STRONG MORITA EQUIVALENCE OF HIGHER-DIMENSIONAL NONCOMMUTATIVE TORI

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Abstract. We show that matrices in the same orbit of the $SO(n,n|\mathbb{Z})$ action on the space of $n \times n$ skew-symmetric matrices give strongly Morita equivalent noncommutative tori, both at the $C^*$-algebra level and at the smooth algebra level. This proves a conjecture of Rieffel and Schwarz.

1. Introduction

Let $n \geq 2$ and $T_n$ be the space of $n \times n$ real skew-symmetric matrices. For each $\theta \in T_n$ the corresponding $n$-dimensional noncommutative torus $A_\theta$ is defined as the universal $C^*$-algebra generated by unitaries $U_1, \ldots, U_n$ satisfying the relation

$$U_k U_j = e((\theta_{kj})) U_j U_k,$$

where $e(t) = e^{2\pi i t}$. Noncommutative tori are one of the canonical examples in noncommutative differential geometry [12,2].

One may also consider the smooth version $A_\theta^\infty$ of a noncommutative torus, which is the algebra of formal series

$$\sum c_{j_1, \ldots, j_n} U_{j_1}^{j_1} \cdots U_{j_n}^{j_n},$$

where the coefficient function $\mathbb{Z}^n \ni (j_1, \ldots, j_n) \mapsto c_{j_1, \ldots, j_n}$ belongs to the Schwartz space $S(\mathbb{Z}^n)$ i.e. the space of $\mathbb{C}$-valued functions on $\mathbb{Z}^n$ which vanish at infinity more rapidly than any polynomial grows. This is the space of smooth elements of $A_\theta$ for the canonical action of $T_n$ on $A_\theta$.

The notion of (strong) Morita equivalence of $C^*$-algebras was introduced by Rieffel [8,10]. Strongly Morita equivalent $C^*$-algebras share a lot of important properties such as equivalent categories of modules, isomorphic $K$-groups, etc., and hence are usually thought to have the same geometry. In [14] Schwarz also introduced the notion of complete Morita equivalence of smooth noncommutative tori (see Section 2 below), which is stronger than strong Morita equivalence and has important application in M(atrix) theory [13,4].

A natural question is to classify noncommutative tori up to strong Morita equivalence. Such results have important application to physics [3,14]. For $n = 2$ this was done by Rieffel [9]. In this case there is a (densely defined) action of the group $GL(2,\mathbb{Z})$ on $T_2$, and two matrices in $T_2$ give strongly Morita equivalent noncommutative tori if and only if they are in the same orbit of this action. The higher dimensional case is much more complicated. In [13] Rieffel and Schwarz found a (densely defined) action of $SO(n,n|\mathbb{Z})$ on $T_n$ generalizing the above $GL(2,\mathbb{Z})$-action. Here $O(n,n|\mathbb{R})$ is the group of linear transformations of the space $\mathbb{R}^{2n}$ preserving...

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the quadratic form \(x_1 x_{n+1} + x_2 x_{n+2} + \cdots + x_n x_{2n}\), and \(SO(n, n|\mathbb{Z})\) is the subgroup of \(O(n, n|\mathbb{R})\) consisting of matrices with integer entries and determinant 1.

Following \[13\] we write the elements of \(O(n, n|\mathbb{R})\) in a block form:

\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

Here \(A, B, C, D\) are \(n \times n\) matrices satisfying

(1) \(A^t C + C^t A = 0 = B^t D + D^t B, \quad A^t D + C^t B = I\).

The action of \(SO(n, n|\mathbb{Z})\) is then defined as

(2) \(g\theta = (A\theta + B)(C\theta + D)^{-1}\),

whenever \(C\theta + D\) is invertible. For each \(g \in SO(n, n|\mathbb{Z})\) this action is defined on a dense open subset of \(T_n\).

Rieffel and Schwarz conjectured that if two matrices in \(T_n\) are in the same orbit of this action then they give strongly Morita equivalent noncommutative tori, both at the \(C^*\)-algebra level and at the smooth algebra level. They proved it for matrices restricted to a certain subset of \(T_n\) of second category. They also showed that the converse of their conjecture at the \(C^*\)-algebra level fails for \(n = 3\) \[13, page 297\], in contrast to the case \(n = 2\), using the classification results of G. A. Elliott and Q. Lin \[6\].

The main goal of this paper is to prove their conjecture:

**Theorem 1.1.** For any \(\theta \in T_n\) and \(g \in SO(n, n|\mathbb{Z})\), if \(g\theta\) is defined then \(A_\theta\) and \(A_{g\theta}\) are strongly Morita equivalent. Also \(A_\infty\) and \(A_{g\infty}\) are completely Morita equivalent.

Schwarz has proved that if two matrices in \(T_n\) give completely Morita equivalent smooth noncommutative tori then they are in the same orbit of the \(SO(n, n|\mathbb{Z})\)-action \[14, Section 5\]. Thus we get

**Theorem 1.2.** Two matrices in \(T_n\) give completely Morita equivalent smooth noncommutative tori if and only if they are in the same orbit of the \(SO(n, n|\mathbb{Z})\)-action.

We have learned recently that using classification theory N. C. Phillips has been able to show that two simple noncommutative tori \(A_\theta\) and \(A_{\theta'}\) are strongly Morita equivalent if and only if their ordered \(K_0\)-groups are isomorphic \[7, Remark 7.9\]. It would be interesting to see directly from the matrices why the ordered \(K_0\)-groups of \(A_\theta\) and \(A_{g\theta}\) are isomorphic.

