BLOW UP AND BLOW UP TIME FOR DEGENERATE KIRCHHOFF-TYPE WAVE PROBLEMS INVOLVING THE FRACTIONAL LAPLACIAN WITH ARBITRARY POSITIVE INITIAL ENERGY

QIANG LIN
College of Automation, Harbin Engineering University
Heilongjiang, Harbin 150001, China

XUETENG TIAN
College of Mathematical Sciences, Harbin Engineering University
Heilongjiang, Harbin 150001, China

RUNZHANG XU*
College of Automation, Harbin Engineering University
College of Mathematical Sciences, Harbin Engineering University
Heilongjiang, Harbin 150001, China

MEINA ZHANG
College of Mathematical Sciences, Harbin Engineering University
Heilongjiang, Harbin 150001, China

Dedicated to Professor Patrizia Pucci on the occasion of her 65th birthday

Abstract. In this paper, we study blow up and blow up time of solutions for initial boundary value problem of Kirchhoff-type wave equations involving the fractional Laplacian

\[
\begin{cases}
  u_{tt} + |u|^2(\theta - 1)\Delta u = f(u), & \text{in } \Omega \times \mathbb{R}^+, \\
  u(x, 0) = u_0, \quad u_t(x, 0) = u_1, & \text{in } \Omega, \\
  u = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times \mathbb{R}^+_t,
\end{cases}
\]

where \([u]_s\) is the Gagliardo seminorm of \(u\), \(s \in (0, 1)\), \(\theta \in [1, 2^*_s/2)\) with \(2^*_s = \frac{2N}{N - 2s}\) \((-\Delta)^s\) is the fractional Laplacian operator, \(f(u)\) is a differential function satisfying certain assumptions, \(\Omega \subset \mathbb{R}^N\) is a bounded domain with Lipschitz boundary \(\partial \Omega\). By introducing a new auxiliary function and an adapted concavity method, we establish some sufficient conditions on initial data such that the solutions blow up in finite time for the arbitrary positive initial energy. Moreover, as \(f(u) = |u|^{p-1}u\), we estimate the upper and lower bounds for blow up time with arbitrary positive energy.

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* Corresponding author: Runzhang Xu.
1. Introduction. In this paper, we investigate the following initial boundary value problem of Kirchhoff-type hyperbolic equations involving the fractional Laplacian

\begin{align}
\begin{cases}
\rho u_{tt} + \frac{p_0}{\lambda} u_s^{p-1} \mathcal{L}_K u = f(u), & \text{in } \Omega \times \mathbb{R}^+, \\
u(x,0) = u_0, \quad u_t(x,0) = u_1, & \text{in } \Omega, \\
u = 0, & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T),
\end{cases}
\end{align}

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $[u]_s = \left( \int_{\mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)dx dy \right)^{1/2}$, space dimension $N > 2s$, $s \in (0,1)$, $\theta \in [1,2^*_s/2]$ with the fractional critical exponent $2^*_s = \frac{2N}{N-2s}$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial \Omega$, $\mathcal{L}_K$ is a nonlocal integro-differential operator, which (up to normalization factors) may be defined by

$$
\mathcal{L}_K u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} (u(x) - u(y)) K(x-y) dy
$$

for any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ is a ball with $x$ as the center and $\varepsilon$ as the radius, $x \in \mathbb{R}^N$, $\varepsilon > 0$, the kernel $K: \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ satisfies

(K) $m(x) K \in L^1(\mathbb{R}^N)$, where $m(x) = \text{min}\{\|x\|^2, 1\}$; there exists $K_0 > 0$ such that $K(x) \geq K_0 |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$.

A typical example for $K$ is given by singular kernel $K(x) = |x|^{-(N+2s)}$. In this way, $\mathcal{L}_K = (-\Delta)^s$ for all $u \in C_0^\infty(\mathbb{R}^N)$ and $[u]_s$ becomes the celebrated Gagliardo seminorm of $u$.

Next we assume that $f: \mathbb{R} \to \mathbb{R}$ in problem (1) satisfies the main condition

(H) $(p+1)F(u) \leq uf(u), \quad p \in (2\theta - 1, 2^*_s - 1)$.

Here, we also assume

$$
F(u) = \int_0^u f(\xi) d\xi.
$$

A simple example is given as $f(u) = |u|^{p-1}u$, see Lemma 3.1 in [25] for more details.

