Relative moduli spaces of complex structures: an example

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Abstract

Let $M$ and $N$ be even-dimensional oriented real manifolds, and $u : M \to N$ be a smooth mapping. A pair of complex structures at $M$ and $N$ is called $u$-compatible if the mapping $u$ is holomorphic with respect to these structures. The quotient of the space of $u$-compatible pairs of complex structures by the group of $u$-equivariant pairs of diffeomorphisms of $M$ and $N$ is called a moduli space of $u$-equivariant complex structures.

The paper contains a description of the fundamental group $G$ of this moduli space in the following case: $N = \mathbb{C}P^1$, $M \subset \mathbb{C}P^2$ is a hyperelliptic genus $g$ curve given by the equation $y^2 = Q(x)$ where $Q$ is a generic polynomial of degree $2g + 1$, and $u(x, y) = y^2$. The group $G$ is a kernel of several (equivalent) actions of the braid-cyclic group $BC_{2g}$ on $2g$ strands.

These are: an action on the set of trees with $2g$ numbered edges, an action on the set of all splittings of a $(4g + 2)$-gon into numbered nonintersecting quadrangles, and an action on a certain set of subgroups of the free group with $2g$ generators. $G_{2g} \subset BC_{2g}$ is a subgroup of the index $(2g + 1)^{2g-2}$.

Key words: Teichmüller spaces, Lyashko–Looijenga map, braid group.

Introduction

Let two even-dimensional oriented real manifolds $M^{2m}$ and $N^{2n}$ be given, and $u : M \to N$ be a smooth mapping. A pair of complex structures at $M$ and $N$ will be called $u$-compatible if the mapping $u$ is holomorphic with respect to these structures. The set of all $u$-compatible pairs of complex structures will be denoted $\mathcal{J}(u)$. Call a pair of diffeomorphisms $\alpha : M \to M$ and $\beta : N \to N$ $u$-equivariant if the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & M \\
\downarrow u & & \downarrow u \\
N & \xrightarrow{\beta} & N
\end{array}
\]
commutes. The $u$-equivariant pairs of diffeomorphisms form a topological group $\mathcal{E}(u)$. The connected component of the unity in the group $\mathcal{E}(u)$ consists of pairs of diffeomorphisms homotopic to identity (inside $\mathcal{E}(u)$). This subgroup will be denoted $\mathcal{E}_0(u)$.

Groups $\mathcal{E}(u)$ and $\mathcal{E}_0(u)$ act on the space $\mathcal{J}(u)$. A quotient $\mathcal{T}(u) = \mathcal{J}(u)/\mathcal{E}_0(u)$ will be called a Teichmüller space of $u$-compatible pairs of complex structures. The smaller quotient $\mathcal{M}(u) = \mathcal{J}(u)/\mathcal{E}(u)$ will be called a moduli space. These spaces are “relative” analogs of the usual Teichmüller space and moduli space of complex structures reducing to them in the case when $u : M \rightarrow N$ is a diffeomorphism.

The article contains a description of the fundamental group $\pi_1(\mathcal{M}(u))$ in the following case. The target manifold $N$ is $\mathbb{CP}^1$. The manifold $M \subset \mathbb{CP}^2$ is a sphere with $g$ handles (denoted $M_g$) given in homogeneous coordinates as $\{[x : y : z] \mid y^2 z^{2g-1} = q(x, z)\}$; here $q(x, z)$ is a homogeneous polynomial of degree $2g+1$ such that all its critical values are different. The mapping $u$ is defined by the formula $u([x : y : z]) = [y^2 : z^2]$.

The article has the following structure. In Section 1 one shows (Theorem 1) that $\pi_1(\mathcal{M}(u)) = \text{Map}(u) \overset{\text{def}}{=} \mathcal{E}(u)/\mathcal{E}_0(u)$ provided $M$ and $N$ have real dimension 2, $u$ is a covering with at least 3 branching values, and there exist complex structures on $M$ and $N$ such that $u$ is holomorphic (i.e., $\mathcal{M}(u) \neq \emptyset$). This result extends the corresponding theorem of the “absolute” Teichmüller theory and reduces the problem to the study of the relative mapping class group $\text{Map}(u)$.

In Section 2 we prove (Theorem 5) that if $u : M_g \rightarrow \mathbb{CP}^1$ is the mapping described above then the group $\text{Map}(u)$ is isomorphic, up to a $\mathbb{Z}_2$-extension, to its projection $\mathcal{L}(u)$ onto the group $G_{2g}$ of diffeomorphisms of $\mathbb{CP}^1$ preserving branching values of $u$. In Subsection 2.2 it is also proved (Theorem 4) that $\mathcal{L}(u)$ is isomorphic to a point stabilizer of a certain action of the braid-cyclic group $BC_{2g}$ on the set of all splittings of a $(4g+2)$-gon into nonintersecting quadrangles.

In Section 3 this action is studied in detail. An algorithm for computing the action is given in Subsection 3.1. In Subsection 3.2 we define an action of $BC_N$ on another object, a set of trees with $N$ edges marked $0, 1, \ldots, N-1$, and show that this two actions are related by some geometric construction. It is proved that the action of $BC_N$ on trees is transitive, which allows to obtain some information about the group $\mathcal{L}(u)$ (Corollary of Statement 10). We formulate a conjecture about presentation of the group $\mathcal{L}(u)$. Our construction has some relations with the theory of Lyashko–Looijenga mapping; these are also considered in Subsection 3.2. The last Subsection 3.3 is a sort of appendix, it is devoted to the algebraic “roots” of the action considered above.

Acknowledgments

The initial inspiration of this work was an article [2] of S. Anisov and S. Lando considering in fact the genus 1 case of the problem (though Teichmüller spaces
do not appear directly in the article). Several important errors in the earlier versions of the text were pointed out by V. Arnold, M. Kontsevich, A. Zvonkin, and D. Zvonkine. The author thanks B. Wajnryb for making him acquainted with the paper [7]. The author is grateful to L. Funar for his letter concerning difficulties in construction of the homomorphisms from the braid groups to $\text{Map}(M_g)$. An elegant proof of the lemma at page 11 was suggested by V. Prasolov. Author is also grateful to F. Cohen for fruitful discussions.

Most of the work was completed during the author’s stay in the University Paris IX supported by a grant of the French Ministry of Foreign Affairs — the author is using an opportunity to express his warm gratitude to both institutions.

1 Fundamental group of the moduli space

1.1 The main players

For convenience we summarize here the definitions of the main objects of the article. Some of them were already described in the Introduction. The reader should refer to the list when necessary.

- $M$ and $N$ — orientable 2-dimensional (real) manifolds.
- $u : M \to N$ — a ramified covering, i.e. a smooth mapping such that the preimage $u^{-1}(x) \subset M$ of any point $x \in N$ is discrete. The set of ramification points of $u$ is denoted $R(u) \subset M$, and the set of branching values (images of ramification points), $B(u) \subset N$.
- $\mathcal{E}(u)$ — a topological group of $u$-equivariant pairs of diffeomorphisms, i.e. of the pairs of diffeomorphisms $\alpha : M \to M$, $\beta : N \to N$ such that diagram (1) commutes.
- $\mathcal{E}_0(u) \subset \mathcal{E}(u)$ — a connected component of the unity.
- $\text{Map}(u) = \mathcal{E}(u)/\mathcal{E}_0(u)$ — the $u$-equivariant mapping class group.
- $\mathcal{J}(u)$ — the set of $u$-compatible pairs of complex structures, i.e. pairs of complex structures at $M$ and $N$ such that the mapping $u$ is holomorphic with respect to them.
- $\mathcal{T}(u) = \mathcal{J}(u)/\mathcal{E}_0(u)$, $\mathcal{M}(u) = \mathcal{J}(u)/\mathcal{E}(u) = \mathcal{T}(u)/\text{Map}(u)$ — relative Teichmüller space and moduli space.
- $\text{Diff}(K)$ — the set of diffeomorphisms of the manifold $K$ (most often $K = M$ or $N$); $\text{Diff}(K; S_1, \ldots, S_n)$ — the set of diffeomorphisms of the manifold $K$ preserving subsets $S_1, \ldots, S_n \subset K$ (each one individually, but
not necessarily pointwise); \( \text{Diff}_0(K) \), \( \text{Diff}_0(K; S_1, \ldots, S_n) \) — corresponding connected components of the unity; \( \text{Map}(K) = \text{Diff}(K)/\text{Diff}_0(K) \), 
\( \text{Map}(K; S_1, \ldots, S_n) = \text{Diff}(K; S_1, \ldots, S_n)/\text{Diff}_0(K; S_1, \ldots, S_n) \) — corresponding mapping class groups.

