SINGULAR R-MATRICES AND DRINFELD’S COMULTIPLICATION

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Abstract. We compute the $R$-matrix which intertwines two dimensional evaluation representations with Drinfeld comultiplication for $U_q(\hat{sl}_2) \ [Dr]$ This $R$-matrix contains terms proportional to the $\delta$-function. We construct the algebra $A(R) \ [RSTS]$ generated by the elements of the matrices $L^\pm(z)$ with relations determined by $R$. In the category of highest weight representations there is a Hopf algebra isomorphism between $A(R)$ and an extension $\mathcal{U}_q(\hat{sl}_2)$ of Drinfeld’s algebra.

1. INTRODUCTION

This note contains two results concerning the algebra $U_q(\hat{sl}_2)$ with Drinfeld comultiplication $\ [Dr]$. This algebra is presented in terms of current generators, and as an algebra, it is isomorphic $\ [B]$ to the Drinfeld-Jimbo $U_q(\hat{sl}_2) \ [Dr, J]$. However it has a different comultiplication and therefore a different Hopf algebra structure.

The first result presented here is a direct calculation of the $R$-matrix which acts as an intertwiner of two dimensional evaluation representations. The second is the construction of the $L$-matrix algebra $A(R)$, presented in terms of relations between generators $L^\pm(z)$, following the construction in $\ [RSTS]$, and the isomorphism between $A(R)$ and Drinfeld’s algebra.

The quantum $R$-matrix which intertwines evaluation representations with Drinfeld comultiplication is completely determined, up to a scalar factor, from the evaluation representation of Drinfeld’s generators and the action of the comultiplication on these generators, from the equation

$$\tilde{R}(\frac{z}{w})(\pi_x \otimes \pi_w)\Delta(x) = (\pi_x \otimes \pi_w)\Delta'(x)\tilde{R}(\frac{z}{w}), \quad x \in U_q(\hat{sl}_2)$$

(see section 2.3 for notation). These relations are considered here as relations between formal power series. It is shown that this implies that the $R$-matrix contains terms proportional to the $\delta$-function.

This matrix is then used to construct the algebra $A(R)$, generated by elements of the triangular matrices $L^\pm(z)$, with relations determined by the $R$-matrix, of the form

$$R(z/w)L^\pm(z)L^\pm(w) = L^\pm(w)L^\pm(z)R(z/w),$$
$$R(z/wq^{-c})L^\pm(z)L^\pm(w) = L^\pm(w)L^\pm(z)R(z/wq^c)$$

(see section 3 for the precise definition), in a construction following the method of $\ [RTF, RSTS]$. This algebra has a natural Hopf algebra structure, and the intertwining relation of the $R$-matrix follows easily from the Yang-Baxter relation for $R$. We construct the Hopf algebra isomorphism between $A(R)$ and an extended version of Drinfeld’s algebra, $\mathcal{U}_q(\hat{sl}_2)$.
In the final section the Poisson limit $q \to 1$ of the quantum algebra $A(R)$ is computed. The relations in this Poisson algebra are determined by the classical $r$-matrix.

It is straightforward to generalize the construction presented here to the case of $U_q(\hat{sl}_n)$ [3]. However most of the new features of the construction are already seen in the case $n = 2$.

2. **Drinfeld’s “new realization” of $U_q(\hat{sl}_2)$**

2.1. Generators and relations. Consider the Drinfeld realization of the algebra $U_q(\hat{sl}_2)$ [Dr2] generated by

$$q^{\pm c}, \phi_{\pm n}^{\pm}, \xi_{m}^{\pm}, \quad n \in \mathbb{Z}_{\geq 0}, \quad m \in \mathbb{Z},$$

with $q^{\pm c}$ central and relations expressed in terms of the generating series

$$\xi^{\pm}(z) = \sum_{n \in \mathbb{Z}} \xi_{n}^{\pm} z^{n}, \quad \phi^{\pm}(z) = \sum_{n \geq 0} \phi_{\pm n}^{\pm} z^{\pm n}$$

as follows. Let $B = \mathbb{C}[[h]]$ be ring formal power series in $h$, where $q = e^{h/2}$. Let

$$g(z) = \frac{q^{2}z - 1}{z - q^{2}} \in B[[z]]$$

be an element of $B$ with coefficients formal power series in $z$. (Throughout this paper the notation $f(z)$ will be used to denote functions in $B[[z]]$, $f(z^{-1}) \in B[[z^{-1}]]$, etc.)

