From Submodule Categories to the Stable Auslander Algebra

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Abstract

We construct two functors from the submodule category of a self-injective representation-finite algebra Λ to the module category of the stable Auslander algebra of Λ. These functors factor through the module category of the Auslander algebra of Λ. Moreover they induce equivalences from the quotient categories of the submodule category modulo their respective kernels and said kernels have finitely many indecomposable objects up to isomorphism. Their construction uses a recollement of the module category of the Auslander algebra induced by an idempotent and this recollement determines a characteristic tilting and cotilting module. If Λ is taken to be a Nakayama algebra, then said tilting and cotilting module is a characteristic tilting module of a quasi-hereditary structure on the Auslander algebra. We prove that the self-injective Nakayama algebras are the only algebras with this property.

1 Introduction

Fix a field $k$, all algebras considered will be algebras over $k$ and all modules will be finite-dimensional left modules unless stated otherwise. For an algebra $A$ we let $A$-mod denote the category of finite-dimensional left $A$-modules. As a shorthand notation we often write $(M,N)_A := \text{Hom}_A(M,N)$ for the homomorphism spaces in an additive category $A$. Similarly we write $(M,N)_A := \text{Hom}_A(M,N)$ for homomorphisms in $A$-mod for an algebra $A$.

For an additive category $A$ let ind($A$) denote the class of indecomposable objects in $A$ and we write ind($A$) := ind($A$-mod). An additive subcategory of $A$ is a full subcategory closed under finite direct sums and direct summands. For an object $A \in A$ let add($A$) denote the smallest additive subcategory of $A$ containing $A$. Denote the algebra of upper triangular $2 \times 2$ matrices with coefficients in $A$ by $T_2(A)$.

The category of morphisms in $A$-mod is the category which has maps $(M_1 \xrightarrow{f_M} M_0)$ of $A$-modules as objects and a morphism of two objects $(M_1 \xrightarrow{f_M} M_0)$ and $(N_1 \xrightarrow{f_N} N_0)$ is a pair $(g_1, g_0) \in (M_1, N_1)_A \times (M_0, N_0)_A$ such that $f_N g_1 = g_0 f_M$. We will identify $T_2(A)$-mod with the category.
of morphisms in $\text{A}-\text{mod}$. The submodule category of $\text{A}$, denoted $\text{S}(\text{A})$, is the full subcategory of monomorphisms in $T_2(\text{A})-\text{mod}$. We denote the full subcategory of epimorphisms by $\text{E}(\text{A})$.

Studies of submodule categories go back to Birkhoff [5]. Recently they have been a subject to active research, including work of Simson about their tame-wild dichotomy [25,27,28]. Also Ringel and Schmidmeier have studied their Auslander-Reiten theory [22], as well as some particular cases of wild type [23]. Moreover they were studied with respect to Gorenstein-projective modules and tilting theory in [20,30]. The homological properties of submodule categories give extensive information on quiver Grassmannians, in particular their isomorphism classes correspond to strata in certain stratifications [6]. If $\text{A}$ is self-injective, the monomorphism category is a Frobenius category, and Chen has shown its stable category is equivalent to the singularity category of $T_2(\text{A})$ [7]. In [16] Kussin, Lenzing and Meltzer give a connection of monomorphism categories to weighted projective lines, which again connects them to singularity categories [17].

We define the kernel of an additive functor $F : \text{A} \to \text{B}$ as the full subcategory of all objects $A$ in $\text{A}$ such that $F(A) \cong 0$. Given an additive category $\text{A}$ and a full subcategory $\text{B}$ of $\text{A}$, the quotient category $\text{A}/\text{B}$ has objects $\text{Ob}(\text{A}/\text{B}) = \text{Ob}(\text{A})$. For $X,Y \in \text{Ob}(\text{A})$, let $R_{\text{B}}(X,Y)$ denote the morphisms of $\text{Hom}_{\text{A}}(X,Y)$ that factor through an object in $\text{B}$. The morphisms spaces of $\text{A}/\text{B}$ are defined as the quotients $\text{Hom}_{\text{A}/\text{B}}(X,Y) := \text{Hom}_{\text{A}}(X,Y)/R_{\text{B}}(X,Y)$.

Fix a basic algebra $\text{A}$ of finite representation type. Let $E$ be the additive generator of $\text{A}-\text{mod}$, i.e. the basic $\text{A}$-module such that $\text{add}(E) = \text{A}-\text{mod}$. The Auslander algebra of $\text{A}$ is $\text{Aus}(\text{A}) := \text{End}_{\text{A}}(E)^{\text{op}}$. Write $\Gamma := \text{Aus}(\text{A})$ and let $e \in \Gamma$ be the idempotent given by the projection onto the summand $\text{A}$ of $E$. Write $\Gamma e \Gamma$ for the two sided ideal generated by $e$, the stable Auslander algebra is defined as the algebra $\Gamma := \Gamma / \Gamma e \Gamma$.

Write $\alpha := \text{coker}(E,-)_{\text{A}} : T_2(\text{A})-\text{mod} \to \Gamma-\text{mod}$, and let $\epsilon : S(\text{A}) \to T_2(\text{A})$ be the quotient functor $\epsilon(M_1 \to M_0) := (M_0 \to M_0/M_1)$. We define the functors $F = \Gamma \otimes_{\Gamma} \alpha(-)$ and $G = \Gamma \otimes_{\Gamma} \alpha(\epsilon(-))$ from $S(\text{A})$ to $\Gamma-\text{mod}$. In [24] Ringel and Zhang studied those functors in the case $\text{A} = k[x]/(x^N)$, and observed that then $\Gamma$ is isomorphic to the preprojective algebra $\Pi_{N-1}$ of type $A_{N-1}$. They show that each of the functors induces an equivalence of categories $S(k[x]/(x^N))/\chi \to \Pi_{N-1}-\text{mod}$, where $\chi$ is the kernel of the respective functor $F$ or $G$. [24] Theorem 1]. Moreover they could also describe those kernels explicitly. We will prove the following generalization of that statement.

**Theorem 1.** Let $\text{A}$ be a basic, self-injective and representation finite algebra. Let $\mathcal{U}$ denote the smallest additive subcategory of $S(\text{A})$ containing $(E \xrightarrow{id} E)$ and all objects of the form $(M \xrightarrow{f} I)$, where $I$ is a projective-injective $\text{A}$-module. Also define $\mathcal{V} := \text{add}((E \xrightarrow{id} E) \oplus (0 \to E))$. Let $m$ be the number
of isomorphism classes of \( \text{ind}(\Lambda) \). Then \( \mathcal{U} \) and \( \mathcal{V} \) have \( 2m \) indecomposable objects up to isomorphism and the following holds.

(i) The functor \( F \) induces an equivalence of categories \( \mathcal{S}(\Lambda)/\mathcal{U} \to \Gamma\text{-mod} \).

(ii) The functor \( G \) induces an equivalence of categories \( \mathcal{S}(\Lambda)/\mathcal{V} \to \Gamma\text{-mod} \).

For now let us assume \( \Lambda \) is self-injective. The idempotent \( e \) yields a recollement of \( \Gamma\text{-mod} \), and there is a tilting and cotilting module \( T \) in \( \Gamma\text{-mod} \) given in terms of that recollement. That recollement and the module \( T \) feature in the construction of \( F \) and \( G \) in Section 4. Let \( \pi: \Gamma\text{-mod} \to \Gamma\text{-mod} \) be the projection to the stable category and let \( \Omega \) denote the syzygy functor on \( \Gamma\text{-mod} \). We prove the following generalization of [24, Theorem 2].

Theorem 2. The functors \( \pi F \) and \( \pi G \) differ by the syzygy functor on \( \Gamma\text{-mod} \), more precisely \( \pi F = \Omega \pi G \).

For the following we don’t need \( \Lambda \) to be self-injective. Let \( \Gamma\text{-torsl} \) denote the full subcategory of \( \Gamma\text{-mod} \) given by objects of projective dimension at most 1. In the final section we prove the following theorem.

