Uniqueness in the Kashiwara-Vergne conjecture

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Abstract. We prove that a universal symmetric solution of the Kashiwara-Vergne conjecture is unique up to order one. In the Appendix by the second author, this result is used to show that solutions of the Kashiwara-Vergne conjecture for quadratic Lie algebras existing in the literature are not universal.

Keywords. Campbell-Hausdorff series, free Lie algebras, differential of the exponential map, Bernoulli numbers.

1. Introduction

Let $G$ be a Lie group over $\mathbb{K} = \mathbb{R}$ (or $\mathbb{K} = \mathbb{C}$), and $\mathfrak{g}$ its Lie algebra. There exists an open neighbourhood $\mathfrak{g}_0$ of the origin $0 \in \mathfrak{g}_0 \subseteq \mathfrak{g}$ such that the restriction to $\mathfrak{g}_0$ of the exponential map $\exp : \mathfrak{g} \to G$ is an analytic diffeomorphism. We denote by $\ln : \exp(\mathfrak{g}_0) \to \mathfrak{g}_0$ the inverse map and by

$$\varphi_1(t) := \frac{t}{e^t - 1} = 1 - \frac{1}{2} t + o(t)$$

the generating series of Bernoulli numbers. It is convenient to have a separate notation for the function $\psi(t) := -(\varphi_1(t) - 1)/2$.

In [4], Kashiwara and Vergne put forward the following conjecture on the properties of the Campbell-Hausdorff series:

Kashiwara-Vergne conjecture. There exists a pair of $\mathfrak{g}$-valued analytic functions $A$ and $B$ defined on an open subset $U \subset \mathfrak{g} \times \mathfrak{g}$ containing $(0,0)$, such that $A(0,0) = B(0,0) = 0$, and for any $(X,Y) \in U$ one has

$$\ln(\exp(Y)\exp(X)) - X - Y = (id - e^{-\text{ad}X})A(X,Y) + (e^{\text{ad}Y} - id)B(X,Y), \quad (1)$$

$$\text{tr} \ (\text{ad}X \circ \delta_1 A(X,Y) + \text{ad}Y \circ \delta_2 B(X,Y)) = \text{tr} (\psi(\text{ad}X) + \psi(\text{ad}Y) - \psi(\text{ad} \ln (\exp(X)\exp(Y)))) , \quad (2)$$

where $\delta_1 A(X,Y), \delta_2 B(X,Y) \in \text{End}(\mathfrak{g})$ are defined as follows,

$$\delta_1 A(X,Y) : Z \mapsto \left. \frac{d}{dt} A(X + tZ,Y) \right|_{t=0}, \quad \delta_2 B(X,Y) : Z \mapsto \left. \frac{d}{dt} A(X,Y + tZ) \right|_{t=0}.$$

Sometimes this statement is referred to as the ‘combinatorial Kashiwara-Vergne conjecture’ (see e.g. [5]). This conjecture was established for solvable Lie
algebras in [4] and for quadratic Lie algebras in [6]. Recently, the general case was settled in [2] based on the earlier work [5].

We denote by $\mathbb{K}[t]$ and $\mathbb{K}[t]$ the ring of formal power series and the ring of polynomials, respectively. We call a solution of the Kashiwara-Vergne conjecture universal if $A$ and $B$ are given by series in Lie polynomials of the variables $X$ and $Y$:

$$A(X,Y) = \rho X + \beta(\text{ad}X)(Y) + o(Y)$$

$$B(X,Y) = \alpha X + \gamma(\text{ad}X)(Y) + o(Y)$$

with $\beta(t), \gamma(t) \in \mathbb{K}[[t]], \alpha, \rho \in \mathbb{K}$, and both $o(Y)$ are of type

$$o(Y) \in \sum_{k \geq 2} \sum_{j_1, \ldots, j_k \geq 0} j_k < j_k \mathbb{K}\text{ad}_{(\text{ad}X)^{j_1}Y} \circ \cdots \circ \text{ad}_{(\text{ad}X)^{j_{k-1}}Y} \circ (\text{ad}X)^{j_k}(Y).$$

If $(A, B)$ is a universal solution, the coefficients of the Taylor expansions of $A$ and $B$ are the same for all Lie algebras over $\mathbb{K}$.

