Let $F$ be a local non-archimedean field and $G$ be the group of $F$-points of a split connected reductive group over $F$. In \cite{4} we define an algebra $\mathcal{J}(G)$ of functions on $G$ which contains the Hecke algebra $\mathcal{H}(G)$ and is contained in the Harish-Chandra Schwartz algebra $\mathcal{C}(G)$. We consider $\mathcal{J}(G)$ as an algebraic analog of the algebra $\mathcal{C}(G)$.

Given a parabolic subgroup $P$ of $G$ with a Levi subgroup $M$ and the unipotent radical $U_P$ we write $X_P := G/U_P$. Let $\mathcal{S}_c(X_P)$ be the space of locally constant functions on $X_P$ with compact support and $\mathcal{S}_{cusp,c}(X_P) \subset \mathcal{S}_c(X_P)$ be subspace of functions whose right shifts span a cuspidal representation of $M$.

In this paper we study two versions of the Schwartz space of $X_P$. The first is $\mathcal{S}(X_P) := \mathcal{J}(\mathcal{S}_c(X_P))$ and the 2nd is the space spanned by functions of the form $\Phi_{Q,P}(\phi)$ where $Q$ is another parabolic with the same Levi subgroup, $\phi \in \mathcal{S}_c(X_Q)$ and $\Phi_{Q,P}$ is a normalized intertwining operator from $L^2(X_Q)$ to $L^2(X_P)$. We formulate a series of conjectures about these spaces; for example, we conjecture that $\mathcal{S}^c(X_P) \subset \mathcal{S}(X_P)$ and that this embedding is an isomorphism on $M$-cuspidal part. We give a proof of some of our conjectures (cf. Theorem \ref{thm:main}).

1. Introduction and Statement of the Results

1.1. Notation. Let $F$ be a non-archimedean local field with ring of integers $\mathcal{O}$; we shall fix a generator $\kappa$ of the maximal ideal of $\mathcal{O}$. Typically, we shall denote algebraic varieties over $F$ by boldface letters (e.g. $G, X$ etc.) and the corresponding sets of $F$-points – by the corresponding ordinary letters (i.e. $G, X$ etc.).

Let $G$ be a connected split reductive group $G$ over $F$ with a Borel subgroup $B$, unipotent radical $U$ be the unipotent radical of $B$ and $T := B/U$ be the Cartan torus. We fix an imbedding of $T$ into $B$.

Let $\Lambda$ be the lattice of cocharacters of $T$ and $\Lambda^\vee$ be the lattice of characters of $T$. We fix a Haar measure $dg$ on $G$ and we denote by $\mathcal{H}(G)$ the Hecke algebra of locally constant compactly supported functions on $G$. A choice of a Haar measure defines a structure of a locally unital associative algebra on $\mathcal{H}(G)$. As well-known the category $\mathcal{M}(G)$ of smooth $G$-modules is equivalent to the category of non-degenerate $\mathcal{H}(G)$-modules.

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1.2. Functions on $X_P$. Let $P$ be a parabolic subgroup of $G$ with a Levi subgroup $M$ and unipotent radical $U_P$. Let $X_P = G/U_P$. This space has a natural $G \times M$ action. Therefore the space $S_c(X_P)$ of locally constant compactly supported functions on $X_P$ becomes a $G \times M$ module; for convenience we are going to twist the $M$ action by the square root of the absolute value of the determinant of the $M$-action on the Lie algebra $u_P$ of $U_P$.

It is easy to see that $X_P$ possesses unique (up to constant) $G$-invariant measure, hence we can talk about $L^2(X_P)$. It has a natural unitary action of $G \times M$ (the action of $M$ is unitary if we twist it by $|\delta_P|^{1/2}$ where $\delta_P : M \to \mathbb{G}_m$ is the determinant of the action of $M$ on the Lie algebra of $U_P$).

1.3. The algebra $\mathcal{J}(G)$ and the space $\mathcal{S}(X_P)$. In [4] we have defined certain algebra $\mathcal{J}(G)$ of functions on $G$ which contains the Hecke algebra $\mathcal{H}(G)$ which can be thought of as an algebraic version of the Harish-Chandra Schwartz space $\mathcal{C}(G)$ (the definition will be recalled in Section 3); in particular, $\mathcal{J}(G)$ is a smooth $G \times G$-module. It is explained in loc. cit. that $\mathcal{J}(G)$ acts on $L^2(X_P)$ for any $P$, so we can set

$$\mathcal{S}(X_P) = \mathcal{J}(G) \cdot S_c(X_P),$$

where $S_c(X_P)$ stands for the space of locally constant functions with compact support on $X_P$. The space $\mathcal{S}(X_P)$ is a smooth $G \times M$-module.

1.4. Intertwining operators – $L^2$-version. The following result is essentially Theorem 2.1 of [1]:

**Theorem 1.5.** Let $P$ and $Q$ be two associate parabolics, i.e. two parabolics with the same Levi subgroup $M$. Then there exists a $G \times M$-equivariant unitary isomorphism $\Phi_{P,Q} : L^2(X_P) \to L^2(X_Q)$. These isomorphisms satisfy the following properties:

1. $\Phi_{P,P} = \text{id}$
2. For 3 parabolic subgroups $P, Q, R$ with the same Levi subgroup $M$ we have $\Phi_{Q,R} \circ \Phi_{P,Q} = \Phi_{P,R}$.

