A Note on Quantum Divide and Conquer for Minimal String Rotation

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Abstract

Lexicographically minimal string rotation is a fundamental problem on string processing that has recently attracted a lot of attention in quantum computing. Near-optimal quantum algorithms have been proposed during its development, with new ideas such as quantum divide and conquer introduced. In this note, we further study its quantum query complexity. Slightly improved quantum algorithms by divide and conquer are proposed:

1. For the function problem, its query complexity is shown to be $\sqrt{n} \cdot 2^{O(\sqrt{\log n})}$, improving the recent result of $\sqrt{n} \cdot 2^{(\log n)^{1/2+\varepsilon}}$ by Akmal and Jin (2022).

2. For the decision problem, its query complexity is shown to be $O\left(\sqrt{n \log^3 n \log \log n}\right)$, improving the recent result of $O\left(\sqrt{n \log^5 n}\right)$ by Childs et al. (2022).

The purpose of this note is to point out some useful algorithmic tricks, e.g., preprocessing and level-wise optimization, that can be used to improve quantum algorithms, especially for those with a divide-and-conquer structure.

Keywords: quantum computing, quantum algorithms, string processing, quantum query complexity, minimal string rotation.

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1 Introduction

Lexicographically minimal string rotation (LMSR) is a fundamental problem on string processing (cf. [Jeu93, CR94, Gus97, CHL07]) that has lots of applications in different areas, e.g., combinatorial mathematics [VLWW01, SWW16], computational geometry [IS89, Mae91], graph theory [CB81, GY03], automata theory [AZ08, Pup10], and computational chemistry [Shi79]. Since Booth [Boo80] proposed the first linear-time algorithm for LMSR based on the KMP string matching algorithm [KMP77], a series of studies have been carried out successively on its sequential [Shi81, Duv83, Cro92, IS94, Ryt03, BCN05] and parallel [AIP87, IS92] algorithms.

Recently, Wang and Ying [WY20] achieved a quantum speedup on LMSR based on an observation, called “exclusion rule” or “Ricochet Property” in the literature, borrowed from parallel string processing algorithms [Vis91, IS92]. Immediately after, Akmal and Jin [AJ22] improved the exclusion rule given by [WY20] and developed a quantum algorithm with a divide-and-conquer structure. The (worst-case) query complexity $\sqrt{n} \cdot 2^{O((\log n)^{2/3})}$ of their quantum algorithm is near-optimal up to $n^{o(1)}$ factors, improving the result of $O(n^{3/4})$ by [WY20]. Very recently, inspired by the structures of quantum algorithms in [AGS19] and [AJ22], Childs, Kothari, Kovacs-Deak, Sundaram, and Wang [CKKD22] further investigated a general framework of quantum divide and conquer. Moreover, they applied the framework to various string problems and obtained near-optimal quantum algorithms for them, including recognizing regular languages, $k$-increasing subsequence, $k$-common subsequence, and LMSR. Especially, they considered the decision problem of LMSR, and showed that its quantum query complexity is $O\left(\sqrt{n \log^{3/2} n}\right)$, improving the result of $\sqrt{n} \cdot 2^{O((\log n)^{1/2+\varepsilon})}$ induced by [AJ22]. However, the result of [CKKD22] does not imply better quantum query complexity of function problems.

In this note, we further study the quantum query complexity of LMSR in both its function and decision problems. The purpose of this note is to point out some useful algorithmic tricks that can be used to improve quantum algorithms, especially for those with a divide-and-conquer structure.

1.1 Main result

Suppose $s$ is a non-empty (0-indexed) string of length $n$. For every $i \in [n]$, let

$$s^{(i)} = s[i..n - 1]s[0..i - 1]$$

denote the rotation of $s$ with start position at index $i$. The function and decision problems of LMSR are formally defined as follows.

**Problem 1** (Function problem of LMSR). Given a string $s$ of length $n$, find the smallest index $k \in [n]$ such that $s^{(k)} \leq s^{(i)}$ for all $i \in [n]$.

**Problem 2** (Decision problem of LMSR). Given a string $s$ of length $n$ and an index $k \in [n]$, determine if $s^{(k)} \leq s^{(i)}$ for all $i \in [n]$.

Inspired by the quantum algorithms for LMSR given in [WY20, AJ22] and [CKKD22], we obtain better quantum query complexity of LMSR by a more refined design of divide and conquer.

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1. Although it was explicitly stated in [AJ22] that the quantum query complexity of LMSR is $\sqrt{n} \cdot 2^{O((\log n)^{2/3})}$, a slightly more detailed analysis will show that it is $\sqrt{n} \cdot 2^{O((\log n)^{1/2+\varepsilon})}$ for any constant $\varepsilon > 0$.

2. It was only stated in [CKKD22] that the quantum query complexity of the decision problem of LMSR is $O(\sqrt{n})$, where $O(\cdot)$ suppresses polylogarithmic factors of $n$. A more detailed analysis following their proof will show that the explicit quantum query complexity is $O\left(\sqrt{n \log^2 n}\right)$.

3. We use the notation $[n] = \{0, 1, 2, \ldots, n - 1\}$ for every positive integer $n$. 

**Theorem 1.1** (Theorem 3.3 and Theorem 4.2 restated). The worst-case quantum query complexity of LMSR is

1. $\sqrt{n} \cdot 2^{O(\sqrt{\log n})}$ for its function problem, and
2. $O\left(\sqrt{n \log^3 n \log \log n}\right)$ for its decision problem.

