Computational approaches to initial-boundary value problems with Neumann boundary conditions

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ABSTRACT
In this paper, recursive solution schemes for different boundary value problems and initial-boundary value problems of partial differential equations with Neumann boundary conditions are proposed. The schemes are based on the Lesnic’s approach and the Advanced Adomian decomposition method (AADM). The Lesnic’s approach to homogeneous and inhomogeneous initial-boundary value problems was generalized. Also, the AADM to initial-boundary value problems was extended. Some examples were presented to demonstrate the high accuracy and efficiency of the proposed schemes.

1. Introduction
Many problems in science and engineering are formulated by initial and or boundary value problems such as in heat diffusion and wave propagation problems among others [1]. Further, many researchers obtained the solutions of initial and boundary value problems by using either initial or boundary conditions. In recent years, there has been significant development in the use of various semi analytical methods for partial differential equations (PDEs) such as the homotopy perturbation method [2], the homotopy analysis method [3], the variational iteration method [4] and the Adomian decomposition method (ADM) [5]. The ADM was applied to a wide class of differential and integral equations [6–13]; even though computation of Adomian polynomial remains tedious at times. However, solutions of these problems are valid only in one-directional domain; that is, either in time or space. Moreover, most of the existing methods are constructed for problems with Dirichlet boundary conditions [14–18], and very few methods with Neumann boundary conditions [19–24] due to their difficulties in dealing with. In [25], Adomian suggested a modified method for various PDEs with initial and boundary conditions. Further, Lesnic and Elliot [26] and Aly et al. [27] have separately proposed inverse linear operators to tackle certain equations with Neumann conditions, respectively. Moreover, in comparison with standard ADM; the authors in [26, 28] utilized definite integral operators that used all of the boundary conditions in a direct way and provide a convergent numerical solution to the correct limit, while the integral operator in the standard ADM is indefinite integral and thus need to calculate the constants of integration. See also [29–35] for convergence of the ADM and [36] for the newly introduce parametrized ADM method with optimum convergence.

However, having analysed the loopholes of the above mentioned methods, we therefore aim to generalize the Lesnic’s approach and the AADM to solve both the boundary value and initial boundary problems (linear and nonlinear) with Neumann boundary conditions. Again, it will be good to note that our proposed methods need no Adomian polynomials at times; while when such polynomials are needed, the steps are really less. The paper is organized as follows: Section 2 outlines the procedures for the existing methods; Section 3 proposes the improved methods; Section 4 presents the application of the proposed methods, and Section 5 is for conclusion.

2. Outline of the methods for boundary value problems
In this section, we give the outline of the existing methods including the Standard ADM, the Lesnic’s approach and the AADM.

2.1. Standard ADM with Neumann conditions
Consider the general second-order nonlinear inhomogeneous temporal-spatial PDE of the form:

\[ L_{xx}u(x, t) + L_{tt}u(x, t) + Nu(x, t) = g(x, t), \]  

(1)
with the Neumann boundary conditions
\[ u_x(a, t) = h_1(t), \quad u_x(b, t) = h_2(t). \] (2)

Where \( L_{xx}u(x, t) = \partial^2 / \partial^2 x^2 u(x, t), \) \( L_{tt}u(x, t) = \partial^2 / \partial^2 t^2 u(x, t), \) \( Nu(x, t) \) is a nonlinear operator which is assumed to be analytic and \( g(x, t) \) is an inhomogeneous term. According to the standard ADM, the inverse operator \( L_{xx}^{-1} \) is applied to Equation (1) and yields:
\[ u(x, t) = \Phi_x + L_{xx}^{-1} g(x, t) - L_{xx}^{-1} L_{tt} u(x, t) - L_{xx}^{-1} Nu(x, t), \] (3)

where \( \Phi_x = c_1(t) + c_2(t)x, c_1(t), \) and \( c_2(t) \) are constants of integration. The ADM decomposes the solution \( u(x, t) \) into an infinite series
\[ u(x, t) = \sum_{n=0}^{\infty} u_n(x, t), \] (4)
and the nonlinear term \( Nu(x, t) \) into a series
\[ Nu(x, t) = \sum_{n=0}^{\infty} A_n(u_0, u_1, u_2, \ldots u_n), \] (5)
where, \( A_n \)'s are called the Adomain polynomials to be obtained by the definitional formula:
\[ A_n = \frac{1}{n!} \frac{d^n}{dx^n} \left[ N\left( \sum_{i=0}^{\infty} \lambda_i u_i \right) \right]_{\lambda = 0}, \quad n \geq 0. \]

