An Evaluation of The Proton Structure Functions $F_2$ and $F_L$ at Small $x$

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We derive the DGLAP independent evolution equations to extract the decoupled structure functions at high-order correction in the small $x$ limit. Determination of the longitudinal structure function is present due to the parameterization of $F_2(x, Q^2)$ and its derivative. Analytical expressions for $\sigma_\perp(x, Q^2)$ in terms of the effective parameters of the parameterization of $F_2(x, Q^2)$ and $F_L(x, Q^2)$ are presented. This analysis is enriched by including the higher-twist effects in calculation of the reduced cross sections which is important at low-$x$ and low-$Q^2$ regions. Numerical calculations and comparison with H1 data demonstrate that the suggested method provides reliable $F_L(x, Q^2)$ and $\sigma_\perp(x, Q^2)$ at low $x$ in a wide range of the low absolute four-momentum transfers squared ($1.5 \text{GeV}^2 < Q^2 < 120 \text{GeV}^2$) at moderate and high inelasticity. Expanding the method to low and ultra low values of $x$ can be considered in the process analysis of new colliders.

I. INTRODUCTION

Some time ago a proposal was published to look for the longitudinal and transverse structure functions in deep inelastic scattering (DIS) [1]. It was shown that a singular $x^{-\lambda}$ input with $\lambda > 0.3$ seems to play a key role with respect to resummation. Authors in Ref.[1] showed that it is possible to obtain scheme independent evolution equations for the structure functions by the following form

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} \sim \Gamma_{22} \otimes F_2(x, Q^2) + \Gamma_{2L} \otimes F_L(x, Q^2), \quad (1)$$

$$\frac{\partial F_L(x, Q^2)}{\partial \ln Q^2} \sim \Gamma_{L2} \otimes F_2(x, Q^2) + \Gamma_{LL} \otimes F_L(x, Q^2). \quad (2)$$

The method is based on physical observables, $F_L$ and $F_2$ [2]. The anomalous dimensions $\Gamma_{ij}$ are computed in perturbative QCD and the symbol $\otimes$ indicates convolution over the variable $x$ by the usual form, $f(x) \otimes g(x) = \int_0^1 x \frac{dz}{z} f(z, \alpha_s) g(x/z)$. The structure functions $F_2$ and $F_L$ are related to the cross sections $\sigma_\perp$ and $\sigma_L$ for interaction of transversely and longitudinally polarized photons with protons. The reduced cross section for deep-inelastic lepton-proton scattering depends on these independent structure functions in the combination

$$\sigma(x, Q^2) = F_2(x, Q^2) - \frac{y^2}{Y_+} F_L(x, Q^2), \quad (3)$$

where $Y_+ = 1 + (1 - y)^2$, $y = Q^2/xs$ denotes the inelasticity and $s$ stands for the center-of-mass squared energy of incoming electrons and protons. As usual $x$ is the Bjorken scaling parameter and $Q^2$ is the four momentum transfer in a deep inelastic scattering process.

The experimental determination of the longitudinal structure function is a realistic prospect at high energy electron-proton colliders. First measurements of $F_L$ at small $x$ were performed at HERA [3]. A next generation of ep colliders is under design, the LHeC [4,5] and the FCC-eh [6] were these measurements will be performed with much increased precision and extended to much lower values of $x$ and high $Q^2$. The electron-proton center of mass energy at the Large Hadron electron Collider (LHeC) reach to $\sqrt{s} \approx 1.3 \text{ TeV}$, which this energy is about 4 times the center-of-mass energy range of ep collisions at HERA [4-5]. The LHeC is designed to become the finest new microscope for exploring new physics, as the kinematic range in the $(x, Q^2)$ plane for electron and positron neutral-current (NC) in the perturbative region is well below $x \approx 10^{-6}$ and extends up to $Q \approx 1 \text{ TeV}$. This behavior will be extended down to $x \approx 10^{-7}$ at FCC-eh that is an option of Future Circular Collider program [6]. The FCC-eh collider would reach a center of mass energy of $\sqrt{s} = 3.5 \text{ TeV}$ at a similar luminosity as the LHeC. Deep inelastic scattering measurements at FCC-eh and LHeC will allow the determination of the parton distribution functions at very small $x$ as they are pertinent in investigations of lepton-hadron processes at ultra-high energy (UHE) neutrino astroparticle physics [7]. Moreover a similar very high energy electron-proton/ion collider (VHEep) [8] has been suggested based on plasma wakefield acceleration, albeit with very low luminosity. The center-of-mass energy, in this collider, is close to 10 TeV which is relevant in investigations of new strong inter-action dynamics related to the high-energy cosmic rays and gravitational physics (the luminosity estimate is about six orders of magnitude below that of LHeC).

Recently several methods of determination of the longitudinal structure function in the nucleon from the proton structure function have been proposed [9-10].

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method is based on a form of the deep inelastic lepton-hadron scattering (DIS) structure function which it was proposed by Block-Durand-Ha (BDH) in Ref.[11]. This new parameterization describes accurate results for the high energy ep and isoscalar $\nu N$ total cross sections. These cross sections obey an analytic expression into $\ln E$ at large energies $E$ of the incident particle. Indeed, this parameterization is relevant to investigations of ultra-high energy processes, such as scattering of cosmic neutrino from hadrons [9].

In QCD, structure functions are defined as convolution of the universal parton momentum distributions inside the proton and coefficient functions, which contain information about the boson-parton interaction [12-13]. The standard and the basic tools for theoretical investigation of DIS structure functions are the DGLAP evolution equations [14-15]. DGLAP equations based on parton model and perturbative QCD theory successfully and quantitatively interpret the proton model and perturbative QCD theory successfully and quantitatively interpret the $Q^2$-dependence of parton distributions (PDFs). It is so successful that most of the PDFs are extracted by using the DGLAP equations up to now. These equations widely can be used to extract the deep inelastic scattering structure functions of proton.

