Numerical methods for the two-dimensional Fokker-Planck equation governing the probability density function of the tempered fractional Brownian motion

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Abstract
In this paper, we study the numerical schemes for the two-dimensional Fokker-Planck equation governing the probability density function of the tempered fractional Brownian motion. The main challenges of the numerical schemes come from the singularity in the time direction. When $0 < H < 0.5$, a change of variables $\partial \left( t^{2H} \right) = 2H t^{2H-1} \partial t$ avoids the singularity of numerical computation at $t = 0$, which naturally results in nonuniform time discretization and greatly improves the computational efficiency. For $H > 0.5$, the time span dependent numerical scheme and nonuniform time discretization are introduced to ensure the effectiveness of the calculation and the computational efficiency. The stability and convergence of the numerical schemes are demonstrated by using Fourier method. By numerically solving the corresponding Fokker-Planck equation, we obtain the mean squared displacement of stochastic processes, which conforms to the characteristics of the tempered fractional Brownian motion.

Keywords Singularity · Nonuniform discretization · Computational efficiency · Mean squared displacement

1 Introduction

Revealing the law of motion of particles in the system is always a hot subject due to its wide applications in physics, biology, chemistry, etc. The modeling of particles’ motion can be traced back to Brownian motion, which is a normal diffusion...
process. With the advance of scientific research, more and more scientists realized that the motion of particles in complex disordered systems generally exhibits anomalous dynamics. The mean squared displacement (MSD) is usually used to distinguish the types of stochastic processes. The MSD of Brownian motion goes like \( (\langle x(t) - \langle x(t) \rangle \rangle)^2 \sim t^{\nu} \) with \( \nu = 1 \); for a long time \( t \) if \( \nu \neq 1 \), it is called anomalous diffusion. For \( \nu < 1 \) and superdiffusion for \( \nu > 1 \); in particular, it is termed as localization diffusion if \( \nu = 0 \) and ballistic diffusion if \( \nu = 2 \) [1, 3, 6].

The notations \( \langle \cdot \rangle \) and \( t \), respectively, stand for the expectation of random variable and the time of evolution of stochastic process. There are two types of typical stochastic processes to model anomalous diffusion: Gaussian processes and non-Gaussian ones [2, 5, 7–9]. In order to obtain MSD, one must know the probability density function (PDF) of stochastic processes, which can be obtained by solving the corresponding Fokker-Planck equations [3, 4, 10–12].

In this paper, we mainly focus on the numerical schemes for the newly developed models, i.e., the two-dimensional Fokker-Planck equations governing the PDF of the tempered fractional Brownian motion (tfBm), and reveal the mechanism of the motion of the particles.

The tfBm is defined as

\[
B_{\alpha, \lambda}(t) = \int_{-\infty}^{+\infty} \left[ e^{-\lambda(t-z)}(t-z)^{-\alpha} - e^{-\lambda(z)}(-z)^{-\alpha} \right] B(dz),
\]

describing anomalous diffusion with exponentially tempered long-range correlations [11, 12], where \( B(z) \) is Brownian motion, \( \alpha < 0.5 \); and the tfBm is a Gaussian process with the corresponding Fokker-Planck equation [3]

\[
\frac{\partial P(x,t)}{\partial t} = -\frac{\Gamma(H+1/2)}{\sqrt{\pi(2\lambda)^H}} \left[ Ht^{H-1}K_H(\lambda t) + t^H \dot{K}_H(\lambda t) \right] \frac{\partial^2 P(x,t)}{\partial x^2}, \tag{1.1}
\]

where \( P(x,t) \) is PDF of the one-dimensional tfBm, \( \dot{K}_H(\lambda t) \) denotes the derivative of \( K_H(\lambda t) \) with respect to \( t \), and \( K_H(\lambda t) \) is the modified Bessel function of the second kind

\[
K_H(\lambda t) = \frac{1}{2} \int_0^\infty z^{H-1} \exp\left[ -\frac{1}{2} \lambda t \left( z + \frac{1}{z} \right) \right] \, dz, \tag{1.2}
\]

with the Hurst index \( H = 0.5 - \alpha \), and \( H > 0, H \neq 0.5, \lambda > 0 \). After doing integration by part and taking the time derivative, there are

\[
Ht^{H-1}K_H(\lambda t) = \frac{\lambda t^H}{4} \int_0^\infty z^H \left( 1 - \frac{1}{z^2} \right) \exp\left[ -\frac{\lambda t}{2} \left( z + \frac{1}{z} \right) \right] \, dz
\]

and

\[
t^H \dot{K}_H(\lambda t) = -\frac{\lambda t^H}{4} \int_0^\infty z^H \left( 1 + \frac{1}{z^2} \right) \exp\left[ -\frac{\lambda t}{2} \left( z + \frac{1}{z} \right) \right] \, dz.
\]

Then (1.1) can be recast as

\[
\frac{\partial P(x,t)}{\partial t} = \frac{\Gamma(H+1/2)}{\sqrt{\pi(2\lambda)^H}} \lambda t^H K_{H-1}(\lambda t) \frac{\partial^2 P(x,t)}{\partial x^2}.
\]
Along the direction of extension of the one-dimensional Fokker-Planck equation, we get
\[
\frac{\partial u(x, y, t)}{\partial t} = \Gamma(H + 1/2) \lambda t^H K_{H-1}(\lambda t) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] u(x, y, t),
\]
(1.3)
with the initial and boundary conditions given by
\[
u(x, y, 0) = u_0(x, y), \quad (x, y) \in \Omega, \\
u(x, y, t) = 0, \quad (x, y, t) \in \partial \Omega \times [0, T].
\]