Also, we decompose \( \Phi_x \) as
\[ \Phi_x = \sum_{n=0}^{\infty} \Phi_{xn} = \sum_{n=0}^{\infty} (c_{1n}(t) + c_{2n}(t)x). \] (6)

Substituting Equations (4)–(6) into Equation (3), yields the following recursion solution scheme
\[ u_0 = \Phi_{x0} + L_{xx}^{-1} g(x, t), \]
\[ u_{n+1} = \Phi_{xn} - L_{xx}^{-1} L_{tt} u_n - L_{xx}^{-1} A_n, \quad n \geq 0. \] (7)

Thus, the approximate solution of Equation (1) is given by
\[ \bar{u}_n = \sum_{m=0}^{n-1} u_m(x, t), \quad n \geq 0, \]
and must satisfy the boundary conditions in Equation (2). Also, to evaluate the values of the constants \( c_{1n}(t) \) and \( c_{2n}(t) \), we begin with
\[ \bar{u}_1 = u_0 = c_{10}(t) + c_{20}(t) + L_{xx}^{-1} g(x, t). \]

Using the boundary conditions in Equation (2) results in
\[ c_{20}(t) + (L_{xx}^{-1} g(x, t))_{x=a} = h_1(t), \]
\[ c_{20}(t) + (L_{xx}^{-1} g(x, t))_{x=b} = h_2(t). \]

It should be noted that by solving these equations with respect to \( c_{20}(t) \) its value is obtained, while \( c_{10}(t) \) remains unknown and consequently \( u_0 \) is not completely determined; hence we cannot proceed with the standard ADM!

### 2.2. Lesnic’s approach with Neumann conditions

Lesnic and Elliot [26] proposed a different inverse linear operator defined by
\[ L_{xx}^{-1} = \int_{x_0}^{x} dx' \int_{x_0}^{x'} dx'' - \frac{(x-x_0)^2}{2(1-x_0)^2} \int_{x_0}^{1} dx', \] (8)
to solve the linear homogeneous heat equation with the Neumann boundary conditions in Equation (2) coupled with the ADM, where \( x' \) and \( x'' \) are dummy variables.

**Theorem 2.1** ([26]): If Neumann boundary conditions in Equation (2) are prescribed to the second-order PDE in Equation (1), then,
\[ L_{xx}^{-1} L_{xx} u(x, t) = (x-x_0) h_1(t) + \frac{(x-x_0)^2}{2(1-x_0)^2} (h_2(t) - h_1(t)) + c(t), \]
where \( c(t) \) is an unknown function to be determined by imposing at sufficiently large value of \( N \) and \( L_{xx}^{-1} \) is given in Equation (8).

Thus, considering the PDE in Equation (1) with Neumann boundary conditions in Equation (2) coupled to the application of Theorem 2.1 we get the recursive solution relations:
\[ u_0 = (x-x_0) h_1(t) + \frac{(x-x_0)^2}{2(1-x_0)^2} (h_2(t) - h_1(t)) + c(t) + L_{xx}^{-1} L_{tt} u_n - L_{xx}^{-1} A_n, \quad n \geq 0. \] (9)

### 2.3. AADM approach with Neumann conditions

Additionally, Aly et al. [27] defined a new different inverse linear operator given by
\[ L_{xx}^{-1} = \int_{\Omega} dx' \int_{a}^{x'} dx'' + \frac{1}{\Omega} \int_{0}^{\Omega} dx' \int_{0}^{x'} dx'' \left( x' \int_{0}^{x'} dx'' \right), \] (10)
where \( \Omega \) is an arbitrary finite constant to solve linear and nonlinear boundary value problems to avoid the restriction in the Lesnic’s approach in Equation (8). However, the basic idea remains the same with that of ADM; but in comparison with the ADM method, this method used definite integral operators and which in turn provide a convergent numerical solution to the correct limit.