For the function problem of LMSR, our quantum algorithm achieves a quasi-polylogarithmic improvement over the result of $\sqrt{n} \cdot 2^{(\log n)^{1/2+\epsilon}}$ by [AJ22]. For the decision problem LMSR, our quantum algorithm makes a polylogarithmic improvement over the result of $O\left(\sqrt{n \log^5 n}\right)$ by [CKKD+22]. For comparison, we collect all known (both classical and quantum) query complexities of LMSR in Table 1.

| Complexity Type | Algorithm Type | Problem Type | Query Complexity | References |
|-----------------|----------------|--------------|------------------|------------|
| Worst-Case      | Classical      | Function     | $\Theta(n)$      | [Boo80, Shi81, Duv83] |
|                 | Parallel       | Function     | $\Theta(\log n)$ | [AIP87, IS92] |
| Quantum         | Function       | $O(n^{3/4})$, $\Omega(\sqrt{n})$ | [WY20] |
|                 |                | $\sqrt{n} \cdot 2^{(\log n)^{1/2+\epsilon}}$ | [AJ22] |
|                 |                | $\sqrt{n} \cdot 2^{O(\sqrt{\log n})}$ | Theorem 3.3 |
| Decision        | $O\left(\sqrt{n \log^3 n}\right)$ | [CKKD+22] |
|                 | $O\left(\sqrt{n \log^3 n \log \log n}\right)$ | Theorem 4.2 |
| Average-Case    | Classical      | Function     | $O(n)$, $\Omega(n / \log n)$ | [IS94, BCN05, WY20] |
| Quantum         | Function       | $O(\sqrt{n \log n})$, $\Omega\left(\sqrt{n / \log n}\right)$ | [WY20] |

† It is the time (depth) complexity of the parallel algorithm.

Theorem 1.1 also implies that the quantum query complexity of minimal suffix, maximal suffix, and maximal string rotation are also improved to the same complexity as in Theorem 1.1 because they can be easily reduced to each other as discussed in [AJ22].

### 1.2 Techniques

In our approach, we consider the function and decision problems of LMSR separately, with different observations and tricks.

**Function problem of LMSR** In the divide-and-conquer approach for LMSR in [AJ22], they obtained a recurrence relation of the form

$$T(n) \leq \tilde{O}\left(\sqrt{\frac{n}{b}}\right) T\left(\frac{n}{b}\right) + \tilde{O}(\sqrt{n})$$

which gives the quantum query complexity

$$T(n) = \sqrt{n} \cdot 2^{(\log n)^{1/2+\epsilon}}$$
for any constant $\varepsilon > 0$, where $\tilde{O}()$ suppresses polylogarithmic factors. Usually, we do not seek to optimize the polylogarithmic factors in the complexity for simplicity. This is because the quantum time complexity of a quantum algorithm often has a polylogarithmic factor overhead compared to its quantum query complexity, even if polylogarithmic factors are eliminated from the quantum query complexity. The most simple example is the Grover search \cite{Gro96}: the quantum query complexity is $O(\sqrt{n})$ without polylogarithmic factors, however, its quantum time complexity is $O(\sqrt{n} \log n)$ (with polylogarithmic factors)\footnote{It should be noted that the best known quantum time complexity of Grover search is $O(\sqrt{n} \log^{*} n)$ \cite{AdW17}.}.

This time, however, optimizing polylogarithmic factors will make a significant difference. In our approach, we try to optimize polylogarithmic factors on each level of the divide and conquer by appropriately using the optimal quantum minimum finding algorithm on bounded-error oracles \cite{WY20} (which is inspired by the improvement on quantum search in \cite{HMdW03}). As a result, we obtain a recurrence relation of the form

$$T(n) \leq \tilde{O}(\sqrt{n} + \sqrt{n}) + \tilde{O}(\sqrt{n}),$$

which, surprisingly, gives the quantum query complexity $T(n) = \sqrt{n} : 2^O(\sqrt{\log n})$.

This is a quasi-polylogarithmic improvement over the quantum query complexity $\sqrt{n} \cdot 2^{(\log n)^{1/2 + \varepsilon}}$ given in \cite{AJ22}. More importantly, our quantum algorithm is quasi-polylogarithmically faster than that of \cite{AJ22} in the sense of quantum time complexity.

### Decision problem of LMSR

In the divide-and-conquer approach for LMSR in \cite{CKKD22}, they directly used the quantum string matching algorithm \cite{RV03} as a subroutine. Specifically, they recursively solved the decision problem $f_n: \Sigma^n \times \Sigma^{n/2} \rightarrow \{0, 1\}$, which determines whether every length-$|t|$ substring in $s$ is lexicographically not smaller than $t$, by deducing that (here, we omit the issues when $n$ is not divisible by 4 and write only rough subscripts)

$$f_n(s, t) = f_{n/2}(s[0..n/2], t[0..n/4]) \land f_{n/2}(s[n/4..3n/4], t[0..n/4]) \land g_n(s, t),$$

where $g_n(s, t)$ requires to find the leftmost and rightmost occurrences of $t[0..n/4]$ in $s$. By the adversary composition trick for quantum divide and conquer in \cite{CKKD22}, this recursion induces a recurrence relation of the form

$$(\text{Adv}(f_n))^2 = 2(\text{Adv}(f_{n/2}))^2 + O(\left(Q(g_n)\right)^2),$$

where Adv($f_n$) is the adversary quantity for $f_n$, and $Q(g_n)$ is the quantum query complexity of $g_n$. Since it was shown in \cite{HLS07,LMR+11} that $Q(f_n) = \Theta(\text{Adv}(f_n))$, solving the recurrence relation will obtain the quantum query complexity $Q(f_n)$ of $f_n$. 

In the approach of \cite{CKKD22}, $Q(g_n) = O(\sqrt{n} \log^2 n)$ is the quantum query complexity of $g_n$ due to the use of quantum string matching algorithm \cite{RV03}, which gives the quantum query complexity

$$Q(f_n) = O\left(\sqrt{n \log^5 n}\right).$$

An obvious improvement can be obtained by using a better quantum string matching algorithm given in \cite{WY20} deducing that $Q(g_n) = O\left(\sqrt{n \log^3 n \log \log n}\right)$, which gives a slightly better query complexity

$$Q(f_n) = O\left(\sqrt{n \log \log n \log^2 n}\right).$$
To further improve the quantum query complexity, our key observation is that every \( g_n'(s', t') \) (in the sub-problems of the recursion) only deals with special cases that \( t' \) is a prefix of \( t \) of length \( n/2^k \) for some \( k \geq 0 \). That is, there are only \( O(\log n) \) different possible \( t' \)'s when calling \( g_n'(s', t') \). On the other hand, quantum string matching algorithm is 2-step: (1) find a deterministic sample of the pattern (see Section 2.6 for more details); and (2) find an occurrence of the pattern in the text with the help of the deterministic sample computed in the last step. From this, we see that every time we call a quantum string matching algorithm, we will always compute a deterministic sample first, and then go to the main part of the matching algorithm. This suggests a structurally different divide-and-conquer strategy:

1. Preprocessing: Since there are only \( O(\log n) \) different possible patterns that appear in the calls of quantum string matching algorithm, we can pre-compute their deterministic samples before getting into the divide-and-conquer procedure.

