Client Selection in Nonconvex Federated Learning: Improved Convergence Analysis for Optimal Unbiased Sampling Strategy

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Abstract

Federated learning (FL) is a distributed machine learning paradigm that selects a subset of clients to participate in training to reduce communication burdens. However, partial client participation in FL causes objective inconsistency, which can hinder the convergence, while this objective inconsistency has not been analyzed in existing studies on sampling methods. To tackle this issue, we propose an improved analysis method that focuses on the convergence behavior of the practical participated client’s objective. Moreover, based on our convergence analysis, we give a novel unbiased sampling strategy, i.e., FedSRC-D, whose sampling probability is proportional to the client’s gradient diversity and local variance. FedSRC-D is provable the optimal unbiased sampling in non-convex settings for non-IID FL with respect to the given bounds. Specifically, FedSRC-D achieves $O\left(\frac{G^2}{\epsilon^2} + \frac{1}{\sqrt{T}}\right)$ higher than SOTA convergence rate of FedAvg, and $O\left(\frac{G^2}{\epsilon^2}\right)$ higher than other unbiased sampling methods. We corroborate our results with experiments on both synthetic and real data sets.

1 Introduction

Federated Learning (FL) has recently emerged as a critical distributed learning paradigm where a number of clients collaborate with a central server to train a model. Edge clients finish the update locally without any data sharing, thus preserving client privacy. Communication is the primary bottleneck of FL since edge devices have limited bandwidth and connection availability (Wang et al., 2021). In practice, only a subset of clients will be selected for training to lessen the communication burden. However, such a uniform sampling strategy, as adapted by standard approaches like FedAvg (McMahan et al., 2017) in each communication round, would result in objective inconsistency under the non-iid setting. Clients selection remains a direct challenge in FL with partial client participation. Considering the crucial unbiased client sampling that may preserve optimization objective, only a few strategies are proposed, e.g. in terms of multinomial distribution(MD) sampling and cluster sampling. Among kinds of sampling methods, importance sampling (IS) shows its superiority due to the unbiased and variance reduction property (Elvira and Martino, 2021; Alain et al., 2015). From original gradient-norm-based sampling to adaptive importance sampling, IS has found its way into distributed learning (Zhao and Zhang, 2014; Stich et al., 2017). When it is introduced in FL, there will inevitably be some challenges to overcome. On the one hand, can the application of IS in...
When most practical applications of FL is under nonconvex settings (Kairouz et al., 2019; Wang et al., 2020; Rizk et al., 2020), it is not apparent what convergence rate can achieve in nonconvex FL, especially when local updates are heterogeneous. Indeed, the standard practice for FL with IS (i.e., FedAvg) is to optimize toward the global ensemble average, which differs from the global update. This update inconsistency causes update gap, resulting in objective inconsistency. We use the term surrogate objective to describe each round’s practical objective and the drift of update direction (gradient) is defined as update gap: $E[\|\nabla F_{1,2}(x_t) - \nabla F_{\text{global}}(x_t)\|^2]$, where $\nabla F_{1,2}(x_t) = \frac{1}{n} (\nabla F_1(x_t) + \nabla F_2(x_t))$.

A few works (Chen et al., 2020; Rizk et al., 2020) try to apply IS on FL in convex/strongly convex setting. Despite some encouraging results on reducing the variance and achieving a better convergence rate (than FedAvg with uniform sampling), this line of work still falls short of answering the aforementioned questions. Specifically, they 1) derive the convergence result under the convex setting, while most practical applications of FL is under nonconvex settings (Karouz et al., 2019; Wang et al., 2021), and 2) remain the sampling probability to be proportional to the gradient norm, which cannot capture FL’s unique property like the correlations between the gradient of clients and server, or client’s local variance. Moreover, in FL, data presented on different clients are typically highly heterogeneous (non-iid-ness), causing drift in each client’s updates, and resulting in slow and unstable convergence (Karimireddy et al., 2020)—the adverse effects of data heterogeneity will be further exacerbated by partial client participation. The optimal client selection technique for mitigating the detrimental effect induced by partial client participation with heterogeneous data is still a non-trivial open challenge in FL.

To tackle these challenges, in this work, we conduct the following in our effort: 1) prove the tight convergence when applying IS in nonconvex FL, 2) identify objective inconsistency caused by partial client participation, and rigorously analyze its impact on convergence, and 3) design a sampling strategy that can select representative clients in FL. Based on our result of 1), we also derive the sampling strategy, FedSRC-G, whose sampling probability is proportional to the updated gradient norm. For 2), in addition to the detailed discussion in Section 4.3, we also provide Fig. 1 for an overview of objective inconsistency, surrogate objective, and update inconsistency. Specifically, the surrogate objective is the weighted sum of the participating clients’ objective, whose update $\nabla F_{\text{global}}(x_t)$ would not be the ideal $\nabla F_{\text{global}}(x_t)$. We use the term surrogate objective to describe each round’s practical objective and the drift of update direction (gradient) is defined as update gap: $E[\|\nabla F_{1,2}(x_t) - \nabla F_{\text{global}}(x_t)\|^2]$, where $\nabla F_{1,2}(x_t) = \frac{1}{n} (\nabla F_1(x_t) + \nabla F_2(x_t))$.

1.1 Contributions

From FedAvg to FedSRC-G and then to FedSRC-D. Despite IS has been applied in FL (Chen et al., 2020; Rizk et al., 2020), it is not apparent what convergence rate can achieve in nonconvex FL, especially when local updates are heterogeneous. Indeed, the standard practice for FL with IS
is to apply the gradient norm based sampling strategy (FedSRC-G). On the other hand, FedSRC-D is motivated by the fact that standard FL with IS only considers the size of the gradient norm but ignores the correlation between the global gradient and local gradient and ignores that the practical participated clients’ surrogate objective also has its own variance like $\sigma_L$. Therefore we propose FedSRC-D.

- We present an unbiased sampling-based algorithm for heterogeneous FL, termed Selective Representative Client for FL (FedSRC). This algorithm is a general framework for arbitrary unbiased client sampling for FL. Particularly, we provide two unbiased sampling strategies, e.g., FedSRC-G and FedSRC-D.
- We propose a novel analysis method to analyze FedSRC and capture the objective inconsistency of partial client participation FL. This is the first theoretical study, to our knowledge, that identifies the objective inconsistency caused by partial client participation, provides the fundamental understanding of partial client objective inconsistency, and solves objective inconsistency by client sampling.
- Based on the improved convergence analysis, we propose a novel unbiased sampling strategy, i.e., FedSRC-D. The sampling probability is proportional to the client’s gradient diversity and local variance. Specifically, FedSRC-D is provable the best sampling in non-convex settings for non-IID FL that achieves $O\left(\frac{G^2}{\epsilon^2} + \frac{1}{\epsilon^2}\right)$ higher than SOTA convergence rate of FedAvg results, and $O\left(\frac{G^2}{\epsilon^2}\right)$ higher than any other unbiased sampling methods including existing IS methods.
- We also derive the nonconvex FL convergence rate based on the global objective convergence analysis without considering the convergence behavior of surrogate objective and update gap as in FedSRC-D. Thus we propose another sampling strategy, FedSRC-G. Its sampling probability is proportional to the updated gradient norm. To the best of our knowledge, this is the first nonconvex FL convergence rate with gradient-norm-based IS. We show FedSRC-G achieves $O\left(\epsilon^{-\frac{2}{3}}\right)$ higher convergence rate than that of SOTA FedAvg.
- We examine the effectiveness of our proposed sampling strategies and compare them with other client sampling strategies in a range of synthetic and real datasets. We also do experiments to show that our sampling strategy is compatible with momentum and also variance reduction methods like adding proximal term (Li et al., 2018).

2 Related Work

FedAvg is first proposed by McMahan et al. (2017) as a de facto algorithm of FL, in which multiple local SGD steps are executed on the available clients to alleviate the communication bottleneck. While communication efficiency, heterogeneity, such as system heterogeneity (Li et al., 2018; Wang et al., 2020; Mitra et al., 2021; Diao et al., 2020), and statistical/objective heterogeneity (Lin et al., 2020; Karimireddy et al., 2020b; Li et al., 2018; Wang et al., 2020; Guo et al., 2021), results in inconsistent optimization objectives and drifted clients models, impeding federated optimization considerably.

Objective inconsistency in FL. Objective inconsistency is not rare in FL due to the heterogeneity of clients’ data and the difference in computing ability. For instance, Wang et al. (2020) first identifies an objective inconsistency caused by heterogeneous local updates. There also exist several works that encounter the difficulty from the objective inconsistency caused by partial client participation (Li et al., 2019; Cho et al., 2020; Balakrishnan et al., 2021). Li et al. (2019); Cho et al. (2020) realized local-global gap $f^*-\frac{1}{m}\sum_{i=1}^{m} F_i^*$ to measure the distance between global optimum and average of all local personal optimum, where the local-global gap results from objective inconsistency at the final optimal point. In fact, objective inconsistency occurs in each training round, not only at the final optimal point. Balakrishnan et al. (2021) also encounters objective inconsistency caused by partial client participation. However, they use $\left\|\frac{1}{n}\sum_{i=1}^{n} \nabla F_i(x_t) − \nabla f(x_t)\right\| \leq \epsilon$ as an assumption to describe such update inconsistency caused by objective inconsistency without any analysis on it. So far, the objective inconsistency caused by partial client participation has not been analyzed though it is prevalent in FL, even in homogeneous local updates. Our work gives the first fundamental convergence analysis on the influence of the objective inconsistency of partial client participation. Specifically, we first analyze the surrogate objective’s convergence, then extend it to the global objective’s convergence by bound the update gap (cf. Section 4.3).
Client selection in FL. In general, the sampling method can be divided into biased and unbiased sampling. Note that unbiased sampling guarantees the same expected value of the client aggregation as the global deterministic aggregation with all clients’ participation. In contrast, biased sampling will lead to converging to sub-optimal. The two most common unbiased sampling strategy in FL is uniform sampling and multinomial sampling (MD), the latter samples according to client data ratio (Wang et al., 2020) [Fraboni et al., 2021]. Many biased sampling strategies have been proposed for accelerating training and reducing communication rounds, such as sampling clients with higher loss (Cho et al., 2020), sampling clients as many as possible under the limitation of threshold (Qu et al., 2021), sampling clients with larger updates (Ribero and Vikalo, 2020) and sampling clients according to client diversity (Balakrishnan et al., 2021). However, all these biased sampling methods can exacerbate the negative effects of objective inconsistency and promise to converge to only a neighbor of optimum. Recently, cluster-based client selection has drawn some attention in FL (Fraboni et al., 2021) [Xu et al., 2021] [Muhammad et al., 2020]). Fraboni et al. (2021) applies cluster method in client selection of FL while omitting the effect of local variance of clients. Notice that cluster-based FL algorithms are different from our setting. The proposed FedSRC-D in Algorithm 1 is a mutated version of this diverse client cluster algorithm and with much less computational complexity as cluster-based methods need additional computation overhead.

3 Preliminaries

Notations: Script $i$ represents the $i_{th}$ client, script $t$ represents the $t_{th}$ training round, and script $k$ represents the $k_{th}$ local epoch. $x_{i,k}^{t}$ and $\xi_{i,k}^{t}$ represent the model parameter and the local dataset at local epoch $k$ of client $i$ in round $t$, respectively.

