Exact non-spherical relativistic star

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We obtain Vaidya-like solutions to include both a null fluid and a string fluid in non-spherical
(plane symmetric and cylindrical symmetric) anti-de Sitter space-times. Assuming that string fluid
diffuse, we find exact solutions of Einstein’s field equations. Thus we extend a recent work of Glass
and Krisch [2] to non-spherical anti-de Sitter space-times.

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I. INTRODUCTION

The string is very important ingredient in many physical-
tical theories and idea of string is fundamental to super-
string theories [1]. The development of the last few years
have opened the possibility to go beyond perturbation
theory and to address the most interesting problems of
quantum gravity. The intense level of activity in string
theory has lead to the idea that many of the classical
vacuum scenario may have atmosphere composed of a
fluid or strings. Recently, Glass and Krisch [2] extended
the classical Vaidya solution [3] to create two fluid atmo-
sphere outside the star, the original null fluid and a new
fluid composed of string, and we shall call it generalized
Vaidya solution. The solution is very important in view
of recent links between black holes and string theories

This solution has been employed to look into the conse-
quence of string fluid on the formation of naked singu-
larities in Vaidya collapse [2, 8]. Indeed, the effect of string
is a shrinkage of the naked singularity initial data space,
or an enlargement of the black hole initial data space of
Vaidya collapse [8] and thus violating cosmic censorship
conjecture (CCC) [8]. However, the conjecture, in it’s
weak version, essentially state that any naked singularity
which is created by evolution of regular initial data will
be shielded from the external view by an event horizon.
According to the strong version of the CCC, naked sin-
gularities are never produced, which in the precise math-
ematical terms demands that space-time should be glob-
ally hyperbolic. Many of the gravitational collapse stud-
ies were motivated by Thorne’s hoop conjecture [10] that
collapse will yield a black hole only if a mass \( M \) is com-
pressed to a region with circumference \( C \leq 4\pi M \) in all
directions. If hoop conjecture is true, naked singularities
may form if collapse can yield \( C \geq 4\pi M \) in some direc-
tion. Most of the studies in collapse were restricted to
spherically symmetric space-times.

On the other-hand, non-spherical collapse not so well
understood. However, non-spherical collapse could occur
in real astrophysical situation, and it is also important
for a better understanding of both cosmic censorship con-
jecture and hoop conjecture. Also, hoop conjecture was
given for space-times with a zero cosmological term and
also Penrose [11] has conjectured that CCC might require
a zero cosmological term. In the presence of negative cos-
nomological term one can expect the occurrence of major
changes. Indeed, Lemos [12] has shown that planar or
cylindrical black holes form rather than naked singular-
ity from gravitational collapse of a planar or cylindrical
matter distribution in an anti-de Sitter space-time, vi-
olating in this way the hoop conjecture but not CCC.
Cai et al. [13] extended the work of Lemos [14] to the
plane symmetric and cylindrically symmetric solutions
in Einstein-Maxwell equations with a negative cosmolog-
ical term and pointed out that the negative cosmological
term plays a crucial role in their solutions, as in the BTZ
black holes.

The purpose of this brief report is to see how the results
that were presented in [2] get modified for non-spherical
(plane symmetric and cylindrical symmetric) space-times
with a negative cosmological constant. We are able to
provide a two-fluid kinematic interpretation of the trans-
verse stresses, and the null-tetrad and the orthogonal vec-
tors for the non-spherical space-times. Exact solutions of
the Einstein field equations are obtained assuming that
the string fluid diffuse.