Note that (1) and (2) together imply that $\Phi_{Q,P} \circ \Phi_{P,Q} = \text{id}$.

**Warning.** The operator $\Phi_{P,Q}$ is not canonical - it depends on various choices. In what follows we are going to choose some operators $\Phi_{P,Q}$ satisfying the above requirements.

1.6. Another version of a space of functions on $X_P$. For a parabolic subgroup $P$ of $G$ with chosen Levi subgroup $M$ let $\text{Ass}(P)$ denote the set of all parabolics $Q$ containing $M$ as a Levi subgroup. We now define another version of the $\mathcal{S}'(X_P)$ of the Schwartz space of functions on $X_P$ by setting

$$\mathcal{S}'(X_P) = \sum_{Q \in \text{Ass}(P)} \Phi_{Q,P}(S_c(X_Q)).$$

The expected relationship between the two versions of the Schwartz space is described by the following conjecture:
Conjecture 1.7.  
(1) \( S'(X_P) \subset S(X_P) \).
(2) \( S'(X_P)_{\text{cusp}} = S(X_P)_{\text{cusp}} \). Here by \( S(X_P)_{\text{cusp}} \) (resp. \( S'(X_P)_{\text{cusp}} \)) we denote the 
\( M \)-cuspidal part of \( S(X_P)_{\text{cusp}} \) (resp. of \( S'(X_P)_{\text{cusp}} \)).
(3) The operator \( \Phi_{P,Q} \) defines an isomorphism between \( S(X_P) \) and \( S(X_Q) \).
(4) The right action of \( J(M) \) on \( L^2(X_P) \) preserves \( S(X_P) \) (thus making \( S(X_P) \)
into a \((J(G), J(M))\)-bimodule (It is easy to see that the Schwartz algebra 
\( \mathcal{C}(G) \) acts on \( L^2(X_P) \) on the right; hence \( L^2(X_P) \) is also a right module over 
\( J(M) \)).

We denote by \( S(X_P)_{\text{cusp}} \) (resp. \( S'(X_P)_{\text{cusp}} \)) the \( M \)-cuspidal part of \( S(X_P)_{\text{cusp}} \) (resp. of \( S'(X_P)_{\text{cusp}} \)).

Below is the main result of this paper:

Theorem 1.8.  
(1) Let \( P \) be a maximal parabolic of \( G \). Then we have \( S'(X_P)_{\text{cusp}} = S(X_P)_{\text{cusp}} \).
(2) Assume that \( S'(X_P)_{\text{cusp}} \subset S(X_P)_{\text{cusp}} \). Then
\( a) \ S'(X_P)_{\text{cusp}} = S(X_P)_{\text{cusp}} \)
\( b) \ The \ operator \( \Phi_{P,Q} \) defines an isomorphism between \( S(X_P)_{\text{cusp}} \) and
\( S(X_Q)_{\text{cusp}} \).

The rest of the paper is devoted to the proof of Theorem 1.8.

1.9. Examples. Let us look at two extreme cases. First, consider the case \( P = G \).
In this case \( \text{Ass}(P) \) consists of 1 element and therefore \( S'(X_P) = \mathcal{H}(G) \); similarly,
\( S(X_P) = J(G) \). Conjecture \( \ref{conj:1.7} \) is obvious in this case. Also, this example shows that
\( S'(X_P) \neq S(X_P) \) in general since \( \mathcal{H}(G) \neq J(G) \).

Second, let us consider the case when \( P \) is a Borel subgroup \( B \) of \( G \). Then \( M \) is a
maximal split torus \( T \) of \( G \) and we have \( \mathcal{H}(T) = J(T) \), so assertion \( 4 \) is obvious. On
the other hand, any representation of \( T \) is cuspidal, so we have \( S(X_B)_{\text{cusp}} = S(X_B) \)
and \( S'(X_B)_{\text{cusp}} = S'(X_B) \). Hence assertion \( 2 \) says in this case that \( S'(X_B) = S(X_B) \).
This statement was conjectured in \( \ref{conj:1.7} \) (note that \( S'(X_B) \) is exactly the Schwartz space of \( \mathcal{B} \)), where we proved this conjecture for the \( I \)-invariant part of both spaces (here \( I \) stands for an Iwahori subgroup of \( G \)).

2. Tempered representations and Harish-Chandra algebra

Definition 2.1. For any smooth representation \((\pi, V)\) of \( G \) and smooth vectors \( v \in V, v^* \in V^* \) we define the function \( m_{v,v^*} \) on \( G \)
\[ m_{v,v^*}(g) := v^*(\pi(g)v), v \in V \]
and say that \( m_{v,v^*} \) is a matrix coefficient of the representation \( \pi \).

The algebra \( \mathcal{H}(G) \) can be embedded into the Harish-Chandra Schwartz algebra \( \mathcal{C}(G) \). The algebra \( \mathcal{C}(G) \) does not act on all smooth representations of \( G \) but only
on tempered ones. Let \( G \) be a locally compact group. In this subsection we only consider unitary representations \((\pi, V, (,))\) of \( G \).
Definition 2.2.  
(1) We say that a representation $(\pi, V, (\, , \, ))$ is in the closure of a representation $(r, W, (\, , \, ))$ if for any vectors $v_1, \ldots, v_n \in V$, a compact $C$ in $G$ and $\epsilon > 0$ there exist vectors $w_1, \ldots, w_n \in W$ such that
\[ |(\pi(g)v_i, v_j) - (r(g)v_i, v_j)| \leq \epsilon \] for all $i, j, 1 \leq i, j \leq n$ and $g \in C$.
(2) A representation $(\pi, V)$ of $G$ is tempered if it is in the closure of the regular representation of $G$.
(3) We denote by $M_t(G)$ the category of tempered representations.
(4) We denote by $\hat{G}_t$ the set of tempered irreducible representations. As follows Schur’s lemma we can consider $\hat{G}_t$ as a subset of the set $\hat{G}$ of smooth irreducible representations of $G$.

