Nonlocal Yang-Mills
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ABSTRACT: We present a very simple and explicit procedure for nonlocalizing the action of any theory which can be formulated perturbatively. When the resulting nonlocal field theory is quantized using the functional formalism — with unit measure factor — its Green’s functions are finite to all orders. The construction also ensures perturbative unitarity to all orders for scalars with nonderivative interactions, however, decoupling is lost at one loop when vector and tensor quanta are present. Decoupling can be restored (again, to all orders) if a suitable measure factor exists. We compute the required measure factor for pure Yang-Mills at order $g^2$ and then use it to evaluate the vacuum polarization at one loop. A peculiar feature of our regularization scheme is that the on-shell tree amplitudes are completely unaffected. This implies that the nonlocal field theory can be viewed as a highly noncanonical quantization of the original, local field equations.
1. Introduction

At any point in the evolution of science there is always a “simplest way” to perceive true results, yet insights of great power sometimes emerge from adopting novel perspectives. The putative finiteness of superstring loop amplitudes is currently best understood through the Polyakov representation [1], a formalism which makes frequent and essential use of the fact that the first quantized dynamical variable is an extended object. Few would care to view finiteness from the perspective of string field theory; indeed it is not even simple to compute tree amplitudes in this way [2]. However when the attempt is made a very curious fact emerges: the finiteness of string field theory would follow trivially from the nonlocality of its interactions [3] were it not for the infinite number of component fields.* Since this last feature is a direct consequence of the infinite number of coordinates needed to describe an extended object we see that the much touted property of “stringyness” is not only of no use in quenching divergences, it is actually the only thing which can threaten the otherwise manifest finiteness of a suitably nonlocal field theory.

To this observation must be added the fact that most of the extra fields of string theory serve no other purpose than to distort its phenomenology in often untestable and sometimes unacceptable ways. So why do we not instead consider nonlocal theories based on a finite number of fields? Until very recently the answer would have been that strings provide the only perturbatively consistent and nonlocal theories which contain interacting vector and tensor quanta.

In fact it was long believed impossible to construct perturbatively consistent, purely scalar theories with nonlocal interactions, despite years of heroic effort [4]. The problem in this case was how to construct an operator formalism through canonical quantization. By

* We have suppressed a necessary qualifier in the interest of rhetorical trenchancy. There are, of course, many sorts of nonlocal vertices and not all of them serve to quench loop divergences. The ultraviolet finiteness of string field theory can be ascribed to the fact that its vertices include nonlocal convergence factors of the form \( \exp(-\alpha' p^2) \).
exploiting the functional formalism directly Polchinski exhibited a satisfactory model in 1984 [5], and a procedure for deriving the associated operator formalism was finally given (in a different context) by Jaén, Llosa and Molina in 1986 [6]. Though highly significant, these developments still do not suffice for the consistent incorporation of vector and tensor quanta because nonlocal interactions disrupt the local gauge symmetries which are used to reconcile Lorentz invariance and perturbative unitarity.

A two-stage procedure for avoiding this problem was discovered in early 1990 [7]. In the first stage a gauge invariant action is nonlocalized and then fortified with an infinite series of carefully chosen higher interactions. These new interactions endow the theory with a strange sort of nonlinear and nonlocal gauge invariance which does the crucial job of reconciling Lorentz invariance and perturbative unitarity at tree order. The second stage consists of finding a measure factor to make the functional formalism invariant under the nonlocal gauge symmetry without compromising perturbative unitarity.* If such a measure factor can be found then the resulting functional formalism defines a set of Green’s functions which are ultraviolet finite and Poincaré covariant to all orders, and which reduce to give perturbatively unitary scattering amplitudes. For QED the first stage was worked out explicitly and a proof was given that the second stage could be carried out as well. For Yang-Mills and gravity only an indirect argument was presented that the first stage could be completed and no results were reported regarding the second stage.

The purpose of this paper is to give a very simple procedure for constructing nonlocal versions of Yang-Mills, gravity or any other local field theory which has a perturbative formulation. This procedure, the work of section 2, has two applications. First, it provides a regularization technique which can be used in any theory. Although the method is no more difficult to implement than dimensional regularization it is also no easier; its selling

* We thank A. Polychronakos for showing us that a measure factor can always be found which restores invariance. The nontrivial thing is to find one whose interactions are entire functions of the derivative operator. Otherwise perturbative unitarity is threatened.
point is that if a suitable measure factor exists then the nonlocalized theory preserves physically equivalent versions of any and all continuous symmetries of its parent. This fact is proven in appendix B. The second application for nonlocalization is the generation of new fundamental theories. If a suitable measure factor exists then the nonlocalized theory appears to meet all the requirements for a fundamental model and there is really no reason to take the local limit.

In section 3 we implement the second stage for Yang-Mills to the first nontrivial order. The resulting Feynman rules are used in section 4 to compute the one loop correction to the gluon propagator. Section 5 discusses the possibility that nonlocalization produces consistent fundamental theories. To do so the resulting models must avoid the nonperturbative problems which occur in most nonlocal theories, including string field theory [8]. These problems derive largely from the existence of higher derivative solutions to the classical field equations; we will show in appendix A that nonlocalization does not engender such solutions. This prompts us to advance a radical proposal for extricating physics from the apparent impasse between gravitation and quantum mechanics. Our conclusions comprise section 6.

Before proceeding we should comment on the distinction between nonlocalization and two other invariant schemes with which it is sometimes confused. In the higher derivative method one attempts to regulate the theory by adding terms with higher covariant derivatives and/or higher powers of the field strength. This makes the propagator fall off faster for large Euclidean momenta but it also induces interactions with more factors of the momenta. The two effects balance at one loop so that divergences remain; it is only at higher loops that this method regulates [9]. Higher derivatives must therefore be used in conjunction with some other technique to provide complete regularization. One example of such a hybrid scheme has been given by Slavnov [10].

Nonlocalization results in propagators which fall off exponentially in Euclidean mo-
mentum space, as opposed to the power law fall off of the higher derivative method. A direct consequence is that the higher derivative propagator contains new poles, some of which necessarily represent ghost particles. Nonlocalization introduces no new singularities. This is why nonlocalization can preserve perturbative unitarity before the unregulated limit is taken while the higher derivative method cannot. Although both methods induce new interactions those from nonlocalization are no worse behaved than the interactions of the original theory; where they have more derivatives there are also compensating inverse powers of the regularization scale. (This is why nonlocalized loop amplitudes are consistent with power counting in the local theory.) Another important difference between the two methods is that higher derivatives preserve local gauge invariance whereas nonlocalization preserves only its physical import, namely decoupling. The actual transformation law is generally distorted into a nonlocal and nonlinear rule.

The second scheme with which we should contrast nonlocalization is continuum regularization [11]. In this method spacetime is endowed with an extra dimension and the fields are determined classically in terms of stochastic fields, of which there is one for each of the old fields. One takes expectation values of products of fields on the higher dimensional spacetime by simply expanding the fields in terms of the stochastic variables and then substituting the appropriate stochastic correlation functions. The regularization is imposed by nonlocalizing the correlation functions. Regulated Green’s functions in the lower dimensional theory are obtained by taking the extra coordinate to infinity.

Although the role of the stochastic correlation functions is somewhat similar to the smearing operator of nonlocalization the two methods have many differences. Nonlocalization leaves the tree amplitudes of any theory unchanged whereas continuum regularization does not. Nonlocalization distorts gauge invariance into a nonlocal and nonlinear symmetry which only closes on shell whereas continuum regularization preserves the original transformation rule. Nonlocalization produces an action in the original dimension whereas
continuum regularization is not derivable from an action on this spacetime. Finally, if a suitable measure factor exists then nonlocalization gives a theory which is perturbatively unitary before the unregulated limit is taken whereas this is not the case for continuum regularization. We wish to emphasize that our point in making this comparison is that the two methods differ, not that one is superior to the other. In fact it seems to us that which method is preferable depends very much on what one desires to do; and it is in any case useful to have different perspectives from which the same problem can be viewed.

2. General first stage construction

We will describe our method in terms of a generic action $S$ which is a functional of generalized fields $\phi_i$. These fields may be commuting or anticommuting, or there may be some of each type. The index which we write as $i$ may represent any sort of label, including Lorentz tensor or spinor indices. The only restriction is that the theory must make sense perturbatively, that is, the action can be decomposed into free and interacting parts:

$$S[\phi] = F[\phi] + I[\phi]$$  \hspace{1cm} (2.1a)

$$F[\phi] = \frac{1}{2} \int d^D x \; \phi_i \; F_{ij} \; \phi_j$$  \hspace{1cm} (2.1b)

where $I[\phi]$ is analytic in $\phi$ around the point $\phi = 0$.

Our construction for nonlocalizing $S[\phi]$ depends upon the specification of a smearing operator $\mathcal{E}_{ij}$. For simplicity we shall take this to have the simple exponential form:

$$\mathcal{E} \equiv \exp \left[ \frac{\mathcal{F}}{2\Lambda} \right]$$  \hspace{1cm} (2.2)

but the construction would work as well for any entire function of $\mathcal{F}\Lambda^{-2}$ which is unity for $\mathcal{F} = 0$ and which falls faster than any power of $\mathcal{F}$. We settled upon (2.2) only because it makes loop integrals simple. The regularization parameter is $\Lambda$; taking $\Lambda \to \infty$ gives the unregulated limit.
Two useful quantities which follow from the smearing operator are:

\[ \hat{\phi}_i \equiv E_{ij}^{-1} \phi_j \]  

\[ O \equiv (E^2 - 1) F^{-1} = \int_0^1 d\tau \frac{1}{\Lambda^2} e^{2\tau} \]  

Note that \( O \) is always well defined and invertible, even for \( F = 0 \). It is especially significant that \( O \) still makes sense when \( F \) is the kinetic operator of a gauge theory.

For each field \( \phi_i \) we now introduce an auxiliary field \( \psi_i \) of the same type. To avoid confusion with other auxiliaries which may be present we shall call the \( \psi \)’s, “shadow fields.”