In recent years, more and more attentions have been drawn to various models involving fractional Laplacian and nonlocal operators. Such operators arise in a quite natural way in many different applications, such as physics, mechanics, biology, materials, control and so on. In [1], the fractional Laplacian is viewed as the infinitesimal generators of stable radially symmetric Lévy processes. In [17], Laskin deduced the fractional Schrödinger equation, which is the result of extending the Feynman path integral from Brownian motion to Lévy quantum mechanics paths. We would like to point out that $(-\Delta)^s$ can be reduced to the classical Laplace operator $-\Delta$ as $s \to 1$, see Proposition 4.4 in [24] for more details. In particular, thanks to the pioneering works of Caffarelli and Silvestre in [9], many interesting results in some classic elliptic problems have been extended in the fractional Laplacian setting. For more recent results involving the fractional Laplacian, for example, we refer the readers to [3, 4, 5, 6, 10, 24, 27, 31] and the references therein.

Recently, Kirchhoff-type problems have attracted more and more attentions. In 1883, Kirchhoff model was proposed by Kirchhoff as follows:

$$
\rho u_{tt} - \left( \frac{p_0}{\lambda} + \frac{E}{2L} \int_0^L |\nabla u|^2 dx \right) \Delta u = 0,
$$

where $\rho, p_0, \lambda, E, L$ are constants which represent some physical meanings, respectively. This equation is a generalization of the well-known D’Alembert’s wave equation for free vibrations of elastic strings. Kirchhoff’s model takes into account the
changes in length of the string produced by transverse vibrations. It is worth pointing out that Fiscella and Valdinoci in [12] proposed a stationary Kirchhoff model involving the fractional Laplacian

$$\left\{ \begin{array}{ll}
M([u]_s^2)(-\Delta)^s u = f, & \text{in } \Omega, \\
u = 0, & \text{in } \mathbb{R}^N \setminus \Omega,
\end{array} \right.$$  

by taking into account the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string. Here, the Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$ was assumed as follows:

(a) $M$ is an increasing and continuous function;
(b) there exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for any $t \in \mathbb{R}^+$.

In order to obtain the existence of solutions for Kirchhoff-type problems, for example, the authors in [2, 12, 29] always assume that $M : \mathbb{R}^+_0 \to \mathbb{R}^+$ is a continuous and nondecreasing function satisfying the condition (b). A typical example for $M$ is given by $M(t) = a + bt^{m-1}$, $t \in \mathbb{R}^+_0$, with $a \geq 0, b \geq 0$. However, it is worthy to emphasize that the Kirchhoff function $M$ is not always monotone, let us refer to [11, 28]. Naturally, we distinguish the Kirchhoff-type problems into nondegenerate and degenerate cases according to $M(0) > 0$ and $M(0) = 0$, respectively. For some recent results in the nondegenerate case, see for instance [43, 44] and the references therein. It is worth mentioning that the degenerate case is rather interesting in Kirchhoff theory, see, for example, [11, 23]. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero. For some recent results in the degenerate case, we refer to [7, 34] and the references therein.

In order to better demonstrate the motivation of our research, it is necessary to present a blueprint of the applications of variational methods to the nonlinear evolution equations. For the celebrated and classical parabolic model it is a convenient choice to refer [14], where Gazzola and Weth summarized the main conclusions about the global well-posedness and finite time blowup of solutions for semilinear parabolic equation in the subcritical and critical initial energy case $J(u_0) \leq d$; especially, the global existence and finite time blowup of solutions for the arbitrary positive initial energy, i.e., $J(u_0) > 0$, were firstly discussed. Then the pseudo-parabolic equation was considered by Xu et al. in [36]. An interesting version of nonlinear parabolic equations involving fractional diffusion $u_t + (-\Delta)^s \Phi(u) = 0$ ($0 < s < 1$) was considered in [33] on the aspects of limit solutions, self-similar solutions and asymptotic attractors concerning the existence of solutions for singular nonlinear diffusion, including $\Phi(u) = log(u)$ as important special case. Then for the space-fractional diffusion equation $u_t + (-\Delta)^s u = |u|^{p-1}u$, a threshold result of global existence and finite time blowup for the subcritical initial energy case, i.e., $J(u_0) < d$ was established by Fu et al. in [13]. The Kirchhoff type parabolic model $u_t + M([u]_p^2)(-\Delta)^p u = f(x,t)$ as an anomalous diffusion one was considered in [30] to show the global well-posedness and large time behavior of solutions. For more recent progress on nonlinear fractional parabolic equations of porous medium type, we refer to an overview [32]. Turn to the wave equation involving the fractional Laplacian operator, we like to skip the research on standard and classical wave model [21, 22, 18, 26, 35, 38, 40] to directly go to the very recent work on the fractional wave equation considered by Pan et al. [25], where the global existence and finite time blowup of solutions for problem (1) in the cases $E(0) \leq d$ were proved, and the high initial energy case is still unsolved until the present paper.
To our best knowledge, there are a lot of papers dedicated to the blowup phenomenon of solution for evolution equations without fractional Laplacian under high energy conditions, let us refer the readers to [15, 19, 20, 37, 39, 41, 42]. Inspired by the above works, we attempt to study blow up of solutions with arbitrary positive energy and estimate the upper and lower bounds of blow up time for problem (1) in the setting of fractional Laplacian by the adapted concavity method. There is no doubt that we have encountered serious difficulties because of the appearance of the Kirchhoff function and fractional Laplacian. In fact, in order to investigate blow up at high energy level and estimate blow up time upper and lower bounds, we have to deal with the degeneracy of the Kirchhoff function and the nonlocal feature of the fractional Laplacian operator. For this, we establish appropriate auxiliary functions and use the concave function method to obtain the results of invariant set, blow up at high energy level, upper and lower bounds of blow up time.

