On the Kadomtsev–Petviashvili equation with combined nonlinearities

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Abstract

In this paper, we study the generalized KP equation with combined nonlinearities. First we show the existence of solitary waves of this equation. Then, we consider the associated Cauchy problem and obtain conditions under which solutions blow-up in finite time or are uniformly bounded in the energy space. We also prove the strong instability of the ground states.

1 Introduction

The Kadomtsev-Petviashvili (KP) equation
\begin{equation}
(u_t + u_{xxx} + uu_x)_x + \varepsilon u_{yy} = 0, \quad \varepsilon = \pm 1,
\end{equation}
were derived by Kadomtsev and Petviashvili in [19] to study the transverse stability of the solitary wave solution of the KdV equation
\begin{equation}
u_t + u_x + \left(\frac{1}{3} - B\right)u_{xxx} + uu_x = 0,
\end{equation}
with different surface tension effect (Bond number) $B \geq 0$ in the context of surface hydrodynamical waves. Actually, Equation (1.1) with $\varepsilon = -1$ (KP-II) models gravity surface waves (of depth smaller than 0.46 cm) in a shallow water channel, while (1.1) with $\varepsilon = 1$ (KP-I) models capillary waves on the surface of a liquid or oblique magneto-acoustic waves in plasma.

When the higher order nonlinearity effects are considered, the KdV equation turns into
\begin{equation}
u_t + u_x + \left(\frac{1}{3} - B\right)u_{xxx} + uu_x - \varsigma u^2 u_x = 0.
\end{equation}

This equation is well-known as the Gardner equation describing of large-amplitude internal waves (see [16], and references therein). Depending on the physical problem under consideration, the coefficient $\varsigma$ in (1.3) can be positive or negative. In the context of internal waves, this depends on the stratification [16]; it is always positive in the particular case of a two-layer fluid. In addition to its appearance in plasma physics [35, 41], Equation (1.3) was also derived in asymptotic theory for internal waves in a two-layer liquid with a density jump at the interface [20, 30]. The linear term $u_x$ can be dropped by a linear transformation, and the resulting equation possesses an explicit solitary wave
\begin{equation}
u(x, t) = \frac{6A}{1 + R \cosh\left(\sqrt{A}(x - x_0 - AB_0 t)\right)}.
\end{equation}

where $B_0 = \frac{1}{3} - B$ and $R = \pm \sqrt{1 - 6A}$ (see [17]).
Figure 1: Solitary waves of the Gardner equation (1.3) given by (1.4) with various values of \( \varsigma \).

Similar to (1.1), a high-dimensional Gardner equation was derived in [32] describing the propagation of weakly nonlinear and weakly dispersive dust ion acoustic wave in a collisionless unmagnetized plasma consisting of warm adiabatic ions, static negatively charged dust grains, nonthermal electrons, and isothermal positrons.

This paper is concerned with the following generalized Kadomtsev–Petviashvili (KP) equation with combined nonlinearities [32, 36, 38]

\[
(u_t - D_x^{2\alpha}u_x + (f(u))_x)_x + \varepsilon u_{yy} = 0, \quad \varepsilon = \pm 1,
\]

(1.5)

where \( u = u(x,y,t) \) is a real valued-function and

\[ f(u) = \mu_1 f_1(u) + \mu_2 f_2(u), \]

with \( \mu_1, \mu_2 \in \mathbb{R} \), and \( f_1(u) = |u|^{p_1-1}u \) and \( f_2(u) = |u|^{p_2-1}u \) are \( C^2 \) real valued functions. Here, \( D_x^{2\alpha} \) with \( \alpha > 0 \) denotes the Riesz potential of order \(-2\alpha\) in the \( x \)-direction, defined by the usual Fourier multiplier operator with the symbol \( |\xi|^{2\alpha} \).

When \( \alpha = 0 \), the above equation reduces to

\[
(u_t - u_x + (f(u))_x)_x + \varepsilon u_{yy} = 0, \quad \varepsilon = \pm 1,
\]

(1.6)

where the operator \( \partial_x^{-1} \) is defined via Fourier transform \( \hat{\partial_x^{-1}} = (i\xi)^{-1} \). This equation models the propagation of short pulses in some media and is known as the Khokhlov–Zabolotskaya (or the dispersionless) equation (see [34] and the references therein). Note also that (1.5) with \( \alpha = 1/2 \) is the relevant KP version of the Benjamin–Ono equation and have been derived from the two-fluid system in the weakly nonlinear regime (see [8]). It was also derived in [16] for long weakly nonlinear internal waves in stratified fluids of large depth. Thus, equation (1.5) has some links with the full dispersion KP equation derived in [22] and considered in [23] as an alternative model to KP (with fewer unphysical shortcomings) for gravity-capillary surface waves in the weakly transverse regime. It is well-known that the KP-I has the Zaitsev traveling waves which is localized in \( x \) and periodic in \( y \):

\[
\psi(x - ct, y) = 12\beta_0^2 \frac{1 - \beta \cosh(\beta_0 x - ct) \cos(\delta y)}{(\cosh(\beta_0 x - ct) - \beta \cos(\delta y))^2}
\]

where \( (\beta_0, \beta) \in (0, +\infty) \times (-1, 1) \) and \( c = \beta_0^2 \frac{a - b^2}{1 - \beta^2} \). Suitable transformation produces solutions which are periodic in \( x \) and localized in \( y \):

\[
\psi(x, y, t) = \frac{2}{3} \frac{-(x + \frac{a}{\sqrt{3}}y + 3(a^2 - b^2)t)^2 + b^2(\frac{1}{\sqrt{3}}y + 6at)^2 + 1/b^2}{(x + \frac{a}{\sqrt{3}}y + 3(a^2 - b^2)t)^2 + b^2(\frac{1}{\sqrt{3}}y + 6at)^2 + 1/b^2}
\]
Notice that (1.5) conserves formally the energy (Hamiltonian) $E$, and momentum quantities $M$ and $\mathbb{P}$, where

$$M(u) = \|u\|_{L^2(\mathbb{R}^2)}^2,$$

$$\mathbb{P}(u) = \int_{\mathbb{R}^2} u(\partial_x^{-1}u_y) \, dx dy,$$

and

$$E(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( |\partial_x^2 u|^2 - \varepsilon |\partial_x^{-1}u_y|^2 \right) \, dx dy - K(u),$$

and $K(u) = \mu_1 K_1(u) + \mu_2 K_2(u) = \mu_1 \int_{\mathbb{R}^2} F_1(u) \, dx dy + \mu_2 \int_{\mathbb{R}^2} F_2(u) \, dx dy$ and $F_j$ for $j = 1, 2$ is the primitive function of $f_j$ with $F_j(0) = 0$. In the current paper, we study the existence and instability of solitary waves of (1.5). By a solitary wave, we mean a solution of the form $u(x, y, t) = v(x - t, y)$. In is clear from the ideas of [39] that (1.5) has no nontrivial solitary wave if $\varepsilon = -1$. So that, $v$ in the case $\varepsilon = 1$ satisfies

$$(v + D_2^{2\alpha} v - f(v))_{xx} + v_{yy} = 0.$$  

In the rest, we assume in (1.5) that $\varepsilon = 1$. We should note that some well-posedness results on the Cauchy problem associated to (1.5) with $\mu_2 = 0$ and $p_1 = 2$ were obtained in [25, 33]. In the case of homogeneous nonlinearity $\mu_2 = 0 < \mu_1$ with $\alpha = 2, 4$, de Bouard and Saut in [9] minimize the associated norm of (1.1) under the constraint $K(u) = \lambda > 0$ and use the concentration-compactness principle to show the existence of solitary wave. Indeed, by a suitable scaling of Lagrange multiplier, they show the minimizer satisfies (1.5). In the nonhomogeneous nonlinearity $\mu_1, \mu_2 \neq 0$, due to the lack of scaling invariance this method does not work, specially when $\mu_1 \mu_2 < 0$. To overcome this difficulty, we use the Pankov-Nehari manifold [83]. Although this result may be obtained by the Mountain-pass argument (see [42]), but one feature of our argument is that the stability of solitary wave can be deduced. It is easy to check that there is a natural scaling invariance associated to (1.5) with $\mu_2 = 0$. More precisely, the scaling

$$u_\lambda(x, y, t) = \lambda^{2\alpha+1} u(\lambda^\alpha x, \lambda y, \lambda^{\alpha+1} t), \quad \lambda > 0$$

leaves (1.5) invariant. This means that $u_\lambda$ is also a solution provided that $u$ is a solution of (1.5). In our consideration with combined nonlinearities, there is no scaling that leaves (1.5) invariant. This arises some difficulty to study systematically the Cauchy problem associated with (1.5). Similar situation appears for the following nonlinear Schrödinger equation with combined power-type nonlinearities

$$iu_t + \Delta u = \mu_1 |u|^{p_1-1} u + \mu_2 |u|^{p_2-1} u.$$  

Tao et al. in [39] presented a comprehensive study for equation (1.11). They obtained the local and global well-posedness, the finite time blow-up in the weighted space $H^1 \cap L^2(\|x\|\, dx)$, and also the asymptotic behavior of the solutions in both energy space $H^1$ and also aforementioned weighted space. The scattering versus blow-up for some particular cases of (1.11) was studied in [4, 29]. Inspired by these works, in this paper, we are interested in establishing sharp criterion concerned with the dichotomy global existence versus blow-up in finite time. We should notice that the global existence is deduced under the assumption of the local existence in the energy space. For this reason, the uniform bound in the energy space is replaced. Such a criterion is an application of the ground states of (1.5). This is necessary taking into account we want to obtain a sharp embedding-type inequality, in which case the best constant depends on such solutions. This also helps us to study the strong instability of ground states of (1.5). The strong instability of ground states of (1.5) with $\mu_2 = 0$ was proved by Liu [29]. Indeed, by the scaling property of the equation, he defined some subsets which are invariant under the flow of the Cauchy problem associated with (1.5) and used the virial identity to show the existence of finite time blow-up in the transverse direction. In our case where $\mu_1 \mu_2 \neq 0$, lack of scaling again arises difficulty to get this method directly. But we apply the ideas of [27] to introduce two families of invariant subsets that the solutions of (1.5) are global (uniformly bounded in the energy space) or blow-up in finite time. Then by choosing appropriate members of these family, we show the ground states are unstable by the mechanism of blow-up.
Notations

We denote $\langle \cdot , \cdot \rangle$ as $L^2(\mathbb{R}^2)$-inner product. We shall also denote by $\hat{\varphi}$ the Fourier transform of $\varphi$, defined as

$$\hat{\varphi}(\zeta) = \int_{\mathbb{R}^2} \varphi(x)e^{-ix\cdot \zeta} \, dx.$$ 

For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$, the nonhomogeneous Sobolev space defined by

$$H^s(\mathbb{R}^2) = \{ \varphi \in \mathcal{S}'(\mathbb{R}^2) : \| \varphi \|_{H^s(\mathbb{R}^2)} < \infty \},$$

where

$$\| \varphi \|_{H^s(\mathbb{R}^2)} = \left\| (1 + |\zeta|^2)^{\frac{s}{2}} \hat{\varphi}(\zeta) \right\|_{L^2(\mathbb{R}^2)},$$

and $\mathcal{S}'(\mathbb{R}^2)$ is the space of tempered distributions. Let $X_\alpha$ be the closure of $\partial_x(C_0^\infty(\mathbb{R}^2))$ for the norm

$$\| \varphi \|_{X_\alpha}^2 = \| D^\alpha_x \varphi \|_{L^2(\mathbb{R}^2)}^2 + \| \varphi_y \|_{L^2(\mathbb{R}^2)}^2 + \| \varphi_x \|_{L^2(\mathbb{R}^2)}^2.$$ (1.12) 

The homogeneous space $X^\alpha$ is defined by the norm

$$\| \varphi \|_{X^\alpha}^2 = \| D^\alpha_x \varphi \|_{L^2(\mathbb{R}^2)}^2 + \| \partial_x^{-1} \varphi_y \|_{L^2(\mathbb{R}^2)}^2.$$ (1.13) 

Let

$$X^s = \left\{ u \in H^s(\mathbb{R}^2); \ (\xi^{-1} \widehat{\alpha}(\xi, \eta)) \vee \in H^s(\mathbb{R}^2) \right\},$$

with the norm

$$\| u \|_{X^s} = \| u \|_{H^s(\mathbb{R}^2)} + \left\| (\xi^{-1} \widehat{\alpha}) \vee \right\|_{H^s(\mathbb{R}^2)},$$

where $'\vee'$ is the Fourier inverse transform.

