Concurrent Learning Based Dual Control for Exploration and Exploitation in Autonomous Search

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Abstract—In this paper, a concurrent learning framework is developed for source search in an unknown environment using autonomous platforms equipped with onboard sensors. Distinct from the existing solutions that require significant computational power for Bayesian estimation and path planning, the proposed solution is computationally affordable for onboard processors. A new concept of concurrent learning using multiple parallel estimators is proposed to learn the operational environment and quantify estimation uncertainty. The search agent is empowered with dual capability of exploiting current estimated parameters to track the source and probing the environment to reduce the impacts of uncertainty, namely Concurrent Learning for Exploration and Exploitation (CLEE). In this setting, the control action not only minimises the tracking error between future agent’s position and estimated source location, but also the uncertainty of predicted estimation. More importantly, the rigorous proven properties such as the convergence of CLEE algorithm are established under mild assumptions on sensor noises, and the impact of noises on the search performance is examined. Simulation results are provided to validate the effectiveness of the proposed CLEE algorithm. Compared with the information-theoretic approach, CLEE not only guarantees convergence, but produces better search performance and consumes much less computational time.

Index Terms—Autonomous search, optimisation and learning, dual control, exploration and exploitation, path planning, source search and estimation.

I. INTRODUCTION

PUBLIC concern towards safety and health issues has been significantly exacerbated during the past few decades, due to rapid industrial development and increasing risks of terrorism. Identifying sources of airborne release (including chemical, biological, radiological and nuclear (CBRN) materials) is one of the most important tasks in disaster management and environment protection [1]. In the early literature, source term estimation (STE) is mainly supported by onsite measurement using static sensor networks that are deployed beforehand in some specific areas of potential risks [1-4]. This type of strategy is very costly, and only feasible for high-risk industry, e.g. nuclear power plants [8]. Recently, significant research efforts have been dedicated to the development of dynamic estimation of airborne release assisted by mobile platforms, for example, autonomous ground robots [6,9] and unmanned aerial vehicles (UAVs) [10,12]. Compared with conventional static methods, autonomous search is much more flexible and cost-effective in emergent accident management. Comprehensive surveys of recent progress on source localisation using autonomous vehicles have been reported in several review papers [1,13,14].

There are various methods dealing with this problem [1,7]. Among them, informative path planning (IPP) becomes increasingly popular, e.g. Infotaxis [8] and Entrotaxis [15]. Vergassola et al. [8] proposed an informative search approach, referred as Infotaxis, by which the agent moves to the next position that is expected to minimise uncertainties of the posterior distribution. More recently, Hutchinson et al. [15] developed Entrotaxis algorithm that steers the agent to search over the most uncertain area in the next movement. Essentially, information-theoretic approaches aim to reduce uncertainties of estimated source location and unknown environment parameters. Therefore, the reward function, which the agent targets to maximise while making its next movement, is defined according to the information gain using some informative measures, for example, entropy, Kullback-Leibler divergence and variance.

Recently, Chen et al. [7] have reformulated the autonomous search problem from a control-theoretic perspective, referred as Dual Control for Exploration and Exploitation (DCEE). The ultimate goal of autonomous search is to design a control strategy that can navigate the agent to an unknown release in an unknown environment, which is a well-posed goal-oriented control problem. Distinct from traditional control settings where operational systems are manipulated by following predefined references or setpoints, the autonomous search problem does not have such a given reference or path that can directly lead the agent to the source. Instead, the agent is required to explore the operational environment to learn the source parameters, and at the same time exploit its belief to move towards the source. From the perspective of dual control, information-theoretic approach is a pure exploration strategy aiming at collecting information from unknown environment while model predictive control (MPC) is a pure exploitation approach targeting at making full use of current uncertain estimation. This novel dual control framework achieves a natural balance between the two objectives, and has demonstrated superior performance in real experiments compared with MPC and IPP.

Although DCEE offers a conceptually promising framework in autonomous search, similar to all the existing information-theoretic approaches, currently it still suffers from two drawbacks: computational burden and no rigorous analysis of its properties such as stability and convergence. All the aforemen-
tioned approaches demand massive computational burden imposed by nonlinear particle filters and optimisation based path planning [15] [16]. More specifically, the Bayesian inference engine is involved in the optimisation loop for informative path planning since the influence of the control action on the predicted posterior of the estimated source and environment parameters is evaluated at each iteration. Consequently, the search algorithm is restricted to be a set of certain moving directions with a fixed step size, and currently carried out remotely in a control centre, rather than onboard on the mobile robot in the existing experimental studies [6] [7] [10]. This significantly restricts their practical applications in autonomous search in a wide area or an extreme environment. Hence, less computationally demanding alternatives are needed so that online computation can be achieved by portable processors on mobile platforms. Currently there are no proven properties of IPP autonomous search strategies. This was actually a main motivation of formulating it from a control-theoretic approach in [7] since it enables to get access to rich knowledge and tools available in control theory. No rigorous analysis on convergence is provided but mainly illustrated through extensive simulation and experimental studies. In IPP and DCEE, the agent’s movement (path planning) and source estimation (environment acquisition) are strongly coupled: the agent takes actions according to the current estimation of source parameters and the estimators update their knowledge by using the concentration collected at agent’s new position determined path planning. This coupling, together with noisy measurement, environment turbulence, complicated particle filtering and optimisation involved in implementation of the search strategy, makes the rigorous analysis of theoretic properties of these search strategies quite challenging. Those two critical limitations motivate this study.

In this paper, inspired by the concept of DCEE, we propose a concurrent learning based DCEE scheme with multiple estimators that encompasses dual effects: driving the agent to the believed location by exploiting current estimation, and reducing uncertainties by exploring the unknown operational environment, which is referred as Concurrent Learning for Exploration and Exploitation (CLEE) for the sake of simplicity. The underlying principle of the concurrent learning scheme advocated in this paper is distinct from the classic dual control in handling the two intricate coupling elements of the system and the environment. Existing dual control approaches impose a probing effect on the system itself, for example, state estimation in stochastic control [17] [18] and parameter estimation in adaptive control [19] [20]. On the other hand, the dual effect introduced in our formulation is used to explore the operational environment, as our objective is to acquire a better understanding of the unknown environment such that the agent is able to approach to the true source location.

To address the two challenges of computational burden and proven properties, two approaches are proposed in this paper. First, instead of implementing computationally demanding particle filtering, an efficient multi-estimator scheme is proposed for source and environment learning. The number of estimators used in CLEE is much smaller than that of particles required for Bayesian filters in information-theoretic algorithms. These estimators run in parallel from a set of randomly started initial estimates. Compared with employing a single estimator, this multi-estimator approach provides a means to quantify uncertainty associated with source estimators, which is of great importance to empower the search agent with dual capability of exploration and exploitation. Furthermore, it significantly improves performance and robustness over a single estimator. The performance of single estimator (such as observer and learning machine) is often severely influenced by the initialisation and setting of the individual estimator, for example, state estimation [21], disturbance observer [22] and parameter adaptation [23]. To our knowledge, there is few result on multi-estimator assisted control algorithms. Devising multiple parallel estimators for the source parameters is conducive to eliminating undesirable behaviour caused by random initialisation of an individual, and also it allows us to take advantage of a priori probability density function (PDF) of source parameters. Secondly, to further reduce the computational load, effective gradient-based optimisation algorithms are utilised to replace the complicated path planning process in the existing methods. It is shown that by combining these two techniques, we are able to reduce the computational load by 100 times while significantly increasing the admissible control set. Most importantly, we establish theoretical guarantee of convergence of the CLEE algorithm, and analyse the steady-state performance of the estimation and search by directly linking them with sensor and environment characters.

