Equation $x^i y^j x^k = u^i v^j u^k$ in words

Jana Hadravová and Štěpán Holub
Faculty of Mathematics and Physics, Charles University
186 75 Praha 8, Sokolovská 83, Czech Republic
hadravova@ff.cuni.cz, holub@karlin.mff.cuni.cz

Abstract. We will prove that the word $a^i b^j a^k$ is periodicity forcing if $j \geq 3$ and $i + k \geq 3$, where $i$ and $k$ are positive integers. Also we will give examples showing that both bounds are optimal.

1 Introduction

Periodicity forcing words are words $w \in A^*$ such that the equality $g(w) = h(w)$ is satisfied only if $g = h$ or both morphisms $g, h : A^* \rightarrow \Sigma^*$ are periodic. The first analysis of short binary periodicity forcing words was published by J. Karhumäki and K. Culik II in [2]. Besides proving that the shortest periodicity forcing words are of the length five, their work also covers the research of the non-periodic homomorphisms agreeing on the given small word $w$ over a binary alphabet. What in their work attracts attention the most, is the fact, that even short word equations can be quite difficult to solve. The intricacies of the equation $x^2 y^3 x^2 = u^2 v^3 u^2$, proved to have only periodic solution [3], nothing but reinforced the perception of difficulty. Not frightened, we will extend the result and prove that the word $a^i b^j a^k$ is periodicity forcing if $j \geq 3$ and $i + k \geq 3$, where $i$ and $k$ are positive integers. Also we will give examples showing that both bounds are optimal.

2 Preliminaries

Standard notation of combinatoric on words will be used: $u \leq_p v$ ($u \leq_s v$ resp.) means that $u$ is a prefix of $v$ ($u$ is a suffix of $v$ resp.). The maximal common prefix (suffix resp.) of two word $u, v \in A^*$ will be denoted by $u \wedge_v v$ ($u \wedge_s v$ resp.). By the length of a word $u$ we mean the number of its letters and we denote it by $|u|$. A (one-way) infinite word composed of infinite number of copies of a word $u$ will be denoted by $u^\omega$. It should be also mentioned that the primitive root of a word $u$, denoted by $p_u$, is the shortest word $r$ such that $u = r^k$ for some positive $k$. A word $u$ is primitive if it equals to its primitive root. Words $u, v$ are conjugate if there are words $\alpha, \beta$ such that $u = \alpha \beta$ and $v = \beta \alpha$. For further reading, please consult [6].
We will briefly recall a few basic and a few more advanced concepts which will be needed in the proof of our main theorem. Key role in the proof will be played by the Periodicity lemma [6]:

**Lemma 1 (Periodicity lemma).** Let \( p \) and \( q \) be primitive words. If \( p^\omega \) and \( q^\omega \) have a common factor of length at least \( |p| + |q| - 1 \), then \( p \) and \( q \) are conjugate. If, moreover, \( p \) and \( q \) are prefix (or suffix) comparable, then \( p = q \).

Reader should also recall that if two word satisfy an arbitrary non-trivial relation, then they have the same primitive root. Another well-known result is the fact that the maximal common prefix (suffix resp.) of any two different words from a binary code is bounded (see [6, Lemma 3.1]). We formulate it as the following lemma:

**Lemma 2.** Let \( X = \{x, y\} \) and let \( \alpha \in xX^* \), \( \beta \in yX^* \) be words such that \( \alpha \land \beta \geq |x| + |y| \). Then \( x \) and \( y \) commute.

The previous lemma can be formulated also for the maximal common suffix:

**Lemma 3.** Let \( X = \{x, y\} \) and let \( \alpha \in X^*x \), \( \beta \in X^*y \) be words such that \( \alpha \land \beta \geq |x| + |y| \). Then \( x \) and \( y \) commute.

The most direct and most well known case is the following.

**Lemma 4.** Let \( s = s_1s_2 \) and let \( s_1 \leq_p s \) and \( s_2 \leq_p s \). Then \( s_1 \) and \( s_2 \) commute.

