A consistency result on long cardinal sequences✩

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ABSTRACT

For any regular cardinal \( \kappa \) and ordinal \( \eta < \kappa^{++} \) it is consistent that \( 2^\kappa \) is as large as you wish, and every function \( f : \eta \rightarrow [\kappa, 2\kappa] \cap \text{Card} \) with \( f(\alpha) = \kappa \) for \( cf(\alpha) < \kappa \) is the cardinal sequence of some locally compact scattered space.

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1. Introduction

If \( X \) is a locally compact, scattered Hausdorff (in short: LCS) space and \( \alpha \) is an ordinal, we let \( I_\alpha(X) \) denote the \( \alpha \)-th Cantor-Bendixson level of \( X \). The cardinal sequence of \( X \), \( CS(X) \), is the sequence of the cardinalities of the infinite Cantor-Bendixson levels of \( X \), i.e.

\[
CS(X) = \langle |I_\alpha(X)| : \alpha < \text{ht}^+(X) \rangle,
\]

where \( \text{ht}^+(X) \), the reduced height of \( X \), is the minimal ordinal \( \beta \) such that \( I_\beta(X) \) is finite. The height of \( X \), denoted by \( \text{ht}(X) \), is defined as the minimal ordinal \( \beta \) such that \( I_\beta(X) = \emptyset \). Clearly \( \text{ht}^+(X) \leq \text{ht}(X) \leq \text{ht}^+(X) + 1 \).

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If $\alpha$ is an ordinal, let $C(\alpha)$ denote the class of all cardinal sequences of LCS spaces of reduced height $\alpha$ and put

$$C_\lambda(\alpha) = \{ s \in C(\alpha) : s(0) = \lambda \land \forall \beta < \alpha \ s(\beta) \geq \lambda \}.$$  

Let $\langle \kappa \rangle_\alpha$ denote the constant $\kappa$-valued sequence of length $\alpha$.

In [4] it was shown that the class $C(\alpha)$ is described if the classes $C_\kappa(\beta)$ are characterized for every infinite cardinal $\kappa$ and ordinal $\beta \leq \alpha$. Then, under GCH, a full description of the classes $C_\kappa(\alpha)$ for infinite cardinals $\kappa$ and ordinals $\alpha < \omega_2$ was given.

The situation becomes, however, more complicated for $\alpha \geq \omega_2$. In [9] we gave a consistent full characterization of $C_\kappa(\alpha)$ for any uncountable regular cardinals $\kappa$ and ordinals $\alpha < \kappa^{++}$ under GCH.

If GCH fails, much less is known on $C_\kappa(\alpha)$ even for $\alpha < \kappa^{++}$.

In [10] it was proved that $\langle \omega \rangle_{\omega_1} \sim \langle \omega_2 \rangle \in C_{\omega_1}(\omega_1 + 1)$ is consistent.

In [5] a similar result was proved for uncountable cardinals instead of $\omega$: if $\kappa$ is a regular cardinal with $\kappa^{< \kappa} = \kappa > \omega$ and $2^\kappa = \kappa^+$, then in some cardinality preserving generic extension of the ground model we have

$$\langle \kappa \rangle_{\kappa^+} \sim \langle \kappa^{++} \rangle \in C(\kappa^+ + 1).$$

In [10] we proved that if $\kappa$ and $\lambda$ are regular cardinals with $\kappa \leq \lambda$, $\kappa^{< \kappa} = \kappa$, $2^\kappa = \kappa^+$, and $\delta < \kappa^{++}$ with $\operatorname{cf}(\delta) = \kappa^+$, then in some cardinality preserving generic extension of the ground model we have

$$\langle \kappa \rangle_\delta \sim \langle \lambda \rangle \in C(\delta + 1).$$

In this paper we will prove a much stronger result than the above mentioned one.

**Theorem 1.1.** Assume that $\kappa$ and $\lambda$ are regular cardinals, $\kappa^{++} \leq \lambda$, $\kappa^{< \kappa} = \kappa$, $2^\kappa = \kappa^+$, $\lambda^{\kappa^+} = \lambda$ and $\delta < \kappa^{++}$. Then, in some cardinality preserving generic extension of the ground model, we have $2^\kappa = \lambda$ and

$$\{ f \in \delta([\kappa, \lambda] \cap \operatorname{Card}) : f(\alpha) = \kappa \text{ whenever } \operatorname{cf}(\alpha) < \kappa \} \subset C_\kappa(\delta).$$

**Definition 1.2.** Let $C$ be a family of sequences of cardinals. We say that an LCS space $X$ is universal for $C$ iff $CS(X) \in C$ and for each $s \in C$ there is an open subspace $Z \subset X$ with $CS(Z) = s$.

**Remark.** The assumption $\delta < \kappa^{++}$ is essential in the construction as we will explain in a Remark on page 8.

So, we do not know whether Theorem 1.1 can be generalized to $\delta = \kappa^{++}$. In fact, if $\kappa$ is a specific uncountable cardinal, the problem whether it is relatively consistent with ZFC that $\langle \kappa \rangle_{\kappa^{++}} \in C(\kappa^{++})$ is a long-standing open question. Nevertheless, by a well-known result of Baumgartner and Shelah, it is known that it is relatively consistent with ZFC that $\langle \omega \rangle_{\omega_2} \in C(\omega_2)$ (see [2]).

Instead of Theorem 1.1 we prove the following stronger result:

**Theorem 1.3.** Assume that $\kappa$ and $\lambda$ are regular cardinals, $\kappa^{++} \leq \lambda$, $\kappa^{< \kappa} = \kappa$, $2^\kappa = \kappa^+$, $\lambda^{\kappa^+} = \lambda$ and $\delta < \kappa^{++}$. Then, in some cardinal preserving generic extension, we have $2^\kappa = \lambda$ and there is an LCS space $X$ which is universal for

$$C = \{ f \in \delta([\kappa, \lambda] \cap \operatorname{Card}) : f(\alpha) = \kappa \text{ whenever } \operatorname{cf}(\alpha) < \kappa \}.$$
Definition 1.4. Let $\kappa < \lambda$ be cardinals, $\delta$ be an ordinal, and $A \subset \delta$. An LCS space $X$ of height $\delta$ is called $(\kappa, \lambda, \delta, A)$-good iff there is an open subspace $Y \subset X$ such that

1) $CS(Y) = \langle \kappa \rangle^\delta$,
2) $I_\zeta(Y) = I_\zeta(X)$, and so $|I_\zeta(X)| = \kappa$, for $\zeta \in \delta \setminus A$,
3) $|I_\zeta(X)| = \lambda$ for $\zeta \in A$,
4) for $\zeta \in A$ the set $Z_\zeta = I_{<\zeta}(Y) \cup I_\zeta(X)$ is an open subspace of $X$ such that
   a) $I_\zeta(Z_\zeta) = I_\zeta(Y)$ for $\zeta < \zeta$,
   b) $I_\zeta(Z_\zeta) = I_\zeta(X)$.

Theorem 1.3 follows immediately from Koszmider’s Theorem, Theorem 1.6 and Proposition 1.7 below.

The following result of Koszmider can be obtained by putting together [7, Fact 32 and Theorem 33]:

Definition 1.5 (See [6, 7]). Assume that $\kappa < \lambda$ are infinite cardinals. We say that a function $F : [\lambda]^2 \rightarrow \kappa^+$ is a $\kappa^+$-strongly unbounded function on $\lambda$ iff for every ordinal $\vartheta < \kappa^+$ and for every family $\mathcal{A} \subset [\lambda]^{<\kappa}$ of pairwise disjoint sets with $|\mathcal{A}| = \kappa^+$, there are different $a, b \in \mathcal{A}$ such that $F(\alpha, \beta) > \vartheta$ for every $\alpha \in a$ and $\beta \in b$.

Koszmider’s Theorem. If $\kappa, \lambda$ are infinite cardinals such that $\kappa^{++} \leq \lambda$, $\kappa^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$ and $\lambda^{\kappa^+} = \lambda$,
then in some cardinal preserving generic extension $\kappa^{<\kappa} = \kappa$, $\lambda^\kappa = \lambda$ and there is a $\kappa^+$-strongly unbounded function on $\lambda$.

For an ordinal $\delta < \kappa^{++}$ let

$$L^\delta_\kappa = \{ \alpha < \delta : \text{cf}(\alpha) \in \{ \kappa, \kappa^+ \} \}.$$ 

Theorem 1.6. If $\kappa < \lambda$ are regular cardinals with $\kappa^{<\kappa} = \kappa$, $\lambda^\kappa = \lambda$, and there is a $\kappa^+$-strongly unbounded function on $\lambda$, then for each $\delta < \kappa^{++}$ there is a $\kappa$-complete $\kappa^+$-c.c poset $\mathcal{P}$ of cardinality $\lambda$ such that in $V^\mathcal{P}$ we have $2^\kappa = \lambda$ and there is a $(\kappa, \lambda, \delta, L^\delta_\kappa)$-good space.

We will prove Theorem 1.6 in Section 2.

