BASE POINT FREE THEOREM OF REID-FUKUDA TYPE

OSAMU FUJINO

Abstract. Let $(X, \Delta)$ be a proper dlt pair and $L$ a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$ for some $a \in \mathbb{Z}_{>0}$. Then $|mL|$ is base point free for every $m \gg 0$.

0. Introduction

The purpose of this paper is to prove the following theorem. This type of base point freeness was suggested by M. Reid in [Re, 10.4].

Theorem 0.1 (Base point free theorem of Reid-Fukuda type). Assume that $(X, \Delta)$ is a proper dlt pair. Let $L$ be a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$ for some $a \in \mathbb{Z}_{>0}$. Then $|mL|$ is base point free for every $m \gg 0$, that is, there exists a positive integer $m_0$ such that $|mL|$ is base point free for every $m \geq m_0$.

This theorem was proved by S. Fukuda in the case where $X$ is smooth and $\Delta$ is a reduced simple normal crossing divisor in [Fk2]. In [Fk3], he proved it on the assumption that dim $X \leq 3$ by using log Minimal Model Program. Our proof is similar to [Fk3]. However, we do not use log Minimal Model Program even in dim $X \leq 3$. He also treated this problem under some extra conditions in [Fk4].

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Notation. (1) We will make use of the standard notations and definitions as in [KoM].

(2) A pair $(X, \Delta)$ denotes that $X$ is a normal variety over $\mathbb{C}$ and $\Delta$ is a $\mathbb{Q}$-divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

(3) Diff denotes the different (See [Utah, Chapter 16]).
1. Preliminaries

In this section, we make some definitions and collect the necessary results.

**Definition 1.1.** (cf. [Ka2, Definition 1.3]) A subvariety $W$ of $X$ is said to be a *center of log canonical singularities* for the pair $(X, \Delta)$, if there exists a proper birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor $E$ on $Y$ with the discrepancy $a(E, X, \Delta) \leq -1$ such that $\mu(E) = W$.

**Definition 1.2.** Let $(X, \Delta)$ be lc and $D$ a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. The divisor $D$ is called *nef and log big* on $(X, \Delta)$ if $D$ is nef and big, and $(D^{\dim W} \cdot W) > 0$ for every center of log canonical singularities $W$ of the pair $(X, \Delta)$.

**Remark 1.3.**
1. Our definition of nef and log big is equivalent to that of Reid and Fukuda (See [Fk3, Definition]).
2. In [Fj], the center of log canonical singularities of a dlt pair was investigated (See [Fj, Definition 4.6, Lemma 4.7]).

The following proposition is [Fk3, Proposition 2] (for the proof, see [Fk2, Proof of Theorem 3] and [Ka1, Lemma 3]).

**Proposition 1.4.** Let $(X, \Delta)$ be a proper dlt pair and $L$ a nef Cartier divisor such that $aL - (K_X + \Delta)$ is nef and big for some $a \in \mathbb{Z}_{>0}$. If $Bs|mL| \cap \Delta = 0$ for every $m \gg 0$, then $|mL|$ is base point free for every $m \gg 0$, where $Bs|mL|$ is the base locus of $|mL|$.

**Lemma 1.5.** (cf. [Fk3, Lemma]) Let $X$ be a proper smooth variety and $\Delta = \sum_i d_i \Delta_i$ a sum of distinct prime divisors such that $\text{Supp} \Delta$ is a simple normal crossing divisor and $d_i$ is a rational number with $0 \leq d_i \leq 1$ for every $i$. Let $D$ be a Cartier divisor on $X$. Assume that $D - (K_X + \Delta)$ is nef and log big on $(X, \Delta)$. Then $H^i(X, \mathcal{O}_X(D)) = 0$ for every $i > 0$.

This is a generalization of Kawamata-Viehweg vanishing theorem.

