UNIQUENESS OF HYPERSURFACES OF CONSTANT HIGHER ORDER MEAN CURVATURE IN HYPERBOLIC SPACE

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Abstract. We study the uniqueness of horospheres and equidistant spheres in hyperbolic space under different conditions. First we generalize the Bernstein theorem by Do Carmo and Lawson [12] to the embedded hypersurfaces with constant higher order mean curvature. Then we prove two Bernstein type results for immersed hypersurfaces under different assumptions. Last, we show the rigidity of horospheres and equidistant spheres in terms of their higher order mean curvatures.

1. Introduction

In 1927, S. N. Bernstein proved that the only entire minimal graphs in $\mathbb{R}^3$ are planes. The analogous problem in higher dimension is known as Bernstein problem. Namely, given $u : \mathbb{R}^n \to \mathbb{R}$ a minimal graph, is the graph of $u$ a flat hyperplane? It turns out that the answer is yes for $n \leq 7$ and that has been settled down by a series of very significative papers [4, 11, 16, 24]. On the contrary, for $n \geq 8$, one has the famous counterexamples by E. Bombieri, E. De Giorgi and E. Giusti [6]. Afterwards, many generalizations of Bernstein theorem have arised. As an example, we mention [23] where R. Schoen, L. Simon and S. T. Yau studied a Bernstein type theorem for stable minimal hypersurfaces.

Later on, Bernstein type theorems for constant mean curvature hypersurfaces in Euclidean and in hyperbolic space $\mathbb{H}^{n+1}$ have been studied. Let us give a very simple example in the Euclidean space. Does it exist an entire graph $M$ in $\mathbb{R}^{n+1}$ with constant mean curvature $H \neq 0$? It is well known that the answer is no and here is the proof. Assume $M$ exists. Without loss of generality, we can assume that the mean curvature vector of $M$ points upward. Consider a sphere $S$ in $\mathbb{R}^{n+1}$ of mean curvature $H$. As $S$ is compact, up to an ambient isometry, we can assume that $S$ is above $M$. Then, translate down $S$. Clearly, there will be a first contact point $p$ between $M$ and $S$. At $p$, $S$ and $M$ are tangent and applying the maximum principle, one gets that $S$ and $M$ should coincide, that is a contradiction.

In the hyperbolic space, there is more variety of constant mean curvature hypersurfaces. We point out three important results that have been proved throughout history. A complete hypersurface $\Sigma$ of $\mathbb{H}^{n+1}$ with constant mean curvature is a horosphere, provided:

1. $\Sigma$ is properly embedded and has exactly one point in its asymptotic boundary [12, Theorem A].
2. $n = 2$ and $\Sigma$ is properly immersed between two horospheres in $\mathbb{H}^3$ with the same asymptotic point [1, Theorem 1].
3. $\Sigma$ is immersed, has all the principal curvatures uniformly are larger than $-1$ and has exactly one point in its asymptotic boundary [8, Theorem 1.5].

Key words and phrases. Bernstein theorem, rigidity, immersed, higher order mean curvature, hyperbolic space.
We will study analogous problems for hypersurfaces with constant higher order mean curvature functions (\(H_r\)-hypersurface in the following). Our results are generalizations of the three statements above. Moreover, motivated by the recent work by R. Souam [25], we show the \(r\)-mean curvature rigidity of horospheres and equidistant spheres (notice that \(H_r\) may be zero).

The article is organized as follows. In Section 2, we fix notations and collect some preliminary results. The result of Section 3 is the following uniqueness theorem for horospheres and equidistant spheres, which is a generalization of [11].

**Theorem 3.1.** Let \(\Sigma\) be a complete \(H_r\)-hypersurface properly embedded in the hyperbolic space \(\mathbb{H}^{n+1}\), \(r \geq 2\). Denote the asymptotic boundary of \(\Sigma\) by \(\partial_\infty \Sigma\). Then we have:

1. if \(\partial_\infty \Sigma\) is a point, then \(\Sigma\) is a horosphere;
2. if \(\partial_\infty \Sigma\) is a sphere and \(\Sigma\) separates poles, then \(\Sigma\) is a equidistant sphere.

For the definition of horosphere separating poles, see Section 3. For \(r = 1\) the result is contained in [12, Theorem B].

In Section 4, we consider the Bernstein problem for immersed hypersurfaces, either with constant \(r\)-mean curvature or satisfying a Weingarten equation. In particular, we generalize results by L. Alías and M. Dajczer, [1], which concerns a more general problem in warped products, by L. Alías, D. Impera and M. Rigoli [3] and by Bonini, Qing and the second author [8]. Let us mention our main results in Section 4.

**Theorem 4.3.** Let \(\Sigma\) be a \(r\)-admissible, \(L_{r-1}\)-parabolic \(H_r\)-hypersurface properly immersed in \(\mathbb{H}^{n+1}\). If \(\Sigma\) is contained in a slab and the angle function does not change sign, then \(\Sigma\) is a horosphere.

**Theorem 4.4.** Let \(\Sigma\) be an immersed, complete, uniformly admissible Weingarten hypersurface in \(\mathbb{H}^{n+1}\). Then \(\Sigma\) is a horosphere provided its asymptotic boundary is a single point.

A slab is the space between two horospheres that share the same asymptotic point. For the definition of \(r\)-admissibility, \(L_{r-1}\)-parabolicity, uniformly admissible Weingarten hypersurface, see Section 2. Finally, in Section 5, we show the \(r\)-mean curvature rigidity of horospheres and equidistant spheres, which generalizes [25].

**Theorem 5.1.** Let \(M\) be a horosphere or a equidistant sphere in hyperbolic space \(\mathbb{H}^{n+1}\), \(n \geq 2\) and \(H_M > 0\) denote its \(r\)-mean curvature, \(r \geq 1\), with respect to the orientation given by the mean curvature vector. Let \(\Sigma\) be a connected properly embedded \(C^2\) hypersurface in \(\mathbb{H}^{n+1}\) which coincides with \(M\) outside a compact subset \(B\) in \(\mathbb{H}^{n+1}\). Choose the orientation on \(\Sigma\) such that the \(r\)-mean curvature \(H_r\) of \(\Sigma\) is equal to \(H_M\) outside the compact set \(B\). With respect to this orientation, if either \(H_r \geq H_M\) or \(|H_r| \leq H_M\), then \(\Sigma = M\).

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2. Preliminaries

2.1. Models for the hyperbolic space \(\mathbb{H}^{n+1}\). We will work in different well-known models for the hyperbolic space. For the sake of completeness we briefly describe them.
The half-space model. Consider the upper half-space

$$\mathbb{R}^{n+1}_+ = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} > 0\}$$

with the metric $\frac{dx_1^2 + \cdots + dx_{n+1}^2}{x_{n+1}^2}$. In this model, horospheres are either horizontal hyperplanes or Euclidean spheres tangent at some point to the hyperplane $\{x_{n+1} = 0\}$. Moreover, the intersections of the upper half-space with Euclidean spheres not contained in the upper half-space are totally umbilical hypersurfaces, whose absolute values of the principal curvatures are strictly less than 1. Such hypersurfaces are usually called *equidistant spheres* when the principal curvatures are not zero. The ones with centers on the hyperplane $\{x_{n+1} = 0\}$ are *(totally geodesic)* hyperplanes.

