Stable space-like singularity formation for axi-symmetric and polarized near-Schwarzschild black hole interiors.

Spyros Alexakis and Grigorios Fournodavlos

Abstract

We show a stability result for the Schwarzschild singularity (inside the black hole region) for the Einstein vacuum equations. The result is proven in the class of polarized axial symmetry, under perturbations of the Schwarzschild data induced on a hypersurface \( \{ r = \epsilon \} \), \( \epsilon < 2M \). Our result is only partly a stability result, in that we show that while a (space-like) singularity persists under perturbations as above, the behaviour of the metric approaching the singularity is much more involved than for the Schwarzschild solution. Indeed, we find that the solution displays asymptotically-velocity-term-dominated dynamics and approaches a different Kasner solution at each point of the singularity. These Kasner-type asymptotics are very far from isotropic, since (as in Schwarzschild) there are two contracting directions and one expanding one. Our proof relies on energy methods and on a new approach to the EVE in axial symmetry, which we believe has wider applicability: In this symmetry class and under a suitable geodesic gauge, the EVE can be studied as a free wave coupled to (nonlinear) ODEs, which couple the geometry of the projected, 2+1 space-time to the free wave. The fact that the nonlinear part of the Einstein equations is described by ODEs lies at the heart of how one can overcome a certain linear instability exhibited by the singularity.

Contents

1 Introduction 3

1.1 The result. 3
1.1.1 AVTD behaviour of our solutions. 4
1.2 Singularity formation in black holes and cosmological singularities: Predictions and results. 5
1.2.1 Singularity formation in black hole interiors and the instability of the Schwarzschild singularity. 5
1.2.2 Big Bang type singularities. 6
1.2.3 The strength of the singularity. 7
1.2.4 Outlook: Results beyond two degrees of symmetry? 9
1.3 An outline of the ideas. 11
1.3.1 The reduced Einstein vacuum equations under polarized axi-symmetry. 11

2 The precise formulation of the result. 16

2.1 The Schwarzschild metric. 16
2.2 The initial data for our problem. 16
2.3 The result, properly formulated. 17
2.4 The geodesic gauge: Reduction of the EVE to free wave-ODE system. 18
2.4.1 The orthonormal frames and their propagation. 18
2.5 The space-time metric, expressed in terms of the orthonormal frame. 19
2.5.1 Gauge fixing for the frame \( e_0, e_1, e_2 \), relative to the singularity and the initial data hypersurface. 20
2.6 The matching of the prescribed initial data. 21
2.6.1 The system for the initial data. 22
2.7 The reduced Einstein equations in geodesic gauge, normalized at the singularity. 23
2.8 The theorem re-cast in terms of the REVESNGG. 24
3 The Iteration scheme.
  3.1 Overview.
  3.2 The recursive equations for $\gamma^m$ and $K_{ij}^m$.
    3.2.1 Rotation formulas and some useful calculations.
  3.3 Determination of the initial data hypersurface $r^m_+(t, \theta)$ and the connection and curvature components adapted to that hypersurface.
    3.3.1 The system of unknowns and the system of equations.

4 The function spaces and bounds for the key variables of our reduced system.
  4.1 Regularity spaces for the parameters.
  4.2 Key Constants.
  4.3 Preparatory steps: The interpolating function $\rho^m$ and its adapted frames.
  4.4 Regularity spaces.
  4.5 The Inductive claim for all parameters in the REVESNGG.
    4.5.1 Inductive claim for $\gamma^{m-1}$.
    4.5.2 Inductive claims on $r^m_{+}^{-1}(t, \theta)$, $K_{ij}^{m-1}(t, \theta)$.
    4.5.3 Inductive claims for $K^m$.
    4.5.4 The asymptotically CMC property of level sets of $r$.
    4.5.5 Remark on the regularity spaces.
    4.5.6 The passage to the limit $m \to \infty$.
    4.5.7 The base case of the inductive step.
  4.6 Basic Analysis tools.
    4.6.1 A generalized Hardy inequality.
    4.6.2 A generalized Gronwall inequality.
    4.7 Fuchsian ODEs and transport equations: Basic estimates.
  4.8 The spatial geometry parameters and their control by the inductive assumption.
    4.8.1 Propagation of vanishing conditions at the poles.
    4.8.2 The new coordinate system: Bounds on metric components and Christoffel symbols, by virtue of our inductive assumptions.
    4.8.3 Consequences of the energy estimates on the free wave $\gamma^m$.

5 The estimates for the next iterate: The free wave $\gamma^m$.
  5.0.1 Some estimates on the geometry of the level sets of $\rho^m$.
  5.0.2 General framework for the energy estimates: The weighted multiplier.
  5.0.3 Language conventions.
  5.1 The wave equation expanded: An inhomogenous equation for $\gamma^m_{\text{rest}}$.
    5.1.1 Bounds on the inhomogenous term in the wave equation (5.16).
    5.1.2 Formulas and bounds for commutations of $\gamma^m_{\text{rest}}$.
  5.2 Lower order energy estimates: $E[\gamma^m_{\text{rest}}(t, \theta), |I| \leq s - 3 - 4c]$.
  5.3 Middle order energy estimates: $E[\gamma^m_{\text{rest}}(t, \theta), s - 3 - 4c < |I| \leq s - 4]$.
    5.3.1 Estimate on the sum of all middle-order energies.
    5.3.2 Improved estimates at orders $s - 4 - k$, $k \in \{1, \ldots, 4c - 1\}$.
  5.4 Top order estimates for $\gamma^m$.
  5.5 Renormalized energy estimates at the low orders: Proof of (4.25).
  5.6 The AVTD behaviour of $\gamma^m_{\text{rest}}$ in the lower orders, via a descent scheme: The inductive step (4.29).
    5.6.1 Control of forcing terms in the Ricatti equations: The lower and higher orders.
  5.7 The estimates for $\gamma^m$ re-cast on level sets of $r$.

6 The estimates for the next iterate meta $m$.
  6.1 Energy estimates for $K^m$: Proof of (4.40) for $K^m_{12}(r, t, \theta)$,
    6.1.1 The functions $K^m_{12}(r, t, \theta)$ and their low derivatives as integrals from the singularity.
    6.1.2 Asymptotic expansion of $K^m_{12}(r, t, \theta)$ at the lower orders.
  6.2 The bounds on $K^m_{12}(r, t, \theta)$, $K^m_{12}(r, t, \theta)$ at the higher orders.
  6.3 The inductive step for $K_{12}^{m+1}$, $K_{12}^{m+1}$ at the top order.
  6.4 Capturing the hypersurface that carries the initial data:
    Determining the functions $r^m_+(t, \theta)$, $K_{12}^m(t, \theta)$.
  6.5 The inductive step on the functions $r^m_+(t, \theta)$, $K_{12}^m(t, \theta)$.
We study the problem of the stability of the singularity of the Schwarzschild black hole from the point of view of the forwards-in-time initial value problem for the Einstein vacuum equations (EVE): We consider perturbations of the initial data of Schwarzschild, along a space-like hypersurface in the black hole interior, and wish to understand the maximal future hyperbolic development of the solution, up to any singularities that might form, with detailed asymptotics at the singularity.

We restrict our attention to polarized axially symmetric perturbations. Within that class we find that the maximal hyperbolic development of any sufficiently small perturbation of the Schwarzschild initial data terminates at a space-like singularity with very rich dynamics. As we review below, at each point on its “final” singular hypersurface, the Schwarzschild solution exhibits a collapsing behaviour of the metric in two principal directions and an expanding behaviour in a remaining, third principal direction. In this regard, the behaviour of our solutions is qualitatively similar to that of the Schwarzschild solution, in that the singularity it forms is still space-like, and moreover at each point there are still two collapsing and one expanding principal directions. From this point of view, our result can be seen as a stability result. However the rates of the two contractions and expansion are different at each point on the final singular hypersurface and also generically different from those of Schwarzschild. Thus, the result we derive should be thought of as a stability property, albeit holding only in a broad sense.

We next present a rough version of our result, and then situate it in the context of singularity formation in black hole interiors and cosmological space-times. We then provide a broad outline of some of the ideas in this paper.

1.1 The result.

We will exclusively be studying space-times \((M^{1+1}, g)\) which are axially symmetric and the axial symmetry is polarized, with the Killing field corresponding to a rotation. In particular for our space-times there exists a system of coordinates \(r \in (0, 2\epsilon), t \in (-\infty, \infty), \theta \in (0, \pi), \phi \in [0, 2\pi)\) with \(\partial_\phi\) being the Killing field. The polarization condition is equivalent to the requirement that:

\[ \partial_\phi \perp \text{Span}(\partial_t, \partial_\tau, \partial_\nu) \]

Equivalently, the metric components \(g_{t\phi}, g_{\phi\phi}, g_{r\phi}\) all vanish.

For the sake of comparison, recall the Schwarzschild metric in the standard coordinates \(r, t, \theta, \phi\):

\[ g_s = -\left(\frac{2M}{r} - 1\right)^{-1}dt^2 + \left(\frac{2M}{r} - 1\right)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad M > 0. \]  

(1.1)

We note that in the interior of the black hole, \(r < 2M\), the characters of the coordinate vector fields \(\partial_\tau, \partial_\nu\) are reversed, namely, \(\partial_\tau\) is timelike and \(\partial_\nu\) is space-like. The (true) singularity is thus at \(r = 0\), where the metric components \((g_{s})_{\phi\phi} = r^2 \sin^2 \theta, (g_{s})_{\theta\theta} = r^2 \text{ collapse}, \) as \(r \to 0^+\), while \((g_{s})_{\tau \tau} = \left(\frac{2M}{r} - 1\right) \text{ expands as } r \to 0^+\).

Now, consider the hypersurface \(\Sigma_\epsilon := \{r = \epsilon\}\), for some constant \(\epsilon > 0\) that will be chosen suitably small further down. In the Schwarzschild space-time, let us denote by \(g_s\) be the induced metric on this hypersurface and by \(K_S\) its the second fundamental form. It follows straightforwardly from (1.1)
that the second fundamental form $K_S$ on these slices is of magnitude $\epsilon^{-3/2}$, see \[2.3\] below. The metric components $(g_S)_{\theta\theta}, (g_S)_{\phi\phi}, (g_S)_{r\theta}$ are of magnitude $\epsilon^2, \epsilon^2\sin^2\theta, (\frac{2M}{r^2}) - 1$ respectively.

The initial data $(g_{\text{init}}, K_{\text{init}})$ that we consider in this paper will be (polarized and axially symmetric) perturbations of the Schwarzschild background data $(g_S, K_S)$. The closeness will be measured in suitable Sobolev spaces, and the closeness (in these spaces) will be captured by a parameter $\eta > 0$. The precise assumption will be formulated further down in subsection 4.5.

It is for the space of initial data for which both $\epsilon, \eta$ are small enough (to be determined later) that we obtain our result. Our main finding will be that in a suitable coordinate system $r, \tilde{t}, \tilde{\theta}$, the solution exists all the way up to a space-like singularity that occurs at $\{ r = 0 \}$.

We provide a first, rough formulation of our main result here:

**Theorem 1.1.** Consider a perturbation $(g, K)$ of the Schwarzschild initial data $(g_S, K_S)$ on the hypersurface $r = \epsilon$. Let $\eta > 0$ capture the size of the perturbation (in a re-normalized sense to be specified in subsection 4.5), in $H^s \times H^{s-1}(\mathbb{S}^2 \times \mathbb{R})$ spaces, for some $s \in \mathbb{N}$, to be chosen suitably large below. Assume the perturbation preserves the polarized axisymmetric structure of the Schwarzschild background and solves the vacuum constraint equations within this symmetry class.

Then for $\epsilon, \eta > 0$ small enough this initial data admits a future maximal hyperbolic development which terminates at a space-like singularity, where the Kretschmann scalar blows up. Near the singularity, the space-time metric $g^{3+1}$ has an asymptotic profile of the following form, in suitable coordinates $r, \phi, \tilde{t}, \tilde{\theta}$:

$$g = -(\frac{2M}{r} - 1)^{-1}dr^2 + g_{\phi\phi}(r, \tilde{t}, \tilde{\theta})d\phi^2 + g_{\tilde{t}\tilde{t}}(r, \tilde{t}, \tilde{\theta})d\tilde{t}^2 + g_{\tilde{t}\tilde{\theta}}(r, \tilde{t}, \tilde{\theta})d\tilde{t}d\tilde{\theta} + g_{r\theta}(r, \tilde{t}, \tilde{\theta})drd\theta + g_{r\tilde{t}}(r, \tilde{t}, \tilde{\theta})drd\tilde{t},$$

where the metric components admit the asymptotic expansions:

$$g_{\phi\phi}(r, \tilde{t}, \tilde{\theta}) = A(\tilde{t}, \tilde{\theta})r^{2\alpha(\tilde{t}, \tilde{\theta})}(1 + O(r^{-1/2}) \sin^2\theta), \quad g_{\tilde{t}\tilde{t}}(r, \tilde{t}, \tilde{\theta}) = B(\tilde{t}, \tilde{\theta})r^{2\beta(\tilde{t}, \tilde{\theta})(2M + O(r^{1/4}))},$$

$$g_{\tilde{t}\tilde{\theta}}(r, \tilde{t}, \tilde{\theta}) = C(\tilde{t}, \tilde{\theta})r^{2\beta(\tilde{t}, \tilde{\theta})(1 + O(r^{1/4}))}, \quad g_{\tilde{t}\theta}(r, \tilde{t}, \tilde{\theta}) = D(\tilde{t}, \tilde{\theta})r^{2\beta(\tilde{t}, \tilde{\theta})(1 + O(r^{1/4}))},$$

as $\to 0^+$. Here the function $\alpha(\tilde{t}, \tilde{\theta})$ is everywhere close to 1, $\delta(\tilde{t}, \tilde{\theta})$ is also close to 1, $\beta(\tilde{t}, \tilde{\theta})$ is everywhere close to $-1/2$. Moreover the coefficients $A(\tilde{t}, \tilde{\theta}), B(\tilde{t}, \tilde{\theta}), C(\tilde{t}, \tilde{\theta})$ are everywhere close to 1 also.

In fact, the value of $\alpha$ at each point $(t, \theta)$ uniquely determines the value of the other powers $\beta(t, \theta), \delta(t, \theta)$. The remaining metric components are less singular, in the sense that the corresponding exponents satisfy $\beta_1(\tilde{t}, \tilde{\theta}) \geq \frac{1}{2}, \beta_2(\tilde{t}, \tilde{\theta}) \leq \frac{1}{2}$.

**Remark 1.2.** The same expansions hold for up to a certain number of $\partial_t, \partial_{\tilde{t}}$ derivatives of the metric, which henceforth we will denote by low, low $\ll s$. (At these orders, the Kretschmann scalar blows up like $r^{-6}$, as $r \to 0$). We expand on this in the stricter formulation of our result.

**Remark 1.3.** Note that the metric $g$ in this form is evidently axisymmetric and polarized, due to the absence of cross-terms $g_{\tilde{t}\phi}, g_{\phi\phi}, g_{\tilde{t}\tilde{\theta}}$, and also since all metric coefficients are independent of $\phi$. However there is an extra gauge normalization, captured in this form: The integral curves of $\partial_\theta$ are geodesics which are (asymptotically as $r \to 0^+$) orthogonal to $\partial_t, \partial_{\tilde{t}}, \partial_{\tilde{\theta}}$. Purely for comparison reasons, the parameter $r$ has been chosen to agree with the corresponding parameter in the Schwarzschild background.

**Remark 1.4.** Observe that as in the Schwarzschild background, the directions $\partial_\phi, \partial_{\tilde{\theta}}$ are collapsing, while the direction $\partial_t$ is expanding. The terms of order $r^{\beta_1(\tilde{t}, \tilde{\theta})}, r^{\beta_2(\tilde{t}, \tilde{\theta})}$ should be seen to be less singular off-diagonal terms (which vanish for the Schwarzschild metric).

While our paper is entirely concerned with black hole interiors, one can state a corollary that links it with recent studies of perturbations of the Schwarzschild black hole exterior regions (for two-ended initial data).

In particular, our main theorem complements the recent breakthrough stability result of the exterior region by Klainerman-Szeftel [20] and that of the inner red-shift region [17] announced by Dafermos-Luk [16], which combined give the full picture of near-Schwarzschild (double-ended) space-times in polarized axi-symmetry.

**Corollary 1.5.** Dynamical space-times, arising from sufficiently regular and small perturbations of the Schwarzschild initial data of mass $M_0$, on a global Cauchy hypersurface $\Sigma$, have the Penrose diagram depicted in Figure 7.

\[1\] In fact, the stability of the inner red-shift region, announced in [16], concerns general space-times that converge to a Kerr along the horizon, which is only simpler in the absence of rotation.
In the exterior regions, they are globally defined, having complete null infinities, and converge to Schwarzschild metrics at timelike infinities, of masses $M_1, M_2$ that are close to that of the background mass $M_0$. Moreover, the inner boundary of their black hole is entirely space-like, singular, and the asymptotic behaviour of the metric towards the singularity are as in Theorem §2. In particular, both the weak and strong cosmic censorship conjectures are valid in the axi-symmetric, polarized, near-Schwarzschild regime.

The proof of this corollary is carried out in §4.3

1.1.1 AVTD behaviour of our solutions.

Let us comment here on the asymptotically velocity term dominated (AVTD) behaviour of our solutions. One can formulate this property in different (essentially equivalent) ways. We here present the property purely in terms of the behaviour of the kinetic part of the energy of the metric components (gravitational field components) relative to the potential part of the same energy. The term was originally coined in the first construction of AVTD space-times by Kichenassamy-Rendal '98 [27], where they constructed analytic Gowdy space-times with Big Bang singularities exhibiting such behaviour, see also section 3 in [24].

In our space-times, $\partial_t$ is time-like and $\partial_\theta, \partial_\phi$ are space-like. Let us denote by $e_0, e_1, e_2$ the unit length vector fields in those directions.

In the Schwarzschild solution (where $t = \tilde{t}, \theta = \tilde{\theta}$), the gravitational field component $(g_\theta)_{\phi\phi} = r^2 \sin^2 \theta$ has the following property:

$$e_0(g_\theta)_{\phi\phi} \sim r^{1/2} \sin^2 \theta,$$

while $e_1(g_\theta)_{\phi\phi} = 0, e_2(g_\theta)_{\phi\phi} = O(r)$. For the rest of the principal gravitational fields components $(g_\theta)_{\theta\theta}, (g_\theta)_{tt}$ we also have that:

$$e_0(g_\theta)_{\theta\theta} \sim r^{1/2}, e_1(g_\theta)_{\theta\theta} = e_2(g_\theta)_{\theta\theta} = 0,$$

and also

$$e_0(g_\theta)_{tt} \sim r^{-5/2}, e_1(g_\theta)_{tt} = e_2(g_\theta)_{tt} = 0.$$

In particular the kinetic (the $e_0$)-part of the energy dominates the potential parts (the $e_1, e_2$-parts).

For the space-times that we construct we have a similar behaviour. We illustrate this for the function

$$\tilde{\gamma} := \frac{1}{2} \log g_{\phi\phi} - \log \sin \tilde{\theta},$$

up to lower-order terms in $r$ in the RHSs, we will see that:

$$e_0(\tilde{\gamma}) \sim -\alpha(\tilde{t}, \tilde{\theta}) \sqrt{2M} r^{-3/2}, |e_1(\tilde{\gamma})| \leq C r^{-1-\frac{1}{4}}, |e_2(\tilde{\gamma})| \leq C r^{-1-\frac{1}{4}}. \quad (1.4)$$

In terms of the energy, this implies that

$$E_{(r=r_0)}(\tilde{\gamma}) = \int_{r=r_0} |e_0(\tilde{\gamma})|^2 + |e_1(\tilde{\gamma})|^2 + |e_2(\tilde{\gamma})|^2 \sin \tilde{\theta} d\tilde{\theta} d\tilde{\phi} = (r_0)^{-3} \int_{r=r_0} 2M |\alpha(\tilde{t}, \tilde{\theta})|^2 \sin \tilde{\theta} d\tilde{\theta} d\tilde{\phi} + O(r_0^{-2-1/2}). \quad (1.5)$$

In particular, the energy of $\tilde{\gamma}$ is all asymptotically concentrated in the $e_0$ direction, i.e. the spatial (potential) components of the energy $e_1(\tilde{\gamma}), e_2(\tilde{\gamma})$ are strictly less singular than the time-like (kinetic) component $e_0(\tilde{\gamma})$ of the energy.

---

*This is a modification of the logarithm of the axi-symmetric gravitational field component.*
This captures the AVTD behaviour of the component $g_{\phi\phi}$ of the gravitational field.

As we will see, a consequence of this behaviour of $g_{\phi\phi}$ is that the principle components $g_{tt}, g_{\theta\theta}$ and their derivatives display a behaviour that is consistent with (1.3). This in particular implies that the derivatives in the $e_0 \parallel \partial_r$ and $e_1, e_2 \perp e_0$ directions of $\log g_{tt}, \log g_{\theta\theta}$ have the property that:

$$
e_0(\log g_{tt}) \sim r^{-3/2}, \quad |e_1(\log g_{tt})| \leq Cr^{-1-\frac{1}{6}}, \quad |e_2 g_{tt}| \leq Cr^{-1-\frac{1}{6}},$$

$$
e_0(\log g_{\theta\theta}) \sim r^{-3/2}, \quad |e_1(\log g_{\theta\theta})| \leq Cr^{-1-\frac{1}{6}}, \quad |e_2 g_{\theta\theta}| \leq Cr^{-1-\frac{1}{6}}.$$  

Notably letting $f$ to stand for any of the components $g_{AA}$ of the gravitational field $g$ (where $A$ takes on one of the values $t, \theta, \phi$), the energy of that field component

$$E[f](r = r_0) \int_{-\infty}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} \left( |e_0 f|^2 + |e_1 f|^2 + |e_2 f|^2 \right) (r = r_0) d\psi \sin \theta d\theta dt$$

is asymptotically, as $r_0 \to 0$, this energy is concentrated entirely in the $e_0$-direction. In other words the kinetic part of the energy dominates the potential parts of the energy. In particular in the evolution equations further down, if one were to drop terms involving spatial derivatives of all fields, one would still derive the correct leading order behaviour of all gravitational fields in $r$, from the resulting (dramatically simplified) equations.

This velocity-term dominated behaviour of the gravitational field for the solutions that we obtain is consistent with the predictions in the physics literature, as we explain next:

### 1.2 Singularity formation in black holes and cosmological singularities: Predictions and results.

The question of whether and how singularities form in the evolution of smooth initial data is a central question for all non-linear evolutionary PDEs. In the Einstein equations specifically, it is intimately linked to the question of strong cosmic censorship; in the usual formulation this predicts that inside black holes, generically, the space-time metric terminates at a final singularity, past which it is inextendible. The nature of the singularity is not formally part of the conjecture.

We recall some results on singularity formation in black hole interiors and in Big Bang settings in the next few sections. A more extensive discussion of these examples can be found in [37] and the references therein.

#### 1.2.1 Singularity formation in black hole interiors and the instability of the Schwarzschild singularity.

A brief comment is in order concerning the possibilities of viewing our result beyond polarized axial symmetry: On one hand, one sees that our stability result ceases to hold by merely removing the condition of polarization from the perturbations: Indeed, one can consider the family of Kerr solutions $g_{K,M,a}, a \neq 0$ bifurcating off of a background Schwarzschild solution $g_{K,M,0}$; as is well-known, the future maximal hyperbolic development of the data on $\{r = \epsilon\}$ is then smooth. (In fact, that maximal hyperbolic development even admits a smooth extension past a Cauchy horizon $\mathcal{CH}^+$, which can be attached as a boundary to this maximal hyperbolic development). In other words, the Schwarzschild singularity entirely disappears under (still axially symmetric!) perturbations that introduce angular momentum. To our knowledge, the only known examples of vacuum space-times that exhibit a Schwarzschild-type singularity, without any symmetry assumptions, are the ones constructed by the second author [19]. These spacetimes contain a singularity at a collapsed 2-sphere, where their asymptotic behaviour agrees with that of Schwarzschild at a suitably high order. This special requirement provided an early indication of the thinness of the set of perturbed initial data, which can lead to Schwarzschild-type singularity.

Therefore, one comes to the conclusion that the Schwarzschild singularity, as it appears in the maximal analytic (two-ended) extension of the Schwarzschild solution, is unstable from the point of view of the initial value problem in full generality (i.e. with no symmetry assumptions imposed). The stability of the (double-ended) Kerr maximal hyperbolic development was also studied in the recent breakthrough paper of Dafermos-Luk [13]. It was shown there that general perturbations of a rotating Kerr solution $g_{K,M,a}, a \neq 0$, when the perturbation is small enough to rule out proximity to a (non-rotating) Schwarzschild solution, still form Cauchy horizons in the interior, along which the metric is $C^0$-extendible. A very

---

3The relation with the Penrose inextendibility theorem is discussed in [37].
interesting, weaker type of singularity, is expected to emerge in that context, see also [14] and references therein for an earlier result on such weak null singularities in spherical symmetry.

A brief comparison of the two types of singularities (space-like and null) is in order: The space-like singularity in Schwarzschild (and in our solutions also) has a locality property, in that each compact set on the final hypersurface \( \{ r = 0 \} \) depends on a compact subset \( D \) of any Cauchy hypersurface in the entire space-time, as in the next picture. In contrast, any given point on the weak null singularity depends on a non-compact set of a Cauchy hypersurface. In particular, it depends on the entire future event horizon to which the weak null hypersurface is “attached”.

![Figure 2: The region to the future of \( \Sigma' \).](image)

This in particular allows for the possibility of studying space-like singularities locally, by prescribing initial data (which perturbs the Schwarzschild data within polarized axi-symmetry) on an incomplete initial data hypersurface; one can reduce oneself to the setting of our theorem by constructing an artificial extension to a complete initial data set (satisfying the constraints) which asymptotes to the Schwarzschild data on \( \{ r = \epsilon \} \) so as to fulfil the assumptions of our theorem (in particular it would be asymptotically cylindrical). If one can do this, then our result would yield a space-like singularity, a portion of which is independent of the extension we constructed.

The construction of such an extension is a matter of solving the constraint equations with asymptotically cylindrical data. We are not aware that this has been done in the literature; pursuing this here is beyond the scope of our paper.

This locality property also makes the space-time singularity indistinguishable (up to a time reversal), whether it occurs in a black hole or at an initial Big-Bang type-singularity. This latter class has been extensively studied and we wish to make some connections of our result to that part of the literature:

### 1.2.2 Big Bang type singularities.

The nature of the Big-Bang type singularities is in general a wide-open and very interesting question. Many of the beliefs surrounding this question stem from the explicit family of Kasner-like solutions, which

\[
g = -dt^2 + (\omega_1)^2 t^{2p_1} + (\omega_2)^2 t^{2p_2} + (\omega_3)^2 t^{2p_3}, \quad \lim_{t \to 0} \omega_i = \sum_{j=1}^3 c_{ij} dx_j, \tag{1.6}
\]

where \( t \in (0, T] \) is a time function synchronizing the singularity at \( t = 0 \) and \( p_i, c_{ij} : \Sigma \to \mathbb{R} \) are functions of the spatial coordinates \( x_1, x_2, x_3 \). Moreover, in vacuum, the \( p_i \)'s must satisfy the Kasner relations:

\[
\sum_{i=1}^3 p_i = \sum_{i=1}^3 p_i^2 = 1. \tag{1.7}
\]

Note that in view of the space-like nature of the singularity, for \( t \) small enough, different points \( (t, x_1, x_2, x_3) \) near different points on the singularity cannot be joined by time-like curves. In our case, different points have Kasner dynamics with exponents close to their Schwarzschild counterparts\( \{ p_1 = -\frac{1}{3}, p_2 = p_3 = \frac{2}{3} \} \), where the topology of \( \Sigma \) is \( S^2 \times \mathbb{R} \).

Note also that locally, up to a time reversal \( t \to (-t) \) one cannot “see” whether the singularity occurs as an initial Big Bang singularity or terminally, inside a black hole. In 1 + 3 vacuum, at least one of the

---

\(^4\)Setting \( ds = (\frac{2M}{r} - 1)^{-1/2} dr, s \sim r^{\frac{2}{3}} \), the Schwarzschild metric takes the form \( g_{tt} = s^{-1}, g_{\theta \theta} \sim s^2 \), \( g_{\phi \phi} \sim s^2 \sin^2 \theta \).
$p_i$'s has to be negative, in view of (1.7). So in particular, isotropic solutions or nearly-isotropic (where all $p_i$'s in (1.8) are close to equal) are not possible in $1 + 3$ vacuum.

The question of how general solutions that exhibit Kasner-type behaviour are within the class of all big-bang singularities has been studied in the mathematical literature in two main directions: In one direction one constructs classes of solutions with the prescribed asymptotics at the singularity. All of the examples constructed in this way directly display an AVTD behaviour towards a (different) Kasner solution at each point on the singularity hypersurface:

Constructions of AVTD singularities: There are various constructions in the literature of AVTD space-times, applying Fuchsian techniques to a first order reduction of the Einstein equations in order to produce Kasner-type singularities, where the Kasner exponents depend on the spatial coordinates of every point on the singularity. All such examples are either in the analytic class, or with extra symmetries imposed (or both):

The first such construction of AVTD space-times was given by Kichenassamy-Rendall [27], in the analytic Gowdy class. Other such results in the literature include: analytic AVTD space-times, without symmetries, by Anderson-Rendall [2] for the Einstein-scalar field model or a stiff-fluid; smooth AVTD space-times in the Gowdy class by Rendall [51]; analytic $U(1)$-symmetric polarized, and half-polarized AVTD space-times in vacuum by Isenberg-Moncrief [25] and Choquet-Bruhat-Isenberg-Moncrief [9]; higher dimensional, analytic, AVTD space-times in vacuum, without symmetries, by Damour et al. [18], for space-time dimensions $1 + d, d \geq 10$. In the other direction, one studies the stability of Kasner-like singularities. Such results, which go beyond the previous constructions via Fuchsian techniques, have only fairly recently been obtained:

Stability of Big Bang singularities without symmetries: In breakthrough works by Rodnianski-Speck [35], [36], it was proved that for models with certain special matter models (massless scalar fields and stiff fluids), isotropic FLRW-Big Bang singularities are non-linearly stable, for $\mathbb{T}^3$ topologies and later by Speck [39] for $S^3$. Moreover, Rodnianski-Speck [37] verified the stable Big Bang formation of Kasner type, in vacuum, for an open set of Kasner exponents in space-time dimensions $1 + d \geq 39$, perturbing off of close-to-isotropic explicit solutions with spatial topology $\mathbb{T}^d$. These works did not include any symmetry assumptions. In particular, the authors consider any sufficiently small perturbation of Kasner data at constant $t_0 > 0$ hypersurface and then solve towards the singularity at $t = 0$. The space-times one thus obtains (within the models considered) are therefore unrestricted stability results, in that the results hold for open sets of data that are prescribed on a hypersurface $\Sigma$ off of the singularity. In contrast, the space-times obtained from constructions by Fuchsian techniques are not apriori known to cover such an open set of data on a hypersurface off of the singularity, even if they enjoy all the gravitational degrees of freedom from a function counting point of view— in principle they could be a very thin set in the moduli space of allowable initial data on $\Sigma$.

The results in [35, 36, 39, 37] are perturbative results (as is ours), but the techniques used differ from ours substantially. The main difference of our setting with those considered in [35, 39, 39, 37] on the face of them, is that our background is highly anisotropic; in fact it contains an expanding direction in addition to the two contracting ones. One could thus speculate on the universality of this extra instability in $(3+1)$ Kasner vacua, which are of necessity highly an-isotropic. One can also wonder whether stability results within more restricted symmetry classes (like the ones considered here) might be true for other $(3+1)$ Kasner vacua. While these questions are not pursued here, we note that the methods we employ seem very robust in that regard.

On the other hand, the stability results in [35, 36, 39, 37] encompass all sufficiently close perturbations of the background solution. In our setting such a result is not true, as we saw by virtue of the Kerr examples. The fact that we are able to prove a stability result in the axial-symmetry class considered here utilizes many geometric features of this symmetry class, but most essentially a way to re-write the equations as a free wave coupled with 1st order ODEs. (This special free-wave-ODE system which we will introduce and exploit here is not available in settings outside (polarized) axial symmetry).

A brief comparison of some of the methods in [35, 36, 39, 37] and this work is also due: The former works use a CMC foliation of the space-times and utilize that gauge in the study of Einstein’s equations. We use instead a geodesic gauge (not used before in the study of the Einstein’s equations, as far as we are aware); as we will explain below, we derive an approximate CMC property of the geodesic parameter which is central in obtaining our result. (We are unable to explain on prior grounds why our geodesic gauge should display this additional behaviour). On a more analytic level, in the former works [35, 36, 39], the authors relied on an approximate monotonicity, i.e., good signs of error terms in the main estimates

---

5Polarization here is defined at the singularity in a function counting sense and is different from ours. However, from a function counting point of view, the gravitational degrees of freedom are the same.
after the use of a combination of identities. In [37] this approximate monotonicity is not available, and the authors allow a much more singular behaviour of the solutions and higher derivatives, coupled with a weights-descent scheme to derive optimal estimates at the lower ones. This is in fact similar to what we perform here (and indeed descent techniques have been used in other problems in nonlinear waves). But the method presented here is different; the source of the descent scheme here is traced directly back to the AVTD behaviour displayed in some geometric parameters of the space-time.

The fact that our result (the only such singularity formation result in (3+1)-vacua with just one degree of symmetry) holds in (polarized) axial symmetry, but manifestly is false with no symmetry assumptions also relates to predictions by Belinskii-Khalatnikov-Lifshitz (BKL) [3, 4].

A long-standing, if controversial, proposal on the generic behaviour of Big Bang (and, by extension, black hole interior spacelike) singularities was put forward in [3, 4]. The prediction there was that generically (in 1+3 vacuum), the space-time should experience rapid oscillations around different Kasner epochs, as one approaches any fixed point on the singular hypersurface; this proposed generic behaviour is often called ‘mix-master’ type. The “generic” part of the statement is based on a formal analysis that identifies settings where this mixmaster behavior should not be true, but instead AVTD behavior occurs. This class of solutions where the mix-master behavior is “turned off”, due to eliminating gravitational degrees of freedom, for example, includes the polarized models studied here, cf. [25].

Finally, we should note that the mix-master type dynamics have only been rigorously derived in the spatially isotropic (meaning three degrees of symmetry) setting of Bianchi IX space-times by Ringström [32]. Further numerical investigations have appeared in [20, 31]. We refer to [25] for a more detailed discussion of such results. The extent to which the mix-master proposal should be trusted to be generic is a matter of discussion, and we do not take a position here. One must note, however that non-mix-master (but rather, AVTD-type behavior) was predicted in the literature for all the settings for which it has now been proven, in particular [35, 36, 39, 37] and the present paper–see [37] for a discussion of the relevant literature.

We close the discussion of the literature by comparing the “strength” of the singularities occurring here with those that have been studied in the literature:

1.2.3 The strength of the singularity.

We present a brief comparison with singularities in black holes and big bangs in vacuum from the literature.

The “strength” of a singularity can be measured with respect to the behaviour of the space-time curvature, measured against a suitably propagated orthonormal frame. Alternatively, as often used in spherical symmetry one can consider the blow-up of the Kretschmann scalar $R^{abcd}(g) \cdot R_{abcd}(g)$. The blow-up of such components against the frame we introduce below signifies the inextendibility of the $C^0$ sense. In fact we strongly expect that the proof of Sbierski [39] extends to this setting to show the $C^0$ inextendibility also.

The choice of the parameter relative to which the blow-up rate is determined can be made in different ways: If one had a CMC of our space-time foliation that terminates at the singularity, one could measure curvature components or the Kretschmann scalar with respect to that parameter; this is done in [35, 36, 39, 37] and the present paper–see [37] for a discussion of the relevant literature.

We here have an asymptotically CMC foliation, given by a parameter $r$ in Theorem 1.1 as we will see the mean curvature $\text{tr}_gK$ of level sets of $r$ satisfies:

$$\text{tr}_gK = \frac{3}{2} r^{-3/2} + O(r^{-\frac{3}{2} + \frac{1}{8}}).$$

Relative to this parameter the Kretschmann scalar blows up like $r^{-6}$. This is in complete agreement with the asymptotic behaviour of this scalar in the higher-dimensional vacuums in space-times in [37]. (The parameter $r$ referred to there corresponds to $r^{3/2}$ here). In fact each of the curvature components $R_{0101}, R_{0202}, R_{0303}$ for an orthonormal frame $<e_0, e_1, e_2, e_3>$ where $e_0$ is time-like blow up like $r^{-3}$, where $r$ is the parameter that appears in Theorem 1.1.

It is also useful to make an analogy with the parameter usually used in spherical symmetry. There, the natural parameter is the area radius of the spheres of symmetry; in our settings the analogues of these are $\{S_{\tau=r_0}: \ell = \tau, r = r_0\}$. Given the extreme an-isotropy of these spheres, one can instead consider the area element at any point $t, \theta$ on such spheres (with a suitable renormalization to account for the degeneracy of the induced metrics at the poles), as the correct localized analogue of the area radius. It follows from (1.3) that the area element $(\sin \theta)^{-1} \sqrt{g_{\theta\theta}} - (g_{\theta})^2 := \text{rad}(t, \theta)$ coefficient behaves like $(\sqrt{r^{3/2}(t, \theta) + \delta(t, \theta)}$. In particular, the blow-up rate of the curvature components $R_{0101}, R_{0202}, R_{0303}$ will behave like:
\[[(\sin \theta)^{-1} \cdot \sqrt{g_{\theta \theta} g_{\phi \phi} - (g_{\theta \phi})^2}]^{6} = \alpha(t, \theta)\delta(t, \theta)\]

As we will see below the function \(\delta(t, \theta)\) is an explicit function of \(\alpha\): \(\delta(t, \theta) = d_2(\alpha(t, \theta))\), given by the explicit formula \([12]\) below. In particular, since the function \(\alpha(t, \theta)\) is close to 1 everywhere, the rate of blow-up relative to this (localized) area radius is a function, whose rate of blow up depends on the point \((t, \theta)\) on the final singularity; we note that this exponent function \(\text{rad}^{-\zeta(t, \theta)}\) has \(\zeta(t, \theta)\) going both above and below 3, as \(\alpha(t, \theta)\) attains values above and below 1.

We note that this is in contrast to the blow-up behaviour exhibited by the massless scalar fields in spherical symmetry considered by Christodoulou in \([11][12]\), where the rate of blow-up of these components is at least like \(\text{rad}^{-6}\). (See also \([1]\) where upper bounds for the same matter model, again in spherical symmetry are established). Thus the vacuum space-times here allow for a broader range of blow-up rates of the basic curvature components in the same parametrization, compared to the massless scalar field in spherical symmetry.

1.2.4 Outlook: Results beyond two degrees of symmetry?

In the classical \((1 + 3)\)-dimensional space-times, many of the settings in which an understanding of the entire maximal hyperbolic developments of solutions to the Einstein equations (including in black hole interior regions) has been obtained, concern space-times with two degrees of symmetry imposed, such as the spherical symmetry or \(T^2\) and Gowdy symmetry classes. In the first case, this is always in the presence of matter fields, in view of Birkhoff’s theorem.

A wealth of literature on such space-times exists over the past decades. Mathematically, two degrees of symmetry result in a quotient space-time of \(1+1\) dimensions. These are especially well-suited for analysis since the resulting quotient space-time is locally described by two scalar-valued function (the area radius (usually denoted by \(r\)–not to be confused with the function \(r\) here–and the conformal factor \(\Omega\) in the \((1+1)\)-quotient). Moreover, in many matter models one is able to close estimates at the level of first-derivative norms of the matters fields and of \(r, \Omega\). This allows for large-data results in these symmetry classes, capturing the behaviour up to (frequently space-like) singularities, with remarkably ingenious techniques.

In the cosmological setting, we single out the resolution of strong cosmic censorship for unpolarized \(T^3\)-Gowdy space-times, in the seminal work of Ringstrom \([33]\), previously known in the polarized\(^7\) case by Chruściel-Isenberg-Moncrief \([13]\). Towards the expanding direction, global existence is known in the polarized \(T^2\) class \([5]\), even for weakly regular spacetimes \([28, 29]\), including precise late-time asymptotics, see also \([34]\).

In spherical symmetry, we recall the seminal work of Christodoulou \([11][12]\) for the Einstein-massless scalar field model, where he provided a complete classification of all solutions arising from one-ended initial data. In particular, he showed that under open conditions on the initial data, a black hole forms, containing a space-like singularity at \(r = 0\), where the Kretschmann scalar \(|\text{Riem}|^2\) blows up no slower than \(r^{-6}\), \(r\) being the area radius function.\(^6\) Large portions of the singularities there, are expected to be very similar to the Schwarzschild one. Of course, one also has the classical Oppenheimer-Snyder solution and the Vaidya space-times display portions of their final singularities to be isometric to a portion of the Schwarzschild space-time.

Naturally, one would like to be able to obtain results of this nature outside the 2-degree symmetry class. We believe the techniques developed here should be very helpful in studying perturbations of any of the solutions obtained in spherical symmetry in just the axi symmetric setting, where the Einstein equations admit a wave map formulation \([7]\).

We note also that the setting of polarized axi-symmetry has recently attracted attention in other settings. We recall the problem of full non-linear stability of the Schwarzschild exterior in vacuum, in \([20]\), as well as the construction of solutions to the Einstein-Vlasov system out of suitable limits of pure vacuum solutions by Luk-Huneau \([22, 23]\).

It is hoped that the methods developed herein can serve as a powerful tool to study many of the other questions surrounding Einstein’s equations in the presence of just one Killing field.

\(^6\) In particular, the structure equations themselves can be studied directly, without considering further derivatives thereof.

\(^7\) Polarized here has a different meaning than ours.

\(^8\) See recent work \([1]\), where the authors derived upper bounds for the Kretschmann scalar.
1.3 An outline of the ideas.

We briefly outline some of the main ideas in this paper and how we overcome the central challenges that come up. First we recall the reduced Einstein equations.

1.3.1 The reduced Einstein vacuum equations under polarized axi-symmetry

Let \((M^{1+3}, g)\) be a Lorentzian manifold. We work in the class of polarized axial symmetry, that is, in the class of metrics which admit a hypersurface orthogonal, spatial Killing vector field \(\partial_\phi\) with closed \(S^3\) orbits. Under this symmetry assumption, the space-time metric takes the form

\[
g = \sum_{a,b=0,1,2} h_{ab} dx^a dx^b + e^{2\gamma} d\phi^2,
\]

where \(h_{ij}, \gamma\) are independent of \(\phi\). Define the projected 1 + 2 metric on the Lorentzian hypersurfaces orthogonal to \(\partial_\phi\):

\[
h = \sum_{a,b=0,1,2} h_{ab} dx^a dx^b
\]

In this context, the EVE

\[
R_{ab}(g) = 0, \quad a, b = 0, 1, 2, 3,
\]

are equivalent to the system \([10, Appendix VII]\)

\[
\Box g \gamma = 0 \quad (1.11)
\]

\[
R_{ab}(h) = \nabla_{ab} \gamma + \partial_a \gamma \partial_b \gamma, \quad (1.12)
\]

for \(a, b = 0, 1, 2\), where \(\nabla\) is the Levi-Civita connection of the \((1 + 2)\)-metric \(h\). Note that \((1.11)\) can also be written as a non-linear wave equation with respect to \(h\):

\[
\Box h \gamma = -|\nabla \gamma|^2. \quad (1.13)
\]

The following lemma plays a central role in our approach.

**Lemma 1.6.** In the reduction \((1.11), (1.12)\) of the EVE, the Riemann curvature of the 1 + 2 metric \(h\) is locally determined by its Ricci tensor, due to the vanishing of the Weyl tensor in 3 dimensions. In particular, the following identities are valid:

\[
R_{abc} = 0 \quad (1.14)
\]

\[
R_{abcd}(h) = 2(h_{a[c} R_{d]b}(h) - h_{b[c} R_{d]a}(h)) \quad (1.15)
\]

for all indices \(a, b = 0, 1, 2\).

The proof is presented in \([A.1]\).

The reduced system \((1.11), (1.12)\) consists of seven equations, when \((1.12)\) is expressed with respect to an \(h\)-orthonormal frame \(\{e_0, e_1, e_2\}\) with \(h_{00} = -1\). As we will see, the \(\{0b\}\)-components of \((1.12)\) should be thought of as constraint equations, which satisfy separate propagations equations (see \([A.2]\), leaving four genuine evolution equations: 1) The scalar wave equation for \(\gamma\) and 2) the equation \((1.12)\) for \((a; b) = (1; 1), (1; 2), (2; 2)\) which we replace by corresponding formulas for \(R_{00ij}(h)\), \(i, j = 1, 2\), using \((1.15)\):

\[
R_{00ij}(h) = -R_{ij}(h) + h_{ij} R_{00}(h) = -\nabla_{ij} \gamma - e_i \gamma e_j \gamma + \delta_{ij} (\nabla_{00} \gamma + (\nabla_0 \gamma)^2). \quad (1.16)
\]

**Choice of gauge:** We wish to use a geodesic gauge, with time-like geodesics that end at different points on the singularity surface \(\{r = 0\}\). The main advantage of this choice is that the second set of equations \((1.12)\) gives rise to a system of (non-linear) ODEs, whose forcing terms are determined from the polarized field \(\gamma\). This uses in an essential way the fact that \(h\) is a \((1 + 2)\)-dimensional metric, and thus the curvature tensor is locally expressible in terms of the Ricci curvature via \((1.15)\). (The latter being directly expressible in terms of the field \(\gamma\)).

---

Here we adopt the notation \(T_{a[bc]} = \frac{1}{2} (T_{abcd} - T_{acbd})\) for the antisymmetrization of indices between square brackets.
We put down some key equations schematically here for the purposes of this outline of ideas: We will be choosing the vector field $e_0$ to be affine:

$$\nabla e_0 e_0 = 0,$$

and associate to it a parameter $r$ which we choose to agree with the one in Schwarzschild, in the sense that:

$$e_0 = -(\frac{2M}{r} - 1)^\frac{3}{2} \partial_r.$$

We will be considering an $h$-orthonormal frame $e_0, e_1, e_2$ which is propagated along $e_0$ through a modification of parallel transport, which we specify in (1.18).

The metric $h$ (and thus also the metric $g$, in view of formula (1.18)) will then be encoded in the connection coefficients of this frame. Certain key connection coefficients we wish to highlight for this introduction are the components

$$K_{ij}(r, t, \theta) := h(\nabla e_i e_0, e_j), \quad i, j = 1, 2.$$

These satisfy the Ricatti equations—see (1.20)-(1.22) below.

Our frame will be partly “initialized” at the singularity, $r = 0$, and partly on the initial data hypersurface. The two key directions $e_1, e_2$ will be normalized by requiring that:

- $e_2$ should capture the collapsing direction, i.e. $K_{22}(r, t, \theta) < 0$ as $r \to 0^+$, and
- $K_{11}(r, t, \theta)$ should capture the normal to $e_2$ expanding direction, i.e. $K_{11} > 0$ as $r \to 0^+$.
- The mixed component $K_{12}(r, t, \theta)$ should be less singular, capturing thus an asymptotic diagonalization of $K_{ij}$, and in particular will satisfy:

$$K_{12}(r, t, \theta) \cdot K_{11}^{-1}(r, t, \theta) = o(1), \quad K_{12}(r, t, \theta) \cdot K_{22}^{-1}(r, t, \theta) = o(1)$$

as $r \to 0^+$.

- $e_2$ should be “tangent” to the singularity in a suitable sense, and $e_1$ should be tangent to the initial data hypersurface.

One would expect this geodesic gauge to not be a suitable framework for the Einstein equations, due to the apparent loss of derivatives of the metric relative to the curvature in the directions $e_1, e_2$ in such a gauge. In particular, one might expect to not be able to prove energy estimates that close for the reduced Einstein equations. This however turns out not to be the case, as we will explain below.

The upshot of all this is that the EVE system in this symmetry class and in the geometric parameters that we introduce below can be seen as a coupled system of a free wave with a system of transport equations for the connection coefficients, the most important of which are the non-linear Ricatti-type system (1.20), (1.21), (1.22) below. The metric $g$ in the wave equation (1.11) of $\gamma$ is of course coupled to $\gamma$, since $g$ is determined by the Ricatti equation with the forcing term depending on $\nabla \gamma, \nabla^2 \gamma$. So (1.11) can schematically be expressed as:

$$\Box g(\gamma) \gamma = 0.$$

This is the main equation where the quasi-linearity of the Einstein equations is manifested in our setting.

A first difficulty appears here already, in that due to the contracting direction $\partial_t$ of the space-time metric $g$, one expects a generic family of time-like geodesics would develop caustics long before they reach the singularity. So in particular, the solutions to the Ricatti equations would blow up before $\{r = 0\}$, resulting in a gauge breakdown which would impede the study of the true singularity which lies at $\{r = 0\}$.

Secondly, we note that both the Schwarzschild metric itself and the solutions we eventually obtain by perturbing its initial data are highly an-isotropic: There are two contracting directions $\partial_\phi, \partial_\theta$ and an expanding direction $\partial_t$. If one hopes to obtain uniform estimates consistent with the desired conclusion (1.3), one must control the metric strongly enough (at least in lower norms), in order to capture the collapsing directions and separate them from the expanding direction. We review this in the overview of the Ricatti equations.
The wave equation on very singular backgrounds, and the “asymptotically CMC, for free” property of our space-times

Our analysis of the wave equation proceeds via energy estimates, using a weighted version of the affine vector field $e_0 \parallel \theta$, as a multiplier. A first important observation at this point involves the mean curvature of level sets of $r$: It follows easily from the asymptotics in (1.19)-(1.20) that the mean curvature of each level set $\Sigma_r$ will be of the form:

$$H(r, t, \theta) \sim G(t, \theta)r^{-3/2}.$$  

However, we will show that $G(t, \theta)$ is in fact a constant, independently of the value of $\alpha(t, \theta)$. In particular, as we see below, in this geodesic gauge, the mean curvature of $\Sigma_r$ is automatically constant to leading order.

This “asymptotically CMC” feature of the geodesic gauge is absolutely essential in deriving suitable estimates for the free wave $\gamma$ and its derivatives. If this property had not held, then for the linear equation $\Box_\gamma \gamma = 0$, the energy of $\gamma$ would not be as predicted by the first term in (1.3), but would rather blow up exponentially.

In order to take advantage of this feature, whenever we study the wave equation $\Box_\gamma \psi = F$ (for $\psi$ being $\gamma$ and its suitable derivatives), we will always be using energy currents, whose associated multiplier vector fields will always be $r$-dependent re-scalings of $e_0$. The vector fields by which we seek to commute the equation are (for the most part) chosen to commute with $e_0$, so as to take advantage of the asymptotically CMC behaviour of the mean curvature that provides some key cancellations.

The analysis of the wave equation is carried out in a separate section, and further details of the ideas are given in the (brief) introduction of that section. Notably, the use of the AVTD behaviour of the CMC behaviour of the mean curvature that provides some key cancellations.

The Ricatti equations and the singular branch of the solutions.

A very central challenge to the stability result we derive, appears at the level of the (non-linear) Ricatti equations. Requiring that the frame $e_1, e_2$ is transported according to the law

$$\nabla_\gamma e_0 = 0, \quad \nabla_\gamma e_1 = -K_{12}e_2, \quad \nabla_\gamma e_2 = K_{12}e_1, = 0;$$  

the corresponding connection coefficients $K_{ij}$, $i, j = 1, 2$, solve the following system

$$e_0K_{11} + (K_{11})^2 + 3(K_{12})^2 = e_0\gamma K_{11} = \nabla_\gamma \gamma + (e_1\gamma)^2 - e_0^2 - (e_0\gamma)^2$$  

$$e_0K_{22} + (K_{22})^2 + 6(e_0\gamma K_{22} = \nabla_\gamma \gamma + (e_2\gamma)^2 - e_0^2 - (e_0\gamma)^2$$  

$$e_0K_{12} + 2(K_{22} + e_0\gamma)K_{12} = \nabla_\gamma \gamma + e_1\gamma e_2 = 0.$$  

(Here $\nabla$ stands for a connection on the space spanned by of $e_1, e_2$—this connection is defined to be the projection of the Levi-Civita connection $\nabla$ onto $\text{Span}(e_1, e_2)$).

In terms of singular behaviour in $r$, we will see below that given the expected asymptotic behaviour of $\gamma$, as $r \to 0^+$, and the AVTD property that implies that all $e_0$-derivatives are more singular in $r$ than spatial derivatives, these equations admit formal solutions with the following asymptotic expansion:

$$K_{22}(r, t, \theta) \sim d_2(t, \theta)\sqrt{2Mr}^{-\frac{1}{2}}, \quad K_{11}(r, t, \theta) \sim d_1(t, \theta)\sqrt{2Mr}^{-3/2}, \quad |K_{12}(r, t, \theta)| \lesssim r^{1-\frac{1}{2}},$$  

where the functions $d_1(t, \theta) = d_1[\alpha(t, \theta)]$ and $d_2(t, \theta) = d_2[\alpha(t, \theta)]$ are given by the explicit formulas of the parameter $\alpha(t, \theta)$ in (1.4):

$$d_1(t, \theta) := \frac{\alpha(t, \theta) - \frac{1}{2} + \sqrt{\left(\alpha(t, \theta) - \frac{1}{2}\right)^2 + 6\alpha(t, \theta) - 4\alpha(t, \theta)^2}}{2},$$  

$$d_2(t, \theta) := \frac{\alpha(t, \theta) - \frac{1}{2} - \sqrt{\left(\alpha(t, \theta) - \frac{1}{2}\right)^2 + 6\alpha(t, \theta) - 4\alpha(t, \theta)^2}}{2}.$$  

\[10\]It is easy to see that (1.18) defines an orthonormal frame, provided $e_0, e_1, e_2$ are orthonormal initially.

\[11\]The Ricatti system (1.19)-(1.22) is a consequence of (1.16) and (1.18), implying the formulas

$$R_{0101}(h) = -e_0K_{11} - (K_{11})^2 - 3(K_{12})^2, \quad R_{0202}(h) = -e_0K_{22} - (K_{22})^2 + (K_{12})^2,$$  

$$R_{0102}(h) = -e_0K_{12} - 2K_{22}K_{12}.$$  

(1.19)
In fact, these two possible leading-order branches correspond to the (unique) "collapsing" direction, which we will choose to be $e_2$ and the dual (principal) "expanding" direction $e_1$. In the above derivation we have implicitly normalized the expanding direction by requiring it to be (asymptotically) orthogonal to the collapsing direction, to suitably high order, implicitly imposing (1.17).

Given that the connection coefficients $K_{ij}$ appear in the wave equation $\Box g = 0$, and especially in the derivatives of this equation, we need to control the tensor $K_{ij}$ in higher order Sobolev spaces $H^s_{\alpha}(\Sigma_r^m)$. In particular, we need to consider derivatives (with respect to $\partial_r, \partial_\theta$) of $K_{ij}$ and derive bounds for them that are consistent with the asymptotic behaviour (1.3) (at least for a low enough number of derivatives).

It is here that an essential (and unexpected) difficulty in this problem arises: Assume that $\gamma$ and its (low enough) derivatives' display a behaviour in $r$ that is consistent with the asymptotics of $g_{\alpha\beta}(r, t, \theta)$ in (1.3). We then need to derive bounds for $K_{ij}$ and its (low enough) derivatives that would be consistent with the asymptotics for (1.23). In particular (low enough) derivatives of $K_{ij}$ should behave (in Sobolev and the $L^\infty$ spaces) as in (1.3).

Here there is a dichotomy: For the un-differentiated terms $K_{ij}$, indeed assuming that up to two of the derivatives of $\gamma$ satisfy pointwise bounds that are consistent with (1.3), we can derive the asymptotics of $K_{ij}$ consistent with (1.3), via a Fuchsian-type analysis of the nonlinear ODEs.

However, once we consider the differentiated equations (here $\partial$ stands for $\partial_r, \partial_\theta$)

\begin{equation}
\begin{aligned}
\partial_\rho K_{11} + (2K_{11} + \rho_0) \partial_\rho K_{11} + 6K_{22} \partial_\rho K_{11} &= \partial_\rho (\bar{\nabla}^2 \gamma + (e_1 \gamma)^2 - \rho_0 (e_0 \gamma)^2) - \rho_0 (e_0 \partial_\gamma) K_{11} \\
\partial_\rho K_{22} + (2K_{22} + \rho_0) \partial_\rho K_{22} - 2K_{12} \partial_\rho K_{12} &= \partial_\rho (\bar{\nabla}^2 \gamma + (e_2 \gamma)^2 - \rho_0 (e_0 \gamma)^2) - \rho_0 (e_0 \partial_\gamma) K_{22} \\
\partial_\rho K_{12} + (2K_{22} + \rho_0) \partial_\rho K_{12} &= \partial_\rho (\bar{\nabla}^2 \gamma + e_1 e_2 \gamma) - (2\partial K_{22} + \rho_0 \partial_\gamma) K_{12},
\end{aligned}
\end{equation}

we find that the free branches of the solutions of these linear equations are

\begin{equation}
\begin{aligned}
\partial K_{11}^{\text{free}} &= c_{11}(t, \theta) r^{2d_1(t, \theta) - \alpha(t, \theta)}, \\
\partial K_{22}^{\text{free}} &= c_{22}(t, \theta) r^{2d_2(t, \theta) - \alpha(t, \theta)},
\end{aligned}
\end{equation}

Recalling that $\alpha(t, \theta)$ is close to 1 in $L^\infty$ and thus $d_2(t, \theta)$ is close to $-1$ and $d_1(t, \theta)$ close to $1$, we note that the first of these free branches (for $K_{11}$) is less singular than the leading order behaviour $r^{-3/2}$, and thus does not impede the proof that $K_{11}$ and its high derivatives satisfy bounds consistent with (1.3).

The second branch of the free solution of $K_{22}$ is potentially detrimental: If the $\partial_r, \partial_\theta$ derivatives of $K_{22}$ do indeed behave in this much more singular way $c(t, \theta) r^{-3 + \epsilon(t, \theta)}$, $|\epsilon(t, \theta)| \leq \frac{1}{2}$, then one has absolutely no hope of eventually proving asymptotics of the form (1.3), and hence, our result. (We remark that this would even kill the hope of deriving estimates for the linearized Einstein equations which would be consistent with the asymptotics that we prove here)–thus this feature of the equation can be termed a linear instability of the EVE around the Schwarzschild singularity, at least in this gauge.

The only hope therefore is that this very singular branch, allowed by the differentiated equations, is somehow not there. This hope is actually validated. It is to prove this part that the ODE character of the second branch of our equations (in this gauge) is used in an essential way:

Whereas if one were to solve the equations (1.21), (1.22) forwards, towards the singularity, one cannot rule out the possibility of this very singular behaviour, one can set this singular branch to zero when one solves the Riccatti equations backwards from the singularity. Skipping some technical issues, that is possible to do if one considered the Riccatti equations (1.20), (1.22), (1.21) separately, taking the RHS as being given, and consistent with the behaviour (1.3).

The iteration scheme: Taking this challenge into account, we resort to an iteration scheme for solving the system (1.11), (1.12), producing a sequence of solutions $(\gamma^m, h^m)$. Taking the previous step $(\gamma^{m-1}, h^{m-1})$ as given, we need to produce a new pair $(\gamma^m, h^m)$.

We first solve the free wave equation $\Box g^m = 0$ (1.29) forwards, towards the singularity; we next solve for the metric $h^m$ via its connection coefficients $K_{ij}^m$. We solve the two of the connection coefficients $K_{22}^m(r, t, \theta), K_{12}^m(r, t, \theta)$ backwards from the singularity, setting the singular branches of the solutions for $\partial K_{22}^m, \partial K_{12}^m$ to zero. This is completed by showing that it is possible to solve for the remaining connection coefficients so that the metric and second fundamental form induced on a suitable hypersurface $\Sigma_{r^m} = \{r = r^m(t, \theta)\}$ (which is to be determined) matches the initial data that we have prescribed.

In other words, we prove our result not by a bootstrap argument, but by a (forwards-backwards) Picard-type iteration. (We note that in recent works by Hintz-Vasy, for example in their breakthrough
proof of the Kerr-de Sitter stability problem [21], the authors solved the Einstein equations via an iteration—however the underlying reasons there are entirely different). Certain technical difficulties that this gives rise to will be discussed in the main body of the paper.

For now, we wish to discuss the final main challenge that we need to overcome, to eventually prove the result, and establish the asymptotics [13].

**Closing the EVE in a geodesic gauge**

Working in a geodesic gauge presents certain challenges in terms of deriving suitable bounds for all the quantities that govern our space-time. In particular, there is a clear danger of losing derivatives, which would not allow the derivation of our estimates in the next step of our iteration. This is in fact true even for the local-in-time problem, independently of singularity formation. To distinguish these two challenges (singular behaviour in r and regularity in fixed Sobolev spaces), we introduce the following convention:

**Language Convention:** Given any parameter \( f(r, t, \theta) \) in our problem, we will use the term regularity to refer to suitably many derivatives of \( f \) lying in \( L^2_{t, \theta} \). The term singularity will refer to the behaviour in \( r \) of different norms of \( f(r, t, \theta) \) that we keep track of (e.g. \( L^\infty_{t, \theta} \) norms, Sobolev norms \( H^k \), as \( r \to 0^+ \)).

The well-known loss of derivatives that occurs in a geodesic gauge, e.g. in Fermi or exponential coordinates, does not make the geodesic gauge suitable, in general, for a study of this initial value problem. Nonetheless, the special structure of the equations makes this possible in our case:

In our approach to this problem, the wave \( \gamma \) is treated as the main part of the evolution. The equation (1.11) is however non-linear in \( \gamma \) since the metric \( g \) is related to \( \gamma \) via the Ricci curvature of \( h \) (1.12).

The relation between the curvature of the projected \((2+1)\)-metric \( h \) and the free wave \( \gamma \) is utilized via the Ricatti equations, and also becomes manifest whenever we commute the equation with derivatives. It is important at this point that we always use the vector field \( e_0 \) as a multiplier. Also, at the top order of derivatives, \( e_0 \) is necessarily one of the commutation vector fields for our equation.

From the point of view of regularity, it is clear that the direction \( e_0 \) is privileged: The Ricatti equations show that \( e_0 K_{ij} \) is on the same level as \( \nabla^2 \gamma \), while the derivatives \( \partial_t K_{ij}, \partial_\theta K_{ij} \) are on the same level as \( \partial_t \nabla^2 \gamma, \partial_\theta \nabla^2 \gamma \).

A delicate balance is struck here: From the point of view of not losing derivatives, use of the \( e_0 \) vector field is good, because it brings out (differentiated) metric and connection terms that are at the correct number of derivatives in terms of the wave \( \gamma \). However, from the point of view of deriving asymptotics up to the singularity it is dangerous, since (in view of the asymptotics [13], that we seek to establish) it generates terms that are more singular in terms of powers of \( r \). How this balance is achieved is explained in more detail in the main body of the proof. The closing occurs in function spaces which at the very top orders use the vector field \( e_0 \) as a commutator up to two times.

**The location of the initial data, and regularity at the poles**

Two more novel aspects of our technique that we wish to highlight here relate to the issue of identifying the position of the initial data hypersurface in the geodesic gauge we have chosen, and also some (technical) issues related to the regularity of our space-times at the axes.

In contrast to [25] [36] [37], or the spherical symmetry setting we are not in a position to choose a space-like foliation which “synchronizes” our approach to the singularity (via for example CMC surfaces). Nor do we have the option of using an area radius parameter for 2-spheres as is often done in spherical symmetry. Rather, the approach to the singularity in our gauge is governed by a (non-affine) parameter \( r \) along our time like \( e_0 \)-geodesics. This in particular implies that the location of the hypersurface (expressed in terms of the coordinate \( r \)) that is to carry the initial data must be solved for. This reduces to a 2x2 system, which relies on connection coefficients that are solved for starting from the singularity. The solvability of the resulting system is far from evident (at least to the authors); in fact it is to obtain such a soluble system that the requirement of tangency of \( e_1 \) (but not of \( e_2 \)) to the initial data hypersurface was imposed.

A further challenge is related to the fact that we split our analysis between the \((3+1)\)-dimensional wave equation \( \Box_{h^{2+1}} \gamma = 0 \) and the quotient metric \( h^{2+1} \). Indeed, the metric \( h^{2+1} \) lives over a manifold-with-boundary (over the coordinates \( \theta, t, r \)), with the boundary being at \( \theta = 0, \theta = \pi \). For various parameters in our inductive procedure we must impose or derive a certain vanishing of transverse derivatives to those two boundaries. These vanishing conditions capture the regularity of the resulting \((3+1)\)-dimensional space-time.
More technical aspects of our analysis will be discussed in separate introductions of the separate sections.

Acknowledgements. We would like to thank Jonathan Luk, Igor Rodnianski, Jared Speck for useful discussions. S.A. was supported by an NSERC discovery grant and an Ontario ER Award. G.F. was supported by the ERC grant 714408 GEOWAKI, under the European Union’s Horizon 2020 research and innovation program.

2 The precise formulation of the result.

We will be introducing the precise gauge in which the theorem is proven. It is useful to consider the Schwarzschild metric and a canonical frame associated to that metric.

2.1 The Schwarzschild metric

The Schwarzschild solution \( \text{(1.1)} \), being spherically symmetric, belongs in the axi-symmetric polarized class and satisfies the EVE \( \text{(1.11), (1.12)} \) for
\[
\gamma_\Sigma = \log r + \log \sin \theta, \quad h_\Sigma = -\left(\frac{2M}{r} - 1\right)^{-1}dr^2 + \left(\frac{2M}{r} - 1\right)dt^2 + r^2d\theta^2. \tag{2.1}
\]

The interior region \( \{r < 2M\} \) is naturally foliated by space-like hypersurfaces \( \Sigma_r, r \in (0, 2M) \), the level sets of the coordinate (and area radius) function \( r \). The limiting slice \( \Sigma_0 \) is the hypersurface \( \{r = 0\} \), where the singularity occurs and where the curvature invariants, such as the Kretschmann scalar, blow up. Also, across \( \Sigma_0 \), the space-time metric is \( C^0 \)-inextendible \[38\].

Consider the orthonormal frame
\[
e_0 = -\left(\frac{2M}{r} - 1\right)^{\frac{1}{2}}\partial_r, \quad e^1 = \left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}}\partial_t, \quad e^2 = \frac{1}{r}\partial_\theta, \quad e^3 = \frac{1}{r\sin \theta}\partial_\phi. \tag{2.2}
\]

In this frame, the second fundamental form \( K_\Sigma \) of the constant \( r \) hypersurfaces \( \Sigma_r \) is given by
\[
(K_\Sigma)_{11} = \frac{M}{r^2}\left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}}, \quad (K_\Sigma)_{22} = (K_\Sigma)_{33} = -\frac{1}{r}\left(\frac{2M}{r} - 1\right)^{\frac{1}{2}}. \tag{2.3}
\]

A direct computation also shows that
\[
-R_{0101}(g_\Sigma) = 2R_{0202}(g_\Sigma) = 2R_{0202}(g_\Sigma) = \frac{2M}{r^3}. \tag{2.4}
\]

2.2 The initial data for our problem.

The space-times we will study in this paper will arise as the future maximal hyperbolic developments of initial data sets that correspond to perturbations of the initial data set \((g_\Sigma, K_\Sigma)\) on \( \Sigma_0 \); the latter corresponds to the metric and second fundamental form induced on the hypersurface \( \Sigma_\epsilon := \{r = \epsilon\} \) in the Schwarzschild space-time.

The closeness of our data to the Schwarzschild background will be encoded in a parameter \( \eta > 0 \), whose smallness will also be specified below. We frequently denote by \( g, K \) the initial data for brevity; also, all quantities in bold-faced letters will be related to the abstract initial data.

The perturbation will also be taken suitably small in appropriate Sobolev spaces:

Specifically, consider a spatial metric \( g \) and a second fundamental form \( K \) (expressed in coordinates \( t, \theta, \phi \)) which satisfy the vacuum constraint equations and the following polarized-axisymmetric condition, for any fixed component \( g_{ab}, K_{ab} \), expressed with respect to the coordinate vector fields \( \partial_t, \partial_\theta, \partial_\phi \):
\[
\partial_\phi g_{ab} = \partial_\phi K_{ab} = 0, \quad a, b = t, \theta, \phi, \quad K_{\theta a} = g_{\theta a} = 0, \quad a = t, \theta.
\]

For definiteness, we will be normalizing the coordinates \( t, \theta \) by requiring that
\[
K_{t \theta}(t, \theta) = 0, \quad g_{t \theta} = 0.
\]

This requirement only specifies the level sets of the coordinates \( t, \theta \). We impose an extra gauge normalization to ensure that the poles occur at \( \theta = 0, \theta = \pi \), and that for each fixed \( t = t_0 \) the set \( \{\theta \in (0, \pi), \phi \in [0, 2\pi]\} \) should extend to a smooth sphere at the poles \( \theta = 0, \theta = \pi \).
We note that this coordinate normalization implies that the vector field \( \partial_\theta \) must be normal to \( \partial_t \). At the poles \( \theta = 0, \pi \) it also implies that \( \partial_\theta \) must be mapped to \(-\partial_\theta\) after flowing by \( \pi \) along \( \partial_\theta \), and moreover the flow of \( \partial_\theta \) at any of the two poles defines a 2-dimensional space. Then \( \partial_\theta \) must be invariant under the flow of \( \partial_\theta \) at the two poles \( \theta = 0, \pi \), since it must be is the unique vector field (up to choice of direction) that is normal to the 2-dimensional space that is left invariant under the flow of \( \partial_\theta \).

Within this gauge normalization, we require that this initial data be close to the corresponding Schwarzschild data in a suitably high Sobolev norm \( H^s \).

Our assumptions on the initial data \((g,K)\) will be formulated in terms of Sobolev spaces defined relative to the coordinates \( t, \theta, \phi \). Also, the assumption on the component \( g_{\theta\theta} = e^{2\gamma} \) will be separate from that of the normal-to-\( \partial_\theta \) part of the metric \( g \): For the former we will impose initial data on \( \gamma = \frac{1}{2} \log(g_{\theta\theta}) \), while initial data on the latter will be treated in terms of the relevant components of \( g,K \) in the coordinates \( t, \theta \). We write \( \gamma_{\text{init}} \) for the initial datum of this parameter, for notational simplicity.

**Remark 2.1.** We recall that for the abstract initial data, we also consider the (abstract) normal vector field \( n \), which is normal to our initial data set. This vector field is used in defining the initial energy of waves on the initial data set.

In view of this, it makes sense to consider the formal jet of the solution metric (given the prescribed \((g,K)\)) off of \( \Sigma \)—this makes sense in a coordinate \( t, \theta, \phi \), and consider the components of \( K \) and also for the function \( \gamma \). We present a more descriptive version of our result in coordinates here:

The energy of a function is defined using formula (2.5)—note that we use the volume form \( \sin \theta d\theta dt \) makes sense to consider the energy \( \int_0^\pi \int_{-\infty}^{\infty} |\partial_{t,\theta}^{k_1,k_2} \left[ (g_{\theta\theta})^{2}\right] \theta d\theta dt \leq \eta^2 (\log(\epsilon))^2 \).

For the remaining two non-zero components of the metric \( g \) on the initial data set we assume that the components \( g_{tt}, g_{\theta\theta} \) satisfy the bounds, for all \( k_1 + k_2 \leq s \):

\[
\int_{-\infty}^{\infty} \int_0^\pi \left| \partial_{t,\theta}^{k_1,k_2} g_{\theta\theta} \left( \log \left( \frac{g_{tt}}{g_{\theta\theta}} \right) \right) \right|^2 \sin \theta d\theta dt \leq \eta^2 (\log(\epsilon))^2, \tag{2.7}
\]

Next, we define

\[
E_1 = g^{-1/2}_{tt} \partial_t, \quad E_2 = g^{-1/2}_{\theta\theta} \partial_\theta \tag{2.8}
\]

and consider the components of \( K \) with respect to this frame, \( K_{22}, K_{11} \). We require then for all \( k_1 + k_2 \leq s - 2 \):

\[
\int_{-\infty}^{\infty} \int_0^\pi \left| \partial_{t,\theta}^{k_1,k_2} \left( K_{ab} - (K_{\theta\theta})_{ab} \right) \right|^2 \sin \theta d\theta dt \leq \eta^2 \epsilon^{-3}, \quad (a;b) = (1;1), (2;2). \tag{2.9}
\]

The four conditions above capture the \( \eta \)-closeness of our initial data to \((g_{\Sigma}, K_{\Sigma})\) on \( \{r = \epsilon\} \).

### 2.3 The result, properly formulated.

The space-times \((M, g^{3+1})\) that we construct will be considered both in terms of coordinates (and the metric components expressed in terms of these coordinates), but also in terms of connection coefficients of certain special frames.

