Conformal reference frames for Lorentzian manifolds

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Abstract

The definition of a conformal reference frame is given, that is, of a special projection of the six-dimensional skies bundle of a Lorentzian manifold (or five-dimensional twistor space) to a three-dimensional manifold. An example is constructed—conformal compactification—for the Minkowski space. The celestial transform of Lorentzian vectors is defined, a kind of spinor correspondence, based on the complex structure on the skies. An 1-form generating the contact structure in the twistor space (when the latter is smooth) is expressed explicitly as a line bundle-valued form. A theorem is proved on the projection of the said 1-form to the fiber-wise normal bundle of a reference frame. It entails the flow of time equation that expresses the space-time derivative of sky images through the celestial transform of 4-vectors.

The Appendix discusses formulations of the causal relation based on the “natural” conformal reference frame of the FLRW cosmology.

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0.1 Introduction

Lorentzian manifolds are the standard framework for the space-time in physics, and there are some mathematical techniques to explore their geometry. The
twistor approach to Lorentzian manifolds is focused on the five-dimensional space \( \mathfrak{N} \) of null geodesics. In the case of Minkowski space, that space can be embedded to complex projective space, said the \textit{twistor space}, but in general (curved) case any canonical complex structure on twistors isn’t possible. However, as \( \mathfrak{N} \) could be defined, for each point \( x \in X \) its sky \( \mathcal{S}_x \) is embedded (or immersed) into \( \mathfrak{N} \), and \( \mathfrak{N} \) is endowed with the natural contact structure \( \mathcal{M} \). A sky \( \mathcal{S}_x \) has natural conformal structure and is isomorphic to \( \mathbb{C}P^1 \). This allows us to formulate the well-known twistor correspondence in a geometrized way agreeable to the curved case; it’s free from aforementioned drawback – inability to use any complex structure globally.

A conformal reference frame, defined in this paper, gives a description of (four-dimensional) Lorentzian manifold \( X \) in terms of immersion of its skies into a three-dimensional manifold. It is known that in the case of a globally hyperbolic \( X \) its \( \mathfrak{N} \) identifies with the spherical cotangent bundle over a Cauchy surface \( M \subset X \). Generalization to certain weaker conditions on \( X \) are possible as well. Then, instead of spheres—submanifolds of \( \mathfrak{N} \)—we can consider the image of \( \mathcal{S}_x \) in \( M \); an immersion in non-singular case. Conformal reference frames generalize Cauchy surfaces in some way. Ignoring the technical difficulties of differential geometry on \( \mathfrak{N} \), conformal reference frame should be understood as a smooth mapping of \( \mathfrak{N} \), or part thereof, to an arbitrary 3-manifold \( M \), the mapping compatible with the contact structure and having certain non-degeneracy condition (i.e., for every \( x \in X \) the mapping \( T\mathcal{S}_x \rightarrow TM \) has rank 2 on a non-empty open subset of \( \mathcal{S}_x \)). In this paper we presents the \textit{flow of time equation} expressing the dependence of the sky image \( \mathcal{S}_x \) on the point \( x \), and how it is related to the contact structure and causality, not from the topological point of view, but from the differential one. It is shown that holomorphic sections of the line bundle \( \mathcal{O}(1,0) \) over the sky should be considered spinors, and the “bundle of sizes” \( \mathcal{L}^R \) (with the weight-\((\frac{1}{2},\frac{1}{2})\) representation of \( \text{SL}(2,\mathbb{C}) \)) is the natural range of the contact form on the twistor space \( \mathfrak{N} \).

1 The bundle of skies

1.1 Preliminaries

This paragraph sets out the facts known about Lorentzian manifolds. Usually, they are understood as pseudo-Euclidean manifolds of signature \((1,3)\)
This paper requires an additional structure, namely:

**Definition.** A Lorentzian manifold is a pseudo-Euclidean four-dimensional manifold with the metric $g$ with the signature $(+-----)$ and the time orientation at each point $x \in X$ (i.e. one of the two connected components of the cone $\{ v \in T_xX \mid g(v) > 0 \}$ is chosen as “chronological future”), with a continuous fashion.

Such manifolds are referred to as space-times in [1]. All manifolds are assumed to be smooth ($C^\infty$).

**Definition.** For each $x \in X$ the boundary of its “chronological future” cone in $T_xX$ is called the future light cone and is denoted by $C^+_x$.

**Definition.** The sky $\mathcal{S}_x$ is the base of the cone $C^+_x$ in $T_xX$, and elements of the former will be denoted $Pv$, where $v \in C^+_x \setminus \{0\}$. The disjoint union of all skies (of all points of $X$) forms a smooth locally trivial bundle over $X$, denoted by $\mathcal{S}X$, with the projection map $x : \mathcal{S}X \rightarrow X$.

**Remark.** An element of $\mathcal{S}_x$ is virtually a null direction at $x$.

**Definition.** Let $T_v\mathcal{S}X$ denote the disjoint union of all tangent bundles $T\mathcal{S}_x$ for all $x \in X$. In other words: the vertical subbundle $\ker dx$ in $T(\mathcal{S}X)$.

