A NOTE ON 2-GENERATED SYMMETRIC AXIAL ALGEBRAS OF MONSTER TYPE

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ABSTRACT. In [10], Yabe gives an almost complete classification of primitive symmetric 2-generated axial algebras of Monster type. In this note, we construct a new infinite-dimensional primitive 2-generated symmetric axial algebra of Monster type \((2, \frac{1}{2})\) over a field of characteristic 5, and use this algebra to complete the last case left open in Yabe’s classification.

1. INTRODUCTION

Axial algebras of Monster type have been introduced in [4] by Hall, Rehren and Shpectorov, in order to generalise some subalgebras of the Griess algebra (defined Majorana algebras by Ivanov [5]) and create a new tool for better understanding, and possibly unifying, the classification of finite simple groups. They have recently appeared also in other branches of mathematics (see [8, 9]). In [6, 7] Rehren started a systematic study of primitive 2-generated axial algebras of Monster type, constructing several classes of new algebras. More examples have been found by Galt et al. [3], and, independently, by Yabe [10]. In [10], Takahiro Yabe obtained an almost complete classification of the primitive 2-generated symmetric axial algebras of Monster type. Yabe left open only the case of algebras over a field of characteristic 5 and axial dimension greater than 5. Indeed something surprising happens in the latter case, namely we show that, over a field \(F\) of characteristic 5, there exists a new infinite-dimensional primitive 2-generated symmetric axial algebra \(H\) of Monster type \((2, \frac{1}{2})\) such that any primitive 2-generated symmetric axial algebras of Monster type \((2, \frac{1}{2})\) is isomorphic to a quotient of \(H\) (in particular, the Highwater algebra \(H\) [1] is a proper factor of \(H\) over an infinite-dimensional ideal).

As a corollary we complete Yabe’s classification.

For the definitions and further motivation refer to [4, 6, 7], for the notation and the basic properties of axial algebras refer to [1]. In particular, throughout this paper, \(F\) is a field of characteristic 5. For a 2-generated symmetric axial algebra \(V\) of Monster type \((\alpha, \beta)\) over \(F\), let \(a_0, a_1\) be the generating axles of \(V\). For \(i \in \{0, 1\}\), let \(\tau_i\) be the Miyamoto involution associated to \(a_i\). Set \(\rho := \tau_0 \tau_1\), and, for \(i \in \mathbb{Z}\), \(a_{2i} := a_i^\rho\) and \(a_{2i+1} := a_i^\rho\). Note that, since \(\rho\) is an automorphism of \(V\), for every \(j \in \mathbb{Z}\), \(a_j\) is an axis. Denote by \(\tau_j := \tau_{a_j}\) the corresponding Miyamoto involution.

The algebra \(V\) is symmetric if it admits an algebra automorphism \(f\) that swaps \(a_0\) and \(a_1\), whence, for every \(i \in \mathbb{Z}\), \(a_i^f = a_{-i+1}\). Let \(\sigma_i\) be the element of \(\langle \tau_0, f \rangle\) that swaps \(a_0\) with \(a_i\). Since, by [1] Lemma 4.2, for \(n \in \mathbb{Z}_+\) and \(i, j \in \mathbb{Z}\) such that \(i \equiv n j\),

\[ a_i a_i + \beta(a_i + a_{i+n}) = a_i a_{j+n} - \beta(a_j + a_{j+n}), \]
we can define

\[ s_{i,n} := a_i a_{i+n} - \beta(a_i + a_{i+n}) \]

where \( \bar{i} \) denotes the congruence class \( i + n \mathbb{Z} \). Since \( V \) is primitive, there is a linear function \( \lambda_{a_i} : V \to \mathbb{F} \) such that every \( v \) can be written in a unique way as 
\[ v = \lambda_{a_0}(v)a_0 + v_0 + v_1 + v_2, \] 
where \( v_0, v_1, v_2 \) are 0-, \( \alpha \)-, \( \beta \)-eigenvectors for \( \text{ad}_{a_0} \), respectively. For \( i \in \mathbb{Z} \), set \( \lambda_i := \lambda_{a_0}(a_i) \) and let

\[ a_i = \lambda_{a_0}(a_i)a_0 + u_i + v_i + w_i \]

be the decomposition of \( a_i \) into \( \text{ad}_{a_0} \)-eigenvectors, where \( u_i \) is a 0-eigenvector, \( v_i \) is an \( \alpha \)-eigenvector and \( w_i \) is a \( \beta \)-eigenvector.

Following Yabe [10], we denote by \( D \) the positive integer such that \( \{a_0, \ldots, a_D-1\} \) is a basis for the linear span \( \langle a_i \mid i \in \mathbb{Z} \rangle \) of the set of the axes \( a_i \)'s. \( D \) is called the axial dimension of \( V \). Our results are the following

**Theorem 1.** Let \( V \) be a primitive 2-generated symmetric axial algebra of Monster type \( (2, \frac{1}{3}) \) over a field of characteristic 5. If \( \lambda_1 = \lambda_2 = 1 \), then \( V \) is isomorphic to a quotient of the algebra \( \mathcal{H} \).

**Corollary 2.** Let \( V \) be a primitive 2-generated symmetric axial algebra of Monster type \( (\alpha, \beta) \) over a field of characteristic 5. If \( D \geq 6 \), then \( V \) is isomorphic to a quotient of one of the following:

1. the algebra \( 6A_\alpha \), as defined in [7];
2. the algebra \( V_\delta(\alpha) \), as defined in [3];
3. the algebra \( \mathcal{H} \), as defined in Section 4.