We present a more descriptive version of our result in coordinates here:

Consider the coordinates \( \phi \in [0, 2\pi], t \in (\infty, \infty), \theta \in (0, \pi) \) constructed on our initial data \((\Sigma, g, K)\) above. Our maximal future hyperbolic development will involve a fourth (time) coordinate \( r \).

---

Footnote: These assumptions can in fact be weakened. The requirements imposed here should only hold for what we will later call the lower derivatives of the parameters; the derivatives beyond this are allowed to be more singular (in terms of powers of \( \epsilon \)). This follows from the proof further down straightforwardly, but we do not make this weakening of the assumptions here for sake of brevity.
The future maximal hyperbolic development will live over a domain:
$$\{ \phi \in [0, 2\pi), t \in (-\infty, \infty), \theta \in (0, \pi), r \in (0, r_\star(t, \theta)) \},$$
where the function $r_\star(t, \theta)$ is one of the parameters that will be solved for in the problem. The abstract initial data $(g, K)$ are induced by $g$ on the hypersurface $\Sigma_{r_\star(t, \theta)} := \{ r = r_\star(t, \theta) \}$. In particular, the restriction of the metric $g$ to $\Sigma_{r_\star}$, expressed in these same coordinates $t, \theta, \phi$, will match exactly the prescribed initial metric $g$. In other words, the metric $g|_{\Sigma_{r_\star}}$ expressed in these coordinates is assumed to be equal (not just up to a coordinate transformation) to our prescribed $g$.

**Theorem 2.2.** Consider an abstract initial data set $(\Sigma, g, K)$ which is an $\eta$-perturbation of the Schwarzschild space-time data on $\Sigma^*_\epsilon = \{ r = \epsilon \}$, as defined in (2.9), (2.7), (2.5). Assume $s \in \mathbb{N}$ is large enough and that $\epsilon, \eta > 0$ are small enough. (How large and small these parameters is derived below).

Then the maximal future hyperbolic development $(\mathcal{M}^{1+3}, g)$ of this initial data set can be described as follows: There exists a fourth coordinate $\tau$ so that $g$ lives over
$$\phi \in [0, 2\pi), t \in (-\infty, \infty), \theta \in (0, \pi), r \in (0, r_\star(t, \theta)),$$
and it acquires the form:
\begin{equation}
\begin{aligned}
g &= -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + g_{\phi\phi}(r, t, \theta) d\phi^2 + g_{\theta\theta}(r, t, \theta) d\theta^2 + g_{t\tau}(r, t, \theta) dt^2 \\
&\quad + g_{\tau r}(r, t, \theta) d\tau dr + g_{\tau \theta}(r, t, \theta) d\tau d\theta + g_{\tau \phi}(r, t, \theta) d\tau d\phi.
\end{aligned}
\end{equation}
Here the integral curves of $\partial_\tau$ are time-like geodesics. Also the abstract initial data $(\Sigma, g, K)$ are induced by $g$ onto $\Sigma_{r_\star} := \{ r = r_\star(t, \theta) \}$.

The metric $g$ exists as an $H^s$-smooth Lorentzian metric until $\{ r = 0 \}$. The asymptotic expansion of the components in the $C^k, k \leq \text{low} - 2$ (for some low $< s$ to be specified below) norms are as follows: There will exist a change of coordinates $(\tilde{t}, \tilde{\theta}) = F(t, \theta)$ so that with respect to the coordinate system
$$\tilde{t}, \tilde{\theta}, \phi, r$$
the components of the metric $g$ have the expansion:
\begin{equation}
\begin{aligned}
g_{\phi\phi}(r, \tilde{t}, \tilde{\theta}) &= A(\tilde{t}, \tilde{\theta}) r^{2\alpha(\tilde{t}, \tilde{\theta})} (1 + O(r^{\frac{1}{2}})) \sin^2 \tilde{\theta}, \\
g_{\theta\theta}(r, \tilde{t}, \tilde{\theta}) &= B(\tilde{t}, \tilde{\theta}) r^{2\beta(\tilde{t}, \tilde{\theta})} (1 + O(r^{\frac{1}{4}})), \\
g_{tt} &= C(\tilde{t}, \tilde{\theta}) r^{2\delta(\tilde{t}, \tilde{\theta})} (2M + O(r^{\frac{1}{4}})), \\
g_{\tau r} &= O(r^{1/4}), \\
g_{\tau \theta} &\equiv 0, \quad r \in (0, \frac{\epsilon}{2}).
\end{aligned}
\end{equation}
Here the exponent functions $\alpha(\tilde{t}, \tilde{\theta}), \beta(\tilde{t}, \tilde{\theta}), \delta(\tilde{t}, \tilde{\theta})$ depend on the point $\tilde{t}, \tilde{\theta}$; they are all close to their values for the Schwarzschild metric, in particular:
\begin{equation}
|\alpha(\tilde{t}, \tilde{\theta}) - 1|, |\beta(\tilde{t}, \tilde{\theta}) - 1|, |\delta(\tilde{t}, \tilde{\theta}) + \frac{1}{2}| \leq \frac{1}{8}.
\end{equation}

The coefficients $A(\tilde{t}, \tilde{\theta}), B(\tilde{t}, \tilde{\theta}), C(\tilde{t}, \tilde{\theta})$ are also pointwise close to their values for the Schwarzschild metric:
\begin{equation}
|A(\tilde{t}, \tilde{\theta}) - 1|, |B(\tilde{t}, \tilde{\theta}) - 1|, |C(\tilde{t}, \tilde{\theta}) - 1| \leq \frac{1}{8}.
\end{equation}

Moreover all these functions $\alpha, \beta, \delta, A, B, C$ are $C^{\text{low} - 2}$-functions, and satisfy similar bounds in the norms $C^k, k \leq \text{low} - 2$.

At the higher norms the behaviour of the metric components is more singular; however we do not write those bounds out here.

**Remark 2.3.** The claims we made above are *optimal* for the leading orders of the first three terms in (2.11). For the other terms they are in fact not optimal, yet they are sufficient for our purposes. Essentially, the principal directions of the metric $g$ are captured by the directions $\partial_r, \partial_\tau, \partial_\phi, \partial_\theta$, and the components of the off-diagonal terms of $g$, with respect to these coordinate fields, are all strictly less singular in terms of the behaviour in $r$.

To prove our result, we find it more convenient to introduce frames in conjunction with coordinates. We will make our claim in terms of connection coefficients corresponding to a gauge-normalized frame, see Theorem 2.9 below. We will then derive Theorem 2.2 from Theorem 2.9 as a consequence of estimates derived in in Section 4, see [3].

We introduce our frame and its gauge normalization (and the corresponding equations that stem from the EVE) in the next subsection.
2.4 The geodesic gauge: Reduction of the EVE to free wave-ODE system

2.4.1 The orthonormal frames and their propagation.

Given a \(g\)-orthonormal frame \(\{e_0, e_1, e_2\}\), on a 3-dim space-like hypersurface \(\Sigma\) with \(e_0\) transversal to \(\Sigma\), we may extend the frame off of \(\Sigma\) via the propagation rule \[1.18\]. Along a fixed \(e_0\) geodesic, this uniquely determines the orthonormal frame, once the frame has been prescribed at one point on the geodesic.

We chose \(e_3 = e^{-\gamma} \partial_\phi\), in which case the equation \(D_{e_0} e_3 = 0\) is automatic, where \(D\) is the Levi-Civita connection of \(g\). We may thus restrict the frame \(\{e_0, e_1, e_2\}\) in the 1 + 2 projected manifold \((\mathcal{M}^{1+3}/S^1, h)\).

**Coordinate Normalization:** We will be expressing the metric \(h\) in a system of coordinates \(t, \theta, r\). The coordinates \(t, \theta\) exist on the initial data set and give rise to a coordinate system on \((\mathcal{M}^{1+3}/S^1, h)\) as follows:

- The coordinates \(t, \theta\) are required to satisfy
  \[ e_0(t) = e_0(\theta) = 0. \] \[ (2.13) \]

- The coordinate function \(r\) satisfies
  \[ e_0 = -\left(\frac{2M}{r} - 1\right)^{\frac{1}{2}} \partial_r, \]
  and is normalized so that \(r = 0\) on the singularity.

Some key non-trivial connection coefficients of \(h\) are defined via:

\[ K_{ij} := h(\nabla e_i e_j) = K_{ji}. \] \[ (2.15) \]

These connection coefficients must satisfy the system \[1.20\], \[1.22\], \[1.21\]; the \(\gamma\) on the RHS solves the wave equation \[1.11\].

Now, the rest of the \((2 + 1)\)-metric \(h\) will be captured in coordinates:

2.5 The space-time metric, expressed in terms of the orthonormal frame.

To complete our set of unknowns, we will be fixing a system of coordinates \((\rho, t, \theta)\), where \(\rho\) is a re-parametrization of \(r\), defined by the equation:

\[ \rho = \rho(r, t, \theta) = r + \chi(r)(r_s - \epsilon), \quad \chi \in C^\infty([0, 2\epsilon]), \quad \chi|_{[0, \frac{\epsilon}{2}] \cup [2\epsilon, 4\epsilon]} \equiv 0, \quad \chi|_{[\frac{\epsilon}{2}, 2\epsilon]} \equiv 1, \] \[ (2.16) \]

The three functions \(\rho, t, \theta\) define a system of coordinates; for this section \(\partial_\rho, \partial_t, \partial_\theta\) will be the coordinate vector fields for this system of coordinates.

We will seek to express the space-time metric (with respect to this system of coordinates) in terms of the frame \(\{e_0, e_1, e_2, e_3\}\) constructed above. First, let us introduce a modification of \(e_1, e_2\) into vector fields \(\mathcal{T}_1, \mathcal{T}_2\) which are tangent to the level sets of \(\rho\):

\[ \mathcal{T}_i := e_i - e_i(\rho) \partial_\rho \]
\[ = e_i + e_i(\rho)\left(\frac{2M}{r} - 1\right)^{\frac{1}{2}} (1 + \partial_\rho \chi(r)(r_s - \epsilon))^{-1} e_0 \in T\Sigma_\rho, \] \[ (2.17) \]

for \(i = 1, 2\).

Now, the scalars that will connect our space-time metric \(h\) in terms of \(e_0, e_1, e_2\) are the scalars that define the coordinate vector fields \(\partial_\rho, \partial_t, \partial_\theta\) as lines combinations of \(\mathcal{T}_1, \mathcal{T}_2\).

In particular we define the functions \(a_{Ai}(\rho, t, \theta), A = t, \theta, i = 1, 2\) via the equations:

\[ \partial_t = a_{t1} \mathcal{T}_1 + a_{t2} \mathcal{T}_2, \quad \partial_\theta = a_{\theta1} \mathcal{T}_1 + a_{\theta2} \mathcal{T}_2. \] \[ (2.18) \]
We also remark how the scalar-valued functions $a_{A_i}$ along with the scalar functions $e_i(\rho)$ determine the metric $h$ (the $(2 + 1)$-part of $g$):

\[
\begin{align*}
 h_{\theta\theta} & = \sum_{i=1,2}[a_1]_i^2 \cdot \left[ 1 - [e_i(\rho)(\frac{2M}{r} - 1)]^{-1}[1 + \partial_r \chi(r)(r_* - \epsilon)]^{-1} \right]^2 \\
 h_{tt} & = \sum_{i=1,2}[a_1]_i^2 \cdot \left[ 1 - [e_i(\rho)(\frac{2M}{r} - 1)]^{-1}[1 + \partial_r \chi(r)(r_* - \epsilon)]^{-1} \right]^2 \\
 h_{t\theta} & = \sum_{i=1,2}[a_1]_i \cdot \left[ 1 - [e_i(\rho)(\frac{2M}{r} - 1)]^{-1}[1 + \partial_r \chi(r)(r_* - \epsilon)]^{-1} \right] \\
 h_{\rho t} & = \sum_{i=1,2}(\frac{2M}{r} - 1)^{-\frac{1}{2}} a_1 \cdot [1 + \partial_r \chi(r)(r_* - \epsilon)]^{-1}, \\
 h_{rt} & = \sum_{i=1,2}(\frac{2M}{r} - 1)^{-\frac{1}{2}} a_1 \cdot [1 + \partial_r \chi(r)(r_* - \epsilon)]^{-1}.
\end{align*}
\] (2.19)

We will also note an evolution equation on the functions $a_{A_i}(\rho,t,\theta)$, which is derived below, in the iterative step, just above (4.89).

\[
\begin{align*}
e_0 a_{t1} - K_{11} a_{t1} & = 0, & e_0 a_{t2} - K_{22} a_{t2} & = 2K_{12} a_{t1} \\
e_0 a_{\theta1} - K_{11} a_{\theta1} & = 0, & e_0 a_{\theta2} - K_{22} a_{\theta2} & = 2K_{12} a_{\theta1}.
\end{align*}
\] (2.20)

2.5.1 Gauge fixing for the frame $e_0, e_1, e_2$, relative to the singularity and the initial data hypersurface:

First, we specify a parameter $r$ along each integral curve of $e_0$ via the condition (2.14). This parameter commences at $r = 0$, for each of the integral curves of $e_0$. The family of integral curves (geodesics) itself is parametrized by two functions $t, \theta$ with $t \in (-\infty, +\infty)$ and $\theta \in (0, \pi)$. We denote these geodesics by $l_t, \theta$; considering them as parametrized curves with parameter $r$ we denote them by $l_{t, \theta}(r)$. We will be requiring that any two different geodesic segments $l_{t, \theta}(r), r \in (0, 2c)$, for some $c > 0$ small enough that will be fixed later, do not intersect. (This will turn out to hold for all the space-times we construct, and thus, $r, t, \theta$ and $\phi$ define a system of coordinates for our space-time).

As we will remark below, imposing this condition is in fact a partial gauge normalization of the affine vector fields $e_0$, in that they do not form focal points before the singularity; equivalently, there is no break-down of the geodesic gauge prior to the singularity. Concretely, we require that all connection coefficients $K_{1i}(r, t, \theta)$, $K_{22}(r, t, \theta)$ are smooth up to $r = 0$, and that $e_1, e_2$ diagonalize $K_{ij}(r, t, \theta)$ asymptotically as $r \to 0$; in particular:

\[
K_{12}(r, t, \theta) \cdot K_{11}^{-1}(r, t, \theta) \to 0, \quad K_{12}(r, t, \theta) \cdot K_{22}^{-1}(r, t, \theta) \to 0
\] (2.21)

as $r \to 0$. The choice of $e_1, e_2$ is finally fixed by requiring that:

\[
K_{11} > 0, \quad K_{22} < 0.
\] (2.22)

Beyond these normalizations of the connection coefficients $K_{ij}$, we impose certain normalizations to the frame elements $e_2, e_1$ themselves:

We impose that $e_2$ is asymptotically tangent to the singularity, as $r \to 0$, along any of the curves $l_{t, \theta}(r)$. In particular, we require (recall the definition (1.24) of $d_2(t, \theta)$):

\[
e_2(r) = o(r^{-\frac{1}{2} + d_2(t, \theta)}).
\] (2.23)

In conjunction with the Ricatti equations and (1.18), we will see that this implies that $e_2(r) = 0$ on the entire space-time that we construct. (This is equivalent to requiring that $e_2$ is tangent to all level sets of $r$).
Moreover we require that the vector field $e_1$ be tangent to the entire hypersurface $\Sigma_{r_\ast}$, on which the initial data will be induced (how this hypersurface is found is made precise in the next subsection). We note that these normalizations, along with the requirement that $e_0 \perp \text{Span}(e_1, e_2)$, uniquely specifies the locations of the geodesics $l_t, \theta$ in the solved-for space-time.

We will see that (2.21) implicitly imposes initial conditions on $K_{12}, K_{22}$ on the singularity; as explained, these conditions should be thought of purely a gauge-fixing requirements. Now, the rest of the parameters need to be prescribed on some hypersurface $\Sigma_{r_\ast} := \{r = r_\ast(t, \theta)\}$, for some unique function $r_\ast(t, \theta)$.

We discuss this next:

2.6 The matching of the prescribed initial data.

The prescribed initial data $(\Sigma, g, K)$ in our problem must be induced on a hypersurface

$$\Sigma_{r_\ast} := \{r = r_\ast(t, \theta)\},$$

for some unique function $r_\ast(t, \theta)$.

The induced connection on $\Sigma_{r_\ast}$.

To make this requirement precise, we firstly identify a canonical rotation of the frame $e_0, e_2$ to yield a new orthonormal frame $\tilde{e}_0, \tilde{e}_2$ that will be adapted to the hypersurface $\Sigma_{r_\ast}$, on which the initial data are to live. (Recall that $e_1$ is required to be tangent to $\Sigma_{r_\ast}$).

Figure 3: The frame $e_1, e_2$ normalized so that $e_2$ is “tangent to the singularity” and $e_1$ is tangent to the initial data hypersurface.

Definition 2.4. An orthonormal frame $e_0^\sharp, e_1^\sharp, e_2^\sharp$, defined over a hypersurface $\Sigma_{r_\ast}$, is called adapted to the hypersurface iff $e_1^\sharp, e_2^\sharp$ are both tangent to the hypersurface (and thus $e_0^\sharp$ is normal to the hypersurface).
We consider the (unique, within small rotation angles) rotation of the frame \((e_0, e_2)\) to a new frame \((\tilde{e}_0, \tilde{e}_2)\) that makes \(\tilde{e}_0\) normal to \(\Sigma_r\) and \(\tilde{e}_2\) tangent to \(\Sigma_r\) (and we preserve \(e_1\)). The new frame \(\tilde{e}_0, \tilde{e}_2\) on \(\Sigma_r\) is given by the following formulas:

\[
\tilde{e}_2 := q e_2 - \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} (\tilde{e}_2 r_*) e_0, \quad \tilde{e}_0 = q e_0 - \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} (\tilde{e}_2 r_*) e_2,
\]

(2.25)

where

\[
q = \sqrt{1 + \left(\frac{2M}{r_*} - 1\right)^{-1}(\tilde{e}_2 r_*)^2}.
\]

(2.26)

Inverting (2.25), we obtain:

\[
e_2 = q \tilde{e}_2 + \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} \tilde{e}_2 r_* e_0, \quad e_0 = q \tilde{e}_0 + \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} (\tilde{e}_2 r_*) \tilde{e}_2.
\]

(2.27)

The connection coefficients \(\tilde{K}_{ij} := g(D_{\tilde{e}_j} \tilde{e}_i, \tilde{e}_k)\), \(\tilde{A}_{ij,l} := g(D_{\tilde{e}_j} \tilde{e}_i, \tilde{e}_l)\), for this new frame \(\tilde{e}_0, e_1, \tilde{e}_2\) on \(\Sigma_r\), are related to the connection coefficients \(K_{ij}, A_{ij,l}\) on \(\Sigma\) as follows:

\[
q^2 K_{22} = q \tilde{K}_{22} + \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} \tilde{e}_2 r_* \tilde{e}_2 r_* + \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} \frac{2M}{r_*} (\tilde{e}_2 r_*)^2
\]

(2.28)

\[
K_{11} = q \tilde{K}_{11} + \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} \tilde{e}_2 r_* \tilde{A}_{11,2} - \frac{2M}{r_*} (\tilde{e}_2 r_*)^2
\]

(2.29)

\[
K_{21} = \tilde{K}_{21} + q^{-1} \left(\frac{2M}{r_*} - 1\right)^{-\frac{1}{2}} (\tilde{e}_2 r_*) \tilde{A}_{22,1}
\]

(2.30)

Next, the requirement that \(\tilde{K}, \tilde{A}\) should “match” the prescribed initial data \((\Sigma, g, K)\) on \(\Sigma_r\) needs to be imposed.

This requirement will implicitly determine the hypersurface \(\Sigma_r\). Before imposing this condition we will study how our initial data can be realized with respect to different frames:

The abstract initial data realized in different frames.

First, we recall some standard formulas: Consider the initial data \((\Sigma, g, K)\) in terms of the background coordinates \(t, \theta\) and the background frame \(n, E_1, E_2\) defined in (2.8). This initial data can equivalently be expressed in terms of the connection coefficients \(A_{ij,l}, K_{ij}, i, j, l \in \{1, 2\}\) of the background frame \(n, E_1, E_2\). Furthermore, we can consider rotations of the frame elements \(E_1, E_2\) (tangent to the initial data surface \(\Sigma\)). This will yield a new frame \((n, E_1', E_2')\), on the initial data hypersurface given by the formulas

\[
E_1' := \cos \varphi E_1 + \sin \varphi E_2, \quad E_2' = -\sin \varphi E_1 + \cos \varphi E_2.
\]

(2.31)

We then consider the connection coefficients, denoted by \(A^\varphi, K^\varphi\) for short, in this new rotated frame. The components of \(K^\varphi\) relative to \(E_1', E_2'\) are given by standard transformation formulas:

\[
K_{11}^\varphi = \cos^2 \varphi K_{11} + \sin^2 \varphi K_{22} + 2 \sin \varphi \cos \varphi K_{12}
\]

(2.32)

\[
K_{22}^\varphi = \sin^2 \varphi K_{11} + \cos^2 \varphi K_{22} - 2 \sin \varphi \cos \varphi K_{12}
\]

\[
K_{12}^\varphi = K_{12} + \sin \varphi \cos \varphi [K_{22} - K_{11}]
\]

On the other hand, the spatial connection coefficients are given by the following. (Recall that \(K_{12} = 0\) by construction).

\[
A_{11,2}^\varphi = g(D_{\cos \varphi E_1 + \sin \varphi E_2} (\cos \varphi E_1 + \sin \varphi E_2), -\sin \varphi E_1 + \cos \varphi E_2)
\]

(2.33)

\[
= \cos \varphi \sin^2 \varphi (E_1 \varphi) + \cos^3 \varphi A_{11,2} + \cos^3 \varphi (E_1 \varphi) - \cos \varphi \sin^2 \varphi A_{12,1}
\]

\[
+ \sin^3 \varphi (E_2 \varphi) + \sin \varphi \cos^2 \varphi A_{21,2} + \sin \varphi \cos^2 \varphi (E_2 \varphi) - \sin^3 \varphi A_{22,1}
\]

\[
= \cos \varphi E_1 \varphi + \sin \varphi E_2 \varphi + \cos \varphi A_{11,2} - \sin \varphi A_{22,1}
\]

\[
= E_1' (\varphi) + \cos \varphi A_{11,2} - \sin \varphi A_{22,1}.
\]

(2.34)
\[
= \sin \varphi (E_1 \varphi) - \cos \varphi (E_2 \varphi) + \cos \varphi A_{22,1} + \sin \varphi A_{11,2} \\
= -E_2^\varphi (\varphi) + \cos \varphi A_{22,1} + \sin \varphi A_{11,2}.
\]

In view of these transformation laws, we now define:

**Definition 2.5.** Consider a symmetric 2x2-matrix valued function \( \tilde{K}_{ij}(t, \theta) \) and a 3x2-matrix valued function \( \tilde{A}_{ij,k}(t, \theta) \), \( i, j, k \in \{1, 2\} \).

We say that these matrix-valued data agree with the prescribed initial data \((\Sigma, A, K)\) up to a gauge transformation, provided there exists a function \( \varphi(t, \theta) \) so that:

\[
\tilde{K}_{11} = \cos^2 \varphi K_{11} + \sin^2 \varphi K_{22} + 2 \sin \varphi \cos \varphi K_{12} \\
\tilde{K}_{22} = \sin^2 \varphi K_{11} + \cos^2 \varphi K_{22} - 2 \sin \varphi \cos \varphi K_{12} \\
\tilde{K}_{12} = K_{12} + \sin \varphi \cos \varphi [K_{22} - K_{11}]
\]

(2.35)

(in the last equation we recall that \( K_{12} = 0 \), yet we include it for completeness), and for the spatial connection coefficients:

\[
\tilde{A}_{11,2} = \tilde{e}_1(\varphi) + \cos \varphi A_{11,2} - \sin \varphi A_{22,1} \\
\tilde{A}_{22,1} = -\tilde{e}_2(\varphi) + \cos \varphi A_{22,1} + \sin \varphi A_{11,2}
\]

It is possible to make these relations explicit in terms of \( \varphi(\Sigma, A, K) \), as shown in Figure 5:

\[
\theta = \pi \\
E_1 \\
E_2 \\
\tilde{e}_1 = e_1 \\
\tilde{e}_2
\]

Figure 5: The adapted frame \((\tilde{e}_1, \tilde{e}_2)\) as a rotation of the fixed background frame \((E_1, E_2)\).

We also make a remark for future reference:

**Remark 2.6.** The value of \( \varphi(t, \theta) \) is (uniquely, up to adding an integer multiple of \( \pi \)) fixed by the value of the tensor \( \tilde{K}_{12}(t, \theta) = [\tilde{K}(\tilde{e}_1, \tilde{e}_2)](t, \theta) \), see (2.35). In particular:

\[
\varphi(t, \theta) = \frac{1}{2} \sin^{-1} \left( \frac{2\tilde{K}_{12}}{K_{22} - K_{11}} \right). \quad (2.37)
\]

(Recall that \( K_{22} - K_{11} \) is a fixed, smooth function, which is fully determined by our initial data).

Note that if we consider connection coefficients \( \tilde{A}, \tilde{K} \) solving (2.35) and (2.36) for some function \( \varphi(t, \theta) \), then the metric \( g \) induced by \( \tilde{A}_{ij,k} \) is identical to the prescribed \( g \) in the \( t, \theta \) coordinates. Also, the prescribed second fundamental form \( \tilde{K} \) is the same (as a tensor) with the prescribed second fundamental form \( K \) of our problem. In particular, the value of the component \( \tilde{K}_{12}(t, \theta) = K(E^*_1, E^*_2) \) uniquely forces the values of the connection coefficients \( \tilde{K}_{11}(t, \theta), \tilde{K}_{22}(t, \theta), \tilde{A}_{11,2}(t, \theta), \tilde{A}_{22,1}(t, \theta) \) of our initial data, relative to the frame \((E^*_1, E^*_2)\), \((\tilde{e}_1, \tilde{e}_2)\).

In other words, for any trace of the form \((2.31)\) in our abstract initial data then the value of the component \( \tilde{K}_{12}(t, \theta) = [K(e_1, e_2)](t, \theta) \) uniquely specifies the gauge-rotation angle \( \varphi(t, \theta) \). Therefore, all other components of the initial data, relative to the frame \((\tilde{e}_1, \tilde{e}_2, \tilde{e}_0)\), should be expressible in terms of \( \tilde{K}_{12}(t, \theta) \).

We obtain these relations in the next subsection.

**Relations between geometric quantities on the initial data set.**

Assume that we have a pair of matrix-valued functions \([\tilde{K}_{ij}](t, \theta), [\tilde{A}_{ij,k}](t, \theta)\), which matches the prescribed initial data \((\Sigma, g, K)\) up to a gauge transformation, in the sense of definition (2.2).

In particular, \([\tilde{K}_{ij}](t, \theta) \equiv [K^*_ij](t, \theta)\) for some (apriori not specified) function \( \varphi(t, \theta) \).
Then there exists a fixed function $F_{t,\phi}^{11} : (-\delta, \delta) \to \mathbb{R}$, $\delta > 0$, so that:

$$\hat{K}_{11}(t, \theta) = K_{11}^{11}(t, \theta) = F_{t,\phi}^{11}(K_{12}(t, \theta)) = F_{t,\phi}^{11}(\hat{K}_{12}(t, \theta)).$$

(2.38)

There is also another fixed function $F_{t,\phi}^{22} : (-\delta, \delta) \to \mathbb{R}$, so that:

$$\hat{K}_{22}(t, \theta) = K_{22}^{22}(t, \theta) = F_{t,\phi}^{22}(K_{12}(t, \theta)) = F_{t,\phi}^{22}(\hat{K}_{12}(t, \theta)).$$

(2.39)

These functions can be in fact be calculated explicitly, using the formulas (2.35), (2.37) and the trigonometric identities $\cos^2 \varphi = \frac{1}{2}(1 + \cos 2\varphi)$, $\sin^2 \varphi = \frac{1}{2}(1 - \cos 2\varphi)$ to find:

$$F_{t,\phi}^{11}(\hat{K}_{12}(t, \theta)) = \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{4\hat{K}_{12}^2}{(K_{22} - K_{11})^2}} \right] K_{11} + \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{4\hat{K}_{12}^2}{(K_{22} - K_{11})^2}} \right] K_{22},$$

(2.40)

$$F_{t,\phi}^{22}(\hat{K}_{12}(t, \theta)) = \frac{1}{2} \left[ 1 + \sqrt{1 - \frac{4\hat{K}_{12}^2}{(K_{22} - K_{11})^2}} \right] K_{22} + \frac{1}{2} \left[ 1 - \sqrt{1 - \frac{4\hat{K}_{12}^2}{(K_{22} - K_{11})^2}} \right] K_{11}.$$

(2.41)

In fact, the converse is also seen to be true: Assuming that a symmetric tensor $\hat{K}_{ij}(t, \theta)$ satisfies the properties (2.40), (2.41), then it agrees with the prescribed second fundamental form $K$ up to a gauge transformation, encoded in a function $\varphi(t, \theta)$. Moreover, that gauge function $\varphi(t, \theta)$ can be determined from the component $\hat{K}_{12}(t, \theta)$ via the formula (2.35).

Using formulas (2.33), (2.34) and (2.37), we then observe that $A_{11,2}^\varphi$, $A_{22,1}^\varphi$ can also be expressed in terms of $K_{12}$ via explicit formulas.

Thus, if a pair $A_{11,2}, A_{22,1}$ arises via (2.36) from the background frame via a rotation by $\varphi(t, \theta)$ (and $\varphi(t, \theta)$ is given from $K_{12}(t, \theta)$ via (2.37)), then $A_{11,2}, A_{22,1}$ can also be expressed via the formulas:

$$\hat{A}_{11,2} = \left( 1 - \frac{4\hat{K}_{12}^2}{(K_{22} - K_{11})^2} \right)^{-\frac{1}{2}} \hat{e}_1 \left( \frac{\hat{K}_{12}}{K_{22} - K_{11}} \right) + F_1 \left( \frac{\hat{K}_{12}}{K_{22} - K_{11}} \right),$$

(2.42)

$$\hat{A}_{22,1} = \left( 1 - \frac{4\hat{K}_{12}^2}{(K_{22} - K_{11})^2} \right)^{-\frac{1}{2}} \hat{e}_2 \left( \frac{\hat{K}_{12}}{K_{22} - K_{11}} \right) + F_2 \left( \frac{\hat{K}_{12}}{K_{22} - K_{11}} \right).$$

Here the functions $F_1, F_2 : (-\delta, \delta) \to \mathbb{R}$ are explicit smooth functions that depend only on the prescribed initial data:

$$F_1(x) = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2}} A_{11,2} - \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2}} A_{22,1},$$

(2.43)

$$F_2(x) = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2}} A_{22,1} + \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2}} A_{11,2}.$$ 

(2.44)

We also note that the vector fields $\hat{e}_1$, $\hat{e}_2$ on the hypersurface $\Sigma_{t=0}$ can also be expressed in terms of the fixed background coordinates $\partial_t, \partial_\theta$, with coefficients that are determined by the value of $K_{12}$.

This follows merely from our choice (2.8) of the initial abstract frame $E_1, E_2$, along with the formula (2.31):

$$\hat{e}_1 = E_1^\varphi = \cos \varphi(g_{1t})^{-1/2} \partial_t + \sin \varphi(g_{1\theta})^{-1/2} \partial_\theta,$$

(2.45)

$$\hat{e}_2 = E_2^\varphi = \cos \varphi(g_{2\theta})^{-1/2} \partial_\theta - \sin \varphi(g_{1\theta})^{-1/2} \partial_t,$$

which after replacing $\varphi$ in favour of $K_{12}$, via (2.37), yields the formulas:

$$\hat{e}_1 = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2}(g_{1t})^{-1/2}} \partial_t + \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2}(g_{1\theta})^{-1/2}} \partial_\theta,$$

(2.46)

$$\hat{e}_2 = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2}(g_{2\theta})^{-1/2}} \partial_\theta - \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2}(g_{1\theta})^{-1/2}} \partial_t,$$

where $x = \hat{K}_{12}(K_{22} - K_{11})^{-1}$.

For future reference, let us note here that the values of $a_{11}, a_{10}$ defined in (2.18) on $\Sigma_{t=0}$ are precisely determined from the coefficients that appear in (2.46) (see also (3.25)-(3.26), for the step).
\[
\begin{align*}
    a_{t1}(r_*(t, \theta), t, \theta) &= \frac{1}{2} \left[ 1 + \sqrt{1 - 4 \left( \frac{K_{12}}{K_{22} - K_{11}} \right)^2 (g_{tt})^{1/2} } \right] \\
    a_{t2}(r_*(t, \theta), t, \theta) &= -\frac{\text{sign}(K_{12})}{2 \left( \frac{K_{22} - K_{11}}{K_{22} - K_{11}} \right)^2} \left[ 1 - \sqrt{1 - 4 \left( \frac{K_{12}}{K_{22} - K_{11}} \right)^2 (g_{tt})^{1/2} } \right], \\
    a_{\theta2}(r_*(t, \theta), t, \theta) &= \frac{1}{2} \left[ 1 + \sqrt{1 - 4 \left( \frac{K_{12}}{K_{22} - K_{11}} \right)^2 (g_{\theta\theta})^{1/2} } \right] \\
    a_{\theta1}(r_*(t, \theta), t, \theta) &= \frac{\text{sign}(K_{12})}{2 \left( \frac{K_{22} - K_{11}}{K_{22} - K_{11}} \right)^2} \left[ 1 - \sqrt{1 - 4 \left( \frac{K_{12}}{K_{22} - K_{11}} \right)^2 (g_{\theta\theta})^{1/2} } \right].
\end{align*}
\]

(2.47)

This leads to a notion of solutions to the system of the equations \[1.20\], \[1.21\], \[1.22\], and \[2.20\], capturing the prescribed initial data on some hypersurface:

### 2.6.1 The system for the initial data

**Definition 2.7.** We say that a solution \( K_{12}(r, t, \theta), K_{22}(r, t, \theta), K_{11}(r, t, \theta) \) to the set of equations \[1.21\], \[1.22\], \[1.20\] in the gauge introduced above, captures our prescribed initial data \((\Sigma, g, K)\) on some hypersurface \( \Sigma_{r_*} \), as in \[2.24\], provided:

In addition to the function \( r_*(t, \theta) \) there exists a function \( \bar{K}_{12}(t, \theta) \) so that:

- if we define the functions \( \bar{K}_{22}(t, \theta), \bar{A}_{22,1}(t, \theta), \bar{A}_{11,2}(t, \theta) \) on \( \Sigma_{r_*} \) via the formulas \[2.38\], \[2.39\], \[2.42\], then on \( \Sigma_{r_*} \) the formulas \[2.28\], \[2.30\], \[2.29\] hold,

- The coefficients \( a_{Ai}(r_*(t, \theta), t, \theta), A = t, \theta, i = 1, 2 \) on \( \Sigma_{r_*} \) satisfy the relations \[2.47\].

The above requirements ensure that the first and second fundamental forms induced by our solution to the system \[1.21\], \[1.22\], \[1.20\], \[2.20\] onto \( \Sigma_{r_*} \) agree with the prescribed initial data \((\Sigma, g, K)\) up to a gauge transformation, as defined in Definition 2.5. The gauge transformation is captured precisely in \( \bar{K}_{12}(t, \theta) \), via \(2.37\).

### 2.7 The reduced Einstein equations in geodesic gauge, normalized at the singularity.

What we have studied so far is a solution \( g \) of the EVE under polarized axial symmetry with abstract initial conditions \((\Sigma, g, K)\), expressed in a special geodesic gauge. This gauge exists provided the space-time admits a non-singular congruence of time-like geodesics, which emanate from the singularity at \( \{ r = 0 \} \), are normal to the collapsing direction \( e_2 \) on the singularity (in the sense that \( e_2(r) = 0 \)), and normal to the hypersurface \( \Sigma_{r_*} \) (on which the abstract initial data live) in the direction \( e_1 \perp \text{Span}(e_2, e_0) \).

Such a space-time yields a solution to the equation

\[ \Box g = 0, \]

along with a system of transport equations in the connection and coordinates-to-frame parameters

\[ K_{22}, K_{11}, K_{12}, a_{t1}, a_{\theta2}, a_{t1}, a_{t2} \]

which are functions in \( r, t, \theta \). These functions satisfy initial conditions either at \( r = 0 \) \((K_{12}, K_{22} \text{ satisfy conditions there})\), or at \( r = r_*(t, \theta) \) (all the rest of the parameters satisfy conditions there in terms of \( \bar{r}_*(r_*, \bar{K}_{12}) \)). The initial data at \( \{ r = r_*(t, \theta) \} \) for \( g \) are given by \( \frac{1}{2} \log(g_{\phi\phi}), K_{33} \).

On the other hand, the initial data at \( \{ r = r_*(t, \theta) \} \) for the variables \( K_{11}, a_{Ai} \), satisfying the evolution equations \[1.20\], \[2.20\], are prescribed via the relations \[2.29\], \[2.47\], through the explicit formulas \[2.35\], \[2.40\], \[2.47\] (replacing \( \varphi \) by \( K_{12} \) via \[2.37\]).

Finally, the equations that determine the values of \( r_*(t, \theta) \) (that defines the hypersurface \( \Sigma_{r_*} \)) on which the initial data are induced) and of \( K_{12}(t, \theta) \) (which determines the gauge parameter \( \varphi(t, \theta) \) on \( \Sigma_{r_*} \)) are \[2.28\], \[2.30\], coupled to \[2.39\] the latter being coupled to \[2.41\]. The initial data for \( K_{22}, K_{12} \) have been fixed at \( r = 0 \), thus, the values of \( K_{22}(r, t, \theta), K_{12}(r, t, \theta) \) are in principle determined by \( \gamma \) alone,
via the Ricatti equations (1.21), (1.22). Therefore, the system of equations (2.28), (2.30), with these substitutions of terms, becomes a 2x2 system on the unknowns \( r_*(t, \theta) \), \( K_{12}(t, \theta) \), if we could treat the RHSs of the Ricatti equations (1.21), (1.22) as “given”.

For any solution of this 2x2 system in the \((2+1)\)-metric \( h \) to yield a smooth hypersurface in the \((3+1)\)-dimensional picture we note that the condition \( \tilde{e}_2(r_*) = 0 \) must be imposed at the poles \( \theta = 0, \pi \).

Furthermore, if this 2x2 system could be solved separately (equivalently, if the functions \( r_*(t, \theta), K_{12}(t, \theta) \) were known to us), then, as we discussed in the second paragraph above, the values of \( K_{11}, A_{11}, A_{21} \) on the hypersurface \( \Sigma_{r_*} \), are determined from \( K_{12} \) on \( \Sigma_{r_*} \). In turn, these values, together with \( \tilde{e}_2(r_*)(t, \theta) \), determine \( K_{11}, a_{91}, a_{82}, a_{11}, a_{12} \) on \( \Sigma_{r_*} \).

**Remark 2.8.** We note that \( \lim_{r \to 0} K_{22}(r, t, \theta) \) has not been prescribed an initial value, as opposed to \( K_{12} \) which has been prescribed an asymptotic expansion at \( r = 0 \) via (2.21), and \( K_{11} \) which has been prescribed one on the hypersurface \( \Sigma_{r_*} \), via the requirement (2.40). However, we are requiring that \( K_{22}(r, t, \theta) \) be smooth all the way to \( r = 0 \). This is the prescription of data on \( K_{22}(r, t, \theta) \) at the singularity; in fact, from the point of view of solving the Ricatti equation (1.21) forward in time, there is a unique (but implicitly defined) initial datum for \( K_{22} \) at \( r = r_* \), from which the solution to that equation does not blow up prior to \( r = 0 \).

It follows readily that a solution of the system \((1.11), (1.20), (1.21), (1.22), (2.20), (2.29), (2.30), (2.40), (2.41), (2.42), (2.46)\), that also satisfies the conditions \( (2.28), (2.29), (2.30), (2.38), (2.39), (2.47)\) that involve the additional functions \( r_*(t, \theta), K_{12}(t, \theta) \) gives rise to a (unique) axially symmetric solution of the EVE with the prescribed initial data; this is shown in the Appendix, \( \text{A.2} \). In addition, the existence of such a solution shows that a smooth congruence of time-like geodesics, which terminates at the singularity \( r = 0 \), exists. This is the system we will study in this paper. Proving an existence result for this system will prove Theorem 2.2 in the geodesic-normalized gauge we have imposed.

In sum, the initial value problem for the EVE, under polarized axial symmetry, has been reduced to the system of equations \((1.11), (1.20), (1.21), (1.22), (2.20), (2.29), (2.30), (2.40), (2.41), (2.42), (2.46)\) with respect to the new system of coordinates \( r, T, \Omega \), \( \Theta \) instead of the system \( r, t, \theta \) constructed above. In particular we will be constructing a new coordinate \( T = T(t, \theta) \) and preserving the old coordinate \( \theta, \) so \( \Theta = \theta \). We will then be expressing the metric \( g \) with respect to the new system of coordinates \( \{r, T, \Theta, \phi\} \) as opposed to the old one \( \{r, t, \theta, \phi\} \). The frame \((e_0, e_1, e_2)\) will still be the same. However in view of the change of coordinates, the coordinate-to-frame scalars \( a_{z1}, a_{A1}, i = 1, 2 \) will now change, as will the expression of the space-time metric \( g \) with respect to the new coordinates.

This, however should be seen as a gauge transformation of our REVESNGG system; in particular the new system of equations thus obtained is manifestly \emph{equivalent} to the original system. The reason this change of gauge is performed is to allow for the optimal estimates for the free wave to be derived; this requires the suitable \emph{adaptation} of one of the coordinate vector fields to the direction of collapse at the singularity. The coordinates \( T, \Theta \) achieve an \emph{alignment} of \( \partial \phi \) with \( e_2 \) at the singularity.

Now, in addition to the new coordinate \( T(t, \theta) \) certain other parameters (notably the scalar valued functions \( e_1(r), e_2(r) \)) will enter our analysis below. However these parameters are readily solved for in terms of the “main variables” in the REVESNGG system; in this sense they are of secondary importance and are not recorded along with the main variables.

### 2.8 The theorem re-cast in terms of the REVESNGG.

Theorem 2.2 refers to metric quantities expressed in terms of a coordinate system; in particular, it refers to a system of coordinates \( t, \theta, r, \phi \).

Here, we present our theorem in terms of the connection coefficients and coordinate-to-frame components of the REVESNGG system. This is the result we show in the bulk of this paper. We show in 31 in the appendix, how the next formulation implies our original Theorem 2.2.

**Theorem 2.9.** Consider polarized and axi-symmetric initial data \((g, K)\) which are perturbations of the Schwarzschild data \((g_{\text{S}}, K_{\text{S}})\) at \( r = \epsilon \), in the sense that assumptions presented in Theorem 2.2 hold.

Then there exists a coordinate function \( T(t, \theta) \), and a coordinate \( \Theta = \theta \) so that the REVESNGG system (where the coordinate-to-frame components \( a_{A1} \) are defined with respect to these coordinates) has
a unique solution

\[ \gamma(r, t, \theta), K_{ij}(r, t, \theta), a_{A_i}(r, t, \theta), i, j = 1, 2, A = T, \Theta \]

and \( r_m(t, \theta), K_{12}(t, \theta) \). These variables satisfy the bounds presented in subsection 4.5 below.

This solution uniquely determines the expression for the vector fields \( e_1, e_2 \) in terms of the coordinate vector fields \( \partial_r, \partial_t, \partial_\theta \) via the parameters \( a_{A_i}(r, t, \theta) \), \( i = 1, 2, A = T, \Theta \); for \( r \leq \epsilon/2 \) these are defined by formulas (4.71), (4.81).\(^{13}\)

In all these estimates the parameters that appear are \( s, C, c, \eta, \epsilon \). The constant \( \epsilon > 0 \) determines the hypersurface \( \{ r = \epsilon \} \) in Schwarzschild, over which we consider the (re-normalized) perturbation of the Schwarzschild data. \( \eta > 0 \) captures the (post-renormalization) closeness of our initial data to that of the Schwarzschild background. \( s \in \mathbb{N} \) denotes the (Sobolev space) order at which we measure the initial data and our solution. \( c > 0 \) captures the order \( \alpha := s - 3 - 4c \) at which we provide optimal estimates for our key parameters \( \gamma, K_{ij}, a_{A_i} \) that are fully in agreement with the claim of Theorem 2.2. The constant \( C > 1 \) captures the growth multiple of the (renormalized) norms of the evolution parameters \( \gamma, K \) between the initial data and the final singularity at \( r = 0 \). The parameters \( \eta, \epsilon \) must satisfy certain smallness conditions which we list out in detail in [4.2]. Here we highlight that \( C \cdot \eta > 0 \) must be small enough to ensure that the explicit function \( d_2(\alpha) \), defined in [1.24], is bounded in absolute value by \( 1 + \frac{1}{2} \) for \( \alpha = 1 + C \cdot \eta \). Moreover, \( \epsilon > 0 \) must satisfy the inequality \( \epsilon < \left( \frac{c_0}{\gamma} \right)^4 \). The full set of bounds we impose on our parameters is spelled out in the subsection “Key constants” below.

3 The Iteration scheme.

3.1 Overview.

Our method is to solve the system REVESNGG using an iteration scheme rather than treating it as a coupled system directly. In particular, we produce a sequence of metrics \( g^m \) in the form (3.1) and these will converge, as the parameter \( m \to \infty \), to a solution of the system REVESNGG. Thus, we obtain a solution of the EVE only in the limit \( m \to \infty \). (Note in particular that the individual metrics \( g^m \) do not solve the vacuum Einstein equations).

Let us spell out a few features of the iterated metrics:

We set \( \gamma^0 = \gamma_\Sigma \) (the value of the function in Schwarzschild), and \( h^0 = h_\Sigma \) (the value of the metric in Schwarzschild), then the subsequent iterates \( \gamma^m, h^m, m \geq 1 \), define a sequence of space-time metrics

\[ g^m = e^{2\gamma^m} d\phi^2 + h^m(r, t, \theta). \] (3.1)

These solve a recursive system that we discuss next.

- \( \gamma^m \) is solved for first at each step of the iteration. It is required to solve the (linear) free wave equation (3.6), relative to the previous metric in the iteration. The (abstractly prescribed) initial data for \( \gamma^m \) live on a hypersurface \( \Sigma^m_{m-1} \), defined at the previous step in the iteration.

- The geometry of the \((2+1)\)-metrics \( h^m \) is encoded in a suitable orthonormal frame \( e_0, e_1, e_2 \). This frame is expressible in terms of fixed background coordinates \( (r, t, \theta) \), as specified by formulas (4.81) below. \( e_0^m \) is required to be tangent to the hypersurface \( \Sigma^m_{m-1} := \{ r = r^m_{\infty}(t, \theta) \} \), on which the initial data will live. \( e_2^m \) is required to satisfy

\[ e_2^m(r) = o(r^{-\frac{1}{2} + d_2^m(t, \theta)}). \] (3.2)

(See the discussion further down in subsection 3.3 on how \( r^m_{\infty}(t, \theta) \) is to be determined).

- In addition to the metric component \( \gamma^m \), the “remaining” parts \( h^m \) of the metric \( g^m \) are encoded in the independent connection and coordinate-to-frame coefficients of this frame. These connection coefficients (that we will solve for) are

\[ K_{11}^m(r, t, \theta), K_{22}^m(r, t, \theta), K_{12}^m(r, t, \theta). \] (3.3)

The coordinate-to-frame coefficients are \( a_{A_i}^m, A = T, \Theta \) and \( i = 1, 2 \). In fact we will be solving for an equivalent system for coefficients \( a_{A_i}^m, A = T, \Theta \), where \( \Theta, T \) are coordinate vector fields constructed out of the old coordinates by an explicit transformation.

\(^{13}\)With the index \( m - 1 \) suppressed.
• The connection coefficients $K^m_{ij}$ solve first order ODEs which are an iterated version of the propagation equations (1.20), (1.21), (1.22); all these evolution equations involve forcing terms in their RHSs that contain covariant derivatives for the just-solved-for $\gamma^m$, evaluated against the previous metric $h^{m-1}$ and its associated previous frame. In particular, these connection coefficients are scalar-valued functions over the coordinates $\{r, t, \theta\}$. The restriction we impose on these is that $K^m_{12} = o(\varepsilon_0^m(t, \theta) - \alpha^m(t, \theta))$ as $r \to 0$, for all $t, \theta$, and that both $K^m_{21}, K^m_{12}$ remain smooth until $r = 0$. (This is in fact a gauge normalization on $K^m_{21}$, as discussed in the previous section).

• The coordinate-to-frame coefficients $a^m_{i\alpha}$ themselves solve first order ODEs, with coefficients depending on the just-solved-for connection coefficients $K^m_{ij}$.

Thus, in particular, we de-couple the free wave $\gamma$ from the $(2+1)$-metric $h$ by performing an iteration, to find a sequence of free waves and $(2+1)$-metrics, $(\gamma^m, h^m)$ (which are indexed by a parameter $m \in \mathbb{N}$). We note that the equations described above are all evolution equations, yet we have not prescribed initial data for these parameters $K^m_{11}, a^m_{A1}, a^m_{A2}, a^m_{i\alpha}$, nor prescribed the hypersurface where this initial data are to be induced.

To determine the solution one needs to prescribe the initial data $(\Sigma, g, K)$ of these variables somewhere. In particular, the prescribed initial data $(\Sigma, g, K)$ are to live on an $m$-dependent hypersurface $\Sigma_{r^m} = \{r = r^m_0(t, \theta)\}$, for some function $r^m_0(t, \theta)$ that is to be solved-for. The equations that prescribe the function $r^m_0(t, \theta)$ are coupled to a special component $K^m_{12}(t, \theta) = K^m_{21}(t, \theta)$ solve the requirements (3.23), (3.27) (which are indexed by a parameter $m \in \mathbb{N}$). We note that the equations described above are all evolution equations, yet we have not prescribed initial data for these parameters $K^m_{11}, a^m_{A1}, a^m_{A2}, a^m_{i\alpha}$, nor prescribed the hypersurface where this initial data are to be induced.

The prescribed initial data $(\Sigma, g, K)$ are induced on a to-be-determined hypersurface

$$\Sigma_{r^m} := \{r = r^m_0(t, \theta)\}. \tag{3.4}$$

"Induced" here means the following: The frame element $e^1_0$ is tangent to $\Sigma_{r^m}$; and then if one considers the rotation $\{e^1_0, e^m_1, e^m_2\}$ of the frame $\{e_0, e^m_1, e^m_2\}$, which makes $e^m_2$ tangent to $\Sigma_{r^m}$ and specifies connection coefficients $K^m_1, \tilde{K}^m_2$ on this rotated frame, according to the formulas (3.18), (3.19), (3.20), (3.21), (3.22) below, then the so-defined $K^m_1, \tilde{K}^m_2, A^m_{ij,k}$ solve the requirements (3.23), (3.27) (which are extrapolations of (2.28), (2.39), (2.42) in the coupled case). Thus, both $K^m_1(t, \theta)$ and $A^m_{ij,k}(t, \theta)$ correspond to the prescribed initial data $(\Sigma, g, K)$ up to a gauge transformation (as described in Definition 2.3 above).

The initial data for the connection coefficient $K^m_{11}$, as well as for $a^m_{11}, a^m_{A1}, a^m_{A2}, a^m_{i\alpha}$, are determined on the just-solved-for hypersurface $\Sigma_{r^m}$, in terms of the parameters $K^m_{12}(t, \theta)$, and $e^m_2(r^m_0)$ and the abstract initial data $(\Sigma, g, K)$ by precisely the (iterated analogues of) formulas (2.29), (2.47), where now $r^m_0(t, \theta), \tilde{K}^m_{12}(t, \theta)$ have just been solved for.

In particular, the parameters that we will be solving for at each step in the iteration are functions of $r, t, \theta$, and some functions of $t, \theta$. The former are:

$$\gamma^m, K^m_{11}, K^m_{22}, K^m_{12}, A_{i\alpha}, A = t, \theta, i = 1, 2, \tag{3.5}$$

while the latter are the function $r^m_0(t, \theta)$ that determines the hypersurface $\Sigma_{r^m}$, on which the initial data will be induced, as well as the function $\tilde{K}^m_{12}(t, \theta)$ on $\Sigma_{r^m}$. The latter encodes the (gauge) choice of the orthonormal frame $e^1_0, e^m_2$ on the initial data hypersurface.

We list out the equations that govern the evolutions of the parameters in (3.5) that depend on $r, t, \theta$. These will be (3.6), (3.10), (3.11), (3.12), (4.98). Notably, the equation (3.6) of $\gamma^m$ is a free wave equation, which we call the free wave part of the system. The equations (3.10), (3.11), (3.12), (4.98) on the time-like connection coefficients and (spatial) frame-to-coordinate components of $h^m$ are 1st order ODEs.

**Definition 3.1.** We call the set of first three equations (which is de-coupled from the remaining ones) the Ricci part of the system; we note that they are non-linear first order ODEs. Equations (4.98) are linear first-order ODEs. We call these the spatial components part of the system.

We now explain our iteration scheme in more detail:
3.2 The recursive equations for $\gamma^m$ and $K^m_{ij}$

Recall from above that at each step $m$ of the iteration, there is a hypersurface $\Sigma_{r^m}$, as in [3.4], on which the prescribed initial data $(\Sigma, g, K)$ live. In particular, at the $m^{th}$ step of the iteration, there exists a hypersurface $\Sigma_{r^{m-1}} := \{ r = r^{m-1}_d(t, \theta) \}$ on which the previous metric $h^{m-1}$ induces the initial data $(\Sigma, g, K)$.

The component $\gamma^m$ is required to solve:

$$\Box_{g^{m-1}} \gamma^m = 0,$$

and the initial Cauchy data $(\gamma_{\text{init}}, n(\gamma_{\text{init}}))$ for $\gamma^m$ live on $\Sigma_{r^{m-1}}$. Next we determine $h^m$:

Each iterated metric $h^m, g^m$, comes equipped with an $m$-dependent frame $e_0, e_1, e_2^m$ (and $e_3^m := e^{-\gamma^m} \partial_3$ for $g^m$). This orthonormal frame is fixed by the requirement that $e_2^m(r) = o(\lceil r^{-\frac{1}{2}} + d^2(t, \theta) \rceil)$ and $e_1^m$ should be tangent to the hypersurface $\Sigma_{r^m}$, on which the initial data are to be induced, along with the following propagation conditions:

- $e_0$ will be time-like and affine for each iterate:

$$\nabla^m e_0 e_0 = 0, \quad e_0 = -\left( \frac{2M}{r} - 1 \right)^{\frac{1}{2}} \partial_r$$

and the vector fields $e_1^m, e_2^m$ will be transported along $e_0^m$ according to the rule:

$$\nabla^m e_0^m e_1^m = -K_{12} e_2^m, \quad \nabla^m e_0^m e_2^m = K_{12} e_1^m,$$

where $\nabla^m$ is the connection intrinsic to $h^m$. Each such frame $e_0, e_1, e_2^m$ can be expressed in terms of the fixed background coordinates $\{ r, t, \theta \}$. This is encoded via coefficients, defined in formulas (3.10) below). The vector fields $e_1^m, e_2^m$ are normalized by the requirements that $K_1^m$ should be tangent to $\Sigma_{r^m} := \{ r = r^m(t, \theta) \}$ and $e_2^m$ by the requirement $e_2^m(r) = o(\lceil r^{-\frac{1}{2}} + d^2(t, \theta) \rceil)$.

Denote by $K^m_{ij}$ the connection coefficients

$$K^m_{ij} := \frac{\partial}{\partial r^i} (\nabla^m e_0^m e_0^m e_0^m + e_0^m e_0^m e_0^m) = K^m_{ij}.$$

For these connection coefficients we impose the equations:

$$e_0 K^m_{11} + (K^m_{11})^2 + 3(K^m_{12})^2 + e_0 \gamma^m K^m_{11} = \nabla^m_{11} \gamma^m + \nabla^m_{17} \gamma^m - \nabla^m_{17} \gamma^m - e_0 \gamma^m + (e_0 \gamma^m)^2,$$

$$e_0 K^m_{22} + (K^m_{22})^2 - (K^m_{12})^2 + e_0 \gamma^m K^m_{22} = \nabla^m_{22} \gamma^m + \nabla^m_{27} \gamma^m - \nabla^m_{27} \gamma^m - e_0 \gamma^m + (e_0 \gamma^m)^2,$$

$$e_0 K^m_{12} + (2K^m_{22} + e_0 \gamma^m) K^m_{12} = \nabla^m_{12} \gamma^m + \frac{1}{2} [\nabla^m_{17} \gamma^m - \nabla^m_{27} \gamma^m + \nabla^m_{27} \gamma^m - \nabla^m_{27} \gamma^m]$$

As before $\nabla^m$ stands for the projection of the connection $\nabla^m$ onto $\text{Span} \langle e_1^m, e_2^m \rangle$. It is thus the “spatial part” of the Levi-Civita connection.

In these ODEs, $K^m_{ij}$ are seen as simply scalar-valued functions of $r, t, \theta$. In the RHS we again consider the function $\gamma^m$ that was just solved for. The covariant derivatives and vector indices 1, 2 that appear in the above RHS are with respect to the connection of $h^m$, projected onto $\text{Span}(e_1^m, e_2^m)$. and with respect to the frame $e_0^m, e_2^m$, associated to the metric $h^m$. The equations above are evaluated at points $(r, t, \theta)$, we note also that in the second equation above we have entered the previously-solved for scalar $K^m_{12}$ instead of $K^m_{12}$; this is for technical convenience only, as the equations (3.11) and (3.12) then become completely de-coupled.

Note that (3.6) is linear in $\gamma^m$, whereas the decoupled Ricatti ODEs (3.10) - (3.12) that we impose remain non-linear in $K^m_{ij}$, $i = 1, 2, 3$.

We will be restricting our attention to solutions of the above equations that do not blow up in the $C^0$ norm prior to $\{ r = 0 \}$. In addition, we will be imposing the asymptotic diagonalisation condition that

$$K^m_{12} \cdot (K^m_{11})^{-1} \rightarrow 0, \quad K^m_{12} \cdot (K^m_{22})^{-1} \rightarrow 0,$$

14These correspond to $\frac{1}{2} \log(g_{\phi \phi}), K_{33}$ respectively.

15Note that for $g^m$, we also have automatically the propagation relation $D^m_{e_0^m} e_3^m = 0$, due to the symmetry, $D^m$ being the connection intrinsic to $g^m$.

16We denote this connection by $\nabla^m$ and omit the index $m$ where it is evident for simplicity.

17Solving a non-linear system of ODEs for $K^m_{ij}$ is crucial for our argument to close. An attempt to work with a linearised version of (3.10) - (3.12) would fail to capture the correct rate of the blow up, and would make it impossible to close the required estimates.
as \( r \to 0 \), as well as suitable variants of this for derivatives of \( K^m_12 \). To distinguish the two directions \( e^1_m, e^2_m \), we choose that \( K^m_{11} > 0, K^m_{22} < 0 \) near the singularity.

We will see that these requirements allow us to uniquely solve for \( K^m_{22}(r,t,\theta), K^m_{12}(r,t,\theta) \) via the ODEs (3.11), (3.12) above, subject to the initial condition at \( r = 0 \), \( K^m_{12}(r,t,\theta) = \sigma(r^2e^2_2(t,\theta) - K^m_1(\theta)) \).

Having solved for these parameters \( K^m_{22}, K^m_{12} \) separately from all other connections coefficients, we next solve for the hypersurface \( \Sigma_{r^m} \) on which the initial data are to be induced. This hypersurface is defined by a function \( r^m(t,\theta) \), which is solved for along with the component \( \dot{K}^m_{12}(t,\theta) \) on \( \Sigma_{r^m} \). As discussed above, \( \dot{K}^m_{12}(t,\theta) \) “sees” the frame \( e^1_m, e^2_m \) that is induced by \( e^1_m, e^2_m \) onto \( \Sigma_{r^m} \) by rotation.

We discuss this system in the next subsection, after a brief remark:

### 3.2.1 Rotation formulas and some useful calculations.

Recall that at each step in our iteration we will have \( e^2_m r = 0 \). On the to-be-determined hypersurface \( \Sigma_{r^m} \) we will evaluate \( \dot{K}^m \) against the following frame, which is adapted to \( \Sigma_{r^m} \):

\[
\begin{align*}
\tilde{e}^m_1 &:= e^m_1 \in T\Sigma_{r^m}, \quad \tilde{e}^m_2 := q[e^m_2 - \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_2 r^m_0)] \in T\Sigma_{r^m}, \\
\tilde{e}^m &:= q[e^m_0 - \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_0 r^m_0)] \in T\Sigma_{r^m}, \quad q = \left(1 - \left(\frac{2M}{r^m r^m_0} - 1\right)^{-1}(e^m_2 r^m_0)^2\right)^{-\frac{1}{2}} \tag{3.14}
\end{align*}
\]

In the latter two formulas \( r^m_0 \) is extended to be constant along the integral curves of \( e^m_0 \), therefore, \( e^m_2 r^m_0 \) makes sense. Note that

\[
\begin{align*}
\tilde{e}^m_2 r^m_0 &= q(e^m_2 r^m_0), \\
q &= \sqrt{1 + \left(\frac{2M}{r^m r^m_0} - 1\right)^{-1}(e^m_2 r^m_0)^2} \tag{3.15}
\end{align*}
\]

which implies that

\[
\begin{align*}
\tilde{e}^m_2 &= q[e^m_2 - \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_2 r^m_0)] e^m_0, \\
\tilde{e}^m_2 r^m_0 &= q[e^m_0 - \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_2 r^m_0)] e^m_2. \tag{3.16}
\end{align*}
\]

Inverting (2.25) we obtain:

\[
\begin{align*}
e^m_2 &= q\tilde{e}^m_2 + \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_0 r^m_0)\tilde{e}^m_2, \\
e^m_0 &= q\tilde{e}^m_0 + \frac{2M}{r^m r^m_0} - 1)^{-\frac{1}{2}}(e^m_2 r^m_0)\tilde{e}^m_2. \tag{3.17}
\end{align*}
\]

The connection coefficients \( K^m_{ij}, A^m_{ij,k} \) are then fixed by requiring that they should induce the required initial data \( (\Sigma, g, K) \) on the hypersurface \( \Sigma_{r^m} \). Inducing here means the following:

We require that if we consider the orthonormal frame \( e^m_0, e^m_2, e^m_1 \) defined by the formulas (3.16), and define connection coefficients \( \dot{K}^m, \dot{A}^m \) on that frame given by the formulas:

\[
\begin{align*}
q^2(\ddot{K}^m_{22}) &= q\tilde{K}^m_{22} + \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_2 e^m_2 r^m_0) + \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{3}{2}}(r^m e^m_2 r^m_0)^2, \tag{3.18} \\
\ddot{K}^m_{11} &= q\tilde{K}^m_{11} + \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(r^m e^m_2 r^m_0) \tilde{A}^m_{11,2}, \tag{3.19} \\
\ddot{K}^m_{21} &= q\tilde{K}^m_{21} + q^{-\frac{1}{2}}(e^m_0 r^m_0) \tilde{A}^m_{12,1}, \tag{3.20} \\
\ddot{A}^m_{11,2} &= q\tilde{A}^m_{11,2} - \left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_2 r^m_0) \tilde{K}^m_{11}, \tag{3.21} \\
\ddot{A}^m_{22,1} &= q^2\tilde{A}^m_{22,1} + 2q\left(\frac{2M}{r^m r^m_0} - 1\right)^{-\frac{1}{2}}(e^m_2 r^m_0) \tilde{K}^m_{21}. \tag{3.22}
\end{align*}
\]

Then the first and second fundamental forms induced by \( \dot{A}^m, \dot{K}^m \) on \( \Sigma_{r^m} \), should both be equivalent to the background metric and second fundamental form (in the sense of definition 2.5) via a rotation by a function \( \varphi^m(t,\theta) \). (The function \( \varphi^m(t,\theta) \) is also to be solved for).

Next, we derive the equations that determine the initial data hypersurface \( \Sigma_{r^m} \), along with the connection coefficients \( \dot{K}^m_{ij}, \dot{A}^m_{ij,k} \) on \( \Sigma_{r^m} \).
3.3 Determination of the initial data hypersurface \( r^m_s(t, \theta) \) and the connection and curvature components adapted to that hypersurface.

The sought-after parameters \( r^m_s(t, \theta) \) and \( \tilde{K}^m_{ij}(t, \theta) \) will be fixed by imposing the equations (3.18), (3.20), with suitable substitutions for certain terms, which are derived by making use of other necessary conditions.

We define \( \tilde{A}^m_{11,2}, \tilde{A}^m_{22,1} \) on \( \Sigma_r^m \) as functions of \( \tilde{K}^m_{ij} \), via the formulas:

\[
\tilde{A}^m_{11,2} = \left( 1 - \frac{4(\tilde{K}^m_{12})^2}{(K_{22} - K_{11})^2} \right)^{-\frac{1}{2}} \tilde{e}^m_1 \left( \frac{\tilde{K}^m_{12}}{K_{22} - K_{11}} \right) + F_1 \left( \frac{\tilde{K}^m_{12}}{K_{22} - K_{11}} \right) ,
\]

\[
\tilde{A}^m_{22,1} = \left( 1 - \frac{4(\tilde{K}^m_{12})^2}{(K_{22} - K_{11})^2} \right)^{-\frac{1}{2}} \tilde{e}^m_2 \left( \frac{\tilde{K}^m_{12}}{K_{22} - K_{11}} \right) + F_2 \left( \frac{\tilde{K}^m_{12}}{K_{22} - K_{11}} \right),
\]

(3.23)

where the functions \( F_1, F_2 \) are given by formulas (2.43), (2.44).

Moreover, the frame elements \( \tilde{e}^m_1(t, \theta), \tilde{e}^m_2(t, \theta) \) are given from the background frame \( E_1(t, \theta), E_2(t, \theta) \) by a rotation of angle \( \varphi^m(t, \theta) \). The rotation angle \( \varphi^m(t, \theta) \) is given by the value of \( \tilde{K}^m_{12}(t, \theta) \) via the formula:

\[
\varphi^m(t, \theta) = \frac{1}{2} \sin^{-1} \left( \frac{2\tilde{K}^m_{12}}{K_{22} - K_{11}} \right),
\]

(3.24)

Thus, making use of formula (2.46), we find that the frame elements \( \tilde{e}^m_1(t, \theta), \tilde{e}^m_2(t, \theta) \) are given from the background frame \( E_1(t, \theta), E_2(t, \theta) \) and the value of \( \tilde{K}^m_{12}(t, \theta) \) by the formulas:

\[
\tilde{e}^m_1 = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} + \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} \right} ,
\]

\[
\tilde{e}^m_2 = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} - \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} ,
\]

where \( x = \tilde{K}^m_{12}(K_{22} - K_{11})^{-1} \). We also write, for future reference, the inverse transformation:

\[
\tilde{e}^m_1 = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} - \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} ,
\]

\[
\tilde{e}^m_2 = \frac{1}{2} \sqrt{1 + \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} + \frac{\text{sign}(x)}{2} \sqrt{1 - \sqrt{1 - 4x^2(g_{tt})^{-1}2\theta} .
\]

(3.26)

**Determination of \( \tilde{K}^m_{ij}(t, \theta) \) and \( r^m_s(t, \theta) \).** The tensor \( \tilde{K}^m_{ij} \) is required to satisfy that \( \tilde{K}_{11}, \tilde{K}_{22} \) are determined in terms of \( \tilde{K}_{12} \) via the formulas:

\[
\tilde{K}^m_{12}(t, \theta) = F_{1,1}^{22}[\tilde{K}^m_{12}], \quad \tilde{K}^m_{11}(t, \theta) = F_{1,1}^{11}[\tilde{K}^m_{12}(t, \theta)].
\]

(3.27)

Here the functions \( F_{1,1}^{22}[], \), \( F_{1,1}^{11}[] \) are given by (2.41), (2.40).

**Remark 3.2.** Observe that in the language of Definition 2.5, the tensor \( [\tilde{K}^m_{ij}](t, \theta) \) obtained as above, is gauge-equivalent to the prescribed initial second fundamental form \( K \).

Then \( \tilde{K}^m_{12} \) and \( r^m_s(t, \theta) \) are determined via a system of two equations in these two unknowns. The equations arise from imposing (3.18), (3.20). Plugging the first formula of (3.27) into (3.18) gives:

\[
-q^{-2} \frac{2M}{r^{n+1}} - 1 - \frac{1}{2} (m^2 \tilde{e}_2^m \tilde{e}_2^m) - q^{-2} \frac{2M}{r^{n+1}} - 1 - \frac{2M}{r^{n+2}} (\tilde{m}^2 \tilde{e}_2^m)^2 + (K^m_{12}(r^m_s(t, \theta), t, \theta)
\]

(3.28)

In the above equation, \( K^m_{12}(r, t, \theta) \) solves the evolution equation (3.11) with “zero free data” at \( r = 0 \). In particular, it is the unique solution of this first order ODE, and thus, (given that the RHS of this equation has already been solved for at this stage), for each fixed \( t, \theta, K^m_{12}(r^m_s(t, \theta), t, \theta) \) is a function of \( r^m_s(t, \theta) \) alone.

Thus, (3.28) is a second order equation on the sought-after \( r^m_s \), with derivatives in the direction of the vector field \( \tilde{e}_2^m \). We also note that the smoothness of \( \Sigma_r^m \) in the resulting (3+1)-dimensional space-time forces that \( \partial t r^m_s = 0 \) at the two poles \( \theta = 0, \pi \). The equation itself then forces that all \( \partial t_k r^m_s = 0 \) with \( k_1 + r_2 \leq s \) and \( k_3 \) odd must vanish at those two poles, in the weighted \( L^2 \) sense defines in our
assumptions. \cite{3.28} is complemented by the unknown $\tilde{K}_{12}^m$ which also appears in the equation. However, we can relate this quantity to the sought-after $r_m^a$ via equation \cite{3.20}, plugging in \cite{3.23}:

$$\tilde{K}_{12}^m(r_m^a(t, \theta), t, \theta) + q^{-1} \left( \frac{2M}{r_m^a} - 1 \right)^{-\frac{1}{2}} (\tilde{e}_2^m r_m^a) \left( 1 - \frac{4(\tilde{K}_{12}^m)^2}{(K_{22} - K_{11})^2} \right)^{-\frac{1}{2}} \tilde{e}_2^m \left( \frac{\tilde{K}_{12}^m}{K_{22} - K_{11}} \right)$$

(3.29)

We note that the equation \cite{3.12}, coupled with the imposed initial condition $K_{12}^m \sim o(r^{2m}(t, \theta) - \alpha^m)$, implies that $K_{12}^m(r_m^a(t, \theta), t, \theta)$ is a function of $r_m^a(t, \theta)$ alone.

Therefore, the system of equations \cite{3.28}, \cite{3.29} provides a system of two equations in the two unknowns $r_m^a(t, \theta), \tilde{K}_{12}^m(t, \theta)$. The next step in the iteration process is to produce a unique solution of this $2 \times 2$ system.

Thus, at this point, the variables $K_{12}^m(r, t, \theta), K_{11}^m(r, t, \theta)$ have been solved for everywhere. Moreover, the initial parameters $\tilde{K}_{12}^m(r, t, \theta), \tilde{K}_{11}^m(r, t, \theta)$, as well as $r_m^a(t, \theta)$, have been determined on the initial data hypersurface $\Sigma_{<\psi}(t, \theta)$. Next, we define the functions $\tilde{K}_{12}^m(t, \theta) a_{\alpha i}^{m}(\rho^m = \epsilon, t, \theta), \Sigma_{<\psi}(t, \theta), \Sigma_i^{m}$, via the second equation in \cite{3.22} and the formula \cite{3.23}.

We then define $K_{11}^m(r_m^a(t, \theta), t, \theta)$ via the formula \cite{3.19} on $\Sigma_i^{m}$. With this initial value, we determine $K_{11}^m(r, t, \theta)$ everywhere by solving the Ricatti equation \cite{3.10} forwards-in-time.

Having solved for the components $K_{11}^m(r, t, \theta)$ we can solve for $a_{\alpha i}^{m}(\rho, t, \theta)$ via \cite{4.98}, and the initial conditions for these parameters. This will complete the determination of the next metric iterate $h^m$ (and thus the next $g^m$ also).

3.3.1 The system of unknowns and the system of equations.

To summarize, the system of functions that we solve for at the $m$th step is as follows: The parameters $\gamma^m(r, t, \theta), K_{12}^m(r, t, \theta), K_{11}^m(r, t, \theta), a_{\alpha i}^{m}(\rho, t, \theta), a_{\alpha 2}^{m}(\rho, t, \theta), a_{\alpha 2}^{m}(\rho, t, \theta)$, which all depend on $r, t, \theta$; the parameters $\tilde{K}_{12}^m, r_m^a$ that only depend on $(t, \theta)$.

