Dynamical analysis of hollow-shaft dual-rotor system with circular cracks

Yang Yongfeng¹ ² *, Wang Jianjun¹, Wang Yanlin¹ ², Fu Chao¹, Zheng Qingyang¹ and Lu Kuan¹

Abstract
In this paper, we considered a dual-rotor system with crack in shaft. The influence of circular crack in hollow shaft on dynamical response was studied. The equations of motion of 12 elements dual-rotor system model were derived. Harmonic balance method was employed to solve the equations. The critical speed and sub-critical speed responses were investigated. It was found that the circular crack in hollow shaft had greater influence on the first-backward critical speed than the first-forward critical speed. Owing to the influence of crack, the vibration peaks occurred at the 1/2, 1/3 and 1/4 critical speeds of the rotor system, along with a reduction in sub-critical speeds and critical speeds. The deeper crack away from the bearing affected the rotor more significantly. The whirling orbits, the time-domain responses and the spectra were obtained to show the super-harmonic resonance phenomenon in hollow-shaft cracked rotor system.

Keywords
Rotor, hollow-shaft, circular cracks, critical speed, super-harmonic

Introduction
Rotating machines represent the maximal and most important class of machinery used for fluid media transportation, metal working and forming, energy generation, providing aircraft propulsion and other purposes.¹ ² High speed and heavy power are the development directions of modern rotating machineries.³ ⁴ In the past decades, there are a lot of literatures that focus on the study of unbalance,⁷ clearance,⁸ base motions,⁹ damping ratio identification,¹⁰ rubbing¹¹ ¹³ and viscoelastic properties of rotor system. Especially, crack and misalignment effects in rotor dynamic characteristics are frequently investigated.¹⁴ ¹⁶

Fatigue crack of the rotor shaft observed in the rotating machinery should be avoided. It may lead to catastrophic failure.¹⁷ In this situation, there are non-linear and non-stationary responses of the rotor system. However, the strong non-linearity can make the system possess characteristics that are substantially different from those of the linear system, such as self-excited oscillations and jump discontinuities. Detailed investigation into the non-linear dynamic response prediction of cracked shaft is very important for diagnosing and preventing rotor cracks.¹⁸ ²⁰

The influence of transverse crack on a rotating shaft has been the attention of many researchers. Extensive reviews of the dynamic response of cracked rotor systems were published by Dimarogonas²¹ and Wauer.²² Pennacchi et al.²³ proposed a model-based transverse crack identification method suitable for industrial machines. The excellent accuracy obtained at defined position and depth of different cracks demonstrated the effectiveness and reliability of the proposed method. Patel and Darpe¹⁷ investigated the influence of the crack-breathing models

¹Institute of Vibration Engineering, Northwestern Polytechnical University, Xi’an, China
²Key Laboratory of Vibration and Control of Aero-Propulsion System Ministry of Education, Northeastern University, Shenyang, China
³CRRC Zhuzhou Electric Locomotive Institute Co., Ltd, Hunan, China

Corresponding author:
Yang Yongfeng, Northwestern Polytechnical University, P.O. Box 264 127, West Youyi Road, Xi’an, Shaanxi 710072, China.
Email: yyf@nwpu.edu.cn