2 Existence of ground state

In this section we prove existence of traveling wave solutions of equation (1.5). Let

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} \left( u_x^2 + |D^\alpha_x u|^2 + |\partial_x^{-1} u_y|^2 \right) \, dx dy.$$ 

Then $u$ is a weak solution of (1.5) if and only if $I'(u) = K'(u)$, or equivalently if $u$ is a critical point of $S$, where $S = I - K$. In [10] the existence of critical points of $S$ was obtained for homogeneous nonlinearities by showing that there exist minimizers of $I(u)$ subject to the constraint $K(u) = \lambda > 0$. Minimizers satisfy the equation $I'(u) = \theta K'(u)$ for the Lagrange multiplier $\theta \in \mathbb{R}$. By homogeneity, it could be scaled away to obtain a solution of (1.9).

One can use the same methods to prove that there exist minimizers of the same variational problem for more general nonlinearities that satisfy appropriate growth conditions. However, for non-homogeneous nonlinearities the Lagrange multiplier cannot be scaled away. In general, the dependence of the Lagrange multiplier $\theta$ on the constraint parameter $\lambda$ may be rather complicated, in particular when $\mu_1 \mu_2 < 0$. We will instead consider a different constrained minimization problem. It is motivated by the fact that, if $u$ is a critical point of $S$, then $P(u) = 0$, where

$$P(u) = \langle S'(u), u \rangle = 2I(u) - N(u)$$

and

$$N(u) = \int_{\mathbb{R}^2} \left( \mu_1 u^{p_1+1} + \mu_2 u^{p_2+1} \right) \, dx dy.$$
Then if \( u \in X_\alpha \) achieves the minimum
\[
m = \inf_{u \in N_0} S(u),
\]
where
\[
N_0 = \{ u \in X_\alpha, \ u \neq 0, \ P(u) = 0 \},
\]
it satisfies \( S'(u) = \theta P'(u) \) for some \( \theta \), which by the homogeneity of the constraint \( P(u) = 0 \) will be shown to be zero, and thus \( S'(u) = 0 \). These solutions are actually ground states. We recall that a ground state solution is thus a solution of (1.9) that minimizes the action \( S \) among all nontrivial solutions of (1.9).

We have from the well-known result of [3, 27] for any \( q \) in \([2, 2^*]\) that the embedding
\[
X_\alpha \hookrightarrow L^q(\mathbb{R}^2)
\]
is continuous, where
\[
2^* = \begin{cases} \infty & \alpha \geq 2, \\ \frac{3\alpha + 2}{2 - \alpha} & \alpha < 2. \end{cases}
\]
Moreover the embedding \( X_\alpha \hookrightarrow L^q_{loc}(\mathbb{R}^2) \) is compact if \( q \in (2, 2^*) \).

Related to (2.2), we first recall the following anisotropic Sobolev-type inequality from the results of Besov, Il'in and Nikolskii [3], together with an interpolation:
\[
\|u\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \leq \rho_p \|u\|_{L^2(\mathbb{R}^2)}^{2c_p} \|D_x^\alpha u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}, \quad u \in X_\alpha,
\]
provided
\[
c_p = \frac{3\alpha + 2 + p(\alpha - 2)}{4\alpha} > 0.
\]
See also [4] Lemma 2.1. We start showing that the best constant \( \rho_p \) is obtained in terms of the minimax value.

**Remark 2.1.** Equation (1.9) in the case \( \mu_2 = 0, \ p_1 = p \) and \( \mu_1 = 1 \) turns into
\[
D_x^{2\alpha} \varphi + \partial_x^{-2} \varphi_{yy} + \varphi = |\varphi|^{p-1} \varphi.
\]
By using the concentration-compactness principle and following the same lines of proof of Theorem 3.1 in [10] joint with with a commutator estimates [31], one can show the existence of nontrivial solitary waves of (2.4). See also [2]. Moreover, they are continuous and tend to zero at infinity (see Theorem 1.1 in [11]).

**Theorem 2.2.** Let \( p > 1 \). Then the optimal constant \( \rho_p \) in (2.3) is such that
\[
\rho_p^{-1} = \alpha^{-1} \left( \frac{p-1}{p+1} \right) k_p^{\frac{p-1}{2}} \|\varphi\|_{L^2(\mathbb{R}^2)}^{p-1} = \alpha^{-1} \left( \frac{p-1}{p+1} \right) k_p^{\frac{p}{2}} \left( \frac{2}{\alpha} \right)^{\frac{1}{2}} m \frac{2}{\alpha},
\]
where \( c_p = c_p - \frac{p-1}{2} \),
\[
k_p = \frac{3\alpha + 2 + p(\alpha - 2)}{2(p - 1)}
\]
and \( \varphi \) is a ground state of (2.4).

**Proof.** The proof is based on the ideas of [13] Theorem 1.2], so we omit the details.

We obtain some Pohozaev-type identities related to the solutions of (2.4).

**Lemma 2.3.** Let \( \varphi \) be a ground state of (2.4), then
\[
2\|\partial_x^{-1} \varphi_y\|_{L^2(\mathbb{R}^2)}^2 = \alpha \|D_x^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2 = \left( \frac{p-1}{p+1} \right) \|\varphi\|_{L^{p+1}(\mathbb{R}^2)}^{p+1},
\]
\[
\|D_x^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2 = k_p^{-1} \|\varphi\|_{L^2(\mathbb{R}^2)}^2.
\]
Proof. The proof follows from the ideas of [10].

**Theorem 2.4.** Let $\mu_1 > 0$ and $p_1 > p_2 \geq 1$ such that $c_{p_1} > 0$. Suppose that $\{u_n\}$ is a minimizing sequence of (2.1). Then there exists a subsequence, renamed by the same, a sequence $\{z_n\} \subset \mathbb{R}^2$, and $u \in X_\alpha$ such that $u_n(\cdot - z_n) \to u$ strongly in $X_\alpha$, $P(u) = 0$ and $S(u) = m$.

![Figure 2: Numerical solitary wave of (1.5) with $\alpha = 1, p_1 = 2, p_2 = 2, \mu_1 = 1$ and $\mu_2 = -0.1$.](image)

To prove, for any $\sigma \in \mathbb{R}$, we set

$$N_\sigma = \{u \in X_\alpha, u \neq 0, P(u) = \sigma\}$$

and define

$$m_\sigma = \inf_{u \in N_\sigma} S_0(u),$$

where

$$S_0 = S - \frac{1}{r + 1} P,$$

$r = p_1$ if $\mu_2 < 0$, and $r = p_2$ if $\mu_2 > 0$. Hereafter in this section, we consider the case $\mu_2 < 0$ in the proof of Theorem 2.4. The argument also works for the case $\mu_2 > 0$ and is simpler.

**Lemma 2.5.** For any $\sigma \in \mathbb{R}$, the set $N_\sigma$ is nonempty. Moreover, $N_\sigma$ is bounded away from zero if $\sigma \leq 0$.

*Proof.* Let $\sigma$ be positive and $u \in X_\alpha$. For $C \in \mathbb{R}$, we define

$$u_C(x, y) = u(x, Cy).$$

Then it is easy to see that

$$\lim_{C \to +\infty} P(u_C) = +\infty.$$ 

Hence, there exists $C > 0$ such that $P(u_C) > \sigma$. On the other hand,

$$\lim_{A \to 0} P(Au) = 0.$$ 

Thus, there exists $A > 0$ such that $P(Au_C) = \sigma$. Next we consider the case $\sigma \leq 0$. We show that there exists $u \in X_\alpha$ such that $N(u) > 0$. This is clear when $\mu_2 > 0$, so we assume that $\mu_2 < 0$. In this case, we consider $Q \in X_\alpha$ which is the ground state of

$$(u + D^2_x u - |\mu_2| f_2(u))_{xx} + u_{yy} = 0.$$ (2.9)
We obtain from the interpolation $L^{p+1}(\mathbb{R}^2) = (L^2(\mathbb{R}^2), L^{p+1}(\mathbb{R}^2))_{[r]}$ and Remark 2.1 that
\[
\|Q\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \leq \|Q\|_{L^2(\mathbb{R}^2)}^{p+1} \|Q\|_{L^{p+1}(\mathbb{R}^2)}^r, \quad r = \frac{p+1}{\frac{p+1}{2} - 1}.\]
Lemma 2.3 implies that
\[
\|Q\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \leq \left(\frac{k_p}{\alpha} \frac{p-1}{p+1} \right)^r \|Q\|_{L^{p+1}(\mathbb{R}^2)}^{p+1},
\]
But it is easy to check that the constant of the right hand side of the above inequality is less than one. This means that $N(Q) > 0$.

Now we show that there exists a unique $r > 0$ such that $P(rQ) = \sigma \leq 0$. First we note that $g(r) = P(rQ) < \sigma$ for $r$ large enough. We observe that $g(r) > 0$ for all sufficiently small $r > 0$. Hence, there is $r > 0$ such that $g(r) = \sigma$. On the other hand, by
\[
g(r) = r^2 (2I(Q) - r^{-2}N(rQ)),
\]
we have $N(rQ) > 0$ for any $r$ such that $g(r) \leq 0$. Thusly, $\frac{d}{dr}(r^{-2}N(rQ)) > 0$ for all $r > 0$ such that $g(r) \leq 0$. This shows that there is at most one $r$ such that $g(r) = \sigma$. In both cases of $\mu_2$, we proved $N_{\sigma}$ is not empty.

Now, let $\sigma \leq 0$ and $u \in N_\sigma$. Then, we have from (2.2) that
\[
0 \geq P(u) \geq C_1 \|u\|_{X_\sigma}^2 - C_2(\|u\|_{X_\sigma}^{p_1+1} + \|u\|_{X_\sigma}^{p_2+1}),
\]
for some constants $C_1, C_2 > 0$. Therefore, $\|u\|_{X_\sigma} \geq C > 0$, where $C$ is a constant depending only on $p_1$, $p_2$, $\mu_1$ and $\mu_2$.

In the following lemma we show that $m_{\sigma}$ is decreasing on $\sigma \leq 0$ which shows the strict subadditivity condition of $m_{\sigma}$.

**Lemma 2.6.** We have $m_{\sigma} \geq 0$ for all $\sigma \in \mathbb{R}$. Moreover, $m_{\sigma}$ positive and strictly decreasing on $(-\infty, 0]$.

