RESEARCH ARTICLE

A Reliability Test of a Complex System Based on Empirical Likelihood

Yan Zhou1*, Liya Fu2*, Jun Zhang3, Yongchang Hui2*

1 College of Mathematics and Statistics, Institute of Statistical Sciences, Shenzhen University, Shenzhen, China, 2 School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an, China, 3 College of Mathematics and Computational Science, Institute of Statistical Sciences, Shen Zhen-Hong Kong Joint Research Center for Applied Statistical Sciences, Shenzhen University, Shenzhen, China

☯ These authors contributed equally to this work.
* huiyc180@xjtu.edu.cn

Abstract

To analyze the reliability of a complex system described by minimal paths, an empirical likelihood method is proposed to solve the reliability test problem when the subsystem distributions are unknown. Furthermore, we provide a reliability test statistic of the complex system and extract the limit distribution of the test statistic. Therefore, we can obtain the confidence interval for reliability and make statistical inferences. The simulation studies also demonstrate the theorem results.

Introduction

In practice, little information can be directly derived from a complex system (CS). What we usually obtain is information about subsystems. Therefore, system reliability, which is based on subsystem data, is a very important research topic and has been a concern for a long time. However, due to the complexity of the system, life distribution, subsystem reliability, and diversity of the data distribution, there are many difficulties studying of a CS.

In recent years, many researchers have provided various methods for calculating and estimating system reliability under the assumption that the distribution of the subsystems (family) is known. For example, for a series system with two subsystems and with binomially distributed pass-fail failure data, Buehler [1] proposed a model and derived the exact lower confidence limit of the system reliability. Rosenblatt [2] proposed an approximate method for system reliability confidence limits based on the asymptotic normality of a U-statistic. Weaver [3] derived a simple and accurate ordering method to calculate the system reliability confidence limits, but it requires the same sample size for all subsystems. Rice and Moore [4] presented a Monte Carlo model for binomial distributed data that is valid for a zero failure case. Coit [5] provided a method which does not require any parametric assumptions for component reliability or time to failure. Tian [6] summarized and compared the advantages and limitations of the parametric methods.
However, only a few studies have investigated the reliability test of a CS. The main reasons are that the life distributions of the subsystems may vary, and it is difficult to find out which one it is. It is not easy to find a test statistic and its asymptotic distribution because of the complexity of the system structures.

In recent studies, regarding the series system in which the subsystems’ life distribution is an index distribution, Yu et al. [7] considered the system failure rate of testing and gave an accurate unbiased test. Based on the method proposed by Yu et al. [7], Li [8] further considered subsystem lives following different distributions for the series system and completed the corresponding test by approximately transforming the non-index distribution to index distribution.

In a normal situation, it is difficult to determine the life distribution of a subsystem, and the structure of the system may be very complicated. In this paper, we assume the following: (i) A complex system is described by minimal paths, and these minimal paths are known; (ii) The subsystems of the complex system are independent, and the distribution of life is unknown. Under these two assumptions, we provide a reliability test statistic for a complex system using the empirical likelihood method [9, 10]. Furthermore, we also extract the limit distribution of the test statistic. Therefore, we can obtain the confidence interval and make statistical inferences for the system reliability based on the limit distribution.

This paper is organized as follows. In Section 2, we describe the reliability test problem of the complex system. In Section 3, we provide a test statistic and derive its asymptotic distribution. In Section 4, we carry out simulation studies for a bridge system. Finally, we draw several conclusions. The proof of the asymptotic distribution is given in the Appendix.

Materials and Methods

Notations

The life of a complex system is $Z = \max_{1 \leq j \leq k} \min_{r \in \Omega_j} Z_{jr}$, where $k$ is the number of minimal paths, $\Omega_j$ is the $j$th minimal path, and $Z_{jr}$ is the life of the $r$th subsystem. The specific expression of complex system reliability has been derived in [11, 12]. For convenience, we first define some operations and give some notions.

**Definition 2.1.** Let $\alpha = (a_1, a_2, \ldots, a_m)^T$ and $\beta = (b_1, b_2, \ldots, b_m)^T$ be two column vectors with $m$ elements.