**Definition.** The geodesic flow $FX$ is a distribution of 1-subspaces in the tangent bundle $T(\mathcal{S}X)$ of the total space of the bundle $\mathcal{S}X$, defined by the equations $dx \parallel v$ (the differential of $x$ is collinear to $v$), $\nabla v = 0$ (constant along the Levi-Civita connection), where $v \in C^+_x \setminus \{0\}$ is a vector representing a given point of the sky.

**Remark.** Integrating the flow $FX$ gives the “light” foliation of the total space $\mathcal{S}X$. Its leafs represent null (light-like) geodesic curves on $X$, maximally extended in both time directions and raised to $\mathcal{S}X$ naturally, by the $P$ mapping of tangent vectors to respective skies.

**Definition.** The equivalence relation $[(x_1, Pv_1)] = [(x_2, Pv_2)]$ on the total space of $\mathcal{S}X$ applies to such pairs $v_1 \in C^+_x \setminus \{0\}$ and $v_2 \in C^+_x \setminus \{0\}$ of null vectors (representing null directions) that lie on the same null geodesic, i.e. a leaf of the foliation; see previous remark.

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1 Many authors use a ($-++$) metric. The difference may affect algebraic aspects of the theory, but the geometry remains the same.
From [4] and [1] we know that \( \mathfrak{N} \), that is defined as the quotient space of \( \mathfrak{S}X \) by the equivalence relation introduced above, possesses a natural contact structure (when \( \mathfrak{N} \) is smooth). In this article a weaker version of this statement will be used, adapted to the fact that \( \mathfrak{N} \) does not always admit a manifold structure.

**Definition.** Let’s choose for each point \( w \in \mathfrak{S}X \) its representative \( v \in \mathcal{C}_x^+ \setminus \{0\} \), where \( x := x(w) \), in a smooth fashion. Then \( \vartheta = (v.dx) \) is an 1-form on the total space \( \mathfrak{S}X \).

**Remark.** Obviously, the 1-form defined so is meaningful only up to multiplication by a positive function. However, there is a real line bundle defined globally over \( \mathfrak{S}X \) that is the range of the form \( \vartheta \) invariantly defined. Moreover, that line bundle is oriented (that is, has the positive side marked). This construction will be postponed until paragraph 2.2.

**Remark.** The 1-form \( \vartheta \) is smooth and never vanishes. But \( \vartheta \) nullifies all the \( FX \) (follows from the fact that any Lorentzian null direction is orthogonal to itself) and \( T_0 \mathfrak{S}X \) (it follows from the equation \( dx = 0 \)). Or differently: five-dimensional bundle \( T(\mathfrak{S}X) \) / \( FX \) possesses a continuous distribution of homogeneous co-oriented hyperplanes \( \vartheta = 0 \) containing \( T\mathfrak{S}_x \) for all \( x \in X \).

### 1.2 Defining a conformal reference frame

**Definition.** A conformal reference frame \( (\Omega, M, j) \) of the Lorentzian manifold \( X \) is defined as:

- such open subset \( \Omega \subset \mathfrak{S}X \) that each its fiber \( \Omega_x := \Omega \cap \mathfrak{S}_x \), \( x \in X \) is not empty;
- a 3-dimensional smooth manifold \( M \);
- such smooth map \( j \) of \( \Omega \) to \( M \) that satisfies following conditions:

  (f) \( \forall w_1, w_2 \in \Omega \quad [w_1] = [w_2] \Rightarrow j(w_1) = j(w_2) \)
  
  (i.e. the map is constant along leaves of the “light” foliation)

  (c) \( \forall w \in \Omega \quad \vartheta_w \in J^*(T^*_j(w)M) \)
  
  (i.e. the oriented distribution of 1-subspaces in \( T^*\Omega \), represented by the form \( \vartheta \), belongs everywhere to \( J^*(T^*M) \), the inverse image of the
cotangent bundle), as well as:

(d) the derivative of \( j \) along \( T\Omega_x \) (fibers of \( \Omega \)) is not degenerate (i.e. the pushforward \( j_* : T_v\Omega \to TM \) of vertical tangent vectors has rank 2 everywhere).

The image of \( \Omega_x \) by \( j \)—a surface immersed to \( M \)—will be denoted by \( \mathfrak{M}_x \) and called a sky image.

**Example.** Let \( M \) be a Cauchy surface in a globally hyperbolic Lorentzian manifold. Denote by \( X \) the part of said manifold that lies after \( M \), and let \( j \) be the projection (along null geodesics) to the twistor bundle \( ST^*M \) (see [1]) followed by projection of the latter onto \( M \). Also, let \( \Omega \) be the nonsingular subset of \( \mathcal{S}X \) with respect to mapping of the skies, i.e. where the mapping \( j_* : T_v\mathcal{S} \to TM \) has rank 2. If everywhere on \( X \) \( \Omega_x \) isn’t empty, then \((\Omega, M, j)\) is a conformal reference frame for the manifold \( X \).

### 1.3 The derivative of sky images

In this paragraph we intend to define the derivative of the sky image \( \mathfrak{M}_x := j(\Omega_x) \) for \( x \in X \) with respect to \( x \). We denote by the differential of a map \( j \) the homomorphism \( dj : T\Omega \to j^*(TM) \) of bundles over \( \Omega \), from the tangent bundle of \( \Omega \) to the inverse image of the tangent bundle of \( M \). It must be distinguished from the pushforward homomorphism.

