Proposed all-versus-nothing violation of local realism in the Kitaev spin-lattice model

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We investigate the nonlocal property of the fractional statistics in Kitaev’s toric code model. To this end, we construct the Greenberger-Horne-Zeilinger paradox which builds a direct conflict between the statistics and local realism. It turns out that the fractional statistics in the model is purely a quantum effect and independent of any classical theory. We also discuss a feasible experimental scheme using anyonic interferometry to test this contradiction.

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Quantum theory can predict results which are never achievable from the local realism (LR) [1]. By definition, LR consists of two constraints of realism and locality: Any observable has a predetermined value, regardless of whether it is measured or not, and the choice of which observable to measure on one party of a multiparticle system does not affect the results of the other parties. These constraints lead not only to the well-known Bell inequalities [2] which put bounds on the correlations and are violated statistically by certain quantum states, but also to the so-called Greenberger-Horne-Zeilinger (GHZ) paradox [3] which derives directly the inconsistent values of the correlations from LR and quantum mechanics. Such paradox is tested by the nonstatistical measurements, yielding succinctly an all-versus-nothing proof between LR and quantum mechanics. In this work we investigate LR in the context of Kitaev’s toric code spin-lattice model, which is an exactly solvable model and is crucial for fault-tolerant topological quantum computation (TQC) [4, 5]. This model has the merits: Its degenerate ground states yield a topologically protected subspace that provides robustness against noise and quasilocal perturbations, arousing much interest in condensed matter and quantum optical physics to realize and control it [6, 7, 8]: Excitations of the ground states known as anyons possess a class of fractional statistics [9] intervening between the bosonic and fermionic statistics, that is, the quantum state of anyons can acquire an unusual phase factor when one anyon is exchanged with another one, in contrast to usual values +1 for bosons and −1 for fermions.

Since anyons are at the heart of TQC [10], it is natural and important to ask whether the anyonic statistics, though defined in quantum mechanics can be described by LR. With this aim, we construct the GHZ paradox by using the string operators that are used in the model to move anyons on the lattice. According to this paradox, the results derived from anyonic statistics will contradict irreconcilably that derived from LR. In this way we conclude straightforwardly that the fractional statistics in Kitaev’s model is purely a quantum effect and independent of any classical theory. In experiment, the GHZ paradox was tested only by using multi-photon systems [11, 12]. The model discussed here also provides a potential platform to test the GHZ paradox in the future and this is discussed briefly at the end.

The Kitaev’s toric code spin-lattice model [4] is introduced as follows. Considering a \( k \times k \) square lattice on a torus \( T^2 \) (see Fig. 1), one spin or qubit is attached to each edge of the lattice. Thus there are \( 2k^2 \) qubits. For each vertex \( V \) and each face \( F \), consider operators of the following forms:

\[
A_V = \prod_{j \in V} \sigma_j^x, \quad B_F = \prod_{j \in F} \sigma_j^z,
\]

where the \( \sigma_j^x \) denotes the Pauli matrix with \( X = x, y, z \) and it acts on the \( j \)-th qubit of a vertex \( V \) or face \( F \). These four-body interacting operators commute with each other because a vertex \( V \) and a boundary \( F \) consist of either 0 or 2 common qubits. From their definitions, we know that operators \( A_V \) and \( B_F \) have eigenvalues \( \pm 1 \). Summing them together, it constructs the model Hamiltonian as

\[
H_0 = - \sum_{V \in T^2} A_V - \sum_{F \in T^2} B_F,
\]

of which the ground states \( |g\rangle \) satisfy \( A_V |g\rangle = |g\rangle \), \( B_F |g\rangle = |g\rangle \) for all \( V, F \). Due to the topological property of torus, the ground states are four-fold degenerate and construct a four-dimensional Hilbert subspace, one basis of which can be explicitly written as

\[
|g_0\rangle = \mathcal{J} \prod_{V \in T^2} (1 + A_V)|0\rangle^{\otimes 2k^2}, \tag{1}
\]

with a normalization constant \( \mathcal{J} \), while the remaining three can be given after we introduce the concept of string operators [4, 12].