The set of solutions of the Kashiwara-Vergne conjecture carries a natural $\mathbb{Z}/2\mathbb{Z}$-action,

$$(A(X,Y), B(X,Y)) \mapsto (B(-Y,-X), A(-Y,-X)).$$

A solution is called symmetric if it is stable with respect to this action. Averaging of any solution produces a symmetric solution. Hence, without loss of generality we can restrict our attention to symmetric solutions.

It is well-known (see e.g. [5]) that $\alpha, \rho$ and $\beta(t)$ are uniquely determined by the Kashiwara-Vergne equations and by the symmetry condition. In this note we prove the uniqueness statement for the function $\gamma(t)$. Thus, the symmetric universal solution of the Kashiwara-Vergne conjecture is unique up to order one in $Y$.

In the Appendix by the second author, this result is applied to show that solutions of the Kashiwara-Vergne conjecture for quadratic Lie algebras obtained in [6] and [1] are not universal.

2. Preliminaries

In this Section, we collect some elementary properties of Lie algebras.

Remark 2.1. (Free Lie algebras with two generators). We denote by $L_\mathbb{K}(x,y)$ the free Lie $\mathbb{K}$-algebra with generators $x$ and $y$. In this section we use the Hall basis $H$ of $L_\mathbb{K}(x,y)$ defined in [3] (Definition 2, page 27).

$H$ consists of Lie words with the following order relation: $x, y \in H$ and $x < y$; if the number of Lie brackets in $a \in H$ is smaller than the number of Lie brackets in $b \in H$ then $a < b$; and we omit the description of the order relation for $a$ and $b$ of equal length. The basis $H$ is built inductively starting with $x, y, [x, y]$, and one adds the elements of the form $[a, [b, c]]$ such that $a, b, c, [b, c] \in H$, $b \leq a \leq [b, c]$, and $b < c$. 

Proof. It is sufficient to show that

\[ \forall n \geq 0 \quad (\text{ad}x)^n(y) \in H. \]

In fact, the cases \( n = 0 \) and \( n = 1 \) are trivial, and for \( n \geq 1 \) we use \( (\text{ad}x)^{n+1}(y) = [x, [x, (\text{ad}x)^n(y)]] \). Furthermore,

\[ \forall n \geq 1 \quad \{(\text{ad}x)^j(y), (\text{ad}x)^{n-j}(y), 0 \leq j < n - j, j \leq n - 1\} \subset H. \quad (3) \]

Here it is sufficient to observe that \( (\text{ad}x)^n(y) = [x, (\text{ad}x)^{n-1}(y)] \).

**Proposition 2.2.** Let \( \xi(t) \in \mathbb{K}[t] \). The following statements are equivalent:

i) for any Lie \( \mathbb{K} \)-algebra \( g \) we have \( \xi(\text{ad}X)(Y) = 0 \) \( \forall X,Y \in g \);

ii) \( \xi(t) = 0 \).

**Proof.** It is sufficient to show that i) implies ii). Let \( n \in \mathbb{N} \). By rescaling \( X \mapsto tX \) and applying \( \frac{d^n}{dt^n}|_{t=0} \) we get \( \xi_n(\text{ad}X)^n(Y) = 0 \). Choosing \( g = L_\mathbb{K}(x, y) \), \( X = x \) and \( Y = y \) we get \( \xi_n = 0 \).

The following will be a very useful notation.

**Definition 2.3.** Let \( W, X, Y \in g \). For any pair \( i, j \in \mathbb{N} \), we set

\[ (t^i u^j : [W, X])_Y := [(\text{ad}Y)^i(W), (\text{ad}Y)^j(X)]. \]

This notation is extended by linearity to any formal power series \( \xi(t, u) \in \mathbb{K}[t, u] \). Then \( (\xi(t, u) : [W, X])_Y \in g[[g]] \) is a formal power series with coefficients in \( g \).

**Remark 2.4.**

i) \( (t^i u^j : [W, X])_Y = -(u^i t^j : [X, W])_Y \).

ii) Jacobi’s identity gives \( (t + u : [W, X])_Y = (\text{ad}Y)([W, X]) \).

**Proposition 2.5.** Let \( \xi(u) = -\xi(-u) \) be a series in \( \mathbb{K}[[u]] \). The following statements are equivalent:

i) for any Lie \( \mathbb{K} \)-algebra \( g \) we have \( (\xi(u) : [X, X])_Y = 0 \) \( \forall X,Y \in g \);

ii) \( \xi(u) = 0 \).

