Gelfand pairs admit an Iwasawa decomposition

Nicolas Monod

Received: 13 August 2019 / Revised: 8 June 2020 / Published online: 3 August 2020
© Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract

Every Gelfand pair \((G, K)\) admits a decomposition \(G = KP\), where \(P \triangleleft G\) is an amenable subgroup. In particular, the Furstenberg boundary of \(G\) is homogeneous. Applications include the complete classification of non-positively curved Gelfand pairs, relying on earlier joint work with Caprace, as well as a canonical family of pure spherical functions in the sense of Gelfand–Godement for general Gelfand pairs.

Let \(G\) be a locally compact group. The space \(\mathcal{M}^b(G)\) of bounded measures on \(G\) is an algebra for convolution, which is simply the push-forward of the multiplication map \(G \times G \rightarrow G\).

Definition

Let \(K \triangleleft G\) be a compact subgroup. The pair \((G, K)\) is a Gelfand pair if the algebra \(\mathcal{M}^b(G)^{K,K}\) of bi-\(K\)-invariant measures is commutative.

This definition, rooted in Gelfand’s 1950 work [14], is often given in terms of algebras of functions [12]. This is equivalent, by an approximation argument in the narrow topology, but has the inelegance of requiring the choice (and existence) of a Haar measure on \(G\).

Examples of Gelfand pairs include notably all connected semi-simple Lie groups \(G\) with finite center, where \(K\) is a maximal compact subgroup. Other examples are provided by their analogues over local fields [18], and non-linear examples include automorphism groups of trees [2,23].

All these “classical” examples also have in common another very useful property: they admit a co-compact amenable subgroup \(P \triangleleft G\). In the semi-simple case, \(P\) is a minimal parabolic subgroup. Moreover, the Iwasawa decomposition implies that \(G\) can be written as \(G = KP\). This note shows that this situation is not a coincidence:
**Theorem** Let $(G, K)$ be a Gelfand pair. Then $G$ admits a co-compact amenable subgroup $P < G$ such that $G = KP$.

The mere existence of $P$ has a number of strong consequences discussed below. Most immediate is that $G$ belongs to the exclusive club whose members boast a homogeneous Furstenberg boundary:

**Corollary 1** Choose a maximal subgroup $P < G$ as in the Theorem.

Then the Furstenberg boundary of $G$ is the homogeneous space $\partial G = G/P \cong K/(K \cap P)$.

In particular, $P$ is unique up to conjugacy.

Another general consequence is that $G$ is exact in the sense of C*-algebras [20, Sect. 7.1].

**Remark** The proof of the Theorem is easy. What surprises us (besides the fact that it went unnoticed during decades of harmonic analysis on Gelfand pairs) is that the unique group $P$ of Corollary 1 is obtained by purely existential methods. Indeed, the author is unaware of a constructive proof—or even of a heuristic based on the classical Iwasawa decomposition, explaining $(G, K) \mapsto P$.

We next derive a more geometric illustration of how consequential the existence of $P$ is. The above classical examples of Gelfand pairs are all CAT(0) groups in the sense that they occur as cocompact isometry groups of non-positively curved spaces: either Riemannian symmetric spaces or Euclidean buildings. General CAT(0) groups constitute a much more cosmopolitan category populated by all sorts of exotic spaces hailing from combinatorial group theory, Kac–Moody theory, etc. Using the “indiscrete Bieberbach theorem” established with Caprace [9], the Theorem of this note leads to a complete classification of CAT(0) Gelfand pairs:

**Corollary 2** Let $(G, K)$ be a Gelfand pair and assume that $G < \text{Isom}(X)$ acts cocompactly on a geodesically complete locally compact CAT(0) space $X$.

Then $X$ is a product of Euclidean spaces, Riemannian symmetric spaces of non-compact type, Bruhat–Tits buildings and biregular trees.

In particular, $G$ lies in a product of Gelfand pairs belonging to the classical sets of examples above.

This statement contains for instance a result by Caprace–Ciobotaru [5], namely: let $X$ be an irreducible locally finite thick Euclidean building. If $G = \text{Aut}(X)$ (or any co-compact subgroup $G < \text{Isom}(X)$) is a Gelfand pair for some compact $K < G$, then $X$ is Bruhat–Tits.

