Solution of the radial Schrödinger equation for the potential family $V(r) = \frac{A}{r^2} - \frac{B}{r} + Cr^\kappa$ using the asymptotic iteration method

M Aygun$^1$, O Bayrak$^{2,3}$ and I Boztosun$^3$

$^1$ Faculty of Arts and Sciences, Department of Physics, Ataturk University, Erzurum, Turkey
$^2$ Faculty of Arts and Sciences, Department of Physics, Bozok University, Yozgat, Turkey
$^3$ Faculty of Arts and Sciences, Department of Physics, Erciyes University, Kayseri, Turkey

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Abstract
We present the exact and iterative solutions of the radial Schrödinger equation for a class of potentials, $V(r) = \frac{A}{r^2} - \frac{B}{r} + Cr^\kappa$, for various values of $\kappa$ from $-2$ to $2$, for any $n$ and $l$ quantum states by applying the asymptotic iteration method. The global analysis of this potential family by using the asymptotic iteration method results in exact analytical solutions for the values of $\kappa = 0, -1$ and $-2$. Nevertheless, there are no analytical solutions for the cases $\kappa = 1$ and $2$. Therefore, the energy eigenvalues are obtained numerically. Our results are in excellent agreement with previous work.

1. Introduction
We search for the exact solution of a new class of potential as in the following form:

$$V(r) = \frac{A}{r^2} - \frac{B}{r} + Cr^\kappa. \quad (1)$$

This potential includes various potentials according to the values of the following potential parameters.

(a) By choosing $C = 0$, this potential turns into the Kratzer potential, which is analytically solvable by using various methods [1–3]. This potential has been used extensively so far in order to describe molecular structure and interactions [1–3].

(b) The Gol’dman–Krivchenkov potential is obtained by choosing $A \neq 0$, $B = 0$ and $\kappa = 2$ and it is solved analytically by using various methods [2, 4]. Furthermore, this potential becomes the spiked harmonic oscillator potential [4] when $A = 0$, $B = 1$ and $\kappa = 2$.

(c) The Coulomb plus linear potential form is obtained by choosing $A = 0$ and $\kappa = 1$ and is solved by methods such as the envelope and variational methods [4]. This potential has raised great interest in atomic and molecular physics ([5] and references therein).
(d) The Coulomb plus harmonic oscillator potential form is obtained by choosing $A = 0$ and $\kappa = 2$. This form is solved by using the moment method [6], the shifted $1/N$ expansion method [7] and the envelope method [4]. In addition, it is applied to examine the Zeeman quadratic effect [8] and the magnetic field effect in the hydrogen atom [9].

These potentials have been extensively used to describe the bound and the continuum states of interaction systems and a great number of papers have been published for the exact and numerical solutions of these potentials (see [1–13] and references therein). Thus, it would be interesting and important to solve the non-relativistic radial Schrödinger equation for a new solvable potential family for any $n$ and $l$ quantum states. Recently, an alternative method, called the asymptotic iteration method (AIM), has been developed by Çiftçi et al [13] for solving second-order homogeneous linear differential equations and it has been applied to solve the non-relativistic radial Schrödinger equation as well as the relativistic wave equations [13–18]. In this paper, we aim to show that AIM could give the energy eigenvalues for potentials that have analytical solutions obtained by using different methods. Moreover, AIM could provide energy eigenvalues for potentials that have no analytical solutions for any $n$ and $l$ quantum states with various $\kappa$ values.

In the next section, we introduce AIM and then in section 3, the analytical solution of the Schrödinger equation is obtained by using AIM for the potentials with $\kappa = 0, -1$ and $-2$ and closed forms for the energy eigenvalues and corresponding eigenfunctions for any $n$ and $l$ quantum numbers are given. Then, in section 4, the energy eigenvalues for a new solvable potential with $\kappa = 1$ and $2$ cases by using the AIM iteration procedure are obtained. Finally, in section 5, we remark on these results.

