ℓ-WEIGHTS AND FACTORIZATION OF TRANSFER OPERATORS

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We analyze the ℓ-weights of the evaluation and q-oscillator representations of the quantum loop algebras $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ for $l = 1$ and $l = 2$ and prove factorization relations for the transfer operators of the associated quantum integrable systems.

Keywords: quantum groups, quantum integrable systems, transfer operators

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1. Introduction

This paper is devoted to the investigation of quantum integrable systems associated with the quantum loop algebras $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$. The central object of the quantum group approach is the universal $R$-matrix, which is an element of the tensor product of two copies of the quantum loop algebra. The integrability objects are constructed by choosing representations for the factors of that tensor product. The consistent application of the method based on the quantum group theory was initiated by Bazhanov, Lukyanov, and Zamolodchikov [1]–[3]. They considered the quantum version of the KdV theory. Subsequently, this method proved to be efficient in studying other quantum integrable models. Within the framework of this approach, $R$-operators [4]–[10], monodromy operators, and $L$-operators were constructed [9]–[14]. The corresponding sets of functional relations were discovered [1], [14]–[18].1 Recently, the quantum group approach was used to derive and investigate equations satisfied by the reduced density operators of the quantum chains related to an arbitrary loop algebra [19], [20].

Functional relations satisfied by commuting integrability objects are known to be a powerful tool in solving quantum integrable models. Typically, they are obtained by using the appropriate fusion rules for representations of the quantum loop algebra [21]–[27].

In fact, the most important functional relation is the factored representation of the transfer operator. All other relations appear to be its consequences (see [17], [28] for $l = 1$ and $l = 2$, where a direct operator approach was used to prove the factorization relations). For higher ranks, the computational difficulties that arise seem to be almost insurmountable. In this paper, we propose to use a different approach based on the analysis of the ℓ-weights of the representations. We demonstrate the effectiveness of the method in the cases $l = 1$ and $l = 2$.

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1For the terminology used here, we refer the reader to [14] and Sec. 5 below.
In Sec. 2, we define the quantum group $U_q(\mathfrak{gl}_{l+1})$ and discuss its Verma modules. These modules are then used to define the evaluation representations of the quantum loop algebras $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$ in Sec. 3. We give two definitions, the first in terms of the Drinfeld–Jimbo generators, and the second in terms of Drinfeld’s second realization. Two definitions are needed because the first is convenient for defining evaluation representations, and the generators used in the second definition contain an infinite commutative subalgebra used to define $\ell$-weights. In Sec. 4, we describe the category $\mathcal{O}$ of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$-modules, introduce the concept of $q$-characters, and define the Grothendieck ring of $\mathcal{O}$. In the same section, we define the $q$-oscillator representations of the Borel subalgebra $U_q(\mathcal{L}(\mathfrak{b}_{l+1}))$ of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$. They are most convenient for our purposes. In [29], the prefundamental representations introduced in [30] were used. We use the $q$-oscillator representations to construct $Q$-operators. In Sec. 5, we define various types of integrability objects and discuss their properties. The factorization of the transfer operators for $\ell=1$ and $\ell=2$ is proved in Sec. 6.

We fix the deformation parameter $\hbar$ such that $q = e^{\hbar}$ is not a root of unity, and define $q$-numbers by the equation

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \quad m \in \mathbb{Z}.$$ 

In this paper, an algebra, unless it is a Lie algebra, is a unital associative algebra. All algebras and vector spaces are assumed to be complex. By a tuple $(s_i)_{i \in I}$, we mean a map from a finite ordered set $I$ to some set of objects $S$. When a tuple has only one component or is used as a multi-index, we omit the parentheses in the notation.

2. Quantum group $U_q(\mathfrak{gl}_{l+1})$

2.1. Definition. We start with a short reminder about some basics facts on the Lie algebras $\mathfrak{gl}_{l+1}$ and $\mathfrak{sl}_{l+1}$. The standard basis of the standard Cartan subalgebra $\mathfrak{k}$ of the general linear Lie algebra $\mathfrak{gl}_{l+1}$ consists of the $(l+1) \times (l+1)$ matrices $K_a$, $a = 1, \ldots, l+1$, defined by the equation

$$K_a = E_{aa},$$

We let $\epsilon_a$, $a = 1, \ldots, l+1$, denote the elements of the dual basis. Below, we often identify an element $\mu \in \mathfrak{k}^*$ with the $(l+1)$-tuple formed by its components $\mu_a = \langle \mu, K_a \rangle$, $a = 1, \ldots, l+1$, with respect to this basis.

There are $l$ simple roots

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i = 1, \ldots, l.$$ 

The full system $\Delta^+$ of positive roots is formed by the roots

$$\alpha_{ij} = \sum_{k=i}^{j-1} \alpha_k = \epsilon_i - \epsilon_j, \quad 1 \leq i < j \leq l+1.$$ 

The system of negative roots is $\Delta_- = -\Delta_+$, and the full root system is $\Delta = \Delta_+ \cup \Delta_-$. The special Lie algebra $\mathfrak{sl}_{l+1}$ is a subalgebra of $\mathfrak{gl}_{l+1}$ formed by traceless matrices. The standard basis of the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_{l+1}$ is formed by the matrices

$$H_i = K_i - K_{i+1}, \quad i = 1, \ldots, l.$$ 

As the positive and negative roots, we take the restriction of $\alpha_{ij}$ and $-\alpha_{ij}$ to $\mathfrak{h}$. For the simple roots, we have

$$\langle \alpha_i, H_j \rangle = a_{ji},$$

\footnote{We let $E_{ab}$ denote the usual matrix units.}
where

\[ a_{ij} = -\delta_{i,j+1} + 2\delta_{ij} - \delta_{i+1,j}. \]

The matrix \( A = (a_{ij})_{i,j=1}^l \) is the Cartan matrix of the Lie algebra \( \mathfrak{sl}_{l+1} \).

We define the quantum group \( U_q(\mathfrak{gl}_{l+1}) \) as the algebra generated by the elements

\[ E_i, \quad F_i, \quad i = 1, \ldots, l, \quad q^X, \quad X \in \mathfrak{k}, \]

satisfying the defining relations

\[
q^0 = 1, \quad q^X q^{X_2} = q^{X_1 + X_2}, \quad q^X E_i q^{-X} = q^{(i,i)} E_i, \quad q^X F_i q^{-X} = q^{-(i,i)} F_i, \quad [E_i, F_j] = \delta_{ij} \frac{q^{K_{i,j} - K_{i+1,j}} - q^{-K_{i,j} + K_{i+1,j}}}{q - q^{-1}}.
\]

In addition, there are the Serre relations. However, we do not use their explicit form in this paper and do not therefore present them here.

2.2. Poincaré–Birkhoff–Witt basis. The Abelian group

\[ Q = \bigoplus_{i=1}^l \mathbb{Z} \alpha_i \]

is called the root lattice of \( \mathfrak{gl}_{l+1} \). Assuming that

\[ E_i \in U_q(\mathfrak{gl}_{l+1})_{\alpha_i}, \quad F_i \in U_q(\mathfrak{gl}_{l+1})_{-\alpha_i}, \quad q^X \in U_q(\mathfrak{gl}_{l+1})_0 \]

for any \( i = 1, \ldots, l \) and \( X \in \mathfrak{k} \), we endow the algebra \( U_q(\mathfrak{gl}_{l+1}) \) with a \( Q \)-gradation. An element \( x \) of \( U_q(\mathfrak{gl}_{l+1}) \) is called a root vector corresponding to a root \( \gamma \) of \( \mathfrak{gl}_{l+1} \) if \( x \in U_q(\mathfrak{gl}_{l+1})_{\gamma} \). In particular, \( E_i \) and \( F_i \) are root vectors corresponding to the roots \( \alpha_i \) and \( -\alpha_i \). Following Jimbo [31], we introduce the elements \( E_\gamma \) and \( F_\gamma \) corresponding to all roots \( \gamma \in \Delta \) with the help of the relations

\[
E_{\alpha_{i,j+1}} = E_i, \quad E_{\alpha_{i,j}} = E_{\alpha_{i,j-1}} E_{\alpha_{j-1,j}} - q E_{\alpha_{j-1,j}} E_{\alpha_{i,j-1}}, \quad j - i > 1,
\]

and

\[
F_{\alpha_{i,j+1}} = F_i, \quad F_{\alpha_{i,j}} = F_{\alpha_{i,j-1}} F_{\alpha_{j-1,j}} - q^{-1} F_{\alpha_{j-1,j}} F_{\alpha_{i,j-1}}, \quad j - i > 1.
\]

It is clear that the elements \( E_{\alpha_{ij}} \) and \( F_{\alpha_{ij}} \) are root vectors corresponding to the respective roots \( \alpha_{ij} \) and \( -\alpha_{ij} \). They are linearly independent, and together with the elements \( q^X, X \in \mathfrak{k} \) are called the Cartan–Weyl generators of \( U_q(\mathfrak{gl}_{l+1}) \). It can be shown that the ordered monomials constructed from the Cartan–Weyl generators form a Poincaré–Birkhoff–Witt basis of the quantum group \( U_q(\mathfrak{gl}_{l+1}) \). In this paper, we choose the following ordering. We first endow \( \Delta \) with the colexicographical order. It means that \( \alpha_{ij} \preceq \alpha_{mn} \) if \( j < n \), or \( j = n \) and \( i < m \). This is a normal ordering of roots in the sense of [32], [33] (also see [34]). In fact, this normal ordering is also a normal ordering of the \( \mathfrak{sl}_{l+1} \) roots. We then say that a monomial is ordered if it has the form

\[ F_{\alpha_{1,j_1}} \cdots F_{\alpha_{r,j_r}} q^X E_{\alpha_{m_1,n_1}} \cdots E_{\alpha_{m_s,n_s}}, \]

where \( \alpha_{i,j} \preceq \cdots \preceq \alpha_{s,r}, \alpha_{m_1,n_1} \preceq \cdots \preceq \alpha_{m_s,n_s} \) and \( X \) is an arbitrary element of \( \mathfrak{k} \). It is demonstrated in [35] that such monomials do form a basis of \( U_q(\mathfrak{gl}_{l+1}) \) (also see [36] for an alternative ordering).
2.3. Verma modules. We use the standard terminology of the representation theory. In particular, we say that a $U_q(\mathfrak{gl}_{l+1})$-module $V$ is a weight module if

$$V = \bigoplus_{\mu \in \mathfrak{t}^*} V_\mu,$$

where

$$V_\mu = \{ v \in V \mid q^X v = q^{(\mu, X)} v \text{ for any } X \in \mathfrak{t} \}.$$

The space $V_\mu$ is called the weight space of weight $\mu$, and a nonzero element of $V_\mu$ is called a weight vector. We say that $\mu \in \mathfrak{t}^*$ is a weight of $V$ if $V_\mu \neq \{0\}$. A weight $U_q(\mathfrak{gl}_{l+1})$-module $V$ is called a highest-weight module of highest weight $\mu$ if there exists a weight vector $v \in V$ of weight $\mu$ such that

$$E_i v = 0, \quad i = 1, \ldots, l, \quad \text{and} \quad V = U_q(\mathfrak{gl}_{l+1}) v.$$

The vector with the above properties is unique up to a scalar factor. We call it the highest-weight vector of $V$. Given $\mu \in \mathfrak{t}^*$, we let $\tilde{V}_\mu$ denote the corresponding Verma $U_q(\mathfrak{gl}_{l+1})$-module. This is a highest-weight module of highest weight $\mu$. We let $\tilde{\pi}^\mu$ denote the representation of $U_q(\mathfrak{gl}_{l+1})$ corresponding to $\tilde{V}^\mu$. Let $m$ be the $l(l+1)/2$-tuple of nonnegative integers $m_{ij}, 1 \leq i < j \leq l+1$, arranged in the colexicographic order of $(i, j)$. More explicitly,

$$m = (m_{12}, m_{13}, m_{23}, \ldots, m_{1j}, \ldots, m_{j-1,j}, \ldots, m_{1,l+1}, \ldots, m_{l,l+1}).$$

The vectors

$$v_m = F_{12}^{m_{12}} F_{13}^{m_{13}} F_{23}^{m_{23}} \ldots F_{1,j}^{m_{1,j}} \ldots F_{j-1,j}^{m_{j-1,j}} \ldots F_{1,l+1}^{m_{1,l+1}} \ldots F_{l,l+1}^{m_{l,l+1}} v_0$$

(1)

where the highest-weight vector is denoted by $v_0$ for consistency, form a basis of $\tilde{V}^\mu$. The explicit relations describing the action of the generators $q^X, E_i,$ and $F_i$ of the quantum group $U_q(\mathfrak{gl}_{l+1})$ on a general basis vector $v_m$ are obtained in [35].

