THE EXTENDER ALGEBRA AND $\Sigma^2_1$-ABSOLUTENESS

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Abstract. We present a self-contained account of Woodin’s extender algebra and its use in proving absoluteness results, including a proof of the $\Sigma^2_1$-absoluteness theorem.

This note provides an introduction to Woodin’s extender algebra and a proof (due to Steel and Woodin independently) of Woodin’s $\Sigma^2_1$-absoluteness theorem, using the extender algebra, from a large cardinal assumption. Unlike the published accounts of this proof, the present account should be accessible to a set theorist familiar with forcing and basic large cardinals (e.g., [8], [4]). In particular, no familiarity with the inner model theory is required. A more comprehensive account of the extender algebra can also be found in [18] §7.2 and [13] §4 and the reader is invited to consult these excellent sources for more information and other applications. A strengthening of the $\Sigma^2_1$ absoluteness in terms of determinacy of games of length $\omega_1$ was proved by Neeman ([14]) also using the extender algebra. In [10], Larson used the extender algebra to prove the consistency of Woodin’s $\Omega$-conjecture.

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1. Logics

1.1. An infinitary propositional logic. For regular cardinals $\gamma \leq \delta$ we shall define the infinitary propositional logic $L_{\delta,\gamma}$. The interesting cases are $\gamma = \omega$ and $\gamma = \delta$, but it will be easier to develop the basic theory in the two cases parallelly. Let $L_{\delta,\gamma}$ be the propositional logic with $\gamma$ variables $a_\xi$, for $\xi < \gamma$, which in addition to the...
standard propositional connectives $\lor, \land, \to, \leftrightarrow,$ and $\neg$ allows infinitary conjunctions of the form $\bigwedge_{\xi<\kappa} \varphi_{\xi}$ and infinitary disjunctions of the form $\bigvee_{\xi<\kappa} \varphi_{\xi}$ for all $\kappa<\delta$.

In addition to the standard axioms and rules of inference for the finitary propositional logic, for each $\kappa<\delta$ and formulas $\varphi_{\xi}$, for $\xi<\kappa$ the logic $L_{\delta,\gamma}$ has axioms $\vdash \bigvee_{\xi<\kappa} \neg \varphi_{\xi} \leftrightarrow \neg \bigwedge_{\xi<\kappa} \varphi_{\xi}$ and $\vdash \bigwedge_{\xi<\kappa} \varphi_{\xi} \to \varphi_{0}$, for every $\eta<\kappa$, as well as the infinitary rule of inference: from $\vdash \varphi_{\xi}$ for all $\xi<\kappa$ infer $\vdash \bigwedge_{\xi<\kappa} \varphi_{\xi}$. The provability relation for $L_{\delta,\gamma}$ will be denoted by $\vdash_{\delta,\gamma}$ or simply $\vdash$ if $\delta$ and $\gamma$ are clear from the context. Each proof in $L_{\delta,\gamma}$ is a well-founded tree and the assertion that $\varphi$ is provable in $L_{\delta,\gamma}$ is upwards absolute between transitive models of ZFC. Since adding a new bounded subset of $\delta$ adds new formulas and new proofs to $L_{\delta,\gamma}$, it is not obvious that the assertion that $\varphi$ is not provable in $L_{\delta,\gamma}$ is upwards absolute between transitive models of ZFC. This is, nevertheless, true: see Lemma 1.2.

Every $x \in \mathcal{P}(\gamma)$ naturally defines a model for $L_{\delta,\gamma}$ via $v_x(a_\xi) = \text{true}$ if $x(\xi) = 1$ for $\xi < \gamma$. Define $A_{\varphi} = A_{\varphi,\delta,\gamma}$ via

$$A_{\varphi} = \{ x \in \mathcal{P}(\gamma) \mid x \models \varphi \}.$$

When $\gamma = \omega$ then these are the so-called $\infty$-Borel sets (see e.g., §91). Note that the sets of the form $A_{\varphi}$ for $\varphi \in L_{\omega_1,\omega}$ are exactly the Borel sets. Also note that $x \models \varphi$ is absolute between transitive models of ZFC containing $x$ and $\varphi$.

1.2. Completeness of $L_{\delta,\gamma}$. The following two lemmas are standard.

Lemma 1.1. (1) For every formula $\varphi$ in $L_{\delta,\gamma}$ we have that $\vdash \varphi$ implies $\models \varphi$.

(2) If $\delta > 2^\omega$ then there is a formula $\varphi$ in $L_{\delta,\gamma}$ such that $\models \varphi$ but not $\vdash \varphi$.

Proof. Clause (1) can be proved by recursion on the rank of the proof.

(2) For $x \subseteq \gamma$ let $\varphi_x = \bigvee_{\xi < \gamma} a_{\xi}^x$, where $a_{\xi}^x = a_\xi$ if $\xi \in x$ and $a_{\xi}^x = \neg a_\xi$ if $\xi \notin x$. Then $y \models \varphi_y$ if and only if $y = x$, and therefore the formula $\varphi = \bigwedge_{\xi \subseteq \gamma} \neg \varphi_{\xi}$ is not satisfiable. However, $\varphi$ is satisfiable in every forcing extension in which there exists a new subset of $\gamma$.

Lemma 1.2. For every $\varphi$ in $L_{\delta,\gamma}$ the following are equivalent.

(1) $\vdash \varphi$.

(2) $A_{\varphi} = \mathcal{P}(\gamma)$ in all generic extensions.

(3) $A_{\varphi} = \mathcal{P}(\gamma)$ in the extension by $\text{Coll}(\omega, \kappa)$ for a large enough cardinal $\kappa$.

Proof. Clause (1) is upwards absolute and by recursion on the rank of the proof it easily implies (2). Also, (2) trivially implies (3).

Assume (1) fails for $\varphi$ and let $\kappa$ be the cardinality of the set of all subformulas of $\varphi$. We may assume $a_\xi$ is a subformula of $\varphi$ if and only if $\xi < \kappa$.

We claim that for any two formulas $\psi_1$ and $\psi_2$ in $L_{\delta,\gamma}$ such that $\psi_1 \not\vDash \varphi$ we have either $\psi_1 \land \psi_2 \not\vDash \varphi$ or $\psi_1 \land \neg \psi_2 \not\vDash \varphi$. Otherwise we have $\psi_1 \vdash \psi_2 \to \varphi$ and $\psi_1 \vdash \neg \psi_2 \to \varphi$ and therefore $\psi_1 \vdash \varphi$, a contradiction.

In the extension by $\text{Coll}(\omega, \kappa)$, enumerate all subformulas of $\varphi$ as $\psi_n$, for $n \in \omega$, and also enumerate $\{ a_\xi : \xi \in \gamma \}$ as $b_n$, for $n \in \omega$. Recursively pick an increasing sequence of $L_{\delta,\gamma}$-theories $T_n$, for $n \in \omega$, satisfying the following requirements: (i) $T_0 = \{ \neg \varphi \}$ and each $T_n$ is a finite consistent set of ground-model formulas in $L_{\delta,\gamma}$, (ii) either $b_n \in T_n$ or $\neg b_n \in T_n$. (iii) if $\psi_n$ is of the form $\bigwedge_{\xi<\lambda} \sigma_\xi$ for some $\lambda$ then

\footnote{Keep in mind that, in spite of the connection with Borel sets, this logic is not the ‘usual’ $L_{\omega_1,\omega}$ (3). Our $L_{\omega_1,\omega}$ happens to be the propositional fragment of the latter. This is really an accident since in each of the two notations ‘$\omega$’ signifies a different constraint.}
either \( \psi_n \in \mathcal{T}_n \) or \( -\sigma_\xi \in \mathcal{T}_n \) for some \( \xi < \lambda \). (iv) if \( \psi_n \) is of the form \( \bigvee_{\xi < \lambda} \sigma_\xi \) for some \( \lambda \) then either \( -\psi_n \in \mathcal{T}_n \) or \( \sigma_\xi \in \mathcal{T}_n \) for some \( \xi < \lambda \).

By the above claim, if \( \mathcal{T}_n \) satisfies the requirements then \( \mathcal{T}_{n+1} \) as required can be chosen. After all \( \mathcal{T}_n \) have been chosen define \( x \subseteq \gamma \) by \( x = \{ \xi : a_\xi \in \bigcup_n \mathcal{T}_n \} \). Then \( x \models \neg \varphi \) can be proved by recursion on the rank of \( \varphi \), using the fact that every infinitary conjunction and every infinitary disjunction appearing in \( \varphi \) is computed correctly. Therefore in this extension \( \Box \) fails.

Recall that the Lindenbaum algebra of a \( \mathcal{L}_{\delta, \gamma} \)-theory \( \mathcal{T} \) is the Boolean algebra of all of equivalence classes of formulas in \( \mathcal{L}_{\delta, \gamma} \) with respect to the equivalence relation \( \sim_\mathcal{T} \) defined by \( \varphi \sim_\mathcal{T} \psi \) if and only if \( \mathcal{T} \vdash \varphi \iff \psi \). We shall denote this algebra by \( \mathcal{B}_{\delta, \gamma} / \mathcal{T} \) and consider its positive elements as a forcing notion (see Lemma 1.4 below).

**Lemma 1.3.** For every \( \mathcal{L}_{\delta, \gamma} \)-theory \( \mathcal{T} \) such that \( \mathcal{B}_{\delta, \gamma} / \mathcal{T} \) has the \( \delta \)-chain condition \( \mathcal{B}_{\delta, \gamma} / \mathcal{T} \) is a complete Boolean algebra.

**Proof.** Immediate, since both \( \mathcal{B}_{\delta, \gamma} \) and \( \mathcal{T} \) are \( \delta \)-complete.

For \( x \subseteq \gamma \) such that \( x \models \mathcal{T} \) define an ultrafilter of \( \mathcal{B}_{\delta, \gamma} / \mathcal{T} \) by (here \( [\varphi] \) stands for the equivalence class of \( \varphi \))

\[
\Gamma_x = \{ [\varphi] \in \mathcal{B}_{\delta, \gamma} / \mathcal{T} \mid x \models \varphi \}.
\]

Note that for every generic \( \Gamma \subseteq \mathcal{B}_{\delta, \gamma} / \mathcal{T} \) there is the unique \( x \subseteq \gamma \) such that \( \Gamma_x = \Gamma \), defined as \( x = \{ \xi : a_\xi \in \Gamma \} \).