2. Adversary quantity with advice: When deriving the recurrence relation, we introduce a variant notion of adversary quantity \( \text{Adv}'(f_n) \), for which certain advice (namely, extra information that helps to reduce the complexity) is given beforehand. In our case, the advice for \( \text{Adv}'(f_n) \) is the pre-computed deterministic samples of \( t[0..n/2^k] \) for all \( k \leq \log n \). When computing \( g_n'(s', t') \), we only need to run the second step of quantum string matching with the help of pre-computed deterministic samples.

This new quantum divide-and-conquer structure offers a different type recurrence relation

\[
\left\{ \begin{array}{l}
Q(f_n) \leq Q_{\text{pre}} + O(\text{Adv}'(f_n)), \\
(\text{Adv}'(f_n))^2 \leq 2(\text{Adv}'(f_{n/2}))^2 + O((Q'(g_n))^2),
\end{array} \right.
\]

where \( Q_{\text{pre}} \) stands for the quantum query complexity of preprocessing, and \( Q'(g_n) \) is the quantum query complexity of \( g_n \) with certain advice pre-computed in preprocessing. In our approach, we use a slightly better quantum algorithm for deterministic sampling and string matching given in [WY20], which deduces that \( Q_{\text{pre}} = O\left( \sqrt{n \log^3 n \log \log n} \right) \) and \( Q'(g_n) = (\sqrt{n \log n}) \). Finally, solving the recurrence relation gives that

\[
\text{Adv}'(f_n) = O(\sqrt{n \log n}), \quad \text{and} \quad Q(f_n) = O\left( \sqrt{n \log^3 n \log \log n} \right).
\]

### 1.3 Discussion

In this note, we obtained better quantum query complexities for both the function and decision problems of LMSR. We hope it would bring new inspiration to discover and improve more quantum divide-and-conquer algorithms, especially for quantum algorithms for string processing [CILG+12, AM14, Mon17, BEG+21, LGS22]. We summarize the main points of our results below.

- **A baby step is a giant step:** Small factors are considerable in (quantum) divide and conquer. Sometimes an improvement on each level of the divide and conquer will yield a significant speedup, even beyond polynomial time (with respect to the improvement made on each level). Our quantum algorithm for the function problem of LMSR makes a polylogarithmic improvement on each level of the divide and conquer, and finally yields a quasi-polynomial speedup with respect to the improvement, thereby a quasi-polylogarithmic speedup with respect to the size of the problem, in the overall complexity. An interesting question is: can we obtain polynomial and even exponential speedups (with respect to the size of the problem) by quantum divide and conquer in certain problems?
Preprocessing before divide and conquer can save the amount of computation. Pre-computed information can help to improve the overall complexity in two folds:

1. The complexity on each level of the divide and conquer can be reduced with the help of pre-computed information;
2. The complexity of preprocessing can be optimized individually.

A more interesting question is: what kind of information pre-computed in preprocessing can help to improve the complexity most, and how to find the trade-offs between preprocessing and divide-and-conquer recursion?

As can be seen in our result of quantum query complexity, there is still a quasi-polylogarithmic gap between the function and decision problems of LMSR. The reason is that the divide-and-conquer tool given in [CKKD+22] does not apply to non-Boolean cases. If we could obtain a similar identity for adversary composition in the general case, i.e., allowing \( f \) to be a general computational task rather than a 0/1-valued function, then we might improve the quantum query complexity of the function problem of LMSR and other problems.

We note that in our algorithm for the decision problem of LMSR, the quantum query complexity is dominated by \( Q_{\text{pre}} \), which is the complexity of preprocessing. This urgently requires a faster quantum algorithm for deterministic sampling (as part of quantum string matching), which might be of independent interest.

### 1.4 Organization of this paper

In the rest of this note, we first introduce the necessary preliminaries in Section 2. Then, in Section 3 and 4, we study the quantum query complexity of function and decision problems of LMSR, respectively.

## 2 Preliminaries

### 2.1 Strings

Let \( \Sigma \) be a finite alphabet with a total order \(<\). A string \( s \in \Sigma^n \) of length \( |s| = n \) is a function \( s: [n] \to \Sigma \). We write \( s[i] \) to denote the \( i \)-th character of \( s \) for \( i \in [n] \), and \( s[i..j] \) to denote the substring \( s[i]s[i+1] \ldots s[j] \) for \( 0 \leq i \leq j < n \). The period of string \( s \) is the minimal positive integer \( d \) such that \( s[i] = s[i + d] \) for every \( i \in [n - d] \). String \( s \) is called periodic if its period \( \leq n/2 \), and aperiodic otherwise. Two strings \( s \) and \( t \) are equal, denoted by \( s = t \), if \( |s| = |t| \) and \( s[i] = t[i] \) for every \( i \in [|s|] \); \( s \) is lexicographically smaller than \( t \), denoted by \( s < t \), if either \( s \) is a proper prefix of \( t \) or there is an index \( i \in \min\{|s|, |t|\} \) such that \( s[i] < t[i] \). We write \( s \leq t \) if \( s = t \) or \( s < t \).

### 2.2 Quantum query complexity

We assume quantum access to the input string \( s \) of length \( n \). More precisely, there is a quantum unitary oracle \( O_s \) such that

\[
O_s: |i\rangle|0\rangle \mapsto |i\rangle|s[i]\rangle
\]

for every \( i \in [n] \). A quantum query algorithm \( A \) with \( T \) queries to quantum oracle \( O_s \) is a sequence of unitary operators

\[A: U_0 \to O_s \to U_1 \to O_s \to \cdots \to O_s \to U_T,\]
where \( U_0, U_1, \ldots, U_T \) are uniform quantum unitary operators that are determined independent of \( O_s \) (but depend on \( n \)). Let \( \{ P_y \} \) be a quantum measurement in the computational basis. The probability that \( A \) outputs \( y \) on input \( s \) is defined by

\[
\Pr[A(s) = y] = \|P_y U_T O_s \ldots O_s U_1 O_s U_0 |0\rangle\|^2.
\]

The quantum query complexity of computational task \( f \), denoted by \( Q(f) \), is a function of \( n \) that is the minimal possible number of queries to \( O_s \) used in a quantum query algorithm \( A \) such that \( \Pr[A(s) = f(s)] \geq 2/3 \) for every input \( s \) of length \( n \).

