Finite Just Non-Dedekind Groups

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Abstract: A group is just non-Dedekind (JND) if it is not a Dedekind group
but all of whose proper homomorphic images are Dedekind groups. The aim
of the paper is to classify finite JND-groups.

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1 Introduction

A group is called Dedekind if all its subgroups are normal. By [1], a group is Dedekind if and only if it is abelian or the direct product of a quaternion group of order 8, an elementary abelian 2-group and an abelian group with all its elements of odd order (one can also see its proof in [10, 5.3.7, p.143]).

Given a group theoretical property $P$, a just non-$P$-group is a group which is not $P$-group but all of whose proper homomorphic images are $P$-groups; for brevity we shall call these JNP-groups. M.F. Newman studied just nonabelian (JNA) groups in [8, 9]. S. Franciosi and others studied solvable just nonnilpotent (JNN) groups in [3] and D.J.S. Robinson studied solvable just non-T (JNT) groups in [11](here the group with property T means in the group normality is a transitive relation).

The aim of this paper is to classify finite JND-groups. In Section 2, we prove that JND-groups are monolithic group. Section 3 deals with solvable JND-groups and Section 4 shows that nonsolvable JND-groups are semisimple. Theorem 4.4 gives complete classification of finite semisimple JND-groups.

Let $G$ be a group. For, sets $X, Y$ of $G$, let $[X, Y]$ denote the subgroup of $G$ generated by $[x, y] = x y x^{-1} y^{-1}$, $x \in X$, $y \in Y$. The derived series of $G$ is

$$G = G^{(0)} \geq G^{(1)} \geq \cdots \geq G^{(n)} \geq \cdots,$$

where $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$, the commutator subgroup of $G^{(n-1)}$. The lower central series of $G$ is

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G) \geq \cdots,$$

where $\gamma_{n+1}(G) = [\gamma_n(G), G]$. The group $G$ is called solvable of derived length $n$ (respectively nilpotent of class $n$) if $n$ is the smallest nonnegative integer such that $G^{(n)} = \{1\}$ (respectively $\gamma_{n+1}(G) = \{1\}$).

2 Some basic properties of JND-groups

We recall that a group is called monolithic if it has smallest nontrivial normal subgroup, called the monolith of $G$. In this section, we study some basic
properties of JND-groups.

**Proposition 2.1.** Let $G$ be a JND-group. Then $G$ is not contained in a direct product of Dedekind groups.

*Proof.* Let $\{H_i\}_{i \in I}$ denote a family of Dedekind groups, where $I$ is an indexing set. Assume that $G$ is contained in $H = \prod_{i \in I} H_i$. Since $G$ is nonabelian, there exists $i \in I$ such that $H_i$ is nonabelian. By the classification Theorem for nonabelian Dedekind groups [10, 5.3.7, p.143], square of each element of a nonabelian Dedekind group is central and its commutator subgroup is isomorphic to $\mathbb{Z}_2$. This implies that $G$ can not be simple. Take any non-trivial element $x \in G$. Then $x \in Z(G)$ if $x^2 = 1$ and $x^2 \in Z(G)$ if $x^2 \neq 1$ (for $G$ is contained in $H$). This proves that each subgroup of $G$ contains a nontrivial central element of $H$. Let $N$ be a nontrivial subgroup of $G$. Let $x \in N \cap Z(H)$, $x \neq 1$. Since $G$ is JND, $G/\langle x \rangle$ is Dedekind, so $N/\langle x \rangle \leq G/\langle x \rangle$, which proves that $N \leq G$. Hence $G$ is Dedekind. \qed 

**Corollary 2.2.** Let $G$ be a JND-group. Then $G$ is monolithic.

*Proof.* If $G$ is a JNA-group, there is nothing to prove for $G^{(1)}$ will be contained in each nontrivial normal subgroup of $G$. Assume that $G$ is not JNA. Let $\mathcal{A}$ denote the set of all nontrivial normal subgroups of $G$. Then $G/H$ is Dedekind for all $H \in \mathcal{A}$. Further, since $G$ is not JNA, there exists $H \in \mathcal{A}$ such that $G/H$ is nonabelian. Therefore by Proposition 2.1, the homomorphism from $G$ to $\prod_{H \in \mathcal{A}} G/H$ which sends $x \in G$ to $(xH)_{H \in \mathcal{A}}$ is not one-one. This proves that $\bigcap_{H \in \mathcal{A}} H \neq \{1\}$. \qed 

**Corollary 2.3.** Let $G$ be as in Corollary 2.2. Assume that $G^{(2)} \neq \{1\}$. Then the monolith of $G$ is $G^{(2)}$.