II. NON-SPHERICAL TWO FLUID MODEL

In this section, we extend the work of the Glass and
Krisch [2] to non-spherical (plane symmetric and cylin-
drical symmetric) anti-de Sitter space-times. Let us first
consider the case of plane symmetry. The metric of gen-
eral plane symmetric space-time, expressed in terms of
Eddington retarded time coordinate (ingoing coordinate)
where \( \infty \leq x, y \leq \infty \) are coordinates which describe two-dimensional zero-curvature space which has topology \( R \times R \). \( -\infty \leq u \leq \infty \) is null coordinate called the retarded time, and \( 0 \leq r \leq \infty \) is the radial coordinate. Further, \( e^{\psi(u,r)} \) is an arbitrary function and where \( 3\alpha^2 = -\Lambda > 0 \) denote negative cosmological constant. It is useful to introduce a local mass function \( m(u,r) \) defined by \( f = 1 - 2m(u,r)/r \). For \( f = m(u)/r \) and \( \psi = 0 \), the metric reduces to the plane symmetric Vaidya-like metric. Initially \( f = M_0/r \) (with \( \psi = 0 \)) provides the vacuum Taub solution \( \psi = 0 \).

It is the field equation \( G^0_0 = 0 \) that leads to the string worldsheet we have the string space-like vector such that \( \psi(u,r) = 0 \). Hence, the metric takes the form

\[
ds^2 = \left[ 1 - \frac{2m(u,r)}{r} \right] du^2 - 2dudr - \alpha^2r^2(dx^2 + dy^2). \tag{2}
\]

The use of a Newman-Penrose null tetrad formalism leads to Einstein tensor of the form

\[
G_{ab} = -2\Psi_{11}(l_al_b + l_bl_a + m_am_b + m_bm_a) - 2\Psi_{11}l_al_b - 6\Lambda. \tag{3}
\]

Here the null tetrad Ricci scalars are

\[
\Psi_{11} = \frac{1}{r^2} \left[ \frac{\partial m}{\partial r} - \frac{\partial^2 m}{\partial r^2} - 1 \right], \tag{4}
\]

\[
\Psi_{22} = -\frac{1}{r^2} \frac{\partial m}{\partial r}, \tag{5}
\]

\[
\Lambda = \frac{1}{r^2} \left[ \frac{\partial^2 m}{\partial r^2} - \frac{\partial m}{\partial r} - 1 \right], \tag{6}
\]

the principal null geodesic vectors are \( l_a, n_a \) of the form

\[
l_a = \delta^u_a, \quad n_a = f/2\delta^u_a + \delta^r_a, \tag{7}
\]

where \( l_al^a = n_an^a = 0, \ l_an^a = -1 \). The metric admits an orthonormal basis defined by four unit vectors

\[
\hat{u}_a = f^{1/2}\delta^u_a + f^{-1/2}\delta^r_a, \quad \hat{r}_a = f^{-1/2}\delta^r_a, \quad \hat{x}_a = \alpha x\delta^u_a, \quad \hat{y}_a = \alpha y\delta^u_a, \tag{8}
\]

where \( \hat{u}_a \) is a timelike unit vector and \( \hat{r}_a, \hat{x}_a, \hat{y}_b \) are unit space-like vector such that

\[
g_{ab} = \hat{u}_a\hat{u}_b - \hat{r}_a\hat{r}_b - \hat{x}_a\hat{x}_b - \hat{y}_a\hat{y}_b. \tag{10}
\]

Associated with the string worldsheet we have the string bivector defined by

\[
\Sigma^{ab} = \epsilon^{AB} \frac{dx^a}{dk^A} \frac{dx^b}{dk^B}, \tag{11}
\]

where \( \epsilon^{AB} \) is two dimensional Levi-Civita symbol. It is useful to write the bivector, in terms of the unit vectors, as

\[
\Sigma^{ab} = \hat{r}^a\hat{r}^b - \hat{u}^a\hat{u}^b. \tag{12}
\]

and the condition that the worldsheet are timelike, i.e., \( \gamma = 1/2\Sigma^{ab}\Sigma_{ab} < 0 \) implies that only the \( \Sigma^{ur} \) component is non-zero, therefore one obtains:

\[
\Sigma^{ur}\Sigma_{ur} = \hat{u}^a\hat{u}^b - \hat{r}^a\hat{r}^b. \tag{13}
\]