Global objective formulation. The standard FL solves a sum-structured optimization problem:

$$ f^* = \min_{x \in \mathcal{R}^d} \left[ f(x) := \sum_{i=1}^{m} p_i F_i(x) \right], \quad (1) $$

where $F_i(x) = E_{\xi_i \sim D_i}[F_i(x; \xi_i)]$ represents the local objective function of client $i$ with data distribution $D_i$. $m$ is the total number of clients and $p_i$ represents the weight of client $i$. With partial client participation, FedAvg (McMahan et al., 2017) randomly selects $|S_t| = n$ clients ($n \leq m$) to communicate and update model. Then the loss function of actual participating users in each round can be expressed as:

$$ f(x_t) = \frac{1}{n} \sum_{i \in S_t} F_i(x_t). \quad (2) $$

IS property. IS uses an simple proposal distribution $p(x)$ to draw a sample, and attaches it with a set of importance weights that are proportional to the probability ratio $q(x)/p(x)$. In this way, the expectation is guaranteed to stay the same: $E_{q(x)}[F_i(x)] = E_{p(x)}[q(x)/p(x)F_i(x)]$.

According to the Monte Carlo method, when $q(x)$ follows uniform distribution, we can estimate $E_{q(x)}[F_i(x)]$ by $1/m \sum_{i=1}^{m} F_i(x)$ and $E_{p(x)}[q(x)/p(x)F_i(x)]$ by $1/n \sum_{i \in S_t} 1/mp_i F_i(x)$, respectively. $m$ and $n$ are sample sizes, $|S_t| = n$ is sampled from the sampling distribution $p(x)$.

Surrogate objective FL formulation. According to IS property, we formulate a surrogate objective function for an arbitrary unbiased sampling strategy:

$$ \tilde{f}(x_t) = \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} F_i(x_t). $$

where $m$ is the total number of clients, $|S_t| = n$ is the number of participating clients in each round, and $p_i$ is the probability that user $i$ is selected at round $t$. We will analyze the convergence behavior
of this surrogate objective in Section 4.3. This convergence result makes it possible to use sampling to reduce the variance introduced by objective inconsistency.

Objective inconsistency of partial client participation. In each training round, as shown in Figure 1(a), the global objective of FL is to optimize toward the direction with full clients participation:

\[ x_{t+1} = \arg\min_{x_t} F_t(x_t) = \arg\min_{x_t} \frac{1}{m} \sum_{i=1}^{m} F_i(x_t). \] (4)

While with partial client participation, it converge toward the optimum of surrogate objective:

\[ x_{t+1} = \arg\min_{x_t} \tilde{f}(x_t) = \arg\min_{x_t} \frac{1}{m} \sum_{i \in S_t} \frac{n_i}{n_{tp}} F_i(x_t). \] (5)

Due to converging to a different objective, there can be an arbitrary large gap in each round’s update between the global and surrogate objective. This is called update inconsistency, and the gap is defined as update gap, as shown in Fig 1(b). We postpone to Section 4.3 to analyze the impact of objective inconsistency on convergence.

Algorithm description. Algorithm 1 is a general framework that enables the representation of any unbiased sampling FL algorithm. A subset of clients is selected in each round with sampling distribution \( p_i^t \). The server communicates the global model to the selected clients. The selected clients perform \( K \) local SGD updates in the form: \( x_{t+1} = x_{t} + \eta_t \nabla F_t(x_t), \). Then, the selected clients communicate the updates \( \Delta_i^t = x_{K,t} - x_{t} \), and the server aggregates the updates for parameter updating \( x_{t+1} = x_t + \frac{1}{K} \sum_{i \in S_t} \frac{n_i}{n_{tp}} \Delta_i^t \). FedSRC-G and FedSRC-D are two sampling strategies which will be introduced in Section 4.2.1 and Section 4.3.1. In Section 4.6, we show that both FedSRC-D and FedSRC-G can be used efficiently and does not need to obtain all clients’ gradient in each round.

4 Theoretical analysis

We first state the key assumptions used for theoretical analysis. In Section 4.2, we give the SOTA convergence rate of nonconvex FL with IS and then propose FedSRC-G by minimizing the convergence variance \( \Phi \). In Section 4.3, we propose an improved analysis method to analyze the convergence behavior of surrogate objective and global objective. Then we propose FedSRC-D by minimizing the convergence variance \( \Phi \).

4.1 Assumptions

To ease the theoretical analysis of FedSRC, we use the following widely used assumptions.

Assumption 1 (L-Smooth). The client’s local objective function is Lipschitz smooth, i.e., there is a constant \( L > 0 \), such that

\[ \| \nabla F_i(x) - \nabla F_i(y) \| \leq L \| x - y \|, \forall x, y \in \mathbb{R}^d, \text{ and } i = 1, 2, \ldots, m. \]
Assumption 2 (Unbiased Local Gradient Estimator and local variance). Let $\xi^i_t$ be a random local data sample in the $i_{th}$ round at $t_{th}$ client: $E[\nabla F_i(x_t, \xi^i_t)] = \nabla F_i(x_t), \forall i \in [m]$, where the expectation is over the local datasets sample. The function $F_i$ have $\sigma_{L,i} > 0$ bounded local variance, i.e., $E[\|\nabla F_i(x_t, \xi^i_t) - \nabla F_i(x_t)\|^2] = \sigma_{L,i}^2 \leq \sigma_L^2$.

Assumption 3 (Bound Dissimilarity). There exists constant $\sigma_G \geq 0$ and $A > 0$ s.t. $E[\|\nabla f_i(x)\|^2] \leq (A^2 + 1)\|\nabla f_i(x)\|^2 + \sigma_G^2$. When all local loss functions are identical, $A^2 = 0$ and $\sigma_G = 0$.

Assumption 4 (Gradient bound). The stochastic gradient’s expected squared norm is uniformly bounded, i.e., $E[\|\nabla F_i(x_{t,k}, \xi_{i,k})\|^2] \leq G^2$ for all $i$.

The above assumptions are commonly used in both non-convex optimization and FL literature, see (Karimireddy et al., 2020; Yang et al., 2021; Koloskova et al., 2020; Wang et al., 2020; Cho et al., 2020; Li et al., 2019). For Assumption 3 if all local loss functions are identical, then we have $A = 0$ and $\sigma_G = 0$.

4.2 Convergence rate of global objective non-convex FL with IS

Theorem 4.1 (Convergence rate of global objective). Under Assumptions 2–3, the output of any sampling strategy that follows Algorithm 1 will converge to a stationary point of the global objective, where the gradient is 0. More specifically, for a pre-defined communication rounds $T$ and setting learning rate $\eta$ and $\eta_L$ to $\eta_L = \frac{1}{\sqrt{TKL}}$ and $\eta = \sqrt{Kn}$, the expected error will be bounded as follows:

$$\min_{t \in [T]} E[\|\nabla f_i(x_t)\|^2] \leq \frac{c}{\sqrt{mKL}} + \frac{\Phi}{c^2},$$

(6)

where $f^0 = f(x_0)$, $f^* = f(x^*)$, $\Phi$ represents combinations of local variance, global variance, and variance of gradient updates, we name it as combination of variance with the following expression:

$$\Phi = 5\sigma^2_MKL^2 M^2 + mK\sigma^2_L^2 + \frac{LmK}{2m} Var\left(\frac{1}{mp_i} \hat{g}_i\right),$$

(7)

where $M^2 = \sigma_L^2 + 4K\sigma_G^2$. The expectation is over the local datasets samples among workers. $\sigma_G$ is the global gradient variance, and $\sigma_L$ is the local variance. In the next Section 4.3 we will see a twin expression $\bar{\Phi}$ that is derived from the surrogate objective’s convergence. In Table 1, for the sake of consistency with other existing works, we transform convergence rates to the $\epsilon$ error bound.

$Var\left(\frac{1}{mp_i} \hat{g}_i\right)$ represents the update variance, i.e. the variance of gradient updates, and defined as

$$Var\left(\frac{1}{mp_i} \hat{g}_i\right) = E[\frac{1}{mp_i} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i]^2,$$

(8)

where $\hat{g}_i = \sum_{k=1}^{K-1} \hat{g}_{i,k}$ and $\hat{g}_{i,k} = \nabla F_i(x_{i,k}, \xi_{i,k})$. Update variance is a critical part of $\Phi$, and we will get sampling strategy FedSRC-G by minimizing it in the next section.

4.2.1 Gradient-norm-based sampling: FedSRC-G

In this section, we analyze the impact of client sampling on convergence rates, specifically, on $\Phi$. We then propose FedSRC-G, which is consistent with standard SGD with IS but flexible enough to handle heterogeneous local updates.

Recall Theorem 4.1 only update variance in $\Phi$ is affected by sampling probability $p_i$. In other words, we need to minimize the variance term with respect to probability for getting optimal sampling probability. This can be formulated as an optimization problem, similar to the problem in SGD (Zhao and Zhang, 2015):

$$\min_{p_i \in [0,1], \sum_{i=1}^{m} p_i = 1} \sum_{p_i \in [0,1]} Var\left(\frac{1}{mp_i} \hat{g}_i\right).$$

(9)
Corollary 4.2 (Optimal sampling probability for FedSRC-G). Solving the above optimization problem, we give the expression of optimal sampling probability:

\[ p_t^i = \frac{\|\hat{g}_i\|}{\sum_{j=1}^{K} \|g_j\|}, \]  

where \( \hat{g}_i = \sum_{k=0}^{K-1} g_k \) is the gradient updates sum of multiple updates.

Compare FedSRC-G with other FL with IS algorithm. 1) Chen et al. (2020) and Rizk et al. (2020) applied IS in FL to solve a convex/strongly convex problem, while we solve a nonconvex problem. 2) In (Rizk et al., 2020), their analysis result and sampling expression rely on knowing the optimum \( x_* \), while we only need to approximate \( x_* \) at the end of training. 3) In (Chen et al., 2020), all clients need to participate in local training. Instead of the server actively sampling, the clients decide to communicate with the server with a probability \( p_t^i \). Instead, the user decides whether to upload with a probability—this client participation pattern could be beyond the server’s capabilities. Nevertheless, in our algorithm, the server in each round can select a fixed number of clients to participate. 4) Our sampling strategy can be easily extended to heterogeneous local updates by changing \( \sum_{k=0}^{K-1} g_k \) to \( \sum_{k=0}^{K_i-1} g_k \) which other does not mention this.

Convergence rate of FedSRC-G. With sampling strategy FedSRC-G and assumption \( [\chi] \) update variance can be bounded by \( KG^2 \). Therefore we can conclude the optimal convergence rate under the FedSRC-G sampling strategy:

\[ \min_{t \in [T]} E \|\nabla f(x_t)\|^2 \leq O \left( \frac{f^0 - f^*}{\sqrt{nKT}} \right) + O \left( \frac{\sigma^2_L}{\sqrt{nKT}} \right) + O \left( \frac{M^2}{T} \right) + O \left( \frac{KG^2}{nKT} \right). \]  

Compare FedSRC-G with FedAvg. Our convergence is faster than the SOTA partial client participation FedAvg convergence rate (Yang et al., 2021), speeding up from \( O \left( \frac{1}{T} + \frac{1}{\sqrt{nKT}} + \frac{1}{T^{3/2}} \right) \) to \( O \left( \frac{1}{T} + \frac{1}{\sqrt{nKT}} \right) \). By setting \( p = \frac{1}{m} \) in Theorem 4.1, we recover the convergence rate of standard FedAvg with partial client participation.