Note that, for every \( \alpha \), Rehren’s algebra \( 6A_\alpha \) and Yabe’s algebra \( V_\delta(\alpha, \frac{3}{2}) \) coincide and the 8-dimensional algebra \( V_\delta(\alpha) \) of type \( (\alpha, \frac{3}{2}) \) constructed in [3] coincides with Yabe’s algebra \( V_\delta(\alpha, \frac{2}{3}) \). Furthermore, remarkably, over a field of characteristic 5, the Highwater algebra \( \mathcal{H} \) (see [2] and [11]) is isomorphic to a quotient of \( \mathcal{H} \), Yabe’s algebras \( V_1(2, \frac{1}{3}) \) and \( V_2(2, \frac{1}{3}) \), and Rehren’s algebra \( 5A_2 \) are all isomorphic, and are in turn a quotient of \( \mathcal{H} \). Finally, also the algebra \( 6A_2 \) is a quotient algebra of \( \mathcal{H} \).

2. The algebra \( \mathcal{H} \)

In this section, for every \( i \in \mathbb{Z} \), denote by \( \bar{i} \) the congruence class \( i + 3\mathbb{Z} \). Let \( \mathcal{H} \) be an infinite-dimensional \( \mathbb{F} \)-vector space with basis \( B := \{\hat{a}_i, \hat{s}_0,j, \hat{s}_1,3k, \hat{s}_2,3k \mid i \in \mathbb{Z}, j, k \in \mathbb{Z}_+\} \),

\[ \mathcal{H} := \bigoplus_{i \in \mathbb{Z}} \mathbb{F}\hat{a}_i \oplus \bigoplus_{j \in \mathbb{Z}_+} \mathbb{F}\hat{s}_0,j \oplus \bigoplus_{k \in \mathbb{Z}_+} (\mathbb{F}\hat{s}_1,3k \oplus \mathbb{F}\hat{s}_2,3k). \]

Set \( \hat{s}_0,0 := 0 \) and, if \( j \neq 3 \), \( \hat{s}_1,j := \hat{s}_0,j =: \hat{s}_2,j \). Let \( \hat{a}_0 \) and \( \hat{f} \) be the linear maps of \( \mathcal{H} \) defined on the basis elements by

\[ \hat{a}_0^{\hat{a}} = \hat{a}_{-1}, \quad (\hat{s}_0,j)^{\hat{a}} = \hat{s}_0,j, \quad (\hat{s}_1,3k)^{\hat{a}} = \hat{s}_2,3k, \quad \text{and} \quad (\hat{s}_2,3k)^{\hat{a}} = \hat{s}_1,3k, \]

\[ \hat{a}_i^{\hat{f}} = \hat{a}_{i+1}, \quad (\hat{s}_0,j)^{\hat{f}} = \hat{s}_0,j \quad \text{if} \quad j \neq 3 \]

\[ \hat{a}_i^{\hat{f}} = \hat{a}_{i-1}, \quad (\hat{s}_1,3k)^{\hat{f}} = \hat{s}_1,3k, \quad (\hat{s}_2,3k)^{\hat{f}} = \hat{s}_2,3k. \]
Define a commutative non-associative product on $\mathcal{H}$ extending by linearity the following values on the basis elements (where $\delta_{ir}$ denotes the Kronecker delta and $(-1) * 1 := -1$, $(-1) * 2 := 1$, and $0 * t := 0$ for every $t \in \mathbb{Z}$):

$$\begin{align*}
(\mathcal{H}_1) \quad & \hat{a}_i \hat{a}_j := -2(\hat{a}_i + \hat{a}_j) + \hat{s}_{i,j}, \\
(\mathcal{H}_2) \quad & \hat{a}_i \hat{s}_{r,j} := -2\hat{a}_i + (\hat{a}_{i-r} + \hat{a}_{i+j}) - \hat{s}_{r,j} - (\delta_{ir} - 1) \cdot (i - r)(\hat{s}_{r, i-j} - \hat{s}_{r+1, i-j}), \\
(\mathcal{H}_3) \quad & \hat{s}_{r,i} \hat{s}_{r,j} := 2(\hat{s}_{r,i} + \hat{s}_{r,j}) - 2(\hat{s}_{0, i-j} + \hat{s}_{1, i-j} + \hat{s}_{2, i-j} + \hat{s}_{0, i+j} + \hat{s}_{1, i+j} + \hat{s}_{2, i+j}), \\
& \text{if } \{i, j\} \not\subseteq 3\mathbb{Z}, \\
(\mathcal{H}_4) \quad & \hat{s}_{0,3h} \hat{s}_{0,3k} := 2(\hat{s}_{0,3h} + \hat{s}_{0,3k}) - (\hat{s}_{0,3|h-k|} + \hat{s}_{0,3(h+k)}), \\
(\mathcal{H}_5) \quad & \hat{s}_{0,3h} \hat{s}_{1,3k} := 2(\hat{s}_{0,3h} + \hat{s}_{1,3h} - \hat{s}_{2,3h} + \hat{s}_{0,3k} + \hat{s}_{1,3k} - \hat{s}_{2,3k}) - (\hat{s}_{0,3|h-k|} + \hat{s}_{1,3|h-k|} - \hat{s}_{2,3|h-k|} + \hat{s}_{0,3(h+k)} + \hat{s}_{1,3(h+k)} - \hat{s}_{2,3(h+k)}), \\
(\mathcal{H}_6) \quad & \hat{s}_{0,3h} \hat{s}_{2,3k} := 2(\hat{s}_{0,3h} + \hat{s}_{1,3h} + \hat{s}_{2,3h} + \hat{s}_{0,3k} - \hat{s}_{1,3k} + \hat{s}_{2,3k}) - (\hat{s}_{0,3|h-k|} - \hat{s}_{1,3|h-k|} + \hat{s}_{2,3|h-k|} + \hat{s}_{0,3(h+k)} + \hat{s}_{1,3(h+k)} - \hat{s}_{2,3(h+k)}), \\
(\mathcal{H}_7) \quad & \hat{s}_{1,3h} \hat{s}_{2,3k} := 2(-\hat{s}_{0,3h} + \hat{s}_{1,3h} + \hat{s}_{2,3h} - \hat{s}_{0,3k} + \hat{s}_{1,3k} + \hat{s}_{2,3k}) - (\hat{s}_{0,3|h-k|} + \hat{s}_{1,3|h-k|} + \hat{s}_{2,3|h-k|} - \hat{s}_{0,3(h+k)} + \hat{s}_{1,3(h+k)} + \hat{s}_{2,3(h+k)}). 
\end{align*}$$

