S-DUALITY FOR 2-d GRAVITY

J. A. Nieto

Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma
de Sinaloa, 80010 Culiacán Sinaloa, México

Abstract

We investigate the analogue of S-duality for 2-d gravity. Our analysis is based in a partition function associated to the Katanaev-Volovich type action for 2-d gravity. We find a S-dual 2-d gravitational action which is related to the original 2-d gravitational action by strong-weak duality transformation.

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1.- INTRODUCTION

The possibility to associate the analogue of S-duality to Einstein gravitational theory in four dimensions has been pursuing by a number of authors [1]-[6]. The picture which emerge from these works is that although technically one can follow a similar procedure as in the case of Abelian gauge theory and non-Abelian gauge theories [7]-[16] the final gravitational theory does not possess an exact strong-weak duality symmetry, as in the case of Yang-Mills theory. The idea can be realized as an exact symmetry, however, for linearized gravity [6] essentially because such a theory is indeed an Abelian gauge theory. From this scenario for linearized gravity arises the interesting possibility to have a strong coupling phase and small-large duality for the cosmological constant [6]. Linearized gravity may also be used as an inspiration to study the strong coupling limit for gravity assuming the possibility of some decompactified dimension. In this case, it seems that the strong coupling limit for gravity in four dimensions is a (4,0) conformal field theory in six dimensions [17]-[18].

In this work, we show that 2-d gravity also possesses an exact strong-weak duality symmetry. Specifically, we derive a S-dual action for 2-d gravity which is connected to the original 2-d gravitational action by strong-weak duality transformation. Our work may be of physical interest for at least two reasons. First, it might be helpful to understand some aspects of S-duality for gravity in four dimensions and second it might be useful to understand some aspects of 2-d gravity itself. Due to the fact that 2-d gravity is deeply related to string theory [19] and black hole physics [20] the present work may also be of special physical interest in such theories.

The plan of this work is as follows: In section 2, we mention some general aspects of 2-d gravity and in particular we discuss a 2-d gravity as a gauge theory. In section 3, we show that it is possible to understand 2-d gravity as an Abelian gauge theory. In section 4, we apply standard techniques to derive the S-dual action for 2-d gravity. Finally, in section 5, we make some comments.
2.- 2-d GRAVITY AS A GAUGE THEORY

It is a fact that interest in studying 2-d gravity has been growing in the last few years. Perhaps the main reason for this growing interest is the deeply connection between quantum 2-d gravity, string theory and black holes physics in two dimensions. One of the hopes is that 2-d gravity may help to shed some light about quantum gravity in four dimensions. There are a variety of models of 2-d gravity (see [21] and references therein). The simplest consistent theories seem to be the models of Jackiw and Teitelboim [22] and Callan, Giddings, Harvey and Strominger [20]. In fact, it has been shown that these two models can be derived through a dimensional reduction from a Chern-Simons theory [23]. Another interesting proposal is the one due to Katanaev and Volovich [24], which proposed action is quadratic in the curvature and in the torsion. Kummer [25] and Hehl [26] have shown that the action of Katanaev and Volovich has a very interesting black holes solutions.

Here, in order to discuss the analogue of S-duality for 2-d gravity we shall consider the de Sitter version of Katanaev-Volovich action, which seems to be due to Solodukhin [27]. This author showed that the equations of motion derived from such an action are exactly integrated with asymptotic de Sitter solution which for some assumptions corresponds to charged black holes solution.