Remark 4.3 (The improvement space of sampling strategies in FL (via our improved analysis)). FedSRC accelerates the convergence by reducing the variance term. Notice that in \( \Phi \), only \( \text{Var} \left( \frac{1}{mp} g_i \right) \) is affected by sampling probability \( p_i \), and an optimal sampling probability can be attained when minimizing the variance of gradient updates w.r.t \( p_i \) as shown in Section 4.2.

However, there exist other variance terms in \( \Phi \) that will also slow down the convergence in practice. We can further speed up convergence by minimizing the \( \Phi \) to minimal if a connection between sampling probability and other variance in \( \Phi \), specifically, local variance \( \sigma_L \), and global variance \( \sigma_G \), can be established.

4.3 Convergence rate with improved analysis method for getting FedSRC-D

In order to further reduce the variance in \( \Phi \) (cf. 7), i.e., local variance \( \langle \sigma_L \rangle \) and global variance \( \langle \sigma_G \rangle \), we divide the convergence of the global objective into a surrogate objective and an update gap, and analyze each term separately. The convergence result of surrogate objective in Theorem 4.5 makes it possible to further minimize the local variance and global variance with sampling probability. While for the update gap, as inspired by the expression form of update variance (cf. 8), we formally define it as below.

Definition 4.4 (Update gap). In order to measure the update inconsistency, we define the update gap:

\[ \chi_t^2 = E \left[ \left\| \nabla \hat{f}(x_t) - \nabla f(x_t) \right\|^2 \right]. \]  

Here the expectation is over all clients’ distribution. When full clients participate, we have \( \chi_t^2 = 0 \). The update inconsistency exists as long as partial client participation.

The update gap is a direct embodiment of the objective inconsistency in the update process. The existence of update gap makes the analysis of global objective different from the analysis of surrogate
objective. However, once we promise the convergence of the update gap, we can re-derive the convergence result for the global objective. Formally, the update gap can help us to connect global objective convergence and surrogate objective convergence as follows:

\[ 
\mathbb{E}[\|\nabla f(x_t)\|^2] = \mathbb{E}[\|\nabla \tilde{f}(x_t)\|^2] + \chi_t^2 .
\] (13)

The equation follows from the unbiased property of FedSRC, see Lemma A.1. In order to deduce the order to reduce variance, we need to solve the following optimization problem:

\[ 
\min_{t \in [T]} \mathbb{E}[\|\nabla \tilde{f}(x_t)\|^2] \leq \frac{f^0 - f^*}{\eta KL T} + \frac{\Phi}{c'} ,
\] (14)

where \( f^0 = f(x_0) \), \( f^* = f(x_*) \), and \( \Phi \) is the new combination of variance, representing combinations of local variance and client gradient diversity. For sampling without replacement:

\[ 
\Phi = \frac{5L^2 K \eta^2}{2mn} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_{L,i}^2 + 4K \zeta_{G,i}) + \frac{L \eta n}{2n} \sum_{i=1}^{m} \frac{1}{m^2 p_i} \sigma_{L,i}^2 ,
\] (15)

where \( \zeta_{G,i} \) represents client gradient diversity: \( \zeta_{G,i} = \|\nabla F_i(x_t) - \nabla f(x_t)\| \). The proof details of Theorem 4.5 and the exact expression of \( \Phi \) for both w/ and w/o replacement can be found in Appendix C.

**Remark 4.6.** We notice that there is no update variance in \( \Phi \), but the local variance and global variance remain in it. Furthermore, the new combination of variance \( \Phi \) can be minimized by optimizing w.r.t sampling probability, as shown in Section 4.3.1.

**Derive the convergence from surrogate objective to global objective.** Under an unbiased sampling strategy, the update gap promises to converge to zero. As shown in Lemma A.1, unbiased sampling promises partial client updates in expectation are equal to the participation of all clients. With enough training rounds, unbiased sampling can guarantee that the update gap \( \chi_t \) will converge to zero. However, we still need the convergence speed of \( \chi_t \) to recover the convergence rate of the global objective. Fortunately, we can bound the convergence behavior of \( \chi_t \) by the convergence rate of surrogate objective according to Definition 4.4 and Lemma A.2. Therefore, the update gap can achieve at least the same convergence rate as the surrogate objective.

**Corollary 4.7** (New convergence rate of global objective). Under Assumption 1–3 and based on the above analysis that update variance is bounded, the global objective will converge to a stationary point. Its gradient is bounded as:

\[ 
\min_{t \in [T]} \mathbb{E}[\|\nabla f(x_t)\|^2] = \min_{t \in [T]} \mathbb{E}[\|\nabla \tilde{f}(x_t)\|^2 + \mathbb{E}[\chi_t^2]] \leq \min_{t \in [T]} \mathbb{E}[\|\nabla \tilde{f}(x_t)\|^2] \leq \frac{f^0 - f^*}{\eta KL T} + \frac{\Phi}{c'} .
\] (16)

**4.3.1 Diversity-based sampling: FedSRC-D**

FedSRC-D has a different expression than typical IS solutions, in which the sampling probability is not constrained to be proportional to the gradient norm. Based on the convergence result of Corollary 4.7, we notice that different from \( \Phi \), where the sampling probability can only minimize \( \text{Var}(\frac{1}{mp}, \tilde{g}_t) \) to minimal, in \( \Phi \), the sampling probability is related to all variance terms. Therefore, in order to reduce variance, we need to solve the following optimization problem:

\[ 
\min_{p_t} \Phi \quad \text{s.t.} \sum_{l=1}^{m} p_l = 1 ,
\]

where \( \Phi \) is a linear combination of local variance \( \sigma_{L,i} \) and gradient diversity \( \zeta_{G,i} \) (cf. Theorem 4.5).
Corollary 4.8 (Optimal sampling probability for FedSRC-D). Solving the above optimization problem, we can find the optimal sampling probability to be:

$$p_i^* = \frac{\sqrt{\alpha_1 \|\nabla F_i(x) - \nabla f(x)\|^2 + \alpha_2 \sigma_{L,i}^2}}{\sum_{j=1}^m \sqrt{\alpha_1 \|\nabla F_j(x) - \nabla f(x)\|^2 + \alpha_2 \sigma_{L,j}^2}};$$  

(17)

where $\alpha_1$ and $\alpha_2$ are constants defined as $\alpha_1 = 20K^2L\eta_L$ and $\alpha_2 = 5KL\eta_L + \frac{n}{n}$.  

Convergence rate of FedSRC-D. Corollary 4.7 concludes the convergence rate of FedSRC-D:

$$\min_{t \in \mathbb{N}} \mathbb{E}\|\nabla f(x_t)\|^2 \leq O \left( \frac{\lambda + \sqrt{\lambda} + \sqrt{\lambda} + \sqrt{\lambda} + \sqrt{\lambda}}{\sqrt{nKT}} \right) + O \left( \frac{\sigma_L^2}{\sqrt{nKT}} \right) + O \left( \frac{\sigma_G^2 + 4K^2\sigma_G^2}{KT} \right).$$

(18)  

Remark 4.9. Compared with the convergence result given in Section 4.2, our novel theoretical result, Corollary 4.7 has these two main differences: 1) sampling probability is not limited to where $\alpha$ model to gain more knowledge in each round. Specifically, the server will give more weight to clients converge to a stationary point without any gap. Our results can be viewed as a rigorous theoretical explanation for the heuristic diversity sampling algorithm of FL. FedSRC-D motivates the global model to gain more knowledge in each round. Specifically, the server will give more weight to clients.

5 Discussion on FedSRC-D and FedSRC-G

The difference between $\Phi$ and $\tilde{\Phi}$. In $\Phi$, only the term $\text{Var} \left( \frac{1}{m_i p_i} \tilde{g}_i \right)$ contains sampling probability $p_i$. For comparison, in $\tilde{\Phi}$, sampling probability $p_i$ is related to gradient diversity $\zeta_{C,i} = \|\nabla F_i(x_i) - \nabla f(x_i)\|$ and local variance $\sigma_{L,i}$. Thus, optimizing $\Phi$ and $\tilde{\Phi}$ w.r.t $p_i$ leads to two different sampling strategies FedSRC-G and FedSRC-D respectively. FedSRC-D reduces variance terms more efficiently since it can reduce all variance terms in $\tilde{\Phi}$, while in $\Phi$ only last term $\text{Var} \left( \frac{1}{m_i p_i} \tilde{g}_i \right)$ can be optimized w.r.t $p_i$. 

Compare FedSRC with uniform sampling. According to Cauchy-Schwarz inequality, we show FedSRC is at least better than uniform sampling by reducing variance:

$$\frac{\text{Var}_{\text{uniform}}}{\text{Var}_{\text{FedSRC-G}}} = \frac{m \sum_{i=1}^m \|\tilde{g}_i\|^2}{\text{Var}_{\text{FedSRC-D}}} \geq 1, \quad \Phi_{\text{uniform}} = \frac{m \sum_{i=1}^m (\sqrt{\alpha_1 \sigma_{L,i}^2 + \alpha_2 \sigma_{G,i}^2})^2}{\sum_{i=1}^m \sqrt{\alpha_1 \sigma_{L,i}^2 + \alpha_2 \sigma_{G,i}^2}} \geq 1.$$ 

This implies that FedSRC does reduce the variance, and the larger the client number $m$ is, the more effective FedSRC is.

Discussion on $\alpha_1$ and $\alpha_2$. Notice that $K$ primarily dominates the value of $\alpha_1$ and $\alpha_2$. When the number of local updates $K$ is large enough, the gradient diversity term in FedSRC-D becomes more critical, and the influence of local variance decreases. This is consistent with our intuition: More updates indicate that more data was used in training, resulting in a slight local variance. We can assume there is no local variance after enough local updates. Thus, only client diversity will impact the sampling probability.

Difference between FedSRC-D and FedSRC-G. Optimizing different variance terms leads to different sampling results. FedSRC-G aims to reduce the effect of update variance, and FedSRC-D aims to reduce the effect of local variance and client diversity. When sampling clients, as illustrated in Fig1, FedSRC-D samples Client 1 and Client 3 to participate, whereas FedSRC-G selects Client 2 and Client 3. As the figure shows, FedSRC-D obtains a smaller variance than FedSRC-G. FedSRC-D achieves the optimal convergence rate with $O(\frac{C}{\sqrt{T}})$ higher than other unbiased sampling methods like FedSRC-G and MD sampling.

What are the FedSRC-D benefits? Our proposed FedSRC-D can promise the objective to converge to a stationary point without any gap. Our results can be viewed as a rigorous theoretical explanation for the heuristic diversity sampling algorithm of FL. FedSRC-D motivates the global model to gain more knowledge in each round. Specifically, the server will give more weight to clients.
(a) $\nu = 20$
(b) $\nu = 30$
(c) $\nu = 40$

Figure 2: Performance of different algorithms on the regression model. The loss is calculated by $f(x, y) = \left\| y - \log\left(\frac{(Ax - b)^2}{2}\right) \right\|^2$. We report the logarithm of global loss with different degrees of gradient noise $\nu$. All methods are well-tuned, and we report the best result of each algorithm under each setting.

with larger diversity and local variance. These clients provide more useful data for the model to learn. Thus these clients are more representative, and sampling them can assist reduce the training round by avoiding redundant sampling clients.

6 Practical implementation for FedSRC.

Gradient-norm-based IS method requires the computation of the full gradient in each iteration (Elvira and Martino, 2021; Zhao and Zhang, 2015). However, obtaining each client’s gradient in advance is generally inadmissible in FL. It is necessary to give a practical algorithm for applying IS in FL. For practical purposes, a series of important sampling algorithms estimate the current round’s gradient by the historical gradient (Cho et al., 2020; Katharopoulos and Fleuret, 2017). Similarly, we utilize the gradient from the previous training iteration to estimate the gradient of the current round, where the previous iteration refers to the one in which the client participates. By using this approximation method, we can save computing resources (Rizk et al., 2020).