The relations between the generating series (2) are the following formal power series relations:

$$\phi^{+}(0)\phi^{-}(0) = \phi^{-}(0)\phi^{+}(0) = 1,$$

$$[\phi^{\pm}(z), \phi^{\pm}(w)] = 0,$$

$$\phi^{+}(z)\phi^{-}(w) = \frac{g\left(\frac{z}{w}q^{-c}\right)}{g\left(\frac{w}{z}q^{-c}\right)} \phi^{-}(w)\phi^{+}(z)$$

$$\phi^{\pm}(z)\xi^{\pm}(w) = g\left(\frac{(\pm)q^{c}}{w}\right)^{\pm 1} \xi^{\pm}(w)\phi^{\pm}(z)$$

$$\phi^{\pm}(w)\xi^{\pm}(z) = g\left(\frac{(\pm)q^{c}}{w}\right)^{\pm 1}^{-1} \xi^{\pm}(w)\phi^{\pm}(z)$$

$$\left(1 - q^{\pm}w\right)\xi^{\pm}(z)\xi^{\pm}(w) = (q^{\pm}z - w)\xi^{\pm}(w)\xi^{\pm}(z)$$

$$\left[\xi^{+}(z), \xi^{-}(w)\right] = (q - q^{-1}) \left(\delta\left(\frac{z}{w}q^{-c}\right)\phi^{-}(w) - \delta\left(\frac{w}{z}q^{-c}\right)\phi^{+}(z)\right).$$

Here, the function $\delta(z)$ is a formal series,

$$\delta(z) = \sum_{n \in \mathbb{Z}} z^{n}.$$

It can also be regarded as a distribution which acts on the space of functions $f(z)$ regular at $z = 1$.

2.2. **Hopf algebra structure.** As an algebra, Drinfeld’s algebra [Dr] is isomorphic to the Drinfeld-Jimbo $U_q(\hat{sl}_2)$ [Dr, J, B]. However it can be endowed with the Hopf
algebra structure of $\mathfrak{dr}_2$ which is different from that found in $[Dr, J]$. In this paper, the notation $U_q(\widehat{sl}_2)$ refers to Drinfeld’s algebra as a Hopf algebra. **Co-product:**

\[
\Delta(q^c) = q^c \otimes q^c,
\]
\[
\Delta(\xi^+(z)) = \xi^+(z) \otimes 1 + \phi^+(z) \otimes \xi^+(zq^{c_1}),
\]
\[
\Delta(\xi^-(z)) = 1 \otimes \xi^-(z) + \xi^-(zq^{c_2}) \otimes \phi^-(z),
\]
\[
\Delta(\phi^+(z)) = \phi^+(z) \otimes \phi^+(zq^{c_2}),
\]
\[
\Delta(\phi^-(z)) = \phi^-(zq^{c_2}) \otimes \phi^-(z).
\]

Here, $c_i$, $i = 1, 2$ is the value of the central charge acting on the $i$-th factor in the tensor product.

**Co-unit:**

\[
\varepsilon(q^c) = 1, \quad \varepsilon(\phi^\pm(z)) = 1, \quad \varepsilon(\xi^\pm(z)) = 0.
\]

**Antipode:**

\[
S(q^c) = q^{-c},
\]
\[
S(\phi^+(z)) = \phi^+(zq^{-c})^{-1},
\]
\[
S(\phi^-(z)) = -\xi^-(zq^{-c})^{-1} \xi^+(zq^{-c}),
\]
\[
S(\xi^+(z)) = -\phi^+(zq^{-c}) \phi^-(zq^{-c})^{-1}.
\]

These satisfy the condition $m \circ (S \otimes \text{id}) \circ \Delta = \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta$, where $m(x \otimes y) = xy$.

**Remark 2.1** Normal ordering. Let $O$ be the category of highest weight representations, where the positive modes $\phi^+_n$, $\xi^+_n$, $\phi^-_n$, $n > 0$ act nilpotently to the right. The product of two generating functions in $U_q(\widehat{sl}_2)$ is well defined when it is normally ordered, i.e. all positive modes on the right and the negative on the left. Then in the category $O$ normally ordered products of generating functions act as Laurent series.

