Particle Mechanics Models with $\mathcal{W}$-symmetries

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Abstract

We introduce a particle mechanics model with Sp($2M$) gauge invariance. Different partial gauge-fixings by means of sl(2) embeddings on the gauge algebra lead to reduced models which are invariant under diffeomorphisms and classical nonlinear $\mathcal{W}$-transformations as the residual gauge symmetries thus providing a set of models of gauge and matter fields coupled in a $\mathcal{W}$-invariant way. The equations of motion for the matter variables give Lax operators in a matrix form. We examine several examples in detail and discuss the issue of integration of infinitesimal $\mathcal{W}$-transformations.
1 Introduction

Extended conformal symmetries play an important role in two-dimensional conformal field theories, 2d gravity models and integrable hierarchies of non-linear differential equations. The study of non-linear extensions of the Virasoro algebra with bosonic conformal primary fields was first developed by Zamolodchikov [1]. Such algebras are known as \( \mathcal{W} \)-algebras —– for recent reviews on \( \mathcal{W} \)-algebras see [2, 3]. Classical \( \mathcal{W} \)-algebras are obtained by a contraction of \( \mathcal{W} \)-algebras through a \( c \to \infty, \hbar \to 0 \) limit, keeping \( \hbar c \) constant. Two methods have enjoyed much success to construct these algebras: the Drinfel’d-Sokolov (DS) Hamiltonian reduction for Kac-Moody current algebras [4, 5, 6, 7] and the zero-curvature approach [8, 9, 10, 11].

The Kac-Moody Hamiltonian reduction consists in introducing a set of first-class constraints in the space generated by the affine currents \( J^a(x) \) equipped with the Kac-Moody Lie-Poisson bracket:

\[
\{ J^a(x), J^b(y) \}_{\text{KM}} = f_{ac}^b J^c(x) \delta(x - y) + \kappa \tilde{g}^{ab} \partial_x \delta(x - y).
\] (1.1)

Here \( \tilde{g}_{ab} \) is proportional to the Cartan-Killing metric and \( f_{ab}^c \) are the structure constants of the underlying Lie algebra \( G \). These constraints generate gauge transformations on the restricted space due to its first-class nature. This gauge freedom is fixed by introducing a second set of constraints and so a Dirac bracket can be defined on the reduced space. The Dirac bracket algebra on this space is the (classical) \( \mathcal{W} \)-algebra. The choice of the whole set of constraints is inspired by the different inequivalent sl(2) embeddings into \( G \).

The Kac-Moody currents generate infinitesimal transformations on the original space via the Poisson bracket (1.1):

\[
\delta_a f(x) \equiv \int dy \epsilon_a(y) \{ f(y), J^a(x) \}_{\text{KM}}.
\] (1.2)

These are the infinitesimal Kac-Moody transformations. After the reduction the remaining currents generate infinitesimal transformations on the reduced space via the Dirac bracket in a similar way: the infinitesimal (classical) \( \mathcal{W} \)-transformations.

The zero-curvature approach starts with a \( G \)-valued field \( \Lambda(x) = \Lambda^a(x) T_a \) (\( T_a \) form a basis of \( G \)) transforming à la Yang-Mills:

\[
\delta \Lambda = \dot{\beta} - [\Lambda, \beta], \quad \beta(x) = \beta^a(x) T_a,
\] (1.3)

where \( \beta^a(x) \) are infinitesimal parameters. By means of a sl(2) embedding into \( G \) the components of \( \Lambda \) are partially constrained. The residual transformations preserving these constraints are the classical \( \mathcal{W} \)-transformations.

Both approaches are equivalent since (1.2) becomes (1.3) once \( \Lambda(t) \) is identified with the Kac-Moody holomorphic current \( J(x) \), but the second one circumvents the language of Poisson manifolds.
In this paper we want to analyze classical \( W \)-symmetries in the context of particle mechanics within the zero-curvature approach. Specifically we present a model containing gauge and matter degrees of freedom \([2]\). The transformations of the model are \([1,3]\) for the gauge variables and \( G \) is a \( \text{sp}(2M) \) algebra. We call \([1,3]\) gauge transformations because they emerge from the study of the constraint structure in the phase space of the particle mechanics model itself, therefore the word \textit{gauge} here and throughout the rest of the paper should be distinguished from what is usually called gauge transformations in the context of the Kac-Moody Hamiltonian reduction described above. According to the previous discussion, after partially fixing this gauge freedom by means of a \( \text{sl}(2) \) embedding we get a model which exhibits infinitesimal transformations associated with any of the classical \( W \)-algebras obtainable from the \( C_n \) series of Lie algebras via the DS reduction. Furthermore, the equations of motion of the matter variables give rise to the DS equations associated with those \( W \)-algebras. We therefore obtain such equations in a dynamical context. The corresponding Lax operators are given as \( M \times M \) matrices.

Models exhibiting \( W \)-symmetries associated with other series of Lie algebras can be also constructed within this framework. For instance, if we consider some embeddings of the \( \text{sl}(N) \) into the \( \text{sp}(2N) \) algebras, we obtain the transformation laws and DS equations associated with the \( A_{N-1} \) classical \( W \)-algebras.

The organization of the paper is as follows. In sect. 2 we formulate the \( \text{Sp}(2M) \) gauge particle model for general \( M \). In sect. 3 we consider the \( \text{Sp}(2) \) model as the simplest example and show how reparametrization invariance arises in our model. We also study the finite gauge transformation leading to the finite diffeomorphism invariance of the model. In sect. 4 we study the \( \text{Sp}(4) \) action with its three possible partial gauge-fixings corresponding to the three inequivalent \( \text{sl}(2) \) embeddings in \( \text{sp}(4) \). We integrate the gauge transformations for one of the three models (sect. 4.1) and perform secondary reductions of it, ending up with systems exhibiting the non-local matrix algebra \( V_{2,2} \[3\] \) and the local \( W(2,4) \) algebra which is also associated with the principal \( \text{sl}(2) \) embedding model (sect. 4.2). In sect. 5 we analyze how \( W_3 \) and \( W_2^3 \) invariant models can be obtained as reductions of a \( \text{sl}(3) \) embedding in the \( \text{Sp}(6) \) gauge particle model. Some comments and discussions are addressed in the last section.

We also include two appendices: one about \( \text{sl}(2) \) embeddings and DS reductions and the other about finite transformations of the \( \text{Sp}(2M) \) model before performing the gauge-fixing.

## 2 Particle model with \( \text{Sp}(2M) \) symmetry

Let us consider a reparametrization-invariant model of \( M \) relativistic particles with a \( \text{Sp}(2M) \) gauge group living in a Minkowskian \( d \)-dimensional space-time \([2]\). The dimension \( d \) satisfies \( d > 2M + 1 \) so the constraints do not trivialize the model. The
canonical action is given by

\[ S = \int dt \left( p_i \dot{x}_i - \lambda_{Ai,j} \phi_{Ai,j} \right), \quad i, j = 1, \ldots, M, \quad A = 1, 2, 3. \]  

(2.1)

The variable \( x_i^\mu(t) \) is the world-line coordinate of the \( i \)-th particle and \( p_i^\mu(t) \) is its corresponding momentum. The Lagrange multipliers \( \lambda_{Ai,j}(t) \) implement the constraints \( \phi_{Ai,j} = 0 \) and satisfy

\[ \lambda_{1,j} = \lambda_{i,j}, \quad \lambda_{3,j} = \lambda_{3,j}. \]

The explicit form of \( \phi_{Ai,j} \) is

\[ \phi_{1,ij} = \frac{1}{2} p_i p_j, \quad \phi_{2,ij} = p_i x_j \quad \text{and} \quad \phi_{3,ij} = \frac{1}{2} x_i x_j. \]  

(2.2)

These \( 2M^2 + M \) constraints close under the usual Poisson bracket \( \{ x_i, p_j \} = \delta_{ij} \) giving a realization of the \( \text{sp}(2M) \) algebra.

It is useful to introduce a matrix notation for the coordinates and momenta of the particles

\[ R = \begin{pmatrix} r \\ p \end{pmatrix}, \quad \text{with} \quad r = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}, \quad p = \begin{pmatrix} p_1 \\ \vdots \\ p_M \end{pmatrix}. \]  

(2.3)

The conjugate of \( R \) is given by

\[ \bar{R} = R^\top J_{2M} = \begin{pmatrix} p^\top & -r^\top \end{pmatrix}, \]  

(2.4)

where \( J_{2M} \) is the \( 2M \times 2M \) symplectic matrix \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). The Lagrange multipliers can be written in a form of \( 2M \times 2M \) symplectic matrix

\[ \Lambda = \begin{pmatrix} B & A \\ -C & -B^\top \end{pmatrix}, \]  

(2.5)

where the components of the \( M \times M \) matrices \( A, B, C \) are the Lagrange multipliers \( \lambda_{1,ij}, \lambda_{2,ij}, \lambda_{3,ij} \) respectively.

The canonical action (2.1) can be written in a matrix form as

\[ S = \int dt \frac{1}{2} \bar{R} \mathcal{D} R, \]  

(2.6)

where \( \mathcal{D} \) is the covariant derivative

\[ \mathcal{D} = \frac{d}{dt} - \Lambda. \]
In this formulation the gauge invariance of the action is expressed in a manifestly
invariant form of Yang-Mills type\footnote{For a previous discussion of geometrical models with Yang-Mills gauge theories see [14].} with the gauge group $\text{Sp}(2M)$\footnote{The supersymmetric version has been studied in [15].}: 
\begin{align}
\delta R &= \beta R, \\
\delta \Lambda &= \dot{\beta} - [\Lambda, \beta],
\end{align}
where $\beta$ is the $2M \times 2M$ matrix form of gauge parameter
\begin{equation}
\beta = \begin{pmatrix} \beta_B & \beta_A \\ -\beta_C & -\beta_B^T \end{pmatrix}
\end{equation}
and the components $\beta_A, \beta_B, \beta_C$ are the $M \times M$ matrices associated with the constraints $\phi_{1ij}, \phi_{2ij}, \phi_{3ij}$. The equations of motion of the matter fields are
\begin{equation}
\mathcal{D}R = \dot{R} - \Lambda R = 0.
\end{equation}

The infinitesimal transformation law (2.8) is the compatibility condition of the pair of equations (2.7) and (2.10):
\begin{equation}
0 = (\delta - \beta, D)R = -(\delta \Lambda - \dot{\beta} + [\Lambda, \beta])R,
\end{equation}
and it can be regarded as a zero-curvature condition. The presence of a zero-curvature condition allows us to apply the ‘soldering’ procedure to reduce the original symmetry of the model to a chiral classical $W$-symmetry by means of a partial gauge-fixing of the $\Lambda$ fields. In appendix A we review this reduction method and display the criteria for choosing the gauge-fixing.

It is useful to express the model in terms of the Lagrangian variables. If we write the momenta $p$ in terms of the Lagrangian variables
\begin{equation}
p = A^{-1}(\dot{r} - Br) \equiv K,
\end{equation}
the action is now rewritten as
\begin{equation}
S = \int dt \frac{1}{2} \left( K^T AK - r^T Cr \right).
\end{equation}

The gauge transformations become
\begin{align}
\delta r &= \beta_A K + \beta_B r, \\
\delta \Lambda &= \dot{\beta} - [\Lambda, \beta].
\end{align}
A characteristic feature of these Lagrangian transformations is that the algebra is open, except for $\text{sp}(2)$,
\begin{equation}
[\delta_1, \delta_2] r = \delta_{1\beta^*} r - (\beta_A^{(2)} A^{-1} \beta_A^{(1)} - \beta_A^{(1)} A^{-1} \beta_A^{(2)}) [L] r,
\end{equation}
where $\beta^* = \beta_B$.\footnote{For a previous discussion of geometrical models with Yang-Mills gauge theories see [14].}
\[ [\delta_1, \delta_2] \Lambda = \delta_{\beta^*} \Lambda, \]  
(2.14)

where \( \beta^* = [\beta^{(2)}, \beta^{(1)}] \) and \([L]_r\) is the Euler-Lagrange equation of motion of \( r \). There are two reasons for the appearance of an open algebra: 1) the transformations of the momenta at the Lagrangian and Hamiltonian level do not generally coincide, 2) there are more than one first-class constraints quadratic in the momenta.