In particular, at iteration 0, all probabilities are set to $\frac{1}{m}$, then during the $i_{th}$ iteration, after the participating clients $i \in S_t$ send the server their updated gradients, the sampling probabilities are updated as: $p^*_{i,t+1} = \frac{\|g^*_i\|}{\sum_{i \in S_t} \|g^*_i\|} (1 - \sum_{i \in S_t} p^*_{i,t})$, where the multiplicative factor follows from ensuring all the probabilities sum to 1. Specifically, we use the average of the latest participated clients’ gradients to approximate the true gradient of the global model for FedSRC-D. In this way, it is not necessary to obtain all clients’ gradients in each round. The validity of the practical algorithm is verified in the next section.

7 Experiments

In this section, we use synthetic dataset to demonstrate our theoretical results. To show the validity of the practical algorithm, we run experiments on real dataset: FEMNIST and CIFAR-10, and show that under the same communication round, our sampling method can achieve higher accuracy than other unbiased sampling strategies.

Synthetic datasets. We firstly examine our theoretical results through logistic regression on synthetic datasets. In details, we randomly generate $(x, y)$ by $y = \log(\frac{(Ax - b)^2}{2})$ with given $A_i$ and $b_i$ as training data for 20 clients, and each client’s local dataset contain 1000 samples. To simulate the gradient noise, in each training step, we calculate the gradient of client $i$ by $g_i = \nabla f_i(A_i, b_i, D_i) + \nu_i$, where $A_i$ and $b_i$ are model parameters, $D_i$ is the local dataset of client $i$, and $\nu_i$ is a zero-mean random variable which control the heterogeneity of client $i$. The larger the $\mathbb{E}\|\nu_i\|^2$, the larger the heterogeneity of client $i$.

Figure 3 demonstrates that these empirical results are align with our theoretical analysis. Additional experiments of different functions can be found in Appendix E. In detail, • FedSRC outperforms other biased and unbiased methods in convergence speed. We can see FedSRC convergence faster than both FedAvg and Power-of-choice sampling. The larger the noise...
Table 2: **Performance of algorithms.** We run 500 communication rounds on FEMNIST and CIFAR10 for each algorithm. We report the mean of maximum 5 accuracies for test datasets and the number of communication rounds to reach the threshold accuracy.

| Algorithm          | FEMNIST | CIFAR10 |
|--------------------|---------|---------|
|                    | Acc (%) | Rounds for 70% | Acc (%) | rounds for 54% |
| FedAvg (w/ uniform sampling) | 70.35  | 426      | 54.28  | 338      |
| FedSRC-G           | 71.69   | 404      | 55.05  | 313      |
| FedSRC-D           | 72.10   | 322      | 55.20  | 303      |

(variance), the more obvious the convergence speed advantage of FedSRC. For $\nu = 30$, FedSRC can achieve near twice faster than FedAvg, and for $\nu = 40$, FedSRC-D can achieve nearly $4 \times$ times faster than FedAvg.

- **FedSRC-D outperforms FedSRC-G.** In experiments, FedSRC-D converges about twice faster as FedSRC-G in Figure 2(a). As all results show, FedSRC-D can reduce more variance than FedSRC-G and thus get a smaller loss.

- **Difference between unbiased and biased sampling.** As is shown in Fig 2(a) biased sampling: power of choice will converge to sub-optimum even in $\nu = 20$, away from the optimum of FedSRC-D and FedSRC-G.

**FEMNIST and CIFAR-10.** We also verify our practical algorithm on real dataset FEMNIST and CIFAR-10. We summarize our numerical results in Table 2: FedSRC-D has better accuracy than FedSRC-G, while FedSRC-D and FedSRC-G both outperform FedAvg with the same communication round. To reach the threshold accuracy, FedSRC-D needs fewer communication rounds. For example, FedSRC-D reduces the number of communication rounds required by FedAvg by a quarter in FEMNIST. In the meantime, FedSRC-D reduces more than 30 communication rounds to reach the threshold compared with FedAvg. This demonstrates the practicality of our method.

Besides, we show our sampling strategies work with the momentum and proximal methods. We also experiment with different numbers of participants $n$ and different levels of heterogeneity alpha and observe the consistent improvement of FedSRC-D. The detailed setting and additional experiments are in Appendix E.
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A Techniques

Here we show some technical lemmas which are helpful in the theoretical proof. We substitute $\frac{1}{m}$ for $\frac{n}{N}$ to simplify writing in all following proofs. $\frac{n_i}{N}$ is the data ratio of client $i$. All our proof can be easily extended from $f(x_i) = \frac{1}{m} \sum_{i=1}^{m} F_i(x_i)$ to $f(x_i) = \sum_{i=1}^{m} \frac{n_i}{N} F_i(x_i)$.

**Lemma A.1.** (Unbiased Sampling). Importance sampling is unbiased sampling. $E(\frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \nabla F_i(x_i)) = \frac{1}{m} \sum_{i=1}^{m} \nabla F_i(x_i)$, no matter whether the sampling is with replacement or without replacement.

Lemma A.1 proves that the importance sampling is an unbiased sampling strategy, either in sampling with replacement or sampling without replacement.

**Proof.** with replacement:

$$E \left( \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \nabla F_i(x_i) \right) = \frac{1}{n} \sum_{i \in S_t} E \left( \frac{1}{mp_i} \nabla F_i(x_i) \right) = \frac{1}{n} \sum_{i \in S_t} E \left( E \left( \frac{1}{mp_i} \nabla F_i(x_i) \mid M \right) \right)$$

$$= \frac{1}{n} \sum_{i \in S_t} E \left( \sum_{l=1}^{m} p_l \frac{1}{mp_l} \nabla F_l(x_i) \right) = \frac{1}{n} \sum_{i \in S_t} \nabla f(x_i) = \nabla f(x_i),$$  
(19)

without replacement:

$$E \left( \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \nabla F_i(x_i) \right) = \frac{1}{n} \sum_{l=1}^{m} E \left( \frac{1}{mp_l} \nabla F_l(x_i) \right) = \frac{1}{n} \sum_{l=1}^{m} E(\mathbb{I}_m) \times E \left( \frac{1}{mp_l} \nabla F_l(x_i) \right)$$

$$= \frac{1}{n} \sum_{l=1}^{m} np_l \times \frac{1}{mp_l} \nabla f(x_l) = \frac{1}{m} \sum_{l=1}^{m} \nabla f(x_i) = \nabla f(x_i),$$  
(20)

where $\mathbb{I}_m \triangleq \begin{cases} 1 & \text{if } x_l \in S_t, \\ 0 & \text{otherwise}. \end{cases}$

**Lemma A.2** (update gap bound).

$$\chi^2 = E\left[ \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \nabla F_i(x_i) - \frac{1}{m} \frac{1}{mp_i} \nabla f(x_i) \right] = E\left[ \nabla \tilde{f}(x_i) \right]^2 - E\left[ \nabla f(x_i) \right]^2 \leq E\left[ \nabla \tilde{f}(x_i) \right]^2.$$  
(22)

where the first equation follows from $E[x - E(x)]^2 = E[x^2] - [E(x)]^2$ and Lemma A.1.

Following Lemma follows from Lemma 4 of [Reddi et al., 2020], but with a looser condition Assumption 3 instead of $\sigma_0^2$ bound. With some effort, we can derive following lemma:

**Lemma A.3** (Local updates bound.). For any step-size satisfying $\eta_L \leq \frac{1}{8K}$, we can have the following results:

$$E\|x_{t,k} - x_t\|^2 \leq 5K(\eta_L^2 \sigma_L^2 + 4K\eta_L^2 \sigma_0^2) + 20K^2(A^2 + 1)\eta_L^2 \| \nabla f(x_t) \|^2.$$  
(23)
Proof.

\[ E_t \| x_{t,k} - x_t \|^2 \]

\[ = E_t \| x_{t,k-1} - x_t - \eta_t g_{t,k-1} \|^2 \]

\[ = E_t \| x_{t,k} - x_t - \eta_t (g_{t,k-1} - \nabla F(x_{t,k-1}) + \nabla F_i(x_{t,k-1}) - \nabla F_i(x_t) + \nabla F_i(x_t)) \|^2 \]

\[ \leq (1 + \frac{1}{2K-1}) E_t \| x_{t,k-1} - x_t \|^2 + E_t \| \eta_t (g_{t,k-1} - \nabla F_i(x_{t,k})) \|^2 \]

\[ + 4K E_t \| \eta_t (\nabla F_i(x_{t,k-1}) - \nabla F_i(x_t)) \|^2 \]

\[ \leq (1 + \frac{1}{2K-1}) E_t \| x_{t,k-1} - x_t \|^2 + \eta_t^2 \sigma_i^2 + 4K \eta_t^2 L^2 E_t \| x_{t,k-1} - x_t \|^2 \]

\[ + 4K \eta_t^2 \sigma_G^2 + 4K \eta_t^2 (A^2 + 1) \| \nabla f(x_t) \|^2 \]

\[ \leq (1 + \frac{1}{K-1}) E_t \| x_{t,k-1} - x_t \|^2 + \eta_t^2 \sigma_i^2 + 4K \eta_t^2 \sigma_G^2 + 4K (A^2 + 1) \| \eta_t \nabla f(x_t) \|^2. \]  \hspace{1cm} (24)

Unrolling the recursion, we get:

\[ E_t \| x_{t,k} - x_t \|^2 \leq \sum_{p=0}^{k-1} (1 + \frac{1}{K-1})^p \left[ \eta_t^2 \sigma_i^2 + 4K \eta_t^2 \sigma_G^2 + 4K (A^2 + 1) \| \eta_t \nabla f(x_t) \|^2 \right] \]  \hspace{1cm} (26)

\[ \leq (K-1) \left[ (1 + \frac{1}{K-1})^K - 1 \right] \left[ \eta_t^2 \sigma_i^2 + 4K \eta_t^2 \sigma_G^2 + 4K (A^2 + 1) \| \eta_t \nabla f(x_t) \|^2 \right] \]  \hspace{1cm} (27)

\[ \leq 5K (\eta_t^2 \sigma_i^2 + 4K \eta_t^2 \sigma_G^2) + 20K^2 (A^2 + 1) \eta_t^2 \| \nabla f(x_t) \|^2. \]  \hspace{1cm} (28)

\[ \Box \]

B Proof of Theorem 4.1

**Theorem B.1.** (Convergence rate of the global objective of FedSRC) Under Assumptions 1 – 3, any sampling strategy of FL that follows algorithm I will converge to a stationary point of a global objective. More specifically, if communication rounds \( T \) is pre-determined and the learning rate \( \eta \) and \( \eta_L \) are constant learning rates, then the expected error will be bounded as follows:

\[ \min_{t \in [T]} E \| \nabla f(x_t) \|^2 \leq \frac{F}{c \eta_L KT} + \Phi, \]  \hspace{1cm} (29)

where \( F = f(x_0) - f(x_*) \), \( M^2 = \sigma_i^2 + 4K \sigma_G^2 \), and the expectation is over the local datasets samples among workers.