The auxiliary action which couples \( \phi_i \) and \( \psi_i \) is:

\[ S[\phi, \psi] \equiv F[\hat{\phi}] - A[\psi] + I[\phi + \psi] \]  

\[ A[\psi] \equiv \frac{1}{2} \int d^D x \, \psi_i \, \mathcal{O}_{ij} \, \psi_j \]

We prove in appendix A that the classical shadow field equations:

\[ \frac{\delta S[\phi, \psi]}{\delta \psi_i(x)} = 0 \]  

uniquely determine the shadow fields as functionals, \( \psi_i[\phi] \), of the original fields \( \phi_i \) — up to terms which vanish with the \( \phi_i \) field equations. Multiplying by the invertible operator \( O \):

\[ \psi_i = O_{ij} \frac{\delta I[\phi + \psi]}{\delta \psi_j} \]

gives an expression from which we can develop a series expansion for \( \psi_i[\phi] \). In fact the quantity \( (\phi + \psi) \) is nothing more than the classical field for the action \(-A + I\) in the presence of the background \( \phi \).

The nonlocalized action is obtained by substituting this classical solution into the auxiliary action:

\[ \hat{S}[\phi] \equiv S[\phi, \psi[\phi]] \]
Expanding $\hat{S}$ in powers of $\phi$ gives the kinetic term $F[\hat{\phi}]$ plus an infinite series of interaction terms, the first of which is just $I[\phi]$. There is a simple graphical representation for the higher interaction vertices: they are the connected trees which follow from using the local interaction vertex but replacing the propagator by $-iO$. Note that since $O$ is an entire function of $F$ so too are these higher interactions. This point is crucial in seeing that the method can preserve perturbative unitarity.

We show in appendix B that if the infinitesimal transformation:

$$\delta \phi_i = T_i[\phi]$$

(2.8a)

generates a symmetry of $S[\phi]$ then the following transformation generates a symmetry of $\hat{S}[\phi]$:

$$\hat{\delta} \phi_i = \mathcal{E}_{ij}^2 T_j \left[ \phi + \psi[\phi] \right]$$

(2.8b)

Therefore nonlocalization preserves generally distorted versions of any and all continuous symmetries. Another important result derived in appendix B is that the shadow field transforms as follows:

$$\hat{\delta} \psi_i [\phi] = \left( 1 - \mathcal{E}^2 \right)_{ij} T_j \left[ \phi + \psi[\phi] \right] - K_{ij} \left[ \phi + \psi[\phi] \right] \delta T_k \left[ \phi + \psi[\phi] \right] \mathcal{E}_{k\ell} \frac{\delta \hat{S}[\phi]}{\delta \phi_{\ell}}$$

(2.9a)

$$K_{ij}^{-1}[\phi] \equiv O_{ij}^{-1} - \frac{\delta^2 I[\phi]}{\delta \phi_i \delta \phi_j}$$

(2.9b)

A direct consequence of the second term in (2.9a) is that closure of the local symmetry — by which we mean $[\delta_1, \delta_2] = \delta_3$ — only implies on shell closure:

$$[\hat{\delta}_1, \hat{\delta}_2] \phi_i = \hat{\delta}_3 \phi_i + \mathcal{E}_{ij}^2 \Omega_{jk}^{12} \left[ \phi + \psi[\phi] \right] \mathcal{E}_{k\ell} \frac{\delta \hat{S}[\phi]}{\delta \phi_{\ell}}$$

(2.10a)

$$\Omega_{i\ell}^{12}[\phi] \equiv \frac{\delta T_i^1[\phi]}{\delta \phi_j} K_{jk}[\phi] \frac{\delta T_j^2[\phi]}{\delta \phi_k} - \frac{\delta T_i^2[\phi]}{\delta \phi_j} K_{jk}[\phi] \frac{\delta T_j^1[\phi]}{\delta \phi_k}$$

(2.10b)

It follows that enforcing off shell closure generally requires the inclusion of generators which vanish with the field equations. In this sense the nonlocalized symmetry algebra is not only distorted but also somewhat enlarged.
When a field of the local theory does not appear in the transformation functional $T_i[\phi]$ there is no reason it need be endowed with a shadow. The nonlocalization of these fields can be accomplished merely by smearing the kinetic term. An example is provided by the vector potential of QED. The great advantage of this shortcut in QED is that it allows one to solve explicitly for the shadow electron field. This is why a closed form for the action of nonlocal QED could be given [7]. One disadvantage of the shortcut is that the associated nonlocalized field equations possess nonperturbative, higher derivative solutions of the same sort that wreck string theory [8].

Quantization is accomplished through the definition:

$$\langle T^* (O[\phi]) \rangle_\mathcal{E} \equiv \int [D\phi] \mu[\phi] \left( \text{Gauge fixing} \right) O[\phi] \exp \left( i \hat{S}[\phi] \right)$$  \hspace{1cm} (2.11)

The object on the left is the nonlocally regulated vacuum expectation value of the $T^*$-ordered product of an arbitrary operator $O[\phi]$. It has a subscript $\mathcal{E}$ to signify the dependence of nonlocalization upon the choice of smearing operator. The two undefined quantities on the right are the measure factor, $\mu[\phi]$, and the gauge fixing corrections. Both of these are needed to maintain perturbative unitarity in gauge theories; for scalars with nonderivative interactions there is no gauge symmetry to fix and the measure factor can be set to unity.*

The issue of perturbative unitarity was discussed extensively in the treatment of QED [7]. The basic idea is that the Cutkosky rules apply to nonlocal theories whose propagators have poles only where $\mathcal{F} = 0$ and whose interaction vertices are entire functions of $\mathcal{F}$. Since $\hat{S}[\phi]$ meets both requirements this guarantees unitarity on a large space of states which includes unphysical polarizations. If the quantum theory is gauge invariant then these unphysical polarizations decouple and it is also unitary on the physical space of states. We will explain in the next section that it is simplest to gauge fix the local action and then

* Experience with the nonlinear sigma model leads us to expect that scalars with derivative interactions require a nontrivial measure factor in order to maintain perturbative unitarity.
nonlocalize the resulting BRS theory. Since our procedure makes no distinction between invariant actions and BRS gauge-fixed ones it follows that a nonlocalized version of BRS invariance is induced.

If the local action is invariant under a continuous symmetry $\delta$ we have seen that the nonlocalized action will be invariant under the symmetry $\hat{\delta}$ which is given in equation (2.8b). The quantum theory will possess this symmetry if and only if a suitable measure factor can be found such that $\hat{\delta}\left([D\phi]\mu[\phi]\right) = 0$. This condition can be reexpressed as follows using simple matrix manipulations and relation (B.12b):

\begin{equation}
\hat{\delta}\left\{\ln\left(\mu[\phi]\right)\right\} = -\text{Tr}\left\{\frac{\delta\hat{\delta}\phi_i}{\delta\phi_m}\right\}
\end{equation}

\begin{equation}
= -\text{Tr}\left\{\mathcal{E}^{-2}_{ij}\frac{\delta T_j}{\delta\phi_k}\left[\phi + \psi[\phi]\right]K_{k\ell}\left[\phi + \psi[\phi]\right]O^{-1}_{\ell m}\right\}
\end{equation}

In addition the interaction vertices of the measure factor must be entire functions of the operator $\mathcal{F}_{ij}$, and of course they must not spoil the regularization. Such a choice may not exist. In this case one of the local symmetries is potentially anomalous; one must renormalize and take the unregulated limit to be sure. An explicit example of the appearance of such an anomaly in nonlocal regularization has been described by Hand in the chiral Schwinger model [12].

A curiosity of definition (2.11) is that the field functions on its right hand side are hatted while the field operators on the left are not. The reason for this can be seen by taking $O[\phi]$ to be a simple product:

\begin{equation}
O[\phi] \equiv \phi_{i_1}\phi_{i_2}\ldots\phi_{i_n}
\end{equation}

and then acting $\mathcal{F}_{ji_1}$ on both sides of (2.11). The zeroth order result on the left is just the various commutator terms; the right hand side gives a sum over pairings of propagators, plus interaction terms. Correspondence at zeroth order implies that a factor of $\mathcal{E}^{-1}$ resides on each field in the functional integrand.
We emphasize that it is important to eliminate the shadow fields at the classical level. Functionally integrating over them would engender divergences from shadow loops; nor does perturbative unitarity require us to quantize these fields because their propagators contain no poles. Nevertheless, the Feynman rules of $\hat{S}[\phi]$ are very cumbersome because of all the induced interactions while the Feynman rules of $S[\phi, \psi]$ are essentially as simple as those of the local theory. We have therefore chosen to use the simple Feynman rules of $S[\phi, \psi]$ and to enforce the condition that $\psi_i$ obey its classical field equation ($\psi_i = \psi_i[\phi]$) by merely omitting closed loops composed solely of shadow lines.

The resulting nonlocalized Feynman rules are a trivial extension of the local ones. Except for the measure factor the vertices are unchanged, but every leg can now connect either to a smeared propagator:*  

$$\frac{i\mathcal{E}^2}{F + i\epsilon} = -i \int_{1}^{\infty} \frac{d\tau}{\Lambda^2} \exp\left(\frac{\tau F}{\Lambda^2}\right)$$  

or to a shadow propagator:

$$-iO = \frac{i(1 - \mathcal{E}^2)}{F} = -i \int_{0}^{1} \frac{d\tau}{\Lambda^2} \exp\left(\frac{\tau F}{\Lambda^2}\right)$$  

The two sorts of propagators are represented graphically for Yang-Mills theory in figures 1a and 1b. We place a bar across the shadow line because its propagator lacks a pole and so carries no quanta. For this reason all external lines must be unbarred. (This more than compensates the factors of $\mathcal{E}^{-1}$ which arise because we use $O[\hat{\phi}]$ rather than $O[\phi]$ on the right hand side of equation (2.11).) Vertices from the measure factor are represented graphically with a circled “M,” as in figure 2a. Measure vertices can be connected only

* A digression is necessary to comment on the various factors of $i$ which appear in what the reader was assured would be a Euclidean formulation of perturbation theory. From the earliest work on nonlocal regularization [7] it has been convenient to set perturbation theory up in Minkowski space and define loop integrations by formal Wick rotation. This convention will be followed here as well. We stress that it is only a trick to facilitate the analytic continuation. If rigor is desired then the whole thing can be done in Euclidean space where $F$ is a nonpositive operator.
to barred lines. The symmetry factor of any diagram is computed without distinguishing between barred and unbarred lines.

