Riesz Projection and Essential $S$-spectrum in Quaternionic Setting

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Received: 18 August 2021 / Accepted: 20 August 2022
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Abstract
This paper is devoted to the investigation of the Weyl and the essential $S$-spectra of a bounded right quaternionic linear operator in a right quaternionic Hilbert space. Using the quaternionic Riesz projection, the $S$-eigenvalue of finite type is both introduced and studied. In particular, we have shown that the Weyl and the essential $S$-spectra do not contain eigenvalues of finite type. We have also described the boundary of the Weyl $S$-spectrum and the particular case of the spectral theorem of the essential $S$-spectrum.

Keywords Quaternions · Quaternionic Riesz projection · Essential $S$-spectrum · Weyl $S$-spectrum

Mathematics Subject Classification 46S10 · 47A60 · 47A10 · 47A53 · 47B07

Communicated by Fabrizio Colombo.

This article is part of the topical collection “Spectral Theory and Operators in Mathematical Physics” edited by Jussi Behrndt and Fabrizio Colombo.

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1 Introduction

Over the recent years, the spectral theory for quaternionic operators has piqued the interest and attracted the attention of multiple researchers, see for instance [1, 4, 12–17, 29] and references therein. Research in this topic is motivated by application in various fields, including quantum mechanics, fractional evolution problems [14], and quaternionic Schur analysis [3]. The concept of spectrum is one of the main objectives in the theory of quaternionic operators acting on quaternionic Hilbert spaces. The obscurity that seemed to surround the precise definition of the quaternionic spectrum of a linear operator was deemed a stumbling block. However, in 2006, Colombo and Sabadini succeeded in devising a new notion conducive to the complete development of the quaternionic operator theory, namely the $S$-spectrum. We refer to [13, Subsection 1.2.1] for a precise history. After several more years of formulating the spectral theorem that stemmed from the $S$-spectrum, Alpay, Colombo, Kimsey, see [6], supplied ample evidence in 2016 to further establish this fundamental theorem for both bounded and unbounded operators. In the book [13], see also [14], the authors briefly explain the concept of $S$-spectrum and give the systematic basis of quaternionic spectral theory. We refer to the book [3] and the references therein for the spectral theory on the $S$-spectrum for Clifford operators and to [9] for some results on operators perturbation.

Motivated by the new concept of $S$-spectrum in the quaternionic setting, Muraleetharan and Thirulogasanthar, in [29, 30], introduced the Weyl and essential $S$-spectra and gave a characterization using Fredholm operators. We refer to [7] for the study of the general framework of the Fredholm element with respect to a quaternionic Banach algebra homomorphism. In general, the set of all operators acting on right Banach space is not quaternionic Banach algebra with respect to the composition operators. By [19, Theoreme 7.1 and Theorem 7.3], if $V^R_H$ is a separable quaternionic Hilbert space, then $B(V^R_H)$ (the set of all right bounded operators) is a quaternionic two-sided $C^*$-algebra and the set of all compact operators $K(V^R_H)$ is a closed two-sided ideal of $B(V^R_H)$ which is closed under adjunction. In this regard, in [29], the author defined the essential $S$-spectrum as the $S$-spectrum of quotient map image of bounded right linear operator on the Calkin algebra $B(V^R_H)/K(V^R_H)$.

In order to explain the objective of this work, we start by recalling a few results concerning the discrete spectrum and the Riesz projection in the complex setting. Let $T$ be a linear operator acting on a complex Banach space $V_C$. We denote the spectrum of $T$ by $\sigma(T)$. Let $\sigma$ be an isolated part of $\sigma(T)$. The Riesz projection of $T$ corresponding
to $\sigma$ is the operator

$$P_\sigma = \frac{1}{2\pi i} \int_{C_\sigma} (z - T)^{-1} \, dz$$

where $C_\sigma$ is a smooth closed curves belonging to the resolvent set $\mathbb{C}\setminus\sigma(T)$ such that $C_\sigma$ surrounds $\sigma$ and separates $\sigma$ from $\sigma(T)\setminus\sigma$. The discrete spectrum of $T$, denoted $\sigma_d(T)$, is the set of isolated point $\lambda \in \mathbb{C}$ of $\sigma(T)$ such that the corresponding Riesz projection $P_{[\lambda]}$ are finite dimensional, see [21, 28]. Note that in general we have $\sigma_e(T) \subset \sigma(T)\setminus\sigma_d(T)$, where $\sigma_e(T)$ denotes the set of essential spectrum of $T$. We refer to [8, 25, 34] for more properties of $\sigma_e(T)$. Note that, if $A$ is a self-adjoint operator on a Hilbert space, then $\sigma_e(T) = \sigma(T)\setminus\sigma_d(T)$. In particular, the essential spectrum is empty if and only if $\sigma(T) = \sigma_d(T)$. We point out that this point, namely the absence of the essential spectrum, has been studied in many works, e.g., [22, 27].

In the quaternionic setting, if $T \in B(V_R\mathbb{H})$ and $q \in \sigma_S(T)\setminus\mathbb{R}$ (where $\sigma_S(T)$ denote the $S$-spectrum of $T$), then $q$ is not an isolated point of $\sigma_S(T)$. Indeed, $[q] := \{hqh^{-1} : h \in \mathbb{H}^*\} \subset \sigma_S(T)$, see [13]. By the compactness of $[q]$, $[q]$ is not an isolated part of $\sigma_S(T)$. However, if we set

$$\Omega := \sigma_S(T)/\sim$$

where $p \sim q$ if and only if $p \in [q]$, and $E_T$ the set of representative, then if $[q]$ is an isolated part of $\sigma_S(T)$ then $q$ is an isolated point of $E_T$.

The first aim of this work is to study the isolated part of the $S$-spectrum of a bounded right quaternionic operators and its relation with the essential $S$-spectrum. To begin with, we consider the Riesz projector associated with a given quaternionic operator $T$ which was introduced in [11]. We refer to [2, 4, 9, 13] for more details on this concept. We treat the decomposition of the essential $S$-spectrum of $T$ as a function of the Riesz projector, and more generally as a function of a projector which commutes with $T$, see Theorem 3.1. The technique of the proof is inspired from [4]. We also discuss the Riesz decomposition theorem [32, Theorem 6] in quaternionic setting. More precisely, we prove that this decomposition is unique. Motivated by this, we study the quaternionic version of the discrete $S$-spectrum. Following the complex formalism given in [21, 28], we show that the essential $S$-spectrum of a given right operator acting on right Hilbert space does not contain discrete element of the $S$-spectrum. The second aim of this work is to give new results concerning the Weyl and essential $S$-spectra in quaternionic setting. First off, Theorem 4.5 gives a description of the boundary of the Weyl $S$-spectrum. The proof is based on the study of the minimal modulus of the right quaternionic operators. We also deal with the particular case of the spectral theorem of essential $S$-spectra. The technique of the proof is inspired from [11].

