A differential graded Lie algebra controlling the Poisson deformations of an affine Poisson variety

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ABSTRACT
We construct a differential graded Lie algebra $\mathfrak{g}$ controlling the Poisson deformations of an affine Poisson variety. We analyze $\mathfrak{g}$ in the case of affine Gorenstein toric Poisson varieties. Moreover, explicit description of the second and third Hochschild cohomology groups is given for three-dimensional affine Gorenstein toric varieties.

1. Introduction
In the last decades, differential graded Lie algebras have become a very important tool in deformation theory. A deformation problem is controlled by a differential graded Lie algebra $\mathfrak{h}$ if its corresponding functor of Artin rings is isomorphic to the deformation functor of $\mathfrak{h}$. In characteristic 0 every deformation functor is controlled by a differential graded Lie algebra, due to Quillen, Deligne, Drinfeld, and Kontsevich. It is well known that associative non-commutative (resp. commutative) deformations of affine varieties are controlled by the Hochschild (resp. Harrison) differential graded Lie algebra.

In recent years, there has been a lot of interest in Poisson deformations, that is, in deformations of a pair consisting of a variety and a Poisson structure on it (see [5,7,9–11]).

In this paper, we construct a differential graded Lie algebra $\mathfrak{g}$ controlling the Poisson deformations of an affine Poisson variety $\text{Spec}(A)$. We see that the Poisson cohomology groups $H^k(\mathfrak{g})$ are related to some parts of the Hodge decomposition of Hochschild cohomology groups $HH^n(A)$ (see, e.g. [6] for definition of the Hodge decomposition). In the case of affine toric varieties, we gave a convex geometric description of these parts in [3].

The paper is organized as follows. In Section 2, we recall basic deformation theory via differential graded Lie algebras and basic results about the Hodge decomposition of Hochschild cohomology groups. The first main result of this paper is a construction of a differential graded Lie algebra $\mathfrak{g}$ controlling the Poisson deformations, which is done in Section 3. We notice that for the computation of the Poisson cohomology groups $H^k(\mathfrak{g})$ parts of the Hodge decomposition of the Hochschild cohomology are relevant (see (1)). The Poisson cohomology groups for affine 

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Poisson Gorenstein toric surfaces are computed in Subsection 4.2. Using results in [3], we explicitly compute some parts of the Hochschild cohomology groups in the case of three-dimensional affine Gorenstein toric varieties, which is our second main result (see Theorem 4.11) obtained in Subsection 4.3. This result also reproves and generalizes [1, Theorem 4.4]. In particular, a complete description of the second (which describes the first order deformations) and third Hochschild cohomology groups (which contains the obstructions for extending deformations to larger base spaces) is given (see Corollary 4.12).

2. Preliminaries

2.1. Deformation theory via differential graded Lie algebras

Let \( k \) be a field of characteristic 0 and let \( \mathcal{A} \) be the category of local Artinian \( k \)-algebras with residue field \( k \) (with local homomorphisms as morphisms). By \( S \) we denote the category of sets. If not otherwise specified a tensor product \( \otimes \) means \( \otimes_k \). Let \( \mathfrak{g} \) be a differential graded Lie algebra over \( k \) (dgla for short). By \( \mathfrak{g}^i \) we denote the degree \( i \) elements of \( \mathfrak{g} \):

**Definition 1.** For a dgla \( \mathfrak{g} \) we define the functor \( MC_\mathfrak{g} : \mathcal{A} \to S \) by

\[
B \mapsto \left\{ x \in \mathfrak{g}^1 \otimes m_B \mid d(x) + \frac{1}{2}[x, x] = 0 \right\}.
\]

\( MC_\mathfrak{g} \) is said to be the Maurer–Cartan functor associated to \( \mathfrak{g} \). Elements in \( MC_\mathfrak{g}(B) \) are the Maurer–Cartan elements of the dgla \( \mathfrak{g} \otimes B \).

**Definition 2.** Let \( G \) denote the category of groups. Let \( \mathfrak{g} \) be a dgla and define the functor \( G_\mathfrak{g} : \mathcal{A} \to G \) given by

\[
B \mapsto \exp (\mathfrak{g}^0 \otimes m_B),
\]

where \( \exp \) is the standard exponential functor on Lie algebras. \( G_\mathfrak{g} \) is said to be the gauge functor associated to \( \mathfrak{g} \).

Fix a dgla \( \mathfrak{g} \) over \( k \); the gauge functor \( G_\mathfrak{g} \) acts naturally on the Maurer–Cartan functor \( MC_\mathfrak{g} \) by the formula

\[
G_\mathfrak{g}(B) \times MC_\mathfrak{g}(B) \to MC_\mathfrak{g}(B)
\]

\[
(e^b, x) \mapsto x + \sum_{n=0}^{\infty} \frac{[b, \cdot]^n}{(n+1)!} ([b, x] - d(b)).
\]

This action is called the gauge action.

**Definition 3.** Let \( \mathfrak{g} \) be a dgla over \( k \). The deformation functor associated to \( \mathfrak{g} \) is the functor \( \text{Def}_\mathfrak{g} : \mathcal{A} \to S \) given by

\[
B \mapsto \frac{MC_\mathfrak{g}(B)}{G_\mathfrak{g}(B)}.
\]

We say that a dgla \( \mathfrak{g} \) controls a functor \( F \) if \( \text{Def}_\mathfrak{g} \cong F \) holds.

2.2. The Hodge decomposition of the Hochschild cohomology

Let \( A \) be a finitely generated \( k \)-algebra. Let \( C^*(A) \) be the Hochschild cochain complex, that is, \( C^n(A) \) is the space of \( k \)-linear maps \( f : A^{\otimes n} \to A \) (or \( A \)-module homomorphisms \( A \otimes A^{\otimes n} \to A \)) with the differential given by...
The Gerstenhaber bracket equips \( p \) the equation \( e_{C_n}C_m \) of \( f \) denoted by \( \text{HH} \) the module of \( K \).

Lemma 2.1.



An easy computation, see also [12].

Proof. One defines

\[
\text{Hochschild } df(a_1 \otimes \cdots \otimes a_n) := a_1f(a_2 \otimes \cdots \otimes a_n) + \\
\sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_ia_{i+1} \otimes \cdots \otimes a_n) + \\
(-1)^n f(a_1 \otimes \cdots \otimes a_{n-1})a_n.
\]

The Gerstenhaber bracket equips \( C^*(A)[1] \) with the structure of a dgla.

