HAUSDORFF DIMENSION FOR THE IMAGES OF FELLER PROCESSES

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Abstract. Let \((X_t)_{t\geq 0}\) be a Feller process generated by a pseudo-differential operator whose symbol satisfies \(\|p(\cdot, \xi)\|_{\infty} \leq c(1 + |\xi|^2)\) and \(p(\cdot, 0) \equiv 0\). The Hausdorff dimension of the set \(\{X_t : t \in E\}\) for any analytic set \(E \subset [0, \infty)\) is almost surely bounded above by \(\beta_{\infty} \dim H E\), and for a large class of examples we establish the lower bound \(\delta_{\infty} \dim H E\) where

\[
\beta_{\infty} := \inf \left\{ \delta > 0 : \lim_{|\xi| \to \infty} \sup_{|\xi| \leq |\xi|} \sup_{z \in \mathbb{R}^d} |p(z, \eta)| = 0 \right\},
\]

\[
\delta_{\infty} := \sup \left\{ \delta > 0 : \lim_{|\xi| \to \infty} \inf_{z \in \mathbb{R}^d} \Re p(z, \xi) = \infty \right\}.
\]

Our result extends the dimension estimates for Lévy processes of Blumenthal and Getoor (1961) and Millar (1971).

Keywords: Feller process, pseudo-differential operator, symbol, Hausdorff dimension, Blumenthal–Getoor index

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1. Background and Main Result

A Feller process \((X_t)_{t\geq 0}\) with state space \(\mathbb{R}^d\) is a strong Markov process such that the associated operator semigroup \((T_t)_{t\geq 0}\),

\[
T_t u (x) = E^x (u (X_t)), \quad u \in C_{c} (\mathbb{R}^d), \; t \geq 0, \; x \in \mathbb{R}^d,
\]

\((C_{c} (\mathbb{R}^d)\) is the space of continuous functions vanishing at infinity) enjoys the Feller property, i.e.

it maps \(C_{c} (\mathbb{R}^d)\) into itself. A semigroup is said to be a Feller semigroup, if \((T_t)_{t\geq 0}\) is a one-parameter semigroup of linear contraction operators \(T_t : C_{c} (\mathbb{R}^d) \to C_{c} (\mathbb{R}^d)\) which is strongly continuous: \(\lim_{t \to 0} \|T_t u - u\|_{\infty} = 0\) for any \(u \in C_{c} (\mathbb{R}^d)\), and has the sub-Markov property: \(0 \leq T_t u \leq 1\) whenever \(0 \leq u \leq 1\).

The infinitesimal generator \((A, D(A))\) of the semigroup \((T_t)_{t\geq 0}\) (or of the process \((X_t)_{t\geq 0}\)) is given by the strong limit

\[
Au := \lim_{t \to 0} \frac{T_t u - u}{t}
\]
on the set \(D(A) \subset C_{c} (\mathbb{R}^d)\) of all \(u \in C_{c} (\mathbb{R}^d)\) for which the above limit exists with respect to the uniform norm. We will call \((A, D(A))\) Feller generator for short.

Let \(C_{c}^\infty (\mathbb{R}^d)\) be the space of smooth functions with compact support. Under the assumption that the test functions \(C_{c}^\infty (\mathbb{R}^d)\) are contained in \(D(A)\), Ph. Courrège

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[6, Theorem 3.4] proved that the generator $A$ restricted to $C_c^\infty(\mathbb{R}^d)$ is a pseudo-differential operator,

$$Au(x) = -p(x, D)u(x) := -\int e^{i(x, \xi)} p(x, \xi) \hat{u}(\xi) \, d\xi, \quad u \in C_c^\infty(\mathbb{R}^d),$$

with symbol $p : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where $\hat{u}$ is the Fourier transform of $u$, i.e. $\hat{u}(x) = (2\pi)^{-d} \int e^{-i(x, \xi)} u(\xi) \, d\xi$. The symbol $p(x, \xi)$ is locally bounded in $(x, \xi)$, measurable as a function of $x$, and for every fixed $x \in \mathbb{R}^d$ it is a continuous negative definite function in the co-variable. This is to say that it enjoys the following Lévy-Khintchine representation,