The following results are proven in [5] and [9].

Claim 2.3.  
(1) A unitary irreducible representation $(\pi, V, (\, , \, ))$ of a reductive group $G$ over a local field is tempered iff matrix coefficients of $\pi$ belong to $L^{2+\epsilon}(G/Z(G))$ for any $\epsilon > 0$.
(2) For any tempered representation $V$ of $G$ the action of the algebra $\mathcal{H}(G)$ on $V$ extends to a continuous action of the Harish-Chandra algebra $\mathcal{C}(G)$.
(3) Let $P$ be a parabolic subgroup of $G$ with a Levi group $M$ and $\sigma$ be a tempered irreducible representation of $M$. Then the unitary induced representation $\pi_\sigma = \text{ind}_G^G(\sigma)$ is tempered.
(4) For a generic unitary character $\chi : M \to S^1$ the representation $\pi_\sigma \otimes \chi$ (which is tempered) is irreducible.
(5) Any representation which is a Hilbert space integral of tempered representations is tempered.

Let us now discuss the Harish-Chandra algebra $\mathcal{C}(G)$.

Definition 2.4.  
(1) Recall that for any $g \in G$, there exists a unique dominant coweight $\lambda(g)$ of $T$ such that $g \in G(\mathcal{O})\lambda(g)(\kappa)G(\mathcal{O})$. We define a function $\Delta$ on $G$ by $\Delta(g) := q^{\langle \lambda, \rho \rangle}$.
(2) We say that a function $f : G \to \mathbb{C}$ is a Schwartz function if
(a) There exists an open compact subgroup $K$ of $G$ such that $f$ is two-sided $K$-invariant.
(b) For any polynomial function $p : G \to F$ and $n > 0$, there exists a constant $C = C_{p,n} \in \mathbb{R}_{>0}$ such that
\[ \Delta(g)|f(g)| \leq C \ln^{-n}(1 + |p(g)|) \] for all $g \in G$.

We denote by $\mathcal{C}(G)$ the space of Schwartz functions.

Obviously we have an inclusion $\mathcal{H}(G) \hookrightarrow \mathcal{C}(G)$ and $\mathcal{H}(G) \subset \mathcal{C}(G)$ is dense in the natural topology of $\mathcal{C}(G)$.

The following statements are well known (see for example [9]).
Claim 2.5.  
1. $C(G)$ has an algebra structure with respect to convolution.
2. Any tempered representation of $\mathcal{H}(G)$ extends to a continuous representation of $C(G)$.

Lemma 2.6. The natural unitary representation of $G \times M$ on the smooth part of $L^2(X_P)$ is tempered.

Proof. Since the right action of $M$ on $X_P$ is free we can write the space $L^2(X_P)$ as a Hilbert space integral

$$L^2(X_P) = \int_{\rho \in \hat{M}} \pi(\rho) \otimes \rho \, d\mu_M$$

where $\mu_M$ is the Plancher measure on $\hat{M}$ and $\pi(\rho)$ is a unitary representation of $G$. It is easy to see that $\pi(\rho) = i_{GP}(\rho)$. Now Lemma 2.6 follows from Claim 2.5. □

Corollary 2.7. The natural representation of $\mathcal{H}(G)$ on the smooth part of $L^2(X_P)$ extends to a representation of $C(G)$.

3. Paley-Wiener theorems and the definition of the algebra $J(G)$

3.1. The Paley-Wiener theorem for $\mathcal{H}(G)$. Let $P$ be a parabolic subgroup of $G$ with a Levi group $M$. The set $\Psi(M)$ of unramified characters of $M$ is equal to $\Lambda_M^\vee \otimes \mathbb{C}^\times$ where $\Lambda_M^\vee \subset \Lambda^\vee$ is the subgroup of characters of $T$ trivial on $T \cap [M, M]$. So $\Psi(M)$ has a structure of a complex algebraic variety; the algebra of polynomial functions on $X_M$ is equal to $\mathbb{C}[\Lambda_M]$ where $\Lambda_M$ is the lattice dual to $\Lambda_M^\vee$. We denote by $\Psi_t(M) \subset \Psi(M)$ the subset of unitary characters.

For any $(\sigma, V) \in \mathcal{M}(M)$ we denote by $i_{GP}(\sigma)$ the corresponding unitarily induced object of $\mathcal{M}(G)$. As a representation of $G(O)$ this representation is equal to $\text{ind}_{P(O)}^{G(O)}(\sigma)$. So for any unramified character $\chi : M \to \mathbb{C}^*$ the space $V_\chi$ of the representation $i_{GP}(\sigma \otimes \chi)$ is isomorphic to the space $V_\sigma$ of the representation $i_{GP}(\sigma)$ and is independent on a choice of $\chi$. Since $X_M$ has a structure of an algebraic variety over $\mathbb{C}$ it make sense to say that a family $\eta_\chi \in \text{End}(V_\chi), \chi \in \Psi(M)$ is regular or smooth.