2.3 Quantum adversary composition

Through quantum adversary method \([HLˇS07, Rei11, LMR+11]\), quantum adversary composition technique that applies to quantum divide and conquer was proposed in \([CKKD+22]\).

For a computational task \( f: D \to E \), the Gram matrix \( F \) of \( f \) is defined by \( F_{x,y} = \delta_{f(x),f(y)} \) for every \( x, y \in D \), where the Kronecker delta \( \delta_{a,b} = 1 \) if \( a = b \) and 0 otherwise. Suppose \( D \subseteq \Sigma^n \) is the set of all possible inputs, where \( \Sigma \) is a finite alphabet. For every \( j \in [n] \), matrix \( \Delta_j \) is defined by \( (\Delta_j)_{x,y} = 1 - \delta_{x_j,y_j} \) for every \( x, y \in D \).

**Definition 2.1.** Using the notations above, the adversary quantity for \( f \) is defined by

\[
\text{Adv}(f) = \max \left\{ \|\Gamma\| : \|\Gamma \circ \Delta_j\| \leq 1, \forall j \in [n] \right\},
\]

where the maximization is over all \( |D| \times |D| \) real and symmetric matrix \( \Gamma \) satisfying \( \Gamma \circ F = 0 \). Here, \( A \circ B \) denotes the Hadamard product of matrices \( A \) and \( B \) of the same size, i.e., \( (A \circ B)_{x,y} = A_{x,y}B_{x,y} \).

The following result shows the connection between adversary quantity and quantum query complexity.

**Theorem 2.1** (Quantum query complexity by adversary quantity, \([HLS07, LMR+11]\)). For every computational task \( f \), we have

\[
Q(f) = \Theta(\text{Adv}(f)),
\]

where \( Q(f) \) is the quantum query complexity of \( f \).

Quantum divide-and-conquer algorithms split a large problem into several small sub-problems. To upper bound the query complexity of such quantum algorithms, we need the following composition lemma given in \([CKKD+22]\).

**Theorem 2.2** (Quantum adversary composition, \([CKKD+22\text{ Lemma 1}]\)). Let \( f_1(x) \) and \( f_2(y) \) be decision problems, and \( g(x,y) = f_1(x) \land f_2(y) \). Then,

\[
(\text{Adv}(g))^2 \leq (\text{Adv}(f_1))^2 + (\text{Adv}(f_2))^2.
\]

2.4 Master Theorem

For the convenience of readers, we present the Master Theorem for divide-and-conquer recurrences used in this note as follows.
**Theorem 2.3** (Master Theorem, [BHS80]). Let $T: \mathbb{N} \to \mathbb{R}$ satisfying

$$T(n) = aT\left(\frac{n}{b}\right) + O(n^c \log^p n),$$

where $a, b, c > 0$ and $p \geq 0$. Then,

$$T(n) = \begin{cases} O\left(n^{\log_b a}\right), & \log_b a > c, \\ O\left(n^c \log^{p+1} n\right), & \log_b a = c, \\ O\left(n^c \log^p n\right), & \log_b a < c. \end{cases}$$

### 2.5 Quantum minimum finding

Quantum minimum finding [DH96, AK99, DHHM06] is a basic subroutine of query complexity $O(\sqrt{n})$ widely used in quantum algorithms. However, when the quantum oracle is bounded-error, we need $O(\log n)$ primitive queries to reduce the error probability for one logical query, which results in a quantum algorithm for minimum finding with query complexity $O(\sqrt{n} \log n)$. Specifically, unitary operator $O_{\text{cmp}}$ is said to be a bounded-error oracle with respect to comparator $\text{cmp}: [n] \times [n] \to \{0, 1\}$ if

$$O_{\text{cmp}}|i\rangle|j\rangle|0\rangle|0\rangle_w = \sqrt{p_{ij}}|i\rangle|j\rangle|\text{cmp}(i, j)\rangle|\phi_{ij}\rangle_w + \sqrt{1 - p_{ij}}|i\rangle|j\rangle|1 - \text{cmp}(i, j)\rangle|\phi_{ij}\rangle_w,$$

where $p_{ij} \geq 2/3$ for every $i \in [n]$, and $|\phi_{ij}\rangle_w$ and $|\phi_{ij}\rangle_w$ are ignorable work qubits for every $i, j \in [n]$. Intuitively, bounded-error quantum oracles are understood as a quantum generalization of probabilistic bounded-error oracles that return the correct answer with probability at least $2/3$. Inspired by the error reduction for quantum search in [HMdW03], it was shown in [WY20] that quantum minimum finding can also be solved with query complexity $O(\sqrt{n} \log n)$, even if the quantum oracle is bounded-error. Specifically, we can find the minimum element with probability $\geq 2/3$ using $O(\sqrt{n})$ queries to $O_{\text{cmp}}$. We formally state the result as follows.

**Lemma 2.4** (Quantum minimum finding on bounded-error oracles, [WY20, Lemma 3.4]). There is a quantum algorithm for minimum finding on bounded-error oracles with query complexity $O(\sqrt{n})$.

### 2.6 Quantum deterministic sampling and string matching

We introduce the quantum string matching algorithm [RV03] and its improved version [WY20], which is based on deterministic sampling [Vis91]. We first introduce the notion of deterministic sampling. The following definition of deterministic sample is a generalized version in [WY20].

**Definition 2.2** (Deterministic samples, [Vis91, WY20]). Suppose $s$ is a string of length $n$ and period $d$. A deterministic sample of $s$ is a tuple $(\delta; i_0, i_1, \ldots, i_{l-1})$ such that

1. $0 \leq \delta < \lfloor n/2 \rfloor$;
2. $i_k - \delta \in [n]$ for $k \in [l]$;
3. For every $j \in \lfloor n/2 \rfloor$ but $j \not\equiv \delta$ (mod $d$), there exists $k \in [l]$ such that $i_k - j \in [n]$ and $s[i_k - j] \neq s[i_k - \delta]$.

