ON LOCALIZATION OF CERTAIN UNIFORM SELECTION PRINCIPLES

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Abstract. We intend to localize the selection principles in uniform spaces (Kočinac, 2003) by introducing their local variations, namely locally Υ-bounded spaces (where Υ is Menger, Hurewicz or Rothberger). It has been observed that the difference between uniform selection principles and the corresponding local correlatives as introduced here is reasonable enough to discuss about these new notions. Certain observations using the critical cardinals (on the uniform selection principles which have not studied before) as well as preservation like properties (on the local versions) are presented. The interrelationships between the notions considered in this paper are outlined into an implication diagram. Certain interactions between these local variations are also investigated. We present several examples to illustrate the distinguishable behaviour of the new notions.

Key words and phrases: Uniform space, selection principles, M-bounded, H-bounded, R-bounded, locally M-bounded, locally H-bounded, locally R-bounded, locally precompact, locally pre-Lindelöf.

1. Introduction

There was a long illustrious history of study of selection principles in set-theoretic topology. This vast field in topology became more popular and had attracted a lot of researcher’s attention in the last twenty five years after Scheepers’ seminal paper [18] (see also [6]), where a systematic study in this fascinating field was initiated. Since then, various topological notions have been defined or characterized in terms of the classical selection principles. Interested readers may explore the survey papers [8, 9, 17] for more information on this topic.

In 2003, Kočinac [7] introduced the study of selection principles in uniform spaces by defining uniform analogues of Menger, Hurewicz and Rothberger covering properties, namely M-bounded, H-bounded and R-bounded respectively.
and differentiated these uniform variations from classical Menger, Hurewicz and Rothberger properties. Interestingly it was observed that these uniform covering properties can also be defined in terms of star selection principles in uniform spaces.

Later in 2013, Končinac and Künzi [11] further extended the study in quasi-uniform spaces. For more information about the uniform selection principles, we refer the reader to consult the papers [8–10, 12] and references therein.

This paper is a continuation of study of uniform selection principles started in [7] and is organised as follows. In Section 3, we present certain observations on uniform selection principles (that were not at all investigated earlier in uniform structures), which seem to be effective in our context. In Section 4, we make an effort to extend the concept of uniform selection principles by introducing local variations of these selection principles, namely locally $M$-bounded, locally $H$-bounded and locally $R$-bounded spaces. Certain situations are described which witness that these local variations behave much differently from the uniform selection principles. Later in this section, preservation like properties of the new notions are investigated carefully and the interrelationships between these new notions are also discussed. Section 5 is the final portion of this article, which is devoted to present illustrative examples. It is shown that the class consisting of each of these local variations is strictly larger than the class containing the corresponding uniform counterparts. We also present exemplary observations of their perceptible behaviours.

2. Preliminaries

For undefined notions and terminologies, see [4]. We start with some basic information about uniform spaces.

Let $X$ be a set and let $A, B \subseteq X \times X$. We define $A^{-1} = \{(x, y) : (y, x) \in A\}$ and $A \circ B = \{(x, y) : \exists z \in X \text{ such that } (x, z) \in A \text{ and } (z, y) \in B\}$. The diagonal of $X \times X$ is the set $\Delta = \{(x, x) : x \in X\}$. A set $U \subseteq X \times X$ is said to be an entourage of the diagonal if $\Delta \subseteq U$ and $U^{-1} = U$. The family of all entourages of the diagonal $\Delta \subseteq U$ and $U^{-1} = U$. The family of all entourages of the diagonal $\Delta \subseteq X \times X$ will be denoted by $E_X(\Delta)$. If $F \subseteq X$ and $U \in E_X(\Delta)$, then $U[F] = \cup_{x \in F} U[x]$ where $U[x] = \{y \in X : (x, y) \in U\}$. Recall that a uniform space can be described equivalently in terms of either a diagonal uniformity or a covering uniformity [4] (see also [3,20]). In this paper we use diagonal uniformity to define a uniform space. A uniformity on a set $X$ is a subfamily $\mathcal{U}$ of $E_X(\Delta)$ which satisfies the following conditions. (i) If $U \in \mathcal{U}$ and $V \in E_X(\Delta)$ with $U \subseteq V$, then $V \in \mathcal{U}$; (ii) If $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$; (iii) For every $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V \circ V \subseteq U$; and (iv) $\cap \mathcal{U} = \Delta$. The pair $(X, \mathcal{U})$ is called a uniform space [4]. Clearly every uniform space $(X, U)$ is a topological space. The family $\tau_U = \{O \subseteq X : \text{ for each } x \in O \text{ there exists a } U \in \mathcal{U} \text{ such that } U[x] \subseteq O\}$ is the topology on $X$ generated by the uniformity $U$. It is well known that
the topology of a space $X$ can be induced by a uniformity on $X$ if and only if $X$ is Tychonoff (see [4, Theorem 8.1.20]). If $(X, d)$ is a metric space, then the family $\{U_\varepsilon : \varepsilon > 0\}$, where $U_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}$, is a base for the uniformity $U$ induced by the metric $d$. Moreover, the topologies induced on $X$ by the uniformity $U$ and by the metric $d$ coincide. By a subspace $Y$ of a uniform space $(X, U)$ we mean the uniform space $(Y, U_Y)$, where $Y \subseteq X$ and $U_Y = \{(Y \times Y) \cap U : U \in U\}$ (which is called the relative uniformity on $Y$). Let $F$ be a family of subsets of $X$. We say that $F$ contains arbitrarily small sets if for every $U \in U$ there exists a $F \in F$ such that $F \times F \subseteq U$ (see [4]). We say that $X$ is complete if every family $F$ of closed subsets of $X$ which has the finite intersection property and contains arbitrarily small sets has nonempty intersection. A uniformity $U$ on a set $X$ is complete [4]. A function $f : (X, U) \to (Y, V)$ between two uniform spaces is uniformly continuous if for every $V \in V$ there exists a $U \in U$ such that for all $x, y \in X$ we have $(f(x), f(y)) \in V$ whenever $(x, y) \in U$ (see [4]).

$(X, U)$ is said to be precompact or totally bounded (resp. pre-Lindelöf) if for each $U \in U$ there exists a finite (resp. countable) $A \subseteq X$ such that $U[A] = X$ and $A$ is a sequence of finite subsets of $X$. Moreover, for a complete uniform space precompactness (resp. pre-Lindelöf compactness) and completeness (resp. Lindelöf compactness) are equivalent. It is easy to observe that if the uniformity $U$ on a set $X$ is induced by a metric $d$, then $(X, U)$ is complete (resp. precompact, pre-Lindelöf) if and only if $(X, d)$ is complete (resp. precompact, pre-Lindelöf).

We now recall some definitions of topological spaces formulated in terms of classical selection principles from [0,15]. A topological space $X$ is said to be Menger (resp. Rothberger) if for each sequence $(U_n)$ of open covers of $X$ there is a sequence $(V_n)$ (resp. $(U_n)$) such that for each $n V_n$ is a finite subset of $U_n$ (resp. $U_n \in U_n$) and $\bigcup_{n \in \mathbb{N}} V_n$ (resp. $\{U_n : n \in \mathbb{N}\}$) is an open cover of $X$. A topological space $X$ is said to be Hurewicz if for each sequence $(U_n)$ of open covers of $X$ there is a sequence $(V_n)$ such that for each $n V_n$ is a finite subset of $U_n$ and each $x \in X$ belongs to $\bigcup_{n \in \mathbb{N}} V_n$ for all but finitely many $n$. A topological space $X$ is said to be locally compact (resp. locally Menger, locally Hurewicz, locally Rothberger, locally Lindelöf) if for each $x \in X$ there exist an open set $U$ and a compact (resp. Menger, Hurewicz, Rothberger, Lindelöf) subspace $Y$ of $X$ such that $x \in U \subseteq Y$.

