Thermodynamics in quasi-spherical Szekeres space-time

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Abstract – We have considered that the universe is the inhomogeneous \((n+2)\)-dimensional quasi-spherical Szekeres space-time model. We consider the universe as a thermodynamical system with the horizon surface as a boundary of the system. To study the generalized second law (GSL) of thermodynamics through the universe, we have assumed the trapped surface is the apparent horizon. Next we have examined the validity of the generalized second law of thermodynamics on the apparent horizon by two approaches: i) using the first law of thermodynamics on the apparent horizon and ii) without using the first law. In the first approach, the horizon entropy has been calculated by the first law. In the second approach, first we have calculated the surface gravity and temperature on the apparent horizon and then the horizon entropy has been found from the area formula. The variation of internal entropy has been found by Gibb’s law. Using these two approaches separately, we find the conditions for validity of GSL in the \((n+2)\)-dimensional quasi-spherical Szekeres model.

Introduction. – In Einstein gravity, the evidence of the connection between gravity and thermodynamics was first discovered in [1] by deriving the Einstein equation from the proportionality of entropy and horizon area together with the first law of thermodynamics. The horizon area of a black hole is associated with its entropy, the surface gravity is related with its temperature in black-hole thermodynamics [2]. Then Padmanabhan [3] was able to formulate the first law of thermodynamics on the horizon, starting from Einstein equations for a general static spherically symmetric space-time. Frolov et al. [4] calculated the energy flux of a background slow-roll scalar field through the quasi-de-Sitter apparent horizon and used the first law of thermodynamics \(-dE = TdS\), where \(dE\) is the amount of the energy flow through the apparent horizon. Using the Hawking temperature \(T_A = \frac{1}{2\pi R_A}\) and Bekenstein entropy \(S_A = \frac{\pi R_A^2}{2G}\) (\(R_A\) is the radius of apparent horizon) at the apparent horizon, the first law of thermodynamics (on the apparent horizon) is shown to be equivalent to Friedmann equations [5] and the generalized second law of thermodynamics is obeyed at the horizon. The thermodynamics in de Sitter space-time was first investigated by Gibbons and Hawking in [6]. When the apparent horizon and the event horizon of the Universe are different, it was found that the first law and generalized second law (GSL) of thermodynamics hold on the apparent horizon, while they break down if one considers the event horizon [7]. On the basis of the well-known correspondence between the Friedmann equation and the first law of thermodynamics of the apparent horizon, Gong et al. [8] argued that the apparent horizon is the physical horizon in dealing with thermodynamics problems. Considering the FRW model of the universe, most studies deal with validity of the generalized second law of thermodynamics starting from the first law when universe is bounded by the apparent horizon [9]. But there are few works on the justification of the first and second laws of thermodynamics on the event horizon [10]. The validity of thermodynamical laws in generalized gravity theories have also been discussed in [11].

Usually, for cosmological phenomena over galactic scale or in the smaller scale, it is reasonable to consider inhomogeneous solutions to Einstein equations. Szekeres [12] in 1975, gave a class of inhomogeneous solutions representing irrotational dust. The space-time represented by these solutions has no killing vectors and it has an invariant family of spherical hypersurfaces. Hence this space-time is referred to as quasi-spherical space-time. Subsequently, the solutions have been extended by Szafirn [13] and Szafirn and Wainwright [14] for a perfect fluid and they studied the asymptotic behaviour for different choice of the parameters involved. Later

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Barrow and Stein-Schabes [15] gave solutions for the dust model with a cosmological constant and showed the validity of the cosmic “no-hair” conjecture. Recently, Chakraborty et al. [16] have extended the Szekeres solution to \((n+2)\)-dimensional space-time and generalized it for matter containing heat flux [17]. Recently, several works have been done on gravitational collapse using this higher-dimensional Szekeres solution [18,19].

In this work, we consider the \((n+2)\)-dimensional quasi-spherical Szekeres space-time. Next we shall examine the validity of the generalized second law of thermodynamics (GSL) on the apparent horizon by two approaches: i) using the first law of thermodynamics on the apparent horizon and ii) without using the first law. In the first approach, we do not need the horizon temperature. So the horizon entropy can be calculated from the first law. In the second approach, first we calculate surface gravity and then the horizon entropy can be found from the area formula. Using these two approaches, we find the conditions for validity of the GSL in the quasi-spherical Szekeres model.