**Proof.** Directly, we obtain \( s = s_1s_2 = s_2s_1 \).

Next, let us remind the following property of conjugate words:

**Lemma 5.** Let \( u, v, z \in A^* \) be words such that \( uz = zv \). Then \( u \) and \( v \) are conjugate and there are words \( \sigma, \tau \in A^* \) such that \( \sigma \tau \) is primitive and

\[
\begin{align*}
u &\in (\sigma \tau)^*, \\
z &\in (\sigma \tau)^*\sigma, \\
v &\in (\tau \sigma)^*.
\end{align*}
\]

We will also need not so well-know, but interesting, result by A. Lentin and M.-P. Schützenberger [3].

**Lemma 6.** Suppose that \( x, y \in A^* \) do not commute. Then \( xy^+ \cup x^+y \) contains at most one imprimitive word.

We now introduce some more terminology. Suppose that \( x \) and \( y \) do not commute and let \( X = \{x, y\} \), i.e. we suppose that \( X \) is a binary code. We say that a word \( u \in X^* \) is \( X \)-primitive if \( u = v^i \) with \( v \in X^* \) implies \( u = v \). Similarly, \( u, v \in X^* \) are \( X \)-conjugate, if \( u = \alpha \beta \) and \( v = \beta \alpha \) and the words \( \alpha \) and \( \beta \) are from \( X^* \).

In the following lemma, first proved by J.-C. Spehner [7], and consequently by E. Barbin-Le Rest and M. Le Rest [1], we will see that all words that are imprimitive but \( X \)-primitive are \( X \)-conjugate of a word from the set \( x^*y \cup xy^* \).
Source of the inspiration of both articles was an article by A. Lentin and M.-P. Schützenberger [4] with its weaker version stating that if the set of \( X \)-primitive words contains some imprimitive words, then so does the set \( x^*y \cup xy^* \). As a curiosity, we mention that Lentin and Schützenberger formulated the theorem for \( x^*y \cap y^*x \) instead of \( x^*y \cup y^*x \) (for which they proved it). Also, the Le Rests did not include in the formulation of the theorem the trivial possibility that the word \( x \) or the word \( y \) is imprimitive.

**Lemma 7.** Suppose that \( x, y \in A^* \) do not commute and let \( X = \{x, y\} \). If \( w \in X^* \) is a word that is \( X \)-primitive and imprimitive, then \( w \) is \( X \)-conjugate of a word from the set \( x^*y \cup y^*x \). Moreover, if \( w \notin \{x, y\} \), then primitive roots of \( x \) and \( y \) are not conjugate.

Putting together Lemma [5] with Lemma [7] we get the following result:

**Lemma 8.** Suppose that \( x, y \in A^* \) do not commute and let \( X = \{x, y\} \). Let \( C \) be the set of all \( X \)-primitive words from \( X^+ \setminus X \) that are not primitive. Then either \( C \) is empty or there is \( k \geq 1 \) such that

\[
C = \{x^iyx^{k-i}, 0 \leq i \leq k\} \quad \text{or} \quad C = \{y^iyx^{k-i}, 0 \leq i \leq k\}.
\]

The previous lemma finds its interesting application when solving word equations. For example, we can see that an equation \( x^iy^jx^k = z\ell \), with \( \ell \geq 2 \), \( j \geq 2 \) and \( i + k \geq 2 \) has only periodic solutions. (This is a slight modification of a well known result of Lyndon and Schützenberger [5]). Notice, that we can use the previous lemma also with equations which would generate notable difficulties if solved “by hand”. E.g. equation

\[
(yx)^iyx(xy)^jxy(xy)^k = z^m,
\]

with \( m \geq 2 \), has only periodic solutions.

We formulate it as a special lemma:

**Lemma 9.** Suppose that \( x, y \in A^* \) do not commute and let \( X = \{x, y\} \). If there is an \( X \)-primitive word \( \alpha \in X^* \) and a word \( z \in A^* \), such that

\[
\alpha = z^i,
\]

with \( i \geq 2 \), then \( \alpha = x^kyx^\ell \) or \( \alpha = y^kxy^\ell \), for some \( k, \ell \geq 0 \).