The system of equations is \cite{3.6}, \cite{3.10}, \cite{3.11}, \cite{3.12}, \cite{4.98} (these are the evolution equations), \cite{3.29}, \cite{3.27} (these are the equations used to capture the initial data); the latter two equations imply the system \cite{3.28}, \cite{3.29}. And finally, the two systems are linked by the equations \cite{3.19}, \cite{3.21}, \cite{3.22}, \cite{3.23}, which provide initial data on the hypersurface $\Sigma_i^{m}$ for the parameters $K_{11}^m(r, t, \theta), a_{\alpha i}^{m}(r, t, \theta)$.

4 The function spaces and bounds for the key variables of our reduced system.

4.1 Regularity spaces for the parameters.

We present the spaces in which we will derive estimates for the variables that describe the space-time metric $g^m$ we deal with. (The field $\gamma^m$, and the connection coefficients $K^m$ of the metric $h^m$ along with the coordinate-to-frame coefficients.

It is well-known that expressing the space-time metric $g$ in a geodesic (Fermi-type) gauge leads to a loss of derivatives, in that one expects the metric components to enjoy less regularity in the spatial directions $(c_1, c_2)$ relative to the special, affine time direction $c_0$ that defines our Fermi coordinates. From this point of view, we can think of the affine direction $c_0$ as being privileged in terms of regularity. While one would worry that this would impede the closure of our estimates in a fixed function space, we do find function spaces that allow us to close our estimates. The algebraic structure of our equations, with a free wave and transport equations (where the free wave supplies the forcing term) is very important in this regard.

More specifically, the relevant variables $\gamma^m, K_{ij}^m, a_{\alpha i}^m$ and a suitable number of derivatives thereof, will be shown to lie in $L^2$-based energy spaces on level sets of the function $r$ and of certain variants $\rho^m$ of the function $r$ that we introduce. These are properly defined in the next subsection. Here we highlight a few features of the bounds we derive:

Hierarchy of Regularities: In view of the loss of spatial regularity for the geometric parameters $K^m, a^m$, in our geodesic gauge, the derivatives of the key parameters $\gamma^m, K^m, a^m$ that we control come in a certain hierarchy: The free wave $\gamma^m$ will have a total of $s - 1$ derivatives in the energy space, however,
at the top order, two of those derivatives must be the “privileged” $e_0$-direction. The variables $K^m$ and $a^m$ will have $s - 3$ derivatives lying in $L^2$, while the Christoffel symbols $\Gamma_{AB}^m$ will have just $s - 4$ derivatives lying in $L^2$. Moreover because of the singular nature of the functions $\gamma^m$ at the poles $\theta = 0, \pi$, certain derivatives of these parameters will lie in the same spaces, but with an additional singular weight $\cot \theta$ which blows up at the poles. (Regularity with respect to these enhanced spaces captures the smoothness of the resulting space-time at those poles).

How we are able to close the energy estimates for $\gamma^m$ in a higher regularity class, relative to that of the coefficients $a^m_{\alpha_1}, K^m_{\alpha_0}$, and how we can recover these singularity-weighted space estimates for the latter parameters will be described in Sections 3, 4, which deal with the iterates $\gamma^m, h^m$ respectively.

More singular estimates for the higher derivatives: As described earlier, at the very top orders, the estimates we derive for the energies of $\partial^I e_0 e_0 \gamma^m$, $\partial^I K^m$ and $\partial^I a^m$, $|I| = s - 3$ are much worse (in terms of their singular behaviour in $r$) than the bounds we derive at the lower orders. As we will see, beyond the number low $= s - 3 - 4c$ of derivatives, where we obtain the optimal behaviour (fully consistent with the asymptotics (1.3)), there is a descent scheme where for each order of regularity $|I|$, the bounds we derive are improved by a power $r^{1/4}$ relative to the order $|I| + 1$. This order-dependent behaviour beyond the lower orders is captured precisely in the function spaces we introduce in the next subsection.

4.2 Key Constants.

We discuss here certain key parameters that will be appearing below, in our claims on the various parameters that we keep track of, and in our derivation of the bounds further down. These constants will be universal and in particular, independent of $m \in \mathbb{N}$. We recall the parameters here, and also the inequalities that we will be imposing on them.

We have already introduced the (small) parameter $\epsilon > 0$, which captures the hypersurface $\{r = \epsilon\}$ in the Schwarzschild space-time, whose induced data we are perturbing.

We have also introduced a second (small) parameter $\eta > 0$, which captures the closeness of our abstract initial data to the Schwarzschild data. In particular, $\eta > 0$ captures the smallness of the difference in a renormalized energy space between our initial data and those of the Schwarzschild background. We refer to this quantity as the perturbation size.

A further constant that will appear is some fixed, large number $B \gg 1$. $B$ depends on the algebraic forms of the equations, (via, for example, the number of terms generated upon commuting our equations with suitable vector fields below). It is also allowed to depend on the mass parameter $M$ of the background Schwarzschild solution that we perturb. This constant will never be explicitly calculated, although in principle this is certainly possible.

The next key constant is $C > 1$, which captures the growth factor of the norms of key parameters in the REVESNGG system. $C$ depends on $\epsilon$ and $B$ in an explicit way

$$C = e^{\int_0^B 10\beta^2 r^{-1+\frac{1}{2}} dr}.$$  \hspace{1cm} (4.1)

Since $B \gg 1$ is fixed and independent of any other choice we make, we think of $C$ as a function of $\epsilon$: $C = C(\epsilon)$; $C(\epsilon = 0) = 1$ and $C(\epsilon)$ is a continuous increasing function in $\epsilon$.

The first key inequality that we demand on $C, \eta$, is that the product $C \cdot \eta$ should satisfy an absolute smallness bound. To present this bound, let us recall the explicit functions of a parameter $\alpha \in \mathbb{R}$

$$d_2(\alpha) := \frac{\alpha - \frac{3}{4} - \sqrt{(\alpha - \frac{3}{4})^2 + 6\alpha - 4\alpha^2}}{2},$$ \hspace{1cm} (4.2)

We also consider the parameter $d_1(\alpha)$

$$d_1(\alpha) := \frac{\alpha - \frac{3}{4} + \sqrt{(\alpha - \frac{3}{4})^2 + 6\alpha - 4\alpha^2}}{2}.$$ \hspace{1cm} (4.3)

(Note that $d_2(1) = -1, d_1(1) = \frac{1}{2}$; the significance of $d_2(\alpha)$ in terms of the asymptotics of $K_{22}(r, t, \theta)$ has been highlighted in the introduction). We then require that for all $\alpha \in [1 - C\eta, 1 + C\eta],$

$$|d_2(\alpha) + 1| \leq 1/8, |d_1(\alpha) - \frac{1}{2}| \leq 1/8.$$ \hspace{1cm} (4.4)

We then let $D$ to be the sup of the Lipschitz norms of $d_2, d_1$ over $[1 - C\eta, 1 + C\eta]$, so in particular:

$$|d_2(1 + x) + 1| \leq D|x|, \hspace{1cm} |d_1(1 + x) - \frac{1}{2}| \leq D|x|.$$ \hspace{1cm} (4.5)
As will become manifest in the proof, it is this requirement (in fact the first of the two) that ensures the AVTD behaviour of our solutions holds. It is also responsible for the “gain” of a power at least \( r^{\frac{3}{2}} \) of various less singular terms in our inductive estimates below, relative to the “principal” singular behaviour of the same terms.

**Remark 4.1.** The requirement \((4.4)\) is in fact stronger than what really needs to be imposed, to derive the AVTD behaviour of our solution (and to show our result); however, a bound of this type does need to be imposed; in particular the methods here do not work for any (polarized, axi-symmetric) large perturbation of Schwarzschild. In particular, it is necessary for our methods to impose that for some fixed \( \delta < \frac{1}{2} \), the inequality \( |d_{2}(\alpha) + 1| \leq \delta \), holds for all \( \alpha \in [1 - C\eta, 1 + C\eta] \). Up to some technical modifications, we believe this follows by essentially the proof we have here, but we do not pursue it in this paper.

The final key constant that plays a role in our analysis is a constant \( c > 0 \): The constant \( c > 0 \) captures the growth of the renormalization power (in \( r \)) at the higher norms (in particular at the top norm). In particular, \( c > 0 \) is chosen large enough in order to absorb certain dangerous terms at the top order and ‘close the estimates’. How large \( c > 0 \) is taken depends on the coefficients of the equations and it is determined further down. In particular, we will require:

\[
c > 20. \tag{4.6}
\]

\( c > 0 \) also determines the energy space in which we will need to bound our initial data, and also in which we will derive bounds for our parameters. In particular, the number \( s \in \mathbb{N} \) that determines the Sobolev spaces \( H^s, H^{s-1} \), in which our initial data metric \( g \) and \( K \) are to live, is chosen so that:

\[
\frac{s - 3}{2} < s - 3 - 4c \iff s > 8c + 3 > 163. \tag{4.7}
\]

**Definition 4.2.** We let \( \text{low} = s - 3 - 4c \); we use \( \text{low}, s - 3 - 4c \) interchangeably.

A second bound on \( \epsilon \) that we need to impose (relative to the other parameters \( B, \eta \) we have already introduced) is:

\[
\frac{C\eta e^{-\frac{3}{2}}}{2} > B\epsilon^{1 - \frac{1}{4}} \iff \frac{C\eta}{2} > B\epsilon^{1/4}. \tag{4.8}
\]

In fact, for various technical reasons we will strengthen the bound to:

\[
B^{3} \epsilon^{1/8} < 10^{-1} C\eta. \tag{4.9}
\]

### 4.3 Preparatory steps: The interpolating function \( \rho^{m} \) and its adapted frames.

**Orthonormal frame and coordinates:** Recall that we have chosen \( \epsilon_{0} \) at every step \( m \) in the iteration to satisfy the same relation relative to the coordinate \( r \), as for the Schwarzschild space-time:

\[
\epsilon_{0}(r) = -\left(\frac{2M}{r} - 1\right)^{\frac{1}{2}}. \tag{4.10}
\]

Also, given \( t, \theta \) coordinate functions on the initial hypersurface, identified for every step \( m \), we extended them via \((2.13)\).

These two coordinates \( t, \theta \) along with the coordinate \( r \) provide a coordinate system for the \( 2+1 \) metric \( h^{m} \). However, for technical reasons we sometimes need to replace the coordinate \( r \) by an \( m \)-dependent modification:

**The coordinate function \( \rho^{m} \), and the regularity spaces on its level sets.**

The space-like hypersurfaces \( \Sigma_{r} \) (level sets of \( r \)) are suitable for deriving energy estimates in a neighborhood of the singularity at \( r = 0 \). However, at each step in the iteration, we must adjust our foliation to include the hypersurface \( \Sigma_{r^{m}, \epsilon}, r_{m}^{m} := r_{m}(r, t, \theta) \sim \epsilon \), on which the initial data are to live, and where we are to ‘start’ most of our estimates. For this reason, we introduce a modification of the function \( r \) near \( \{ r = \epsilon \} \) to capture this. Let:

\[
\rho^{m} = \rho^{m}(r, t, \theta) = r + \chi(r)(r_{m}^{m} - \epsilon), \quad \chi \in C^{\infty}(0, 2\epsilon), \quad \chi|_{[0, \frac{\epsilon}{2}], [\frac{3\epsilon}{2}, 2\epsilon]} \equiv 0, \quad \chi|_{[\frac{\epsilon}{2}, \frac{3\epsilon}{2}]} \equiv 1, \tag{4.11}
\]
for \( m \geq 0, \rho^0 = r, \) and use the level sets of \( \rho^m, \) denoted by \( \Sigma_{\rho^m}, \) to foliate the region \( \{0 < r < 2\epsilon\}. \) We will often consider the coordinates \( \{\rho^m, t, \theta\}. \) Note that for \( r^m \sim \epsilon, \) the correspondence \( \rho^m \leftrightarrow r \) is one to one, for fixed \( t, \theta. \)

By definition (2.13), the functions \( \rho^m, t, \theta \) also constitute a coordinate system. In this coordinate system:

\[
e_0(t) = 0, e_0(\theta) = 0.
\]

Note that by definition \( \Sigma_{\rho^m} = \Sigma_r, \) for \( r \in [0, \frac{\epsilon}{2}) \cup [\frac{3\epsilon}{2}, 2\epsilon] \) and \( \{r = r^m\} = \{\rho^m = \epsilon\}. \) Also, along an \( e_0 \) geodesic we have

\[
\partial_r \rho^m = [1 + \partial_r \chi(r)(r^m - \epsilon)]^{-1} \partial_r = -\left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}}[1 + \partial_r \chi(r)(r^m - \epsilon)]^{-1} e_0.
\]  

(4.12)

On the other hand, the future directed \( g^m \)-unit normal to \( \Sigma_{\rho^m} \) is given by the \( g^m \)-normalised gradient of \( \rho^m:\)

\[
n^m = -\frac{\nabla \rho^m}{\sqrt{-g^m(\nabla \rho^m, \nabla \rho^m)}}.
\]  

(4.13)

In the regions \( r \in [0, \frac{\epsilon}{2}) \cup [\frac{3\epsilon}{2}, 2\epsilon], \) \( n^m \) coincides with the unit normal to \( \Sigma_r, \) which can be viewed as a perturbation of \( e_0:\)

\[
n^m\big|_{\{0 < r < \frac{\epsilon}{2}\} \cup \{\frac{3\epsilon}{2} < r < 2\epsilon\}} = e_0 - \left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}}(e_1^m r)e_1^m - \left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}}(e_2^m r)e_2^m
\]

\[
\sqrt{1 - \left(\frac{2M}{r} - 1\right)^{-1}(e_1^m r)^2 - \left(\frac{2M}{r} - 1\right)^{-1}(e_2^m r)^2}
\]

(4.14)

Generally the future \( g^m \)-unit normal to \( \Sigma_{\rho^m} \) reads:

\[
n^m = \frac{e_0 - (e_0 \rho^m)^{-1}(e_1^m \rho^m)e_1^m - (e_0 \rho^m)^{-1}(e_2^m \rho^m)e_2^m}{\sqrt{1 - (e_0 \rho^m)^{-2}(e_1^m \rho^m)^2 - (e_0 \rho^m)^{-2}(e_2^m \rho^m)^2}}.
\]  

(4.15)

while the lapse of the foliation \( \Sigma_{\rho^m} \) equals:

\[
\Phi^m := \frac{1}{\sqrt{(e_0 \rho^m)^2 - (e_1^m \rho^m)^2 - (e_2^m \rho^m)^2}}.
\]  

(4.16)

We note here that one of the reasons for requiring the tangency of \( e_2^m \) to the singularity (in the asymptotic sense (3.2) is already apparent here. Had that condition not been imposed, then the coefficient of \( e_2 \) would have been much too singular, and would in fact be more dominant in the energy of \( \gamma^m \) than the vector field \( e_0, \) making impossible (and in fact false!) the derivation of our inductive claims. Thus our (gauge) condition forces out this potentially more singular coefficient.

### 4.4 Regularity spaces.

We introduce the spaces in which the various parameters will be measured. Recall that \( \gamma^m \) is studied in the \((3 + 1)\)-dimensional space-time and the bounds will be using \( L^2 \)-based energies on that space \( \mathbb{R} \times S^2 \times (0, 2\epsilon). \) The parameters \( K_{ij}, \alpha^m_{ij}, \) will be studied on the \((2 + 1)\)-dimensional space \( \mathbb{R} \times (0, \pi) \times (0, 2\epsilon). \) These parameters also will be bounded in \( L^2 \)-based spaces, with respect to the volume form \( \sin \theta d\theta d\Omega. \) In the instances where we use a different volume form, we will spell it out explicitly.

**Notation:** We will defining \( L^2 \)-based spaces on level sets of \( \rho^m. \) Thus, the functions will depend on \( t \in (-\infty, +\infty), \theta \in (0, \pi), \phi \in [0, 2\pi); \) all functions will be \( \phi \)-independent, so sometimes we will omit \( \phi \) altogether. We refer to the volume form will be sin \( \theta d\theta d\Omega, \) unless otherwise stated (the canonical volume form on \( S^2 \times \mathbb{R}, \) for \( \phi \)-independent functions). We sometimes denote this also by \( \text{vol}_{EUC}. \)

Given a smooth function \( \psi(\rho^m, t, \theta) : \{\Sigma_{\rho^m}\}_{\rho^m \in (0, 2\epsilon)} \rightarrow \mathbb{R}, \) we define the energy

\[
E[\psi(\rho^m, t, \theta)] = \int_{\Sigma_{\rho^m}} \left[ (e_0 \psi)^2 + |\nabla \psi|^2 \right] \text{vol}_{EUC}.
\]  

(4.17)

Here \( |\nabla \psi|^2 \) stands for \( |e_1^m \psi|^2 + |e_2^m \psi|^2, \) for any \( h^m \)-orthonormal frame \( e_1^m, e_2^m \) orthogonal to \( e_0. \) We also define the \( H^1 \) norm

\[
\|\psi\|_{H^1[\rho^m]} := \left( \sum_{|l| \leq 1} \int_{\Sigma_{\rho^m}} (\partial^l \psi)^2 \text{vol}_{EUC} \right)^{\frac{1}{2}},
\]  

(4.18)
where $\partial^I$ stands for a combination of $\partial_t, \partial\theta$ derivatives dictated by the multi-index $I$.

For some of our parameters we will be using a slight variant of these standard Sobolev spaces; the variants are tailored to capture some delicate behaviour of our parameters at the poles $\theta = 0, \theta = \pi$. (Had we been studying our system away from the poles, the Sobolev spaces introduced just above would have been sufficient).

Let us introduce the 1-st order operator

$$\partial_\theta = \partial_\theta + \frac{\cos \theta}{\sin \theta}$$

as well as the second order operator

$$\Delta_{\hat{S}^2} = \partial_\theta \partial_\theta.$$ 

Note that $\Delta_{\hat{S}^2}$ agrees with the standard round Laplacian acting on $\phi$-independent functions on $\hat{S}^2$.

Based on this, we will introduce the higher-order operators $\partial_\theta^{k_1} \partial_t^{k_2} ...$ for any multi-index $I$ consisting of an even number of $\theta$‘s and any number of $t$‘s: Letting $2k_1$ be the (even) number of $\theta$‘s and $k_2$ be the number of $t$‘s in the multi-index $I$, we define:

$$\partial^I_{\theta t} v = (\Delta_{\hat{S}^2})^{k_1} \partial_t^{k_2} \partial_\theta^{k_1} v.$$ 

In other words, $\partial^I$ differs from $\partial$ only for the $\theta$-indices; for the $t$-indices it agrees with $\partial_t$. We will see further down how $L^2_{\sin \theta \cos \theta \sin \theta d\theta d\rho dt}$ control of the $\partial^I$ derivatives of a function (usually $\gamma^{rel}_{\text{est}}$ for most of this paper) yields control of the same function in the standard Sobolev spaces $H^k_{\sin \theta \cos \theta \sin \theta}$, $k = |I|$. We also introduce the associated norm to this operator:

$$\|\psi\|_{\Omega^k} = \sum_{|I| \leq k, I = (k_1, k_2)} \int \Omega \left(\partial^I_\psi \right)^2 \sin \theta d\theta d\rho dt.$$ 

The definition of the analogous homogenous norm $\Omega^k$ is immediate. We will note in Lemma 4.9 the equivalence of this $\Omega^k$ norm with the standard norm $H^k$.

Finally, we make a final convention: For technical reasons, we will be requiring that $s \in \mathbb{N}$ be an odd number, and in particular $s - 3$ is an even number.

4.5 The Inductive claim for all parameters in the REVESNGG.

We present here the inductive claims on the parameters we solve for in the REVESNGG. These claims are verified trivially at the 0-th step by the Schwarzschild variables $\gamma_S = \gamma^0$, $(K_S)_{ij} = K^0_{ij}, (a_S)_{Ai} = a^0_{Ai}$, $i, j = 1, 2$, where $e^0_1 \parallel \partial_t, e^0_2 \parallel \partial_\theta$, cf. [4.21] and [4.5.7].

In all parameters that are functions of $r, t, \theta$, there will be a key distinction between the lower orders $k \leq l$ and the higher derivatives.

At the lower orders, the inductive claim is substantially stronger and it involves proving optimal asymptotic behaviours, as $r \to 0$. At the higher derivatives, the bounds we claim are weaker; in fact, for each derivative beyond the lower ones, the bounds we claim become more singular by a fixed amount. Even in these very singular spaces, the closeness to the Schwarzschild background is part of what is being claimed, albeit in a weaker sense compared to the lower norms.

A few general comments: Firstly, we present the claims for the step $m - 1$. We will then verify the validity of the claim for $m \in \mathbb{N}$. Secondly, most of the estimates in our inductive assumptions will be broken in three categories, depending on the number of derivatives on the various quantities:

There will be the low orders where there are a total of low $:= s - 3 - 4c$ derivatives on the various quantities. At those orders we claim what is (for $\gamma^{m-1}_0$ and $K^{m-1}_0$) the optimal behaviour (for their leading orders). At the next ‘higher’ orders, where $s - 3 - 4c < l \leq s - 4$, we claim bounds which are more singular in terms of powers of $r$; each derivative beyond the optimal orders “costs” a power $-\frac{1}{2}$ in $r$. Finally, at the top orders, we take $s - 3$ coordinate derivatives, and then up to two $e_0$ derivatives. We note that for the step $m - 1$, these vector fields are the ones defined in the previous step $m - 2$.

**Convention:** All the inductive statements we write below will be for the step $m - 1$ in the induction. The statements will then have to be verified for the $m^{th}$ step. We note that all estimates below (for the step $m - 1$) will be assumed to hold on level sets of the function $\rho^{m-1}$. Moreover, we will be taking the difference of these parameters from the corresponding values in the Schwarzschild space-time. Here $\gamma^S(t, \theta, \rho^{m-1})$.
evaluated at values \( t = a, \theta = b, \rho^{m-1} = c \) is identified with the value \( \gamma_S \) at \( (t = a, \theta = b, r = c) \) in the standard \( t, \theta, r \) coordinates. The same convention applies to all other quantities \( (K^{m-1}_{ij}) \) below.

### 4.5.1 Inductive claim for \( \gamma^{m-1} \)

We assume certain energy estimates are satisfied by \( \gamma^{m-1} \) across all level sets of the functions \( r \) and \( \rho^{m-2} \), \( m \geq 2 \). Precisely the same estimates are true on level sets of \( \rho^{m-1} \), with \( r \) replaced by \( (\rho^{m-2}) \) in the RHSs.

\[
\sqrt{\mathcal{E}[\partial^I (\gamma^{m-1} - \gamma^S)]} \leq C \cdot \eta \cdot r^{-\frac{3}{2}}, \quad \text{for } |I| \leq s - 3 - 4c, \quad (4.22)
\]

\[
\sqrt{\mathcal{E}[\partial^I (\gamma^{m-1} - \gamma^S)]} \leq C \cdot \eta \cdot r^{-\frac{3}{2} + (s - 3 - |I|) \frac{1}{4} - c}, \quad \text{for } s - 3 - 4c < |I| \leq s - 4, \quad (4.23)
\]

\[
\sqrt{\mathcal{E}[\partial^I e^0_{ij} (\gamma^{m-1} - \gamma^S)]} \leq C \cdot \eta \cdot r^{-\frac{3}{2} - 2|J_0|-c}, \quad |J_0| \leq 2, \quad \text{for } |I| = s - 3, \forall |T, T, \ldots, T|.
\]

**Note:** At the very last estimates, we exclude the top order derivatives, when all the \( s - 3 \) derivatives are in one “less regular” direction \( \partial_T = \partial_{T^{m-1}} \) direction that we will introduce in \( 4.8.2 \) below.

**Remark 4.3.** We note that there is an \( r^{-c}, c > 0 \), loss in the blow up behavior of \( \gamma^{m-1} \) at the top derivatives, which is improved at each lower order until the order \( s - 3 - 4c \). At that (low) order the behaviour in terms of powers of \( r \) is optimal.

**Definition 4.4.** Below we will be using the notation

\[ \gamma^{m-1}_{\text{rest}} (\rho^{m-1}, t, \theta) := \gamma^{m-1} (\rho^{m-1}, t, \theta) - \gamma^S (\rho^{m-1}, t, \theta). \]

(This implicitly also defines \( \gamma^{m-1}_{\text{rest}} \) in the coordinates \( r, t, \theta \) also).

Furthermore, at the lower orders we make a stronger claim: We claim that \( \gamma^{m-1} \) has the following expansion at the lower orders \( t \leq s - 3 - 4c \):

\[
\gamma^{m-1} (r, t, \theta) = \alpha^{m-1} (t, \theta) \log r + \gamma_1^{m-1} (r, t, \theta),
\]

\[
e_0 \gamma^{m-1}_1 (r, t, \theta) = -\left( \frac{2M}{r} - 1 \right) \frac{1}{r} \alpha^{m-1} (t, \theta) + e_0 \gamma^{m-1}_1 (r, t, \theta),
\]

for a function \( \alpha^{m-1} (t, \theta) \in H^{s-3-4c} \) that verifies the pointwise bound

\[
\| \alpha^{m-1} (t, \theta) - 1 \|_{L^\infty} \leq C \eta \ll 1.
\]

Moreover \( \gamma_1^{m-1} (r, t, \theta) \) (the ‘leftover term’) satisfies

\[
r^3 \| e_0 \gamma^{m-1}_1 \|^2_{H^{s-3-4c}} \leq B^2 r^\frac{3}{2},
\]

for all \( r \in (0, 2c) \). In particular, the ‘leftover term’ is strictly less singular (at the lower derivatives) than the ‘main term’ \( \alpha^{m-1} (t, \theta) \log r \) in (4.25).

**Improved behaviour of the Hessian terms**

We make certain further claims on the functions \( \gamma^{m-1} \), which stem from the AVTD behaviour of these solutions (at the orders below the top ones). These will also be important in making the optimal inductive claims on the asymptotic behaviours of the connection coefficients \( K^{m-1}_{ij} \), since they appear in the RHS of the Ricatti system (3.10) – (3.12).

The terms we will seek to bound at the step \( m \) are:

\[
\text{Hess}(\gamma^m) \in \left\{ \nabla^{m-1}_{22} \gamma^m + (e^{2-1}_{m} \gamma^m)(e^{m-1}_{\gamma^m}), \nabla^{m-1}_{11} \gamma^m + (e^{m-1}_{\gamma^m})(e^{m-1}_{\gamma^m}), \nabla^{m-1}_{12} \gamma^m + \frac{1}{2} [(e^{m-1}_{\gamma^m})(e^{m-1}_{\gamma^m}) + (e^{m-1}_{\gamma^m})(e^{m-1}_{\gamma^m})] \right\}.
\]

We seek to bound these expressions in the spaces \( H^l, l \leq s - 4 \).

The bounds we claim, for the corresponding quantities of the step \( m - 1 \), are as follows:

\[
\| \text{Hess}(\gamma^{m-1}) \|_{H^{l-3-4c}} \leq Br^{-2-\frac{3}{4}},
\]

\[
\| \text{Hess}(\gamma^{m-1}) \|_{\dot{H}^l} \leq Br^{-2-\frac{3}{4} + (s-3-\frac{1}{4})} c, \quad s - 3 - 4c < l \leq s - 4,
\]

cf. Lemma 5.15

\footnote{This \( \partial_{T^{m-1}} \) is to illustrate that the direction depends on the step \( m - 1 \) in our induction}
We remark that these estimates are not optimal, but they suffice for us to prove the results we want. We note for example that we can prove that at the lower orders \( l \leq \text{low} - 2 \), the third term in (4.28) above satisfies the stronger estimate:

\[
\|\mathcal{V}_{12}^{m-1} \cdot r_{*,m}^1 + 1/2 (e_1^{m-1} \Gamma_{12}^{m-1} + e_2^{m-1} \Gamma_{12}^{m-1})\|_{H_{\text{low}} - 2} \leq Br^{-1/2 - 1/4},
\]

which would yield a much more improved behaviour for \( K_{12}^{m-1} \) than the one we claim below, cf. (4.30).

However, this is not needed to close our estimates (i.e., to prove our induction).

### 4.5.2 Inductive claims on \( r_{*,m}^{m-1}(t, \theta) \), \( K_{12}^{m-1}(t, \theta) \).

The functions \( r_{*,m}^{m-1}(t, \theta) \) are required to satisfy \( \partial_\theta r_{*,m}^{m-1} = 0 \) at the two poles \( \theta = 0, \pi \). Then the functions \( r_{*,m}^{m-1}(t, \theta) \), \( K_{12}^{m-1}(t, \theta) \) will satisfy the following bounds at the lower derivatives:

For \( k_1 + k_2 = k \leq \text{low} \) we have:

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (r_{*,m}^{m-1} - \epsilon)|^2 \sin \theta d\theta dt} \leq (D + 1)C\eta \epsilon, \quad k_1 + k_2 = k \leq \text{low},
\]

(4.31)

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (K_{12}^{m-1})|^2 \sin \theta d\theta dt} \leq C\eta \epsilon^{-3/2 + 1/4}, \quad k_1 + k_2 = k \leq \text{low}.
\]

(4.32)

For the higher and top derivatives \( k \leq \text{low} - s - 3 \), letting \( h = k - (s - 3 - 4\epsilon) \) we have:

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (r_{*,m}^{m-1} - \epsilon)|^2 \sin \theta d\theta dt} \leq 3C\eta \epsilon^{1/4 - 1/4} h,
\]

(4.33)

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (K_{12}^{m-1})|^2 \sin \theta d\theta dt} \leq C\eta \epsilon^{-3/2 - 1/4} h.
\]

(4.34)

Moreover we claim the following top-order estimate, when \( k_1 + k_2 = s - 4 \):

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (r_{*,m}^{m-1} - \epsilon) \cdot \cot \theta|^2 \sin \theta d\theta dt} \leq 3C\eta \epsilon^{1/4 - 1/4} h,
\]

(4.35)

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (K_{12}^{m-1}) \cdot \cot \theta|^2 \sin \theta d\theta dt} \leq C\eta \epsilon^{-3/2 - 1/4} h.
\]

(4.36)

The bounds claimed for \( K_{12}^{m-1}(t, \theta) \) are also claimed for the expression \( \bar{c}_2^{m-1} K_{12}^{m-1}(t, \theta) \cdot \bar{c}_1^{m-1}(r_{*,m}^{m-1})(t, \theta) \), with a factor of 2 in the RHS.

We also have some inductive claims on the \( \bar{c}_1^{m-1} \)-derivative of the function \( r_{*,m}^m \), at the low derivatives \( k \leq \text{low} \) our claim is as follows:

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (\bar{c}_2^{m-1} (r_{*,m}^{m-1} - \epsilon))|^2 \sin \theta d\theta dt} \leq 3C\eta \epsilon^{-1/2}.
\]

(4.37)

At the high derivatives \( k \leq \text{low} - s - 3 \) the corresponding claim is:

\[
\sqrt{\int_{-\infty}^{\infty} \int_0^\pi |\partial_{t, \theta} (\bar{c}_2^{m-1} (r_{*,m}^{m-1} - \epsilon))|^2 \sin \theta d\theta dt} \leq 3C\eta \epsilon^{1/2 - 1/4} h.
\]

(4.38)

Note that the equations for \( r_{*,m}^m \) in the REVESNGG then imply that the function \( r_{*,m}^m \) must have an even expansion (in \( \theta \)) at the poles \( \theta = 0, \pi \). We also note that as a consequence of the equations, the function \( K_{12}^{m-1} \) will also vanish at the poles, along with all its derivatives which are even in \( \theta \).
4.5.3 Inductive claims for $K^{m-1}$

We commence here with the behaviour of the components $K_{ij}^{m-1}(r, t, \theta)$ at the lower orders.

We observe that in view of the AVTD-type assumption \[4.29\] at the end of the inductive claim on $\gamma^{m-1}$, the most singular terms in the RHSs of \[3.10], \[3.11], \[3.12\] (in terms of behaviour in $r$) are the terms $\varepsilon_0 \gamma^m, (\varepsilon_0 \gamma^m)^2$ (provided we can confirm the above inductive claims for $\gamma^m$). This is true at all orders below the top ones.

At the lowest orders, the validity of the inductive claims \[4.25\], \[4.27\], as well as \[4.29\], at the $m^{th}$ step, imply that the RHSs of these equations satisfy the following asymptotic expansion in $\mathcal{H}^{low}$:

\[
\|\text{RHS}[3.10]\| - \frac{3}{2} \alpha^m - (\alpha^m)^2 2M r^{-3} \|\mathcal{H}^{low} = O(r^{-3 + \frac{1}{4}}),
\]

\[
\|\text{RHS}[3.11]\| - \frac{3}{2} \alpha^m - (\alpha^m)^2 2M r^{-3} \|\mathcal{H}^{low} = O(r^{-3 + \frac{1}{4}}),
\]

\[
\|\text{RHS}[3.12]\| \|\mathcal{H}^{low} = O(r^{-3 + \frac{1}{4}}).
\]

In particular, this implies that formally, the equations \[3.10\], \[3.11\], \[3.12\] admit solutions of the form:

\[
K_{11}^{m-1}(r, t, \theta) = \frac{d_1^{m-1}(t, \theta) \sqrt{2M}}{r^{\frac{3}{2}}} + u_{11}^{m-1}(r, t, \theta), \quad (4.39)
\]

\[
K_{22}^{m-1}(r, t, \theta) = \frac{d_2^{m-1}(t, \theta) \sqrt{2M}}{r^{\frac{3}{2}}} + u_{22}^{m-1}(r, t, \theta), \quad (4.40)
\]

\[
K_{12}^{m-1}(r, t, \theta) = u_{12}^{m-1}(r, t, \theta), \quad (4.41)
\]

where

\[
d_1^{m-1}(t, \theta) := \frac{\alpha^{m-1} - \frac{3}{2} + \sqrt{\left(\alpha^{m-1}(t, \theta) - \frac{3}{2}\right)^2 + 6\alpha^{m-1}(t, \theta) - 4\alpha^{m-1}(t, \theta)^2}}{2}, \quad (4.42)
\]

\[
d_2^{m-1}(t, \theta) := \frac{\alpha^{m-1}(t, \theta) - \frac{3}{2} - \sqrt{\left(\alpha^{m-1}(t, \theta) - \frac{3}{2}\right)^2 + 6\alpha^{m-1}(t, \theta) - 4\alpha^{m-1}(t, \theta)^2}}{2}, \quad (4.43)
\]

and furthermore, the “remainder” terms $u_{22}^{m-1}(r, t, \theta), u_{11}^{m-1}(r, t, \theta), u_{12}^{m-1}(r, t, \theta)$ are $O(r^{-\frac{3}{4} + \frac{1}{4}})$ in $\mathcal{H}^{low}$.

Our claim at the lower orders is that this formal solution is in fact true. This will be part of our inductive claim at the lower orders\[19\].

Then the inductive claim that we make for $K_{ij}^{m-1}(r, t, \theta)$, $i, j = 1, 2$, is at the lower orders:

\[
\|u_{ij}^{m-1}(r, t, \theta)\|_{\mathcal{H}^l} \leq 5 \cdot B r^{-\frac{3}{4} + \frac{1}{4}}, \quad i, j = 1, 2, \quad l \leq s - 3 - 4c,
\]

while at the higher orders we claim instead:

\[
\|(K_{ij}^{m-1} - K_{ij}^S)(r, t, \theta)\|_{\mathcal{H}^l} \leq 5 \cdot C \eta \cdot r^{-\frac{3}{4} + (s-3-\frac{3}{4} - c), \quad \text{for} \quad s - 3 - 4c < l \leq s - 4. \quad (4.45)
\]

(In particular there is no division into a “principal term” $d_0^{m-1}(t, \theta) r^{-\frac{3}{4}}$ and a “remainder” $u_{ij}^{m-1}$ beyond The top order estimates for $K^{m-1}$ are when $|l| = s - 3$ and we allow the possibility of allowing the singular weight $\cot \theta$ in our norm.

Also, at the top order we make separate claims for $K_{22}^{m-1}, K_{22}^{m-1},$ and $K_{11}^{m-1}$. For the first two we claim:

\[
\|e_0^{jh}(K_{22}^{m-1} - K_{22}^S)(r, t, \theta)\|_{\mathcal{H}^{s-3}} \leq 8 \cdot C \eta \cdot r^{-\frac{3}{4} - \frac{3}{2}(\text{boundary}) - c}, \quad (4.46)
\]

the enhanced version of this claim with the singular weight at the poles is as follows:

\[
\|e_0^{jh} \partial_\theta (K_{22}^{m-1} - K_{22}^S)(r, t, \theta) \cdot \cot \theta\|_{\mathcal{H}^{s-4}} \leq 8 \cdot C \eta \cdot r^{-\frac{3}{4} - \frac{3}{2}(\text{boundary}) - c}, \quad (4.47)
\]

\[
\|e_0^{jh} K_{22}^{m-1}(r, t, \theta) \cdot \cot \theta\|_{\mathcal{H}^{s-3}} \leq 8 \cdot C \eta \cdot r^{-\frac{3}{4} - \frac{3}{2}(\text{boundary}) - c}, \quad (4.47)
\]

\[19\] At the higher orders, our estimates do not distinguish between a leading order and a remainder.
The claim at the top order for $K_{11}^{m-1}(r, t, \theta)$ is slightly weaker: In particular we claim the above bounds, for all top-order derivatives $\partial^1 K_{11}^{m-1}$, $|I| = s - 3$ except when all $s - 3$ derivatives are in a special direction $\partial_r = \partial_{r_{m-1}}$ which will be specified in \[4.8.2\] in particular we claim:

$$
\| e_3 \{ \partial^1 [K_{11}^{m-1} - K_{11}^S] \} \|_{L^2} \leq 8 \cdot C \eta \cdot r^{-\frac{3}{2} - \frac{3}{2}|J_{01}| - c},
$$

(4.48)

for all $|J_0| \leq 2$ and for all derivatives of order $s - 3$ except for the case where all $s - 3$ directions are in the direction $\partial_r = \partial_{r_{m-1}}$ introduced in \[4.8.2\].

Furthermore, in analogy with the enhanced top order estimates \[4.47\] for $K_{22}^{m-1}$ we have the inductive claim:

$$
\| e_3 \partial_\Theta (K_{11}^{m-1} - K_{11}^S) (r, t, \theta) \cdot \cot \theta \|_{H^{r-4}} \leq 8 \cdot C \eta \cdot r^{-\frac{3}{2} - \frac{3}{2}(|J_{01}|) - c}.
$$

(4.49)

Let us make a few remarks here about the top order energy estimates on these key components to our analysis:

**Remark 4.5.** We note that at the top orders for $K_{m-1}^{m-1}$ we include the singular weight $\cot \theta$ in certain of our top order estimates.

This is in contrast to the estimates for $\gamma_{m-1}^{m-1}$ where this singular weight is absent, even at the top orders. The reason we are able to control $K_{m-1}^{m-1}$ with this extra weight is because of the energy of $\gamma$ that we control at the top orders involves the spatial direction $e_2$; this will allow us to control the singular weight with an application of the Hardy inequality further down.

Even beyond this issue, we note the absence of control of *spatial* parts of the metric iterates $g_{m-1}$ from our inductive assumptions. This “spatial” control (including a suitable version of the singular weights at the top orders) will be obtained in the next section, where the control of the spatial components of the metric $g_{m-1}$ will be derived from the bounds be have on $K_{ij}^{m-1}$ and suitable transport equations in an $m$-dependent gauge that we control.

The derivation of how control of $K_{m-1}^{m-1}$ and the initial data parameters $r_{m-1}^{m-1}$, $K_{12}^{m-1}$ yields spatial regularity of $g_{m-1}$ will be obtained in sections \[4.8\] and \[4.8.2\].

**Remark 4.6.** We note that at the very top orders, the factors $5$, $8$, which are to be compared with the absence of such factors in \[4.23\]-\[4.26\], is due to the algebraic structure of the (differentiated) Ricatti equations, at the middle and top orders. In particular, the number and coefficients of the most singular terms\[20\] are what leads to this extra factor.

### 4.5.4 The asymptotically CMC property of level sets of $r$.  

We remark that by the Sobolev embedding $H^2(S^2_{\theta, \phi} \times \mathbb{R}) \hookrightarrow L^\infty(S^2_{\theta, \phi} \times \mathbb{R})$ and \[4.44\] we also have the pointwise bound $\| u_{m-1}^{m-1} \|_{L^\infty} \leq C_{\text{Sob}} \cdot 5B^{r - \frac{3}{2} + \frac{1}{2}}$, since $s \geq 3 + 4c + 2$. ($C_{\text{Sob}}$ is the constant in the above Sobolev embedding).

On the other hand, by the formula $K_{33}^{m-1} = e_0 \gamma_{m-1}$ in polarized axial symmetry and the claim \[4.27\], we also deduce a bound on $K_{33}^{m-1}$\[21\]

$$
K_{33}^{m-1}(r, t, \theta) = e_0 \gamma_{m-1}(r, t, \theta) =: -\frac{\alpha_{m-1}(t, \theta)}{r^{\frac{3}{2}}} + u_{33}^{m-1}(r, t, \theta)
$$

(4.50)

where

$$
\| u_{33}^{m-1} \|_{H^{s-3-4c}} \leq \| e_0 \gamma_{m-1} \|_{H^{s-3-4c}} + \frac{2}{\sqrt{2M}} \| \alpha_{m-1} \|_{H^{s-3-4c}} r^{-\frac{3}{2}} \leq 5B^{r - \frac{3}{2} + \frac{1}{2}},
$$

(4.51)

and in particular, $\| u_{33}^{m-1} \|_{L^\infty} \leq C_{\text{Sob}} \cdot B^{r - \frac{3}{2} + \frac{1}{2}}$, for all $r \in (0, 2e)$.

\[20\] These are related to the notion of “borderline terms” we introduce below.

\[21\] $K_{31}^{m-1}, K_{32}^{m-1}$ are automatically zero.
Observe, in view of the behavior of $K_{ij}^{m-1}$ at lower orders, that by taking the 3-dim trace of $K^{m-1}$ we obtain
\[ \text{tr}_{m-1} K^{m-1} = -\frac{3}{2} \frac{\sqrt{2M}}{r^2} + \text{tr}_{m-1} u^{m-1}. \] (4.52)

Thus, by (4.44), (4.51), $\text{tr}_{m-1} K^{m-1}$ is constant to order $r^{-\frac{3}{2} + \frac{1}{4}}$ in the norm $\| \cdot \|_{H^{3-\epsilon}}$, as $r \to 0$. This uniformity of $\text{tr}_{m-1} K^{m-1}$ plays a central role in the derivations of the energy estimates for $\gamma^m$ below and it is one of the key ingredients to deriving its logarithmic blow up, see \[\ref{6.2}\]

In particular, a consequence of our inductive assumptions is that:
\[ \| \text{tr}_{m-1} u^{m-1} \|_{H^{3-\epsilon}} \leq 15 B r^{-\frac{3}{4} + \frac{1}{4}}, \] (4.53)
for all $r \in (0, 2\epsilon]$.

**Improved behaviour of $K_{12}^{m-1}$**: Up to order low $-2$, $K_{12}^{m-1}$ in fact satisfies the stronger bound
\[ \| K_{12}^{m-1} \|_{H^{low-2}} \leq 5 \cdot Br^{-\frac{1}{2} - \frac{1}{4}} \] (4.54)

Although we do not need the improved behaviour of $K_{12}^{m-1}$ to close our estimates below, we find it convenient to put it down here, in order to infer directly the better behaviour of the metric in the $r, \tilde{t}, \tilde{\theta}$ coordinates, see Theorem [2.9] and its proof in [13]. We verify (4.54) for the step $m$ in [6.1.2].

**4.5.5 Remark on the regularity spaces**
As seen in our inductive statements, this choice of function spaces comes at the cost of more singular estimates at the top order (relative to two orders below the top). This can be seen, for example, by comparing (4.22) with (4.23). The worse behaviour in $r$ at the higher orders is remedied by a descent scheme in the $r$-weights, used at many points in this paper, which in turn exploits the AVTD behaviour of our solution in an essential way.

**4.5.6 The passage to the limit $m \to \infty$.**
The above estimates suffice to show the boundedness of the iterates in the REVESNGG system. We can then consider differences between corresponding variables in successive steps in our iteration, establishing that the iteration defines a contraction mapping for the terms in the REVESNGG system in the corresponding spaces ($\gamma^m - \gamma^{m-1}$ in $H^{\epsilon-1}$, etc). This then furnishes a solution to the coupled REVESNGG system with our prescribed (smooth) initial data. Then, the standard uniqueness result for the EVE implies that this is the unique (smooth) solution to our problem. Moreover, at the lower derivatives $H^{low-1}$, the solution will display the (optimal) asymptotic behaviour that was claimed in our theorems.

This contraction mapping argument is a straightforward modification of our argument to derive the claimed bounds; we just subtract the corresponding equations for each of the parameters in the REVESNGG. We will not perform this here, since it would be notationally very cumbersome and essentially a straightforward modification of our arguments for boundedness.

**4.5.7 The base case of the inductive step.**
We proceed to prove the above estimates by induction, for all $m \in \mathbb{N}$. In particular we assume, that all claims listed above hold for all steps up to $m - 1$ and we seek to derive the same claims for step $m$. We need to check that the claims hold at the zeroth step also:

At the zeroth step, $\gamma^0, K^0$ are equal to their Schwarzschild counterparts:
\[ \gamma^0 = \log r + \log \sin \theta, \quad K^0_{11} = \frac{M}{r^2} \left( \frac{2M}{r} - 1 \right)^{-\frac{1}{2}}, \quad K^0_{22} = K^0_{33} = -\frac{1}{r} \left( \frac{2M}{r} - 1 \right)^{\frac{1}{2}} \] (4.55)

Also, we have
\[ a^0_{01} = a^0_{12} = 0, \quad a^0_{02} = r, \quad a^0_{11} = \left( \frac{2M}{r} \right)^{\frac{1}{2}}. \] (4.56)

The initial hypersurface is $\{ r = r^0 = \epsilon \}$, and $(\rho^0 = r, t, \theta)$ are the classical coordinates in Schwarzschild. Hence, the above claims hold trivially at the zeroth step in our induction.

Prior to proceeding with verifying the inductive step $m$, we will note certain consequences of the inductive assumptions for the step $m - 1$. (These consequences will be used in the verification of the inductive step $m$).
4.6 Basic Analysis tools.

We put down some very basic tools on which we rely, such as the Sobolev and generalized Gronwall inequalities and certain Fuchsian-type ODE and transport-equation type estimates, which are used frequently throughout this paper.

4.6.1 A generalized Hardy inequality.

We will frequently use the following Hardy-type inequality, whose proof is in the Appendix of [24]. Recall that $\partial_\theta := \partial_\theta + \frac{\cot \theta}{\sin \theta}$. Then:

**Lemma 4.7.** For any function $v(\theta) \in H^1_{\text{loc}}(0, \pi)$ the following holds:

$$
\int_0^\pi |\partial_\theta v|^2 \sin \theta d\theta + \int_0^\pi |v|^2 \sin \theta d\theta \geq C_{\text{Hardy}} \int_0^\pi |v|^2 (\sin \theta)^{-1} d\theta
$$

(4.57)

We also note a consequence of the above, which follows by Cauchy-Schwarz:

$$
\int_0^\pi |\partial_\theta v|^2 \sin \theta d\theta + \int_0^\pi |v|^2 \sin \theta d\theta \geq C'_{\text{Hardy}} \int_0^\pi |\partial_\theta v|^2 \sin \theta d\theta
$$

(4.58)

4.7 A generalized Gronwall inequality.

We will frequently use, sometimes without particular mention, the classical Sobolev inequality

$$
\|F(t, \theta)\|_{L^2_{\text{loc}}(\mathbb{R})} \leq C_{\text{Sob}} \|F(t, \theta)\|_{L^2(\mathbb{R})},
$$

(4.59)

where $C_{\text{Sob}} > 0$ is a universal constant.

We also recall the following variant of the standard Gronwall inequality:

**Lemma 4.8.** Let $F, F_0, G, H : (0, \delta) \to \mathbb{R}$ be continuous functions, $F_0$ non-increasing, satisfying

$$
F^2(r) \leq F_0^2(r) + \int_r^\delta |H(\tau)|F^2(\tau) d\tau + \int_r^\delta |G(\tau)|F(\tau) d\tau, \quad r \in (0, \delta).
$$

(4.60)

Then $F$ verifies the bound:

$$
|F(r)| \leq e^{\int_r^\delta \frac{1}{2} |H(\tau)| d\tau} \left( |F_0(r)| + \frac{1}{2} \int_r^\delta |G(\tau)| d\tau \right),
$$

(4.61)

for all $r \in (0, \delta]$.

**Proof.** Fix $r_0 \in (0, \delta]$ and let $A(r) = F_0^2(r_0) + \int_r^\delta |H(\tau)|F^2(\tau) d\tau + \int_r^\delta |G(\tau)|F(\tau) d\tau, r \in [r_0, \delta]$. Since $F_0^2(r)$ is non-increasing, we have $F^2(r) \leq A(r)$, for all $r \in [r_0, \delta]$ and hence

$$
\partial_r A(r) = -|H(r)||F^2(r) - |G(r)||F(r)| \geq -|H(r)|A(r) - |G(r)|\sqrt{A(r)}, \quad r \in [r_0, \delta],
$$

or

$$
2\partial_r \sqrt{A(r)} \geq -|H(r)|\sqrt{A(r)} - |G(r)|, \quad r \in [r_0, \delta].
$$

Hence, integrating in $[r, \delta]$ we obtain

$$
\sqrt{A(r)} \leq |F_0(r_0)| + \frac{1}{2} \int_r^\delta |G(\tau)| d\tau + \frac{1}{2} \int_r^\delta |H(\tau)|\sqrt{A(\tau)} d\tau, \quad r \in [r_0, \delta].
$$

The standard Gronwall’s inequality now implies

$$
|F(r)| \leq \sqrt{A(r)} \leq e^{\int_r^\delta \frac{1}{2} |H(\tau)| d\tau} \left( |F_0(r_0)| + \frac{1}{2} \int_r^\delta |G(\tau)| d\tau \right),
$$

for all $r \in [r_0, \delta]$. Evaluating the preceding inequality at $r = r_0$, we validate (4.61) for $r = r_0$. Since $r_0 \in (0, \delta]$ is arbitrary, the conclusion follows. □
Let us put down some standard elliptic estimates on $S^2$, which help us in the operator $\Delta_{S^2}$ to obtain our derived estimates in the usual Sobolev spaces, instead of $\partial_\theta$. First we note that for any $\phi$-independent function $v \in H^k(S^2)$ we have:

$$\int_{S^2} |\partial_\theta v|^2 dV_{S^2} = - \int_{S^2} \Delta_{S^2} v \cdot v dV_{S^2} \leq \delta \int_{S^2} |\Delta_{S^2} v|^2 dV_{S^2} + 4\delta^{-1} \int_{S^2} |v|^2 dV_{S^2},$$  \hspace{1cm} (4.62)

for any $\delta > 0$. We also recall the standard elliptic estimate (for $\phi$-independent functions $v$ over $S^2$):

$$\int_{S^2} |\partial^2_\theta v|^2 dV_{S^2} = \int_{S^2} |\Delta_{S^2} v|^2 dV_{S^2} + \int_{S^2} |\partial v|^2 dV_{S^2}.$$  \hspace{1cm} (4.63)

Combining this with (4.62) with $\delta = \frac{1}{4}$, we find that control of $\int_{S^2} |\Delta_{S^2} v|^2 dV_{S^2}$ and $\int_{S^2} |v|^2 dV_{S^2}$ implies the $H^2(S^2)$ norm. In particular for $\gamma^m_{\text{rest}}$ and for each multi-index $I = (2k_1, k_2)$ (where there is an even number of $\phi$-indices), it suffices to derive our claimed bounds for $\overline{\partial}^I_{\gamma^m_{\text{rest}}} = (\Delta_{S^2})^{k_1} \partial^{k_2}_{\theta \ldots \theta} \gamma^m_{\text{rest}}$.

**Lemma 4.9.** On any level set $\Sigma_\gamma^m(\ell_1, \phi)$ let $\overline{\partial}^k$ be the Sobolev spaces built with respect to the operators $\Delta_{S^2}, \partial_\theta$ (with the volume form $\sin \theta d\theta d\phi$). Let $H^k$ be the standard Sobolev spaces built out of $\partial, \partial_\theta$ (with the same volume form). Consider a function $v(\theta, \phi)$ and which is bounded in $H^k(\Sigma_{\gamma^m}; R)$, where $k$ is even. Then the same function is bounded in $H^k(\Sigma_{\gamma^m}; R)$, with the same bounds, up to a universal multiplicative constant.

**Proof.** The proof for all derivatives of order $\leq k - 1$ follows by an iterated application of (4.62) and (4.63). We also thus obtain the desired bound for all derivatives of order $k$, provided an even number of them are $\theta$-derivatives. The missing ones are obtained by the standard interpolation inequality:

$$2\|\partial^2_\theta v\|^2_{L^2(S^2 \times R)} \leq \|\partial^2_\theta v\|^2_{L^2(S^2 \times R)} + \|\partial v\|^2_{L^2(S^2 \times R)} + \|v\|_{L^2(S^2 \times R)}.$$

\[ \square \]

We will apply the above to $\gamma^m_{\text{rest}}$ also.

For each multi-index $I = (2k_1, k_2)$ Lemma 4.9 implies that it suffices to derive our claimed bound for $\overline{\partial}^I_{\gamma^m_{\text{rest}}}$ (and $\overline{\partial}^I_{(\ell_0, \phi, m)_{\text{rest}}}$ at the top orders) instead of $\partial^I_{\gamma^m_{\text{rest}}}$ (and $\partial^I_{(\ell_0, \phi, m)_{\text{rest}}}$ at the top orders).

An iterated application of the above Lemma shows that $\|\overline{\partial}^I_{\theta \ldots \theta} \gamma^m_{\text{rest}}\|$ controls $\|\partial^I_{\theta \ldots \theta} \gamma^m_{\text{rest}}\|$.

This still leaves the challenge of deriving our bounds when $I = (2k_1 + 1, k_2)$. For those, we use the bounds on the orders $(2k_1 + 2, k_2), (2k_1, k_2)$; if $2k_1 + 2 \leq 4$ we use (4.62) with $\delta = \frac{1}{4}$. In the remaining cases we use $\delta = (\rho^{m-2})^{1/4}$. We easily verify that if we can check our inductive claims for $\overline{\partial}^I_{\gamma^m_{\text{rest}}} = (\Delta_{S^2})^{k_1+1} \partial^{k_2}_{\theta \ldots \theta} \gamma^m_{\text{rest}}$ then our full inductive claim follows.

Finally, we put down some useful bounds that generalize the Hardy inequality.

We will seek to bound $\int_{S^2} |\frac{\partial_v}{\sin \theta}^2 \sin \theta d\theta$ for certain functions $v(\theta)$ which are even at the poles $\theta = 0, \pi$. In particular we will derive bounds on quantities:

$$\|\partial^I_{\theta} \frac{\partial_v}{\sin \theta}\|_{L^2(S^2)} \hspace{1cm} (4.64)$$

by regular Sobolev norms on $S^2$. Let us distinguish the two cases $I = (2k_1, k_2)$ and $I = (2k_1 + 1, k_2)$. Let us consider the first case first, where we derive:

$$\|\partial^I_{\theta} \frac{\partial_v}{\sin \theta}\|_{L^2(S^2)} \leq \|\Delta_{S^2}^{k_1} \partial^{k_2}_{\theta \ldots \theta} \frac{\partial_v}{\sin \theta}\|_{L^2(S^2)} + \|\Delta_{S^2}^{k_1+1} \partial^{k_2}_{\theta \ldots \theta} \frac{\partial_v}{\sin \theta}\|_{L^2(S^2)}.$$  \hspace{1cm} (4.65)

Thus it suffices to bound expressions $\|\Delta_{S^2}^{k_1} \partial^{k_2}_{\theta \ldots \theta} \frac{\cos \theta}{\sin \theta} \frac{\partial_v}{\sin \theta}\|_{L^2(S^2)}$, by regular Sobolev norms (the introduction of $\cos \theta$ in the numerator above instead of the factor 1 makes no difference, clearly). We do this by merely writing:

$$\Delta_{S^2}^{k_1} \partial^{k_2}_{\theta \ldots \theta} \frac{\cos \theta}{\sin \theta} \frac{\partial_v}{\sin \theta} = (\Delta_{S^2})^{k_1+1} \partial^{k_2}_{\theta \ldots \theta} v - (\Delta_{S^2})^{k_1} \partial^{k_2}_{\theta \ldots \theta} v$$  \hspace{1cm} (4.66)

The $L^2(S^2)$-norm of the RHS is clearly bounded by:

$$\|((\Delta_{S^2})^{k_1+1} \partial^{k_2}_{\theta \ldots \theta} v)_{L^2(S^2)} + \|((\Delta_{S^2})^{k_1} \partial^{k_2}_{\theta \ldots \theta} v)_{L^2(S^2)} \|_{L^2(S^2)} \leq \|\theta v_{\theta \ldots \theta} v\|_{L^2(S^2)} + \|\theta v_{\theta \ldots \theta} v\|_{L^2(S^2)}$$

For the case where $I = (2k_1 + 1, k_2)$ above is treated in exactly the same way, except that we keep $\frac{\partial_v}{\sin \theta}$ at the left in all formulas and all substitutions above.
Remark 4.10. Below when we apply the Leibnitz rule to terms $\Delta_{\mathbb{S}^2}[F \cdot G]$ and $\overline{\partial}^1[F \cdot G]$ we will denote the terms on the RHS by $\sum_{k_1 \cup k_2 = 1} \partial^1 F \cdot \partial^{2k_1} F$. This is a slight abuse of notation, since the RHS in fact contains derivatives of the form $(\Delta_{\mathbb{S}^2})^{k_1} \partial_{\theta, \overline{\theta}}^{k_2} \partial_{r, \overline{r}}^{k_3} F$ (involving $\Delta_{\mathbb{S}^2}$ directly). However the Lemma 4.49 implies that the $L^2$ norms of those terms are bounded by that of $\partial_{r, \overline{r}}^{k_1} \partial_{\theta, \overline{\theta}}^{k_2} \partial_{r, \overline{r}}^{k_3} F$.

Since we bound $L^2$ norms in this paper, this abuse of notation will not cause any confusion.

We also frequently use the following classical product inequality (for $s > 2$ always below), often without mention:

$$\|F_1(t, \theta) \cdot F_2(t, \theta)\|_{H^s(\mathbb{S}^2 \times \mathbb{R})} \leq C_{\text{product}}\|F_1\|_{L^2(\mathbb{S}^2 \times \mathbb{R})} \cdot \|F_2\|_{H^s(\mathbb{S}^2 \times \mathbb{R})} + C_{\text{product}}\|F_1\|_{H^{s-2}(\mathbb{S}^2 \times \mathbb{R})} \cdot \|F_2\|_{L^2(\mathbb{S}^2 \times \mathbb{R})}$$

(4.67)

4.7.1 Fuchsian ODEs and transport equations: Basic estimates.

We will be frequently encountering equations of the form:

$$\partial_r u(r, t, \theta) + f(r, t, \theta) \cdot u(r, t, \theta) = G(r, t, \theta),$$

with the coefficient $f(r, t, \theta)$ satisfying an an asymptotic expansion:

$$f(r, t, \theta) \sim \zeta(t, \theta) r^{-1},$$

(4.69)

in the sense that:

$$|f(r, t, \theta) - \zeta(t, \theta) r^{-1}| \leq B r^{-1+\delta},$$

for some $\delta > 0$ and $r \in (0, r_*]$ \footnote{Usually in this paper $\delta = 1/4$.} We then note that the general solution of this equation is of the form:

$$u(r, t, \theta) = e^{-\int_{r_*}^{r} f(s, t, \theta) ds} \int_{r_*}^{r} e^{\int_{s}^{r} f(s, t, \theta) ds} G(s, t, \theta) ds + c(t, \theta) e^{-\int_{r_*}^{r} f(s, t, \theta) ds},$$

for any $r_* > 0$ we wish to choose. The first term arises from the forcing term in (4.68), while the second corresponds to the general solution of the corresponding homogeneous equation. In particular, $c(t, \theta)$ is a function that we are free to choose. However, specifying an initial condition for the function $u(r, t, \theta)$ at some point $t = t_0, \theta = \theta_0$ and $r = r_0$, or specifying the limit

$$\lim_{r \to 0^+} e^{\int_{r_*}^{r} f(s, t, \theta) ds} \left[u(r, t, \theta) - e^{-\int_{r_*}^{r} f(s, t, \theta) ds} \int_{r_*}^{r} e^{\int_{s}^{r} f(s, t, \theta) ds} G(s, t, \theta) ds\right],$$

uniquely fixes the value of $c(t_0, \theta_0)$. (So the point $r_0$ can be chosen arbitrarily, including $r_0 = 0$. In the latter case, however, one needs to know that the integral $\int_{r_*}^{r} f(s, t, \theta) ds$ is convergent, for this formula to make sense).

We also note that this formula, along with an initial data prescription of the form:

$$c(t, \theta) = u_{\text{init}}(r_*(t, \theta), t, \theta)$$

can be used to derive the following energy estimate for the solution

Lemma 4.11. Assuming $|\zeta(t, \theta)| < c_0$ and $r_* > 0$ is small enough so that for all $r \in (0, r_*]$: $|f(r, t, \theta)| \leq \frac{c_0}{r}$, then thinking of $u(r) := u(r, t, \theta)$ as a map from $r$ to $L^2(\mathbb{S}^2 \times \mathbb{R})$, we derive that for every $r \in (0, r_*]$: $\|u(r)\|_{L^2}^2 \leq C r^{-2c_0} \|u_{\text{init}}\|_{L^2}^2 + C r^{-2c_0} \int_{r_*}^{r} s^{2c_0} \|G(s, t, \theta)\|_{L^2}^2 ds.$

(4.70)

The proof follows straightforwardly.

We are now ready to introduce certain key parameters that capture the spatial geometry of the metric iterates $h^m, g^m$. The control of these parameters via the inductive assumptions we are making at step $m - 1$ will enable us to derive the step $m$ of our inductive claim.
4.8 The spatial geometry parameters and their control by the inductive assumption.

We will show how the inductive assumption implies certain bounds on secondary quantities. We start by introducing these quantities:

We will frequently need to modify the spatial orthonormal frame $e_i^{m-1}, e_2^{m-1}$ into a new frame, which is tangential to the level sets of $\rho^{m-1}$, see (4.11)–(4.12):

**Definition 4.12.** Let:

$$e_i^{m-1} := e_i^{m-1} - e_i^{m-1}(\rho^{m-1})\partial_{\rho^{m-1}}$$

$$= e_i^{m-1} + e_i^{m-1}(\rho^{m-1})(\frac{2M}{r} - 1)^{-\frac{1}{2}}[1 + \partial_r \chi(r)(\rho^{m-1} - e)]^{-1} e_0 \in T\Sigma_{\rho^{m-1}},$$

for $i = 1, 2$.

Let us put down some bounds on the coefficients that appear in the above equation:

Recall our gauge normalisation assumption (3.2):

$$\lim_{r \to 0} e_i^{m-1} = o(r^{-\frac{1}{2} + \epsilon}(t, \theta)), \quad m \geq 1.$$  

(4.72)

In the next lemma, we will show that in fact (3.2), together with the gauge law (1.18), implies that $e_i^{m-1}$ annihilates $r$, i.e., it is tangent to the level sets $\Sigma_r$. On the other hand, $e_1^{m-1}$ acting on $r$ gives a non-zero, but much less singular term.

We recall also the inductive assumptions (4.31), (4.33), as well as the expression for $e_1^{m-1}$ on the initial data in terms of $K_{12}^{m-1}$, (3.25). Combining these bounds, we control the initial data of $e_1^{m-1}(\rho^{m-1})$ on $\Sigma_{\rho^{m-1}}$ as follows:

$$\| e_1^{m-1}(\rho^{m-1}) \|_{H^s - 3 - 4c} \leq C \eta^{-D} \tilde{\eta},$$

$$\| e_1^{m-1}(\rho^{m-1}) \|_{H^l} \leq C \eta^{-D} \eta^{-\frac{1}{2}|J_0| + (s-3-4c)\frac{1}{2} - c}, \quad \text{for all } l \in \{s-3-4c+1, \ldots, s-4\}.$$  

(4.73)

**Lemma 4.13.** The vector field $e_1^{m-1}$ annihilates the function $r$, i.e., $e_1^{m-1} r \equiv 0$, while $e_1^{m-1} r$ satisfies the bounds over level sets of $r$:

$$\| e_1^{m-1} r \|_{L^\infty} \leq C \eta^{-D} \eta^{-1}, \quad \| e_1^{m-1} r \|_{H^s - 3 - 4c} \leq C \eta^{-D} \eta^{-1},$$

$$\| e_0^{m-1} e_1^{m-1} r \|_{H^l} \leq C \eta^{-D} \eta^{-\frac{1}{2}|J_0| + (s-3-4c)\frac{1}{2} - c},$$

(4.74)

for all $r \in (0, 2c], |J_0| \leq 1, s - 3 - 4c < l \leq s - 4$.

The same bounds hold on the level sets of $\rho^{m-1}$, with $r$ replaced by $\rho^{m-1}$.

For that parameter, we also have a bound at the top order:

$$\| e_0^{m-1} \partial f^{m-1} \|_{L^2} \leq C \eta^{-D} \eta^{-\frac{1}{2}|J_0| + (s-3-4c)\frac{1}{2} - c},$$

(4.75)

for all $I, |I| = s - 3$ except the case where $I = (T, T, \ldots, T)$. In that case we make no claim.

**Proof.** Using the propagation rule (1.18), we compute:

$$e_0^{m-1} r = -K_{12}^{m-1} e_2^{m-1} r - (K_1^{m-1})^b e_b^{m-1} r + \frac{1}{2}(\frac{2M}{r} - 1)^{-\frac{1}{2}} \frac{2M}{r^2} e_1^{m-1} r,$$

$$e_0^{m-1} r = -2K_{12}^{m-1} e_2^{m-1} r - K_1^{m-1} e_1^{m-1} r + \frac{1}{2}(\frac{2M}{r} - 1)^{-\frac{1}{2}} \frac{2M}{r^2} e_1^{m-1} r,$$

$$e_0^{m-1} r = K_{12}^{m-1} e_2^{m-1} r - (K_1^{m-1})^b e_b^{m-1} r + \frac{1}{2}(\frac{2M}{r} - 1)^{-\frac{1}{2}} \frac{2M}{r^2} e_2^{m-1} r,$$

(4.76)

$$e_0^{m-1} r = -K_{12}^{m-1} e_2^{m-1} r + \frac{1}{2}(\frac{2M}{r} - 1)^{-\frac{1}{2}} \frac{2M}{r^2} e_2^{m-1} r$$

and thus by (4.39), (4.40), (4.41):

$$\partial_r (e_l^{m-1} r) + \left[ \frac{1}{2} - \frac{d_{l-1}^{m-1}}{r} - \left( \frac{2M}{r} - 1 \right)^{-\frac{1}{2}} u_{l+1}^{m-1} + \frac{1}{2} \frac{2M}{r} - r \right] e_l^{m-1} r$$

$$= 2(\frac{2M}{r} - 1)^{-\frac{1}{2}} K_{12}^{m-1} e_2^{m-1} r$$

(4.77)
\[ \partial_t (e_2^{-1} r) + \left[ \frac{1}{2} - \frac{d_2^{-1}}{r} - \left( \frac{2M}{r} - 1 \right) - \frac{1}{2} u_{22}^{-1} \right] e_2^{-1} r = 0. \]  
\text{(4.78)}

Recall that \( d_1^{-1} - \frac{1}{2} \| L \| \leq D \cdot C \eta \leq \frac{1}{2} \) and \( d_2^{-1} + \frac{1}{2} \| L \| \leq D \cdot C \eta \leq \frac{1}{2} \). The equation \( \text{(4.78)} \), together with the initial assumption \( \text{(4.72)} \) and \( \text{(4.44)} \), imply that \( e_2^{-1} r = 0 \) everywhere. Hence, \( \text{(4.77)} \) reduces to a homogeneous ODE for \( e_1^{-1} r \), whose general solution has the following behaviour (in \( L^\infty \)):

\[ e_1^{-1} r = (e_1^{-1} r_{\ast}) \left[ \frac{d_1^{-1} - \frac{1}{2}}{2} + O(r d_1^{-1} - \frac{1}{2} + \frac{1}{2}) \right]. \]

\text{(4.79)}

The lower order energy bounds in the second line of \( \text{(4.74)} \) follow by directly differentiating \( \text{(4.77)} \), utilising \( \text{(4.44)} \) and the initial data bounds \( \text{(4.73)} \), cf. Lemma \( \text{4.11} \). On the other hand, for the higher order energy estimates, we commute \( \text{(4.76)} \) instead with \( \partial^I \), \( |I| = l \), and use the expansion \( \text{(4.39)} \) only for the coefficients of the top order terms in the resulting equation:

\[ \partial_t \partial^I (e_1^{-1} r) + \left[ \frac{1}{2} - \frac{d_1^{-1}}{r} - \left( \frac{2M}{r} - 1 \right) - \frac{1}{2} u_{11}^{-1} + \frac{1}{2} \frac{2M - r}{2} \right] \partial^I e_1^{-1} r = 0. \]

\text{(4.80)}

The higher order estimate in \( \text{(4.74)} \), \( |J_0| = 0 \), follows from Lemma \( \text{4.11} \) the initial data bounds \( \text{(4.73)} \) and the inductive assumptions \( \text{(4.44)-(4.45)} \) for \( K^{-1}_{11} \); by finite induction in \(|I| \in \{s - 3; s - 4; \ldots \}, s \in -4 \). The case \( |J_0| = 1 \) follows from applying the \( \text{4.70} \) to \( J_0 = 0 \) estimate to \( \text{(4.80)} \) after solving for \( c_0 \partial^I e_1^{-1} r \).

To derive the claims for \( c_1 e^{-1} (\rho e^{-1}) \), we repeat the same commutation argument (up to adding an inconsequential terms involving \( \chi \). The top order estimate for that parameter follows, since now \( e_1^{-1} e^{-1} = 0 \) on \( \Sigma_\rho \), by invoking the tangency of \( e^{-1} \) to that hypersurface.

\( \square \)

We will often use the frame \( \tau_1^{-1}, \tau_2^{-1} \) introduced in \( \text{(4.71)} \), instead of \( \xi_1^{-1}, \xi_2^{-1} \). In order to go to-and-fro between coordinate vector fields and frames it is also useful to express the vector fields \( \tau_1^{-1}, \tau_2^{-1} \) in terms of coordinate vector fields for some system of coordinates on the level sets \( \Sigma_{\rho e^{-1}} \).

We will in fact be using different coordinate systems on these level sets. All of our coordinate systems \( T, \Theta \) will be extended by requiring \( c_0 (T) = c_0 (\Theta) \). For now let us introduce the transformation formulas for coordinate vector fields to frames, and backwards. These formulas are universal; we can use them for any system of coordinates \( T, \Theta \) propagated according to \( c_0 (T) = c_0 (\Theta) = 0 \). This will follow from our derivation of the relevant evolution equations.

**Definition 4.14.** Consider the coordinate vector fields \( \partial_t, \partial_\theta \) on any level set of \( \rho e^{-1} \). Let us define \( a_{ti}^{-1}, a_{\theta i}^{-1}, (a e^{-1} i)^{ij}, (a e^{-1} i)^{\theta j} \), \( i = 1, 2 \), via the relations

\[ \partial_t = a_{11}^{-1} \tau_1^{-1} + a_{12}^{-1} \tau_2^{-1}, \quad \partial_\theta = a_{91}^{-1} \tau_1^{-1} + a_{92}^{-1} \tau_2^{-1} \]

\text{(4.81)}

\[ \begin{align*}
\tau_1^{-1} &= (a e^{-1} i)^{11} \partial_t + (a e^{-1} i)^{\theta 1} \partial_\theta, \\
\tau_2^{-1} &= (a e^{-1} i)^{21} \partial_t + (a e^{-1} i)^{\theta 2} \partial_\theta
\end{align*} \]

Let us note that the values of the coordinate-to-frame coefficients also determine the form of the metric \( h e^{-1} \) in analogy to \( \text{(2.19)} \), just adding indices \( m - 1 \) to all the terms there.

We will in fact not be using the coordinate-to-frame coefficients defined by these background coordinates, for reasons that we review after the next formulas. However, we put down the equations on the evolution of these parameters and the bounds we can derive on their initial data. This is because our evolution equations are universal (meaning they hold for all choices of coordinate systems \( T, \Theta \) with \( c_0 (T) = c_0 (\Theta) = 0 \)), and to illustrate how the metric \( h e^{-1} \) can be reconstructed from these coefficients.

For future reference, let us note here that the values of \( a_{11}^{-1}, a_{12}^{-1}, a_{91}^{-1}, a_{92}^{-1} \) on \( \Sigma_{\rho e^{-1}} \) are precisely the coefficients that appear in \( \text{(3.25)-(3.26)} \), for the step \( m - 1 \).