Creative Commons CC BY: This article is distributed under the terms of the Creative Commons Attribution 4.0 License (https://creativecommons.org/licenses/by/4.0/) which permits any use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
(switching-crack model and response-dependent breathing-crack model) on the non-linear vibration characteristics of the cracked rotor. Switching-crack modeling reveals chaotic, quasi-periodic and sub-harmonic vibration response for deeper cracks, and more realistic breathing crack model reveals no evidence of them. Rubio and Fernandez-Saez\textsuperscript{34} proposed a new procedure to analyze the non-linear dynamics of cracked rotors using an iterative technique that transformed the full non-linear problem into a succession of time-dependent linear ones. The calculations using the proposed method were over a 100 times faster than the corresponding to integrate the full non-linear problem, being very helpful in on-line crack-identification procedures. Sinou\textsuperscript{25} adopted harmonic balance method to study the stability of the rotor system presenting a transverse breathing crack by considering the effects of crack depth, crack position and the rotating speed. The areas of instability will increase considerably when the crack deepened and that the crack’s position and depth were the main factors affecting not only the non-linear behaviour of the rotor system but also the different zones of dynamic instability in the periodic solution for the cracked rotor. Ishida and Inoue\textsuperscript{26} used harmonic excitation force to investigate the non-linear response of cracked rotor. The occurrence of various types of non-linear resonances due to crack was clarified, and types of these resonances, their resonance points and dominant frequency component of these resonances were clarified numerically and experimentally. Sawicki et al.\textsuperscript{27} investigated the modelling and analysis of machines with breathing cracks, which open and close due to the self-weight of the rotor, producing a parametric excitation. Penny and Friswell\textsuperscript{28} considered a cracked asymmetric Jeffcott rotor and studied the influence of small out-of-balance forces and cracks. Al-Shudeifat\textsuperscript{29,30} proposed a new breathing function which could describe the breathing mechanism of the crack more precisely. The process of solving the stiffness matrix of the crack element is given in detail, and the dynamic characteristics of the rotor system with breathing crack or open crack are solved, respectively. These dynamic phenomena are verified by experiments. Moreover, Floquet theory is used to analyse the influence of crack and damping on the stability of the cracked rotor, and the changing law of the instability speed region is obtained. Cavalini et al.\textsuperscript{31} used a crack-identification methodology based on a non-linear approach, which uses external applied diagnostic forces at certain frequencies, to estimate the location and depth of the crack. In the above two new diagnostic methods, the relationship between the response frequency and the crack is more explicit, which is expected to improve the accuracy of the crack identification.

In recent years, the hollow structures are widely used in aero-engine to improve efficiency. More attention is needed about the dynamic characteristics of cracked rotor system with hollow shaft in the engineering rotor.\textsuperscript{32,33} In this work, the influence of circular crack in hollow shaft, the critical speed and sub-critical speed responses is studied. We are devoted to provide some guidance for the detection and identification of hollow-shaft dual-rotor system crack faults.

The modeling of the cracked rotor system

The finite element modeling of open crack is addressed in this part. The crack leads to a reduction of stiffness where the synchronous breathing of the crack between compression and tension stress fields on the crack faces of contact may lead to a permanent plastic deformation by which the breathing mechanism becomes dominated by the permanently open crack state.\textsuperscript{29}

Circular crack is a typical form of transverse crack models. As far as the author knows, fewer literatures study this crack model compared with the huge literatures of straight crack model. However, as a rotating shaft, it is easier for cracks to propagate along the circumference than along the axis. In this part, the crack stiffness model is established by the neutral axis theory.\textsuperscript{34–36}

When the shaft is subject to stress concentration, especially in the position where the cross-section suddenly changes or material defects exist, the fatigue cracks will propagate faster on both sides of the crack edge under long-term action of the alternating load. As a result, the area of the crack element section will become an annular one.

To show this characteristic accurately, a finite element model of the rotor system with 12 elements and 13 nodes is established as shown in Figure 1. The bearing is located at nodes 1 and 13, and the disk is located at nodes 4 and 10. There is unbalance in disk 1, and the crack is located in element 4.

For a cracked rotor system with hollow shaft, with the propagation of crack, penetration of the shaft’s inner wall may occur. The cross-section of the crack element before penetration is shown in Figure 2. The shadow area represents the crack. The initial angle of the crack is taken as 0. The $o-xy$ is a fixed coordinate system. The crack region is symmetric about the $oy$ axis, and the depth of the cracks is same on each section. $o$ is the centroid of cross-section when there is no crack, $c$ is the centroid when the crack appears, $e = \partial c$ represents the change of centroid position and $\Omega$ is the rotational speed of the rotor. The outer radius and inner radius of the shaft are $R$.
and \( r \), respectively. \( \theta \) is the crack angle, \( h \) is the depth of the crack and the non-dimensional crack depth is given by \( u = h/R \).

The area moments of inertia of the cracked element about its centroidal \( x \) and \( y \) axes are constant quantities during the rotation of the shaft while the area moments of inertia of the cracked element about its fixed \( x \) and \( y \) axes are time-varying quantities during the rotation of the shaft. The cracked element stiffness matrix in the rotating \( x \) and \( y \) axes can be written in a form similar to that of the asymmetric rod in space in Pilkey.\(^{37}\) In the circular crack model, crack propagation appears as it increases in depth and crack angle, but \( h \) and \( u \) are relatively independent parameters. In order to study the crack propagation process, let \( \cos(\theta) = (R-h)/R \).