Gerstenhaber and Schack described the Hodge decomposition of the Hochschild (co-)homology that we will briefly recall (see [6] for more details). In the group ring of the permutation group \( S_n \) one defines \( s_{i,n-i} \) to be \( \sum (\text{sgn}\pi)\pi \), where the sum is taken over those permutations \( \pi \in S_n \) such that \( \pi(1) < \pi(2) < \cdots < \pi(i) \) and \( \pi(i+1) < \pi(i+2) < \cdots < \pi(n) \). Let \( s_n = \sum_{i=1}^{n-1} s_{i,n-i} \). It holds that \( C^n(A) = C^n_{(1)}(A) \oplus \cdots \oplus C^n_{(n)}(A) \), where \( C^n_{(i)}(A) = \{ f \in C^n(A) \mid f \circ s_n = (2^i - 2)f \} \). The Hodge decomposition is

\[
\text{HH}^n(A) \cong \text{H}^n_{(1)}(A) \oplus \cdots \oplus \text{H}^n_{(n)}(A),
\]

where \( \text{H}^n_{(i)}(A) \) is the \( i \)-th cohomology of \( C^*_{(i)}(A) \).

We denote the projectors of \( \text{HH}^n(A) \) to \( \text{H}^n_{(i)}(A) \) by \( e_n(i) \).

Lemma 2.1. For an element \( p \in H^2_{(2)}(A) \) and an element \( q \in H^1_{(1)}(A) \) we have the following:

- the equation \( e_3(3)[p,p] = 0 \) is the Jacobi identity, \( e_3(2)[p,p] = 0 \)
- \([p,q] = e_3(2)[p,q] \) and \([q,q] = e_3(1)[q,q] \).

Proof. An easy computation, see also [12].

3. Poisson deformations

Poisson deformations are deformations of a pair consisting of a variety and a Poisson structure on it. Lately, there has been a lot of interest in these deformations, see for example results of Namikawa [9–11] or Kaledin and Ginzburg [7].

Definition 4. A skew-symmetric Hochschild 2-cocycle \( p \) (i.e. \( p \in C^2_{(2)}(A) \) with \( dp = 0 \)) that satisfies the Jacobi identity

\[
p(a \otimes (b \otimes c)) + p(b \otimes (c \otimes a)) + p(c \otimes (a \otimes b)) = 0
\]

is called an (algebraic) Poisson structure (or a Poisson bracket). A commutative algebra together with a Poisson bracket is called a Poisson algebra. Its spectrum is called an affine Poisson variety.

Note that \( p \in H^2_{(2)}(A) \cong \text{hom}_A(\Omega^2_{A/k}, A) \) (see e.g. [8]), where \( \Omega^2_{A/k} \) is the 2nd exterior power of the module of Kähler differentials. Using Lemma 2.1 we can equivalently define the Poisson structure as an element \( p \in H^2_{(2)}(A) \) with \( e_3(3)[p,p] = 0 \).
Definition 5. A Poisson deformation of a Poisson algebra $A$ over an Artin ring $B$ is a pair $(A', \pi)$, where $A'$ is a Poisson $B$-algebra and $\pi : A' \otimes_B k \to A$ is an isomorphism of Poisson $k$-algebras. Two such deformations $(A', \pi_1)$ and $(A'', \pi_2)$ are equivalent if there exists an isomorphism of Poisson $B$-algebras $\phi : A' \to A''$ such that it is compatible with $\pi_1$ and $\pi_2$, i.e. such that $\pi_1 = \pi_2 \circ (\phi \otimes_B k)$.

3.1. dgla that controls the deformation problem

A functor that encodes this deformation problem is

$$ PDef_A : A \to \mathcal{S} $$

$$ B \mapsto \{\text{Poisson deformations of } A \text{ over } B\}/\sim. $$

In the following, we define a dgla that controls the above deformation problem.

Consider the double complex 1 (see Figure 1).

![Double complex 1](image-url)

The map $d_p$ is defined as $d_p := -[\mu, _p] : C^n(A) \to C^{n+1}(A)$, where $\mu_p \in C^2(A)$ is a Poisson structure on $A$. In the double complex 1, we restrict $d_p$ on the chosen domains and codomains. Note that we have $d[\mu_p, f] = [\mu_p, df]$ (since $d\mu_p = 0$) and thus we really obtain a double complex. We denote its total complex by $D^*$. The cohomology groups of $D^*$ are called Poisson cohomology groups.

We define the bracket $[\cdot, \cdot]_p$ on $D^*$ as follows: let $C^n(A) = C^n_1(A) \oplus \cdots \oplus C^n_n(A)$ and define

$$ [\cdot, \cdot]_p : C^n(A) \times C^n(A) \to C^{n+1}(A) $$

$$ [(f_1, \ldots, f_m), (g_1, \ldots, g_n)]_p := \left( [f_1, g_1], \ldots, \sum_{i+j=k} [f_i, g_j], \ldots, [f_m, g_n] \right), $$

where we restrict $[f_i, g_j]$ to $C^{n+1}_{i+j-1}(A)$.

This bracket defines a dgla structure on $D^*[1] :$ the shifted differential $d_p[1]$ is equal to $[\mu, _p]_p$ and the shifted differential $d[1]$ is equal to $[\mu, _p]_p$, where $\mu$ is the commutative multiplication on $A$. We denote the shifted differential of $D^*[1]$ by $\bar{d}$. It is given by $\bar{d} = [\mu + \mu_p, _p]$. We can immediately check that the bracket $[\cdot, \cdot]_p$ and differential $\bar{d}$ equip $D^*[1]$ with the structure of a dgla. We denote this dgla by $C_p(A)[1]$. 
Remark 1. Note that the Gerstenhaber bracket is in general not graded with respect to the Hodge decomposition and thus the above product is not the Gerstenhaber bracket. From Lemma 2.1, we have \([\mu, \mu]_p = [\mu, \mu], [\mu, \mu_p]_p = [\mu, \mu_p]_p = e_3(3)[\mu_p, \mu_p].\)

After applying the differentials \(d\) on the double complex 1, we obtain for \(j, k \geq 1\) the first spectral sequence

\[
E_{1}^{j, k} = H_{(j)}^{j+k-1}(A) \Rightarrow H_{(j)}^{j+k-1}(C_p^*(A)[1]),
\]

where \(d_1 = -[\mu_p, \cdot] : E_{1}^{j, k} \rightarrow E_{1}^{j+1, k}.