Let \( \eta \) and \( \eta_L \) be chosen as such that \( \eta_L \leq \frac{1}{8K^3} \), \( \eta \leq 1 \) and \( \frac{1}{2} - 10L^2 \frac{1}{m} \sum_{i=1}^{m} K^2 \eta_i^2 (A^2 + 1) > 0. \) It then holds that:

\[ \Phi = \frac{1}{c} \left[ 5 \eta_L^2 L^2 K^2 \sum_{i=1}^{m} (\sigma_i^2 + 4K^2 \sigma_G^2) + \eta_L L \frac{\eta_L}{2m} \sigma_i^2 + \frac{L \eta_L}{2nK} V \left( \frac{1}{m \eta_i} \right) \right]. \]  \hspace{1cm} (30)

**Corollary B.2.** Suppose \( \eta_L \) and \( \eta \) are such that the conditions mentioned above are satisfied. \( \eta_L = \mathcal{O} \left( \frac{1}{\sqrt{K} T} \right) \) and \( \eta = \mathcal{O} \left( \sqrt{K \eta} \right) \). Then for sufficiently large \( T \), the iterates of Theorem 4.1 satisfy:

\[ \min_{t \in [T]} E \| \nabla f(x_t) \|^2 = \mathcal{O} \left( \frac{\sigma_i^2}{\sqrt{nKT}} + \frac{KG^2}{\sqrt{nKT}} + \frac{\sigma_G^2}{K^2} \right). \]  \hspace{1cm} (31)

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Proof.

\[ E_t[f(x_{t+1})] \overset{(a1)}{=} f(x_t) + \langle \nabla f(x_t), E_t[x_{t+1} - x_t] \rangle + \frac{L}{2} E_t[\|x_{t+1} - x_t\|^2] \]

\[ = f(x_t) + \langle \nabla f(x_t), E_t[\eta \Delta_t + \eta \eta L \nabla f(x_t) - \eta \eta L \nabla f(x_t)] \rangle + \frac{L}{2} \eta^2 E_t[\|\Delta_t\|^2] \]

\[ = f(x_t) - \eta \eta L \|\nabla f(x_t)\|^2 + \eta \langle \nabla f(x_t), E_t[\Delta_t + \eta \eta L \nabla f(x_t)] \rangle + \frac{L}{2} \eta^2 E_t[\|\Delta_t\|^2], \]

\[ \tag{32} \]

where (a1) follows from Lipschitz continuous condition.

Firstly we consider \( A_1 \):

\[ A_1 = \langle \nabla f(x_t), E_t[\Delta_t + \eta \eta L \nabla f(x_t)] \rangle \]

\[ = \left\langle \nabla f(x_t), E_t[-\frac{1}{|S_t|} \sum_{i \in S_t} \frac{1}{m p_i} \sum_{k=0}^{K-1} \eta \eta L g_{i,k} + \eta \eta L \nabla f(x_t)] \right\rangle \]

\[ \overset{(a2)}{=} \left\langle \nabla f(x_t), E_t[-\frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \eta \eta L \nabla F_i(x_{i,k}) + \eta \eta L \nabla f(x_t)] \right\rangle \]

\[ = \left\langle \sqrt{\eta \eta L} \nabla f(x_t), -\frac{\sqrt{\eta \eta L}}{K} E_t\left[ \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} (\nabla F_i(x_{i,k}) - \nabla F_i(x_t)) \right] \right\rangle \]

\[ \overset{(a3)}{=} \frac{\eta \eta L K}{2} \|\nabla f(x_t)\|^2 + \frac{\eta \eta L}{2K} E_t \left\| \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} (\nabla F_i(x_{i,k}) - \nabla F_i(x_t)) \right\|^2 \]

\[ - \frac{\eta \eta L}{2K} E_t \left\| \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \nabla F_i(x_{i,k}) \right\|^2 \]

\[ \overset{(a4)}{\leq} \frac{\eta \eta L}{2} \|\nabla f(x_t)\|^2 + \frac{\eta \eta L^2 K}{2m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} E_t \|x_{i,k} - x_t\|^2 - \frac{\eta \eta L}{2K} E_t \left\| \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \nabla F_i(x_{i,k}) \right\|^2 \]

\[ \leq \left( \frac{\eta \eta L}{2} + 10K^3 L^2 \eta L^4 (A^2 + 1) \right) \|\nabla f(x_t)\|^2 + \frac{5L^2 \eta^2 L^4}{2} K^2 \sigma^2 \]

\[ - \frac{\eta \eta L}{2K} E_t \left\| \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \nabla F_i(x_{i,k}) \right\|^2, \]

\[ \tag{33} \]

where (a2) follows from Assumption 2 and Lemma A.1, (a3) is due to \( \langle x, y \rangle = \frac{1}{2} \left[ \|x\|^2 + \|y\|^2 - \|x - y\|^2 \right] \) and (a4) comes from Assumption 4.
Next consider $A_2$. Let $\hat{g}_i = \sum_{k=0}^{K-1} g_{i,k}^*$

$$A_2 = E_t \| \Delta_t \|^2$$

$$= E_t \left\| \eta \frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{p_i}} \sum_{k=0}^{K-1} g_i^* \right\|^2$$

$$= \eta^2 L_n \frac{1}{n} E_t \left\| \frac{1}{m_{p_i}} \sum_{i=1}^{m} \sum_{k=0}^{K-1} g_i^* \right\|^2$$

$$+ \eta^2 L_n \frac{1}{n} \sum_{i=1}^{m} \sum_{k=0}^{K-1} g_i^* (x_i^* - \nabla F_i(x_i^*) + \nabla F_i(x_i^*)) \|^2$$

$$\leq \frac{\eta^2 L_n}{n} V \left( \frac{1}{m_{p_i}} \hat{g}_i \right)$$

$$+ \frac{\eta^2 L_n}{m_{p_i}} \sum_{i=1}^{m} \sum_{k=0}^{K-1} E \left\| g_i (x_i^*) - \nabla F_i(x_i^*) \right\|^2 + \eta^2 L_n \frac{1}{n} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \nabla F_i(x_i^*) \|^2$$

$$\leq \frac{\eta^2 L_n}{n} V \left( \frac{1}{m_{p_i}} \hat{g}_i \right) + \frac{\eta^2 L_n \sigma^2}{m} + \eta^2 L_n \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \nabla F_i(x_i^*) \|^2. \quad (34)$$

The third equality follows from independent sampling. Specifically, for sampling with replacement, due to every index being independent, we utilize $E \| x_1^2 + \ldots + x_n \|^2 = E \| x_1 \|^2 + \ldots + \| x_n \|^2$.

For sampling without replacement:

$$E \left\| \frac{1}{n} \sum_{i \in S_t} \left( \frac{1}{m_{p_i}} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i \right) \right\|^2$$

$$= \frac{1}{n^2} E \left\| \sum_{i=1}^{m} I_i \left( \frac{1}{m_{p_i}} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i \right) \right\|^2$$

$$= \frac{1}{n^2} E \left( \left\| \sum_{i=1}^{m} I_i \left( \frac{1}{m_{p_i}} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i \right) \right\|^2 \mid I_i = 1 \right) \times P(I_i = 1) \quad (37)$$

$$+ \frac{1}{n^2} E \left( \left\| \sum_{i=1}^{m} I_i \left( \frac{1}{m_{p_i}} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i \right) \right\|^2 \mid I_i = 0 \right) \times P(I_i = 0) \quad (38)$$

$$= \frac{1}{n} \sum_{i=1}^{m} p_i \left\| \frac{1}{m_{p_i}} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i \right\|^2 \quad (39)$$

$$= \frac{1}{n} \left\| \frac{1}{m_{p_i}} \hat{g}_i - \frac{1}{m} \sum_{i=1}^{m} \hat{g}_i \right\|^2. \quad (40)$$

From above, we observe that it is possible to gain a speedup by sampling from the distribution that minimizes $V \left( \frac{1}{m_{p_i}} \hat{g}_i \right)$. Moreover, as we have discussed before, the optimal sampling probability is $p_i^* = \frac{\| \hat{g}_i \|}{\sum_{i=1}^{m} \| \hat{g}_i \|}$. For MD sampling [Li et al., 2019], which samples according to date ratio, the optimal sampling probability is $p_i^* = \frac{q_i \| g_i \|}{\sum_{i=1}^{m} q_i \| g_i \|}$, where $q_i = \frac{m_i}{N}$.
Now substitute the expression of $A_1$ and $A_2$:

$$
E_t[f(x_{t+1})] \leq f(x_t) - \eta_t L \| \nabla f(x_t) \|_2^2 + \eta_t \langle \nabla f(x_t), E_t[\Delta_t + \eta_t L \nabla f(x_t)] \rangle + \frac{L}{2} \eta_t^2 E_t[\Delta_t]^2
$$

$$
\leq f(x_t) - \eta_t KL \left( \frac{1}{2} - 10L^2K^3\eta_t^2(A^2 + 1) \right) \| \nabla f(x_t) \|_2^2 + \frac{5\eta_t^2 L^2 K^2}{2} (\sigma_L^2 + 4K^2\sigma_G^2)
$$

$$
+ \frac{\eta_t^2 \eta_G^2 K L^2}{2m} \sum_{l=1}^{m} V(\frac{1}{mp_l} \hat{g}_l) - \left( \frac{\eta_t}{2K} - \frac{L\eta_t^2 \eta_G^2}{2} \right) E_t \left\| \frac{1}{m} \sum_{i=1}^{m} \sum_{k=0}^{K-1} \nabla F_t(x_{t,k}) \right\|_2^2
$$

$$
\leq f(x_t) - c \eta_t KL \| \nabla f(x_t) \|_2^2 + \frac{5\eta_t^2 L^2 K^2}{2} (\sigma_L^2 + 4K^2\sigma_G^2) + \frac{\eta_t^2 \eta_G^2 K L^2}{2m} \sum_{l=1}^{m} V(\frac{1}{mp_l} \hat{g}_l),
$$

(41)

where the last inequality follows from $\left( \frac{\eta_t}{2K} - \frac{L\eta_t^2 \eta_G^2}{2} \right) \geq 0$ if $\eta_t \leq \frac{1}{KL}$, and (a9) holds because there exists a constant $c > 0$ (with some $\eta_t$) satisfying $\frac{1}{2} - 10L^2\frac{1}{m} \sum_{i=1}^{m} K^2\eta_i^2(A^2 + 1) > c > 0$.

Rearranging and summing from $t = 0, \ldots, T-1$, we have:

$$
\sum_{t=1}^{T-1} c \eta_t KL E_t \| \nabla f(x_t) \|_2^2 \leq f(x_0) - f(x_T) + T(\eta_t KL) \Phi.
$$

(42)

Which implies:

$$
\min_{t \in [T]} E_t \| \nabla f(x_t) \|_2^2 \leq \frac{f_0 - f^*}{c \eta_t KL T} + \Phi,
$$

(43)

where

$$
\Phi = \frac{1}{c} \left[ \frac{5\eta_t^2 KL^2}{2} (\sigma_L^2 + 4K^2\sigma_G^2) + \frac{\eta_t KL}{2m} \sum_{l=1}^{m} V\left( \frac{1}{mp_l} \hat{g}_l \right) \right].
$$

(44)

C Proof of Theorem 4.5

In this section, we analyze the surrogate objective $\tilde{f}(x)$ defined in 3. To simplify the writing mode, in this section, we only use $\nabla f(x_t)$ instead of $\tilde{f}(x_t)$.

