THE NUMBER OF HECKE EIGENVALUES OF SAME SIGNS

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Abstract. We give the best possible lower bounds in order of magnitude for the number of positive and negative Hecke eigenvalues. This improves upon a recent work of Kohnen, Lau & Shparlinski. Also, we study an analogous problem for short intervals.

1. Introduction

Let \( k \geq 2 \) be an even integer and \( N \geq 1 \) be squarefree. Among all holomorphic cusp forms of weight \( k \) for the congruence subgroup \( \Gamma_0(N) \), there are finitely many of them whose Fourier coefficients in the expansion at the cusp \( \infty \),

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(k-1)/2}e^{2\pi inz} \quad (\Im z > 0),
\]

are the Hecke eigenvalues. Up to scalar multiples, these forms are the only simultaneous eigenfunctions of all Hecke operators. We call them the primitive forms, and write \( H^*_{k}(N) \) for the set of all primitive forms of weight \( k \) for \( \Gamma_0(N) \). One central problem in modular form theory is to study the Hecke eigenvalues \( \lambda_f(n) \). (We omit the factor \( n^{(k-1)/2} \) to avoid its uneven amplifying effect.) Classically it is known that the arithmetical function \( \lambda_f(n) \) is real multiplicative, and verifies Deligne’s inequality

\[
|\lambda_f(n)| \leq d(n)
\]

for all \( n \geq 1 \), where \( d(n) \) is the divisor function. Furthermore we have

\[
\lambda_f(p^\nu) = \lambda_f(p)^\nu \quad \text{and} \quad \lambda_f(p) = \varepsilon_f(p)/\sqrt{p}
\]

for all primes \( p \mid N \) and integers \( \nu \geq 1 \), where \( \varepsilon_f(p) \in \{\pm 1\} \). (See [5] and [10].) The distribution of the Hecke eigenvalues \( \lambda_f(n) \) is delicate. The Lang-Trotter conjecture concerns the frequency of \( \lambda_f(p) \) taking a value in the admissible range where \( p \) runs over primes. This conjecture is still open but there are progress made on itself or the pertinent questions, for instance, [6], [18], [16], [17], [2], [4], [15], etc. In this regard, various techniques and tools are applied, such as \( \ell \)-adic representations, Chebotarev density theorem, sieve-theoretic arguments, Rankin-Selberg \( L \)-functions and the method of \( \mathcal{B} \)-free numbers. In [15], Kowalski, Robert & Wu investigated the nonvanishing problem and gave the sharpest upper estimate to-date on the gaps between consecutive nonzero Hecke eigenvalues. Another wide belief is Sato-Tate’s conjecture, asserting that \( \lambda_f(p) \)'s are equidistributed on \([-2, 2]\) with respect to the Sato-Tate measure.

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In this paper, we are concerned with the Hecke eigenvalues of the same sign. Kohnen, Lau & Shparlinski [14, Theorem 1] proved

\[ N_f^\pm(x) := \sum_{\substack{n \leq x, (n,N) = 1 \atop \lambda_f(n) \geq 0}} \frac{x}{(\log x)^{17}} \]

for \( x \geq x_0(f) \). Very recently Wu [21, Corollary] improved this result by reducing the exponent 17 to \( 1 - \frac{1}{2 \sqrt{3}} \), as a simple application of his estimates on power sums of Hecke eigenvalues. The exponent \( 1 - \frac{1}{2 \sqrt{3}} \) can be improved to \( 2 - \frac{16}{\pi} \) if one assumes Sato-Tate’s conjecture.

Our first result is to remove the logarithmic factor by the \( \mathcal{B} \)-free number method, which is the best possible in order of magnitude.

**Theorem 1.** Let \( f \in H_k^*(N) \). Then there is a constant \( x_0 \) such that the inequality

\[ N_f^\pm(x) \gg_f x \]

holds for all \( x \geq x_0 \).

Remarks. 1. It is clear from the proof that our method gives the stronger result

\[ \sum_{\substack{n \leq x, (n,N) = 1 \atop n \text{ squarefree}, \lambda_f(n) \geq 0}} 1 \gg_f x \]

for every \( x \geq x_0(f) \).

2. The method is robust and applies to, for example, modular forms of half-integral weight. We return to this problem in another occasion.

By coupling [13] with Alkan & Zaharescu’s result in [11, Theorem 1], it is shown in [14, Theorem 2] (see also [13, Theorem 3.4]) that there are absolute constants \( \eta < 1 \) and \( A > 0 \) such that for any \( f \in H_k^*(N) \) the inequality

\[ N_f^\pm(x + x^\eta) - N_f^\pm(x) > 0 \]

holds for \( x \geq (kN)^A \), but no explicit value of \( \eta \) is evaluated. Apparently it is interesting and important to know how small \( \eta \) can be, in order for a better understanding of the local behaviour. A direct consequence of (1.5) is that \( \lambda_f(n) \) has a sign-change in a short interval \([x, x + x^\eta]\) for all sufficiently large \( x \). The sign-change problem was explored in [11, 14, 21] on different aspects. Here we prove that there are plenty of eigenvalues of the same signs in intervals of length about \( x^{1/2} \). More precisely, we have the following.