**The Szekeres’ model.** – The metric ansatz for the \((n+2)\)-dimensional Szekeres’ space-time [12,16] is of the form

\[
\text{d}s^2 = -\text{d}t^2 + e^{2\alpha}dr^2 + e^{2\beta} \sum_{i=1}^{n} \text{d}x_i^2,
\]

where the metric coefficients \(\alpha\) and \(\beta\) are functions of all space-time co-ordinates i.e.,

\[\alpha = \alpha(t, r, x_1, \ldots, x_n), \quad \beta = \beta(t, r, x_1, \ldots, x_n).\]

Now considering both radial and transverse stresses the energy momentum tensor has the structure

\[T_{\mu\nu} = \text{diag}(-\rho, p_r, p_r, \ldots, p_r).\]

Now for the choice namely \(\beta' \neq 0\), \(\beta'' = 0\) we have from the field equations the explicit form of the metric coefficients are as follows [16]:

\[e^\beta = R(t, r) \quad e^{\nu(r,x_1,\ldots,x_n)}\]

and

\[e^\alpha = \frac{R' + R\nu'}{\sqrt{1 + f(r)}}\]

and the evolution equation for \(R\) gives

\[R\dot{R} + \frac{1}{2}(n-1)\dot{R}^2 = \frac{p_r}{n} \frac{R^2}{\dot{R}} = \frac{n}{2} f(r),\]

where \(f(r)\) is the function of \(r\). Also the function \(\nu\) satisfies

\[e^{-2\nu} \sum_{i=1}^{n} [(n-2)\nu_{x_i}^2 + 2\nu_{x_i,x_i}] = -n,\]

which has a solution of the form

\[e^{-\nu} = A(r) \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} B_i(r) x_i + C(r)\]

with the restriction,

\[\sum_{i=1}^{n} B_i^2 - 4AC = -1\]

for the arbitrary functions \(A(r), B_i(r), (i = 1, 2, \ldots, n)\) and \(C(r)\).

Now from conservation equation \(T_{\mu\nu} = 0\) we get [19]

\[
\dot{\rho} + \dot{\alpha}(\rho + p_r) + n\beta'(p_r - p_r) = 0,
\]

\[
p' + n\beta'(p_r - p_r) = 0,
\]

\[
\alpha x_i (p_r - p_r) = \frac{2}{\nu'} \rho x_i (i = 1, 2, \ldots, n).\]

In the general case when both radial and tangential pressures are non-zero and distinct then from the Einstein equations \(G_{\mu\nu} = T_{\mu\nu}\) (choosing \(8\pi G = c = 1\)) they can be obtained in compact form as [19]

\[\rho = \frac{F'}{\zeta^2}, \quad p_r = -\frac{F}{\zeta^2}, \quad p_r = p_r + \frac{\zeta p_r'}{n\zeta^2}\]

where \(\zeta = e^\beta\), \(F(t, r) = \frac{n}{2} R^{n-1} e^{(n+1)\nu}(\dot{R}^2 - f(r))\).

Now consider that the radial and tangential pressures are equal i.e., \(p_r = p_r = p\), so from (8), we get that the isotropic pressure is a function of \(t\) only i.e., \(p = p(t)\). As there is no restriction on the energy density then \(\rho\) is in general a function of all the \((n+2)\) variables i.e., \(\rho = \rho(t, r, x_1, \ldots, x_n)\) and hence no equation of state is imposed. So the conservation equations (8) yields

\[\dot{\rho} + (\dot{\alpha} + n\beta') (\rho + p) = 0.\]

Also the metric (1) can be written as

\[\text{d}s^2 = -\text{d}t^2 + \frac{(R^2 + R\nu')^2}{1 + f(r)} \text{d}r^2 + R^2 e^{2\nu} \sum_{i=1}^{n} \text{d}x_i^2.\]

Here \(R\) is the radius of the non-concentric spheres. If the spheres are concentric i.e., if \(\nu' = 0\), then the above Szekeres metric reduces to a \((n+2)\)-dimensional spherically symmetric Lamaître-Tolman-Bondi (LTB) metric [16].