\footnote{Since \( e^{-1} \) is tangent to \( \Sigma_{\rho e^{-1}} \) \( \rho e^{-1} \) in our gauge, \( e_1^{-1} (\rho e^{-1}) = e_1^{-1} (\epsilon) = 0 \), and \( e_1^{-1} = \epsilon_1^{-1} \).}
We also note the initial data for the variables $a^{AI}$, $A = t, \theta$ and $i = 1, 2$ on the initial data set, as a consequence of (3.25):

\[
(a^{m-1})^I(r^{m-1}_*(t, \theta), t, \theta) = \frac{1}{2} \left[ 1 + \sqrt{1 - 4(\bar{K}_{12}^{m-1}(K_{22} - K_{11})^{-1})^2(g_{t\theta})^{-1/2}} \right].
\]

\[
a^{m-1}_{12}(r^{m-1}_*(t, \theta), t, \theta) = -\frac{\text{sign}(\bar{K}_{12}^{m-1}(K_{22} - K_{11})^{-1})}{2} \sqrt{1 - 4(\bar{K}_{12}^{m-1}(K_{22} - K_{11})^{-1})^2(g_{t\theta})^{-1/2}},
\]

\[
a^{m-1}_{\theta 2}(r^{m-1}_*(t, \theta), t, \theta) = \frac{1}{2} \left[ 1 + \sqrt{1 - 4(\bar{K}_{12}^{m-1}(K_{22} - K_{11})^{-1})^2(g_{t\theta})^{-1/2}} \right].
\]

\[
a^{m-1}_{\theta 1}(r^{m-1}_*(t, \theta), t, \theta) = \frac{\text{sign}(\bar{K}_{12}^{m-1}(K_{22} - K_{11})^{-1})}{2} \sqrt{1 - 4(\bar{K}_{12}^{m-1}(K_{22} - K_{11})^{-1})^2(g_{t\theta})^{-1/2}}.
\]

\[\tag{4.82}\]

\[\tag{4.83}\]

In particular, given the bounds [4.32], [4.36], [4.34] for $\bar{K}_{12}^{m-1}(t, \theta)$, and the assumption [2.7] on $g_{t\theta}$, we derive the following bounds on these initial data, first at the lower orders $l \leq 0$:

\[
\|a^{m-1}_{12}(r^{m-1}_*(t, \theta), t, \theta) - a^{S}_1(c, t, \theta)\|_{H^l} \leq C\epsilon^{1/2 - DC\eta - \frac{1}{4}}, \quad \|a^{m-1}_{21}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^l} \leq C\epsilon^{1 - DC\eta},
\]

\[
\|a^{m-1}_{\theta 2}(r^{m-1}_*(t, \theta), t, \theta) - a^{S}(c, t, \theta)\|_{H^l} \leq C\epsilon^{1 - DC\eta}, \quad \|a^{m-1}_{\theta 1}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^l} \leq C\epsilon^{1/2 - DC\eta}.
\]

At the higher orders, the worse behaviour of $\bar{K}_{12}^{m-1}$ yields a more singular behaviour for the above quantities, in terms of the power of $\epsilon$. In particular for low $+1 \leq l \leq s - 4$, $h = l - \omega$:

\[
\|a^{m-1}_{12}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^l} \leq C\epsilon^{1/2 - DC\eta - \frac{1}{4}}, \quad \|a^{m-1}_{21}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^l} \leq C\epsilon^{1 - DC\eta - \frac{1}{4}},
\]

\[
\|a^{m-1}_{\theta 2}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^l} \leq C\epsilon^{1 - DC\eta - \frac{1}{4}}, \quad \|a^{m-1}_{\theta 1}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^l} \leq C\epsilon^{1/2 - DC\eta - \frac{1}{4}}.
\]

Finally, at the top orders we have the most singular behaviour:

\[
\|a^{m-1}_{12}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^{l+3}} \leq C\epsilon^{1/2 - DC\eta - c}, \quad \|a^{m-1}_{21}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^{l+3}} \leq C\epsilon^{1 - DC\eta - c}, \quad \text{for } J_1 \leq 1, i = 1, 2,
\]

\[
\|a^{m-1}_{\theta 2}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^{l+3}} \leq C\epsilon^{1/2 - DC\eta - c}, \quad \|a^{m-1}_{\theta 1}(r^{m-1}_*(t, \theta), t, \theta)\|_{H^{l+3}} \leq C\epsilon^{1/2 - DC\eta - c}.
\]

Recall the equation:

\[
\partial_{\mu^m} = [1 + \partial_{r}(\chi(r)(r_m^m - \epsilon))^{-1}] \partial_{r} = -(\frac{2M}{r} - 1)^{-\frac{1}{2}} [1 + \partial_{r}(\chi(r)(r_m^m - \epsilon))^{-1}] \epsilon_0 \tag{4.84}
\]

Let us calculate $\nabla_\theta \partial_{\mu^m}$:

\[
\nabla_\theta \partial_{\mu^m} = -\nabla_\theta (\frac{2M}{r} - 1)^{-\frac{1}{2}} [1 + \partial_{r}(\chi(r)(r_m^m - \epsilon))^{-1}] \epsilon_0 = (\frac{2M}{r} - 1)^{-\frac{1}{2}} \partial_{r} [\frac{2M}{r} - 1]^{-\frac{1}{2}} [1 + \partial_{r}(\chi(r)(r_m^m - \epsilon))^{-1}] \epsilon_0 \tag{4.85}
\]
Let us use this in evaluating \( \nabla_{\gamma^{m-1}} \partial_{\rho^{m-1}} \):

\[
\nabla_{\gamma^{m-1}} \partial_{\rho^{m-1}} = \nabla_{\gamma^{m-1}} \partial_{\rho^{m-1}} + [\epsilon^{m-1}_{\rho^{m-1}}(\rho^{m-1}) \left( \frac{2M}{r} - 1 \right) \frac{1}{2} \frac{1}{2} [1 + \partial_{r}\chi(r)(r^m - \epsilon)]^{-1} \nabla_{e_0} \partial_{\rho^{m-1}}
\]

\[
= \epsilon^{m-1}_{\rho^{m-1}} \left[ \left( \frac{2M}{r} - 1 \right) \frac{1}{2} [1 + \partial_{r}\chi(r)(r^m - \epsilon)]^{-1} \right] e_0 - \left( \frac{2M}{r} - 1 \right) \frac{1}{2} [1 + \partial_{r}\chi(r)(r^m - \epsilon)]^{-1} \nabla_{e_0} \partial_{\rho^{m-1}}
\]

\[
+ [\epsilon^{m-1}_{\rho^{m-1}}(\rho^{m-1}) \left( \frac{2M}{r} - 1 \right) \frac{1}{2} [1 + \partial_{r}\chi(r)(r^m - \epsilon)]^{-1} \nabla_{e_0} \partial_{\rho^{m-1}}
\]

(4.86)

For simplicity, in the following derivations, we omit the index \( m - 1 \) from all the relevant variables. The commutation relations \([\partial_{r}, \partial_{\theta}] = [\partial_{r}, \partial_{\phi}] = 0\) yield an ODE for \( a_{t_1}, a_{\theta_1}, i = 1, 2\). In particular, we have:

\[
0 = [\partial_{r}, \partial_{\theta}] = [\partial_{r}, a_{t_1} \vec{e}_1 + a_{t_2} \vec{e}_2],
\]

(4.87)

Taking the inner product of the previous equation with respect to \( e_1 \) and using (4.84), we obtain:

\[
e_{0} a_{t_1} = -a_{t_1} h(\nabla_{e_0} \vec{e}_1, e_1) - a_{t_2} h(\nabla_{e_0} \vec{e}_2, e_1) + a_{t_1} h(\nabla_{e_0} \vec{e}_0, e_1) + a_{t_2} h(\nabla_{e_0} \vec{e}_0, e_1)
\]

(4.88)

The analogous computation for \( a_{t_2} \) (multiplying with \( e_2 \) the first equation instead) is similar and so are the ones for \( a_{\theta_1}, \) derived from the identity \( 0 = [\partial_{r}, \partial_{\theta}] \), which yield the following ODE system:

\[
0 = a_{t_1} h(\nabla_{e_0} \vec{e}_1, e_1) - a_{t_2} h(\nabla_{e_0} \vec{e}_2, e_1) + a_{t_1} h(\nabla_{e_0} \vec{e}_0, e_1) + a_{t_2} h(\nabla_{e_0} \vec{e}_0, e_1)
\]

(4.89)

Then for the above system of equations, with initial data prescribed on \( \Sigma_{\gamma^{m-1}} \) we can explicitly write the solutions to the above system:

\[
a_{t_1}^{m-1}(r, t, \theta) = \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{t_1}^{m-1}(r_{m-1}(r, \theta), t, \theta),
\]

\[
a_{t_2}^{m-1}(r, t, \theta) = \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{t_2}^{m-1}(r_{m-1}(r, \theta), t, \theta),
\]

\[
a_{\theta_1}^{m-1}(r, t, \theta) = \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{\theta_1}^{m-1}(r_{m-1}(r, \theta), t, \theta),
\]

\[
a_{\theta_2}^{m-1}(r, t, \theta) = \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{\theta_2}^{m-1}(r_{m-1}(r, \theta), t, \theta) + \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{\theta_2}^{m-1}(r_{m-1}(r, \theta), t, \theta) + \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{\theta_2}^{m-1}(r_{m-1}(r, \theta), t, \theta) + \int_{-\infty}^{r} \int_{\tau_{m-1}^{m-1}(s, \tau_s (s, \theta))}^{m-1}(1 - \frac{2M}{s})^{-\frac{1}{2}} ds \cdot a_{\theta_2}^{m-1}(r_{m-1}(r, \theta), t, \theta)
\]

(4.90)

A key remark is in order here: The coordinate expression on the metric (in terms of \( t, \theta, r \)) that we can obtain from the above will turn out to not be adequate to derive our desired estimates for \( \gamma^{m} \). The moral reason for this is that these coordinates 
emanate
from the initial data hypersurface via extension along \( e_0 \). They thus fail to capture the principal directions of collapse/expansion at the singularity \( \{ r = 0 \} \). This would manifest itself in non-optimal behaviour (in terms of powers of \( r \)) for certain Christoffel symbols in this coordinate system.

The remedy to this issue is to consider new coordinates which are adapted to the principal collapsing directions at the singularity. It is with respect to these new coordinates that the Christoffel symbols will have a suitable behaviour that allows us to close our estimates.

In fact, there are two possible choices of coordinates that we can make. The first system is where the optimal behaviour of the spatial part of the metric near the singularity becomes apparent: In this system of coordinates \((\theta, \ell)\), the coordinate vector fields are tangent (in an asymptotic sense) to the principal
collapsing and expanding directions at the singularity. These are the directions that are referenced in Theorem 11 they are introduced in the Appendix, in the proof of that theorem. We will not work with these (very rigid) coordinates in our main proof however. Instead, for our main proof we use a “hybrid” coordinate system; one that is in between the one that emanates from the initial data set and the one that emanates entirely from the singularity, which captures both principal collapsing/expanding directions: This “hybrid” coordinate system is used to achieve two goals: First to capture only the collapsing direction $e_2^{m-1}$ by one of the coordinate vector fields. Secondly, to provide sufficient spatial regularity of the metric when expressed with respect to this coordinate system.

We will introduce this new coordinate system shortly, after a useful remark on the vanishing of certain parameters at the poles $\theta = 0, \pi$.

### 4.8.1 Propagation of vanishing conditions at the poles.

In the analysis we perform below, we will at times invoke the generalized Hardy inequality in Lemma 4.7. This will apply to functions of $\theta, t$ that vanish at the two poles $\theta, \pi$. We present here how certain key parameters vanish to first order at those two poles at each step in our iteration. This ensures that whenever the generalized Hardy inequality in Lemma 4.7 is invoked, the assumed vanishing of the function at the poles will hold.

We have imposed the condition $\tilde{e}_2^m(r^m) = 0$ at the poles, which in view of the smoothness of the vector field and function implies: $\tilde{e}_2^m(r^m) = O(\sin \theta)$.

We will show that this vanishing condition for this and some other parameters is propagated off of the initial data hypersurface:

**Lemma 4.15.** For each step of our iteration, $\tilde{K}_{12}^m(t, \theta), \tilde{e}_2^m r_2^m(t, \theta)$ both vanish to first order at the poles, in other words

$$
\tilde{K}_{12}^m(t, \theta), \tilde{e}_2^m r_2^m(t, \theta) = O(\sin \theta), \quad \forall r \in (0, 2\epsilon], t \in \mathbb{R}.
$$

Moreover the following vanishing conditions hold off the initial data hypersurface:

$$
K_{12}^m(r, t, \theta) = O(\sin \theta), \forall r \in (0, 2\epsilon], t \in \mathbb{R}.
$$

Furthermore $e_1^m(r, t, \theta = 0), e_2^m(r, t, \theta = \pi), e_1^m(r, t, \theta = \pi), e_2^m(r, t, \theta = \pi)$ are parallel to $\partial_t, \partial_\theta$ for all $r \in (0, 2\epsilon)$, which is captured by the requirements:

$$
(a^{m-1})^\theta, (a^{m-1})^\gamma, (a^{m-1})^r = O(\sin \theta),
$$

**Proof.** We prove the above by an induction on $m$: We assume it is true at step $m-1$ and derive the statement at step $m$. We have derived that $\gamma_{12}^m$ is a $C^{\text{low-2}}$ function over $S^2 \times \mathbb{R}$. As discussed above, this implies that $\partial_{\gamma_{12}^m} = O(\sin \theta)$.

From this we can derive our claim as follows: First we note that in view of the regularity of the metric $g_{m-1}$ we have that:

$$
(\nabla^2)_{12} g_{m-1} = O(\sin \theta).
$$

We can then invoke the tensorial nature of the LHS and choose normal coordinates at each of the poles to derive that:

$$
(\nabla^2)_{12} g_{m-1} = O(\sin \theta).
$$

Now, to derive our claim (4.93), we express $e_2^{m-1}$ in terms of the coordinate vector fields $\partial_t, \partial_\theta, \partial_\gamma$ using the functions $(a^{m-1})^\gamma, g_{m-1}^\gamma$. In view of our inductive assumptions on these parameters $(a^{m-1})^\gamma, \gamma_{12}^{m-1}$ we derive:

$$
\nabla_{12} g_{m-1}^\gamma - \frac{1}{2} [\nabla_1 g_{m-1}^\gamma \nabla_2 g_{m-1}^\gamma + \nabla_1 \gamma_{12}^{m-1} \nabla_2 \gamma_{12}^{m-1}] = O(\sin \theta).
$$

Considering the evolution equation (4.12, 4.89), we then derive (4.92) as well as the expression (4.14) for $e_2^m$ in terms of $K_{12}^m$; this then implies (4.91) since the LHS of that equation is of the form $O(\sin \theta)$ and the LHS is a smooth function of $K_{12}^m$.

With these conditions verified, we proceed to confirm (4.93). This is initially verified on the initial data hypersurface (4.91) and the formula (4.24). Then the evolution equations (4.78) along with (4.92) confirm (4.93).

Although not needed, we note that the above proof implies the vanishing of the even $\theta$-derivatives of $K_{12}^m, K_{12}^m$ as well as $e_2^m(r^m), e_2^m(r^m)$ and $(a^m)^\gamma, (a^m)^\gamma$ at the poles $\theta = 0, \theta = \pi$.
4.8.2 The new coordinate system: Bounds on metric components and Christoffel symbols, by virtue of our inductive assumptions.

We will consider a new coordinate function $T = T(t, \theta)$ so that the coordinate system $\{T = T(t, \theta), \Theta = \theta\}$ has the coordinate vector field $\partial_\theta$ capturing the direction $\epsilon_2^{m-1}$ at the singularity. (Here, as above $T = T^{m-1}$, but we suppress the suffix $m-1$ for notational simplicity).

We do this as follows: Let $\partial_\theta, \partial_T$ be the sought-after coordinate vector fields that correspond to the sought-after coordinates. Let us express these sought-after vector fields as linear combinations of the $\epsilon_1, \epsilon_2$ on each level set of $\rho^{m-1}$. They will be expressed as linear combinations, given by a formula as follows:

$$\partial_T = a_1^{m-1} \epsilon_1^{m-1} + a_2^{m-1} \epsilon_2^{m-1}, \quad \partial_\theta = a_3^{m-1} \epsilon_1^{m-1} + a_4^{m-1} \epsilon_2^{m-1},$$

(A.97)

$$\epsilon_1^{m-1} = (a^{m-1} \epsilon_1^{m-1}) \partial_T + (a^{m-1} \epsilon_2^{m-1}) \partial_\theta,$$

$$\epsilon_2^{m-1} = (a^{m-1} \epsilon_1^{m-1}) \partial_T + (a^{m-1} \epsilon_2^{m-1}) \partial_\theta.$$  

As in the case for the coordinate vector fields $\partial_\theta, \partial_T$ these coefficients are then governed by the evolution equations:

$$e_0a_1^{m-1} - K_{11}^{m-1} a_1^{m-1} = 0, \quad e_0a_2^{m-1} - K_{22}^{m-1} a_2^{m-1} = 2K_{12}^{m-1} a_1^{m-1},$$

$$e_0a_3^{m-1} - K_{11}^{m-1} a_3^{m-1} = 0, \quad e_0a_4^{m-1} - K_{22}^{m-1} a_4^{m-1} = 2K_{12}^{m-1} a_1^{m-1},$$

(A.98)

where the coefficients $a_1^{m-1}$ can be thought of as functions of $r, t, \theta$ or of $\rho^{m-1}, r, \theta$.

We can solve for $a_1^{m-1}, a_2^{m-1}, a_3^{m-1}, a_4^{m-1}$ after we prescribe suitable initial conditions somewhere. We will solve for $a_1^{m-1}$ backwards from the singularity setting the free branch equal to zero. This condition captures that $\partial_\theta$ is parallel (in an asymptotic sense) to the direction of $\epsilon_2^{m-1}$ at the singularity. As a consequence of the evolution equation, we derive that $a_1^{m-1} = 0$ everywhere. The requirement $\Theta = \theta$ is captured by requiring $\partial_\theta \theta = 1$. Thus recalling (4.99), we have on $\Sigma_{r^{m-1}}$:

$$1 = \epsilon_2^{m-1}(\theta) \cdot c_2^{m-1}(t, \theta),$$

(A.99)

where $c_2^{m-1}(t, \theta) = a_2^{m-1}(r^{m-1}, t, \theta).$ Note that $\epsilon_2^{m-1}(t, \theta)$ on $\Sigma_{r^{m-1}}$ is given in terms of the already solved-for $K_{12}^{m-1}$, and that:

$$\epsilon_2^{m-1}(t, \theta) = \sqrt{1 - \left(\frac{2M}{r} - 1\right)^{-1}[\epsilon_2^{m-1}(r^{m-1})]^2[1 + \partial_r \chi(r)(r^{m-1} - \epsilon)]} \epsilon_2^{m-1}(t, \theta).$$

This then implies bounds on $\epsilon_2^{m-1}(t, \theta)$, and thus on $a_2^{m-1}(r^{m-1}, t, \theta)$ at $\{\rho^{m-1} = \epsilon\}$. (We put these down right below).

So far we have solved for the vector field $\tilde{\Theta}$ which is meant to be the coordinate vector field $\partial_\theta$, once we specify the coordinate function $T = T(t, \theta)$. To obtain this function, we must impose the necessary relation:

$$\tilde{\Theta}(T) = 0,$$

(A.100)

which on $\Sigma_{r^{m-1}}$ translates into:

$$\tilde{\Theta}(T) = 0 \implies \sum_{i=1,2} a_i^{m-1} \epsilon_i^{m-1} [T(t, \theta)] = 0.$$  

(A.101)

We note also that the vector field $\tilde{\Theta}$ (which will equal $\partial_\theta$ in the coordinates system $\{\Theta, T\}$ will equal:

$$\partial_\theta = \tilde{\Theta} = \partial_\theta + \frac{\partial T}{\partial \theta} \partial_T;$$

(A.102)

moreover the vector field $\partial_T$ in the same coordinate system will equal:

$$\partial_T = \frac{\partial T}{\partial t} \partial_t.$$  

(A.103)

Now, the coefficients $a_2^{m-1}$ have already been solved for at this point, and $a_1^{m-1} = 0$; we will see that they are $H^{*-3}$ regular (plus allowing an extra singular weight at the poles). Also, recall that the vector fields $\epsilon_2^{m-1}, \epsilon_1^{m-1}$ are expressible via the values of $K_{12}^{m-1}(t, \theta)$, which has already been solved for, in terms of $\partial_\theta, \partial_T$. This implies that $H^{*-3}$ regularity holds for the function $\frac{\partial T}{\partial \theta}$ (with the some extra singular weights at the poles).
We now impose the initial conditions \( T(t, \theta) = t \) on \( \{ \theta = 0 \} \). Equation (4.101) then be seen as a 1-parameter family of transport equations on \( \{ (t, \theta) \in [0, \pi] \times \mathbb{R} \} \). Coupled with the imposed initial condition, we can obtain a unique solution \( T(t, \theta) \).

We can derive regularity for \( \frac{\partial T}{\partial \theta} \): Refer to (4.100) and take another \( \partial \theta \) derivative. We can then take up to another \( s - 4 \) derivatives in the directions \( \partial \Theta \) or \( \partial r \). We recall that \( a_T^{m-1} \) is parallel to \( \partial \Theta \). Thus the resulting equation yields estimates on up to \( s - 4 \) derivatives of \( \partial \theta \Theta \), provided at least one of them is in the \( \partial \Theta \)

Now, on the initial data surface \( \Sigma_{r=0} \) we recall formulas (3.25) that link \( \partial_t, \partial_\theta \) to \( \tilde{e}_1^{m-1}, \tilde{e}_2^{m-1} \) on this surface. Combined with the above formula, these give explicit formulas for \( a_T^{m-1}, A_T = T, \Theta \) and \( i = 1, 2 \) on the initial data hypersurface \( \Sigma_{r=0} \). These appear at the lower, higher and top orders in the Lemma right below, for \( r = r_{m-1}(t, \theta) \).

Off of the initial data hypersurface the regularity of this solution is obtainable from the transport equation (4.98). We put these down in the next Lemma.

Prior to this, we introduce one piece of notation, which is necessary to single out a special case at the top orders: Let \( \tilde{H}^{\frac{1}{2}} \) stand for the homogenous Sobolev space consisting of all iterated \( \partial T, \partial \Theta \) derivatives, except for the one where all derivatives are taken in the \( \partial r \)-direction:

\[
\| f \|_{\tilde{H}^{\frac{1}{2}}[\mathbb{T}]} = \left( \sum_{l=1}^{l \leq s} \int_{\Sigma_r} \int_{\mathbb{T}^s} \| \partial^l f \|^{2} \sin^l \theta d\theta dt. \right)^{\frac{1}{2}}
\]

**Lemma 4.16.** The coefficients \( a_T^{m-1}, a_\Theta^{m-1}, \tilde{e}_i^{m-1}, \tilde{e}_i^{m-1}, i = 1, 2 \) in the transformations (4.81) that we just constructed have the following regularity properties in the Sobolev spaces \( \tilde{H}^{l} \), defined with respect to the coordinates \( T, \Theta = T^{m-1}, \Theta^{m-1} \):

At the lower orders we claim, for \( l \leq s \),

\[
\| a_T^{m-1}(r, t, \theta) \|_{\tilde{H}^l} \leq Cr^{-\frac{1}{2}-DC_\eta}, \quad \| a_\Theta^{m-1}(r, t, \theta) \|_{\tilde{H}^l} \leq Cr^{-\frac{1}{2}-DC_\eta},
\]

(4.104)

At the higher orders, for \( l+1 \leq s-4 \), \( h = l-1 \),

\[
\| a_T^{m-1}(r, t, \theta) \|_{\tilde{H}^l} \leq Cr^{-\frac{1}{2}-DC_\eta-h}, \quad \| a_\Theta^{m-1}(r, t, \theta) \|_{\tilde{H}^l} \leq Cr^{-\frac{1}{2}-DC_\eta-h},
\]

(4.105)

Finally, at the top orders our claims are as follows:

\[
\| a_T^{m-1}(r, t, \theta) \|_{\tilde{H}^{s-3}} \leq Cr^{-\frac{1}{2}-DC_\eta-c}, \quad \| \partial_\theta a_T^{m-1}(r, t, \theta) \|_{\tilde{H}^{s-4}} \leq Cr^{-\frac{1}{2}-DC_\eta-c-\frac{1}{2}},
\]

(4.106)

\[
\| a_\Theta^{m-1}(r, t, \theta) \|_{\tilde{H}^{s-3}} \leq Cr^{-\frac{1}{2}-DC_\eta-c}, \quad \| \partial_\theta a_\Theta^{m-1}(r, t, \theta) \|_{\tilde{H}^{s-4}} = 0,
\]

\[
\| a_T^{m-1}(r, t, \theta) \|_{\tilde{H}^{s-3}} \leq Cr^{-\frac{1}{2}-DC_\eta-c}, \quad \| \partial_\theta a_T^{m-1}(r, t, \theta) \|_{\tilde{H}^{s-4}} \leq Cr^{-\frac{1}{2}-DC_\eta-c-\frac{1}{2}}.
\]

(We note the second set of estimates at the top order involves an extra weight \( \cot \theta \) which is singular at the two poles.)

**Remark 4.17.** The top order derivatives that are not captured in the above are:

\[
\| \partial^l T \partial^l_\theta a_T^{m-1}(r, t, \theta) \|_{L^2} \| \partial^l T \partial^l_\theta a_T^{m-1}(r, t, \theta) \|_{L^2}
\]

These terms will appear when trying to derive some inductive steps at the top orders below; they will be dealt with by using the special algebraic structure of the terms where they appear, to re-express them in terms of other top-order terms which are bounded.

51
We note that the above implies the following bounds on the components of the metric $g$ with respect to the coordinates $T, \Theta$, via the formulas

$$h^{m-1}_{TT} = \sum_{i=1,2} \left[ a_{T_i}^{m-1} \left( \rho^{m-1}(r^{m-1} - 1) \right)^{-1} \left( 1 + \partial_r \chi(r)(r_* - \epsilon) \right)^{-2} \right],$$

$$h^{m-1}_{mT} = \sum_{i=1,2} \left[ a_{\Theta_i}^{m-1} \left( \rho^{m-1}(r^{m-1} - 1) \right)^{-1} \left( 1 + \partial_r \chi(r)(r_* - \epsilon) \right)^{-2} \right],$$

$$h^{m-1}_{T\Theta} = \sum_{i=1,2} \left[ a_{T_i}^{m-1} \left( \rho^{m-1}(r^{m-1} - 1) \right)^{-1} \left( 1 + \partial_r \chi(r)(r_* - \epsilon) \right)^{-2} \right].$$

At the low orders $l \leq 2$:

$$\|g^{m-1}_{TT}(r, t, \theta)\|_{H^l} \leq C r^{-1-2DC\eta}, \|g^{m-1}_{mT}(r, t, \theta)\|_{H^l} \leq C r^{-1-2DC\eta}, \|g^{m-1}_{T\Theta}(r, t, \theta)\|_{H^l} \leq C r^{-1-2DC\eta}, \|g^{m-1}_{\Theta\Theta}(r, t, \theta)\|_{H^l} \leq C r^{-1-2DC\eta}.$$

At the higher orders, for $l + 1 \leq l \leq s-4$, $h = l - 2$:

$$\|g^{m-1}_{TT}(r, t, \theta)\|_{H^{s-3/2}} \leq C r^{-1-2DC\eta-c}, \|g^{m-1}_{mT}(r, t, \theta)\|_{H^{s-3/2}} \leq C r^{-1-2DC\eta-c}, \|g^{m-1}_{T\Theta}(r, t, \theta)\|_{H^{s-3/2}} \leq C r^{-1-2DC\eta-c}, \|g^{m-1}_{\Theta\Theta}(r, t, \theta)\|_{H^{s-3/2}} \leq C r^{-1-2DC\eta-c}.$$

Proof of Lemma 4.10: Let us first derive the claimed bounds on $\Sigma^{m-1}_r$. We start with the decoupled quantity $a_{T_1}^{m-1}(r, \theta)$. This is defined by (4.99); the expression (3.25) for $a_{T_1}^{m-1}$ in terms of the background coordinates $t, \theta$ together with the bounds on $K^{12}$ implies our claim on $\Sigma^{m-1}_r$ for this parameter.

For the parameters $a_{T_1}^{m-1}, a_{T_2}^{m-1}$ on $\Sigma^{m-1}_r$, we outlined how the estimates in the claimed spaces follow using the estimates we are assuming on $K^{12}$ and $a_{T_2}^{m-1}$ on $\Sigma^{m-1}_r$. We derive the claimed bounds by simply applying the inductive assumptions on these parameters along with the product inequality.

Now, we can obtain our bounds for the parameters $a_{T_1}^{m-1}, a_{T_2}^{m-1}, a_{T_1}^{m-1}$ off of $\Sigma^{m-1}_r$ by using the integral representations

$$a^{m-1}_{T_1}(r, t, \theta) = e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_1}(t, \theta)} e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_1}(t, \theta)},$$

$$a^{m-1}_{T_2}(r, t, \theta) = e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_2}(t, \theta)} e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_2}(t, \theta)},$$

$$a^{m-1}_{T_3}(r, t, \theta) = e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_3}(t, \theta)} e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_3}(t, \theta)},$$

$$a^{m-1}_{T_4}(r, t, \theta) = e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_4}(t, \theta)} e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_4}(t, \theta)},$$

$$a^{m-1}_{T_5}(r, t, \theta) = e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_5}(t, \theta)} e^{\int_{r^{m-1}(t, \theta)}^r a^{m-1}_{T_5}(t, \theta)},$$

Then our claim follows straightforwardly by just differentiating the above equations, and using our assumed bounds on $a^{m-1}$ on $\Sigma^{m-1}_r$ and those on $K^{12}$: The required bounds and regularity for
the coefficients $a_{\alpha i}^{m-1}(r^{m-1}_*(t, \theta), t, \theta)$ on the initial data hypersurface $\Sigma_{r^{m-1}_*(t, \theta)}$ have already been established. Furthermore the functions $K^{m-1}_2(r, t, \theta), K^{m-1}_{12}(r, t, \theta)$ have the required regularity, as part of the inductive assumption, which allows us to derive our claim by differentiating the evolution equations and invoking Lemma 4.11.

We will also need to derive bounds on the metric components $g_{\mu A}^{m-1}, A = T, \Theta$ also. Let us find an expression for these mixed components $g_{\mu T}, g_{\mu \Theta}$.

Using (4.84), we derive:

$$g_{\mu T}^{m-1} = \sum_{i=1, 2} a_{\mu i}^{m-1} \cdot e_i^{m-1}(\rho^{m-1})(\frac{2M}{r} - 1)^{-1}[1 + \partial r\chi(r)(r^{m-1} - e)]^{-1}$$

$$g_{\mu \Theta}^{m-1} = \sum_{i=1, 2} a_{\mu i}^{m-1} \cdot e_i^{m-1}(\rho^{m-1})(\frac{2M}{r} - 1)^{-1}[1 + \partial r\chi(r)(r^{m-1} - e)]^{-1}.$$  \hspace{1cm} (4.112)

Thus the bounds in Lemmas 4.13, 4.16 directly imply the following bounds for $l \leq l_0, J_1 \leq 1$:

$$\|((\partial^J \jmath_1) g_{\mu T}^{m-1})\|_{H^l} \leq B\rho^{\frac{3}{2} - 2DC\eta - J_1}, \|((\partial^J \jmath_1) g_{\mu T}^{m-1})\|_{H^l} \leq B\rho^{\frac{3}{2} - J_1}.$$  \hspace{1cm} (4.113)

For $l \in \{l_0 + 1, s - 4\}$ and $h = l - l_0$:

$$\|((\partial^J \jmath_1) g_{\mu T}^{m-1})\|_{H^l} \leq B\rho^{\frac{3}{2} - 2DC\eta - \frac{3}{4} - J_1}, \|((\partial^J \jmath_1) g_{\mu T}^{m-1})\|_{H^l} \leq B\rho^{\frac{3}{4} - \frac{3}{4} - J_1}.$$  \hspace{1cm} (4.114)

We also note that $(g^{m-1})_{\mu \phi}$ vanishes for $\{|r \leq e/2|\}$

We next derive analogues of these bounds for the components of the inverse of $g^{m-1}$. Note first that the vector fields $e_i^{m-1}$ can also be expressed in terms of the coordinates $\partial r, \partial \theta$ via formulas:

$$e_i^{m-1} = (a^{m-1})^{iT} \partial r + (a^{m-1})^\Theta \partial \theta.$$  \hspace{1cm} (4.115)

The components of the 2x2 matrix $(a^{m-1})^{AB}, i = 1, 2, A = T, \Theta$ are then just the inverse of the matrix $a_{\alpha i}^{m-1}$. In particular we derive the following bounds:

**Lemma 4.18.**

$$\|((a^{m-1})^{iT}(r, t, \theta))\|_{H^l} \leq Cr^{\frac{1}{2} - DC\eta} \|\|((a^{m-1})^\Theta(r, t, \theta))\|_{H^l} \leq Cr^{1 - DC\eta},$$  \hspace{1cm} (4.116)

$$\|((a^{m-1})^{2T}(r, t, \theta))\|_{H^l} = 0, \|((a^{m-1})^{1\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{\frac{3}{4} - DC\eta}, \text{ for } l \leq l_0,$$

At the higher orders, for $l + 1 \leq l_0 \leq s - 4, h = l - l_0$:

$$\|((a^{m-1})^{AB}(r, t, \theta))\|_{H^l} \leq Cr^{\frac{1}{2} - DC\eta - \frac{3}{4}}, \|((a^{m-1})^{2\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{1 - DC\eta - \frac{3}{4}},$$  \hspace{1cm} (4.117)

$$\|((a^{m-1})^{APT}(r, t, \theta))\|_{H^l} = 0, \|((a^{m-1})^{P\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{1 - \frac{3}{4} - DC\eta},$$

Finally, at the top orders our claims are:

$$\|((a^{m-1})^{1T}(r, t, \theta))\|_{H^{s-3}} \leq r^{\frac{1}{2} - DC\eta} \|((a^{m-1})^{2\Theta}(r, t, \theta))\|_{H^{s-3}} \leq Cr^{1 - DC\eta - c},$$

$$\|\partial r((a^{m-1})^{2\Theta}(r, t, \theta) \cdot \cot \theta)\|_{H^{s-3}} \leq Cr^{1 - DC\eta - c} \|((a^{m-1})^{2T}(r, t, \theta))\|_{H^{s-3}} = 0, \|((a^{m-1})^{1\Theta}(r, t, \theta))\|_{H^{s-3}} \leq r^{\frac{1}{2} - DC\eta},$$

This yields some bounds on the components of $(g^{m-1})^{AB}$ (with raised indices) in the components with respect to this system of coordinates, for $l \leq l_0$:

$$\|((g^{m-1})^{TT}(r, t, \theta))\|_{H^l} \leq Cr^{1 - 2DC\eta}, \|((g^{m-1})^{\Theta\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{2 - 2DC\eta}, \|((g^{m-1})^{T\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{\frac{3}{4} - 2DC\eta},$$

while at the higher orders for $l + 1 \leq l_0 \leq s - 4, h = l - l_0$:

$$\|((g^{m-1})^{TT}(r, t, \theta))\|_{H^l} \leq Cr^{1 - 2DC\eta - \frac{3}{4}}, \|((g^{m-1})^{\Theta\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{2 - 2DC\eta - \frac{3}{4}},$$

$$\|((g^{m-1})^{T\Theta}(r, t, \theta))\|_{H^l} \leq Cr^{\frac{3}{4} - 2DC\eta - \frac{3}{4}},$$  \hspace{1cm} (4.118)
and at the top orders when $J_1 \leq 2$:

$$
\| (\partial_\rho )^{J_1} (g^{m-1}) \phi r, t, \theta \|_{H^{s-3}} \leq C r^{-2} D \eta - c - J_1,
\| (\partial_\rho )^{J_1} (g^{m-1}) \phi (r, t, \theta) \|_{H^{s-3}} \leq C E^{-2} D \eta - c - J_1.
$$

We also obtain bounds on the cross metric components:

$$
\| (\partial_\rho )^{J_1} (g^{m-1}) \phi \|_{H^l} \leq B \rho^{-\frac{1}{2} - D \eta - \frac{4}{7} J_1}, \quad \| (\partial_\rho )^{J_1} (g^{m-1}) \phi T \|_{H^l} \leq B \rho^{-\frac{1}{2} - D \eta - \frac{4}{7} J_1}.
$$

(4.119)

(The first term vanishes for $\rho^{m-1} \leq \epsilon/2$). The bounds at higher orders $l \in \{l_0 + 1, \ldots, s - 4\}$ are analogous, letting $h = l - l_0$:

$$
\| (\partial_\rho )^{J_1} (g^{m-1}) \phi \|_{H^l} \leq B \rho^{-\frac{1}{2} - D \eta - \frac{4}{7} J_1 - \frac{4}{2}}, \quad \| (\partial_\rho )^{J_1} (g^{m-1}) \phi T \|_{H^l} \leq B \rho^{-\frac{1}{2} - D \eta - \frac{4}{7} J_1 - \frac{4}{2}}.
$$

(4.120)

We also put down some estimates for the Christoffel symbols $(\Gamma^{m-1})^C_{AB}$, $C = T, \Theta$ which will be useful:

**Lemma 4.19.** At the lower and higher orders our claimed bounds for these Christoffel symbols are:

$$
\| (\Gamma^{m-1})^\Theta_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - \frac{4}{7}}, \quad \| (\Gamma^{m-1})^\Theta_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-3 - 2 CD \eta},
$$

$$
\| (\Gamma^{m-1})^T_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{1/4 - 2 CD \eta - \frac{4}{7}}, \quad \| (\Gamma^{m-1})^T_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - \frac{4}{7}},
$$

(4.121)

for all $l \leq l_0 - 1$.

At the higher orders, the corresponding bounds are as follows, where $l \in \{l_0, \ldots, s - 5\}$:

$$
\| (\Gamma^{m-1})^\Theta_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - \frac{4}{7}}, \quad \| (\Gamma^{m-1})^\Theta_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-3 - 2 CD \eta - \frac{4}{7}},
$$

$$
\| (\Gamma^{m-1})^T_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{1/4 - 2 CD \eta - \frac{4}{7}}, \quad \| (\Gamma^{m-1})^T_{AB} \|_{H^l [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - \frac{4}{7}},
$$

(4.122)

We also have the following extra bounds at the top orders, where $J_1 \leq 2$

$$
\| (\partial_\rho )^{J_1} (\Gamma^{m-1})^\Theta_{AB} \cdot \cot \Theta \|_{H^{s-4} [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - c - \frac{4}{7} J_1},
$$

$$
\| (\partial_\rho )^{J_1} (\Gamma^{m-1})^\Theta_{AB} \cdot \cot \Theta \|_{H^{s-4} [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-3 - 2 CD \eta - c - \frac{4}{7} J_1},
$$

$$
\| (\partial_\rho )^{J_1} (\Gamma^{m-1})^T_{AB} \cdot \cot \Theta \|_{H^{s-4} [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-1 - \frac{4}{7} CD \eta - c - \frac{4}{7} J_1},
$$

$$
\| (\partial_\rho )^{J_1} (\Gamma^{m-1})^T_{AB} \cdot \cot \Theta \|_{H^{s-4} [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-1 - \frac{4}{7} CD \eta - c - \frac{4}{7} J_1},
$$

$$
\| (\partial_\rho )^{J_1} (\Gamma^{m-1})^T_{AB} \cdot \cot \Theta \|_{H^{s-4} [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - c - \frac{4}{7} J_1},
$$

$$
\| (\partial_\rho )^{J_1} (\Gamma^{m-1})^T_{AB} \cdot \cot \Theta \|_{H^{s-4} [\Sigma, \rho, m-1]} \leq C (\rho^{m-1})^{-2 CD \eta - c - \frac{4}{7} J_1}.
$$

We will also need to put down some estimates on the Christoffel symbols $(\Gamma^{m-1})^C_{AB}$, where at least one of the indices $A, B$ (say $A$ wlog) equals $\rho$. Given our bounds on $\partial (g^{m-1}_{AB})$ and $(g^{m-1})^{CD}$ right above, we can obtain the following bounds on these Christoffel symbols:

**Lemma 4.20.**

$$
\| (\Gamma^{m-1})^\rho_{AB} \|_{H^{l_0}} \leq B \rho^{m-1} (\rho^{m-1})^{-\frac{1}{2}}, \quad A = T, \Theta,
$$

$$
\| (\Gamma^{m-1})^T_{AB} \|_{H^{l_0}} \leq B \rho^{m-1} (\rho^{m-1})^{-\frac{1}{2} - \frac{1}{2}}, \quad \| (\Gamma^{m-1})^{\Theta}_{AB} \|_{H^{l_0}} \leq B \rho^{m-1} (\rho^{m-1})^{-\frac{1}{2} - \frac{1}{2}},
$$

(4.123)

$$
\| (\Gamma^{m-1})^\rho_{AB} \|_{H^{l_0}} \leq B \rho^{m-1} (\rho^{m-1})^{-\frac{1}{2} - \frac{1}{2}}, \quad \| (\Gamma^{m-1})^T_{AB} \|_{H^{l_0}} \leq B \rho^{m-1} (\rho^{m-1})^{-\frac{1}{2} - \frac{1}{2}}.
$$
The analogues of these estimates at the higher derivatives are as follows, for \( h = l - \text{low}, \ l \in \{\text{low} + 1, \ldots, s - 4\} \):

\[
\begin{align*}
\| (\Gamma^{m-1}_\rho A)_H^l \|_{H^1} & \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}}, A = T, \Theta, \\
\| (\Gamma^{m-1}_\rho T)_H^l \|_{H^1} & \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3}, \| (\Gamma^{m-1}_\rho \Theta)_H^l \|_{H^1} \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}}, \\
\| (\Gamma^{m-1}_\rho \rho)_H^l \|_{H^1} & \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3}, \| (\Gamma^{m-1}_\rho \rho \Theta)_H^l \|_{H^1} \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3}, \\
\| (\Gamma^{m-1}_\rho \rho \Theta \rho)_H^l \|_{H^1} & \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3}.
\end{align*}
\] (4.124)

At the top derivatives, our bounds are:

\[
\begin{align*}
\| \partial_\rho (\Gamma^{m-1}_\rho \rho \Theta) \cdot \cot \Theta \|_{H^1} & \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3}, \| (\Gamma^{m-1}_\rho \rho \Theta \rho \Theta) \|_{H^1} \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3} - c, \\
\| (\Gamma^{m-1}_\rho \rho \Theta \rho \rho \Theta) \|_{H^1} \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3} - c, \| (\Gamma^{m-1}_\rho \rho \Theta \rho \rho \rho \Theta) \|_{H^1} \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3} - c, \\
\| (\Gamma^{m-1}_\rho \rho \Theta \rho \rho \rho \rho \Theta) \|_{H^1} \leq BC\eta(\rho^{m-1})^{-\frac{1}{3}} - \frac{1}{3} - c.
\end{align*}
\] (4.125)

Proof of Lemmas 4.12, 4.20: The proof is straightforward, from the definition of the Christoffel symbols, the bounds obtained for \( g_{\alpha \beta}^{m-1} \) and \( (g^{m-1})_{\alpha \beta} \) directly above, as well as the product inequality.

For the remaining Christoffel symbols as they will appear (combined) in the wave equation,

\[
\| \sum_{A, B = T, \Theta, \phi} (g^{m-1})_{\alpha \beta} \Gamma_{\alpha \beta}^A \|_{H^1}, A, B = T, \Theta, \phi
\]

we make the following connections to the (frame) connection coefficients \( K_{\alpha \beta}^{m-1} \), using the definitions of the frame coefficients and the definition of the Christoffel symbol (For the terms after the top order in the RHS, we use generic notation).

\[
\begin{align*}
& - \sum_{A, B = T, \Theta, \phi} (g^{m-1})_{\alpha \beta} \Gamma_{\alpha \beta}^A \partial_\rho v = \text{tr}_{\rho m-1} K_{\alpha \beta}^{m-1} \cdot e_{0} v + O(e^{m-1}(\rho^{m-1})) \cdot K_{\alpha \beta}^{m-1} \cdot e_{0} v
\end{align*}
\] (4.126)

We note here again that \( e_{\rho m-1}(\rho) = 0 \) for \( \rho \leq \epsilon/2 \); also \( e_{\rho m-1}(\rho) \) satisfies the bounds in Lemma 4.13 and notably in the \( L^\infty \) norm the coefficient \( O(e^{m-1}(\rho^{m-1})) \) is bounded by \( \rho^{\frac{1}{2}} \cdot DC\eta \); the higher derivatives are bounded by the corresponding bounds for \( e^{m-1}(\rho^{m-1}) \). In particular the second term in the RHS is less singular near the singularity. (The tangency of \( e_{\rho m-1}^{m-1} \) to the level sets of \( \rho \) is crucial here; had this choice not been made, the second term would have been singular than the first term).

### 4.8.3 Consequences of the energy estimates on the free wave \( \gamma^m \).

We will also make frequent use below of certain basic implications of our inductive claim. One key such estimate encodes the basic implication of the bounds on \( (a^{m-1})_{\alpha \beta}^{\mu}(r, t, \theta), (a^{m-1})_{\alpha \beta}^{\mu}(r, t, \theta) \); it shows that the behaviour of the parameters \( e_{\rho m-1}(\rho^{m-1}) \cdot \text{tr}_{\gamma m-1} \) is better than what the energy bounds [4.22], [4.23], [4.24] imply, at all orders below the top order.

We first prove the following:

**Lemma 4.21.** For all \( |I| \leq s - 3 \) the quantity

\[
\| \partial_{\rho m-1} \gamma^m_{\alpha \beta} \|_{H^k(S_{\rho m-1})}
\]

satisfies the estimate

\[
\begin{align*}
(\rho^{m-1})^{\frac{1}{2}} \| \gamma^m_{\alpha \beta} \|_{H^k(S_{\rho m-1})} & \leq \epsilon^{\frac{1}{2}} \| \gamma^m_{\alpha \beta}(\epsilon, t, \theta) \|_{H^k(S_{\rho m-1})} + \int_{\rho m-1} C \gamma^m_{\alpha \beta} \| e_{\rho m-1} \|_{H^k(S_{\rho m-1})} d\tau,
\end{align*}
\] (4.127)

**Proof:**

\[
\begin{align*}
- \frac{1}{2} \partial_{\rho m-1} \| \gamma^m \|_{H^1}^2 & \leq \| \gamma^m \|_{H^1} \| \partial_{\rho m-1} \gamma^m \|_{H^1}
\end{align*}
\]
(by C-S)
Moreover, the same bounds hold on the level sets of $\rho^{m-1}$.

Remark 4.24. This from we derive:

Lemma 4.22. The lower order energy estimates \((4.22), \ (4.44)\) as well as Lemma 4.18 imply the bounds:
\[
\|e_0 \partial^I \gamma_{\text{rest}}^m\|_{L^\infty} \leq C \cdot C \eta^{r-2} \cdot \|\partial^I \gamma_{\text{rest}}^m\|_{L^\infty} \leq 2C \cdot C \eta^{r-1-DC\eta} \log r, \tag{4.128}
\]
for all $r \in (0, 2e]$, provided $s \geq 3 + 4c + |I| + 3$.

Proof. The bounds on $e_0 \partial^I \gamma_{\text{rest}}^m$ follow by applying \((4.59)\) to $e_0 \partial^I \gamma_{\text{rest}}^m$, see \((4.27)\), and using the assumption \((4.27)\), using \((4.7)\). For the spatial gradient of $\partial^I \gamma_{\text{rest}}^m$ we have $|\nabla \partial^I \gamma_{\text{rest}}^m| \leq |e_0^{m-1} \partial^I \gamma_{\text{rest}}^m| + |e_0^{m-1} \partial^I \gamma_{\text{rest}}^m|$, where
\[
|e_0^{m-1} \partial^I \gamma_{\text{rest}}^m| \leq |(m-1) e_0 \partial^I \gamma_{\text{rest}}^m| + |(e_0^{m-1} \partial^I \gamma_{\text{rest}}^m)| = |e_0^{m-1} \partial^I \gamma_{\text{rest}}^m| \leq |e_0^{m-1} \partial^I \gamma_{\text{rest}}^m| + |e_0^{m-1} \partial^I \gamma_{\text{rest}}^m|
\]
for all $r \in (0, 2e]$.

An extension of the above estimate on the spatial components of the energy of $\gamma_{\text{rest}}^m$ is as follows:

Lemma 4.23. Assume the energy estimates \((4.22), (4.23), (4.24)\), for the $m^{th}$ step of our induction (i.e. for the function $\gamma^m$), up to order $|I| = l + 1$, as that Lemma 4.18 also holds. The coordinate derivatives $\partial^I$ of order $|I| = l$ will be bounded as follows:
\[
|\partial^I \gamma_{\text{rest}}^m| \leq C \eta^{r-2} \cdot \|\partial^I \gamma_{\text{rest}}^m\|_{L^\infty} \leq C \cdot C \eta^{r-1} \cdot \|\partial^I \gamma_{\text{rest}}^m\|_{L^\infty} \leq 2C \cdot C \eta^{r-1-DC\eta} \log r, \tag{4.129}
\]
for all $r \in (0, 2e]$. On the other hand, the parameters $e_0^{m-1} \partial^I \gamma_{\text{rest}}^m$, $e_0^{m-1} \partial^I \gamma_{\text{rest}}^m$ satisfy the following bounds, for $b = 1, 2$:
\[
|e_0^{m-1} \partial^I \gamma_{\text{rest}}^m| \leq C \cdot C \cdot \eta^{r-2} \cdot \|\partial^I \gamma_{\text{rest}}^m\|_{L^\infty} \leq C \cdot C \cdot \eta^{r-1} \cdot \|\partial^I \gamma_{\text{rest}}^m\|_{L^\infty} \leq 2C \cdot C \cdot \eta^{r-1-DC\eta} \log r, \tag{4.133}
\]
Moreover, the same bounds hold on the level sets of $\rho^{m-1}$.

Proof. For the first claim, the proof is immediate, using the inductive assumption on the energy of $\gamma_{\text{rest}}^m$ (whichever is relevant for our order), as well as \((4.127)\).

We can then derive \((4.130)\), \((4.131)\): We use formulas \((4.81), (4.71)\) Lemma 4.13 to express the LHSs of \((4.130), (4.131)\) in terms of $\partial^I \gamma_{\text{rest}}^m$, $|I| = |I| + 1$:
\[
e_0^{m-1} \partial^I \gamma_{\text{rest}}^m = \sum_{A = T, \theta} (a^{m-1} b A \partial_A \partial^I \gamma_{\text{rest}}^m) \tag{4.132}
\]
in view of the bounds in Lemmas 4.13 and 4.18 our claim follows, on the level sets of $\rho^{m-1}$. This the claim \((4.129)\) also follows by directly involving the assumed $L^2$ bounds of $e_0 \partial^I \gamma_{\text{rest}}^m$ in view of the energy bounds on $\partial^I \gamma_{\text{rest}}^m$ that we are assuming.

The claim on level sets of $r$ also follows, as will be explained at the end of this section.

Remark 4.24. The proof of \((4.130), (4.131)\) is the realization of the descent scheme mentioned in the introduction. We note it can be applied to all orders below the top ones. In particular (as we will see) it implies that at the orders below top, the spatial derivatives $e_0^{m-1} \partial^I \gamma_{\text{rest}}^m$, $b = 1, 2$ of $\gamma_{\text{rest}}^m$ that appear in the RHSs of the Riccati equations are less singular than the derivatives $e_0 \gamma_{\text{rest}}^m$, and the main terms that contribute to the asymptotics of the connection coefficients $K_{ij}$ are the $e_0$-derivatives. This is a manifestation of the AVTD behaviour of the fields.
Following the control on the spatial part of the metric \( h^{m-1} \) and \( g^{m-1} \) and the refined control of \( \gamma^m \) using the AVTD behaviour of the solution, we are ready to derive the next step of our inductive claims. One final note prior to doing this: Our inductive claims for the derivatives of the key parameters in the REVESNNG system were with respect to the vector fields \( \partial_t, \partial_\theta \). However in view of the construction of the new coordinate \( T(t, \theta) \) in this section, we see that it suffices to derive our inductive claims on the derivatives \( \partial^k_{t, \theta} \) of our parameters instead of the derivatives \( \partial^k_{\phi} \).

5 The estimates for the next iterate: The free wave \( \gamma^m \).

Here we study the free wave equation:

\[
\Box_{g^{m-1}}(\gamma^m) = 0, \quad (5.1)
\]

which holds on the entire region \([r \in (0, 2\epsilon)] \times S^2 \times \mathbb{R}\).

We recall that the prescribed initial data for this equation are required to live on a hypersurface \( \Sigma_{r^{m-1}} \) given graphically by:

\[
\Sigma_{r^{m-1}} := \{ r = r^{m-1}(t, \theta) \}.
\]

The function \( r^{m-1}(t, \theta) \) is here assumed to satisfy the inductive assumption on regularity given by \( 4.31 \), \( 4.33 \), \( 4.37 \), \( 4.38 \).

We then proceed to solve this equation in the entire region \( r \in (0, 2\epsilon) \times S^2 \times \mathbb{R} \) using energy estimates. As noted, we will not be controlling \( \gamma^m \) directly but instead \( \gamma^m_{\max} = (\gamma^m - \gamma^3) \).

To do this, we recall the function \( \rho^{m-1} \); our energy estimates will live over level sets of this function. We also recall that it suffices to use the operators \( \partial^k_{\phi} \) instead of \( \partial^k_{t, \theta} \) to derive our claimed estimates on the next iterate \( \gamma^m \). We will be employing these operators, since their commutation properties with the wave operator are more favorable.

Now, to derive the inductive step we will be considering the wave equation commuted with coordinate and frame vector fields, considering the equations:

\[
\Box_{g^{m-1}} \partial^k_{t,\ldots,\theta} e_0 J_0 (\gamma^m - \gamma^S) = [\Box_{g^{m-1}}, \partial^k_{t,\ldots,\theta} e_0 J_0 ] (\gamma^m - \gamma^S) - \partial^k_{t,\ldots,\theta} \partial^k_{t,\ldots,\theta} e_0 J_0 \Box_{g^{m-1}} (\gamma^S), \quad (5.2)
\]

here \( k_1 + k_2 \leq s - 3, \ J_0 \leq 2; \) in fact \( J_0 = 0 \) unless \( k_1 + k_2 = s - 3 \).

In particular, we will be deriving energy estimates for the equation:

\[
\Box_{g^{m-1}} v = G(r, t, \theta), \quad (5.3)
\]

and then replacing the RHS from \( 5.2 \). We distinguish three cases which are treated separately:

1. The first case is where \( k = k_1 + k_2 \leq \text{low} \). In this case, the inductive claim \( 4.22 \) must confirm (for the value of \( m \) of the index) the optimal behaviour for \( \gamma^m_{\max} \); in fact the claim \( 4.25 \) is yet more refined (also optimal) information. The second case is when \( \text{low} + 1 \leq k_1 + k_2 \leq s - 4 \). The last one is \( k_1 + k_2 = s - 3 \) and \( J_0 = 0, 1, 2 \). At orders higher than low the bounds claimed are non-optimal.

These claims are asserted on level sets of the function \( \rho^{m-1} \) and also on level sets of \( r \). We derive the claims on the level sets of \( \rho^{m-1} \) and then discuss at the end of this section how to extend these to the level sets of \( r \).

It is useful to put down some facts about the intrinsic and extrinsic geometry of these level sets in the metric \( g^{m-1} \). We do this right below:

5.0.1 Some estimates on the geometry of the level sets of \( \rho^{m-1} \).

The lapse \( \Phi^{m-1} \) \( 4.16 \) behaves like

\[
\Phi^{m-1} = \frac{1}{\sqrt{(e_0 \rho^{m-1})^2 - (e_1 \rho^{m-1})^2 - (e_2 \rho^{m-1})^2}} (2M(r) - 1)^{-\frac{1}{2}} (1 + Br^{\frac{1}{2}}). \quad (5.4)
\]

We determine the behaviour of the volume form \( \text{vol}_{\Sigma_{r^{m-1}}} \) by integrating the mean curvature \( 4.52 \) (using the first variation of area formula):

\[
\text{vol}_{\Sigma_{r^{m-1}}} \sim e^{-\int \varrho \gamma^m} e^{\int^{t^*} (2M(r) - 1)^{-\frac{1}{2}} (k_1 + k_2) dr} \text{vol}_{Euc} \sim (2M)^{-\frac{1}{2}} r^2 \text{vol}_{Euc}, \quad \text{vol}_{Euc} = \sin \theta d\theta d\phi. \quad (5.5)
\]
5.0.2 General framework for the energy estimates: The weighted multiplier.

We will use $r$-weighted $e_0$ multipliers to obtain energy estimates for the free wave $\gamma^m$ (which solves (5.1)).

For now, for any function $v(r,t,\theta,\phi)$, with $\partial_r v = 0$, define the weighted $e_0$-current:

$$J_a = Q_{ab}[v](e_0)^b f(r) = f(r)(e_0)^b(\partial_a v \partial_b v) - \frac{1}{2} g^{m-1}_{ab} \partial^d v \partial_d v. \tag{5.6}$$

Note that $Q_{00}[v] = \frac{1}{2} (e_0 v)^2 + |\nabla^{m-1} v|^2$. The divergence of (5.6) (with respect to the $(3+1)$-dimensional metric $g^{1+1}$) reads:

$$m^{-1} \nabla^a J_a = m^{-1} \nabla^a [Q_{ab}[v](e_0)^b f(r)]$$

$$= - e_0 f(r) Q_{00}[v] + f(r) (K^{1+1})^{ab} \partial_b v \partial_a v - \frac{1}{2} g^{m-1}_{ab} \partial^d v \partial_d v + f(r) e_0 v \Box_{m-1} g v$$

$$= - e_0 f(r) \left[ \frac{1}{2} (e_0 v)^2 + |\nabla v|^2 \right] - \frac{1}{2} \text{tr}_{m-1} g (K^{m-1}) f(r) |\nabla v|_{m-1}^2$$

$$+ f(r) (K^{1+1}_{m-1} (e_1 v)^2 + K^{m-1}_{12} (e_2 v)^2 + K^{m-1}_{12} e_1 v e_2 v) + f(r) e_0 v \Box_{m-1} g v$$

$$= - \frac{1}{2} \text{tr}_{m-1} g m^{-1} f(r) |\nabla v|_{m-1}^2 - \left[ \frac{3 \sqrt{2} M}{r} \right] f(r) + e_0 f(r) \left[ \frac{1}{2} (e_0 v)^2 \right]$$

$$+ \left[ \frac{3 \sqrt{2} M}{r} \right] f(r) - e_0 f(r) \left[ \frac{1}{2} \nabla v|_{m-1}^2 + \left( \frac{d_{m-1}^{1+1} \sqrt{2} M}{r^2} + u_{m-1}^{1+1} \right) f(r) (e_1 v)^2 \right]$$

$$+ \left( \frac{d_{m-1}^{1+1} \sqrt{2} M}{r^2} + u_{m-1}^{1+1} \right) f(r) (e_2 v)^2 + f(r) u_{m-1}^{1+1} e_1 v e_2 v + f(r) e_0 v \Box_{m-1} g v. \tag{5.7}$$

Note that we have made use of the asymptotically CMC property; indeed $\text{tr}_{m-1} g (K^{m-1})$ to leading order contributes the (constant in $t, \theta$) factor $\frac{3 \sqrt{2} M}{r^2}$ above.

We integrate the above over the domain defined by $\{ t \in (-\infty, +\infty), \theta \in (0, \pi), \phi \in [0, 2\pi) \}$ and two level sets of $\rho^{m-1}$: one is denoted merely by $\Sigma_{\rho^{m-1}}$ and the value of $\rho^{m-1}$ can be arbitrarily small, and the other is $\{ \rho^{m-1} = \epsilon \}$. (Recall that $\{ \rho^{m-1} = \epsilon \}$ is the hypersurface $\{ r = r^{m-1}_{\epsilon} \} = \Sigma_{\rho^{m-1}}$ on which the initial data live.

Integrating (5.7) over this domain, and employing Stokes’ theorem, we write the LHS of (5.7) as a boundary integral to obtain the energy identity:

$$\int_{\Sigma^{m-1}_{\rho^{m-1}}} f(r^{m-1} \rho) Q_{ab}[v](e_0)^b n^a \mathrm{vol}_{\Sigma^{m-1}_{\rho^{m-1}}} - \int_{\Sigma^{m-1}_{\rho^{m-1}}} f(m^{-1} r^{*}) Q_{ab}[v](e_0)^b n^a \mathrm{vol}_{\Sigma^{m-1}_{\rho^{m-1}}} = 0 \tag{5.8}$$

where we recall that by virtue of our inductive assumption $|v_{ij}^{m-1}|_{L^\infty} \leq B r^{-1/4}$ and $e_0 \rho^{m-1} = -(\frac{2}{r} - 1) \frac{1}{2} [1 + \partial_r \chi(r) (m^{-1} r_{*} - \epsilon)]$. On the other hand, by (4.15) and (4.74) we find:

$$Q_{ab}[v](e_0)^b n^a = \frac{1}{2} (e_0 v)^2 + |\nabla v|^2_{m-1}^2 - \frac{m^{-1} e_0 v e_1 v e_2 v}{e_0 v e_1 v + e_0 v e_2 v}$$

$$2 \sqrt{1 - \left( \frac{m^{1+1} e_0 v e_1 v e_2 v}{e_0 v e_1 v + e_0 v e_2 v} \right)^2} \leq \frac{1}{4} (1 \pm O(m^{-1})) |(e_0 v)^2 + |\nabla v|^2|_{m-1}^2. \tag{5.9}$$

where $|\nabla v|^2_{m-1} = |e_1^{-1} v|^2 + |e_2^{-1} v|^2$. Here the term $O(m^{-1})$ satisfies the bound:

$$|O(m^{-1})| \leq C \eta \rho^{m-1} \tag{5.10}$$

We here recall the bound (4.9) which implies
was called "approximate monotonicity".

The reason for the factor $\text{in terms of deriving our desired estimates.}

of $W(r) \sqrt{\tau} dr$ is finite, then we can utilize the Gronwall lemma to obtain a uniform energy estimate for $v$ over the hypersurfaces $\Sigma_{m-1}^\rho$, $\rho^{m-1} \in (0,2e^{\frac{\rho}{\tau^2}}]$. 

With this strategy in mind, we consider the coefficients of the terms in the RHS of (5.8), and observe that the $W(r)$ that naturally arises is of the form $r^{-\frac{2}{3}}$. This is precisely at the borderline where we do not obtain a finite integral $\int_0^w |W(r)\sqrt{\tau} dr$. In fact, the most dangerous integrand in the RHS of (5.8) is $\Phi^{m-1} \frac{\sqrt{\tau}}{r^2} f(\tau) \frac{1}{2} (e_0 v)^2$, which corresponds to the leading order behaviour of $\text{tr}_{m-1} \gamma^{m-1}$, see (4.32).

It is the asymptotically CMC property of our geodesic parameter that yields the constant multiple of $r^{-\frac{2}{3}}$. Thus $f(\tau)$ must be chosen to cancel this particular term out; we must then choose $f(\tau) = \tau^\frac{2}{3}$. As we shall see below, the latter weight choice is also exactly consistent with the logarithmic behaviour of $v = \partial^J \gamma^m$. At $r = 0$ that we will derive for the lower derivatives.

Having chosen $f(\tau) = \tau^\frac{2}{3}$ to cancel out the most dangerous term in (5.8), we must consider the terms

\[
\begin{align*}
- & \left[ \frac{3}{2} \frac{\sqrt{M}}{\tau^2} f(\tau) - e_0 f(\tau) \right] \frac{1}{2} \nabla v |_{m-1}^2 
- (d_1^{m-1}(t,\theta) \sqrt{2M} \frac{1}{\tau^2}) f(\tau)(e_1 v)^2 

\end{align*}
\]

in the bulk estimate. We note that in principle these are bounded by an expression

\[
\int_\rho^w \frac{\text{Const}}{s} [f(\tau) Q_{ab} (v) (e_0 b, n^a) \text{vol}_{\Sigma^s} ds.
\]

In principle this term would again spell trouble, since the integrating factor would be $e_0^\frac{1}{2} ds$ which is not uniformly bounded as $\rho \to 0$. However, in view of the expressions for $d_1^{m-1}(t,\theta) \sim -\frac{1}{4}, d_2^{m-1}(t,\theta) \sim 1$ and the bounds on $|d_1^{m-1}(t,\theta) + \frac{1}{4} |, |d_2^{m-1}(t,\theta) - 1 | < \frac{1}{3}$ the sign of $\text{Const}$ will in fact be negative, thus favorable for us.\footnote{The reason for the factor $\sqrt{\tau}$ in $\int_0^w |W(r)\sqrt{\tau} dr$ is the form (5.3) of the lapse function.} The remaining bulk terms from (5.8) can be bounded by the energy we are controlling times a coefficient that is integrable in $r$, and are thus not dangerous in terms of deriving our desired estimates.

Thus throughout our analysis of equation $\Box^{m-1} v = F$, $v = \partial^{j} (e_0 b^j) \gamma^m$ for all orders $|I|$ the multiplier that we choose in forming the energy current $J$ will be $\tau^{3/2} e_0$. The resulting identity is then:

\[
\int_{\Sigma_{\rho}^{m-1}} f(r^{m-1} \rho) Q_{ab}(v)(e_0 b, n^a) \text{vol}_{\Sigma_{\rho}^{m-1}} - \int_{\Sigma_{r}^{m-1}} f(r^{m-1} r^{\ast}) Q_{ab}(v)(e_0 b, n^a) \text{vol}_{\Sigma_{r}^{m-1}} \]

\[
\leq \int_{\rho}^{w} \int_{\Sigma_{r}} \Phi^{m-1} \left[ \frac{1}{2} (\text{tr}_{m-1} \gamma^{m-1} + O(\tau^{-1\frac{1}{2}})) f(\tau) (e_0 v)^2 + (e_1^{m-1} v)^2 \right] \text{vol}_{\Sigma_{r}} ds.
\]

This analysis of the free wave equation will guide us in deriving the claimed estimates (1.22) at the lowest orders. We also need to understand the commutation of our equations with derivatives $\partial_{\tau}^{k_1} \partial_{\theta}^{k_2}$, $2k_1 + 2k_2 = k \leq s - 3$. In preparation of performing this commutation, we introduce some language and notational conventions to describe the terms generated by these commutations:

\subsection{Language conventions.}

Motivated by the $r$-weighted multiplier estimates we discussed for the free wave equation, we consider the broad class of estimates that we will be deriving below for the various derivatives of $\gamma^m_{\text{rest}}$.

Consider any inequality of the form (5.14), for all $r \leq \epsilon$, where $w$ is a constant (which will vary depending on the context below):

\footnote{In [59] certain analogous borderline terms appeared with a favourable sign; the definite, favourable sign of such terms was called "approximate monotonicity."}
\[
\sum_{|l| \leq b} r^w E[|\partial^l \gamma_{\text{rest}}^m|(r)] \leq \sum_{|l| \leq b} \epsilon^w E[|\partial^l \gamma_{\text{rest}}^m|(e)] + \cdots + \int_r^\epsilon Q_1(\tau) \cdot \sqrt{\tau^w E[|\partial^l \gamma_{\text{rest}}^m|](\tau)} d\tau \\
+ \int_r^\epsilon Q_2(\tau) \cdot \tau^w E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau.
\]

(5.14)

Here \(Q_1(\tau), Q_2(\tau)\) will be fixed functions, which depend on the context below. We recall the estimates in Lemma 4.3. We then call terms of the form

\[
\int_r^\epsilon Q_1(\tau) \cdot \sqrt{\tau} E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau, \int_r^\epsilon Q_2(\tau) \cdot \tau^w E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau
\]

below borderline if we have apriori bounds on \(Q_1(\tau), Q_2(\tau)\) which yield:

\[
\int_r^\epsilon |Q_1(\tau)| d\tau, \int_r^\epsilon |Q_2(\tau)| d\tau < \infty.
\]

(In fact we will always have uniform bounds on the RHSs, depending on \(B, \epsilon\) this is not important for this discussion now).

If we only have bounds \(Q_1(\tau) \lesssim \tau^{-1}\) or \(Q_2(\tau) \lesssim \tau^{-1}\) we call such terms borderline. In particular for such terms the Gronwall inequality in Lemma 4.8 does not yield finite energy bounds as \(r \to 0^+\).

We extend this notion to estimates of the form:

\[
\sum_{|l| \leq b} r^w E[|\partial^l \gamma_{\text{rest}}^m|(r)] \leq \sum_{|l| \leq b} \epsilon^w E[|\partial^l \gamma_{\text{rest}}^m|(e)] \\
+ \cdots + \int_r^\epsilon | \int_{\Sigma_\gamma} F^{l, \text{bd}}(\tau, t, \theta) \cdot \tau^{w/2} E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau | + \int_r^\epsilon \int_{\Sigma_\gamma} F^{q, \text{bd}}(\tau) \cdot \tau^w E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau |
\]

(5.15)

where \(\mathbf{e}\) stands for one of the vector fields \(e_0, e_1, e_2\).

In this setting, we will be assuming that the functions \(F^{l, \text{bd}}\) satisfy bounds in \(L^2_{L, \theta}[\tau]\) for all \(\tau \in (0, 2\epsilon)\); “assuming” here will mean either because of the inductive assumptions we are making on the parameters from step \(m - 1\), or from bounds on derivatives of \(\gamma_{\text{rest}}^m\) at orders below \(|l|\), as well as straightforward combinations of such estimates for products of such factors. (By straightforward here we mean estimates that can be obtained by applying Cauchy-Schwarz, the product and Sobolev inequalities). If from either direct invocations of our inductive bounds of straightforward combinations of such bounds we obtain bounds of the form:

\[
\sqrt{\int_{\Sigma_\tau} |F^{l, \text{bd}}(\tau, t, \theta)|^2 d\text{vol}_{\text{Eucl}}} \leq B^2 C^2 \tau^{-1+\delta}
\]

for some \(\delta > 0\) we call the expression

\[
\int_r^\epsilon | \int_{\Sigma_\gamma} F^{l, \text{bd}}(\tau, t, \theta) \cdot \tau^{w/2} E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau |
\]

below borderline. Observe that such functions can be bounded by a term of the form

\[
\int_r^\epsilon Q_1(\tau) \cdot \sqrt{\tau} E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau
\]

which are below borderline in the language of the previous definition. If we can only obtain bounds of the form

\[
\sqrt{\int_{\Sigma_\tau} |F^{l, \text{bd}}(\tau, t, \theta)|^2 d\text{vol}_{\text{Eucl}}} \leq D \tau^{-1}
\]

for some constant \(D > 0\) then we call the expression

\[
\int_r^\epsilon | \int_{\Sigma_\gamma} F^{l, \text{bd}}(\tau, t, \theta) \cdot \tau^{w/2} E[|\partial^l \gamma_{\text{rest}}^m|](\tau) d\tau |
\]

borderline. We use similar language conventions for the terms:
\[ \int_{r} F^{0}(\tau, t, \theta) \cdot \tau^{w} e(\partial^{l}\gamma_{\text{rest}}^{m}(\tau)) \cdot E[\partial^{l}\gamma_{\text{rest}}^{m}(\tau)] d\tau \]

In this case, we will be assuming that \( F^{0}(\tau, t, \theta) \) depends on parameters of the step \( m-1 \) or on lower derivatives of \( \gamma_{\text{rest}}^{m} \). Moreover we will be assuming that we have \( L^{\infty}_{\rho}(\tau) \) bounds on these parameters on each \( \Sigma_{\tau} \). If these bounds then yield bounds on \( \| F^{0}(\tau, t, \theta) \|_{L^{\infty}_{\rho}} \) of the form:

\[ |F^{0}(\tau, t, \theta)| \leq B^{3} \tau^{-1 + \delta} \]

for all \( \tau \in (0, 2\epsilon) \) and for some fixed \( \delta > 0 \) then the term

\[ \int_{r}^{\epsilon} F^{0}(\tau, t, \theta) \cdot \tau^{w} E[\partial^{l}\gamma_{\text{rest}}^{m}(\tau)] d\tau \]

is called below borderline. If these bounds yield bounds on \( \| F^{0}(\tau, t, \theta) \|_{L^{\infty}_{\rho}} \) of the form:

\[ |F^{0}(\tau, t, \theta)| \leq D \tau^{-1} \]

for all \( \tau \in (0, 2\epsilon) \), for some \( D > 0 \), then the term is called borderline.

These notions will be useful in deriving our energy estimates for \( \gamma_{\text{rest}}^{m} \) in the rest of this section.

Remark 5.1. We note that the bulk of our analysis will be in deriving the desired estimates from the initial data set \( \Sigma_{\rho}^{m-1} = \{ \rho^{m-1} = \epsilon \} \) towards the singularity at \( \{ r = 0 \} = \{ \rho^{m-1} = 0 \} \). The estimates for the remaining region \( \rho^{m-1} \in (\epsilon, 2\epsilon) \) are just easier versions of these estimates and we do not write them out explicitly.