In [7] (see also [8]), Kočinac introduced the following uniform selection principles. A uniform space $(X, U)$ is Menger-bounded (in short, $M$-bounded) if for each sequence $(U_n)$ of members of $U$ there is a sequence $(F_n)$ of finite subsets of $X$ such that $\bigcup_{n \in \mathbb{N}} U_n[F_n] = X$. A uniform space $(X, U)$ is said to be Hurewicz-bounded (in short, $H$-bounded) if for each sequence $(U_n)$ of members of $U$ there is a sequence $(F_n)$ of finite subsets of $X$ such that each $x \in X$ belongs to $U_n[F_n]$.
for all but finitely many $n$. Also a uniform space $(X, \mathcal{U})$ is Rothberger-bounded (in short, R-bounded) if for each sequence $(U_n)$ of members of $\mathcal{U}$ there is a sequence $(x_n)$ of members of $X$ such that $\bigcup_{n \in \mathbb{N}} U_n[x_n] = X$. The above properties are also known as uniformly Menger, uniformly Hurewicz and uniformly Rothberger respectively. We also say that a metric space $(X, d)$ is M-bounded (resp. H-bounded, R-bounded) if $X$ with the induced uniformity is M-bounded (resp. H-bounded, R-bounded).

Throughout the paper $(X, \mathcal{U})$ (or in short, $X$, when $\mathcal{U}$ is clear from the context) stands for a uniform space, where $\mathcal{U}$ is a diagonal uniformity on $X$.

The following two results are useful in our context.

**Theorem 2.1** (cf. [7]). Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces.

1. $(X \times Y, \mathcal{U} \times \mathcal{V})$ is H-bounded if and only if both $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are H-bounded.
2. If $(X, \mathcal{U})$ is M-bounded and $(Y, \mathcal{V})$ is precompact, then $(X \times Y, \mathcal{U} \times \mathcal{V})$ is M-bounded.

**Theorem 2.2** (cf. [7]).

1. M-bounded, H-bounded and R-bounded properties are hereditary and preserved under uniformly continuous mappings.
2. Let $(X, \mathcal{U})$ be a uniform space and $Y \subseteq X$. If $(Y, \mathcal{U}_Y)$ is H-bounded, then $(\overline{Y}, \mathcal{U}_{\overline{Y}})$ is also H-bounded.
3. If $(X, \mathcal{U})$ is a complete uniform space, then $(X, \mathcal{U})$ is Hurewicz, if and only if it is H-bounded.
4. If $(X, \mathcal{U})$ is Menger, Hurewicz, Rothberger, then it is also M-bounded, H-bounded and R-bounded respectively.

### 3. Few observations on uniform selection principles

We present few more observations on uniform selection principles that will be useful subsequently. Throughout the paper we use the symbol $\Upsilon$ to denote any of the Menger, Hurewicz or Rothberger properties. Accordingly $\Upsilon$-bounded denotes any of the M-bounded, H-bounded or R-bounded properties.

We start with two basic observations (without proof) about uniform selection principles.

**Lemma 3.1.** Let $(X, \mathcal{U})$ be a uniform space. A subspace $Y$ of $X$ is

1. M-bounded if and only if for each sequence $(U_n)$ of members of $\mathcal{U}$ there exists a sequence $(F_n)$ of finite subsets of $Y$ such that $Y \subseteq \bigcup_{n \in \mathbb{N}} U_n[F_n]$.
2. H-bounded if and only if for each sequence $(U_n)$ of members of $\mathcal{U}$ there exists a sequence $(F_n)$ of finite subsets of $Y$ such that each $y \in Y$ belongs to $U_n[F_n]$ for all but finitely many $n$. 
Theorem 3.1. Let \( (X, \mathcal{U}) \) be a uniform space with \( X = \bigcup_{n \in \mathbb{N}} X_n \). Then \( X \) is \( \mathcal{U} \)-bounded if and only if \( \mathcal{V} \)-bounded.

**Proof.** We only present proof of the converse part for the case of \( H \)-bounded. Let \( (U_n) \) be a sequence of members of \( \mathcal{U} \). By Lemma 3.1(2) for each \( k \in \mathbb{N} \) we can choose a sequence \( (F_n^{(k)}) : n \geq k \) of finite subsets of \( X_n \) such that each \( x \in X_k \) belongs to \( U_n[F_n^{(k)}] \) for all but finitely many \( n \geq k \). For each \( n \) let \( F_n = \bigcup_{k \leq n} F_n^{(k)} \). Then \( (F_n) \) is a sequence of finite subsets of \( X \). We show that each \( x \in X \) belongs to \( U_n[F_n] \) for all but finitely many \( n \). Let \( x \in X \). Choose a \( k_0 \in \mathbb{N} \) such that \( x \in X_{k_0} \). Clearly \( x \in U_n[F_n^{(k_0)}] \) for all but finitely many \( n \geq k_0 \) and hence \( x \in U_n[F_n] \) for all but finitely many \( n \) as \( F_n^{(k_0)} \subseteq F_n \) for all \( n \geq k_0 \). Hence the result. \( \square \)

Let \( (Y, \mathcal{V}) \) be a uniform space and \( X \) be a set. If \( f : X \to Y \) is an injective mapping, then there is a natural uniformity on \( X \) given by the collection \( f^{-1}(\mathcal{V}) = \{ f^{-1}(V) : V \in \mathcal{V} \} \). The uniformity \( f^{-1}(\mathcal{V}) \) on \( X \) is generated by the base \( \{ g^{-1}(V) : V \in \mathcal{V} \} \), where \( g : X \times X \to Y \times Y \) is defined as \( g(x, y) = (f(x), f(y)) \).

**Theorem 3.2.** Let \( f : X \to Y \) be an injective mapping from a set \( X \) onto a uniform space \( (Y, \mathcal{V}) \). If \( Y \) is \( \mathcal{V} \)-bounded, then \( X \) is also \( \mathcal{V} \)-bounded.

**Proof.** Suppose that \( Y \) is \( H \)-bounded. Let \( (U_n) \) be a sequence of members of \( f^{-1}(\mathcal{V}) \). Observe that \( \{ g^{-1}(V) : V \in \mathcal{V} \} \) is a base for the uniformity \( f^{-1}(\mathcal{V}) \), where \( g : X \times X \to Y \times Y \) such that \( g(x, y) = (f(x), f(y)) \). For each \( n \) choose a \( V_n \in \mathcal{V} \) such that \( g^{-1}(V_n) \subseteq U_n \). Next choose a sequence \( (F_n) \) of finite subsets of \( Y \) such that for each \( y \in Y \) there exists a \( n_y \in \mathbb{N} \) such that \( y \in V_n[F_n] \) for all \( n \geq n_y \). For each \( n \) choose \( F_n = \{ y_1^{(n)}, y_2^{(n)}, \ldots, y_{k_n}^{(n)} \} \) and \( F'_n = \{ x_1^{(n)}, x_2^{(n)}, \ldots, x_{k_n}^{(n)} \} \subseteq X \), where \( f(x_i^{(n)}) = y_i^{(n)} \) for each \( 1 \leq i \leq k_n \). Let \( x \in X \). Then there exists an \( n_{f(x)} \in \mathbb{N} \) such that \( f(x) \in V_n[F_n] \) for all \( n \geq n_{f(x)} \). For each \( n \geq n_{f(x)} \) there exists a \( t_n \) with \( 1 \leq t_n \leq k_n \) such that \( f(x) \in V_n[y_i^{(n)}] \). Thus for \( n \geq n_{f(x)} \) we have \( g(x, x_i^{(n)}) \in V_n \) i.e. \( x \in U_n[x_i^{(n)}] \subseteq U_n[F'_n] \) for all but finitely many \( n \). This completes the proof. \( \square \)