**Theorem 2.** Let \( f \in H_k^*(N) \). There is an absolute constant \( C > 0 \) such that for any \( \varepsilon > 0 \) and all sufficiently large \( x \geq N^2 x_0(k) \), we have

\[ N_f^\pm(x + C_N x^{1/2}) - N_f^\pm(x) \gg_{\varepsilon} (Nx)^{1/4 - \varepsilon}, \]

where

\[ C_N := C N^{1/2} \Psi(N)^{3}, \quad \Psi(N) := \sum_{d \mid N} d^{-1/2} \log(2d) \]

and \( x_0(k) \) is a suitably large constant depending on \( k \) and the implied constant in \( \gg_{\varepsilon} \) depends only on \( \varepsilon \).

\(^\dagger\)It is worthy to indicate that they gave explicit values for the implied constant in \( \gg \) and \( x_0(f) \).
The result in Theorem 2 is uniform in the level $N$, and its method of proof is based on Heath-Brown & Tsang [8]. The exponent of $\Psi(N)$ in $C_N$ can be easily reduced to any number bigger than $3/2$, which however may not be essential as $\Psi(N)$ is already very small - $\log \Psi(N) = o(\sqrt{\log N})$. The range of $x \geq N^{2}\theta(k)$ can also be refined to $x \geq N^{1+\epsilon}k^A$ for some constant $A > 0$, but we save our effort.

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2. Proof of Theorem

Let $p'$ be the least prime such that $p' \mid N$ and $\lambda_f(p') < 0$. Introduce the set

$$B = \{ p : \lambda_f(p) = 0 \} \cup \{ p : p \mid N \} \cup \{ p' \} \cup \{ p^2 : p \mid p'N \text{ and } \lambda_f(p) \neq 0 \} = \{ b_i \}_{i \geq 1} \text{ (with increasing order).}$$

By virtue of Serre’s estimate [18] (181):\

$$| \{ p \leq x : \lambda_f(p) = 0 \} | \ll_{f, \delta} \frac{x}{(\log x)^{1+\delta}}$$

for $x \geq 2$ and any $\delta < \frac{1}{2}$, we infer that\

$$\sum_{i \geq 1} \frac{1}{b_i} < \infty \quad \text{and} \quad (b_i, b_j) = 1 \quad (i \neq j).$$

Let $\mathcal{A} := \{ a_i \}_{i \geq 1} \text{ (with increasing order) be the sequence of all } B\text{-free numbers, i.e. the integers indivisible by any element in } B$. According to [7], $\mathcal{A}$ is of positive density

$$(2.1) \quad \lim_{x \to \infty} \frac{|\mathcal{A} \cap [1,x]|}{x} = \prod_{i=1}^{\infty} \left( 1 - \frac{1}{b_i} \right) > 0.$$

From the definition of $B$ and the multiplicativity of $\lambda_f(n)$, we have $\lambda_f(a) \neq 0$ for all $a \in \mathcal{A}$. Then we partition

$$\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-,$$

where

$$\mathcal{A}^\pm := \{ a_i \in \mathcal{A} : \lambda_f(a_i) \geq 0 \}.$$

Without control on the sizes of $\mathcal{A}^\pm$, we construct a set from $\mathcal{A}^+ \cup \mathcal{A}^-$ such that the sign of $\lambda_f(a)$ is switched on the counterpart. Consider

$$\mathcal{N}^\pm := \mathcal{A}^\pm \cup \{ a_ip' : a_i \in \mathcal{A}^\mp \}.$$

\footnote{According to [11], we have $p' \ll (k^2N)^{29/60}$.}
Clearly \( \lambda_f(a) \geq 0 \) and \((a,N) = 1\) for all \( a \in \mathcal{N}^\pm \) and
\[
\mathcal{N}^\pm(x) \geq |\mathcal{N}^\pm \cap [1,x]| \geq |\mathcal{A} \cap [1,x/p']|
\]
for all \( x \geq 1 \). The desired result follows with the inequality (2.1).

3. PROOF OF THEOREM 2

The method of proof is based on the investigation of
\[
S_f^*(x) := \sum_{n \leq x, (n,N)=1} \lambda_f(n).
\]
Since the \( L \)-function associated to \( f \) is belonged to the Selberg class and of degree 2, we apply the standard complex analysis to derive truncated Voronoi formulas for \( S_f^*(x) \).