In summary, the key contributions of this paper are enumerated as follows.

1) Autonomous search of a hazardous release is formulated as a learning-based control problem. By this formulation, the search agent is able to approach to an unknown target, by actively and autonomously learning the operational environment.

2) A concurrent learning algorithm with multiple environment estimators is developed, which achieves a balanced trade-off between exploitation of believed source location and exploration of uncertain environment, that is, simultaneously navigating the agent to the source and reducing uncertainties associated with the acquired environment knowledge.

3) For the first time, the convergence of the proposed autonomous search algorithm, CLEE, is rigorously established under sensor noises and turbulence disturbances. Possible countermeasures to alleviate the influence of sensor noises are discussed.

4) The proposed CLEE provides a computationally efficient solution for autonomous search. Simulation results are provided to demonstrate the performance of the proposed method in comparison with information-theoretic approaches. Our solution shows superior performance with a significant reduce in computational time and memory storage.

The remainder of this paper is organised as follows. In Section II we formulate the autonomous search problem and develop feasible value functions for the path planning and estimation. In Section III CLEE algorithm is proposed by de-
employing multiple environment estimators. Section IV provides simulation results and detailed discussions in comparison with existing approaches. Section V concludes this paper.

II. PROBLEM FORMULATION

A. Agent Modelling

The searching agent is considered as a fully autonomous vehicle, for example, a mobile robot or a UAV, which is equipped with chemical/biological sensors. We assume that the agent has been devised with a low-level controller that can steer the agent to the desired position directed by high-level decision-making process. Therefore, the dynamics of the search agent can be simplified as

\[ p_{k+1} = p_k + u_k \]  

where \( p_k \) is the position of the searching agent at current step \( k \) with \( \Omega \) being the domain of searching space, and \( u_k \in \mathcal{U} \subseteq \mathbb{R}^3 \) is the control action with \( \mathcal{U} \) being the admissible set of actions. It is worth mentioning that in this paper the admissible set \( \mathcal{U} \) can be continuous, which is distinct from the existing results in [6] [7] [10] where the search is restricted to certain directions with a fixed step size.

B. Sensor Modelling

Information collection in autonomous search of an airborne release is mainly from onboard chemical/biological sensors. As the search agent moves to a new position, concentration measurement will be taken. The agent is required to remain at current position for a short period to obtain a reliable reading, referred as the sampling time. It is likely that the sensors fail to detect any meaningful readings. Sensors deployed on mobile platforms are usually low-price portable devices whose performances are seriously constrained by limited power supply and local turbulence. As summarised in [11], there are several scenarios that may cause a non-detection event: 1) there is no chemical material present in the current position, or present but out of the detection range of the sensor; 2) the source is present in the position but the detector fails to receive any material due to intermittent turbulence; 3) the concentration present in the position but the detector fails to any meaningful readings. Sensors deployed on mobile platforms are usually low-price portable devices whose performances are seriously constrained by limited power supply and local turbulence. As summarised in [11], there are several scenarios that may cause a non-detection event: 1) there is no chemical material present in the current position, or present but out of the detection range of the sensor; 2) the source is present in the position but the detector fails to receive any material due to intermittent turbulence; 3) the concentration present in the position but the detector fails to any meaningful readings. Sensors deployed on mobile platforms are usually low-price portable devices whose performances are seriously constrained by limited power supply and local turbulence. As summarised in [11], there are several scenarios that may cause a non-detection event: 1) there is no chemical material present in the current position, or present but out of the detection range of the sensor; 2) the source is present in the position but the detector fails to receive any material due to intermittent turbulence; 3) the concentration present in the position but the detector fails to receive any meaningful readings.

In summary, the sensor reading can be modelled as

\[ z(p_k, \Theta_s) = \begin{cases} M(p_k, \Theta_s) + v_k, & D = 1 \\ \bar{v}_k, & D = 0 \end{cases} \]

where \( M \) is the true chemical concentration, \( D \) denotes either a detection event \( D = 1 \) or a non-detection event \( D = 0 \), and \( v_k \) and \( \bar{v}_k \) represent additive Gaussian noises imposed on the sensor readings. The probability distribution of \( D \) is a random process, usually characterised by Poisson distribution.

C. Objective Function Construction

Atmospheric transport and dispersion model (ATDM), governing the spatial-temporal diffusion of the pollutant materials, is utilised to predict the concentration in space and time, given the parameters of a release. In this paper, we denote true source parameters as \( \Theta_s = [s^T, q]^T \in \mathbb{R}^4 \) with \( s = [s_x, s_y, s_z]^T \in \mathbb{R}^3 \) being the position of the source and \( q \in \mathbb{R}^+ \) representing a positive release rate. Expectation of the concentration, \( E[z(p_k, \Theta_s)] \), at agent’s position \( p_k \) can be obtained by

\[ E[z(p_k, \Theta_s)] = M(p_k, \Theta_s) \]

where \( M(p_k, \Theta_s) = \frac{q}{4\pi \zeta} \left( \left\| p_k - s \right\| \right)^{-\frac{3}{2}} \exp \left( -\frac{\left\| p_k - s \right\|}{\zeta} \right) \times \exp \left( \frac{-(p_k,x - s_x) u_x \cos \phi_s}{2\zeta s} \right) \times \exp \left( \frac{-(p_k,y - s_y) u_y \sin \phi_s}{2\zeta s} \right) \]

with \( u_x \), \( u_y \), \( \phi_s \), \( \zeta \), and \( s \) being the wind speed, wind direction, particle lifetime, and the distance to the centre (in terms of expectation). Thus, it can be used to formulate an optimisation objective for the autonomous agent, by taking the position \( p_k \) as the optimisation variable. Following the convention in optimisation theory, the objective function is defined as

\[ g(p_k, \Theta_s) = (M_0 - M(p_k, \Theta_s))^2 \]

where \( M_0 \) is a predefined upper bound of the concentration measurement. It is clear that the optimal solution of \( g \) is \( p_k^* = s \), where \( M(p_k^*, \Theta_s) \) is maximised.

To estimate the source location \( s \) and release rate \( q \) based on available measurements, we may define an additional value function taking the source term as the decision variable. Least square methods can serve for this purpose, given by

\[ F(\Theta) = \sum_{i=1}^{k} f_i(\Theta) = \sum_{i=1}^{k} [M(p_i, \Theta) - z_i(p_i)]^2 \]

where \( z_i(p_i) \) denotes the measured concentration at agent position \( p_i, \forall i = 1, \ldots, k \). We assume that there is a single source of release in the environment, and an optimal estimation of the source term is available, denoted by \( \Theta_k = [s_k^T, q_k]^T \) at the \( k \)-th step. Then, minimising \( F(\Theta) \) results in \( \Theta_k \rightarrow \Theta_s \), as the sampling instance \( k \rightarrow \infty \). However, in the real situations, only local optimal solutions can be expected, due to the presence of measurement noises and model uncertainties, as we will analyse in the sequel.

