Conservation laws for the classical Toda field theories.

Erling G. B. Hohler and Kåre Olaussen
Institutt for fysikk, NTH,
Universitetet i Trondheim
N–7034 Trondheim, Norway.

July 9, 1993

Abstract

We have performed some explicit calculations of the conservation laws for classical (affine) Toda field theories, and some generalizations of these models. We show that there is a huge class of generalized models which have an infinite set of conservation laws, with their integrated charges being in involution. Amongst these models we find that only the $A_m$ and $A_m^{(1)} (m \geq 2)$ Toda field theories admit such conservation laws for spin-3. We report on our explicit calculations of spin-4 and spin-5 conservation laws in the (affine) Toda models. Our perhaps most interesting finding is that there exist conservation laws in the $A_m$ models ($m \geq 4$) which have a different origin than the exponents of the corresponding affine theory or the energy-momentum tensor of a conformal theory.
1 Introduction

There is an intimate connection between the integrability of a Hamiltonian system and the existence of a sufficient number of conserved quantities. Liouville proved that if a system with $N$ degrees of freedom (i.e., with a $2N$-dimensional phase space) has $N$ independent first integrals in involution (i.e. with mutually vanishing Poisson brackets), then the system is integrable by quadratures\cite{1}. Conversely, an integrable system with $N$ degrees of freedom has $N$ independent conservation laws in involution. During the last quarter of a century there has been great progress in the understanding of integrable systems with *infinitely* many degrees of freedom, e.g. certain 1+1 dimensional non-linear partial differential equations\cite{2}. Such systems must necessarily have an infinite number of conserved quantities in involution. Since they are governed by local evolution equations the corresponding conservation laws can be written in a local form

$$\partial J_S + \partial \overline{J}_S = 0, \quad S = 1, 2, \ldots, \infty$$

(1)

where $\partial \equiv \partial/\partial t + \partial/\partial x$ and $\overline{\partial} \equiv \partial/\partial t - \partial/\partial x$. Since there are many ways to count to infinity it is a more delicate question to decide when the existence of an infinite number of independent conservation laws imply complete integrability.

We here report on some investigations of the conservation laws for models defined by Lagrangians of the type

$$L = \frac{1}{2} \partial \varphi \cdot \overline{\partial} \varphi - V(\varphi) \equiv \frac{1}{2} \partial \varphi \cdot \overline{\partial} \varphi - \sum_{r=1}^{n} v_r(\varphi),$$

(2)

where $\varphi$ is a real $m$-component field, and $v_r(\varphi) = a_r \exp(\alpha_r \cdot \varphi)$. Here the $\alpha_r$’s are distinct real $m$-component vectors, and the $a_r$’s are real numbers. To this class of models belong the so-called Toda field theories, exemplified by the potential (in appropriate dimensionless units)

$$V(\varphi) = \sum_{r=1}^{m-1} \exp(\varphi_r - \varphi_{r+1}) \equiv W_m,$$

(3)

and the affine Toda field theories, exemplified by the potential

$$V(\varphi) = W_m + \exp(\varphi_m - \varphi_1),$$

(4)

i.e. the model (3) with periodic boundary conditions. We note that with an $x$-independent field $\varphi$ these models reduce to respectively the open and periodic Toda lattices\cite{3}, hence their names.

For a simple physical interpretation these models can be viewed as a collection of (infinitely long) parallel strings, with a (somewhat strange) interaction potential between adjacent strings. Another possible physical interpretation is
Table 1: Some potentials of (affine) Toda field theories, with the nomenclature for their associated algebras. There are no universally accepted names for the affine Lie algebras, we are following the conventions of Kac\cite{Kac}. Here \( m \geq 2 \) for \( A_{2m}^{(2)} \) and \( A_{2m-1}^{(2)} \). To avoid equivalences one may also make the restrictions \( m \geq 2 \) for \( B_m \), \( m \geq 3 \) for \( C_m \), and \( m \geq 4 \) for \( D_m \).