A number of points deserve mention. First, tree order Green’s functions are unchanged except for external line factors which are unity on shell. This follows because every internal line of a tree graph can be either barred or unbarred. Hence it is only the sum of (2.14a) and (2.14b) which enters, and this sum just gives the local propagator. Second, the fact that on-shell tree amplitudes are unchanged by nonlocalization means that $\hat{S}[\phi]$ preserves any and all symmetries of $S[\phi]$ on shell. A direct proof of this fact is given for continuous symmetries in appendix B. Third, all loops contain at least one of the smeared propagators (2.14a) and are therefore convergent in the ultraviolet. Finally, when applying the cutting rules one allows only unbarred lines to be cut because they alone contain poles. This automatically implies unitarity on the extended space of states which includes unphysical polarizations.

When the measure factor is unity nonlocal regularization is no more difficult to use than dimensional regularization. We have shown this by carrying out the full one and two loop renormalization program for scalar $\phi^4$ theory in four spacetime dimensions and for $\phi^3$ theory in six dimensions [13]. We refer the interested reader to this work for a very detailed study of the method in operation. In particular it should be noted that renormalization proceeds in nonlocal regularization as in any other method: the bare parameters are determined as functions of the renormalized ones and the scale $\Lambda$ such that the limit $\Lambda \rightarrow \infty$ is finite for all noncoincident Green’s functions. Even though $\hat{S}[\phi]$ possesses nonlocal interactions its bare parameters are still those of a local theory. Once a smearing operator has been chosen the nonlocal interactions are completely determined by the local ones. We remark also that the method correctly handles overlapping divergences and that power counting works as it would if the scale $\Lambda$ had been a momentum cutoff.
3. The second stage for Yang-Mills

When vector and/or tensor quanta are allowed to interact some form of gauge invariance must be present in order to reconcile Poincaré invariance and perturbative unitarity. For this invariance to continue operating in loops the functional formalism must respect it. This generally requires that the functional measure factor should depend nontrivially upon the fields. The measure factor must also meet the requirements for being an acceptable interaction in the regulated theory. This means that it must have manifest Poincaré invariance, that it can involve only entire functions of the kinetic operator, and that diagrams involving it must be well damped in Euclidean momentum space. If no such measure factor exists then the regulated theory has an anomaly.* For the appearance of such an anomaly in nonlocal regularization we refer the reader to Hand’s study of the chiral Schwinger model [12].

We shall take pure Yang-Mills for the unregulated theory. Its Lagrangian and field strength tensor are:

\[ L = -\frac{1}{4} F_{a\mu\nu} F_{a}^{\mu\nu} \]  
\[ F_{a\mu\nu} \equiv A_{a\nu,\mu} - A_{a\mu,\nu} - gf_{abc} A_{b\mu} A_{c\nu} \]

where \( g \) is the coupling constant and the \( f_{abc} \) are the structure constants. It is invariant under the familiar transformation:

\[ \delta_\theta A_{a\mu} = -\theta_{a,\mu} + gf_{abc} A_{b\mu} \theta_c \]  

Note that the local symmetry algebra closes for all field configurations:

\[ [\delta_{\theta_1}, \delta_{\theta_2}] A_{a\mu} = -\Theta_{a,\mu} + gf_{abc} A_{b\mu} \Theta_c \]

\[ \Theta_a \equiv gf_{abc} \theta_{1b} \theta_{2c} \]

* The anomaly may, however, vanish in the unregulated limit.
To nonlocalize we identify the kinetic operator:

\[ F^{\alpha\beta}_{ab} \equiv \delta_{ab} (\partial^2 \eta^{\alpha\beta} - \partial^\alpha \partial^\beta) \]  

(3.4a)

and the interaction:

\[ I[A] \equiv \int d^D x \left\{ \frac{gf_{abc} A_{\alpha\nu,\mu} A_{\beta}^\mu A_{\gamma}^\nu - 1}{4} g^2 f_{abc} f_{cde} A_{\alpha\mu} A_{\beta\nu} A_{\mu}^\nu A_{\nu}^\gamma \right\} \]  

(3.4b)

Comparison with (2.5) and (2.6) gives the following expansion for the shadow field:

\[ B_{a\alpha}^\alpha [A] = \mathcal{O}_{ab}^{\alpha\beta} \frac{\delta I[A + B]}{\delta A_{b}^\beta} \]

\[ = \mathcal{O}_{ab}^{\alpha\beta} g f_{bcd} \left[ A_{c\beta}^\gamma A_{d,\gamma}^\alpha + A_{c\gamma} A_{d}^\alpha \gamma - 2 A_{c\gamma} A_{d}^\beta \gamma \right] + O(g^2 A^3) \]  

(3.5)

The resulting nonlocalized action follows immediately from (2.5) and (2.7):

\[ \tilde{S}[A] = \frac{1}{2} \int d^D x \left\{ \tilde{A}_{a\alpha} F^{\alpha\beta}_{ab} \tilde{A}_{b\beta} - B_{a\alpha} [A] \mathcal{O}^{-1}_{ab} \delta B_{b\beta} [A] \right\} + I[A + B[A]] \]  

(3.6)

We see from (2.8) that the nonlocalized gauge symmetry is:

\[ \hat{\delta} \theta A_{a}^\alpha = \mathcal{E}^2 \mathcal{O}_{ab}^{\alpha\beta} \left\{ -\theta_{a}^\alpha + g f_{bcd} \left( A_{c\beta} + B_{c\beta} [A] \right) \theta_{d}^\beta \right\} \]

\[ = -\theta_{a}^\alpha + \mathcal{E}^2 \mathcal{O}_{ab}^{\alpha\beta} g f_{bcd} \left( A_{c\beta} + B_{c\beta} [A] \right) \theta_{d}^\beta \]  

(3.7)

As was the case for QED [7], nonlocalizing Yang-Mills results in a gauge transformation which is neither linear nor local. Another point in common with QED is the failure of the symmetry generators to close under commutation, except for field configurations which obey the equations of motion. Comparison with relations (2.9) and (2.10) shows:

\[ [\hat{\delta}_{\theta_1}, \hat{\delta}_{\theta_2}] A_{a}^\alpha = \hat{\delta}_{\Theta} A_{a}^\alpha + \mathcal{E}^2 \mathcal{O}_{ab}^{\alpha\beta} \Omega_{bc\beta\gamma} [A + B[A]] \mathcal{E}^2 \gamma_{cd} \frac{\delta \tilde{S}[A]}{\delta A_{d}^\beta} \]  

(3.8a)

\[ \Omega_{ab}^{\alpha\beta} [A] \equiv g^2 f_{a\mu} f_{b\nu} \left\{ \theta_{1d} K_{c\mu}^{\alpha\beta} [A] \theta_{2f} - \theta_{2d} K_{c\nu}^{\alpha\beta} [A] \theta_{1f} \right\} \]  

(3.8b)

\[ \left( K_{ab}^{\alpha\beta} \right)^{-1} [A] = \mathcal{O}^{-1}_{ab} \alpha^\beta - \frac{\delta^2 I[A]}{\delta A_{a\alpha} \delta A_{b\beta}} \]  

(3.8c)
This means that the space of generators (3.7) is only a subset of a true algebra which includes generators that vanish with the field equations.

The most obvious application of the procedure laid out in section 2 would be to functionally quantize as follows:

\[
\langle T^* \left( O[A] \right) \rangle_G = \int [DA] \mu[A] \left( \text{Gauge Fixing} \right) O[\tilde{A}] \exp \left( i \tilde{S}[A] \right)
\]  

(3.9)

The trivial problem with this approach is that the measure factor defined by equation (2.12) would be singular on account of the unsuppressed, longitudinal components of transformation law (3.7). This could be circumvented by the simple expedient of redefining what we called the “gauge parameter” so as to absorb the unsuppressed terms into the zeroth order piece.

The harder problem is how to fix the gauge. It is here that the nonclosure of \( \hat{\delta} \) has its only significant impact. We might parallel the usual derivation of Faddeev and Popov by introducing unity in the form:

\[
1 = \int [D\theta] \delta \left( \partial \cdot A^\theta_a - f_a \right) \det \left( \frac{\delta \partial \cdot A^\theta_a}{\delta \theta_c} \right)
\]  

(3.10)

where by \( A^\theta \) we mean the transformed vector potential. One would then change variables \( A \rightarrow A' = A^\theta \) and exploit the gauge invariance of \( \tilde{S}[A] \) and \( [DA] \mu[A] \), and the presumed invariance of the determinant, to factor out the functional integration over \( \theta \). The final step would be to functionally average over \( f_a \) with the desired weighting factor, usually a simple Gaussian. Unfortunately the Faddeev-Popov determinant is not invariant owing to the failure of two \( \delta \)-transformations to give another \( \delta \)-transformation. Hence the integration over \( \theta \) does not quite factor out and the naive derivation fails.

This problem is a familiar one from supergravity and the answer is well known [14]. The physically essential symmetry of a gauge-fixed theory is BRS invariance. When the algebra of gauge transformations closes for all field configurations the ghost structure
of the BRS action comes from exponentiating the Faddeev-Popov determinant, and the BRS transformation of the gauge field is just a gauge transformation with the parameter equal to a constant Grassmann number times the ghost field. For an algebra which fails to close off shell one needs to introduce higher ghost terms into both the action and the BRS transformation. There will also generally need to be a correction to the measure factor.

One could probably derive the necessary changes by formally integrating the Faddeev-Popov determinant with respect to \( \theta \), however a much simpler approach is to nonlocalize the BRS Lagrangian directly.