We denote by $\text{Forg} : \mathcal{M}(G) \to \text{Vect}$ the forgetful functor, by $\widehat{\mathcal{E}(G)} = \{e(\pi)\}$ the ring of endomorphisms of $\text{Forg}$ and define $\mathcal{E}(G) \subset \widehat{\mathcal{E}(G)}$ as the subring of endomorphisms $\eta_\pi$ such that

1) For any Levi subgroup $M$ of $G$ and $\sigma \in \text{Ob}(\mathcal{M}(M))$, the endomorphisms $\eta_{i_{GP}(\sigma \otimes \chi)}$ are regular functions of $\chi$.

2) There exists an open compact subgroup $K$ of $G$ such that $\eta_\pi$ is $K \times K$-invariant for every $\pi$.

By definition, we have a homomorphism

$$PW : \mathcal{H}(G) \to \mathcal{E}(G), \quad f \mapsto \pi(f).$$
The following is usually called "the matrix Paley-Wiener theorem" (cf. [2], Theorem 25):

**Theorem 3.2.** The map $PW$ is an isomorphism.

### 3.3. The Paley-Wiener $C(G)$. As follows from the Claim 2.3 the representations $\pi_{\sigma,\chi}$ of $G$ belong to $\mathcal{M}_t(G)$. for any tempered representation of $M$ and a unitary character $\chi$ of $M$.

Let $\mathcal{E}_t(G)$ be the subring of endomorphisms $\{\eta\}$ of the forgetful functor $\text{For}t: \mathcal{M}_t(G) \rightarrow \text{Vect}$ such that

1. The function $\chi \mapsto \eta(\pi_{\sigma,\chi})$ is a smooth function of $\chi \in \Psi_t(M)$ for any Levi subgroup $M$ of $G$ and $\sigma \in \mathcal{M}_t$.

2. There exists an open compact subgroup $K$ of $G$ such that $\eta$ is $K \times K$-invariant.

The following version of the matrix Paley-Wiener theorem is contained in the last section of [10].

**Claim 3.4.** The map $f \mapsto \pi(f)$ defines an isomorphism between algebras $C(G)$ and $\mathcal{E}_t(G)$.

### 3.5. The definition of the algebra $J(G)$. Let $P$ be a parabolic subgroup with Levi group $M$. We say that an unramified character $\chi: M \rightarrow \mathbb{C}^*$ is (non-strictly) positive if for any coroot $\alpha$ of $G$, such that the corresponding root subgroup lies in the unipotent radical $U_P$ of $P$ (which in particular defines a homomorphism $\alpha: F^* \rightarrow Z(M)$), we have $|\chi(\alpha(x))| \geq 1$ for $|x| \geq 1$.

Let $\mathcal{E}_{J}(G)$ be ring of collections $\{\eta_\sigma \in \text{End}_C(V)\}$ for tempered irreducible $(\pi, V)$ which extend to a rational function $E_{iG_P(\omega \otimes \chi)} \in \text{End}_C(\sigma \otimes \chi)$ for every tempered irreducible representation $\sigma$ of $M$ and which are

a) regular on the set of characters $\chi$ such that $\chi^{-1}$ is (non-strictly) positive.

b) $K \times K$-invariant for some open compact subgroup $K$ of $G$.

As follows from the definition, we have an embedding $\mathcal{E}_{J}(G) \rightarrow \mathcal{E}_t(G)$.

**Definition 3.6.** We define $J(G)$ to be the preimage of $\mathcal{E}_{J}(G)$ in $C(G)$. Note that we have natural embeddings $\mathcal{H}(G) \subset J(G) \subset C(G)$.

Next, let us explain certain direct sum decompositions of algebras $C(G)$ and $J(G)$.

**Definition 3.7.** (1) Let $(M, \sigma), (M', \sigma')$ be a pair of square-integrable representations of Levi subgroups of $G$. We write $(M, \sigma) \sim (M', \sigma')$ if there exists an element $g \in G$ and a unramified character $\chi \in \Psi(M)$ such that $M' = M^g$ and $(\sigma')^g$ is equivalent to $\sigma \otimes \chi$.

(2) We denote by $R$ the set of equivalent classes of such representations $(M, \sigma)$.

(3) For any $r = (M, \sigma)$ we denote by $\mathcal{M}_r(G) \subset \mathcal{M}_t(G)$ the subcategory of representations in the closure of $\bigcup_{\chi \in \Psi(M)} i_{G_P}(\sigma \otimes \chi)$ where $P = MU_P$ is a parabolic subgroup and by $C_r(G) \subset C(G)$ the corresponding subalgebra.
For any \( r \in R \) we define \( J_r(G) := \mathcal{C}_r(G) \cap \mathcal{J}(G) \)

The following statement is contained in [10].

**Claim 3.8.** (1) The subcategory \( \mathcal{M}_r(G) \) does depends neither on a representative \((M, \sigma)\) of \( r \) nor on the choice of a parabolic \( P \).
(2) \( \mathcal{M}(G) = \bigoplus_{r \in R} \mathcal{M}_r(G) \)

**Corollary 3.9.** \( \mathcal{J}(G) = \bigoplus_{r \in R} \mathcal{J}_r(G) \)

4. **Intertwining operators**

Until the end of this section we fix \( r \in R \) and choose a representative \((M, \sigma)\) of \( r \) and a parabolic subgroup \( P = MU_P \subset G \). We write \( X_P := G/U_P \) and denote by \( dx \) a \( G \)-invariant measure on \( X_P \).