Here we describe anyons as the quasiparticle excitations of the spin-lattice system with \( H_0 \). There are two types of anyons: \( z \)-particles living on the vertices and \( x \)-particles living on the faces of the lattice. These anyons
are created in pairs (of the same type) by string operators:

\[ S^z_{P_x} = \prod_{r \in P_x} \sigma^z_r, \quad S^z_{P_z} = \prod_{r \in P_z} \sigma^z_r, \tag{2} \]

are string operators associated with string \( P_x \) on the lattice and string \( P_z \) on the dual lattice, respectively (Fig. 1). Note that two anyons of the same type would annihilate each other when they meet and this is so-called fusion rule. Then we can see that \( A_V \) and \( B_F \) are just two closed string operators. Especially, there are four nonequivalent classes of closed strings that are not contractible, e.g., \( \{ C_{x_1}, C_{z_1}, C_{x_2}, C_{z_2} \} \) in Fig. 1. The corresponding string operators \( \{ S^z_{C_{x_1}}, S^z_{C_{z_1}}, S^z_{C_{x_2}}, S^z_{C_{z_2}} \} \) have the same commutation relations as \( \{ \sigma^z_1, \sigma^z_2, \sigma^z_3, \sigma^z_4 \} \) and all of them commute with \( A_V \) and \( B_F \) and thus \( H_0 \), which consequently give out the remaining three bases of the ground state subspace through \( \{ S^z_{C_{x_1}}|g\rangle, S^z_{C_{z_1}}|g\rangle, S^z_{C_{x_2}}|g\rangle, S^z_{C_{z_2}}|g\rangle \} \). [4][13].

In this viewpoint of local operators, string operators \( D_1 = S^z_{L_{y_1}}S^z_{L_{y_2}} \) are written as \( D_1 = S^z_{L_{y_1}}S^z_{L_{y_2}} \). When it acts on a ground state \( |g\rangle \), anyons will be created, moved and annihilated as follows: First, a pair of \( z \)-particles are created by the string operator \( S^z_{L_z} = \sigma^z_2 \) with \( L_z \) denoting the edge where the \( j \)-th qubit crosses the loop \( L_z \) as shown in Fig. 2(a); Then a pair of \( x \)-particles are created, moved and annihilated along the loop \( L_x \) by the string operator \( S^z_{L_x} = \sigma^z_1 \). At last, the string operator \( S^z_{L_{y_1}} = \sigma^z_1 \sigma^z_2 \sigma^z_1 \) moves one of the \( z \)-particles to meet and hence annihilate another one. As a result, we can see from Fig. 2(a) that the loops \( L_x \) and \( L_z \) construct a link. According to the anyonic statistics, a global phase factor \(-1\) is picked up in front of the ground state, i.e., \( D_1|g\rangle = -|g\rangle \). Likewise, for the operations \( D_2 = S^z_{L_{y_2}}S^z_{L_{y_3}}S^z_{L_{y_4}} \) and \( D_3 = S^z_{L_{y_5}}S^z_{L_{y_6}}S^z_{L_{y_7}} \), they have the same interpretations as \( D_1 \) but with different loops \( L_{x_2} \). The links constructed by \( L_{x_2} \) for \( D_{2,3} \) are shown in Fig. 2(b) and 2(c), respectively. As for the anyonic statistics, we have \( D_2,3|g\rangle = -|g\rangle \). As for the operation \( D_4 = S^z_{L_{y_5}} \), it has a straightforward correspondence with \( A_V \) operator and its loop \( L_c \) constructs a simple unknot shown in Fig. 2(d). Therefore, when acting it on the ground state, no change would appear, i.e., \( D_4|g\rangle = |g\rangle \).