The longitudinal structure function $F_L(x,Q^2)$ of proton in terms of coefficient function is given by [14]

$$x^{-1}F_L(x,Q^2) = C_{L,ns}^{(\alpha_s)}(x) \otimes q_{ns}(x,Q^2) + \frac{e^2}{2} > [C_{L,s}^{(\alpha_s)}(x) \otimes q_{s}(x,Q^2) + C_{L,g}^{(\alpha_s)}(x) \otimes g(x,Q^2)],$$

where $q_{ns}, q_s,$ and $g$ are the flavour non singlet, flavour singlet and gluon density respectively. $< e^2 > = \frac{\sum_{N_f}^N e_i^2}{N_f}$ is the average squared charge for $N_f$ where $N_f$ denotes the number of effectively massless flavours. The coefficient functions $C_{L,a}(a = q, g)$’s can be written by the perturbative expansion as follows [16]:

$$C_{L,a}^{(\alpha_s)}(x) = \sum_{n=1}^{\infty} (\frac{\alpha_s}{4\pi})^n c_{L,a}^{(n)}(x),$$

where $n$ denotes the order in running coupling constant.

The coupled DGLAP evolution equations for the singlet quark structure function $F_i(x,Q^2) = \sum_i x_i q_i(x,Q^2) + \overline{q_i}(x,Q^2)$ and the gluon distribution $G(x,Q^2) = xg(x,Q^2)$ can be written as

$$\frac{\partial G(x,Q^2)}{\partial \ln Q^2} = P_{gg}(\alpha_s,x) \otimes G(x,Q^2) + 18 \frac{\overline{q}_i}{3} P_{qg}(\alpha_s,x) \otimes F_i(x,Q^2),$$

$$\frac{\partial F_i(x,Q^2)}{\partial \ln Q^2} = P_{gg}(\alpha_s,x) \otimes F_i(x,Q^2) + 2N_fP_{qg}(\alpha_s,x) \otimes G(x,Q^2).$$

The splitting functions $P_{ij}$’s are the Altarelli-Parisi kernels at LO up to NNLO loops corrections [17]

$$P_{ij}(\alpha_s,x) = \sum_{n=1}^{\infty} (\frac{\alpha_s}{4\pi})^n P_{ij}^{(n)}(x),$$

where $\alpha_s(Q^2)$ is the running coupling constant.

The main purpose of this study is to give purely dynamical independent structure functions, which is expected to be more reliable at small $x$. The second purpose is to connect the longitudinal structure function to the proton structure function at the coupled DGLAP structure functions equations. This has the prospect to resolve the origin of longitudinal structure function at LHeC and FCC-eh scales. The third purpose is to understand the Higher-twist (HT) effects of the reduced cross section at low-$x$ and low-$Q^2$ values. The organization of this paper is as follows. In section II we introduce the basic formula used for the definition of decoupling DGLAP evolution equations into the proton and longitudinal structure functions. In section III we present the longitudinal structure function with respect to the parameterization of $F_2$. In section IV we also present the effective exponent for the singlet structure function in an independent method. Finally the formalism of HT effects used in this analysis in section V. Section VI contains our order analysis. The main results and finding of the present longitudinal structure function and reduced cross section at moderate and high inelasticity are discussed in details in section VII. In the same section, we present the extracted HT effects at low $Q^2$ values and detailed comparisons with the experimental data. We also expanded the available energy to the range of new collider energies (i.e., LHeC and FCC-eh). This section also includes a brief discussion of the implication of the finding to future research.

II. BASIC FORMULA

In the limit of the high center-of-mass energy, or equivalently at small values of Bjorken $x$, the DGLAP evolution equations can be written as

$$\frac{\partial G(x,Q^2)}{\partial \ln Q^2} = P_{gg}(\alpha_s,x) \otimes G(x,Q^2) + \frac{18}{5} P_{qg}(\alpha_s,x) \otimes F_2(x,Q^2),$$

$$\frac{\partial F_2(x,Q^2)}{\partial \ln Q^2} = P_{qg}(\alpha_s,x) \otimes F_2(x,Q^2) + \frac{10N_f}{18} P_{qg}(\alpha_s,x) \otimes G(x,Q^2).$$
\[ F_L(x, Q^2) = \frac{18}{5N_f}C_{L,s}(\alpha_s, x) \otimes F_2(x, Q^2) \]
\[ + < e^2 > C_{L,g}(\alpha_s, x) \otimes G(x, Q^2). \quad (9) \]

The standard parameterization of the singlet and gluon distribution functions observed by the following forms as 
\[ x \to 0, \]
\[ F_2'(x, Q^2)_{x \to 0} = A_s(Q^2)x^{-\lambda_s(Q^2)}, \]
\[ G(x, Q^2)_{x \to 0} = A_g(Q^2)x^{-\lambda_g(Q^2)}. \quad (10) \]

Where \( A_s \) and \( A_g \) are \( Q^2 \) dependent and \( \lambda_s \)s are strictly positive. This behavior of the singlet structure function proposed by Lopez and Yndurain [18], then the inclusive electroproduction on the proton studied at low \( x \) and low 
\( Q^2 \) using a soft and hard Pomeron in Ref.[19]. The gluon exponent at low values of \( x \) at \( Q^2 = 1 \text{ GeV}^2 \) for MSTW08 NLO computed which the fitted value with its uncertainties is obtained to be \(-0.428^{+0.066}_{-0.057} \) [20]. The effective exponent for the gluon distribution at \( Q^2 = 10 \text{ GeV}^2 \) and \( x = 10^{-4} \) by NNPDF3.0, CT14, MMHT14, ABM12 and CJ15 parameterizations determined with the values of -0.20, -0.15, -0.29, -0.15 and -0.14 respectively. This value by the fixed coupling LLx BFKL solution gives the value \( \lambda_g \approx -0.5 \), which is the so-called hard-Pomeron exponent. For singlet structure function an effective exponent based on HERA combined data and a phenomenological model parametrized in Refs.[21] and [22] respectively. In Ref.[23] these intercepts are determined and applied to the deep inelastic lepton nucleon scattering at low values of \( x \). Ref.[24] was used a form inspired by double asymptotic \( x, Q^2 \) scaling. The hard Pomeron behavior of the photon-proton cross section based on a simple power-law behavior and double asymptotic scaling at low \( x \) values for \( 10 < W < 10^4 \text{ GeV} \) in the kinematic range of VHEep is shown in Ref.[8].

