SINGULARITY FORMATION IN THE YANG-MILLS FLOW

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1. Introduction

The Yang-Mills heat flow was shown to exist for all time over compact Kähler surfaces by Donaldson [2] (see also [3]). Over a compact Riemannian manifold, Råde [10] proved that for dimensions two and three, the flow exists for all time and converges to a Yang-Mills connection. However, his methods fail for higher dimensions. In dimensions \( n > 4 \) it has been shown by Naito [9] that singularities can occur in finite time (see also [6]). The case of \( n = 4 \) is still unknown (see [11] and [12]). The Yang-Mills heat flow and the harmonic map heat flow appear to share similar properties. In [5], Grayson and Hamilton show using a monotonicity formula from [7] that rapidly forming singularities in the harmonic map heat flow converge, after a blow-up process, to homothetically shrinking solitons.

The same technique works in the case of the Yang-Mills flow. There is, however, a complication arising from the presence of the gauge group. For this reason, we must allow the freedom to make gauge transformations. We use Hamilton's monotonicity formula for the Yang-Mills flow [7] to prove that a sequence of blow-ups of a rapidly forming singularity will converge, modulo the gauge group, to a non-trivial homothetically shrinking soliton. In the last section, explicit examples of such solitons are given in the case of trivial bundles over \( \mathbb{R}^n \) for \( 5 \leq n \leq 9 \).

We will first introduce some notation. Let \( M \) be a compact \( n \)-dimensional Riemannian manifold. Let \( E \) be a real vector bundle of rank \( r \) over \( M \) with compact structure group \( G \) and assume the fibres carry an inner product and that \( G \subset SO(r) \) respects this inner product.

We will often work in local coordinates, using Greek letters for the bundle indices and Latin letters for the manifold indices. A connection \( A \) is locally an...
endomorphism valued 1-form given by \( A^\alpha_{i\beta} \). The curvature of the connection is given by
\[
F^\alpha_{ij\beta} = \partial_i A^\alpha_{j\beta} - \partial_j A^\alpha_{i\beta} + A^\alpha_{i\gamma} A^\gamma_{j\beta} - A^\alpha_{j\gamma} A^\gamma_{i\beta}.
\]

The Yang-Mills heat flow is a flow of connections \( A = A(t) \) given by
\[
\frac{\partial}{\partial t} A^\alpha_{j\beta} = D_p F^\alpha_{pj\beta},
\]
where \( D \) denotes covariant differentiation with respect to the connection. In normal coordinates, this right hand side is given by
\[
D_p F^\alpha_{pj\beta} = \partial_p F^\alpha_{pj\beta} + A^\alpha_{p\gamma} F^\gamma_{pj\beta} - F^\alpha_{pj\gamma} A^\gamma_{p\beta}.
\]

The Yang-Mills flow is a gradient flow for the Yang-Mills functional
\[
E = \frac{1}{2} \int_M |F|^2.
\]

Let \( s = s^\alpha_{\beta} \) be a gauge transformation. Then \( s \) acts on a connection matrix \( A \) by
\[
s \cdot A = s As^{-1} - (ds)s^{-1}.
\]

Define a flow of connections \( A = A(x, t) \) on the trivial bundle over \( \mathbb{R}^n \times (-\infty, 0) \) to be a homothetically shrinking soliton if it is a solution to the Yang-Mills heat equation and satisfies the dilation condition
\[
A^\alpha_{i\beta}(x, t) = \lambda A^\alpha_{i\beta}(\lambda x, \lambda^2 t).
\]

The main theorem is as follows.