**Lemma 3.1.** Let \( f \in H_k^*(N) \). Then for any \( A > 0 \) and \( \varepsilon > 0 \), we have
\[
S_f^*(x) = \frac{\eta_f}{\pi \sqrt{2}} (Nx)^{1/4} \sum_{d \mid N} \frac{(-1)^{\omega(d)} \lambda_f(d)}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{\frac{nx}{dN}} - \frac{\pi}{4} \right)
+ O \left( N^{1/2} \left\{ 1 + \left( \frac{x}{M} \right)^{1/2} + \left( \frac{N}{x} \right)^{1/4} \right\} (Nx)^\varepsilon \right)
\]
uniformly for \( 1 \leq M \leq x^A \) and \( x \geq N^{1+\varepsilon} \), where \( \eta_f = \pm 1 \) depends on \( f \) and the implied \( O \)-constant depends on \( A, \varepsilon \) and \( k \) only. The function \( \omega(d) \) counts the number of all distinct prime factors of \( d \).

Remark. The case \( N = 1 \) and \( A = 1 \) of (3.1) is covered in [12, Theorem 1.1] with \( h = k = 1 \) therein. Our proof follows closely Section 3.2 of [9], and we first evaluate the case without the constraint \((n,N) = 1\): for any \( A > 0 \) and \( \varepsilon > 0 \), we have uniformly in \( 1 \leq M \leq x^A \),
\[
S_f(x) := \sum_{n \leq x} \lambda_f(n)
= \frac{\eta_f(Nx)^{1/4}}{\pi \sqrt{2}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{\frac{nx}{N}} - \frac{\pi}{4} \right)
+ O \left( N^{1/2} \left\{ 1 + \left( \frac{x}{M} \right)^{1/2} + \left( \frac{N}{x} \right)^{1/4} \right\} (Nx)^\varepsilon \right).
\]

**Proof.** As usual, denote by \( \mu(N) \) the Möbius function. (3.1) follows from (3.2) because
\[
S_f^*(x) = \sum_{d \mid N} \mu(d) \sum_{n \leq x/d} \lambda_f(dn)
= \sum_{d \mid N} (-1)^{\omega(d)} \lambda_f(d) \sum_{n \leq x/d} \lambda_f(n)
\]
(3.3)
by the multiplicativity of $\lambda_f(n)$ and the first equality in (1.2). Note that $x/d \geq x^{1/(1+\varepsilon)}$ when $x \geq N^{1+\varepsilon}$ and $d|N$, we can keep the same range of $M$ for all inner sums over $n$ by selecting a suitable $A$. Inserting (3.2) into (3.3), the main term of (3.1) comes up immediately. The effect of summing the $O$-terms over $d|N$ is negligible in light of the second formula in (1.2), and hence the result.

To prove (3.2), we consider $M \in \mathbb{N}$ without loss of generality. As usual write

$$L(s, f) := \sum_{n \geq 1} \lambda_f(n)n^{-s} \quad (\Re s > 1).$$

Let $\kappa := 1 + \varepsilon$ and $T > 1$ be a parameter, chosen as

$$T^2 = \frac{4\pi^2 (M + \frac{1}{2})}{N}.$$  

By the truncated Perron formula (see [20, Corollary II.2.4] with the choice of $\sigma_a = 1$, $\alpha = 2$ and $B(n) = C_e n^\varepsilon$), we have

$$(3.5) \quad S_f(x) = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} L(s, f) \frac{x^s}{s} ds + O\left(N^{1/2} \left\{ \left( \frac{x}{M} \right)^{1/2} + 1 \right\} (Nx)^\varepsilon \right).$$

We shift the line of integration horizontally to $\Re s = -\varepsilon$, the main term gives

$$(3.6) \quad \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} L(s, f) \frac{x^s}{s} ds = L(0, f) + \frac{1}{2\pi i} \int_{\mathcal{L}} L(s, f) \frac{x^s}{s} ds,$$

where $\mathcal{L}$ is the contour joining the points $\kappa \pm iT$ and $-\varepsilon \pm iT$. Using the convexity bound

$$L(\sigma + it, f) \ll (\sqrt{N(k + |t|)})^{\max\{0,1-\sigma\}+\varepsilon} \quad (-\varepsilon \leq \sigma \leq \kappa),$$

the integrals over the horizontal segments and the term $L(0, f)$ can be absorbed in $O((NTx)^\varepsilon (N^{1/2} + T^{-1}x))$. The $O$-constant depends on $k$ and $\varepsilon$, and in the sequel, such a dependence in implied constants will be tacitly allowed.