**Theorem 2.2** ([27]): If Neumann boundary conditions in Equation (2) are prescribed to the second-order PDE in Equation (1), then,
\[ L_{xx}^{-1} L_{xx} u(x, t) = u(x, t) - (x-\Omega) h_1(t) - \frac{\Omega}{2} h_2(t) \]
\[ - \frac{1}{\Omega} \int_{0}^{\Omega} u(x, t) dx, \]
where \( a \leq x \leq b, \Omega \) is a finite constant, and \( L_{xx}^{-1} \) is in Equation (9).
3. Outline of the proposed methods for initial-boundary value problems

With the aim to devise better methods that take into account both the initial and boundary conditions comprising of both the Dirichlet and Neumann boundary conditions; we propose in this section the following improvements based on the Adomian’s idea of averaging [25].

3.1. Improved Lesnic’s approach

Consider the PDE given in Equation (1) with the Neumann conditions Equation (2) and the initial conditions

\[ u(x, 0) = p_1(x), \quad u_t(x, 0) = p_2(x). \]  

(12)

Firstly, we consider the \( t \) partial solution and applying the inverse operator \( L_{tt}^{-1} = \int_0^t \int_0^t u(x,t)dt \) to both sides of Equation (1), we get

\[ u(x, t) = u(x, 0) + tu_t(x, 0) + L_{tt}^{-1} g(x, t) - L_{tt}^{-1} L_{xx} u(x, t) - L_{tt}^{-1} N u(x, t). \]  

(13)

Secondly, we consider the \( x \) partial solution of Equation (1) via the application of Theorem 2.1

\[ u(x, t) = (x - x_0) h_1(t) + \frac{(x - x_0)^2}{2(1 - x_0)^2} h_2(t) - h_1(t) + c(t) + L_{xx}^{-1} g(x, t) - L_{xx}^{-1} L_{tt} u(x, t) - L_{xx}^{-1} N u(x, t). \]  

(14)

Finally, averaging Equations (13) and (14) we get a new recurrence solution scheme as follows:

\[
\begin{align*}
  u_0 &= \frac{1}{2} \left( p_1(x) + tp_2(x) + (x - x_0)h_1(t) \right) \\
  &\quad + \frac{(x - x_0)^2}{2(1 - x_0)^2} (h_2(t) - h_1(t)) \\
  &\quad + c(t) + L_{xx}^{-1} g(x, t) + L_{xx}^{-1} g(x, t), \\
  u_{n+1} &= -\frac{1}{2} \left( L_{tt}^{-1} L_{xx} u_n + L_{tt}^{-1} L_{tt} u_n + L_{xx}^{-1} A_n + L_{tt}^{-1} A_n \right), \\
  n &\geq 0. 
\end{align*}
\]  

(15)

Therefore, from our recursive solution scheme in Equation (15), we can see that all of the initial and boundary conditions are taken into account.

3.2. Improved AADM

Considering Equations (1)–(2) and the initial conditions in Equation (12). Taking the \( t \) partial solution and \( x \) partial solution of Equation (1) via the application of Theorem 2.2 and averaged as described above, we get other recurrence solution scheme as follows:

\[
\begin{align*}
  u_0 &= \frac{1}{2} \left( p_1(x) + tp_2(x) + (x - \Omega)h_1(t) + \frac{\Omega}{2} h_2(t) \right) \\
  &\quad + \frac{1}{2} \left( L_{tt}^{-1} g(x, t) + L_{xx}^{-1} g(x, t) \right), \\
  u_{n+1} &= -\frac{1}{2} \left( -\frac{1}{\Omega} \int_0^\Omega u_n(x,t)dx + L_{tt}^{-1} L_{xx} u_n \\
  &\quad + L_{xx}^{-1} L_{tt} u_n \right) \\
  &\quad - \frac{1}{2} \left( L_{xx}^{-1} A_n + L_{tt}^{-1} A_n \right), \quad n \geq 0, 
\end{align*}
\]  

(16)

which also takes into account all the initial and boundary conditions.