Similarly, the statement contains some cases of results by Abramenko–Parkinson–Van Maldeghem [1] and Lécureux [21, Sect. 7], [22] establishing the non-commutativity of Hecke algebras associated to certain Coxeter groups. Namely, when Kac–Moody theory associates to them a locally finite thick building, Corollary 2 implies that the Hecke algebra can only be commutative in the affine case.

The “Iwasawa decomposition” $G = KP$ is stronger yet than the existence of $P$. For instance, it is a key ingredient for results of Furman [13, Thm. 10] and it could
shed some light on the spherical dual of $G$, see below. It should also impose further restrictions on the centraliser lattice in case $G$ is a compactly generated simple group, see [10]. Already the existence of $P$ implies that this lattice is at most countable: see [10, pp. 11–12] and use that $G/P$ is metrisable in this setting.

We now contemplate some of the analytic legacy that the decomposition $G = KP$ bestows upon a general Gelfand pair $(G, K)$. Following Gelfand and Godement [17], the fundamental building block of non-commutative Fourier–Plancherel theory is given by positive definite spherical functions on Gelfand pairs, namely continuous $\varphi: G \to \mathbb{C}$ satisfying

$$\varphi(x) \varphi(y) = \int_K \varphi(xky) \, dk \quad \forall x, y \in G$$

where the integration is with respect to the unique Haar probability measure on $K$; see also [11] and [26]. This is the abstract generalisation of addition formulas for special functions such as Legendre functions [25].

Here is how $P$ enters the picture:

Let $\nabla_P$ be the modular function of $P$, which is non-trivial unless $G$ itself is amenable and $G = P$. Then $\varphi(kp) = \nabla_P(p)$ gives a well-defined continuous function $\varphi: G \to \mathbb{R}_{>0}$ when $k \in K$, $p \in P$ because $\nabla_P$ is identically one on $K \cap P$. For every parameter $s \in \mathbb{C}$, define

$$\varphi_s(g) = \int_K \varphi(g^{-1}k)^{\frac{1}{2} + is} \, dk.$$ 

In view of Corollary 1, $\varphi_s$ is actually canonically attached to the pair $(G, K)$ up to conjugation. On the other hand, we claim that $\varphi_s$ is the matrix coefficient of the (projectively) unique $K$-fixed vector in a parabolically induced representation from $P$. In particular, $\varphi_s$ is a pure positive definite spherical function on $G$ for each real $s$.

This claim is a classical fact for semi-simple groups, where $\varphi_s$ above is the Harish-Chandra formula. The Theorem makes it available for general Gelfand pairs, as desired by Godement [16, Sect. 16]. Of course this only gives a principal series and suggests to investigate fully the characters of $P$.

To justify our claim, recall that Weil’s integration formula [3, VII Sect. 2.5] implies that the push-forward to $G/P$ of the Haar measure of $K$ has a Radon–Nikodým cocycle given at $(g, xP)$ by $\varphi(g^{-1}x)/\varphi(x)$.

This formula uses the fact that $G$ is unimodular in any Gelfand pair [24, 24.8.1]. Therefore, the unitary induction $\pi_s$ of the character $\nabla^{is}_P$ is given on various spaces of functions $f$ on $G/P$ by

$$(\pi_s(g)f)(xP) = f(g^{-1}xP) \left( \frac{\varphi(g^{-1}x)}{\varphi(x)} \right)^{\frac{1}{2} + is}.$$ 

The only $K$-invariant vectors $v$ are constant functions on $G/P$ and hence the associated matrix coefficient $\varphi_s(g) = \langle \pi_s(g)v, v \rangle$ is uniquely defined once $v$ has unit norm. The fact that $\varphi_s$ is pure and spherical (for $s \in \mathbb{R}$) now follows from the general theory of Gelfand pairs, specifically I.II.6 and I.III.2 in [12].
Proof of the Theorem and of Corollary 1  We recall that an affine $G$-flow is a non-empty compact convex set $C$ in some locally convex topological vector space over $\mathbb{R}$, endowed with a jointly continuous $G$-action preserving the affine structure of $C$. An affine flow is called irreducible if it does not contain any proper affine subflow. An argument due to Furstenberg implies that $G$ admits an irreducible flow $\Delta G$ which is universal in the sense that it maps onto every irreducible flow. Moreover, $\Delta G$ is unique up to unique isomorphisms. It turns out that $\Delta G$ is the simplex of probability measures $\mathcal{P}(\partial G)$ over the Furstenberg boundary $\partial G$ of $G$, and that this is actually one of the possible definitions of $\partial G$. For all this, we refer to [15].

