Dynamic Modulation Yields One-Way Beam Splitting

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This article demonstrates the realization of an extraordinary beam splitter based on nonreciprocal and synchronized photonic transitions in obliquely illuminated space-time-modulated (STM) slabs which impart the coherent temporal frequency and spatial frequency shifts. As a consequence of such unusual photonic transitions, a one-way beam splitting and amplification is exhibited by the STM slab. Beam splitting is a vital operation for various optical and photonic systems, ranging from quantum computation to fluorescence spectroscopy and microscopy. Despite the beam splitting is conceptually a simple operation, the performance characteristics of beam splitters significantly influence the repeatability and accuracy of the entire optical system. As of today, there has been no approach exhibiting a nonreciprocal beam splitting accompanied with transmission gain and an arbitrary splitting angle. Here, we show that oblique illumination of a periodic and semi-coherent dynamically-modulated slab results in coherent photonic transitions between the incident light beam and its counterpart space-time harmonic (STH). Such photonic transitions introduce a unidirectional synchronization and momentum exchange between two STHs with same temporal frequencies, but opposite spatial frequencies. Such a beam splitting technique offers high isolation, transmission gain and zero beam tilting, and is expected to drastically decrease the resource and isolation requirements in optical and photonic systems. In addition to the analytical solution, we provide a closed-form solution for the electromagnetic fields in STM structures, and accordingly, investigate the properties of the wave isolation and amplification in subluminal, superluminal and luminal ST modulations.

I. INTRODUCTION

Beam splitters are ubiquitous in the optics and photonics systems, including the linear optical quantum computers [1, 2] and optical interferometers [3–5], range-finders [6], and fluorescence spectroscopy and microscopy devices [7, 8]. Recent advances in optics and photonics have led to the invention of a range of linear beam splitting techniques beyond the conventional realization by a semi-reflective optical glass. This includes beam splitting based on periodic structures and photonic crystals [9, 10], artificial atoms in superconducting circuits [5], slow light with electromagnetically induced transparency in atomic vapor cells [11], Landau-Zener transitions of electronic spin states [12], modulated superconducting quantum interference devices in superconducting cavities [13]. However, all these approaches provide a passive and reciprocal beam splitting which in turn increase the resource requirements of the overall system, including demand for high power light sources and isolators.

This letter presents the application of space-time-modulated (STM) photonic structures to extraordinary beam splitting. As of today, various applications of STM structures have been reported, where normal incidence of the light beam to the STM structure yields unusual interaction with electromagnetic wave [14–18]. These applications include but not limited to the parametric traveling-wave amplifiers [19–22], optical isolators [17, 23–27], nonreciprocal metasurfaces [28–30], pure frequency mixer [31], circulators [32–34], and mixer-duplexer-antenna system [35, 36]. Nevertheless, there has been a lack of investigation on the properties of STM media under oblique incidence and its applications.

Here, we introduce a one-way, tunable and highly efficient beam splitter and amplifier based on coherent photonic transitions through the oblique illumination of space-time-modulated (STM) photonic structures. The contributions of this paper are as follows.

1) In contrast to conventional beam splitters which are restricted to reciprocal response with more than 3 dB insertion loss, the proposed STM beam splitter is capable of providing nonreciprocal response with transmission gain. Such a nonreciprocal response and amplification may decrease the resource and isolation requirements in optical and photonic systems. It can be also used in antenna applications, where the transmitted and received waves are engineered appropriately.

2) We show that the STM beam splitter presents an efficient performance for both collimated and non-collimated incidence beam with no output beam tilt. This is very interesting as conventional passive beam splitters suffer from poor performance for non-collimated beams and provide an undesired output beam tilt.

3) The presented STM beam splitter may be easily tuned and provide an arbitrary angle of difference for the two output beams. In addition, an unequal power division between the output beams may be achieved.

4) Here, we present the first application of obliquely illuminated STM slabs. Consequently, for the first time, the scheme and results for the FDTD simulation results for oblique incidence to a STM slab is presented. 5) A closed-form solution is presented that provides a deep insight into the wave propagation inside the STM beam splitters and the difference between the subluminal, lu-
mininal and superluminal ST modulations. 6) The analysis of the STM beam splitter is further accomplished by investigation of its analytical three dimensional dispersion diagrams, achieved by Bloch-Floquet decomposition of STHs. Accordingly, the rest of the paper is structured as follows. Section II presents the operation principle of the proposed STM beam splitter. In Sec. III, we derive the analytical solution for oblique electromagnetic wave propagation inside the STM beam splitter based on the Bloch-Floquet representation of the electromagnetic fields. Then, Sec. IV presents the time and frequency domains numerical simulation results for the beam splitting and amplification in the STM beam splitter. Next, the closed form solution will be provided in Sec. V, which gives a leverage for understanding the wave propagation and photonic transitions in STM structures. A short discussion on practical realization of superluminal STM structures at different frequencies will be presented in Sec. VI. Finally, Sec. VII concludes the paper.