Theorem 3. Let \( \Lambda \) be a basic representation-finite algebra and let \( \Gamma \) be its Auslander algebra. Then \( \Gamma \) has a quasi-hereditary structure such that the objects of \( \Gamma\text{-torsl} \) are precisely the \( \Delta \)-filtered \( \Gamma\text{-modules} \) if and only if \( \Lambda \) is uniserial.

This was already observed in [24] in the particular case of the Auslander algebra of \( k[x]/\langle x^N \rangle \). The theorem implies that \( T \) arises as the characteristic tilting module of a quasi-hereditary structure on \( \Gamma \) if and only if \( \Lambda \) is uniserial, i.e. if \( \Lambda \) is a self-injective Nakayama algebra.

The content is organized as follows. In Section 2 we recall the notion of a category of finitely presented functors, and that the module category \( \Gamma\text{-mod} \) is equivalent to the category of finitely presented additive contravariant functors from \( \Lambda\text{-mod} \) to the category of abelian groups. We also give the basic properties of the functor \( \alpha \), and recall characterizations of the projective and injective objects in the category of finitely presented functors.

In Section 3 we restrict our attention to the Auslander algebras of self-injective algebras. Then we study the recollement induced by the idempotent \( e \in \Gamma \) and introduce the tilting and cotilting module \( T \). Moreover we recall some properties of the stable Auslander algebra \( \Gamma \).

In Section 4 we consider the functors \( F \) and \( G \) that arise as compositions of functors studied in the previous sections. We prove Theorems 1 and 2 for these functors and thereby generalise the situation in [24].

Section 5 is dedicated to the proof of Theorem 3. First we give some properties of the Nakayama algebras and recall the notion of a left strongly quasi-hereditary structure. In Subsection 5.3 we introduce a quasi-hereditary
structure on the Auslander algebras of the Nakayama algebras which fulfills the conditions of Theorem 3. In Subsection 5.4 we prove that no other Auslander algebras satisfy those conditions.

2 Submodule Categories

Recall that $\Lambda$ is a basic representation-finite algebra.

**Definition 2.1.** We define functors:

- $\eta: S(\Lambda) \to T_2(\Lambda)\text{-mod}$, $f \mapsto f$,
- $\epsilon: S(\Lambda) \to T_2(\Lambda)\text{-mod}$, $f \mapsto \text{coker}(f)$.

The functor $\eta$ is simply the inclusion of $S(\Lambda)$ in $T_2(\Lambda)\text{-mod}$. On morphisms, $\epsilon$ is given by the induced maps of cokernels. Notice that $\epsilon$ is full and faithful and its essential image is the full subcategory $E(\Lambda)$ of $T_2(\Lambda)$, hence we can consider $\epsilon$ as a composition of an equivalence $S(\Lambda) \to E(\Lambda)$ followed by $\eta$.

Now we recall some well known facts on representable functors. These go back to Auslander [2], Freyd [13, 14] and Gabriel [15], while [18] contains a handy summary of those techniques.

Let $\mathcal{A}$ be an additive category. We consider the category $\text{Fun}(\mathcal{A})$ of additive functors from $\mathcal{A}^{op}$ to the category $\text{Ab}$ of abelian groups, with morphisms given by natural transformations. We say a functor $F \in \text{Fun}(\mathcal{A})$ is representable if $F$ is isomorphic to $(-,M)_\mathcal{A}$ for some $M \in \text{Ob}(\mathcal{A})$. We say $F$ is finitely presented if there exist representable functors $(-,M)_\mathcal{A}$, $(-,N)_\mathcal{A}$ and an exact sequence

$$(-,M)_\mathcal{A} \to (-,N)_\mathcal{A} \to F \to 0.$$  

We denote the full subcategory of finitely presented functors by $\text{fun}(\mathcal{A})$. The category $\text{fun}(\mathcal{A})$ is abelian, cf. [13 Theorem 5.11]. To reduce encumbrance we write $\text{fun}(\Lambda) := \text{fun}(\Lambda\text{-mod})$.

The following lemma comes from applying [13 Theorem 5.35] to the opposite of $\Lambda\text{-mod}$.

**Lemma 2.2.** The functor $M \mapsto (-,M)_\Lambda$ from $\Lambda\text{-mod}$ to $\text{fun}(\Lambda)$ induces an equivalence of categories from $\Lambda\text{-mod}$ to the full subcategory of representable functors in the category $\text{fun}(\Lambda)$. Moreover, the representable functors are the projective objects of $\text{fun}(\Lambda)$.

Let $D := (-,k)_k$ be the vector space duality. If we apply [13 Theorem 5.35] to $\Lambda\text{-mod}$ and then apply the vector space duality we obtain the following dual statement to Lemma 2.2, cf. [13 Exercise A. Chapter 5].

**Lemma 2.3.** The functor $M \mapsto D(M,-)_\Lambda$ from $\Lambda\text{-mod}$ to $\text{fun}(\Lambda)$ induces an equivalence from $\Lambda\text{-mod}$ to the full subcategory of injective objects in $\text{fun}(\Lambda)$. 

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2.1 A Functor to Representations of the Auslander Algebra

Any functor in \( \text{fun}(\Lambda) \) is determined by its value on the Auslander generator and its endomorphisms, so \( \text{fun}(\Lambda) \) is equivalent to \( \Gamma\text{-mod} \), this is [15, Chapitre II, Proposition 2]. Note that the representable functors \( (\cdot, M)_\Lambda \) correspond to the right \( \text{End}(E) \)-modules \( (E, M)_\Lambda \) acted upon by pre-composition, but these may also be viewed as left \( \Gamma \)-modules.

We will consider the functor
\[
\alpha := \text{coker}(E, -)_\Lambda : T_2(\Lambda)\text{-mod} \to \Gamma\text{-mod},
\]
which was already studied by Auslander and Reiten in [4].

**Remark.** The Gabriel quiver of \( \Gamma \) is the opposite quiver of the Auslander-Reiten quiver of \( \Lambda\text{-mod} \) with relations given by the Auslander-Reiten translate. The indecomposable projective \( \Gamma \)-modules are represented by the indecomposable objects of \( \Lambda\text{-mod} \). More precisely, given \( M \in \text{ind}(\Lambda) \), then \( (E, M)_\Lambda \) is the indecomposable projective \( \Gamma \)-module arising as the projective representation of the opposite of the Auslander-Reiten quiver of \( \Lambda\text{-mod} \) generated at the vertex of \( M \).

**Definition 2.4.** We call a category a **Krull-Schmidt category** if every object decomposes into a finite direct sum of indecomposable objects in a unique way up to isomorphism.

A functor \( F : \mathcal{A} \to \mathcal{B} \) between Krull-Schmidt categories is called **objective** if the induced functor \( \mathcal{A}/\ker(F) \to \mathcal{B} \) is faithful.

Our notion of an objective functor is equivalent to that used in [24]. For more information on this property we refer to [25].

**Proposition 2.5.** The functor \( \alpha \) is full, dense and objective. Its kernel is \( \text{add}(((E \xrightarrow{\text{id}} E) \oplus (E \to 0)) \).

**Remark.** The indecomposable objects of \( \ker(\alpha) \) are either of the form \( (M \xrightarrow{\text{id}} M) \) or \( (M \to 0) \) for \( M \in \text{ind}(\Lambda) \). Since \( \Lambda \) is of finite representation type, say with \( m \) indecomposable objects up to isomorphism, this means \( \ker(\alpha) \) has exactly \( 2m \) indecomposable objects up to isomorphism.

