On Non-Abelian Extensions of 3-Lie Algebras

Li-Na Song (宋丽娜),1 Abdenacer Makhlouf,2,† and Rong Tang (唐荣)1

1Department of Mathematics, Jilin University, Changchun 130012, China
2University of Haute Alsace, Laboratoire de Mathématiques, Informatique et Applications, Mulhouse, France

(Received August 14, 2017; revised manuscript received January 15, 2018)

Abstract In this paper, we study non-abelian extensions of 3-Lie algebras through Maurer-Cartan elements. We show that there is a one-to-one correspondence between isomorphism classes of non-abelian extensions of 3-Lie algebras and equivalence classes of Maurer-Cartan elements in a DGLA. The structure of the Leibniz algebra on the space of fundamental objects is also analyzed.

PACS numbers: 02.30.Ik, 02.10.-v, 02.10.Xm, 11.25.Yb, 45.20.Jj DOI: 10.1088/0253-6102/69/4/347

Key words: 3-Lie algebras, Leibniz algebra, non-abelian extension, Maurer-Cartan element

1 Introduction

Ternary Lie algebras (3-Lie algebras) or more generally $n$-ary Lie algebras are a natural generalization of Lie algebras. They were introduced and studied first by Filippov in Ref. [1]. This type of algebras appeared also in the algebraic formulation of Nambu Mechanics,[2] generalizing Hamiltonian mechanics by considering two hamiltonians,[3] and by Bagger and Lambert in a construction of an $N = 2$ supersymmetric version of the worldvolume theory.[4] For more applications, see also Refs. [5–11].

Several algebraic aspects of $n$-Lie algebras were studied in the last years. The concept of Lie algebra representation was extended naturally to 3-Lie algebras first in Ref. [12]. In this framework, adjoint representations were extended naturally to 3-Lie algebras first considered by Basu and Harvey for lifted Nahm equations,[5] and by Bagger and Lambert in a construction of an $N = 2$ supersymmetric version of the worldvolume theory.[4] For more applications, see also Refs. [7–11].

In Sec. 2, we provide a summary of non-abelian extensions of Leibniz algebras and cohomologies of 3-Lie algebras. A characterization of non-abelian extensions of a 3-Lie algebra by another 3-Lie algebra is given in Sec. 3 and several examples provided. In Sec. 4, we show that there is a one-to-one correspondence between isomorphism classes of non-abelian extensions of 3-Lie algebras and equivalence classes of Maurer-Cartan elements. Finally, we analyze in Sec. 5 the corresponding Lie algebra structure on the space of fundamental objects and show that it is a non-abelian extension of Leibniz algebras.

2 Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and all the vector spaces in this paper considered over $\mathbb{K}$.

2.1 Non-Abelian Extensions of Liebniz Algebras

A Leibniz algebra is a vector space $\mathfrak{t}$ endowed with a linear map $[\cdot, \cdot]_{\mathfrak{t}} : \mathfrak{t} \otimes \mathfrak{t} \to \mathfrak{t}$ satisfying

$$[x, [y, z]_{\mathfrak{t}}]_{\mathfrak{t}} = [[x, y]_{\mathfrak{t}}, z]_{\mathfrak{t}} + [y, [x, z]_{\mathfrak{t}}]_{\mathfrak{t}}, \quad \forall x, y, z \in \mathfrak{t}. \quad (1)$$

This is in fact a left Leibniz algebra. In this paper, we only consider left Leibniz algebras which we call Leibniz algebras.

Let $(\mathfrak{t}, [\cdot, \cdot]_{\mathfrak{t}})$ be a Leibniz algebra. We denote by $\text{Der}^{L}(\mathfrak{t})$ and $\text{Der}^{R}(\mathfrak{t})$ the set of left derivations and the works in Refs. [22–23], we find a suitable approach which uses Maurer-Cartan elements to study non-abelian extensions of 3-Lie algebras. We also show that the Leibniz algebra on the space of fundamental objects is a non-abelian extension of Leibniz algebras.

The paper is organized as follows. In Sec. 2, we provide a summary of non-abelian extensions of Leibniz algebras and cohomologies of 3-Lie algebras. A characterization of non-abelian extensions of a 3-Lie algebra by another 3-Lie algebra is given in Sec. 3 and several examples provided. In Sec. 4, we show that there is a one-to-one correspondence between isomorphism classes of non-abelian extensions of 3-Lie algebras and equivalence classes of Maurer-Cartan elements. Finally, we analyze in Sec. 5 the corresponding Lie algebra structure on the space of fundamental objects and show that it is a non-abelian extension of Leibniz algebras.

In Ref. [22], the authors introduced the notion of a generalized representation of a 3-Lie algebra, by which abelian extensions of 3-Lie algebras are studied. Due to its difficulty and less of tools, non-abelian extensions of 3-Lie algebras are not studied. In this paper, motivated by

*Supported by National Natural Science Foundation of China under Grant No. 11471139 and National Natural Science Foundation of Jilin Province under Grant No. 20170101050JC
†Corresponding author, E-mail: abdenacer.makhlouf@uha.fr
© 2018 Chinese Physical Society and IOP Publishing Ltd

http://www.iopscience.iop.org/ctp http://ctp.itp.ac.cn
set of right derivations of $g$ respectively:

$$\text{Der}^L(t) = \{D \in \mathfrak{gl}(t) | D[x, y]_t = \text{ad}^L_x(y) = [x, D]_t, \forall x, y \in t\},$$

$$\text{Der}^R(t) = \{D \in \mathfrak{gl}(t) | D[x, y]_t = \text{ad}^R_x(y) = [y, D]_t, \forall x, y \in t\}.$$ 

Note that the right derivations are called anti-derivations in Refs. [24–25]. It is easy to see that for all $x \in t$, $\text{ad}^L_x : t \rightarrow t$, which is given by $\text{ad}^L_x(y) = [x, y]_t$, is a left derivation; $\text{ad}^R_x : t \rightarrow t$, which is given by $\text{ad}^R_x(y) = [y, x]_t$, is a right derivation.

**Definition 1**

(i) Let $t, s, \hat{t}$ be Leibniz algebras. A non-abelian extension of Leibniz algebras is a short exact sequence of Leibniz algebras:

$$0 \rightarrow s \rightarrow \hat{t} \rightarrow t \rightarrow 0.$$ 

We say that $\hat{t}$ is a non-abelian extension of $t$ by $s$.