1. Define $\alpha \prec \beta$ if $a_i \leq b_i$, $i = 1, 2, \ldots, m$;
2. Define $\alpha \setminus \beta = (\delta_1, \delta_2, \ldots, \delta_m)^T$, where $\delta_i = \begin{cases} 1 & a_i = 1, b_i = 0, \\ 0 & \text{others} \end{cases}$, $i = 1, 2, \ldots, m$;
3. Define $\alpha \oplus \beta = (\gamma_1, \gamma_2, \ldots, \gamma_m)^T$, where $\gamma_1 = a_1 + b_1$ and $\gamma_i = \max\{a_i, b_i\}$ for $i = 2, \ldots, m$.

**Definition 2.2.** Let $A = (a_1, a_2, \ldots, a_k)$ be an $m \times k$ matrix, and $A_j = (a_1 \setminus \alpha_j, a_2 \setminus \alpha_j, \ldots, a_{j-1} \setminus \alpha_j, a_j)$ for a given $j$, where $j = 2, \ldots, k$. If $a_{t_j} \setminus \alpha_j < a_j \setminus \alpha_j$ for all $t_j < t \leq j$, then we cut out the column $\alpha_j \setminus \alpha_j$ of $A$, which is an $m \times n_j$ matrix, where $n_j \leq j - 1$ and $j = 2, 3, \ldots, k$. Define $V(A_j) = \max_{1 \leq s \leq k} \sum_{t=1}^{n_j} A_{js}(s, t)$, where $A_{js}(s, t)$ is the $s$th row and the $t$th column element of $A_j$.

**Definition 2.3.** Let $C = (c_1, c_2, \ldots, c_k)$ be an $m \times k$ matrix. For a fixed index $j, j = 1, 2, \ldots, k$, define $f_{i_1, i_2, \ldots, i_j} = c_{i_1} \oplus c_{i_2} \oplus \ldots \oplus c_{i_j}$, where $1 \leq i_1 < i_2 < \ldots < i_j \leq k$. Let $C_j = (f_{i_1, i_2, \ldots, i_j})$ be an $m \times C_j$ matrix whose column vectors are made of all vectors $f_{i_1, i_2, \ldots, i_j}$, where $C_j$ is the combination number of $j$ in $k$. 

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Definition 2.4: Let $\alpha = (a_1, a_2, \ldots, a_m)^T$ and $R = (r_1, r_2, \ldots, r_m)^T$ be vectors with $m$ elements, and $D = (d_1, d_2, \ldots, d_k)$ be an $m \times k$ matrix. Define

$$R^v = \prod_{j=1}^{m} r_j^v \quad \text{and} \quad R^D = \sum_{j=1}^{k} R^j.$$

Based on these definitions, the reliability function for a system can be calculated with the following theorem [11].

Theorem 2.5: Let $F_i(t)$ be a life distribution function for an independent subsystem $S_i$, and $R_i(t) = 1 - F_i(t)$ is the reliability of the $i$th subsystem, $r = 1, 2, \ldots, m$. The minimal path matrix of a complex system is $A_{n \times k}$ and $F_{1A}(t)$ is the life function for the complex system. $\Psi_{1A}(t) = 1 - F_{1A}(t)$ is the reliability function. Define

$$C = \left( \begin{array}{c} 1^T \\ A \end{array} \right) = (c_1, c_2, \ldots, c_k),$$

where $1$ is a vector of 1s. Thus,

$$\Psi_{1A}(t) = -\sum_{j=1}^{k} \tilde{R}(t)^{c_j},$$

where $\tilde{R}(t) = \{ -1, R_1(t), R_2(t), \ldots, R_m(t) \}^T$.

In view of the characteristics of the minimal path matrix, the calculation procedure can be cut short; Thus, we can obtain the following corollary.