**The GSL of thermodynamics on the apparent horizon.** – Now we consider the metric (8) in the following form:

\[\text{d}s^2 = h_{ab} \text{d}x^a \text{d}x^b + R^2 e^{2\nu} \sum_{i=1}^{n} \text{d}x_i^2, \quad a, b = 0, 1, \]

where \(h_{ab} = \text{diag}(-1, (R^2 + R\nu')^2)\).

The formation of the event horizon depends greatly on the computation of null geodesics which is almost impracticable for the present space-time geometry.
So a closely related concept of a trapped surface (a space-like 2-surface whose normals on both sides are future-pointing converging null-geodesic families) will be considered. The dynamical apparent horizon $R_A$, a marginally trapped surface with vanishing expansion, is determined by the relation [5,19,20] (see also the appendix)

$$\kappa^{ab}\partial_b(Re')\partial_a(Re') = 0. \quad (14)$$

This implies

$$\dot{R}_A^2 = 1 + f(r). \quad (15)$$

So from eq. (10), we obtain (on the apparent horizon)

$$F(t, r) = \frac{n}{2} R_A^{-n} e^{(n+1)\nu} \quad (16)$$

Now Gibb’s law of thermodynamics states that [7]

$$T_A dS_I = p dV + d(E_I), \quad (17)$$

where, $S_I, p, V$ and $E_I$ are respectively entropy, pressure, volume and internal energy within the apparent horizon. Here the expression for internal energy can be written as $E_I = \rho V$. Here $T_A$ is the temperature on the apparent horizon. Now the volume of the $(n+1)$-dimensional space is [5]

$$V = \Omega_{n+1} R_A^{n+1} e^{(n+1)\nu}, \quad \text{where} \quad \Omega_{n+1} = \frac{\pi^{n+2}}{\Gamma\left(\frac{n+2}{2}\right)} \quad (18)$$

The time variation of internal entropy is obtained as (using (2), (3), (11), (17) and (18))

$$\dot{S}_I = \frac{\Omega_{n+1} R_A^{n+1} e^{(n+1)\nu}}{T_A} (\rho + p) \left( \frac{\dot{R}_A}{R_A} - \frac{\dot{R}_A'}{R_A'} + \frac{R_A''}{R_A} + \frac{R_A'\nu'}{R_A' + R_A}\right). \quad (19)$$

**Validity conditions of the GSL using the first law of thermodynamics.** The unified first law is defined by [21]

$$dE = A\Psi + W dV, \quad (20)$$

where

$$A = (n+1)\Omega_{n+1} R^n e^{n\nu}$$

is the area [5] and the volume $V$ is defined in (18). The work density function is given by

$$W = -\frac{1}{2} \kappa^{ab} T_{ab} = \frac{1}{2} (\rho - p). \quad (22)$$

The energy supply vector is given by

$$\Psi_a = h_{ac} \kappa_c^b \partial_b(Re') + W \partial_a(Re') = \left(-\frac{1}{2} (\rho + p) \dot{R}e' + \frac{1}{2} (\rho + p) (R' + \nu') e' \right) \quad (23)$$

So

$$\Psi = \Psi_a dx^a = -\frac{1}{2} (\rho + p) e' [\dot{R} dt - (R' + \nu') dr]. \quad (24)$$

The total energy inside the quasi-spherical surface is given by

$$E = \frac{n(n+1)}{2} \Omega_{n+1} R^{n-1} e^{(n-1)\nu} \times \left[ e^{2\nu} - \kappa^{ab} \partial_a(Re') \partial_b(Re') \right] = \frac{n(n+1)}{2} \Omega_{n+1} R^{n-1} e^{(n+1)\nu} [\dot{R}^2 - f(r)]. \quad (25)$$

Comparing (10) and (25), we get

$$E = (n+1) \Omega_{n+1} F. \quad (26)$$

From this, we can say that $F(t, r)$ represents the mass function within the quasi-spherical surface. Now using (18), (21), (22) and (24), we get

$$A\Psi + W dV = (n+1) \Omega_{n+1} R^n e^{(n+1)\nu} \times [-p\dot{R} dt + \rho (R' + \nu') dr]. \quad (27)$$