*Proof.* By Corollary 2.2, $G$ is monolithic. Let $K$ denote the monolith of $G$. Then $K \subseteq G^{(2)}$. If $G$ is JNA, then $K = G^{(1)}$ and so $K = G^{(2)}$. If $G$ is JND but not JNA, then $G/K$ is nonabelian Dedekind. Now by [10, 5.3.7, p.143], the commutator subgroup $G^{(1)}/K$ of $G/K$ is of order 2. So $G^{(2)} \subseteq K$. \qed
3 Finite solvable JND-groups

In this section, we classify finite solvable JND-groups. Solvable JNA-groups with nontrivial center is characterized in [9] and centerless solvable JNA-groups have been classified in [8, Theorem 5.2, p.360]. So, it only remains to classify finite solvable JND-groups which are not JNA-groups.

Proposition 3.1. Let $G$ be a JND-group. Let $Z(G)$, the center of $G$ be nontrivial. Then $G$ is a solvable JNA-group.

Proof. Suppose that $G$ is JND but not JNA. By Corollary 2.2, $G$ is monolithic. Let $K$ denote the monolith of $G$. Since every subgroup of $Z(G)$ is normal subgroup of $G$, $K$ is central subgroup of order $p$ for some prime $p$.

We claim that $p = 2$. Since $G$ is JND but not JNA, $G/K$ is nonabelian Dedekind. By the structure theorem for nonabelian Dedekind groups [10, 5.3.7, p.143], the commutator $(G/K)^{(1)} = G^{(1)}/K$ is of order 2. Thus $|G^{(1)}| = 2p$. Let $x$ be an element of $G^{(1)}$ of order 2. If $x \in Z(G)$, then $K = \langle x \rangle$, so $p = 2$. Assume that $x \notin Z(G)$. Since $|G^{(1)}/K| = 2$ and $x \notin K$, so $G^{(1)} = \langle x \rangle K$. Let $g \in G$ such that $g x g^{-1} \neq x$. Then there exists a nontrivial element $h \in K$ such that $g x g^{-1} = x h$. Now since $h \in Z(G)$, $h^2 = x^2 h^2 = (x h)^2 = (g x g^{-1} h)^2 = 1$ implies that $p = 2$.

Next, we show that $G$ does not contain an element of odd prime order. Assume that $x \in G$ is of odd prime order $q$. Since $\langle x \rangle K$ has a unique subgroup of order $q$ and $\langle x \rangle K \trianglelefteq G$ (for $G/K$ is Dedekind), $\langle x \rangle \trianglelefteq G$. But, then $K \subseteq \langle x \rangle$, a contradiction.

Further, since $G/K$ is a nonabelian Dedekind, by [10, 5.3.7, p.143], $G$ does not contain any element of infinite order. Thus we have shown that $G$ is a 2-group. Lastly, since $G/K$ is a nonabelian Dedekind, by [10, 5.3.7, p.143], $G$ contains a nonabelian subgroup $H$ of order 16 such that $K \subseteq H$ and $H/K \cong Q_8$. But this is not possible [2, 118, p.146].

Lemma 3.2. A finite centerless solvable JND-group is a JNT-group.

Proof. Let $G$ be a finite centerless solvable JND-group. Since a Dedekind group is also a T-group, it is sufficient to show that $G$ is not a T-group.

Suppose that $G$ is a T-group. Let $K$ denote the monolith of $G$ (Corollary 2.2). Since $G$ is a finite solvable T-group, $K$ is a cyclic group of order $p$.
for some prime $p$. Since $G/K$ is nonabelian Dedekind group, by \[10\] 5.3.7, p.143, $|G^{(1)}/K| = 2$. Further, since a solvable $T$-group is of derived length at most two \[10\] 13.4.2, p.403, $G^{(1)}$ is abelian. Now since $G^{(1)}$ is an abelian group of order $2p$ and $G$ is a $T$-group, $p = 2$. But then $K \subseteq Z(G) = \{1\}$. This is a contradiction. Therefore $G$ is a JNT-group.