In Feynman gauge the local BRS Lagrangian is:

\[
L_{\text{BRS}} = -\frac{1}{2} A_{\alpha,\mu} A^\alpha_{\nu,\mu} - \bar{\eta}_a \eta^\alpha \eta_{a,\mu} + g f_{abc} \bar{\eta}_a \eta^\beta \eta^\mu A_{b\mu} \eta_c \\
+ g f_{abc} A_{a\nu,\mu} A^\beta_{b\mu} A^\nu_{c} - \frac{1}{4} g^2 f_{abc} f_{cde} A_{a\mu} A_{b\nu} A^\mu_{d} A^\nu_{e}
\]

(3.11)

It is invariant under the following global symmetry transformation:

\[
\delta A_{a\alpha} = \left( \eta_{a,\alpha} - g f_{abc} A_{b\alpha} \eta_c \right) \delta \zeta
\]

(3.12a)

\[
\delta \eta_a = -\frac{1}{2} g f_{abc} \eta_b \eta_c \delta \zeta
\]

(3.12b)

\[
\delta \bar{\eta}_a = -A^\mu_{a,\mu} \delta \zeta
\]

(3.12c)

where \( \zeta \) is a constant anticommuting \( \mathbb{C} \)-number.

To nonlocalize this action we identify the gluon and ghost kinetic operators:

\[
F_{\alpha\beta} = \delta_{ab} \eta^{\alpha\beta} \partial^2
\]

(3.13a)

\[
F_{ab} = \delta_{ab} \partial^2
\]

(3.13b)

Since the indices and derivatives separate we have found it convenient to use a smearing operator with no indices:

\[
\mathcal{E} \equiv \exp \left( \frac{\partial^2}{2\Lambda^2} \right)
\]

(3.14a)

\[
\mathcal{O} \equiv \frac{\partial^2}{\partial^2 - 1}
\]

(3.14b)
The BRS interaction is:

\[ I[A, \eta, \eta] = g f_{abc} \int d^D x \left\{ \eta_a^{\mu} A_{b\mu} \eta_c + A_{a\nu,\mu} A_{b}^{\mu} \eta_{\nu} - \frac{1}{2} g f_{cde} A_{a\mu} A_{b\nu} A_{d}^{\mu} A_{c}^{\nu} \right\} \]  \hspace{1cm} (3.15)

Comparison with (2.5) and (2.6) gives the following expansions for the shadow fields:

\[ B_a^{\alpha}[A, \eta, \eta] = \overline{\mathcal{O}} \frac{\delta I}{\delta A_{a\alpha}} [A + B, \eta + \bar{\psi}, \eta + \psi] \]
\[ = \overline{\mathcal{O}} g f_{abc} [A^\alpha A_c^{\beta, \beta} + A_{b\beta} A_c^{\beta, \alpha} - 2 A_{b\beta} A_c^{\alpha, \beta} - \eta_b^{\alpha} \eta_c] + O(g^2) \]  \hspace{1cm} (3.16a)

\[ \bar{\psi}_a[A, \eta, \eta] = -\overline{\mathcal{O}} \frac{\delta I}{\delta \eta_a} [A + B, \eta + \bar{\psi}, \eta + \psi] \]
\[ = -\overline{\mathcal{O}} g f_{abc} A_{b\alpha} \eta_c^{\alpha} + O(g^2) \]  \hspace{1cm} (3.16b)

\[ \psi_a[A, \eta, \eta] = \overline{\mathcal{O}} \frac{\delta I}{\delta \eta_a} [A + B, \eta + \bar{\psi}, \eta + \psi] \]
\[ = -\overline{\mathcal{O}} g f_{abc} \partial^\alpha (A_{b\alpha} \eta_c) + O(g^2) \]  \hspace{1cm} (3.16c)

To economize space we will henceforth suppress the functional argument list on the shadow fields. From expressions (2.5) and (2.7) we see that the nonlocalized BRS action is:

\[ \hat{S}[A, \eta, \eta] = \int d^D x \left\{ -\frac{1}{2} \hat{A}_{a\alpha,\beta} \hat{A}_a^{\alpha,\beta} - \frac{1}{2} B_a^{\alpha} \overline{\mathcal{O}}^{-1} B_a^{\alpha} - \hat{\eta}_a^{\alpha} \hat{\eta}_{a,\alpha} - \bar{\psi}_a \overline{\mathcal{O}}^{-1} \psi_a \right\} + I[A + B, \eta + \bar{\psi}, \eta + \psi] \]  \hspace{1cm} (3.17)

The nonlocalized BRS symmetry which follows from applying (2.8) to (3.12) is easy to give in terms of the shadow fields:

\[ \hat{\delta} A_{a\alpha} = \overline{\mathcal{E}}^2 \left\{ (\eta_{a,\alpha} + \psi_{a,\alpha}) - g f_{abc} (A_{b\alpha} + B_{c\alpha}) (\eta_c + \psi_c) \right\} \delta \zeta \]  \hspace{1cm} (3.18a)

\[ \hat{\delta} \eta_a = -\frac{1}{2} g f_{abc} \overline{\mathcal{E}}^2 (\eta_b + \psi_b) (\eta_c + \psi_c) \delta \zeta \]  \hspace{1cm} (3.18b)

\[ \hat{\delta} \bar{\eta}_a = -\overline{\mathcal{E}}^2 (A_{a}^{\alpha,\alpha} + B_{a}^{\alpha,\alpha}) \delta \zeta \]  \hspace{1cm} (3.18c)

Because the shadow fields — \( B_{a\alpha}[A, \eta, \eta], \bar{\psi}_a[A, \eta, \eta] \) and \( \psi_a[A, \eta, \eta] \) — depend upon the ghosts not every occurrence of \( \eta_a \) in (3.18a) derives from the replacement \( \theta_a \rightarrow -\eta_a \delta \zeta \) in (3.7). These “extra” terms appear as well in the BRS transformations of supergravity models whose algebras only close on shell [14].
The transformation law (3.18) is inconvenient because the variation of the antighost no longer produces just a longitudinal gluon. This property plays a special role in using BRS invariance to prove decoupling. We therefore fold in a dynamically irrelevant symmetry of the class described in equations (B.16) and (B.17):

\[
\hat{\delta}^{0} A_{a\alpha} = (1 - E^2) \eta_{a,\mu} \delta \zeta - E^2 \psi_{a,\mu} \delta \zeta \quad (3.19a)
\]

\[
\hat{\delta}^{0} \eta_{a} = 0 \quad (3.19b)
\]

\[
\hat{\delta}^{0} \eta_{a} = -(1 - E^2) A_{a,\mu} \delta \zeta + E^2 B_{a,\mu} \delta \zeta \quad (3.19c)
\]

Combining the two gives a physically equivalent but more convenient transformation:

\[
\hat{\delta}^{1} A_{a\alpha} = \left\{ \eta_{a,\alpha} - g f_{abc} E^2 (A_{b\alpha} + B_{c\alpha}) (\eta_{c} + \psi_{c}) \right\} \delta \zeta \quad (3.20a)
\]

\[
\hat{\delta}^{1} \eta_{a} = -\frac{1}{2} g f_{abc} E^2 (\eta_{b} + \psi_{b}) (\eta_{c} + \psi_{c}) \delta \zeta \quad (3.20b)
\]

\[
\hat{\delta}^{1} \eta_{a} = -A_{a,\mu} \delta \zeta \quad (3.20c)
\]

That such a form can be reached also follows from the general argument of de Wit and van Holten [14].

Since the formalism of section 2 applies to gauge fixed theories as well as to invariant ones we can read off the rule for functional quantization directly from expression (2.11):

\[
\langle T^{*} (O[A, \bar{\eta}, \eta]) \rangle_{E} \equiv \int [DA][D\bar{\eta}][D\eta] \mu[A, \bar{\eta}, \eta] O[A, \bar{\eta}, \eta] \exp \left( i \hat{S}[A, \bar{\eta}, \eta] \right) \quad (3.21)
\]

Note, however, that it should be the transformation \( \hat{\delta}^{1} \), not \( \hat{\delta} \), which goes into expression (2.12) to partially determine the measure factor. Of course one can freely modify \( \mu[A, \bar{\eta}, \eta] \) by terms that are invariant under \( \hat{\delta}^{1} \) but with a natural sort of minimality condition the answer is:

\[
\ln \left( \mu[A, \bar{\eta}, \eta] \right) = -\frac{i}{2} g^2 f_{acd} f_{bcd} \int d^{D} x \ A_{a\mu} \mathcal{M} A_{b}^{\mu} + O(g^3) \quad (3.22a)
\]
\[ M \equiv \frac{1}{2^D \pi^{D/2}} \int_0^1 d\tau \frac{\Lambda^{D-2}}{(\tau + 1)^{D/2}} \exp\left(\frac{\tau}{\tau + 1} \frac{\partial^2}{\Lambda^2}\right) \times \left\{ -\frac{2}{\tau + 1} - (D - 1) + 2(D - 1) \frac{\tau}{\tau + 1} \right\} \] 

(3.22b)

There is no proof that the requisite higher order terms exist, although this is what we expect because the local theory is free of anomalies.

The presumed existence of a suitable measure factor implies Slavnov-Taylor identities in the usual way. Consider, for example, the following choice for the operator \( O[A, \eta, \eta] \) of (3.21):

\[ O[A, \eta, \eta] = \eta_a(x) A_{\mu}^{\nu}(y) \] 

(3.23)

Of course the functional integral vanishes by ghost number counting, but a BRS transformation gives a nontrivial relation between Green’s functions:

\[ \delta^1 \left( \eta_a(x) A_{\mu}^{\nu}(y) \right) = -A_{\mu}^{\nu}(x) \frac{\delta \tilde{S}}{\delta \eta_a(y)} \delta \zeta \] 

(3.24)

Upon functionally integrating by parts we infer the transversality of the vacuum polarization at one loop.* This result will be explicitly verified in the next section.