**Proof of Proposition 2.5.** We imitate the proof of [24, Proposition 3]. Let \( X \) be an object in \( \Gamma\text{-mod} \), it has a projective presentation
\[
(E, M_1)_\Lambda \xrightarrow{p_1} (E, M_0)_\Lambda \xrightarrow{p_0} X \to 0.
\]
By Lemma 2.2 there is \( f \in (M_1, M_0)_\Lambda \) such that \( p_1 = (E, f)_\Lambda \). But then \( \alpha(f) \simeq X \), so \( \alpha \) is dense. Let \( \Phi \in \text{Hom}_\Gamma(X, Y) \) and let \( f \in (M_1, M_0)_\Lambda \) and \( g \in (N_1, N_0)_\Lambda \) be such that \( \alpha(f) \cong X \) and \( \alpha(g) \cong Y \). Now \( \Phi \) can be extended to a map \( (\Phi_1, \Phi_0) \) of the projective presentations of \( X \) and \( Y \). There are \( \phi_i \).
for $i = 0, 1$ such that $(E, \phi_i) \cong \Phi_i$. But then clearly $\alpha(\phi_1, \phi_0) \cong \Phi$, thus $\alpha$ is full.

Clearly $\alpha(M \xrightarrow{id} M) \cong 0 \cong \alpha(M \to 0)$. Let

$$(g_1, g_0) \in \text{Hom}_{T_2(\Lambda)}((M_1 \xrightarrow{f_M} M_0), (N_1 \xrightarrow{f_N} N_0))$$

be such that $\alpha(g_1, g_0) = 0$. We want to show that $(g_1, g_0)$ factors through a $T_2(\Lambda)$-module of the form $(M \xrightarrow{id} M) \oplus (N \to 0)$.

Consider the following commutative diagram:

$$(E, M_1)_\Lambda \xrightarrow{(E, f_M)_\Lambda} (E, M_0)_\Lambda \xrightarrow{\alpha(f_M)} 0$$

$$(E, N_1)_\Lambda \xrightarrow{(E, f_N)_\Lambda} (E, N_0)_\Lambda \xrightarrow{\alpha(f_N)} 0.$$  

The rows are projective presentations. Now $c \circ (E, g_0)_\Lambda = 0$ and hence there is $h'$ such that $(E, g_0)_\Lambda = (E, f_N)_\Lambda \circ h'$. Since the functor $(E, -)_\Lambda$ is full there is a map $h: M_0 \to N_1$ such that $h' = (E, h)_\Lambda$ and $g_0 = f_N h$. Then the following diagram in $\Lambda$-mod is commutative:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{[f_M, id]} & M_0 \oplus M_1 & \xrightarrow{[h, g_1 - h f_M]} & N_1 \\
\downarrow{f_M} & & \downarrow{[id, 0]} & & \downarrow{f_N} \\
M_0 & \xrightarrow{id} & M_0 & \xrightarrow{g_0} & N_0.
\end{array}
$$

Note that the compositions of the rows are $g_1$ and $g_0$, and hence $(g_1, g_0)$ factors through the $T_2(\Lambda)$-module $(M_0 \oplus M_1 \xrightarrow{[id, 0]} M_0)$. \hfill \square

**Remark.** The functors $\epsilon$ and $\eta$ are faithful and hence objective. The composition $\alpha \eta$ is also objective since it is just a restriction of the objective functor $\alpha$ to an additive subcategory. Moreover $\alpha \epsilon$ is objective because $\epsilon$ is fully faithful and the image of $\epsilon$ contains all objects of $\ker \alpha$.

The following corollary of Proposition 2.5 describes the composition $\alpha \eta$.

**Corollary 2.6.** Let $\chi := \text{add}(E \xrightarrow{id} E)$. Let $\Gamma$-torsl denote the full subcategory of $\Gamma$-mod consisting of objects of projective dimension $\leq 1$. The functor $\alpha \eta$ induces an equivalence of categories

$$S(\Lambda)/\chi \to \Gamma\text{-torsl}.$$  

**Proof.** We know already that $\alpha \eta$ is full and objective and by Proposition 2.5 the kernel of $\alpha \eta$ is $\chi$. It remains to show that the essential image of $\alpha \eta$
is Γ-torsl. Let $f \in (M_1, M_0)_\Lambda$ be a monomorphism. Since Hom-functors are left-exact, $(E, f)$ is a monomorphism and $\alpha(f)$ has a projective resolution

$$
0 \longrightarrow (E, M_1)_\Lambda \xrightarrow{(E, f)_\Lambda} (E, M_0)_\Lambda \longrightarrow \alpha(f) \longrightarrow 0.
$$

Using that $(E, -)_\Lambda$ is an equivalence when considered as a functor to Γ-proj, we see any object in Γ-torsl has a projective resolution of this form where $f : M_1 \to M_0$ is a monomorphism.

**Remark.** We know $\alpha$ behaves really well with respect to the additive structure on $T_2(\Lambda)$-mod and Γ-mod, and these are both abelian categories. However $\alpha$ is far from being exact, in fact it preserves neither epimorphisms nor monomorphisms. Take for example $\Lambda = k[x]/(x^2)$ and let $\Lambda k$ be the simple $\Lambda$ module. Consider a monomorphism $f : (0 \to \Lambda \Lambda) \to (\Lambda \Lambda \xrightarrow{id} \Lambda \Lambda)$. Since $\alpha(\Lambda \Lambda \xrightarrow{id} \Lambda \Lambda) = 0$ but $\alpha(0 \to \Lambda \Lambda) \neq 0$, $\alpha(f)$ is not a monomorphism. Also there is an epimorphism $g : (\Lambda \Lambda \xrightarrow{id} \Lambda \Lambda) \to (\Lambda \Lambda \to \Lambda k)$, but $\alpha(\Lambda \Lambda \to \Lambda k) \neq 0$, thus $\alpha(g)$ is no epimorphism.

There are several characterizations of the subcategory Γ-torsl, one of which also justifies the notation we use for it.

**Proposition 2.7.** The following are equivalent for an object $X \in \Gamma$-mod.

(i) $X$ is in Γ-torsl.

(ii) The injective envelope of $X$ is projective.

(iii) $X$ is torsionless, i.e. a submodule of a projective module.

**Proof.** (i) $\implies$ (ii). Let $X$ be of projective dimension $\leq 1$, so it has a projective resolution $0 \to P_1 \xrightarrow{u} P_0 \to X \to 0$. Let $v_i : P_i \to I(P_i)$ be the injective envelope of $P_i$ for $i = 0, 1$ and consider the following diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & P_1 \\
\downarrow v_1 & & \downarrow v_0 \\
0 & \longrightarrow & I(P_1) \\
& & \downarrow f \\
\end{array}
\quad
\begin{array}{ccc}
P_0 & \longrightarrow & X \\
\downarrow f & & \downarrow f \\
I(P_0) & \longrightarrow & X' \\
\end{array}
\longrightarrow 0.
$$

Here $X'$ is defined as the module making the diagram commutative with exact rows. Since $v_0$ is injective the snake lemma yields a monomorphism $\ker(f) \to \coker(v_1)$, but since any Auslander algebra has dominant dimension $\geq 2$ we can embed $\coker(v_1)$ into a projective-injective module. Thus $\ker(f)$ embeds in a projective-injective module $I(\ker(f))$. Again using that the dominant dimension of $\Gamma$ is $\geq 2$, we know $I(P_i)$ is projective for $i = 0, 1$. Thus the lower sequence splits and $X'$ is projective-injective. The inclusion $\ker(f) \to I(\ker(f))$ factors through $X$, because $I(\ker(f))$ is injective, and thus we get a monomorphism $X \to X' \oplus I(\ker(f))$.
(ii) \implies (iii). Clear.

(iii) \implies (i). We have an exact sequence \[ 0 \to X \to P \xrightarrow{\pi} C \to 0, \]
where \( P \) is projective. Then \( C \) has a projective resolution
\[ 0 \to P_2 \to P_1 \to P \xrightarrow{\pi} C \to 0, \]
with \( \text{im}(p_1) \cong X \). Thus \( X \) has a projective resolution of length \( \leq 1 \). \( \square \)

We say a module is divisible if it is a factor module of an injective module. We denote the full subcategory of divisible \( \Gamma \)-modules by \( \Gamma\text{-divbl} \). We get the following dual statement to Proposition 2.7.