(ii) A linear section of $\hat{t}$ is a linear map $\sigma : t \rightarrow \hat{t}$ such that $p \circ \sigma = \text{id}$.

Let $\hat{t}$ be a non-abelian extension of $t$ by $s$, and $\sigma : t \rightarrow \hat{t}$ a linear section. Define $\omega : t \otimes t \rightarrow s, l : t \rightarrow \mathfrak{gl}(s)$ and $r : t \rightarrow \mathfrak{gl}(s)$ respectively by

$$\omega(x, y) = [\sigma(x), \sigma(y)]_t - \sigma(x, y)_t, \quad \forall x, y \in t,$$

$$l_x(\beta) = [\sigma(x), \beta]_t, \quad \forall x \in t, \quad \beta \in s,$$

$$r_y(\alpha) = [\alpha, \sigma(y)]_t, \quad \forall y \in t, \quad \alpha \in s.$$ 

Given a linear section, we have $\hat{t} \cong t \oplus s$ as vector spaces, and the Leibniz algebra structure on $\hat{t}$ can be transferred to $t \oplus s$:

$$[x + \alpha, y + \beta]_{(l, r, \omega)} = [x, y]_t + \omega(x, y) + l_x(\beta) + r_y(\alpha) + [\alpha, \beta]_t.$$ 

**Proposition 1** With the above notations, $(t \oplus s, [\cdot, \cdot]_{(l, r, \omega)})$ is a Leibniz algebra if and only if $l, r, \omega$ satisfy the following equalities:

$$l_x(\alpha) - [\alpha, \sigma(x)]_t = 0, \quad r_x(\beta) = [\sigma(x), \beta]_t - \beta_\sigma(x)x_\sigma, \quad (6)$$

$$l_x(\alpha) + r_x(\beta) = 0, \quad (7)$$

$$[l_x, l_y] - l_{[x, y]} = \text{ad}^L_{\omega(x, y)}, \quad (9)$$

$$[l_x, r_y] - r_{[x, y]} = \text{ad}^R_{\omega(x, y)}, \quad (10)$$

$$r_y(\alpha) + l_x(\alpha) = 0, \quad (11)$$

$$l_x\omega(x, z) - l_y\omega(x, z) - r_x\omega(x, y) = \omega([x, y]_t, z) + \omega(x, [y, z]_t). \quad (12)$$

Equation (6) means that $l_x \in \text{Der}^L(s)$ and Eq. (7) means that $r_x \in \text{Der}^R(s)$. See Ref. [26] for more details about non-abelian extensions of Leibniz algebras.

**2.2 3-Lie Algebras and Their Representations**

We recall in this section definitions, representations and cohomology of 3-Lie algebras.

**Definition 2** A 3-Lie algebra is given by a vector space $g$ together with a skew-symmetric linear map $[\cdot, \cdot, \cdot]_g : \wedge^3 g \rightarrow g$ such that the following identity holds:

$$F_{x_1, x_2, x_3, x_4, x_5} \triangleq [x_1, x_2, [x_3, x_4, x_5]_g]_g - [x_1, x_2, x_3]_g - [x_4, x_5]_g - [x_1, x_2, x_4]_g + [x_3, x_5]_g - [x_1, x_2, x_5]_g = 0. \quad (13)$$

The identity (13) is called Fundamental Identity (FI) or sometimes Nambu identity.

**Definition 3** A morphism of 3-Lie algebras $f : (g, [\cdot, \cdot, \cdot]_g) \rightarrow (h, [\cdot, \cdot, \cdot]_h)$ is a linear map $f : g \rightarrow h$ such that

$$f[x, y, z]_g = [f(x), f(y), f(z)]_h.$$ 

We call elements in $\wedge^2 g$ fundamental objects of the 3-Lie algebra $(g, [\cdot, \cdot, \cdot]_g)$.

One defines a bilinear operation $[\cdot, \cdot]_g : \wedge^2 g \rightarrow \wedge^2 g$, given by

$$[X, Y]_g = [x_1, x_2, y_1]_g \wedge y_2 + y_1 \wedge [x_1, x_2, y_2]_g, \quad \forall X = x_1 \wedge x_2, \quad Y = y_1 \wedge y_2.$$ 

It turns out that $(\wedge^2 g, [\cdot, \cdot]_g)$ is a Leibniz algebra, and it plays an important role in the theory of 3-Lie algebras.

The concept of representation introduced by Kasymov is defined as follows.

**Definition 4** (Ref. [12]) A representation $\rho$ of a 3-Lie algebra $g$ on a vector space $V$ is given by a linear map $\rho : \wedge^3 g \rightarrow \mathfrak{gl}(V)$, such that for all $x_1, x_2, x_3, x_4 \in g$ there holds:

$$\rho([x_1, x_2, x_3]_g, x_4) = \rho(x_1, [x_2, x_3, x_4]_g) \quad \rho([x_1, x_2, x_4]_g, x_3) = \rho(x_1, [x_2, x_3, x_4]_g) \quad \rho([x_1, x_3, x_4]_g, x_2) = \rho(x_1, [x_2, x_3, x_4]_g).$$

It leads to the following semidirect product for 3-Lie algebras.

**Lemma 1** Let $g$ be a 3-Lie algebra, $V$ a vector space and $\rho : \wedge^3 g \rightarrow \mathfrak{gl}(V)$ a skew-symmetric linear map. Then $(V; \rho)$ is a representation of $g$ if and only if there is a 3-Lie algebra structure on the direct sum of vector spaces $g \oplus V$, defined by

$$[x_1, v_1, x_2, v_2, x_3, v_3]_g = [x_1, x_2, x_3]_g + \rho(x_1, v_1) + \rho(x_2, x_3) + \rho(x_3, x_1),$$

for $x_i \in g, v_i \in V, 1 \leq i \leq 3$. The previous 3-Lie algebra structure is called semidirect product and denoted by $g \rtimes V$.

Furthermore, there is a cohomology of 3-Lie algebras with coefficients in a representation $(V; \rho)$. First $p$-cochains on $g$ are defined to be linear maps

$$\alpha : \wedge^p g \otimes \cdots \otimes \wedge^2 g \otimes g \rightarrow V, \quad p = 0, 1, \ldots$$

Denote the space of $p$-cochains by $C^p(g, V)$. Then the coboundary operator $\delta : C^{p-1}(g, V) \rightarrow C^p(g, V)$ is defined as
here $X_i = x_i \land y_i$ for $i = 1, \ldots, p$ and $z \in \mathfrak{g}$.

