DIPPA: An improved Method for Bilinear Saddle Point Problems

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Abstract

This paper studies bilinear saddle point problems \( \min_x \max_y g(x) + x^\top Ay - h(y) \), where the functions \( g, h \) are smooth and strongly-convex. When the gradient and proximal oracle related to \( g \) and \( h \) are accessible, optimal algorithms have already been developed in the literature \([4, 27]\). However, the proximal operator is not always easy to compute, especially in constraint zero-sum matrix games \([39]\). This work proposes a new algorithm which only requires the access to the gradients of \( g, h \). Our algorithm achieves a complexity upper bound \( \tilde{O}\left(\frac{\|A\|_2^2}{\mu_x \mu_y} + \sqrt{\kappa_x \kappa_y (\kappa_x + \kappa_y)}\right) \) which has optimal dependency on the coupling condition number \( \|A\|_2^2 \) up to logarithmic factors.

1 Introduction

We consider the convex-concave bilinear saddle point problem of the following form

\[
\min_{x \in X} \max_{y \in Y} f(x, y) \triangleq g(x) + \langle x, Ay \rangle - h(y). \tag{1}
\]

This formulation arises in several popular machine learning applications such as matrix games \([2, 3, 14]\), regularized empirical risk minimization \([41, 31]\), AUC maximization \([38, 30]\), prediction and regression problems \([32, 35]\), reinforcement learning \([10, 8]\).

We study the most fundamental setting where \( g \) is \( L_x \)-smooth and \( \mu_x \)-strongly convex, and \( h \) is \( L_y \)-smooth and \( \mu_y \)-strongly convex. For the first-order algorithms which iterate with gradient and proximal point operation of \( g, h \), it has been shown that the upper complexity bounds \([4, 27]\) of \( \mathcal{O}\left(\left(\frac{\|A\|_2^2}{\mu_x \mu_y} + 1\right) \log \left(\frac{1}{\varepsilon}\right)\right) \) match the lower complexity bound \([30]\) for solving bilinear saddle point problems. However, finding the exact proximal point of \( g \) and \( h \) could be very costly and impractical in some actual implementations. One example is matrix games with extra cost functions \( g \) and \( h \). In this case, the total cost function of the first player is \( g(x) + \langle x, Ay \rangle \), and that of the second player is \( h(y) - \langle x, Ay \rangle \). The forms of \( g \) and \( h \) could be complicated. Moreover, Kanzow & Steck \([15]\) proposed an augmented Lagrangian-type algorithm for solving generalized Nash equilibrium problems. Applying this algorithm to solve equality constrained matrix games \([39]\), we obtain a subproblem of the form \([1]\). Even if the constraints are linear in \( x \) and \( y \), the complexity of calculating the exact proximal point of \( g \) and \( h \) could be unacceptable.

In this paper, we consider first-order algorithms for solving the bilinear saddle point problem \([1]\) with assuming that only the gradients of \( g \) and \( h \) are available. In this setting, Ibrahim et al. \([14]\),

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Figure 1: Comparison of previous upper bounds [34], lower bound [40] and the results in this paper where $L_x = L_y, \mu_x < \mu_y$, ignoring logarithmic factors. We demonstrate the upper bounds and the lower bounds as a function of $\|A\|_2$ with other fixed parameters.

Zhang et al. [40] proved a gradient complexity lower bound $\Omega \left( \left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x + \kappa_y} \right) \log \left( \frac{1}{\varepsilon} \right) \right)$. On the other hand, we can modify the algorithm in [4] to adapt to this setting with employing Accelerated Gradient Descent (AGD) to approximately solve each proximal point of $g, h$. This inexact version achieves an upper bound of $\tilde{O} \left( \left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x + \kappa_y} \right) \log \left( \frac{1}{\varepsilon} \right) \right)$ where $L = \max \{\|A\|_2, L_x, L_y\}$ (see Section F in Appendix for more details). And recently, Wang & Li [34] showed a same upper bound in a more general case. This upper bound is tight when $x$ and $y$ are approximately decoupled (i.e., $\|A\|_2 < \max\{\mu_x, \mu_y\}$) or the coupling matrix $A$ is dominant in Problem (1) (i.e., $\|A\|_2 > \max\{L_x, L_y\}$). However, in the intermediate state, this upper bound would no longer be tight (see Figure 1 for illustration).

In this work, we propose a new algorithm Double Inexact Proximal Point Algorithm (Algorithm 4) and prove a convergence rate of $\tilde{O} \left( \left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x \kappa_y (\kappa_x + \kappa_y)} \right) \log \left( \frac{1}{\varepsilon} \right) \right)$. Our upper bound enjoys a tight dependency on the coupling condition number $\frac{\|A\|_2}{\sqrt{\mu_x \mu_y}}$ while suffers from an extra factor $\sqrt{\kappa_x \kappa_y}$ on the condition numbers of $g$ and $h$. Our method is better than previous upper bounds for $\|A\|_2 = \Omega(\sqrt{L_x \mu_y + L_y \mu_x})$ and matches lower bound for $\|A\|_2 = \Omega(\sqrt{L_x \mu_y (L_x \mu_y + L_y \mu_x)})$ (see Figure 1 for illustration). However, our method does not perform well in the case of weak coupling where $\|A\|_2 = \mathcal{O}(\sqrt{L_x \mu_y + L_y \mu_x})$.

The remainder of the paper is organized as follows. We present preliminaries of the saddle point problems in Section 2. Then we review some related work in Section 3 and two algorithm AGD and APFB which will be employed to solve subproblems of our method in Section 4. In Section 5, we present the details of our method and provide a brief sketch of the analysis. We conclude our work in Section 6 and all the details of the proof can be found in Appendix. Moreover, we also provide an inexact version of APFB in Section F in Appendix.
2 Preliminaries

In this paper, we use \( \|A\|_2 \) to denote the spectral norm of \( A \), i.e., the largest singular value of \( A \).

Then we review some standard definitions of strong convexity and smoothness. For a differentiable function \( \varphi : X \rightarrow \mathbb{R} \), \( \varphi \) is said to be \( \ell \)-smooth if its gradient is \( \ell \)-Lipschitz continuous; that is, for any \( x_1, x_2 \in X \), we have
\[
\|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2 \leq \ell \|x_1 - x_2\|_2.
\]
Moreover, \( \varphi \) is said to be \( \mu \)-strongly convex, if for any \( x_1, x_2 \in X \) we have
\[
\varphi(x_2) \geq \varphi(x_1) + \langle \nabla \varphi(x_1), x_2 - x_1 \rangle + \frac{\mu}{2} \|x_2 - x_1\|_2^2.
\]

The following lemma is useful in our analysis.

Lemma 1. Let \( \varphi \) be \( \ell \)-smooth and \( \mu \)-strongly convex on \( \mathbb{R}^d \). Then for all \( x_1, x_2 \in \mathbb{R}^d \), one has
\[
\langle \nabla \varphi(x_1) - \nabla \varphi(x_2), x_1 - x_2 \rangle \geq \frac{\ell \mu}{\ell + \mu} \|x_1 - x_2\|_2^2 + \frac{1}{\ell + \mu} \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2.
\]

In this work, we are interested in the class of bilinear functions \( f : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R} \) of the form
\[
f(x, y) = g(x) + \langle x, Ay \rangle - h(y),
\]
where \( g : \mathbb{R}^{d_x} \rightarrow \mathbb{R} \) is \( L_x \)-smooth and \( \mu_x \)-strongly convex, and \( h : \mathbb{R}^{d_y} \rightarrow \mathbb{R} \) is \( L_y \)-smooth and \( \mu_y \)-strongly convex. And we denote \( \kappa_x = L_x/\mu_x \) and \( \kappa_y = L_y/\mu_y \).

Without loss of generality, we can assume that \( L_x = L_y \). Otherwise, one can rescale the variables and take \( \hat{f}(x, y) = f(x, \sqrt{L_x/L_y} y) \). It is not hard to check that this rescaling will not change condition numbers \( \kappa_x, \kappa_y \) and coupling condition number \( \sqrt{\kappa_x \kappa_y} \).

Let the proximal operator related to \( f \) at point \((x, y)\) be
\[
\text{prox}_f(x, y) = \min_{u} \max_{v} f(u, v) + \frac{1}{2} \|u - x\|_2^2 - \frac{1}{2} \|v - y\|_2^2.
\]
If \( f(x, y) = \langle x, Ay \rangle \), we use \( \text{prox}_A(x, y) \) for simplicity.

The optimal solution of the convex-concave minimax optimization problem \( \min_x \max_y f(x, y) \) is the saddle point \((x^*, y^*)\) defined as follows.

Definition 1. \((x^*, y^*)\) is a saddle point of \( f : X \times Y \rightarrow \mathbb{R} \) if for any \( x \in X \) and \( y \in Y \), there holds
\[
f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*)
\]

For strongly convex-strongly concave functions, it is well known that such a saddle point exists and is unique. Given a tolerance \( \varepsilon > 0 \), our goal is to find an \( \varepsilon \)-saddle point which is defined as follows.

Definition 2. \((\hat{x}, \hat{y})\) is called an \( \varepsilon \)-saddle point of \( f \) if
\[
\|\hat{x} - x^*\|_2 + \|\hat{y} - y^*\|_2 \leq \varepsilon.
\]

In this work, we focus on first-order algorithms which have access to oracle \( H_f \). For an inquiry on any point \((x, y)\), the oracle returns
\[
H_f(x, y) \triangleq \{ \nabla g(x), \nabla h(y), Ay, A^Tx \}.
\]
(2)

Given an initial point \((x_0, y_0)\), at the \( k \)-th iteration, a first-order algorithm calls the oracle on \((x_{k-1}, y_{k-1})\) and then obtains a new point \((x_k, y_k)\). And \( x_k \) and \( y_k \) lie in two different vector spaces:
\[
x_k \in \text{span}\{x_0, \nabla g(x_0), \ldots, \nabla g(x_{k-1}), Ay_0, \ldots, Ay_{k-1}\},
\]
\[
y_k \in \text{span}\{y_0, \ldots, y_k, \nabla h(y_0), \ldots, \nabla h(y_{k-1}), A^Tx_0, \ldots, A^Tx_{k-1}\}.
\]
Table 1: Comparison of gradient complexities to find an $\varepsilon$-saddle point of Problem (1) where $L_x = L_y$, $\kappa_x = L_x/\mu_x$, $\kappa_y = L_y/\mu_y$ and $L = \max\{\|A\|_2, L_x\}$. The notations $\tilde{O}$ and $\tilde{\Omega}$ have ignored some logarithmic factors.

| References              | Gradient Complexity                                                                 |
|-------------------------|-------------------------------------------------------------------------------------|
| Nesterov & Scrima [26]  | $\tilde{O}\left(\frac{\|A\|_2}{\mu_x} + \frac{\|A\|_2}{\mu_y} + \kappa_x + \kappa_y\right)$ |
| Mokhtari et al. [23]    | $\tilde{O}\left(\frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x \kappa_y}\right)$ |
| Lin et al. [19]         | $\tilde{O}\left(\frac{\|A\|_2 L_x}{\mu_x \mu_y} + \sqrt{\kappa_x + \kappa_y}\right)$ |
| Wang & Li [34]          | $\tilde{O}\left(\sqrt{\frac{\|A\|_2 L_y}{\mu_x \mu_y}} + \sqrt{\kappa_x + \kappa_y}\right)$ |
| Chambolle & Pock [4]    | $\tilde{O}\left(\frac{\|A\|_2 L_x}{L_x \mu_y} + \sqrt{\kappa_x \kappa_y}\right)$ |
| Inexact version (Theorem 15) | $\tilde{O}\left(\frac{\|A\|_2 L_x}{L_x \mu_y} + \sqrt{\kappa_x \kappa_y}\right)$ |
| This paper (Theorem 9)  | $\tilde{O}\left(\frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \frac{\sqrt{\kappa_x \kappa_y}}{\kappa_x + \kappa_y}\right)$ |
| Lower bound             | $\tilde{\Omega}\left(\frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x + \kappa_y}\right)$ |

3 Related Work

There are many algorithms designed for the convex-concave saddle point problems, including extragradient (EG) algorithm [17, 33, 22, 23], reflected gradient descent ascent [19, 20, 36], optimistic gradient descent ascent (OGDA) [9, 22, 23], and other variants [28, 29, 21].