To handle the integral over the vertical segment $\mathcal{L}_v := [-\varepsilon - iT, -\varepsilon + iT]$, we invoke the functional equation

$$\left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma\left( s + \frac{k - 1}{2} \right) L(s, f) = i^k \eta_f \left( \frac{\sqrt{N}}{2\pi} \right)^{1-s} \Gamma\left( 1 - s + \frac{k - 1}{2} \right) L(1 - s, f),$$

where $\eta_f := \mu(N)\lambda_f(N)\sqrt{N} \in \{\pm 1\}$ (see [10, p.375] with an obvious change of notation). Then we deduce that

$$(3.7) \quad \frac{1}{2\pi i} \int_{\mathcal{L}_v} L(s, f) \frac{x^s}{s} ds = i^k \eta_f \sum_{n \geq 1} \frac{\lambda_f(n)}{n} I_{\mathcal{L}_v}(nx),$$

where

$$I_{\mathcal{L}_v}(y) := \frac{1}{2\pi i} \int_{\mathcal{L}_v} \left( \frac{4\pi^2}{N} \right)^{s-1/2} \frac{\Gamma(1 - s + (k - 1)/2)y^s}{\Gamma(s + (k - 1)/2)} \frac{ds}{s}.$$

The quotient of the two gamma factors is

$$|t|^{1-2\sigma} e^{-2i(t \log |t| - t) + i\text{sgn}(t)\pi(k-1)/2} \{1 + O(t^{-1})\}$$
for bounded $\sigma$ and any $|t| \geq 1$, where the implied constant depends on $\sigma$ and $k$. Together with the second mean value theorem for integrals (see [20, Theorem I.0.3]), we obtain

\[
I_{\mathcal{L}_v}(n) \ll N^{1/2} \left( \frac{N}{nx} \right)^\varepsilon \left( \left| \int_1^T t^{2\varepsilon} e^{-i\theta(t)} \, dt \right| + T^{2\varepsilon} \right)
\]

(3.8)

\[
\ll N^{1/2} \left( \frac{NT^2}{nx} \right)^\varepsilon \left( \left| \int_a^b e^{-i\theta(t)} \, dt \right| + 1 \right)
\]

for some $1 \leq a \leq b \leq T$, where $g(t) := t \log \left( Nt^2/\left( 4\pi^2 nx \right) \right) - 2t$. In view of (3.4), we have

\[
g'(t) = -\log \left( 4\pi^2 nx/(Nt^2) \right) < 0 \quad \text{and} \quad |g'(t)| \geq |\log (n/(M + 1/2))|
\]

for $n \geq M + 1$ and $1 \leq t \leq T$. Using (1.1) and [20, Theorem I.6.2], we infer that

\[
\sum_{n>M} \frac{\lambda_f(n)}{n} I_{\mathcal{L}_v}(n) \ll N^{1/2} \left( \frac{NT^2}{x} \right)^\varepsilon \sum_{n>M} \frac{d(n)}{n^{1+\varepsilon}} \left( \left| \log \frac{n}{M + \frac{1}{2}} \right|^{-1} + 1 \right)
\]

\[
\ll N^{1/2} \left( \frac{NT^2}{x} \right)^\varepsilon \left\{ \sum_{M<n<2M} \frac{d(n)(M + \frac{1}{2})}{n^{1+\varepsilon}|n-M-\frac{1}{2}|} + \frac{1}{M^{\varepsilon/2}} \right\}
\]

(3.9)

\[
\ll N^{1/2} \left( \frac{NT^2}{\sqrt{Mx}} \right)^\varepsilon
\]

\[
\ll N^{1/2} (Nx)^\varepsilon.
\]

For $n \leq M$, we extend the segment of integration $\mathcal{L}_v$ to an infinite line $\mathcal{L}_v^*$ in order to apply Lemma 1 in [3]. Write

\[
\mathcal{L}_v^\pm := [\frac{1}{2} + \varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm i\infty], \quad \mathcal{L}_h^\pm := [-\varepsilon \pm iT, \frac{1}{2} + \varepsilon \pm iT]
\]

and define $\mathcal{L}_v^*$ to be the positively oriented contour consisting of $\mathcal{L}_v$, $\mathcal{L}_v^\pm$ and $\mathcal{L}_h^\pm$. The contribution over the horizontal segments $\mathcal{L}_h^\pm$ is

\[
I_{\mathcal{L}_h^\pm}(n) \ll \int_{-\varepsilon}^{1/2-\varepsilon} \left( \frac{4\pi^2}{N} \right)^{\sigma-1/2} T^{1-2\sigma} (nx)^\sigma \frac{d\sigma}{\sigma} \int_0^1 \frac{nx}{NT^2} d\sigma
\]

\[
\ll N^{1/2} \left( \frac{nx}{NT^2} \right)^{1/2+\varepsilon} \left( \left| \log \frac{M + \frac{1}{2}}{n} \right|^{-1} + 1 \right)
\]