5.1 The wave equation expanded: An inhomogeneous equation for \( \gamma_{\text{rest}}^{m} \).

Our claims concern the function \( \gamma_{\text{rest}}^{m} = \gamma^{m} - \gamma^{S} \) instead of the free wave \( \gamma^{m} \) which satisfies (5.1). Thus we study the inhomogeneous wave equation:

\[ \Box_{\rho^{m-1}} \gamma_{\text{rest}}^{m} = -\Box_{\rho^{m-1}} \gamma^{S}. \] (5.16)

It is useful to express the wave operator with respect to the coordinates \( \{ \rho^{m-1}, t, \theta, \phi \} \). There are two “main parts” of the wave operator—the most important involve the more singular derivatives in the \( e_{0} \) (parallel to \( \partial_{\rho^{m-1}} \)) direction. There are also the terms in the spatial directions \( \partial_{T}, \partial_{\theta} \). These appear in the last line of the next equation. There are some cross terms, due to the fact that \( \partial_{\rho^{m-1}} \) is not normal to \( \partial_{\theta} \) and \( \partial_{\phi} \) in the region \( \rho^{m-1} \in [\epsilon/2, 3\epsilon/2] \). The origin of these terms is manifested in the transition between the vector fields \( e_{i}^{m-1}, i = 1, 2 \) and the vector fields \( \tau^{i}^{m-1} \) via formulas (5.7). (The latter vector fields are tangent to the level sets of \( \rho^{m-1} \), recall.) We then have, for any function \( v(t, \theta, \rho^{m-1}) \):

\[ \Box_{\rho^{m-1}} v = \left[ \left( \frac{2M}{r} - 1 \right) [1 + \partial_{r} \chi(r)] (m^{-1} r_{*} - \epsilon) \right] \partial_{\rho^{m-1}} v + \left[ (M - 1) \right] \partial_{\rho^{m-1}} v + O(\epsilon^{m-1}(\rho^{m-1})) \cdot K^{m-1} \left( \frac{2M}{r} \right) \partial_{\rho^{m-1}} v + \sum_{A = T, \Theta} (g^{m-1})^{A\rho} \partial_{A} v - 2 \sum_{A = \rho, \phi, \theta} (g^{m-1})^{\rho A} (m^{-1} \Gamma)^{\rho A} \partial_{A} v \]

\[ - \sum_{B = \phi, T, \Theta} (m^{-1} g)^{A B} (m^{-1} \Gamma)^{A B} \partial_{B} v - \sum_{B = \phi, T, \Theta} (m^{-1} g)^{A B} (m^{-1} \Gamma)^{A B} \partial_{B} v \]

Here \( (m^{-1} \Gamma)^{A B} \) are the Christoffel symbols of \( g^{m-1} \) with respect to the system of coordinates \( \{ \rho^{m-1}, T, \Theta, \phi \} \).

Let us make a comment here on the importance of choosing \( e^{m-1}_{2} \) being “tangent” to the singularity, which implies that \( e^{m-1}_{2}(r) \) and \( e^{m-1}_{2}(\rho^{m-1}) \) vanish for \( r \leq \epsilon/2 \):

61
Remark 5.2. Had we not made the choice $e_2^{m-1}(r) = o(r^{-\frac{1}{2} + d_2^{m-1}(t, \theta)})$, we would have had $e_2^{m-1}(r) = O(r^{-\frac{1}{2} + d_2^{m-1}(t, \theta)})$. In that case, the terms in the third line of the above would have been more singular than the terms in the second line. In particular we would not have been able to derive our claimed bounds for $\gamma^m$ that we are claiming. Thus this (gauge) choice for the frame element $e_2^{m-1}$ is essential for our argument here.

Note that for the Schwarzschild metric all derivatives, metric components and Christoffel symbols which involve both the “spatial” directions $T, \Theta$ and the “time-like” direction $\rho$ all vanish. In the wave equation above there are such terms, and we will call them “mixed” terms:

**Definition 5.3.** Consider the terms in (5.17) which involve either a “mixed” metric component $g_{A\rho}$, $A = T, \Theta$ and/or a “mixed” Christoffel symbol $\Gamma^S_{AB}$ where at least one of $A, B, C$ is of the form $T, \Theta$ and at least one other in the form $\rho$. We denote the sum of such terms in the operator by $\Box^{\text{mixed}}$.

We also consider the sum of all terms involving only derivatives in the directions $\Theta, T$ and metric and Christoffel symbols $g_{AB}, g^{AB}, \Gamma^C_{AB}$ have all indices taking values among $\Theta, T$; the sum of those terms is denoted by $\Box^{\text{partial}}$.

### 5.1.1 Bounds on the inhomogeneous term in the wave equation (5.16).

Our first step will be to derive some bounds on the inhomogeneous term of (5.16). Prior to stating our claim, we single out one exceptional case: At the top order estimates $|I| = s - 3, |J_0| = 2$ on $\gamma^m_{\text{fast}}$ we observe that since $e_0(\theta) = 0$, it suffices to derive our claimed bounds (4.24) on $\gamma^m$ instead of on $\gamma^m_{\text{fast}}$. Thus we will not need to subtract $\gamma^S$ at the most top order; so in the Lemma below we will be making no claim on the terms that would have been generated had we made that subtraction.

**Lemma 5.4.** There exists a universal constant $B = B|I| > 0$ so that for all $I$, with $|I| \leq s - 3 - 4c$:

$$
\|\overline{D}(\Box_{m-1} \gamma^S)\|_{L^2_t(S_{r^m-1} = r)} \leq B^2 r^{-\frac{1}{2} + \frac{1}{4}}.
$$

For $s - 3 - 4c < |I| \leq s - 4$ the estimate we have is:

$$
\sum_{s - 3 - 4c < |I| \leq s - 4} \|\overline{D}(\Box_{m-1} \gamma^S)\|_{L^2_t(S_{r^m-1} = r)} \leq 5\sqrt{2}MC\eta r^{-3 - \frac{|I| - (s - 3 - 4c)}{4}} + B^2 r^{-\frac{1}{2} + \frac{1}{4} + \frac{|I| - (s - 3 - 4c)}{4}}.
$$

Moreover, the only terms in $\overline{D}(\Box_{m-1} \gamma^S)$ that contribute to the first (more singular in $\tau$) term in the RHS of (5.19) is the term $\overline{D}(\tau \gamma^m - K^{m-1}) \cdot e_0 \gamma^S$ in (5.17) (with $\gamma^S$ being the function $v$ that is being acted on).

**Proof.** We commence with the bounds on

$$
\Box_{m-1} \gamma^S = \Box_{m-1} \log \rho^m - \Box_{m-1} \log \sin \theta.
$$

The two terms on the RHS will also be treated and bounded separately. We commence with the term $\Box_{m-1} \log \rho^m$. We write $\rho = r^m$ for brevity.

We subtract from this the quantity $\Box_{\rho} \log \rho = 0$ (where the wave operator is expressed in terms of the coordinates $\rho = r^m, t, \theta, \phi$-here $\rho = r$); thus we are reduced to bounding the following terms:

$$
-(\tau \gamma^m - \tau \gamma^S) e_0 \log \rho, \Box_{m-1} \log \rho
$$

The remaining terms involve derivatives $\partial_\tau, \partial_\theta$ that annihilate $\log \rho$.

Let us commence with the first term in (5.20):

The term $-(\tau \gamma^m - \tau \gamma^S) e_0 (\log \rho)$ is first expanded using the expression (4.12) to first calculate

$$
e_0 (\rho^{m-1}) = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} [1 + \partial_r \chi (r^m - 1)]^{-1}.
$$
Therefore we will derive our bounds as follows:

\[
\| \overline{\partial} \left( -\text{tr}_{m-1}\mathcal{P}(K^{m-1} - K^S) e_{\Theta} \log \rho \right) \|_{L^2[\Sigma_{m-1}^n]}
\]

\[= \| \overline{\partial} \left( -\text{tr}_{m-1}\mathcal{P}(K^{m-1} - K^S)(1 - \frac{2M}{r})^{\frac{1}{2}} [1 + \partial_r \chi(r)(r_s^{m-1}(t, \theta) - \epsilon)]^{-1}(\rho^{m-1})^{-1} \right) \|_{L^2[\Sigma_{m-1}^n]}
\]

\[\leq \sum_{t_1 \cup t_2 = t} \| \overline{\partial}^t \left( (-\text{tr}_{m-1}\mathcal{P}(K^{m-1} - K^S)) \partial^2_t \left( 1 - \frac{2M}{r} \right)^{\frac{1}{2}} [1 + \partial_r \chi(r)(r_s^{m-1}(t, \theta) - \epsilon)]^{-1}(\rho^{m-1})^{-1} \right) \|_{L^2[\Sigma_{m-1}^n]}
\]

\[\leq \| \overline{\partial} \left( -\text{tr}_{m-1}\mathcal{P}(K^{m-1} - K^S) \right) \|_{L^2} \cdot \| (1 - \frac{2M}{r}) \frac{1}{2} [1 + \partial_r \chi(r)(r_s^{m-1}(t, \theta) - \epsilon)]^{-1}(\rho^{m-1})^{-1} \|_{L^\infty}
\]

\[+ \| (\text{tr}_{m-1}\mathcal{P}(K^{m-1} - K^S)) \|_{L^\infty},
\]

\[\| \overline{\partial} \left( 1 - \frac{2M}{r} \right)^{\frac{1}{2}} [1 + \partial_r \chi(r)(r_s^{m-1}(t, \theta) - \epsilon)]^{-1}(\rho^{m-1})^{-1} \|_{L^2[\Sigma_{m-1}^n]}.
\]

(5.22)

Now, we consider the case where \(|t| \leq \) low first. In that setting in both summands in the RHS of the above, the second factor is bounded by \(C((\rho^{m-1})^{-3/2})\).

Note that by our inductive assumption (4.53) at the low orders the first factor in those summands is bounded by \(4C\eta(\rho^{m-1})^{-3/2} + \frac{\epsilon}{4}\). Thus this term satisfies our claimed bounds at the lower orders. At the higher orders, we have the same bound for the second factor. However, for the first summand, by our inductive assumption (4.45) the RHS of (5.22) is bounded by a more singular power or \(\rho^{m-1}\); it has an additional factor \((\rho^{m-1})^{-\frac{1}{2}}\), and it does not have the extra power of \((\rho^{m-1})^{\frac{1}{4}}\). In either case our claims are satisfied for this term at the higher orders, since no extra power \((\rho^{m-1})^{1/4}\) is claimed there.

We now consider the other term in (5.20), with \(v = \rho \log \rho\) which does not vanish when \(v = \log \rho\); this is

\[O(e_i(\rho^{m-1})) \cdot K^{m-1} \cdot \left( \frac{2M}{r} \right)^{\frac{1}{2}} \partial_{\rho^{m-1}} v = O(e_i(\rho^{m-1})) \cdot K^{m-1} \cdot \left( \frac{2M}{r} \right)^{\frac{1}{2}} \rho^{m-1}. \]

We then invoke the bounds we have on \(e_i(\rho^{m-1})\) in Lemma (4.13) using Cauchy-Schwarz and the product inequality, we easily see that all terms involving \(e_i(\rho^{m-1})\) are bounded a claimed in our Lemma.

The remaining terms in the RHS of (5.20) are handled in a similar manner, using the inductive assumptions on the various parameters and the product inequality.

Next we derive the claimed bounds for \(\Box_{m-1, \rho} \log \sin \theta\).

For this term we proceed by expanding the wave operator as in (5.17); among coordinate derivatives, the only non-zero terms are then those involving derivatives in the \(\Theta\)-direction, since \(v = \log \sin \theta\). These are:

\[\sum_{a, b = T, \Theta, \phi} (m^{-1})^a \Gamma^{\Theta}_{\phi \Theta} \partial_{\phi \Theta} (\log \sin \theta) = \sum_{a, b = T, \Theta, \phi} (m^{-1})^a \Gamma^{\Theta}_{\phi \Theta} \partial_{\phi \Theta} (\log \sin \theta). \]

(5.24)

(Note that \(\gamma^a_{\phi \Theta} = 0\) for \(a, b = T, \Theta, \phi\).

The first term of terms is more singular (in terms of powers \(\Theta^{-1}, (\tau - \theta)^{-1}\)) than the second, so we commence with the sum of those two terms.

\[\sum_{a, b = T, \Theta, \phi} (m^{-1})^a \Gamma^{\Theta}_{\phi \Theta} \partial_{\phi \Theta} (\log \sin \theta) = \sum_{a, b = T, \Theta, \phi} (m^{-1})^a \Gamma^{\Theta}_{\phi \Theta} \partial_{\phi \Theta} (\log \sin \theta).
\]

(5.25)

A key thing to observe here is a certain cancellation of two singular terms in the right hand side (the first and the last, taken together), using that \(\partial_{\phi \Theta} = 1\), since \(\Theta = \theta\) in these coordinates:

\[\Box_{m-1, \rho} \log \sin \theta = \Box_{m-1, \rho} \log \sin \theta.
\]

(5.26)

The extra power \(r^{\frac{1}{4}}\) captures the asymptotically CMC property at the lower orders.
\[(m-1)g^{\Theta} \partial_{\Theta} (\log \sin \theta) - [(g^S)^{\phi} (\Gamma^{m-1})] : (m-1)g^{\Theta} \partial_{\Theta} (\log \sin \theta) \]
\[(= (m-1)g^{\Theta} \frac{1}{\sin \theta} + (cot \theta)^2) \partial_{\Theta} (\log \sin \theta)^2 + (m-1)g^{\Theta} \partial_{\Theta} \theta \frac{\cos \theta}{\sin \theta} = -(m-1)g^{\Theta}. \]

Recall the bounds in previous subsection on \((g^{m-1})^{\Theta}\); these imply that the RHS term above satisfies the bounds claimed for the LHS of our Lemma 5.3.

The remaining terms in (5.25) can be controlled as follows:

\[-[(m-1)g^{\phi} \cdot (\Gamma^{m-1})_{\phi,6} - (g^S)^{\phi} (\Gamma^{S})_{\phi,0}] : (m-1)g^{\Theta} \partial_{\Theta} (\log \sin \theta)\]
\[= \frac{1}{2}[(m-1)g^{\phi} \cdot (-\partial_{\Theta} (e^{2s})) - (g^S)^{\phi} (-\partial_{\Theta} e^{2s})] : (m-1)g^{\Theta} \partial_{\Theta} (\log \sin \theta) \]
\[= \partial_{\Theta} (\gamma^m - 1) \cdot (m-1)g^{\Theta} \partial_{\Theta} (\log \sin \theta) \]
\[= \partial_{\Theta} (\gamma^m - 1) \cdot (m-1)g^{\Theta} \cdot \cot \theta. \]

Now consider \(\tilde{\partial} = (\Delta_g)^{k_1} \partial^{k_2}\) acting on the above. We derive that:

\[\|\tilde{\partial} [\partial_{\Theta} \gamma_{\gamma-1} \cdot (g^{m-1})^{\Theta} \cdot \cot \theta]\|_{L^2[\Sigma_{m-1}]} \leq \|\tilde{\partial} \left( \partial_{\Theta} \gamma_{\gamma-1} \cdot (g^{m-1})^{\Theta} \right)\|_{L^2[\Sigma_{m-1}]} \]
\[\leq \sum_{j \leq |I|} \sum_{I_1 \cup I_2 = I} \|\tilde{\partial} \left( \partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta \right) \cdot (\partial^{j_2} g^{m-1})^{\Theta}\|_{L^2[\Sigma_{m-1}]} \]

Let us see how the above can be bounded, since the same argument will be used frequently in the rest of the paper. We recall first that by the product inequality the RHS is bounded by:

\[B \|\tilde{\partial} [\partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta]\|_{L^2[\Sigma_{m-1}]} \leq \|\tilde{\partial} \left( \partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta \right)\|_{L^2[\Sigma_{m-1}]} \]

The factors:

\[\|\tilde{\partial} [\partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta]\|_{L^2[\Sigma_{m-1}]} \leq \|\tilde{\partial} \left( \partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta \right)\|_{L^2[\Sigma_{m-1}]} \]

are bounded respectively by \(B \|\partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta\|_{L^2[\Sigma_{m-1}]} \leq C \eta \rho^{m-1/8}\). The other factors are bounded by our inductive assumptions, after also using the Hardy inequality for the first one; in particular we invoke the inequality:

\[\|\tilde{\partial} [\partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta]\|_{L^2[\Sigma_{m-1}]} \leq 2 \|\tilde{\partial} \left( \Delta_g \gamma_{\gamma-1} \right)\|_{L^2[\Sigma_{m-1}]} + \|\tilde{\partial} \left( \partial_{\Theta} \gamma_{\gamma-1} \right)\|_{L^2[\Sigma_{m-1}]} \leq C \eta \log \rho^{m-1} \]

for \(|I| \leq 2 - 2\),

\[\|\tilde{\partial} [\partial_{\Theta} \gamma_{\gamma-1} \cdot \cot \theta]\|_{L^2[\Sigma_{m-1}]} \leq 2 \|\tilde{\partial} \left( \Delta_g \gamma_{\gamma-1} \right)\|_{L^2[\Sigma_{m-1}]} + \|\tilde{\partial} \left( \partial_{\Theta} \gamma_{\gamma-1} \right)\|_{L^2[\Sigma_{m-1}]} \]

\[\leq C \rho^{m-1} \eta^{-1} \frac{|I| - 4 - (|I|-1)}{2}, \text{if low} \leq |I| \leq s - 4. \]

On the other factors we have already derived the bounds:

\[\|\tilde{\partial} (g^{m-1})^{\Theta}\|_{L^2[\Sigma_{m-1}]} \leq B^2 \rho^{m-1} \frac{1}{2}, \text{if low} \leq |I| \leq \frac{1}{2} \]

Combining these bounds proves that the term (5.27) satisfies the desired bound. All remaining terms can be bounded by our assumptions on Christoffel symbols and the argument above. This completes our proof of the bounds in (5.18), (5.19).
5.1.2 Formulas and bounds for commutations of $\Box g^{m-1}$ with $\bar{\partial}^I$.

Having bounded the RHS of (5.16) in suitable spaces, we next need to act on the LHS of that equation by $\bar{\partial}^I$, and commute the derivatives past $\Box g^{m-1}$. So, to proceed with our analysis of the wave equation at higher orders, we wish to commute (5.17) with $\bar{\partial}^I$, $|I|=l \leq s-4$.

We will put down some formulas that will help us in calculating the required commutation. We proceed in two steps: First we consider the terms in the wave operator that have coefficients that are singular at the two poles $\Theta = 0, \pi$ and proceed to calculate and bound the commutation terms that arise from those with some care. Next, we study the remaining terms that arise in the commutation, and bound those also; the latter terms are almost straightforward applications of our inductive assumptions. The second step will be performed in the subsequent subsections.

Initially, let us consider the sum of terms in $\Box g^{m-1}$ that corresponds to the Laplacian on the 2-spheres $\Theta \in (0, \pi), \phi \in [0, 2\pi), \rho^{m-1} = \text{fixed}$. This operator $\Delta^{m-1, T}_{g^{m-1}}$ is defined via:

$$\Delta^{m-1, T}_{g^{m-1}} v = (m-1)^2 g^{T\Theta} [\partial_\Theta^2 v - \frac{\cos \Theta}{\sin \Theta} \partial_\Theta v] - \frac{1}{2} [(m-1) g^{T\Theta}]^2 \partial_\Theta (m-1) g^{T\Theta} \partial_\Theta v + (m-1)^2 g^{T\Theta} \cdot \partial_\Theta \gamma^{m-1}_{\text{rest}} \cdot \partial_\Theta v.$$  (5.31)

Note that this operator corresponds to a special sum of terms.

Using this formula, the commutation of $\Delta^{m-1, T}_{g^{m-1}}$ with $\Delta_{g^2}$ is easily calculated, in (5.33) right below.

$$[\Delta_{g^2}, \Delta^{m-1, T}_{g^{m-1}}](v) = \Delta_{g^2} (m-1)^2 g^{T\Theta} \cdot \Delta_{g^2} v + 2 \partial_\Theta (m-1)^2 g^{T\Theta} \cdot \partial_\Theta \Delta_{g^2} v$$

$$- \Delta_{g^2} [(m-1)^2 g^{T\Theta}]^2 \partial_\Theta (m-1) g^{T\Theta} \cdot \partial_\Theta (m-1) g^{T\Theta} \cdot \partial_\Theta v$$

$$- \Delta_{g^2} \frac{1}{2} [(m-1)^2 g^{T\Theta}]^2 \partial_\Theta (m-1) g^{T\Theta} \cdot \partial_\Theta (m-1) g^{T\Theta} \cdot \partial_\Theta v$$

$$+ \Delta_{g^2} [(m-1)^2 g^{T\Theta}]^2 \partial_\Theta \gamma^{m-1}_{\text{rest}} \cdot \partial_\Theta (m-1) g^{T\Theta} \cdot \partial_\Theta \gamma^{m-1}_{\text{rest}} \cdot \partial_\Theta v$$

$$+ [(m-1)^2 g^{T\Theta}]^2 \partial_\Theta \gamma^{m-1}_{\text{rest}} \cdot \partial_\Theta \gamma^{m-1}_{\text{rest}} \cdot \partial_\Theta v.$$  (5.33)

We note that $[\Delta_{g^2}, \sin \Theta \partial_\Theta] = 2 \cos \Theta \cdot \Delta_{g^2}$. Thus the last factors in the last two lines can be replaced by $2 \cos \Theta \cdot \Delta_{g^2} v$. We can iterate the above formula and then act by $\partial_\Theta$ repeatedly on the resulting formulas. We can use this formula to calculate and bound:

$$[\bar{\partial}^I, \Delta^{m-1, T}_{g^{m-1}}](\gamma^{m-1}_{\text{rest}})$$
in \(L^2(\sin \Theta d\Theta dT)\). In that norm, these terms will be bounded as in the RHS of the equations in \(5.5\).

We next wish to see how \(\Delta_{\rho^2}\) commutes with the remaining part of \(\Box_{\rho^{m-1}}\). Again recall formula (5.17) (denoting \(\gamma^m\) by \(v\)); the terms that require special treatment due to their singular behaviour at the poles are precisely:

\[
\Delta_{\rho^{m-1}} \cdot v
\]

(which was already calculated above) \textit{and} the terms \(\langle m-1 \Gamma_{TT}^\Theta \rangle \partial_\Theta v\), \(\langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta v\), and the terms in the third line of (5.17). These will be treated directly below. For the \textit{rest} of the terms in (5.17), the commutation formula is straightforward: We use the commutation of \(\partial_{\rho^{m-1}}, \partial_T\) with \(\partial_\Theta\). Thus the commutation terms generated can be calculated from the Leibnitz rule, and in particular are \textit{not} singular at the poles. These commutations will give rise to terms that are written out (in generic notation) in (5.38) directly below.

Let us now calculate:

\[
[\Delta_{\rho^2}, \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta, [\Delta_{\rho^2}, \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta].
\]  

(5.34)

We will explain how the terms in the third line of (5.17) can be computed and bounded by the same argument we present for the ones here.

For both these terms, we observe that the coefficient of \(\partial_\Theta\) of the second operator in \([\cdot, \cdot]\) \textit{vanishes} at the two poles.

To make use of this, we re-write:

\[
\left\langle \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta \right\rangle = \left\langle \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \sin \Theta \right\rangle \cdot (\sin \Theta \partial_\Theta),
\]

\[
\left\langle \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta \right\rangle = \left\langle \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \sin \Theta \right\rangle \cdot (\sin \Theta \partial_\Theta).
\]  

(5.35)

In particular given our derived bounds on the Christoffel symbols (as well as the Hardy inequality), the RHS is uniformly bounded at orders \(|\nu| \leq \text{low} - 2\) by

\[
B(\rho^{m-1})^{-2-\frac{3}{4}}.
\]

At each order beyond that, the power of \(\rho^{m-1}\) becomes more singular by \(-1/4\).

With these formulæ in hand, we bound the first term in (5.34) as follows:

\[
\left\| \Delta_{\rho^2}, \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta \sin \Theta \right\|_{L^2(\sin \Theta d\Theta dT)} \leq \left\| \Delta_{\rho^2} \right\|_{L^2(\sin \Theta d\Theta dT)} \left\| \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \sin \Theta \right\|_{L^2(\sin \Theta d\Theta dT)} + 2 \left\| \partial_\Theta \left( \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \sin \Theta \partial_\Theta \right) \right\|_{L^2(\sin \Theta d\Theta dT)} + 2 \left\| \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \sin \Theta \partial_\Theta \partial_\Theta \right\|_{L^2(\sin \Theta d\Theta dT)}.
\]  

(5.36)

(In the very last term we applied the commutation formula:

\[
[\Delta_{\rho^2}, \sin \Theta \partial_\Theta] = 2 \cos \Theta \cdot \Delta_{\rho^2}.
\]

We note that in view of Lemma \(4.16\)

\[
\left\| \Delta_{\rho^2} \right\|_{L^2(\sin \Theta d\Theta dT)} \left\| \Sigma_{m-1} \right\| \leq (\rho^{m-1})^{-\frac{3}{4}} E[v].
\]  

(5.37)

This calculation can be iteratively applied to calculate \(\Delta_{\rho^2} \langle m-1 \Gamma_{TT} \rangle \langle m-1 \Gamma_{\Theta}^\Theta \rangle \partial_\Theta \sin \Theta\) and to bound this term in \(L^2(\sin \Theta d\Theta dT)\) by the RHSs of the equations in Proposition \(5.5\). An entirely analogous calculation (and derivation of resulting bounds) can be performed on

\[
\left\| \partial_T \langle \sin \Theta \rangle \right\|_{L^2(\rho^{m-1})} \left\| \partial_T \right\|_{L^2(\rho^{m-1})} \left\| m-1 \Gamma_{TT} \right\|_{L^2(\rho^{m-1})}.
\]

(5.36)

For the two terms in the third line of (5.17) we use the same argument, “creating” the vector field \(\sin \Theta \partial_\Theta\) by multiplying each expression by \(1 = \sin \Theta / \sin \Theta\), and using the above argument for commutations. We also use the vanishing of \(\epsilon^2_{m-1} (\rho^{m-1})\) at the poles to bound expressions

\[
\left\| \partial_T \left[ \epsilon^2_{m-1} \rho^{m-1} \right] \sin \Theta \right\|_{L^2(\rho^{m-1})} \left\| \partial_T \right\|_{L^2(\rho^{m-1})}.
\]

(5.36)
These terms are in fact easy to handle, since they vanish identically for $\rho^{m-1} \leq \epsilon/2$.

What remains is to find all the other commutation terms in $[\bar{\partial}^\gamma, \Box_{g,m-1}]$. These we write out (schematically, using the notational conventions introduced earlier in this section) in the RHS of the next equation, from the third line of the RHS onwards:

$$\Box_{m-1} \bar{\partial}^\gamma m_{\text{rest}} =$$

(5.38)

$$+ \sum_{l \leq 3} \left[ \bar{\partial}^l \left( \text{tr}_{m-1} \rho^{m-1} \right) \right]_{l \leq 3} \Box_{m-1} \left( \text{tr} \left( e^{m-1} \rho^{m-1} \right) \right) \cdot \Box_{m-1} \left( \text{tr} \left( \rho^{m-1} \right) \right) .$$

(5.39)

$$(\text{with respect to the volume form } \sin^2 \Theta \Theta d^3 t \phi).$$

We note that the last term in the last line has been controlled already by virtue of Lemma 5.3. Let us denote by $RHS[i_{\text{rest}}^{m-1} \bar{\partial}^\gamma m]$ the RHS of the equation (5.38), except for the last line. We claim:

**Proposition 5.5.** At step $\gamma$ assume all inductive claims concerning the REVESNG parameters of step $m - 1$ hold true. Also, for each order $|l| \leq s - 4$ assume that the inductive claims on $\bar{\partial}^\gamma m$ for orders $|l'| < |I| \leq |I|$ hold true.

Then on each level set $\Sigma_{l'}$ of $\rho^{m-1}$ the $L^2_{\text{rest}} \bar{\partial}^\gamma m$ norm of $\text{RHS}[i_{\text{rest}}^{m-1} \bar{\partial}^\gamma m]$ (with respect to the volume form $\sin^2 \Theta \Theta d^3 t \phi$) is bounded as follows: For $|l| \leq 3$:

$$\| \text{RHS}[5.38] \|_{L^2_{\text{rest}}[\gamma]} \leq B^{2} C_{\gamma}^{-1} + B^{2} C_{\gamma}^{-1} - D C \eta . E \left[ \bar{\partial}^{m-1} \gamma_{\text{rest}} \right] .$$

(5.40)

While for $|l| \geq 4$:

$$\| \text{RHS}[5.38] \|_{L^2_{\text{rest}}[\gamma]} \leq B^{2} C_{\gamma}^{-1} + B^{2} C_{\gamma}^{-1} - D C \eta . E \left[ \bar{\partial}^{m-1} \gamma_{\text{rest}} \right] .$$

(5.41)

**Proof of Proposition 5.5.**

We refer to (5.38). We note that the term $\bar{\partial}^\gamma \left( \Box_{m-1} \gamma^\delta \right)$ already satisfies the bounds required by our Lemma, in view of (5.18), which has already been derived. (Note that for the higher orders, that term contributes to the first summand in the RHS. Thus matters are reduced to controlling the terms on the RHS of (5.38), except for the term in the last line.

A few terms that require a few extra estimates are treated separately, after a brief remark on the importance of a gauge choice that was made in obtaining the bound in Lemma 5.5.

Now, we proceed with the proof. We first show our claim for the lower orders $|l| = 3 \leq 4$.

The most important term in the RHS of (5.38) is:

$$\sum_{l \leq 3} \bar{\partial}^l \left( \text{tr}_{m-1} \rho^{m-1} \right) \cdot \Box_{m-1} \left( \text{tr} \left( \rho^{m-1} \right) \right) .$$

(5.42)

To control this term we use the crucial fact that $\text{tr}_{m-1} \rho^{m-1}$ is asymptotically constant along $\Sigma_{l'}$ to leading order, as $r \rightarrow 0$, as captured in (4.52). This yields:

$$\partial^l \left( \text{tr}_{m-1} \rho^{m-1} \right) = O \left( \chi_{[\frac{5}{2}, \frac{3}{2}]} \left( \rho^{m-1} \right) \right) + \Gamma^l \left( \text{tr}_{m-1} \rho^{m-1} \right),$$

(5.43)

for every $l \leq 3$. We thus observe that this term will contribute a below-boundary bulk term in the final energy estimate, since using our inductive assumption and the product inequality:

$$\| \partial^l \left( \text{tr}_{m-1} \rho^{m-1} \right) \|_{L^2_{\text{rest}}[\gamma]} \leq \frac{3B}{r^{1+l}},$$

(5.44)

for every $l \leq s - 3 - 4c$. We thus observe that this term will contribute a below-boundary bulk term in the final energy estimate, since using our inductive assumption and the product inequality:

$$\| \partial^l \left( \text{tr}_{m-1} \rho^{m-1} \right) \|_{L^2_{\text{rest}}[\gamma]} \leq \frac{3B}{r^{1+l}},$$

(5.45)

for every $l \leq s - 3 - 4c$. We thus observe that this term will contribute a below-boundary bulk term in the final energy estimate, since using our inductive assumption and the product inequality:
\[ \| \sum_{I \cup J = l, |I| \leq l} \partial^{1+\epsilon_{m-1}} \| L^2[\gamma_{\text{rest}}^{m-1} \cdot \gamma_{\text{rest}}^{m-1}] \| L^2[\gamma_{\text{rest}}^{m-1} \cdot \gamma_{\text{rest}}^{m-1}] \| L^2[\gamma_{\text{rest}}^{m-1} \cdot \gamma_{\text{rest}}^{m-1}] \| \leq \frac{4B}{T^{3+\frac{1}{2}}} \]  

(5.44)

We again stress here the importance of the asymptotically CMC property of the level sets of \( r \): Had this cancellation in \( \partial r \) not been present we would have not been able to obtain the desired energy estimates, since we would have obtained a borderline term already at the lower order energies.

We next consider the commutation of \( \mathcal{D} \) with \( \Box^\text{partial}_{g^{m-1}} \) and with \( \Box^\text{mixed}_{g^{m-1}} \), and bound the resulting expressions. We distinguish the part \( \Delta^{m-1}_{\text{g}^{m-1}} \) in \( \Box^{m-1}_{\text{g}^{m-1}} \) which involves derivatives in the \( \Theta \)-direction: we will treat these terms afterwards, using the formulas (5.33) we derived.

We first consider the following terms in the RHS of (5.38):

\[ \sum_{I, \cup J = l, |I| \leq l} \sum_{A, B = T, \Theta, \rho, C = T, \rho} \partial^{1+\epsilon_{m-1}} (g^{AB} T^{ABC}_{AB}) \partial^{1+\epsilon_{m-1}} \partial_{C \gamma_{\text{rest}}}^{m-1} \| L^2[\Sigma_{\rho^{m-1}, \tau}] \]  

(5.45)

We claim that the first two terms in the above are bounded in \( L^2_{\sin \Theta, \Theta, \rho, \tau} \) by virtue of our inductive assumptions by \( B \tau^{-1-\frac{1}{2}} \). We claim that the second term is bounded in the same space by \( B \tau^{-1-\frac{1}{2}} E_{\mathcal{D} [\gamma_{\text{rest}}^{m-1}]} \).

These two claims follow readily from the inequality:

\[ \| \sum_{I, \cup J = l, |I| \leq l} \sum_{A, B = T, \Theta, \rho, C = T, \rho} \partial^{1+\epsilon_{m-1}} (g^{AB} T^{ABC}_{AB}) \partial^{1+\epsilon_{m-1}} \partial_{C \gamma_{\text{rest}}}^{m-1} \| L^2[\Sigma_{\rho^{m-1}, \tau}] \]  

\[ \leq \sum_{A, B = T, \Theta, \rho, C = T, \rho} \| \partial^{1+\epsilon_{m-1}} (g^{AB} T^{ABC}_{AB}) \| L^2[\Sigma_{\rho^{m-1}, \tau}] \| \partial_{C \gamma_{\text{rest}}}^{m-1} \| L^2[\Sigma_{\rho^{m-1}, \tau}] \| \]  

(5.46)

Now, recall the bounds on the Christoffel symbols that we have derived. Recall also the Sobolev embedding, which bounds the \( L^\infty \) norms above by the corresponding \( H^2 \) bound on those quantities; thus combining these estimates we derive that the above quantity is bounded by \( B \tau^{-1-\frac{1}{2}} \).

Let us now consider the third term in (5.45). In this term, the total number of derivatives on \( \gamma_{\text{rest}}^{m-1} \) equals the number of derivatives that we are trying to bound (in our claimed energy bound). On the other hand, when \( |I| = l - 1 \) this term can be re-expressed and bounded as follows:

\[ \| \sum_{A, C = T, \rho} \partial g^{AC} \partial_{AC} \partial^{1+\epsilon_{m-1}} \| L^2 \]  

\[ \leq \sum_{A, C = T, \rho} \| \partial g^{AC} \| L^2[\Sigma_{\rho^{m-1}, \tau}] \| (a^{m-1}) A \| L^2[\Sigma_{\rho^{m-1}, \tau}] \| \epsilon^{m-1}_{\partial C} \partial^{1+\epsilon_{m-1}} \| L^2[\Sigma_{\rho^{m-1}, \tau}] \| \]  

(5.47)

where \( |I'| = l \). The RHS bound is achieved when \( C = A = \Theta, \rho = i = 2 \). When either of \( A, C \) takes the value \( T \) and/or \( i \) takes the value 1 the bound is much less singular (in terms of powers of \( \tau \)).

Now, the commutation terms in \( \mathcal{D}^{m-1, T}, \mathcal{D}^{m-1, \Theta}, \mathcal{D}^{m-1, \rho} \) satisfy the bounds in our Proposition by the same simple application of the product inequality, by invoking the formulas we derived for these terms using the formulas (5.33), (5.36), our inductive assumptions on the components of \( g^{m-1} \) and the inductive assumptions on the lower-order derivatives of \( \gamma_{\text{rest}}^{m-1} \).

We can now derive \( L^2 \) bounds to the rest of the terms in the RHS of (5.38). The remaining terms that were not treated are those commutation terms that arise from \( \mathcal{D}^{i} \) acting on \( \Box^{m-1}_{g^{m-1}, \gamma_{\text{rest}}^{m-1}} \), when some
of the derivatives hit terms in the first four lines, (except for \(\text{tr}_{\gamma^{m-1}} K^{m-1}\) which was the first to be considered above). Among the remaining terms, all terms involving one derivative \(\partial_{\rho^{m-1}}\) or \(e^{m-1}(\rho^{m-1})\) are straightforwardly bounded as claimed, invoking the bounds in Lemma 4.13 on \(e^{m-1}(\rho^{m-1})\).

We are left with one term: The term on the RHS, involving two derivatives \(\partial_{\rho^{m-1}, \rho^{m-1}}\gamma_{\text{rest}}^m\).

The first of these terms yields terms:

\[
\partial^I \partial^J \partial^I_{\rho^{m-1}, \rho^{m-1}} \gamma_{\text{rest}}^m \left[ \left( \frac{2M}{r} - 1 \right) [1 + \partial_r \chi(r) (r^{m-1} - c)]^2 \right],
\]

where \(I_1 \cup I_2 = I, \ |I_1| \leq |I| - 1\). These terms do not immediately fall under our inductive assumptions since they involve two derivatives \(\partial_r\). For those terms we invoke the inhomogenous wave equation (5.10) and the expression (5.17) for the wave operator to express these derivatives in terms of terms involving at most one \(\partial_{\rho^{m-1}}\) derivatives. Our desired bound then follows from the already-derived bounds on the RHS of the resulting equation. In particular we obtain the following bounds, for \(|I| \leq s - 3 - 4c\):

\[
|\overline{\mathcal{D}}^I \partial^J \partial^I_{\rho^{m-1}} \gamma_{\text{rest}}^m|_{L^2(\Sigma_{\rho^{m-1}})} \leq BC\eta^{-2} + C\eta^{-\frac{1}{2}} E[\partial^J \gamma_{\text{rest}}^m], |I| = |I| - 1.
\]

(5.48)

While for \(|I| = k \in \{s - 3 - 4c + 1, \ldots, s - 4\}\):

\[
|\overline{\mathcal{D}}^I \partial^J \partial^I_{\rho^{m-1}} \gamma_{\text{rest}}^m|_{L^2(\Sigma_{\rho^{m-1}})} \leq BC\eta^{-2 + \frac{k - 1}{4}} + C\eta^{-\frac{1}{2}} E[\partial^J \gamma_{\text{rest}}^m], |I| = |I| - 1.
\]

(5.49)

Thus these commutation terms also satisfy the required bounds.

Having controlled the commutation terms in the norm \(L^2[\sin \Theta d\Theta d\tau]\) we can now derive our inductive claims on \(\overline{\mathcal{D}}^I \gamma_{\text{rest}}^m\), at lower, and higher, and top orders:

### 5.2 Lower order energy estimates: \(|I| \leq s - 3 - 4c\)

We summarize the energy estimate for \(\partial^I \gamma_{\text{rest}}^m\) \(\leq s - 3 - 4c\) in the next proposition.

**Proposition 5.6.** If the inductive assumptions in \(\ref{4.4}\) on the metric \(g^{m-1}\) hold true, then the following energy inequality on level sets \(\{\rho^{m-1} = \rho\}\) is valid, for some universal constant \(B > 0\), and for the function \(O(\cdot)\) (satisfying the bounds in \(\ref{5.11}\)):

\[
\sum_{|I| \leq s - 3 - 4c} \rho^3 (1 + O(\rho^{m-1})) E[\partial^I \gamma_{\text{rest}}^m(\Sigma_{\rho^{m-1} = \rho})] \leq \sum_{|I| \leq s - 3 - 4c} (1 + O(\epsilon)) \epsilon^3 E[\partial^I \gamma_{\text{rest}}^m(\epsilon, t, \theta)]
\]

\[
+ \int_{\rho}^\infty \frac{B^2}{\tau^{1-\frac{3}{4}}} \sum_{|I| \leq s - 3 - 4c} \tau^3 E[\partial^I \gamma_{\text{rest}}^m] d\tau + \int_{\rho}^\infty \frac{5B \cdot C\eta}{\tau^{1-\frac{3}{4}}} \sqrt{\sum_{|I| \leq s - 3 - 4c} \tau^3 E[\partial^I \gamma_{\text{rest}}^m]} d\tau.
\]

(5.50)

Note that the inductive bound \(\ref{4.22}\) for \(\gamma_{\text{rest}}^m\) follows readily from \(\ref{5.50}\) and Gronwall’s inequality 4.8. In particular for \(F^2(r) = \sum_{|I| \leq s - 3 - 4c} \tau^3 E[\partial^I \gamma_{\text{rest}}^m]\), as well as \(H(\tau) \equiv 10B^2 \cdot \frac{1}{1-\frac{3}{4}}\) and \(G(\tau) = \|\text{RHS}(\ref{5.35})\|_{L^2[\tau]}\) we find that \(\ref{4.16}\) implies:

\[
\sqrt{\left(1 + O(\tau)\right) \sum_{|I| \leq s - 3 - 4c} \tau^3 E[\partial^I \gamma_{\text{rest}}^m(\Sigma_{\rho^{m-1}})]}
\]

\[
\leq e^{10B^2} \int_{r_1}^r \tau^{m-1} \tau^{1-\frac{3}{4}} dr \left(1 + O(\epsilon)\right) \sum_{|I| \leq s - 3 - 4c} \epsilon^3 E(\partial^I \gamma_{\text{rest}}^m(\epsilon, t, \theta)) + \frac{1}{2} \int_{\rho^{m-1}}^\infty \frac{B \cdot C\eta}{\tau^{1-\frac{3}{4}}} d\tau\right)^2 (1 + O(\epsilon)) \eta + BCE^{\frac{1}{4}}\eta.
\]

(5.51)

Here the terms \(O(\tau), O(\epsilon)\) satisfy the bounds \(\ref{5.11}\). In particular the above inequality implies:

\[
\sqrt{\sum_{|I| \leq s - 3 - 4c} \tau^3 E[\partial^I \gamma_{\text{rest}}^m(\Sigma_{\rho^{m-1}})]} \leq (1 + \epsilon^{1/4} 10B^2) \left(\frac{C\eta}{4} + BCE^{1/4}\eta\right).
\]

(5.52)

Thus invoking the bounds \(\ref{4.9}, \ 5.11\) we derive that:

\[
\sqrt{\sum_{|I| \leq s - 3 - 4c} \tau^3 E[\partial^I \gamma_{\text{rest}}^m(\Sigma_{\rho^{m-1}})]} < C\eta,
\]

69
as desired. So our claim in the lower orders follows, provided we can show Proposition 5.6. We do this next:

**Proof of Proposition 5.6.** The Proposition will be proven by finite induction on $|I| = l$. In particular the claimed bounds (4.22) are assumed to hold for all $\partial^l \gamma^m_{\text{rest}}$ with $|I| \leq l - 1$. Recall the identity (5.55) and set $v = \partial^l \gamma^m_{\text{rest}}$, $|I| = l \leq s - 3 - 4c$, $f(v) = \tau^2$ to obtain the energy inequality: [using also $\|u^{m-1}_{\gamma} - \|_{L^\infty} \leq 5B r^1 - 4$, $\|d^v_{m-1}(|t, \theta) - 1\|_{L^\infty} \leq DC^n \leq \frac{1}{2}$, $\|d^v_{m-1}(|t, \theta) + 1\|_{L^\infty} \leq DC^n \leq \frac{1}{2}$]

$$
\int_{\Sigma_{\rho m-1}} \tau^2 Q_{ab} [\partial^l \gamma^m_{\text{rest}}(e_0)] n^a \text{vol}_{\Sigma_{\rho m-1}} - \int_{\Sigma_{\rho m-1}} (n^v - r_s)^2 Q_{ab} [\partial^l \gamma^m_{\text{rest}}(e_0)] n^a \text{vol}_{\Sigma_{\rho m-1}} \quad (5.53)
$$

$$
\leq \int_{\rho m-1}^{\rho m} \int_{\Sigma_r} \Phi^{m-1} \left[ \frac{\sqrt{2M}}{\tau^2} (DC^n - 2)^{1/4} (e_0 \partial^l \gamma^m_{\text{rest}} e^2 + (DC^n - 1/2) \tau^2 (e_0 \partial^l \gamma^m_{\text{rest}})^2) \right]

+ C \frac{1}{\rho m-1} \int_{\Sigma_r} \Phi^{m-1} \left( e_0 \partial^l \gamma^m_{\text{rest}} e^2 + \nabla \partial^l \gamma^m_{\text{rest}} \right) \text{vol}_{\Sigma_r} d\tau
$$

Now, use (5.9), (5.4), (5.5) as well as the estimates in Lemma 4.13 and using the bounds $DC^n \leq \frac{1}{2}$ just above, we note that the terms in the second line of the above are in fact negative and can thus be dropped to yield:

$$
(1 + O(\rho^{m-1})) (1 + O(\rho^{m-1})^3) E[\partial^l \gamma^m_{\text{rest}}(e_0)] - (1 + O(\epsilon)) \epsilon^3 E[\partial^l \gamma^m_{\text{rest}}(e, t, \theta)] \quad (5.54)
$$

$$
\leq \int_{\rho m-1}^{\rho m} \int_{\Sigma_r} \Phi^{m-1} \left( e_0 \partial^l \gamma^m_{\text{rest}} e^2 + \nabla \partial^l \gamma^m_{\text{rest}} \right) \text{vol}_{\Sigma_r} d\tau
$$

It remains to estimate the last term in (5.54). For this we invoke Proposition 5.5 which we have already proven. Our claim follows.

### 5.3 Middle order energy estimates: $E[\partial^l \gamma^m_{\text{rest}}(e_0)]$, $s - 3 - 4c < |I| \leq s - 4$.

Next we derive the middle order energy estimates for $\gamma^m_{\text{rest}}$ (4.22). We begin with proving a 'summed up' estimate which involves the sum of all energies for $\partial^l \gamma^m_{\text{rest}}$ for $|I|$ between orders $s - 3 - 4c + 1$ and $s - 4$. In particular this will confirm (4.22) is valid for $\gamma^m_{\text{rest}}$ in the case $|I| = s - 4$.

We will then derive the (stronger) inductive step claims for the lower derivatives $\partial^l \gamma^m_{\text{rest}}$, $|I| \in \{s - 3 - 4c + 1, \ldots, s - 4\}$.

#### 5.3.1 Estimate on the sum of all middle-order energies.

Our claim for the sum of all middle-order energies is as follows:

**Proposition 5.7.** Assuming the inductive bounds in (4.4) and the lower-order energy estimate (4.22), then the following energy inequality on level sets $\{\rho^{m-1} = \rho\}$ is valid for some universal constant $B > 0$:

$$
\sum_{s - 3 - 4c < |I| \leq s - 4} (1 + O(\rho)) \rho^3 E[\partial^l \gamma^m_{\text{rest}}(e_0)] \leq \sum_{s - 3 - 4c < |I| \leq s - 4} (1 + O(\epsilon)) \epsilon^3 E[\partial^l \gamma^m_{\text{rest}}(e, t, \theta)] + C \epsilon^4 |B^2 C^2 \eta^2 r^{-2c} |
$$

$$
+ \int_{r}^{10B^2} \frac{\tau^{3/4}}{\tau^{1/4}} \sum_{s - 3 - 4c < |I| \leq s - 4} \tau^2 E[\partial^l \gamma^m_{\text{rest}}(e_0)] d\tau

+ \int_{r}^{10B^2} \left( \frac{5C^n + 5(Cn)^2}{\tau} + \frac{BCC^n \eta}{\tau^{1/4}} \right) \tau^{-c} \sum_{s - 3 - 4c < |I| \leq s - 4} \tau^2 E[\partial^l \gamma^m_{\text{rest}}(e_0)] d\tau
$$

for all $\rho \in (0, 2\epsilon)$.

**Remark 5.8.** We note that our claim concerns the energies of orders between $s - 3 - 4c + 1$ and $s - 4$, and only such energies appear also in the RHS. The contribution of the lower order energies (for which the inductive claim has already been proven) appear in the term $+C \epsilon^4 |B^2 C^2 \eta^2 r^{-2c}|$.

**Remark 5.9.** The main difference of (5.55) from (5.50) is the additional factor of $\tau^{-c}$ in the last term in the RHS of (5.55), which is responsible for the (weaker) bounds that we can derive at the middle orders; the borderline term is $$(5C^n + 5(Cn)^2) \tau^{-c} \sqrt{\sum_{s - 3 - 4c < |I| \leq s - 4} \tau^2 E[\partial^l \gamma^m_{\text{rest}}(e_0)]}.$$
We highlight in the proof the terms that contribute to the key borderline coefficient \((\frac{5C\eta + 5C\eta^2}{\tau})^{28}\).

Let us check how this Proposition implies our claim: Given (5.55), we employ Lemma (4.8) to derive:

\[
\sqrt{(1 + O(r)) \sum_{s - 3 - 4c < |I| < s - 4} r^3 E[\partial^{|I|}m]|\Sigma_r] \leq e^{\gamma_m} \sum_{s - 3 - 4c < |I| < s - 4} (1 + O(\epsilon)) e^{3 E[\partial^{|I|}m]}(\epsilon, t, \theta)|\Sigma_r].
\]

Thus, (recalling that \(C\eta < \frac{1}{4}\) and our choice of \(c > 20\)) (4.16) and since \(\epsilon\) is appropriately small so that \(2e^{1/4}BCS < \frac{C}{4}\), we deduce the bound

\[
\sum_{s - 3 - 4c < |I| < s - 4} r^3 E[\partial^{|I|}m]|\Sigma_r] \leq \frac{9}{10} C\eta\epsilon^{-c + \frac{1}{4}}, \quad \text{for all } r \in (0, 2\epsilon].
\]

Now, directly below we will derive the improved bounds for for all terms at lower orders \(|I| < s - 4\): letting \(h = s - 3 - |I|\) the energies of those terms will be bounded by \(10c^2\epsilon^2\). In view of the bounds we have imposed on \(\epsilon\), (5.57) then confirms the inductive assumption (4.23) for \(\gamma_m\) in the case \(|I| = s - 4\).

**Proof of Proposition 5.7.** We repeat the argument from the lower order case on invoking (5.53), and discarding the borderline bulk terms in the second line using their favorable sign.

Thus (5.54) is still valid for \(\partial^{|I|}m, |I| = l \leq s - 4\). Our claim then follows by invoking Proposition 5.3 to these middle orders.

\[
\square
\]

### 5.3.2 Improved estimates at orders \(s - 4 - k, k \in \{1, \ldots, 4c - 1\}\).

Now that we have (5.57) at our disposal, we proceed to show that \(\gamma_m\) satisfies the stronger claims in (4.23), for all the lower derivatives \(I, s - 3 - 4c < |I| = l \leq s - 3\).

**Proposition 5.10.** Assuming the inductive assumptions in (4.23) hold true and the energy estimates (4.23), (5.57) hold, then the following stronger energy inequality is valid for the (sum of the) first \(c - k\) of the higher derivatives:

\[
\sum_{s - 3 - 4c < |I| \leq s - 3 - k} (1 + O(\rho^{m-1})) r^3 E[\partial^{|I|}m] |\Sigma_{\rho^{m-1}}| \leq \sum_{s - 3 - 4c < |I| \leq s - 4 - k} (1 + O(\epsilon)) e^{3 E[\partial^{|I|}m]}(\epsilon, t, \theta)]
\]

\[
+ \epsilon^{2 - DC\eta} BC^2 \eta^{-2c + \frac{1}{4}} + \int_{\rho^{-1}}^{\epsilon} \sum_{s - 3 - 4c < |I| \leq s - 4 - k} \tau^3 E[\partial^{|I|}m] d\tau
\]

\[
+ \int_{\rho^{-1}}^{\epsilon} \int \sum_{s - 3 - 4c < |I| \leq s - 4 - k} \tau^3 E[\partial^{|I|}m] d\tau d\tau
\]

\[
\text{for all } r \in (0, 2\epsilon] \text{ and every } 0 < k < 4c. \text{ The exponent } p \text{ in the last term equals } p = 1, \text{ for } k \leq 2c, \text{ and } p = 1 - \frac{1}{4}, \text{ for } k > 2c. \text{ The coefficient } L_p \text{ equals } 5C^2\eta^2 \text{ in the case } k \leq 2c; \text{ it equals } B^2C\eta \text{ when } k > 2c.
\]

Let us check how the above Proposition implies our desired inductive step at the lower middle-derivatives. Lemma (4.8) applied to the energy inequality (5.58) yields:

\[
\sqrt{(1 + O(\rho^{m-1}))} \sum_{s - 3 - 4c < |I| \leq s - 4 - k} r^3 E[\partial^{|I|}m] |\Sigma_{\rho^{m-1}}| \leq e^{\gamma_m} \sum_{s - 3 - 4c < |I| \leq s - 4 - k} e^{3 E[\partial^{|I|}m]}(\rho^{m-1}, t, \theta)
\]

\[
\leq e^{\gamma_m} B^2 \epsilon^{2 - \frac{1}{4}} d\epsilon \left[ \sqrt{(1 + O(\epsilon))} \sum_{s - 3 - 4c < |I| \leq s - 4 - k} e^{3 E[\partial^{|I|}m]}(\epsilon, t, \theta) \right]
\]

\[\square\]
Then in the case of lower middle-order derivatives \((4c - 1 \geq k > 2c; p = 1 - \frac{1}{k})\) we note that \(c - \frac{k}{4} + \frac{3}{2} > c - \frac{k}{2} + \frac{1}{4} > 1\), and that the power of \(r\) is \(-c + \frac{3}{2} + \frac{1}{4}\); in particular the power is by \(\frac{1}{4}\) larger than we need. Using that \(r \in (0, 2c)\), we derive that:

\[
\sqrt{\sum_{s-3-4c < |I| \leq s-3-k} r^{3}E[\partial^{j-\gamma}I_{\text{rest}}]}(r, t, \theta) \leq \frac{C}{2} (\eta + \epsilon^{\frac{1}{4}} CB r^{-c + \frac{3}{2} + \frac{1}{4}} + 2BC \eta r^{-c + \frac{3}{2} + \frac{1}{4}}). \tag{5.61}
\]

In view of the bounds \([4.9]\) we have imposed on \(\epsilon\) in terms of \(C, B\) we derive:

\[
\sqrt{\sum_{s-3-4c < |I| \leq s-3-k} r^{3}E[\partial^{j-\gamma}I_{\text{rest}}]} \leq \frac{3}{4} C \eta r^{-c + \frac{3}{2}}. \tag{5.62}
\]

So by a finite induction on \(k\) we derive:

\[
\sqrt{r^{3}E[\partial^{j-\gamma}I_{\text{rest}}]} \leq C \eta r^{-c + \frac{3}{2}}, \quad \text{for every } |I| = s - 4 - k, 2c < k < 4c. \tag{5.63}
\]

This confirms our inductive claim in this case.

On the other hand, for the case \((k \leq 2c; p = 1)\) we do not have the gain of a power of \(r^{-\frac{1}{4}}\) more than we need. In particular our bound \([5.59]\) implies:

\[
\sqrt{\sum_{s-3-4c < |I| \leq s-3-k} r^{3}E[\partial^{j-\gamma}I_{\text{rest}}]}(r, t, \theta) \leq \frac{C}{2} (\eta + \epsilon^{\frac{1}{4}} CB r^{-c + \frac{3}{2}} + \sqrt{\frac{5C}{\epsilon} \eta r^{-c + \frac{3}{2}}}) \tag{5.64}
\]

So in this case to derive our inductive claim we invoke the lower bounds \([4.6]\) we imposed on \(c > 1\) coupled with \([4.9]\) for the second term to show that the RHS is \(\lesssim C \eta r^{-c + \frac{3}{2}}\).

Thus matters are reduced to proving Proposition \([5.10]\). We do this next:

**Proof of Proposition \([5.10]\)** We argue by finite induction. Starting from the estimate \([5.57]\), \(|I| = s - 4\), that we proved above, assume \([5.58]\) and hence \([5.62]\) are valid for \(|I| = s - 3 - 1, \ldots, s - 3 - k + 1\). We will derive \([5.59]\) for the fixed \(0 < k < 4c\). Recall that \([5.54]\) is valid for any multi-index \(I\). We proceed to estimate the last term in the RHS of \([5.54]\) for \(s - 3 - 4c < |I| \leq s - 3 - k\) by plugging in the wave equation \([5.38]\) and arguing similarly to \([5.55]\) to obtain:

\[
\left| \int_{\Sigma_{r}} \Phi^{m-1} \int_{I_{1}^{m-1} I_{2}^{m-1}} e_{0} \partial^{j-\gamma}I_{\text{rest}} \partial^{j-\gamma}I_{\text{rest}} \text{vol}_{\Sigma}, d\tau \right| 
\leq \int_{\Sigma_{r}} \Phi^{m-1} \int_{I_{1}^{m-1} I_{2}^{m-1}} e_{0} \partial^{j-\gamma}I_{\text{rest}} \sum_{I_{1}^{m-1} I_{2}^{m-1}} \partial^{j+1} \text{tr}_{m-1} K^{m-1} e_{0} \partial^{j+2} \gamma \text{vol}_{\Sigma}, d\tau 
+ \int_{\Sigma_{r}} \Phi^{m-1} \int_{I_{1}^{m-1} I_{2}^{m-1}} e_{0} \partial^{j-\gamma}I_{\text{rest}} \sum_{I_{1}^{m-1} I_{2}^{m-1}} \partial^{j+1} (\text{tr}_{m-1} K^{m-1} - \text{tr}_{m-1} K^{\frac{5}{2}}) e_{0} \partial^{j+2} \gamma \text{vol}_{\Sigma}, d\tau 
+ \int_{\Sigma_{r}} \Phi^{m-1} \int_{I_{1}^{m-1} I_{2}^{m-1}} e_{0} \partial^{j-\gamma}I_{\text{rest}} \text{RHS}[5.38] \text{vol}_{\Sigma}, d\tau.
\tag{5.65}
\]

Here \(\text{RHS}[5.38]\) stands for all the other terms in the RHS of \([5.38]\) except for the two we wrote out explicitly (involving \(\text{tr}_{m-1} K^{m-1}\)). Given the estimates we have derived for all these terms, we find their contribution to be below-borderline; thus invoking Cauchy-Schwartz we find they contribute to all the terms in the RHS of \([5.58]\). Thus matters are reduced to controlling the first two lines in the RHS of \([5.65]\).

At this point our method proof deviates from that of Proposition \([5.7]\) due to the more careful handling needed for the first two terms in the RHS of \([5.65]\) in order to derive the desired stronger conclusion. These are the only borderline terms in the middle order energy estimates and combined, these give the last term in the RHS of the energy inequality \([5.58]\), instead of the borderline coefficient \(\sim \eta^{-1}\) in the
third line of (5.55). In fact the borderline terms correspond to $I_1 = I, I_2 = \emptyset$, $|I| = s - 3 - k$, while the rest of the summands can be easily seen by the inductive assumption (4.45), (4.52) on $K^{m-1}$ to be below borderline. We control these other, below-borderline terms as follows:

$$\left| \int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \cdot \sum_{I_1 \cup I_2, |I_1| < |I|, |I_2| < |I|} \partial^I \text{tr}_{m-1} \gamma K^{m-1} e_0 \partial^J \gamma^m_{\text{rest}} \text{vol}_{\Sigma_\tau} d\tau \right|$$

(5.66)

$$\leq C_{\text{Sub}} \int_{\rho^{m-1}}^{\epsilon} \sqrt{\tau} \left\| \text{tr}_m - \gamma K^{m-1} \right\|_{H^{s-3+4c}} \sum_{s-3+4c < |I| \leq s-3-k} \tau^3 E[\partial^I \gamma^m_{\text{rest}}] d\tau$$

$$+ C_{\text{Sub}} \int_{\rho^{m-1}}^{\epsilon} \sqrt{\tau} \left\| e_0 \gamma^m_{\text{rest}} \right\|_{H^{s-3+4c}} \left\| \text{tr}_m - \gamma K^{m-1} \right\|_{H^{s-3-k}} \tau^\frac{3}{2} \left\| e_0 \partial^J \gamma^m_{\text{rest}} \right\|_{L^2} d\tau$$

The same argument can be used to bound

$$\left| \int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \cdot \sum_{I_1 \cup I_2, |I_1| < |I|, |I_2| < |I|} \partial^I \text{tr}_{m-1} \gamma K^{m-1} e_0 \partial^J \gamma^S \text{vol}_{\Sigma_\tau} d\tau \right|$$

by the same bounds.

We proceed now to the borderline term with $I_1 = I, I_2 = \emptyset$, $|I| = s - 3 - k$. The key observation that allows us to handle this term in a more refined manner is the following: having at our disposal the already derived energy bound (5.55), we may integrate by parts and view the resulting term as an inhomogeneous term that we have already controlled. Moreover, in the case $k > 2c$, we may also exploit the splitting of the mean curvature (4.52) at the lower derivatives, after performing $4c - k$ consecutive integrations by parts to offload derivatives from $\partial^I \left( [\text{tr}_{m-1} K^{m-1} - \text{tr}_g S] \right)$. More precisely, we treat this term as follows:

Case $k \leq 2c$: Integrate by parts once the $I_1 = I, I_2 = \emptyset$ term in (5.65), where $|I| = s - 3 - k$, and use the induction step together with the assumption (4.45) to derive

$$\int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \partial^I \text{tr}_{m-1} \gamma K^{m-1} e_0 \gamma^m_{\text{rest}} \text{vol}_{\Sigma_\tau} d\tau$$

(5.67)

$$- e_0 \int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \partial^I \left( [\text{tr}_{m-1} \gamma K^{m-1} - \text{tr}_g S] \right) \partial^J \left( \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \cdot e_0 \gamma^m_{\text{rest}} \text{vol}_{\Sigma_\tau} / \text{vol}_{Euc} \right) \text{vol}_{Euc} d\tau$$

$$\left( I_1 \cup I_2 = I, |I_2| = 1 \right)$$

$$\leq \int_{\rho^{m-1}}^{\epsilon} \frac{3 C^2 \eta_1^3}{\tau^{3/2 + k/2}} d\tau.$$

$$\left( \Phi \sim \sqrt{\tau}, \text{vol}_{\Sigma_\tau} \sim \tau^{3/2} \text{vol}_{Euc}, e_0 \gamma^m_{\text{rest}} \sim \tau^{-3/2} \right)$$

We analogously treat the borderline terms from the second line RHS of (5.65), which arose from from $\Box_{m-1} = \gamma^S$.

$$\int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \partial^I \left( \text{tr}_{m-1} \gamma K^{m-1} - \text{tr}_g S \right) e_0 \gamma^S \text{vol}_{\Sigma_\tau} d\tau$$

(5.68)

$$- e_0 \int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \partial^I \left( [\text{tr}_{m-1} \gamma K^{m-1} - \text{tr}_g S] \right) \partial^J \left( \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \cdot e_0 \gamma^S \text{vol}_{\Sigma_\tau} / \text{vol}_{Euc} \right) \text{vol}_{Euc} d\tau$$

$$\left( I_1 \cup I_2 = I, |I_2| = 1 \right)$$

$$\leq \int_{\rho^{m-1}}^{\epsilon} \frac{4 C^2 \eta^2}{\tau^{2 - 2c + 3/2}} d\tau.$$

$$\left( \Phi \sim \sqrt{\tau}, \text{vol}_{\Sigma_\tau} \sim \tau^{3/2} \text{vol}_{Euc}, e_0 \gamma^m_{\text{rest}} \sim \tau^{-3/2} \right),$$

with the factor of $2M$ cancelling out.

Case $k > 2c$: We need to bound the same two terms as in the previous case. In this setting, we integrate by parts $4c - k$ the $I_1 = I, I_2 = \emptyset$ written-out terms in (5.65), where $|I| = s - 3 - k$:

$$\int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \partial^I \text{tr}_{m-1} \gamma K^{m-1} e_0 \gamma^m_{\text{rest}} \text{vol}_{\Sigma_\tau} d\tau$$

(5.69)

$$= (-1)^{4c-k} \int_{\rho^{m-1}}^{\epsilon} \int_{\Sigma_\tau} \partial^I \left( \text{tr}_{m-1} \gamma K^{m-1} \right) \partial^J \left( \Phi^{m-1} \tau^\frac{3}{2} e_0 \partial_t \gamma^m_{\text{rest}} \cdot e_0 \gamma^m_{\text{rest}} \text{vol}_{\Sigma_\tau} / \text{vol}_{Euc} \right) \text{vol}_{Euc} d\tau$$

$$\left( I_1 \cup I_2 = I, |I_2| = 4c - k \right)$$

73
\[
\leq \int_{\mu_{m-1}}^{\epsilon} \| \partial^{j_1} \text{tr}_{m-1} \eta^{K^{m-1}} \|_{L^2 C_4 \eta \sqrt{\tau}} \| \tau^{\frac{3}{4}} e_0 \partial \gamma_{\text{rest}}^{m} \|_{H^{4c-k}} \text{vol}_{Euc} d\tau \quad (|I_1| = s - 3 - 4c, \text{using } s > 8c + 3)
\]
\[
\leq \int_{\mu_{m-1}}^{\epsilon} \frac{BC^2 \eta^2}{\tau^{1 - \frac{3}{k}}} \tau c e_0 - e_0^{2k+4} d\tau
\]
(by (4.52) and the induction step for \( \partial^{j_2} \gamma^{m} \), \(|I_2| + |I| = s - 3 - (2k - 4c)\))
\[
= \int_{\mu_{m-1}}^{\epsilon} \frac{BC^2 \eta^2}{\tau^{1 - \frac{3}{k}}} \tau c e_0 - e_0^{2k+4} d\tau
\]

We also apply the same argument to the second borderline term with the factor \( \gamma^S \) again to find:

\[
\int_{\mu_{m-1}}^{\epsilon} \int_{\Sigma_{r}} \Phi^{m-1} \frac{\tau^{3}}{\tau^{2}} e_0 \partial^{j_1} (\gamma_{\text{rest}}^{m}) \partial^{j_1} \text{tr}_{m-1} \gamma^{K^{m-1}} \text{vol}_{Euc} d\tau
\]
\[
= (-1)^{4c-k} \int_{\mu_{m-1}}^{\epsilon} \int_{\Sigma_{r}} \partial^{j_1} (\text{tr}_{m-1} \gamma^{K^{m-1}} \text{tr}_{m-1} \gamma^{K^{S}}) \partial^{j_2} \left( \Phi^{m-1} \frac{\tau^{3}}{\tau^{2}} e_0 \partial \gamma_{\text{rest}}^{m} \text{vol}_{Euc} \right)_{I_1 \cup I_2 = I} \text{vol}_{Euc} d\tau \quad (|I_1| = s - 3 - 4c, \text{using } s > 8c + 3)
\]
\[
\leq C_2 \int_{\mu_{m-1}}^{\epsilon} \| \partial^{j_1} (\text{tr}_{m-1} \gamma^{K^{m-1}} \text{tr}_{m-1} \gamma^{K^{S}}) \|_{L^2 C_4 \eta \sqrt{\tau}} \| \tau^{\frac{3}{4}} e_0 \partial \gamma_{\text{rest}}^{m} \|_{H^{4c-k}} d\tau
\]
\[
= \int_{\mu_{m-1}}^{\epsilon} C_2 BC \eta^2 \tau c e_0 - e_0^{2k+4} d\tau
\]
(by (4.52) and the induction step for \( \partial^{j_2} \gamma^{m} \), \(|I_2| + |I| = s - 3 - (2k - 4c)\))
\[
= \int_{\mu_{m-1}}^{\epsilon} B^2 \eta^2 \tau c e_0 - e_0^{2k+4} d\tau
\]

The RHSs of (5.67), (5.68), (5.69), (5.70) correspond exactly to the last term in the claimed energy inequality (5.58). Thus, combining (5.54), (5.67), (5.68), (5.69), (5.70) and summing in \( s - 3 - 4c < |I| \leq s - 3 - k \), we arrive at (5.58). This completes the proof of the proposition and hence the higher order energy estimates (4.24) for \( \gamma_{\text{rest}}^{m} \).

\[\square\]

6.4 Top order estimates for \( \gamma^{m} \)

The top order inductive assumptions (4.24) that we wish to derive for \( \gamma_{\text{rest}}^{m} \) are divided into cases based on \( J_0 \), \( J_0 = 0, 1, 2 \). We only study the top-order case \( J_0 = 2 \), since the remaining cases follow from it by taking integrals of the top order estimates in the \( e_0 \) direction.

We also note that at this top order it suffices to derive our claim for \( \gamma^{m} \) directly instead of \( \gamma_{\text{rest}}^{m} \). This is just because

\[
\partial^{j} e_0 \partial \gamma_{\text{rest}}^{m} = \partial^{j} e_0 \gamma^{m} + \partial^{j} e_0 \left( \frac{2M}{r} - 1 \right)^{-\frac{1}{2}} \cdot 1
\]

The second term in the RHS has already been bounded in Lemma (4.13) and the bounds are better than those claimed on the LHS. So it suffices to derive the claimed bounds on \( \gamma^{m} \). We also recall that our claim (4.24) is for all derivatives \( \partial^{j} \) but where \( I \neq (T, \ldots, T) \). This part will be used in deriving bounds on some top-order Christoffel symbols.

We will prove our claimed estimate for \( \partial^{j} \partial_{p_{r}} \gamma^{m} \), with the RHS in our claim having an extra factor \( (\frac{2M}{r} - 1)^{-\frac{1}{2}} \). In view of the relation (4.10), this clearly implies our claim with two \( e_0 \)-derivatives in place of the two \( \partial_{r} \)-derivatives.

So we use \( \partial_{p_{r}} \) instead of \( e_0 \) as a commutator field. We derive the equation:

\[
\square \gamma_{m-1} \partial^{j} \partial_{p_{r}} \gamma^{m} = \sum_{l_1 \cup l_2 = l, j_1 + j_2 = |I_2| + |J_2| < s-1} \partial^{j_1} \partial^{j_1} \left[ (\frac{2M}{r} - 1) [1 + \partial_{r} \chi(r)(m-1-r_{s}-e)] \cdot \partial^{j_2} \partial_{p_{r}} \partial^{m-1} \gamma^{m} \right] + \sum_{l_1 \cup l_2 = l, j_1 + j_2 = |I_2| + |J_2| < s-1} \left[ \partial^{j_1} \partial^{j_1} \left[ - \text{tr}_{m-1} \gamma^{K^{m-1}} - \frac{M}{r^2} \left( \frac{2M}{r} - 1 \right)^{-\frac{1}{2}} + \left( \frac{2M}{r} - 1 \right)^\frac{1}{2} \right] \right]
\]
\[
\partial_r^2 \chi(r)^{(m-1)r^* - \epsilon} [1 + \partial_r \chi(r)^{(m-1)r^* - \epsilon}] \left[ \partial^2 \gamma^m \right]
\]
Proposition 5.12. Assuming the inductive estimates in \((5.4), (5.77)\) and the lower order estimates \((5.57), (5.63)\), then the following energy inequality is valid at the top order:

\[
\sum_{|I|=s-3, \ell=1,2} (m-1) \rho^3 (1 + O(\rho^{-1})) E[\overline{\partial}^{\ell} \rho_{\rho \gamma}^m \rho_{\rho \gamma}^m] \leq \sum_{|I|=s-3, \ell=1,2} \left[ \epsilon^2 (1 + O(\epsilon)) E[\overline{\partial}^{\ell} \rho_{\rho \gamma}^m (e, t, \theta)] \Sigma \right]
\]

\[
+ \epsilon^{1/4} B \sqrt{\rho} \tau^{-2c} + \int_{m-1}^\epsilon \frac{BC \epsilon}{\tau} \rho^3 [\overline{\partial}^{\ell} \rho_{\rho \gamma}^m] d\tau + \int_{m-1}^\epsilon \left( \frac{C \eta}{\tau} + \frac{B^2}{\tau} \right) \sqrt{\rho} \tau^{-2c} \sqrt{\rho} \tau^{-2c} E[\overline{\partial}^{\ell} \rho_{\rho \gamma}^m] d\tau
\]

for all \(\rho^{m-1} \in (0, 2\epsilon)\).

Applying Lemma 4.8 to \((5.77)\) and arguing as in \((5.56)\), we obtain the desired top order estimate:

\[
\sqrt{(\rho^{-1})^3 E[\overline{\partial}^{\ell} \rho_{\rho \gamma}^m]} \leq C \eta^{m-1} \rho^{-c}, \quad \text{for all } \rho^{m-1} \in (0, \epsilon),
\]

in view of the bounds \((4.6), (4.9)\) that we have imposed. So we proceed to prove the Proposition:

Proof of Proposition 5.12. It suffices to prove the validity of the splitting \((5.73)\):

\[
\int_{m-1}^\epsilon \int_{\Sigma_{\tau}} \Phi^{m-1} \tau^2 e_0 \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \frac{\partial}{\rho \gamma} \rho_{\rho \gamma}^m \text{vol}_{\Sigma}, d\tau
\]

\[
= \int_{m-1}^\epsilon \int_{\Sigma_{\tau}} (\tilde{\delta}_{\text{Gron}} + \tilde{\delta}_{\text{bound}} + \tilde{\delta}_{\text{IBP}}) \text{vol}_{\Sigma}, d\tau
\]

where the \(\tilde{\delta}_{\text{Gron}}, \tilde{\delta}_{\text{IBP}}\) terms satisfy \((5.74)\), after possibly integrating parts, while the borderline terms included in \(\tilde{\delta}_{\text{bound}}\) satisfy \((5.75)\).