The bilinear case has also been studied extensively [24, 4, 13]. And, Kolossoski & Monteiro [16] introduced convergence results even when the feasible space is non-Euclidean and Chen et al. [5, 6] proposed optimal algorithms for solving a special class of stochastic saddle point problems.

For a class of matrix games where $g, h$ are zero functions, Carmon et al. [2, 3] developed variance reduction and Coordinate methods to solve the games.

For the strongly convex-strongly concave minimax problems, Tseng [33], Nesterov & Scrima [26] provided upper bounds based on a variational inequality. Moreover, Gidel et al. [11], Mokhtari et al. [23] derived upper bounds for the OGDA algorithm. Recently, Lin et al. [19], Wang & Li [34] proposed algorithms based on the approximately proximal technique and improved the upper bounds. On the other hand, Ibrahim et al. [14], Zhang et al. [40] established a lower complexity bound among all the first-order algorithms. We provide a comparison between our results and existing results in the literature in Table 1.

4 Algorithm Components

In this section, we present two main algorithm components. Both of them are crucial for our algorithms.
4.1 Nesterov’s Accelerated Gradient Descent

We present a version of Nesterov’s Accelerated Gradient Descent (AGD) in Algorithm 1 which is widely-used to minimize an $\ell$-smooth and $\mu$-strongly convex function $f$ \cite{Nesterov2018}. Moreover, AGD is shown to be optimal among all the first-order algorithms for smooth and strongly convex optimization.

Algorithm 1 AGD
1: Input: function $\varphi$, initial point $x_0$, smoothness $\ell$, strongly convex module $\mu$, run-time $T$.
2: Initialize: $\tilde{x}_0 = x_0$, $\eta = 1/\ell$, $\kappa = \ell/\mu$ and $\theta = \sqrt{\kappa - 1}/\sqrt{\kappa + 1}$.
3: for $k = 1, \ldots, T$ do
4: $x_k = \tilde{x}_{k-1} - \eta \nabla \varphi(\tilde{x}_{k-1})$.
5: $\tilde{x}_k = x_k + \theta(x_k - x_{k-1})$.
6: end for
7: Output: $x_T$.

The following theorem provides the convergence rate of AGD.

Theorem 2. Assume that $\varphi$ is $\ell$-smooth and $\mu$-strongly convex. Then the output of Algorithm 1 satisfies

$$\varphi(x_T) - \varphi(x^*) \leq \frac{\ell + \mu}{2} \|x_0 - x^*\|_2^2 \exp \left( - \frac{T}{\sqrt{\kappa}} \right),$$

where $\kappa = \ell/\mu$ is the condition number, and $x^*$ is the unique global minimum of $\varphi$.

A standard analysis of Theorem 2 based on estimating sequence can be found in \cite{Nesterov2018}. AGD will be used as a basic component for acceleration in this paper.

4.2 Accelerated Proximal Forward-Backward Algorithm

The Accelerated Proximal Forward-Backward Algorithm (APFB, Algorithm 2) is proposed by Chambolle & Pock \cite{Chambolle2011} which is an optimal method when the proximal oracle of $g, h$ is available \cite{Pock2016}. APFB takes the alternating order of updating $x$ and $y$ and employs momentum steps as well as AGD which yields acceleration.

Algorithm 2 APFB
1: Input: function $g, h$, coupling matrix $A$, initial point $x_0, y_0$, strongly convex module $\mu_x, \mu_y$, run-time $T$.
2: Initialize: $\tilde{x}_0 = x_0$, $\gamma = 1/\|A\|_2 \sqrt{\mu_x/\mu_y}$, $\sigma = 1/\|A\|_2 \sqrt{\mu_y/\mu_x}$ and $\theta = \|A\|_2/\sqrt{\mu_x \mu_y + \|A\|_2}$.
3: for $k = 1, \ldots, T$ do
4: $y_k = \arg \min_v h(v) + \frac{1}{2\gamma} \|v - y_{k-1} - \sigma A^\top \tilde{x}_{k-1}\|_2^2$.
5: $x_k = \arg \min_u g(u) + \frac{1}{2\gamma} \|u - x_{k-1} + \gamma A y_k\|_2^2$.
6: $\tilde{x}_k = x_k + \theta(x_k - x_{k-1})$.
7: end for
8: Output: $x_T, y_T$.

A theoretical guarantee for the APFB algorithm is presented in the following theorem. The proof of Theorem 3 can be found in Appendix Section B.

5
Theorem 3. Assume that $g$ is $\mu_x$-strongly convex and $h$ is $\mu_y$-strongly convex. Then the output of Algorithm 2 satisfies

$$
\mu_x \|x_T - x^*\|_2^2 + \mu_y \|y_T - y^*\|_2^2 \leq \theta^{T-1} \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} \left( \mu_x \|x_0 - x^*\|_2^2 + \mu_y \|y_0 - y^*\|_2 \right),
$$

where $x^*, y^*$ is the unique saddle point of $f(x, y) = g(x) + \langle x, Ay \rangle - h(y)$ and $\theta = \frac{\|A\|_2}{\sqrt{\mu_x \mu_y} \|A\|_2}$.

5 Methodology

In this section, we first consider balanced cases where $\kappa_x = \kappa_y$. We introduce a Double Proximal Point Algorithm (DPPA, Algorithm 3) with assuming that each subproblem can be solved exactly. Then we present an inexact version of DPPA as Double Inexact Proximal Point Algorithm (DIPPA, Algorithm 4) with solving the subproblems iteratively and show its theoretical guarantee for solving balanced bilinear saddle point problems. At last, we apply Catalyst framework with DIPPA to solve unbalanced bilinear saddle point problems.

5.1 Double Proximal Point Algorithm for Balanced Cases

We first consider balanced cases where $\mu_x = \mu_y$. This implies $\kappa_x = \kappa_y$. Our method is inspired from the algorithm Hermitian and skew-Hermitian splitting (HSS) which is designed for the non-Hermitian positive definite system of linear equations. We present the DPPA algorithm for balanced cases in Algorithm 3. DPPA split the function $f$ into two parts $f = f_1 + f_2$ where $f_1(x, y) = g(x) - h(y)$ and $f_2(x, y) = \langle x, Ay \rangle$. Moreover there are two proximal steps at each iteration of DPPA: Line 6 performs a proximal step with respect to the function $f_1$, while Line 7 performs another proximal step related to the function $f_2$.

Algorithm 3 DPPA for balanced cases

1: **Input:** function $g, h$, coupling matrix $A$, initial point $(x_0, y_0)$, smoothness $L$, strongly convex module $\mu$, run-time $K$.
2: **Initialize:** $\alpha = 1/\sqrt{L\mu}$.
3: for $k = 1, \ldots, K$ do
4:   $z_k = x_{k-1} - \alpha A y_{k-1}$.
5:   $w_k = y_{k-1} + \alpha A^T x_{k-1}$.
6:   $(\tilde{x}_k, \tilde{y}_k) = \text{prox}_{\alpha(g - h)}(z_k, w_k)$.
7:   $(x_k, y_k) = \text{prox}_{\alpha A}(2\tilde{x}_k - z_k, 2\tilde{y}_k - w_k)$.
8: end for
9: **Output:** $x_K, y_K$.

The theoretical guarantee for the algorithm DPPA in balanced cases is given in the following theorem.

Theorem 4. Assume that $g, h$ are both $L$-smooth and $\mu$-strongly convex. Denote $(x^*, y^*)$ to be the saddle point of the function $f(x, y) = g(x) + x^T Ay - h(y)$. Then the sequence $\{z_k, w_k\}_{k \geq 1}$ in Algorithm 3 satisfies

$$
\|z_{k+1} - z^*\|_2^2 + \|w_{k+1} - w^*\|_2^2 \leq \eta \left( \|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2 \right)
$$
where \( \eta = \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2 \), \( \kappa = L/\mu \) and \( z^* = x^* - \alpha A y^* \), \( w^* = y^* + \alpha A^T x^* \).

Using the Theorem 4, we directly obtain that \( \| x_k - x^* \|_2^2 + \| y_k - y^* \|_2 \) converges to 0 linearly. We present our result in Corollary 5.

Corollary 5. Based on the same notations and assumptions of Theorem 4, the output of Algorithm 4 satisfies

\[
\| x_K - x^* \|_2^2 + \| y_K - y^* \|_2 \leq \frac{\| A \|_2^2 + L \mu}{L \mu} \left( \| x_0 - x^* \|_2^2 + \| y_0 - y^* \|_2 \right) \exp \left( -\frac{2K}{\sqrt{\kappa}} \right).
\]

Proof. Firstly, with \( \alpha = \frac{1}{\sqrt{L \mu}} \), we note that

\[
\| z_k - z^* \|_2^2 + \| w_k - w^* \|_2^2 = \| x_k - x^* \|_2^2 + \| y_k - y^* \|_2 + \alpha^2 \left( \| A^T (x_k - x^*) \|_2^2 + \| A (y_k - y^*) \|_2^2 \right)
\]

\[
\leq \left( 1 + \frac{\| A \|_2^2}{L \mu} \right) \left( \| x_k - x^* \|_2^2 + \| y_k - y^* \|_2 \right).
\]

Therefore, we can conclude that

\[
\| x_T - x^* \|_2^2 + \| y_T - y^* \|_2 \leq \| z_{T+1} - z^* \|_2^2 + \| w_{T+1} - w^* \|_2 \leq \eta^T \left( \| z_1 - z^* \|_2^2 + \| w_1 - w^* \|_2 \right)
\]

\[
\leq \exp \left( -\frac{2 T \mu}{\sqrt{\kappa}} \right) \frac{\| A \|_2^2 + L \mu}{L \mu} \left( \| x_0 - x^* \|_2^2 + \| y_0 - y^* \|_2 \right),
\]

where we have used that \( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \leq 1 - \frac{1}{\sqrt{\kappa}} \leq \exp \left( -\frac{1}{\sqrt{\kappa}} \right). \]

5.2 Double Inexact Proximal Point Algorithm for Balanced Cases

We provide the details of Double Inexact Proximal Point Algorithm (DIPPA) for balanced cases in Algorithm 4. We iteratively solve the two proximal steps in DPPA. More specifically, we may employ AGD to approximately find the proximal point of the function \( f_1(x, y) = g(x) - h(y) \) where the variables \( x \) and \( y \) are completely decoupled. The second proximal point subproblem \( [3] \) can be solved by APFB since the proximal operator of \( \tilde{g}_k \) is easy to obtain. We note that subproblem \( [3] \) is quadratic, and finding the saddle point of \( \tilde{f}_k \) is equivalent to solving the system of linear equations with coefficient matrix \( \begin{bmatrix} I & -\alpha A^T \\ -\alpha A & I \end{bmatrix} \), which also can be solved by some Krylov subspace methods \([12] [7]\).