- \( p_1, p_2 \) — generic names for mappings relating to a pair its first, resp., second, term. These mappings will be applied to the pairs of complex structures, diffeomorphisms, mapping classes, etc.

- \( \mathcal{E}^{(1)}(u) \triangleq p_1(\mathcal{E}(u)) \subset \text{Diff}(M, R(u)) \subset \text{Diff}(M) \) (apparently a \( u \)-equivariant mapping sends ramification points to ramification points), 
  \( \mathcal{E}^{(2)}(u) \triangleq p_2(\mathcal{E}(u)) \subset \text{Diff}(N, B(u)) \subset \text{Diff}(N) \) (the same is true for branching values). 
  \( \mathcal{E}_{0}^{(1)}(u) \triangleq p_1(\mathcal{E}_0(u)), \mathcal{E}_{0}^{(2)}(u) \triangleq p_2(\mathcal{E}_0(u)) \).

- The quotient \( \mathcal{E}^{(1)}(u)/\mathcal{E}_{0}^{(1)}(u) \subset \text{Map}(M, R(u)) \) will be called a group of projectable mapping classes (at \( M \)); it will be denoted \( \mathcal{P}(u) \). Similarly, 
  \( \mathcal{L}(u) \triangleq \mathcal{E}^{(2)}(u)/\mathcal{E}_{0}^{(2)}(u) \subset \text{Map}(N, B(u)) \) will be called a group of liftable mapping classes (at \( N \)).

### 1.2 Homotopy structure

Our aim in this Subsection is the following theorem:

**Theorem 1** Let \( B(u) \) contain at least three points. Then \( \pi_1(M(u)) \) is isomorphic to the relative mapping class group \( \text{Map}(u) \triangleq \mathcal{E}(u)/\mathcal{E}_0(u) \).

To prove it, we are to prove first some auxiliary statements about homotopy structure of diffeomorphism groups and Teichmüller spaces. Almost everything we do is based on the classical lemma due to Alexander:

**Lemma** The group of orientation-preserving diffeomorphisms of the \( n \)-dimensional disk with pointwise fixed boundary is contractible.

See [1] for proof.

**Corollary** Let \( S \) be a noncompact 2-manifold, and \( a, b \in S \). Then any connected component of the space of paths in \( S \) beginning at \( a \) and ending in \( b \) is contractible.

**Proof** The manifold \( S \) can be contracted to its 1-dimensional subcomplex (graph) such that \( a \) and \( b \) are its vertices. A connected component in the space of paths joining \( a \) and \( b \) in this graph is obviously contractible. ■

**Statement 1** Let the set \( B(u) \) of branching values contain at least three points. Then the set \( \mathcal{E}_0(u) \) is contractible.
Proof Since $u$ is a ramified covering, the mapping $p_2 : \mathcal{E}_0(u) \to \mathcal{E}_0^{(2)}(u)$ is a covering, too (nonramified). So it suffices to prove that the set $\mathcal{E}_0^{(2)}(u)$ is contractible.

The set $B(u) \subset N$ is discrete. Join its points with a network, a set of smooth nonintersecting arcs such that the complement of these arcs in $M$ is a union of discs. Apparently a liftable diffeomorphism homotopic to the identity maps each branching value to itself. A network is mapped to another network homotopic to the original one. Thus $\mathcal{E}_0^{(2)}(u)$ is fibered over the connected component of the space of networks. The fiber is a direct product of several copies of the group of diffeomorphisms of a 2-disk, fixed on the boundary. The fiber is contractible by Alexander’s lemma, and the base is contractible by its Corollary given in the beginning of this Subsection.

Corollary Spaces $\mathcal{E}_0(u)$ and $\mathcal{E}_0^{(2)}(u)$ are homeomorphic (the covering $p_2$ is trivial).

Statement 2 Let the set $B(u)$ contain at least three points. Then the relative Teichmüller space $\mathcal{T}(u) = J(u)/\mathcal{E}_0(u)$ is weakly contractible (i.e. all its homotopy groups are trivial).

Proof Apparently, for any complex structure $I$ on $N$ there exists exactly one $u$-compatible pair $I \in J(u)$ such that $I = p_2(I)$. This, together with the Corollary of Statement 1, allows to consider the space $\mathcal{T}(u)$ as a quotient of the space $J$ of complex structures at $N$ by action of the group $\mathcal{E}_0^{(2)}(u)$. The point stabilizer of this action consists of diffeomorphisms of $N$ homotopic to identity, preserving branching values of $u$ and holomorphic with respect to the relevant complex structure. Since $B(u)$ contains at least three points, the stabilizer is trivial.

The natural quotient mapping $J \to \mathcal{T}(u)$ is thus a fibration with the fiber $\mathcal{E}_0^{(2)}(u)$. The fiber is contractible by Statement 1 and the total space is contractible, too. An exact homotopy sequence of the fibration $J \to \mathcal{T}(u)$ shows now that $\mathcal{T}(u)$ is weakly contractible.

Remark In fact, $\mathcal{T}(u)$ is contractible. What we need, though, is only the fact that it is simply connected (see the proof of Theorem 1 below).

Proof of Theorem 1 follows immediately from Statement 2: the Teichmüller space $\mathcal{T}(u)$ is simply connected, and one has only to observe that the action of the group $\text{Map}(u)$ in $\mathcal{T}(u)$ is faithful and discrete.

2 Equivariant mapping class group

From now on we restrict ourselves to the model example, a ramified covering $u : M_g \to \mathcal{C}P^1$ described in Introduction. Recall that $q$ is a homogeneous
polynomial of two complex variables, of degree $2g+1$. The associated polynomial of one variable $Q$ is defined by the equation $q(x, z) = z^{2g+1}Q(x/z)$. We suppose that all the $2g$ critical values of the polynomial $Q$ are distinct.

The full preimage $u^{-1}(a)$ of a generic point $a \in \mathcal{CP}^1$ consists of $4g+2$ points. There are $2g + 2$ branching values of $u$ having less preimages. Namely, $u^{-1}(\infty)$ contains only one point, $u^{-1}(0)$ contains $2g + 1$, and for each of the $2g$ critical values $P_0, \ldots, P_{2g-1}$ of the polynomial $Q$, the set $u^{-1}(P_i)$ has cardinality $4g$. We suppose that $g > 0$, and thus mapping $u$ satisfies hypotheses of Theorem 1.

In this Section we study the structure of the group $\text{Map}(u) = \pi_1(\mathcal{M}(u))$.

To avoid confusion we will always denote the target manifold of the mapping $u$ as $\mathcal{CP}^1$. One should remember, though, that we do not assume a complex structure on it to be fixed.

### 2.1 Mapping classes of sphere with marked points

Consider the projection $p_2 : \mathcal{E}(u) \to \mathcal{E}^{(2)}(u)$. Mappings of $\mathcal{CP}^1$ to itself that belong to the image of $p_2$ preserve the branching order. So, they map points $0$ and $\infty$ to themselves, and points $P_0, \ldots, P_{2g-1}$ are permuted in some way. In other words, $\mathcal{E}^{(2)}(u)$ is a subgroup of $\text{Diff}(\mathcal{CP}^1, \{0\}, \{\infty\}, \{P_0, \ldots, P_{2g-1}\})$.

The mapping class group $\text{Map}(\mathcal{CP}^1, \{0\}, \{\infty\}, \{P_0, \ldots, P_{2g-1}\})$ will be called $G_{2g}$ for short. We are now to study the group $G_{2g}$.

A simple curve $\varphi$ joining the points $0$ and $\infty$ of $\mathcal{CP}^1$ and avoiding the points $P_0, \ldots, P_{2g-1}$ will be called a meridian. A system $\varphi_0, \ldots, \varphi_{2g-1}$ of $2g$ meridians (numbered) will be called a slicing if the meridians do not intersect each other (except at $0$ and $\infty$), and cut $\mathcal{CP}^1$ into “slices” each containing exactly one point $P_k$. The group $G_{2g}$ acts on the set of homotopy classes of slicings. It follows immediately from Alexander’s lemma that this action is faithful (a point stabilizer is trivial).