We note that $\tilde{V}^\mu$ is an infinite-dimensional $U_q(\mathfrak{gl}_{l+1})$-module. However, if $\mu_i - \mu_{i+1} \in \mathbb{Z}_{\geq 0}$ for all $i = 1, \ldots, l$, there is a unique maximal submodule of $\tilde{V}^\mu$, such that the corresponding quotient module is simple and finite dimensional. We let $V^\mu$ and $\pi^\mu$ respectively denote this $U_q(\mathfrak{gl}_{l+1})$-module and the corresponding representation. We note that any finite-dimensional $U_q(\mathfrak{gl}_{l+1})$-module can be constructed in this way.

The weights $\omega_a \in \mathfrak{t}^*, a = 1, \ldots, l+1$, defined as

$$\omega_a = \sum_{b=1}^a e_b = (1, \ldots, 1, 0, \ldots, 0),$$

correspond to finite-dimensional representations called fundamental ones. The restriction of $\omega_i, i = 1, \ldots, l,$ to $\mathfrak{h}$ are the fundamental weights of $\mathfrak{sl}_{l+1}$, and hence

$$\langle \omega_i, H_j \rangle = \delta_{ij}$$

for all $i, j = 1, \ldots, l$. 

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3. Quantum loop algebra $U_q(\mathcal{L}(g))$

3.1. Definition in terms of Drinfeld–Jimbo generators. Let $g$ be a complex finite-dimensional simple Lie algebra of rank $l$, $\mathfrak{h}$ a Cartan subalgebra of $g$, and $\Delta$ the root system of $g$ relative to $\mathfrak{h}$ (see, e.g., books [37], [38]). We fix a system of simple roots $\alpha_i$, $i = 1, \ldots, l$. The corresponding coroots $\check{\alpha}_i$ form a basis of $\mathfrak{h}$, and hence

$$\mathfrak{h} = \bigoplus_{i=1}^{l} \mathbb{C}\check{\alpha}_i.$$ 

The Cartan matrix $A = (a_{ij})_{i,j=1}^{l}$ of $g$ is defined by the equation

$$a_{ij} = \langle \alpha_j, \check{\alpha}_i \rangle.$$ 

We note that any Cartan matrix is symmetrizable. This means that there exists a diagonal matrix $D = \text{diag}(d_1, \ldots, d_l)$ such that the matrix $DA$ is symmetric and $d_i$, $i = 1, \ldots, l$, are positive integers. Such a matrix is defined up to a nonzero scalar factor. We fix the integers $d_i$ assuming that they are relatively prime.

Following Kac [39], we use the notation $\mathcal{L}(g)$ for the loop algebra of $g$, $\tilde{\mathcal{L}}(g)$ for its standard central extension by the one-dimensional center $\mathbb{C}k$, and $\hat{\mathcal{L}}(g)$ for the Lie algebra obtained from $\tilde{\mathcal{L}}(g)$ by adding a natural derivation $d$. By definition,

$$\hat{\mathcal{L}}(g) = \mathcal{L}(g) \oplus \mathbb{C}k \oplus \mathbb{C}d,$$

and as the Cartan subalgebra of $\hat{\mathcal{L}}(g)$ we use the space

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k \oplus \mathbb{C}d.$$

The Lie algebra $\hat{\mathcal{L}}(g)$ is isomorphic to the affine algebra associated with the extended Cartan matrix of $g$. Here, the basis coroots are $\check{\alpha}_i$, $i = 1, \ldots, l$, and

$$\check{\alpha}_0 = k - \sum_{i=1}^{l} a_i \check{\alpha}_i.$$ 

The integers $\check{a}_i$, $i = 1, \ldots, l$, together with $\check{a}_0 = 1$, are the dual Kac labels of the Dynkin diagram associated with the extended Cartan matrix of $g$. Thus, we have

$$\hat{\mathfrak{h}} = \bigoplus_{i=0}^{l} \mathbb{C}\check{\alpha}_i \oplus \mathbb{C}d.$$ 

To introduce the corresponding simple roots, we identify the space $\mathfrak{h}^*$ with the subspace of $\hat{\mathfrak{h}}^*$ defined as

$$\{ \lambda \in \hat{\mathfrak{h}}^* \mid \langle \lambda, k \rangle = 0, \langle \lambda, d \rangle = 0 \},$$

and let $\delta$ be the element of $\hat{\mathfrak{h}}^*$ defined by the equations

$$\langle \delta, \check{\alpha}_i \rangle = 0, \quad i = 0, 1, \ldots, l, \quad \langle \delta, d \rangle = 1.$$
Then the simple roots are \( \alpha_i, \ i = 1, \ldots, l \), and \( \alpha_0 = \delta - \theta \), where

\[
\theta = \sum_{i=1}^{l} a_i \alpha_i
\]

is the highest root of \( \Delta \). The integers \( a_i, \ i = 1, \ldots, l \), together with \( a_0 = 1 \) are the Kac labels of the Dynkin diagram associated with the extended Cartan matrix of \( \mathfrak{g} \).

It can be shown that the equation

\[
a_{ij} = \langle \alpha_j, \hat{\alpha}_i \rangle, \quad i, j = 0, 1, \ldots, l,
\]

gives the entries of the extended Cartan matrix \( A^{(1)} \) of the Lie algebra \( \mathfrak{g} \). Complementing the integers \( d_i, i = 1, \ldots, l \), with a suitable integer \( d_0 \), it can be shown that the matrix \( A^{(1)} \) is symmetrizable. We note that for \( \mathfrak{g} = \mathfrak{sl}_{l+1}, d_i = 1 \) for all \( i = 0, 1, \ldots, l \).

The system of positive roots of the affine algebra \( \hat{\mathcal{L}}(\mathfrak{g}) \) is

\[
\hat{\Delta}_+ = \{ \gamma + n\delta | \gamma \in \Delta_+, n \in \mathbb{Z}_{\geq 0} \} \cup \{ n\delta | n \in \mathbb{Z}_{\geq 0} \} \cup \{ (\delta - \gamma) + n\delta | \gamma \in \Delta_+, n \in \mathbb{Z}_{\geq 0} \},
\]

where \( \Delta_+ \) is the system of positive roots of the Lie algebra \( \mathfrak{g} \). The system of negative roots \( \hat{\Delta}_- \) of \( \hat{\mathcal{L}}(\mathfrak{g}) \) is \( \hat{\Delta}_- = -\hat{\Delta}_+ \), and the full root system is

\[
\hat{\Delta} = \hat{\Delta}_+ \cup \hat{\Delta}_- = \{ \gamma + n\delta | \gamma \in \Delta, n \in \mathbb{Z} \} \cup \{ n\delta | n \in \mathbb{Z} \setminus \{0\} \}.
\]

It is convenient for our purposes to set

\[
\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}k = \bigoplus_{i=0}^{l} \mathbb{C} \tilde{\alpha}_i.
\]

It is easy to show that for any \( \lambda \in \mathfrak{h}^* \), there is a unique element \( \tilde{\lambda} \in \tilde{\mathfrak{h}}^* \) such that

\[
\langle \tilde{\lambda}, k \rangle = 0, \quad \langle \tilde{\lambda}, h \rangle = \langle \lambda, h \rangle, \quad h \in \mathfrak{h}.
\]

For each \( i = 0, 1, \ldots, l \), we set \( q_i = q^{\tilde{\alpha}_i} \) and assume that \( q^\nu = e^{\nu} \) for any \( \nu \in \mathbb{C} \). We define the quantum loop algebra \( U_q(\hat{\mathcal{L}}(\mathfrak{g})) \) as the algebra generated by the elements \( e_i, f_i, i = 0, 1, \ldots, l \), and \( q^h, h \in \tilde{\mathfrak{h}} \) satisfying the relations

\[
q^{\nu k} = 1, \quad \nu \in \mathbb{C}, \quad q^{h_1} q^{h_2} = q^{h_1 + h_2},
\]

\[
q^h e_i q^{-h} = q^{\langle \alpha_i, h \rangle} e_i, \quad q^h f_i q^{-h} = q^{-\langle \alpha_i, h \rangle} f_i,
\]

\[
[e_i, f_j] = \delta_{ij} \frac{q_{i}^{\tilde{\alpha}_i} - q_{i}^{-\tilde{\alpha}_i}}{q_{i} - q_{i}^{-1}}
\]

for all \( i = 0, 1, \ldots, l \). There are also the Serre relations, whose explicit form is not used in this paper.

The quantum loop algebra \( U_q(\hat{\mathcal{L}}(\mathfrak{g})) \) is a Hopf algebra. The comultiplication \( \Delta \), the antipode \( S \), and the counit \( \varepsilon \) are given by the relations

\[
\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes 1 + q_i^{-\tilde{\alpha}_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q_i^{\tilde{\alpha}_i} + 1 \otimes f_i,
\]

\[
S(q^h) = q^{-h}, \quad S(e_i) = -q_i^{-\tilde{\alpha}_i} e_i, \quad S(f_i) = -f_i q_i^{\tilde{\alpha}_i},
\]

\[
\varepsilon(q^h) = 1, \quad \varepsilon(e_i) = 0, \quad \varepsilon(f_i) = 0.
\]

We do not use these relations in this paper. They are given here to fix our conventions.
To distinguish from the tensor product of maps, we write the tensor product of representations $\varphi$ and $\psi$ of $U_q(\mathcal{L}(g))$ as

$$\varphi \otimes \Delta \psi = (\varphi \otimes \psi) \circ \Delta$$

and, similarly, the tensor product of the corresponding $U_q(\mathcal{L}(g))$-modules $V$ and $W$ as $V \otimes_\Delta W$.

The Abelian group

$$\hat{Q} = \bigoplus_{i=0}^{l} \mathbb{Z} \alpha_i$$

is called the root lattice of $\hat{E}(g)$. Assuming that

$$e_i \in U_q(\mathcal{L}(g))_{\alpha_i}, \quad f_i \in U_q(\mathcal{L}(g))_{-\alpha_i}, \quad q^h \in U_q(\mathcal{L}(g))_0$$

for any $i = 0, 1, \ldots, l$ and $h \in \mathfrak{h}$, we endow the algebra $U_q(\mathcal{L}(g))$ with a $\hat{Q}$-gradation. An element $x$ of $U_q(\mathcal{L}(g))$ is called a root vector corresponding to a root $\gamma \in \hat{\Delta}$ if $x \in U_q(\mathcal{L}(g))_\gamma$. Linearly independent root vectors corresponding to all roots from $\hat{\Delta}$ can then be constructed (see, e.g., [4], [40]-[42] or [43], [44] for an alternative approach).

### 3.2. Drinfeld’s second realization

The quantum loop algebra $U_q(\mathcal{L}(g))$ can be realized in a different way [45], [46] as the algebra with the generators $\xi^\pm_{i,m}$ with $i = 1, \ldots, l$ and $m \in \mathbb{Z}$, $q^h$ with $h \in \mathfrak{h}$, and $\chi_{i,m}$ with $i = 1, \ldots, l$ and $m \in \mathbb{Z} \setminus \{0\}$. They satisfy the defining relations

$$q^0 = 1, \quad q^{h_1} q^{h_2} = q^{h_1 + h_2},$$

$$[q^h, \chi_{j,m}] = 0, \quad [\chi_{i,m}, \chi_{j,n}] = 0,$$

$$q^h \xi^\pm_{i,m} q^{-h} = q^{h(a_{i,j} - \delta_{i,j})} \xi^\pm_{j,n}, \quad [\chi_{i,m}, \xi^\pm_{j,n}] = \pm \frac{1}{m} [ma_{ij}] q^{\xi^\pm_{j,n+1}},$$

$$\xi^\pm_{i,m+1} \xi^\pm_{j,n} - q^{(a_{i,j})} \xi^\pm_{j,n} \xi^\pm_{i,m+1} = q^{a_{i,j}} \xi^\pm_{i,m} \xi^\pm_{j,n+1} - \xi^\pm_{j,n+1} \xi^\pm_{i,m};$$

$$[\xi^+_{i,m}, \xi^-_{j,n}] = \delta_{ij} \frac{q^{\alpha_i}}{q_i - q_i^-};$$

$$[\xi^+_{i,m}, \xi^-_{j,n}] = \delta_{ij} \frac{q^{\alpha_i}}{q_i - q_i^-}, \quad m + n > 0,$$

$$[\xi^+_{i,m}, \xi^-_{j,n}] = -\delta_{ij} \frac{q^{\alpha_i}}{q_i - q_i^-}, \quad m + n < 0,$$

and the Serre relations, whose explicit form is not important for our consideration. The quantities $\phi^\pm_{i,m}$, $i = 1, \ldots, l$, $m \in \mathbb{Z}_{>0}$, are given by the formal power series

$$1 + \sum_{m=1}^{\infty} \phi^\pm_{i,m} u^\pm = \exp \left( \pm (q - q^{-1}) \sum_{m=1}^{\infty} \chi_{i,\pm m} u^\pm \right).$$

We emphasize that our definition of $\phi^\pm_{i,m}$ is slightly different from the commonly used one.