III. CONCURRENT LEARNING FOR DUAL CONTROL WITH EXPLORATION AND EXPLOITATION

A. Framework of Concurrent Learning with Dual Effects

Suppose that the search agent aims to minimise the objective function defined in \( g \) at each step. Then, it will move to the position where the maximum concentration measurement is
expected based on the current belief of the source location, i.e., its estimate \( \Theta_k \). This type of method can be understood as a pure exploitation strategy, which aims to make full use of the current estimation of the source parameters \( \Theta_k \). If the source estimators were perfect without any noisy disturbances or uncertainties, then exploiting the trustworthy estimator would accelerate the speed of finding the source and its associated parameters. Nonetheless, the operational environment is often uncertain, and perfect estimation is never available. This motivates the development of active learning methods to autonomously explore the unknown environment so as to construct more accurate estimators.

The dual control framework for autonomous source search and estimation was firstly introduced by Chen et al. \(^1\) recently. The goal is to drive the agent towards the believed position of a release and in the meanwhile reduce uncertainty associated with the estimation of the target position. Generally speaking, uncertainty is often measured in a stochastic sense for probability density function of a variable. In \(^4\), particle filters are utilised to estimate the source parameters and the uncertainty associated with the estimated source target. However, it requires a large number of particles to support the Bayesian inference engine, which incurs heavy computational burden. Quantifying the uncertainty is of great importance, as demonstrated in our previous works \(^6\). To alleviate this problem, we thus introduce a set of \( N \) source term estimators and they are initialised according to the prior knowledge of the source parameters. It is worth emphasising that the number of estimators \( N \) is much smaller than that of particles in Bayesian filters as shown later.

The concentration information collected up to time step \( k \) is denoted by \( Z_k := \{ z_1 (p_1), z_2 (p_2), \ldots, z_k (p_k) \} \). Let \( \Theta_k^i \) be the source term estimation of the \( i \)th estimator at the \( k \)th measurement, and \( \Theta_k := \frac{1}{N} \sum_{i=1}^{N} \Theta_k^i \) as the nominal estimation, i.e., the mean, of the source parameters. The posterior distribution of source estimation can be represented as \( p(\Theta | Z_k) \) at time \( k \). When the search agent moves to a new position directed by the control input \( u_k \), the hypothetical posterior distribution of source estimation will be updated as \( \hat{p}(\Theta | Z_{k+1}) = p(\Theta | Z_k) \) where \( Z_{k+1} = \{ \hat{z}_{k+1} | u_k \} \), and consequently the future belief of concentration can be regarded as a random variable conditional on \( u_k \), denoted as \( z_{k+1} \sim \hat{p}(\hat{z}_{k+1} | u_k) \). As a result, the control input \( u_k \) will not only affect the future concentration measurement at agent’s new position but also affect the belief of future measurement distribution.

Motivated by the above discussion, the control input \( u_k \) should be designed to navigate the agent to the position where the predicted posterior of the measurement \( \hat{z}_{k+1} | u_k \) is close to the predefined threshold \( M_0 \). Therefore, the conditional cost function can be formulated as

\[
\min_{u_k \in U} J(u_k) = \min_{u_k \in U} \mathbb{E }_\theta [ \mathbb{E }_{\hat{z}_{k+1} | u_k} (M_0 - \hat{z}_{k+1} | u_k)]
\]

subject to \( p_{k+1} = p_k + u_k \).

The physical interpretation is, based on all the available information including priors and available measurements, we would like the robot moving to a place where the predicted maximum concentration is located. This mechanism is behind Chemotaxis, a widely adopted search strategy in nature from bacteria to human being \(^23\). We show that the control action \( u_k \), obtained from the optimisation problem in (6), implicitly carries dual effects. We define \( \tilde{z}_{k+1} | k \) as the nominal predicted concentration of the future virtual measurements, i.e. the mean of \( p(\hat{z}_{k+1} | u_k) \), written as

\[
\tilde{z}_{k+1} | k := \mathbb{E } \hat{z}_{k+1} | k Z_{k+1} | k.
\]

Based on the definition of \( \tilde{z}_{k+1} | k \), we can further define \( \hat{z}_{k+1} | k = \tilde{z}_{k+1} | k - \tilde{z}_{k+1} | k \). Therefore, the objective function can be reformulated as

\[
J(u_k) = \mathbb{E }_{\Theta, \tilde{z}_{k+1} | k} \left[ (M_0 - \tilde{z}_{k+1} | k )^2 \right] Z_{k+1} | k.
\]

Expanding (6) leads to

\[
J(u_k) = \mathbb{E } \left[ (M_0 - \tilde{z}_{k+1} | k )^2 \right] Z_{k+1} | k
-2 \mathbb{E } \tilde{z}_{k+1} | k (M_0 - \tilde{z}_{k+1} | k ) Z_{k+1} | k
+ \mathbb{E } \tilde{z}_{k+1} | k Z_{k+1} | k.
\]

Since \( M_0 \) and \( \tilde{z}_{k+1} | k \) are deterministic variables and \( \mathbb{E } \tilde{z}_{k+1} | k = 0 \), it follows that

\[
J(u_k) = \mathbb{E } \left[ (M_0 - \tilde{z}_{k+1} | k )^2 \right] Z_{k+1} | k
+ \mathbb{E } \tilde{z}_{k+1} | k Z_{k+1} | k.
\]

In case of \( N \) estimators, we have \( \tilde{z}_{k+1} | k = \frac{1}{N} \sum_{i=1}^{N} \hat{z}_{k+1} | k, \) with \( \hat{z}_{k+1} | k \) being the predicted measurement at agent’s future position \( p_k | 1 \) based on the \( i \)th source estimator \( \Theta_k^i \). Then, the optimisation problem for CLEE can be formulated as

\[
\min_{u_k \in U} J(u_k) = \min_{u_k \in U} \left[ (M_0 - \tilde{z}_{k+1} | k )^2 + p_{k+1} | k \right]
\]

subject to \( p_{k+1} = p_k + u_k \).

**Remark 1:** According to the definition of \( p_{k+1} | k \) in (11b), it is clear that \( p_{k+1} | k \) is the predicted covariance of \( z_{k+1} | k, \) where \( V_i = 1, \ldots, N, \) given that each estimator has a uniform weight of \( 1/N \). The value function in (11a) consists of two parts: the first part exploits current information by navigating the agent towards the believed position of higher concentration, and the second part aims to gather more information by reducing the covariance of future virtual measurements. In short, exploitation drives the agent to the position where high concentration is expected by minimising \( (M_0 - \tilde{z}_{k+1} | k)^2, \) while the exploration effect makes the agent search over some position that can reduce the predicted covariance of the concentration by minimising \( p_{k+1} | k. \)

**Remark 2:** Recently, how to balance between exploration and exploitation has aroused extensive discussions and arguments in many areas, in particular artificial intelligence, optimisation and decision-making \(^17\). In some cases, artificial weights are introduced on purpose to impose both effects \(^17\). From the above formulation process of our framework, a natural balance between the two effects is derived.
from a physically meaningful cost function. Accordingly, our framework eliminates the requirement for choosing trade-off weights.