The warped product model. $\mathbb{H}^{n+1}$ can be viewed as the warped product $\mathbb{R} \times \mathbb{R}^n$, that is the product manifold $\mathbb{R} \times \mathbb{R}^n$ endowed with the following metric

$$\langle \cdot, \cdot \rangle = \pi_1^*(dt^2) + e^{2t} \pi_2^*(\langle \cdot, \cdot \rangle_{\mathbb{R}^n}),$$

where $\pi_1$ and $\pi_2$ denote the projections onto the two factors and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is the Euclidean metric. Notice that the leafs $\mathbb{R}_t = \{t\} \times \mathbb{R}^n$ are horospheres with $r$-mean curvature one, for every $r = 1, \ldots, n$ with respect to $-T$, where $T$ is the lift of $\frac{\partial}{\partial t}$. All the $\mathbb{R}_t$’s share the same point at infinity. For an immersed hypersurface $\Sigma$ of $\mathbb{R} \times \mathbb{R}^n$, oriented by $\nu$, we define the height function $h \in C^\infty(\Sigma)$ to be the restriction of $\pi_1$ to $\Sigma$ and the angle function by $\Theta = \langle \nu, T \rangle$.

The hyperboloid model. For $n \geq 2$, The Minkowski space $\mathbb{L}^{n+2}$, is the vector space $\mathbb{R}^{n+2}$ endowed with the Lorentzian metric $\langle \cdot, \cdot \rangle$ given by

$$\langle \bar{x}, \bar{x} \rangle = -x_0^2 + \sum_{i=1}^{n+1} x_i^2,$$

where $\bar{x} = (x_0, x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+2}$. Then hyperbolic space, de Sitter spacetime and the positive null cone are given by

$$\mathbb{H}^{n+1} = \{\bar{x} \in \mathbb{L}^{n+2} | \langle \bar{x}, \bar{x} \rangle = -1, x_0 > 0\},$$

$$\mathbb{S}^{1,n} = \{\bar{x} \in \mathbb{L}^{n+2} | \langle \bar{x}, \bar{x} \rangle = 1\},$$

$$\mathbb{N}^{n+1} = \{\bar{x} \in \mathbb{L}^{n+2} | \langle \bar{x}, \bar{x} \rangle = 0, x_0 > 0\},$$

respectively. We identify the ideal boundary at infinity of hyperbolic space $\mathbb{H}^{n+1}$ with the unit round sphere $\mathbb{S}^n$ sitting at height $x_0 = 1$ in the null cone $\mathbb{N}^{n+1}$ of Minkowski space $\mathbb{L}^{n+2}$. Here, horospheres are the intersections of affine *null hyperplanes* of $\mathbb{L}^{n+2}$ with $\mathbb{H}^{n+1}$. A *null hyperplane* is such that its normal vector field belongs to $\mathbb{N}^{n+1}$.

2.2. The *k*-mean curvatures $H_k$. Let $\Sigma$ be an orientable, connected, immersed hypersurface in hyperbolic space $\mathbb{H}^{n+1}$. Let $\nu$ be an orientation on $\Sigma$ and denote by $A$ the second fundamental form of the immersion with respect to $\nu$. Denote by $\kappa_1, \cdots, \kappa_n$ the principal curvature of $\Sigma$, that is the eigenvalues of $A$. The *k*-mean curvatures $H_k$ of $\Sigma$, $1 \leq k \leq n$, is defined by

$$\binom{n}{k} H_k(x) = \sigma_k(\kappa(x)).$$
where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the $k$-elementary symmetric function defined by

$$\sigma_k(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}$$

(2.3)

Thus, $H_1$ is the mean curvature, $H_n$ is the Gauss-Kronecker curvature and $H_2$ is a multiple of the scalar curvature, when the ambient space is Einstein. Functions like $\sigma_k$ are a particular case of hyperbolic polynomials (see [19]).

It was proved by R. Reilly in [22] that the study of the $k$-mean curvatures is related to the study of the classical Newton transformations $P_k$, that are defined inductively as follows.

$$P_0 = I,$$
$$P_k = \sigma_r I - AP_{k-1},$$

where $I$ is the identity matrix and $A$ is a symmetric matrix. Each $P_k$ is a self-adjoint operator that has the same eigenvectors of $A$.

Before establishing the relation between $P_k$ and $H_k$, let us recall that J. L. Barbosa and G. Colares extended the relation to space forms [5] and M. F. Elbert to any Riemannian manifold [13] (see also [14]).

Let $f : \Sigma \to \mathbb{H}^{n+1}$ be an isometric immersion of a connected oriented Riemannian $n$-manifold into the hyperbolic space and let $A$ its second fundamental form with respect to an orientation $\nu$. Let $D \subset \Sigma$ be a domain. A variation of $D$ is a differentiable map $F : (-\varepsilon, \varepsilon) \times D \to \mathbb{H}^{n+1}$, $\varepsilon > 0$, such that for each $t \in (-\varepsilon, \varepsilon)$ the map $F_t : \{t\} \times D \to \mathbb{H}^{n+1}$ defined by $F_t(p) = F(t, p)$ is an immersion and $F_0 = f|_D$. Define $V_t(p) = \partial F_t(p)/\partial t$ and $u(t) = (V_t, \nu_t)$, where $\nu_t$ is the unit normal vector field in $F_t(D)$ such that $\nu_0 = \nu$. We say that a variation $F$ of $D$ has compact support if supp($F_t$) $\subset K$, for all $t \in (-\varepsilon, \varepsilon)$, where $K \subset D$ is a compact domain. Let $H'_k$ the $k$-mean curvature of $F_t$, and $\sigma'_k = \binom{n}{k} H'_k$. Then one has

$$\frac{\partial}{\partial t}(\sigma'_{k+1})|_{t=0} = L_k(u) + u(\sigma_1 \sigma_{k+1} - (k+2) \sigma_{k+2} - (n-k) \sigma_k) + V^T \sigma_{k+1}$$

where $L_k(u) = tr(P_k(Hess(u)))$ and $V^T$ is the projection of $V$ on $T\Sigma$. Notice that, in the case $\sigma_{k+1}$ constant, then the left-hand side and the last term in the previous equality are zero.

### 2.3. Ellipticity of $L_k$ and $L_k$-parabolicity.

As $L_k(u) = tr(P_k(Hess(u)))$, $L_k$ is an elliptic operator if and only if $P_k$ is a positive definite matrix. In particular, $L_0$ is the Laplace-Beltrami operator $\Delta$. Let us establish a geometric condition that guarantees the ellipticity of $L_k$.