We finish this preliminary part with the following useful lemmas:

**Lemma 10.** Let \( u, v, z \in A^* \) be words such that \( z \leq_s v \) and \( uv \leq_p zv^i \), for some \( i \geq 1 \). Then \( uv \in zp^*_v \).

**Proof.** Let \( 0 \leq j < i \) be the largest exponent such that \( zv^j \leq_p uv \) and let \( r = (zv^j)^{-1}uv \). Then \( r \) is a prefix of \( v \). Our assumption that \( z \leq_s v \) yields that \( v \leq_s vr \) and

\[
r(r^{-1}v) = v = (r^{-1}v)r.
\]

From the commutativity of words \( r^{-1}v \) and \( r \), it follows that they have the same primitive root, namely \( p_v \). Since \( uv = (zv^j)r \) we have \( uv \in zp^*_v \), which concludes the proof.
Lemma 10 has the following direct corollary.

Lemma 11. Let $w, v, t \in A^*$ be words such that $|t| \leq |w|$ and $wv \leq_{p} tv^i$, for some $i \geq 1$. Then $w \in tp^*_v$.

Proof. Lemma 10 with $u = t^{-1}w$ and $z$ empty yields that $wv \in p^*_v$. Then $wv \in tp^*_v$ and from $|t| \leq |w|$, we obtain that $w \in tp^*_v$.

Lemma 12. Let $u, v \in A^*$ be words such that $|u| \geq |v|$. If $\alpha u$ is a prefix of $v^i$ and $u\beta$ is a suffix of $v^i$, for some $i \geq 1$, then $\alpha u \beta$ and $v$ commute.

Proof. Since $\alpha u$ is a prefix of $v^+$ and $|u| \geq |v|$, we have $\alpha^{-1}v\alpha \leq_{p} u \leq_{p} u\beta$. Our assumption that $u\beta$ is a suffix of $v^i$ yields that $u\beta$ has a period $|v^i|$. Then, $u\beta \leq_{p} (\alpha^{-1}v\alpha)^i$ and, consequently, $\alpha u \beta \leq_{p} v^i$. From $v \leq_{s} u\beta$ and Lemma 10, it follows that $\alpha u \beta \in p^*_v$, which concludes the proof.

Lemma 13. Let $u, v \in A^*$ be words such that $|u| \geq |v|$. If $\alpha u$ and $\beta u$ are prefixes of $v^i$, for some $i \geq 1$, and $|\alpha| \leq |\beta|$, then $\alpha$ is a suffix of $\beta$, and $\beta \alpha^{-1}$ commutes with $v$.

Proof. Since $\alpha u$ is a prefix of $v^+$ and $|u| \geq |v|$, we have $\alpha^{-1}v\alpha \leq_{p} u$. Similarly, $\beta^{-1}v\beta \leq_{p} u$. Therefore, $\alpha^{-1}v\alpha = \beta^{-1}v\beta$.

and $|\alpha| \leq |\beta|$ yields $\alpha \leq_{s} \beta$. From $\beta \alpha^{-1}v = v\beta \alpha^{-1}$ we obtain commutativity of $v$ and $\beta \alpha^{-1}$.

Notice that the previous result can be reformulated for suffixes of $v^i$:

Lemma 14. Let $u, v \in A^*$ be words such that $|u| \geq |v|$. If $u\alpha$ and $u\beta$ are suffixes of $v^i$, for some $i \geq 1$, and $|\alpha| \leq |\beta|$, then $\alpha$ is a prefix of $\beta$, and $\alpha^{-1} \beta$ commutes with $v$.

3 Solutions of $x^iy^jx^k = u^iv^juk$

Theorem 1. Let $x, y, u, v \in A^*$ be words such that $x \neq u$ and

$$x^iy^jx^k = u^iv^juk,$$

where $i + k \geq 3$, $ik \neq 0$ and $j \geq 3$. Then all words $x, y, u$ and $v$ commute.