| \( V(\varphi) \) | Algebra | \( V(\varphi) \) | Algebra |
|-------------------|--------|-----------------|--------|
| \( W_m + \exp(\varphi_m) \) | \( B_m \) | \( W_3 + \exp(\varphi_2 + \varphi_3 - 2\varphi_1) \) | \( D_4^{(1)} \) |
| \( W_m + \exp(2\varphi_m) \) | \( C_m \) | \( W_m + \exp(\varphi_m) + \exp(-\varphi_1 - \varphi_2) \) | \( B_m^{(1)} \) |
| \( W_m + \exp(\varphi_{m-1} + \varphi_m) \) | \( D_m \) | \( W_m + \exp(2\varphi_m) + \exp(-2\varphi_m) \) | \( C_m^{(1)} \) |
| \( W_m + \exp(\varphi_m - \varphi_1) \) | \( A_{m-1}^{(1)} \) | \( W_{2m} + \exp(2\varphi_{2m}) + \exp(-\varphi_1) \) | \( A_{2m} \) |
| \( W_2 + \exp\left(\frac{1}{2}\varphi_2 - \frac{1}{2}\varphi_1\right) \) | \( A_2^{(2)} \) | \( W_{2m-1} + \exp(2\varphi_{2m-1}) + \exp(-\varphi_1 - \varphi_2) \) | \( A_{2m-1}^{(2)} \) |
| \( W_m + \exp(\varphi_m) + \exp(-\varphi_1) \) | \( D_m^{(2)} \) | \( W_m + \exp(\varphi_{m-1} + \varphi_m) + \exp(-\varphi_1 - \varphi_2) \) | \( D_m^{(1)} \) |

as a collection of parallell wires, with exponential inductance for the leakage currents between adjacent wires. The fields then represent the voltages on the wires. However, the main interest in these models stems from the fact that they after quantization may represent a huge class of conformal field theories, or integrable perturbations of such theories. Thus, they have the potential of describing the relevant degrees of freedom of many 2-dimensional statistical models at—and close to—their critical points. They also provide interesting examples of interacting relativistic quantum field theories.

In a general Toda field theory the \( \alpha_i \)'s constitute the simple positive roots of a Lie algebra, and in a general affine Toda field theory they constitute these roots augmented with an additional vector which is some linear combination of the roots of the Lie algebra\footnote{In many, but not all, cases this vector is the negative of the maximal root of the Lie algebra.}. In mathematicians nomenclature the examples \( (3, 4) \) correspond to the algebras \( A_{m-1} \equiv SU(m) \) and \( A_{m-1}^{(1)} \). Readers who are unfamiliar with the language of Lie algebras should not feel intimidated by these statements,—they are just a peculiar (but concise) way of describing some of the possible potentials \( V(\varphi) \). Some explicit examples of model potentials with their associated algebras are listed in Table 1.

By simply looking at the potentials in Table 1 it is rather difficult to see any direct significance of the Lie algebras for the models they specify. Although the existence of these algebras have been instrumental in proving integrability of the (classical) models\footnote{In many, but not all, cases this vector is the negative of the maximal root of the Lie algebra.}, and the existence of an infinite set of (classical) conservation laws in involution\footnote{In many, but not all, cases this vector is the negative of the maximal root of the Lie algebra.}, this does not rule out that there may exist additional models in the class \((3, 4)\) which are either integrable or at least have an infinite set of local conservation laws. We thus set out to investigate the
conservation laws of models (2) in more generality, and indeed found additional examples which—as a consequence of conformal symmetry—have an infinite set of (classical) conservation laws.

2 Field equations and spin-1 conservation laws

We now turn to the Lagrangian (2). The components of the vectors $\alpha_r$ may be collected into a $n \times m$ matrix $A_{ri} \equiv (\alpha_r)_i$. Not all such matrices represent truly different models, since we may make orthogonal transformations on the field components, $\varphi_i = O_{ij} \varphi'_j$, and permute the order of the potentials $v_r$. I.e., models described by the matrices $A_{ri}$ and (summation convention is used unless noted otherwise)

$$A'_{ri} = P_{rs} A_{sj} O_{ji},$$

where $P_{rs}$ is a permutation matrix and $O_{ji}$ is an orthogonal matrix, are essentially equivalent. Such transformations with $A_{ri} = A'_{ri}$ are symmetries of the model. Further, depending on the solvability of the equations

$$A_{ri} \Delta \varphi_i + \log \left( |L^2 a_r| \right) = 0,$$

the coefficients in front of the exponentials may be transformed to a simpler form by a constant shift of the fields $\varphi \rightarrow \varphi + \Delta \varphi$ (while remaining proportional to a dimensional quantity $L^{-2}$, where $L$ is some length).