**Definition.** For a conformal reference frame, let the fiber-wise normal bundle denote

\[
N_M \Omega := j^*(TM) / dj(T_v\Omega), \quad \text{where} \quad T_v\Omega := T_v\mathcal{S}X|_\Omega.
\]

**Remark.** Restriction of the fiber-wise normal bundle \( N_M \Omega \) to \( \Omega_x \), i.e. its “fiber” \( N_M \Omega_x = j|_{\Omega_x}^*(TM) / dj(T\Omega_x) \) is, away of self-intersections, the normal bundle \( N\mathfrak{M}_x \) of the surface \( \mathfrak{M}_x \) in \( M \) raised to \( \Omega_x \) by \( j : \Omega \to M \).

**Remark.** The concept of the normal bundle does not require any structure but differentiability; it’s merely a quotient space.

**Definition.** Given a local trivialization of \( x \) (that is, such domain \( U \subset X \) that \( U \times \Delta \) is a subdomain of \( \Omega \), where \( \Delta \) is the standard disk along the

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2 With respect to the Big Bang cosmology (Robertson–Walker spaces), \( M \) can be, more generally, a conformal boundary of \( X \); see [7] 6.8 and 10.
fibers and $x$ becomes a projection onto $U$) and a mapping to $M$ specified as $j : U \times \Delta \to M$, derivative of the family $\mathcal{M}_x$ with respect to $x \in X$ is defined (locally) as the homomorphism of vector bundles from $\Delta \times T X$ to the corresponding piece of $N M \Omega$:

$$d \mathcal{M}_x := \frac{\partial j}{\partial x} / dj(T \Delta)$$

where “/” denotes the quotient (by the image of $T \Delta$ by $j$). In other words, the vectors $\frac{\partial j}{\partial x}(\xi) \in j^*(TM)$ for all $(x, z) \in U \times \Delta, \xi \in T_x X$ represent $d \mathcal{M}_x(\xi) \in N M \Omega$ (at the same $(x, z)$).

The local trivialization exists everywhere and, obviously, the value of the derivative is independent of its choice.

**Definition.** $N \Omega := T \Omega / V \Omega$ is a natural (independent of trivialization) domain of the mapping $d \mathcal{M}_x$; see the diagram. Similarly, for all $\mathcal{S} X$ by $N \mathcal{S} X := x^*(T X) = T(\mathcal{S} X) / V \mathcal{S} X$ we denote the tangent bundle to $X$ raised to $\mathcal{S} X$. Also, $d x$ always denotes the projection of $T$ onto respective $N$.

**Definition.** Under the same conditions we define $p := dj / dj(T, \Omega) : T \Omega \to N M \Omega$ – the tautological projection of $T \Omega$ onto the fiber-wise normal bundle.

### 2 Example: Minkowski space

**2.1 Definition and the spinor correspondence**

**Definition.** The algebraic spinor correspondence or the Pauli transform maps a vector from $\mathbb{R}^4$, defined with components $(x^0, x^1, x^2, x^3)$, to the following Hermitian $2 \times 2$ matrix:

$$x^{\alpha \beta} = \frac{1}{2} \begin{pmatrix} x^0 + x^3 & x^1 + i x^2 \\ x^1 - i x^2 & x^0 - x^3 \end{pmatrix}.$$

\[3\] The real factor in the formula is more often chosen as $1/\sqrt{2}$ or 1. Its value hasn’t importance for this paper.
Remark. It is easy to verify that every null (with respect to the 
\( \eta = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \) metric) vector \( x \in \mathbb{R}^4 \) with the condition \( x^0 \geq 0 \) can be expressed as
\[
x^{A\bar{A}} = \psi^{A} \bar{\psi}^{\bar{A}}, \quad \psi \in \mathbb{S}^{1,0}, \quad \text{where} \quad \mathbb{S}^{1,0} := \mathbb{C}^2
\]
and conversely, any \( \psi \in \mathbb{S}^{1,0} \) gives a vector from \( \mathbb{C}^+ \) in this way.

**Definition.** The Minkowski space \( \mathbb{M} \) is the \( \mathbb{R}^4 \) coordinate space endowed with the standard pseudo-Euclidean metric \( \eta \) specified above.

Remark. Here the action of the group \( \text{SO}^+(1,3) \) on \( \mathbb{M} \) will induce a representation of the group \( \text{SL}(2, \mathbb{C}) \) on \( \text{Herm}(2) \) defined by
\[
\text{SL}(2, \mathbb{C}) \ni C : x^{A\bar{A}} \mapsto C^A_L x^{LL'} \bar{C}^{\bar{A}}_{L'}
\]
(or, as matrix multiplication, \( x^{A\bar{A}} \) is multiplied by \( C \) on the left and by \( C^* \) on the right); that is the tensor product of two 2-dimensional representations on the spaces \( \mathbb{S}^{1,0} \) and \( \mathbb{S}^{0,1} \) (see below) respectively.

**Definition.** The dual space \( \mathbb{S}^{*}_{1,0} \) of spinors is the linear complex space dual to \( \mathbb{S}^{1,0} \), and it’s equipped with a representation of the group \( \text{SL}(2, \mathbb{C}) \) that multiplies row vectors by the inverse matrix \( C^{-1} \) on the right.

