Abstract

We characterize an arbitrary de Branges space with bi-Lipschitz phase for large distances as a subspace of a weighted Paley–Wiener space, consisting of the elements square-integrable against a heavier weight on the real line. In particular, such a space is embedded with equivalence of norms in a classical Paley–Wiener space.

1 Introduction

1.1 Preliminaries

The inverse Fourier transform, defined by

$$\mathcal{F}^{-1}[\phi](z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{izt} \phi(t) \, dt, \quad z \in \mathbb{C}$$

maps $L^2[-\pi, \pi]$ to a space of entire functions known as the classical Paley–Wiener space and denoted by $L^2_{\pi}$. By the Paley–Wiener theorem, $L^2_{\pi}$ consists of all entire functions of exponential type at most $\pi$ that are square-integrable on $\mathbb{R}$. In other words,

$$L^2_{\pi} = \{ f \text{ entire} : \| f \|_2 < \infty, \ |f(z)| \leq C e^{(\pi+\epsilon)|z|} \},$$

where the norm, $\| \cdot \|_2$, is induced by the usual Hermitian product $\langle f, g \rangle_2 = \int_{-\infty}^{\infty} f(t)\overline{g(t)} \, dt$.

The Hilbert space $L^2_{\pi}$ admits a reproducing kernel at each $\zeta \in \mathbb{C}$, that is, a function $k_\zeta \in L^2_{\pi}$ satisfying $\langle f, k_\zeta \rangle_2 = f(\zeta)$. It even admits an orthonormal basis of reproducing kernels, namely $\{ k_n \}_{n \in \mathbb{Z}}$. Explicitly,

$$k_\zeta(z) = \frac{\sin(\pi(z - \zeta))}{\pi(z - \zeta)},$$

yielding the following expansion [1, p.150]: for $f \in L^2_{\pi}$,

$$f(z) = \sum_{n=-\infty}^{\infty} d_n \frac{\sin(\pi(z - n))}{\pi(z - n)}.$$
In [2], de Branges has identified a large class of spaces of entire functions which also admit orthogonal bases of reproducing kernels [2, p.55] and can be seen as a generalization of $L_2^\pi$. A Hilbert space $\mathcal{H}$ of entire functions is a de Branges space if it satisfies the following properties [2, p.57]:

1. The linear functional $\mathcal{H} \to \mathbb{C}$, $f \mapsto f(z_0)$ is bounded for all $z_0 \in \mathbb{C}$;

2. If $f(z) \in \mathcal{H}$, then $f^*(z)$ also belongs to $\mathcal{H}$ and has the same norm as $f(z)$, where $f^*(z) = \overline{f(\bar{z})}$;

3. If $f(z) \in \mathcal{H}$ and $f(z_0) = 0$, then $f(z)\frac{z - z_0}{z - \bar{z}_0}$ also belongs to $\mathcal{H}$ and has the same norm as $f(z)$.

By the Riesz lemma, the first property ensures that $\mathcal{H}$ admits a reproducing kernel at each $\zeta \in \mathbb{C}$.

Let us see how to construct a de Branges space starting from an Hermite–Biehler function, that is, an entire function $E(z)$ such that $|E(\bar{z})| < |E(z)|$ for $\Im z > 0$. Recall that a function $f(z)$ is of bounded type in a domain $D \subseteq \mathbb{C}$ if it is the ratio of two analytic, bounded functions in $D$, itself analytic in $D$. By a theorem of Nevanlinna [2, p.22], assuming that the zeroes of $f(z)$ do not accumulate at the origin, $f(z)$ is of bounded type in the upper half-plane if and only if

$$f(z) = B(z)e^{-ih\cdot e^{G(z)}},$$

where $B(z)$ is a Blaschke product, $h \in \mathbb{R}$, and $G(z)$ is an analytic function in $\mathbb{C}^+$ whose real part is the Poisson transform of a signed measure on $\mathbb{R}$. The mean type of $f(z)$ is then defined as $h$. It is characterized as

$$h = \limsup_{y \to \infty} \frac{\log |f(iy)|}{y}.$$

Let us denote by $\mathcal{N}_h^+$ the class of functions of bounded type in $\mathbb{C}^+$ whose mean type does not exceed $h$. For an Hermite–Biehler function $E(z)$,

$$\mathcal{H}(E) = \{ f \text{ entire} ; \|f/E\|_2 < \infty \text{ and } f/E, f^*/E \in \mathcal{N}_h^+ \}$$

is a concrete example of a de Branges space. Indeed, de Branges proved that each de Branges space is isometrically equal to a space of the form $\mathcal{H}(E)$ [2, p.57]. Notice that $L_2^\pi$ itself is a de Branges space, obtained from the Hermite–Biehler function $E(z) = e^{-i\pi z}$.