II. OPERATION PRINCIPLE

Figure 1 sketches the nonreciprocal beam transmission and splitting in a STM slab. By appropriate design of the band structure, that is, the ST modulation format and its associated temporal and spatial modulation frequencies, unidirectional energy and momentum exchange between the incident wave-under angle of incidence and transmission \( \theta_1 = \theta_{T,0} = 45^\circ \) and temporal frequency \( \omega_0 \) to the first lower STH-under angle of transmission \( \theta_{T,-1} = -45^\circ \) and temporal frequency \( \omega_0 \) will occur. Assuming TM\(_y\) or \( E_y \) polarization, the electric field of the incident light beam in the forward \( +z \)-direction may be expressed as

\[
E_1^S(x, z, t) = E_0 e^{-i[k_x x + k_z z - \omega_0 t]},
\]

is traveling in the \( +z \)-direction under the angle of incidence \( \theta_1 = 45^\circ \) and impinges to the periodic STM slab. The \( x \)- and \( z \)-components of the spatial frequency read \( k_x = k_0 \sin(\theta_1) \) and \( k_z = k_0 \cos(\theta_1) \), respectively, in which \( k_0 = \omega_0/\nu_b = \omega_0 \sqrt{\varepsilon_r}/c \), with \( \omega_0 \) being the temporal frequency of the incident wave, \( \nu_b \) denoting the phase velocity in the background medium, \( \varepsilon_r \) representing the relative electric permittivity of the background medium, and \( c \) denoting the speed of light in vacuum. The STM slab assumes a sinusoidal ST-varying permittivity, as

\[
\varepsilon(z, t) = \varepsilon_{av} + \delta \varepsilon \sin(qz - \Gamma t),
\]

where \( \varepsilon_{av} = \varepsilon_r + \delta \varepsilon \) is the average permittivity of the slab, \( \delta \varepsilon \) denotes the modulation strength, \( \Gamma = 2\omega_0 \) is the modulation temporal frequency, and

\[
q = \frac{2k_0}{\gamma},
\]

represents the spatial frequency of the modulation, with \( \gamma = v_m/v_b \) being the ST velocity ratio, where \( v_m \) and \( v_b \)

\[
E_{T,0}^S(x, z, t) = \hat{y} \sum_{m=\pm M} A_m e^{-i(k_x x + k_z z + \omega_m t)} = \frac{1}{\eta_S} \left[ \hat{k}_S \times E_S(x, z, t) \right]
\]

and

\[
E_S(x, z, t) = \frac{1}{\eta_S} \sum_{m=\pm M} \left[ -\hat{x} \frac{k_{z,m}}{\mu_0 \omega_m} + \hat{z} \frac{\sin(\theta_1)}{\eta_S} \right] A_m e^{-i(k_x x + k_z z - \omega_m t)},
\]

where \( M \to \infty \) is the number of STHs. In Eq. (4), \( \eta_S = \sqrt{\mu_0/(\varepsilon_0 \epsilon_r)} \), and \( A_m \) represents the unknown amplitude of the \( m \)th STH, characterized by the spatial frequency

\[
k_{z,m} = \beta_0 + mq,
\]

and the temporal frequency

\[
\omega_m = \omega_0 + m\Omega = (1 + 2m)\omega_0,
\]

with \( \beta_0 \) being the unknown spatial frequency of the fundamental harmonic. The unknowns of the electric field, that is, \( A_m \) and \( \beta_0 \), will be found through satisfying Maxwell’s equations.

The transmission angle of the \( m \)th transmitted STH, \( \theta_{T,m} \), satisfies the Helmholtz relation as

\[
k_0^2 \sin^2(\theta_1) + k_m^2 \cos^2(\theta_{T,m}) = k_m^2,
\]

are, the phase velocity of the modulation and the background medium, respectively. Since the slab permittivity is periodic in space and time, with spatial frequency \( q \) and temporal frequency \( 2\omega_0 \), the electric field inside the slab may be decomposed into ST Bloch-Floquet waves as

\[
E_S(x, z, t) = \hat{y} \sum_{m=\pm M} A_m e^{-i(k_x x + k_z z + \omega_m t)},
\]
where \( k_m = \omega_m/v_0 \) denotes the wavenumber of the \( m \)th transmitted STH outside the STM slab. Solving Eq. (5) for \( \theta_{T,m} \) yields

\[
\theta_{T,m} = \sin^{-1} \left( \frac{k_x}{k_m} \right) = \sin^{-1} \left( \frac{\sin(\theta_1)}{1 + m\Omega/\omega_0} \right) = \sin^{-1} \left( \frac{\sin(\theta_1)}{1 + 2m} \right).
\]

Equation (6a) demonstrates the spectral decomposition of the transmitted wave. Consequently, the fundamental STH, \( m = 0 \), and the first lower STH, \( m = -1 \), with equal temporal frequency \( \omega_0 \), will be respectively transmitted under the angles of transmission of

\[
\theta_{T,0} = \theta_1 = 45^\circ,
\]

\[
\theta_{T,-1} = -\theta_1 = -45^\circ.
\]

so that they are transmitted under 90° angle difference, presenting the desired beam splitting. The scattering angle of the \( m \)th STH inside the STM slab reads

\[
\theta_{S,m} = \tan^{-1} \left( \frac{k_x}{k_{z,m}} \right).
\]

In addition, the transmitted electric field from the slab may be found as

\[
\mathbf{E}_T(x, z, t) = \mathbf{E}_S(x, z, t)e^{-ik_{x,m,z}z} = \hat{y} \sum_{m=-M}^{M} A_m e^{-i(k_x x + k_{z,m}[d+z] - \omega_m t)}.
\]

The sourceless wave equation reads

\[
\nabla^2 \mathbf{E}_S(x, z, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} [\mathbf{E}(z, t) \mathbf{E}_S(x, z, t)] = 0.
\]