The crack depth is assumed to be constant. The moment of inertia of the crack section in two directions is definite, but the crack will cause a certain offset in the sectional centroid position. The crack region in Figure 2(b) is divided into three parts, which is \( A_{\text{crack}} = A_1 + A_2 - A_3 \), where \( A_1 \) refers to the arcuate region between line \( y = R \) and line \( y = R-h \), \( A_2 \) represents the trapezoidal area between line \( y = R-h \) and line \( y = (R-h)\cos(\theta) \) and \( A_3 \) means the un-shaded arched area between line \( y = R-h \) and line \( y = (R-h)\cos(\theta) \). The moment of inertia of the crack area on the axes \( ox \) \( I_{ox} \) is given by

\[
I_{ox} = I_{ox}^A + I_{ox}^B - I_{ox}^C
\]  

where

\[
I_{ox}^A = \int_{A_1} y^2 dA = \int_{R-h}^R 2y^2 \sqrt{R^2 - y^2} dy
\]

\[
I_{ox}^B = \int_{A_2} y^2 dA = \int_{(R-h)\cos(\theta)}^{R-h} 2y^2 \tan(\theta) dy
\]

\[
I_{ox}^C = \int_{A_3} y^2 dA = \int_{(R-h)\cos(\theta)}^{R-h} 2y^2 \left[ (R-h)^2 + y^2 \right] dy
\]

The moment of inertia of the crack area on the axes \( oy \) \( I_{oy} \) is given by

\[
I_{oy} = I_{oy}^A + I_{oy}^B - I_{oy}^C
\]  

Figure 1. Finite element model of the cracked rotor system.

Figure 2. The crack sketch before penetration. (a) Cross-section of the crack element; (b) Geometry of the crack area.
The remaining area of the crack element $A_{ce}$ is given by

$$A_{ce} = A - A_{crack} = \pi \cdot (R^2 - r^2) - \theta \cdot R^2 - \theta \cdot (R - h)^2$$

(3)

The offset distance $e$ of the centroid $o$ is given by

$$e = \left( \int_{A_1} y \, dA + \int_{A_2} y \, dA - \int_{A_3} y \, dA \right) / A_{ce}$$

(4)

For the situation of crack penetration, conditions and parameter settings remain the same as those before penetration. The crack region in Figure 3(b) is divided into three parts, where $A_1$ represents to the arcuate region between line $y = R$ and line $y = R-h$, $A_2$ represents to the trapezoidal area between line $y = R-h$ and line $y = r \cos(\theta)$ and $A_3$ means the unshaded arched area between line $y = r$ and line $y = r \cos(\theta)$. The moment of inertia of the crack area on the axes $ox$ $I_{ox}$ is given by

$$I_{ox} = I_{ox}^{A_1} + I_{ox}^{A_2} - I_{ox}^{A_3}$$

(5)

where

$$I_{ox}^{A_1} = \int_{A_1} y^2 \, dA = \int_{R-h}^{R} 2y^2 \sqrt{R^2 - y^2} \, dy$$

$$I_{ox}^{A_2} = \int_{A_2} y^2 \, dA = \int_{r \cos(\theta)}^{R-h} 2y^3 \tan(\theta) \, dy$$

$$I_{ox}^{A_3} = \int_{A_3} y^2 \, dA = \int_{r \cos(\theta)}^{r} 2y^2 \sqrt{r^2 - y^2} \, dy$$

Figure 3. The crack sketch after penetration. (a) Cross-section of the crack element; (b) Geometry of the crack area.
The moment of inertia of the crack area on the axes \( oy \) is given by

\[
I_{oy} = I_{oy}^A + I_{oy}^A - I_{oy}^A
\]

where

\[
I_{oy}^A = \int_{A_1} x^2 dA = \int_0^{R \sin(\theta)} 2x^2(\sqrt{R^2 - x^2} - (R - h)) dx
\]

\[
I_{oy}^A = \int_{A_2} x^2 dA = \int_0^{r \sin(\theta)} 2x^2(R \cdot \sin(\theta) - x)/\tan(\theta) dx
\]

\[
+ \int_0^{r \sin(\theta)} 2x^2((R - h) - r \cdot \cos(\theta)) dx
\]

\[
I_{oy}^A = \int_{A_3} x^2 dA = \int_0^{r \sin(\theta)} 2x^2(\sqrt{r^2 - x^2} - r \cdot \cos(\theta)) dx
\]

\[
A_{cy} = A - A_{crack} = \pi \cdot (R^2 - r^2) - (\theta \cdot R^2 - \theta \cdot r^2)
\]

\[
e = \left( \int_{A_1} y dA + \int_{A_2} y dA - \int_{A_3} y dA \right) / A_{cy}
\]