It is possible to define the notion of interpolating sequence in a Hilbert space of analytic functions admitting reproducing kernels [3, p.21], in particular in a de Branges space. Let $\mathcal{H}$ be a de Branges space and $k_z \in \mathcal{H}$ be the reproducing kernel at $z \in \mathbb{C}$. The sequence of complex number $\{z_j\}$ is interpolating for $\mathcal{H}$ if there exists an $f \in \mathcal{H}$ satisfying $f(z_j) = a_j$ for any choice of interpolation data $\{a_j/\|k_z\|\} \in \ell^2(\mathbb{C})$. It is complete interpolating if in addition $f$ is unique.

In the sequel we shall say that two positive functions $f$ and $g$ are comparable and write $f \simeq g$ if there exist constants $A, B > 0$ such that $Af \leq g \leq Bf$. 

2
A sequence \( \{z_j\} \) is sampling for \( \mathcal{H} \) if \( \|f\|^2 \simeq \sum |f(z_j)|^2/\|k_{z_j}\|^2 \). The notions of sampling and interpolating sequences are often presented as dual. It is well known [4, p.3] that \( \{z_j\} \) is complete interpolating if and only if it is interpolating and sampling.

In [5], Lyubarskii and Seip characterized geometrically the complete interpolating sequences for a large class of de Branges spaces that includes \( L^2_\pi \), defined as follows. Given a de Branges space \( \mathcal{H} \), let \( M(z) = \|k_z\| \). \( \mathcal{H} \) is a weighted Paley–Wiener space if the restriction of \( M \) to the real axis is a majorant-weight, that is, if

1. \( M(x) > 0 \) for all \( x \in \mathbb{R} \);
2. \( \|f\| \simeq \|f/M\|_2 \) for all \( f \in \mathcal{H} \).

Concrete examples of weighted Paley–Wiener spaces are produced as follows. Let \( m(t) \simeq 1 \) be a measurable function on \( \mathbb{R} \). The potential of the measure \( m(t) \, dt \) is defined as

\[
\omega_m(z) = \int_{-\infty}^{\infty} \log^* \left| 1 - \frac{z}{t} \right| \, m(t) \, dt,
\]

where

\[
\log^* \left| 1 - \frac{z}{t} \right| = \begin{cases} 
\log |1 - z/t| + (\Re z)/t & \text{if } |t| \leq 1 \\
\log |1 - z/t| & \text{otherwise}.
\end{cases}
\]

Then,

\[
PW(m) = \{ f \text{ entire } ; \|fe^{-\omega_m}\|_2 < \infty \text{ and } |f(z)e^{-\omega_m(z)}| \leq C e^{\varepsilon|\Im z|} \}
\]
is a weighted Paley–Wiener space. For \( g(z) \) real-entire, \( e^gPW(m) \) is also a weighted Paley–Wiener space. Indeed, Lyubarskii and Seip proved that any weighted Paley–Wiener space is equal with equivalence of norms to a space of the form \( e^gPW(m) \). As an example, \( PW(1) \), the simplest weighted Paley–Wiener space, is equal to \( L^2_\pi \).

1.2 Main result

Studying classes of de Branges spaces is partly motivated by an attempt to prove the Feichtinger conjecture in special cases. In our settings, this last asserts that each sampling sequence can be written as a finite union of interpolating sequences.

This conjecture appears to be true for any weighted Paley–Wiener space. The Hermite–Biehler function of such a space may be chosen with zeroes equidistributed on the same line [5], in the sense that the distance between two consecutive zeroes is comparable with 1. At the other extreme, de Branges spaces coming from an Hermite–Biehler function with sparse zeroes have also been investigated: for them, the Feichtinger conjecture is also true [6].

The question is to identify a class of de Branges space broader than the weighted Paley–Wiener spaces, but still opposite to the sparse case, in which
some structural results may be obtained [4, p.5]. Indeed, a class of de Branges spaces larger than the weighted Paley–Wiener spaces is already studied in [5]. Its definition and the results implicitly present in [5] are given in the appendix. These spaces are called MC-spaces in the expository paper [7]. For the moment, let us mention that some weighted Paley–Wiener spaces are obtained from an Hermite–Biehler function whose zeroes get arbitrarily close to the real axis. However, severe restrictions about the location of these zeroes are then obtained: taken as axioms, such limitations define the MC-spaces.

The present paper propose to study a class of de Branges spaces larger than the class of weighted Paley–Wiener spaces, even larger than the class of MC-spaces, but for which a structural result is derived. Let $\mathcal{H}(E)$ be a de Branges space whose Hermite–Biehler function $E$ does not have any real zeroes. Then $E(x)$ admits a polar decomposition $|E(x)|e^{-i\varphi(x)}$ on the real axis, where $\varphi(x)$, the so-called phase, is real-analytic and well-defined up to the addition of $2k\pi$. It is well-known that $\varphi(x)$ is also increasing [2, p.54]. We are interested in the case where $\varphi$ is bi-Lipschitz for large distances: there exist positive constants $N$, $C_1$, and $C_2$ such that

$$C_1(x_2 - x_1) \leq \varphi(x_2) - \varphi(x_1) \leq C_2(x_2 - x_1)$$

whenever $x_2 - x_1 \geq N$, where $C_1$, $C_2$, and $N$ are independent of $x_1$ and $x_2$.