Substituting Eqs. (S1) and (4) into Maxwell’s equations, yields a matrix equation as

\[
[K][\mathbf{A}] = 0,
\]

where \( [K] \) is a \((2M+1) \times (2M+1)\) matrix with elements

\[
K_{m,m} = \epsilon_{av} - \frac{k_x^2 + k_{z,m}^2}{k_0^2},
\]

\[
K_{m,-1} = i \frac{\delta_x}{2},
\]

\[
K_{m,+1} = -i \frac{\delta_x}{2},
\]

and where \( \mathbf{A} \) represents a \((2M+1) \times 1\) vector containing \( A_m \) coefficients. Equation (10a) has nontrivial solution if

\[
\text{det} \{[K]\} = 0.
\]

Equation (11) represents the dispersion relation of the STM beam splitter which provides the unknown spatial frequency of the fundamental ST harmonic for a given frequency, i.e., \( \beta_0(\omega_0) \). After finding the \( \beta_0(\omega_0) \), the \( [K] \) matrix in Eq. (10a) is known and therefor, the unknown amplitude of the STHs \( A_m \) will be calculated using Eq. (10a).

III. ANALYTICAL 3D DISPERSION DIAGRAM

Figure 2 presents a qualitative illustration of the three-dimensional dispersion diagram for the periodic STM slab in Fig. 1. This diagram is formed by \( 2M+1 \) periodic set of double cones (here, only \( m = 0 \) and \( m = -1 \) harmonics are shown), each of which representing a STH, with apexes at \( k_x = 0, k_z = -mq \) and \( \omega = -2m\omega_0 \), and the slope of \( v_m \) with respect to \( k_x - k_z \) plane. Consider oblique incidence of a wave, representing the fundamental harmonic \( m = 0 \) with temporal frequency \( \omega_0 \), propagating along \([+x, z]\) direction. It is characterized by \( x \) and \( z \) components of the spatial frequency, \( k_x = \hat{x} k_x \) and \( k_z = \hat{z} k_z \). The wave impinges to the medium under the angle of incidence \( \theta_1 = 45^\circ \) and excites an infinite number of (we truncate it to \( 2M+1 \)) STH waves, with different spatial and temporal frequencies of \([k_x, k_{z,m}] \) and \( \omega_m \). However, interestingly, the first lower STH \( m = -1 \) offers similar characteristics as the fundamental harmonic, that is, the identical temporal frequency of \( \omega_0 \) and identical \( z \)-component of the spatial frequency of \( k_{z,-1}^x = k_{z,0}^x \), but opposite \( x \)-component of the spatial frequency of \( k_{z,-1}^z = -k_{z,0}^z \). Hence, \( m = -1 \) harmonic propagates along \([-x, z]\) direction. In general the \( x \)-component of the \( m \)th STH reads \( k_{x,m} = -k_{x,-m-1} \).
Moreover, since \( \omega_m = \omega_{-m-1} \), the undesired STHs acquire temporal frequency of \( 2m\omega_0 \), and far away from the fundamental harmonic. Thus, most of the incident energy is residing in \( m = 0 \) and \( m = -1 \) harmonics, both at \( \omega_0 \), respectively transmitted under \( \theta_{T,0} = \theta_1 \) and \( \theta_{T,-1} = - \theta_1 \) transmission angles with \( 2\theta_1 \) angle difference.

The exchange of the energy and momentum between the fundamental and first lower harmonic occurs only for the forward, \( +z \), wave incidence. This may be observed from Fig. 2, as the forward harmonics (red circles, where \( \partial \omega / \partial k_z > 0 \)) are very close, whereas the backward harmonics (grey circles, where \( \partial \omega / \partial k_z < 0 \)) are far apart from each other. Therefore, a nonreciprocal transition of energy is achieved from the incident wave under \( \theta_1 = 45^\circ \) to the first STH under \( \theta_{T,-1} = -45^\circ \), through the ST modulation under \( \theta_{mod} = 0^\circ \).

Figure 3(a) shows the analytical solution for three dimensional dispersion diagram of the STM medium in Fig. 1, computed using Eq. (11) for \( \gamma = 1.2 \). For a given frequency, this three dimensional diagram provides the two dimensional \( k_z/q - k_x/q \) isofrequency diagram of the medium. Figure 3(b) plots the isofrequency diagram at \( \omega/2\omega_0 = 0.5 \) (or \( \omega = \omega_0 \)), containing an infinite periodic set of circles centered at \( (k_z/q, k_x/q) = (-m, 0) \) with radius \( \gamma (0.5 + m) \).

It may be seen from Figs. 3(a) and 3(b) that at \( \omega = \omega_0 \), the \( m = 0 \) and \( m = -1 \) STHs offer identical isofrequency circles. However, their associated forward harmonics (red circles) are very close to each other whereas their associated backward harmonics (grey circles) are significantly separated. For a nonzero velocity ratio (\( \gamma > 0 \)), the forward and backward STHs acquire different distances, i.e. \( \Delta \beta^\pm = k_{z,m+1}^\pm - k_{z,m}^\pm \) [17]. Particularly, as \( \gamma \) increases, \( \Delta \beta^- \) increases and \( \Delta \beta^+ \) decreases. As a result, at the limit of \( \gamma = 1 \) the forward harmonics acquire distances \( \Delta \beta^+/q = 0 \), and the backward harmonics acquire distances \( \Delta \beta^-/q = 2 \). Hence, increasing \( \gamma \) results in the significant enhancement in the nonreciprocity of the medium, so that the forward harmonic waves tend to merge together (\( \Delta \beta^+ \to 0 \)) and exchange their energy and momentum, whereas the backward harmonics tend to separate from each other (\( \Delta \beta^+ \to 2 \)) (Fig. 3(a)). Hence, such a dynamic modulation has nearly no effect on the backward incident beam.