Corollary: According to Zhang et al. [11], suppose that there are $r_i$ zeroes in the $i$th row of $A$ in Theorem 2.5, where $i = 1, \ldots, m$. Let $l = \max_{1 \leq i \leq m} r_i$. Then

$$\Psi_{1A}(t) = -\sum_{j=1}^{l} \tilde{R}^j(t) - \sum_{j=l+1}^{k} (-1)^j C_j \prod_{i=1}^{m} R_i(t).$$

A complex system reliability test based on the empirical likelihood

To infer the reliability lower confidence limit or the confidence limit of a CS using subsystem data, we can construct the following hypothesis tests: For a given $t$, test whether $\Psi_{1A}(t)$ is not less than $\Psi_0$ or not; that is,

1. $H_0 : \Psi_{1A}(t) = \Psi_0$ vs $H_1 : \Psi_{1A}(t) \neq \Psi_0$;

2. $H_0 : \Psi_{1A}(t) \geq \Psi_0$ vs $H_1 : \Psi_{1A}(t) < \Psi_0$.

We use empirical likelihood (EL), which is a nonparametric method introduced by Owen [9, 10, 13], to test the two hypotheses. Here, we first give a short introduction to empirical likelihood. The definitions of the empirical distribution function and the empirical accumulate function are given as follows.

Definition 3.1: Let $X_1, X_2, \ldots, X_n \in \mathbb{R}$ be independently identically distributed, then the empirical cumulative distribution function of $X_1, X_2, \ldots, X_n$ is

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(x_i \leq x),$$

for $-\infty < x < \infty$. 
**Definition 3.2.** Let $X_1, X_2, \ldots, X_n \in \mathbb{R}$ be independent and with a common cumulative distribution $F$, the nonparametric likelihood of the $F$ is

$$L(F) = \Pi_{i=1}^{n}(F(X_i) - F(X_i^{-})).$$

Define

$$R(F) = \frac{L(F)}{L(F_n)}.$$ 

Then the empirical likelihood ratio statistic is defined to be

$$R(\mu_0) = \max \left\{ \Pi_{i=1}^{n}nw_i | \sum_{i=1}^{n} w_i (X_i - \mu_0) = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}.$$ 

The resulting empirical likelihood confidence region for the mean $\mu_0$ is

$$\{\mu | R(\mu) \geq r_0 \} = \arg \max \left\{ \sum_{i=1}^{n} w_i X_i | \Pi_{i=1}^{n}nw_i \geq r_0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}.$$ 

For a given hypothesis test, we can obtain a confidence region with the empirical likelihood method without assuming the specification of the data distribution family. In the following two subsections, we will derive the test statistics for two hypothesis tests for the reliability of a complex system using the empirical likelihood method.

**Two-sided hypothesis test for the reliability of a CS**

In this subsection, we consider a two-sided hypothesis test for the reliability of a complex system. For a given $t$,

$$H_0 : \Psi_{1,t}(t) = \Psi_0 \quad \text{vs} \quad H_1 : \Psi_{1,t}(t) \neq \Psi_0.$$ 

According to Theorem 2.5, we have $\Psi_{1,t}(t) = -\sum_{j=1}^{k} \sum_{l=1}^{c_j} (-1)^{l} \Pi_{i=1}^{m} [R_i(t)]^{l_j}$, where $\hat{c}_{ij}$ equals 0 or 1, and $\Psi_{1,t}(t)$ is a smoothing function of $R_1(t), \ldots, R_m(t)$. We know that $R_1(t), \ldots, R_m(t)$ are independent of each other for the independent subsystems, and use $I(Z_{ir} > t)$ to replace $R_i(t)$, where $Z_{ir}$ is the observed lifetime of the $r$th subsystem of the $i$th sample. For any $t > 0$, we have

$$E\Psi_{1,t}(Z_{i}, t) = -E \left\{ \sum_{j=1}^{k} \sum_{l=1}^{c_j} (-1)^{l} \Pi_{i=1}^{m} [\hat{R}_i(t)]^{l_j} \right\}$$

$$= -\sum_{j=1}^{k} \sum_{l=1}^{c_j} (-1)^{l} \Pi_{i=1}^{m} E\{[\hat{R}_i(t)]^{l_j}\}$$

$$= -\sum_{j=1}^{k} \sum_{l=1}^{c_j} (-1)^{l} \Pi_{i=1}^{m} E\{I(Z_{ir} > t)]^{l_j}\}$$

$$= -\sum_{j=1}^{k} \sum_{l=1}^{c_j} (-1)^{l} \Pi_{i=1}^{m} [R_i(t)]^{l_j}$$

$$= \Psi_{1,t}(t).$$
Under the null hypothesis, we obtain

$$\sum_{i=1}^{n} w_i [\Psi_{1,a}(Z_i, t) - \Psi_0] = 0.$$ 

Therefore, a statistic for the two-sided test based on the EL is given as follows:

$$\mathcal{R}(\Psi_0) = \max \left\{ \prod_{i=1}^{n} w_i [\Psi_{1,a}(Z_i, t) - \Psi_0] = 0, \ w_i \geq 0, \ \sum_{i=1}^{n} w_i = 1 \right\}.$$ 