The string energy-momentum tensor for a cloud of string, by analogy with the one for the perfect fluid, is written as:

\[
T_{ab} = \rho\Sigma_{ab} - p_{\perp}h_{ab}. \tag{14}
\]

The energy-momentum of two fluid system is

\[
T_{ab} = T_{ab}^{(n)} + T_{ab}^{(s)}, \tag{15}
\]

It is the null fluid tensor corresponding to the component of the matter field that moves along the null hypersurfaces \( u = \text{const} \). The effective energy momentum tensor for two fluid system, in terms of the unit vectors, can be cast as:

\[
T_{ab} = \psi l_al_b + \rho\hat{u}_a\hat{u}_b + p_{\perp}\hat{r}_a\hat{r}_b + p_{\perp}(\hat{x}_a\hat{x}_b + \hat{y}_a\hat{y}_b). \tag{16}
\]

For \( \rho = p_r = p_{\perp} = 0 \), Eq. \( \ref{16} \) reduces to stress-energy tensor which gives Vaidya metric. The Einstein field equations (\( G^a_b - 3\alpha^2\delta^a_b = T^a_b \)) now take the form:

\[
\psi = \frac{1}{4\pi r^2} \frac{\partial m}{\partial u}, \tag{17}
\]

\[
\rho = -p_r = \frac{1}{8\pi r^2} \frac{\partial m}{\partial r} + \frac{1}{4\pi} \frac{\partial^2 m}{\partial r^2} - \frac{1}{8\pi} 3\alpha^2, \tag{18}
\]

\[
p_{\perp} = -\frac{1}{4\pi} \frac{\partial^2 m}{\partial r^2} - \frac{1}{8\pi} 3\alpha^2. \tag{19}
\]

### A. Diffusive mass solutions

A typical approach to characterizing diffusive transport of the particles begins with Fick’s law of diffusion. The Fick’s law, widely used in transport theory, is a phenomenological statement of a macroscopic nature about the relationship between the mass current \( \vec{J} \) and the particle density \( n \) of a fluid. More precisely put,

\[
J_{(n)} = -D \nabla n, \tag{20}
\]

introducing the diffusion constant \( D \), called Fick’s diffusion constant and where \( \nabla \) is a purely spatial gradient. Then 4-current conservation \( J_{(n)}^{\mu} = 0 \), where

\[
J_{(n)}^\mu \partial_\mu = (n, \vec{J}_{(n)}) = n\partial_n - D \left( \frac{\partial n}{\partial r} \right) \partial_r.
\]
If Fick’s law is combined with the continuity equation, there results the diffusion equation
\[ \partial_t n = D \nabla^2 n. \] (21)

Here we assume that the string diffuse and that diffusion is the movement of particle form higher number to lower number. By rewriting the \( T_{\alpha \beta} \) components as
\[ \frac{\partial m}{\partial u} = 4\pi r^2 \psi, \] (22)
\[ \frac{\partial m}{\partial r} = 4\pi r^2 \rho - \frac{3}{2} \alpha^2 r^2 + \frac{1}{2}, \] (23)

The integrability condition,
\[ \frac{\partial}{\partial r} \left( \frac{\partial m}{\partial u} \right) = \frac{\partial}{\partial u} \left( \frac{\partial m}{\partial r} \right), \] (24)

with the help of Eqs. (22) and (23), takes the form
\[ \frac{\partial \rho}{\partial u} = r^{-2} \partial_u (r^2 \psi). \] (25)

Rewriting the diffusion equation (21) \( (n \text{ replaced by } \rho) \)
\[ \frac{\partial \rho}{\partial u} = Dr^{-2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \rho}{\partial r} \right), \] (26)

and comparing with \( \partial \rho / \partial u \) in Eq. (25), we obtain
\[ \frac{\partial m}{\partial u} = 4\pi Dr^2 \frac{\partial \rho}{\partial r}. \] (27)