Let us consider a SO(1,2) one form gravitational gauge field \( \omega = \omega_\mu dx^\mu = \frac{1}{2} \omega^{AB} J_{AB} dx^\mu \) in two dimensions, where \( J_{AB} = -J_{BA} \) are the generators of SO(1,2) and \( \mu = 0, 1 \). We shall split \( \omega^{AB}_\mu \) as a SO(1,1) connection \( \omega^{ab}_\mu \) and the \( \omega^{2a}_\mu = e^a_\mu \) Zweibein field, with \( a, b = 0, 1 \). Thus, the de Sitter curvature

\[
F^{AB}_{\mu\nu} = \partial_\mu \omega^{AB}_\nu - \partial_\nu \omega^{AB}_\mu + \omega^{AC}_{\mu} \omega^{B}_{\nu C} - \omega^{AC}_{\nu} \omega^{B}_{\mu C}
\]  

leads to the Macdowell-Mansourri type curvature

\[
F^{ab}_{\mu\nu} = R^{ab}_{\mu\nu} + \Sigma^{ab}_{\mu\nu}
\]  

and
\[ F^2_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a_{\mu c} e^c_\nu - \omega^a_{\nu c} e^c_\mu, \]  
\[ (3) \]

where

\[ R^{ab}_{\mu\nu} = \partial_\mu \omega^{ab}_\nu - \partial_\nu \omega^{ab}_\mu + \omega^{ac}_{\mu} \omega^{b}_{\nu c} - \omega^{ac}_{\nu} \omega^{b}_{\mu c}, \]
\[ (4) \]
is the 2-d curvature and

\[ \Sigma^{ab}_{\mu\nu} = e^a_\mu e^b_\nu - e^a_\nu e^b_\mu. \]
\[ (5) \]

Of course,

\[ T^a_{\mu\nu} \equiv F^2_{\mu\nu} \]
\[ (6) \]

can be identified with the torsion.

Let us consider the action

\[ S = \frac{1}{4\alpha^2} \int d^2 x \sqrt{-g} g^{\alpha\beta} F_{\mu\nu}^{AB} F_{\alpha\beta}^{CD} \eta_{AC} \eta_{BD}, \]
\[ (7) \]

where \( g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \), \( g = \det(g_{\mu\nu}) \), \( (\eta_{AB}) = (-1, 1, 1) \) is the Killing-Cartan metric associated to SO(1,2) and \( \alpha^2 \) is a dimensionless coupling constant.

Using (2)-(6), one may develop the action (7) to obtain

\[ S = \frac{1}{4\alpha^2} \int d^2 x \sqrt{-g} (g^{\mu\alpha} g^{\nu\beta} R^{ab}_{\mu\nu} \eta_{ac} \eta_{bd} + g^{\mu\alpha} g^{\nu\beta} T^a_{\mu\nu} T^b_{\alpha\beta} \eta_{ab} + 2 g^{\mu\alpha} g^{\nu\beta} \Sigma^{ab}_{\mu\nu} \eta_{ac} \eta_{bd} + g^{\mu\alpha} g^{\nu\beta} \Sigma^{cd}_{\mu\nu} \eta_{ac} \eta_{bd}), \]
\[ (8) \]

where we consider that \( \eta_{22} = 1 \). In order to compare the action (8) with previous actions which are quadratic in torsion and in curvature, let us introduce the completely antisymmetric quantities \( \varepsilon^{\mu\nu} \) and \( \varepsilon^{ab} \), with \( \varepsilon^{01} = 1 \) and \( \varepsilon_{01} = 1 \), associated to the space-time and the gauge group SO(1,1) respectively. We have

\[ \varepsilon^{\mu\nu} \varepsilon_{ab} = e (e^a_\mu e^b_\nu - e^a_\nu e^b_\mu), \]
\[ (9) \]

where \( e \) is the determinant of \( (e^a_\mu) \). Introducing the density tensors \( \epsilon^{\mu\nu} = -\frac{1}{e} \varepsilon^{\mu\nu} \) and \( \epsilon_{\mu\nu} = e \varepsilon_{\mu\nu} \) and the completely antisymmetric quantities \( \epsilon^{ab} = -\varepsilon^{ab} \) and \( \epsilon_{ab} = \varepsilon_{ab} \) the relation (9) gives
\[ e^\mu_\nu e^\alpha_\beta = -(e^\mu_a e^\nu_b - e^\nu_a e^\mu_b) \]  

and therefore by virtue of (5) we find that

\[ \Sigma^\mu_\nu = -e^\mu_\nu e^\alpha_\beta. \]  

