Critical behaviour of the dilute O\((n)\),
Izergin-Korepin
and dilute \(A_L\) face models: Bulk properties

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Abstract

The analytic, nonlinear integral equation approach is used to calculate the finite-size corrections to the transfer matrix eigen-spectra of the critical dilute O\((n)\) model on the square periodic lattice. The resulting bulk conformal weights extend previous exact results obtained in the honeycomb limit and include the negative spectral parameter regimes. The results give the operator content of the 19-vertex Izergin-Korepin model along with the conformal weights of the dilute \(A_L\) face models in all four regimes.

1 Introduction

Among other physical phenomena, the integrable dilute O\((n)\) model on the square lattice \([1]\) is relevant to self-avoiding polymer chains in the bulk \([2,3]\). The partition function of the dilute O\((n)\) model is defined by \([1,4]\)

\[
Z = \sum_{\mathcal{G}} \rho_1^{m_1} \cdots \rho_9^{m_9} n^P,
\]

where the sum is over all configurations \(\mathcal{G}\) of non-intersecting closed loops which cover some (or none) of the lattice bonds. The possible loop configurations at a vertex are shown in Fig. 1, with a vertex of type \(i\) carrying a Boltzmann weight \(\rho_i\). In configuration \(\mathcal{G}\); \(m_i\) is the number of occurrences of the vertex of type \(i\), while \(P\) is the total number of closed loops of fugacity \(n\).

The loop weights in (1.1) are \([1,4]\)

\[
\rho_1(u) = 1 + \frac{\sin u \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda}
\]
Fig. 1. The 9 vertices of the dilute O(n) model.

\[ \rho_2(u) = \rho_3(u) = \frac{\sin(3\lambda - u)}{\sin 3\lambda} \]
\[ \rho_4(u) = \rho_5(u) = \frac{\sin u}{\sin 3\lambda} \]
\[ \rho_6(u) = \rho_7(u) = \frac{\sin u \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \]
\[ \rho_8(u) = \frac{\sin(2\lambda - u) \sin(3\lambda - u)}{\sin 2\lambda \sin 3\lambda} \]
\[ \rho_9(u) = -\frac{\sin u \sin(\lambda - u)}{\sin 2\lambda \sin 3\lambda} . \]

Here \( n = -2 \cos 4\lambda \). These weights were originally constructed via a mapping involving the Potts model [1] and later seen to satisfy the Yang-Baxter equation for loop models [4,5]. On the other hand, when mapped to a 3-state vertex model [1], the dilute O(n) model is seen to be related to the integrable 19-vertex model of Izergin and Korepin [6]. The Nienhuis O(n) model on the honeycomb lattice [7–9] follows from either of the special values \( u = \lambda \) and \( u = 2\lambda \) of the spectral parameter [1,10]. In the appropriate region the model thus contains the essential physics of the self-avoiding polymer problem at \( n = 0 \) [2,7,8,11,12].

The dilute O(n) model has recently been used to construct a family of dilute A–D–E lattice models [13–15]. These models are restricted solid–on–solid models with a finite number of heights built on the A–D–E Dynkin diagram. At criticality, the face weights are [13–15]

\[
W\left(\frac{d}{a} \frac{c}{b} \right| u \right) = \rho_1(u)\delta_{a,b,c,d} + \rho_2(u)\delta_{a,b,c}A_{a,d} + \rho_3(u)\delta_{a,c,d}A_{a,b} + \sqrt{\frac{S_a}{S_b}}\rho_4(u)\delta_{b,c,d}A_{a,b} + \sqrt{\frac{S_c}{S_a}}\rho_5(u)\delta_{a,b,d}A_{a,c} + \rho_6(u)\delta_{a,b,c}A_{a,d} + \sqrt{\frac{S_a S_c}{S_b S_d}}\rho_7(u)\delta_{c,b,d}A_{a,b}A_{a,c} + \rho_8(u)\delta_{a,c,d}A_{a,b}A_{a,d} + \sqrt{\frac{S_a S_c}{S_b S_d}}\rho_9(u)\delta_{b,d,c}A_{a,b}A_{a,c} \]  

(1.3)

where the \( \rho_i \) are as given above. The generalized Kronecker delta is unity if all its arguments take the same value and is zero otherwise. The Perron-Frobenius vectors \( S_a \) in the face weights are the eigenvector of the largest eigenvalue of
the adjacency matrix $A$ of the $A-D-E$ graphs,

$$
\sum_b A_{a,b} S_b = 2 \cos \frac{\pi}{L+1} S_a ,
$$

where for the dilute $A_L$ models, $L$ is the number of graph states, with $a, b, c, d = 1, 2, \cdots, L$.

The dilute $O(n)$ model exhibits various branches of critical behaviour [16–19]. These are reflected in the properties of the dilute $A-D-E$ models, for which there are four physical branches [13]

| Branch | Condition | $\lambda$ | $L$ |
|--------|-----------|----------|-----|
| 1      | $0 < u < 3\lambda$ | $\lambda = \frac{\pi}{4} \frac{L}{L+1}$ | $L = 2, 3, \cdots$ |
| 2      | $0 < u < 3\lambda$ | $\lambda = \frac{\pi}{4} \frac{L+2}{L+1}$ | $L = 3, 4, \cdots$ |
| 3      | $-\pi + 3\lambda < u < 0$ | $\lambda = \frac{\pi}{4} \frac{L+2}{L+1}$ | $L = 3, 4, \cdots$ |
| 4      | $-\pi + 3\lambda < u < 0$ | $\lambda = \frac{\pi}{4} \frac{L}{L+1}$ | $L = 2, 3, \cdots$ |

Recent studies have highlighted the prominence of the dilute $A_L$ face models, which admit an off-critical extension [13,15]. In regime 2 the $A_3$ model lies in the same universality class as the Ising model in a magnetic field and gives the magnetic exponent $\delta = 15$ [13,15,20]. This $A_3$ model also shows the $E_8$ scattering theory for massive excitations over the groundstate [21–23]. Both $su(2)$ and $su(3)$ fusion hierarchies of the dilute $A_L$ face models have been constructed in [24,25].

