Primordial non-Gaussianities of gravitational waves beyond Horndeski

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We clarify the features of primordial non-Gaussianities of tensor perturbations in Gao’s unifying framework of scalar-tensor theories. The general Lagrangian is given in terms of the ADM variables so that the framework maintains spatial covariance and includes the Horndeski theory and Gleyzes-Langlois-Piazza-Vernizzi (GLPV) generalization as specific cases. It is shown that the GLPV generalization does not give rise to any new terms in the cubic action compared to the case of the Horndeski theory, but four new terms appear in more general theories beyond GLPV. We compute the tensor 3-point correlation functions analytically by treating the modification to the dispersion relation as a perturbation. The relative change in the 3-point functions due to the modified dispersion relation is only mildly configuration-dependent. When the effect of the modified dispersion relation is small, there is only a single cubic term generating squeezed non-Gaussianity, which is the only term present in general relativity. The corresponding non-Gaussian amplitude has a fixed and universal feature, and hence offers a “consistency relation” for primordial tensor modes in a quite wide class of single-field inflation models. All the other cubic interactions are found to give peaks at equilateral shapes.

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I. INTRODUCTION

Inflation \cite{1–3}, the accelerated expansion of the early universe, is an almost perfect idea for generating seeds for structure formation as well as resolving several issues in standard Big Bang cosmology. Valuable information about the physics of inflation is carried by the power spectrum and bispectrum of primordial perturbations. The properties of the primordial curvature perturbations, \( R \), can be explored through observations of CMB anisotropies \cite{4} and are characterized primarily by the amplitude and the spectral index, which can be translated to information about the shape of the inflaton potential. Some nonstandard inflation models predict strongly non-Gaussian curvature perturbations \cite{5}, and hence can be constrained by the bispectrum of \( R \) \cite{6}. In addition to the curvature perturbations, tensor modes, \( i.e. \), gravitational waves, are also produced during the phase of the inflationary expansion. Currently, we have only an upper bound on the tensor-to-scalar ratio \( r \), but primordial gravitational waves could be a smoking gun of inflation if observed in future experiments of direct detection or CMB B-mode polarization measurements.

In this paper, we study primordial non-Gaussianity of gravitational waves from inflation. After the seminal work by Maldacena who computed not only the scalar 3-point correlation function but also the tensor 3-point function \cite{7}, several authors have investigated non-Gaussian signatures of primordial tensor modes \cite{8−21}. As tensor non-Gaussianity could in principle be measured \( e.g. \), via the bispectrum of B-mode fluctuations, it offers us yet another discriminant among an enormous number of different inflation models. The purpose of the present paper is therefore to clarify the features of primordial tensor non-Gaussianities within a framework involving as many inflation models as possible.

Generalized G-inflation \cite{22} is the general framework to study single-field inflation models based on the Horndeski theory \cite{23, 24}, which is the most general scalar-tensor theory with second-order field equations and thus is free from Ostrogradsky ghost instabilities. Within this generalized G-inflation framework, the general form of the power spectra of curvature and tensor perturbations has been obtained in Ref. \cite{22}, and the cubic interactions of \( R \) have been derived in Refs. \cite{25−29} to evaluate primordial non-Gaussianity of the curvature perturbations. Tensor 3-point interactions in the Horndeski theory have been classified completely in Ref. \cite{11} and it has been shown that there are only two independent contributions: the “standard” one that is present already in general relativity and generates squeezed non-Gaussianity, and the other “nonstandard” one predicting equilateral non-Gaussianity that arises from the coupling between the Einstein tensor and derivatives of the scalar field. In particular, the former contribution has the fixed non-Gaussian amplitude irrespective of an underlying model.

Recently, it was noticed that the Horndeski theory can further be generalized to higher derivative theories that nevertheless preserve the same propagating degrees of freedom, \( i.e. \), two polarizations of gravitational waves and one

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and the Lapse function $N$ and the presence of the scalar degree of freedom. One notices here that the relations (5) surfaces as constant time hypersurfaces, one obtains the Lagrangian of the form $\mathcal{L}_H^{(4)} = G_2(\phi, X) - G_3(\phi, X) \partial^i \phi + G_4(\phi, X) R^{(4)} + G_5(\phi, X) G_{\mu \nu}^{(4)} \partial^\mu \partial^\nu \phi \cdots$, \hfill (1)

where $R^{(4)}$ and $G_{\mu \nu}^{(4)}$ are the (four-dimensional) Ricci scalar and the Einstein tensor, respectively, and we have four arbitrary functions of $\phi$ and $X := -g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi / 2$. Performing an ADM decomposition by taking $\phi = \text{const}$ hypersurfaces as constant time hypersurfaces, one obtains the Lagrangian of the form

$$\mathcal{L}_H^{(4)} = A_2(t, N) + A_3(t, N) K + A_4(t, N) (K^2 - K_{ij} K^{ij}) + B_4(t, N) R$$

$$+ A_5(t, N) (K^3 - 3K K_{ij} K^{ij} + 2K_{ij} K^{jk} K^{ki}) + B_5(t, N) K^{ij} \left( R_{ij} - \frac{1}{2} g_{ij} R \right),$$ \hfill (2)

where $K_{ij}$ and $R_{ij}$ are the extrinsic and intrinsic curvature tensors on the constant $\phi$ hypersurfaces. The functions of $\phi$ and $X (= \dot{\phi}^2 / 2N^2)$ in the covariant Lagrangian (1) are now regarded as the functions of the time coordinate $t$ and the Lapse function $N$ in the ADM language. We thus should have four functions of $t$ and $N$, and indeed $A_4$, $A_5$, $B_4$, and $B_5$ obey

$$A_4 = -B_4 - N \frac{\partial B_4}{\partial N}, \quad A_5 = \frac{N}{6} \frac{\partial B_5}{\partial N},$$ \hfill (3)

leaving four arbitrary functions as expected. This is the unitary gauge description of the Horndeski theory which is particularly useful in the study of cosmology.

The scalar degree of freedom is apparently hidden in the unitary gauge description. However, variation of (2) with respect to $N$ yields a second-class constraint eliminating one degree of freedom rather than two. This fact signifies the presence of the scalar degree of freedom. One notices here that the relations (3) play no role when counting the degrees of freedom. Therefore, the degrees of freedom remain the same even if one liberates $A_4$ and $A_5$ from (3) and considers the theory with six arbitrary functions of $t$ and $N$. This trick was first used by Gleyzes et al. [30] to construct the GLPV scalar-tensor theory beyond Horndeski. Since the GLPV theory is more general than Horndeski, the covariant equations of motion are of higher order in general. Nevertheless, the true propagating degrees of freedom are one scalar and two tensors as in the Horndeski theory [30, 31, 38]. (Note, however, that counting the degrees of freedom using the unitary gauge Lagrangian involves subtle points [39, 40].) See also Refs. [41, 42] for attempts to extend the Horndeski theory in different directions.

The GLPV theory can further be generalized while retaining the number of propagating degrees of freedom as demonstrated by Gao [32, 43]. The key is to notice that one no longer needs to impose the specific combinations of $K_{ij}$ and $R_{ij}$ as presented in Eq. (2). Now one arrives at the unifying framework of scalar-tensor theories which is given by the sum of three dimensional scalars composed of $K_{ij}$, $R_{ij}$, and $a_i := \partial_i \ln N$, with coefficients depending on $t$ and $N$:

$$\mathcal{L} = d_0(t, N) + d_1(t, N) R + d_2 R^2 + \cdots + d_4 a_i a^i + \cdots + (a_0 + a_1 R + \cdots) K$$

$$+ (a_2 R^{ij} + \cdots) K_{ij} + b_1 K^2 + b_2 K_{ij} K^{ij} + \cdots + c_1 K^3 + c_2 K K_{ij} K^{ij} + c_3 K_{ij} K^{ij} K_{k}^{i} + \cdots.$$ \hfill (4)
The “six-parameter” subclass of the above Lagrangian with
\begin{align*}
a_0 &= A_3, \quad -2a_1 = a_2 = B_5, \quad b_1 = -b_2 = A_4, \\
c_1 &= -c_2/3 = c_3/2 = A_5, \quad d_0 = A_2, \quad d_1 = B_4,
\end{align*}
corresponds to the GLPV theory. By imposing the further restrictions (3) we have the “four-parameter” subclass which reproduces the Horndeski theory. Even (the healthy extension of) Horva gravity \cite{44, 45} is included within this framework. Among possible terms Gao considers 28 operators satisfying the requirements that (i) there are no derivatives higher than two when going back to the covariant Lagrangian and (ii) the number of second-order derivative operators does not exceed three \cite{32}. We shall work in Gao’s framework to cover a fairly wide range of single-field inflation models, and evaluate all possible tensor 3-point functions. As explained below, the number of operators that are relevant to tensor perturbations is in fact not as large as 28.