Since we are interested with homotopy classes of mappings only, we may suppose that $P_k = \exp((2k + 1)\pi i/2g) \in \mathcal{CP}^1$, $k = 0, \ldots, 2g - 1$. We can also fix a “standard” slicing where the meridian $\varphi_k$ is the line $\arg z = 2\pi k/2g$. Thus, an element $\alpha \in G_{2g}$ is completely determined by the homotopy classes of meridians $\alpha(\varphi_0), \ldots, \alpha(\varphi_{2g-1})$.

Consider a rotation $z \mapsto z \exp(2\pi i/2g)$. It shifts cyclically meridians of the standard system: $\varphi_0 \mapsto \varphi_1 \mapsto \ldots \mapsto \varphi_{2g-1} \mapsto \varphi_0$. The corresponding mapping class will be called $\lambda \in G_{2g}$. Obviously, $\lambda$ is an element of order $2g$.

Consider now a subgroup $G^0_{2g} \subset G_{2g}$ consisting of elements mapping the meridian $\varphi_0$ to itself.

**Statement 3** The group $G^0_{2g}$ is isomorphic to a braid group $B_{2g}$ on $2g$ strands.
an action of $u_k$ is shown in Fig. 1 (it is assumed that $u_k$ is identical outside a small neighborhood of the segment $P_{k-1}P_k$).

**Statement 4** The mapping class group $G_{2g}$ is generated by its subgroups $G^0_{2g}$ and $\langle \lambda \rangle \subset G_{2g}$.

See [5] for proof.

Let $s_k$ be a simple curve which starts at $\infty$, encircles the point $P_k$, and then ends at $\infty$ not intersecting the meridians $\varphi_i$, $i = 0, \ldots, 2g - 1$. Homotopy classes of the loops $s_k$ are free generators of the fundamental group $\pi_1(\mathbb{C}P^1 \setminus \{0, P_0, \ldots, P_{2g-1}, \infty\})$ ($\infty$ is a base point). Group $G_{2g}$ acts on this fundamental group in the following way:

1. $\lambda s_k = s_{k+1}$, \hspace{2cm} (2)
2. $u_k s_{k-1} = s_k$, \hspace{2cm} (3)
3. $u_k s_k = s^{-1}_k s_{k-1} s_k$, \hspace{2cm} (4)
4. $u_k s_l = s_l$, if $l \neq k - 1, k$. \hspace{2cm} (5)

Addition in subscripts is made here mod $2g$.

Automorphisms $\lambda$ and $u_k$, $k = 1, \ldots, 2g - 1$ of the free group $F(s_0, \ldots, s_{2g-1})$ given by formulas (2)–(5) generate a subgroup of the group $\text{Aut}(F(s_0, \ldots, s_{2g-1}))$ called *braid-cyclic group* $BC_{2g}$ on $2g$ strands. Thus formulas (2)–(5) define a homomorphism $G_{2g} \to BC_{2g}$. It follows easily from Alexander’s lemma that this homomorphism has no kernel, i.e. is an isomorphism.

Thus we can summarize the results about group $G_{2g}$ in the following

**Theorem 2** The mapping class group $G_{2g} \overset{\text{def}}{=} \text{Map}(\mathbb{C}P^1; \{0\}, \{\infty\}, \{P_0, \ldots, P_{2g-1}\})$ is isomorphic to the braid-cyclic group $BC_{2g}$. It is a product (not direct) $G_{2g} = \langle \lambda \rangle G^0_{2g}$. Here the subgroup $G^0_{2g}$ is a stabilizer of the meridian $\varphi_0$, it is isomorphic to the braid group $B_{2g}$. The element $\lambda$ has order $2g$. 

Figure 1: Mapping class $u_k$
2.2 Liftable mapping classes

In this Subsection we are to answer a question, which mapping classes \( \alpha \in G_{2g} \) are liftable, i.e. belong to the image of the homomorphism \( p_2 : \text{Map}(u) \to G_{2g} \). The subgroup of liftable classes will be denoted \( \mathcal{L}(u) \).

Consider the full preimage \( u^{-1}(\varphi) \subset M_{g} \) of a meridian \( \varphi \) under the mapping \( u \). It is a graph having \( 2g + 2 \) vertices. One of them is a point \( T = [0 : 1 : 0] = u^{-1}(\infty) \) and the other \( 2g + 1 \) are elements \( S_1, \ldots, S_{2g+1} \) of \( u^{-1}(0) \) (zeros of the polynomial \( Q \)). The graph has \( 4g + 2 \) edges, each one joining \( T \) with some \( S_k \). The valence of the vertex \( T \) is \( 4g + 2 \), and the valence of each vertex \( S_k \) is 2. This allows us to ignore vertices \( S_k \) and to consider \( u^{-1}(\varphi) \) as a graph with a single vertex \( T \) and \( 2g + 1 \) loops \( \gamma_1, \ldots, \gamma_{2g+1} \) attached to it. Vertices \( S_k \) look then as midpoints of the loops. Relative positions of the loops are described in

**Statement 5** If one cuts \( M_g \) along the \( 2g \) curves \( \gamma_1, \ldots, \gamma_{2g} \), one obtains a \( 4g \)-gon embedded into \( M_g \) so that its opposite sides are glued together (correspond to the same curve \( \gamma_i \)). The remaining curve \( \gamma_{2g+1} \) joins opposite vertices of the \( 4g \)-gon obtained (looks like its “large diagonal”).

Indeed, numeration of the curves \( \gamma_i \) can be chosen at random, and so any \( 2g \) curves \( \gamma_i \) can become \( \gamma_1, \ldots, \gamma_{2g} \) mentioned in the Statement.

**Proof** Take the “square root” of the mapping \( u \) presenting it as \( u = s \circ v \) where \( v : M_g \to \mathbb{C}P^1 \) maps the point \( [x : y : z] \) to \( [y : z] \), and \( s : \mathbb{C}P^1 \to \mathbb{C}P^1 \) is defined as \( s(\xi) \stackrel{\text{def}}{=} \xi^2 \). Let \( \Phi \) be the full preimage of the meridian \( \varphi \) under \( s \). Without loss of generality one can suppose that \( \Phi \) is the real axis. It divides \( \mathbb{C}P^1 \) into two open disks \( U_{\pm} = \{ \xi \mid \text{Im} \xi <> 0 \} \). Preimages of the disks \( U_{\pm} \) under the mapping \( v \) are the sets \( D_{\pm} = v^{-1}(U_{\pm}) = \{ (x, y) \in \mathbb{C}^2 \mid y^2 = Q(x), \text{Im} y <> 0 \} \). Projection \( (x, y) \mapsto x \) maps homeomorphically each of the sets \( D_{\pm} \) to the set \( A = \{ x \in \mathbb{C} \mid Q(x) \notin \mathbb{R}_+ \} \). The polynomial \( Q \) does not have critical values on the real axis \( \Phi \) (\( \Phi \) is a preimage of the meridian \( \varphi \)), and, so, \( A \) is homeomorphic to a disk, and the same is true for \( D_{\pm} \), too. Thus we have proved that the \( u^{-1}(\varphi) \) cuts \( M_{g} \) into a union of two disks.

The curves \( \gamma_1, \ldots, \gamma_{2g+1} \) constitute the boundary of each disk \( D_{\pm} \). So, orientation of \( D_{\pm} \) fixes a cyclic order of the curves \( \gamma_1, \ldots, \gamma_{2g+1} \), and the orientation of \( D_- \) does the same thing. Consider the involution \( \mu : M_{g} \to M_{g} \) given by the formula \( \mu(x, y) = (x, -y) \). It exchanges the disks \( D_{+} \leftrightarrow D_{-} \), leaves every curve \( \gamma_i \) fixed (as a whole) and preserves the orientation. Thus, the cyclic orderings induced in the set \( \gamma_1, \ldots, \gamma_{2g+1} \) by orientations of the disks \( D_{+} \) and \( D_{-} \) are the same. This finishes the proof. 

The last assertion of Statement 5 was proved using the involution \( \mu \). This involution looks like a central symmetry of the \( 4g \)-gon described in Statement 5. It is clear that \( p_2 \circ \mu = p_2 \), so that \( \mu \) projects to the identity mapping of \( \mathbb{C}P^1 \). By this reason the full preimage \( u^{-1}(\mathcal{U}) \subset M_{g} \) of any subset \( \mathcal{U} \subset \mathbb{C}P^1 \) is centrally
self-symmetric. So, to describe \( u^{-1}(U) \) it suffices to describe its intersection with one disk, say, \( D_+ \).