**Proposition 2.8.** The following are equivalent for an object \( X \in \Gamma\text{-mod} \).

(i) \( X \) has injective dimension \( \leq 1 \).

(ii) The projective cover of \( X \) is injective.

(iii) \( X \) is in \( \Gamma\text{-divbl} \).

Later we will have use for the following lemma, which is due to Auslander and Reiten, see [3, Proposition 4.1]. A proof of the version stated here is found in [24, Section 6].

**Lemma 2.9.** Let \( f \) be a morphism in \( \Lambda\text{-mod} \). Then \( f \) is an epimorphism if and only if \( ((E, P)_\Lambda, \alpha(f))_T = 0 \) for any projective module \( P \) in \( \Lambda\text{-mod} \).

### 2.2 Relative Projective and Injective Objects of \( S(\Lambda) \)

The submodule category is additive and by the snake lemma it is closed under extensions. Thus it is an exact subcategory of \( T_2(\Lambda)\text{-mod} \). The projective and injective objects of \( T_2(\Lambda)\text{-mod} \) are known, a classification can for example be found in [29, Lemma 1.1]. All projective \( T_2(\Lambda)\)-modules are a direct sum of modules of the form \( (P \xrightarrow{id} P) \) or \((0 \to P)\), where \( P \) is a projective \( \Lambda \)-module. In particular all projective \( T_2(\Lambda)\)-modules belong to \( S(\Lambda) \), and they are the relative projective modules of that exact subcategory.

Dually, the injective \( T_2(\Lambda)\)-modules are a direct sum of modules of the form \((I \xrightarrow{id} I)\) or \((I \to 0)\), where \( I \) is an injective \( \Lambda \)-module. The relative injective objects of \( S(\Lambda) \) can be written as direct sums of objects of the form \((I \xrightarrow{id} I)\) or \((0 \to I)\), with \( I \) an injective \( \Lambda \)-module.

If additionally \( \Lambda \) is self-injective, i.e. \( \Lambda\text{-mod} \) is a Frobenius category, the proposition below, found in [7, Lemma 2.1], is an easy consequence.

**Proposition 2.10.** Let \( \Lambda \) be a self-injective algebra of finite representation type. Then \( S(\Lambda) \) is a Frobenius category and the projective-injective objects are exactly those in \( \text{add}((\Lambda \xrightarrow{id} \Lambda) \oplus (0 \to \Lambda)) \).
Remark 2.11. If $\Lambda$ is self-injective the submodule category $\mathcal{S}(\Lambda)$ is precisely the full subcategory of Gorenstein projective $T_2(\Lambda)$-modules, cf. [19 Theorem 1.1]. Thus $\eta$ is the inclusion of the Gorenstein projective modules of $T_2(\Lambda)$-mod.

3 The Auslander Algebra of Self-injective Algebras

In this section we fix $\Lambda$ as a finite-dimensional basic self-injective $k$-algebra of finite representation type.

Let $\nu := D(\mathcal{A}, \Lambda)$ be the Nakayama functor on $\Lambda$-mod. Its restriction to projective modules is an equivalence from the projective $\Lambda$-modules to the injective $\Lambda$-modules with inverse $\nu^{-1} := (D(-), \Lambda)$. Recall that $e$ denotes the idempotent of $\mathcal{A}$ given by the opposite of the projection onto the summand $\Lambda$ of $E$. Let $\mathcal{A}$ denote the left ideal generated by $e$. The following lemma describes the projective-injective objects of $\mathcal{A}$-mod explicitly.

Lemma 3.1. The projective-injective objects of $\mathcal{A}$-mod are precisely the objects of $\text{add}(\mathcal{A})$. Moreover $\mathcal{A} \cong (E, \Lambda) \cong D(\Lambda, E)$.

Proof. It is clear that $\mathcal{A} \cong (E, \Lambda)$. Recall that there is an equivalence $D(\Lambda, -)_\Lambda \cong (-, \nu_\Lambda)_\Lambda$ and, since $\Lambda$ is self-injective, $\nu_\Lambda = \Lambda$. Hence $(E, \Lambda)_\Lambda \cong D(\Lambda, E)_\Lambda$ and by lemmas 2.2 and 2.3 it is a projective-injective module.

Let $(E, M)_\Lambda$ be a projective-injective $\mathcal{A}$-module. Then every monomorphism $(E, M)_\Lambda \rightarrow (E, N)_\Lambda$ is a split monomorphism, but that implies any monomorphism $M \rightarrow N$ in $\Lambda$-mod is a split monomorphism. Thus $M$ is a projective-injective $\Lambda$-module. \qed

Remark. This means the indecomposable projective-injective $\mathcal{A}$-modules are the projective modules at vertices corresponding to indecomposable projective-injective $\Lambda$-modules, when we consider the Gabriel quiver of $\mathcal{A}$ as the opposite of the Auslander-Reiten quiver of $\Lambda$-mod.

3.1 Recollement

Notice that $e\mathcal{A}e = \text{End}(\Lambda)^{\text{op}} \cong \Lambda$, hence $\Lambda$ embeds into $\mathcal{A}$. Let $\mathcal{A}/e\mathcal{A}$ denote the two sided ideal of $\mathcal{A}$ generated by $e$ and denote the quotient $\mathcal{A}/e\mathcal{A}$ by $\mathcal{A}$, we call this the stable Auslander algebra of $\Lambda$.

Consider the diagram

$$
\begin{array}{ccc}
\mathcal{A} \xrightarrow{q} & \mathcal{A} \xleftarrow{p} & \mathcal{A} \\
\xleftarrow{e} & \xrightarrow{l} & \xleftarrow{r}
\end{array}
$$

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of functors, where the functors are defined as follows:

\[ q := \Gamma \otimes \Gamma - , \quad l := \Gamma e \otimes_A - , \]
\[ \iota := \text{Inclusion}, \quad e := (\Gamma e, -)_\Gamma, \]
\[ p := (\Gamma / \Gamma e \Gamma, -)_\Gamma, \quad r := (e \Gamma, -)_A. \]

This construction goes back to Cline, Parshall and Scott \cite{8, 9}, and it gives a recollement of abelian categories. By the definition of a recollement we have the following conditions:

(a) The functor \( l \) is a left adjoint of \( e \) and \( r \) is a right adjoint of \( e \).

(b) The unit \( \text{id}_A \to e l \) and the counit \( e r \to \text{id}_A \) are isomorphisms.

(c) The functor \( q \) is a left adjoint of \( \iota \) and \( p \) is a right adjoint of \( \iota \).

(d) The unit \( \text{id}_\Gamma \to p \iota \) and the counit \( q \iota \to \text{id}_\Gamma \) are isomorphisms.

(e) The functor \( \iota \) is an embedding onto the full subcategory \( \text{ker}(e) \).

Remark. Since \( \Lambda \) is self-injective, \( \Gamma \) can be identified with the projective quotient algebra introduced in \cite{10, Section 5} and the recollement above is the same as the main recollement from \cite{10, Section 4}.

We construct the intermediate extension functor \( c: \Lambda\text{-mod} \to \Gamma\text{-mod} \) as follows. Since the counit \( e r \to \text{id}_A \) is an isomorphism we have an inverse \( \text{id}_A \to e l \). If we apply the adjunction \( (l, e) \) to the inverse we get a natural transformation \( \gamma: l \to r \). Then we define \( c := \text{im}(\gamma) \).

Let us recall the notions of tilting modules and cotilting modules. Let \( X \) be a \( \Gamma \)-module, we call \( X \) a tilting module if the following hold:

1. The projective dimension of \( X \) is at most 1.
2. \( X \) is rigid, i.e. \( \text{Ext}^1_\Gamma(X, X) = 0 \).
3. \( X \) has \( n \) indecomposable summands where \( n \) is the number of indecomposable direct summands of \( \Gamma \).

Dually we say \( X \) is cotilting if it satisfies (2) and (3) and has injective dimension at most 1.