**IV. NUMERICAL SIMULATION RESULTS**

We next verify the above theory by finite difference time-domain (FDTD) numerical simulation of the dynamic process through solving Maxwell’s equations. Figure 4 plots the implemented finite-difference time-domain scheme for numerical simulation of the oblique wave impinging to the STM beam splitter. We first discretize the medium to \( K + 1 \) spatial samples and \( M + 1 \) temporal samples, with the steps of \( \Delta z \) and \( \Delta t \) respectively. Next, the finite-difference discretized form of the first two Maxwell’s equations for the electric and magnetic fields:

\[
\begin{align*}
\frac{\partial E_z}{\partial x} + \frac{\partial E_t}{\partial z} &= 0, \\
\frac{\partial B_z}{\partial x} - \frac{\partial B_t}{\partial z} &= 0.
\end{align*}
\]

**FIG. 3.** Analytical dispersion diagram of the periodic STM slab in Fig. 1 for \( \gamma = 1.2 \) computed using Eq. (11). The forward incidence under \( \theta_1 = 45^\circ \) corresponding to \( k_z/q = 0.4243 \) excites the \( m = -1 \) STH, resulting in a strong exchange of energy between \( m = 0 \) and \( m = -1 \) harmonics, with the identical temporal frequency of \( \Omega_0 \). (a) Three dimensional dispersion diagram constituted of an array of periodic cones [17]. (b) Isofrequency diagram at \( \omega = \omega_0 \) presents an infinite set of circles centered at \( (k_z/q, k_x/q) = (-m, 0) \) with radius \( \gamma (0.5 + m) \).
The presented analytical and numerical results demonstrate that the dynamic beam splitter provides a perfect nonreciprocal beam splitting, in the lack of beam tilting. Moreover, it may be seen that, in contrast to conventional passive beam splitters, the beam splitting is achieved for a non-collimated beam. Other interesting features may be presented by changing the modulation parameters (γ, 0°, and ϵω), including tunable transmission angles, unequal splitting ratio and unequal angles of transmission. Figure 7 compares the analytical and numerical results for the spectrum of the incident and transmitted electric fields in Fig. 5. This figure shows that 3dB transmission gain is achieved for each of transmitted beams in the forward excitation. Moreover, it may be seen from Fig. 7 that the undesired higher order harmonics, at ω = 2mω0, are sufficiently weak so that the beam splitter safely operates at single frequency ω0.

V. CLOSED FORM SOLUTION FOR ELECTROMAGNETIC FIELDS

It is shown in Sec. IV that by proper design of the band structure, a pure unidirectional beam splitting can be achieved in an obliquely illuminated STM slab. The analytical solution of the electromagnetic fields based on the double Bloch-Floquet decomposition of electromagnetic fields, presented in Sec. III, provides an accurate solution for the fields scattered by such a slab, which is very useful. However, such an analytical solution does not provide a deep insight into the wave propagation inside the slab. In particular, it is of great interest to have an intuitive explanation about the effect of different parameters, e.g. x, γ, kx, and k, on the wave propagation and energy exchange between the incident field m = 0 and the excited first lower harmonic m = −1. Moreover, the accurate analytical solution, based on the mathemat-
FIG. 6. Nonreciprocal beam splitting in periodically STM slab. FDTD numerical simulation for the wave incidence to the slab, (a) From the right with \( \theta_I = 45^\circ \). (b) From the top, i.e., \( \theta_I = -45^\circ \).

The electric field inside the STM slab may be represented based on the superposition of the aforementioned two STHs, i.e., \( m = 0 \) and \( m = -1 \). The electric field is then defined by

\[
E_S(x, z, t) = a_0(z)e^{-i(k_xx+k_zz-\omega_0t)} + a_{-1}(z)e^{i(-k_xx+(q-k_z)z-\omega_0t)},
\]

where \( a_0(z) \) and \( a_{-1}(z) \) are the unknown field coefficients. We shall stress that, here the field coefficients are \( z \)-dependent since they include both the amplitude and the change in the spatial frequency (wavenumber) introduced by the ST modulation. Following the procedure provided in the supplemental material in Ref. [37], we insert the electric fields in (13) into the wave equations in (S4), and achieve a coupled differential equation for the field coefficients, i.e.,

\[
\frac{d}{dz} \begin{bmatrix} a_0(z) \\ a_{-1}(z) \end{bmatrix} = \begin{bmatrix} M_0 & C_0 \\ C_{-1} & M_{-1} \end{bmatrix} \begin{bmatrix} a_0(z) \\ a_{-1}(z) \end{bmatrix},
\]

where

\[
M_0 = \frac{ik_0^2}{2k_z}(\epsilon_{av} - \epsilon_r),
\]

\[
M_{-1} = \frac{ik_0^2}{2(k_z - q)} \left[ \epsilon_{av} - \epsilon_r \frac{k_z^2 + (q - k_z)^2}{k_0^2} \right],
\]

\[
C_0 = i\frac{\delta k_0^2}{4k_z},
\]

\[
C_{-1} = i\frac{\delta k_0^2}{4(k_z - q)}.
\]