To show that the functor PDef\(_A\) is controlled by the dgla \(C_p^*(A)[1]\) we first need few Lemmata. For a \(k\)-algebra \(A\), we define the \(k\)-algebra \(A_0\) that is as a \(k\)-vector space isomorphic to \(A\) and it has zero multiplication.

Lemma 3.1. Poisson algebra structures on \(A_0\) are in bijection with Maurer–Cartan elements of \(C_p^*(A_0)[1]\), that is, with elements \((\mu, \mu_p) \in C^2_{(1)}(A_0) \oplus C^2_{(2)}(A_0)\) satisfying \(\frac{1}{2}[\mu, \mu] = [\mu, \mu_p] = \frac{1}{2}[\mu_p, \mu_p] = 0\).

Proof. Let \((\mu, \mu_p)\) be a Maurer–Cartan element of \(C_p^*(A_0)[1]\). For \(a, b \in A_0\), the product given by \(\mu(a, b) := \mu(a \otimes b)\) is commutative and associative if and only if \(\mu \in C^2_{(1)}(A_0)\) and \(\frac{1}{2}[\mu, \mu] = 0\). We define \(\{a, b\} := \mu_p(a, b) := \mu_p(a \otimes b)\) and show that \(\mu_p\) defines a Poisson structure. Since \(\mu_p \in C^2_{(2)}(A_0)\), everything except the Jacobi identity is clear. The Jacobi identity we get from \(\frac{1}{2}[\mu_p, \mu_p] = 0\) as in Lemma 2.1 (note that we have \([\mu_p, \mu_p] = e_3(3)[\mu_p, \mu_p]\). We now show the following claim:

\[
\{a, b \cdot c\} = \{a, b\}c + \{a, c\}b \quad \text{(i.e.,} \mu_p(a, \mu(b, c)) = \mu(\mu_p(a, b), c) + \mu(\mu_p(a, c), b))
\]

holds if and only if \([\mu, \mu_p] = 0\). Assume that

\[
F(a, b, c) := \mu_p(a, \mu(b, c)) - \mu_p(a, b, c) - \mu_p(\mu_p(a, b), c) - \mu_p(\mu_p(a, c), b) = 0
\]

holds. We have

\[
F(a, b, c) + F(c, a, b) = (\mu_p(a, \mu(b, c)) - \mu_p(a, b, c) - \mu_p(\mu_p(a, b), c)) + (\mu_p(c, \mu(a, b)) - \mu_p(\mu_p(c, a), b) - \mu_p(\mu_p(c, b), a)) = -[\mu_p, \mu](a, b, c).
\]

and thus we see one direction. For the other direction we compute

\[
[\mu_p, \mu](a, b, c) + [\mu_p, \mu](a, c, b) - [\mu_p, \mu](b, a, c) =
\]

\[
-\mu_p(ab, c) - \mu_p(ac, b) + \mu_p(ab, c) - \mu_p(ac, b) + \mu_p(ac, b) - \mu_p(ab, c) - \mu_p(ab, c) + \mu_p(ac, b) - \mu_p(ab, c) + \mu_p(ac, b) =
\]

\[
2(-\mu_p(ab, c) + \mu_p(ab, c) - \mu_p(ab, c)) = -2F(a, b, c).
\]

To shorten the notation we wrote \(ab = \mu(a, b)\) and similarly for other elements. Thus the claim is proved. From this we easily conclude the proof. \(\square\)

Definition 6. The Poisson structure on a vector space \(V\) is a pair \((\cdot, \{\ ,\ \})\), such that \((V, \cdot, \{\ ,\ \})\) is a Poisson algebra.

Lemma 3.2. Let \(A\) be a Poisson algebra and let \(B\) be an Artin ring. Maurer–Cartan elements of \(C_p^*(A \otimes m_B)[1]\) are in bijection with Poisson structures on the vector space \(A \otimes_k B\), giving the known Poisson structure on \(A \cong A \otimes_k B/m_B.\)
Proof. Let a Maurer–Cartan element \((\mu, \mu_p)\) of \(C^*_p(A_0)[1]\) represents the Poisson bracket of \(A\). The Poisson structures on the vector space \(A \otimes k B\), giving the known product on \(A \cong A \otimes k B / m_B\) are obtained by \((\xi, \xi_p) \in C^2((A \otimes m_B) \oplus C^2(A \otimes m_B)\) satisfying

\[
[(\mu, \mu_p) + (\xi, \xi_p), (\mu, \mu_p) + (\xi, \xi_p)]_p = 0.
\] (2)

Since \([(\mu, \mu_p), (\mu, \mu_p)]_p = 0\) and the differential on \(C^*_p(A \otimes m_B)[1]\) is given by \([(\mu, \mu_p), \cdot],\) then we see that equation (2) gives us MC elements \((\xi, \xi_p)\) of \(C^*_p(A \otimes m_B)[1]\). \(\square\)

Proposition 3.3. For a Poisson algebra \(A\) the functor \(\text{PDef}_A\) is controlled by the dgla \(C^*_p(A)[1]\).

Proof. We write for short \(\mathfrak{g} := C^*_p(A)[1]\). By Lemma 3.2 there exists a bijection between \(\text{MC}_{\mathfrak{g}}(B)\) and Poisson structures on the vector space \(A \otimes k B\), giving the known Poisson structure on \(A \cong A \otimes k B / m_B\).

To conclude the proof we show that two Poisson structures \((\cdot, \{\cdot, \cdot\}\) and \((\cdot', \{\cdot, \cdot\}')\) on \(A \otimes k B\) are equivalent (in the sense of Definition 5) if and only if the corresponding elements \((\gamma, \gamma_p), (\gamma', \gamma'_p) \in \text{MC}_{\mathfrak{g}}(B)\) are gauge equivalent. The Poisson structures are equivalent if and only if there exists \(x \in C^1(A) \otimes m_B\) such that

\[
a \cdot' b = \exp(x)(\exp(-x)(a) \cdot \exp(-x)(b)),
\]

(3)

\[
\{a, b\}' = \exp(x)(\{ \exp(-x)(a), \exp(-x)(b)\}).
\]

(4)

As above let a Maurer–Cartan element \((\mu, \mu_p)\) of \(C^*_p(A_0)[1]\) represents the Poisson bracket of \(A\). From (3) we obtain

\[
(\mu + \gamma')(a, b) = \exp(x)(\exp(-x)(a) \cdot \exp(-x)(b)) = \exp([x, \cdot])(\mu + \gamma)(a, b),
\]

(5)

where the later equality we get after some elementary computation. In the same way, from (4) we obtain

\[
(\mu_p + \gamma'_p)(a, b) = \exp(x)\{ \exp(-x)(a), \exp(-x)(b)\} = \exp([x, \cdot])(\mu_p + \gamma_p)(a, b).
\]

(6)

Elements \((\gamma, \gamma_p) \in \text{MC}_{\mathfrak{g}}(B)\) and \((\gamma', \gamma'_p) \in \text{MC}_{\mathfrak{g}}(B)\) are gauge equivalent if

\[
(\gamma', \gamma'_p) = (\gamma, \gamma_p) + \sum_{n=0}^{\infty} \frac{[x, \cdot]_p^n}{(n + 1)!} ([x, (\gamma, \gamma_p)]_p - \tilde{d}(x))
\]

(7)

holds.