To obtain the covariance $\mathcal{P}_{k+1|k}$, we resort to the classical principle of predicting covariance estimation in extended Kalman filters [26], which can be formulated as

$$
\mathcal{P}_{k+1|k} = F_k^T \mathcal{P}_{k|k} F_k + \mathcal{Q}_k
$$

(12)

where $\mathcal{P}_{k|k}$ denotes current covariance of estimated measurement, and $F_k = \left[ F_{k+1|k}, \ldots, F_{k+N|k} \right]^T$. In (12), $p_{k+1|k}$ is future position of the agent, which will serve as the optimisation variable in the CLEE algorithm.

Now, we can present the gradient-based optimisation algorithm for the source term estimation and path planning. For notational convenience, we will use $g(p_k) = (M_0 - z_{k+1})^2$ to denote the first term in the dual objective [11a]. The $N$ source estimators can be updated according to

$$
\Theta_{k+1}^i = \Theta_k^i - \eta_k \nabla \Theta f_k(\Theta_k^i), \quad \forall i = 1, 2, \ldots, N
$$

(13)

and the path planning is given by

$$
p_{k+1} = p_k + u_k
$$

$$
u_k = -\delta \left[ \nabla_p g(p_k) + \nabla_p \mathcal{P}_{k+1|k} \right]
$$

(14)

where $\nabla(\cdot)$ denotes the estimated gradients, and $\eta, \delta \in \mathbb{R}^+$ are constant step sizes to be designed. Note that $\nabla_p \mathcal{P}_{k+1|k}$ is pure prediction without noises. Basically, algorithms (13) and (14) use gradient descent method to ensure that the agent moves towards the believed position of a release and the source estimators converge to the true parameters that minimise the least square function in [5].

The approximated gradients can be written as the true gradients buried with random additive noises

$$
\nabla \Theta f_k(\Theta_k) = \nabla \Theta f_k(\Theta_k^i) + \mathcal{N}(\Theta_k^i)
$$

$$
\nabla_p g(p_k) = \nabla_p g(p_k) + \mathcal{H}(p_k)
$$

(15)

where $\mathcal{N}(\Theta_k^i)$ and $\mathcal{H}(p_k)$ denote the gradient noises, which can be regarded as a source of perturbation to the true gradient. There are a number of reasons for the use of noisy gradients. First, in real experiment, the position where measurement is collected might not be exactly the same position as $p_k$, due to mapping errors and turbulence influences, etc. Second, the concentration measurements collected are strongly subject to sensor noises and uncertainties. In this paper, we deploy a random noise term to quantify those impacts, and we will analyse how the agent will behave under uncertain information.

In summary, the implementation structure of CLEE can be encapsulated in Algorithm [1]. The key feature of our algorithm is the adoption of dual effects for exploitation and exploration. Multiple estimators have been developed for source term construction, which provides an effective tool for quantifying information uncertainty. In order to reduce future covariance of the measurement, we need to predict future concentration by using current parameters of ATDM, based on which the exploration objective $\mathcal{P}_{k+1|k}$ can be formulated as a function of optimisation variable $p_{k+1|k}$.

To establish the convergence of Algorithm [1], some basic assumptions on the gradient noises are required.

**Assumption 1:** The gradient noise, $\mathcal{H}(p_k)$, satisfies the following properties:

$$
\mathbb{E} [\mathcal{H}(p_k) | \mathbf{E}_{k-1}] = 0
$$

$$
\mathbb{E} [\left\| \mathcal{H}(p_k) \right\|^2 | \mathbf{E}_{k-1}] \leq \phi^2 \left\| p_k - s \right\|^2 + \varphi^2
$$

(16)

(17)

where $\phi$ and $\varphi$ are non-negative constants, and $\mathbf{E}_{k-1}$ represents the filtration (past history in $\sigma$-field) of $p_j$, for all $j \leq k-1$. Moreover, the noise $\mathcal{N}(\Theta_k^i)$ has similar conditions, given by

$$
\mathbb{E} [\mathcal{N}(\Theta_k^i) | \mathbf{E}_{k-1}] = 0
$$

$$
\mathbb{E} [\left\| \mathcal{N}(\Theta_k^i) \right\|^2 | \mathbf{E}_{k-1}] \leq \rho^2 \left\| \Theta_k^i - \Theta_s \right\|^2 + \sigma^2
$$

(18)

(19)

where $\rho$ and $\sigma$ are non-negative constants, and $\mathbf{E}_{k-1}$ is the filtration of $\Theta_j, \forall j \leq k-1$.

**Remark 3:** Assumption [1] implies that the gradient noises used in the optimisation algorithm [13] and (14) have zero mean and bounded variance. It is reasonable to require zero mean, since at every position the noises should be unbiased. The noise variances can be non-uniform, as in the real situation of autonomous search. It is assumed that the bounds of gradient variance are proportional to the current distance between the agent and the source. When the agent is far away from the source, the noise variances are usually larger because turbulence and model uncertainties will be more significant. This relation is revealed by the second inequalities.
in (17). Alternatively, we can eliminate the explicit dependence on $s$ by defining a new set of constants, $\tilde{\phi}^2 = 2\phi^2$ and $\tilde{\varphi}^2 = 2\phi^2||s||^2 + \varphi^2$ such that
\[
\mathbb{E}\left[||\mathcal{H}(p_k)||^2 | \Xi_{k-1}\right] \leq 2\phi^2||p_k||^2 + 2\phi^2||s||^2 + \varphi^2
\]
(20)
This equivalence provides a different way to quantify the terms of the gradient noises, which is more commonly-used in practice [27]. By setting $\varphi^2$ large enough, the variance bound only relies on the current location $p_k$. Similar conclusions can be drawn for $\mathcal{N}(\Theta_i^k)$.

**Remark 4:** There is an important difference between the traditional gradient-based search methods such as in Chemotaxis and the proposed CLEE algorithm in this study. In the early works (e.g. [28, 29]), mobile robots are equipped with sensors that directly collect the local gradients of concentration, and utilise the measured gradients to plan their next movement. Clearly, this type of search suffers severely from sensor errors and turbulent fluctuations, since the next movement is purely determined by instantaneous gradient measurements. In our framework, the search agent measures the local concentration value, and uses all the available information, including priors and available measurements, to learn the source parameters. Based on the acquired knowledge of source, the search agent uses model evaluated gradients to plan its next movement. This learning process lasts over the entire period of search, and therefore an instant sample, subject to noise and turbulence, will not cause considerable interruption to the path planning. Combining with the multi-estimator scheme, the robustness of CLEEN algorithm is significantly improved, and is able to achieve comparable search performance as information-theoretic methods with less computational load, which will be demonstrated later through theoretical and simulation studies.

**B. Convergence Analysis**

In this subsection, we will show that the path planning algorithm [14] in conjunction with multiple source estimators [13] will lead the agent to a small neighbourhood of the source location $s$. First, we show that $N$ estimators converge to true source parameters with a bounded error at a geometric rate.