4. The gluon self energy at one loop

The Feynman rules can be read off from the functional formalism (3.21) using relations (3.17) and (3.22). The barred and unbarred gluon propagators are represented graphically in figure 1a and have the following forms:

\[ \frac{-i\delta_{\alpha\beta}}{p^2 - i\epsilon} \exp\left(\frac{-p^2}{\Lambda^2}\right) = \frac{-i\delta_{\alpha\beta}}{p^2 - i\epsilon} \int_0^\infty \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2}{\Lambda^2}\right) \] 

(4.1a)

\[ \frac{-i\delta_{\alpha\beta}}{p^2 - i\epsilon} \left\{ 1 - \exp\left(-\frac{p^2}{\Lambda^2}\right) \right\} = \frac{-i\delta_{\alpha\beta}}{p^2 - i\epsilon} \int_0^1 \frac{d\tau}{\Lambda^2} \exp\left(-\tau \frac{p^2}{\Lambda^2}\right) \] 

(4.1b)

*Although we expect transversality to persist to all orders we have no proof of this for lack of knowledge about the higher order terms in the measure factor. In particular we have not been able to decide whether or not the measure depends upon the ghosts.
The barred and unbarred ghost propagators are shown in figure 1b and can be expressed as follows:

\[-i \delta_{ab} \frac{p^2}{p^2 - i \epsilon} \exp \left( -\frac{p^2}{\Lambda^2} \right) = -i \delta_{ab} \int_{1}^{\infty} \frac{d\tau}{\Lambda^2} \exp \left( -\tau \frac{p^2}{\Lambda^2} \right) (4.2a)\]

\[-i \delta_{ab} \left\{ 1 - \exp \left( -\frac{p^2}{\Lambda^2} \right) \right\} = -i \delta_{ab} \int_{0}^{1} \frac{d\tau}{\Lambda^2} \exp \left( -\tau \frac{p^2}{\Lambda^2} \right) (4.2b)\]

The obvious advantage of this gauge is that the tensor and gauge indices decouple from the nonlocal factors. The 2-point measure vertex is represented in figure 2a and has the form:

\[M_{ab}^{\alpha \beta}(p) \equiv -\frac{f_{acd} f_{bcd} \eta^{\alpha \beta}}{2^D (\pi)^{D/2}} \int_{0}^{1} d\tau \frac{\Lambda^{D-2}}{(\tau + 1)^{D/2}} \exp \left( -\frac{\tau}{\tau + 1} \frac{p^2}{\Lambda^2} \right) \left[ (D - 1) - 2 \frac{D - 2}{\tau + 1} \right] (4.3)\]

The local 3-point vertices are shown in figure 2b and can be expressed as follows:

\[I_{abc}^{\alpha \beta \gamma}(p_1, p_2, p_3) \equiv -i f_{abc} \left\{ \eta^{\alpha \beta} (p_2 - p_1)^{\gamma} + \eta^{\beta \gamma} (p_3 - p_2)^{\alpha} + \eta^{\gamma \alpha} (p_1 - p_3)^{\beta} \right\} (4.4a)\]

\[I_{abc}^{\alpha}(p_1, p_2, p_3) \equiv i f_{abc} p_3^\alpha (4.4b)\]

The local 4-point vertex is depicted in figure 2c and has the form:

\[I_{abcd}^{\alpha \beta \gamma \delta}(p_1, p_2, p_3, p_4) \equiv -i f_{abc} f_{dcd} (\eta^{\alpha \gamma} \eta^{\beta \delta} - \eta^{\gamma \beta} \eta^{\delta \alpha})
+ f_{ace} f_{dbe} (\eta^{\alpha \delta} \eta^{\gamma \beta} - \eta^{\delta \gamma} \eta^{\alpha \beta}) + f_{ade} f_{bce} (\eta^{\alpha \beta} \eta^{\delta \gamma} - \eta^{\beta \delta} \eta^{\gamma \alpha}) (4.5)\]

The only thing missing is the higher measure vertices. Since the N-point measure vertex comes in at order \(g^N\) it is clear that only for \(N = 2\) can it contribute to the (order \(g^2\)) one loop vacuum polarization.

The one loop vacuum polarization can be naturally divided into four parts. These correspond to figures 3-6 and we shall term them \(A, B, C\) and \(D\) respectively:

\[\Pi_{ab}^{\alpha \beta} = A_{ab}^{\alpha \beta} + B_{ab}^{\alpha \beta} + C_{ab}^{\alpha \beta} + D_{ab}^{\alpha \beta} (4.6)\]
The three barring variations depicted in figure 3 come from the 3-gluon vertex. They can be summed up to give the following:

\[ iA_{ab}^{\alpha\beta}(p) = \frac{1}{2} \int_{R_1^2} d^2\tau \int \frac{d^Dk}{(2\pi)^D} \ \operatorname{ig} I_{acd}^{\alpha\gamma\delta}(p, -k, -q) \left\{ -i\delta_{ce} \eta_{\gamma\rho} \exp\left(-\tau_1 \frac{k^2}{\Lambda^2}\right) \right\} \]

\[ \times \operatorname{ig} I_{bef}^{\beta\rho\sigma}(-p, k, q) \left\{ -i\delta_{df} \eta_{\delta\sigma} \exp\left(-\tau_2 \frac{q^2}{\Lambda^2}\right) \right\} (4.7a) \]

\[ = \frac{ig^2 f_{acd} f_{bed}}{2^D \pi^{D/2}} \int_{R_1^2} \frac{d^2\tau}{(\tau_1 + \tau_2)^{D/2}} \Lambda^{D-4} \exp\left(-\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2}\right) \left\{ + \frac{3}{2} (D - 1) \frac{\Lambda^2}{\tau_1 + \tau_2} \eta^{\alpha\beta} \right\} \]

\[ - \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \left[ p^2 \eta^{\alpha\beta} + (2D - 3)p^\alpha p^\beta \right] + \left[ \frac{5}{2} p^2 \eta^{\alpha\beta} + \frac{1}{2} (D - 6)p^\alpha p^\beta \right] \left(4.7b\right) \]

The symbol \( R_1^2 \) in this expression denotes the standard 2-parameter region of integration at one loop, namely the first quadrant minus the square \( 0 \leq \tau_1, \tau_2 < 1 \). A very similar contribution comes from the three ghost loops of figure 4:

\[ iB_{ab}^{\alpha\beta}(p) = \int_{R_1^2} d^2\tau \int \frac{d^Dk}{(2\pi)^D} \ \operatorname{ig} I_{acd}^{\alpha}(p, -k, -q) \left\{ -i\delta_{ce} \exp\left(-\tau_1 \frac{k^2}{\Lambda^2}\right) \right\} \]

\[ \times \operatorname{ig} I_{bef}^{\beta}(p, k, q) \left\{ -i\delta_{df} \exp\left(-\tau_2 \frac{q^2}{\Lambda^2}\right) \right\} (4.8a) \]

\[ = \frac{ig^2 f_{acd} f_{bed}}{2^D \pi^{D/2}} \int_{R_1^2} \frac{d^2\tau}{(\tau_1 + \tau_2)^{D/2}} \Lambda^{D-4} \exp\left(-\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2}\right) \left\{ - \frac{1}{2} \frac{\Lambda^2}{\tau_1 + \tau_2} \eta^{\alpha\beta} + \frac{\tau_1 \tau_2}{\tau_1 + \tau_2} p^\alpha p^\beta \right\} \]

\[ (4.8b) \]

The diagram of figure 5 comes from the 4-gluon vertex:

\[ iC_{ab}^{\alpha\beta}(p) = \int \frac{d^Dk}{(2\pi)^D} \ \operatorname{ig}^2 I_{abcd}^{\alpha\beta\gamma\delta}(p, -p, -k, k) \left( \frac{-i\eta_{\gamma\delta}}{k^2 - i\epsilon} \right) \exp\left(-\frac{k^2}{\Lambda^2}\right) \]

\[ = -\frac{ig^2 f_{acd} f_{bed}}{2^D \pi^{D/2}} \eta^{\alpha\beta} 2 \left( \frac{D - 1}{D - 2} \right) \Lambda^{D-2} \]

\[ (4.9a) \]

\[ (4.9b) \]

This contribution would vanish in dimensional regularization; here it plays an essential role in canceling the quadratic divergence. The same comments pertain as well to the measure factor contribution which comprises figure 6:

\[ iD_{ab}^{\alpha\beta}(p) = -\frac{ig^2 f_{acd} f_{bed}}{2^D \pi^{D/2}} \eta^{\alpha\beta} \int_0^1 \frac{d\tau}{(\tau + 1)^{D/2}} \Lambda^{D-2} \exp\left(-\frac{\tau}{\tau + 1} \frac{p^2}{\Lambda^2}\right) \left\{ (D - 1) - \frac{2}{2^D - 1} \right\} \]

\[ (4.10) \]
To check that the vacuum polarization is transverse first add the longitudinal parts of A and B. Now reduce the parameter integral to which the sum is proportional:

\[ \int_{R_1^2} \frac{d^2 \tau}{(\tau_1 + \tau_2)^{D/2}} \exp \left( -\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2} \right) \times \left\{ \left( \frac{3}{2} D - 2 \right) \frac{1}{\tau_1 + \tau_2} - (2D - 3) \frac{\tau_1 \tau_2}{(\tau_1 + \tau_2)^2} \frac{p^2}{\Lambda^2} + \frac{1}{2} (D - 1) \frac{p^2}{\Lambda^2} \right\} \]

\[ = -(D - 1) \int_{R_1^2} \frac{d^2 \tau}{(\tau_1 + \tau_2)^{D/2}} \exp \left( -\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2} \right) \left\{ -\frac{1}{2} \frac{D}{\tau_1 + \tau_2} + \frac{\tau_1 \tau_2}{(\tau_1 + \tau_2)^2} \frac{p^2}{\Lambda^2} - \frac{1}{2} \frac{p^2}{\Lambda^2} \right\} \]

\[ - (D - 2) \int_{R_1^2} \frac{d^2 \tau}{(\tau_1 + \tau_2)^{D/2}} \exp \left( -\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2} \right) \left\{ \frac{1}{2} \frac{D - 2}{\tau_1 + \tau_2} + \frac{\tau_1 \tau_2}{(\tau_1 + \tau_2)^2} \frac{p^2}{\Lambda^2} \right\} \] (4.11a)

\[ = -(D - 1) \int_{R_1^2} d^2 \tau \left\{ \frac{1}{2} \left( \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_2} \right) \right\} \left( \tau_1 + \tau_2 \right)^{-D/2} \exp \left( -\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2} \right) \]

\[ + \left\{ \int_0^1 dx \int_1^\infty dy + \int_1^\infty dx \int_1^\infty dy \right\} \frac{D - 2}{(x + 1)^{D/2 + 1}} \frac{\partial}{\partial y} \left( y^{1-D/2} \exp \left( -\frac{xy}{x + 1} \frac{p^2}{\Lambda^2} \right) \right) \] (4.11b)

\[ = 2 \left( \frac{D - 1}{D - 2} \right) + \int_0^1 d\tau \left( \tau + 1 \right)^{-D/2} \exp \left( -\frac{\tau}{\tau + 1} \frac{p^2}{\Lambda^2} \right) \left\{ (D - 1) - 2 \left( \frac{D - 2}{\tau + 1} \right) \right\} \] (4.11c)

These final terms serve to cancel the longitudinal contributions from C and D, proving transversality.