4. Application of the proposed methods

In this section, we test the efficiency of the proposed methods on certain initial and boundary value problems arising from mathematical physics applications as follows:

4.1. Application of the Improved Lesnic’s approach

Example One: Consider the linear inhomogeneous heat equation

\[ u_t = u_{xx} + xt^2, \quad 0 \leq x \leq 1, \quad t > 0, \]  

(17)

with specified conditions

\[ u(x, 0) = \sin(x), \quad u_x(0, t) = e^{-t} - \frac{1}{3}t^3, \quad u_x(1, t) = \cos(1)e^{-t} + \frac{1}{3}t^3. \]  

(18)

With the exact solution

\[ u(x, t) = e^{-t} \sin(x) + \frac{1}{3}xt^3. \]  

(19)

First, we consider the \( t \) partial solution and applying the inverse operator \( L_{tt}^{-1} = \int_0^t \int_0^t \) to Equation (17) and decompose accordingly, we get the recursive relation

\[
\begin{align*}
  u_0 &= \sin(x) + \frac{1}{3}x t^3, \\
  u_{n+1} &= L_{tt}^{-1} L_{xx} u_n, \quad n \geq 0. 
\end{align*}
\]  

(20)
Few terms of Equation (20) as
\begin{align*}
    u_0 &= \sin(x) + \frac{1}{3}x^3, \\
    u_1 &= -\sin(x)t, \\
    u_2 &= \frac{1}{2} \sin(x)t^2, \\
    \vdots
\end{align*}

Secondly, we consider the x partial solution as in Equation (18) accordingly to obtain the recursive relation
\begin{align*}
    u_0 &= x \left( e^{-t} + \frac{1}{3}t^3 \right) + \frac{x^2}{2} (\cos(1) - 1)e^{-t} + c(t) \\
    &\quad - \frac{1}{6} t^2x^3 + \frac{1}{4} t^2x^2, \\
    u_{n+1} &= L_{xx}^{-1} L_t u_n, \quad n \geq 0. \tag{21}
\end{align*}

Few terms of Equation (22) as
\begin{align*}
    u_0 &= x \left( e^{-t} + \frac{1}{3}t^3 \right) + \frac{x^2}{2} (\cos(1) - 1)e^{-t} + c(t) \\
    &\quad - \frac{1}{6} t^2x^3 + \frac{1}{4} t^2x^2, \\
    u_1 &= \frac{x^2}{120} \left( 5x^2 \cos(1)e^{-t} + 2tx^3 - 5x^2 e^{-t} \\
    &\quad - 20tx^2 - 5tx^2 - 10 \cos(1)e^{-t} \right) \nonumber \\
    &\quad + \frac{x^2}{120} (20xe^{-t} + 30t^2 - 20e^{-t} + 5t), \\
    u_2 &= \frac{x^2}{10080} \left( 14x^4 \cos(1)e^{-t} - 14x^4 e^{-t} - 4x^5 \right. \\
    &\quad - 70x^4 \cos(1)e^{-t} + 84x^3 e^{-t} \right) \\
    &\quad + \frac{x^2}{10080} (168tx^3 + 14x^4 - 142x^2 e^{-t} \\
    &\quad - 420tx^2 + 98 \cos(1)e^{-t} - 35x^2) \\
    &\quad + \frac{x^2}{10080} (112e^{-t} + 420t + 42), \tag{22}
\end{align*}

Next, we average the two recursive relations in Equations (20) and (22) to obtain the improved Lesnic’s approach scheme:
\begin{align*}
    u_0 &= \frac{1}{2} \left( \sin(x) + \frac{1}{3}x^3 + x \left( e^{-t} + \frac{1}{3}t^3 \right) \\
    &\quad + \frac{x^2}{2} (\cos(1) - 1)e^{-t} + c(t) \right) \\
    &\quad - \frac{1}{6} \left( \frac{1}{2} t^2x^3 + \frac{1}{4} t^2x^2 \right), \\
    u_{n+1} &= \frac{1}{2} \left( L_t^{-1} L_{xx} u_n + L_{xx}^{-1} L_t u_n \right), \quad n \geq 0. \tag{23}
\end{align*}