The convolution results for the compact form of the kernels are given by
\[ \Phi_{qq}(x, Q^2) = P_{qq}(x, \alpha_s) \otimes x^{\lambda_s(Q^2)} \]
\[ \Phi_{gg}(x, Q^2) = P_{gg}(x, \alpha_s) \otimes x^{\lambda_g(Q^2)} \]
\[ \Theta_{gq}(x, Q^2) = \frac{18}{5} P_{gq}(x, \alpha_s) \otimes x^{\lambda_s(Q^2)} \]
\[ \Theta_{gg}(x, Q^2) = \frac{10N_f}{18} P_{gg}(x, \alpha_s) \otimes x^{\lambda_s(Q^2)} \]
\[ I_{L,q}(x, Q^2) = \frac{18}{5N_f} C_{L,q}(x, \alpha_s) \otimes x^{\lambda_q(Q^2)} \]
\[ I_{L,g}(x, Q^2) = < e^2 > C_{L,g}(x, \alpha_s) \otimes x^{\lambda_g(Q^2)}. \quad (14) \]

where we have defined the convolution form to be
\[ f(x) \otimes g(x) = \int_x^1 \frac{dz}{z} f(\alpha_s, z) g(z). \quad (15) \]

The idea is to modify the evolution equations in order to satisfy simultaneously the two independent evolution equation based on the structure functions. The singlet and longitudinal structure functions contain the gluon distribution where coming from the perturbative QCD. The solution of Eq.(13) is straightforward and given by
\[ G(x, Q^2) = \frac{F_L(x, Q^2)}{I_{L,q}(x, Q^2)} = \frac{F_2(x, Q^2)}{I_{L,g}(x, Q^2)} \]
\[ (16) \]

We will now seek the DGLAP evaluation equation for the proton structure function using Eqs.(12), (13) and (16) as
\[ \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \Gamma_{22}(x, Q^2) F_2(x, Q^2) \]
\[ + \Gamma_{2L}(x, Q^2) F_L(x, Q^2), \quad (17) \]

where
\[ \Gamma_{22}(x, Q^2) = \Phi_{qq}(x, Q^2) - \Theta_{gq}(x, Q^2) \frac{I_{L,q}(x, Q^2)}{I_{L,g}(x, Q^2)}, \]
\[ \Gamma_{2L}(x, Q^2) = \frac{\Theta_{gg}(x, Q^2)}{I_{L,g}(x, Q^2)}. \quad (18) \]

In order to find solution of the DGLAP equation with the derivative of \( F_L \) we use Eq.(16) in (11) as
\[ \frac{\partial}{\partial \ln Q^2} \left[ \frac{1}{I_{L,g}(x, Q^2)} \left( F_L(x, Q^2) - I_{L,q}(x, Q^2) F_2(x, Q^2) \right) \right] \]
\[ + \frac{1}{I_{L,g}(x, Q^2)} \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} - \frac{\partial I_{L,g}(x, Q^2)}{\partial \ln Q^2} F_2(x, Q^2) \]
\[ - I_{L,q}(x, Q^2) \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} - \Theta_{gq}(x, Q^2) F_2(x, Q^2) \]
\[ - \frac{\Phi_{gg}(x, Q^2)}{I_{L,g}(x, Q^2)} \left( F_L(x, Q^2) - I_{L,q}(x, Q^2) F_2(x, Q^2) \right) = 0 \]
\[ (19) \]
In what follows it is convenient to use the fixed coupling from the above solution as we obtain the DGLAP equation for $F_L$ into $F_2$ by the following form

$$\frac{\partial F_L(x, Q^2)}{\partial \ln Q^2} = T_{LL}(x, Q^2) F_L(x, Q^2) + T_{L2}(x, Q^2) F_2(x, Q^2), \quad (20)$$

where

$$T_{LL}(x, Q^2) = \Phi_{gg}(x, Q^2) + \Theta_{gg}(x, Q^2) \frac{I_{L,g}(x, Q^2)}{I_{L,g}(x, Q^2)},$$
$$T_{L2}(x, Q^2) = I_{L,g}(x, Q^2) [\Phi_{qq}(x, Q^2) - \Phi_{gg}(x, Q^2) - \Theta_{gg}(x, Q^2)] + \Theta_{gg}(x, Q^2) I_{L,g}(x, Q^2). \quad (21)$$

We now pass to the more realistic case with running coupling. In this case the evolution equation for the longitudinal structure function has the form

$$\frac{\partial F_L(x, Q^2)}{\partial \ln Q^2} = \Gamma_{LL}(x, Q^2) F_L(x, Q^2) + \Gamma_{L2}(x, Q^2) F_2(x, Q^2), \quad (22)$$

where

$$\Gamma_{LL}(x, Q^2) = T_{LL}(x, Q^2) + \frac{\partial}{\partial \ln Q^2} \ln I_{L,g}(x, Q^2),$$
$$\Gamma_{L2}(x, Q^2) = T_{L2}(x, Q^2) + I_{L,g}(x, Q^2) \times \frac{\partial}{\partial \ln Q^2} \ln I_{L,g}(x, Q^2). \quad (23)$$

The final form of the DGLAP equations for the structure functions, which takes into account all the modifications mentioned above, is

$$\frac{\partial}{\partial \ln Q^2} \left( \begin{array}{c} F_2(x, Q^2) \\ F_L(x, Q^2) \end{array} \right) = \left( \begin{array}{cc} \Gamma_{22} & \Gamma_{2L} \\ \Gamma_{L2} & \Gamma_{LL} \end{array} \right) \times \left( \begin{array}{c} F_2(x, Q^2) \\ F_L(x, Q^2) \end{array} \right). \quad (24)$$

### III. Determining the Longitudinal Structure Function

One can rewrite Eq.(17) related to the proton structure function $F_2(x, Q^2)$ and its derivative with respect to $\ln Q^2$, as we have

$$F_L(x, Q^2) = \frac{1}{\Gamma_{2L}} \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} - \frac{\Gamma_{22}}{\Gamma_{2L}} F_2(x, Q^2). \quad (25)$$

This relation (i.e., Eq.(25)) helps to estimate the proton longitudinal structure function at high order corrections to the running coupling constant. However, some available methods used for the solution of similar equation have been derived and demonstrated over other techniques in the literature [25-27].