Let us state our main result. Following Lyubarskii and Seip [5], we shall say that $f(z)$ is of $\omega_m$-type if for each $\varepsilon > 0$, there exists a $C_\varepsilon > 0$ such that

$$|f(z)| \leq C_\varepsilon e^{\varepsilon|z|}e^{\omega_m(z)}$$

in the complex plane.

**Theorem 1.1** Let $\mathcal{H} = \mathcal{H}(E)$ be a de Branges space, with $E(x) = |E(x)|e^{-i\varphi(x)}$ on the real line. Assume $\varphi$ is bi-Lipschitz for large distances. Then, there exists a measurable $m \simeq 1$ and a real-entire $g(z)$ such that

$$\mathcal{H} \subseteq e^g PW(m).$$

Namely,

$$e^{-g}\mathcal{H} = \{f \text{ entire } ; \|f(x)e^{-\omega_m(x)}e^{\theta(x)}\|_2 < \infty, \ |f(z)|e^{-\omega_m(z)} \leq C_\varepsilon e^{\varepsilon|\text{Re} z|} \}$$

where $\theta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \left(1 + \frac{1}{(t-x)^2}\right) \varphi'(t) \, dt$.

In their study of weighted Paley–Wiener spaces, Lyubarskii and Seip developed a multiplier lemma in the line of Beurling [8]. It produces for $\tau > \sup m$ an Hermite–Biehler function $E_{\tau-m}$ whose zeroes are equidistributed along the axis $\text{Re} z = -1$. Let $\Sigma_{\tau-m}$ be the set of zeroes of $E_{\tau-m}$. The multiplication by $E_{\tau-m}$ is then a bijection with equivalence of norms from $PW(m)$ to

$$L^2_{\tau\tau}[\Sigma_{\tau-m}] = \{f \text{ entire } ; \|f(x)\|_2 < \infty, \ |f(z)| < C_\varepsilon e^{(\pi\tau+\varepsilon)|\text{Re} z|}, \ f(\Sigma_{\tau-m}) = 0\}.$$
In this way, Lyubarskii and Seip transferred results about sampling and interpolation from $L^2_{\Sigma}[\Sigma_{\tau-m}]$ to $PW(m)$. Our theorem implies that a similar transfer may be done from the following class of spaces,

$$\{ f \text{ entire} : \|f(x)e^{\theta(x)}\|_2 < \infty, \ |f(z)| < C_\varepsilon e^{(\pi\tau+\varepsilon)|\Im z|}, \ f(\Sigma_{\tau-m}) = 0 \}$$

to the class of de Branges spaces with bi-Lipschitz phase for large distances.

For proving our theorem we shall mimick the simplest part of Lyubarskii and Seip’s study of weighted Paley–Wiener spaces with two modifications. Firstly, we shall generalize the definition of $\omega_\gamma$ so that $\gamma(x)$ may be any positive, continuous function whose antiderivative $\phi(x)$ satisfies $|\phi(x) - \phi(0)| \simeq |x|$. Efforts are done for showing that $\omega_\gamma$ satisfies the expected properties of the potential of a measure in $\mathbb{C}$ (see [9]). Secondly, we shall generalize Lyubarskii and Seip’s multiplier lemma so the zeroes of the resulting Hermite–Biehler function may be multiple.

**Notation and terminology** For positive functions $f$ and $g$, we write $f \lesssim g$ if there exists a constant $C$ such that $f \leq Cg$ pointwise. We write $f \simeq g$ and say that $f$ is comparable with $g$ if $f \lesssim g$ and $g \lesssim f$.

For real-valued functions $f$ and $g$, we write $f \gg g$ if $f > g + \varepsilon$ for a certain $\varepsilon > 0$.

For a complex-valued function $f$, we define $f^*(z) = \overline{f(z)}$. The $L^2$-norm of $f(z)$ is defined as $\|f\|^2_2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx$, provided that this last integral converges. Finally, we say that $f(z)$ is real-entire if it is an entire function real on the real line.

**Acknowledgements** The author would like to acknowledge Prof. Kristian Seip for his reading and encouraging comments; and Prof. Dmitry Jakobson for his kind invitation to present the result at McGill’s Analysis Seminar.

## 2 Proofs

### 2.1 Considered potentials

Let $\varphi(x)$ be a continuously differentiable, increasing function satisfying $|\varphi(x)| \simeq |x|$ for $|x|$ large. Our aim is to verify that

$$\omega_\varphi(z) = \int_{-\infty}^{\infty} \log^* \left| 1 - \frac{z}{t} \right| \varphi'(t) \, dt$$

satisfies the expected properties of a potential.

Let us show that the above integral is absolutely convergent. The condition on $\varphi$ ensures that for $R > |z| + 1$,

$$\int_{|t| > R} \frac{\partial}{\partial t} \left( \log^* \left| 1 - \frac{z}{t} \right| \right) \varphi(t) \, dt = \int_{|t| > R} \Re \left( \frac{z^2}{t^2(t-z)} \right) \varphi(t) \, dt$$
is well-defined. The following integral is thus also well-defined,
\[ \int_{|t|> R} \log^+ |1 - \frac{z}{t}| \varphi'(t) \, dt = A_R(z) - \int_{|t|> R} \left( \frac{\partial}{\partial t} \log^+ |1 - \frac{z}{t}| \right) \varphi(t) \, dt, \]
where \( A_R(z) = \log^+ |1 + z/R| \varphi(-R) - \log^+ |1 - z/R| \varphi(R) \). Since its integrand changes sign finitely many times, it is absolutely convergent. The result follows.