Few terms of Equation (22) are
\begin{align*}
    u_0 &= \frac{1}{2} \left( \sin(x) + \frac{1}{3}x^3 + x \left( e^{-t} + \frac{1}{3}t^3 \right) \\
    &\quad + \frac{x^2}{2} (\cos(1) - 1)e^{-t} + c(t) \right) \\
    &\quad - \frac{1}{6} \left( \frac{1}{2} t^2x^3 + \frac{1}{4} t^2x^2 \right), \\
    u_1 &= \frac{-x^2}{240} \left( 5x^2 \cos(1)e^{-t} + 2tx^3 - 5x^2 e^{-t} \\
    &\quad - 20tx^2 - 5tx^2 - 10 \cos(1)e^{-t} \right) \nonumber \\
    &\quad + \frac{x^2}{240} (20xe^{-t} + 30t^2 - 20e^{-t} + 5t - \sin(x)t), \\
    u_2 &= \frac{x^2}{20160} \left( 14x^4 \cos(1)e^{-t} - 14x^4 e^{-t} - 4x^5 \right. \\
    &\quad - 70x^4 \cos(1)e^{-t} + 84x^3 e^{-t} \right) \\
    &\quad + \frac{x^2}{20160} (168tx^3 + 14x^4 - 142x^2 e^{-t} \\
    &\quad - 420tx^2 + 98 \cos(1)e^{-t} - 35x^2) \\
    &\quad + \frac{x^2}{20160} (112e^{-t} + 420t + 42 + \frac{1}{2} \sin(x)t^2), \tag{25}
\end{align*}

Thus, the approximate solution of Equation (17) is obtained by summing the above iterates as follows:
\begin{align*}
    \eta_n &= \sum_{n=0}^{m-1} u_n(x, t), \quad m > 0.
\end{align*}

The absolute errors for this method at different times levels are given in Table 1 and plotted in Figures 1 and 2.

Example Two: Consider the linear homogeneous wave equation
\begin{align*}
    u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{26}
\end{align*}

with specified conditions
\begin{align*}
    u(x, 0) &= 0, \quad u_t(x, 0) = 2 \cos(x) u_x(0, t) = 0, \\
    u_x(1, t) &= -\sin(1) \sin(2t). \tag{27}
\end{align*}

With the exact solution
\begin{align*}
    u(x, t) &= \cos(x) \sin(2t). \tag{28}
\end{align*}

Proceeding as above, we obtain the improved Lesnic’s approach scheme:
\begin{align*}
    \eta_0 &= \frac{1}{2} \left( 2t \cos(t) - \frac{x^2}{2} \sin(1) \sin(2t) + c(t) \right), \\
    \eta_{n+1} &= \frac{1}{2} \left( L_t^{-1} L_{xx} \eta_n + L_{xx}^{-1} L_t \eta_n - 3L_{xx}^{-1} \eta_n - 3L_t^{-1} \eta_n \right), \\
    n &\geq 0. \tag{29}
\end{align*}
Table 1. Absolute errors of the Improved Lesnic’s approach of Example One.

| x  | $\phi_1$  | $\phi_3$  | $\phi_5$  | $\phi_7$  |
|----|-----------|-----------|-----------|-----------|
| 0.0| 1.37786718e−03 | 7.20339386e−06 | 3.03331367e−03 | 1.15724731e−05 |
| 0.1| 1.19205842e−03 | 7.86093846e−06 | 3.15936032e−03 | 7.15422778e−05 |
| 0.2| 8.79066506e−04 | 8.37940843e−06 | 8.90275073e−04 | 1.29831117e−04 |
| 0.3| 6.02655510e−04 | 7.22254200e−06 | 3.32388914e−03 | 9.46694835e−05 |
| 0.4| 3.32664904e−05 | 6.15519070e−06 | 5.48325078e−03 | 2.05762831e−04 |
| 0.5| 5.63705972e−06 | 4.85058232e−06 | 8.27636604e−03 | 2.90714429e−04 |
| 0.6| 1.08909354e−06 | 3.48822771e−06 | 1.06837002e−02 | 3.38814549e−04 |

Thus, we simulate the scheme in Equation (29) and reported the absolute errors at different times levels in Table 2 and plotted in Figures 3 and 4.