The following lemma shows that every string has a deterministic sample of small size. In the rest of this note, a deterministic sample always indicates any deterministic sample of size $O(\log n)$.
Lemma 2.5 (Deterministic sampling, [Vis91, WY20]). For every string $s$ of length $n$, there is a deterministic sample $(\delta; i_0, i_1, \ldots, i_{l-1})$ with $l = O(\log n)$.

The following lemma shows that we can find a deterministic sample efficiently with high probability. Lemma 2.6 is slightly different from but implicit in the quantum deterministic sampling algorithm given in [WY20].

Lemma 2.6 (Quantum deterministic sampling, [WY20, Lemma 4.3]). Given string $s$ of length $n$, there is a quantum algorithm for deterministic sampling with success probability $1 - \delta$ with query complexity $O(\sqrt{n \log n \log \log n \log \max\{1/\delta, n\}})$.

The following lemma shows that how to solve string matching by quantum algorithms when a deterministic sample of the pattern is given.

Lemma 2.7 (Quantum string matching with deterministic sampling, [WY20]). Given string $p$ of length $m$ and its deterministic sample, there is a quantum algorithm that for every string $t$ of length $n$, finds the leftmost and rightmost occurrences of $p$ in $t$ or reports that $p$ does not occur in $t$ with query complexity $O(\sqrt{n \log m})$.

Combining Lemma 2.6 and Lemma 2.7, we obtain an improved quantum algorithm for string matching as follows.

Lemma 2.8 (Quantum string matching, [WY20]). Given text $t$ of length $n$ and pattern $p$ of length $m$, there is a quantum algorithm to find the leftmost and rightmost occurrences of $p$ in $t$ or reports that $p$ does not occur in $t$ with query complexity $O(\sqrt{n \log m} + \sqrt{m \log^2 m \log \log m})$.

3 Function Problem of LMSR

In this section, we study the quantum query complexity of the function problem of LMSR. As in [AJ22], LMSR is reduced to the problem of minimal length-$\ell$ substrings.

Problem 3 (Minimal length-$\ell$ substrings). Given a string of length $n$, find the smallest index $k \in [n - \ell]$ such that $s[k..k + \ell - 1] \leq s[i..i + \ell - 1]$ for all $i \in [n - \ell]$.

Remark 3.1. The problem of minimal length-$\ell$ substrings defined in Problem 3 is slightly different from that in [AJ22]. For our purpose, we only have to focus on the leftmost occurrence of the minimal length-$\ell$ substring.

Here, we recall the exclusion rule for minimal length-$\ell$ substrings used in [AJ22].

Theorem 3.1 (Exclusion rule for minimal length-$\ell$ substrings, [AJ22, Lemma 4.8]). Suppose $s$ is a string of length $n$, and $n/2 \leq \ell \leq n$. Let

$$I = \left\{ k \in [n - \ell] : s[k..k + \ell - 1] = \min_{i \in [n - \ell]} s[i..i + \ell - 1] \right\}$$

be the set of all indexes of minimal length-$\ell$ substrings of $s$, which form an arithmetic progression. For every integer $a \geq 0$ and $m \geq 1$ with $a + m \leq n - \ell$, let

$$J = \left\{ a \leq k < a + m : s[k..k + m - 1] = \min_{a \leq i < a + m} s[i..i + m - 1] \right\}$$

be the set of all indexes of minimal length-$m$ substrings of $s[a..a + 2m - 1]$. If $I \cap \{\min J, \max J\} = \emptyset$, then $I \cap J = \emptyset$.  

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For simplicity as well as completeness, we give a direct reduction from LMSR to minimal length-\(\ell\) substrings as follows.

**Proposition 3.2.** LMSR can be reduced to the problem of minimal length-\(\ell\) substrings defined in Problem 3.

*Proof.* Suppose string \(s\) of length \(n\) is given, and we want to find the LMSR of \(s\). Let \(s' = ss\) be the concatenation of two \(s\)'s. Then it can be shown that the LMSR of \(s\) is the leftmost minimal length-\(n\) substring of \(s'\). \(\square\)

We provide a quantum algorithm for the function problem of LMSR with better quantum query complexity as follows.

**Theorem 3.3** (Quantum query complexity of function problem). The worst-case quantum query complexity of Problem 7 is \(\sqrt{n} \cdot 2^{O(\sqrt{\log n})}\).

*Proof.* By Proposition 3.2 we only have to provide a quantum algorithm that solves the problem of minimal length-\(\ell\) substrings defined in Problem 3. The framework of our algorithm follows from that of [AJ22] but is slightly improved.

Suppose \(s\) is a string of length \(n\), and \(O_\ell\) is the quantum oracle with access to \(s\). Now we want to find the leftmost minimal length-\(\ell\) substring of \(s\). For convenience, we also find the rightmost minimal length-\(\ell\) substring of \(s\). Let \(b\) be some parameter to be determined, and \(m = \lfloor \ell/b \rfloor\). We will recursively reduce the problem of minimal length-\(\ell\) substrings of size \(n\) to \(\lceil n/m \rceil - 1\) sub-problems of size at most \(2m\).

To this end, for every \(i \in \lceil n/m \rceil - 1\), the string of the \(i\)-th sub-problem is
\[
s_i = s[im..\min\{(i + 2)m - 1, n - 1\}],
\]
and we recursively solve the problem of minimal length-\(\ell_i\) substrings with \(\ell_i = m\). Suppose \(x_i\) (resp. \(y_i\)) is the index of the leftmost (resp. rightmost) minimal length-\(\ell_i\) substring of \(s_i\), where \(im \leq x_i \leq y_i \leq \min\{(i + 2)m - 1, n - 1\}\). Note that
\[
s_{x_i..x_i + m - 1} = s_{y_i..y_i + m - 1} = \min_{im \leq j \leq \min\{(i + 2)m - 1, n - 1\}} s_{j..j + m - 1}.
\]

By Theorem 3.1 we know that the index of the minimal length-\(\ell\) substring of \(s\) exists among the indexes \(x_i\) and \(y_i\). Therefore, we only have to find the minimal length-\(\ell\) substring among the \(2(\lceil n/m \rceil - 1)\) indexes \(x_i\) and \(y_i\). To achieve this, we construct a comparator \(cmp: \lceil n/m \rceil - 1 \times \lceil n/m \rceil - 1 \rightarrow \{0, 1\}\) such that
\[
 cmp(i, j) = \begin{cases} 
 1, & \min\{s_{x_i..x_i + \ell - 1}, s_{y_i..y_i + \ell - 1}\} < \min\{s_{x_j..x_j + \ell - 1}, s_{y_j..y_j + \ell - 1}\}, \\
 0, & \text{otherwise.}
\end{cases}
\]

