Fundamental solutions of homogeneous elliptic
differential operators.

Brice Camus
Ruhr-Universität Bochum, Fakultät für Mathematik,
Universitätsstr. 150, D-44780 Bochum, Germany.
Email : brice.camus@univ-reims.fr

Abstract
We compute fundamental solutions of homogeneous elliptic differential operators,
with constant coefficients, on $\mathbb{R}^n$ by mean of analytic continuation of distributions.
The result obtained is valid in any dimension, for any degree and can be extended
to pseudodifferential operators of the same type.

Key words: Fundamental solutions; PDE.

Let be $P := P(D_x)$ a pseudodifferential operator, with constant coefficients,
obtained by mathematical quantization of the function $p$, i.e :

$$Pf(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle \xi, x \rangle} p(\xi) \hat{f}(\xi) d\xi. \quad (1)$$

Here, and in the following, the notation :

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle y, \xi \rangle} f(y) dy,$$

designs the Fourier transform of $f$. We note $S(\mathbb{R}^n)$ the Schwartz space and $S'(\mathbb{R}^n)$ the distributions on $S(\mathbb{R}^n)$. A fundamental solution for $P$ is a distribution $\mathcal{G} \in S'(\mathbb{R}^n)$ such that $P\mathcal{G} = \delta$. Fundamental solutions play a major role in the theory of PDE. For a large overview on this subject, and applications, we refer to [1,2]. Apart in some trivial cases, there is few explicit characterizations of fundamental solutions and most results concern the existence. The case of order 3 homogeneous operators, in dimension 3, was treated in [3].

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Always for $n = 3$, the case of certain elliptic homogeneous operators of degree 4 was also solved in [4]. We are here interested in the case of a definite homogeneous polynomial $p_k$ on $\mathbb{R}^n$, i.e. $p_k(\xi) = 0 \iff \xi = 0$ and:

$$p_k(\lambda \xi) = |\lambda|^k p_k(\xi).$$

Note that $k$ has to be even but we do not impose the spherical symmetry of $p_k$. Strictly speaking, it is not necessary to assume that $k \in \mathbb{N}$ and we can consider operators with a conical singularity at the origin. The main motivation is that such an operator can generalize the Laplacian (positivity and ellipticity) but these operators also play a role in physical optics. To state the main result, we introduce the spherical average of $g \in \mathcal{S}(\mathbb{R}^n)$ w.r.t. the symbol $p$:

$$A(g)(r) = r^{n-1} \int_{\mathbb{S}^{n-1}} g(r\theta)p_k(\theta)^{-1}d\theta. \quad (2)$$

Here $p_k(\theta)$ designs the restriction of $p_k$ on the sphere, which defines a strictly positive function. Clearly, we have $A(g) \in \mathcal{S}(\mathbb{R}^+)$ and we can extend $A(g)$ by 0 for $r < 0$ to obtain a $L^2$ function on the line. The main result is:

**Theorem 1** If $k < n$ a fundamental solution $\mathcal{G} \in \mathcal{S}'(\mathbb{R}^n)$ for $P$ is given by:

$$\langle \mathcal{G}, f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p_k(\theta)^{-1} \hat{f}(r\theta)r^{n-1-k}drd\theta.$$

But, when $k \geq n$, we have:

$$\langle \mathcal{G}, f \rangle = -C_{k,n} \frac{\partial^{k-1}A(\hat{f})}{\partial r^{k-1}}(0) + D_{k,n} \int_{\mathbb{R}_+} \log(u) \frac{\partial^k A(\hat{f})}{\partial r^k}(u)du,$$

where $C_{k,n}$ and $D_{k,n}$ are universal constants given by Eqs.(8,9).

The reader can observe the analogy with the Laplacian, see e.g. [1]. In particular, one has to distinguish the case of an integrable (resp. non-integrable) singularity for $p_k(\xi)^{-1}$ for the $n$-dimensional Lebesgue measure.

**Proof of Theorem 1.** Since $p_k \geq 0$, we define a family $p(z)$ of distributions:

$$\forall f \in \mathcal{S}(\mathbb{R}^n) : \langle p(z), f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p_k(\xi)^z \hat{\check{f}}(\xi) d\xi.$$  

The r.h.s. is holomorphic for $\Re(z) > -1/k$ and, by continuity, we obtain:

$$\lim_{z \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} p_k(\xi)^z \hat{\check{f}}(\xi) d\xi = f(0) = \langle \delta, f \rangle. \quad (3)$$
The Laurent development of $p$ in $z = -1$ can be written:

$$p(z - 1) = \sum_{j=-1}^{-d} z^j \mu_j + \mu_0 + \sum_{j=1}^{\infty} \mu_j z^j. \quad (4)$$

This point is justified in Lemma 2 below. But, according to Eq.(3), we have:

$$\lim_{z \to 0} \langle p(z - 1), P(D)f \rangle = \langle \delta, f \rangle.$$

It is then easy to check that $\mu_0$ is a fundamental solution for $P$. Also, note that Eqs.(3,4) provide the set of non-trivial relations:

$$P(D)\mu_j = 0, \quad \forall j < 0, \quad (5)$$

in the sense of distributions of $S'(\mathbb{R}^n)$. The existence of such non-zero $\mu_j$, for $j < 0$, implies the non-uniqueness of fundamental solutions in $S'(\mathbb{R}^n)$.

**Lemma 2** The distributions $p(z - 1)$ are meromorphic on $\mathbb{C}$ with poles located at rational points $z_{j,k} = -\frac{j}{k}, \ j \in \mathbb{N}$.

