CYCLIC COMPOSITION OPERATORS ON SEGAL-BARGMANN SPACE

G. RAMESH, B. SUDIP RANJAN, AND D. VENKU NAIDU

Abstract. We study the hypercyclic, supercyclic and cyclic properties of composition operator $C_\phi$ on the Segal-Bargmann space $\mathcal{H}(\mathcal{E})$, where $\phi(z) = Az + b$, $A \in \mathcal{B}(\mathcal{E})$, $b \in \mathcal{E}$ with $\|A\| \leq 1$ and $A^*b \in (I - A^*A)^{1/2}$. In this connection we also give a characterization of the symbols $\phi$ which induce the bounded composition operator $C_\phi$ on $\mathcal{H}(\mathcal{E})$ and show that the properties of $\phi$ influence the cyclic behaviour of $C_\phi$.

1. Introduction

It is known that every bounded linear operator on an infinite dimensional complex separable Hilbert space is the sum of two hypercyclic operators [2, p. 50]. It is interesting to note that this result holds true with the summands being cyclic operators. Therefore it is very important to study the cyclic operators in order to study bounded operators. We know that every hypercyclic operator is cyclic and supercyclicity is a property which is intermediate between these two.

Since the closed linear span of $\text{Orb}(T, x)$ is the smallest closed $T$-invariant subspace that contains the vector $x$, the cyclic property is connected with the study of invariant subspaces. Analogously, Hypercyclicity has the same connection with invariant subsets. There does not exist a linear operator on a finite dimensional space. But this is not the case with the bounded linear operators on infinite dimensional spaces. This was first observed by G.D. Birkhoff [6], who showed in 1929 that the translation operator $f(z) \rightarrow f(z + 1)$ is hypercyclic on the Fréchet space of all entire functions. Details on dynamical properties of operators can be found in [4].

Let $\mathcal{E}$ be a separable Hilbert space of complex valued functions on a nonempty set $X$ and $\phi : X \rightarrow X$ be a map. The composition operator
is $C_\phi$ defined by

$$(C_\phi f)(x) = f(\phi(x)) \text{ for all } f \in \mathcal{E}, x \in X.$$  

Such operators are clearly linear. The basic idea in the study of composition operators is to describe the operator theoretic properties of $C_\phi$ with the help of function theoretic properties of $\phi$ and vice versa.

Since $n$th-powers of the composition operator $C_\phi$ is related with the composition induced by the $n$th-iterates of $\phi$, the cyclic property of $C_\phi$ is connected with the dynamics of $\phi$. In [11], Guo, Kunyu; Izuchi, Keiji gave a necessary and sufficient condition for a holomorphic mapping to be a cyclic vector of a composition operator on Fock type space.

In [12], Jiang, Liangying; Prajitura, Gabriel T.; Zhao, Ruhan gave a necessary and sufficient condition for the cyclicity of composition operator on the classical Fock space $F^2(\mathbb{C})$. In [16], Tesfa Mengestie proved that the cyclicity of weighted composition operator $C_{\psi,\varphi}$ on classical Fock space $F^2(\mathbb{C})$ depends on the inducing map $\varphi(z) = az + b$, where $|a| \leq 1, b \in \mathbb{C}$ and the weight function $\psi$.

In the first section, we give brief details of the basic material that we need to prove our main results. In the second section, we study the cyclic behaviour of the composition operator $C_\phi$ and establish the connection between dynamical behaviour of $C_\phi$ and the Hilbert space operator $A^*$. In other words, we are able to show that the cyclic behaviour of a composition operator is strongly influenced by the dynamical properties of its inducing map, $\phi(z) = Az + b$ for $z \in \mathcal{E}$. In the third section we show that there is no supercyclic composition operator on the Segal-Bargmann space under certain condition on the inducing map.

1.1. The space $\mathcal{H}(\mathcal{E})$. Let $\mathcal{E}$ be an infinite dimensional complex Hilbert space. For each integer $m \geq 1$, we write $\mathcal{E}^m$ for the symmetric tensor product of $m$ copies of $\mathcal{E}$. Define $\mathcal{E}^0$ to be $\mathbb{C}$ with its usual inner product, $\mathcal{E}^1 = \mathcal{E}$ and for $m \geq 2$, $\mathcal{E}^m$ is the closed subspace of the tensor product of $m$ copies of $\mathcal{E}$, that is, $\mathcal{E}^{\otimes m}$ which consists of all elements that are invariant under the natural action of the symmetric group $S_m$. That is,

$$\mathcal{E}^m = \{ x \in \mathcal{E}^{\otimes m} : \pi x = x \text{ for all } \pi \in S_m \}.$$  

The action of $S_m$ on $\mathcal{E}^{\otimes m}$ is defined on elementary tensors by

$$\pi(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(m)}.$$  

For any $z \in \mathcal{E}$, we use $z^m = z \otimes \cdots \otimes z \in \mathcal{E}^m$ to denote the tensor product of $m$ copies of $z$. Each $\mathcal{E}^m$ is a Hilbert space with an inner
product \( \langle \cdot, \cdot \rangle_{E^m} \) defined by
\[
\langle z^m, w^m \rangle_{E^m} = \langle z, w \rangle^m_E
\]
where \( \langle \cdot, \cdot \rangle_E \) denotes the inner product on \( E \).

**Definition 1.1.** A function \( p_m : E \to \mathbb{C} \) is called continuous \( m \)-homogeneous polynomial on \( E \) if there exists an element \( \zeta \in E^m \) such that \( p_m(z) = \langle z^m, \zeta \rangle \) for \( z \in E \).