The convergence rate of the algorithm DIPPA in balanced cases is provided in the following theorem.

**Theorem 6.** Assume that \( g, h \) are both \( L \)-smooth and \( \mu \)-strongly convex. Denote \( (x^*, y^*) \) is the saddle point of the function \( f = g(x) + x^T A y - h(y) \). Set

\[
\rho = \frac{1}{2 \sqrt{\kappa}}, \quad C_0 = \| x_0 - x^* \|_2^2 + \| y_0 - y^* \|_2^2,
\]

\[
\varepsilon_k = \frac{C_0 \mu}{16} (1 - \rho)^{k+1}, \quad \delta_k = \frac{C_0 L \mu}{2(1 + \sqrt{\kappa})(L \mu + \| A \|_2^2)} (1 - \rho)^{k+1}.
\]
Algorithm 4 DIPPA for balanced cases

1: **Input:** function $g,h$, coupling matrix $A$, initial point $(x_0,y_0)$, smoothness $L$, strongly convex module $\mu$, run-time $K$, tolerance sequences $\{\varepsilon_k\}_{k \geq 1}$ and $\{\delta_k\}_{k \geq 1}$.

2: **Initialize:** $\alpha = 1/\sqrt{L\mu}$.

3: for $k = 1, \cdots, K$ do

4: $z_k = x_{k-1} - \alpha A y_{k-1}$.

5: $w_k = y_{k-1} + \alpha A^T x_{k-1}$.

6: Let $G_k(x) = g(x) + \frac{1}{2\alpha} \| x - z_k \|^2_2$.

7: Find $\hat{x}_k$ such that $G_k(\hat{x}_k) - \min_x G_k(x) \leq \varepsilon_k$.

8: Let $H_k(y) = h(y) + \frac{1}{2\alpha} \| y - w_k \|^2_2$.

9: Find $\hat{y}_k$ such that $H_k(\hat{y}_k) - \min_y H_k(y) \leq \varepsilon_k$.

10: Obtain $(x_k, y_k)$ to be $\delta_k$-saddle point of the following problem

$$\min_{x} \max_{y} \hat{f}_k(x, y) = \hat{g}_k(x) + \langle x, Ay \rangle - \hat{h}_k(y)$$

$$\triangleq \frac{1}{2\alpha} \| x - 2x_k + z_k \|^2_2 + \langle x, Ay \rangle - \frac{1}{2\alpha} \| y - 2y_k + w_k \|^2_2.$$ 

11: **end for**

12: **Output:** $x_K, y_K$.

Then the output of Algorithm 4 satisfies

$$\| x_K - x^* \|^2_2 + \| y_K - y^* \|^2_2 \leq C(1 - \rho)^K \left( \| x_0 - x^* \|^2_2 + \| y_0 - y^* \|^2_2 \right),$$

where $C = 4\sqrt{\kappa} + 1 + \frac{\|A\|^2}{L\mu}$ and $\kappa = L/\mu$.

Then we upper bound the complexity of solving subproblems to analyze the total complexity of DIPPA.

**Lemma 7.** Consider the same assumption and the same definitions of $\rho$, $\varepsilon_k$, $\delta_k$ and $C$ in Theorem 6. In order to find $\varepsilon_k$-optimal points $x_k$ ($\hat{y}_k$) of $G_k$ ($H_k$), we need to run AGD $K_1$ steps, where

$$K_1 = \left\lfloor \sqrt{\kappa} \log \left( \frac{32C(\sqrt{L} + \sqrt{\mu})^2}{\mu(1 - \rho)} \right) \right\rfloor + 1.$$

And in order to obtain $\delta_k$-saddle point $(x_k, y_k)$ of $\hat{f}_k$, we need to run APFB $K_2$ steps, where

$$K_2 = \left\lfloor \left( \frac{\|A\|^2}{\sqrt{L\mu}} + 1 \right) \log \left( \frac{20C(1 + \sqrt{\kappa})(L\mu + \|A\|^2)}{L\mu(1 - \rho)} \right) \right\rfloor + 2.$$

Now we can provide the upper bound of total complexity of Algorithm 4 for solving balanced bilinear saddle point problems.

**Theorem 8.** The total queries to Oracle $\mathcal{A}$ needed by Algorithm 4 to produce $\varepsilon$-saddle point of $f$ is at most

$$\tilde{O} \left( \left( \frac{\|A\|^2}{\mu} + \kappa^{3/4} \right) \log \left( \frac{\|x_0 - x^*\|^2_2 + \|y_0 - y^*\|^2_2}{\varepsilon} \right) \right).$$
where the notation $\tilde{O}$ have omitted some logarithmic factors depending on $\kappa$ and $\frac{\|A\|_2^2}{L\mu}$. 

**Proof.** By Theorem 6, in order to produce $\varepsilon$-saddle point of $f$, we only need to run DIPPA $K$ steps, where

$$K = \left\lfloor 2\sqrt{\kappa} \log \left( \frac{C(\|x_0 - x^*\|_2^2 + \|y_0 - y^*\|_2^2)}{\varepsilon} \right) \right\rfloor + 1.$$ 

Together with Lemma 7, the total complexity is upper bounded by

$$K(2K_1 + 2) = \tilde{O} \left( \sqrt{\kappa} \left( \frac{\|A\|_2^2}{\sqrt{L\mu}} + \sqrt{\kappa} \right) \log \left( \frac{\|x_0 - x^*\|_2^2 + \|y_0 - y^*\|_2^2}{\varepsilon} \right) \right).$$

\[\blacksquare\]

### 5.3 Catalyst-DIPPA for Unbalanced Cases

Catalyst [18, 37] is a successful framework to accelerate existing first-order algorithms. We present the details of Catalyst-DIPPA in Algorithm 5. The idea is to repeatedly solve the following auxiliary balanced saddle point problems using DIPPA:

$$\min_x \max_y f_k(x, y) \triangleq f(x, y) + \frac{\beta}{2} \|x - \tilde{x}_k\|_2^2,$$

where $\beta = \frac{L_x(\mu_y - \mu_x)}{L_x - \mu_y}$. We remark that the function $f_k$ is balanced: the condition number corresponding to $y$ is $\kappa_y$ and the condition number related to $x$ is

$$\frac{L_x + \beta}{\mu_x} = \frac{L_x(L_x - \mu_y) + L_x(\mu_y - \mu_x)}{\mu_x(L_x - \mu_y) + L_x(\mu_y - \mu_x)} = \kappa_y,$$

where we have recalled that $L_x = L_y$. With the rescaling technique, we can apply DIPPA to solve the following saddle point problem

$$\min_x \max_y \hat{f}_k(x, y) \triangleq \hat{f}_k \left( \sqrt{\frac{L_x + \beta}{L_x}} x, y \right).$$

Note that the coupling matrix of $\hat{f}_k$ is $\sqrt{\frac{L_x + \beta}{L_x}} A$. So the total gradient complexity of Catalyst-DIPPA is

$$\tilde{O} \left( \sqrt{\frac{\mu_x + \beta}{\mu_x}} \right) \tilde{O} \left( \sqrt{\frac{L_x}{L_x + \beta}} \frac{\|A\|_2^2}{\|A\|_2^2 + \kappa_y^{3/4}} \right)$$

$$= \tilde{O} \left( \frac{\|A\|_2^2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x \kappa_y} \right).$$

We formally state the convergence rate in the following theorem.

**Theorem 9.** Assume that $g(x)$ is $L_x$-smooth and $\mu_x$-strongly convex, $h(y)$ is $L_y$-smooth and $\mu_y$-strongly convex and $L_x = L_y$. The total queries to Oracle 3 needed by Algorithm 5 to produce $\varepsilon$-saddle point of $f(x, y) = g(x) + \langle x, A y \rangle - h(y)$ is at most

$$\tilde{O} \left( \left( \frac{\|A\|_2^2}{\sqrt{\mu_x \mu_y}} + \sqrt{\kappa_x \kappa_y (\kappa_x + \kappa_y)} \right) \log \left( \frac{1}{\varepsilon} \right) \right),$$

where $\kappa_x = L_x / \mu_x$, $\kappa_y = L_y / \mu_y$ and the notation $\tilde{O}$ have omitted some logarithmic factors depending on $\kappa_x$, $\kappa_y$ and $\frac{\|A\|_2^2}{\sqrt{\mu_x \mu_y}}$.  

9
Algorithm 5 Catalyst-DIPPA for unbalanced cases

1: **Input:** function \( f \), initial point \((x_0, y_0)\), smoothness \( L_x = L_y \), strongly convex module \( \mu_x < \mu_y \), run-time \( K \), accuracy sequence \( \{\varepsilon_k\}_{k \geq 1} \).

2: **Initialize:** \( \beta = \frac{L_x(\mu_y - \mu_x)}{L_x - \mu_y} \), \( q = \frac{\mu_x}{\mu_x + \beta} \), \( \theta = \frac{1 - \sqrt{q}}{1 + \sqrt{q}} \) and \( \hat{x}_0 = x_0 \).

3: **for** \( k = 1, \ldots, K \) **do**

4: Let \( f_k(x, y) = f(x, y) + \frac{\beta}{2} \| x - \hat{x}_k \|_2^2 \).

5: Obtain \((x_k, y_k)\) to be \( \varepsilon_k \)-saddle point of \( f_k \) by applying DIPPA.

6: \( \hat{x}_k = x_k + \theta(x_k - x_{k-1}) \).

7: **end for**

8: **Output:** \( x_K, y_K \).

6 Conclusion

In this paper, we have proposed a novel algorithm DIPPA to solve bilinear saddle point problems. Our method does not need any additional information about proximal operation of \( g, h \) and achieves a tight dependency on the coupling condition number. There is still a gap between the upper bounds and lower bounds of first-order algorithms for solving bilinear saddle point problems. We wish our technique can be used in a more general case other than the bilinear case.

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A Technique Lemmas

We first present some equivalent statements of the definition of smoothness.

**Lemma 10.** Let $\varphi$ be convex on $\mathbb{R}^d$. Then following conditions below, holding for all $x_1, x_2 \in \mathbb{R}^d$, are equivalent:

1. $\|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2 \leq \ell \|x_1 - x_2\|_2$,
2. $\varphi(x_2) - \varphi(x_1) - \langle \nabla \varphi(x_1), x_2 - x_1 \rangle \leq \frac{\ell}{2} \|x_1 - x_2\|_2^2$,
3. $\langle \nabla \varphi(x_1) - \nabla \varphi(x_2), x_1 - x_2 \rangle \geq \frac{1}{\ell} \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2$.