Denote by $\Gamma_k$ the connected component in $\mathbb{R}^n$ of the set $\{H_k > 0\}$ that contains the vector $(1, \ldots, 1)$. As it is proved in [17] Section 2], for any $k = 1, \ldots, n-1$,

$$\Gamma_{k+1} \subset \Gamma_k$$

(2.4)

Notice that $\Gamma_n$ is the positive cone in $\mathbb{R}^n$. Moreover, since $\Gamma_1$ is the largest cone, the mean curvature is positive at any point where the principal curvatures vector stays in the cone $\Gamma_k$.

Moreover, we recall the classical Gårding inequality [19]:

$$H_1 \geq H^{1/2} \geq \cdots \geq H^{1/k} \geq H^{1/(k+1)} > 0,$$

(2.5)

providing all the $r$-mean curvature involved are positive.
Definition 2.1. A hypersurface \( \Sigma \) of \( \mathbb{H}^{n+1} \) is called \( k \)-admissible if the principal curvatures vector at any point of \( \Sigma \) stays in the cone \( \Gamma_k \), that is,

\[
\lambda(x) = (\kappa_1(x), \ldots, \kappa_n(x)) \in \Gamma_k
\]

for all \( x \in \Sigma \).

It is well known that the existence of an elliptic point on a \( H_k \)-hypersurface with \( H_k > 0 \) yields that the hypersurface is \( k \)-admissible \([19, 17]\). Moreover, \( k \)-admissibility yields that \( L_{k-1} \) is elliptic.

In \([3]\), L. Alías, D. Impera and M. Rigoli assumed \( L_k \)-parabolicity to study the Bernstein type theorems for hypersurfaces with constant \( k \)-mean curvature in warped product spaces.

Definition 2.2. A hypersurface \( \Sigma \) in \( \mathbb{H}^{n+1} \) is \( L_k \)-parabolic if the only bounded above \( C^1 \) solutions \( u : \Sigma \rightarrow \mathbb{R} \), of the inequality

\[
L_k u \geq 0
\]

are constants.

2.4. Weakly horospherically convexity. Intuitively, a hypersurface is weakly horospherically convex at \( p \) if and only if all the principal curvatures of the hypersurface at \( p \) are simultaneously \( < -1 \) or \( > -1 \).

For later use, we recall some basic definitions related to the normal geodesic flow in \([7, 8]\).

Let \( f : M \rightarrow \mathbb{H}^{n+1} \) be an isometric immersion of an orientable connected Riemannian manifold of dimension \( n \), and \( \eta \) a unit normal vector field orienting \( M \). The hyperbolic Gauss map \( G \) of \( M \) is defined as follows: for every \( p \in M \), \( G(p) \) is the point at infinity of the unique geodesic starting at \( f(p) \) with tangent vector \(-\eta(p)\).

Notice that \( G(p) \) coincides with the point at infinity of the unique horosphere in \( \mathbb{H}^{n+1} \) passing through \( f(p) \) whose mean curvature vector coincides with \(-\eta(p)\) at \( f(p) \). Moreover, with our notion of Gauss map, an horosphere oriented by the mean curvature vector \((\kappa_i = 1)\) has injective Gauss map.

Now, we give a notion of weak horospherical convexity, using the definition in \([7, 8]\) (notice that the orientation is different from that in \([15]\)).

Definition 2.3. \([8]\) Let \( f : M^n \rightarrow \mathbb{H}^{n+1} \) be an immersed, oriented hypersurface in \( \mathbb{H}^{n+1} \) with unit normal vector field \( \eta \). Let \( \mathcal{H}_p \) denote the horosphere in \( \mathbb{H}^{n+1} \) that is tangent to the hypersurface at \( f(p) \) and whose mean curvature vector at \( f(p) \) coincides with \(-\eta(p)\). We will say that \( f : M^n \rightarrow \mathbb{H}^{n+1} \) is weakly horospherically convex at \( p \) if there exists a neighborhood \( V \subset M^n \) of \( p \) so that \( f(V \setminus \{p\}) \) does not intersect with \( \mathcal{H}_p \). Moreover, the distance function of the hypersurface \( f : M^n \rightarrow \mathbb{H}^{n+1} \) to the horosphere \( \mathcal{H}_p \) does not vanish up to the second order at \( f(p) \) in any direction.

As we say at the beginning, the formal definition of weakly horospherically convex at a point \( p \) implies that all the principal curvatures of the hypersurface at \( p \) are simultaneously \( < -1 \) or \( > -1 \). By choosing the orientation, we may assume that all the principal curvatures of a weakly horospherically convex hypersurface are \( > -1 \). We say that a hypersurface is uniformly weakly horospherically convex if all the principal curvatures \( \kappa_i \) are uniformly larger than \(-1\), i.e. \( \kappa_i \geq c_0 > -1 \).

It is clear that the Gauss map of a weakly horospherically convex hypersurface is a local diffeomorphism, therefore, such hypersurface can be parametrized by a subset of \( \Omega \subset S^n \).
Now, let \( f : \Omega \subset \mathbb{S}^n \to \mathbb{H}^{n+1} \) be a properly immersed, complete, and uniformly weakly horospherically convex hypersurface.

The (past) normal geodesic flow \( \{f^t\}_{t \in \mathbb{R}} \) in \( \mathbb{H}^{n+1} \) of \( f \) is given by

\[
 f^t(x) := \exp_{f(x)}(-t\eta(x)) = f(x) \cosh t - \eta(x) \sinh t : \Omega \to \mathbb{H}^{n+1} \subset \mathbb{R}^{1,n+1}
\]

It is well known [7, 8, 15] that the principal curvatures \( \kappa_i^t \) of \( f^t \) are given by

\[
 \kappa_i^t = \frac{\kappa_i + \tanh t}{1 + \kappa_i \tanh t}
\]

Moreover, it is easily seen that the hyperbolic Gauss map \( G^t \) is invariant under the normal geodesic flow.

2.5. Admissible Weingarten hypersurfaces. We will briefly introduce the elliptic problem of Weingarten hypersurfaces [7] in our context, that is, restricted to weakly horospherically convex hypersurfaces with the orientation under which all the principal curvatures are simultaneously larger than \(-1\).

Let \( \mathcal{W}(x_1, \ldots, x_n) \) be a symmetric function of \( n \)-variables, such that \( \mathcal{W}(\kappa_0, \ldots, \kappa_0) = 0 \) for some number \( \kappa_0 > -1 \). Moreover, let

\[
 \mathcal{K} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i > -1, i = 1, \ldots, n\},
\]

Let \( \Gamma_+ = \{(x_1, \ldots, x_n) : x_1 > 0, \ldots, x_n > 0\} \) and \( \Gamma^* \) be an open connected component of \( \{(x_1, \ldots, x_n) : \mathcal{W}(x_1, \ldots, x_n) > 0\} \) satisfying

1. \( (\kappa, \ldots, \kappa) \in \Gamma^* \cap \mathcal{K} \), for every \( \kappa \in (\kappa_0, \infty) \),
2. For every \( (x_1, \ldots, x_n) \in \Gamma^* \cap \mathcal{K} \), and \( (y_1, \ldots, y_n) \in \Gamma^* \cap \mathcal{K} \cap ((x_1, \ldots, x_n) + \Gamma_+) \), there exists a curve \( \gamma \) connecting \( (x_1, \ldots, x_n) \) to \( (y_1, \ldots, y_n) \) inside \( \Gamma^* \cap \mathcal{K} \) such that \( \gamma' \in \Gamma_+ \) along \( \gamma \)
3. \( \mathcal{W} \in C^1(\Gamma^*) \) and \( \frac{\partial \mathcal{W}}{\partial x_i} > 0 \) in \( \Gamma^* \).