We now introduce some eigenvectors for $\text{ad}_{a_0}$ and study how they multiply. For $i \in \mathbb{Z}_+$, set

$$\begin{align*}
\hat{u}_i := & -2a_0 + (\hat{a}_i + \hat{a}_{-i}) + 2\hat{s}_{0,i}, \\
\hat{v}_i := & -2a_0 + (\hat{a}_i + \hat{a}_{-i}) - \hat{s}_{0,i}, \\
\overline{\sigma}_i := & -2a_0 + (\hat{a}_{-i} + \hat{a}_i) - (\hat{s}_{0,i} + \hat{s}_{1,i} + \hat{s}_{2,i}), \\
\overline{\omega}_i := & \hat{s}_{1,i} - \hat{s}_{2,i}.
\end{align*}$$

Then, the $\hat{u}_i$'s and $\overline{\sigma}_i$'s are $0$-eigenvectors for $\text{ad}_{a_0}$, the $\hat{v}_i$'s are $2$-eigenvectors for $\text{ad}_{a_0}$, the $\overline{\omega}_i$'s are $-2$-eigenvectors for $\text{ad}_{a_0}$. Note that, if $i \neq 1$, $\overline{\sigma}_i = \hat{u}_i$ and $\overline{\omega}_i = 0$. Moreover, it will be convenient to use the following notation: for $i, j \in \mathbb{Z}_+$, set

$$\begin{align*}
\hat{c}_j := & -2\hat{a} + (\hat{a}_{-j} + \hat{a}_j), \\
\hat{c}_{i,j} := & -2\hat{c}_i - 2\hat{c}_j + \hat{c}_{i-j} + \hat{c}_{i+j}, \\
\hat{\sigma}_{i,j} := & \hat{s}_{0,i} + \hat{s}_{0,j} + \hat{s}_{1,i-j} + \hat{s}_{1,j-i} + \hat{s}_{2,i-j} + \hat{s}_{2,j-i}, \\
\hat{u}_{i,j} := & -2\hat{u}_i - 2\hat{u}_j + \hat{u}_{i-j} + \hat{u}_{i+j}, \\
\hat{v}_{i,j} := & -2\hat{v}_i - 2\hat{v}_j + \hat{v}_{i-j} + \hat{v}_{i+j}.
\end{align*}$$

We collect in the following lemma the main relations among the above vectors.

**Lemma 3.** For all $i, j \in \mathbb{Z}_+$, we have

$$\begin{align*}
(1) \quad & \hat{u}_j = \hat{c}_j + 2\hat{s}_{0,j}, \\
(2) \quad & \hat{v}_j = \hat{c}_j - \hat{s}_{0,j}, \\
(3) \quad & \overline{\sigma}_j = \hat{c}_j - (\hat{s}_{0,j} + \hat{s}_{1,j} + \hat{s}_{2,j}), \\
(4) \quad & \overline{\omega}_{i,j} = \hat{c}_{i,j} + \hat{s}_{0,i} + \hat{s}_{0,j} + 2\hat{s}_{0,i-j} + 2\hat{s}_{0,i+j}, \\
(5) \quad & \hat{v}_{i,j} = \hat{c}_{i,j} + 2(\hat{s}_{0,i} + \hat{s}_{0,j}) - (\hat{s}_{0,i-j} + \hat{s}_{0,i+j}), \\
(6) \quad & \hat{c}_{i,j} = \hat{\sigma}_{i,j}, \\
(7) \quad & \hat{c}_i \hat{c}_{r,j} = \hat{c}_{i,j} = \hat{c}_j \hat{c}_{r,i}.
\end{align*}$$
Proof. The first five assertions are immediate. A straightforward computation gives the sixth:
\[
\hat{c}_i \hat{c}_j = (-2\hat{a}_0 + (\hat{a}_{-i} + \hat{a}_i))(-2\hat{a}_0 + (\hat{a}_{-j} + \hat{a}_j))
\]
\[
= -\hat{a}_0 - 2(-2\hat{a}_{-i} + \hat{a}_0 + \hat{s}_{0,i} - 2\hat{a}_i - 2\hat{a}_0 + \hat{s}_{0,i})
\]
\[
- 2(-2\hat{a}_0 - 2\hat{a}_{-j} + \hat{s}_{0,j} - 2\hat{a}_0 - 2\hat{a}_j + \hat{s}_{0,j})
\]
\[
+ (-2\hat{a}_{-i} - 2\hat{a}_{-j} + \hat{s}_{-i,j} - 2\hat{a}_i - 2\hat{a}_j + \hat{s}_{i+j})
\]
\[
= \hat{s}_{0,i} + \hat{s}_{0,j} + \hat{s}_{-i,j} + \hat{s}_{-i,j} + \hat{s}_{i+j} + \hat{s}_{i+j} = \sigma_{i,j}.
\]
Similarly, for the seventh, we have
\[
\hat{c}_i \hat{s}_{r,j} = -(2\hat{a}_0 - (\hat{a}_{-i} + \hat{a}_i))\hat{s}_{r,j} =
\]
\[
= -2[(-2\hat{a}_0 + (\hat{a}_{-j} + \hat{a}_j) - \hat{s}_{r,j} - (\hat{s}_{0,j} - 1) * (\hat{s}_{r-1,j} - \hat{s}_{r+1,j})]
\]
\[
+ (-2\hat{a}_{-i} + (\hat{a}_{-i} + \hat{a}_{i+j}) - \hat{s}_{r,j} - (\hat{s}_{-i,j} - 1) * (\hat{s}_{r-1,j} - \hat{s}_{r+1,j})]
\]
\[
+ (-2\hat{a}_i + (\hat{a}_{i+j} + \hat{a}_{i+j}) - \hat{s}_{r,j} - (\hat{s}_{i,j} - 1) * (\hat{s}_{r-1,j} - \hat{s}_{r+1,j})).
\]
Here, \( \hat{s}_{r,j}, \hat{s}_{r-1,j}, \) and \( \hat{s}_{r+1,j} \) cancel and the first equality of the last claim follows after the terms are rearranged. Since, by the definition, \( \hat{c}_i \) is symmetric in \( i \) and \( j \), we have \( \hat{c}_i \hat{c}_j = \hat{c}_j \hat{c}_i = \hat{c}_j \hat{s}_{r,i} \).