The article is organised as follows: In Sect. 2, we present general definitions about operators theory in quaternionic setting. In Sect. 3, we discuss the question of decomposition of the essential $S$-spectrum. Finally, in Sect. 4, we provide new results of the Weyl $S$-spectrum.
2 Mathematical Preliminaries

In this section, we review some basic notions about quaternions, right quaternionic Hilbert space, right linear operator (even unbounded and define its $S$-spectrum), and slice functional calculus. For details, we refer to the reader [1, 11, 13, 19, 23].

2.1 Quaternions

Let $\mathbb{H}$ be the Hamiltonian skew field of quaternions. This class of numbers can be written as:

$$q = q_0 + q_1i + q_2j + q_3k,$$

where $q_l \in \mathbb{R}$ for $l = 0, 1, 2, 3$ and $i, j, k$ are the three quaternionic imaginary units satisfying

$$i^2 = j^2 = k^2 = ijk = -1.$$

The real and the imaginary part of $q$ is defined as $\text{Re}(q) = q_0$ and $\text{Im}(q) = q_1i + q_2j + q_3k$, respectively. Then, the conjugate and the usual norm of the quaternion $q$ are given respectively by

$$\bar{q} = q_0 - q_1i - q_2j - q_3k$$

and

$$|q| = \sqrt{q\bar{q}}.$$

The set of all imaginary unit quaternions in $\mathbb{H}$ is denoted by $\mathbb{S}$ and defined as

$$\mathbb{S} = \left\{ q_1i + q_2j + q_3k : q_1, q_2, q_3 \in \mathbb{R}, q_1^2 + q_2^2 + q_3^2 = 1 \right\}.$$

The name imaginary unit is due the fact that, for any $I \in \mathbb{S}$, we have

$$I^2 = -\bar{I}I = -|I|^2 = -1.$$

For every $q \in \mathbb{H}\setminus\mathbb{R}$, we associate the unique element

$$I_q := \frac{\text{Im}(q)}{|\text{Im}(q)|} \in \mathbb{S}$$

such that

$$q = \text{Re}(q) + I_q|\text{Im}(q)|.$$

This implies that

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I.$$
where

\[ \mathbb{C}_I := \mathbb{R} + I\mathbb{R}. \]

We can associate to \( q \in \mathbb{H} \) the 2-dimensional sphere

\[ [q] := \{ \text{Re}(q) + I|\text{Im}(q)| : I \in \mathbb{S} \}. \]

This sphere has center at the real point \( \text{Re}(q) \) and radius \( |\text{Im}(q)| \).

## 2.2 Right Quaternionic Hilbert Space and Operator

In this subsection, we recall the concept of right quaternionic Hilbert space and right linear operator (see, [1, 12, 19]).

**Definition 2.1** [1] Let \( V^R_H \) be a right vector space. The map

\[ \langle ., . \rangle : V^R_H \times V^R_H \rightarrow \mathbb{H} \]

is called an inner product if it satisfies the following properties:

(i) \( \langle f, gq + h \rangle = \langle f, g \rangle q + \langle f, h \rangle \), for all \( f, g, h \in V^R_H \) and \( q \in \mathbb{H} \).

(ii) \( \langle f, g \rangle = \overline{\langle g, f \rangle} \), for all \( f, g \in V^R_H \).

(iii) If \( f \in V^R_H \), then \( \langle f, f \rangle \geq 0 \) and \( f = 0 \) if \( \langle f, f \rangle = 0 \).

The pair \((V^R_H, \langle ., . \rangle)\) is called a right quaternionic pre-Hilbert space. Moreover, \( V^R_H \) is said to be right quaternionic Hilbert space, if

\[ \| f \| = \sqrt{\langle f, f \rangle} \]

defines a norm for which \( V^R_H \) is complete.

In the sequel, we assume that \( V^R_H \) is complete and separable. We now recall the concept of Hilbert basis in the quaternionic case. First, we review the following proposition, the proof of which is similar to its complex version, see [19, 33].

**Proposition 2.2** Let \( V^R_H \) be a right quaternionic Hilbert space and let \( \mathcal{F} = \{ f_k : k \in \mathbb{N} \} \) be an orthonormal subset of \( V^R_H \). The following properties are equivalent:

1. For every \( f, g \in V^R_H \), the series \( \sum_{k \in \mathbb{N}} \langle f, f_k \rangle \langle f_k, g \rangle \) converges absolutely and

\[ \langle f, g \rangle = \sum_{k \in \mathbb{N}} \langle f, f_k \rangle \langle f_k, g \rangle. \]

2. For every \( f \in V^R_H \), we have

\[ \| f \|^2 = \sum_{k \in \mathbb{N}} |\langle f_k, f \rangle|^2 \]
\[(3) \quad \mathcal{F}^\perp := \left\{ f \in V_R^H : \langle f, g \rangle = 0 \text{ for all } g \in \mathcal{F} \right\} = \{0\}.\]

\[(4) \quad \langle \mathcal{F} \rangle := \left\{ \sum_{l=1}^{m} f_l q_l : f_l \in \mathcal{F}, \ q_l \in H, \ m \in \mathbb{N} \right\} \text{ is dense in } V_R^H.\]

**Definition 2.3** Let \( \mathcal{F} \) be an orthonormal subset of \( V_R^H \). \( \mathcal{F} \) is said to be Hilbert basis of \( V_R^H \) if \( \mathcal{F} \) verifies one of the equivalent conditions of Proposition 2.2.

The proof of the following proposition is the same as its complex version, see [19, 33].

**Proposition 2.4** Let \( V_R^H \) be a right quaternionic Hilbert space. Then,

1. \( V_R^H \) admits a Hilbert basis.
2. Two Hilbert basis of \( V_R^H \) have the same cardinality.
3. If \( \mathcal{F} \) is a Hilbert basis of \( V_R^H \), then every \( f \in V_R^H \) can be uniquely decomposed as follows:

\[ f = \sum_{k \in \mathbb{N}} f_k \langle f_k, f \rangle \]

where the series \( \sum_{k \in \mathbb{N}} f_k \langle f_k, f \rangle \) converges absolutely in \( V_R^H \).