**Main Theorem.** Let \( A \) be a smooth solution of the Yang-Mills flow on \( M \) for \( 0 \leq t < T \) with a singularity at some point \( X \) as \( t \to T \). Suppose that it is a rapidly forming singularity, so that
\[
(T - t)|F| \leq C,
\]
for some constant \( C \). Then a sequence of blow-ups around \( (X, T) \),
\[
A_p(i)(x, t) = \lambda_i A_p(\lambda_i x, T + \lambda_i^2 t),
\]
with factor \( \lambda_i \to 0 \), has a subsequence which converges in \( C^\infty \) on compact sets modulo the gauge group to a homothetically shrinking soliton \( A \) defined on \( \mathbb{R}^n \times (-\infty, 0) \) which has non-zero curvature.
In the course of the proof of the theorem, the meaning of the phrase 'modulo the gauge group' will be made more precise.

The outline of the paper is as follows. In section 2, we show that given a bound on the curvature, we can derive bounds on all of the derivatives of the curvature. In section 3, we use these estimates, together with a theorem of Uhlenbeck’s [13], to get bounds on the connections and then convergence of a sequence of blow-ups near the singular point. We use Hamilton’s monotonicity formula [7] to show that, allowing gauge transformations, the sequence converges to a homothetically shrinking soliton. We then use an ‘$\epsilon$-regularity’ result to show that this soliton must have non-zero curvature. Finally, in section 4, we give examples of homothetically shrinking solitons on a trivial $SO(n)$ bundle over $\mathbb{R}^n$ for $5 \leq n \leq 9$.

The results of this paper will form part of my forthcoming PhD thesis at Columbia University. I would like to thank my thesis advisor D.H. Phong, for his constant support and advice, and Richard Hamilton for suggesting an improvement to an initial draft of the result. I am also grateful to the referee for pointing out a couple of errors in an earlier version of this paper.

2. Estimates on derivatives of the curvature

We begin by proving local estimates on the derivatives of the curvature. First define the parabolic cylinder

$$P_r(X, T) = \{(x, t) \in M \times \mathbb{R} \mid d(X, x) \leq r, \ T - r^2 \leq t \leq T\}.$$ 

The following lemma is proved in [5] (the conclusion is slightly different, but it follows easily.)

**Lemma 2.1** There exists a constant $s > 0$ and for every $\gamma < 1$, a constant $C_\gamma$, such that if $h$ is a smooth function on $M$ with

$$\frac{\partial h}{\partial t} \leq \triangle h - h^2$$

when $h \geq 0$ in some parabolic cylinder $P_r(X, T)$ with $r \leq s$ then

$$h \leq \frac{C_\gamma}{r^2}$$

on $P_{\gamma r}(X, T)$.  

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Then we have

**Theorem 2.2** There exists a constant $s > 0$ and constants $C_k$ for $k = 1, 2, \ldots$ depending only on $M$ such that if $A(t)$ is a solution to the Yang-Mills flow with $|F| \leq K$ in some $P_r(X, T)$ for $r \leq s$ for some constant $K \geq 1/r^2$, then with $r_k = r/2^k$ we have

$$|D^k F| \leq C_k K^{k/2+1}$$
onumber

on $P_{r_k}(X, T)$.

**Proof** We will use induction to prove the theorem. The notation $S \ast T$ will mean some algebraic bilinear expression involving the tensors $S$ and $T$. $R$ will denote the Riemannian curvature tensor on the base manifold. $C$ will denote a constant which depends only on $M$. Assume that the constant $K$ is at least 1. We begin with the case $k = 1$. Using commutation formulae and the Bianchi identity, calculate

$$\frac{\partial}{\partial t}|F|^2 = \triangle |F|^2 - 2|DF|^2 + F \ast F \ast F + R \ast F \ast F$$
$$\leq \triangle |F|^2 - 2|DF|^2 + CK^3,$$

and

$$\frac{\partial}{\partial t}|DF|^2 = \triangle |DF|^2 - 2|D^2 F|^2 + DF \ast DF \ast F$$
$$+ DF \ast DF \ast R + DF \ast F \ast DR$$
$$\leq \triangle |DF|^2 - 2|D^2 F|^2 + CK|DF|^2 + CK|DF|$$
$$\leq \triangle |DF|^2 - 2|D^2 F|^2 + CK|DF|^2 + CK^4.$$