Proof. First notice that, by Lemma 10 theorem holds in case that either of the words $x$, $y$, $u$ or $v$ is empty. In what follows, we suppose that $x$, $y$, $u$ and $v$ are non-empty. By symmetry, we also suppose, without loss of generality, that $|x| > |u|$ and $i \geq k$; in particular, $i \geq 2$. Recall that $p_x$ ($p_y$, $p_u$, $p_v$ resp.) denote the primitive root of $x$ ($y$, $u$, $v$ resp.).

We first prove the theorem for some special cases.
(A) Let $p_x = p_u$. Then $p_x^{i} y^j p_x^{-k} = v^j$ for some $n \geq 1$, and we are done by Lemma 5.

Notice that the solution of case (A) allows us to assume the useful inequality

$$(i + k - 1) |u| < |p_x|, \quad (*)$$

since otherwise $p_x^\omega$ and $u^\omega$ have a common factor of the length at least $|p_x| + |u|$, and $u$ and $x$ commute by the Periodicity lemma. From

$$(u^{-i+1} p_x u^{-k}) u = u(u^{-i} p_x u^{-k+1})$$

and Lemma 5 we see that there are words $\sigma$ and $\tau$ such that $\sigma \tau$ is primitive and

$$(u^{-i+1} p_x u^{-k}) \in (\sigma \tau)^m, \quad u = (\sigma \tau)\ell \sigma, \quad u^{-i+1} p_x u^{-k} \in (\tau \sigma)^m,$$

for some $m \geq 1$ and $\ell \geq 0$. Then we have

$$u = (\sigma \tau)\ell \sigma, \quad p_x = u^i (\tau \sigma)^m u^{-k-1} = u^{i-1} (\sigma \tau)^m u^k, \quad (**)$$

for some $m \geq 1$ and $\ell \geq 0$.

(B) Let $p_y$ and $p_v$ be conjugate.

Let $\alpha$ and $\beta$ be such that $p_y = \alpha \beta$ and $p_v = \beta \alpha$. Since $x^i p_y$ is a prefix of $u^i p_v^+$, we can see that $u^{-i} x^i \alpha \beta \leq_p \beta (\alpha \beta)^+$. From Lemma 10 we infer that and $u^{-i} x^i \in \beta (\alpha \beta)^*$. Similarly, by the mirror symmetry, $p_y^{x^k} \leq_p p_v^{+1} u^k$ yields that $x^k u^{-k} \in (\alpha \beta)^* \alpha$. Then

$$x^{i+k} = u^i p_v^{n+1} u^k,$$

for some $n \geq 1$. From $|v| > |y|$, it follows that $|v| \geq |y| + |p_v|$ and, consequently,

$$(i + k) (|x| - |u|) = j (|v| - |y|) \geq 3 |p_v|.$$

Then $n \geq 3$ and we are done by Lemma 9.

(C) Let $p_x$ and $p_v$ be conjugate.

Let $\alpha$ and $\beta$ be such that $p_x = \alpha \beta$ and $p_v = \beta \alpha$. From (C) and $i \geq 2$, it follows that $u^i p_v$ is a prefix of $p_x^*$. Then $u^i (\beta \alpha) \leq_p \alpha (\beta \alpha)^+$ and Lemma 12 yields that $u^i \in \alpha (\beta \alpha)^*$. From $i |u| < |p_x|$, it follows $u^i = \alpha$. Since $p_x$ is a suffix of $\alpha \beta \alpha u^k = p_x u^{i+k}$ and $u$ is a prefix of $p_v$, we deduce from Lemma 3 that $x$ and $u$ commute, case (A).

We will now discuss separately cases when $|x| \geq |v|$ and $|x| < |v|$.

1. **Suppose that** $|x| \geq |v|$.