**Lemma 1.4.** Assume \( \mathcal{T} \) is a theory in \( \mathcal{L}_{\delta, \gamma} \) and \( M \) is a transitive model of \( \text{ZFC}^c \) such that \( \{ \delta, \gamma, \mathcal{T} \} \subseteq M \), and \( \mathcal{B}_{\delta, \gamma} / \mathcal{T} \) has the \( \delta \)-chain condition in \( M \). Then for every \( x \in \mathcal{P}(\gamma) \) we have \( x \models \mathcal{T}^M \) if and only if \( x \) is \( \mathcal{B}_{\delta, \gamma} / \mathcal{T} \)-generic over \( M \).

**Proof.** If \( x \in \mathcal{B}_{\delta, \gamma} / \mathcal{T} \)-generic over \( M \) then a proof by induction on the complexity shows that \( x \models \varphi \) for all \( [\varphi] \in \Gamma \), hence \( \{ \varphi \in \mathcal{L}_{\delta, \gamma} : x \models \varphi \} \) includes \( \mathcal{T}^M \).

Now assume \( x \models \mathcal{T}^M \). Let \( \varphi_\xi (\xi < \kappa) \) be a maximal antichain of \( (\mathcal{B}_{\delta, \gamma} / \mathcal{T})^M \) that belongs to \( M \). By the \( \delta \)-chain condition \( \kappa < \delta \) and therefore \( \bigvee_{\xi < \kappa} \varphi_\xi \) is in \( \mathcal{L}_{\delta, \gamma}^M \). By the maximality of the antichain we have \( \mathcal{T}^M \models \bigvee_{\xi < \kappa} \varphi_\xi \). Therefore \( x \models \bigvee_{\xi < \kappa} \varphi_\xi \).

This implies \( x \models \varphi_\xi \) for some \( \xi < \kappa \) and \( \Gamma_x \) intersects the given maximal antichain. Since the antichain was arbitrary, we conclude \( x \) is generic.

## 2. Elementary embeddings

### 2.1. Extenders I

We only sketch the bare minimum of the theory of extenders. For more details see [12] or [4, §26]. An *extender* \( E \) is a set that codes an elementary embedding \( j_E : V \rightarrow M \). For every elementary embedding \( j : V \rightarrow M \) and \( \lambda > \text{crit}(j) \) there is an extender \( E \) in \( V_{\lambda+1} \) such that \( j_E \) and \( j \) coincide up to \( V_{\lambda} \) in the sense that \( j(A) \cap V_{\lambda} = j_E(A) \cap V_{\lambda} \) for all \( A \subseteq V_{\lambda} \). The model \( M \) is constructed as a direct limit of ultrapowers of \( V \) and it is denoted by \( \text{Ult}(V, E) \).

A *generator* of an elementary embedding \( j : V \rightarrow M \) is an ordinal \( \xi \) such that there are an inner model \( N \) and elementary embeddings \( i_1 \) and \( i_2 \) such that the
Note that the Woodinness of $\delta$ with critical point $\kappa$ commutes and $\text{crit}(i_2) = \xi$. For example, the critical point $\kappa$ is the least generator and an elementary embedding such that $j(\kappa) \geq (2^\kappa)^+$ must have other generators. A generator of an extender $E$ is a generator of $j_E$. The strength of an extender $E$ is the largest $\lambda$ such that $V_\lambda \subseteq M$, where $j_E : V \rightarrow M$.

The only properties of extenders used in the present paper are (E1) and (E2).

(E1) If $E$ is an extender with $\kappa = \text{crit}(j_E)$ then $\text{Ult}(N, E)$ can be formed for every model $N$ such that $(V_{\kappa+1})^N = V_{\kappa+1}$. Also, $\text{Ult}(N, E) \supseteq V_\lambda$, where $\lambda$ is the strength of $E$ and whether or not $\xi$ is a generator of $E$ depends only on $E$ and $V_{\kappa+1}$.

For a proof of (E1) see [12, Lemma 1.5], or note that this is immediate from Definition 2.3 below since all ultrafilters $E_s$ concentrate on $|\kappa|^{<\omega}$. While (E1) is a property of all extenders, (E2) below is not. However, we shall consider only the extenders satisfying (E2).

(E2) The strength of $E$ is greater than the supremum of the generators of $E$.

It is important to note that the strength depends only on $E$, and not on the model to which $E$ was applied.

2.2. Woodin cardinals. If $A$ is a set such that $j : V \rightarrow M$ satisfies $V_\lambda = (V_\lambda)^M$ and $j(A) \cap V_\lambda = A \cap V_\lambda$ then we say $j$ is an $A, \lambda$-strong embedding. If $j_E$ is an $A, \lambda$-strong then we say $E$ is $A, \lambda$-strong. A cardinal $\delta$ is a Woodin cardinal if for every $A \subseteq V_\delta$ there is $\kappa < \delta$ such that there are $A, \lambda$-strong elementary embeddings with critical point $\kappa$ for an arbitrarily large $\lambda < \delta$. We say that $A$ reflects to $\kappa$. Note that the Woodinness of $\delta$ is witnessed by the extenders in $V_\delta$, and therefore $\delta$ is Woodin in $V$ if and only if it is Woodin in $L(V_\delta)$. Moreover, it suffices to consider only the extenders that satisfy property (E2).

2.3. Extenders II. A reader not interested in extenders per se may want to skip the rest of this section on the first reading and take (E1) and (E2) for granted. The actual definition of an extender is, strictly speaking, not necessary for our present purpose. However, this notion is central to the theory and we include it for the reader’s convenience. Every extender is of the following form.

Example 2.1. Assume $j : V \rightarrow M$ is an elementary embedding with $\text{crit}(j) = \kappa$. Fix $\lambda$ such that $\kappa \leq \lambda < j(\kappa)$. Typically, we take $\lambda$ such that $M \supseteq V_\lambda$. For $s \in [\lambda]^{<\omega}$ define $E_s \subseteq [\kappa]^m$ (where $m = |s|$) by

$$X \in E_s \text{ if and only if } s \in j(X).$$

Then $E(j, \lambda) = \{E_s : s \in [\lambda]^{<\omega}\}$ is a $(\kappa, \lambda)$-extender.

Fix a cardinal $\kappa$. For $m \in \omega$ and $s \subseteq m$ with $|s| = n$ consider the projection map $\pi = \pi_{m, s} : [\kappa]^m \rightarrow [\kappa]^n$ defined by

$$\pi((\xi_i : i < m)) = (\xi_i : i \in s).$$

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2after all, it is the extender algebra
More generally, if \( s \subseteq t \) are finite sets of ordinals (listed in the increasing order) then the projection \( \pi = \pi_{t,s}: [k]^t \to [k]^s \) is defined by
\[
\pi([\xi_i: i \in t]) = ([\xi_i: i \in s]).
\]
Recall that if \( \mathcal{U} \) and \( \mathcal{V} \) are ultrafilters on sets \( I \) and \( J \), respectively, then we write \( \mathcal{U} \leq_{\text{RK}} \mathcal{V} \) if and only if there is \( h: J \to I \) such that
\[
X \in \mathcal{U} \text{ if and only if } h^{-1}(X) \in \mathcal{V}.
\]
In this situation we say \( \mathcal{U} \) is Rudin–Keisler reducible to \( \mathcal{V} \) and that \( h \) is the Rudin–Keisler reduction of \( \mathcal{U} \) to \( \mathcal{V} \).

Respecting the notation commonly accepted in the theory of large cardinals, we denote an ultrapower of a structure \( M \) associated to an ultrafilter \( \mathcal{U} \) by \( \text{Ult}(M, \mathcal{U}) \). Its elements are the equivalence classes of \( f \in M^I \cap M \),
\[
[f]_\mathcal{U} = \{ g \in M^I \cap M: (\mathcal{U}i) f(i) = g(i) \}
\]
and the membership relation is defined by \( [f]_\mathcal{U} \in [g]_\mathcal{U} \) if and only if \( (\mathcal{U}i) f(i) \in g(i) \).

If \( \mathcal{U} \) is \( \aleph_1 \)-complete then \( \text{Ult}(M, \mathcal{U}) \) is well-founded whenever \( M \) is well-founded, and we identify \( \text{Ult}(M, \mathcal{U}) \) with its transitive collapse.

Assume \( \mathcal{U} \) and \( \mathcal{V} \) are ultrafilters on index-sets \( I \) and \( J \), respectively, and \( \mathcal{U} \leq_{\text{RK}} \mathcal{V} \) is witnessed by a reduction \( h: J \to I \). Then for any structure \( M \) we can define a map \( j_h: M^I/\mathcal{U} \to M^J/\mathcal{V} \) by
\[
j_h([f]_\mathcal{U}) = [f \circ h]_\mathcal{V}
\]
In the following lemma \( M \) is any structure and \( \mathcal{U} \) and \( \mathcal{V} \) are arbitrary ultrafilters and its proof is straightforward.

**Lemma 2.2.** If \( \mathcal{U} \leq_{\text{RK}} \mathcal{V} \), \( h \) is the Rudin–Keisler reduction, and \( j_h \) is defined as above, then the diagram
\[
\begin{array}{ccc}
M & \xrightarrow{j_\mathcal{U}} & \text{Ult}(M, \mathcal{U}) \\
\downarrow{j_\mathcal{V}} & & \downarrow{j_h} \\
\text{Ult}(M, \mathcal{V})
\end{array}
\]
commutes and \( j_h \) is an elementary embedding. \( \square \)

For a finite set \( s \) and \( i < |s| \) let \( s_i \) denote its \( i \)-th element. As common in set theory, we start counting at 0.