The quaternionic multiplication is not commutative. Afterwards, we recall that if \( V_R^H \) is a right separable quaternionic Hilbert space, we can define the left scalar multiplication on \( V_R^H \) using an arbitrary Hilbert basis on \( V_R^H \). We refer to [19] for an explanation of this construction. Let \( \mathcal{F} = \left\{ f_k : k \in \mathbb{N} \right\} \) be a Hilbert basis of \( V_R^H \). The left scalar multiplication on \( V_R^H \) induced by \( \mathcal{F} \) is defined as the map

\[ \mathbb{H} \times V_R^H \rightarrow V_R^H \quad (q, f) \mapsto qf = \sum_{k \in \mathbb{N}} f_k q \langle f_k, f \rangle. \]

The properties of the left scalar multiplication are described in the following proposition.

**Proposition 2.5** [19, Proposition 3.1] Let \( f, g \in V_R^H \) and \( p, q \in \mathbb{H} \), then

1. \( q(f + g) = qf + qg \) and \( q(fp) = (qf)p \).
2. \( \|qf\| = |q| \|f\| \).
3. \( q(pf) = (qp)f \).
4. \( \langle qf, g \rangle = \langle f, qg \rangle \).
5. \( rf = fr \), for all \( r \in \mathbb{R} \).
6. \( qf_k = f_k q \), for all \( k \in \mathbb{N} \).

It is easy to see that \( (p + q)f = pf + qf \), for all \( p, q \in \mathbb{H} \) and \( f \in V_R^H \). In the sequel, we consider \( V_R^H \) as a right quaternionic Hilbert space equipped with the left scalar multiplication.
Definition 2.6 Let $V^R_H$ be a right quaternionic Hilbert space. A mapping $T : \mathcal{D}(T) \subset V^R_H \rightarrow V^R_H$, where $\mathcal{D}(T)$ denote the domain of $T$, is called quaternionic right linear if

$$T(f + gq) = T(f) + T(g)q, \text{ for all } f, g \in \mathcal{D}(T) \text{ and } q \in \mathbb{H}.$$ 

The operator $T$ is called closed, if the graph $\mathcal{G}(T) := \{(f, Tf) : f \in \mathcal{D}(T)\}$ is a closed right linear subspace of $V^R_H \times V^R_H$.

We call an quaternionic right operator $T$ bounded if

$$\|T\| := \sup \left\{ \|Tf\| : f \in V^R_H, \|f\| = 1 \right\} < +\infty.$$

The set of all bounded right operators on $V^R_H$ is denoted by $\mathcal{B}(V^R_H)$ and the identity operator on $V^R_H$ will be denoted by $I_{V^R_H}$.

Let $T \in \mathcal{B}(V^R_H)$, we denote the null space of $T$ by $N(T)$ and its range space by $R(T)$. A closed subspace $M$ of $V^R_H$ is said to be $T$-invariant subspace if $T(M) \subset M$. Note that the function $f \mapsto Tf - fq$ is not right linear, we refer to [12, 13] for this point of view. The fundamental suggestion of [12] is to define the spectrum using the Cauchy kernel series. We recall these concepts from the book [13].

Definition 2.7 Let $T : \mathcal{D}(T) \subset V^R_H \rightarrow V^R_H$ be a right linear operator. We define the operator $Q_q(T) : \mathcal{D}(T^2) \rightarrow V^R_H$ by

$$Q_q(T) := T^2 - 2\text{Re}(q)T + |q|^2 I_{V^R_H}.$$ 

1. The $S$-resolvent set of $T$ is defined as follows:

$$\rho_S(T) := \left\{ q \in \mathbb{H} : N(Q_q(T)) = \{0\}, \overline{R(T)} = \mathbb{H} \text{ and } Q_q(T)^{-1} \in \mathcal{B}(V^R_H) \right\}.$$ 

2. The $S$-spectrum of $T$ is defined as:

$$\sigma_S(T) = \mathbb{H} \setminus \rho_S(T).$$ 

3. The point $S$-spectrum of $T$ is given by

$$\sigma_{pS}(T) := \left\{ q \in \mathbb{H} : N(Q_q(T)) \neq \{0\} \right\}.$$ 

For $T \in \mathcal{B}(V^R_H)$, the $S$-spectrum $\sigma_S(T)$ is a non-empty compact set, see [13]. We recall that if $T \in \mathcal{B}(V^R_H)$ and $q \in \sigma_S(T)$, then all the elements of the sphere $[q]$ belong to $\sigma_S(T)$, see [3, Theorem 7.2.8].

Let $v \in V^R_H \setminus \{0\}$, then $v$ is a right eigenvalue of $T$ if $T(v)$ is a right quaternionic multiple of $v$. That is

$$T(v) = vq.$$
where \( q \in \mathbb{H} \), known as the right eigenvalue. The set of right eigenvalues coincides with the point \( S \)-spectrum, see [19, Proposition 4.5].

Now, we recall the definition of essential \( S \)-spectrum, we refer to [29, 30] for more details. Let \( V_{\mathbb{H}}^R \) be a separable right quaternionic Hilbert space equipped with a left scalar multiplication. Using [19], \( B(V_{\mathbb{H}}^R) \) is a quaternionic two-sided Banach \( C^* \)-algebra with unity, and the set of all compact operators \( K(V_{\mathbb{H}}^R) \) is a closed two-sided ideal of \( B(V_{\mathbb{H}}^R) \). We consider the natural quotient map:

\[
\pi : B(V_{\mathbb{H}}^R) \longrightarrow C(V_{\mathbb{H}}^R) := B(V_{\mathbb{H}}^R)/K(V_{\mathbb{H}}^R) \quad T \mapsto [T] = T + K(V_{\mathbb{H}}^R).
\]

Note that \( \pi \) is a unital homomorphism, see [29]. The norm on \( C(V_{\mathbb{H}}^R) \) is given by

\[
\| [T] \| = \inf_{K \in K(V_{\mathbb{H}}^R)} \| A + K \|.
\]

**Definition 2.8** [29] The essential \( S \)-spectrum of \( T \in B(V_{\mathbb{H}}^R) \) is the \( S \)-spectrum of \( \pi(A) \) in the Calkin algebra \( C(V_{\mathbb{H}}^R) \). That is,

\[
\sigma^S_e(T) := \sigma_S(\pi(A)).
\]

### 2.3 The Quaternionic Functional Calculus

The quaternionic functional calculus is defined on the class of slice regular function \( f : U \longrightarrow \mathbb{H} \) for some set \( U \subset \mathbb{H} \). We recall this concept and refer to [9, 11, 13] and the references therein on the matter.