Statement 6 allows to view the disk \( D_+ \) as a polygon with \( 2g + 1 \) vertices. Take now into account the \( 2g + 1 \) points \( u^{-1}(0) \) and regard them as vertices, too, to obtain a \((4g + 2)\)-gon. We supply the vertices of this \((4g + 2)\)-gon with alternating coloring, so that the \( 2g + 1 \) vertices are \( u^{-1}(0) \) black, and the other \( 2g + 1 \) vertices \( u^{-1}(\infty) \) are white. All the white vertices are indeed the same point in the \( M_g \), but Statement 6 allows to regard them as different ones. We will do it systematically, without further notice. For example, if two smooth curves \( \gamma_1, \gamma_2 \subset D_+ \) intersect at the point \( u^{-1}(\infty) \) only, and are separated near it by some curve \( \gamma_i \subset u^{-1}(\varphi) \), then we say that \( \gamma_1 \) and \( \gamma_2 \) have no common points. They look as starting at different vertices of the polygon \( D_+ \).

Statement 6 describes the preimage of an individual meridian. The following theorem deals with preimages of slicings.

**Theorem 3** Let \( \varphi_1, \ldots, \varphi_{2g} \) be a slicing. Let the preimage \( u^{-1}(\varphi_1) \) cut \( M_g \) into a union \( D_+ \sqcup D_- \). Let \( \Gamma \overset{\text{def}}{=} u^{-1}(\varphi_2 \cup \ldots \cup \varphi_{2g}) \sqcap D_+ \). Then \( \Gamma \) is a union of simple curves (edges) starting in white vertices and ending at black ones. The edges have no common internal points. They cut \( D_+ \) into parts (faces), each of them being homotopic either to a quadrangle or to a lune (biadgle). Each face contains exactly one preimage of some point \( P_i \in \mathbb{C}P^1 \).

**Proof** Recall that the meridians \( \varphi_0, \ldots, \varphi_{2g-1} \) of the slicing cut \( \mathbb{C}P^1 \) into slices each one containing exactly one point \( P_i \). Connected components of preimages of these slices are the parts into which \( \Gamma \) divides \( M_g \).

Consider, for example, the slice \( U \) bounded by \( \varphi_1 \) and \( \varphi_2 \), and containing \( P_1 \). Split it into a union of curves \( \varphi_\tau, 1 \leq \tau \leq 2 \), where the curves \( \varphi_\tau, \tau \neq 3/2 \) are meridians having no common internal points, and the curve \( \varphi_{3/2} \) starts at \( \infty \), passes \( P_1 \), ends at 0, and has no common internal points with the other curves \( \varphi_\tau \) either. Preimages \( u^{-1}(\varphi_\tau) \cap D_+ \) for \( \tau \neq 3/2 \) are systems of smooth curves joining black and white vertices of \( D_+ \). These curves are homotopic to the respective components of \( u^{-1}(\varphi_1) \) (for \( 1 \leq \tau < 3/2 \)) or \( u^{-1}(\varphi_2) \) (for \( 3/2 < \tau \leq 2 \)).

Preimage of the point \( P_1 \) in the disk \( D_+ \) consists of \( 2g \) points. One of them is a branching point of the multiplicity 2, and the other \( 2g - 1 \) points are not critical. Thus the preimage \( u^{-1}(\varphi_{1/2}) \) consists of \( 2g \) components. \( 2g - 1 \) of them are smooth, and the corresponding part of \( M_g \) is a lune. The last component looks like two smooth curves intersecting transversally at one point. Apparently, the corresponding part of \( M_g \) is a quadrangle. \( \blacksquare \)

Recall that the meridians of the slicing are numbered: \( \varphi_0, \varphi_1, \ldots, \varphi_{2g-1} \). Number the points \( P_i \) also, so that the slice bounded by \( \varphi_k \) and \( \varphi_{(k+1) \mod 2g} \) contain \( P_k \). The last assertion of Theorem 6 allows then to label the faces with symbols \( 0, 1, \ldots, 2g - 1 \) — the face \( F \) containing the preimage of \( P_i \) is labelled \( i \overset{\text{def}}{=} \ell(F) \).
Let us direct each edge, as well as each side of the \((4g + 2)\)-gon, from its white end to its black end. Let \(F_1\) and \(F_2\) be two faces separated by an edge \(e\), \(F_1\) lying to the right of it, and \(F_2\), to the left. Since the mapping \(u\) preserves the orientation, labels of the faces satisfy the equation \(\ell(F_2) = \ell(F_1) + 1 \pmod{2g}\). This remark allows to reproduce positions and labels of the lune-like (biangular) faces uniquely up to a homotopy, if positions and labels of the quadrangular faces are given. Thus, the full preimage of a slicing can be described, up to a homotopy, by two sorts of data: an embedding of the \((2g + 1)\)-loop graph \(u^{-1}(\varphi_1)\) into \(M_g\), and a splitting of the corresponding \((4g + 2)\)-gon \(D_+\) into quadrangles labelled 0, 1, \ldots, \(2g - 1\). Recall, too, that the vertices of the \((4g + 2)\)-gon \(D_+\) are colored alternatingly black and white. Splitting of \(D_+\) into labelled quadrangles together with vertex coloring will be called a (marked) quadrangulation. Quadrangulation corresponding to the slicing \(\Phi\) will be denoted \(\square(\Phi)\).

Fix now a “standard” slicing \(\Phi_0 = \{\varphi^0_0, \ldots, \varphi^0_{2g-1}\}\), for example, the one described in Subsection 2.1. Recall that the mapping class group \(G_{2g} = \text{Map}(\mathcal{C}P^1, \{0\}, \{\infty\}, \{(P_0, \ldots, P_{2g-1})\})\) acts (faithfully) at the set of homotopy classes of slicings. Now we can formulate the required criterion of liftability:

**Theorem 4** The element \(\alpha \in G_{2g}\) is liftable (belongs to the subgroup \(L(u) \subset G\)) if and only if \(\square(\alpha(\Phi_0)) = \square(\Phi_0)\).

**Proof** follows easily from Alexander’s lemma.

### 2.3 A short exact sequence

**Lemma** Let \(Q : \mathcal{C} \to \mathcal{C}\) be a polynomial of odd degree \(2g + 1\) with \(2g\) distinct critical values, and \(f : \mathcal{C} \to \mathcal{C}\) be a diffeomorphism such that \(Q(f(x)) = Q(x)\) for all \(x \in \mathcal{C}\). Then \(f(x) \equiv x\).

**Proof** The mapping \(f\) in question is an algebraic diffeomorphism of \(\mathcal{C}\), and thus a linear function: \(f(x) = px + q\). This function preserves all the \(2g\) critical points (distinct) of the polynomial \(P\), and therefore \(f(x) \equiv x\).

Recall (see Theorem 3) that the group \(G_{2g}\) is an indirect product \(\langle \lambda \rangle G^0_{2g}\). Here the cyclic subgroup \(\langle \lambda \rangle\) generated by the “rotation” \(\lambda\) (see definition at page 2) of order \(2g\). The subgroup \(G^0_{2g}\) is isomorphic to the braid group \(B_{2g}\) and consists of the mapping classes preserving the meridian \(\varphi_0\) of the standard slicing. Denote \(L^0(u) = L(u) \cap G^0_{2g}\).

**Theorem 5** The group \(\text{Map}(u)\) fits into a short exact sequence of groups

\[
0 \to \mathbb{Z}_2 \to \text{Map}(u) \to L(u) \to 0
\]

The generator of the group \(\mathbb{Z}_2\) is mapped to the involution \(\mu\) lying in the center of \(\text{Map}(u)\). The second arrow is the projection \(p_2\). The group \(\text{Map}(u)\) is an
indirect product $RF$ where $p_2(F) = L^0(u)$, and $p_2 : F \to L^0(u)$ is an isomorphism. The group $R$ is a cyclic group generated by the element $(A, \lambda) \in \text{Map}(u)$ of order $4g$.

Recall that the involution $\mu : M_g \to M_g$ acts by the formula $\mu(x, y) = (x, -y)$ (see remark after the proof of Statement 3).

**Proof** Prove first that the kernel of the mapping $p_2 : \text{Map}(u) \to L(u)$ is generated by the involution $\mu$. Projections $p_1(\tau)$ and $p_2(\tau)$ determine an element $\tau \in \text{Map}(u)$ completely. Thus, it suffices to prove that $\mu$ is the only nontrivial element of the group $L^{(1)}(u)$ which projects to the identity mapping of $\mathcal{C}P^1$.