In order to close the gauge algebra we introduce \( M \) auxiliary vectors \((F_1, \ldots, F_M)\) and modify the transformation law of the coordinates \( r \) as

\[ \delta r = \beta_A (K + F) + \beta_B r. \]  
(2.15)

The transformation of \( F \) is determined by the condition that \( K + F \) transforms as \( p \) in the Hamiltonian formalism. Explicitly we get

\[ \delta F = -A^{-1} \left[ \beta_A \partial_t (K + F) + \beta_A B^\top (K + F) + (\delta A - \beta_B A) F + \beta_A C r \right], \]  
(2.16)

while the transformation of \( \Lambda \) remains unchanged

\[ \delta \Lambda = \dot{\beta} - [\Lambda, \beta]. \]  
(2.17)

The new algebra closes off-shell.

The invariant action under the modified gauge transformations is

\[ S = \int dt \frac{1}{2} \left( K^\top AK - r^\top Cr - F^\top AF \right), \]  
(2.18)

The redundancy of the auxiliary variables \( F \) is guaranteed by the action itself which implies \( F = 0 \) as the equation of motion.

This action is also invariant under one-dimensional diffeomorphisms (Diff) \( t \) reparametrizations---, which can be obtained from the above gauge transformations by the change of gauge parameters given in eq. (A.10):

\[ \beta = \tilde{\beta} + \epsilon \Lambda + \dot{\epsilon} H, \]  
(2.19)

where \( H \) is an arbitrary element of the Cartan subalgebra of \( \text{sp}(2M) \).

The Diff transformations of the fields are given by (see appendices A and B for a derivation):

\[ \delta \Lambda^\gamma = \epsilon \dot{\Lambda}^\gamma + (1 + \sum_\alpha (\alpha, \gamma) \tilde{k}_\alpha) \dot{\epsilon} \Lambda^\gamma, \quad \delta \Lambda^\alpha = \epsilon \dot{\Lambda}^\alpha + \dot{\epsilon} k^\alpha + \ddot{\epsilon} \tilde{k}_\alpha, \quad \delta \epsilon r = \epsilon \dot{r} + \dot{\epsilon} N r, \quad \delta \epsilon F = \epsilon \dot{F} - \dot{\epsilon} NF. \]

These Diff transformations may be regarded as realizations of the Virasoro group generated by (improved) Sugawara energy-momentum tensors. The freedom in choosing the different Virasoro realizations is reflected in the arbitrariness of the \( \tilde{k}_\alpha \) constants. When all of them are zero we obtain the usual realization with all the gauge fields having conformal weight equal to one. The \( \tilde{\beta} \) transformations are the same as in (2.14), (2.16) and (2.17) with \( \beta_A, \beta_B \) and \( \beta_C \) replaced by \( \tilde{\beta}_A, \tilde{\beta}_B \) and \( \tilde{\beta}_C \).
Once we perform a partial gauge-fixing of the \( \Lambda \) matrix induced by a \( \text{sl}(2) \) embedding on \( \text{sp}(2M) \), the remnant \( \Lambda \) fields will remain primaries or quasi-primaries. On the other hand, the matter and auxiliary variables will not in general transform as primary fields after the gauge-fixing. This gauge-fixing procedure will be explicitly shown in the next sections by considering several examples. The general discussions are given in Appendices A and B.

### 3 \( \mathcal{W}_2 \) model and finite gauge transformations

Let us now study a particle model with \( \text{sl}(2) \) gauge symmetry [1 6] and show how reparametrization invariance appears in our model. Being \( \text{sl}(2) \approx \text{sp}(2) \) we shall consider the model introduced in the previous section, specialized to \( M = 1 \). In this case the Lagrangian gauge transformations close off-shell and no auxiliary variables have to be introduced. We will also construct the finite form of the residual diffeomorphism transformations from the knowledge of the finite transformations before the gauge-fixing. This may be a useful procedure when direct integration of the infinitesimal residual gauge transformations cannot be performed in a simple way.

First let us write down the model explicitly. It is described by the first-order action (2.1) with gauge transformations (2.7) and (2.8), where \( R, \Lambda \) and \( \beta \) are given by

\[
R = \begin{pmatrix} x \\ p \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_2 \\ -\lambda_3 \\ -\lambda_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_2 \\ -\beta_3 \\ -\beta_2 \end{pmatrix}.
\]

After the elimination of \( p \) via its equation of motion the action reads

\[
S = \int dt \left[ \frac{1}{2\lambda_1} (\dot{x} - \lambda_2 x)^2 - \frac{\lambda_3}{2} x^2 \right]
\]

(3.1)

and the gauge transformations are

\[
\delta x = \beta_2 x + \frac{\beta_1}{\lambda_1} (\dot{x} - \lambda_2 x),
\]

\[
\delta \lambda_1 = \dot{\beta}_1 + 2\lambda_1 \beta_2 - 2\lambda_2 \beta_1,
\]

\[
\delta \lambda_2 = \dot{\beta}_2 + \lambda_1 \beta_3 - \lambda_3 \beta_1,
\]

\[
\delta \lambda_3 = \dot{\beta}_3 + 2\lambda_2 \beta_3 - 2\lambda_3 \beta_2.
\]

(3.2)

The gauge algebra still closes off-shell. That is because, as we have mentioned in section 2, Lagrangian open algebras can only occur in theories which possess more than one constraint quadratic in the momenta.

Let us now study the issues of partial gauge-fixing and remnant gauge transformations along the lines of appendix A. We can rewrite the matrix of Lagrange multipliers \( \Lambda \) as

\[
\Lambda = \lambda_1 E_+ + 2\lambda_2 h - 2\lambda_3 E_-,
\]
which defines the embedding of sl(2) in sp(2). In this simple case the space of remnant fields is generated by $E_-$ alone: $G_W = \text{Ker ad}E_- = \langle E_- \rangle$, the remnant parameter belongs to $\text{Ker ad}E_+$ and the gauge-fixing is given by

$$
\begin{align*}
\lambda_1 &= 1, & \lambda_2 &= 0, & \lambda_3 &\equiv \lambda.
\end{align*}
$$

The associated partially gauge-fixed action is

$$
S_{pgf} = \int dt \left( \frac{\dot{x}^2}{2} - \frac{\lambda x^2}{2} \right),
$$

which produces the matter equation of motion

$$
\ddot{x} + \lambda x = 0.
$$

This is precisely the DS equation $Lx = 0$ where $L$ is the standard KdV operator.

The existence of a diffeomorphism symmetry sector —the only remnant symmetry in this model— can be shown by changing gauge parameters according to (2.19). In the present case $\mathcal{H} \cap G_W = \{0\}$. Hence no arbitrary constants $k_i$ can be introduced. The redefinition (2.19) is here, in components,

$$
\beta_1 = \epsilon, \quad \beta_2 = \sigma, \quad \beta_3 = \rho + \lambda \epsilon.
$$

If the partial gauge-fixing (3.3) is imposed then the remnant transformations are parametrized by $\epsilon$ and the other parameters are written in terms of it:

$$
\sigma = -\frac{1}{2} \dot{\epsilon}, \quad \rho = \frac{1}{2} \ddot{\epsilon}.
$$

Using (3.3), (3.6) and (3.7) in (3.2) one shows that the remnant transformations are indeed (world-line) diffeomorphisms:

$$
\begin{align*}
\delta x &= \epsilon \dot{x} - \frac{1}{2} \ddot{\epsilon} x, \\
\delta \lambda &= \epsilon \dot{\lambda} + 2 \dot{\epsilon} \lambda + \frac{1}{2} \ddot{\epsilon}.
\end{align*}
$$

These infinitesimal transformations can be integrated directly to give their standard finite forms

$$
\begin{align*}
x'(t) &= (\dot{f})^{-1/2} x(f(t)), \\
\lambda'(t) &= (\dot{f})^2 \lambda(f(t)) + \frac{1}{2} \left( \frac{\dddot{f} - \frac{3}{2} (\ddot{f})^2}{(f)^2} \right).
\end{align*}
$$

Now we present an alternative way to find the previous finite transformations. We will find the finite form of the residual transformations from the finite gauge transformations obtained before imposing the partial gauge-fixing conditions. First we consider a redefinition of the gauge parameters that shows the diffeomorphism invariance before
the gauge-fixing. Next we find the finite form of these transformations. Finally we impose the gauge conditions. In this way we obtain the finite form of the remnant transformations from the finite form of the transformations before the gauge-fixing.

Before imposing the gauge-fixing condition let us introduce the following change in the gauge parameters (2.19)

$$\beta_1 = \lambda_1 \epsilon, \quad \beta_2 = \sigma + \lambda_2 \epsilon, \quad \beta_3 = \rho + \lambda_3 \epsilon.$$ (3.10)

The gauge transformations (3.2) in terms of these new parameters read:

$$\delta x = \epsilon \dot{x} + \sigma x,$$

$$\delta \lambda_1 = \epsilon \lambda_1 + \epsilon \dot{\lambda}_1 + 2\sigma \lambda_1,$$

$$\delta \lambda_2 = \epsilon \dot{\lambda}_2 + \epsilon \dot{\lambda}_2 + \sigma + \rho \lambda_1,$$

$$\delta \lambda_3 = \epsilon \lambda_3 + \epsilon \dot{\lambda}_3 - 2\sigma \lambda_3 + \dot{\rho} + 2\rho \lambda_2.$$ (3.11)

The $\epsilon$ transformation is just a world-line reparametrization where $x$ transforms as a scalar and the Lagrange multipliers transform as vectors. The weights of $x$ and $\lambda$ under reparametrization (3.11) are different from the ones of after the gauge-fixing (3.8). In the latter the variable $x$ is no longer a scalar and $\lambda$ transforms as a weight-two tensor with a “central extension” term.

The finite forms of the new transformations (3.11) are found for each $\epsilon, \sigma$ and $\rho$ transformations:

- **Reparametrizations**

  $$x'(t) = x(f(t)), \quad \lambda'_a(t) = \dot{f}(t) \lambda_a(f(t)), \quad a = 1, 2, 3.$$ (3.12)

- **Local scale transformations**

  $$x' = e^\sigma x, \quad \lambda'_1 = e^{2\sigma} \lambda_1, \quad \lambda'_2 = \lambda_2 + \dot{\sigma}, \quad \lambda'_3 = e^{-2\sigma} \lambda_3.$$ (3.13)

- **Local redefinition of Lagrange multipliers**

  $$x' = x, \quad \lambda'_1 = \lambda_1, \quad \lambda'_2 = \lambda_2 + \rho \lambda_1, \quad \lambda'_3 = \lambda_3 + 2\rho \lambda_2 + \lambda_1 \rho^2.$$ (3.14)

Now we impose the gauge-fixing condition (3.3). Notice that (3.10) reduces to (3.4) on the gauge-fixing surface. Any arbitrary configuration in the gauge orbit can be
realized using a composition of the above finite transformations with generic functions f(t), σ(t) and ρ(t). If we consider the following composition

□ → □' → □'' → □'''

the complete finite transformation is given by

\[
\begin{align*}
\tilde{x} &= e^{\sigma(f(t))}x(f(t)), \\
\tilde{\lambda}_1 &= \dot{f}(t)e^{2\sigma(f(t))}\lambda_1(f(t)), \\
\tilde{\lambda}_2 &= \dot{f}(t)\left[\lambda_2(f(t)) + \rho(f(t))\lambda_1(f(t)) + \dot{\sigma}(f(t))\right], \\
\tilde{\lambda}_3 &= \dot{f}(t)e^{-2\sigma(f(t))}\left[\lambda_3(f(t)) + \dot{\rho}(f(t))\right] \\
&\quad \quad + 2\rho(f(t))\lambda_2(f(t)) + \lambda_1(f(t))\rho^2(f(t)).
\end{align*}
\] (3.15)

Imposing the gauge-fixing conditions (3.3) on these transformations we obtain the finite form of the conditions (3.7) for the finite gauge parameters:

\[
\begin{align*}
\sigma(t) &= -\frac{1}{2} \ln \dot{f}(f^{-1}(t)), \\
\rho(t) &= -\dot{\sigma}(t).
\end{align*}
\] (3.16)

Using this restriction in the composition of finite gauge transformations (3.15) we arrive at the finite residual transformations (3.9). The interesting point here is that we have been able to integrate the infinitesimal transformations (3.8) without actually doing it.

### 4 Sp(4) models

Here we will consider the Sp(4) model. In order to obtain W-transformations we need to introduce the appropriate gauge-fixing. For sp(4) we have three different classes of sl(2) embeddings (see appendix A) which will lead to three different gauge-fixings. Notice that not every element of these equivalence classes will produce a gauge-fixed model written in terms of coordinates and velocities. Only those that produce a non-singular matrix A after the gauge-fixing will have this property (see (2.11)).