This paper is organized as follows. Our proof of Theorem\[14\] is constructive, and we shall use the Heisenberg equivalence modules constructed by Rieffel in \[11\]. So we recall briefly Rieffel’s construction first in Section\[2\]. In order to apply Rieffel’s construction we need to reduce an arbitrary matrix in \(T_n\) to one satisfying certain nice properties. This is done in Section\[8\]. We prove Theorem\[14\] in Section\[9\].

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2. Heisenberg Equivalence Modules

In this section we recall Schwarz’s definition of complete Morita equivalence and Rieffel’s construction of Heisenberg equivalence modules for noncommutative tori.

Let $L = \mathbb{R}^n$. We shall think of $\mathbb{Z}^n$ as the standard lattice in $L^*$, and $\theta$ as in $\wedge^2 L$. One may also describe $A_\theta$ as the universal $C^*$-algebra generated by unitaries $\{U_x\}_{x \in \mathbb{Z}^n}$ satisfying the relation

$$ U_x U_y = \sigma_\theta (x, y) U_{x+y}, $$

where we write $x, y$ as column vectors, and $\sigma_\theta (x, y) = e((x \cdot \theta y)/2)$. Under this description the smooth algebra $A_\theta^\infty$ becomes $\mathcal{S}(\mathbb{Z}^n, \sigma_\theta)$, the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ equipped with the convolution induced by (3). There is a canonical action of the Lie algebra $L$ as derivations on $A_\theta^\infty$, which is induced by the canonical action of $T^n$ on $A_\theta$ and is given explicitly by

$$ \delta_X (U_x) = 2\pi i \langle X, x \rangle U_x $$

for all $X \in L$ and $x \in \mathbb{Z}^n$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $L$ and $L^*$.

Given a right $A_\theta^\infty$-module $E$, a connection on $E$ is a linear map $\nabla : L \to \text{Hom}_{\mathbb{C}}(E)$ satisfying the Leibniz rule:

$$ \nabla_X (f U_x) = (\nabla_X f) U_x + f \cdot \delta_X (U_x) $$

for all $X \in L, f \in E$ and $x \in \mathbb{Z}^n$. For each $X \in L$ the connection $\nabla$ induces a derivation $\delta_X$ on $\text{End}_{A_\theta^\infty}(E)$ by

$$ (\delta_X a)(f) = \nabla_X (a f) - a \cdot \nabla_X f $$

for all $a \in \text{End}_{A_\theta^\infty}(E)$ and $f \in E$. If $\nabla$ has constant curvature, i.e. there is skew-symmetric bilinear map $\Omega : L \times L \to \mathbb{C}$ such that $[\nabla_X, \nabla_Y] = \Omega(X, Y) \cdot 1$ for all $X, Y \in L$, then $X \mapsto \delta_X$ is a Lie algebra homomorphism from $L$ to the derivation space $\text{Der}(\text{End}_{A_\theta^\infty}(E))$ of $\text{End}_{A_\theta^\infty}(E)$. When $E$ is equipped with an $A_\theta^\infty$-valued inner product, we shall consider only Hermitian connections, i.e. $\delta_X (\langle f, g \rangle) = \langle \nabla_X f, g \rangle + \langle f, \nabla_X g \rangle$ for $X \in L$ and $f, g \in A_\theta^\infty$.

We refer to [11] for the definition and standard facts about strong Morita equivalence of $C^*$-algebras. Let $E$ be a strong Morita equivalence $A_\theta^\infty - A_{\theta'}^\infty$-bimodule. For clarity we let $L_\theta$ and $L_{\theta'}$ denote the space $L$ for $\theta$ and $\theta'$ respectively. We say that $E$ is a complete Morita equivalence $A_\theta^\infty - A_{\theta'}^\infty$-bimodule [11] page 729] if there is a constant-curvature connection $\nabla$ on $\text{End}_{A_\theta^\infty}$ and a linear isomorphism $\phi : L_\theta \to L_{\theta'}$ such that the induced Lie algebra homomorphism $L_\theta \to \text{Der}(\text{End}_{A_\theta^\infty}(E)) = \text{Der}(A_\theta^\infty)$ coincides with the composition homomorphism $L_\theta \xrightarrow{\phi} L_{\theta'} \to \text{Der}(A_{\theta'}^\infty)$. Intuitively, this means that the equivalence bimodule $E$ is "smooth", i.e. it transfers the tangent spaces $(L_\theta$ and $L_{\theta'})$ of the noncommutative differentiable manifolds $A_\theta^\infty$ and $A_{\theta'}^\infty$ back and forth.

Next we recall Rieffel’s construction of Heisenberg equivalence modules in [11] Sections 2-4]. Let $M$ be a locally compact abelian group, let $\hat{M}$ be its dual group, and let $G = M \times \hat{M}$. There is a canonical Heisenberg cocycle on $G$ defined by

$$ \beta((m, s), (l, t)) = \langle m, t \rangle, $$
where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $M$ and $\hat{M}$. There is also a skew bicharacter, $\rho$, on $G$ defined by

$$\rho(x, y) = \beta(x, y)\bar{\beta}(y, x).$$

We'll concentrate on the case $M = \mathbb{R}^p \times \mathbb{Z}^q \times W$, where $p, q \in \mathbb{Z}_{\geq 0}$ with $2p+q = n$ and $W$ is a finite abelian group. Say $W = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ for some $n_1, \ldots, n_k \in \mathbb{N}$. We shall write $G$ as $\mathbb{R}^p \times \mathbb{R}^* \times \mathbb{Z}^q \times \mathbb{T}^q \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}) \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})$. Let

$$P_1 = \text{diag}(\frac{1}{n_1}, \cdots, \frac{1}{n_k}), \quad J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \quad J_1 = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & I_q & 0 \\ 0 & -I_q & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} 0 & P_1 & 0 \\ -P_1 & 0 & J_2 \end{pmatrix}, \quad J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}.$$