Let \(solve(s, \ell)\) denote the function call of the main problem, and assume that \(solve(s, \ell)\) will return two indexes \(x\) and \(y\) indicating the leftmost and rightmost indexes of the minimal length-\(\ell\) substrings with success probability \(\Omega(1)\). Then, the bounded-error quantum oracle \(O_{cmp}\) with respect to the comparator \(cmp\) works as follows:

1. Compute \(x_i\) and \(y_i\) by calling \(solve(s_i, \ell_i)\) with success probability \(\Omega(1)\).
2. Compute \(x_j\) and \(y_j\) by calling \(solve(s_j, \ell_j)\) with success probability \(\Omega(1)\).
3. Find the minimal length-$\ell$ substring out of the four with indexes $x_i, y_i, x_j$ and $y_j$ by $O(1)$ calls of Grover search [Gro96] with success probability $\Omega(1)$. Each call of Grover search uses $O(\sqrt{\ell})$ queries to $O_s$.

It can be seen that the above process will give a correct answer with probability $\Omega(1)$, and the success probability can be made to $\geq 2/3$ only by $O(1)$ repetitions. Let $T(n)$ be the quantum query complexity of calling $\text{solve}(s, \ell)$ with $|s| = n$. Then, one query to $O_{\text{cmp}}$ has quantum query complexity

$$T(|s_i|) + T(|s_j|) + O(\sqrt{n}) \leq 2T(2m) + O(\sqrt{n}).$$

By Lemma 2.4, we can find the index $k$ such that the minimal length-$\ell$ substring has index either $x_k$ or $y_k$ with success probability $\Omega(1)$ using $O(\sqrt{n/m})$ queries to $O_{\text{cmp}}$. After that, we can find the index of the minimal length-$\ell$ substring out of $x_k$ and $y_k$ by $O(1)$ calls of Grover search with success probability $\Omega(1)$, with each call using $O(\sqrt{\ell})$ queries to $O_s$.

From the above process, with success probability $\Omega(1)$, we obtain an index $z$ of any minimal length-$\ell$ substring, and know that the minimal length-$\ell$ substring is $s[z..z + \ell - 1]$. Finally, by Lemma 2.8, we can find the leftmost and rightmost occurrences of $s[z..z + \ell - 1]$ with success probability $\Omega(1)$ using $O(\sqrt{n\log \ell + \ell\log^3 \ell \log \log \ell})$ queries to $O_s$. To sum up, we obtain a recurrence

$$T(n) \leq O\left(\sqrt{\frac{n}{m}}\right)\left(2T(2m) + O\left(\sqrt{\ell}\right)\right) + O\left(\sqrt{n\log \ell + \ell\log^3 \ell \log \log \ell}\right).$$

To solve the recurrence, we remove the big-$O$ notation by introducing a constant $c > 1$, then the recurrence becomes

$$T(n) \leq c\left(\sqrt{bT\left(\frac{n}{b}\right)} + \sqrt{bm} + \sqrt{n\log^3 n \log \log n}\right).$$

In the following, we assume that $\log n = \log_2 n$ for convenience. Let $d = 2\sqrt{\log c} > 0$ and choose $b(n) = 2^{d\sqrt{\log n}} = \omega(\text{poly}(\log n))$. We will show that $T(n) \leq \sqrt{nb(n)} = \sqrt{n2^{d\sqrt{\log n}}}$ for sufficiently large $n$ by induction. For sufficiently large $n$, it holds that $b(n) \geq \log^3 n \log \log n$, and thus the recurrence becomes

$$T(n) \leq c\left(\sqrt{b(n)T\left(\frac{n}{b(n)}\right)} + \sqrt{nb(n)} + \sqrt{n\log^3 n \log \log n}\right),$$

$$\leq c\left(\sqrt{b(n)T\left(\frac{n}{b(n)}\right)} + 2\sqrt{nb(n)}\right).$$

By induction, we have

$$T(n) \leq c\left(\sqrt{b(n)}\sqrt{\frac{n}{b(n)}}b\left(\frac{n}{b(n)}\right) + 2\sqrt{nb(n)}\right)$$

$$= c\sqrt{n}\left(b\left(\frac{n}{b(n)}\right) + 2\sqrt{b(n)}\right).$$
To see that \( T(n) \leq \sqrt{nb(n)} \) for sufficiently large \( n \), it is sufficient to show that

\[
\lim_{n \to \infty} \frac{b\left( \frac{n}{b(n)} \right) + 2\sqrt{b(n)}}{b(n)} < \frac{1}{c}. \tag{1}
\]

Note that

\[
b\left( \frac{n}{b(n)} \right) = 2^d \sqrt{\log n - d \sqrt{\log n}}, \quad \text{and} \quad \lim_{n \to \infty} \frac{2\sqrt{b(n)}}{b(n)} = 0.
\]

The left hand side of Eq. (1) becomes

\[
\lim_{n \to \infty} \frac{2^d \sqrt{\log n - d \sqrt{\log n}}}{2^d \sqrt{\log n}} = \lim_{n \to \infty} \frac{2^d (\sqrt{\log n - d \sqrt{\log n}})}{2^d \sqrt{\log n - d \sqrt{\log n} + \sqrt{\log n}}}
\]

\[
= \lim_{n \to \infty} 2^{d \frac{\sqrt{\log n}}{\sqrt{\log n}}} = 2^{d \left( 1 + \sqrt{\frac{d}{\sqrt{\log n}}} \right)} = 2^{d \frac{1}{c^2}} = \frac{1}{c^2} < \frac{1}{c}.
\]

Therefore, the solution to the recurrence is

\[
T(n) \leq \sqrt{nb(n)} = \sqrt{n} 2^{O(\sqrt{\log n})}.
\]

Compared to the approach in [AJ22], our speedup mainly comes from the appropriate use of optimal quantum minimum finding on bounded-error oracles (see Lemma 2.4). It can be seen that the quantum minimum finding in Lemma 2.4 with query complexity \( O(\sqrt{n}) \) only achieves a logarithmic improvement over the naïve method by error reduction with query complexity \( O(\sqrt{n \log n}) \). However, it turns out that such \( \text{poly}(\log n) = \tilde{O}(1) \) improvement on the frequently used subroutine can bring a \( 2^{\log n^c} = \omega(\text{poly}(\log n)) \) speedup beyond polylogarithmic factors.