**Proof.** Recall that \( (\xi(u) : [X, X])_Y = [X, \xi(\text{ad}Y)(X)] \). Similar to the proof of Proposition 2.2 it is sufficient to show that in the free Lie algebra \( L_\mathbb{K}(X, Y) \) we have \( [X, (\text{ad}Y)^{2i+1}X] \neq 0 \) for any \( i \in \mathbb{N} \). Indeed, if we rename \( X = y, Y = x \), the elements \( [y, (\text{ad}x)^{2i+1}y] \) belong to the basis \( H \) and, hence, are non-vanishing.

Every formal power series \( \xi(t, u) \in \mathbb{K}[[t, u]] \) can be split into the sum of its symmetric and skew-symmetric parts:

\[ \xi(t, u) = \xi(t, u)_{\text{skew}} + \xi(t, u)_{\text{sym}} = \frac{\xi(t, u) - \xi(u, t)}{2} + \frac{\xi(t, u) + \xi(u, t)}{2}. \]

**Proposition 2.6.** Let \( \xi(t, u) \in \mathbb{K}[[t, u]] \). The following statements are equivalent:

i) for any Lie \( \mathbb{K} \)-algebra \( g \) we have \( (\xi(t, u) : [X, X])_Y = 0 \) \( \forall X,Y \in g \);

ii) \( \xi(t, u) = \xi(u, t) \).
Proof. By skew-symmetry of the Lie bracket, \((\xi(t, u)_{\text{sym}} : [X, X])_Y = 0\) for any \(\xi\), and \((\xi(t, u) : [X, X])_Y = (\xi(t, u)_{\text{skew}} : [X, X])_Y\).

Let \(\xi(t, u)\) be a formal power series with vanishing symmetric part. Then, it can be written as
\[
\xi(t, u) = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{0 \leq j < n-j} \xi_{n,j}(t^j u^{n-j} - u^j t^{n-j}).
\]
Suppose that
\[
(\xi(t, u) : [X, X])_Y \equiv \sum_{n=0}^{\infty} \sum_{0 \leq j < n-j} \xi_{n,j}((\text{ad}Y)^j X, (\text{ad}Y)^{n-j} X) = 0
\]
for every Lie \(\mathbb{K}\)-algebra \(g\) and every \(X, Y \in g\). By rescaling \(X \mapsto tX\) and then applying the \(n\)-th derivative in \(t\) we get
\[
\forall n \geq 1 \sum_{0 \leq j < n-j} \xi_{n,j}((\text{ad}Y)^j X, (\text{ad}Y)^{n-j} X) = 0.
\]
Then we choose \(g = L_\mathbb{K}(x, y)\), \(X = y\) and \(X = y\). Since all Lie words in the sum are linearly independent (recall property (\(n\))) this implies \(\xi_{n,j} = 0\) for all \(n, j\) and \(\xi(t, u) = 0\). 

Lemma 2.7. In the Lie algebra \(L_\mathbb{K}(x, y)\) we have
\[
\forall n \in \mathbb{N}, \quad (u^{2n+1} : [y, y])_x \notin \text{span}_\mathbb{K}\{(((t + u)t^l u^m : [y, y])_y)_x | l, m \in \mathbb{N}\}.
\]

Proof. We want to show that \((u^{2n+1} : [y, y])_x \notin \text{span}_\mathbb{K}\{(((t + u)t^l u^{2n-l} : [y, y])_y)_x | 0 \leq l \leq 2n\}\). If \(n = 0\) this statement is obvious. Let \(n \geq 1\), and suppose that we can some find coefficients \(c_j \in \mathbb{K}\) such that
\[
(u^{2n+1} : [y, y])_x = \sum_{j=0}^{2n} c_j((t + u)u^{2n-j} t^j : [y, y])_x.
\]
Let \(\xi(t, u) := u^{2n+1} - \sum_{j=0}^{2n} c_j(t + u)u^{2n-j} t^j\), then identity (2) can be written as
\((\xi(t, u) : [y, y])_x = 0\). The universal property of a free Lie algebra allows to apply Proposition 2.6 so \(\xi(t, u) = \xi(u, t)\). This means that \(u^{2n+1} - t^{2n+1} = 0\) modulo \((t + u)\), and this is a contradiction. 