**Definition 1.2.** A function \( f : E \to \mathbb{C} \) is called continuous polynomial if \( f \) can be written as a finite sum of continuous homogeneous polynomials. That is there is an integer \( m \geq 0 \) and there are elements \( a_j \in E^j, j = 0, 1, \ldots, m \) such that
\[
f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle.
\]

We denote the space of all continuous \( m \)-homogeneous polynomials and the space of all continuous polynomials on \( E \) by \( P_m(E) \) and \( P(E) \), respectively. For \( f, g \) in \( P(E) \), we can find an integer \( m \geq 0 \) and elements \( a_j, b_j \in E^j \) for \( 0 \leq j \leq m \) such that
\[
f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle \quad \text{and} \quad g(z) = \sum_{j=0}^{m} \langle z^j, b_j \rangle.
\]
Define, \( \langle f, g \rangle = \sum_{j=0}^{m} j! \langle b_j, a_j \rangle \). Then \( \langle \cdot, \cdot \rangle \) defines an inner product on \( P(E) \). The completion of \( P(E) \) in the norm induced by the above inner product is called the Segal-Bargmann space and it is denoted by \( \mathcal{H}(E) \).

**Proposition 1.3.** [15] Each element \( f \) in \( \mathcal{H}(E) \) can be identified as an entire function on \( E \) having a power series expansion of the form
\[
f(z) = \sum_{j=0}^{\infty} \langle z^j, a_j \rangle
\]
for all \( z \in E \), where \( a_j \in E^j, j = 0, 1, 2, \ldots; E^j \) denotes the \( j \) times symmetric tensor product of \( E \). Furthermore, \( \|f\|^2 = \sum_{j=0}^{\infty} j! \|a_j\|^2 \).

Conversely, if \( \sum_{j=0}^{\infty} j! \|a_j\|^2 < \infty \), then the power series \( \sum_{j=0}^{\infty} \langle z^j, a_j \rangle \) defines an element in \( \mathcal{H}(E) \).

Applying Proposition 1.3 to the function
\[
K(z, w) := K_w(z) = \exp(\langle z, w \rangle/2) \quad \text{for all} \ z, w \in E,
\]
we can say that this function is the reproducing kernel function for \( \mathcal{H}(\mathcal{E}) \) and the normalized kernel function is defined by

\[
k_w(z) = \exp\left(\frac{\langle z, w \rangle}{2} - \frac{\|w\|^2}{4}\right).
\]

The linear span of the set \( \{K_w : w \in \mathcal{E}\} \) is dense in \( \mathcal{H}(\mathcal{E}) \). As a result, \( \mathcal{H}(\mathcal{E}) \) is a reproducing kernel Hilbert space. For each \( f \in \mathcal{H}(\mathcal{E}) \), we have \( \langle f, K(x, \cdot) \rangle = f(x) \). For a general theory of these spaces, see Chapter 2 of [1].

1.2. Composition operators. Let \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) be Hilbert spaces and \( K_1 \) and \( K_2 \) be the kernel functions for \( \mathcal{H}(\mathcal{E}_1) \) and \( \mathcal{H}(\mathcal{E}_2) \) respectively. We denote \( K \) as the kernel function for both the spaces \( \mathcal{H}(\mathcal{E}_1) \) and \( \mathcal{H}(\mathcal{E}_2) \) since the kernel functions on these spaces have the same form.

For any mapping \( \phi : \mathcal{E}_1 \to \mathcal{E}_2 \), the composition operator \( C_\phi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1) \) is defined by

\[
C_\phi(h) = h \circ \phi \quad \text{for all } h \in \mathcal{H}(\mathcal{E}_2).
\]

It is clear that \( C_\phi \) is a closed operator. By the closed graph theorem it follows that \( C_\phi \) is bounded if and only if \( h \circ \phi \) belongs to \( \mathcal{H}(\mathcal{E}_1) \) for all \( h \in \mathcal{H}(\mathcal{E}_2) \).

If \( C_\phi \) is bounded, then we have the following identities:

1. \( C_\phi^* K_z = K_{\phi(z)} \) for all \( z \in \mathcal{E}_1 \),
2. Let \( a \in \mathcal{E}_2 \). If \( f(w) = \langle w, a \rangle \) for all \( w \in \mathcal{E}_2 \), then

\[
(C_\phi f)(z) = \langle \phi(z), a \rangle \quad \text{for any } z \in \mathcal{E}_1.
\]

1.3. Cyclicity.

Definition 1.4. [4] Let \( \mathcal{E} \) be an infinite dimensional separable complex Hilbert space. A bounded linear operator \( T : \mathcal{E} \to \mathcal{E} \) is said to be cyclic if there exists a non zero vector \( x \in \mathcal{E} \) such that \( \text{span}(T^n x : n \geq 0) = \mathcal{E} \) and the non zero vector \( x \) is said to be the cyclic vector of the operator \( T \).

We call the set \( \{T^n x : n \geq 0\} \) as the orbit of \( T \) and is denoted by \( \text{Orb}(T, x) \). It may happen that the set \( \{T^n x : n \geq 0\} \) itself is dense or the projective orbit is dense, without the linear span; in this case we have the following definitions.

Definition 1.5. [4] Let \( \mathcal{E} \) be an infinite dimensional complex separable Hilbert space. A bounded operator \( T : \mathcal{E} \to \mathcal{E} \) is said to be hypercyclic if \( \text{Orb}(T, x) \) is dense in \( \mathcal{E} \). In this case \( x \) is said to be a hypercyclic vector.

The operator \( T \) is said to be supercyclic if the set \( \{\lambda T^n x : n \geq 0, \lambda \in \mathbb{C}\} \) is dense in \( \mathcal{E} \).
Definition 1.6. Let $E$ be an infinite dimensional complex Hilbert space and let $T : E \to E$ be a bounded linear operator, the set $\rho(T) := \{ \lambda \in \mathbb{C} : T - \lambda I : E \to E \text{ is invertible and } (T - \lambda I)^{-1} \text{ is bounded} \}$ is called the resolvent set and the complement $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is called the spectrum of $T$.

It is well known that $\sigma(T)$ is non empty compact subset of $\mathbb{C}$.

Theorem 1.7. [15, Theorem 1.3] Let $\phi : E_1 \to E_2$ be a mapping. Then the composition operator $C_\phi : \mathcal{H}(E_2) \to \mathcal{H}(E_1)$ is bounded if and only if $\phi(z) = Az + b$ for all $z \in E_1$, where $A : E_1 \to E_2$ is a bounded linear operator with $\|A\| \leq 1$ and $A^*b$ belongs to the range of $(I - A^*A)^{\frac{1}{2}}$. Furthermore, the norm of $\|C_\phi\|$ is given by

$$\|C_\phi\| = \exp \left( \frac{1}{2} \|v\|^2 + \frac{1}{2} \|b\|^2 \right),$$

where $v$ is the unique vector in $E_1$ of minimum norm satisfying $A^*b = (I - A^*A)^{\frac{1}{2}}v$.