Remark. All the bundle \( \mathcal{S}\mathbb{M} \) has a natural trivialization, arising from the trivialization of \( T\mathbb{M} \) by translations. Correspondence between projectivizations on the cone \( \mathbb{C}^+ \) and the space \( \mathbb{S}^{1,0} \) also enables identification of \( \mathcal{S} \)—the base of the cone—with the Riemann sphere \( \mathbb{C}P^1 = \mathbb{P}\mathbb{S}^{1,0} \). On the other hand, each \( \mathcal{P}\psi, \psi \in \mathbb{S}^{1,0} \setminus \{0\} \) matches a homogeneous line \( \{ \varsigma \in \mathbb{S}^{1,0} \mid \varsigma \psi = 0 \} \), hence the projectivizations of the spaces \( \mathbb{S}^{*}_{1,0} \) and \( \mathbb{S}^{1,0} \) are canonically isomorphic.

For the reasons explained below, we will conveniently present a point of a sky in the Minkowski space as “\( \mathcal{P}\varsigma \)”, where \( \varsigma \in \mathbb{S}^{*}_{1,0} \setminus \{0\} \), and identify it with a complex homogeneous line in \( \mathbb{S}^{*}_{1,0} \).

### 2.2 Complex line bundles

**Definition.** The bundle \( \mathcal{O}(k,l), \ k,l \in \mathbb{Z} \) over \( \mathcal{S} \) is a complex line bundle, whose fiber at a point \( \mathcal{P}\varsigma, \ \varsigma \in \mathbb{S}^{1,0} \setminus \{0\} \) is a 1-dimensional space of all homogeneous bidegree \( (k,l) \) complex-valued functions on the homogeneous line \( \{ \lambda \varsigma \mid \lambda \in \mathbb{C} \} \subset \mathbb{S}^{*}_{1,0} \). The term “homogeneous bidegree” refers to
\[
\forall f \in \mathcal{O}(k,l)_{\mathcal{P}\varsigma} \forall \lambda \in \mathbb{C} : f(\lambda \varsigma) = \lambda^k \bar{\lambda}^l f(\varsigma).
\]
Remark. Every homogeneous bidegree \((k, l)\) function on the whole \(S^*_{1,0}\) defines a section of \(\mathcal{O}(k, l)\) by restriction to all complex homogeneous lines.

Remark. In particular, \(\mathcal{O}(1, 0)\) is a holomorphic bundle, and every \(\omega \in S^*_{1,0}\) defines its holomorphic section as a linear functional on \(S^*_{1,0}\). Thus, the space of spinors \(S^1_{1,0} = \Gamma_{\text{hol}}(\mathcal{O}(1, 0))\) identifies with the space of holomorphic sections.

Definition. \(S^{0,1} := \overline{S^{1,0}} = \Gamma_{\text{antihol}}(\mathcal{O}(0, 1))\) is the complex conjugate space of spinors. Also, it has the dual space \(S^*_{0,1}\).

Remark. As can be seen from the representation, \(2 \times 2\) Pauli matrices should be understood as elements of the tensor product \(S^{1,0} \otimes S^{0,1}\), so they specify sections of the bundle \(\mathcal{O}(1, 1)\).

Definition. The celestial transform \(s_M : M \rightarrow \Gamma(\mathcal{O}(1, 1))\) for Lorentzian vectors denotes the same map from \(M\) to \(S^{1,0} \otimes S^{0,1}\) as for algebraic one, but its values are interpreted as \((1,1)\)-homogeneous nonholomorphic polynomials \(\zeta_A x^{AA'} \bar{\zeta}_{A'}\) on \(S^*_{1,0}\) or, the same, as sections \(\zeta_A \cdot x^{AA'} \cdot \bar{\zeta}_{A'}\) of the nonholomorphic bundle \(\mathcal{O}(1, 1) = \mathcal{O}(1, 0) \otimes \mathcal{O}(0, 1)\) over \(\mathcal{G}\).

We may tell the celestial transform the geometric spinor correspondence. Furthermore, \(s_M\) will also denote the respective mapping from \(M \times \mathcal{G} = \mathcal{G}M\) to the total space of \(\mathcal{O}(1, 1)\).

Remark. Two bundles \(\mathcal{O}(1, 1)\) constructed over \(\mathcal{G}\) and \(\tilde{\mathcal{G}} = P\mathcal{S}^*_{0,1}\), are no different, except the complex structure on their bases. They are canonically isomorphic as complex linear bundles over surfaces. Therefore, the celestial transform withstands exchange \(\mathcal{G}\) with \(\tilde{\mathcal{G}}\).

Definition. \(\mathcal{L}^+\) denotes the subbundle of non-negative functions in \(\mathcal{O}(1, 1)\), whereas its fibers are equivalent to the ray \([0, +\infty)\) up to multiplication by a positive number. In a formula: \(\mathcal{L}^+ = \{ \zeta \cdot \bar{\zeta} \mid \zeta \in \mathcal{O}(1, 0) \}\). An \(\mathbb{R}\)-linear bundle of real-valued \((1,1)\)-homogeneous functions, containing \(\mathcal{L}^+\), will be denoted by \(\mathcal{L}^\mathbb{R}\).