We say a module \( M \) in \( A\text{-mod} \) is generated by \( N \) if there exists an epimorphism \( N^n \to M \) for some \( n \in \mathbb{N} \). Dually we say \( M \) is cogenerated by \( N \) if there is a monomorphism \( M \hookrightarrow N^n \) for some \( n \in \mathbb{N} \). We denote by \( \text{gen}(N) \) (resp. \( \text{cogen}(N) \)) the full subcategory of modules generated by \( N \) (resp. cogenerated by \( N \)).

Consider the \( \Gamma \)-module \( T := c(E) \).

**Lemma 3.2.** The module \( T \) is a tilting and cotilting module. Moreover the following conditions hold.

10
(i) The kernel of $p$ is $\ker(p) = \text{cogen}(T) = \Gamma\text{-torsl}.$

(ii) The kernel of $q$ is $\ker(q) = \text{gen}(T) = \Gamma\text{-divbl}.$

Proof. To see that $T$ is a tilting and cotilting module we refer to [10, Section 5]. There it is also shown that $\ker(p) = \text{cogen}(T)$ and $\ker(q) = \text{gen}(T).$ Since $T$ is tilting, all projective $\Gamma$-modules are in $\text{cogen}(T).$ Hence $\Gamma\text{-torsl}$ is contained in $\text{cogen}(T).$ Since $T$ is a tilting module it is of projective dimension at most 1 and hence torsionless by Proposition 2.7. But $\Gamma\text{-torsl}$ is closed under taking submodules, thus $\text{cogen}(T) \subset \Gamma\text{-torsl}.$ This proves (i), the proof of $\text{gen}(T) = \Gamma\text{-divbl}$ goes dually.

3.2 The Stable Auslander Algebra

The algebra $\Gamma$ has an alternative description. Notice that $\Gamma e\Gamma \subset \Gamma$ is given by all maps in $\text{End}(E)^{op}$ that factor through a projective-injective $\Lambda$-module. Therefore $\Gamma = \text{End}(E)^{op},$ the opposite of the endomorphism ring of $E$ in the stable category $\Lambda\text{-mod}.$ Thus $\Gamma\text{-mod}$ is equivalent to the category of finitely presented additive functors from $(\Lambda\text{-mod})^{op}$ to $k$-vector spaces.

The following proposition is classical. It follows from [14, Theorem 1.7] and the fact that every map in a triangulated category is a weak kernel and weak cokernel.

Proposition 3.3 (Freyd’s Theorem). Let $T$ be a triangulated category. Then $\text{fun}(T)$ is a Frobenius category.

Since $\Lambda$ is self-injective, $\Lambda\text{-mod}$ is a triangulated category. Hence the following corollary.

Corollary 3.4. The category $\Gamma\text{-mod}$ is a Frobenius category.

4 From Submodule Categories to Representations of the Stable Auslander Algebra

Here we follow the story of [24] in a more general setting for any basic self-injective algebra $\Lambda$ of finite representation type. We have already studied the functor $\alpha\eta: S(\Lambda) \to \Gamma\text{-mod}$ in Section 2 and $q: \Gamma\text{-mod} \to \Gamma\text{-mod}$ in Section 3. We use what we have gathered about those functors to study the compositions

$$S(\Lambda) \xrightarrow{\eta} T_2(\Lambda)\text{-mod} \xrightarrow{\alpha} \Gamma\text{-mod} \xrightarrow{q} \Gamma\text{-mod}.$$ 

The functors $F$ and $G$ are given by $F := q\alpha\eta$ and $G := q\alpha\epsilon.$ The functor $F$ was already studied by Li and Zhang in [19]. In [14] Auslander and Reiten considered the composition $G,$ based on previous work by Gabriel [15].
4.1 Induced Equivalences

We have already established that $\eta$ and $\alpha$ as well as the compositions $\alpha \eta$ and $\alpha \epsilon$ are objective. In corollary 2.6 we established that the essential image of $\alpha \eta$ is $\Gamma$-torsl. For now we shall consider $\alpha \eta$ as a functor to $\Gamma$-torsl. Moreover we write $q_t$ for the restriction of $q$ to $\Gamma$-torsl.

**Proposition 4.1.** The functor $q_t$ is objective, i.e. it induces an equivalence from $\Gamma$-torsl/$\text{add}(T)$ to $\Gamma\text{-mod}$.

**Proof.** Since $\ker q = \text{gen}(T)$ and $\Gamma$-torsl = $\text{cogen}(T)$ we know that the kernel of $q_t$ is $\text{cogen}(T) \cap \text{gen}(T) = \text{add}(T)$. First we show that the induced functor $\Gamma$-torsl/$\text{add}(T) \to \Gamma\text{-mod}$ is faithful. By [12, Proposition 4.2] there is an exact sequence of functors

$$le \xrightarrow{\psi} \text{id}_\Gamma \xrightarrow{\phi} \iota q \xrightarrow{} 0.$$  

Let $Y \in \Gamma$-torsl, there is an epimorphism $\phi_Y : Y \to \iota q(Y)$ and the morphism $\psi_Y : le(Y) \to Y$ factors through $\ker(\phi_Y)$ via an epimorphism, in particular $\ker(\phi_Y)$ is in $\text{gen}(T)$. Since $\ker(\phi_Y)$ is a submodule of $Y$ it belongs to $\text{cogen}(T)$, thus $\ker(\phi_Y) \in \text{add}(T)$. Now let $f : X \to Y$ be a morphism in $\Gamma$-torsl such that $q_t(f) = 0$. Then $\phi_Y f = 0$ and thus $f$ factors through $\ker(\phi_Y)$.

To show $q_t$ is full we consider the adjoint pair $(q, \iota)$. Let $X, Y \in \Gamma$-torsl, we want to show the map $q_{XY}$ induced by the functor $q$ in the following sequence is surjective.

$$(X,Y) \Gamma \xrightarrow{(qX,qY)} (qX,qY)_{\Gamma} \xrightarrow{\Phi} (X,\iota qY)_{\Gamma}.$$  

Here $\Phi$ is the isomorphism given by the adjunction. Let $\phi_Y : Y \to \iota q(Y)$, by [12, Proposition 4.2] this is an epimorphism, so we get an exact sequence

$$0 \xrightarrow{} K \xrightarrow{} Y \xrightarrow{\phi_Y} \iota qY \xrightarrow{} 0.$$  

By our argument above we know $K \in \text{add}(T)$. Apply $(X,-)_{\Gamma}$ to the exact sequence above and get the exact sequence

$$0 \xrightarrow{} (X,K)_{\Gamma} \xrightarrow{} (X,Y)_{\Gamma} \xrightarrow{(X,\phi_Y)} (X,\iota qY)_{\Gamma} \xrightarrow{} \text{Ext}^1_T(X,K).$$  

Since $X \in \text{cogen}(T)$ and $K \in \text{add}(T)$ we have $\text{Ext}^1(X,K) = 0$, thus $(X,\phi_Y) = \Phi q_{XY}$ is an epimorphism. Since $\Phi$ is an isomorphism that implies $q_{XY}$ is an epimorphism.

To show denseness we adapt the proof of [24, Proposition 5]. Let $X \in \Gamma\text{-mod}$ and write $X := \iota(I(X))$. We let $u : X \to I(X)$ be the injective envelope
and \( p: PI(X) \to I(X) \) be a projective cover with kernel \( K \). We get an induced diagram

\[
\begin{array}{cccccccc}
0 & \to & K & \to & Y & \to & X & \to & 0 \\
0 & \to & K & \to & PI(X) & \to & I(X) & \to & 0
\end{array}
\]

Since \( u' \) is a monomorphism, \( Y \) embeds into a projective-injective \( \Gamma \)-module, and hence \( Y \in \Gamma\text{-torsl} \). Moreover \( K \) has injective dimension at most 1, hence \( q_t(K) = 0 \). By the defining properties of a recollement we have \( q_t(X) \cong q_t(X) \cong X \), and \( q_t \) is right-exact because \( q_t \) is a left adjoint. It follows that \( q_t(Y) \cong q_t(X) \cong X \). We have shown that \( q_t \) is dense.