Let us now simplify the Lagrangian before computing the quadratic and cubic terms of tensor perturbations. We define the tensor perturbations \( h_{ij} \) by
\begin{align*}
ds^2 &= -dt^2 + \gamma_{ij} dx^i dx^j \\
&= -dt^2 + a^2 (\epsilon^b)_{ij} dx^i dx^j,
\end{align*}
where \((\epsilon^b)_{ij} = \delta_{ij} + h_{ij} + (1/2) h_{ik} h_{kj} + \cdots \) and \( h_{ij} \) is transverse and traceless. We simply have \( \sqrt{-g} = a^3 \) with this definition. As a consequence of the traceless and transverse nature of \( h_{ij} \), the trace of the extrinsic curvature tensor, \( K \sim \gamma^{ij} \partial_t \gamma_{ij} \), can be written solely in terms of the background quantities. One sees that the intrinsic curvature scalar \( R \) does not contain any first-order terms up to total derivatives. These facts lead us to the following 14 terms that are relevant to the tensor perturbations:
\begin{align*}
\frac{\mathcal{L}}{\sqrt{-g}} &= \left( a_1 R + a_4 R_i^j R_j^i \right) K + \left( a_2 R_i^j + a_7 R_i^k R_k^j \right) K^j_i + b_3 R K^2 + (b_2 + b_4) R K_i^j K_i^j \\
&\quad + \left( b_5 KK_i^j + b_6 K_i^k K_j^k \right) R^j_i + c_2 K K_j^j K_i^k + c_3 K_i^k K_j^k K_i^k + d_1 R + d_3 R_i^j R_j^i + d_7 R_i^k R_k^j R_j^i.
\end{align*}
Here, the coefficients may be regarded as the functions of \( t \) since we have set \( N = 1 \). It is convenient to split the extrinsic curvature into the background and perturbation parts as
\begin{equation}
K_i^j = H \delta_i^j + \delta K_i^j,
\end{equation}
where \( H := \dot{a}/a \) and
\begin{equation}
\delta K_i^j = \frac{1}{2} \ddot{h}_{ij} + \frac{1}{4} \left( h_{ik} \dot{h}_{kj} - \dot{h}_{ik} h_{kj} \right) + \mathcal{O}(h^3),
\end{equation}
with a dot standing for differentiation with respect to \( t \). It can be seen directly that \( \delta K_i^j \) is traceless, \( \delta K_i^i = 0 \). Substituting Eq. (8) to the Lagrangian (7) and extracting the perturbations, we obtain
\begin{align*}
\frac{\mathcal{L}}{\sqrt{-g}} &= \ddot{a}_1 R + \ddot{a}_3 R_i^j R_j^i + \ddot{a}_7 R_i^k R_k^j R_j^i + \ddot{b}_2 \delta K_i^j \delta K_i^j + c_3 \delta K_i^j \delta K_i^k \delta K_i^k \\
&\quad + \ddot{a}_2 R_i^j \delta K_j^i + a_7 R_i^k \delta K_j^k + b_6 R_i^j \delta K_j^k + b_6 R_i^j \delta K_j^k \delta K_i^k,
\end{align*}
where the coefficients are defined as
\begin{align*}
\ddot{a}_1 &= d_1 + (3a_1 + a_2) H + (9b_5 + 3b_4 + 3b_5 + b_6) H^2, \\
\ddot{a}_3 &= d_3 + (3a_4 + a_7) H, \\
\ddot{b}_2 &= b_2 + 3 (c_2 + c_3) H, \\
\ddot{a}_2 &= a_2 + (3b_5 + 2b_6) H.
\end{align*}
Those coefficients are determined once the concrete form of the Lagrangian is fixed and the background cosmological evolution is obtained. The reduced Lagrangian (10) is sufficient for deriving the most general quadratic and cubic Lagrangians for the tensor perturbations within Gao’s unifying framework of scalar-tensor theories.

In the GLPV subclass, we have only four non-vanishing coefficients, \( \ddot{a}_1, \ddot{b}_2, c_3, \) and \( \ddot{a}_2 \), all of which are arbitrary. In the Horndeski case, \( \ddot{b}_2 \) and \( c_3 \) are fixed by \( \ddot{a}_1 \) and \( \ddot{a}_2 \) via the relations (3), and hence we have essentially two
functional degrees of freedom, though all four of these coefficients are still non-vanishing. Thus, even if one generalizes the Horndeski theory to GLPV, no new terms appear in the Lagrangian for the tensor perturbations, though the coefficients can be chosen more freely in the GLPV theory. The other four terms appear for the first time when going to Gao’s framework. In particular, the $a_7$ and $b_6$ terms are completely new, while the $\tilde{d}_3$ and $d_7$ terms can be found in the specific case of Hořava gravity as well.

### III. PRIMORDIAL POWER SPECTRUM WITH MODIFIED DISPERSION RELATION

#### A. Linear theory

From Eq. (10) we obtain the quadratic action for the tensor perturbations [32, 34, 36]:

$$S = \frac{1}{8} \int dt d^3 x \ a^3 \left[ G_T \ddot{h}_{ij} - \frac{F_T}{a^2} (\partial_k h_{ij})^2 + 2 \frac{\tilde{d}_3}{a^4} (\partial^2 h_{ij})^2 \right],$$

where

$$G_T := 2\tilde{b}_2, \quad F_T := 2\tilde{d}_1 + \dot{\tilde{a}}_2 + H\tilde{a}_2.$$  \hspace{1cm} (16)

The third term modifies the dispersion relation. This term is absent in the Horndeski theory and even in its GLPV generalization, but it appears in more general theories in the unifying framework. The goal of this section is to present the power spectrum of primordial tensor modes with the modified dispersion relation. Primordial perturbations with this type of the dispersion relation have been studied e.g., in Refs. [46–48], with an emphasis on the curvature perturbation.

Before calculating the linear solution and the power spectrum, let us comment on the stability conditions derived from the quadratic action (15). It is required that $G_T > 0$ to avoid ghost instabilities. We also impose the condition

$$\tilde{d}_3 \leq 0$$

in order for the modes with high momenta to be stable. If $F_T > 0$ then the tensor perturbations are stable at low momenta as well. (However, in fact $F_T$ can be negative for a short period provided that $\tilde{d}_3 < 0$, because the instability grows only at low momenta and hence is not catastrophic.)