We start by grouping together all the terms which are strictly below borderline.

Lemma 5.13. Consider the RHS of \((5.71)\). All terms with \(|I| < s-3\) are placed in \(\mathcal{F}_{\text{Gron}}\).

The further terms from the RHS of \((5.71)\) that fall under \(\mathcal{F}_{\text{Gron}}\) are all terms with \(|I| = s-3\) that do not involve a differentiated Christoffel symbol, nor \(K^{m-1}\).

The remaining term in the RHS of \((5.71)\) are divided as follows: Those with \(|I| = s-3\) involving a differentiated Christoffel symbol are placed in \(\mathcal{F}_{\text{IBP}}\); so are all terms involving \(K\) except the terms involving \(\overline{\partial}^{\ell} [\text{tr} K^{m-1}] e_0^m \gamma\). These latter term are placed in \(\mathcal{F}_{\text{bound}}\).

Proof. We first show that all the terms that we placed in \(\mathcal{F}_{\text{Gron}}\) satisfy the bound \((5.74)\).

This follows by directly invoking all the bounds on the background geometry that are collected in the previous section, as well as the Cauchy-Schwarz and the product inequality straightforwardly applied. For the terms which contain factors \(\overline{\partial}^{\ell} \rho_{\rho \gamma}^m \frac{\partial}{\rho \gamma} \rho_{\rho \gamma}^m (\sin \theta \cdot \rho_\theta \gamma^m)\) with \(|I| < s-3\) we may need to apply the Hardy inequality to the first factor, as was done for the lower-order terms. The restrictions we have imposed on \(I\) for the terms that we placed in \(\mathcal{F}_{\text{Gron}}\) imply that the resulting terms we obtain lie in spaces that have been already bounded, and satisfy the bounds in \((5.74)\).

Let us consider the terms that we placed in \(\mathcal{F}_{\text{IBP}}\). We commence with the most involved such term, which will be treated using integrations by parts. The remaining terms can be treated by a similar integration by parts argument. The most involved term is:

\[
\int_{m-1}^\epsilon \int_{\Sigma_{\tau}} \Phi^{m-1} \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \frac{\partial}{\rho \gamma} \rho_{\rho \gamma}^m \sin \theta \rho_\theta \gamma \cdot e_0 \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \text{dvol}_{\Sigma}, d\tau
\]

This summam require an integration by parts, since the Christoffel terms are bounded up to \(|I| = s-4\) derivatives, as opposed to the \(|I| = s-3\) we have here. We integrate by parts in the derivatives \(\overline{\partial}^{\ell}\) in the first factor, to derive, up to below-borderline terms:

\[
\int_{m-1}^\epsilon \int_{\Sigma_{\tau}} \Phi^{m-1} \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \frac{\partial}{\rho \gamma} \rho_{\rho \gamma}^m \sin \theta \rho_\theta \gamma \cdot e_0 \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \text{dvol}_{\Sigma}, d\tau
\]

\[
- \int_{m-1}^\epsilon \int_{\Sigma_{\tau}} \Phi^{m-1} \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \frac{\partial}{\rho \gamma} \rho_{\rho \gamma}^m \sin \theta \rho_\theta \gamma \cdot e_0 \overline{\partial}^{\ell} \rho_{\rho \gamma}^m \text{dvol}_{\Sigma}, d\tau
\]

\[
(5.80)
\]
where \(|I| = s - 4\). (Note that the second term in the RHS can be bounded as required in (5.76), so we may consider only the first term in the RHS. In that term, the last factor in the RHS has an extra derivative \(\partial_\tau\) from the integration by parts.) Now we integrate by parts the derivative \(e_0\) in the second factor. We derive, up to below-borderline terms:

\[
\int_{\Sigma}^{e_{m-1}} \nabla_t \partial_\rho \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \partial_{\rho \rho} \partial_\tau \gamma^m \, dv \Sigma, d\tau
= \int_{\Sigma}^{e_{m-1}} e_0 \nabla_t \partial_\rho \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \nabla_t \partial_\tau \gamma^m \, dv \Sigma, d\tau
+ \int_{\Sigma_{\rho = m-1}} \partial_\rho \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \nabla_t \partial_\tau \gamma^m \, dv \Sigma, d\tau.
\]

(5.81)

Here \(|I| = s - 4\) and \(|I'| = s - 2\). Note in particular that the integrals \(\int_{\Sigma} \nabla_t \partial_\rho \gamma^m \, dv \Sigma, d\tau\) can be bounded by the top-order energy of \(E[\nabla_t \partial_\rho \gamma^m]\) (where \(|I' = s - 3\), times \(\tau^{-5-\epsilon}\):

\[
\int_{\Sigma} \nabla_t \partial_\rho \gamma^m \, dv \Sigma, d\tau \leq \int_{\Sigma} |(a^{m-1})_L \Gamma^{m-1}_t \nabla_t \partial_\rho \gamma^m| \, dv \Sigma, d\tau \leq \tau^{-5-\epsilon} E[\nabla_t \partial_\rho \gamma^m].
\]

(We have used the pointwise bounds in Lemma 4.16). By this argument we derive that all the terms in the RHS of (5.81) are below borderline and thus they can be bounded as in (5.74). All the remaining terms in \(\mathcal{F}_{IBP}\) can also be treated in this way—integrating one spatial derivative onto \(e_0 \partial_\rho \partial_{\rho \rho} \gamma^m\), followed by another integration in \(e_0\) from the resulting factor. The resulting terms can be bounded as claimed in (5.76).

Next, we treat the borderline terms, as defined in Lemma 5.13. These are when \(|I| = |I_1|\), and all the derivatives in \(\nabla_t\) hit the term \(\text{tr} K^{m-1}\). Thus these borderline terms as they appear in the bulk integral are:

\[
\nabla_t \partial_\rho \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \partial_{\rho \rho} \partial_\tau \gamma^m
\]

\(J_1 + J_2 = 2\). \(|I| = s - 3\). Then all terms with \(J_1 = 0, 1, 2\) all give rise to borderline terms. Then the term involving \(K^{m-1} \text{rest}\) will be placed in \(\mathcal{F}_{IBP}\) and treated right below. For the remaining term, we derive the estimates (using the standard volume element \(\sin \theta d\theta dt\) on \(\{\rho^{m-1} = \tau\}\):

\[
\|\nabla_t \partial_\rho \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \partial_{\rho \rho} \partial_\tau \gamma^m \|_{L^2(\Sigma_{\rho = m-1})}
\]

\[
\leq 4 C \eta \tau^{-5-c}.
\]

(5.82)

This then directly implies that those terms are bounded as claimed in (5.75). The exact same bounds hold for the other borderline terms:

\[
\|\nabla_t \partial_0 \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \partial_{\rho \rho} \partial_\tau \gamma^m \|_{L^2(\Sigma_{\rho = m-1})}
\]

\[
\leq 8 C \eta \tau^{-5-c}.
\]

(5.83)

\[
\|\nabla_t \partial_\rho \partial_\tau \left( \Gamma^{m-1}_t \right)_{ij} \frac{\partial^2}{\partial \rho^2} \frac{\partial_{\rho \rho} \gamma^m}{\sin \Theta} \cdot \partial_{\rho \rho} \partial_\tau \gamma^m \|_{L^2(\Sigma_{\rho = m-1})}
\]

\[
\leq 8 C \eta \tau^{-5-c}.
\]

(5.84)

The term we placed in \(\mathcal{F}_{IBP}\) is treated as above by an integration by parts; the terms resulting from this operation will all be below-borderline.

We have then shown that all terms in the RHS of (5.71) fall under the categories \(\mathcal{F}_{IBP}, \mathcal{F}_{Gron}, \mathcal{F}_{Bord}\), and derived the bounds we claimed on those. We thus derive Proposition 5.77.
5.5 Renormalized energy estimates at the low orders: Proof of (4.25).

The optimal energy bound \( \gamma_{\text{rest}}^m \) for the lower derivatives of \( \gamma_{\text{rest}}^m \) yields a logarithmic upper bound in \( L^\infty \) for \( \gamma_{\text{rest}}^m \) itself. Indeed, integrating \( \partial \gamma^m \) over \( [r, r_m^{m-1}] \), we obtain

\[
\begin{align*}
|\gamma_{\text{rest}}^m(r, t, \theta)| & \leq |\gamma_{\text{rest}}^m(r_m^{m-1}, t, \theta)| + \int_r^{r_m^{m-1}} \partial_r \gamma_{\text{rest}}^m \, dr \leq |\gamma_{\text{rest}}^m(r_m^{m-1}, t, \theta)| + C \int_r^{r_m^{m-1}} \sqrt{2} \| \epsilon_0 \gamma_{\text{rest}}^m \|_{H^2} \, dr \\
& \leq |\gamma_{\text{rest}}^m(r_m^{m-1}, t, \theta)| + (2M)^{-\frac{3}{4}} \eta \int_r^{r_m^{m-1}} \frac{1}{r} \, dr = |\gamma_{\text{rest}}^m(r_m^{m-1}, t, \theta)| + (2M)^{-\frac{3}{4}} \eta \log \frac{r_m^{m-1}}{r}
\end{align*}
\]

However, in order to prove the leading order behaviour \( (4.25) \) for \( \gamma_{\text{rest}}^m \), we need to derive renormalised energy estimates for the variable \( \gamma_{\text{rest}}^m \). For the rest of this subsection we write \( \rho \) instead of \( \rho^{m-1} \), for brevity. We compute

\[
\Box_{m-1}^0 \gamma_{\text{rest}}^m = \nabla \gamma_{\text{rest}}^m \nabla \gamma_{\text{rest}}^m - \frac{1}{\log \rho} \frac{1}{\log \rho} - m \gamma^S_m - \Box_{m-1} \gamma^S
\]

(5.86)

Notice that the most singular zeroth order terms in the RHS cancel, leaving

\[
\Box_{m-1}^0 \gamma_{\text{rest}}^m = -2(2M - 1)^{\frac{3}{4}} \frac{1}{\rho \log \rho} e_0(\gamma_{\text{rest}}^m) - 2 \frac{1}{\rho \log \rho} \sum_{A \in \mathcal{T}, \rho_0} \gamma_{\text{rest}}^m A^\mu A^\mu \rho_{\rho \rho} \rho_{\rho \rho} - \Box_{m-1} \gamma^S
\]

(5.87)

Next we will commute the above equation with \( \tilde{\Theta}^I \), \( |I| \leq s-3-4c \). We obtain an equation on \( \Box_{m-1} \tilde{\Theta}^I \gamma_{\text{rest}}^m \); the terms in the RHS are \( \tilde{\Theta}^I \) acting on the RHS of (5.86) as well as the commutation terms generated by commuting \( \tilde{\Theta}^I \) with the wave operator \( \Box_{m-1} \). The latter have already been computed in section 5.1.2. Combining with the lower-order estimates that we have already obtained on \( \gamma_{\text{rest}}^m \), we derive that those terms are bounded in \( L^2_{[\sin \Theta \cos \theta \sin \theta \sin \theta \sin \theta] \Sigma_0} \) by

\[
B \rho^{-3 \frac{3}{4}} |\log \rho|^{-1}.
\]

(5.88)

Analogously, we note that \( \tilde{\Theta}^I \gamma_{\text{rest}}^m \) has already been bounded, in section 5.1.1, and is bounded by the same quantity. In short, letting \( O_{L^2_{[\sin \Theta \cos \theta \sin \theta \sin \theta \sin \theta] \Sigma_0}}(\rho^{-3 \frac{3}{4}} |\log \rho|^{-1}) \) stand for a general sum of terms that are bounded in \( L^2_{[\sin \Theta \cos \theta \sin \theta \sin \theta \sin \theta] \Sigma_0} \) by (5.88), we obtain:

\[
\Box_{m-1} \tilde{\Theta}^I \gamma_{\text{rest}}^m = O_{L^2_{[\sin \Theta \cos \theta \sin \theta \sin \theta \sin \theta] \Sigma_0}}(\rho^{-3 \frac{3}{4}} |\log \rho|^{-1}) - \tilde{\Theta}^I \left( -2(2M - 1)^{\frac{3}{4}} \frac{1}{\rho \log \rho} e_0(\gamma_{\text{rest}}^m) - 2 \frac{1}{\rho \log \rho} \sum_{A \in \mathcal{T}, \rho_0} \gamma_{\text{rest}}^m A^\mu A^\mu \rho_{\rho \rho} \rho_{\rho \rho} - \Box_{m-1} \gamma^S \right)
\]

(5.89)

In the next proposition we derive improved estimates for \( \gamma_{\text{rest}}^m \) that also confirm (4.25)-(4.26).
Proposition 5.14. The following renormalised estimate for $\frac{\gamma_m}{\log r}$ is valid:

$$ r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \leq C \eta, \quad r \in (0, 2\epsilon). $$

(5.90)

Moreover, $\gamma^m$ has the expansion:

$$ \gamma^m = \alpha^m(t, \theta) \log r + \log \sin \theta + \gamma^m_1(r, t, \theta), $$

(5.91)

where

$$ \alpha^m(t, \theta) - 1 \in H^{1-3-4c}, \quad \|\alpha^m - 1\|_{H^{1-3-4c}} \leq C \eta, $$

(5.92)

and $e_0 \gamma^m_1$ satisfies the estimate

$$ \|e_0 \gamma^m_1\|_{H^{1-3-4c}} \leq B r^{-\frac{3}{2}}|\log r|^{\frac{1}{2}}, \quad |\log r| = 1, 2, $$

(5.93)

for all $r \in (0, 2\epsilon]$.

Proof: Putting $v = \partial^I \frac{\gamma_m}{\log r}$, $f(r) = r^3 \log r |d|^{4}$ in (5.8) and utilising (5.9), (5.4), (5.5) we deduce that

$$ \int_{\Sigma_{\mu-1}} (1 + O(\epsilon)) r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \|\nabla \partial^I \frac{\gamma_m}{\log r} \| \text{vol}_{Euc} $$

(5.94)

$$ \leq \int_{\Sigma_{\mu-1}} (1 + O(\epsilon)) r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \|\nabla \partial^I \frac{\gamma_m}{\log r} \| \text{vol}_{Euc} $$

$$ \leq \int_{\Sigma_{\mu-1}} (1 + O(\epsilon)) r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \|\nabla \partial^I \frac{\gamma_m}{\log r} \| \text{vol}_{Euc} $$

Note that the second term in the RHS of (5.94) has an unfavourable sign for an upper bound and according to (5.4), (5.5), the coefficient of the $\tau^3 |\log r|^4 (e_0 \partial^I \frac{\gamma_m}{\log r})^2$ is of the order $\tau^{-3} |\log r|^{-1}$ which fails to be integrable in $[0, \epsilon]$. Normally, this would prevent us from deriving a uniform Gronwall type energy estimate. However, as we shall see, there is a crucial cancellation coming from the RHS of (5.8) that will allow us to apply the Gronwall inequality and derive the claimed estimate.

$$ \int_{\Sigma_{\mu-1}} (1 + O(\epsilon)) r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \|\nabla \partial^I \frac{\gamma_m}{\log r} \| \text{vol}_{Euc} $$

(5.95)

$$\leq \int_{\Sigma_{\mu-1}} (1 + O(\epsilon)) r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \|\nabla \partial^I \frac{\gamma_m}{\log r} \| \text{vol}_{Euc} $$

$$\leq \int_{\Sigma_{\mu-1}} (1 + O(\epsilon)) r^3 \log r \left| \sum_{|I| \leq 3-4c} E[\partial^I \frac{\gamma_m}{\log r}] \right| \|\nabla \partial^I \frac{\gamma_m}{\log r} \| \text{vol}_{Euc} $$

Note that the second term in the RHS of (5.94) has an unfavourable sign for an upper bound and according to (5.4), (5.5), the coefficient of the $\tau^3 |\log r|^4 (e_0 \partial^I \frac{\gamma_m}{\log r})^2$ is of the order $\tau^{-3} |\log r|^{-1}$ which fails to be integrable in $[0, \epsilon]$. Normally, this would prevent us from deriving a uniform Gronwall type energy estimate. However, as we shall see, there is a crucial cancellation coming from the RHS of (5.8) that will allow us to apply the Gronwall inequality and derive the claimed estimate.
To close the estimate, it remains to bound the $H^{s-3-4c}$ norm of $\gamma^m_{\log \tau}$ by the corresponding energy:

$$-\frac{1}{2} \partial_r \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}}^2 \leq \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} \| \partial_r \gamma^m_{\log \tau} \|_{H^{s-3-4c}}$$

$$\implies \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} \leq \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} \leq \| \partial_r \gamma^m_{\log \tau} \|_{H^{s-3-4c}}$$

$$\implies \| \partial_r \gamma^m_{\log \tau} \|_{H^{s-3-4c}} \leq \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} + \int_{\rho^{m-1}}^c \frac{2}{\tau} \| E[\partial_r \gamma^m_{\log \tau}] \| d\tau$$

Combining (5.95), (5.96) we deduce that

$$\sum \int_{|I| \leq s-3-4c} (1 + O(r)) r^4 \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} + \| \nabla \partial_r \gamma^m_{\log \tau} \| \text{vol}_{Euc}$$

$$\leq \int_{|I| \leq s-3-4c} (1 + O(r)) r^{m-13} \| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} + \| \nabla \partial_r \gamma^m_{\log \tau} \| \text{vol}_{Euc}$$

$$\leq \int_{|I| \leq s-3-4c} \frac{B}{\tau^{1 - \frac{2}{4}}} \sum \frac{1}{\tau^3} \| \nabla \partial_r \gamma^m_{\log \tau} \| d\tau + \epsilon^2 C^2 \eta^2$$

$$+ \epsilon^2 \| \nabla \partial_r \gamma^m_{\log \tau} \|_{H^{s-3-4c}} + \int_{\rho^{m-1}}^c \frac{CB}{\tau \log \tau^2} \sqrt{\tau^3} \| \nabla \partial_r \gamma^m_{\log \tau} \| d\tau$$

Thus, employing Lemma 4.8 and invoking our smallness assumptions on $\epsilon > 0$ relative to the other parameters, we arrive at (5.90).

The renormalised estimate (5.90) implies that the map $\gamma^m_{\log \tau} : \{ 0 < r < 2c \} \rightarrow H^{s-3-4c}(\Sigma_{\rho^{m-1}})$ is uniformly continuous:

$$\| \gamma^m_{\log \tau} \|_{H^{s-3-4c}} \leq \int_{\rho^{m-1}}^c \frac{CB}{\tau \log \tau^2} \sqrt{\tau^3} \| \nabla \partial_r \gamma^m_{\log \tau} \| d\tau$$

Hence, $\gamma^m_{\log \tau}$ has a limit in $H^{s-3-4c}$, as $r \rightarrow 0$, which we denote by $\alpha^m - 1 := \gamma^m_{\log \tau} - 1 \in H^{s-3-4c}$ and which by definition and (5.90) satisfies (5.92). From the previous computations it also follows that (given $s - 3 - 4c > 2$):

$$\| \alpha^m - 1(t, \theta) \|_{H^{s-3-4c}}$$
Thus, invoking the bound (5.90), the assumed bounds on $r^{m-1}$ and by virtue of the initial data assumption on $\gamma_{\text{rest}}$, we derive that:

$$\|\alpha^m(t, \theta) - 1\|_{H^{s-3-4c}} \leq C\eta \quad (5.98)$$

as claimed in our inductive step (4.26).

We next prove the inductive claim (4.25), (5.93) on the remainder term $\gamma_1^m$, which was defined via:

$$\gamma_1^m := \gamma^m - \alpha^m \log r - 2\log \sin \theta.$$

Consider

$$(m-1)^3\|e_0\alpha_1^m\|_{L^2} = (m-1)^3 \int_{\Sigma_{m-1}} [e_0(r_{\text{rest}} - (\alpha^m - 1) \log r)]^2 \text{vol}_{Euc} \quad (5.99)$$

$$= (m-1)^3 \int_{\Sigma_{m-1}} [e_0(\log r \cdot m_{\text{rest}} / log r - (\alpha^m - 1) \log r)]^2 \text{vol}_{Euc}.$$  

$$= 2(m-1)^3 \int_{\Sigma_{m-1}} |\log r|^2 [e_0(\log r_{\text{rest}} / log r)]^2 + (2M/r + O(r^3))\|r_{\text{rest}}^m / \log r\|^2 + |\alpha^m - 1|^2 - 2(\alpha^m - 1)\|r_{\text{rest}}^m / \log r\| \text{vol}_{Euc}.$$  

Therefore by the limit

$$\lim_{\rho \to 1} \frac{r_{\text{rest}}^m}{\log r} - (\alpha^m - 1)\|L^2[\Sigma_{\rho m-1}] = 0$$

derived just above, we conclude that $\rho^{m-1} \|e_0\alpha_1^m\|_{L^2} \to 0$ as $\rho^{m-1} \to 0$.

Note that by definition $\gamma_1^m$ satisfies that $\|\alpha^m \|_{H^{s-3-4c}}$ tends to zero, as $r \to 0$ [recalling that $e_0(\log \sin \theta) = 0$]. This implies that $\rho^{m-1} \|e_0\alpha_1^m\|_{L^2}$ tends to zero, as $r \to 0$. The above computation can be obviously iterated for $\partial^\gamma \gamma_1^m$, yielding

$$\lim_{r \to 0} \rho^{m-1} \|e_0\partial^\gamma \gamma_1^m\|_{L^2} = 0, \quad |I| \leq s - 3 - 4c. \quad (5.100)$$

The latter limit can be improved to a quantitative rate of decay with the use of the wave equation for $r_{\text{rest}}^m$ (5.38), which we rephrase plugging in (5.91). We recall that by the estimates on $r_{\text{rest}}^m$ that we have derived, as well as the bounds on the metric $g^{m-1}$ and its Christoffel symbols, which imply (in bounding the lower derivatives of $\gamma_{\text{rest}}^m$):

$$\|\Box_{g_{m-1}}^\text{mixed} \gamma_{\text{rest}}^m\|_{H^1} + \|\Box_{g_{m-1}}^\text{spatial} \gamma_{\text{rest}}^m\|_{H^1} + \|\Box_{g_{m-1}} \gamma^S\|_{H^1} \leq B\tau^{-2} - \frac{1}{2} - 2DC\eta.$$  

Thus plugging in (5.91) into the wave equation for $\gamma_{\text{rest}}^m$, we find:

$$- \Box_{g_{m-1}}^\text{mixed} \gamma_{\text{rest}}^m - \Box_{g_{m-1}}^\text{spatial} \gamma_{\text{rest}}^m - \Box_{g_{m-1}} \gamma^S \leq B\tau^{-2} - \frac{1}{2} - 2DC\eta.$$  

81
\[ r^2 e_0 \gamma_1^m = \int_{\delta}^{r} \left[ c_{\text{mixed}, m} \gamma_{\text{rest}} + c_{\text{spatial}, m} \gamma_{\text{rest}} \right] d\tau + O(1) \left( \frac{2M}{r} - 1 \right) \frac{1}{r} \text{tr}_{m-1} g_{m-1} u_{m-1} \right] r^2 e_0 \gamma_1^m + \left[ O(1) + \sqrt{r} \text{tr}_{m-1} g_{m-1} u_{m-1} \right] \alpha^m \right] d\tau + \delta \frac{2}{r} e_0 \gamma_1^m (\delta, t, \theta), \]

where \( O(1) \) in the last equality stands for an explicit, analytic function of \( r \). Taking the \( H^{s-3} \) norms of both sides of the last equation and using the inductive assumption for \( u_{m-1} \), the inductive bounds for \( g_{m-1} \) and its Christoffel symbols, the (now-derived) corresponding ones for \( \gamma_{\text{rest}} \), we obtain the inequality on all level sets of \( \rho_{m-1} \) and \( r \):

\[
\| r^2 e_0 \gamma_1^m \|_{H^{s-3-4c} [\Sigma_r]} \leq C (1 + DC \eta) \int_{\delta}^{r} \left[ r^2 \left[ \tau^{-2 - 2DC \eta \frac{1}{2} + \tau^{-1} \frac{1}{4}} \sum \frac{1}{r} \right] d\tau + \| r^2 e_0 \gamma_1^m (\delta, t, \theta) \|_{H^{s-3-4c}} \right]
\]

The estimate (5.93) for \( |J_0| = 1 \) follows by employing Gronwall’s inequality, letting \( \delta \to 0 \) and using (5.100).

For the case \( |J_0| = 2 \), we similarly argue by taking the \( H^{s-3-4c} \) norms of both sides of wave equation for \( \gamma_{\text{rest}} \), re-expressed as

\[
\frac{2}{r} e_0 \gamma_1^m = - \text{tr}_{m-1} g_{m-1} e_0 \gamma_1^m + O(1) \left( \frac{2}{r} \text{tr}_{m-1} g_{m-1} u_{m-1} \right) \left( \frac{2}{r} \right) - 1 \right) \frac{1}{r} \frac{2}{r} e_0 \gamma_1^m + \text{tr}_{m-1} g_{m-1} u_{m-1} + \gamma_{\text{rest}} + \gamma_{\text{rest}} \gamma_{\text{rest}} + \gamma_{\text{rest}} \gamma_{\text{rest}} \gamma_{\text{rest}}
\]

and using in addition the estimate (5.93) for \( |J_0| = 1 \) that we just derived, along with all the inductive assumptions on the metric \( g_{m-1} \). The estimate (5.93) for \( |J_0| = 2 \), then follows. \( \Box \)

### 5.6 The AVTD behaviour of \( \gamma_{\text{rest}}^m \) in the lower orders, via a descent scheme: The inductive step (4.29).

We put down some consequences of the energy estimates we have derived on \( \gamma_{\text{rest}}^m \), proving (4.29). The estimates have been claimed as part of the inductive step, and capture the AVTD behaviour of the solution, at the lower orders; notably we show that the kinetic part of the energy of \( \gamma_{\text{rest}}^m \), \( \int_{I_1} \delta, |e_1(\gamma_{\text{rest}})|^2 \) (and the kinetic energy of the below-top order derivatives of \( \gamma^m \)) dominates the potential part of the energy \( \int_{I_1} \delta, |e_1(\gamma^m)|^2 + |e_2(\gamma^m)|^2 \); their ratio is in fact bounded by a strictly positive power of \( r \).

This behaviour is used in an essential way in deriving the claimed (optimal) bounds for the behaviour of \( K_{\text{int}}^m (r, t, \theta) \) further down. We already derived in Lemma (4.22) the improved behaviour of the derivatives \( e_i^{m-1} \partial^i \gamma_{\text{rest}}^m \) relative to \( e_i \partial^i \gamma_{\text{rest}}^m \), at the lower orders and in the \( L^\infty \) norm. The challenge now is to show the same improved estimates for the suitable combinations of \( e_i^{m-1} \partial^i \gamma_{\text{rest}}^m \), \( i = 1, 2 \) at all norms below the top.

This challenge is imperative in order to derive the claimed inductive bounds in section 4.5.3. To obtain the inductive step \( m \) of those claims, we will need to control the RHSs of equations (3.10), (3.11) in the suitable norms. These RHSs depend on quantities that have already been bounded at this point, notably the function \( \gamma^m \) (that was just solved for) and the previous \((2 + 1)\)-metric \( h_{m-1} \).

However, we do not just use the energy estimates we have just obtained on \( \gamma_{\text{rest}}^m = \gamma^m - \gamma^S \) and its derivatives; such an approach would not see the claimed AVTD behaviour described above. And it would moreover not allow us to derive the claimed bounds on the connection coefficients of \( h^m \). Instead, to capture the AVTD behaviour for \( \gamma_{\text{rest}}^m \) and its lower derivatives, we utilize a descent scheme for the spatial derivatives of the function \( \gamma_{\text{rest}}^m \). The descent scheme relies on the following idea:

Given an order \( k \in \mathbb{N} \) and the energy bounds we have derived on the energy of \( \partial^i (\gamma^m - \gamma^S) \), the bounds do not distinguish between the directions \( e_i^{m-1}, e_i^{m-2}, e_i^{m-1} \) in the energy (4.17). However, at the same time, the bound on the energy of one higher derivatives \( \partial_\theta (\gamma_{\text{rest}}^m), \partial_\theta (\gamma_{\text{rest}}^m) \) yields a much-improved bound for the \( L^2 \) norm of \( \partial_\theta (\gamma_{\text{rest}}^m), \partial_\theta (\gamma_{\text{rest}}^m) \).

Then, if we use the expression (1.81) for the vector fields \( e_i^{m-1}, e_i^{m-2} \) in terms of \( \partial_\theta, \partial_\theta \) we can derive better bounds for the \( L^2 \) norms of the quantities \( \partial^i (e_i^{m-1}, e_i^{m-2}) \) than the ones implied by the energy estimates for \( \partial^i (\gamma_{\text{rest}}^m) \).

We utilize this strategy at all orders below the top in the remainder of this subsection; the estimates derived there will be put to use in the next section where we control the geometry of the metric \( h^m \) via the Riccati system.

Note that these derivatives appear in the RHSs of equations (3.10), (3.11), (3.11) it is essential to prove that the most singular terms in those RHSs are the ones involving the time-derivatives \( e_0 \).

82
We note that this descent scheme clearly does not work at the top order. (Since there is no higher order from which we can descend). The estimates on the RHS of the Ricatti equations are derived separately at the top order later in this section, by a different argument which utilizes the specific algebraic structure of the RHSs of those equations in an essential way.

5.6.1 Control of forcing terms in the Ricatti equations: The lower and higher orders

Consider the RHSs of the Ricatti equations (3.10), (3.12), (3.11). Let us consider the terms there that depend exclusively on \( \epsilon_0 \)-derivatives of the function \( \gamma^m \). In view of the inductive step (4.25), that we have now derived, those terms satisfy the following estimates:

\[
\| e_0(\gamma^m) + \frac{2M}{r} - 1 \|^{3}_H \leq B r^{-3/2 + 1/4}
\]  

Moreover we note that (5.93) can also be re-cast as:

\[
\| e_0 e_0(\gamma^m) - \frac{2M}{r} - 1 \|^{3}_H \leq B r^{-3/4}.
\]  

At the intermediate orders \( k \in \{ \text{low} + 1, \ldots, s - 3 \} \), the energy estimate (4.22) yields:

\[
\| e_0(\gamma^m_{\text{rest}}) \|^{3}_H \leq C \eta r^{-k - 1/4}
\]  

We also note for future reference that the energy estimates, in conjunction with the wave equation imply:

\[
\| e_0 e_0(\gamma^m_{\text{rest}}) \|^{3}_H \leq C \eta r^{-k - 1/4}
\]  

While for the top order terms we recall that the inductive step on the top order energy of \( \gamma^m \) that we have already derived implies the estimates:

\[
\sqrt{\int_{t, \theta} \| \partial_t^s e_0(\gamma^m_{\text{rest}}) \|^2 \| \Sigma_{m - 1, s - 3} \| \leq C \eta r^{-3 + 1/4}, |J| = s - 3.
\]  

5.6.2 Lower order estimates

We next consider the rest of the terms in the RHSs of (3.10), (3.12), (3.11); these all depend on \( \epsilon_1^{m - 1}, \epsilon_2^{m - 1} \)-derivatives of \( \gamma^m \). As explained, our goal is to derive that these terms (below the top order) satisfy better bounds than the energy estimates we obtained for \( \gamma^m \) would suggest:

In particular the terms we seek to bound are:

\[
\nabla^{m - 1}_{22} (\gamma^m) + \epsilon_2^{m - 1} (\gamma^m_{\text{rest}}) \nabla^{m - 1}_{22} (\gamma^m), \nabla^{m - 1}_{11} (\gamma^m) + \epsilon_1^{m - 1} (\gamma^m_{\text{rest}}) \nabla^{m - 1}_{11} (\gamma^m)
\]  

We seek to bound these expressions in the spaces \( H^k, k \leq s - 4 \) in this subsection, in particular proving the inductive step of (4.29). The top order estimates (when \( k = s - 3 \)) are dealt with in the subsequent subsections.

We claim bounds for these quantities as follows:

**Lemma 5.15.** At the optimal orders \( H^k, k \leq \text{low} \), the terms in (5.106) are all bounded by \( B^2 r^{-2 - 1/4 + \frac{1}{2}} \).

**At the higher orders \( k \in \{ \text{low} + 1, \ldots, s - 4 \} \) their \( H^k \) norm is bounded by \( B^2 r^{-2 + \frac{3}{4} - \frac{k - 1/4}{2}} \).**

**Remark 5.16.** We note that the first claim implies directly that the terms in (5.106) are all bounded by \( B^2 r^{-2 - \frac{3}{4}} \). This follows directly from (4.5).

**Proof.** We commence by re-casting the RHSs of the Ricatti equations by using derivatives with respect to frame elements \( m^{-1}\gamma \), via formulas (4.71). The reason for this is that we have formulas (4.81) to express these vector fields in terms of the coordinates \( \partial_t, \partial_{\theta} \).

In particular, we will derive the claimed bounds for the quantities:

\[
\nabla^{m - 1}_{\tau_1 \tau_2} (\gamma^m) + \epsilon_2^{m - 1} (\gamma^m_{\text{rest}}) \nabla^{m - 1}_{\tau_1 \tau_2} (\gamma^m), \nabla^{m - 1}_{\tau_2 \tau_1} (\gamma^m) + \epsilon_1^{m - 1} (\gamma^m_{\text{rest}}) \nabla^{m - 1}_{\tau_2 \tau_1} (\gamma^m)
\]  

(5.107)
Recall also the bounds on \( e^{m-1}(r) \) \( \text{(4.79)} \). Combining with \( \text{(5.101)}, \text{(5.103)} \). We see that if we can prove the bounds for \( \text{(5.107)} \) then the claimed bounds for \( \text{(5.106)} \) follow. So we prove \( \text{(5.107)} \).

Using:

\[
\|\nabla_\gamma (m\gamma) + (m-1)e_2(m\gamma)\|_{H^1} \leq \|\nabla_\gamma (\gamma^S) + (m-1)e_2(m\gamma)\|_{H^1} + \|\nabla_\gamma (m\gamma - \gamma^S)\|_{H^1},
\]

it is clear that it suffices to bound the two terms on the RHS of the above separately by the claim in our Lemma.

Let us commence by bounding the terms in the second line. We will invoke the bounds on \( \gamma^m - \gamma^S = \gamma^m_{\text{rest}} \) at the lower derivatives:

\[
\|\nabla_\gamma (m\gamma - \gamma^S)\|_{H^1} \leq \|\nabla_\gamma (\gamma^S)\|_{H^1} + \|\nabla_\gamma (m\gamma - \gamma^S)\|_{H^1}.
\]

We commence with the first term: We use formulas \( \text{(4.81)} \) to express \( e^{m-1}_{2} \) in terms of the coordinate vector fields \( \partial_t, \partial_\theta \). We are thus reduced to bounding:

\[
\sum_{A,B,T,\theta, C=1,2,4} \|[(a^{m-1})^{2A} \cdot (a^{m-1})^{2B}]\partial_\gamma (\gamma^S)\|_{H^1} + \sum_{A,B,T,\theta, C=1,2,4} \|[(a^{m-1})^{2A} \cdot (a^{m-1})^{2B}]\partial_\gamma (\gamma^S)\|_{H^1},
\]

Now, invoking the bounds in Lemma \( \text{(4.18)} \) on \( (a^{m-1})^T, (a^{m-1})^\Theta \) in \( H^1 \) as well as the estimates in Lemma \( \text{(4.23)} \) on the terms \( \partial_\gamma (\gamma^S) \), as well as the product inequality, our desired bounds for this term follow.

For the second term, we use:

\[
\|[(m-1)e_2(m\gamma - \gamma^S)]\|_{H^1} \leq \|[(m-1)e_2(m\gamma - \gamma^S)]\|_{H^1} + \|[(m-1)e_2(m\gamma - \gamma^S)]\|_{H^1} + \|[(m-1)e_2(m\gamma - \gamma^S)]\|_{H^1}.
\]

The first term in the RHS of the above is controlled by the product inequality, and by recalling the expression \( \text{(4.81)} \) for \( e^{m-1}_{2} \) in terms of the derivatives \( \partial_t, \partial_\theta \) as above, to find:

\[
\|(m-1)e_2(m\gamma - \gamma^S)\|_{H^1} \leq C\eta^{-1/4} \cdot C\eta^{-1/4}.
\]

The last term in \( \text{(5.111)} \) can be controlled by invoking the Hardy and product inequalities to derive:

\[
\|(a^{m-1})^{2\Theta} \cdot (e^{m-1}_{2}(m\gamma - \gamma^S))\|_{H^1} \leq C\|(a^{m-1})^{2\Theta}\|_{H^1} \cdot \|\partial_\gamma (\gamma^S)\|_{L^\infty} + C\|(a^{m-1})^{2\Theta}\|_{H^1} \cdot \|(e^{m-1}_{2}(m\gamma - \gamma^S))\|_{H^1+1} \leq 2CB\eta^{-1/8} \cdot C\eta^{-1/8},
\]

or if \( l = \text{low} \) the power in the RHS has an extra power \(-\frac{1}{4} \). Thus combining the two previous estimates, we derive our claimed bound for the terms in the second line of \( \text{(5.108)} \).

We now bound the first term in the RHS of \( \text{(5.108)} \) in a similar manner, again expressing the vector fields \( m^{-1}\nabla_i \) in terms of the coordinate vector fields \( \partial_t, \partial_\theta \) and using the inductive bounds on \( (a^{m-1})^T, (a^{m-1})^\Theta \).
Again, noting that the logρ term is cancelled by the derivatives, we take this to find:

\[
|m-1|\nabla_{22}^S - (e_{m-1}^2 (\gamma^S))^2 = ((a_{m-1}^2)^2 ((\csc \theta)^2 - (\cot \theta)^2) + ((a_{m-1}^2)^2 \partial_\theta (a_{m-1}^2 \theta) \cot \theta) - \sum_{A,B=t,\theta} (a_{m-1}^2)^2 (1) \partial_\gamma^S \cot \theta). \tag{5.114}
\]

Note that \((a_{m-1}^2)^2 \partial_\theta (a_{m-1}^2 \theta)\) is bounded by \(e_{m-1}^2 (a_{m-1}^2 \theta)\) in all energy norms \(H^k, k \leq s - 4\). Thus, invoking the inductive assumptions and the product inequality we derive our desired bounds.

The terms \(||m-1|\nabla_{22}^S (\gamma^m) + (m-1) e_1 (\gamma^m) | \gamma^m)|_{H^1}, ||m-1|\nabla_{12}^S (\gamma^m) + (m-1) e_1 (\gamma^m) | \gamma^m)|_{H^1}\) are controlled in an analogous (in fact simpler) manner in all cases except at the order \(|I| = s - 4\) where the bounds on the factor

\[
\partial_{T^T...T}^m \nabla_{22}^S (\gamma^m) - c_1 \gamma^m - c_1 \gamma^m - \gamma^m
\]

require a special note, due to the lack of a bound on the norm of \(E[\partial_{T^T...T}^m \gamma^m]\) at the very top order.

In this case, we instead use the wave equation on \(\gamma^m\) to re-express the terms above in terms of other derivatives which we can control; in particular we replace the RHS by

\[
\partial_{T^T...T}^m \nabla_{22}^S (\gamma^m) + \nabla_{22}^S (\gamma^m) + \nabla_{22}^S (\gamma^m) + \nabla_{22}^S (\gamma^m) + \nabla_{22}^S (\gamma^m)
\]

The RHS of the above is then bounded as claimed, in view of the bounds we have already derived on the first (spatial) derivatives in the \(e^2_{m-1}\)-directions, as well as our the inductive assumptions on the \(K_{22}^S\) coefficients of the previous step.

Other than this special case, all other lower-order derivatives follow by the argument we presented for \(K_{22}^S\).

This concludes the proof of our Lemma. □

### 5.7 The estimates for \(\gamma^m\) re-cast on level sets of \(r\).

We make a small extension of our heretofore derived results, in preparation for our analysis of the Ricatti system in the next section.

The inductive claim we have verified proves estimates for \(\gamma^m_{\text{rest}}\) on level sets of \(\rho^m\). We note also that at the top order estimates, the vector fields \(e^m_{b-1}, b = 0, 1, 2\) are also involved.

Our aim is to prove:

**Proposition 5.17.** The inductive steps that we have derived also hold verbatim on level sets of \(r\), with respect to the coordinate vector fields \(\partial_r, \partial_\theta\) defined with respect to the coordinates \(\{r, t, \theta\}\). At the top orders, the vector fields \(e^m_{i-1}, i = 0, 1, 2\) remain the same. The only difference is that the constant on the RHS will be multiplied by a factor of \(\frac{9}{8}\).

**Proof.** For the purposes of this proof let us denote by \(\partial_r, \partial_\theta\) the previous vector fields defined with respect to the coordinate system \(\{\rho^m, t, \theta\}\) and by \(\overline{\partial_r}, \overline{\partial_\theta}\) the ones defined with respect to the coordinate system \(\{r, t, \theta\}\).

We recall that \(\rho^m = r\) for \(r \leq 2\rho\), and thus \(\overline{\partial_r} = \partial_r\) and \(\overline{\partial_\theta} = \partial_\theta\) for \(r \in (0, c/2)\). Thus our argument will be to commence our estimates on \(\{r = c/2\}\) and solve backwards, until \(r = 0\).

We then just need to use the already-derived estimate with respect to the vector fields \(\partial_r, \partial_\theta\). Coupled with the expressions \(\{1.71, 4.81\}\) we can express \(\overline{\partial_r}, \overline{\partial_\theta}\) in terms of \(e_0, \partial_r, \partial_\theta\) to derive the same estimates qualitatively, with the vector fields \(\partial_r\) replaced by vector fields \(\overline{\partial_r}\). We then note that we also have *bulk estimates* for quantities:

\[
\int_{r = c}^{r = c/2} \frac{1}{r} |J_1 + \frac{2}{h} J_0| e_2 (\partial_r e^m_{b-1} e_0 \gamma^m_{\text{rest}}) (\tau) \sin \theta d\tau d\tau \leq \frac{C^2 r^2}{3} \left[ (\epsilon_1)^{2-h(|I|)} - (\epsilon_2)^{2-h(|I|)} \right].
\]

Here \(h(|I|) = 0\) for \(|I| \leq s - 3 - 4c\), and \(h(|I|) = \frac{|I| - (s-3-4c)}{4}\). These also imply the same qualitative estimates:

\[
\int_{r = c}^{r = c/2} \frac{1}{r} |J_1 + \frac{2}{h} J_0| e_2 (\partial_r e^m_{b-1} e_0 \gamma^m_{\text{rest}}) (\tau) \sin \theta d\tau d\tau \leq \frac{9}{8} \frac{C^2 r^2}{3} \left[ (\epsilon_1)^{2-h(|I|)} - (\epsilon_2)^{2-h(|I|)} \right].
\]
Therefore utilising the energy estimates we derived above at all orders, across the surfaces \( \Sigma \), our claim follows in a straightforward (simpler) way, via the Gronwall inequality, where all commutation terms have now already been controlled.

6 The estimates for the next metric iterate \( m \) \( h \).

As explained in the introduction, the next step in the induction is to construct the next iterate of the metric \( m \) \( h \); this involves all the relevant connection coefficients \( \nabla^\alpha_{ij}(r, t, \theta) \), coordinate-to-frame coefficients \( a^\alpha_{ij}(r, t, \theta) \). It also involves determining the initial data hypersurface \( \Sigma_{t, \theta} \) on which the prescribed initial data are to be induced, via the function \( r^\alpha_n(t, \theta) \), and also the component \( K_{ij}^n(t, \theta) \) of the second fundamental form on that hypersurface, which captures the rotation angle between the fixed background canonical frame \( E_i \), \( E_j \) and the frame \( \tilde{e}_i^\alpha, \tilde{e}_j^\alpha \) on \( \Sigma_{t, \theta} \).

This section is split into three parts. In the first we solve for the variables \( K_{ij}^n(r, t, \theta) \), \( K_{22}^m(r, t, \theta) \). These solve the system of equations \((3.11), (3.12)\), with the prescribed initial conditions \((3.13)\) at the singularity \((r = 0)\). At a second step we use the above solution (for all \( r \in (0, 2e) \)) to solve for the two functions \( r^\alpha_m(t, \theta) \), \( K_{ij}^m(t, \theta) \). These are chosen so as to solve the system \((3.28), (3.29)\) and we derive the claimed inductive estimates for these parameters. In the third part we solve for the remaining variables \( K_{22}^m(r, t, \theta), a^\alpha_{ij}(r, t, \theta) \) with initial data suitably defined on \( \Sigma_{t, \theta} \).

The next subsection commences the first part:

6.1 Energy estimates for \( K^m \): Proof of \((4.40), (4.41), (4.46), (4.45), (4.44)\) for \( K_{12}^m(r, t, \theta), K_{22}^m(r, t, \theta) \)

As stated in our inductive claim, our desired estimates for all the parameters are to hold on both level set of \( r \) and level sets of \( \rho^m \). In fact these two parameters agree for \( r \leq \epsilon/2, r \geq 3\epsilon/2 \) and are comparable in the in-between region. Since we are dealing with transport equations the transition from one estimate to the other is straightforward. For completeness, we prove the claim for the \( r \)-level sets for \( K_{12}^m, K_{22}^m \) and for the \( \rho^m \)-parameters for all the other quantities. A straightforward adaptation of the equations in either of the two situations yields the claim for the other level sets. (The only difference in the equations is the introduction of a multiplicative factor involving \( \chi(r) \) which satisfies uniform bounds in all the relevant spaces).

6.1.1 The functions \( K_{22}^m(r, t, \theta), K_{12}^m(r, t, \theta) \) and their low derivatives as integrals from the singularity.

We now prove the existence of solutions \( K_{ij}^m(r, t, \theta) \) to the equations \((4.10), (4.12)\), deriving that they verify the inductive assumptions \((4.40), (4.41), (4.46), (4.45), (4.44)\) for the \( m \)-th step in the iteration. The benefit of having closed the energy estimates for \( \gamma^m \) in the previous subsection is that we may treat \((3.10), (3.12), (3.11)\) as ODEs for \( K_{ij} \), decoupled from the rest of the variables; each of these ODEs we can solve either forwards or backwards.

We recall that the parameters \( K_{22}^m, K_{12}^m(r, t, \theta) \) satisfy: (All \( e_i \) below are short for \( e_i^{m-1} \)).

\[
e_0 K_{22}^m + (K_{22}^m)^2 - (K_{12}^m)^2 + e_0 \gamma K_{22}^m = \nabla_{22}^{-1}(\gamma^m) + (e_2(\gamma^m))(e_2(\gamma^m-1) - e_2(\gamma^m)) - (e_0(\gamma^m))^2 \tag{6.1}
\]

\[
e_0 K_{12}^m + (2K_{22}^m + e_0(\gamma^m))K_{22}^m = \nabla_{12}^{-1}(\gamma^m) + \frac{1}{2}[e_1(\gamma^m-1) \cdot e_2(\gamma^m) + e_2(\gamma^m) \cdot e_2(\gamma^m-1)] \tag{6.2}
\]

We recall that these equations are to be solved backwards from the singularity for \( r \in (0, 2e) \); in the case of \( K_{22}^m \) the requirement is that the solution should be smooth and negative (at least initially close to \( r = 0 \)). For \( K_{12}^m \) the requirement is that the solution should vanish to order \( o(r^{2M^2(t, \theta) - a^m(t, \theta)}) \).

We will now prove that \( K_{22}^m(r, t, \theta) \) and \( K_{12}^m(r, t, \theta) \) satisfy the following expansions in \( r \) (both in the \( L_\infty \), and \( H^m_{low} \) norms):

\[
K_{22}^m(r, t, \theta) := \frac{d_2^m(t, \theta)\sqrt{2M}}{r^2} + u_{22}^m(r, t, \theta) \quad K_{12}^m(r, t, \theta) := u_{12}^m(r, t, \theta) \tag{6.3}
\]

where

\[
d_2^m(t, \theta) := \frac{\alpha^m(t, \theta) - \frac{3}{2} - \sqrt{\left(\alpha^m(t, \theta) - \frac{3}{2}\right)^2 + 6\alpha^m(t, \theta) - 4|\alpha^m(t, \theta)|^2}}{2}.
\]

86
The functions \(u^m_{22}(r,t,\theta)\), \(u^m_{23}(r,t,\theta)\) are claimed to be (in the inductive step \((4.40), (4.41)\) lower-order corrections (in terms of behaviour in \(r\)), as \(r \to 0\). We arrived at this coefficient \(d^m_r(t,\theta)\) by solving for the unique leading-order formal solution of the ODE \((3.11)\). Here invoking Proposition 5.14 (where we derived control on \(\alpha^m\)), we find: \(\|d^m_r(t,\theta)\|_1 + 1\|L^\infty \leq DC\eta \leq \frac{1}{4}\).

Our goal is to derive the claimed estimates \(Br^{-1-\frac{1}{4}}\) on \(u^m_{22}(r,t,\theta), u^m_{23}(r,t,\theta)\) at the lower orders \((H^1, l \leq \text{low})\) as well as in \(L^\infty\). This will verify the claims \((4.44), (4.45), (4.46)\) for \(K^m_{22}(r,t,\theta), K^m_{23}(r,t,\theta)\).

We obtain the higher order estimates on \(K^m_{22}(r,t,\theta)\) and \(K^m_{23}(r,t,\theta)\) in the next subsection.

**Notation:** As in the previous subsection, unless otherwise stated, we will use the symbols \(\nabla, \nabla\) to denote the covariant derivatives intrinsic to \(h^{m-1}\) and the normal space to \(e_0\) in \(h^{m-1}\) respectively. Moreover the notation \(O(r^b), O_\eta(r^b)\) will be used to denote a term bounded by \(Br^b, B\eta r^b\) (where \(B\) is the universal fixed constant we use throughout). The norms in which these bounds will be assumed to hold will be clear from the equation where they appear--unless stated otherwise they will be in the same norm as the LHS of the relevant equation.

**Remark 6.1:** A note is in order on the level sets where we will be deriving our estimates: In this subsection, we will be deriving our claims on the parameters \(K^m_{22}, K^m_{23}\) on level sets of the function \(r^m\). Once the function \(r^m(t,\theta)\) has been solved for further down, we will remark how the exact same estimates hold on level sets of the function \(\rho^m\) which is built out of \(r^m\). The latter step will complete our inductive claim for these two parameters.

**6.1.2 Asymptotic expansion of \(K^m_{22}(r,t,\theta), K^m_{23}(r,t,\theta)\) at the lower orders.**

Recall that \((4.50)\) can be re-expressed as:

\[
e_0 \gamma^m = -[\alpha^m(t,\theta)] \frac{\sqrt{2M}}{r^{3/2}} + u^m_{33}(r,t,\theta).
\]

Using this notation, and substituting the expression \((6.3)\) in the LHS of \((3.12)-(3.11)\), as well as \((5.91), (5.93), (4.51)\) in the \(e_0^{\gamma^m}, e_0^{\gamma^m}\) terms in the RHS we derive the equivalent system:

\[
e_0 u^{m}_{22} + (2d^m_r(t,\theta) - \alpha^m(t,\theta)) \frac{\sqrt{2M}}{r^{3/2}} u^m_{22} + (u^m_{22})^2 - (u^m_{12})^2 + u^m_{33} u^m_{22}
\]

\[
= \nabla_{22} \gamma^m + (\nabla_{22} \gamma^m)(\nabla_{22} \gamma^m) - e_0 u^m_{33} - (u^m_{33})^2 + (2\alpha^m - d^m_r) \frac{\sqrt{2M}}{r^{3/2}} u^m_{33}.
\]

\[
e_0 u^{m}_{12} + (2d^m_r - \alpha^m) \frac{\sqrt{2M}}{r^{3/2}} u^m_{12} + (2u^m_{22} + u^m_{33}) u^m_{12} = \nabla_{12} \gamma^m + [\nabla_{12} \gamma^m + \nabla_{22} \gamma^m + \nabla_{22} \gamma^m] \cdot (6.5)
\]

We recall also that by Lemma \((4.29)\) we have that the expressions in the RHSs of the above are all bounded in \(H^k, k \leq \text{low}\) and in \(L^\infty\) by \(Br^{-1-\frac{1}{4}}, C_{\text{Sob}} Br^{-1-\frac{1}{4}}\).

This then allows us to solve these two equations as a de-coupled system of a non-linear and a linear ODE.

**Proposition 6.2.** There exists a unique solution \(u^m_{22}(r,t,\theta), u^m_{23}(r,t,\theta)\) to \((6.4), (6.5)\), with the additional requirement for \(u^m_{22}(r,t,\theta)\) that as \(r \to 0\) we have the bound \(u^m_{12}(r,t,\theta) = o(r^{2m/3 - \alpha^m})\). These unique solutions satisfy the bounds for all \(r \in (0, 2\epsilon)\):

\[
\sum_{(i,j)=(1,2),(2,2)} \|u^m_{ij}\|_{L^\infty} + \|\partial^I u^m_{ij}\|_{L^2} \leq Br^{-1-\frac{1}{4}}, \forall [I] \in \{1, \ldots, s - 3 - 4c\}. \tag{6.6}
\]

The conclusion of the previous proposition validates the inductive assumption \((4.44)\) for \(u^m_{ij}\), for \((i,j) = (1,2)\) and \((i,j) = (2,2)\).

**Proof.** Rewrite the system \((6.4)-(6.5)\) in the form

\[
\partial_r (r^{\alpha^m - 2\alpha^m} u^m_{12}) = - \left( \frac{2M}{r} - 1 \right) \frac{1}{2} r^{\alpha^m - 2\alpha^m} \left[ \nabla_{12} \gamma^m + \nabla_{12} \gamma^m \right] - e_0 u^m_{33} - (u^m_{33})^2 + O(1) u^m_{22}, \tag{6.7}
\]

\[
\partial_r (r^{\alpha^m - 2\alpha^m} u^m_{22}) = - \left( \frac{2M}{r} - 1 \right) \frac{1}{2} r^{\alpha^m - 2\alpha^m} \left[ \nabla_{22} \gamma^m + \nabla_{22} \gamma^m - 2u^m_{22} + u^m_{33} \right] \tag{6.8}
\]

87
We proceed by integrating (6.7), (6.8) in \([0,2\epsilon]\), imposing the conditions\(^3\)

\[
\lim_{r \to 0} r^{\alpha m - 2d^m} u_{22} = \lim_{r \to 0} r^{\alpha m - 2d^m} u_{12} = 0,
\]

to obtain:

\[
r^{\alpha m - 2d^m} u_{22}^m = - \int_0^r \frac{2M}{\tau} \left(1 - 1^{\frac{1}{\alpha m - 2d^m}} \tau^{\alpha m - 2d^m} \left[ - \frac{1}{2} \nabla_2^m \gamma + (\nabla_2^m \gamma)(\nabla_2^m - 1) - c_0 u_3^m - (u_3^m)^2 + O(1)u_{22}^m \right] + (2\alpha m - d^m) \frac{2M}{\tau} \frac{\alpha m}{d^m} \left( \frac{1}{\alpha m - 2d^m} - (u_{12}^m)^2 - (u_{12}^m)^2 - u_{33}^m u_{22}^m \right) d\tau
\]

\[
r^{\alpha m - 2d^m} u_{12}^m = - \int_0^r \frac{2M}{\tau} \left(1 - 1^{\frac{1}{\alpha m - 2d^m}} \tau^{\alpha m - 2d^m} \left[ \frac{1}{2} \nabla_2^m \gamma - \frac{1}{2} \nabla_2^m \gamma - 1 + \nabla_2^m \gamma \nabla_2^m - 1 \right) \right) \right) \right) - \tau r m u_{12}^m
\]

\[
\alpha m \left( \frac{1}{\alpha m - 2d^m} \right) u_{12}^m \right) \right) d\tau
\]

Utilising the already derived inductive step of (4.29) for the RHS we infer that

\[
u_{22}^m = \frac{1}{r^{\alpha m - 2d^m}} \int_0^r \tau^{\alpha m - 2d^m} O\left( \frac{1}{\tau^{1 - \frac{1}{\alpha m - 2d^m}}} \right) u_{22}^m d\tau
\]

\[
u_{12}^m = \frac{1}{r^{\alpha m - 2d^m}} \int_0^r \tau^{\alpha m - 2d^m} O\left( \frac{1}{\tau^{1 - \frac{1}{\alpha m - 2d^m}}} \right) u_{12}^m d\tau + \frac{1}{r^{\alpha m - 2d^m}} \int_0^r \left( \frac{2M}{\tau} \left(1 - 1^{\frac{1}{\alpha m - 2d^m}} \tau^{\alpha m - 2d^m} \left[ \frac{1}{2} \nabla_2^m \gamma - \frac{1}{2} \nabla_2^m \gamma - 1 + \nabla_2^m \gamma \nabla_2^m - 1 \right) \right) \right) \right) - \tau r m u_{12}^m
\]

A standard Picard iteration argument of iterating linear equations with the prescribed behaviour at at \(r = 0\) for \(u_{12}^m, u_{22}^m\) furnishes a continuous solution to (6.4)–(6.5) satisfying the pointwise bounds:

\[
|u_{12}^m(r, t, \theta)|, |u_{22}^m(r, t, \theta)| \leq Br^{-1 - \frac{1}{4}}
\]

for \(r \in (0, 2\epsilon]\).

\(\square\)

**Remark 6.3.** For the function \(u_{12}^m(r, t, \theta)\) the condition imposed in (6.9) is a standard initial condition to be imposed on a linear first order ODE. However for \(u_{22}^m(r, t, \theta)\) the vanishing imposed in (6.9) is a necessary requirement to produce a solution of (6.1) (re-cast as (6.4)) that remains smooth until \(r = 0\).

Next, we derive \(H^1_{\text{low}}\) estimates for \(u_{12}^m(r, t, \theta), u_{22}^m(r, t, \theta)\), proving (4.44) for \((i, j) = (1, 2), (2, 2)\). We argue by finite induction, assuming the estimate\(^3\)

\[
|\partial^l u_{12}^m(r, t, \theta)|_{L^2_{t, \theta}} \leq B_{1-l} r^{1-l - \frac{1}{4}}, \forall |l| \leq l - 1 \leq l \leq \text{low} - 1.
\]

is valid for \(|l| \leq l - 1 \leq s - 4 - 4c\) and proceed to show that the analogous estimate holds for \(\partial^l u_{12}^m\), where \(|l| = l \leq s - 3 - 4c\).

**Remark 6.4.** Once this has been established, we can use the smallness of \(\epsilon\) in (4.9) to derive that all these quantities are bounded by \(Br^{1 - \frac{3}{4}}\).

Note that for \(|l| = 0\), (4.44) for \((i, j) = (1, 2), (2, 2)\) holds true by virtue of (6.14).

To prove this inductive step consider the variables \(\partial^l K_{12}^m(r, t, \theta), \partial^l K_{22}^m(r, t, \theta)\) (recall that \(\partial^l\) means we differentiate \(|I| = l\) times in either of the directions \(\partial_\theta, \partial_\theta\)): The resulting linear ODE equation for \(\partial^l u_{22}^m(r, t, \theta)\) is of the form:

\[^3\text{Note that these conditions at } r = 0 \text{ are verified by a function satisfying 6.6.}\]

\[^4\text{for some } t\text{-dependent constant } B_{1-l}.\]
The RHS terms \(Q(\partial^e_0 \gamma, \partial^f u^{m-1})\), \(\partial^f u^{m_2}, \partial^f u^{m_1-1}, \partial^f u^{m_1} - \partial^f u^{m_1-1}\) consist of quadratic terms that arise in the differentiation of equation \((6.16)\) involving the variables \(e_0 \gamma_m, u^{m-1}\). The total number of \(\partial\) derivatives in such terms is \(l = |I|\). The terms in \(\sum_{r=1}^{l=1} C_r \cdot \partial^f u^{m_2}, \partial^f u^{m_1-1}, \partial^f u^{m_1}\) contain only lower that \(l\) derivatives of \(u^{m_2}, u_{12}\). Thus we may inductively derive \(L^2\) bounds on \(\partial^f u^{m_2}_{12}\) over the hypersurfaces \(\Sigma_\rho\). We denote the RHS of \((6.16)\) by \(RHS[\partial^f u^{m_2}_{12}] (r, t, \theta)\). Moreover, all such quadratic expressions involve terms that have been previously controlled as part of the inductive step. In particular we have estimates on the \(L^2\) norms of these terms over the hypersurfaces \(\Sigma_{\rho, m-1}, \Sigma_r\) by \(B_{l-1} r^{-\frac{3}{2} - 1 - \frac{3}{2}}\). At this point we make a key observation:

The ODE equation \((6.16)\) is linear in \(\partial^f K_{12}^{m}\) and admits a free branch solution which corresponds to the homogenous equation. Let:

\[
w(r, t, \theta) = w(r, t, \theta) = \int_\rho ^{2M} \left( \frac{2M}{s} - 1 \right) - \frac{1}{2} (2K_{22}(s, t, \theta) + e_0 \gamma_m(s, t, \theta)) ds = r \frac{d_m^m(t, \theta) - 2a^m(t, \theta)}{s} + O(r^{-2 - \frac{1}{2}}).
\]

(6.17)

(the latter expression follows from the bounds on \(d^m_m(t, \theta) - d_2^2, \alpha^m(t, \theta) - \alpha^S\) and the inductive assumptions on \(K^m(r, t, \theta), K_{12}^{m}(r, t, \theta)\) in \(L^\infty\) which we just verified). Then the equation \((6.16)\) admits a general solution of the form:

\[
(\partial^f u^{m}_{12})(r, t, \theta) = c(t, \theta) e^{-w(r, t, \theta)} + e^{-w(r, t, \theta)} \int_0^r e^{w(s, t, \theta)} [RHS[\partial^f u^{m}_{12}]](s, t, \theta) ds.
\]

We note that the free branch:

\[
c(t, \theta) e^{-w(r, t, \theta)}
\]

of the solution is more singular in \(r\) than the solution of the undifferentiated equation obtained above. In particular we recall that \(e^{-w(r, t, \theta)} \sim r^{-3 + 4c}(t, \theta)\), where \([e^m(t, \theta)] < \frac{1}{2}\). The presence of such a free branch would completely invalidate the inductive claims \((4.43)\) and also \((4.45)\). However, since we solve this equation backwards, we are free to set this singular free branch to zero and we do so. Thus the solution that we consider for \(l \leq s = 3 - 4c\) is:

\[
(\partial^f u^{m}_{12})(r, t, \theta) = e^{-w(r, t, \theta)} \int_0^r e^{w(s, t, \theta)} [RHS[\partial^f u^{m}_{12}]](s, t, \theta) ds.
\]

(6.18)

Our claimed bound follows directly by the already-derived bounds on the \(L^2\)-norm of the integrand in the RHS.

We next derive the analogue of this integral expression for \(\partial^f u^{m}_{12}(r, t, \theta) = \partial^f K_{12}^{m}(r, t, \theta)\):

The function \(\partial^f K_{12}^{m}\) is solved-for backwards from \(r = 0\) by considering the ODE \((6.2\) differentiated by \(\partial^f\). The resulting equation is of the form:

\[
e_0 \partial^f K_{12}^{m} + (2K_{22}^{m} + e_0(\gamma^m)) \partial^f u^{m}_{12} = \partial^f [m-1] \nabla_1 (\gamma^m) + e_1 (\gamma^m) e_2 (\gamma^m) + Q(\partial^e_0 \gamma, \partial^f K^{m-1})
\]

+ \(\sum_{r=1}^{1} C_r \cdot \partial^f u^{m_2}, \partial^f u^{m_1}, \partial^f u^{m_1-1}\).

(6.19)

In analogy with the case of \(K_{12}^{m}\), the solution of this that we consider is:

\[
(\partial^f u^{m}_{12})(r, t, \theta) = e^{-w(r, t, \theta)} \int_0^r e^{w(s, t, \theta)} [RHS[\partial^f u^{m}_{12}]](s, t, \theta) ds.
\]

(6.20)

I.e. again the free branch of the solution is set to zero. (Note that this free branch was already set to zero for the undifferentiated equation \((6.8)\).)
The desired estimates then follow by a finite induction, just as for $u_{22}^0$.

We also note that given the definitions

$$\partial^1 K_{m2}^S(t, \theta) = r^{-3/2} \partial^2 d_2^m(t, \theta) + \partial^1 u_{22}^m(t, \theta), \partial^1 K_{12}^S(t, \theta) = \partial^2 u_{12}^m(t, \theta),$$

the bounds \[4.26\] on $\|\partial^I (\alpha^m(t, \theta) - 1)\|_{L^2}$ for all $|I| \leq 1$, the corresponding bound

$$\|\partial^I (d_2^m(t, \theta) - d_2^S(t, \theta))\|_{L^2} \leq DC \eta,$$

the assumed closeness of $K_{22}^S$ to $K_{22}^S$ and the bounds just derived on $u_{12}^m(t, \theta)$, $u_{22}^m(t, \theta)$ as well as the bound fixing the smallness \[4.9\] of $\epsilon$ relative to $C \eta$ imply the following analogue of \[6.21\]:

**Lemma 6.5.** The functions $K_{22}^S(t, \theta)$, $K_{12}^S(t, \theta)$ at $\Sigma_{r=\epsilon} := \{r = \epsilon\}$ thought of as functions in $t, \theta$ satisfy the following bounds for all $k \leq 1$:

$$\int_{\Sigma_{r=\epsilon}} |\partial^I [K_{22}^m (\epsilon, t, \theta) - K_{22}^S(t, \theta)]|^2 \sin \theta d\theta dt \leq 2C^2 \eta^2 e^{-3},$$

$$\int_{\Sigma_{r=\epsilon}} |\partial^I [K_{12}^m (\epsilon, t, \theta)]|^2 \sin \theta d\theta dt \leq B^2 C^2 e^{-2 - \frac{1}{2}} \leq C \eta^2 e^{-3 + \frac{1}{2}}.$$ (6.21)

This estimate will play a key role in solving for $r^m_2(t, \theta)$ and $\tilde{K}_{12}^m(t, \theta)$ via the inverse function theorem in section 6.3. For now, however, let us also derive bounds on $K_{22}^S(t, \theta)$, $K_{12}^S(t, \theta)$ at the higher orders:

### 6.2 The bounds on $K_{22}^m(t, \theta)$, $K_{12}^m(t, \theta)$ at the higher orders.

In order to derive estimates at the higher orders, we subtract the Riccati equation satisfied by $K_{22}^S(t, \theta)$ from our equation, to derive:

$$e_0 [K_{22}^m - K_{22}^S] + 2K_{22}^m \cdot (K_{22}^m - K_{22}^S) + (K_{22}^m - K_{22}^S)^2 - (K_{12}^{m-1})^2 + e_0 \gamma^m (K_{22}^m - K_{22}^S) = [m^{-1} \nabla_{22}^m (\gamma) - (e_2^m (\gamma) - e_0^m (\gamma - \gamma)) + 2e_0 \gamma^m] e_0 \gamma^m,$$

$$e_0 K_{12}^m + (2K_{22}^m + e_0 (\gamma^m)) K_{12}^m = \nabla_{12}^m (\gamma) + \frac{1}{2} [e_1 (\gamma - \gamma) e_2 (\gamma) + e_1 (\gamma) e_2 (\gamma)].$$ (6.22)

We take the derivatives $\partial^I$, $|I| \in \{\text{low + 1, \ldots, s - 4}\}$ of this equation to derive

$$e_0 \partial^I [K_{22}^m - K_{22}^S] + \frac{2(2d_2^m - \alpha^m) \sqrt{2M}}{r^2} \partial^I [K_{22}^m - K_{22}^S] + O \left( \frac{1}{r^{1+\frac{1}{2}}} \right) \partial^I K_{22}^m$$

$$= \partial^I \left[ \nabla_{22}^m (\gamma) + \frac{\sqrt{2M}}{r^2} \right] \partial^I [K_{22}^m - K_{22}^S] + O \left( \frac{1}{r^{1+\frac{1}{2}}} \right) \partial^I K_{22}^m$$

$$- \sum_{I_1 \cup I_2 = I, |I_1| < |I|} \partial^{I_1} K_{22}^m \partial^{I_2} e_0 \gamma^m$$

$$- \sum_{I_1 \cup I_2 = I, |I_1| < |I|, |I_2| < |I|} \partial^{I_1} K_{22}^m \partial^{I_2} K_{22}^m,$$

$$e_0 \partial^I K_{12}^m (2d_2^m (t, \theta) - \alpha^m (t, \theta)) \frac{\sqrt{2M}}{r^2} \partial^I K_{12}^m + O \left( \frac{1}{r^{1+\frac{1}{2}}} \right) \partial^I (6.24)$$

$$= \partial^I \left[ \nabla_{12}^m (\gamma) + \frac{1}{2} [e_1 \gamma^m e_2 \gamma^m + e_1 \gamma^m e_2 \gamma^{m-1}] \right] - \sum_{I_1 \cup I_2 = I, |I_1| < |I|} \partial^{I_1} K_{12}^m \partial^{I_2} (2K_{22}^m + e_0 \gamma^m)$$

$$- \sum_{I_1 \cup I_2 = I, |I_1| < |I|} \partial^{I_1} K_{12}^m \partial^{I_2} (2K_{22}^m + e_0 \gamma^m)$$

Here, we will bound the functions $\partial^I K_{22}^m(t, \theta, \delta)$, $\partial^I K_{12}^m(t, \theta, \delta)$ for functions $\delta(t, \theta)$ that are close (in suitable norms) to $\epsilon$.

We derive our bounds for $K_{22}^S(t, \theta, \delta(t, \theta))$. The bounds for $K_{12}^S(t, \theta, \delta(t, \theta))$ follow by essentially the same argument.

**Remark 6.6.** We also note here that if we were to solve $K_{11}^m$ backwards from $r = 0$ by setting the free branch of that solution to zero, we would derive the same estimate for that parameter as for $K_{22}^m$ by the same proof at all low and high orders $|I| \leq s - 4$.  


We will first bound a suitably weighted \( L^2 \) norm \( \partial^i(K_{22}^m - K_{22}^S)(s,t,\theta) \) in the bulk region \( \{ r \leq 2\epsilon \} \).
In particular for all \( k \geq s - 3 - 4c, k \leq s - 4 \) we will derive the bounds:

\[
\int_0^{2e} \int_{\Sigma^t} [\partial^i[K_{22}^m - K_{22}^S]]^2 r^{\frac{1}{2}(k-(s-3-4c))} \sin \theta dt d\theta dr \leq 4C^2 \eta^2 \epsilon^{-3}.
\] (6.25)

\[
\int_0^{2e} \int_{\Sigma^t} [\partial^i K_{12}^m] r^{\frac{1}{2}(k-(s-3-4c))} \sin \theta dt d\theta dr \leq 4C^2 \eta^2 \epsilon^{3\frac{1}{2}}.
\] (6.26)

For brevity of notation, for each \( k > l \) we denote by \( p(k) \) the term:

\[
p(k) := \frac{1}{2}(k - (s - 3 - 4c)).
\]

Then, after (6.25), (6.26) have been established, we will derive the following energy estimate for \( \int_{\Sigma^t} [\partial^i K_{22}^m]^2 r^{\eta(m)}(\delta(t,\theta),t,\theta) \sin \theta dt d\theta \), where \( \Sigma^t \) is any graphical hypersurface expressed in terms of \( \{ \rho = \delta(t,\theta) \} \) or \( \{ r = \delta(t,\theta) \} )

\[
\int_{\Sigma^t} [\partial^i[K_{22}^m(r(t,\theta),t,\theta) - K_{22}^S(r(t,\theta))]]^2 \sin \theta dt dt \leq C \eta^2 \epsilon^{3-\rho(k)}.
\] (6.27)

(After these bounds have been derived, we will also explain how the same bounds hold on level sets of the new coordinate function \( \rho^m \)).

Proof of (6.25), (6.26): We focus on (6.25) and explain at the end the modification needed in deriving (6.26). We will prove this claim by a finite induction. So we assume our claim has been proven to orders \( l - 1 \leq s - 5 \) and we will derive it for order \( l \leq s - 4 \).

Consider the equation (6.23). Recall this becomes a linear 1st order ODE equation in \( \partial^i K_{22}^m \). Recall the integrating factor for that equation is \( F(r,t,\theta) = e^{(\frac{2M}{r} - 1)^{-1/2}[2K_{22}^m(r,t,\theta)]} \).