For any character \( \chi \in \Psi(M) \) we define \( (\pi_P(\chi), V_P, \chi) := \text{ind}_{G}^{G(\mathcal{O})} \sigma_{M(\mathcal{O})} \).

The restriction of the representation \( \pi_P(\chi) \) to the subgroup \( G(O) \) does not depend on \( \chi \) (and is equal

to \( \text{ind}_{G(\mathcal{O})}^{G(\mathcal{O})} \sigma_{M(\mathcal{O})} \)). We denote this space by \( V_P \).

Of course the image of \( V_P, \chi \in \Psi(M) \) in the space of \( M \)-valued functions on \( X_P \) depends on \( \chi \).

For any \( f \in V_P, \chi \in \Psi(M) \) we denote by \( f \chi \) the corresponding function of \( X_P \).

**Definition 4.1.** (1) We denote \( \mathcal{C}(X_P) \) the space of smooth complex-valued functions on \( X_P \) and \( \mathcal{S}_c(X_P) \subset \mathcal{S}(X_P) \) be the subspace of compactly supported functions.
(2) For any pair \( P = MU_P, Q = MU_Q \) of associated parabolic subgroups we denote by \( I_{P,Q} : \mathcal{S}_c(X_P) \to \mathcal{C}(X_Q) \) the geometric (or non-normalized) intertwining operator given by

\[
I_{P,Q}(f)(g) = \int_{U_Q} f(gu)\,du
\]

The following statement is contained in Section 2 of [1] (Theorem 2.7); it is a slightly stronger version of Theorem [1.5].

**Claim 4.2.** (1) There exists an non-empty open subset \( \Psi_+(M) \subset \Psi(M) \) such that the integral \( I_{P,Q}(f) = \int_{U_Q} f(gu)\,du \) is absolutely convergent for all \( f \in V_{\chi} \).
(2) For any \( f \in V_{\chi} \) the function \( I_{P,Q}(f_{\chi}) \in V_Q \) is a rational function of \( \chi \).
(3) The map \( I_{P,Q} : V_{P,\chi} \to V_{Q,\chi} \) is \( G \)-covariant.
(4) There exist rational \( \mathbb{C} \)-valued functions \( r_{P,Q}(\chi) \) such that the operators \( \Phi_{P,Q} := r_{P,Q}I_{P,Q} \) satisfy the following:
   (a) \( \Phi_{P,Q} \) is regular on the set of non-strongly elements with respect to \( P \).
   (b) \( \Phi_{P,Q} \circ \Phi_{Q,P} = Id \)
   (c) \( \Phi_{P,Q}(\chi) \) is unitary for unitary characters \( \chi \).
   (d) For any three associate parabolic subgroups \( P_i = MU_i, 1 \leq i \leq 3 \) we have \( \Phi_{P_1,P_2} \circ \Phi_{P_2,P_3} = \Phi_{P_1,P_3} \).
4.3. Functions on $X_P$. We define two subspace of $L^2(X_P)$

**Definition 4.4.**

1. $S(X_P) := \mathcal{J}(G) \cdot S_c(X_P)$
2. $S'(X_P) := \sum_Q \Phi_{P,Q}(S_c(X_Q))$

where the sum is over the set of parabolic subgroup $Q = MU_Q$ of $G$ associated with $P$.

**Remark 4.5.** Operators $\Phi_{P,Q}$ are not canonical but it is easy to see that any two choices differ by the multiplication by an regular invertible function on $\Psi(M)$ and therefore the space $S'(X_P)$ is well defined.

**Proposition 4.6.** There exists a surjective (but not necessarily injective) morphism of $G \times M$-modules $\mathcal{J}(G)_{U_P} \rightarrow S(X_P)$.

**Proof.** We define a map $\alpha_P : \mathcal{C}(G) \rightarrow L^2(X_P)$ by

$$\alpha(f)(g) := \int_{U_P} f(gu)\,du$$

where the absolute convergence of the integral follows from Theorem 4.4.3 in [7]. It is clear that $\alpha_P$ factorizes through the map $\zeta_P : \mathcal{C}(G)_{U_P} \rightarrow L^2(X_P)$. It is clearly sufficient to prove the following statement.

**Lemma 4.7.**

1. $\alpha_P(\mathcal{J}(G)) \subset S(X_P)$
2. $\alpha_P$ defines a surjection from $\mathcal{J}(P)_{U_P}$ onto $S(X_P)$.

**Proof.** Since the map $\alpha$ commutes with the action of the algebra $\mathcal{J}(G)$ and $\mathcal{J}(G)\mathcal{H}(G) = \mathcal{J}(G)$ it is sufficient to see that $\alpha_P(\mathcal{H}(G)) = S_c(X_P)$. But the last claim is obviously true. \hfill $\square$

Let us explain why $\zeta$ is not necessarily injective. Let $G = SL(2, F)$ and $P$ be a Borel subgroup of $G$; thus $X_P$ can be naturally identified with $F^2 \setminus \{0\}$. Let $St$ denote the Steinberg representation of $G$. Then $\mathcal{J}(G)$ contains a direct summand isomorphic to $\text{End}_f(St)$ where $\text{End}_f$ stands for endomorphisms of finite rank. It is easy to see that any homomorphism of $G$-modules from $St$ to $L^2(F^2 \setminus \{0\})$ is equal to 0. Hence the above subalgebra must act by 0 on $L^2(F^2 \setminus \{0\})$. On the other hand, $\dim St_{U_P} = 1$ hence $St \otimes St_{U_P} \simeq St$ is a non-zero subspace of $\mathcal{J}(G)_{U_P}$ which lies in the kernel of $\zeta$.