Since \(\tilde{d}(x) = [(\mu, \mu_p), x]_p = -[x, (\mu, \mu_p)]_p\) and \([x, \cdot] = [x, \cdot]_p\), we see that (7) holds if and only if equations (5) and (6) hold. \(\square\)

4. Computation of the Hochschild and Poisson cohomology groups for Gorenstein toric varieties

4.1. Affine Gorenstein toric varieties

Let \(M, N\) be mutually dual, finitely generated, free Abelian groups. We denote by \(M_{\mathbb{R}}, N_{\mathbb{R}}\) the associated real vector spaces obtained via base change with \(\mathbb{R}\). Let \(\sigma \subset N_{\mathbb{R}}\) be a rational, polyhedral cone with apex in 0 and let \(a_1, ..., a_N \in N\) denote its primitive fundamental generators (i.e. none of the \(a_i\) is a proper multiple of an element of \(N\)). We define the dual cone \(\sigma^\vee := \{ r \in M_{\mathbb{R}} \mid \langle \sigma, r \rangle \geq 0 \} \subset M_{\mathbb{R}}\) and denote by \(\Lambda := \sigma^\vee \cap M\) the resulting semi-group of lattice points. Its spectrum Spec(\(k[\Lambda]\)) is called an affine toric variety.
Affine toric Gorenstein varieties are obtained by putting a lattice polytope \( P \subset \mathbb{A} \cong \mathbb{R}^{n-1} \) into the affine hyperplane \( \mathbb{A} \times \{1\} \subset \mathbb{A} \times \mathbb{R} =: \mathbb{N} \) and defining \( \sigma := \text{Cone}(P) \), the cone over \( P \). Then, the canonical degree \( R' \in M \) equals \((0,1)\).

It is a trivial check that Hochschild differentials respect the grading given by the degrees \( R \in M \). Thus, we get the Hochschild subcomplex \( C^i_{(i)} \) and we denote the corresponding cohomology groups by \( H^i_{(i)}(A) \). Then, the canonical degree \( R' \) is the degree \( R \) part of the (higher) André–Quillen cohomology group \( T^n_{(i)}(A) \) (see [3, Section 4]). We will not use general André–Quillen cohomology theory, we will only use the well-known isomorphism \( T^n_{(i)}(A) \cong H^n_{(i)}(A) \) for \( n \geq i \) (see, e.g. [8]).

### 4.2. Poisson cohomology groups of Poisson Gorenstein toric surfaces

Let \( X_\sigma = \text{Spec}(A_\sigma) \) be the Gorenstein toric surface given by \( g(x,y,z) = xy - z^{r+1} \). \( A_\sigma := \sigma_\vee \cap M \) is generated by \( S_1 := (0,1), S_2 := (1,1) \) and \( S_3 := (r+1, r) \), with the relation \( S_1 + S_3 = (r+1)S_2 \).

In order to compute the Poisson cohomology groups we need to analyze the spectral sequence (1). First, we need to understand all the parts of the Hochschild cohomology.

**Proposition 4.1.** It holds that

\[
\dim_k T^i_{(1)}(A) = \dim_k T^i_{(2)}(A) = \begin{cases} 1 & \text{if } R = kS_2 \text{ for } 2 \leq k \leq r+1 \\ 0 & \text{otherwise.} \end{cases}
\]

Moreover, \( T^i_{(1)}(A) \cong H^n_{(i)}(A) = 0 \). For \( i \geq 3 \), we have \( T^i_{(i)}(A) = 0 \) if \( k \neq i - 1, i \) and

\[
T^i_{(1)}(A) \cong T^i_{(1)}(A) \cong A_r/\left( \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3} \right).
\]

The later has \( k \)-dimension equal to \( r \).

**Proof.** [3, Proposition 3.3, Example 3].

**Corollary 4.2.** Since \( T^i_{(1)}(A) \cong H^n_{(i)}(A) \) and \( T^i_{(2)}(A) \cong H^n_{(i)}(A) \) we see that \( E_2^{ik} = E_2^{ik} \) holds for every \( j, k \geq 1 \).

Elements from \( H^2_{(2)}(A) \) define Poisson structures on \( \text{Spec}(A_\sigma) \) by Lemma 2.1, since \( H^3_{(3)}(A) = 0 \). Let \( \mu_\rho \in H^2_{(2)}(A) \) denote a Poisson structure on \( \text{Spec}(A_\sigma) \). Let \( g_\sigma := C^i_{(i)}(A_\sigma)[1] \). From above we have the following description of the spectral sequence (1):

\[
E^1_{1,3} : d_i H^1_{(2)}(A) \rightarrow H^5_{(3)}(A) \rightarrow 0 \rightarrow \ldots
\]

\[
E^1_{1,2} : H^2_{(2)}(A) \rightarrow H^3_{(2)}(A) \rightarrow 0 \rightarrow \ldots
\]

\[
E^1_{1,1} : H^1_{(1)}(A) \rightarrow H^2_{(2)}(A) \rightarrow 0 \rightarrow \ldots
\]

\[
E^1_{1,j} \text{ for } j > 3 \text{ have only two non-vanishing terms } E^{j-1}_{1,j} = H^j_{(j)}(A) \text{ and } E^{j}_{1,j} = H^j_{(j)}(A).
\]

**Corollary 4.3.**

\[
H^0(g_\sigma) \cong \ker(H^1_{(1)}(A) \rightarrow H^2_{(2)}(A)),
\]

\[
H^1(g_\sigma) \cong \text{coker}(H^1_{(1)}(A) \rightarrow H^2_{(2)}(A)) \oplus \ker(H^2_{(1)}(A) \rightarrow H^3_{(2)}(A)),
\]

for \( k \geq 2 \) we have
They appear in lattice Proposition 4.4.