Figure 1. Comparison of the exact, Lesnic and improved Lesnic solutions for $\phi_2$ at $t = 0.5$.

Figure 2. Comparison of the exact, Lesnic and improved Lesnic solutions for $\phi_3$ at $t = 1.0$.

Figure 3. Comparison of the exact, Lesnic and improved Lesnic solutions for $\phi_5$ at $t = 0.5$.

Figure 4. Comparison of the exact, Lesnic and improved Lesnic solutions for $\phi_5$ at $t = 1.0$.

Table 2. Absolute errors of the improved Lesnic’s approach of Example Two.

| x  | $\phi_1$  | $\phi_3$  | $\phi_5$  | $\phi_7$  |
|----|-----------|-----------|-----------|-----------|
| 0.0| 9.78409294e−05 | 1.24461399e−08 | 1.20179533e−02 | 2.50079275e−05 |
| 0.1| 1.01363624e−04 | 5.40703719e−08 | 1.19622480e−02 | 9.46493865e−05 |
| 0.2| 1.11356042e−04 | 1.74673143e−07 | 1.17952322e−02 | 2.46852214e−04 |
| 0.3| 1.27329302e−04 | 3.70894623e−07 | 1.15177768e−02 | 2.42704392e−04 |
| 0.4| 1.46974006e−04 | 6.00156261e−07 | 1.11307074e−02 | 2.36699706e−05 |
| 0.5| 1.68314564e−04 | 8.63002474e−07 | 1.06358432e−02 | 2.25072830e−05 |
| 0.6| 1.89090938e−04 | 1.12581833e−06 | 1.00357213e−02 | 2.18459213e−05 |
| 0.7| 2.06285951e−04 | 1.36279616e−06 | 9.33886460e−03 | 2.05894746e−05 |
| 0.8| 2.18182015e−04 | 1.55065900e−06 | 8.52393620e−03 | 1.90894698e−05 |
| 0.9| 2.22790437e−04 | 1.67090491e−06 | 7.64550673e−03 | 1.73424044e−05 |
| 1.0| 2.18962307e−04 | 1.71165227e−06 | 6.67281480e−03 | 1.53541940e−05 |
Example Three: Consider the nonlinear homogeneous equation

\[ u_t + uu_x = u_{xx}, \quad 0 \leq x \leq 1, \quad t > 0, \]

with specified conditions

\[ u_x(0, t) = \frac{2}{1 + 2t}, \quad u_x(1, t) = \frac{2}{1 + 2t}. \]

The exact solution is given by

\[ u(x, t) = \frac{2}{1 + 2t}x. \]

As in above, we get the improve Lesnic’s approach scheme:

\[ u_0 = \frac{2}{1 + 2t}x, \]

\[ u_{n+1} = L_{xx}^{-1}L_{tt}u_n + L_{xx}^{-1}A_n, \quad n \geq 0, \]

where, \( A_n \)'s are the Adomian polynomials of the nonlinear term \( uu_x \) expressed using Equation (5) with few terms

\[ A_0 = u_0u_{0x}, \]

\[ A_1 = u_1u_0 + u_0u_{1x}, \]

\[ A_2 = u_2u_0 + u_1u_{1x} + u_0u_{2x}, \]

\[ \vdots \]

So that,

\[ u_0 = \frac{2}{1 + 2t}x, \]

\[ u_{n+1} = 0, \quad n \geq 0. \]

Thus we obtain the exact solution in one iteration without calculating the Adomian polynomials.