**Definition 2.9** A set \( U \subset \mathbb{H} \) is called

(i) axially symmetric if \( [x] \subset U \) for any \( x \in U \) and

(ii) a slice domain if \( U \) is open, \( U \cap \mathbb{R} \neq \emptyset \) and \( U \cap C_I \) is a domain in \( C_I \), for any \( I \in \mathbb{S} \).

**Definition 2.10** Let \( U \subset \mathbb{H} \) be an open set. A real differentiable function \( f : U \longrightarrow \mathbb{H} \) is said to be left \( s \)-regular (resp. right \( s \)-regular) if for every \( I \in \mathbb{S} \), the function \( f \) satisfy

\[
\frac{1}{2} \left[ \frac{\partial}{\partial x} f(x + Iy) + I \frac{\partial}{\partial y} f(x + Iy) \right] = 0 \quad (\text{resp.} \quad \frac{1}{2} \left[ \frac{\partial}{\partial x} f(x + Iy) + \frac{\partial}{\partial y} f(x + Iy) I \right] = 0).
\]

We denote the class of left \( s \)-regular (resp. right \( s \)-regular) by \( \mathcal{R}^L(U) \) (resp. \( \mathcal{R}^R(U) \)). We recall that \( \mathcal{R}^L(U) \) is a right \( \mathbb{H} \)-module and \( \mathcal{R}^R(U) \) is a left \( \mathbb{H} \)-module. Let \( V_{\mathbb{H}}^R \) be a separable right quaternionic Hilbert space equipped with a Hilbert basic
and with a left scalar multiplication. We recall that $\mathcal{B}(V^R_{\mathbb{H}})$ is a two-sided ideal quaternionic Banach algebra with respect to the left multiplication given by
\[(q, T)f = \sum_{g \in \mathcal{N}} gq(g, Tf) \text{ and } (Tq)f = \sum_{g \in \mathcal{N}} T(g)q(g, f).\]

**Definition 2.11** Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$ and $q \in \rho_S(T)$. The left $S$-resolvent operator is given by
\[S_L^{-1}(q, T) := -Q_q(T)^{-1}(T - \overline{q}^H_{V^R_{\mathbb{H}}}),\]
and the right $S$-resolvent operator is defined by
\[S_R^{-1}(q, T) := -(T - \overline{q}^H_{V^R_{\mathbb{H}}})Q_q(T)^{-1}.\]

**Definition 2.12** [4, Definition 3.4] Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$ and let $U \subset \mathbb{H}$ be an axially symmetric $s$-domain that contains the $S$-spectrum $\sigma_S(T)$ and such that $\partial(U \cap C_I)$ is union of a finite number of continuously differentiable Jordan curves for every $I \in \mathbb{S}$. We say that $U$ is a $T$-admissible open set.

**Definition 2.13** Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$, $W \subset \mathbb{H}$ be open set. A function $f \in \mathcal{R}^L(W)$ (resp. $\mathcal{R}^R(W)$) is said to be locally left regular (resp. right regular) function on $\sigma_S(T)$, if there is a $T$-admissible domain $U \subset \mathbb{H}$ such that $U \subset W$.

We denote by $\mathcal{R}^L_{\sigma_S(T)}$ (resp. $\mathcal{R}^R_{\sigma_S(T)}$) the set of locally left (resp. right) regular functions on $\sigma_S(T)$. Now, we recall the two versions of the quaternionic functional calculus.

**Definition 2.14** [11, Definition 4.10.4] Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$ and $U \subset \mathbb{H}$ be a $T$-admissible domain. Then, $f(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} S_L^{-1}(q, T) dqI f(q) \forall f \in \mathcal{R}^L_{\sigma_S(T)} \quad (1)$
and $f(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} f(q) dqI S_R^{-1}(q, T) \forall f \in \mathcal{R}^R_{\sigma_S(T)} \quad (2)$
where $dqI = -dqI$.

The two integrals that appear in Eqs (1) and (2) are independent of the choice of imaginary unit $I \in \mathbb{S}$ and $T$-admissible domain, see [11, Theorem 4.10.3].

A set $\sigma$ is called an isolated part of $\sigma_S(T)$ if both $\sigma$ and $\sigma_S(T) \setminus \sigma$ are closed subsets of $\sigma_S(T)$.

**Definition 2.15** [5, 9] Let $T \in \mathcal{B}(V^R_{\mathbb{H}})$. Denote by $U_\sigma$ an axially symmetric $s$-domain that contains the axially symmetric isolated part $\sigma \subset \sigma_S(T)$ but not any other point
of $\sigma_S(T)$. Suppose that the Jordan curves $\partial(U_\sigma \cap \mathbb{C}_I)$ belong to the $S$-resolvent set $\rho_S(T)$, for any $I \in \mathbb{S}$. We define the quaternionic Riesz projection by

$$P_\sigma = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_I)} S_L^{-1}(q, T) dq I.$$ 

**Remark 2.16** [5, 9] The concept of $P_\sigma$ can be given by using the right $S$-resolvent operator $S_R^{-1}(q, T)$, that is

$$P_\sigma = \frac{1}{2\pi} \int_{\partial(U_\sigma \cap \mathbb{C}_I)} dq I S_R^{-1}(q, T).$$

Note that $P_\sigma$ is a projection that commutes with $T$, see [5, Theorem 2.8].

In the sequel, we assume that $V_R^\mathbb{H}$ is a separable right quaternionic Hilbert space with infinite dimensional.

### 3 Riesz Projection and Essential Spectrum

We recall that in [29, 30], the study of the essential $S$-spectrum is established using the theory of Fredholm operators. The aim of this section is to show that the essential $S$-spectrum does not contain discrete element of the $S$-spectrum. In this regard, let $V_R^\mathbb{H}$ be a separable right quaternionic Hilbert space. We note that if $K \in \mathcal{K}(V_R^\mathbb{H})$, then $\sigma^e_S(A + K) = \sigma^S(A)$ for all $A \in \mathcal{B}(V_R^\mathbb{H})$. We start by showing that in general the $S$-spectrum does not satisfy this property. We define $V'_R = \mathcal{B}(V_R^\mathbb{H}, \mathbb{H})$ and call $V'_R$ the right dual space of $V_R^\mathbb{H}$.

**Theorem 3.1** Let $T \in \mathcal{B}(V_R^\mathbb{H})$. Then, $\sigma_S(T + A) \subset \sigma_S(A)$ for all $A \in \mathcal{B}(V_R^\mathbb{H})$ if and only if $T \equiv 0$.