The integral identity we have just established can be used as well to simplify the transverse part. The result is:

\[ \Pi_{ab}^{\alpha \beta}(p) = \frac{g^2}{2^D \pi^{D/2}} f_{acd} f_{bcd} (p^2 \eta_{\alpha \beta} - p^\alpha p^\beta) \Pi(p^2) \] (4.12a)

\[ \Pi(p^2) = \int_{R_1^2} d^2 \tau \left( \frac{\Lambda^{D-4}}{(\tau_1 + \tau_2)^{D/2}} \right) \exp \left( -\frac{\tau_1 \tau_2}{\tau_1 + \tau_2} \frac{p^2}{\Lambda^2} \right) \left\{ 2(D - 2) \frac{\tau_1 \tau_2}{(\tau_1 + \tau_2)^2} - \frac{1}{2} (D - 6) \right\} \]

\[ = 2 \int_0^{1/2} dx \left( 2 - \frac{D - 2}{x} \frac{p^2}{\Lambda^2} \right) \left[ x(1-x)p^2 \right]^{\frac{D-2}{2}} \left\{ 2(D - 2) x (1 - x) - \frac{1}{2} (D - 6) \right\} \] (4.12b)

where the incomplete gamma function is:

\[ \Gamma(n, z) \equiv \int_z^\infty dt \ t^{n-1} e^{-t} \] (4.13)
The dimensionally regulated result is obtained by the replacement:

$$\Gamma\left(2 - \frac{D}{2}, x\frac{P^2}{\Lambda^2}\right) \longrightarrow \Gamma(2 - \frac{D}{2})$$ (4.14)

This is an important check on accuracy. The same relation occurred for the one loop vacuum polarization of nonlocal QED [7]; and the general rule relating the two methods has been inferred for any diagram at any order in nonlocal scalar field theory [13]. We conjecture the same rule applies for nonlocal Yang-Mills as well. If true, this defines the general measure factor and the formalism is complete.

Note that even if our conjecture is true nonlocal regularization would still differ from the dimensional scheme. The latter regulates by deforming the arguments of divergent Gamma functions while the former works by trading these Gamma functions in for incomplete Gamma functions evaluated at predictable functions of the Feynman parameters. One obvious distinction is that the replacement (4.14) discards gauge invariant divergences for $D > 4$. Dimensional regularization is sensitive only to logarithmic divergences whereas nonlocal regularization sees everything.

5. Acceptable nonlocalization as noncanonical quantization

The first stage procedure of section 2 can be used to regulate any theory. The result is basically a systematized version of proper time regularization. At this level nonlocalization functions like any other regulator: the original theory is changed to make it finite, but at the cost of introducing a physically unacceptable property which persists as long as the regulator is on. In order to obtain a physically acceptable theory one must take the unregulated limit after the appropriate renormalizations have been performed.

For the case of nonlocal regularization the problem is that perturbative unitarity is violated by loops which contain vector and tensor quanta. The first stage contrives to enforce tree order decoupling through the agency of a peculiar nonlinear gauge invariance.
Since the naive functional measure does not preserve this symmetry decoupling must fail in loop amplitudes. If we are only interested in developing a noninvariant regulator this is as irrelevant as the ghost states of the Pauli-Villars method; all will be set right in the unregulated limit. However the situation changes radically if the second stage can be carried out — and we have seen some evidence that it can be for Yang-Mills. In this case the regulated theory is not only finite but also Poincaré invariant and perturbatively unitary. It has all the attributes of a fundamental theory; why then must one take the unregulated limit?

One answer is that at fixed loop order our nonlocal amplitudes violate the bounds implied by partial wave unitarity — though of course the trees are those of the local theory. This is perhaps not as damning as it seems because a similar breakdown occurs even at tree order in string theory [15]. The problem there has not been much studied but the feeling is that it can be resolved by an ingenious resummation of perturbation theory [16]. Pending an argument against the existence of the same mechanism in nonlocal field theory it is obviously premature to fret overmuch about the violation of nonperturbative bounds by finite order perturbation theory at extremely high energies.

The same considerations apply to the perturbative, off shell violations of causality which appear as quantum effects in nonlocal field theory. Off shell acausality occurs even at tree order in string field theory [8]. The effect has received almost no attention because it does not give acausal poles in the perturbative S-matrix, because it is extremely small (in nonlocal field theory the strength of a signal propagated an invariant interval \( \ell^2 > 0 \) outside the lightcone would be suppressed by factors of \( \exp[-\ell^2 \Lambda^2] \)), and because essentially mystical considerations have encouraged the belief that very small length scales cannot be probed in string theory. In the rarified atmosphere of critical thinking which now prevails one might advance any of these points with equal smugness as a reason for ignoring the acausality of nonlocal field theory. Of greater import to us is the fact that
it is only through quantum effects that the acausality of nonlocal field theory manifests itself. If this acausality could be viewed as the consequence of a noncanonical quantization procedure — and we will shortly argue that it can be — then it might be both very small and logically consistent. We note in passing the observation of reference [8] that Planck scale acausality might actually be desirable in providing an explanation of the horizon problem of cosmology [17].

A more serious problem would seem to be the nonperturbative instabilities which tend to occur in any fundamentally nonlocal theory. The origin of these instabilities is explained at great length in reference [8]. Basically they are due to the extra canonical degrees of freedom associated with higher time derivatives. It is simple to show that if the Lagrangian contains up to \( N \) time derivatives then the associated Hamiltonian is linear in \( N - 1 \) of the corresponding canonical variables. A fully nonlocal Lagrangian whose interactions are entire functions of the derivative operator can be viewed as an \( N \)th derivative Lagrangian in the limit that \( N \) becomes infinite. Such a nonlocal theory will then inherit the instabilities of its higher derivative limit sequences provided the time dependence of the extra solutions does not become arbitrarily choppy in the limit. The condition for the extra solutions to have smooth limits is that no invertible field redefinition should exist which takes the nonlocal field equations into local ones. That string theory obeys this condition can be inferred from its tree order S-matrix; that the nonlocal actions produced by the construction of section 2 do not obey this condition follows from the fact that their tree amplitudes agree with those of their local ancestors.

In fact we show in appendix A that the solutions of the nonlocal field equations are in one-to-one correspondence with those of the original theory. The relation is very simple:

\[
\phi^{\text{nonloc}}_i = E_{ij}^{2} \phi^{\text{loc}}_j.
\]

Comparison of the actions shows that for solutions we also have \( S[\phi] = \hat{S}[E^2 \phi] \).
stress that these are nonperturbative, albeit classical, results. There are no extra classical solutions. With no extra solutions there is no necessary problem with either stability — although the theory might still be unstable — or with a finite initial value formalism.

Since the on shell amplitudes of our nonlocalized action differ from those of its local parent only at loop order it must be that the nonlocal field theory can be viewed as a highly noncanonical quantization of fields which obey the local equations of motion. The logical possibility of alternate quantization schemes has long been realized but they have attracted little attention because they tend not to result in unitary time evolution. We stress that if the second stage goes through then our construction does maintain unitarity, at least perturbatively.

We close with a discussion of another reason for the lack of interest in noncanonical quantization, namely the spectacular phenomenological success obtained by applying the canonical method to familiar gauge theories. The leftover interaction in all this is gravity. Indeed, one might regard the last thirty years of effort in quantum gravity as a sort of proof by exhaustion of a fundamental incompatibility between general relativity and quantum mechanics. It is usually assumed that the former must be changed to accommodate the latter; we wish here to propose nonlocalization as a specific mechanism for achieving the converse.

This is far too radical a proposal to be fully realized. For one thing it is unaesthetically arbitrary; one can nonlocalize with any smearing operator provided it is both analytic and convergent. Another problem is the background dependence which seems to infiltrate the construction of section 2 from the distinction it makes between the free and interacting parts of the action. Finally, there is the issue of unification. Even if quantum gravity should owe its ultraviolet finiteness to nonlocality we still expect it to form part of a larger structure that includes the other forces in some way more fundamental than as optional matter couplings.
These are all valid criticisms of the proposal — and we have thought it good to raise them in advance of anyone else. Nevertheless we have been unable to dismiss the notion that certain forms of nonlocalization might, when applied to special theories, yield a nonperturbatively acceptable theory. Nor have we been able to rid ourselves of the feeling that noncanonical quantization, as outlined here, might play a decisive role in the quantum theory of gravitation.