Define

$$h = (8K^2 + |F|^2)|DF|^2.$$  

Then

$$\frac{\partial h}{\partial t} \leq (\triangle |F|^2 - 2|DF|^2 + CK^3)|DF|^2 + (8K^2 + |F|^2)(\triangle |DF|^2 - 2|D^2 F|^2$$
$$+ CK|DF|^2 + CK^4).$$

But

$$\triangle h \geq (\triangle |F|^2)|DF|^2 + (8K^2 + |F|^2)\triangle |DF|^2 - 8K|DF|^2|D^2 F|.$$
Hence
\[
\frac{\partial h}{\partial t} \leq \triangle h - 2|DF|^4 + CK^3|DF|^2 + (8K^2 + |F|^2)(-2|D^2F|^2 \\
+ CK|DF|^2 + CK^4) + 8K|DF|^2|D^2F| \\
\leq \triangle h - 2|DF|^4 + CK^3|DF|^2 + CK^6 - 2(8K^2 + |F|^2)|D^2F|^2 \\
+ |DF|^4 + 16K^2|D^2F|^2 \\
\leq \triangle h - \frac{1}{2}|DF|^4 + CK^6 \\
\leq \triangle h - \frac{h^2}{CK^4} + CK^6
\]
for some constant $C$. For that same $C$, define
\[
\hat{h} = \frac{h}{CK^4} - K.
\]
Then for $\hat{h} \geq 0$,
\[
\frac{\partial \hat{h}}{\partial t} \leq \frac{1}{CK^4}(\triangle h - \frac{h^2}{CK^4} + CK^6) \\
= \triangle \hat{h} - (\hat{h} + K)^2 + K^2 \\
\leq \triangle \hat{h} - \hat{h}^2.
\]
Then by Lemma 2.1, we have
\[
\hat{h} \leq \frac{C}{r^2} \leq CK
\]
on $P_{r/2}(X,T)$. Hence
\[
h \leq CK^5
\]
and
\[
|DF| \leq CK^{3/2}.
\]
Assume inductively that we have the estimates for $D^lF$ on $P_{r_{k-1}}(X,T)$ for $1 \leq l \leq k - 1$. Calculating on $P_{r_{k-1}}(X,T)$, first notice
\[
\frac{\partial}{\partial t}|D^2F|^2 = \triangle |D^2F|^2 - 2|D^3F|^2 + D^2F \ast D^2F \ast F + D^2F \ast DF \ast DF \\
+ D^2F \ast D^2F \ast R + D^2F \ast DF \ast DR + D^2F \ast F \ast D^2R \\
\leq \triangle |D^2F|^2 - 2|D^3F|^2 + CK^5.
\]
It is not difficult to see by an induction argument that

\[ \frac{\partial}{\partial t} |D^{k-1}F|^2 \leq \triangle |D^{k-1}F|^2 - 2|D^kF|^2 + CK^{k+2}. \]

and

\[ \frac{\partial}{\partial t} |D^kF|^2 \leq \triangle |D^kF|^2 - 2|D^{k+1}F|^2 + CK|D^kF|^2 + CK^{k+3}. \]

Choose \( B \) to be a constant with \( K^{(k+1)/2} \leq B \leq CK^{(k+1)/2} \) and \( |D^{-1}F| \leq B \).