Proposition 1.7. If $\kappa < \lambda$ are regular cardinals and $\delta < \kappa^{++}$, then a $(\kappa, \lambda, \delta, L^\delta_\kappa)$-good space is universal for

$$\mathcal{C} = \{ f \in \delta([\kappa, \lambda] \cap \text{Card}) : f(\alpha) = \kappa \text{ whenever cf}(\alpha) < \kappa \}.$$ 

Proof. Let $X$ be a $(\kappa, \lambda, \delta, L^\delta_\kappa)$-good space. Fix $f \in \mathcal{C}$. For $\zeta \in L^\delta_\kappa$ pick $T_\zeta \in \lfloor I_\zeta(X) \rfloor^{f(\zeta)}$, and let

$$Z = Y \cup \bigcup \{ T_\zeta : \zeta \in L^\delta_\kappa \}.$$ 

Since $I_{<\zeta}(Y) \cup T_\zeta$ is an open subspace of $X$ for $\zeta \in L^\delta_\kappa$, for every $\alpha < \delta$ we have

$$I_\alpha(Z) = I_\alpha(Y) \cup \bigcup \{ I_\alpha(I_{<\zeta}(Y) \cup T_\zeta) : \zeta \in L^\delta_\kappa \}.$$ 

Since

$$I_\alpha(I_{<\zeta}(Y) \cup T_\zeta) = \begin{cases} I_\alpha(Y) & \text{if } \alpha < \zeta, \\ T_\zeta & \text{if } \alpha = \zeta, \\ \emptyset & \text{if } \zeta < \alpha, \end{cases}$$

we have

$$I_\alpha(Z) = I_\alpha(Y) \cup \bigcup \{ I_\alpha(I_{<\zeta}(Y) \cup T_\zeta) : \zeta \in L^\delta_\kappa \}.$$
we have

\[ I_\alpha(Z) = \begin{cases} 
I_\alpha(Y) & \text{if } \alpha \notin \mathcal{L}_\delta, \\
I_\alpha(Y) \cup T_\alpha & \text{if } \alpha \in \mathcal{L}_\delta.
\end{cases} \]

Since \(|I_\alpha(Y)| = \kappa \) and \(|I_\alpha(Y) \cup T_\alpha| = \kappa + f(\alpha) = f(\alpha)\), we have \(CS(Z) = f\), which was to be proved. \( \square \)

2. Proof of Theorem 1.6

2.1. Graded posets

In [5], [8], [11] and in many other papers, the existence of an LCS space is proved in such a way that instead of constructing the space directly, a certain “graded poset” is produced which guaranteed the existence of the wanted LCS-space. From these results, Bagaria, [1], extracted the notion of s-posets and established the formal connection between graded posets and LCS-spaces. For technical reasons, we will use a reformulation of Bagaria’s result introduced in [12].

If \( \preceq \) is an arbitrary partial order on a set \( X \) then define the topology \( \tau_{\preceq} \) on \( X \) generated by the family

\[ \{ U_{\preceq}(x), X \setminus U_{\preceq}(x) : x \in X \} \]

as a subbase, where \( U_{\preceq}(x) = \{ y \in X : y \preceq x \} \).

In what follows, if \( i \) is a partial function from \( [X]^2 \) to \( X \) where \( X \) is the domain of some poset, for every \( \{s, t\} \in [X]^2 \setminus \text{dom}(i) \) we will write \( i(s, t) = \text{undef} \). So, we will write \( i : \ [X]^2 \to X \cup \{\text{undef}\} \) in order to represent a partial function \( i \) from \( [X]^2 \) to \( X \).

Proposition 2.1 ([12, Proposition 2.1]). Assume that \( \langle X, \preceq \rangle \) is a poset, \( \{X_\alpha : \alpha < \delta\} \) is a partition of \( X \) and \( i : \ [X]^2 \to X \cup \{\text{undef}\} \) is a function satisfying (a)–(c) below:

(a) if \( x \in X_\alpha, y \in X_\beta \) and \( x \preceq y \) then either \( x = y \) or \( \alpha < \beta \),

(b) \( \forall \{x, y\} \in [X]^2 \ ( \forall z \in X (z \preceq x \land z \preceq y) \iff z \preceq i(x, y) \).

(c) if \( x \in X_\alpha \) and \( \beta < \alpha \) then the set \( \{y \in X_\beta : y \preceq x\} \) is infinite.

Then \( \mathcal{X} = \langle X, \tau_{\preceq} \rangle \) is an LCS space with \( I_\alpha(\mathcal{X}) = X_\alpha \) for \( \alpha < \delta \).

Definition 2.2. Let \( \kappa < \lambda \) be cardinals, \( \delta \) be an ordinal, and \( A \subset \delta \). Assume that \( \langle X, \preceq \rangle \) is a poset, \( \{X_\alpha : \alpha < \delta\} \) is a partition of \( X \) and \( i : \ [X]^2 \to X \cup \{\text{undef}\} \) is a function satisfying conditions (a)–(c) from Proposition 2.1.

We say that poset \( \langle X, \preceq \rangle \) is \( (\kappa, \lambda, \delta, A) \)-good iff there is a set \( Y \subset X \) such that:

(d) if \( x_0 \preceq x_1 \), then either \( x_0 = x_1 \) or \( x_0 \in Y \);

(e) \( X_\zeta \in [Y]^< \kappa \) for \( \zeta \in \delta \setminus A \);

(f) \( |X_\zeta \cap Y| = \kappa \) for \( \zeta \in A \).

Proposition 2.3. Let \( \kappa < \lambda \) be cardinals, \( \delta \) be an ordinal, and \( A \subset \delta \). If \( \langle X, \preceq \rangle \) is a \( (\kappa, \lambda, \delta, A) \)-good poset, then \( \mathcal{X} = \langle X, \tau_{\preceq} \rangle \) is a \( (\kappa, \lambda, \delta, A) \)-good space.

Proof. By Proposition 2.1, \( \mathcal{X} = \langle X, \tau_{\preceq} \rangle \) is an LCS space with \( I_\alpha(\mathcal{X}) = X_\alpha \) for \( \alpha < \delta \).

By (d), the subspace \( Y \) is open, and so \( I_\zeta(Y) = I_\zeta(X) \cap Y \). Thus \( |I_\zeta(Y)| = \kappa \) by (e) and (f). So \( CS(Y) = \langle \kappa \rangle_\delta \), i.e. 1.4(1) holds.

If \( \zeta \in \delta \setminus A \), then \( I_\zeta(X) \subset Y \) by (e), so \( I_\zeta(X) = I_\zeta(Y) \). Thus 1.4(2) holds. Moreover \( I_\zeta(Y) = I_\zeta(X) \cap Y \). 1.4(3) follows from (f).
Also, for \( \zeta \in A \) (a) and (d) imply that \( U_\leq(s) \subset Z_\zeta \) for \( s \in Z_\zeta \), and so \( Z_\zeta \) is an open subspace of \( \mathcal{X} \).

Hence \( I_\xi(Z_\zeta) = I_\xi(X) \cap Z_\zeta = X_\xi \cap Z_\zeta \).

Thus \( I_\xi(Z_\zeta) = I_\xi(Y) \) for \( \xi < \zeta \), and \( I_\xi(Z_\zeta) = X_\zeta \). So 1.4(4) also holds.

Thus \( \mathcal{X} \) is a \((\kappa, \lambda, \delta, A)\)-good space. \( \square \)

So, instead of Theorem 1.6, it is enough to prove Theorem 2.4 below.

**Theorem 2.4.** If \( \kappa < \lambda \) are regular cardinals with \( \kappa^{<\kappa} = \kappa \), \( \lambda^\kappa = \lambda \), and there is a \( \kappa^{+}\)-strongly unbounded function on \( \lambda \), then for each \( \delta < \kappa^{++} \) there is a \( \kappa \)-complete \( \kappa^{+}\)-c.c poset \( \mathcal{P} \) of cardinality \( \lambda \) such that in \( \mathcal{V}_\mathcal{P} \) we have \( 2^\kappa = \lambda \) and there is a \((\kappa, \lambda, \delta, \mathcal{L}^\delta_\kappa)\)-good poset.

So, assume that \( \kappa, \lambda \) and \( \delta \) satisfy the hypothesis of Theorem 2.4. In order to construct the required poset \( \mathcal{P} \), first we need to recall some notion from [8, Section 1].

2.2. **Orbits**

If \( \alpha \leq \beta \) are ordinals let

\[ [\alpha, \beta) = \{ \gamma : \alpha \leq \gamma < \beta \} . \]

We say that \( I \) is an **ordinal interval** iff there are ordinals \( \alpha \) and \( \beta \) with \( I = [\alpha, \beta) \). Write \( I^- = \alpha \) and \( I^+ = \beta \).

If \( I = [\alpha, \beta) \) is an ordinal interval let \( E(I) = \{ \varepsilon_\nu^I : \nu < \text{cf}(\beta) \} \) be a cofinal closed subset of \( I \) having order type \( \text{cf}(\beta) \) with \( \alpha = \varepsilon_0^I \) and put

\[ E(I) = \{ [\varepsilon_\nu^I, \varepsilon_{\nu+1}^I) : \nu < \text{cf} \beta \} \]

provided \( \beta \) is a limit ordinal, and let \( E(I) = \{ \alpha, \beta' \} \) and put

\[ E(I) = \{ [\alpha, \beta'), \{ \beta' \} \} \]

provided \( \beta = \beta' + 1 \) is a successor ordinal.

Define \( \{ \mathcal{I}_n : n < \omega \} \) as follows:

\[ \mathcal{I}_0 = \{ [0, \delta) \} \) and \( \mathcal{I}_{n+1} = \bigcup \{ E(I) : I \in \mathcal{I}_n \} . \]

Put \( \mathbb{I} = \bigcup \{ \mathcal{I}_n : n < \omega \}. \)

Note that \( \mathbb{I} \) is a **cofinal tree of intervals** in the sense defined in [8]. So, the following conditions are satisfied:

(i) For every \( I, J \in \mathbb{I}, I \subset J \) or \( J \subset I \) or \( I \cap J = \emptyset \).

(ii) If \( I, J \) are different elements of \( \mathbb{I} \) with \( I \subset J \) and \( J^+ \) is a limit ordinal, then \( I^+ < J^+ \).