In what follows we observe that the longitudinal structure function which is one of the direct observables can be determined from the proton structure function $F_2$ \(^1\).

With this relation it is possible to explore the structure of $F_L(x, Q^2)$ in the low $x$ range. At low $x$, measurements of this observable are very scarce. A first measurement of $F_2$ at low $x$ at HERA has been performed in Ref.[3].

The LHeC will extend the pioneering H1 measurement into a region of much smaller $x$ and higher $Q^2$. In the LHeC design report [5], a simulation study at low $x$ for the longitudinal structure function has been performed in considerable detail. This has recently been updated [28], with still enlarged luminosity and improved detector systems. The $F_L$ measurements with the LHeC will reach a precision of a few per cent. They are considered to be extended down to $x$ below $10^{-6}$ with the FCC-he.

We now employ the $F_2$ parameterization [11] in Eq.(25) which obtained from a combined fit of the HERA data [29] with $x \leq 0.1$ and $W \geq 25$GeV. This parametrization describes fairly good the available experimental data on the proton structure function in agreement with the Froissart [30] bound behavior. The explicit expression for the $F_2$ parametrization [11] is given by the following form

$$F_2^p(x, Q^2) = D(Q^2)(1 - x)^n \sum_{m=0}^{\infty} A_m(Q^2)L^m, \quad (26)$$

where

$$A_0(Q^2) = a_0 + a_0 \ln(1 + \frac{Q^2}{\mu^2}),$$
$$A_1(Q^2) = a_1 + a_1 \ln(1 + \frac{Q^2}{\mu^2}) + a_1 \ln^2(1 + \frac{Q^2}{\mu^2}),$$
$$A_2(Q^2) = a_2 + a_2 \ln(1 + \frac{Q^2}{\mu^2}) + a_2 \ln^2(1 + \frac{Q^2}{\mu^2}),$$
$$D(Q^2) = \frac{Q^2(\frac{Q^2}{\mu^2})}{(Q^2 + M^2)^2},$$
$$L^m = \ln^m(\frac{1}{x} \frac{Q^2}{\mu^2}). \quad (27)$$

Here $M$ and $\mu^2$ are the effective mass and scale factor respectively. The fixed parameters are defined by the

\(^1\) In the formalism here applied, both $F_2$ and $F_L$ are related, at small $x$ mainly through the gluon distribution. In general, however, the two are independent observables. Precision measurements of $F_2$ and $F_L$ as are envisaged for the LHeC, have the main goal to test their relation with maximum accuracy because failure to relate the two would indicate a breakdown of the conventional QCD formalism.
Block-Halzen [31] fit to the real photon-proton cross section. The additional parameters with their statistical errors are given in Table I. Eventually, inserting Eq.(26) and its derivative into (25) one obtains

\[ F_L(x, Q^2) = \mathcal{R}(F_{i,j}(x, Q^2)) D(Q^2)(1-x)^n \]

\[ \sum_{m=0}^{2} A_m(Q^2)L^m, \quad (28) \]

where

\[ \mathcal{R}(F_{i,j}(x, Q^2)) = \frac{1}{\Gamma_{2L}} \left[ \frac{\partial \ln D(Q^2)}{\partial \ln Q^2} \right] \]

\[ + \frac{\partial \ln (\sum_{m=0}^{2} A_m(Q^2)L^m)}{\partial \ln Q^2} - \frac{\Gamma_{22}}{\Gamma_{2L}} \quad (29) \]

We observe that the longitudinal structure function is dependent on the proton structure function in the same form as Eq.(26) with the Froissart boundary condition which evaluated in terms of the function \( \mathcal{R} \).

IV. DETERMINING THE RATIO \( \sigma_{\tau}/F_2 \)

Considering the expressions (3) and (28) we can estimate the ratio \( \sigma_{\tau}/F_2 \) for the longitudinal structure function. This ratio is given by

\[ \frac{\sigma_{\tau}(x, Q^2)}{F_2(x, Q^2)} = 1 - \Delta(x, y, Q^2), \quad (30) \]

where

\[ \Delta(x, y, Q^2) = f(y)\mathcal{R}(F_{i,j}(x, Q^2)). \quad (31) \]

Here \( f(y) = \frac{y^2}{Y_{\tau}} \) and \( \mathcal{R} \) is defined as

\[ \mathcal{R}(F_{i,j}(x, Q^2)) = \frac{1}{\Gamma_{2L}} \left[ \frac{\partial \ln F_2(x, Q^2)}{\partial \ln Q^2} \right] \]

\[ - \frac{\Gamma_{22}}{\Gamma_{2L}} \quad (32) \]

Inserting in Eq.(32) the \( F_2 \) parameterized (26), we get

\[ \mathcal{R}(F_{i,j}(x, Q^2)) = \text{Eq.}(29). \quad (33) \]