### 3.3. Spectral parameters

In applications to the theory of quantum integrable systems, one usually considers families of representations of a quantum loop algebra and its subalgebras, parameterized by a complex parameter called a spectral parameter. We introduce a spectral parameter in the following way. We assume that a quantum loop algebra $U_q(\mathcal{L}(g))$ is $\mathbb{Z}$-graded,

$$U_q(\mathcal{L}(g)) = \bigoplus_{m \in \mathbb{Z}} U_q(\mathcal{L}(g))_m, \quad U_q(\mathcal{L}(g))_m U_q(\mathcal{L}(g))_n \subset U_q(\mathcal{L}(g))_{m+n},$$
such that any element \( x \in U_q(\mathcal{L}(\mathfrak{g})) \) can be uniquely represented as
\[
x = \sum_{m \in \mathbb{Z}} x_m, \quad x_m \in U_q(\mathcal{L}(\mathfrak{g}))_m.
\]

Given \( \zeta \in \mathbb{C}^\times \), we define the grading automorphism \( \Gamma_\zeta \) by the equation
\[
\Gamma_\zeta(x) = \sum_{m \in \mathbb{Z}} \zeta^m x_m.
\]

Now, for any representation \( \varphi \) of \( U_q(\mathcal{L}(\mathfrak{g})) \), we define the family \( \varphi_\zeta \) of representations as
\[
\varphi_\zeta = \varphi \circ \Gamma_\zeta.
\]

If \( V \) is the \( U_q(\mathcal{L}(\mathfrak{g})) \)-module corresponding to the representation \( \varphi \), we let \( V_\zeta \) denote the \( U_q(\mathcal{L}(\mathfrak{g})) \)-module corresponding to the representation \( \varphi_\zeta \).

In this paper, we endow \( U_q(\mathcal{L}(\mathfrak{g})) \) with a \( \mathbb{Z} \)-gradation assuming that
\[
q^h \in U_q(\mathcal{L}(\mathfrak{g}))_0, \quad e_i \in U_q(\mathcal{L}(\mathfrak{g}))_{s_i}, \quad f_i \in U_q(\mathcal{L}(\mathfrak{g}))_{-s_i},
\]
where \( s_i \) are arbitrary integers. We set
\[
s = \sum_{i=0}^l a_i s_i,
\]
where \( a_i \) are the Kac labels of the Dynkin diagram associated with the extended Cartan matrix of \( \mathfrak{g} \).

4. Highest \( \ell \)-weight representations

4.1. \( \ell \)-weights of \( U_q(\mathcal{L}(\mathfrak{g})) \)-modules. A \( U_q(\mathcal{L}(\mathfrak{g})) \)-module \( V \) is called a weight module if
\[
V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,
\]
where
\[
V_\lambda = \{ v \in V \mid q^h v = q^{\langle \lambda, h \rangle} v \text{ for any } h \in \mathfrak{h} \}.
\]
The notation \( \tilde{\lambda} \) for \( \lambda \in \mathfrak{h}^* \) is explained in Sec. 3.1. The space \( V_\lambda \) is called the weight space of weight \( \lambda \), and a nonzero element of \( V_\lambda \) is called a weight vector of weight \( \lambda \). We say that \( \lambda \in \mathfrak{h}^* \) is a weight of \( V \) if \( V_\lambda \neq \{0\} \).

A \( U_q(\mathcal{L}(\mathfrak{g})) \)-module \( V \) is said to be in the category \( \mathcal{O} \) if

a) \( V \) is a weight module all of whose weight spaces are finite dimensional;

b) there exists a finite number of elements \( \mu_1, \ldots, \mu_s \in \mathfrak{h}^* \) such that every weight of \( V \) belongs to the set \( \bigcup_{i=1}^s D(\mu_i) \), where \( D(\mu) = \{ \lambda \in \mathfrak{h}^* \mid \lambda \leq \mu \} \), with \( \leq \) being the usual partial order in \( \mathfrak{h}^* \) (see, e.g., book [38]).

Let a \( U_q(\mathcal{L}(\mathfrak{g})) \)-module \( V \) be in the category \( \mathcal{O} \). The algebra \( U_q(\mathcal{L}(\mathfrak{g})) \) contains an infinite-dimensional commutative subalgebra generated by the elements \( \phi_{i,\pm m}^\pm, \ i = 1, \ldots, l, \ m \in \mathbb{Z}_{>0}, \) and \( q^h, \ h \in \mathfrak{h} \). We can refine the weight decomposition in (2) as follows. Let \( \lambda \) be a weight of \( V \). By definition, the space \( V_\lambda \) is finite dimensional. The restriction of the action of the elements \( \phi_{i,\pm m}^\pm \) to \( V_\lambda \) constitutes a countable set of
pairwise commuting linear operators on $V$. Hence, there is a basis of $V$ that consists of eigenvectors and generalized eigenvectors of all those operators (see, e.g., book [47]). This leads to the following definitions.

An $\ell$-weight is a triple

$$\Lambda = (\lambda, \Lambda^+, \Lambda^-),$$

where $\lambda \in \mathfrak{h}^*$, $\Lambda^+ = (\Lambda^+_i(u))_{i=1}^l$ and $\Lambda^- = (\Lambda^-_i(u^{-1}))_{i=1}^l$ are $\ell$-tuples of formal power series

$$\Lambda^+_i(u) = 1 + \sum_{m \in \mathbb{Z}_{>0}} \Lambda^+_{i,m} u^m \in \mathbb{C}[[u]], \quad \Lambda^-_i(u^{-1}) = 1 + \sum_{m \in \mathbb{Z}_{>0}} \Lambda^-_{i,-m} u^{-m} \in \mathbb{C}[[u^{-1}]].$$

We let $\mathfrak{h}^*_\ell$ denote the set of $\ell$-weights.

We define a surjective homomorphism $\varpi: \mathfrak{h}^*_\ell \to \mathfrak{h}^*$ by the relation

$$\varpi(\Lambda) = \lambda,$$

if $\Lambda = (\lambda, \Lambda^+, \Lambda^-)$. We then have

$$V_\lambda = \bigoplus_{\varpi(\Lambda) = \lambda} V_\Lambda,$$

where $V_\Lambda$ is a subspace of $V_\lambda$ such that for any $v \in V_\Lambda$, there is $p \in \mathbb{Z}_{>0}$ such that

$$(\phi^+_{i,m} - \Lambda^+_{i,m})^p v = 0, \quad (\phi^-_{i,-m} - \Lambda^-_{i,-m})^p v = 0$$

for all $i = 1, \ldots, l$ and $m \in \mathbb{Z}_{>0}$. The space $V_\Lambda$ is called the $\ell$-weight space of $\ell$-weight $\Lambda$. We say that $\Lambda$ is an $\ell$-weight of $V$ if $V_{\Lambda} \neq \{0\}$. A nonzero element $v \in V_\Lambda$ such that

$$\phi^+_{i,m} v = \Lambda^+_{i,m} v, \quad \phi^-_{i,-m} v = \Lambda^-_{i,-m} v$$

for all $i = 1, \ldots, l$ and $m \in \mathbb{Z}_{>0}$ is said to be an $\ell$-weight vector of $\ell$-weight $\Lambda$. Every nontrivial $\ell$-weight space contains an $\ell$-weight vector.

For any two $\ell$-weights $\Lambda = (\lambda, \Lambda^+, \Lambda^-)$ and $\Xi = (\xi, \Xi^+, \Xi^-)$, we define the $\ell$-weight $\Lambda \Xi$ as

$$\Lambda \Xi = (\lambda + \xi, (\Lambda \Xi)^+, (\Lambda \Xi)^-),$$

where

$$(\Lambda \Xi)^+ = (\Lambda^+_i(u) \Xi^+_i(u))_{i=1}^l, \quad (\Lambda \Xi)^- = (\Lambda^-_i(u^{-1}) \Xi^-_i(u^{-1}))_{i=1}^l.$$

The product in Eq. (3) is an associative operation with respect to which $\mathfrak{h}^*_\ell$ is an Abelian group. Here,

$$\Lambda^{-1} = (-\lambda, (\Lambda^+)^{-1}, (\Lambda^-)^{-1}),$$

where

$$(\Lambda^+)^{-1} = (\Lambda^+_i(u)^{-1})_{i=1}^l, \quad (\Lambda^-)^{-1} = (\Lambda^-_i(u^{-1}))_{i=1}^l,$$

and the role of the identity element is played by the $\ell$-weight $(0, (\underbrace{1, \ldots, 1}_l), (\underbrace{1, \ldots, 1}_l))$. We note that by definition each $\Lambda^+_i(u)$ and each $\Lambda^-_i(u^{-1})$ is an invertible power series.

A $U_q(\mathcal{L}(\mathfrak{g}))$-module $V$ in the category $\mathcal{O}$ is called a highest $\ell$-weight module of highest $\ell$-weight $\Lambda$ if there exists an $\ell$-weight vector $v \in V$ of $\ell$-weight $\Lambda$ such that

$$e_i v = 0, \quad i = 1, \ldots, l, \quad \text{and} \quad V = U_q(\mathcal{L}(\mathfrak{g})) v.$$

Let $V$ and $W$ be respective highest $\ell$-weight $U_q(\mathcal{L}(\mathfrak{g}))$-modules in the category $\mathcal{O}$ of highest $\ell$-weights $\Lambda$ and $\Xi$. The submodule of $V \otimes \Lambda W$ generated by the tensor product of the highest $\ell$-weight vectors is a highest $\ell$-weight module of highest $\ell$-weight $\Lambda \Xi$. 

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4.2. Evaluation representations. A common strategy to construct representations of the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{sl}(i+1)))$ is to use Jimbo’s homomorphism $\varepsilon$ from $U_q(\mathcal{L}(\mathfrak{sl}(i+1)))$ to the quantum group $U_q(\mathfrak{gl}(i+1))$ defined by Eqs. [31]:

$$
\varepsilon(q^{\alpha_i}) = q^{\nu(K_i - K_i)},
\varepsilon(q^{\alpha_i}) = q^{\nu(K_i - K_{i+1})},
\varepsilon(e_0) = E_{i+1}q^{K_i + K_{i+1}},
\varepsilon(e_i) = E_{i+1},
\varepsilon(f_0) = E_{i+1}q^{K_i - K_{i+1}},
\varepsilon(f_i) = E_{i+1},
$$

where $i = 1, \ldots, l$. If $\pi$ is a representation of $U_q(\mathfrak{gl}(i+1))$, then $\pi \circ \varepsilon$ is a representation of $U_q(\mathcal{L}(\mathfrak{sl}(i+1)))$ called the evaluation representation. Given $\mu \in \mathfrak{h}^*$, we let $\tilde{\pi}^\mu$ denote the representation $\tilde{\pi}^\mu \circ \varepsilon$, where the representation $\tilde{\pi}^\mu$ is described in Sec. 2.3. Since the representation spaces of $\tilde{\pi}^\mu$ and $\tilde{\pi}^{\mu'}$ are the same, by a slight abuse of the notation we let $\tilde{V}^\mu$ denote both corresponding modules. The same convention is used for the the finite-dimensional counterparts, if there are any. It is common to call $\tilde{V}^\mu$ an evaluation module. It is a highest-weight $U_q(\mathcal{L}(\mathfrak{sl}(i+1)))$-module of highest weight given by the restriction of $\mu$ to $\mathfrak{h}^*$.

In fact, for any $\mu \in \mathfrak{h}^*$, the module $\tilde{V}^\mu_\zeta$ is also a highest $\ell$-weight module in the category $\mathcal{O}$. The $\ell$-weight spaces and corresponding $\ell$-weights for $l = 1, 2$ were found in [48]. It appears that all $\ell$-weight spaces are one-dimensional and are therefore generated by $\ell$-weight vectors. Although the $\ell$-weight vectors do not coincide with the basis vectors $v^i_\mathbf{m}$ defined by Eq. (1), they can also be labeled by the $(l + 1)/2$-tuples $\mathbf{m}$, and for the $\ell$-weights we use the notation

$$
\Lambda^\mu_{\mathbf{m}}(\zeta) = (\lambda^\mu_{\mathbf{m}}, \Lambda^\mu_{\mathbf{m}}^+(\zeta), \Lambda^\mu_{\mathbf{m}}^-(\zeta)).
$$

The highest $\ell$-weight vectors correspond to $\mathbf{m} = \mathbf{0}$.

