Analysis of the Students' Argumentation based on the level of Ability: Study on the Process of Mathematical Proof

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Abstract: Argumentation is one important component in building an understanding of mathematics students. Argumentation is concerned with a person's ability to give reasons based on the facts to make a conclusion and is frequently used in mathematical proofs. Argumentation produce arguments that cannot be categorized right or wrong but it is valid or not, so a person's ability does not guarantee the validity of arguments, depending on his thinking process. This research aims to analyze the arguments using the Toulmin argumentation consists of: data, claim, warrant, backing, qualifier and rebuttal. The researchers grouped students based on skill level high and low then analyze the sixth component of argumentation used students and validitasnya in the process of proof. Qualitative Research methods are used to collect data in the form of the results of the work of students, think aloud, field notes, and the results of the interview. The results showed that students with low ability can make valid arguments with true claims whereas students with high ability can also produce a valid argument with wrong claims. Each component of the argument has different characteristics based on the student's ability level.

1. Introduction
Proof is a central aspect of mathematics [1–3] as in science subjects, and mathematics is seen as proving science [4]. Reference [5], one important part in understanding mathematics is to develop arguments and mathematical proof and evaluate it.  t is that it emphasizes the importance of argumentation and proving in the learning of mathematics. There are three aspects of knowledge that play an important role in mathematical proofs and show that evidence requires knowledge of facts and rules and is supported by the visualization of appropriate issues as well as specific methods as strategic principles [6].

Research on argumentation in proof has been done by many researchers of mathematics education based on the level of education: Magister and doctoral program students in proof [7], teachers [8, 9] undergraduate student [10, 11, 12] and school student [13, 14]. These studies are still done partially, so it can not be compared. In this study, students in a department of mathematics and mathematics education were selected as participants versus the level of ability.
Prior researchers focus on true-value mathematical statements so that one must prove the truth deductively and the inductive method is unacceptable as proof. The interesting thing is if the statement is false so that the student only shows one counterexample to deny the correctness of the given statement, then in this condition, the inductive method can be used as long as it helps the student generate the counterexample. Thus, students with different levels of ability can produce claims that vary depending on the thinking process. This is the focus of the study by describing students' basic thinking when generating claims and processes generating counterexample and compiling them into a deductive proof.

Toulmin's argumentation [15] was chosen to analyze the argumentation of the students in proving because Toulmin's argument provides space for non-formal arguments such as examples, drawings, diagrams, verbal arguments to produce conclusions. Toulmin's argument consists of six elements: data, claim, warrant backing, qualifier and rebuttal. Claims are statements made on the basis of data. Data is the 'foundation' of the argument based, facts relevant to the conclusion. Warrant (reason) such as 'bridge' that connects data and conclusions. Warrant (reason) is the rationale used to generate conclusions. This basic thought can take the form of formulas, definitions, make analogies, drawings or diagrams and graphs. Reasons are reinforced by backing which is further evidence required. A modal qualifier is a level of confidence in a conclusion, such as 'probably' or presumably, necessarily (necessarily). Rebuttal (argument) is a statement that denies the conclusions that are generated if conditions are not met. However, in this research, only the element of warrant will be discussed as the students' mindset in making the claim.

Based on the above explanation, this study aims to analyze the validity of claims generated by students as well as the type of warrant that underlies the claim and the thinking processes of students in generating counterexample and compiling it into valid proof, based on the level of student ability. It is important to be an input for educators in designing appropriate learning so that students with low level of ability can show good learning outcomes in a mathematical proof, by developing argumentation.

2. Research Methods
This research is a qualitative research with phenomenology design which aims to explain the phenomenon of argumentation which arises based on the level of ability in argumentation, related to student argument when constructing the evidence. In phenomenology design, data is collected not only through student work, but also through in-depth interviews to analyze, identify, understand and explain the students' thinking processes underlying their reactions and perceptions [16] to the given problem.

2.1. Material
The problem given is the problem of algebra that is a mathematical statement that is wrong (disproved) so that students are expected to find a counterexample which in Toulmin model called rebuttal. Prior to use in the study, this assignment sheet has been validated by experts. The tasks are given as follows:

Suppose the function \( f: \mathbb{R} \rightarrow \mathbb{R} \) by the formula \( f(x) = x^2 \) and \( g: \mathbb{R} \rightarrow \mathbb{R} \) by the formula \( g(x) = x \). Investigate whether \( f(x) \geq g(x) \) for all \( x \) Real numbers?

