We compute the contribution of the decays $K_L \to \pi^0\bar{Q}Q$ and $K^+ \to \pi^+\bar{Q}Q$, where $Q$ is a dark fermion of the dark sector, to the measured widths for the rare decays $K^+ \to \pi^+\bar{\nu}\nu$ and $K_L \to \pi^0\nu\bar{\nu}$. After taking into account the bound from $K^0$-$\bar{K}^0$ mixing, they can both be larger than the standard model result. The recent experimental limit for $\Gamma(K^+ \to \pi^+\bar{\nu}\nu)$ from NA62 sets a new and very strict bound on the dark-sector parameters. If we evade the Grossman-Nir bound by having $\Gamma(K^+ \to \pi^+\bar{Q}Q)$ to vanish for kinematical reasons, a branching ratio within the reach of the KOTO sensitivity is possible. A discriminating prediction of this scenario is the small momentum of the neutral pion.

This result implies that whatever new physics is contributing to this decay, it must be of the same order or smaller than the SM. The result in Eq. (3) has become the strongest limit in this flavor sector, stronger than that derived from the mass mixing in the neutral Kaon system.

FIG. 1: Summary of the experimental limits (90% CL) on $K_L \to \pi^0\nu\bar{\nu}$ (KOTO) and $K^+ \to \pi^+\bar{\nu}\nu$ (NA62) as well as the single event sensitivity (SES) of KOTO. Also indicated are the GN bound and the SM predictions. The blue region is excluded (assuming the validity of the GN bound). The vertical double arrow depicts the range of possible values for $K_L \to \pi^0\nu\bar{\nu}$ obtained in the scenario discussed in this letter.
to [12]
\[ \text{BR} (K_L \to \pi^0 \nu \bar{\nu}) = 2.1^{+1.4}_{-1.1} \times 10^{-9} \ 95\% \ CL. \]  
(6)
The central value in Eq. (6) is about two orders of magnitude above the SM prediction which is [9]
\[ \text{BR} (K_L \to \pi^0 \nu \bar{\nu}) = (3.00 \pm 0.31) \times 10^{-11}. \]  
(7)
As shown in Fig. 1, it is still possible that new physics dominates this channel and the current sensitivity of KOTO—falling as it does in the interval between the SM prediction and the exclusion limit in Eq. (5)—could find it. Scenarios giving rise to events in the KOTO SES range are discussed in [12].

Yet there is a catch: the central value in Eq. (6) and Eq. (3), when taken together, violate the GN bound [4]
\[ \text{BR} (K_L \to \pi^0 \nu \bar{\nu}) \leq 4.3 \text{BR} (K^+ \to \pi^+ \nu \bar{\nu}), \]  
(8)
which is only based on isospin symmetry and the difference in the kaon respective lifetimes. For this reason, the very stringent new limit in Eq. (3) on the charged kaon decay seems to imply a comparably stronger limit on new physics in the neutral kaon channel, as depicted in Fig. 1. As anticipated, this bound is bypassed in the dark-sector model by the different kinematics of the two decays arising from the choice in Eq. (1).

A model of the dark sector.—Among the many models for the dark sector [13], we use one made to resemble QED—that is, a theory of charged fermions. It has the advantage of being simple. It contains fermions \( Q_i^c \) and \( Q_i^o \), where the index \( i \) runs over generations like in the SM, and these dark fermions are charged only under a gauge group \( U_o(1) \)—a proxy for more general interactions—with different charges for the \( Q^c \) and \( Q^o \) type. The dark photon is massless and directly only couples to the dark sector by the different kinematics of the two decays arising from the choice in Eq. (1).

The dark fermions couple to the SM fermions by means of a Yukawa-like interactions. The Lagrangian contains terms coupling SM fermions of different generations with the dark fermions. In general the interaction is not diagonal and, for the \( s \) and \( d \) SM quarks relevant for kaon physics, is given by
\[ \mathcal{L} \supset g_{\rho} \rho_{1}^{d} S_{R} Q_{s}^{c} + g_{\rho} \rho_{1}^{d} S_{R} Q_{d}^{c} \]  
(9)
In Eq. (9), the fields \( S_{L} \) and \( S_{R} \) are heavy messenger scalar particles, singlets of the SM \( SU(2) \) gauge group as well as \( SU(3) \) color triplets (color indices are implicit in Eq. (9)). The symmetric matrices \( \rho_{1}^{d} \) are the result of the diagonalization of the mass eigenstates of both the SM and dark fermions; they provide the generation mixing (and the CP-violating phases) necessary to have the messengers play a role in flavor physics. The messenger fields are heavier than the dark fermions and charged under the \( U_o(1) \) gauge interaction, carrying the same charges as the dark fermions.