(iii) \( \mathcal{I}_n \) partitions \( [0, \delta) \) for each \( n < \omega \).

(iv) \( \mathcal{I}_{n+1} \) refines \( \mathcal{I}_n \) for each \( n < \omega \).

(v) For every \( \alpha < \delta \) there is an \( I \in \mathbb{I} \) such that \( I^- = \alpha \).

Then, for each \( \alpha < \delta \) we define

\[ n(\alpha) = \min \{ n : \exists ! I \in \mathcal{I}_n \text{ with } I^- = \alpha \} , \]

and for each \( \alpha < \delta \) and \( n < \omega \) we pick
Proposition 2.5. Assume that $\zeta < \delta$ is a limit ordinal. Then, there is an interval

$$J(\zeta) \in \mathcal{I}_{n(\zeta)-1} \cup \mathcal{I}_{n(\zeta)}$$

such that $\zeta$ is a limit point of $E(J(\zeta))$.

If $\text{cf}(\zeta) = \kappa^+$, then $J(\zeta) \in \mathcal{I}_{n(\zeta)}$ and $J(\zeta)^+ = \zeta$.

Proof. If there is an $I \in \mathcal{I}_{n(\zeta)}$ with $I^+ = \zeta$ then $J(\zeta) = I$. If there is no such $I$, then $\zeta$ is a limit point of $E(I(\zeta, n(\zeta) - 1))$, so $J(\zeta) = I(\zeta, n(\zeta) - 1)$.

Assume now that $\text{cf}(\zeta) = \kappa^+$. Then $\zeta \in E(I(\zeta, n(\zeta) - 1))$, but $|E(I(\zeta, n(\zeta) - 1)) \cap \zeta| \leq \kappa$, so $\zeta$ cannot be a limit point of $E(I(\zeta, n(\zeta) - 1))$. Therefore, it has a predecessor $\xi$ in $E(I(\zeta, n(\zeta) - 1))$, i.e. $[\xi, \zeta) \in \mathcal{I}_{n(\zeta)}$, and so $J(\zeta) = [\xi, \zeta)$ and $J(\zeta) \in \mathcal{I}_{n(\zeta)}$. □

If $\text{cf}(J(\zeta)^+) \in \{\kappa, \kappa^+\}$, we denote by $\{\varepsilon_{\nu}^\zeta : \nu < \text{cf}(J(\zeta)^+)\}$ the increasing enumeration of $E(J(\zeta))$, i.e. $\varepsilon_{\nu}^\zeta = \varepsilon_{\nu}^{J(\zeta)}$ for $\nu < \text{cf}(J(\zeta)^+)$. Now if $\zeta < \delta$, we define the basic orbit of $\zeta$ (with respect to $\parallel$) as

$$o(\zeta) = \bigcup \{(E(I(\zeta, m)) \cap \zeta) : m < n(\zeta)\}.$$

We refer the reader to [8, Section 1] for some fundamental facts and examples on basic orbits. In particular, we have that $\alpha \in o(\beta)$ implies $o(\alpha) \subseteq o(\beta)$.

If $\zeta \in \mathcal{L}_\delta^\kappa$, we define the extended orbit of $\zeta$ by

$$\overline{o}(\zeta) = o(\zeta) \cup (E(J(\zeta)) \cap \zeta).$$

Observe that if $J(\zeta) \in \mathcal{I}_{n(\zeta)-1}$ then $\overline{o}(\zeta) = o(\zeta)$.

The underlying set of our poset will consist of blocks. The following set $\mathcal{B}$ below serves as the index set of our blocks:

$$\mathcal{B} = \{S\} \cup \mathcal{L}_\delta^\kappa.$$

Let

$$B_S = \delta \times \kappa$$

and

$$B_\zeta = \{\zeta\} \times [\kappa, \lambda)$$

for $\zeta \in \mathcal{L}_\delta^\kappa$.

The underlying set of our poset will be

$$X = \bigcup \{B_T : T \in \mathcal{B}\}.$$

To obtain a $(\kappa, \lambda, \delta, \mathcal{L}_\delta^\kappa)$-good poset we take $Y = B_S$ and

$$X_\zeta = \begin{cases} 
\{\zeta\} \times \kappa & \text{if } \zeta \in \delta \setminus \mathcal{L}_\delta^\kappa, \\
\{\zeta\} \times \lambda & \text{if } \zeta \in \mathcal{L}_\delta^\kappa.
\end{cases}$$
Define the functions $\pi : X \rightarrow \delta$ and $\rho : X \rightarrow \lambda$ by the formulas

$$\pi((\alpha, \nu)) = \alpha \text{ and } \rho((\alpha, \nu)) = \nu.$$ 

Define

$$\pi_B : X \rightarrow \mathbb{B} \text{ by the formula } x \in B \pi_B(x).$$ 

Finally we define the orbits of the elements of $X$ as follows:

$$o^*(x) = \begin{cases} 
\{o(\pi(x)) \text{ for } x \in B_S, \\
\text{ and } \sigma(\pi(x)) \text{ for } x \in X \setminus B_S. 
\end{cases}$$

Observe that $o^*(x) \in \pi(x) \leq \kappa^+$ and

$$|o^*(x)| \leq \kappa \text{ unless } x \in B_\xi \text{ with } cf(\xi) = \kappa^+.$$ 

To simplify our notation, we will write $o(x) = o(\pi(x))$ and $\sigma(x) = \sigma(\pi(x))$.

### 2.3. Forcing construction

Let $\Lambda \in I$ and $\{x, y\} \in [X]^2$. We say that $\Lambda$ separates $x$ from $y$ if

$$\Lambda^- < \pi(x) < \Lambda^+ < \pi(y).$$

Let $F : [\lambda]^2 \rightarrow \kappa^+$ be a $\kappa^+$-strongly unbounded function.

Define

$$f : [X]^2 \rightarrow [\delta]^\kappa$$

as follows:

$$f\{x, y\} = \begin{cases} 
o(x) \cup \{\epsilon^{\pi(x)} : \zeta < F(\rho(x), \rho(y))\} & \text{if } \pi_B(x) = \pi_B(y) \neq S, \text{ and } cf(\pi(x)) = \kappa^+, \\
o^*(x) \cap o^*(y) & \text{otherwise.}
\end{cases}$$

Observe that

$$|f\{x, y\}| \leq \kappa$$

for all $\{x, y\} \in [X]^2$.

**Definition 2.6.** We define the poset $P = \langle P, \leq \rangle$ as follows: $(A, \leq, i) \in P$ iff the following conditions hold:

1. $(P1)$ $A \in [X]^{<\kappa}$;
2. $(P2)$ $\leq$ is a partial order on $A$ such that $x \leq y$ implies $x = y$ or $\pi(x) < \pi(y)$;
3. $(P3)$ if $x \leq y$ and $\pi_B(x) \neq S$, then $x = y$;
4. $(P4)$ $i : [A]^2 \rightarrow A \cup \{\text{undef}\}$ such that for each $\{x, y\} \in [A]^2$ we have

$$\forall a \in A([a \leq x \land a \leq y] \text{ iff } a \leq i\{x, y\});$$
(P5) for each \( \{x, y\} \in [A]^2 \) if \( x \) and \( y \) are \( \preceq \)-incomparable but \( \preceq \)-compatible, then

\[
\pi(i(x, y)) \in f(x, y);
\]

(P6) If \( \{x, y\} \in [A]^2 \) with \( x < y \), and \( \Lambda \in \mathbb{I} \) separates \( x \) from \( y \), then there is \( z \in A \) such that \( x < z < y \) and \( \pi(z) = \Lambda^+ \).

The ordering on \( P \) is the extension: \( \langle A, \preceq, i \rangle \leq \langle A', \preceq', i' \rangle \) iff \( A' \subset A \), \( \preceq' \preceq \preceq, \) \( (A' \times A') \), and \( i' \subset i \).

**Remark.** Property (P5) will be used to prove that \( \mathcal{P} \) satisfies the \( \kappa^+ \)-chain condition. For this, we will use in an essential way that \( \delta < \kappa^+ \) and \( f: [X]^2 \rightarrow [\delta]^{<\kappa} \). Then, if \( R = \langle r_\nu: \nu < \kappa^+ \rangle \) is a subset of \( P \) of size \( \kappa^+ \) with \( r_\nu = \langle A_\nu, \preceq_\nu, i_\nu \rangle \) for \( \nu < \kappa^+ \), by using the assumption that \( \kappa^{<\kappa} = \kappa \), we can assume that \( \{A_\nu: \nu < \kappa^+ \} \) forms a \( \Delta \)-system with kernel \( A_\delta \) and that the conditions \( r_\nu \) \( (\nu < \kappa^+) \) are pairwise isomorphic. Note that if \( \kappa^+ < \delta < \kappa^{++} \), we can not assume that \( A_\delta \) is an initial segment of each \( A_\nu \) for \( \nu < \kappa^+ \). However, since \( |f(x, y)| \leq \kappa \) for all \( \{x, y\} \in [X]^2 \), we can assume by (P5) that if \( x, y \in A_\delta \) with \( x \neq y \) and \( \nu < \mu < \kappa^+ \), we have that \( i_\nu(x, y) = i_\mu(x, y) \). Then, by using the fact that \( \mathcal{F} \) is a \( \kappa^+ \)-strongly unbounded function, we will be able to find two different conditions \( r_\nu \) and \( r_\mu \) in \( R \) that are compatible in \( \mathcal{P} \). To show that \( r_\nu \) and \( r_\mu \) are compatible, we will be able to define the infimum of pairs of elements \( \{x, y\} \) where \( x \in A_\nu \setminus A_\mu \) and \( y \in A_\mu \setminus A_\nu \) by using the properties of trees of intervals and orbits (specially Proposition 2.5). Note that if \( \delta = \kappa^{++} \), we can not define the notion of a basic orbit of an element \( \zeta < \delta \) on a tree of intervals \( \{I_n: n < \omega \} \) where \( I_0 = \{[0, \delta] \} \) in such a way that \( |o(\zeta)| \leq \kappa \).