Remark. It is easy to see that all values of the transform \(s_M\) defined above are, in fact, sections of \(\mathcal{L}^\mathbb{R}\).

Remark. The bundles have naturally defined operations \(\zeta \in \mathcal{O}(1, 0) \mapsto \zeta \cdot \bar{\zeta}\), denoted by \(\|\cdot\|^2\) (modulus squared) and \(\|\cdot\| : (T\mathcal{G} = \mathcal{O}(2, 0)) \rightarrow \mathcal{L}^+\), both continuous maps of bundles.

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The $L^+$ is called the bundle of sizes, and its sections are fields of sizes. The bundle/sections of $L^R$ are signed bundle/fields of sizes.

Remark. Representations of $SL(2, \mathbb{C})$ on $S^*_1$ and the spaces of spinors generate a (consistent) action of the same group on the sky that corresponds to the covering

$$\{\pm 1\} \hookrightarrow SL(2, \mathbb{C}) \xrightarrow{2:1} PSL(2, \mathbb{C}) \rightarrow 1.$$ 

It preserves $T\mathcal{S}$ and $O(1, 0)$ as holomorphic bundles ($O(1, 1)$ and other – as line bundles), while vector fields and sections of $O(1, 1)$ are transformed as they would be simply by the Riemann sphere’s motions.

Remark. By the definition of $\vartheta$ from 1.1, for each $w \in \mathcal{S}M$, $x := x(w)$ we have to choose such null $v \in C^+_x \setminus \{0\}$ that represents the point $w \in \mathcal{S}_x$. Let’s express $v$ in the covariant form (i.e. to make it an element of $T^*\mathcal{M}$) as $v_A A' = \varsigma_A \varsigma_{A'}$. For each tangent vector $\Xi \in T_w \mathcal{S}M$ we have $\vartheta(\Xi) = v \xi = \varsigma_A \varsigma^A' \xi_A', \xi := x_*(\Xi)$, hence

$$\vartheta \propto \varsigma_A dx^{A'} \varsigma_{A'} : T(\mathcal{S}M) \to L^R.$$ (1)

The symbol “$\propto$” here denotes equality up to multiplication by a positive number.

### 2.3 The construction of Poincaré-invariant conformal reference frame

**Proposition 1.** The triple $(\Omega := \mathcal{S}M, M := L^R, j := s_M)$ is a conformal reference frame for $\mathcal{M}$.

Remark. The total space of the bundle $L^R$ can be represented as $\mathcal{S} \times \mathbb{R}$, and any $\mathcal{M}_x$ is a graph of a section $\varsigma_A \cdot x^{A'} \cdot \xi_{A'}$.

**Proof.** Obviously, a map sending skies to graphs over $\mathcal{S}$ satisfies the condition (d) of the definition of the conformal reference frame. The condition (f) is made from the fact that $s_M(\mathcal{S}_w)$, where $v$ is a null vector, vanishes at the point $Pv$. To prove (c) in an easy way, let’s investigate all $w \in \mathcal{S}_0M$ which doesn’t lose generality since a translation of $\mathcal{M}$ becomes the fiber-wise addition (of a fixed smooth section) in the total space $L^R$. From the triviality of $\mathcal{S}M$ referred to in 2.1, $T_w \mathcal{S}M = T_w \mathcal{S}_0 \oplus T_0 \mathcal{M}$, where $T_w \mathcal{S}_0 \subset \ker \vartheta$ and, moreover, $j_*(T_w \mathcal{S}_0)$ consists of vectors tangent to $\mathcal{M}_0$ (the zero section of $L^R$).
We choose, as in (1) at the end of the preceding paragraph, such \( \varsigma \in S^*_1 \setminus \{0\} \) that \( P_\varsigma = w \). A direct calculation

\[ \vartheta(\Xi) = \varsigma_A \xi^{AA'} \bar{\varsigma}_{A'} = \frac{\partial j}{\partial x} |_{\varsigma}(\xi), \quad \text{where} \quad \xi := x_*(\Xi), \; \Xi \in T_w \mathcal{SM} \]

shows that the desired image of the form \( \vartheta|_w \) by \( j \) is simply the projection of the space \( T_{(P_\varsigma,0)} \mathcal{L}^R \) onto the vertical (tangent to the fibers of \( \mathcal{L}^R \)) direction. The image of the form \( \vartheta \) at an arbitrary point \( w \in \mathcal{SM}, \; x := x(w) \) would differ only in the projection that would go along the tangent to the graph \( \varsigma_A \cdot x^{AA'} \cdot \bar{\varsigma}_{A'} = s_\mathbb{M} x \) instead of the one to the zero section.