**Proof.**

(i) $\Rightarrow$ (ii): Just note that

$$\varphi(x_2) - \varphi(x_1) - \langle \nabla \varphi(x_1), x_2 - x_1 \rangle = \int_0^1 \langle \nabla \varphi(x_1 + t(x_2 - x_1)) - \nabla \varphi(x_1), x_2 - x_1 \rangle \, dt$$

$$\leq \int_0^1 \ell t \|x_2 - x_1\|_2^2 \, dt = \frac{\ell}{2} \|x_2 - x_1\|_2^2,$$

where the inequality follows from (i) and Cauchy–Schwarz inequality.

(ii) $\Rightarrow$ (iii): Consider the function $\psi(x) = \varphi(x) - \langle \nabla \varphi(x_1), x \rangle$ defined on $\mathbb{R}^d$. It is easy to check that $\psi$ is convex and satisfies condition (ii). Furthermore, the optimal point of $\psi$ is $x_1$, which implies

$$\psi(x_1) = \min_{x \in \mathbb{R}^d} \psi(x) \leq \min_{x \in \mathbb{R}^d} \left\{ \psi(x_2) + \langle \nabla \psi(x_2), x - x_2 \rangle + \frac{\ell}{2} \|x - x_2\|_2^2 \right\}$$

$$= \psi(x_2) - \frac{1}{2\ell} \|\nabla \psi(x_2)\|_2^2,$$

where the optimal point of the second problem is $x_2 - \frac{1}{\ell} \nabla \varphi(x_2)$.

Following from the definition of $\psi$ and Equation (4), we have

$$\varphi(x_1) - \langle \nabla \varphi(x_1), x_1 \rangle \leq \varphi(x_2) - \langle \nabla \varphi(x_1), x_2 \rangle - \frac{1}{2\ell} \|\nabla \varphi(x_2) - \nabla \varphi(x_1)\|_2^2, \quad \text{i.e.,}$$

$$\varphi(x_2) - \varphi(x_1) - \langle \nabla \varphi(x_1), x_2 - x_1 \rangle \geq \frac{1}{2\ell} \|\nabla \varphi(x_2) - \nabla \varphi(x_1)\|_2^2. \quad (5)$$

Similarly, there also holds

$$\varphi(x_1) - \varphi(x_2) - \langle \nabla \varphi(x_1), x_1 - x_2 \rangle \geq \frac{1}{2\ell} \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2. \quad (6)$$

Adding both sides of Equation (5) and (6) together, we know that $\varphi$ satisfies condition (iii).

(iii) $\Rightarrow$ (i): By Cauchy–Schwarz inequality and (ii), we have

$$\|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2 \leq \ell \langle \nabla \varphi(x_1) - \nabla \varphi(x_2), x_1 - x_2 \rangle$$

$$\leq \ell \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2 \|x_1 - x_2\|_2,$$

which is our desired result. 

Now, we are ready to prove Lemma 1.
Proof of Lemma 7 Consider function \( \psi(x) = \varphi(x) - \frac{\mu}{2} \|x\|_2^2 \). By \( \mu \)-strongly convexity of \( \varphi \), for any \( x_1, x_2 \in \mathbb{R}^d \), we know that

\[
\psi(x_2) - \psi(x_1) - \langle \nabla \psi(x_1), x_2 - x_1 \rangle = \varphi(x_2) - \varphi(x_1) - \langle \nabla \varphi(x_1), x_2 - x_1 \rangle - \frac{\mu}{2} (\|x_2\|_2^2 - \|x_1\|_2^2 - 2 \langle x_1, x_2 - x_1 \rangle)
\]

which implies that \( \psi \) is convex.

On the other hand, by \( \ell \)-smoothness of \( \varphi \) and Condition (iii) in Lemma 10 there holds

\[
\psi(x_2) - \psi(x_1) - \langle \nabla \psi(x_1), x_2 - x_1 \rangle = \varphi(x_2) - \varphi(x_1) - \langle \nabla \varphi(x_1), x_2 - x_1 \rangle - \frac{\mu}{2} \|x_2 - x_1\|_2^2 \\
\leq \frac{\ell - \mu}{2} \|x_2 - x_1\|_2^2,
\]

which implies that \( \psi \) is \((\ell - \mu)\)-smooth. Consequently, following from Condition (iii) in Lemma 10 we have

\[
\langle \nabla \psi(x_1) - \nabla \psi(x_2), x_1 - x_2 \rangle \geq \frac{1}{\ell - \mu} \|\nabla \psi(x_1) - \nabla \psi(x_2)\|_2^2,
\]

i.e.,

\[
\langle \nabla \varphi(x_1) - \nabla \varphi(x_2), x_1 - x_2 \rangle - \mu \|x_1 - x_2\|_2^2 \geq \frac{1}{\ell - \mu} \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2 - \mu \|x_1 - x_2\|_2^2.
\]

By rearranging above inequality, we get that

\[
\left(1 + \frac{2\mu}{\ell - \mu}\right) \langle \nabla \varphi(x_1) - \nabla \varphi(x_2), x_1 - x_2 \rangle \geq \left(\mu + \frac{\mu^2}{\ell - \mu}\right) \|x_1 - x_2\|_2^2 + \frac{1}{\ell - \mu} \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2,
\]

that is

\[
\frac{\ell + \mu}{\ell - \mu} \langle \nabla \varphi(x_1) - \nabla \varphi(x_2), x_1 - x_2 \rangle \geq \frac{\ell \mu}{\ell - \mu} \|x_1 - x_2\|_2^2 + \frac{1}{\ell - \mu} \|\nabla \varphi(x_1) - \nabla \varphi(x_2)\|_2^2.
\]

We then show Lipschitz continuity of the proximal operator with respect to strongly convex functions.

Lemma 11. Let \( \varphi \) be convex on \( \mathcal{X} \). For all \( x_1, x_2 \in \mathcal{X} \), define

\[
u_i = \arg\min_{u \in \mathcal{X}} \varphi(u) + \frac{1}{2} \|u - x_i\|_2, \quad i = 1, 2.
\]

Then there holds

\[\|u_1 - u_2\|_2 \leq \|x_1 - x_2\|_2.\]

Proof. By strongly convexity of the functions \( \Phi_i(u) \triangleq \varphi(u) + \frac{1}{2} \|u - x_i\|_2^2 \), we have

\[
\varphi(u_2) + \frac{1}{2} \|u_2 - x_1\|_2^2 \geq \varphi(u_1) + \frac{1}{2} \|u_1 - x_1\|_2^2 + \frac{1}{2} \|u_1 - u_2\|_2^2,
\]

\[
\varphi(u_1) + \frac{1}{2} \|u_1 - x_2\|_2^2 \geq \varphi(u_2) + \frac{1}{2} \|u_2 - x_2\|_2^2 + \frac{1}{2} \|u_1 - u_2\|_2^2.
\]
With adding both side of above two inequalities, we obtain that
\[
\frac{1}{2} \| u_2 - x_1 \|^2 + \frac{1}{2} \| u_1 - x_2 \|^2 \geq \frac{1}{2} \| u_1 - x_1 \|^2 + \frac{1}{2} \| u_2 - x_2 \|^2 + \| u_1 - u_2 \|^2, \quad \text{i.e.,}
\]
\[- \langle u_2, x_1 \rangle - \langle u_1, x_2 \rangle \geq - \langle u_1, x_1 \rangle - \langle u_2, x_2 \rangle + \| u_1 - u_2 \|^2, \quad \text{i.e.,}
\]
\[
\langle u_1 - u_2, x_1 - x_2 \rangle \geq \| u_1 - u_2 \|^2.
\]
Then following from Cauchy–Schwarz inequality, there holds
\[
\| u_1 - u_2 \|^2 \leq \| u_1 - u_2 \| \| x_1 - x_2 \|,
\]
which implies that
\[
\| u_1 - u_2 \| \leq \| x_1 - x_2 \|.
\]

\[\square\]

\section{B Proof of Theorem 3}

\textit{Proof.} Note that
\[
y_k = \arg \min_y H_k(y) \triangleq h(y) + \frac{1}{2\sigma} \| y - y_{k-1} - \sigma A^\top x_{k-1} \|^2_2.
\]
By \((\mu_y + 1/\sigma)\)-strongly convexity of \(H_k\), we know that
\[
h(y^*) + \frac{1}{2\sigma} \| y^* - y_{k-1} - \sigma A^\top x_{k-1} \|^2_2 \geq h(y_k) + \frac{1}{2\sigma} \| y_k - y_{k-1} - \sigma A^\top x_{k-1} \|^2_2 + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \| y_k - y^* \|^2_2,
\]
that is
\[
h(y^*) + \frac{1}{2\sigma} \| y_k - y_{k-1} - y^* \|^2_2 + \langle y_k - y^*, A^\top x_{k-1} \rangle \geq h(y_k) + \frac{1}{2\sigma} \| y_k - y_{k-1} \|^2_2 + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \| y_k - y^* \|^2_2. \tag{7}
\]
Similarly, by
\[
x_k = \arg \min_x g(x) + \frac{1}{2\gamma} \| x - x_{k-1} + \gamma A y_k \|^2_2,
\]
we have
\[
g(x^*) + \frac{1}{2\gamma} \| x^* - x_{k-1} + \gamma A y_k \|^2_2 \geq g(x_k) + \frac{1}{2\gamma} \| x_k - x_{k-1} + \gamma A y_k \|^2_2 + \left( \frac{\mu_x}{2} + \frac{1}{2\gamma} \right) \| x_k - x^* \|^2_2,
\]
which implies
\[
g(x^*) + \frac{1}{2\gamma} \| x_{k-1} - x^* \|^2_2 - \langle x_k - x^*, A y_k \rangle \geq g(x_k) + \frac{1}{2\gamma} \| x_k - x_{k-1} \|^2_2 + \left( \frac{\mu_x}{2} + \frac{1}{2\gamma} \right) \| x_k - x^* \|^2_2. \tag{8}
\]
Then we add both sides of the inequalities (7) and (8). Thus we have

\[
\frac{1}{2\gamma} \|x_{k-1} - x^*\|_2^2 + \frac{1}{2\sigma} \|y_{k-1} - y^*\|_2^2 \\
\geq \left(\frac{\mu_x}{2} + \frac{1}{2\gamma}\right) \|x_k - x^*\|_2^2 + \left(\frac{\mu_y}{2} + \frac{1}{2\sigma}\right) \|y_k - y^*\|_2^2 \\
+ \frac{1}{2\gamma} \|x_k - x_{k-1}\|_2^2 + \frac{1}{2\sigma} \|y_k - y_{k-1}\|_2^2 \\
+ g(x_k) + h(y_k) - g(x^*) - h(y^*) + \langle x_k - x^*, Ay_k \rangle - \langle y_k - y^*, A^T \bar{x}_{k-1} \rangle.
\]

Observe that

\[
f(x_k, y^*) - f(x^*, y_k) = g(x_k) + \langle x_k, Ay^* \rangle - h(y^*) - g(x^*) - \langle x^*, Ay_k \rangle + h(y_k).
\]