Let

$$G = \sum_{i=1}^{n} \log n w_i - n \sum_{i=1}^{n} w_i [\Psi_{1,a}(Z_i, t) - \Psi_0] + \mu \left( \sum_{i=1}^{n} w_i - 1 \right),$$

where $\lambda$ and $\mu$ are Lagrange multipliers. The estimating function based on the derivative of $G$ with respect to $w_i$ is

$$\frac{1}{w_i} - n \lambda^T (\Psi_{1,a}(Z_i, t) - \Psi_0) + \mu = 0.$$ 

Because $\sum_{i=1}^{n} w_i \frac{\partial G}{\partial w_i} = 0$, then $\mu = -n$, we have

$$w_i = \frac{1}{n(1 + \lambda (\Psi_{1,a}(Z_i, t) - \Psi_0))}. $$

To solve $\lambda$ from the following equation

$$\sum_{i=1}^{n} \frac{\Psi_{1,a}(Z_i, t) - \Psi_0}{n(1 + \lambda (\Psi_{1,a}(Z_i, t) - \Psi_0))} = 0,$$

let $n_i = \sum_{i=1}^{n} \Psi_{1,a}(Z_i, t)$, then

$$\frac{n_i (1 - \Psi_0)}{1 + \lambda (1 - \Psi_0)} - \frac{(n - n_i) \Psi_0}{1 - \lambda \Psi_0} = 0.$$

From the above equation, we can obtain

$$\lambda = \frac{1}{n} \frac{n_i - n \Psi_0}{\Psi_0 (1 - \Psi_0)}.$$ 

Therefore, we can derive

$$-2 \log \mathcal{R}(\Psi_0) = n_i \log \frac{n_i}{n \Psi_0} + (n - n_i) \log \frac{n_i}{n(1 - \Psi_0)}.$$ 

Then, we can obtain the following theorem from the EL method.

**Theorem 3.1**: Random variables $Z_1, Z_2, \ldots, Z_n$ are independent random vectors of the $m$-dimension with a common distribution $F_0, \ \Psi_{1,a}(Z_i, t) \in \mathbb{R}^1$, and $\Psi_0 \in \mathbb{R}^1$. Let $m(Z, \Psi_0) = \Psi_{1,a}(Z, t) - \Psi_0 \in \mathbb{R}^1$. Under the null hypothesis, $E(m(Z, \Psi_0)) = 0$ and $\text{var}(m(Z, \Psi_0)) < 1$. Then $-2 \log \mathcal{R}(\Psi_0)$ converges to $\chi^2_1$ in distribution as $n \to \infty.$
The proof of Theorem 3.1 is similar to Owen [10]. Let $\chi^2_{(1),.95}$ be the 95% quantile of the chi-square distribution with one degree of freedom. According to Theorem 3.1, we have
\[
\lim_{n \to \infty} P\{-2 \log R(\Psi_0) > \chi^2_{(1),.95}\} = 0.05.
\]

One-sided hypothesis test for the reliability of a CS

For the reliability of a complex system, we are more interested in whether the reliability of the system is no less than a given value, that is, for a given $t$,
\[
H_0 : \Psi_{1,t}(t) \geq \Psi_0 \quad \text{vs} \quad H_1 : \Psi_{1,t}(t) < \Psi_0.
\]
Under the same condition as Theorem 3.1, we can obtain the following theorem.