**Proof.** Since $\frac{1}{p_1+1}N(u) \geq K(u)$, then $S_0(u) \geq 0$ for all $u \in X_\sigma$. This means that $m_{\sigma} \geq 0$ for any $\sigma \in \mathbb{R}$. If $\sigma \leq 0$, then by Lemma 2.3 there exits $C$ such that $\|u\|_{X_\sigma} \geq C$. Then
\[
S_0(u) \geq C \|u\|_{X_\sigma}^2
\]
for any $u \in N_{\sigma}$. Hence, $m_{\sigma} > 0$.

Next, for $\sigma < \sigma_2 \leq 0$, we can consider $u \in N_{\sigma}$ such that $S_0(u) < 2m_{\sigma_2}$. We will get the desired inequality $m_{\sigma_1} > m_{\sigma_2}$, if no such $u$ exists. By the proof of Lemma 2.3, there exists $C_1$ such that $P(r_uu) = \sigma_2$. We show that there is $r_0 < 1$, independent of $u$, such that $r_u \leq r_0$. By using
\[
S_0(u) \geq \left(1 - \frac{2}{p_1+1}\right) I(u) \geq \left(1 - \frac{2}{p_1+1}\right) \|u\|_{X_\sigma}^2,
\]
we obtain that there is $C > 0$ such that $\|u\|_{X_\sigma} \leq C$ for all $u$ such that $S_0(u) < 2m_{\sigma_2}$. Hence, it follows immediately that $g'(r) \geq -C_0$ for all $r < 1$, where $g(r) = P(r_uu)$. Then we have by integrating from $r_u$ to 1 that
\[
\sigma_1 - \sigma_2 = g(1) - g(r_u) \geq -C_0(1 - r_u),
\]
and thereby $r_u \leq 1 - \frac{\sigma_2 - \sigma_1}{C_0} =: r_0$. Now as $P(r_uu) \leq 0$ for $r_u \leq r \leq 1$, then similar to the proof of Lemma 2.3, it holds that $\|ru\|_{X_\sigma} \geq C$. Consequently, $h'(r) \geq 2(1 - \frac{2}{p_1+1})C^2$ for all $r_u \leq r \leq 1$, where $h(r) = S_0(r_uu)$. It follows that $S_0(u) - S_0(r_uu) \geq 2(1 - \frac{2}{p_1+1})C^2(\sigma_2 - \sigma_1)$. The fact $m_{\sigma_2} \leq S_0(r_uu)$ reveals that
\[
S_0(u) \geq m_{\sigma_2}2(1 - \frac{2}{p_1+1})C^2(\sigma_2 - \sigma_1)
\]
for all $u \in N_{\sigma}$ such that $S_0(u) \leq 2m_{\sigma_2}$. Therefore, $m_{\sigma_1} > m_{\sigma_2}$, and the proof is complete. \qed
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get the minimizer function $u \in X_\alpha$, we apply the concentration-compactness principle for the sequence $\chi_n = |u_n|^2 + |D^2_x u_n|^2 + |D^{\alpha}_y (u_n)_y|^2$, and $\lim_{n \to \infty} \int_{\mathbb{R}^2} \chi_n \, dx \, dy = L > 0$, up to a subsequence. The sequence $\chi_n$ can be normalized such that $\int_{\mathbb{R}^2} \chi_n \, dx \, dy = L$. We rule out the vanishing and dichotomy cases. If the vanish case occurs, then by an argument similar to [9, 10], $u_n \to 0$ for any $2 < q < 2^*$ as $n \to \infty$. This implies that $K(u_n) \to 0$ as $n \to \infty$. Hence, $I(u_n) \to 0$ and $S(u_n) \to 0$ as $n \to \infty$, because $P(u_n) \to 0$. This contradicts $S(u_n) \to m > 0$. Suppose that the dichotomy case occurs. Then, there are the bounded sequences $\{v_n\}, \{w_n\} \subset X_\alpha$ such that

$$
\lim_{n \to \infty} \|u_n - v_n - w_n\|_{X_\alpha} = 0, \\
\lim_{n \to \infty} K(u_n) - K(v_n) - K(w_n) = 0
$$

and $N(u_n) - N(v_n) - N(w_n) \to 0$ as $n \to \infty$. These imply that $P(u_n) - P(v_n) - P(w_n) \to 0$ and

$$
S_0(u_n) - S_0(v_n) - S_0(w_n) \to 0
$$

as $n \to \infty$. Suppose (by extracting subsequences if necessary) that $\sigma_1 = \lim_{n \to \infty} P(v_n)$ and $\sigma_2 = \lim_{n \to \infty} P(w_n)$. Then $\sigma_1 + \sigma_2 = 0$. If $\sigma_1 > 0$, then there is $n_0 \in \mathbb{N}$ such that $\sigma_{2,n} = P(w_n) < \sigma_2/2$ for all $n \geq n_0$. Since $m_\sigma$ is strictly decreasing in $\sigma$, so $S_0(w_n) \geq m_{\sigma_{2,n}} > m_{\sigma_2/2}$ for all $n \geq n_0$. As $S_0(v_n) \geq 0$ for all $n$, then

$$
S_0(v_n) + S_0(w_n) \geq \sigma_{2/2}
$$

for all $n \geq n_0$. It is concluded to the contradiction

$$
m = \lim_{n \to \infty} S(u_n) = \lim_{n \to \infty} S_0(u_n) = \lim_{n \to \infty} (S_0(v_n) + S_0(w_n)) \geq \sigma_{2/2} > m.
$$

Similar contradiction holds for the case $\sigma_1 < 0$. Next, we consider $\sigma_0 = \sigma_2 = 0$. In this case, we have from the coercivity of $I$ that $I_1, I_2 > 0$, where $I_1 = \lim_{n \to \infty} I(v_n)$ and $I_2 = \lim_{n \to \infty} I(w_n)$. Then, for any $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $I(v_n) < 2N(v_n)(1 + \epsilon)^{\mu_1 - 1}$ and $I(w_n) < 2N(w_n)(1 + \epsilon)^{\mu_1 - 1}$ for

Figure 3: Numerical solitary waves of (1.5) with $\alpha = 1$, $p_1 = 2$, $p_2 = 2$, $\mu_2 = 1$, $\mu_1 = -0.1$ (left) and $\mu_1 = 1$ (right).
all \( n \geq n_0 \). Since \( \{v_n\} \) is bounded in \( X_\alpha \), then \( K(v_n) - K(\theta v_n) \leq C(\theta - 1) \) for all \( n \) and \( \theta > 1 \), and \( N(v_n) - N(\theta v_n) \leq C(\theta - 1) \) for some \( C > 0 \). If \( P(v_n) > 0 \), then \( P(\theta v_n) = 0 \) for some \( \theta < \frac{2I(v_n)}{N(v_n)} \), because we have for \( \theta > 1 \) that \( P(\theta v_n) \leq 2\theta^2 I(v_n) - \theta^{\sigma + 1} N(v_n) \). An straightforward computation shows for some \( C_1 > 0 \) that

\[
m \leq S_0(\theta v_n) \leq S_0(v_n) + C(\theta - 1) \leq S_0(v_n + C_1 \epsilon).
\]

Hence, \( S_0(v_n) \geq m - C_1 \epsilon \). This inequality also trivially holds if \( P(v_n) \leq 0 \), and also for \( S_0(w_n) \). Therefore, we obtain that

\[
S_0(v_n) + S_0(w_n) \geq 2m - 2C_1 \epsilon
\]

for all \( n \geq n_0 \). This obviously shows that \( m \lim_{n \to \infty} S(u_n) \geq 2m - 2C_1 \epsilon \), and consequently, \( m \geq 2m \) which is a contradiction. Thus the dichotomy does not occur. Finally the compactness case should occur. So, by using Lemma 3.3 in [10], there is a sequence \( \{z_n\} \subset \mathbb{R}^2 \) and some \( u \in X_\alpha \) such that \( \tilde{u}_n = u_n(\cdot - z_n) \to u \) in \( X_\alpha \) and \( \tilde{u}_n(\cdot - z_n) \to u \) in \( L^q_{loc}(\mathbb{R}^2) \) for any \( q \in (2, 2^*) \), and thereupon the strong convergence in \( L^q(\mathbb{R}^2) \) is deduced. Hence \( K(\tilde{u}_n) = K(u_n) \to K(u) \) and \( N(\tilde{u}_n) = N(u_n) \to N(u) \) as \( n \to \infty \). The weak lower semicontinuity of \( I \) shows that

\[
S(u) + K(u) = I(u) \leq \liminf_{n \to \infty} I(u_n) = \liminf_{n \to \infty}(S(u_n) + K(u_n)) = m + K(u).
\]

By a similar computation, \( P(u) \leq 0 \) and \( S(u) \leq m \). But, we have \( S_0(u) < m_\sigma \) for all \( \sigma < 0 \), and thereby \( P(u) = 0 \). Indeed if \( P(u) = \sigma < 0 \) then we get the contradiction \( m_\sigma \leq S_0(u) < m_\sigma \), hence, \( u \) achieves the minimum \( m \). Moreover, as \( \lim_{n \to \infty} S(\tilde{u}_n) = m = S(u) \), then \( \lim_{n \to \infty} I(\tilde{u}_n) = \lim_{n \to \infty}(S(\tilde{u}_n) + K(\tilde{u}_n)) = I(u) \). Consequently, we obtain from \( \tilde{u}_n \to u \) in \( X_\alpha \) that \( \tilde{u}_n \to u \) in \( X_\alpha \).

![Figure 4: The numerical surfaces of the projections of solitary waves of (1.5) on the XZ-plane (left) and the YZ-plane (right) with \( \alpha = 1, p_1 = 2, p_2 = 2, \mu_2 = 1 \) and the various values of \( \mu_1 \).](image)

**Remark 2.7.** Numerical results illustrated in Figures 2-4 show a similar description of the behavior and the polarity change of the solitary waves with different sign of \( \mu_1 \) and \( \mu_2 \) as it was reported for the Gradner equation (1.3) (see Figure 7).

**Theorem 2.8.** Let \( u \in X_\alpha \) satisfy \( P(u) = 0 \) and \( S(u) = m \). Then \( u \) is a solution of (1.9). Moreover, \( m = m' \), and \( u \) achieves \( m \) and only if \( u \) achieves the minimum \( m' \), where

\[
m' = \inf_{u \in N_0} S_0(u)
\]

and \( N_0' = \{u \in X_\alpha, u \neq 0, P(u) \leq 0\} \).
Proof. By the definition of $m$, there is the Lagrange multiplier $\theta \in \mathbb{R}$ such that $S'(u) = \theta P'(u)$. Thus,

$$\theta(P'(u), u) = \langle S'(u), u \rangle = P(u) = 0.$$ 

By the assumption, we obtain

$$\theta(P'(u), u) = 4I(u) - \theta(N'(u), u) = 2 \left( 1 - \frac{1}{p_1 + 1} \right) I(u) < 0.$$ 

Thus, $\theta = 0$ and $S'(u) = 0$.

