)} \Gamma_{22} + \beta_2^{(1)} \Gamma_{21}) + \beta_2^{(2)} \)
\[ + (-2 \Gamma_{11} \Gamma_{22} + v_2 \nu_2) ] + \bar{k}_1 (k_2 \beta_2^{(3)} | \alpha_1^{(1)} (-2 \beta_1^{(1)} \Gamma_{21} + \alpha_2^{(2)} v_2) + (-2 \beta_1^{(1)} \Gamma_{21} + \alpha_2^{(2)} v_2) ] \]
\[ + k_2 (\alpha_1^{(1)} \beta_2^{(j)} + (-2 \beta_2^{(j)} \Gamma_{21} + (-1)^{(3-j)} \alpha_2^{(3-j)} v_2) + \alpha_1^{(3-j)} (-2 \beta_1^{(1)} \beta_2^{(j)} \Gamma_{22} + (-1)^{(3-j)} \nu_2 \]
\[ + \Gamma_{22} + 3(-1)^{(3-j)} \alpha_2^{(3-j)} \beta_2^{(j)} \nu_2) ] \)

\[ \nu_1 = \alpha_2^{(2)} \alpha_2^{(1)} - \alpha_2^{(1)} \alpha_2^{(2)}, \quad \nu_2 = \beta_2^{(2)} - \beta_2^{(1)} \beta_2^{(1)}, \quad \nu_2 = (k_1 - k_2), \quad \nu_1 = (k_1 - k_2), \]
\[ e^{\delta_{11}} = -2g_{12} g_{12} \Gamma_{11} \left( \bar{g}_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{11}^2 \kappa_{21} \kappa_{21}, \]
\[ e^{\delta_{22}} = -2g_{12} g_{12} \Gamma_{21} \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{11}^2 \kappa_{21} \kappa_{21}, \]
\[ e^{\delta_{31}} = -2g_{12} g_{12} \Gamma_{12} \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{11}^2 \kappa_{21} \kappa_{21}, \]
\[ e^{\delta_{12}} = -2g_{12} g_{12} \Gamma_{22} \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{11}^2 \kappa_{21} \kappa_{21}, \]
\[ v_1 = (k_2 \bar{k}_1 + k_1 \bar{k}_1), \quad v_2 = (k_1 k_2 + k_3 k_3), \]
\[ e^{\xi_{11}} = -g_{12} g_{12} \left( -1 \right)^j k_1 \beta_1^{(3-j)} \nu_1 - \bar{k}_2 \alpha_2^{(j)} \Gamma_{11} + \bar{k}_1 \alpha_1^{(j)} \Gamma_{21} \right) \]
\[ \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{11} \kappa_{12}, \]
\[ e^{\xi_{22}} = -g_{12} g_{12} \left( -1 \right)^j k_2 \beta_2^{(3-j)} \nu_1 - \bar{k}_2 \alpha_2^{(j)} \Gamma_{11} + \bar{k}_1 \alpha_1^{(j)} \Gamma_{21} \right) \]
\[ \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{12} \kappa_{21} \]
\[ e^{\xi_{31}} = -g_{12} g_{12} \left( -k_2 \beta_2^{(j)} \Gamma_{11} + k_1 \beta_1^{(j)} \Gamma_{12} + (-1)^{(3-j)} k_1 \alpha_1^{(3-j)} \nu_2 \right) \]
\[ \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{11} \kappa_{21}, \]
\[ e^{\xi_{31}} = -g_{12} g_{12} \left( -k_2 \beta_2^{(j)} \Gamma_{21} + k_1 \beta_1^{(j)} \Gamma_{22} + (-1)^{(3-j)} k_2 \alpha_2^{(3-j)} \nu_2 \right) \]
\[ \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) / k_{12} \kappa_{21}, \]
\[ e^{\xi_{31}} = g_{12} g_{12} \left( g_1 \Gamma_{11} \Gamma_{22} - g_2 v_2 - g_3 \Gamma_{21} \Gamma_{12} \right) \right)^2 / k_{11}^2 \kappa_{12} \kappa_{21} \kappa_{22}, \]
\[ \kappa = k_{11} \kappa_{12} \kappa_{21} \kappa_{22}. \]
We arrive the degenerate two-soliton solution by substituting the expression given in (28a)-(28c) in Eq. (1). The auxiliary functions are found to be

\[ s^{(1)}(x,t) = s^{(2)}(x,t) = \Gamma_{11} e^{\xi_1 - \xi_2} + \Gamma_{21} e^{\xi_1 + \xi_2} + \Gamma_{12} e^{\xi_1 + \xi_2} + \Gamma_{22} e^{\xi_1 - \xi_2} \]

\[ + e^{\xi_1 + 2\xi_2 + 4\phi_1} + e^{2\xi_1 + \xi_2 + 2\phi_2} + e^{\xi_1 + 2\xi_2 + 4\phi_3} + e^{2\xi_1 + \xi_2 + 2\phi_4} \]

where the constants are obtained as

\[ e^{\phi_1} = -\frac{\delta_{12}^2 \Gamma_{11} \Gamma_{12}}{\kappa_{11} \kappa_{21}}, \quad e^{\phi_2} = \frac{\delta_{12}^2 \Gamma_{11} \Gamma_{21}}{\kappa_{11} \kappa_{22}}, \quad e^{\phi_3} = \frac{\delta_{12}^2 \Gamma_{22} \Gamma_{21}}{\kappa_{12} \kappa_{22}}, \quad e^{\phi_4} = \frac{\delta_{12}^2 \Gamma_{22} \Gamma_{21}}{\kappa_{21} \kappa_{22}} \]

\[ \begin{align*}
A_1 &= (2\kappa_{11}(\kappa_{12} + \kappa_{12}^2)) - \kappa_{11} \kappa_{21}^2 (2k_1 + k_2 - k_2) + \kappa_{11} \kappa_{21}^2 (2k_1 - k_2 + k_2) \\
&+ \kappa_{12} \kappa_{22} (k_1 - k_2) + (k_1 - k_2 - 2k_2)k_1 + 2(k_1 + k_2 + k_2)k_2) \\
A_2 &= (\kappa_{11} + 2k_1 - 2k_2 - k_2)k_1 + \kappa_{11} \kappa_{21}^2 (2k_1 - k_2 + k_2) + 2(\kappa_{11} \kappa_{21}^2)k_1 \\
&- \kappa_{12} \kappa_{22} (k_1 + k_2 - k_2) + 2\kappa_{12} \kappa_{22} \kappa_{11} - (2k_1 + k_2 - k_2)k_2) \\
\end{align*} \]

\[ e^{\phi_{11}} = \frac{\delta_{12}^2 \Gamma_{11} \Gamma_{12} \psi}{\kappa_{11} \kappa_{21}}, \quad e^{\phi_{12}} = \frac{\delta_{12}^2 \Gamma_{11} \Gamma_{21} \psi}{\kappa_{11} \kappa_{22}} \]

\[ \psi = \begin{pmatrix} k_1 k_2 \Gamma_{11} \Gamma_{22} + k_1 (-k_2 \Gamma_{21} \Gamma_{12} + k_1 \Gamma_{11} \Gamma_{22} - k_2 \nu_1 \nu_2) - \bar{k}_1 (k_2 \Gamma_{21} \Gamma_{12} + k_2 \nu_1 \nu_2) \end{pmatrix} \]

**Conflicts of interest**

The authors declare that they have no conflict of interest.

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