The solution to the coupled differential equation

FIG. 7. Comparison of the analytical and numerical results for the frequency spectrum of the incident and transmitted electric fields in Fig. 5, i.e., wave incidence to the slab from the left with \( \theta_I = 45^\circ \).
in (14a) is given by [37]
\[
\begin{align*}
a_0(z) &= \frac{E_0}{2\Delta} \left( (M_0 - M_{-1} + \Delta) e^{\frac{M_0 + M_{-1} + \Delta}{2} z} \right) \quad (15a) \\
- a_{-1}(z) &= \frac{E_0 C_{-1}}{\Delta} \left( e^{\frac{M_0 + M_{-1} + \Delta}{2} z} - e^{-\frac{M_0 + M_{-1} - \Delta}{2} z} \right), \quad (15b)
\end{align*}
\]
where \(\Delta = \sqrt{(M_0 - M_{-1})^2 + 4C_0 C_{-1}}\). For a given ST modulation ratio \(\gamma\), the field coefficients in Eq. (15) acquire different forms. In general, ST modulation is classified into three categories, i.e., subluminal \((0 < \gamma < 1)\) or \(v_m < v_b\), luminal \((\gamma \rightarrow 1)\) or \(v_m \rightarrow v_b\), and superluminal \((\gamma > 1)\) or \(v_m > v_b\).

A. Subluminal and Superluminal ST Modulations

Considering \(\epsilon_{av} = \epsilon_z\), the \(a_0(z)\) and \(a_{-1}(z)\) in Eq. (15) would be a periodic sinusoidal function with respect to \(z\), if \(\Delta = \sqrt{(M_0 - M_{-1})^2 + 4C_0 C_{-1}}\) is imaginary, i.e., \((M_0 - M_{-1})^2 + 4C_0 C_{-1} < 0\). By solving this, we achieve an interval for the luminal ST modulation, that is,
\[
\gamma_{\text{sub}} < \frac{1}{\sqrt{\epsilon_{av} + \delta_c}} \leq \gamma_{\text{num}} \leq \frac{1}{\sqrt{\epsilon_{av} - \delta_c}} < \gamma_{\text{sup}}, \quad (16)
\]
where \(\gamma_{\text{sub}}, \gamma_{\text{num}}\) and \(\gamma_{\text{sup}}\) are ST velocity ratio for subluminal, luminal and superluminal ST modulations, respectively. The interval for luminal ST modulation is called sonic regime in analogy with sonic boom effect in acoustics, where an airplane travels with the same speed or faster than the speed of sound. It should be noted that the luminal ST modulation interval in Eq. (16) is exactly same as the one achieved from the exact analytical solution [16, 17, 22].

Figure 8(a) plots the closed form and FDTD numerical simulation results for the absolute electric field coefficient inside the slab, with the wave incidence from the left side (forward incidence), considering superluminal ST modulation of \(\gamma = 1.2\) and \(\delta_c = 0.28\). It is seen from this figure that both \(a_0(z)\) and \(a_{-1}(z)\) possess periodic sinusoidal form and exhibit a substantial transmission gain at \(z = 3\lambda_0\). Such a transmission gain may be tuned through the variation of \(\gamma\) and \(\delta_c\). This result is consistent with the transmission gain achieved in the FDTD numerical simulation results in Figs. 5 and 7. The coherence length \(l_c\), where both \(a_0(z)\) and \(a_{-1}(z)\) acquire their maximum amplitude is found as [37]
\[
l_c = \pi \left( \frac{k_0^4 \epsilon_{av} - \epsilon_z}{k_z - q} \right)^2 + \frac{\delta^2 k_0^4}{4k_z (k_z - q)} \right)^{-1}.
\]

Figure 8(b) plots the result for the superluminal STM slab in Fig. 8(a), except for wave incidence from the right side (backward incidence). It may be seen from this figure that, in contrast to the forward wave incidence where a substantial exchange of the energy and momentum between the \(m = 0\) and \(m = -1\) STHs are achieved, here the incident wave passes through the slab with negligible alteration and minor transition of energy and momentum to the \(m = -1\) ST harmonic. This is obviously in agreement with the nonreciprocal response presented in Figs. 5, 6(a) and 6(b).

FIG. 8. Closed-form solution results and the FDTD numerical simulation results for the \(z\)-dependent absolute field coefficients in Eq. (13), i.e., \(a_0(z)\) and \(a_{-1}(z)\), inside the superluminal STM beam splitter, with \(\gamma = 1.2\) and \(\delta_c = 0.28\). (a) Forward wave incidence, where the wave propagates from left to right. (b) Backward wave incidence, where the wave propagates from right to left.
B. Luminal ST Modulation

It may be shown that the for the luminal ST modulation, where $\gamma \to 1$, the field coefficients in Eq. (15), $a_0(z)$ and $a_{-1}(z)$, acquire pure real (or complex) forms. This yields exponential growth of the electric field amplitude along the STM slab. Hence, considering $\gamma = 1$, the total electric field inside the STM slab reads

$$E_S(x, z, t)|_{\gamma=1} = E_0 \cosh \left( \frac{\delta k_0^2}{4k_z} \right) e^{-i(k_z x + k_z z - \omega_0 t)} (18)$$

$$- i \frac{\delta k_0^2}{2k_z} E_0 \sinh \left( \frac{\delta k_0^2}{4k_z} \right) e^{i(-k_z x + (q-k_z) z - \omega_0 t)}.$$

Figure 9(a) plots the closed form and FDTD numerical simulation results for the absolute value of the electric field coefficients $a_0(z)$ and $a_{-1}(z)$ inside the luminal ($\gamma = 1$ and $\delta_0 = 0.28$) STM slab for forward wave incidence. It may be seen from this figure that both $a_0(z)$ and $a_{-1}(z)$ possess a non-periodic exponentially growing profile and exhibit a substantial transmission gain at $z \geq 3\lambda_0$. It should be noted that, the solution for the field coefficients presented in Eqs. (15) and (S21) are very useful and provide a deep insight into the wave propagation inside the STM slab, especially for the luminal ST modulation (sonic regime), where the Bloch-Floquet-based analytical solution does not exist since the solution does not converge [16, 17, 22].