Proposition 1.8. [12, Proposition 5.1]

(i) If $\varphi(z) = az$ with $|a| = 1$, then $C_\varphi$ is cyclic on the Fock space $\mathcal{F}^2(\mathbb{C})$ if and only if $a^n \neq a$ for every $n > 1$.

(ii) If $\varphi(z) = az + b$ with $|a| < 1$ and $a \neq 0$, then $C_\phi$ is cyclic on $\mathcal{F}^2(\mathbb{C})$.

2. Cyclic properties

Let $E$ be an infinite dimensional Hilbert space. We say that a map $\phi$ on $E$ satisfies the property $\mathcal{P}$, if it satisfies the following.

(i) $\phi(z) = Az + b$ for all $z \in E$.

(ii) $A : E \to E$ is a bounded linear operator with $\|A\| \leq 1$ and $b \in E$.

(iii) $A^*b$ belongs to the range of $(I - A^*A)^{\frac{1}{2}}$.

Remark 2.1. If $\phi : E \to E$ be a mapping such that it satisfies the property $\mathcal{P}$, then by Theorem 1.7, the induced composition operator $C_\phi$ is bounded on the Segal-Bargmann space $\mathcal{H}(E)$.

Here we show that for $z \in E$, the kernel function $K_z$ is the cyclic vector for the bounded composition operator $C_\phi$ if $z$ is the hypercyclic vector for the bounded operator $A^*$, under some additional condition on the spectrum of the operator $A$.

For any $z \in E$, we have $C_\phi f(z) = f(\phi(z))$. For any positive integer $m$, we have, $C_\phi^m f(z) = f(\phi^m(z))$. Here $\phi^m$ denotes the $m$ times composition of $\phi$. So for any $z \in E$, we have the following identity

$$\phi^m(z) = A^m z + A^{m-1} b + \cdots + Ab + b.$$
Therefore,
\[ C^m \phi f(z) = f(A^m z + A^{m-1} b + \cdots + Ab + b). \]
So, for any kernel function \( K_z \), we have
\[
C^m \phi K_z(w) = K_z(A^m w + A^{m-1} b + \cdots + Ab + b) \\
= K_z(A^m w) K_z(A^{m-1} b + \cdots + Ab + b) \\
= K_1(A^*)^m z(w) K_z(A^{m-1} b + \cdots + Ab + b).
\]
From the above relation it is clear that the dynamical properties of the composition operator \( C_{\phi} \) depend on the behaviour of the iterates \( A^n = A \circ A \circ A \cdots \circ A \) (n-times).

In the proceeding result, we determine the necessary condition for a kernel function to be a hypercyclic vector of the composition operator \( C_{\phi} \), where the inducing map \( \phi : E \to E \) satisfies the property \( \mathcal{P} \).

**Theorem 2.1.** Let \( \phi \) be a mapping on \( E \) satisfying the property \( \mathcal{P} \). If \( z \in E \) is a hypercyclic vector for \( A^* \), then \( K_z \in \mathcal{H}(E) \) is a hypercyclic vector of \( C_{\phi} \) in each of the following cases:

1. \( b = 0 \)
2. \( b \neq 0 \) and \( 1 \notin \sigma(A) \).

**Proof.** Suppose \( z \) be a non zero vector in \( E \) such that the set \( \{(A^*)^m z : m \geq 0\} \) is dense in \( E \). We show that \( K_z \) is the hypercyclic vector for \( C_{\phi} \). Let \( g \in \mathcal{H}(E) \), such that \( g \) is orthogonal to the orbit of \( K_z \) under \( C_{\phi} \).

**Proof of (1)**
We claim that \( g \equiv 0 \). By the choice of the function \( g \), we have \( \langle C^m \phi K_z, g \rangle = 0 \) for all \( m \geq 0 \). Since \( b = 0 \), then by Eq. (3), we have \( \langle g, K(A^*)^m z \rangle = 0 \), which implies that \( g((A^*)^m z) = 0 \), for every \( m \geq 0 \). This shows that \( g \) is a zero function in \( \mathcal{E} \), by our assumption and hence this proves our claim.

**Proof of 2**
Let \( g \in \mathcal{H}(E) \) be such that \( g \perp C^m \phi K_z \) for all \( m \geq 0 \). That is,
\[
\langle C^m \phi K_z, g \rangle = 0, \text{ for all } m \geq 0.
\]
By Eq. (3), we have
\[
\langle g, K(A^*)^m z \rangle = 0, \text{ for all } m \geq 0.
\]
Hence
\[
\frac{\langle g, K(A^*)^m z \rangle}{K_z((I - A^m)(I - A)^{-1} b)} g((A^*)^m z) = 0.
\]
By our assumptions that the set \( \{(A^*)^m z : m \geq 0\} \) is dense in \( E \) and \( 1 \notin \sigma(A) \), we conclude that \( g \equiv 0 \) on \( E \). Hence, the kernel function \( K_z \) is the hypercyclic vector for \( C_{\phi} \).
Corollary 2.2. If the operator $A^*$ is hypercyclic then the composition operator $C_\phi$ is cyclic operator on $\mathcal{H}(E)$ in each of the following cases:

1. $b = 0$.
2. $b \neq 0$ and $1 \notin \sigma(A)$.

Example 2.3. We have shown that if the operator $A^*$ is hypercyclic on $E$ then $C_\phi$ is cyclic on $H_{pE_q}$ in each of the following cases:

1. $b = 0$.
2. $b \neq 0$ and $1 \notin \sigma(A)$.

Proposition 2.4. Let $\phi$ be a mapping on $E$ satisfying the property $P$ with $b = 0$. If $K_z$ is hypercyclic for $C_\phi$ for some $z \in E$, then $z$ is a hypercyclic for $A^*$.