For \( p \in P \) write \( p = \langle A_p, \preceq_p, i_p \rangle \).

To complete the proof of Theorem 2.4 we will use the following lemmas which will be proved later:

**Lemma 2.7.** \( \mathcal{P} \) is \( \kappa \)-complete.

**Lemma 2.8.** \( \mathcal{P} \) satisfies the \( \kappa^+ \)-c.c.

**Lemma 2.9.**

(a) For all \( x \in X \), the set

\[
D_x = \{q \in P: x \in A_q\}
\]

is dense in \( \mathcal{P} \).

(b) If \( x \in X, \alpha < \pi(x) \) and \( \zeta < \kappa \), then the set

\[
E_{x, \alpha, \zeta} = \{q \in P: x \in A_q \land \exists b \in A_q \cap (\{\alpha\} \times (\kappa \setminus \zeta)) b \preceq_q x\}
\]

is dense in \( \mathcal{P} \).

Since \( \lambda^{<\kappa} = \lambda \), the cardinality of \( P \) is \( \lambda \). Thus, Lemma 2.7 and Lemma 2.8 above guarantee that forcing with \( P \) preserves cardinals and \( 2^\kappa = \lambda \) in the generic extension.

Let \( G \subset P \) be a generic filter. Put \( A = \bigcup \{A_p: p \in G\}, i = \bigcup \{i_p: p \in G\} \) and \( \preceq = \bigcup \{\preceq_p: p \in G\} \). Then \( A = X \) by Lemma 2.9(a).

We claim that \( \langle X, \preceq \rangle \) is a \( (\kappa, \lambda, \delta, \mathcal{L}_\delta^\kappa) \)-poset.

Recall that we put \( X_\zeta = \{\zeta\} \times \kappa \) for \( \zeta \in \delta \setminus \mathcal{L}_\delta^\kappa \) and \( X_\zeta = \{\zeta\} \times \lambda \) for \( \zeta \in \mathcal{L}_\delta^\kappa \). Then the poset \( \langle X, \preceq \rangle \), the partition \( \{X_\zeta: \zeta < \delta\} \), the function \( i \) and \( Y = \delta \times \kappa \) clearly satisfy conditions 2.1(a,b) and 2.2(d,e,f) by the definition of the poset \( \mathcal{P} \).
Finally condition 2.1(c) holds by Lemma 2.9(b).
So to complete the proof of Theorem 2.4 we need to prove Lemmas 2.7, 2.8 and 2.9.
Since \( \kappa \) is regular, Lemma 2.7 clearly holds.

**Proof of Lemma 2.9.** (a) Let \( p \in P \) be arbitrary. We can assume that \( x \notin A_p \).
Let \( A_q = A_p \cup \{ x \} \), \( \preceq_q = \preceq_p \cup \{ (x, x) \} \), and define \( i' \triangleright i \) such that \( i'\{a, x\} = \text{undef} \) for \( a \in A_p \). Then \( q = \langle A_q, \preceq_q, i_q \rangle \in D_3 \) and \( q \leq p \).
(b) Let \( p \in P \) be arbitrary. By (a) we can assume that \( x \in A_p \). Write \( \beta = \pi(x) \).
Let \( m \) be the natural number such that \( I(\alpha, m) = I(\beta, m) \) and \( I(\alpha, m + 1) \neq I(\beta, m + 1) \). We put \( I_k = I(\alpha, k) \) for \( k \geq m + 1 \). Let \( K = \{ \alpha \} \cup \{ \langle i_k^j, k \rangle : m + 1 \leq k < n(\alpha) \} \).
For each \( \gamma \in K \) pick \( b_\gamma \in (\{ \gamma \} \times (\kappa \setminus \zeta)) \setminus A_p \). So \( \pi(b_\gamma) = \gamma \).
Let \( A_q = A_p \cup \{ b_\gamma : \gamma \in K \} \).

\[
\preceq_q = \preceq_p \cup \{ (b_\gamma, b_\gamma') : \gamma, \gamma' \in K, \gamma \leq \gamma' \} \cup \{ (b_\gamma, z) : \gamma \in K, z \in A_p, x \preceq_p z \}.
\]

We let \( i_q(\{y, z\}) = i_q(y, z) \) if \( \{y, z\} \in [A_p]^2 \), \( i_q(b_\gamma, b_\gamma') = b_\gamma \) if \( \gamma, \gamma' \in K \) with \( \gamma < \gamma' \), \( i_q(b_\gamma, z) = b_\gamma \) if \( \gamma \in K \) and \( x \preceq_p z \), and \( i_q(b_\gamma, z) = \text{undef} \) otherwise.

Let \( q = \langle A_q, \preceq_q, i_q \rangle \). Next we check that \( q \in P \). Clearly (P1), (P2), (P3) and (P5) hold for \( q \). (P4) also holds because if \( y \in A_p \) and \( \gamma \in K \) then either \( b_\gamma \preceq_p y \) or they are \( \preceq_q \)-incompatible.

To check (P6) assume that \( b_\gamma \preceq_q y \) and \( \Lambda \) separates \( b_\gamma \) from \( y \). If \( \Lambda^+ < \beta \), then \( z = b_\Lambda^+ \) meets the requirements of (P6). If \( \Lambda^+ = \beta \), we have \( b_\gamma \preceq_q x \preceq_q y \) and \( \pi(x) = \beta \), and so we are done. And if \( \Lambda^+ > \beta \), we apply condition (P6) for \( p \), and so there is \( z \in A_p \) such that \( x \preceq_p z \preceq_p y \) and \( \pi(z) = \Lambda^+ \), and hence \( b_\gamma \preceq_q z \preceq_q y \).

By the construction, \( q \leq p \).
Finally \( q \in E_{x, \alpha, \zeta} \) because \( b_\alpha \in A_q \cap (\{ \alpha \} \times (\kappa \setminus \zeta)) \) and \( b_\alpha \preceq_q x \). \( \square \)

The rest of the paper is devoted to the proof of Lemma 2.8.

**Proof of Lemma 2.8.** Assume that \( \langle r_\nu : \nu < \kappa^+ \rangle \subset P \) with \( r_\nu \neq r_\mu \) for \( \nu < \mu < \kappa^+ \).

In the first part of the proof, till Claim 2.16, we will find \( \nu < \mu < \kappa^+ \) such that \( r_\nu \) and \( r_\mu \) are twins in a strong sense, and \( r_\nu \) and \( r_\mu \) form a good pair (see Definition 2.15). Then, in the second part of the proof, we will show that if \( \{ r_\nu, r_\mu \} \) is a good pair, then \( r_\nu \) and \( r_\mu \) are compatible in \( P \).

Write \( r_\nu = \langle A_\nu, \preceq_\nu, i_\nu \rangle \) and \( A_\nu = \{ x_\nu,i : i < \sigma_\nu \} \).

Since we are assuming that \( \kappa^\kappa = \kappa \), by thinning out \( \langle r_\nu : \nu < \kappa^+ \rangle \) by means of standard combinatorial arguments, we can assume the following:

(A) \( \sigma_\nu = \sigma \) for each \( \nu < \kappa^+ \).
(B) \( \{ A_\nu : \nu < \kappa^+ \} \) forms a \( \Delta \)-system with kernel \( A_\Delta \).
(C) For each \( \nu < \mu < \kappa^+ \) there is an isomorphism \( h_{\nu, \mu} : (A_\nu, \preceq_\nu, i_\nu) \rightarrow (A_\mu, \preceq_\mu, i_\mu) \) such that for every \( i, j < \sigma \) the following holds:
  (a) \( h_{\nu, \mu} \upharpoonright A_\Delta = \text{id} \),
  (b) \( h_{\nu, \mu}(x_\nu,i) = x_\mu,i \),
  (c) \( \pi_B(x_\nu,i) = \pi_B(x_\mu,i) \) iff \( \pi_B(x_\nu,i) = \pi_B(x_\mu,j) \),
  (d) \( \pi_B(x_\nu,i) = S \) iff \( \pi_B(x_\mu,i) = S \),
  (e) if \( \{ x_\nu,i, x_\nu,j \} \in [A_\Delta]^2 \) then \( x_\nu,i = x_\mu,i, x_\nu,j = x_\mu,j \) and \( i_\nu(x_\nu,i, x_\nu,j) = i_\mu(x_\mu,i, x_\mu,j) \),
  (f) \( \pi(x_\nu,i) \in o(x_\nu,j) \) iff \( \pi(x_\mu,i) \in o(x_\mu,j) \),
  (g) \( \pi(x_\nu,i) \in \pi(x_\nu,j) \) iff \( \pi(x_\mu,i) \in \pi(x_\mu,j) \),
  (h) \( \pi(x_\nu,i) \in o^*(x_\nu,j) \) iff \( \pi(x_\mu,i) \in o^*(x_\mu,j) \),
(i) \( \pi(x_{\nu,k}) \in f\{x_{\nu,i}, x_{\nu,j}\} \) iff \( \pi(x_{\nu,k}) \in f\{x_{\mu,i}, x_{\mu,j}\} \).

(j) \( \text{cf}(\pi(x_{\nu,i})) = \kappa^+ \) iff \( \text{cf}(\pi(x_{\mu,i})) = \kappa^+ \).