In order to generate chirality-changing processes we also need in the Lagrangian the mixing terms
\[ \mathcal{L} \supset \lambda S S_{0} \left( S_{L}^{\dagger} \tilde{H} S_{L} + S_{R}^{\dagger} S_{R} H \right), \]  
(10)
where \( H \) is the SM Higgs boson, \( \tilde{H} = i \sigma_{2} H^{*} \), and \( S_{0} \) a scalar singlet. The Lagrangian in Eq. (10) gives rise to the mixing after the scalars \( S_{0} \) and \( H \) take a vacuum expectation value (VEV), respectively, \( \mu_{S} \) and \( v \)—the electroweak VEV. After diagonalization, the messenger fields \( S_{\pm} \) couple both to left- and right-handed SM fermions with strength \( g_{L} / \sqrt{2} \) and \( g_{R} / \sqrt{2} \), respectively. We can assume that the size of this mixing—proportional to the product \( \mu_{S} v \) of the VEVs—is large and of the same order of the masses of the heavy fermions and scalars.

This model (see [6] for more details) has been used to discuss processes with the emission of dark photons in Higgs physics [16], flavor changing neutral currents [17], kaon [18, 19] and \( Z \) boson [20] decays.

Then, SM fermions couple to the dark photon only via non-renormalizable interactions [21] induced by loops of dark-sector particles. The corresponding effective Lagrangian relevant for the rare decays of the Kaons is equal to
\[ \mathcal{L} = \frac{e_{\rho}^{d}}{2 \Lambda} \bar{s} \sigma_{\mu \nu} (\mathcal{D}_{M} + i \gamma_{5} \mathcal{D}_{E}) \rho_{\mu \nu} + H.c. \]  
(11)
where \( B_{\mu \nu} \) is the strength of the dark photon field, \( \Lambda \) the effective scale of the dark sector. The magnetic- and electric-dipole are given by
\[ \mathcal{D}_{M} = \frac{\rho_{\mu \nu} \rho_{\mu \nu}^{s}}{2} \text{ Re} \frac{g_{L} g_{R}}{(4 \pi)^{2}} \]  
and \[ \mathcal{D}_{E} = \frac{\rho_{\mu \nu} \rho_{\mu \nu}^{s}}{2} \text{ Im} \frac{g_{L} g_{R}}{(4 \pi)^{2}} \]  
respectively. For simplicity we take \( g_{L} = g_{R} \) real and \( \mathcal{D}_{E} = 0 \). A CP-violating phase arises from the mixing parameters:
\[ \rho_{sd} \rho_{d}^{*} \rho_{dd}^{*} = 2 i \sin \delta_{CP}. \]  
(13)

Constraint from the Kaon mass difference.—A direct constraint on the parameters of the model arises because the same term driving the meson decay also enters the box diagram that gives rise to the mass difference of the
neutral meson. This quantity is given by
\[
\Delta m_{K^0} = \left[ g^4_L(p_{d4}^L p_{d4}^L + g^4_K(p_{d4}^K p_{d4}^K) \right] f_{K^0}^2 m_{K^0} \quad \text{192\pi}^2,
\]
where we have used the leading vacuum insertion approximation (\(B_{K^0} = 1\)) to estimate the matrix element
\[
\langle K^0|\bar{q}_L(\gamma_{\mu}\gamma_5)q_L|K^0\rangle = \frac{1}{3} m_{K^0} f_{K^0}^2 B_{K^0}\eta_{QCD}
\]
and a similar one for right-handed fields. Since we are just after an order of magnitude estimate, we neglect the running (and contributions from mixing) of the Wilson coefficient \(\eta_{QCD}\) of the 4-fermion operator. Given the long-distance uncertainties, to satisfy the experimental bound on the mass difference, we only impose that the new contribution does not exceed the measured value.