Note that also \( \sigma_{i,j} \) is symmetric in \( i \) and \( j \). This evident by Lemma 3 (4), since \( \hat{H} \) is commutative. Two more relations will be useful in the sequel.

Lemma 4. For all \( i, j \in \mathbb{Z}_+ \), we have
\[
(1) \quad \hat{s}_{0,i}(\hat{s}_{0,j} + \hat{s}_{1,j} + \hat{s}_{2,j}) = \hat{s}_{0,i} + \hat{s}_{0,j} + 2(\hat{s}_{0,|i-j|} + \hat{s}_{0,i+j}),
\]
\[
(2) \quad \hat{u}_i - \hat{u}_i = -2\hat{s}_{0,i} + \hat{s}_{1,i} + \hat{s}_{2,i}.
\]
Proof. A straightforward computation gives the claims. \( \square \)

Now are now ready to compute the products of the vectors \( \hat{u}_j, \hat{v}_j, \) and \( \hat{v}_j \).

Lemma 5. For all \( i, j \in \mathbb{Z}_+ \), we have
\[
(1) \quad \hat{u}_i \hat{u}_j = -\hat{u}_{i+j} - 2(\hat{u}_{|i-j|} - \hat{u}_{|i-j|}) - 2(\hat{u}_{|i+j|} - \hat{u}_{|i+j|}),
\]
\[
(2) \quad \hat{u}_i \hat{v}_j = \hat{v}_{i+j},
\]
\[
(3) \quad \hat{v}_i \hat{v}_j = -2\hat{u}_{i+j} - (\hat{u}_{|i-j|} - \hat{u}_{|i-j|}) - (\hat{u}_{i+j} - \hat{u}_{i+j}),
\]
\[
(4) \quad \hat{u}_i \hat{v}_j = \hat{u}_{i+j},
\]
\[
(5) \quad \hat{v}_i \hat{v}_j = \hat{v}_{i+j}.
\]
Proof. By Lemma 3
\[
\hat{u}_i \hat{u}_j = (\hat{c}_i + 2\hat{s}_{0,i})(\hat{c}_j + 2\hat{s}_{0,j})
\]
\[
= \hat{c}_i \hat{c}_j + 2\hat{c}_i \hat{s}_{0,j} + 2\hat{s}_{0,i} \hat{c}_j - \hat{s}_{0,i} \hat{s}_{0,j}
\]
\[
= \hat{s}_{i,j} + 2\hat{c}_i \hat{c}_j - \hat{s}_{0,i} \hat{s}_{0,j}
\]
\[
= \hat{c}_i \hat{c}_j - \hat{s}_{0,i} \hat{s}_{0,j}.
\]
Assume \( i \equiv 3 0 \). Then \( \hat{s}_{i,j} = \hat{s}_{0,i} + \hat{s}_{0,j} + 2(\hat{s}_{0,|i-j|} + \hat{s}_{0,i+j}) \). Moreover,
\[
(3) \quad \hat{s}_{0,i} \hat{s}_{0,j} = 2(\hat{s}_{0,i} + \hat{s}_{0,j}) - (\hat{s}_{0,|i-j|} + \hat{s}_{0,i+j}).
\]
This is immediate if \( j \equiv 3 0 \), since in this case \( (\hat{H}_4) \) holds. If \( j \not\equiv 3 0 \), then \( |i-j| \not\equiv 3 0 \) and \( i + j \not\equiv 3 0 \). Thus \( \hat{s}_{0,|i-j|} = \hat{s}_{0,|i-j|} = \hat{s}_{2,|i-j|}, \hat{s}_{0,i+j} = \hat{s}_{i,i+j} = \hat{s}_{0,i+j} \) and \( (\hat{H}_3) \)
reduces to \(3\). Hence, by Lemma \(3\) (4)
\[
\hat{u}_i \hat{u}_j = -\hat{c}_{i,j} + \hat{s}_{0,i} + \hat{s}_{0,j} + 2(\hat{s}_{0,i} + \hat{s}_{0,j}) - 2(\hat{s}_{0,i} + \hat{s}_{0,j})
\]
\[
+ (\hat{s}_{0,i} + \hat{s}_{0,j})
\]
\[
= -\hat{c}_{i,j} - (\hat{s}_{0,i} + \hat{s}_{0,j}) - 2(\hat{s}_{0,i} + \hat{s}_{0,j}) = -\hat{u}_{i,j}.
\]
Assume \(i \neq 3\) and \(j \neq 3\). Then \(\hat{s}_{i,j} = \hat{s}_{0,i} + \hat{s}_{0,j} + \hat{s}_{1,i-j} + \hat{s}_{2,i-j} + \hat{s}_{1,i+j} + \hat{s}_{2,i+j}\), while the product \(\hat{s}_{0,i} \hat{s}_{0,j}\) is given by \((\hat{H}_2)\). Hence
\[
\hat{u}_i \hat{u}_j = -\hat{c}_{i,j} + \hat{s}_{0,i} + \hat{s}_{0,j} + \hat{s}_{1,i-j} + \hat{s}_{2,i-j} + \hat{s}_{1,i+j} + \hat{s}_{2,i+j} - 2(\hat{s}_{0,i} + \hat{s}_{0,j})
\]
\[
+ 2(\hat{s}_{0,i} + \hat{s}_{0,j})
\]
\[
= -\hat{u}_{i,j} - 2(\hat{u}_{i-j} - \varpi_{i-j}) - 2(\hat{u}_{i+j} - \varpi_{i+j}).
\]
This proves the first assertion. The remaining assertions follow with similar computations, using Lemma \(4\).