Theorem C.1 (Convergence rate of global and surrogate objective of FedSRC). Under Assumption 7 and the same conditions as theorem 4.4, the minimal gradient norm of surrogate objective will be bounded as follows by setting $\eta_t = \frac{1}{\sqrt{KL}}$ and $\eta \sqrt{Kn}$ Let local and global learning rates $\eta$ and $\eta_t$ satisfy $\eta_t < \frac{1}{\sqrt{30KL}} \sqrt{\sum_{i=1}^{m} \frac{1}{\eta_t^2 \sigma_G^2}}$ and $\eta_t \leq \frac{1}{KL}$. Under Assumption 7 and partial worker participation, the sequence of outputs $x_k$ generated by Algorithm 2 satisfies:

$$
\min_{t \in [T]} E_t \| \nabla f(x_t) \|_2^2 \leq \frac{F}{c \eta_t KL T} + \frac{1}{c} \Phi,
$$

(45)

where $F = f(x_0) - f(x^*)$, and the expectation is over the local dataset samplings among workers. $\zeta_{G,i}$ is defined as client gradient diversity: $\zeta_{G,i} = \| \nabla F_i(x_t) - \nabla f(x_t) \|_2$.

For sample with replacement: $\Phi = \frac{5K^2 L^2 \eta_t^2}{2m^2} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L^2 + 4K^2 \zeta_{G,i}^2) + \frac{L\eta_t \eta_t}{2m} \sum_{i=1}^{m} \frac{1}{m^2 p_i} \sigma_G^2$.

For sampling without replacement: $\Phi = \frac{5K^2 L^2 \eta_t^2}{2mn} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L^2 + 4K^2 \zeta_{G,i}^2) + \frac{L\eta_t \eta_t}{2n} \sum_{i=1}^{m} \frac{1}{m^2 p_i} \sigma_G^2$.
Corollary C.2. Suppose $\eta_L$ and $\eta$ are such that the conditions mentioned above are satisfied. \(\eta_L = O\left(\frac{1}{\sqrt{TKL}}\right)\) and $\eta = O\left(\sqrt{Kn}\right)$. Then for sufficiently large $T$, the iterates of Theorem 4.5 satisfy:

\[
\min_{t \in [T]} E\|\nabla f(x_t)\|^2 \leq O\left(\frac{F}{\sqrt{nKT}}\right) + O\left(\frac{\sigma^2_L}{\sqrt{nKT}}\right) + O\left(\frac{\sigma^2_L}{KT}\right).
\]  

(46)

C.1 Sample with replacement

\[
\min_{t \in [T]} E\|\nabla f(x_t)\|^2 \leq \frac{f_0 - f_*}{c\eta_L KT} + \frac{1}{c} \Phi,
\]  

(47)

where $\Phi = \frac{5L^2K\eta^2}{2m^2} \sum_{l=1}^{m} \frac{1}{p_l} (\sigma^2_L + 4K\zeta_{G,l}^2) + \frac{L^2 n}{2n} \sum_{l=1}^{m} \frac{1}{m_p l} \sigma^2_L$.

Proof.

\[
\mathbb{E}[f(x_{t+1})] \overset{(a)}{=} f(x_t) + \langle \nabla f(x_t), E[x_{t+1} - x_t] \rangle + \frac{L}{2} E_t[\|x_{t+1} - x_t\|^2]
\]

\[
= f(x_t) + \langle \nabla f(x_t), E_t[\eta \Delta_t + \eta \eta_L K \nabla f(x_t) - \eta \eta_L K \nabla f(x_t)] \rangle + \frac{L}{2} \eta^2 E_t[\|\Delta_t\|^2]
\]

\[
= f(x_t) - \eta \eta_L K \|\nabla f(x_t)\|^2 + \eta \langle \nabla f(x_t), E_t[\Delta_t + \eta \eta_L K \nabla f(x_t)] \rangle + \frac{L}{2} \eta^2 E_t[\|\Delta_t\|^2].
\]  

(48)

Where (a) follows from Lipschitz continuous condition.

Firstly consider $A_1$:

\[
A_1 = \langle \nabla f(x_t), E_t[\Delta_t + \eta \eta_L K \nabla f(x_t)] \rangle
\]

\[
= \left\langle \nabla f(x_t), E_t[\frac{1}{|S_t|} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \eta_L g^i_{t,k} + \eta \eta_L K \nabla f(x_t)] \right\rangle
\]

\[
= \left\langle \nabla f(x_t), E_t[\frac{1}{|S_t|} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \eta L \nabla F_i(x_{t,k}^i) + \eta_L K \nabla f(x_t)] \right\rangle
\]

\[
= \left\langle \sqrt{K} \eta L \nabla f(x_t), \frac{\sqrt{\eta L}}{K} E_t[\frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x_{t,k}^i) + K \nabla f(x_t)] \right\rangle
\]

\[
= \frac{K \eta L}{2} \|\nabla f(x_t)\|^2 + \frac{\eta L}{2K} E_t \left\langle \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x_{t,k}^i) + K \nabla f(x_t) \right\|^2
\]

\[
- \frac{\eta L}{2K} E_t \| \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x_{t,k}^i) \|^2,
\]  

(49)

where (a2) follows from Assumption 2 and (a3) is due to $\langle x, y \rangle = \frac{1}{2} \left(\|x\|^2 + \|y\|^2 - \|x - y\|^2\right)$ for $x = \sqrt{K \eta L} \nabla f(x_t)$ and $y = \frac{\sqrt{\eta L}}{K} \left[\frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x_{t,k}^i) + K \nabla f(x_t)\right]$. 

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In order to bound $A_1$, we need to bound the following part:

\[
E_t \| \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) - K \nabla f(x_t) \|^2 \\
= E_t \| \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) - \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x_t) \|^2 \\
\leq \frac{K}{n} \sum_{i \in S_t} \sum_{k=0}^{K-1} E_t \| \frac{1}{mp_i} (\nabla F_i(x^i_{t,k}) - \nabla F_i(x_t)) \|^2 \\
= \frac{K}{n} \sum_{i \in S_t} \sum_{k=0}^{K-1} E_t \{ E_t(\| \frac{1}{mp_i} (\nabla F_i(x^i_{t,k}) - \nabla F_i(x_t)) \|^2 | M) \} \\
= \frac{K}{n} \sum_{i \in S_t} \sum_{k=0}^{K-1} E_t \{ \frac{1}{m^2p_i} \| \nabla F_i(x^i_{t,k}) - \nabla F_i(x_t) \|^2 \} \\
\leq \frac{K^2}{m^2} \sum_{i=1}^{m} \frac{L^2}{p_l} E_t \| x^i_{t,k} - x_t \|^2 \\
\leq \frac{L^2K^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_l} \left( 5K^2\eta_l^2\sigma_L^2 + 4K\eta_l^2\zeta_G^2, i \right) + 20K^2(A^2 + 1)\eta_l^2 \| \nabla f(x_t) \|^2 \\
= \frac{5L^2K^3\eta_l^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_l} \left( \sigma_L^2 + 4K\sigma_L^2 \right) + \frac{20L^2K^4\eta_l^2(A^2 + 1)}{m^2} \sum_{i=1}^{m} \frac{1}{p_l} \| \nabla f(x_t) \|^2, \quad (50)
\]

where (a4) follows from the fact that $E\| x_1 + \cdots + x_n \|^2 \leq nE \left( \| x_1 \|^2 + \cdots + \| x_n \|^2 \right)$, (a5) is due to Assumption 1, and (a6) is due to Lemma A.3.

Combining the above formulations, we have:

\[
A_1 \leq \frac{K\eta_L}{2} \| \nabla f(X_t) \|^2 + \frac{\eta_L}{2K} \left( \frac{5L^2K^3\eta_l^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_l} \left( \sigma_L + 4K\zeta_G^2, i \right) + \frac{20L^2K^4\eta_l^2(A^2 + 1)}{m^2} \sum_{i=1}^{m} \frac{1}{p_l} \| \nabla f(x_t) \|^2 \right) \\
- \frac{\eta_L}{2K} E_t - \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \|^2. \quad (51)
\]
Next we consider to bound $A_2$:

$$A_2 = E_t \| \Delta_t \|^2$$

$$= E_t \left\| -\eta_n \frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sum_{k=0}^{K-1} g^i_{t,k} \right\|^2$$

$$= \eta_n^2 E_t \left\| \frac{1}{n} \sum_{i \in S_t} \sum_{k=0}^{K-1} \left( \frac{1}{m_{pi}} g^i_{t,k} - \frac{1}{m_{pi}} \nabla F_i(x^i_{t,k}) \right) \right\|^2 + \eta_n^2 E_t \left\| -\frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2$$

$$= \eta_n^2 \frac{1}{n^2} \sum_{i \in S_t} \sum_{k=0}^{K-1} E_t \left\| \frac{1}{m_{pi}} g^i_{t,k} - \frac{1}{m_{pi}} \nabla F_i(x^i_{t,k}) \right\|^2 + \eta_n^2 E_t \left\| -\frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2$$

$$= \eta_n^2 \frac{1}{n^2} \sum_{i \in S_t} \sum_{k=0}^{K-1} E_t \left( E \left\| \frac{1}{m_{pi}} (g^i_{t,k} - \nabla F_i(x^i_{t,k})) \right\|^2 \right) + \eta_n^2 E_t \left\| -\frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2$$

$$= \eta_n^2 \frac{1}{n^2} \sum_{k=0}^{K-1} \left( \sum_{i \in S_t} \frac{1}{m_{pi}} \right) \left\| g^i_{t,k} - \nabla F_i(x^i_{t,k}) \right\|^2 + \eta_n^2 E_t \left\| -\frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2$$

$$\leq \eta_n^2 \frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sigma_L^2 + \eta_n^2 E_t \left\| -\frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{pi}} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2$$

(52)

where $M$ represents the whole sample space and (a7) is due to Assumption

Now substituting the expression of $A_1$ and $A_2$:

$$E_t[f(x_{t+1})]$$

$$\leq f(x_t) - \eta_n L K \| \nabla f(x_t) \|^2 + \eta_n \langle \nabla f(x_t), E_t[\Delta_t + \eta_n L K \nabla f(x_t)] \rangle + \frac{L^2}{2} \eta_n^2 E_t \| \Delta_t \|^2$$

$$\leq f(x_t) - K \eta_n L \left( \frac{1}{2} - \frac{10 K^2 \eta_n^2 L^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_i} \right) \| \nabla f(x_t) \|^2 + \frac{5 L^2 K^2 \eta_n^3}{2m^2} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L + 4 K \zeta_G^{(i)})$$

$$\quad + \frac{L \eta_n^2 K^2}{2n} \sum_{i=1}^{m} \frac{1}{m^2 p_i} \sigma^2$$

$$\quad \left( \frac{\eta_n}{KL} - \frac{L \eta_n^2}{2} \right) \left\| \nabla f(x_t) \right\|^2$$

$$\leq f(x_t) - K \eta_n \left( \frac{1}{2} - \frac{10 K^2 \eta_n^2 L^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_i} \right) \| \nabla f(x_t) \|^2 + \frac{5 L^2 K^2 \eta_n^3}{2m^2} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L + 4 K \zeta_G^{(i)})$$

(53)

$$\leq f(x_t) - K \eta_n \left( \frac{1}{2} - \frac{10 K^2 \eta_n^2 L^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_i} \right) \| \nabla f(x_t) \|^2 + \frac{5 L^2 K^2 \eta_n^3}{2m^2} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L + 4 K \zeta_G^{(i)})$$

(54)

$$\leq f(x_t) - c K \eta_n \| \nabla f(x_t) \|^2 + \frac{5 L^2 K^2 \eta_n^3}{2m^2} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L^2 + 4 K \zeta_G^{(i)^2}) + \frac{L \eta_n^2 K^2}{2n} \sum_{i=1}^{m} \frac{1}{m^2 p_i} \sigma_L^2$$

(55)

where (a8) follows from \( \left( \frac{\eta_n}{KL} - \frac{L \eta_n^2}{2} \right) \geq 0 \) if \( \eta_n \leq \frac{1}{KL} \), and (a9) holds because there exists a constant \( c > 0 \) satisfying \( \left( \frac{1}{2} - \frac{15 K^2 \eta_n^2 L^2}{m^2} \sum_{i=1}^{m} \frac{1}{p_i} \right) > c \) if \( \eta_n < \frac{1}{\sqrt{300KL} \sqrt{\frac{2}{\sum_{i=1}^{m} \frac{1}{p_i}}}} \).