Writing the above statements of composite string operators into more convenient forms, we have

\[
D_1|g\rangle = \sigma^x_1 \sigma^x_2 \sigma^x_1 \sigma^x_2 \sigma^y_1 \sigma^y_1 |g\rangle = -|g\rangle, \tag{3a}
\]

\[
D_2|g\rangle = \sigma^x_1 \sigma^x_2 \sigma^x_1 \sigma^x_2 \sigma^y_3 \sigma^y_3 |g\rangle = -|g\rangle, \tag{3b}
\]

\[
D_3|g\rangle = \sigma^x_1 \sigma^x_3 \sigma^x_4 \sigma^x_1 \sigma^y_1 \sigma^y_1 |g\rangle = -|g\rangle, \tag{3c}
\]

\[
D_4|g\rangle = \sigma^x_2 \sigma^x_2 \sigma^y_1 |g\rangle = |g\rangle, \tag{3d}
\]

in which the algebraic relation \( \sigma^x_1 \sigma^y_1 = i \sigma^y_1 \) has been used. Noting that Eqs. (3) contain only local operators, the operators in Eqs. (3) are measured, their results must satisfy the same functional relations satisfied by the corresponding operators in Eqs. (4).

\[
m^2_0 m^2_1 m^2_2 m^2_3 m^2_4 = -1, \tag{4a}
\]

\[
m^2_0 m^2_1 m^2_2 m^2_3 m^2_4 = -1, \tag{4b}
\]

\[
m^2_0 m^2_2 m^2_3 m^2_4 = -1, \tag{4c}
\]

\[
m^2_0 m^2_2 m^2_3 m^2_4 = -1. \tag{4d}
\]

After obtaining this, we next reveal how it produces the contradiction according to LR.

Noting that Eqs. (3) contain only local operators, the operators in each equation thereby commute and can all simultaneously have their eigenvalues. Thus, from LR we can associate an element of reality to each of the eigenvalues in Eqs. (4). For instance, the observers on particles (2, 3, 4, 5, 6) measure, without disturbing each other, the observables \( \{ \sigma^x_0, \sigma^x_2, \sigma^x_3, \sigma^x_4 \} \), respectively and if the multiplier of their results is 1 (or \(-1\)), then from Eq. (3a) they can predict with certainty that the result of measuring \( \sigma^y_1 \) will be \(-1 \) (or 1). That is, they can predict with certainty the value of quantity \( \sigma^y_1 \) by measuring other particles without disturbing particle 1, and therefore an element of reality can be associated to the physical quantity \( \sigma^y_1 \). Analogously, we can associate elements of reality to all the physical quantities in Eqs. (3). Then we can suppose that this result was somehow predetermined and initially hidden in the original state of the system. Such predictions with certainty would lead us to assign values \(+1\) or \(-1\) to all the observables in Eqs. (3). However, such assignment cannot be consistent with rules of quantum
mechanics because if we multiply Eqs. (4a)–(4c) together, it will lead to, \(m_i^2m_j^2m_k^2m_l^2 = -1\), which directly contradicts Eq. (4d). Therefore, we conclude that the four predictions of quantum mechanics given by Eqs. (3) cannot be reproduced by LR. This completes the construction of the GHZ paradox in the context of the Kitaev spin-lattice model.

Further, the above GHZ paradox applies to more general situations. We can enlarge the loops \(L_{x,z}\) in Fig. 2 to generalize these \(D_1,2,3,4\) operations. For each set of \(D_1,2,3,4\) when acting on a ground state, it can admit the GHZ paradox only if they satisfy all of the following requirements: (i) the loop \(L_x\) for all of them should be the same, no matter how large area they enclose; (ii) there is a loop \(L_z\) for each of \(D_1,2,3\) that should construct a link when combined with \(L_x\); (iii) when we merge the \(L_z\) of \(D_1\) with the \(L_z\) of \(D_2\) together with overlapping edges vanishing, the resultant loop should be the same as the \(L_z\) of \(D_3\). In this case, the string operators of anyons give us a simple yet effective approach to look for various sets of \(D_1,2,3,4\) operators to construct the GHZ paradox.

To sum up, it turns out that the GHZ paradox is very common in the Kitaev's toric code model. The all-versus-nothing violation of LR above well shows the anyonic statistics in the model as a pure quantum effect. In a way, it also indicates that the anyonic statistics may be at the conflictive regime between LR and quantum mechanics, which still needs an investigation in the future.