**Theorem 6.** The algebra \(\hat{H}\) defined above is a primitive 2-generated symmetric axial algebra of Monster type \((2, \frac{1}{2})\) over any field of characteristic 5.

**Proof.** Remind that in characteristic 5, \(\frac{1}{2} = -2\). It is easy to see that the maps \(\tau_0\) and \(\hat{f}\) are algebra automorphisms of \(\hat{H}\) and that the map \(\hat{\theta} := \tau_0 \hat{f}\) induces on the set \(\{\hat{a}_i \mid i \in \mathbb{Z}\}\) the translation \(\hat{a}_i \mapsto \hat{a}_{i+1}\). Let \(H := \langle \hat{a}_0, \hat{a}_1 \rangle\) be the subalgebra of \(\hat{H}\) generated by \(\hat{a}_0\) and \(\hat{a}_1\). Note that \(\hat{s}_{0,1} = \hat{a}_0 \hat{a}_1 + 2(\hat{a}_0 + \hat{a}_1) \in H\). Also, \(\hat{a}_{-1} = \hat{a}_0 \hat{s}_{0,1} + 2 \hat{a}_0 - \hat{a}_1 + \hat{s}_{0,1} \in H\). This gives us \(\hat{a}_{-1} \in H\). Clearly, \(H = \langle \hat{a}_0, \hat{a}_1 \rangle\) is invariant under \(\hat{f}\) and also \(H = \langle \hat{a}_{-1}, \hat{a}_0, \hat{a}_1 \rangle\) is invariant under the involution \(\hat{\tau}_0\). Thus \(H\) is invariant under \(\hat{\theta}\) and so \(H\) contains all the \(\hat{a}_i\)'s. It follows that \(H\) contains all the \(sf_{e,j}\), that is \(H = \hat{H}\).

Since, for every \(i \in \mathbb{Z}\), \(\hat{a}_i = \hat{a}_0^i\), to show that \(\hat{H}\) is an axial algebra of Monster type \((2, \frac{1}{2})\) it is enough to prove that \(\hat{a}_0\) is an axis with respect to the Monster fusion law in Table 3.

Further, for every \(i \in \mathbb{Z}_+\), we have
\[
\langle \hat{a}_0, \hat{a}_i, \hat{a}_{-i}, \hat{s}_{0,i} \rangle = \langle \hat{a}_0, \hat{u}_i, \hat{v}_i, \hat{w}_i \rangle \text{ if } i \neq 3 0
\]
and
\[
\langle \hat{a}_0, \hat{a}_i, \hat{a}_{-i}, \hat{s}_{0,i}, \hat{s}_{1,i}, \hat{s}_{2,i} \rangle = \langle \hat{a}_0, \hat{u}_i, \hat{v}_i, \hat{w}_i, \varpi_i, \varpi_i \rangle \text{ if } i \equiv 3 0.
\]

Hence, a basis of \(\text{ad}_{\hat{a}_0}\)-eigenvectors for \(\hat{H}\) is given by
\[
\hat{a}_0, \hat{u}_i, \hat{v}_i, \hat{w}_i, \varpi_i, \varpi_i, \varpi_i, \varpi_i, \text{ with } i, k \in \mathbb{Z}_+.\]
In particular, since \( \hat{a}_0 \) is the unique element of this basis that is a 1-eigenvector for \( \text{ad}_{\hat{a}_0} \), the algebra is primitive. Finally, set

\[
H_0 := \langle \hat{u}_i, \hat{v}_i | i \in \mathbb{Z}_+ \rangle, \quad H_2 := \langle \hat{v}_i | i \in \mathbb{Z}_+ \rangle, \quad H_{-2} := \langle \hat{u}_i, \hat{w}_i | i \in \mathbb{Z}_+ \rangle.
\]

Then, for \( z \in \{0, 2, -2\} \), \( H_z \) is the \( z \)-eigenspace for \( \text{ad}_{\hat{a}_0} \) and \( \hat{r}_0 \) acts as the identity on \( \langle \hat{a}_0 \rangle \oplus H_0 \oplus H_2 \) and as the multiplication by \(-1\) on \( H_{-2} \). Since \( \tau_0 \) is an algebra automorphism, we have \( H_z H_{-2} \subseteq H_{-2} \) for every \( z \in \{0, 2\} \) and \( H_{-2} H_{-2} \subseteq \langle \hat{a}_0 \rangle \oplus H_0 \oplus H_2 \). By Lemma 5, we also have \( H_0 H_0 \subseteq H_0 \), \( H_0 H_2 \subseteq H_2 \), and \( H_2 H_2 \subseteq H_0 \). Hence \( \mathcal{H} \) respects (a restricted version of) the Monster fusion law and the result is proved.

\[ \square \]

Note that \( \langle \hat{s}_{0,3k} - \hat{s}_{1,3k}, \hat{s}_{0,3k} - \hat{s}_{2,3k}, \hat{s}_{1,3k} - \hat{s}_{2,3k} | k \in \mathbb{Z}_+ \rangle \) is an \( f \)-invariant ideal of \( \mathcal{H} \) and the corresponding factor algebra is isomorphic to the Highwater algebra \( \mathcal{H} \). Moreover, the algebra \( 6A_2 \) is isomorphic to the factor of \( \mathcal{H} \) over the ideal linearly spanned by the vectors

\[ \hat{a}_i - \hat{a}_{i-6}, \text{ for } i \geq 3; \quad \hat{s}_{0,4} - \hat{s}_{0,2}, \quad \hat{s}_{0,5} - \hat{s}_{0,1}, \quad \hat{s}_{0,3} - \hat{s}_{0,4} - \hat{s}_{0,2} + \hat{a}_2 - \hat{a}_{i-1} + \hat{a}_1 + \hat{a}_2 - 2(\hat{a}_0 + \hat{a}_3). \]