The eventual dominance relation \( \leq^* \) on the Baire space \( \mathbb{N}^\mathbb{N} \) is defined by \( f \leq^* g \) if and only if \( f(n) \leq g(n) \) for all but finitely many \( n \). A subset \( A \) of \( \mathbb{N}^\mathbb{N} \) is said to be dominating if for each \( g \in \mathbb{N}^\mathbb{N} \) there exists a \( f \in A \) such
that \( g \leq^* f \). A subset \( A \) of \( \mathbb{N}^\mathbb{N} \) is said to be bounded if there is a \( g \in \mathbb{N}^\mathbb{N} \) such that \( f \leq^* g \) for all \( f \in A \). Moreover a set \( A \subseteq \mathbb{N}^\mathbb{N} \) is said to be guessed by \( g \in \mathbb{N}^\mathbb{N} \) if \( \{ n \in \mathbb{N} : f(n) = g(n) \} \) is infinite for all \( f \in A \). The minimum cardinality of a dominating subset of \( \mathbb{N}^\mathbb{N} \) is denoted by \( \mathfrak{d} \), and the minimum cardinality of a unbounded subset of \( \mathbb{N}^\mathbb{N} \) is denoted by \( \mathfrak{b} \). Let \( \text{cov}(\mathcal{M}) \) be the minimum cardinality of a family of meager subsets of \( \mathbb{R} \) that covers \( \mathbb{R} \). In [1] (see also [2, Theorem 2.4.1]), \( \text{cov}(\mathcal{M}) \) is described as the minimum cardinality of a \( F \subseteq \mathbb{N}^\mathbb{N} \) such that for every \( g \in \mathbb{N}^\mathbb{N} \) there is \( f \in F \) such that \( f(n) \neq g(n) \) for all but finitely many \( n \). Thus we can say that if \( F \subseteq \mathbb{N}^\mathbb{N} \) and \( |F| < \text{cov}(\mathcal{M}) \), then \( F \) can be guessed by a \( g \in \mathbb{N}^\mathbb{N} \). It is to be noted that the Baire space is also a uniform space with the uniformity \( \mathcal{B} \) induced by the Baire metric.

**Theorem 3.3.** Every pre-Lindelöf space \( (X, \mathcal{U}) \) with \( |X| < \mathfrak{d} \) is M-bounded.

**Proof.** Let \( (U_n) \) be a sequence of members of \( \mathcal{U} \). Apply the pre-Lindelöf property to obtain a sequence \( (A_n) \) of countable subsets of \( X \) such that \( U_n[A_n] = X \) for each \( n \). Say \( A_n = \{ x^{(m)}_n : m \in \mathbb{N} \} \) for each \( n \). Now for each \( x \in X \) define \( f_x \in \mathbb{N}^\mathbb{N} \) by \( f_x(n) = \min \{ m \in \mathbb{N} : x \in U_n[x^{(m)}_n] \} \), \( n \in \mathbb{N} \). Since the cardinality of \( \{ f_x \in \mathbb{N}^\mathbb{N} \} \) is less than \( \mathfrak{d} \), there is a \( g \in \mathbb{N}^\mathbb{N} \) and for \( x \in X \) a \( n_x \in \mathbb{N} \) such that \( f_x(n_x) < g(n_x) \). For each \( n \) define \( F_n = \{ x^{(m)}_n : m \leq g(n) \} \). Observe that if \( x \in X \), then \( x \in U_{n_x}[F_{n_x}] \). Clearly \( \{ U_n[F_n] : n \in \mathbb{N} \} \) covers \( X \) and hence \( X \) is M-bounded. \( \blacksquare \)

Similarly we obtain the following.

**Theorem 3.4.** Every pre-Lindelöf space \( (X, \mathcal{U}) \) with \( |X| < \mathfrak{b} \) is H-bounded.

**Theorem 3.5.** Every pre-Lindelöf space \( (X, \mathcal{U}) \) with \( |X| < \text{cov}(\mathcal{M}) \) is R-bounded.

**Proof.** Let \( (U_n) \) be a sequence of members of \( \mathcal{U} \). Consider \( A_n \) and \( f_x \) as in Theorem 3.3 and proceed as follows.

Since the cardinality of \( \{ f_x : x \in X \} \) is less than \( \text{cov}(\mathcal{M}) \), choose a \( g \in \mathbb{N}^\mathbb{N} \) such that \( \{ n : f_x(n) = g(n) \} \) is infinite for all \( x \in X \).

Observe that if \( x \in X \), then \( f_x(n_x) = g(n_x) \) for some positive integer \( n_x \) i.e. \( x \in U_{n_x}[x^{(n_x)}_{g(n_x)}] \). Thus \( \{ U_n[x^{(n)}_g] : n \in \mathbb{N} \} \) is a cover of \( X \). Which shows that \( X \) is R-bounded. \( \blacksquare \)

**Theorem 3.6.** If \( (X, \mathcal{U}) \) is M-bounded, then any uniformly continuous image of \( X \) into \( \mathbb{N}^\mathbb{N} \) is non-dominating.

**Proof.** In view of Theorem 2.2(1) we can assume that \( X \) is a subspace of \( \mathbb{N}^\mathbb{N} \) and that \( X \) is M-bounded. For each \( n \in \mathbb{N} \) consider \( U_n = \{ (\varphi, \psi) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} : \varphi(n) = \psi(n) \} \) \( \in \mathcal{B} \). Let \( \{ P_n : n \in \mathbb{N} \} \) be a partition of \( \mathbb{N} \) into pairwise disjoint infinite subsets. For each \( n \in \mathbb{N} \), apply the M-bounded property of \( X \) to \( (U_k : k \in P_n) \) to obtain a sequence \( (F_k : k \in P_n) \) of finite subsets of \( X \) such that
$X \subseteq \cup_{k \in P_n} U_k[F_k]$. Thus $F_n$ is defined for each $n \in \mathbb{N}$. Now define $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = 1 + \max\{f(n) : f \in F_n\}$. To complete the proof we show that $X$ is not dominating (which is witnessed by $g$).

Let $f \in X$. For each $k \in \mathbb{N}$ choose a $n_k \in P_k$ such that $f \in U_{n_k}[F_{n_k}]$. Again choose for each $k \in \mathbb{N}$ a $f_k \in F_{n_k}$ such that $(f, f_k) \in U_{n_k}$ i.e. $f(n_k) = f_k(n_k)$ for all $k \in \mathbb{N}$. Consequently $f(n_k) < g(n_k)$ for all $k \in \mathbb{N}$ and hence the set \{n \in \mathbb{N} : g(n) \notin f(n)\} is infinite. \hfill \Box

**Theorem 3.7.** If $(X, U)$ is H-bounded, then any uniformly continuous image of $X$ into $\mathbb{N}^\mathbb{N}$ is bounded (with respect to $\leq^*$).

**Proof.** The proof is modelled in Theorem 3.6. It remains only to observe that $g$ dominates every element of $X$. \hfill \Box

**Theorem 3.8.** If $(X, U)$ is R-bounded, then any uniformly continuous image of $X$ into $\mathbb{N}^\mathbb{N}$ can be guessed.

**Proof.** We closely follow the proof of Theorem 3.6. Choose $U_n$'s as in Theorem 3.6 and proceed with the following modifications. For each $n \in \mathbb{N}$ choose a sequence $(f_k : k \in P_n)$ of members of $X$ such that $X \subseteq \cup_{k \in P_n} U_k[f_k]$. Thus $f_n$ is defined for each positive integer $n$. Define $g : \mathbb{N} \to \mathbb{N}$ by $g(n) = f_n(n)$. We now show that $X$ is guessed by $g$. Choose any $f \in X$ and for each $k \in \mathbb{N}$ choose a $n_k \in P_k$ such that $(f, f_{n_k}) \in U_{n_k}$. Clearly the set \{n \in \mathbb{N} : f(n) = g(n)\} is infinite (as it contains all $n_k$'s) and this completes the proof. \hfill \Box

4. Local variations of uniform selection principles

4.1. Locally $\Upsilon$-bounded spaces. We now introduce the main definition of this paper.

**Definition 4.1.** Let $(X, U)$ be a uniform space. Then $X$ is said to be locally $\Upsilon$-bounded if for each $x \in X$ there exists a $U \in U$ such that $U[x]$ is a $\Upsilon$-bounded subspace of $X$.