**Theorem 3.2:** Let $Z_1, Z_2, \ldots, Z_n$ be independent random vectors with a common and unknown distribution $F_0$ and $Z_i \in \mathbb{R}^m$ for $i = 1, \ldots, n$. Let $I(Z_i < t) = [I(Z_{i1} < t), \ldots, I(Z_{im} < t)]^T$ and $EI(Z_i < t) = 1 - R(t)$, where $R(t) = [R_1(t), \ldots, R_m(t)]$. Let $\Psi_{1,t}(Z, t)$ be a function that maps $\mathbb{R}^m$ to $\mathbb{R}$. Let $E(\Psi_{1,t}(Z, t)) = \Psi_1$ and $\text{var}(\Psi_{1,t}(Z, t)) < \infty$. Then under the null hypothesis, for any $t > 0$,
\[
\lim_{n \to \infty} P\{-2 \log R(\Psi_0) \leq c\} = \begin{cases} 
\frac{1}{2} + \frac{1}{2}P\{\chi_i^2 \leq c\} & \Psi_{1,t}(t) = \Psi_0 \\
1 & \Psi_{1,t}(t) > \Psi_0,
\end{cases}
\]
where $c$ is a constant.

The proof of Theorem 3.2 is given in the Appendix. According to Theorem 3.2, if we take 0.05 as a remarkable level, first, we should make a judgment of the $\Psi_{1,t}(X, t)$ average value according to the certificate process: If $\sum_{t=1}^n \Psi_{1,t}(Z, t)/n > \Psi_0$, then we should accept $H_0$; otherwise, we should take $c = \chi^2_{(1),.95}$ as a threshold value and then check whether $-2\log R(\Psi_0)$ is larger than $c$. If $-2\log R(\Psi_0) < c$, then accept the null hypothesis; otherwise, we will reject the null hypothesis.

Simulation studies

In this section, we carry out simulation studies to assess the performance of the proposed method. We take $\alpha = 0.05$ and $\alpha = 0.1$ as remarkable levels, compare the power under the alternative hypothesis under controlling type I error, and check whether the result is close to the theory result as the sample size increases. We consider ten different sample sizes, specifically, $n = 10, 15, 20, 30, 50, 60, 80, 100, 150$, and $300$. For each case, 10000 replications are carried out. We generate samples of each subsystem from $\chi^2_{(1),.95}$ and take $t = 15$ as an example. This test is not related to the system's degree of complexity, and we select a simple bridge-type complex system for the simulation studies. The expression of the bridge-type complex system used by Zhang et al. [11] is given in Fig 1.

There are four minimal paths in the above system: $\{s_1, s_2\}, \{s_3, s_4\}, \{s_1, s_4, s_5\}, \{s_2, s_3, s_5\}$. Then, we have
\[
A = \begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix} = (z_1, z_2, z_3, z_4)
\]
and

\[ C = \begin{pmatrix} \mathbf{1}^T \\ \mathbf{A} \end{pmatrix} = (c_1, c_2, c_3, c_4) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \]

The maximum of the number of element zero is two in every row of \( A \). It is easy to calculate

\[ \tilde{C} = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}. \]

and

\[ \hat{R}(t) = (-1, R_1(t), R_2(t), \ldots, R_5(t))^T. \]

According to the Corollary, the system reliability \( \Psi_{1,A}(t) \) is given as follows:

\[ \Psi_{1,A}(t) = R_1(t)R_2(t) + R_3(t)R_4(t) + R_1(t)R_2(t)R_5(t) + R_3(t)R_4(t)R_5(t) + R_1(t)R_2(t)R_5(t)R_6(t) - R_1(t)R_2(t)R_3(t)R_4(t)R_5(t) - R_1(t)R_2(t)R_3(t)R_6(t)R_5(t) - R_1(t)R_3(t)R_4(t)R_5(t)R_6(t) - R_1(t)R_3(t)R_5(t)R_6(t)R_5(t) - R_1(t)R_3(t)R_4(t)R_5(t) + 2R_1(t)R_2(t)R_3(t)R_4(t)R_5(t)R_6(t). \]

When \( t \) takes a value of 15, \( \Psi_{1,A}(t) = 0.889 \). We consider the following two hypotheses.
Simulation 1: $H_0: \Psi_{1,t} = 0.889$ vs $H_1: \Psi_{1,t} \neq 0.889$

Empirical likelihood does not depend on the distribution of samples but is related to $\Psi_0$, which will be tested. The first simulation checks whether the type I error is controlled or not. Here, we take $n = 10, 20, 30, 50, 80, 100, 50, 200, 250, 300$. It is easy to find that the accepted rate is well controlled about 0.95 when the level of significance is 0.05 (see the left panel in Fig 2), and the accepted rate is well controlled about 0.9 when the level of significance is 0.1 (see the right panel in Fig 2). The results are more accurate as the sample size increases.