**Lemma 1:** Consider a dispersion described by ATDM [1] and the measurement errors and disturbances satisfy Assumption [1]. All the $N$ source estimators in [13] converge to a neighbourhood of the true position of the release $s$ from a random initialisation set if the learning rate $\eta$ of each estimator satisfies
\[
0 < \eta < \min\left\{\frac{2\gamma}{2 + \rho^2}, \frac{2\gamma}{\gamma^2 + \rho^2}\right\}
\]
(21)
where $0 \leq \gamma \leq \overline{\gamma}$ are positive constants of Hessian bounds of the least square function in [5]. Moreover, the expected mean-square-errors (MSE), $\mathbb{E}[||\Theta_i^k - \Theta_s||^2], \forall i = 1, \ldots, N$, converge at a geometric rate to a bounded neighbourhood of zero, given by
\[
\lim_{k \to \infty} \mathbb{E}[||\Theta_i^k - \Theta_s||^2] \leq \frac{\eta^2\rho^2}{1 - \max\{(1 - \overline{\gamma})^2, (1 - \overline{\gamma})^2\} - \eta^2\rho^2}
\]
(22)
The expectation of estimation error converges zero, i.e.,
\[
\lim_{k \to \infty} \mathbb{E}[\Theta_i^k - \Theta_s] = 0, \forall i = 1, 2, \ldots, N.
\]
(23)
**Proof:** It follows from (13) and (15) that
\[
\Theta_i^{k+1} = \Theta_i^k - \eta \left[\nabla f_k(\Theta_i^k) + \mathcal{N}(\Theta_i^k)\right], \forall i = 1, 2, \ldots, N
\]
(24)
Now, let $\tilde{\Theta}_i^k = \Theta_i^k - \Theta_s$ denote the error of the agent’s estimation relative to source parameters. Then, substituting $\tilde{\Theta}_k$ into (24) results in the error dynamics as
\[
\tilde{\Theta}_i^{k+1} = \tilde{\Theta}_i^k - \eta \nabla f_k (\Theta_i^k) - \eta \mathcal{N}(\Theta_i^k).
\]
(25)
To relate the gradient term with $\tilde{\Theta}_i^{k+1}$, we resort to the mean value theorem [30]. For a twice-differentiable function $H(x) : \mathbb{R}^m \to \mathbb{R}$, the following relation holds, for any $a, b \in \mathbb{R}^m$,
\[
\nabla_x H(b) - \nabla_x H(a) = \left[\int_a^b \nabla_x^2 H(t) \, dt\right] (b-a).
\]
(26)
Therefore, applying the above theorem leads to
\[
\nabla f_k (\Theta_i^k) = \nabla f_k (\Theta_s) + \left[\int_0^1 \nabla^2 f_k (\Theta_s + \tau \tilde{\Theta}_i^k) \, d\tau\right] \tilde{\Theta}_i^k.
\]
(27)
Let us denote
\[
\mathcal{T}_k := \int_0^1 \nabla^2 f_k (\Theta_s + \tau \tilde{\Theta}_i^k) \, d\tau.
\]
(28)
Then, it follows from the bounds of the Hessian matrix of $\nabla^2 f_k (\Theta_s + \tau \tilde{\Theta}_i^k)$ that
\[
\gamma I_4 \leq \mathcal{T}_k \leq \overline{\gamma} I_4.
\]
(29)
Consequently, substituting (29) and (27) into (25) yields
\[
\tilde{\Theta}_i^{k+1} = (I_4 - \eta \mathcal{T}_k) \tilde{\Theta}_i^k - \eta \mathcal{N}(\tilde{\Theta}_i^k)
\]
(30)
where $\nabla f_k (\Theta_s) = 0$ has been used. Taking the square of the Euclidean norm of the error dynamics (30) gives
\[
||\tilde{\Theta}_i^{k+1}||^2 = ||(I_4 - \eta \mathcal{T}_k) \tilde{\Theta}_i^k - \eta \mathcal{N}(\tilde{\Theta}_i^k)||^2
\]
\[
\leq ||(I_4 - \eta \mathcal{T}_k) \tilde{\Theta}_i^k||^2 + \eta^2 ||\mathcal{N}(\tilde{\Theta}_i^k)||^2
\]
\[
+ 2\eta \left[||I_4 - \eta \mathcal{T}_k||\tilde{\Theta}_i^k||\mathcal{N}(\tilde{\Theta}_i^k)||\right]
\]
(31)
Let $Q_i^k := \mathbb{E}[||\tilde{\Theta}_i^k||^2]$ denote the expected mean-square-error of the variable $\tilde{\Theta}_i$. Then, taking the expectation of (31) results in
\[
Q_i^{k+1} \leq ||I_4 - \eta \mathcal{T}_k||^2 Q_i^k + \eta^2 (\rho^2 Q_i^k + \overline{\gamma}^2)
\]
\[
+ \mathbb{E}\left[2\eta \left[||I_4 - \eta \mathcal{T}_k||\tilde{\Theta}_i^k||\mathcal{N}(\tilde{\Theta}_i^k)||\right]\right]
\]
(32)
where conditions of the gradient noise in Assumption [1] have been utilised. Revisiting (32), we have
\[
0 \leq ||I_4 - \eta \mathcal{T}_k||^2 \leq \kappa^2
\]
(33)
where $\kappa^2 = \max\{(1 - \overline{\gamma})^2, (1 - \overline{\gamma})^2\}$. Hence, it follows from (32) that
\[
Q_i^{k+1} \leq (\kappa^2 + \eta^2 \rho^2) Q_i^k + \eta^2 \overline{\gamma}^2.
\]
(34)
Recursively iterating (34) gives
\[ Q_k^i \leq (\kappa^2 + \eta^2 \rho^2)^{k-1} Q_0^i + \sum_{j=0}^{k-1} (\kappa^2 + \eta^2 \rho^2)^j \eta^2 \vartheta^2 \] (35)
where \( Q_0^i \) is the initial mean-square-error of the \( i \)th estimator, which is bounded. To guarantee the convergence of the estimators, it is required that
\[ 0 < \kappa^2 + \eta^2 \rho^2 < 1 \] (36)
which implies
\[ 0 < \eta < \min \left\{ \frac{2\gamma}{\gamma^2 + \rho^2}, \frac{2\gamma}{\gamma^2 + \rho^2} \right\}. \] (37)

Under the above condition, we have
\[ \lim_{k \to \infty} \sum_{j=0}^{k-1} (\kappa^2 + \eta^2 \rho^2)^j \eta^2 \vartheta^2 = \frac{\eta^2 \vartheta^2}{1 - (\kappa^2 + \eta^2 \rho^2)} \] (38)

Therefore, combining (35), (37) and (38) yields
\[ \lim_{k \to \infty} Q_k^i \leq \frac{\eta^2 \vartheta^2}{1 - (\kappa^2 + \eta^2 \rho^2)} \] (39)
where \( \lim_{k \to \infty} (\kappa^2 + \eta^2 \rho^2)^k Q_0^i = 0 \) has been applied. In view of (34) and (36), it can be concluded that the estimator MSE converges to a small neighborhood of zero at a geometric rate, which is given by \( O((\kappa^2 + \eta^2 \rho^2)^k) \), with \( 0 < \kappa^2 + \eta^2 \rho^2 < 1 \). Because the estimators are unbiased, we have
\[ \lim_{k \to \infty} \mathbb{E}(\Theta_k^i) = 0, \quad \forall i = 1, 2, \ldots, N. \] (40)
This completes the proof.