Then $J$ is a square matrix of size $n+q+2k$, and we shall think of it as a 2-form on $H^* := \mathbb{R}^p \times \mathbb{R}^* \times \mathbb{R}^q \times \mathbb{R}^* \times \mathbb{R}^k \times \mathbb{R}^k$. Let $J'$ be the matrix obtained by replacing negative entries of $J$ by 0. Then $J = J' - (J')^t$. For any $x, y \in G$ we have

$$\beta(x, y) = \epsilon(x \cdot J'y) \quad \text{and} \quad \rho(x, y) = \epsilon(x \cdot Jy),$$

where we use the natural covering map $\mathbb{R}^p \times \mathbb{R}^* \times \mathbb{R}^q \times \mathbb{R}^* \times \mathbb{Z}^k \times \mathbb{Z}^k \to G$ to write $x$ and $y$ as column vectors in $\mathbb{R}^{n+q+2k}$ (notice that though $J'y$ depends on the choice of the representative of $y$ in $\mathbb{R}^p \times \mathbb{R}^* \times \mathbb{R}^q \times \mathbb{R}^* \times \mathbb{Z}^k \times \mathbb{Z}^k$, the values $\epsilon(x \cdot J'y)$ and $\epsilon(x \cdot Jy)$ do not depend on such choice).

**Definition 2.1.** [11 Definition 4.1] By an embedding map for $\theta \in T_n$ we mean a linear map $T$ from $L^*$ to $H^*$ such that:

1. $T(\mathbb{Z}^n) \subseteq \mathbb{R}^p \times \mathbb{R}^* \times \mathbb{Z}^q \times \mathbb{R}^* \times \mathbb{Z}^k \times \mathbb{Z}^k$. Then we can think of $T(\mathbb{Z}^n)$ as in $G$ via composing $T|_{\mathbb{Z}^n}$ with the natural covering map $\mathbb{R}^p \times \mathbb{R}^* \times \mathbb{Z}^q \times \mathbb{R}^* \times \mathbb{Z}^k \times \mathbb{Z}^k \to G$.

2. $T(\mathbb{Z}^n)$ is a lattice in $G$.

3. The form $J$ on $H^*$ is pulled back by $T$ to the form $\theta$ on $L^*$, i.e. $T^*J T = \theta$.

The condition (2) above is equivalent to

2'. The map $\hat{T} := \gamma \circ T : L^* \to \mathbb{R}^p \times \mathbb{R}^* \times \mathbb{R}^q$ is invertible, where $\gamma$ is the projection of $H^*$ onto $\mathbb{R}^p \times \mathbb{R}^* \times \mathbb{R}^q$.

The bimodule Rieffel constructed is the Schwartz space $S(M)$, i.e. the space of smooth functions on $M$ which, together with all their derivatives, vanish at infinity more rapidly than any polynomial grows.

**Proposition 2.2.** [11 Theorem 2.15, Corollary 3.8] Let $\theta, \theta' \in T_n$, and let $T, S$ be embedding maps of $L^*$ into $H^*$ for $\theta$ and $-\theta'$ respectively such that $S(\mathbb{Z}^n) = (T(\mathbb{Z}^n))^\perp$, where $(T(\mathbb{Z}^n))^\perp = \{z \in G : \rho(z, y) = 1 \text{ for all } y \in T(\mathbb{Z}^n)\}$. Let $T'$ and $T''$ be the composition maps $\mathbb{Z}^n \xrightarrow{T} G \to M$ and $\mathbb{Z}^n \xrightarrow{T} G \to \hat{M}$ respectively. Define $S'$ and $S''$ similarly. Fix a Haar measure on $M$. Then $S(M)$ is a strong Morita equivalence $A_S^{\infty} \cdot A_S^{\infty}$-bimodule with the module structure and inner products defined by:

$$\langle fU_x \rangle(m) = \epsilon(-T(x) \cdot J'T(x)/2 \langle m, T''(x) \rangle f(m - T'(x)),$$

$$\langle fg \rangle_S(z, \sigma) \langle x \rangle = \epsilon(-T(x) \cdot J'T(x)/2 \int_G \langle m, -T''(x) \rangle g(m + T'(x))f(m) dm,$$

$$\langle Vz f \rangle(m) = \epsilon(-S(x) \cdot J'S(x)/2 \langle m, -S''(x) \rangle f(m + S'(x)),$$

$$\langle S \sigma g \rangle \langle x \rangle = K \cdot \epsilon(S(x) \cdot J'S(x)/2 \int_G \langle m, S''(x) \rangle f(m)g(m + S'(x)) dm,$$
where $K$ is a positive constant and for clarity $V_x$ denotes the unitary in $\mathcal{S}(\mathbb{Z}^n, \sigma_{\theta'})$. Moreover, there is a linear map $Q : \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \to \text{Hom}_C(S(M))$ such that $\nabla_X = Q(_{\tilde{\phi}}^{\phi})_{\phi}(X)$ and $\nabla_X = Q(_{\tilde{\psi}}^{\psi})_{\psi}(-X)$ are connections with respect to $S(M)_{\theta_\theta'}$ and $S(M)_{\tilde{\phi}_\phi}$ respectively. The connection $\nabla$ has constant curvature