The following example shows that there exists a solvable JND-group which is not a JNA-group.

**Example 3.3.** Consider an elementary abelian 3-group $A$ of order 9. Let $\psi$ denote the homomorphism from $Q_8$ to $Aut A = Gl_2(3)$ defined as $i \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $j \mapsto \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, where $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ is the quaternion group of order 8. It is easy to check that $\psi$ is injective. Let $G = AQ$ denote the natural semidirect product of $A$ by $Q_8$. Then $G$ is a JND-group with monolith $A$.

The following proposition classifies all finite solvable JND-groups which are not JNA-groups.

**Lemma 3.4.** A finite solvable group $G$ is JND but not JNA if and only if there exists an elementary abelian normal $p$-subgroup $A$ of $G$ for some prime $p$ which is also monolith of $G$ and a nonabelian Dedekind group $X$ of $G$ such that $A \cap X = \{1\}$, $G = AX$ and the conjugation action of $X$ on $A$ is faithful and irreducible.

**Proof.** Suppose that $G$ is a finite solvable JND-group but not JNA-group. By Corollary 2.2 $G$ is monolithic. Let $K$ be the monolith of $G$. Then $G/K$ is a nonabelian Dedekind. Thus by \[10\] 5.3.7, p.143, $|G^{(1)}/K| = 2$. Since $K$ is characteristically simple and abelian, it is an elementary abelian $p$-group of order $p^n$ for some prime $p$ \[10\] 3.3.15 (ii), p.87.

Assume that $G^{(1)}$ is abelian. If $p \neq 2$, then $G^{(1)}$ contain unique element of order 2 and so $Z(G) \neq 1$. By Proposition 3.1 this is a contradiction. Thus $p = 2$. Now by Proposition 3.1 $G$ is not nilpotent and by Lemma 3.2 $G$ is a JNT-group. So by Case 6.2 and its Subcases 6.211, 6.212, 6.22 and 6.222 in \[11\] pp.202-208, there is no finite JNT-group which is not JNA and has a minimal normal subgroup isomorphic to an elementary abelian 2-group.
Assume that \( G^{(1)} \) is nonabelian. Then \([G^{(1)}, K] \neq 1\), for \(|G^{(1)}/K| = 2\). Now since \( G \) is a finite nonnilpotent JNT-group and \([G^{(1)}, K] \neq 1\), by Case 6.1 of [11, p.202], there is a nontrivial normal subgroup \( A \) of \( G \), a solvable \( T \)-subgroup \( X \) of \( G \) such that \( A \cap X = \{1\} \), \( G = AX \) and the conjugation action of \( X \) on \( A \) is faithful and irreducible. Further, since \( K \subseteq A \) and the conjugation action of \( X \) on \( A \) is irreducible, \( K = A \). So \( X \cong G/A = G/K \) is a nonabelian Dedekind group.

Conversely, suppose that \( G = AX \), \( A \cap X = \{1\} \), \( X \) is a nonabelian Dedekind subgroup, \( A \) is an elementary abelian \( p \)-group and also the monolith of \( G \). Since \( A \) is solvable and \( G/A \cong X \) is nonabelian Dedekind and so solvable, \( G \) is solvable. Further, since \( A \) is the monolith of \( G \) and \( G/A \cong X \) is nonabelian Dedekind, \( G \) is JND but not JNA.

The following proposition lists some more properties of finite solvable JND-groups which are not JNA-groups.

**Proposition 3.5.** Let \( G, A \) and \( X \) be as in the Lemma 3.4. Then

(i) The stabilizer of any nontrivial element of \( A \) is trivial.

(ii) \(|X| \) divides \( p^n - 1 \), in particular \( p \) and \(|X|\) are coprime.

(iii) \( X \cong Q_8 \times A_o \), where \( A_o \) is a cyclic group of odd order.