$$p(x, \xi) = c(x) - i(b(x), \xi) + \frac{1}{2} i^2(b(\xi), \xi) + \int_{\mathbb{R}^d \setminus \{0\}} (1 - e^{i(z, \xi)} + i(z, \xi) 1_{\{|z| \leq 1\}}) \nu(x, dz),$$

where $(c(x), b(x), a(x), \nu(x, dz))_{x \in \mathbb{R}^d}$ are the Lévy characteristics: $c(x)$ is a nonnegative measurable function, $b(x) := (b_j(x)) \in \mathbb{R}^d$ is a measurable function, $a(x) := (a_{jk}(x)) \in \mathbb{R}^{d \times d}$ is a nonnegative definite matrix-valued function, and $\nu(x, dz)$ is a nonnegative, $\sigma$-finite kernel on $\mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d \setminus \{0\})$ such that for every $x \in \mathbb{R}^d$, $\int_{\mathbb{R}^d \setminus \{0\}} (1 + |z|^2) \nu(x, dz) < +\infty$. For details and a comprehensive bibliography we refer to the monographs [10] by N. Jacob and the survey [3]. Since we will only consider the case where $c \equiv 0$, we will from now on use the Lévy triplet $(b(x), a(x), \nu(x, dz))$.

It is instructive to have a brief look at Lévy processes which are a particular but important subclass of Feller processes. Our standard reference for Lévy processes is the monograph by K. Sato [20]. A Lévy process $(Y_t)_{t \geq 0}$ is a stochastically continuous random process with stationary and independent increments. The characteristic exponent or the symbol $\psi : \mathbb{R}^d \to \mathbb{C}$ of a Lévy process is a continuous negative definite function, i.e. it is given by a Lévy-Khintchine formula of the form (1.2) with characteristics $(b, a, \nu(dz))$ which do not depend on $x$.

The notion of Hausdorff dimension is very useful in order to characterize the irregularity of stochastic processes. The Hausdorff dimension of the image sets of a Lévy process has been extensively studied, see [2, 19, 18] and also the survey paper [29, 30] for details. Recall that the Hausdorff dimension of a set $A \subset \mathbb{R}^d$ is the unique number $\lambda$, where the $\lambda$-dimensional Hausdorff measure $\mathcal{H}^\lambda(A)$, defined by

$$\mathcal{H}^\lambda(A) = \sup_{\varepsilon > 0} \inf \left\{ \sum_{n=1}^{\infty} (\text{diam } A_n)^{\lambda} : A_n \text{ Borel, } \bigcup_{n=1}^{\infty} A_n \supset A \text{ and diam } A_n \leq \varepsilon \right\},$$

changes from $+\infty$ to a finite value.

For the study of the Hausdorff dimension for the sample paths of Lévy processes, various indices were introduced in [2, Sections 2, 3 and 5]:

$$\beta'' = \sup \left\{ \delta > 0 : \lim_{|\xi| \to \infty} \frac{\text{Re } \psi(\xi)}{|\xi|^{\delta}} = \infty \right\},$$

$$\beta = \inf \left\{ \delta > 0 : \lim_{|\xi| \to \infty} \frac{|\psi(\xi)|}{|\xi|^{\delta}} = 0 \right\}.$$
Remark 1.1. Let \((Y_t)_{t \geq 0}\) be a \(d\)-dimensional Lévy process with indices \(\beta''\) and \(\beta\) given above. For every analytic set \(E \subset [0, 1]\) we have, almost surely
\[
\min\{d, \beta'' \dim_HE\} \leq \dim_HE \leq \min\{d, \beta \dim_HE\}.
\]

The purpose of this paper is to estimate the Hausdorff dimension for the image of a Feller process. Throughout we will make the following assumptions on the symbol \(p(x, \xi)\):
\[
\|p(\cdot, \xi)\|_\infty \leq c(1 + |\xi|^2) \quad \text{and} \quad p(\cdot, 0) \equiv 0.
\]

The first condition means that the generator has only bounded ‘coefficients’, see, e.g. [24, Lemma 2.1] or [25, Lemma 6.2]; the second condition implies that the Feller process is conservative in the sense that the life time of the process is almost surely infinite, see [23, Theorem 5.2].

We can now state the first result of our paper, which partly extends Remark 1.1.