## 3 Non-Abelian Extensions of 3-Lie Algebras

We discuss in this section non-abelian extensions of 3-Lie algebras. First we recall some basics.

**Definition 5** A non-abelian extension of a 3-Lie algebra $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ by a 3-Lie algebra $(\mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{h}})$ is a short exact sequence of 3-Lie algebra morphisms: $0 \rightarrow \mathfrak{h} \xrightarrow{s} \mathfrak{g} \xrightarrow{p} \mathfrak{g} \rightarrow 0$, where $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ is a 3-Lie algebra.

A section of a non-abelian extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{h}$ is a linear map $s : \mathfrak{h} \rightarrow \tilde{\mathfrak{g}}$ such that $p \circ s = \text{id}$.

**Definition 6** Two extensions of $\mathfrak{g}$ by $\mathfrak{h}$, $(\mathfrak{g}, [\cdot, \cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{g}_2, [\cdot, \cdot, \cdot]_{\mathfrak{g}_2})$, are said to be isomorphic if there exists a 3-Lie algebra morphism $\theta : \mathfrak{g}_2 \rightarrow \mathfrak{g}_1$ such that we have the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow \mathfrak{h} \xrightarrow{s_2} \mathfrak{g}_2 \xrightarrow{p_2} \mathfrak{g} \rightarrow 0 \\
\| \\
0 \rightarrow \mathfrak{h} \xrightarrow{s_1} \mathfrak{g}_1 \xrightarrow{p_1} \mathfrak{g} \rightarrow 0.
\end{array}
\]

Given a section $s$ of a non-abelian extension $\tilde{\mathfrak{g}}$ of $\mathfrak{g}$ by $\mathfrak{h}$, we can define $\rho : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}(\mathfrak{h})$, $\nu : \mathfrak{g} \rightarrow \text{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{h})$ and $\omega : \wedge^3 \mathfrak{g} \rightarrow \mathfrak{h}$ respectively by

\[
\rho(x, y)(u) = [s(x), s(y), u]_{\tilde{\mathfrak{g}}}, \\
\nu(x)(u, v) = [s(x), u, v]_{\mathfrak{g}}, \\
\omega(x, y, z) = [s(x), s(y), s(z)]_{\mathfrak{g}} - s[x, y, z]_{\mathfrak{g}}.
\]

Obviously, $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{h}$ as vector spaces. Transfer the 3-Lie algebra structure on $\tilde{\mathfrak{g}}$ to that on $\mathfrak{g} \oplus \mathfrak{h}$, we obtain a 3-Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{h}})$, where $[\cdot, \cdot]_{(\mathfrak{p}, \mathfrak{s}, \mathfrak{w})}$ is given by

\[
[x_1 + v_1, x_2 + v_2, x_3 + v_3]_{\mathfrak{g} \oplus \mathfrak{h}} = [x_1, x_2, x_3]_{\mathfrak{g}} + \\
\omega(x_1, x_2, x_3) + \rho(x_1, x_2, x_3) + \nu(x_1)(v_1)(x_2, x_3) + \nu(x_2)(v_2)(x_1, v_3) + \\
\nu(x_3)(v_3)(v_1, v_2) + [v_1, v_2, v_3]_{\mathfrak{h}}.
\]

The following proposition provides the conditions on $\rho, \nu$ and $\omega$ such that $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{h}})$ is a 3-Lie algebra.

**Proposition 2** The pair $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot, \cdot]_{\mathfrak{g} \oplus \mathfrak{h}})$, defined above, is a 3-Lie algebra if and only if $\rho, \nu$ and $\omega$ satisfy, for all $x_1, \ldots, x_5 \in \mathfrak{g}$, $v_1, \ldots, v_5 \in \mathfrak{h}$, the following conditions
Proof The proof is obtained by straightforward computations of the fundamental identity for different combinations of elements and the converse is direct. □

Example 1 We consider \( g \) to be the simple 4-dimensional 3-Lie algebra defined with respect to a basis \( \{x_1, x_2, x_3, x_4\} \) by the skew-symmetric brackets

\[
[x_1, x_2, x_3] = x_4, \quad [x_1, x_2, x_4] = x_3, \quad [x_1, x_3, x_4] = x_2, \quad [x_2, x_3, x_4] = x_1,
\]

and \( h \) to be the 3-dimensional 3-Lie algebra defined with respect to basis \( \{v_1, v_2, v_3\} \) by

\[
[v_1, v_2, v_3] = v_1.
\]

Then every non-abelian extension of \( g \) by \( h \) is given by \( \rho = 0 \). The following families of \( \nu \) and \( \omega \) provide non-abelian extensions of \( g \) by \( h \)

- \( \nu(x_1)(v_1, v_2) = r_1 v_1, \nu(x_1)(v_1, v_3) = r_2 v_1, \nu(x_1)(v_2, v_3) = r_3 v_1, \nu(x_2)(v_1, v_2) = (r_1 r_4/r_2) v_1, \nu(x_2)(v_1, v_3) = (r_3 r_4/r_2) v_1, \nu(x_2)(v_2, v_3) = (r_3 r_6/r_2) v_1, \nu(x_3)(v_1, v_2) = (r_1 r_5/r_2) v_1, \nu(x_3)(v_1, v_3) = (r_3 r_5/r_2) v_1, \nu(x_3)(v_2, v_3) = (r_3 r_6/r_2) v_1,

- \( \omega(x_1, x_2, x_3) = -(r_3 r_6/r_2) v_1 + r_5 v_2 - (r_3 r_6/r_2) v_3, \omega(x_1, x_2, x_4) = -(r_3 r_5/r_2) v_1 + r_5 v_2 - (r_1 r_5/r_2) v_3, \omega(x_1, x_3, x_4) = -(r_3 r_4/r_2) v_1 + r_4 v_2 - (r_1 r_4/r_2) v_3, \omega(x_2, x_3, x_4) = -r_3 v_1 + r_2 v_2 - r_1 v_3,

where \( r_i \) are parameters in \( K \).

Example 2 We consider \( g \) to be the 3-dimensional 3-Lie algebra defined with respect to a basis \( \{x_1, x_2, x_3\} \) by the skew-symmetric bracket \( [x_1, x_2, x_3] = x_1 \) and \( h \) to be the same 3-Lie algebra which we write with respect to basis \( \{v_1, v_2, v_3\} \), that is \( [v_1, v_2, v_3] = v_1 \). Then every non-abelian extension of \( g \) by \( h \) is given by one of the following triples \((\rho, \nu, \omega)\).