$$\Omega = 2\pi i T^{-1} \left( \sum_{j=1}^p \tilde{e}_j \wedge e_j \right),$$

where $e_1, \ldots, e_p$ are the standard basis of $\mathbb{R}^p$ and $\tilde{e}_1, \ldots, \tilde{e}_p$ are the dual basis of $\mathbb{R}^p$. Thus $S(M)$ is a complete Morita equivalence $A_{\theta_\theta'}^\theta - A_{\theta_\theta'}^\theta$-bimodule. When completed with the norm $\| f \| := \| \langle f, f \rangle \|_{A_{\theta}}$, $\| f \|_{A_{\theta}} = \| \langle f, f \rangle \|_{A_{\theta}}$, $S(M)$ becomes a strong Morita equivalence $A_{\theta'} - A_{\theta'}$-bimodule.

**Remark 2.3.** (1) The definition of embedding maps in Definition 2.1 differs from that in [11, Definition 4.1] by a sign of $\theta$. This is because Rieffel’s $A_\theta$ is our $A_{-\theta}$ (see the discussion at the end of page 285 of [11]).

(2) In [11, Section 4] the definition of embedding maps and the part of Proposition 2.2 above concerning connections and curvatures are only given for the case $W = 0$ [11, Definition 4.1] [11, pages 290-291]. The general case was discussed there in terms of tensor products with finite dimensional representations [11, Section 5]. For our purpose it’s better to deal with $\mathbb{R}^p \times \mathbb{Z}^q \times W$ directly. The proofs in [11, pages 290-291] for the case $W = 0$ are easily checked to hold for the general case.

## 3. Decomposition of Matrices

In Proposition 3.1 we shall use the construction in [11] to find the appropriate finite abelian group $W$. To this goal we need the matrix $g \in SO(n, n|\mathbb{R})$ to be of the special form in Lemma 3.3 below. We shall prove in Lemma 3.3 that every $g$ can be reduced to such a special one.

**Lemma 3.1.** Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n, n|\mathbb{R})$. Then $DC^t$ is skew-symmetric.

**Proof.** Since $g \in O(n, n|\mathbb{R})$ we have that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Hence

$$g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D^t & B^t \\ C^t & A^t \end{pmatrix}.$$

Since $O(n, n|\mathbb{R})$ is a group we have that $g^{-1} \in O(n, n|\mathbb{R})$. By [11] the matrix $(D^t)^tC^t = DC^t$ is skew-symmetric.

Using Lemma 3.1 simple calculations yield:

**Lemma 3.2.** Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in O(n, n|\mathbb{R})$. Let $\theta \in T_n$ with $C\theta + D$ invertible. Then $(C\theta + D)^{-1}C$ is skew-symmetric.

**Lemma 3.3.** Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n|\mathbb{Z})$, and let $p \in \mathbb{Z}_{\geq 0}$. Then the following are equivalent:
(i) there is some $\theta \in T_n$ such that $(C\theta + D)^{-1}C$ is of the form \( \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} \) for some $F_{11} \in GL(2p|\mathbb{R})$;

(ii) there exists a $Z \in T_{2p}$ such that

\[
C = \begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -C_{11}Z & D_{12} \\ -C_{21}Z & D_{22} \end{pmatrix},
\]

where $C_{11} \in M_{2p}(\mathbb{Z})$.

In this event, the matrix \( \begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix} \) is invertible. The matrix $Z$ is unique, and its entries are all rational numbers. Also for any $\theta' \in T_n$ in the block form

\[
\begin{pmatrix} \theta_{11}' & \theta_{12}' \\ \theta_{21}' & \theta_{22}' \end{pmatrix},
\]

where $\theta_{11}'$ has size $2p \times 2p$, the matrix $C\theta' + D$ is invertible if and only if $\theta_{11}' - Z$ is invertible. In this case

\[
(C\theta' + D)^{-1}C = \begin{pmatrix} (\theta_{11}' - Z)^{-1} & 0 \\ 0 & 0 \end{pmatrix}.
\]

**Proof.** (i)⇒(ii). From the assumption we have $C \begin{pmatrix} I_{2p} & 0 \\ 0 & 0 \end{pmatrix} = C$. Thus $C$ has the desired form in (ii). Notice that

\[
\begin{pmatrix} C_{11} & 0 \\ C_{21} & 0 \end{pmatrix} = C = (C\theta + D) \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} (C_{11}\theta_{11} + D_{11})F_{11} & 0 \\ (C_{21}\theta_{11} + D_{21})F_{11} & 0 \end{pmatrix},
\]

where we are writing both $\theta$ and $D$ in block forms. Thus $C_{j1} = (C_{j1}\theta_{11} + D_{j1})F_{11}$ for $j = 1, 2$. Let $Z = \theta_{11} - (F_{11})^{-1}$. Then $D_{j1} = -C_{j1}Z$. By Lemma 3.2, the matrix $F_{11}$ is skew-symmetric. Then so is $Z$.