**Definition 2.3.** Assume \( \kappa < \lambda \) are uncountable cardinals. A \( (\kappa, \lambda) \)-extender is \( E: [\lambda]^{<\omega} \to V_{\kappa+2} \) such that for all \( s \) and \( t \) in \( [\lambda]^{<\omega} \) we have
(a) \( E_s \) is a nonprincipal \( \kappa \)-complete ultrafilter on \( [\kappa]^{<\omega} \).
(b) If \( s \subseteq t \) then \( \pi_{t,s} \) is a Rudin–Keisler reduction of \( E_s \) to \( E_t \).
(c) Normality: If \( s \in [\lambda]^{<\omega} \) and \( f: [\kappa]^{<\omega} \) is such that \( f(u) < u_i \) for \( E_s \) many \( u \),
then there exist \( \xi < s_t \) and \( j \) such that \( f \circ \pi_{s \cup \{\xi\}, j} = u_j \) for \( E_{s \cup \{\xi\}} \) many \( u \).
(d) Countable completeness: if \( s(n) \in [\lambda]^{<\omega} \) and \( X(n) \in E_{s(n)} \) for all \( n < \omega \)
then there is increasing \( h: \bigcup_n s(n) \to \kappa \) such that \( h''s(n) \in X(n) \) for all \( n \).

If \( E \) is an extender, then the models
\[
M_s = \text{Ult}(V, E_s)
\]
are, by the \( \kappa \)-completeness of \( E_s \), well-founded. By Lemma \ref{lem:2.2} they form a directed system under the embeddings (writing \( j_{t,s} \) for \( j_{\pi_{t,s}} \))

\[
j_{t,s} : M_t \to M_s.
\]

The direct limit of this system will be denoted by \( \text{Ult}(V,E) \) and identified with its transitive collapse if it is well-founded.

It is not difficult to show that if \( E \) satisfies (a), (b) and (c) of Definition \ref{def:2.3} then the corresponding ultrapower \( \text{Ult}(V,E) \) is well-founded if and only if \( E \) satisfies (d) as well.

The following facts can be found e.g., in \cite{18}. For an extender \( E \) we write \( \kappa_E = \text{crit}(E) \) (the critical point of \( E \), i.e., the least ordinal moved by \( j_E \)) and \( \lambda_E = \sup \{ \eta \mid V_\eta \subseteq (V_\eta)^M \} \) (the strength of \( E \)). Note that, with the above notation for a \((\kappa,\lambda)\)-extender, we have \( \kappa_E = \kappa \) but not necessarily \( \lambda_E = \lambda \). However, if \( j : V \to M \) is an elementary embedding such that \( \text{crit}(j) = \kappa \), the strength of \( j \) is \( \lambda \), and \( \lambda \) is a strong limit cardinal, then the \((\kappa,\lambda)\) extender \( E \) defined in Example \ref{ex:2.1} satisfies \( \text{Ult}(V,E) \cap V_\lambda = M \cap V_\lambda \) and \( j(A) \cap V_\lambda = j_E(A) \cap V_\lambda \) for all \( A \) and is therefore \( \lambda \)-strong. The assumption that \( \lambda \) is strong limit is needed to code subsets of \( P(\alpha) \) by sets of ordinals for every \( \alpha < \lambda \).

If \( E \) is a \((\kappa,\lambda)\)-extender, let \( M \) be a model to which \( E \) can be applied and let \( \kappa < \xi \leq \lambda \). One defines \( E \upharpoonright \xi = \langle E_s : s \in [\xi]^{<\omega} \rangle \).

Like in Lemma \ref{lem:2.2} one can define an elementary embedding \( i \) such that the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{j E \upharpoonright \xi} & \text{Ult}(M, E \upharpoonright \xi) \\
& \searrow j_E \downarrow \nearrow i & \\
& & \text{Ult}(M, E)
\end{array}
\]

commutes. If \( \text{crit}(i) = \xi \) then we say \( \xi \) is a generator of \( E \).

**Lemma 2.4.** If \( j : V \to M \) is a \( \lambda \)-strong embedding with \( \text{crit}(j) = \kappa \). Let \( E = E(j, \lambda) \) be as in Example \ref{ex:2.1} Then

(1) \( E \) satisfies (E1) and (E2).

(2) For every \( X \subseteq \kappa \) we have \( j(X) \cap \lambda = j_E(X) \cap \lambda \).

In particular, if \( \delta \) is a Woodin cardinal then there is a family \( \vec{E} \subseteq V_\delta \) of extenders satisfying (E1) and (E2) such that \( \delta \) is Woodin in \( L[\vec{E}] \).

**Proof.** Clause (E1) is the immediate consequence of the fact that \( \text{Ult}(N,E) \) depends only on \( V_{\kappa + 1} \cap N \). Clearly every generator of a \((\kappa,\lambda)\)-extender is \( \leq \lambda \), and therefore (E2) follows. \( \square \)

Lemma \ref{lem:2.5} below will not be needed elsewhere in the present paper. It is included only as an illustration that the countable completeness of the extenders is a necessary requirement for wellfoundedness of the ultrapower.

**Lemma 2.5.** Assume \( \kappa \) is a measurable cardinal. Then there is \( \lambda > \kappa \) and \( E : [\lambda]^{<\omega} \to V_{\kappa + 2} \) such that

(a) \( E_s \) is a nonprincipal \( \kappa \)-complete ultrafilter on \( \kappa^s \),

(b) If \( s \subseteq t \) then \( \pi_{t,s} \) is a Rudin–Keisler reduction of \( E_s \) to \( E_t \) such that the ultrapower \( \prod_E V \) is ill-founded.
Proof. Let $j: V \rightarrow M$ be an elementary embedding with $\text{crit}(j) = \kappa$. Let $\lambda = \sup\{n \in \omega : j^n(\kappa)\}$. For all $X$, $n$ and $k$ we have the following.

$$j^{n+k}(X) \cap j^k(\kappa) = j^k(j^n(X) \cap \kappa) = j^k(X \cap \kappa) = j^k(X) \cap j^k(\kappa).$$

Because of this for a finite $s \subseteq \lambda$ the following defines a subset $E_s$ of $\kappa^{[s]}$:

$$X \in E_s \text{ if and only if } s \subseteq \lambda \Rightarrow \sup X \subseteq \lambda$$

where $n$ is such that $\max(s) < j^n(\kappa)$. Then $E_s$ is clearly a $\kappa$-complete ultrafilter.

The map $E, [\lambda]^{< \omega} \ni s \mapsto E_s \in V_{\kappa+1}$ satisfies (E1) and (E2). We can therefore form the direct limit of the ultrapowers $\prod_{E_s} V$, $s \in [\lambda]^{< \omega}$. However, this ultrapower is not well-founded. Let $s(n) = \{j^n(\kappa) : i < n\}$ and $M_n = \text{Ult}(V, E_{s(n)})$. Then $j_{s(n), s(n+1)}(\kappa) > \kappa$, and therefore in the direct limit we have a decreasing $\omega$-sequence of ordinals.

The problem with $E$ defined in Lemma 2.5 is that the ultraproducts are iterated the ‘wrong way.’ Let us consider this example a little more closely. If $s(n) = \{j^m(\kappa) : m < n\}$ then $E_{s(n)}$ is the set of all $X \subseteq \kappa^n$ such that

$$(\langle \xi_0(\xi_0) \xi_1 \xi_2 \rangle \ldots \langle \xi_{n-1} \xi_n \rangle, \xi_1, \ldots, \xi_n) \in X$$

Then the set of all decreasing $n$-tuples of ordinals $< \kappa$ belongs to $E_{s(n)}$ for each $n$.

3. The Extender Algebra

Assume $\delta$ is a Woodin cardinal and $\gamma \leq \delta$ is regular. Fix $\vec{E}$, a system of extenders in $V_\delta$ such that for every $A \subseteq V_\delta$ there is $\kappa < \delta$ such that for all $\lambda < \delta$ above $\kappa$ there is an $A, \lambda$-strong elementray embedding $j: V \rightarrow M$ with $\text{crit}(j) = \kappa$ coded by some $E \in \vec{E}$. Let $T_{\delta, \gamma}(\vec{E})$ be the deductive closure in $L_{\delta, \gamma}$ of all sentences of the form

$$\Psi(\vec{\varphi}, \kappa, \lambda): \bigvee_{\xi < \kappa} \varphi_\xi \leftrightarrow \bigvee_{\xi < \lambda} \varphi_\xi$$

for a sequence $\vec{\varphi} = \langle \varphi_\xi : \xi < \delta \rangle$ in $L_{\delta, \gamma}$, $\kappa$ such that $\vec{\varphi}$ reflects to $\kappa$, $\lambda \in [\kappa, \delta)$, and this is witnessed by an extender in $\vec{E}$.

**Remark 3.1.** This is not the standard definition of $T_{\delta, \gamma}(\vec{E})$. One usually considers $T_{\delta, \gamma}(\vec{E})$ to be the set of all sentences $\Psi(\vec{\varphi}, \kappa, \lambda)$ such that for some extender $E \in \vec{E}$ such that $\text{crit}(j_E) = \kappa$ we have $j_E(\vec{\varphi}) \upharpoonright \lambda = \vec{\varphi} \upharpoonright \lambda$. I don’t know whether there is a real difference between the two versions of the extender algebra and the only reason for using the alternate definition is that I initially got the definition wrong. Luckily, it all works out.

**Lemma 3.2.** If $\vec{\varphi} = \langle \varphi_\xi : \xi < \delta \rangle$ reflects to $\kappa$ then $\varphi_\xi \in V_\kappa$ for all $\xi < \kappa$.

**Proof.** If $j: V \rightarrow M$ is $\kappa, \lambda$-strong then $j$ moves all ordinals in the interval $[\kappa, \lambda)$. Hence if $\varphi$ reflects to $\kappa$ then $\varphi_\xi$ for $\xi < \kappa$ are fixed by elementary embeddings that move arbitrarily large ordinals in the interval $[\kappa, \delta)$. Therefore $\alpha(\xi) = \min\{\gamma : \varphi_\xi \in V_\gamma\}$ is less than $\kappa$ for every $\xi < \kappa$.

In the following it may be worth emphasizing that $x$ is assumed to belong to the same inner model as $\vec{E}$ (cf. Theorem 3.3).