Plugging Equality (10) into Inequality (9), we have

\[
\frac{1}{2\gamma} \|x_{k-1} - x^*\|_2^2 + \frac{1}{2\sigma} \|y_{k-1} - y^*\|_2^2 \\
\geq \left(\frac{\mu_x}{2} + \frac{1}{2\gamma}\right) \|x_k - x^*\|_2^2 + \left(\frac{\mu_y}{2} + \frac{1}{2\sigma}\right) \|y_k - y^*\|_2^2 \\
+ \frac{1}{2\gamma} \|x_k - x_{k-1}\|_2^2 + \frac{1}{2\sigma} \|y_k - y_{k-1}\|_2^2 \\
+ f(x_k, y^*) - f(x^*, y_k) + \langle x_k - \bar{x}_{k-1}, A(y_k - y^*) \rangle.
\]

With recalling the definition of \(\bar{x}_{k-1}\), the last term of Inequality (11) can be rewritten as

\[
\langle x_k - x^*, A(y_k - y^*) \rangle = \langle x_k - x_{k-1} - \theta(x_{k-1} - x_k - 2), A(y_k - y^*) \rangle \\
= \langle x_k - x_{k-1}, A(y_k - y^*) \rangle - \theta \langle x_{k-1} - x_k - 2, A(y_k - y^*) \rangle - \theta \langle x_{k-1} - x_k - 2, A(y_k - y_{k-1}) \rangle.
\]

Furthermore, we have

\[
- \theta \langle x_{k-1} - x_k - 2, A(y_k - y_{k-1}) \rangle \\
\geq -\theta \|A\|_2 \|x_{k-1} - x_k - 2\|_2 \|y_k - y_{k-1}\|_2 \\
\geq -\frac{\theta}{2} \|A\|_2 \sqrt{\frac{\mu_x}{\mu_y}} \|x_{k-1} - x_k - 2\|_2^2 - \frac{\theta}{2} \|A\|_2 \sqrt{\frac{\mu_y}{\mu_x}} \|y_k - y_{k-1}\|_2^2 \\
\geq -\frac{\theta}{2\gamma} \|x_{k-1} - x_k - 2\|_2^2 - \frac{1}{2\sigma} \|y_k - y_{k-1}\|_2^2,
\]

where we have recalled that \(\gamma = \frac{1}{\|A\|_2} \sqrt{\frac{\mu_x}{\mu_y}}\), \(\sigma = \frac{1}{\|A\|_2} \sqrt{\frac{\mu_y}{\mu_x}}\) and \(\theta < 1\). Similarly, there also holds

\[
\langle x_k - x_{k-1}, A(y_k - y^*) \rangle \geq -\frac{1}{2\gamma} \|x_k - x_{k-1}\|_2^2 - \frac{1}{2\sigma} \|y_k - y^*\|_2^2.
\]

Plugging Equation (12) and (13) into Inequality (11), we know that

\[
\frac{1}{2\gamma} \|x_{k-1} - x^*\|_2^2 + \frac{1}{2\sigma} \|y_{k-1} - y^*\|_2^2 + \frac{\theta}{2\gamma} \|x_{k-1} - x_k - 2\|_2^2 + \theta \langle x_{k-1} - x_k - 2, A(y_k - y^*) \rangle \\
\geq \left(\frac{\mu_x}{2} + \frac{1}{2\gamma}\right) \|x_k - x^*\|_2^2 + \left(\frac{\mu_y}{2} + \frac{1}{2\sigma}\right) \|y_k - y^*\|_2^2 + \frac{1}{2\gamma} \|x_k - x_{k-1}\|_2^2 \\
+ f(x_k, y^*) - f(x^*, y_k) + \langle x_k - x_{k-1}, A(y_k - y^*) \rangle.
\]
Therefore we have
\[
\frac{\mu_x}{2} \| x_k - x^* \|^2_2 + \frac{\mu_y}{2} \| y_k - y^* \|^2_2 \\
\leq \left( \frac{\mu_x}{2} + \frac{1}{2\gamma} \right) \| x_k - x^* \|^2_2 + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \| y_k - y^* \|^2_2 + \frac{1}{2\gamma} \| x_k - x_{k-1} \|^2_2 + \langle x_k - x_{k-1}, A(y_k - y^*) \rangle \\
\leq \theta^k \left( \frac{\mu_x}{2} + \frac{1}{2\gamma} \| x_0 - x^* \|^2_2 + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \| y_0 - y^* \|^2_2 \right) \\
= \frac{\| A \|^2}{2} \theta^{k-1} \left( \sqrt{\frac{\mu_x}{\mu_y} \| x_0 - x^* \|^2_2} + \sqrt{\frac{\mu_y}{\mu_x} \| y_0 - y^* \|^2_2} \right).
\]

\[\Box\]

C Proof of Theorem 4

Proof. Since \((x^*, y^*)\) is the saddle point of \(f\), there holds \(\nabla f(x, y) = 0\), that is
\[
\nabla g(x^*) + Ay^* = 0, \quad \nabla h(y^*) - A^T x^* = 0.
\]

Note that \((x_k, y_k) = \text{prox}_{\alpha A}(2\tilde{x}_k - z_k, 2\tilde{y}_k - w_k)\) which implies that
\[
(x_k, y_k) = \arg\min_x \max_y \tilde{f}_k(x, y) \triangleq \frac{1}{2} \| x - 2\tilde{x}_k + z_k \|^2_2 - \frac{1}{2} \| y - 2\tilde{y}_k + w_k \|^2_2 + \langle x, Ay \rangle.
\]

Hence, we have \(\nabla \tilde{f}_k(x_k, y_k) = 0\), that is
\[
x_k - (2\tilde{x}_k - z_k) + \alpha Ay_k = 0, \\
y_k - (2\tilde{y}_k - w_k) - \alpha A^T x_k = 0.
\]

(16)

Similarly, according to \((\tilde{x}_k, \tilde{y}_k) = \text{prox}_{\alpha(g-h)}(z_k, w_k)\), we have
\[
\alpha \nabla g(\tilde{x}_k) + \tilde{x}_k - z_k = 0, \\
\alpha \nabla h(\tilde{y}_k) + \tilde{y}_k - w_k = 0.
\]

(17)

Therefore, we can conclude that
\[
\| z_{k+1} - z^* \|^2_2 + \| w_{k+1} - w^* \|^2_2 \\
= \| x_k - x^* - \alpha A(y_k - y^*) \|^2_2 + \| y_k - y^* + \alpha A^T (x_k - x^*) \|^2_2 \\
= \| x_k - x^* \|^2_2 + \| y_k - y^* \|^2_2 + \alpha^2 \| A(y_k - y^*) \|^2_2 + \alpha^2 \| A^T (x_k - x^*) \|^2_2 \\
= \| x_k - x^* + \alpha A(y_k - y^*) \|^2_2 + \| y_k - y^* - \alpha A^T (x_k - x^*) \|^2_2 \\
= \| 2\tilde{x}_k - z_k - x^* - \alpha Ay^* \|^2_2 + \| 2\tilde{y}_k - w_k - y^* + \alpha A^T x^* \|^2_2 \\
= \| \tilde{x}_k - \alpha \nabla g(\tilde{x}_k) - x^* + \alpha \nabla g(x^*) \|^2_2 + \| \tilde{y}_k - \alpha \nabla h(\tilde{y}_k) - y^* + \alpha \nabla h(y^*) \|^2_2,
\]

where the forth equality is based on Equation (16) and the last equality follows from Equation (15) and (17).
On the other hand, by Equation (15) and (17), we also have
\[
\|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2 = \|\tilde{x}_k + \alpha \nabla g(\tilde{x}_k) - x^* - \alpha \nabla g(x^*)\|_2^2 + \|\tilde{y}_k + \alpha \nabla h(\tilde{y}_k) - y^* - \alpha \nabla h(y^*)\|_2^2
\]

Now, we only need to prove that
\[
\|\tilde{x}_k + \alpha \nabla g(\tilde{x}_k) - x^* - \alpha \nabla g(x^*)\|_2^2 \leq \eta \|\tilde{x}_k + \alpha \nabla g(\tilde{x}_k) - x^* - \alpha \nabla g(x^*)\|_2^2.
\]
(18)

In fact, above inequality is equivalent to
\[
(1 - \eta) \left( \|\tilde{x}_k - x^*\|_2^2 + \alpha^2 \|\nabla g(\tilde{x}_k) - \nabla g(x^*)\|_2^2 \right) \leq 2(1 + \eta)\alpha \langle \tilde{x}_k - x^*, \nabla g(\tilde{x}_k) - \nabla g(x^*) \rangle.
\]
(19)

Recalling the definition of \(\eta\), we have \(\frac{1 - \eta}{2(1 + \eta)} = \frac{\sqrt{L\mu}}{L + \mu}\). Then together with \(\alpha = 1/\sqrt{L\mu}\), inequality (19) is just
\[
\frac{L\mu}{L + \mu} \|\tilde{x}_k - x^*\|_2^2 + \frac{1}{L + \mu} \|\nabla g(\tilde{x}_k) - \nabla g(x^*)\|_2^2 \leq \langle \tilde{x}_k - x^*, \nabla g(\tilde{x}_k) - \nabla g(x^*) \rangle,
\]
which holds according to Lemma 1.

\[\square\]

**Remark.** By the proof of inequality (18), there also holds
\[
\|2\tilde{x}_k - z_k - 2x^* + z^*\|_2^2 \leq \eta \|z_k - z^*\|_2^2.
\]
(20)

### D Proof of Theorem 6

**Proof.** Denote \(\tilde{x}_k^* = \arg \min_x G_k(x)\). Note that
\[
\|2\tilde{x}_k - z_k - 2x^* + z^*\|_2^2 \\
\leq 4(1 + \beta) \|\tilde{x}_k - \tilde{x}_k^*\|_2^2 + (1 + 1/\beta) \|2\tilde{x}_k^* - z_k - 2x^* + z^*\|_2^2 \\
\leq 4(1 + \beta) \|\tilde{x}_k - \tilde{x}_k^*\|_2^2 + (1 + 1/\beta) \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^2 \|z_k - z^*\|_2^2 \\
\leq 2(\sqrt{\kappa} + 1) \|\tilde{x}_k - \tilde{x}_k^*\|_2^2 + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \|z_k - z^*\|_2^2,
\]
where \(\beta = \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\) and the second inequality is according to Equation (20).

Observe that \(G_k\) is \((\mu + 1/\alpha)\)-strongly convex, hence we have
\[
\left( \frac{\mu + \sqrt{L\mu}}{2} \right) \|\tilde{x}_k - \tilde{x}_k^*\|_2^2 \leq G_k(\tilde{x}_k) - G_k(\tilde{x}_k^*) \leq \varepsilon_k.
\]

Therefore, there holds
\[
\|2\tilde{x}_k - z_k - 2x^* + z^*\|_2^2 \leq \frac{4}{\mu} \varepsilon_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \|z_k - z^*\|_2^2.
\]
(21)

On the other hand, let \((x_k^*, y_k^*)\) be the saddle point of \(\tilde{f}_k\), which satisfies
\[
\begin{cases}
x_k^* + \alpha A y_k^* = 2\tilde{x}_k - z_k, \\
y_k^* - \alpha A^\top x_k^* = 2\tilde{y}_k - w_k.
\end{cases}
\]
(22)
Then we have
\[
\|x_k - x^* + \alpha A(y_k - y^*)\|_2^2 \\
\leq (1 + \sqrt{\kappa}) \|x_k - x_k^* + \alpha A(y_k - y_k^*)\|_2^2 + (1 + 1/\sqrt{\kappa}) \|x_k^* - x^* + \alpha A(y_k^* - y^*)\|_2^2 \\
\leq (1 + \sqrt{\kappa}) \left( 2 \|x_k - x_k^*\|_2^2 + \frac{2 \|A\|_2^2}{L\mu} \|y_k - y_k^*\|_2^2 \right) + \frac{8}{\mu} \varepsilon_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa}} \|z_k - z^*\|_2^2,
\]
where the second inequality is according to Equation (21) and (22).