Here \(\Omega = (0, L) \times (0, L')\) is the spatial domain, \(\partial \Omega\) is the boundary of \(\Omega\), and \(u(x, y, t)\) is the PDF of the two-dimensional tfBm. The analytical solution of (1.3) is difficult to find and one has to resort to numerical schemes. In the literatures [22–29], the explicit method, implicit method, and ADI method, etc, have been discussed; however, the diffusion coefficient is usually constant. In (1.3), the diffusion coefficient is \(\Gamma(H + 1/2) \lambda t^H K_{H-1}(\lambda t)\), which tends to infinity as \(t \to 0\) and \(0 < H < 0.5\); and the diffusion coefficient increases first and then decreases with time \(t\) if \(H > 0.5\). For these two reasons, it is difficult to solve the equation directly using classical methods. We hope to obtain effective numerical methods by analyzing and applying the properties of diffusion coefficients.

This paper is organized as follows. In Section 2, for \(0 < H < 0.5\), we derive a modified implicit method to circumvent the singularity of numerical calculation at \(t = 0\) and use nonuniform time stepsizes to improve computational efficiency and ensure the accuracy of the numerical solution. As \(H > 0.5\), a time span dependent numerical method with nonuniform time stepsizes is proposed to improve the accuracy of numerical solution and computational efficiency. In Section 3, we present numerical results and show the characteristics of diffusion process corresponding to (1.3). Finally, a brief conclusion is provided in Section 4.

### 2 Numerical schemes with nonuniform time stepsizes

Now, we introduce the numerical schemes that not only eliminate the singularity, but also improve the computational efficiency and ensure the accuracy of the numerical solution. Generally in designing numerical schemes, one first needs to get a mesh in the space-time region where one wants to acquire the numerical approximation \(u^k_{m,n}\) of the exact solution \(u(x_m, y_n, t_k)\), where \((x_m, y_n, t_k)\) is the coordinate of the \((m, n, k)\) node of the mesh, and \(U^k_{M,N} = \{u^k_{m,n}, 0 \leq m \leq M, 0 \leq n \leq N, 0 \leq k \leq K\}\). In order to facilitate the numerical calculations, we rewrite the matrix \(U^k_{M,N}\) as a vector \(\tilde{U}^k_{M,N} = (u^k_{0,0} \ldots u^k_{M,0} \ldots u^k_{0,1} \ldots u^k_{M,1} \ldots u^k_{0,N} \ldots u^k_{M,N})^T\).

Since in space the solution of (1.3) has homogeneous properties, the sizes of the mesh \(\Delta x = x_{m+1} - x_m = h\) and \(\Delta y = y_{n+1} - y_n = l\) are taken as constants, and we
discretize the operators \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial y^2} \) by means of the three-point centered formulas, i.e.,

\[
h^2 \frac{\partial^2 u(x, y_n, t_k)}{\partial x^2} \bigg|_{x=x_m} = \delta_x^2 u_{m,n}^k = u_{m+1,n}^k - 2u_{m,n}^k + u_{m-1,n}^k + O(h^4),
\]

\[
l^2 \frac{\partial^2 u(x_m, y, t_k)}{\partial y^2} \bigg|_{y=y_n} = \delta_y^2 u_{m,n}^k = u_{m,n+1}^k - 2u_{m,n}^k + u_{m,n-1}^k + O(l^4).
\]

In the following, we sufficiently make use of the properties of the time-dependent diffusion coefficients to do the time discretizations. For \( t^H K_H(\lambda t) \), there exist the estimates

\[
t^H K_H(\lambda t) \leq t^H (\lambda t)^{-H} 2^{H-1} \Gamma(H)
\]

\[
= \frac{2^{H-1}}{\lambda^H} \Gamma(H),
\]

which uses the fact that \( K_H(t) \leq t^{-H} 2^{H-1} \Gamma(H) \), and

\[
t^H K_H(\lambda t) \geq \frac{t^H}{2} \int_1^\infty z^{H-1} \exp \left[ -\frac{1}{2} \lambda t \left( z + \frac{1}{z} \right) \right] dz
\]

\[
\geq \frac{t^H}{2} \int_1^\infty z^{H-1} \exp \left[ -\frac{1}{2} \lambda t (z + 1) \right] dz
\]

\[
= \frac{2^{H-1} e^{-\frac{1}{2} \lambda t}}{\lambda^H} \Gamma \left( H, \frac{\lambda t}{2} \right),
\]

where

\[
\Gamma \left( H, \frac{\lambda t}{2} \right) = \int_{\frac{1}{2}}^\infty e^{-x} x^{H-1} dx.
\]

Consequently, we have

\[
\frac{2^{H-1} e^{-\frac{1}{2} \lambda t}}{\lambda^H} \Gamma \left( H, \frac{\lambda t}{2} \right) \leq t^H K_H(\lambda t) \leq \frac{2^{H-1}}{\lambda^H} \Gamma(H),
\]

which leads to

\[
\lim_{t \to 0} t^H K_H(\lambda t) = \frac{2^{H-1}}{\lambda^H} \Gamma(H). \tag{2.1}
\]