We now pass to the proof of Theorem 1.8. First we are going to discuss the explicit form of normalized intertwining operators for maximal parabolics.

5. INTERTWIXING OPERATORS IN THE CUSPIDAL CORANK 1 CASE

In this section we assume that $M \subset G$ be a Levi subgroup of semi-simple corank 1. Then there exist two parabolic subgroups $P = MU_P$ and $P_- = MU_-$ containing $M$. We would like to give an explicit description of the normalized intertwining operator $\Phi_{P,P_-}$.
We denote by $M_+ \subset M$ the subset of elements $m$ such that the map $u \to mum^{-1}$ contracts $U_P$ to $e$. To simplify notations we assume that $G$ is semisimple. We can identify $\mathbb{C}^*$ with $\Psi(M), z \to \chi_z$ in such a way that that $|\chi_z(m)| < 1$ for $|z| \leq 1, m \in M_+$.

Let $\sigma$ be an irreducible unitary cuspidal representation of $M$. We write $(\pi_z, \mathcal{V}_z)$ for the representation of $G$ on $i_{GP}(\sigma \otimes \chi_z)$. The following statement is contained in [2].

We denote by $W_\sigma \subset N_G(M)/M$ the subgroup of elements $x$ such that $\sigma^x = \sigma \otimes \chi$ for some $\chi \in \Psi(M)$. Since $|N_G(M)/M| \leq 2$ we see that either $W_\sigma = \{e\}$ or $W_\sigma = \{S_2\}$.

**Claim 5.1.** (1) If $W_\sigma = \{e\}$ then representations $\mathcal{V}_z$ are irreducible for all $z \in \mathbb{C}^*$.

(2) If $|z| \neq 1$ and the representation $(\pi_z, \mathcal{V}_z)$ is reducible then $\mathcal{V}_z$ has unique irreducible $G$-submodule $W_z \subset \mathcal{V}_z$ and the quotient representation $\overline{\mathcal{V}_z} := \mathcal{V}_z/W_z$ is irreducible.

(3) If $|z| = 1$ then either $\pi_z$ is irreducible or is the direct sum of two tempered irreducible representations $\mathcal{V}_z = \mathcal{V}_z^+ \oplus \mathcal{V}_z^-$. We consider now the case when $W_\sigma = \{S_2\}$. We shall write $I(z) : \mathcal{V}_{P,\chi} \to \mathcal{V}_{Q,\chi}$ instead of $I_{P,\chi}$. The following statement is proven in [3].

**Claim 5.2.** Either the geometric intertwining operator $I(z)$ is regular and invertible for all $z \in \mathbb{C}^*$ of there exist $z_0 \in \mathbb{C}^*, |z_0| = 1$ such that $I(z)$ has a first order pole at $z_0$.

In the first case we have $\Phi = I$.

We consider now the case when $I$ has a pole. Clearly, we can assume without loss of generality that $z_0 = 1$. The next statement is also contained in [3].

**Claim 5.3.** (1) $I(z)$ has a first order pole at $z = 1$ and $(z-1)I(z)(1) = a \cdot Id$, where $a \in \mathbb{C}^*$.

(2) There exists $c > 1$ such that operators $I^{\pm 1}(z)$ are regular and invertible for $z \notin \{1, c^{\pm 1}\}$.

(3) $I$ is regular at $c$, $\ker(I(c)) = \mathcal{W}_c$ and $I(c)$ defines an isomorphism $\overline{\mathcal{W}_c} \to \mathcal{W}_{-c}$.

(4) $I^{-1}$ is regular at $c^{-1}, \ker(I(c^{-1})) = \mathcal{W}_{c^{-1}}$ and $I(c^{-1})$ defines an isomorphism $\overline{\mathcal{W}_{c^{-1}}} \to \mathcal{W}_c$.

Now we define $\Phi(z) = I(z) \frac{(1-z)}{(1-cz)^{-1}}$.

It is now clear that $\Phi$ can be considered as a normalized intertwining operator $\Phi_{P,P}$; if we define a similar operator $\Phi_{P,-P}$ then these operators satisfy the conditions of Claim [1].

To formulate the next statement we need to slightly change the point of view. Namely, we would like to modify $\Phi$ so that it becomes an operators from $\mathcal{V}_{P,z}$ to $\mathcal{V}_{P,z^{-1}}$. For this let us choose an element $n \in N_G(M)$ so that $n^2$ belongs to the center of $M$ and $nPn^{-1} = P_-$. Multiplying $\sigma$ by some element of $\Psi(M)$ we can assume that $\text{Ad}(n)(\sigma) \simeq \sigma$. Then the right multiplication by $n$ defines an isomorphism between $S_\sigma(X_P)e$ and $S_\sigma(X_{P_-})e$ which commutes with $G$ and commutes with $M$ up to the action of $\text{Ad}(n)$. Hence, for every $z \in \mathbb{C}^*$ it defines an isomorphism between $\mathcal{V}_{P,z}$ and $\mathcal{V}_{P,z^{-1}}$.
and $V_{P,z}$. Composing $\Phi$ with the inverse of this isomorphism we get a (rational) isomorphism $\tilde{\Phi}(z)$ between $V_{P,z}$ and $V_{P,z-1}$.