(i) \( \begin{align*}
\rho(x_1, x_2)(v_1) &= 0, \rho(x_1, x_2)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \rho(x_1, x_3)(v_1) = 0, \rho(x_1, x_3)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \\
\nu(x_1)(v_1, v_2) &= r_1 v_1, \nu(x_1)(v_1, v_3) = r_2 v_1, \nu(x_1)(v_2, v_3) = r_3 v_1, \nu(x_2)(v_1, v_2) = (r_1 r_4/r_2) v_1, \nu(x_2)(v_1, v_3) = (r_3 r_4/r_2) v_1, \nu(x_2)(v_2, v_3) = (r_3 r_6/r_2) v_1, \\
\omega(x_1, x_2, x_3) &= 0,
\end{align*} \)

(ii) \( \begin{align*}
\rho(x_1, x_2)(v_1) &= 0, \rho(x_1, x_2)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \rho(x_1, x_3)(v_1) = 0, \rho(x_1, x_3)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \\
\rho(x_2, x_3)(v_1) &= -r_3 r_4 v_1, \rho(x_2, x_3)(v_2) = r_1 v_1, \rho(x_2, x_3)(v_3) = r_2 v_1, \\
\nu(x_1)(v_1, v_2) &= 0, \nu(x_1)(v_1, v_3) = 0, \nu(x_1)(v_2, v_3) = 0, \nu(x_2)(v_1, v_2) = r_1 v_1, \nu(x_2)(v_1, v_3) = r_3 v_1, \nu(x_2)(v_2, v_3) = (r_1 r_4/r_2) v_1, \\
\nu(x_3)(v_1, v_2) &= r_4 v_1, \nu(x_3)(v_1, v_3) = r_5 v_1, \nu(x_3)(v_2, v_3) = (r_2 r_4/r_3 r_5/r_4) v_1, \\
\omega(x_1, x_2, x_3) &= 0,
\end{align*} \)

(iii) \( \begin{align*}
\rho(x_1, x_2)(v_1) &= 0, \rho(x_1, x_2)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \rho(x_1, x_3)(v_1) = 0, \rho(x_1, x_3)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \\
\rho(x_2, x_3)(v_1) &= 0, \rho(x_2, x_3)(v_2) = r_1 v_1, \rho(x_2, x_3)(v_3) = (r_3 r_1/r_2) v_1, \\
\nu(x_1)(v_1, v_2) &= 0, \nu(x_1)(v_1, v_3) = 0, \nu(x_1)(v_2, v_3) = 0, \nu(x_2)(v_1, v_2) = r_2 v_1, \nu(x_2)(v_1, v_3) = r_3 v_1, \\
\nu(x_2)(v_2, v_3) &= r_4 v_1, \nu(x_3)(v_1, v_2) = r_5 v_1, \nu(x_3)(v_1, v_3) = (r_3 r_5/r_2) v_1, \nu(x_3)(v_2, v_3) = (r_4 r_5 + r_1/r_2) v_1, \\
\omega(x_1, x_2, x_3) &= 0,
\end{align*} \)

(iv) \( \begin{align*}
\rho(x_1, x_2)(v_1) &= 0, \rho(x_1, x_2)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \rho(x_1, x_3)(v_1) = 0, \rho(x_1, x_3)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \\
\nu(x_1)(v_1, v_2) &= r_1 v_1, \nu(x_1)(v_1, v_3) = r_2 v_1, \nu(x_1)(v_2, v_3) = r_3 v_1, \nu(x_2)(v_1, v_2) = (r_2 r_4/r_1) v_1, \nu(x_2)(v_1, v_3) = (r_3 r_4/r_1) v_1, \\
\nu(x_2)(v_2, v_3) &= (r_2 r_4/r_1) v_1, \nu(x_2)(v_2, v_3) = (r_3 r_4/r_1) v_1, \nu(x_3)(v_1, v_2) = r_5 v_1, \nu(x_3)(v_1, v_3) = (r_2 r_5/r_1) v_1, \\
\nu(x_3)(v_2, v_3) &= (r_3 r_5/r_1) v_1, \\
\omega(x_1, x_2, x_3) &= -r_3 v_1 + r_2 v_2 - r_1 v_3,
\end{align*} \)

(v) \( \begin{align*}
\rho(x_1, x_2)(v_1) &= 0, \rho(x_1, x_2)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \rho(x_1, x_3)(v_1) = 0, \rho(x_1, x_3)(v_2) = 0, \rho(x_1, x_3)(v_3) = 0, \\
\rho(x_2, x_3)(v_1) &= 0, \rho(x_2, x_3)(v_2) = r_1 v_1, \rho(x_2, x_3)(v_3) = (r_3 r_1/r_2) v_1,
\end{align*} \)
Any non-abelian extension, by choosing a section, is isomorphic to \((g \oplus h, \cdot, \cdot, [\cdot, \cdot, \cdot]_{\rho, \nu, \omega})\). Therefore, we only consider in the sequel non-abelian extensions of the form \((g \oplus h, [\cdot, \cdot, \cdot]_{\rho, \nu, \omega})\).

**Proposition 3** Let \((g \oplus h, [\cdot, \cdot, \cdot]_{\rho, \nu, \omega})\) and \((g \oplus h, [\cdot, \cdot, \cdot]_{\rho, \nu, \omega})\) be two non-abelian extensions of \(g\) by \(h\). Then the two extensions are isomorphic if and only if there is a linear map \(\xi: g \rightarrow h\) such that the following equalities hold:

\[
\begin{aligned}
\nu^2(x)(v_2, v_3) - \nu^1(x)(v_2, v_3) &= -[\xi(x_1), v_2, v_3]_h, \\
\rho^2(x_1, x_2, x_3) &= -\nu^1(x)(\xi(x_2), v_3) - \nu^1(x)(v_3, \xi(x_1)) \\
\omega^2(x_1, x_2, x_3) - \omega^1(x_1, x_2, x_3) &= -\rho^1(x_1, x_2)\xi(x_3) - \rho^1(x_2, x_3)\xi(x_1) - \rho^1(x_3, x_1)\xi(x_2) + \nu^1(x)(\xi(x_2), \xi(x_3))
\end{aligned}
\]