We shall be more interested in the convex subset $\mathcal{P}(G)$ of $\mathcal{M}^b(G)$ consisting of the probability measures, as well as in the corresponding subset $\mathcal{P}(G)^K.K$. We note the following straightforward facts:

- $\mathcal{P}(G)$ is closed under the multiplication given by convolution.
- The monoid $\mathcal{P}(G)$ contains $G$ via the identification of points with Dirac masses.
- The normalised Haar measure $\kappa$ of $K$ is an idempotent belonging to $\mathcal{P}(G)^K.K$.
- $\mathcal{P}(G)^K.K = \kappa \mathcal{P}(G)\kappa$; it is a monoid with $\kappa$ as identity.

By generalised vector-valued integration [4, IV Sect. 7.1], any affine $G$-flow $C$ is endowed with an action of the monoid $\mathcal{P}(G)$ which is affine in both variables. It will be crucial below that this action is moreover continuous for the variable in $G$. One way to see this is to check first that any $\mu \in \mathcal{P}(G)$ induces a continuous map $C \to \mathcal{P}(C)$ by push-forward on orbits, using that the $G$-action on $C$ is equicontinuous over compact subsets of $G$. Then observe that the action of $\mu$ is obtained by composing this map $C \to \mathcal{P}(C)$ with the continuous barycenter map $\mathcal{P}(C) \to C$.

Since $K$ is compact, it has a non-empty fixed-point set $C^K$; better yet, the idempotent $\kappa$ provides a continuous projection $\kappa : C \to C^K$. In particular, the monoid $\mathcal{A} = \kappa \mathcal{P}(G)\kappa$ preserves the convex compact set $C^K$.

Only now do we use the assumption that we have a Gelfand pair: the monoid $\mathcal{A}$ is commutative. Since $\mathcal{A}$ acts by continuous operators, the Markov–Kakutani theorem therefore implies that $\mathcal{A}$ fixes a point $p$ in $C^K$. From now on, we assume that $C$ is irreducible. The convex set $\mathcal{P}(G)p$ is $G$-invariant and hence must be dense. It follows that $\kappa \mathcal{P}(G)p$ is dense in $C^K$, but $\kappa \mathcal{P}(G)p$ is $\mathcal{A}p$ which is reduced to $p$. In conclusion, we have shown that $K$ has a unique fixed point in $C$.

We now apply this to the case where $C = \Delta G$ is the simplex of probability measures on $\partial G$ and deduce that $K$ fixes a unique such measure on $\partial G$. Since $K$ is compact, every $K$-orbit supports an invariant measure: the push-forward of $\kappa$. This implies that $K$ has a single orbit in $\partial G$. In particular, $\partial G = G/P$ for some co-compact subgroup $P < G$ and moreover $G = KP$.

Next, we observe that $P$ is relatively amenable in $G$, which means by definition that every affine $G$-flow has a $P$-fixed point. Indeed, this property characterises the subgroups that fix a point in $\Delta G$: this follows from the universal property of $\Delta G$. This characterisation also implies that this $P$ is already maximal relatively amenable. Indeed, if $P' < G$ is relatively amenable and contains $P$, it also fixes a point in $\Delta G$; this induces an affine $G$-map $\mathcal{P}(G/P') \to \mathcal{P}(G/P)$, which must be the identity by universality of $\mathcal{P}(G/P) = \Delta G$. 

S Springer
We recall that relative amenability is equivalent to amenability in a wide class of ambient locally compact groups \( G \) including all exact groups, but it is only \textit{a posteriori} that the Theorem implies that \( G \) is exact, see [20, Sect. 7.1]. In the locally compact setting, it is still an open question to exhibit an example where the weaker relative notion does not coincide with amenability [8]. In the co-compact case, however, we can settle the question with the Proposition below and conclude that \( P \) is amenable. Thus the Proposition will complete the proof.