Remark 2.8. Here we explain that Propositions 2.2, 2.5, and 2.6 still apply if we restrict to finite-dimensional Lie algebras.

In their proofs, at some point we choose \(g\) equal to the free Lie algebra \(L_\mathbb{K}(x, y)\). Let \(N \geq 2\). We introduce \(g_N := L_\mathbb{K}(x, y)/I_N\), where \(I_N\) is an ideal of \(L_\mathbb{K}(x, y)\) such that \(g_N\) is an \(N\)-nilpotent Lie algebra. In particular \(g_N\) is a finite-dimensional Lie algebra with basis \(H/I_N\). To modify the proofs it is sufficient to replace \(L_\mathbb{K}(x, y)\) with \(g_N\), for a good choice of \(N\): \(n \leq N - 1\) in Proposition 2.2, \(2i + 2 \leq N - 1\) in Proposition 2.5, and \(n \leq N - 2\) in Proposition 2.6.

In the previous theorems we do not use Lie groups. We end this section by computing some derivatives of the exponential map of a Lie group \(G\) with Lie algebra \(g\).

Let \(g \in G\). We use the notation \(R_g : G \to G\) for the right translation. In the following lemma we denote by \(1_G\) the group unit of \(G\).
Lemma 2.9. Let $X, Y \in \mathfrak{g}$, then
\begin{enumerate} \item \[d(\ln \circ R_{\exp(X)})(1_G) = \varphi_1(\text{ad}X),\]
\item \[\frac{d}{ds} \ln(\exp(sY)\exp(X)) = \varphi_1(\text{ad}_{\ln(\exp(sY)\exp(X))}(Y)),\]
\item \[\frac{d^2}{ds^2}|_{s=0} \ln(\exp(sY)\exp(X)) = \left(\frac{\varphi_1(t+u) - \varphi_1(u)}{t}\right) \varphi_1(t) : [Y, Y]\] \end{enumerate}

Proof. i) The formula of this differential is a consequence of the well-known formula for the differential of the exponential map:
\begin{equation}
d(\exp)_X = (dL_{\exp(X)})_{1_G} \circ \varphi_1(-\text{ad}X)^{-1}.
\end{equation}
ii) Using part i) and $R_{\exp(X)} R_{\exp(-X)} = id$ we get
\[\frac{d}{ds} \ln \circ R_{\exp(X)}(\exp(sY)) \equiv d(\ln \circ R_{\exp(X)})_{\exp(sY)} \frac{d}{ds}(\exp(sY)) = \varphi_1(\text{ad}_{\ln(\exp(sY)\exp(X))}) \circ (dR_{\exp(-sY)})_{\exp(sY)} \frac{d}{ds}(\exp(sY)).\]
Formula (5) gives $\frac{d}{ds}(\exp(sY)) = d(L_{\exp(sY)})_{1_G}(Y)$, so
\[d(R_{\exp(-sY)})_{\exp(sY)} \frac{d}{ds}(\exp(sY)) = e^{s \text{ad}Y}(Y) = Y.\]
iii) Using ii), i) and a direct calculation we get
\begin{align*}
\frac{d^2}{ds^2}|_{s=0} \ln \circ R_{\exp(X)}(\exp(sY)) &= \left(\varphi_1(\beta(t) - \varphi_1(u)) : \left[\frac{d}{ds}|_{s=0} \ln \circ R_{\exp(X)}(\exp(sY)), Y\right]\right)_X \\
&= \left(\varphi_1(t + u) - \varphi_1(u) : \left[\varphi_1(\text{ad}X)(Y), Y\right]\right)_X.
\end{align*}

3. The Campbell-Hausdorff series

In this section we only make use of Equation (1), and we derive formulas for $\beta(t)$, $\gamma(t)_{\text{odd}}$, and $A_2(X, Y) := \frac{d^2}{ds^2}|_{s=0} A(X, sY)$.