Theorem 1.2. Let \((X_t)_{t \geq 0}\) be a Feller process with the generator \((A, D(A))\) such that \(C_c^\infty(\mathbb{R}^d) \subset D(A)\), i.e. \(A|_{C_c^\infty(\mathbb{R}^d)} = -p(\cdot, D)\) is a pseudo-differential operator with symbol \(p(x, \xi)\). Assume that the symbol satisfies (1.3). Then, for every bounded analytic set \(E \subset [0, \infty)\),
\[
(1.4) \quad \dim_H X(E) \leq \min\{d, \beta_{\infty} \dim_HE\}
\]
holds almost surely, where the generalized Blumenthal-Getoor indices (at infinity) are given by
\[
\beta_{\infty} := \inf \left\{ \delta > 0 : \lim_{|\xi| \to \infty} \sup_{|y| \leq |\xi|} \sup_{z \in \mathbb{R}^d} \frac{|p(z, \eta)|}{|\xi|^\delta} = 0 \right\}.
\]

Remark 1.3. The index \(\beta_{\infty}\) coincides with the index \(\beta_{\infty}^{\mathbb{R}^d}\) from [3, Remark 5.14b].

The inequality (1.4) with \(E = [0, 1]\) can also be deduced from the variation of sample functions for Feller processes. Recall that, for a \(p \in (0, \infty)\) and a function \(f\) defined on the interval \([0, T]\) and taking values in \(\mathbb{R}^d\), its \(p\)-variation is given by
\[
V_p(f, [0, T]) = \sup \left\{ \sum_{j=0}^{m-1} |f(t_{j+1}) - f(t_j)|^p : 0 = t_0 < t_1 < \ldots < t_m = T, \ m \geq 1 \right\}.
\]

Then, we have the following assertion.

Proposition 1.4. Let \((X_t)_{t \geq 0}\) be a Feller process with the generator \((A, D(A))\) such that \(C_c^\infty(\mathbb{R}^d) \subset D(A)\), i.e. \(A|_{C_c^\infty(\mathbb{R}^d)} = -p(\cdot, D)\) is a pseudo-differential operator with symbol \(p(x, \xi)\). Assume that the symbol satisfies (1.3). Then, for any \(p > \beta_{\infty}\), the \(p\)-variation of the sample function \((X_t)_{t \geq 0}\) is finite almost surely. In particular,
\[
\mathcal{H}^p X([0, 1]) \leq 2^{p-1}V_p(X, [0, 1]) < \infty
\]
holds almost surely, and so \(\dim_H X([0, 1]) \leq \beta_{\infty}\).

To show the lower bound we need more assumptions on the Feller process.

Theorem 1.5. Let \((X_t)_{t \geq 0}\) be a Feller process in \(\mathbb{R}^d\) with the transition probability density \(p(t, x, y)\), which satisfies
\[
(1.5) \quad p(t, x, y) \leq ct^{-d/\alpha}, \quad t \in (0, 1], \ x, y \in \mathbb{R}^d,
\]
for some \( \alpha \in (0, 2) \). Then, for any analytic set \( E \subset [0, 1] \) we have
\[
\dim_H X(E) \geq (\alpha \land d) \dim_H E.
\]

Let us give a few examples where the conditions of Theorem 1.5 are satisfied.

**Example 1.6.**

(a) For a Lévy process condition (1.5) is easy to check; for example, it holds true the characteristic exponent \( \psi \) satisfies
\[
\Re \psi(\xi) \geq c|\xi|^\alpha, \quad |\xi| > 1.
\]

(b) For a symmetric Markov process condition (1.5) is equivalent to the following Nash type inequality
\[
\|f\|_{L^2(\mathbb{R}^d; dx)}^{2+2\alpha/d} \leq C \left[ D(f, f) + \delta \|f\|_{L^2(\mathbb{R}^d; dx)}^2 \right] \|f\|_{L^1(\mathbb{R}^d; dx)}, \quad f \in C_c^\infty(\mathbb{R}^d),
\]
for some positive constants \( C \) and \( \delta \), where \( D(f, f) = -\langle f, Af \rangle_{L^2(\mathbb{R}^d; dx)} \). See [10, Vol. II, Section 3.6], also the original paper [4] and [1, 26] for more recent developments. The reader can refer to [5, Proposition II.1] for general (non-symmetric) semigroups satisfying (1.5) in terms of functional inequalities.

c) Sufficient conditions when a Lévy type process satisfies (1.5) are given in [12, 13], where the approach relies on the parametrix construction of a Markov process.