We move to the Fourier space,

$$h_{ij}(t, \mathbf{x}) = \int \frac{d^3 k}{(2\pi)^3} \tilde{h}_{ij}(t, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}},$$

and solve the equation of motion for $\tilde{h}_{ij}(t, \mathbf{k}),$

$$\frac{1}{a^2} \frac{d}{d\eta} \left( a^2 G_T \frac{d}{d\eta} \tilde{h}_{ij} \right) + \left( F_T k^2 - \frac{2\tilde{d}_3}{a^2} k^4 \right) \tilde{h}_{ij} = 0,$$

where we have introduced the conformal time defined by $d\eta = dt/\dot{a}.$

To obtain an exact solution to Eq. (19) and manipulate the non-Gaussianity in a tractable way, we make the following assumptions on the background quantities. First, the background cosmological evolution is assumed to be exact de Sitter, so that the scale factor is given by $a = 1/H(\eta).$ Second, $F_T, G_T, \tilde{d}_3,$ and also the other coefficients in the Lagrangian (10) are all assumed to be constant. The second assumption is natural and consistent in view of the first assumption, though strictly speaking the coefficients of the terms involving $R_{ij},$ i.e., $\tilde{a}_1, \tilde{d}_3, \cdots,$ can be dependent on time as the background evolution is insensitive to those coefficients and is determined by the terms constructed only from $K_{ij}.$

With the above assumptions, the dispersion relation can be written as

$$\omega^2 = c_h^2 k^2 + c^2 k^4 \eta^2,$$

where we defined the dimensionless quantities

$$c_h^2 := \frac{F_T}{G_T}, \quad c^2 := -2H^2 \frac{\tilde{d}_3}{G_T}.$$  \hspace{1cm} (21)
The normalized mode solution to the perturbation equation (19) is given in terms of the Whittaker function \( W \) by

\[
\psi_k(\eta) = \frac{\sqrt{\pi}}{a} \frac{e^{-\pi^2/k^2/8x}}{\sqrt{-G_T k^2 \eta}} W \left( \frac{i c_k^2}{4 \epsilon}, \frac{3}{4}, -i \epsilon k^2 \eta^2 \right). 
\]  

(22)

In the case of the standard dispersion relation, \( \epsilon \to 0 \), which is the case for the GLPV theory as well as Horndeski, we have

\[
\psi_k \to \frac{\sqrt{\pi}}{a} \frac{1}{\sqrt{2\omega}} \exp \left[ -i \int \eta(\eta') d\eta' \right],
\]

(24)

up to a phase factor, where \( H^{(1)}_{3/2} \) is the Hankel function of the first kind. Thus, the familiar mode function is reproduced. For \( \epsilon k^2 \eta^2 \gg 1 \) with \( \epsilon \neq 0 \), \( a \sqrt{G_T/2} \psi_k \) reduces to the positive frequency WKB solution,

\[
\frac{a \sqrt{G_T}}{2} \psi_k \approx \frac{1}{\sqrt{2\omega}} \exp \left[ -i \int \omega(\eta') d\eta' \right],
\]

with \( \omega \approx -\epsilon k^2 \eta \), showing that \( a \sqrt{G_T/2} \psi_k \) is indeed canonically normalized. In the opposite, superhorizon limit, \( -k \eta \to 0 \), we find

\[
\psi_k \to \sqrt{\frac{\pi}{2}} \frac{e^{-\pi^2/k^2/8x}}{\Gamma(5/4 - i c_k^2/4 \epsilon) \sqrt{G_T} k^{3/4} \eta^{3/2} / e^{3/4} k^{3/2}}
\]

(25)

up to a phase factor.

Using the mode function \( \psi_k \), the quantized tensor perturbation can be expanded as

\[
\tilde{h}_{ij}(t, \mathbf{k}) = \sum_s \left[ \psi_k e^{(s)}_{ij}(\mathbf{k}) a_s(\mathbf{k}) + \psi^*_{-k} e^{(s)}_{ij}(-\mathbf{k}) a^+_s(-\mathbf{k}) \right],
\]

(26)

where the transverse and traceless polarization tensor with the helicity states \( s = \pm 2 \), \( e^{(s)}_{ij}(\mathbf{k}) \), is normalized as \( e^{(s)}_{ij}(\mathbf{k}) e^{(s')}_{ij}(\mathbf{k}) = \delta_{ss'} \) and has the properties such that \( e^{(s)}_{ij}(\mathbf{k}) = e^{(-s)}_{ij}(\mathbf{k}) = e^{(s)}_{ij}(-\mathbf{k}) \). The commutation relation is given by \([a_s(\mathbf{k}), a^+_s(\mathbf{k}')] = (2\pi)^3 \delta_{ss'} \delta^3(\mathbf{k} - \mathbf{k}') \). Now we are ready to compute the primordial power spectrum. First we define \( P_{ij,kl}(\mathbf{k}) \) by

\[
\langle \tilde{h}_{ij}(\mathbf{k}) \tilde{h}_{kl}(\mathbf{k}') \rangle = (2\pi)^3 \delta^3(\mathbf{k} + \mathbf{k}') P_{ij,kl}(\mathbf{k}) .
\]

(27)

We have

\[
P_{ij,kl}(\mathbf{k}) = |\psi_k|^2 \Pi_{ij,kl}(\mathbf{k}),
\]

(28)

with

\[
\Pi_{ij,kl}(\mathbf{k}) := \sum_s e^{(s)}_{ij}(\mathbf{k}) e^{(s)}_{kl}(\mathbf{k}).
\]

(29)

The power spectrum, \( P_h := (k^3/2\pi^2) P_{ij,ij} \), is then given by

\[
P_h(k) = \frac{2}{\pi^2} \frac{H^2 G_T^{1/2}}{F_{T}^{3/2}} |F(\epsilon/c_k^2)|^2,
\]

(30)

where we defined

\[
F(x) := \frac{x^{-3/4} e^{-\pi/(8x)}}{2 \Gamma(5/4 - i/(4x))}.
\]

(31)

Since \( |F(x)| = 1 - 5x^2/8 + O(x^4) \), the above expression includes, as the \( x \to 0 \) limit, the general result with the standard dispersion relation derived from generalized G-inflation [22].
FIG. 1: The real part of the exact solution (22) (solid lines) versus that of the approximate expression (32) (dashed lines) as functions of $y = c_h k \eta$. We take (a) $\delta = 10^{-2}$ and (b) $\delta = 10^{-3}$. It can be seen that the approximation is good for $\delta \times y^2 \lesssim 1$.

B. Small $\epsilon$ expansion of the mode function with modified dispersion relation

In the next section we compute the 3-point functions by means of the in-in formalism [7]. To do so, one has to integrate products of $\psi_k$ with respect to time. Unfortunately, this turns out to be impossible in an analytic way if one directly uses $\psi_k$ written in terms of the Whittaker function. Therefore, we assume that the modification to the dispersion relation is small, and expand $\psi_k$ in terms of $\epsilon$. The approximation allows us to calculate the 3-point functions analytically in a perturbative manner.

To second order in $\epsilon$, we find, up to a phase factor,

$$\frac{a\sqrt{G}}{2} \psi_k = F(\delta) e^{-iy+i\delta y^2/2} \sqrt{2c_h k} \left[ -i \frac{y}{2} + 1 - \frac{\delta}{2} \left( y + iy^2 \right) - \delta^2 \left( \frac{5}{12} y^2 + i \frac{y^3}{24} + \frac{1}{8} y^4 \right) + O(\delta^3) \right],$$

(32)

where $y := c_h k \eta$ and $\delta := \epsilon/c_h^2$. The approximation is valid as long as $\delta \times y^2 = \epsilon k^2 \eta^2 \lesssim 1$, as is clearly seen from Fig. 1. We use this formula in the following calculations of the 3-point functions of the tensor perturbations. The same approximation was used in Ref. [46] to evaluate the effects of nonstandard dispersion relations on non-Gaussianities of the curvature perturbations.

IV. CUBIC INTERACTIONS AND 3-POINT CORRELATION FUNCTIONS

A. The cubic action

Expanding Eq. (10) to third order in $h_{ij}$, we obtain the following cubic action,

$$S = \int dt d^3 x \left\{ \frac{c_8}{8} h_{ij}^2 h_{ij}^2 h_{ij}^2 + \frac{F_T}{4a^2} \left( h_{ik} h_{jl} - \frac{1}{2} h_{ij} h_{kl} \right) h_{ij,kl} + \frac{a_7}{8a^4} h_{ik}^2 h_{ij}^2 \partial^2 h_{jk}^2 \partial^2 h_{li}^2 - \frac{b_6}{8a^2} h_{ik}^2 h_{ij}^2 \partial^2 h_{lk}^2 \partial^2 h_{kj}^2 \right\}.$$

(33)

At this stage, several comments are in order. It is easy to see that $\tilde{b}_2 \delta K_j^i \delta K_j^j$ does not contribute to the cubic action. It is also noted that $\tilde{d}_1 R$ and $\tilde{a}_2 R^l_j \delta K_j^l$ give rise to the identical terms even at cubic order, leading to the single combination having the coefficient $F_T$. We are thus left with the above 6 combinations out of the 8 terms in
Eq. (10). If the gravitational sector of the theory is described only by the Einstein-Hilbert term, only the terms with the coefficient $\mathcal{F}_T$ remain, and therefore all the other terms signal theories beyond general relativity.