The following statement is a very basic case of much more general results by Andy Zucker [27, Thm. 7.5]; the elementary proof below is inspired by reading his preprint.

\begin{prop}
Let \( G \) be a Hausdorff topological group and \( P < G \) a closed subgroup such that \( \partial G = G/P \). Then \( P \) is amenable.
\end{prop}

\begin{warning}
A subgroup of \( G \) fixing a point in \( \partial G \) is not necessarily amenable. However, in the locally compact case and assuming \( \partial G \) homogeneous, this follows from the Proposition because amenability of locally compact groups passes to subgroups.
\end{warning}

\begin{proof}[Proof of the Proposition]
We know that \( P \) is co-compact and relatively amenable. The latter is equivalent to the existence of a \( P \)-invariant mean \( \mu \) on the space \( C^b_{ru}(G) \) of right uniformly continuous bounded functions (cf. Thm. 5 in [8]). It suffices to show that \( \mu \) descends to \( C^b_{ru}(P) \), viewed as a quotient of \( C^b_{ru}(G) \) under restriction (by Katetov extension [19]). Let thus \( f \in C^b_{ru}(G) \) be any map vanishing on \( P \); we need to show \( \mu(f) = 0 \) and can assume \( f \geq 0 \). Given \( \epsilon > 0 \) there is an identity neighbourhood \( U \) in \( G \) such that \( f \leq \epsilon \) on \( UP \). By Urysohn’s lemma in \( G/P \), there is \( h \in C(G/P) \) vanishing on a neighbourhood of \( P \) but taking constant value \( \|f\|_\infty \) outside \( UP \). Viewing \( h \) as an element of \( C^b_{ru}(G) \), we thus have \( f \leq \epsilon 1_G + h \). We now claim \( \mu(h) = 0 \), which finishes the proof since \( \epsilon \) is arbitrary. The claim follows from the fact that \( \mu \) is mapped to a \( P \)-invariant probability measure on \( G/P \) under the inclusion of \( C(G/P) \) in \( C^b_{ru}(G) \). Indeed, the only \( P \)-invariant probability measure on \( G/P \cong \partial G \) is the Dirac mass at \( P \) by strong proximality of \( P \) on \( \partial G \), see [15, II.3.1].
\end{proof}

\begin{proof}[Proof of Corollary 2]
Consider \( G < \text{Isom}(X) \) as in the statement. We first recall that \( X \) is \textit{minimal} in the sense that it does not contain a closed convex \( G \)-invariant proper subset, see [6, 3.13]. Next, we recall that general splitting results (1.9 together with 1.5(iii) in [6]) allow us to reduce to the case where \( X \) has no Euclidean factor. In any Gelfand pair, \( G \) is unimodular [24, 24.8.1]; this, together with the elements collected thus far, allows us to apply Theorem M in [7]. That result states that \( G \) has no fixed point at infinity. On the other hand, our Theorem above provides a subgroup \( P < \text{Isom}(X) \) acting co-compactly on \( X \). We are now in position to apply the indiscrete Bieberbach theorem [9, Thm. B], which identifies \( X \) with a product of classical spaces as desired.
\end{proof}

\begin{remark}
A part of the proof of the Theorem is reminiscent of the fact that any irreducible \textit{unitary} representation of \( G \) has at most a one-dimensional subspace of \( K \)-fixed vectors, a fact that actually characterises Gelfand pairs. We recall that the corresponding statement fails for \textit{real} Hilbert spaces, whereas our affine flows are always over the reals.
\end{remark}
Acknowledgements I am grateful to Andy Zucker for sending me his preprint and to Pierre-Emmanuel Caprace for several insightful comments on a preliminary version.

References

Selberg ne fait aucune espèce d’allusion à l’existence possible d’une littérature mathématique.,

Godement [17], 1957.