Define \( h = (8B^2 + |D^{-1}F|^2)|D^kF|^2 \). Then

\[
\frac{\partial h}{\partial t} \leq (8B^2 + |D^{-1}F|^2)(\triangle |D^kF|^2 - 2|D^{k+1}F|^2 + CK|D^kF|^2 + CK^{k+3})
+ |D^kF|^2(\triangle |D^{-1}F|^2 - 2|D^kF|^2 + CK^{k+2})
\leq (8B^2 + |D^{-1}F|^2)\triangle |D^kF|^2 + |D^kF|^2\triangle |D^{-1}F|^2 - 16B^2|D^{k+1}F|^2
- 2|D^kF|^4 + CK^{k+2}|D^kF|^2 + CK^{2k+4}.
\]

But

\[ \triangle h \geq (8B^2 + |D^{-1}F|^2)\triangle |D^kF|^2 + |D^kF|^2\triangle |D^{-1}F|^2 - 8B|D^kF|^2|D^{k+1}F|. \]

Hence

\[
\frac{\partial h}{\partial t} \leq \triangle h - 16B^2|D^{k+1}F|^2 - 2|D^kF|^4 + CK^{k+2}|D^kF|^2 + CK^{2k+4}
+ 16B^2|D^{k+1}F|^2 + |D^kF|^4
\leq \triangle h - |D^kF|^4 + CK^{k+2}|D^kF|^2 + CK^{2k+4}
\leq \triangle h - \frac{1}{2}|D^kF|^4 + CK^{2k+4}
\leq \triangle h - \frac{h^2}{CK^{2k+2}} + CK^{2k+4}.
\]

Now define

\[ \hat{h} = \frac{h}{CK^{2k+2}} - K. \]

Then for \( \hat{h} \geq 0 \) we have

\[ \frac{\partial \hat{h}}{\partial t} \leq \triangle \hat{h} - \hat{h}^2. \]

Hence \( \hat{h} \leq CK \) on \( P_{\tau_\delta}(X, T) \) from which it follows that \( h \leq CK^{2k+3} \) and

\[ |D^kF| \leq CK^{k/2+1}. \]
3. Formation of singularities

In this section, we will prove the main theorem. Suppose that the flow has a singularity at \( X \in M \) at time \( T \). We suppose that the singularity is rapidly forming, that is, we have the dilation invariant estimate

\[
|F| \leq \frac{C}{T-t},
\]

for some constant \( C \). Then Theorem 2.2 gives the dilation invariant estimates

\[
|D^k F| \leq \frac{C_k}{(T-t)^{k/2+1}}.
\]

We will perform a blow-up procedure around the point \( X \) at time \( T \). Choose a small geodesic ball \( B(r) \) of radius \( r \) around \( X \) over which \( E \) is trivial. We identify \( B(r) \) with the ball of radius \( r \) in \( \mathbb{R}^n \). In these coordinates, the connections \( A = A(x,t) \) are given by matrix-valued one-forms \( A = A_p dx^p \). Now choose a sequence of real numbers \( \lambda_i \) tending to zero, and define a sequence of flows of connection one forms \( A(i) = A_p(i) dx^p \) on balls of increasing size in \( \mathbb{R}^n \) by setting

\[
A_p(i)(x,t) = \lambda_i A_p(\lambda_i x, T + \lambda_i^2 t)
\]

for \( x \in B_{1/\lambda_i}(0) \) and \( t \in [-T/\lambda_i^2, 0) \).

We need the following monotonicity formula of Hamilton’s [7] for the Yang-Mills flow. Similar results have been obtained by Naito [9] and Chen and Shen [1].

**Theorem 3.1** If \( A(t) \) solves the Yang-Mills heat flow on \( 0 \leq t < T \) and if \( k \) is any positive backward solution to the scalar heat equation on \( M \) with \( \int_M k = 1 \), then the quantity

\[
Z(t) = (T-t)^2 \int_M |F|^2 k
\]

is monotone decreasing in \( t \) when \( M \) is Ricci parallel (that is, \( D_i R_{jk} = 0 \)) with weakly positive sectional curvatures; while on a general \( M \) we have

\[
Z(t) \leq CZ(\tau) + C(t-\tau)^2 E_0
\]

whenever \( T - 1 \leq \tau \leq t \leq T \), where \( E_0 \) is the initial energy and \( C \) is a constant depending only on \( M \). Moreover, if

\[
W(t) = (T-t) \int_M |D_i F^i_{\alpha \beta} + \frac{D_i k}{k} F^i_{\alpha \beta}|^2 k
\]
then
\[ \int_{T}^{T} W(t) dt \leq CZ(\tau) + C E_0. \]