We will examine these three embeddings using the following labeling of gauge parameters:

\[
\beta_A = \begin{pmatrix} \beta_2 & \beta_{10} \\ \beta_{10} & \beta_5 \end{pmatrix}, \quad \beta_B = \begin{pmatrix} \beta_3 & \beta_9 \\ \beta_8 & \beta_6 \end{pmatrix}, \quad \beta_C = \begin{pmatrix} \beta_1 & \beta_7 \\ \beta_7 & \beta_4 \end{pmatrix}.
\] (4.1)

#### 4.1 (0, 1) embedding

Consider the gauge-fixing induced by the sl(2)-embedding with characteristic (0, 1) given by (A.22). The remnant fields after the gauge-fixing are T, C, G and H. Explicitly
the gauge-fixing is given by
\[ \Lambda_r = \begin{pmatrix} H & 0 & 0 & 1 \\ 0 & -H & 1 & 0 \\ C & T & -H & 0 \\ T & G & 0 & H \end{pmatrix}. \tag{4.2} \]

In this gauge the action (2.1) becomes
\[ S = \int dt \left[ (\dot{x}_1 - Hx_1)(\dot{x}_2 + Hx_2) + \frac{1}{2} \left( Cx_1^2 + Tx_1x_2 + Gx_2^2 \right) - F_1F_2 \right]. \tag{4.3} \]

The equations of motion for the matter variables from this action are:
\[ \begin{pmatrix} L_{x_1} \\ L_{x_2} \end{pmatrix} = \begin{pmatrix} C & -\left(\frac{d}{dt} + H\right)^2 + \frac{1}{2}T \\ -(\frac{d}{dt} + H)^2 + \frac{1}{2}T & G \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0. \tag{4.4} \]

These can be regarded as the DS equations for this embedding. The corresponding Lax operator is given in a 2 \times 2 matrix form.

There are four residual gauge transformations. The remnant parameters live in \( \ker \text{ad}E \) and are \( \beta_{10}, \beta_5, \beta_2 \) and \( \beta_3 - \beta_6 \). The change given by eq. (2.19) yields the following redefinition of these parameters after the gauge-fixing:
\[ \begin{align*}
\tilde{\beta}_2 &= \beta_2, \\
\tilde{\beta}_5 &= \beta_5, \\
\epsilon &= \beta_{10}, \\
\alpha &= \tilde{\beta}_3 - \tilde{\beta}_6 = \beta_3 - \beta_6 - 2\epsilon H - k\dot{\epsilon}.
\end{align*} \tag{4.5} \]

Here \( k \equiv \frac{1}{6}(\tilde{k}_\alpha - \tilde{k}_\beta) \) is an arbitrary constant. We keep it to stress a freedom in choosing the weight of the fields, though it could be absorbed into the definition of \( \alpha \).

The four residual transformations are:

- \( \epsilon \)-sector (Diff).
  \[ \begin{align*}
\delta H &= \epsilon\dot{H} + H\dot{\epsilon} + \frac{k}{2}\dot{\epsilon}, \\
\delta C &= \epsilon\dot{C} + (2 - k)C\dot{\epsilon}, \\
\delta x_1 &= \epsilon\dot{x}_1 + \frac{\epsilon}{2}(k - 1)x_1\dot{\epsilon}, \\
\delta x_2 &= \epsilon\dot{x}_2 - \frac{\epsilon}{2}(k + 1)x_2\dot{\epsilon}, \\
\delta F_1 &= \epsilon\dot{F}_1 - \frac{\epsilon}{2}(k - 1)F_1\dot{\epsilon}, \\
\delta F_2 &= \epsilon\dot{F}_2 + \frac{\epsilon}{2}(k + 1)F_2\dot{\epsilon}.
\end{align*} \tag{4.6} \]

In the transformations of matter and auxiliary variables we have introduced an antisymmetric combination of the equations of motion (see eq. (B.2)).

The matter and auxiliary variables \( x_1, x_2, F_1 \) and \( F_2 \) transform as primary fields under diffeomorphisms with weights \( \frac{1}{2}(k - 1), -\frac{1}{2}(k + 1), \frac{1}{2}(1 - k) \) and \( \frac{1}{2}(1 + k) \) respectively. The gauge variables \( C \) and \( G \) transform also as primary fields with weights \( 2 - k \) and \( 2 + k \). Instead, \( T \) is a quasi-primary field with weight 2 and \( H \) transforms as a field of weight 1 with a \( \ddot{\epsilon} \) term.
• \( \alpha \)-sector (Dilatations).

\[
\begin{align*}
\delta H &= \frac{1}{2} \alpha, & \delta T &= 0, & \delta C &= -\alpha C, & \delta G &= \alpha G, \\
\delta x_1 &= \frac{1}{2} \alpha x_1, & \delta x_2 &= -\frac{1}{2} \alpha x_2, & \delta F_1 &= -\frac{1}{2} \alpha F_1, & \delta F_2 &= \frac{1}{2} \alpha F_2. \tag{4.7}
\end{align*}
\]

• \( \beta_2 \) (= \( \tilde{\beta}_2 \))-sector.

\[
\begin{align*}
\delta H &= \frac{1}{2} C \beta_2, & \delta T &= \beta_2 (\dot{C} - 2CH) + 2 \dot{\beta}_2 C, & \delta C &= 0, \\
\delta G &= \beta_2 (4H^3 - 2HT - 6H \dot{H} + \frac{1}{2} \dot{T} + \ddot{H}) - \dot{\beta}_2 (6H^2 - T - 3\dot{H}) + 3H \ddot{\beta}_2 - \frac{1}{2} \dddot{\beta}_2, \\
\delta x_1 &= \beta_2 (2Hx_2 + \dot{x}_2) - \frac{1}{2} x_2 \dddot{\beta}_2 + \beta_2 F_1, & \delta x_2 &= 0, \\
\delta F_1 &= 0, & \delta F_2 &= -\beta_2 (\dot{F}_1 - [L]_{x_1}) - \frac{1}{2} \dddot{\beta}_2 F_1. \tag{4.8}
\end{align*}
\]

• \( \beta_5 \) (= \( \tilde{\beta}_5 \))-sector. The residual \( \beta_5 \) transformations can be obtained from the \( \beta_2 \) transformations by the following replacements:

\[
\begin{align*}
\beta_2 &\leftrightarrow \beta_5, & H &\leftrightarrow -H, & C &\leftrightarrow G, & x_1 &\leftrightarrow x_2, & F_1 &\leftrightarrow F_2. \tag{4.9}
\end{align*}
\]

The algebra of these residual transformations is:

\[
\begin{align*}
[\delta_\epsilon, \delta_\epsilon'] &= \delta_\epsilon''; & \epsilon'' &= \epsilon' \dot{\epsilon} - \epsilon \dot{\epsilon}', \\
[\delta_\epsilon, \delta_\alpha] &= \delta_\alpha'; & \alpha' &= -\epsilon \dot{\alpha}, \\
[\delta_\epsilon, \delta_{\beta_2}] &= \delta_{\beta_2'}; & \beta_2' &= (1 - k) \beta_2 \dot{\epsilon} - \epsilon \dot{\beta}_2, \\
[\delta_\epsilon, \delta_{\beta_5}] &= \delta_{\beta_5'}; & \beta_5' &= (1 + k) \beta_5 \dot{\epsilon} - \epsilon \dot{\beta}_5, \\
[\delta_\alpha, \delta_{\beta_2}] &= \delta_{\beta_2'}; & \beta_2' &= -\alpha \beta_2, \\
[\delta_\alpha, \delta_{\beta_5}] &= \delta_{\beta_5'}; & \beta_5' &= \alpha \beta_5, \\
[\delta_{\beta_2}, \delta_{\beta_5}] &= \delta_{\epsilon'} + \delta_{\alpha'} + \delta_{\gamma'}; & \epsilon' &= \gamma' = -\frac{1}{2} (\beta_2 \dot{\beta}_5 - \dot{\beta}_2 \beta_5) - 2H \beta_2 \beta_5, \\
\alpha' &= -\frac{1}{2} (\dot{\beta}_2 \dot{\beta}_5 - (1 + k) \beta_2 \dot{\beta}_5 + (k - 1) \beta_5 \dot{\beta}_2) + 2(2 + k) \beta_2 \dot{\beta}_2 \dot{\beta}_5 H - \\
&\quad - 2(2 - k) \beta_5 \dot{\beta}_2 H - 2 \beta_2 \beta_5 (\frac{1}{2} \dot{T} - 5 \dot{H}^2 - k \ddot{H}), \\
[\delta_\alpha, \delta_{\alpha'}] &= [\delta_\alpha, \delta_{\gamma}] = [\delta_{\beta_2}, \delta_{\beta_2'}] = [\delta_{\beta_5}, \delta_{\beta_5'}] = [\delta_{\beta_2}, \delta_{\gamma}] = [\delta_{\beta_5}, \delta_{\gamma}] = 0. \tag{4.10}
\end{align*}
\]
where the $\gamma$ transformation is a trivial transformation, i.e. it is proportional to the equations of motion. It is explicitly given by

$$
\begin{align*}
\delta H &= \delta T = \delta C = \delta G = 0, \\
\delta x_1 &= \gamma F_2, \quad \delta x_2 = \gamma F_1, \\
\delta F_1 &= -2\gamma(\dot{F}_1 + H F_1) - \gamma L_{x_1}, \\
\delta F_2 &= -2\gamma(\dot{F}_2 - H F_2) - \gamma L_{x_2}.
\end{align*}
$$

Notice that we have an open algebra with field-dependent structure functions. The non-closure of the present algebra is due to the introduction of equations of motion in the definition of the $\epsilon$ transformation in terms of the original $\beta$ transformations.

Let us present the finite forms of previous infinitesimal transformations. The strategy is to perform the gauge-fixing on the finite $\text{Sp}(2M)$ transformations as it was done in the case of $\text{sl}(2)$ (see sect. 3). In appendix B we display the expressions of these finite transformations. Explicitly, we can find the following four sets of finite transformations:

- **Diff. sector:** The residual finite diffeomorphisms are obtained by the following composition of finite transformations, $(\omega := \tilde{\beta}_3 + \tilde{\beta}_6)$:

$$
X \xrightarrow{\tilde{\beta}_7} \square \xrightarrow{\omega} \square \xrightarrow{\text{diff}} \tilde{X},
$$

where $X$ stands for any variable. We can express $\tilde{\beta}_7$ and $\omega$ in terms of the Diff parameter $f(t)$ obtaining the residual transformations:

$$
\begin{align*}
T &\rightarrow f^2 T(f) - \left( \frac{f^{(3)}}{f} - \frac{3\dot{f}^2}{2f^2} \right), \\
H &\rightarrow \dot{f} H(f) + \frac{k}{2f^2}, \\
C &\rightarrow \dot{f}^{2-k} C(f), \\
G &\rightarrow \dot{f}^{2+k} G(f)
\end{align*}
$$

\text{matter variables} \quad \begin{cases} 
 x_1 &\rightarrow \dot{f}^{\frac{1}{2}(k-1)} x_1(f) \\
 x_2 &\rightarrow \dot{f}^{\frac{1}{2}(k+1)} x_2(f)
\end{cases}
$$

\text{auxiliary variables} \quad \begin{cases} 
 F_1 &\rightarrow \dot{f}^{\frac{1}{2}(1-k)} F_1(f) \\
 F_2 &\rightarrow \dot{f}^{\frac{1}{2}(1+k)} F_2(f)
\end{cases}
$$

(4.11)

- **$\alpha$-sector (dilatations):** The finite transformations corresponding to the $\alpha$-sector, $(\alpha := \tilde{\beta}_3 - \tilde{\beta}_6)$ are the same as before and after the gauge-fixing:

$$
\begin{align*}
T &\rightarrow T, \\
H &\rightarrow H + \frac{1}{2}\dot{\alpha}, \\
C &\rightarrow e^{-\alpha} C, \\
G &\rightarrow e^{\alpha} G.
\end{align*}
$$
matter variables \[
\begin{align*}
x_1 &\rightarrow e^{\frac{i}{2} \alpha} x_1 \\
x_2 &\rightarrow e^{\frac{-i}{2} \alpha} x_2
\end{align*}
\]

auxiliary variables \[
\begin{align*}
F_1 &\rightarrow e^{-\frac{i}{2} \alpha} F_1 \\
F_2 &\rightarrow e^{\frac{1}{2} \alpha} F_2.
\end{align*}
\]