\section{Proofs of the main results}

In the next lemma we recall some basic properties of the elements \( s_{\bar{r}, n} \) that will be used throughout the proof of Theorem 1 without further reference.

\begin{lemma}
Let \( V \) be a primitive 2-generated symmetric axial algebra of Monster type. For every \( n \in \mathbb{Z}_+ \) and \( i \in \mathbb{Z} \) the following hold

(1) \( s_{\bar{r}, n} \) is \( \lambda_{a_0} \)-eigenvector for \( \text{ad}_{a_0} \), with \( r + i \equiv k \mod n \);

(2) the group \( \langle \tau_0, \bar{f} \rangle \) acts transitively on the set \( \{ s_{\bar{r}, n} | \bar{r} \in \mathbb{Z}/n\mathbb{Z} \} \), for each \( n \in \mathbb{Z}_+ \);

(3) \( \lambda_{a_0}(s_{0,n}) = \lambda_n - \beta - \beta \lambda_n \).
\end{lemma}

\begin{proof}
The first assertion is Lemma 4.2 in \( \text{[1]} \) and (2) and (3) follow immediately. For the last one, note that, for every \( x \in V \), we have \( \lambda_{a_0}(a_0 x) = \lambda_{a_0}(x) \) (this follows immediately from the linearity of \( \lambda_{a_0} \), decomposing \( x \) into a sum of \( \text{ad}_{a_0} \)-eigenvectors). Hence, by the definition of \( s_{0,n} \), we get

\[ \lambda_{a_0}(s_{0,n}) = \lambda_{a_0}(a_0 a_n - \beta(a_0 + a_n)) = \lambda_n - \beta - \beta \lambda_n. \]
\end{proof}

\begin{proof}[Proof of Theorem 1]
Let \( V \) be a primitive 2-generated symmetric axial algebra of Monster type \( (2, \frac{1}{2}) \) over a field of characteristic 5 such that \( \lambda_1 = \lambda_2 = 1 \). By \( \text{[1]} \) Lemma 4.4], for \( h \in \mathbb{Z} \), the \( \text{ad}_{a_0} \)-eigenvectors \( u_h \) and \( v_h \), defined in Section 1, are as follows (remind that \( \frac{1}{2} = -2 \) in characteristic 5)

\begin{align*}
(6) \quad u_h &= -2a_0 + (a_h + a_{-h}) + 2s_{0,h}, \\
(7) \quad v_h &= a_0 + 2(a_h + a_{-h}) - 2s_{0,h}.
\end{align*}

By the fusion law, for every \( h, k \in \mathbb{Z} \), the following identities hold

\begin{align*}
(8) \quad a_0(u_h u_k - v_h v_k + \lambda_{a_0}(v_h v_k)a_0) &= 0
\end{align*}

\end{proof}
and

\[ a_0(u_h u_k + u_h v_k) = 2u_h v_k. \]

By \cite[Lemma 4.3]{1}, for every \( j \in \mathbb{Z} \), we have \( a_0 s_{0,j} = -2a_0 + (a_{-j} + a_j) - s_{0,j} \).

Using the action of the group of automorphisms \( \langle \tau_0, f \rangle \), we get, for every \( j, k \in \mathbb{Z} \), \( r \in \mathbb{Z} \),

\[ a_{r+j} s_{r,f,j} = -2a_{r+j} + (a_{r-j} + a_{r+j}) - s_{0,j}. \]

Set \( V_0 := \{ (a_i, s_{r,n} \mid i \in \mathbb{Z}, n \in \mathbb{Z}, r \in \mathbb{Z} \} \) and, for \( t \in \mathbb{Z} \), denote by \([t]_3\) the congruence class \( t + 3\mathbb{Z}, 0 \equiv [t]_3 := 0, (-1) \equiv [t]_3 := -1, \) and \((1) \equiv [t]_3 := 1. \)

**Claim.** For every \( i \in \mathbb{Z}^+ \), \( r \in \mathbb{Z} \), and \( t \in \{0, 1, 2\} \)

1. \( \lambda_i = 1 \) and \( \lambda_{a_0}(s_{r,i}) = 0; \)
2. if \( i \equiv 0 \), \( s_{r,i} = s_{0,i}; \)
3. if \( i \equiv 2 \) and \( r \equiv 3 \), \( s_{r,i} = s_{r,i}; \)
4. for every \( l \in \mathbb{Z}, j \leq i, m \in \mathbb{Z} \), the products \( a_i s_{m,j} \) belong to \( V_0 \) and satisfy the formula

\[ a_i s_{m,j} = -2a_i + (a_{-j} + a_{j}) - s_{m,j} + (\delta_{[l]_3[m]_3} - 1) \ast [l - m]_3 s_{m-1,j} - s_{m+1,j}. \]

Note that, by the symmetries of \( V \), part (iv) of Claim holds if and only if, for every \( r \in \{0, \ldots, j-1\} \), the products \( a_0 s_{r,f,j} \) satisfy the corresponding formula in Equation \( (11) \).

We proceed by induction on \( i \). Let \( i = 1 \). By the hypothesis \( \lambda_1 = 1 \), hence (i) holds by Lemma \cite[(4)]{7} and (ii) holds trivially.

Let \( i = 2 \). Again by the hypothesis, \( \lambda_2 = 1 \), hence (i) holds by Lemma \cite[(4)]{7}.

Equation (1) in \cite[Lemma 4.8]{1} becomes \( -2(s_{0,2} - s_{1,2}) = 0 \), whence

\[ s_{1,2} = s_{0,2} \]

and parts (ii) and (iv) hold.