Direct interpretation of the constructed reference frame: the total space \( \mathcal{L}^R \) is the space of null hyperplanes in \( \mathbb{M} \), and \( j \)-preimage of an element of \( \mathcal{L}^R \) is given by a constant null direction—an element of \( \mathcal{G} \)—and a hyperplane in \( \mathbb{M} \). Choosing \( P_\varsigma \in \mathcal{G} \) and \( \chi \in \mathcal{L}^R|_{P_\varsigma} \) arbitrarily, we have the set \( x(j^{-1}(\chi)) \) of all points \( \mathbb{M} \), \( s_\mathbb{M} \)-images of whose skies pass through \( \chi \), to become a hyperplane—solution of the linear equation \( \varsigma_A x^{AA'} \bar{\varsigma}_{A'} = \chi(\varsigma) \) at \( x^{AA'} \). Presence of the natural action of the Poincaré group on \( \mathcal{L}^R \) ensures that the set of all null hyperplanes in \( \mathbb{M} \) can be obtained in such a way. These null hyperplanes can also be understood as the light cones of “points at infinity” and the very three-dimensional total space \( \mathcal{L}^R \) —as the piece \( I \) of the conformal compactification \( \mathcal{M} \) of the Minkowski space \( \mathbb{M} \) which brings the present example closer to the one given in 1.2 and will be founded in the next paragraph.

### 2.4 Interpretation through the twistor correspondence

The manifold of null lines in the space \( \mathcal{M} \) permits a known description as the projective null twistor space \( \mathcal{PN} \subset \mathbb{P}(S^*_0 \times S^1) \simeq \mathbb{CP}^3 \). \( \mathcal{N} \) is given by \( \pi_L \omega^L + \pi_{L'} \bar{\omega}^{L'} = 0 \) and it’s a smooth real hypersurface in \( S^*_0 \times S^1 \), and \( \mathcal{PN} \) lies in the respective complex projective space. If we restrict to the Minkowski space proper \( (X := \mathbb{M}) \), then \( \mathfrak{N} = \mathcal{PN}_a \subset \mathcal{PN} \) (hereinafter referred to as the affine part of \( \mathcal{PN} \)). Moreover, \( |\mathfrak{G}_x| \) is given by the following equation:

\[ \omega^L = i x^{LL'} \pi_{L'} \quad [2], \quad \text{equation (2)} \]

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4 The conformal compactification \( \mathcal{M} \) is explained in [6] Chap. 9, [8] paragraphs 2.1, 2.2, and [5]. A more general construction of the conformal boundary of a Lorentzian manifold is given in [7] section 6.8.
where $S^0_{1,1} \ni \pi = \zeta$ are complex projective coordinates in the fibers of $\mathcal{S}M$. $N_a$ is the union of all solutions of equation (2) for all $x \in \mathbb{M}$, and $N_a = N \setminus \{\pi = 0\}$. The smoothness of the mapping $[] : \mathcal{S}M \rightarrow P\mathbf{N}_a$ is obvious.

**Definition.** The twistor contraction $\tau : P(S^0_{1,1} \setminus \{0\} \times S^{1,0}) \rightarrow \mathcal{O}(1,1)$ maps a point $P(\pi, \omega)$ to an element of the total space of the bundle $\mathcal{O}(1,1)$ over $\mathcal{S}$ (identical to $\mathcal{O}(1,1)$ over $\mathcal{S}$), expressed with the point $P\pi$ on the base and the “complex size” $\lambda\pi \mapsto -i|\lambda|^2 \bar{\pi}_L \omega_L$.

**Remark.** Correctness of the definition of the twistor contraction requires that specified $(1,1)$-homogeneous function on $\{\lambda\pi\}$, depending on the pair $(\pi, \omega)$ as on a parameter, did not change when projective coordinates multiplied by a non-zero complex number, that is easy to check.

**Remark.** Definitions of $\tau$ and $N$, as well as the explicit form of $N_a$ imply that

$$N_a = \{(\pi \in S^0_{1,1} \setminus \{0\}, \omega \in S^{1,0}) \mid \tau(\pi, \omega) \in L^R\}$$

(in other words, $P\mathbf{N}_a = \tau^{-1}(L^R)$). Obviously, the $\tau$-preimage of every element of $L^R$ is the affine part of a projective line in $P\mathbf{N}$ (or a plane in $N$), but with a constant $P\pi$ unlike ones defined by the equation (2).

**Remark.** The map $j$ introduced in the previous paragraph coincides with the composition of the projection $[] : \mathcal{S}M \rightarrow P\mathbf{N}_a$ with the twistor contraction $\tau$; it’s demonstrated through contraction of (2) with $-i\bar{\pi}_L$. So, $\tau$-preimage of each element $\chi \in L^R$ specifies a point at infinity in $\hat{\mathbb{M}}$, where null lines comprising the null hyperplane $x(j^{-1}(\chi))$ intersect.

**Remark.** From the expression of $j$ from $\tau$ and the fact that the projective line $\omega = 0$ is $|\mathcal{S}_0| \subset P\mathbf{N}_a$, we obtain, similarly to the Proposition 1, that the contact form on $P\mathbf{N}_a$, restricted to $|\mathcal{S}_0|$, admits the expression

$$\vartheta |_{|\mathcal{S}_0|} = d\varpi |_{\{\omega = 0\}} = -i\bar{\pi}_L d\omega |_{\{\omega = 0\}}.$$ 

In view of the homogeneity of $\tau$, the expression $d\tau$ correctly specifies a mapping $TP\mathbf{N}_a|_{|\mathcal{S}_0|} \rightarrow \mathcal{O}(1,1)$.