(ii)⇒(i). For any $\theta' \in T_n$ we have

\[
C\theta' + D = \begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix} \begin{pmatrix} \theta_{11}' - Z & \theta_{12}' \\ 0 & I \end{pmatrix}.
\]

Take $\theta \in T_n$ such that $C\theta + D$ is invertible. Then \( \begin{pmatrix} C_{11} & D_{12} \\ C_{21} & D_{22} \end{pmatrix} \) is invertible. Therefore $C\theta' + D$ is invertible if and only if $\theta_{11}' - Z$ is invertible. In this case simple computations yield $\theta'$. In particular $(C\theta + D)^{-1}C$ has the form described in (i). By varying $\theta$ slightly we may assume that $\theta$ is rational, i.e. the entries of $\theta$ are all rational numbers. Then so are $F_{11}$ and $Z = \theta_{11} - (F_{11})^{-1}$. □

**Notation 3.4.** For any $R \in GL(n|\mathbb{Z})$ we denote by $\rho(R)$ the matrix $\begin{pmatrix} R & 0 \\ 0 & (R^{-1})^t \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$. For any $N \in T_n \cap M_n(\mathbb{Z})$ we denote by $\mu(N)$ the matrix $\begin{pmatrix} I & N \\ 0 & I \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$.

Notice that the noncommutative tori corresponding to the matrices $\rho(R)\theta = R\theta R^t$ and $\mu(N)\theta = \theta + N$ are both isomorphic to $A_0$.

**Lemma 3.5.** Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $SO(n, n|\mathbb{Z})$. Then there exists an $R \in GL(n|\mathbb{Z})$ such that $g \cdot \rho(R)$ satisfies the condition (1) in Lemma 3.6 for some $p \in \mathbb{Z}_{\geq 0}$. 
Let \( V = \{ X \in \mathbb{R}^n \mid CX = 0 \} \), and let \( K = V \cap \mathbb{Z}^n \). Since the entries of \( C \) are all integers, \( K \) spans \( V \). By the elementary divisors theorem \( [3] \) page 153, Theorem III.7.8] we can find a basis \( \beta_1, \ldots, \beta_n \) of \( \mathbb{Z}^n \), some integer \( 1 \leq k \leq n \) and positive integers \( c_k, \ldots, c_n \) such that \( K \) is generated by \( c_k \beta_k, \ldots, c_n \beta_n \). Then \( V = \text{span}(\beta_k, \ldots, \beta_n) \). Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{Z}^n \). Then 
\[
(\beta_1, \ldots, \beta_n) = (e_1, \ldots, e_n)R \text{ for some } R \in GL(n|\mathbb{Z}).
\]

Let
\[
\begin{pmatrix}
A' & B' \\
C' & D'
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \rho(R) \in SO(n, n|\mathbb{Z}).
\]

Choose \( \theta \in \mathcal{T}_n \) such that \( C\theta + D \) is invertible. Let \( \theta' = R^{-1} \theta (R^{-1})^t \in \mathcal{T}_n \). Now we need

**Lemma 3.6.** \((C' + D')^{-1} C'\) is of the form \( \begin{pmatrix} F_{11} & 0 \\ 0 & 0 \end{pmatrix} \) for some \( F_{11} \in GL(k-1|\mathbb{R}) \).

*Proof.* In view of Lemma \( \ref{lemma3.2} \) this is clearly equivalent to saying that the vectors \( X = (x_1, \ldots, x_k)^t \in \mathbb{R}^n \) satisfying \((C' + D')^{-1} C' X = 0\) are exactly those with \( x_1 = \cdots = x_{k-1} = 0 \). Notice that \((C' + D')^{-1} C' = R^{-1} (C\theta + D)^{-1} C R\). Hence \((C' + D')^{-1} C' X = 0\) if and only if \( CRX = 0\), if and only if \( RX \in V\), if and only if \( (\beta_1, \ldots, \beta_n) X \in V\), if and only if \( x_1 = \cdots = x_{k-1} = 0 \).

Back to the proof of Lemma \( \ref{lemma3.6} \). By Lemma \( \ref{lemma3.2} \) the matrix \( F_{11} \) is skew-symmetric. Since \( F_{11} \in GL(k-1|\mathbb{Z}) \) we see that \( k - 1 \) is even. This completes the proof of Lemma \( \ref{lemma3.6} \). \( \square \)

### 4. Strong Morita Equivalence

In this section we prove Theorem \( \ref{thm1.1} \). We shall employ the notation in Section \( \ref{section2} \) and Lemma \( \ref{lemma3.3} \). In view of Proposition \( \ref{prop2.2} \) the key is to find embedding maps. This is established in the following

**Proposition 4.1.** Let \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n|\mathbb{Z}) \) satisfying the conditions (1) and (2) in Lemma \( \ref{lemma3.3} \) for some \( p \in \mathbb{Z}_{\geq 0} \). Then there exist an \( N \in \mathcal{T}_n \cap M_p(\mathbb{Z}) \), an \( R \in GL(n|\mathbb{Z}) \), a \( g' \in SO(n, n|\mathbb{Z}) \) and a finite abelian group \( W \) such that \( g = \nu(N) \rho(R) g' \) and for any \( \theta \in \mathcal{T}_n \) with \( C\theta + D \) invertible there are embedding maps \( T, S : L^* \to H^* \) for \( \theta \) and \(-g'\theta\) respectively satisfying \( S(\mathbb{Z}^n) = (T(\mathbb{Z}^n))^\perp \) (see Definition \( \ref{def2.1} \) and Proposition \( \ref{prop2.2} \) for the meaning of \( (T(\mathbb{Z}^n))^\perp \)).