**Proof.** Let \( a \in A, a \neq 1 \). Assume that the stabilizer \( \text{stab}_X(a) \) of \( a \) in \( X \) is nontrivial. Assume that \( x \in \text{stab}_X(a), x \neq 1 \). Since \( G/A \) is a Dedekind group, \( \langle x \rangle A \subseteq G \). Thus \( Z(\langle x \rangle A) \) is a nontrivial normal subgroup of \( G \) (for \( a \in Z(\langle x \rangle A) \)) and so \( A \subseteq Z(\langle x \rangle A) \), for \( A \) is the monolith of \( G \). But this is a contradiction, for the conjugation action of \( X \) on \( A \) is faithful. This proves (i). Now (ii) is implied by the class equation for the action of \( X \) on \( A \).

Further, by [11, Lemma 1, p.185], there is an extension field \( E \) of \( \mathbb{Z}_p \) such that \( Z(X) \cong Y \leq E^* \) and \( E = \mathbb{Z}_p(Y) \), where \( E^* \) denote the multiplicative group of \( E \). Clearly \( E \) is a finite field, so \( E^* \) is a cyclic group. This implies \( X \cong Q_8 \times A_o \), where \( A_o \) is a cyclic group of odd order [10, 5.3.7, p.143]. This proves (iii).
4 Finite nonsolvable JND-groups

Recall that a group is semisimple [10, p.89] if its maximal solvable normal subgroup is trivial. Also a maximal normal completely reducible subgroup is called the CR-radical [10, p.89].

**Proposition 4.1.** Let $G$ be a finite nonsolvable JND-group. Then $G$ is a semisimple group.

**Proof.** Assume that $G$ has a nontrivial normal solvable subgroup $N$. Then $G/N$ is a Dedekind group. Hence by [10, 5.3.7, p.143], $G/N$ is solvable. But then $G$ is solvable, a contradiction. □

Now we fix some notations for the rest of the section. For a group $G$, we denote $\text{Inn } G$ for the inner automorphism subgroup of $\text{Aut } G$, the automorphism group of $G$ and $\text{Out } G$ for the outer automorphism group of $G$. Let $H$ denote a finite nonabelian simple group. Consider the semidirect product $(\text{Aut } H \times \ldots \times \text{Aut } H) \rtimes S_r$ and $(\text{Out } H \times \ldots \times \text{Out } H) \rtimes S_r$, where $S_r$ acts on $(\text{Aut } H \times \ldots \times \text{Aut } H)$ as well as on $(\text{Out } H \times \ldots \times \text{Out } H)$ by permuting the coordinates. Let

$$\tilde{\nu} : (\text{Aut } H \times \ldots \times \text{Aut } H) \rtimes S_r \longrightarrow (\text{Out } H \times \ldots \times \text{Out } H) \rtimes S_r$$

be the homomorphism defined by $\tilde{\nu}(x_1, x_2, \ldots, x_r, x_{r+1}) = (x_1 \text{Inn } H, \ldots, x_r \text{Inn } H, x_{r+1})$. We denote by $\beta$ the projection of $(\text{Out } H \times \ldots \times \text{Out } H) \rtimes S_r$ onto the $(r + 1)$-th factor $S_r$, which is obviously a homomorphism.

**Lemma 4.2.** Let $H$ be a finite nonabelian simple group. Then $\text{Out } H$ does not contain a subgroup isomorphic to the quaternion group $Q_8$ of order 8.

**Proof.** If $H$ is isomorphic to either alternating group $\text{Alt}_n$ of degree $n$ or to a Sporadic simple group, then $|\text{Out}(H)| \leq 4$ (see [12, 2.17, 2.19, p.299] and [6, Table 2.1C, p.20]), so the Lemma follows in this case. If $H$ is isomorphic
to a finite simple group of Lie type, then the Lemma follows by [4, Theorem 2.5.12, p.58].

**Corollary 4.3.** Let $H$ be a finite nonabelian simple group. Then for any $m \in \mathbb{N}$, $Out H \times \ldots \times Out H$ does not contain a subgroup isomorphic to the quaternion group $Q_8$ of order 8.