It may be interesting to confront our results for the effective exponent with the properties of the power-like behavior of the proton structure function. H1 Collaboration [32] reports a measurement of the exponent \( \lambda(x, Q^2) \) in the low \( x \) domain of deeply inelastic positron-proton scattering. \( \lambda(x, Q^2) \) was found to be independent of \( x \) and to increase linearly with \( \ln Q^2 \) for \( 10^{-5} \leq x \leq 0.01 \) and \( Q^2 \geq 1.5 \text{ GeV}^2 \). Indeed the function \( \lambda(Q^2) \) was determined from fits of the form 

\[ F_2(x, Q^2) = c(Q^2)x^{-\lambda(Q^2)} \]

to the H1 data. Where the coefficients \( c(Q^2) \) are approximately independent of \( Q^2 \) and \( \lambda(Q^2) \) rises linearly with \( \ln Q^2 \) as

\[ \lambda(Q^2) = a_{\lambda} \ln(Q^2/\Lambda^2), \quad (34) \]

with

\[ a = 0.048 \pm 0.0013(\text{stat}) \pm 0.0037(\text{syst}). \]

Also this behavior for the effective exponent was given in Refs. [33] and [34] by the following forms respectively

\[ \lambda_{eff}^s(Q^2) = 0.13 + 0.1 \left( \frac{Q^2}{10} \right)^{0.35}, \quad (35) \]

\[ \lambda_{phn}^s(Q^2) = 0.329 + 0.1 \log \left( \frac{Q^2}{90} \right), \quad (36) \]

Now let us use this behavior for evolution of \( \frac{\partial \ln F_2(x, Q^2)}{\partial \ln Q^2} \) in accordance with the effective exponents [32-35], as we have

\[ \frac{\partial \ln F_2(x, Q^2)}{\partial \ln Q^2} \approx -\frac{d\lambda(Q^2)}{d\ln Q^2} \ln x, \quad (37) \]

where the coefficients are approximately independent of \( Q^2 \) values. Therefore the \( \mathcal{R} \) function from the effective exponent is found to be

\[ \mathcal{R}(F_{i,j}(x, Q^2)) = \frac{1}{\Gamma_{2L}} \left[ -\frac{d\lambda(Q^2)}{d\ln Q^2} \ln x \right] - \frac{\Gamma_{22}}{\Gamma_{2L}}. \quad (38) \]

In order to find the ratio \( \sigma_{\tau}/F_2 \), the \( F_2 \) parameterization and effective exponent methods are used which our conclusions are presented by the following forms respectively

\[ \frac{\sigma_{\tau}(x, Q^2)}{F_2(x, Q^2)} = 1 - \frac{y^2}{Y_{\tau}} \frac{1}{\Gamma_{2L}} \left[ \frac{\partial \ln D(Q^2)}{\partial \ln Q^2} \right] \]

\[ + \frac{\partial \ln (\sum_{m=0}^{2} A_m(Q^2)L^m)}{\partial \ln Q^2} - \frac{\Gamma_{22}}{\Gamma_{2L}} \quad (39) \]

and

\[ \frac{\sigma_{\tau}(x, Q^2)}{F_2(x, Q^2)} = 1 - \frac{y^2}{Y_{\tau}} \frac{1}{\Gamma_{2L}} \left[ -\frac{d\lambda(Q^2)}{d\ln Q^2} \ln x \right] - \frac{\Gamma_{22}}{\Gamma_{2L}}. \quad (40) \]

V. HIGHER TWIST CORRECTION

In a wide kinematic region in terms of \( x \) and \( Q^2 \), one can describe the deeply inelastic structure functions using leading-twist corrections in QCD. At low \( Q^2 \) values there are constraints on the structure function \( F_i(x, Q^2) \) which follow from eliminating the kinematical singularities at \( Q^2 = 0 \) from the hadronic tensor \( W_{\mu\nu} \). In this region the higher twists (HT) concept introduced, which the operator product expansion leads to the representation [2]

\[ F_2(x, Q^2) = \sum_{n=0}^{\infty} \frac{C_n(x, Q^2)}{(Q^2)^n}. \quad (41) \]
where the function $C_0$ refers to as leading twist (LT) and $C_{>1}$ refers to as higher twist (HT). Higher twist corrections emerge both in the region of large and small values of $x$. These corrections arise from the struck proton's interaction with target remnants where reflecting confinement. Introduction of higher-twist terms is one possible way to extend the DGLAP framework to low $Q^2$ values. Indeed the conventional DGLAP evolution does not describe the DIS data in the low $x - Q^2$ region very well [36]. It is customary to correct the leading-twist structure function by adding a sentence that is inversely related to $Q^2$. The higher-twist effects parameterize in the form of a phenomenological unknown function, and then they obtain the values of the unknown parameters using the fitting to the experimental data [37-38]. The phenomenological power correction to the structure function from the HT effects is considered by the following form

$$F_2(x,Q^2) = F_2^{LT}(x,Q^2)(1 + \frac{C_{HT}(x)}{Q^2}), \quad (42)$$

where the $F_2^{LT}$ is the leading twist contribution to the structure function $F_2$ and the higher-twist coefficient function $C_{HT}(x)$ is determined from fit to the data. Here $C_{HT} = 0.12 \pm 0.07 \text{ GeV}^2$ is a free parameter [37-38]. Note that in such a parametrization the power correction does not depend on $x$. Recently the HT coefficient function parameterized has been reported as follows [39]

$$C_{HT}(x) = h_0 x^{h_1} (1 + h_2 x).$$

The $h_i\{i = 0, 1, 2\}$ parameters, where represent the HT effects in diffractive DIS in the perturbative domain, fixed at the initial scale $Q^2_0$.