The continuity of \( \omega_{\varphi'} \) also follows from the previous relation, by applying the dominated convergence theorem to the integral on its right-hand side.

Let us show that for \( y \neq 0 \), \( z = x + iy \), \( \frac{\partial}{\partial y} \omega_{\varphi'} \) may be calculated by interchanging the derivative and the integral. The dominated convergence theorem implies
\[ \frac{\partial}{\partial y} \int_{|t|> R} \log^+ |1 - \frac{z}{t}| \varphi'(t) \, dt = \frac{\partial}{\partial y} A_R(z) - \int_{|t|> R} \frac{\partial}{\partial y} \left( \frac{\partial}{\partial t} \log^+ |1 - \frac{z}{t}| \right) \varphi(t) \, dt. \]
The last integral in the above relation may be evaluated by parts, yielding
\[ \frac{\partial}{\partial y} \int_{|t|> R} \log^+ |1 - \frac{z}{t}| \varphi'(t) \, dt = \int_{|t|> R} \frac{\partial}{\partial y} \log^+ |1 - \frac{z}{t}| \varphi'(t) \, dt. \]
The dominated convergence theorem may also be used for the interval of integration \([−R, R] \), yielding in total
\[ \frac{\partial}{\partial y} \omega_{\varphi'}(z) = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} \log^+ |1 - \frac{z}{t}| \varphi'(t) \, dt = \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} \varphi'(t) \, dt = \pi \mathcal{P}_{\varphi'}(z), \]
where \( \mathcal{P} \) denotes the Poisson transform.

Finally, let us point out that \( \Delta \omega_{\varphi'}(x + iy) = 2\pi \varphi'(x) \, dx \, d\delta_0(y) \) in the sense of distribution, where \( \delta_0 \) denotes the Dirac measure at 0. The proof is exactly the same as when \( \varphi'(x) \simeq 1 \).

### 2.2 Multiplier lemma

We now develop a version of Lyubarskii and Seip’s multiplier lemma dealing with a positive, continuous \( \gamma(t) \) whose antiderivative is bi-Lipschitz for large distances. We shall obtain an equivalence \( e^{\omega_{\varphi'}(z)} \simeq |F_\gamma(z)| \) for a real-entire function \( F_\gamma(z) \) whose zeroes are equidistributed on the real line, but such zeroes will have multiplicity. It will not be possible to shift them to \( \mathbb{C}^- \) without breaking the equivalence, which thus holds on an upper half-plane \( \Im z > \varepsilon > 0 \) only.

Let \( \varphi(x) = \int_0^x \gamma(t) \, dt \). By the hypothesis, there exist \( C_1, C_2, \) and \( N \) such that
\[ C_1 (x_2 - x_1) \leq \varphi(x_2) - \varphi(x_1) \leq C_2 (x_2 - x_1) \quad \text{whenever} \quad x_2 - x_1 \geq N. \quad (2.1) \]
Consider a sequence \( \mathcal{A} = \{\alpha_k\}_{k \in \mathbb{Z}} \) of natural numbers satisfying \( 2C_2 N < \alpha_k < B \) for a certain bound \( B \). Let \( \cdots < x_{-1} < x_0 < x_1 < \cdots \) be the partition of \( \mathbb{R} \) defined by \( x_0 = 0 \) and \( \int_{x_k}^{x_{k+1}} \gamma(t) \, dt = \alpha_k. \)
We claim that $x_{k+1} - x_k \simeq 1$. Indeed, for $x_k \leq x \leq x_k + 2N$

$$\varphi(x) - \varphi(x_k) \leq \varphi(x_k + 2N) - \varphi(x_k) \leq 2C_2N < \alpha_k,$$

which yields $x_{k+1} > x_k + 2N$. Similarly, for $x - x_k \geq B/C_1$

$$\varphi(x) - \varphi(x_k) \geq B > \alpha_k,$$

which yields $x_{k+1} < x_k + (B/C_1)$. Therefore, $2N < x_{k+1} - x_k < B/C_1$, as desired.

Let $\xi_k = \frac{1}{\alpha_k} \int_{x_k}^{x_{k+1}} \gamma(t) \, dt$. We claim that $\xi_k$, which belongs to $(x_k, x_{k+1})$, is bounded away from $x_k$, and hence $\xi_{k+1} - \xi_k \simeq 1$. In fact,

$$\xi_k \geq \frac{1}{\alpha_k} \int_{x_k}^{x_k + N} x_k \gamma(t) \, dt + \frac{1}{\alpha_k} \int_{x_k + N}^{x_{k+1}} (x_k + N) \gamma(t) \, dt = x_k + \frac{N}{\alpha_k} (\varphi(x_{k+1}) - \varphi(x + N)).$$

Moreover, $\varphi(x_{k+1}) - \varphi(x + N) \geq C_1 (x_{k+1} - x_k - N) > C_1 N$. Therefore, $\xi_k - x_k \geq C_1 N^2/\alpha_k \geq C_1 N^2/B$, as desired.