2. Overview of the asymptotic iteration method (AIM)

AIM is proposed to solve the second-order differential equations of the form [13, 16]

$$y'' = \lambda_0(x) y' + s_0(x) y,$$

(2)

where $\lambda_0(x) \neq 0$. The variables, $s_0(x)$ and $\lambda_0(x)$, are sufficiently differentiable. The differential equation (2) has a general solution [13]

$$y(x) = \exp \left( - \int x \alpha(x') \, dx' \right) \left[ C_2 + C_1 \int x \exp \left( \int x' \left[ \lambda_0(x'') + 2\alpha(x'') \right] \, dx'' \right) \, dx' \right]$$

(3)

if $k > 0$, for sufficiently large $k$, we obtain the $\alpha(x)$ values from

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \quad k = 1, 2, 3, \ldots,$$

(4)

where

$$\lambda_k(x) = \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x) \lambda_{k-1}(x)$$

$$s_k(x) = s_{k-1}(x) + s_0(x) \lambda_{k-1}(x), \quad k = 1, 2, 3, \ldots.$$  

(5)

It should be noted that one can also start the recurrence relations from $k = 0$ with the initial conditions $\lambda_{-1} = 1$ and $s_{-1} = 0$ [15]. For a given potential, the radial Schrödinger equation is converted to the form of equation (2). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by equation (5).

The energy eigenvalues are obtained from the roots of the quantization condition, given by the termination condition of the method in equation (4). The quantization condition of the method together with equation (5) can also be written as

$$\delta_k(x) = \lambda_k(x) s_{k-1}(x) - \lambda_{k-1}(x) s_k(x) = 0 \quad k = 1, 2, 3, \ldots.$$  

(6)
The energy eigenvalues are obtained from this equation if the problem is exactly solvable. If not, for a specific \( n \) principal quantum number, we choose a suitable \( x_0 \) point, determined generally as the maximum value of the asymptotic wavefunction or the minimum value of the potential, and the approximate energy eigenvalues are obtained from the roots of this equation for sufficiently great values of \( k \) with iteration.

The wavefunctions are determined by using the following wavefunction generator:

\[
y_n(x) = C_2 \exp \left( - \int x \, \frac{s_k(x')}{\lambda_k(x')} \, dx' \right),
\]

where \( k \geq n \), \( n \) represents the radial quantum number and \( k \) shows the iteration number. For exactly solvable potentials, the radial quantum number \( n \) is equal to the iteration number \( k \) and the eigenfunctions are obtained directly from equation (7). For non-trivial potentials that have no exact solutions, \( k \) is always greater than \( n \) in these numerical solutions and the approximate energy eigenvalues are obtained from the roots of equation (6) for sufficiently great values of \( k \) by iteration.

3. \( \kappa = 0, -1 \) and \(-2 \) cases: analytical solutions

Inserting the potential given by equation (1) into the Schrödinger equation gives

\[
\frac{d^2 R_{nl}}{dr^2} + \frac{2m E}{\hbar^2} \left[ E - \frac{A}{r^2} + \frac{B}{r} - Cr^\kappa - \frac{l(l+1)\hbar^2}{2mr^2} \right] R_{nl} = 0,
\]

where \( n \) and \( l \) are radial and orbital angular momentum quantum numbers, \( A, B \) and \( C \) are strictly positive constants. By using the following ansätze

\[
-\epsilon^2 = \frac{2mE}{\hbar^2}, \quad -\epsilon_{nl}^2 = -\epsilon^2 - \tilde{C}, \quad \tilde{A} = \frac{2mA}{\hbar^2}, \quad \tilde{B} = \frac{2mB}{\hbar^2}, \quad \tilde{C} = \frac{2mC}{\hbar^2},
\]

equation (8) becomes

\[
\frac{d^2 R_{nl}}{dr^2} + \left[ -\epsilon^2 - \frac{\tilde{A}}{r^2} + \frac{\tilde{B}}{r} - \frac{\tilde{C}r^\kappa - l(l+1)}{r^2} \right] R_{nl} = 0.
\]

The aim of this paper is to show how to obtain the analytical and numerical solutions of \( V(r) = \frac{A}{r^2} - \frac{B}{r} + Cr^\kappa \) potential with different \( \kappa \) values. \( \kappa = 0, -1 \) and \(-2 \) values of this potential have analytical solutions and have been extensively studied so far [1–3, 18]. Therefore, we do not give details of the calculations and only show the analytical solutions. By inserting \( \kappa = 0, -1 \) and \(-2 \) into equation (10) and by doing some simple algebra described in the previous section, the energy eigenvalues and regular eigenfunctions are obtained as follows:

| \( \kappa \) | Eigenvalues | Eigenfunctions |
|------------|-------------|----------------|
| 0, \( E_{nl} = C - \frac{mB}{2\hbar^2} \left( n + \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \frac{2mA}{\hbar^2}} \right)^{-2} \), \( R_{nl}(r) = N e^{-\epsilon^2 r^2} r^l F_1 \left( -n, 2A^\kappa = 2; 2\epsilon^2 r \right) \) |
| -1, \( E_{nl} = -\frac{mB - C^2}{2\hbar^2} \left( n + \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \frac{2mA}{\hbar^2}} \right)^{-2} \), \( R_{nl}(r) = N e^{-\tilde{A} r^2} r^l F_1 \left( -n, 2A^\kappa = 1; 2\epsilon^2 r \right) \) |
| -2, \( E_{nl} = -\frac{mB}{2\hbar^2} \left( n + \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \frac{2mA}{\hbar^2}(A + C)} \right)^{-2} \), \( R_{nl}(r) = N e^{-\tilde{C} r^2} r^l F_1 \left( -n, 2A^\kappa = 2; 2\epsilon^2 r \right) \) |
with
\[ \epsilon_{nl}^{\pm 0} = -\frac{\bar{B}}{2(n + \Lambda + 1)}, \quad \Lambda_{nl}^{\pm 0} = - \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \tilde{A}} \]
\[ \epsilon_{nl}^{\pm 1} = -\frac{\bar{B} - \tilde{C}}{2(n + \Lambda + 1)}, \quad \Lambda_{nl}^{\pm 1} = - \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \tilde{A}} \]
\[ \epsilon_{nl}^{\pm 2} = -\frac{\bar{B}}{2(n + \Lambda + 1)}, \quad \Lambda_{nl}^{\pm 2} = - \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \tilde{A} + \tilde{C}}. \] (11)

These results are in excellent agreement with the previous results obtained by using different methods (see [1–3, 18] and references therein).

4. \( \kappa = 1 \) and 2 cases: iterative solutions

For cases \( \kappa = 1 \) and 2, the exact analytical solutions cannot be found and in this section, we present how to find the energy eigenvalues by applying the asymptotic iteration method. If we consider \( V(r) = \frac{\Delta}{r} - \frac{\bar{B}}{r} + Cr^\kappa \) potential in the three-dimensional radial Schrödinger equation, we obtain equation (8). The straightforward application of AIM to solve this equation gives us the energy eigenvalues; however, we have observed that the energy eigenvalues oscillate and do not converge within a reasonable number of iterations. The sequence appears to converge when the number of iterations \( k \approx 30 \), but then it begins to oscillate as the iteration number \( k \) increases. This result violates the principle behind the AIM; as the number of iterations increases, the method should converge and should not oscillate. In order to overcome this problem and obtain a rapid convergence, we make a change of variables as \( r = r_0 \rho \), where \( r_0 = \frac{\hbar^2}{2mB} \), then we obtain
\[ \frac{d^2 R_{nl}}{d\rho^2} + \left[ -\epsilon + \frac{1}{\rho} - \gamma^2 \rho^2 - \frac{l'(l' + 1)}{\rho^2} \right] R_{nl} = 0 \] (12)
with the following ansatz
\[ \epsilon = \frac{\hbar^2 E}{2mB^2}, \quad \gamma = \frac{2mCr^\kappa+2}{\hbar^2}, \quad l' = - \frac{1}{2} + \sqrt{\left( l + \frac{1}{2} \right)^2 + \tilde{A}}, \quad \tilde{A} = \frac{2mA}{\hbar^2} \] (13)
we can transform equation (12) to another Schrödinger equation form by changing the variable to \( \rho = u^2 \) and then by inserting \( R(u) = u^{l' + 1} \phi(u) \) into the transformed equation. Therefore, we obtain the Schrödinger equation as follows:
\[ \frac{d^2 \phi(u)}{du^2} + \left[ 4\epsilon u^2 + 4 - 4\gamma^2 u^{2\kappa+2} - \frac{\Lambda(\Lambda + 1)}{u^2} \right] \phi(u) = 0, \] (14)
where \( \Lambda = 2l' + \frac{1}{2} \).