6. Conclusions

The major result of this paper is the procedure of section 2 for constructing a regulated, nonlocal action $\widehat{S}[\phi]$ from any local action $S[\phi]$ which can be formulated perturbatively. The nonlocalized theory can either be thought of as a regularization of the local theory or as a candidate fundamental model in its own right. Though $\widehat{S}[\phi]$ contains an infinite tower of induced interactions these can be completely subsumed into an auxiliary field line which couples using only the local interaction. Loop calculations in this formulation of the theory are no more difficult than with dimensional regularization but the great thing about nonlocalization is its potential for preserving symmetries. We prove in appendix B that $\widehat{S}[\phi]$ preserves suitably nonlocalized versions of any and all continuous symmetries of $S[\phi]$.

Whether or not these symmetries are preserved in path integral quantization depends upon the existence of an otherwise suitable measure factor $\mu[\phi]$ having the property that $[D\phi]\mu[\phi]$ is invariant. One must not expect that all symmetries can be preserved in all theories; for example, gauge invariance is broken in the chiral Schwinger model [12]. However, some symmetries are certainly preserved in some theories; for example, the required measure factor has been shown to exist for nonlocal QED [7]. No conclusion has been reached yet for nonlocal Yang-Mills, although we did compute both the invariant measure and the BRS measure at order $g^2$. 
Quantization also requires that the gauge be fixed. A peculiarity of the nonlocalized gauge symmetry is that its generators typically fail to close except on shell. This means that the Faddeev-Popov determinant is not gauge invariant, thus complicating the usual functional gauge fixing procedure. The simplest solution to this problem is to nonlocalize the BRS theory directly and then functionally quantize using a BRS invariant measure. When this is done one finds both higher ghost interactions and higher ghost contamination in the nonlocalized BRS transformation. Similar effects have been noted with formulations of local supersymmetry which fail to close off shell [14].

Our partial determination of the measure factor was sufficient to evaluate the vacuum polarization at one loop, which we did in section 4. As was the case with nonlocal QED [7], the result bears a striking resemblance to that obtained from dimensional regularization. One merely replaces the divergent Gamma function of the latter method with an incomplete Gamma function whose argument depends upon the Schwinger parameters and ratios of momentum invariants to the regularization parameter $\Lambda^2$. For the vacuum polarization the necessary replacement is given in equation (4.14). In another work we have presented the general relation for any amplitude at any order in scalar field theory [13]. If this same rule applies as well to nonlocal Yang-Mills then the measure factor is determined.

The construction of section 2 can be applied to supersymmetric gauge theories and to supergravity. Since it preserves the on shell tree amplitudes it will necessarily preserve global supersymmetry (untouched since it is linearly realized) and some sort of nonlocal and nonlinear generalization of gauge invariance and local supersymmetry. If a supersymmetric measure factor can be found as well then we will have produced the long-sought gauge and supersymmetric invariant regulator.

Perhaps the most stimulating feature of our method is that it leaves unchanged the tree amplitudes of the local theory. A consequence is that one ought to be able to view nonlocal regularization as a noncanonical quantization of fields obeying the local equa-
tions of motion. The connection can be made explicit and we have sketched it in section 5. Though the idea is obviously incomplete at this point, we feel it may play a role in reconciling gravitation and quantum mechanics.

Appendix A: Classical Solutions

Here we prove a series of theorems concerning classical solutions to the Euler-Lagrange equations associated with the local action, $S[\chi]$ — for the form of which see (2.1) — the auxiliary action, $S[\phi, \psi]$ — see (2.5) — and the nonlocalized action, $\tilde{S}[\phi]$ — see (2.7).

**Theorem A1:** The shadow fields $\psi_i[\phi]$ can be expressed as follows:

$$\psi_i[\phi] = -\left(\frac{\mathcal{E}^2 - 1}{\mathcal{E}^2}\right)_{ij} \phi_j + \mathcal{O}_{ij} \frac{\delta \tilde{S}[\phi]}{\delta \phi_j}$$  \hspace{1cm} (A.1)

To see this consider the variations of $S[\phi, \psi]$:

$$\frac{\delta S[\phi, \psi]}{\delta \phi_i} = \mathcal{E}^2 F_{jk} \phi_k + \frac{\delta I[\phi + \psi]}{\delta \phi_i}$$ \hspace{1cm} (A.2a)

$$\frac{\delta S[\phi, \psi]}{\delta \psi_i} = -\mathcal{O}_{ij}^{-1} \psi_j + \frac{\delta I[\phi + \psi]}{\delta \phi_i}$$ \hspace{1cm} (A.2b)

Subtracting (A.2b) from (A.2a) and multiplying by the invertible operator $\mathcal{O}$ gives:

$$\psi_i = -\left(\frac{\mathcal{E}^2 - 1}{\mathcal{E}^2}\right)_{ij} \phi_j + \mathcal{O}_{ij} \left\{ \frac{\delta S[\phi, \psi]}{\delta \phi_j} - \frac{\delta S[\phi, \psi]}{\delta \psi_j} \right\}$$ \hspace{1cm} (A.3)

We can neglect the rightmost term because $\psi_i[\phi]$ is defined to enforce the vanishing of $\psi_i$ field equations. Since any such extra terms can be absorbed into a field redefinition —
which only changes the still undetermined measure factor — it follows that the first stage of nonlocalization is unique.

**Theorem A2:** If the fields $\phi_i$ and $\psi_i$ obey the Euler-Lagrange equations of $S[\phi, \psi]$ then the field $\chi_i = \phi_i + \psi_i$ obeys the Euler-Lagrange equations of $S[\chi]$.

Multiplying (A.2a) by $\mathcal{E}^2$, adding it to $(1 - \mathcal{E}^2)$ times (A.2b), and setting the sum to zero gives the equation:

$$\mathcal{F}_{ij}(\phi_i + \psi_i) + \frac{\delta I[\phi + \psi]}{\delta \phi_i} = 0 \quad (A.5)$$

Substituting $\chi_i = \phi_i + \psi_i$ proves the theorem.

**Theorem A3:** If $\chi_i$ obeys the Euler-Lagrange equation of $S[\chi]$ then the following fields:

$$\phi_i = \mathcal{E}^2_{ij} \chi_j \quad (A.6a)$$
$$\psi_i = (1 - \mathcal{E}^2)_{ij} \chi_j \quad (A.6b)$$

obey the Euler-Lagrange equations of $S[\phi, \psi]$.

By adding (A.6a) and (A.6b) we see that $\phi_i + \psi_i = \chi_i$. Substitution into (A.2a) gives:

$$\frac{\delta S}{\delta \phi_i} \left[ \mathcal{E}^2 \chi, (1 - \mathcal{E}^2) \chi \right] = \mathcal{F}_{ij} \chi_j + \frac{\delta I[\chi]}{\delta \chi_i} \quad (A.7a)$$

Doing the same for (A.2b) gives:

$$\frac{\delta S}{\delta \psi_i} \left[ \mathcal{E}^2 \chi, (1 - \mathcal{E}^2) \chi \right] = \mathcal{F}_{ij} \chi_j + \frac{\delta I[\chi]}{\delta \chi_i} \quad (A.7b)$$

The result follows from imposing the $\chi_i$ field equations.

**Theorem A4:** If $\phi_i$ obeys the Euler-Lagrange equation of $\tilde{S}[]$ then $\chi_i = \phi_i + \psi_i[\phi]$ obeys the Euler-Lagrange equation of $S[\chi]$.

First note that $\phi_i$ and $\psi_i = \psi_i[\phi]$ enforce the vanishing of (A.2a) — by relation (A.4) — and of (A.2b) — from the definition of $\psi_i[\phi]$. The result follows from theorem A2.

**Theorem A5:** If $\chi_i$ obeys the Euler-Lagrange equation of $S[\chi]$ then the field $\phi_i$, defined by (A.6a), obeys the Euler-Lagrange equation of $\tilde{S}[]$. 
This result follows from (A.4) and theorem A3. Theorems A4 and A5 together prove the very important fact that the solutions of the nonlocalized Euler-Lagrange equations are in one-to-one correspondence with those of the local action. It follows that the nonlocalized theory is free of the sorts of higher derivative solutions that wreck string field theory. Note also that the smearing operator in (A.6a) generally smooths out small scale phenomena. Therefore it is entirely possible for a singular solution of the local theory to give a nonsingular solution in the nonlocalized theory.

Appendix B: Classical Symmetries

Here we prove a series of theorems concerning classical symmetries of the local action, \( S[\chi] \) — for the form of which see (2.1) — the auxiliary action, \( S[\phi, \psi] \) — see (2.5) — and the nonlocalized action, \( \tilde{S}[\phi] \) — see (2.7).

**Theorem B1:** If \( S[\phi] \) is invariant under the infinitesimal transformation:

\[
\delta \phi_i = T_i[\phi] \tag{B.1}
\]

then the following transformation is a symmetry of \( S[\phi, \psi] \):

\[
\Delta \phi_i \equiv \mathcal{E}_{ij}^2 T_j[\phi + \psi] \tag{B.2a}
\]

\[
\Delta \psi_i \equiv (1 - \mathcal{E}^2)_{ij} T_j[\phi + \psi] \tag{B.2b}
\]

First note that by adding (B.2a) and (B.2b) we find:

\[
\Delta (\phi + \psi)_i = T_i[\phi + \psi] \tag{B.3}
\]

Now from the definition of \( S[\phi, \psi] \) — expression (2.5) — it follows that:

\[
\Delta S[\phi, \psi] = \int d^Dx \left\{ \left( \phi_i + \psi_i \right) \mathcal{F}_{ij} T_j[\phi + \psi] + \frac{\delta I[\phi + \psi]}{\delta \phi_i} T_i[\phi + \psi] \right\} = \left( \delta S \right)[\phi + \psi] \tag{B.4}
\]
Therefore we see that $\Delta S[\phi,\psi] = 0$ as a consequence of the assumed relation $\delta S[\phi] = 0$. A significant corollary is that $[\delta_1,\delta_2] = \delta_3$ implies $[\Delta_1,\Delta_2] = \Delta_3$.