When \( 0 < H < 0.5 \), (1.2) shows that

\[
K_{H-1}(\lambda t) = \frac{1}{2} \int_0^\infty z^{H-1-1} \exp \left[ -\frac{1}{2} \lambda t \left( z + \frac{1}{z} \right) \right] dz
\]

\[
= \frac{1}{2} \int_0^\infty \left( \frac{1}{z} \right)^{2-H} \exp \left[ -\frac{1}{2} \lambda t \left( z + \frac{1}{z} \right) \right] z^2 d \left( \frac{1}{z} \right)
\]

\[
= \frac{1}{2} \int_0^\infty \left( \frac{1}{z} \right)^{1-H} \exp \left[ -\frac{1}{2} \lambda t \left( z + \frac{1}{z} \right) \right] d \left( \frac{1}{z} \right)
\]

\[
= K_{1-H}(\lambda t). \tag{2.2}
\]
Combining (2.1) and (2.2) results in
\[
\lim_{t \to 0} t^{1-2H} \left( t^H K_{H-1}(\lambda t) \right) = \lim_{t \to 0} t^{1-H} K_{H-1}(\lambda t) = \lim_{t \to 0} t^{1-H} K_{1-H}(\lambda t) = \frac{2^{-H}}{\lambda^{1-H}} \Gamma(1 - H).
\]

For (1.3), as \(2H - 1 < 0\), the diffusion coefficient is singular at \(t = 0\) and decreases monotonously; while \(2H - 1 > 0\), the coefficient firstly increases then decreases with the increase of \(t\). Therefore, we need to, respectively, design the difference schemes of (1.3) in two different cases, i.e., \(2H - 1 < 0\) and \(2H - 1 > 0\).

**Case I:** As \(0 < H < 0.5\), \(\lim_{t \to 0} t^H K_{H-1}(\lambda t)\) diverges. In order to eliminate the singularity, multiplying both sides of (1.3) by \(t^{1-2H}\), we get
\[
\frac{\partial u(x, y, t)}{\partial t^{2H}} = \frac{\Gamma(H + 1/2)}{2H\sqrt{\pi(2\lambda)^H}} t^{1-H} K_{H-1}(\lambda t) \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] u(x, y, t). \tag{2.3}
\]
Generally, the bigger coefficient implies the large variation of the solution. To make the distribution of numerical errors uniform, smaller time stepsizes should be taken for the case of bigger coefficient. In this case of (1.3), with the increase of the time, the diffusion coefficient decays approximately as power law at a small time and exponentially at a relatively large time. To balance the decay of diffusion coefficient and make the variation of the solution approximately stationary, by taking \(t^{2H}\) as a whole variable, we get a nonuniform discretization of \([0, T]\) with \(t_k = (\tau k)^{1/2H}\), \(\tau = \Delta(t_k^{2H}) = (t_{k+1})^{2H} - (t_k)^{2H}, \tau > 0\), which greatly reduces the computation cost while keeping the accuracy.

From now on, the finite difference scheme can be obtained by discretizing (2.3):
\[
\frac{u_{m,n}^{k+1} - u_{m,n}^k}{\Delta(t_k^{2H})} = \frac{\Gamma(H + 1/2)}{2H\sqrt{\pi(2\lambda)^H}} t_{k+1}^{1-H} K_{H-1}(\lambda t_{k+1}) \left[ \frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} \right] u_{m,n}^{k+1}, \tag{2.4}
\]
which can be rearranged as
\[
\left( 1 + \frac{2r}{(\Delta x)^2} + \frac{2r}{(\Delta y)^2} \right) u_{m,n}^{k+1} - \frac{r}{(\Delta x)^2} \left( u_{m+1,n}^{k+1} + u_{m-1,n}^{k+1} \right) - \frac{r}{(\Delta y)^2} \left( u_{m,n+1}^{k+1} + u_{m,n-1}^{k+1} \right) = u_{m,n}^k, \tag{2.5}
\]
where
\[
r = \frac{\Gamma(H + 1/2)}{2H\sqrt{\pi(2\lambda)^H}} \left[ t_{k+1}^{1-H} K_{H-1}(\lambda t_{k+1}) \right].
\]
The coupling form of (2.5) can be written as
\[
C(t_{k+1})\tilde{U}_{M,N}^{k+1} = \tilde{U}_{M,N}^k,
\]
implies that
\[
\tilde{U}_{M,N}^{k+1} = \prod_{j=0}^{k} C^{-1}(t_{j+1})\tilde{U}_{M,N}^0,
\]

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where $C(t_{k+1})$ is a growth matrix, being symmetrical. Let $c_{i,j}$ be the element of row $i$ and column $j$ of matrix $C(t_{k+1})$. Then
\[
c_{i,j} = \begin{cases} 
1 + \frac{2r}{(\Delta x)^2} + \frac{2r}{(\Delta y)^2}, & i = j, \\
-\frac{r}{(\Delta y)^2}, & j = i - 1 \text{ or } j = i + 1, \\
-\frac{r}{(\Delta x)^2}, & j = i - M - 1 \text{ or } j = i + M + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