As in (3.8), for $n \leq M$ we get that

\[
I_{\mathcal{L}_h^\pm}(n) \ll N^{1/2} \left( \frac{nx}{N} \right)^{1/2+\varepsilon} \left( \int_T^\infty t^{-1/2} e^{-i\theta(t)} \, dt + \frac{1}{T^{1+2\varepsilon}} \right)
\]

\[
\ll N^{1/2} \left( \log \frac{M + \frac{1}{2}}{n} \right)^{-1} + 1). \]
So
\[ \sum_{n \leq M} \frac{\lambda_f(n)}{n} (I_{\mathcal{L}^\pm}(nx) + I_{\mathcal{L}^\pm}(nx)) \ll \sum_{n \leq M} \frac{d(n)}{n} (|I_{\mathcal{L}^\pm}(nx)| + |I_{\mathcal{L}^\pm}(nx)|) \]
\[ \ll N^{1/2}(Nx)^{\varepsilon}. \]

Now all the poles of the integrand in
\[ I_{\mathcal{L}^\pm}(y) = \frac{\sqrt{N}}{2\pi} \frac{1}{2\pi i} \int_{\mathcal{L}^\pm} \frac{\Gamma(1-s+(k-1)/2)\Gamma(s)}{\Gamma(1-s+(k-1)/2)\Gamma(1+s)} \left( \frac{4\pi^2 y}{N} \right)^s ds \]
lie on the right of the contour $\mathcal{L}^\pm$. After a change of variable $s$ into $1-s$, we see that
\[ I_{\mathcal{L}^\pm}(y) = \frac{\sqrt{N}}{2\pi} I_0 \left( \frac{4\pi^2 y}{N} \right), \]
with
\[ I_0(t) := \frac{1}{2\pi i} \int_{\mathcal{L}_\varepsilon} \frac{\Gamma(s+(k-1)/2)\Gamma(1-s)}{\Gamma(1-s+(k-1)/2)\Gamma(2-s)} e^{t-s} ds. \]

Here $\mathcal{L}_\varepsilon$ consists of the line $s = \frac{1}{2} - \varepsilon + it$ with $|t| \geq T$, together with three sides of the rectangle whose vertices are $\frac{1}{2} - \varepsilon - iT, 1 + \varepsilon - iT, 1 + \varepsilon - iT$, and $\frac{1}{2} - \varepsilon + iT$. Clearly our $I_0$ is a particular case of $I_\rho$ defined in [3 Lemma 1], corresponding to the choice of parameters $\rho = 0, \delta = A = 1, \omega = 1, h = 2, k_0 = -(2k+1)/4$. It hence follows that
\[ I_{\mathcal{L}^\pm}(nx) = \frac{i^k(nNx)^{1/4}}{\pi \sqrt{2}} \cos \left( 4\pi \sqrt{\frac{nx}{N}} - \frac{\pi}{4} \right) + O \left( \frac{N^{3/4+\varepsilon}}{(nx)^{1/4}} \right), \]

The value of $c'_0$ in Lemma 1 of [3] is $1/\sqrt{\pi}$ by direct computation. We conclude
\[ \sum_{n \leq M} \frac{\lambda_f(n)}{n} I_{\mathcal{L}^\pm}(nx) = \frac{i^k(nNx)^{1/4}}{\pi \sqrt{2}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{\frac{nx}{N}} - \frac{\pi}{4} \right) \]
\[ + O \left( N^{1/2} \left\{ \left( \frac{N}{x} \right)^{1/4} + 1 \right\} (Nx)^{\varepsilon} \right), \]
from (3.10) and (3.11), and finally the asymptotic formula (3.2) by (3.5)-(3.7), (3.9) and (3.12).

Following Theorem 1 of [8], we have the next lemma.

**Lemma 3.2.** Let $f \in \mathcal{H}_k^1(N)$. There exist positive absolute constants $C, c_1, c_2$ such that for all sufficiently large $X \geq N^2 X_0(k)$, we can find $x_1, x_2 \in [X, X + CNX^{1/2}]$ for which
\[ S_f^*(x_1) > c_1(NX)^{1/4} \quad \text{and} \quad S_f^*(x_2) < -c_2(NX)^{1/4}, \]
where $C_N := CN^{1/2} \Psi(N)^2$ and $X_0(k)$ is a constant depending only on $k$. The same result also holds for $S_f(x)$.
The last error term in (3.13) appears only when \( \beta = 1 \) or \(-1\) and \( \alpha \) is a (large) parameter, both chosen at our disposal. Consider the following integral

\[
r_{\beta} = r_{\beta}(\alpha, \tau, t) := \int_{-1}^{1} K_{\tau}(u) \cos \left( 4\pi(t + \alpha u)\sqrt{\beta - \frac{\pi}{4}} \right) du,
\]

where \( t \in \mathbb{N} \) and \( \beta > 0 \). Because

\[
w(\xi) := \int_{-1}^{1} (1 - |u|) e^{i2\pi \xi u} du = \left( \frac{\sin \pi \xi}{\pi \xi} \right)^{2} = \begin{cases} 1 & \text{if } \xi = 0, \\ O \left( \min(1, \xi^{-2}) \right) & \text{if } \xi \neq 0,
\end{cases}
\]