Note that in order to obtain (C)(c) we use condition (P5) and the fact that \( |f\{x, y\}| \leq \kappa \) for all \( x \neq y \).

Also, we may assume the following:

(D) There is a partition \( \sigma = K \cup F \cup D \cup M \) such that for each \( \nu < \mu < \kappa^+ \):

(a) \( \forall i \in K \) \( x_{\nu,i} \in A_\delta \) and so \( x_{\nu,i} = x_{\mu,i} \). \( A_\delta = \{x_{\nu,i} : i \in K\} \).

(b) \( \forall i \in F \) \( x_{\nu,i} \neq x_{\mu,i} \) but \( \pi_B(x_{\nu,i}) = \pi_B(x_{\mu,i}) \neq S \).

(c) \( \forall i \in D \) \( x_{\nu,i} \notin A_\delta \), \( \pi_B(x_{\nu,i}) = S \) and \( \pi(x_{\nu,i}) \neq \pi(x_{\mu,i}) \).

(d) \( \forall i \in M \) \( \pi_B(x_{\nu,i}) \neq S \) and \( \pi(x_{\nu,i}) \neq \pi(x_{\mu,i}) \).

(E) If \( \pi(x_{\nu,i}) = \pi(x_{\nu,j}) \) then \( \{i, j\} \in [K \cup F]^2 \cup [D \cup M]^2 \).

By [3, Corollary 17.5], if \( \sigma < \kappa = \kappa^{<\kappa} \) then the following partition relation holds:

\[ \kappa^+ \rightarrow (\kappa^+,(\omega)_\sigma)^2. \]

(i.e. given any function \( c : [\kappa^+]^2 \rightarrow 1 + \rho \) either there is a set \( A \in [\kappa^+]^{\kappa^+} \) such that \( c''[A]^2 = \{0\} \), or for some \( \xi < \sigma \) there is a set \( B \in [\kappa^+]^\omega \) such that \( c''[B]^2 = \{1 + \xi\} \).

Hence we can assume:

(F) \( \pi(x_{\nu,i}) \leq \pi(x_{\mu,i}) \) for each \( i \in \sigma \) and \( \nu < \mu < \kappa^+ \).

For \( i \in \sigma \) let

\[ \delta_i = \begin{cases} \pi(x_{\nu,i}) & \text{if } i \in K \cup F, \\ \sup\{\pi(x_{\nu,i}) : \nu < \kappa^+\} & \text{if } i \in D \cup M. \end{cases} \]

**Claim 2.10.** (a) If \( i \in D \cup M \), then the sequence \( \{\pi(x_{\nu,i}) : \nu < \kappa^+\} \) is strictly increasing, \( \text{cf}(\delta_i) = \kappa^+ \) and \( \sup(J(\delta_i)) = \delta_i \). Moreover for every \( \nu < \kappa^+ \) we have \( \pi(x_{\nu,i}) < \delta_i \).

(b) If \( \{i, j\} \in [M]^2 \) and \( x_{\nu,i} \leq x_{\nu,j} \), then \( x_{\nu,i} = x_{\nu,j} \).

**Proof.** If \( i \in D \cup M \), then (F) and (D)(c-d) imply that the sequence \( \{\pi(x_{\nu,i}) : \nu < \kappa^+\} \) is strictly increasing. Hence \( \text{cf}(\delta_i) = \kappa^+ \) and \( \pi(x_{\nu,i}) < \delta_i \) for \( i \in D \cup M \).

Thus Proposition 2.5 implies \( \sup(J(\delta_i)) = \delta_i \). So (a) holds.

(D)(d) and condition (P3) imply (b). □

We put

\[ Z_0 = \{\delta_i : i \in \sigma\}. \]

Since \( \pi''A_\delta = \{\delta_i : i \in K\} \) we have \( \pi''A_\delta \subset Z_0 \). Then, we define \( Z \) as the closure of \( Z_0 \) with respect to \( I \):

\[ Z = Z_0 \cup \{I^+ : I \in I, I \cap Z_0 \neq \emptyset\}. \]

Observe that

\[ |Z| < \kappa. \]
By Claim 2.10(a), the sequence \( (\pi(x_{\nu,i}) : \nu < \kappa^+) \) is strictly increasing for \( i \in D \cup M \). Since \( |Z| < \kappa \), and \( |o^*(x_{\nu,k})| \leq \kappa \) for \( x_{\nu,k} \in B_S \cap A_\Delta \), we can assume that

\[(G) \; \pi(x_{\nu,i}) \notin o^*(x_{\nu,k}) \text{ for } x_{\nu,k} \in B_S \cap A_\Delta \text{ and } i \in D \cup M.\]

Our aim is to prove that there are \( \nu < \mu < \kappa^+ \) such that the forcing conditions \( r_\nu \) and \( r_\mu \) are compatible. However, since we are dealing with infinite forcing conditions, we will need to add new elements to \( A_\nu \cup A_\mu \) in order to be able to define the infimum of pairs of elements \( \{x,y\} \) where \( x \in A_\nu \setminus A_\mu \) and \( y \in A_\mu \setminus A_\nu \). The following definitions will be useful to provide the room we need to insert the required new elements.

Let

\[ \sigma_1 = \{ i \in \sigma \setminus K : \text{cf}(\delta_i) = \kappa \} \]

and

\[ \sigma_2 = \{ i \in \sigma \setminus K : \text{cf}(\delta_i) = \kappa^+ \} \]

Assume that \( i \in \sigma_1 \cup \sigma_2 \). Let

\[ \xi_i = \min \{ \zeta \in \text{cf}(\delta_i) : \epsilon^{J(\delta_i)}_\zeta > \sup(\delta_i \cap Z) \} \]

Since \( |Z| < \kappa \leq \text{cf}(\delta_i) \), the ordinal \( \xi_i \) is defined and \( \delta_i > \epsilon^{J(\delta_i)}_{\xi_i} \).

Then, if \( i \in \sigma_1 \) we put

\[ \gamma(\delta_i) = \epsilon^{J(\delta_i)}_{\xi_i} \text{ and } \gamma(\delta_i) = \delta_i, \]

and if \( i \in \sigma_2 \) we put

\[ \gamma(\delta_i) = \epsilon^{J(\delta_i)}_{\xi_i} \text{ and } \gamma(\delta_i) = \epsilon^{J(\delta_i)}_{\xi_i + \kappa}. \]

For \( i \in \sigma_2 \), since \( \gamma(\delta_i) < \delta_i \) and \( \delta_i = \lim \{ \pi(x_{\nu,i}) : \nu < \kappa^+ \} \) by Claim 2.10(a) for all \( i \in D \cup M \), we can assume that

\[(H) \; \pi(x_{\nu,i}) \in J(\delta_i) \setminus \gamma(\delta_i), \text{ and so } \pi(x_{\nu,i}) \notin Z, \text{ for all } i \in D \cup M.\]

We will use the following fundamental facts.

**Claim 2.11.** If \( x_{\nu,i} \preceq_\nu x_{\nu,j} \) then \( \delta_i \leq \delta_j \).

**Proof.** \( x_{\nu,i} \preceq_\nu x_{\nu,j} \) implies \( \pi(x_{\nu,i}) \leq \pi(x_{\nu,j}) \) by \( (P2) \). \( \square \)

**Claim 2.12.** Assume \( i, j \in \sigma \). If \( x_{\nu,i} \preceq_\nu x_{\nu,j} \) then either \( \delta_i = \delta_j \) or there is \( a \in A_\Delta \cap B_S \) with \( x_{\nu,i} \preceq_\nu a \preceq_\nu x_{\nu,j} \).

**Proof.** Assume that \( i, j \notin K \) and \( \delta_i \neq \delta_j \). By Claim 2.11, we have \( \delta_i < \delta_j \). Since \( i \in F \cup M \) and \( x_{\nu,i} \preceq_\nu x_{\nu,j} \) imply \( x_{\nu,i} = x_{\nu,j} \) and so \( \delta_i = \delta_j \), we have that \( i \in D \), and so \( \pi(x_{\nu,j}) < \pi(x_{\nu,i}) \), \( \text{cf}(\delta_i) = \kappa^+ \) and \( J(\delta_i)^+ = \delta_i \) by Proposition 2.5.

Since \( \delta_i < \delta_j \), we have \( \delta_i < \gamma(\delta_j) < \pi(x_{\nu,j}) \) by \( (H) \), and so \( J(\delta_i) \) separates \( x_{\nu,i} \) from \( x_{\nu,j} \). By \( (P6) \), we infer that there is an \( a = x_{\nu,k} \in A_\nu \) such that \( \pi(a) = \delta_i \) and \( x_{\nu,i} \preceq_\nu a \preceq_\nu x_{\nu,j} \).

Since \( x_{\nu,k} \neq x_{\nu,j} \), we have \( x_{\nu,k} \in B_S \), and so \( k \in K \cup D \). But as \( \pi(x_{\nu,k}) = \delta_i \in Z \) we obtain \( k \notin D \) by \( (H) \), and so \( k \in K \), which implies \( a = x_{\nu,k} \in A_\Delta \cap B_S \). \( \square \)
Claim 2.13. If \( x_{\nu,i} \in A_\delta \cap B_S \) and \( x_{\nu,j} \in A_\nu \) are compatible but incomparable in \( r_\nu \), then \( x_{\nu,k} = i_\nu \{ x_{\nu,i}, x_{\nu,j} \} \in A_\delta \cap B_S \).

Proof. First, (P2) implies \( x_{\nu,k} \in B_S \).