Consider the auxiliary measure

$$d\nu(t) = \gamma(t) \, dt - \sum_{k = -\infty}^{\infty} \alpha_k \, d\delta_{\xi_k}(t),$$

where $d\delta_{\xi}$ denotes the Dirac measure at $\xi \in \mathbb{R}$. Let $f_\nu(x) = \int_{[0,x]} \, d\nu(t)$ and $g_\nu(x) = \int_0^x f_\nu(t) \, dt$. For $a < b$ and $z \notin \mathbb{R}$, two integrations by parts give

$$\int_{[a,b]} \log |1 - z/t| \, d\nu(t) = \int_a^b g_\nu(t) \frac{\partial^2}{\partial t^2} \log |1 - z/t| \, dt + R(a,b),$$

where $R(a,b) = (f_\nu(t) \log |1 - z/t| - g_\nu(t) \frac{\partial}{\partial t} \log |1 - z/t|) \big|_a^b$.

Since $\xi_k \in (x_k, x_{k+1})$, clearly $f_\nu(x_k) = 0$ for all $k$. Since in addition $A$ is bounded, $f_\nu$ is a bounded function on $\mathbb{R}$. Moreover, $g_\nu(x_k) = 0$ for all $k \in \mathbb{Z}$ by choice of $\xi_k$. In addition, $g_\nu$ is bounded on $\mathbb{R}$.

We deduce on the one hand that $R(a,b) \to 0$ when $a \to -\infty$ and $b \to \infty$. On the other hand, using any appropriate branch of the logarithm,

$$\int_a^b \frac{\partial^2}{\partial t^2} \log |1 - z/t| \, dt = \int_a^b g_\nu(t) \Re \frac{\partial^2}{\partial t^2} \log (1 - z/t) \, dt$$

$$= - \int_a^b g_\nu(t) \Re \frac{1}{(t - z)^2} \, dt + \int_a^b \frac{g_\nu(t)}{t^2} \, dt.$$}

Since $g_\nu$ is bounded, and since $g_\nu(t)$ is nonnegative and $\simeq t^2$ in a neighborhood of $x_0 = 0$, we conclude that $\int_{-\infty}^\infty \log |1 - z/t| \, d\nu(t)$ is well-defined and satisfies

$$\left| \int_{-\infty}^\infty \log |1 - z/t| \, d\nu(t) \right| \lesssim 1.$$
when \( \Im z \gg 0 \) (i.e., when \( \Im z > \varepsilon > 0 \) for a certain \( \varepsilon > 0 \)).

An integration by parts also gives
\[
\int_{-\infty}^{\infty} \frac{\chi(t)}{t} \, d\nu(t) = -f_\nu(-1) - f_\nu(1) + \int_{|t| > 1} \frac{f_\nu(t)}{t^2} \, dt.
\]
Since \( f_\nu \) is bounded, the above integral is just a real constant \( C \). In total,
\[
\left| \int_{-\infty}^{\infty} \log^* |1 - z/t| \, d\nu(t) - C x \right| \lesssim 1
\]
when \( \Im z \gg 0 \). Letting \( \alpha = C - \sum_{|\xi| \leq 1} \alpha_k/\xi_k \), it follows that
\[
|\omega_\gamma(z) - \alpha x - \sum_k \alpha_k (\log |1 - z/\xi_k| + x/\xi_k) | \lesssim 1.
\]
In other words, for \( F_\gamma(z) = e^{\alpha z} \prod_k (1 - z/\xi_k)^{\alpha_k} e^{z \alpha_k/\xi_k} \),
\[
|F_\gamma(z)| \simeq e^{\omega_\gamma(z)}
\]
when \( \Im z \gg 0 \).

In total, we have proven the following multiplier lemma:

**Proposition 2.1** Let \( \gamma(x) \) be a positive, continuous function whose antiderivative \( \varphi(x) \) satisfies the condition \( \square \). Let \( A = \{\alpha_k\}_{k \in \mathbb{Z}} \) be a bounded sequence of positive integers larger than \( 2C_2 N \). Then, \( F_\gamma(z) = e^{\alpha z} \prod_k (1 - z/\xi_k)^{\alpha_k} e^{z \alpha_k/\xi_k} \)
is a real-entire function satisfying
\[
|F_\gamma(z)| \simeq e^{\omega_\gamma(z)} \quad \text{when } \Im z \gg 0.
\]
The consecutive zeroes \( \{\xi_k\} \) of \( F_\gamma \) are real, have respective multiplicity \( \alpha_k \), and satisfy \( \xi_{k+1} - \xi_k \simeq 1 \).