Assume \( i \geq 3 \) and the result true for every \( l \leq i \). By the fusion law, \( u_1 u_i, u_1 v_i \) are 0- and 2-eigenvectors for \( \text{ad}_{a_0} \), respectively. Further, since

\[ (s_{1,i+1})^{\tau_0} = s_{-1,i+1} = s_{i,i+1}, \]

we get that \( s_{i,i+1} - s_{i,i+1} \) is negated by the map \( \tau_0 \), in particular \( s_{i,i+1} - s_{i,i+1} \) is a \(-2\)-eigenvector for \( \text{ad}_{a_0} \). It follows that

\[ \lambda_{a_0}(u_1 t_j) = \lambda_{a_0}(u_1 v_j) = \lambda_{a_0}(s_{i,i+1} - s_{i,i+1}) = 0. \]

By Equations \( (9) \) and \( (10) \) and linearity of \( \lambda_{a_0} \), we get

\[ 0 = \lambda_{a_0}(u_1 u_i + u_1 v_i) - 2\lambda_{a_0}(s_{i,i+1}) - 2\lambda_{i+1}, \]

and

\[ 0 = \lambda_{a_0}(u_1 u_i) = \lambda_{a_0}(s_{0,1}s_{0,i}) - 2\lambda_{a_0}(s_{i,i+1}) + 2\lambda_{i+1} - 2, \]

whence

\[ \lambda_{a_0}(s_{i,i+1}) = 1 - \lambda_{i+1} \quad \text{and} \quad \lambda_{a_0}(s_{0,1}s_{0,i}) = \lambda_{i+1} - 1. \]

As above, substituting \( u_1 \), \( u_i \), \( v_1 \), and \( v_i \) in Equation \( (8) \), with \( h = 1 \) and \( k = i \), we get

\[ a_0(s_{i,i+1} + s_{i,i+1}) = -2(1 + \lambda_{i+1})a_0 + 2(a_{i+1} + a_{-i-1}) - 2s_{0,i+1}. \]

On the other hand, since \( s_{i,i+1} - s_{i,i+1} \) is a \(-2\)-eigenvector for \( \text{ad}_{a_0} \),

\[ a_0(s_{i,i+1} - s_{i,i+1}) = -2(s_{i,i+1} - s_{i,i+1}). \]
Hence, in order to prove (iv), we just need to check that Equation (11) holds.

Assume first that \( i + 1 \equiv 3 0 \). Substituting the expressions (6) and (7) in Equation (9), with \( h = 1 \) and \( k = i \), and using Equation (11), we get

\[
s_{0,1} s_{0,i} = 2(\lambda_{i+1} - 1) a_0 - 2(s_{1,i+1} + s_{i,i+1} + s_{0,i+1}) + 2s_{0,1} - s_{0,i-1} + 2s_{0,i}.
\]

Since, \( s_{0,1} \) and, by the inductive hypothesis, \( s_{0,i} \), and \( s_{0,j} \) are \( \sigma_j \)-invariant for every \( j \in Z \), subtracting to both members of Equation (18) their images under \( \sigma_{i+1} \) we get

\[
0 = s_{0,1} s_{0,i} - (s_{0,1} s_{0,i})^\sigma_{i+1} = 2(\lambda_{i+1} - 1)(a_0 - a_{i+1}),
\]

whence either \( \lambda_{i+1} = 1 \) or \( a_0 = a_{i+1} \). But, again, in the latter case, \( \lambda_{i+1} = \lambda_{a_0}(a_0) = 1 \). Hence, by Equation (13), giving (i). In particular Equation (18) becomes

\[
s_{0,1} s_{0,i} = -2(s_{1,i+1} + s_{i,i+1} + s_{0,i+1}) + 2s_{0,1} - s_{0,i-1} + 2s_{0,i}.
\]

Again, taking the image under \( \sigma_i \) of both members of the above equation, we get

\[
0 = s_{0,1} s_{0,i} - (s_{0,1} s_{0,i})^\sigma_i = -2(s_{1,i+1} - s_{i-1,i+1}),
\]

whence \( s_{1,i+1} = s_{i-1,i+1} \). Since, by Lemma (7), the group \( (\tau_0, f) \) is transitive on the set \( \{s_{r,i+1} \mid 0 \leq r \leq i\} \), it follows that (iii) holds, in particular

\[
s_{i+1,i+1} = s_{2,i+1}.
\]

Hence, in order to prove (iv), we just need to check that Equation (11) holds for \( l = 0 \) and \( m \in \{0, 1, 2\} \). The case \( m = 0 \) follows from Equation (10), cases \( m \in \{1, 2\} \) follow from Equations (10), (17), and (21).

Assume now \( i + 1 \equiv 3 1 \). Substituting the expressions (6) and (7) in Equation (9), with \( h = 2 \) and \( k = i - 1 \), since by the inductive hypothesis (iv), for every \( l \in Z \) and \( j \leq i + 1, r \in Z \), the products \( a s_{r,j} \) are given by Equation (11), we get

\[
s_{0,2} s_{0,i-1} = 2(\lambda_{i+1} - 1) a_0 - 2(s_{1,i-3} + s_{2,i-3} + s_{0,i-3}) + 2s_{0,2} - s_{0,i+1} + 2s_{0,i-1}.
\]

Since, \( s_{0,2} \) and, by the inductive hypothesis, \( s_{0,i-1} \) and \( s_{1,i-3} + s_{2,i-3} + s_{0,i-3} \) are \( \sigma_j \)-invariant for every \( j \in Z \), as above we obtain

\[
0 = s_{0,2} s_{0,i-1} - (s_{0,2} s_{0,i-1})^\sigma_{i+1} = 2(\lambda_{i+1} - 1)(a_0 - a_{i+1}),
\]

whence, as in the previous case, \( \lambda_{i+1} = 1 \) giving (i) by Equation (13). Similarly,

\[
0 = s_{0,2} s_{0,i-1} - (s_{0,2} s_{0,i-1})^\sigma_i = s_{1,i+1} - s_{0,i+1},
\]

whence \( s_{1,i+1} = s_{0,i+1} \). Since the group \( (\tau_0, f) \) is transitive on the set of all pairs \( (s_{r,i+1}, s_{r,i+1}) \), it follows that \( s_{r,i+1} = s_{0,i+1} \), for every \( r \in Z \).