Since \( \pi(x_{\nu,k}) = \pi(i_\nu \{ x_{\nu,i}, x_{\nu,j} \}) \in \{ x_{\nu,i}, x_{\nu,j} \} = o^*(x_{\nu,i}) \cap o^*(x_{\nu,j}) \subseteq o^*(x_{\nu,i}) \) by (P5), and \( x_{\nu,i} \in A_\delta \cap B_S \), we have \( k \notin D \cup M \) by (G). Thus \( k \in K \), and so \( x_{\nu,k} \in A_\delta \).

Hence \( x_{\nu,k} = i_\nu \{ x_{\nu,i}, x_{\nu,j} \} \in A_\delta \cap B_S \). \( \square \)

Claim 2.14. Assume that \( x_{\nu,i} \) and \( x_{\nu,j} \) are compatible but incomparable in \( r_\nu \). Let \( x_{\nu,k} = i_\nu \{ x_{\nu,i}, x_{\nu,j} \} \). Then either \( x_{\nu,k} \in A_\delta \) or \( \delta_i = \delta_j = \delta_k \).

Proof. If \( \delta_k \neq \delta_i \), we infer that there is \( b \in A_\delta \cap B_S \) with \( x_{\nu,k} \leq_\nu b \leq_\nu x_{\nu,i} \) by Claim 2.12. So \( x_{\nu,k} = i_\nu \{ b, x_{\nu,j} \} \) and thus \( x_{\nu,k} \in A_\delta \) by using Claim 2.13.

Similarly, \( \delta_k \neq \delta_j \) implies \( x_{\nu,k} \in A_\delta \). \( \square \)

Definition 2.15. \( \{ r_\nu, r_\mu \} \) is a good pair

iff the following holds:

(a) for all \( i \in F \) with \( \text{cf}(\delta_i) = \kappa^+ \) we have

\[
f\{x_{\nu,i}, x_{\mu,i}\} \supset \delta(i) \cap \gamma(i),
\]

(b) for all \( \{i,j\} \subseteq \delta \) with \( \delta_i = \delta_j \) and \( \text{cf}(\delta_i) = \kappa^+ \) we have

\[
f\{x_{\nu,i}, x_{\mu,j}\} \supset \sigma(i) \cap \gamma(i).
\]

Claim 2.16. There are \( \nu < \mu < \kappa^+ \) such that the pair \( \{ r_\nu, r_\mu \} \) is good.

Proof. Let

\[
\vartheta = \sup\{ \xi + \kappa : \ell \in \sigma_2 \cap F \}.
\]

Since \( F \) is a \( \kappa^+ \)-strongly unbounded function on \( \lambda \) we can find \( \nu < \mu < \kappa^+ \) such that

for all \( i \in F \) we have

\[
F\{ \rho(x_{\nu,i}), \rho(x_{\mu,i}) \} \geq \vartheta,
\]

and for all \( \{i,j\} \subseteq \delta \) with \( \delta_i = \delta_j \) and \( \text{cf}(\delta_i) = \kappa^+ \) we have

\[
F\{ \rho(x_{\nu,i}), \rho(x_{\mu,j}) \} \geq \vartheta.
\]

Hence, \( \{ r_\nu, r_\mu \} \) is good. \( \square \)

To finish the proof of Lemma 2.8 we will show that

If \( \{ r_\nu, r_\mu \} \) is a good pair, then \( r_\nu \) and \( r_\mu \) are compatible. (f)

So, assume that \( \{ r_\nu, r_\mu \} \) is a good pair.

Write \( \delta_{x_{\nu,i}} = \delta_{x_{\mu,i}} = \delta_i \).

If \( s = x_{\nu,i} \) write \( s \in K \) iff \( i \in K \). Define \( s \in F \), \( s \in M \), \( s \in D \) similarly.
In order to amalgamate conditions \( r_\nu \) and \( r_\mu \), we will use a refinement of the notion of amalgamation given in [8, Definition 2.4].

Let \( A' = \{ x_{\nu,i} : i \in F \cup D \cup M \} \). For \( x \in (A_\nu \setminus A_\mu) \cup (A_\mu \setminus A_\nu) \) define the twin \( x' \) of \( x \) in a natural way: \( x' = h_{\nu,\mu}(x) \) for \( x \in A_\nu \setminus A_\mu \), and \( x' = h_{\nu,\mu}^{-1}(x) \) for \( x \in A_\mu \setminus A_\nu \).

Let \( \text{rk} : (A', \preceq_\nu) \to \theta \) be an order-preserving injective function for some ordinal \( \theta < \kappa \), and for \( x \in A' \) let

\[
\beta_x = e^{\delta_x + \text{rk}(x)}.
\]

Since \( \text{cf}(\gamma(\delta_x)) = \kappa \) and \( |A'| < \kappa \) we have

\[
\beta_x \in (\overline{\gamma}(\delta_x) \cap \gamma(\delta_x)) \setminus \{ \beta_z : \text{rk}(z) < \text{rk}(x) \}.
\]

For \( x \in A' \) let

\[
y_x = (\beta_x, 0),
\]

and put

\[
Y = \{ y_x : x \in A' \}.
\]

So, for every \( x \in A' \), \( y_x \in B_S \) with \( \pi(y_x) < \pi(x) \).

Define the functions \( g : Y \to A_\nu \) and \( \bar{g} : Y \to A_\mu \) as follows:

\[
g(y_x) = x \quad \text{and} \quad \bar{g}(y_x) = x',
\]

where \( x' \) is the “twin” of \( x \) in \( A_\mu \).

Now, we are ready to start to define the common extension \( r = \langle A, \preceq, i \rangle \) of \( r_\nu \) and \( r_\mu \). First, we define the universe \( A \) as

\[
A = A_\nu \cup A_\mu \cup Y.
\]

Clearly, \( A \) satisfies (P1). Now, our purpose is to define \( \preceq \).

Extend the definition of \( g \) as follows: \( g : A \to A_\nu \) is a function,

\[
g(x) = \begin{cases} 
  x & \text{if } x \in A_\nu, \\
  x' & \text{if } x \in A_\mu \setminus A_\nu, \\
  s & \text{if } x = y_s \text{ for some } s \in A'.
\end{cases}
\]

We introduce two relations on \( A_p \cup A_q \cup Y \) as follows:

\[
\preceq^{R1} = \{ (y, x) \in Y \times A : g(y) \preceq_\nu g(x) \},
\]

\[
\preceq^{R2} = \{ (x, z) \in A \times A : \exists a \in A_\delta \ g(x) \preceq_\nu a \preceq_\nu g(z) \}.
\]

Then, we put

\[
\preceq = \preceq_\nu \cup \preceq_\mu \cup \preceq^{R1} \cup \preceq^{R2}.
\]

The following claim is well-known and straightforward.
Claim 2.17. $\leq_{\nu, \mu} = \leq | (A_\nu \cup A_\mu)$ is the partial order on $A_\nu \cup A_\mu$ generated by $\leq_\nu \cup \leq_\mu$.

The following straightforward claim will be used several times in our arguments.

Claim 2.18. If $x \leq z$ then $g(x) \leq_\nu g(z)$.

Sublemma 2.19. $\leq$ is a partial order on $A_\nu \cup A_\mu \cup Y$.

**Proof.** We should check that $\leq_\nu$ is transitive, because it is trivially reflexive and antisymmetric.

So let $s \leq t \leq u$. We should show that $s \leq u$.

Since $x \leq z$ implies $g(x) \leq_\nu g(z)$, we have $g(s) \leq_\nu g(t) \leq_\nu g(u)$ and so

$$g(s) \leq_\nu g(u).$$

If $(s, u) \in (Y \times A) \cup (A_\nu \times A_\nu) \cup (A_\mu \times A_\mu)$, then (*) implies $s \leq R_1 u$ or $s \leq_\nu u$ or $s \leq_\mu u$, which implies $s \leq u$ by (★).

So we can assume that $s \in A_\nu$ (the case $s \in A_\mu$ is similar), and so $u \in Y$ or $u \in A_\mu$.

**Case 1.** $u \in A_\mu$.

If $t \in A_\nu \cup A_\mu$, then $s \leq_{\nu, \mu} t \leq_{\nu, \mu} u$, and so $s \leq_{\nu, \mu} u$ by Claim 2.17. So $s \leq u$.

Assume that $t \in Y$. Then $s \leq R_2 t$, and so there is $a \in A_\delta$ such that $g(s) \leq_\nu a \leq_\nu g(t)$. Since $t \leq u$ implies $g(t) \leq_\nu g(u)$, we have $g(s) \leq_\nu a \leq_\nu g(u)$, and so $s \leq R_2 u$. Thus $s \leq u$.

**Case 2.** $u \in Y$.

If $t \in Y$, then $s \leq R_2 t$, and so there is $a \in A_\delta$ such that $g(s) \leq_\nu a \leq_\nu g(t)$. Since $t \leq u$ implies $g(t) \leq_\nu g(u)$, we have $g(s) \leq_\nu a \leq_\nu g(u)$, and so $s \leq R_2 u$. Thus $s \leq u$.

Assume that $t \in A_\nu \cup A_\mu$. Then $t \leq R_2 u$, and so there is $a \in A_\delta$ such that $g(t) \leq_\nu a \leq_\nu g(u)$. Then $g(s) \leq_\nu a \leq_\nu g(u)$, and so $s \leq R_2 u$. Thus $s \leq u$. □

So, by the previous Sublemma 2.19 and by the construction, (P2) and (P3) hold for $\leq$.