*Proof.* Let \( Z \) be as in Lemma \( \ref{lemma3.3} \) for \( g \). Then \( Z \) is rational, and hence there is some \( m \in \mathbb{N} \) such that \( mZ \) is integral. Thinking of \( mZ \) as a bilinear alternating form on \( \mathbb{Z}^n \), by \([3] \) page 598, Exercise XV.17] we can find an \( R \in GL(2p|\mathbb{Z}) \), some integer \( 1 \leq k \leq p \) and integers \( h_1, \ldots, h_k \) such that 
\[
mZ = R^t \begin{pmatrix} 0 & P & 0 \\ -P & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R,
\]
where \( P = \text{diag}(h_1, \ldots, h_k) \). Let \( m_j/n_j = h_j/m \) with \( (m_j, n_j) = 1 \) and \( n_j > 0 \) for each \( 1 \leq j \leq k \). Set \( W = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k} \).

Let \( \theta \in \mathcal{T}_n \) with \( C\theta + D \) invertible. We are ready to construct an embedding map for \( \theta \) now. Our method is similar to that in the proof of the proposition in \( \ref{lemma3.3} \). But our situation is more complicated since we have to deal with the torsion part \( W \). Write \( \theta \) in block form as in Lemma \( \ref{lemma3.3} \). By Lemmas \( \ref{lemma3.3} \) and \( \ref{lemma3.2} \) the
matrix \(\theta_{11} - Z\) is invertible and skew-symmetric. So we can find a \(T_{11} \in GL(2p|\mathbb{R})\) such that \(T_{11}J_0T_{11} = \theta_{11} - Z\), where \(J_0\) is defined in \(\text{(3)}\). Let \(T_{31} = \theta_{31}\), and let \(T_{32}\) be any \(q \times q\) matrix such that \(T_{32} - T_{32}^t = \theta_{22}\), where \(q = n - 2p\). Let \(P_2 = \text{diag}(m_1, \ldots, m_k)\), and let
\[
T_1 = \begin{pmatrix} T_{11} & 0 \\ 0 & I_q \\ T_{31} & T_{32} \end{pmatrix}, \quad T_2 = \begin{pmatrix} P_2 & 0 \\ 0 & I_k \\ R & 0 \\ 0 & I_q \end{pmatrix}, \quad T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}.
\]

Then \(T_1, T_2\) and \(T\) have sizes \((n+q) \times n, 2k \times n\) and \((n+q+2k) \times n\) respectively. Simple calculations yield \(T^tJ \theta = \theta\), where \(J\) is defined in \(\text{(5)}\). Notice that as a linear map from \(L^* = \mathbb{R}^n\) to \(H^* = \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^q \times \mathbb{R}^k \times \mathbb{R}^k\), \(T\) carries the lattice \(\mathbb{Z}^n = \mathbb{Z}^{2p} \times \mathbb{Z}^q\) into \(\mathbb{R}^p \times \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{R}^q \times \mathbb{Z}^k \times \mathbb{Z}^k\). Also observe that \(\tilde{T}\) (see Definition \(\text{(2)}\)) is given by the invertible matrix \(\begin{pmatrix} T_{11} & 0 \\ 0 & I_q \end{pmatrix}\). Thus the conditions in Definition \(\text{(2)}\) are satisfied and hence \(T\) is an embedding map for \(\theta\).

Let \(D = T(\mathbb{Z}^n)\). By Definition \(\text{(1)}\) we may think of \(D\) as in \(G = \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{Z}^q \times \mathbb{T}^q \times (\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k})\). We need to find some embedding map of \(\mathbb{Z}^n\) into \(G\) with image being exactly \(D^+ = \{z \in G : \rho(z, y) = 1\text{ for all } y \in D\}\), where \(\rho\) is defined in \(\text{(4)}\).

For any \(x \in G\), it is in \(D^+\) exactly if \(x \cdot JTz \in \mathbb{Z}\) for all \(z \in \mathbb{Z}^n\), exactly if \(T^tJx \in \mathbb{Z}^n\). Let \(T_3 = \begin{pmatrix} 0 \\ -I_q \end{pmatrix}\) be an \((n+q) \times q\) matrix. Let \(T_4 = \text{diag}(n_1, \ldots, n_{k}, n_1, \ldots, n_k)\). Set
\[
\tilde{T} = \begin{pmatrix} T_1 & T_3 & 0 \\ T_2 & 0 & T_4 \end{pmatrix},
\]
a square matrix of size \(n + q + 2k\). It is easy to check that \(T^tJx \in \mathbb{Z}^n\) exactly if \(\tilde{T}^tJx \in \mathbb{Z}^{n+q+2k}\). Also it is easy to see that \(\tilde{T}\) is invertible. Thus
\[
D^+ = (\tilde{T}^tJ)^{-1}(\mathbb{Z}^{n+q+2k}).
\]
Recall the matrices \(J_0\) and \(J_1\) defined in \(\text{(5)}\). Straight-forward calculations show that
\[
(\tilde{T}^tJ)^{-1} = \begin{pmatrix} -J_1 & 0 \\ T_{11} & T_{13} \\ T_{31} & T_{32} \end{pmatrix}^{-1} = \begin{pmatrix} -J_1(T_{11})^{-1} & 0 \\ J_1(T_{11})^{-1} & T_{13}^{-1} \\ 0 & -I_q \end{pmatrix}^{-1} \begin{pmatrix} T_{11} & 0 \\ 0 & I_q \\ T_{31} & T_{32} \end{pmatrix},
\]
and
\[
J_1(T_{11})^{-1} = \begin{pmatrix} J_0(T_{11})^{-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -J_0(T_{11})^{-1}T_{31} \\ 0 \end{pmatrix}.
\]
Thus
\[
(\tilde{T}^tJ)^{-1}(\mathbb{Z}^q \times \mathbb{Z}^q) = 0^p \times \mathbb{Z}^q \times \mathbb{Z}^q \times \mathbb{Z}^q \times \mathbb{Z}^2,
\]
which is \(0\) in \(G\). So \((\tilde{T}^tJ)^{-1}(\mathbb{Z}^{n+q+2k}) = (\tilde{T}^tJ)^{-1}(\mathbb{Z}^{2p} \times \mathbb{Z}^q \times \mathbb{Z}^{2k}).\) Let \(\Delta\) be the set of all vectors \(y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\) in \(\mathbb{Z}^{2p} \times \mathbb{Z}^q \times \mathbb{Z}^{2k}\).
Lemma 4.2. Let \( \varphi \) be the embedding \( \mathbb{Z}^n \hookrightarrow \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} \) sending \((x_1, \ldots, x_n)^t\) to