The decrease of the moment of inertia due to the crack can be defined by

\[
I_1 = I_{ox} + A_{cy} \cdot e^2, \quad I_2 = I_{oy}.
\]

The moment of inertia of the crack element relative to the new centroid axis \( c_x \) and \( c_y \) can be obtained by

\[
\begin{align*}
I_{cx}^c &= I - I_1 \\
I_{cy}^c &= I - I_2
\end{align*}
\]

where \( I_1 \) and \( I_2 \) are very important parameters to calculate the stiffness reduction matrix. In order to show the effect of different models, the moment of inertias for solid shaft and hollow shaft, circular crack, and straight crack are compared here. The physical parameters for a rotor system are shown in Table 1. Assume that the solid shaft has the same outer diameter as the hollow axis. Figures 4 and 5 show \( I_1, I_2 \) and the relative reduction of \( I_1 \) with variety of crack depth. With the increasing of crack depth, the loss of moment of inertia in \( x \) and \( y \) directions increases. The effect of hollow shaft and circular crack is greater than the solid shaft and straight crack.

Timoshenko beam-axis model is used to calculate the crack stiffness of the shaft. The coordinates of the \( i \)th element in rotor system can be expressed as \([x_i, y_i, \theta_{xi}, \theta_{yi}, x_{i+1}, y_{i+1}, \theta_{x(i+1)}, \theta_{y(i+1)}]\). \( x \) and \( y \) are the nodal displacements and \( \theta_x \) and \( \theta_y \) are the rotating angular displacements. When the crack is fully open, the stiffness reduction

### Table 1. Value of the physical parameters.

| Parameter                          | Value          |
|-----------------------------------|----------------|
| Length of the rotor shaft, L      | 0.724 m        |
| Outer radius of rotor shaft, R     | 7.9 mm         |
| Inner radius of rotor shaft, r     | 4.74 mm        |
| Density, \( \rho \)                | 7800 kg/m³     |
| Stiffness of bearing \((k_{xx}, k_{yy})\) | \(5 \times 10^7\) N/m |
| Damping of bearing \((c_{xx}, c_{yy})\) | \(5 \times 10^2\) N s/m |
matrix $K_{\text{crack}}$ caused by the crack can be written as

$$
K_{\text{crack}} = \begin{bmatrix}
12l_1 \frac{1}{1 + \phi_1} & 0 & 0 & -12l_1 \frac{1}{1 + \phi_1} & 0 & 0 & 6l_1 \frac{1}{1 + \phi_1} \\
0 & 12l_2 \frac{1}{1 + \phi_2} & -6l_2 \frac{1}{1 + \phi_2} & 0 & 0 & -12l_2 \frac{1}{1 + \phi_2} & 0 \\
0 & -6l_2 \frac{1}{1 + \phi_2} & F l_2 \frac{4}{1 + \phi_2} & 0 & 0 & 6l_2 \frac{1}{1 + \phi_2} & 0 \\
6l_1 \frac{1}{1 + \phi_1} & 0 & 0 & \hat{\rho} l_1 \frac{4 + \phi_1}{1 + \phi_1} & -6l_1 \frac{1}{1 + \phi_1} & 0 & 0 \\
-12l_1 \frac{1}{1 + \phi_1} & 0 & 0 & \hat{\rho} l_1 \frac{2 - \phi_1}{1 + \phi_1} & -6l_1 \frac{1}{1 + \phi_1} & 0 & 6l_1 \frac{1}{1 + \phi_1} \\
0 & -12l_2 \frac{1}{1 + \phi_2} & 6l_2 \frac{1}{1 + \phi_2} & 0 & 0 & 12l_2 \frac{1}{1 + \phi_2} & 0 \\
0 & -6l_2 \frac{1}{1 + \phi_2} & \hat{\rho} l_2 \frac{2 - \phi_2}{1 + \phi_2} & 0 & 0 & 6l_2 \frac{1}{1 + \phi_2} & 0 \\
6l_1 \frac{1}{1 + \phi_1} & 0 & 0 & \hat{\rho} l_1 \frac{2 - \phi_1}{1 + \phi_1} & -6l_1 \frac{1}{1 + \phi_1} & 0 & 6l_1 \frac{1}{1 + \phi_1} \\
\end{bmatrix}
$$

(10)
where \( \phi_1 = \frac{12EI_1}{w_1A_eG_{mp}}, \phi_2 = \frac{12EI_2}{w_2A_eG_{mp}} \) and \( \mu_1 \) is the shear coefficient.

