Moduli spaces of Calabi–Yau complete intersections

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Abstract

In this short note, based on the work [7] in 1994, we describe compactifications of moduli spaces of Calabi–Yau complete intersections in Gorenstein toric Fano varieties. © 2015 The Author. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP3.

1. Introduction

In the early stage of the study of mirror symmetry of Calabi–Yau manifolds, Klemm, Theisen, S.-T. Yau and the present author [8,7] have shown the power of Gel’fand–Kapranov–Zelevenski (GKZ) hypergeometric systems [5] in determining Picard–Fuchs differential equations satisfied by period integrals. In case of Calabi–Yau hypersurfaces, Lian, S.-T. Yau and the author [10,9] have studied compactifications of the moduli spaces and the existence of special degeneration points called large complex structure limits (LCSL).

In this short note, we will describe compactifications of the moduli spaces of Calabi–Yau complete intersections, i.e., the parameter spaces of GKZ systems studied in [7]. Recent study [11] indicates that interesting birational geometries of complete intersections, which do not come from ambient toric varieties, arise from the boundary points of the compactifications. We will also define GKZ systems for reflexive Gorenstein cones [3] in general.

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2. Calabi–Yau complete intersections

A projective variety is called Fano variety if its anti-canonical class is an ample Cartier divisor. Toric Fano varieties with at worst Gorenstein singularities are called Gorenstein Toric Fano varieties. It is known that they are classified by the so-called reflexive polytopes. Let $M$ and $N$ be lattices of rank $d$ which are dual to each other with the pairing $(\ , \ ) : M \times N \to \mathbb{Z}$. A reflexive polytope $\Delta$ in $M_{\mathbb{R}}$ is an integral polytope which contains the origin $0$ in its interior, and the dual polytope

$$\Delta^* := \{ y \in N_{\mathbb{R}} \mid (x, y) \geq -1 \ (x \in \Delta) \}$$

is an integral polytope in $N_{\mathbb{R}}$. By definition, when $\Delta \subset M_{\mathbb{R}}$ is reflexive, $\Delta^* \subset N_{\mathbb{R}}$ is reflexive and vice versa. Given a reflexive polytope, we have a Gorenstein toric Fano variety $P_{\Delta} := \text{Proj} \ S_{\Delta}$ with the graded ring $S_{\Delta}$ associated to semigroup $\mathbb{R}_{\geq 0}(1, \Delta) \cap M$ with $M := \mathbb{Z} \oplus M$. Similarly we have $P_{\Delta^*}$ for the dual polytope $\Delta^*$, and hence we have a pair of Gorenstein toric Fano varieties $(P_{\Delta}, P_{\Delta^*})$. Considering suitable resolutions $\hat{P}_{\Delta}$ and $\hat{P}_{\Delta^*}$ (called MPF resolutions) of these toric Fano varieties, we have a pair of smooth Calabi–Yau manifolds for $d \leq 4$, which are mirror dual to each other [2]. We denote by $T = \text{Spec} \ \mathbb{C}[M] = \text{Spec} \ \mathbb{C}[u_1^{\pm}, \ldots, u_d^{\pm}]$ and $T^* = \text{Spec} \ \mathbb{C}[N] = \text{Spec} \ \mathbb{C}[t_1^{\pm}, \ldots, t_d^{\pm}]$, respectively, the dense $(\mathbb{C}^*)^d$-orbits in $P_{\Delta}$ and $P_{\Delta^*}$.

Anti-canonical class $-K_{P_{\Delta}}$ is given by the sum of all torus invariant Cartier divisors with multiplicity one for each. Then, a family of complete intersection Calabi–Yau varieties in $P_{\Delta}$ is specified by the data of the so-called nef-partition of the anti-canonical class into nef divisors, which may be represented combinatorially by the Minkowski sum decomposition

$$\Delta = \Delta_1 + \Delta_2 + \cdots + \Delta_r.$$

Each component $\Delta_i$ is identified with the supporting polytope of the global sections of the corresponding semi-ample line $T$-linear line bundle $L(\Delta_i)$ on $P_{\Delta}$. Generic Laurent polynomials $f_{\Delta_i} = \sum_{m \in \Delta_i} c_m u^m \in \mathbb{C}[M]$ determine a complete intersection

$$Z = \{ f_{\Delta_1} = f_{\Delta_2} = \cdots = f_{\Delta_r} = 0 \} \subset T$$

in the torus $T$, and its closure in $P_{\Delta}$ determines a Calabi–Yau variety. When, $d - r \leq 3$, the closure $X$ of $Z$ in $P_{\Delta}$ gives a smooth Calabi–Yau complete intersection [4].