The rest of the paper is organized as follows. In Section 2, we recall some necessary definitions and properties of the fractional Sobolev spaces. In Section 3, we prove the finite time blow up of the solution under some sufficient conditions on initial data with $E(0) > 0$. In Section 4, we investigate the upper and lower bounds for blow up time with $E(0) > 0$.

2. Preliminary. In this section, we give some lemmas and notations which will be used throughout this paper. In what follows, $\| \cdot \|_p$ denotes the usual $L^p(\Omega)$ norm. Specially, we denote $\| \cdot \|$ the $L^2(\Omega)$ norm.

Put $Q = \mathbb{R}^{2N} \setminus \mathcal{U}$ where $\mathcal{U} = C(\Omega) \times C(\Omega) \subset \mathbb{R}^{2N}$, and $C(\Omega) = \mathbb{R}^N \setminus \Omega$. Let $W$ be the linear space of Lebesgue measurable functions $u$ from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to $\Omega$ of any function $u$ in $W$ belongs to $L^2(\Omega)$ and

$$\int_{Q} |u(x) - u(y)|^2 K(x-y)dxdy < \infty. \quad (3)$$

Here, we remark that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, so that it enjoys the compact property, making the map

$$\mathbb{R}^{2N} \ni (x, y) \mapsto |u(x) - u(y)|^2 K(x-y)$$
belongs to $L^1(\mathbb{R}^{2N})$. Since $\Omega$ is bounded, the restriction to $\Omega$ of any function in $W$ belongs to $L^2(\Omega)$, so that all the assumptions on $W$ are satisfied. Further, we study the weak solution of problem (1) in the Hilbert space, refer to [27, 31]. We also note that, when $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, then it has the so-called extension property from $\Omega$ to $\mathbb{R}^N$, see [8]. The space $W$ is equipped with the norm:

$$\|u\|_{W} = \left( \|u\|_{L^2(\Omega)}^2 + \int_{Q} |u(x) - u(y)|^2 K(x-y)dxdy \right)^{\frac{1}{2}}. \quad (4)$$

We shall work in the closed linear subspace:

$$W_0 = \{ u \in W : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \}, \quad (5)$$
endowed with the norm

$$\|u\|_{W_0} = \left( \int_{Q} |u(x) - u(y)|^2 K(x-y)dxdy \right)^{\frac{1}{2}}, \quad u \in W_0, \quad (6)$$

$$(u, v)_{W_0} = \int_{Q} [u(x) - u(y)] \cdot [v(x) - v(y)] K(x-y)dxdy,$$
and we also introduce $\langle \cdot, \cdot \rangle$ to denote the duality pairing between $W_0$ and its dual space $W^{-s}$. Recall that the fractional Sobolev space $H^s(\Omega) = W^{s,2}(\Omega)$ is

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{N/2+s}} \in L^2(\Omega \times \Omega) \right\},$$

(7)

endowed with the norm

$$\|u\|_{H^s(\Omega)} = \left( \|u\|_{L^2(\Omega)}^2 + \int_Q |u(x) - u(y)|^2 K(x-y)\,dxdy \right)^{\frac{1}{2}}.$$  

(8)

From now on, without further mentioning, we assume that $K : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^+$ satisfies assumption (K) and we state the main functional properties proved in detail in [24, 27, 31].

**Lemma 2.1.** ([31, Lemma 6]) For any $v \in [1, 2^*_s]$, there exists a positive constant $C_0 = C_0(N, v, s)$ such that for any $u \in W_0$

$$\|u\|_{L^v(\Omega)}^2 \leq C_0 \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \,dxdy$$

(9)

$$\leq C \int_Q |u(x) - u(y)|^2 K(x-y)\,dxdy,$$

where $C = C_0/K_0$.