For $l = 1$, we have $\mathbf{m} = (m_{12})$. The component $\lambda^\mu_{\mathbf{m}}$ of the $\ell$-weight $\Lambda^\mu_{\mathbf{m}}(\zeta)$ is given by the equation

$$
\lambda^\mu_{\mathbf{m}} = (\mu_1 - \mu_2 - 2m_{12})\omega_1,
$$

and for the only components of $\Lambda^\mu_{\mathbf{m}}^+(\zeta)$ and $\Lambda^\mu_{\mathbf{m}}^-(\zeta)$, we have the expression

$$
\Lambda^\mu_{\mathbf{m}}^+(\zeta, u) = \frac{(1 - q^{2\mu_1 + 1}\zeta u)(1 - q^{2\mu_1 + 1 - 2m_{12}}\zeta u)}{(1 - q^{2\mu_2 + 1}\zeta u)(1 - q^{2\mu_1 + 1 - 2m_{12}}\zeta u)}
$$

and

$$
\Lambda^\mu_{\mathbf{m}}^-(\zeta, u) = \frac{(1 - q^{2\mu_1 - 1}\zeta u - 1)(1 - q^{2\mu_1 - 1 - 2m_{12}}\zeta u - 1)}{(1 - q^{2\mu_2 - 1}\zeta u - 1)(1 - q^{2\mu_1 - 1 - 2m_{12}}\zeta u - 1)}.
$$

For $l = 2$, we have $\mathbf{m} = (m_{12}, m_{13}, m_{23})$. In this case,

$$
\lambda^\mu_{\mathbf{m}} = (\mu_1 - \mu_2 - 2m_{12} - m_{13} + m_{23})\omega_1 + (\mu_2 - \mu_3 + m_{12} - m_{13} - 2m_{23})\omega_2,
$$

and for the components of $\Lambda^\mu_{\mathbf{m}}^+(\zeta)$ and $\Lambda^\mu_{\mathbf{m}}^-(\zeta)$, we have

$$
\Lambda^\mu_{\mathbf{m}}^+(\zeta, u) = \frac{(1 - q^{2\mu_1 - 2m_{13} + 1}\zeta u)(1 - q^{2\mu_1 - 2m_{12} - 2m_{13} + 2}\zeta u)}{(1 - q^{2\mu_2 - 2m_{23}}\zeta u)(1 - q^{2\mu_1 - 2m_{12} - 2m_{13}}\zeta u)}
$$

and

$$
\Lambda^\mu_{\mathbf{m}}^-(\zeta, u) = \frac{(1 - q^{2\mu_1 - 2m_{13} - 1}\zeta u - 1)(1 - q^{2\mu_1 - 2m_{12} + 2m_{13} - 2}\zeta u - 1)}{(1 - q^{2\mu_2 - 2m_{23}}\zeta u - 1)(1 - q^{2\mu_1 - 2m_{12} + 2m_{13}}\zeta u - 1)}.
$$

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4.3. $\ell$-weights of $U_q(\mathcal{L}(b_+))$-modules. There are two standard Borel subalgebras of the quantum loop algebra $U_q(\mathcal{L}(g))$. In terms of the Drinfeld–Jimbo generators, they are defined as follows. The Borel subalgebra $U_q(\mathcal{L}(b_+))$ is the subalgebra generated by $e_i$ with $0 \leq i \leq l$ and $q^h$ with $h \in \bar{\mathfrak{h}}$, and the Borel subalgebra $U_q(\mathcal{L}(b_-))$ is the subalgebra generated by $f_i$ with $0 \leq i \leq l$ and $q^h$ with $h \in \bar{\mathfrak{h}}$. It is clear that these subalgebras are Hopf subalgebras of $U_q(\mathcal{L}(g))$. The description of $U_q(\mathcal{L}(b_+))$ and $U_q(\mathcal{L}(b_-))$ in terms of the Drinfeld generators is more intricate.

The category $\mathcal{O}$ of representations of $U_q(\mathcal{L}(b_+))$ is defined by exactly the same words as it was defined in Sec. 4.1 for $U_q(\mathcal{L}(g))$. By definition, any $U_q(\mathcal{L}(b_+))$-module $V$ in the category $\mathcal{O}$ admits the weight decomposition

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$  \hspace{1cm} (12)

which can be again refined by considering $\ell$-weights.

The Borel subalgebra $U_q(\mathcal{L}(b_+))$ does not contain the elements $\phi_{i,-m}^-$, $i = 1, \ldots, l$, $m \in \mathbb{Z}_{>0}$, and its infinite-dimensional commutative subalgebra is generated only by the elements $\phi_{i,m}^+$, $i = 1, \ldots, l$, $n \in \mathbb{Z}_{>0}$, and $q^h$, $h \in \bar{\mathfrak{h}}$. Accordingly, an $\ell$-weight $\Lambda$ is now a pair

$$\Lambda = (\lambda, \Lambda^+),$$  \hspace{1cm} (13)

where $\lambda \in \mathfrak{h}^*$ and $\Lambda^+$ is an $l$-tuple $\Lambda^+ = (\Lambda_i^+(u))_{i=1}^l$ of the formal series

$$\Lambda^+(u) = 1 + \sum_{m=1}^{\infty} \Lambda_i^+ u^m \in \mathbb{C}[[u]].$$

We let $\mathfrak{h}^*_\ell$ denote the set of $\ell$-weights (13).

For any two $\ell$-weights $\Lambda = (\lambda, \Lambda^+)$ and $\Xi = (\xi, \Xi^+)$, we define the $\ell$-weight $\Lambda \Xi$ as

$$\Lambda \Xi = (\lambda + \xi, (\Lambda \Xi)^+),$$  \hspace{1cm} (14)

where

$$(\Lambda \Xi)^+ = (\Lambda_i^+(u) \Xi_i^+(u))_{i=1}^l.$$

The product in (14) is an associative operation with respect to which $\mathfrak{h}^*_\ell$ is an Abelian group. Here,

$$\Lambda^{-1} = (-\lambda, (\Lambda^+)^{-1}),$$

where

$$(\Lambda^+)^{-1} = (\Lambda_i^+(u)^{-1})_{i=1}^l,$$

and the role of the identity element is played by the $\ell$-weight $(0, (1, \ldots, 1))$.

We define a surjective homomorphism $\varpi^+: \mathfrak{h}^*_\ell \rightarrow \mathfrak{h}^*$ by the relation

$$\varpi^+(\Lambda) = \lambda,$$

if $\Lambda = (\lambda, \Lambda^+)$. For any $V_\lambda$ entering decomposition (12), we now have

$$V_\lambda = \bigoplus_{\varpi^+(\Lambda) = \lambda} V_\lambda.$$
where $V_\Lambda$ is the subspace of $V_\Lambda$ such that for any $v$ in $V_\Lambda$ there is $p \in \mathbb{Z}_{>0}$ such that

$$(\phi_{i,m}^+ - \Lambda_{i,m}^+)v = 0$$

for all $i = 1, \ldots, l$ and $m \in \mathbb{Z}_{>0}$. The space $V_\Lambda$ is called the $\ell$-weight space of $\ell$-weight $\Lambda$. We say that $\Lambda$ is an $\ell$-weight of $V$ if $V_\Lambda \neq \{0\}$. A nonzero element $v \in V_\Lambda$ such that

$$\phi_{i,m}^+ v = \Lambda_{i,m}^+ v$$

for all $i = 1, \ldots, l$ and $m \in \mathbb{Z}_{>0}$ is said to be an $\ell$-weight vector of $\ell$-weight $\Lambda$. Every nontrivial $\ell$-weight space contains an $\ell$-weight vector.

A $U_q(\mathcal{L}(b_+))$-module $V$ is called a highest $\ell$-weight module of highest $\ell$-weight $\Lambda$ if there exists an $\ell$-weight vector $v \in V$ of $\ell$-weight $\Lambda$ such that

$$e_i v = 0, \quad 1 \leq i \leq l, \quad \text{and} \quad V = U_q(\mathcal{L}(b_+))v.$$ 

The vector with the above properties is unique up to a scalar factor. We call it the highest $\ell$-weight vector of $\ell$-weight $\Lambda$. Let $V$ and $W$ be highest $\ell$-weight $U_q(\mathcal{L}(b_+))$-modules in the category $\mathcal{O}$ of highest $\ell$-weights $\Lambda$ and $\Xi$ respectively. The submodule of $V \otimes_\Delta W$ generated by the tensor product of the highest $\ell$-weight vectors is a highest $\ell$-weight module of highest $\ell$-weight $\Lambda \Xi$.

For any $U_q(\mathcal{L}(b_+))$-module $V$ in the category $\mathcal{O}$ and an element $\xi \in \mathfrak{h}^*$, we define the shifted $U_q(\mathcal{L}(b_+))$-module $V[\xi]$ by shifting the action of the generators $q^h$. Namely, if $\varphi$ is the representation of $U_q(\mathcal{L}(b_+))$ corresponding to the module $V$ and $\varphi[\xi]$ is the representation corresponding to the module $V[\xi]$, then

$$\varphi[\xi](e_i) = \varphi(e_i), \quad i = 1, \ldots, l, \quad \varphi[\xi](q^h) = q^{\xi(h)\varphi(q^h)}, \quad h \in \mathfrak{h}.$$ 

(15)

It is clear that the module $V[\xi]$ is in the category $\mathcal{O}$.

For an arbitrary $U_q(\mathcal{L}(g))$-module $V$, we define the shifted module $V[\xi]$, $\xi \in \mathfrak{h}^*$, as a $U_q(\mathcal{L}(b_+))$-module obtained by first restricting $V$ to $U_q(\mathcal{L}(b_+))$ and then shifting the obtained $U_q(\mathcal{L}(b_+))$-module.

### 4.4. Oscillator representations.

Representations of $U_q(\mathcal{L}(b_+))$ can be constructed as restrictions of the representations of $U_q(\mathcal{L}(g))$. However, no less important for the theory of integrable systems are representations that cannot be obtained by such a procedure.

For $\mathfrak{g} = \mathfrak{sl}_{l+1}$ in this paper, we use the family of $U_q(\mathcal{L}(b_+))$ representations, called the $q$-oscillator representations, which cannot be extended to representations of $U_q(\mathcal{L}(\mathfrak{sl}_{l+1}))$. We define them in two steps. First, we introduce a homomorphism of $U_q(\mathcal{L}(b_+))$ into the $l$th tensor power of the $q$-oscillator algebra $\text{Osc}_q$, and then for each factor of the tensor product we choose a suitable representation of $\text{Osc}_q$.

The $q$-oscillator algebra $\text{Osc}_q$ is an algebra with generators $b^+, b$ and $q^\nu N$, $\nu \in \mathbb{C}$, and the relations

$$q^0 = 1, \quad q^{\nu_1 N} q^{\nu_2 N} = q^{(\nu_1 + \nu_2)N},$$

$$q^{-\nu_2 N} b^+ q^{\nu_1 N} = q^{\nu_1 N} b^+, \quad q^\nu b^- q^{-\nu N} = q^{-\nu} b^-,$$

$$b^+ b = \frac{q^N - q^{-N}}{q - q^{-1}}, \quad b b^+ = \frac{q^{N-1} - q^{-N}}{q - q^{-1}}.$$ 

Two basic representations of $\text{Osc}_q$ are of particular interest. First, let $W^+$ be the free vector space on the sequence $(w_n)_{n \in \mathbb{Z}_{>0}}$. It can be shown that the relations

$$q^\nu N w_n = q^{\nu n} w_n,$$

$$b^+ w_n = w_{n+1}, \quad b w_n = [n]_q w_{n-1},$$
where we set $w_{-1} = 0$, endow $W^+$ with the structure of an $\text{Osc}_q$-module. We let $\chi^+$ denote the corresponding representation of the algebra $\text{Osc}_q$. Further, we again let $W^-$ be a free vector space on the sequence $(u_n)_{n \in \mathbb{Z}_{\geq 0}}$. The relations

$$q^{\nu N} w_n = q^{-\nu(n+1)} u_n, \quad bw_n = w_{n+1}, \quad b^\dagger w_n = -[n]_q w_{n-1},$$

where we again set $w_{-1} = 0$, endow $W^-$ with the structure of an $\text{Osc}_q$-module. We let $\chi^-$ denote the corresponding representation of $\text{Osc}_q$.

We consider the tensor product of $l$ copies of the $q$-oscillator algebra, and set

$$b_i = \bigotimes_{i=1}^{l-i} \bigotimes_{i=1}^{l-i} b \otimes 1 \otimes \cdots \otimes 1, \quad b_i^\dagger = \bigotimes_{i=1}^{l-i} \bigotimes_{i=1}^{l-i} b^\dagger \otimes 1 \otimes \cdots \otimes 1,$$

$$q^{\nu N} = \bigotimes_{i=1}^{l-i} q^{\nu N} \otimes \cdots \otimes \otimes_1.$$

In [35], the homomorphism $o$ of $U_q(\mathcal{L}(\mathfrak{b}_+))$ into $(\text{Osc}_q)^{\otimes l}$ described by the equations

$$o(q^{\rho_{i0}}) = q^{\rho(2N_1+\sum_{j=2}^{l} N_j)}, \quad o(e_0) = b_1^\dagger q^{\sum_{j=2}^{l} N_j},$$

$$o(q^{\rho_{i1}}) = q^{\rho(N_i+1-N_1)}, \quad o(e_i) = -b_i b_{i+1}^\dagger q^{N_i-N_1-1},$$

$$o(q^{\rho_{il}}) = q^{-\rho(2N_i+\sum_{j=1}^{l-1} N_j)}, \quad o(e_i) = -(q-q^{-1})^{-1} b_i q^{N_i},$$

where $1 \leq i < l$, was obtained by some limiting procedure starting from the evaluation representations $\tilde{\lambda}$ of $U_q(\mathcal{L}(\mathfrak{g}_{l+1}))$. We define

$$\theta = \left( \bigotimes_{i=1}^{l} \chi^{+} \right) \circ o.$$

The basis of this representation is formed by the vectors

$$w_n = b_1^{n_1} \cdots b_l^{n_l} w_0,$$

where $n_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq l$, and we use the notation $\mathbf{n} = (n_1, \ldots, n_l)$. Here, the vector $w_0$ is the vacuum vector, satisfying the equations

$$b_i w_0 = 0, \quad 1 \leq i \leq l.$$

The representation $\theta$ is in the category $\mathcal{O}$. It is a highest $\ell$-weight representation with the highest $\ell$-weight vector $w_0$.