The last statement of Theorem 3.1 gives
\[ \int_{T-1}^{T} (T - t)^2 \int_{M} |\text{div} F_A + \frac{Dk}{k}.F_A|^2 k \, dV \, dt \leq C < \infty. \]

Hence for \( \epsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \int_{T-\delta}^{T} (T - t)^2 \int_{M} |\text{div} F_A + \frac{Dk}{k}.F_A|^2 k \, dV \, dt \leq \epsilon. \]

Now let \( k \) be the solution to the backwards heat equation which becomes a delta function at the point \( X \) at time \( T \). In the geodesic ball, we get
\[ \int_{T-\delta}^{T} (T - t)^2 \int_{B(r)} \|\text{div} F_A + \frac{Dk}{k}.F_A\|^2 k \, dV \, dt \leq \epsilon. \]

Let \( A(i) \) be the dilated connections and let \( k(i) \) be the dilations of \( k \). Making a change of variables,
\[ \int_{-\delta/\lambda_i^2}^{0} \|t\|^2 \int_{B_{r/\lambda_i}(0)} \|\text{div} A(i) F_A + \frac{Dk(i)}{k(i)}.F_A\|^2 k(i) \, dV \, dt \leq \epsilon. \]

Fix a compact time interval \([t', t'']\) in \((-\infty, 0)\) and a compact set \( D \) in \( \mathbb{R}^n \). Suppose \( i \) is large enough so that \( t' \) is contained in the interval \((-\delta/\lambda_i^2, 0)\) and that \( D \) is contained in the ball \( B_{r/\lambda_i}(0) \). Fix a point \( x_0 \) in \( D \). We will now use a theorem of Uhlenbeck’s \([13]\), which states that if the \( L^n \) norm of the curvature is small enough, then there exists a local Coulomb gauge, in which the \( L^p \) norm of the connection matrix is bounded by the \( L^p \) norm of the curvature, for any \( p \geq 2n \). Since by assumption the curvature is uniformly bounded by \( C/(T - t) \), we can apply Uhlenbeck’s theorem in a small enough ball \( B \) containing \( x_0 \). Thus there exists, for each \( i \), a gauge transformation \( s(i) \) over \( B \) such that \( s(i) \cdot A(i) \) at time \( t' \) is bounded in \( L^p \) over \( B \) for any \( p \). The bounds do not depend on \( i \). We can also get uniform bounds on the \( C^k \) norms of \( s(i) \cdot A(i) \), for large enough \( i \), at \( t' \) on a slightly smaller ball \( B' \) from the bounds on the derivatives of the curvature given in Theorem 2.2, using the argument of Lemma 2.3.11 in \([3]\). Since \( s(i) \cdot A(i) \) is a solution of the Yang-Mills flow, we can bound the connection matrices and
their derivatives uniformly in time on the interval \([t', t'']\) over the ball \(B'\). By passing to a subsequence, we can get uniform convergence in \(C^\infty\). Since this works for a small ball around any point, we can apply the patching argument of [2], Corollary 4.4.8, so that after passing to another subsequence, \(s(i) \cdot A(i)\) converges in \(C^\infty\) on \(D \times [t', t'']\) to a solution \(\overline{A}\) of the Yang-Mills flow. Now consider a sequence of increasing compact sets \(D_j\) and time intervals \([t'_j, t''_j]\) which exhaust \(\mathbb{R}^n \times (-\infty, 0)\). By the usual diagonal argument, after passing to a subsequence, we get a sequence of gauge transformations \(s(i)\) such that \(s(i) \cdot A(i)\) converges in \(C^\infty\) on compact sets to a solution \(\overline{A}\) of the Yang-Mills flow on \(\mathbb{R}^n \times (-\infty, 0)\). Also, a subsequence of the \(k(i)\) will converge to \(k\) on \(\mathbb{R}^n \times (-\infty, 0)\) given by

\[
\overline{F}(x, t) = \frac{1}{(-4\pi t)^{n/2}} e^{\frac{|x|^2}{4t}}.
\]