**Remark 4.1.** Locally precompact and locally pre-Lindelöf spaces can be similarly defined.

From the above definition and [7], we obtain the following implication diagram (where the abbreviations C, H, L, M, R denote respectively compact, Hurewicz, Lindelöf, Menger and Rothberger spaces, the prefixes l, p stand respectively for ‘locally’ and ‘pre-’ and the suffix b stands for ‘-bounded’).
We now present equivalent formulations of the new notions.

**Theorem 4.1.** If \((X, U)\) is a uniform space, then the following assertions are equivalent.

1. \(X\) is locally \(\Upsilon\)-bounded.
2. For each \(x \in X\) and \(V \in U\) there exist a \(U \in U\) and a \(\Upsilon\)-bounded subspace \(Y\) of \(X\) such that \(U[x] \subseteq Y \subseteq V[x]\).
3. For each \(x \in X\) there exist a \(U \in U\) and a \(\Upsilon\)-bounded subspace \(Y\) of \(X\) such that \(U[x] \subseteq Y\).
4. For each \(x \in X\) there exists a \(U \in U\) such that \(U[x]\) is a \(\Upsilon\)-bounded subspace of \(X\).

**Proof.** We only present proof of (1) \(\Rightarrow\) (2). Let \(x \in X\) and \(V \in U\). We can find a \(U \in U\) such that \(U[x]\) is a \(\Upsilon\)-bounded subspace of \(X\). Clearly \(U \cap V \in U\) and \(Y = U[x] \cap V[x]\) is a \(\Upsilon\)-bounded subspace of \(X\) with \((U \cap V)[x] \subseteq Y \subseteq V[x]\). Hence (2) holds.

We say that a metric space \((X, d)\) is locally \(\Upsilon\)-bounded if for each \(x \in X\) there exists an open set \(V\) in \(X\) such that \(x \in V\) and \(V\) is a \(\Upsilon\)-bounded subspace of \((X, d)\). Likewise locally precompact and locally pre-Lindelöf metric spaces can also be defined.

**Remark 4.2.** It is easy to observe that if the uniformity \(U\) on a set \(X\) is induced by a metric \(d\), then the uniform space \((X, U)\) is locally \(\Upsilon\)-bounded if and only if the metric space \((X, d)\) is locally \(\Upsilon\)-bounded. Similar assertion holds for locally precompact and locally pre-Lindelöf metric spaces.

Using Theorem 3.1, we have the following observation.

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**Figure 1.** Diagram of local properties in uniform spaces
Proposition 4.1. Let \((X, U)\) be a uniform space such that \(X\) is Lindelöf. Then \(X\) is \(\Upsilon\)-bounded if and only if \(X\) is locally \(\Upsilon\)-bounded.

Proposition 4.2. Let \((X, U)\) be a uniform space. If \(X\) is locally \(\Upsilon\), then \(X\) is locally \(\Upsilon\)-bounded.

Proof. Let \(x \in X\). Choose an open set \(U\) and a \(\Upsilon\) subspace \(Y\) of \(X\) such that \(x \in U \subseteq Y\). By Theorem 2.2(4), \(Y\) is a \(\Upsilon\)-bounded subspace of \(X\). Since \(x \in U\) is open in \(X\), choose a \(V \in U\) such that \(V[x] \subseteq U\). We thus obtain a \(\Upsilon\)-bounded subspace \(V[x]\) of \(X\), which shows that \(X\) is locally \(\Upsilon\)-bounded. □

It can also be observed that locally compact (resp. locally Lindelöf) implies locally precompact (resp. locally pre-Lindelöf).

Theorem 4.2. If \((X, U)\) is a complete uniform space, then \(X\) is locally Hurewicz if and only if \(X\) is locally \(H\)-bounded.

Proof. The forward implication follows from Proposition 4.2. For the other direction, let \(x \in X\). Choose \(U \in U\) such that \(U[x]\) is \(H\)-bounded. Clearly \(x \in \text{Int}U[x] \subseteq U[x]\). By Theorem 2.2(4), \(U[x]\) is a \(H\)-bounded subspace of \(X\). Again by Theorem 2.2(3), \(U[x]\) is a Hurewicz subspace of \(X\) as \(\overline{U[x]}\) is complete. Thus \(X\) is locally Hurewicz. □

Theorem 4.3. Let \((X, U)\) be locally \(\Upsilon\)-bounded. An element \(U \in U\) is open in \(X \times X\) if and only if \((Y \times Z) \cap U\) is open in \(Y \times Z\) for any two \(\Upsilon\)-bounded subspaces \(Y\) and \(Z\) of \(X\).

Proof. We only present the proof of converse part. Let \(U \in U\). Let \((x, y) \in U\). Since \(X\) is locally \(\Upsilon\)-bounded, there exist \(V, W \in U\) such that \(V[x]\) and \(W[y]\) are \(\Upsilon\)-bounded subspaces of \(X\). Clearly \((\text{Int} V[x] \times \text{Int} W[y]) \cap U\) is open in \(\text{Int} V[x] \times \text{Int} W[y]\) as \((V[x] \times W[y]) \cap U\) is open in \(V[x] \times W[y]\). Evidently \((x, y) \in (\text{Int} V[x] \times \text{Int} W[y]) \cap U\) is open in \(X \times X\) and hence \(U\) is open in \(X \times X\) as required. □

The following observation is due to Theorems 3.3, 3.4, and 3.5 with necessary modifications.

Proposition 4.3. Every locally pre-Lindelöf space with cardinality less than \(\mathfrak{d}\) (resp. \(\mathfrak{b}, \text{cov}(\mathcal{M})\)) is locally \(M\)-bounded (resp. locally \(H\)-bounded, locally \(R\)-bounded).

4.2. Some observations on locally \(\Upsilon\)-bounded spaces. We now present some preservation like properties of these local variations under certain topological operations.

Proposition 4.4. Locally \(\Upsilon\)-bounded property is hereditary.
**Proposition 4.5.** Let \( f : X \to Y \) be an injective mapping from a set \( X \) onto a uniform space \((Y, \mathcal{V})\). If \( Y \) is locally \( \Upsilon \)-bounded, then \( X \) is also locally \( \Upsilon \)-bounded.

**Proof.** First recall that the uniformity \( f^{-1}(\mathcal{V}) \) on \( X \) is generated by the base \( \{g^{-1}(V) : V \in \mathcal{V}\} \), where \( g : X \times X \to Y \times Y \) is given by \( g(x, y) = (f(x), f(y)) \). Now let \( x \in X \). Since \( Y \) is locally \( \Upsilon \)-bounded, there exists a \( U \in \mathcal{V} \) such that \( (Z, \mathcal{V}_Z) \) is a \( \Upsilon \)-bounded subspace of \( Y \), where \( Z = U[f(x)] \). Let \( W = f^{-1}(Z) \) and define \( h : W \to Z \) by \( h(u) = f(u) \). By Theorem 3.2, \((W, \mathcal{S})\) is \( \Upsilon \)-bounded, where \( \mathcal{S} = h^{-1}(\mathcal{V}_Z) \). Also define \( \tilde{g} : W \times W \to Z \times Z \) by \( \tilde{g}(u, v) = (h(u), h(v)) \). Now observe that \( \mathcal{B} = \{\tilde{g}^{-1}(V) : V \in \mathcal{V}_Z\} \) is a base for the uniformity \( \mathcal{S} \) on \( W \) and \( \mathcal{B}' = \{(W \times W) \cap g^{-1}(V) : V \in \mathcal{V}\} \) is a base for the uniformity \( \mathcal{S}' = f^{-1}(\mathcal{V})_W \) on \( W \). It is easy to verify that \( \mathcal{B} = \mathcal{B}' \) and hence \( \mathcal{S} = \mathcal{S}' \). Thus \((W, \mathcal{S})\) is a \( \Upsilon \)-bounded subspace of \( X \). Clearly \( g^{-1}(U) \in f^{-1}(\mathcal{V}) \) and \( g^{-1}(U)[x] = W \). Consequently \( g^{-1}(U)[x] \) is a \( \Upsilon \)-bounded subspace of \( X \) and the proof is now complete. \( \Box \)