Now let us look at the Horndeski and GLPV subclass by taking $a_7 = b_6 = \tilde d_3 = d_7 = 0$:

$$S \to \int \text{d}t \text{d}^3x \, a^3 \left[ \frac{c_3}{8} \tilde h^{ij} \tilde h_{ik} \tilde h_{jk} + \frac{\mathcal{F}_T}{4a^2} \left( h_{ik} h_{jl} - \frac{1}{2} h_{ij} h_{kl} \right) h_{ij,kl} \right]. \quad (34)$$

As shown in Ref. [11], we have two independent cubic interactions in the Horndeski theory. Those correspond to the two contributions displayed in the above action. We see that no new contributions appear and we have only the above two even in the GLPV theory beyond Horndeski. Thus, we conclude that the tensor bispectrum in the GLPV theory reduces to a combination of the shapes obtained within the Horndeski theory [11].

The other four interaction terms newly appear in Gao’s framework. Among them the $\tilde d_3$ and $d_7$ terms have also been studied in the context of Hořava gravity [13], though the modification of the mode function due to the nonstandard dispersion relation has been ignored for simplicity. It is worth noting that in the case of the standard dispersion relation, $\tilde d_3 = 0$, the $b_6$ term can be recast in the form of the $c_3$ term by the use of the linear equation of motion and integration by parts.

B. 3-point correlation functions

We turn to evaluate the 3-point correlation functions of the tensor modes. Although some of the 3-point functions have already been obtained in the literature [11, 13], below we present all the results for completeness. We also evaluate the corrections arising from the nonstandard dispersion relation. Using the in-in formalism [7], the 3-point functions can be computed as

$$\langle \tilde h_{ij,1}(k_1) \tilde h_{ij,2}(k_2) \tilde h_{ij,3}(k_3) \rangle = -i \int t_0^t \text{d}t' \left\{ \left[ \tilde h_{ii,jj}(t, k_1) \tilde h_{ii,jj}(t, k_2) \tilde h_{ii,jj}(t, k_3), H_{\text{int}}(t') \right] \right\}, \quad (35)$$

where $H_{\text{int}}$ is the interaction Hamiltonian obtained from the cubic action (33). Actually, integration with respect to time is to be performed using the conformal time rather than $t$ from $\eta = -\infty$ to $\eta = 0$. As mentioned earlier, this cannot be done analytically using the exact mode function written in terms of the Whittaker function (22). To make this step feasible, we use instead the approximate expression (32).

For convenience we introduce the non-Gaussian amplitude $A_{i_1j_1i_2j_2i_3j_3}$ defined by [11]

$$\langle \tilde h_{ij,1}(k_1) \tilde h_{ij,2}(k_2) \tilde h_{ij,3}(k_3) \rangle = (2\pi)^7 \delta^{(3)}(k_1 + k_2 + k_3) - \frac{p_h^2}{k_1^3 k_2^3 k_3^3} A_{i_1j_1i_2j_2i_3j_3}, \quad (36)$$

where $A_{i_1j_1i_2j_2i_3j_3}$ is written as a sum of each contribution, which we compute to order $\epsilon^2$. It turns out that the corrections due to the nonstandard dispersion relation start at $\mathcal{O}(\epsilon^2)$, so that we write

$$A_{i_1j_1i_2j_2i_3j_3} = \sum_{\bullet = c_3, a_7, \cdots} \left( A^{(\bullet)}_{i_1j_1i_2j_2i_3j_3} + \epsilon^2 \frac{\tilde c^{(\bullet)}}{c_h^2} A^{(\bullet)}_{i_1j_1i_2j_2i_3j_3} \right), \quad (37)$$

where $\bullet = c_3, a_7, \cdots$ denotes the corresponding term in the cubic action.

First, the non-Gaussian amplitude arising from the cubic interaction with the coefficient $\mathcal{F}_T$ is given by

$$A^{(\mathcal{F}_T)}_{i_1j_1i_2j_2i_3j_3} = \bar A \left\{ \Pi_{ij,1,ik}(k_1) \Pi_{ij,2,jl}(k_2) \left[ k_{3k} k_{3l} \Pi_{i_3,j_3,ij}(k_3) - \frac{1}{2} k_{3k} k_{3l} \Pi_{i_3,j_3,ij}(k_3) \right] + 5 \text{ permutations of } 1, 2, 3 \right\}, \quad (38)$$

$$C^{(\mathcal{F}_T)}_{i_1j_1i_2j_2i_3j_3} = \bar C \left\{ \Pi_{ij,1,ik}(k_1) \cdots \right\}, \quad (39)$$

where in $C^{(\mathcal{F}_T)}_{i_1j_1i_2j_2i_3j_3}$ the structure constructed from $\Pi_{ij,kl}$ is identical to that in $A^{(\mathcal{F}_T)}_{i_1j_1i_2j_2i_3j_3}$ and

$$\bar A = \frac{K_t}{16} \left( -1 + \frac{K_2}{K_t^2} + \frac{K_3}{K_t} \right), \quad (40)$$

$$\bar C = \frac{K_t}{8} \left( 1 - \frac{K_2}{K_t^2} + \frac{K_3}{K_t^2} - \frac{2}{3} \frac{K_2^2}{K_t^4} - \frac{6}{3} \frac{K_2 K_3}{K_t^4} + \frac{6}{3} \frac{K_3^2}{K_t^6} \right). \quad (41)$$
Here, we introduced the notations $K_i := k_1 + k_2 + k_3$, $K_2 := k_1k_2 + k_2k_3 + k_3k_1$, and $K_3 := k_1k_2k_3$. Note that $A^{(F_T)}_{i_1j_1i_2j_2i_3j_3}$ is the only term present in the case of general relativity. Interestingly, this term has the fixed and universal form and is insensitive to an underlying theory and the inflationary energy scale. This result strengthens the statement originally made in the Horndeski theory in Ref. [11].

The concrete expressions for the other amplitudes are obtained as

\[ A^{(c_3)}_{i_1j_1i_2j_2i_3j_3} = \frac{3c_3}{8} \frac{H^2}{\mathcal{G}_T^2} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} \Pi_{123}, \tag{42} \]
\[ A^{(a_7)}_{i_1j_1i_2j_2i_3j_3} = \frac{3a_7}{8} \frac{H^5}{\mathcal{G}_T^2 c_h} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} \Pi_{123}, \tag{43} \]
\[ A^{(b_0)}_{i_1j_1i_2j_2i_3j_3} = -\frac{3b_0}{4} \frac{H^2}{\mathcal{G}_T c_h^2} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} \Pi_{123}, \tag{44} \]
\[ A^{(d_7)}_{i_1j_1i_2j_2i_3j_3} = \frac{3d_7}{2} \frac{H^4}{\mathcal{G}_T^2 \epsilon_h^2} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} \Pi_{123}, \tag{45} \]