Next define the function $i : [A]^2 \to A \cup \{\text{undef}\}$ as follows:

$$i = i_\nu \cup i_\mu,$$

and for $\{s, t\} \in [A]^2 \setminus ([A_\nu]^2 \cup [A_\mu]^2)$ such that $s$ and $t$ are $\leq$-compatible, put $i\{s, t\} = i\{s, y_s\} = i\{t, y_s\} = y_s$ if $s \in A'$ and $t = s'$, and otherwise consider the element

$$v = i_\nu \{g(s), g(t)\},$$

and let

$$i\{s, t\} = \begin{cases} v & \text{if } v \in A_\delta, \\ y_v & \text{if } v \notin A_\delta. \end{cases}$$

Let

$$i\{s, t\} = \text{undef}.$$
if $s$ and $t$ are not $\leq$-compatible.

If $s$ and $t$ are compatible, then so are $g(s)$ and $g(t)$ because $x \leq y$ implies $g(x) \leq g(y)$ by Claim 2.18. Moreover $i_\nu\{s,t\} = i_\mu\{s,t\}$ for $\{s,t\} \in [A_\Delta]^2$ by condition (C)(e), so the definition above is meaningful, and gives a function $i$.

**Claim 2.20.** If $v \in A_\Delta$ and $s \in A$, then $\pi(v) \in o^*(g(s))$ iff $\pi(v) \in o^*(s)$.

**Proof.** If $s \in A_\nu \cup A_\mu$ then $g(s) = s$ or $g(s) = s'$, and so $\pi(v) \in o^*(g(s))$ iff $\pi(v) \in o^*(s)$ by (C)(b) and (C)(h).

Consider now the case $s = y_x \in Y$. Then $\pi(s) \in E(J(\delta_x)) \cap [\gamma(\delta_x), \gamma(\delta_x))$, and so

$$o^*(s) = o(\pi(s)) = \bigcup \{E(I) : I \in \mathbb{I}, I^- < \pi(s) < I^+\} \cap \pi(s) = \bigcup \{E(I) : I \in \mathbb{I}, J(\delta_x) \subset I\} \cap \pi(s).$$

We distinguish the following two cases.

**Case 1.** $\pi(x) < \delta_x$.

If $x \in B_S$ then $\gamma(\delta_x) < \pi(x) < \delta_x$ by (H), and so

$$o^*(x) \cap \pi(s) = o(\pi(x)) \cap \pi(s) = \bigcup \{E(I) : J(\delta_x) \subset I\} \cap \pi(s) = o^*(s).$$

If $x \notin B_S$ then $x \in M$ and $\gamma(\delta_x) < \pi(x) < \delta_x$ by (H), and so

$$o^*(x) \cap \pi(s) = \mathbb{O}(\pi(x)) \cap \pi(s) = \left(\bigcup \{E(I) : J(\delta_x) \subset I\} \cup E(J(\pi(x)))\right) \cap \pi(s) = \bigcup \{E(I) : J(\delta_x) \subset I\} \cap \pi(s) = o^*(s).$$

**Case 2.** $\pi(x) = \delta_x$.

Then $x \in F$ and so

$$o^*(x) = \mathbb{O}(\pi(x)) = o(x) \cup (E(J(\delta_x)) \cap \delta_x) = \left(\bigcup \{E(I) : I^- < \pi(x) < I^+\} \cup E(J(\delta_x))\right) \cap \pi(x) = \bigcup \{E(I) : J(\delta_x) \subset I\} \cap \pi(x),$$

so $o^*(s) = o^*(x) \cap \pi(s)$.

So in both cases $o^*(s) = o^*(x) \cap \pi(s)$. Also, note that as $v \in A_\Delta$, we have that $\pi(v) \notin (\gamma(\delta_x), \delta_x)$, and hence if $v \in o^*(g(s))$ then $\pi(v) < \pi(s)$. So, $\pi(v) \in o^*(x) = o^*(g(s))$ iff $\pi(v) \in o^*(s)$. □

**Claim 2.21.** If $\{s,t\} \in [A]^2$, $v \in A_\Delta$ and $\pi(v) \in f\{g(s), g(t)\}$ then $\pi(v) \in f\{s, t\}$.

**Proof.** We should distinguish two cases.

**Case 1.** $f\{g(s), g(t)\} = o^*(g(s)) \cap o^*(g(t))$.

As $\pi(v) \in f\{g(s), g(t)\}$, we have $\pi(v) \in o^*(g(s)) \cap o^*(g(t))$. Since $\pi(v) \in o^*(g(s))$ implies $\pi(v) \in o^*(s)$ and $\pi(v) \in o^*(g(t))$ implies $\pi(v) \in o^*(t)$ by Claim 2.20, we have $\pi(v) \in o^*(s) \cap o^*(t) = f\{s, t\}$.

**Case 2.** $f\{g(s), g(t)\} = o(g(s)) \cup \{\varepsilon_{\pi(g(s))} : \zeta < F\{\rho(g(s)), \rho(g(t))\}\}$. 
So $\pi_B(g(s)) = \pi_B(g(t)) \neq S$ and $cf(\pi(g(s))) = \kappa^+$. We can assume that $s \in A_\nu \setminus A_\mu$ and $t \in A_\mu \setminus A_\nu$. If $g(s) \in M$, then $g(t) \in M$ by (E). Then as $[\gamma(\delta_{g(s)}) \cup (J(\delta_{g(s)}))^+ \cap \pi''A_\delta = \emptyset$, we infer that $\pi(v) \in o(g(s)) = o(g(t))$, and thus $\pi(v) \in o(s) \cap o(t) \subset \varphi\{s, t\}$. Now assume that $g(s), g(t) \in F$. So $s, t \in F$, and $\delta' = \delta_{g(s)} = \delta_{g(t)}$ has cofinality $\kappa^+$. So,

$$
\pi(v) \in \varphi\{g(s), g(t)\} = o(\delta') \cup \{\varepsilon_\zeta : \zeta < F\{\rho(g(s)), \rho(g(t))\}\}. \quad (\triangle)
$$

Since $\pi''A_\delta \cap (\gamma(\delta'), \delta') = \emptyset$, $(\triangle)$ implies

$$
\pi(v) \in \overline{o}(\delta') \cap \gamma(\delta').
$$

But, by $(\Delta)$

$$
\overline{o}(\delta') \cap \gamma(\delta') \subset \varphi\{s, t\},
$$

and so $\pi(v) \in \varphi\{s, t\}$. □

Sublemma 2.22. $(A, \leq, i)$ satisfies $(P4)$ and $(P5)$.

Proof. Let $\{s, t\} \in [A]^2$ be a pair of $\leq$-incomparable and $\leq$-compatible elements. We distinguish the following cases.

Case 1. $\{s, t\} \in [A_\nu]^2$. (The case $\{s, t\} \in [A_\mu]^2$ is similar)

Since $\leq_\nu \subseteq \leq$, we have $i_\nu\{s, t\} \leq s, t$, so to check (P4) we should show that $x \leq s, t$ implies $x \leq i_\nu\{s, t\}$. We can assume that $x \notin A_\nu$.

If $x \in Y$, then $x \leq_{R1} s$ and $x \leq_{R1} t$, i.e. $g(x) \leq_\nu g(s), g(t)$ and so $g(x) \leq_\nu i_\nu\{g(s), g(t)\} = \triangle i_\nu\{s, t\} = g(i_\nu\{s, t\})$, and so $x \leq_{R1} i_\nu\{s, t\}$. Thus $x \leq i_\nu\{s, t\}$.

If $x \notin A_\nu$, then $x \leq_{R2} a$ and $x \leq_{R2} b$, i.e. $g(x) \leq_\nu a \leq_\nu g(s)$ and $g(x) \leq_\nu b \leq_\nu g(t)$ for some $a, b \in A_\delta$. Then $c = i_\nu\{a, b\} \in A_\delta$, and so $g(x) \leq_\nu c \leq_\nu i_\nu\{g(s), g(t)\} = \triangle i_\nu\{s, t\} = g(i_\nu\{s, t\})$, and so $x \leq_{R2} i_\nu\{s, t\}$. Thus $x \leq i_\nu\{s, t\}$.

Finally (P5) holds in Case 1 because $r_\nu$ satisfies (P5).

Case 2. $\{s, t\} \notin [A_\nu]^2 \cup [A_\mu]^2$.

To check (P4) we should prove that $i\{s, t\}$ is the greatest common lower bound of $s$ and $t$ in $(A, \leq)$.

Assume first that $s$ and $t$ are not twins. Note that by Claim 2.18, $g(s)$ and $g(t)$ are $\leq_\nu$-compatible. Write $v = i_\nu\{g(s), g(t)\}$.

Case 2.1. $v \in A_\delta$, and so $i\{s, t\} = v$.

Since $v = g(v) \leq_\nu g(s)$ and $v \in A_\delta$, we have $v \leq_{R2} s$. Similarly $v \leq_{R2} t$. Thus $v$ is a common lower bound of $s$ and $t$.

To check that $v$ is the greatest lower bound of $s, t$ in $(A, \leq)$ let $w \in A$, $w \leq s, t$. Then $g(w) \leq_\nu g(s), g(t)$.

Thus $g(w) \leq_\nu i_\nu\{g(s), g(t)\} = v$.

Since $v \in A_\delta$, $g(w) \leq_\nu v$ implies $w \leq_{R2} v$. Thus $w \leq v$. Thus (P4) holds.

To check (P5) observe that $g(s)$ and $g(t)$ are incomparable in $A_\nu$. Indeed, $g(s) \leq_\nu g(t)$ implies $v = g(s) \in A_\delta$ and so $g(s) \leq_\nu g(t)$ implies $s \leq_{R2} t$, which contradicts our assumption that $s$ and $t$ are $\leq$-incomparable.
Thus, by applying (P5) in $r_ν$, $$\pi (v) \in f \{g(s), g(t)\}.$$ Thus $\pi (v) \in f \{s, t\}$ by Claim 2.21, and so (P5) holds.