We now observe that locally \( \Upsilon \)-bounded property remains invariant under certain mappings. First we recall the following definitions from [15]. A surjective continuous mapping \( f : X \to Y \) is said to be weakly perfect if \( f \) is closed and \( f^{-1}(y) \) is Lindelöf for each \( y \in Y \). Also a surjective continuous mapping \( f : X \to Y \) is said to be bi-quotient if whenever \( y \in Y \) and \( U \) is a cover of \( f^{-1}(y) \) by open sets in \( X \), then finitely many \( f(U) \) with \( U \in U \) cover some open set containing \( y \) in \( Y \). It is immediate that surjective continuous open (and also perfect) mappings are bi-quotient.

**Theorem 4.4.**

1. If \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a uniformly continuous, bi-quotient mapping from a locally \( \Upsilon \)-bounded space \( X \) onto \( Y \), then \( Y \) is also locally \( \Upsilon \)-bounded.

2. If \( f : (X, \mathcal{U}) \to (Y, \mathcal{V}) \) is a uniformly continuous, weakly perfect mapping from a locally \( \Upsilon \)-bounded space \( X \) onto \( Y \), then \( Y \) is also locally \( \Upsilon \)-bounded.

**Proof.** (1). For each \( x \in X \) choose a \( U_x \in \mathcal{U} \) such that \( U_x[x] \) is \( \Upsilon \)-bounded. Consider the open cover \( \{\text{Int}U_x[x] : x \in X\} \) of \( X \). Let \( y \in Y \). Since \( f \) is a bi-quotient mapping, there is a finite set \( \{\text{Int}U_{x_i}[x_i] : 1 \leq i \leq k\} \subseteq \{\text{Int}U_x[x] : x \in X\} \) and an open set \( V \) in \( Y \) such that \( y \in V \subseteq \bigcup_{i=1}^k \text{Int}f(U_{x_i}[x_i]) \) i.e. \( V \subseteq \bigcup_{i=1}^k f(U_{x_i}[x_i]) \). By Theorem 2.2(1) and Theorem 3.1, \( \bigcup_{i=1}^k f(U_{x_i}[x_i]) \) is a \( \Upsilon \)-bounded subspace of \( Y \). Next we can choose a \( W \in \mathcal{V} \) with \( W[y] \subseteq V \) as \( V \) is open in \( Y \) containing \( y \). Eventually \( W[y] \) is the required \( \Upsilon \)-bounded subspace of \( Y \).

(2). Let \( y \in Y \) and say \( A = f^{-1}(y) \). For each \( x \in A \) choose a \( U_x \in \mathcal{U} \) such that \( U_x[x] \) is \( \Upsilon \)-bounded. Consider the cover \( \{\text{Int}U_x[x] : x \in A\} \) of \( A \) by open sets in \( X \). Since \( A \) is Lindelöf, there is a countable collection \( \{\text{Int}U_{x_n}[x_n] : n \in \mathbb{N}\} \) that covers \( A \) i.e. \( A \subseteq \bigcup_{n \in \mathbb{N}} \text{Int}U_{x_n}[x_n] \). Observe that \( y \in Y \setminus f(X \setminus \bigcup_{n \in \mathbb{N}} \text{Int}U_{x_n}[x_n]) \subseteq f(\bigcup_{n \in \mathbb{N}} \text{Int}U_{x_n}[x_n]) \). By Theorem 3.1 and Theorem 2.2(1), \( f(\bigcup_{n \in \mathbb{N}} \text{Int}U_{x_n}[x_n]) \) is a
Let $\mathbf{X}$ be a uniform space $\mathbf{X}$, and $\mathbf{Y}$ be the uniformly continuous open mapping from a locally $\mathbf{Y}$-bounded subspace of $\mathbf{Y}$ containing $\mathbf{U}$. Thus, we obtain a $V \in \mathcal{V}$ such that $V[y] \subseteq \mathbf{Y} \setminus f(X \setminus \cup_{n \in \mathbb{N}} \text{Int} U_x[x_n])$. Clearly $V[y]$ is a $\mathbf{Y}$-bounded subspace of $\mathbf{Y}$. This completes the proof.

\textbf{Corollary 4.1.} If $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is a uniformly continuous, perfect (or a uniformly continuous, open) mapping from a locally $\mathbf{Y}$-bounded space $X$ onto $Y$, then $Y$ is also locally $\mathbf{Y}$-bounded.

Recall that a bijective mapping $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is said to be a uniform isomorphism if both $f$ and $f^{-1}$ are uniformly continuous \[^{[4]}\]. We say that two uniform spaces $X$ and $Y$ are uniformly isomorphic if there exists a uniform isomorphism of $X$ onto $Y$. It is clear that every uniform isomorphism is an open mapping.

Let $\Lambda$ be an indexed set and $\{(X_{\alpha}, U_{\alpha}) : \alpha \in \Lambda\}$ be a family of uniform spaces. Let $X = \bigoplus_{\alpha \in \Lambda} X_{\alpha}$, where $\bigoplus_{\alpha \in \Lambda} X_{\alpha} = \cup_{\alpha \in \Lambda} Y_{\alpha}$ with each $Y_{\alpha} = X_{\alpha} \times \{\alpha\}$ and $U = \cup_{\alpha \in \Lambda} U_{\alpha} : U_{\alpha} \in U_{\alpha} \times \mathcal{V}_{\alpha}$ where each $\mathcal{V}_{\alpha} = \{(\alpha, \alpha)\}$ is the uniformity on $\{\alpha\}$. Then the uniform space $(X, U)$ is called the sum of the family $\{(X_{\alpha}, U_{\alpha}) : \alpha \in \Lambda\}$. For each $\alpha \in \Lambda$, $(X_{\alpha}, U_{\alpha})$ is uniformly isomorphic to the subspace $(Y_{\alpha}, U_{Y_{\alpha}})$ of $(X, U)$ through the uniform isomorphism $\varphi_{\alpha} : X_{\alpha} \to Y_{\alpha}$ given by $\varphi_{\alpha}(x) = (x, \alpha)$. For each $\alpha \in \Lambda$, $Y_{\alpha}$ is a clopen subset of $X$. Moreover if for each $\alpha \in \Lambda$ the space $(X_{\alpha}, U_{\alpha})$ is uniformly isomorphic to a fixed uniform space $(Y, \mathcal{V})$, then the sum $(X, U)$ is uniformly isomorphic to the uniform space $(Y \times \Lambda, \mathcal{V} \times \mathcal{D})$, where $\mathcal{D}$ is the discrete uniformity on $\Lambda$.

The following folk lemma is useful for our next result.

\textbf{Lemma 4.1 (Folklore).} If $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are two uniform spaces, then $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is open if and only if for each $U \in \mathcal{U}$ and each $x \in X$ there exists a $V \in \mathcal{V}$ such that $V[f(x)] \subseteq f(U[x])$.

\textbf{Theorem 4.5.} Let $\{(X_{\alpha}, U_{\alpha}) : \alpha \in \Lambda\}$ be a family of uniform spaces. The sum $\bigoplus_{\alpha \in \Lambda} X_{\alpha}$ is locally $\mathbf{Y}$-bounded if and only if each $X_{\alpha}$ is locally $\mathbf{Y}$-bounded.