Suppose \( \Sigma \) is a hypersurface of \( \mathbb{H}^{n+1} \) satisfying the following general Weingarten equation

\[
 \mathcal{W}(\kappa_1, \ldots, \kappa_n) = K \quad \text{and} \quad (\kappa_1, \ldots, \kappa_n) \in \Gamma^* \cap \mathcal{K} \text{ on } \Sigma,
\]

for some positive constant \( K \), where \( (\kappa_1, \ldots, \kappa_n) \) are the principal curvatures of the \( \Sigma \).

**Definition 2.4.** In (2.10), a positive number \( K \) is admissible for a given curvature function \( \mathcal{W} \) if \( \mathcal{W}(\bar{\kappa}_0, \ldots, \bar{\kappa}_0) = K \), \( \frac{\partial \mathcal{W}}{\partial x_i}(\bar{\kappa}_0, \ldots, \bar{\kappa}_0) > 0 \), and \( \bar{\kappa}_0 > \kappa_0 \).

A hypersurface \( \Sigma \) such that (2.10) is satisfied for an admissible constant is called admissible Weingarten hypersurface. If the principal curvatures have a uniform lower bound which is strictly bigger than \(-1\), then we call it uniformly admissible Weingarten hypersurface. In particular if \( \mathcal{W} \) is an elementary symmetric function of the principal curvatures, all the assumptions are satisfied. Hence a weakly horospherically convex, \( r \)-admissible \( H_r \)-hypersurface is an admissible Weingarten hypersurface. We will always chose \( K \) such that

\[
 \mathcal{W}(\bar{\kappa}_0, \ldots, \bar{\kappa}_0) = K, \quad \frac{\partial \mathcal{W}}{\partial x_i}(\bar{\kappa}_0, \ldots, \bar{\kappa}_0) > 0
\]

for some \( \bar{\kappa}_0 > \kappa_0 \).
3. Bernstein theorem for embedded hypersurfaces

In this section, we extend the Bernstein type theorem by Do Carmo and Lawson in [12] to hypersurfaces with constant $r$-mean curvature in the hyperbolic space. Recall that $\partial_\infty \mathbb{H}^{n+1}$ has a natural conformal structure of a sphere $S^n(\infty)$. When the asymptotic boundary of a hypersurface $\Sigma$ is a sphere in $S^n(\infty)$, we can assume that it is an equator. We say that $\Sigma$ separates poles if the north and the south poles with respect to such equator are in distinct connected components of $\mathbb{H}^{n+1} \cup S^n(\infty) \setminus (\Sigma \cup \partial_\infty \Sigma)$.

**Theorem 3.1.** Let $\Sigma$ be a complete hypersurface properly embedded in hyperbolic space $\mathbb{H}^{n+1}$ with constant $r$-mean curvature ($r \geq 2$). Denote by $\partial_\infty \Sigma \subset S^n(\infty)$ the asymptotic boundary of $\Sigma$. Then we have the following:

1. if $\partial_\infty \Sigma$ is a point, then $\Sigma$ is a horosphere;
2. if $\partial_\infty \Sigma$ is a sphere and $\Sigma$ separates poles, then $\Sigma$ is an equidistant sphere.

**Proof.** (1) Suppose the asymptotic boundary of $\Sigma$ is only one point $q_\infty \in S^n(\infty)$. First, inspired by [21], we prove that $\Sigma$ has a strictly convex point. We consider the half-space model for $\mathbb{H}^{n+1}$ so that $q_\infty$ corresponds to the infinity point. In this model, the horospheres whose asymptotic boundary is $q_\infty$ are given by the equations $x_{n+1} = \text{constant}$. We write the coordinates as $(\bar{x}, x_{n+1})$, where $\bar{x} = (x_1, \ldots, x_n)$. Then the geodesics orthogonal to the horospheres with $q_\infty$ as asymptotic point, are one-to-one correspondence with the horospheres $\bar{x} \in \mathbb{R}^n$ and can be written as $\gamma_{\bar{x}}(s) = (\bar{x}, s)$ for $s > 0$. Each such geodesic $\gamma = \gamma_{\bar{x}}$ determines a family of hyperplanes orthogonal to $\gamma$, $h_t(\bar{x}) = \{x \in \mathbb{R}^{n+1} : \|x - (\bar{x}, 0)\| = t\}$, where $\|\cdot\|$ denotes the standard Euclidean norm. Let $E_t$ be a family of equidistant spheres such that for any $t$, the asymptotic boundary is $\partial_\infty E_t = \{(\bar{x}, 0) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 = t^2\}$ and that the mean curvature vector at the highest point, points upward. Notice that every principal curvature of $E_t$ is equal to a constant $0 < H_0 < 1$ at any point. For $t$ small, $E_t \cap \Sigma = \emptyset$. Then increase $t$ till the first $t$ such that $E_t \cap \Sigma$ contains a point $p$. Notice that, in a neighborhood of $p$, the hypersurface $\Sigma$ lies above $E_t$. For any tangent vector $X$ at $p$ (tangent to $\Sigma$ and $E_t$), consider the 2-plane $P_X$ generated by $X$ and the $x_{n+1}$-axis. In a neighborhood of $p$, $P_X \cap \Sigma$ is a regular curve, that lies above the regular curve $P_X \cap E_t$ that has curvature $H_0$. Hence the curvature of $P_X \cap \Sigma$ is larger or equal to $H_0$. Then, $p$ is a strictly convex point of $\Sigma$.

Since $\partial_\infty \Sigma = \{q_\infty\}$, if $t$ small enough, we have

$$h_t \cap \Sigma = \emptyset$$

For any $t > 0$, we denote by $H_{+}^{n+1}(t)$ and $H_{-}^{n+1}(t)$ the half-spaces determined by $h_t = h_t(\bar{x})$. We set

$$\Sigma_{\pm}(t) = \Sigma \cap H^{n+1}(t)$$

and $\Sigma_{\pm}(t) = \emptyset$ for any $\bar{x}$ and $t$ sufficiently small.

Note also that $\Sigma$ separates $\mathbb{H}^{n+1}$ into two connected components $\Omega_{+}$ and $\Omega_{-}$ where $\partial_\infty \Omega_{+} = \{q_\infty\}$ and $\partial_\infty \Omega_{-} \cong \{\mathbb{R}^n \times \{0\}\}$.