For an affine Gorenstein toric variety $X$.

4.3. The Hochschild cohomology of three-dimensional affine Gorenstein toric varieties

From Corollary 4.3 we know that $H^1(A_r) = \ker(H^{1+1}(A_r) \to H^2(A_r))$. In [4], we proved that the Gerstenhaber product $H^2(A_r) \times H^2(A_r) \to H^4(A_r)$ is the zero map. Since by definition $d_1 = -[\mu_p, ]$, we see that $d_1$ is the zero map. Thus, $H^2(g_r) \cong H^2(A_r) \cong A_r/(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$

\[ H^k(g_r) \cong \begin{cases} \text{coker}(H^k(\frac{1}{2})/(A_r) \xrightarrow{d_1} H^{k+1}(\frac{1}{2}+1)(A_r)) & \text{if } k \text{ is even} \\ \ker(H^{k+1}(\frac{1}{2}+1)(A_r) \xrightarrow{d_1} H^{k+2}(\frac{1}{2}+1)(A_r)) & \text{if } k \text{ is odd}. \end{cases} \]

**Proposition 4.4.** It holds that $H^2(g_r) \cong A_r/(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$

**Proof.** From Corollary 4.3 we know that $H^2(g_r) \cong \ker(H^2(A_r) \xrightarrow{d_1} H^3(A_r))$. In [4], we proved that the Gerstenhaber product $H^2(A_r) \times H^2(A_r) \to H^4(A_r)$ is the zero map. Since by definition $d_1 = -[\mu_p, ]$, we see that $d_1$ is the zero map. Thus, $H^2(g_r) \cong H^2(A_r) \cong A_r/(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$

\[ H^k(g_r) \cong \begin{cases} \text{coker}(H^k(\frac{1}{2})/(A_r) \xrightarrow{d_1} H^{k+1}(\frac{1}{2}+1)(A_r)) & \text{if } k \text{ is even} \\ \ker(H^{k+1}(\frac{1}{2}+1)(A_r) \xrightarrow{d_1} H^{k+2}(\frac{1}{2}+1)(A_r)) & \text{if } k \text{ is odd}. \end{cases} \]

**Example 1.** For every hypersurface given by a polynomial $g(x, y, z)$ in $k^3$, we can define a Poisson structure $\pi_g$ on the quotient $k[x, y, z]/g$, namely:

\[ \pi_g := \partial_x(g) \partial_y \partial_z + \partial_y(g) \partial_z \partial_x + \partial_z(g) \partial_x \partial_y, \]

that is, we contract the differential 1-form $dg$ to $\partial_x \partial_y \partial_z$. In the case of Gorenstein toric surfaces $X_n = \text{Spec}(A_r)$ we have that

\[ \pi_g = f_0(\lambda_1, \lambda_2)x^{x_1-\lambda_1+x_2}, \]

where $f_0$ is skew-symmetric and bi-additive with $f_0(S_1, S_3) = -(r + 1)$ (see [3, Example 4]). Thus, we see that $\pi_g \in H^2(g_r) \cong H^2(A_r)$. In this case, we see that $H^1(g_r) \cong H^1(A_r) \cong A_r/(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ by the proof of Proposition 4.4 and Corollary 4.3 since $H^1(A_r) \to H^2(A_r)$ is surjective. This is special case of [7, Lemma 3.1].

4.3. The Hochschild cohomology of three-dimensional affine Gorenstein toric varieties

For an affine Gorenstein toric variety $X_n = \text{Spec}(A)$ we will explicitly compute $T^i_{(1)}(A)$ for all $i \geq 1$. They appear in $L^*_{1,2}$ (see (1) and note that $H^{i+1}_{(1)}(A) \cong T^i_{(1)}(A)$) and they are also important ingredients for understanding $HH^2(A)$ and $HH^3(A)$ (see Corollary 4.12). This subsection reproves and generalizes [1, Theorem 4.1].

In [3], we obtained a convex geometric description of $T^i_{(1)}(A)$ for $i \geq 1$, which we recall now. Let the cone $\sigma \subset N_R$ with primitive generators $a_1, \ldots, a_N \in N$ represent an $n$-dimensional toric variety $X_n = \text{Spec}(A), n \geq 3$. For $R \in M$ we define the affine space

\[ \mathcal{A}(R) := \{ a \in N_R \mid \langle a, R \rangle = 1 \} \subset N_R \]

and consider the polyhedron $Q(R) := \sigma \cap \mathcal{A}(R) \subset \mathcal{A}(R)$. Vertices of $Q(R)$ are $\bar{a}_j := a_j/\langle a_j, R \rangle$, for all $j$ satisfying $\langle a_j, R \rangle \geq 1$. We denote $T^1_{(1)}(A)$.

Let $d_{jk} := \bar{a}_j \bar{a}_k$ denote the compact edges of $Q(R)$ (for $\langle a_j, R \rangle \geq 1, \langle a_k, R \rangle \geq 1$). We denote the lattice $N \cap \text{Span}_R\{a_j, a_k\}$ by $\bar{N}_{jk}$ and its dual with $\bar{M}_{jk}$. Let $\bar{R}_{jk}$ denote the projection of $R$ to $\bar{M}_{jk}$. By $T^1_{(a_j, a_k)}(-R_{jk})$ we denote the degree $-R_{jk}$ part of $T^1_{(1)}(A)$ of the toric surface given by the cone spanned by $a_j, a_k$ in $\bar{N}_{jk} \otimes \mathbb{Z}$. We define $\text{Span}_kK^R_{jk}$ to be

\[ \text{Span}_kK^R_{jk} := \text{Span}_k(K^R_{aj} \cap K^R_{ak}), \]

with $K^R_{aj} = \{ r \in \Lambda \mid \langle a_j, r \rangle < \langle a_j, R \rangle \}$. Let
Let a cone $\mathcal{C}$.

**Lemma 4.6.** Using the previous notation we see that the polytope generators of the dual cone $\mathcal{C}^*$.

**Proof.** See [3, Lemma 4.3].

We obtain $\mathcal{C}$.