There is an automorphism of $U_q(\mathcal{L}(\mathfrak{g}_{l+1}))$ defined by the equations

$$\sigma(q^{\rho_{i1}}) = q^{\rho_{i1}^{-1}}, \quad \sigma(e_i) = e_{i+1}, \quad \sigma(f_i) = f_{i+1}, \quad 0 \leq i \leq l,$$

where it is assumed that $q^{\rho_{i1}^{l+1}} = q^{\rho_{00}}, e_{l+1} = e_0$ and $f_{l+1} = f_0$. We can restrict $\sigma$ to an automorphism of $U_q(\mathcal{L}(\mathfrak{b}_+))$. It is useful to bear in mind that $\sigma^{l+1} = 1$. We define a collection of homomorphisms from $U_q(\mathcal{L}(\mathfrak{b}_+))$ into $\text{Osc}_q^{\otimes l}$ as

$$o_{\alpha} = o \circ \sigma^{-\alpha},$$

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and a family of representations

\[ \theta_a = \chi_a \circ o_a, \quad a = 1, \ldots, l + 1, \]  

where

\[ \chi_a = \chi^− \otimes \cdots \otimes \chi^− \otimes \chi^+ \otimes \cdots \otimes \chi^+. \]

We note that \( \theta = \theta_{l+1}. \) The corresponding basis vectors are

\[ w_{a,n} = b_1^n \cdots b_{l-a+1}^{n_{l-a+1}} b_{l-a+2}^{n_{l-a+2}} \cdots b_l^{n_{l-1}} w_{a,0}. \]

The vacuum vector \( w_{a,0} \) satisfies the equations

\[ b_i^l w_{a,0} = 0, \quad 1 \leq i \leq l - a + 1, \]

\[ b_i w_{a,0} = 0, \quad l - a + 2 \leq i \leq l. \]

All the representations \( \theta_a \) are highest \( \ell \)-weight representations in the category \( \mathcal{O}. \) The vectors \( w_{a,n} \) are the \( \ell \)-weight vectors, and \( w_{a,0} \) is the highest \( \ell \)-weight vector of the representation \( \theta_a \).

The explicit form of the \( \ell \)-weights \( \Psi_{a,n}(\zeta) = (\psi_{a,n}, \Psi_{a,n}^+(\zeta)) \) for the representations \((\theta_a)_\zeta\) was found in [49]. For the future use, we give the relevant expressions for \( l = 1 \) and \( l = 2. \) For \( l = 1 \), we have \( n = (n_1) \) and

\[ \psi_{1,n} = -(2n_1 + 2)\omega_1, \]

\[ \Psi_{1,n,1}(\zeta, u) = \frac{1 - q\zeta^* u}{(1 - q^{-2n_1 - 1}\zeta^* u)(1 - q^{-2n_1 - 1}\zeta^* u)}. \]

\[ \psi_{2,n} = -2n_1 \omega_1, \]

\[ \Psi_{2,n,1}(\zeta, u) = 1 - q\zeta^* u. \]

For \( l = 2 \), we have \( n = (n_1, n_2) \) and

\[ \psi_{1,n} = -(2n_1 - n_2 - 3)\omega_1 + (n_1 - n_2)\omega_2, \]

\[ \Psi_{1,n,1}(\zeta, u) = \frac{1 - q^{-2n_2}\zeta^* u}{(1 - q^{-2n_1 - 2n_2}\zeta^* u)(1 - q^{-2n_1 - 2n_2 - 2}\zeta^* u)}. \]

\[ \Psi_{1,n,2}(\zeta, u) = \frac{(1 - q\zeta^* u)(1 - q^{-2n_1 - 2n_2 - 1}\zeta^* u)}{(1 - q^{-2n_2 - 1}\zeta^* u)(1 - q^{-2n_2 + 1}\zeta^* u)}, \]

\[ \psi_{2,n} = (n_1 - 2n_2 + 1)\omega_1 + (-2n_1 + n_2 - 2)\omega_2, \]

\[ \Psi_{2,n,1}(\zeta, u) = 1 - q^{-2n_1}\zeta^* u, \]

\[ \psi_{2,n,1}(\zeta, u) = \frac{1 - q\zeta^* u}{(1 - q^{-2n_1 - 1}\zeta^* u)(1 - q^{-2n_1} + 1)}. \]

\[ \psi_{3,n} = (-n_1 + n_2)\omega_1 + (-n_1 - 2n_2)\omega_2, \]

\[ \Psi_{3,n,1}(\zeta, u) = 1, \]

\[ \Psi_{3,n,2}(\zeta, u) = 1 - q\zeta^* u. \]

4.5. \( q \)-characters and Grothendieck ring. Let \( V \) be a \( U_q(\mathcal{L}(g)) \)-module in the category \( \mathcal{O}. \) We define the character of \( V \) as a formal sum

\[ \text{ch}(V) = \sum_{\lambda \in \hat{g}^*} \dim V_\lambda [\lambda]. \]
By the definition of the category $\mathcal{O}$, $\dim V_{\lambda} = 0$ for $\lambda$ outside the union of a finite number of sets of the form $D(\mu)$, $\mu \in \mathfrak{h}^*$. For any two $U_q(\mathcal{L}(\mathfrak{g}))$-modules $V$ and $U$ in the category $\mathcal{O}$, we have

$$ \text{ch}(V \oplus U) = \text{ch}(V) + \text{ch}(U). $$

More generally, if $U_q(\mathcal{L}(\mathfrak{g}))$-modules $V$, $W$, and $U$ in the category $\mathcal{O}$ can be included in a short exact sequence

$$ 0 \to V \to W \to U \to 0, \quad (17) $$

then

$$ \text{ch}(W) = \text{ch}(V) + \text{ch}(U). $$

It can be also shown that

$$ \text{ch}(V \otimes \Delta U) = \text{ch}(V) \text{ch}(U) $$

for any $U_q(\mathcal{L}(\mathfrak{g}))$-modules $V$ and $U$ in the category $\mathcal{O}$. To multiply characters, we here assume that $[\lambda][\mu] = [\lambda + \mu]$ for any $\lambda, \mu \in \mathfrak{h}$.

The Grothendieck group of the category $\mathcal{O}$ of $U_q(\mathcal{L}(\mathfrak{g}))$-modules is defined as the quotient of the free Abelian group on the set of all isomorphism classes of objects in $\mathcal{O}$ by the relations

$$ \langle V \rangle = \langle U \rangle + \langle W \rangle, $$

if the objects $V$, $W$, and $U$ can be included in short exact sequence (17). For any object $V$, $\langle V \rangle$ denotes the isomorphism class of $V$. Setting

$$ \langle V \rangle \langle W \rangle = \langle V \otimes \Delta W \rangle, $$

we arrive at the Grothendieck ring of $\mathcal{O}$. It is a commutative unital ring, for which the role of the unit is played by the trivial $U_q(\mathcal{L}(\mathfrak{g}))$-module. We see that the character can be regarded as a map from the Grothendieck ring of $\mathcal{O}$. However, it is not injective and does not therefore uniquely distinguish its elements. That job is done by the $q$-character.

We define the $q$-character of a $U_q(\mathcal{L}(\mathfrak{g}))$-module $V$ in the category $\mathcal{O}$ as

$$ \text{ch}_q(V) = \sum_{\Lambda \in \mathfrak{h}^*_+} \dim(V_{\Lambda})[\Lambda]. $$

It is easy to demonstrate that

$$ \varpi(\text{ch}_q(V)) = \text{ch}(V). $$

Here, we assume that

$$ \varpi([\Lambda]) = [\varpi(\Lambda)], $$

and extend this rule by linearity. The $q$-character has the same properties as the usual character. Namely, if $U_q(\mathcal{L}(\mathfrak{g}))$-modules $V$, $W$, and $U$ in the category $\mathcal{O}$ can be included in short exact sequence (17), then

$$ \text{ch}_q(W) = \text{ch}_q(V) + \text{ch}_q(U), $$

and for any two $U_q(\mathcal{L}(\mathfrak{g}))$-modules $V$ and $U$ in the category $\mathcal{O}$, we have

$$ \text{ch}_q(V \otimes \Delta W) = \text{ch}_q(V) \text{ch}_q(W) \quad (18) $$

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To define the product of \( q \)-characters, we assume that
\[
[\Lambda][\Xi] = [\Lambda \Xi]
\]
for any \( \Lambda, \Xi \in \mathfrak{h}_+^* \). Thus, the \( q \)-character can also be regarded as a map from the Grothendieck ring of \( \mathcal{O} \), and it can be shown that this map is injective. In other words, different elements of the Grothendieck ring of \( \mathcal{O} \) have different \( q \)-characters.

It follows from (18) that
\[
\text{ch}_q(V \otimes \Delta W) = \text{ch}_q(V) \text{ch}_q(W) = \text{ch}_q(W) \text{ch}_q(V) = \text{ch}_q(W \otimes \Delta V).
\]

It means that the \( \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \)-modules \( V \otimes \Delta W \) and \( W \otimes \Delta V \) belong to the same equivalence class in the Grothendieck ring of \( \mathcal{O} \).

All the foregoing can be naturally extended to the categories \( \mathcal{O} \) of modules over \( \mathcal{U}_q(\mathcal{L}(\mathfrak{b}_+)) \) and \( \mathcal{U}_q(\mathcal{L}(\mathfrak{b}_-)) \).

5. Universal \( R \)-matrix and integrability objects

5.1. Quantum group as a \( \mathbb{C}[[\hbar]] \)-algebra. As any Hopf algebra, the quantum loop algebra \( \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \) has another comultiplication called the opposite comultiplication. It is defined by the equation
\[
\Delta^\prime = \Pi \circ \Delta,
\]
where
\[
\Pi(x \otimes y) = y \otimes x
\]
for all \( x, y \in \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \).

There are several different definitions of quantum groups. In particular, \( \hbar \) can be not only a complex number \([31],[44],[51]\) but also an indeterminate, such that the quantum group is a \( \mathbb{C}[[\hbar]] \)-algebra \([4],[40]–[42],[45]\). We assume temporally that this is the case. Then \( \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \) is a quasitriangular Hopf algebra. This means that up to a central element, there exists a unique invertible element \( \mathcal{R} \) of the completed tensor product \( \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \hat{\otimes} \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \), called the universal \( R \)-matrix, such that
\[
\Delta^\prime(x) = \mathcal{R} \Delta(x) \mathcal{R}^{-1}
\]
for all \( x \in \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \), and\(^3\)
\[
(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}^{(13)} \mathcal{R}^{(23)}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}^{(13)} \mathcal{R}^{(12)}.
\]

These last equations are regarded as equalities in the completed tensor product of three copies of \( \mathcal{U}_q(\mathcal{L}(\mathfrak{g})) \).

In fact, it follows from the explicit expression for the universal \( R \)-matrix \([4],[40]–[42]\) that it is an element of a completed tensor product of two Borel subalgebras \( \mathcal{U}_q(\mathcal{L}(\mathfrak{b}_+)) \) and \( \mathcal{U}_q(\mathcal{L}(\mathfrak{b}_-)) \).

A general integrability object is defined as follows. Let \( \varphi \) be a representation of \( \mathcal{U}_q(\mathcal{L}(\mathfrak{b}_+)) \) on a vector space \( V \), and \( \psi \) a representation of \( \mathcal{U}_q(\mathcal{L}(\mathfrak{b}_-)) \) on a vector space \( U \).\(^4\) The corresponding integrability object \( \mathbf{X}_{\varphi|\psi} \) is defined by the equation
\[
\rho_{\varphi|\psi} \mathbf{X}_{\varphi|\psi} = [(\varphi \otimes \psi)(\mathcal{R})],
\]
\(^3\)For the explanation of the notation, see, e.g., \([52]\).
\(^4\)In this paper, we always assume that the representations used are in the category \( \mathcal{O} \).
where \( \rho_{\varphi|\psi} \) is a scalar normalization factor. It is evident that this object is an element of \( \text{End}(V) \otimes \text{End}(U) \).