• \( \beta_2 (= \tilde{\beta}_2) \)-sector: The residual finite \( \beta_2 \) transformations are obtained by the following composition of finite transformations:

\[
X \xrightarrow{\beta_4, \tilde{\beta}_2} \Box \xrightarrow{\tilde{\beta}_3} \tilde{X},
\]

and are given by:

\[
\begin{align*}
T &\rightarrow T + \beta_2 \dot{\dot{C}} + 2C \beta_2 - 2\beta_2 C H - \frac{1}{2} \beta_2^2 C^2 \\
C &\rightarrow C \\
H &\rightarrow H + \frac{1}{2} \beta_2 C \\
G &\rightarrow G + \beta_2 \left( \frac{i}{2} \dot{T} + 6 H \dot{H} + \ddot{H} - 2 H T + 4 H^3 \right) + \\
&\quad + \beta_2 \left( T - 6 H^2 + 3 \dot{H} \right) + 3 \beta_2 H - \frac{1}{2} \beta_2^3 + \\
&\quad + \beta_2^2 \left( 5 C H^2 - \frac{1}{2} C T - 2 C \dot{H} - 3 \dot{C} H + \frac{1}{2} \dot{C} \right) + \\
&\quad + \beta_2^3 \left( \frac{3}{2} \dot{C} - 8 C H \right) + \frac{7}{4} \beta_2^2 C + \frac{3}{2} \beta_2 \beta_3 C + \\
&\quad + \beta_2^3 \left( 2 H C^2 - C \dot{C} \right) - 2 \beta_2^2 \dot{\beta}_2 C^2 + \frac{1}{4} \beta_2^4 C^3
\end{align*}
\]

matter variables \[
\begin{align*}
x_1 &\rightarrow x_1 + \beta_2 \left( \dot{x}_2 + F_1 + 2 H x_2 \right) - \frac{1}{2} \beta_2 x_2 + \frac{1}{2} \beta_2^2 C x_2 \\
x_2 &\rightarrow x_2
\end{align*}
\]

auxiliary variables \[
\begin{align*}
F_1 &\rightarrow F_1 \\
F_2 &\rightarrow F_2 - \beta_2 \left( \dot{F}_1 - [L] x_1 \right) - \frac{1}{2} \beta_2 F_1 + \frac{1}{2} \beta_2^2 C F_1.
\end{align*}
\]

• \( \beta_5 (= \tilde{\beta}_5) \)-sector: Again, the residual finite \( \beta_5 \) transformations can be obtained from the \( \beta_2 \) transformations with the replacements displayed above (4.9).

Notice the appearance of the Schwarzian derivative in the Diff transformation of \( T \). These transformations are actually finite symmetry transformations of the action, under which the Lagrangian changes by a total derivative term and the set of equations of motion remains invariant. The finite transformations parametrized by \( \alpha \), \( \beta_2 \) and \( \beta_5 \) are a parametrization of the specific \( \mathcal{W} \)-transformations. One might expect, according to the algebra of the infinitesimal transformations, that the composition of \( \beta_2 \) and \( \beta_5 \) transformations should give a finite Diff transformation but clearly this is not the case. In this sense, the above form of finite \( \mathcal{W} \)-transformations is parametrized in a rather non-standard way.
We will comment on a secondary reduction of this model. When we further require $H = 0$ on the gauge field matrix (4.2) we get a system whose symmetry transformations realize a non-local algebra discussed by Bilal [13]. This is expected from the form of equation of motion (4.4). Let us see how the residual symmetry satisfies a non-local algebra. The condition $H = 0$ further requires

$$\delta H = \frac{1}{2} \partial (\alpha + k \dot{\epsilon}) + \frac{1}{2} (C \beta_2 + G \beta_5) = 0. \quad (4.14)$$

If we solve it for $\alpha$, assuming a suitable boundary condition, as

$$(\alpha + k \dot{\epsilon}) = -\partial^{-1} (C \beta_2 + G \beta_5), \quad (4.15)$$

there remain three residual transformations. The fields $C$ and $G$ transform as weight 2 primaries under Diff. They transform in a non-local way under $\beta_2$ and $\beta_5$ transformations due to (4.15):

$$\delta C = \partial^{-1} (\beta_2 C - \beta_5 G) C + (\beta_5 \frac{\dot{T}}{2} + \dot{\beta}_5 T - \frac{\dot{\beta}_5}{2}),$$
$$\delta G = -\partial^{-1} (\beta_2 C - \beta_5 G) G + (\beta_2 \frac{\dot{T}}{2} + \dot{\beta}_2 T - \frac{\dot{\beta}_2}{2}),$$
$$\delta T = 2(\dot{\beta}_2 C + \dot{\beta}_5 G) + (\beta_2 \dot{C} + \beta_5 \dot{G}). \quad (4.16)$$

They are equivalent to the non-local and non-linear algebra $V_{2,2}$ discussed in [13]. The matter fields are also transformed non-locally,

$$\delta x_1 = -\frac{1}{2} \partial^{-1} (C \beta_2 + G \beta_5) x_1 + \beta_2 \dot{x}_2 - \frac{1}{2} x_2 \dot{\beta}_2 + \beta_2 F_1,$$
$$\delta x_2 = \frac{1}{2} \partial^{-1} (C \beta_2 + G \beta_5) x_2 + \beta_5 \dot{x}_1 - \frac{1}{2} x_1 \dot{\beta}_5 + \beta_5 F_2,$$
$$\delta F_1 = \frac{1}{2} \partial^{-1} (C \beta_2 + G \beta_5) F_1 - \beta_5 \left( \dot{F}_2 - [L]_{x_2} \right) - \frac{1}{2} \dot{\beta}_5 F_2,$$
$$\delta F_2 = -\frac{1}{2} \partial^{-1} (C \beta_2 + G \beta_5) F_2 - \beta_2 \left( \dot{F}_1 - [L]_{x_1} \right) - \frac{1}{2} \dot{\beta}_2 F_1. \quad (4.17)$$

We can further impose the condition $G = 1$ in addition to $H = 0$. The residual algebra becomes local again, i.e. $\alpha$ and $\beta_5$ are solved in local forms. The resulting system is shown to have $\mathcal{W}(2,4)$ symmetry which will be discussed in the next subsection.

A systematic study of secondary reductions of $\mathcal{W}$-algebras has been carried out in ref. [17].
4.2 (1, 1) (principal) embedding

The gauge-fixing induced by the principal \( sl(2) \) embedding (A.23) in \( sp(4) \) is given by

\[
\Lambda_r = \begin{pmatrix}
0 & 0 & 0 & \sqrt{3} \\
0 & 0 & \sqrt{3} & \frac{1}{5} T \\
\frac{1}{6} W & \frac{\sqrt{3}}{10} T & 0 & 0 \\
\frac{\sqrt{3}}{10} T & 2 & 0 & 0
\end{pmatrix}.
\] (4.18)

The two remnant fields are \( T \) and \( W \). Here numerical factors are taken for convention.

The action (2.18) is given in this case by:

\[
S_{pgf} = \int dt \left[ \frac{\dot{x}_1 \dot{x}_2}{\sqrt{3}} - \frac{T}{10} \left( \frac{\dot{x}_1^2}{3} - \sqrt{3} x_1 x_2 + F_2^2 \right) + \frac{W}{12} x_1^2 + x_2^2 - \sqrt{3} F_1 F_2 \right].
\] (4.19)

The residual transformations are parametrized by \( \epsilon \) and \( \rho \), related to the remnant \( \beta \) parameters in the following way:

\[
\epsilon - \frac{1}{20} \delta + \frac{3}{100} \rho T = \frac{1}{2\sqrt{3}} \beta_{10} - \frac{1}{4} \beta_4, \quad \rho = \frac{1}{6} \beta_2.
\] (4.20)

\( \epsilon \) parametrizes the diffeomorphism sector. \( T \) is a quasi-primary weight-two field and \( W \) is a primary weight-four field. The matter \((x_i)\) and auxiliary \((F_i)\) fields are not primary fields. Indeed we can mix them to obtain a set of primary fields \((\tilde{x}_i \text{ and } \tilde{F}_i)\):

\[
\tilde{x}_1 = x_1, \quad \tilde{x}_2 = x_2 + \frac{3}{20\sqrt{3}} T x_1 - \frac{1}{2\sqrt{3}} \tilde{x}_1, \\
\tilde{F}_1 = F_1 + \frac{1}{10\sqrt{3}} T F_2, \quad \tilde{F}_2 = F_2.
\] (4.21)

The action (4.19) then becomes (neglecting total derivative terms):

\[
S_{pgf} = \int dt \left[ -\frac{1}{30} T \dddot{x}_1^2 - \frac{1}{12} \left( \dddot{x}_1 - \frac{3}{10} T \ddot{x}_1 \right)^2 + \frac{1}{12} W x_1^2 + \ddot{x}_2^2 - \sqrt{3} F_1 F_2 \right].
\] (4.22)

Notice that this action is of a higher order in \( \tilde{x}_1 \). Its equation of motion is

\[
\dddot{x}_1^{(4)} - T \dddot{x}_1 - \dot{T} \ddot{x}_1 - \left( W + \frac{3}{10} T - \frac{9}{100} T^2 \right) \ddot{x}_1 = 0.
\] (4.23)

On the other hand, the matter variable \( \tilde{x}_2 \) decouples and disappears on-shell. There are the same number of physical degrees of freedom as in (4.19). It has the following two residual symmetries (up to equations of motion):

- \( \epsilon \) sector (Diff).

\[
\delta T = \epsilon \dot{T} + 2 \epsilon T - 5 \epsilon, \quad \delta W = \epsilon \dot{W} + 4 \epsilon W, \\
\delta \tilde{x}_1 = \epsilon \dddot{x}_1 - \frac{3}{2} \ddot{x}_1 \epsilon, \quad \delta \tilde{x}_2 = \epsilon \dddot{x}_2 + \frac{1}{2} \ddot{x}_2 \epsilon, \\
\delta \tilde{F}_1 = \epsilon \dddot{F}_1 + \frac{1}{2} \ddot{F}_1 \epsilon, \quad \delta \tilde{F}_2 = \epsilon \dddot{F}_2 + \frac{1}{2} \ddot{F}_2 \epsilon.
\] (4.24)
\[ \delta T = 3\rho \dot{W} + 4\rho W, \]
\[ \delta W = -\frac{1}{20} \rho \dot{\rho}^{(7)} + \frac{7}{25} \rho \dot{\rho}^{(5)} T + \frac{7}{10} \rho \dot{\rho}^{(4)} \dot{T} + \]
\[ + \rho \left( \frac{21}{25} \dot{T} - \frac{49}{125} T^2 + \frac{3}{5} \dot{W} \right) + \rho \left( \frac{14}{25} \dot{T} - \frac{147}{125} T \dot{T} + \frac{9}{10} \dot{W} \right) + \]
\[ + \rho \left( \frac{3}{100} T^{(5)} - \frac{177}{500} T \dot{T} - \frac{39}{250} T^2 + \frac{108}{625} T^2 \dot{T} + \frac{1}{10} \dot{W} - \frac{7}{25} (TW) \right), \]
\[ \delta \tilde{x}_1 = -\frac{1}{20} \rho \ddot{x}_1 + \frac{1}{5} \rho \dot{x}_1 + \rho \left( \frac{23}{100} T \ddot{x}_1 - \frac{1}{2} \ddot{x}_1 \right) + \rho \left( -\frac{27}{100} \ddot{T} \ddot{x}_1 - \frac{41}{50} T \ddot{x}_1 + \dddot{x}_1 \right), \]
\[ \delta \tilde{x}_2 = 0, \quad \delta \tilde{F}_1 = 0, \quad \delta \tilde{F}_2 = 0. \] (4.25)

They show that \( T \) and \( W \) transform according to the infinitesimal transformations induced by the classical \( \mathcal{W}(2,4) \) algebra. Thus the action (4.22) provides a particle model in which the \( \mathcal{W}(2,4) \) symmetry is implemented.