Here we have omitted the function of integration, because it can be set to zero by suitable redefinition of coordinate. Thus solving the diffusion equation for \( \rho \) and then integrating only those solutions to obtain \( m \) that provides exact Einstein solutions for diffusing string fluids. There are many analytic solutions of Eq. (26) and three of them are
\[ \rho = \rho_0 + \frac{k_1}{r} \] \( (28) \)
\[ \rho = \rho_0 + (k_2/6)r^2 + k_2 Du, \] \( (29) \)
\[ \rho = \rho_0 + k_3 (Du)^{-3/2} \exp \left( -\frac{r^2}{4Du} \right). \] \( (30) \)

\[ m(u, r) = m_0 + 4\pi \int r^2 \rho dr + 4\pi r^2 D \int \frac{\partial \rho}{\partial r} dr - \frac{1}{2} \alpha^2 r^3 + \frac{1}{2} r. \] \( (31) \)

The corresponding masses with the density solutions above are:
\[ m(u, r) = m_0 + \frac{4}{3} \pi r^3 \rho_0 + 2\pi k_1 (r^2 - 2Du) - \frac{1}{2} \alpha^2 r^3 + \frac{1}{2} r, \] \( (32) \)
\[ m(u, r) = m_0 + \frac{4}{3} \pi r^3 \rho_0 + \frac{4}{3} \pi k_2 r^2 Du (r + 1) + \frac{2}{15} \pi k_2 r^5 - \frac{1}{2} \alpha^2 r^3 + \frac{1}{2} r, \] \( (33) \)
\[ m(u, r) = m_0 + \frac{4}{3} \pi r^3 \rho_0 + \frac{2}{3} \pi k_3 r^3 (Du)^{-3/2} \exp \left( -\frac{r^2}{4Du} \right) - \frac{1}{2} \alpha^2 r^3 + \frac{1}{2} r. \] \( (34) \)

Here the family of solutions discussed above, in general, belongs to Type II fluid defined in [13]. However, it is interesting to note that if parameters \( k_1, k_2, \) and \( k_3 \) are set to zero then all the mass solutions above become the static solution:
\[ m(r) = m_0 + \frac{4}{3} \pi r^3 \rho_0 - \frac{1}{2} \alpha^2 r^3 + \frac{1}{2} r. \] (35)

Substituting this mass function Eq. (35), in Eqs. (17), (18), and (19), yields
\[ \psi = 0, \] (36)
\[ \rho = -p_r = \rho_0, \] (37)
\[ p_\perp = -\rho_0. \] (38)

Thus, we have \( m = m(r) \) and \( \psi = 0 \), so the matter field degenerates to type I isotropic string fluid [20]. We turn our attention to the cylindrical symmetric anti-de Sitter space-time. The metric [2] in the Cylindrical space-time has the form [18]:
\[ ds^2 = \left( 1 - \frac{2m(u, r)}{r} \right) du^2 + 2du dr + r^2 d\theta^2 + \alpha^2 r^2 dz^2, \] (39)

where \(-\infty < u, z < \infty, 0 \leq r < \infty \) and \( 0 \leq \theta \leq 2\pi \). The topology of two dimensional zero curvature space-time is \( R \times S^1 \). The analysis given above, with suitable modifications, is valid in the cylindrical anti-de Sitter space-time as well. Hence, to conserve space, we avoid
for the mass function: This leads to a slightly complicated expressions \( C_p = \text{constant} \). If we consider static mass function \( m(r) \) gives corresponding static solution: \[ m(r) = c_1 + c_2 \frac{r^3}{3} + \frac{r}{2} \] (42)

If we consider static mass function \( m(r) \), then \( p_r = p_\perp = \frac{1}{4\pi}S(u) - \frac{1}{8\pi}3\alpha^2 \) (41)

Thus, we have \( m = m(r) \) and \( \psi = 0 \), so again the matter field degenerates to type I isotropic string fluid \[ 20 \].