**Lemma 3.3.** For every real $x$ in $L[\vec{E}]$ we have $x \models T_{\delta, \gamma}(\vec{E})$. In particular, $T_{\delta, \gamma}(\vec{E})$ is a consistent theory.
Proof. Fix a sequence $\vec{\varphi}$ that reflects to $\kappa$ and this is witnessed by extenders in $\vec{E}$. We need to check $\Psi(\vec{\varphi}, \kappa, \lambda)$ for all $\lambda > \kappa$. We may assume $x \models \varphi_\xi$ for some $\xi$, since otherwise $\Psi(\vec{\varphi}, \kappa, \lambda)$ vacuously holds for all $\lambda$. Pick an extender $E \in \vec{E}$ such that $j_E$ is $(\vec{\varphi}, \lambda)$-strong for some $\lambda > \xi$. Since $x$ is a real it is not moved by any $j$ and therefore by elementarity we have $x \models \varphi_\eta$ for some $\eta < \kappa$ and the conclusion follows.

Note that if a cardinal $\gamma$ is $\delta$-strong (or equivalently, in the terminology introduced above, if $\delta$ reflects to $\gamma$) and this is witnessed by extenders in $\vec{E}$ then $\gamma \not\models T_{\delta, \gamma}(\vec{E})$, as can be seen by taking $\varphi_\xi(x)$ to be $a_\xi$ if $x \in x$ and $\not\models_\xi$ if $\xi \notin x$. However, if $\text{crit}(E) = \gamma$ then by the minimality of $\gamma$ and elementarity we have $\gamma \models j_E(T_{\delta, \gamma}(\vec{E}))$.

The extender algebra with $\gamma$ generators corresponding to $\vec{E}$ is the algebra

$$W_{\delta, \gamma}(\vec{E}) = B_{\delta, \gamma}/T_{\delta, \gamma}(\vec{E}).$$

Again, the important instances of the extender algebra are given by $\gamma = \omega$ and $\gamma = \delta$ but for convenience we develop theory of $W_{\delta, \gamma}(\vec{E})$ for an arbitrary $\gamma \leq \delta$.

Lemma 3.4. If $\delta$ is a Woodin cardinal and $\vec{E}$ is a system of extenders witnessing its Woodinness, then $W_{\delta, \gamma}$ has the $\delta$-chain condition and is therefore complete.

Proof. Assume the contrary and let $A = \{[\varphi_\xi] \mid \xi < \delta\}$ be an antichain. Let $\kappa$ be the minimal cardinal such that $A$ reflects to $\kappa$. Therefore

$$T_{\delta, \gamma}(\vec{E}) \not\models \bigvee_{\xi < \kappa} \varphi_\xi \rightarrow \bigvee_{\xi < \lambda} \varphi_\xi$$

for $\lambda > \kappa$, contradicting the assumption that $A$ was an antichain.

The completeness of $W_{\delta, \gamma}$ now follows by Lemma 1.3.

The following two facts will not be needed in this note. Ketchersid and Zoble ([7]) proved a converse of Lemma 3.4 that if $W_{\delta, \gamma}(\vec{E})$ has $\delta$-cc then $\delta$ is Woodin. Hjorth proved that $W_{\delta, \omega}(\vec{E}) \times W_{\delta, \omega}(\vec{E})$ has the $\delta$-chain condition ([8] Lemma 3.6)). It is not known whether $W_{\delta, \omega}(\vec{E}) \times B$ has the $\delta$-chain condition whenever $B$ has the $\delta$-chain condition.

The following fact (not needed in proofs of the main results of this note) was pointed out to me by Paul Larson.

Lemma 3.5. Assume $\gamma$ is any cardinal less than the least critical point of each extender in $\vec{E}$ and $\mathbb{P}$ is a forcing notion of cardinality $\gamma$. Then in $W_{\delta, \gamma}(\vec{E})$ there is a condition $p$ that forces that $\mathbb{P}$ is a regular subordering of $W_{\delta, \gamma}(\vec{E})$.

The first proof of Lemma 3.5. Pick a bijection between $\mathbb{P}$ and $\gamma$ and hence identify $\mathbb{P}$ with $\langle \gamma, \leq_{\mathbb{P}} \rangle$ for some partial ordering $\leq_{\mathbb{P}}$ on $\gamma$. Let the sentence $\varphi$ be the conjunction of axioms expressing the following.

(a) The order on $\mathbb{P}$: $a_\xi \rightarrow a_\eta$ whenever $\xi \leq_{\mathbb{P}} \eta$.
(b) The incompatibility relation on $\mathbb{P}$: $\bigwedge_{\xi < \gamma} (\neg (a_\xi \rightarrow a_\eta) \vee \neg (a_\xi \rightarrow a_\zeta))$, if $\xi \not\leq_{\mathbb{P}} \xi$.
(c) For every maximal antichain $A$ of $\mathbb{P}$, $\bigvee_{\xi \in A} a_\xi$.

Then $\varphi$ is in $L_{\delta, \gamma}$ and since its size is below the least critical point of an extender in $\vec{E}$, it is consistent with $T_{\delta, \gamma}(\vec{E})$. Therefore we may take $p$ to be (the equivalence class of) $\varphi$. 

□
3.1. **Iteration trees.** We say that $T = (\zeta, \leq_T)$ is a *tree order* on an ordinal $\zeta$ if

1. $T = (\zeta, \leq_T)$ is a tree with root $0$,
2. $\leq_T$ is coarser than $\leq$,
3. every successor ordinal is a successor in $\leq_T$,
4. if $\xi \leq \zeta$ is a limit ordinal then the set of $\leq_T$-predecessors of $\xi$ is cofinal in $\xi$.

Consider a transitive model $M$ of a large enough fragment of ZFC and a system $\vec{E}$ of extenders in $M$. We allow $M$ to be a proper class. As a matter of fact, it is typically going to be a proper class. Nevertheless, we omit the (straightforward) nuisances involved in formalization of the notion of an iteration tree in ZFC.

An $\vec{E}$-iteration tree is a structure consisting of $(T, M_\eta, E_\xi \mid \eta \leq \zeta, \xi < \zeta)$, together with a commuting system of elementary embeddings $j_{\xi\eta} : M_\xi \rightarrow M_\eta$ for $\xi \leq_T \eta$ such that

1. $\leq_T$ is a tree order on $\zeta$,
2. $E_\xi$ is an extender in $\vec{E}M_\xi$,
3. if $\xi = \text{crit}(E_\xi)$ then the immediate $\leq_T$-predecessor of $\xi + 1$ is the least ordinal $\eta$ such that $M_\eta \cap V_{\xi+1} = M_\xi \cap V_{\xi+1}$,
4. if $\xi + 1$ is the immediate $\leq_T$ successor of $\eta$ then $M_{\xi+1} = \text{Ult}(M_\eta, E_\xi)$, hence $j_{\eta,\xi+1} = j_{E_\xi}$ as computed with respect to $M_\eta$,
5. if $\xi$ is a limit ordinal then $M_\xi$ is the direct limit of $M_\eta$, for $\eta <_{T} \xi$, and
6. each $M_\xi$ is well-founded.

It is usually not required that an iteration tree satisfies condition (7), and the iteration trees satisfying this condition are called *normal* iteration trees. If $E_\xi$ was always applied to $M_\xi$, then we would have a linear iteration that is moreover internal—i.e., each extender used in the construction belongs to the model to which it is applied. The wellfoundedness of such an iteration follows from a rather mild additional condition about the extenders. On the other hand, the choice of condition (7) is behind the power of the iteration trees (see the proof of Theorem 3.9). This condition also prevents the obstacle to wellfoundedness of the iteration exposed in Lemma 2.5 (see Lemma 3.3). It will be important that the extenders in $E$ have the property (E2), that for every generator $\xi$ of $E$ we have $\text{Ult}(V, E) \cap V_\xi = V_\xi$, or in other words, that every extender is $\xi$-strong for each of its generators $\xi$.

**Lemma 3.6.** Assume $(T, M_\eta, E_\xi \mid \eta \leq \zeta, \xi < \zeta)$, is an iteration tree such that every extender used in the construction of $T$ satisfies (E2). Assume $E_0$ and $E_1$ are extenders used along the same branch of $T$ and $E_1$ was used after $E_0$. Then $\kappa = \text{crit}(E_1)$ is greater than the supremum of all generators of $E_0$.

**Proof.** Assume the contrary. Let us first consider the case when $E_1$ was applied to $M_\beta = \text{Ult}(M_\alpha, E_0)$ for some $\alpha$ along the branch. By our assumptions $E_0$ is $\xi$-strong and therefore $M_\beta \cap V_\xi = M_\alpha \cap V_\xi$. Therefore, since $E_1$ could be applied to $M_\beta$ it could be applied to $M_\alpha$ as well, contradicting the rule for constructing an iteration tree.

We may therefore assume $E_1$ was applied to a model $M_\eta$ that is a direct limit of other models on the branch, including $\text{Ult}(M_\alpha, E_0)$ for some $\alpha$. The above an an induction argument show that $\text{crit}(j_{E_1})$ is greater than any generator of any extender used in the construction of this branch, including the generators of $E_0$. □
Note that an iteration tree \((T, M_\eta, E_\xi \mid \eta \leq \omega_1, \xi < \omega_1)\) has a branch of length \(\omega_1\) by (4). In Lemma 3.7 below, and elsewhere, we assume that all critical points of elementary embeddings \(j_\eta\) used in building an iteration tree are countable ordinals. Lemma 3.7 is an attempt to extract one of the key ideas from the proof of Theorem 3.9. It is really a version of the comparison lemma for mice due to Steel (see [18]). Preprint [17] was very helpful during the extraction of this lemma.

Lemma 3.7. Assume \((T, M_\eta, E_\xi \mid \eta \leq \omega_1, \xi < \omega_1)\) is an iteration tree and \(b \subseteq \omega_1\) is its cofinal branch. Assume \(H \prec H^{(2^{\omega_1})^+}\) is countable and it contains the iteration tree. Let \(\bar{H}\) be its transitive collapse and let \(\pi : H \to \bar{H}\) be the collapsing map.

With \(\alpha = H \cap \omega_1\) we have the following.