Apparently any such mapping $\tau : M_g \to M_g$ acts by the following way: $\tau(x, y) = (f(x), \pm y)$ where $f : \mathcal{C} \to \mathcal{C}$ is a diffeomorphism preserving the polynomial $Q$: $Q(f(x)) = Q(x)$. By the lemma, $f(x) \equiv x$.

Fix now a standard slicing $\Phi_0$ at $\mathcal{C}P^1$, and let $M_g = D_+ \cup D_-$ like in Theorem 3. Define $F$ as the following subgroup of $\text{Map}(u)$: $f \in F \iff p_1(f)$ preserves $D_+$ (and therefore $D_-$, too). Apparently, $p_2(F) = L^0(u)$. Since $\mu \notin F$, the restriction $p_2|_F$ is an isomorphism.

Let now $(A, \lambda)$ be a lifting of the element $\lambda \in G_{2g}$. Such lifting exists (i.e. $\lambda \in L(u)$): the mapping $A : M_g \to M_g$ is a rotation of the $4g$-gon $D_+ \cup D_-$ to the $1/4g$ of the full cycle (more exactly, it is one of two mappings projecting to $\lambda$, the second in $A\mu$). Obviously, $A$ is an element of order $4g$. Theorem 3 implies that the group $L(u)$ is generated by the element $\lambda = p_2((A, \lambda))$ and the subgroup $L^0(u) = p_2(F)$; the kernel of $p_2$ is generated by $(\mu, \text{Id}) = (A, \lambda)^{2g}$. So, the group $\text{Map}(u)$ is generated by $(A, \lambda)$ and the subgroup $F$. $\blacksquare$

### 3 Actions of the braid-cyclic group

#### 3.1 Action on marked quadrangulations

The liftability criterion, as formulated in Theorem 3, does not answer directly a question when an element $\alpha \in G_{2g} = BC_{2g}$ written in generators $\lambda, u_1, \ldots, u_{2g-1}$ is liftable. In this Section we will try to do it giving an algorithmic description of the action of $\alpha$ at marked quadrangulations.

Consider a quadrangulation $\Delta$ of the $(2N + 2)$-gon $S$ with vertices colored alternatingly black and white. It is easy to see that any quadrangle has two black and two white vertices, and that its opposite vertices have the same color. Fix now a vertex $v$. Orientation of $S$ determines a linear ordering of the quadrangles (faces of $\Delta$) having $v$ their vertex. On the other hand, this set is ordered by the labels $0, 1, \ldots, N - 1$ of faces. We call quadrangulation $\Delta$ *monotone* if these two orderings coincide for all white vertices $v$ and are opposite for all black vertices.

**Statement 6** For any slicing $\Phi$ the quadrangulation $\square(\Phi)$ of the $(4g + 2)$-gon $S = D_+$ is monotone.
Proof Consider the quadrangulation \( \square(\Phi) \) together with the biangular faces. Mapping \( u : M_g \to \mathbb{CP}^1 \) preserves the orientation, and therefore the cyclic ordering of edges in every white vertex of the graph \( \Gamma = u^{-1}(\Phi) \) coincides with the ordering of the meridians: \( \varphi_0, \varphi_1, \ldots, \varphi_{2g-1} \). So, if the quadrangulation is not monotone then there exists a pair of faces (biangular or quadrangular) attached to the same vertex and separated by the preimage of the meridian \( \varphi_0 \). This is impossible because \( u^{-1}(\varphi_0) \) forms the boundary of the \((4g + 2)\)-gon \( S \). The same argument applies to black vertices.

Note now that for any monotone quadrangulation the face labelled 1 has only one or two sides adjacent to other faces, and if there are two such sides then they are opposite. The other two or three sides lie in the boundary of the polygon \( S \). The same remark applies to the face labelled \( N \). Now we are ready to define an action of the braid-cyclic group \( BC_N \) on the set of monotone quadrangulations.

**Statement 7** The following rules define an action of the group \( BC_N \) on the set of quadrangulations of the \((2N + 2)\)-gon, with faces labelled \( 0, 1, \ldots, N - 1 \).

1. Element \( u_k \) acts only on the faces labelled \( k \) and \( k - 1 \). Here are two cases:
   
   (a) If the faces labelled \( k - 1 \) and \( k \) are not adjacent then they exchange their labels (the quadrangulation remains the same).
   
   (b) If these faces are adjacent, and thus form a hexagon, then this hexagon is rotated by \( 1/3 \) of the full cycle.

2. Element \( \lambda \) acts as follows. First, it shifts the face labelling cyclically: \( 1 \mapsto 2 \mapsto \ldots \mapsto N \mapsto 1 \). Let \( F \) be the face labelled 1, and \( AB \) and \( CD \) be its opposite side lying on the boundary of \( S \) (they exist by the previous remark), and let \( P \) and \( Q \) are vertices of \( S \) such that \( A, B, P \) are subsequent vertices, and \( C, D, Q \), too. Then the face \( F = ABCD \) is replaced by the face \( PBQD \).

Action of \( \lambda \) on quadrangulations (“the flip”) is shown in Fig. 2. Action of \( u_k \) is shown at Fig. 3 below.

**Proof** To prove that we have defined an action, we should check the defining relations of the group \( BC_N \) (see [4] for proof; it follows easily from relations (3)–(6), too):

\[
\begin{align*}
\lambda^N &= 1 \quad & (7) \\
\lambda u_k &= u_{k+1} \lambda, \quad k = 1, \ldots, N - 2. \quad & (8) \\
u_k u_l &= u_l u_k, \quad k, l = 1, \ldots, N - 1, |k - l| \geq 2; \quad & (9) \\
u_k u_{k+1} u_k &= u_{k+1} u_k u_{k+1}, \quad k = 1, \ldots, N - 2. \quad & (10)
\end{align*}
\]

The checking is routine. ■

Let \( a \mapsto a^{inv} \) be an anti-homomorphous involution of the group \( BC_N \) sending each \( u_k \) to itself, and \( \lambda \) to \( \lambda^{-1} \).
Theorem 6 Actions of the group $BC_{2g}$ on the set of slicings and on the set of monotone quadrangulations are connected by the following formula:

$$\square(\alpha(\Phi_0)) = \alpha^{inv}(\square(\Phi_0)).$$

Proof Write $\alpha = s_1s_2\ldots s_M$ where each $s_i$ is either $\lambda$ or $u^{\pm1}_k$, and denote $\alpha' = s_2\ldots s_M$. To prove the theorem by induction on $M$ suppose that for $\alpha'$ the theorem is proved, and consider three cases. The symbol $F_i$ denotes the face labelled $i$ in the quadrangulation $\square(\Phi)$ where $\Phi = \alpha'(\Phi_0)$.

Case 1. $s_1 = u^{\pm1}_k$, and the faces $F_k$ and $F_{k+1}$ are not adjacent. When $u^{\pm1}_k$ acts on $\Phi_0$, then all the meridians except $\varphi_k$ remain the same, points $P_k$ and $P_{k+1}$ exchange places, and each point $P_i$ goes to itself. Therefore the quadrangulations $\square(\alpha'(\Phi_0))$ and $\square(\alpha(\Phi_0))$ can differ in the faces labelled $k$ and $k + 1$ only. These faces are not adjacent, so the quadrangulation remains geometrically the same. Labels $k$ and $k + 1$ exchange places because the points $P_k$ and $P_{k+1}$ do so.

Case 2. $s_1 = u_k$, and the faces $F_k$ and $F_{k+1}$ are adjacent, thus forming a hexagon $H$. Like in the previous case, one shows that the quadrangulations $\square(\alpha'(\Phi_0))$ and $\square(\alpha(\Phi_0))$ can differ inside the hexagon $H$ only. Denote $\varrho'$ the border between $F_k$ and $F_{k+1}$. The image $u(\varrho')$ is the meridian $\alpha'(\varphi_k)$. Image of some other diagonal of $H$, say, $\varrho$, is the meridian $\alpha(\varphi_k)$.