Thus, using (11) we find that (8) becomes

\[
S = \frac{1}{4\alpha^2} \int d^2x \epsilon \left( \frac{1}{2} e^{\mu\nu} R^{ab} \epsilon^{\alpha\beta} R_{\alpha\beta ab} - \frac{1}{2} e^{\mu\nu} T^{ab}_{\mu\nu} \epsilon^{\alpha\beta} T_{\alpha\beta a} + 2 e^{ab} e^a \right) + 2 \epsilon^{\mu\nu} e^a \epsilon^b, 
\]

where we also used the fact that \( e^2 = -g, \epsilon^{\mu\nu} \epsilon^{\alpha\beta} = -(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}), \epsilon^{ab} e^a = -2 \) and \( \epsilon^{\mu\nu} = e^a_\mu e^b_\nu e^{ab} \).

Our final goal is to write (12) in abstract notation. For this purpose we shall use the differential form notation \( R^{ab} = \frac{1}{2} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu \) and \( T^a = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu \), with \( dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu \). We shall also use the Hodge dual \( *R^{ab} = \frac{1}{2} e^{\mu\nu} R^{ab}_{\mu\nu} \) and \( *T^a = \frac{1}{2} e^{\mu\nu} T^a_{\mu\nu} \). Note that \( dx^\mu \wedge dx^\nu = -e e^{\mu\nu} dx^0 \wedge dx^1 \). Thus, we find that (12) can be written as

\[
S = \frac{1}{\alpha^2} \int_{M^2} \left( \frac{1}{2} Tr^* R + \frac{1}{2} Tr^* T + R + \frac{1}{2} \epsilon_{ab} e^a \wedge e^b \right)
\]

Here \( M^2 \) is a two dimensional manifold, \( R = \frac{1}{2} e_{ab} R^{ab}_{\mu\nu} dx^\mu \wedge dx^\nu \) and \( e^a = e^a_\mu dx^\mu \). We recognize in the third and fourth terms of (13) the Euler number and the cosmological constant term respectively. In fact, rescaling the Zweibein field \( e^a_\mu \) in the form \( \lambda e^a_\mu \), with \( \lambda \) a constant, we discover that the action (13) is just the action given in the expression (1) of reference [27] (see also the expression (4.45) of reference [26]). Therefore, we have shown that (8) is a different form to write (13). The main goal in showing this equivalence is to be sure that (7) may provide a possible starting point. At the same time, the procedure allowed us to introduce some notation and useful relations which will become important in the next sections.
3.- 2-d GRAVITY AS ABELIAN GAUGE THEORY

We shall consider (8) assuming a vanishing torsion. In such a case, the action (8) is reduced to

$$S = \frac{1}{4\alpha^2} \int d^2x \sqrt{-g} (g^{\mu\alpha} g^{\nu\beta} R_{\mu\nu} R_{\alpha\beta} \delta^{ac}\delta^{bd} + 2g^{\mu\alpha} g^{\nu\beta} \Sigma_{\mu\nu} R_{\alpha\beta} \delta^{ac}\delta^{bd} + g^{\mu\alpha} g^{\nu\beta} \Sigma_{\mu\nu} \Sigma_{\alpha\beta} \eta^{ac}\eta^{bd}) \tag{14}$$