4 Decision Problem of LMSR

In this section, we study the quantum query complexity of the decision problem of LMSR. Because a slower quantum string matching algorithm was used in the approach of [CKKD+22], there is still room to improve the quantum query complexity of the decision problem of LMSR obtained in [CKKD+22]. Directly replacing it by the quantum algorithm given in Lemma 2.8 can already obtain a slightly better query complexity \( O(\sqrt{n \log \log n \log^2 n}) \). Nevertheless, we can obtain even better quantum query complexity of the decision problem of LMSR with the preprocessing of certain deterministic samples.

As in [CKKD+22], we will introduce the decision problem of minimal length-\( \ell \) substrings.

**Problem 4** (Minimal length-\( \ell \) substrings, decision version). *Given string \( s \) of length \( n \) and string \( t \) of length \( \ell \leq n \), determine if \( t \leq s[i..i+\ell-1] \) for every \( i \in [n-\ell] \).*

For completeness, we show that the decision problem of LMSR can be reduced to that of minimal length-\( \ell \) substrings.

**Proposition 4.1.** The decision problem of LMSR can be reduced to the decision problem of minimal length-\( \ell \) substrings.
Proof. Suppose string $s$ of length $n$ and an index $k \in [n]$ are given, and we want to determine if $s^{(k)} \leq s^{(i)}$ for every $i \in [n]$. Let $s' = ss$ be the concatenation of two $s$'s. Then it can be shown that the problem can be reduced to check if $t' \leq s'[i..i + \ell - 1]$ by taking $t' = s^{(k)}$ and $\ell = n$. \hfill $\blacksquare$

**Theorem 4.2** (Quantum query complexity of decision problem). *The worst-case quantum query complexity of Problem 2 is $O\left(\sqrt{n \log^2 n \log \log n}\right)$.*

Proof. By Proposition 4.1 we only have to solve the decision problem of minimal length-$\ell$ substrings defined in Problem 4 under the case that $\ell = \lceil n/2 \rceil$. The framework of our algorithm follows from that of Lemma 3.3 but is slightly improved. Intuitively, this also follows Theorem 3.3 by taking $b = 2$. We first analyze the query complexity if directly replacing the quantum string matching by a better one given in Lemma 2.8, which gives a slightly better quantum query complexity $O\left(\sqrt{n \log \log n \log^2 n}\right)$; then we illustrate how preprocessing can help to improve the quantum query complexity of divide and conquer, and finally obtain an even better query complexity $O\left(\sqrt{n \log^3 n \log \log n}\right)$.

Let $\Sigma$ be the finite alphabet and $f_n: \Sigma^n \times \Sigma^{\lceil n/2 \rceil} \rightarrow \{0,1\}$ defined by $f_n(s, t) = 1$ if $t \leq s[i..i + \lceil n/2 \rceil - 1]$ for every $i \in [n - \lceil n/2 \rceil]$; and 0 otherwise. Then, we have

$$f_n(s, t) = f_{\lceil n/2 \rceil}(s[0..2\lceil n/4 \rceil - 1], t[0..\lceil n/4 \rceil - 1])$$

$$\wedge f_{\lceil n/2 \rceil}(s[\lceil n/4 \rceil..3\lceil n/4 \rceil - 1], t[0..\lceil n/4 \rceil - 1])$$

$$\wedge g_n(s, t),$$

where $g_n(s, t)$ checks if $t$ is lexicographically smaller than the minimal length-$\lceil n/2 \rceil$ substring of $s$. $g_n(s, t)$ can be computed as follows.

1. By Lemma 2.8, we can find the leftmost and rightmost occurrences, denoted $x_1$ and $y_1$, of $t[0..\lceil n/4 \rceil - 1]$ in $s[0..2\lceil n/4 \rceil - 1]$ with success probability $\Omega(1)$ with quantum query complexity $O\left(\sqrt{n \log^3 n \log \log n}\right)$. Note that we only need to consider the case that $x_1$ and $y_1$ exist; otherwise, $f_{\lceil n/2 \rceil}(s[0..2\lceil n/4 \rceil - 1], t[0..\lceil n/4 \rceil - 1]) = 0$, then $f_n(s, t)$ always holds the correct answer regardless of how $g_n(s, t)$ works.

2. By Lemma 2.8, we can find the leftmost and rightmost occurrences, denoted $x_2$ and $y_2$, of $t[0..\lceil n/4 \rceil - 1]$ in $s[\lceil n/4 \rceil..3\lceil n/4 \rceil - 1]$ with success probability $\Omega(1)$ with quantum query complexity $O\left(\sqrt{n \log^3 n \log \log n}\right)$. Note that we only need to consider the case that $x_2$ and $y_2$ exist; otherwise, $f_{\lceil n/2 \rceil}(s[\lceil n/4 \rceil..3\lceil n/4 \rceil - 1], t[0..\lceil n/4 \rceil - 1]) = 0$, then $f_n(s, t)$ always holds the correct answer regardless of how $g_n(s, t)$ works.

3. We only have to guarantee that $g_n(s, t)$ works correctly when

$$f_{\lceil n/2 \rceil}(s[0..2\lceil n/4 \rceil - 1], t[0..\lceil n/4 \rceil - 1]) = f_{\lceil n/2 \rceil}(s[\lceil n/4 \rceil..3\lceil n/4 \rceil - 1], t[0..\lceil n/4 \rceil - 1]) = 1.$$

In this case, the four indexes $x_1, y_1, x_2$ and $y_2$ exist and they represent the minimal length-$\lceil n/4 \rceil$ substring of $s$. By Theorem 3.1 we only have to check if four substrings of length $\lceil n/2 \rceil$ with start indexes $x_1, y_1, x_2$ and $y_2$ are lexicographically not smaller than $t$. This can be done by Grover search [Gro96] with success probability $\Omega(1)$ with quantum query complexity $O(\sqrt{n})$. 