In this paper we both generalise and extend earlier calculations of the critical properties, such as the central charges and bulk scaling dimensions (the conformal spectra), of the dilute $O(n)$ model and the related dilute $A_L$ and Izergin-Korepin models. After outlining the necessary preliminaries in Section 2, our calculations are presented in Section 3 for branches 1 and 2 and in Section 4 for branches 3 and 4. The method employed involves the extension of the nonlinear integral equation approach [26,27,18] to obtain the complete conformal spectra, as has been done for the six-vertex model [28,29] and most recently [30] for the Andrews-Baxter-Forrester (ABF) model [31]. Having read Section 2, those readers not specifically interested in the technical details may prefer to skip to Section 5 where a discussion of our results for the various models concludes the paper.
2 Bethe equations and known results

As we are interested in bulk critical behaviour, we consider periodic boundary conditions across a finite lattice of width $N$, where for convenience we take $N$ even. The eigenvalues $T(u)$ for the row-transfer matrix $T(u)$ of the dilute $O(n)$ model are given by [17–19]

\[ T(u) = e^{-i\phi} \frac{s(2\lambda - u)s(3\lambda - u)}{s(2\lambda)s(3\lambda)} \frac{Q(u + \lambda)}{Q(u - \lambda)} + \frac{s(u)s(3\lambda - u)}{s(2\lambda)s(3\lambda)} \frac{Q(u)Q(u - 3\lambda)}{Q(u - \lambda)Q(u - 2\lambda)} + e^{i\phi} \frac{s(u)s(\lambda - u)}{s(2\lambda)s(3\lambda)} \frac{Q(u - 4\lambda)}{Q(u - 2\lambda)}, \]  

(2.1)

where

\[ s(u) = \sin^N(u), \quad Q(u) = \prod_{j=1}^m \cosh(iu - u_j) \]  

(2.2)

and the $m$ zeros $\{u_j\}$ satisfy the Bethe equations

\[ e^{i\phi} \left[ \frac{\cosh(u_j + i\lambda)}{\cosh(u_j - i\lambda)} \right]^N = -\prod_{k=1}^m \frac{\sinh(u_j - u_k + 2i\lambda)\sinh(u_j - u_k - i\lambda)}{\sinh(u_j - u_k - 2i\lambda)\sinh(u_j - u_k + i\lambda)} \]  

(2.3)

for $j = 1, \ldots, m$. It is convenient to label the sectors of $T(u)$ by $\ell = N - m$, where $\ell = 0$ for the largest (groundstate) sector, $\ell = 1$ for the next largest, etc.

The Bethe equations ensure that the eigenvalues $T(u)$ are analytic functions of $u$. Apart from the phase factors $\phi$ these equations are the Bethe equations of the Izergin-Korepin model [32–34]. In general $\phi$ is a continuous variable associated with a “seam” to ensure that loops which wrap round the cylinder carry the correct weight $n$. Thus

\[ \phi = \pi - 4\lambda \]  

(2.4)

for the dilute O($n$) model in the largest ($\ell = 0$) sector of $T(u)$ with $\phi = 0$ in all other sectors. For the Izergin-Korepin model, $\phi = 0$ in all sectors.

On the other hand, for the dilute $A_L$ face models there is a fixed number of Bethe roots ($\ell = 0$) and

\[ \phi = \pi s/(L + 1) \]  

(2.5)
with \( s = 1, \ldots, L \). In this case the transfer matrix \( T(u) \) has elements

\[
\langle \sigma | T(u) | \sigma' \rangle = \prod_{j=1}^{N} W \left( \begin{array}{c} \sigma'_j \\ \sigma_j \end{array} \middle| \sigma'_{j+1} \sigma_{j+1} \right| u \right),
\]

(2.6)

where the paths \( \sigma = \{ \sigma_1, \sigma_2, \ldots, \sigma_N \} \) and \( \sigma' = \{ \sigma'_1, \sigma'_2, \ldots, \sigma'_N \} \) are allowed configurations of heights along a row with periodic boundary conditions \( \sigma_{N+1} = \sigma_1 \) and \( \sigma'_{N+1} = \sigma'_1 \). The face weights are those defined in (1.3). The expression for the eigenvalues \( T(u) \) was given in [21] in terms of the more general elliptic functions. As we are also interested in the dilute \( O(n) \) model, we do not restrict the crossing parameter \( \lambda \) to the values given in (1.5). This may lead to unphysical regimes in the dilute \( A_L \) face model for which, however, the finite-size corrections to the transfer matrix eigenspectra are still of interest from the viewpoint of statistical mechanics and conformal field theory.

### 2.1 Central charge

Some exact results are known for the dilute \( O(n) \) model [17–19]. In particular, the central charge is found to be

\[
c = 1 - \frac{3\phi^2}{\pi(\pi - 2\lambda)} \quad \text{branches 1 & 2},
\]

(2.7)

\[
c = \frac{3}{2} - \frac{3\phi^2}{2\pi \lambda} \quad \text{branches 3 & 4}.
\]

(2.8)

These results follow from the finite-size behaviour of the largest eigenvalue. The result (2.7) had already been obtained from the Bethe equations in the honeycomb limit [35–37]. However, the result (2.8) [18] had to await the development of the more sophisticated nonlinear integral equation approach [26,27,18] (see also [38,39]).

The reason for this is that the distribution of Bethe roots for the largest eigenvalue differs significantly in each case. In the limit of infinite size \( N \) the Bethe roots are distributed on the lines [17–19]

\[
\begin{align*}
\text{branches 1 and 2} & \quad \Re m \ (u_j) = \frac{1}{2} \pi, \\
\text{branches 3 and 4} & \quad \Re m \ (u_j) = \pm \frac{1}{2} \pi \lambda.
\end{align*}
\]

(2.9) (2.10)

Whereas there are no finite-size deviations from the line (2.9), the finite-size deviations from (2.10) are severe enough to render the more standard root
density approach\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{\textsuperscript{1}}}}}}}}}}}}}}}} invalid. Here we extend the analytic, nonlinear integral equation approach in the dilute $O(n)$ model \cite{18,19} to the calculation of the conformal weights in all four branches. Our treatment follows that given in the recent study of the ABF model \cite{30}.