Consider $\text{Conv}(\Delta_1, \Delta_2, \ldots, \Delta_r)$, then this turns out to be a reflexive polytope (contained in $\Delta \subset M_{\mathbb{R}}$), and defines a reflexive polytope $\nabla = \text{Conv}(\Delta_1, \ldots, \Delta_r)^*$ in $N_{\mathbb{R}}$. Support functions $\varphi_i$ describing $\Delta_i = \{ x \in M_{\mathbb{R}} \mid (x, y) \geq -\varphi_i(y), \ (y \in N_{\mathbb{R}}) \}$, and similar expression for $\Delta$ with $\varphi = \sum \varphi_i$, determine the dual nef partition,

$$\nabla = \nabla_1 + \nabla_2 + \cdots + \nabla_r,$$

and also the Laurent polynomials $f_{\nabla_i} = \sum_{n \in \nabla_i} a_i t^n$. Considering the closure of the zero locus $\{ f_{\nabla_1} = \ldots = f_{\nabla_r} = 0 \} \subset T^*$ in $\hat{P}_\nabla$ as above, we have (a family of) smooth Calabi–Yau complete intersections $X^*$ when $d - r \leq 3$ and $a_{n_i}$’s are general.

Due to Batyrev–Borisov [4], $X$ and $X^*$ are mirror dual to each other.

3. GKZ systems and moduli spaces of mirror families

Consider the mirror family $X^* = \{ X^*_a \}_a$ described in the preceding section. We can write the period integrals of a holomorphic three form (see [7]),
\[
\Pi(a) = \int \frac{1}{f_{\nu_1}(a)f_{\nu_2}(a) \cdots f_{\nu_r}(a)} \prod_{i=1}^{d} \frac{dt_i}{t_i}.
\] (3.1)

Let us introduce \( \mathbb{Z}' = \bigoplus_{i=1}^{r} \mathbb{Z} e_i \) and define \( \mathcal{N} := \mathbb{Z}' \oplus N \). Then \( e_i \times (N \cap \nabla_i) =: A_i \) is the set of integral points of \( \nabla_i \) with \( e_i \) added for the first \( r \)-entries. We arrange these integral vectors in column vectors of \( (r + d) \times |N \cap \nabla_i| \) matrix \( A_i \), and define a \( (r + d) \times p \) matrix with \( p := \sum_i |N \cap \nabla_i| \),

\[
A = (A_1 A_2 \cdots A_r).
\]

One of the main results in [7] was

**Proposition 3.1.** The period integrals \( \Pi(a) \) satisfies GKZ hypergeometric system [5] of \( A \) and exponent \( \beta = -(e_1 + \cdots + e_r) \times 0 \in \mathcal{N}_\mathbb{C} \).

It was noted there that the GKZ system above is highly reducible, but suffices to determine the period integrals of the family \( \mathcal{X}^* \) after finding suitable factorizations of differential operators.

The parameter space of the GKZ system is naturally compactified by a toric variety associated with the so-called secondary fan of the point configuration \( A \) given in \( \mathcal{N}_\mathbb{R} \). To describe the secondary fan, let us turn \( A \) to standard form,

\[
\mathcal{A} = \phi \circ A
\]

by a linear transformation \( \phi \in \text{GL}(\mathcal{N}) \) defined by \( \phi(e_1) = e_1, \phi(e_i) = e_1 + e_i (i = 2, \ldots, r) \) and \( \phi|_N = id_N \). Note that the first entry of the column vectors of \( \mathcal{A} \) are now set to one. This amounts to considering hypersurfaces

\[
\{ F_{\mathbf{\lambda}}(a) = f_{\mathbf{\nu}_1}(a) + \lambda_2 f_{\mathbf{\nu}_2}(a) + \cdots + \lambda_r f_{\mathbf{\nu}_r}(a) = 0 \} \subset (\mathbb{C}^*)^{r-1} \times T^*.
\]

with the corresponding period integral

\[
\tilde{\Pi}(a) = \int \frac{1}{F_{\mathbf{\lambda}}(a)} \prod_{i=2}^{r} \frac{d\lambda_i}{\lambda_i} \prod_{i=1}^{d} \frac{dt_i}{t_i}
\]

extending the torus \( T^* \) to \( (\mathbb{C}^*)^{r-1} \times T^* \). \( \tilde{\Pi}(a) \) satisfies the GKZ system of \( \mathcal{A} \) with exponent \( \tilde{\beta} = -1 \times 0_{r-1} \times 0_N \subset \mathcal{N}_\mathbb{C} \) as have been studied in detail in [8,9].