Figure 9(b) plots the result for the luminal STM slab in Fig. 9(a), except for wave incidence from the right side (backward incidence). It may be seen from this figure that, in contrast to the forward wave incidence, here the incident wave passes through the slab with negligible alteration and minor transition of energy and momentum to the $m = -1$ ST harmonic.

VI. DISCUSSION ON PRACTICAL REALIZATION OF SUPERLUMINAL ST MODULATION

To practically realize superluminal ST modulation (here $\gamma = 1.2$), the phase velocity of the modulation should be greater than the velocity of the incident wave in the background (unmodulated) medium and not the velocity of light in vacuum [14]. Considering a glass as the background medium with permittivity $> 1.5$, achieving $\gamma = 1.2$ would be realistic. For instance, one may use coupled structures with two different lines (possessing different phase velocities) for the input wave and the modulation [27, 38, 39]. In such structures a modulation velocity greater than at least one of the characteristic velocities involved is required. In general, the way of achieving the fast pumping depends on the frequency range, as follows.

- At low frequencies, one may use filter constants which are appropriately selected for two weakly coupled transmission lines [17, 27, 38, 39], one for the pump and one for the main incident wave.

- At ultra-high frequencies, a serpentine transmission line may be used for supporting the propagation of the main incident wave [40], which lowers the phase velocity relative to the modulation velocity.

- At microwave frequencies the pump wave may be supported in a closed waveguide [41], thereby achieving a fast phase velocity.

- At optical frequencies, birefringence may be
used [42], providing two wave systems of different phase velocities where the main incident may be carried on different axes of polarization.

In addition, recently, there has been an experimental demonstration of time-modulated structure [43], where the medium is periodically modulated in time only, representing the limiting case of an infinite modulation velocity, i.e., \( q = 0 \) and hence \( v_m = \Omega/q \rightarrow \infty \).

VII. CONCLUSION

We have introduced a unidirectional beam splitter and amplifier based on optical and photonic time-modulated transistions in obliquely illuminated space-time-modulated (STM) media. The operation of this dynamic beam splitter is demonstrated by both the analytical, closed-form and numerical simulation results. While the normally illuminated STM media have been previously used for the realization of optical and photonic devices, including insulators, parametric amplifiers and nonreciprocal frequency generators, this paper presented the first study investigating the oblique illumination of STM media. Accordingly, this paper proposed a forward-looking application of such dynamic media. The proposed unidirectional beam splitter is endowed with unique functionalities, including adjustable one-way transmission gain, tunable splitting angle and arbitrary unequal splitting power ratio, as well as high isolation, which may substantially reduce the source and isolation requirements is optics and photonics systems. The practical realization of this beam splitter may be accomplished based on self-biased STM structures [27] in combination with optical second harmonic generators.