We have defined \( w(r,t,\theta) = m^m(r,t,\theta) \) via \( F(r,t,\theta) = e^{-w(r,t,\theta)} \). Notably, using the expressions we have for the asymptotics of \( K_{22}^m(\rho,t,\theta), c^0(m) \) we see that in \( L^\infty \) integrating factor is asymptotic to \( r^{3+\epsilon^m(t,\theta)} \) as \( r \to 0 \) in the sense that:

\[
\|F(r,t,\theta) - r^{3+\epsilon^m(t,\theta)}\|_{L^\infty} \leq Cr^{-2-3/4}.
\]

moreover the function \( \epsilon^m(t,\theta) \) satisfies \( |\epsilon^m(t,\theta)| \leq DC\eta \leq 1/8 \).

Now, consider the equation (6.23); using the integrating factor \( F(r) \) and shorthand notation this can be re-expressed as:

\[
r^{\rho(k)+1}\partial_r(F(r)\partial_r[K_{22}^m - K_{22}^S])(r,t,\theta)) = (\frac{2M}{r} - 1)^{-1/2}\text{RHS\[(6.23)\]}_r, r^{\rho(k)+1}.
\] (6.28)

In fact, let us recall the bounds on \( \text{RHS\[(6.23)\]}_r \) that we derived in the inductive step of (4.29).

We will integrate this equation over \( t,\theta, r \) (with the volume form \( \sin \theta dt d\theta dr \)) and then apply the standard Hardy inequality on the RHS: The first integration gives:

\[
\int_0^{2e} \int_{t,\theta} r^{\rho(k)+1+3} [F^{-1}(r)\partial_r(F(r)\partial_r[K_{22}^m - K_{22}^S])(r,t,\theta))^2 drdtsin\theta d\theta \leq \int_0^{2e} \int_{t,\theta} r^{\rho(k)+1+3} (\frac{2M}{r} - 1)^{-1}\text{RHS\[(6.23)\]}_r(s,t,\theta))]^2 drdtsin\theta d\theta.
\] (6.29)

The standard weighted 1-d Hardy inequality (which can be invoked, in view of the vanishing of the functions \( \partial^i K_{22}^m \) and the choice of \( p(k) \)), then implies a lower bound for the LHS of the above:

\[
\int_0^{2e} \int_{t,\theta} r^{\rho(k)+1+6}[F^{-1}(r)\partial_r(F(r)\partial_r[K_{22}^m - K_{22}^S])(r,t,\theta))^2 drdtsin\theta d\theta \geq C \int_0^{2e} \int_{t,\theta} r^{\rho(k)+1+3-2}[F^{-1}(r)\partial_r(r^{3+\epsilon^m(t,\theta)}(r^{\rho(k)+1+3-2}[F^{-1}(r)\partial_r[r^{3+\epsilon^m(t,\theta)}(r^{\rho(k)+1+3-2}[F^{-1}(r)\partial_r[K_{22}^m - K_{22}^S])(r,t,\theta)])^2 drdtsin\theta d\theta \geq C r^{\rho(k)} \int_0^{2e} \int_{t,\theta} r^{\rho(k)+1+3-2}[r^{3+\epsilon^m(t,\theta)}(r^{\rho(k)+1+3-2}[F^{-1}(r)\partial_r[K_{22}^m - K_{22}^S])(r,t,\theta)])^2 drdtsin\theta d\theta.
\] (6.30)
(The constant $C(p(k))$ has come from the classical Hardy inequality. Note that $C(p(k)) \sim [p(k)]^2$. Thus we derive:

$$
\int_0^{2\epsilon} \int_{t,\theta} r^{p(k)-1+3-2} \left[ \partial^t [K_{22}^m - K_{22}^S](r, t, \theta) \right]^2 drdtsin\theta d\theta \\
\leq C(p(k))^{-1} \int_{t,\theta} \left[ \frac{2M}{r} - 1 \right]^{-1} r^{3+1+2p(k)} \left[ \text{RHS}[(6.23)] \right]^2 drdtsin\theta d\theta. 
$$

(6.31)

Let us now derive a bound on the RHS of the above: We note that the terms involving spatial derivatives $\nabla \gamma^m$ have all been bounded in Lemmas 5.15. Lemma 5.15 directly implies that the contribution of those terms to the total norm is bounded by $\frac{2M}{C(p(k))} \epsilon^3$. (The denominator $C(p(k))$ comes from the classical Hardy inequality, both here and below). The terms involving $e_0 \epsilon_0 (\gamma^m), e_0(\gamma^m)$ have been bounded in (5.103), (5.104). The contribution of those terms is thus bounded by $\frac{22}{C(p(k))} \epsilon^3$. Finally, there are all the terms that involve lower derivatives of $K_{22}^m, K_{22}^S$. Since we are assuming that those terms are already bounded as in (6.25), we find that the contributions of those terms is bounded by $\frac{B_{1/4}^{22,22,22}}{C(p(k))} \epsilon^3$.

In sum, using (4.9), we conclude that the RHS of the above is bounded by $\frac{B_{1/4}^{22,22,22}}{C(p(k))} \epsilon^3$ for all $|I| \leq s - 4$. Thus we derive a bound for the LHS of the above by $\frac{22}{C(p(k))} C^2 \eta^2 \epsilon^3$.

Having bounded this bulk term, we can now bound

$$
\int_{\Sigma_{r=\epsilon}} r^{p(k)+6} \left[ \partial^t [K_{22}^m - K_{22}^S](r, t, \theta) \right]^2 sin\theta dr d\theta.
$$

By the fundamental theorem of calculus we find:

$$
\int_{\Sigma_{r=\epsilon}} r^{p(k)} \left[ \partial^t [K_{22}^m - K_{22}^S] \right] dt sin\theta d\theta = \int_{t,\theta} \int_0^{\delta} \partial_t [r^{p(k)} r^{3+1/2} |J_{1}| |\partial^t [K_{22}^m - K_{22}^S]|]^2 drdtsin\theta d\theta

\leq (p(k) + 4) \int_{t,\theta} \int_0^{\delta} r^{p(k)-1} \left[ |r^{3+3/2} |J_{1}| |\partial^t [K_{22}^m - K_{22}^S]| \right]^2 drdtsin\theta d\theta

+ \int_{t,\theta} \int_0^{\delta} \left( 1 - \frac{2M}{r} \right)^{-1/2} r^{p(k) + 3/2 |J_{1}| |\text{RHS}[(6.23)]|} \cdot |\partial^t [K_{22}^m - K_{22}^S]| drdtsin\theta d\theta. \tag{6.32}
$$

(The coefficient $+4$ in the first term in the RHS has come from incorporating the second and third terms in the LHSs of (6.23) into that term).

Then the first term in the RHS of (6.32) has already been bounded by $\frac{B_{1/4}^{22,22,22}}{C(p(k))} C^2 \eta^2 \epsilon^3$. The second term can be controlled by Cauchy-Schwarz:

$$
\int_{t,\theta} \int_0^{\delta} \left( 1 - \frac{2M}{r} \right)^{-1/2} r^{p(k)} \left( r^6 \text{RHS} |\partial^t [K_{22}^m - K_{22}^S] \cdot \partial^t [K_{22}^m - K_{22}^S]| drdtsin\theta d\theta

\leq \kappa \int_{t,\theta} \int_0^{\delta} \left( \frac{2M}{r} - 1 \right)^{-1/2} r^{p(k) + 1} r^6 |\text{RHS}[(6.23)]|^2 drdtsin\theta d\theta

+ (4\kappa)^{-1} \int_{t,\theta} \int_0^{\delta} \left( \frac{2M}{r} - 1 \right)^{-1/2} r^{p(k) - 1} r^6 |\partial^t [K_{22}^m - K_{22}^S]|^2 drdtsin\theta d\theta. \tag{6.33}
$$

Choose $\kappa = 1$. Then given the bounds we have obtained on the second term in the RHS (the bulk bound derived above) and the bounds we have derived on the bulk integral of RHS[(6.23)], we derive the bound, for every fixed $\delta$ (factoring out the $r^6$ from the LHS, and recalling the bound on the constant $C(p(k))$):

$$
\int_{\Sigma_{r=\delta}} |\partial^t [K_{22}^m - K_{22}^S]|^2 (\delta, t, \theta) dsin\theta d\theta \leq \delta^{-p(k)} 10C^2 \eta^2 \delta^{-3}. \tag{6.34}
$$

If the hypersurface $\{r = \delta\}$ is replaced by a function $\{r = \delta(t, \theta)\}, \delta(t, \theta) \sim \epsilon$ the same argument applies to derive:

$$
\int_{\Sigma_{\delta(t, \theta)}} |(\partial^t [K_{22}^m - K_{22}^S])(\delta(t, \theta), t, \theta)|^2 dsin\theta d\theta \leq 2\epsilon^{-p(k)} 10C^2 \eta^2 \epsilon^{-3}. \tag{6.35}
$$
In particular as we will see below (once we have defined the function \( \rho^m \) via \( r^m_0(t, \theta) \)) these bounds imply:

\[
\int_{\Sigma_p} |\partial^j [K_{22}^m - K_2^{S2}] (r = \rho^m (t, \theta))|^2 dt \sin \theta d\theta \leq 5 \epsilon^{-p(k)} C^2 \eta^2 \epsilon^{-3}.
\]

(6.36)

Moreover the same proof applies to \( K_{12}^m \) to yield:

\[
\int_{\Sigma_{r=t}} |\partial^j K_{12}^m (\delta(t, \theta), t, \theta)|^2 dt \sin \theta d\theta \leq \epsilon^{-p(k)} C^2 \eta^2 \delta^{-3+\frac{1}{2}}.
\]

(6.37)

(The reason for the smaller coefficient in the RHS is that there is only one top-order term in the RHS of the Ricatti equation (3.12) for \( K_{12}^m \)). We also note that stronger bounds can be derived at the orders below the top, in view of Lemma 5.15 but these are not needed and so we do not put them down.

And in analogy with (6.35), we have:

\[
\int_{\Sigma_{r=t}} |\partial^j (K_{12}^m (\delta(t, \theta)))|^2 dt \sin \theta d\theta \leq 3 \epsilon^{-p(k)} C \eta^2 \epsilon^{-3}.
\]

(6.38)

Again, once we have derived bounds for \( \rho^m \) via \( r^m_0(t, \theta) \) these bounds will imply:

\[
\int_{\Sigma_p} |\partial^j (K_{12}^m (r = \rho^m (t, \theta)))|^2 dt \sin \theta d\theta \leq 3 \epsilon^{-p(k)} C^2 \eta^2 \epsilon^{-3+\frac{1}{2}}.
\]

(6.39)

6.3 The inductive step for \( K_{22}^m, K_{12}^m \) at the top order.

We here prove the inductive claim for \( K_{22}^m, K_{12}^m \) at the top orders. We note that to derive our claim, we will use that the variables \( K_{22}, K_{12} \) are de-coupled from the remaining variable \( K_{11} \) in the equations (1.20), (1.21), (1.22). This remains true in the iterative variables \( K_{22}^m (r, t, \theta), K_{12}^m (r, t, \theta) \), via equations (3.10), (3.11).

We note that the proof of the top order claims would have been simpler, had we been treating directly the coupled system in section 2.7. The setting of the iteration that we use here requires us to use some tricks; these make use of this de-coupling of \( K_{22}^m, K_{12}^m \) from \( K_{11}^m \).

We just treat the top order case in the case of extra singular weights, with the unknowns \( \partial^j \partial_b [K_{22}^m - K_2^{S2}] \cot \theta, |J| = s - 4, \partial^j [K_{12}^m \cdot \cot \theta] \); the cases \( |J| = s - 3 \), are an easier version of this case.

Our use of the de-coupling of the two variables that we are interested in here, is by constructing a metric \( \overline{h}^m \) which is partly artificial, in that its components \( K_{22}^m (r, t, \theta), K_{12}^m (r, t, \theta) \) agree with \( K_{22}(r, t, \theta), K_{12}(r, t, \theta) \); but its connection coefficients where \( e_1 \) appears twice do not agree with that of the metric iterate \( h^{m} \).

We define a new tensor \( K_{ij}^m \) in place of \( K_{ij} \) as follows:

\[
K_{ij}^m = K_{ij}^{22}, K_{ij}^{12} = K_{ij}^m, K_{ij}^{11} = (K_{ij}^{11})^{\text{reg}},
\]

(6.40)

where the latter is defined to be the unique solution of (3.10) with the “free branch” (with behaviour \( O(r) \)) of the solution set to zero.

In particular proving our claim for \( K_{22}^m, K_{12}^m \) will imply our claim for \( K_{22}^m, K_{12}^m \). We also note that having solved for the variables \( K_{ij}^m (r, t, \theta) \), we can define a metric \( \overline{h}^m \) once we specify functions \( e_2(r), e_1(r) \), as well as (asymptotic) initial data for the corresponding coordinate-to-frame coefficients \( a^{m}_{ij}(r, t, \theta) \). To define this new (artificial) metric \( \overline{h}^m \) we set \( e_2(r) = o(r^{-\frac{1}{2} + d_2(t, \theta)}), e_1(r) = o(r^{\frac{1}{2} - a_1(t, \theta)}) \); equations (1.76) then imply that \( e_1(r) = 0, e_2(r) = 0 \). We also prescribe the solutions of the system in (4.89) by requiring the free branches of \( a^{m}_{ij}, a^{m}_{ij} \) to zero, and also the coefficients of the free branches of \( a^{m}_{ij}, a^{m}_{ij} \) as follows: \( a^{m}_{ij} \) is prescribed asymptotically as \( r \to 0 \) as before. (In particular \( \partial \delta \) captures the direction of \( e_1^m \) at the singularity). \( a^{m}_{ij} \) is also chosen asymptotically as \( r \to 0 \) to ensure the (asymptotic) commutation of the (asymptotic) vector fields \( a^{m}_{ij} e_1 + o^{m}_{ij} e_2 \). We now note in particular we have the formulas (4.90), by replacing the initial data factors, and we commence the integrals in the exponentials by \( \epsilon \).

---

35 The latter is for convenience only, and inconsequential since we are only proving claims on \( K_{22}^m (r, t, \theta), K_{12}^m (r, t, \theta) \) here.

36 A fuller discussion of this relation appears in section [here] we just employ this aspect of the construction in that subsection for technical convenience.
We note (see remark 6.6) that we can derive the same bounds for $K_{i1}^m$ as those claimed for $K_{i1}^1$ at all orders below the top. We also note that we have $a_{i2}^2 = a_{i1}^2 = 0$, and for the variables $a_{i2}^m, a_{i1}^m$ we have the same bounds as for the ‘real’ metric $h^m$.

Now, let us derive the claimed bounds on $K_{i2}^m = K_{i2}^1, K_{i2}^m = K_{i2}^1$ at the top orders.

We do this via the Lemma:

**Lemma 6.7.** Under the inductive assumptions on $\gamma^m, g^{m-1}$, we also claim for all $|J| = s - 3$, and on any level set $\Sigma_r, \Sigma_{\nu}^m$ we have the estimate:

$$\int_{\Sigma_\tau} r^{2c + 3 + |J|} |\partial^J K_{i1}^m|^2 \sin d\theta d\delta d\tau \leq (C \eta)^2 r^3. \tag{6.41}$$

We also have analogues of these estimates at the other top order terms: In particular for each multi-index $J, |J| = s - 4$ and for $i = 1, 2$ we claim:

$$\int_{\Sigma_\tau} r^{2c + 3 + 2|J|} |\partial^J \partial_\theta K_{i1}^m|^2 \sin d\theta d\delta d\tau \leq (C \eta)^2 r^3, \int_{\Sigma_\tau} r^{2c + 3 + 2|J|} |\partial^J \partial_\theta K_{i2}^m|^2 \sin d\theta d\delta d\tau \leq (C \eta)^2 r^{3 + \frac{1}{2}}. \tag{6.42}$$

We note that in view of the definition of $K^m$, the above implies (6.40) for $K_{i2}^m, K_{i2}^1$.

**Proof.** We will prove the slightly harder case of (6.42). The claim (6.41) follows by an easier adaptation of this argument. We will first derive our bounds for $\partial^J \partial_\theta \partial_{\gamma^{m-1}} K_i^m, \cot \theta K_i^m$ and invoke the wave equation (6.42) on $\gamma^m$, then we will see further down how this implies our claim for the full second fundamental form $K_i^m$ at the top orders, as claimed.

We recall the Riccati equations (3.10), (3.12), (3.11), which hold verbatim for the components of $K^m$, with all occurrences of $K^m$ replaced by $K_m^m$. Note that if we add the evolution equations with $e_0 K_{i1}^m, e_0 K_{i2}^m$ and invoke the wave equation (77) on $\gamma^m$, then we multiply the the equation by $e^{c + \frac{1}{2}}$ we derive an equation of the form:

$$e_0 [e^{c + \frac{1}{2}} \partial_{\gamma^m} K_{i1}^m] = \left(\frac{2M}{r} - 1\right) \frac{1}{2} (c + \frac{3}{2}) + \frac{1}{2} \partial_{\gamma^m} K_{i1}^m + r^{c + \frac{1}{2}} \left\{-\frac{1}{2} |\partial_\theta \partial_{\gamma^m} K_{i1}^m|^2 - |K_{i1}^m|^2 \right\}

- [e_0^2 \gamma^m + (e_0 \gamma^m)^2 + \sum_{i=1,2} (K_{ii}^m - K_{i1}^{m-1}) e_0 \gamma^m + (e_0 \gamma^m - e_0 \gamma^{m-1}) e_0 \gamma^m + (K_{i1}^m)^2 - (K_{i1}^{m-1})^2)] \right\} \right) \right). \tag{6.43}$$

In particular we note that all forcing terms involving $\gamma^m$ in the RHS now satisfy bounds in our desired energy spaces; in particular the terms with second spatial derivatives of $\gamma^m$ do not appear. This is the upshot of invoking the wave equation on $\gamma^m$ to re-express spatial derivatives of $\gamma^m$ in terms of time $e_0$-derivatives.

We then take the $\partial^J$ derivative of the above equation, with $|J| = s - 3$ (for the first claim in our Lemma), or act on it by the operator $\cot \theta \cdot \partial^J \partial_\theta$ with $|J| = s - 4$, for the second case. We consider only the second case, since the first is an easier adaptation of this.

We obtain commutation terms which will not be top-order in the resulting equation; these we denote by l.o.t.’s. Our differentiated equation then yields:

$$e_0 [\cot \theta \cdot \partial^J \partial_\theta \partial_{\gamma^{m-1}} K_{i1}^m] + \sum_{a,b=1,2,3} \left\{[\cot \theta \cdot \partial^J \partial_\theta K_{i1}^m] \cdot ((K_{i1}^m)^{ab} + g^{m}_{ab} e_0 \gamma^m) - \cot \theta \cdot \partial^J \partial_\theta ([K_{i1}^{m-1}]^2 - 3(K_{i1}^{m-1})^2) \right\}

= -2 \cot \theta \cdot \partial^J \partial_\theta [e_0 e_0 \gamma^m] + \partial^J [\nabla \gamma^{m-1}, \nabla \gamma^m], \tag{6.44}$$

where in the last term we have used generic notation for products of first derivatives.

Using the usual Hardy inequality in $r$ from $r = 0$, we derive the bound:

$$\int_0^\tau \int_{\Sigma_\tau} \tau^{2c + 3} [\partial^J \partial_\theta \partial_{\gamma^m} K_{i1}^m - \cot \theta]^2 d\sin \theta d\delta d\tau \leq C_{\text{Hardy}} (c + 9)^{-2} \int_0^\tau \int_{\Sigma_\tau} \tau^{2c + 3 + 2} [\partial^J \partial_\theta \partial_{\gamma^m} K_{i1}^m - \cot \theta]^2 d\sin \theta d\delta d\tau \tag{6.45}$$

Now combine this with the equation (6.44), to substitute the RHS.

---

37 Here $\bar{h}^{m-1}$ is the 3-dimensional metric on the space $e_0^{1,0}$, spanned by $e_0^{m-1}, e_0^{m-1}, e_0^{m-1}$. 

94
Let us note a bound in \( L^2[\Sigma_r] \) on \( \partial^j \partial_a e_0 \gamma^m \cot \theta \). We express \( \partial^j \partial_a \) by \( \partial^j \) with \( |i| = s - 3 \). We then use the Hardy inequality to bound:

\[
\|\partial^j \partial_a e_0 \gamma^m \cot \theta\|_{L^2[\Sigma_r]} \leq \|(a^{m-1})^2\|_{L^\infty} \cdot \|e_0^{-1} \partial^j \partial_a e_0 \gamma^m\|_{L^2[\Sigma_r]} \leq \|(a^{m-1})^2\|_{L^\infty} \cdot E[\partial^j \partial_a e_0 \gamma^m].
\]

Let us recall the bound on the top order term \( \partial^j (\gamma^m), i = 0, 1, 2 \in L^2[\Sigma_r] \) by \( C \eta r^{-3-c-\frac{2}{s}} \). Using this we will be able to prove our claim (using (4.6) as before) provided we can prove the bound:

\[
\int_0^r \sum_{d=1,2,3} \int_{\Sigma_r} r^{2c+6+1} \left( \frac{2M}{r} - 1 \right) \sum_{a,b=1,2,3} \left[ (K^{m})^{ab} \partial_a \partial_b K^{m} \right] \left[ \partial^j \partial_a \partial_b (K^{m})^{ab} \right] (\cot \theta)^2 d\sin \theta dt d\tau \notag
\]

\[
\leq \int_0^r \int_{\Sigma_r} 4M \tau^{2c+3} |\partial^j \partial_a \partial_b K^m| \cot \theta^2 \sin \theta d\theta d\tau + O(\tau^{\frac{1}{2}}). \tag{6.46}
\]

This inequality can be proven by first invoking the pointwise bounds on \( \|K^{m}\|_{L^\infty[\Sigma_r]} \) (as noted, these are the same as those for \( K_i^m \)), as well as (4.10), which reduce matters to bounding

\[
\int_0^r \sum_{d=1,2,3} \int_{\Sigma_r} r^{2c+3} \sum_{a,b=1,2,3} \left[ \partial^j \partial_a \partial_b K^{m} \right] \left[ \partial^j \partial_a \partial_b (K^{m})^{ab} \right] (\cot \theta)^2 d\sin \theta dt d\tau
\]

by the RHS of (6.46). In particular we will show that:

\[
\left| \sum_{a,b=1,2} \int_0^r \int_{\Sigma_r} r^{2c+3} \left[ \partial^j \partial_a \partial_b K^{m} \right] \left[ \partial^j \partial_a \partial_b (K^{m})^{ab} \right] (\cot \theta)^2 d\sin \theta dt d\tau \right|
\]

\[
- \int_0^r \tau^{2c+3} \int_{\Sigma_r} |\partial^j \partial_a \partial_b K^m| \cot \theta^2 \sin \theta d\theta d\tau = O(\tau^{1+\frac{1}{t}}). \tag{6.47}
\]

This is done by integration by parts: We commence with the term \( \partial^j \partial_a \partial_b K^m \) to which we apply the Codazzi equation to write:

\[
\partial^j \partial_a \partial_b K^m = \partial^j \nabla^a K^m_{\partial b} + \partial^j \nabla^{\partial b} K^m_{a}.
\]

We then invoke the standard coordinate expression for the Ricci curvature components \( \nabla_{\partial b} \), \( \nabla^{\partial b} \); win particular we express this in terms of coordinate derivatives of the metric components and Christoffel symbols. We note that the terms with most derivatives will have a derivative \( \partial_0 \); this is since \( e_0 \) is normal to \( e_1, e_2 \) everywhere. [Recall that in this proof we choose \( e_1 \) so that \( e_1(r) = 0 \) . Thus, the bounds on \( \partial^j \nabla^a K^m_{\partial b} \) follow just from the estimates we have on the metric components and Christoffel coefficients, as part of the implications of the inductive assumptions.

In particular, we will have at most \( s - 3 \) spatial derivatives hitting any metric component. These metric components are expressible in terms of the coordinate-to-frame component, the latter having been expressed as integrals from \( r = 0 \) involving the components \( K^m_i \). We find in short:

\[
\sum_{b,a=1,2,3} \int_0^r \int_{\Sigma_r} r^{2c+9} \partial^j (R^m_{b\partial 0}) \partial^j (R^m_{b\partial 0}) (\cot \theta)^2 \sin \theta d\theta d\tau \notag
\]

\[
\leq \int_0^r O(\tau^{\frac{1}{2}}) \int_{\Sigma_r} \tau^{2c+9} |\partial^j \partial_a K^m_{ab}|^2 (\cot \theta)^2 \sin \theta d\theta d\tau. \tag{6.48}
\]

Thus this terms can be absorbed into the LHS.

The “main term” is the one with \( \nabla_a K^m_{\partial b} \). We then integrate by parts the derivative \( \nabla_a \). The resulting main term is:

\[
\int_0^r \sum_{d=1,2,3} \int_{\Sigma_r} r^{2c+9} \sum_{a,b=1,2,3} [\partial^j K^m_{ab}] [\partial^j \nabla_a (K^m)]_{ab} \sin \theta d\theta d\tau. \tag{6.49}
\]

We can then use the curvature identity to switch \( \nabla_a, \nabla^b \) in the second factor (the resulting term involves a curvature term that is again bounded like the curvature term (6.48) above); after this integration by parts the term we invoke the Codazzi equations again, to obtain:
\[
\int_0^\tau \sum_{d=1,2,3} \int_{\Sigma_r} \tau^{2c+39} \sum_{a,b=1,2,3} [\partial^j \nabla^a (K^m_{ab})][\partial^j \nabla^b (K^m_{ab})] \sin \theta d\theta dt d\tau \\
= \int_0^\tau \sum_{d=1,2,3} \int_{\Sigma_r} \tau^{2c+9} \sum_{a,b=1,2,3} [\partial^j \partial \epsilon (K^m_{ab})][\partial^j \partial \epsilon (K^m_{ab})] \sin \theta d\theta dt d\tau \\
+ \int_0^\tau \sum_{d=1,2,3} \int_{\Sigma_r} \tau^{2c+9} \sum_{a,b=1,2,3} [\partial^j R_{ab\epsilon} \epsilon][\partial^j R_{ab\epsilon} \epsilon] \sin \theta d\theta dt d\tau.
\] (6.50)

The first term on the RHS are the term we wanted to obtain for our claim. the curvature term is bounded as in (6.48). This completes our proof of (6.42); for \( i = 2 \) we thus derive our desired estimates for \( K^m_{12} \) at the top orders.

The desired top-order estimates for \( K^m_{12} = \Delta^m_{12} \) are obtained from (6.47) by virtue of and the already desired bonds on \( \text{tr} K^m \).

\[ \square \]

6.4 Capturing the hypersurface that carries the initial data: Determining the functions \( r^m_{*}(t, \theta), K^m_{12}(t, \theta) \).

We here seek to identify the hypersurface \( \Sigma_r^m \) on which our prescribed initial data will be induced. As noted in the introduction, in the gauge we have chosen two key parameters related to the initial data are not fixed apriori by us, but must be solved for at each stage of the iteration: These are:

a. the function \( r^m_{*}(t, \theta) \) which defines the graphical hypersurface \( \Sigma_r^m(t, \theta) := \{ r = r^m_{*}(t, \theta) \} \) on which the prescribed initial data will live.

b. The direction \( e^m_2 \) on the initial data hypersurface (expressed as a rotation of the background fixed frame \( E_0, E_1, E_2 \)) which upon transport along \( e_0 \) according to equation (6.8) yields the collapsing direction at the singularity; moreover \( e^m_2 \) and the normal direction \( e^m_0 \) to \( e^m_2 \) is asymptotically a principal direction, in the sense that \( |m| K_{12}(r, t, \theta) \cdot |K^m_{12}(r, t, \theta)|^{-1} = o(1), |K^m_{12}(r, t, \theta)| \cdot |K_{12}(r, t, \theta)|^{-1} = o(1) \) as \( r \to 0 \). This direction \( e^m_2 \) is in 1-1 correspondence with a special direction \( \hat{e}^m_2 \) on \( \Sigma_r^m \); the latter is captured by \( \hat{K}^m_{12} \).

As we have seen in formula (3.21), the rotation angle \( \varphi^m(t, \theta) \) to transition from the background frame \( (E_1(t, \theta), E_2(t, \theta)) \) to \( (\hat{e}^m_1(t, \theta), \hat{e}^m_2(t, \theta)) \) is in 1-1 correspondence with the value of the initial data tensor \( \hat{K} \) evaluated against the frame \( (\hat{e}^m_1(t, \theta), \hat{e}^m_2(t, \theta)) \) at \( (t, \theta) \). Thus \( K(\hat{e}^m_1, \hat{e}^m_2) = \hat{K}^m_{12}(t, \theta) \) is the key parameter that “sees” the rotation of the background frame.

6.5 The inductive step on the functions \( r^m_{*}(t, \theta), \hat{K}^m_{12}(t, \theta) \).

We will be proving the inductive claims for \( r^m_{*}(t, \theta) \) and \( \hat{K}^m_{12}(t, \theta) \). We find it technically more convenient (for technical reasons related to the poles at \( \theta = 0, \pi \)) to obtain our estimates in the spaces \( L^2(d\theta dt) \) instead of \( L^2(\sin \theta d\theta dt) \), for all \( \partial \theta, \partial \epsilon \), derivatives up to order \( low - 1 \). In fact in this space the boundary condition \( \hat{e}^m_2(r^m_{*}) = 0 \) at the two poles is more readily imposed. At the orders higher than that we will obtain our estimates in the spaces defined by \( L^2(\sin \theta d\theta dt) \). To obtain our strengthened bound at the lower orders, we note that by virtue of the standard Hardy inequality, the bounds (6.21) (which hold for \( k \leq low \) imply the bounds:

\[
\int |\partial^I [K^{m}_{22}(\epsilon, t, \theta) - K_{22}(t, \theta)]|^2 d\theta dt \leq 2C^2 \eta^2 \epsilon^{-3},
\]

\[
\int |\partial^I K^m_{12}(\epsilon, t, \theta)|^2 d\theta dt \leq B^2 C^2 \epsilon^{-2-\frac{1}{2}} \leq C^2 \eta^2 \epsilon^{-3},
\]

for all multi-indices \( I, |I| \leq low - 1 \).

To distinguish estimates obtained with respect to this volume form, we will denote this space by \( L^2(\sin \theta d\theta dt) \). When we consider hypersurfaces \( \Sigma \) below on which coordinates \( \theta, t \) naturally live, we will denote this volume form by \( L^2(\Sigma, \sin \theta d\theta dt) \). The above notation will also extend to the standard Sobolev spaces \( H^k(d\theta dt) \). Outside this subsection, when we write \( L^2 \) or \( H^k \) we will mean that \( L^2 \) or \( H^k \) is with respect to the usual volume form \( \sin \theta d\theta dt \). We then claim:
**Proposition 6.8.** Consider the functions $K_{22}^m(t, r, \theta), K_{12}^m(t, r, \theta)$ defined by the formulas (6.18) (using (6.3), (6.20)). Then there exists a unique pair of functions $r_*^m(t, \theta), \tilde{K}_{12}^m(t, \theta)$ close to $(e, 0)$ in the norms below so that the functions $\tilde{K}_{12}^m[\Sigma_{\eta^m}], \tilde{K}_{12}^m[\Sigma_{\eta^m}]$ defined via the relations (3.18), (3.20) (and invoking (3.23) to express the second term in (3.20) in terms of the unknown $\tilde{K}_{12}^m$) in terms of $r_*^m(t, \theta), (\tilde{K}_{12}^m(t, \theta) = K_{12}^m(r_*^m(t, \theta), t, \theta))$ satisfy the requirement:

\[
\tilde{K}_{22}^m(t, \theta) = F_{t, \theta}^{22}[\tilde{K}_{12}^m(t, \theta)], \text{ subject to } \partial_3 r_*^m = 0 \text{ at } \theta = 0, \pi. \quad (6.52)
\]

Moreover the functions $r_*^m(t, \theta), \tilde{K}_{12}^m(t, \theta)$ then satisfy the following estimates:

At the lower derivatives \( |I| = k \leq \text{low} - 1 := s - 4 - 4c \):

\[
\sum_{i=0}^k ||\partial_i^j (r_*^m - \epsilon)|| L^2(\Sigma_{\eta^m}, d\theta dt) \leq (D + 1) C \eta \epsilon_{\eta^m} \sum_{i=0}^k ||\partial_i^j (r_*^m - \epsilon)|| L^2(\Sigma_{\eta^m}, d\theta dt) \leq 2(D + 1) C \eta \epsilon^{-1/2}. \quad (6.53)
\]

\[
\sum_{i=0}^l ||\partial_i^j \tilde{K}_{12}^m|| L^2(\Sigma_{\eta^m}, d\theta dt) \leq DC \eta^{-3/2 + 4}. \quad (6.54)
\]

At the higher derivatives \( |I| = k \in \{\text{low}, \ldots, s - 3\} \) we have the bounds:

\[
\int_{-\infty}^\infty \int_0^\pi |\tilde{e}_2^m (r_*^m - \epsilon)|^2 \sin \theta d\theta dt \leq C^2 \eta^2 \epsilon^{-1 - \frac{k - 1}{2}}, \quad (6.55)
\]

\[
\int_{-\infty}^\infty \int_0^\pi |(r_*^m - \epsilon)|^2 \sin \theta d\theta dt \leq 4C^2 \eta^2 \epsilon^{-2 - \frac{k - 1}{2}}, \quad (6.56)
\]

\[
\int_{-\infty}^\infty \int_0^\pi |\partial_i^j \tilde{K}_{12}^m|^2 \sin \theta d\theta dt \leq 2C^2 \eta^2 \epsilon^{-3 - \frac{k - 1}{2}}. \quad (6.57)
\]

\[
\int_{-\infty}^\infty \int_0^\pi |\partial_i^j (\tilde{e}_2^m (r_*^m - \epsilon))|^2 \sin \theta d\theta dt \leq 4C^2 \eta^2 \epsilon^{-3 - \frac{k - 1}{2}}. \quad (6.58)
\]

Once we have proven this proposition, the inductive claim for $r_*^m(t, \theta), \tilde{K}_{12}^m(t, \theta)$ will be verified, and we may define $\tilde{K}_{11}^m$ on $\Sigma_{\eta^m}$ as an explicit function of $\tilde{K}_{12}^m$. This will be done in the later subsections.

### 6.6 Solving for $r_*^m(t, \theta), \tilde{K}_{12}^m(t, \theta)$ at the lower orders, via the inverse function theorem.

Here we prove Proposition 6.8 producing the desired bounds on $r_*^m(t, \theta), \tilde{K}_{12}^m(t, \theta)$ only at the lower $\leq \text{low} - 1$ orders. The higher order bounds will be obtained in the next subsection.

We will prove our result by an application of the inverse function theorem, once we suitably express the requirement (6.52) (with the boundary condition at the poles) in terms of the sought-after functions $r_*^m(t, \theta)$ and $\tilde{K}_{12}^m(t, \theta)$.

In particular, we expand out the equation (6.52), making use of the formulas (3.18), (3.20), (6.18), as well as (3.23).

Writing $\tilde{e}_2$ for $\tilde{e}_2^m$ for short, we derive:

\[
- \left( \frac{2M}{r_*^m} - 1 \right) \tilde{e}_2 \tilde{e}_2 r_* - \left( \frac{2M}{r_*^m} - 1 \right) \frac{2M}{r_*^m} \left( \tilde{e}_2 r_* \right)^2 + \left( \frac{m}{q} \right)^2 K_{22}^m(t, \theta, r_*) = \left( \frac{m}{q} \right) F_{t, \theta}^{22}[\tilde{K}_{12}^m] \quad (6.59)
\]

Recall the formula (6.18):

\[
K_{22}^m(t, \theta, r_*) = e^{-w(t, \theta)} \int_0^t e^{w(s, t, \theta)} \text{RHS}[K_{22}^m(s, t, \theta)] ds \quad (6.60)
\]

Next, we recall the formula for $F_{t, \theta}^{22}(-)$:

\[
F_{t, \theta}^{22}(\tilde{K}_{12}^m(t, \theta)) = (1 - \frac{\tilde{K}_{12}^m}{(K_{11} - K_{22})^2}) K_{22} + \frac{\tilde{K}_{12}^m}{(K_{11} - K_{22})^2} K_{11}. \quad (6.61)
\]

Next, we subtract $K_{22}^m(\epsilon, t, \theta)$ from both sides of the above equation. To analyze the resulting differences, we introduce the notation:

\[
r_*^m - \epsilon = \delta r_*^m. \quad (6.62)
\]
We remark that we will be seeking the variables \( \delta r_m^{\ast} \), \( \tilde{K}_{22}^m(t, \theta) \) in the certain suitably small balls in the Banach space \( H^{\text{low}-1}(d\sigma dt) \times H^{\text{low}-1}(d\sigma dt) \), defined in (6.88) below.

We also introduce a piece of notation:

**Definition 6.9.** \( O_{\text{low}}^{\text{low}-1}(f) \) below will stand for a quantity bounded in the relevant norm \( \| \cdot \|_{H^{\text{low}-1}} \) by \( B\| f\|_{H^{\text{low}-1}} \) for \( B \) the universal constant introduced in our introduction. We let \( O_{\text{low}}^{\text{low}-1}(f) \) stand for a generic function \( G \) of the variable \( f \) (and possibly other parameters too, such as \( t, \theta \), which satisfies a uniform bound for all \( l \leq \text{low} - 1 \)):

\[
\sqrt{\int_{t, \theta} |\partial f|^2 d\sigma dt} \leq B\eta \epsilon^p \sum |l| \sqrt{\int_{t, \theta} |\partial f|^2 d\sigma dt},
\]

where \( B > 0 \) is the uniform constant from the introduction.

We next make the simple but key observation that:

\[
K_{22}^m(r_\ast^m(t, \theta), t, \theta) - K_{22}^m(\epsilon, t, \theta) = d_2^m(t, \theta)[(r_\ast^m(t, \theta))^{-\frac{3}{2}} - \epsilon^{-\frac{3}{2}}] + O_{\epsilon^{-5/2 + \frac{q}{4}}}(\delta r_m^\ast).
\]  

(6.63)

Prior to deriving this, let us expand out the first term on the RHS and make a note about its derivatives:

\[
\partial^I[d_2^m(t, \theta)[(r_\ast^m(t, \theta))^{-\frac{3}{2}} - \epsilon^{-\frac{3}{2}}]] = \partial^I[d_2^m(t, \theta)\epsilon^{-\frac{3}{2}I} (1 + \frac{\delta r_m^\ast}{\epsilon})^{-\frac{3}{2} - 1}]
\]

= \[I_1 \cup I_2 = I\] \( \partial^I[d_2^m(t, \theta)]\epsilon^{-\frac{3}{2}I} \left[ \sum_{q=1}^{\infty} \left( -\frac{3}{q} \right) \delta r_m^\ast \right] \)

(6.64)

Let us note that one term (the “main term” for us) in the RHS is when \( I_2 = I \) and \( q = 1 \). For those values the “main term” we obtain is:

\[-\frac{3}{2} d_2^m(t, \theta) \partial^I(\delta r_m^\ast).
\]

Then the sum of all terms inside the brackets \([\ldots]\) except the main terms in the ball \( B \) is bounded in \( L^2 \) by \( B\epsilon^{1+\frac{q}{4}} \| \delta r_m^\ast \|_{H^{\text{low}-1}} \).

Now let us derive (6.63):

**Proof of (6.63):** The thing that needs proof is that if we define \( D(\delta r_m^\ast) \) via the formula:

\[D^I(\delta r_m^\ast)(t, \theta) := \partial^I[u_2^m(r_\ast^m(t, \theta), t, \theta) - u_2^m(\epsilon, t, \theta)]\]

(6.65)

then we need to show:

\[D^I(\delta r_m^\ast) = O_{\epsilon^{-5/2 + \frac{q}{4}}}(\delta r_m^\ast).
\]  

(6.66)

To show this, let us express:

\[D(\partial^I(\delta r_m^\ast))(t, \theta) = e^{w(r_m^\ast(t, \theta))} \int_{r_m^\ast(t, \theta)}^{r_m^\ast(t, \theta)} e^{-w(s)} \text{RHS}[\partial^I \partial_s[u_2^m - u_2^m]](s, t, \theta)ds\]

We can then replace the terms \( \text{RHS}[\partial^I \partial_s[u_2^m - u_2^m]] \) using the expression in the RHS of (6.16). Each of these RHSs (evaluated in \( L^2 \) on any hypersurface \( \Sigma_{\gamma}(t, \theta), \epsilon(t, \theta) \) is uniformly bounded by \( B\epsilon^{-3/2} \) by virtue of the inductive bounds we have verified on \( \gamma_m \) and \( u_2^m \). Recalling the expression (6.17) on \( w(s, t, \theta) \), we derive the claim (6.66). \( \square \)

In particular, for all \( |I| \leq \text{low} - 1 \) we have derived the equation:

\[\partial^I K_{22}^m(r_\ast^m(t, \theta), t, \theta) - \partial^I K_{22}^m(\epsilon, t, \theta) = -\frac{3}{2} d_2^m(t, \theta)\epsilon^{-\frac{3}{2}} \partial^I(\delta r_m^\ast) + O_{\epsilon^{-5/2 + \frac{q}{4}}}(\delta r_m^\ast).
\]  

(6.67)

98
We define a key term in (6.63) to be a function \( V(t, \theta) \), and note a pointwise lower bound for it:
\[
V(t, \theta) := -(2M)^{-1/2} \frac{3}{2} d^{m}_{\infty}(t, \theta) e^{-\frac{\epsilon}{2}} \geq (2M)^{-1/2} \frac{3}{2} (1 - \frac{1}{8}) e^{-\frac{\epsilon}{2}}.
\] (6.68)

(The last inequality follows from the expression (4.48) for \( d^{m}_{\infty}(t, \theta) \) in terms of \( a^{m}(t, \theta) \) as well as the \( L^\infty \) bound \( |a^{m}(t, \theta) - 1| \leq C \eta \) and the Lipschitz bound for the function \( d^{m}_{\infty}(a^{m}) \)).

With this remark, we observe that equation (6.52) (with its LHS given by (6.70)) is one equation involving only the two unknowns \( \hat{K}_{12}^{m}(\epsilon, t, \theta) \) here the operator on the LHS is precisely:
\[
\Phi^1[\delta r^m, \hat{K}_{12}^{m}] = F_{t, \theta}^{22}[\hat{K}_{12}^{m}(t, \theta)] - K_{22}^{m}(\epsilon, t, \theta); \tag{6.69}
\]

here the operator on the LHS is precisely:
\[
\Phi^1[\delta r^m, \hat{K}_{12}^{m}] = -[(2M \frac{r^m}{r^m} - 1)^{-\frac{1}{2}} \tilde{e}_2 \tilde{e}_2 r^m_* + (\frac{2M}{r^m} - 1)^{-\frac{3}{2}} \frac{2M}{r^2} (\tilde{e}_2(r^m_*))^2 \epsilon + V(t, \theta) \cdot (\delta r^m) + O^{low}_{\epsilon^{-5/2 + \frac{3}{4}}}[\delta r^m]]. \tag{6.70}
\]

We note also that the RHS of (6.69) can be decomposed into a fixed term, and a term that depends on \( K_{22}^{m}(\epsilon, t, \theta) \), by recalling the form of \( F^{22}[\ldots] \) from (2.41):
\[
F_{t, \theta}^{22}[\hat{K}_{12}^{m}(t, \theta)] - K_{22}^{m}(\epsilon, t, \theta) = [K_{22}(t, \theta) - K_{22}^{m}(\epsilon, t, \theta)] - \frac{(\hat{K}_{12}^{m})^2}{(K_{11} - K_{22})^2} K_{11} + \frac{(\hat{K}_{12}^{m})^2}{(K_{11} - K_{22})^2} K_{22}. \tag{6.71}
\]

Thus we can re-write (6.69) can be re-expressed by moving all terms that depend on \( \hat{K}_{12}^{m} \) to the LHS:
\[
\Phi^1[\delta r^m, \hat{K}_{12}^{m}] = -[(2M \frac{r^m}{r^m} - 1)^{-\frac{1}{2}} \tilde{e}_2 \tilde{e}_2 r^m_* + (\frac{2M}{r^m} - 1)^{-\frac{3}{2}} \frac{2M}{r^2} (\tilde{e}_2(r^m_*))^2 \epsilon + V(t, \theta) \cdot (\delta r^m) + O^{low}_{\epsilon^{-5/2 + \frac{3}{4}}}[\delta r^m]] + \frac{(\hat{K}_{12}^{m})^2}{(K_{11} - K_{22})^2} K_{11} - \frac{(\hat{K}_{12}^{m})^2}{(K_{11} - K_{22})^2} K_{22} = [K_{22}(t, \theta) - K_{22}^{m}(\epsilon, t, \theta)]. \tag{6.72}
\]

We should recall that \( \tilde{e}_2 = \tilde{e}_2^m \) is a vector field that depends on the unknown \( \hat{K}_{12}^{m} \), via the formula (3.25). Thus (6.69) (with its LHS given by (6.70)) is one equation involving only the two unknowns \( r^m_*, \hat{K}_{12}^{m}(t, \theta) \), the LHS, and the RHS being fixed.

We also recall the bound derived in Lemma 6.6 which implies:
\[
\int_{t, \theta} |\partial^I (\hat{K}_{22}^{m}(\epsilon, t, \theta) - K_{22}(t, \theta))|^2 dt d\theta \leq 2C^2 \eta^2 \epsilon^{-3},
\]
for all \(|I| \leq 1 - 1\). \[38\]

Now, we perform the same analysis on the second equation (3.20).

Our first aim is to express \( u^{m}_2(r^m_*(t, \theta), t, \theta) = K^{m}_2(r^m_*(t, \theta), t, \theta) \) as a function of \( \delta r^m_\cdot \). For this, we refer to the integral representation (6.20) and again utilize (6.62), along with Taylor’s theorem and (6.2) to obtain for all \(|I| \leq 1 - 1\):
\[
[K_{12}^{m}(r^m_*, t, \theta) - K^{m}_1(\epsilon, t, \theta)] =
= (\delta r^m_* \cdot \sqrt{\epsilon} M^{1/2} [-2K_{22}^{m} + e_0(m \gamma)] \cdot K_{12}^{m} + [m^{-1} \nabla^{2}_{12}(m \gamma) + e_1(\gamma^{m-1})e_2(\gamma^m) + e_2(\gamma^{m-1})e_1(\gamma^m)](\epsilon, t, \theta)
+ O^{low}_{\epsilon^{-5/2 + \frac{3}{4}}}[\delta r^m_*]]. \tag{6.73}
\]

The fact that the “remainder term” is of the form \( O^{low}_{\epsilon^{-5/2 + \frac{3}{4}}}[\delta r^m_*] \) follows readily by the same argument as for \( \partial^I K_{22}^{m} \), applied this time to (6.20) and invoking the (verified for the step \( m \)) inductive claim on the quantities \( \partial^I \nabla_{ij}(m \gamma), \partial^I (e_1(m \gamma)), i, j \in \{1, 2\} \).

In particular, in analogy with (6.67) we derive the following expression for derivatives \( \partial^I, |I| \leq 1 - 1 \):
\[38\]Below \( e_i \) is shorthand for \( e_i^{m-1} \).
\[ \partial^I [K_{12}^m(r^m_*, t, \theta) - K_{12}^m(\epsilon, t, \theta)] = \]

\[ = \partial^I (\delta r^m_*) \cdot \sqrt{M^{1/2}} \left( [-2K_{22}^m \cdot + e_0(\gamma_m)] \cdot K_{12}^m + \partial^I [-M^2 \nabla_{12}^2(\gamma_m)] + e_1(\gamma_m)e_2(\gamma_m) + e_1(\gamma_m)e_2(\gamma_m-1)(\epsilon, t, \theta) + C_{\text{low}} \right) \]

\[ = [\delta r^m_*(\epsilon, t, \theta)] \]

Now, we recall our second equation (6.29), which we re-express as:

\[ \tilde{K}_{12}^m(r^m_*(t, \theta), t, \theta) = K_{12}^m(r^m_*(t, \theta), t, \theta) + \tilde{e}_2^m(\delta r^m_*) \cdot \tilde{e}^m_2(\gamma_m)[(K_{22} - K_{11})^{-1} \cdot \tilde{K}_{12}^m](t, \theta) + F_2([K_{22} - K_{11}]^{-1} \tilde{K}_{12}^m)(t, \theta) \]

Using the expression (6.73) as well as (3.23) we derive that our second equation (6.75) can be re-expressed in the form:

\[ \tilde{K}_{12}^m(r^m_*(t, \theta), t, \theta) - \tilde{e}_2^m(\delta r^m_*) \cdot \tilde{e}^m_2(\gamma_m)[(K_{22} - K_{11})^{-1} \cdot \tilde{K}_{12}^m] = K_{12}^m(r^m_*(t, \theta), t, \theta) - K_{12}^m(\epsilon, t, \theta) \]

\[ = [\tilde{K}_{12}^m(r^m_*(t, \theta), t, \theta) - \tilde{e}_2^m(\delta r^m_*) \cdot \tilde{e}^m_2(\gamma_m)[(K_{22} - K_{11})^{-1} \cdot \tilde{K}_{12}^m]](t, \theta) + F_2([K_{22} - K_{11}]^{-1} \tilde{K}_{12}^m) \]

Thus matters are reduced to solving the system of equations

\[ \tilde{V}(t, \theta) = -\sqrt{M^{1/2}} \left( [-2K_{22}^m \cdot + e_0(\gamma_m)] \cdot [K_{12}^m + \delta r^m_*(\delta r^m_*) \cdot \tilde{e}_2(\gamma_m)^{-1} \cdot \tilde{K}_{12}^m] \right) \]

Then our equation can be expressed in short as:

\[ \tilde{K}_{12}^m(r^m_*(t, \theta), t, \theta) - \tilde{V}(t, \theta) \cdot (\delta r^m_*) + O_{\text{low}} \left[ \frac{m-1}{2} \right] \tilde{K}_{12}^m = K_{12}^m(\epsilon, t, \theta) \]

We denote the LHS of the above by

\[ \Phi^2[\delta r^m_*, \tilde{K}_{12}^m]. \]

Thus matters are reduced to solving the system of equations

\[ \Phi^2[\delta r^m_*, \tilde{K}_{12}^m] = K_{22}^m(\epsilon, t, \theta) - K_{22}^m(\epsilon, t, \theta), \]

\[ \Phi^2[\delta r^m_*, \tilde{K}_{12}^m] = K_{12}^m(\epsilon, t, \theta). \]

We next solve this 2x2 system of equations via a variant of the inverse function theorem. We will use the inverse function theorem on \( H_{\text{low}}^{\text{low}}(dt \partial \theta) \), which requires us to also consider the differentiated version of these equations. To do this we first we introduce a piece of notation:

**Remark 6.10.** Given a multi-index \( I \) with \( |I| = k \in \{ 1, \ldots, \text{low}-1 \} \), the terms \( (l.o.t.'s)_I = (l.o.t.'s)_I(\delta r^m_*, \tilde{K}_{12}^m) \) in any equation appearing below will be a linear combination of products of lower-order terms \( \partial^i(\delta r^m_*) \) with \( p < k \) multiplied by terms of the form \( \partial^i K_{22}^m(\epsilon, t, \theta), \partial^i K_{12}^m(\epsilon, t, \theta), \partial^i \nabla(\gamma_m)(\epsilon, t, \theta) \).

Unless mentioned otherwise, the \( L^2 \) norm of these terms relative to the volume form \( dt \partial \theta \) will be bounded by
when appearing in the equation $\Phi^1 = \ldots$ and by

$$B\epsilon^{-1/4}\|\tilde{e}_2\delta r^m\|_{H^{k-1}} + B\epsilon\|\tilde{K}^m_{12}\|_{H^{k-1}},$$

when appearing in the equation $\Phi^2 = \ldots$.

Now, the differentiated version of (6.80), (6.81) is:

$$\partial^I[-\frac{2M}{\overline{r}^m} - 1 - \frac{2M}{\overline{r}^m} - 1 - \frac{2M}{\overline{r}^m}] + V(t, \theta)\partial^I(\delta r^m) + \partial^I\tilde{K}^m_{12} \frac{O(\tilde{K}^m_{12})}{K_{11} - K_{22}} \quad (6.82)$$

$$+ (l.o.t.'s)_1(\delta r^m, \tilde{K}^m_{12}) = \partial^I[\tilde{K}^m_{22}(t, \theta) - K^m_{22}(e, t, \theta)],$$

$$\partial^I\tilde{K}^m_{12}(t, \theta) - \tilde{e}_2^m(\delta r^m) \cdot \partial^I\tilde{e}_2^m[(\tilde{K}^m_{22} - K^m_{11})^{-1}, \tilde{K}^m_{12}](t, \theta)$$

$$- \{(\frac{2M}{\overline{r}^m} - 1 - \frac{2M}{\overline{r}^m}[(\tilde{K}^m_{22} - K^m_{11})^{-1} \cdot \tilde{K}^m_{12}] \cdot \partial^I(\tilde{e}^m_2(\delta r^m) + (F^3)'[(\tilde{K}^m_{22} - K^m_{11})^{-1} \tilde{K}^m_{12}]) (\tilde{K}^m_{22} - K^m_{22})^{-1}\delta^I\tilde{K}^m_{12}(t, \theta))$$

$$+ V(t, \theta) \cdot \partial^I(\delta r^m)(t, \theta) + (l.o.t.'s)_2(\delta r^m, \tilde{K}^m_{12}) = \partial^I[e^{-w(s)} e^{-w(s)} \int_0^{\pi} e^{w(s)} d\gamma = e^{1-1}(\gamma^{-1})e^{1-1}(\gamma^{1-1}) + e^{1-1}(\gamma^{-1})e^{1-1}(\gamma^{1-1})]ds].$$

(6.83)

(The terms $(l.o.t.'s)_1(\ldots), (l.o.t.'s)_2(\ldots)$ are quadratic terms that arise from the product rule. We do not record their precise form here but note that they satisfy the bounds in remark (6.10)).

We will solve the pair of equations (6.80), (6.81) in the two unknowns by an application of the inverse function theorem below. To do this, we recall that the RHSs in (6.80), (6.81) satisfy the following bounds in $H^k(\partial\Omega dt)$, $k = 0$:

$$\int_0^{\pi} \int_0^{\pi} |\partial^I\text{RHS}[6.80]|^2 d\theta dt \leq 2C^2 \eta^2 \epsilon^{-3}, \int_0^{\pi} \int_0^{\pi} |\partial^I\text{RHS}[6.81]|^2 d\theta dt \leq C^2 \eta^2 \epsilon^{-3+\frac{1}{2}}.$$

(6.84)

We now specify the notion of solution that we will use: The solution to the system of equations (6.80), (6.81) is in the sense of integration by parts. Specifically if we expand out those two equations in terms of the sought-after variables $\delta r^m, \tilde{K}^m_{12}$, multiply against test functions $v, u$ and integrate by parts one of the derivatives $\tilde{e}^2_2$ from the first term in (6.82) and the one derivative $\tilde{e}^2_2$ from the term $\tilde{K}^m_{22} - K^m_{22})^{-1} \cdot \tilde{K}^m_{12}$(t, $\theta$) from (6.83) then the corresponding integral identity should hold.

To derive the weak formulation of this system of equations, let us derive the integration by parts of the vector fields $\tilde{e}^2_2$ that we perform, as well as the boundary terms at the poles $\theta = 0, \theta = \pi$ that it gives rise to.

We recall that:

$$\tilde{e}^m_2[F] = (\partial_\theta (a^m)^{2\theta} + (a^m)^{2\theta} \partial_\theta)[F].$$

(6.85)

Recall that $(a^m)^{2\theta}$, $(a^m)^{2\theta}$ are determined by $\tilde{K}^m_{12}$ via the coefficients in the formula (4.25).

Thus, integration by parts of the outside $\tilde{e}^m_2$ (integrating against a general test function $v$) works as follows:

$$- \int_{-\infty}^{\infty} \int_0^{\pi} \tilde{e}^m_2 \tilde{e}^m_2[F] \cdot v d\theta dt = \int_0^{\pi} \tilde{e}^m_2[F] \cdot \tilde{e}^m_2 v d\theta dt - \int_{-\infty}^{\infty} \int_0^{\pi} \partial_\theta (a^m)^{2\theta} + \partial_\theta (a^m)^{2\theta} \tilde{e}^m_2[F] \cdot v d\theta dt$$

$$+ \int_{-\infty}^{\infty} (a^m)^{2\theta} \tilde{e}^m_2 \partial^I[F](\theta = 0, t) \cdot v d\theta dt - \int_{-\infty}^{\infty} (a^m)^{2\theta} \tilde{e}^m_2 \partial^I[F](\theta = \pi, t) \cdot v d\theta dt.$$

(6.86)

In view of this and the system (6.80), (6.81), then up to low $-1$ derivatives of the solution should satisfy the following $weak$ version of the two equations, where

$$(\text{l.o.t.'s})_1(\delta r^m, \tilde{K}^m_{12}), (\text{l.o.t.'s})_2(\delta r^m, \tilde{K}^m_{12})$$

stand for a general linear combination of lower-order terms as above, which now include such terms that arise from the commutations of derivatives in order to perform the integrations by parts we just described.
**Definition 6.11.** We consider functions \((\delta r_m, \tilde{K}_{12}^m) \in H^{\text{low-1}} \times H^{\text{low-1}}\). We call this pair a weak solution of the system \((6.80), (6.81)\) if any only if for any multi-index \(I, |I| = k \leq \text{low} - 1\) and any fixed pair of functions \((v, u) \in H^1((0, \pi) \times \mathbb{R}) \times H^1((0, \pi) \times \mathbb{R})\). We have \(39\)

\[
\int_{t, \theta} \left( \frac{2M}{r_s} - 1 \right)^{-\frac{1}{2}} \tilde{w}_2 \partial^i (\delta r_m) \cdot \tilde{w}_2 v - \left( \frac{2M}{r_s} - 1 \right)^{-\frac{1}{2}} \frac{2M}{r_s} \partial^i [(\tilde{w}_2 r_m^m)] v \\
+ [V - \tilde{w}_2] \frac{2M}{r_s} \left( \frac{2M}{r_s} - 1 \right)^{-\frac{1}{2}} + \left( \frac{2M}{r_s} - 1 \right)^{-\frac{1}{2}} [\text{div}(\tilde{w}_2^m)]^2 + \tilde{w}_2 [\text{div}(\tilde{w}_2^m)] : [\partial^i (\delta r_m^m)] \cdot v \\
+ \left[ \frac{O(\tilde{K}_{12})^{m}}{K_{22} - K_{11}} \right] \partial^i \tilde{K}_{12}^m \cdot v d\theta dt + \int_{t, \theta} \left( \text{l.o.t.} \right) \partial^i \tilde{r}_m \cdot \tilde{K}_{12}^m v d\theta dt \\
= \int_{t, \theta} \tilde{q}_m \partial^i [K_{22}(t, \theta) - K_{12}^m (\epsilon, t, \theta)] v d\theta dt,
\]

\[
\int_{t, \theta} \partial^i \tilde{K}_{12}^m (t, \theta) \cdot u d\theta dt + \int_{t, \theta} \tilde{w}_2^m (\delta r_m^m) \cdot \partial^i ([K_{22} - K_{11}]^{-1} \cdot \tilde{K}_{12}^m)(t, \theta) \cdot \tilde{w}_2^m u d\theta dt \\
+ \int_{t, \theta} \left[ \tilde{w}_2^m \tilde{w}_2^m (\delta r_m^m) + \text{div}(\tilde{w}_2^m) \tilde{w}_2^m (\delta r_m^m) \right] \cdot \partial^i ([K_{22} - K_{11}]^{-1} \cdot \tilde{K}_{12}^m)(t, \theta) \cdot u d\theta dt \\
- \int_{t, \theta} \tilde{w}_2^m (\delta r_m^m) \cdot \tilde{K}_{12}^m \cdot \partial^i \tilde{w}_2^m (\delta r_m^m) \\
+ (F^{'1})'(K_{22} - K_{11})^{-1} \tilde{K}_{12}^m (K_{22} - K_{11})^{-1} \partial^i \tilde{K}_{12}^m (t, \theta) \cdot u d\theta dt \\
+ \int_{t, \theta} \tilde{V}(t, \theta) \cdot \partial^i (\delta r_m^m)(t, \theta) \cdot u + \left( \text{l.o.t.} \right)^2 (\delta r_m^m, \tilde{K}_{12}^m) \cdot u d\theta dt \\
= \int_{t, \theta} \partial^i \tilde{w}_2^m (\delta r_m^m) \cdot v(\theta, t) dt + \int_{-\infty}^{\infty} \tilde{w}_2^m (\delta r_m^m) \cdot \partial^i \tilde{K}_{12}^m \cdot u(\theta, t) dt
\]

at the boundaries \(\theta = 0, \theta = \pi\).

We will be deriving a solution in the weak sense described above. This solution will be obtained by combining the inverse function theorem with a viscosity-type modification of the system \((6.80), (6.81)\) [re-cast as \((6.87)\)].

We start with setting the ground for the inverse function theorem. In particular, we will be seeking solutions \((\delta r_m^m)(t, \theta), \tilde{K}_{12}^m(\theta, t)\) in the following subsets of the Banach space \(H^{\text{low-1}}:\)

\[
\mathcal{B} := \left\{ \left( \| \delta r_m^m \|_{L^2(\Sigma)}, \| \tilde{w}_2^m (r_m^m - \epsilon) \|_{H^1(\Sigma)} \right) \leq DC\eta, \| \tilde{w}_2^m (r_m^m - \epsilon) \|_{H^1(\Sigma)} \leq 2DC\eta^{-1/2}, \right. \\
\left. \left. \right\| \tilde{w}_2^m \right\|_{H^1(\Sigma)} \right\} \leq 5DC\eta^{-2}, \| \tilde{K}_{12}^m \|_{H^1(\Sigma)} \leq DC\eta^{-3/2 + \frac{1}{4}, \forall \theta \leq \text{low} - 1} \right\}
\]

(6.88)

(Note again that the vector field \(\tilde{w}_2^m(t, \theta)\) is implicitly defined by \(\tilde{K}_{12}^m(t, \theta)\) via formula \((3.25)\)).

We start by recalling that the definition of the space \(\mathcal{B}\) and the Sobolev embedding show that over \(\mathcal{B}\):

\[
\| \tilde{w}_2^m (\delta r_m^m) \| \leq DC\eta^{-1/2}, \| \tilde{w}_2^m (\delta r_m^m) \| \leq DC\eta^{-2}
\]

(6.89)

We note that over \(\mathcal{B}\) \(\| \tilde{w}_2^m - 1 \| \leq \frac{1}{\xi}, \text{in view of the bounds on } \eta. \)

\(39\) We write \(\tilde{w}_2^m\) instead of \(\tilde{w}_2^m\), in some heavily labeled terms, for notational simplicity.
Note also that \( 	ilde{d}B(e_m^o) \) as defined in remark 6.12 can be bounded as follows, letting \( x = \tilde{K}_1^{m_m}(K_{22} - K_{11})^{-1} \):

\[
d\tilde{B}(e_m^o) = \partial_{\theta} \left\{ \frac{1}{2} \sqrt{1 + \frac{1}{4 \pi \eta} (g_{oo})^{-1/2}} \right\} + \partial_t \left\{ \frac{\text{sign}(x)}{2} \sqrt{1 - \frac{1}{4 \pi \eta} (g_{TT})^{-1/2}} \right\}
\]

(6.90)

Thus \( \tilde{d}B(e_m^o) \) is pointwise bounded by \( 2CC_{\text{Sob}} \eta \cdot \epsilon^{-1-1/4} \) in \( B \).

Let us note that these bounds imply that any pair of functions \( (\delta r_u^m, \tilde{K}_1^m(t, \theta)) \) in the above space we have the bound:

\[
[V(t, \theta) - e_m^o e_2^m - \frac{2M}{r^2} - 1]^{-\frac{1}{2}} + \frac{2M}{r^2} - 1)\cdot \tilde{d}B(e_m^o) \geq \sqrt{2M} \frac{3}{2} - \frac{1}{4} \epsilon^{-5/2}.
\]

We will use the above momentarily to derive the existence of a solution to the system (6.80), (6.81), in the weak sense described in (6.87), in the space (6.88). Thus we will be deriving the claimed bounds (6.53), (6.54).

**Remark 6.14.** We note that the standard Sobolev embedding on \( S^2 \times \mathbb{R} \) yields that any weak solutions belong to \( \mathcal{C}_{\text{low}}^{2-3} \), and the fact that they satisfy the solutions weakly (in the sense of (6.87)) then implies that they also solve the equations in the classical sense in that space. This follows by an approximation of the identity argument for the test functions, and an integration by parts off of \( v, u \).

To obtain our solutions and our bounds, we work not with the equations (6.87) directly, but rather with a viscosity version of these equations. (We do this to make use of the Lax-Maslov theorem further down—it is more convenient to apply this theorem instead of working on spaces that are dependent on the solution-dependent vector field \( e_m^o \).)

Letting \( \Delta \) be the standard Laplacian on \( S^2 \times \mathbb{R} \), (with \( t \in \mathbb{R}, \theta, \phi \in S^2 \)) \( \partial^2_t + \Delta_{\text{3d}} \), for any \( \zeta > 0 \) we consider the modified equations that arise from (6.80), (6.81) by adding a term \( -\zeta \cdot \Delta \) leading order term to each side. We do not write out the modified equations here, but rather skip to the (modified) weak solutions that we consider for the new equations:

Note that the solutions depend on \( \zeta > 0 \) now, and in particular the vector fields \( \tilde{e}_m^o \) (being given by \( m \tilde{K}_1^{m} \) via formula (3.25)) depend on \( \zeta \) also, since \( m \tilde{K}_1^{m} \) depends on \( \zeta \).

We again use the same notation as in definition 6.10 for the lower-order terms (l.o.t.’s) \( I_t, \epsilon \).

The notion of solution for this viscosity-altered system of equations is then again the following weak sense, for any \( k \leq \text{low} \) and any \( (v, u) \in [H^1 \times H^1](S^2 \times \mathbb{R}) \):

\[
\int_{t, \theta} \zeta \cdot \sum_{i=1}^3 \partial^i \partial_t (m \delta r_v^m) \partial_t v dt d\theta dt + \int_{t, \theta} \left( \frac{2M}{r^2} - 1 \right)^{-\frac{1}{2}} \tilde{e}_m^o \cdot \partial^i (\delta r_v^m) \cdot e_2^m v - \frac{2M}{r^2} - 1) \cdot \tilde{e}_m^o \cdot \partial^i (\delta r_v^m) \cdot v
\]

\[
+ [\mathcal{O}(\tilde{K}_1^{m})] \partial^i \tilde{K}_1^{m} \cdot v dt d\theta dt + \int_{t, \theta} \left( \text{l.o.t.’s} \right) [m \delta r_v^m, \tilde{K}_1^{m}] v dt d\theta dt
\]

\[
= \int_{t, \theta} \partial^i [K_{22}(t, \theta) - K_{12}(\epsilon, t, \theta)] v dt d\theta dt,
\]

\[
\int_{t, \theta} \zeta \cdot \sum_{i=1}^3 \partial^i \partial_t (m \tilde{K}_1^{m}) \partial_t u dt d\theta dt + \int_{t, \theta} \left[ 1 + \left( \frac{2M}{r^2} - 1 \right)^{-\frac{1}{2}} \right] \left( [(K_{22} - K_{11})^{-1} e_2^m e_2^m \delta r_v^m] + \text{dive} e_2^m \cdot (K_{22} - K_{11})^{-1} e_2^m \delta r_v^m \right) \partial^i \tilde{K}_1^{m} \cdot u
\]

\[
+ \partial^i [(K_{22} - K_{11})^{-1} e_2^m (\delta r_v^m)] \tilde{K}_1^{m} \cdot u + [(K_{22} - K_{11})^{-1} e_2^m (\delta r_v^m)] \partial^i \tilde{K}_1^{m} \cdot e_2^m u dt d\theta dt
\]

\[
+ \int_{t, \theta} \text{L} \partial^i (\delta r_v^m) \cdot u dt d\theta dt + \int_{t, \theta} \left( \text{l.o.t.’s} \right) [m \delta r_v^m, \tilde{K}_1^{m}] u dt d\theta dt = \int_{t, \theta} \partial^i K_{12}(\epsilon, t, \theta) \cdot u dt d\theta dt \]

(6.91)

We note that in the second equation, the coefficient in front of \( \partial^i \tilde{K}_1^{m} \cdot u \) is of the form \( (1 + \mathcal{O}(\eta)) \geq \frac{7}{8} \).

Let us derive the existence of a weak solution to this viscosity-enhanced system in detail, for completeness:

We will be considering the LHS of the above (for all \( k \)) as providing a map \( \Psi_{\zeta} \) from the space \( \Omega := \mathcal{B}(H_t, \delta) \times H_{t, \theta} \) (where \( m \delta r_v^m, m \tilde{K}_1^{m} \) will belong) to the space of bounded bi-linear forms acting.
on $H^1_{\text{low}}$. Denote the norm associated to this latter space by $\| \cdot \|_B$. Recall the duality between bi-linear forms on $L^2_{t,\theta} \times L^2_{t,\theta}$ and the space $L^2_{t,\theta} \times L^2_{t,\theta}$ itself. In particular for each $k \in \mathbb{N}$, $k \leq \text{low}$ we identify the bilinear form $\Psi_\zeta [\partial^I(m \delta_r \zeta), \partial^I(m \tilde{K}_{12}^\zeta)]$, $|I| = k$, with the factors of the integrands in (6.91) that depend on $\partial^I(m \delta_r \zeta), \partial^I(m \tilde{K}_{12}^\zeta)$.

Moreover the norm of $\Psi_\zeta [\partial^I(m \delta_r \zeta), \partial^I(m \tilde{K}_{12}^\zeta)]$ in $(L^2_{t,\theta} \times L^2_{t,\theta})^*$ is identified with the sum of the norms of the factors in (6.91) in $H^1_{\text{low},k}$, $|I| = k$.

In particular if we choose $v := \partial^I(m \delta_r m)$ and $u := \partial^I(m \tilde{K}_{12}^\zeta)$ as test functions, for all $|I| \leq \text{low} - 1$ and then sum in $I$ then the norms of the components of $\Psi_\zeta = (\Psi_1^{\zeta}, \Psi_2^{\zeta})$ evaluated against those test functions will be:

$$
\|\Psi_1^{\zeta}\|^2_B = \sum_{|I|=0}^{\text{low}-1} \int_{t,\theta} \zeta [\partial^I(m \delta_r \zeta) v] + [\partial^I(m \delta_r \zeta) u]^2 + [\partial^I(m \delta_r \zeta) v L^2_{t,\theta} + [\partial^I(m \delta_r \zeta) u] L^2_{t,\theta}]
\|\Psi_2^{\zeta}\|^2_B = \sum_{|I|=0}^{\text{low}-1} \int_{t,\theta} \zeta [\partial^I(m \tilde{K}_{12}^\zeta) v] + [\partial^I(m \tilde{K}_{12}^\zeta) u]^2 + [\partial^I(m \tilde{K}_{12}^\zeta) v L^2_{t,\theta} + [\partial^I(m \tilde{K}_{12}^\zeta) u] L^2_{t,\theta}]
$$

(6.92)

We will consider the space of functions $m \delta_r m \tilde{K}_{12}^\zeta$ which lie in the space $\Omega$ and derive the existence of a weak solution $m \delta_r m \tilde{K}_{12}^\zeta$ to the system (6.91) as follows: We observe that the map $\Psi_\zeta$ is uniformly (independently of $\zeta > 0$) bounded in $C^1$ between $H_{\text{low},k} \times H_{\text{low},k} \to (H_{\text{low},k} \times H_{\text{low},k})^*$. Moreover this map satisfies certain coercivity bounds over the domain $\Omega$: Let $D \Psi$ be the linearization of the map around any fixed element $(m \delta_r m, m \tilde{K}_{12}^\zeta) \in \Omega$. Then the above expression implies:

$$
\|D\Psi^1[(v, u), (v, u)]\|^2 \geq \zeta \|v\|^2_{H_{\text{low},1}} + \|\tilde{e}^2 v\|^2_{H_{\text{low}}} + (3 - \frac{1}{8}) \zeta \|v\|^2_{H_{\text{low}}},
\|D\Psi^2[(v, u), (v, u)]\|^2 \geq \zeta \|u\|^2_{H_{\text{low},1}} + (1 - \frac{1}{8}) \zeta \|u\|^2_{H_{\text{low}}}
$$

(6.93)

We prove these bounds momentarily. For now let us note how they imply the existence of our solution:

The RHSs of the two equations in (6.91) are bounded (respectively) as follows:

$$
\partial^I[K_{22}^\zeta(e, t, \theta) - K_{22}^\zeta(e, t, \theta)]_{L^2} \cdot \|v\|_{L^2} \leq C \eta \gamma^{-3/2} \cdot \|v\|_{L^2}, \|\partial^I K_{12}^\zeta(e, t, \theta)\|_{L^2} \cdot \|u\|_{L^2} \leq C \eta \gamma^{-1 - \frac{3}{2} + \frac{1}{4}} \cdot \|u\|_{L^2}
$$

(6.94)

On the other hand, the coercivity estimates and a simple Cauchy-Schwarz estimate for the cross terms imply that $\Psi_\zeta[\Omega]$ contains the product of balls $B_0(D \gamma \eta \gamma^{-3/2} \times B_0(D \gamma \eta \gamma^{-3/2} + 3/4}$; (6.94) implies that the pair of RHSs of our equations is contained in that product of balls.