Letting \(\epsilon \to 0\), we get

\[
\text{div}_{\overline{A}} F_{\overline{A}} + \frac{x}{2t} F_{\overline{A}} = 0.
\]

Hence

\[
\frac{\partial}{\partial t} A^\alpha_{i\beta} + \frac{x^p}{2t} (\partial_p A^\alpha_{i\beta} - \partial_i A^\alpha_{p\beta} + A^\gamma_p A^\alpha_{i\gamma} - A^\gamma_i A^\alpha_{p\gamma}) = 0.
\]

Now choose an exponential gauge for \(\overline{A}\), in which \(x^p A^\alpha_{p\beta} = 0\), at time \(t = -1\). Then notice that, from the equation above, the condition \(x^p \overline{A}^\alpha_{p\beta} = 0\) holds for all \(t\). Then \(\overline{A}\) satisfies

\[
\frac{\partial}{\partial t} \overline{A}^\alpha_{i\beta} + \frac{x^p}{2t} \partial_p \overline{A}^\alpha_{i\beta} + 2t \overline{A}_i^\alpha = 0,
\]

and by the following lemma \(\overline{A}\) is a homothetically shrinking soliton.

**Lemma 3.2** If a flow of one-forms \(A_i = A_i(x, t)\) satisfies

\[
\frac{\partial}{\partial t} A_i + \frac{x^p}{2t} \partial_p A_i + 2t A_i = 0,
\]

on \(\mathbb{R}^n \times (-\infty, 0)\) then

\[
A_i(x, t) = \lambda A_i(\lambda x, \lambda^2 t).
\]
Proof First notice that if $A$ satisfies the dilation invariant condition then by differentiating with respect to $\lambda$ and setting $\lambda$ equal to 1, it is immediate that it satisfies the differential equation
\[
\frac{\partial}{\partial t} A_i + \frac{x^p}{2t} \partial_p A_i + \frac{1}{2t} A_i = 0,
\]
on $\mathbb{R}^n \times (-\infty, 0)$. The lemma then follows from Holmgren’s uniqueness theorem for linear partial differential equations (see for example [8], p.125), since the hyperplanes \(\{t = \text{constant}\}\) are non-characteristic.

We will now show that the soliton has non-zero curvature. First we have the following ‘$\epsilon$-regularity’ result which is almost the same as one given in [5] for the harmonic map heat flow (a similar result for the Yang-Mills flow is proved in [1]). Write $k_{(X,T)}$ for the backwards solution of the heat equation on $M$ which becomes a delta function at $X$ at time $T$.

**Theorem 3.3** There exist constants $\epsilon > 0$ and $\beta > 0$ depending only on $M$ and $E_0$ such that for any $X \in M$, any $T > 0$ and any $\alpha \geq 0$ with $T - \beta \leq \alpha < T$ we can find $\rho > 0$ and $B < \infty$ such that if $A$ is any solution to the Yang-Mills heat flow on $0 \leq t \leq T$ with energy bounded by $E_0$ and
\[
(T - \alpha)^2 \int_M |F_A(x, \alpha)|^2 k_{(X,T)}(x, \alpha) dV \leq \epsilon,
\]then $|F_A(x, t)| \leq B$ for all $(x, t) \in P_{\rho}(X, T)$.

**Proof** We omit the proof, since it is almost identical to that of Theorem 3.2 in [5].

We also have the following corollary.