2.2. Participants and Procedure of Research
Participants were students of Pattimura University of Ambon, Maluku Indonesia, numbering 125 people from mathematics education and department of mathematics, consisting of 56 students with high ability level and 69 students with low ability level. The grouping of students is based on the value of the field of science (not common), namely: students are said to have high ability level if all grades of science subjects at least B and the contrary are categorized low ability.
The researcher gives the proof problem to the participants to be solved individually. During work, students are asked to voice what they think (think aloud). Students are given the opportunity to explore, write and say all their thoughts and ideas without being limited by time. They will be finished when they feel they are not able to finish it or have no idea anymore. During the problem-solving, the researchers recorded, observed and recorded all behaviors including think-aloud students. Students are then interviewed individually to explain their arguments when constructing a proof. During the interview, students are given the opportunity to improve their answers, without researcher intervention. Researchers then analyze all data (observation data, interviews, think aloud and field notes) and reduce things that are considered not important. Analysis of interview results using a multi-case study approach developed by Bromley [17]. Researchers classify students’ ability level into high ability and low ability, then discuss the argumentation that happened to each group of special student on warrant element and validity of claim.

3. Results and Discussion

Inglish [7] distinguishes warrants into inductive warrants, deductive warrants, and intuitive structural warrants. Inductive warrant occurs when the truth of the claim is based on an alleged evaluation in one or more specific cases. A deductive warrant is a conclusion based on the definition of axioms, properties, and mathematical rules. While an intuitive warrant is a conclusion based on the result of intuition about the structure of a person’s internal representation. Students can generate some initial claims before they arrive at a final claim that must be based on multiple warrant uses. The following shows data on the number of students who use the three types of the warrant on the first claim based on the level of ability.

| Warrant (%) | Inductive | Deductive | Structural Intuitive |
|-------------|-----------|-----------|----------------------|
| High Ability | 26 (20.8) | 18 (14.4) | 12 (9.6) |
| Low Ability  | 47 (37.6) | 5 (4)     | 17 (13.6) |
| Sum          | 73 (54.6) | 23 (18.4) | 29 (23.2) |

Figure 1 shows a surprising result because the percentage of students based on the type of warrant used by each group is very large. Figure 1 shows that more than 50 percent of students still use inductive warrants as the primary premise for generating claims, and this number is largely derived from low ability groups. The deductive warrant is still the smallest part of the mathematical proofing process. In the high ability group, the percentage of students using deductive warrant is only one-half of the percentage of students using inductive warrant, whereas the percentage of students with intuitive warrant structural is between inductive and deductive warrant. The big difference occurs in the low ability group. The percentage of students using inductive warrant in low ability group is more than three times the percentage of students who use deductive warrant and twice more than an intuitive
structural warrant. Percentage of use of deductive warrant is the least in low ability group. The percentage of students using intuitive structural warranties on high ability and low abilities showed similar numbers.

The three types of warrants generate claims with different validities that are valid (true) and invalid (false) claims. Invalid claims can also be changed to valid if the student realizes his mistake and renews the warrant previously used. The following will explain the three types of warrants based on the level of ability.

3.1. Warrant at Low Ability Level
From the 69 students at low ability level (Table 1), there were 47 students (37.6%) using inductive warrant, 5 students (4%) using deductive warrant and 17 students (13.6%) using intuitive warrant structural. These three types of warrants result in different claim validity. The percentage of students generating valid and invalid claims is described in Table 2.

| Warrant             | Valid (%) | Not valid (%) |
|---------------------|-----------|---------------|
| Inductive           | 18 (26.09)| 29 (42.03)    |
| Deductive           | 0 (0)     | 5 (7.24)      |
| Structural Intuitive| 0 (0)     | 17 (24.64)    |
| Sum                 | 18 (26.09)| 51 (73.91)    |

The results in Table 2 indicate that of 69 students with low abilities, the percentage of students who generate valid claims based on an inductive warrant is only one-half (50%) of the invalid claims. Whereas claims based on intangible deductive and structural warrants are entirely invalid.

Based on students' work and interviews, it was found that inductive warrants used by students with low abilities and producing true claims are all examples of real numbers. Invalid claims based on an inductive warrant consist of two examples of real numbers and arrows. Students try to use the arrow diagram as a function representation and describe the mapping result, but it does not produce a valid claim because they do not register all real numbers in the codomain region which leads them to conclude that the true value statement, that is for all real numbers apply \( f(x) \geq g(x) \) or \( x^2 \geq x \). The same thing happens to inductive warrants in the form of numerical examples when taking some sample numbers to test, students do not choose fractions, they only focus on integers that generate claims \( x^2 \geq x \).