Although the path planning and environment acquisition are coupled, it has been shown that under Assumption 1 source estimators can converge to true parameters independent of the path planning when measurement samples \( k \to \infty \). This important property allows us to employ the well-known separation principle for the convergence analysis of the overall algorithm. Such an analytical principle has been widely used to establish the stability of disturbance observer based control (DOBC) [21, 31], where design of the controller is separated from design of the observer. In addition, we will further analyse the composite search performance (steady-state performance) in relation to the noise characters.

**Theorem 1:** Consider a dispersion described by ATDM [3] and the measurement errors and disturbances satisfy Assumption 1. For a set of properly designed learning rates \( \eta \) and \( \delta \), the search agent converges to a bounded neighbourhood of the source location using the proposed CLEE in Algorithm 1. Moreover, the steady-state MSE bound between agent and true source is given by
\[ \lim_{k \to \infty} \mathbb{E}\left\| p_k - s \right\|^2 \leq \frac{\delta^2 (\varphi^2 + 2 \bar{\nu}^2)}{1 - 2 \max\{(1 - \delta \mu)^2, (1 - \delta \bar{\mu})^2\} - \delta^2 \vartheta^2} \] (41)
where \( \bar{\nu} > 0 \) denotes the upper bound of the gradient norm of the estimators’ variance \( \left\| \nabla_p P_{k+1} \right\| \).

**Proof:** Under perfect atmospheric dispersion model in [3], the concentration function \( z(p_k, \Theta_k) \) is a continuously differentiable and concave function of the relative position of the agent to the source. The Hessian matrix of \( g(p_k) \) with respect to the agent’s position \( p_k \) is bounded, written as
\[ \mu I_3 \leq \nabla^2 g(p_k) \leq \mu I_3 \] (42)
where \( 0 \leq \mu \leq \bar{\mu} \), since \( g(p_k) \) is convex for any \( M_0 \geq M(p_k, \Theta_k) \). According to the path update law (14) and the approximated gradient (15), we have
\[ p_{k+1} = p_k - \delta \left[ \nabla_p g(p_k) + \mathcal{H}(p_k) + \nabla_p P_{k+1} \right]. \] (43)
Denote \( \bar{p}_k = p_k - s \) as the error of the agent’s position relative to the source position. Consequently, the error dynamics of \( \bar{p}_k \) can be written as
\[ \bar{p}_{k+1} = \bar{p}_k - \delta \left[ \nabla_p \mathcal{H}(p_k) + \nabla_p P_{k+1} \right]. \] (44)
Following a similar argument as in Lemma 1, we have
\[ \bar{p}_{k+1} = (I_3 - \delta \mathcal{L}_k) \bar{p}_k - \delta \mathcal{H}(p_k) - \delta \nabla_p P_{k+1} \] (45)
where
\[ \mathcal{L}_k := \int_0^1 \nabla^2 g(s + \tau \bar{p}_k) d\tau \] (46)
with \( \mu I_3 \leq \mathcal{L}_k \leq \bar{\mu} I_3 \). Then, taking the square of the Euclidean norm for both sides of the error dynamics (45) leads to
\[ \left\| \bar{p}_{k+1} \right\|^2 \leq \left\| (I_3 - \delta \mathcal{L}_k) \bar{p}_k - \delta \mathcal{H}(p_k) - \delta \nabla_p P_{k+1} \right\|^2 \] (47)
Let \( \mathbb{P}_k := \mathbb{E}\left\| \bar{p}_k \right\|^2 \) denote the expected mean-square-error between the agent’s position and the source location. Taking the expectation of (47) and further applying the noise conditions in Assumption 1, we have
\[ \mathbb{E}\left[ \left( I_3 - \delta \mathcal{L}_k \right) \bar{p}_k - \delta \mathcal{H}(p_k) - \delta \nabla_p P_{k+1} \right] \] (48)
where the following three relationships have been applied to derive the second inequality,
\[ \mathbb{E}\left[ 2\delta \left( \left( I_3 - \delta \mathcal{L}_k \right) \bar{p}_k \right)^T \mathcal{H}(p_k) \right] = 0 \]
\[ \mathbb{E}\left[ 2\delta \left( \nabla_p \mathcal{P}_{k+1} \right)^T \mathcal{H}(p_k) \right] = 0 \]
\[ \mathbb{E}\left[ 2\delta \left( I_3 - \delta \mathcal{L}_k \right) \bar{p}_k \left\| \nabla_p \mathcal{P}_{k+1} \right\| \right] \leq \mathbb{E} \left( \left\| \nabla_p \mathcal{P}_{k+1} \right\|^2 \right) \] (49)
It follows from (46) that
\[ 0 \leq \left\| I_3 - \delta \mathcal{L}_k \right\|^2 \leq \kappa^2_1 \] (50)
where \( \kappa^2 = \max \{1 - \delta \mu^2, (1 - \delta \bar{\mu})^2\} \). In view of the definition of \( P_{k+1|k} \), it is known that the last term in (48), \( 2 \mathbb{E}[\delta^2 \| \nabla_p P_{k+1|k} \|^2] \), is a measure of the variance of the estimator errors, and thereby is a function of \( Q_k := [Q_k^1, \ldots, Q_k^N]^T \). Note that the ATDM is a smooth function with respect to the source estimators \( \Theta_k \), and thus \( F_{k+1} \) is bounded. Therefore, we can always find an upper bound \( \bar{\nu}^2 > 0 \) such that \( \| \nabla_p P_{k+1|k} \|^2 \leq \bar{\nu}^2 \), and consequently there always exists a bounded mapping function \( h_k(Q_k) \) such that \( h_k^T(Q_k) Q_k = 2 \mathbb{E}[\delta^2 \| \nabla_p P_{k+1|k} \|^2] \). Now, applying (50) to (48) gives

\[
P_{k+1} \leq (2\kappa^2 + \delta^2 \varphi^2) P_k + h_k^T(Q_k) Q_k + \delta^2 \varphi^2.
\]  

(51)

Combining (34) and (51), the closed-loop error dynamics of \( P_{k+1} \) and \( Q_{k+1} \) can be written as

\[
\begin{bmatrix}
P_{k+1} \\
Q_{k+1}
\end{bmatrix} \leq \begin{bmatrix}
2\kappa^2 + \delta^2 \varphi^2 & h_k^T(Q_k) \\
0 & \kappa^2 + \eta^2 \rho^2
\end{bmatrix} \begin{bmatrix}
P_k \\
Q_k
\end{bmatrix} + h_k(Q_k) Q_k
\]

(52)

Note that the coupling between the path planning and environment acquisition is reflected by the term \( h_k^T(Q_k) Q_k \), which converges to a neighbourhood of zero at a geometric rate according to Lemma 1. Now we show the convergence of the closed dynamics in (52) when \( 2\kappa^2 + \delta^2 \varphi^2 \) and \( \kappa^2 + \eta^2 \rho^2 \) are within the unit circle. Consider a Lyapunov candidate \( V_{k+1} \) as

\[
V_{k+1} = P_{k+1} + \lambda^T Q_{k+1}
\]