It follows from (19) that

\[
(\varphi \otimes \psi)(\Pi(\Delta(x))) = X_{\varphi|\psi}(\varphi \otimes \psi)(\Delta(x))(X_{\varphi|\psi})^{-1},
\]

and Eq. (20) gives

\[
X_{\varphi_1 \otimes \Delta \varphi_2 | \psi} = X_{(13)}(15) \quad X_{\varphi_1 \otimes \Delta \varphi_2 | \psi} = X_{(13)}(12). \tag{22}
\]

In fact, hereinafter, we assume that

\[
\rho_{\varphi_1 \otimes \Delta \varphi_2 | \psi} = \rho_{\varphi_1 | \psi} \rho_{\varphi_2 | \psi}, \quad \rho_{\varphi_1 | \psi_1 \otimes \Delta \varphi_2} = \rho_{\varphi_1 | \psi_1} \rho_{\varphi_2 | \psi}.
\]

5.2. Quantum group as a \( \mathbf{C} \)-algebra. The expression for the universal \( R \)-matrix of a quantum loop algebra \( U_q(\mathcal{L}(\mathfrak{g})) \) considered as a \( \mathbb{C}[[\hbar]] \)-algebra can be constructed using the procedure proposed by Khoroshkin and Tolstoy \[40\]–\[42\]. However, in this paper, we define the quantum loop algebra \( U_q(\mathcal{L}(\mathfrak{g})) \) as a \( \mathbf{C} \)-algebra. In fact, one can use the expression for the universal \( R \)-matrix from \[40\]–\[42\] to construct the integrability objects in this case as well, having in mind that \( U_q(\mathcal{L}(\mathfrak{g})) \) is quasitriangular only in some restricted sense (see \[53\], p. 327 in book \[54\], and the discussion below).

We restrict ourselves to the following case. Let \( \varphi \) be a representation of \( U_q(\mathcal{L}(\mathfrak{b}_+)) \) on a vector space \( V \), and \( \psi \) a representation of \( U_q(\mathcal{L}(\mathfrak{b}_-)) \) on a vector space \( U \). We define an integrability object \( X_{\varphi|\psi} \) as an element of \( \text{End}(V) \otimes \text{End}(U) \) by the equation

\[
\rho_{\varphi|\psi} X_{\varphi|\psi} = (\varphi \otimes \psi)(\mathcal{R}_{\prec} \circ \mathcal{R}_{\sim} \circ \mathcal{R}_{\succ}) \mathcal{K}_{\varphi|\psi}, \tag{23}
\]

where \( \mathcal{R}_{\prec} \), \( \mathcal{R}_{\sim} \), and \( \mathcal{R}_{\succ} \) are elements of \( U_q(\mathcal{L}(\mathfrak{b}_+)) \otimes U_q(\mathcal{L}(\mathfrak{b}_-)) \), \( \mathcal{K}_{\varphi|\psi} \) is an element of \( \text{End}(V) \otimes \text{End}(U) \), and \( \rho_{\varphi|\psi} \) a scalar normalization factor.

The explicit expressions for the elements \( \mathcal{R}_{\prec} \), \( \mathcal{R}_{\sim} \), and \( \mathcal{R}_{\succ} \) in the case of \( \mathfrak{g} = \mathfrak{sl}_{l+1} \) are given in \[52\]. For the general case, the reader is referred to \[40\]–\[42\]. The element \( \mathcal{K}_{\varphi|\psi} \) for a general \( \mathfrak{g} \) is given by the equation

\[
\mathcal{K}_{\varphi|\psi} = \sum_{\lambda \in \mathfrak{h}^*} \varphi(q^{-\sum_{i,j=1}^{l} a_i c_{ij} d_j^{-1}(\lambda, \alpha_i)}) \otimes \Pi_{\lambda}, \tag{24}
\]

where \( \Pi_{\lambda} \in \text{End}(U) \) is the projector on the component \( U_{\lambda} \) of the weight decomposition

\[
U = \bigoplus_{\lambda \in \mathfrak{h}^*} U_{\lambda},
\]

and \( c_{ij} \) are the matrix entries of the matrix \( C \) inverse to the Cartan matrix \( A \) of the Lie algebra \( \mathfrak{g} \). Using formulas in \[55\], for \( \mathfrak{g} = \mathfrak{sl}_{l+1} \) we obtain

\[
c_{ij} = \frac{i(l-j+1)}{l+1}, \quad i \leq j,
\]

\[
c_{ij} = \frac{(l-i+1)j}{l+1}, \quad i > j.
\]

It can be shown that the integrability objects defined by Eq. (23) satisfy Eqs. (21) and (22). Hence, the integrability objects defined by Eq. (23) behave as if they were constructed from the real universal \( R \)-matrix.

Each integrability object \( X_{\varphi|\psi} \in \text{End}(V) \otimes \text{End}(U) \) generates an integrability object \( Y_{\varphi|\psi} \in \text{End}(U) \) defined as

\[
Y_{\varphi|\psi} = (\text{tr}_{\varphi} \otimes \text{id}_{\text{End}(U)})(X_{(13)}(15) \otimes (\varphi(q^t) \otimes 1_{\text{End}(U)})), \tag{25}
\]
where \( q \in U_q(\mathcal{L}(g)) \) is a group-like\(^5\) twisting element necessary for the convergence of the trace, and \( \text{tr}_\varphi \) is the trace on \( U_q(\mathcal{L}(b_+)) \) defined by the equation

\[
\text{tr}_\varphi = \text{tr}_{\text{End}(V)} \circ \varphi.
\]  

(26)

Here, \( \text{tr}_{\text{End}(V)} \) is the usual trace on the endomorphism algebra of the vector space \( V \). By definition, the integrability object \( Y_{\varphi|\psi} \) depends only on the equivalence class the representation \( \varphi \) in the Grothendieck ring.

It is productive to define the corresponding universal integrability objects. They are defined as formal objects with specific rules of use. If \( \varphi \) is a representation of \( U_q(\mathcal{L}(b_+)) \) on a vector space \( V \), the universal integrability object \( X_\varphi \) corresponding to the integrability objects of type \( X \) behaves as an element of \( \text{End}(V) \otimes U_q(\mathcal{L}(b_-)) \) and obeys the rule

\[
(id_{\text{End}(V)} \otimes \psi)(X_\varphi) = \rho_{\varphi|\psi} X_{\varphi|\psi}
\]

for any representation \( \psi \) of \( U_q(\mathcal{L}(b_-)) \). The universal integrability object \( Y_\varphi \) corresponding to the integrability objects of type \( Y \) behaves as an element of \( U_q(\mathcal{L}(b_-)) \) and obeys the rule

\[
\psi(Y_\varphi) = \rho_{\varphi|\psi} Y_{\varphi|\psi}.
\]

It follows from (25) that

\[
Y_\varphi = (\text{tr}_\varphi \otimes \text{id}_{U_q(\mathcal{L}(g))})(X_\varphi) (\varphi(q^t) \otimes 1_{U_q(\mathcal{L}(g))})
\]

and the product of two universal integrability objects \( Y_{\varphi_1} \) and \( Y_{\varphi_2} \) as an element of \( U_q(\mathcal{L}(b_-)) \), by the rule

\[
\psi(Y_{\varphi_1} Y_{\varphi_2}) = \rho_{\varphi_1|\psi} \rho_{\varphi_2|\psi} Y_{\varphi_1|\psi} Y_{\varphi_2|\psi}.
\]

Using the first equation in (22), we obtain

\[
Y_{\varphi_1 \otimes \Delta \varphi_2|\psi} = Y_{\varphi_1|\psi} Y_{\varphi_2|\psi}
\]

or, in terms of universal integrability objects,

\[
Y_{\varphi_1 \otimes \Delta \varphi_2|\psi} = Y_{\varphi_1|\psi} Y_{\varphi_2|\psi}.
\]

(27)

Because the representations \( \varphi_1 \otimes \Delta \varphi_2 \) and \( \varphi_2 \otimes \Delta \varphi_1 \) belong to the same equivalence class in the Grothendieck ring of \( \mathcal{O} \), we have

\[
Y_{\varphi_1 \otimes \Delta \varphi_2|\psi} = Y_{\varphi_2 \otimes \Delta \varphi_1|\psi},
\]

and therefore

\[
Y_{\varphi_1|\psi} Y_{\varphi_2|\psi} = Y_{\varphi_2|\psi} Y_{\varphi_1|\psi}
\]

(28)

\(^5\)An element \( x \in U_q(\mathcal{L}(g)) \) is called group-like if \( \Delta(x) = x \otimes x \).
or, in terms of universal integrability objects,
\[ \mathcal{Y}_{\phi_1} \mathcal{Y}_{\phi_2} = \mathcal{Y}_{\phi_2} \mathcal{Y}_{\phi_1}. \]

Let now \( x \in U_q(\mathcal{L}(h_+)) \cup U_q(\mathcal{L}(h_-)) \) be a group-like element commuting with the twisting element \( q^t \). Starting with Eq. (21), we obtain
\[ \mathcal{Y}_{\phi|\psi}(x) = \psi(x) \mathcal{Y}_{\phi|\psi}. \]

In particular, for any \( h \in \widehat{h} \), we have
\[ \mathcal{Y}_{\phi|\psi}(q^h) = \psi(q^h) \mathcal{Y}_{\phi|\psi}. \]

Hence, in terms of universal integrability objects,
\[ \mathcal{Y}_{\phi}x = x \mathcal{Y}_{\phi}, \]

and
\[ \mathcal{Y}_{\phi} q^h = q^h \mathcal{Y}_{\phi}. \]

We consider the behavior of the integrability objects \( \mathcal{X}_{\phi|\psi} \) and \( \mathcal{X}_{\phi|\psi} \) under a shift of the homomorphism \( \phi \), Eq. (15). First, because the elements \( R_\prec \delta, R_\sim \delta, \) and \( R_\succ \delta \) do not depend on the generators \( q^h \) and \( h \in \widehat{h} \) (see [40], [42]), we can verify that
\[ (\phi \otimes \psi)(R_\prec \delta R_\sim \delta R_\succ \delta) = (\phi \otimes \psi)(R_\prec \delta R_\sim \delta R_\succ \delta). \]

Then, using Eqs. (15) and (24), we obtain
\[ K_{\phi[\xi]|\psi} = K_{\phi|\psi}(1_{\text{End}(V)} \otimes \psi(q^{-\sum_{i,j=1}^{l} (\xi,\tilde{\alpha}_i)c_{ij}\tilde{\alpha}_j})) \]

and arrive at the equations
\[ \mathcal{X}_{\phi[\xi]|\psi} = \mathcal{X}_{\phi|\psi}(1_{\text{End}(V)} \otimes \psi(q^{-\sum_{i,j=1}^{l} (\xi,\tilde{\alpha}_i)c_{ij}\tilde{\alpha}_j})), \]
\[ \mathcal{X}_{\phi[\xi]} = \mathcal{X}_{\phi}(1_{\text{End}(V)} \otimes q^{-\sum_{i,j=1}^{l} (\xi,\tilde{\alpha}_i)c_{ij}\tilde{\alpha}_j}). \]

For the integrability objects \( \mathcal{Y}_{\phi|\psi} \) and \( \mathcal{Y}_{\phi} \), we have the equations
\[ \mathcal{Y}_{\phi[\xi]|\psi} = \mathcal{Y}_{\phi|\psi}(q^{-\sum_{i,j=1}^{l} (\xi,\tilde{\alpha}_i)c_{ij}\tilde{\alpha}_j}), \]
\[ \mathcal{Y}_{\phi[\xi]} = \mathcal{Y}_{\phi}(q^{-\sum_{i,j=1}^{l} (\xi,\tilde{\alpha}_i)c_{ij}\tilde{\alpha}_j}), \]

where \( \tilde{\alpha}_i = \tilde{\alpha}_i - t_i \). Here and below, we set
\[ t = \sum_{i=1}^{l} t_i \tilde{\alpha}_i, \]

where \( t_i, i = 1, \ldots, l \) are complex twisting parameters.
In fact, the integrability objects depend on spectral parameters. To define such objects, we use the representation \( \varphi_\zeta \) as the representation \( \varphi \), and the \( n \)th tensor power of the homomorphism \( \psi \) as \( \psi \).

The corresponding universal integrability objects are denoted as

\[
X_\varphi(\zeta) = X_{\varphi_\zeta}, \quad Y_\varphi(\zeta) = Y_{\varphi_\zeta},
\]

while for the usual integrability objects, we use the notation

\[
X_{n\varphi|\psi}(\zeta) = X_{\varphi_\zeta|\psi \otimes \cdots \otimes \psi}, \quad Y_{n\varphi|\psi}(\zeta) = Y_{\varphi_\zeta|\psi \otimes \cdots \otimes \psi}.
\]

When \( n = 1 \), we usually write just \( X_{\varphi|\psi}(\zeta) \) and \( Y_{\varphi|\psi}(\zeta) \). In accordance with our conventions, we set

\[
\rho_{n\varphi|\psi}(\zeta) = \rho_{\varphi_\zeta|\psi \otimes \cdots \otimes \psi} = (\rho_{\varphi_\zeta|\psi})^n = \rho_{\varphi|\psi}(\zeta)^n.
\]

All integrability objects that we use in this paper are constructed as described above. However, depending on the role they play in the integration procedure, they are given different names. Below we describe the main classes of integrability objects. It is worth noting that the proposed nomenclature is somewhat conventional, although is widespread.