In particular if we can derive the coercivity estimates (6.93) then we will have found a solution to the problem, with the bounds (6.96) in $H_{\text{low}}, H_{\text{low},-1}$. The coercivity bounds (6.93) also imply that our solutions are unique in the domain $\Omega$.

Moreover, we claim certain bounds on this solution:

**Lemma 6.15.** The solutions $m \delta_r m \tilde{K}_{12}^\zeta$ to the above system satisfy the bounds, for every $I, |I| \in \{0, 1, \ldots, \text{low}\}$:

$$
\int_{t,\theta} |\partial^I(m \delta_r \zeta)|^2 d\theta dt \leq 2C^2 \eta^2 \gamma^2 \int_{t,\theta} |\partial^I(m \tilde{K}_{12}^\zeta)|^2 d\theta dt \leq 2C^2 \eta^2 \gamma^2 \gamma^{-3 + \frac{1}{2}}.
$$

(6.95)

Moreover, our solution satisfies the following bound (also independent of $\zeta > 0$), for all $I, |I| \leq \text{low} - 1 \leq l$:

$$
\|\tilde{e}^2 \partial^I(m \delta_r \zeta)\|^2_{L^2} \leq 2C^2 \eta^2 \gamma^{-1}, \|\partial^I \tilde{e}^2 \partial^I(m \delta_r \zeta)\|^2_{L^2} \leq 5(D + 1)^2 C^2 \eta^2 \gamma^{-4}.
$$

(6.96)

We prove this Lemma together with (6.93). The proof is in fact essentially the same.
Proof. The proof we provide below for Lemma 6.15 proves (6.93) by replacing \( m \delta r \), \( K_{12} \) by \( v, u \).

The Lemma follows by using the functions \( m \delta r \), \( K_{12} \) themselves as test functions \( v, u \) (respectively) in the definition of weak solution, (6.87), and using a finite induction to move the lower-order terms (which have already been controlled) to the RHS.

One potentially problematic term is the one with \( \tilde{e}^m_n u \) in the RHS of the second equation in (6.91), due to the extra derivative on the test function \( u \) (where now \( u = K_{12} \)). Using integrations by parts (with respect to the volume form \( dvdt \)) on the \( \tilde{e} \) derivative in the second factor, we derive (writing \( \tilde{e} \) instead of \( \tilde{e}^m_n \), for brevity):

\[
\int_{t_0}^T \left| \frac{2M}{r^m} - 1 \right| - \frac{1}{2} (K_{22} - K_{11})^{-1} \cdot \tilde{e}_2 (m \delta r^e) \partial^i (m \tilde{K}_{12}) \tilde{e}_2 \partial^j (m \tilde{K}_{12}) dvdt \]

\[
= \int_{t_0}^T (K_{22} - K_{11})^{-1} \cdot \left[ \tilde{e}_2 (\frac{2M}{r^m} - 1) - \frac{1}{2} \tilde{e}_2 (m \delta r^e) + \frac{2M}{r^m} - 1 \right] - \frac{1}{2} \tilde{e}_2 (m \delta r^e) \cdot \tilde{e}_2 (m \tilde{K}_{12}) \eta \right| \partial^i (m \tilde{K}_{12})^2 d\sigma dt. \tag{6.97}
\]

Note the following bound on the coefficient of \( \partial^i (m \tilde{K}_{12})^2 \) in the RHS of the above:

Using the pointwise bounds (6.95), (6.96), (6.90) as well as the expression (3.25) on \( \tilde{e}_2 \) and the definition of our ball \( B \) we derive:

\[
(K_{22} - K_{11})^{-1} \cdot \frac{2M}{r^m} \cdot \tilde{e}_2 (m \delta r^e) + (\frac{2M}{r^m} - 1) \cdot \frac{1}{2} \tilde{e}_2 (m \delta r^e) \cdot \tilde{e}_2 (m \tilde{K}_{12}) \]

\[
\leq 2DC\eta + 4(D\eta)^2 \leq \frac{1}{2}. \tag{6.98}
\]

We wish to see how the RHSs of the equations (6.80), (6.81) we are solving, are thought of as linear operators on \( L^2 \) satisfy the following bounds when we test them on functions \( v = \partial^i (m \tilde{r}^e)(t, \theta), u = \partial^i K_{12}^m (t, \theta) \):

\[
\int_{t_0}^T \partial^i (K_{22}^m (t, \theta) - K_{22}^m (t, \theta)) \cdot \partial^j (m \tilde{K}_{12}) dvdt \leq \frac{1}{4} \cdot \epsilon^{-5/2} \int_{t_0}^T \partial^j (m \tilde{K}_{12})^2 2dvdt + 2\epsilon^{5/2} \int_{t_0}^T \partial^j (K_{22}^m (t, \theta) - K_{22}^m (t, \theta))^2 dvdt, \tag{6.99}
\]

\[
\int_{t_0}^T \partial^i (K_{12}^m (t, \theta)) \cdot \partial^j (m \tilde{K}_{12}) dvdt \leq (1/4) \cdot \int_{t_0}^T \partial^j (m \tilde{K}_{12})^2 dvdt + \int_{t_0}^T \partial^j (K_{12}^m (t, \theta))^2 dvdt. \tag{6.100}
\]

We now recall the bounds from Lemma 6.21

\[
\int_{t_0}^T \partial^j (K_{22}^m (t, \theta) - K_{22}^m (t, \theta))^2 dvdt \leq 2C^2 \eta^2 \epsilon^{-3}, \tag{6.101}
\]

\[
\int_{t_0}^T \partial^j (m \tilde{K}_{12})^2 dvdt \leq C^2 \eta^2 \epsilon^{-3} \tag{6.102}
\]

Thus, considering the two equations in (6.91) with \( v = (m \delta r^e)(t, \theta) \) and \( u = K_{12}^m (t, \theta) \), the above two inequalities give bounds on the RHSs of (6.91). We also use the Cauchy-Schwarz inequalities to absorb all cross terms in addition to (6.98) into the main positive quadratic terms

\[
\int_{t_0}^T \left| \frac{2M}{r^m} - 1 \right| - \frac{1}{2} \tilde{e}_2 (m \delta r^e) \cdot \tilde{e}_2 (m \tilde{K}_{12})^2 dvdt. \tag{6.99}
\]

in the LHSs of the two equations in (6.91).

This directly yields the first bounds claimed in Lemma 6.15 as well as the first bound in (6.96).

We note that the equation (6.82) coupled with the already derived bound on \( \partial^i (m \tilde{r}^e) \) implies directly that:

\[
\int_{t_0}^T \left| \frac{2M}{r^m} - 1 \right| - \frac{1}{2} \tilde{e}_2 (m \delta r^e) \cdot \tilde{e}_2 (m \tilde{K}_{12})^2 dvdt \leq \int_{t_0}^T \partial^j (m \tilde{K}_{12})^2 dvdt \leq 5C^2 \eta^2 \epsilon^{-3}.
\]
This completes the proof of our Lemma. Thus we have derived the existence of a weak solution and of the desired bounds.

Finally, we derive a solution of the original system \(^6.57\) by a limiting argument:

Given the uniform bounds in the spaces \(^6.96\), we have (independently of \(\zeta > 0\)), we can pass to a limit \(\zeta \to 0\) and derive the existence of a solution to the system \(^6.87\), which holds for all \(v, u \in H^{low} \times H^{low}\).

These solutions satisfy the bounds \(^6.53\), \(^6.54\), and these are independent of \(\zeta > 0\). Moreover they solve the system \(^6.87\) for all \(I, |I| \leq low\).

6.7 Estimates for the solved-for \(r^m_*(t, \theta), \tilde{K}^m_{12}\) at the higher derivatives.

We now prove the bounds \(^6.55\), \(^6.56\), \(^6.57\) in this subsection. For these orders our claims are with respect to the usual volume form \(\sin \theta d\theta dt\).

We recall that the functions \(r^m_*(t, \theta), K^m_{12}(t, \theta)\) have already been solved for at this point, and we have also obtained the bounds \(^6.53\), \(^6.54\) at the lower orders. We now seek to derive bounds for them at the higher orders.

We derive these bounds here, however in the reverse order of what we did before:

We will first bound the functions \(\partial^I K^m_{12}(t, \theta, r^m_*(t, \theta)), \partial^I K^m_{12}(t, \theta, r^m_*(t, \theta))\) on \(\Sigma_{e,\pi}\). In the \(L^2(\sin \theta d\theta dt)\) norm. After those bounds have been derived, we will then bound the sought-after parameters \(\partial^I (r^m_*(t, \theta)), \partial^I (\tilde{K}^m_{12}(t, \theta, r_*(t, \theta)))\).

**Bounds on** \(K^m_{12}(t, \theta, r^m_*(t, \theta)), K^m_{12}(t, \theta, r^m_*(t, \theta))\) **at the higher norms:**

Having already solved for \(r^m_*(t, \theta)\) in the previous subsection, we can now invoke the bounds \(^6.55\), \(^6.58\) applied to \(\delta_*(t, \theta) = r^m_*(t, \theta)\). This then implies \(^6.36\), \(^6.39\).

**Derivation of the bounds on** \(\partial^I \tilde{e}^m_2(r^m_*, \partial^I (r^m_*) - e, \partial^I \tilde{K}^m_{12}, k \geq low\):** To achieve this, we use the same argument as in the previous subsection, referring to the equations \(^6.82\), \(^6.83\).

Instead of treating equations \(^6.80\), \(^6.81\) directly we first consider a suitable integrated version of this equation, and then derive our desired estimates using that re-cast equation. (As we will see, using the integrated version of the equation will also capture the imposed \(\partial_0 (\delta r^m_*) = 0\) condition at the two poles \(\theta = 0, \theta = \pi\).

In view of the expression \(^6.85\) for the vector field \(\tilde{e}^m_2\), we also put down the formula \(\theta\) for the integral along the integral curves of \(\tilde{e}^m_2\): We consider integrals originating at \(t = 0\) until \(\theta = \pi/2\). Denote this integral operator by \((\tilde{e}^m_2)^{-1}\). We let \(s\) stand for the parameter with \(\tilde{e}^m_2(s) = 1, s = 0\) at \(\theta = 0\). In particular for any \(s \in \{0, \pi/2\}, t = t_*, t_\in \mathbb{R}\) we let \(\theta_*(s), t_*(s)\) be the point that arises by flowing along \(\tilde{e}^m_2\) from \((0, t_*)\) for parameter \(s\). Then define:

\[
(\tilde{e}^m_2)^{-1}[F](\theta, t_*) = \int_0^\theta F(\theta_*(s), t_*(s))ds.
\]

Note of course that with this definition:

\[
(\tilde{e}^m_2)^{-1}[\tilde{e}^m_2 \tilde{e}^m_2 \partial^I (e^m_1 - 1) \partial^I (\delta r^m_*)](\theta, t_*) = \tilde{e}^m_2 \partial^I (e^m_1 - 1) \partial^I (\delta r^m_*)(\theta, t_*)
\]

\[
+ (\tilde{e}^m_2)^{-1}[\partial^I \tilde{K}^m_{12} \frac{O(\tilde{K}^m_{12})}{K_{11} - K_{22}}] + (\tilde{e}^m_2)^{-1}[\partial^I \tilde{K}^m_{12}] = (\tilde{e}^m_2)^{-1}[\partial^I [K_{22}(t, \theta) - K^m_{22}(t, \theta, \theta)]].
\]

\((6.103)\)
We will then be multiplying the above equation by $-e_2^m \partial^j (\delta r_i^m)$ and integrating over $[0, \pi] \times \mathbb{R}$ with respect to the volume form $\sin \theta d\theta dt$. (Note that strictly speaking, we should break up the interval $[0, \pi]$ into $[0, \pi/2]$ and $[\pi/2, \pi]$ and add the resulting two expressions—we skip the obvious details here).

Let us first consider the “main terms” in the resulting integral identity, first in the LHS and then in the RHS. We will subsequently briefly discuss how all the remaining terms can be absorbed into these main terms.

The first main term we obtain from the procedure just outlined is:

$$\int_{-\infty}^{\infty} \int_{0}^{\pi} |e_2^m \partial^j (\delta r_i^m)|^2 \sin \theta d\theta dt.$$

This is one of the terms we are seeking to bound, and we keep this term as is. The next key term in the integral is:

$$-\int_{-\infty}^{\infty} \int_{0}^{\pi} [V \cdot \partial^j (\delta r_i^m)] e_2^m \partial^j (\delta r_i^m) \sin \theta d\theta dt$$

For this term, we perform an integration by parts with respect to $e_2^m$. We obtain the main term:

$$+ \int_{-\infty}^{\infty} \int_{0}^{\pi} V |\partial^j (e_2^m)^{-1} \partial^j (\delta r_i^m)|^2 \sin \theta d\theta dt. \quad (6.104)$$

This term we wish to keep. There is also a second term arising from $-e_2^m (\sin \theta) = (a^m)^{2\theta} \cos \theta$; that term is of the form:

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\pi} (a^m)^{2\theta} \cdot V^{-1} \cdot e_2^m [(e_2^m)^{-1} (V \partial^j (\delta r_i^m))]^2 \cos \theta \sin \theta d\theta dt.$$

In this term we again integrate by parts the $e_2^m$ derivative, and we derive:

$$\frac{1}{2} \int_{-\infty}^{\infty} \int_{0}^{\pi} (a^m)^{2\theta} \cdot V^{-1} \cdot e_2^m [(e_2^m)^{-1} (V \partial^j (\delta r_i^m))]^2 \cos \theta \sin \theta d\theta dt$$

The first term in the RHS has a favourable sign. For the second term, we recall the pointwise bounds on $(a^m)^{2\theta}$ and its first $\partial \theta$-derivative from formula (3.25) as well as the bounds on $V(t, \theta)$ and its first derivatives. Combining these with the standard Hardy inequalities on the interval $[0, \pi]$ , we find that the RHS of the above is bounded in absolute value by:

$$(a^{1/4}) \int_{-\infty}^{\infty} \int_{0}^{\pi} \epsilon^{-3} |\partial^j (\delta r_i^m)|^2 \sin \theta d\theta dt.$$

Thus this term can be absorbed into the main term (6.104).

Let us also derive some useful bounds on the product term appearing in the RHS of the weak equation (6.103).

For brevity, let $v := \partial^j ([\delta r_i^m](t, \theta) - (\delta r_i^m)(t, 0))$. We must control the term:

$$\int_{-\infty}^{\infty} \int_{0}^{\pi} (e_2^m)^{-1} (V \partial^j (\delta r_i^m) - K_{22}(\cdot, t)) (t, \theta) \cdot e_2^m (v) \sin \theta d\theta dt. \quad (6.106)$$

Again we perform an integration by parts of the derivative $e_2^m$. The main term we obtain is:

$$\int_{-\infty}^{\infty} \int_{0}^{\pi} (\partial^j [K_{22}^m(\cdot, t) - K_{22}(\cdot, t)] (t, \theta) \cdot v \sin \theta d\theta dt.$$

This term can be controlled by Cauchy-Schwarz in absolute value by:

$$2e^2 \int_{-\infty}^{\infty} \int_{0}^{\pi} |(\partial^j (K_{22}^m(\cdot, t) - K_{22}(\cdot, t)) (t, \theta) |^2 \sin \theta d\theta dt + \frac{1}{8} e^2 \int_{-\infty}^{\infty} \int_{0}^{\pi} |v|^2 \sin \theta d\theta dt.$$

There is a correction term from our previous integration by parts, which is of the form:
\[ \int_{-\infty}^{\infty} \int_{0}^{\pi} a^{2\theta} (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot (v) \cos \theta d\theta dt. \]

To control this term, we break the inner integral \( \int_{0}^{\pi/2} \) into \( \int_{0}^{\pi/2} + \int_{\pi/2}^{\pi/2} \) and write \( v = \varepsilon_{2}^{m} (\varepsilon_{2}^{m})^{-1} v \), where now in the first interval \( \varepsilon_{2}^{m} [F][\theta, t_{s}] = \int_{0}^{t_{s}} F(\theta(t), t_{s}, s) ds \) and in the second interval \( \varepsilon_{2}^{m} [F][\theta, t_{s}] = \int_{0}^{t_{s}} F(\theta(t), t_{s}, s) ds \). In both of these intervals we integrate by parts again. The resulting expression in each of the intervals is essentially the same, so we just perform it on the first interval \( \theta \in [0, \pi/2] \) and we derive:

\[ \int_{-\infty}^{\infty} \int_{0}^{\pi/2} a^{2\theta} (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot (v) \cos \theta d\theta dt \]

\[ = \int_{-\infty}^{\infty} \int_{0}^{\pi/2} [a^{2\theta}] (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot ((\varepsilon_{2}^{m})^{-1} v) \sin \theta d\theta dt \]  

(6.107)

\[ + \int_{-\infty}^{\infty} \int_{0}^{\pi/2} a^{2\theta} (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot ((\varepsilon_{2}^{m})^{-1} v) \cos \theta d\theta dt. \]

(Note that the two boundary terms vanish, since at \( \theta = 0 \) the factor \( (\varepsilon_{2}^{m})^{-1} v \) vanishes, while at \( \theta = \pi/2 \) \( \cos \theta \) vanishes.) Now the first term can be controlled by Cauchy-Schwarz:

\[ \left| \int_{-\infty}^{\infty} \int_{0}^{\pi/2} [a^{2\theta}] (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot ((\varepsilon_{2}^{m})^{-1} v) \sin \theta d\theta dt \right| \]

\[ \leq \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left[ a^{2\theta} \right] (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \sin \theta d\theta dt \]

(6.108)

\[ + \frac{1}{4} \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left[ a^{2\theta} \right] (\varepsilon_{2}^{m})^{-1} v^{2} \sin \theta d\theta dt. \]

Using Hardy’s inequality, (recalling that \( \varepsilon_{2}^{m}(\theta) = a^{2\theta} \)) both terms can be controlled by:

\[ \frac{1}{4} \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left| \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right|^{2}(t, \theta) \sin \theta d\theta dt \]

(6.109)

Finally, we now bound the second term in the RHS of (6.107) again by Cauchy-Schwarz and then Hardy, and introducing \( \sin \theta \) into the volume form of the term involving \( K_{22}^{m} - K_{22}^{m} \):

\[ \int_{-\infty}^{\infty} \int_{0}^{\pi/2} a^{2\theta} (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot ((\varepsilon_{2}^{m})^{-1} v) \cos \theta d\theta dt \]

\[ \leq 4 \varepsilon^{2} \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)^{2}(t, \theta) \sin^{2} \theta d\theta dt + \frac{\varepsilon^{2}}{4} \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left[ a^{2\theta} \right] (\varepsilon_{2}^{m})^{-1} v^{2} \sin^{2} \theta d\theta dt \]

\[ \leq 4 \varepsilon^{2} \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)^{2}(t, \theta) \sin \theta d\theta dt + \frac{\varepsilon^{2}}{4} \int_{-\infty}^{\infty} \int_{0}^{\pi/2} \left[ a^{2\theta} \right] (\varepsilon_{2}^{m})^{-1} v^{2} \sin^{2} \theta d\theta dt \]

(6.110)

We separately consider the product

\[ \int_{-\infty}^{\infty} \int_{0}^{\pi} (\varepsilon_{2}^{m})^{-1} \left( \partial^{2} (K_{22}^{m}(\cdot, \cdot, t) - K_{22}^{m}(\cdot, \cdot, t)) \right)(t, \theta) \cdot (\varepsilon_{2}^{m})^{-1} \sin \theta d\theta dt \]

which arises in the integrated equation out of (6.103); this term (using the \( L^{\infty} \) bounds for \( \tilde{K}_{12}^{m} \) in our space) is also bounded via Cauchy-Schwarz by:

\[ \frac{\varepsilon^{1/4}}{2} \int_{-\infty}^{\infty} \int_{0}^{\pi/4} \left| \partial^{2} (K_{22}^{m}) \right|^{2} \sin \theta d\theta dt + \frac{\varepsilon^{1/4}}{2} \int_{-\infty}^{\infty} \int_{0}^{\pi/4} |(\varepsilon_{2}^{m})^{-1} \sin \theta|^{2} \sin \theta d\theta dt \]

This concludes our treatment of the main terms in the integral identity we derived from (6.103). We note the secondary terms that also arise from the same equation, which arise from the terms we have not considered. These are straightforwardly bounded by Cauchy-Schwarz and absorbed into our main terms;
also all products of lower-order terms with $v$ are bounded by Cauchy-Schwarz, with the $L^2$-norm of the lower-order term placed in the right-hand side.

Our claimed bounds then follow directly from the system [6.80], [6.81] as described above: From [6.82] and all estimates derived after it, we derive for all $\kappa > 0$:

$$
\int_{t,\theta} \left( \frac{2M}{r^*} - 1 \right) - \frac{1}{2} \| \partial^t \tilde{e}_2 r^m \|^2 + \| V(t, \theta) + o(\epsilon^2 - 2 \frac{\tilde{K}_{12}^m}{K_{22} - K_{11}}) \| \partial^t (\delta r^m) \|^2  + o(1) \| \partial^t \tilde{K}_{12}^m \cdot \partial^t (\delta r^m) \| \sin \theta d\theta dt \\
\leq 4\kappa \int_{\Sigma_t} \| \partial^t (K_{22}^m(t, \theta) - K_{22}(t, \theta))^2 dt \sin \theta d\theta + \int_{\Sigma_t} \| (l.o.t.'s)^2 dt \sin \theta d\theta + \kappa^{-1} \int_{\Sigma_t} \| \partial^t (\delta r^m) \|^2 \sin \theta d\theta.
$$

(6.111)

In this case we recall the bounds from Lemma [6.55] we now have on the terms

$$
\int_{\Sigma_t} \| \partial^t (K_{22}^m(t, \theta) - K_{22}(t, \theta))^2 dt \sin \theta d\theta + \int_{\Sigma_t} \| (l.o.t.'s)^2 dt \sin \theta d\theta \leq C^2 \eta^2 \epsilon^{-3\frac{1}{2}}.
$$

(6.112)

On the other hand, we consider [6.83], and multiply it by \( \partial^t \tilde{K}_{12}^m \) and again integrate in \( t, \theta \) with respect to the volume form \( \sin \theta d\theta dt \). We derive:

$$
\begin{align*}
\int_{t,\theta} \| \partial^t \tilde{K}_{12}^m \|^2 - (\tilde{e}_2 \delta r^m) (K_{22} - K_{11})^{-1} \partial^t \tilde{e}_2 \tilde{K}_{12}^m \cdot \partial^t \tilde{K}_{12}^m \sin \theta d\theta dt = \\
\int_{t,\theta} o(1) \partial^t \tilde{K}_{12}^m \cdot \partial^t (K_{22}^m(t, \theta) + (l.o.t.'s)) | \sin \theta d\theta dt.
\end{align*}
$$

(6.113)

The delicate term is the second one in the LHS, which we deal with via integration by parts (and commutation terms which give rise to \( (l.o.t.'s) \)):

$$
\begin{align*}
- \int_{t,\theta} (\tilde{e}_2 \delta r^m) (K_{22} - K_{11})^{-1} \partial^t \tilde{e}_2 \tilde{K}_{12}^m \cdot \partial^t \tilde{K}_{12}^m \sin \theta d\theta dt \\
= \int_{t,\theta} [\text{div} (\tilde{e}_2 \delta r^m) (K_{22} - K_{11})^{-1} + \tilde{e}_2 \delta r^m (K_{22} - K_{11})^{-1}] \| \partial^t \tilde{K}_{12}^m \|^2 + (l.o.t.'s) \cdot (K_{22} - K_{11})^{-1}
\end{align*}
$$

(6.114)

The coefficient \([\text{div} (\tilde{e}_2 \delta r^m) (K_{22} - K_{11})^{-1} + \tilde{e}_2 \delta r^m (K_{22} - K_{11})^{-1}]\) is \( o(1) \); hence the term can be absorbed in the first term in (6.113). The RHS can be controlled by Cauchy-Schwarz as before.

Then adding (6.111), (6.113) and choosing \( \kappa = \frac{1}{2} \epsilon^{-3/2} \) and absorbing the first term into the LHS, we derive our claims (6.55), (6.56), (6.57). \qed

We have thus derived the existence of our the hypersurface \( \Sigma_{r^m} \) on which our initial data will be induced, along with the key component \( \tilde{K}_{12}^m(t, \theta) \) on that hypersurface.

The rest of this section is devoted to deriving the inductive assumptions for the remaining term \( K_{11}^m(r, \theta) \) in the REVESNGG system. We commence by deriving the data for these quantities on the just-solved-for hypersurface \( \Sigma_{r^m} \).

### 6.8 The initial data for the connection coefficients and curvature components on the initial data hypersurface \( \Sigma_{r^m} \).

Having solved for \( r^m(t, \theta) \) and \( \tilde{K}_{12}^m(t, \theta) \) and derived the claimed inductive estimates in the relevant spaces, we now proceed to impose the required initial conditions on the remaining parameter \( \tilde{K}_{12}^m(t, \theta) \), on the initial data hypersurface \( \Sigma_{r^m} \). (All these parameters are with respect to the adapted frame \( \tilde{m} \tilde{e}_0, \tilde{m} \tilde{e}_1, \tilde{m} \tilde{e}_2 \) on \( \Sigma_{r^m} \) they are in fact determined purely for the value of \( \tilde{K}_{12}^m(t, \theta) \) as we will recall.)

After these, we shall also impose the required initial conditions on the corresponding parameter that depend on the frame \( \tilde{m} \tilde{e}_0, \tilde{m} \tilde{e}_1, \tilde{m} \tilde{e}_2 \). (This will depend also on \( r^m(t, \theta) \) and on \( r^m \tilde{e}_2 (r^m(t, \theta)) \).)

We also express the new frames \( \tilde{m} \tilde{e}_0, \tilde{m} \tilde{e}_1, \tilde{m} \tilde{e}_2 \) and \( m \tilde{e}_0, \tilde{e}_1, \tilde{e}_2 \) with respect to the background coordinate vector fields \( \partial_t, \partial_\theta \) and find initial values for the parameters \( a^m_{\alpha i} (a^m)^{\alpha i} \): the vector fields \( \tilde{m} \tilde{e}_1, \tilde{m} \tilde{e}_2 \) are prescribed in terms of \( \partial_t, \partial_\theta \) via the requirements (6.46).
The desired bound at the top orders is derived in the last subsection of this section.

We use (3.19), along with (6.119) and the assumed closeness of \( K_{11} \) to \( K_{11}^* \) to derive for all \( I, |I| \leq s - 4 \):

\[
||\partial^I (K_{11}(r_m(t, \theta), t, \theta) - K_{11}(t, \theta))||_{L^2_{t,\theta}} \leq B \epsilon^{-3/2 + \frac{b}{4} + \frac{h}{4}} \leq \eta \epsilon^{-3/2}.
\]

The key to deriving the above four bounds is the control (by the same bounds) of the terms \( \tilde{\epsilon}^m_1, \tilde{\epsilon}^m_2 \) in terms of \( \partial_1, \partial_2 \) using (6.115). This gives a gain of a power \( \epsilon^{3/8} \) at the minimum. The rest of the terms are easily seen to also be bounded by the RHSs of (6.121), (6.122).

Then the function \( K_{11}^m(r_m(t, \theta), t, \theta) \), on the hypersurface \( \Sigma_{r_m}^{(t, \theta)} \), is defined by the equation (3.19).

In view of the inductive bounds (4.31), (4.32) on the quantities \( r_m(t, \theta), \tilde{K}_{12}^m(t, \theta) \) that were verified in the previous subsection, we next derive bounds on \( K_{11}^m(r_m(t, \theta), t, \theta) \).

We use (3.19), along with (6.119) and the assumed closeness of \( K_{11} \) to \( K_{11}^* \) to derive for all \( I, |I| \leq s - 4 \):

\[
||\partial^I (K_{11}(r_m(t, \theta), t, \theta) - K_{11}(t, \theta))||_{L^2_{t,\theta}} \leq (D + 1)C|\eta| \epsilon^{-3 + \frac{a}{4} + \frac{h}{4}} + \eta^2 \epsilon^{-3} \leq 2\eta^2 \epsilon^{-3}.
\]

While at the higher orders \( k > h, k \leq s - 4 \) (invoking the closeness of \( K_{11} \) to \( K^S \)) we similarly derive:

\[
||\partial^I (K_{11}(r_m(t, \theta), t, \theta) - K_{11}(t, \theta))||_{L^2_{t,\theta}} \leq 2(C|\eta|)^2 \epsilon^{-3 - \frac{k-1}{4}} \leq 2\eta^2 \epsilon^{-3 - \frac{k-1}{4}}.
\]

The desired bound at the top orders is derived in the last subsection of this section.

We use thus derived the desired bounds for \( K_{11}^m \) on the initial data hypersurface \( \Sigma_{r_m} \).

We proceed in the next subsections to derive the claimed bounds for \( K_{11}^m(r, t, \theta) \) off of this hypersurface, for all \( r \in (0, 2\epsilon) \).
6.9 Energy estimates for $K_{11}^m(r, t, \theta)$ and the asymptotically CMC property of the surfaces $\Sigma_{r, m}$.

Here we verify the inductive assumption (4.45), (4.44), (4.46), for the connection coefficient $K_{11}^m(r, t, \theta)$, and also verify (4.52).

We just imposed the initial data for $K_{11}^m$ on $\Sigma_{r, m}$ and also derived the desired bounds for it on that hypersurface. The evolution equation we have imposed is the equation (3.10). Having already solved for and bounded $K_{12}^m(r, t, \theta), K_{11}^m(r, t, \theta)$, this equation is purely a non-linear ODE in $K_{11}^m(r, t, \theta)$. Contrary to $K_{12}^m, K_{12}^m$, this ODE will be solved forwards, towards the singularity.

Remark 6.16. The reason why $K_{11}^m(r, t, \theta)$ can be solved forwards without potentially violating our “asymptotically CMC” claim at the level of derivatives is that in contrast to $K_{12}^m(r, t, \theta)$ the non-linear term $(K_{11}^m)^2$ in the Ricatti-type equation is non-focusing (given the sign of the initial datum is close to $K_{11}$). This implies that the solution remains smooth for all $r > 0$, and becomes singular at $r = 0$ purely because of the singular behaviour of the RHS in the Ricatti equation. Moreover, all solutions that we obtain (regardless of the initial condition we impose, provided it is close enough to $K_{11}$) will have a “free branch” of the form $O(r^{\gamma_m}(t, \theta))$ with $|\gamma_m(t, \theta)| \leq \frac{1}{2}$. In particular, it is much less singular than the leading-order behaviour $d_1^m(t, \theta) r^{-3/2}$ which is the contribution of the singular forcing terms on the RHS; this is in contrast to the behaviour of $K_{11}^m(r, t, \theta)$. Crucially this same free branch appears for the differentiated variable $\partial K_{11}^m(r, t, \theta)$; this is in contrast with $K_{12}^m(r, t, \theta)$, which admitted a singular branch of the form $r^{-3/2} + r^{\gamma_m(t, \theta)}$ for the differentiated variable $\partial K_{12}^m(r, t, \theta)$. Recall that it was to set these two singular branches (which would destroy our desired “asymptotically CMC” property) to zero that we solved for $K_{12}^m(r, t, \theta)$ and $K_{22}^m(r, t, \theta)$ backwards from the singularity.

Beyond the question of where initial data are imposed, the derivation of the inductive claim follows the same outline as for $K_{12}^m, K_{22}^m$.

At the bottom-lower orders, we show (4.39), recalling the expression:

$$K_{11}^m =: \frac{d_1^m(t, \theta) \sqrt{2M}}{r^2} + u_{11}^m$$

(6.125)

where according to Proposition 5.14 $\|d_1^m - \frac{1}{2}\|_{L^\infty} \leq DC\eta$. The function $u_{11}^m(r, t, \theta)$ is again claimed to be less singular, as $r \to 0$, see (4.44).

Recall that from the already verified inductive step for $\gamma_m$, (5.93) it follows that

$$|\epsilon_0^m u_{11}^m| \leq Br^{-1 - \frac{5}{2} \epsilon}$$

(6.126)

$r \leq 0, 2\epsilon, |\eta_0| \leq 1$. For the energy estimates of $u_{11}^m, K_{11}^m$, $i, j = 1, 2$, we start at the lower orders validating (6.3) for $K_{11}^m$ and work ourselves up to the top order energy estimates.

6.9.1 Lower-order estimates for $K_{11}^m(r, t, \theta)$: The asymptotic expansion.

Substitute the (6.3) in the LHS of (3.10)-(3.12) and (4.51) in the $c_0^2 \gamma_m, c_0 \gamma_m$ terms:

$$c_0 \gamma_m + (2d_1^m - \alpha_m) \sqrt{\frac{2M}{r^2}} u_{11}^m + (u_{11}^m)^2 + 3(u_{12}^m - 1)^2 + u_{33}^m u_{11}^m$$

(6.127)

$$= \nabla_{11} \gamma_m + (\nabla_{11} \gamma_m)^2 - c_0 \gamma_m - (u_{33}^m)^2 + (2\alpha_m - d_1^m) \sqrt{\frac{2M}{r^2}} u_{33}^m + \alpha_m O(1)$$

(6.128)

where $O(1/r^2)$ are explicit, analytic functions of $r$.

Notice that considering the model homogenous equation

$$c_0 y + (2d_1^m - \alpha_m) \sqrt{\frac{2M}{r^2}} y = 0$$

we observe that the solutions of this equation behave to leading order as $r^{2d_1^m(t, \theta) - \alpha_m(t, \theta)}$. Note that in view of the formula (4.42) this behaviour is much less singular than $r^{2d_1^m(t, \theta) - \alpha_m(t, \theta)}$. In fact this is the real asymptotic behaviour for the “homogenous free part” of the true equation (6.127).
Proposition 6.17. Given \( u^m_{11}(r^m, t, \theta) \in H^{s-3-4c} \), there exists a unique solution \( u^m_{11}(r, t, \theta) \in H^{s-3-4c} \), \( i, j = 1, 2 \), to the equation (6.127). This satisfies the bounds:

\[
\| u^m_{11} \|_{H^{s-3-4c}} \leq \frac{C}{2} r^{-\Delta C \eta} \| u^m_{11}(r^m, t, \theta) \|_{H^{s-3-4c}} + Br^{-\frac{1}{4}},
\]

for all \( r \in (0, 2c] \).

Observe that the conclusion of the previous proposition validates the inductive assumption (4.44) for \( u^m_{11} \), in view of the bounds on \( K^m_{11}(r^m, t, \theta) \), \( t, \theta \) in the previous subsection.

**Proof.** Write the equation (6.127) in the form

\[
\partial_r (r^{m-2d_1} m_{11}^m) = -\left( \frac{2M - 1}{r} \right)^{\frac{1}{2}} r^{m-2d_1} \left[ \nabla_1 \gamma^m + (\nabla_1 \gamma^m)^2 - c_0 u^m_{33} - (u^m_{33})^2 + O(1) u^m_{11} \right] + (2\alpha - d_1) \sqrt{\frac{2M}{r^2}} u^m_{33} + \alpha m O\left( \frac{1}{r^2} \right) - (u^m_{11})^2 - 3(u^m_{12})^2 - u^m_{33} u^m_{11}.
\]

(6.130)

We proceed by integrating (6.130) in \([r, r^m]\) to obtain:

\[
r^{m-2d_1} u^m_{11} \big|_{r^m} - \left[ (\frac{2M - 1}{r})^{\frac{1}{2}} r^{m-2d_1} \left[ \nabla_1 \gamma^m + (\nabla_1 \gamma^m)^2 - c_0 u^m_{33} - (u^m_{33})^2 + O(1) u^m_{11} \right] + (2\alpha - d_1) \sqrt{\frac{2M}{r^2}} u^m_{33} + \alpha m O\left( \frac{1}{r^2} \right) - (u^m_{11})^2 - 3(u^m_{12})^2 - u^m_{33} u^m_{11} \big] \right|_{r = r^m} d\tau.
\]

(6.131)

Utilising the estimates (6.126) for \( \gamma^m, u^m_{33} \), as well as the estimates in Lemmas (5.15) we infer that

\[
\| u^m_{11} \|_{L^\infty} \leq \frac{C}{2} r^{-\Delta C \eta} \| u^m_{11}(r^m, t, \theta) \|_{L^\infty} + Br^{-\frac{1}{4}},
\]

(6.135)

for \( r \in (0, 2c] \).

Next, we derive the lower order order energy estimates for \( u^m_{11}, i, j = 1, 2 \), proving (6.129). We argue by finite induction, assuming the estimate

\[
\| u^m_{11} \|_{H^{l+1}} \leq C r^{-\Delta C \eta} \| u^m_{11}(r^m, t, \theta) \|_{H^{l+1}} + Br^{-\frac{1}{4}},
\]

(6.136)

is valid for \( 0 \leq l \leq s - 3 - 4c \) and proceed to show that the analogous estimate holds for \( \partial^l u^m_{11} \), where \( |l| = l \leq s - 3 - 4c \). Note that for \( l = 1 \), (6.136) holds true by virtue of (6.14).

We derive:

\[
- \partial_{\mu m} \int_{\Sigma_{t+1} r^m} (\partial^l l^m) \| \vol_{Euc} = \int_{\Sigma_{t+1} r^m} 2\partial^l u^m_{11} \partial^l \left[ \frac{c_0 u^m_{11}}{(2M - 1)(1 + \partial_{\mu} (r^m - \theta))} \right] \| \vol_{Euc}
\]

(6.137)

\[
\leq \int_{\Sigma_{t+1} r^m} 2\alpha m \frac{d_{11}}{r} (\partial^l l^m)^2 \| \vol_{Euc} + \frac{B}{r^{1+2\Delta C \eta}} \| \partial^l l^m \|_{L^2} \leq \frac{B}{r^{1+2\Delta C \eta}} \| \partial^l l^m \|_{L^2} \| \partial^l l^m \|_{H^1} \| \partial^l l^m \|_{H^1} \| \partial^l l^m \|_{H^1} \| \partial^l l^m \|_{H^1} \| \partial^l l^m \|_{H^1},
\]

by (5.57), (4.51), and (4.29)
and hence

\[-\partial_{\rho m} \left( (m-1) \rho D\partial_{\rho} \left[ \partial^I u_{11} \right]_2 \right) \leq C\| \partial^I u_{11} \|_2 \leq C\| \partial^I u_{11} \|_2 \]  

\[+ \frac{B}{r^{1/4}} \| u_{11} \|_{H^1} + \frac{B}{r^{1/4}} \| u_{12} \|_{H^1} \]  

(6.138)

Then applying Lemma 4.8 to the above we derive the claim (6.136) for \(|I| = l\). Given the bounds we have imposed on \(\epsilon, \eta\) in terms of \(C\), this yields the inductive claim (4.44) for \(u_{11}^m\).

### 6.9.2 Higher order estimates for \(K_{11}^m\)

The higher order estimates for \(\partial^I K_{11}^m\), \(s - 3 - 4c < |I| \leq s - 4\), are derived in a similar manner to those for \(u_{11}^m\) in the previous subsubsection; We differentiate the Ricatti equation (3.10) to the required higher orders. We subtract from this the corresponding equation for the Schwarzschild component \(K_{11}^m(r, t, \theta)\) for \(\rho\). We show (6.140) is valid in increasing order in \(K\) of the highest order terms \(\partial^I (K_{11}^m - K_{11}^m)\) we replace the expressions (6.3)-(5.91). This enables us to distinguish the leading-order behaviour of those coefficients from the lower-order terms; the latter can be readily absorbed into the main estimate. We derive:

\[e_0 \partial^I [K_{11}^m - K_{11}^m] + \frac{2M}{r^2} \partial^I [e_0^{(m-1)}]_1 [K_{11}^m - K_{11}^m] + \frac{1}{r^{1/4}} \partial^I u_{11} + \frac{1}{r^{1/4}} \partial^I u_{12} \leq C\| \partial^I u_{11} \|_2 \leq C\| \partial^I u_{11} \|_2 \]  

(6.139)

\[= \partial^I \left[ (\nabla \partial_{\rho} + (\nabla \rho S - \epsilon_0^2 \rho - S) - \left( (e_0^{(m-1)})^2 - (e_0^{(m-1)})^2 \right) \right] \]

\[+ \sum_{I_1 + I_2 = I} \partial^I K_{11}^m \partial^I e_0^{(m-1)} + \sum_{I_1 + I_2 = I} \partial^I K_{11}^m \partial^I e_0^{(m-1)} \]  

This equation holds for all orders.

Our estimate at the higher orders is then the following:

**Proposition 6.18.** Given the value \(K_{11}^m(r, t, \theta)\) prescribed via (4.19), there exists a unique solution \(K_{11}^m\), to (6.139) until \(r = 0\). At the higher orders it satisfies the following estimates on level sets of \(\rho^m\):

\[\| \partial^I [K_{11}^m - K_{11}^m] \|_{L^2} \leq C \| \partial^I [K_{11}^m - K_{11}^m] \|_{H^1} \leq C \| \partial^I [K_{11}^m - K_{11}^m] \|_{H^1} \]  

(6.140)

for all \(r \in [0, 2\epsilon]\), \(s - 3 - 4c < |I| \leq s - 4\). The same estimate holds for \(|I| = s - 3\), but for \(\partial^I K_{11}^m\) main instead of \(\partial^I K_{11}^m\) as claimed in (4.48); analogously the claimed estimate.

The estimates (6.140), confirm the inductive claim on \(K_{11}^m\) at the higher derivatives. At the top derivatives we will prove (4.48) right below, using the already-derived (6.41) below.

**Proof.** We show (6.140) is valid in increasing order in \(|I| = l\), assuming the estimate (6.140) is valid for every \(|I| < l\). Note that in the case \(|I| < l = s - 3 - 4c + 1\), the estimate (6.140) is valid by (6.3), (6.129).

In the derivations below we make use of the energy estimates (5.57), (5.78) for \(\rho^m\) and (4.29). We also make the convention that wherever \(e_0\) appears it refers to \(e_0^m\) of the \(m\)th (current) step in the induction.

The result of these (and the product inequality) is that the RHS of (6.139) is bounded in the \(L^2\) norm by \(\rho^{s-3-4c}\).

Thus, the energy inequality for \(\partial^I K_{11}^m\) reads:

\[-\partial_{\rho} \int_{\Sigma_{\rho}} (\partial^I K_{11}^m)^2 \text{vol}_{Euc} = \int_{\Sigma_{\rho}} 2(\partial^I K_{11}^m) \partial^I \left[ \frac{e_0 K_{11}^m}{(2M - 1)^{1/2}} \left[ 1 + \partial_{\rho} \chi(r)(\rho^m - e) \right] \right] \text{vol}_{Euc} \]  

\[\leq \int_{\Sigma_{\rho}} \frac{2M - 4d_{11}^m}{r} (\partial^I K_{11}^m)^2 \text{vol}_{Euc} + \frac{B}{r^{1/4}} \| \partial^I K_{11}^m \|_{L^2} \]  

(plugging in (6.139))

\[+ \left( \frac{B}{r^{1/4} + |D\eta|} \right)^{1/2} \| \partial^I K_{11}^m \|_{L^2} + \frac{B}{r^{1/4} + |D\eta|} \| \partial^I K_{11}^m \|_{L^2} \| \partial^I K_{11}^m \|_{L^2} \]  

(113)
In view of the bound (6.58), we observe that the above term is bounded by the factor $\tilde{e}$ we spell out in more detail. The parts which differ are precisely analogous to the one performed in 6.3; these parts we just outline. The parts which differ is of the form $|\rho, \epsilon|$. However, recall that ($\partial_\epsilon\partial_\theta$) we note that if one of those derivatives is of the form $\tilde{e}_2^m$ we can be re-expressed in terms of the derivatives $\partial_\epsilon, \partial_\theta$: $\tilde{e}_1^m (a^m)_{\epsilon\theta} = (a^m)_{\epsilon\theta} = (a^m)_{\epsilon\theta}$. Then, the term (6.142) can be expressed as follows, where $|J| = 1 - 4$: $\partial_\epsilon(\tilde{e}_1^m (\hat{K}_{22} - \hat{K}_{11})) = (a^m)_{\epsilon\theta} (a^m)_{\epsilon\theta} (a^m)_{\epsilon\theta}$. Thus, the term (6.142) can be expressed as follows, where $|J| = 1 - 4$: $\partial_\epsilon(\tilde{e}_1^m (\hat{K}_{22} - \hat{K}_{11})) = (a^m)_{\epsilon\theta} (a^m)_{\epsilon\theta} (a^m)_{\epsilon\theta}$. In view of the bound (6.58), we observe that the above term is bounded by $\epsilon^{-3/2 - \epsilon}$. In particular the initial data for $\partial_\epsilon \hat{K}_{11}$ at the top order satisfy the required bounds.

For all these top-order derivatives we can then repeat the proof in section 6.3 for the true connection coefficients $K_{11}^m, K_{12}^m, K_{22}^m$; we use the already-derived bounds for $K_{12}^m, K_{22}^m$ and we derive the claim for $trK^m$, as in that section. Since $trK^m = K_{11}^m + K_{22}^m$ and the claim has already been derived for $K_{22}^m$, we derive our claim for these top-order terms for $K_{11}^m$ also.

$$\partial_\epsilon(\tilde{e}_1^m (\hat{K}_{22} - \hat{K}_{11})) = (a^m)_{\epsilon\theta} (a^m)_{\epsilon\theta} (a^m)_{\epsilon\theta}.$$
A.2 Solutions to REVESNGG yield solutions of the vacuum Einstein equations.

Let us show how a solution to the REVESNGG system, with initial data that satisfy the vacuum constraint equations yield a metric $g$ that satisfies the vacuum Einstein equations:

We are given initial data $g, K$ for the EVE on an initial 3-dim hypersurface $\Sigma$ satisfying the constraint equations

$$
\begin{align*}
\{ (R_{0i}(g)|_{\Sigma} = 0 ) & \} D^bK_{ab} - D_b\text{tr}_gK = 0, & b = 1, 2, 3 \\
(2R_{00}(g)|_{\Sigma} + R(g)|_{\Sigma} = 0 ) & \text{R}(g) - |K|^2 + (\text{tr}_g K)^2 = 0
\end{align*}
$$

(A.3)

where $D$ is Levi-Civita connection of $g$. Let $\gamma, K_{ij}$, $i, j = 1, 2$, solve the wave-Ricatti system of equations (1.11), (1.20)-(1.22), for an orthonormal frame $\{e_i\}$ satisfying (1.18).

These parameters, together with the initial configurations satisfying (A.3), produce a 1 + 3-metric $g$, which see the location of the initial data hypersurface in our chosen gauge, along with the position of our chosen frame $e_1, e_2$ on our initial data surface. We will show that $g$ is in fact a solution to the EVE (1.10).

Axi-symmetric and polarized metrics (A.8) satisfy (10) Appendix VII the relations $R_{ai}(g) = 0$, $a = 0, 1, 2$, $R_{33}(g) = -\Box_\eta \gamma$. Since the wave equation is part of the system (1.11)-(1.12), it remains to show the vanishing of the Ricci components $R_{ab}(g)$, $a, b = 0, 1, 2$. For this purpose, we make use of the general geometric formula:

$$
R_{ab}(g) = R_{ab}(h) - \nabla_{ab}\gamma - \nabla_a\gamma \nabla_b\gamma, \quad a, b = 0, 1, 2.
$$

(A.4)

By (1.20)-(1.22) and (1.19), we obtain the identities:

$$
R_{00ij} = -\nabla_i \gamma - \epsilon_{i\epsilon_j} \epsilon_{\epsilon_j \gamma + \nabla_i \gamma + (\nabla_\gamma)^2), \quad i, j = 1, 2.
$$

(A.5)

Hence, $R_{12}(h) = -R_{0102}(h) = \nabla_1 \gamma + \epsilon_{1\epsilon_2} \epsilon_{\epsilon_2 \gamma}$, giving $R_{12}(g) = 0$. Contracting indices in (A.5), we obtain:

$$
R_{00}(h) = R_{0101}(h) + R_{0202}(h) = -h_{\Box} \gamma - (\nabla_\gamma)^2 + \nabla_0 \gamma + (\epsilon_\gamma)^2 + (\epsilon_0)^2 \nabla_\gamma \gamma + (\epsilon_0)^2,
$$

(A.6)

verifying that $R_{00}(g) = 0$. We also have from (A.1) the identity:

$$
R_{00ij}(h) = -R_{ij}(h) + \delta_{ij} R_{00}(h) + \frac{1}{2} \delta_{ij} R(h), \quad i, j = 1, 2.
$$

(A.7)

Evaluating (A.7) for $i = j = 1$, and plugging in (1.16), (A.6), we deduce that

$$
R_{11}(h) - \frac{1}{2} R(h) = \nabla_{11} \gamma + (\epsilon_1)^2,
$$

(A.8)

and similarly for $i = j = 2$:

$$
R_{22}(h) - \frac{1}{2} R(h) = \nabla_{22} \gamma + (\epsilon_2)^2.
$$

(A.9)

Hence, by (A.4), the vanishing $R_{11}(g), R_{22}(g)$ reduces to proving the vanishing of $R(h)$. Note that by tracing (A.4), we also have

$$
R(g) = R(h).
$$

(A.10)

We may thus rewrite (A.8)-(A.9) in the form

$$
R_{11}(g) = \frac{1}{2} R(g), \quad R_{22}(g) = \frac{1}{2} R(g).
$$

(A.11)

Next, we derive evolution equations for the Ricci components $R_{0i}(g), R_{02}(g)$, utilising the contracted second Bianchi identity: $[i, i_* = 1, 2, i \neq i_*]$

$$
e_0 R_{0i}(g) = \Box_\epsilon_0 R_{0i}(g) + (-1)^{i*} K_{12} R_{0i} - \frac{3}{2} D_j R_{ji}(g) - \frac{1}{2} D_i R(g) + (-1)^{i*} K_{12} R_{0i},
$$

(A.12)

and similarly for $i = j$.

We have:

$$
R_{ij}(g) = 0, \quad \text{for } i \neq j.
$$
Since by (A.3), \( R_{01}(g) = R_{02}(g) = 0 \) on \( \Sigma \) and \( R_{01}(g), R_{02}(g) \) satisfy the homogeneous ODE system (A.12), they vanish everywhere.

Lastly, by virtue of the second Bianchi identity and the vanishing of the components \( R_{0a}(g), R_{ab}(g) \), \( a \neq b \), it follows that

\[
e_0 R(g) = 2D^a R_{0a}(g) = -2K_{11} R_{11}(g) - 2K_{22} R_{22}(g) = -(K_{11} + K_{22}) R(g),
\]

where in the last equality we made use of (A.11). Note that according to (A.3) (and \( R_{00}(g) = 0 \)), \( R(g) \) vanishes on \( \Sigma \). This implies that \( R(g) \) vanishes everywhere, which in turn also yields the vanishing of \( R_{11}(g), R_{22}(g) \).

### A.3 Proof of Corollary 1.5

We sketch the proof of this Corollary.

The conditions imposed on the initial data in [20], along the hypersurface \( \Sigma \) (see Figure 1), induce initial data along the event horizons \( \mathcal{H}^\pm \) that are compatible with the ones in [16]. Next, the stability of the inner red-shift regions [10] induces initial data on space-like pieces in the interior of the black hole, emanating from the timelike infinities. In particular, given the (small) \( \epsilon > 0 \) and \( \eta > 0 \) that are needed for our theorem, if the initial perturbation is chosen small enough, the data induced on \( \Sigma \) will be \( \frac{\epsilon}{\eta} \)-close to (two different, in principle) Schwarzschild initial data near each (asymptotically cylindrical) end.

In particular, we obtain a hypersurface \( \Sigma_c \) covered by coordinates \( t, \theta, \phi \) where the data on \( \{ t \leq -H \} \) is a \( \eta/6 \)-perturbation (in the norms of our main theorem) of the Schwarzschild data with mass \( M_1 \) on the corresponding portion of \( \{ r = \epsilon \} \), and on \( \{ t \geq H \} \) is an \( \eta/6 \)-perturbation of the Schwarzschild data with mass \( M_1 \) on the corresponding portion of \( \{ r = \epsilon \} \). Moreover, \( M_1, M_2 \) can be taken to be \( \eta/10 \)-close to the background \( M \) by taking the initial perturbations small enough.

We now wish to apply Theorem 1.1 to this initial data set to proceed further towards the singularity.

Although Theorem 1.1 is stated in the case of a single Schwarzschild metric, i.e. \( M_1 = M_2 \), it can be easily adapted to the more general two-mass-limits scenario \( M_1 \neq M_2, M_1, M_2 \sim M_0 \), by arguing as follows:

Let \( (g, K) \) denote the perturbed initial data for the EVE on \( \Sigma \cong (-\infty, +\infty) \times S^2 \). Also, let \( (g_{S1}, K_{S1}) \) be the Schwarzschild initial data on \( \Sigma' \) of mass \( M_i \), \( i = 1, 2 \). Consider the subset of \( \Sigma' \), \([-C, C] \times S^2, C > 0 \), whose domain of dependence in Schwarzschild of mass \( M_0 \) intersects the singularity at \([-B, B] \times S^2, B > 0 \), see Figure 2. Then, we define

\[
\begin{align*}
g_1 &= h_{[-C-1, +\infty]} g + h_{(-\infty, -C]} g_{S1}, & K_1 &= h_{[-C-1, +\infty]} K + h_{(-\infty, -C]} K_{S1}, \\
g_2 &= h_{(-\infty, +\infty]} g + h_{[C, +\infty]} g_{S2}, & K_2 &= h_{(-\infty, +\infty]} K + h_{[C, +\infty]} K_{S2},
\end{align*}
\]

where the pairs of functions \( h_{[-C-1, +\infty]}, h_{(-\infty, -C]} \) and \( h_{(-\infty, +\infty]}, h_{[C, +\infty]} \) are both partitions of unity, satisfying

\[
\begin{align*}
h_{[-C-1, +\infty]} &= \begin{cases} 
1, & [-C, +\infty) \\
0, & (-\infty, -C - 1]
\end{cases}, & h_{(-\infty, +\infty]} &= \begin{cases} 
1, & (-\infty, C] \\
0, & [C + 1, +\infty)
\end{cases}.
\end{align*}
\]

In particular, the two pairs of initial data agree with \( (g, K) \) on \([-C, C] \times S^2 \).

The pairs \( (g_i, K_i) \) are not initial data for the EVE, since they obviously do not satisfy the constraint equations in the regions \([-C - 1, -C]\) and \([C, C + 1]\). However, we note that the singularity is \( \eta \)-close to the background Schwarzschild data, given that \( M_1, M_2 \) are both \( \eta/10 \)-close to \( M \), the initial data for these “cut-and-paste” initial data sets. Thus, we can run our iteration algorithm in Section 3 since each pair of initial data converges to Schwarzschild of the same mass \( M_i \) at both ends \( \vert t \vert = +\infty \). Hence, we may pass to the limit, producing space-time (1 + 3)-metrics \( g_i \), having induced data \( (g_i, K_i) \) on \( \Sigma \), but do not exactly solve the EVE. On the other hand, by (A.11), the initial data for the two metrics \( g_1, g_2 \) agree on \( \Sigma_C := \Sigma' \cap \{-C \leq t \leq C\} \): \( (g_1, K_1) = (g_2, K_2) = (g, K) \), verifying as well the constraint equations on this portion of the initial hypersurface \( \Sigma' \). Hence, they are both the same solution to the EVE in the domain of dependence region \( \mathcal{D}_{\text{com}} := \mathcal{D}(\Sigma' \cap \{-C \leq t \leq C\}) \), arising from \( (g, K_{\Sigma_C}) \).

Hence, the metric

\[
g := \begin{cases} 
g_1, & \mathcal{D}(\Sigma' \cap \{t \geq -C\}) \\
g_2, & \mathcal{D}(\Sigma' \cap \{t \leq C\})
\end{cases}
\]

\[\text{The domain of dependence considered with respect to } g_1 = g_2.\]
is well-defined and its induced data on Σ′ are g, K. Moreover, it satisfies the constraint equations on Σ and the reduced equations (1.11), (1.20)-(1.22). Thus, by the derivations in (A.2) we conclude that g is in fact a solution to the EVE, consistent with Theorem 1.1 in the future of Σ′, which we desired to prove.

**Remark A.1.** In the previous proof, we conveniently exploited the fact that our treatment of the reduced equations (1.11), (1.20)-(1.22), via the iteration scheme outlined in Section 3 does not make any further use of the constraint equations (A.3) for the EVE. Otherwise, one would have to adapt our derivations to explicitly deal with the different two-mass-limits at |t| = +∞, see Figure 1.

## B The optimal coordinates at the singularity: Derivation of Theorem 2.2

In order to derive Theorem 2.2 from Theorem 2.9 we need to establish the following Lemma:

**Lemma B.1.** There exist new coordinates ρ, ĥ, θ, with values ĥ ∈ (0, π), ī ∈ R whose coordinate fields ∂ī, ∂ī ∈ TΣα define frame-to-coordinates and coordinate-to-frame coefficients (a)ī, (a)ī, âi, ai, i = 1, 2, as in (4.81), which satisfy the following improved estimates: Let Σ1 be the Sobolev space of order l with respect to the volume form sinnψdψ on Σ. Then For all orders l ≤ −2 we have:

\[ \|a_{11}(r, t, θ) - a^S_{11}(r, t, θ)\|_{H^l[Σ'ν]} ≤ DCηr^{-d_1(t, θ)}, \quad \|a_{22}(r, t, θ) - a^S_{22}(r, t, θ)\|_{H^l} ≤ DCr^{-d_2}, \quad a_1 ≡ 0 \]

(B.1)

\[ \|a_{12}\|_{H^l} ≤ r^{1 - DCη}, \]

for all r ∈ (0, 2ε).

The bounds for the same quantities at the higher derivatives l ∈ {l − 1, 2, ..., s − 4} are as follows, where h = l − (l − 2):

\[ \|a_{12} - a_{12}^S\|_{H^l[Σ'ν]} \leq DCr^{1 - DCη - \frac{h}{2}}, \quad \|a_{11} - a_{11}^S\|_{H^l[Σ'ν]} \leq r^{-\frac{h}{2} - DCη - \frac{h}{2}}, a_1 ≡ 0, \|a_{12}\|_{H^l[Σ'ν]} \leq r^{1 - DCη - \frac{h}{2}} \]

(B.2)

where l ∈ {l, 2, ..., s − 4} and h = l − l.

**Proof.** The frame coefficients âi, ai, i = 1, 2, satisfy the ODEs (4.89), with ī, ĥ in place of t, θ. In this case, however, we will solve for all these parameters backwards from the singularity.

Notice that a_1, a_2 satisfy separate, homogeneous ODEs. Hence, setting there free branches equal to zero, eliminates the variable a_1 ≡ 0. This choice implicitly imposes that that ∂ī is parallel to unique collapsing direction, and that ∂ī should be the corresponding principal dual direction.

Then, the equations for a_1, a_2 decouple as well, having general solutions of the form:

\[ a_{11}(ρ, t, θ) = c_{11}(t, θ)e^{\int_0^t (1 - 2M/R_{K11})ds}, a_{22}(ρ, t, θ) = c_{22}(t, θ)e^{\int_0^t (1 - 2M/R_{K22})ds}, \]

\[ a_1 = 0, a_{12}(ρ, t, θ) = e^{-\int_0^t (1 - 2M/R_{K22})ds}K_{12}(t) \int_0^t e^{\int_0^t (1 - 2M/R_{K22})ds}2K_{12}(t)k_{11}(t, θ)\cdot (1 - 2M/R)^{1/2}dτ \]

(B.3)

(In the last term we have used the function a_1(ρ, t, θ) that was solved for first).

For future reference, we note that at the lowest orders, l ≤ −2, the parameter K_{12}(r, t, θ) satisfies the stronger bound, for all l ≤ l − 2:

\[ \|K_{12}\|_{H^l[Σ'ν]} \leq DCr^{1 - DCη}. \]

This is stronger than the inductive claim at the lower orders; We note that this follows by controlling the RHS in (1.22) in H^l by |Cη|^2r^{−2}DCη. This in turn follows by re-expressing the RHS of that equation in terms of the frame-to-coordinates coefficients a^4(r, t, θ). At all orders l ∈ {l − 1, 2, ..., s − 4} the bound worsens by a factor of r^{−l/4} for each order beyond low − 2.

We are free to choose the coefficients c_{11}(t, θ), c_{22}(t, θ); the only restriction is that the resulting vector fields

\[ \partialī = \sum_{i=1,2} a_{iī}e_ī, \partialī = \sum_{i=1,2} a_{iī}e_ī \]
should commute; it suffices to check this condition on \( \Sigma_{r_+} \), where we recall that by construction \( a_{t1} = c_{t1} \) and \( a_{\theta2} = c_{\theta2} \); in particular on that hypersurface we will be requiring:

\[
\begin{bmatrix}
c_{\theta2}(t, \theta) \hat{e}_2, c_{t1}(t, \theta) \hat{e}_1 + a_{t2}[c_{t1}, K_{12}] \cdot \hat{e}_2
\end{bmatrix} = 0. \tag{B.4}
\]

(We use the notation \( a_{t2}[c_{t1}, K_{12}] \) to highlight the dependence of the variable \( a_{t2} \) only on the two parameters \( c_{t1}, K_{12} \).)

Our freedom comes in choosing the values of the two functions \( c_{\theta2}(t, \theta), c_{t1}(t, \theta) \) along two curves \( \{ t = 0 \} \) and \( \{ \theta = 0 \} \) respectively. In fact that requirement fixes the values of the coordinates \( t, \theta \) on the lines \( \{ \theta = 0 \}, \{ t = 0 \} \). For definiteness, we will set \( t = t \) and \( \theta = \theta \) on those lines; the resulting solution to \( \{ B.4 \} \) then specifies the values of the functions \( c_{\theta2}(t, \theta), c_{t1}(t, \theta) \) everywhere.

Let us in fact expand \( \{ B.4 \} \) into:

\[
\begin{align*}
\left( c_{\theta2}(t, \theta) \cdot \hat{e}_2[c_{t1}(t, \theta)] \right) \cdot \hat{e}_1 - \left( c_{t1}(t, \theta) \hat{e}_1[c_{\theta2}(t, \theta)] + c_{\theta2}(t, \theta) \cdot \hat{e}_2[c_{t1}c_{t1}, K_{12}] \right) \cdot \hat{e}_2 \\
+ \left[ c_{\theta2}(t, \theta) \cdot \hat{e}_2[c_{t1}(t, \theta)] \right] \cdot \hat{e} \cdot \left[ \hat{A}_{1212} \right] = 0.
\tag{B.5}
\end{align*}
\]

Thus we derive a system of 1st order transport equations:

\[
\begin{align*}
\left( c_{\theta2}(t, \theta) \cdot \hat{e}_2[c_{t1}(t, \theta)] \right) & = c_{\theta2}(t, \theta) \cdot c_{t1}(t, \theta) \hat{A}_{1212}, \\
\left( c_{t1}(t, \theta) \hat{e}_1[c_{\theta2}(t, \theta)] + c_{\theta2}(t, \theta) \cdot \hat{e}_2[c_{t1}c_{t1}, K_{12}] \right) & = c_{\theta2}(t, \theta) \cdot c_{t1}(t, \theta) \cdot \hat{A}_{1212}.
\tag{B.6}
\end{align*}
\]

At this point we recall the expressions \( (3.23) \) for \( \hat{A}_{212}, \hat{A}_{121}, \hat{A}_{211} \) in terms of \( (0\text{th} \text{ and 1st derivatives of}) \) the variable \( K_{12} \); using also the expressions \( (3.25) \) for \( \hat{e}_1, \hat{e}_2 \) in terms of the background coordinates \( t, \theta \) we can view the above as a \( 2 \times 2 \) 1st order system in the two parameters \( c_{t1}, c_{\theta2} \).

Now, we will use the above system to solve for \( c_{t1}(t, \theta), c_{\theta2}(t, \theta) \). To do this, we need to impose conditions on these functions; we choose to do so on the two curves \( \{ \theta = 0 \} \) and \( \{ t = 0 \} \). As noted, we set \( t = t \) and \( \theta = \theta \) respectively on those two curves.

Implicitly using the formulas \( (4.81) \) this prescribes the values of \( c_{t1}(t, \theta), c_{\theta2}(t, \theta) \) on these two curves, respectively. Then using the expressions second line formulas in \( (4.81) \), we solve for \( c_{t1}(t, \theta) \) first along the integral curves of \( \hat{e}_2 \), and then for \( c_{\theta2} \) along the integral curves of \( \hat{e}_1 \).

From these formulas we can directly derive our claimed bounds on the parameters \( a_{Ai} \) on \( \Sigma_{r_+} \), utilizing our derived bounds on \( K_{12} \). Given the bounds on the components of \( K_{ij}(r, t, \theta) \) (in particular with the improved bound on \( K_{12}(r, t, \theta) \)), we then invoke the evolution equations and repeat the analysis for the coordinates \( T, \Theta \) to derive the claimed bounds for these parameters off of \( \Sigma_{r_+} \).

\[ \square \]

Invoking formulas \( (2.19) \), (for the coordinates \( \tilde{t}, \tilde{\theta} \) in the place of \( t, \theta \)) the above Lemma proves our claim for the metric components \( g_{\tilde{t}\tilde{t}}, g_{\tilde{\theta}\tilde{\theta}} \). For the metric components \( g_{\tilde{t}r}, g_{\tilde{\theta}r} \) we invoke the definitions \( (2.17), (2.18) \) (again, for the coordinates \( \tilde{t}, \tilde{\theta} \) in the place of \( t, \theta \)) as well as the bounds in Lemma \( 4.13 \) for \( e_1(r) \) and the fact that \( e_2(r) = 0 \) we derive our claim for those components also.

References

[1] Xinliang An and Ruixiang Zhang, *Polynomial blow-up upper bounds for the Einstein-scalar field system under spherical symmetry*, preprint. (Personal communication with the authors).

[2] L. Anderson and A. D. Rendall, *Quiescent cosmological singularities*, Commun. Math. Phys. 218 (2001), 479-511.

[3] A. V. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, *Oscillatory approach to the singular point in relativistic cosmology*, Adv. Phys. 19 (1970), 525-573.

[4] A. V. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, *A general solution of the Einstein equations with a timelike singularity*, Adv. Phys. 31 (1982), 639-667.

[5] B. K. Berger, P. T. Chrusciel, J. Isenberg and V. Moncrief, *Global foliations of vacuum spacetimes with \( T^2 \)-isometry*, Ann. Physics 260:1 (1997), 117-148.
[6] B. K. Berger and V. Moncrief, Exact $U(1)$ symmetric cosmologies with local Mixmaster dynamics, Phys. Rev. D62 023509 (2000).

[7] M. Cantor, The existence of non-trivial asymptotically flat initial data for vacuum spacetimes, Comm. Math. Phys. 57 (1977), no. 1, 83-96.

[8] Y. Choquet-Bruhat, Maximal submanifolds and submanifolds with constant mean extrinsic curvature of a Lorentzian manifold, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 3 (1976), no. 3, 361-376.

[9] Y. Choquet-Bruhat, J. Isenberg and V. Moncrief, Topologically general $U(1)$ symmetric vacuum space-times with AVTD behavior, Nuovo Cimento (2004), 119B, 625-638.

[10] Y. Choquet-Bruhat, General relativity and the Einstein equations, Oxford University Press 2009.

[11] D. Christodoulou, A mathematical theory of gravitational collapse. Comm. Math. Phys. 109 (1987), no. 4, 613-647.

[12] D. Christodoulou, The formation of black holes and singularities in spherically symmetric gravitational collapse, Comm. Pure Appl. Math. 44 (1991), no. 3, 339-373.

[13] P. T. Chruściel, J. Isenberg and V. Moncrief, Strong cosmic censorship in polarized Gowdy spacetimes, Class. Quantum Grav. 7 (1990), 1671-1680.

[14] M. Dafermos, Black holes without spacelike singularities, Commun. Math. Phys. 332 (2014), 729-757.

[15] M. Dafermos and J. Luk, The interior of dynamical vacuum black holes I: The $C^0$-stability of the Kerr Cauchy horizon, arXiv:1710.01722.

[16] M. Dafermos and J. Luk, The interior of dynamical vacuum black holes II: event horizon data and the stability of the red-shift region, in preparation.

[17] M. Dafermos and J. Luk, The interior of dynamical vacuum black holes III: The $C^0$-stability of the bifurcation sphere of the Kerr Cauchy horizon, in preparation.

[18] T. Damour, M. Henneaux, A. D. Rendall and M. Weaver, Kasner-like behaviour for subcritical Einstein-matter systems. Annales of Henri Poincaré 3 (2002), 1049-1111.

[19] G. Fournodavlos, On the backward stability of the Schwarzschild black hole singularity, Comm. Math. Phys. 345 (2016), no. 3, 923-971.

[20] D. Garfinkle, Numerical simulations of singular spacetimes, Class. Quantum Grav. 29 (2012), 7 pp.

[21] P. Hintz and A. Vasy, The global non-linear stability of the Kerrde Sitter family of black holes, Acta Math. 220 (2018), no. 1, 1-206.

[22] C. Huneau and J. Luk, Trilinear compensated compactness and Burnett’s conjecture in general relativity, arXiv:1907.10743.

[23] C. Huneau and J. Luk, High-frequency backreaction for the Einstein equations under polarized $U(1)$ symmetry, Duke Math. J. 167 (2018), no. 18, 3315-3402.

[24] A. Ionescu and S. Klainerman, On the global stability of the new-map equation in Kerr spaces with small angular momentum, Annals of PDE 1 (2015) 1–78.

[25] J. Isenberg and V. Moncrief, Asymptotic behavior in polarized and half-polarized $U(1)$-symmetric vacuum spacetimes, Class. Qu. Grav. 19 (2002), no. 21, 5361-5386.

[26] S. Klainerman and J. Szeftel, Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations, arXiv:1711.07597.

[27] S. Kichenassamy and A. D. Rendall, Analytic description of singularities in Gowdy spacetimes, Class. Quantum Grav. 15 (1998), no. 5, 1339-1355.

[28] P. G. LeFloch and J. Smulevici, Weakly regular $T^2$-symmetric spacetimes: the future geometry of future Cauchy developments, J. Eur. Math. Soc. 17:5 (2015), 1229-1292.

[29] P. G. LeFloch and J. Smulevici, Future asymptotics and geodesic completeness of polarized $T^2$-symmetric spacetimes, Anal. PDE 9 (2016), no. 2, 363-395.

[30] J. Luk, Weak null singularities in general relativity, J. Amer. Math. Soc. 31 (2018), no. 1, 1-63.

[31] A. D. Rendall, Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity, Class. Qtm. Grav. 17, 3305-3316 (2000).

[32] H. Ringström, The Bianchi IX attractor, Ann. Henri Poincaré 2 (2001), no. 3, 405-500.
[33] H. Ringström, *Strong cosmic censorship in $T^3$-Gowdy spacetimes*, Ann. of Math. (2) **170** (2009), no. 3, 1181-1240.

[34] H. Ringström, *Instability of spatially homogeneous solutions in the class of $T^2$-symmetric solutions to Einstein’s vacuum equations*, Comm. Math. Phys. 334:3 (2015), 1299-1375.

[35] I. Rodnianski and J. Speck, *A regime of linear stability for the Einstein-scalar Field system with applications to nonlinear Big Bang formation*, Annals of Mathematics **187** (2018), 65-156.

[36] I. Rodnianski and J. Speck, *Stable Big Bang Formation in Near-FLRW Solutions to the Einstein-Scalar Field and Einstein-Stiff Fluid Systems*, Selecta Math. (N.S.) **24** (2018), no. 5, 4293-4459.

[37] I. Rodnianski and J. Speck, *On the nature of Hawking’s incompleteness for the Einstein-vacuum equations: The regime of moderately spatially anisotropic initial data*, arXiv:1804.06825.

[38] J. Sbierski, *The $C^0$-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry*, J. Differential Geom. **108** (2018), no. 2, 319-378.

[39] J. Speck, *The maximal development of near-FLRW data for the Einstein-scalar field system with spatial topology $S^3$*, Comm. Math. Phys. **364** (2018), no. 3, 879-979.

[40] R. M. Wald, *General Relativity*, The University of Chicago Press, 1984.

[41] M. Weaver, J. Isenberg and B.K. Berger, *Mixmaster behavior in inhomogeneous cosmological spacetimes*, Phys. Rev. Lett. **80** (1998), 2984-2987.

[42] G. Weinstein, *On rotating black holes in equilibrium in general relativity*, Comm. Pure Appl. Math. **43** (1990), no. 7, 903-948.

Spyros Alexakis  
Address: Department of Mathematics, University of Toronto, Room 6290, 40 St. George Street, Toronto, Ontario M5S 2E4, Canada  
Email: alexakis@math.utoronto.ca

Grigorios Fournodavlos  
Address: Laboratoire Jacques-Louis Lions, Sorbonne Université, 4 place Jussieu, 75005 Paris, France  
Email: grigorios.fournodavlos@sorbonne-universite.fr