If $i \geq 3$ or $x \neq p_x$, then $(u^{-i} x^i) x^{i-1}$ is a prefix of $v^j$ that is longer than $|p_x| + |x|$ by (C). By the Periodicity lemma, $p_x$ is a conjugate of $p_v$ and we are in case (C). The remaining cases deal with $i = k = 2$ and $i = 2, k = 1$. 
1a) First suppose that \( i = k = 2 \). Since \((u^{-i}x)x\) is a prefix of \(v^i\) and \((xu^{-k})x\) is a suffix of \(v^j\), we get, by Lemma 12, that \((u^{-i}x)x(xu^{-k})x\) commutes with \(v\). Then
\[
x^3 = u^i p_u u^k,
\]
for some \( n \geq 0 \). From \((i + k - 1)|u| < |p_u| \leq |x|\) and \(|p_v| \leq |v| \leq |x|\) we infer that \( n \geq 2 \). Therefore, \( p_u = p_x \) holds by Lemma 9 and we have case (A).

1b) Suppose now that \( i = 2 \) and \( k = 1 \). We will have a look at the words \( u \) and \( x = p_x \) expressed by \( \ast \ast \). Let \( h = (\sigma \tau)^m \) and \( h' = (\tau \sigma)^m \). Then \( \ast \ast \) yields
\[
u = (\sigma \tau)^l \sigma, \quad x = u^2 h' = uh_u.
\]

1b.i) Suppose now that \(|p_v| \leq |uh|\). Since \( h'u'h \) is a prefix of \(v^i\) and \( u'h \) is a suffix of \(v^j\), we obtain by Lemma 12 that \( h'u'h = p_u^n \). From \(|p_v| \leq |uh|\), we infer \( n \geq 2 \) and, according to Lemma 9, \( \sigma \) and \( \tau \) commute. Then also \( x \) and \( u \) commute and we have case (A).

1b.ii) Suppose that \(|p_v| > |uh|\). From \(|x| \geq |v| \geq |p_v|\), it follows that \( p_v = h'u'h_1 \) for some prefix \( u_1 \) of \( u \). We can suppose that \( u_1 \) is a proper prefix of \( u \), otherwise \( x \) and \( v \) are conjugate and we have case (C). Then \( u_1 h' \leq p \) \( u'h \leq p (\sigma \tau)^+ \) and, by Lemma 13, we obtain \( u u_1^{-1} \in (\sigma \tau)^+ \). Therefore, \( u_1 \in (\sigma \tau)^+ \). Since \( h \leq p v \), we can see that \( \sigma \tau \leq p \tau \sigma^+ \). Lemma 8 then implies commutativity of \( \sigma \) and \( \tau \). Therefore, the words \( x \) and \( u \) also commute and we are in case (A).

2. Suppose that \(|x| < |v|\) and \( i|x| = i|u| + |v|\).

From \( x \leq_s v \), we have \( x \leq_s x u^k \). Since \( u \leq_p x \) we deduce from Lemma 3 that \( x \) and \( u \) commute, thus we have case (A).

3. Suppose that \(|x| < |v|\) and \( i|x| > i|u| + |v|\).

Let \( r \) be a non-empty word such that \( u^i v r = x^i \). Notice that \(|r| < |p_x|\) otherwise the words \( p_x \) and \( p_u \) are conjugate and we have case (C). Considering
the words $u$ and $p_x$ expressed by ([13]), we can see that $(\tau\sigma)^m u^{k-1}u^i$ is a prefix of $v$ and $u^{i-1}(\sigma\tau)^m$ is a suffix of $v$. Notice also that we have case (A) if $\sigma$ and $\tau$ commute.

3a) Consider first the special case when $r = u^k$.

3a.i) If $i = k$, then $v^{j-2} = u^i y^i u^i$. If $j \geq 4$, we have case (B) by Lemma [9] If $j = 3$, then the equality $u^j v^i = x_1$ implies $x = u^{2i} y^i u^{2i}$ and we get case (A) again by Lemma [9].