Theorem B2: If $S[\phi,\psi]$ is invariant under:

$$
\Delta \phi_i = \tau_i[\phi,\psi] \tag{B.5a}
$$

$$
\Delta \psi_i = \sigma_i[\phi,\psi] \tag{B.5b}
$$

then the following transformation is a symmetry of $\hat{S}[\phi]$:

$$
\hat{\delta} \phi_i = \tau_i[\phi,\psi[\phi]] \tag{B.6}
$$

From definition (2.7) and the fact that $\psi_i[\phi]$ is constructed to enforce equation (A.2b) we see that:

$$
\hat{\delta} S[\phi] = \frac{\delta S[\phi,\psi[\phi]]}{\delta \phi_i} \hat{\delta} \phi_i + \frac{\delta S[\phi,\psi[\phi]]}{\delta \psi_j} \frac{\delta \psi_j[\phi]}{\delta \phi_i} \hat{\delta} \phi_i \tag{B.7}
$$

$$
= (\Delta S)[\phi,\psi[\phi]]
$$

Note that one generally obtains a different result from applying (B.6) to $\psi_i[\phi]$ than by first transforming $\psi$ according to (B.5b) and then setting $\psi_i = \psi_i[\phi]$:

$$
\hat{\delta} \psi_i[\phi] = \frac{\delta \psi_i[\phi]}{\delta \phi_j} \tau_j[\phi,\psi[\phi]] \tag{B.8a}
$$

$$
\neq \sigma_i[\phi,\psi[\phi]] \tag{B.8b}
$$

The equality of (B.8a) and (B.8.b) was unnecessary in proving theorem B2 because $\psi_i[\phi]$ is defined to make the variation of $S[\phi,\psi]$ with respect to $\psi_i$ vanish. The same is not the case for closure: a simple exercise shows that $[\Delta_1,\Delta_2] = \Delta_3$ implies $[\hat{\delta}_1,\hat{\delta}_2] = \hat{\delta}_3$ if and only if $\hat{\delta} \psi_i[\phi] = \sigma_i[\phi,\psi[\phi]]$.

Theorem B3: If $S[\phi]$ is invariant under (B.1) then $\hat{S}[\phi]$ is invariant under:

$$
\hat{\delta} \phi_i = \mathcal{E}_{ij}^2 T_j[\phi + \psi[\phi]] \tag{B.9}
$$
The same transformation takes the shadow field to:

$$\hat{\delta \psi}_i[\phi] = \left(1 - \mathcal{E}^2\right)_{ij} T_j \left[ \phi + \psi[\phi] \right] - K_{ij} \left[ \phi + \psi[\phi] \right] \frac{\delta T_k}{\delta \phi_j} \left[ \phi + \psi[\phi] \right] \mathcal{E}^2_{k\ell} \frac{\delta \hat{S}[\phi]}{\delta \phi_\ell} \quad (B.10a)$$

where the operator $K_{ij}[\phi]$ is:

$$K_{ij}^{-1}[\phi] \equiv O_{ij}^{-1} - \frac{\delta^2 I[\phi]}{\delta \phi_i \delta \phi_j} \quad (B.10b)$$

Of course relation (B.9) is a trivial consequence of the two preceding theorems. To prove (B.10) first apply the transformation to expression (A.1):

$$\hat{\delta \psi}_i[\phi] = \left(1 - \mathcal{E}^2\right)_{ij} \hat{\delta \phi}_j - O_{ij} \frac{\delta \delta \phi_k}{\delta \phi_j} \frac{\delta \hat{S}[\phi]}{\delta \phi_k}$$

$$= \left(1 - \mathcal{E}^2\right)_{ij} T_j \left[ \phi + \psi[\phi] \right] - O_{ij} \left\{ \delta_j^k + \frac{\delta \psi_k[\phi]}{\delta \phi_j} \right\} \frac{\delta T_k}{\delta \phi_k} \left[ \phi + \psi[\phi] \right] \mathcal{E}^2_{k\ell} \frac{\delta \hat{S}[\phi]}{\delta \phi_\ell} \quad (B.11b)$$

Now functionally differentiate the $\psi$ equation of motion:

$$O_{ij}^{-1} \frac{\delta \psi_j[\phi]}{\delta \phi_k} = \frac{\delta^2 I}{\delta \phi_i \delta \phi_j} \left[ \phi + \psi[\phi] \right] \left\{ \delta_j^k + \frac{\delta \psi_k[\phi]}{\delta \phi_j} \right\} \quad (B.12a)$$

This can be rearranged to obtain an expression for $\delta \psi_i / \delta \phi_k$ involving only $\phi$ and $\psi$:

$$\frac{\delta \psi_i[\phi]}{\delta \phi_k} = \left\{ O_{ij}^{-1} - \frac{\delta^2 I}{\delta \phi_i \delta \phi_j} \left[ \phi + \psi[\phi] \right] \right\}^{-1} \frac{\delta^2 I}{\delta \phi_j \delta \phi_k} \left[ \phi + \psi[\phi] \right] \quad (B.12b)$$

Substitution into (B.11b) and a few simplifications gives (B.10). A significant corollary is that if the local symmetry obeys $[\delta_1, \delta_2] = \delta_3$ then the nonlocalized one obeys:

$$\left[ \hat{\delta}_1, \hat{\delta}_2 \right] \phi_i = \hat{\delta}_3 \phi_i + 2 \mathcal{E}^2_{ij} \sum_{jk} \frac{\mathcal{E}^2_{\ell m}}{\delta \phi_\ell} \frac{\delta \hat{S}[\phi]}{\delta \phi_m} \quad (B.13a)$$

$$= \left[ \hat{\delta}_1, \hat{\delta}_2 \right] \phi_i = \hat{\delta}_3 \phi_i + \mathcal{E}^2_{ij} \sum_{jk} \left( \frac{\delta T_{ij}^1[\phi]}{\delta \phi_j} K_{jk}[\phi] \frac{\delta T_{ij}^2[\phi]}{\delta \phi_k} - \frac{\delta T_{ij}^2[\phi]}{\delta \phi_j} K_{jk}[\phi] \frac{\delta T_{ij}^1[\phi]}{\delta \phi_k} \right) \quad (B.13b)$$

Therefore a local symmetry group nonlocalizes to a symmetry which must include generators proportional to the field equations in order to give a closed commutation algebra.
As with so much else these generators are more easily studied at the level of the auxiliary action.

**Theorem B4:** The following transformation generates a symmetry of $S[\phi, \psi]$:

\[
\Delta \phi_i \equiv A_{ij}[\phi, \psi] \left\{ \frac{\delta S[\phi, \psi]}{\delta \phi_j} - \frac{\delta S[\phi, \psi]}{\delta \psi_j} \right\} \tag{B.14a}
\]

\[
\Delta \psi_i \equiv -A_{ij}[\phi, \psi] \left\{ \frac{\delta S[\phi, \psi]}{\delta \phi_j} - \frac{\delta S[\phi, \psi]}{\delta \psi_j} \right\} \tag{B.14b}
\]

provided $A_{ij}[\phi, \psi] = -A_{ji}[\phi, \psi]$.

Of course $\Delta \psi_i = -\Delta \phi_i$. By simply transforming the auxiliary action:

\[
\Delta S[\phi, \psi] = \left( \frac{\delta S}{\delta \phi_i} - \frac{\delta S}{\delta \psi_i} \right) \Delta \phi_i
\]

\[
= \left( \frac{\delta S}{\delta \phi_i} - \frac{\delta S}{\delta \psi_i} \right) A_{ij}[\phi, \psi] \left( \frac{\delta S}{\delta \phi_j} - \frac{\delta S}{\delta \psi_j} \right) \tag{B.15}
\]

we see that the theorem follows trivially from the antisymmetry of $A_{ij}$. Since $\psi_i[\phi]$ is defined to make $\delta S/\delta \psi_i = 0$ the image of this transformation under theorem B2 gives the most general off shell symmetry of $\hat{S}[\phi]$. Such symmetries occur in all field theories — even conventional, local ones. Of course generators which have no effect upon solutions of the field equations are without dynamical significance. We consider them here only because some subset of these symmetries must be added to the dynamically significant generators in order to obtain a closed commutation algebra.

Comparison with relation (A.2) reveals that transformation (B.14a) can be recast in the following simple form:

\[
\Delta \phi_i = A_{ij}[\phi, \psi] \mathcal{E}^{-2}_{jk} \mathcal{O}^{-1}_{k\ell} \left[ (1 - \mathcal{E}^2) \right]_{\ell m} \phi_m - \mathcal{E}^2_{\ell m} \psi_m \tag{B.16}
\]

An important special case is given by the choice:

\[
A_{ij}[\phi, \psi] = M_{ij} \mathcal{O}_{jk} \mathcal{E}^2_{k\ell} \tag{B.17}
\]

where $M_{ij}$ is an antisymmetric operator which commutes with $\mathcal{F}_{ij}$. When $\delta$ is the BRS symmetry the nonlocalized transformation rule induced by theorems B1 and B2 turns out to be simple:

\[
\Delta \phi_i = M_{ij} \mathcal{O}_{jk} \mathcal{E}^2_{k\ell} \]

\[
\Delta \psi_i = -M_{ij} \mathcal{O}_{jk} \mathcal{E}^2_{k\ell} \]
not to agree with the local rule even at lowest order. Instead the lowest order terms differ by a factor of $\varepsilon^2$. We have found it convenient to absorb this factor by using the image of one of the dynamically trivial symmetries of type (B.17).

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FIGURE CAPTIONS

Fig. 1a: The smeared and shadow gluon propagators of nonlocal Yang-Mills.

Fig. 1b: The smeared and shadow ghost propagators of nonlocal Yang-Mills.

Fig. 2a: The 2-point measure vertex of nonlocal Yang-Mills.

Fig. 2b: The 3-point vertices of local Yang-Mills.

Fig. 2c: The 4-point vertex of local Yang-Mills.

Fig. 3: The component $A^\alpha_\beta(p)$ of the one loop vacuum polarization.

Fig. 4: The component $B^\alpha_\beta(p)$ of the one loop vacuum polarization.

Fig. 5: The component $C^\alpha_\beta(p)$ of the one loop vacuum polarization.

Fig. 6: The component $D^\alpha_\beta(p)$ of the one loop vacuum polarization.