The equations of motion become

$$-\overline{\partial} \varphi_i = v_r A_{ri}. \quad (5)$$

To each right eigenvector $\chi^{(k)}$ of $A$ with zero eigenvalue, the field $\varphi^{(k)} = \varphi \cdot \chi^{(k)}$ satisfies a free field equation, $\overline{\partial} \varphi^{(k)} = 0$. The corresponding spin-1 currents $\mathcal{J} = \partial \varphi^{(k)}$ and $\mathcal{K} = \overline{\partial} \varphi^{(k)}$ are conserved,

$$\overline{\partial} \mathcal{J} = 0, \quad \partial \mathcal{K} = 0. \quad (6)$$

By dimensional analysis and (1+1-dimensional) Lorentz invariance it follows that a general spin-$S$ conservation law is expressible in the form (4), with

$$J_S = P(\partial \varphi, \partial^2 \varphi, \ldots), \quad \overline{\mathcal{J}}_S = v_r Q_r(\partial \varphi, \partial^2 \varphi, \ldots), \quad (7)$$

or the parity transformation of this (i.e. $\partial \leftrightarrow \overline{\partial}, J_S \rightarrow \overline{K}_S, \overline{\mathcal{J}}_S \rightarrow K_S$). In equation (4) $P$ is a polynomial of dimension $S$, and $Q_r$ is a polynomial of dimension $S - 2$. Here $\partial$ and $\overline{\partial}$ have (inverse length) dimensions 1, $\varphi$ has dimension 0, and $v_r$ has dimension 2. Then, by ordinary rules of differentiation, equations (4) automatically imply two classes of spin-$S$ conservation laws,

$$\overline{\partial} P(\mathcal{J}, \partial \mathcal{J}, \ldots) = 0, \quad \partial P(\mathcal{K}, \overline{\partial} \mathcal{K}, \ldots) = 0. \quad (8)$$
However, these are rather obvious consequences of the free fields present in the model. To maintain focus on the more interesting conservation laws we shall assume that all free fields have been separated from the model, so that the matrix $A_{ri}$ have no right eigenvectors with zero eigenvalue. This means in particular that $n \geq m$, and that all $m \times m$ submatrices of $A_{ri}$ are invertable.

3 Conservation laws due to conformal invariance

In this section we search for spin-2 conservation laws, essentially the conservation of the energy momentum tensor. We are in particular searching for those models where the conservation can be written in a form similar to (6),

$$\partial T = 0, \quad \partial \overline{T} = 0,$$

as is typical for the energy-momentum tensor of a conformal field theory. Again, by ordinary rules of differentiation this will imply the existence of an infinite set of higher spin conserved currents,

$$\partial P(T, \partial T, \ldots) = 0, \quad \partial P(T, \partial \overline{T}, \ldots) = 0,$$

where $P$ is a polynomial of dimension $S$, with $T, \overline{T}$ being of dimension 2. The general form of the spin-2 currents can be written as

$$J_2^{\alpha} = \beta_{ij} \partial \varphi_i \partial \varphi_j + \rho_i \partial^2 \varphi_i, \quad \overline{J}_2 = v_r C_r,$$

(9)

(or their parity transforms) where $\beta_{ij}$ is symmetric. Inserting this ansatz into equation (6) and using the equations of motion (5) we find that the equation (no sum over $r$)

$$C_r A_{rj} = 2 A_{ri} \beta_{ij} + A_{ri} \rho_i A_{rj}$$

must hold. With $\beta_{ij} = \frac{1}{2} \delta_{ij}$ we always have the solution $C_r = 1 + A_{ri} \rho_i$. With $\rho_i = 0$ this leads to the canonical energy-momentum tensor, as i.e. calculated from Noether’s theorem. Here we instead search for models where $C_r = 0$ is a possible solution, since these will be conformally invariant. Let $u$ denote the $n$-component vector with all entries equal to $-1$. Then the equation to be solved is $A_{ri} \rho_i = u_r$.