Finally, assume \( i + 1 \equiv 3 2 \). Substituting the expressions (9) and (11) in Equation (9), with \( h = 1 \) and \( k = i \), using the inductive hypothesis as above (in particular \( s_{i-2,i-1} = s_{2,i-1} \)), we get

\[
s_{0,1} s_{0,i} = 2(\lambda_{i+1} - 1)a_0 - 2(s_{0,i-1} + s_{1,i-1} + s_{2,i-1}) + 2s_{0,1} - s_{0,i+1} + 2s_{0,i}.
\]
Since $s_{0,1}$ and, by the inductive hypothesis, $s_{0,i}$ and $s_{0,i-1} + s_{1,i-1} + s_{2,i-1}$ are $\sigma_j$-invariant for every $j \in \mathbb{Z}$, we have

$$0 = s_{0,1}s_{0,i} - (s_{0,1}s_{0,i})^{T_{i+1}} = 2(\lambda_{i+1} - 1)(a_0 - a_{i+1}),$$

whence, as above, it follows $\lambda_{i+1} = 1$. Hence, by Equation (13), giving (i). Then,

$$0 = s_{0,1}s_{0,i} - (s_{0,1}s_{0,i})^{T_1} = s_{1,i+1} - s_{0,i+1},$$

whence we conclude as in the previous case. This finishes the inductive step and the Claim is proved. As a consequence, we get that the subspace $V_0$ is closed with respect to the multiplication by any axes $a_i$.

We now consider the products $s_{r,i}s_{t,j}$, for $i, j \in \mathbb{Z}_+$, $r, t \in \mathbb{Z}$. Proceeding as above, by Equation (9) with $h = i$ and $k = j$, we obtain

$$s_{0,i}s_{0,j} = 2(s_{0,i} + s_{0,j}) - 2(s_{0,i | i-j} + s_{1,i | i-j} + s_{2,i | i-j} + s_{0,i+j} + s_{1,i+j} + s_{2,i+j}),$$

if $\{i, j\} \not\subseteq 3\mathbb{Z}$, and

$$s_{0,i}s_{0,j} = 2(s_{0,i} + s_{0,j}) - (s_{0,i | i-j} + s_{0,i+j}), \quad \text{if} \quad i \equiv 3 \quad j \equiv 3 \quad 0.$$

If $i \not\equiv 3 \quad 0$, then $s_{0,i} = s_{1,i} = s_{2,i}$, thus, applying $f$ and $T_1$ to Equation (24), we get

$$s_{0,i}s_{1,j} = 2(s_{0,i} + s_{1,j}) - 2(s_{0,i | i-j} + s_{1,i | i-j} + s_{2,i | i-j} + s_{0,i+j} + s_{1,i+j} + s_{2,i+j}),$$

and

$$s_{0,i}s_{2,j} = 2(s_{0,i} + s_{2,j}) - 2(s_{0,i | i-j} + s_{1,i | i-j} + s_{2,i | i-j} + s_{0,i+j} + s_{1,i+j} + s_{2,i+j}).$$

If $i \equiv 3 \quad j \equiv 3 \quad 0$, in a similar way, from Equation (25), we get, for any $r \in \mathbb{Z}$,

$$s_{r,i}s_{r,j} = 2(s_{r,i} + s_{r,j}) - (s_{r,i | i-j} + s_{r,i+j}).$$

The last products needed are $s_{r,3h}s_{t,3k}$, for $h, k \in \mathbb{Z}_+$, $r, t \in \mathbb{Z}$ with $r \not\equiv 3 \quad t$. For $k \in \mathbb{Z}_+$, set

$$\pi_{3h} := a_0 + 2(a_{-3k} + a_{3k}) - 2(s_{0,3k} + s_{1,3k} + s_{2,3k}).$$

Then, by Equation (11), $\pi_{3h}$ is a 0-eigenvector for $ad_{a_0}$. Hence, by the fusion law, we have

$$a_0(\pi_{3h}u_{3h} + \pi_{3h}v_{3h}) = 2\pi_{3h}v_{3h}.$$
Using the maps $\tau_0$ and $f$, we derive the formulas for the products $s_{0,3h}s_{2,3k}$ and $s_{1,3h}s_{3,3k}$. Hence $V_0$ is a subalgebra of $V$, and since $a_0, a_1 \in V_0$, we get $V_0 = V$. Therefore, the map

$$\phi : \hat{H} \to V$$

$$\hat{a_i} \mapsto a_i$$

$$\hat{s}_{r,j} \mapsto s_{r,j}$$

from the basis $B$ of $H$ and $V$, extends to a surjective linear map $\bar{\phi} : \hat{H} \to V$. $\bar{\phi}$ is actually an algebra homomorphism, since $V$ satisfies the multiplication table of the algebra $\hat{H}$ and the result follows.

**Proof of Theorem 1.** Let $V$ be a primitive 2-generated axial algebra of Monster type $(\alpha, \beta)$ over a field $F$ of characteristic 5, such that $D \geq 6$. If $\alpha = 4\beta$, condition $D \geq 6$ and the proof of Proposition 4.12 in [1] yield $(\alpha, \beta) = (2, \frac{1}{2})$ and $\lambda_1 = \lambda_2 = 1$. Then, by Theorem 1 V satisfies (3). If $\alpha = 2\beta$, the proof of Claim 5.18 in [10] is still valid in characteristic 5, proving (2). Similarly, if $\alpha \neq 4\beta, 2\beta$, the proof of Claim 5.19 in [10] is also valid in characteristic 5 and gives (1).

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