\textbf{Proof.} Let $\mathcal{U}$ be the uniformity on the sum $X = \bigoplus_{\alpha \in \Lambda} X_{\alpha}$ as inherited from the $U_{\alpha}$’s. Let $X$ be locally $\mathbf{Y}$-bounded. Since each $X_{\alpha}$ is uniformly isomorphic to the subspace $X_{\alpha} \times \{\alpha\}$ of $X$, the conclusion immediately follows from Proposition\[^{[4]}\,\[^{[3]}\]$ and Corollary\[^{[1]}\,\[^{[1]}\]$.

Conversely suppose that each $X_{\alpha}$ is locally $\mathbf{Y}$-bounded. Choose a $x \in X$ and say $x = (u, \beta)$, where $\beta \in \Lambda$ and $u \in X_{\beta}$. Clearly there exists a $U \in \mathcal{U}_{\beta}$ with a $\mathbf{Y}$-bounded subspace $U[u]$ of $X_{\beta}$. Consider the uniform isomorphism $\varphi_{\beta} : (X_{\beta}, U_{\beta}) \to (Y_{\beta}, U_{Y_{\beta}})$ given by $\varphi_{\beta}(y) = (y, \beta)$. Now $(X_{\beta}, U_{\beta})$ is uniformly isomorphic to the subspace $(Y_{\beta}, U_{Y_{\beta}})$ of $X$, where $Y_{\beta} = X_{\beta} \times \{\beta\}$. By Lemma\[^{[1]}\,\[^{[1]}\]$ choose a $V \in \mathcal{U}_{Y_{\beta}}$ such that $V[\varphi_{\beta}(u)] \subseteq \varphi_{\beta}(U[u])$ i.e. $V[x]$ is a $\mathbf{Y}$-bounded subspace of $(Y_{\beta}, U_{Y_{\beta}})$. Since $x \in \text{Int} Y_{\beta} V[x]$ is open in $X$, there is a $W \in \mathcal{U}$ such
that $W[x] \subseteq \text{Int}_{\Upsilon} V[x]$. Consequently $W[x]$ is $\Upsilon$-bounded and the proof is now complete. \hfill \square

**Theorem 4.6.** Let $\{X_{\alpha} : \alpha \in \Lambda\}$ be a family of open subspaces of a uniform space $(X, \U)$ satisfying $X = \cup_{\alpha \in \Lambda} X_{\alpha}$. Then $X$ is locally $\Upsilon$-bounded if and only if each $X_{\alpha}$ is locally $\Upsilon$-bounded.

**Proof.** We only prove the converse part. Choose a $x \in X$ and say $x \in X_{\beta}$. Since $(X_{\beta}, U_{X_{\beta}})$ is locally $\Upsilon$-bounded, choose a $V \in U_{X_{\beta}}$ such that $V[x]$ is a $\Upsilon$-bounded subspace of $X_{\beta}$. Since $x \in \text{Int}_{X_{\beta}} V[x]$ is open in $X$, there exists an $U \in \U$ such that $U[x] \subseteq \text{Int}_{X_{\beta}} V[x]$. It follows that $U[x]$ is $\Upsilon$-bounded. Thus $X$ is locally $\Upsilon$-bounded. \hfill \square

**Theorem 4.7.** Let $\{X_{\alpha} : \alpha \in \Lambda\}$ be a family of closed subspaces of a uniform space $(X, \U)$ satisfying $X = \cup_{\alpha \in \Lambda} X_{\alpha}$. If the collection $\{X_{\alpha} : \alpha \in \Lambda\}$ is locally finite in $X$, then $X$ is locally $\Upsilon$-bounded if and only if each $X_{\alpha}$ is locally $\Upsilon$-bounded.

**Proof.** Forward implication is easy and we prove only the other direction. Let $\V$ be the uniformity on the sum $Y = \oplus_{\alpha \in \Lambda} X_{\alpha}$. Define $f : Y \to X$ by $f(x, \alpha) = x$. We claim that $f$ is a uniformly continuous, perfect mapping. Let $U \in \U$. For each $\alpha$ choose a $U_{\alpha} \in U_{X_{\alpha}}$ such that $\cup_{\alpha \in \Lambda} U_{\alpha} \subseteq U$. Also for each $\alpha$ define $V_{\alpha} = \{(x, \alpha), (y, \alpha) : (x, y) \in U_{\alpha}\}$. Clearly $V = \cup_{\alpha \in \Lambda} V_{\alpha} \in \V$. Let $(u, v) \in V$ and choose a $\beta$ so that $(u, v) \in V_{\beta}$. Thus $u = (x, \beta)$ and $v = (y, \beta)$ for some $(x, y) \in U_{\beta}$ and hence $(f(u), f(v)) \in U_{\beta}$ i.e. $(f(u), f(v)) \in U$. Clearly $f$ is uniformly continuous.

To show that $f$ is perfect, for each $\alpha$ we define a continuous mapping $\varphi_{\alpha} : X_{\alpha} \to Y$ by $\varphi_{\alpha}(x) = (x, \alpha)$. Let $F$ be closed in $Y$. Since each $\varphi^{-1}_{\alpha}(F)$ is closed and $X_{\alpha}$’s are locally finite in $X$, $f(F) = \cup_{\alpha \in \Lambda} \varphi^{-1}_{\alpha}(F)$ is also closed in $X$. Let $x \in X$. Now choose a $W \in \U$ such that $W[x]$ intersects only finitely many $X_{\alpha}$’s, say $W[x] \cap X_{\alpha} \neq \emptyset$ for $1 \leq i \leq k$. Clearly $f^{-1}(x) = \oplus_{\alpha \in \Lambda} \{x\}$ is compact. Thus $f$ is a perfect mapping. Now by Corollary 4.1 $X$ is locally $\Upsilon$-bounded. \hfill \square

The next two results concern respectively the locally $H$-bounded and locally $M$-bounded property in product spaces.

**Theorem 4.8.** Let $(X, \U)$ and $(Y, \V)$ be two uniform spaces. Then $(X \times Y, \U \times \V)$ is locally $H$-bounded if and only if both $(X, \U)$ and $(Y, \V)$ are locally $H$-bounded.

**Proof.** By considering the projection mappings, the forward implication follows from Corollary 4.1

Conversely assume that both $X$ and $Y$ are locally $H$-bounded. Let $(x, y) \in X \times Y$. Choose a $U \in \U$ and a $V \in \V$ such that $U[x]$ and $V[y]$ are $H$-bounded subspaces of $X$ and $Y$ respectively. By Theorem 2.1(1) $U[x] \times V[y]$ is a $H$-bounded subspace of $X \times Y$. Clearly $(x, y) \in \text{Int}_X U[x] \times \text{Int}_Y V[y]$ is an open set
in $X \times Y$. Thus there is a $W \in U \times V$ such that $W[\{(x, y)\}] \subseteq \text{Int}_X U[x] \times \text{Int}_Y V[y]$. Consequently $W[\{(x, y)\}]$ is an $H$-bounded subspace of $X \times Y$ and hence $X \times Y$ is locally $H$-bounded.

It is to be noted that locally $R$-bounded property is not productive (see Example 5.7). We are unable to verify whether the above result holds for locally $M$-bounded spaces. Still, we have the following observation that can be easily verified. Moreover, the same example also demonstrates that ‘locally $M$-bounded’ cannot be replaced by ‘locally $R$-bounded’ in the following result.

Theorem 4.9. Let $(X, U)$ and $(Y, V)$ be two uniform spaces. If $X$ is locally $M$-bounded and $Y$ is locally precompact, then $(X \times Y, U \times V)$ is locally $M$-bounded.

5. Examples

We now present examples to illustrate the distinction between the behaviours of the local variations as introduced in this article.