**Proof.** Let $g = \hat{f}$. Using standard polar coordinates we obtain:

$$\langle p(z - 1), f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_+ \times S^{n-1}} (r^k p_k(\theta))^{z-1} g(r\theta) r^{n-1} dr d\theta.$$

Here $p_k(\theta)$ is the restriction of $p_k$ to $S^{n-1}$. Next, if we define:

$$y = (y_1, ..., y_n) = (r p_k(\theta)^{\frac{k}{n}}, \theta),$$

we obtain a very elementary formulation:

$$\langle p(z - 1), f \rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_+} y_1^{k(z-1)} G(y_1) dy_1.$$

This new amplitude $G$ is obtained by pullback and integration:

$$G(y_1) = \int y^* (g(r, \theta) r^{n-1} |Jy|) dy_2 ... dy_n. \quad (6)$$

We have $G \in S(\mathbb{R}^+) \ and \ G(y_1) = O(y_1^{n-1}) \ in \ y_1 = 0$. Starting from:

$$\frac{\partial^k}{\partial y_1^{k}} y_1^{kz} = \prod_{j=0}^{k-1} (kz - j) y_1^{k(z-1)},$$

after integrations by parts, we accordingly obtain that:

$$\langle p(z - 1), f \rangle = \frac{1}{(2\pi)^n} \prod_{j=0}^{k-1} \frac{1}{(kz - j)} \int_{\mathbb{R}_+} y_1^{kz} \frac{\partial^k}{\partial y_1^{k}} G(y_1) dy_1. \quad (7)$$
The integral of the r.h.s. is holomorphic in the strip $\Re(z) > -\frac{1}{k}$. Finally, we can iterate the previous construction to any order to obtain the result. ■

Since $z = 0$ is a simple pole, the constant term of the Laurent series is:

$$\frac{1}{(2\pi)^n k} \lim_{z \to 0} \partial_z \left( \prod_{j=1}^{k-1} \frac{1}{(kz - j)} \right) \int_{\mathbb{R}^+} y_1^k \partial_{y_1} G(y_1) dy_1.$$  

For the calculations, we distinguish integrable and non-integrable singularities.

**Case of** $k < n$. The derivative of the rational function provides:

$$C_{k,n} = \frac{1}{(2\pi)^n k} \left( \frac{\partial_z \left( \prod_{j=1}^{k-1} \frac{1}{(kz - j)} \right)}{z=0} \right) = (-1)^{k+1} \left( \frac{\gamma + \psi(k)}{(2\pi)^n k} \right),$$  

(8)

where $\gamma$ is Euler’s constant and $\psi$ the poly-gamma function of order 0, i.e.:

$$\psi(z) = \frac{\partial_z (\log(\Gamma(z)))}{\Gamma(z)}.$$  

Accordingly, since $G$ vanishes up to the order $n - 1$ at the origin, we have:

$$C_{k,n} \int_{\mathbb{R}^+} \partial_{y_1} G(y_1) dy_1 = -C_{k,n} \partial_{y_1} G(0) = 0.$$  

On the other side, by derivation of the integral, we find the term:

$$\frac{(-1)^{k-1}}{(k-1)!} \int_{\mathbb{R}^+} \log(y_1) \partial_{y_1} G(y_1) dy_1.$$  

Since $k < n$, we can integrate by parts to finally obtain:

$$\int_{\mathbb{R}^+} G(y_1) \frac{dy_1}{y_1^n} = \int_{\mathbb{R}^+ \times S^{n-1}} p_k(\theta)^{-1} g(r\theta) r^{n-1-k} dr d\theta.$$  

Here, we have replaced the expression for our amplitude, via inversion of our diffeomorphism. This proves the first statement of Theorem 1. ■

**Case of** $k \geq n$. The term attached to the derivative of the rational function is non-zero and provides:

$$-C_{k,n} \partial_{y_1} G(0) + \frac{1}{(2\pi)^n (k-1)!} \int_{\mathbb{R}^+} \log(y_1) \partial_{y_1} G(y_1) dy_1.$$  

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We compute first the derivative of $G$. Contrary to the case $k < n$ we cannot take the limit directly but we will reach the result with the Schwartz kernel technic. By Fourier inversion formula we have:

$$\partial_{y_1}^{k-1}G(0) = \frac{1}{2\pi} \int e^{-i\xi u} (i\xi)^{k-1}G(u) du d\xi.$$ 

We can extend the integral w.r.t. $u$ on the whole line by extending $G$ by zero for $u < 0$. Going back to initial coordinates provides:

$$\partial_{y_1}^{k-1}G(0) = \frac{1}{2\pi} \int e^{-i\xi r} (i\xi)^{k-1}r^{n-1} \int g(r\theta)p_k(\theta)^{-1}d\theta dr d\xi.$$ 

The r.h.s. is exactly the derivative of order $k-1$ of $A(g)$ defined in Eq.(2). By the same technic, the logarithmic integral gives the non-local contribution:

$$\frac{1}{(k-1)!} \int \log(u) \frac{\partial^k A(g)}{\partial r^k}(u) du.$$ 

In particular the second universal constant of Theorem 1 is:

$$D_{k,n} = \frac{1}{(2\pi)^n (k-1)!} (-1)^{k-1}. \quad (9)$$

This proves the second assertion of Theorem 1.

The main result holds also holds for elliptic pseudo-differential operators with homogeneous symbol of degree $\alpha \in ]0, \infty[$. Using $k = E(\alpha) + 1$ integrations by parts in Eq.(7), where $E(x)$ is the integer part of $x$, the construction remains the same and all formulas can be analytically continued w.r.t. the degree $\alpha > 0$. This continuation is trivial since the functions of Eqs.(8,9), defining respectively $C_{n,k}$ and $D_{n,k}$, are analytic w.r.t. the variable $k > 0$.

References

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