The cosine switching function \( f(t) = (1 + \cos(\Omega t))/2 \) for breathing cracks is used to describe the opening and closing of cracks during rotation. The dynamic equations of the cracked rotor can be given by

\[
M\ddot{q}(t) + [C + G]\dot{q}(t) + (K - f(t)K_c)q(t) = F_1\cos(\Omega t) + F_2\sin(\Omega t) + F_g
\]  

(11)

where \( M, C, G \) and \( K \) are the mass matrix, damping matrix, gyro matrix and stiffness matrix of a non-crack rotor, respectively. For the crack-element stiffness matrix

\[
K_c = K_{\text{crack}}
\]

For other elements, \( K_c \) equals 0 and \( F_1 \) and \( F_2 \) represent the unbalanced excitation vector. For the node contains unbalance

\[
F_1^j = m_c d \cdot \Omega^2 [\cos\beta, \sin\beta, 0, 0]^T
\]

\[
F_2^j = m_c d \cdot \Omega^2 [-\sin\beta, \cos\beta, 0, 0]^T
\]

where \( m_c \) is the eccentric mass, \( d \) is the eccentricity distance and \( \beta \) is the angle of unbalanced mass. For other nodes, \( F_1 \) and \( F_2 \) equal 0. \( F_g \) is the gravity force vector of the rotor system. The value of each node is given by \( F_g^j = [-m_i g, 0, 0, 0] \), where \( m_j \) is the mass of node \( j \).

Harmonic balance method is used to solve equation (11), so the solution is assumed to be expressed in Fourier series as

\[
q(t) = A_0 + \sum_{i=1}^{n} [A_i \cos(i\Omega t) + B_i \sin(i\Omega t)]
\]  

(12)

where \( n \) represents the retained harmonic number. From our experience, \( n = 4 \) is sufficient to reveal the dynamic characteristics of cracked rotor system. By substituting equation (12) into equation (11), it is obtained that

\[
\begin{bmatrix}
\tilde{K} & C_1 & K_1 & 0 \\
-C_1 & \tilde{K} - \Omega^2 M & 0 & K_1 \\
K_1 & 0 & \tilde{K} - (2\Omega)^2 M & C_2 \\
0 & K_1 & -C_2 & \tilde{K} - (2\Omega)^2 M \\
\end{bmatrix}
\begin{bmatrix}
A_1 \\
B_1 \\
A_2 \\
B_2
\end{bmatrix} =
\begin{bmatrix}
\bar{F} \\
A_0 \\
A_0 \\
A_0
\end{bmatrix}
\]

(13)

where \( K_1 = -K_c/4, \quad \bar{F} = F_1 + K_cK^{-1}F_g/2, \quad \tilde{K} = \tilde{K} - \Omega^2 M - K_c(K^{-1})K_c/8, \quad C_s = s\Omega C, \quad s = 1, 2, \ldots, n, \quad A_0 = \tilde{K}^{-1}(F_g + K_cA_1/4). \) By solving the linear equation (13), the steady-state response of the rotor system can be obtained based on harmonic balance method.

**Numerical simulations**

For the rotor system shown in Figure 1, the physical parameters are the same as shown in Table 1. Figure 6 shows the vibration amplitude of a non-crack rotor system with asymmetric stiffness and symmetric bearing stiffness. The first-forward critical speed is \( w_{f1} = 281.8 \text{ rad/s} \), and the first-backward critical speed is \( w_{b1} = 272.9 \text{ rad/s} \). In addition, we can obtain the critical speed by the Eigenvalue Method, and the critical speed is \( w_{f1} = 281.8 \text{ rad/s} \) and \( w_{b1} = 272.8 \text{ rad/s} \). For asymmetric-cracked shaft with isotropic bearing or symmetric intact shaft with anisotropic
Figure 6. Vibration amplitude at node 4 for non-crack rotor with asymmetric/symmetric stiffness.