The comparison requires the introduction of the full effective Lagrangian \([23]\) inclusive of the new operators induced by the dark sector. By using the results in \([24]\), we obtain \([18, 19]\)
\[
\frac{D_M^2}{\Lambda^2} \leq \frac{3}{32\pi^2} \frac{\Delta m_{K^0}^{\text{exp}}}{f_{K^0} m_{K^0}} = 2.6 \times 10^{-21} \text{MeV}^{-2}, \quad (14)
\]

In Eq. (15) we have taken for the hadron matrix element
\[
\langle 0|\bar{q} \sigma \mu v d|K^0\rangle = (g_{\pi\pi}^L p_{\pi}^L p_{\pi}^L + g_{\pi\pi}^K p_{\pi}^K p_{\pi}^K) \sqrt{2f_{\pi\pi}^K(q^2)} m_{\pi} + m_{K^0}, \quad (18)
\]
with \(f_K = 159.8 \text{MeV} \) and \(\Delta m_{K^0}^{\text{exp}} = 3.52 \times 10^{-18} \text{MeV}\) \([25]\).

\textbf{The decay width.}— This process is experimentally seen as two photons (from the decay of the pion) plus the missing energy and momentum carried away from by the neutrinos. In the presence of the dark sector, the same signature is provided by \(K \rightarrow \pi \gamma\), where \(\gamma\) is a dark photon, but this decay is forbidden by the conservation of the angular momentum when the dark photon is massless. This means that the decay we are interested in can only proceed if the dark photon is off shell and decays into a pair of dark fermions.

The box diagrams are sub-leading—doubly suppressed by a mass factor \(O(m_{K^0}/\Lambda)\) and an additional factor \(O(g_{\pi\pi}^L g_{\pi\pi}^2/4\pi^4/\alpha_D)\).

Assigning the momenta as \(K(p_K) \rightarrow \pi(p_\pi)Q(q_1)Q(q_2)\), we find
\[
d\Gamma(K_L \rightarrow \pi^0 QQ) \approx \frac{2\alpha_D^2 \xi(z_1, z_2)^2}{\Lambda^2} \frac{m_K}{1 + r_\pi^2} \Omega_C(z_1, z_2) \sin^2 \delta_{\text{CP}} \left[ q^2 + 4z_1z_2 + r_\pi^2(2z_1 + 2z_2 - 1) \right], \quad (15)
\]
where the tensor form factor is given by
\[
f_{\pi\pi}^K(q^2) = \frac{f_{\pi\pi}^K(0)}{1 - \frac{q^2}{s_{\pi\pi}}} q^2, \quad (19)
\]
with \(q^2\) as before and \(f_{\pi\pi}^K(0) = 0.417(15)\) and \(s_{\pi\pi} = 1.10(14) \text{GeV}^{-1}\) on the lattice \([27]\).

The phase-space integration is between
\[
\frac{m_{12}^2}{m_K^2} \cos \gamma = \frac{(m_{12}^2)_{\min} - m_{\pi}^2 - m_Q^2}{2m_K^2}, \quad \frac{m_{12}^2}{m_K^2} \cos \gamma = \frac{(m_{12}^2)_{\max} - m_{\pi}^2 - m_Q^2}{2m_K^2},
\]
where
\[
(m_{12}^2)_{\max} = (m_{Q} + m_{\pi})^2, \quad (m_{K} - m_{Q})^2,
\]
and
\[
E_2 = \frac{E_2^2 - m_\pi^2 + m_\pi^2}{2\sqrt{m_{12}^2}} = \frac{E_3^2 - m_\pi^2}{2\sqrt{m_{12}^2}}.
\]
with $m_{\tau_d}^2 = 2m_K^2 z_1 + m_{\pi}^2 + m_Q^2$.

The result for $\Gamma(K^+ \to \pi^+ Q\bar{Q})$ is the same as that in Eq. (15) but for the absence of the CP-violating $\sin^2 \delta_{CP}$ and for a factor 0.954 coming from the isospin rotation and the difference in the masses. The two widths together satisfy the GN relationship in Eq. (8) once the different lifetimes of the $K^+$ and the $K_L$ are taken into account in the BRs.