Proof. Let $z \in E$ be such that $\{C_\phi^m K_z : m \geq 0\} = \mathcal{H}(E)$. If $\{(A^*)^m z : m \geq 0\} \notin E$, then there exists a non zero vector $p \in E$ such that $\langle (A^*)^m z, p \rangle = 0$.

We have $C_\phi^m K_z(w) = K_{(A^*)^m(z)}(w)$, as $\phi(z) = Az$ for all $z \in E$ and $\|A\| \leq 1$.

Since $p \in E$, we have $K_p \in \mathcal{H}(E)$. Then by our assumption we have a subsequence $\{C_\phi^m K_z\}$ such that $\lim_{k \to \infty} C_\phi^m K_z = K_p$. Then

$$\lim_{k \to \infty} \langle C_\phi^m K_z, K_p \rangle = \langle K_p, K_p \rangle.$$

That is,

$$(5) \quad 1 = \lim_{k \to \infty} \langle K_{(A^*)^m(z)} \rangle, K_p \rangle = \langle K_p, K_p \rangle = \exp(\|p\|^2).$$

Hence, this implies that $p = 0$, which is a contradiction. Therefore, we have the set $\{(A^*)^m z : m \geq 0\}$ is dense in $E$.

Definition 2.5. Let $(X, d)$ be a complete metric space. Then a map $\varphi : X \to X$ is called a contraction if there is a number $0 \leq c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq cd(x, y)$$

for all $x, y \in X$.

It is very well known that in a complete metric space $X$, every contraction has a unique fixed point. That is, there is a unique $x \in X$ such that $\varphi(x) = x$.

Theorem 2.6. \cite[68]{10} If $T$ is compact and $\lambda \neq 0$, then $\text{ran}(\lambda I - T)$ is closed and $\ker(\lambda I - T)$ is finite dimensional.
Theorem 2.7. [13, Theorem 3.3] Let $H$ be a Hilbert space and $C$ be a nonempty closed convex subset of $H$ and $f : C \to C$ be a mapping such that $\|f(x) - f(y)\| \leq \|x - y\|$ and if $\{f^n(x)\}$ is bounded for some $x \in C$, then $f$ has a fixed point in $C$.

The next result describes a property of the symbol $\phi$ which induces a bounded composition operator $C_\phi$ on $\mathcal{H}(\mathcal{E})$.

Theorem 2.8. Let $\phi$ be a mapping on $\mathcal{E}$ satisfying the property $\mathcal{P}$. Then $\phi$ has a fixed point in each of the following cases:

1. $\|A\| < 1$.
2. $A$ is compact with $\|A\| \leq 1$.
3. $A$ is non-compact with $\|A\| = 1$ and $1 \notin \sigma(A)$.

Proof. Note that $C_\phi$ is bounded and $\phi(z) = Az + b$ for all $z \in \mathcal{E}$. If $A = 0$ then $\phi(z) = b$ fixes the point $b \in \mathcal{E}$. Now assume that $A \neq 0$.

Proof of 1:
In this case, we have
$$\|\phi(z) - \phi(w)\| = \|Az - Aw\| \leq \|A\| \|z - w\|.$$ As $\|A\| < 1$, the function $\phi$ is contraction and hence it has a fixed point, by the Banach contraction principle.

Proof of 2:
Since $A$ is compact, by Theorem 2.6, we have $\text{ran}(I - A)$ and $\text{ran}(I - A^*A)^{\frac{1}{2}}$ both are closed. We claim that $\phi$ has a fixed point. That is, to show there exists a point $z_0 \in \mathcal{E}$ such that $Az_0 + b = z_0$. This means $b \in \text{ran}(I - A)$. Since $\text{ran}(I - A)$ is closed, to prove our claim it is enough to prove $b \in \ker(I - A^*A)^{\frac{1}{2}}$.

Suppose $v \in \ker(I - A^*)$ with $\|v\| = 1$, then $A^*v = v$. As $\|A\| \leq 1$, we have
\[
1 \geq \|Av\| = \|Av\| \|v\| \geq |\langle Av, v \rangle| = |\langle A^*v, v \rangle| \|v\| = 1
\]
That is,
\[
|\langle Av, v \rangle| = \|Av\| \|v\|.
\]
So, this implies that $Av = \lambda v$ for some $\lambda \in \mathbb{C}$. We can write $\lambda$ as
\[
\lambda = \langle \lambda v, v \rangle = \langle Av, v \rangle = \langle v, A^*v \rangle = \langle v, v \rangle = 1.
\]
This shows that $Av = v$. Hence, we have $A^*v = v$ and $Av = v$. From this we can get that $A^*Av = A^*v = v$, which implies that $v \in \ker(I - A^*A)^{\frac{1}{2}}$. Now by Theorem 1.7, we know that $A^*b \in \text{ran}(I - A^*A)^{\frac{1}{2}} = (\ker(I - A^*A)^{\frac{1}{2}})^\perp$, it follows that $b$ is orthogonal to $v$. Hence, there exists a point $z_0 \in \mathcal{E}$ such that $\phi(z_0) = z_0$, and this proves our claim. That is, $\phi$ fixes a point in $\mathcal{E}$.
**Proof of 3:**
From Theorem 1.7, there exists a unique vector \( u \) in \( E \) of smallest norm such that

\[
(I - A^*A)^{\frac{1}{2}} u = b
\]

and \( \|C_\phi\| = \exp \left( \frac{\|u\|^2}{2} \right) \).

By Eq. (1), as \( A \) is invertible, we have

\[
\phi^n(z) = A^n z + A^{(n-1)} b + \cdots + Ab + b = A^n z + (I - A^n)(I - A)^{-1} b.
\]

In particular,

\[
\phi^n(0) = (I - A^n)(I - A)^{-1} b.
\]

Hence

\[
|\phi^n(0)| \leq 2 \|I - A^{-1}\| < \infty.
\]

Hence, by Theorem 2.7, \( \phi \) has a fixed point. \( \square \)

The assumption in (3) of Theorem 2.8 that \( 1 \notin \sigma(A) \) cannot be removed and this is illustrated with the following examples.