and

\[ C^{(c_3)}_{i_1j_1i_2j_2i_3j_3} = A^{(c_3)}_{i_1j_1i_2j_2i_3j_3} \cdot 2 \left( 1 - 12 \frac{K_2}{K_i} + 15 \frac{K_3}{K_i^2} \right), \tag{46} \]
\[ C^{(a_7)}_{i_1j_1i_2j_2i_3j_3} = A^{(a_7)}_{i_1j_1i_2j_2i_3j_3} \cdot 2 \left( 27 - 12 \frac{K_2}{K_i} + 33 \frac{K_3}{K_i^2} - 240 \frac{K_2^2}{K_i^3} + 210 \frac{K_2K_3}{K_i^4} \right) \left( 3 + 4 \frac{K_2}{K_i} \right)^{-1}, \tag{47} \]
\[ C^{(b_0)}_{i_1j_1i_2j_2i_3j_3} = A^{(b_0)}_{i_1j_1i_2j_2i_3j_3} \cdot 2 \left( 7 - 28 \frac{K_2}{K_i} + 30 \frac{K_3}{K_i^2} \right), \tag{48} \]
\[ C^{(d_7)}_{i_1j_1i_2j_2i_3j_3} = A^{(d_7)}_{i_1j_1i_2j_2i_3j_3} \cdot 20 \left( 1 + 3 \frac{K_2}{K_i} + 51 \frac{K_3}{K_i^2} - 18 \frac{K_2^2}{K_i^3} - 126 \frac{K_2K_3}{K_i^4} + 126 \frac{K_3^2}{K_i^5} \right) \left( 1 + 3 \frac{K_2}{K_i} + 15 \frac{K_3}{K_i^2} \right)^{-1}, \tag{49} \]

where we introduced the shortened notation for the common factor: $\Pi_{123} := \Pi_{i_1j_1i_2j_2j_3}(k_1)\Pi_{i_1j_1i_2j_2j_3}(k_2)\Pi_{i_1j_1i_2j_2j_3}(k_3)$. We see that $A^{(c_3)}_{i_1j_1i_2j_2i_3j_3}$ and $A^{(b_0)}_{i_1j_1i_2j_2i_3j_3}$ have the same momentum dependences as expected. Finally, the $d_7$ term itself is a small correction of $\mathcal{O}(e^2)$ in our approximation, so that $A^{(d_7)}_{i_1j_1i_2j_2i_3j_3} = 0$ and

\[ C^{(d_7)}_{i_1j_1i_2j_2i_3j_3} = \frac{1}{4K_i} \left[ 1 + \frac{K_2}{K_i} + 3 \frac{K_3}{K_i^2} \right] \left\{ -k_3^2 \Pi_{i_1j_1i_2j_2j_3}(k_3) \left[ \frac{1}{2} \Pi_{i_2j_2j_3}(k_2) \Pi_{i_1j_1i_2j_2j_3}(k_1) k_1k_2 \right. \right. \right. \right. \]
\[ + \Pi_{i_1j_1i_2j_2j_3}(k_1) k_1k_2 + \left. \frac{1}{2} \Pi_{i_1j_1i_2j_2j_3}(k_1) k_1k_2 \right\} + 5 \text{ permutations of } 1, 2, 3 \} \tag{50} \]

Let us investigate the 3-point correlation functions between each polarization mode of gravitational waves. The two polarization modes are expressed as

\[ \xi^{(s_1)}(\mathbf{k}) = \tilde{h}_{ij}(\mathbf{k}) e^{s_1(s_2)}_{ij}(\mathbf{k}) \tag{51} \]

For the 3-point functions $\langle \xi^{(s_1)} \xi^{(s_2)} \xi^{(s_3)} \rangle$ its amplitude can be evaluated by computing

\[ \begin{align*}
\left\{ \begin{array}{c}
A^{(s_1s_2s_3)}_{(c_3)} \\
C^{(s_1s_2s_3)}_{(s_1s_2s_3)}
\end{array} \right\} & = e^{(s_1)(s_2)}_{i_1j_1i_2j_2}(k_1) e^{(s_2)(s_3)}_{i_1j_1i_2j_2}(k_2) e^{(s_3)}_{i_1j_1i_2j_2}(k_3) \times \left\{ \begin{array}{c}
A^{(s_1s_2s_3)}_{i_1j_1i_2j_2j_3}
\end{array} \right\}.
\end{align*} \tag{52} \]

It is straightforward to obtain the following amplitudes,

\[ A^{s_1s_2s_3}_{(F_T)} = \tilde{A} \cdot F^{s_1s_2s_3}_{(GR)}, \tag{53} \]
\[ A^{s_1s_2s_3}_{(c_3)} = \frac{3c_3}{8} \frac{H^2}{\mathcal{G}_T^2} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} F^{s_1s_2s_3}_{(H)}, \tag{54} \]
\[ A^{s_1s_2s_3}_{(a_7)} = \frac{3a_7}{8} \frac{H^5}{\mathcal{G}_T^2 c_h^3} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} F^{s_1s_2s_3}_{(H)}, \tag{55} \]
\[ A^{s_1s_2s_3}_{(b_0)} = -\frac{3b_0}{4} \frac{H^2}{\mathcal{G}_T c_h^3} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} F^{s_1s_2s_3}_{(H)}, \tag{56} \]
\[ A^{s_1s_2s_3}_{(d_7)} = \frac{3d_7}{2} \frac{H^4}{\mathcal{G}_T \epsilon_h^2} \left( 3 + 4 \frac{K_2}{K_i} \right) \frac{K_2^2}{K_i^2} F^{s_1s_2s_3}_{(H)}. \tag{57} \]
and

\[ C^{s_1, s_2, s_3}_{F_t} = \tilde{C} \cdot F^{s_1, s_2, s_3}_{(GR)}, \]

(58)

\[ C^{s_1, s_2, s_3}_{(c_3)} = A^{s_1, s_2, s_3}_{(c_3)} \cdot 2 \left( 1 - 12 \frac{K_2}{K_t^2} + 15 \frac{K_3}{K_t^3} \right), \]

(59)

\[ C^{s_1, s_2, s_3}_{(c_7)} = A^{s_1, s_2, s_3}_{(c_7)} \cdot 2 \left( 27 - 14 \frac{K_2}{K_t^2} + 30 \frac{K_3}{K_t^3} - 240 \frac{K_2^2}{K_t^4} + 210 \frac{K_2 K_3}{K_t^5} \right) \left( 3 + 4 \frac{K_2}{K_t^2} \right)^{-1}, \]

(60)

\[ C^{s_1, s_2, s_3}_{(b_6)} = A^{s_1, s_2, s_3}_{(b_6)} \cdot 2 \left( 7 - 29 \frac{K_2}{K_t^2} + 30 \frac{K_3}{K_t^3} \right), \]

(61)

\[ C^{s_1, s_2, s_3}_{(d_7)} = A^{s_1, s_2, s_3}_{(d_7)} \cdot 20 \left( 1 + 3 \frac{K_2}{K_t^2} + 51 \frac{K_3}{K_t^3} - 18 \frac{K_2^2}{K_t^4} - 126 \frac{K_2 K_3}{K_t^5} + 126 \frac{K_3^2}{K_t^6} \right) \left( 1 + 3 \frac{K_2}{K_t^2} + 15 \frac{K_3}{K_t^3} \right)^{-1}, \]

(62)

\[ C^{s_1, s_2, s_3}_{(d_3)} = \frac{1}{4 K_t} \left( 1 + \frac{K_2}{K_t^2} + 3 \frac{K_3}{K_t^3} \right) F^{s_1, s_2, s_3}_{(d_3)}, \]

(63)

where we defined

\[ F_{(GR)}^{+++} := \frac{1}{128} \cdot \frac{K_t^8}{K_t^3} \left( 1 - 4 \frac{K_2}{K_t^2} + 8 \frac{K_3}{K_t^3} \right), \]

(64)

\[ F_{(H)}^{+++} := \frac{1}{64} \cdot \frac{K_t^6}{K_t^3} \left( 1 - 4 \frac{K_2}{K_t^2} + 8 \frac{K_3}{K_t^3} \right), \]

(65)

\[ F_{(d_3)}^{+++} := \frac{1}{256} \cdot \frac{K_t^{10}}{K_t^3} \left( 1 - 10 \frac{K_2}{K_t^2} + 25 \frac{K_3}{K_t^3} + 24 \frac{K_2^2}{K_t^4} - 128 \frac{K_2 K_3}{K_t^5} + 126 \frac{K_3^2}{K_t^6} \right), \]