Let $t_0$ be the smallest $t$ for which $h_t(\bar{x}) \cap \Sigma \neq \emptyset$. Then for all $t > t_0$ such that $t - t_0$ sufficiently small, consider the reflected hypersurfaces $\Sigma_{-}(t) = r_{h_t}(\Sigma_{-}(t))$, where $r_{h_t}$ is the hyperbolic isometry fixing $h_t(\bar{x})$.

The hypersurfaces $\Sigma_{-}(t)$ have the following properties:

$$\Sigma_{-}(t) \subset \Omega_{+},$$

Equation (3.1)
\[ \Sigma'_-(t) - \partial \Sigma'_-(t) \subset \text{int}(\Omega_+). \]

Let us now suppose that condition (3.1) does not hold for some large \( t \). Then there must be some \( \bar{t} > t_0 \) such that the surfaces \( \Sigma'_-(\bar{t}) \) and \( \Sigma'_+(\bar{t}) \) have a point \( p \) of common tangency (possibly at the boundary), and that \( \Sigma'_-(\bar{t}) \) lies above \( \Sigma'_+(\bar{t}) \) in a neighborhood of \( p \). By the tangency principle in [17, Theorem 1.1], we conclude that these hypersurfaces coincide. From this, it easily follows that \( \Sigma \) is compact, which is a contradiction. Thus we conclude that (3.1) holds for all \( t \) and all \( \bar{x} \in \mathbb{R}^n \).

Relation (3.1) yields that at any of its points, \( \Sigma \) is tangent to the (horizontal) horosphere passing through the point. If this was not the case, then it is easily seen that there is a geodesic hyperplane \( h_t(\bar{x}) \) (for an appropriate choice of \( \bar{x} \) and \( t \)) such that \( \Sigma'_-(t) \nsubseteq \Omega_+ \) and this contradicts (3.1). It follows that the function \( x_{n+1} \) must be constant on \( \Sigma \) and \( \Sigma \) is a horosphere.

(2) First we prove that \( \Sigma \) has a strictly convex point. By the embeddedness, \( \Sigma \) divides the hyperbolic space into two connected components. Also, \( \partial_{\infty} \Sigma \) separates \( S^n(\infty) \) into two components: \( S^n_+ \cup S^n_- = S^n(\infty) - \partial_{\infty} \Sigma \). Denote by \( \nu \) the unit normal orienting \( \Sigma \). We first prove that \( \Sigma \) has a strictly convex point, unless \( \Sigma \) is a hyperplane.

Let \( N_\varepsilon(t) \) \( t \in [0, 1] \), be a family of equidistant spheres with the following properties.

1. The mean curvature vector of \( N_\varepsilon(t) \) points upward for any \( t \in [0, 1] \).
2. The angle between \( N_\varepsilon(0) \) and \( \{x_{n+1} = 0\} \), is \( \frac{\pi}{2} + \varepsilon \) with \( \varepsilon > 0 \) small and \( \Sigma \) is contained in the mean-convex side of \( N_\varepsilon(0) \).
3. \( N_\varepsilon(t) \) is obtained from \( N_\varepsilon(0) \) by a homothety from the euclidean center of \( N_\varepsilon(0) \). By construction, the angle \( \theta_\varepsilon(t) \) between \( N_\varepsilon(t) \) and \( \{x_{n+1} = 0\} \) satisfies \( \theta_\varepsilon(t) > \frac{\pi}{2} \) hence the mean curvature vector of \( N_\varepsilon(t) \) points towards \( \Sigma \).
4. \( \partial_{\infty} N_\varepsilon(1) = \partial_{\infty} \Sigma \).

Increasing \( t \), there exists a first \( \bar{t} < 1 \) such that \( N_\varepsilon(\bar{t}) \) and \( \Sigma \) has a contact point \( p \), then \( p \) is a strictly convex point. If such point does not exists, then \( \Sigma \) lies above \( N_\varepsilon(1) \). Notice that, letting \( \varepsilon \to 0 \), \( N_\varepsilon(1) \) tends to the hyperplane whose asymptotic boundary coincides with \( \partial \Sigma \). Hence \( \Sigma \) lies above such hyperplane.

Let \( S_\varepsilon(t) \) \( t \in [0, 1] \), be a family of equidistant spheres with the following properties.

1. The mean curvature vector of \( S_\varepsilon(t) \), at the highest point, points downward for any \( t \in [0, 1] \).
2. The angle between \( \partial_{\infty} S_\varepsilon(0) \) and \( \{x_{n+1} = 0\} \), is \( \frac{\pi}{2} - \varepsilon \) with \( \varepsilon > 0 \) small and \( \Sigma \) is contained in the mean-convex side of \( S_\varepsilon(0) \).
3. \( S_\varepsilon(t) \) is obtained from \( S_\varepsilon(0) \) by a homothety from the euclidean center of \( S_\varepsilon(0) \). By construction, the angle \( \theta_\varepsilon(t) \) between \( S_\varepsilon(t) \) and \( \{x_{n+1} = 0\} \) satisfies \( \theta_\varepsilon(t) < \frac{\pi}{2} \) hence the mean curvature vector of \( S_\varepsilon(t) \) points towards \( \Sigma \).
4. \( \partial_{\infty} S_\varepsilon(1) = \partial \Sigma \).

Increasing \( t \), there exists a first \( \bar{t} < 1 \) such that \( S_\varepsilon(\bar{t}) \) and \( \Sigma \) has a contact point \( p \), then \( p \) is a strictly convex point. If such point does not exists, then \( \Sigma \) lies below \( S_\varepsilon(1) \). Notice that, letting \( \varepsilon \to 0 \), \( S_\varepsilon(1) \) tends to the hyperplane whose asymptotic boundary coincides with \( \partial \Sigma \). Hence \( \Sigma \) lies below such hyperplane.

We conclude that either there is a strictly convex point on \( \Sigma \) or \( \Sigma \) is a hyperplane.

Hence we may assume that there is a strictly convex point. By the argument after (2.4), \( H_1 \) is positive on \( \Sigma \). Moreover, since \( \Sigma \) separates poles, we can select \( S^n_\varepsilon \) as asymptotic boundary of the region into which the mean curvature vector of \( \Sigma \) points.
Now let us use the half-space model. We take the center of $S^+_n$ to be the origin of the half-space model and then $S^n_-$ is the component which is unbounded in the Euclidean topology.

First we prove that the $r$-mean curvature of $\Sigma$ satisfies $H_r < 1$. Fix a point $x \in S^n_-$ and consider the family of horospheres having $x$ as asymptotic point. There is a horosphere first touches $\Sigma$. At this contact point, the horosphere and $\Sigma$ are tangent. Moreover, with respect to that normal vector, $\Sigma$ is below the horosphere. Therefore $r$-mean curvature $H_r$ of $\Sigma$ is strictly less than 1 by the tangency principle in [17].