1. Abramenko, P., Parkinson, J., Van Maldeghem, H.: A classification of commutative parabolic Hecke algebras. J. Algebra 385, 115–133 (2013)
2. Amann, O. É.: Groups of Tree-Automorphisms and their Unitary Representations. PhD thesis, ETHZ, Dissertation n° 15292 (2003)
3. Bourbaki, N.: Éléments de mathématique: Intégration. Chapitre 7 et 8. Actualités Scientifiques et Industrielles, No. 1306. Hermann, Paris, (1963)
4. Bourbaki, N.: Éléments de mathématique: Intégration. Chapitres 1, 2, 3 et 4. Deuxième édition revue et augmentée. Actualités Scientifiques et Industrielles, No. 1175. Hermann, Paris, (1965)
5. Caprace, P.-E., Ciobotaru, C.: Gelfand pairs and strong transitivity for Euclidean buildings. Ergodic Theory Dyn. Syst. 35(4), 1056–1078 (2015)
6. Caprace, P.-E., Monod, N.: Isometry groups of non-positively curved spaces: structure theory. J. Topol. 2(4), 661–700 (2009)
7. Caprace, P.-E., Monod, N.: Fixed points and amenability in non-posituve curvature. Math. Ann. 356(4), 1303–1337 (2013)
8. Caprace, P.-E., Monod, N.: Relative amenableability. Groups Geom. Dyn. 8(3), 747–774 (2014)
9. Caprace, P.-E., Monod, N.: An indiscrete Bieberbach theorem: from amenable CAT(0) groups to Tits buildings. J. Éc. polytech. 2, 333–383 (2015)
10. Caprace, P.-E., Reid, C.D., Willis, G.A.: Locally normal subgroups of totally disconnected groups. Part II: Compactly generated simple groups. Forum Math. Sigma 5(e12), 1–89 (2017)
11. Dieudonné, J.: Gelfand pairs and spherical functions. Internat. J. Math. Math. Sci. 2(2), 153–162 (1979)
12. Faraut, J.: Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques. In: Analyse harmonique, Les Cours du C.I.M.P.A., pp. 315–446. Nice (1983)
13. Furman, A.: On minimal strongly proximal actions of locally compact groups. Israel J. Math. 136, 173–187 (2003)
14. Gelfand, I.M.: Spherical functions in symmetric Riemann spaces. Dokl. Akad. Nauk SSSR 70, 5–8 (1950)
15. Glasner, S.: Proximal flows. In: Lecture Notes in Mathematics, vol. 517. Springer-Verlag (1976)
16. Godement, R.: A theory of spherical functions. I. Trans. Am. Math. Soc. 73, 496–556 (1952)
17. Godement, R.: Introduction aux travaux de A. Selberg. In: exposés 137–168, volume 4 of Séminaire Bourbaki, pp. 95–110. Société mathématique de France (1957)
18. Gross, B.H.: Some applications of Gelfand pairs to number theory. Bull. Am. Math. Soc. (N.S.) 24(2), 277–301 (1991)
19. Katětov, M.: On real-valued functions in topological spaces. Fund. Math. 38, 85–91 (1951)
20. Kirchberg, E., Wassermann, S.: Permanence properties of C*-exact groups. Doc. Math. 4, 513–558 (1999)
21. Lécureux, J.: Automorphismes et compactifications d’immeubles: moyennabilité et action sur le bord. PhD thesis, Université de Lyon, No d’ordre: 261–2009 (2009)
22. Lécureux, J.: Hyperbolic configurations of roots and Hecke algebras. J. Algebra 323(5), 1454–1467 (2010)
23. Of’shanskii, G.I.: Classification of the irreducible representations of the automorphism groups of Bruhat–Tits trees. Funct. Anal. Appl. 11(1), 26–34 (1977)
24. Simonnet, M.: Measures and probabilities. Universitext. Springer, Berlin (1996)
25. Vilenkin, N.Y.: Special functions and the theory of group representations. In: Singh, V.N. (ed.) Translations Mathematical Monographs, vol. 22. American Mathematical Society, Providence, RI (1968)
26. Wolf, J.A.: Harmonic analysis on commutative spaces. In: Mathematical Surveys and Monographs, vol. 142. American Mathematical Society, Providence, RI (2007)
27. Zucker, A.: Maximally highly proximal flows. Preprint, arXiv:1812.00392v2, (2019)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.