The above results for the central charge have already been used to obtain the central charges of the dilute $A-D-E$ models \cite{13,15,19}. In particular, for the dilute $A_L$ face models, either (2.5) or the $O(n)$ value (2.4) with the appropriate value of $\lambda$ in (1.5) gives

\begin{equation}
\begin{aligned}
c &= \begin{cases} 
1 - \frac{6}{h(h-1)}, & \text{branches 1 & 2}, \\
\frac{3}{2} - \frac{6}{h(h-1)}, & \text{branches 3 & 4},
\end{cases} 
\end{aligned}
\tag{2.11}
\end{equation}

where

\begin{equation}
h = \begin{cases} 
L + 1, & \text{branches 2 & 4}, \\
L + 2, & \text{branches 1 & 3}.
\end{cases}
\tag{2.12}
\end{equation}

The first two branches give realisations of the unitary minimal series, while the other two branches involve a product of the unitary minimal series and an Ising model.

The $O(n)$ model had earlier been identified \cite{41,42} in the conformal classification scheme \cite{43,44} with the aid of the Nienhuis Coulomb gas results \cite{7,8}. In particular, $c = 1 - 6(g-1)^2/g$, where $g \in [1,2]$, with $g = h/(h-1)$, in the high temperature phase (branch 1) and $g \in [0,1]$, with $g = (h-1)/h$, in the low temperature phase (branch 2). Here $g = 2(1 - 2\lambda/\pi)$. The Ising value $c = \frac{1}{2}$ thus occurs both for the dilute $A_2$ model ($n = 1$ in the high temperature $O(n)$ phase) and the dilute $A_3$ model ($n = \sqrt{2}$ in the low temperature $O(n)$ phase). The central charges of the dilute $A_L$ face models have recently been estimated numerically from the finite-size diagonalisation of the dilute $A_L$ model transfer matrix for various $L$ on all four branches \cite{45}. The central charge has also been derived by solving the transfer matrix functional relations of the dilute $A_L$ model on branches 2 and 4 \cite{46}. The calculation confirms the result (2.11) obtained via the dilute $O(n)$ model \cite{13,15,19}.

\subsection{Scaling dimensions}

Various scaling dimensions have been calculated via the Bethe equations for the dilute $O(n)$ model. Again in the honeycomb limit for branches 1 and 2,

\footnote{See, for example, \cite{40} and references therein.}
the ‘magnetic’ set of scaling dimensions is found to be [35–37]

\[ X^\sigma_\ell = \frac{\ell^2(\pi - 2\lambda)^2 - (\pi - 4\lambda)^2}{4\pi(\pi - 2\lambda)} = \frac{1}{8} g \ell^2 - \frac{(g - 1)^2}{2g}. \]  

(2.13)

Alternatively, this result is written as

\[ X^\sigma_\ell = \begin{cases} 2\Delta_{\ell/2,0}, & \text{branch 1} \\ 2\Delta_{0,\ell/2}, & \text{branch 2} \end{cases} \]  

(2.14)

where

\[ \Delta_{r,s}^{(h)} = \frac{[hr - (h - 1)s]^2 - 1}{4h(h - 1)} \]  

(2.15)

is the Kac formula. This result had earlier been obtained via Coulomb gas calculations [47,48].

On the other hand, both numerical evidence [37] and root-density calculations [40] revealed the set of ‘thermal’ dimensions to be

\[ X^\epsilon_j = \begin{cases} 2\Delta_{1,2j+1}, & \text{branch 1} \\ 2\Delta_{2j+1,1}, & \text{branch 2} \end{cases} \]  

(2.16)

Both the results (2.13) and (2.16) generalized earlier results via the Coulomb gas [7,8,11].\(^2\) The thermal dimensions follow from

\[ X^\epsilon_j = \begin{cases} 2\Delta_{1,2j+1}, & \text{branch 1} \\ 2\Delta_{2j+1,1}, & \text{branch 2} \end{cases} \]  

(2.17)

in the Kac formula [37].

On the other hand, the situation is not so clear for branches 3 and 4 of the dilute square lattice model. Numerical evidence [18] indicates that the magnetic dimensions are given by

\[ X^\sigma_\ell = \frac{\chi \ell^2}{2\pi} - \frac{(\pi - 4\lambda)^2}{8\pi \lambda}. \]  

(2.18)

The only known thermal result is \( X^\epsilon_1 = 1 \) [18]. There are no Coulomb gas results for these branches.

\(^2\) The leading thermal dimension had been conjectured earlier for the O(\(n\)) model by Cardy and Hamber [49].
The conformal weights of the dilute $A_L$ face models have been estimated numerically from the finite-size diagonalisation of the transfer matrix (for $L = 3$ and $L = 4$ at $u = 3\lambda/2$ on all four branches) [45] and from numerical solutions to the Bethe equations for $L = 3$ [23]. For branches 1 and 2, the results fulfil the expectation that the scaling dimensions reflect the conformal weights of the unitary minimal series. For branches 3 and 4, they reflect a product of the Ising and unitary minimal series. The related modular invariant partition function has been discussed at length in [45].

As mentioned above, the finite-size corrections to the transfer matrix eigen-spectra have been obtained for the dilute $A_L$ face models in branches 2 and 4 in [46] via the functional relation method [38,39,50]. The analytic calculation confirms the conformal weights obtained via the calculation of the local height probabilities for $L$ odd [20]. Here we consider the dilute models in all four branches with more general crossing parameter $\lambda$ and calculate the conformal spectra for each branch.

3 Branches 1 and 2

We consider branches 1 and 2 defined by

$$0 < u < 3\lambda \quad \pi/6 \leq \lambda < \pi/3.$$  \hfill (3.1)

This regime covers the $\lambda$ values (1.5) for the dilute $A_L$ models. However, the derivation below is also valid for the dilute O($n$) and Izergin-Korepin models in the larger interval $0 < \lambda < \pi/3$. Let us introduce the new variable $v = iu$ with a shift $v_j = u_j - \frac{1}{2}i\pi$. The Bethe equations (2.3) are then of the form

$$p(v_j) = -1,$$  \hfill (3.2)

where

$$p(v) = e^{-i\phi} \frac{\Phi(v - i\lambda)q(v - i\lambda)q(v + 2i\lambda)}{\Phi(v + i\lambda)q(v + i\lambda)q(v - 2i\lambda)},$$  \hfill (3.3)

$$\Phi(v) = \sinh^N v, \quad q(v) = \prod_{j=1}^m \sinh(v - v_j).$$  \hfill (3.4)