### 2.3 Proof of the theorem

Let \( \mathcal{H} = \mathcal{H}(E) \) be a de Branges space whose phase \( \varphi \) is bi-Lipschitz for large distances. Since \( E \) is an Hermite–Biehler function, \( \log |E(x + iy)| \) is subharmonic. The computation its distributional Laplacian is the same as for weighted Paley–Wiener spaces. Indeed, for \( H(x, y) = \log |E(x + iy)| \),
\[
\frac{\partial^2}{\partial y^2} (H(x, |y|)) = \frac{\partial^2 H}{\partial y^2}(x, |y|) + 2 \frac{\partial H}{\partial y}(x, |y|) \, d\delta_0(y) \, dx
\]
in the sense of distribution, where \( \delta_0 \) denotes the Dirac measure at 0. The harmonicity of \( H \) in the neighborhood of the closed upper half-plane implies
\[
\Delta(H(x, |y|)) = (\Delta H)(x, |y|) + 2 \frac{\partial H}{\partial y}(x, |y|) \, d\delta_0(y) \, dx = 2 \frac{\partial H}{\partial y}(x, 0) \, dx \, d\delta_0(y).
\]
Since log \(E(x) = H(x, 0) - i \varphi(x)\) has an analytic extension in the neighborhood of \(\mathbb{R}\), the Cauchy–Riemann equations yield \(\Delta(H(x, |y|)) = 2 \varphi'(x) \, dx \, d\delta_0(y)\). Consequently,

\[
\log |E(x + i|y|)| = h(z) + \omega_{\varphi'/\pi}(z),
\]

where \(z = x + iy\) and \(h\) is harmonic. Since \(h(\bar{z}) = h(z)\), \(h\) is indeed the real part of a real-entire function \(g(z)\). It follows that for \(\gamma = \varphi'/\pi\)

\[
e^{-\gamma \mathcal{H}} = \{f \text{ entire } ; \|fe^{-\omega_{\gamma/\pi}}\|_2 < \infty, \ f^\# e^{-\omega_{\gamma/\pi} - i\omega_{\gamma/\pi}} \in \mathcal{N}_{0+}^\#\},
\]

where \(f^\# \in \{f, f^*\}\).

Observe that \(\omega_{\gamma}(z+i) > \omega_{\gamma}(z)\) on the upper half-plane. In fact, \(e^{\omega_{\gamma}(z) - \omega_{\gamma}(z+i)}\) is a bounded function whose mean type is 0, since the mean value theorem and the dominated convergence theorem imply

\[
\limsup_{y \to \infty} \left| \frac{\omega_{\gamma}(iy) - \omega_{\gamma}(i(y+1))}{y} \right| \leq \limsup_{y \to \infty} \frac{y+1}{y} \int_{-\infty}^{\infty} \frac{1}{\gamma(t) \, dt} = 0.
\]

Therefore, \(\omega_{\gamma}(z)\) may be replaced with \(\omega_{\gamma}(z+i)\) in the condition \(f^\# e^{-\omega_{\gamma} - i\omega_{\gamma}} \in \mathcal{N}_{0+}^\#\). Moreover, the multiplier lemma ensures that \(e^{\omega_{\gamma}(z)} \approx F_{\gamma}(z)\) on \(\Im z > 0\) for an entire function \(F_{\gamma}(z)\) with equidistributed (but multiple) zeroes on the real axis. Obviously, there exists an \(m(x) \approx 1\) generating the same zeroes with the same multiplicities, so \(e^{\omega_{\gamma}(z)} \approx F_{\gamma}(z) \approx e^{\omega_{\gamma}(z)}\) on \(\Im z > 0\). In particular, \(e^{\omega_{\gamma}(z+i)} \approx e^{\omega_{\gamma}(z+i)} \approx e^{\omega_{\gamma}(z)}\) for \(\Im z > 0\), where the last equivalence is justified by the mean value theorem. In total, the condition \(f^\# e^{-\omega_{\gamma} - i\omega_{\gamma}} \in \mathcal{N}_{0+}^\#\) may thus be replaced with \(f^\# e^{-\omega_{\gamma} - i\omega_{\gamma}} \in \mathcal{N}_{0+}^\#\) in the characterization of \(e^{-\gamma \mathcal{H}}\).

As for weighted Paley–Wiener spaces, we may replace this last condition with an omega type condition. In fact, let \(\tau > \sup m\) and \(E_{\tau-m}(z) = F_{\tau-m}(z+1)\), where \(F_{\tau-m}\) is given by the multiplier lemma (with zeroes of multiplicity 1). Then, \(E_{\tau-m}(z) \approx e^{-\omega_{\gamma}}\) for \(\Im z > 0\), while \(\omega_{\gamma}(z+i) = \pi \tau |y|\). The condition \(f^\# e^{-\omega_{\gamma} - i\omega_{\gamma}} \in \mathcal{N}_{0+}^\#\) is thus equivalent to \(f^\# E_{\tau-m} e^{-\pi \tau |y|} \in \mathcal{N}_{0+}^\#\), which is in turn equivalent to \(f^\# E_{\tau-m} \in \mathcal{N}_{\tau+}^\#\). Observe that \(E_{\tau-m}\) is an Hermite–Biehler function satisfying \(E_{\tau-m}(\bar{z}) = E_{\tau-m}(z-2i)\), and hence the mean value theorem implies

\[
\frac{E_{\tau-m}(iy)}{E_{\tau-m}(iy)} \approx e^{\omega_{\gamma}(i(y-2))-\omega_{\gamma}(iy)} \simeq 1.
\]

