Coarsening with a frozen vertex

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Abstract

In the standard nearest-neighbor coarsening model with state space \( \{-1, +1\}^{\mathbb{Z}^2} \) and initial state chosen from symmetric product measure, it is known (see [2]) that almost surely, every vertex flips infinitely often. In this paper, we study the modified model in which a single vertex is frozen to +1 for all time, and show that every other site still flips infinitely often. The proof combines stochastic domination (attractivity) and influence propagation arguments.

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1 Introduction

As in our earlier paper [1], we study and compare the long time behavior of two continuous time Markov coarsening models with state space \( \Omega = \{-1, +1\}^{\mathbb{Z}^d} \). One, \( \sigma(t) \), is the standard model in which at time zero \( \{\sigma_x(0) : x \in \mathbb{Z}^d\} \) is an i.i.d. set with \( \theta \equiv P(\sigma_x(0) = +1) = 1/2 \) and then vertices update to agree with a strict majority of their \( 2d \) nearest neighbors or, in case of a tie, choose their value by tossing a fair coin. The modified model, \( \sigma'(t) \), is the same except that \( \sigma' \) at the origin \((0,0,...,0)\) is frozen to +1 for all \( t \geq 0 \).

For \( d = 2 \), it is an old result [2] that in the standard \( \sigma(t) \) model, almost surely, every vertex changes sign infinitely many times as \( t \to \infty \). The main result of this paper (see Theorem 2.7) is that the same is true for the frozen model \( \sigma'(t) \) on \( \mathbb{Z}^2 \). It is believed (see, for example, Sec. 6.2 of [3]), but not proved, that the \( d = 2 \) behavior of \( \sigma \) remains valid at least for some values of \( d > 2 \). If this were so, then the arguments of this paper would show the same for the corresponding \( \sigma' \) model.

In the previous paper [1] we considered models with infinitely many frozen vertices and in this paper a model with a single frozen vertex. It would be of interest to study models with finitely many, but more than one, frozen vertices; in this regard, see the remark following the proof of Theorem 2.8 below.

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2 Results

In this section we fix $d = 2$. We also use the standard convention that the updates are made when independent rate one Poisson process clocks at each vertex ring.

Let $A_T$ denote the event that the “right” neighbor of the origin (at $x = (1,0)$) is $-1$ for some $t \geq T$. Let $A_T' \subset A_T$ denote the event that the right neighbor of the origin is the first neighbor to be $-1$ at some time $t \geq T$ (more precisely, that no other neighbor is $-1$ at an earlier time in $[T, \infty]$). Let $B_{L,s}$ for $s \in \{-1, +1\}^\Lambda_L$ (where $\Lambda_L = \{-L, -L + 1, \ldots, L\}^2$) denote the event that $\sigma'(0)|_{\Lambda_L} = s$ and write $B_{L,+}$ when $s \equiv +1$. We denote the probability measure for the frozen origin $\sigma'(\cdot)$ model by $P'$ and that for the regular coarsening model $\sigma(\cdot)$ by $P$.

**Lemma 2.1.** For all $L$, 

$$P(A_T'|B_{L,+}) \geq 1/4.$$  

**Proof.** The result is an easy consequence of symmetry among the four neighbors of the origin and the fact that $P(A_T) = 1$ (indeed, for all $T$, $P(A_T) = 1$ — see [2]).

Let $\Sigma_L^T$ denote the sigma-field generated by the initial spin values and clock rings and coin tosses up to time $T$ inside the box $\Lambda_L$.

**Proposition 2.2.** For any $T$, $L$,

$$P'(A_T|\Sigma_L^T) \geq 1/4 \ a.s.$$ 

**Proof.** Let $\tilde{\sigma}_T^L(\cdot)$ denote the model with the spin values at all sites in $\Lambda_L$ frozen to $+1$ from time $0$ up to time $T$ and with the spin value at the origin remaining frozen at $+1$ thereafter. Denote the corresponding probability measure by $\tilde{P}_T^L$. Under the standard coupling, $\tilde{\sigma}(\cdot)$ stochastically dominates $\sigma'\sigma(\cdot)$, so we have

$$P'(A_T|\Sigma_L^T) \geq \tilde{P}_T^L(A_T) \geq \tilde{P}_T^L(A_T^1).$$

To continue the proof, we will use the following result about the “propagation speed” of influence between different spatial regions:

**Lemma 2.3.** Let $D_T^L$ denote the event that $\sigma_x(t) = +1 \forall x \in \Lambda_L, \forall t \in [0,T]$. Then

$$\forall L, T, \varepsilon, \exists L' \text{ such that } P(D_T^L|B_{L',+}) \geq 1 - \varepsilon.$$  

**Proof.** Let $L' \gg L$ and note that given $B_{L',+}$, $(D_T^L)^c$ can occur only if there is a nearest neighbor (self-avoiding) path between the boundaries of the two sets, $Z^2 \setminus \Lambda_{L'}$ and $\Lambda_L$, along which there are clock rings occurring in succession between times $0$ and $T$. Any such path is at least of length $L' - L$ (i.e., contains at least $L' - L$ vertices besides the starting one).

Consider a particular path $\gamma$ of length $m \geq L' - L$. For each $m$ there are no more than $3^m$ such paths from each boundary point and the time it takes for successive clock rings along $\gamma$ is at least $S_m = \sum_{i=1}^m t_i$ where the $t_i$ are i.i.d. exponential random variables with parameter $1$. By the exponential Markov inequality, for any $\alpha > 0$,

$$P(\sum_{i=1}^m t_i < T) = P(\sum_{i=1}^m t_i > -T) \leq \frac{E(e^{-\alpha \sum_{i=1}^m t_i})}{e^{-\alpha T}} = e^{\alpha T}E(e^{-\alpha t_i})^m = \frac{e^{\alpha T}}{(1 + \alpha)^m}.$$ 

Therefore, since there are at most $C L'$ possible starting points (for some constant $C$),

$$P((D_T^L)^c|B_{L',+}) \leq C L' \sum_{m = L' - L}^{\infty} 3^m \frac{e^{\alpha T}}{(1 + \alpha)^m} = C(\alpha, T, L)L' \left(\frac{3}{1 + \alpha}\right)^L,$$

where $C(\alpha, T, L)$ is a constant depending on $\alpha$, $T$ and $L$. Taking $\alpha > 2$ and the limit as $L' \to \infty$ completes the proof of the lemma. 

ECP 21 (2016), paper 9. 

Page 2/4  http://www.imstat.org/ecp/
Coarsening with a frozen vertex

Proof. (Continuation of proof of Proposition 2.2.)

Pick \(\epsilon > 0\) and fix \(T\) and \(L\). By Lemma 2.3, \(\exists L'\) such that

\[
P(D_{L'}^T|B_{L',+}) \geq 1 - \epsilon.
\]

Therefore, given \(B_{L',+}\), with probability at least \(1 - \epsilon\), \(\sigma_t(\cdot)\) positively dominates \(\sigma_L^T(\cdot)\) for \(0 \leq t < S\), where \(S = \inf\{t > 0|\sigma_t(0,0) = -1\}\), and so

\[
\hat{P}_L^T(A_L^T) \geq P(A_L^T|B_{L',+}) - \epsilon \geq 1/4 - \epsilon.
\]

Taking the limit as \(\epsilon \to 0\) completes the proof of Proposition 2.2. \(\square\)

Now let \(\Sigma_T\) denote the sigma field generated by the initial assignment of spins on \(\mathbb{Z}^2\) and the clock rings and coin tosses on \(\mathbb{Z}^2\) up to time \(T\).

**Proposition 2.4.** For all \(T\),

\[
P'(A_T|\Sigma_T) \geq 1/4 \text{ a.s.}
\]

**Proof.** For \(L \geq 1\) let \(X_L = P'(A_T|\Sigma_T)\). \(\{X_L^T, L \geq 1\}\) is an increasing filtration of sigma fields, and \(E(X_{L+1}|\Sigma_T^T) = X_L\). By the martingale convergence theorem, \(\lim_{L \to \infty}(X_L) = X_\infty = P'(A_T|\Sigma_T)\) and since \(X_L \geq 1/4\) for all \(L\), we have \(P'(A_T|\Sigma_T) \geq 1/4\). \(\square\)

Let \(A_{T,T'}\) denote the event that the right neighbor of the origin is \(-1\) for some time \(t \in [T,T']\). The following is immediate from Proposition 2.4.