Draw curves $\gamma_1$ and $\gamma_2$ like in Fig. 3. The sets $\ell_1 = u^{-1}(\gamma_1) \cap H$ and $\ell_2 = u^{-1}(\gamma_2) \cap H$ are homotopic to diagonals of $F_1$ and $F_2$, respectively. Then, diagonal $\varrho$ has a common point with the curve $\ell_1$, and another one, with $\ell_2$, and these points are preserved under any homotopy of curves $\ell_1$ and $\ell_2$ with fixed endpoints. It means that $\varrho$ is the diagonal obtained from $\varrho'$ by a counterclockwise rotation of $H$ to the $1/3$ of the full cycle. To determine the labelling of the quadrangles obtained we can use the same arguments as in the previous case.
Case 3. $s_1 = \lambda$. In this case the meridians are mapped cyclically: $\varphi_0 \mapsto \varphi_1 \mapsto \ldots \mapsto \varphi_{2g} \mapsto \varphi_0$. Thus the splitting of the whole $M_g$ into quadrangles does not change, but the face which was labelled $i$ is now labelled $i - 1$. Besides this, the splitting $M_g = D_+ \cup D_-$ changes: we cut $M_g$ along the preimage of the new $\varphi_0$ which was previously $\varphi_1$. One can easily observe that this corresponds exactly to the “flip” of quadrangulation defining the action of $\lambda^{-1}$ in Statement 7.

Remark Indeed, we can choose any slicing $\Phi$ as a “standard” slicing $\Phi_0$. Another choice of $\Phi_0$, though, leads to another system of generators $\lambda, u_1, \ldots, u_{2g-1}$, and thus to another involution $a \mapsto a^{\Phi_0}$, so that Theorem 3 remains true. If one changes $\Phi_0$ only (but involution remains the same), the theorem fails. See [10] for more information on systems of generators of the braid group.

3.2 Action on trees with labelled edges

In this Subsection we define an action of the group $BC_N$ on the set of all trees with $N$ edges labelled $0, 1, \ldots, N - 1$, and investigate its relation with the action of Statement 7. This action is of some independent interest, and also allows us to obtain some information about the group $L(u)$.

Statement 8 The following rules define an action of the group $BC_N$ on the set of all trees with $N$ edges labelled $0, 1, \ldots, N - 1$.

1. Element $u_k$ acts only on the edges labelled $k - 1$ and $k$. Here are two cases:
   
   (a) If the edges labelled $k - 1$ and $k$ do not have common vertices then they exchange their labels (the tree remains the same).
   
   (b) If the edge labelled $k - 1$ joins the vertices $A$ and $B$, and the edge labelled $k$ joins $A$ and $C$ then the edge $AB$ receives label $k$, edge $AC$ is erased, and the edge $BC$ is drawn and labelled $k - 1$.

2. Element $\lambda$ does not affect the tree but shifts the edge labelling cyclically: $1 \mapsto 2 \mapsto \ldots \mapsto N \mapsto 1$. 

The proof copies that of Statement 7.

**Statement 9** Action of the braid-cyclic group $BC_N$ on the set of trees with edges labelled $0, 1, \ldots, N - 1$ is transitive.

**Proof** We prove that already the action of the subgroup $B_N \subset BC_N$ generated by $u_1, \ldots, u_{N-1}$ is transitive.

Take some vertex $a$ of the tree as a root, and orient every edge away from the root (downwards). We can speak now about upper and lower end of an edge. Two edges having the same upper end will be called brothers. If the lower end of one edge is the upper end of another edge, then such edges will be called parent and child, respectively.

Call the complexity of a tree $\Gamma$ (notation $C(\Gamma)$) the sum of lengths of all strictly decreasing paths starting at the root. The minimal possible value for the complexity is $N$, the number of edges. Only one tree has complexity $N$, this is the “bush” tree where all $N$ edges are attached to the root. We will prove that for any other tree $\Gamma$ there exists an element $\alpha \in B_N$ such that $C(\alpha(\Gamma)) < C(\Gamma)$. This will mean that $\Gamma$ can be mapped to the “bush” tree by some element of the braid group, and thus the action is transitive.

The generator $u_k$ of the braid group affects the tree complexity in the following way (the edge labelled $i$ is denoted $e_i$):

1. If $e_{k-1}$ and $e_k$ are not adjacent then $C(u_k(\Gamma)) = C(\Gamma)$.
2. If $e_{k-1}$ and $e_k$ are brothers then $C(u_k(\Gamma)) > C(\Gamma)$.
3. If $e_{k-1}$ is the parent of $e_k$ then $C(u_k(\Gamma)) < C(\Gamma)$.
4. If $e_k$ is the parent of $e_{k-1}$ then any inequality between $C(u_k(\Gamma))$ and $C(\Gamma)$ is possible, but $C(u_k^2(\Gamma)) = C(u_k^{-1}(\Gamma)) < C(\Gamma)$.

We will look for such element $\alpha = u_{i_1} \ldots u_{i_M}$ that for $s = 1, \ldots, M - 1$ action of $u_{i_s}$ does not change complexity, and action of $u_{i_M}$ reduces it. To find $\alpha$ consider several cases.

1. There exists an edge $e_k$ with $k < N$ and having a child.
   1.1 $e_{k+1}$ is a child of $e_k$. In this case we apply $u_{k+1}$ and decrease complexity.
   1.2 $e_{k+1}$ is not a child of $e_k$ but there is a child $e_s$ with $s < k + 1$. Take the greatest such $s$. The edge $e_{s+1}$ is neither a brother nor a parent of $e_s$, so that we can apply $u_s$ not increasing the complexity. Thus we have reproduced the same situation but with $s \mapsto s + 1$. Repeating the process several times we arrive to the situation of Case 1.1.
   1.3 All the children of $e_k$ have labels greater than $k + 1$. Then take the smallest label $s$. Here are two more cases:
1.3.1 $e_{s-1}$ is not a child of $e_s$. Then we apply $u_s$ not changing the complexity. Thus we have reproduced the situation of Case 1.3 but with $s \mapsto s - 1$. Repeating the process several times we arrive to the situation of Case 1.3.

1.3.2 $e_{s-1}$ is not a child of $e_s$. Here we apply $u_s^2$ decreasing the complexity.

2. Only the edge $e_N$ has children (the “shrub” tree). Here are two cases:

2.1 $e_{N-1}$ is a child of $e_N$. Here we can apply $u_N^2$ to decrease complexity.

2.2 $e_{N-1}$ is attached to the root and is not thus a child of $e_N$. Let $e_k$ be a child of $e_N$ with the greatest $k$. So, $e_{k+1}$ is neither a child nor a parent nor a brother of $e_k$, and we can apply $u_{k+1}$ not changing the complexity. Thus we have reproduced the same situation but with $k \mapsto k + 1$. Repeating the process several times we arrive to the situation of Case 2.1.

Establish now a link between the action of the braid-cyclic group on trees and the action of Statement 7.

Let now $\Delta$ be a monotone marked quadrangulation of the $(2N+2)$-gon with vertices colored alternatingly black and white. Recall that any quadrangle has two black and two white vertices, its opposite vertices having the same color. Draw a graph $\Gamma(\Delta)$ joining two black vertices of every quadrangle. Edges of $\Gamma(\Delta)$ are naturally labelled $0, 1, \ldots, N-1$.

**Statement 10** The graph $\Gamma(\Delta)$ is a tree. For any tree $T$ with edges labelled $0, 1, \ldots, N-1$ there exists exactly one monotone quadrangulation $\Delta$ such that $\Gamma(\Delta) = T$. Actions of the braid-cyclic group on monotone quadrangulations and on trees are compatible: $\Gamma(\alpha(\Delta)) = \alpha(\Gamma(\Delta))$ for any $\alpha \in BC_N$ and any monotone quadrangulation $\Delta$.

**Proof** Suppose that $\Gamma$ has a cycle formed by edges $d_1, \ldots, d_k$. Since $d_1, \ldots, d_k$ do not intersect, they form a polygon $P$. At least one vertex of the original $(2N+2)$-gon lies inside $P$ which is impossible. Thus, $\Gamma$ is a tree.

The second assertion is proved by induction on $N$. Consider the face $F$ of $\Delta$ marked $N-1$. The quadrangulation $\Delta$ is monotone, and therefore two opposite sides of $F$ are sides of the $(2N+2)$-gon. Take now the tree $T$. The edge $e_{N-1}$ divides it into two subtrees, the first containing $k$ edges, and the second, $N-k$ edges. This allows to determine uniquely, up to a rotation of the $(2N+2)$-gon, the position $F$ — its black diagonal divides the polygon into parts containing $2k + 3$ and $2N - 2k + 1$ vertices, respectively. Positions of other faces are then fixed uniquely, too, by the induction hypothesis.

The third assertion is proved by the obvious comparison of definitions, see Statement 9 and Statement 8. ■
Corollary \( \mathcal{L}(u) \) is a subgroup of the braid-cyclic group of finite index \((2g + 1)^{2g-2}\).

Proof Indeed, there exist \((n + 1)^{n-2}\) different trees with \(n\) numbered edges.