The relations above describe a Hopf algebra structure in the category $O$. The antipode should have the property $(xv, w) = \langle v, S(x)w \rangle$, $x \in U_q(\widehat{sl}_2)$, $v \in M^*, w \in M$, where $M \in O$, $M^* \in O^*$ the right dual to $M$, and $O^*$ is the category of lowest weight modules. Therefore acting by $S$ on a normally ordered product of generating functions gives an anti-normally ordered product which acts in $O^*$ as Laurent series.

### 2.3. The $R$-matrix.

Let $V = \text{End}(\mathbb{C}^2)$ be the two dimensional representation of $\mathfrak{sl}_2$ with basis $v_1, v_2$. Consider the two-dimensional evaluation representation of $U_q(\widehat{sl}_2)$, $V_z = V \otimes_\mathbb{C} \mathbb{C}[z, z^{-1}]$. The map

\[
\pi_z : U_q(\widehat{sl}_2)|_{c=0} \to V[z, z^{-1}]
\]

is defined by $[Dl]$

\[
\xi^\pm(w) \mapsto (q - q^{-1}) \delta(\frac{w}{z}) \sigma^\pm \in V[z, z^{-1}][[w, w^{-1}]]
\]
\[
\phi^+(w) \mapsto \begin{pmatrix}
(d(q^2 \frac{w}{z})) & 0 \\
0 & (d(q^2 \frac{w}{z}))^{-1}
\end{pmatrix} \in V[z^{-1}][[w]],
\]
\[
\phi^-(w) \mapsto \begin{pmatrix}
(d(\frac{z}{w}))^{-1} & 0 \\
0 & (d(q^2 \frac{z}{w}))
\end{pmatrix} \in V[z][[w^{-1}]],
\]

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where
\[ d(z) = \frac{1 - z}{q - q^{-1}z} \]  
(9)

and \( \sigma^\pm \) are the Pauli matrices. Here, the notation \( \mathbb{C}[z][w] \) indicates formal power series in \( w \) with coefficients polynomials in \( z \), etc.

**Proposition 2.1.** There is a unique, up to a scalar multiple, \( 4 \times 4 \) matrix \( \tilde{R}(z) \) with diagonal entries in \( \mathcal{B}[z] \) and off diagonal elements in \( \mathcal{B}[z, z^{-1}] \) which satisfies the intertwining relation
\[ \tilde{R}(z/z')(\pi_z \otimes \pi_{z'})\Delta(x) = (\pi_z \otimes \pi_{z'})\Delta'(x)\tilde{R}(z/z'), \quad x \in U_q(\hat{sl}_2). \]  
(10)

This unique solution is \( \tilde{R}(z) = R_{12}^{-1}(z) \), where
\[ R_{12}(z) = f(z) \begin{pmatrix} 1 & d(z) & 0 & \gamma(z) \delta(z) \\ \gamma(z) \delta(z) & d(q^2z) & 0 & 1 \end{pmatrix}, \]  
(11)

where \( \gamma(z) = (q - q^{-1})(1 - z) \).

**Remark 2.2** Here, \( (1 - z)^n \delta(z), \quad n \in \mathbb{Z}_{\geq 0} \) can be thought of as a distribution acting on the space of functions with poles of order less than or equal to \( n \).

The proof of proposition (2.1) is provided in the appendix.

The matrix \( R \) will be used in the next section to define the algebra \( A(R) \). The scalar factor \( f(z) \) is uniquely determined from the requirement that the quantum determinant is central (see next section), which gives
\[ f(z) = q^{1/2}(1 - z)\tilde{g}(z)^2 \in \mathcal{B}[z], \]  
(12)

where
\[ \tilde{g}(z) = \frac{(q^4z; q^4)_\infty}{(q^2z; q^2)_\infty}, \quad (z, q)_n = \prod_{j=0}^{n} (1 - zq^j). \]

The function \( f(z) \) in (12) is the unique solution to the difference equation
\[ f(z)f(q^2z) = d(q^2z)^{-1}, \quad f(z) \in \mathcal{B}[z]. \]  
(13)

With this choice of \( f(z) \), the matrix \( R \) is equal to the evaluation of the universal \( R \)-matrix \( [R] \).