**Definition 2.2.** A function $u = u(x,t)$ is said to be a weak solution of problem (1), if

$$u \in L^\infty(0,\infty; W_0), \quad u_t \in L^\infty(0,\infty; L^2(\Omega)), $$

$$u(\cdot,0) = u_0 \text{ in } W_0, \quad u_t(\cdot,0) = u_1 \text{ in } L^2(\Omega) \quad \text{and}$$

$$\langle u_t(\cdot,t), \phi(\cdot,t) \rangle + \int_0^t \langle u(\cdot,\tau), \phi(\cdot,\tau) \rangle_{W_0} d\tau = \int_0^t \langle f(u(\cdot,\tau)), \phi(\cdot,\tau) \rangle d\tau + \langle u_1, \phi \rangle,$$

for any $\phi \in L^1(0,\infty; W_0)$ and for any $t \in \mathbb{R}_0^+$, where

$$\langle u(\cdot,\tau), \phi(\cdot,\tau) \rangle_{W_0} = \|u(\cdot,\tau)\|^{2(\theta-1)}_{W_0^\theta} \int_Q [u(x,t) - u(y,t)] \phi(x,t) - \phi(y,t)] K(x-y)\,dxdy.$$  

Moreover, we define the total energy functional, potential energy functional and Nehari functional as follows:

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2\theta} \|u\|_{W_0}^{2\theta} - \int_{\Omega} F(u)\,dx,$$

(10)

$$A(u) = \frac{1}{2\theta} \|u\|_{W_0}^{2\theta} - \int_{\Omega} F(u)\,dx,$$

(11)

$$B(u) = \|u\|_{W_0}^{2\theta} - \int_{\Omega} uf(u)\,dx,$$

(12)

and the unstable set

$$\mathcal{M} = \{ u \in W_0 | B(u) < 0 \}$$

(13)

for problem (1). We define the depth of the potential well as follows:

$$d = \inf \{ A(u) | u \in W_0 \setminus \{0\} \text{ and } B(u) = 0 \}.$$

**Lemma 2.3.** If $u(x,t)$ is the weak solution of problem (1) satisfying $u_0 \in W_0$ and $u_1 \in L^2(\Omega)$, then $E(t) = E(0)$. 


Proof. Multiplying the both sides of the first equation of (1) by $u_t$ and then integrating over $\mathbb{R}^N$, we obtain
\[
\int_{\mathbb{R}^N} u_t(u_{tt} + [u]_{s}^{2(\theta-1)}(-\Delta)^{s} u - f(u))dx = 0.
\]
Since
\[
\int_{\mathbb{R}^N} [u]_{s}^{2(\theta-1)}(-\Delta)^{s} u u_t dx = \frac{1}{2\theta} \frac{d}{dt} \|u\|_{W_0}^{2\theta},
\]
we get
\[
\int_{\mathbb{R}^N} u_t(u_{tt} + [u]_{s}^{2(\theta-1)}(-\Delta)^{s} u u_t dx
\]
\[
= \int_{\Omega} u_t u u_t dx + \int_{\mathbb{R}^N} [u]_{s}^{2(\theta-1)}(-\Delta)^{s} u u_t dx - \int_{\Omega} f(u) u_t dx
\]
\[
= \frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2\theta} \|u\|_{W_0}^{2\theta} - \int_{\Omega} F(u) dx \right).
\] (14)
Integrating (14) on $[0,t]$, we obtain that $E(t) = E(0)$. The proof is thus finished. \(\square\)

3. Finite time blow up at high energy level. To prove the blowup result to problem (1) with $E(0) > 0$, we first prove that the set $\mathcal{M}$ is invariant for problem (1) when $E(0) > 0$.

Lemma 3.1. Let $f(u)$ satisfy assumption (H). If $(u_0, u_1) \in W_0 \times L^2(\Omega)$ satisfies that
\[
\|u_0\|^{2\theta} \geq \frac{2\theta(p+1)C^\theta}{(p+1-2\theta)} E(0),
\] (15)
\[
u_0 \in \mathcal{M},
\] (16)
\[
\int_{\Omega} u_0 u_1 dx > 0,
\] (17)
where $C$ is the constant given in (9), then the corresponding solution to problem (1) belongs to $\mathcal{M}$.