Let us see the moduli spaces of the complete intersections and the hypersurfaces. Both are parametrized by the affine parameters \( a \in \mathbb{C}^p \) up to the torus actions,

\[
\mathcal{M}_f = \left\{ ([f_{\mathbf{\nu}_1}(a)], [f_{\mathbf{\nu}_2}(a)], \ldots, [f_{\mathbf{\nu}_r}(a)]) \mid a \in \mathbb{C}^p \right\} \times T_i t_i \mid t_i \in \mathbb{C}^* ,
\]

\[
\mathcal{M}_F = \left\{ [F_{\mathbf{\lambda}}(a)] \mid a \in \mathbb{C}^p \right\} \times T_i t_i, \lambda_k \rightarrow A_k \lambda_k, t_i, A_k \in \mathbb{C}^* ,
\]

where \( [f] \) represents the class of non-vanishing scalar multiples of \( f \). The quotient \( \mathcal{M}_F \) is the space of Laurent polynomials up to torus actions, and there is a natural compactification in terms of the secondary fan [6]. It is easy to see the following proposition by writing the torus actions on the affine parameter \( a \in \mathbb{C}^p \).

**Proposition 3.2.** The quotients \( \mathcal{M}_f \) and \( \mathcal{M}_F \) are isomorphic, and they are compactified by the GKZ secondary fan for the point configuration \( \mathcal{A}(\mathcal{A} \approx \mathcal{A}) \).
The secondary fan is the normal fan of the secondary polytope, and the latter is related to the Newton polytope of the discriminant of \( F_\ast (a) \) [6]. Then, the isomorphism of \( \mathcal{M}_f \) and \( \mathcal{M}_F \) may be deduced from the correspondence of the critical points,

\[
\text{Crit}(f_{V_1}(a), f_{V_2}(a), \ldots, f_{V_r}(a)) \cong \text{Crit}(F_\ast (a)),
\]

with \( \lambda_k(k = 2, \ldots, r) \) being regarded as the Lagrange multipliers.

**Example 3.3.** Complete intersection of five general \((1, 1)\) divisors in \( \mathbb{P}^4 \times \mathbb{P}^4 \) defines a Calabi–Yau threefolds. The data of nef partitions \( \Delta = \Delta_1 + \cdots + \Delta_5 \) and the dual nef partition \( \nabla = \nabla_1 + \cdots + \nabla_5 \) may be found in [11, (2-2)] (with slightly different notation for polytopes). In [7, Sect. 4], Picard–Fuchs differential operator of the mirror family \( \mathcal{X}^\ast \) has been determined from Proposition 3.1 by finding factorization of differential operators. The compactification \( \overline{\mathcal{M}}_f = \overline{\mathcal{M}}_F \) defined in the above proposition turns out to be \( \mathbb{P}^2 \). Interestingly, we see that all toric boundary points show the property of the large complex structure limit (LCSL) (see [11, Prop. 2.6]).

**Remark 3.4.** When we consider a hypersurface \( X \) in \( \mathbb{P}_\Delta \) (i.e., \( r = 1 \)), the moduli space \( \overline{\mathcal{M}}_f \) for the mirror family \( \mathcal{X}^\ast \) has been studied [9] in detail. In particular, the correspondence of LCSLs to MPCPs \( \mathbb{P}_{\Delta} \) of \( \mathbb{P}_\Delta \) are shown using Gröbner fan, a refinement of the secondary fan. Different MPCPs \( \mathbb{P}_{\Delta} \) of \( \mathbb{P}_\Delta \) are often referred to topology changes in physics [1]. The above example shows that for complete intersections \((r > 1)\) in general, we have LCSLs which do not correspond to the topology changes of the ambient spaces \( \mathbb{P}_\Delta \). In particular, in the case of Example 3.3, it has been observed that the LCSLs nicely corresponds to the birational models of complete intersections [11, Sect. 3]. This indicates that we can study the birational geometry of complete intersection \( X \) through the compactification \( \overline{\mathcal{M}}_f \) of the mirror family \( \mathcal{X}^\ast \).