(66)

and when spins are mixed the corresponding functions can be derived from \( F^{+++}_{(GR,H,d_3)}(k_1, k_2, k_3) = F^{+++}_{(GR,H,d_3)}(k_1, k_2, -k_3) \). Note the relations

\[ F^{+++}_{(GR)} = \frac{K_t^2}{2} F^{+++}_{(H)}, \]

(67)

\[ F^{+++}_{(d_3)} = \frac{K_t^{10}}{4} \left( 1 - 6 \frac{K_2}{K_t^2} + 20 \frac{K_3}{K_t^3} \right) F^{+++}_{(H)}. \]

(68)

Since theories in Gao’s framework do not violate the parity symmetry, we have the properties \( F^{+++}_{(GR,H,d_3)} = F^{+++}_{(GR,H,d_3)} \) and \( F^{+++}_{(GR,H,d_3)} = F^{+++}_{(GR,H,d_3)} \). In the equilateral configuration, we have

\[ F^{+++}_{(H)} = F^{+++}_{(H)} = F^{+++}_{(H)} = \frac{1}{9} F^{+++}_{(H)}, \]

(69)

\[ F^{+++}_{(GR)} = F^{+++}_{(GR)} = F^{+++}_{(GR)} = \frac{1}{81} F^{+++}_{(GR)}, \]

(70)

\[ F^{+++}_{(d_3)} = F^{+++}_{(d_3)} = F^{+++}_{(d_3)} = \frac{13}{189} F^{+++}_{(d_3)}, \]

(71)

while in the squeezed limit \( k_3 \to 0 \) we find

\[ F^{+++}_{(GR,d_3)} \approx F^{+++}_{(GR,d_3)} \approx -\frac{k_3^2}{2}, \]

(72)

\[ F^{+++}_{(GR)} \approx F^{+++}_{(GR)} \approx -\frac{k_3^4}{32 k_1^2}, \]

(73)

\[ F^{+++}_{(d_3)} \approx F^{+++}_{(d_3)} \approx \frac{7 k_3^4}{32 k_1^2}. \]

(74)
FIG. 2: $A^{+++}_{(3,3)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 3: $A^{+++}_{(2,3)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 4: $A^{+++}_{(0,2)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$. 
FIG. 5: $A^{+++}_{(k_0)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 6: $A^{+++}_{(F)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 7: $A^{+++}_{(F)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$. 
FIG. 8: $A^{++}_{c_{3}}(1, k_{2}/k_{1}, k_{3}/k_{1}) (k_{2}/k_{1})^{-1} (k_{3}/k_{1})^{-1}$ as a function of $k_{2}/k_{1}$ and $k_{3}/k_{1}$. The plot is normalized to 1 for equilateral configurations $k_{2}/k_{1} = k_{3}/k_{1} = 1$.

FIG. 9: $A^{++}_{c_{7}}(1, k_{2}/k_{1}, k_{3}/k_{1}) (k_{2}/k_{1})^{-1} (k_{3}/k_{1})^{-1}$ as a function of $k_{2}/k_{1}$ and $k_{3}/k_{1}$. The plot is normalized to 1 for equilateral configurations $k_{2}/k_{1} = k_{3}/k_{1} = 1$.

FIG. 10: $A^{++}_{c_{b_{6}}}(1, k_{2}/k_{1}, k_{3}/k_{1}) (k_{2}/k_{1})^{-1} (k_{3}/k_{1})^{-1}$ as a function of $k_{2}/k_{1}$ and $k_{3}/k_{1}$. The plot is normalized to 1 for equilateral configurations $k_{2}/k_{1} = k_{3}/k_{1} = 1$. 
FIG. 11: $A^{++}_{(d_2)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 12: $C^{+++}_{(e_3)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 13: $C^{+++}_{(e_3)}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$. 


FIG. 14: $C_{(a_2)}^{+++}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 15: $C_{(b_6)}^{+++}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 16: $C_{(d_7)}^{+++}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$. 
FIG. 17: $C_{(d_3)}^{+++} (1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 18: $C_{(e_3)}^{+++} (1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 19: $C_{(e_3)}^{++-} (1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$. 
FIG. 20: $C_{(a^+_7)}^{++-}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 21: $C_{(a^+_6)}^{++-}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$.

FIG. 22: $C_{(d^+_7)}^{++-}(1, k_2/k_1, k_3/k_1)(k_2/k_1)^{-1}(k_3/k_1)^{-1}$ as a function of $k_2/k_1$ and $k_3/k_1$. The plot is normalized to 1 for equilateral configurations $k_2/k_1 = k_3/k_1 = 1$. 
The configuration dependences of all the “+++” and “++−” amplitudes and the corresponding $O(\epsilon^2)$ corrections are shown in Figs. 2–23. At leading order in $\epsilon$ expansion, we find that only the contribution from $\mathcal{F}_T$ term, i.e., $A_{(\mathcal{F}_T)}^{+++}$, peaks in the squeezed limit, while all the other $A_{(\mathcal{F}_T)}^{++-}$’s peak in the equilateral configuration. We emphasize again that $A_{(\mathcal{F}_T)}^{+++}$ is a fixed and universal feature even for single-field inflation models beyond Horndeski. In other words, it is impossible to suppress or enhance $A_{(\mathcal{F}_T)}^{+++}$ even in the present general framework beyond Horndeski as long as the effect of the modified dispersion relation is small. It would therefore be of great interest to explore this “consistency relation” with CMB B-mode observations such as LiteBIRD [49]. When spins are mixed, the momentum dependences of $A_{(\mathcal{F}_T)}^{++-}$ ($\bullet = c_3, a_7, b_0, d_2$) are similar to each other and their peaks are located between the equilateral and squeezed configurations. As for the corrections of $O(\epsilon^2)$, it can be seen that they have momentum dependences similar to their leading order counterparts, namely, the relative changes in the amplitudes have mild momentum dependences, as is also the case for the curvature perturbation [46]. However, $C_{(d_3)}^{+++}$ shows the peak at a squeezed shape, breaking the uniqueness of $A_{(\mathcal{F}_T)}^{+++}$’s momentum dependence at this order. Therefore, it would be interesting to investigate the case where the effect of the modification to the dispersion relation cannot be treated perturbatively.$^1$

**V. CONSTRAINTS FROM CMB OBSERVATIONS**

We can roughly estimate the constraints on the parameters of the theory by comparing the results obtained in the previous section with the Planck results [4, 6]. Schematically, one has

$$\frac{\langle hhh \rangle}{\langle hh \rangle \langle hh \rangle} \sim \sum_{k_1 k_2 k_3} \frac{A_{(\mathcal{F}_T)}^{s_1 s_2 s_3} (1, k_2/k_1, k_3/k_1)}{k_1 k_2 k_3} = \sum_{k_1 k_2 k_3} \frac{A_{(\mathcal{F}_T)}^{s_1 s_2 s_3} (1, k_2/k_1, k_3/k_1)}{(k_2/k_1) (k_3/k_1)}$$

(75)

where the quantities in the right hand side are presented in the previous section. It is reasonable to assume that

$$\langle hhh \rangle \lesssim \langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle$$

(76)

where $\mathcal{R}$ is the curvature perturbation. (Otherwise, the dominant source of non-Gaussianity in the CMB comes from the tensor modes, but if so $\langle hhh \rangle$ can be constrained directly from CMB observations.) Then, using the tensor-to-scalar ratio $r = \langle hh \rangle / \langle \mathcal{R} \mathcal{R} \rangle$ and the nonlinearity parameters $f_{NL}^\mathcal{R}$, we have

$$\frac{A_{(\mathcal{F}_T)}^{s_1 s_2 s_3}}{k_1 k_2 k_3} \lesssim \frac{\langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle}{\langle hh \rangle \langle hh \rangle} = \frac{1}{r^2} \frac{\langle \mathcal{R} \mathcal{R} \mathcal{R} \rangle}{\langle \mathcal{R} \mathcal{R} \rangle \langle \mathcal{R} \mathcal{R} \rangle} \sim \frac{f_{NL}^\mathcal{R}}{r^2}$$