\[
\begin{aligned}
\nu^1(x)(v_1, v_2) &= 0, \quad \nu^1(x)(v_1, v_3) = 0, \quad \nu^1(x)(v_2, v_3) = 0, \quad \nu^1(x_1)(v_2, v_3) = 0, \quad \nu^1(x_2)(v_1, v_3) = 0, \\
\nu^1(x_2)(v_2, v_3) &= -(r_1/r_2)v_1, \quad \nu^1(x_3)(v_1, v_2) = r_2v_1, \quad \nu^1(x_3)(v_1, v_3) = r_3v_1, \quad \nu^1(x_3)(v_2, v_3) = r_4v_1,
\end{aligned}
\]

\[
\begin{aligned}
\omega^1(x_1, x_2, x_3) &= 0, \\
\rho^1(x_1, x_2)(v_1) &= 0, \quad \rho^1(x_1, x_2)(v_2) = 0, \quad \rho^1(x_1, x_2)(v_3) = 0, \quad \rho^1(x_1, x_3)(v_1) = 0, \quad \rho^1(x_1, x_3)(v_2) = 0, \\
\rho^1(x_1, x_3)(v_3) &= r_1v_1, \\
\rho^1(x_2)(v_1, v_2) &= 0, \quad \rho^1(x_2)(v_1, v_3) = 0, \quad \rho^1(x_2)(v_2, v_3) = 0, \quad \rho^1(x_2)(v_1, v_3) = r_2v_1, \\
\rho^1(x_2)(v_2, v_3) &= r_3v_1, \quad \rho^1(x_3)(v_1, v_2) = 0, \quad \rho^1(x_3)(v_1, v_3) = r_4v_1, \quad \rho^1(x_3)(v_2, v_3) = (r_3r_4 + r_1)/r_2v_1,
\end{aligned}
\]

\[
\begin{aligned}
\omega^1(x_1, x_2, x_3) &= 0, \\
\rho^1(x_1, x_2, x_3) &= 0, \\
\omega^1(x_1, x_2, x_3) &= -r_2v_1 + r_1v_2,
\end{aligned}
\]

\[
\begin{aligned}
\omega^1(x_1, x_2, x_3) &= r_2v_2 - r_1v_3,
\end{aligned}
\]

\[
\begin{aligned}
\rho^1(x_1, x_2, x_3) &= 0, \\
\nu^1(x_1)(v_1, v_2) &= 0, \quad \nu^1(x_1)(v_1, v_3) = r_1v_1, \quad \nu^1(x_1)(v_2, v_3) = r_2v_1, \quad \nu^1(x_2)(v_1, v_2) = 0, \quad \nu^1(x_2)(v_1, v_3) = r_3v_1, \\
\nu^1(x_2)(v_2, v_3) &= (r_2r_3/r_1)v_1, \quad \nu^1(x_3)(v_1, v_2) = 0, \quad \nu^1(x_3)(v_1, v_3) = r_4v_1, \quad \nu^1(x_3)(v_2, v_3) = (r_2r_4/r_1)v_1,
\end{aligned}
\]

\[
\begin{aligned}
\omega^1(x_1, x_2) &= r_2v_1 - r_1v_3,
\end{aligned}
\]

\[
\begin{aligned}
\rho^1(x_1, x_2)(v_1) &= 0, \quad \rho^1(x_1, x_2)(v_2) = 0, \quad \rho^1(x_1, x_2)(v_3) = 0, \quad \rho^1(x_1, x_3)(v_1) = 0, \quad \rho^1(x_1, x_3)(v_2) = 0, \\
\rho^1(x_1, x_3)(v_3) &= r_1v_1, \\
\rho^1(x_1, x_2)(v_1) &= 0, \quad \rho^1(x_2)(v_1, v_2) = 0, \quad \rho^1(x_2)(v_1, v_3) = 0, \quad \rho^1(x_2)(v_2, v_3) = 0, \\
\rho^1(x_2)(v_1, v_3) &= 0, \quad \rho^1(x_2)(v_2, v_3) = (r_2r_3/r_1)v_1, \\
\nu^1(x_1)(v_1, v_2) &= 0, \quad \nu^1(x_1)(v_1, v_3) = 0, \quad \nu^1(x_2)(v_1, v_2) = 0, \quad \nu^1(x_2)(v_1, v_3) = (r_2r_3/r_1)v_1, \\
\nu^1(x_2)(v_2, v_3) &= 0, \quad \nu^1(x_3)(v_1, v_2) = 0, \quad \nu^1(x_3)(v_1, v_3) = (r_2r_4/r_1)v_1, \quad \nu^1(x_3)(v_2, v_3) = 0,
\end{aligned}
\]

\[
\begin{aligned}
\omega^1(x_1, x_2) &= 0,
\end{aligned}
\]
we can deduce that Eq. (29) holds. By
\[ \theta(x_1, x_2, v_3) = [\theta(x_1), \theta(x_2), \theta(v_3)]_{(p, q, p, q)} \]
we can deduce that Eq. (30) holds. By
\[ \theta(x_1, x_2, x_3) = [\theta(x_1), \theta(x_2), \theta(x_3)]_{(p, q, p, q)} \]
we can deduce that Eq. (31) holds.

4 Non-Abelian Extensions in Terms of Maurer-Cartan Elements

In Ref. [19], the author constructed a graded Lie algebra for $n$-Lie algebras. Here, we give the precise formulas for the 3-Lie algebra case.

We define $C^p(g, g) = \text{Hom}(\Lambda^2 g \otimes \cdots \otimes \Lambda^2 g \otimes g, g)$

\[ \alpha \circ \beta(\bar{x}_1, \ldots, \bar{x}_{p+q}, x) = \sum_{k=0}^{p-1} (-1)^k \left( \sum_{\sigma \in \text{unsh}(k, q)} (-1)^n (\alpha(\bar{x}_{\sigma(1)}, \ldots, \bar{x}_{\sigma(k)}, \beta(\bar{x}_{\sigma(k+1)}, \ldots, \bar{x}_{\sigma(k+q)}), y_{k+q+1}, \bar{x}_{k+q+2}, \ldots, \bar{x}_{p+q}, x) \right. \]

Furthermore, $(C^*(g, g), [\cdot, \cdot]^{3\text{Lie}}, \delta)$ is a DGLA, where $\delta$ is given by $\delta P = (-1)^p P$ for all $P \in C^p(g, g)$, and $\delta$ is the coboundary operator of $g$ with coefficients in the adjoint representation. See Ref. [22] for more details.