### 4. GKZ systems for Gorenstein cones

For nef partitions \( \Delta = \Delta_1 + \cdots + \Delta_r \) and \( \nabla = \nabla_1 + \cdots + \nabla_r \) of reflexive polytopes, we define cones

\[
C_\Delta = \mathbb{R}_{\geq 0}(1, 0, \ldots, 0, \Delta_1) + \cdots + \mathbb{R}_{\geq 0}(0, 0, \ldots, 1, \Delta_r) \subset \overline{M}_\mathbb{R}, \\
C_\nabla = \mathbb{R}_{\geq 0}(1, 0, \ldots, 0, \nabla_1) + \cdots + \mathbb{R}_{\geq 0}(0, 0, \ldots, 1, \nabla_r) \subset \overline{N}_\mathbb{R}.
\]  \hspace{1cm} (4.1)

\( C_\Delta \) and \( C_\nabla \) are dual cones to each other under the natural (standard) pairing \( \overline{M}_\mathbb{R} \times \overline{N}_\mathbb{R} \rightarrow \mathbb{R} \). Furthermore, all primitive generators of one dimensional cones of \( C_\Delta \) line on \( \{ x \mid (x, n_\Delta) = 1 \} \) with \( n_\Delta = (1, \ldots, 1, 0) \in \overline{N} \) and similarly for \( C_\nabla \) with \( m_\nabla = (1, \ldots, 1, 0) \). Since \((m_\nabla, n_\Delta) = r\), these are called reflexive Gorenstein cones with index \( r \) [3].

In general, a rational polyhedral cone \( C_\sigma \subset \overline{M}_\mathbb{R} \) is called Gorenstein if there exists \( n_\sigma \in \mathbb{N} \) such that all primitive generators of one dimensional cones lie on \( \{ x \mid (x, n_\sigma) = 1 \} \) for some \( n_\sigma \in \mathbb{N} \). If the dual cone \( C_\sigma \subset \overline{N}_\mathbb{R} \) is also a Gorenstein cone with \( m_\sigma \in \mathbb{M} \), then \( C_\sigma \) and \( C_\sigma \) are called reflexive Gorenstein cone with index \((m_\sigma, n_\sigma) \in \mathbb{Z}_{\geq 0} \). The dual cones (4.1) are an example of reflexive Gorenstein cones of index \( r \), both of which are of type complete splitting (see [3] for definition). There are examples where one of the cones does not split while the other does. Such cones are used to explain the mirror symmetry of rigid Calabi–Yau manifolds [3].

Our definition of GKZ system in Proposition 3.1 generalizes to reflexive Gorenstein cones.
**Definition 4.1 (GKZ systems for Gorenstein cones).** Consider a reflexive Gorenstein cone $C_\sigma$ with $m_\sigma$ (and $n_\sigma$ for $C_\sigma$) as above. Let $\mathcal{A} = C_\sigma \cap \{ y \in \overline{N} \mid (m_\sigma, y) = 1 \}$ be the point configuration, and define a GKZ system with $\mathcal{A}$ and exponent $\beta = -n_\sigma \in \overline{N}_C$. 

This definition naturally generalizes the GKZ system in Proposition 3.1. The compactification of the parameter space of the GKZ system is given by the secondary fan, which we can calculate by making a similar coordinate change $\hat{\mathcal{A}} = \varphi \circ \mathcal{A}$ with suitable linear transformation $\varphi$ using $m_\sigma$ (or directly by $\mathcal{A}$).

**Remark 4.2.**

1. Reflexive Gorenstein cones and also the above GKZ system can be recognized in recent developments of the so-called two sphere partition functions of gauged linear sigma models in physics (see e.g. [12]).

2. When we represent the point configuration $\mathcal{A}$ explicitly by the corresponding $\text{rk} \overline{N} \times |\mathcal{A}|$ matrix $\mathcal{A}$ (we are using the same letter for the matrix), the solution $\gamma$, up to Ker$\mathcal{A}$, of the linear equation $\mathcal{A}\gamma = \beta$ plays a fundamental role when writing formal solution of the system [10]. This $\gamma$ coincides with the $R$-charge $q_R$ by $q_R = -2\gamma$ in the description of gauged linear sigma models, and is one of the fundamental data of the two sphere partition functions [12].

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