After the shift, the Bethe roots $v_j$ are distributed along the real axis, with

$$\overline{q}(v) = q(\overline{v}) \quad \text{and} \quad \overline{p}(v) = 1/p(v)$$  \hfill (3.5)

where the overbar denotes complex conjugation.
3.1 Nonlinear integral equation

Define two functions that are Analytic and Non-Zero (ANZ) in the strips around the real axis:

\[
\alpha(v) = e^{i\omega}g(v)p(v + i\xi),
A(v) = 1 + \alpha(v)/g(v). \quad (3.6)
\]

The phase factor \(\omega\) has been introduced for taking different branches of the log function involved in the subsequent Fourier transforms. We take

\[
\omega = \begin{cases} 
\text{sgn}(v)(\ell - r)\pi & \text{dilute } O(n) \text{ model, } \ell \neq 0 \\
r\pi & \text{dilute } O(n) \text{ model, } \ell = 0 \\
\pi(r - s) & \text{dilute } A_L \text{ face model}
\end{cases} \quad (3.7)
\]

where the function \(\text{sgn}(v) = -1\) for \(\Re(v) > 0\) and \(+1\) otherwise. For the \(O(n)\) model the integers \(r, s\) are restricted (as discussed further in section 5). For the moment we leave them as arbitrary integers. The function \(g\) is introduced for compensating the anticipated bulk behaviour of \(p(v + i\xi)\) and is given by

\[
g(v) = \left(\frac{\text{th}\rho(v + i\lambda + i\xi)}{\text{th}\rho(v - i\lambda + i\xi)}\right), \quad \rho = \pi/6\lambda, \quad (3.8)
\]

where \(0 < \xi \leq \frac{1}{2}\pi\). The \(i\pi\)-periodic function \(\alpha\) can be rewritten as

\[
\alpha(v) = e^{i\omega - i\phi}g(v)\frac{\Phi(v - i\lambda + i\xi)q(v - i\lambda + i\pi + i\xi)q(v + 2i\lambda + i\xi)}{\Phi(v + i\lambda + i\xi)q(v + i\lambda + i\pi + i\xi)q(v - 2i\lambda + i\pi + i\xi)}. \quad (3.9)
\]

The above treatment results in the function \(\alpha\) representing finite-size corrections. To see this we consider the Fourier transform pair

\[
\alpha(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\ln \alpha(v)\right)^'' e^{-ikv} dv
\]

\[
\left(\ln \alpha(v)\right)^'' = \int_{-\infty}^{\infty} \alpha(k)e^{ikv} dk \quad (3.10)
\]

for \(\alpha\) and similarly for \(A\). The Fourier transform of \(q(v)\) is defined to be

\[
q(k) = \frac{1}{2\pi} \int_{-\infty+ir}^{\infty+ir} \left(\ln q(v)\right)^'' e^{-ikv} dv \quad 0 < r < \pi,
\]
\[
\left( \ln q(v) \right)'' = \int_{-\infty}^{\infty} q(k)e^{ikv} \, dk \quad 0 < \Im(v) < \pi. \quad (3.11)
\]

To represent \( \alpha(k) \) by \( A(k) \) and \( \overline{A}(k) \) we also need another relation, which can be given by applying Cauchy’s theorem to the auxiliary function
\[
h(v) = \frac{1 + p(v)}{p(v)q(v)}, \quad (3.12)
\]

which satisfies the non-trivial analyticity property
\[
\int_{-\infty+i\xi}^{\infty+i\xi} \left( \ln h(v) \right)'' e^{-ikv} \, dv = \int_{-\infty-i\xi}^{\infty-i\xi} \left( \ln h(v) \right)'' e^{-ikv} \, dv. \quad (3.13)
\]

From the equations following Fourier transforming (3.9) and inserting (3.12) into (3.13) we obtain
\[
q(k) = \frac{Nk e^{1/2\pi k} \cosh(1/2\lambda k)}{2 \sinh(1/2\pi k) \cosh(1/2\lambda k)} \\
\quad + \frac{e^{1/2\pi k} \cosh(3/2\lambda k)}{2 \sinh(1/2\pi k - \lambda k) \cosh(3/2\lambda k)} \left( e^{\xi k}A(k) - e^{-\xi k}\overline{A}(k) \right), \quad (3.14)
\]
\[
\alpha(k) = F(k)A(k) - F_\xi(k)\overline{A}(k), \quad (3.15)
\]

where
\[
F_\xi(k) = -e^{-2\xi k} \frac{\sinh(\lambda k) \cosh(1/2\pi k - 3/2\lambda k)}{\cosh(3/2\lambda k) \sinh(1/2\pi k - \lambda k)} \quad (3.16)
\]

and \( F(k) = F_0(k) \). Transforming back and integrating twice we obtain the nonlinear integral equation
\[
\ln \alpha(v) = F * \ln A - F_\xi * \ln \overline{A} + C + C'v, \quad (3.17)
\]

where the convolution is defined by
\[
(f * g)(v) = \int_{-\infty}^{\infty} f(w)g(v - w) \, dw. \quad (3.18)
\]

The constant \( C' \) is chosen to be \( C' = 0 \) for all terms to remain finite. The other constant \( C \) is scaling-dependent and is fixed after taking the scaling
limit defined by

\[ a_\pm(x) = \lim_{N \to \infty} a(\pm v)/g(\pm v), \]
\[ A_\pm(x) = \lim_{N \to \infty} A(\pm v) = 1 + a_\pm(x). \]  

(3.19)

The nonlinear integral equation then becomes

\[ \left( \ln a_\pm \over \ln \alpha \right) = 2i\sqrt{3} e^{-x} \left( \pm e^{2\rho i \xi} \right) + K * \left( \ln A_\pm \over \ln \alpha \right) + C_\pm \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \] 

(3.20)

in which the kernel \( K \) is given by

\[ K = \begin{pmatrix} F_1 & -F_2 \\ -F_2 & F_1 \end{pmatrix}, \] 

(3.21)

where

\[ F_1(x) = F_1(-x) = \frac{1}{2\rho} F \left( \pm \frac{1}{2\rho} x \right), \]
\[ F_2(x) = F_2(-x) = \frac{1}{2\rho} F \left( \pm \frac{1}{2\rho} x + 2i\xi \right). \] 

(3.22)

We can see that \( K^T(x) = K(-x) \), a key property to be used in the derivation of the finite-size corrections. Taking \( x \to \infty \) we obtain

\[ C_\pm = i\pi (\omega_\pm - \phi)/(\pi - 2\lambda), \] 

(3.23)

where

\[ \omega_\pm = \begin{cases} \omega \mp 2\ell \lambda & \text{dilute O}(n) \text{ model}, \\ \omega & \text{dilute } A_L \text{ face model} \end{cases} \] 

(3.24)

The nonlinear integral equation (3.20) is equivalent to the Bethe equations for the largest eigenvalue, as given in [18,19]. The key difference is the change in the integration constants \( C_\pm \) for the low-lying excited states. This is similar to the nonlinear integral equation approach in the six-vertex [28] and ABF [30] models. In each case the constants contain the necessary information to extract the conformal weights.