Recall that a collection $A$ of subsets of $\mathbb{N}$ is said to be an almost disjoint family if each $A \in A$ is infinite and for any two distinct elements $B, C \in A$, $|B \cap C| < \aleph_0$. Also $A$ is said to be a maximal almost disjoint (in short, MAD) family if $A$ is not contained in any larger almost disjoint family. For an almost disjoint family $A$, let $\Psi(A) = A \cup \mathbb{N}$ be the Isbell-Mrówka space. It is well known that $\Psi(A)$ is a locally compact zero-dimensional Hausdorff space (and hence is a Tychonoff space) (see [5]).

Note that $\Upsilon$-bounded implies locally $\Upsilon$-bounded. The following example shows that the class of locally $\Upsilon$-bounded uniform spaces properly contains the class of $\Upsilon$-bounded uniform spaces.

Example 5.1 (locally $\Upsilon$-bounded $\not\Rightarrow$ $\Upsilon$-bounded). Let $\Psi(A)$ be the Isbell-Mrówka space with $|A| > \aleph_0$. Let $U$ be the corresponding uniformity on $\Psi(A)$. Since $A$ is an uncountable discrete subspace of $\Psi(A)$, it follows that $\Psi(A)$ is not pre-Lindelöf. Thus $\Psi(A)$ cannot be $\Upsilon$-bounded. Since for each member of
Thus \( R \) is not locally \( \Upsilon \)-bounded by Proposition 4.2.

The preceding example can also be used to show the existence of locally pre-Lindelöf (resp. locally precompact) space which is not pre-Lindelöf (resp. precompact).

**Remark 5.1.** The Isbell-Mrówka space \( \Psi(A) \) is \( \Upsilon \)-bounded if and only if \(|A| \leq \aleph_0\).

We now give an example of a locally \( H \) (or, \( M \))-bounded (and hence locally \( M \))-bounded) space which is not locally \( R \)-bounded. First we recall that a set \( A \subseteq \mathbb{R} \) has strong measure zero if for every sequence \((\varepsilon_n)\) of positive reals there exists a sequence \((I_n)\) of intervals such that \(|I_n| < \varepsilon_n\) for all \( n \) and \( A \subseteq \bigcup_{n \in \mathbb{N}} I_n \).

**Example 5.2** (locally \( H \) (or, \( M \))-bounded \( \iff \) locally \( R \)-bounded). Consider \( \mathbb{R} \) with the uniformity induced by the Euclidean metric \( d \). Clearly \( \mathbb{R} \) is locally precompact and hence is locally \( H \)-bounded and locally \( M \)-bounded as well. Now if possible suppose that \( \mathbb{R} \) is locally \( R \)-bounded. By Proposition 4.1, \( \mathbb{R} \) is \( R \)-bounded.

Let \((\varepsilon_n)\) be a sequence of positive real numbers. Apply \( R \)-bounded property of \( \mathbb{R} \) to \((\varepsilon_n)\) to obtain a sequence \((x_n)\) of reals such that \( \mathbb{R} = \bigcup_{n \in \mathbb{N}} B_d(x_n, \varepsilon_n) \). Which in turn implies that \( \mathbb{R} \) has strong measure zero, a contradiction. Thus \( \mathbb{R} \) fails to be locally \( R \)-bounded.

The following is an example of a locally pre-Lindelöf space which is not locally \( \Upsilon \)-bounded.

**Example 5.3** (locally pre-Lindelöf \( \not\Rightarrow \) locally \( \Upsilon \)-bounded). Let \( \mathbb{R}^\omega \) be the Tychonoff product of \( \omega \)-copies of \( \mathbb{R} \). The topology of \( \mathbb{R}^\omega \) is induced by the metric \( d(x, y) = \sup_{i \in \mathbb{N}} \min\{|x_i - y_i|, 1\} \) on \( \mathbb{R}^\omega \), where \( x = (x_i) \) and \( y = (y_i) \). Let \( U \) be the uniformity on \( \mathbb{R}^\omega \) induced by \( d \). We claim that \( \mathbb{R}^\omega \) is not \( M \)-bounded.

On the contrary, assume that \( \mathbb{R}^\omega \) is \( M \)-bounded. For each \( n \in \mathbb{N} \) choose \( U_n = \{(x, y) \in \mathbb{R}^\omega \times \mathbb{R}^\omega : x = (x_i), y = (y_i), and |x_i - y_i| < n\} \). We now show that for each \( n \), \( U_n \subseteq U \). Fix \( n \in \mathbb{N} \). Choose \( 0 < \varepsilon < 1 \). Clearly each \( U_n \) contains the basic member of the form \( U_{\varepsilon, n} = \{(x, y) \in \mathbb{R}^\omega \times \mathbb{R}^\omega : d(x, y) < \varepsilon\} \). Thus \( U_n \subseteq U \) for each \( n \).

Apply \( M \)-bounded property of \( \mathbb{R}^\omega \) to \((U_n)\) to obtain a sequence \((F_n)\) of finite subsets of \( \mathbb{R}^\omega \) such that \( \mathbb{R}^\omega = \bigcup_{n \in \mathbb{N}} U_n[F_n] \). Say \( F_n = \{x^{(n,j)} = (x_i^{(n,j)}) : 1 \leq j \leq k_n\} \) for each \( n \).

Choose \( a = (x_i) \in \mathbb{R}^\omega \) such that \( x_n = n + \sum_{j=1}^{k_n} |x_i^{(n,j)}| \) for each \( n \). Also choose a \( n_0 \in \mathbb{N} \) such that \((x, x_{(n_0, j)})\) \( \in U_{n_0} \) for some \( x_{(n_0, j)} \) \( \in F_{n_0} \). By the construction of \( U_{n_0} \), we obtain \(|x_{n_0} - x_{(n_0, j)}| < n_0\), which is a contradiction as \( x_{n_0} = n_0 + \sum_{j=1}^{k_{n_0}} |x_i^{(n_0,j)}| \). Thus \( \mathbb{R}^\omega \) is not \( M \)-bounded. Now apply Proposition 4.1 to conclude that \( \mathbb{R}^\omega \) is not locally \( M \)-bounded (and hence it is not locally \( \Upsilon \)-bounded by Figure 4).
We now give an example of a locally $H$-bounded (and hence a locally $M$-bounded) space which is not locally precompact. First we recall that a subset $A$ of a metric space $(X,d)$ is said to be uniformly discrete if $A$ is countably infinite and there exists a $\varepsilon > 0$ such that if $x, y \in A$ with $x \neq y$, then $d(x, y) > \varepsilon$.

**Example 5.4** (locally $H$ (or, $M$)-bounded $\nRightarrow$ locally precompact). Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be an injective function such that $f(\mathbb{N} \times \mathbb{N})$ is a uniformly discrete subset of $\mathbb{R}$. Define a metric $\rho$ on $\mathbb{N} \times \mathbb{N}$ by

$$
\rho((m, n), (m', n')) = \begin{cases} 
\infty & \text{if } m \neq m', \\
\frac{1}{m} & \text{if } m = m' \text{ but } n \neq n', \\
0 & \text{if } m = m', n = n'.
\end{cases}
$$

Let $\sigma$ be a metric on $\mathbb{R}$ satisfying $\sigma(x, y) \leq d(x, y)$, where $d$ is the usual metric on $\mathbb{R}$, and $\sigma(f(m, n), f(m', n')) \leq \rho((m, n), (m', n'))$. Let $\mathcal{U}_d$ and $\mathcal{U}_\sigma$ be the uniformity on $\mathbb{R}$ induced by $d$ and $\sigma$ respectively. Since $(\mathbb{R}, d)$ is $\sigma$-compact, $(\mathbb{R}, \mathcal{U}_d)$ is $H$-bounded. By Lemma 3.2, $(\mathbb{R}, \mathcal{U}_\sigma)$ is also $H$-bounded (and hence it is locally $H$-bounded). Clearly $(\mathbb{R}, \sigma)$ is complete. Since for each $m$ the ball $B_{\infty}(f(m, n), \frac{1}{m})$ contains the infinite discrete set $\{f(m, n) : n \in \mathbb{N}\}$, $(\mathbb{R}, \sigma)$ is not locally compact. Also since every complete precompact space is compact, a similar observation as in Theorem 4.2 shows that a complete locally precompact space is locally compact. We can now conclude that $(\mathbb{R}, \mathcal{U}_\sigma)$ is not locally precompact.