Simulation 2: $H_0: \Psi_{1,t} \geq \Psi_0$ vs $H_1: \Psi_{1,t} < \Psi_0$

The true value is 0.889 for the one-sided hypothesis test. In Fig 3, the sample size is fixed, and different values of $\Psi_0 = 0.80, 0.83, 0.86, 0.88, 0.89, 0.90, 0.92, 0.94, 0.96$, and 0.98, are considered. The sample size takes $n = 10, 15, 20, 30, 50, 60, 80, 100, 150, 300$. Then, we compare the accepted rates of the different sample sizes in Fig 3. Fig 4 shows the power of the different sample sizes. Fig 3 shows that the accepted rate increases under the null hypothesis when the
sample size increases and vice versa. Fig 4 shows that the rejected rate approaches 0 under the null hypothesis. The power increases when $C_0$ increases under the alternative hypothesis.

**Conclusion**

This study on the reliability of a complex system described by minimal paths has theory significance, as well as practical application. However, there are few research results. Based on the life samples of subsystems, we use the empirical likelihood method to solve the reliability test problem, provide a reliability test statistic for a CS, and extract the limit distribution of the test statistic.

We carry out some simulation studies about a simple bridge-type complex system and obtain results. The simulation results are consistent with the theorem. Therefore, we use the EL to propose a statistic for the test of a complex system. This test statistic does not depend on the distribution of the samples, but is related to $\Psi_0$, which will be tested, and is irrelevant to the degree of complexity of the system. In addition, it can control type I error well with a small sample size, and the power is consistent with the result when the sample size increases.
Appendix

Proof of Theorem 3.2

Let \( w = (w_1, w_2, \ldots, w_n) \). It is easy to prove that function \( f(w) = -2 \log \left( \prod_{i=1}^{n} w_i \right) \) is a convex function of \( w \), and

\[
S_1 = \left\{ w \left| \sum_{i=1}^{n} w_i [\Psi_{1,A}(X_i, t) - \Psi_0] = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\} \right.
\]

\[
S_2 = \left\{ w \left| \sum_{i=1}^{n} w_i [\Psi_{1,A}(X_i, t) - \Psi_0] \geq 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\} \right.
\]

are closed convex sets; therefore there exist minimums in \( S_1 \) and \( S_2 \), respectively. Therefore, the minimums of \( f(w) \) exist in \( S_1 \) and \( S_2 \) by themselves. Let \( g(w) = \sum_{i=1}^{n} w_i [\Psi_{1,A}(X_i, t) - \Psi_0] \), where \( \Psi_{1,A}(X, t) = -\sum_{j=1}^{k} \sum_{i=1}^{c_j} (-1)^{j} \Pi_{i=1}^{n} E \{ I_{(X_i \geq t)} \} \). Let \( h(w) = \sum_{i=1}^{n} w_i - 1 \) and

\[
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Fig 4. Test \( H_0: \Psi_{1,A}(\theta) \geq \Psi_0 \) vs \( H_1: \Psi_{1,A}(\theta) < \Psi_0 \). Numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 denote the powers of 0.80, 0.83, 0.86, 0.88, 0.90, 0.92, 0.94, 0.96, and 0.98, respectively. The x-coordinate is the sample size, and the y-coordinate is power. The level of significance in the left panel is 0.05, and the level of significance in the right panel is 0.1.
From Eq (2), we can derive

\[
L(w) = \prod_{i=1}^{n} w_i, \text{ we can obtain }
\]

\[
w^{(1)} = \arg_{\omega} \sup \left\{ L(w) \sum_{i=1}^{n} w_i | \Psi_{1,i}(X_i, t) - \Psi_0 | = 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\},
\]

\[
w^{(2)} = \arg_{\omega} \sup \left\{ L(w) \sum_{i=1}^{n} w_i | \Psi_{1,i}(X_i, t) - \Psi_0 | \geq 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}.
\]