we can write, with the notation \( \alpha_{\beta} := 2\alpha \sqrt{\beta} \) and \( \alpha_{\beta}^\pm := 2\alpha(\sqrt{\beta} \pm 1) \),

\[
r_{\beta} = \int_{-1}^{1} (1 - |u|) \left( 1 + \tau \frac{e^{i4\pi \alpha u} + e^{-i4\pi \alpha u}}{2} \right) \Re \left( e^{i4\pi(t + \alpha u)\sqrt{\beta - \pi/4}} \right) du
\]

\[
= \Re e^{i(4\pi t \sqrt{\beta - \pi/4})} \int_{-1}^{1} (1 - |u|) \left( e^{i2\pi \alpha_{\beta} u} + \frac{\tau}{2} e^{i2\pi \alpha_{\beta}^+ u} + \frac{\tau}{2} e^{i2\pi \alpha_{\beta}^- u} \right) du
\]

\[
= \left( w(\alpha_{\beta}) + \frac{\tau}{2} w(\alpha_{\beta}^+) + \frac{\tau}{2} w(\alpha_{\beta}^-) \right) \cos \left( 4\pi t \sqrt{\beta - \frac{\pi}{4}} \right)
\]

\[
= \delta_{\beta=1} \frac{\tau}{2\sqrt{2}} + O \left( \min \left( 1, \frac{1}{\alpha_{\beta}^2} \right) + \delta_{\beta\neq1} \min \left( 1, \frac{1}{(\alpha_{\beta}^\pm)^2} \right) \right),
\]

where the \( O \)-constant is absolute,

\[
\delta_{\beta=1} := \begin{cases} 1 & \text{if } \beta = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \delta_{\beta\neq1} := 1 - \delta_{\beta=1}.
\]

The last error term in (3.13) appears only when \( \beta \neq 1 \).

For all \( X \geq N^{2}X_{0}(k) \) (whose value will be specified below), we write \( T = (X/N)^{1/2} \) and \( t = \lfloor T \rfloor + 1 \in \mathbb{N} \), and consider the convolution

\[
J_{\tau} = \int_{-1}^{1} F_{f}(t + \alpha u) K_{\tau}(u) du,
\]

where

\[
F_{f}(t + \alpha u) := \frac{\sqrt{2} S_{f}^{*}(N(t + \alpha u)^{2})}{\eta_{f} \sqrt{N(t + \alpha u)}}.
\]

By Lemma 3.1 with \( M = NT^{2} = X \), we deduce that

\[
F_{f}(t + \alpha u) = \sum_{d \mid N} \frac{(-1)^{\omega(d)}}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_{f}(n)}{n^{3/4}} \cos \left( 4\pi(T + \alpha u) \sqrt{\frac{n}{d}} - \frac{\pi}{4} \right) + O_{k} \left( \frac{1}{T^{1/4}} \right),
\]

and

\[
J_{\tau} = \sum_{d \mid N} \frac{(-1)^{\omega(d)}}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_{f}(n)}{n^{3/4}} r_{n/d} + O_{k} \left( \frac{1}{T^{1/4}} \right)
\]

by (1.2).
Next we estimate the contribution of the $O$-term in (3.13) to $J_\tau$. Using (1.2) and (1.1) again, its contribution to $J_\tau$ is

\begin{equation}
\ll \sum_{d \mid N} \frac{1}{d^{3/4}} \left\{ \sum_{n \leq M} \frac{d(n)}{n^{3/4}} R'_d(n) + \sum_{n \leq M, n \neq d} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \right\},
\end{equation}

where

\[ R'_d(n) := \min \left( 1, \frac{d}{\alpha^2 n} \right), \quad R''_{d,n}(\alpha) := \min \left( 1, \frac{d}{\alpha^2 n^2 n - \sqrt{d^2}} \right). \]

Consider the second sum in the curly braces. We separate $n$ into

\[ n \leq \alpha_--d, \quad \alpha_--d < n < \alpha_+d \quad \text{or} \quad \alpha_+d \leq n \]

where $\alpha := (1-\alpha^{-1/2})^{1/2}$, and $R''_{d,n}(\alpha)$ is $\leq 1/\alpha$, $1$ or $d/(\alpha n)$ accordingly. Therefore,

\[ \sum_{n \leq M, n \neq d} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \leq \frac{1}{\alpha} \sum_{n \leq \alpha--d} \frac{d(n)}{n^{3/4}} + \sum_{\alpha--d < n < \alpha_+d} \frac{d(n)}{n^{3/4}} + \frac{d}{\alpha} \sum_{n > \alpha_+d} \frac{d(n)}{n^{3/4}}. \]