Using (25) and (27), comparing the coefficients of $dt$ and $dr$ in (20), we can recover the field equations (4) and (9). Now from the unified first law (20) and using (27), we get

$$dE = (n+1) \Omega_{n+1} R^n e^{(n+1)\nu} [-\rho p + \rho \dot{R} dt + \rho e^{-\nu} d(R\nu')]. \quad (28)$$

We know that heat is one of the forms of energy. Therefore, the heat flow $dQ$ through the apparent horizon is just the amount of energy crossing it during the time interval $dt$. That is, $dQ = -dE$ is the change of the energy inside the apparent horizon. So the amount of the energy crossing on the apparent horizon is given by [22]

$$-dE_A = (n+1) \Omega_{n+1} R^n \dot{R}_A \dot{R}_A e^{(n+1)\nu} (\rho + p) dt = A \dot{R}_A e^{\nu} T_{\nu\rho} k^\rho k^\nu dt. \quad (29)$$

The first law of thermodynamics (Clausius relation) on the apparent horizon is defined as follows:

$$T_A dS_A = dQ = -dE_A. \quad (30)$$

So using (29) and (30), we obtain the time variation of the entropy on the apparent horizon as

$$\dot{S}_A = \frac{(n+1) \Omega_{n+1} R^n \dot{R}_A \dot{R}_A e^{(n+1)\nu}}{T_A} (\rho + p). \quad (31)$$

Combining (19) and (31), we obtain

$$\dot{S}_I + \dot{S}_A = \frac{\Omega_{n+1} R^{n+1} e^{(n+1)\nu}}{T_A} (\rho + p) \times \left( \frac{n+2}{\rho + p} \frac{\dot{R}_A}{R_A} - \frac{\dot{R}_A'}{R_A'} + \frac{R_A''}{R_A} + \frac{R_A'\nu'}{R_A' + R_A}\right). \quad (32)$$

Using (2), (3), (9), (15), (16) and (32), after manipulation we get the rate of change of total entropy as

$$\dot{S}_I + \dot{S}_A = \frac{\Omega_{n+1} F}{T_A} \left[ \left( \frac{n}{2} + \frac{1}{\nu} \frac{(n-1)F' \sqrt{1 + \frac{f}{F'} - 2F'} F^{-\frac{n}{n+1}} e^{\frac{n+1}{n+1}\nu}}{2(1 + f) F'} \right) \times \left( \frac{n+2}{F'} \frac{(n+1)(f' + 2(1 + f) \nu')}{2(1 + f) F'} \right) \right]. \quad (33)$$
If the expression inside the square bracket is non-negative then the GSL will be justified. For a marginally bound case, i.e., for a \( f(r) = 0 \), the GSL is satisfied if the following conditions hold:

- **i)** \( F' \geq \frac{3(n+1)}{n+2} F \nu' \) and
  \[ \dot{F} \leq 3(n+1) \left( \frac{n}{2} \right)^{\frac{1}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu} \]
  or

- **ii)** \( 2F \nu' < F' < \frac{3(n+1)}{n+2} F \nu' \) and
  \[ \dot{F} > 3(n+1) \left( \frac{n}{2} \right)^{\frac{1}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu}. \]

For the TBL model (\( \nu' = 0 \)), the GSL is valid if \( \dot{F} \leq (n-1) \left( \frac{2}{n} \right)^{\frac{n}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu} \) which also satisfies the above validity condition i) (since \( \nu' > 0 \)).