In the following we study on the determination of the higher twist contributions in deeply-inelastic structure function [40-41]. To better illustrate our analysis for the longitudinal structure function at low $Q^2$ values, we added a higher twist term in the description of the parameterization of $F_2^{BDH}$ [11]. To elaborate further, $F_2$ can be expressed in terms of the higher twist coefficients as

$$F_2 = D(Q^2)(1 - x)^n(1 + \frac{C_{HT}}{Q^2}) \sum_{m=0}^{2} A_m(Q^2) L^m. \quad (43)$$

Now we present an analytical analysis equation for the longitudinal structure function with considering the HT effects. In this case a correction of the approximate form $(C_{HT}/Q^2)F_2(x,Q^2)$ is expected. For $F_L(x,Q^2)$ we have

$$F_L(x,Q^2) = R_{HT}(F_{i,j}(x,Q^2)) D(Q^2)(1 - x)^n \sum_{m=0}^{2} A_m(Q^2) L^m. \quad (44)$$

where

$$R_{HT}(F_{i,j}(x,Q^2)) = \frac{1}{\Gamma_{2L}} \left(1 + \frac{C_{HT}}{Q^2}\right) \frac{\partial \ln D(Q^2)}{\partial \ln Q^2}$$

$$+ \frac{\partial \ln (\sum_{m=0}^{2} A_m(Q^2) L^m)}{\partial \ln Q^2} - \frac{\Gamma_{22}}{\Gamma_{2L}} \frac{C_{HT}}{Q^2} \frac{1 + \Gamma_{22}}{\Gamma_{2L}}. \quad (45)$$

We also represent a simple form of the HT correction to the ratio $\sigma_t/F_2$ by the following form

$$\frac{\sigma_t(x,Q^2)}{F_2(x,Q^2)} = 1 - f(y) \frac{\partial \ln F_2(x,Q^2)}{\partial \ln Q^2} - \frac{\Gamma_{22}}{\Gamma_{2L}} \frac{C_{HT}}{Q^2} \left\{1 + f(y) \frac{\partial \ln F_2(x,Q^2)}{\partial \ln Q^2} - \frac{1}{\Gamma_{2L}} \frac{\partial \ln F_2(x,Q^2)}{\partial \ln Q^2} \right\}. \quad (46)$$

**VI. ORDER OF ANALYSIS**

The strong coupling constant, $\alpha_s(Q^2)$, is related with the $\beta$-function as [42]

$$\beta(\alpha_s) = -\frac{\beta_0}{4\pi} \alpha_s^2 - \frac{\beta_1}{16\pi^2} \alpha_s^3 - \frac{\beta_2}{64\pi^2} \alpha_s^4 + \mathcal{O}(\alpha_s^5), \quad (47)$$

where

$$\beta_0 = 11 - \frac{2}{3} N_f,$$

$$\beta_1 = 102 - \frac{38}{3} N_f,$$

$$\beta_2 = \frac{2857}{6} - \frac{6673}{18} N_f + \frac{325}{54} N_f^2,$$

are one-loop, two-loop and three-loop corrections to the QCD $\beta$-function. The running coupling constant has the following forms at leading-order (LO) up to next-to-next-to-leading order (NNLO) analysis respectively

$$\alpha_s^{LO} = \frac{4\pi}{\beta_0 t},$$

$$\alpha_s^{NLO} = \frac{4\pi}{\beta_0 t} \left[1 - \frac{\beta_1 \ln t}{\beta_0^2 t} \right],$$

$$\alpha_s^{NNLO} = \frac{4\pi}{\beta_0 t} \left[1 - \frac{\beta_1 \ln t}{\beta_0^2 t} + \frac{1}{(\beta_0 t)^2} \left(\frac{\beta_1}{\beta_0} \right)^2 \left(\ln^2 t - \ln t + 1 + \frac{\beta_2}{\beta_0} \right) \right]. \quad (48)$$

The variable $t$ is defined as $t = \ln(\frac{Q^2}{\Lambda^2})$ and $\Lambda$ is the QCD cut-off parameter for each heavy quark mass threshold as we take the $N_f = 4$ for $m_c^2 < \mu^2 < m_b^2$. The $\Lambda$ parameter has been extracted from the running coupling $\alpha_s$ normalized at $Z$-boson mass. The running coupling constant is $\alpha_s(M_Z^2) = 0.116$ [11,43-44].
The parton-parton splitting function $P_{ij}(i,j=q,g)$ [17] and the coefficient functions $C_{L,(q,g)}$ [16] can be expressed as

$$P_{ij}(x,Q^2) = \frac{\alpha_s(Q^2)}{4\pi} p_{ij}^{(0)}(x) + \frac{\alpha_s(Q^2)}{4\pi} p_{ij}^{(1)}(x) + \frac{\alpha_s(Q^2)}{4\pi} p_{ij}^{(2)}(x) + \mathcal{O}(p_{ij}^{(3)}(x)), \quad (49)$$

and

$$C_{L,(q,g)}(x,Q^2) = \frac{\alpha_s(Q^2)}{4\pi} c_{L,(q,g)}^{(0)}(x) + \frac{\alpha_s(Q^2)}{4\pi} c_{L,(q,g)}^{(1)}(x) + \mathcal{O}(c_{L,(q,g)}^{(3)}(x)) \quad (50)$$

where $n = \{0, 1, 2\}$ are LO, NLO and NNLO splitting and coefficient functions respectively.

VII. RESULTS AND DISCUSSIONS

The singlet structure and gluon distribution functions are controlled by Pomeron exchange. The exponent $\lambda_s$ and $\lambda_g$ are found to be $\simeq 0.33$ and $\simeq 0.43$ respectively [45-46]. Recently, authors in Ref.[47] presented a tensor-Pomeron model where it is applied to low-x deep inelastic lepton-nucleon scattering and photoproduction processes. In this model, in addition to the soft tensor Pomeron interct determined is made to be 0.3008($^{+0.073}_{-0.048}$) with the latest HERA data for $x < 0.01$.

The data reported by the H1 collaboration [3] provide the average of the reduced cross section and the longitudinal structure function, in a range of the kinematical variables $2.910^{-5} < x < 0.01$ and $1.5 \text{ GeV}^2 < Q^2 < 120 \text{ GeV}^2$, and extends to high inelasticity up to $y = 0.85$. This measurement performed by H1 collaboration for neutral current $e^+p$ scattering cross section using data at proton beam energies of $E_p = 920 \text{ GeV}$, $575 \text{ GeV}$, and $460 \text{ GeV}$ and for the longitudinal structure function using data at proton beam energies of $E_p = 920 \text{ GeV}$ and $820 \text{ GeV}$ respectively. The lepton beam energy used at this experiment is $E_e = 27.6 \text{ GeV}$.