It is also easy to show that on the whole $N_a$:

$$\vartheta |_{N_a} = -i(\bar{\pi}_L d\omega + \bar{\omega}' d\pi_L').$$
3 The flow of time equation

In the general Lorentzian case we have to consider line bundles over different skies. Let’s denote by \( O_x(1,0) \), \( L_x^+ \), and so on respective bundles over the sky \( S_x \). Similarly (with lower “\( x \)”) the spaces of spinors, depending on \( x \), will be indicated.

Remark. It is easy to see that the inverse image \( x^*(T_xX) = NS_x \) of the tangent space is \( S_x \times T_xX \) that, after exchanging the base with the fiber, turns to \( S (T_xX) \). Moreover, \( S (T_xX) \) is isomorphic to \( S M \) (which was considered in section 2) and the isomorphism is defined up to the action of the Lorentz group.

Definition. Let’s denote by \( L^R_X \) the disjoint union of the signed bundles of sizes \( L^R_x \) for all \( x \in X \), that gives a smooth line bundle over \( S X \). Similarly, \( L^+X \subset L^R_X \) is the disjoint union of the bundles of sizes \( L^+_x \) for all \( x \in X \).

Definition. The map \( s : NS \to L^R_X \) is defined fiber-wise through \( s_{T_xX} : NS_x \to L^R_x \) for all \( x \in X \), where \( s_{T_xX} \) is identical to the map \( s_M \) introduced in paragraph 2.2.

Remark. When selected (at random) orientation on \( T_xX \), correctness of the definition of \( s_{T_xX} \) follows from the \( \text{SL}(2, \mathbb{C}) \)-invariance of the space \( S^1_{x,0} \) and the weights of the respective representation of the group in \( O_x(1,1) \). Reversing the orientation on \( T_xX \) interchanges \( S_x \) and \( \varsigma A \in S^1_{x,0} \) with \( \bar{S_x} \) and \( \pi A' \in S^0_{x,1} \), respectively, but \( L^R_x \) and the mapping of the bundle \( NS_x \) to it doesn’t change.

Now we are able to supersede the definition of the form \( \vartheta \), given in 1.1, with

Definition. \( \vartheta := s \circ dx \) is an 1-form on the total space \( S X \) with values in \( L^R_X \), defined by the composition of the horizontal projection \( dx : T(S X) \to NS \) with the map \( s \) introduced above.

Remark. The “new” form \( \vartheta \) : \( T\Omega \to L^R_X \) and \( \mathbb{R} \)-valued form defined in 1.1 up to a positive factor, coincide as distributions of oriented co-directions on \( S X \).

Remark. Directly from the definition of \( \vartheta \) follows that the celestial transform is expressed as \( s = dx_\ast(\vartheta) \).
Theorem 1. For any conformal reference frame \((\Omega, M, j)\) for a Lorentzian manifold \(X\) the projection \(p\) (see 1.3) satisfies

\[
\ker p = \ker \vartheta|_\Omega.
\]

Proof. The kernel of \(dj\) in \(T\Omega\), according to the definition of the conformal reference frame, is three-dimensional, lies entirely in \(\ker \vartheta\), and has the zero intersection with \(T_v\Omega\), hence \(\ker \vartheta|_\Omega = \ker dj \oplus T_v\Omega\). Since the fiber-wise normal bundle \(N_M\Omega_x\) — the range of \(p\) — is obtained from \(j|_{\Omega_x}^*(TM)\) as the quotient by the image of \(dj(T\Omega_x)\), the theorem is proved.

Corollary. The map \(dM_x\) (derivative of the sky image with respect to \(x\)) nullifies \(dx(\ker \vartheta)\), and only this. See the diagram at the end of 1.3.

Corollary. The direct image of the 1-form \(\vartheta\) by \(p\) is well-defined, and vanishes nowhere on \(\Omega\). That is, the bundles \(L^\mathbb{R}_\Omega := L^\mathbb{R}_X|_\Omega\) and \(N_M\Omega\) are naturally isomorphic.

Theorem 2. Let \(X\) be an oriented manifold with a conformal reference frame, satisfying the conditions of Theorem 1. We define, based on the last Corollary, the homomorphism of line bundles \(a_j := p_*|_{\vartheta}^{-1} : L^\mathbb{R}_\Omega \to N_M\Omega^x\). Then the derivative (see 1.3) of the family \(\{M_x\} = \{j|_{\Omega_x}\}\) will satisfy the so named “flow of time equation”:

\[
d_{AA'}M_x = a_j(\varsigma_A \cdot \pi_{A'}) \iff p_*|_{\vartheta} \cdot d_{AA'}M_x = \varsigma_A \cdot \pi_{A'},
\]

where \(\varsigma_A\) and \(\pi_{A'}\) are bases in \(S_{x,1}^1\) and \(S_{x,0}^1\) complex conjugate to each other (or, equivalently, the coordinates in \(S_{x,1}^*\) and \(S_{x,0}^*\), respectively, in which the algebraic spinor correspondence for the differential \(d_{AA'}M_x\) is expressed.

Proof. The identity \(p = dM_x \circ dx\) presented in the diagram in 1.3 and the definition of \(\vartheta\) entail that \(p_*|_{\vartheta} = (dM_x)_*(s)\). And \(s\xi = \varsigma_A \cdot \xi_{AA'} \cdot \pi_{A'}\) by construction, where \(\xi \in T_xX\), which proves the theorem.