**Example 2.** A typical example of a non-isolated, three-dimensional toric Gorenstein singularity is the affine cone $\mathcal{X}_0$ over the weighted projective space $\mathbb{P}(1,2,3)$. The cone $\mathcal{C}$ is given by $\mathcal{C} = \langle a_1, a_2, a_3 \rangle$, where

\[
a_1 = (-1, -1, 1), \quad a_2 = (2, -1, 1), \quad a_3 = (-1, 1, 1).
\]

We obtain $\mathcal{C} = \langle s_1, s_2, s_3 \rangle$ with

\[
s_1 = (0, 1, 1), \quad s_2 = (-2, -3, 1), \quad s_3 = (1, 0, 1).
\]

We need to better understand $s^i_{\mathcal{C}(R)}$ and $\dim_k \mathcal{T}_{\langle a_i, a_j \rangle}(\mathcal{C})$ that appears in $\mathcal{C}^i_{\mathcal{C}, j+1}(R)$.

**Lemma 4.6.** Let a cone $\mathcal{C} = \langle a_j, a_{j+1} \rangle \subset \mathcal{N}_{j, j+1}$ define a toric surface given by the edge $\mathcal{C}_j$. We have

\[
\dim_k \mathcal{T}_{\langle a_i, a_j \rangle}(\mathcal{C}) = \max \{0, W_j(R) + W_{j+1}(R) - 2 - \dim_k \mathcal{T}_{\langle a_i, a_j \rangle}(\mathcal{C}) \}.
\]

**Proof.** See [3, Lemma 4.3].

\[
W_j(R) := \begin{cases} 2 & \text{if } \langle a_j, R \rangle > 1 \\ 1 & \text{if } \langle a_j, R \rangle = 1 \\ 0 & \text{if } \langle a_j, R \rangle \leq 0. \end{cases}
\]

**Proposition 4.5.** If the compact part of $\mathcal{Q}(R)$ lies in a two-dimensional affine space we have

\[
\dim_k \mathcal{T}_{(i)}^i(-R) = \max \left\{ 0, \sum_{j=1}^{N} V_j(R) - \sum_{d_j \in \mathcal{Q}(R)} Q_j(R) - \left( \binom{n}{i} + s_{\mathcal{C}(R)}^i \right) \right\},
\]

where

\[
V_j(R) := \begin{cases} \binom{n}{i} & \text{if } \langle a_j, R \rangle > 1 \\ \binom{n-1}{i} & \text{if } \langle a_j, R \rangle = 1 \\ 0 & \text{if } \langle a_j, R \rangle \leq 0, \end{cases}
\]

\[
Q_j(R) := \begin{cases} \left( W_j(R) + W_k(R) + n - 4 - \dim_k \mathcal{T}_{\langle a_i, a_k \rangle}(\mathcal{C}) \right) \left( \binom{n}{i} + s_{\mathcal{C}(R)}^i \right) & \text{if } \langle a_j, R \rangle, \langle a_k, R \rangle \neq 0 \\ 0 & \text{otherwise}, \end{cases}
\]

\[
s_{\mathcal{C}(R)}^i := \begin{cases} \dim_k \mathcal{T}_{\langle a_i, a_j \rangle}(\mathcal{C}) \cap \mathcal{K}_{\mathcal{C}(R)} & \text{if } \mathcal{Q}(R) \text{ is compact} \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** See [3, Proposition 4.14].
Lemma 4.7. \[
\dim_k T^1_{(a_j, a_{j+1})}(-\bar{R}_{j,j+1}) = \begin{cases} 
1 & \text{if } 2 \leq \langle a_j, R \rangle = \langle a_{j+1}, R \rangle \leq \ell(j) \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. The toric surface \( \langle a_j, a_{j+1} \rangle \) is isomorphic to the Gorenstein toric surface \( \text{Spec}(A_{\ell(j)-1}) \) and then the proof follows from Proposition 4.1 (more precisely from equation (8)). \( \Box \)

Lemma 4.8. If there exists \( a_j \) such that \( \langle R, a_j \rangle \leq 0 \), then \( s_{Q(R)}^i = 0 \), otherwise \( s_{Q(R)}^i \leq \binom{3}{i} \).

Proof. Follows trivially from definitions. \( \Box \)

The next lemma establishes a useful criterion when \( T^1_{(i)}(-R) \) is zero in the Gorenstein three-dimensional case.

Lemma 4.9. Assume that \( \langle a_j, R \rangle \neq \langle a_{j+1}, R \rangle \) for all \( j = 1, \ldots, N \) \( (a_{N+1} := a_1) \). Then \( T^1_{(i)}(-R) = 0 \).

Proof. We will use Proposition 4.5. By the assumption and Lemma 4.7 we know that \( \dim_k T^1_{(a_j, a_{j+1})}(-\bar{R}_{j,j+1}) = 0 \) for all \( j \). Since \( \langle a_j, R \rangle = 1 \) for at most two \( j \in \{1, \ldots, N\} \) we can using Lemma 4.8 easily see that
\[
\sum_{j=1}^N V_j^i(R) - \sum_{j=1}^N \left( W_j^i(R) + W_{j+1}^i(R) - 1 \right) - \binom{3}{i} + s_{Q(R)}^i \leq 0,
\]
which implies that \( T^1_{(i)}(-R) = 0 \) for all \( i \). \( \Box \)

Lemma 4.10. Let \( R = qR^* \) for \( q \geq 2 \). It holds that \( \dim_k \cap \text{Span}_k K^R_{j,j+1} = 3 \) if \( \ell(j) < q \) for all \( j \). Moreover, \( \dim_k \cap \text{Span}_k K^R_{j,j+1} = 2 \) if \( \ell(j) < q \) for all \( j \) except two (denoted by \( j_1 \) and \( j_2 \)), for which it holds that \( d_{j_1} \) and \( d_{j_2} \) are parallel (the case \( j_1 = j_2 \) is included). Otherwise it holds that \( \dim_k \cap \text{Span}_k K^R_{j,j+1} = 1 \).

Proof. By definition we easily see that
\[
\text{Span}_k K^R_{j,j+1} = \begin{cases} 
M \otimes \mathbb{Z} & \text{if } \ell(j) < q \\
\text{Span}_k \{a^j_j \cap a^j_{j+1}, R^*\} & \text{if } \ell(j) \geq q.
\end{cases}
\]

Since \( \text{Span}_k \{a^j_j \cap a^j_{j+1}, R^*\} = \{c \in M \otimes \mathbb{Z} \mid \langle c, a_j \rangle = \langle c, a_{j+1} \rangle\} \) we see that \( \text{Span}_k \{a^j_j \cap a^j_{j+1}, R^*\} = \text{Span}_k \{a^j_j \cap a^j_{j+1}, R^*\} \) for \( j_1, j_2 \in \{1, \ldots, N\} \) if and only if \( j_1 = j_2 \) or \( d_{j_1} \) is parallel to \( d_{j_2} \). Thus, we can easily conclude the proof. \( \Box \)

Let \( \text{int}(\sigma^\nu) \) denotes the interior of \( \sigma^\nu \).