\[
(-d_1x_1, \ldots, -d_kx_k, 0, \ldots, 0, x_{2k+1}, \ldots, x_2)^t \times 0^q \times (x_{2p+1}, \ldots, x_n, c_1x_1, \ldots, c_kx_k, x_{k+1}, \ldots, x_{2k})^t.
\]

Let \( \varphi \) be the composition of \( \varphi_1 \) and \( \begin{pmatrix} R^t & 0 \\ 0 & I_{2q+2k} \end{pmatrix} : \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} \to \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} \). Then \( \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} = \triangle \oplus \varphi(\mathbb{Z}^n) \).

Proof. Let \( y = \begin{pmatrix} y_1 \\ 0 \\ y_3 \end{pmatrix} \in \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^q \times \mathbb{Z}^{2k} \) satisfying \( y_3 \in n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z} \times n_1\mathbb{Z} \times \cdots \times n_k\mathbb{Z} \). Say \( y_3 = (n_1z_1, \ldots, n_kz_k, n_1z_{k+1}, \ldots, n_kz_{2k})^t \). Then it is easy to see that \( y \in \triangle \) exactly if

\[
(R^t)^{-1}y_1 = (m_1z_1, \ldots, m_kz_k, z_{k+1}, \ldots, z_{2k}, 0, \ldots, 0)^t \quad \text{and} \quad y_2 = 0.
\]

It is clear from this that \( \mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k} = \triangle \oplus \varphi(\mathbb{Z}^n) \). \( \square \)

Back to the proof of Proposition 4.1. Putting \( \varphi : \mathbb{Z}^n \to (\mathbb{Z}^{2p} \times 0^q \times \mathbb{Z}^{q+2k}) \) and \( (T^tJ)^{-1} : \mathbb{Z}^{n+q+2k} \to H^* \) together, we get a map \( S := (T^tJ)^{-1} \circ \varphi : \mathbb{Z}^n \to H^* \) with \( S(\mathbb{Z}^n) = D^\perp \). Let

\[
Q_1 = \text{diag}(d_1, \ldots, d_k), \quad Q_2 = \text{diag}(c_1, \ldots, c_k).
\]

A routine calculation shows that

\[
S = \begin{pmatrix} W_1 & 0 \\ 0 & W_2 \end{pmatrix}
\]

where \( W_1 \) and \( W_2 \) have sizes \((n+q) \times 2p\) and \((n+q) \times q\) respectively:

\[
W_1 = \begin{pmatrix} J_0(T^t_{11})^{-1}R^t & P_1 \\ 0 & P_1 \end{pmatrix}, \quad W_2 = \begin{pmatrix} -J_0(T^t_{11})^{-1}T^t_{31} \\ I_q \\ T^t_{32} \end{pmatrix}.
\]

Clearly \( S \) satisfies Definition 4.1(2'). Then \( S \) is an embedding map for

\[
-\theta' = S^*JS = \begin{pmatrix} \theta_{11}' & \theta_{12}' \\ \theta_{21}' & \theta_{22}' \end{pmatrix}.
\]
where

\[
\theta'_{11} = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} R F_{11} R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix},
\]

\[
+ \begin{pmatrix} 0 & -Q_2 P_1 & 0 \\ Q_2 P_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
\theta'_{12} = \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} R F_{11} \theta_{12},
\]

\[
\theta'_{21} = -\theta_{21} F_{11} R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix},
\]

\[
\theta'_{22} = -\theta_{21} F_{11} \theta_{12} + \theta_{22}.
\]

Proposition 2.2 tells us that \( S(M) \) is a complete Morita equivalence \( A_0^\infty - A_0^\infty \)-bimodule. Clearly the dual \( \phi^*: L_0^\infty \to L_0^\infty \) of \( \phi: L_0 \to L_0 \) is just \( -T^{-1} \circ \tilde{S} \). A routine calculation shows that \( \phi^* \) is given by the matrix

\[
\mathcal{A} = -\begin{pmatrix} F_{11} R^t \begin{pmatrix} P_1 & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} & -F_{11} \theta_{12} \end{pmatrix}.
\]

It is also easy to see that the matrix form of the normalized curvature \( \frac{1}{2\pi} \Omega \) is

\[
\Phi = \begin{pmatrix} F_{11} \\ 0 \\ 0 \end{pmatrix}.
\]