We know \( \alpha \eta \) is objective and dense when considered as a functor to \( \Gamma\text{-torsl} \) and thus the functor \( F \) is objective. Moreover \( F \) is full and dense because \( q_t \) and \( \alpha \eta \) are full and dense.

Let \( f \in T_2(\Lambda)\text{-mod} \), then Lemmas 2.9 and 3.1 imply that \( f \) is an epimorphism if and only if \( e(\alpha(f)) = (\Gamma e, \alpha(f))_\Gamma = 0 \). This means the essential image of \( \alpha \epsilon \) is ker(\( e \)), but we can identify ker(\( e \)) with \( \Sigma \text{-mod} \) via \( \iota \). We have established \( \alpha \epsilon \) is an objective functor so we conclude \( G \) is objective. The functor \( G \) is also full and dense because \( \alpha \epsilon \) is full and dense when considered as a functor to ker(\( e \)).

Now we can prove our first main theorem.

**Theorem 1.** Let \( \Lambda \) be a basic, self-injective and representation finite algebra. Let \( U \) denote the smallest additive subcategory of \( S(\Lambda) \) containing \( (E \to E) \) and all objects of the form \( (M \to I) \), where \( I \) is a projective-injective \( \Lambda \)-module. Also define \( V := \text{add}((E \to E) \oplus (0 \to E)) \). Let \( m \) be the number of isomorphism classes of \( \text{ind}(\Lambda) \). Then \( U \) and \( V \) have \( 2m \) indecomposable objects up to isomorphism and the following holds.

(i) The functor \( F \) induces an equivalence of categories \( S(\Lambda)/U \to \Sigma\text{-mod} \).

(ii) The functor \( G \) induces an equivalence of categories \( S(\Lambda)/V \to \Sigma\text{-mod} \).

**Proof.** The indecomposable objects of \( V \) are \((M \to M)\) and \((0 \to M)\) for any \( M \in \text{ind}(\Lambda) \). Hence \( V \) clearly has \( 2m \) indecomposable objects up to isomorphism. The indecomposable objects of \( U \) are \((M \to M)\) and the injective envelope \((M \to I(M))\) for each \( M \in \text{ind}(\Lambda) \), as well as the objects \((0 \to I)\) for each injective object \( I \in \text{ind}(\Lambda) \). Since the objects \((I \to I)\) for \( I \in \text{ind}(\Lambda) \) injective appear twice in this list, \( U \) has \( 2m \) indecomposable objects up to isomorphism.
Next we prove (i). Let \((M \xrightarrow{f} N) \in S(\Lambda)\), and assume \(F(f) = 0\). Consider the diagram

\[
\begin{array}{c}
0 & \xrightarrow{} & P & \xrightarrow{\sim} & P & \xrightarrow{} & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \xrightarrow{} & (E, M)_\Lambda & \xrightarrow{(E, f)_\Lambda} & (E, N)_\Lambda & \xrightarrow{\alpha(f)} & 0 \\
\downarrow & & \| & & \| & & \| \\
0 & \xrightarrow{} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{\alpha(f)} & 0.
\end{array}
\]

Here the bottom row is a minimal projective resolution and all rows and columns are exact, thus \(P\) is projective and the first two columns are split exact sequences of projective modules. Since \(\Lambda\text{-mod}\) is equivalent to the full subcategory of projective \(\Gamma\text{-modules}\) this shows \(f\) is a direct sum of an isomorphism \((M' \xrightarrow{f'} N')\), corresponding to \(P \cong P\), and a monomorphism \((M'' \xrightarrow{f''} N'')\), corresponding to the map \(p_1\). Now \(F(f) = q(\alpha(f)) = 0\) if and only if \(P_1\) is projective-injective by Lemma 3.2 which is if and only if \(N''\) is projective-injective by Lemma 3.1.

We already know \(F\) is objective, and thus the functor \(S(\Lambda)/U \to \Gamma\text{-mod}\) induced by \(F\) is faithful. Since \(F\) is full and dense the induced functor is also full and dense.

Now to (ii). The kernel of \(\alpha\) is \(\text{add}((E \xrightarrow{id} E) \oplus (E \to 0))\). But

\[
\epsilon((E \xrightarrow{id} E) \oplus (0 \to E)) = (E \xrightarrow{id} E) \oplus (E \to 0),
\]

hence \(\mathcal{V} = \text{add}((E \xrightarrow{id} E) \oplus (0 \to E)) = \ker(\alpha \epsilon)\). Moreover the restriction of \(q\) to the essential image \(\ker(\epsilon)\) of \(\alpha \epsilon\) is an equivalence by the defining properties of a recollement. We have shown \(G\) is full, dense and objective, thus (ii) holds.

4.2 Interplay with Triangulated Structure

We have already established that the categories \(S(\Lambda)\) and \(\Gamma\text{-mod}\) are Frobenius categories. Then it is natural to ask whether the triangulated structure of the stable category \(\Gamma\text{-mod}\) interacts nicely with those functors \(F\) and \(G\).

Let \(\pi : \Gamma\text{-mod} \to \Gamma\text{-mod}\) be the projection to the stable category. We denote the syzygy functor on \(\Gamma\text{-mod}\) by \(\Omega\). The following was proven in a special case in [24, Section 7], and we prove this more general statement analogously.

**Theorem 2.** The functors \(\pi F\) and \(\pi G\) differ by the syzygy functor on \(\Gamma\text{-mod}\), more precisely \(\pi F = \Omega \pi G\).
Proof. Let \((L \xrightarrow{f} M)\) be an object in \(S(\Lambda)\). We have the corresponding exact sequence

\[
0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0.
\]

Notice that \(g = \epsilon(f)\). Apply \((E, -)_{\Lambda}\) to this sequence and obtain the exact sequence

\[
0 \longrightarrow (E, L)_{\Lambda} \xrightarrow{(E, f)} (E, M)_{\Lambda} \xrightarrow{(E, g)} (E, N)_{\Lambda}.
\]

The cokernel of \((E, f)\), and hence the image of \((E, g)\), is by definition \(\alpha \eta(f)\). Also the cokernel of \((E, g)\) is \(\alpha \epsilon(f)\). Thus we get an exact sequence

\[
0 \longrightarrow \alpha \eta(f) \xrightarrow{\text{im}(E, g)} (E, N)_{\Lambda} \longrightarrow \alpha \epsilon(f) \longrightarrow 0.
\]

From \([12, \text{Proposition 4.2}]\) we know there is an exact sequence of functors \(le \longrightarrow \text{id}_{\Gamma} \longrightarrow \nu q \longrightarrow 0\). We obtain a commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& \phi & & & & & \\
0 & \longrightarrow & \alpha \eta(f) & \longrightarrow & (E, N)_{\Lambda} & \longrightarrow & \alpha \epsilon(f) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \alpha \eta(f) & \longrightarrow & (E, N)_{\Lambda} & \longrightarrow & \alpha \epsilon(f) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\nu F(f) & \xrightarrow{\nu q(\text{im}(E, g))} & \nu q(E, N)_{\Lambda} & \longrightarrow & \nu G(f) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & & \\
\end{array}
\]

Since \(e\) is exact and \(e \alpha \epsilon = 0\) the map \(\phi\) is an isomorphism. But then we can extend the top row to a short exact sequence and apply the snake lemma to see that \(\nu q(\text{im}(E, g))\) is a monomorphism.