Let $\mathcal{E}$ be the equidistant sphere of $r$-mean curvature $H_r$, such that $\partial_\infty \mathcal{E} = \partial_\infty \Sigma$ and the mean curvature vector of $\mathcal{E}$ also points to $S^n_+$. By applying to $\mathcal{E}$ the isometries of $\mathbb{H}^{n+1}$ given by homotheties with respect to the center of $S^n_+$, we get a foliation of $\mathbb{H}^{n+1}$ consisting of equidistant spheres, denoted by $\mathcal{E}_t$, $t \in \mathbb{R}$. Choose the parameter $t$ such that $\mathcal{E}_0 = \mathcal{E}$ and $\mathcal{E}_t$ goes to the origin as $t \to -\infty$ (to the infinity point as $t \to +\infty$). Since $\partial_\infty \mathcal{E} = \partial_\infty \Sigma$, we have that $\Sigma \cap \mathcal{E}_t$ is compact for all $t \neq 0$ and $\Sigma \cap \mathcal{E}_t = \emptyset$ for all $|t|$ sufficiently large. If $\Sigma \neq \mathcal{E}$, then $\Sigma \cap \mathcal{E}_t \neq \emptyset$ for some $t \neq 0$. Suppose $t > 0$, let $t_1 = \sup\{t: \Sigma \cap \mathcal{E}_t \neq \emptyset\}$. Then $\Sigma$ is below $\mathcal{E}_{t_1}$ with respect to $\nu$ near the contact point. Thus, by the tangency principle in [17], $\Sigma = \mathcal{E}_{t_1}$, which contradicts $\partial_\infty \mathcal{E} = \partial_\infty \Sigma$. The case of $t < 0$ is similar. Hence, $\Sigma = \mathcal{E}$.

\[\square\]

4. Bernstein theorem for immersed hypersurface

In this section, we consider the Bernstein theorem for immersed hypersurfaces, either with constant $r$-mean curvature, or satisfying a general elliptic Weingarten equation.

4.1. The case of constant $r$-mean curvature hypersurface contained in a slab. In this section, we consider the Bernstein theorem for immersed hypersurfaces with constant $r$-mean curvature contained in a slab. The starting point is the following non-existence theorem.

**Theorem 4.1.** In any slab of hyperbolic space $\mathbb{H}^{n+1}$, there is no complete properly immersed hypersurface $\Sigma$ with $r$-mean curvature satisfying $H := \sup_{\Sigma} |H_r| < 1$ for any $r \geq 1$.

**Proof.** In the upper half-space model of hyperbolic space $\mathbb{H}^{n+1}$, we assume that the slab is between two horospheres given by two horizontal Euclidean hyperplanes. We can foliate the whole space by a family of equidistant spheres $\mathcal{E}(t)$ with mean curvature being $H_\frac{1}{2}$ with respect to the normal vector field that points upward at the highest point, for $t \in [0, \infty)$. When $t$ is small, $\mathcal{E}(t)$ and $\Sigma$ are disjoint. Then, consider $t_0$ such that $\mathcal{E}(t_0)$ and the hypersurface $\Sigma$ first touch at some point $p$. In a neighborhood of $p$, $\Sigma$ is above $\mathcal{E}(t_0)$ with respect to the upward normal vector of $\mathcal{E}(t_0)$ at $p$. We also have $H_r(\mathcal{E}(t_0)) \geq H_r(\Sigma)$ in that neighborhood and $\mathcal{E}(t_0)$ is $r$-admissible. Therefore, by the tangency principle [17, Theorem 1.1], $\Sigma = \mathcal{E}(t_0)$, which is a contradiction, because $\mathcal{E}(t_0)$ is not contained in any slab. \[\square\]

As a consequence of the previous Theorem, we are able to prove the analogous of [1, Theorem 1] for surfaces of constant Gaussian curvature.

**Corollary 4.2.** If $\Sigma$ is a properly immersed complete surface in $\mathbb{H}^3$ with constant 2-mean curvature $0 < H_2 \leq 1$ contained in a slab then $\Sigma$ is a horosphere.

**Proof.** By Theorem [4.1] we have $H_2 = 1$, which implies that $\Sigma$ is a complete flat immersion. Then the result follows from [18, Theorem 5]. \[\square\]
In higher dimension, we need to add $L_k$-parabolicity (see Definition 2.2) and a geometric assumption, in order to get a Bernstein type Theorem.

**Theorem 4.3.** Let $\Sigma$ be a complete, $r$-admissible, $L_{r-1}$-parabolic properly immersed hypersurface with constant $r$-mean curvature. If $\Sigma$ is contained in a slab and the angle function does not change sign, then $\Sigma$ is a horosphere.

**Proof.** Define $\phi = e^h H^{1/r}_r + e^h \Theta$ where $\Theta$ is the angle function and $h$ is the height, as we defined in Section 2.1. It follows from the proof of Theorem 32 in [3] that

\[
L_{r-1}\phi \geq c_k e^h (H^{1/r}_r (H_{r-1} - H^{r+1}_r))
- \left(\frac{n}{k}\right) e^h \Theta(n H_1 H_r - (n - r) H_{r+1} - r H^{r+1}_r) \geq 0,
\]

where we have used the Garding inequality (2.5) in the last inequality. Since $\Sigma$ is $L_{r-1}$-parabolic, $\phi$ is a constant. In particular, $\Delta \phi = 0$. Then, equation (3.8) in [2] gives that

\[
0 = \Delta \phi = e^h \Theta(||A||^2 - (n - 1)H^2)
\]

Notice that, in the notation of (3.8) in [2], $\rho(t) = e^t$, $H(t) = 1$ and $Ric_P(\hat{N}) = 0$ as $P = \mathbb{R}^n$.

We conclude that $||A||^2 = (n - 1)H^2$, that yields $\Sigma$ is a totally umbilical hypersurface. Thus, all the principal curvatures are equal to a constant. Moreover, since the hypersurface is contained in a slab, then all the principal curvatures are larger or equal than 1 by Theorem 4.1. Then it is either a horosphere or sphere by [10, Theorems A,B]. However, in the latter case, the angle function changes sign. Therefore, it has to be a horosphere.

\[\Box\]

### 4.2. The case of admissible Weingarten hypersurfaces.

In this section, we get a Bernstein type theorem for admissible Weingarten hypersurfaces. Note that uniformly weakly horospherically convex hypersurfaces with injective hyperbolic Gauss map become embedded under the (past) normal geodesic flow (see Theorem 1.3 in [8]).

**Theorem 4.4.** Suppose that $\Sigma$ is an immersed, complete, uniformly weakly horospherically convex admissible Weingarten hypersurface in $\mathbb{H}^{n+1}$. Then $\Sigma$ is a horosphere provided its asymptotic boundary is a single point.