Theorem 3.1. A universal solution of the Kashiwara-Vergne conjecture has
\[\beta(t) = \beta_\alpha(t) \equiv \varphi_1(-t) \left(\frac{\varphi_1(t) - 1}{t} + \alpha\right).\]

Proof. In (1) we rescale $Y$ by $tY$ and we compute the derivative in $t = 0$:
\[d(\ln \circ R_{\exp(X)})(1_G) = (id - e^{-\text{ad}X}) \circ \beta(\text{ad}X)(Y) - \alpha(\text{ad}X)(Y).\]
Using Lemma 2.9 part i) and $(id - e^{-\text{ad}X}) \equiv \varphi_1(-\text{ad}X)^{-1} \circ \text{ad}X$, we get
\[\varphi_1(-\text{ad}X) \circ (\varphi_1(\text{ad}X) - id + \alpha \text{ad}X)(Y) = \text{ad}X \circ \beta(\text{ad}X)(Y).\]
As this identity has to be verified for any $\mathbb{K}$-Lie algebra $\mathfrak{g}$ and for any $X, Y \in \mathfrak{g}$, Proposition 2.2 implies $\varphi_1(-t)(\varphi_1(t) - 1 + \alpha t) = t \beta(t).$
The following theorem uses the notation

\[ A_2(X,Y) = (\pi(t,u) : [Y,Y])_X, \]

where the formal power series \( \pi(t,u) \in \mathbb{K}[[t,u]] \) is skew-symmetric (i.e. \( \pi(t,u) = \pi(t,u)_{skew} \)).

**Theorem 3.2.** A universal solution of the Kashiwara-Vergne conjecture has

i) \( \gamma(t)_{odd} = \gamma_{\alpha}(t)_{odd} := \frac{\alpha}{2} t + \frac{1}{2} \left( \frac{\varphi_1(t)-1}{t} \varphi_1(-t) \right)_{odd}, \)

ii) \( \pi(t,u) = \pi_{\gamma}(t,u) := -\varphi_1(t) + \frac{\varphi_1(t) + \varphi_1(t+u)}{t+u} \varphi_1(t+u)_{skew}. \)

**Proof.** By rescaling \( Y \mapsto sY \) and then applying the second derivative in \( s \) to Equation (1) we obtain,

\[ \frac{d^2}{ds^2}|_{s=0} \ln(e^sY) e^sX) = (id-e^{adX})A_2(X,Y)+(-\alpha u + 2\gamma(u) : [Y,Y])_X. \] (6)

Here we have used that

\[ \frac{d^2}{ds^2}|_{s=0}(e^{adY})s - id)B(X,sY) = \alpha(adY)^2(X) + 2adY \circ \gamma(adX)(Y) = (-\alpha u + 2\gamma(u) : [Y,Y])_X. \]

Let \( s(t,u) := \frac{\varphi_1(t+u)-\varphi_1(u)}{t} \varphi_1(t) + \alpha u - 2\gamma(u). \) The comparison of part iii) of Lemma 2.9 with Equation (6) gives

\[ (s(t,u) : [Y,Y])_X = (1-e^{adX})A_2(X,Y) \equiv ((1-e^{t+u})\pi(t,u) : [Y,Y])_X \]

for any \( X,Y \in \mathfrak{g} \) and any Lie \( \mathbb{K} \)-algebra \( \mathfrak{g} \).

Let \( g(t,u) := s(t,u) - (1-e^{t+u})\pi(t,u). \) Proposition 2.6 gives \( g(t,u)_{skew} = 0 \), in particular

\[ s(t,u)_{skew} = (1-e^{t+u})\pi(t,u). \]

Putting \( t+u = 0 \) we get statement i). To get statement ii) it is sufficient to remark that \( (1-e^{t+u}) = -\varphi_1(t+u)^{-1}(t+u). \) \( \square \)

4. The equation with traces

In this section we derive formulas for \( \rho \) and \( \gamma(t). \) We begin with a technical remark.

**Remark 4.1.** Let \( \lambda, \mu \in \mathbb{K} \setminus \{0\} \) be two distinct numbers, and \( \mathfrak{g}_{\lambda,\mu} = \mathbb{K}a \oplus \mathbb{K}b \oplus \mathbb{K}c \) be the 3-dimensional Lie algebra with Lie brackets \([a,b] = 0, [a,c] = \lambda c, [b,c] = \mu c. \)