**Proof** Assume that $T$ is non-zero operator and $\sigma_S(T + A) \subset \sigma_S(A)$ for all $A \in \mathcal{B}(V_R^\mathbb{H})$. Let $0_{V_R^\mathbb{H}} \neq x \in V_R^\mathbb{H}$ such that

$$Tx = y \neq 0_{V_R^\mathbb{H}}.$$

**First step:** There exists $f \in V'_R$ such that

$$f(x) = 1 \text{ and } f(y) \neq 0.$$

Indeed, if $x = yq$ for some $q \in \mathbb{H}$, the result follows from Hahn-Banach theorem. Now assume that $x$ and $y$ are linearly independent. Then, there exists a right basis $Z$ of $V_R^\mathbb{H}$ such that $\{x, y\} \subset Z$. We consider the following map

$$f : u = \sum_{z \in Z} zq_z \mapsto q_x + q_y.$$
It is clear that $f$ is right linear and $f(x) = 1$ and $f(y) \neq 0$.

**Second step:** Take

$$A := (x - y) \otimes f$$

the rank one operator on $V^R_H$ given by

$$u \mapsto (x - y)f(u).$$

We have:

$$(T + A)^2x - 2(T + A)x = (T + A)x - 2x = -x,$$

So, $1 \in \sigma_S(A + T)$. Since $\dim V^R_H > 1$, then $0 \in \sigma_S(A)$. In the sequel, we assume that $x \neq y$. Let $q \in \sigma_S(A)$. By [3, Lemma 4.2.3], we have

$$(f(x - y))^2 - 2\text{Re}(q)f(x - y) + |q|^2 = 0$$

if and only if

$$q \in \left\{hf(x - y)h^{-1} : h \in H \setminus \{0\}\right\}.$$ 

This implies that $1 \notin \sigma_S(A)$. Indeed, if $1$ is an $S$-eigenvalue of $A$, there exists $h \in H^*$ such that

$$hf(x - y)h^{-1} = 1$$

and so $f(y) = 0$, contradiction. $\square$

**Definition 3.2** A bounded operator $S \in B(V^R_H)$ is called a *quasi-inverse* of the operator $T \in B(V^R_H)$ if there exists $K_1, K_2 \in K(V^R_H)$ such that

$$ST = I_{V^R_H} - K_1 \text{ and } TS = I_{V^R_H} - K_2.$$

**Lemma 3.3** Let $T \in B(V^R_H), q \in H\setminus \sigma^S_e(T)$. Let $R_q(T)$ be a quasi-inverse of $Q_q(T)$ and $A$ be an operator that commute with $T$. Then, there exist $K \in K(V^R_H)$ such that

$$AR_q(T) = R_q(T)A + K.$$

**Proof** Since $AT = TA$, then $Q_q(T)A = A Q_q(T)$. Let $K_1, K_2 \in K(V^R_H)$ such that

$$R_q(T)Q_q(T) = I_{V^R_H} - K_1 \text{ and } Q_q(T)R_q(T) = I_{V^R_H} - K_2.$$
Then,

\[ AR_q(T) = R_q(T)A + K \]

where \( K = K_1AR_q(T) - R_q(T)AK_2 \in \mathcal{K}(V^R_{\mathbb{H}}) \).

**Theorem 3.4** Let \( V^R_{\mathbb{H}} \) be a quaternionic Hilbert space and \( T \in \mathcal{B}(V^R_{\mathbb{H}}) \). Let \( P_1 \) be a projector in \( \mathcal{B}(V^R_{\mathbb{H}}) \) commuting with \( T \) and let \( P_2 = \mathbb{I}_{V^R_{\mathbb{H}}} - P_1 \). Take \( T_j := TP_j = P_jT, \ j = 1, 2 \). Then,

\[ \sigma^S_e(T) = \sigma^S_e(T_1|_{R(P_1)}) \cup \sigma^S_e(T_2|_{R(P_2)}), \]

where \( T_i|_M \) denotes the restriction of \( T_i \) to \( M \).

**Proof** Let \( q \notin \sigma^S_e(T) \). Then, there exist \( A_q(T) \in \mathcal{B}(V^R_{\mathbb{H}}) \) and \( K_1, K_2 \in \mathcal{K}(V^R_{\mathbb{H}}) \) such that

\[ A_q(T)Q_q(T) = \mathbb{I}_{V^R_{\mathbb{H}}} - K_1 \quad \text{and} \quad Q_q(T)A_q(T) = \mathbb{I}_{V^R_{\mathbb{H}}} - K_2. \]

Using Lemma 3.3, we infer that there exists \( K_3 \in \mathcal{K}(V^R_{\mathbb{H}}) \) such that

\[ A_q(T)P_1 = P_1A_q(T) + K_3 \quad \text{and} \quad A_q(T)P_2 = P_2A_q(T) - K_3. \]

Therefore,

\[ A_q(T) = P_1A_q(T)P_1 + P_2A_q(T)P_2 + K_4, \tag{3} \]

where \( K_4 = (\mathbb{I}_{V^R_{\mathbb{H}}} - 2P_1)K_3 \in \mathcal{K}(V^R_{\mathbb{H}}) \). Now, we consider the following identity

\[ Q_q(T) = (T_1^2 - 2\text{Re}(q)T_1 + |q|^2P_1) + (T_2^2 - 2\text{Re}(q)T_2 + |q|^2P_2). \tag{4} \]

We multiply the identity (4) by \( A_q(T) \) on the left and on the right, we obtain

\[
\mathbb{I}_{V^R_{\mathbb{H}}} - K_1 = (P_1A_q(T)P_1 + P_2A_q(T)P_2 + K_4)((T_1^2 - 2\text{Re}(q)T_1 + |q|^2P_1)) \\
+ (P_1A_q(T)P_1 + P_2A_q(T)P_2 + K_4)(T_2^2 - 2\text{Re}(q)T_2 + |q|^2P_2)]
\]

and

\[
\mathbb{I}_{V^R_{\mathbb{H}}} - K_2 = ((T_1^2 - 2\text{Re}(q)T_1 + |q|^2P_1))(P_1A_q(T)P_1 + P_2A_q(T)P_2 + K_4) \\
+ ((T_2^2 - 2\text{Re}(q)T_2 + |q|^2P_2))(P_1A_q(T)P_1 + P_2A_q(T)P_2 + K_4).
\]