Consider the triplet
\[
(b(x), 0, m(x, z) \mu(\text{d}z))
\]
where the functions \( b(\cdot) \) and \( m(\cdot, z) \) are bounded and Hölder continuous with \( m(x, z) \geq c > 0 \) for all \( x, z \in \mathbb{R}^d \); \( \mu \) is the Lévy measure: \( \int_{\mathbb{R}^d} |z|^2 \land 1 \mu(\text{d}u) < \infty \), moreover it satisfies the following condition: There exists \( \beta > 1 \) such that
\[
\sup_{t \in S^d} q^U(t \ell) \leq \beta \inf_{t \in S^d} q^L(t \ell), \quad r \geq 1,
\]
where \( S^d \) is the unit sphere in \( \mathbb{R}^d \), and
\[
q^U(\xi) := \int_{\mathbb{R}^d} (|\xi, z|^2 \land 1) \mu(\text{d}z), \quad q^L(\xi) := \int_{|\xi, z| \leq 1} |\xi, z|^2 \mu(\text{d}z).
\]
Note that the Lévy–Khintchine exponent
\[
q(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi, z)) \mu(\text{d}z),
\]
always satisfies the inequalities \( (1 - \cos 1)q^L(\xi) \leq q(\xi) \leq 2q^U(\xi) \); moreover, we have for large \(|\xi|\) the relations
\[
q(\xi) \asymp q^L(\xi) \asymp q^U(\xi), \quad |\xi| > 1,
\]
i.e. the exponent does not oscillate "too much". In particular, condition (1.9) on the Lévy measure \( \mu(\text{d}z) \) together with the boundedness of \( m(x, z) \) implies that
\[
\inf_{x \in \mathbb{R}^d} \Re p(x, \xi) \geq c|\xi|^\alpha, \quad |\xi| > 1,
\]
holds for \( \alpha = 2/\beta \in (0, 2) \) and \( c > 0 \), see [11, 13]. It was proved in [13] that starting with the Lévy characteristics as in (1.8) there exists a Feller process \( (X_t)_{t \geq 0} \) associated with symbol \( p(x, \xi) \) given by (1.2) and, moreover, this process \( (X_t)_{t \geq 0} \) possesses the transition probability density \( p(t, x, y) \). Similar conditions (for a slightly different Lévy triplet) are given in [12].
d) Condition (1.5) holds true for the transition probability density of the stable-like process associated with the Lévy triplet $(0, 0, |z|^{-1-\alpha(x)} d|z| \tilde{\mu}(x, d\ell))$, where $\ell := z/|z|$ for $z \in \mathbb{R}^d$; here, the index function $\alpha(x)$ and the kernel $\tilde{\mu}(x, d\ell)$ are bounded and continuous such that

$$C_1 \leq \int_{S^d} |(v, \ell)|^{\alpha(x)} \tilde{\mu}(x, d\ell) \leq C_2, \quad v \in S^d, x \in \mathbb{R}^d,$$

holds with some positive constants $C_1$ and $C_2$, see [14, Theorem 5.1]. In this case, $\alpha$ in (1.5) is equal to $\min_{x \in \mathbb{R}^d} \alpha(x)$.

**Remark 1.7.** Note that in the cases a), c) and d) (cf. Example 1.6) the characteristic exponent and the symbol satisfy, respectively, (1.7) and (1.10) where, for d), $\alpha = \min_{x \in \mathbb{R}^d} \alpha(x)$. This allows to state the lower bound in terms of the Blumenthal–Getoor index:

$$\dim_H X(E) \geq (\delta_\infty \wedge d) \dim_H E,$$

where

$$\delta_\infty := \sup \left\{ \delta > 0 : \lim_{|\xi| \to \infty} \inf_{z \in \mathbb{R}^d} \text{Re} p(z, \xi) = \infty \right\}$$

is the generalized Blumenthal–Getoor index at infinity. This index coincides with the index $\delta_\infty^{\mathbb{R}^d}$ from [3, Remark 5.14b].