Theorem 4.11. Let \( X_\sigma \) be a three-dimensional affine toric Gorenstein variety. The following holds:

(1) \( T^1_{(1)}(-R) \) is non-trivial in the following cases:
   (a) \( R = R^* \) with \( \dim_k T^1_{(1)}(-R) = N - 3 \),
   (b) \( R = qR^* \) (for \( q \geq 2 \)) with \( \dim_k T^1_{(1)}(-R) = \max \{0, \{j \mid q \leq \ell(j)\} - 2\} \),
   (c) \( R = qR^* - ps \) with \( 2 \leq q \leq \ell(j) \) and \( p \in \mathbb{Z} \) sufficiently large such that \( R \notin \text{int}(\sigma^\nu) \). In this case \( \dim_k T^1_{(1)}(-R) = 1 \).
   Additional degrees exist only in the following two (overlapping) exceptional cases:
   (d) \( P \) contains a pair of parallel edges \( d_p \) \( d_o \), both longer than every other edge. Then \( \dim_k T^1_{(1)}(-qR^*) = 1 \) for \( q \) in the range
max{ℓ(l) | l ≠ j, k} < q ≤ min{ℓ(j), ℓ(k)},

(e) P contains a pair of parallel edges d_p, d_k (with distance d = \langle a_i, s_k \rangle = \langle a_k, s_j \rangle). In this case, dim_k T^i_{(1)}(−R) = 1 for R = q R^* + ps_j with 1 ≤ q ≤ ℓ(j) and 1 ≤ p ≤ (ℓ(k) − q)/d.

(2) T^i_{(2)}(−R) is non-trivial in the following cases:
(a) R = R^* with dim_k T^i_{(2)}(−R) = N − 3,
(b) R = q R^* (for q ≥ 2) with dim_k T^i_{(2)}(−R) = max{0, 2 · {j | q ≤ ℓ(j)} − 3},
(c) R = q R^* − ps_j with 2 ≤ q ≤ ℓ(j) and p ∈ \mathbb{Z} sufficiently large such that R ∉ int(σ^v). In this case, dim_k T^i_{(2)}(−R) = 2.
Additional degrees exist only in the following two (overlapping) exceptional cases:
(d) P contains a pair of parallel edges d_p, d_k both longer than every other edge. Then, dim_k T^i_{(2)}(−q R^*) = 2 for q in the range
max{ℓ(l) | l ≠ j, k} < q ≤ min{ℓ(j), ℓ(k)},

(e) P contains a pair of parallel edges d_p, d_k (with distance d = \langle a_i, s_k \rangle = \langle a_k, s_j \rangle). In this case, dim_k T^i_{(2)}(−R) = 2 for R = q R^* + ps_j with 1 ≤ q ≤ ℓ(j) and 1 ≤ p ≤ (ℓ(k) − q)/d.

(3) T^i_{(3)}(−R) is non-trivial in the following cases:
(b) R = q R^* (for q ≥ 2) with dim_k T^i_{(3)}(−R) = max{0, {j | q ≤ ℓ(j)} − 1},
(c) R = q R^* − ps_j with 2 ≤ q ≤ ℓ(j) and p ∈ \mathbb{Z} sufficiently large such that R ∉ int(σ^v). In this case, dim_k T^i_{(3)}(−R) = 1.
Additional degrees exist only in the following two (overlapping) exceptional cases:
(d) P contains a pair of parallel edges d_p, d_k both longer than every other edge. Then, dim_k T^i_{(3)}(−q R^*) = 1 for q in the range
max{ℓ(l) | l ≠ j, k} < q ≤ min{ℓ(j), ℓ(k)},

(e) P contains a pair of parallel edges d_p, d_k (with distance d = \langle a_i, s_k \rangle = \langle a_k, s_j \rangle). In this case, dim_k T^i_{(3)}(−R) = 1 for R = q R^* + ps_j with 1 ≤ q ≤ ℓ(j) and 1 ≤ p ≤ (ℓ(k) − q)/d.

(4) We have T^i_{(i)}(−R) = 0 for i ≥ 4.

**Proof.** We distinguish the following cases.

- Let R = R^*. In this case, we have \langle a_i, R \rangle = 1 and Span_k R^k_a = (a_i)^⊥ for all i. Thus, s^i_{Q(R)} = 0 for all i and by Lemma 4.7 we have T^i_{\langle a_i, a_{i+1} \rangle}(−R_{j+1}) = 0 = 0 for all j. Moreover,

\[
\sum_{j=1}^{N} V^i_j(R) − \sum_{d_k \in Q(R)} Q^i_{d_k}(R) = N \left( \begin{array}{c} 2 \\ i \end{array} \right) − N \left( \begin{array}{c} 1 \\ i \end{array} \right).
\]

From Proposition 4.5 it follows that dim_k T^i_{(1)}(−R^*) = dim_k T^i_{(2)}(−R^*) = N − 3 and T^i_{(i)}(−R^*) = 0 for i ≥ 2. Thus, we proved (a) cases (note that T^i_{(3)}(−R^*) = 0 and thus (a) case does not appear in the case (3)).

- Let R = q R^*, where q ≥ 2.
In this case, we have \langle a_i, R \rangle ≥ 2. Thus, \sum_{j=1}^{N} V^i_j(R) = \left( \begin{array}{c} 3 \\ i \end{array} \right)N. Let us define v := \# \{j | q ≤ ℓ(j)\}. For i = 1, we have \sum_{j=1}^{N} Q^i_{d_k}(R) = 3N − v (since for q ≤ ℓ(j) we have dim_k T^i_{\langle a_i, a_{i+1} \rangle}(−R_{j+1}) = 1 by Lemma 4.7). Thus, dim_k T^i_{(1)}(−R) = v − 3 + s^i_{Q(R)} holds by Proposition 4.5. If s^i_{Q(R)} = 1 (see Lemma 4.10 when this holds) we obtain the case (1b). There is an exceptional case for which dim_k T^i_{(2)}(−R) ≠ 0: this appears if v = 2 and s^i_{Q(R)} = 2. From Lemma 4.10 we see that this happens for q in the range...
Let $\dim_k T^1_{(i)}(R) = 2(v-3) + s^2_{Q(k)}$. As in the case $i=1$ we obtain from Theorem 4.10 the cases (2b) and (2d).