We now make a quick observation which reflects that Lemma 3.2 does not hold for locally $\Upsilon$-bounded spaces.

**Example 5.5.** Consider the complete metric space $\ell^\infty$ with the supremum metric. Observe that $\ell^\infty$ is not locally Lindelöf. Let $\mathcal{U}$ be the uniformity on $\ell^\infty$ induced by the sup. metric. Since every complete pre-Lindelöf space is Lindelöf, similarly as Theorem 4.2 it is easy to see that every complete locally pre-Lindelöf space is locally Lindelöf. Consequently $(\ell^\infty, \mathcal{U})$ is not locally $\Upsilon$-bounded. It follows that $(\ell^\infty, \mathcal{U})$ is not locally $\Upsilon$-bounded. But the uniform space $(\ell^\infty, \mathcal{D})$ with discrete uniformity $\mathcal{D}$ is locally $\Upsilon$-bounded.

Recall that product of a $M$-bounded (resp. $H$-bounded) space with a precompact space is again $M$-bounded (resp. $H$-bounded), but if we replace ‘precompact’ by ‘locally precompact’, then the product need not be $M$-bounded (resp. $H$-bounded).

**Example 5.6** ($M$ (resp. $H$)-bounded $\times$ locally precompact $\nRightarrow$ $M$ (resp. $H$)-bounded). Let $X$ be the Isbell-Mrówka space $\Psi(A)$, where $A$ is a MAD family. Let $\mathcal{U}$ be the corresponding uniformity on $X$. By Example 5.4, $X$ is locally precompact but not pre-Lindelöf. Let $Y = \mathbb{R}$ with the usual uniformity $\mathcal{V}$. Since $Y$ is $\sigma$-compact, $Y$ is $M$-bounded as well as $H$-bounded. If possible suppose that $X \times Y$ is $M$-bounded. Using the projection mapping and then by applying
Theorem 2.2(1), we have $X$ is $M$-bounded, which is a contradiction. Thus $X \times Y$ is not $M$-bounded and hence it fails to be $H$-bounded.

We now observe that locally $R$-bounded property behaves somewhat differently from locally $M$-bounded and locally $H$-bounded property as well.

Example 5.7 (locally $R$-bounded $\times$ locally precompact $\not\Rightarrow$ locally $R$-bounded). This example is about the product of a locally $R$-bounded space and a locally precompact space which fails to be locally $R$-bounded. To prove this, first consider a $R$-bounded uniform space $X$ and also consider $R$ with usual uniformity. Suppose if possible that $X \times R$ is locally $R$-bounded. By Corollary 4.1 and by means of projection mapping, it follows that $R$ is locally $R$-bounded, which is a contradiction (see Example 5.2). Thus the product $X \times R$ fails to be locally $R$-bounded.

Remark 5.2. Product of a $R$-bounded space and a precompact space need not be $R$-bounded. The reason is as follows. If possible, suppose that the assertion is true. Now consider Example 5.7. Since $R$ is a countable union of its precompact subspaces, our supposition together with Theorem 3.1 imply $X \times R$ is $R$-bounded. We then again arrive at a contradiction as in Example 5.7. Thus the product of a $R$-bounded space and a precompact space need not be $R$-bounded.

We now present an example of a $\Upsilon$-bounded space for which Theorem 4.5 fails to hold.

Example 5.8. Consider the sum $\bigoplus_{\alpha \in \omega_1} X$ of $\omega_1$ copies of a $\Upsilon$-bounded space $(X, U)$. The sum is uniformly isomorphic to $(X \times \omega_1, U \times D)$, where $D$ is the discrete uniformity on $\omega_1$. From Theorem 2.2(1), $X \times \omega_1$ is not $\Upsilon$-bounded and consequently $\bigoplus_{\alpha \in \omega_1} X$ is not $\Upsilon$-bounded.

Remark 5.3.

(1) It is interesting to observe that Theorem 4.6 also cannot be recovered for $\Upsilon$-bounded spaces. Consider the Isbell-Mrówka space $\Psi(A)$ as in Example 5.1. As noted in that example, observe that for each member of $\Psi(A)$ we can always find a countable basic open set containing it. Thus $\Psi(A)$, which is itself not $\Upsilon$-bounded, is a union of its $\Upsilon$-bounded open subspaces.

(2) Also note that Theorems 2.3, 2.4 and 2.5 can not be extended to locally pre-Lindelöf spaces. Assume that $\omega_1 < \min\{b, \text{cov}(\mathcal{M})\}$ (which implies $\omega_1 < d$ too). Let $\Psi(A)$ be the Isbell-Mrówka space with $|A| = \omega_1$. By Example 5.1 $\Psi(A)$ is locally pre-Lindelöf. Unfortunately by Remark 5.1 $\Psi(A)$ is not $\Upsilon$-bounded.

References

[1] T. Bartoszyński, Combinatorial aspects of measure and category, Fund. Math., 127 (3) (1987), 225–239.
[2] T. Bartoszyński, H. Judah, Set Theory: On the Structure of the Real Line, A.K. Peters, 1995.
[3] A.A. Borubaev, Uniform Spaces and Uniformly Continuous Mappings, Ilim, Frunze, 1990.
[4] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[5] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, NJ, 1960.
[6] W. Just, A.W. Miller, M. Scheepers, P.J. Szeptycki, The combinatorics of open covers (II), Topology Appl., 73 (1996), 241–266.
[7] Lj.D.R. Kočinac, Selection principles in uniform spaces, Note Mat., 22 (2) (2003), 127–139.
[8] Lj.D.R. Kočinac, Selected results on selection principles, in: Proc. Third Seminar on Geometry and Topology, Tabriz, Iran, July 15-17, 2004, 71–104.
[9] Lj.D.R. Kočinac, Some covering properties in topological and uniform spaces, Proc. Steklov Inst. Math., 252 (2006), 122–137.
[10] Lj.D.R. Kočinac, Variations of classical selection principles: An overview, Quaest. Math., 43 (8) (2020), 1121-1153.
[11] Lj.D.R. Kočinac, H.-P.A. Künzi, Selection properties of uniform and related structures, Topology Appl., 160 (2013), 2495–2504.
[12] G. Di Maio, Lj.D.R. Kočinac, Statistical convergence in topology, Topology Appl., 156 (2008), 28–45.
[13] H.-P.A. Künzi, M. Mršević, I.L. Reilly, M.K. Vamanamurthy, Pre-Lindelöf quasi-pseudometric and quasi-uniform spaces, Mat. Vesnik, 46 (3-4) (1994), 81–87.
[14] S. Mrówka, On completely regular spaces, Fund. Math., 41 (1954), 105–106.
[15] E.A. Michael, Bi-quotient maps and cartesians product of quotient maps, Ann. Inst. Fourier (Grenoble), 18 (2) (1968), 287–302.
[16] I. Rechaw, Every Lusin set is undetermined in the point-open game, Fund. Math., 144 (1994), 43–54.
[17] M. Sakai, M. Scheepers, The combinatorics of open covers, in: Recent Progress in General Topology III, K.P. Hart, J. van Mill, P. Simon (eds.), 751-800, Atlantis Press, Amsterdam/Paris, 2014.
[18] M. Scheepers, Combinatorics of Open Covers I: Ramsey Theory, Topology Appl., 69 (1996), 31–62.
[19] L.A. Steen, J.A. Seebach Jr., Counterexamples in Topology, Springer-Verlag New York, Heidelberg, Berlin, (1978).
[20] J.W. Tukey, Convergence and Uniformity in Topology, Ann. of Math. Stud., Princeton University Press, 1940.

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