At first sight this action looks as an action for a non-Abelian gauge field. However, this is an illusion because in fact considering that $\omega^{ab}_{\mu} = -\epsilon^{ab}\omega_{\mu}$ the curvature (4) becomes

$$R^{ab}_{\mu\nu} = -\epsilon^{ab} R_{\mu\nu}, \tag{15}$$

where

$$R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \tag{16}$$

and therefore the action (14) is reduced to

$$S = -\frac{1}{\alpha^2} \int d^2x \sqrt{-g} \left(\frac{1}{2} R^{\mu\nu} R_{\mu\nu} + \epsilon^{\mu\nu} R_{\mu\nu} - 1\right), \tag{17}$$

which is an action associated to an Abelian gauge theory, with $\omega_\mu$ as a gauge field. The second term is the Euler number and classically do not contribute to the dynamics. So, in a sense the action (17) is a Maxwell-type action with field strength (16) and gauge field $\omega_\mu$. The difference, however, is that, due to the vanishing of the torsion $T^a = de^a - \epsilon^{ab}\omega_\Lambda e_b$, $\omega_\mu$ is related to the Zweibein field $e^a_\mu$ by the formula $\omega_\mu = -\epsilon^{\alpha\beta} \partial_\alpha e^a_\beta e_{\mu a}$ (see Ref. [28]).

At this stage, it is convenient to rescale the Zweibein field $e^a_\mu$ in the form $\lambda e^a_\mu$, where $\lambda$ is a constant. So the metric changes to $g_{\mu\nu} \rightarrow \lambda^2 g_{\mu\nu}$ and therefore $\sqrt{-g} \rightarrow \lambda^2 \sqrt{-g}$ and $g^{\mu\nu} \rightarrow \lambda^{-2} g^{\mu\nu}$. We also need to rescale the curvature as $R_{\mu\nu} \rightarrow \frac{1}{G} R_{\mu\nu}$ where $G$ is also a constant. Considering these rescalings the action (17) can be written as

$$S = -\int d^2x \sqrt{-g} \left( \frac{1}{2G^2 \lambda^2 \alpha^2} R^{\mu\nu} R_{\mu\nu} + \frac{1}{G \alpha^2} \epsilon^{\mu\nu} R_{\mu\nu} - \frac{\lambda^2}{\alpha^2} \right). \tag{18}$$
Of course, $G_N = G \alpha^2$ and $\Lambda = \frac{\lambda^2}{\alpha^2}$ can be identified with the gravitational Newton constant and cosmological constant in two dimensions, respectively. For our purpose, it turns out more convenient to use the constants $\gamma = G \lambda \alpha$ and $\theta = \frac{1}{G \alpha^2}$ which, in analogy to the four dimensional Maxwell theory, can be identified as the gravitational ”electric” coupling constant and the $\theta$ parameter respectively. Thus, we have

$$S = -\int d^2x \sqrt{-g}(\frac{1}{2\gamma^2} R_{\mu\nu} R_{\mu\nu} + \theta \epsilon^{\mu\nu} R_{\mu\nu} - \Lambda).$$ \hspace{1cm} (19)

It turns out that this action is equivalent to the action

$$I = -\frac{1}{4} \int d^2x \sqrt{-g}(+\tau^+ R_{\mu\nu} + R_{\mu\nu} + ^-\tau^- R_{\mu\nu} - R_{\mu\nu}),$$ \hspace{1cm} (20)

where

$$\pm R_{\mu\nu} = R_{\mu\nu} \pm \tilde{\lambda} \epsilon_{\mu\nu}$$ \hspace{1cm} (21)

and

$$\pm \tau = \frac{1}{\gamma^2} \pm \frac{\theta}{\tilde{\lambda}},$$ \hspace{1cm} (22a)

with $\tilde{\lambda} = \frac{\lambda}{\alpha}$. If we give to $\pm \tau$ the typical complex form of a modular transformation

$$\pm \tau = \frac{1}{\gamma^2} \pm i\frac{\theta}{\tilde{\lambda}},$$ \hspace{1cm} (22b)

we note that (20) also leads to (19) but with the second term now as a complex quantity $i\theta \epsilon^{\mu\nu} R_{\mu\nu}$.

### 4.- S-DUALITY FOR 2-d GRAVITY

The computation of the S-dual 2-d gravitational action is now straightforward. Let us first introduce the two form $G$ and let us set

$$\pm P_{\mu\nu} \equiv \pm R_{\mu\nu} + \pm G_{\mu\nu},$$ \hspace{1cm} (23)
with $\pm G_{\mu\nu} = - \pm G_{\nu\mu}$.