3.2 Conformal spectra

The eigenvalues of the transfer matrix are given by (2.1). For small positive values of \( u \) the first term in the eigenvalue expression dominates exponentially.
For small positive imaginary \( v \) we therefore have

\[
T(v) \sim e^{-i\phi} \Phi(v) \Phi(v - 2i\lambda) \frac{q(v + i\lambda)}{q(v - i\lambda)}
\]  
(3.25)

for the finite-size corrections. Taking Fourier transforms and integrating twice yields

\[
\ln T(v) = -N f_{\infty}(v) + \frac{2\sqrt{3}\rho}{\pi} \int_{-\infty}^{\infty} \left( \frac{\sinh 4\rho(v - w - i\xi)}{\sinh 6\rho(v - w - i\xi)} \ln A(w) \right) dw,
\]

(3.26)

where the free energy is given by

\[
f_{\infty}(v) = 2 \int_{-\infty}^{\infty} dk \frac{\sinh(kiv) \sinh(3k\lambda + ivk) \cosh(5k\lambda - k\pi) \cosh(k\lambda)}{k \cosh(3k\lambda) \sinh(k\pi)}.
\]

(3.27)

The integration constants have been fixed again by the limit \( v \to \infty \). Taking the thermodynamic limit \( N \to \infty \) in (3.26) and using the definitions (3.19) gives

\[
\ln T(v) = -N f_{\infty}(v) + \frac{2\sqrt{3}\rho}{N\pi} e^{2\rho v} \Im \left( e^{-2\rho i\xi} \int_{-\infty}^{\infty} \ln A_+(x) e^{-x} \right)
\]

\[
- \frac{2\sqrt{3}\rho}{N\pi} e^{2\rho v} \Im \left( e^{2\rho i\xi} \int_{-\infty}^{\infty} \ln A_-(x) e^{-x} \right) dx
\]

(3.28)

up to order \( 1/N \). To proceed further we consider the expression

\[
\int_{-\infty}^{\infty} \left[ \left( \frac{\ln a_+}{\ln \overline{a}_+} \right)' (\ln A_+, \ln \overline{A}_+) - \left( \frac{\ln a_-}{\ln \overline{a}_-} \right)' (\ln A_-, \ln \overline{A}_-) \right] dx
\]

(3.29)

which can be written exactly as

\[
L(z) + L(1/z) = \frac{\pi^2}{3}
\]

(3.30)
in terms of the Rogers dilogarithmic function

\[ L(x) = \int_0^x \left( \frac{\ln(1 + y)}{y} - \frac{\ln y}{1 + y} \right) dy. \] (3.31)

On the other hand, making use of (3.20) in (3.29) and using \( a_\pm(-\infty) = \mp\infty \) and \( a_\pm(\infty) = \frac{1}{\mp\infty} = e^{i(\omega_\pm - \phi)} \), we arrive at the result

\[ \mp 8\sqrt{3}m \left( e^{\mp2\rho \xi} \int_{-\infty}^\infty \ln A_\pm(x)e^{-x} \right) + \frac{\pi(\omega_\pm - \phi)^2}{\pi - 2\lambda}. \] (3.32)

Equating (3.32) and (3.30) gives the integral in (3.28). Thus inserting this integral into the expression (3.28) we obtain

\[ \ln T(v) = -Nf_\infty(v) - \frac{\pi \sin(2\rho v)}{6N}(c - 24\Delta) \] (3.33)

to leading order in \( 1/N \). This is our final result, from which the central charge and conformal weights can be read-off \([51-53]\) as

\[ c = 1 - \frac{3\phi^2}{\pi(\pi - 2\lambda)}, \] (3.34)

\[ \Delta = \begin{cases} \frac{(\omega - \phi \pm 2\lambda)^2 - (\pi - 4\lambda)^2}{8\pi(\pi - 2\lambda)} & \text{dilute O}(n) \text{ model}, \\
\frac{(\omega - \phi)^2 - (\pi - 4\lambda)^2}{8\pi(\pi - 2\lambda)} & \text{dilute } A_L \text{ face model}. \end{cases} \] (3.35)

### 4 Branches 3 and 4

On branches 3 and 4 the spectral and crossing parameters are specialized in the regime

\[ -\pi + 3\lambda < u < 0 \quad \frac{1}{6\pi} \leq \lambda < \frac{1}{3\pi}. \] (4.1)

The following computation of the finite-size corrections to the transfer matrix eigenspectra for each of the models is valid for the larger interval \( 0 < \lambda < \frac{1}{3\pi} \).
We proceed in a similar manner as for branches 1 and 2 and introduce a new parameter $v = iu$ and set $v_j = u_j$. The function $p(v)$ is defined by

$$p(v) = e^{i(\omega - \phi)} \frac{\Phi(v - i\lambda + \frac{1}{2}\pi i)q(v - i\lambda)q(v + 2i\lambda)}{\Phi(v + i\lambda + \frac{1}{2}\pi i)q(v + i\lambda)q(v - 2i\lambda)},$$

(4.2)

with $\Phi$ and $q$ as given in (3.4). In [18,19] it has been checked that the Bethe ansatz roots are (almost) located on the lines $\Im m (v) = \pm \frac{1}{2} \lambda$ in the complex $v$-plane. As a consequence we still have the symmetries of equation (3.5).