We will show how the matrix of gauge fields (4.18) can be transformed into the form corresponding to the (0,1) embedding (4.2) with \( H = 0 \) and \( G = 1 \). This is achieved by performing finite \( \text{Sp}(4) \) transformations shown in Appendix B. First we make a \( B_\beta \) transformation with
\[ e^{B_\beta} = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{6}} & \sqrt{2} \end{pmatrix}, \]
in order to set the \( A \) submatrix in the form \( A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). Next we realize a \( C_\beta \) transformation with
\[ C_\beta = \begin{pmatrix} \frac{1}{5} T & 0 \\ 0 & 0 \end{pmatrix}. \]

After these gauge transformations the form of the matrix of gauge fields becomes
\[ \Lambda'' = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ W + \frac{4}{25} T^2 - \frac{4}{25} T \dot{T} & \frac{1}{5} T & 0 & 0 \\ \frac{1}{5} T & 1 & 0 & 0 \end{pmatrix}, \] (4.26)

This shows the equivalence of the (4.19) system and that given by the action (4.3) with \( H = 0, \ G = 1 \). After the secondary reduction the weight 4 field \( C \) is no longer primary but is given in terms of the weight 4 primary field \( W \) and the weight 2 quasi-primary field \( T \) as shown in (4.26).

### 4.3 \((1/2,0)\) embedding
The gauge-fixing induced by this embedding \((A.23)\) has six remnant fields, namely \(T, B, C, D_1, D_2\) and \(D_3\) and is given by

\[
\Lambda_r = \begin{pmatrix} 0 & 0 & 1 & 0 \\ B & -D_1 & 0 & D_3 \\ T & C & 0 & -B \\ C & D_2 & 0 & D_1 \end{pmatrix}.
\] (4.27)

The action \((2.1)\) becomes:

\[
S_{\text{pgf}} = \frac{1}{2} \int dt \left[ \dot{x}_1^2 + \dot{x}_2^2 + \frac{2}{D_3} \left( D_1 x_2 \dot{x}_2 - B x_1 \dot{x}_2 - BD_1 x_1 x_2 + \frac{1}{2} D_1^2 x_2^2 \right) + \frac{B^2 x_1^2}{D_3} + \frac{1}{2} T x_1^2 + 2 C x_1 x_2 + D_2 x_2^2 - F_1^2 - D_3 F_2^2 \right].
\] (4.28)

A characteristic feature of this embedding is that the primary field \(D_3\) appears in denominators. The equations of motion for \(x_1\) and \(x_2\) are:

\[
\begin{pmatrix} [L] x_1 \\ [L] x_2 \end{pmatrix} = Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\] (4.29)

\[
Q = \begin{pmatrix} -\frac{d^2}{dt^2} + \frac{B^2}{D_3} + \frac{1}{2} T & -\frac{B}{D_3} \frac{d}{dt} + C - \frac{BD_1}{D_3} \\ \frac{B}{D_3} \frac{d}{dt} + C - \frac{BD_1}{D_3} & -\frac{1}{D_3} \frac{d^2}{dt^2} + \frac{D_2}{D_3} \frac{d}{dt} + D_2 + \frac{D_1^2}{D_3} - \frac{D_1}{D_3} + \frac{D_2}{D_3} \end{pmatrix}.
\]

There are six residual transformations parametrized by \(\epsilon = \beta_2, \beta_4, \beta_5, \beta_6, \beta_9\) and \(\beta_{10}\), which are (up to equations of motion):

- \(\epsilon\)-sector (Diff).

\[
\begin{align*}
\delta T &= \epsilon \dot{T} + 2 \epsilon \dot{t} - \epsilon, \\
\delta B &= \epsilon \dot{B} + (\frac{3}{2} - k) \epsilon B, \\
\delta C &= \epsilon \dot{C} + (\frac{3}{2} + k) \epsilon C, \\
\delta D_1 &= \epsilon \dot{D}_1 + \epsilon D_1 + k \epsilon, \\
\delta D_2 &= \epsilon \dot{D}_2 + (1 + 2k) \epsilon D_2, \\
\delta D_3 &= \epsilon \dot{D}_3 + (1 - 2k) \epsilon D_3, \\
\delta x_1 &= \epsilon \dot{x}_1 - \frac{1}{2} \epsilon x_1 \epsilon, \\
\delta x_2 &= \epsilon \dot{x}_2 - k \epsilon x_2 \epsilon, \\
\delta F_1 &= \epsilon \dot{F}_1 + \frac{1}{2} \epsilon F_1 \epsilon, \\
\delta F_2 &= \epsilon \dot{F}_2 + k \epsilon F_2 \epsilon;
\end{align*}
\] (4.30)

\(k \equiv \frac{1}{6} (\bar{k}_\beta - \bar{k}_\alpha)\) is the arbitrary constant.

- \(\beta_4\)-sector.

\[
\begin{align*}
\delta T &= 0, & \delta B &= 0, & \delta C &= -\beta_4 B, \\
\delta D_1 &= -\beta_4 D_3, & \delta D_2 &= 2 \beta_4 D_1 - \beta_4, & \delta D_3 &= 0, \\
\delta x_1 &= 0, & \delta x_2 &= 0, & \delta F_1 &= 0, & \delta F_2 &= 0.
\end{align*}
\] (4.31)

- \(\beta_5\)-sector.

\[
\begin{align*}
\delta T &= 0, & \delta B &= \beta_5 C, & \delta C &= 0, \\
\delta D_1 &= -\beta_5 D_2, & \delta D_2 &= 0, & \delta D_3 &= 2 \beta_5 D_1 + \beta_5, \\
\delta x_1 &= 0, & \delta x_2 &= \beta_5 \left( \frac{1}{D_3} \left( \dot{x}_2 + D_1 x_2 - B x_1 \right) + F_2 \right), \\
\delta F_1 &= 0, & \delta F_2 &= -\frac{1}{D_3} \beta_5 F_2 - \beta_5 \left( \dot{F}_2 + D_1 F_2 - [L]_2 \right).
\end{align*}
\]
\section*{5 Sl(3) models}

The $W(2, 4, \ldots, 2M)$ gauge transformations can be obtained by considering the principal $\text{sl}(2)$ embedding in a general $\text{sp}(2M)$ algebra. It is also possible to construct particle-like models having symmetries related to other $W$ algebras. If we want to obtain models related to the $A_{N-1}$ series (for instance, the $W_N$ algebras) we have to look for embeddings of the $\text{sl}(N)$ algebras in the symplectic algebras. There is a canonical embedding, namely,

\begin{equation}
\text{sl}(N) \oplus u(1) \subset \text{sp}(2N).
\end{equation}
Explicitly, the set of matrices of the form
\[
\begin{pmatrix}
B & 0 \\
0 & -B^\top
\end{pmatrix}
\]  
(5.2)
are a subalgebra of \(\text{sp}(2N)\) isomorphic to \(\text{sl}(N) \oplus \text{u}(1)\). We may construct the particle model taking this specific form of the gauge fields matrix but we cannot follow the procedure outlined in section 3 because \(A = 0\) so eq. (2.11) is no longer valid to eliminate the \(p\) variables through their equations of motion. Indeed, when we put all the momenta on-shell, we obtain a null Lagrangian.

However we can deal with other embeddings and obtain particle-like actions. For example, let us consider a canonical action (2.1) with \(M = 3\) taking \(\Lambda\) as:
\[
\Lambda = \begin{pmatrix}
\lambda_7 & 0 & 0 & 0 & \lambda_1 & \lambda_3 \\
0 & \lambda_7 - \lambda_8 & \lambda_5 & \lambda_1 & 0 & 0 \\
0 & \lambda_2 & \lambda_8 & \lambda_3 & 0 & 0 \\
0 & \lambda_4 & \lambda_6 & -\lambda_7 & 0 & 0 \\
\lambda_4 & 0 & 0 & 0 & \lambda_8 - \lambda_7 & -\lambda_2 \\
\lambda_6 & 0 & 0 & 0 & -\lambda_5 & -\lambda_8
\end{pmatrix}.
\]  
(5.3)
We are considering only a part of the \(\phi_{A_{ij}}\) constraints (see appendix A). They still close under Poisson bracket giving a realization of the \(\text{sl}(3)\) algebra (we then have a \(\text{sl}(3)\) subalgebra of \(\text{sp}(6)\)). The gauge transformations still are (2.8) and (2.7) with the following \(\beta\) matrix:
\[
\beta = \begin{pmatrix}
\beta_7 & 0 & 0 & 0 & \beta_1 & \beta_3 \\
0 & \beta_7 - \beta_8 & \beta_5 & \beta_1 & 0 & 0 \\
0 & \beta_2 & \beta_8 & \beta_3 & 0 & 0 \\
0 & \beta_4 & \beta_6 & -\beta_7 & 0 & 0 \\
\beta_4 & 0 & 0 & 0 & -\beta_7 + \beta_8 & -\beta_2 \\
\beta_6 & 0 & 0 & 0 & -\beta_5 & -\beta_8
\end{pmatrix} = \bar{\beta} + \epsilon \Lambda.
\]  
(5.4)
Let us perform the following gauge-fixing (induced by the principal \(\text{sl}(2)\) embedding in \(\text{sl}(3)\)):
\[
\lambda_1 = \lambda_2 = 1, \quad \lambda_3 = \lambda_7 = \lambda_8 = 0, \quad \lambda_4 = \lambda_5 = \frac{1}{2} T, \quad \lambda_6 = W.
\]  
(5.5)
Again we cannot use (2.11) because \(\text{det } A = 0\) but we can get a (higher order) particle-like Lagrangian once we eliminate the momenta variables:
\[
S_{\text{pgf}} = \int dt \left[ \frac{1}{2} (\ddot{x}_1 \dot{x}_3 - \dot{x}_1 \ddot{x}_3) - \frac{1}{2} T (x_1 \dot{x}_3 - \dot{x}_1 x_3) - W x_1 x_3 \right].
\]  
(5.6)
The equations of motion for the $x_1$ variables are:

$$[L]_{x_1} = \ddot{x}_3 - T\dot{x}_3 - (W + \frac{1}{2}\dot{T})x_3, \quad [L]_{x_3} = -\ddot{x}_1 + T\dot{x}_1 + (-W + \frac{1}{2}\dot{T})x_1. \quad (5.7)$$

They are two copies of the DS equation for $W_3$.

This action exhibits a (classical) $W_3$ symmetry, being the remnant parameters $\epsilon$ and $\rho = \tilde{\beta}_3$:

- $\epsilon$-sector (Diff).
  
  $$\delta T = \epsilon \dot{T} + 2\epsilon T - 2\epsilon, \quad \delta W = \epsilon \dot{W} + 3\epsilon W, \quad \delta x_1 = \epsilon \dot{x}_1 - \epsilon x_1, \quad \delta x_3 = \epsilon \dot{x}_3 - \epsilon x_3. \quad (5.8)$$

- $\rho$-sector.
  
  $$\delta T = 2\rho \dot{W} + 3\rho W, \quad \delta W = \frac{1}{6} \rho^{(5)} - \frac{5}{6} \rho \dot{T} - \frac{5}{4} \rho \dot{\rho} T + \rho \left( -\frac{3}{4} \dot{T} + \frac{2}{3} T^2 \right) + \rho \left( -\frac{1}{6} \dot{T} + \frac{2}{3} T \dot{T} \right), \quad \delta x_1 = \frac{1}{2} \left( \frac{3}{4} \rho \dot{x}_1 - \rho \dot{x}_1 + 2\rho \ddot{x}_1 - \frac{4}{3} \rho T x_1 \right), \quad \delta x_3 = \frac{1}{2} \left( \frac{3}{4} \rho \dot{x}_3 - \rho \dot{x}_3 + 2\rho \ddot{x}_3 - \frac{4}{3} \rho T x_3 \right). \quad (5.9)$$

We can also present a model exhibiting the symmetry associated with the $W$-algebra generated through the only non-principal sl(2) embedding into sl(3), namely $W_3^3$. By performing the following gauge-fixing:

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = -1, \quad \lambda_4 = B, \quad \lambda_5 = C, \quad \lambda_6 = -T, \quad \lambda_7 = -\lambda_8 = H, \quad (5.10)$$

we obtain the following action:

$$S_{pgf} = \int dt \left[ H x_1 \dddot{x}_2 - \dot{x}_1 \dddot{x}_2 + \left( H + \frac{\dot{C}}{C} \right) \dddot{x}_1 - \dot{x}_2 + 2 \left( \dot{H} - \frac{H C}{C} + H^2 \right) \dddot{x}_1 x_2 \right] \dddot{x}_1 x_2 + 2H \left( T - \dot{H} + \frac{H C}{C} - H^2 \right) x_1 - BC x_1 x_2 - ] \quad (5.11)$$