Consider the partition function

$$Z = \int d^+Gd^-Gd\omega d\bar{\omega} dV \exp(-I_E),$$

(24)

where $I_E$ is the extended action

$$I_E = -\frac{1}{4} \int d^2x \sqrt{-g} (\tau^+P_{\mu\nu} + \tau^- P^{\mu\nu} + \bar{\lambda} \epsilon_{\mu\nu})$$

- $\frac{1}{2} \int d^2x \sqrt{-g} (W^\mu{}_{\nu} + G_{\mu\nu} + W^{\mu\nu} - G_{\mu\nu}).$

Here, $\pm W_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \pm \bar{\lambda} \epsilon_{\mu\nu}$ is the dual field strength. The tensor $\Xi_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ satisfies the Dirac quantization law

$$\int \Xi \in 2\pi \mathbb{Z}.$$  

(26)

We note that the partition function $Z$ is invariant not only under the typical gauge invariance

$$\omega \rightarrow \omega + d\lambda, G \rightarrow G$$

(27)

transformation, but also under

$$\omega \rightarrow \omega + B \text{ and } G \rightarrow G + dB,$$

(28)

where $B$ is an arbitrary one form.

We first need to show that (25) is equivalent to (20). For this purpose, let us consider the partition function $Z$ containing the dual field $V$:

$$\int DV \exp\left(-\frac{1}{2} \int d^2x \sqrt{-g} (\tau^+P_{\mu\nu} + \tau^- P^{\mu\nu} + \bar{\lambda} \epsilon_{\mu\nu})\right).$$

(29)

Integrating out the dual connection $V$ at the classical level, we get a delta function setting $dG = 0$. Thus, the gauge invariance (28) allows us to set $G = 0$, reducing (25) to the original action (20). Therefore, the actions (25) and (20) are, in fact, classically equivalents.

We next note that the gauge invariance (28) enables one to fix a gauge with $\omega = 0$. The action (25) is then reduced to
\[ I_E = -\frac{1}{4} \int d^2x \sqrt{-g}(\tau^+ Q^{\mu\nu} + Q_{\mu\nu} - \tau^- Q^{\mu\nu} - Q_{\mu\nu}) \]

(30)

\[ -\frac{1}{2} \int d^2x \sqrt{-g}(^+ W^{\mu\nu} + G_{\mu\nu} + ^- W^{\mu\nu} - G_{\mu\nu}), \]

where

\[ \pm Q_{\mu\nu} \equiv \pm \Omega_{\mu\nu} - \pm G_{\mu\nu}, \quad (31) \]

with

\[ \pm \Omega_{\mu\nu} = \pm \tilde{\lambda} \epsilon_{\mu\nu}. \quad (32) \]

By virtue of (31) the extended action (30) becomes

\[ I_E = -\frac{1}{4} \int d^2x \sqrt{-g}(\tau^+ G^{\mu\nu} + G_{\mu\nu} + \tau^- G^{\mu\nu} - G_{\mu\nu}) \]

(33)

\[ -\frac{1}{4} \int d^2x \sqrt{-g}(^+ \Omega^{\mu\nu} + \Omega_{\mu\nu} + ^- \Omega^{\mu\nu} - \Omega_{\mu\nu}) \]

\[ -\frac{1}{2} \int d^2x \sqrt{-g}(^+ L^{\mu\nu} + G_{\mu\nu} + ^- L^{\mu\nu} - G_{\mu\nu}), \]

where

\[ \pm L^{\mu\nu} = \pm W^{\mu\nu} - \pm \tau^\pm \Omega^{\mu\nu}. \quad (34) \]