Rearranging and summing from \( t = 0, \ldots, T - 1 \), we have:

$$\sum_{t=1}^{T-1} c_{tKL} E_t \| \nabla f(x_t) \|^2 \leq f(x_0) - f(x_T) + T(\eta_n L K \left( \frac{5 L^2 K^2 \eta_n^3}{2m^2} \sum_{i=1}^{m} \frac{1}{p_i} (\sigma_L^2 + 4 K \zeta_G^{(i)^2}) + \frac{L \eta_n^2 K^2}{2n} \sum_{i=1}^{m} \frac{1}{m^2 p_i} \sigma_L^2 \right) )$$

(56)
Which implies:

\[
\min_{t \in [T]} E \| \nabla f(x_t) \|^2 \leq \frac{f_0 - f_*}{c \eta L KT} + \frac{1}{c} \tilde{\Phi},
\]

where \( \tilde{\Phi} = \frac{5L^2K^2\eta^2}{2m^2} \sum_{l=1}^{m} \frac{1}{p_l} (\sigma_L^2 + 4K\zeta_{C_l,i}^2) + \frac{L \eta n}{2m} \sum_{l=1}^{m} \frac{1}{m^p_l} \sigma_L^2. \)

C.2 Sample without replacement

\[
\min_{t \in [T]} E \| \nabla f(x_t) \|^2 \leq \frac{f_0 - f_*}{c \eta L KT} + \frac{1}{c} \tilde{\Phi},
\]

where \( \tilde{\Phi} = \frac{5L^2K^2\eta^2}{2m^2} \sum_{l=1}^{m} \frac{1}{p_l} (\sigma_L^2 + 4K\zeta_{C_l,i}^2) + \frac{L \eta n}{2m} \sum_{l=1}^{m} \frac{1}{m^p_l} \sigma_L^2. \)

Proof.

\[
\mathbb{E}[f(x_t)] \leq f(x_t) + \langle \nabla f(x_t), E[x_{t+1} - x_t] \rangle + \frac{L}{2} E_t[\|x_{t+1} - x_t\|]
\]

\[
= f(x_t) + \langle \nabla f(x_t), E_t[\eta \Delta_t + \eta \eta L K \nabla f(x_t) - \eta \eta L K \nabla f(x_t)] \rangle + \frac{L}{2} \eta^2 E_t[\|\Delta_t\|^2]
\]

\[
= f(x_t) - \eta \eta L K \| \nabla f(x_t) \|^2 + \eta \langle \nabla f(x_t), E_t[\Delta_t + \eta \eta L K \nabla f(x_t)] \rangle + \frac{L}{2} \eta^2 E_t[\|\Delta_t\|^2].
\]

Similarly, we consider to bound \( A_1 \) first:

\[
A_1 = \langle \nabla f(x_t), E_t[\Delta_t + \eta \eta L K \nabla f(x_t)] \rangle
\]

\[
= \langle \nabla f(x_t), E_t \left[ - \frac{1}{|S_t|} \sum_{i \in S_t} \frac{1}{m_{p_i}} \sum_{i=0}^{K-1} \eta \eta L g_{i,k} + \eta \eta L K \nabla f(x_t) \right] \rangle
\]

\[
= \langle \nabla f(x_t), E_t \left[ - \frac{1}{|S_t|} \sum_{i \in S_t} \frac{1}{m_{p_i}} \sum_{i=0}^{K-1} \eta \eta L \nabla F_i(x_{i,k}) + \eta \eta L K \nabla f(x_t) \right] \rangle
\]

\[
= \left \langle \sqrt{K \eta L \nabla f(x_t)}, \sqrt{\eta \eta L} E_t \left[ - \frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{p_i}} \sum_{k=0}^{K-1} \nabla F_i(x_{i,k}) + K \nabla f(x_t) \right] \right \rangle
\]

\[
= \frac{\eta \eta L}{2K} \| \nabla f(x_t) \|^2 + \frac{\eta \eta L}{2K} E_t \left[ - \frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{p_i}} \sum_{k=0}^{K-1} \nabla F_i(x_{i,k}) + K \nabla f(x_t) \right] \| ^2
\]

\[
- \frac{\eta \eta L}{2K} E_t \left[ - \frac{1}{n} \sum_{i \in S_t} \frac{1}{m_{p_i}} \sum_{k=0}^{K-1} \nabla F_i(x_{i,k}) \right] ^2.
\]

Since \( x_t \) are sampled from \( S_t \) without replacement, this causes pairs \( x_{t1}, x_{t2} \) to no longer be independent. We introduce the activation function by:

\[
\Im_m \triangleq \begin{cases} 
1 & \text{if } x \in S_t, \\
0 & \text{otherwise}.
\end{cases}
\]
The we get the following bound:

\[
E_t \left\| \frac{1}{n} \sum_{i \in S_t} \frac{1}{m p_i} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) - K \nabla f(x_t) \right\|^2
\]

\[
= E_t \left\| \frac{1}{n} \sum_{i \in S_t} \frac{1}{m p_i} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) - \frac{1}{n} \sum_{i \in S_t} \frac{1}{m p_i} \sum_{k=0}^{K-1} \nabla F_i(x_t) \right\|^2
\]

\[
\leq \frac{1}{n^2} \sum_{i_1 \neq i_2} E_t \left\| \left( \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) - \frac{1}{m p_i} \nabla F_i(x_t) \right) \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2
\]

where (b1) follows from $\| \sum_{i=1}^{m} t_i \|^2 = \sum_{i \in [m]} \| t_i \|^2 + \sum_{i \neq j} \langle t_i, t_j \rangle \leq \sum_{i \in [m]} m \| t_i \|^2 - \frac{1}{2} \sum_{i \neq j} \| t_i - t_j \|^2$ (where (c1) is due to $\langle x, y \rangle = \frac{1}{2} [\| x \|^2 + \| y \|^2 - \| x - y \|^2]$), and (b2) is due to $E \| x_1 + \cdots + x_n \|^2 \leq n E \left( \| x_1 \|^2 + \cdots + \| x_n \|^2 \right)$, and (b3) is from Lemma 1.

Therefore, we have the bound of $A_1$:

\[
A_1 \leq \frac{K \eta_L}{2} \| \nabla f(x_t) \|^2 + \frac{\eta_L L^2 K}{2n} \left( 5K \eta_L \sum_{l=1}^{m} \frac{1}{p_l} (\sigma_L^2 + 4K \zeta_{G,i}) + 10K^2 \eta_L^2 \| \nabla f(x_t) \|^2 \frac{1}{m} \sum_{l=1}^{m} \frac{1}{p_l} \right)
\]

\[
- \frac{\eta_L}{2K} E_t \left\| \frac{1}{n} \sum_{i \in S_t} \frac{1}{m p_i} \sum_{k=0}^{K-1} \nabla F_i(x^i_{t,k}) \right\|^2.
\]
And $A_2$ has the following expression:

$$A_2 = E_t \| \Delta_t \|^2$$

$$= E_t \left[ -\eta_L \frac{1}{n} \sum_{i \in S_t} \frac{1}{mp_i} \sum_{k=0}^{K-1} g_{t,k}^i \right]^2$$

$$= \eta_L^2 E_t \left[ \frac{1}{n} \sum_{i \in S_t} \sum_{k=0}^{K-1} \left( \frac{1}{mp_i} g_{t,k}^i - \frac{1}{mp_i} \nabla F_i(x_{t,k}^i) \right) \right]^2$$

$$= \eta_L^2 \frac{1}{n^2} \sum_{i \in S_t} \sum_{k=0}^{K-1} \left( \frac{1}{mp_i} g_{t,k}^i - \nabla F_i(x_{t,k}^i) \right)^2$$

$$\leq \eta_L^2 \frac{K}{n} \sum_{i \in S_t} \frac{1}{m} \sigma_L^2 + \eta_L^2 E_t \left[ -\frac{1}{n} \sum_{i \in S_t} \sum_{k=0}^{K-1} \nabla F_i(x_{t,k}^i) \right]^2$$

(65)

Since we have the expression of $A_1$ and $A_2$, we can derive:

$$E_t[f(x_{t+1})]$$

$$\leq f(x_t) - \eta_L K \| \nabla f(x_t) \|^2 + \eta \langle \nabla f(x_t), E_t[\Delta_t + \eta_L K \nabla f(x_t)] \rangle + \frac{L}{2} \eta_L^2 E_t \| \Delta_t \|^2$$

(b4)

$$\leq f(x_t) - \eta_L K \left( \frac{1}{2} - \frac{10 L^2 K^2 \eta_L^2}{nm} \sum_{l=1}^{m} \frac{1}{p_l} \right) \| \nabla f(x_t) \|^2$$

$$\leq \frac{L \eta_L^2 K}{2n} \sum_{l=1}^{m} \frac{1}{p_l} \sigma_L^2 - \left( \frac{1}{2} - \frac{10 L^2 K^2 \eta_L^2}{2nm} \right) \sum_{l=1}^{m} \frac{1}{p_l} (\sigma_L^2 + 4 K \zeta_{G,i})$$

$$+ \frac{L \eta_L^2 K}{2n} \sum_{l=1}^{m} \frac{1}{p_l} \sigma_L^2 - \left( \frac{1}{2} - \frac{10 L^2 K^2 \eta_L^2}{2nm} \right) \sum_{l=1}^{m} \frac{1}{p_l} (\sigma_L^2 + 4 K \zeta_{G,i})$$

(66)

Also, for (b4), step sizes need to satisfy \( \left( \frac{10 L^2 K^2 \eta_L^2}{2nm} \sum_{l=1}^{m} \frac{1}{p_l} \right) \geq 0 \) if \( \eta_L \leq \frac{1}{K \mathcal{L}_T} \), and there exists a constant \( c > 0 \) satisfying \( \left( \frac{1}{2} - \frac{15 L^2 K^2 \eta_L^2}{mn} \frac{1}{p_l} \right) \geq c \) if \( \eta_L < \frac{1}{\sqrt{30 K L^{2/3} \mathcal{L}_T}} \).