To determine $F_L$, we use Eq.(28) which corresponds to the parameterization of $F_2$. The results for the longitudinal structure function are presented in Fig.1 and compared with the H1 data [3] as accompanied with total errors. In this figure and the rest, the error bands represent the uncertainty estimation coming from the parameterization of $F_2$. Table I contain the uncertainties in the present work. As can be seen in this figure, the results are comparable with the H1 data in the interval $1.5 \text{ GeV}^2 < Q^2 < 120 \text{ GeV}^2$. At all $Q^2$ values, the extracted longitudinal structure functions are in good agreement with the simulated data were published.

Recently simulated data for $F_L$ measurement published by the LHeC study group [28]. In this simulated method, the $F_L$ values are what one may expect but they are in themselves meaningless. Here the errors yet have an absolute meaning. That means, if the value of $F_L$ is different, for example scaled by a factor $k$, then $F_L \rightarrow k$ times $F_L$. However, delta $F_L$ (i.e. statistical and total) stays the same. Here we use our predictions for the longitudinal structure function with LHeC uncertainties Delta $F_L$. In Fig.2 these comparisons are shown in the interval $2.5 \text{ GeV}^2 \leq Q^2 \leq 2000 \text{ GeV}^2$ as described in [28].

In Fig.3 the ratio of the longitudinal to transverse cross sections $R(x,Q^2) = \sigma_L(x,Q^2)/\sigma_T(x,Q^2)$ is plotted in a wide range of $Q^2$ values. This ratio is expressed in terms of the longitudinal-to-transverse ratio of structure functions as defined by

$$R(x,Q^2) = \frac{F_L(x,Q^2)}{F_T(x,Q^2) - F_L(x,Q^2)} = \frac{R(f_{ij}(x,Q^2))}{1 - R(f_{ij}(x,Q^2))} = \left\{ \begin{array}{l} \text{Eq.}(29) \text{ parameterization of } F_2 \text{ effective exponent} \end{array} \right\}
$$

In this figure we present this ratio in comparison with the color dipole model (CDM) results [48-51] and experimental data [52-53]. The value of the ratio cross sections $R$ predicted to be 0.5 or 0.375 related to the color-dipole cross sections in Refs.[48-51]. H1 and ZEUS collaborations in Refs.[52-53] show that the ratio $R$ is found to be $R = 0.260 \pm 0.050$ at $3.5 \leq Q^2 \leq 45 \text{ GeV}^2$ and $0.105 \pm 0.037$ at $5 \leq Q^2 \leq 110 \text{ GeV}^2$ respectively. In this figure, we compared our results with the other results mentioned in the literature [48-53]. We observe that the behavior of $R$ is very little dependent on $x$ and $Q^2$ in a wide range of $x$ and $Q^2$ values. This behavior is observable in comparison with the experimental data. In figures 4 and 5, the ratio of $R$ have been depicted at fixed value of the center-of-mass energy $s$ (i.e., $\sqrt{s} = 1.3 \text{ TeV}$). As can be seen in these figures, the results are comparable with other constant values in the interval $0.1 < y < 0.5$. In Fig.5 we observe that the ratio $R$ for $\sqrt{s} = 1.3 \text{ TeV}$ is consistent with a constant behavior with respect to inelasticity $y$ for fixed values of $Q^2$.

The determination of the structure function $F_L$ can be used to determine the reduced cross section $\sigma_T$. The $Q^2$-evolution results of the reduced cross section $\sigma_T$ are depicted in Fig.6. The NNLO results compared with the H1 data [3] correspond to the parameterization of $F_2$ [11] and the effective exponent [32] respectively. In both methods ($F_2$ parameterized and effective exponent) we find that the results are comparable with the H1 data at fixed value of the inelasticity $y = 0.49$ and at a
center-of-mass energy $\sqrt{s} = 225$ GeV. This method persuades us that the obtained results in the NNLO approximation can be pertinent in future analysis of the ultra-high energy neutrino data. The result of this study is shown in Fig.7. The center-of-mass energy $\sqrt{s} = 1.3$ TeV used in accordance with the LHeC center-of-mass energy. Also the averaged parameter $y$ is constrained by the equality $<y> = 0.49$. The longitudinal structure function and the reduced cross section are predicted in this energy. The longitudinal structure function accompanied with the statistical errors of the parameterization of $F_2$ at $2.5 \leq Q^2 \leq 2000$ GeV$^2$ has been shown in this figure (i.e., Fig.7). Also the values of the reduced cross section at $\sqrt{s} = 1.3$ TeV and at $y = 0.49$ obtained with the parameterization of $F_2$ and effective exponent in this figure (i.e., Fig.7) in a wide range of $Q^2$ values. As can be seen, the depletion and enhancement in these obtained results reflect the behavior of experimental data and this is comparable with the HERA data. In Fig.8, our reduced cross sections have been shown as a function of $Q^2$ values at low $x$ values with and without the HT corrections. The effect of the HT contribution to $\sigma_r$ can be seen from this figure. Indeed, the HT corrections lead to large corrections for low $Q^2$ values. We compare our NNLO prediction for the reduced cross section with the H1 data [3] for some selected values of low $Q^2$ at a center-of-mass energy $\sqrt{s} = 319$ GeV and a fixed inelasticity value $y = 0.8$. From the data versus model comparisons, deviation between our results with and without the HT corrections can be clearly be observed. Also, the results have been depicted with and without HT corrections at fixed value of the invariant mass $W$ (i.e, $W = 230$ GeV) at NNLO analysis. As can be seen in this figure, the results are comparable with the H1 data [54] at all $Q^2$ values. From the data versus theory comparisons, deviation between our results with and without the HT corrections at low values of $Q^2$ can be observed. One can see that the differences between theory and data are decreased by including HT effects in the analysis.