Using the partition function (24) with (33) as an action and integrating over \(^+ G_{\mu\nu} \) and \(^- G_{\mu\nu} \) we get

\[ S_E = -\frac{1}{4} \int d^2x \sqrt{-g}((-\frac{1}{\tau})^+ L^{\mu\nu} + L_{\mu\nu} + (-\frac{1}{\tau})^- L^{\mu\nu} - L_{\mu\nu}) \]

(35)

\[ -\frac{1}{4} \int d^2x \sqrt{-g}(^+ \tau^+ \Omega^{\mu\nu} + \Omega_{\mu\nu} + ^- \tau^\pm \Omega^{\mu\nu} - \Omega_{\mu\nu}). \]

Substituting (34) into (35) and using the fact that \( \pm W_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \pm \tilde{\lambda} \epsilon_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + \pm \Omega^{\mu\nu} \) we finally get the dual action

\[ S_D = -\frac{1}{4} \int d^2x \sqrt{-g}((-\frac{1}{\tau})^+ W^{\mu\nu} + W_{\mu\nu} + (-\frac{1}{\tau})^- W^{\mu\nu} - W_{\mu\nu}) + 2\tilde{\lambda}^2 \int d^2x \sqrt{-g}. \quad (36) \]
Note that the action (36) has the dual gauge invariance $V \rightarrow V + d\alpha$ and, up to cosmological constant term, is of the general type (20) but with $\tau$ replaced by $-\frac{1}{\tau}$, where $\tau$ can be either $+\tau$ or $-\tau$. Therefore, we have shown that the coupling transforms as $\tau \rightarrow -\frac{1}{\tau}$ when one changes from the action (20) to the action (36).

5.-FINAL COMMENTS

In this article, we have shown that it is possible to associate the analogue of S-duality to 2-d gravity. The main observation is that 2-d gravity can have an interpretation of an Abelian gauge theory and therefore one can apply standard procedure to find the S-dual action. Specifically, we proved that the partition function for 2-d gravity has the exact S-dual symmetry

$$Z_\omega(\tau) = Z_V(-\frac{1}{\tau}),$$

where $Z_\omega(\tau)$ is the partition function associated to the action (20), while $Z_V(-\frac{1}{\tau})$ is the partition function associated to the action (36).

A natural question is now to investigate what is the strong coupling limit of 2-d gravity. In this limit we should expect a decompactified dimension. So, we are searching for a gravitational theory in 2+1 dimensions. But as it was mentioned in the introduction from Chern-Simons theory it is possible to obtain the action of Jackiw-Teitelboim and Callan-Giddings-Harvey-Strominger, which are not quadratic in the curvature. On the other hand, it is known that 2+1 dimensional gravity can be obtained from Euler and Pontrjagin topological invariant terms in 3+1 dimensions. Therefore, since this two topological invariants can be written as a Chern-Simons actions in 2+1 dimensions one should expect that we must go to 3+1 dimensions but with an additional term quadratic in the curvature. Fortunately, topological gravity in 3+1 dimensions seems to accomplish the required combinations in the curvature. In fact, a typical form of 3+1 topological gravity is a term of the form
\[ S = -\int d^4x\sqrt{-g}\frac{1}{2\gamma^2}(R_{\dot{\mu}\dot{\nu}} + \frac{1}{2}\epsilon^{\dot{\mu}\dot{\nu}\dot{\alpha}\dot{\beta}}R_{\dot{\alpha}\dot{\beta}})(R_{\dot{\mu}\dot{\nu}} + \frac{1}{2}\epsilon_{\dot{\mu}\dot{\nu}}\dot{\alpha}\dot{\beta}R_{\dot{\alpha}\dot{\beta}}) \]  

which under dimensional reduction one should expect to lead to the 2-d gravitational action (8). Thus, this roughly observations lead us to conjecture that the strong coupling limit of 2-d gravity must be (something like) topological gravity in 3+1 dimensions. It is interesting that our conclusion is in agreement with stochastic quantization theory where a similar connection between 2-d gravity and 4-d topological gravity is found [29].