2. PROOFS

**Proof of Theorem 1.2.** (1) We first claim that for any $p > \beta_\infty$,

$$(2.11) \quad \mathbb{P}^x \left( \sup_{|s-t| \leq h} |X_s - X_t| > u \right) \leq c h u^{-p}$$

holds for all $x \in \mathbb{R}^d$, $h > 0$ and $u \in (0, u_0)$, where $u_0$ and $c$ are two positive constants independent of $h$. By the Markov property, it suffices to verify that for sufficiently small $u > 0$

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}^x \left( \sup_{0 \leq r \leq t-s} |X_r - x| \geq (t-s)^{1/p} u \right) \leq c u^{-p}.$$

Note that, according to [27, Proposition 4.3],

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}^x \left( \sup_{0 \leq r \leq t-s} |X_r - x| \geq u \right) \leq c (t-s) \sup_{z \in \mathbb{R}^d} \sup_{|\xi| \leq 1/u} |p(z, \xi)|.$$

This along with the very definition of $\beta_\infty$ yields the required assertion (2.11).

(2) Now we will follow the proofs of [28, Lemma 4.7 and Proposition 4.8] and [22, Theorem 4] (with some significant modifications) to get the desired upper bound. We first suppose that $\dim_H E < 1$. For any constant $\gamma$ with $\gamma \in (\dim_H E, 1)$, there exists a sequence of balls $\{B(t_{j,k}, h_{j,k})\}_{j,k \in \mathbb{N}}$ such that

$$E \subset \bigcup_{k=1}^\infty B(t_{j,k}, h_{j,k}) \text{ for all } j \in \mathbb{N}, \quad \lim_{j \to \infty} \sup_{k \in \mathbb{N}} h_{j,k} = 0 \text{ and } \sup_{j,k \in \mathbb{N}} \sum_{k=1}^\infty (h_{j,k})^\gamma < \infty.$$

Without loss of generality, we further assume that $h_{j,k} \leq 1/j$ for all $j, k \in \mathbb{N}$. For $j \in \mathbb{N}$, let

$$\Omega_j := \left\{ \omega : \sup_{|s-t| \leq 1/j} |X_s - X_t| \leq u_0 \right\},$$
where \( u_0 \) is the constant in (2.11). Note that \( X(\Omega) \subseteq \bigcup_{k=1}^{\infty} B(X_{t,j,k}, D(t_{j,k}, h_{j,k})) \), where \( D(t, h) = \sup_{|s-t| \leq h} |X_s - X_t| \). Then, for any \( j \geq j_0 \) and \( p > \beta_\infty \),

\[
\sum_{k=1}^{\infty} \mathbb{E}(\gamma^p(t_{j,k}, h_{j,k}) I_{\Omega_{j_0}}) \leq \gamma p \sum_{k=1}^{\infty} \int_{0}^{u_0} \gamma^{p-1} \mathbb{P}(D(t_{j,k}, h_{j,k}) \geq u) \, du
\]

\[
= \gamma p \sum_{k=1}^{\infty} \int_{0}^{u_0} \gamma^{p-1} \mathbb{P} \left( \sup_{|s-t_{j,k}| \leq h_{j,k}} |X_s - X_{t_{j,k}}| \geq u \right) \, du
\]

\[
\leq c \gamma p \sum_{k=1}^{\infty} \int_{0}^{u_0} \gamma^{p-1} (1 \wedge (h_{j,k} u^{-p})) \, du,
\]

where the last inequality follows from (2.11). It is elementary to verify that, up to a constant, the integral in the last term is bounded by

\[
\int_{0}^{h_{j,k}^{1/p}} \gamma^{p-1} \, du + h_{j,k} \int_{h_{j,k}^{1/p}}^{u_0} \gamma^{p-1} \, du \leq c_1 h_{j,k}^\gamma
\]

with some constant \( c_1 \) independent of \( j_0 \) and \( h_{j,k} \). Therefore,

\[
\sup_{j \geq j_0} \sum_{k=1}^{\infty} \mathbb{E}(\gamma^p(t_{j,k}, h_{j,k}) I_{\Omega_{j_0}}) \leq c_2 \sup_{j \geq j_0} \sum_{k=1}^{\infty} h_{j,k}^\gamma < \infty,
\]

which yields \( \sup_{j \geq j_0} \mathbb{E}^x(\mathcal{F}_{\gamma/j}^p(X(\Omega) \cap \Omega_{j_0})) < \infty \). Using monotone convergence we get that

\[
\mathbb{E}^x(\mathcal{F}_{\gamma/j}^p(X(\Omega) \cap \Omega_{j_0})) < \infty \quad \text{and} \quad \mathcal{F}_{\gamma/j}^p(X(\Omega) \cap \Omega_{j_0}) < \infty \quad \text{a.s.}
\]