**Validity conditions of the GSL without using the first law of thermodynamics.** The surface gravity is defined as

\[ \kappa = \frac{1}{2\sqrt{-h}} \partial_a \left( \sqrt{-h} h^{ab} \partial_b (Re^\nu) \right), \]

where \( h = \det(h_{ab}) \). So on the apparent horizon, we get

\[ \kappa = -\frac{\dot{R}_A e^\nu}{2} - \frac{\dot{R}_A (\dot{R}_A' + \dot{R}_A \nu') e^\nu}{2(R^2 + R \nu')} \]

\[ + \frac{\sqrt{1+f}}{2(R^2 + R \nu')} \frac{\partial}{\partial \nu} (e^\nu \sqrt{1+f}). \]

Now apparent horizon temperature is (using (3)–(5), (9), (10), (15) and (16))

\[ T_A = \frac{\kappa}{2 \pi} \frac{e^\nu}{8 \pi} \left[ (n-1) \left( \frac{n}{2F} \right)^{\frac{1}{n+1}} e^{\frac{n+1}{n+4} \nu} - \frac{\dot{F}}{F \sqrt{1+f}} \right]. \]

Since one can relate the entropy with the surface area of the apparent horizon through \( S_A = \frac{A}{4G} \). Therefore using (21) we have

\[ S_A = 2\pi(n+1) \Omega_{n+1} R_A^2 e^{\nu}. \]

The variation of entropy on the apparent horizon is obtained as (using (15) and (16))

\[ \dot{S}_A = 4\pi(n+1) \Omega_{n+1} F \sqrt{1+f} \dot{f} e^{-\nu}. \]

Using (9), (10), (15), (16), (19), (36) and (38), finally we obtain (after manipulation) the rate of change of total entropy as

\[ \dot{S}_I + \dot{S}_A = \frac{\Omega_{n+1} F}{T_A} \left[ \left( \frac{n}{2} \right)^{\frac{1}{n+1}} (n-1) \left( \frac{n}{2F} \right)^{\frac{1}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu} - \dot{F} \right] \]

\[ \times \left( \frac{1}{F} \right) - \frac{(n-1)(F' + 2(1+f)\nu')}{2(1+f)(F' - 2F \nu')} + \frac{(n+1) \sqrt{1+f}}{2} \]

\[ \times \left( n-1 \right) \left( \frac{n}{2F} \right)^{\frac{1}{n+1}} e^{\frac{n+1}{n+4} \nu} - \frac{\dot{F}}{F \sqrt{1+f}} \right]. \]

If the expression inside the square brackets is non-negative, then the GSL will be justified. Now, here we impose the validity conditions of the GSL using the first law. For the marginally bound case \( (f(r) = 0) \),

- **i)** \( F' > \frac{3(n+1)}{n+2} F \nu' \) and
  \[ \dot{F} \leq 3(n+1) \left( \frac{n}{2} \right)^{\frac{1}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu} \]
  then we see that GSL using first law may be satisfied but GSL without using the first law may not be satisfied. But if **ii)** \( 2F \nu' < F' < \frac{3(n+1)}{n+2} F \nu' \) and
  \[ \dot{F} > 3(n+1) \left( \frac{n}{2} \right)^{\frac{1}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu}, \]
  then the GSL using the first law and the GSL without using the first law may both be satisfied. So we may conclude that the validity of the GSL using the first law is much stronger than the GSL without using the first law in Szekeres’ model. For the TBL model \( (\nu' = 0) \), if \( \dot{F} \leq (n-1) \left( \frac{2}{n} \right)^{\frac{n}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu} \) we see that the GSL using the first law and the GSL without using the first law are both satisfied but if \( \dot{F} > (n-1) \left( \frac{2}{n} \right)^{\frac{n}{n+1}} F^{\frac{n+2}{n+1}} e^{\frac{n+1}{n+4} \nu} \) we see that the GSL using the first law cannot be satisfied and the GSL without using the first law may be satisfied. So there is no restriction for the validity of the GSL without using the first law in the TBL model with a marginally bound case.