**Case II:** As $H > 0.5$, the function $\lambda^{1-H} t^H K_{H-1}(\lambda t)$ increases first and then decreases with time. We define
\[
t_{\text{max}} := \max_{t \in [0, T]} \left\{ \lambda^{1-H} t^H K_{H-1}(\lambda t) \geq \lambda^{1-H} t^H K_{H-1}(\lambda t) \quad \text{for any } t \in [0, t_\ast] \right\}.
\]
The maximum point of the function $\lambda^{1-H} t^H K_{H-1}(\lambda t)$ is (if it is not $T$)
\[
t_{\text{max}} \approx \frac{0.7442 H - 0.148 H^{-1.3075}}{\lambda},
\]
which is got by plotting the function and fitting parameters. In the interval $t \in [0, t_{\text{max}}]$, the diffusion coefficient increases approximately as power law, while in the interval $t \in [t_{\text{max}}, T]$, the diffusion coefficient decays exponentially. Thus, in order to balance the trend of diffusion coefficient and improve the accuracy of the numerical solution and computational efficiency, we introduce the time span dependent difference schemes to solve equation. We choose a nonuniform partition of $[0, t_{\text{max}}]$ with $t_k = (\tau k)^{1/2H}$ being the power law decay and a nonuniform partition of $[t_{\text{max}}, T]$ with $t_k = (\tau k)^{1/H}$, which is power law increasing.

One can set up the difference scheme of (1.3) as
\[
\begin{align*}
\frac{u_{m,n}^{k+1} - u_{m,n}^k}{\Delta(t_k^H)} &= \frac{\Gamma(H+1/2)}{2H \sqrt{\pi}} (\lambda t_{k+1})^{1-H} K_{H-1}(\lambda t_{k+1}) \left[ \frac{\delta_x^2}{(\Delta x)^2} + \frac{\delta_y^2}{(\Delta y)^2} \right] u_{m,n}^{k+1}, \quad k \leq k_1, \\
\frac{u_{m,n}^{k+1} - u_{m,n}^k}{\Delta(t_k^H)} &= \frac{\Gamma(H+1/2)}{H \sqrt{\pi}} (\lambda t_{k+1})^{1-H} K_{H-1}(\lambda t_{k+1}) \left[ \frac{\delta_x^2}{(\Delta x)^2} + \frac{\delta_y^2}{(\Delta y)^2} \right] u_{m,n}^{k+1}, \quad k \geq k_1.
\end{align*}
\]
The coupled form of (2.7) can be written as, when $k \leq k_1$,
\[
C_1(t_{k+1}) \tilde{U}_{M,N}^{k+1} = \tilde{U}_{M,N}^k, \quad \tilde{U}_{M,N}^{k+1} = \prod_{j=0}^{k} C_1^{-1}(t_{j+1}) \tilde{U}_{M,N}^{0},
\]
where the matrix $C_1(t_{k+1})$ is the same as $C(t_{k+1})$. When $k \geq k_1$,
\[
C_2(t_{k+1}) \tilde{U}_{M,N}^{k+1} = \tilde{U}_{M,N}^k, \quad \tilde{U}_{M,N}^{k+1} = \prod_{j=1}^{k} C_2^{-1}(t_{j+1}) \prod_{j=0}^{k-1} C_1^{-1}(t_{j+1}) \tilde{U}_{M,N}^{0},
\]
where $C_2(t_{k+1}) = 2t^H C_1(t_{k+1})$. Growth matrices $C_1(t_{k+1})$ and $C_2(t_{k+1})$, respectively, correspond to two equations in (2.7).

By solving (2.4) or (2.7), one can get all $u_{m,n}^k$, the approximations of the exact solution. Checking stability and convergence is critical to understand the effectiveness of the numerical schemes.
convergence of the schemes (2.4) and (2.7) (for the details, see Appendix). Let
\[ e_{m,n}^k = u_{m,n}^k - u(x_m, y_n, t_k) \]
be the difference between the numerical solution and the exact solution. The local truncation error is
\[ R_{m,n}^k = O \left( \tau + h^2 + l^2 \right). \]

Then we have the results.

**Theorem 1** Assume that \( U(x, y, t) \in C^{4,4,3} (\bar{\Omega} \times [0, T]) \) and \( H > 0, H \neq 0.5 \).
Then the schemes (2.4) and (2.7) are unconditionally stable, that is,
\[ \| U^k(x, y) \|_{L^2} < \| U^0(x, y) \|_{L^2}, \]

**Theorem 2** Assume that \( U(x, y, t) \in C^{4,4,3} (\bar{\Omega} \times [0, T]) \) and \( H > 0, H \neq 0.5 \).
The schemes (2.4) and (2.7) have first-order convergence in time and second-order convergence in space, i.e.,
\[ \| e^k(x, y) \|_{L^2} < O \left( \tau + h^2 + l^2 \right). \]

### 3 Numerical results

The numerical verification is divided into two sub-sections. The Section 3.1 describes the rationality of the numerical solution, and the Section 3.2 verifies convergence of the numerical solution.