This implies that \( \dim_H X(\Omega) \leq \gamma p \) almost surely on \( \Omega_{j_0} \). Note that, according to (2.11), we have \( \lim_{j \to \infty} \mathbb{P}(\Omega_j) = 1 \). Letting first \( j_0 \to \infty \), then \( \gamma \to \dim_H E \), and finally \( p \to \beta_\infty \) along countable sequences proves the desired assertion. \( \fbox{1} \)

(3) Next, we consider \( \dim_H E = 1 \). We may assume that \( E = [0, 1] \). In this case, for any \( j \in \mathbb{N} \), we cover \( [0, 1] \) by finitely many set \( E_{j,k} := [(k-1)/j, k/j] \) for \( k = 1, 2, \ldots, j \). Hereafter we will adopt the notations used in part (2). As in the calculations here, we can obtain that for any \( j \geq j_0 \) and \( p > \beta_\infty \),

\[
\sup_{j \geq j_0} \sum_{k=1}^{j} \mathbb{E} \left( D^p \left( \frac{k-1}{j}, \frac{1}{j} \right) I_{\Omega_{j_0}} \right) \leq c p \sup_{j \geq j_0} \sum_{k=1}^{j} \frac{1}{j} = c p
\]

holds with some constant \( c \) independent of \( j_0 \). Then, \( \dim_H X(\Omega) \leq p \) almost surely on \( \Omega_{j_0} \). The required assertion follows by letting \( j_0 \to \infty \) then \( p \to \beta_\infty \). \( \square \)

\( \fbox{1} \) has proven the following conclusion for the upper bound of the Hausdorff dimension for the images of Markov process \((X_t)_{t \geq 0}\) on \( \mathbb{R}^d \): Suppose that there exist positive and finite constants \( H_1 \) and \( \beta \) such that

\[
\mathbb{P} \left( \sup_{|s-t| \leq h} |X_s - X_t| \geq h^{H_1} u \right) \leq c u^{-\beta}
\]

for all \( t \geq 0, h \in (0, h_0) \) and \( u \geq u_0 \), where \( h_0, u_0 \) and \( c \) are positive constants. Then, for every analytic set \( E \subset [0, \infty) \) with \( \dim_H E \leq \beta H_1, \dim_H X(\Omega) \leq d \wedge (\dim_H E/H_1) \) almost surely. The sufficient condition here is completely different from our sufficient condition (2.11), which is key to yield our required upper bounded via the index \( \beta_\infty \).
Proof of Proposition 1.4. (1) According to [16, Theorem 1.3] and [17, Theorem 3], we know that if there exist two positive constants $r_0$ and $C$ such that for all $t > 0$ and $0 < r < r_0$,

$$a(t, r) := \sup_{0 < s \leq t, x \in \mathbb{R}^d} \mathbb{P}^x(\{X_s - x \geq r\}) \leq C\epsilon^\beta r^{-\alpha}$$

with two constants $\alpha > 0$ and $\beta > (3 - \epsilon)/(\epsilon - 1)$, then for any $p > \alpha/\beta$, the $p$-variation of the sample function $(X_t)_{t \geq 0}$ is finite almost surely. It is clear that for any $t, r > 0$,

$$a(t, r) \leq \sup_{x \in \mathbb{R}^d} \mathbb{P}^x(\sup_{s \leq t} |X_s - x| \geq r).$$

Applying [3, Corollary 5.2] yields that

$$a(t, r) \leq c t \sup_{x \in \mathbb{R}^d} \sup_{|\xi| \leq 1/r} |p(x, \xi)|$$

holds for some constant $c > 0$. By the very definition of $\beta_\infty$, for any $p > \beta_\infty$, there exists an $r_0$ small enough such that for any $0 < r \leq r_0$,

$$\sup_{x \in \mathbb{R}^d} \sup_{|\xi| \leq 1/r} |p(x, \xi)| \leq r^{-p},$$

which proves the finiteness of the $p$-variation because of Manstavičius' results mentioned earlier.