Remark 1 The coboundary operator $\delta$ associated to the adjoint representation of the 3-Lie algebra $g$ can be written as $\delta P = (-1)^p [\mu_g, P]^{3\text{Lie}}$, for all $P \in C^p(g, g)$, where $\mu_g \in C^1(g, g)$ is the 3-Lie algebra structure on $g$, i.e., $\mu_g(x, y, z) = [x, y, z]_g$. Thus, we have $\delta P = [\mu_g, P]^{3\text{Lie}}$.

Now, we describe non-abelian extensions using Maurer-Cartan elements. Let $(L, [\cdot, \cdot, \cdot])$ be a differential graded Lie algebra, with $L_0$ abelian.\(^1\) The set $\text{MC}(L)$ of Maurer-Cartan elements of the DGLA $(L, [\cdot, \cdot, \cdot])$ is defined by

\[ \text{MC}(L) = \left\{ P \in L_1 | dP + \frac{1}{2} [P, P] = 0 \right\}. \]

Moreover, $P_0, P_1 \in \text{MC}(L)$ are called gauge equivalent if and only if there exists an element $\xi \in L_0$ such that

\[ P_1 = e^{\text{ad}_\xi} P_0 - \frac{e^{\text{ad}_\xi} - 1}{\text{ad}_\xi} d\xi. \]

\(^1\)This condition guarantees that the right hand side of Eq. (35) makes sense. In general, we should assume that $L$ is equipped with a descending filtration:

\[ L = F_1 L \supset F_2 L \supset \cdots \supset F_k L \supset \cdots, \]

which is compatible with the Lie bracket, and such that $L$ is complete with respect to this filtration, i.e.

\[ L = \lim L/F_k L. \]

See Refs. [23, 27] for more details.
Proof By the definition of the bracket $[\cdot, \cdot]^{3\text{Lie}}$ and $\bar{\delta}$, we obtain that $(C(g \oplus h, h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$ is a sub-DGLA of $(C(g \oplus h, g \oplus h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$. For $a \in C^p(g \oplus h, h)$, we can regard it as $a \in C^p(g \oplus h, h)$ such that $a|_{C^p(h, h)} = 0$. Moreover, for $a \in C^p(g \oplus h, h), \beta \in C^q(g \oplus h, h)$, we have $[\alpha, \beta]^{3\text{Lie}}|_{C^{p+q}(h, h)} = 0$ and $\bar{\delta}(a)|_{C^{p+1}(h, h)} = [\mu g + \eta h, \alpha]^{3\text{Lie}}|_{C^{p+1}(h, h)} = 0$. Thus, we obtain $(C(g \oplus h, h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$ is a sub-DGLA of $(C(g \oplus h, h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$. Therefore, $(C^\infty(g \oplus h, h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$ is a sub-DGLA of $(C(g \oplus h, g \oplus h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$. Obviously, $C^\infty(g \oplus h, h) = \text{Hom}(g, h)$ is abelian.

Proposition 4 The following two statements are equivalent:

(a) $(g \oplus h, [\cdot, \cdot], (\rho, \nu, \omega))$ is a 3-Lie algebra, which is a non-abelian extension of $g$ by $h$;

(b) $\rho + \nu + \omega$ is a Maurer-Cartan element of the DGLA $(C^\infty(g \oplus h, h), [\cdot, \cdot]^{3\text{Lie}}, \delta)$.

Proof By Proposition 2, $(g \oplus h, [\cdot, \cdot], (\rho, \nu, \omega))$ is a 3-Lie algebra if and only if Eqs. (18)–(20) hold.

$c(e_1 e_2 e_3 \wedge e_4 e_5) = \rho([x_1, x_2, x_3]_g, x_4)(e_5) + \rho([x_1, x_3, x_2]_g)(e_4) + \rho([x_2, x_1, x_3]_g)(e_4) + \rho([x_3, x_1, x_2]_g)(e_4) + \rho([x_2, x_3, x_1]_g)(e_4) + \rho([x_3, x_2, x_1]_g)(e_4) + \rho([x_1, x_2, x_3]_g)(e_4) + \rho([x_1, x_3, x_2]_g)(e_4) + \rho([x_3, x_1, x_2]_g)(e_4) + \rho([x_2, x_2, x_1]_g)(e_4) + \rho([x_3, x_3, x_1]_g)(e_4) + \rho([x_3, x_1, x_3]_g)(e_4) + \rho([x_2, x_1, x_3]_g)(e_4) + \rho([x_1, x_2, x_2]_g)(e_4) + \rho([x_1, x_1, x_3]_g)(e_4) + \rho([x_2, x_3, x_3]_g)(e_4)$

If $c = \rho + \nu + \omega$ is a Maurer-Cartan element, we have

$$\left(\frac{\bar{\delta}c + \frac{1}{2}[c, c]^{3\text{Lie}}}{\bar{\delta}c + \frac{1}{2}[c, c]^{3\text{Lie}}}(e_1 \wedge e_2, e_3 \wedge e_4, e_5) = 0, \quad \forall e_i \neq x_1 + y_1 \in g \oplus h. \right.$$
Thus, $c = \rho + \nu + \omega$ is a Maurer-Cartan element if and only if Eqs. (18)--(28) hold.

**Corollary 1** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two 3-Lie algebras. Then there is a one-to-one correspondence between non-abelian extensions of the 3-Lie algebra $\mathfrak{g}$ by $\mathfrak{h}$ and Maurer-Cartan elements in the DGLA $(C_\succ(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}), [\cdot, \cdot]^{\text{3Lie}}, \partial)$.

**Theorem 2** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two 3-Lie algebras. Then the isomorphism classes of non-abelian extensions $\mathfrak{g}$ by $\mathfrak{h}$ one-to-one correspond to the gauge equivalence classes of Maurer-Cartan elements in the DGLA $(C_\succ(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}), [\cdot, \cdot]^{\text{3Lie}}, \partial)$.