4.1. \( \kappa = 1 \) case

For \( \kappa = 1 \), we obtain the following equation by using equation (14):
\[ \frac{d^2 \phi(u)}{du^2} + \left[ 4\epsilon u^2 + 4 - 4\gamma^2 u^4 - \frac{\Lambda(\Lambda + 1)}{u^2} \right] \phi(u) = 0. \] (15)
If we take the wavefunction in the following form:
\[ \phi(u) = u^{\Lambda+1} \exp \left( -\frac{\gamma\beta u^4}{2} \right) f(u), \] (16)
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where $\beta$ is an arbitrarily introduced constant to improve the convergence speed of the method [15]. We also introduce $\gamma$ in the asymptotic wavefunction since the parameter $\gamma$ appears explicitly in the potential. It is known that when changing the value of $\gamma$ in the potential, it affects the eigenvalues and the shape of the wavefunction. Therefore, by introducing $\gamma$ into equation (16), we control the change of the potential parameters and, as a result, obtain a better convergence. If we insert this wavefunction into equation (15), we obtain the second-order homogeneous linear differential equation as follows:

$$\frac{d^2 f(u)}{du^2} = \left[ 2 \left( 2\beta \gamma u^3 - \frac{\Lambda + 1}{u} \right) \right] \frac{df(u)}{du} + \left[ (4\beta \gamma \Lambda + 10\beta \gamma - 4\epsilon) u^2 - 4\beta^2 \gamma^2 u^6 - 4 + 4\gamma^2 u^4 \right] f(u)$$

which is now amenable to an AIM solution. By comparing this equation with equation (2), we can write the $s_0(u)$ and $\lambda_0(u)$ values as

$$s_0(u) = (4\beta \gamma \Lambda + 10\beta \gamma - 4\epsilon) u^2 - 4\beta^2 \gamma^2 u^6 - 4 + 4\gamma^2 u^4$$

$$\lambda_0(u) = 2 \left( 2\beta \gamma u^3 - \frac{\Lambda + 1}{u} \right).$$

In order to obtain the energy eigenvalues from equation (17) by using equation (5), we obtain $s_k(r)$ and $\lambda_k(r)$ in terms of $s_0(r)$ and $\lambda_0(r)$. Then, by using the quantization condition of the method given by equation (4), we obtain the energy eigenvalues. Therefore, we have to choose a suitable $u_0$ point to solve the equation $\delta_n(u_0, \epsilon) = 0$ iteratively in order to find $\epsilon$ values. In this study, we obtain $u_0$ from the maximum point of the asymptotic wavefunction, which is the same as the root of $\lambda_0(u) = 0$, thus $u_0 = \left( \frac{\Lambda + 1}{2\beta^2} \right)^{1/4}$. This straightforward application of AIM gives us the energy eigenvalues as the sequence appears to converge when the number of iterations ($k$) are $k \simeq 30$ as shown in table 1.

In this table, we also present the convergence rate of AIM calculations. The energy eigenvalues appear as the number of iterations are $k \simeq 30$. However, the speed of the convergence depends on the arbitrarily introduced constant, $\beta$. We have investigated the optimum values of $\beta$ that give the best convergence and have kept the one that appears to yield the best convergence rate. Therefore, for case $\kappa = 1$, we have performed calculations for the different values of $\beta$. It is seen from table 1 that the best convergence is obtained when the constant $\beta$ values are $\beta = 0.4, 0.5$ and $0.6$. For other values of $\beta$, the convergence needs more iteration.

Having determined the value of $\beta$ in an empirical way, for case $\kappa = 1$, the energy eigenvalues by using AIM are shown for different values of $n$ and $l$ for convergence constant $\beta = 0.5, \Lambda = 1$ and $\gamma = 1$ in table 2.

4.2. $\kappa = 2$ case

For $\kappa = 2$, we obtain the following equation by using equation (14):

$$\frac{d^2 \phi(u)}{du^2} + \left[ 4\epsilon u^2 + 4 - 4\gamma^2 u^6 - \frac{\Lambda(\Lambda + 1)}{u^2} \right] \phi(u) = 0.$$  \hspace{1cm} (20)

If we take the wavefunction in the following form:

$$\phi(u) = u^{\Lambda + 1} \exp \left( -\frac{\gamma \beta u^4}{2} \right) f(u).$$

\hspace{1cm} (21)
Table 1. For $\kappa = 1$ case, the energy eigenvalues ($\varepsilon$) for the $n = 0$ and $l = 0$ states by means of several $\beta$ values. We take $\tilde{A} = 1$ and $\gamma = 1$.