#### 3.1 Localization diffusion

First, we show the rationality of the numerical solution. One only needs to check whether the MSD obtained by numerical solution is consistent with tfBm. So, we test the schemes (2.4) and (2.7) by solving (1.3) and calculating the MSD of tfBm. Although particles diffuse in unbounded domain, it is of course impossible to use boundary conditions at infinity in numerical calculations. Thus, we solve (1.3) in a large enough two-dimensional domain \( \Omega = (-100, 100) \times (-100, 100) \) by the proposed method. The numerical results \( U_{M,N}^K \) with an initial data \( u(x, y, 0) = e^{-x^2-2y^2} \) are obtained, where \( M, N, \) and \( K \) are bounded positive integers, and their bounds depend on the stepsizes in space and time.

From Fig. 1, we can see that the nonuniform time stepsizes method is more computationally efficient. The number of time steps \( K \) for nonuniform time stepsizes method is

\[ K = \begin{cases} \left\lceil \frac{T^{2H}}{\tau} \right\rceil, & 0 < H < 0.5, \\ \left\lceil \frac{t_{\text{max}}^{2H}}{\tau} + \left( T - t_{\text{max}} \right)^H \right\rceil, & H > 0.5. \end{cases} \]
where \([x]\) denotes the smallest integer that is bigger than or equal to \(x\), while the number of time steps for uniform time stepsize method is

\[
K = \frac{T}{\tau}.
\]

The complete algorithm is shown in Algorithm 1. First, we choose the time mesh through nonuniform time stepsizes. Next, Module 1 computes \(\tilde{U}_{M,N}^K\) by the scheme (2.4) or (2.7). From Module 2, one can get the MSD of stochastic process by expectation formula.

**Algorithm 1** Numerical scheme with nonuniform discretization.

request: \(H, t_{max}\)

1: \(t_1 = (\tau k_1)^{1/2H}\), \(t_2 = (\tau k_1)^{1/H}\)
2: if \(H < 0.5\) then
3: \(t = t_1\)
4: else
5: \(t_{max} = (\tau k_1)^{1/2H} = (\tau k_2)^{1/H}\)
6: \(t = [t_1(1 : k_1), t_2(k_2 + 1 : end)]\)

Module 1-Running of scheme (2.4) or (2.7)

request: \(C(t_k), C_1(t_k), C_2(t_k), U_{M,N}^0\)

7: \(\tilde{U}_{M,N}^0 = \text{reshape}(U_{M,N}^0, (M + 1)(N + 1), 1)\)
8: if \(H < 0.5\) then
9: \(\tilde{U}_{M,N}^{k+1} = C_{-1}(t_k)\tilde{U}_{M,N}^k\)
10: else
11: while \(k < k_1\) do
12: \(\tilde{U}_{M,N}^{k+1} = C_1^{-1}(t_k)\tilde{U}_{M,N}^k\)
13: while \(k > k_1\) do
14: \(\tilde{U}_{M,N}^{k+1} = C_2^{-1}(t_k)\tilde{U}_{M,N}^k\)
15: All \(U_{M,N}^k\) are known

Module 2-Calculating MSD

16: \(Pr_{M,N}^k = \frac{U_{M,N}^k}{\sum_{m=0}^{M} \sum_{n=0}^{N} U_{m,n}^k}\)
17: MSD = \(\sum_{m=0}^{M} \sum_{n=0}^{N} (x_m^2 + y_n^2) Pr_{m,n}^k - \left(\sum_{m=0}^{M} \sum_{n=0}^{N} x_m Pr_{m,n}^k\right)^2 - \left(\sum_{m=0}^{M} \sum_{n=0}^{N} y_n Pr_{m,n}^k\right)^2\)

In order to study the motion of the particles, it is necessary to know the MSD of diffusion process tfBm. We define the MSD of two-dimensional stochastic process as

\[
\langle [x(t) - \langle x(t) \rangle]^2 + [y(t) - \langle y(t) \rangle]^2 \rangle.
\]

The MSD of the stochastic process can be obtained by the numerical solution \(U_{M,N}^K\). The discrete probability distribution \(Pr_{M,N}^K = \)
\{P_{m,n}^k \mid 0 \leq m \leq M, 0 \leq n \leq N, 0 \leq k \leq K\} of particles is captured at each moment, after normalizing $U_{M,N}^k$. The expectation formula implies that

$$\langle x(t_k) \rangle = \sum_{m=0}^{M} \sum_{n=0}^{N} x_m P_{m,n}^k,$$

$$\langle y(t_k) \rangle = \sum_{n=0}^{N} \sum_{m=0}^{M} y_n P_{m,n}^k,$$

and

$$\langle x^2(t_k) \rangle = \sum_{m=0}^{M} \sum_{n=0}^{N} x_m^2 P_{m,n}^k,$$

$$\langle y^2(t_k) \rangle = \sum_{n=0}^{N} \sum_{m=0}^{M} y_n^2 P_{m,n}^k,$$

From Fig. 2, one can see that $[x(t) - \langle x(t) \rangle]^2 + [y(t) - \langle y(t) \rangle]^2 \sim t^0$ for a long time $t$, which implies that the stochastic process is a localization diffusion.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Number of time steps for uniform and nonuniform ($H = 0.2, 0.3, 0.7, 0.8$) time stepsizes with $\tau = 0.05$ and $\lambda = 0.1$.}
\end{figure}