(77)

where we have made the second assumption that different $A_{(\mathcal{F}_T)}^{s_1 s_2 s_3}$’s do not cancel each other. Thus, we obtain the constraint

$$\frac{A_{(\mathcal{F}_T)}^{s_1 s_2 s_3}}{k_1 k_2 k_3} \lesssim \frac{f_{NL}^\mathcal{R}}{r^2} \sim 10^3 \left( \frac{r}{0.1} \right)^{-2} \left( \frac{f_{NL}^\mathcal{R}}{10} \right)$$

(78)

$^1$ The 3-point function of the curvature perturbations has been evaluated when the $k^4$ term is dominant in the context of ghost inflation [50].
which is translated to
\[
\frac{c_3 H}{\mathcal{G}_T}, \frac{a_7 H^3}{\mathcal{G}_T c_h^4}, \frac{b_6 H^2}{\mathcal{G}_T c_h^2}, \frac{d_7 H^4}{\mathcal{G}_T c_h^6} \lesssim 10^3 \left( \frac{r}{0.1} \right)^{-2} \left( \frac{f_{\text{NL}}^R}{10} \right),
\] (79)

or, equivalently,
\[
\frac{c_3 c_h}{H}, \frac{a_7 H}{c_h^4}, \frac{b_6}{c_h^2}, \frac{d_7 H^2}{c_h^6} \lesssim 10^{1.3} \left( \frac{r}{0.1} \right)^{-3} \left( \frac{\mathcal{P}_R}{10^{-9}} \right)^{-1} \left( \frac{f_{\text{NL}}^R}{10} \right),
\] (80)

where \( \mathcal{P}_R \) is the power spectrum of the curvature perturbation. Note the mass dimensions of the coefficients: \([c_3] = 1, [b_6] = 0, [a_7] = -1, \) and \([d_7] = -2 \). Thus, the current constraints are very weak.

VI. DISCUSSIONS AND CONCLUSIONS

In this paper, we have clarified primordial non-Gaussianities of tensor modes by computing the tensor 3-point functions within the unifying framework of scalar-tensor theories \cite{32}. The framework includes the Horndeski theory and recent GLPV generalization as specific cases. We have shown that no new terms appear in the cubic Lagrangian even if one goes to the GLPV theory beyond Horndeski. In more general theories beyond GLPV, we have found four new interactions, one of which is related to the modification of the dispersion relation in the linear theory. The impact of this modification of the dispersion relation can be parametrized using a small parameter \( \epsilon \), and we have computed the 3-point functions analytically by treating \( \epsilon \) as a small expansion parameter. Two of the four new terms beyond GLPV have already been studied to leading order in \( \epsilon \) in the context of Ho\'rava gravity \cite{13}, and our results are in agreement with those in Ref. \[13\] where they overlap. In Ref. \[11\] it was found that there are only two independent terms in the cubic Lagrangian within Horndeski: the standard one present already in general relativity generating squeezed non-Gaussianity with the fixed amplitude and the other nonstandard one predicting equilateral non-Gaussianity. We have strengthened this statement by showing that at leading order in the \( \epsilon \) expansion the squeezed non-Gaussianity is only generated by this “standard” cubic term and has the fixed amplitude, \( A_{(\mathcal{F}_T)}^{++} \), even in the general unifying framework of scalar-tensor theories. At leading order in \( \epsilon \) all the other non-Gaussian amplitudes peak at equilateral shapes. The “standard” interaction is quite likely to be present in the cubic Lagrangian because it can easily be generated from the term linear in the Ricci scalar. The fixed and universal nature of \( A_{(\mathcal{F}_T)}^{++} \) thus provides us the “consistency relation” in the primordial tensor sector. Any detection of the equilateral tensor non-Gaussianity would imply the nonstandard interactions between gravity and the scalar degree of freedom, though it is difficult to distinguish among different contributions. Note that the inverse is not true; even in the absence of equilateral non-Gaussianity, gravity could be modified significantly from general relativity, because one can consider various nonstandard interactions that do not affect the tensor sector, as well as \( R_{ij} \delta K_{ij} \).

We have found that the effects of the modified dispersion relation appear in the non-Gaussian amplitudes at \( O(\epsilon^2) \). The momentum dependences of the correction \( C_{(\mathcal{F}_T)}^{s_1 s_2 s_3} \) are similar to their leading order counterparts \( A_{(\mathcal{F}_T)}^{s_1 s_2 s_3} \). It should be noted that the correction to the squeezed non-Gaussianity, \( C_{(\mathcal{F}_T)}^{s_1 s_2 s_3} \), breaks its fixed and universal nature. Therefore, detection of squeezed tensor non-Gaussianity whose amplitude is different from \( A_{(\mathcal{F}_T)}^{++} \) would imply a significant higher order term in the dispersion relation or the tensor modes of non-inflationary origin.

Let us mention here some other sources of gravitational waves from the early universe, as it would be interesting to explore tensor non-Gaussian signatures of such origin in order to contrast them with the results in this paper. A spectator scalar field other than inflaton can produce gravitational waves \cite{51}, though it is difficult to expect large amplitudes \cite{52, 53}. Large tensor modes are produced, \( e.g. \), by self-ordering of multi-component scalar fields after a global phase transition \cite{54, 55, 56, 57, 58}. Vector fields can also source gravitational waves during inflation \cite{59, 60}. We hope that we will come back to the issue of non-Gaussian signatures of those gravitational waves in future publications.

Finally, it should be noted that by construction the present analysis does not cover multi-field inflation models. Though the multi-field effects in the tensor sector are expected to be small, it would be interesting to investigate whether one can discriminate single- and multi-field models using primordial non-Gaussianity of tensor modes.

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Appendix A: Disformal transformation

Let us investigate the transformation properties of the quadratic and cubic actions under the disformal transformation,

\[ \hat{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\partial_\mu\phi\partial_\nu\phi. \]  

(A1)

It is more convenient to write this in the unitary gauge as

\[ \hat{g}_{\mu\nu} = e^{2\alpha} \left[ g_{\mu\nu} + (1 - e^{2\beta}) n_\mu n_\nu \right], \]  

(A2)

where \( n_\mu \) is the unit normal to the constant time hypersurfaces, \( n_\mu = (-N, 0) \). Since we are interested in the tensor perturbations on a cosmological background, we may only consider the case where both \( \alpha \) and \( \beta \) are functions of time and do not fluctuate. Under the above disformal transformation, the ADM variables transform as

\[ \hat{\gamma}_{ij} = e^{2\alpha} \gamma_{ij}, \quad \hat{N} = e^{\alpha + \beta} N, \quad \hat{N}^i = N^i, \]  

(A3)

yielding

\[ \hat{K}^i_j = e^{-\alpha - \beta} \left( K^i_j + \frac{\alpha}{N} \delta^i_j \right), \]  

(A4)

\[ \hat{R}^i_j = e^{-2\alpha} R^i_j. \]  

(A5)

Substituting these to the Lagrangian (10), we find that the disformal transformation maintains the form of the Lagrangian for the tensor perturbations:

\[ \frac{\mathcal{L}}{\sqrt{-g}} \rightarrow \frac{\mathcal{L}}{\sqrt{-\hat{g}}} = \hat{d}_1 \hat{R} + \hat{d}_3 \hat{R}^i_j \hat{R}^j_i + \hat{d}_7 \hat{R}^i_j \hat{R}^j_k \hat{R}^k_i + \hat{b}_2 \delta \hat{K}^i_j \delta \hat{K}_j^i + \hat{c}_3 \delta \hat{K}^i_j \delta \hat{K}_j^i \delta \hat{K}_k^i + \hat{c}_2 \delta \hat{K}^i_j \delta \hat{K}_j^i \delta \hat{K}_k^i, \]  

(A6)

where

\[ \hat{d}_1 = e^{-2\alpha - \beta} \hat{d}_1, \quad \hat{d}_3 = e^{-\beta} \hat{d}_3, \quad \hat{d}_7 = e^{2\alpha - \beta} \hat{d}_7, \quad \hat{b}_2 = e^{-2\alpha + \beta} \hat{b}_2, \]

\[ \hat{c}_3 = e^{-\alpha + 2\beta} \hat{c}_3, \quad \hat{c}_2 = e^{-\alpha} \hat{c}_2, \quad \hat{c}_7 = e^\alpha \hat{c}_7, \quad \hat{b}_6 = e^\beta \hat{b}_6, \]  

(A7)

and we dropped the terms that are irrelevant to the tensor perturbations. This, in particular, implies that

\[ \mathcal{G}_T \rightarrow \hat{\mathcal{G}}_T = e^{-2\alpha + \beta} \mathcal{G}_T, \quad \mathcal{F}_T \rightarrow \hat{\mathcal{F}}_T = e^{-2\alpha - \beta} \mathcal{F}_T. \]  

(A8)

Using the two time-dependent functions \( \alpha \) and \( \beta \), one can fit \( \mathcal{G}_T \) and \( \mathcal{F}_T \) to the standard form, i.e., \( \hat{\mathcal{G}}_T = \hat{\mathcal{F}}_T = M^2_F \) \[61\]. However, the higher order term in the dispersion relation cannot be removed \[36\]. It is also clear that one cannot make further simplifications in the cubic action by the use of the disformal transformation.