**Proof** Two elements $c = \rho + \nu + \omega$ and $c' = \rho' + \nu' + \omega'$ in $\text{MC}(\mathcal{L})$ are equivalent if there exists $\xi \in \text{Hom}(\mathfrak{g}, \mathfrak{h})$ such that $c' = e^{\text{ad}_\xi}c - \left(e^{\text{ad}_\xi} - 1\right)/\text{ad}_\xi \delta \xi$. More precisely, for all $e_i = x_i + v_i \in \mathfrak{g} \oplus \mathfrak{h}$, we have

$$c'(e_1 \wedge e_2, e_3) = \left(\left(\text{id} + \text{ad}_\xi + \frac{1}{2!} \text{ad}_\xi^2 + \frac{1}{3!} \text{ad}_\xi^3 + \cdots + \frac{1}{n!} \text{ad}_\xi^n + \cdots\right) \cdot \delta \xi\right) (e_1 \wedge e_2, e_3).$$

Furthermore, by the bracket in Theorem 1, we have

$$[\xi, c]^{\text{3Lie}} (e_1 \wedge e_2, e_3) = -[\xi, c]^{\text{3Lie}} (e_1 \wedge e_3, e_2) - [\xi, c]^{\text{3Lie}} (e_2 \wedge e_1, e_3) - [\xi, c]^{\text{3Lie}} (e_3 \wedge e_2, e_1),$$

Thus, we have

$$[\xi, c]^{\text{3Lie}} (e_1 \wedge e_2, e_3) = -[\xi, c]^{\text{3Lie}} (e_1 \wedge e_3, e_2) - [\xi, c]^{\text{3Lie}} (e_2 \wedge e_1, e_3) - [\xi, c]^{\text{3Lie}} (e_3 \wedge e_2, e_1).$$
Moreover, we have
\[
[\xi, [\xi, \xi]^\text{3Lie}]^\text{3Lie}(e_1 \wedge e_2, e_3) = 0.
\]
More generally, for \( n \geq 3 \)
\[
ad^n_\xi c = 0.
\]
For all \( e_i = x_i + v_i \in \mathfrak{g} \oplus \mathfrak{h} \), we have
\[
\tilde{\delta}\xi(e_1 \wedge e_2, e_3) = \left[\mu_\xi + \mu_\mathfrak{h}, \xi\right]^\text{3Lie}(e_1 \wedge e_2, e_3) = \left(\mu_\mathfrak{h} \circ \xi\right)(e_1 \wedge e_2, e_3) - \left(\xi \circ (\mu_\mathfrak{h} + \mu_\mathfrak{g}\right))(e_1 \wedge e_2, e_3)
\]
Thus, we have
\[
[\xi, \tilde{\delta}\xi]^\text{3Lie}(e_1 \wedge e_2, e_3) = -2\delta\xi(e_1 \wedge e_2, e_3) - \tilde{\delta}\xi(e_1 \wedge \xi(e_2), e_3) - \tilde{\delta}\xi(e_1 \wedge e_2, \xi(e_3))
\]
Moreover, we have
\[
[\xi, [\xi, \tilde{\delta}\xi]^\text{3Lie}]^\text{3Lie}(e_1 \wedge e_2, e_3) = 0.
\]
More generally, for \( n \geq 3 \)
\[
ad^n_\xi \tilde{\delta}\xi = 0.
\]
5 Non-Abelian Extensions of Leibniz Algebras

In this section, we always assume that \((\mathfrak{g} \oplus \mathfrak{h}, \{\cdot, \cdot\}_{(\mathfrak{g} \oplus \mathfrak{h})})\) is a non-abelian extension of the 3-Lie algebra \(\mathfrak{g}\) by \(\mathfrak{h}\). We aim to analyze the corresponding Leibniz algebra structure on the space of fundamental objects. Note that \(\wedge^2(\mathfrak{g} \oplus \mathfrak{h}) \cong ([\wedge^2(\mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{h})) \oplus (\wedge^2(\mathfrak{g})\) naturally. We use \(\{\cdot, \cdot\}_F\) to denote the Leibniz bracket on the space of fundamental objects of the 3-Lie algebra \((\mathfrak{g} \oplus \mathfrak{h}, \{\cdot, \cdot\}_{(\mathfrak{g} \oplus \mathfrak{h})})\).

First we introduce a Leibniz algebra structure on \((\wedge^2(\mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{h}))\). Define a linear map \(\{\cdot, \cdot\} : ([\wedge^2(\mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{h})) \oplus ([\wedge^2(\mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{h})) \rightarrow (\wedge^2(\mathfrak{h}) \oplus (\mathfrak{g} \oplus \mathfrak{h})\) by
\[
\{u_1 \wedge v_1 + x_1 \wedge y_1, u_2 \wedge v_2 + x_2 \wedge y_2\} = [u_1, v_1, u_2]_h \wedge v_2 + u_2 \wedge [u_1, v_1, u_2]_h + \nu(x_2)(u_1, v_1) \wedge w_2 + x_2 \wedge [u_1, v_1, w_2]_h
\]
\[
+ \nu(x_1)(w_1, u_2) \wedge v_2 + u_2 \wedge \nu(x_1)(w_1, u_2) - \rho(x_1, x_2)(w_1) \wedge w_2 + x_2 \wedge \nu(x_1)(w_1) u_2)
\]
Proposition 5 With the above notations, \((\wedge^2(\mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{h}), \{\cdot, \cdot\})\) is a Leibniz algebra.

Proof By direct computation, we have
\[
\{u_1 \wedge v_1 + x_1 \wedge y_1, u_2 \wedge v_2 + x_2 \wedge y_2\} = [u_1 \wedge v_1 + x_1 \wedge y_1, u_2 \wedge v_2 + x_2 \wedge y_2]_F.
\]
Thus, \((\wedge^2(\mathfrak{g}) \oplus (\mathfrak{g} \oplus \mathfrak{h}), \{\cdot, \cdot\})\) is a Leibniz subalgebra of the Leibniz algebra \((\wedge^2(\mathfrak{g} \oplus \mathfrak{h}), \{\cdot, \cdot\}_F)\).

We define \(\varpi : (\wedge^2(\mathfrak{g}) \oplus (\wedge^2(\mathfrak{g}) \rightarrow (\wedge^2(\mathfrak{g} \oplus (\mathfrak{g} \oplus \mathfrak{h}))\) by
\[
\varpi(x \wedge y, z \wedge t) = -t \otimes \omega(x, y, z) + z \otimes \omega(x, y, t),
\]
\[
l(x \wedge y)(u \wedge v + z \wedge w) = \rho(x, y)(u) \wedge v + u \otimes \rho(x, y)(v) + [x, y, z]_g \otimes w + \omega(x, y, z) \wedge w + z \otimes \rho(x, y)(w),
\]
\[
r(x \wedge y)(u \wedge v + z \wedge w) = -y \otimes \nu(x)(u, v) + x \otimes \nu(y)(u, v) + \rho(z, x)(w) - x \otimes \rho(z, y)(w),
\]
for all \( x, y, z, t \in \mathfrak{g}, u, v, w \in \mathfrak{h} \).