It is easy to see that \([\text{ad}a, \text{ad}b] = 0\), and as a consequence \( \text{ad} \ln(exp(a)exp(b)) = \text{ad}a + \text{ad}b \). Moreover, for any \( \xi(t,u) \in \mathbb{K}[[t,u]] \) one has

\[ \text{tr}(\xi(\text{ad}a, \text{ad}b)) = \xi(\lambda, \mu) + 2\xi(0,0). \]
Theorem 4.2. A universal solution of the Kashiwara-Vergne conjecture has
\[ \rho = 0, \]
\[ \gamma(t) = \gamma_a(t) := \beta_a(t) - \beta_a(0) + \psi'(0) - \psi'(t). \]

Proof. Let \( \epsilon \in \mathbb{K} \), we have \((\text{ad}\, Y \epsilon) \circ \delta_2 B(X, Y \epsilon) = \text{ad} \circ \gamma(\text{ad} X) \epsilon + o(\epsilon)\) and
\[ \delta_1 A(X, Y \epsilon) = \rho \text{ id} - \sum_{n \geq 1} \beta_n \sum_{j=0}^{n-1} (\text{ad} X)^j \circ \text{ad} ((\text{ad} X)^{n-1-j} Y) \epsilon + o(\epsilon). \]

Let \( g \) be the Lie algebra in Remark 4.1, \( X = a \), and \( Y = b \). We get
\[ \text{tr}(\text{ad} b \circ \delta_2 B(a, b \epsilon)) \epsilon = \text{tr}(\text{ad} b \circ \gamma(\text{ad} a) \epsilon + o(\epsilon)), \]
\[ \text{tr}(\text{ad} a \circ \delta_1 A(a, b \epsilon)) = \text{tr}(\rho \text{ ad} a - \sum_{n \geq 1} \epsilon \beta_n (\text{ad} a)^n \circ \text{ad} b + o(\epsilon)), \]
\[ \text{tr}(\psi(\text{ad} a) + \psi(\epsilon \text{ ad} b) - \psi(\text{ad} a + \epsilon \text{ ad} b)) = \]
\[ = \text{tr}(-\epsilon(\psi'(\text{ad} a) - \psi'(0)\text{ ad} a) \circ \text{ad} b + o(\epsilon)). \]

In particular Equation (2) gives
\[ \text{tr}(\rho \text{ ad} a) = 0, \]
\[ \text{tr}\left(\left(\gamma(\text{ad} a) - \beta(\text{ad} a) + \beta(0) + \psi'(\text{ad} a) - \psi'(0)\right) \circ \text{ad} b\right) = 0. \]

Rescaling \( a, b \) and using the properties of \( g_{\lambda, \mu} \) we get
\[ \rho = 0, \]
\[ \gamma(t) - \beta(t) + \beta(0) + \psi'(t) - \psi'(0) = 0. \]

Remark 4.3. If one replaces \( \psi \) by a series \( f = f_0 + f_1 t + \cdots \in \mathbb{K}[[t]] \) one gets another conjecture that one can call an \( f \)-Kashiwara-Vergne conjecture. Then,

i) Theorem 4.2 is modified by replacing \( \psi \) by \( f \) and adding \( f_0 = 0 \) in the conclusion.

ii) Theorems 3.1, 3.2 and part i) imply \( f(t)_{\text{even}} = \psi(t)_{\text{even}} \), otherwise a universal solution of the \( f \)-Kashiwara-Vergne conjecture does not exist.

iii) If a universal solution of the \( f \)-Kashiwara-Vergne conjecture has \( A(X, Y) = B(-Y, -X) \) then one can show that \( f(t)_{\text{odd}} = f_1 t \) (we stress that \( \psi(t)_{\text{odd}} = \psi'(0) t \)). To get an easy proof one can use the Lie algebra of Remark 4.1.

A Comparison with quadratic solutions (by E. Petracci)

In the previous sections we did not determine the value of the constant \( \alpha \). Imposing the symmetry condition we obtain \( \beta_{\alpha}(\text{ad} X) Y + o(Y) = -\alpha Y - \gamma(-\text{ad} Y) X + o(X) \), so \( \alpha - \frac{1}{2} =: \beta_{\alpha}(0) = -\alpha \). Hence a universal symmetric solution has \( \alpha = \frac{1}{4} \).
Vergne and Alekseev-Meinrenken both considered a quadratic Lie algebra and obtained symmetric solutions. It is natural to ask whether these solutions are universal. In fact, quadratic Lie algebras have the special property \( \text{tr}((\text{ad}X)^{2n} \circ \text{ad}Y) = 0 \) for any \( n \in \mathbb{N} \) and any couple of vectors \( X, Y \), which simplifies the equation with traces \([2]\).