**Proof.** Assume that $\alpha$ is an injective homomorphism from $Q_8$ to $Out H \times \ldots \times Out H$. Let $u = (x_1, x_2, \ldots, x_m)$ denote an element of $\alpha(Q_8)$ of order 4. Then there is $t$ ($1 \leq t \leq m$) such that $x_t$ is of order 4. Let $p_t$ denote the projection of $Out H \times \ldots \times Out H$ onto the $t$-th factor. Then $(p_t \circ \alpha)(Q_8)$ is a subgroup of $Out H$ which contains an element of order 4. Since a homomorphic image of $Q_8$ containing an element of order 4 is isomorphic to $Q_8$, $(p_t \circ \alpha)(Q_8) \cong Q_8$. By Lemma [1.2], this is impossible. □

**Theorem 4.4.** A finite nonsolvable group $G$ is JND-group if and only if there exists a finite nonabelian simple group $H$, a natural number $r$ and a Dedekind group $D \subseteq (Out H \times \ldots \times Out H) \rtimes S_r$ such that

(i) the usual action of $\beta(D)$ on the set $\{1, 2, \ldots, r\}$ is free and transitive, and

(ii) $G \cong \tilde{\nu}^{-1}(D)$,

where all the notations have meaning described as after the Proposition [4.1]. Further, $G$ is JND but not JNA if and only if $D$ is a nonabelian Dedekind group and $r$ is even.

**Proof.** Suppose that $G$ is a nonsolvable JND-group. By Corollary [2.2], $G$ is monolithic. Let $K$ denote the monolith of $G$. Since $G$ is nonsolvable and $K$ is characteristically simple, by [10, 3.3.15 (ii), p.87], there exists a
finite nonabelian simple group $H$ and a natural number $r$ such that $K \cong (H \times \ldots \times H)$.

By Proposition 1, $G$ is semisimple. We show that $K$ is the CR-radical of $G$. Let $N$ be the CR-radical of $G$ containing $K$. Then $N$ is semisimple [5, Lemma, p.205]. Assume that $N \neq K$. Then there exists nontrivial completely reducible normal subgroup $L$ of $N$ which is complement of $K$ in $N$. Now since $L \cong N/K$ and $G/K$ is a Dedekind group, $L$ is solvable [10, 5.3.7, p.143]. Further, since nontrivial normal subgroup of a semisimple group is semisimple [5, Lemma, p.205], $L$ is also semisimple. This is a contradiction.

Now by [10, 3.3.18 (i), p.89], there exists $G^* \cong G$ such that $(\text{Inn} H \times \ldots \times \text{Inn} H)^r \leq G^* \leq (\text{Aut} H \times \ldots \times \text{Aut} H)^r \rtimes S_r$. We identify $G$ with $G^*$ and $H$ with \text{Inn} H. Thus $K$ is identified with $(\text{Inn} H \times \ldots \times \text{Inn} H)^r$.

Take $D = G/K \subseteq (\text{Out} H \times \ldots \times \text{Out} H)^r \rtimes S_r$. Then $D$ is a Dedekind group and $G \cong \mathcal{v}^{-1}(D)$. This proves (ii).

Next, we claim that $\beta(D)$ acts transitively on the set of symbols $\{1, 2, \ldots, r\}$. Let $O$ denote an orbit of the natural action of $\beta(D)$ on $\{1, 2, \ldots, r\}$. Consider the subgroup $M_O = \{(x_1, x_2, \ldots, x_r, 1) | x_i \in \text{Inn} H \text{ and } x_i = 1 \text{ if } i \notin O\} \subseteq (\text{Aut} H \times \ldots \times \text{Aut} H)^r \rtimes S_r$. It is easy to observe that for each element of $G$, the $(r + 1)$-th coordinate is an element of $\beta(D)$. This implies that $M_O$ is a normal subgroup of $G$ contained in $K$. But $K$ is the monolith of $G$, so $M_O = K$. This proves that $O = \{1, 2, \ldots, r\}$.

Now, we show that the action of $Z(\beta(D))$ on $\{1, 2, \ldots, r\}$ is free. Suppose that an element $u$ of $Z(\beta(D))$ fixes a symbol $a$ under the natural action of $\beta(D)$ on $\{1, 2, \ldots, r\}$. Clearly $u$ fixes each element of the orbit $\beta(D).a$ of $a$ which is $\{1, 2, \ldots, r\}$. This implies $u = 1$. So, no nontrivial element of $Z(\beta(D))$ will fix any symbol in $\{1, 2, \ldots, r\}$.