1. \(\alpha \in b\) and \(M_\alpha\) is the direct limit of \(M_\xi, \xi \in b \cap \alpha\).
2. \(M_\alpha \cap H\) is the direct limit of \(\langle M_\xi \cap H, \xi \in b \cap \alpha\rangle\).
3. \(\pi^{-1} J_\alpha \cap H\) agree on \(M_\alpha \cap \bar{H}\) and in particular
   a. \(\pi^{-1}(M_\alpha \cap \bar{H})\) is included in \(M_\omega_1 \cap H\), and
   b. \(\text{crit}(\bar{J}_\alpha) \geq \alpha\).
4. \(\pi^{-1}(M_\alpha \cap \bar{H})\) agrees with \(j_{E_\alpha}\) on \(M_\alpha \cap V_\gamma\), where \(\gamma\) is the strength of the extender \(E_\alpha\) such that \(\text{Ult}(M_\alpha, E_\alpha)\) is the successor of \(M_\alpha\) along \(b\).
5. \(M_\omega_1 \cap V_{\alpha+1} \cap H = M_\alpha \cap \cap V_{\alpha+1}\cap \bar{H}\).

Proof. Clause (11) follows by (4) and (9). By elementarity in \(H\) it holds that \(M_\omega_1\) is the direct limit of \(M_\xi\), for \(\xi \in b\), and therefore (12) follows. By applying this and (11), (13) follows as well. Let \(\gamma\) be as in (14). We have \(J_\omega_1 = j_{E_\alpha} \circ i\) for some \(i\). By Lemma 3.6 we have \(\text{crit}(\bar{J}_i) \geq \gamma\) and (14) follows.

Clause (13) is a consequence of (13). \(\square\)

3.2. The iteration game. In what follows elements of an iteration tree will be proper classes instead of sets. The arguments can be formalized within ZFC by using the standard reflection and compactness devices. We leave out the well-known details. We define a two-player game of transfinite length \(\zeta\) in which the players build an iteration tree, starting from a model \(M\) and a system of extenders \(\vec{E}\) in \(M\). Let \(M_0 = M\). In his \(\alpha\)-th move player \(I\) picks an extender \(E_\alpha\) in \(\vec{E}M_\alpha\) such that the strength of \(E_\alpha\) is greater than the strength of \(E_\beta\) for all \(\beta < \alpha\). Then the referee finds the minimal \(\beta \leq \alpha\) such that \((\text{writing} \text{crit}(E) \text{for the critical point of the elementary embedding } j_E) V_{\text{crit}(E_\alpha) + 1} \cap M_\beta = V_{\text{crit}(E_\alpha) + 1} \cap M_\alpha\). Hence \(M_\beta\) is the earliest model in the iteration to which \(E_\alpha\) can be applied. Referee then defines

\[
M_{\alpha+1} = \text{Ult}(M_\beta, E_\alpha),
\]

with \(j_{\beta+1}\) being the corresponding embedding. The referee also extends the tree order \(T\) by adding \(\alpha + 1\) as an immediate successor to \(\beta\). At a limit stage \(\alpha\)-th player \(I\) picks a maximal branch \(\langle M_\xi \mid \xi \in b\rangle\) of \(T\) such that \(b\) is cofinal in \(\alpha\) and lets \(M_\alpha\) be the direct limit of the system \(\langle M_\xi, j_\xi \eta \mid \xi \in b\rangle\). If \(M_\alpha\) is well-founded then we identify it with its transitive collapse.

The first player who disobeys the rules loses. Assume both players obeyed the rules of the iteration game. If \(M_\alpha\) is ill-founded then the game is over and \(I\) wins. If all \(M_\alpha\) are well-founded, then \(I\) wins, and otherwise \(I\) wins.

For definiteness, we call the above game the \((\vec{E}, \zeta)\)-iteration game in \(M\). We shall suppress \(\vec{E}\) and \(M\) whenever they are clear from the context. An \((\vec{E}, \zeta)\)-iteration strategy is a winning strategy for \(I\) in the iteration game of length \(\zeta\). A pair \((M, \vec{E})\) is \((\vec{E}, \zeta)\)-iterable if \(I\) has a \(\vec{E}, \zeta\)-winning strategy. An \((\vec{E}, \zeta)\)-iteration of \(M\) is an
elementary embedding $j_{\mathcal{O}} : M \rightarrow M_\zeta$ extracted from an $\vec{E}$-iteration tree on $\zeta$. A model is **fully iterable** if it is $(\vec{E}, \zeta)$-iterable for every ordinal $\zeta$.

**Theorem 3.8** (Martin–Steel, [12]). Assume there exist $n$ Woodin cardinals and a measurable above them all. For every $a \in \mathbb{R}$ and every $m \leq n$ there exists an inner model containing $a$ and $m$ Woodin cardinals, denoted by $M_m(a)$. It is $(\omega_1 + 1)$-iterable and its Woodin cardinals are countable ordinals in $V$.

If there are class many Woodin cardinals then every $M_n(a)$ is fully iterable in every forcing extension. \hfill $\square$

The assumption that there are class many Woodin cardinals is not optimal. For the case when $n = 1$ see the proof of Theorem 4.4 below.

3.3. **Genericity iterations.** Assuming $M$ is sufficiently iterable, we may talk about iteration strategies for player I. These are the strategies that, when played against II’s winning strategy, produce models (necessarily well-founded) with desirable properties.

**Theorem 3.9.** Assume $(M, \vec{E})$ is $(\omega_1 + 1)$-iterable and $\vec{E}$ witnesses a countable ordinal $\delta$ is a Woodin cardinal in $M$. Then for every $x \subseteq \omega$ there is a (well-founded) countable iteration $j : M \rightarrow M^*$ such that $x \in j(\mathcal{W}_{\delta, \omega}(\vec{E}))$-generic over $M^*$.

**Proof.** By Lemma 3.4 and Lemma 3.3 we only need to assure $x \models j(T_{\delta, \omega}(\vec{E}))$. Define a strategy for player I for building an $\vec{E}$-iteration tree with $M_0 = M$ as follows. Assume $\langle T, M_\xi, E_\xi \mid \xi \leq \alpha \rangle$ has been constructed. If $x \models j_{\alpha}(T_{\delta, \omega}(\vec{E}))$ then $j_{\alpha} : M \rightarrow M_\alpha$ is the required iteration and we stop. Otherwise, let $\lambda$ be the minimal cardinal such that there are $\vec{\varphi}, \kappa$, and a $\vec{\varphi}, \lambda$-strong extender $E \in j_{\alpha}(\vec{E})$ with $\text{crit}(E) = \kappa$ such that $x \models \exists j(\vec{\varphi}, \kappa, \lambda)$. Fix such $\vec{\varphi}, \lambda$ and $E$. We have $\forall \lambda < j_{\alpha}(\delta)$. Then let player I play $E_\alpha = E$. Note that $j_{E_\alpha}(\kappa) \geq \lambda$.

This describes the iteration strategy for I. We claim that if II responds with his winning strategy then the process of building the iteration tree terminates at some countable stage. Assume otherwise. Let $\langle T, M_\xi, E_\xi \mid \xi < \omega_1 \rangle$ be the resulting iteration tree and let $b \subseteq \omega_1$ be its cofinal branch such that $M_{\omega_1}$, the direct limit of $\langle M_\xi \mid \xi \in b \rangle$, is well-founded.

Fix a countable $H < H^{(2n_1)+}$ containing everything relevant. Let $\alpha = H \cap \omega_1$ and let $\vec{H}$ be the transitive collapse of $H$, with $\pi^{-1} : H \rightarrow H(\theta)$ the inverse of the collapsing map. Since the iteration did not stop at stage $\alpha$, we can consider the extender $E_\alpha$ chosen in $M_\alpha$. By the choice of the iteration, $E_\alpha$ is $\vec{\varphi}, \lambda$-strong for some $\vec{\varphi}$ and $\lambda$ such that $\vec{\varphi}$ reflects to $\alpha$ but $x \models \forall \xi < \alpha \exists \xi, \vec{\varphi}_\xi$, and $x \models \forall \xi < \alpha \exists \xi, \vec{\varphi}_\xi$.

By Lemma 3.2 we have $\varphi_\eta \in M_\alpha \cap V_\alpha$ for all $\eta < \alpha$, hence the formula $\forall \eta < \alpha \varphi_\eta$ belongs to $V_{\alpha+1}$. By (11) of Lemma 3.7 there exist $\xi < T \alpha$ and $\vec{\psi} \in M_\xi$ such that $\vec{\varphi} = j_{\xi}(\vec{\psi})$. By (15) of Lemma 3.7 we have that $\forall \eta < \alpha \varphi_\eta$ belongs to $\vec{H}$ and $\varphi_\eta$ belongs to $\vec{H} \cap M_\alpha$ for all $\eta < \alpha$.

Let $E_{\alpha'}$ be the extender applied to $M_\alpha$, in order so that $\text{Ult}(M_\alpha, E_{\alpha'})$ is its successor on the branch $b$. We don’t necessarily have $E_\alpha = E_{\alpha'}$. However, the strength of $E_{\alpha'}$ is not smaller than the strength of $E_\alpha$ by the minimality of $\lambda$ chosen at the $\alpha$-th stage of the construction of the iteration tree. Therefore by Lemma 3.7 we have that $\pi^{-1}$ and $j_{\omega_1}$ agree on $M_\alpha \cap \vec{H}$ and that $j_E$ and $j_{\omega_1}$ agree on $M_\alpha \cap V_\lambda$. Since $j_{\omega_1}$ and $j_E$ agree on $M_\alpha \cap V_\lambda$, we have that $j_{\omega_1}(\vec{\varphi})$
implies $\bigvee_{\xi<\lambda} \varphi_\xi$, hence $x \models j_{\text{new}}(\vec{\psi})$ holds in $H$. Thus for some $\xi \in H \cap \omega_1$ we have $x \models \varphi_\xi$ and therefore $x \models \bigvee_{\xi<\alpha} \varphi_\xi$, a contradiction. \qed

Two extensions of Theorem 3.9 in different directions are in order (their obvious common generalization is omitted).

**Theorem 3.10.** Assume $(M, \vec{E})$ is $(\omega_1+1)$-iterable and $\vec{E}$ witnesses a countable ordinal $\delta$ is a Woodin cardinal in $M$. Then for every $x \subseteq \omega_1$ there is a (well-founded) countable iteration $j: M \to M^*$ such that $x \cap j(\delta)$ is $j(\mathcal{W}_{\delta,\delta}(\vec{E}))$-generic over $M^*$.