### C. Separable solutions

Next, we see how spherically symmetric solutions of Govender and Govender \[ 7 \] gets modified in our non-spherical space-times. Let us assume that

\[ \rho = R(u)U(u), \] (43)

then, it is easy to see that:

\[ \rho = e^{u\lambda} \left( \frac{C_1}{e^{r\sqrt{\frac{\pi}{D}}} r} + \frac{C_2 e^{r\sqrt{\frac{\pi}{D}}}}{2r\sqrt{\frac{\pi}{D}}} \right), \] (44)

where \( C_1 \) and \( C_2 \) are constants and \( \lambda \) is the separation constant. This leads to a slightly complicated expressions for the mass function:

\[ m(u, r) = m_0 + \frac{4}{3} \pi r^2 e^{\lambda u} \frac{C_1}{e^{r\sqrt{\frac{\pi}{D}}}} \zeta_\perp (u) \]
\[ + \frac{4}{3} \pi r^2 \frac{C_2}{2\sqrt{\frac{\pi}{D}}} \zeta_\parallel (u) - \frac{1}{2} \alpha^2 r^3 + \frac{r}{2}. \] (45)

where,

\[ \zeta_\parallel (u) = \left( 1 \pm \frac{3D u}{r} \sqrt{\frac{\lambda}{D}} - \frac{3D u}{r^2} \right). \]

On the other hand, if we require that \( \rho = R(r) + U(u) \), then

\[ \rho = C_0 + \lambda u + \frac{C_1}{r} + \frac{\lambda r^2}{6D}, \] (46)

with corresponding mass function

\[ m(u, r) = m_0 + \frac{4}{3} \pi r^3 (C_0 + 2\lambda u) + 2\pi C_1 r^2 + \]
\[ + \frac{2}{15} \frac{\lambda r^5}{D} - 4\pi C_1 Du - \frac{1}{2} \alpha^2 r^3 + \frac{r}{2}. \] (47)

Here, \( M(u) = -4\pi C_1 Du \) can be considered as Vaidya mass and other terms in the above equation have contribution from the string fluid and the cosmological term.

### D. Constant density string fluid (\( \rho = \rho_0 \))

Although strictly constant density is not completely realistic, the particular analytic solution to Einstein field equation with this equation of state has provided some insights concerning stars in general relativity. For example, the star represented by this solution has property that it cannot remain in equilibrium if the size-to-mass ratio is less than \( 9/4 \) \[ 21 \]. To obtain the constant density solution, we take \( \rho = \rho_0 \). With this it can be seen that Eq. (18) can be easily integrated to yield:

\[ m(u, r) = \tilde{M}(u) + \beta^2 r^3 + \frac{r}{2} \] (48)

where \( \beta^2 = (8\pi\rho_0 - 3\alpha^2)/6 \) is a constant. Because \( \beta \) is the constant, the physical parameter can also be trivially evaluated to give

\[ \psi = \frac{1}{4\pi r^2} \frac{\tilde{M}(u)}{\partial u} \quad p_r = p_\perp = -\rho_0, \] (49)

and in this case also we have isotropic string fluid.

### III. CONCLUDING REMARKS

In this work we have discussed Vaidya-like solution in non-spherical (planar and cylindrical) anti-de Sitter space-times for a two-fluid system: a null fluid and a string fluid. By assuming that string diffuse analytical solutions of Einstein’s field equations have been obtained. The Vaidya’s radiating star metric is today commonly used for various purposes: As a testing ground for various formulations of the CCC. As an exterior solution for models of objects consisting of heat-conducting matter. Recently, it has also proved to be useful in the study of Hawking radiation, the process of black-hole evaporation \[ 22 \], and in the stochastic gravity program \[ 23 \]. The solutions presented here can be useful to get insights of the string effects in these physical process that too in non-spherical space-time.
Since, comparatively a very few studies have been done in non-spherical collapse, it should be very interesting use these to study string effects in the final fate of collapse. This work is under progress. 

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