For the one-sided hypothesis test,

\[
H_0 : \Psi_{1,i}(t) \geq \Psi_0 \quad \text{vs} \quad H_1 : \Psi_{1,i}(t) < \Psi_0,
\]

the nonparametric maximum likelihood ratio is

\[
\mathcal{R}(\Psi_0) = \frac{\sup \left\{ L(w) \sum_{i=1}^{n} w_i | \Psi_{1,i}(X_i, t) - \Psi_0 | \geq 0, w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}}{\sup \left\{ L(w) | w_i \geq 0, \sum_{i=1}^{n} w_i = 1 \right\}}.
\]

The feasible region of problem (MP) is \( D = \{ w | g(w) \geq 0 \} \), also

\[
\frac{\partial h(w)}{\partial w_i} \bigg|_{w_i=\omega^{(2)}} = 1, \quad i = 1, \ldots, n,
\]

\[
\frac{\partial g(w)}{\partial w_i} \bigg|_{w_i=\omega^{(2)}} = \Psi_{1,i}(X_i, t) - \Psi_0.
\]

Because vector \((1, 1, \ldots, 1)^T\) is independent of \([\Psi_{1,i}(X_i, t) - \Psi_0], \ldots, [\Psi_{1,m}(X_m, t) - \Psi_0])^T\), and functions \(f(w), g(w)\) and \(h(w)\) are first-order continuous differentiable and satisfy the Kuhn–Tucker theorem (necessary condition), then there exist \(\lambda^* \geq 0\) and \(\mu^* \in \mathbb{R}^1\), which satisfy the following equations:

\[
\frac{\partial f}{\partial w_i} \bigg|_{w_i=\omega^{(2)}} - \lambda^* \frac{\partial g}{\partial w_i} \bigg|_{w_i=\omega^{(2)}} - \mu^* \frac{\partial h}{\partial w_i} \bigg|_{w_i=\omega^{(2)}} = 0, \quad (1)
\]

\[
\lambda^* g(w^{(2)}) = 0. \quad (2)
\]

From these equations, we can obtain \(\mu^* = -2n\). Let \(\lambda^{(2)} = -\lambda^*/2n \leq 0\), then

\[
w^{(2)}_i = \frac{1}{n} \frac{1}{1 + \lambda^{(2)} [\Psi_{1,i}(X_i, t) - \Psi_0]}.
\]

From Eq (2), we can derive

\[
\lambda^{(2)} g(w^{(2)}) = 0. \quad (4)
\]
If \( \lambda(2) = 0 \), we have \( w_i^{(2)} = 1/n \), and
\[
g(w^{(2)}) = \sum_{i=1}^{n} \frac{1}{n} [\Psi_{1,A}(X_i, t) - \Psi_0] = (\Psi_{1,A}(X, t) - \Psi_0) \geq 0. \tag{5}
\]

If \( \lambda(2) < 0 \), we obtain from Eq (4)
\[
g(w^{(2)}) = 0,
\]
\[
\Rightarrow \sum_{i=1}^{n} \frac{1}{n + \lambda(2)} [\Psi_{1,A}(X_i, t) - \Psi_0] = 0,
\]
\[
\Rightarrow \sum_{i=1}^{n} \frac{1}{n} [\Psi_{1,A}(X_i, t) - \Psi_0] - \sum_{i=1}^{n} \frac{\lambda(2)}{n + \lambda(2)} [\Psi_{1,A}(X_i, t) - \Psi_0]^2 = 0,
\]
\[
\Rightarrow \lambda(2) \sum_{i=1}^{n} \frac{1}{n + \lambda(2)} [\Psi_{1,A}(X_i, t) - \Psi_0]^2 = (\Psi_{1,A}(X, t) - \Psi_0).
\]

Because \( \lambda(2) < 0 \) and \( w_i^{(2)} > 0 \), we have
\[
\lambda(2) \sum_{i=1}^{n} w_i^{(2)} [\Psi_{1,A}(X_i, t) - \Psi_0]^2 < 0 \quad \text{and} \quad (\Psi_{1,A}(X, t) - \Psi_0) < 0,
\]

hence
\[
\lambda(2) = 0 \iff (\Psi_{1,A}(X, t) - \Psi_0) \geq 0. \tag{7}
\]