Since \(\iota\) is fully faithful and exact this implies we have the following exact sequence in \(\Gamma\)-\text{mod}:

\[
0 \longrightarrow F(f) \longrightarrow q((E, N)_{\Lambda}) \longrightarrow G(f) \longrightarrow 0.
\]

Now \(\iota\) preserves epimorphisms and \(q\) is its left adjoint, thus \(q\) preserves projective objects. We know \((E, N)_{\Lambda}\) is a projective \(\Gamma\)-module and hence \(q((E, N)_{\Lambda})\) is projective, this shows \(\pi F(f) \cong \Omega \pi G(f)\) in \(\Gamma\)-\text{mod}. \(\blacksquare\)

Remark. By Proposition 2.10 \(S(\Lambda)\) is also a Frobenius category, so the stable category \(\Sigma(\Lambda)\) is a triangulated category. Hence one might ask whether \(F\) and \(G\) induce a triangle functor from \(\Sigma(\Lambda)\) to \(\Gamma\)-\text{mod}. However all maps factoring through projective objects in \(\Sigma(\Lambda)\) factor through both \(\mathcal{U}\) and \(V\), thus any induced triangle functor would have to factor through the abelian category \(\Gamma\)-\text{mod}, which renders any such functor trivial.
5 Auslander Algebras of Nakayama Algebras

A finite length module is said to be uniserial if it has a unique composition series. We say an algebra $A$ is uniserial, or a Nakayama algebra, if all indecomposable $A$-modules have a unique composition series. In this section we prove Theorem 3.

**Theorem 3.** Let $\Lambda$ be a basic representation-finite algebra and let $\Gamma$ be its Auslander algebra. Then $\Gamma$ has a quasi-hereditary structure such that the objects of $\Gamma$-torsl are precisely the $\Delta$-filtered $\Gamma$-modules if and only if $\Lambda$ is uniserial.

The only if part is proven in Subsection 5.4, but before that we describe a quasi-hereditary structure with the properties from Theorem 3 for the Auslander algebras of Nakayama algebras. First, however, we consider the example of self-injective Nakayama algebras over an algebraically closed field explicitly, to get some picture of the situation.

5.1 Self-injective Nakayama Algebras

The classification of Nakayama algebras over algebraically closed fields is well known, and can for example be found in [1, V.3]. We recall the self-injective case in this subsection to get an explicit description of an example. We are particularly interested in the self-injective Nakayama algebras, so we consider the structure of their Auslander algebras in explicit terms here. Let $\tilde{A}_m$ denote the quiver with vertices $\mathbb{Z}/m\mathbb{Z}$ and arrows $i \to i + 1$ for all $i \in \mathbb{Z}/m\mathbb{Z}$. Write $k\tilde{A}_m$ for the path algebra of this quiver and let $J(k\tilde{A}_m)$ denote the ideal generated by the arrows. For $m, N \in \mathbb{N}$ we define $A(m, N) := k\tilde{A}_m / J(k\tilde{A}_m)^{N+1}$, these are precisely the basic connected self-injective Nakayama algebras. Notice that $A(1, N) \cong k[x]/(x^{N+1})$, and hence the case studied in detail in [24] is included. We parametrize the simple $A(m, N)$ modules by the vertices $j \in \mathbb{Z}/m\mathbb{Z}$ of $\tilde{A}_m$. The category $A(m, N)$-mod has indecomposable objects $[i]_j$ for $j \in \mathbb{Z}/m\mathbb{Z}$ and $i = 1, \ldots, N + 1$, where $\text{soc}([i]_j) = S(j)$ and $[i]_j$ has Loewy length $i$.

To get an idea of the general shape of the Auslander-Reiten quiver, take for example the Auslander-Reiten quiver of $A(4, 3)$ in figure 1. The Gabriel quiver of $\Gamma$ is the opposite of this quiver.

We may consider $A(m, N)$-mod as a $\mathbb{Z}/m\mathbb{Z}$-fold cover of $A(1, N)$-mod. Namely, if we give $A(1, N) = k[x]/(x^{N+1})$ the $\mathbb{Z}$-grading given by monomial degrees, it induces a $\mathbb{Z}/m\mathbb{Z}$ grading and the categories $k[x]/(x^{N+1})$-mod$^{\mathbb{Z}/m\mathbb{Z}}$ and $A(m, N)$-mod are isomorphic.

5.2 Quasi-hereditary Algebras

We follow the approach of [11] to quasi-hereditary algebras. Let $A$ be a basic finite-dimensional algebra and let $\Xi$ be a set parametrizing the isomorphism
classes of simple objects in $A$-mod. We write $S(\xi)$ for the simple $A$-module corresponding to $\xi \in \Xi$, $P(\xi)$ for the projective cover of $S(\xi)$, and $I(\xi)$ for the injective envelope of $S(\xi)$. Give $\Xi$ a partial ordering $(\Xi, \leq)$. For each $\xi \in \Xi$ the standard module $\Delta(\xi)$ is the maximal factor module of $P(\xi)$ such that for all composition factors $S(\rho)$ of $\Delta(\xi)$, we have $\rho \leq \xi$. Dually the costandard module $\nabla(\xi)$ is the maximal submodule of $I(\xi)$ such that for all composition factors $S(\rho)$, we have $\rho \leq \xi$. A partial ordering on the simple $A$-modules is called adapted if for every $A$-module $X$ with $\top(X) = S(\xi)$ and $\soc(X) = S(\xi')$, such that $\xi$ and $\xi'$ are incomparable, there is $\rho \in \Xi$ such that $\rho > \xi$ and $S(\rho)$ is in the composition series of $X$. If the partial ordering is adapted, then all the standard and costandard modules are the same for any refinement of the partial ordering cf. \cite{11, Section 1].

The full subcategory of standard (resp. costandard) modules in $A$-mod will be denoted by $\Delta$ (resp. $\nabla$), and $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$) denotes the full subcategory of all objects that have a filtration by standard (resp. costandard) modules. If the standard modules are Schurian and $A \in \mathcal{F}(\Delta)$, we say $A$ is quasi-hereditary. The characteristic tilting module of a quasi-hereditary algebra is determined up to isomorphism as the basic module $C$ such that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{add}(C)$. By \cite{11, Proposition 3.1} it is indeed a tilting module.

Recall that a left strongly quasi-hereditary algebra is a quasi-hereditary algebra such that all standard modules have projective dimension at most 1, or equivalently that $C$ has projective dimension at most 1. Ringel has shown that any Auslander algebra of a representation-finite algebra has a left strongly quasi-hereditary structure cf. \cite{21, Section 5].

The isomorphism classes of simple $\Gamma$-modules are in a canonical bijection with the isomorphism classes of $\text{ind}(\Lambda)$, denoted by $\text{ind}(\Lambda)/\sim$. For any $M \in \text{ind}(\Lambda)$, let $p_M: E \to M$ be the projection, and $i_M: M \to E$ be the inclusion. Then $M$ corresponds to the idempotent $(i_M p_M)_{\text{op}} \in \Gamma$, which corresponds to an isomorphism class of simple $\Gamma$-modules. We
use this bijection to parametrize the simple $\Gamma$-modules. For $M \in \text{ind}(\Lambda)$ we let $[M]$ denote its isomorphism class in $\text{ind}(\Lambda)/\sim$, although we write $S(M), P(M), I(M), \Delta(M), \nabla(M)$ resp. instead of $S([M]), P([M]), I([M]), \Delta([M]), \nabla([M])$ resp.

5.3 Auslander Algebras of Nakayama Algebras

Let $\Lambda$ be a Nakayama algebra and let $\Gamma$ be its Auslander algebra. Let $\ell(M)$ denote the Loewy length of a $\Lambda$-module $M$. We consider a partial ordering on $\text{ind}(\Lambda)/\sim$ given by the Loewy length: For $M, N \in \text{ind}(\Lambda)$, say $[M] > [N]$ if $\ell(M) < \ell(N)$, but $[M], [N]$ are incomparable if $\ell(M) = \ell(N)$.

Remark. Notice that modules with greater Loewy-length are smaller in our partial ordering. Thus the simple modules are maximal.