**Proof.** As the Gauss map of $\Sigma$ is locally injective, we can apply Theorem 4.2 in [8]. Then, for $t$ large enough, the past normal geodesic flow defined in [2],[3], deforms $\Sigma$ into a properly embedded, uniformly weakly horospherically convex hypersurface $\Sigma_t$ with single point boundary at infinity. Moreover, the principal curvatures of $\Sigma_t$ are given by (see (2.9)):

\[
\kappa_i^t = \frac{\kappa_i + \tanh(t)}{1 + \kappa_i \tanh(t)} \quad \text{and} \quad \kappa_i = \frac{\kappa_i - \tanh(t)}{1 - \kappa_i \tanh(t)}
\]

Let

\[
W^t(x_1, \ldots, x_n) := W\left(\frac{x_1 - \tanh(t)}{1 - x_1 \tanh(t)}, \ldots, \frac{x_n - \tanh(t)}{1 - x_n \tanh(t)}\right).
\]

Then it follows from the definition of $W$ that $W^t$ is a symmetric function of $n$-variables with

\[
W^t\left(\frac{\kappa_0 + \tanh(t)}{1 + \kappa_0 \tanh(t)}, \ldots, \frac{\kappa_0 + \tanh(t)}{1 + \kappa_0 \tanh(t)}\right) = W(\kappa_0, \ldots, \kappa_0) = 0
\]
and \( \frac{\kappa_0 + \tanh(t)}{1 + \kappa_0 \tanh(t)} > -1 \).

Define

\[
(4.5) \quad \mathcal{T}(x_1, \ldots, x_n) = \left( \frac{x_1 + \tanh(t)}{1 + x_1 \tanh(t)}, \ldots, \frac{x_n + \tanh(t)}{1 + x_n \tanh(t)} \right).
\]

We then have

\[
(4.6) \quad \Gamma^*_t \cap \mathcal{K} = \mathcal{T}(\Gamma^*_t \cap \mathcal{K})
\]

and

\[
(4.7) \quad \mathcal{T}((x_1, \ldots, x_n) + \Gamma_n) = \mathcal{T}(x_1, \ldots, x_n) + \Gamma_n.
\]

For ellipticity, one can easily compute

\[
(4.8) \quad \frac{\partial \mathcal{W}}{\partial x_i} = \frac{1 - \tanh^2(t)}{(1 - x_i \tanh(t))^2} \frac{\partial \mathcal{W}}{\partial y_i}.
\]

Therefore, \((\mathcal{W}_t, \Gamma^*_t)\) satisfies (1)-(3). Thus, [7, Theorem 4.4] yields that \(\Sigma_t\) is a horosphere and so is \(\Sigma\), since \(\Sigma\) is a time-slice of the foliation formed by a horosphere under the normal geodesic flow.

\[\square\]

**Corollary 4.5.** Suppose that \(\Sigma\) is an immersed, complete, uniformly weakly horospherically convex, \(r\)-admissible \(H_r\)-hypersurface in \(\mathbb{H}^{n+1}\). Then \(\Sigma\) is a horosphere provided its asymptotic boundary is a single point.

**5. \(r\)-MEAN CURVATURE RIGIDITY OF HOROSPERES AND EQUIDISTANT SPHERES**

Motivated by the following result of M. Gromov [20]:

A hyperplane in a Euclidean space \(\mathbb{R}^n\) cannot be perturbed on a compact set so that its mean curvature satisfies \(H \geq 0\).

R. Souam proved the following extension to hyperbolic space [25]:

Let \(M\) denote a horosphere, an equidistant sphere or a hyperplane in a hyperbolic space \(\mathbb{H}^{n+1}\), \(n \geq 2\) and \(H_M \geq 0\) its constant mean curvature. Let \(\Sigma\) be a connected properly embedded \(C^2\)-hypersurface in \(\mathbb{H}^{n+1}\) which coincides with \(M\) outside a compact subset of \(\mathbb{H}^{n+1}\). If the mean curvature of \(\Sigma\) is \(\geq H_M\), then \(\Sigma = M\).

The proof in [25] is based on the tangency principle for mean curvature. We are able to extend Souam’s result to the case of \(r\)-mean curvature by the tangency principle in [17].

**Theorem 5.1.** Let \(M\) be a horosphere or an equidistant sphere in hyperbolic space \(\mathbb{H}^{n+1}\), \(n \geq 2\) and denote by \(H_M > 0\) its \(r\)-mean curvature, \(r \geq 1\), with respect to the orientation given by the mean curvature vector. Let \(\Sigma\) be a connected properly embedded \(C^2\) hypersurface in \(\mathbb{H}^{n+1}\) which coincides with \(M\) outside a compact subset \(B\) in \(\mathbb{H}^{n+1}\). Choose the orientation on \(\Sigma\) such that the \(r\)-mean curvature \(H_r\) of \(\Sigma\) is equal to \(H_M\) outside the compact set \(B\). With respect to this orientation, if either \(H_r \geq H_M\) or \(|H_r| \leq H_M\), then \(\Sigma \equiv M\).
Proof. We take the upper half-space model of hyperbolic space.

(1) The case of horospheres. Consider the family of horospheres \( \mathcal{O}_t = \{ x \in \mathbb{R}^{n+1} | x^{n+1} = t \} \), \( t > 0 \) and assume that \( M = \mathcal{O}_1 \). Notice that, \( H_M = 1 \).

Let \( \Sigma \) be a connected properly embedded \( C^2 \) hypersurface in \( \mathbb{H}^{n+1} \) with \( r \)-mean curvature either \( H_r \geq 1 \) or \( H_r \leq 1 \).

Now, assume that \( H_r \geq 1 \).

Since \( \Sigma \) coincides with \( M \) outside a compact subset \( B \) of \( \mathbb{H}^{n+1} \), the mean convex side of \( \Sigma \), that is the component where the mean curvature vector points towards, coincides with the domain \( \{ x \in \mathbb{R}^{n+1} | x^{n+1} > 1 \} \), outside a compact set.

We consider the largest \( T \geq 1 \) such that \( \Sigma \cap \mathcal{O}_T \neq \emptyset \) and let \( p \in \Sigma \cap \mathcal{O}_T \). At the point \( p \), \( \Sigma \) and \( \mathcal{O}_T \) are tangent, in a neighborhood of \( p \), the horosphere \( \mathcal{O}_T \) lies above \( \Sigma \), while the \( r \)-mean curvature of \( \Sigma \) is larger or equal than the \( r \)-mean curvature of \( \mathcal{O}_T \) (with respect to the upward normal vector). By the tangency principle \([17, \text{Theorem } 1.1]\), \( \Sigma \) coincide with \( \mathcal{O}_T \) in a open neighborhood of \( p \). Hence the subset \( \Sigma \cap \mathcal{O}_T \) is open. As it is also closed, we get that \( \Sigma \) coincides with \( \mathcal{O}_T \) and \( T = 1 \).