This leads us to conclude that

\[
\mathbb{I}_{V^R_{\mathbb{H}}} - K_1 = P_1A_q(T)P_1(T_1^2 - 2\text{Re}(q)T_1 + |q|^2P_1) \\
+ P_2A_q(T)P_2(T_2^2 - 2\text{Re}(q)T_2 + |q|^2P_2) + K_5
\]
and
\[
\| V \|_R - K_2 = (T_1^2 - 2\text{Re}(q)T_1 + |q|^2 P_1)P_1 A_q(T) P_1 \\
+ (T_2^2 - 2\text{Re}(q)T_2 + |q|^2 P_2)P_2 A_q(T) P_2 + K_6,
\]
where
\[
K_5 = K_4 Q_q(T) \in \mathcal{K}(V^R) \quad \text{and} \quad K_6 = Q_q(T) K_4 \in \mathcal{K}(V^R).
\]

Take
\[
A_{q,j}(T) = P_j A_q(T) P_j, \quad \text{for } j = 1, 2.
\]
Then,
\[
P_1(\| V \|_R - K_1 - K_5) P_1 = A_{q,1}(T)(T_1^2 - 2\text{Re}(q)T_1 + |q|^2 P_1)
\]
and
\[
P_1(\| V \|_R - K_2 - K_6) P_1 = (T_1^2 - 2\text{Re}(q)T_1 + |q|^2 P_1) A_{q,1}(T).
\]

As a consequence, \( A_{q,1}(T) \mid_{\mathcal{R}(P_1)} \) is a quasi-inverse of \((T_1^2 - 2\text{Re}(q)T_1 + |q|^2 P_1) \mid_{\mathcal{R}(P_1)}\). Similarly, we have \( A_{q,2}(T) \mid_{\mathcal{N}(P_1)} \) is a quasi-inverse \((T_2^2 - 2\text{Re}(q)T_2 + |q|^2 P_2) \mid_{\mathcal{N}(P_1)}\).
Therefore, we conclude that
\[q \in \mathbb{H}\setminus(\sigma_e^S(T_1 \mid_{\mathcal{R}(P_1)}) \cup \sigma_e^S(T_2 \mid_{\mathcal{R}(P_2)})).\]

Conversely, let \( q \notin \sigma_e^S(T_1 \mid_{\mathcal{R}(P_1)}) \cup \sigma_e^S(T_2 \mid_{\mathcal{R}(P_2)}) \). Then, there exists \( A_{q,i}(T) \in \mathcal{B}(\mathcal{R}(P_1)) \) and\( K_{i,j} \in \mathcal{K}(\mathcal{R}(P_1))\), for \( i, j = 1, 2 \), such that
\[
A_{q,i} Q_q(T_i \mid_{\mathcal{R}(P_1)}) = I_{\mathcal{R}(P_1)} - K_{1,i} \quad \text{and} \quad Q_q(T_i) A_{q,i}(T) = I_{\mathcal{R}(P_1)} - K_{2,i}.
\]
Let us define the operator
\[
B_q(T) := P_1 A_{q,1}(T) P_1 + P_2 A_{q,2}(T) P_2.
\]

We get:
\[
B_q(T) Q_q(T) = P_1 A_{q,1}(T) Q_q(T) P_1 + P_2 A_{q,2}(T) Q_q(T) P_2 \\
= P_1(I_{\mathcal{R}(P_1)} - K_{1,1}) P_1 + P_2(I_{\mathcal{R}(P_2)} - K_{1,2}) P_2 \\
= P_1 + P_2 - P_1 K_{1,1} P_1 - P_2 K_{1,2} P_2 \\
= I_{\| V \|_R} - K_6.
\]
where $K_6 = P_1K_{1,1}P_1 + P_2K_{1,2}P_2 \in \mathcal{K}(V^R_H)$. By a similar argument, we obtain

$$Q_q(T)B_q(T) = \mathbb{I}_{V^R_H} - K_7,$$

where $K_7 = P_1K_{2,1}P_1 + P_2K_{2,2}P_2 \in \mathcal{K}(V^R_H)$. Therefore, $B_q(T)$ is a quasi-inverse of $Q_q(T)$. We deduce that $q \notin \sigma^S(T)$. □

The result of [32, Theorem 6] can be reformulate as follows:

**Theorem 3.5** Let $T \in \mathcal{B}(V^R_H)$ and let $\sigma$ be an isolated part of $\sigma_S(T)$. Put $V^R_{1,H} = R(P_\sigma)$ and $V^R_{2,H} = N(P_\sigma)$. Then, $V^R_H = V^R_{1,H} \oplus V^R_{2,H}$, the spaces $V^R_{1,H}$ and $V^R_{2,H}$ are $T$-invariant subspaces and

$$\sigma_S(T|_{V^R_{1,H}}) = \sigma \text{ and } \sigma_S(T|_{V^R_{2,H}}) = \sigma_S(T)\setminus \sigma. \quad (5)$$

In the next result, we discuss the uniqueness of the decomposition (5).

**Theorem 3.6** Let $T \in \mathcal{B}(V^R_H)$ and $P$ be a projection in $\mathcal{B}(V^R_H)$ such that $TP = PT$. Set

$$T_1 := TP|_{R(P)} \text{ and } T_2 := T(\mathbb{I}_{V^R_H} - P)|_{N(P)}.$$

If $\text{dist}(\sigma_S(T_1), \sigma_S(T_2)) > 0$, then

$$R(P) = R(P\sigma_S(T_1)) \text{ and } N(P) = N(P\sigma_S(T_1)).$$

**Proof** First and foremost, we have by [4, Theorem 4.4]

$$\sigma_S(T) = \sigma_S(T_1) \cup \sigma_S(T_2).$$

Take $\sigma_1 = \sigma_S(T_1)$ and $\sigma_2 = \sigma_S(T_2)$ and assume that $\text{dist}(\sigma_1, \sigma_2) > 0$. According to the proof of [32, Theorem 6], there exist a pair of a non-empty disjoint axially symmetric domains $(U_{\sigma_1}, U_{\sigma_2})$ such that

$$\sigma_j \subset U_{\sigma_j}, \quad \overline{U}_{\sigma_1} \cap \overline{U}_{\sigma_2} = \emptyset$$

and the boundary $\partial(U_{\sigma_j} \cap \mathbb{C}_I)$ is the union of finite number of continuously differentiable Jordan curves for $j = 1, 2$ and for all $I \in \mathbb{S}$. In this way, we see that $U_{\sigma_j}$ is $T_j$-admissible open set for $j = 1, 2$. So by the quaternionic functional calculus, we deduce that

$$\frac{1}{2\pi} \int_{\partial(U_{\sigma_j} \cap \mathbb{C}_I)} S_L^{-1}(s, T_j)ds_I = \mathbb{I}_{V^R_{j,H}} \text{ for } j = 1, 2.$$
and

\[ \frac{1}{2\pi} \int_{\partial(U_{\sigma_i} \cap \mathbb{C}^I)} S^{-1}_L(s, T_j) ds_I = 0 \text{ for } i \neq j \]

where

\[ V^R_{1,\mathbb{H}} := R(P) \text{ and } V^R_{2,\mathbb{H}} := N(P). \]