### 2.4. Extension of Drinfeld’s algebra.
For what follows it will be useful to describe a slightly extended version of Drinfeld’s algebra, \( U_q(\hat{sl}_2) \) \[ FR \]. Consider the generators \( \alpha^{\pm}_{\pm n} \),
\[ \alpha^{\pm}(z) = \sum_{n \geq 0} \alpha^{\pm}_{\pm n} z^{\pm n}, \]

such that
\[ \phi^{\pm}(z) = \alpha^{\pm}(z)^{-1} \alpha^{\pm}(q^2z)^{-1}. \]  
(14)
The algebra $\mathcal{U}_q(\hat{sl}_2)$ is generated by $\xi^\pm(z), \alpha^\pm(z), q^c$ with relations as in (4) and

\[
[\alpha^\pm(z), \alpha^\pm(w)] = 0, \quad \alpha^\pm(z)\xi^\pm(w) = d(zw^\pm)^{-1}\xi^\pm(w)\alpha^\pm(z), \\
\alpha^\pm(z)\xi^\pm(w) = d(q^2w^\pm)^{-1}\xi^\pm(w)\alpha^\pm(z), \quad \alpha^\pm(z)\xi^\pm(w) = d(\xi^\pm z^{-1})\xi^\pm(w)\alpha^\pm(z), \\
\alpha^\pm(z)\alpha^\pm(w) = \frac{f(zq^c/w)}{f(zq^{-c}/w)}\alpha^\pm(w)\alpha^\pm(z).
\]

The Hopf algebra structure extends to $\mathcal{U}_q(\hat{sl}_2)$:

\[
\Delta(\alpha^\pm(z)) = \alpha^\pm(z) \otimes \alpha^\pm(zq^c), \\
\Delta(\alpha^\mp(z)) = \alpha^\mp(zq^c) \otimes \alpha^\mp(z), \\
S(\alpha^\pm(z)) = \alpha^\pm(zq^{-c})^{-1}, \\
\varepsilon(\alpha^\pm(z)) = 1.
\]

The two dimensional evaluation representation of $\alpha^\pm(z)$ is obtained by solving the difference equation (13). The result is

\[
\pi_z\alpha^+(w) = f(\frac{w}{z}) \begin{pmatrix} 1 & 0 \\ 0 & d(\frac{w}{z}) \end{pmatrix}, \\
\pi_z\alpha^-(w) = \frac{1}{f(\frac{z}{w})} \begin{pmatrix} 1 & 0 \\ 0 & d(q^2\frac{z}{w})^{-1} \end{pmatrix}.
\]

3. The Quantum Current Algebra $A(R)$

We define the Hopf algebra $A(R)$ using the same methods introduced in [RSTS]. Consider the algebra generated by the coefficients of $L_{ij}^\pm(z)$, with $i, j = 1, 2$, $L_{ij}^\pm(z) \in A(R)[[z^\pm1]]$, and the off diagonal elements in $A(R)[[z, z^{-1}]]$. The matrix $L^+_1(L^-)$ is lower (upper) triangular.

The determining relations are

\[
R_{12}(\frac{z}{w})L^+_1(z)L^+_2(w) = L^+_2(w)L^+_1(z)R_{12}(\frac{z}{w}), \\
R_{12}(\frac{z}{w}q^{-c})L^-_1(z)L^-_2(w) = L^-_2(w)L^-_1(z)R_{12}(\frac{z}{w}q^c).
\]

Here the matrix $R(z)$ is the same as in (11) with $f(z)$ as in (12). We choose $f(z)$ by requiring that the quantum determinants

\[
\mathcal{D}^\pm(z) = L^\pm_{11}(z)L^\pm_{22}(zq^c) - L^\pm_{12}(z)L^\pm_{21}(zq^c)
\]

are central. The choice of the shift $q^{-2}$ in the definition of $\mathcal{D}^\pm(z)$ is motivated by the identification of the evaluation of $L^+_1(z)$ with $R(z)$ (see below).

Finally, the algebra $A(R)$ is defined as the quotient of the algebra generated by $L_{ij}^\pm(z)$ with relations (18) by the relation $\mathcal{D}^\pm(z) = 1$.