Next, suppose that $S(u) = m$ and $P(u) = 0$. Then clearly, $m' \leq S(u) = m$. Since $m < m_o$ for all $\sigma < 0$, we have $m < S_0(u)$ for all $u$ such that $P(u) < 0$ and thereby $m \leq m'$. This means that $m = m'$. Now if $u$ achieves the minimum $m'$, then $P(u) \leq 0$ and $S_0(u) = m' = m$. Suppose that $P(u) < 0$. Then $m_o \leq S_0(u) = m$ which contradicts the fact $m < m_o$. Hence, $P(u) = 0$ and $u$ achieves $m$. \hfill \Box

A direct corollary of Theorem 2.8 is the following stability result whose proof is similar to one in [15], and we omit the details. Recall that the set $D \subset X_\alpha$ is said to be stable with respect to (1.5) if for any $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $u_0 \in B_\delta(D)$ ($\delta$-neighborhood of $D$), the solution $u$ of (1.5) with $u(0) = u_0$ satisfies $u(t) \in B_\epsilon(D)$ for all $t > 0$. Otherwise we say $D$ is unstable.

Theorem 2.9. If $d''(1) > 0$ then the set $\{ \varphi \in X_\alpha \setminus \{ 0 \}, S(\varphi) = m, P(\varphi) = 0 \}$ is stable, where $d(c) = E(\varphi) + cM(\varphi)$.

3 Blowing-up and uniform boundedness

In this section, we derive the sharp threshold for blowing-up and global existence (uniformly boundedness in the energy space). First we state the following local well-posedness for the initial-value problem

$$u \text{ and if }$$

and we omit the details. Recall that the set $D \subset X_\alpha$ is said to be stable with respect to (1.5) if for any $\epsilon > 0$ there exists some $\delta > 0$ such that, for any $u_0 \in B_\delta(D)$ ($\delta$-neighborhood of $D$), the solution $u$ of (1.5) with $u(0) = u_0$ satisfies $u(t) \in B_\epsilon(D)$ for all $t > 0$. Otherwise we say $D$ is unstable.

Theorem 3.1. Let $u_0 \in X^s$, $s \geq \alpha + 1$. Then there exists $T > 0$ such that (1.5) has a unique solution $u(t)$ with $u(0) = u_0$ satisfying

$$u \in C([0, T); X^s) \cap C^1 ([0, T); H^{s-\alpha-1}(\mathbb{R}^2)), \quad \partial_x^{-1}u_y \in C ([0, T); H^{s-1}(\mathbb{R}^2));$$

and if $\partial_x^{-2}(u_0)_{yy} \in L^2(\mathbb{R}^2)$, one has

$$u_t \in L^\infty ([0, T); X^0), \quad \partial_x^{-1}u_{yt} \in L^\infty ([0, T); H^{-1}(\mathbb{R}^2)).$$

Furthermore we have $M(u(t)) = M(u_0)$ and $E(u(t)) = E(u_0)$.

Next, we obtain the conditions under which the local solutions are uniformly bounded in the energy space.

Theorem 3.2. Let $s \geq \alpha + 1$, $\mu_1, \mu_2 > 0$ and $p_1 > p_2$. Let $u \in C([0, T); H^{s}(\mathbb{R}^2))$ with $\partial_x^{-1}u_y \in C([0, T); H^{s-1}(\mathbb{R}^2))$ be the solution obtained in Theorem 3.1 corresponding to the initial data $u_0$.

(i) If $p_1, p_2 < s_c$, then $u(t)$ is uniformly bounded in $X_\alpha$ for $t \in [0, T]$.

(ii) If $p_1 = s_c = 1 + \frac{4\alpha}{2s+\alpha}$ and

$$1 - \frac{2\mu_1\mu_2}{p_1 + 1} \|u_0\|_{L^2(\mathbb{R}^2)}^{2p_1} > 0,$$ \hspace{1cm} (3.1)

then $u(t)$ is uniformly bounded in $X_\alpha$ for $t \in [0, T]$. Particularly, $u(t)$ is uniformly bounded in $X_\alpha$ if

$$\|u_0\|_{L^2(\mathbb{R}^2)}^{2p_1} < \frac{1}{\mu_1^2} \frac{2}{2 + 3\alpha} \left( \frac{\alpha}{2} \right)^{\frac{3\alpha}{2}} \|\varphi_{s_c}\|_{L^2(\mathbb{R}^2)}^{\frac{3\alpha}{2}},$$

where $\varphi_{s_c}$ is the ground state of

$$\left( u + D_x^{2\alpha} u - u^s \right)_{xx} + u_{yy} = 0.$$ \hspace{1cm} (3.2)
Proof. Let \( u \in C([0,T); \mathbb{X}^* ) \) be the solution of (1.5) with the initial data \( u(0) = u_0 \). Then by using the invariants \( E \) and \( M \), we have that

\[
E(u_0) \geq \frac{1}{2} \| u \|_{\mathbb{X}_\alpha}^2 - \frac{\mu_1}{p_1+1} \| u \|_{L^{p_1+1}(\mathbb{R}^2)}^{p_1+1} - \frac{\mu_2}{p_2+1} \| u \|_{L^{p_2+1}(\mathbb{R}^2)}^{p_2+1}
\]

\[
\geq \frac{1}{2} \| u \|_{\mathbb{X}_\alpha}^2 - \frac{\mu_1 \rho_{p_1}}{p_1+1} \| u \|_{L^2(\mathbb{R}^2)}^{2c_{p_1}} \| D_u^0 u \|_{L^2(\mathbb{R}^2)}^{(p_1-1)/\alpha} \| \partial_x^{-1} u_y \|_{L^2(\mathbb{R}^2)}^{(p_1-1)/2}
\]

\[
- \frac{\mu_2 \rho_{p_2}}{p_2+1} \| u \|_{L^2(\mathbb{R}^2)}^{2c_{p_2}} \| D_u^0 u \|_{L^2(\mathbb{R}^2)}^{(p_2-1)/\alpha} \| \partial_x^{-1} u_y \|_{L^2(\mathbb{R}^2)}^{(p_2-1)/2}
\]

\[
= h(\| u \|_{\mathbb{X}_\alpha}),
\]

where

\[
h(z) = \frac{1}{2} z^2 - \frac{\mu_1 \rho_{p_1}}{p_1+1} \| u_0 \|_{L^2(\mathbb{R}^2)}^{2c_{p_1}} z^{(\alpha+2)(p_1-1)/2} - \frac{\mu_2 \rho_{p_2}}{p_2+1} \| u_0 \|_{L^2(\mathbb{R}^2)}^{2c_{p_2}} z^{(\alpha+2)(p_2-1)/2}.
\]

and \( \rho_{p_1}, \rho_{p_2} > 0 \) are the same as in Theorem 2.2. This immediately implies for \( p_1, p_2 < s_c \), that \( \| u \|_{\mathbb{X}_\alpha} \) is uniformly bounded for all \( t \in [0,T) \). The uniform bound still holds for \( p_1 = s_c > p_2 \) provided

\[
1 - \frac{2 \mu_1 \rho_{p_1}}{p_1+1} \| u_0 \|_{L^2(\mathbb{R}^2)}^{2c_{p_1}} > 0.
\]

On the other hand, (2.5) gives the condition of boundedness of \( u \) in terms of the ground state of (2.4).

For the supercritical nonlinearities or in the case of the combined supercritical and critical nonlinearities, we need to impose more additional conditions in terms of the best constant of (2.3).

**Theorem 3.3.** Let \( p_1, p_2 > 0, p_2 > p_1 \geq s_c \). Suppose that \( u \) is the solution of (1.5) as in Theorem 3.2 corresponding to the initial data \( u_0 \).

(i) If \( p_1 = s_c, \| u_0 \|_{\mathbb{X}_\alpha} < z_0, E(u_0) < h(z_0) \) and (3.1) holds, then \( u(t) \) is uniformly bounded in \( \mathbb{X}_\alpha \) for \( t \in [0,T) \), where \( h \) is as in the proof of Theorem 3.2.

\[
z_0 = \left( \frac{A}{B} \right)^{\frac{s_c-1}{p_2-s_c+1}},
\]

and

\[
A = 1 - \frac{2 \rho_{p_1} \mu_1}{p_1+1} \| u_0 \|_{L^2(\mathbb{R}^2)}^{2c_{p_1}}, \quad B = \frac{\rho_{p_2} \mu_2}{p_2+1} (p_2-1) \left( \frac{\alpha+2}{2\alpha} \right) \| u_0 \|_{L^2(\mathbb{R}^2)}^{2c_{p_2}}.
\]

(ii) If \( p_1 > s_c, \) there exists

\[
z_1 = z_1 \left( \alpha, p_1, p_2, \rho_{p_1}, \rho_{p_2}, \mu_1, \mu_2, \| u_0 \|_{L^2(\mathbb{R}^2)} \right) > 0
\]

such that if

\[
E(u_0) < \frac{(p_1-1)(\alpha+2) - 4\alpha}{2(p_1-1)(\alpha+2)} z_1^2,
\]

and \( \| u_0 \|_{\mathbb{X}_\alpha} < z_1 \), then the solution \( u(t) \) is uniformly bounded in the energy space.

Proof. (i) Let \( h \) be as the same in the proof of 3.2. Then,

\[
E(u(t)) \geq h(\| u(t) \|_{\mathbb{X}_\alpha}).
\]

The function \( h \) is continuous on \([0, \infty)\) and

\[
h'(z) = Az - Bz^{(p_2-1)(\alpha+2)-1}.
\]
Inequality \(3.1\) shows that \(h'(z) = 0\) has only a positive root \(z_0\). Hence, \(h\) is increasing on the interval \([0, z_0)\), decreasing on \([z_0, +\infty)\) and

\[
h_{\max} = h(z_0) = A \frac{(p_2 - 1)(\alpha + 2) - 4\alpha}{2(p_2 - 1)(\alpha + 2)} z_0^2.
\]

The invariant \(E(u(t)) = E(u_0)\) and \(E(u_0) < h(z_0)\) imply that

\[
h(\|u(t)\|_{X_\alpha}) \leq E(u(t)) = E(u_0) < h(z_0)
\]

for all \([0, T)\). We show that \(\|u(t)\|_{X_\alpha} < z_0\) for all \([0, T)\), if \(\|u_0\|_{X_\alpha} < z_0\). This means that \(u(t)\) is uniformly bounded in the energy space. Suppose by contradiction that \(\|u(t)\|_{X_\alpha} \geq z_0\) for some \(t_1 < T\). Then, by continuity of \(\|u(t)\|_{X_\alpha}\), there exists \(t_0 \in (0, T)\) such that \(\|u(t_0)\|_{X_\alpha} = z_0\). Thus,

\[
h(\|u(t_0)\|_{X_\alpha}) = h(z_0) = h_{\max}.
\]

Now, \((3.4)\) yields with \(t = t_0\) that

\[
h(\|u(t_0)\|_{X_\alpha}) = h(z_0) = h_{\max} \leq E(u(t_0)) = E(u_0) < h_{\max}.
\]

This contradiction gives the desired result. We note by a similar argument that if \(\|u(t)\|_{X_\alpha} > z_0\) for all \([0, T)\), if \(\|u_0\|_{X_\alpha} > z_0\).