**Case 2.2.** $v \notin A_δ$, and so $i \{s, t\} = y_ν$.

First, we show that $δ_ν = δg(s) = δg(t)$. Note that if $g(s)$ and $g(t)$ are $≤_ν$-comparable, then $v = g(s)$ or $v = g(t)$, and we have that $δg(s) = δg(t)$, because otherwise we would infer from Claim 2.12 that $s, t$ are $≤$-comparable, which is impossible.

Now assume that $g(s)$ and $g(t)$ are $≤_ν$-incomparable.

If $δ_ν < δg(s)$, then there is $a ∈ A_δ \cap B_δ$ with $v ≤_ν a ≤_ν g(s)$ by Claim 2.12. Thus $v = i_ν \{a, g(t)\}$ and so $v ∈ A_δ$ by Claim 2.13, which is impossible. Thus $δ_ν = δg(s)$, and similarly $δ_ν = δg(t)$. Hence

$$δg(s) = δg(t) = δ_ν.$$ And we have $$\pi (y_ν) \in E(J(δ_ν)) \cap [\gamma (δ_ν), \gamma (δ_ν)].$$

Then, if $s, t ∈ F$ and $cf(δ_ν) = κ^+$, by condition (▲), we deduce that $E(J(δ_ν)) \cap \gamma (δ_ν) \subset f \{s, t\}$, and so as $\pi (y_ν) < \gamma (δ_ν)$, we have $\pi (y_ν) \in f \{s, t\}$. Otherwise,

$$E(J(δ_ν)) \cap min(\pi (s), \pi (t)) \subset f \{s, t\}.$$ Then as $v = i_ν \{g(s), g(t)\}$, we have $\pi (v) < \pi (g(s)), \pi (g(t))$, hence $\pi (y_ν) < \pi (s), \pi (t)$ and thus $\pi (y_ν) \in f \{s, t\}$.

Thus (P5) holds.

To check (P4) first we show that $y_ν ≤ s, t$. Indeed $g(v) ≤_ν g(s)$ implies $y_ν ≤^{R_1} s$. We obtain $y_ν ≤^{R_1} t$ similarly.

Let $w ≤ s, t$.

Assume first that $δg(w) < δ_ν$. Since $w ≤ s, t$ we have $g(w) ≤_ν g(s), g(t)$ by Claim 2.18 and hence $g(w) ≤_ν i_ν \{g(s), g(t)\} = v$. By Claim 2.12 there is $a ∈ A_δ$ such that $g(w) ≤_ν a ≤_ν v$. Thus $w ≤^{R_2} y_ν$.

Assume now that $δg(w) = δ_ν$.

Then, we have that $w ∈ Y$. To check this fact, assume on the contrary that $w ∈ A_ν \cup A_μ$. So, we have $δ_w = δg(w) = δ_ν = δg(s) = δg(t)$. Note that if $s ∈ Y$, then $\pi (s) ∈ [\gamma (δ_w), \gamma (δ_w)]$, which contradicts the assumption that $w ≤ s$. So $s /∈ Y$, and analogously $t /∈ Y$.

Assume that $w ∈ A_ν$. If $s ∈ A_μ$, as $w ≤ s$ there is $b ∈ A_δ$ such that $w ≤ b ≤ s$, which is impossible because $\pi (w) > \gamma (δ_w) = γ(δ_ν)$ and $[\gamma (δ_ν), J(δ_ν)^+) \cap π''A_δ = \emptyset$. Thus $s /∈ A_μ$. And by means of a parallel argument, we can show that $t /∈ A_μ$. So $s, t ∈ A_ν$, which was excluded. Analogously, $w ∈ A_μ$ implies $s, t ∈ A_μ$.

Therefore, $w = y_z$ for some $z ∈ A'$. Then $z ≤_ν g(s) and z ≤_ν g(t)$, and so $z ≤_ν i_ν \{g(s), g(t)\} = v$. Thus $y_z ≤^{R_1} y_ν$.

Now, assume that $s$ and $t$ are twins. So $t = s'$ and $i \{s, s'\} = y_s$. If $s ∈ F$ and $cf(π(s)) = κ^+$, we have that $π(y_s) ∈ π(δ_s) \cap γ(δ_s) \subset f \{s, s'\}$ by (▼). Otherwise, $π(y_s) ∈ π^*(π(s)) \cap π^*(π(s')) = f \{s, s'\}$. Thus (P5) holds. To check (P4), it is clear that $y_s < s, s'$. So, assume that $w < s, s'$. If $w = y_u ∈ Y$, then as $w < s$ we infer that $w ≤ s, and thus $w ≤ y_s$. Now, suppose that $w ∈ A_ν \cup A_μ$. Then, there is $b ∈ A_δ$ such that either $w ≤ b ≤ s or w ≤ b ≤ s'$. In both cases, we have $w ≤ y_s$.

So we proved Sublemma 2.22. □
Sublemma 2.23. \((A, \leq, i)\) satisfies (P6).

Proof. Assume that \(\{s, t\} \in [A]^2\), \(s \leq t\) and \(\Lambda\) separates \(s\) from \(t\), i.e.,

\[\Lambda^- < \pi(s) < \Lambda^+ < \pi(t)\]

We should find \(v \in A\) such that \(s \leq v \leq t\) and \(\pi(v) = \Lambda^+\).

Note that since \(s \leq t\), we have \(\delta_{g(s)} \leq \delta_{g(t)}\) by Claim 2.11.

We can assume that \(\{s, t\} \notin [A_{\nu}]^2 \cup [A_{\mu}]^2\) because \(r_{\nu}\) and \(r_{\mu}\) satisfy (P6).

We distinguish the following cases.

Case 1. \(\delta_{g(s)} < \delta_{g(t)}\).

As \(g(s) \leq_{\nu} g(t)\), there is \(a \in A_{\Lambda} \cap B_{\delta}\) with \(g(s) \leq_{\nu} a \leq_{\nu} g(t)\) by Claim 2.12.

Case 1.1. \(\pi(a) \in \Lambda\).

Thus \(\Lambda\) separates \(a\) from \(g(t)\).

Applying (P6) in \(r_{\nu}\) for \(a\) and \(g(t)\) and \(\Lambda\) we obtain \(b \in A_{\nu}\) such that \(a \leq_{\nu} b \leq_{\nu} g(t)\) and \(\pi(b) = \Lambda^+\).

Note that as \(\pi(a) \in \Lambda, a \in A_{\Lambda}\) and \(\pi(b) = \Lambda^+\), we have that \(\pi(b) \in Z\). Thus \(b \in A_{\Lambda}\) by \((H)\).

Thus \(g(s) \leq_{\nu} b \leq_{\nu} g(t)\) implies \(s \leq_{R^2} b \leq_{R^2} t\), and so \(s \leq b \leq t\).

Case 1.2. \(\pi(a) \notin \Lambda\).

If \(\Lambda^+ = \pi(a)\), then we are done because \(g(s) \leq_{\nu} a \leq_{\nu} g(t)\) implies \(s \leq a \leq t\).

So we can assume that \(\Lambda^+ < \pi(a)\).

Since \(r_{\nu}\) and \(r_{\mu}\) satisfy (P6) and \(\Lambda\) separates \(s\) from \(a\), we can assume that \(s \notin A_{\nu} \cup A_{\mu}\).

Hence \(s = y_{g(s)}\) and \(\Lambda\) separates \(g(s)\) from \(a\) because \(\pi(s) \in J(\delta_{g(s)}) \subset \Lambda\). (If \(\Lambda \subsetneq J(\delta_{g(s)})\), then \(\Lambda^- < \pi(s) < \Lambda^+\) is not possible.)

Thus there is \(b \in A_{\nu}\) such that \(g(s) \leq_{\nu} b \leq_{\nu} a\) and \(\pi(b) = \Lambda^+\).

Since \(\delta_{g(s)} \in Z_0\), we have \(\pi(b) \in Z\), and so \(b \in A_{\Lambda}\) by \((H)\).

Thus \(s = y_{g(s)} \leq_{R^1} b \leq_{R^2} t\), and so \(s \leq b \leq t\).

Case 2. \(\delta_{g(s)} = \delta_{g(t)}\).

We will see that this case is not possible.

Case 2.1. \(s \in A_{\nu}\).

Note that if \(t \in A_{\mu}\), then since \(s \leq t\) there is \(b \in A_{\Lambda}\) such that \(s \leq b \leq t\), which is impossible because \(\pi(s) > \gamma(\delta_s)\) and \(\gamma(\delta_s) = J(\delta_s^+) \cap \pi^\prime A_{\Lambda} = \emptyset\). Thus \(t \notin A_{\mu}\).

Since \(s \in A_{\nu}\), \(s \leq t\) and \(\delta_s = \delta_{g(t)}\) we have \(t \notin Y\), and so \(t \in A_{\nu}\), which was excluded.

By means of a similar argument, we can show that \(s \in A_{\mu}\) is also impossible.

Case 2.2. \(s = y_{g(s)}\).

Then \(\pi(s) \in E(J(\delta_{g(s)}))\) and so \(\Lambda^- < \pi(s) < \Lambda^+\) implies \(J(\delta_{g(s)}) \subset \Lambda\). But then \(\pi(t) \leq \Lambda^+\), so \(\Lambda\) cannot separate \(s\) from \(t\).

Thus (P6) holds.

So we proved Sublemma 2.23. \(\square\)
Thus we proved that $r$ is a common extension of $r_\nu$ and $r_\mu$. This completes the proof of Lemma 2.8, i.e. $\mathcal{P}$ satisfies $\kappa^+\text{-c.c.}$.

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