(2) For any $\epsilon > 0$, let $\{E_j\}_{j=1}^n$ be a sequence of closed, non-overlapping intervals such that $[0, 1] \subset \bigcup_{j=1}^n E_j$, $E_j \cap E_k = \emptyset$ and $\text{diam } E_j \leq \epsilon$ for all $1 \leq j, k \leq n$. Without loss of generality, we further assume that $\{E_j\}_{j=1}^n$ is arranged in the natural order, i.e. for any $1 \leq j \leq n - 1$, if $s \in E_j$ and $t \in E_{j+1}$, then $s \leq t$. For $1 \leq j \leq n$, let $t_j \in E_j$ be the right endpoint of the interval $E_j$. We may assume that $t_n = 1$. Then,

$$\sum_{j=1}^n \sup_{s, t \in E_j} |X(s) - X(t)|^p \leq \sum_{j=1}^n \left( \sup_{s \in E_j} |X(s) - X(t_j)| + \sup_{t \in E_j} |X(t) - X(t_j)| \right)^p \leq 2^{p-1} \sum_{j=1}^n \sup_{s \in E_j} |X(s) - X(t_j)|^p.$$

Since the paths of a Feller process are right-continuous with finite left-limits, we can choose for every $\eta > 0$ some $\tau_j = \tau^{n, \eta}_{j}(\omega) \in E_j$ such that

$$\sup_{t \in E_j} |X(t) - X(t_j)|^p \leq |X(\tau_j) - X(t_j)|^p + \frac{\eta}{n}.$$

Note that the set of points $\{0, \tau_1, \tau_1, \ldots, \tau_n, t_n\}$ is a partition of $[0, 1]$. Therefore, combining all the estimates above, we arrive at

$$\mathcal{H}_p^\infty(X([0, 1])) \leq \sum_{j=1}^n \sup_{s, t \in E_j} |X(s) - X(t)|^p \leq 2^{p-1} \left( V_p(X, [0, 1]) + \eta \right).$$

Letting $\epsilon \to 0$ and then $\eta \to 0$ yields the assertion. \hfill $\Box$

The proof of Theorem 1.4 relies on several results, which for the reader's convenience we quote below. First, define the $\lambda$-capacity of a Borel set $B \subset \mathbb{R}^d$ as follows:

$$\text{Cap}_\lambda(B) := \left( \inf \left\{ \int_B \int_B |x - y|^{-\lambda} \varpi(dy) \varpi(dx) : \varpi \in \mathcal{M}_1^+(B) \right\} \right)^{-1}$$

(2.12)
Lemma 2.1. If $F \subset \mathbb{R}^d$ is a closed set with strictly positive Hausdorff measure $\mathcal{H}^\lambda(F) > 0$ for some $\lambda > 0$, then $\text{Cap}_\lambda(F) > 0$ for all $\lambda' < \lambda$.

The lemma below is taken from [2, Lemma 2.2], see also [15].

Lemma 2.2. Let $f : \mathcal{X} \mapsto \mathbb{R}^d$ be a measurable function on a metric space $(\mathcal{X}, d(\cdot, \cdot))$, and $E \subset \mathcal{X}$ be a Borel set. If there exists a probability measure $\varpi \in \mathcal{M}_1^+(E)$ such that

$$\int_E \int_E |f(x) - f(y)|^{-\lambda} \varpi(dx)\varpi(dy) < \infty$$

for some $\lambda > 0$, then $\mathcal{H}^\lambda(f(E)) > 0$.

Let $(Y_t)_{t \geq 0}$ be a Markov process in $\mathbb{R}^d$, and

$$\beta'(Y, x) := \sup \{\lambda \geq 0 : \mathbb{E}^x(|Y_t - Y_s|^{-\lambda}) = O(|t-s|^{-1}) \text{ as } t-s \to 0\} ,$$

This index was introduced in [21].