The criterium for solvability is seemingly modest: that all left eigenvectors $\xi^{(k)}$ of $A$ with eigenvalue 0 must be orthogonal to $u$. However, this is enough to rule out all the affine Toda theories. To investigate which matrices $A$ lead to conformal models, assume first that $n = m$. Then $A$ is invertible by assumption, and a solution $\rho$ always exist. Next we may extend $A$ to an arbitrary $n \times m$, matrix by adding interaction potentials $v_r = a_r \exp (\alpha_r \cdot \varphi)$, $r = m + 1, \ldots, n$ with
\[ \alpha_r = \gamma_r - \rho/\rho^2, \] where the \( \gamma_r \)'s are distinct vectors such that \( \gamma_r \cdot \rho = 0 \). In fact, it seems that all potentials

\[ V(\varphi) = \exp \left( -\rho \cdot \varphi / \rho^2 \right) U(\varphi_\perp), \]  

with \( U \) arbitrary, lead to a conformal model. Here \( \varphi_\perp = \varphi - (\rho \cdot \varphi / \rho^2) \rho \). The conserved current is a light-cone component of the (traceless) energy-momentum tensor,

\[ T = \frac{1}{2} \partial \varphi \cdot \partial \varphi + \rho \cdot \partial^2 \varphi \]  

(or its parity transformation \( \overline{T} \)).

To investigate whether the conservation laws in the above conformal models are in involution, we have to turn to a canonical formulation. Contrary to our notation we prefer to use ordinary space-time \((x,t)\) variables. I.e., our basic canonical variables are the fields \( \varphi(x,t) \) and \( \pi(x,t) = \partial \varphi(x,t) / \partial t \), with equal time Poisson brackets \( \{ \pi_i(x,t), \varphi_j(y,t) \} = \delta_{ij} \delta(x - y) \). Expressions like \( \partial^n \varphi \) should thus be interpreted as shorthand notation for a perhaps lengthy expression in the fields \( \varphi, \pi \), and their spatial derivatives. This expression can be obtained by eliminating all time derivatives of fields with the use of the equations of motion. It is convenient to introduce the combinations

\[ \psi_i(x) = \partial \varphi_i(x) = \pi_i(x) + \varphi'_i(x), \quad \overline{\psi}_i(x) = \overline{\partial} \varphi_i(x) = \pi_i(x) - \varphi'_i(x). \]  

Here and in the following we drop explicit references to time dependence. All fields are assumed to be evaluated at the same fixed time. A \( ' \) denotes differentiation with respect to the remaining (space) coordinate. The following analysis is much inspired by some remarks by Freeman and West [7]. The fields \( (12) \) have Poisson brackets

\[ \{ \psi_i(x), \psi_j(y) \} = - \{ \overline{\psi}_i(x), \overline{\psi}_j(y) \} = -\delta_{ij} \delta'(x - y), \quad \{ \psi_i(x), \overline{\psi}_j(y) \} = 0, \]  

and we further find that \( \{ \psi_i(x), V(\varphi(y)) \} = \{ \overline{\psi}_i(x), V(\varphi(y)) \} = -f_i(x) \delta(x - y) \), where \( f_i = \nu, A_{ri} \). Now we rewrite the expression \( (11) \) for \( T \) in terms of canonical fields, and evaluate the Poisson bracket

\[ \{ T(x), T(y) \} = 2 \left[ T(x) \partial_x + \partial_x T(x) - 4\rho^2 \partial_x^3 \right] \delta(x - y). \]  

There is an identical expression with \( T \to \overline{T} \), while the Poisson bracket between \( T \) and \( \overline{T} \) is zero. Equation \( (14) \) is the spatial version of the Virasoro algebra, but it is also a Poisson bracket for the KdV hierarchy of equations. I.e., with the bracket \( (14) \) and \( H = \int dx T(x)^2 \) as the Hamiltonian, the equation \( T_t = \{ T, H \} \) becomes the KdV-equation. Since this equation is known to have an infinite set of conserved quantities in involution, see ref. [8], the first two of which are \( \int dx T(x) \)
and $\int dx T(x)^2$, the same will necessarily be true when the same quantities are viewed as coming from a conformal field theory.