3a.ii) Suppose therefore that $k < i$. Notice that $u = \sigma$, otherwise, from $\tau\sigma \leq_p v$ and $u^k = r \leq_p v$, we get commutativity of $\sigma$ and $\tau$. Therefore,

$$v \in (\tau\sigma)^m \sigma^{k-1} p_x^k \sigma^{i-1}(\sigma\tau)^m.$$ 

We have

$$v u^k x^{i-k} = u r x^{i-k} = u^{-i} x^{i-k}.$$

From $i > k$ and ([13]) we get $|u^{-i} x^{i-k}| > 0$ and, consequently, $|v u^k| > |x^k|$. Let $v'$ denote the word $v u^k x^{i-k}$. Then $v^{j-2} v' = r y^i$, and $j \geq 3$ together with $|v| > |x| > |u^k| = |v|$ yields that $v'$ is a suffix of $y^i$. According to ([13]), $v' = u^{-i} x^{i-k} \in (\tau\sigma)^m \sigma^{k-1} p_x^k$. Then, $\sigma^k$ is a suffix of $y^j$ and we have

$$(\sigma^k y \sigma^{-k}) = \sigma^k y^j \sigma^{-k} = v^{j-2} v' \sigma^{-k}.$$ 

This is a point where Lemma [9] turns out to be extremely useful. Direct inspection yields that $v^{j-2} v' \sigma^{-k}$ is not a power of a word from $\{\sigma, \tau\}^*$. One can verify, for example, that the expression of $v^{j-2} v' \sigma^{-k}$ in terms of $\sigma$ and $\tau$ contains exactly $j-2$ occurrences of $\tau \sigma$. Therefore, Lemma [9] yields that $\sigma$ and $\tau$ commute, a contradiction.

3b) We first show that $r = u^k$ holds if $k \geq 2$. Indeed, if $k \geq 2$ then $u^k p_x u^{-k}$ is a suffix of $v$ and, consequently, $u^k p_x u^{-k} r$ is a suffix of $x^i$. Since $u^k p_x u^{-k} u^k$ is also a suffix of $x^i$, we can use Lemma [14] and get commutativity of $x$ with one of the words $u^{-k} r$ or $r^{-1} u$. From $|r| < |p_x|$ and $|u^k| < |p_x|$, we get $r = u^k$.

3c) Suppose that $k = 1$ and $r \neq u$.

3c.i) If $|r| < |u|$, then $r$ is a suffix of $u$ and $|x r^{-1} u| > |x|$. Since $x r^{-1} \leq_p v$ and $k = 1$, the word $x = x r^{-1} r$ is a suffix of $x r^{-1} u$. Therefore, $x r^{-1}$ is a suffix of $(x r^{-1})^+$. Since $u^2 \leq_p x$ and $|x r^{-1}| \geq |u| + (|u| - |r|)$, the Frequency lemma implies that the primitive root of $u r^{-1}$ is a conjugate of $p_u$. But since $p_u$ is prefix comparable with $x r^{-1}$, we obtain that $x r^{-1} \in p_u^+$ Then also $r \in p_u^+$ and $x r^{-1} \in p_u^+$ Consequently, $x$ and $u$ commute, and we have case (A).

3c.ii) Suppose therefore that $|r| > |u|$. Then $u$ is a suffix of $r$. Since $r$ is a suffix of $p_x$ and $p_x = u^i (\tau\sigma)^m$, the word $r$ is a suffix of $u^i (\tau\sigma)^m$. From $|v| > |x|$ we obtain $u^{-i} x u^i \leq_p v$. Consequently, from $p_x = u^i (\tau\sigma)^m$ and $r \leq_p v$, it follows that $r$ is a prefix of $(\tau\sigma)^m u^i$. Consider first the special case when $r \in (\tau\sigma)^m p_u^*$. If $r \in (\tau\sigma)^m p_u^+$, then $r \leq_a u^i (\tau\sigma)^m$ yields that $(\tau\sigma)^m$ and $u$ commute by Lemma [9] Consequently, $\sigma$ and $\tau$ commute, and we have case (A). Therefore, $r = (\tau\sigma)^m$, $p_x = u^i r$ and $v = u^{-i} x r^{-1} \in (r u^i)^+$. We have proved that $x$ and $v$ have conjugate primitive roots, which yields case (C). Consider now the general case.
If \( m \leq \ell \), then \((\tau \sigma)^m\) is a suffix of \( u \). Since \( r \) is a prefix of \((\tau \sigma)^m u^i\), and \( u \preceq_r r \), we get from Lemma 10 the case \( r \in (\tau \sigma)^m p^*_u \).