Figure 7. The first critical speed with variety of crack depth.

Figure 8. The first critical speed with crack position in different elements.
bearings, the first-forward and backward whirl speeds could be excited by unbalance force. As shown in Figure 6, the results obtained by two methods are in good agreement with each other.

Figure 6. Rotation speed–crack depth–vibration amplitude waterfall at node 4.

Figure 7. First critical speed of a hollow-shaft rotor system with a circular crack decreases with the crack depth going deeper.

Figure 10. Rotation speed–vibration amplitude diagram of rotor system at node 4 with crack located in different elements when crack depth \( u = 0.5 \).

bearings, the first-forward and backward whirl speeds could be excited by unbalance force. As shown in Figure 6, the results obtained by two methods are in good agreement with each other.

The first critical rotational speed of a hollow-shaft rotor system with a circular crack is shown in Figure 7. It can be seen that the first critical speed of the cracked rotor decreases with the crack depth going deeper.
Figure 11. Rotation speed–vibration amplitude diagram of rotor system at node 4 with no crack.

Figure 12. Phase orbits near 138.3 rad/s of node 4 when crack depth $u = 0.4$.

Figure 13. Phase orbits near 92.1 rad/s of node 4 when crack depth $u = 0.4$.

Figure 14. Phase orbits near 69 rad/s of node 4 when crack depth $u = 0.4$.

Figure 15. Power spectrum of node 4 when crack depth $u = 0.4$. 
Figure 13. Phase orbits near 92.1 rad/s of node 4 when crack depth $u = 0.4$.

Figure 14. Phase orbits near 69 rad/s of node 4 when crack depth $u = 0.4$.

Figure 15. Power spectrum of node 4 when crack depth $u = 0.4$. 
In particular, the first-backward critical speed is very sensitive to the crack depth while the first-forward critical speed is not.

For the above study, the crack position is constant in element 4. Here, we investigate the effect of crack position. Figure 8 shows the first critical speeds at different element locations. When the crack position is close to the bearing position (Element 1), the first critical speed decreases slowly with increasing crack depth. When the crack position is far away from the bearing, especially for the element near the middle of the shaft (Element 4, Element 5, Element 6), and the first-order critical speed drops faster. Element 6 is the fastest one.

Figure 9 shows the rotation speed–crack depth–vibration amplitude waterfall at node 4 when the circular crack is located in element 4. Due to the circular crack, vibration peaks appear at the 1/2, 1/3 and 1/4 first-forward and backward critical rotational speeds. Since we study the response of a cracked rotor system, the basic characteristics of response are the same as given in Ishida and Yamamoto. When the crack depth is small ($u = 0.1$ and $u = 0.2$), the vibration peaks near 1/3 and 1/4 critical speeds are insignificant. With the crack expansion and deepening, the vibration peaks near 1/3 and 1/4 critical speeds become very obvious. At the same time, the sub-critical rotational speeds of the cracked rotor system also decrease with the increasing of the crack depth. In short, the crack will cause the rotor system to have vibration peaks near the first critical speed and $1/n$ ($n = 2, 3, 4$) first critical speed. The amplitude of vibration peak near 1/4 critical speed is very sensitive to the crack depth and position. There are vibration peaks between the 1/2 first critical speed and the first critical speed. The rotation speed is near 215 rad/s. With the crack depth going deeper and the element closer to crack position, the rotation speed will go down and the amplitude will go up. This is an important dynamic characteristic to detect the crack fault in rotor system.

Figure 10 shows the rotation speed–vibration amplitude diagram of node 4 with the crack locating in different elements when the circular crack depth $u = 0.5$. It can be seen that when the crack is located near the support position of element 1, the vibration peaks near 1/3 and 1/4 critical speeds are not obvious. While the crack is far away from the bearing support, the vibration peaks near 1/3 and 1/4 critical speed become obvious. There is a peak near 215 rad/s. With the crack depth going deeper and the measurement point closer to crack position, the speed will go down and the amplitude will go up. As a result, the crack located in the middle shaft of the rotor has a greater influence on the dynamic characteristics than in the supporting position. Figure 11 shows the rotation speed–vibration amplitude diagram of rotor system at node 4 with no crack. It can be seen that the vibration peak only occurred at the critical speed of the rotor system, and there is no super-harmonic resonance phenomenon in no-cracked rotor systems.