Similarly, we also have
\[
\|y_k - y^* - \alpha A^\top (x_k - x^*)\|_2^2 \\
\leq (1 + \sqrt{\kappa}) \left( 2 \|y_k - y_k^*\|_2^2 + \frac{2 \|A\|_2^2}{L\mu} \|x_k - x_k^*\|_2^2 \right) + \frac{8}{\mu} \varepsilon_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa}} \|w_k - w^*\|_2^2.
\]

Therefore, we can conclude that
\[
\|z_{k+1} - z^*\|_2^2 + \|w_{k+1} - w^*\|_2^2 \\
= \|x_k - x^* - \alpha A(y_k - y^*)\|_2^2 + \|y_k - y^* + \alpha A^\top (x_k - x^*)\|_2^2 \\
= \|x_k - x^* + \alpha A(y_k - y^*)\|_2^2 + \|y_k - y^* - \alpha A^\top (x_k - x^*)\|_2^2 \\
\leq 2(1 + \sqrt{\kappa}) \left( 1 + \frac{\|A\|_2^2}{L\mu} \right) \delta_k + \frac{16}{\mu} \varepsilon_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa}} \left( \|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2 \right) \\
\leq 2C_0(1 - \rho)^{k+1} + (1 - 2\rho) \left( \|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2 \right),
\]
where we have recalled the definition of \(\varepsilon_k\) and \(\delta_k\).

Let \(a_k = \|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2\). Then we have
\[
\frac{a_{k+1}}{(1 - 2\rho)^{k+1}} - \frac{a_k}{(1 - 2\rho)^k} \leq 2C_0 \left( \frac{1 - \rho}{1 - 2\rho} \right)^{k+1}, \quad \text{i.e.,}
\]
\[
\frac{a_{k+1}}{(1 - 2\rho)^{k+1}} - \frac{a_1}{1 - 2\rho} \leq 2C_0 \sum_{i=2}^{k+1} \left( \frac{1 - \rho}{1 - 2\rho} \right)^i = 2C_0 \left( \frac{1 - \rho}{1 - 2\rho} \right)^2 \left( \frac{1 - \rho}{1 - 2\rho} \right)^{k+1} - 1 \\
\leq 2C_0(1 - \rho)^2 \left( \frac{1 - \rho}{1 - 2\rho} \right)^k.
\]

Consequently, we have
\[
\|x_K - x^*\|_2^2 + \|y_K - y^*\|_2^2 \leq \|z_{K+1} - z^*\|_2^2 + \|w_{K+1} - w^*\|_2^2 \\
\leq \frac{2C_0}{\rho} (1 - \rho)^{K+2} + a_1(1 - 2\rho)^K \leq (4C_0\sqrt{\kappa} + a_1)(1 - \rho)^{K+1},
\]
where the last inequality is according to \((1 - 2\rho)^k \leq (1 - \rho)^{k+1}\) for \(k \geq 1\) and \(\rho = \frac{1}{2\sqrt{\kappa}}\). Then, together with
\[
a_1 \leq \frac{\|A\|_2^2 + L\mu}{L\mu} \left( \|x_0 - x^*\|_2^2 + \|y_0 - y^*\|_2^2 \right) = C_0(\|A\|_2^2 + L\mu),
\]
we obtain the desired result.
Hence, we have
\[ x^* = \arg \min_x g(x) + \frac{1}{2\alpha} \|x - z^*\|_2^2. \]
Observe that \( x^* = \arg \min_x g(x) + \frac{1}{2\alpha} \|x - z^*\|_2^2 \). Then by Lemma \ref{lemma11} we have
\[ \|\tilde{x}_k^* - x^*\|_2^2 \leq \|z_k - z^*\|_2^2. \]
Hence, we have
\[ \|x_{k-1} - \tilde{x}_k^*\|_2^2 \leq 2 \|x_{k-1} - x^*\|_2^2 + 2 \|\tilde{x}_k^* - x^*\|_2^2 \leq 4 \left( \|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2 \right). \]
Note that the condition number of function \( G_k \) is
\[ \frac{L + 1/\alpha}{\mu + 1/\alpha} = \sqrt{\kappa}. \]
Suppose the sequence \( \{\tilde{x}_k\}_{k=1}^K \) is obtained by AGD for optimizing \( G_k \) where \( \tilde{x}_{k,0} = x_{k-1}, \tilde{x}_{k,K_1} = \tilde{x}_k \). Then following from Theorem \ref{theorem2} there holds
\[ G_k(\tilde{x}_k) - G_k(\bar{x}_k) \leq \frac{L + \mu + 2/\alpha}{2} \|x_{k-1} - \tilde{x}_k^*\|_2 \exp \left( -\frac{K_1}{\sqrt{\kappa}} \right) \]
\[ \leq 2C_0(L + \mu + 2\sqrt{L\mu}C(1 - \rho)^k \frac{\mu(1 - \rho)}{32C(\sqrt{L} + \sqrt{\mu})^2} \]
\[ \leq \frac{C_0\mu}{16} (1 - \rho)^{k+1} = \varepsilon_k. \]
Similarly, we also need to run AGD \( K_1 \) steps for optimizing \( H_k \) with initial point \( y_{k-1} \).
Now, we turn to consider \( \tilde{f}_k \). Let \( (x_k^*, y_k^*) \) be the saddle point of \( \tilde{f}_k \). Then we have
\[ \|x_k^* - x^* + \alpha A(y_k^* - y^*)\|_2^2 \leq \frac{8}{\mu} \varepsilon_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \|z_k - z^*\|_2^2, \]
\[ \|y_k^* - y^* - \alpha A^\top(x_k^* - x^*)\|_2^2 \leq \frac{8}{\mu} \varepsilon_k + \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \|w_k - w^*\|_2^2. \]
Therefore, we have
\[ \|x_{k-1} - x_k^*\|_2^2 + \|y_{k-1} - y_k^*\|_2^2 \]
\[ \leq 2 \left( \|x_{k-1} - x^*\|_2^2 + \|y_{k-1} - y^*\|_2^2 \right) + 2 \left( \|x_k^* - x^*\|_2^2 + \|y_k^* - y^*\|_2^2 \right) \]
\[ \leq 4 \left( \|z_k - z^*\|_2^2 + \|w_k - w^*\|_2^2 \right) + \frac{16}{\mu} \varepsilon_k \]
\[ \leq C_0(4C(1 - \rho)^k + (1 - \rho)^{k+1}). \]
Suppose the sequence \( \{(x_{k,t}, y_{k,t})\}_{t=0}^{K_2} \) is obtained by APFB for solving the subproblem \( \bar{f}_k \) where \( (x_{k,0}, y_{k,0}) = (x_{k-1}, y_{k-1}), (x_{k,K_2}, y_{k,K_2}) = (x_k, y_k) \).
Then by Theorem \ref{theorem3} we have
\[ \|x_k - x_k^*\|_2^2 + \|y_k - y_k^*\|_2^2 \leq \left( \frac{\|A\|_2}{\|A\|_2 + \sqrt{L\mu}} \right)^{K_2+1} \left( \|x_{k-1} - x_k^*\|_2^2 + \|y_{k-1} - y_k^*\|_2^2 \right) \]
\[ \leq C_0 \left( 4C(1 - \rho)^k + (1 - \rho)^{k+1} \right) \exp \left( -\frac{K_2 - 1}{\|A\|_2 + \sqrt{L\mu} + 1} \right) \]
\[ \leq 5C_0C(1 - \rho)^k \frac{L\mu(1 - \rho)}{20C(1 + \sqrt{\kappa})(L + \|A\|_2^2)} \leq \delta_k. \]
Algorithm 6 AIPFB

1: **Input:** function $g, h$, coupling matrix $A$, initial point $x_0, y_0$, strongly convex module $\mu_x, \mu_y$, run-time $T$, tolerance sequence $\{\varepsilon_k\}_{k \geq 1}$.

2: **Initialize:** $\bar{x}_0 = x_0$, $\gamma = \frac{1}{\|A\|_2} \sqrt{\frac{\mu_x}{\mu_y}}$, $\sigma = \frac{1}{\|A\|_2} \sqrt{\frac{\mu_x}{\mu_y}}$ and $\theta = \frac{\|A\|_2}{\sqrt{\mu_x \mu_y + \|A\|_2}}$.

3: **for** $k = 1, \ldots, T$ **do**

4: \quad Let $h_k(y) = h(y) + \frac{1}{2\gamma} \|y - y_{k-1} - \sigma A^T \bar{x}_{k-1}\|^2_2$.

5: \quad Find $y_k$ such that $h_k(y_k) - \min_y h_k(y) \leq \varepsilon_k$.

6: \quad Let $g_k(x) = g(x) + \frac{1}{2\gamma} \|x - x_{k-1} + \gamma A y_k\|^2_2$.

7: \quad Find $x_k$ such that $g_k(x_k) - \min_x g_k(x) \leq \varepsilon_k$.

8: \quad $\bar{x}_k = x_k + \theta (x_k - x_{k-1})$.

9: **end for**

10: **Output:** $x_T, y_T$.

\[ \square \]

F Accelerated Inexact Proximal Forward Backward Algorithm

In this section, we provide an inexact version of APFB, called Accelerated Inexact Proximal Forward Backward, in Algorithm 6 for completeness. Similar to DIPPA, we employ AGD to solve subproblems. And a theoretical guarantee is given in following theorem.

**Theorem 12.** Assume that $g(x)$ is $L_x$-smooth and $\mu_x$-strongly convex and $h(y)$ is $L_y$-smooth and $\mu_y$-strongly convex. The total queries to Oracle needed by Algorithm 6 to produce $\varepsilon$-saddle point of $f(x, y) = g(x) + \langle x, Ay \rangle - h(y)$ is at most

\[ \tilde{O}\left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \sqrt{\frac{\|A\|_2}{\mu_x \mu_y} (\kappa_x + \kappa_y)} + \sqrt{\kappa_x + \kappa_y} \bigg), \right. \]

\[ \log \left( \frac{\mu_x \|x_0 - x^*\|_2^2 + \mu_y \|y_0 - y^*\|_2^2}{\varepsilon} \right) \bigg), \]

where $\kappa_x = L_x/\mu_x$, $\kappa_y = L_y/\mu_y$ and the notation $\tilde{O}$ have omitted some logarithmic factors depending on $L_x, L_y, \|A\|_2, \mu_x$ and $\mu_y$.

We first present the convergence rate of the outer loop of Algorithm 6.