(ii) By an argument similar to (i), we see that

\[
h'(z) = z - B_1 z c_1 - B_2 z c_2 - 1,
\]

where

\[
e_j = (p_j - 1) \left(\frac{\alpha + 2}{2\alpha}\right)
\]

and

\[
B_j = \frac{\rho p_j \mu_j}{p_j + 1} e_j \|u_0\|_{L^2(R^2)}^{2c_1 \rho_j}, \quad j = 1, 2.
\]

Set \(\tilde{h}(z) = \frac{h(z)}{2}\). Then,

\[
\tilde{h}'(z) = -B_1 (e_1 - 1) z c_1 - B_2 (e_2 - 1) z c_2 - 2.
\]

The assumption \(p_2 > p_1 > s_c\) shows for \(z > 0\) that \(\tilde{h}'(z) < 0\). So, \(\tilde{h}\) is decreasing on \([0, +\infty)\). By the fact \(\tilde{h}(0) = 1\), there exists \(z_1 > 0\) such that \(\tilde{h}(z_1) = 0\), and then

\[
h(z_1) = \left(\frac{1}{2} - \frac{1}{e_1}\right) z_1^2 + \frac{\rho p_2 \mu_2}{e_1 (p_2 + 1)} \left(\frac{\alpha + 2}{2\alpha}\right) (p_2 - p_1) z_1 c_2.
\]

We now obtain from the conservation of energy together with \(E(u_0) < \frac{s_c - 2 z_1^2}{2}\) that

\[
h(\|u(t)\|_{X_\alpha}) \leq E(u(t)) = E(u_0) \leq \left(\frac{1}{2} - \frac{1}{e_1}\right) z_1^2 + \frac{\rho p_2 \mu_2}{e_1 (p_2 + 1)} \left(\frac{\alpha + 2}{2\alpha}\right) (p_2 - p_1) z_1 c_2 = h(z_1).
\]

By the same argument as in (i), we conclude that \(\|u(t)\|_{X_\alpha} < z_1\) for all \(t \in [0, T)\), if \(\|u_0\|_{X_\alpha} < z_1\). This gives the uniformly boundedness of \(u\).

\[\Box\]

**Proposition 3.4.** Let \(\phi \in C^1(\mathbb{R})\) be a nonnegative measurable function satisfying \(|\phi'(y)| \lesssim \phi(y)\) and \(u\) be the solution of Theorem \(3.7\). Then \(\phi^{1/2}(y) u \in L^\infty((0, T); L^2(\mathbb{R}^2))\), if \(\phi^{1/2}(y) u_0 \in L^2(\mathbb{R}^2)\).

The proof of this proposition is similar to one of Theorem 3.3 in \(37\). Now by Proposition \(3.3\), the quantity

\[
\mathcal{J}(u) = \int_{\mathbb{R}^2} \phi(y) u^2(t) \, dx dy
\]

is well defined as soon as \(u_0 \in X^*\) and \(\phi^{1/2}(y) u_0 \in L^2(\mathbb{R}^2)\).
Theorem 3.5. Let \( \phi \in C^4(\mathbb{R}) \) as in Proposition 3.4. Suppose that \( u \) is a solution, obtained from Theorem 3.1, such that \( \phi^{1/2}(y)u_0 \in L^2(\mathbb{R}^2) \). Then \( u(t) \) satisfies
\[
\frac{1}{2} \frac{d^2}{dt^2} \mathcal{I}(u) = \frac{1}{2} \int_{\mathbb{R}^2} \phi^{(4)}(\partial_x^{-1} u)^2 \, dx \, dy
+ 2 \int_{\mathbb{R}^2} \phi''(y) \left( (\partial_x^{-1} u_y)^2 - \kappa_1 F_1(u) - \kappa_2 F_2(u) \right) \, dx \, dy,
\]
where \( \kappa_j = (p_j - 1)/2 \).

Remark 3.6. If \( \int_{\mathbb{R}^2} |x|^2 u_0^2 \, dx \, dy \) is finite, then \( t \mapsto \int_{\mathbb{R}^2} xu^2(t) \, dx \, dy \) is a \( C^1 \) function of time and
\[
\frac{d}{dt} \int_{\mathbb{R}^2} xu^2(t) \, dx \, dy = (2\alpha - 1)\|D_x^2 u\|_{L^2(\mathbb{R}^2)}^2 - \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^2 + 2 \int_{\mathbb{R}^2} (F(u) - uf(u)) \, dx \, dy. \tag{3.5}
\]
We use the Young inequality for the next result. Recall from the Young inequality for any \( a > 0 \) and \( p_2 > p_1 > 0 \) that for any \( \tau > 0 \) there exists \( C_\tau > 0 \) such that
\[
a^{p_1} \leq \tau a^{p_2} + C_\tau a^{p_2}. \tag{3.6}
\]
Now we state our blow-up result in terms of the energy of initial data and the sign of \( \mu_1 \) and \( \mu_2 \).

Theorem 3.7. Let \( u \in C([0, T); H^s(\mathbb{R}^2)) \) with \( \partial_x^{-1} u_y \in C([0, T); H^{s-1}(\mathbb{R}^2)) \) be the solution obtained in Theorem 3.1 corresponding to the initial data \( u_0 \). Then the solution \( u(t) \) blows up in finite time in the sense that \( T < +\infty \) must hold in each of the following three cases:

(i) \( \mu_2 > 0, E(u_0) \leq 0 \) and \( p_2 \geq p_1 \geq 5 \);

(ii) \( \mu_1 < 0, E(u_0) \leq 0 \) and \( p_2 \geq \max\{5, p_1\} \);

(iii) \( \mu_2 < 0 < \mu_1, p_2 \geq \max\{\theta + 1, p_1\} \) with \( \theta \in (0, 1) \), and there is \( \tau > 0 \) such that
\[
p_2 - 1 \theta E(u_0) + A_{p_1} \tau M(u_0) \leq 0 \tag{3.7}
\]
and
\[
A_{p_1} C_\tau + A_{p_2} \leq 0,
\]
where \( C_\tau \) is as in (3.7) and
\[
A_{p_1} = \frac{\mu_1}{2(p_1 + 1)} (\theta(p_2 - 1) - p_1 + 1), \quad A_{p_2} = \frac{\mu_2 K_2}{p_2 + 1} (1 - \theta).
\]

Proof. In what follows, we will show for \( \phi(y) = y^2 \) in Theorem 3.3 that the second derivative of \( \mathcal{I}(u) \) is negative for positive times \( t \). More precisely, in each of the described three cases, it follows that \( \mathcal{I}(u(t_0)) \) for some \( t_0 < T \), and the blow-up result can be deduced from the conserved momentum \( M \) and the classical Weyl-Heisenberg inequality.

(i) and (ii) We obtain from (3.3) and the conservation of energy \( E \) that
\[
\frac{1}{8} \frac{d^2}{dt^2} \mathcal{I}(u) = \|u\|_{L^2(\mathbb{R}^2)}^2 - \dot{K}(u)
= \frac{5 - p_1}{4} \|u\|_{L^2(\mathbb{R}^2)}^2 - \frac{p_2 - p_1}{2} K_2(u) \, dx \, dy
- \frac{p_1 - 1}{4} \|D_x u\|_{L^2(\mathbb{R}^2)}^2 - \kappa_1 E(u_0)
= \frac{5 - p_2}{4} \|u\|_{L^2(\mathbb{R}^2)}^2 - \frac{p_1 - p_2}{2} K_1(u) - \frac{p_2 - 1}{4} \|D_x u\|_{L^2(\mathbb{R}^2)}^2 - \kappa_2 E(u_0),
\]
where
\[
\dot{K}(u) = \kappa_1 K_1(u) + \kappa_2 K_2(u). \tag{3.9}
\]
(iii) In this case, we have from (3.7) and the mass conservation that
\[
\frac{1}{8} \frac{d^2}{dt^2} \mathcal{F}(u) = \|u\|_{L^2(\mathbb{R}^2)}^2 - (\theta + (1 - \theta))k_2 K_2(u) - \kappa_1 K_1(u)
\]
\[
= \left(1 - \frac{p_2 - 1}{4}\right) \|u\|_{L^2(\mathbb{R}^2)}^2 - \kappa_2 \theta \|D_x^2 u\|_{L^2(\mathbb{R}^2)}^2 + \kappa_2 \theta E(u_0)
\]
\[+ \kappa_2 (1 - \theta) K_2(u) - \frac{1}{2} (p_1 - 1 - \theta (p_2 - 1)) K_1(u)
\]
\[\leq \frac{4 - (p_2 - 1)\theta}{4} \|u\|_{L^2(\mathbb{R}^2)}^2 - \kappa_2 \theta \|D_x^2 u\|_{L^2(\mathbb{R}^2)}^2 + \kappa_2 \theta E(u_0)
\]
\[+ A_{p_1} \tau M(u_0) + (A_{p_1} C_\tau + A_{p_2}) \int_{\mathbb{R}^2} u^{p_2+1} \, dx dy.
\]
Now, if \( \tau > 0 \) is in such a way that \( A_{p_1} C_\tau + A_{p_2} \leq 0 \), then \( \frac{d^2}{dt^2} \mathcal{F}(u) < 0 \) provided (3.8) holds.

We have seen in Theorem 3.2 for the critical case \( p_1 = s_c \) that the solutions of (1.7) is uniformly bounded if the initial data is almost less that the \( L^2 \)-norm of the ground state of (2.9). In the following, we study the conditions under which \( L^2 \)-prescribed solutions of (1.9) exist in the critical case.

For \( \varrho > 0 \), we consider the minimizing problem
\[
d_{\varrho} = \inf \{ E(u); u \in X_\alpha, M(u) = \varrho \} \tag{3.10}
\]
and the set
\[
\Sigma_\varrho = \{ u \in X_\alpha, E(u) = d_{\varrho}, M(u) = \varrho \}.
\]

**Theorem 3.8.** Let \( p_2 = s_c, \mu_2 = 1 \) and \( Q \) be a ground state of (2.9). If \( \mu_1 < 0 \) and \( p_1 > s_c \), then for any \( \varrho \in (M(Q), +\infty) \), the set \( \Sigma_\varrho \) is nonempty. If \( \mu_1 > 0 \) and \( p_1 < s_c \), then for any \( \varrho \in (0, M(Q)) \), the set \( \Sigma_\varrho \) is not empty.

**Proof.** First we consider the case \( \mu_1 < 0 \) and \( p_1 > s_c \). The minimization value \( d_{\varrho} \) is bounded from below. Indeed, we have from (3.7) for any \( u \in X_\alpha \) with \( M(u) = \varrho \) that
\[
E(u) \geq \frac{1}{2} \|u\|_{X_\alpha}^2 + \left( \frac{\mu_1}{p_1 + 1} - \frac{C_\tau}{p_2 + 1} \right) \int_{\mathbb{R}^2} u^{p_1+1} \, dx dy - \tau \varrho
\]
\[
\geq \frac{1}{2} \|u\|_{X_\alpha}^2 + \left( \frac{\mu_1}{p_1 + 1} - \frac{C_\tau}{p_2 + 1} \right) \theta^{p_1} \rho_{p_1} \|u\|_{X_\alpha}^{\frac{2(p_1+1)}{p_1+1}} - \tau \varrho
\tag{3.11}
\]
If we choose \( \tau > 0 \) such that \( \frac{\mu_1}{p_1 + 1} - \frac{C_\tau}{p_2 + 1} > 0 \), then \( d_{\varrho} > -\infty \). We show that if \( \varrho \in (0, M(Q)) \), then \( d_{\varrho} \geq 0 \) while \( d_{\varrho} < 0 \) for all \( \varrho \in (M(Q), +\infty) \). Let \( u \in X_\alpha \) and \( M(u) = \varrho \leq M(Q) \). Then, we have that
\[
E(u) \geq \frac{1}{2} \|u\|_{X_\alpha}^2 - \rho_{p_2} \varrho^{\frac{p_2}{p_2+1}} \|u\|_{X_\alpha}^2 = \left( \frac{1}{2} - \rho_{p_2} \varrho^{\frac{p_2}{p_2+1}} \right) \|u\|_{X_\alpha}^2.
\]
We note that
\[
\rho_{p_2}^{-1} = \alpha^{-1} \left( \frac{\alpha}{2} \right)^{\frac{\alpha^2}{\alpha + 2}} \frac{2^{1-\alpha}}{\alpha + 2} \|Q\|_{L^2(\mathbb{R}^2)}^{\frac{4\alpha}{\alpha^2}}.
\]
Then, \( E(u) \geq 0 \) if
\[
\|u\|_{L^2(\mathbb{R}^2)} \leq \ell_\alpha \|Q\|_{L^2(\mathbb{R}^2)},
\]
where
\[
\ell_\alpha = \alpha^+ (\alpha + 2)^{-(s_c - 1)} 2^{\frac{1}{\alpha}}.
\]
Thus, it follows that \( d_{\varrho} \geq 0 \) for all \( \varrho \in (0, \ell_\alpha M(Q)) \). Next, for \( \epsilon > 0 \) we set
\[
u_\epsilon(x, y) = \frac{\sqrt{\varrho}}{\|Q\|_{L^2(\mathbb{R}^2)}} \epsilon^{\frac{\alpha + 2}{2}} Q(\epsilon x, \epsilon^{\alpha+1} y),
\]
and
Then $\|u_n\|_{L^2(\mathbb{R}^2)}^2 = \varrho$. Moreover, it is straightforward to check that