We have seen that a universal symmetric solution of the Kashiwara-Vergne conjecture has

\[
B(X, Y) = \frac{1}{4} X + \left( \beta_{1/4}(\text{ad}X) - \psi'(\text{ad}X) + \frac{1}{2} \text{id} \right)(Y) + o(Y),
\]

\[
A(X, Y) = \beta_{1/4}(\text{ad}X)(Y) + \frac{1}{2} (\pi \gamma_{1/4}(t, u) : [Y, Y]) x + o(Y^2)
\]

with \( \beta_\alpha(t) \) given in Theorem 3.1 and \( \pi \gamma_{1/4}(t, u) \) given in Theorem 3.2.

**Remark A1. (Vergne’s solution for quadratic Lie algebras)**

We denote by \( B_V(X, Y) \) the \( B \) found by M. Vergne in \([6]\). Following her paper we find \( B_V(X, Y) = \frac{1}{4} X + \gamma_V(\text{ad}X)(Y) + o(Y) \). Let

\[
R(t) := \frac{e^t - e^{-t} - 2t}{t^2},
\]

after a bit long calculation we see that the series \( \gamma_V(t) \) is given by

\[
t \gamma'_V(t) + 2 \gamma_V(t) = \frac{1}{8} t - \frac{1}{2} t \varphi_1(-t) R(t) \varphi_1'(t) = \frac{1}{8} t + \frac{1}{12} t^2 + \frac{1}{72} t^3 - \frac{1}{360} t^4 + o(t^4).
\]

This differential equation gives \( \gamma_V(t)_{\text{odd}} = \gamma_{1/4}(t)_{\text{odd}} \). A universal solution has

\[
t \gamma'_{1/4}(t) + 2 \gamma_{1/4}(t) = \frac{1}{8} t + \frac{1}{12} t^2 + \frac{1}{72} t^3 - \frac{1}{480} t^4 + o(t^4).
\]

In particular the symmetric solution found by M. Vergne for a quadratic Lie algebra is not universal.

**Remark A2. (Alekseev-Meinrenken’s solution for quadratic Lie algebras)**

Let \( B_{AM}(X, Y) = \alpha_{AM} + \gamma_{AM}(\text{ad}X)(Y) + o(Y) \) the \( B \) found by Alekseev and Meinrenken, \( \beta_{AM}(t) \) their series \( \beta(t) \), etc.

Following their paper \([11]\) and the paper \([6]\) of M. Vergne, after some efforts we find the following formulas. Let \( g(t) = \frac{1}{2} R(t) \), and \( \Pi(t) \) be the formal power series such that

\[
t \Pi'(t) + 2 \Pi(t) = \frac{1}{2} \varphi_1(t)^{-1} - g(t) \varphi_1(-t)(1 - \varphi_1(t)).
\]

Then

\[
\beta_{AM}(t) = \Pi(t) - \frac{1}{4} \left( g(t) \varphi_1(-t) - \frac{1}{2} \varphi_1(t)^{-1} \right) t - \frac{1}{2} \varphi_1(t)^{-1} + g(t) \varphi_1(-t)(1 - \varphi_1(t)),
\]

\[
\rho_{AM} = 0,
\]

\[
\gamma_{AM}(-t) + \gamma_{V}(t) = \varphi_1(-t) g(t) t \left( \beta_{AM}(-t) - \frac{1}{4} \varphi_1(t) - \varphi_1'(t) \right) + \frac{1}{2} \varphi_1(t)^{-1} t \beta_{AM}(-t) - \frac{1}{8} t,
\]

\[
\alpha_{AM} = \frac{1}{4}.
\]
Using Maple we get $\beta_{AM}(t) = 2 \beta_1(t)$, $\gamma_{AM}(t)_{odd} = \gamma_V(t)_{odd}$, and

$$t\gamma'_{AM}(t) + 2\gamma_{AM}(t) = \frac{1}{8} t + \frac{1}{12} t^2 + \frac{1}{72} t^3 - \frac{1}{720} t^4 + o(t^4).$$

In particular the symmetric solution of Alekseev and Meinrenken is not universal, and it is different from the solution of Vergne.

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