Proof. We claim that $u(\cdot, t) \in \mathcal{M}$ for $t \in [0, T_{\text{max}})$, where $T_{\text{max}}$ is the maximum existence time of $u(t)$. Arguing by contradiction, we suppose that $t_0 \in (0, T_{\text{max}})$ is the first time such that
\[
B(u(t_0)) = 0,
\] (18)
and
\[
B(u(t)) < 0 \text{ for } t \in [0, t_0).
\] (19)
We first introduce an auxiliary function
\[
M(t) := \|u\|^2,
\] (20)
and directly
\[
M'(t) = 2(u, u_t),
\] (21)
and
\[
M''(t) = 2(u, u_{tt}) + 2\|u_t\|^2.
\] (22)
Multiplying the first equation of (1) both sides by $u$ and then integrating over $\mathbb{R}^N$, we derive

$$2\langle u, u_t \rangle = -2\|u\|_{W_0^2}^2 + 2\int_\Omega f(u)udx.$$  \hfill (23)

Furthermore, substituting (23) into (22), by the definition of $B(u)$, we get

$$M''(t) = 2\|u_t\|^2 + 2\int_\Omega f(u)udx - 2\|u\|_{W_0^2}^2$$

$$= 2\|u_t\|^2 - 2B(u).$$  \hfill (24)

By (19), we obtain that $M''(t) > 0$ for every $t \in [0, t_0)$, which implies that $M'(t)$ is strictly increasing on $[0, t_0)$. Then by (17) we see that $M'(t) > M'(0) > 0$ for every $t \in [0, t_0)$. In other words, we obtain that $M(t)$ is also strictly increasing on $[0, t_0)$. Then by (20) we have that $M(t) > 0$ for every $t \in [0, t_0)$. Since

$$\left(M^\theta(t)\right)' = \theta(M(t))^{\theta-1}M'(t) > 0,$$  \hfill (25)

which means that $M^\theta(t)$ is also strictly increasing on $[0, t_0)$. Combining (15) and (25), we obtain

$$M^\theta(t) > M^\theta(0) \geq \frac{2\theta(p+1)C^\theta}{(p+1-2\theta)}E(0)$$  \hfill (26)

for every $t \in (0, t_0)$. From the continuity of $u(t)$ at $t = t_0$ it follows that

$$M^\theta(t_0) = \|u(t_0)\|_{W_0^2}^2 > \frac{2\theta(p+1)C^\theta}{(p+1-2\theta)}E(0).$$  \hfill (27)

On the other hand, from assumption (H) and Lemma 2.3, we have

$$E(0) = E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2\theta}\|u\|_{W_0^2}^2 - \int_\Omega F(u)dx$$

$$\geq \frac{1}{2}\|u_t\|^2 + \frac{1}{2\theta}\|u\|_{W_0^2}^2 - \frac{1}{p+1}\int_\Omega f(u)udx$$

$$= \frac{1}{2}\|u_t\|^2 + \left(\frac{1}{2\theta} - \frac{1}{p+1}\right)\|u\|_{W_0^2}^2 + \frac{1}{p+1}B(u),$$

which together with (18) and Lemma 2.1 shows that

$$E(0) \geq \frac{1}{2}\|u_t(t_0)\|^2 + \left(\frac{1}{2\theta} - \frac{1}{p+1}\right)\|u(t_0)\|_{W_0^2}^2$$

$$\geq \frac{1}{2}\|u_t(t_0)\|^2 + \frac{C^{-\theta}(p+1-2\theta)}{2\theta(p+1)}\|u(t_0)\|_{W_0^2}^{2\theta}$$

$$\geq \frac{C^{-\theta}(p+1-2\theta)}{2\theta(p+1)}\|u(t_0)\|_{W_0^2}^{2\theta},$$

which contradicts (27). So the proof is completed. 

Next, we prove blow up results of the corresponding solution to problem (1) when $E(0) > 0$.

**Theorem 3.2.** Let $u_0 \in W_0$ and $u_1 \in L^2(\Omega)$. Assume that $u_0 \in \mathcal{M}$, $E(0) > 0$ and (17) holds, then the corresponding solution to problem (1) blows up in finite time.
Proof. Firstly, Lemma 3.1 tells that the corresponding solution \( u(t) \in \mathcal{M} \). Arguing by contradiction, we suppose that \( u(t) \) is global. For any \( t \in [0, \infty) \), we recall the auxiliary function \( M(t) \) defined by (20). Furthermore, from the proof of Lemma 3.1 and \( u(t) \in \mathcal{M} \) it follows that

\[
M(t) > 0, \tag{30}
\]