4 The other dimension of space

As a generalization of Lorentzian manifolds we can consider pseudo-Euclidean manifolds of the signature \((1, d)\), i.e. one plus and \(d\) minuses (where \(d\) is

\(^5\) In other words, \(a_j = \vartheta_*|_\Omega\) and the solution of the equation \(a_j|_\vartheta = p\).
a natural number), and the time orientation at each point. Section 1 is applicable to this case with obvious changes on the dimension: a sky becomes $S^{d-1}$, $\Delta$ becomes the $d-1$-dimensional ball, and so on. The complex algebra of Section 2 does not admit the obvious generalization, because only the case $(1,3)$ has the spinor group isomorphic to a complex special linear group. Nevertheless, the space $\mathbb{M}^{1+d}$ admits the conformal compactification, and the invariant frame of reference built in paragraph 2.3, making it possible to define the bundle of sizes $L^R X$ as the range of the contact form. Theorem 1 also remains true for any $d$.

5 Appendix – The causality

This part of the paper isn’t included to the peer-reviewed text ready for publication.

The causal relation in Relativity is understood in terms of Lorentzian manifolds. This is fairly convenient in Special Relativity, but has several shortcomings for arbitrary Lorentzian space-times, and in General Relativity when geometry (as the metric tensor) effectively becomes one of the field variables. The manifold substantionalism leads to causal relation expressed in terms of causal paths and sets that can be seen in [1] and elsewhere, but it’s presently unknown whether this concept matches the fundamental physical one.

The aim of this section is to propose a construction of the causality consistent with space-time that is a Lorentzian manifold $X$, but based on other, more quantum-friendly foundations. This construction of the causality will somewhere differ from predictions of General Relativity. More precisely, let’s assume $X$ as some approximation to the physical world. Now we want to:

- Give a description of the causal relation in terms external to $X$.
- Propose a kind of absolute reference frame suitable for our universe.
- Permit for interpretation of Quantum Mechanics along the lines of [9].

As defined in paragraph 1.2, there may be several (essentially) different conformal reference frames for the same $X$. For Minkowski space, we obviously identify our $L^R$ with $I = I^+ = I^-$, whereas for another similar $X$, that is only asymptotically Minkowski space, projections to the future null infinity $I^+$ and to the past null infinity $I^-$ will result in different reference frames.
Does the current physical cosmology propose any hint about the natural conformal reference frame? Fortunately, the “cosmological censorship” principle implies that any null geodesic extended to the past must meet the Cosmological Singularity of the FLRW cosmology. Since the latter is 3-dimensional conformal boundary of our space-time (see e.g. \[7\]), it’s an obvious candidate for \(M\). Let \(M\) be the Cosmological Singularity from now on and let \(j\) denote the natural operation of geodesic continuation towards the Singularity and taking the intersection point then, similarly to the Example from 1.2. This conformal reference frame will be termed \textit{absolute}.

For such a universe without enough curvature to make gravitational lensing, as a pure Robertson–Walker space, the picture will not differ greatly from the one explained in paragraph 2.3 for the Minkowski space. \(\Omega\) should be taken as the whole \(\mathcal{G}X\), and all sky images will be smooth spheres embedded to \(M\). Causality relation will admit a simple geometric description. Let \(\mathcal{P}_x\) denote the union of \(\mathcal{M}_x\) with its interior. Then any event belongs to the causal past of \(x\) if and only if its sky image lies in \(\mathcal{P}_x\). This can be formulated for the Minkowski case as well (albeit with little merit), replacing “interior” with “the negative side in \(\mathcal{L}^{\mathbb{R}^n}\”).

For a more realistic cosmology \(j_*|_{\mathcal{T}_x}\Omega\) will degenerate somewhere. Similarly to the Example, \(\Omega\) can be defined as the open set where \(j\) doesn’t degenerate. We can now hope (albeit I currently have no theorem) that the closure of \(\mathcal{M}_x\) will be a topologically closed surface in \(M\) despite singularities (but not necessarily a submanifold). Then, \(\mathcal{P}_x\) can be taken as a 3-dimensional subset of \(M\) as well: the union of \(\mathcal{M}_x\)’s closure with its interior. A redefined “causal relation” depending of \(\mathcal{P}_x\) may be weaker that the one of General Relativity even in some globally hyperbolic cases. Moreover, a realistic model of the universe can’t be globally hyperbolic. Particularly, the (absolute reference frame)-based causality relation abolishes the event horizon in a black hole (which could shed light to the “information paradox”).

Connecting this to the “Locale of Time” concept requested in \[9\] 1.3, a possible construction is the space of closed subsets of \(M\). It’s a bounded join-semilattice (with the set union “\(\cup\)” as the join) and has a natural non-Hausdorff topology specified with the base

\[
\{ \{ B \text{ is a closed subset of } M \mid B \cap K = \emptyset \} \mid K \in M \}.
\]

Substituting \(K := \mathcal{P}_x\) for any space-time event \(x\), we have an open set in the Locale of Time necessary for definition of open balls in the Space of Ultimations discussed in that paper.
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