**Example 2.9.** Consider the right shift operator \( R \) on \( \ell^2(\mathbb{N}) \), defined by

\[
R(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots) \quad \text{for all } \{x_n\}_{n=1}^\infty \in \ell^2(\mathbb{N}).
\]

The adjoint \( R^* \) is given by

\[
R^* (x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots), \quad \forall \{x_n\}_{n=1}^\infty \in \ell^2(\mathbb{N}).
\]

Observe that \( \|R\| = 1 \), \( R^* R = I \) and \( \sigma(R) = \{\lambda : \|\lambda\| \leq 1\} \).

Now consider the affine map \( \psi \) on \( \ell^2(\mathbb{N}) \) defined by \( \psi(x) = Rx + b \) with \( R^* b \in \text{ran}(I - R^*R)^{\frac{1}{2}} = 0 \). Then we see that \( R^*b = 0 \). Hence by Theorem 1.7, the corresponding composition operator \( C_\psi \) is bounded on \( \mathcal{H}(\ell^2(\mathbb{N})) \).

Take \( b = e_1 \), where \( e_1 = (1, 0, 0, 0, \ldots) \). Clearly \( C_\psi \) is bounded with \( \|C_\psi\| = \sqrt{e} \).

Next, consider

\[
\psi^n(z) = R^n z + R^{(n-1)} e_1 + \cdots + R e_1 + e_1,
\]

for all \( z \in \ell^2(\mathbb{N}) \). In particular,

\[
\psi^n(0) = R^{(n-1)} e_1 + \cdots + R e_1 + e_1.
\]

It can be seen that

\[
\|\psi^n(0)\|^2 = n.
\]
Explicitly the map $\psi$ can be written as
\begin{equation}
\psi(x_1, x_2, x_3, \ldots) = R(x_1, x_2, x_3, \ldots) + (1, 0, 0, 0, \ldots) \\
= (1, x_1, x_2, x_3, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N}).
\end{equation}

It is very clear that the mapping $\psi$ on $\ell^2(\mathbb{N})$ only fixes the point $(1,1,1,\ldots)$, which is not in $\ell^2(\mathbb{N})$. Hence $\psi$ has no fixed points in $\ell^2(\mathbb{N})$.

Remark 2.2. B. Carswell, B. MacCluer, and A. Schuster proved that $C_\phi$ is bounded on $\mathcal{F}^2(\mathbb{C})$ if and only if $\phi(z) = az + b$, where $|a| \leq 1$, and $b = 0$ if $|a| = 1$ (See [8], for details). But this is not true in the case of composition operator $C_\psi$ on $\mathcal{H}(\mathcal{E})$, where $\mathcal{E}$ is infinite dimensional Hilbert space. In the above example we have $\|R\| = 1$ and $R^*b = 0 \in \ker(I - R^*R)^{\frac{1}{2}}$ such that the composition operator $C_\psi$ is bounded on $\mathcal{H}(\ell^2(\mathbb{N}))$, where $\psi(z) = Rz + b$ but these two conditions does not necessarily imply that $b = 0$.

Corollary 2.10. Let $\phi : \mathbb{C}^n \to \mathbb{C}^n$ be a map and $C_\phi$ be the associated composition operator on $\mathcal{F}^2(\mathbb{C}^n)$, then $\phi$ has a fixed point.

Proof. As $\ker(I - A)$ and $\ker(I - A^*A)^{\frac{1}{2}}$ both are closed, by (2) of Theorem 2.8, the conclusion follows.

Example 2.11. Let $\mu$ be a real number such that $0 < \mu < 1$. For $\{x_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ define the weighted unilateral shift on $\mathcal{B}(\ell^2(\mathbb{N}))$ by
\begin{equation}
S(x_1, x_2, x_3, \ldots) = (0, \mu x_1, x_2, x_3, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N}).
\end{equation}
The adjoint $S^*$ of $S$ is given by
\begin{equation}
S^*(x_1, x_2, x_3, \ldots) = (\mu x_2, x_3, x_4, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N}).
\end{equation}

Now we see that $S^*S \neq SS^*$ and neither of them are equal to the identity operator $I$ on $\ell^2(\mathbb{N})$. The spectrum $\sigma(S)$ is given by $\sigma(S) = \{\lambda : |\lambda| \leq 1\}$. Observe that $1 \in \sigma(S)$.

Note that $S^*S(x_1, x_2, x_3, \ldots) = (\mu^2 x_1, x_2, x_3, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N})$. It can be seen that
\begin{equation}
(I - S^*S)(x_1, x_2, x_3, \ldots) = ((1 - \mu^2)x_1, 0, 0, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N}).
\end{equation}

Hence
\begin{equation}
(I - S^*S)^{\frac{1}{2}}(x_1, x_2, x_3, \ldots) = (\sqrt{1 - \mu^2}x_1, 0, 0, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N}).
\end{equation}

Let $\hat{b} = (1, \sqrt{\frac{1 - \mu^2}{\mu}}, 0, 0, \ldots)$. Then we have
\begin{equation}
(I - S^*S)^{\frac{1}{2}}e_1 = S^*\hat{b}.
\end{equation}
where \( e_1 = (1, 0, 0, \ldots) \). We define the map \( \hat{\psi}(x) = Sx + \hat{b} \). Since \( \|S\| = 1 \) and by the condition in Eq. (16), by applying Theorem 1.7, the corresponding composition operator \( C_{\hat{\psi}} \) is bounded on \( \mathcal{H}(\ell^2(\mathbb{N})) \).

Explicitly the map \( \hat{\psi} \) can be written as

\[
\hat{\psi}(x_1, x_2, x_3, \ldots) = (1, \mu x_1 + \frac{\sqrt{1 - \mu^2}}{\mu}, x_2, x_3, \ldots), \forall \{x_n\} \in \ell^2(\mathbb{N}).
\]

It can be easily verified that \( \hat{\psi} \) has a fixed point \( (1, \mu + \frac{\sqrt{1 - \mu^2}}{\mu}, \mu + \frac{\sqrt{1 - \mu^2}}{\mu}, \ldots) \), which is not in \( \ell^2(\mathbb{N}) \). That means \( \hat{\psi} \) has no fixed point.