**Proof.** The strategy for player I is identical to the strategy used in the proof of Theorem 3.9, and the proof of the latter shows that the iteration terminates at some countable stage at which $x \cap j(\delta) \models j(\mathcal{T}_{\delta,\delta}(\vec{E}))$. \qed

Assume an extender $E$ and a forcing $\mathbb{P}$ are such that for some $\kappa$ we have $\text{crit}(j_E) > \kappa$ and $\mathbb{P}$ is in $V_\kappa$. Then $E$ still defines an extender in the extension by $\mathbb{P}$ and this extender is $\lambda$-strong for every $\lambda$ for which $E$ is $\lambda$-strong. This is essentially a consequence of the Levy-Solovay result that a measurable cardinal cannot be destroyed by a small forcing \cite{4}. In the following lemma and elsewhere we shall slightly abuse the notation and denote this extender by $E$.

**Theorem 3.11.** Assume $(M, \vec{E})$ is $(\omega_1+1)$-iterable and $\vec{E}$ witnesses a countable ordinal $\delta$ is a Woodin cardinal in $M$. Assume moreover $\kappa < \min\{\text{crit}(j_E): E \in \vec{E}\}$ and $\mathbb{P} \in V_\kappa \cap M$ is a forcing notion. If $G \subseteq \mathbb{P}$ in $V$ is $M$-generic then for every $x \subseteq \omega$ there is a (well-founded) countable iteration $j: M[G] \to M^*[G]$ such that $x$ is $j(\mathcal{W}_{\delta,\omega}(\vec{E}))$-generic over $M^*[G]$.

**Proof.** By the above discussion $\delta$ is still a Woodin cardinal in $M[G]$ as witnessed by $\vec{E}$. The proof is similar to the proof of Theorem 3.9. In his strategy, player I computes $\mathbb{P}(\vec{\gamma}, \kappa, \lambda)$ in $M[G]$ instead of $M$. Since $G \subseteq \mathbb{P}$ in $V$, this describes an iteration strategy of I in $V$. Thus II can respond to it by using his winning iteration strategy and the other details of the proof are identical.

As a matter of fact, a simple proof shows that the full iterability of $M$ implies the full iterability of $M[G]$ (essentially using the same strategy). The point is that, since $\mathbb{P}$ is small, the ultrapowers of $M$ lift to ultrapowers of $M[G]$. Therefore this is a consequence of Theorem 3.9. \qed

**The second proof of Lemma 3.9.** We now provide a different proof that if $\gamma$ is any cardinal less than the least critical point of each extender in $\vec{E}$ and $\mathbb{P}$ is a forcing notion of cardinality $\gamma$ then in $\mathcal{W}_{\delta,\gamma}(\vec{E})$ there is a condition $p$ that forces that $\mathbb{P}$ is a regular subordering of $\mathcal{W}_{\delta,\gamma}(\vec{E})$. The present proof will also show that there exist condition $q \in \mathcal{W}_{\delta,\gamma}(\vec{E})$ and condition $r \in \mathbb{P}$ that force that $\mathbb{P}$ is forcing-equivalent to $\mathcal{W}_{\delta,\gamma}(\vec{E})$. Assume $G \subseteq \mathbb{P}$ is generic over $M$. We may assume $\mathbb{P} = (\gamma, <_E)$ for some ordering $<_E$ on $\gamma$. Since $\delta$ is a countable ordinal and $\gamma < \delta$, by using Theorem 3.9 we can find an iteration $j: M \to M^*$ such that $G$ is $j(\mathcal{W}_{\delta,\gamma}(\vec{E}))$ generic over $M^*$. By our assumption $\gamma$ is smaller than the critical point of $j$ and therefore $j(\mathbb{P}) = \mathbb{P}$. Hence in $M^*$ there is $q_0 \in j(\mathcal{W}_{\delta,\gamma}(\vec{E}))$ and $r \in \mathbb{P}$ that forces $\mathbb{P}$ and $j(\mathcal{W}_{\delta,\gamma}(\vec{E}))$ are forcing equivalent. By the elementarity, this is true in $M$ for $\mathbb{P}$ and $\mathcal{W}_{\delta,\gamma}(\vec{E})$. \qed
Lemma 3.7. In [16] it was proved that if the forcing $P$ is true when $P$ a countable generic iteration. They also pointed out that this is not necessarily the case. 

Theorem 3.13. Assume $P$ is a forcing notion of cardinality $\kappa$ and $x$ is a $P$-name for a real. If $(M, E)$ is $(\kappa^+ + 1)$-iterable and $E$ witnesses a countable ordinal $\delta$ is a Woodin cardinal in $M$. Then there is a (well-founded) iteration $j : M \rightarrow M^*$ of length $< \kappa^+$ such that $P$ forces $x$ is $j(\mathcal{W}_{\delta, \omega}(E))$-generic over $M^*$. 

A proof of this theorem has the similar structure as the proof of Theorem 1.8. By Theorem 3.12. Showing that the constructions terminates before the $\kappa^+$-th stage requires taking the elementary submodel $H$ of cardinality $\kappa$ and an appropriate analogue of Lemma 3.7. In [16] it was proved that if the forcing $P$ is proper then one can find a countable genericity iteration. They also pointed out that this is not necessarily true when $P$ is only assumed to be semiproper.

The following is an analogue of Theorem 3.9 for an arbitrary set of ordinals and it will be used in the proof of Theorem 4.1. A more useful version (Theorem 4.5) requires a stronger large cardinal assumption.

Theorem 3.13. Assume $(M, E)$ is fully iterable and $E$ witnesses a countable ordinal $\delta$ is a Woodin cardinal in $M$. Then for every set of ordinals $x$ there is a (well-founded) iteration $j : M \rightarrow M^*$ of length $< |x|^+$ such that $x$ is $j(\mathcal{W}_{\delta, \omega}(E))$-generic over $M^*$.

Proof. Let $E_0 \in E$ be an extender with minimal strength. Iterate $E_0$ to obtain an iteration $j_0 : M \rightarrow M_0$ such that $x \subseteq j(\delta)$. Since this is a linear iteration, it is well-founded. By the minimality, the strength of every extender in $j(\delta)$ is greater than the strength of all iterates of $E_0$ used in $j_0$. Therefore for every iteration $i : M_0 \rightarrow M^*$ of $(M_0, j_0(E))$ we have that $i \circ j_0 : M \rightarrow M^*$ is an iteration of $(M, E)$, and therefore $(M_0, j_0(E))$ is fully iterable.

From this point on the proof is analogous to the proof of Theorem 3.9. Define a strategy for player I for building an iteration tree starting with $M_0$ as follows. Assume $(T, M, E, \xi | \xi \leq \alpha)$ has been constructed. If $x \models (j_{\alpha_0} \circ j_0)(T_{\delta, \omega}(E))$ then $j_{\alpha_0} \circ j_0 : M \rightarrow M_0$ is the required iteration and we stop. Otherwise, let $\lambda$ be the minimal cardinal such that there are $\bar{\varphi}$, $\kappa$, and a $\bar{\varphi}$, $\lambda$-strong extender $E \in (j_{\alpha_0} \circ j)(E)$ with $\text{crit}(E) = \kappa$ such that $x \not\models \Psi(\bar{\varphi}, \kappa, \lambda)$. Fix such $\bar{\varphi}$, $\kappa$, and $E$. We have $x \not\models \bigvee_{\xi < \kappa} \varphi_\xi$ and $x \models \bigvee_{\xi < \lambda} \varphi_\lambda$. and note that $\lambda < (j_{\alpha_0} \circ j_0)(\delta)$. Then let player I play $E_\alpha = E$. Note that $(j_{\alpha_0} \circ j_0)(\kappa) \geq \lambda$.

This describes the iteration strategy for I. A proof that if II responds with his winning strategy then the process of building the iteration tree terminates at some stage before $|x|^+$ is identical to the proof of Theorem 3.9 using the variant of Lemma 3.7 for an iteration tree of height $|x|^+$.

4. Absoluteness

4.1. Absoluteness in $L(\mathbb{R})$. The following result and its proof are a prototype for the main result of this note, Theorem 1.8.

Theorem 4.1. Assume $M_1(a)$ is fully iterable in all forcing extensions for all $a \in \mathbb{R}$. Then all $\Sigma_2^1$ statements are forcing absolute.

Proof. A $\Sigma_2^1$ statement $\varphi(a)$ with parameter $a \in \mathbb{R}$ is of the form $(\exists x) \psi(x, a)$ where $\psi$ is $\Pi_1^1$. To $\varphi$ we associate a sentence $\varphi^*$ of $M_1(a)$ (see Theorem 3.8) stating that
there exists a forcing notion $P$ forcing $\varphi$. We claim that $\varphi$ holds in $V$ if and only if $\varphi^*$ holds in $M_1(a)$. This will suffice since the definition of $M_1(a)$ is not changed by forcing. If $\varphi^*$ holds in $M_1(a)$ then since $M_1(a) \cap V_{\delta + 1}$ is countable we can find an $M_1(a)$-generic filter $G \subseteq P$. If $M_1(a)[G] \models \psi(x, a)$ then by Shoenfield’s absoluteness theorem $\psi(x, a)$ holds in $V$. For the converse implication, assume $\psi(x, a)$ holds in some forcing extension of $V$ for some $x \in \mathbb{R}$. Since $M_1(a)$ is fully iterable in all forcing extensions, apply Theorem 3.9 to find a countable iteration $j: M_1(a) \rightarrow M^*$ such that $x$ is generic over $M^*$. Then (again using Shoenfield) $M^* \models \varphi^*$, and by elementarity $M_1(a) \models \varphi^*$.

The assumptions of Theorem 4.1 are far from optimal. By a result of Martin and Solovay (11), if $\kappa$ is a measurable cardinal then $\Sigma^1_3$ sentences are forcing absolute for forcing notions in $V_\kappa$. As a matter of fact, all $\Sigma^1_3$ sentences are absolute between all forcing extensions of $V$ if and only if all sets have sharps (Martin–Steel, Woodin; see e.g., [18] for terminology). The important fact about Theorem 4.1 and its extension, Theorem 4.2 below, is that its proof is susceptible to far-reaching generalizations.