$$E(u_n) = \epsilon^{2\alpha} \left( \frac{\varrho}{2M(Q)} \|Q\|_{X,\alpha}^2 - \left( \frac{\varrho}{M(Q)} \right)^{\frac{3\alpha+2}{2}} K_2(Q) \right) + \epsilon^{(\alpha+2)\frac{p-1}{p}} \left( \frac{\varrho}{M(Q)} \right)^{\frac{p-1}{p}(\alpha+2)} K_1(Q)$$

provided $\varrho > M(Q)$ as $\epsilon \ll 1$, where in the above we use the following Pohozaev identity

$$\|Q\|_{X,\alpha}^2 = 2K_2(Q).$$

Next we show that every minimizing sequence for $d_\varrho$ is bound in $X_\alpha$ and bound from below in $L^{p_1+1}(\mathbb{R}^2)$. Let $\{u_n\}$ be a minimizing sequence. Since $M(u_n) = \varrho$, then we obtain from (3.11) that $\{u_n\}$ is bounded in $X_\alpha$. Furthermore, since $d_\varrho$, we have $E(u_n) \leq d_\varrho/2$ for $n$ large enough. The definition of $E$ and (3.7) show that

$$\|u_n\|_{L^{p_1+1}(\mathbb{R}^2)} \geq -E(u_n) \geq -\frac{d_\varrho}{2}.$$

Now suppose that $\{u_n\}$ is a minimizing sequence for $d_\varrho$, i.e., $\|u_n\|_{L^2(\mathbb{R}^2)}^2 = \varrho$ and $E(u_n) \to d_\varrho$ as $n \to \infty$. We first prove

$$d_{\varrho \theta} < \varrho d_\varrho$$

for all $\varrho > M(Q)$ and $\theta > 1$. Since $\varrho > M(Q)$, there exists $\delta > 0$ such that $\|u_n\|_{X,\alpha}^2 \geq \delta$ for sufficiently large $n$. If this is not true, then $\|u_n\|_{X,\alpha} \to 0$ as $n \to \infty$. By (2.3), this implies that $E(u_n)$ tends to zero as $n \to \infty$ which contradicts $d_\varrho < 0$. Therefore, the minimization problem (3.10) can be rewritten as

$$d_{\varrho \theta} = \inf\{E(u); \; u \in X_\alpha, \; M(u) = \varrho, \; \|u\|_{X,\alpha}^2 \geq \delta\}. \quad \text{(3.15)}$$

If we set $\tilde{u}(x,y) = u(\theta^{-\frac{2\alpha}{m+2}} x, \theta^{-\frac{2\alpha}{m+2}} y)$ with $\theta > 1$, then we can see that $M(\tilde{u}) = \theta M(u)$ and

$$d_{\varrho \theta} \leq \inf\{E(\tilde{u}); \; u \in X_\alpha, \; M(u) = \varrho, \; \|u\|_{X,\alpha}^2 \geq \delta\}, \quad \text{(3.16)}$$

where

$$E(\tilde{u}) = \frac{\theta^{\frac{2\alpha}{m+2}}}{2} \|u\|_{X,\alpha}^2 - \theta K(u) = \theta E(u) - \frac{1}{2} \left( \theta - \theta^{\frac{2\alpha}{m+2}} \right) \|u\|_{X,\alpha}^2 \leq \theta E(u) - \frac{\delta}{2} \left( \theta - \theta^{\frac{2\alpha}{m+2}} \right).$$

We have by taking the infimum that

$$d_{\varrho \theta} \leq \theta d_\varrho - \frac{\delta}{2} \left( \theta - \theta^{\frac{2\alpha}{m+2}} \right) < \theta d_\varrho \quad \text{for all } \varrho > M(Q) \text{ and } \theta > 1. \quad \text{(3.17)}$$

This implies that

$$d_\varrho < d_\beta + d_{\varrho - \beta}$$

for all $\varrho > M(Q)$ and $\beta \in (0, \varrho)$. More precisely, if $\beta > M(Q)$ and $0 < \varrho - \beta \leq M(Q)$, then we obtain that

$$d_\varrho = d_{\beta \varrho / \beta} < \frac{\varrho}{\beta} d_\beta = d_\beta + \frac{\varrho - \beta}{\beta} d_\beta \leq d_\beta + d_{\varrho - \beta}.$$
Now we apply the concentration-compactness principle in $X_{\alpha}$ for $\{u_n\}$, similar to \[9,10\]. The vanishing case cannot occur by \[3.11\]. The dichotomy case is rules out, similar to \[10\] by using the subadditivity property \[3.17\] with the help of the fractional Leibniz rule together with the fractional commutator estimate \[18\] Lemma 2.5 \[44\] Lemma 2.12 (see also \[24\] for the estimates of higher order fractional derivatives) and the fractional Poincaré inequality (see \[12\] Proposition 2.1, \[21\] Lemma 2.2 and \[5,28\]). Finally, we can conclude by ruling out vanishing and dichotomy that indeed compactness occurs. Thereby, we infer that there exist a subsequence $\{u_{n_k}\}$, $\{z_k\} \subset \mathbb{R}^2$ and some $u \in X_{\alpha}$ such that

$$u_{n_k}(\cdot - z_k) \rightarrow u$$

in $L^q(\mathbb{R}^2)$ for all $q < 2^\ast$. Thus, we deduce from this fact combined with the weak lower semicontinuity of the $X^\alpha$-norm that

$$E(u) \leq \lim_{k \to \infty} E(u_{n_k}) = d_\varrho.$$ 

We have from the definition $d_\varrho$ of that $E(u) = d_\varrho$. Particularly, $E(u_{n_k}) \rightarrow E(u)$, and it indicate that $\|u_{n_k}\|_{X_{\alpha}} \rightarrow \|u\|_{X_{\alpha}}$, which implies that $u_{n_k}(\cdot - z_k) \rightarrow u$ strongly in $X_{\alpha}$.

Next we consider the case $\mu_1 > 0$ and $p_1 < s_c$. We first note for $u \in X_{\alpha}$ with $u \neq 0$ and

$$u_\varepsilon(x,y) = \varepsilon^{\frac{\alpha+2}{2}} u(\varepsilon x, \varepsilon^{\alpha+1} y) \quad (3.18)$$

that

$$d_\varrho \leq E(u_\varepsilon) = \frac{\varepsilon^{2\alpha}}{2} \|u\|^2_{X_{\alpha}} - \frac{\varepsilon^{(\alpha+2)(p_1-1)}}{2} K_1(u) - \frac{\varepsilon^{(\alpha+2)(p_2-1)}}{2} K_2(u) < 0$$

for sufficiently small $\varepsilon > 0$. Moreover, any minimizing sequence if bounded in $X_{\alpha}$ by \[\|u\|_{X_{\alpha}} \leq M\]. Furthermore, it follows from $\varrho < M(Q)$ that there exists $\delta > 0$ such that $K_1(u_n), K_2(u_n) \geq \delta$ for all sufficiently large $n$. We also observe for $\varrho > M(Q)$ and $V = \frac{\sqrt{q_0}}{\|Q\|_{L^2(\mathbb{R}^2)}}$ that $M(U) = \varrho$ and

$$\frac{1}{2} \|U\|^2_{X_{\alpha}} - K_2(Q) = \frac{1}{2} \frac{\|Q\|^2_{X_{\alpha}}}{M(Q)} \varrho - \frac{\varrho^{p_2+1}}{(M(Q))^{\frac{p_2+1}{2}}} K_2(Q) < 0.$$ 

This together with $p_2 < p_2$ with \[3.18\] reveals that

$$\lim_{\varepsilon \rightarrow \infty} E(u_\varepsilon) = -\infty.$$ 

Second, let

$$u_\theta(x,y) = \theta^{\frac{\alpha}{p_1-1}} u(\theta x, \theta^{\alpha+1} y),$$

so that

$$\frac{d}{d\theta} E(u_\theta) \bigg|_{\theta=1} = E(u) = \theta_1 \left( \frac{1}{2} \|u\|^2_{X_{\alpha}} - K_1(u) \right) - \theta_2 K_2(u) = \theta_1 E(u) - \theta_3 K_2(u),$$

where

$$\theta_1 = \frac{4\alpha}{p_1-1} + \alpha - 2, \quad \theta_2 = \frac{4\alpha}{p_2-1} + \alpha - 2, \quad \theta_3 = \frac{2\alpha}{p_1-1}(p_2 - p_1).$$

Together with $\|u_\theta\|^2_{L^2(\mathbb{R}^2)} = \theta^{\frac{4\alpha}{p_1-1}-\alpha-2} \|u\|^2_{L^2(\mathbb{R}^2)}$, which implies that

$$\frac{d}{d\theta} \bigg|_{\theta=1} \|u_\theta\|_{L^2(\mathbb{R}^2)} > 0,$$

and this shows that $d_\varrho$ is decreasing in $\varrho$. As a consequence, we have the strict subadditivity condition. But we alternatively show this directly. Let $\{U_n\}$ and $\{V_n\}$ be sequences of functions in $X_{\alpha}$ such
Thus, we deduce from the Brezis-Lieb lemma that $E$ gives that $\theta > \nu$, which implies for any $\nu > 0$. To do so, we define some submanifolds of $X$, which together with $d_\nu$ can obtain from the (2.3) that $\vartheta \leq \delta(\tilde{\nu}_n) + (\nu_0 - 1)K_1(u_n) + (\nu_0)^2 - 1)K_2(u_n)$

This means that $d_{\nu} + \nu'' < d_{\nu} + \nu'$. This inequality holds similarly when $\nu' > \nu''$. If $\nu' = \nu''$, then there exists $\nu > 0$ such that $E(\tilde{\nu}_n) \leq \tilde{\vartheta}E(U_n) - \delta$ for sufficiently large $n$. This in turn implies that

$$E(\tilde{\nu}_n) \leq \tilde{\vartheta}E(U_n) - \delta$$

for sufficiently large $n$. Thus, we obtain that

$$d_{\nu} + \nu'' \leq \tilde{\vartheta} \lim_{n \to \infty} E(U_n) - \delta = \tilde{\vartheta}E(U_n) - \delta = \nu' + \nu' - \delta.$$