| $k$   | $\beta = 0.2$ | $\beta = 0.4$ | $\beta = 0.5$ | $\beta = 0.6$ | $\beta = 0.7$ | $\beta = 0.9$ | $\beta = 2.0$ |
|-------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 20    | 2.360 684 41  | 2.360 711 58  | 2.360 713 87  | 2.360 713 87  | 2.360 802 71  | 2.361 682 03  | 2.464 171 11  |
| 30    | 2.360 714 40  | 2.360 712 39  | 2.360 712 39  | 2.360 714 35  | 2.360 766 26  | 2.391 337 69  |
| 40    | 2.360 712 34  | "             | "             | 2.360 712 45  | 2.360 716 12  | 2.370 217 99  |
| 50    | 2.360 712 62  | "             | "             | 2.360 712 39  | 2.360 712 69  | 2.363 724 51  |
| 60    | 2.360 712 53  | "             | "             | 2.360 712 39  | 2.360 712 41  | 2.361 682 45  |
| 70    | 2.360 712 68  | "             | "             | 2.360 712 39  | 2.360 712 39  | 2.361 029 92  |
| 80    | 2.360 712 40  | "             | "             | 2.360 712 38  | 2.360 712 39  | 2.360 816 31  |
| 90    | 2.360 712 38  | "             | "             | 2.360 712 36  | 2.360 714 18  | 2.360 764 66  |

Table 2. For $\kappa = 1$ case, the energy eigenvalues ($\varepsilon$) for several quantum states. We take $\tilde{A} = 1$, $\gamma = 1$ and convergence constant $\beta = 0.5$.

| $k$   | $n = 0$, $l = 0$ | $n = 1$, $l = 0$ | $n = 1$, $l = 1$ | $n = 2$, $l = 0$ | $n = 2$, $l = 1$ | $n = 2$, $l = 2$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 20    | 2.360 713 87    | 4.112 474      | 4.724 5997      | 5.570 24        | 6.097 373       | 6.739 515       |
| 30    | 2.360 713 39    | 4.112 295      | 4.724 2771      | 5.552 92        | 6.074 553       | 6.706 985       |
| 40    | 2.360 713 09    | 4.112 290      | 4.724 2690      | 5.552 11        | 6.073 379       | 6.704 987       |
| 50    | "              | 4.112 291      | 4.724 2688      | 5.552 08        | 6.073 327       | 6.704 887       |
| 60    | "              | "              | 5.552 07        | 6.073 324       | 6.704 883       |
| 70    | "              | "              | "              | 5.552 07        | 6.073 324       | 6.704 883       |

Table 3. For $\kappa = 2$ case, the energy eigenvalues ($\varepsilon$) for several quantum numbers. We take $\tilde{A} = 1$ and convergence constant $\beta = 1$.

| $n$   | $l$   | $\gamma = 0.1$ | $\gamma = 1$ | $\gamma = 10$ |
|-------|-------|---------------|--------------|---------------|
| 0     | 0     | 0.122 004 3681 | 3.358 248 3393 | 39.649 597 3187 |
| 1     | 0.325 860 2332 | 4.896 387 8137 | 53.842 882 5862 |
| 2     | 0.547 330 2077 | 6.796 402 5443 | 72.004 705 1977 |
| 1     | 0.751 813 5075 | 8.963 936 8105 | 94.044 333 8946 |
| 2     | 0.962 181 3797 | 10.837 987 2414 | 112.131 419 5487 |
| 1     | 1.168 433 3950 | 11.542 758 5197 | 120.187 908 2158 |
| 2     | 1.372 953 445 | 13.010 225 5287 | 134.185 012 0876 |
| 1     | 1.168 433 3950 | 11.542 758 5197 | 120.187 908 2158 |
| 2     | 1.372 953 445 | 13.010 225 5287 | 134.185 012 0876 |

When we insert this wavefunction into equation (20), we obtain the second-order homogeneous linear differential equation as follows:

$$\frac{d^2 f(u)}{du^2} = 2 \left[ \left( 2\beta \gamma u^3 - \frac{\Lambda + 1}{u} \right) \right] \frac{df(u)}{du} + \left[ (4\beta \gamma \Lambda + 10\beta \gamma - 4\epsilon)u^2 - 4\beta^2 \gamma^2 u^6 - 4 + 4\gamma^2 u^6 \right] f(u)$$

which is now amenable to an AIM solution. By comparing this equation with equation (2), we obtain $s_0(u)$ and $\lambda_0(u)$ values as

$$s_0 = 4\beta \gamma u^2 \Lambda - 4\beta^2 \gamma^2 u^6 + 10\beta \gamma u^2 - 4\epsilon u^2 - 4 + 4\gamma^2 u^6$$

$$\lambda_0 = 2 \left( 2\beta \gamma u^3 - \frac{\Lambda + 1}{u} \right).$$
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Table 4. Comparison of our results with the shifted $1/N$ expansion, moment and perturbative asymptotic iteration methods for the energy eigenvalues, where $A = 0, B = 1, C = 1, \kappa = 2, m = 1$ and $\hbar = 1$.