The most famous integrability objects are the \( R \)-operators. They form a special class of integrability objects of type \( X \) used to permute integrability objects of type \( Y \). However, there is a more general method for demonstrating the commutativity of integrability objects of type \( Y \), described in the preceding section, and we do not define and use \( R \)-operators in this paper.

5.3. Monodromy operators and transfer operators. When \( \varphi \) is a representation of the quantum loop algebra \( U_q(\mathcal{L}(g)) \) on a vector space \( V \) and \( \psi \) is a representation of \( U_q(\mathcal{L}(g)) \) on a vector space \( U \), the integrability object \( X_{n\varphi|\psi}(\zeta) \) is called the monodromy operator and is denoted by \( M_{n\varphi|\psi}(\zeta) \).

The type-\( Y \) companion of the monodromy operator \( M_{n\varphi|\psi}(\zeta) \) is called the transfer operator and is denoted by \( T_{n\varphi|\psi}(\zeta) \). Explicitly, we have

\[
T_{n\varphi|\psi}(\zeta) = (\text{tr}_{\text{End}(V)} \otimes \text{id}_{\text{End}(W \otimes n)})(M_{n\varphi|\psi}(\zeta)(\varphi_\zeta(t) \otimes 1_{\text{End}(W \otimes n)})),
\]

where \( t \) is the twisting element defined in (30).

Let \( \varphi_1, \varphi_2, \) and \( \psi \) be representations of \( U_q(\mathcal{L}(g)) \). It follows from Eq. (28) that

\[
T_{n\varphi_1|\psi}(\zeta_1)T_{n\varphi_2|\psi}(\zeta_2) = T_{n\varphi_2|\psi}(\zeta_2)T_{n\varphi_1|\psi}(\zeta_1)
\]

for any \( \zeta_1, \zeta_2 \in \mathbb{C}^X \). Similar commutativity relations hold for the universal transfer operators:

\[
T_{\varphi_1}(\zeta_1)T_{\varphi_2}(\zeta_2) = T_{\varphi_2}(\zeta_2)T_{\varphi_1}(\zeta_1).
\]

In fact, it is important for commutativity that the twist element be group-like.

In this paper, we construct the monodromy operators using the evaluation representations \( \varphi^\lambda \) and \( \varphi^{\omega_1} \) defined in Sec. 4.2 as \( \varphi \) and using the \((\ell + 1)\)-dimensional evaluation representation \( \varphi^{\omega_1} \) as \( \psi \). The following notation is then used:

\[
T_{n}^\lambda(\zeta) = T_{n\varphi^\lambda|\varphi^{\omega_1}}(\zeta), \quad T_{n}^{\omega_1}(\zeta) = T_{n\varphi^{\omega_1}|\varphi^{\omega_1}}(\zeta).
\]

Similarly, for the corresponding universal transfer operators, we use the notation

\[
\tilde{T}^\lambda(\zeta) = T_{\varphi^\lambda|\varphi^{\omega_1}}(\zeta), \quad T^{\omega_1}(\zeta) = T_{\varphi^{\omega_1}|\varphi^{\omega_1}}(\zeta).
\]

\textsuperscript{6}One can also use the tensor product \( \psi_{\eta_1} \otimes \cdots \otimes \psi_{\eta_n} \) as \( \psi \). However, we do not consider such a generalization in this paper.
5.4. **L-opera tors and Q-opera tors.** We now let \( \varphi \) be a representation of \( U_q(\mathcal{L}(b_+)) \) on a vector space \( W \), which cannot be extended to a representation of \( U_q(\mathcal{L}(g)) \), and \( \psi \) be a representation of \( U_q(\mathcal{L}(g)) \) on a vector space \( U \). In this case, the integrability object \( X_{\Delta}^n_{\psi}(\zeta) \) defined by Eq. (31) is called an \( L \)-operator and is denoted by \( L_{\psi}^n(\zeta) \).

The companion of the \( L \)-operator \( L_{\psi}^n(\zeta) \) of type \( Y \) is called a \( Q \)-operator and is denoted by \( Q_{\psi}^n(\zeta) \).

Explicitly, we have

\[
Q_{\psi}^n(\zeta) = (\mathrm{tr}_{\mathrm{End}(W)} \otimes \mathrm{id}_{\mathrm{End}(W \otimes n)})(L_{\psi}^n(\zeta)(\varphi^\dagger(t) \otimes 1_{\mathrm{End}(W \otimes n)})),
\]

where \( t \) is the twisting element defined in (30).

Let \( \varphi_1 \) and \( \varphi_2 \) be representations of \( U_q(\mathcal{L}(b_+)) \) that cannot be extended to representations of \( U_q(\mathcal{L}(g)) \), and \( \psi \) be a representation of \( U_q(\mathcal{L}(g)) \). It follows from Eq. (28) that

\[
Q_{\psi}^{\varphi_1}(\zeta_1)Q_{\psi}^{\varphi_2}(\zeta_2) = Q_{\psi}^{\varphi_2}(\zeta_2)Q_{\psi}^{\varphi_1}(\zeta_1)
\]

for any \( \zeta_1, \zeta_2 \in \mathbb{C}^*, \) and similarly for universal \( Q \)-operators,

\[
Q_{\psi_2}(\zeta_1)Q_{\varphi_2}(\zeta_2) = Q_{\varphi_2}(\zeta_2)Q_{\psi_2}(\zeta_1).
\]

In this paper, we work with the \( Q \)-operators defined as

\[
Q_{\psi}^a(\zeta) = Q_{\theta_a}(\varphi^\dagger(\zeta)), \quad a = 1, \ldots, l + 1,
\]

where the representations \( \theta_a \) are defined by Eq. (16). The corresponding universal \( Q \)-operators are denoted as

\[
Q_{\theta}^a(\zeta) = Q_{\theta_a}(\zeta).
\]

We use a prime to indicate that we redefine these operators below.

6. **Factorization of transfer operators**

6.1. **Case \( l = 1 \).** We consider the tensor product \( (W_1)_{\zeta_1} \otimes (W_2)_{\zeta_2} \) of two oscillator modules and introduce an independent labeling for oscillators, using \( n_{11} \) and \( n_{21} \) for the respective first and second factors. In other words, we label the basis vectors of the tensor product by a 2-tuple of nonnegative integers \((n_{11}, n_{21})\) denoted as \( n \). We hope that this slight abuse of notation does not lead to confusion. We compare the product of the highest \( \ell \)-weights of the modules \( (W_1)_{\zeta_1} \) and \( (W_2)_{\zeta_2} \) with the highest \( \ell \)-weight of the evaluation \( U_q(\mathcal{L}(\mathfrak{sl}_2)) \)-module \((\tilde{V}^\mu)^\zeta\).

The explicit form of these \( \ell \)-weights is

\[
\Psi_{1,0}(\zeta_1)\Psi_{2,0}(\zeta_2) = (-2\omega_1, ((1 - q\zeta_2u)(1 - q^{-1}\zeta_1u)^{-1})),
\]

\[
A_0^\mu(\zeta) = ((\mu_1 - \mu_2)\omega_1, (1 - q^{2\mu_2}\zeta u)(1 - q^{2\mu_1}\zeta^{-1}u)^{-1})
\]

(see Secs. 4.4 and 4.2). Setting

\[
\zeta_1 = \zeta_1^\mu = q^{2(\mu_1 + 1/2)/s} \zeta, \quad \zeta_2 = \zeta_2^\mu = q^{2(\mu_2 - 1/2)/s} \zeta
\]

in (32), we obtain

\[
\Psi_{1,0}(\zeta_1)\Psi_{2,0}(\zeta_2) = (-2\omega_1, (1 - q^{2\mu_2}\zeta u)(1 - q^{2\mu_1}\zeta^{-1}u)^{-1})).
\]

Thus, the product of the highest \( \ell \)-weights of the modules \( (W_1)_{\zeta_1} \) and \( (W_2)_{\zeta_2} \) coincides with the highest \( \ell \)-weight of the evaluation module \((\tilde{V}^\mu)^\zeta\) shifted by \((-2 - \mu_1 + \mu_2)\omega_1\).
We let \( \Xi_n^\mu(\zeta) = (\xi_n^\mu, \Xi_n^{\mu+}(\zeta)) \) be the \( \ell \)-weights of the module
\[
W_\mu(\zeta) = (W_1)_\zeta^\mu \otimes_\Delta (W_2)_\zeta^\mu.
\]
Using expressions from Sec. 4.4, we can see that
\[
\xi_n^\mu = (-2 - 2n_{11} - 2n_{21})\omega_1
\]
and \( \Xi_n^{\mu+}(\zeta) = (\Xi_{n,1}^{\mu+}(\zeta, u)) \), where
\[
\Xi_n^{\mu+}(\zeta, u) = \frac{(1 - q^{2\mu_1+2\zeta s} u)(1 - q^{2\mu_1-2n_{11}+2})\zeta s u)}{(1 - q^{2\mu_2}\zeta u)(1 - q^{2\mu_1-2n_{11}}\zeta s u)}.
\]
It is remarkable that \( \Xi_n^{\mu+}(\zeta) \) is independent of \( n_{21} \). In fact, changing \( n_{11} \) to \( m_{12} \), we obtain the \( \Lambda_n^{\mu+}(\zeta) \) component of the \( \ell \)-weight \( \Lambda_n^\mu(\zeta) \) of the evaluation \( U_q(\mathcal{L}(\mathfrak{g}l_2)) \)-module \((\tilde{V}_\mu')_\zeta\). Analyzing the expression for the \( \xi^\mu_n \) component and identifying \( n_{11} \) with \( m_{12} \), we see that for any fixed \( n_{21} \), the \( \ell \)-weights \( \Xi_n^{\mu+}(\zeta) \) coincide with the corresponding \( \ell \)-weights of the module \((\tilde{V}_\mu')_\zeta\) shifted by\(^8\)
\[
\delta_n^\mu = (-2 - \mu_1 + \mu_2 - 2n_{21})\omega_1
\]
(see Eqs. (4) and (5)). Thus, we have the relation satisfied by \( q \)-characters
\[
\text{ch}_q(W_\mu(\zeta)) = \sum_{n'} \text{ch}_q(\tilde{V}_\zeta^{\mu}[\delta_n'])
\]
which is equivalent to the relation in the Grothendieck ring
\[
(W_\mu(\zeta)) = \sum_{n'} (\tilde{V}_\zeta^{\mu}[\delta_n']) = \left\langle \bigoplus_{n'} \tilde{V}_\zeta^{\mu}[\delta_n'] \right\rangle.
\]
Here, the summation over \( n' \) means summation over \( n_{21} \) from 0 to \( \infty \). It follows that
\[
\mathcal{Y}_{W_\mu(\zeta)} = \sum_{n'} \mathcal{Y}_{\tilde{V}_\zeta^{\mu}[\delta_n']},
\]
and, using (29), we obtain
\[
\mathcal{Y}_{W_\mu(\zeta)} = q^{(\mu_1-\mu_2+2)\delta_1/2} \sum_{n_{12}=0}^\infty q^{n_{12}\delta_1} \bar{T}_\mu(\zeta) = q^{(\mu_1-\mu_2)\delta_1/2} \frac{q^{\delta_1}}{1 - q^{\delta_1}} \bar{T}_\mu(\zeta).
\]
On the other hand, taking (27) into account, we see that
\[
\mathcal{Y}_{W_\mu(\zeta)} = \mathcal{Y}_{\phi_1(\varsigma_1^\mu)} \mathcal{Y}_{\phi_2(\varsigma_2^\mu)} = \mathcal{Q}_1'(\varsigma_1^\mu) \mathcal{Q}_2'(\varsigma_2^\mu),
\]
and thus arrive at the factorization formula
\[
q^{(\mu_1-\mu_2)\delta_1/2} \frac{q^{\delta_1}}{1 - q^{\delta_1}} \bar{T}_\mu(\zeta) = \mathcal{Q}_1'(\varsigma_1^\mu) \mathcal{Q}_2'(\varsigma_2^\mu).
\]
\( ^8 \)We set \( n' = (n_{21}) \). It may seem that we are using unnecessarily cumbersome notation. This is justified by the fact that we use the same notation for all values of \( l \).
Introducing the new universal $Q$-operators

$$Q_1(\zeta) = \zeta^{-\hat{a}_1 s/4}Q'_1(\zeta), \quad Q_2(\zeta) = \zeta^{\hat{a}_1 s/4}Q'_2(\zeta),$$

we rewrite the factorization formula as

$$C_1 \overline{T}^\mu(\zeta) = Q_1(\zeta^\mu)Q_2(\zeta^\mu),$$

where

$$C_1 = \frac{q^{\hat{a}_1/2}}{1 - q^{\hat{a}_1}}.$$ 

The advantage of this formula over (34) is that the factor $C_1$ id independent of $\mu$, and this is necessary in order to obtain the determinant formula.