**Theorem 2.12.** Let \( \phi \) be a mapping on \( \mathcal{E} \) satisfying the property \( \mathcal{P} \). If \( 1 \notin \sigma(A) \) and \( C_{\phi} : \mathcal{H}(\mathcal{E}) \to \mathcal{H}(\mathcal{E}) \) is hypercyclic then \( A^* : \mathcal{E} \to \mathcal{E} \) is hypercyclic.

**Proof.** Suppose \( A^* \) is not hypercyclic. Hence for every \( \zeta \in \mathcal{E}\setminus\{0\}, \mathcal{E} \{ (A^*)^n \zeta : n \geq 0 \} \neq \mathcal{E} \).

So, for any \( \zeta \in \mathcal{E} \) there exists \( P_\zeta \in \mathcal{E} \) non zero such that \( \langle (A^*)^n \zeta, P_\zeta \rangle = 0 \) for all \( n \geq 0 \).

Now for \( f \in \mathcal{H}(\mathcal{E}) \) we have

\[
C_{\phi}^n f(z) = f(\phi^n(z)) = f(A^n z + A^{n-1}b + \cdots + Ab + b).
\]

By Theorem 2.8, there exists \( z_0 \in \mathcal{E} \) such that \( \phi(z_0) = z_0 \). Hence \( \phi^n(z_0) = z_0 \). That is,

\[
A^n z_0 + A^{n-1}b + \cdots + Ab + b = z_0
\]

or,

\[
A^{n-1}b + \cdots + Ab + b = (I - A^n)z_0.
\]

Then

\[
C_{\phi}^n f(z) = f(A^n z + (I - A^n)z_0) = f(A^n z + z_0 - A^n z_0) = f(A^n(z - z_0) + z_0).
\]

Now consider the power series expansion of \( f \) around \( z_0 \),

\[
f(z) = \sum_{j=0}^{\infty} \langle (z - z_0)^j, a_j \rangle, a_j \in \mathcal{E}^j.
\]
Then
\[ f \left( A^n(z - z_0) + z_0 \right) = \sum_{j=0}^{\infty} \langle (A^n(z - z_0) + z_0 - z_0)^j, a_j \rangle \]
\[ = \sum_{j=0}^{\infty} \langle (A^n(z - z_0))^j, a_j \rangle \]
\[ = \sum_{j=0}^{\infty} \langle (z - z_0)^j, ((A^*)^n)^j a_j \rangle. \]

Hence
\[ C_\phi^n f(z) = \sum_{j=0}^{\infty} \langle (z - z_0)^j, ((A^*)^n)^j a_j \rangle \]
\[ = \mathbb{C} + \langle (z - z_0), (A^*)^n a_1 \rangle + \cdots, \]

Since \( a_1 \in \mathcal{E} \), there exists \( \zeta \neq 0 \) in \( \mathcal{E} \) such that \( \langle (A^*)^n a_1, \zeta \rangle = 0 \).

Now consider the function defined on \( \mathcal{E} \) by
\[ g(z) = \langle z - z_0, \zeta \rangle, \ \forall z \in \mathcal{E}. \]

Thus \( g \in \mathcal{H}(\mathcal{E}) \). Now,
\[ \langle C_\phi^n f, g \rangle = \langle \zeta, (A^*)^n a_1 \rangle = 0. \]

Hence \( g \in \{ C_\phi^n f : n \geq 0 \}^\perp \).

Now if \( f \) hypercyclic for \( C_\phi \) then we get \( g = 0 \), a contradiction. \( \square \)

**Proposition 2.13.** Suppose the mapping \( \phi : \mathcal{E} \to \mathcal{E} \) has two fixed points, then the bounded composition operator \( C_\phi \) cannot be cyclic operator on \( \mathcal{H}(\mathcal{E}) \).

**Proof.** To prove this, we use the fact that the adjoint of a cyclic operator can have only simple eigenvalues (See [7, Proposition 2.7]). Suppose that there are two fixed points, namely \( \alpha \) and \( \beta \). Then we have \( C_\phi^* K_\alpha = K_{\phi(\alpha)} = K_\alpha \) and \( C_\phi^* K_\beta = K_{\phi(\beta)} = K_\beta \). This shows that 1 is the eigenvalue for \( C_\phi^* \) with multiplicity at least two and hence \( C_\phi \) is cannot be cyclic. \( \square \)

**Proposition 2.14.** Let \( \phi \) be a mapping on \( \mathcal{E} \) satisfying the property \( \mathcal{P} \) and \( 1 \not\in \sigma(A) \). If \( C_\phi \) is hypercyclic, then \( \phi \) is injective.

**Proof.** From Theorem 1.7, the composition operator \( C_\phi \) is bounded. By Theorem 2.12, the operator \( A^* \) is cyclic. Therefore the dimension of the orthogonal complement of range of \( A^* \) is at most one (See [7, Page 33]), that is, \( \dim \ker(A) = \dim \text{ran}(A^*)^\perp \leq 1. \)
First we suppose that $\dim \ker(A) = 0$. If possible let the holomorphic map $\phi$ is not injective on $E$, then there exists a pair of points $\alpha, \beta$ with $\alpha \neq \beta$ such that $\phi(\alpha) = \phi(\beta)$. This implies that $\alpha - \beta \in \ker(A) = \{0\}$, a contradiction to the fact that $\alpha \neq \beta$. Hence we conclude that $\phi$ is injective map on $E$.

Now suppose that $\dim \ker(A) = 1$. Then there is a non zero vector $x$ in $E$ such that $\ker(A) = \text{span}\{x\}$. Again if possible let the mapping $\phi$ is not injective, then there exists distinct pair of points $\eta, \zeta$ such that $\phi(\eta) = \phi(\zeta)$. This implies that $\eta - \zeta \in \ker(A)$ and hence $\eta - \zeta = \mathbb{C}x$. From this we observe that there are infinitely many such pair of distinct points. As a consequence the set $G = \{(z, w) : z \neq w, \phi(z) = \phi(w)\}$ contains infinitely many elements.