Define the operators:

\[ R_1 = \begin{bmatrix} \mathbb{I}_{R(P)} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } R_2 = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{I}_{N(P)} \end{bmatrix}. \]

We have

\[
P_{\sigma_1} = \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} S^{-1}_L(s, T) ds_I = \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} ds_I S^{-1}_R(s, T)
\]

\[
= \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} ds_I S^{-1}_R(s, T) P + \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} ds_I S^{-1}_R(s, T) (\mathbb{I}_{V^R_{H}} - P)
\]

\[
= \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} ds_I S^{-1}_R(s, T_1) P + \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} ds_I S^{-1}_R(s, T_2) (\mathbb{I}_{V^R_{H}} - P)
\]

\[
= \frac{1}{2\pi} \int_{\partial(U_{\sigma_1} \cap \mathbb{C}^I)} ds_I S^{-1}_R(s, T_1) P = P_{\sigma_1} P = (I - P_{\sigma_2}) P = R_1.
\]

We conclude that

\[ R(P) = R(P_{\sigma_1}). \]

By the similar arguments, we achieve that

\[ P_{\sigma_2} = R_2 \]

and so, \( N(P) = R(P_{\sigma_2}) = N(P_{\sigma_1}). \quad \Box \)

We now analyze the Riesz projection associated with the isolated spheres. To start off, we give the following definition:

**Definition 3.7** \( T \in \mathcal{B}(V^R_{\mathbb{H}}). \) A point \( q \in \sigma_S(T) \) is called an eigenvalue of finite type if \( V^R_{\mathbb{H}} \) is a direct sum of \( T \)-invariant subspaces \( V^R_{1,\mathbb{H}} \) and \( V^R_{2,\mathbb{H}} \) such that

1. \( \dim(V^R_{1,\mathbb{H}}) < \infty, \)
2. \( \sigma_S(T|_{V^R_{1,\mathbb{H}}}) \cap \sigma_S(T|_{V^R_{2,\mathbb{H}}}) = \emptyset, \)
3. \( \sigma_S(T|_{V^R_{1,\mathbb{H}}}) = [q]. \)
Remark 3.8 (1) In complex spectral theory, in a complex Hilbert space $V_C$, for a continuous linear operator, $T$, the condition $(H3)$ is replaced by $\sigma(T|_{V_1,C}) = \{q\}$ (where $V_C$ is a direct sum of $T$-invariant subspaces $V_1,C$ and $V_2,C$), see [21]. In the quaternionic setting, we must take the whole 2-sphere $[q]$ because if $q \in \sigma_S(T|_{V_{1,R}})$, then $[q] \subseteq \sigma_S(T|_{V_{1,R}})$.

(2) Let $T \in B(V_{R,H}^C)$. If $q \in \sigma_S(T) \setminus \mathbb{R}$, then $q$ is not an isolated point of $\sigma_S(T)$. Take
\[
\Omega := \sigma_S(T) / \sim
\]
where $p \sim q$ if and only if $p \in [q]$. Let $E_T$ denote the set of representatives of $\Omega$. (3) By [24, Proposition 4.44 and Theorem 4.47], if $\dim(V_{H}) < \infty$, then the $S$-spectrum of $T$ consists of right eigenvalues only and $\#E_T < \infty$. In particular, if $T$ satisfied the assumptions $(H1)$ and $(H3)$, then $q \in \sigma_{pS}(T)$.

Lemma 3.9 Let $T \in B(V_{R,H}^C)$. Let $E_T$ denote the set of representatives as above. Then, $q$ is an isolated point of $E_T$ if and only if $[q]$ is an isolated part of $\sigma_S(T)$.

Proof Assume that $[q]$ is isolated part of $\sigma_S(T)$ and let $U_q \subset \mathbb{H}$ be an open set such that
\[
[q] = \sigma_S(T) \cap U_q.
\]
If there exists $q \neq p \in E_T \cap U_q$, then $p \notin [q]$. This implies that
\[
E_T \cap U_q = \{q\}.
\]
Conversely, if $q$ is an isolated point of $E_T$, then $[q]$ is an open subset of $\sigma_S(T)$. Since $[q]$ is compact, then $[q]$ is isolated part of $\sigma_S(T)$. \qed

Let $V_C$ be a complex Hilbert space and $T$ be a continuous linear operator acting on $V_C$. It follows from [21, Theorem 1.1] that if $q \in \sigma(T)$ (where $\sigma(T)$ denotes the complex spectrum of $T$), then $q$ is an eigenvalue of finite type if and only if $\dim R(P_{\{q\}}) < \infty$, where $P_{\{q\}}$ is the Riesz projection corresponding to the isolated point $q$. In the next theorem, we show that the same is true for the right eigenvalue of finite type in the quaternionic setting.

Theorem 3.10 Let $T \in B(V_{R,H}^C)$ and $q \in \sigma_{pS}(T)$. Then, $q$ is a right eigenvalue of finite type if and only if $[q]$ is an isolated part in $E_T$ and $\dim R(P_{\{q\}}) < \infty$.

Proof Assume that $q$ is a right eigenvalue of finite type and consider the direct sum
\[
V_{H}^R = V_{1,H}^R \oplus V_{2,H}^R
\]
with the properties $(H1) - (H3)$. By Theorem 3.6, this decomposition is unique and so
\[
V_{1,H}^R = R(P_{\{q\}}).
\]
The converse comes from Theorem 3.5. \qed
Corollary 3.11 Let \( T \in \mathcal{B}(V_{\mathbb{H}}^R) \). Then,
\[
\sigma^S_e(T) \subset \sigma_S(T) \setminus \sigma^S_d(T)
\]
where \( \sigma^S_d(T) \) denotes the set of all right eigenvalue of finite type.