For \( N = 2g \) the graph \( \Gamma(\Delta) \) can be also obtained by the following construction. Consider a mapping \( v : M_g \to \mathbb{C}P^1 \) defined by the formula \( v([x : y : z]) = [x : z] \). This mapping is not defined in the point \([0 : 1 : 0]\) but outside it there is an equality \( u = Q \circ v \) where \( Q : \mathbb{C}P^1 \to \mathbb{C}P^1 \) sends \( \xi \) to \( Q(\xi) \), \( Q \) being a polynomial used in definition of \( M_g \). Consider a slicing \( \Phi \) such that \( \Delta = u^{-1}(\Phi) \), and join every point \( P_k \) with 0 by a simple curve \( \gamma_k \) inside the corresponding slice.

The preimage \( Q^{-1} \left( \bigcup_{k=0}^{2g-1} \gamma_k \right) \subset \mathbb{C}P^1 \) was considered in [8, 9, 3]. It looks like a graph with two types of edges. Edges of the first type join different preimages of the point 0. The midpoints of these edges are preimages of the points \( P_i \), one edge for each \( i \). Edges of the second type are "hanging" edges joining a preimage of 0 with a preimage of some \( P_i \) (different from the midpoints considered earlier). It was shown in [3] that the edges of the first type form a tree \( \mathcal{G} \). It is easy to see that \( \Gamma(\Delta) = v^{-1}(\mathcal{G}) \). This gives another proof of the first assertion of Statement 10.

Statement 9 implies that there exists a slicing \( \Phi \) such that the \( \square(\Phi) \) is a “trivial” quadrangulation, where all the quadrangles have a common black vertex. Consider such slicing and the corresponding generators \( \lambda, u_1, \ldots, u_{2g-1} \) of the group \( G_{2g} \). It follows from Theorem [3], Statement [7] and Theorem [3] that the element \( U = u_1 \ldots u_{2g-1} \) is liftable.

Conjecture Elements \( \lambda \) and \( U \) generate the group \( \mathcal{L}(u) \). There are no relations between them except \( \lambda^{2g} = 1 \), and thus \( \mathcal{L}(u) \) is isomorphic to the free product \( \mathbb{Z}_{2g} * \mathbb{Z} \).

3.3 Action on subgroups of the free group

In this Subsection we are to reveal the algebraic origin of the actions of Statement [7] and Statement [8]. We show that it is a part of the general action of the group \( \text{Aut}(F_N) \) of automorphisms of the free group \( F_N \) on the set of its subgroups. Embedding \( BC_N \hookrightarrow \text{Aut}(F_N) \) is given by the formulas (3)–(5).

To establish the link we identify first certain subgroups of \( F_N \) with trees with labelled edges.

Consider \( F_N \) as the fundamental group of the union of \( N \) circles \( \mathcal{U} = \bigvee_{k=0}^{N-1} (S^1)_k \). Coverings spaces (nonramified) \( \Upsilon \) of \( \mathcal{U} \) are graphs where all vertices have valence 2\( N \), and edges are labelled 0, 1, \ldots, \( N - 1 \). The label \( k \) means that the corresponding edge is mapped to the \( k \)-th circle of the union \( \mathcal{U} \) by the covering map \( r : \Upsilon \to \mathcal{U} \). The number of vertices of \( \Upsilon \) is equal to the number of sheets of the covering \( r \). If the circles of the union \( \mathcal{U} \) are oriented, then the graph \( \Upsilon \) receives an orientation, too.
Recall that a \textit{path} in the graph $\Upsilon$ is a sequence of edges (two successive edges in the sequence have a common vertex) passing every edge not more than once; \textit{circuit} is a closed sequence of edges passing every vertex not more than once. Covering $\Upsilon$ will be called \textit{tree-like} if it possesses the following properties:

1. All circuits in it have length 1 or 2.
2. If a circuit has length 2 then both edges in it have the same label.
3. For any number $k \in \{0, 1, \ldots, N - 1\}$ there exists exactly one circuit of length 2 whose edges are labelled $k$.

For a tree-like covering space $\Upsilon$ a graph $\Gamma(\Upsilon)$ is defined as follows. Vertices of $\Gamma(\Upsilon)$ are vertices of $\Upsilon$. Two vertices, $A$ and $B$ of $\Gamma(\Upsilon)$ are connected by the edge labelled $k$ if $\Upsilon$ contains edges $AB$ and $BA$ both labelled $k$ (labels must coincide by Property 2). Loops of $\Upsilon$ are ignored. By Property 3 it has $N$ edges so that every label $0, 1, \ldots, N - 1$ is used exactly once.

The graph $\Upsilon$ can be restored uniquely by $\Gamma(\Upsilon)$. Indeed, positions and labels of loops are determined unambiguously by $\Gamma(\Upsilon)$ and the condition that for every vertex $v$ of $\Upsilon$ and every $k \in \{0, 1, \ldots, N - 1\}$ exactly two edge ends incident to $v$ should be labelled $k$.

Choose a base point $B$ in the covering space $\Upsilon$; without loss of generality let $B$ be a vertex. Now we can relate the covering $r : \Upsilon \to \mathcal{U}$ to the subgroup $g(\Upsilon) \defeq r_* (\pi_1 (\Upsilon, B)) \subset \pi_1 (\mathcal{U}, O) = F_N$ of the free group; here $O$ is the vertex of the union. This correspondence between subgroups and coverings with base point is one-to-one, by the following classical theorem of homotopic topology (see e.g. [6] for proof):

\textbf{Theorem 7} For any subgroup $G \subset F_N$ there exists covering $r : \Upsilon \to \mathcal{U}$ with the base point $B$ such that $g(\Upsilon) = G$. Any two coverings $r_1, r_2$ possessing this property are equivalent in the following sense: there exist a homeomorphism $f : \Upsilon_1 \to \Upsilon_2$ such that $f(B_1) = B_2$ and $r_2 \circ f = r_1$. The number of sheets of the covering $r$ is equal to the index $[F_N : G]$ of the subgroup $G$.

Generators of the fundamental group $\pi_1(\mathcal{U}, O) = F_N$ are in one-to-one correspondence with the circles of the union. This allows us to think that the edges of the graphs $\Upsilon$ and $\Gamma(\Upsilon)$ are labelled by generators $e_0, \ldots, e_{N-1}$ of the group $F_N$, so that it is possible to speak about products of labels, etc. By Property 3 for every $k$ there is exactly one edge of $\Gamma(\Upsilon)$ labelled $e_k$, and we will call it simply “edge $e_k$”.

For any sequence of edges $\nu$ in the graph $\Upsilon$ define the element $\tau_\nu \in F_N$ as $\tau_\nu = r_\nu(\nu)$. In other words, one should multiply the labels of the edges passed by $\nu$. The label should be taken with the exponent $+1$ if $\nu$ passes the edge in positive direction, and with the exponent $-1$ (the inverse element) otherwise.
The group \( g(\Upsilon) \) consists of all the elements \( \tau_\nu \) for all closed sequences of edges \( \nu \) starting and ending at the base vertex \( B \). We will also use the notation \( \tau_\nu \) where \( \nu \) is a sequence of edges of the graph \( \Gamma(\Upsilon) \); here all the exponents are taken +1 since the edges of \( \Gamma(\Upsilon) \) bear no orientation.

It is easy to see that the following elements constitute a system of generators for the group \( g(\Upsilon) \):

1. Elements \( a_{\nu,e}^\Gamma(\Upsilon) \) defined as \( \tau_\nu e \tau_\nu^{-1} \) where \( \nu \) is an arbitrary path in \( \Gamma(\Upsilon) \) starting at the base vertex \( B \), and \( e \) is a generator different from the last letter in the word \( \tau_\nu \).

2. Elements \( b_{\nu,e}^\Gamma(\Upsilon) \) defined as \( \tau_\nu e^2 \tau_\nu^{-1} \) where \( \nu \) is an arbitrary path in \( \Gamma(\Upsilon) \) starting at the base vertex \( B \), not passing the edge labelled \( e \), and ending in the vertex \( A \) attached to that edge.

These elements are just \( \tau_\varphi \) for the following sequences of edges \( \varphi \) in the graph \( \Upsilon \):

1. For \( a_{\nu,e}^\Gamma(\Upsilon) \) the sequence \( \varphi \) goes first along the path \( \nu \) in \( \Gamma(\Upsilon) \). Every two successive vertices of the path \( \nu \) are connected in \( \Upsilon \) by two edges having the same label (by Property 3). Apparently, these edges must have opposite orientations. The path \( \varphi \) takes each time the edge with the positive direction. After this, \( \varphi \) passes the loop labelled \( e \) and attached to the final vertex of the path \( \nu \), and then returns along the path \( \nu \) taking the same edges as for the first time.