\begin{figure}[h]
\centering
\begin{subfigure}{0.5\textwidth}
\centering
\includegraphics[width=\textwidth]{fig2a.png}
\caption{Simulations of MSD with $T = 200, \lambda = 0.1$ (left).}
\end{subfigure} \hspace{0.5cm}
\begin{subfigure}{0.5\textwidth}
\centering
\includegraphics[width=\textwidth]{fig2b.png}
\caption{Simulations of MSD with $T = 600, \lambda = 0.01$ (right).}
\end{subfigure}
\caption{Simulations of MSD with $T = 200, \lambda = 0.1$ (left), $T = 600, \lambda = 0.01$ (right), $\tau = 0.5$, and $h = l = 0.5$.}
\end{figure}
Fig. 3 Probability density function at time $t = 5$ (a), 150 (b), 450 (c), 600 (d), with $H = 0.7$, $\lambda = 0.02$, $\tau = 0.5$, and $h = l = 0.5$. Comparing the four subgraphs, one can see that the probability density function no longer changes when the time is large enough, which implies that the MSD of tBm is a constant after a long time. Therefore, the stochastic process is a localization diffusion.

As $H > 0$, the larger the $H$ is, the longer the time required for MSD $\sim t^0$ and the more dispersed the particles are; the result is opposite when $\lambda$ becomes large. It is consistent with the effect of the parameter $\lambda$, which moderates the length of the jump. Fractional Brownian motion is recovered when $\lambda = 0$, and its MSD is like $t^{2H}$. Figure 3 depicts the evolution of the discrete probability distribution when the time is 5, 150, 450, and 600, respectively. Figures 2 and 3 show that

| $M = N$ | $H = 0.2$ | Rate | $H = 0.4$ | Rate | $H = 0.6$ | Rate | $H = 0.8$ | Rate |
|---------|-----------|------|-----------|------|-----------|------|-----------|------|
| 10      | 5.48e−05  |      | 3.44e−03  |      | 1.46e−02  |      |           |      |
| 20      | 1.23e−13  | 2.16 | 1.04e−05  | 2.04 | 8.51e−04  | 2.02 | 3.66e−03  | 2.00 |
| 40      | 3.00e−14  | 2.04 | 2.57e−06  | 2.02 | 2.12e−04  | 2.01 | 9.14e−04  | 2.00 |
most of the particles diffuse within the bounded domain, and its size is related to $H$ and $\lambda$.

### 3.2 Convergence

Finally, we numerically test the convergences of the schemes (2.4) and (2.7). We solve (1.3) in the two-dimensional domain $\Omega = (0, 1) \times (0, 1)$ by the proposed method up to time $T = 2$, with the parameter $\lambda = 1$. The numerical results with a smooth initial data $u(x, y, 0) = (x - 10)^2 + (2y - 5)^2$ are presented in tables, where $u_{\tau, M, N}$ denotes the numerical solution with fixed sizes $\tau, h = 1/M$ and $l = 1/N$ at time $t = 2$. Since the exact solutions of (1.3) are unknown, we use the following formulas to calculate the orders of the numerical solutions:

**convergence rate of space**

$$\text{convergence rate of space} = \frac{\ln \left( \frac{\|u_{\tau, 2M, 2N} - u_{\tau, M, N}\|}{\|u_{\tau, M, N} - u_{\tau, M/2, N/2}\|} \right)}{\ln 2},$$

**convergence rate of time**

$$\text{convergence rate of time} = \frac{\ln \left( \frac{\|u_{\tau, 2\tau} - u_{\tau, M, N}\|}{\|u_{\tau, M, N} - u_{\tau/2, M, N}\|} \right)}{\ln 2}.$$  

When $0 < H < 0.5$, we use the scheme (2.4) to solve (1.3). As $H > 0.5$, we apply the scheme (2.7) to solve (1.3). From Tables 1 and 2, one can see that the proposed methods have second-order convergence in space. Tables 3 and 4 demonstrate that the proposed methods have first-order convergence in time, furthermore, Theorem 2 is numerically confirmed.
Table 4  Time convergence rates with $M = N = 60$ and discrete $L_{\infty}$-norm

| $\tau$  | $H = 0.2$  | Rate | $H = 0.4$  | Rate | $H = 0.6$  | Rate | $H = 0.8$  | Rate |
|---------|-------------|------|-------------|------|-------------|------|-------------|------|
| 1/1200  | 8.13e−13    | 3.06e−05 | 1.93e−03    | 8.45e−03 |
| 1/2400  | 3.60e−13    | 1.18  | 1.51e−05    | 1.02  | 9.64e−04    | 1.00  | 4.26e−03    | 0.99  |
| 1/4800  | 1.69e−13    | 1.09  | 7.49e−06    | 1.01  | 4.81e−04    | 1.00  | 2.14e−03    | 0.99  |

4 Conclusion

Anomalous diffusion is widely observed in the nature world, the types of which are abundant, and the mechanisms of different types of anomalous diffusions sometimes are fundamentally different. This paper focuses on providing the numerical methods for the Fokker-Planck equation governing the PDF of the tfBm, and simulating the corresponding dynamics. The main challenges come from the variable coefficient of the model, and even its singularity at the starting point $t = 0$. By introducing the nonuniform time stepsizes, the efficient numerical schemes are designed, and the numerical stability and convergence are theoretically proved. The simulation results, using the proposed schemes, further reveal the dynamics of the localization diffusion of tfBm.