(53)

where

\[
\lambda \geq \max \left\{ \frac{h_k(Q_k)}{\alpha_p - \alpha_q}, \frac{h_k(Q_k)}{1 - 2\alpha_q} \right\}
\]

(54)

with \( \alpha_p := 2\kappa^2 + \delta^2 \varphi^2 \) and \( \alpha_q := \kappa^2 + \eta^2 \rho^2 \). It should be noted that the variables, \( P_{k+1}, Q_{k+1}, \lambda \) and \( h_k(Q_k) \), are all non-negative, and thus \( V_{k+1} \geq 0 \). Substituting (52) into (53) yields

\[
V_{k+1} \leq \alpha_p P_k + [h_k(Q_k) + \alpha_q \lambda^T] Q_k + \beta_p + \lambda^T \beta_q
\]

(55)

where \( \beta_p := \delta^2 \varphi^2 \) and \( \beta_q := \eta^2 \rho^2 \). According to the values of convergence rates \( \alpha_p \) and \( \alpha_q \), we consider the following two cases: \( \alpha_p > \alpha_q \) and \( \alpha_p \leq \alpha_q \). When \( \alpha_p > \alpha_q \) and \( \lambda > \frac{h_k(Q_k)}{\alpha_p - \alpha_q} \), (55) can be written as

\[
V_{k+1} \leq \alpha_p [P_k + \alpha_p^{-1} h_k(Q_k) + \alpha_q \lambda^T] Q_k] + \beta_p + \lambda^T \beta_q
\]

\[
\leq \alpha_p V_k + \beta_p + \lambda^T \beta_q,
\]

(56)

Similarly, when \( \alpha_p \leq \alpha_q \) and \( \lambda > \frac{h_k(Q_k)}{1 - 2\alpha_q} \), we have

\[
V_{k+1} \leq \alpha_p^{-1} [h_k(Q_k) + \alpha_q \lambda] [h_k(Q_k) + \alpha_q \lambda]^{-1} \lambda P_k + \lambda^T \beta_q
\]

\[
+ \beta_p + \lambda^T \beta_q
\]

\[
\leq \alpha_p^{-1} [h_k(Q_k) + \alpha_q \lambda] V_k + \beta_p + \lambda^T \beta_q.
\]

(57)

Under (54), it can be concluded from (56) and (57) that \( V_k \) is a convergent sequence as \( k \to \infty \). This completes the bounded stability analysis of Algorithm 1.

Now, we analyse the steady-state search performance. Recursively iterating (51) gives

\[
P_k \leq (2\kappa^2 + \delta^2 \varphi^2) P_0 + \sum_{j=0}^{k-1} (2\kappa^2 + \delta^2 \varphi^2) \delta^2 \varphi^2 + h_k^T(Q_k) Q_k
\]

where \( P_0 \) is the initial mean-square-error, which is bounded for any initialisation of \( p_0 \). It follows from the definition of \( h_k^T(Q_k) Q_k \) that

\[
\lim_{k \to \infty} h_k(Q_k) Q_k \leq 2\delta^2 \varphi^2.
\]

(59)

Thus, the following relation can be obtained

\[
\lim_{k \to \infty} \sum_{j=0}^{k-1} (2\kappa^2 + \delta^2 \varphi^2) \delta^2 \varphi^2 + h_k^T(Q_k) Q_k
\]

\[
\leq \frac{\delta^2 (\varphi^2 + 2\varphi^2)}{1 - (2\kappa^2 + \delta^2 \varphi^2)}.
\]

(60)

Then, it follows from (58) and (60) that

\[
\lim_{k \to \infty} \mathbb{E} \| p_k - s \|^2 \leq \frac{\delta^2 (\varphi^2 + 2\varphi^2)}{1 - 2\max \{1 - \delta \mu^2, (1 - \delta \bar{\mu})^2\} - \delta^2 \varphi^2}
\]

(61)

where \( \lim_{k \to \infty} (2\kappa^2 + \delta^2 \varphi^2) \mathbb{E} P_0 = 0 \) has been applied. Similarly, it can be obtained from (58) that the agent converges to a bounded mean-square-error-at a geometric rate, given by \( O((2\kappa^2 + \delta^2 \varphi^2)^k) \), with \( 0 < 2\kappa^2 + \delta^2 \varphi^2 < 1 \). This completes the steady-state performance analysis.

**Remark 5:** We have proven that the searching agent will eventually converge to a neighbourhood of the true source position in Theorem 1. As having been discussed, the gradient noise is used to quantify the impacts of sensor noises, mapping errors and turbulence disturbances. Under extreme environment conditions, the noise will impose more significant influence on the proposed algorithm due to the increase of the noise terms, \( \rho^2, \varphi^2, \rho^2 \) and \( \varphi^2 \). Theorem 1 shows that if there are larger uncertainties, the learning rates, \( \eta \) and \( \delta \), should be reduced in order to guarantee convergence. Moreover, smaller \( \eta \) and \( \delta \) will consequently lead to enhanced accuracy in terms of steady-state error, as illustrated by (61). However, due to limited budget of power supply and time constraints in emergent events, faster response is often desirable. As a result, there is a trade-off between the steady-state performance and convergence speed. It is worth mentioning that the proposed algorithm exhibits a geometric convergence rate, owing to the implementation of constant learning rates, and consequently it can obtain a satisfactory performance within a limited number of steps.

**Remark 6:** There is a significant difference between the existing dual control formulation and our framework in this paper. Previous studies introduce the exploration effect on the system for purposes of state or parameter estimation [17, 19, 20], while in our work the probing effect is used to explore the environment (in this case, learn the source location and release rate). This crucial distinction allows us to learn the unknown environment by reducing estimation uncertainty.

**Remark 7:** If we remove the second term \( P_{k+1|k} \) in the path planning objective in (11a), then our algorithm reduces to the
TABLE I: Operational parameters and source knowledge.

|                | UAV    | Source | Prior |
|----------------|--------|--------|-------|
| Measurement budget | 180    | -      | -     |
| Flight budget    | 2,000s | -      | -     |
| Velocity         | 2m/s   | -      | -     |
| Maximum step size| 4m     | -      | -     |
| Start position   | [2, 2, 1] | -    | -     |
| $x$ position     | -      | 80m    | $U(x_{\min}, x_{\max})$ |
| $y$ position     | -      | 80m    | $U(y_{\min}, y_{\max})$ |
| $z$ position     | -      | 1m     | $U(z_{\min}, z_{\max})$ |
| Release rate     | -      | 10g/s  | $N(11, 2)$ |

pure exploitation strategy, which solely relies on the current estimators of the source parameters. It should be emphasised that the learning process of pure exploitation is passive or accidental, since source parameters are updated when the agent makes full use of current belief. In Algorithm 1 the probing effect is included in the value function, by which the agent makes full use of current belief. In Algorithm 1, the accidental, since source parameters are updated when the agent makes full use of current belief. In this sense, our CLEE framework is closely related to active learning in MPC [7, 17, 32]. Generally speaking, dual control belongs to a much wider class of machine learning problems, of exploration and exploitation in an uncertain environment.