Therefore, similar to the case $\mu_1 > 0$, the boundedness of the minimizing sequence $\{u_n\}$ now implies that there exists $u \in X_1$ such that $u_n \to u$ in $X_1$, up to a subsequence. The limit function is nontrivial. If not, then $K(u_n) \to 0$ which contradicts to $E(u_n) = \frac{1}{2}\|u_n\|_{X_1}^2 \to d_\nu < 0$. Next, we prove $M(u) = \vartheta$. If $M(u) < \vartheta$, we get

$$\kappa_0 = \frac{\sqrt{\vartheta}}{\|u\|_{L^2(\mathbb{R}^2)}}, \quad \kappa_n = \frac{\sqrt{\vartheta}}{\|u_n - u\|_{L^2(\mathbb{R}^2)}} > 1.$$

Note that

$$E(\theta u) = \theta^2 E(u) + \theta^2(1 - \theta^p)K_1(u) + \theta^2(1 - \theta^p)K_2(u),$$

which implies for any $\theta > 0$ that

$$E(u) = \theta^{-2} E(\theta u) + (\theta^p - 1)K_1(u) + (\theta^p - 1)K_2(u).$$

Thus, we deduce from the Brezis-Lieb lemma that

$$E(u_n) = E(u) + E(u_n - u) + o_n(1),$$

$$= \kappa_0^{-2} E(\kappa_0 u) + (\kappa_0^p - 1)K_1(\kappa_0 u) + (\kappa_0^p - 1)K_2(\kappa_0 u)$$

$$+ \kappa_0^{-2} E(\kappa_n(u_n - u)) + (\kappa_n^p - 1)K_1(\kappa_0(u_n - u)) + (\kappa_n^p - 1)K_2(\kappa_0(u_n - u))$$

$$\geq \frac{M(u)}{\vartheta} d_\nu + \frac{M(u_n - u)}{\vartheta} d_\nu + o_n(1).$$

Taking the limit $n \to \infty$, we get

$$d_\nu \geq \frac{M(u)}{\vartheta} d_\nu,$$

which together with $d_\nu < 0$, implies that $M(u) = \vartheta$. Hence, $M(u) = \vartheta$ and then $u_n \to u$ in $L^p(\mathbb{R}^2)$. We can obtain from the [2,3] that $u_n \to u$ in $L^p(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Consequently, we obtain

$$E(u) \leq \liminf_{n \to \infty} E(u_n) = d_\nu.$$

This means that $E(u) = d_\nu$ and $u \in \Sigma_\vartheta$. 

4 Instability of ground states

In this section, we prove the instability of ground state solutions of (1.3) by the mechanism of blow-up. To do so, we define some submanifolds of $X_\alpha$ which are invariant under the flow of the Cauchy problem.
associated with (1.5), and the solution of (1.5) will blow-up in finite time if we choose the associated initial data suitably from these sets. The instability of the ground states is derived from the blow-up phenomenon.

Let \( b, d \geq 0 \), and define

\[
R_{b,d}(u) = ab\|D_x^2 u\|_{L^2(\mathbb{R}^2)}^2 + (d-b)\|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^2 - (b+d)(\mathcal{E}_1(u) + \mathcal{E}_2(u)).
\]

We denote

\[
\Lambda_{b,d} = \{ u \in X_\alpha \setminus \{0\}, S(u) < m, R_{b,d}(u) = 0 \},
\]

and set

\[
m_{R_{b,d}} = \inf_{u \in \Lambda_{b,d}} S(u).
\]

**Lemma 4.1.** Let \( \mu_1 > 0 \). If \( \mu_2 < 0, p_1 \geq p_2 \) and

\[
p_1 \geq 4 \max \{ d-b, ab \} \frac{b+d}{b+d} + 1.
\]

Then

\[
m = m_{R_{b,d}}.
\]

Identity (4.1) holds when \( \mu_2 > 0 \) if

\[
p_1 \geq p_2 \geq 4 \max \{ d-b, ab \} \frac{b+d}{b+d} + 1.
\]

Proof. By Theorem 2.4 we show that \( \varphi \) is ground state of (1.9) if and only if \( \varphi \in \Lambda_{b,d} \) and \( S(\varphi) = m_{R_{b,d}} \).

First we note that if \( \varphi \in \Lambda_{b,d} \), then \( \varphi \in \Lambda_{b,d} \) by

\[
R_{b,d}(\varphi) = \left< S'(\varphi), \frac{b+d}{2} \varphi + b x \varphi_x + d y \varphi_y \right> = 0. \tag{4.2}
\]

Next, suppose that \( R_{b,d}(u) < 0 \). Then \( R_{b,d}(\lambda u) > 0 \) for some sufficiently small \( \lambda > 0 \), and hereby there exists \( \lambda_0 \in (0,1) \) such that \( R_{b,d}(\lambda_0 u) = 0 \). Hence, \( m_{R_{b,d}} \leq \tilde{S}(u) \), where \( \tilde{S} = S - \frac{2}{r(2+b)}R_{b,d} \) with \( r = p_1 - 1 \) if \( \mu_2 < 0 \) and \( r = p_2 - 1 \) when \( \mu_2 > 0 \). This means that

\[
m_{R_{b,d}} = \tilde{m}_{R_{b,d}} := \inf_{u \in \Lambda_{b,d}} \tilde{S}(u),
\]

where

\[
\tilde{\Lambda}_{b,d} = \{ u \in X_\alpha \setminus \{0\}, R_{b,d}(u) \leq 0 \}.
\]

So it is enough to find the ground state \( \varphi \) such that \( \tilde{m}_{R_{b,d}} = \tilde{S}(\varphi) \). Notice the assumptions on \( p_1 \) and \( p_2 \) shows that \( \tilde{S}(\varphi) > 0 \). Hence, there exists a minimizing sequence \( \{ u_n \} \subset \tilde{\Lambda}_{b,d} \) of \( \tilde{m}_{R_{b,d}} \) that is bounded in \( X_\alpha \), and in \( L^q(\mathbb{R}^2) \) for \( q \in [2,2^*] \) from (2.2), and \( \lim_{n \to \infty} \tilde{S}(u_n) = \tilde{m}_{R_{b,d}} \). This implies that there exist a subsequence of \( \{ u_n \} \), still denoted by the same \( \{ u_n \} \), and \( u \in X_\alpha \) such that \( u_n \) converges to \( u \) weakly in \( X_\alpha \). It follows from Lemma 3.3 in (10) that \( u_n \to u \) a.e. in \( \mathbb{R}^2 \). We show that \( \tilde{S}(u) = \tilde{m}_{R_{b,d}} \) and \( u \in \Lambda_{b,d} \).

If we assume that \( i_{p_1} := \inf \|u_n\|_{L^{p_1+1}(\mathbb{R}^2)} > 0 \), then Lemma 4 in (27) shows that \( u \not\equiv 0 \) a.e. in \( \mathbb{R}^2 \). Now if \( i_{p_1} = 0 \), then exists a subsequence of \( \{ u_n \} \), still denoted by the same, such that \( \|u_n\|_{L^{p_1+1}(\mathbb{R}^2)} \to 0 \) as \( n \to \infty \), so that \( \|u_n\|_{L^{p_2+1}(\mathbb{R}^2)} \to 0 \) from the boundedness of \( \{ u_n \} \) in \( X_\alpha \). Since \( u_n \subset \tilde{\Lambda}_{R_{b,d}} \), \( \|u_n\|_{\tilde{X}_\alpha} \to 0 \) as \( n \to \infty \). Moreover, \( \|D_x^2 u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^{2(d-b)} \to 0 \). Consequently, we have from (23) that

\[
\|D_x^2 u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^{2(d-b)} \lesssim \|u_n\|_{L^{p_1+1}(\mathbb{R}^2)}^{p_1} \lesssim \|D_x^2 u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^{2(d-b)}.
\]
for all $p \leq p_1$, and equivalently,
\[ \|D_x^p u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)} \left( 1 - C\|D_x^p u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)} \right) \leq 0. \]

This turns into
\[ \|D_x^p u\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)} \geq 1. \]

This contradicts $\|u_n\|_{X_n} \to 0$ by the assumptions on $p_1$. Consequently, $i_{p_1} > 0$. Now if $R_{b,d}(u) > 0$, then the Brezis-Lieb lemma and the fact $\{u_n\} \subset \tilde{\Lambda}_{b,d}$ reveals that $R_{b,d}(u_n - u) \leq 0$ as $n \to \infty$, so that $\tilde{S}(u_n - u) \geq \tilde{m}_{R_{b,d}}$. Since $\tilde{S}(u_n - u) \to \tilde{m}_{R_{b,d}}$ as $n \to \infty$, we have again from the Brezis-Lieb lemma that $\tilde{S}(u) \leq 0$ which contradicts $u \not\equiv 0$ a.e. Consequently, $u \in \tilde{\Lambda}_{b,d}$. This shows that $\tilde{S}(u) = \tilde{m}_{R_{b,d}}$. It is easy to check that $R_{b,d}(u) = 0$. Indeed, since $R_{b,d}(\lambda u) > 0$ for sufficiently small $\lambda > 0$, so $R_{b,d}(u) < 0$ implies that $\lambda_0 u \in \Lambda_{b,d}$, which is a contradiction to the definition of $\tilde{m}_{R_{b,d}}$. Finally, we prove that $u$ is ground state. We have from $m_{R_{b,d}} = \tilde{m}_{R_{b,d}} = S(u)$ that there is $\theta \in \mathbb{R}$ such that $S'(u) + \theta R'_{b,d}(u) = 0$. The fact $u \in \Lambda_{b,d}$ reveals from
\[
\left\langle S'(u) + \theta R'_{b,d}(u), \frac{b + d}{2} \varphi + bx \varphi_x + dy \varphi_y \right\rangle = 0
\]
that $\theta = 0$. Now if $S'(w) = 0$, then $w \in \Lambda_{b,d}$ from (4.2). Therefore, it follows from the definition of $m_{R_{b,d}}$ such that $S(u) \leq S(w)$. \(\square\)

Consider the submanifolds
\[ \Lambda^+_{b,d} = \{ u \in X_\alpha, S(u) < m, R_{b,d}(u) \geq 0 \}, \]
and
\[ \Lambda^-_{b,d} = \{ u \in X_\alpha, S(u) < m, R_{b,d}(u) < 0 \}. \]

**Lemma 4.2.** Let $d \geq b$ and $p_j$ satisfy Lemma \[1.1\]. Then the sets $\Lambda^+_{b,d}$ and $\Lambda^-_{b,d}$ are invariant under the flow generated by the Cauchy problem associated with (1.5).

**Proof.** We only prove that $R_{b,d}^+$ is invariant under the flow generated by the Cauchy problem associated with (1.5) since the proofs of the other set is similar. Let $u(t)$ be the solution of (1.5) with initial data $u_0 \in R_{b,d}^+$. We first note from Theorem 3.1 that $u(t) < m$. Next, we show that $R_{b,d}(u(t)) \geq 0$ for $t \in [0, T)$. If it is not true, the continuity of $R_{b,d}$ implies that there exists $t_1 \in (0, T)$ such that $R_{b,d}(u(t_1)) = 0$. This means that $u(t_1) \in \Lambda$. So that $S(u(t_1)) \geq m_{R_{b,d}} \geq m$, which contradicts with $S(u(t)) < m$ for all $t \in (0, T)$. Therefore, $R_{b,d}(u(t)) > 0$ for $t \in [0, T)$. \(\square\)

The following theorem gives another conditions under which the uniformly boundedness of solutions in the energy space is guaranteed.