The equations of motion for $x_1$ and $x_2$ provide DS equations for $W_3^3$:

$$\dddot{x}_2 - 2\dddot{x}_2 + \left( \frac{2H C}{C} - 3\dot{H} - 3H^2 - T - \frac{\dot{C}}{C} + \frac{2C^2}{C^2} \right) \dddot{x}_2 + \left( \frac{4H \dot{C}}{C} + \frac{4H^2 \dot{C}}{C} + 2H \dddot{C} \right) \dddot{x}_2 - 6H \dddot{H} - 2H^3 + 2HT + BC - 2\dddot{H} \right) x_2 = 0, \quad (5.12)$$

$$\dddot{x}_1 - \dddot{x}_1 + \left( \frac{2H \dot{C}}{C} - 3\dot{H} - 3H^2 - T \right) \dddot{x}_1 + \left( \frac{H C}{C} - \frac{H^2 \dot{C}}{C} + 2H^3 - 2HT - BC - \dot{H} - \dot{T} + \frac{TC}{C} \right) x_1 = 0. \quad (5.13)$$
The previous action exhibits $W_2^2$ symmetry:

- $\epsilon$-sector (Diff).

\[
\begin{align*}
\delta H &= \epsilon \dot{H} + H \epsilon + k \epsilon, \\
\delta T &= \epsilon \dot{T} + 2k \dot{T} - \frac{1}{2} \epsilon, \\
\delta B &= \epsilon \dot{B} + (\frac{3}{2} - 3k) \dot{B} \epsilon, \\
\delta C &= \epsilon \dot{C} + (\frac{3}{2} + 3k) \dot{C} \epsilon, \\
\delta x_1 &= \epsilon x_1 + (k - \frac{1}{2}) x_1 \dot{x} , \\
\delta x_2 &= \epsilon x_2 + 2k x_2 \dot{x} .
\end{align*}
\]

- $\alpha(= \frac{1}{2} \beta_5)$-sector.

\[
\begin{align*}
\delta H &= \dot{\alpha}, \\
\delta T &= 0, \\
\delta B &= -3 \alpha B, \\
\delta C &= 3 \alpha C, \\
\delta x_1 &= \alpha x_1, \\
\delta x_2 &= 2 \alpha x_2.
\end{align*}
\]

- $\beta_2(= \tilde{\beta}_2)$-sector.

\[
\begin{align*}
\delta H &= \frac{1}{2} B \beta_2, \\
\delta T &= \beta_2 \left( \frac{1}{2} \dot{B} - 3B H \right) + \frac{3}{2} \dot{\beta}_2 B, \\
\delta C &= \beta_2 \left( 9H^2 - 3H - T \right) - 6 \dot{\beta}_2 H + \dot{\beta}_2, \\
\delta x_1 &= \beta_2 \left( \frac{1}{C} \left( H^2 - T - \dot{H} \right) x_1 - \frac{2H}{C} \dot{x}_1 + \dot{x}_1 \right), \\
\delta x_2 &= \frac{\beta_2}{C} \left( 8H^2 - \frac{2H^2}{C} + 2\dot{H} \right) x_2 + \left( \frac{\dot{C}}{C} - 2H \right) \dot{x}_2 - \ddot{x}_2 + \dot{x}_2 + \frac{\beta_2}{C} (\dot{x}_2 - 2H x_2).
\end{align*}
\]

- $\beta_6(= \tilde{\beta}_6)$-sector.

\[
\begin{align*}
\delta H &= -\frac{1}{2} C \beta_6, \\
\delta T &= \beta_6 \left( \frac{1}{2} \dot{C} + 3CH \right) + \frac{3}{2} \dot{\beta}_6 C, \\
\delta C &= 0, \\
\delta B &= \beta_6 \left( T - 9H^2 - 3\dot{H} \right) - 6 \dot{\beta}_6 H - \dot{\beta}_6, \\
\delta x_1 &= 0, \\
\delta x_2 &= 0.
\end{align*}
\]

6 Conclusions

We have introduced a particle mechanics model in phase space which can be recast as a one-dimensional $Sp(2M)$ gauge theory. Different partial gauge-fixings of this model by $sl(2)$ embeddings in $sp(2M)$ yield reduced theories in which the remnant Lagrange multiplier variables correspond to generators of classical $W$-algebras associated with the $C_n$ simple Lie algebras. We have also shown how to obtain models invariant under $W$-algebras related to other series, such as the $A_n$.

In relation with the issue of finite $W$-transformations, the simplest reduced theory, with $W_2$ symmetry, has only one remnant Lagrange multiplier which transforms as a weight-two quasi-primary field. In this case, the finite form of its symmetry transformations is easily obtained by using the finite transformations of the $sp(2)$ model and restricting them to those satisfying the gauge-fixing condition.
Application of this procedure to the model associated with the \((0, 1)\) sl(2) embedding in sp(4) yields finite symmetry transformations of its action. These finite transformations are perfectly acceptable as a parametrization of the gauge freedom of the system and they are actually useful for building the general solution of the model. However they cannot be regarded as standard finite \(\mathcal{W}\)-diffeomorphism transformations because their composition does not give ordinary diffeomorphisms. In order to obtain the expected form of finite \(\mathcal{W}\)-diffeomorphism transformations one might introduce a non-linear change of infinitesimal gauge parameters before the gauge-fixing by modifying the Yang-Mills transformations in a similar way as it was done to extract the ordinary diffeomorphism. Indeed, other approaches \[18, 19\] seem to point in the direction of treating all \(\mathcal{W}\)-transformations as Diff of an extended space.

We have also shown a derivation of a non-local algebra in the course of a secondary reduction of the Sp(4) model. This secondary reduction does not come from a sl(2) embedding in sp(4) because the non-remnant gauge parameter is not solved algebraically. However the non-local algebra \[13\] appears in the dynamical context. It may be interesting to study various non-local algebras arising from the secondary reductions of Sp(2M) models.

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A Zero-curvature condition and \(\mathcal{W}\)-transformations

In this appendix we review the ‘soldering’ procedure to construct classical chiral \(\mathcal{W}\)-transformations \[8\] and its relation with the zero-curvature approach and sl(2, \(R\))-embedding technique \[8, 11, 10\].

Let \(\Lambda(t)\) be a Lie-algebra valued field transforming

\[
\delta \Lambda = \dot{\beta} - [\Lambda, \beta].
\]  
(A.1)

This can be regarded as a zero-curvature condition:

\[
[\delta - \beta, \partial_t - \Lambda] = 0
\]  
(A.2)

Let us now consider a partial gauge-fixing on the matrix of Lagrange multipliers \(\Lambda(t)\):

\[
\Lambda = M + W;
\]  
(A.3)
where $M$ is a non-zero constant element of $\mathcal{G}$ and $W = W^b T'_b$. The Lie algebra elements $T'_b$ span $\mathcal{G}_W$, a subspace of $\mathcal{G}$ ($b = 1, \ldots, \dim \mathcal{G}_W < \dim \mathcal{G}$) and $W^b$ are the remnant fields living in $\mathcal{G}_W$. We are looking for residual gauge transformations (2.8) preserving the partial gauge-fixing \((A.3)\). The zero-curvature condition is now the gauge-slice conservation condition:

$$[M, \beta] + \delta W = \dot{\beta} - [W, \beta]. \tag{A.4}$$

The possible partial gauge-fixings, i.e. the choices of $M$ and $W$, are restricted once we impose the following two requirements:

- We want to express a subset of the gauge parameters $\beta^a$ as a function of another subset (remnant parameters) and the remnant fields $W^b$ in a purely algebraic way.

- The residual transformations should include a diffeomorphism (Virasoro) sector in such a way that we could identify a weight-two quasi-primary field to it.

The first requirement is algebraically equivalent to the condition of the total set of constraints being second-class in the Kac-Moody Hamiltonian reduction. Both requirements are satisfied if the partial gauge-fixing \((A.3)\) is induced by a sl(2, $\mathbb{R}$) embedding \([6, 20]\), \(S\), of the original Lie algebra $\mathcal{G}$,

$$M = E_+, \quad \mathcal{G}_W = \ker \text{ad} E_-, \tag{A.5}$$

where $E_+$, $E_-$ and $h$ are the defining elements of the sl(2, $\mathbb{R}$) embedding:

$$[h, E_\pm] = \pm E_\pm, \quad [E_+, E_-] = h. \tag{A.6}$$

The mapping $\text{ad}S$ given by $\text{ad}S : a \to \text{ad}a$ where $a \in S$ and

$$\text{ad}a : \mathcal{G} \rightarrow \mathcal{G}, \quad g \rightarrow [a, g], \quad a \in \mathcal{G},$$

is a representation of $S$ on $\mathcal{G}$. This representation is completely reducible so $\mathcal{G}$ (as a vector space) decomposes to a direct sum of invariant subspaces of spin $j$ (integer or half-integer) and multiplicity $n_j$ (branching):

$$\mathcal{G} = \sum_{j \geq 0} \sum_{i=1}^{n_j} \mathcal{G}^{(i)}_j, \quad \mathcal{G}^{(i)}_j = \sum_{m=-j}^{j} \mathcal{G}^{(i)}_{j,m}, \quad \sum_{j \geq 0} n_j (2j + 1) = \dim \mathcal{G}. \tag{A.7}$$

The $\mathcal{G}^{(i)}_{j,m}$ are one-dimensional eigenspaces of $\text{ad}h$ with eigenvalue $m$. A spin 1 subspace is always present in the branching, namely, $S$ itself (denoted by $\mathcal{G}_1$). They define a gradation of $\mathcal{G}$:

$$\hat{\mathcal{G}}_m = \begin{cases} \sum_{j \geq m} \sum_{i=1}^{n_j} \mathcal{G}^{(i)}_{j,m}, & \text{if } m \text{ is an eigenvalue of } \text{ad}h \\ \{0\}, & \text{otherwise} \end{cases} \Rightarrow \quad \mathcal{G} = \sum_m \hat{\mathcal{G}}_m, \quad \left[\hat{\mathcal{G}}_m, \hat{\mathcal{G}}_n\right] \subset \hat{\mathcal{G}}_{m+n}. \tag{A.8}$$
According to (A.5), every remnant field lives in $\hat{G}_{m=−j}$ and there are
\begin{equation}
N(S) = \sum_{j \geq 0} n_j
\end{equation}
such fields.

The presence of this gradation ensures that the first requirement is satisfied. Indeed, remnant parameters live in $\ker \text{ad} E_+$, i.e., in $\hat{G}_{m=j}$. Restrictions of the zero-curvature condition (A.4) to subspaces $\hat{G}_j$ allow us to express in an algebraic way parameters living in $\hat{G}_{j−1}$ as functions of parameters living in $\hat{G}_j$ (and fields) because $M$ lives in $\hat{G}_1$.

So, as we go down on the spectrum of $m$’s, we have an algebraic algorithm to express all the gauge parameters as functions of those living in $\ker \text{ad} E_+$ and fields. Finally, restrictions of (A.4) to subspaces $\hat{G}_{m=−j}$ give the transformations of the remnant fields, $\delta W^b$.

The existence of a Virasoro sector can be shown by performing a decomposition of parameters: $\beta \rightarrow \tilde{\beta}, \epsilon$. Consider the following change:
\begin{equation}
\beta = \tilde{\beta} + \epsilon \Lambda + \epsilon H,
\end{equation}
where $H = \sum_\alpha \tilde{k}_\alpha H_\alpha$ is a general element of the Cartan subalgebra $\mathcal{H}$ of $\mathcal{G}$, with constant coefficients and $\tilde{\beta} = \tilde{\beta}^c T^c_c$ ($c = 1, \ldots, \dim \mathcal{G} − 1$). With this change Virasoro transformations appear both before and after the gauge-fixing. However, the transformation laws of the remnant fields after the gauge-fixing are generally different from the original ones.