To rewrite (14) in the conventional Virasoro form, we assume the $x$-coordinate to be periodic with period $\ell$, and introduce the Fourier modes

$$L_m = 2\pi \rho^2 \delta_{m0} + \frac{\ell}{4\pi} \int_0^\ell dx \exp \left( \frac{2\pi i m x}{\ell} \right) T(x).$$

(15)

Then equation (14) implies the Virasoro algebra

$$i \{L_m, L_n\} = (m-n) L_{m+n} + 4\pi \rho^2 m \left( m^2 - 1 \right) \delta_{m+n,0},$$

(16)

which shows that the model has central charge $c = 48\pi \rho^2$.

## 4 Spin-3 conservation laws

We now make a general search for spin-3 conservation laws. Since we may always shift the current by the gradient of an antisymmetric tensor, $J \rightarrow J + \partial X$, $\mathcal{J} \rightarrow \mathcal{J} - \partial X$, it suffices to search for expressions of the form

$$J_3 = \alpha_{ijk} \partial \varphi_i \partial \varphi_j \partial \varphi_k + \beta_{ij} \partial \varphi_i \partial^2 \varphi_j,$$

(17)

(or their parity transforms). Here $\alpha_{ijk}$ is completely symmetric, and $\beta_{ij}$ is antisymmetric. By performing the differentiations and using the equations of motion (5) we find that equation (1) implies that

$$B_{ri} = A_{rk} \beta_{ki},$$

and in turn (no summation convention)

$$3 \sum_{k=1}^m A_{rk} \alpha_{kij} = \sum_{k=1}^m \left( A_{ri} A_{rk} \beta_{kj} + A_{rj} A_{rk} \beta_{ki} \right).$$

(18)

To proceed we now restrict the index $r$ to an arbitrary $m$-component subset of \{1, \ldots, $n$\}. The matrix $A_{ri}$ is invertible in the corresponding subspace. Multiplying from the left by this inverse matrix $A^{-1}$, utilizing the complete symmetry of $\alpha_{kij}$, and finally multiplying the resulting equation by $A_{sk} A_{ti} A_{uj}$ and summing over $kij$, we obtain (no summation convention)

$$K_{st} \tilde{\beta}_{su} + K_{su} \tilde{\beta}_{st} = K_{ut} \tilde{\beta}_{us} + K_{us} \tilde{\beta}_{ut},$$

(19)

where $K = AA^T$ and $\tilde{\beta} = A \beta A^T$, with $A^T$ denoting the transpose of $A$. This equation must hold for the full set of indices \{1, \ldots, $n$\}, since it is supposed to hold for arbitrary $m$-component subsets. Note that (19) is invariant under orthogonal transformations of the fields, $\varphi_i = O_{ij} \varphi'_j$. When two indices (say $t$ and $u$) are equal equation (19) simplifies to (no summation convention)

$$(K_{tt} + 2 K_{st}) \tilde{\beta}_{st} = 0.$$  (20)
From this equation, and the (anti)symmetry of the matrices, we immediately see that $\beta_{st} \neq 0$ imply $K_{ss} = K_{tt} = -2 K_{st}$, and if also $\beta_{tu} \neq 0$ then in addition $K_{uu} = K_{ss} = -2 K_{tu} = -2 K_{ut}$. Equipped with this information one may start analyzing the general situation by considering (19) for all combinations of indices $stu$ within larger and larger subsets (which by a permutation may be mapped into $\{1, \ldots, \ell\}$). Modulo permutation of indices and multiplication by constants we only find the solutions

$$K_{rs} = (2 \delta_{r,s} - \delta_{r+1,s} - \delta_{r-1,s}), \quad \tilde{\beta}_{rs} = (\delta_{r+1,s} - \delta_{r-1,s}), \quad \text{with } r, s \in \{1, \ldots, n\}. \tag{21}$$