Remark 3.1 This construction is motivated by the following considerations. Suppose there exists a universal $R$-matrix for $\mathcal{U}_q(\hat{sl}_2)$ such that $L^+_1(z) = (id \otimes \pi_z)R$, $L^-_1(z) = (id \otimes \pi_z)R^{-1}_2$ and $R(z/w) = (\pi_z \otimes \pi_w)R \otimes R$. Then the relations (18) follow from the Yang-Baxter relation for $R$. 
The algebra $A(R)$ is a Hopf algebra with comultiplication
\[ \Delta L^+(z) = L^+(z) \hat{\otimes} L^+(zq^2z), \quad \Delta L^-(z) = L^-(zq^2z) \hat{\otimes} L^-(z) \] (19)
which is easily seen to be an algebra homomorphism of (18). (Here the notation \( \hat{\otimes} \) indicates matrix multiplication in $V$ and tensor product in $A(R)$.) The co-unit acts as $\varepsilon(L^\pm) = 1$, and the action of the antipode is
\[ S(L^\pm(z)) = L^\pm(zq^{-c})^{-1}. \]

**Remark 3.2** Since $D^\pm(z) = 1$, $L^\pm_{ii}(z) \in A(R)$. The matrices $S(L^\pm(z)) = L^\pm(z)^{-1}$ are well defined in the category $O^*$, as they should be according to remark (2.1.1).

Define the map
\[ \mu : A(R) \to \overline{U}_q(\hat{sl}_2) \]
as
\[
L^+(z) \mapsto \begin{pmatrix}
\alpha^+(z) & 0 \\
\xi^+(z)\alpha^+(z) & \alpha^+(q^2z)^{-1}
\end{pmatrix}, \\
L^-(z) \mapsto \begin{pmatrix}
\alpha^-(z) & -\alpha^-(z)\xi^-(z) \\
0 & \alpha^-(q^2z)^{-1}
\end{pmatrix}.
\]

**Proposition 3.1.** The map $\mu$ is a Hopf algebra isomorphism.

The proof is by direct calculation.

By using the evaluation representation for the Drinfeld generators, it is easily shown that
\[ \pi_w L^+(z) = R(z/w), \quad \pi_w L^-(z) = R_{21}^{-1}(w/z). \]
(Here, we use the notation $(\pi_w L^+(z)_{ij})_{kl} = R_{ij,kl}(z/w)$ where the action of $R$ on the basis vectors of $V$ is $R(z)v_i \otimes v_j = \sum_{k,l=1}^2 R_{ij,kl}(z)v_k \otimes v_l$.

The relations (13) in the evaluation representation are therefore equivalent to the Yang-Baxter equation for $R$:
\[ R_{12}(z)R_{13}(wz)R_{23}(w) = R_{23}(w)R_{13}(wz)R_{12}(z). \] (20)

The intertwining relation (14) with $x = L^\pm(w)_{ij}$ is again a simple consequence of the Yang-Baxter relations (20), with the co-product as in (19).

Note that this Hopf algebra isomorphism provides a trivial proof that $\Delta$ is an algebra homomorphism of Drinfeld’s algebra (c.f. [DI]).

### 4. The Poisson Algebra Limit

The Poisson limit $h \to 0$ of Drinfeld’s algebra is obtained by keeping $p = q^c$ and the generators $\xi^\pm$ and $\phi^\pm$ or $\alpha^\pm$ fixed. In this limit the algebra $\overline{U}_q(\hat{sl}_2)$ becomes a Poisson algebra with
\[ \{a, b\} = \lim_{h \to 0} \frac{a \cdot b - b \cdot a}{h}, \quad a, b \in \overline{U}_q(\hat{sl}_2) \]
Explicitly,
\[ \phi^\pm(z) = \alpha^\pm(z)^{-2} \]
and
\[
\{\alpha^+(z), \alpha^+(w)\} = 0,
\{\xi^+(z), \xi^-(w)\} = \left(\delta(z^{-1})\phi^- (w) - \delta(p)\phi^+(z)\right),
\{\xi^+(z), \xi^+(w)\} = \pm \frac{1}{2} \left(\lambda(z) - \lambda(z^{-1})\right) \xi^+(z)\xi^+(w),
\{\alpha^+(z), \alpha^-(w)\} = \frac{1}{4} \lambda(z) \alpha^+(z)\alpha^-(w),
\{\alpha^\pm(z), \alpha^\pm(w)\} = -\frac{1}{2} \lambda(z) \alpha^\pm(z)\xi^\pm(w),
\{\alpha^\pm(z), \xi^\mp(w)\} = \frac{1}{2} \lambda(z) \alpha^\pm(z)\xi^\mp(w).
\]