and (21)-(22) hold for \( t \in [0, \infty) \). Then from (21) and the Cauchy-Schwartz inequality we get

\[
M'(t)^2 = 4(u, u_t)^2 \leq 4\|u\|^2 \|u_t\|^2, \quad t \in [0, \infty), \tag{31}
\]

which together with (24) implies that

\[
M''(t)M(t) - (1 + \alpha)(M'(t))^2 \geq \left( 2 \|u_t\|^2 - 2\|u\|_{W_0^{2, p}}^2 + 2 \int_\Omega f(u)udx \right) M(t) - 4(1 + \alpha)(u, u_t)^2 \geq \left( 2 \|u_t\|^2 - 2\|u\|_{W_0^{2, p}}^2 + 2 \int_\Omega f(u)udx \right) M(t) - 4(1 + \alpha)\|u\|^2 \|u_t\|^2 = M(t) \left( -2(1 + 2\alpha)\|u_t\|^2 - 2\|u\|_{W_0^{2, p}}^2 + 2 \int_\Omega f(u)udx \right), \quad t \in [0, \infty),
\]

where \( \alpha > 0 \). By Lemma 2.1, Lemma 2.3 and (10), we notice that

\[
\xi(t) := -2(1 + 2\alpha)\|u_t\|^2 - 2\|u\|_{W_0^{2, p}}^2 + 2 \int_\Omega f(u)udx \geq -2(1 + 2\alpha)\|u_t\|^2 - 2\|u\|_{W_0^{2, p}}^2 + 2(p + 1) \int_\Omega F(u)dx \geq -2(1 + 2\alpha)\|u_t\|^2 - 2\|u\|_{W_0^{2, p}}^2 + (p + 1)\|u_t\|^2 + \left( \frac{p + 1}{\theta} \right) \|u\|_{W_0^{2, p}}^2 - 2(p + 1)E(0) = -2(1 + 2\alpha)\|u_t\|^2 + \left( \frac{p + 1}{\theta} \right) \|u\|_{W_0^{2, p}}^2 - 2(p + 1)E(0) \geq -2(1 + 2\alpha)\|u_t\|^2 + C^{-\theta} \left( \frac{p + 1}{\theta} \right) \|u\|_{W_0^{2, p}}^2 - 2(p + 1)E(0)
\]

for \( t \in [0, \infty) \). Set \( \alpha = (p - 1)/4 > 0 \), then \( (4\alpha + 1 - p)\|u_t\|^2 = 0 \). So from (33) we further get

\[
\xi(t) \geq C^{-\theta} \left( \frac{p + 1}{\theta} \right) \|u\|_{W_0^{2, p}}^2 - 2(p + 1)E(0) \geq 0, \quad t \in [0, \infty). \tag{34}
\]

At this point, by (30)-(34), we can derive that

\[
M''(t)M(t) - (1 + \alpha)(M'(t))^2 > 0, \quad t \in [0, \infty), \tag{35}
\]

where \( \alpha > 0 \). This implies that

\[
(M^{-\alpha})' = -\alpha M^{-\alpha-1} M'(t) < 0 \tag{36}
\]

and

\[
(M^{-\alpha})'' = -\alpha M^{-\alpha-2} (M''(t)M(t) - (1 + \alpha)(M'(t))^2) < 0 \tag{37}
\]
for every $t \in [0, \infty)$, which means that the function $M^{-\alpha}(t)$ is concave. Obviously, $M(0) > 0$, then from (36) it follows that there must exist a $T_{\max} > 0$ such that

$$\lim_{t \to T_{\max}} M^{-\alpha}(t) = 0,$$

i.e.,

$$\lim_{t \to T_{\max}} \|u\|^2 = \infty. \quad (39)$$

Thus, the proof is completed. \qed

4. Upper and lower bounds of the blow up time. In this section, we give the proofs of the upper and lower bounds for blow up time with arbitrary positive energy. In the following, we are concerned with a special nonlinear source term given as $f(u) = |u|^{p-1}u$. Consequently, for $p \in (2\theta - 1, 2\theta^* - 1]$, we obtain the upper bound of the blow up time; for $p \in (2\theta - 1, 2\theta^*/2]$, we obtain the lower bound of the blow up time. For convenience, we redefine energy functional as follows:

$$E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2\theta} \|u\|_{W_0}^{2\theta} - \frac{1}{p+1} \|u\|^{p+1}_{p+1}.$$

To prove the upper bound for blow up time of problem (1), we will utilize the following result.