**Corollary 2.5.**

\[
\lim_{T' \to \infty} P'(A_{T,T'}|\Sigma_T) \geq 1/4 \text{ a.s.}
\]

**Lemma 2.6.** For any \(T \geq 0\) and \(\gamma > 0\), \(\exists\) a deterministic \(T'\) such that

\[
P'\{\omega: P'(A_{T,T'}|\Sigma_T) \geq 1/8\} \geq 1 - \gamma.
\]

**Proof.** This is a straightforward consequence of the preceding corollary. \(\square\)

**Theorem 2.7.** For any \(T\),

\[
P'(A_T) = 1, \text{ and hence } P'(\cap_{T > 0} A_T) = 1.
\]

It follows that with probability one, \(\sigma'(1,0)(t)\) changes sign infinitely many times as \(t \to \infty\).

**Proof.** Given \(T\) and \(\epsilon > 0\) construct a sequence of deterministic times \(\{T_i: i \geq 0\}\) so that

1. \(T_0 = T\), and
2. \(P'(\omega: P'(A_{T_{i-1},T_i}|\Sigma_{T_{i-1}}) \geq 1/8) \geq 1 - \frac{\epsilon}{2^i}\).

Condition now on the event (of probability at least \(1 - \sum_{i=1}^\infty \frac{\epsilon}{2^i} = 1 - \epsilon\)) that

\[
P'(A_{T_{i-1},T_i}|\Sigma_{T_{i-1}}) \geq 1/8 \text{ for all } i.\]

On this conditioned probability space, letting \(\tilde{W}_i = 1\) (and otherwise 0) if \(A_{T_{i-1},T_i}\) occurs, we note that the \(\tilde{W}_i\)’s stochastically dominate i.i.d. \(\{0,1\}\)-valued \(W_i\)’s with \(\text{Prob}(W_i = 1) = 1/8\). Thus

\[
P'(A_{T_{i-1},T_i} \text{ occurs for only finitely many } i) \leq \epsilon.
\]

Letting \(\epsilon \to 0\) completes the proof of the first part of the theorem. The second part then follows because by stochastic domination (attractivity) and the results of [2], \(\sigma'(0,0)(t_i)\) equals +1 for an infinite sequence of \(t_i \to \infty\). \(\square\)

The next theorem follows from a modified version of the proof of Theorem 2.7.

**Theorem 2.8.** Every site in \(\mathbb{Z}^2 \setminus \{(0,0)\}\) flips infinitely many times in \(\sigma'(\cdot)\) with probability one.
Coarsening with a frozen vertex

Proof. For any site $z$ other than the origin, and for $L$ much larger than say the Euclidean norm of $z$, we consider the unfrozen $\sigma$ model in which at time zero all the vertex values are set to $+1$ in the box of side length $2L$, centered at $z/2$ (so that the origin and $z$ are located symmetrically with respect to this box). Then with probability $1/2$ the vertex at $z$ flips to $-1$ before the one at the origin flips and until just after that time, there is no difference between the frozen (at the origin) $\sigma'$ model and the unfrozen $\sigma$ model. Hence there is probability at least $1/2$ in $\sigma'$ that $z$ will flip to minus. By applying the methods used in the proof of Theorem 2.7 (but with $1/4$ now replaced by $1/2$), we conclude that $z$ will flip infinitely many times with probability one. \hfill $\square$

We note that the line of reasoning in the proof of the last theorem could have also been used to give a modified proof of Theorem 2.7 with $1/4$ replaced by $1/2$. A more interesting remark is the following.

Remark 2.1. For the process $\sigma''$ with some finite set $S$ of vertices frozen to $+1$, it is possible to show by an extension of the arguments used in this paper that there is a finite deterministic $S' \supseteq S$ such that all sites in $\mathbb{Z}^2 \setminus S'$ flip infinitely many times in $\sigma''(\cdot)$ with probability one. In some cases, $S'$ must be strictly larger than $S$ — e.g., when $S = \{(-L, -L), (-L + L), (+L, -L), (+L, +L)\}$, $S'$ includes all of $\Lambda_L$. One may also consider processes where some vertices are frozen to $-1$ and some to $+1$. We expect to pursue these issues in a future paper.

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