Obviously the first and last terms on the right-hand side are $\ll \alpha^{-1/3} d^{1/4} \log(2d)$. Note that $n \gg d$ in the second sum. So, by using Shiu’s Theorem 2 in [19] it follows

\[ \sum_{\alpha_--d < n < \alpha_+d} \frac{d(n)}{n^{3/4}} \ll \frac{d^{3/4}}{\alpha} \sum_{n \neq d} \frac{d(n)}{n^{3/4}} \ll \alpha^{-1/2} d^{1/4} \log(2d) \]

if $d > \alpha$. Otherwise (i.e. $d \leq \alpha$), pulling out $d(n) \ll n^\varepsilon \ll d^\varepsilon \ll \alpha^\varepsilon$, we have

\[ \sum_{\alpha_--d < n < \alpha_+d} \frac{d(n) n^{-3/4}}{n^{3/4}} \ll \alpha^{\varepsilon} d^{-3/4} \sum_{\alpha_--d < n < \alpha_+d} \frac{1}{n^{3/4}} \ll \alpha^{\varepsilon} d^{-3/4} \alpha^{-1/2} \]

\[ \ll \alpha^{\varepsilon} d^{-3/4} \alpha^{-1/2} \ll \alpha^{1/3} d^{1/4} \log(2d). \]

(We can assume that $(\alpha_+ - \alpha_-)d \geq \alpha^{-1/2} d \geq c'$ for a small constant $c'$, otherwise the last sum is empty.) Hence

\[ \sum_{n \leq M, n \neq d} \frac{d(n)}{n^{3/4}} R''_{d,n}(\alpha) \ll \alpha^{-1/3} d^{1/4} \log(2d). \]

The first sum in the bracket of (3.13) can be treated in the same fashion (even more easily). Thus, (3.15) is bound by

\[ \ll \alpha^{-1/3} \sum_{d \mid N} \frac{\log(2d)}{d^{1/2}} =: \alpha^{-1/3} \Psi(N). \]
We conclude from (3.14) with (3.13) and (1.2) that
\[ J_\tau = \frac{\tau}{2\sqrt{2}} \sum_{d|N} \frac{(-1)^{\omega(d)}}{d^2} + O\left(\frac{\Psi(N)}{\alpha^{1/3}}\right) + O_k\left(\frac{1}{T^{1/4}}\right), \]
where the implied constant is absolute in the first $O$-term, but depends on $k$ in the second. Noticing that
\[ \sum_{d|N} \frac{(-1)^{\omega(d)}}{d^2} = \prod_{p|N} \left(1 - \frac{1}{p^2}\right) \geq \frac{6}{\pi^2} \]
and $T \geq \sqrt{NX_0(k)}$, we take $\alpha = C\Psi(N)^3$ with a large absolute constant $C$ and a large $X_0(k)$ so that both $O$-terms $O(\alpha^{-1/3}\Psi(N))$ and $O_k(T^{-1/4})$ are $\leq \cos(\pi/4)/\pi^2 = 1/(\pi^2 \sqrt{2})$. Therefore
\[ J_{-1} < -1/(\pi^2 \sqrt{2}) \quad \text{and} \quad J_1 > 1/(\pi^2 \sqrt{2}). \]

With the nonnegativity of $K_\tau(u)$ and the estimate
\[ 1 - (2\pi\alpha)^{-2} \leq \int_{-1}^{1} K_\tau(u) \, du \leq 2 \quad (\tau = \pm 1), \]
we have
\[ 2F_f(t + \alpha \eta_+) \geq 1/(\pi^2 \sqrt{2}) \quad \text{and} \quad (1 - (2\pi\alpha)^{-2})F_f(t + \alpha \eta_-) \leq -1/(\pi^2 \sqrt{2}) \]
for some $\eta_+, \eta_- \in [-1,1]$. Let $C_N = C N^{1/2}\Psi(N)^3$. As
\[ X - 3C_N \sqrt{X} \leq N(t + \alpha \eta_\pm)^2 \leq X + 3C_N \sqrt{X}, \]
our assertion follows from the definition of $F_f$ and replacing $X - 3C_N \sqrt{X}$ by $X$. \qed