[1] Knill, E., Laflamme, R. & Milburn, G. J. A scheme for efficient quantum computation with linear optics. Nature 409, 46 (2001).
[2] Kok, P. et al. Linear optical quantum computing with photonic qubits. Rev. Mod. Phys. 79, 135 (2007).
[3] Zeilinger, A. General properties of lossless beam splitters in interferometry. Am. J. Phys. 49, 882–883 (1981).
[4] Keith, D. W., Ekstrom, C. R., Turchette, Q. A. & Pritchard, D. E. An interferometer for atoms. Phys. Rev. Lett. 66, 2693 (1991).
[5] Oliver, W. D. et al. Mach-zehnder interferometry in a strongly driven superconducting qubit. Science 310, 1653–1657 (2005).
[6] Macneille, S. M. Beam splitter (1946). US Patent 2,403,731.
[7] Valeur, B. & Brochon, J.-C. New trends in fluorescence spectroscopy: applications to chemical and life sciences, vol. 1 (Springer Science & Business Media, 2012).
[8] Lichtman, J. W. & Conchello, J.-A. Fluorescence microscopy. Nat. Methods 2, 910 (2005).
[9] Bayindir, M., Temelkuran, B. & Ozbay, E. Photonic-crystal-based beam splitters. Appl. Phys. Lett. 77, 3902–3904 (2000).
[10] Shi, S., Sharkawy, A., Chen, C., Pustai, D. M. & Prather, D. W. Dispersion-based beam splitter in photonic crystals. Opt. Lett. 29, 617–619 (2004).
[11] Xiao, Y. et al. Slow light beam splitter. Phys. Rev. Lett. 101, 043601 (2008).
[12] Petta, J., Lu, H. & Gossard, A. A coherent beam splitter for electronic spin states. Science 327, 669–672 (2010).
[13] Chirolli, L., Burdak, G., Kumar, S. & DiVincenzo, D. P. Superconducting resonators as beam splitters for linear-optics quantum computation. Phys. Rev. Lett. 104, 230502 (2010).
[14] Cassidy, E. S. Dispersion relations in time-space periodic media: part II-unstable interactions. Proc. IEEE 55, 1154–1168 (1967).
[15] Yu, Z. & Fan, S. Complete optical isolation created by indirect interband photonic transitions. Nat. Photonics 3, 91 – 94 (2009).
[16] Taravati, S. Giant linear nonreciprocity, zero reflection, and zero band gap in equilibrated space-time-varying media. Phys. Rev. Appl. 9, 064012 (2018).
[17] Taravati, S., Chamanara, N. & Caloz, C. Nonreciprocal electromagnetic scattering from a periodically space-time modulated slab and application to a quasisonic isolator. Phys. Rev. B 96, 165144 (2017).
[18] Salary, M. M., Jafar-Zanjani, S. & Mosallaei, H. Time-varying metamaterials based on graphene-wrapped microwaves: Modeling and potential applications. Phys. Rev. B 97, 115421 (2018).
[19] Cullen, A. L. A travelling-wave parametric amplifier. Nature 181, 332 (1958).
[20] Tien, P. K. Parametric amplification and frequency mixing in propagating circuits. J. Appl. Phys. 29, 1347–1357 (1958).
[21] Tien, P. & Suhl, H. A traveling-wave ferromagnetic amplifier. Proceedings of the IRE 46, 700–706 (1958).
[22] Cassidy, E. S. & Oliner, A. A. Dispersion relations in time-space periodic media: part I-stable interactions. Proc. IEEE 51, 1342 – 1359 (1963).
[23] Wentz, J. A nonreciprocal electrooptic device. Proceedings of the IEEE 54, 97–98 (1966).
[24] Bhandare, S. et al. Novel nonmagnetic 30-dB traveling-wave single-sideband optical isolator integrated in III/V material. IEEE J. Sel. Top. Quantum Electron. 11, 417–421 (2005).
[25] Lira, H., Yu, Z., Fan, S. & Lipson, M. Electrically driven nonreciprocity induced by interband photonic transition on a silicon chip. Phys. Rev. Lett. 109, 033901 (2012).
[26] Chamanara, N., Taravati, S., Deck-Léger, Z.-L. & Caloz, C. Optical isolation based on space-time engineered asymmetric photonic bandgaps. Phys. Rev. B 96, 155409 (2017).
[27] Taravati, S. Self-biased broadband magnet-free linear isolator based on one-way space-time coherency. Phys. Rev. B 96, 235150 (2017).
[28] Hadad, Y., Sonnens, D. L. & Alù, A. Space-time gradient metasurfaces. Phys. Rev. B 92, 100304 (2015).
SUPPLEMENTAL MATERIAL

The beam splitter is placed between \( z = 0 \) and \( z = d \), and represented by the space-time-varying permittivity of

\[
\epsilon(z,t) = \epsilon_{av} + \delta \epsilon \sin(qz - \Omega t),
\]

where \( \Omega = 2\omega_0 \) and

\[
q = \frac{2k_z}{\gamma}.
\]

The electric field inside the beam splitter is defined based on the superposition of the \( m = 0 \) and \( m = -1 \) space-time harmonics fields, i.e.,

\[
E_{S}(x, z, t) = a_0(z)e^{-i(k_zx+k_zz-\omega_0t)} + a_{-1}(z)e^{i(-k_zx+(q-k_z)z-\omega_0t)},
\]

and the corresponding wave equation reads

\[
\frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2[\epsilon(t, z) \mathbf{E}]}{\partial t^2}.
\]

Inserting the electric field in (S3) into the wave equation in (S4) results in

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) \left[ a_0(z)e^{-i(k_zx+k_zz-\omega_0t)} + a_{-1}(z)e^{i(-k_zx+(q-k_z)z-\omega_0t)} \right]
\]

\[
= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left[ \left( \epsilon_{av} + \frac{\delta}{2} e^{i(qz-2\omega_0t)} + \frac{\delta}{2} e^{-i(qz-2\omega_0t)} \right) \left( a_0(z)e^{-i(k_zx+k_zz-\omega_0t)} + a_{-1}(z)e^{i(-k_zx+(q-k_z)z-\omega_0t)} \right) \right],
\]

and applying the space and time derivatives, while using a slowly varying envelope approximation, yields

\[
\left[ (k_z^2 + k_z^2) a_0(z) - 2i k_z \frac{\partial a_0(z)}{\partial z} \right] e^{-i(k_zx+k_zz-\omega_0t)}
\]

\[
+ \left[ (k_z^2 + (q-k_z)^2) a_{-1}(z) - 2i (k_z - q) \frac{\partial a_{-1}(z)}{\partial z} \right] e^{i(-k_zx+(q-k_z)z-\omega_0t)}
\]