**Lemma 4.1.** ([16, Theorem 2.1]) Assume that $\Phi \in C^2([0, T])$ satisfying

$$\Phi_{tt} - \alpha \Phi_t^2 + \gamma \Phi_t + \beta \Phi \geq 0, \quad \alpha > 1, \beta \geq 0, \gamma \geq 0, \quad (40)$$

and

$$\Phi_t(0) > \frac{\gamma}{\alpha - 1} \Phi(0), \quad \left(\Phi_t(0) - \frac{\gamma}{\alpha - 1} \Phi(0)\right)^2 > \frac{2\beta}{2\alpha - 1} \Phi(0). \quad (41)$$

Then

$$\limsup_{t \to T^*} \Phi(t) = +\infty,$$

where

$$T \leq \frac{\Phi^{-\alpha}(0)}{A}, \quad (42)$$

$$A^2 = (\alpha - 1)^2 \Phi^{-2\alpha}(0) \left(\left(\Phi_t(0) - \frac{\gamma}{\alpha - 1} \Phi(0)\right)^2 - \frac{2\beta}{2\alpha - 1} \Phi(0)\right).$$

Moreover, $\Phi(t)$ satisfies

$$\Phi(t) \geq \frac{e^{\frac{\gamma t}{\alpha - 1}}}{(\Phi^{-\alpha}(0) - At)^\frac{1}{\alpha - 1}}.$$

With the above lemma in mind, we can give the proof of the upper bound for blow up time.

**Theorem 4.2.** Assume that $E(0) > 0$. Let $u_0 \in W_0$ and $u_1 \in L^2(\Omega)$ such that (17) holds and

$$\left(\int_{\Omega} u_0 u_1 dx\right)^2 > 2E(0)\|u_0\|^2.$$

Then there exists

$$T^* \leq \frac{1}{A} \|u_0\|^{\frac{1}{1+\theta}},$$

where

$$A = (\alpha - 1)^2 \Phi^{-2\alpha}(0) \left(\left(\Phi_t(0) - \frac{\gamma}{\alpha - 1} \Phi(0)\right)^2 - \frac{2\beta}{2\alpha - 1} \Phi(0)\right).$$
such that
\[
\limsup_{t \to T^*} \|u(\cdot, t)\| = +\infty,
\]
where
\[
A^2 = \frac{(p-1)^2}{4} \|u_0\|^{-\frac{p+3}{2}} \left( \left( \int_{\Omega} u_0 u_1 dx \right)^2 - 2E(0) \|u_0\|^2 \right).
\]

**Proof.** Multiplying the both sides of the first equation of (1) by $u$ and integrating over $\mathbb{R}^N$, we have
\[
\frac{1}{2} \frac{d}{dt} \Phi(t) - J(t) + \|u\|^{2\theta}_{W_\theta} = \|u\|^{p+1}_{p+1},
\]
where
\[
\Phi(t) = M(t) = \|u\|^2, \quad J(t) = \int_{\Omega} |u|^2 dx.
\]
Similarly, we multiply the both sides of the first equation of (1) by $u_t$ and integrate over $\mathbb{R}^N$ to obtain
\[
\frac{d}{dt} \left( \frac{1}{2} J(t) + \frac{1}{2\theta} \|u\|^{2\theta}_{W_\theta} \right) = \frac{d}{dt} \left( \frac{1}{p+1} \|u\|^{p+1}_{p+1} \right).
\]
Integrating (45) over $(0, t)$, we obtain
\[
\frac{1}{2} J(t) + \frac{1}{2\theta} \|u\|^{2\theta}_{W_\theta} - E(0) = \frac{1}{p+1} \|u\|^{p+1}_{p+1}.
\]
Combining (44) and (46), we get
\[
\frac{1}{2} \frac{d^2}{dt^2} \Phi(t) + (p+1)E(0) = \frac{p+3}{2} J(t) + \frac{p+1 - 2\theta}{2\theta} \|u\|^{2\theta}_{W_\theta}.
\]
Since $p+1 > 2\theta$, there holds
\[
\frac{1}{2} \frac{d^2}{dt^2} \Phi(t) + (p+1)E(0) > \frac{p+3}{2} J(t).
\]
By the Cauchy-Schwarz inequality, we know
\[
\Phi_t^2 \leq 4J \Phi.
\]
Then, we obtain
\[
\Phi_t \Phi + 2(p+1)E(0) \Phi - \frac{p+3}{4} \Phi_t^2 \geq 0.
\]
Comparing (49) with (40), it is easy to see that
\[
\gamma = 0,
\]
\[
\alpha = \frac{p+3}{4} \geq 1,
\]
\[
\beta = 2(p+1)E(0) > 0,
\]
hence
\[
\frac{\gamma}{\alpha-1} = 0
\]
and
\[
\frac{2\beta}{2\alpha-1} = 8E(0).
\]
Thus, by Lemma 4.1, there exists
\[ T^* \leq \frac{1}{A} \|u_0\|^{-\frac{2}{p+1}}. \]
such that
\[ \limsup_{t \to T^*} \|u(\cdot, t)\| = +\infty, \]
where $A$ is given by (43). Moreover, we have
\[ \|u\|^2 \geq \left(\|u_0\|^{-\frac{2}{p+1}} - At\right)^{-\frac{4}{p+1}}. \]
This completes the proof. \(\square\)