4.1 Nonlinear integral equation

We proceed by defining functions that are ANZ in the strips around the real axis:

$$A(v) = 1 + \alpha(v)/g(v) \quad \alpha(v) = g(v)p(v - i\lambda)[1 + p(v)]$$

$$B(v) = 1 + \beta(v)/g(v) \quad \beta(v) = g(v)p(v - i\lambda)\frac{1 + p(v - i\lambda)}{1 + p(v - i\lambda)}$$

$$C(v) = 1 + \gamma(v)/g(v) \quad \gamma(v) = g(v)p(v - i\lambda)$$

$$\delta(v) = p(v).$$

(4.3)

The function $g(v) = \text{th}^{N}(v + i\lambda - \frac{1}{2}\pi i)$, with $\rho = \pi/(2\pi - 6\lambda)$, is introduced to compensate the anticipated bulk behaviour of the functions $\alpha, \beta, \gamma$.

We define the Fourier transform of the functions $\alpha, \beta, \gamma$ as in (3.10). For $q$ we have

$$q(k) = \frac{1}{2\pi} \int_{-\infty + ir}^{\infty + ir} \left( \ln q(v) \right)^n e^{-ikv} dv \quad -\pi + \frac{1}{2} \pi < r < -\frac{1}{2} \pi$$

$$\left( \ln q(v) \right)^n = \int_{-\infty}^{\infty} q(k)e^{ikv} dk \quad -\pi + \frac{1}{2} \pi < \Im m (v) < -\frac{1}{2} \pi$$

(4.4)

$$q_1(k) = \frac{1}{2\pi} \int_{-\infty + ir}^{\infty + ir} \left( \ln q(v) \right)^n e^{-ikv} dv \quad -\frac{1}{2} \pi < r < \frac{1}{2} \pi$$

$$\left( \ln q(v) \right)^n = \int_{-\infty}^{\infty} q_1(k)e^{ikv} dk \quad -\frac{1}{2} \pi < \Im m (v) < \frac{1}{2} \pi.$$  

(4.5)

To solve the functional relations we need the relations of the Fourier transforms
of $\alpha, \beta, \gamma$. First we can see that not all functions are independent and thus we have

$$
\begin{align*}
\beta(k) - \gamma(k) - \delta(k) + C(k) &= 0 \\
\alpha(k) - \overline{\alpha}(k) - \gamma(k) + \overline{\gamma}(k) - \delta(k) &= 0 \\
A(k) - B(k) - C(k) &= 0.
\end{align*}
$$

(4.6)

Applying the Fourier transform to the $\delta, \gamma$ gives

$$
\begin{align*}
\gamma(k) &= Nk \sinh \lambda k / \sinh \pi k / 2 - \frac{1}{2}Nke^{\frac{\lambda k}{2}} / \cosh \frac{k}{2}(3\lambda - \pi) \\
&\quad + (e^{-\lambda k + \frac{\pi k}{2}} + e^{2\lambda k} - e^{\lambda k}) q(k) - q_1(k) \\
\delta(k) &= Nk \sinh \lambda k / \sinh \pi k / 2 - 4e^{\frac{\lambda k}{2}} \sinh \lambda k / 2 \cosh \frac{k}{2}(3\lambda - \pi) q(k).
\end{align*}
$$

(4.7)

(4.8)

Other relations follow by applying Cauchy’s theorem to the auxiliary functions

$$
\begin{align*}
h_1(v) &= p(v - \frac{1}{2}i\lambda)[1 + p(v + \frac{1}{2}i\lambda)], \\
h_2(v) &= \frac{1 + p(v - \frac{1}{2}i\lambda)[1 + p(v + \frac{1}{2}i\lambda)]}{p(v + \frac{1}{2}i\lambda)}, \\
h_3(v) &= [1 + p(v - \frac{1}{2}i\lambda)[1 + p(v + \frac{1}{2}i\lambda)]]/q(v - \frac{1}{2}i\lambda),
\end{align*}
$$

(4.9)

(4.10)

(4.11)

which all satisfy the non-trivial analyticity property

$$
\int_{-\infty+\frac{1}{2}i\lambda}^{\infty+\frac{1}{2}i\lambda} \left(\ln q(v)\right)'' e^{-ikv} dv = \int_{-\infty-\frac{1}{2}i\lambda}^{\infty-\frac{1}{2}i\lambda} \left(\ln q(v)\right)'' e^{-ikv} dv.
$$

(4.12)

It follows, respectively, that

$$
\begin{align*}
\alpha(k) &= -e^{\lambda k} \overline{\alpha}(k), \\
A(k) - \delta(k) &= e^{\lambda k} \overline{A}(k) + \delta(k),
\end{align*}
$$

(4.13)

(4.14)

and

$$
\begin{align*}
e^{-\frac{1}{2}\lambda k} \left(A(k) - e^{\lambda k} q(k)\right)
&= e^{\frac{1}{2}\lambda k} \left(B(k) - \overline{\beta}(k) - q_1(k) + \frac{1}{2}Nke^{-\frac{1}{2}\lambda k} \cosh \left(\frac{3}{2}\lambda k - \frac{1}{2}\pi k\right)\right).
\end{align*}
$$

(4.15)
Now solving (4.6)–(4.15) and their complex conjugates in terms of the functions \( A \) and \( B \), we find

\[
\alpha(k) + \gamma(k) = F(k)A(k) + G(k)\overline{A}(k) + H(k)B(k) + \overline{H}(k)\overline{B}(k)
\]
\[
\beta(k) - \gamma(k) = \overline{F}(k)A(k) + \overline{G}(k)\overline{A}(k) + B(k)
\]
\[
C(k) = A(k) - B(k) \tag{4.16}
\]
\[
q(k) = \frac{Nke^{-\frac{1}{2}\pi k}}{2 \sinh \frac{1}{2}\pi k \cosh \frac{k}{2}(3\lambda - \pi)} - \frac{e^{-\frac{1}{2}(\pi + \lambda)k}A(k) - e^{-\frac{1}{2}(\pi - \lambda)k}\overline{A}(k)}{4 \sinh \lambda k \cosh \frac{k}{2}(3\lambda - \pi)}
\]

with

\[
F(k) = \frac{\sinh \frac{k}{2}(\pi - 3\lambda) - 2 \sinh \frac{k}{2}(\pi - 5\lambda)}{2 \sinh \lambda k \cosh \frac{k}{2}(3\lambda - \pi)}
\]
\[
G(k) = \frac{3e^{-\frac{1}{2}(\pi - 5\lambda)k} - 2e^{-\frac{1}{2}(\pi - 7\lambda)k} - 2e^{-\frac{1}{2}(\pi - 3\lambda)k} + e^{\frac{1}{2}(\pi - \lambda)k}}{4 \sinh \lambda k \cosh \frac{k}{2}(3\lambda - \pi)} \tag{4.17}
\]
\[
H(k) = -\frac{e^{-\frac{1}{2}\lambda k}}{2 \cosh \frac{1}{2}\lambda k}. \tag{4.18}
\]