**Lemma 13.** Assume that $g(x)$ is $L_x$-smooth and $\mu_x$-strongly convex and $h(y)$ is $L_y$-smooth and $\mu_y$-strongly convex. Set

\[ \theta = \frac{\|A\|_2}{\sqrt{\mu_x \mu_y} + \|A\|_2}, \quad \rho = \frac{\sqrt{\mu_x \mu_y}}{2 \sqrt{\mu_x \mu_y} + 4 \|A\|_2}, \]

\[ C_0 = \mu_x \|x_0 - x^*\|_2^2 + \mu_y \|y_0 - y^*\|_2^2, \quad \varepsilon_k = \frac{C_0 \rho(1 - \theta)}{16} (1 - \rho)^{k-1}. \]

For $T \geq 2$, the output of Algorithm 6 satisfies

\[ \mu_x \|x_T - x^*\|_2^2 + \mu_y \|y_T - y^*\|_2 \leq C(1 - \rho)^T \left( \mu_x \|x_0 - x^*\|_2^2 + \mu_y \|y_0 - y^*\|_2^2 \right), \]

where $C$ is a constant depending on $\mu_x, \mu_y, L_x, L_y, \kappa_x, \kappa_y$.\[ \square \]
where

\[ C = \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + 1, \]

and \((x^*, y^*)\) is the saddle point of the function \(f(x, y) = g(x) + x^\top A y - h(y)\).

**Proof.** Denote \(y_k^* = \arg\min_y h_k(y),\) \(x_k^* = \arg\min_x g_k(x),\) \(w_k = y_{k-1} + \sigma A^\top \tilde{x}_{k-1}\) and \(z_k = x_{k-1} - \gamma A y_k\). Since \(h_k(y)\) is \((\mu_y + 1/\sigma)\)-strongly convex and \(h_k(y_k) - h_k(y_k^*) \leq \varepsilon_k\), we know that

\[ h_k(y^*) \geq h_k(y_k^*) + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y^* - y_k^*\|_2^2 \geq h_k(y_k) + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y^* - y_k^*\|_2^2 - \varepsilon_k. \]

Equivalently, we have

\[ h(y^*) \geq h(y_k) + \frac{1}{2\sigma} \|y_k - w_k\|_2^2 - \frac{1}{2\sigma} \|y^* - w_k\|_2 + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y^* - y_k^*\|_2^2 - \varepsilon_k \]

\[ = h(y_k) + \frac{1}{\sigma} \langle y_k - w_k, y_k - y^* \rangle - \frac{1}{2\sigma} \|y_k - y^*\|_2^2 + \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y^* - y_k^*\|_2^2 - \varepsilon_k. \]

On the other hand, note that

\[ \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y^* - y_k^*\|_2^2 - \frac{1}{2\sigma} \|y_k - y^*\|_2^2 \]

\[ = \frac{\mu_y}{2} \|y_k - y^*\|_2^2 - \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y_k - y_k^*\|_2^2. \]

Using Young’s inequality yields

\[ \left( \mu_y + \frac{1}{\sigma} \right) \langle y_k - y^*, y_k - y_k^* \rangle \leq \frac{\mu_y}{4} \|y_k - y^*\|_2^2 + \left( \mu_y + \frac{1}{\sigma} \right) \left( 1 + \frac{1}{\mu_y \sigma} \right) \|y_k - y_k^*\|_2^2. \]

It follows that

\[ h(y^*) \geq h(y_k) + \frac{1}{\sigma} \langle y_k - w_k, y_k - y^* \rangle + \frac{\mu_y}{4} \|y_k - y^*\|_2^2 - \left( \mu_y + \frac{1}{\sigma} \right) \left( 1 + \frac{1}{\mu_y \sigma} \right) \|y_k - y_k^*\|_2^2 - \varepsilon_k. \]

By \((\mu_y + 1/\sigma)\)-strongly convexity of \(h_k\), we have

\[ \left( \frac{\mu_y}{2} + \frac{1}{2\sigma} \right) \|y_k - y_k^*\|_2^2 \leq h_k(y_k) - h_k(y_k^*) \leq \varepsilon_k. \]

Putting these pieces together yields

\[ h(y^*) \geq h(y_k) + \frac{1}{\sigma} \langle y_k - w_k, y_k - y^* \rangle + \frac{\mu_y}{4} \|y_k - y^*\|_2^2 - \left( 2 + \frac{2}{\mu_y \sigma} \right) \varepsilon_k. \tag{23} \]

Plugging \(w_k = y_{k-1} + \sigma A^\top \tilde{x}_{k-1}\) and \(2 \langle y_k - y_{k-1}, y_k - y^* \rangle = \|y_k - y_{k-1}\|_2^2 + \|y_k - y^*\|_2^2 - \|y_{k-1} - y^*\|_2^2\) into Inequality (23), we have

\[ h(y^*) + \frac{1}{2\sigma} \|y_{k-1} - y^*\|_2^2 + \langle y_k - y^*, A^\top \tilde{x}_{k-1} \rangle \]

\[ \geq h(y_k) + \frac{1}{2\sigma} \|y_k - y_{k-1}\|_2^2 + \left( \frac{\mu_y}{4} + \frac{1}{2\sigma} \right) \|y_k - y^*\|_2^2 - \left( 2 + \frac{2}{\mu_y \sigma} \right) \varepsilon_k. \tag{24} \]
Similarly, we can obtain
\[ g(x^*) + \frac{1}{2\gamma} \|x_{k-1} - x^*\|^2_2 - \langle x_k - x^*, A y_k \rangle \]
\[ \geq g(x_k) + \frac{1}{2\gamma} \|x_k - x_{k-1}\|^2_2 + \left( \frac{\mu_x}{4} + \frac{1}{2\gamma} \right) \|x_k - x^*\|^2_2 - \left( 2 + \frac{2}{\mu_x\gamma} \right) \epsilon_k. \] (25)

Observe that \( \mu_x\gamma = \mu_y\sigma = \sqrt{\frac{\mu_x\mu_y}{\|A\|_2}} = 1/\theta - 1 \). Adding both sides of Inequalities (24) and (25) yields
\[ \frac{1}{2\gamma} \|x_{k-1} - x^*\|^2_2 + \frac{1}{2\sigma} \|y_{k-1} - y^*\|^2_2 \]
\[ \geq \left( \frac{\mu_x}{4} + \frac{1}{2\gamma} \right) \|x_k - x^*\|^2_2 + \left( \frac{\mu_y}{4} + \frac{1}{2\sigma} \right) \|y_k - y^*\|^2_2 + \frac{1}{2\gamma} \|x_k - x_{k-1}\|^2_2 + \frac{1}{2\sigma} \|y_k - y_{k-1}\|^2_2 \]
\[ + g(x_k) + h(y_k) - g(x^*) - h(y^*) + \langle x_k - x^*, A y_k \rangle - \langle y_k - y^*, A^T x_{k-1} \rangle - \frac{4}{1 - \theta} \epsilon_k. \] (26)

Plugging Equations (10), (12) and (13) into Inequality (26), we have
\[ \frac{1}{2\gamma} \|x_{k-1} - x^*\|^2_2 + \frac{1}{2\sigma} \|y_{k-1} - y^*\|^2_2 + \frac{\theta}{2\gamma} \|x_{k-1} - x_{k-2}\|^2_2 + \theta \langle x_{k-1} - x_{k-2}, A(y_{k-1} - y^*) \rangle \]
\[ \geq \left( \frac{\mu_x}{4} + \frac{1}{2\gamma} \right) \|x_k - x^*\|^2_2 + \left( \frac{\mu_y}{4} + \frac{1}{2\sigma} \right) \|y_k - y^*\|^2_2 + \frac{1}{2\gamma} \|x_k - x_{k-1}\|^2_2 \]
\[ + f(x_k, y^*) - f(x^*, y_k) + \langle x_k - x_{k-1}, A(y_k - y^*) \rangle - \frac{4}{1 - \theta} \epsilon_k. \]

By Definition 1 we have \( f(x_k, y^*) - f(x^*, y_k) \geq 0 \). Recall that \( \epsilon_k = \frac{C_0\rho(1-\theta)}{16} (1 - \rho)^{k-1} \) where \( \rho = \frac{\sqrt{\mu_x\mu_y}}{2\sqrt{\mu_x\mu_y + \|A\|_2}} \). Denoting
\[ a_k = \left( \frac{\mu_x}{4} + \frac{1}{2\gamma} \right) \|x_k - x^*\|^2_2 + \left( \frac{\mu_y}{4} + \frac{1}{2\sigma} \right) \|y_k - y^*\|^2_2 + \frac{1}{2\gamma} \|x_k - x_{k-1}\|^2_2 + \langle x_k - x_{k-1}, A(y_k - y^*) \rangle, \]
we have
\[ \frac{a_k}{(1 - 2\rho)^k} \leq \frac{a_{k-1}}{(1 - 2\rho)^{k-1}} + \frac{C_0\rho}{4(1 - \rho)} \left( \frac{1 - \rho}{1 - 2\rho} \right)^k, \ i.e., \]
\[ \frac{a_k}{(1 - 2\rho)^k} \leq a_0 + \frac{C_0\rho}{4(1 - \rho)} \sum_{i=1}^{k} \left( \frac{1 - \rho}{1 - 2\rho} \right)^i = a_0 + \frac{C_0\rho}{4(1 - \rho)} \frac{1 - \rho}{1 - 2\rho} - 1 \]
\[ \leq a_0 + \frac{C_0}{4} \left( \frac{1 - \rho}{1 - 2\rho} \right)^k. \]

Moreover, Inequality (14) implies \( a_T \geq \frac{\mu_x}{4} \|x_T - x^*\|^2_2 + \frac{\mu_y}{4} \|y_T - y^*\|^2_2 \). Consequently, for \( T \geq 2 \)
we have
\[
\mu_x \|x_T - x^*\|_2^2 + \mu_y \|y_T - y^*\|_2^2 \\
\leq 4a_0(1 - 2\rho)^T + C_0(1 - \rho)^T \\
= (1 - 2\rho)^T \left( \mu_x + \frac{2}{\gamma} \right) \|x_0 - x^*\|_2^2 + \left( \mu_y + \frac{2}{\sigma} \right) \|y_0 - y^*\|_2^2 + C_0(1 - \rho)^T \\
\leq (1 - 2\rho)^T \left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} \rho C_0 + C_0(1 - \rho)^T \right) \\
\leq C_0(1 - \rho)^T \left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + 1 \right).
\]

where the last inequality is according to \((1 - 2\rho)^T - 1 \leq (1 - \rho)^T\) for \(T \geq 2\).

For the inner loop, we have the following lemma.