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To sum up, \( g_n(s,t) \) can be solved with quantum query complexity \( O(\sqrt{n \log^3 n \log \log n}) \). By Theorem 2.2 we have

\[
(\text{Adv}(f_n))^2 \leq 2(\text{Adv}(f_{\lfloor n/2 \rfloor}))^2 + (\text{Adv}(g_n))^2 \\
\leq 2(\text{Adv}(f_{\lfloor n/2 \rfloor}))^2 + O((Q(g_n))^2) \\
= 2(\text{Adv}(f_{\lfloor n/2 \rfloor}))^2 + O(n \log^3 n \log \log n).
\]

Let \( T(n) = (\text{Adv}(f_n))^2 \). Then, we have

\[
T(n) = 2T\left(\frac{n}{2}\right) + O(n \log^3 n \log \log n).
\]

In the following, we assume that \( \log n = \log_2 n \) for convenience. To solve the recurrence, we note that

\[
\frac{T(n)}{n} = \frac{T(n/2)}{n/2} + O(\log^3 n \log \log n) \\
= O\left(\sum_{k=0}^{\lfloor \log n \rfloor} \log^3 \frac{n}{2^k} \log \log \frac{n}{2^k}\right) \\
\leq O\left(\sum_{k=0}^{\lfloor \log n \rfloor} \log^3 n \log \log n\right) = O(\log^4 n \log \log n).
\]

This implies \( T(n) = O(n \log^4 n \log \log n) \), and thus \( \text{Adv}(f_n) = O(\sqrt{n \log \log n \log^2 n}) \). By Theorem 2.1 we conclude that the quantum query complexity of the above algorithm is \( O(\sqrt{n \log \log n \log^2 n}) \).

In fact, we can improve the query complexity of \( f_n \) by noting that the computation of \( g_n \) calls a complete quantum string matching algorithm, however, \( g_n \) only involves special patterns that are prefixes of \( t \). Specifically, \( g_n \) only involves prefixes of \( t \) of length \( \lceil n/2^k \rceil \) for every integer \( k \geq 1 \), there are only \( O(\log n) \) possible patterns in total. Looking into the framework of quantum string matching (see Section 2.6), we find that it is 2-step: (1) find a deterministic sample of the pattern; and (2) find an occurrence of the pattern in the text with the help of the deterministic sample. The first step takes \( O(\sqrt{m \log^3 m \log \log m}) \) quantum query complexity, and the second step takes \( O(\sqrt{n \log m}) \), which is much less than the former. So our strategy is to preprocess the deterministic samples of all possible patterns, which are prefixes of \( t \) of length \( \lceil n/2^k \rceil \). To achieve this, we use Lemma 2.6 to find the deterministic sample of \( t[0..\lceil n/2^k \rceil - 1] \) for every \( k \in [[\log n] + 1) \). This can be done with probability \( \Omega(1) \) as follows:

- **Preprocessing:** For every \( k \in [[\log n] + 1) \), we pre-compute the deterministic sample of \( t[0..\lceil n/2^k \rceil - 1] \) with success probability \( 1 - O(1/\log(n)) \). By Lemma 2.6 this can be done with quantum query complexity

\[
O\left(\sqrt{\frac{n}{2^k} \log \frac{n}{2^k} \log \log \frac{n}{2^k} \log n}\right).
\]

The combined success probability for all cases for \( k \in [[\log n]] \) is \( \Omega(1) \), and the total quantum query complexity is

\[
Q_{\text{pre}} = O\left(\sum_{k=0}^{\lfloor \log n \rfloor} \sqrt{\frac{n}{2^k} \log \frac{n}{2^k} \log \log \frac{n}{2^k} \log n}\right) = O\left(\sqrt{n \log^3 n \log \log n}\right).
\]
Then, we modify the first and second steps of the computation of $g_n$, with the third step unchanged, as follows.

1'. By Lemma 2.7, we can find the leftmost and rightmost occurrences, denoted $x_1$ and $y_1$, of $t[0..\lceil n/4 \rceil - 1]$ in $s[0..2\lceil n/4 \rceil - 1]$ with success probability $\Omega(1)$ with quantum query complexity $O(\sqrt{n \log n})$, with the help of the pre-computed deterministic sample of $t[0..\lceil n/4 \rceil - 1]$.

2'. By Lemma 2.7, we can find the leftmost and rightmost occurrences, denoted $x_2$ and $y_2$, of $t[0..\lceil n/4 \rceil - 1]$ in $s[\lceil n/4 \rceil..3\lceil n/4 \rceil - 1]$ with success probability $\Omega(1)$ with quantum query complexity $O(\sqrt{n \log n})$, with the help of the pre-computed deterministic sample of $t[0..\lceil n/4 \rceil - 1]$.

In this way, $g_n(s, t)$ can be solved with quantum query complexity $O(\sqrt{n \log n})$. With certain information (the deterministic samples) given beforehand, let $\text{Adv}'(f_n)$ be the adversary quantity for $f_n$, and let $Q'(g_n)$ be the quantum query complexity for computing $g_n$. Here, we note that $Q'(g_n) = O(\sqrt{n \log n})$. Similar to the previous analysis, we have

$$ (\text{Adv}'(f_n))^2 = 2(\text{Adv}'(f_{n/2}))^2 + O((Q'(g_n))^2) $$

$$ = 2(\text{Adv}'(f_{n/2}))^2 + O(n \log n). $$

Let $T'(n) = (\text{Adv}'(f_n))^2$. Then, we have

$$ T'(n) = 2T'(\frac{n}{2}) + O(n \log n). $$

By Theorem 2.3, the solution to $T'(n)$ is $T'(n) = O(n \log^2 n)$. That is, $\text{Adv}'(f_n) = O(\sqrt{n \log n})$. Let $Q'(f_n)$ be the adversary quantity for $f_n$ with certain information given beforehand to compute all $g_n$’s. Then, by Theorem 2.1, we have $Q'(f_n) = O(\text{Adv}'(f_n)) = O(\sqrt{n \log n})$. Finally, the quantum query complexity of $f_n$ is

$$ Q(f_n) = Q_{\text{pre}} + Q'(f_n) $$

$$ = O(\sqrt{n \log^3 n \log \log n}) + O(\sqrt{n \log n}) $$

$$ = O(\sqrt{n \log^3 n \log \log n}). $$

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