Figures 12 to 15 show the phase orbits and the power spectrum of node 4 near the sub-critical speeds when the circular crack depth is 0.4. It can be seen that when the rotating speed is close to the 1/2 first-forward critical speed, it appears as two overlapping ellipses in the phase orbits and $2\times$ component in the frequency domain. Actually, the $2\times$ component is the first-forward critical speed. In the same way, for 1/3 and 1/4 critical speeds, the phase orbits are three and four overlapping ellipses, respectively. Similarly, the frequency components are dominated by the $3\times$ component or $4\times$ component. For the backward critical speed, we can observe the same phenomenon. It shows the super-harmonic resonance phenomenon in cracked rotor systems.

Conclusion

A hollow-shaft rotor system with circular cracks is studied in this paper. The time-varying stiffness matrix of the crack element is deduced. The influence of the crack on the critical speed and sub-critical speed is shown. It is found that the circular cracks could reduce both the first-forward and backward critical rotational speeds of the rotor system, especially the latter one. Owing to the influences, the vibration peaks occur near $1/n$ ($n = 1, 2, 3$, and 4) critical rotational speed. As the crack increases, the peaks become more prominent. When the crack is located in the middle of shaft, the effect will be greatest. Super-harmonic resonance phenomena can be observed in the cracked rotor system. The results of this paper can provide some guidance for detection and identification of crack fault in hollow-shaft dual-rotor system.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.
Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This study was funded by National Natural Science Foundation of China (grant number 11972295), the Key Laboratory of Vibration and Control of Aero-Propulsion System Ministry of Education, Northeastern University (grant number VCAME201803), and Graduate Innovation Fund of Northwestern Polytechnical University (grant number CX2020124).