Lemma 2.3. Let $(Y_t)_{t \geq 0}$ be a Markov process with values in $\mathbb{R}^d$, and $E \subset [0,1]$ be an analytic set with Hausdorff dimension $\dim_H E$. Then,

$$\dim_H Y(E) \geq \beta'(Y, x) \dim_H E \quad \mathbb{P}^x - \text{a.e.}$$

Proof. Let $0 < \lambda < \beta'(Y, x)$ and $0 < \alpha < \alpha' < \dim_H E$. We find, using Jensen’s inequality for concave functions, that there exists a constant $C > 0$ such that

$$\mathbb{E}^x |Y_t - Y_s|^{-\lambda} \leq \left(\mathbb{E}^x |Y_t - Y_s|^{-\lambda'}\right)^\alpha \leq C |t-s|^{-\alpha}, \quad |t-s| \leq 1.$$  

Since $\alpha' < \dim_H E$, we have $\mathcal{H}^{\alpha'}(E) = \infty$, where $\mathcal{H}^{\alpha'}(E)$ is the Hausdorff measure of $E$ with dimension $\alpha'$. We will use the following result, proved in [7, p. 489, Corollary]: Let $A \subset \mathbb{R}^d$ be an analytic set with $\mathcal{H}^\lambda(A) = \infty$ for some $\lambda > 0$. Then for any $r > 0$ there exists a closed subset $F_r \subset A$ such that $\mathcal{H}^\lambda(F_r) \geq r$. By this, there exists a closed set $F \subset E$ such that $\Lambda^\alpha(F) > 0$, which implies by Lemma 2.1 that $\text{Cap}_\alpha(F) > 0$ for all $\alpha < \alpha'$. By the definition of the capacity, there exists a probability measure with support on $F$ such that

$$\int_F \int_F |t-s|^{-\alpha} \mu(dt)\mu(ds) < \infty.$$  

Thus, the above inequality, together with (2.16) and the Fubini theorem, gives us

$$\mathbb{E}^x \left(\int_F \int_F |Y_t - Y_s|^{-\alpha} \mu(dt)\mu(ds)\right) < \infty,$$

which in turn yields that

$$\int_F \int_F |Y_t - Y_s|^{-\alpha} \mu(dt)\mu(ds) < \infty \quad \mathbb{P}^x - \text{a.s.}$$

According to Lemma 2.2, we get

$$\Lambda^\alpha(Y(E, \omega)) \geq \Lambda^\alpha(Y(F, \omega)) > 0 \quad \mathbb{P}^x - \text{a.s.}$$

Therefore, we derive the statement of the lemma by passing to the limit as $\alpha \uparrow \dim_H E$ and $\lambda \uparrow \beta'(Y, x)$. \hfill $\square$
Now, we are in a position to present the proof of Theorem 1.4.

**Proof of Theorem 1.4.** By the strong Markov property and (1.5), we have for any \( \lambda < d \wedge \alpha, x \in \mathbb{R}^d \) and \( 0 < s < t \) with \( t - s \leq 1 \)

\[
E^x|X_t - X_s|^{-\lambda} = E^x|X_t - X_s|^{-\lambda} \left[ \mathbb{1}_{\{|X_t - X_s| \leq (t-s)^{1/\alpha}\}} + \mathbb{1}_{\{|X_t - X_s| > (t-s)^{1/\alpha}\}} \right] \\
\leq E^x \left\{ E^x \left[ |X_{t-s} - X_0|^{-\lambda} \mathbb{1}_{\{|X_{t-s} - X_0| \leq (t-s)^{1/\alpha}\}} \right] \right\} + (t-s)^{-\lambda/\alpha} \\
= E^x \int_{|X_s - y| \leq (t-s)^{1/\alpha}} |X_s - y|^{-\lambda} p(t-s, X_s, y) \, dy + (t-s)^{-\lambda/\alpha} \\
\leq c_1 (t-s)^{-\lambda/\alpha} \int_{|X_s - y| \leq (t-s)^{1/\alpha}} \left( (t-s)^{-1/\alpha} |X_s - y| \right)^{-\lambda} (t-s)^{-d/\alpha} \, dy \\
+ (t-s)^{-\lambda/\alpha} \\
= c_2 (t-s)^{-\lambda/\alpha} \int_{|y| \leq 1} |y|^{-\lambda} \, dy + (t-s)^{-\lambda/\alpha} \\
\leq c_3 (t-s)^{-\lambda/\alpha} \\
\leq c_3 (t-s)^{-1}.
\]

Thus, we have the lower estimate for the index \( \beta'(X, x) \) defined in (2.14):

\[ \beta'(X, x) \geq \lambda \quad \mathbb{P}^x \text{– a.e.} \]

Letting \( \lambda \to d \wedge \alpha \) in the inequality above, we prove (1.6) by Lemma 2.3. \( \square \)

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