In particular, \(E_{\tau-m}/E_{\tau-m} \in \mathcal{N}_{0+}^\#\). The condition \(f^\# E_{\tau-m} \in \mathcal{N}_{\tau+}^\#\) is thus equivalent to \(f^\# E_{\tau-m} \in \mathcal{N}_{\tau+}^\#\) and \((f^\# E_{\tau-m})^* \in \mathcal{N}_{\tau+}^\#\) simultaneously (where \(f^\#\) varies in \(\{f, f^*\}\)). In other words, \(f^\# E_{\tau-m}\) is an entire function of bounded type not exceeding \(\pi \tau\) on both the upper and lower half-planes. By Krein’s theorem [2, p.38], it is equivalent to say that \(f^\# E_{\tau-m}\) is an entire function of exponential type at most \(\pi \tau\). Furthermore, for \(f \in e^{-\gamma \mathcal{H}}\)

\[
\|f E_{\tau-m}\|_2 \simeq \|f e^{\omega_{\gamma}}\|_2 = \|f e^{-\omega_{\gamma}}\|_2 \simeq \|f(x)e^{-\omega_{\gamma}(x+i)}\|_2 \leq \|f e^{-\omega_{\gamma}}\|_2 < \infty.
\]

In particular, \(f^\# E_{\tau-m}\) belongs to the classical Paley–Wiener space \(L_{\pi \tau}^2\), so the exponential type condition \(|f^\#(z)E_{\tau-m}(z)| \leq C e^{\pi \tau |z|}\) may be replaced with
$|f^\#(z)E_{\tau-m}(z)| \leq Cz e^{(\pi^2+c)|y|}$ for $z = x + iy \in \mathbb{C}$. Since $E_{\tau-m}$ is an Hermite–Biehler function, it is equivalent to say that this last inequation holds on \{Re z \geq 0\} only (where $f^\#$ varies in \{f, f^\ast\}). There, \[|E_{\tau-m}(z)| \simeq e^{\pi\tau|y|}e^{-\omega_m(z)},\] and hence the last inequation is equivalent to $|f^\#(z)|e^{-\omega_m(z)} \leq Cz e^{c|y|}$ for $\Re z \geq 0$, that is, to $|f(z)|e^{-\omega_m(z)} \leq Cz e^{c|y|}$ for all $z \in \mathbb{C}$. Therefore,

$$e^{-g}\mathcal{H} = \{f \text{ entire } ; \|fe^{-\omega}\|_2 < \infty, \ |f(z)|e^{-\omega_m(z)} \leq Cz e^{c|y|}\}.$$

Let $\theta(x) = \omega_\gamma(x + i) - \omega_\gamma(x)$, so $e^{-\omega_\gamma(x)} \simeq e^{-\omega_m(x)}e^{\theta(x)}$. This extra-weight is easily computed:

$$\theta(x) = \int_{-\infty}^{\infty} \log \left| 1 - \frac{i}{t - x} \right| \gamma(t) \, dt = \frac{1}{2} \int_{-\infty}^{\infty} \log \left( 1 + \frac{1}{(t - x)^2} \right) \frac{\varphi'(t)}{\pi} \, dt,$$

where the integral is well-defined. In conclusion,

$$e^{-g}\mathcal{H} = \{f \text{ entire } ; \|f e^{-\omega_m(x)}e^{\theta(x)}\|_2 < \infty, \ |f(z)|e^{-\omega_m(z)} \leq Cz e^{c|y|}\} \subseteq PW(m),$$

with the above expression for $\theta(x)$.

### 3 Appendix: MC-spaces

In [3], Lyubarskii and Seip studied structural properties of a larger class of de Branges spaces than the weighted Paley–Wiener spaces. Let us define this larger class through a list of postulates [7].

A piecewise continuous function $\mathbb{R} \to \mathbb{R}$ is a mountain chain if its graph consists of a succession of continuous pieces, called mountains and plateaux, satisfying the following conditions:

- Each mountain has a Poissonian shape $\frac{\eta}{(x - \xi)^2 + \eta^2}$ with two sides and a summit, $(\xi, 1/\eta)$;
- The bases of the mountains are $\simeq 1$;
- The summits have level more than 1; horizontally, they are bounded away from the endpoints of the mountain bases;
- The plateaux consist of horizontal segments of level 1, without restriction on their lengths (finite or infinite).