**Lemma 14.** Consider the same assumption and the same definitions of \(\varepsilon_k\), \(\theta\), \(\rho\), \(C\) and \(C_0\) in Theorem 13. Denote \(\kappa_x = L_x/\mu_x\), \(\kappa_y = L_y/\mu_y\) and \(\bar{\kappa} = \|A\|_2 / \sqrt{\mu_x \mu_y}\). In order to find \(\varepsilon_k\)-optimal points \(x_k\) of \(g_k\), we need to run AGD \(K_1\) steps, where

\[
K_1 = \left\lceil \sqrt{\frac{\kappa_y + \bar{\kappa}}{1 + \bar{\kappa}}} \log \left( \frac{320C(1 - \rho)\kappa_y + 2\bar{\kappa} + 1}{\rho(1 - \theta)} \right) \right\rceil + 1.
\]

And in order to obtain \(\varepsilon_k\)-optimal point \(y_k\) of \(h_k\), we need to run AGD \(K_2\) steps, where

\[
K_2 = \left\lceil \sqrt{\frac{\kappa_x + \bar{\kappa}}{1 + \bar{\kappa}}} \log \left( \frac{80C(\kappa_x + 2\bar{\kappa} + 1)}{\rho(1 - \theta)} \right) \right\rceil + 1.
\]

**Proof.** By Lemma 11 and Cauchy-Schwarz inequality, we have
\[
\|y_k - y^*\|_2^2 \leq \|w_k - w^*\|_2^2 \\
\leq 2 \left( \|y_{k-1} - y^*\|_2^2 + \sigma^2 \|A\|_2^2 \|\bar{x}_{k-1} - x^*\|_2^2 \right) \\
= 2 \left( \|y_{k-1} - y^*\|_2^2 + 2(1 + \theta)^2 \frac{\mu_x}{\mu_y} \|x_{k-1} - x^*\|_2^2 + 2\theta^2 \frac{\mu_x}{\mu_y} \|x_{k-2} - x^*\|_2^2 \right) \\
\leq 2 \|y_{k-1} - y^*\|_2^2 + \frac{16\mu_x}{\mu_y} \|x_{k-1} - x^*\|_2^2 + \frac{4\mu_x}{\mu_y} \|x_{k-2} - x^*\|_2^2.
\]

It follows that
\[
\|y_{k-1} - y_k\|_2^2 \leq 2 \|y_{k-1} - y^*\|_2^2 + 2 \|y_k - y^*\|_2^2 \\
\leq 6 \|y_{k-1} - y^*\|_2^2 + \frac{32\mu_x}{\mu_y} \|x_{k-1} - x^*\|_2^2 + \frac{8\mu_x}{\mu_y} \|x_{k-2} - x^*\|_2^2.
\]

By Theorem 13, we have \(\|y_{k-1} - y_k\|_2^2 \leq 40CC_0(1 - \rho)^{k-2}/\mu_y\). Denote \(\bar{\kappa} = \|A\|_2 / \sqrt{\mu_x \mu_y}\). Then the condition number of function \(h_k\) is
\[
\frac{L_y + 1/\sigma}{\mu_y + 1/\sigma} = \frac{\kappa_y + \bar{\kappa}}{1 + \bar{\kappa}}.
\]
Thus, by Theorem 2, the first subproblem in step $k$ will need to run AGD with initial point $y_{k-1}$ at most $K_1$ steps where $K_1$ satisfies

$$K_1 = \left\lceil \sqrt{\frac{\kappa_y + \tilde{\kappa}}{1 + \tilde{\kappa}}} \log \left( \frac{20CC_0(1 - \rho)^{k-2}(L_y + \mu_y + 2/\sigma)}{\mu_y \epsilon_k} \right) \right\rceil + 1$$

On the other hand, Lemma 11 and Cauchy-Schwarz inequality also imply

$$\|x^*_k - x^*\|^2_2 \leq \|z_k - z^*\|^2_2 \leq 2 \left( \|x_{k-1} - x^*\|^2_2 + \gamma^2 \|A\|^2_2 \|y_k - y^*\|^2_2 \right) = 2 \left( \|x_{k-1} - x^*\|^2_2 + \frac{\mu_y}{\mu_x} \|y_k - y^*\|^2_2 \right).$$

It follows that

$$\|x_{k-1} - x^*_k\|^2_2 \leq 2 \|x_{k-1} - x^*\|^2_2 + 2 \|x^*_k - x^*\|^2_2 \leq 6 \|x_{k-1} - x^*\|^2_2 + \frac{4\mu_y}{\mu_x} \|y_k - y^*\|^2_2 \leq \frac{10}{\mu_x} CC_0(1 - \rho)^{k-1}.$$  

The condition number of function $g_k$ is

$$L_x + 1/\gamma = \frac{\kappa_x + \tilde{\kappa}}{1 + \tilde{\kappa}}.$$

Thus, by Theorem 2, the second subproblem in step $k$ will need to run AGD with initial point $x_{k-1}$ at most $K_2$ steps where $K_2$ satisfies

$$K_2 = \left\lceil \sqrt{\frac{\kappa_x + \tilde{\kappa}}{1 + \tilde{\kappa}}} \log \left( \frac{8CC_x(\kappa_x + 2\tilde{\kappa} + 1)}{\rho(1 - \theta)} \right) \right\rceil + 1.$$

Then we can provide the proof of Theorem 12.

**Proof of Theorem 12.** Denote $\tilde{\kappa} = \|A\|_2/\sqrt{\mu_x \mu_y}$.

By Lemma 13 in order to find $\epsilon$-saddle point of $f$, we need to run AIPFB

$$K = \left\lceil (4\tilde{\kappa} + 2) \log \left( \frac{CC_0}{\epsilon} \right) \right\rceil + 1$$

steps. Therefore, the number of total queries to Oracle (2) is upper bounded by

$$K(K_1 + K_2) = \tilde{O} \left( (1 + \tilde{\kappa}) \sqrt{\frac{\kappa_x + \kappa_y + \tilde{\kappa}}{1 + \tilde{\kappa}}} \log \left( \frac{\mu_x \|x_0 - x^*\|^2_2 + \mu_y \|y_0 - y^*\|^2_2}{\epsilon} \right) \right)$$

$$= \tilde{O} \left( \sqrt{\tilde{\kappa}^2 + \kappa_x + \kappa_y + \tilde{\kappa} (\kappa_x + \kappa_y)} \log \left( \frac{\mu_x \|x_0 - x^*\|^2_2 + \mu_y \|y_0 - y^*\|^2_2}{\epsilon} \right) \right).$$

\[\square\]
We can apply Catalyst framework to accelerate the Algorithm 6. The details are presented in Algorithm 7 Catalyst-AIPFB. Again we remark that the function to produce is

\[
\beta = \frac{L_x(\mu_y - \mu_x)}{L_x - \mu_y}, \quad q = \frac{\mu_x}{\mu_x + \beta}, \quad \theta = \frac{1 - \sqrt{q}}{1 + \sqrt{q}} \text{ and } \tilde{x}_0 = x_0.
\]

For \( L = \max\{\|A\|_2, L_x\} \). In this section, we improve term \( \sqrt{\frac{\|A\|_2}{\mu_x \mu_y}} \) to be \( \sqrt{\frac{\|A\|_2 L}{\mu_x \mu_y}} \) by Catalyst framework.

Without loss of generality, we can assume that \( L_x = L_y \). Otherwise, one can rescale the variables and take \( \tilde{f} = f(\sqrt{L_y/L_x}x, \sqrt{L_x/L_y}y) \). It is not hard to check that this rescaling will not change condition numbers \( \kappa_x, \kappa_y, \mu_x \mu_y \), the coupling matrix \( A \) and increase max\{\|A\|_2, L_x, L_y\}.

We first consider the special case where \( \mu_x = \mu_y \). The total queries to Oracle needed by Algorithm 6 to produce \( \varepsilon \)-saddle point is at most

\[
O \left( \left( \frac{\sqrt{\|A\|_2}}{\mu_y} + \sqrt{\kappa_y} \right) \log \left( \frac{\mu_x \|x_0 - x^*\|_2^2 + \mu_y \|y_0 - y^*\|_2^2}{\varepsilon} \right) \right).
\]

For the general case, without loss of generality we assume \( \mu_x \leq \mu_y \). Similar to Catalyst-DIPPA, We can apply Catalyst framework to accelerate the Algorithm 6. The details are presented in Algorithm 7. Again we remark that the function \( f_k \) in each subproblem is balanced: the condition number corresponding to \( y \) is \( \kappa_y \) and the condition number related to \( x \) is

\[
\frac{L_x + \beta}{\mu_x + \beta} = \frac{L_x(L_x - \mu_y) + L_x(\mu_y - \mu_x)}{\mu_x(L_x - \mu_y) + L_x(\mu_y - \mu_x)} = \kappa_y,
\]

where we have recalled that \( L_x = L_y \).

By results of Catalyst \cite{beam2017catalyst}, the number of total queries to Oracle needed by Algorithm 6 is upper bounded by

\[
O \left( \frac{\sqrt{\mu_x + \beta}}{\mu_x} \right) O \left( \sqrt{\frac{\|A\|_2 L}{\mu_y^2} + \sqrt{\kappa_y}} \right)
\]

\[
= O \left( \sqrt{\frac{\|A\|_2 L}{\mu_x \mu_y}} + \sqrt{\kappa_x + \kappa_y} \right).
\]

A formal statement of theoretical guarantee for Catalyst-AIPFB is presented as follows.

\[\text{Algorithm 7 Catalyst-AIPFB}\]

1: Input: function \( f \), initial point \((x_0, y_0)\), smoothness \( L_x, L_y \), strongly convex module \( \mu_x < \mu_y \), run-time \( T \), accuracy sequence \( \{\varepsilon_k\}_{k \geq 1} \).

2: Initialize: \( \beta = \frac{L_x(\mu_y - \mu_x)}{L_x - \mu_y}, q = \frac{\mu_x}{\mu_x + \beta}, \theta = \frac{1 - \sqrt{q}}{1 + \sqrt{q}} \text{ and } \tilde{x}_0 = x_0. \)

3: for \( k = 1, \ldots, K \) do

4: Let \( f_k(x, y) = f(x, y) + \frac{\beta}{q} \|x - \tilde{x}_k\|^2 \).

5: Obtain \( \varepsilon_k \)-saddle point \((x_k, y_k)\) of \( f_k(x, y)\) by applying AIPFB.

6: \( \tilde{x}_k = x_k + \theta(x_k - x_{k-1}) \).

7: end for

8: Output: \( x_T, y_T \).

F.1 An Improved upper bound for AIPFB

For \( L_x = L_y \) and \( \mu_x \leq \mu_y \), the complexity in Theorem \[\text{12}\] becomes

\[
O \left( \frac{\|A\|_2}{\sqrt{\mu_x \mu_y}} + \frac{\|A\|_2 L}{\mu_x^{3/2} \mu_y^{1/2}} + \sqrt{\kappa_x + \kappa_y} \right),
\]

where \( L = \max\{\|A\|_2, L_x\} \).
Theorem 15. Assume that \( g(x) \) is \( L_x \)-smooth and \( \mu_x \)-strongly convex and \( h(y) \) is \( L_y \)-smooth and \( \mu_y \)-strongly convex. The total queries to Oracle (2) needed by Algorithm 7 to produce \( \varepsilon \)-saddle point of \( f(x,y) = g(x) + \langle x, Ay \rangle - h(y) \) is at most

\[
\tilde{O}\left( \left( \sqrt{\frac{\|A\|_2 L}{\mu_x \mu_y}} + \sqrt{\kappa_x + \kappa_y} \right) \log \left( \frac{1}{\varepsilon} \right) \right),
\]

where \( L = \max \{ \|A\|_2, L_x, L_y \}, \kappa_x = L_x/\mu_x, \kappa_y = L_y/\mu_y \) and the notation \( \tilde{O} \) have omitted some logarithmic factors depending on \( L_x, L_y, \|A\|_2, \mu_x \) and \( \mu_y \).