Again, these are formal power series relations with
\[
\lambda(z) = \frac{1}{4} \frac{1 + z}{1 - z}.
\]

The algebra $A(R)$ in this limit is a Poisson algebra generated by the elements $L^\pm_{ij}(z)$ as above, and the quantum determinant is the usual determinant. The relations (21) become, in this limit,
\[
\begin{align*}
\{L^+_1(z), L^+_2(w)\} &= [L^+_1(z)L^+_2(w), r(z)], \\
\{L^-_1(z), L^-_2(w)\} &= L^-_1(z)L^-_2(w)r(z)p^{-1} - r(z)p\,L^-_1(z)L^-_2(w).
\end{align*}
\]

The classical $r$-matrix is obtained from $R(z)$ by taking the limit $h \to 0$ of the elements of $R$. We find
\[
R(z) = 1 + h \, r(z) + O(h^2).
\]

Here we consider the off diagonal element of $R$ to be in $B[[z, z^{-1}]]$. This means that the coefficient in front of the $\delta$ function in $r$ is the coefficient of $h$ in the expansion of $(q - q^{-1})f(z)/d(z)$, which is 1. Thus, the classical $r$-matrix is
\[
r(z) = \begin{pmatrix}
\lambda(z) & 0 & 0 & 0 \\
0 & -\lambda(z) & 0 & 0 \\
0 & \delta(z) & -\lambda(z) & 0 \\
0 & 0 & 0 & \lambda(z)
\end{pmatrix}.
\]

The isomorphism $\mu$ gives, in this limit,
\[
L^+ = \begin{pmatrix}
\alpha^+(z) & 0 \\
\alpha^+(z)x^+(z) & \alpha^+(z)^{-1}
\end{pmatrix}, \quad L^- = \begin{pmatrix}
\alpha^-(z) & \alpha^-(z)x^-(z) \\
0 & \alpha^-(z)^{-1}
\end{pmatrix}.
\]

Since all the operators commute there is no problem with normal ordering in this limit.

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Appendix A. The Intertwining Relation

The following is a proof of Proposition (2.1).

Let

\[ \bar{R}(z) = \begin{pmatrix} a_1(z) & 0 & 0 & 0 \\ 0 & b_1(z) & \gamma_1 \delta(z) & 0 \\ 0 & \gamma_2 \delta(z) & b_2(z) & 0 \\ 0 & 0 & 0 & a_2(z) \end{pmatrix}, \]

with the diagonal elements in \( \mathbb{C}[[z]] \) and the off-diagonal elements in \( \mathbb{C}[[z, z^{-1}]] \). Then, up to a scalar multiple, the unique solution to the intertwining relation (10) is \( R_{12}^{-1}(z) \).

The proof consists of using the evaluation representation (8) to explicitly compute

\[ \bar{R}(z/z')(\pi_z \otimes \pi_{z'} \Delta(\xi^\pm(w))) = (\pi_z \otimes \pi_{z'} \Delta'(\xi^\pm(w))) \bar{R}(z/z'). \]

Writing out these relations, it is immediately apparent that the off-diagonal terms of the matrix \( \bar{R} \) must be proportional to the \( \delta \) function if they are nonzero. Cancelling an overall factor of \( \delta(w/z) \) and setting \( z' = 1 \), this results in eight relations between formal power series in \( z \) and \( z^{-1} \):

\[ \begin{align*}
    a_1(z)d(q^2z^{-1}) &= b_1(z) + d(q^2z)\gamma_2(z)\delta(z), \\
    a_1(z) &= \gamma_1(z)\delta(z) + d(q^2z)b_2(z), \\
    b_1(z) + \gamma_1(z)\delta(z)/d(z^{-1}) &= a_2(z)/d(z), \\
    \gamma_2(z)\delta(z) + b_2(z)/d(z^{-1}) &= a_2(z), \\
    b_1(z) + \gamma_1(z)\delta(z)/d(z^{-1}) &= a_1(z)/d(z), \\
    \gamma_2(z)\delta(z) + b_2(z)/d(z^{-1}) &= a_1(z), \\
    a_2(z)d(q^2z^{-1}) &= b_1(z) + d(q^2z)\gamma_2(z)\delta(z), \\
    a_2(z) &= \gamma_1(z)\delta(z) + b_2(z)d(q^2z). 
\end{align*} \]

The first four relations come from considering \( \Delta(\xi^+) \) in (11) and the last four from \( \Delta(\xi^-) \).