Transforming back and integrating twice, we obtain a coupled set of nonlinear integral equations,

\[
\ln \alpha(v) + \ln \gamma(v) = F* \ln A + G* \ln \overline{A} + H* \ln B + \overline{H}* \ln \overline{B} + C,
\]
\[
\ln \beta(v) - \ln \gamma(v) = \overline{F}* \ln A + \overline{G}* \ln \overline{A} + \ln B, \tag{4.19}
\]

where we have introduced

\[
F(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)e^{ikv} dk,
\]
\[
G(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k)e^{ikv} dk, \tag{4.20}
\]
\[
H(v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(k)e^{ikv} dk.
\]

Taking the scaling limit as in (3.19), the nonlinear integral equations become

\[
\begin{pmatrix}
\ln a_+ + \ln c_+ \\
\ln b_+ - \ln c_+ \\
\ln \overline{a}_+ + \ln \overline{c}_+ \\
\ln \overline{b}_+ - \ln \overline{c}_+
\end{pmatrix} = \pm 4ie^{-x} \begin{pmatrix}
e^{-\pm \rho \lambda} \\
0 \\
-e^{\pm \rho \lambda} \\
0
\end{pmatrix} + K * \begin{pmatrix}
\ln A_+ \\
\ln B_+ \\
\ln \overline{A}_+ \\
\ln \overline{B}_+
\end{pmatrix} + C_+ \begin{pmatrix}1 \\
0 \\
0 \\
-1\end{pmatrix}, \tag{4.21}
\]
where the kernel $K$ again satisfies the very useful symmetry $K^T(x) = K(x)$. The integration constant in (4.19) follows from the limit $x \to \infty$. We have
\[ C_\pm = \pi i (\omega_\pm - \phi) / (2\lambda). \] (4.22)

Again the integration constants $C_\pm$ contain the essential information to obtain the conformal spectra.

4.2 Conformal spectra

For branches 3 and 4 the transfer matrix eigenvalues in (2.1) are dominated by
\[ T(v + i\lambda - \frac{1}{2}\pi i) \sim e^{i\phi} \Phi(v + i\lambda + \frac{1}{2}\pi i) \Phi(v + \frac{1}{2}\pi i) \frac{q(v - 3i\lambda)}{q(v - i\lambda)} A(v). \] (4.23)

Taking Fourier transforms and using the solution (4.16) for $q(k)$, we obtain
\[
\ln T(v + i\lambda - \frac{1}{2}\pi i) = -Nf_\infty(v) + \frac{\rho}{\pi} \int_{-\infty}^{\infty} \left( \frac{\ln A(w)}{\cosh 2\rho(v - w + \frac{1}{2}i\lambda)} - \frac{\ln A(w)}{\cosh 2\rho(v - w - \frac{1}{2}i\lambda)} \right) dw, \] (4.24)

where the bulk free energy is given by
\[
f_\infty(v) = 2 \int_{-\infty}^{\infty} dk \frac{\sinh(kv) \sinh(3k\lambda + ivk - \pi k) \cosh(5k\lambda - k\pi) \cosh(k\lambda)}{k \cosh(3k\lambda - \pi k) \sinh(k\pi)}. \] (4.25)

Taking the thermodynamic limit $N \to \infty$ and using the definition as in (3.19) gives
\[
\ln T(v) = -Nf_\infty(v) + \frac{2i}{N\pi} e^{2\rho v} \Im m \left( e^{i\lambda} \int_{-\infty}^{\infty} \ln A_+(x) e^{-x} \right)
+ \frac{2i}{N\pi} e^{-2\rho v} \Im m \left( e^{-i\lambda} \int_{-\infty}^{\infty} \ln A_-(x) e^{-x} \right) dx. \] (4.26)

To calculate the integral, we consider the expression
\[
\int_{-\infty}^{\infty} \left[ \begin{array}{c}
\ln a_\pm + \ln c_\pm \\
\ln b_\pm - \ln c_\pm \\
\ln \pi_\pm + \ln \tau_\pm \\
\ln \tilde{a}_\pm - \ln \tilde{c}_\pm
\end{array} \right] \left( \begin{array}{c}
\ln A_\pm \\
\ln B_\pm \\
\ln \tilde{A}_\pm \\
\ln \tilde{B}_\pm
\end{array} \right) \right] ^T dx.
\]

(4.27)

which can be evaluated exactly using the Rogers dilogarithmic function relation (3.30). We thus arrive at

\[
L \left( a_\pm (\infty) \right) + L \left( 1/a_\pm (\infty) \right) + L \left( 1/b_\pm (\infty) \right) + L \left( b_\pm (\infty) \right) + L \left( c_\pm (\infty) \right) + L \left( 1/c_\pm (\infty) \right) = \pi^2 - 8k\pi^2,
\]

(4.28)

where \( k = 0, 1 \) [28]. We have also used the asymptotics of the functions

\[
a_\pm (\infty) = e^{i(\omega_\pm - \phi)}(e^{i(\omega_\pm - \phi)} + 1),
\]

\[
b_\pm (\infty) = e^{2i(\omega_\pm - \phi)}/(e^{i(\omega_\pm - \phi)} + 1),
\]

\[
c_\pm (\infty) = e^{i(\omega_\pm - \phi)}
\]

and

\[
a_\pm (-\infty) = c_\pm (-\infty) = c_\pm (-\infty) = 0.
\]

On the other hand, substituting (4.21) into (4.27) we arrive at

\[
\pm 16\Im m \left( e^{\pm i\rho \lambda} \int_{-\infty}^{\infty} \ln A_\pm (x)e^{-x} \right) + \frac{\pi(\omega_\pm - \phi)^2}{\lambda}.
\]

(4.29)

Combining the results (4.28) and (4.29) we are left with

\[
\Im m \left( e^{\pm i\rho \lambda} \int_{-\infty}^{\infty} \ln A_\pm (x)e^{-x} \right) = \frac{\pi^2}{24} \left( \frac{3}{2} - \frac{3(\omega_\pm - \phi)^2}{2\pi \lambda} - 12k \right).
\]

(4.30)