Other numerical schemes and more wide applications will be considered in future work. For instance, obtain a high-order approximation scheme or study wavelet variational method. In many applications, it is useful to add a drift $vt$. The $v$ describes velocity of water, when particles diffuse in water. Then the Fokker-Planck equation with drift can be explored. And the numerical schemes to solving the equation with drift will be studied.

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Appendix A: Numerical stability

For $0 < H < 0.5$, the Fourier series of $u^k(x, y)$ is

$$ u^k(x, y) = \sum_{p_1 = -\infty}^{+\infty} \sum_{p_2 = -\infty}^{+\infty} \hat{u}_{p_1, p_2}^k \exp \left( i \frac{2p_1 \pi x}{L} + i \frac{2p_2 \pi y}{L'} \right), \quad \text{(A.1)} $$

where

$$ \hat{u}_{p_1, p_2}^k = \frac{1}{LLL'} \int_0^L \int_0^{L'} u^k(x, y) \exp \left( -i \frac{2p_1 \pi x}{L} - i \frac{2p_2 \pi y}{L'} \right) \, dx \, dy \quad p_1, p_2 = 0, \pm 1, \ldots. $$
There exists Parseval equation

\[ \|u^k(x, y)\|_{L^2}^2 = LL' \sum_{p_1 = -\infty}^{+\infty} \sum_{p_2 = -\infty}^{+\infty} |\hat{u}^k_{p_1, p_2}|^2. \]

From (2.4), we get

\[ u^{k+1}(x + x_m, y + y_n) - u^k(x + x_m, y + y_n) = \frac{r}{h^2} \delta_x u^{k+1}(x + x_m, y + y_n) + \frac{r}{l^2} \delta_y u^{k+1}(x + x_m, y + y_n). \]  

(A.2)

Substituting (A.1) into (A.2) leads to

\[ \sum_{p_1 = -\infty}^{+\infty} \sum_{p_2 = -\infty}^{+\infty} \hat{u}^k_{p_1, p_2} Q(p_1, p_2) = \sum_{p_1 = -\infty}^{+\infty} \sum_{p_2 = -\infty}^{+\infty} \hat{u}^{k+1}_{p_1, p_2} Q(p_1, p_2) \left\{ \frac{1}{1 + \frac{2r}{h^2}} + \frac{2r}{l^2} \right\}, \]

where

\[ Q(p_1, p_2) = \exp\left( i \frac{2p_1 \pi x}{L} + i \frac{2p_2 \pi y}{L'} \right) \exp\left( i \frac{2p_1 \pi m h}{L} + i \frac{2p_2 \pi n l}{L'} \right). \]

(A.3)

Since the two sides of (A.3) are the Fourier series, we have

\[ \hat{u}^{k+1}_{p_1, p_2} = G_1(p_1 h, p_2 l) \hat{u}^k_{p_1, p_2}, \]

(A.4)

where

\[ G_1(p_1 h, p_2 l) = \frac{1}{1 + \frac{2r}{h^2} \left( 1 - \cos \frac{2p_1 \pi h}{L} \right) + \frac{2r}{l^2} \left( 1 - \cos \frac{2p_2 \pi l}{L'} \right)}. \]

This implies that

\[ 0 \leq G_1(p_1 h, p_2 l) \leq 1. \]

Combining Parseval equation and (A.4) results in

\[ \|u^k(x, y)\|_{L^2}^2 = LL' \sum_{p_1 = -\infty}^{+\infty} \sum_{p_2 = -\infty}^{+\infty} |\hat{u}^k_{p_1, p_2}|^2 < \|u^0(x, y)\|_{L^2}^2. \]

As \( H > 0.5 \), for \( t \geq t_{\text{max}} = t_{k_1} \), using the same process, we have

\[ \hat{u}^{k+1}_{p_1, p_2} = G_2(p_1 h, p_2 l) \hat{u}^k_{p_1, p_2}, \]

(A.5)

where

\[ G_2(p_1 h, p_2 l) = \frac{1}{1 + \frac{2r_1}{h^2} \left( 1 - \cos \frac{2p_1 \pi h}{L} \right) + \frac{2r_1}{l^2} \left( 1 - \cos \frac{2p_2 \pi l}{L'} \right)}. \]
\[ r_1 = \frac{\Gamma(H + 1/2)}{H \sqrt{\pi(2\lambda)^H}} \left[ \lambda t_{k+1} K_{H-1}(\lambda t_{k+1}) \tau \right]. \]

For \( k \leq k_1 \), with the proof being completely the same as the case that \( 0 < H < 0.5 \), there exists
\[
\left\| u^k(x, y) \right\|_{L^2}^2 < \left\| u^0(x, y) \right\|_{L^2}^2. \quad (A.6)
\]

For \( k > k_1 \), combining (A.5) and (A.6) leads to
\[
\left\| u^k(x, y) \right\|_{L^2}^2 = LL' \sum_{p_1=-\infty}^{+\infty} \sum_{p_2=-\infty}^{+\infty} [G_2(p_1 h, p_2 l)]^{2k-2k_1} \left| \hat{u}_{p_1, p_2}^{k_1} \right|^2 < \left\| u^0(x, y) \right\|_{L^2}^2.
\]