In order to examine the Virasoro transformations of the gauge field $\Lambda$, we decompose it as
\[
\Lambda = \sum_\gamma \Lambda^\gamma E_\gamma + \sum_\alpha \Lambda^\alpha H_\alpha,
\]
where $\{E_\gamma, H_\alpha\}$ form a Cartan-Weyl basis of the Lie algebra $\mathcal{G}$. The zero-curvature condition together with the definition (A.10) produces the following Virasoro transformations before the gauge-fixing:
\begin{align*}
\delta \Lambda^\gamma &= \epsilon \Lambda^\gamma + (1 + \sum_\alpha (\alpha, \gamma) \tilde{k}_\alpha) \epsilon \Lambda^\gamma, \\
\delta \Lambda^\alpha &= \epsilon \Lambda^\alpha + \epsilon \Lambda^\alpha + \tilde{k}_\alpha \epsilon.
\end{align*}
(11)

Notice that the fields living on the Cartan subalgebra (indices $\alpha$) transform as weight-one tensors, generally with an inhomogeneous extension term. Instead the fields living in the root spaces (indices $\gamma$) transform as tensors of weight $1 + \sum_\alpha (\alpha, \gamma) \tilde{k}_\alpha$. Had we considered a change of parameters (A.10) without the $\epsilon H$ term then all fields would have transformed as weight-one tensors. This is equivalent to taking the usual Sugawara energy-momentum tensor as the generator of the Virasoro transformations in the Kac-Moody Hamiltonian reduction framework. Then the addition of the $\epsilon H$ term
in (A.10) corresponds to changing the realization of the Virasoro group by considering an improved Sugawara energy-momentum tensor as the generator.

The transformations generated by \( \epsilon \) remain to be Virasoro transformations after the gauge-fixing procedure. Indeed, the parameter \( \epsilon \) lives in the subspace generated by \( E_+ \) so it is one of remnant parameter (we can take the \( T_0' \) Lie algebra elements as the generators of all the \( \mathcal{G}_{j,m}^{(i)} \) subspaces except the one of \( \mathcal{G}_{1,1} \), i.e. \( E_+ \)). The zero-curvature condition (A.4) after the gauge-fixing reads:

\[
[M, \tilde{\beta}] = \dot{\beta} - \delta W - [W, \tilde{\beta}] + \epsilon \dot{W} + \dot{\epsilon}(M + W) + \tilde{\epsilon}H + \epsilon[H, M + W].
\] (A.12)

To solve the zero-curvature condition we have to expand \( H \) and \( W \) as follows:

\[
H = k_0h + \sum_i k_i H_i + \sum_{\sigma} k_{\sigma} H_{\sigma},
\] (A.13)

\[
W = \sum_i W^i H_i + \sum_{\alpha} W^\alpha e_\alpha + \sum_{\rho} W^\rho e_\rho.
\] (A.14)

In the expansion of \( H \), \( h \) is the sl(2)-embedding element, \( \{H_i\} \) span \( \mathcal{G}_W \cap \mathcal{H} \) and \( \{H_\sigma\} \) form a basis of the rest of \( \mathcal{H} \). In the expansion of \( W \), \( W^i \) are the fields living in \( \mathcal{H} \), \( W^\alpha \) are the fields living in \( \hat{\mathcal{G}}_W \) but not in \( \mathcal{H} \) and \( W^\rho \) are the rest of remnant fields. The following relations hold:

\[
[h, e_\rho] = -j(\rho)e_\rho, \quad [h, e_\alpha] = 0, \quad [H_i, e_\rho] = r_i(\rho)e_\rho, \quad [H_i, e_\alpha] = r_i(\alpha)e_\alpha.
\]

One can study the propagation of the parameter \( \epsilon \) through the equations imposed by the zero-curvature condition at each level in the gradation of \( \mathcal{G} \). The result of this analysis is:

\[
\tilde{\beta} = -(1 + k_0)\dot{\epsilon}h - \dot{\epsilon}k_\sigma H_{\sigma} - \tilde{\epsilon}E_- + \text{(terms without } \epsilon) \] (A.15)

Once we introduce (A.14) in (A.12) we get the residual infinitesimal transformations of the remnant fields, \( \delta W \), under the \( \epsilon \) sector. There are some cancellations due to the presence of the term \( \epsilon H \) which cut off the propagation of the \( k_0 \) and \( k_\sigma \) parameters. Hence the only surviving arbitrariness comes from the \( k_i \) parameters. The result is summarized as:

- The field \( T \) living in the subspace generated by \( E_- \), which is one of the \( e_\rho \) generators, transforms as a quasi-primary field of weight two:

\[
\delta T = \epsilon \dot{T} + 2\dot{\epsilon}T - \ddot{\epsilon}.
\] (A.16)

- Fields living in the subspace spanned by \( H_i, W^i \), transform as weight-one fields plus a term \( \ddot{\epsilon} \):

\[
\delta W^i = \epsilon \dot{W}^i + \dot{\epsilon}W^i + k_i \ddot{\epsilon}, \quad (i = 1, \ldots, \dim \mathcal{H} \cap \mathcal{G}_W).
\] (A.17)
• The rest of remnant fields living in $\mathcal{G}_{m=-j}$, $W^\rho$ and $W^\alpha$, are primary fields:

$$\delta W^\rho = \epsilon \dot{W}^\rho + \left(1 + j(\rho) + \sum_i k_ir_i(\rho)\right)\dot{\epsilon}W^\rho,$$

$$\delta W^\alpha = \epsilon \dot{W}^\alpha + \left(1 + \sum_i k_ir_i(\alpha)\right)\dot{\epsilon}W^\alpha.$$  \hspace{1cm} (A.17)

In general, the field living in $\mathcal{G}_{m=-j}$ has weight $1 + j$ apart from possible shifts, which exist in case the subspace $\mathcal{H} \cap \mathcal{G}_W$ is non-trivial. The following relation holds:

$$\sum \text{weights} := \sum_{j \geq 0} n_j (1 + j) = \frac{1}{2} (\dim \mathcal{G} + N(\mathcal{G})).$$ \hspace{1cm} (A.18)

There is no explicit general formula for the transformations generated by the other remnant parameters. They are precisely specific chiral $\mathcal{W}$-transformations because we have a set of infinitesimal transformations with closed algebra and containing a Virasoro sector with the weight-two quasi-primary field $T$.

### A.1 Inequivalent $\text{sl}(2, R)$ embeddings

It is useful to separate the set of all possible sl$(2, R)$ embeddings in $\mathcal{G}$ into classes of equivalent embeddings. Two embeddings $S_1$ and $S_2$ are said to be equivalents if there exists an automorphism of $\mathcal{G}$ mapping $S_1$ onto $S_2$. There will be as many admissible gauge-fixings [A.3] as classes of equivalent $\text{sl}(2, R)$ embeddings.

Given a canonical decomposition of $\mathcal{G}$ (i.e. given a Cartan subalgebra of $\mathcal{G}$, $\mathcal{H}$, a set of positive roots, $\Delta_+$, and a set of simple roots, $\Pi$),

$$\mathcal{G} = \sum_{\alpha \in \Delta_+} \dot{\mathcal{G}}_{-\alpha} \dot{\mathcal{H}} \dot{\sum}_{\alpha \in \Delta_+} \dot{\mathcal{G}}_{\alpha},$$ \hspace{1cm} (A.19)

and a sl$(2, R)$ embedding in $\mathcal{G}$, $S$, we can always choose a member of the same class of equivalence of $S$ such that:

$$h \in \mathcal{H}, \quad \text{i.e.} \quad h = H_\delta \quad \text{where} \quad \delta = \sum_{\beta \in \Pi} c_\beta \beta,$$ \hspace{1cm} (A.20)

$$E_\pm = \sum_{\gamma \in \Gamma_\delta} e_{\pm \gamma}, \quad e_\gamma \in \mathcal{G}_\gamma, \quad \Gamma_\delta = \{\gamma \in \Delta_+ \mid (\gamma, \delta) = 1\};$$ \hspace{1cm} (A.21)

$\delta$ is the defining vector of such an embedding. Let us consider the Dynkin diagram of $\mathcal{G}$. We construct the characteristic of this $\text{sl}(2, R)$ embedding by writing down the number $(\beta, \delta)$ under the dot of the Dynkin which represents the root $\beta$ for each $\beta \in \Pi$. Two important results follow [21, 22]:

• Two sl$(2, R)$ embeddings are equivalent if and only if their characteristics coincide.
If a characteristic is associated with a \( \text{sl}(2, R) \) embedding then it exhibits numbers of the set \( \{0, \frac{1}{2}, 1\} \).

It can be shown that the potential characteristic which exhibits a 1 under every dot always gives rise to a \( \text{sl}(2, R) \) embedding, which is known as the principal \( \text{sl}(2, R) \) embedding.

As an example we present here the case \( \mathcal{G} = \text{sp}(4, R) \). The Dynkin diagram, normalizations and positive roots set for \( \text{sp}(4, R) \) are:

\[
\begin{align*}
\alpha & \quad \beta \\
\circ & \quad \circ \\
\alpha & = \frac{1}{6}  \\
\beta & = \frac{1}{3} \quad (\alpha, \beta) = \frac{-1}{6}.
\end{align*}
\]

\( \Delta_+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\} \).

There are only three classes of non-equivalent \( \text{sl}(2, R) \) embeddings. Their characteristics are:

\[
\begin{align*}
\alpha & \quad \beta \\
\frac{1}{2} & \quad 0 \\
\alpha & = \frac{1}{\sqrt{12}}  \\
\beta & = \frac{1}{\sqrt{12}}.
\end{align*}
\]

Our matrix conventions for the generators of \( \text{sp}(4, R) \) are:

\[
H_\alpha = \frac{1}{12} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
H_\beta = \frac{1}{6} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

\[
E_\alpha = \frac{1}{\sqrt{12}} \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix},
E_\beta = \frac{1}{\sqrt{12}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

\[
E_{\alpha + \beta} = \frac{1}{\sqrt{12}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
E_{-(\alpha + \beta)} = \frac{1}{\sqrt{12}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
E_{2\alpha + \beta} = \frac{1}{\sqrt{6}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
E_{-(2\alpha + \beta)} = \frac{1}{\sqrt{6}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
Here we display a representative of every class of equivalent embeddings and the corresponding branchings:

• \((0, 1)\) embedding:
  \[
  h = 6H_\alpha + 6H_\beta \quad E_\pm = \sqrt{6}E_{\pm(\alpha + \beta)}
  \]
  \[
  G = G_1^{(1)} + G_1^{(2)} + G_1^{(3)} + G_0
  \]
  \[
  G_1^{(2)} = \langle E_{2\alpha + \beta}, E_\alpha, E_{-\beta} \rangle \quad G_1^{(3)} = \langle E_\beta, E_{-\alpha}, E_{-(2\alpha + \beta)} \rangle
  \]
  \[
  G_0 = \langle H_\alpha \rangle . \tag{A.22}
  \]

• \(\left(\frac{1}{2}, 0\right)\) embedding:
  \[
  h = 6H_\alpha + 3H_\beta \quad E_\pm = \sqrt{3}E_{\pm(2\alpha + \beta)}
  \]
  \[
  G = \hat{G}_1 + \hat{G}_1^{(1)} + \hat{G}_1^{(2)} + \hat{G}_1^{(3)} + \hat{G}_0
  \]
  \[
  \hat{G}_1^{(1)} = \langle E_{\alpha + \beta}, E_{-\alpha} \rangle \quad \hat{G}_1^{(2)} = \langle E_\alpha, E_{-(\alpha + \beta)} \rangle
  \]
  \[
  \hat{G}_0^{(1)} = \langle E_\beta \rangle \quad \hat{G}_0^{(2)} = \langle E_{-\beta} \rangle \quad \hat{G}_0^{(3)} = \langle H_{-\beta} \rangle . \tag{A.23}
  \]

• \((1, 1)\) (principal) embedding:
  \[
  h = 18H_\alpha + 12H_\beta \quad E_\pm = \sqrt{18}E_{\pm\alpha} + \sqrt{12}E_{\pm\beta}
  \]
  \[
  G = \hat{G}_1 + \hat{G}_3
  \]
  \[
  \hat{G}_3 = \langle E_{2\alpha + \beta}, E_{\alpha + \beta}, \sqrt{3}E_{-\beta} - \sqrt{2}E_{\alpha}, H_{\alpha} - H_{\beta}, \sqrt{3}E_{-\beta} - \sqrt{2}E_{-\alpha}, \]
  \[
  E_{-(\alpha + \beta)}, E_{-(2\alpha + \beta)} \rangle . \tag{A.24}
  \]

We consider another element of this conjugacy class because the previous one produces a gauge fixing such that there is no gauge-fixed Lagrangian in terms of coordinates and velocities:

\[
  h = 18H_\alpha + 6H_\beta \quad E_\pm = \sqrt{18}E_{\pm(\alpha + \beta)} + \sqrt{12}E_{\pm\beta}
  \]
  \[
  G = \hat{G}_1 + \hat{G}_3
  \]
  \[
  \hat{G}_3 = \langle E_{2\alpha + \beta}, E_\alpha, \sqrt{3}E_{-\beta} - \sqrt{2}E_{\alpha + \beta}, H_\alpha + 2H_\beta, \sqrt{3}E_{-\beta} - \sqrt{2}E_{-(\alpha + \beta)}, \]
  \[
  E_{-\alpha}, E_{-(2\alpha + \beta)} \rangle . \tag{A.25}
  \]