\[
= \frac{\omega_0^2}{c^2} \left( \left( \epsilon_{av} + \frac{\delta}{2} e^{i(qz-2\omega_0t)} + \frac{\delta}{2} e^{-i(qz-2\omega_0t)} \right) a_0(z)e^{-i(k_zx+k_zz-\omega_0t)} + a_{-1}(z)e^{i(-k_zx+(q-k_z)z-\omega_0t)} \right).
\]
We then multiply both sides of Eq. (S6) with $e^{i(k_xx + k_zz - \omega_0t)}$, which gives
\[
\left[(k_x^2 + k_z^2)a_0(z) - 2ik_x \frac{\partial a_0(z)}{\partial z}\right] + \left[(k_x^2 + (q - k_z)^2)a_{-1}(z) - 2i(k_x - q) \frac{\partial a_{-1}(z)}{\partial z}\right] e^{i(qz - 2\omega_0t)} = \frac{\omega_0^2}{c^2} \left[\epsilon_{sv} + \frac{\delta}{2} e^{i(qz - 2\omega_0t)} + 9 \frac{\delta}{2} e^{-i(qz - 2\omega_0t)} a_0(z) + \left[\epsilon_{sv} e^{i(qz - 2\omega_0t)} + 9 \frac{\delta}{2} e^{-i(qz - 2\omega_0t)} + \frac{\delta}{2}\right] a_{-1}(z)\right],
\]
(S7)
and next, applying $\int_0^\pi dz$ to both sides of (S7) yields
\[
\frac{da_0(z)}{dz} = \frac{ik_0^2}{2k_x} \left[\left[\epsilon_{sv} - \epsilon_t\right] a_0(z) + \frac{\delta}{2} a_{-1}(z)\right],
\]
(S8)
which may be cast as
\[
\frac{da_0(z)}{dz} = M_0 a_0(z) + C_0 a_{-1}(z),
\]
(S9)
where
\[
M_0 = \frac{ik_0^2}{2k_x} (\epsilon_{sv} - \epsilon_t), \quad C_0 = \frac{i\delta k_0^2}{4k_x}.
\]
(S10)
Following the same procedure, we next multiply both sides of (S7) with $e^{-i(qz - 2\omega_0t)}$, and applying $\int_0^\pi dz$ in both sides of the resultant, which reduces to
\[
\left[(k_x^2 + (q - k_z)^2)a_{-1}(z) - 2i(k_x - q) \frac{\partial a_{-1}(z)}{\partial z}\right] e^{i(qz - 2\omega_0t)} = \frac{\omega_0^2}{c^2} \left[\epsilon_{sv} + \frac{\delta}{2} e^{i(qz - 2\omega_0t)} + 9 \frac{\delta}{2} e^{-i(qz - 2\omega_0t)} a_{-1}(z) + \left[\epsilon_{sv} e^{i(qz - 2\omega_0t)} + 9 \frac{\delta}{2} e^{-i(qz - 2\omega_0t)} + \frac{\delta}{2}\right] a_{-1}(z)\right],
\]
(S11)
which may be cast as
\[
\frac{da_{-1}(z)}{dz} = C_{-1} a_0(z) + M_{-1} a_{-1}(z),
\]
(S12)
where
\[
M_{-1} = \frac{ik_0^2}{2(k_x - q)} \left[\epsilon_{sv} - \epsilon_t, k_x^2 + (q - k_z)^2\right], \quad C_{-1} = \frac{i\delta k_0^2}{4(k_x - q)}.
\]
(S13)
Equations (S9) and (S12) form a matrix differential equation. We then look for independent differential equations for $a_0(z)$ and $a_{-1}(z)$, which is expressed by
\[
\frac{d^2 a_0(z)}{dz^2} - (M_0 + M_{-1}) \frac{da_0(z)}{dz} + (M_0M_{-1} - C_0C_{-1}) a_0(z) = 0,
\]
(S14)
\[
\frac{d^2 a_{-1}(z)}{dz^2} - (M_0 + M_{-1}) \frac{da_{-1}(z)}{dz} + (M_0M_{-1} - C_0C_{-1}) a_{-1}(z) = 0.
\]
(S15)
which are second order differential equations. Using the initial conditions of $a_0(0) = E_0$ and $a_{-1}(0) = 0$ gives
\[
a_0(z) = \frac{E_0}{2\Delta} \left((M_0 - M_{-1} + \Delta) e^{\frac{M_0 + M_{-1} + \Delta}{2}z} - (M_0 - M_{-1} - \Delta) e^{\frac{M_0 + M_{-1} - \Delta}{2}z}\right),
\]
(S16)
\[
a_{-1}(z) = C_{-1} \frac{E_0}{\Delta} \left(e^{\frac{M_0 + M_{-1} + \Delta}{2}z} - e^{\frac{M_0 + M_{-1} - \Delta}{2}z}\right),
\]
(S17)
where
\[
\Delta = \sqrt{(M_0 + M_{-1})^2 - 4(M_0M_{-1} - C_0C_{-1})} = \sqrt{(M_0 - M_{-1})^2 + 4C_0C_{-1}}.
\]
(S18)
Assuming an imaginary result for $\Delta$ (which occurs for subluminal and superluminal space-time modulations), $a_0(z)$ and $a_{-1}(z)$ acquire a periodic sinusoidal form, where the maximum amplitude of them occur at the coherence length $z = l_c$, where
\[
\frac{d}{dz} a_0(z)|_{z=l_c} = \frac{d}{dz} a_{-1}(z)|_{z=l_c} = 0,
\]
(S19)
which corresponds to

\[
I_c = \pi \left( \left( \frac{k_0^2 [\epsilon_{av} - \epsilon_z]}{k_z - q} \right)^2 + \frac{\delta^2 k_0^4}{4 k_z (k_z - q)} \right)^{-1}.
\]

For luminal space-time modulation, where \( \gamma \to 1 \) and \( q \to 2k_z \), \( a_0(z) \) and \( a_{-1}(z) \) acquire exponentially growing profile, and hence, the total electric field inside the slab reads

\[
E(x, z, t) = E_0 \cosh \left( \frac{\delta k_0^2}{4k_z} z \right) e^{-i(k_z x + k_z z - \omega_0 t)} - i \frac{\delta k_0^2}{2k_z} E_0 \sinh \left( \frac{\delta k_0^2}{4k_z} z \right) e^{i(k_z x + (q - k_z) z - \omega_0 t)},
\]

which demonstrates the wave amplification of both \( m = 0 \) and \( m = -1 \) ST harmonics, for the luminal space-time modulation.