2. For \( b_{\nu,e}^\Gamma(\Upsilon) \) the sequence \( \varphi \) goes along the path \( \nu \) in \( \Gamma(\Upsilon) \) taking each time the edge with the positive direction. The final vertex \( A \) is joined with some vertex \( C \) by two edges labelled \( e \). \( \varphi \) passes them both in the positive direction, and then returns to \( B \) taking the same edges as for the first time.

The automorphism group \( \text{Aut}(F_N) \) acts on the set of subgroups of \( F_N \). The correspondence of Theorem 7 allows to consider this action as an action on the set of coverings of the union \( \mathcal{U} \), with a marked vertex chosen in each covering.

**Theorem 8** For any tree-like covering \( \Upsilon \) and any element \( s \in BC_N \subset \text{Aut}(F_N) \) the covering \( s(\Upsilon) \) is tree-like, too. Actions of the group \( BC_N \) on the set of coverings and on the set of trees are compatible: \( \Gamma(s(\Upsilon)) = s(\Gamma(\Upsilon)) \).

Note that \( \Gamma(\Upsilon) \) is a tree with one marked vertex. The group \( BC_N \) acts on such trees, too: the action of Subsection 3.2 preserves the set of vertices of the tree, so that we can assume that the marked vertex simply remains unchanged.

**Proof** Let \( \Upsilon' \) be a covering such that \( \Gamma(\Upsilon') = s(\Gamma(\Upsilon)) \). To prove the statement we must show that the elements \( s(a_{\nu,e}(\Upsilon')) \) and \( s(b_{\nu,e}(\Upsilon')) \) belong to the subgroup...
Case 3.  

\(g(Y')\). Since the subgroups \(s(g(Y))\) and \(g(Y')\) have the same index, they coincide, and thus \(s(Y) = Y'\).

It is enough to consider the case when \(s\) is either \(\lambda\) or some \(u_k\). For brevity we will show in detail why \(s(a_{\nu,e}^{\Gamma(Y)}) \in g(Y')\); the proof for \(b_{\nu,e}\) is similar.

Case 1. \(s = \lambda\). Here \(s(a_{\nu,e}^{\Gamma(Y)}) = a_{\nu,s(e)}^{\Gamma(Y)} \in g(Y')\).

Case 2. \(s = u_k\), and the edges \(e_{k-1}\) and \(e_k\) in \(\Gamma(Y)\) are not adjacent. The following situations may occur:

2.1. \(e\) is neither \(e_k\) nor \(e_{k-1}\). In this case \(s(a_{\nu,e}^{\Gamma(Y)}) = s(\tau_\nu)es(\tau_\nu)^{-1}\). The word \(s(\tau_\nu)\) differs from \(\tau_\nu\) by substitutions \(e_{k-1} \mapsto e_k, e_k \mapsto e_k^{-1}e_{k-1}e_k\). Let \(C, D\) be the ends of the edge \(e_{k-1}\), and \(E, F\) be the ends of the edge \(e_k\) in the graph \(\Gamma(Y)\). Then in \(s(\Gamma(Y))\) the edge \(CD\) is labelled \(e_k\), and \(EF\) is labelled \(e_{k-1}\). Since the vertices \(C, D, E, F\) are all different, the graph \(Y'\) contains loops \(\ell_E\) and \(\ell_F\) labelled \(e_k\) and attached to vertices \(E\) and \(F\), respectively. Thus, \(s(\tau_\nu)\) is just \(\tau_{\nu'}\) (in the graph \(Y'\)) where \(\nu'\) coincides with \(\nu\) except for the edge \(EF\) which is replaced by the edge sequence \(\ell_E^{-1}, EF, \ell_F\). The last edge of the path \(\nu\) in \(\Gamma(Y)\) is not \(e\), and thus, the last edge of \(\nu'\) in \(\Gamma(Y')\) is not \(e\) either, and therefore in \(Y'\) there exists a loop labelled \(e\) and attached to the last vertex of \(\nu'\). So, \(s(a_{\nu,e}^{\Gamma(Y)}) \in g(Y')\).

2.2. \(e = e_{k-1}\). In this case \(s(a_{\nu,e}^{\Gamma(Y)}) = s(\tau_\nu)e_k s(\tau_\nu)^{-1}\), and the subsequent proof is exactly the same as in the case 2.1.

2.3. \(e = e_k\), and the last edge of the path \(\nu\) is not \(e_{k-1}\). In this case \(s(a_{\nu,e}^{\Gamma(Y)}) = s(\tau_\nu) e_k^{-1} e_{k-1} e_k s(\tau_\nu)^{-1}\). The element \(s(\tau_\nu)\) is \(\tau_{\nu''}\) where \(\nu''\) is exactly as in the case 2.1. The last edge of the path \(\nu''\) in \(Y'\) is labelled neither \(e_k\) nor \(e_{k-1}\), and thus there exist loops \(\ell_k\) and \(\ell_{k-1}\) with these labels attached to the last vertex of \(\nu''\). So, \(s(a_{\nu,e}^{\Gamma(Y)}) = \tau_{\nu''} \tau_{\nu} \tau_{\nu''}^{-1}\) where \(\varphi = \ell_k^{-1} \ell_{k-1} \ell_k\). Therefore \(s(a_{\nu,e}^{\Gamma(Y)}) \in g(Y')\).

2.4. \(e = e_k\), and the last edge of the path \(\nu\) is \(e_{k-1}\). Thus the last edge (say, \(EF\)) of the path \(\nu''\) in \(Y'\) is \(e_k\), and there is a loop \(\ell\) labelled \(e_{k-1}\) and attached to \(E\). The subsequent proof is just the same as in the case 2.3 but the path \(\varphi\) is taken to be \(FE, \ell, EF\).

Case 3. \(s = u_k\), and the edges \(e_{k-1}\) and \(e_k\) in \(\Gamma(Y)\) are adjacent: \(e_{k-1} = CD, \ e_k = DE\). The following situations may occur here:

3.1. The path \(\nu\) passes neither \(e_k\) nor \(e_{k-1}\). In this case \(s(\tau_\nu) = \tau_{\nu}\), and \(s(e)\) is either \(e\) (if \(e \not= e_k, e_{k-1}\)), or \(e_k\) (if \(e = e_{k-1}\)), or \(e_k^{-1} e_{k-1} e_k\) (if \(e = e_k\)). In all cases the loops labelled \(e\), \(e_k\) and \(e_{k-1}\) are attached to the final vertex \(F\) of the path \(\nu\) (both in \(Y\) and \(Y'\) because the edges of \(\nu\) does not change their labels), and \(s(e)\) is \(\tau_{\nu}\) for some path
φ starting and ending at F. Thus $s(a_{\nu,e}^{\Gamma(\Upsilon)}) = \tau_{\nu} \tau_{\nu}^{-1} \in g(\Upsilon')$. The same reasoning applies to the case when $\nu$ passes $e_{k-1}$ (the edge $CD$) but not $e_k$.

3.2. $\nu$ passes $e_k$ (the edge $DE$) but not $e_{k-1}$. Graph $\Upsilon'$ does not contain the edge $DE$ but contains the edge $CE$ labelled $e_{k-1}$ (and the edge $CD$ labelled $e_k$). Thus there exists a loop $\ell$ (in $\Upsilon$) attached to $E$ and labelled $e_k$. So, $s(\tau_{\nu}) = \tau_{\nu'}$ where the path $\nu'$ coincides with $\nu$ everywhere except the edge $DE$ which is replaced by the edge sequence $DC, CE, \ell$. The word $s(e)$ is handled exactly like in the case 3.1. Thus, $s(a_{\nu,e}^{\Gamma(\Upsilon)}) = \tau_{\nu} s(e) \tau_{\nu}^{-1} \in g(\Upsilon')$.

3.3. $\nu$ passes both $e_{k-1}$ and $e_k$. Since $\Gamma(\Upsilon)$ is a tree, it passes these edges subsequently, either as $CD, DE$, or as $ED, DC$. In the first case the reasoning is the same as in the case 3.2 but $\nu'$ is obtained from $\nu$ by a change of the segment $CD, DE$ to the segment $CE, \ell$. The second case is similar. ■

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