We take $m_{K_L} = 497.61$, $m_{\pi^0} = 134.98$ MeV [25] and span the possible values within the window in Eq. (1) $178 < m_Q < 181$ MeV assuming maximal CP violation ($\sin \delta_{CP} = 1$). We vary the dark-photon coupling constant: $0.05 < \alpha_D < 0.15$. After enforcing the limit in Eq. (14), we obtain that the integration of Eq. (15) over the kinematical variables leads to

$$3.9 \times 10^{-12} < \text{BR} (K_L \to \pi^0 \nu \bar{\nu}) < 3.7 \times 10^{-8},$$

(20)
a range that covers the entire region from below the SM prediction to above the KOTO SES region (as depicted in Fig. 1).

The result in Eq. (20) only depends in a significative manner on

- the choice of $m_Q$ and $\alpha_D$. By taking $m_Q$ closer to the upper end of the window in Eq. (1) we close the phase space and in the end, Sommerfeld-Fermi enhancement notwithstanding, the width goes to zero. Notice that the window in Eq. (1) can be (slightly) enlarged by having the $\Gamma(K^+ \to \pi^+ \nu \bar{\nu})$ not closed but only suppressed by the kinematics below the experimental limit in Eq. (3) (and still above the SM prediction).

There may be a question about so light a mass for the dark fermion because of the impact of the dark sector on the CMB. This needs to be investigated further [28].

The size of $\alpha_D$ is currently only weakly constrained by galaxy dynamics and cosmology (see [19] and references therein). Even large values, close to the perturbative limit, are not excluded.

- $\sin \delta_{CP}$. The whole decay width is proportional to the size of CP violation. Its size can be modulated by taking $\sin \delta_{CP}$ smaller than one.

- the limit in Eq. (14). This can be made stronger thus proportionally suppressing the BR in Eq. (20).

Notice that the new limit in Eq. (3) would imply a stronger bound on the dark-sector parameters if the channel were to be open and not kinematically restricted.

The transverse momentum of the pions.—The particular kinematic window we chose constrains the possible transverse momenta $p_T^{\pi^0}$ of the $\pi^0$ and we have

$$p_T^{\pi^0} < \sqrt{[m_K^2 - (2m_Q - m_{\pi^0})^2][m_K^2 - (2m_Q + m_{\pi^0})^2]} / 2m_K$$

which gives $p_T^{\pi^0} < 36$ MeV—for the most favorable case of taking $m_Q = 178$ MeV. This value can be increased to around 60 MeV if we allow $m_Q$ to drop below the threshold for the $K^+$ decay while still suppressing the width of this channel by the smallness of the phase space.

The signal region of KOTO cuts off pions with momenta smaller than 130 MeV to reduce the background from $K_L \to \pi^+ \pi^- \pi^0$ [10]. For this reason, the three events observed [11] cannot come from the decay into the dark sector. It is a prediction of the scenario here discussed that the pions have small transverse momentum; this requirement puts them in a kinematical region unfortunately obscured by the background and perhaps of difficult experimental resolution.

Other models.—The same kinematics exploited for the dark sector can also work, mutatis mutandis, in a two-body decay $K \to \pi^0 \nu \bar{\nu}$, with $\pi^0$ a scalar or a massive dark photon. The signature of this process is the monochromatic transverse momentum of the pion. This discriminating feature will be decided when (and if) a sufficient number of events is collected.

Conclusions.—The recently announced new limit on the Kaon decay $K^+ \to \pi^+ \nu \bar{\nu}$ [8] implies that very little room is left in this channel for new physics. If the GN bound is applied, the decay $K_L \to \pi^0 \nu \bar{\nu}$ is constrained to be lower than the current KOTO sensibility [11]. The potential tension between events possibly seen by the KOTO collaboration and the GN bound can be resolved in a model of the dark sector with light dark fermions if we exploit the mass difference between the charged and neutral Kaons and choose the dark fermion mass as to kinematically close the decay of the charged Kaon while leaving open that of the neutral Kaon—for which there is still room above the SM prediction in the region currently probed by KOTO. The low momentum of the $\pi^0$ in the decay is the signature of this scenario.

MF is affiliated to the Physics Department of the University of Trieste and the Scuola Internazionale Superiore di Studi Avanzati—the support of which is gratefully acknowledged. MF and EG are affiliated to the Institute for Fundamental Physics of the Universe, Trieste, Italy.

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