Let $E$ be an Hermite–Biehler function without real zero. Then $E(x) = |E(x)|e^{-i\varphi(x)}$ on the real axis, where $\varphi(x)$, the so-called phase, is real-analytic and well-defined up to the addition of $2k\pi$. For $x \in \mathbb{R}$, let $\xi_x - i\eta_x$ be the zero of $E(z)$ closest to $x$ (with the smallest $x$-coordinate in case of equality). We postulate the existence of a $\delta > 0$ such that the function

$$\gamma(x) = \begin{cases} \frac{\eta_x}{(x - \xi_x)^2 + \eta_x^2} & \text{if } \eta_x < \delta, \\ 1 & \text{otherwise} \end{cases}$$
is a mountain chain, satisfying in addition
\[ \varphi'(x) \simeq \gamma(x). \]

Observe that each mountain of \( \gamma \) lies over a zero of \( E(z) \) in the critical strip \(-\delta < \Im z < 0\), and this, bijectively. We postulate that each such zero is simple. Finally, let \( \{ (\xi_k, 1/\eta_k) \}_{k \in \mathbb{Z}} \) be an indexation of the summits of \( \gamma \) in order of \( x \)-coordinates, so
\[ \cdots < \xi_{-1} < \xi_0 < \xi_1 < \cdots \]

We postulate the following, weak limitation on the growth of the summits:
\[ \log(\eta_k) - \log(\eta_l) = O(|\xi_k - \xi_l|^1) \]
uniformly in \( l \) when \( |k - l| \to 0 \), where \( \varepsilon > 0 \) is arbitrarily small.

A de Branges space \( \mathcal{H} \) shall be called an MC-space if \( \mathcal{H} = \mathcal{H}(E) \) for an Hermite–Biehler function \( E \) without real zero, whose phase satisfies the aforementioned, postulated properties. In their profound study of weighted Paley–Wiener spaces, Lyubarskii and Seip proved the following:

If \( \mathcal{H} \) is an MC-space, then there exists a real-entire \( g(z) \) and a measurable, positive \( m(x) \) such that
\[ e^{gP W}(m) \subseteq \mathcal{H}, \]
while the majorants of these two spaces are both comparable with \( e^{g(x)} e^{\omega_m(x)} \) on the real axis. In particular, if \( \mathcal{H} \) is a weighted Paley–Wiener space, \( e^{gP W}(m) = \mathcal{H} \) (equality with equivalence of norms).

For a full equality \( e^{gP W}(m) = \mathcal{H}(E) \), in addition to the properties of \( E(z) \) (more precisely, of \( \varphi'(x) \)) making \( \mathcal{H}(E) \) an MC-space, the authors established that we need the following, rather complicated condition.

For \( \alpha \in \mathbb{R} \), let \( \Lambda_\alpha \) be the zero set of \( \sin(\varphi(x) - \alpha) \). Let \( g \) and \( m \) be given by the previous theorem. Then, \( \mathcal{H}(E) = e^{gP W}(m) \) if and only if the following conditions are satisfied: Firstly, \( e^{-g(z)} e^{-\omega_m(z)} E(z) \) is a (nonentire) function of exponential type on \( \mathbb{C} \), which satisfies
\[ \int_{-\infty}^{\infty} \log^+ \frac{|e^{-g(x)} e^{-\omega_m(x)} E(x)|}{x^2 + 1} \, dx < \infty. \]

Secondly, for two \( \alpha \in [0, \pi) \), \( \Lambda_\alpha \) is separated, while
\[ v(x) = \frac{\sin^2(\varphi(x) - \alpha)}{\varphi'(x) \text{dist}(x, \Lambda_\alpha)^2} \]
satisfies the Muckenhoupt \( (A_2) \) condition.

Namely, this last condition means that
\[ \frac{1}{|I|} \int_I v(x) \, dx \cdot \frac{1}{|I|} \int_I \frac{1}{v(x)} \, dx \lesssim 1 \]
when \( I \) ranges among the finite intervals.
References

[1] B.Y. Levin, *Lectures on Entire Functions*, Translations of Mathematical Monographs vol.150, American Mathematical Society, 1996.

[2] L. de Branges, *Hilbert Spaces of Entire Functions*, Prentice-Hall, 1968.

[3] K. Seip, *Interpolation and Sampling in Spaces of Analytic Functions*, University Lecture Series vol.33, American Mathematical Society, 2004.

[4] J. Marzo, S. Nitzan, J.-F. Olsen, *Sampling and interpolation in de Branges spaces with doubling phase*, Journal d’analyse mathématique vol.117, 2012.

[5] Y. Lyubarskii, K. Seip, *Weighted Paley–Wiener spaces*, Journal of the American Mathematical Society vol.15 no 4, 2002, 979-1006.

[6] Y. Belov, T.Y. Mengestie, K. Seip, *Discrete Hilbert transforms on sparse sequences*, Proceedings of the London Mathematical Society vol.103 no.1, 2011, 73-105.

[7] P. Poulin, *Weighted Paley-Wiener spaces and mountain chain axioms: a detailed exposition*, pre-print, arXiv:1305.0184.

[8] A. Beurling, *The collected works of Arne Beurling*, Contemporary Mathematicians, Birkhäuser Boston Inc., Boston, MA, 1989, Edited by L. Carleson, P. Malliavin, J. Neuberger and J. Wermer.

[9] T. Ransford, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts 28, Cambridge University Press, 1995.