Rearranging and summing from \( t = 0, \ldots, T - 1 \), we have:

$$\sum_{t=1}^{T-1} c \eta_L K E \| \nabla f(x_t) \|^2 \leq f(x_0) - f(x_T) + T(\eta_L K) \mathcal{F}.$$ (67)

Which implies:

$$\min_{t \in [T]} E \| \nabla f(x_t) \|^2 \leq \frac{f_0 - f_T}{c \eta_L K T} + \frac{1}{c} \mathcal{F},$$ (68)

where \( \Phi = \frac{5 L^2 K^2 \eta_L^2}{2nm} \sum_{l=1}^{m} \frac{1}{p_l} (\sigma_L^2 + 6 K \zeta_{G,i}) \) + \( \frac{L \eta_L^2 K}{2n} \sum_{l=1}^{m} \frac{1}{m^2 p_l} \sigma_L^2 \).
D  Proof of Optimal sampling probability

D.1 optimal sampling probability of FedSRC-G

Recall theorem 1, only the last variance term in the convergence term $\Phi$ is affected by sampling. In other words, we need to minimize the variance term with respect to probability. We formalized it as below:

$$
\min_{p_i \in [0, 1]} \sum_{i=1}^{m} p_i V\left( \frac{1}{m p_i} \hat{g}_i \right) \Leftrightarrow \min_{p_i \in [0, 1]} \sum_{i=1}^{m} \frac{1}{m^2 i} \sum_{i=1}^{m} \frac{1}{p_i} \| \hat{g}_i \|^2 .
$$

This problem can be solved in closed form by the KKT condition. It is easy to verify that the solution of the above optimization is:

$$
p^*_i = \frac{\| \sum_{k=0}^{K-1} g_{i,k} \|}{\sum_{i=1}^{m} \| \sum_{k=0}^{K-1} g_{i,k} \|}, \forall i \in 1, 2, ..., m .
$$

Under optimal sampling probability, the variance can be minimized to:

$$
V\left( \frac{1}{m p_i} \hat{g}_i \right) = \left( \frac{\sum_{i=1}^{m} \hat{g}_i}{m} \right)^2
$$

Remark: If the uniform distribution is adopted $p_i = \frac{1}{m}$, it is easy to observe that the variance of the stochastic gradient is bounded by $\frac{\sum_{i=1}^{m} \| g_i \|^2}{m}$.

According to Cauchy-Schwarz inequality,

$$
\frac{\sum_{i=1}^{m} \| \hat{g}_i \|^2}{m} \left( \frac{\sum_{i=1}^{m} \| \hat{g}_i \|}{m} \right)^2 = \frac{m \sum_{i=1}^{m} \| \hat{g}_i \|^2}{(\sum_{i=1}^{m} \| \hat{g}_i \|)^2} \geq 1 ,
$$

This implies that importance sampling does improve convergence rate, especially when $\frac{(\sum_{i=1}^{m} \| g_i \|)^2}{\sum_{i=1}^{m} \| g_i \|^2} << m$.

D.2 optimal sampling probability of FedSRC-D

Our result is of the following form:

$$
\min_{t \in [T]} E[\| \nabla f(x_t) \|^2] \leq \frac{f_0 - f_*}{c \eta \eta L} + \Phi ,
$$

it’s easy to see that the sampling strategy only affects $\Phi$, so to enhance the convergence, we need to minimize $\Phi$ with respect to $p$. As shown, the expression of $\Phi$ in with and without replacement are similarly, only differ in number $n$ and $m$. Here we just consider with replacement case. Exactly, we need to solve this optimization problem:

$$
\min_{p_l \in [0, 1]} \Phi = \frac{1}{c} (\frac{5L^2K \eta^2}{2m^2} \sum_{i=1}^{m} \frac{1}{p_l} (\sigma_{L,i}^2 + 6K \zeta_{G,i}^2)\frac{L^2 \eta \eta}{2n} \sum_{i=1}^{m} \frac{1}{m^2 p_l} \sigma_{L,i}^2) \quad \text{s.t.} \quad \sum_{i=1}^{m} p_l = 1 .
$$

Solving this optimization problem, we can find the optimal sampling probability to be:

$$
p^*_l = \frac{\sqrt{5KL^2 \eta L (\sigma_{L,i}^2 + 6K \zeta_{G,i}^2) + \frac{n}{\alpha_1} \sigma_{L,i}^2}}{\sum_{i=1}^{m} \sqrt{5KL^2 \eta L (\sigma_{L,i}^2 + 6K \zeta_{G,i}^2) + \frac{n}{\alpha_1} \sigma_{L,i}^2}} .
$$

For simplicity’s sake, we rewrote the optimal sampling probability as :

$$
p^*_i = \frac{\sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2}}{\sum_{i=1}^{m} \sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2}} .
$$
where $\alpha_1 = 20K^2L\eta_L$, $\alpha_2 = 5KL\eta_L + \frac{n}{n}$.

**Remark:** Now we compare with the uniform sampling strategy:

$$
\Phi_{\text{importance}} = \frac{L\eta}{2c} \left( \sum_{l=1}^{m} \sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2} \right)^2 \tag{76}
$$

For uniform $p_l = \frac{1}{m}$:

$$
\Phi_{\text{uniform}} = \frac{L\eta}{2c} \sum_{l=1}^{m} \left( \sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2} \right)^2 \tag{77}
$$

According to Cauchy-Schwarz inequality:

$$
\frac{\sum_{l=1}^{m} (\sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2})^2}{m} \left( \frac{\sum_{l=1}^{m} \sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2}}{m} \right)^2 = \frac{m \sum_{l=1}^{m} \left( \sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2} \right)^2}{m} \geq 1 \tag{78}
$$

implies that importance sampling does improve convergence rate (importance sampling based approach might be $n$-times faster in convergence than uniform), especially when

$$
\frac{\sum_{l=1}^{m} (\sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2})^2}{m} \frac{\sum_{l=1}^{m} \sqrt{\alpha_1 \zeta_{G,i}^2 + \alpha_1 \sigma_{L,i}^2}}{m} << 1
$$

**E** Experiment details.

**E.1** Additional Experiments

**Synthetic dataset** We demonstrate the experiment in different functions with different $A$ and $b$. Each function is set with the noise of 20, 30, 40 to illustrate our theoretical results. As for constructing different functions, we assign $A = 8$, $10$ and $b = 2, 1$ respectively to see the convergence behavior of different functions.

We choose 10 out of 150 clients in each round. All the algorithms run on the same environment with a fixed learning rate of $0.001$. We train each experiment for 2000 rounds to make global loss have a stable convergence performance. We display the log of global loss in Fig 3, where the Power-of-Choice is a biased sampling strategy which selects clients with higher loss (Cho et al., 2020).

**FEMNIST and CIFAR-10** Specifically, we train a two-layer MLP on the split-FEMNIST and a resnet 18 on split-CIFAR-10, respectively. The "split" follows the idea introduced in (Yu et al., 2019; Hsu et al., 2019; Reddi et al., 2020), where we leverage the Latent Dirichlet Allocation (LDA) to control the distribution drift with the dirichlet parameter $\alpha$. Larger $\alpha$ indicates smaller drifts. Unless otherwise stated, we set dirichlet parameter $\alpha = 0.5$.

Unless specifically mentioned otherwise, our studies use the following protocol. All datasets are split with parameter $\alpha = 0.5$, the server choose $n = 20$ clients according to our proposed probability from the total of $m = 300$ clients, and each is trained for $T = 500$ communication rounds with $K = 5$ local epochs. The default local dataset batch size is 20. The learning rates are well-tuned and set the same in all algorithms, specifically $lr_{\text{global}} = 1$ and $lr_{\text{local}} = 0.01$.

All algorithms use FedAvg as the backbone. We compare FedSRC-D and FedSRC-G with FedAvg in different dataset with different setting.

In Fig 4, it shows that FedSRC-D converges faster to more accurate solutions than both FedSRC-G and FedAvg.

For CIFAR-10, we report the mean of the best 10 test accuracies on global test data here. In Table 2, we compare the performance of FedSRC-D, FedSRC-G, and FedAvg on non-IID FEMNIST and CIFAR-10. Specifically, we use $\alpha = 0.1$ for FEMNIST and $\alpha = 0.5$ for CIFAR-10 to split dataset.
Figure 3: Performance of different algorithms on the regression model. The loss is calculated by $f(x, y) = \|y - \log(\frac{A_i x - b_i}{2})\|_2^2$, we report the logarithm of global loss with different degree of gradient noise $\nu$. All methods are well-tuned, and we report the best result of each algorithm under each setting.

Figure 4: Accuracy of different FedSRC-D compared with FedSRC-G and FedAvg on FEMNIST.

As for Multinomial Distribution (MD) sampling (Li et al., 2018), it samples based on clients’ data ratio and average aggregates. It is symmetric in sampling and aggregation with FedAvg, with similar performance. It can be seen that FedSRC-D has better accuracy than FedSRC-G, while FedSRC-D and FedSRC-G both outperform FedAvg with the same communication round.

In Table 3, we demonstrate that FedSRC is compatible with other FL optimization algorithms, e.g., Fedprox (Li et al., 2018) and FedMime (Karimireddy et al., 2020a). Moreover, FedSRC-D keeps its superiority in this setting.

We also experiment with different choices of heterogeneity $\alpha$ in CIFAR-10. The parameter of heterogeneity $\alpha$ changes from 0.1 to 0.5 to 1. We observe the consistent improvement of FedSRC-D in Table 4.

Besides, we also experiment with various client numbers to examine the efficiency of FedSRC in FEMNIST dataset. Here we set $\alpha = 1$, and participated client number choose from $n = 10, 30, 50$. As shown in Table 5, FedSRC-D maintains its supremacy with different participated client numbers.
Table 3: **Performance of algorithms with momentum and prox.** We run 500 communication rounds on CIFAR10 for each algorithm. We report the mean of maximum 5 accuracies for test datasets and the number of communication rounds to reach the threshold accuracy.

| Algorithm                      | CIFAR-10 + momentum | CIFAR-10 + prox |
|-------------------------------|---------------------|-----------------|
|                               | Acc (%) | Rounds for 65% | Acc (%) | rounds for 65% |
| FedAvg (w/ uniform sampling)  | 0.6567  | 390            | 0.6596  | 283            |
| FedSRC-G                      | 0.671   | 283            | 0.681   | 252            |
| FedSRC-D                      | 0.6604  | 283            | 0.6677  | 252            |

Table 4: **Performance of algorithms under different $\alpha$.** We run 500 communication rounds on CIFAR10 for each algorithm. We report the mean of maximum 5 accuracies for test datasets and the number of communication rounds to reach the threshold accuracy.

| Algorithm                      | $\alpha = 0.1$ | $\alpha = 0.5$ | $\alpha = 1.0$ |
|-------------------------------|----------------|----------------|----------------|
|                               | Acc (%) | Rounds for 42% | Acc (%) | rounds for 65% | Acc (%) | rounds for 71% |
| FedAvg (w/ uniform sampling)  | 0.4209  | 263            | 0.6567  | 283            | 0.7183  | 246            |
| FedSRC-G                      | 0.427   | 305            | 0.6571  | 252            | 0.7218  | 239            |
| FedSRC-D                      | 0.4311  | 209            | 0.6604  | 283            | 0.7248  | 221            |

Table 5: **Performance of algorithms under different participated client number $n$.** We run 500 communication rounds on FEMNIST for each algorithm. We report the mean of maximum 5 accuracies for test datasets and the number of communication rounds to reach the threshold accuracy.

| Algorithm                      | $n = 10$ | $n = 30$ | $n = 50$ |
|-------------------------------|----------|----------|----------|
|                               | Acc (%) | Rounds for 85% | Acc (%) | rounds for 85% | Acc (%) | rounds for 85% |
| FedAvg (w/ uniform sampling)  | 0.8717  | 263      | 0.8727  | 267            | 0.8729  | 239            |
| FedSRC-G                      | 0.8739  | 305      | 0.8734  | 286            | 0.8751  | 222            |
| FedSRC-D                      | 0.8741  | 209      | 0.8746  | 270            | 0.8747  | 212            |