If $D$ is abelian, then $Z(\beta(D)) = \beta(D)$ and so the action of $\beta(D)$ is free. If $D$ is nonabelian Dedekind group, then by the structure Theorem for Dedekind groups [10, 5.3.7, p.143], there exists a nonnegative integer $t$ and an abelian group $A_o$ of odd order such that we can write $D = Q_8 \times (\mathbb{Z}_2)^t \times A_o$. Thus for any $x \in \beta(D)$, either $x \in Z(\beta(D))$ or $1 \neq x^2 \in Z(\beta(D))$. This implies that a noncentral element $x$ also does not fix any symbol of set
\{1, 2, \ldots, r\} (for then \(1 \neq x^2 \in \mathbb{Z}(\beta(D))\) will fix that symbol). Thus action of \(\beta(D)\) on \{1, 2, \ldots, r\} is free and transitive. In particular \(r = |\beta(D)|\). This proves (i).

Now, assume that \(G\) is JND but not JNA. Then by Corollary 4.3, \(\beta(Q_8) \neq 1\) and so 2 divides \(|\beta(D)| = r\).

Conversely, suppose that there exists a Dedekind group \(D \subseteq \langle \text{Out } H \times \ldots \times \text{Out } H \rangle \rtimes S_r\) for some \(r \in \mathbb{N}\) and a nonabelian finite simple group \(H\) such that, the usual action of \(\beta(D)\) on \{1, 2, \ldots, r\} is free and transitive. Let \(G = \widetilde{\nu}^{-1}(D)\). We will show that \(G\) is a JND-group. By [10, 3.3.18 (ii), p.89], \(G\) is semisimple with CR-radical \(K = \langle \text{Inn } H \times \ldots \times \text{Inn } H \rangle\) and \(\text{Inn } H \times \ldots \times \text{Inn } H \leq G \leq \langle \text{Aut } H \times \ldots \times \text{Aut } H \rangle \rtimes S_r\). We will show that \(K\) is the monolith of \(G\).

Since \(K = K^{(1)}\), \(K\) is contained in all terms of the derived series of \(G\). Further, since \(G\) is semisimple, there is smallest nonnegative integer \(n\) such that \(G^{(n)} = G^{(n+i)}\) for all \(i \in \mathbb{N}\). This implies that \(G^{(n)} / K\) is a perfect group. But since \(G^{(n)} / K\) is Dedekind and so solvable [10, 5.3.7, p.143], \(G^{(n)} = K\). Let \(N\) be a nontrivial normal subgroup of \(G\). Since a nontrivial normal subgroup of a semisimple group is semisimple [5, Lemma, p.205] and a semisimple group has trivial center, \(N \cap K \neq \{1\}\). By [7, Theorem 2, p.156], \(N \cap K = N_1 \times N_2 \times \ldots \times N_r\), where \(N_i \leq \text{Inn } H\) and at least one \(N_i \neq 1\).

Now since \(N_i = \text{Inn } H\) and \(\beta(D)\) acts transitively on \(\text{Inn } H \times \ldots \times \text{Inn } H\), so \(N \cap K = \text{Inn } H \times \ldots \times \text{Inn } H = K\), that is \(K \subseteq N\). This proves that \(K\) is the monolith of \(G\). Thus \(G\) is JND-group. Further, if \(D\) is nonabelian Dedekind group, then \(G\) is JND but not JNA.}

\[\square\]

**Remark 4.5.** Let \(G\) be finite just nonsolvable (JNS) (respectively just non-nilpotent (JNN)) group. Let \(n\) be the smallest nonnegative integer such that \(G^{(n)} = G^{(n+k)}\) (respectively \(\gamma_n(G) = \gamma_{(n+k)}(G)\)) for all \(k \in \mathbb{N}\). Then it is easy to see that \(G^{(n)}\) (respectively \(\gamma_{(n+1)}(G)\)) is the monolith of \(G\).
The idea of the proof of the above theorem can be used to show that:
A finite nonsolvable group $G$ is JNS-group (respectively JNN-group) if and only if there exists a finite nonabelian simple group $H$, a natural number $r$ and a solvable (respectively nilpotent) group $D \subseteq (\text{Out } H \times \ldots \times \text{Out } H) \rtimes S_r$ such that

(i) the usual action of $\beta(D)$ on the set $\{1, 2, \ldots, r\}$ is transitive, and
(ii) $G \cong \tilde{\nu}^{-1}(D)$,

where all the notations have meaning described as after the Proposition 4.1.

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