For $i=3$, we have $\sum_{j=1}^N Q^3_{k,j+1}(R) = N - v$. By Proposition 4.5, we have $\dim_k T^1_{(3)}(-R) = v-1 + s^3_{Q(k)}$. As in the case $i=1$, we obtain from Theorem 4.10 the cases (3b) and (3d).

- Let $R \notin \text{int}(\sigma^\nu)$. By Lemma 4.9, we see that the only possible cases for having a non-zero $T^1_{(i)}(-R)$ occur when $\langle a_j, R \rangle = \langle a_{j+1}, R \rangle > 0$ for some $j \in \{1, \ldots, N\}$ and $\langle a_i, R \rangle \leq 0$ for all other $i$. This happens for $R = qR^* - ps^j$ with $q \geq 1$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin \text{int}(\sigma^\nu)$. In this case, we have $\langle a_j, R \rangle = \langle a_{j+1}, R \rangle = q$ and $\langle a_i, R \rangle \leq 0$ for other $i$. If $q=1$, then by Lemma 4.7 it holds that $T^1_{(a_j, a_{j+1})}(-R) = 0$ and thus by Proposition 4.5 we have

$$\dim_k T^1_{(i)}(-R) = \max\{0, 2\left(\frac{3}{i}\right) - \left(\frac{1}{i}\right) - \left(\frac{3}{i}\right)\} = 0$$

for all $i$. If $q \geq 2$, then using Lemma 4.7 we see that

$$\dim_k T^1_{(i)}(-R) = \left\{\begin{array}{ll}
2\left(\frac{3}{i}\right) - \left(\frac{2}{i}\right) - \left(\frac{3}{i}\right) & \text{if } 2 \leq q \leq \ell(j) \\
2\left(\frac{3}{i}\right) - \left(\frac{3}{i}\right) - \left(\frac{3}{i}\right) = 0 & \text{if } q > \ell(j).
\end{array}\right.$$}

In the cases $2 \leq q \leq \ell(j)$, we see that $\dim_k T^1_{(1)}(-R) = \dim_k T^1_{(3)}(-R) = 1$, $\dim_k T^1_{(2)}(-R) = 2$ and $\dim_k T^1_{(i)}(-R) = 0$ for $i \geq 4$. This proves (c) cases.

- Let $R \in \text{int}(\sigma^\nu)$ and $R \neq qR^*$ for some $q \geq 1$. By Lemma 4.9, it follows that $T^1_{(i)}(-R) = 0$ for all $R$, except maybe for $R = qR^* + ps^j$ for some $j$ since in this case we have $\langle a_j, R \rangle = \langle a_{j+1}, R \rangle = q$.

Let us first assume that $q \geq 2$. In this case, we have that

$$\dim_k T^1_{(i)}(-R) = N\left(\frac{3}{i}\right) - \sum_{l=1}^N \left(3 - \dim_k T^1_{(a_l, a_{l+1})}(\bar{R}_{l,l+1})\right) - \left(\frac{3}{i}\right) + s^2_{Q(k)}.$$}

We see that $T^1_{(i)}(-R) = 0$ if $T^1_{(a_l, a_{l+1})}(\bar{R}_{l,l+1}) = 0$ for all $l \in \{1, \ldots, N\}$ since $s^2_{Q(k)} \leq \left(\frac{3}{i}\right)$. If only $T^1_{(a_j, a_{j+1})}(\bar{R}_{j,j+1}) \neq 0$ we still obtain $T^1_{(i)}(-R) = 0$ since in this case $\dim_k T^1_{(a_j, a_{j+1})}(\bar{R}_{j,j+1}) = 1$ by Lemma 4.7. Thus, we have

$$\dim_k T^1_{(i)}(-R) = N\left(\frac{3}{i}\right) - \left(2\left(\frac{2}{i}\right) + (N-2)\left(\frac{3}{i}\right)\right) - \left(\frac{3}{i}\right) + \left(\frac{2}{i}\right) = \left(\frac{3}{i}\right) - \left(\frac{2}{i}\right).$$}

We see that in this case $\dim_k T^1_{(1)}(-R) = \dim_k T^1_{(3)}(-R) = 1$ and $\dim_k T^1_{(2)}(-R) = 2$. Similarly, we can treat the case $q = 1$ and thus finish the proof.
Remark 2. Note that in the case \( i = 1 \) our formulas agree with the ones given in [1, Theorem 4.4], which were obtained by different methods.

Example 3. Let \( X_\sigma \) be as in Example 2. From Theorem 4.11 we obtain that if \( R \in \{ 2R^\alpha - \alpha \gamma_3, 2R^\beta - \beta \gamma_1, 2R^\gamma - \gamma \delta_1 \mid \alpha \geq 1, \beta \geq 1, \gamma \geq 2 \} \), then \( \dim_k T_{(1)}^1(-R) = \dim_k T_{(3)}^1(-R) = 1 \) and \( \dim_k T_{(2)}^1(-R) = 2 \). For other degrees \( S \in M \) we have \( T_{(i)}^1(-S) = 0 \) for all \( i \geq 1 \).

Corollary 4.12. Let \( \text{Spec}(A) \) be a three-dimensional affine toric Gorenstein variety. The Hodge decomposition gives us

\[
\begin{align*}
\text{HH}^2(A) &\cong T_{(1)}^1(A) \oplus T_{(2)}^0(A), \\
\text{HH}^3(A) &\cong T_{(1)}^2(A) \oplus T_{(3)}^0(A) \oplus T_{(2)}^1(A).
\end{align*}
\]

Descriptions of \( T_{(2)}^0(A) \) and \( T_{(3)}^0(A) \) were given in [3]. The module \( T_{(1)}^1(A) \) was analyzed in [2, Corollary 5.4]. Theorem 4.11 gives us an explicit description of \( T_{(1)}^1(A) \) and \( T_{(2)}^1(A) \) and thus we complete understanding of the second (which describes the first order associative non-commutative deformations) and third Hochschild cohomology group (which contains the obstructions for extending associative non-commutative deformations to larger base spaces).

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