Here the $\delta_{rs}$'s should either be interpreted to vanish when the indices $r, s$ are outside the range $\{1, \ldots, n\}$, in which case $n = m$, $m \geq 2$ and the model is equivalent to the $A_m$ Toda field theory (3), or to be periodically extended (so that index values $n+k$ and $k$ are equivalent), in which case $n = m+1$ and the model is equivalent\cite{footnote} to the $A_{m+1}^{(1)}$ affine Toda field theory (4). In both cases the model have fields $\varphi_1, \ldots, \varphi_{m+1}$ obeying the constraint $\sum_{i=1}^{m+1} \varphi_i = 0$. It is already known from the work of Olive and Turok\cite{footnote} that the $A_{m+1}^{(1)}$-models ($n \geq 2$) are the only affine Toda field theories with spin-3 conservation laws. We have shown that there is no other such models (apart from the expected $A_n$ ones) within the larger class (2).

To find explicit expressions for the conserved currents it is convenient to introduce new fields, $\theta_r \equiv \sum_{i=1}^r \varphi_i$. These are defined such that $\theta_r A_{ri} = \theta_i - \theta_{i-1} = \varphi_i$, which imply

$$\partial \varphi_i \beta_{ij} \partial^2 \varphi_j = \partial \theta_r \tilde{\beta}_{rs} \partial^2 \theta_s, \quad v_r B_{ri} \partial \varphi_i = v_r \tilde{\beta}_{rs} \partial \theta_s,$$

and, with the use of (18),

$$\sum_{ijk} \alpha_{ijk} \partial \varphi_i \partial \varphi_j \partial \varphi_k = \frac{1}{3} \sum_{rst} \left( K_{rs} \tilde{\beta}_{rt} + K_{rt} \tilde{\beta}_{rs} \right) \partial \theta_r \partial \theta_s \partial \theta_t.$$

Inserting the explicit expressions for $K$ and $\tilde{\beta}$ we finally obtain

$$J_3 = \sum_{r=1}^m \left[ 2 (\partial \theta_r)^2 (\partial \theta_{r+1} - \partial \theta_{r-1}) + (\partial \theta_r) \left( \partial^2 \theta_{r+1} - \partial^2 \theta_{r-1} \right) \right], \tag{22}$$

with $\theta_0 = \theta_{m+1} = 0$. This expression looks identical for the $A_m$ and $A_{m+1}^{(1)}$ models. However, when rewritten in terms of canonical variables $\varphi(x), \pi(x)$ they become different—since the fields obey different equations of motion in the two cases. Further

$$J_3 = \sum_{r=1}^n v_r (\partial \theta_{r+1} - \partial \theta_{r-1}), \tag{23}$$

\footnote{In these two cases the matrices $A_{ri}$ allows one to transform all the coefficients $a_r$ (in $v_r(\varphi)$) to the same absolute value. In principle there is still an arbitrary choice of sign for each of them. However, all $a_r$ must be positive for the potential $V(\varphi)$ to be bounded from below.}
where the indices should be interpreted cyclically modulo $m + 1$. This expression is different for the $A_m$ and $A_m^{(1)}$ models, due to the additional $(m + 1)$st term in the latter case.

5 Spin-4 and spin-5 conservation laws

We have repeated the analysis of the preceding section for the spin-4 and spin-5 conservation laws, so far under less general assumptions. In the spin-4 case one may write
\[ J_4 = \beta_{ij} \partial^2 \varphi_i \partial^2 \varphi_j + \cdots, \]
with $\beta$ symmetric, and in the spin-5 case one may write
\[ J_5 = \beta_{ij} \partial^2 \varphi_i \partial^3 \varphi_j + \cdots, \]
with $\beta$ antisymmetric. Similar ansätze for $J_4$ and $J_5$ can be written down. There are many possible terms in each expressions, also when all opportunities of subtracting gradients of antisymmetric tensors have been used. Inserting the general expressions for $J$ and $\mathbf{J}$ into (1) and using (5) one obtains equations involving the different coefficients. These may be solved successively until one is left with a set of equations involving components of the matrices $\tilde{\beta}$ and $K$. When a solution matrix $\tilde{\beta}$ is found the complete expressions for the conserved currents can be generated by back substitution. Thus, the problem of finding conservation laws reduces to finding solution matrices $\tilde{\beta}$.