[1] A. H. Guth, “The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems,” Phys. Rev. D 23, 347 (1981). doi:10.1103/PhysRevD.23.347
[2] K. Sato, “First Order Phase Transition of a Vacuum and Expansion of the Universe,” Mon. Not. Roy. Astron. Soc. 195, 467 (1981).
[3] A. A. Starobinsky, “A New Type of Isotropically Expanding Models Without Singularity,” Phys. Lett. B 91, 99 (1980). doi:10.1016/0370-2693(80)90670-X
[4] P. A. R. Ade et al. [Planck Collaboration], “Planck 2015 results. XX. Constraints on inflation,” arXiv:1502.02114 [astro-ph.CO].
[5] X. Chen, M. x. Huang, S. Kachru and G. Shiu, “Observational signatures and non-Gaussianities of general single field inflation,” JCAP 0701, 002 (2007) [hep-th/0605045].
[6] P. A. R. Ade et al. [Planck Collaboration], “Planck 2015 results. XVII. Constraints on primordial non-Gaussianity,” arXiv:1502.01592 [astro-ph.CO].
[7] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 0305, 013 (2003) [astro-ph/0210603].
[39] C. Deffayet, G. Esposito-Farese and D. A. Steer, “Counting the degrees of freedom of generalized Galileons,” Phys. Rev. D 92, 084013 (2015) [arXiv:1506.01974 [gr-qc]].

[40] D. Langlois and K. Noui, “Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability,” arXiv:1510.06930 [gr-qc].

[41] M. Zamalacárrregui and J. García-Bellido, “Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian,” Phys. Rev. D 89, 064046 (2014) doi:10.1103/PhysRevD.89.064046 [arXiv:1308.4695 [gr-qc]].

[42] S. Ohashi, N. Tanahashi, T. Kobayashi and M. Yamaguchi, “The most general second-order field equations of bi-scalar-tensor theory in four dimensions,” JHEP 1507, 008 (2015) doi:10.1007/JHEP07(2015)008 [arXiv:1505.06029 [gr-qc]].

[43] D. Langlois and K. Noui, “Degenerate higher derivative theories beyond Horndeski: evading the Ostrogradski instability,” arXiv:1510.06930 [gr-qc].

[44] M. Zumalacárregui and J. García-Bellido, “Transforming gravity: from derivative couplings to matter to second-order scalar-tensor theories beyond the Horndeski Lagrangian,” Phys. Rev. D 90, no. 10, 104046 (2014) doi:10.1103/PhysRevD.90.104046 [arXiv:1308.4695 [gr-qc]].

[45] X. Gao, “Hamiltonian analysis of spatially covariant gravity,” Phys. Rev. D 90, no. 10, 104033 (2014) [arXiv:1409.6708 [gr-qc]].

[46] P. Hořava, “Quantum Gravity at a Lifshitz Point,” Phys. Rev. D 79, 084008 (2009) [arXiv:0901.3775 [hep-th]].

[47] D. Blas, O. Pujolas and S. Sibiryakov, “Consistent Extension of Hořava Gravity,” Phys. Rev. Lett. 104, 181302 (2010) [arXiv:0909.3525 [hep-th]].

[48] A. Ashoorioon, D. Chialva and U. Danielsson, “Effects of Nonlinear Dispersion Relations on Non-Gaussianities,” JCAP 1106, 034 (2011) [arXiv:1104.2338 [hep-th]].

[49] J. Martin and R. H. Brandenberger, “The Corley-Jacobson dispersion relation and transPlanckian inflation,” Phys. Rev. D 65, 103514 (2002) [hep-th/0201189].

[50] M. Biagetti, M. Fasiello and A. Riotto, “Enhancing Inflationary Tensor Modes through Spectator Fields,” Phys. Rev. D 88, 103518 (2013) [arXiv:1305.7241 [astro-ph.CO]].

[51] K. Jones-Smith, L. M. Krauss and H. Mathur, “A Nearly Scale Invariant Spectrum of Gravitational Radiation from Global Phase Transitions,” Phys. Rev. Lett. 100, 131302 (2008) doi:10.1103/PhysRevLett.100.131302 [arXiv:0712.0778 [astro-ph]].

[52] E. Fenu, D. G. Figueroa, R. Durrer and J. García-Bellido, “Gravitational waves from self-ordering scalar fields,” JCAP 1504, 011 (2015) doi:10.1088/1475-7516/2015/04/011 [arXiv:1411.3029 [astro-ph.CO]].

[53] T. Fujita, J. Yokoyama and S. Yokoyama, “Can a spectator scalar field enhance inflationary tensor mode?,” PTEP 2015, 043E01 (2015) [arXiv:1411.3658 [astro-ph.CO]].

[54] K. Jones-Smith, L. M. Krauss and H. Mathur, “A Nearly Scale Invariant Spectrum of Gravitational Radiation from Global Phase Transitions,” Phys. Rev. Lett. 100, 131302 (2008) doi:10.1103/PhysRevLett.100.131302 [arXiv:0712.0778 [astro-ph]].

[55] E. Fenu, D. G. Figueroa, R. Durrer and J. García-Bellido, “Gravitational waves from self-ordering scalar fields,” JCAP 0910, 005 (2009) doi:10.1088/1475-7516/2009/10/005 [arXiv:0908.0425 [astro-ph.CO]].

[56] J. T. Giblin, Jr., L. R. Price, X. Siemens and B. Vlcek, “Gravitational Waves from Global Second Order Phase Transitions,” JCAP 1211, 006 (2012) doi:10.1088/1475-7516/2012/11/006 [arXiv:1111.4014 [astro-ph.CO]].

[57] D. G. Figueroa, M. Hindmarsh and J. Urrestilla, “Exact Scale-Invariant Background of Gravitational Waves from Cosmic Defects,” Phys. Rev. Lett. 110, no. 10, 101302 (2013) doi:10.1103/PhysRevLett.110.101302 [arXiv:1212.5458 [astro-ph.CO]].

[58] S. Kuroyanagi, T. Hiramatsu and J. Yokoyama, “Reheating signature in the gravitational wave spectrum from self-ordering scalar fields,” arXiv:1509.08264 [astro-ph.CO].

[59] S. Mukohyama, R. Namba, M. Peloso and G. Shiu, “Blue Tensor Spectrum from Particle Production during Inflation,” JCAP 1408, 036 (2014) [arXiv:1405.0346 [astro-ph.CO]].

[60] N. Barnaby and M. Peloso, “Large Nongaussianity in Axion Inflation,” Phys. Rev. Lett. 106, 181301 (2011) [arXiv:1011.1500 [hep-ph]].

[61] P. Creminelli, J. Gleyzes, J. Noreia and F. Vernizzi, “Resilience of the standard predictions for primordial tensor modes,” Phys. Rev. Lett. 113, no. 23, 231301 (2014) doi:10.1103/PhysRevLett.113.231301 [arXiv:1407.8439 [astro-ph.CO]].