In this work, we also calculated the longitudinal structure function and reduced cross section using the effective exponent measured by the H1 collaboration. At low $x$, the exponent $\lambda$ has been observed to increase linearly with $\ln Q^2$. In this regard, data has been collected in the kinematic range $3.10^{-5} \leq x \leq 0.2$ and $1.5 \leq Q^2 \leq 150$ GeV$^2$. To determine $F_L$ and $\sigma$ at any arbitrary $Q^2$ scale, we only need to know the $F_2$ parameterization and its derivative with respect to $\ln Q^2$, which is provided in the kinematic range $10^{-6} \leq x \leq 0.1$ and $0.1 \leq Q^2 \leq 5000$ GeV$^2$. We observed that the general solutions are in satisfactory agreements with the available experimental data at a center-of-mass energy $\sqrt{s} = 225$ GeV and a fixed value of inelasticity $y = 0.49$. Our analysis is also enriched with the higher twist (HT) corrections to the reduced cross section at a center-of-mass energy $\sqrt{s} = 319$ GeV and a fixed value of inelasticity $y = 0.8$, which extend to small values of $Q^2$. It has been demonstrated that the HT terms are required for kinematic coverage. This persuades us that the obtained results can be extended to high energy regime in new colliders (like in the proposed LHeC and FCC-eh colliders). The $F_L$ expectations, as for LHeC, are much more precise than the initial H1 measurement and extend to lower $x$ and higher $Q^2$. These measurements will shed light on the parton dynamics at small $x$. When confronted with the DGLAP based predictions here derived, one will explore the evolution dynamics deeply and more reliably than HERA measurements did allow.

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VIII. Summary

In conclusion, we have computed the longitudinal structure function $F_L$ of the proton using the DGLAP independent evolution equations. The obtained explicit expression for the longitudinal structure function is determined by parameterized the transversal proton structure function. Then we present a further development of the method of extraction of the reduced cross section from the $F_2$ and $F_L$ parameterization. The calculations are consistent with the H1 data from HERA collider.
TABLE I: The effective Parameters at low $x$ for $0.15 \text{ GeV}^2 < Q^2 < 3000 \text{ GeV}^2$ provided by the following values. The fixed parameters are defined by the Block-Halzen fit to the real photon-proton cross section as $M^2 = 0.753 \pm 0.068 \text{ GeV}^2$ and $\mu^2 = 2.82 \pm 0.290 \text{ GeV}^2$.

| parameters | value |
|------------|-------|
| $a_{00}$   | $2.550 \times 10^{-1} \pm 1.60 \times 10^{-2}$ |
| $a_{01}$   | $1.475 \times 10^{-1} \pm 3.025 \times 10^{-2}$ |
| $a_{10}$   | $8.205 \times 10^{-4} \pm 4.62 \times 10^{-5}$ |
| $a_{11}$   | $-5.148 \times 10^{-2} \pm 8.19 \times 10^{-3}$ |
| $a_{12}$   | $-4.725 \times 10^{-3} \pm 1.01 \times 10^{-3}$ |
| $a_{20}$   | $2.217 \times 10^{-3} \pm 1.42 \times 10^{-4}$ |
| $a_{21}$   | $1.244 \times 10^{-2} \pm 8.56 \times 10^{-4}$ |
| $a_{22}$   | $5.958 \times 10^{-2} \pm 2.32 \times 10^{-3}$ |
| $n$        | $11.49 \pm 0.99$ |
| $\lambda$  | $2.430 \pm 0.153$ |
| $\chi^2$(goodness of fit) | 0.95 |

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FIG. 1: The longitudinal structure function extracted at NNLO analysis in comparison with the H1 data [3] as accompanied with total errors. The results are presented at fixed value of the inelasticity $y$ ($y = 0.49$). Combined data for $E_p = 460$ and 575 GeV give $\sqrt{s} = 225$ GeV. The band corresponds to the uncertainty in the parameterization of $F_2$ in [11].
FIG. 2: The longitudinal structure function extracted at NNLO analysis in comparison with central our data as accompanied with the LHec total errors as described in [28]. The results are presented in the interval values of $Q^2$ (i.e., $2.5\text{ GeV}^2 \leq Q^2 \leq 2000\text{ GeV}^2$). The band corresponds to the uncertainty in the parameterization of $F_2$ in [11].
FIG. 3: Predictions of the ratio $R(x, Q^2)$ over a wide range of $Q^2$ values (i.e., $2.5 \leq Q^2 \leq 2000$ GeV$^2$) compared with other results. Dash, Dot, Dash-Dot and Dash-Dot Dot lines are the constant values in [48-50, 52-53].

FIG. 4: Ratio $R(x, Q^2)$ plotted as a function of $x$ variable at $\sqrt{s} = 1.3$ TeV in a wide range of $0.1 < y < 0.5$ compared with the dipole upper bounds [48-50, 52-53].
FIG. 5: The same as Fig.4 for the ratio $R$ vs $Q^2$.

FIG. 6: The reduced cross section extracted at NNLO analysis in comparison with the H1 data [3] as accompanied with total errors. The results are presented at fixed value of the inelasticity $y$ ($y = 0.49$). Combined data for $E_p = 460$ and 575 GeV give $\sqrt{s} = 225$ GeV. These results are based on two methods: the parameterization of $F_2$ [11] and the effective exponent [33].
FIG. 7: The longitudinal structure function (Left: $F_L$) and the reduced cross section (Right: $\sigma_r$) extracted at NNLO analysis at $\sqrt{s} = 1.3$ TeV. The averaged value of inelasticity has the value $y = 0.49$.

FIG. 8: The reduced cross section extracted at NNLO analysis in comparison with the H1 data [3] at low $Q^2$ values with and without the HT corrections. The results are presented at $y = 0.8$ and $\sqrt{s} = 319$ GeV. The band corresponds to the uncertainty in the parameterization of $F_2$ in [11].
FIG. 9: The longitudinal structure function extracted at NNLO analysis with and without HT corrections in comparison with the H1 data [54] as accompanied with total errors. The results are presented at fixed value of the invariant mass $W$ ($W = 230$ GeV) at low values of $x$. 