Due to \( L(w^{(1)}) = \sup \{ L(w) | \sum_{i=1}^{n} w_i [\Psi_{1,A}(X_i, t) - \Psi_0] = 0, \ w_i \geq 0, \ \sum_{i=1}^{n} w_i = 1 \} \), by the Lagrange multiplier
\[
w_i^{(1)} = \frac{1}{n + \lambda(1)} \frac{1}{\Psi_{1,A}(X_i, t) - \Psi_0},
\]

where \( \lambda(1) \) satisfies the following equation,
\[
g(\lambda(1)) = \sum_{i=1}^{n} \frac{1}{n + \lambda(1)} [\Psi_{1,A}(X_i, t) - \Psi_0] = 0.
\]

For \( \lambda(2) < 0 \), we have
\[
w_i^{(2)} = \frac{1}{n + \lambda(2)} \frac{1}{\Psi_{1,A}(X_i, t) - \Psi_0} \quad \text{and} \quad g(w^{(2)}) = 0,
\]

hence
\[
g(\lambda(2)) = \sum_{i=1}^{n} \frac{1}{n + \lambda(2)} [\Psi_{1,A}(X_i, t) - \Psi_0] = 0.
\]

Derivative with respect to \( \lambda \), we have
\[
g'(\lambda) = -\sum_{i=1}^{n} \frac{1}{n} \frac{[\Psi_{1,A}(X_i, t) - \Psi_0]^2}{(1 + \lambda(2)[\Psi_{1,A}(X_i, t) - \Psi_0])^2} < 0.
\]

Therefore, \( g(\lambda) \) is strictly monotone about \( \lambda \); hence, when \( \lambda(2) < 0 \) and \( \lambda(2) = \lambda(1) \), we have
\[ w^{(2)}_i = w^{(1)}_i. \] Thus, \[ w^{(2)}_i = 1/n. \] When \( \lambda^{(2)} = 0 \), we have \( -2\log R(\Psi_0) = 0. \) According to Theorem 3.1, when \( \lambda^{(2)} < 0 \), \( -2\log R(\Psi_0) \) converges to \( \chi^2_1 \) in distribution. According to Eq. (7), when \( \Psi_{1A}(X, t) - \Psi_0 > 0 \), then \( w^{(2)}_i = 1/n \). When \( \Psi_{1A}(X, t) - \Psi_0 = 0 \), then \( -2\log R(\Psi_0) \) converges to \( \chi^2_1 \) in distribution. Otherwise, \( \Psi_{1A}(X, t) \) converges to \( N(\Psi_1, \sigma^2) \) under \( H_0 \).

1. When \( \Psi_1 > \Psi_0 \), \( p(\Psi_{1A}(X, t) - \Psi_0 \geq 0) \to 1. \)

2. When \( \Psi_1 = \Psi_0 \), \( p(\Psi_{1A}(X, t) - \Psi_0 \geq 0) \to \frac{1}{2}. \)

For any \( c > 0 \),

\[
\lim_{n \to \infty} p(-2\log R(\Psi_0) \leq c) = \lim_{n \to \infty} p(-2\log R(\Psi_0) \leq c \mid \Psi_{1A}(X, t) - \Psi_0 \geq 0)p(\Psi_{1A}(X, t) - \Psi_0 \geq 0) + \lim_{n \to \infty} p(-2\log R(\Psi_0) \leq c \mid \Psi_{1A}(X, t) - \Psi_0 < 0)p(\Psi_{1A}(X, t) - \Psi_0 < 0)
\]

\[ = \frac{1}{2} + \frac{1}{2}p(\chi^2_1 \leq c). \]

The proof is finished.

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**Author Contributions**

**Formal analysis:** YZ LF JZ YH.

**Funding acquisition:** YZ LF YH.

**Methodology:** YZ LF YH.

**Project administration:** YH.

**Software:** YZ LF JZ.

**Supervision:** YZ LF YH.

**Writing – original draft:** YZ LF JZ YH.

**Writing – review & editing:** YZ LF JZ YH.

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