For each pair $(\eta, \zeta) \in G$, consider the non zero function $f = K_\eta - K_\zeta$, then $C_\phi^* f = K_{\phi(\eta)} - K_{\phi(\zeta)} = 0$. This shows that the function $f$ in the $\ker(C_\phi^*) = \text{ran}(C_\phi)^\perp$. Hence the orthogonal complement of the range of $C_\phi$ has infinite dimension as $G$ is an infinite set. This is a contradiction as we know that the orthogonal complement of the range of a cyclic operator has dimension at most one. Thus $\phi$ is injective.

THEOREM 2.15. [4, Proposition 1.17] If $T$ be a bounded hypercyclic operator on an infinite dimensional complex separable Hilbert space $H$, then the point spectrum $\sigma_p(T^*) = \emptyset$.

PROPOSITION 2.16. Let $\phi$ be a mapping on $E$ satisfying the property $\mathcal{P}$. If $\phi$ has a fixed point then $C_\phi$ cannot be hypercyclic.

Proof. If possible there exist $\alpha \in E$ such that $\phi(\alpha) = \alpha$. Now we have the following identity;

$$\langle f, C_\phi^* K_\alpha \rangle = \langle f, K_{\phi(\alpha)} \rangle = \langle f, K_\alpha \rangle.$$ 

This shows that the kernel function $K_\alpha$ is the eigenvector of $C_\phi^*$ corresponding to the eigenvalue $1$. By Theorem 2.15, this is a contradiction to the fact that $C_\phi$ is hypercyclic. 

COROLLARY 2.17. If the bounded composition operator $C_\phi$ is hypercyclic on $\mathcal{H}(E)$, then the mapping $\phi$ cannot have a fixed point.

Observe that not every cyclic operator on a Hilbert space has a dense range, but there are certain composition operators which has dense range. For example, composition operators on the Hardy space $H^2(\mathbb{D})$ has dense range [7, Theorem 1.4]. The problem of determining which composition operators have dense range is nontrivial. This forces us to investigate the proceeding Theorem.
Theorem 2.18. Let $\phi$ be a mapping on $E$ satisfying the property $P$. Assume that $1 \notin \sigma(A)$. If $C_\phi$ is hypercyclic, then the range of $C_\phi$ is dense in $H(E)$.

Proof. From Theorem 2.12, we get that $A^*$ is cyclic operator. Therefore $\dim \text{ran}(A^*)^\perp \leq 1$. Let $M$ be the closure of the range of $C_\phi$. Then $\dim M^\perp \leq 1$. We prove the result by considering the following cases which exhaust all possibilities.

Case 1. $\dim M^\perp = 0$:
In this case clearly, the range of $C_\phi$ is dense in $H(E)$.

Case 2. $\dim M^\perp = 1$:
Choose a non-zero function $h \in H(E)$ such that $M^\perp = \text{span}\{h\}$. By Proposition 1.3, the function $h$ can be written as $h(z) = \sum_{j=0}^\infty \langle z^j, a_j \rangle,$ where $a_j \in E^j$ with $\sum_{j=0}^\infty j! \|a_j\|^2 < \infty$. By Theorem 2.12, we get that $A^*$ is cyclic operator. Hence $\dim \text{ran}(A^*)^\perp \leq 1$. By Proposition 2.14, $\phi$ is injective and hence $A$ is injective, therefore $\dim \text{ran}(A^*)^\perp \neq 1$. Hence $\dim \text{ran}(A^*)^\perp = 0$. Thus $\ker(A) = \{0\}$. Let $\zeta$ be any element in $E$. Consider the function $P_m(z) = \langle (z - b)^m, \zeta^m \rangle \in H(E)$. Then $\langle C_\phi P_m, h \rangle = 0$ for every $m \geq 0$. This implies that $\langle a_m, (A^*)^{\otimes m} \zeta^m \rangle = 0$. Since $a_m \in E^m$, there is an element $p \in E$ such that $p^m = p^{\otimes m} = a_m$. Hence $\langle p, A^* \zeta \rangle = 0$ and this implies that $\langle Ap, \zeta \rangle = 0$ for all $\zeta \in E$. Therefore, the element $p$ is in the $\ker(A)$, hence $p = 0$. Using this fact we can conclude that $a_j = 0$ and this implies that $h = 0$, a contradiction.

3. Supercyclicity

We know that if a bounded operator on a separable Hilbert space is hypercyclic then it is supercyclic. In this section we show that there is no supercyclic composition operator on the Segal-Bargmann space. In [12, Theorem 5.4], it is shown that there is no bounded supercyclic composition operator on $F^2(\mathbb{C}^N)$. This also holds true in Segal-Bargmann space $H(E)$ under some suitable conditions, where $E$ is any infinite dimensional separable complex Hilbert space.

Theorem 3.1. [3, Theorem 2.2] Let $T$ be a bounded supercyclic operator on a Hilbert space $H$, and the set $\{T^n : n \in \mathbb{N}\}$ is uniformly bounded. Then for each $x \in H$,

$$\lim_{n \to \infty} T^n x = 0.$$
Proposition 3.2. Let $\phi$ be a mapping on $E$ satisfying the property $\mathcal{P}$. Then $C_\phi$ cannot be supercyclic operator in each of the following cases:

(i) $b = 0$.
(ii) $b \neq 0$ and $1 \notin \sigma(A)$.

Proof. If possible, let $C_\phi$ be supercyclic. By Theorem 1.7, we have $\phi(z) = Az + b$, where $A : E \to E$ is a bounded linear operator with $\|A\| \leq 1$ and $b \in \text{ran}(I - AA^*)^\perp$.

**Proof of (i):** $b = 0$:
For any $f \in \mathcal{H}(E)$ we have,

$$f(z) = \sum_{j=0}^{\infty} \langle z^j, a_j \rangle, \quad a_j \in E^j.$$

with $\|f\|^2 = \sum_{j=0}^{\infty} j! \|a_j\|^2 < \infty$. Therefore,

$$C^n_\phi f(z) = f(A^n z) = \sum_{j=0}^{\infty} \langle (A^n z)^j, a_j \rangle,
= \sum_{j=0}^{\infty} \langle z^j, ((A^*)^n)^\otimes j a_j \rangle,
= \sum_{j=0}^{\infty} \langle z^j, B^{\otimes j} a_j \rangle,$$

where $(A^*)^n = B$. Note that $\|B\| \leq 1$, and

$$\|C^n_\phi f\|^2 = \sum_{j=0}^{\infty} j! \|B^{\otimes j} a_j\|^2 \leq \sum_{j=0}^{\infty} j! \|a_j\|^2 = \|f\|^2 < \infty.$$

This implies that,

$$\|C^n_\phi\| \leq 1, \text{ for all } n \geq 0, 1, 2, \ldots.$$

The set $\{C^n_\phi : n \in \mathbb{N}\}$ is uniformly bounded and $C^n_\phi (1) = 1$. Hence by Theorem 3.1, $C_\phi$ cannot be supercyclic.