4.2. Application of the Improved AADM

Example One: Consider the linear inhomogeneous wave equation

\[ u_{tt} = u_{xx} + 2\pi^2 e^{-t} \sin(\pi x), \quad 0 \leq x \leq 1, \quad t > 0, \quad (30) \]

with specified conditions

\[ u(x, 0) = \sin(\pi x), \quad u_t(x, 0) = -\pi \sin(\pi x), \quad u_x(0, t) = \pi e^{-\pi t}, \quad u_x(1, t) = -\pi e^{-\pi t}. \quad (31) \]

With the exact solution

\[ u(x, t) = e^{-\pi t} \sin(\pi x). \quad (32) \]

As described in the improved AADM procedure, we get the solution of Equations (30)–(31) given the recursive solution scheme:

\[ u_0 = \frac{1}{2} ((1 - \pi t) \sin(\pi x) + \pi (x - \Omega) e^{-\pi t} - \frac{\Omega \pi}{2} e^{-\pi t}) + \frac{1}{2} \left( l_{tt}^{-1}(2\pi^2 e^{-\pi t} \sin(\pi x)) + l_{xx}^{-1}(2\pi^2 e^{-\pi t} \sin(\pi x)) \right), \]

\[ u_{n+1} = \frac{1}{2} \left( -\frac{1}{\Omega^2} \int_0^\Omega u_n(x, t) dx + l_{tt}^{-1}l_{xx}u_n + l_{xx}^{-1}l_{tt}u_n \right), \quad n \geq 0. \quad (33) \]

Hence, simulating the scheme in Equation (33); we report the absolutes errors at different times levels in Table 3 and plotted in Figures 5 and 6.

Example Two: Consider the nonlinear inhomogeneous wave equation

\[ u_{tt} = u_{xx} + u + u^2 - xt - x^2t^2, \quad 0 \leq x \leq 1, \quad t > 0, \quad (34) \]

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**Figure 5.** Comparison of the exact, AADM and improved AADM solutions for \( \beta_2 \) at \( t = 0.5 \).

**Table 3.** Absolute errors of the Improved AADM of Example One.

| x   | \( \beta_1 \) | \( \beta_2 \) | \( \beta_3 \) | \( \beta_4 \) |
|-----|---------------|---------------|---------------|---------------|
| 0.0 | 0.00000000e+00 | 0.00000000e+00 | 0.00000000e+00 | 0.00000000e+00 |
| 0.1 | 1.24602435e-08 | 1.53119317e-14 | 2.58611018e-09 | 3.15423094e-15 |
| 0.2 | 1.58534032e-06 | 3.12994919e-11 | 3.29650561e-07 | 6.50650998e-12 |
| 0.3 | 2.69110415e-05 | 2.69738794e-09 | 5.59146029e-06 | 6.61023260e-10 |
| 0.4 | 1.99609734e-04 | 6.36178795e-08 | 4.11519581e-05 | 1.32248578e-08 |
| 0.5 | 9.40625502e-04 | 7.36430921e-07 | 1.95536737e-04 | 1.53088948e-07 |
| 0.6 | 3.32064156e-03 | 5.43414960e-06 | 6.90293560e-04 | 1.12948728e-06 |
| 0.7 | 9.59556565e-03 | 2.93779078e-05 | 1.95555175e-03 | 6.10706702e-06 |
| 0.8 | 2.39571776e-02 | 1.26439037e-04 | 4.98073595e-04 | 2.62841496e-05 |
| 0.9 | 5.34250805e-02 | 4.57080210e-04 | 1.11059831e-02 | 9.50176405e-05 |
| 1.0 | 1.08938027e-01 | 1.43962234e-03 | 2.26459908e-02 | 2.99268082e-04 |
with specified conditions
\[ u(x, 0) = 0, \quad u_t(x, 0) = x, \quad u_x(0, t) = t, \quad u_x(1, t) = t. \] (35)