For completeness we also present here a representative of each of the two classes of equivalent \(\text{sl}(2, R)\) embeddings in \(\text{sl}(3, R)\):

\[
\begin{array}{c}
\alpha \\
\bigcirc \\
\beta
\end{array}
\]

\((\alpha, \alpha) = (\beta, \beta) = \frac{1}{3} \quad (\alpha, \beta) = -\frac{1}{6}, \quad \Delta_+ = \{\alpha, \beta, \alpha + \beta\} .
\]
• principal \((1, 1)\) embedding:

\[
\begin{align*}
    h &= 6H_\alpha + 6H_\beta \\
    E_\pm &= \sqrt{6}E_{\pm\alpha} + \sqrt{6}E_{\pm\beta} \\
    G &= G_1 + G_2 \\
    G_2 &= \langle E_{\alpha+\beta}, E_\beta - E_\alpha, H_\alpha - H_\beta, E_{-\alpha} - E_{-\beta}, E_{-(\alpha+\beta)} \rangle \\
\end{align*}
\quad \text{(A.26)}
\]

• non-principal \((\frac{1}{2}, \frac{1}{2})\) embedding:

\[
\begin{align*}
    h &= 3H_\alpha + 3H_\beta \\
    E_\pm &= \sqrt{3}E_{\pm(\alpha+\beta)} \\
    G &= G_\frac{1}{2} + G_\frac{1}{2}^{(1)} + G_\frac{1}{2}^{(2)} + G_0 \\
    G_\frac{1}{2}^{(1)} &= \langle E_\alpha, E_{-\beta} \rangle \\
    G_\frac{1}{2}^{(2)} &= \langle E_\beta, E_{-\alpha} \rangle \\
    G_0 &= \langle H_\alpha - H_\beta \rangle.
\end{align*}
\]

The \(\mathfrak{sl}(3, R)\) subalgebra of \(\mathfrak{sp}(6, R)\) that we have considered in sect. 5 is realized by taking the following subset of the \(\phi_{A_{ij}}\) quadratic constraints in the \(M = 3\) case:

\[
\begin{align*}
    E_\alpha &= \frac{1}{\sqrt{6}}p_1p_2, \\
    E_\beta &= \frac{1}{\sqrt{6}}p_3x_2, \\
    E_{\alpha+\beta} &= -\frac{1}{\sqrt{6}}p_1p_3, \\
    E_{-\alpha} &= -\frac{1}{\sqrt{6}}x_1x_2, \\
    E_{-\beta} &= \frac{1}{\sqrt{6}}p_2x_3, \\
    E_{-(\alpha+\beta)} &= \frac{1}{\sqrt{6}}x_1x_3, \\
    H_\alpha &= \frac{1}{6}(p_2x_2 + p_1x_1), \\
    H_\beta &= \frac{1}{6}(p_3x_3 - p_2x_2).
\end{align*}
\]

B Diffeomorphism invariance and finite transformations of the \(\text{Sp}(2M)\) model

Here we show first the invariance of the \(\text{Sp}(2M)\) model under ordinary diffeomorphism transformations, along the lines of Appendix A. We shall later present the model’s gauge symmetry transformations in their finite form.

According to Appendix A we can perform the change of parameters \([A.10]\) with \(H = \sum_{i=1}^{M} \tilde{k}_\alpha H_{\alpha_i}\), where \(\alpha_i\) are the simple roots of \(\mathfrak{sp}(2M, R)\) and \(\tilde{k}_\alpha\) are constants. When \(G = \mathfrak{sp}(2M, R)\) then \(H\) is the diagonalized matrix:

\[
H = \begin{pmatrix} N & 0 \\ 0 & -N^T \end{pmatrix},
\]

\[
N = \frac{1}{4(M + 1)} \begin{pmatrix} \tilde{k}_{\alpha_1} & 0 & & \\
\tilde{k}_{\alpha_2} - \tilde{k}_{\alpha_1} & \ddots & & \\
& \ddots & \tilde{k}_{\alpha_{M-1}} - \tilde{k}_{\alpha_{M-2}} & \\
0 & & 2\tilde{k}_{\alpha_M} - \tilde{k}_{\alpha_{M-1}} \end{pmatrix}.
\quad \text{(B.1)}
\]
where $\alpha_M$ is the longest root.

As stated in appendix A (see (A.11)), the Lagrange multipliers transform as primary fields with, eventually, $\ddot{\epsilon}$ terms under the $\epsilon$ sector infinitesimal transformations. For the matter and auxiliary variables, this change of parameters produces the following infinitesimal transformations:

$$
\tilde{\delta}_\epsilon r = \epsilon \dot{r} + \dot{\epsilon} N r + \epsilon A F,
$$

$$
\tilde{\delta}_\epsilon F = -\epsilon \dot{F} - \dot{\epsilon} (F + NF) + \epsilon \left( A^{-1} B A F - A^{-1} \dot{A} F - B^\top F - \dot{K} - B^\top K - C r \right).
$$

These transformations are equivalent to diffeomorphism transformations. In order to show this let us introduce an antisymmetric combination of the equations of motion:

$$
\delta_\epsilon q^i(t) = \tilde{\delta} q^i(t) + \int dt' M^{ij}(t, t') [L]_{q^j(t')},
$$

where

$$
\begin{align*}
q^i &= x_i \quad (i = 1, \ldots, M) \\
q^i &= F_i \quad (i = M + 1, \ldots, 2M)
\end{align*}
$$

and

$$
M(t, t') = \begin{pmatrix}
0 & \epsilon(t) \delta(t - t') I \\
-\epsilon(t) \delta(t - t') I & M(t, t')
\end{pmatrix},
$$

$$
\tilde{M}(t, t') = -\epsilon(t) \left( B^\top(t) A^{-1}(t) - A^{-1}(t) B(t) \right) \delta(t - t') + 
\epsilon(t') A^{-1}(t') \frac{d}{dt'} \delta(t - t') - \epsilon(t) A^{-1}(t) \frac{d}{dt} \delta(t - t').
$$

It can be shown that

$$
M^\top(t', t) = -M(t, t');
$$

so $\delta_\epsilon$ is a symmetry transformation of the action too and

$$
\delta_\epsilon r = \epsilon \dot{r} + \dot{\epsilon} N r, \quad \delta_\epsilon F = \epsilon \dot{F} - \dot{\epsilon} N F,
$$

which are diffeomorphism transformations for the matter and auxiliary variables. They transform as primary fields.

In summary, the infinitesimal gauge transformations of the Sp(2M) model before the gauge-fixing are:

- Diffeomorphism transformations:

$$
\delta \Lambda^\gamma = \epsilon \dot{\Lambda}^\gamma + (1 + \sum_\alpha (\alpha, \gamma) \tilde{k}_\alpha) \dot{\epsilon} \Lambda^\gamma, \quad \delta \Lambda^\alpha = \epsilon \dot{\Lambda}^\alpha + \dot{\epsilon} \Lambda^\alpha + \tilde{k}_\alpha \dot{\epsilon},
$$

$$
\delta_\epsilon r = \epsilon \dot{r} + \dot{\epsilon} N r, \quad \delta_\epsilon F = \epsilon \dot{F} - \dot{\epsilon} N F,
$$

- Yang-Mills type transformations:

$$
\delta \Lambda = \dot{\beta} - [\Lambda, \tilde{\beta}], \quad \delta r = \tilde{\beta}_A(K + F) + \dot{\beta}_B r,
$$

$$
\delta F = -A^{-1} \left[ \tilde{\beta}_A(K + \dot{F}) + \tilde{\beta}_A B^\top K + \dot{\beta}_A C r - B \tilde{\beta}_A F + \tilde{\beta}_A F \right] - \tilde{\beta}_B^\top F.
$$
where:

$$\Lambda = \sum_{\gamma} \Lambda^\gamma E_{\gamma} + \sum_{\alpha} \Lambda^\alpha H_{\alpha} = \begin{pmatrix} B & A \\ -C & -B^\top \end{pmatrix}, \quad \tilde{\beta} = \begin{pmatrix} \tilde{\beta}_B \\ -\tilde{\beta}_C \\ -\tilde{\beta}_B^\top \end{pmatrix}.$$  

After performing the gauge-fixing, Lagrange multipliers still transform as primary or quasi-primary fields (see appendix A) whereas matter and auxiliary fields do not have, in general, a nice behavior under Diff transformations. For instance, if the $E_{\gamma}$ element of the $\text{sl}(2, R)$ embedding is taken to live in the $A$ or $B$ sectors of a general $\text{sp}(2M, R)$ matrix, then non-desired $\tilde{\epsilon}$ terms appear in the residual $\epsilon$ transformations coming from the algorithm described in appendix A (see (A.14)) through the $\tilde{\beta}_A$ or $\tilde{\beta}_B$ factors of (2.15) and (2.16). In any case, the only undetermined constants that remain in the infinitesimal transformations after the gauge-fixing from those in the (A.13) decomposition of $H$ are the constants $k_i$ as in the residual transformations for the Lagrange multipliers.

Finite transformations\footnote{For a recent discussion on finite gauge transformations see [23].} can be obtained by exponentiating the infinitesimal ones as $X'(t) = \exp(\theta^\alpha \Gamma_{\alpha}) X^i$, where the generators $\Gamma_{\alpha} = R_{\alpha} \frac{\partial}{\partial X^i}$ satisfy $[\Gamma_{\alpha}, \Gamma_{\beta}] = f_{\gamma}^{\alpha\beta} \Gamma_{\gamma}$ and $X^i$ represents any of the variables. The coefficients $f_{\alpha\beta}^{\gamma}$ are the structure functions of the $\text{sp}(2M)$ gauge algebra.

It is useful to perform the integration using the matrix notation. The explicit form of the finite gauge transformations is considered in the following four sets of transformations. Any finite transformations may be obtained by the composition of them.

- The diffeomorphism transformations:

  \begin{align}
  \Lambda^\gamma(t) &= \hat{f}(t)^1 + \sum_{\alpha} \kappa_{\alpha} \Lambda^\gamma(f(t)), \\
  \Lambda^\alpha(t) &= \hat{f}(t) \Lambda^\alpha(f(t)) + \hat{k}_\alpha \frac{\hat{f}(t)}{\hat{f}(\tilde{t})}, \\
  r'_i(t) &= \hat{f}(t) N_{i\alpha} r(f(t)), \\
  F'_i(t) &= \hat{f}(t) - N_{i\alpha} F(f(t)), \quad i = 1, \ldots, M. \quad (B.4)
  \end{align}

  where $N = (N_{ij})$ is the diagonalized constant matrix given in eq. (B.1).

- Transformations generated by $\tilde{\beta}_A$:

  \begin{align}
  A' &= A + \{ \tilde{\beta}_A - \tilde{\beta}_A B^\top - B \tilde{\beta}_A \} + \tilde{\beta}_A C \tilde{\beta}_A, \\
  B' &= B - \tilde{\beta}_A C, \\
  C' &= C, \\
  r' &= r + \tilde{\beta}_A (K + F), \\
  F' &= A'^{-1} \left[ AF - \tilde{\beta}_A \{ \partial_t (K + F) + B^\top (K + F) + Cr \} \right]. \quad (B.5)
  \end{align}

- Transformations generated by $\tilde{\beta}_B$:

  \begin{align}
  A' &= e^{\tilde{\beta}_B} A e^{\tilde{\beta}_B^\top}, \\
  B' &= e^{\tilde{\beta}_B} (B - \partial_t) e^{-\tilde{\beta}_B}, \\
  C' &= e^{-\tilde{\beta}_B} C e^{-\tilde{\beta}_B},
  \end{align}

  \footnote{For a recent discussion on finite gauge transformations see [23].}
\[ r' = e^{\tilde{\beta} \bar{y}} r, \quad F' = e^{-\tilde{\beta}^\top} F. \] (B.6)

- Transformations generated by \( \tilde{\beta}_C \):

\[ A' = A, \quad r' = r, \quad F' = F, \]
\[ B' = B + A\tilde{\beta}_C, \quad C' = C + \{\tilde{\beta}_C + \tilde{\beta}_C B + B^\top \tilde{\beta}_C \} + \tilde{\beta}_C A\tilde{\beta}_C. \] (B.7)

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