At the end, let us discuss briefly a feasible experimental implementation of the above consideration. The Kitaev spin-lattice model could be realized through dynamic laser manipulation of trapped atoms \(\oplus\) or molecules \(\oplus\) in an optical lattice. In addition, an approach of anyonic interferometry in atomic systems was as well suggested recently by Jiang et al. \(\oplus\) to measure topological degeneracy and anyonic statistics, enabling the measurement of the statistical phase associated with arbitrary braiding paths. By using this approach, it suffices to implement the operations \(D_{1,2,3,4}\) in Eqs. (3) and to detect the sign. We briefly introduce this in the following.

Consider a spin lattice of trapped atoms or molecules inside an optical cavity (as shown in Fig. 2a of Ref. \(\oplus\)), which provides a model Hamiltonian \(H_0\) and on which the spins are called memory qubits. Except for the memory qubits, an additional ancilla spin is needed to probe the sign change before ground states and hence is called the probe qubit. To achieve controlled-string operations, an optical cavity associated with the quantum nondemolition interaction between the common cavity mode and selected spins is used to implement, e.g., a z-type string operation

\[
\Lambda |S_0^\pm\rangle = |1\rangle_A \otimes |S_0^\pm\rangle + |0\rangle_A |0\rangle \otimes I, \tag{5}
\]

where the probe qubit is spanned by \(|0_A\rangle, |1_A\rangle\). It means: If the ancilla spin is in state \(|0_A\rangle\), no operation is applied to the memory qubits; If the ancilla spin is in state \(|1_A\rangle\), the operation \(S_0^\pm\) is applied to the topological memory. For our spin-lattice system with \(H_0\), we prepare its initial state \(|\Psi_{\text{initial}}\rangle\) to be a ground state \(|g\rangle\) and \(D_j|\Psi_{\text{initial}}\rangle = \pm|\Psi_{\text{initial}}\rangle\). Here the sign in front of the ground state \(|g\rangle\) is what we need to observe. The following interference experiment can be used to measure the sign. First, we prepare the probe qubit in a superposition state \((|0_A\rangle + |1_A\rangle)/\sqrt{2}\). We then use controlled-string operations to achieve interference of the following two possible evolutions: If the probe qubit is in state \(|0_A\rangle\), no operation is applied to the memory qubits; If the probe qubit is in state \(|1_A\rangle\), the operation \(D_j\) is applied to the topological memory, which picks up the extra phase factor \(e^{i\theta}\) we want to measure. After the controlled-string operations, the probe qubit will be in state \((|0_A\rangle + e^{i\theta}|1_A\rangle)/\sqrt{2}\). Finally, we project the probe qubit to the basis of \(|\xi_\pm\rangle \equiv (|0_A\rangle \pm e^{i\phi}|1_A\rangle)/\sqrt{2}\) with \(\phi \in [0, 2\pi]\), and measure the operator \(\sigma_\phi \equiv \langle \xi_+ |(\xi_+ - |\xi_-\rangle)(\xi_-)\). The measurement of \(\langle \sigma_\phi \rangle\) versus \(\phi\) should have fringes with perfect contrast and a maximum shifted by \(\phi = \theta\), for \(\langle \sigma_\phi \rangle = \cos(\phi - \theta)\). In other words, for sigma operations \(D_j\) (\(j = 1, 2, 3\)), the \(\phi\) shifts of the maximal \(\langle \sigma_\phi \rangle\) will differ from those for \(D_4\) by \(\pi\).

In summary, we have shown the GHZ paradox in the context of Kitaev's toric code spin-lattice model by using the anyonic string operations. It shows that the anyonic statistics in the model cannot be described by LR but be a purely quantum effect. In return, the Kitaev model provides a potential platform for testing the GHZ para-
dox or LR in the future. A feasible experimental consideration by using the anyonic interferometry is discussed to test such a contradiction at the end. It is worth noting that the measurement employed in the above experimental scheme is non-destructive and can be repeated without disturbing the ground state. It is a predominant advantage of using Kitaev’s model compared with the experimental tests by using multi-photon systems. Also recent experiments demonstrated string operations on small networks of interacting NMR qubits [14] and non-interacting optical qubits [15, 16], on the basis of which it is possible to realize our construction in advance on a small-scale qubit system.

Besides, the ground states of the Kitaev model belong to graph states, which is crucial in quantum information application. Bell inequalities have been shown to discuss LR for graph states [17]. Our construction here actually also contributes to the subject.

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