**Proof of (ii):** $b \neq 0$ and $1 \notin \sigma(A)$. By Theorem 2.8, there exists a point $z_0 \in E$ such that $\phi(z_0) = z_0$. Now for any $f \in \mathcal{H}(E)$, consider the power series expansion of $f$ around $z_0$,

$$f(z) = \sum_{j=0}^{\infty} \langle (z - z_0)^j, \zeta_j \rangle, \quad \zeta_j \in E^j,$$
with \( \|f\|^2 = \sum_{j=0}^{\infty} j! \|\zeta_j\|^2 < \infty \). Then
\[
C_\phi^n f(z) = \sum_{j=0}^{\infty} \langle (z - z_0)^j, ((A^*)^n) \otimes \zeta_j \rangle.
\]

Following similar steps as in **Proof of (i)**, we can get the desired result. \(\square\)

From the preceding result, we have seen that for a mapping \( \phi : \mathcal{E} \to \mathcal{E} \) with \( \phi(0) = 0 \) such that the composition operator \( C_\phi \) is bounded operator on \( \mathcal{H}(\mathcal{E}) \), then \( C_\phi \) cannot be supercyclic operator.

The above conclusion remains true in the case of \( b \neq 0 \) and \( 1 \notin \sigma(A) \).

Next we can ask the following question.

**Question 3.3.** Let \( \phi \) be a mapping on \( \mathcal{E} \) satisfying the property \( P \). Describe the supercyclicity of \( C_\phi \) when \( b \neq 0 \) and \( 1 \in \sigma(A) \).

We have the following example to hope for an affirmative answer.

**Example 3.4.** Let us consider the map \( \varphi(z) = Az + b \) on \( \mathbb{C}^3 \), where \( A \) is the \( 3 \times 3 \) matrix defined by
\[
A = \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and \( b = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \).

Then we see that \( 1 \in \sigma(A) \) and \( b \neq 0 \) but the corresponding bounded composition operator \( C_{\varphi} \) is not supercyclic on \( \mathcal{F}^2(\mathbb{C}^3) \) (See, [12, Theorem 5.4]).

**References**

[1] J. Agler and J. E. McCarthy, *Pick interpolation and Hilbert function spaces*, Graduate Studies in Mathematics, 44, American Mathematical Society, Providence, RI, 2002.
[2] W. Arveson, Subalgebras of \( C^* \)-algebras. III. Multivariable operator theory, Acta Math. **181** (1998), no. 2, 159–228.
[3] S. I. Ansari and P. S. Bourdon, Some properties of cyclic operators, Acta Sci. Math. (Szeged) **63** (1997), no. 1-2, 195–207.
[4] F. Bayart and Matheron, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, 179, Cambridge University Press, Cambridge, 2009.
[5] S. K. Berberian, *Introduction to Hilbert space* (Spanish), translated from the English by Joaquín Sánchez Guillén, Editorial Teide, Barcelona, 1970.
[6] G.D. Birkhoff, *Démonstration d’un théorème élémentaire sur les fonctions entières*, C. R. Acad. Sci. Paris, 189 (1929), 473-475.
[7] P. S. Bourdon and J. H. Shapiro, Cyclic phenomena for composition operators, Mem. Amer. Math. Soc. **125** (1997), no. 596, x+105 pp.
[8] B. J. Carswell, B. D. MacCluer and A. Schuster, Composition operators on the Fock space, Acta Sci. Math. (Szeged) 69 (2003), no. 3-4, 871–887.
[9] S. R. Foguel, Weak Limits of Powers of a Contraction in Hilbert Space, Proceedings of the American Mathematical Society, Vol. 16, No. 4 (Aug., 1965), pp. 659-661.
[10] T. Furuta and R. Nakamoto, Some theorems on certain contraction operators, Proc. Japan Acad. 45 (1969), 565–567.
[11] K. Guo and K. Izuchi, Composition operators on Fock type spaces, Acta Sci. Math. (Szeged) 74 (2008), no. 3-4, 807–828.
[12] L. Jiang, G. T. Prajitura and R. Zhao, Some characterizations for composition operators on the Fock spaces, J. Math. Anal. Appl. 455 (2017), no. 2, 1204–1220.
[13] P. Kocourek, W. Takahashi and J.-C. Yao, Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces, Taiwanese J. Math. 14 (2010), no. 6, 2497–2511.
[14] C. S. Kubrusly, The elements of operator theory, second edition, Birkhäuser/Springer, New York, 2011.
[15] T. Le, Composition operators between Segal-Bargmann spaces, J. Operator Theory 78 (2017), no. 1, 135–158.
[16] Tesfa Mengestie, Cyclic and supercyclic weighted composition operators on the fock space (arxiv print).
[17] J. Mujica, Complex analysis in Banach spaces, North-Holland Mathematics Studies, 120, North-Holland Publishing Co., Amsterdam, 1986.
[18] J. R. Retherford, Hilbert space: compact operators and the trace theorem, London Mathematical Society Student Texts, 27, Cambridge University Press, Cambridge, 1993.
[19] J. H. Shapiro, Composition operators and classical function theory, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY - HYDERABAD, KANDI, SANGAREDDY, TELANGANA, INDIA 502 285.
Email address: rameshg@math.iith.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY - HYDERABAD, KANDI, SANGAREDDY, TELANGANA, INDIA 502 285.
Email address: ma16resch11003@iith.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY - HYDERABAD, KANDI, SANGAREDDY, TELANGANA, INDIA 502 285.
Email address: venku@math.iith.ac.in