Now that we have the matrices \( \theta, \theta', \mathcal{A}, \) and \( \Omega \), Schwarz \[14\] page 733] has shown how to find \( g' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in SO(n, n|\mathbb{Z}) \) such that \( \theta' = g' \theta \). Actually we have the formulas:

\[
\begin{align*}
C' &= \mathcal{A}^{-1} \Phi, & D' &= \mathcal{A}^{-1} - C' \theta, \\
A' &= \mathcal{A} + \theta C', & B' &= \theta' \mathcal{A}^{-1} - A' \theta.
\end{align*}
\]

Our formulas \( \mathbf{11} \) are exactly the equation (53) of \[14\], in slightly different form. Straight-forward calculations yield

\[
C' = \begin{pmatrix} T_4 & 0 & 0 \\ 0 & -I_{2p-2k} & (R^t)^{-1} \end{pmatrix}, & D' = \begin{pmatrix} 0 & -P_2 & 0 \\ P_2 & 0 & 0 \\ 0 & 0 & R \end{pmatrix},
\]

\[
A' = \begin{pmatrix} 0 & -Q_2 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} (R^t)^{-1}, & B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & Q_1 & 0 \\ 0 & 0 & -I_{2p-2k} \end{pmatrix} R.
\]

Let

\[
\tilde{g} = \begin{pmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \end{pmatrix} := g(g')^{-1}.
\]
Then \( \tilde{g} \in SO(n, n|\mathbb{Z}) \). A routine calculation shows that \( \tilde{C} = 0 \). By \[\text{Lemma 1.4}\] we have \( I = \tilde{A}^t \tilde{D} + \tilde{C}^t \tilde{B} = \tilde{A}^t \tilde{D} \). Then \( \tilde{A} \) is invertible. Recall the matrix \( \rho(\tilde{A}) \) in Notation \[\text{3.3}\]. We get
\[
\tilde{g} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ 0 & \tilde{D} \end{pmatrix} = \begin{pmatrix} I & \tilde{B} \tilde{A}^t \\ 0 & I \end{pmatrix} \rho(\tilde{A}).
\]
Hence \( \begin{pmatrix} I & \tilde{B} \tilde{A}^t \\ 0 & I \end{pmatrix} = \tilde{g}(\rho(\tilde{A}))^{-1} \in SO(n, n|\mathbb{Z}) \). By \[\text{Lemma 1.4}\] the matrix \( I^r(\tilde{B} \tilde{A}^t) = \tilde{B} \tilde{A}^t \) is skew-symmetric. So we get
\[
\tilde{g} = \nu(\tilde{B} \tilde{A}^t) \rho(\tilde{A}), \quad g = \tilde{g} g' = \nu(\tilde{B} \tilde{A}^t) \rho(\tilde{A}) g'.
\]
Notice that \( g', \tilde{B} \tilde{A}^t \) and \( \tilde{A} \) do not depend on \( \theta \). This finishes the proof of Proposition \[\text{1.1}\].

**Remark 4.3.** (1) We would like to point out that the argument right after (52) of \[\text{1.4}\] is not quite complete. When \( n = 2 \) the fact that \( W \) transforms the integral lattice \( \wedge^e(D^*) \) into itself does not imply that \( W \) has integral entries. In other words, \[\text{1.4}\] above may not give integral matrices when \( n = 2 \). So the case \( n = 2 \) in \[\text{1.4}\] has to be dealt separately, and it does follow from \[\text{9}\]. In our situation we don’t need to separate the case \( n = 2 \) since \( g' \) obviously has integral entries.

(2) Given \( g' \) explicitly, one can also check directly that \( g' \in SO(n, n|\mathbb{Z}) \) and \( \theta' = g' \theta \): straight-forward calculations show that \( g' \) satisfies \[\text{1.4}\] and \( \theta' = g' \theta \). Then \( g', \tilde{g} \in O(n, n|\mathbb{Z}) \) and hence we still have \( I = \tilde{A}^t \tilde{D} \). Thus \( det(g') = det(\tilde{g}^{-1}) = 1 \). Therefore \( g' \in SO(n, n|\mathbb{Z}) \).

Proof of Theorem \[\text{1.4}\]. We may think of \( \mathcal{A}_\theta \) as the universal \( C^* \)-algebra generated by unitaries \( \{U_{x, \rho}\}_{x \in \mathbb{Z}} \) satisfying the relation \( U_{x, \rho} U_{y, \rho} = e(x \cdot \theta y) U_{y, \rho} U_{x, \rho} \). For any \( R \in GL(n|\mathbb{Z}) \) and \( \theta \in T_n \) there is a natural isomorphism \( A_{\rho}^\infty \rightarrow A_{\rho(R)\theta}^\infty \). Given by \( U_{x, \rho} \mapsto U_{(R^{-1})^* x, \rho(R)\theta} \). Under this isomorphism \( \delta_{X, \theta} \) becomes \( \delta_{RX, \rho(R)\theta} \) for any \( X \in L^* \). Similarly, for any \( N \in T_n \cap M_n(\mathbb{Z}) \) and \( \theta \in T_n \) there is a natural isomorphism \( A_{\rho}^\infty \rightarrow A_{\rho(N)\theta}^\infty \). Given by \( U_{x, \rho} \mapsto U_{x, \rho(N)\theta} \), under which \( \delta_{X, \theta} \) becomes \( \delta_{X, \rho(N)\theta} \) for any \( X \in L^* \). Now Theorem \[\text{1.4}\] follows from Lemma \[\text{3.6}\] and Propositions \[\text{1.1}\] and \[\text{2.2}\].

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