**Corollary 5.4.**

1. $S'$ is the space of regular functions $f : \mathbb{C}^* - \{c\} \to V$ such that the function $(z - c)f(z)$ is regular at $z = c$, $((z - c)f(z))(c) \in W_c$ and such that $f(z)$ is $K$-invariant for some open compact subgroup $K$ of $G$.

2. $J_\sigma$ is the space of regular functions $h : \mathbb{C}^* - \{c\} \to \text{End}(V)$ such that
   
   (a) the function $(z - c)h(z)$ is regular at $z = c$
   (b) $((z - c)h(z))(c) \in \text{Hom}(W_c, W_c)$
   (c) $f(z)$ is two-sided $K$-invariant for some open compact subgroup $K$ of $G$
   (d) $h(z^{-1}) = \tilde{\Phi}(z)h(z)\tilde{\Phi}^{-1}(z)$.

**Proof.** The first assertion follows immediately from the definition of $S'$ and the construction of $\Phi(z)$. Let us prove the 2nd assertion. First of all, it is clear that any $h$ satisfying (a)-(d) belongs to $J_\sigma$. On the other hand, let $h$ be any element of $J_\sigma$. Then by definition it defines a rational function $h : \mathbb{C}^* - \{c\} \to \text{End}(V)$ which satisfies (c) and (d) and which does not have poles when $|z| \leq 1$. Since $\tilde{\Phi}(z)$ is an isomorphism for $z \neq c^{-1}$, it follows that $h$ can have a pole only at $c$. Now (d) and Claim 5.3 imply conditions (a) and (b). \qed

### 6. Proof of Theorem 1.8

The assertion (1) of Theorem 1.8 is equivalent to the following

**Lemma 6.1.** Let $G, P, Q, \sigma$ be as Corollary 5.4. Let $S = S(X_P)_\sigma, S' = S'(X_P)_\sigma$. Then $S' = S$.

**Proof.** It is clear from Corollary 5.4 that for any $j \in J_\sigma$ and $f \in S_c(X_P)_\sigma$ we have $j(f) \in S'$. It also clear that the quotient $S'/S_c(X_P)_\sigma$ is isomorphic to $W_c$ as a representation of $H$. So to prove the equality $S' = S$ it is sufficient to find $j \in J_\sigma$ which is not regular at $c$. To find such $j$ choose a function $r$ in Corollary 5.4 which satisfies the first two conditions of Corollary, has a pole at $c$ and has a second order zero at $c^{-1}$. Now take $h(z) = r(z) + \Phi(z)r(z)\Phi^{-1}(z)$.

\qed

We now pass to the 2nd assertion of Theorem 1.8

#### 6.2

In this subsection we show that the following statement

**Conjecture 6.3.** $S'(X_P)_{\text{cusp}} \subset S(X_P)_{\text{cusp}}$.

implies the validity of the part (2) of Theorem 1.8.

First we show that this Conjecture implies the validity of the 2(b). For this it is enough to show that for every $\phi \in S(X_Q)$ we have $\Phi_{Q,P}(\phi) \in S(X_P)$. By definition of $S(X_Q)$ we have

$$\phi = j \cdot f$$
where \( f \in S_c(X_Q) \) and \( j \in \mathcal{J}(G) \). Then we have
\[
\Phi_{Q,P}(\phi) = j \cdot \Phi_{Q,P}(f) \in \mathcal{J}(G) \cdot S'(X_P) \subset \mathcal{J}(G) \cdot S(X_P) = S(X_P).
\]

So, to complete the proof it is enough to show that the inclusion \( S'(X_P)_\text{cusp} \subset S(\hat{X}_P)_\text{cusp} \) implies that \( S'(X_P)_\text{cusp} = S(\hat{X}_P)_\text{cusp} \). Let us as before choose a unitary cuspidal representation \( \sigma \) of \( M \) and let \( S(X_P)_\sigma, S'(X_P)_\sigma \) be the corresponding direct summands of \( S(\hat{X}_P)_\text{cusp} \), \( S'(X_P)_\text{cusp} \). These are modules over \( \mathbb{C}[\Psi(M)] \) and it is clear that if we take \( K \)-invariant vectors for some open compact subgroup \( K \) of \( G \), these modules become finitely generated. Hence it is enough to show that the embedding \( S'(X_P)_\text{cusp} \subset S(\hat{X}_P)_\text{cusp} \) is surjective in the formal neighbourhood of every \( \chi \in \Psi(M) \).

Let us first assume that \( \chi \) is non-strictly negative with respect to \( P \). Then by definition we have \( S(\hat{X}_P)_{\sigma,\chi} = S_c(\hat{X}_P)_{\sigma,\chi} \) where \( S(\hat{X}_P)_{\sigma,\chi} \) and \( S_c(\hat{X}_P)_{\sigma,\chi} \) denote the formal completions of the corresponding spaces at \( \chi \). Since \( S_c(X_P) \subset S'(X_P) \) the desired surjectivity follows. Let now \( \chi \) be arbitrary. Then there exists an associate parabolic \( Q \) such that \( \chi \) is non-strictly negative with respect to \( Q \). Since we have \( S'(X_P) = S'(X_Q) \) and \( S(X_P) = S(X_Q) \), it follows from the above argument that the map \( S'(X_P)_{\sigma,\chi} \to S(\hat{X}_P)_{\sigma,\chi} \) is surjective for every \( \chi \).