With the exact solution
\[ u(x, t) = xt. \] (36)

proceeding as above, the improved AADM recursive solution scheme for Equations (35)–(36) is given by:
\[
\begin{align*}
\alpha_0 &= \frac{1}{2} \left( x t + (x - \Omega) t + \frac{\Omega}{2} t - L_{\Omega}^{-1}(x t + \Omega^2 t^2) \right) - L_{\Omega}^{-1}(x t + \Omega^2 t^2), \\
\alpha_n &= \frac{1}{2} \left( \frac{1}{\Omega} \int_0^\Omega u_t(x, t) dx + L_{\Omega}^{-1} L_{\Omega} u_n + L_{\Omega}^{-1} L_{\Omega} u_n \right) + \frac{1}{2} \left( L_{\Omega}^{-1} A_n + L_{\Omega}^{-1} A_n + L_{\Omega}^{-1} u_n + L_{\Omega}^{-1} u_n \right), \\
n &\geq 0,
\end{align*}
\] (37)

where, \( A_n \)'s are the Adomian polynomials of the nonlinear term \( u^2 \) expressed using Equation (5) with few terms
\[
\begin{align*}
A_0 &= u_0^2, \\
A_1 &= 2u_0u_1, \\
A_2 &= 2u_0u_2 + u_1^2, \\
A_3 &= 2u_1u_2 + 2u_0u_3, \\
&\vdots
\end{align*}
\]

Hence, simulating the scheme in Equation (37); we report the absolutes errors at different times levels in Table 4 and plotted in Figures 7 and 8.

5. Conclusion

In this research, boundary value problems and initial-boundary value problems have been investigated. Improved methods for solving PDEs with Neumann conditions are presented and provided with the recurrence scheme formulae for both the boundary value and initial-boundary value problems. The formulae presented were based on the Lesnic’s approach and AADM. Further, these schemes reduced the size of errors at the same time increase the accuracy of the solutions as shown in the given tables; and required less computational work since they do not need to calculate the constants of integration and regard Adomian polynomials meaningless at times. It is also important to mention here that, the propose recursive solution schemes take into account the entire initial and boundary conditions to make possible physically realistic solutions. In future work, further types of problems will be investigated.

Table 4. Absolute errors of the Improved AADM of Example Two.

| x     | \( \vartheta_3 \) | \( \vartheta_5 \) | \( \vartheta_3 \) | \( \vartheta_5 \) |
|-------|-------------------|-------------------|-------------------|-------------------|
|       | \( t = 0.5 \)     | \( t = 0.5 \)     | \( t = 1.0 \)     | \( t = 1.0 \)     |
| 0.0   | 0.00000000e+00    | 0.00000000e+00    | 0.00000000e+00    | 0.00000000e+00    |
| 0.1   | 6.43399997e-12    | 6.91549754e-19    | 2.66666666e-11    | 6.5234098e-19     |
| 0.2   | 3.42085723e-10    | 4.12363452e-15    | 3.877713e-09      | 1.773477e-15      |
| 0.3   | 2.98029447e-09    | 6.9030533e-13     | 8.0598656e-08     | 1.5563443e-13     |
| 0.4   | 9.29515458e-08    | 2.68378558e-11    | 7.28836349e-07    | 2.9016631e-12     |
| 0.5   | 8.01884656e-07    | 4.6822023e-10     | 4.16765332e-06    | 1.5919835e-11     |
| 0.6   | 4.23896826e-06    | 4.91557088e-09    | 1.78086039e-05    | 6.36894219e-11    |
| 0.7   | 1.67341058e-05    | 3.63247546e-08    | 6.21706559e-05    | 1.46104651e-09    |
| 0.8   | 5.41244863e-05    | 2.0754766e-07     | 1.87154966e-04    | 1.07511856e-08    |
| 0.9   | 1.51694697e-04    | 9.74287135e-07    | 5.02763454e-04    | 4.85612001e-08    |
| 1.0   | 3.77909382e-04    | 3.91713845e-06    | 1.23449974e-03    | 1.39014474e-07    |

Figure 6. Comparison of the exact, AADM and improved AADM solutions for \( \vartheta_3 \) at \( t = 1.0 \).

Figure 7. Comparison of the exact, AADM and improved AADM solutions for \( \vartheta_5 \) at \( t = 0.5 \).

Figure 8. Comparison of the exact, AADM and improved AADM solutions for \( \vartheta_5 \) at \( t = 1.0 \).
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