In this subsection, the sensor measurement noises are set as additive Gaussian white noises. We assume that there is no non-detection event, which means the sensor can always obtain concentration values buried in noises. Each algorithm has been repeated for 200 times with the same configurations. The obtained mean-square-errors of the CLEE and Entrotaxis algorithms are displayed in Fig. 1. MSE evaluates the performance of the source estimators, calculated by $\mathcal{E}(s_k - s)^2$. It is clear that all algorithms can gradually achieve acceptable estimation of the source position within limited budgets. Uniform distribution of the source location has been applied in the initialisation process, as it is assumed that there is no prior information regarding source position. As a result, the initial guess of the source position is around the centre of the search space. In general, Entrotaxis requires a large number of measurements to update its particle filter, which leads to a slow acquisition rate of source estimation. The acquisition process continues until the flight is terminated at the maximum measurement budget. On the other hand, our proposed algorithms allow quick update of the source estimators by using instantaneous measurements. CLEE algorithm converges to bounded MSE at approximately 1,000s. This property is helpful to conducting emergent identification of the source parameters.

Apart from Entrotaxis and DCEE, the step size of movement can be any value in the range of $[1\text{m}, 4\text{m}]$, and the direction of movement can be arbitrary by taking advantage of low computational load of CLEE. The learning rates are set as $\eta = 5$ and $\delta = 4$. The initial conditions of estimators are randomly chosen in accordance with the prior knowledge in Table I. It should be noted that measurement signals from sensors are converted in dB scale. When implementing the proposed algorithms, the gradients are normalised to keep agent’s movement within a meaningful range.
to closely monitor the status of the release or take further remedy actions. In Fig. 2 the distance between the agent and the source is displayed. A noticeable phenomenon is that agent’s position using Entrotaxis is quite far from the source position. The proposed CLEE in this paper can keep the agent in the neighbourhood of the true source, and the steady-state distances are around 35.96m, 20.28m, 2.36m, 2.22m and 1.61m, for $N = 5, 10, 50, 100, 200$ and 1,000, respectively. Illustrative examples of search paths of CLEE with $N = 10$ and $N = 100$ have been demonstrated in Figs. 3a and 3b.

To show the influence of the number of estimators, we have presented the average performance using different values of $N$, as shown in Fig. 4. Initially, increasing $N$ can significantly enhance the performance in terms of estimators’ MSE and the agent distance to the source ($N$ ranging from 5 to 50). For $N \geq 50$, increasing $N$ is no longer able to provide much performance improvement ($N$ ranging from 50 to 1,000). Therefore, the proposed CLEE framework does not require a large number of estimators, and tens of them will be sufficient for autonomous search problem.

Entrotaxis aims to minimise the estimation error of source parameters, and it does not necessarily imply that the agent has to move close to the source position. Correspondingly, there is a substantial distance between the agent and the source, even though the estimated source position has been fairly satisfactory. A representative search path using Entrotaxis has been presented in Fig. 5c. In essence, Entrotaxis and other informative path planning methods utilise pure exploration strategy [7], which drives the agent to the most informative position in order to reduce information uncertainties. As can be seen from Fig. 5c the agent ignores the current estimation of the source position but probes the uncertain areas around the estimated source. This consequently leads to large errors between the agent and the source.

An important advantage of the proposed methods in this paper is the computational efficiency. For clear comparison, we have summarised time consumed by different algorithms, as shown in Table II. The simulations are carried out using Matlab with a processor of 2.8 GHz Quad-Core Intel Core i7. It can be seen that our algorithm is much faster than Entrotaxis. It only consumes less than 1% of the time for Entrotaxis ($N \leq 100$). As a result, CLEE also occupies much less memory storage since the number of estimators is much smaller. This is a very important and advantageous feature because processors used on mobile platforms are usually lower-price portable chips that cannot offer intensive computational power or large memory.

**TABLE II: Time consumed by running different algorithms for 200 trials.**

| Estimators/Particles | CLEE | Entrotaxis |
|----------------------|------|------------|
| Time (second)        |      |            |
| 5                    | 21.8 | 23.1       |
| 10                   | 24.3 | 26.3       |
| 50                   | 30.5 | 56.8       |
| 100                  | 56.8 | 2940.4     |
| 200                  |      |            |
| 1000                 |      |            |

**B. Unknown Environment with Sensor Non-detection Events**

In this subsection, operational environment is also assumed to be unknown. Consequently, this necessitates estimating a full list of parameters of ATDM, i.e. $\Theta_s = [s^T, q, \phi_s, u_s, \zeta_{s1}, \zeta_{s2}]^T$. In addition, the sensor may fail to return any readings in a random manner. This type of behaviour does happen quite often in the real experiment, in particular, for portable sensors. Because of sensor dropouts, the noises are not longer unbiased, and consequently the performance of proposed algorithms will be degraded.

We have run each algorithm for 200 times with random sensor dropouts. Fig. 5 shows the MSE of source estimation. It is noticeable that the information-theoretic method exhibits quite strong resilience to the sensor failures and environment uncertainties. In Fig. 6 distance between the agent and the true source is displayed. Compared with previous results in Fig. 2, it is clear that the performance of our method has been influenced by the sensor faults and unknown environment, whereas Entrotaxis provides similar result as before. Nevertheless, all algorithms with different number of estimators can gradually navigate the agent move close to the source position. Noticeably, the estimation performance of the CLEE algorithm is degraded in the later period of search, in particular, for $N = 5$ and 10. This behaviour is probably due to the presence of non-detection events, which leads to biased measurement information. As the estimation process continues, such a biased measurement is gradually accumulated and worsens the search performance. The overall search performance and robustness to sensor dropouts are significantly enhanced by increasing the number of estimators.

By the simulation comparison, it can be concluded that the information-theoretic algorithm is computationally intensive, but it demonstrates strong robustness to sensor dropouts and unknown environment. On the other hand, the proposed algorithm in this paper is much cheap to evaluate, but more sensitive to failures and uncertainties. By comparing the obtained results in Sections IV-A and IV-B we notice that knowing the environment can significantly enhance the performance of CLEE algorithm, but has quite limited contribution for Entrotaxis. In general, our algorithm is expected to produce
Fig. 3: Representative runs of different algorithms: (a) CLEE with $N = 10$, (b) CLEE with $N = 100$, (c) Entrotaxis. Red lines are the paths of the UAV, the green dots represent the estimated source position, and the black dots represent the true source position.

Fig. 4: Performance of CLEE algorithm with different number of estimators.

Fig. 5: Mean-square-error between estimated and true source positions with unknown environment and sensor failures.

V. Conclusion

This paper has developed a computationally efficient solution for autonomous search of airborne release with proven more accurate estimation under good sensor performance and less uncertain environment.
properties like convergence. A new learning framework, inspired by dual control for exploration and exploitation (DCEE), has been formulated to solve this goal-oriented control problem in an unknown environment with an unknown target. Gradient-based optimisation algorithms have been proposed to estimate the source parameters, and to plan next movement by formulating suitable value functions. Theoretical guarantee for convergence and steady-state performance are analysed under measurement noises and uncertain turbulence. From the simulation studies, the effectiveness of the proposed solution has been validated. It has been demonstrated that our algorithm achieves superior performance comparing with informative path planning, and it also consumes much less computation time.

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