**Discussions.** – We have considered that the universe is the inhomogeneous \((n+2)\)-dimensional quasi-spherical Szekeres space-time model. We consider the universe as a thermodynamical system with the horizon surface as a boundary of the system. To study the generalized second law (GSL) of thermodynamics through the universe, we have assumed the trapped surface is the apparent horizon. Next we have examined the validity of the generalized second law of thermodynamics (GSL) on the apparent horizon by two approaches: i) using the first law of thermodynamics on the apparent horizon and ii) without using the first law. In the first approach, the horizon entropy has been calculated by the first law. In the second approach, first we have calculated the surface gravity and temperature on the apparent horizon and then the horizon entropy has been found from the area formula. The variation of internal entropy has been found by Gibb’s law. Using these two approaches separately, we find the conditions for validity of the GSL in the \((n+2)\)-dimensional quasi-spherical Szekeres model. Also for a marginally bound case, we have found the bounds on the derivatives of the mass function \( F \). Now in the FRW universe, if we calculate the surface gravity \( \kappa \) on the apparent horizon, then the temperature on the apparent horizon will be \( T_A = \frac{\kappa}{2 \pi} = \frac{1}{2 \pi R_A} (1 - \frac{R}{2HR_A}) \), where \( R_A \) is the radius of the apparent horizon of the FRW universe and \( H \) is the Hubble parameter. If we use this temperature expression then the rate of change of entropy on the apparent horizon using two methods (using the first law and without using the first law) cannot be equal. But if we consider the approximation \( \frac{R}{2HR_A} < 1 \), the temperature on the apparent horizon will be \( T_A = \frac{1}{2 \pi R_A} \) and in this case, the rate of change of entropy on the
apparent horizon using the two methods (using the first law and without using the first law) must be equal. In our analysis for the Szekeres’ model, we have not considered any approximation for the calculation of temperature. Hence the rate of change of the total entropy on the apparent horizon using the two methods (using the first law and without using the first law) i.e., eqs. (33) and (39) cannot be equal. Therefore we have found different conclusions for the validity of the GSL in the two methods. We have found that the validity of the GSL using the first law is much stronger than the GSL without using the first law in Szekeres’ model. Also in a marginally bound case, the GSL without using the first law in the TBL model is always satisfied but the validity of the GSL using the first law in the TBL model depends on the mass function $F$.

**Appendix**

Define, $X = \frac{R^2 + Rr'}{\sqrt{1 + f(r)}}$ and $Y = R e^\epsilon$. Let us consider the 2-surface $S_{r,t}$ ($r = \text{const}$, $t = \text{const}$) is a trapped surface and $K^\mu$ denotes the tangent vector field to the null geodesics which is normal to $S_{r,t}$. So on the apparent horizon (on $S_{r,t}$) we have [20]

$$K_\mu K^\mu = 0, \quad K^{\alpha \nu}_{\gamma \nu} K^{\nu} = 0$$

and

$$K^2 = K^3 = 0, \quad (K^0)^2 - X^2 (K^1)^2 = 0.$$  \hspace{1cm} (40)

Now on $S_{r,t}$, the choice of affine parameter may clearly be such that

$$K^0 = X, \quad K^1 = \epsilon = \pm 1.$$  \hspace{1cm} (41)

Now on $S_{r,t}$,

$$K^\mu_{\mu} = K^\mu_{\mu} + \Gamma^\mu_{\mu \nu} K^\nu =$$

$$K^0_{,0} + K^1_{,1} + X \left( \frac{X'}{X} + \frac{Y'}{Y} \right) + \epsilon \left( \frac{X'}{X} + \frac{Y'}{Y} \right).$$  \hspace{1cm} (42)

Since $K^2_2 = K^3_3 = 0$ on $S_{r,t}$. On the other hand, forming $\partial / \partial t$ of the first equation of (32) and setting $\mu = 1$ in the second equation gives on $S_{r,t}$

$$K^0_{,0} - \epsilon X K^1_{,0} - \dot{X} = 0$$

and

$$K^0_{,1} X + \epsilon (K^1_{,1} + 2 \dot{X}) + \frac{X'}{X} = 0.$$  \hspace{1cm} (43)

Eliminating $K^0_{,1}$ between these two equations and substituting in (35) gives

$$K^\mu_{\mu} = \frac{2}{Y} (XY' + eY') = \frac{2(R' + Rr')}{R} \left( \frac{R}{\sqrt{1 + f(r)}} + \epsilon \right).$$

On the apparent horizon, $K^\mu_{\mu} = 0$ gives

$$\frac{\dot{R}}{\sqrt{1 + f(r)}} + \epsilon = 0 \quad \text{(since, $R' + Rr' \neq 0$)} \Rightarrow$$

$$\dot{R}^2 = \epsilon^2 (1 + f(r)) = 1 + f(r)$$

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