ORCID iD
Yang Yongfeng https://orcid.org/0000-0003-0402-4440

References
1. Li CF, She HX, Tang QS, et al. The effect of blade vibration on the nonlinear characteristics of rotor-bearing system supported by nonlinear suspension. *Nonlin Dyn* 2017; 89: 987–1110.
2. Hou L, Chen YS, Fu YQ, et al. Application of the HB-AFT method to the primary resonance analysis of a dual-rotor system. *Nonlin Dyn* 2017; 88: 2531–2551.
3. Luo Z, Zhu YP, Zhao XY, et al. Determining dynamic scaling laws of geometrically distorted scaled models of a cantilever plate. *J Eng Mech* 2016; 142: 04015108.
4. Zhang GH and Ehmann KF. Dynamic design methodology of high speed micro-spindles for micro/meso-scale machine tools. *Int J Adv Manuf Technol* 2015; 76: 229–246.
5. Dai HH, Jing XJ, Wang Y, et al. Post-capture vibration suppression of spacecraft via a bio-inspired isolation system. *Mech Syst Signal Process* 2018; 105: 214–240.
6. Qin ZY, Yang ZB, Zu J, et al. Free vibration analysis of rotating cylindrical shells coupled with moderately thick annular plate. *Int J Mech Sci* 2018; 142-143: 127–139.
7. Fu C, Xu YD, Yang YF, et al. Response analysis of an accelerating unbalanced rotating system with both random and interval variables. *J Sound Vib* 2020; 466: 115047.
8. Li HG, Meng G, Meng ZQ, et al. Effects of boundary conditions on a self-excited vibration system with clearance. *Int J Nonlin Sci Num* 2007; 8: 571–579.
9. Han QK and Chu FL. Parametric instability of flexible rotor-bearing system under time-periodic base angular motions. *Appl Math Model* 2015; 39: 4511–4522.
10. Wang WM, Li QH, Gao JJ, et al. An identification method for damping ratio in rotor systems. *Mech Syst Signal Process* 2016; 68-69: 536–554.
11. Ma H, Shi CY, Han QK, et al. Fixed-point rubbing fault characteristic analysis of a rotor system based on contact theory. *Mech Syst Signal Process* 2013; 38: 137–153.
12. Chu FL and Lu WX. Experimental observation of nonlinear vibrations in a rub-impact rotor system. *J Sound Vib* 2005; 283: 621–643.
13. Zhang WM and Meng G. Stability, bifurcation and chaos analyses of a high-speed micro-rotor system with rub-impact. *Sensor Actuat A-Phys* 2006; 127: 163–178.
14. Yang YF, Chen H and Jiang TD. Nonlinear response prediction of cracked rotor based on EMD. *J Franklin Inst* 2015; 352: 3378–3393.
15. Yang YF, Wu QY, Wang YL, et al. Dynamic characteristics of cracked uncertain hollow-shaft. *Mech Syst Signal Process* 2019; 124: 36–48.
16. Ma H, Zeng J, Feng RJ, et al. Review on dynamics of cracked gear systems. *Eng Fail Anal* 2015; 55: 224–245.
17. Patel TH and Darpe AK. Influence of crack breathing model on nonlinear dynamics of a cracked rotor. *J Sound Vib* 2008; 311: 953–972.
18. Qin WY, Meng G and Zhang T. The swing vibration, transverse oscillation of cracked rotor and the intermittence chaos. *J Sound Vib* 2003; 259: 571–583.
19. Lu YJ, Zhang YF, Shi XL, et al. Nonlinear dynamic analysis of a rotor system with fixed-tilting-pad self-acting gas-lubricated bearings support. *Nonlin Dyn* 2012; 69: 877–890.
20. Meng G, Zhang WM, Huang H, et al. Micro-rotor dynamics for micro-electro-mechanical systems (MEMS). *Chaos Soliton Fract* 2009; 40: 538–562.
21. Dimarogonas AD. Vibration of cracked structures: a state of the art review. *Eng Fract Mech* 1996; 55: 831–857.
22. Wauer J. Dynamics of cracked rotors: literature survey. *Appl Mech Rev* 1990; 43: 13–17.
23. Pennacchi P, Bachschmid N and Vania A. A model-based identification method of transverse cracks in rotating shafts suitable for industrial machines. *Mech Syst Signal Process* 2006; 20: 2112–2147.
24. Rubio L and Fernandez-Saez J. A new efficient procedure to solve the nonlinear dynamics of a cracked rotor. *Nonlin Dyn* 2012; 70: 1731–1745.
25. Sinou JJ. Effects of a crack on the stability of a non-linear rotor system. *Int J Nonlin Mech* 2007; 42: 959–972.
26. Ishida Y and Inoue T. Detection of a rotor crack using a harmonic excitation and nonlinear vibration analysis. *J Vib Acoust* 2006; 128: 741–749.
27. Sawicki JT, Friswell MI, Kulesza Z, et al. Detecting cracked rotors using auxiliary harmonic excitation. *J Sound Vib* 2011; 330: 1365–1381.
28. Penny JET and Friswell MI. The dynamics of rotating machines with cracks. *MSF* 2003; 440–444: 311–318.
29. Al-Shudeifat MA. On the finite element modeling of the asymmetric cracked rotor. *J Sound Vib* 2013; 332: 2795–2807.
30. Al-Shudeifat MA. Stability analysis and backward whirl investigation of cracked rotors with time-varying stiffness. *J Sound Vib* 2015; 348: 365–380.
31. Cavalini AA Jr, Sanches L, Bachschmid N, et al. Crack identification for rotating machines based on a nonlinear approach. *Mech Syst Signal Process* 2016; 79: 72–85.
32. Lu ZY, Hou L, Chen YS, et al. Nonlinear response analysis for a dual-rotor system with a breathing transverse crack in the hollow shaft. *Nonlin Dyn* 2016; 83: 169–185.
33. Guo CZ, Al-Shudeifat MA, Vakakis AF, et al. Vibration reduction in unbalanced hollow rotor systems with nonlinear energy sinks. *Nonlin Dyn* 2015; 79: 527–538.
34. Du Y, Zhou SX, Jing XJ, et al. Damage detection techniques for wind turbine blades: a review. *Mech Syst Signal Process* 2020; 141: 106445.
35. Liu Y, Zhao YL, Li JT, et al. Application of weighted contribution rate of nonlinear output frequency response functions to rotor rub-impact. *Mech Syst Signal Process* 2020; 136: 106518.
36. Yang YF, Zheng QY, Wang JJ, et al. Dynamics response analysis of airborne external storage system with clearance between missile-frame. *Chin J Aeronaut.* Epub ahead of print 20 June 2020. DOI: 10.1016/j.cja.2020.06.008.
37. Pilkey WD. *Analysis and design of elastic beams*. 1st ed. New York: John Wiley and Sons, 2012.
38. Ishida Y and Yamamoto T. *Linear and nonlinear rotor dynamics*. New York: Wiley, 2012.