**Theorem 4.2.** Assume $M_n(a)$ is fully iterable in all forcing extensions for every $a \in \mathbb{R}$. Then all $\Sigma^1_{n+2}$ statements are forcing absolute.

**Proof.** We first prove the case $n = 2$. A $\Sigma^1_1$ sentence $\varphi$ has the form $(\exists x)(\forall y)\psi(x, y, a)$ for some real parameter $a$ and a $\Pi^1_2$ formula $\psi$. Let $M_2(a)$ be the minimal model for two Woodin cardinals containing $a$ fully iterable in all forcing extensions (Theorem 3.8), and let $\delta_0 < \delta_1$ be its Woodin cardinals.

To $\varphi$ associate a sentence $\varphi^*$ stating that there is a forcing $P$ in $V_{\delta_0 + 1}$ and a $P$-name $\hat{x}$ for a real such that for every forcing $Q \in V_{\delta_1 + 1}$ and a $\hat{Q}$-name $\hat{y}$ for a real we have

$$\Vdash_{P \times \hat{Q}} \psi(\hat{x}, \hat{y}, a).$$

We claim that $\varphi$ holds in $V$ if and only if $\varphi^*$ holds in $M_2(a)$.

Assume $M_2(a) \models \varphi^*$. Find $G \subseteq P$ generic over $M_2(a)$ and let $x = \text{int}_G(\hat{x})$. Let $y$ be any real. Let $E_1$ be the system of extenders witnessing $\delta_1$ is Woodin in $M_2(a)$. We may assume $\min(\text{crit}(j_E)): E \in E_1 > \delta_0$. Using the full iterability of $M_2(a)$ and Theorem 3.9 find an iteration $j: M_2(a)[G] \rightarrow M^*[G]$ such that $y$ is $j(W_{\delta, \omega}(E_1))$-generic over $M^*[G]$. Then by elementarity we have $M^*[G][y] \models \psi(x, y, a)$ and by Shoenfield’s absoluteness theorem $\psi(x, y, a)$ holds. Since $y$ was arbitrary, we have proved $\varphi$ holds in $V$.

Now assume $\varphi$ holds in $V$ and let $x \in \mathcal{P}(\omega)$ be such that $\langle \forall y \rangle \psi(x, y, a)$. By Theorem 3.9 find an iteration $j_0: M_2(a) \rightarrow M^*$ such that $x$ is $j_0(W_{\delta, \omega}(E_0))$-generic over $M^*$. Fix $y \in \mathcal{P}(\omega)$. By Theorem 4.1 there is an iteration $j_1: M^*[x] \rightarrow M^{**}[x]$ such that $y$ is $(j_1 \circ j_0)(W_{\delta, \omega}(E_1))$-generic over $M^{**}[x]$. Then $V \models \varphi(x, y, a)$ and by the Shoenfield’s absoluteness theorem $M^{**}[x] \models \varphi(x, y, a)$. Since $y$ was arbitrary, this shows $\varphi^*$ holds in $M_2(a)$.

This concludes proof of the theorem in the case when we have two Woodin cardinals. In general case, to a $\Sigma^1_{n+2}$ formula $(\exists x_1)(\forall x_2) \ldots (\varphi(x_1, x_2, \ldots, x_n, a)$ with $\psi$ being $\Sigma^1_1$ one associates a formula $\varphi^*$ of $M_n(a)$ stating

$$(\exists P_1 \in V_{\delta_0 + 1})(\exists x_1)(\forall P_2 \in V_{\delta_1 + 1})(\forall x_2) \ldots \Vdash_{P_1 \times P_2 \times \ldots \times P_n} \psi(\hat{x}_1, \ldots, x_n, a)$$

and proves that $V \models \varphi$ is equivalent to $M_n(a) \models \varphi^*$ as above. \qed
The iterability assumptions of Theorem 4.8 and Theorem 4.2 may sound awkward. However, (1) implies (2) of Theorem 4.4 and the following example show that some assumption of this sort is necessary.

**Remark 4.3.** The following was pointed out by Menachem Magidor. In Theorem 4.2 it is not sufficient to assume that there are $n$ Woodin cardinals and a measurable above. This is because if $V = L[\vec{E}]$ for some system of extenders then some forcing extension of $V$ is of the form $L[x]$, for a real $x$. Such an extension satisfies the projective statement ‘there exists a real $x$ and a $\Delta^1_2(x)$ well-ordering of $\mathbb{R}$.’

Therefore the existence of $M_\alpha(a)$ is a strictly weaker assumption than its iterability in all forcing extensions, needed in Theorem 4.1. We write $M_\alpha$ for $M_\alpha(0)$. Recall that a set of ordinals $X$ has a sharp if there is a nontrivial elementary embedding of $L[X]$ into itself.

**Theorem 4.4.** The following are equivalent.

1. $M_1$ exists and it is fully iterable in all forcing extensions.
2. Every set has a sharp and there is a proper class model with a Woodin cardinal.

**Proof.** We only prove the implication from (1) to (2). We only need to show that the full iterability of $M_1$ implies that every set has a sharp. Assume the contrary, and let $X$ be a set without a sharp. Then the Covering Lemma holds in $L[X]$, hence the successor $\lambda^+$ of some singular cardinal $\lambda$ such that $X \subseteq V_\lambda$ is correctly computed in $L[X]$. By Theorem 3.13 there is an iteration $j : M_1 \rightarrow M^*$ such that $X$ is generic over $M^*$ and $j(\delta) < \lambda^+$. Then $M^*[X]$ correctly computes $\lambda^+$, since it includes $L[X]$. On the other hand, by the chain condition of the extender algebra $j(\delta)$ remains a cardinal in $M^*[X]$. A contradiction.

I learned the above proof that (1) implies (2) from Ralf Schindler. This proof also shows that if $M_{n+1}$ is fully iterable then for every set $X$ there is an inner model including $L[X]$ with $n$ Woodin cardinals that has a sharp.

### 4.2. Genericty iterations for subsets of $\omega_1$

We finally turn to applications of the algebra $W_{\delta,\delta}$ with $\delta$ generators. The assumption of the existence of an $(\omega_1 + 1)$-iterable model with a measurable Woodin cardinal is presently beyond reach of the inner model theory.

**Theorem 4.5.** Assume $(M, \vec{E})$ is $(\omega_1 + 1)$-iterable and $\vec{E}$ witnesses a countable ordinal $\delta$ is a measurable Woodin cardinal in $M$. Then for every $x \subseteq \omega_1$ there is a (well-founded) $\omega_1$-iteration $j : M \rightarrow M^*$ such that $x$ is $j(W_{\delta,\delta}(\vec{E}))-generic over $M^*$.

**Proof.** This proof is an extension of the proof of Theorem 3.10. We need to assure $x \models j(T_{\delta,\delta}(\vec{E}))$ for an $\omega_1$-iteration $j$. Define a strategy of player I for building an iteration tree starting with $M_0 = M$ as follows. Assume $\langle T, M_\eta, E_\xi \mid \eta \leq \alpha, \xi < \alpha \rangle$ has been constructed.

(a) Assume there is a sequence $\vec{\varphi}$ in $M_\alpha$ that reflects to some $\kappa$ but $x \not\models \Psi(\vec{\varphi}, \kappa, \lambda)$ for some $\lambda$ satisfying $\lambda < j_{00}(\delta)$ and $\lambda > \sup_{\xi < \alpha}(\lambda_{E_\xi})$. Choose the minimal $\lambda$ with this property, fix the appropriate $\vec{\varphi}$ and $\kappa$ so that $x \not\models \Psi(\vec{\varphi}, \kappa, \lambda)$. Then let $E_\alpha$ be a $(\vec{\varphi}, \lambda)$-strong extender in $M_\alpha$ such that $\text{crit}(E_\alpha) = \kappa$. Note that $j_{E_\alpha}(\kappa) \geq \lambda$. 


(b) Now assume $\alpha$ is countable and $x \models j_{0\alpha}(T_{3\delta}(E))$. Since $\delta$ is a measurable Woodin cardinal, use normal measure $U$ on $\alpha$ to define $M_{\alpha+1} = \text{Ult}(M_\alpha, U)$ and let $j_{\alpha,\alpha+1}: M_\alpha \to M_{\alpha+1}$.

This describes the iteration strategy for $I$. Now consider the iteration tree formed when $\Pi$ plays his winning strategy against the iteration strategy just defined for player I. By Theorem 3.10 the set $C$ of all stages $\alpha$ in the iteration such that $x \models j_{0\alpha}(T_{3\delta}(E))$ and $\alpha = j_{0\alpha}(\delta)$ is unbounded, and it is therefore a club. Note that $x \cap \alpha = j_{0\alpha}(W_{3\delta}(E))$-generic for all $\alpha \in C$.

By the choice of extenders and Lemma 3.6 for $\alpha \in C$ the critical points of embeddings constructed after $\alpha$ stage will never drop below $j_{0\alpha}(\delta)$.

The critical sequence defines a club in $\omega_1$ and an $\omega_1$-iteration $\langle N_\xi, j_{\xi\eta} \mid \xi, \eta \leq \omega_1 \rangle$ such that $N_0 = M$, for $\xi < \eta$ the embedding $j_{\xi\eta}: N_\xi \to N_\eta$ has $\alpha_\xi$ as its critical point, and $x \cap \alpha_\xi \models j_{0\xi}(T_{3\delta}(E))$ for all $\xi$. Since the critical points are increasing, this implies that $x \models j_{0\omega_1}(T_{3\delta}(E))$, as required.

Theorem 4.8 below was proved independently by Steel and Woodin and a proof of its strengthening due to Neeman can be found in [13]. Theorem 4.8 implies that this implies that $\text{theorem } 4.8 \text{ implies that } \text{the sentence } \exists X \subseteq \mathbb{R} \psi(X) \text{ is fully iterable in all forcing extensions would provide an another proof of Woodin's } \Sigma_2^1 \text{-absoluteness theorem } [19] \text{; see also } [1], [9], \text{ or } [2].$ The proof of this theorem will use the following standard forcing fact. The assumption that $V_{\omega_1 + 1} \cap M$ is countable is used only to assure the existence of generic objects.