Suppose that \( m > \ell \). Then \( u \) is a suffix of \((\tau \sigma)^m\). Let \( s \) denote the word \((\tau \sigma)^m u^{-1} = (\tau \sigma)^{m-\ell-1} r \).

If \( |r| \geq |(\tau \sigma)^m| \), then \( r = s'su \) for some \( s' \). From \( r \preceq_p (\tau \sigma)^m u^i \), it follows that \( s'su \) is a prefix of \( su^{i+1} \). Lemma 11 then yields \( s's \in sp^*_u \). Therefore \( r \in su^{i+1} \) and from \( su = (\tau \sigma)^m \), we have the case \( r \in (\tau \sigma)^m p^*_u \).

Let \( |r| < |(\tau \sigma)^m| \). From \( |r| > |u| \) and \((\tau \sigma)^m = su \), we obtain that there are words \( s_1, s_2 \) such that \( s = s_1 s_2 \), \( r = s_2 u \preceq_p v \) and \( s_1 \preceq_s v \). Since \( s \) is both a prefix and a suffix of \( u \), Lemma 4 implies that \( s_1 \) and \( s_2 \) have the same primitive root, namely \( p_s \).

Note that \( p_x = u^i s_2 s_1 \). We now have

\[
(\alpha \beta \gamma)^{i-1} u^i s_2 s_1 = (\alpha \beta \gamma)^i u^i s_2 s_1. 
\]

From \( i \geq 2 \), it follows that \( u^i s_2 s_1 \) is a suffix of \((u^i s u)^{i-1} u^i \) for some \( n \geq 1 \). Lemma 3 then yields commutativity of \( s \) and \( u \). Hence, words \( x \) and \( u \) also commute and we are in case (A).

4. Suppose now that \( |x| < |v| \) and \( i|x| < i|u| + |v| \).

First notice that in this case also \( k|x| < k|u| + |v| \). If \( j|y| \geq |v| + |p_y| \), then, by the Periodicity lemma, \( p_y \) and \( p_y \) are conjugate, and theorem holds by (B). Assume that \( j|y| < |v| + |p_y| \). Then, since \( i|x| < i|u| + |v| \) and \( k|x| < k|u| + |v| \), we can see that \( j = 3 \) and there are non-empty words \( \alpha \), \( \beta \) and \( \gamma \) for which \( y = \alpha \beta \gamma \) and \( v = (\beta \gamma)(\alpha \beta \gamma)(\alpha \beta \gamma) \). Therefore, \( \gamma x \leq |v| \) and \( u^i \gamma x \) is a prefix of \( x^2 \). Therefore, by Lemma 10, \( u^i \gamma \) commutes with \( x \). We obtain the following equalities:

\[
 v = \gamma p^*_x \alpha, \quad y^\dagger = \alpha v \gamma = (\alpha \gamma)p^*_x (\alpha \gamma), \]

where \( n \geq 1 \). If \( n \geq 2 \), then \( x \) and \( y \) commute by Lemma 4. If \( n = 1 \), then \( p_x = x \) and \( i = 2 \). Since \( \gamma x^k = v u^k = \gamma x \alpha u^k \) and \( |\alpha u^k| \leq |x| \), also \( k = 2 \) and \( \alpha u^k = x \). Then \( |\alpha| = |\gamma| \) and \( u^2 \gamma = x = \alpha u^2 \). If \( |\alpha| \geq |\gamma| \), then \( u \) and \( \gamma \) commute, a contradiction with \( p_x = x \). Therefore, \( |x| < 3|\gamma| \) and \( |v| = |\gamma x| < 5|\gamma| \). Since \( \gamma \) is a suffix of \( x \) and \( \alpha \) is a prefix of \( x \), \( (\gamma \alpha \beta)^2 \gamma \alpha \) is a factor of \( v^3 \) longer than \( |y| + |v| \). Therefore, by the Periodicity lemma, words \( y \) and \( v \) are conjugate, and
we have case (B).