Now we are ready to give the main result of this section.
Theorem 3 Let \((g,[\cdot,\cdot],\cdot)_g\) and \((h,[\cdot,\cdot],\cdot)_h\) be two 3-Lie algebras and \((g \oplus h,[\cdot,\cdot],\cdot)_{(g,h)}\) a non-abelian extension of the 3-Lie algebra \(g\) by \(h\). Then the Leibniz algebra \((\Lambda^2(g \oplus h),[\cdot,\cdot],\cdot)_{\Lambda^2}\) is a non-abelian extension of the Leibniz algebra \((\Lambda^2g,[\cdot,\cdot],\cdot)_{\Lambda^2g}\) by the Leibniz algebra \(((\Lambda^2h) \oplus (g \otimes h),[\cdot,\cdot],\cdot)_{\Lambda^2}\).

Proof One can show that conditions (6)–(12) in Proposition 1 hold directly. Thus, \((\Lambda^2(g \oplus h),[\cdot,\cdot],\cdot)_{\Lambda^2}\) is a non-abelian extension of the Leibniz algebra \((\Lambda^2g,[\cdot,\cdot],\cdot)_{\Lambda^2g}\) by the Leibniz algebra \(((\Lambda^2h) \oplus (g \otimes h),[\cdot,\cdot],\cdot)_{\Lambda^2}\). Here we use a different approach to prove this theorem. Using the isomorphism between \((\Lambda^2g,[\cdot,\cdot],\cdot)_{\Lambda^2g}\) and \(((\Lambda^2h) \oplus (g \otimes h),[\cdot,\cdot],\cdot)_{\Lambda^2}\), the Leibniz algebra structure on \((\Lambda^2g \oplus h)\) is given by

\[
[u_1 \wedge v_1 + x_1 \otimes w_1 + y_1 \wedge z_1, u_2 \wedge v_2 + x_2 \otimes w_2 + y_2 \wedge z_2]_F
\]

\[
= [u_1 \wedge v_1 + x_1 \otimes w_1, u_2 \wedge v_2 + x_2 \otimes w_2]_F + [y_1 \wedge z_1, u_2 \wedge v_2 + x_2 \otimes w_2]_F
\]

\[
+ [u_1 \wedge v_1 + x_1 \otimes w_1, y_2 \wedge z_2]_F + [y_1 \wedge z_1, y_2 \wedge z_2]_F
\]

\[
= \{u_1 \wedge v_1 + x_1 \otimes w_1, u_2 \wedge v_2 + x_2 \otimes w_2\} + \{y_1 \wedge z_1\}(u_2 \wedge v_2 + x_2 \otimes w_2)
\]

\[
+ r(y_2 \wedge z_2)(u_1 \wedge v_1 + x_1 \otimes w_1) + \rho(y_1 \wedge z_1, y_2 \wedge z_2) + [y_1 \wedge z_1, y_2 \wedge z_2]_F.
\]

Thus, by Eq. (5), we deduce that \((\Lambda^2(g \oplus h),[\cdot,\cdot],\cdot)_{\Lambda^2}\) is a non-abelian extension of the Leibniz algebra \((\Lambda^2g,[\cdot,\cdot],\cdot)_{\Lambda^2g}\) by the Leibniz algebra \(((\Lambda^2h) \oplus (g \otimes h),[\cdot,\cdot],\cdot)_{\Lambda^2}\). □

References

[1] V. T. Filippov, Sib. Mat. Zh. 26 (1985) 126.
[2] Y. Nambu, Phys. Rev. D 7 (1973) 2405.
[3] L. Takhtajan, Commun. Math. Phys. 160 (1994) 295.
[4] P. Gautheron, Lett. Math. Phys. 37 (1996) 103.
[5] A. Basu and J. A. Harvey, Nucl. Phys. B 713 (2005) 136.
[6] J. Bagger and N. Lambert, Phys. Rev. D 77 (2008) 065008.
[7] J. Bagger and N. Lambert, Phys. Rev. D 79 (2009) 025002.
[8] J. Gomis, D. Rodriguez-Gómez, M. Van Raamsdonk, and H. Verlinde, J. High Energy Phys. 8 (2008) 094.
[9] P. M. Ho and Y. Matsuo, J. High Energy Phys. 06 (2008) 105.
[10] P. M. Ho, R. Hou, and Y. Matsuo, J. High Energy Phys. 6 (2008) 020.
[11] G. Papadopoulos, J. High Energy Phys. 5 (2008) 054.
[12] Sh. M. Kasymov, Algebra i Logika 26 (1987) 277.
[13] Y. Daletskii and L. Takhtajan, Lett. Math. Phys. 39 (1997) 127.
[14] J. Figueroa-O’Farrill, J. Math. Phys. 50 (2009) 113514.
[15] J. A. de Azcárraga and J. M. Izquierdo, J. Phys. Conf. Ser. 175 (2009) 012001.
[16] L. Takhtajan, St. Petersburg Math. J. 6 (1995) 429.
[17] R. Bai, C. Bai, and J. Wang, J. Math. Phys. 51 (2010) 063505.
[18] R. Bai, G. Song, and Y. Zhang, Front. Math. China 6 (2011) 581.
[19] M. Rotkiewicz, Extracta Math. 20 (2005) 219.
[20] J. A. de Azcárraga and J. M. Izquierdo, J. Phys. A: Math. Theor. 43 (2010) 293001.
[21] A. Makhlof, Chapter 4 in Non Associative & Non Commutative Algebra and Operator Theory, eds. C.T. Gueye and M.S. Molina, Springer, Mulhouse 160 (2016).
[22] J. Liu, A. Makhlof, and Y. Sheng, Algebr Represent Theor. 20 (2017) 1415.
[23] Y. Fregier, J. Algebra 398 (2014) 243.
[24] J. L. Loday, Enseign. Math. 39 (1993) 269.
[25] J. L. Loday and T. Pirashvili, Math. Ann. 296 (1993) 139.
[26] J. Liu, Y. Sheng, and Q. Wang, Commun. Algebra. 46 (2018) 574.
[27] V. A. Dolgushev, Stable Formality Quasi-Isomorphisms for Hochschild Cochains, arXiv:math.K-Theory and Homology/1109.6031.