Now we are ready to prove Theorem 2
We exploit the consecutive sign changes of $S^*_f(x)$. Let $x \geq N^2 X_0(k)$ where $X_0(k)$ takes the value as in Lemma 3.2. We apply Lemma 3.2 to the intervals $[x, x + C_N x^{1/2}]$ and $[y, y + C_N y^{1/2}]$ where $y = x + C_N x^{1/2}$. Over each of the intervals, $S^*_f(x)$ attains in magnitude $(N x)^{1/4}$ in both positive and negative directions. Hence, we can find three points $x < x_1 < x_2 < x_3 < x + 3C_N x^{1/2}$ such that $S^*_f(x_i)$ $(i = 1, 2, 3)$ takes alternate signs and their absolute values are $\gg (N x)^{1/4}$. (Note that $2\sqrt{x} \geq \sqrt{x + C_N \sqrt{x}}$.) It follows that the two differences
\[ S^*_f(x_2) - S^*_f(x_1) = \sum_{x_1 < n \leq x_2} \lambda_f(n) \]
and
\[ S^*_f(x_3) - S^*_f(x_2) = \sum_{x_2 < n \leq x_3} \lambda_f(n) \]
have absolute values $\gg (N x)^{1/4}$ but are of opposite signs. This implies (1.6), since for example, if
\[ \sum_{a < n < b} \lambda_f(n) < -c'(N x)^{1/4} \]
for some constant $c' > 0$ and $b \ll x$, then we have

$$c'(Nx)^{1/4} < \sum_{a<n<b, (n,N)=1 \atop \lambda_f(n)<0} (-\lambda_f(n)) \ll x^\varepsilon \sum_{a<n<b, (n,N)=1 \atop \lambda_f(n)<0} 1.$$ 

This completes the proof of Theorem 2. \hfill \Box

## References

[1] E. Alkan & A. Zaharescu, *Nonvanishing of Fourier coefficients of newforms in progressions*, Acta Arith. **116** (2005), no. 1, 81–98.

[2] A. Balog & K. Ono, *The Chebotarev density theorem in short intervals and some questions of Serre*, J. Number theory **91** (2001), 356–371.

[3] K. Chandrasekharan & R. Narasimhan, *The approximate functional equation for a class of zeta-functions*, Math. Ann. **152** (1963), 30–64.

[4] A. C. Cojocaru, E. Fouvry & M. Ram Murty, *The square sieve and the Lang-Trotter conjecture*, Canad. J. Math. **57** (2005), no. 6, 1155–1177.

[5] P. Deligne, *La conjecture de Weil, I, II*, Publ. Math. IHES **48** (1974), 273–308, **52** (1981), 313–428.

[6] N. Elkies, *Distribution of supersingular primes*, Journées Arithmétiques, 1989 (Luminy, 1989). Astérisque No. 198-200 (1991), 127–132 (1992).

[7] P. Erdős, *On the difference of consecutive terms of sequences, defined by divisibility properties*, Acta Arith. **12** (1966), 175–182.

[8] D. R. Heath-Brown & K.-M. Tsang, *Sign changes of $E(T)$, $\Delta(x)$, and $P(x)$*, J. Number Theory **49** (1994), 73-83.

[9] A. Ivić, *The Riemann zeta-function. The theory of the Riemann zeta-function with applications*, John Wiley & Sons, 1985.

[10] H. Iwaniec & E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications **53**, American Mathematical Society, Providence, RI, 2004, xii+615 pp.

[11] H. Iwaniec, W. Kohnen & J. Sengupta, *The first sign change of Hecke eigenvalue*, Int. J. Number Theory **3** (2007), no. 3, 355–363.

[12] M. Jutila, *Lectures on a method in the theory of exponential sums*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics **80**, Published for the Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin, 1987, viii+134 pp.

[13] W. Kohnen, *Sign changes of Fourier coefficients and eigenvalues of cusp forms*, in: Number theory, 97–107, Ser. Number Theory Appl., **2**, World Sci. Publ., Hackensack, NJ, 2007.

[14] W. Kohnen, Y.-K. Lau & I. E. Shparlinski, *On the number of sign changes of Hecke eigenvalues of newforms*, J. Austral. Math. Soc., to appear.

[15] E. Kowalski, O. Robert & J. Wu, *Small gaps in coefficients of $L$-functions and $\mathfrak{B}$-free numbers in short intervals*, Revista Matemática Iberoamericana **23** (2007), No. 1, 281–322.

[16] M. R. Murty, V. K. Murty & N. Saradha, *Modular forms and the Chebotarev density theorem*, Amer. J. Math. **110** (1988), 253–281.

[17] V. K. Murty, *Modular forms and the Chebotarev density theorem, II*, in: Analytic number theory (Kyoto, 1996), 287–308, London Math. Soc. Lecture Note Ser. **247**, Cambridge Univ. Press, Cambridge, 1997.

[18] J.-P. Serre, *Quelques applications du théorème de densité de Chebotarev*, Inst. Hautes Études Sci. Publ. Math. **54** (1981), 323–401.
[19] P. Shiu, *A Brun-Titchmarsh theorem for multiplicative functions*, J. reine angew Math. 313 (1980), 161–170.

[20] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Translated from the second French edition (1995) by C. B. Thomas. Cambridge Studies in Advanced Mathematics, 46. Cambridge University Press, Cambridge, 1995. xvi+448 pp.

[21] J. Wu, *Power sums of Hecke eigenvalues and application*, Preprint, 2008.

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