4b) Suppose that \(|u^i\gamma| > |x|\), denote \(z = x^{-1} u^i\gamma\) and \(z' = \gamma^{-1} v\alpha^{-1} = x^k u^{-k}\alpha^{-1}\). From 
\[ |y| + |\gamma| + |\alpha| < |v| = |\gamma'\alpha|, \]
we deduce \(|y| < |z'|\). Since \(x^{i-1} = zz'\) and \(z'\) is a prefix of \(x^k\), the word \(zz'\) has a period \(|z| < |\gamma|\). Since \(zz'\) is a factor of \(v\) greater than \(|z| + |y|\) and \(v\) has a period \(|p_y|\), the Periodicity lemma implies \(|p_y| \leq |z| < |\gamma|\), a contradiction with \(|\gamma| < |p_y|\).

\(\square\)

4 Conclusion

The minimal bounds for \(i, j, k\) in the previous theorem are optimal. In case that \(i = k\) and \(j\) is even, Eq. (1) splits into two separate equations, which have a solution if and only if either \(i = k\) and \(j = 2\), or \(i = k = 1\), see [2].

Apart from these solutions, we can find non-periodic solutions also in case that \(i \neq k\). Namely, for \(j = 2\) and \(i = k + 1\), we have
\[
\begin{align*}
x &= \alpha^{2k+1} (\beta\alpha^k)^2, & u &= \alpha, \\
y &= \beta\alpha^k, & v &= (\alpha^k \beta)^2 (\alpha^{3k+1} \beta\alpha^k \beta^k). \\
\end{align*}
\]
So far this seems to be the only situation when the equation
\[
x^i y^2 x^k = u^i v^2 u^k
\]
with \(i > k\) has a non-periodic solution. We conjecture that if \(|i - k| \geq 2\), then Eq. (2) has only periodic solutions.

If \(i = k = 1\) and \(j\) is odd, then Eq. (1) has several non-periodic solutions, for example:
\[
\begin{align*}
x &= \alpha \beta \alpha, & u &= \alpha, \\
y &= \gamma, & v &= \alpha \gamma^j \alpha, \\
\end{align*}
\]
where \(\beta^2 = v^{j-1}\).

References

1. Evelyne Barbin-Le Rest and Michel Le Rest. Sur la combinatoire des codes à deux mots. *Theor. Comput. Sci.*, 41:61–80, 1985.
2. Karel Culik II and Juhani Karhumäki. On the equality sets for homomorphisms on free monoids with two generators. *ITA*, 14(4):349–369, 1980.
3. Elena Czeizler, Štěpán Holub, Juhani Karhumäki, and Markku Laine. Intricacies of simple word equations: an example. *Internat. J. Found. Comput. Sci.*, 18(6):1167–1175, 2007.
4. A. Lentin and M.-P. Schützenberger. A combinatorial problem in the theory of free monoids. In G. Pollak, editor, Algebraic theory of semigroups / edited by G. Pollak. North-Holland Pub. Co Amsterdam ; New York, 1979.

5. R. C. Lyndon and M.-P. Schützenberger. The equation $a^m = b^n c^p$ in a free group. The Michigan Mathematical Journal, 9(4):289–298, 12 1962.

6. Grzegorz Rozenberg and Arto Salomaa, editors. Handbook of formal languages, vol. 1: word, language, grammar. Springer-Verlag New York, Inc., USA, 1997.

7. J.-P. Spehner. Quelques problèmes d’extension, de conjugaison et de présentation des sous-monoïdes d’un monoïde libre. PhD thesis, Université Paris VII, Paris, 1976.