Let $X$ be an indecomposable $\Gamma$-module with $\text{top}(X) = S(M)$ and $\text{soc}(X) = S(N)$. If $M \not\cong N$ and $\ell(M) = \ell(N)$ there is a non-trivial map $f: N \to M$ such that for every indecomposable summand $M'$ of $\text{im}(f)$, $S(M')$ is in the composition series of $X$. We have $\ell(M) > \ell(M')$ for every such summand $M'$ of $\text{im}(f)$, i.e. $[M'] > [M]$. Thus our partial order is adapted.

Let $M \in \text{ind}(\Lambda)$. If $M$ is simple then there are no non-trivial homomorphisms from other simple $\Lambda$ modules to $M$. Hence $\text{Hom}_{\Gamma}(P(N), P(M)) = 0$ for all simple $N \not\cong M$, which implies $\Delta(M) = P(M)$. Similarly we have $\text{Hom}_{\Gamma}(I(M), I(N)) = 0$ for all simple $N \not\cong M$, and thus $\nabla(M) \cong I(M)$.

Now $\Lambda$ is uniserial, so if $M$ is not simple it has a unique maximal proper submodule $M'$, and clearly $\ell(M') = \ell(M) - 1$. Recall that $P(M) \cong (E, M)_\Lambda$. Let $N \in \text{ind}(\Lambda)$, for any map in $(N, M')_\Lambda$, composition with the inclusion of $M'$ in $M$ gives a map in $(N, M)_\Lambda$. In this way $P(M')$ embeds in $P(M)$ as a $\Gamma$-submodule. Any non-surjective map to $M$ factors through $M'$, in particular, if $N \in \text{ind}(\Lambda)$ with $\ell(N) < \ell(M)$, then any map in $(N, M)_\Lambda$ factors through $M'$. This shows that $\Delta(M) \cong P(M)/P(M')$ and thus $\Delta(M)$ has projective dimension 1. Consequently all modules in $F(\Delta)$ have projective dimension at most 1.

We proceed in a similar way for the costandard modules. If $M$ is non-simple, then it has a unique maximal proper factor module $M''$. The projection $M \to M''$ induces an epimorphism $I(M) \to I(M'')$. We know any non-injective map from $M$ to $N$ factors through $M''$. In particular, if $\ell(N) < \ell(M)$ and $N \in \text{ind}(\Lambda)$, then any map in $(M, N)_\Lambda$ factors through the projection $M \to M''$. Thus the kernel of the map $I(M) \to I(M'')$ has no composition factors $S(N)$ such that $\ell(N) < \ell(M)$. Together this implies that we have an exact sequence

$$
0 \longrightarrow \nabla(M) \longrightarrow I(M) \longrightarrow I(M'') \longrightarrow 0.
$$

In particular $\nabla(M)$ has injective dimension 1 and thus any module in $F(\nabla)$ has injective dimension at most 1. Taking everything together we get the following proposition.
**Proposition 5.1.** Let $\Gamma$ be the Auslander algebra of a Nakayama algebra. The partial ordering above gives $\Gamma$ a quasi-hereditary structure with $\mathcal{F}(\Delta) = \Gamma\text{-torsl}$ and $\mathcal{F}(\nabla) = \Gamma\text{-divbl}$.

**Proof.** The $\Gamma$-module $\Gamma$ is in $\mathcal{F}(\Delta)$ and all the standard modules are Schurian. Thus our partial ordering gives a quasi-hereditary structure on $\Gamma$. Since all objects of $\mathcal{F}(\Delta)$ have projective dimension at most 1 we see $\mathcal{F}(\Delta) \subset \Gamma\text{-torsl}$. Also all the costandard modules have injective dimension at most 1, so by [11, Lemma 4.1*] and Proposition 2.7 we have $\Gamma\text{-torsl} \subset \mathcal{F}(\Delta)$.

We show $\mathcal{F}(\nabla) = \Gamma\text{-divbl}$ dually using [11, Lemma 4.1] and Proposition 2.8. 

**5.4 Other Representation-finite Algebras**

Quasi-hereditary structures on Auslander algebras of representation-finite algebras that have the property given in Proposition 5.1 are rare in general. Indeed, the examples illustrated in Subsection 5.3 are the only cases.

**Proposition 5.2.** Let $\Lambda$ be a basic representation-finite algebra and let $\Gamma$ be its Auslander algebra. If $\Gamma$ has a quasi-hereditary structure such that the $\Delta$-filtered modules coincide with $\Gamma\text{-torsl}$, then $\Lambda$ is uniserial.

**Proof.** We keep the notation from Subsection 5.2. Let $\Gamma$ have a quasi-hereditary structure such that $\mathcal{F}(\Delta) = \Gamma\text{-torsl}$. It suffices to show that all indecomposable projective and all indecomposable injective $\Lambda$-modules have a unique composition series. Let $M \in \text{ind}(\Lambda)$ be a submodule of an indecomposable projective module. Let $\Delta(M)$ be the standard module generated by $(E,M)$. By assumption we have a projective resolution

$$
0 \longrightarrow P_1 \overset{\pi_1}{\longrightarrow} (E,M)_{\Lambda} \overset{\pi_0}{\longrightarrow} \Delta(M) \longrightarrow 0.
$$

Since $\Gamma\text{-proj}$ is equivalent to $\Lambda\text{-mod}$ we get a monomorphism $f: N \rightarrow M$ in $\Lambda\text{-mod}$ such that $\pi_1 = (E,f)_{\Lambda}$. Let $M'$ be any proper submodule of $M$ and let $\iota$ denote its inclusion. Then $\alpha(\iota)$ is $\Delta$-filtered, hence it has $\Delta(M)$ as a factor module. Thus there is a commutative diagram

$$
0 \longrightarrow (E,M')_{\Lambda} \overset{(E,\iota)}{\longrightarrow} (E,M)_{\Lambda} \longrightarrow \alpha(\iota) \longrightarrow 0
$$

$$
\begin{array}{ccc}
0 & \xrightarrow{\psi} & (E,N)_{\Lambda} \overset{\pi_1}{\longrightarrow} (E,M)_{\Lambda} \overset{\pi_0}{\longrightarrow} \Delta(M) \longrightarrow 0.
\end{array}
$$

We get an induced map $\psi: (E,M')_{\Lambda} \rightarrow (E,N)_{\Lambda}$ making the diagram above commutative. This yields a monomorphism $g: M' \rightarrow N$ such that $\iota = fg$. If we identify $N$ with its image in $M$ via $f$ this shows every proper submodule $M'$ of $M$ is a submodule of $N$. Since $N$ is a submodule of an indecomposable...
injective module it is also indecomposable. This shows that any indecomposable injective \( \Lambda \)-module has a unique composition series. Dually, using that \( \Gamma \)-divbl = \( \mathcal{F}(\nabla) \) we can show all indecomposable projective \( \Lambda \)-modules have a unique composition series.

Combining Proposition 5.2 with Proposition 5.1 now yields Theorem 6.

Given a quasi-hereditary structure on \( \Gamma \) the condition \( \mathcal{F}(\Delta) = \Gamma \)-torsl is the same as condition (i) in \([11, \text{Lemma 4.1}]\) combined with condition (iv) in \([11, \text{Lemma 4.1*}]\). Thus all the conditions of these two lemmas hold, in particular \( \mathcal{F}(\nabla) = \Gamma \)-divbl and thus \( \text{add}(T) = \text{add}(C) \). Since both \( T \) and \( C \) are basic this implies they are isomorphic. Conversely, if \( T \cong C \), then \( \mathcal{F}(\Delta) = \text{cogen}(C) = \text{cogen}(T) = \Gamma \)-torsl. Hence Theorem 6 yields the following corollary.

**Corollary 5.3.** Let \( \Lambda \) be a basic self-injective algebra of finite representation type and let \( \Gamma \) be the Auslander algebra of \( \Lambda \). We let \( T = c(E) \) be the canonical tilting and cotilting module as defined in Subsection 3.1. Then \( T \) is a characteristic tilting module of a quasi-hereditary structure on \( \Gamma \) if and only if \( \Lambda \) is a Nakayama algebra.

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