### 7. Some further questions

#### 7.1. The \( \mathcal{J} \)-version of the Jacquet functor.

Let us assume the 4th assertion of Conjecture \( \text{[7.1]} \). Then we can define a functor
\[
r_{GP}^{\mathcal{J}} : \text{Right } \mathcal{J}(G)\text{-modules} \to \text{Right } \mathcal{J}(M)\text{-modules}
\]
by setting
\[
r_{GP}^{\mathcal{J}}(\pi) = \pi \otimes_{\mathcal{J}(G)} S(X_P). \tag{7.1}
\]

It would be interesting to investigate exactness properties of this functor and compute it in some examples. Note that manifestly we have \( r_{GP}^{\mathcal{J}}(\mathcal{J}(G)) = S(X_P) \). Also note that Conjecture \( \text{[7.1]} \) implies that at least non-canonically the functor \( r_{GP}^{\mathcal{J}} \) depends only on \( M \) and not on \( P \) (the non-canonicty comes from the fact that the operators \( \Phi_{P,Q} \) are not canonically defined).

#### 7.2. The spherical part.

Let us define the spherical part of \( S(X_P) \) by setting
\[
S_{\text{sph}}(X_P) = S(X_P)^{G(\mathcal{O}) \times M(\mathcal{O})}. \tag{7.2}
\]

We would like to describe this space explicitly. For this note that set-theoretically we have
\[
G(\mathcal{O}) \backslash X_P / M(\mathcal{O}) = M(\mathcal{O}) \backslash M / M(\mathcal{O}).
\]

Hence elements of \( S_{\text{sph}}(X_P) \) can be thought of as \( M(\mathcal{O}) \times M(\mathcal{O}) \)-invariant functions on \( M \). Recall that the Satake isomorphism (for \( M \)) says that the spherical Hecke algebra \( \mathcal{H}_{\text{sph}}(M) \) consisting of compactly supported \( M(\mathcal{O}) \times M(\mathcal{O}) \)-invariant functions
on $M$ is isomorphic to the complexified Grothendieck ring of the category of finite-dimensional representations of the Langlands dual group $M^\vee$ (considered as a group over $\mathbb{C}$). We shall denote the corresponding map $K_0(\text{Rep}(M^\vee)) \to \mathcal{H}_{\text{sph}}(M)$ by $\text{Sat}_M$.

Let $G^\vee$ denote the Langlands dual group of $G$ and let $P^\vee$ be the corresponding parabolic subgroup of $G^\vee$ with unipotent radical $U_{P^\vee}$. Let $u_{P^\vee}$ denote the Lie algebra of $U_{P^\vee}$; it has a natural action of $M^\vee$.

Now let

$$f_P = \sum_{i=0}^{\infty} \text{Sat}_M([\text{Sym}^i(u_{P^\vee})]). \quad (7.3)$$

As was discussed above, we can regard $f_P$ as a $G(\mathcal{O}) \times M(\mathcal{O})$-invariant function on $X_P$.

**Conjecture 7.3.** $\mathcal{S}_{\text{sph}}(X_P)$ is a free right $\mathcal{H}_{\text{sph}}(M)$-module generated by $f_P$.

In the case when $P$ is a Borel subgroup this conjecture is proved in [3].

### 7.4. Iwahori part and K-theory

Let $\mathcal{H}_{\text{aff}}(G)$ denote the affine Hecke algebra of $G$. This is an algebra over $\mathbb{C}[v, v^{-1}]$; its specialization at $v = q^{1/2}$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}(G, I)$ of $G$.

Let $\mathcal{N}_{G^\vee}$ (resp. $\mathcal{N}_{M^\vee}$) denote the nilpotent cone in the Lie algebra of $G^\vee$ (resp. in the Lie algebra of $M^\vee$). Let also $\mathcal{B}_{G^\vee}, \mathcal{B}_{M^\vee}$ denote the corresponding flag varieties. The cotangent bundle $T^*\mathcal{B}_{G^\vee}$ maps naturally to $\mathcal{N}_{G^\vee}$. Thus we can define

$$\text{St}_{G^\vee, M^\vee} = T^*\mathcal{B}_{G^\vee} \times_{\mathcal{N}_{G^\vee}} T^*\mathcal{B}_{M^\vee}. \quad (\text{St})$$

This variety is acted on by the group $G^\vee \times \mathbb{C}^\times$ (where the second factor acts on $\mathcal{N}_{G^\vee}$ by the formula $t(x) = t^2x$ where $t \in \mathbb{C}^\times$ and $x \in \mathcal{N}_{G^\vee}$). Thus we can consider the complexified equivariant $K$-theory $K_{M^\vee \times \mathbb{C}^\times}^{}(\text{St}_{G^\vee, M^\vee})$. This is a vector space over $\mathbb{C}[v, v^{-1}] = K_{\mathbb{C}^\times}(pt)$. It is easy to see (generalizing the standard construction of [6]) that it has a structure of $\mathcal{H}_{\text{aff}}(G) \otimes \mathcal{H}_{\text{aff}}(M)$.

**Conjecture 7.5.** The specialization of $K_{M^\vee \times \mathbb{C}^\times}^{}(\text{St}_{G^\vee, M^\vee})$ at $v = q^{1/2}$ is isomorphic to $\mathcal{S}'(X_P)^I$.

It would be interesting to extend this Conjecture to $\mathcal{S}(X_P)$ instead of $\mathcal{S}'(X_P)$.

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