Now, assume that \( H_r \leq 1 \). We consider the smallest \( \tau \leq 1 \) such that \( \Sigma \cap \mathcal{O}_\tau \neq \emptyset \) and let \( p \in \Sigma \cap \mathcal{O}_\tau \). At the point \( p \), \( \Sigma \) and \( \mathcal{O}_\tau \) are tangent, in a neighborhood of \( p \), the horosphere \( \mathcal{O}_\tau \) lies below \( \Sigma \), while the \( r \)-mean curvature of \( \Sigma \) is less than or equal to the \( r \)-mean curvature of \( \mathcal{O}_\tau \) (with respect to the upward normal vector). Notice that, we do not know in advance whether the normal vector to \( \Sigma \) at \( p \) points upward or downward. However, by the assumption \(|H_r| \leq H_M\), one can check that the \( r \)-mean curvature of \( \Sigma \) with respect to the upward normal is always less than or equal to \( H_M \). Then it follows from the tangency principle \([17, \text{Theorem } 1.1]\) that \( \Sigma \) coincide with \( \mathcal{O}_\tau \) in a open neighborhood of \( p \). Hence the subset \( \Sigma \cap \mathcal{O}_\tau \) is open. As it is also closed, we get that \( \Sigma \) coincides with \( \mathcal{O}_\tau \) and \( \tau = 1 \). Notice that, in spite of the fact that \( \Sigma \) may have non positive \( r \)-mean curvature, we can apply \([17, \text{Theorem } 1.1]\) because the principal curvature vector of \( \mathcal{O}_\tau \) lies in the positive cone.

(2) The case of equidistant spheres.

We may assume that the mean curvature vector of \( M \) points upward and that the asymptotic boundary of \( M \) is a \((n-1)\)-sphere of \( \partial_\infty \mathbb{H}^{n+1} \) centered at the origin of \( \mathbb{R}^{n+1} \). As in the previous case, since \( \Sigma \) coincides with \( M \) outside a compact subset \( B \) of \( \mathbb{H}^{n+1} \), the mean convex side of \( \Sigma \), coincides with the mean convex side of \( M \) outside a compact set. Let \( \mathcal{E}(t) \) be a foliation in equidistant spheres, obtained rescaling \( M \), with respect to the origin, such that \( \mathcal{E}(0) = M \) and \( \mathcal{E}(t) \) is above \( M \) for positive \( t \) and below \( M \) for negative \( t \). Notice that, all the \( \mathcal{E}(t) \) has the same \( r \)-mean curvature. Similar as in (1), when \( H_r \geq H_M \), \( \Sigma \) touches \( \mathcal{E}(t), t \geq 0 \) from below. When \(|H_r| \leq H_M \), \( \Sigma \) touches \( \mathcal{E}(t), t \leq 0 \) from above. In both cases, by tangency principle, \( \Sigma = M \).

\[ \square \]

References

1. Luis J. Alías and Marcos Dajczer, *Uniqueness of constant mean curvature surfaces properly immersed in a slab*, Commentarii Mathematici Helvetici 81 (2006), no. 3, 653–663.
2. ________, *Constant mean curvature hypersurfaces in warped product spaces*, Proceedings of the Edinburgh Mathematical Society 50 (2007), no. 3, 511–526.
3. Luis J. Alías, Debora Impera, and Marco Rigoli, *Hypersurfaces of constant higher order mean curvature in warped products*, Transactions of the American Mathematical Society 365 (2013), no. 2, 591–621.
4. Frederick J. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of Bernstein’s theorem*, Annals of Mathematics (1966), 277–292.
5. João Lucas Marques Barbosa and Antônio Gervasio Colares, *Stability of hypersurfaces with constant r-mean curvature*, Annals of Global Analysis and Geometry 15 (1997), no. 3, 277–297.
6. Enrico Bombieri, Ennio De Giorgi, and Enrico Giusti, *Minimal cones and the Bernstein problem*, Inventiones Mathematicae 7 (1969), 243–268.
7. Vincent Bonini, José M. Espinar, and Jie Qing, *Hypersurfaces in hyperbolic space with support function*, Advances in Mathematics 280 (2015), 506–548.
8. Vincent Bonini, Jie Qing, and Jingyong Zhu, *Weakly horospherically convex hypersurfaces in hyperbolic space*, Annals of Global Analysis and Geometry 52 (2017), no. 2, 201–212.
9. Robert Bryant, *Surfaces of mean curvature one in hyperbolic space*, Astérisque 154 (1987), no. 155, 321–347.
10. Robert J. Currier, *On hypersurfaces of hyperbolic space infinitesimally supported by horospheres*, Transactions of the American Mathematical Society 313 (1989), no. 1, 419–431.
11. Ennio De Giorgi, *An extension of Bernstein’s theorem*, Annals of the Scuola Normale Superiore of Pisa-Class of Sciences 19 (1965), no. 1, 79–85.
12. Manfredo P. Do Carmo and H. Blaine Lawson, *On Alexandrov-Bernstein theorems in hyperbolic space*, Duke Math. J. 50 (1983), no. 4, 995–1003.
13. Maria Fernanda Elbert, *Constant positive 2-mean curvature hypersurfaces*, Illinois J. of Math. 1 (2002), no. 46, 247–267.
14. Maria Fernanda Elbert and Barbara Nelli, *A note on the stability for constant higher mean curvature hypersurfaces in a Riemannian manifold*, arXiv preprint arXiv:2012.12103 (2019).
15. José M. Espinar, José A. Gálvez, and Pablo Mira, *Hypersurfaces in \( \mathbb{R}^{n+1} \) and conformally invariant equations: the generalized Christoffel and Nirenberg problems*, Journal of the European Mathematical Society 11 (2009), no. 4, 903–939.
16. Wendell H. Fleming, *On the oriented Plateau problem*, Rendiconti del Circolo Matematico di Palermo 11 (1962), no. 1, 69–90.
17. Francisco Fontenele and Sérgio L. Silva, *A tangency principle and applications*, Illinois Journal of Mathematics 45 (2001), no. 1, 213–228.
18. José A. Gálvez, *Surfaces of constant curvature in 3-dimensional space forms*, Mat. Contemp 37 (2009), 1–42.
19. Lars Gårding, *An inequality for hyperbolic polynomials*, Journal of Mathematics and Mechanics (1959), 957–965.
20. Misha Gromov, *Mean curvature in the light of scalar curvature*, arXiv preprint arXiv:1812.09731 (2018).
21. Barbara Nelli and Harold Rosenberg, *Some remarks on embedded hypersurfaces in hyperbolic space of constant curvature and spherical boundary*, Annals of Global Analysis and Geometry 13 (1995), no. 1, 23–30.
22. Robert C. Reilly, *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, Journal of Differential Geometry 8 (1973), no. 3, 465–477.
23. Richard Schoen, Leon Simon, and Shing-Tung Yau, *Curvature estimates for minimal hypersurfaces*, Acta Mathematica 134 (1975), no. 1, 275–288.
24. James Simons, *Minimal varieties in Riemannian manifolds*, Annals of Mathematics (1968), 62–105.
25. Rabah Souam, *Mean curvature rigidity of horospheres, hyperspheres and hyperplanes*, arXiv preprint arXiv:1912.02669 (2019).

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