From (27), \( \gamma_1(z) = 0 \) since all other terms are in \( \mathbb{C}[[z]] \) and there is nothing to cancel terms in \( \mathbb{C}[[z^{-1}]] \) coming from \( \delta(z) \). Therefore, from (28), \( b_1(z) = \frac{a_2(z)}{d(z)} \), and from (30), \( b_1(z) = \frac{a_1(z)}{d(z)} \), so \( a_2 = a_1 \). From (33), \( a_1(z) = b_2(z)d(q^2z) \).

As formal power series,

\[ \frac{1}{1 - z^{-1}} = \sum_{n \leq 0} z^n = \delta(z) - \frac{z}{1 - z}, \]

and therefore

\[ d^{-1}(z^{-1}) = d(q^2z) + (q - q^{-1})\delta(z). \]

From (29),

\[ \gamma_2(z)\delta(z) = (q^{-1} - q)b_2(z)\delta(z). \]

These relations determine \( R \) only up to a scalar multiple. Let

\[ a_1(z) = \frac{a(z)}{1 - z}. \]

\[ 1 \]In the sense the coefficient in front of \( \delta(w/z) \), since it is independent of \( w \), must be zero.
Then
\[ b_1(z) = \frac{a(z)}{(1-z)d(z)}, \quad b_2(z) = \frac{a(z)}{(1-z)d(q^2z)}, \quad \gamma_2(z) = \frac{(q^{-1} - q)a(z)}{(1-z)d(q^2z)}. \]

Thus \( \tilde{R} \) is identified with the matrix \( R^{-1}(z) \)
\[
R^{-1}_{12}(z) = q^{-1/2}g(z)^{-2} \begin{pmatrix}
\frac{1}{1-z} & \frac{q^{-1} - q}{(1-z)d(z)} & 0 \\
\gamma'(z)d(z) & \frac{1}{q - q^{-1}} & 0 \\
0 & 1 & 1 - z
\end{pmatrix},
\]
with
\[
\gamma'(z) = \frac{q^{-1} - q}{(1-z)d(q^2z)} = \frac{q^{-1} - q}{q^{-1} - qz},
\]
with \( a(z) = q^{-1/2}g(z)^{-2} \).

References

[B] J. Beck, Braid group action and quantum affine algebras. Comm. Math. Phys. 165 (1994), 555–568.

[DI] J. Ding and K. Iohara, Drinfeld comultiplication and vertex operators. Preprint q-alg/9608003, 1996.

[Dr] V. Drinfeld, Hopf algebra and the quantum Yang-Baxter equation. Dokl. Akad. Nauk. SSSR 283, 1985, 1060–1064.

[Dr1] V. Drinfeld, Quantum groups, Proc. ICM 1986: Berkeley CA, 798–820.

[Dr2] V. Drinfeld, New realization of Yangian and quantum affine algebra. Sov. Math. Dokl. 36, 1988, 212–216. 1 (1990), 193–225.

[FR] E. Frenkel and N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and \( W \)-algebras. Comm. Math. Phys. 178, 1996, 237–264.

[IK] A. Izergin and V. Korepin, Sov. Phys. Doklady 26, 1981, 653.

[J] M. Jimbo, A \( q \)-difference analogue of \( U(g) \) and the Yang-Baxter equation. Letts. Math. Phys. 10, 1985, 63–69.

[K] R. Kedem, in preparation.

[R] N. Reshetikhin, unpublished.

[RTF] N. Reshetikhin, L. Takhtajan and L. Faddeev, Quantization of Lie groups and Lie algebras. Leningrad Math. J.

[RSTS] N. Reshetikhin and M. Semenov-Tian-Shansky, Central extensions of quantum current groups. Letts. Math. Phys. 19 (1990), 133–142.

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