**Lemma 4.6.** Assume $M$ is a transitive model of ZFC such that $M \cap V_{\omega_1 + 1}$ is countable. Assume $\mathbb{P}$ and $\mathbb{Q}$ belong to $M \cap V_\delta$ and $\mathbb{P}$ is a regular subalgebra of $\mathbb{Q}$. If $G \subseteq \mathbb{P}$ is $M$-generic, then there is an $M$-generic $H \subseteq \mathbb{Q}$ such that $G \in M[H]$.

**Lemma 4.7.** Assume $\mathbb{P}$ is a forcing notion in $M$ with $\delta$-cc of cardinality $\delta$. If $j: M \to M^*$ is an elementary embedding with $\text{crit}(j) = \delta$, then $\mathbb{P}$ is a regular subordering of $j(\mathbb{P})$ in $M$.

**Proof.** We may assume $\mathbb{P} \subseteq V_\delta$. Let $\mathcal{A}$ be a maximal antichain in $\mathbb{P}$. By the $\delta$-cc we have $\mathcal{A} \in V_\delta$, and therefore $j(\mathcal{A}) = \mathcal{A}$. By the elementarity, $\mathcal{A}$ is a maximal antichain in $j(\mathbb{P})$ in $M^*$. Since being a maximal antichain is absolute, $\mathcal{A}$ is a maximal antichain of $j(\mathbb{P})$ in $M$. $\Box$

**Theorem 4.8.** Assume there exists a model $\mathcal{M}_1$ with a countable ordinal $\delta$ that is a measurable Woodin cardinal in $\mathcal{M}_1$ which is fully iterable in all forcing extensions. Then to every $\Sigma_2^1$ statement $\varphi$ we can associate a statement $\varphi^*$ such that if $V \models \varphi$ then $\mathcal{M}_1 \models \varphi^*$ and if $\mathcal{M}_1 \models \varphi^*$ and $\text{CH}$ holds then $V \models \varphi$.

**Proof.** The sentence $\varphi$ is of the form $(\exists X \subseteq \mathcal{R})\psi(X)$ where $\psi$ is a statement of $L(\mathcal{R})$. To it we associate $\varphi^*$ stating that below some condition $\mathcal{W}_{3\delta}$ forces $\varphi$ and $|\delta| = \aleph_1$.

In order to prove that $\varphi$ implies $\mathcal{M}_1 \models \varphi^*$, assume $X \subseteq \mathcal{R}$ such that $\psi(X)$ holds. Go to a forcing extension of $V$ with the same reals that satisfies CH. Fix $Y \subseteq \omega_1$ that codes $X$ and all reals. Using Lemma 4.5 find an iteration $j: \mathcal{M}_1 \to \mathcal{M}^*$ of length $\omega_1$ such that $Y = j(\mathcal{W}_{3\delta}(E))$-generic over $\mathcal{M}$. This forcing has $\text{has } j(\delta)$-chain condition and it collapses all cardinals below $j(\delta)$ to $\omega$. Therefore $\mathcal{R}^{\mathcal{M}_1}[Y] = \mathcal{R}$, hence $\mathcal{M}_1[Y] \models \psi(X)$.

We now assume $\text{CH}$ and $\mathcal{M}_1 \models \varphi^*$ and prove $\varphi$. In $\mathcal{M}_1$ fix a condition $p$ in $\mathcal{W}_{3\delta}$ and a name $\dot{X}$ such that $p$ forces $\psi(\dot{X})$. By using CH we can enumerate $\mathcal{P}(\omega)$ as
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$r_\xi$, for $\xi < \omega_1$. In this proof we shall write $M_0$ for $\mathfrak{M}_1$. We shall now describe an iteration strategy for player I. Along with the iteration, player I constructs generic filters. After player I’s strategy is described, we shall run it against player II’s winning strategy for the iteration game and argue that the run of the game produces a well-founded iteration $j : M_0 \rightarrow M^*$ and $G \subseteq j(W_{\delta,\delta}(\bar{E}))$ generic over $M^*$ such that $p = j(p)$ is in $G$ and that $M^*[G]$ contains all reals. All extenders $E$ used in player I’s strategy defined below will satisfy crit$(E) \geq \delta$, therefore assuring $j(p) = p$.

Since $M_0 \cap V_{\delta+1}$ is countable, in $V$ we can find a $G_0 \subseteq W_{\delta,\delta}(\bar{E})$ generic over $M_0$ and containing $p$. Now use the extender in $M_0$ with critical point $\delta$ to find $j_0 : M_0 \rightarrow M_1$. By Lemma 4.7, $G_0$ can be extended to a generic filter for $j_0(W_{\delta,\delta}(\bar{E}))$.

We want to find a generic $G_1 \subseteq j_0(W_{\delta,\delta}(\bar{E}))$ extending $G_0$ and such that $r_0 \in M_1[G_1]$. Let $\delta_0$ be the least Woodin cardinal in $M_1$ greater than $\delta$ whose Woodinness is witnessed by (an initial segment of) $j_0(\bar{E})$. Let $\bar{E}_1$ consist of generators in $j_0(\bar{E})$ witnessing Woodinness of $\delta_0$ whose critical points exceed $\delta$. Now player I attempts to find an iteration $i_0 : M_1 \rightarrow M^*_1$ using the extenders in $\bar{E}_1$ such that $r_0$ is generic over $i_0(W_{\delta_0,\delta_0}(\bar{E}_1))$. (Recall that player II continues playing his winning strategy for the iteration game corresponding to $\bar{E}$ from $\mathfrak{M}_1$, and therefore by Theorem 3.9 after countably many stages we will assure that $r_0$ is generic.) Assume I has succeeded in finding $i_0$. Since the critical point never drops below $\delta$, this is an iteration of $M_1$ resulting in some $M^*_1$. Since $p$ forces that $(i_0 \circ j_0)(W_{\delta,\delta}(\bar{E}))$ collapses $2^{\delta_0}$ to $\aleph_0$, $i_0(W_{\delta_0,\delta_0}(\bar{E}_1))$ can be embedded as a regular subalgebra of the former below $\delta$. We can therefore use Lemma 4.7 to find a generic $G_1 \subseteq (i_0 \circ j_0)(W_{\delta,\delta}(\bar{E}))$ including $G_0$ such that $r_0 \in (i_0 \circ j_0)(M_1[G_1])$.

We proceed in this manner. At the $\alpha$th stage we have an iteration $j^0_{\alpha} : M_0 \rightarrow M_{\alpha}$ and $G_{\alpha}$ is a generic filter for $j^0_{\alpha}(W_{\delta,\delta}(\bar{E}))$. We then find an ultrapower $j^1_{\alpha} : M_{\alpha} \rightarrow M^*_{\alpha}$ with critical point $j^0_{\alpha}(\delta)$ and write $j_{\alpha} = j^1_{\alpha} \circ j^0_{\alpha}$. By Lemma 4.7 the filter $G_{\alpha}$ can be extended to a generic ultrafilter included in $j_{\alpha}(W_{\delta,\delta}(\bar{E}))$.

Fix the least Woodin cardinal in $M^*_{\alpha}$ above $j^0_{\alpha}(\delta)$. Using the algebra with $\omega$ generators on this cardinal and Theorem 3.9, find an iteration of $i_{\alpha} : M_{\alpha} \rightarrow M^*_{\alpha}$ that makes $r_{\alpha}$ generic for an algebra that is a regular subalgebra of $(i_\alpha \circ j_\alpha)(W_{\delta,\delta}(\bar{E}))$. Again player I plays only the extenders $E$ with crit$(E) \geq j^0_{\alpha}(\delta)$ so the responses of player II result in an iteration $i_\alpha : M^*_{\alpha} \rightarrow M^*_{\alpha+1}$ as required. By Lemma 4.7 we can find a generic filter $G_{\alpha+1} \subseteq (i_\alpha \circ j_\alpha)(W_{\delta,\delta}(\bar{E}))$ extending $G_{\alpha}$ such that $r_{\alpha}$ belongs to $M^*_{\alpha+1}[G_{\alpha+1}]$.

At a limit stage of the construction player II chooses a maximal branch of the iteration tree constructed so far. By the $\delta$-cc, every maximal antichain of the image of $W_{\delta,\delta}(\bar{E})$ in this model belongs to some earlier model. Therefore the direct limit of the $G_\xi$ corresponding to the models on the branch is generic.

This describes a game which produces an iteration $j : M_0 \rightarrow M^*$ such that $j(\delta) = \delta_1$ and a $G \subseteq j(W_{\delta,\delta}(\bar{E}))$ generic over $M^*$ and containing $j(p) = p$. By the choice of $p$ and elementarity $\psi(X)$ holds in $M^*[G]$. Since the model $M^*[G]$ also contains all reals, $\psi(X)$ holds in $V$. 

\begin{corollary}
(Steel, Woodin). Assume there exists a fully iterable model $\mathfrak{M}_1$ with a countable ordinal $\delta$ that is a measurable Woodin cardinal in $\mathfrak{M}_1$. Then every $\Sigma^2_1$ statement true in some forcing extension of $V$ is true in every forcing extension of $V$ that satisfies CH.
\end{corollary}
5. $\Sigma_2^2$ ABSOLUTENESS

A positive answer to the following (modulo sufficient large cardinals) was conjectured by John Steel (see also [21]).

**Question 5.1.** Assume $\varphi$ is a $\Sigma_2^2$ sentence such that CH+$\varphi$ holds in some forcing extension. Is it true that $\varphi$ holds in every forcing extension that satisfies ♦?

By a result of Woodin (see [6]), if there is a measurable Woodin cardinal then there is a forcing $\mathbb{P}$ that forces every $\Sigma_2^2$ sentence $\varphi$ that holds in some forcing extension satisfying CH. The forcing $\mathbb{P}$ is the iteration of the collapse of $2^{\aleph_0}$ to $\aleph_1$ and another forcing notion. It is not known whether the collapse of $2^{\aleph_0}$ to $\aleph_1$ alone suffices for this conclusion.

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