**Theorem 4.3.** Let $\mu_1 > 0$ and $u_0 \in \Lambda^+_{b,d}$. Suppose that $p_1 \geq p_2$ and
\[ p_1 > 4 \max\{d - b, \alpha b\} \frac{b + d}{b + d} + 1, \]
if $\mu_2 < 0$ while
\[ p_1 \geq p_2 > 4 \max\{d - b, \alpha b\} \frac{b + d}{b + d} + 1 \]
when $\mu_2 > 0$. Then the solution $u(t)$ in Theorem 3.1 is uniformly bounded in the energy space.

**Proof.** We first note that $\Lambda^+_{b,d} \neq \emptyset$. Let $u_0 \in \Lambda^+_{b,d}$ and $u(t)$ be the corresponding solution of (1.5) for $t \in [0, T)$ with the initial data $u_0$. Suppose by contradiction that $T < +\infty$. Then by Theorem 3.1
\[ \lim_{t \to T^-} \|u\|^2_{X_n} = +\infty. \]
Hence, by the assumptions on \( p_j \) and using the conservation laws \( E(u(t)) = E(u_0) \) and \( M(u(t)) = M(u_0) \) for \( 0 \leq t < T \) that

\[
S(u_0) - \frac{2}{r(b + d)} R_{b,d}(u(t)) = \hat{S}(u(t)) - \frac{2}{r(b + d)} Q(u(t)) \\
\lesssim \hat{S}(u(t)) \approx \|u\|_\hat{X}_\alpha^2,
\]

where \( \hat{S} \) and \( r \) are as in the proof of Lemma 4.1. Thus, we deduce from (4.3) that

\[
\lim_{t \to T^-} R_{b,d}(u(t)) = -\infty.
\]

Now, we infer from the continuity that there is \( t_0 \in (0, T) \) such that \( R_{b,d}(u(t_0)) = 0 \). Lemma 4.1 then implies \( S(u(t_0)) \geq m \), which contradicts the fact \( S(u(t)) = S(u_0) < m \).

To apply the concavity method, we need to show that \( R_{b,d}(u(t)) \) is negative which is fundamental in our instability analysis.

**Theorem 4.4.** Let \( \varphi \in N_0, b \geq 0, d \geq b + 1 \) and \( u_0 \in \Lambda_{b,d}^- \cap \Lambda_{b,d}^+ \). Suppose that \( p_1 \geq p_2 \) and

\[
p_1 > 4 \max \left\{ \frac{d - b}{b + d}, \frac{\alpha b}{d + b - 1} \right\} + 1,
\]

if \( \mu_2 < 0 \) while

\[
p_1 \geq p_2 > 4 \max \left\{ \frac{d - b}{b + d}, \frac{\alpha b}{d + b - 1} \right\} + 1,
\]

when \( \mu_2 > 0 \). Then the solution \( u(t) \) of (1.3), corresponding to the initial data \( u(0) = u_0 \), satisfies

\[
R_{b,d}(u(t)) < \frac{(1 - r)(b + d)}{2} (S(\varphi) - S(u_0))
\]

for \( 0 \leq t < T \), where \( r \) is as the same in the proof of Lemma 4.1. Moreover, \( u(t) \) blows up in finite time. More precisely, there exists \( 0 < \tau < \infty \) such that

\[
\lim_{t \to \tau^-} \|u(t)\|_{L^2(\mathbb{R}^2)} = +\infty.
\]

**Proof.** Note from the virial identity (3.5) that

\[
\frac{1}{8} \frac{d^2}{dt^2} \mathcal{I}(u) = R_{b,d}(u) - R_{b,d-1}(u).
\]

Then proof of theorem is similar to the one of Theorem 3.4 by using the above identity and combining Lemmas 4.1 and 4.2.

The following equivalence is useful to work with the critical points of \( m \).

**Lemma 4.5.** Let \( s \geq \alpha + 1 \). Then \( \tilde{m} = m \), where \( \tilde{m} = \inf_{u \in \tilde{N}_0} S(u) \) and

\[
\tilde{N}_0 = \{ u \in X^s \setminus \{0\}, \ P(u) = 0 \}.
\]

**Proof.** Clearly \( \tilde{m} \geq m \). To see the converse, it suffices to show that for any \( \epsilon > 0 \) and \( u^* \in \tilde{N}_0 \), there holds

\[
S(u^*) \geq \inf_{u \in \tilde{N}_0} S(u) - \epsilon.
\]

Since \( X^s \) is dense in \( X_\alpha \), we find a sequence \( \{u_n\} \subset X^s \) such that \( u^* = u_n + w_n \) such that \( w_n \to 0 \) in \( X_\alpha \) as \( n \to \infty \). Then we obtain from \( u^* \neq 0 \) and \( u^* \in \tilde{N}_0 \) that \( \lim_{n \to \infty} P(u_n) = 0 \) and

\[
\lim_{n \to \infty} \|u\|_{L^q(\mathbb{R}^2)}^q \neq 0.
\]
for any $q \leq 2^*$. It is easy to check that there exists a sequence $\{v_n\} \subset \mathbb{R}$ such that $v_n u_n \in \tilde{N}_0$ and $v_n \to 1$ as $n \to \infty$. Denoting $u^* = v_n u_n + (1 - v_n) u_n + w_n$, we have from $1 - v_n \to 0$ and $w_n \to 0$ in $X_\alpha$ as $n \to \infty$ that
\[
\frac{1}{2} \|u^*\|_{X_\alpha}^2 \geq \frac{1}{2} \|v_n u_n\|_{X_\alpha}^2 - \epsilon \frac{1}{3}
\]
and
\[
K(u^*) \leq K(v_n u_n) + \epsilon \frac{1}{3}.
\]
On the other hand, the fact $\|w_n\|_{X_\alpha} \to 0$ as $n \to \infty$ implies that $S(u^*) \geq S(v_n u_n) - \epsilon \geq \tilde{m} - \epsilon$. \hfill \QED

**Lemma 4.6.** Let $u, w \in X_\alpha$ be fixed and $\|w\|_{X_\alpha} \leq C$. Then there exists positive numbers $C_j$ with $j = 1, 2, 3$, independent of $w$, such that
\[
S(u + w) < S(u) + C_1 \|w\|_{X_\alpha},
\]
\[
R_{b,d}(u) - C_3 \|w\|_{X_\alpha} < R_{b,d}(u + w) < R_{b,d}(u) + C_2 \|w\|_{X_\alpha}.
\]

**Proof.** The proof is similar to Lemma 4.7 in [13], so we omit the details. \hfill \QED

No we are in the position to show the strong instability of solitary waves.

**Theorem 4.7.** Let $\mu_1 > 0$ and $\varphi$ be a solitary wave solution. Suppose that $p_1 \geq p_2$ and $p_1 > s_\alpha$ if $\mu_2 < 0$ and $p_1 > p_2 > s_\alpha$ when $\mu_2 > 0$. Then for any $\delta > 0$, there exists $u_0 \in X_\alpha$ ($s \geq \alpha + 1$) with $\|u_0 - \varphi\|_X < \delta$, such that the solution $u(t)$ of (1.5) with initial data $u(0) = u_0$ satisfies
\[
\lim_{t \to T^-} \|u(t)\|_{X_\alpha} = +\infty
\]
for some $T > 0$.

**Proof.** The proof is based on the application of Theorem 4.4. For $B, D > 0$ we set $w(x, y) = \sqrt{BD} \varphi(Bx, Dy)$, and
\[
r_j = 4 + \frac{p_j + 3}{2} (\alpha - 2), \quad j = 1, 2.
\]
Then, we have from the Pohojaev identities
\[
\alpha \|D^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2 = 2 \|\partial_x^{-1} \varphi_y\|_{L^2(\mathbb{R}^2)}^2, \\
\|\partial_x^{-1} \varphi_y\|_{L^2(\mathbb{R}^2)}^2 = k_1 K_1(\varphi) + k_2 K_2(\varphi), \\
k_1 \|\varphi\|_{L^2(\mathbb{R}^2)}^2 = \frac{r_1}{2} \|D^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2 - (p_2 - p_1) K_2(\varphi), \\
k_2 \|\varphi\|_{L^2(\mathbb{R}^2)}^2 = \frac{r_2}{2} \|D^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2 - (p_1 - p_2) K_1(\varphi)
\]
that
\[
S(\varphi) = \left( \frac{r_j}{2\alpha k_j} + \frac{1}{2} + \frac{1}{\kappa_j} \right) \|\partial_x^{-1} \varphi_y\|_{L^4(\mathbb{R}^2)}^2.
\]
Moreover, we consider $d = (1 + \alpha)b$ to observe that
\[
\frac{1}{b} R_{b,d}(w) = \alpha \left( \frac{B^{2\alpha} + \frac{D^2}{2B^2} - k_2(\alpha + 2)(BD)^{\kappa_j}}{2} \right) \|D^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2
\]
\[
- (\alpha + 2)(-1)^j (k_2(\alpha + 2)(BD)^{\kappa_j} - k_3(\alpha + 2)(BD)^{\kappa_j}) K_j(\varphi)
\]
and
\[
S(w) = \frac{\alpha}{2} \left( \frac{r_j}{2\alpha k_j} + \frac{B^{2\alpha}}{\alpha} + \frac{D^2}{2B^2} - (BD)^{\kappa_j} \right) \|D^\alpha \varphi\|_{L^2(\mathbb{R}^2)}^2
\]
\[
- (\alpha + 2) \left( \frac{(-1)^j(p_2 - p_1)}{2k_j} + (BD)^{\kappa_j} - \frac{\kappa_2 k_j}{k_3} \right) K_j(\varphi)
\]
\[21\]
for \( j = 1, 2 \). Now, if \( \mu_2 < 0 \), then by using \( j = 1 \) in the above equations and choosing \( BD > 1 \) and near to 1, we find after some calculations for \( b > (\alpha + 2)^{-1} \) that \( R_{b,d}(w) < 0 \) and \( S(w) < m \) provided

\[
p_1 > 1 + \frac{4\alpha b}{(\alpha + 2)b - 1}
\]

and \( p_2 \leq p_1 \). In the case \( \mu_2 > 0 \), we get the similar inequalities if

\[
p_1 \geq p_2 > 1 + \frac{4\alpha b}{(\alpha + 2)b - 1}
\]

We note from (4.6) and \( R_{b,d}(w) < 0 \) that \( R_{b,d-1}(w) > 0 \) if we choose \( b > 0 \) sufficiently large. Now, by choosing for \( \delta > 0 \) the above function \( w \) such that \( \|w - \varphi\|_{X^s} < \delta/2 \), we can find from the density \( X^s \hookrightarrow X_\alpha \) the initial data \( u_0 \in X^s \) such that

\[
\|u_0 - w\|_{X_\alpha} < \min\{\delta/2, \frac{\delta_1}{C_1 + C_2 + C_3}\},
\]

where \( C_j \) with \( j = 1, 2, 3 \) are as in Lemma 4.6 and \( \delta_1 = \min\{m - S(w), -R_{b,d}(w), R_{b,d-1}(w), \delta/2\} \). Therefore,

\[
\|\varphi - u_0\|_{X_\alpha} \leq \|u_0 - w\|_{X_\alpha} + \|w - \varphi\|_{X_\alpha} < \delta,
\]

so that \( u_0 \in \Lambda_{b,-d}^- \cap \Lambda_{b,d-1}^+ \). Theorem 4.4 shows that the solution \( u(t) \) associated with the initial data \( u_0 \) blows-up in a finite time.

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