In analysing these equations we have (so far) assumed that the $K$’s come from a Toda field theory or an affine Toda field theory. For the affine models Olive and Turok[6] found that there should exist (non-trivial) conservation laws for all spins $S$ such that $S - 1 = m \mod h$, where $m$ is an exponent of the algebra and $h$ is its Coxeter number. We have explicitly calculated the spin-4 and spin-5 conservation laws for all affine Lie algebras, relying heavily on algebraic manipulation programs[9]. For the cases treated by Olive and Turok we find no additional conservation laws. They did however not treat the $D_{2n}^{(1)}$ theories, which have some exponents occurring with multiplicity 2. In particular, this is true for exponent 3 of $D_4^{(1)}$. One may suspect that the corresponding model has two independent spin-4 conservation laws. This indeed turns out to be the case. Thus, there seems to be a conservation law for each exponent, counting multiplicities.

The conservation laws for the affine Toda theories carry over to the Toda theories. However, we have found that there occur additional conservation laws in the latter models. For the spin-4 case this can be interpreted as a consequence of conformal symmetry. In this case the additional law can be written as $\partial^2 T^2 = 0$, where $T$ is (a part of) the energy-momentum tensor. For all Lie algebras except $A_1$ the conserved densities of the form
\[ \mathcal{D} \mathcal{P}(T, \partial T, \ldots) = 0, \] (24)
are different from those that correspond to the exponents of the affine Lie algebras. But we have also found an additional conservation law in the spin-5 case for the $A_m$ models when $m \geq 4$. This one cannot be accounted for by expressions like (24). Thus, there exist Toda field theories with conservation laws of a different origin than the exponents of the corresponding affine Lie algebra or the energy-momentum tensor of a conformal theory.

More details of our results and calculations will be presented elsewhere. The published version of this letter has appeared in \[10\]
References

[1] J. Liouville, Jour. de Math. 20 (1855) 137. See e.g. V. I. Arnold, *Mathematical Methods of Classical Mechanics*, Ch. 10, 271–278, Springer-Verlag 1979.

[2] R. M. Miura, C. S. Gardner and M. D. Kruskal, *KdV equations and generalizations II. Existence of conservation laws and constants of motion*, J. Math. Phys. 9, (1968) 1204–1209.

[3] M. Toda, *J. Phys. Soc. Japan*, 22 (1967), 431. M. Toda, Theory of Nonlinear Lattices, Springer-Verlag 1981.

[4] V. G. Kac, *Infinite Dimensional Lie Algebras*, Progress in Mathematics Vol. 44, Birkhäuser 1984.

[5] A. N. Leznov and M. V. Saveliev, *Representation of zero curvature for the system of nonlinear partial differential equations $x_\alpha,zz = \exp (kx)_\alpha$ and its integrability*, Lett. Math. Phys. 3, (1979) 489–494.

[6] D. Olive and N. Turok, *Local conserved densities and zero curvature conditions for Toda lattice field theories*, Nucl. Phys. B 257 (FS 14), (1985) 277–301.

[7] M. D. Freeman and P. West, *On the relation between integrability and infinite-dimensional algebras*, preprint KCL-TH-93-1, january 1993.

[8] C. S. Gardner, *Korteweg - de Vries Equation and Generalizations. IV The Korteweg - de Vries Equation as a Hamiltonian System*, J. Math. Phys. 12 (1971) 1548–1551.

[9] S. Wolfram, *Mathematica, A System for Doing Mathematics by Computer*, Second Edition, Addison-Wesly 1991.

[10] E.G.B. Hohler and K. Olaussen, *Conservation laws for the classical Toda field theories*, Mod. Phys. Lett. A8 (1993) 3377–3385