**Corollary 3.4** There exist $\epsilon > 0$ and $\beta > 0$ depending only on the bundle and $E_0$ such that if $A$ is any solution to the Yang-Mills flow on $0 \leq t < T$ and if
\[
(T - t)^2 \int_M |F_A(x, t)|^2 k_{(X,T)}(x, t) dV \leq \epsilon
\]for some $t$ in $T - \beta \leq t < T$ then $A(x, t)$ extends smoothly on $0 \leq t \leq T$ in some neighbourhood of $X$. 

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Proof Apply the above theorem to \( \tilde{A}(x,t) = A(x,t - \zeta) \) for \( \zeta > 0 \) and let \( \zeta \to 0 \). We then get uniform estimates on \( |F| \) in some small neighbourhood of \( X \) for \( t < T \). From Theorem \( 2.2 \) we get bounds on the higher derivatives, and the result follows.

We can now show that the limiting flow \( \tilde{A} \) does not have zero curvature. Since we have assumed that \( X \) is a singular point, we know that for all \( t \) in \( T - \beta \leq t < T \) we have

\[
(T - t)^2 \int_M |F_A(x,t)|^2 k_{(X,T)}(x,t) dV \geq \epsilon
\]

for a fixed \( \epsilon > 0 \). This estimate is dilation invariant. Now we have the bound

\[
(T - t)^2 |F|^2 \leq C^2.
\]

Note that there exists \( \rho > 0 \) such that, with \( \overline{k} \) as above,

\[
\int_{|x| > \rho \sqrt{|t|}} \overline{k}(x,t) dV \leq \frac{\epsilon}{2C^2}.
\]

Then after making the change of coordinates as before, and taking the limit, we see that

\[
|t|^2 \int_{|x| \leq \rho \sqrt{|t|}} |F_{\tilde{A}}(x,t)|^2 \overline{k}(x,t) dV \geq \frac{\epsilon}{2},
\]

and hence the soliton has non-zero curvature.

4. Examples of homothetically shrinking solitons

We give examples of homothetically shrinking solitons on \( \mathbb{R}^n \times SO(n) \) for \( 5 \leq n \leq 9 \). It was shown by Gastel [4] that such solitons exist in these dimensions. We follow [6] and consider \( SO(n) \)-equivariant connections given by

\[
A_i(x) = -\frac{h(r)}{r^2} \sigma_i(x),
\]

where \( r = |x| \), \( h \) is a real-valued function on \([0, \infty)\) and \( \{\sigma_i\}_{i=1}^n \) in \( so(n) \) are given by

\[
(\sigma_i)^\alpha \beta = \delta^\alpha \beta x^\beta - \delta_\beta^\alpha x^\alpha, \quad \text{for } 1 \leq \alpha, \beta \leq n.
\]

A long but straightforward calculation shows that the Yang-Mills heat equation becomes

\[
h_t = h_{rr} + (n - 3) \frac{h_r}{r} - (n - 2) \frac{h(h - 1)(h - 2)}{r^2}.
\]
Define a function
\[ \phi(\rho) = \frac{\rho^2}{a_n \rho^2 + b_n}, \]
where \(a_n\) and \(b_n\) are positive constants given by
\[ a_n = \frac{\sqrt{n - 2}}{2 \sqrt{2}} \quad \text{and} \quad b_n = \frac{1}{2} (6n - 12 - (n + 2) \sqrt{2n - 4}). \]
Notice that \(b_n\) is positive if and only if \(5 \leq n \leq 9\). Now for \(t\) in \((-\infty, 0)\), let
\[ h(r, t) = \phi(\rho), \quad \text{where} \quad \rho = \frac{r}{\sqrt{-t}}. \]
Then it is easy to check that this \(h(r, t)\) gives a smooth solution \(A = A(x, t)\) of the Yang-Mills heat equation on \(\mathbb{R}^n \times (-\infty, 0)\) and that
\[ A_i(x, t) = \lambda A_i(\lambda x, \lambda^2 t). \]

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