So far, our discussion has focused on pure 2-d gravity. But what about 2-d gravity interacting with matter fields. For the case of a scalar field we do not even need to consider an action with quadratic curvature to find the analogue of S-dual for 2-d gravity. In fact, consider the action [30]

\[ S = -\int d^2x\sqrt{-g}\left(\kappa D^\mu\varphi D_\mu\varphi + \Lambda^{\mu\nu}R_{\mu\nu}\right), \]  

where

\[ D_\mu = \partial_\mu + \omega_\mu, \]  

\[ \kappa \] is a constant and \( \Lambda^{\mu\nu} \) is a Lagrange multiplier. If we perform an integration with respect the Lagrange multiplier \( \Lambda^{\mu\nu} \) we obtain \( R_{\mu\nu} = 0 \). Therefore, this result together with the gauge invariance \( \omega_\mu \to \omega_\mu + \partial_\mu f \) allows us to set \( \omega_\mu = 0 \), reducing (39) to

\[ S = -\kappa \int d^2x\sqrt{-g}\partial^\mu\varphi\partial_\mu\varphi \]  

which is the typical action for a scalar field.

Let us go back to (39). Since (39) is also invariant under \( \varphi \to \varphi + b \) and \( \omega \to \omega + db \) we can set \( \varphi = 0 \). Integrating by parts the resultant action and integrating out \( \omega \) we get the dual action for the variable \( \Phi = \frac{1}{2}\varepsilon_{\mu\nu}\Lambda^{\mu\nu}; \)

\[ S = -\left(-\frac{1}{\kappa}\right)\int d^2x\sqrt{-g}\partial^\mu\Phi\partial_\mu\Phi, \]  

showing the typical dual transformation \( \kappa \to -\frac{1}{\kappa} \) for the coupling constant.
For the case of 2-d supergravity [31] and [28] we can write the partition function for 2-d gravity as a product of two partition functions: one corresponding to 2-d gravity and the other corresponding to Rarita-Schwinger field. This way, one should expect to get the S-dual 2-d supergravity symmetry
\[ Z_{\omega,\psi}(\tau) = Z_{V,\varphi}(-\frac{1}{\tau}) \]
where \( \psi \) is the Rarita-Shwinger field and \( \varphi \) its dual.

Some time ago, Ikeda and Izawa [32] (see also [33]) used nonlinear Poincaré algebra to construct a gauge theory for 2-d gravity which turns out to be equivalent to most general Poincaré gauge theory for 2-d gravity with dynamical torsion:
\[
S = \int d^2x \sqrt{-g} \left( \frac{1}{4\alpha^2} g^{\mu\alpha} g^{\nu\beta} R_{\mu\nu} R_{\alpha\beta} \eta_{ac} \eta_{bd} + \frac{1}{2\beta^2} g^{\mu\alpha} g^{\nu\beta} T_{\mu\nu} T_{\alpha\beta} \eta_{ab} + \Lambda \right). 
\] (43)
Here, \( \alpha, \beta, \) and \( \Lambda \) are different constants. Clearly, dropping from this action the torsion and including the \( \theta \) term we obtain our action (19). In a similar way, it has been shown that the quadratic \( W_3 \) algebra leads to the \( W_3 \) gravity [34]. So it seems that from our work follows that it must also be possible to associate the analogue of S-duality to \( W_3 \) gravity.

Another interesting observation is that when a 4-d Abelian gauge theory is coupled to gravity, it has been found that the partition function is not a modular-invariant function but transform as a modular form [7]. In view of the result of the present work it seems that in two dimensions the partition function of an Abelian gauge field coupled to gravity must be modular-invariant function.

Finally, it is known that Liouville 2-d gravity theory has a very interesting features in the strong coupling regime [35]. It may be interesting to understand such a features from the point of view of the present work. It is also known that the S-duality gauge invariance is deeply related to bosonization [16], 2-d ADS / CFT correspondence [36], and noncommutative gauge theory [37]. It may be also very interesting to see whether the present work can be useful in those directions.
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