2. Proof of Theorem

**Proof of Theorem (0.1).** By using [Sz, Resolution Lemma] as in the proof of the Divisorial Log Terminal Theorem of [Sz], we have a log resolution $f : Y \to X$ of $(X, \Delta)$, which satisfies the following conditions:

1. $K_Y + f^{-1}_*\Delta = f^*(K_X + \Delta) + \sum_i a_i E_i$ with $a_i > -1$ for every $i$, where $E_i$'s are irreducible exceptional divisors,
(2) $f$ induces isomorphism at every generic point of center of log canonical singularities of $(X, \Delta)$.

We define $E := \sum_i a_i E_i \geq 0$ and $F := f_*^{-1} \Delta + E - \sum_i a_i E_i$. Then

$$K_Y + F = f^*(K_X + \Delta) + E.$$ 

If $\Delta \cdot S = 0$, then $(X, \Delta)$ is klt. So we may assume that $\Delta \cdot S \neq 0$. We take an irreducible component $S$ of $\Delta$. Then $(S, \text{Diff}((\Delta - S)))$ is dlt. It can be checked easily by [KoM, Corollary 5.52, Definition 2.37] and [Utah, 17.2 Theorem]. We put $S_0 := f_*^{-1} S$ and $M := f^* L$. We consider the following exact sequence;

$$0 \rightarrow \mathcal{O}_Y(-S_0) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{S_0} \rightarrow 0.$$ 

Tensoring with $\mathcal{O}_Y(m M + E)$ for $m \geq a$, we have the exact sequence;

$$0 \rightarrow \mathcal{O}_Y(m M + E - S_0) \rightarrow \mathcal{O}_Y(m M + E) \rightarrow \mathcal{O}_{S_0}(m M + E) \rightarrow 0.$$ 

By Lemma (1.3), $H^1(Y, \mathcal{O}_Y(m M + E - S_0)) = 0$. Note that $M$ is nef and $m M + E - S_0 - (K_Y + F - S_0) = f^*(m L - (K_X + \Delta))$ is nef and log big on $(Y, F - S_0)$. Then we have that

$$H^0(Y, \mathcal{O}_Y(m M + E)) \rightarrow H^0(S_0, \mathcal{O}_{S_0}(m M + E))$$

is surjective. By the projection formula, we have that

$$H^0(Y, \mathcal{O}_Y(m M + E)) \simeq H^0(X, f_* \mathcal{O}_Y(m M + E)) \simeq H^0(X, \mathcal{O}_X(m L))$$

and

$$H^0(S_0, \mathcal{O}_{S_0}(m M + E)) \supset H^0(S, \mathcal{O}_{S_0}(m M)) \simeq H^0(S, \mathcal{O}_S(m L)).$$

Note that $E$ is effective and $f$-exceptional and that $E|_{S_0}$ is effective but not necessarily $f|_{S_0}$-exceptional, where $f|_{S_0} : S_0 \rightarrow S$. We consider the following commutative diagram;

$$
\begin{array}{ccc}
H^0(Y, \mathcal{O}_Y(m M + E)) & \longrightarrow & H^0(S_0, \mathcal{O}_{S_0}(m M + E)) \longrightarrow 0 \\
\uparrow \simeq & & \uparrow \iota \\
H^0(X, \mathcal{O}_X(m L)) & \longrightarrow & H^0(S, \mathcal{O}_S(m L)).
\end{array}
$$

Then $H^0(X, \mathcal{O}_X(m L)) \rightarrow H^0(S, \mathcal{O}_S(m L))$ is surjective and $\iota$ is isomorphism since the left vertical arrow is isomorphism and $\iota$ is injective by the above argument. By induction on dimension, $|m L|_S$ is base point free for every $m \gg 0$ since $(a L - (K_X + \Delta))|_S = a L|_S - (K_S + \text{Diff}(\Delta - S))$ is nef and log big on $(S, \text{Diff}(\Delta - S))$. So we have that $Bs|m L| \cap \Delta \neq \emptyset$. By Proposition (1.3), we get the result. \hfill \Box
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Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502 Japan
E-mail address: fujino@kurims.kyoto-u.ac.jp