| $n$ | $l$ | $B$ | $C$ | Shifted $1/N E$ [7] | Moment $E$ [6] | AIM $E$ [14] | Present Work $E$ |
|-----|-----|-----|-----|-----------------------|----------------|----------------|----------------|
| 0   | 0   | 1   | 1   | 0.600 25              | 0.593 77       | 0.593 65       | 0.593 77       |

Table 5. Comparison of our results with the ground state energy eigenvalues of the hydrogen atom in the $V(r) = Cr^\kappa$ potential, where $A = 0, B = 1, \kappa = 2, m = 1$ and $\hbar = 1$.

| $C$ | $E$ [6] | $E_{\text{AIM}}$ |
|-----|---------|------------------|
| 0.1 | −0.296 088 | −0.296 087 76 |
| 0.5 | 0.179 668 3 | 0.179 668 48 |
| 1.0 | 0.593 771 1 | 0.593 771 26 |
| 2.0 | 1.223 705 10 | 1.223 705 10 |
| 5.0 | 2.561 732 68 | 2.561 732 68 |
| 10.0 | 4.150 123 6 | 4.150 123 6 |
| 20.0 | 6.479 950 56 | 6.479 950 56 |
| 50.0 | 11.265 447 48 | 11.265 447 48 |
| 100.0 | 16.805 247 84 | 16.805 247 84 |
| 1000.0 | 59.375 469 04 | 59.375 469 04 |
| 2000.0 | 85.734 803 86 | 85.734 803 86 |
| 5000.0 | 138.557 197 64 | 138.557 197 64 |

Similar to the case where $\kappa = 1$, in order to obtain the energy eigenvalues from equation (22) by using equation (5), we obtain $s_k(r)$ and $\lambda_k(r)$ in terms of $s_0(r)$ and $\lambda_0(r)$. Then, by using the quantization condition of the method given by equation (4), we obtain the energy eigenvalues. For this case, we choose $u = \left(\frac{n+1}{2\beta \gamma}\right)^{\frac{1}{2}}$ and we obtain the energy eigenvalues after $k = 30$ iterations. The AIM results are presented in table 3 for different values of $n$ and $l$ quantum numbers for different values of $\gamma$. In tables 4 and 5, we compare our results with previous works conducted for special forms of our potential. Our results are in excellent agreement with the moment method [6], which is the closest one to exact solution. They are better than perturbation results.

5. Conclusion

In this paper, we have presented the exact and iterative solutions of the radial Schrödinger equation for a class of potentials, $V(r) = \frac{A}{r} - \frac{B}{r^2} + Cr^\kappa$, for various values of $\kappa$ from $-2$ to 2 for any $n$ and $l$ quantum states by applying the asymptotic iteration method. According to the value of $\kappa$, this potential family includes Kratzer, modified Kratzer, Goldman–Krivchenkov or spiked harmonic oscillator, Coulomb plus linear and Coulomb plus harmonic oscillator potentials.

These potentials have been extensively used to describe the bound and continuum states of interaction systems and a great number of papers have been published on the exact and numerical solutions of these potentials (see [1–13] and references therein). We have examined these potentials in this paper and have attempted to obtain exact or numerical solutions.

The global analysis of this potential family by using the asymptotic iteration method has resulted in exact analytical solutions for the values of $\kappa = 0, -1$ and $-2$. In addition, closed forms for the energy eigenvalues as well as the corresponding eigenfunctions are obtained,
but for $\kappa = 1$ and 2, there are no analytical solutions. Therefore, the energy eigenvalues are obtained numerically.

The advantage of the asymptotic iteration method is that it gives the eigenvalues directly by transforming the second-order differential equation into a form of $y'' = \lambda_0(r)y' + s_0(r)y$. The wavefunctions are easily constructed by iterating the values of $s_0(r)$ and $\lambda_0(r)$. The asymptotic iteration method results in exact analytical solutions if they exist and provides the closed forms for the energy eigenvalues as well as the corresponding eigenfunctions. Where there is no such solution, the energy eigenvalues are obtained by using an iterative approach [14–19]. As is presented, AIM puts no constraints on the potential parameter values involved and it is easy to implement. The results are sufficiently accurate for practical purposes. It is worth extending this method to examine other interacting systems.

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