Inserting this in the expression \( \ln T(v) \) we obtain the final result

\[
\ln T(v + i\lambda - \frac{1}{2} \pi i) = -N f_\infty (v) + \frac{\pi \sin(2i\rho v)}{6N} (c - 24\Delta).
\]

(4.31)

The central charges and conformal weights are given by

\[
c = \frac{3}{2} - \frac{3(\pi - 4\lambda)^2}{2\pi \lambda},
\]

(4.32)

\[
\Delta = \left\{ \begin{array}{ll}
\frac{(\omega - \phi \pm 2\epsilon \lambda)^2 - (\pi - 4\lambda)^2}{16\pi \lambda} + \Delta_{\text{Ising}} & \text{dilute O(n) model} \\
\frac{(\omega - \phi)^2 - (\pi - 4\lambda)^2}{16\pi \lambda} + \Delta_{\text{Ising}} & \text{dilute A_L face model}
\end{array} \right.
\]

(4.33)

with \( \Delta_{\text{Ising}} = 0, \frac{1}{2} \).
5 Summary and discussion

We have calculated the finite-size corrections to the transfer matrix eigen-spectra of the intimately related dilute O\((n)\), dilute \(A_L\) and Izergin-Korepin models at criticality via the nonlinear integral equation approach. The resulting conformal weights defining the critical exponents are seen to follow from appropriate branches of the log functions appearing in the Bethe equations. For the dilute O\((n)\) model the integration constants appearing in the key nonlinear integral equations (3.20) and (4.21) differ in the distinct limits \(v \to \pm \infty\), leading to two constants, \(C_{\pm}\), in the scaling limit. However, they satisfy |\(C_+\)| = |\(C_-\)|, allowing the calculation to go through in a similar manner as for the ABF model [30].

5.1 Dilute O\((n)\) model

Consider first the dilute O\((n)\) model in branches 1 and 2 in the \(u\) positive regime. Our final results are the central charge (3.34) and the conformal weights (3.35). These are in agreement with the previous results outlined in Sections 2.1 and 2.2 with the O\((n)\) \(\phi\) value (2.4). In particular, the magnetic dimensions (2.13) follow with the parameters \(\ell \neq 0\) and \(r = 0\) in (3.24). Similarly, the thermal dimensions (2.16) follow with \(\ell = 0\) and \(r = 2j\). As expected, in this regime the conformal dimensions are seen to be in agreement with the results obtained in the honeycomb limit.

On the other hand, in branches 3 and 4 of the \(u\) negative regime our final results are (4.32) and (4.33). Here the conformal dimensions are new. The conjectured magnetic dimensions (2.18) [18] are associated with \(\ell \neq 0\) and \(r = \ell\). The similarly conjectural thermal dimension \(X_1 = 2\Delta_{\text{Ising}} = 1\) follows from the choice \(\ell = 0\) and \(r = 0\). More generally we see that this dimension belongs to the thermal set

\[
X_{j+1} = \frac{j^2 \pi - j(\pi - 4\lambda)}{2\pi \lambda} + \Delta_{\text{Ising}},
\]

which is given by setting \(\ell = 0\) and \(r = 2j\) in (4.33).

5.2 Dilute \(A_L\) face model

Recall that the Bethe equations of the dilute \(A_L\) face model follow from the choice of crossing parameter given in (1.5) with seam \(\phi\) as given in (2.5). In
this case the branches 1 and 2 results (3.34) and (3.35) give the central charge (2.11) and conformal weights (2.15) of the unitary minimal series.

In a similar manner, the branches 3 and 4 results are indicative of the product of the unitary minimal series with the Ising model. The central charge is given by (2.11) and the conformal weights again by given by (2.15), with however, the additional $\Delta_{\text{Ising}}$ component. Here $\Delta_{\text{Ising}} = 0, \frac{1}{2}, \frac{1}{16}$ is expected, in accordance with the results of [20]. However, $\frac{1}{16}$ does not appear in our results. Our method needs further refinement to reveal this conformal dimension.

For the more general crossing parameter

$$\lambda = \frac{\pi}{4}(1 + \frac{k}{L+1})$$

with $|k| < [(L+1)/3]$, the integer part of the fraction $(L+1)/3$, our results (3.34) and (4.32), and (3.35) and (4.33) imply

$$c = 1 - \frac{6k^2}{(L+1)(L+1-k)}$$ branches 1 and 2 (5.35)

$$c = \frac{3}{2} - \frac{6k^2}{(L+1)(L+1-k)}$$ branches 3 and 4 (5.36)

for the central charges and

$$\Delta = \frac{[(L+1)t - (L+1-k)s]^2 - k^2}{4(L+1)(L+1-k)}$$ branches 1 and 2 (5.37)

$$\Delta = \frac{[(L+1)t - (L+1-k)s]^2 - k^2}{4(L+1)(L+1-k)} + \Delta_{\text{Ising}}$$ branches 3 and 4 (5.38)

for the conformal weights, where $s = 1, 2, \cdots, L$ and $t = 1, 2, \cdots, L - k$. These results indicate the non-unitary minimal models for the dilute $A_L$ models in branches 1 and 2 and the non-unitary minimal models plus an Ising model in branches 3 and 4. For the case $k = 1$ our results confirm the conformal weights presented in [20]. For $k > 1$ our results show that the dilute $A_L$ models in branches 1 and 2 can be classified by the same universality classes as for the ABF models [54,30].

5.3 Izergin-Korepin model

Our results for the $O(n)$ model reduce to those of the Izergin-Korepin model when the seam $\phi = 0$. In this way the central charge and conformal weights
are given by
\[ c = 1, \quad X_{\ell,r} = \frac{\ell^2 (\pi - 2\lambda)}{4\pi} + \frac{r^2 \pi}{\pi - 2\lambda} \quad (5.39) \]
for \(0 < u < 3\lambda\) with \(0 < \lambda < \pi/3\). On the other hand,
\[ c = \frac{3}{2}, \quad X_{\ell,m} = \frac{\lambda \ell^2}{2\pi} + \frac{m^2}{2\pi \lambda} + 2\Delta_{\text{Ising}} \quad (5.40) \]
for \(-\pi + 3\lambda < u < 0\) with \(0 < \lambda < \pi/3\). The result (5.39) is in agreement with the previous results in the honeycomb limit, while the result (5.40), reflecting also the additional Ising content, is new.

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