Appendix B: Convergence

We use notations
\[
L u(x, y, t) = \frac{\partial u(x, y, t)}{\partial(t^{2H})} - \frac{\Gamma(H+1/2)\lambda t_1^{1-H} K_{H-1}(\lambda t_1)}{2H \sqrt{\pi(2\lambda)^H}} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] u(x, y, t),
\]
\[
L^{(1)} u^k_{m, n} = \frac{u^{k+1}_{m, n} - u^k_{m, n}}{\Delta(t_{k+1}^{2H})} - \frac{\Gamma(H+1/2)\lambda t_k^{1-H} K_{H-1}(\lambda t_k+1)}{2H \sqrt{\pi(2\lambda)^H}} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] u^k_{m, n}.
\]

As \( 0 < H < 0.5 \), performing the Taylor expansion at \( t_k^{2H} \), there exist
\[
\frac{\Delta t_k}{\Delta(t_{k+1}^{2H})} \tau = \frac{t_k^{1-2H}}{t_{k+1}^{1-2H}} - \frac{(1-2H)_{k+1}^{-1}}{8H^2} \tau + O\left(\tau^2\right) \quad (B.1)
\]
and
\[
\frac{(\Delta t_k)^2}{\Delta(t_{k+1}^{2H})} \tau = \frac{t_k^{-4H}}{4H^2} \tau + O\left(\tau^2\right) \quad (B.2)
\]

Letting \( R^k_{m, n} = L^{(1)} u^k_{m, n} - [L u(x, y, t)]^k_{m, n} \), and using (B.1) and (B.2) lead to
\[
R^k_{m, n} = -\frac{(1-2H)_{k+1}^{1-4H}}{8H^2} \tau \left( \frac{\partial u(x, y, t)}{\partial t} \right)_{m, n}^k - \frac{t_k^{-4H}}{8H^2} \tau \left( \frac{\partial^2 u(x, y, t)}{\partial t^2} \right)_{m, n}^k + O\left(\tau^2 + h^2 + l^2\right).
\]
\[
= O\left(\tau + h^2 + l^2\right) \quad (B.3)
\]

For \( e^k_{m, n} = u^k_{m, n} - u(x_m, y_n, t_k) \), from (1.3), (2.4), and (B.3), we have
\[
e^{k+1}(x + x_m, y + y_n) - e^k(x + x_m, y + y_n) = r \left[ \frac{\delta^2}{h^2} + \frac{\delta^2}{l^2} \right] e^{k+1}(x + x_m, y + y_n) + \tau R^k(x + x_m, y + y_n).
\]

Following the proof process of numerical stability and using the expansion similar to (A.3), there exists
\[
\left\| e^{k+1} \right\|_{L^2}^2 < \left\| e^k + \tau R^k \right\|_{L^2}^2.
\]
leading to
\[ \left\| e^k \right\|_{L^2} < \left\| e^{k-1} \right\|_{L^2} + \tau \left\| R^{k-1} \right\|_{L^2} \]
\[ \leq \left\| e^0 \right\|_{L^2} + k\tau \max_{0 \leq i \leq k} \left\| R^i \right\|_{L^2} \]
\[ \leq t_k^{2H} \max_{0 \leq i \leq k} \left\| R^i \right\|_{L^2} \]
\[ = O \left( \tau + h^2 + l^2 \right). \]

For \( H > 0.5 \), when \( t > t_{\text{max}} \), by Taylor expansion at \( t_k^H \), we have
\[ \frac{\Delta t_k}{\Delta (t_k^H)} = \frac{t_k^{1-H}}{H} - \frac{(1-H)t_k^{1-2H}}{2H^2} \tau + O \left( \tau^2 \right) \] (B.4)
and
\[ \frac{(\Delta t_k)^2}{\Delta (t_k^H)} = \frac{t_k^{2-2H}}{H^2} \tau + O \left( \tau^2 \right), \] (B.5)
which implies that
\[ R_{m,n}^k = -\frac{(1-H)t_k^{1-2H}}{2H^2} \tau \left( \frac{\partial u(x,y,t)}{\partial t} \right)^k_{m,n} - t_k^{2-2H} \tau \left( \frac{\partial^2 u(x,y,t)}{\partial t^2} \right)^k_{m,n} + O \left( \tau^2 + h^2 + l^2 \right) \]
\[ = O \left( \tau + h^2 + l^2 \right). \]

For \( k \leq k_1 \),
\[ \left\| e^k \right\|_{L^2} < \left\| e^0 \right\|_{L^2} + t_k^{2H} \max_{0 \leq i \leq k} \left\| R^i \right\|_{L^2} = O \left( \tau + h^2 + l^2 \right). \] (B.6)

When \( k \geq k_1 \), combining (B.6) leads to
\[ \left\| e^k \right\|_{L^2} < \left\| e^{k_1} \right\|_{L^2} + \tau (k - k_1) \left\| R^{k-1} \right\|_{L^2} \]
\[ \leq \left\| e^0 \right\|_{L^2} + t_k^{2H} \max_{0 \leq i \leq k} \left\| R^i \right\|_{L^2} + t_k^H \max_{k_1 \leq i \leq k} \left\| R^i \right\|_{L^2} \]
\[ \leq \left( t_k^H + t_{\text{max}}^H \right) \max_{0 \leq i \leq k} \left\| R^i \right\|_{L^2} \]
\[ = O \left( \tau + h^2 + l^2 \right). \]

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