In terms of the usual integrability objects, Eq. (35) takes the form

$$\rho_{\varphi^1}(\zeta)^n C_n \overline{T}^\mu(\zeta) = \rho_{\varphi_1}(\zeta^\mu)^n \rho_{\varphi_2}(\zeta^\mu)^n Q_1(\zeta^\mu)Q_2(\zeta^\mu),$$

where

$$C_n = (\varphi^1 \otimes \cdots \otimes \varphi^1)(C_1).$$

If we choose

$$\rho_{\varphi^1}(\zeta) = \rho_{\varphi_1}(\zeta^\mu)\rho_{\varphi_2}(\zeta^\mu),$$

we obtain a simpler expression

$$C_n \overline{T}^\mu(\zeta) = Q_1(\zeta^\mu)Q_2(\zeta^\mu).$$

6.2. Case $l = 2$. We now consider the tensor product $(W_1)_{\zeta_1} \otimes (W_2)_{\zeta_2} \otimes (W_3)_{\zeta_3}$ of three oscillator modules and introduce an independent labeling for oscillators, using the tuple $n = (n_{11}, n_{12}, n_{21}, n_{22}, n_{31}, n_{32})$ for it. The product of the highest $\ell$-weights of the modules $(W_1)_{\zeta_1}, (W_2)_{\zeta_2},$ and $(W_3)_{\zeta_3}$ is

$$\Psi_{1,0}(\zeta_1)\Psi_{2,0}(\zeta_2)\Psi_{3,0}(\zeta_3) = (2\zeta_1 - 2\zeta_2, (1 - \zeta_3 u)(1 - q^{-2}\zeta_3^u)^{-1}, (1 - q^{-2}\zeta_3^u)(1 - q^{-1}\zeta_3^u)^{-1})$$

(see Sec. 4.4), while the highest $\ell$-weight of the evaluation $U_q(\mathcal{L}(\mathfrak{sl}_3))$-module $($\overline{T}^\mu)_{\zeta}$ has the form

$$\Lambda_0^\mu(\zeta) = (\mu_1 + \mu_2)\omega_1 - (\mu_2 + \mu_3)\omega_2, ((1 - q^{2\mu_2}\zeta^u)(1 - q^{2\mu_1}\zeta^u)^{-1}, (1 - q^{2\mu_3}\zeta^u)(1 - q^{2\mu_2}\zeta^u)^{-1})$$

(see Sec. 4.2). Setting

$$\zeta_1 = \zeta_1^\mu = q^{2(\mu_1 + 1)/s}\zeta, \quad \zeta_2 = \zeta_2^\mu = q^{2\mu_2/s}\zeta, \quad \zeta_3 = \zeta_3^\mu = q^{2(\mu_3 - 1)/s}\zeta,$$

we see that in this case the product of the highest $\ell$-weights of the modules $(W_1)_{\zeta^\mu_1}, (W_2)_{\zeta^\mu_2},$ and $(W_3)_{\zeta^\mu_3}$ coincides with the highest $\ell$-weight of the evaluation $U_q(\mathcal{L}(\mathfrak{sl}_3))$-module $($\overline{T}^\mu)_{\zeta}$ shifted by $(-2 - \mu_1 + \mu_2)\omega_1 + (-2 - \mu_2 + \mu_3)\omega_2$.

Similarly to the above, we let $\Xi_n^\mu(\zeta) = (\zeta_n^\mu, \Xi_n^\mu(\zeta))$ denote the $\ell$-weights of the module

$$W^\mu(\zeta) = (W_1)_{\zeta^\mu_1} \otimes (W_2)_{\zeta^\mu_2} \otimes (W_3)_{\zeta^\mu_3}.$$
Using expressions from Sec. 4.4, we can see that

\[
\xi^\mu_n = (-2 - 2n_{11} - n_{12} + n_{21} - 2n_{22} - n_{31} + n_{32})\omega_1 + \\
+ (-2 + n_{11} - n_{12} - 2n_{21} + n_{22} - n_{31} - 2n_{32})\omega_2
\]

(36)

and \(\Xi^\mu_n(\zeta) = (\Xi^\mu_{n,1}(\zeta, u), \Xi^\mu_{n,2}(\zeta, u))\), where

\[
\Xi^\mu_{n,1}(\zeta, u) = \frac{(1 - q^{2\mu_1 - 2n_{12} + 2\zeta u})(1 - q^{2\mu_1 - 2n_{11} - 2n_{12} + 2\zeta u})}{(1 - q^{2\mu_2 - 2n_{21} - 2\zeta u})(1 - q^{2\mu_1 - 2n_{11} - 2n_{12} + 2\zeta u})},
\]

\[
\Xi^\mu_{n,2}(\zeta, u) = \frac{(1 - q^{2\mu_1 - 2n_{11} - 2n_{12} + 1\zeta u})(1 - q^{2\mu_1 - 2n_{12} + 1\zeta u})}{(1 - q^{2\mu_1 + 3\zeta u})(1 - q^{2\mu_1 - 2n_{12} + 3\zeta u})} \times \\
\times \frac{(1 - q^{2\mu_2 + 1\zeta u})(1 - q^{2\mu_2 - 2n_{21} + 1\zeta u})}{(1 - q^{2\mu_3 - 1\zeta u})(1 - q^{2\mu_2 - 2n_{21} + 1\zeta u})}.
\]

We see that \(\Xi^\mu_n(\zeta)\) is independent of \(n_{22}, n_{31}\) and \(n_{32}\). In fact, identifying \(n_{11}\) with \(m_{12}, n_{12}\) with \(m_{13},\) and \(n_{21}\) with \(m_{23}\), we obtain the \(A^\mu_m(\zeta)\) component of the \(\ell\)-weight \(A^\mu_m(\zeta)\) of the evaluation \(U_q(\mathcal{L}(\mathfrak{sl}_3))\)-module \(V^\mu_\zeta\) (see Eqs. (8) and (9)). Using the above identification, we rewrite Eq. (36) as

\[
\xi^\mu_n = (\mu_1 - \mu_2 - 2n_{12} - m_{13} + m_{23})\omega_1 + (\mu_2 - \mu_3 + m_{12} - m_{13} - 2m_{23})\omega_2 + \\
+ (-2 - \mu_1 + \mu_2 - 2n_{22} - n_{31} + n_{32})\omega_1 + (-2 - \mu_2 + \mu_3 + n_{22} - n_{31} - 2n_{32})\omega_2.
\]

We see that under the above identifications, for any fixed \(n_{22}, n_{31},\) and \(n_{32}\), the \(\ell\)-weight components \(\Xi^\mu_n(\zeta)\) coincide with the corresponding \(\ell\)-weight components of the evaluation \(U_q(\mathcal{L}(\mathfrak{sl}_3))\)-module \(V^\mu_\zeta\) shifted by\(^9\)

\[
\delta_{n'} = (-2 - \mu_1 + \mu_2 - 2n_{22} - n_{31} + n_{32})\omega_1 + (-2 - \mu_2 + \mu_3 + n_{22} - n_{31} - 2n_{32})\omega_2
\]

(see Eq. (7)). Thus, the \(q\)-characters satisfy the relation

\[
\text{ch}_q(W^\mu(\zeta)) = \sum_{n'} \text{ch}_q(V^\mu_\zeta[\delta_{n'}]),
\]

which is equivalent to the relation in the Grothendieck ring

\[
(W^\mu(\zeta)) = \bigoplus_{n'} V^\mu_\zeta[\delta_{n'}] = \left\langle \bigoplus_{n'} V^\mu_\zeta[\delta_{n'}] \right\rangle.
\]

Here, the summation over \(n'\) means summation over \(n_{22}, n_{31},\) and \(n_{32}\) from 0 to \(\infty\). It follows that

\[
\mathcal{Y}_{W^\mu(\zeta)} = \sum_{n'} \mathcal{Y}_{V^\mu_\zeta[\delta_{n'}]},
\]

and, using (29), we obtain

\[
\mathcal{Y}_{W^\mu(\zeta)} = q^{(2\mu_1 + \mu_2 - \mu_3)\delta_1 + (\mu_1 + \mu_2 - 2\mu_3)\delta_2}/3 \quad q^{\delta_1}/(1 - q^{\delta_1}) \quad q^{\delta_1 + \delta_2}/(1 - q^{\delta_1 + \delta_2}) \quad q^{\delta_2}/(1 - q^{\delta_2}) \quad \mathcal{Y}_\zeta^\mu(\zeta).
\]

\(^9\)We now set \(n' = (n_{22}, n_{31}, n_{32}).\)
On the other hand, again taking Eq. (27) into account, we see that
\[ \mathcal{V}_{\lambda + (\mu + \rho)/s}(\zeta) = \mathcal{V}_{\lambda + (\mu + \rho)/s}(\zeta) = Q_1(\zeta_1^\mu)Q_2(\zeta_2^\mu)Q_3(\zeta_3^\mu), \]
and thus obtain the factorization formula
\[ q^{(2\mu_1 + \mu_2 - \mu_3)\hat{\alpha}_1 + (\mu_1 + \mu_2 - 2\mu_3)\hat{\alpha}_2}/3 \quad \frac{q^{\hat{\alpha}_1}}{1 - q^{\hat{\alpha}_1}} \quad \frac{q^{\hat{\alpha}_1 + \hat{\alpha}_2}}{1 - q^{\hat{\alpha}_1 + \hat{\alpha}_2}} \quad \frac{q^{\hat{\alpha}_2}}{1 - q^{\hat{\alpha}_2}} \quad \mathcal{T}^\mu(\zeta) = Q_1(\zeta_1^\mu)Q_2(\zeta_2^\mu)Q_3(\zeta_3^\mu). \]
Introducing new universal \( Q \)-operators
\[ Q_1(\zeta) = \zeta^{-(2\hat{\alpha}_1 + \hat{\alpha}_2)s/6}Q_1'(\zeta), \quad Q_2(\zeta) = \zeta^{(\hat{\alpha}_1 - \hat{\alpha}_2)s/6}Q_2'(\zeta), \quad Q_3(\zeta) = \zeta^{(\hat{\alpha}_1 + 2\hat{\alpha}_2)s/6}Q_3'(\zeta), \]
we obtain the factorization formula
\[ C_2\mathcal{T}^\mu(\zeta) = Q_1(\zeta_1^\mu)Q_2(\zeta_2^\mu)Q_3(\zeta_3^\mu), \quad (37) \]
where
\[ C_2 = \frac{q^{\hat{\alpha}_1/2}q^{(\hat{\alpha}_1 + \hat{\alpha}_2)/2}q^{\hat{\alpha}_2/2}}{1 - q^{\hat{\alpha}_1}1 - q^{\hat{\alpha}_1 + \hat{\alpha}_2}1 - q^{\hat{\alpha}_2}}. \]
We again obtain a factorization formula with the coefficient \( C_2 \) that is independent of \( \mu \).

Choosing an appropriate normalization, we write (37) as
\[ C_2\mathcal{T}^\mu(\zeta) = Q_1(\zeta_1^\mu)Q_2(\zeta_2^\mu)Q_3(\zeta_3^\mu). \]

6.3. Determinant formula. The Weyl group \( W \) of the root system of \( \mathfrak{gl}_{l+1} \) is isomorphic to the symmetric group \( S_{l+1} \). It is generated by simple reflections \( r_i, \quad i = 1, \ldots, l \). The minimal number of generators \( r_i \) necessary to represent an element \( w \in W \) is said to be the length of \( w \) and is denoted by \( l(w) \). It is assumed that the identity element has the length equal to 0.

Using a quantum version of the Bernstein–Gelfand–Gelfand resolution for the quantum group \( \mathcal{U}_q(\mathfrak{gl}_{l+1}) \) (see, e.g., [56]–[58]), we obtain the equation
\[ \mathcal{T}^\mu(\zeta) = \sum_{w \in W} (-1)^{l(w)}\mathcal{T}^{w \cdot \mu}(\zeta) = \sum_{w \in S_{l+1}} \text{sgn}(w)\mathcal{T}^{w \cdot \mu}(\zeta), \]
where \( w \cdot \lambda \) denotes the affine action of \( w \) defined as
\[ w \cdot \mu = w(\mu + \rho) - \rho, \]
with
\[ \rho = \frac{1}{2} \sum_{i,j=1}^{l+1} \alpha_{ij}. \]
Now, using (35) and (37), for \( l = 1 \) and \( l = 2 \) we arrive at the determinant formula
\[ C_l\mathcal{T}^\mu(\zeta) = \det(Q_a(q^{(2\mu_1 + \mu_2)/s}\zeta))_{a,b=1}^{l+1}. \]
Assuming the appropriate normalization, in terms of the usual integrability objects we have
\[ C_l\mathcal{T}^\mu(\zeta) = \det(Q_a(q^{(2\mu_1 + \mu_2)/s}\zeta))_{a,b=1}^{l+1}. \]
7. Conclusion

Analyzing $\ell$-weights of the evaluation and $q$-oscillator representations, we have proved factorization relations for the transfer operators of quantum integrable systems associated with the quantum loop algebras $U_q(L(\mathfrak{sl}_{l+1}))$ for $l = 1$ and $l = 2$. The results obtained are entirely consistent with the results obtained in [14], [17], [28]. More details and the proof of factorization in the case of an arbitrary rank will be given in a forthcoming paper.

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