Finally, we estimate the lower bound of blow up time for problem (1) with arbitrary positive energy.

**Theorem 4.3.** Assume that $p \in (2\theta - 1, 2\theta/2]$ and $E(0) > 0$. Let $u(\cdot, t)$ be the solution of problem (1), which blows up at a finite time $T$. Then
\[ T \geq \int_{G(0)}^\infty \left( \frac{p+1}{p+1}E(t) + \frac{2}{p+1}\|u\|_{p+1}^2 + \frac{2}{p+1}\|u_0\|_{p+1}^2 \right)^{-\frac{1}{2}} dy, \]
where $C$ is the constant given in (9),
\[ G(0) = \|u_0\|_{p+1}^2 \]
and
\[ E(0) = \frac{1}{2}\|u_1\|^2 + \frac{1}{2\theta}\|u_0\|_{W_0}^2 - \frac{1}{p+1}\|u_0\|_{p+1}^2. \]

**Proof.** From (10) and Lemma 2.3, we derive
\[ \|u_1\|^2 + \frac{1}{\theta}\|u\|_{W_0}^2 = 2E(t) + \frac{2}{p+1}\|u\|_{p+1}^2 = 2E(0) + \frac{2}{p+1}\|u\|_{p+1}^2. \]
Let $G(t) = \int_\Omega |u|^{p+1} dx$. Then by Cauchy's inequality, inequalities of arithmetic and geometric means, (50) and Lemma 2.1, we have
\begin{align*}
G'(t) \leq & (p+1) \left( \int_\Omega |u_1|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega |u|^{2p} dx \right)^{\frac{1}{2}} \\
\leq & \frac{p+1}{2} \left( \int_\Omega |u_1|^2 dx + \int_\Omega |u|^{2p} dx \right) \\
\leq & \frac{p+1}{2} \left( \|u_1\|^2 + C\|u\|_{W_0}^2 \right) \\
\leq & \frac{p+1}{2} \left( 2E(t) + \frac{2}{p+1}\|u\|_{p+1}^2 + C\left( \frac{2\theta E(t) + \frac{2\theta}{p+1}\|u\|_{p+1}^2}{\theta} \right)^{\frac{\theta}{p+1}} \right) \\
\leq & (p+1)E(t) + G(t) + C\left( \frac{2\theta E(t) + \frac{2\theta}{p+1}\|u\|_{p+1}^2}{\theta} \right)^{\frac{\theta}{p+1}} \\
= & (p+1)E(0) + G(t) + C\left( \frac{2\theta E(t) + \frac{2\theta}{p+1}\|u\|_{p+1}^2}{\theta} \right)^{\frac{\theta}{p+1}} \\
& + C\left( \frac{2\theta E(t) + \frac{2\theta}{p+1}\|u\|_{p+1}^2}{\theta} \right)^{\frac{\theta}{p+1}} G(t)^{\frac{\theta}{p+1}}. \end{align*}
From (39) and the continuous embedding from $L^{p+1}(\Omega)$ into $L^2(\Omega)$, we obtain that $\lim_{t \to T^*} G(t) = \infty$. Hence from (51) we can get the desired result. The proof is therefore complete. \(\square\)
Remark 1. In this section, we just consider a special case of the nonlinear source term $f(u)$. Then we estimate the upper bound for $\|u\|^2$ and the lower bound estimate for $\|u\|^{p+1}$. However, there is no estimate of blow up upper and lower bounds for more general source terms $f(u)$ with arbitrary positive energy. Moreover, for the lower bound estimate of blow up time, we only get the corresponding results for the case $p \in (2\theta - 1, 2^*_s/2]$. Therefore, the corresponding results for the case $p \in (2^*_s/2, 2^*_s - 1]$ have still not been resolved.

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E-mail address: Linqiang.edu@126.com
E-mail address: tianxueteng617@163.com
E-mail address: xurunzh@163.com
E-mail address: meina_zhang@163.com