Of the 55 students who produced false claims, there were 9 students (4 with deductive warrants and 5 with intuitive structural warrants) who changed initial claims and generated new valid claims due to the realization of errors in the initial warrant. New valid claims are all based on an inductive warrant that is deductive proof. This process occurs when the student realizes that the number he used at the beginning as a warrant is incomplete, there are still other real numbers, that is, fractions with one numerator that produce \( x^2 < x \).

3.2. Warrant at Level High Ability
From 56 students at high ability level (Table 1), there were 26 students (20.8%) using inductive warrant, 18 students (14.4%) using deductive warrant and 12 students (9.6%) using intuitive warrant structural. The validity of the resulting claims is also different, as in Table 3.

| Warrant       | Valid (%) | Not valid (%) |
|---------------|-----------|---------------|
| Inductive     | 12 (21.43)| 14 (25)       |
| Deductive     | 8 (14.28) | 10 (17.86)    |
Students at a high ability level have claim validity varying by type of warrant used. Although the number of students generating more invalid claims is more than valid claims, the difference in the number of students is only between two and three percent, but the intuitive structural warrant is 7 percent but the difference is not significant.

The inductive warrant used by high ability students consists of examples of real numbers and graphs of linear and quadratic functions. An inductive warrant with examples of real numbers generates valid claims but there are also invalid claims. An invalid claim is caused because the student enrolls only negative and positive integers and forgets the fractional number. In contrast, inductive warrants in the form of function graphs all produce valid claims. Valid claims occur when graphs of linear and squared functions are depicted in a Cartesian coordinate system, students find that the quadratic function graph \( f(x) = x^2 \) is not always above the linear function graph \( g(x) = x \). Based on prior knowledge, (the graph above shows the function value more than the function graph below) the student concludes that \( f(x) \geq g(x) \) is not true for all real numbers. Students give counterexample based on graph, that is for \( 0 < x < 1 \) value \( f(x) < g(x) \) or \( x^2 < x \). Students who use inductive warrant in the form of function graph generates claim smoothly, without making a mistake because it is supported by the correct understanding of function graph.

The intuitive structural warrant also contributes to the truth of the resulting claim. Students who use intuitive structural warrant have awareness () on the given problem. Interview results show that students "suspect" misjudged statements based on problem language 'investigating' so that they try to find counter-examples. There are also students who quickly (intuition) claim that there are numbers that do not meet the given inequality. Most students who generate invalid claims based on intuitive structural warrant say that "the square of a number must be positive and always more than itself".

The deductive warrant that underlies the claim takes the form of the properties of the real numbers. Students classify real numbers into positive, negative and fractional integers. Next, the students use the properties of numbers (negative numbers squared positive, positive squared positive numbers), then for fractions, they need backing to determine the relationship \( x^2 \) with \( x \). Based on the results obtained they claim that the statement is false because there is a fractional number with a numerator that does not meet the given specificness. The deductive warrant that produces a false claim is caused because the student does not use the properties of real numbers correctly, ie the students use the analytic method in proof but when grouping their numbers using only two groups, that is greater than zero and less than zero.

| Warrant       | Valid | Not valid |
|---------------|-------|-----------|
| Structural    | 4 (7.14) | 8 (14.29) |
| Intuitive     | 20 (36.85) | 28 (51.15) |
| Sum           | 24 (42.85) | 32 (57.15) |

4. Conclusion
Based on the results and discussion, it can be concluded several things. First, students with low abilities tend to use one type of warrant inductive warrant in the form of examples of real numbers, while students with high ability tend to use more of warrant types. Second, the function graph as one of the function representation, only used by the student in high ability group. Third, when finding counterexample students with high ability to perform systemization (compiling in deductive evidence) as a condition of proof, on the contrary, the low ability group realizes that there are examples that do not meet but cannot conclude because it does not properly understand the proof by counterexample. Fourth, a deductive warrant is only used by students with high ability with backing example number. Fifth, both groups restructured their claims when they realized the warrant's error.

The inductive warrant does not always produce invalid claims, otherwise deductive warrants do not always produce valid claims while intuitive structural warrants tend to generate invalid claims. The inductive warrant can generate a valid claim for a disproved statement if the student uses the complete sample number and for the student function graph must have an initial knowledge of the concept of a
good functional value so as to make a correct conclusion based on the resulting image. The intuitive structural warrant is basically correct, but weak because it is incomplete.

Invalid claims may become valid if students are aware of their mistakes and restructure. The process of generating a new valid claim is based on an inductive and deductive warrant. In this condition, students may use the same warrant type or renew warrant types. In low ability, students still use an inductive warrant of numbers by taking a number different from the initial warrant. In high ability, the majority of students replace warrant types with deductive warrant can produce correct claim even though initially make the wrong claim.

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