Proof Assume that \( q \) is a right eigenvalue of finite type. Then, \([q]\) is an isolated part of \( \sigma_S(T) \) and \( \dim R(P_{[q]}) < \infty \). Therefore,
\[
\dim R(T P_{[q]}) < \infty \text{ and so } \sigma^S_e(T |_{R(P_{[q]})}) = \emptyset.
\]
Using Theorem 3.4, we infer that
\[
\sigma^S_e(T) = \sigma^S_e(T |_{R(I_{\mathbb{H}}^R - P_{[q]})}).
\]
On the other hand,
\[
\sigma^S_e(T |_{R(I_{\mathbb{H}}^R - P_{[q]})}) \subset \sigma_S(T |_{R(I_{\mathbb{H}}^R - P_{[q]})}) = \sigma_S(T) \setminus [q].
\]
This leads to conclude that \( q \notin \sigma^S_e(T) \). \( \square \)

Example 3.12 We consider the right quaternionic Hilbert space:
\[
\ell^2_{\mathbb{H}}(\mathbb{Z}) := \left\{ x : \mathbb{Z} \rightarrow \mathbb{H} \text{ such that } \|x\|^2 := \sum_{i \in \mathbb{Z}} |x_i|^2 < \infty \right\}.
\]
with the right scalar multiplication
\[
x a = (x_i a)_{i \in \mathbb{Z}}
\]
for \( x = (x_i)_{i \in \mathbb{Z}} \) and \( a \in \mathbb{H} \). The associated scalar product is given by
\[
\langle x, y \rangle := \langle x, y \rangle_{\ell^2_{\mathbb{H}}(\mathbb{Z})} := \sum_{i \in \mathbb{Z}} \overline{x_i} y_i.
\]
The right shift is the map:
\[
T : \ell^2_{\mathbb{H}}(\mathbb{Z}) \rightarrow \ell^2_{\mathbb{H}}(\mathbb{Z})
\]
\[
x \longmapsto y = (y_i)_{i \in \mathbb{Z}}
\]
where \( y_i = x_{i+1} \) if \( i \neq -1 \) and \( 0 \) if \( i = -1 \). We have
\[
\|T(x)\|^2 = \sum_{i \neq -1} |x_i|^2 \leq \|x\|^2.
\]
The S-spectrum of $T$ was studied in [7, 31]. In particular, we have

$$\sigma_S(T) = \sigma_{pS}(T) = \nabla_H(0, 1),$$

where $\nabla_H(0, 1)$ is the closed quaternionic unit ball. By Theorem 3.10, none of these $S$-eigenvalues are of finite type.

We recall:

**Lemma 3.13** [18, Corollary 2.22] Let $V_{\mathbb{H}}^R$ be a quaternionic right vector space and $E$ be a right linear independent subspace of $V_{\mathbb{H}}^R$. Then, there exists a right basis $B$ of $V_{\mathbb{H}}^R$ such that $E \subset B$. In particular, every quaternionic right vector space has a right basis.

**Proposition 3.14** [24, Proposition 4.44] Let $V_{\mathbb{H}}^R$ be a quaternionic Hilbert space and $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. If $q_1, \ldots, q_n \in \mathbb{H}$ are right eigenvalues of $T$ such that $[q_i] \neq [q_j]$, $\forall 1 \leq i < j \leq n (n \geq 2)$, and $Q_{q_j}(T)x_j = 0$, $0 \neq x_j \in V_{\mathbb{H}}^R$, $\forall 1 \leq j \leq n$, then $x_1, x_2, \ldots, x_n$ are right-linearly independent in $V_{\mathbb{H}}^R$.

We have the following lemma:

**Lemma 3.15** Assume that $\dim(V_{\mathbb{H}}^R) < \infty$ and let $T \in \mathcal{B}(V_{\mathbb{H}}^R)$. Then, $\#E_T < \infty$. In this case, set

$$E_T = \{q_1, q_2, \ldots, q_n\},$$

then $V_{\mathbb{H}}^R$ is a direct sum of $T$-invariant right subspaces $V_{1, \mathbb{H}}^R$, $V_{2, \mathbb{H}}^R$, $\ldots$, $V_{n, \mathbb{H}}^R$. Moreover, if $T_i := T|_{V_{i, \mathbb{H}}^R} : V_{i, \mathbb{H}}^R \to V_{i, \mathbb{H}}^R$ for $i = 1, \ldots, n$, then

$$\sigma_S(T_i) = \{hq_i h^{-1} : h \in \mathbb{H}^*\}.$$

**Proof** Combine Lemma 3.13, Proposition 3.14 and Theorem 3.10 \qed

Let $V_{\mathbb{H}}^R$ be a quaternionic right Hilbert space, $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $q$ be a right eigenvalue of $T$ of finite type. In this way, we see that

$$V_{\mathbb{H}}^R = R(P_{\{q\}}) \oplus R(P_{\sigma_S(T)\setminus\{q\}}).$$

**Definition 3.16** The algebraic multiplicity of the right eigenvalue $q$ is, by definition, the dimension of the space $R(P_{\{q\}})$.

In the next, we write:

$$m_T(q) := \dim R(P_{\{q\}}).$$

We refer to [21, 25] for the definition in complex setting. Now, inspired by the description of the Riesz projection in the complex setting, see [21], we give a quaternionic version of a result describing the finite part of right eigenvalues of finite type.
Theorem 3.17 Let $V_{\mathbb{H}}^R$ be a quaternionic Hilbert space, $T \in \mathcal{B}(V_{\mathbb{H}}^R)$ and $\sigma$ be an axially symmetric isolated part of $\sigma_S(T)$. Set $P_{\sigma}$ the Riesz projection correspond to $\sigma$ and take

$$E_T^\sigma = \sigma/ \cong$$

where $q \cong p$ if and only if $p \in [q]$. Then, $\dim R(P_{\sigma}) < \infty$ if and only if $\# E_T^\sigma < \infty$ and $q$ is a right eigenvalue of finite type for all $q \in E_T^\sigma$. Besides, if so, then

$$\dim R(P_{\sigma}) = \sum_{q \in E_T^\sigma} \dim R(P_{[q]}).$$

Proof If $\dim R(P_{\sigma}) < \infty$, then $P_{\sigma} T$ is a finite rank operator. In this way, we see that

$$\# E_T^\sigma|_{R(P_{\sigma})} = n < \infty.$$  

By Lemma 3.15,

$$R(P_{\sigma}) = V_{1,\mathbb{H}}^R \oplus V_{2,\mathbb{H}}^R \oplus ... \oplus V_{n,\mathbb{H}}^R$$

where $V_{j,\mathbb{H}}^R$ is $T$-invariant with the properties

$$\sigma_S(T|_{V_{j,\mathbb{H}}^R}) = [q_j], \ j = 1, ..., n.$$  

Let $i \in \{1, ..., n\}$. Since $P_{\sigma}$ is a projection, then

$$V_{j,\mathbb{H}}^R = N(P_{\sigma}) \oplus R(P_{\sigma}).$$

In this fashion, we have

$$V_{\mathbb{H}}^R = V_{i,\mathbb{H}}^R \oplus W_{i,\mathbb{H}}^R$$

where

$$W_{i,\mathbb{H}}^R = V_{1,\mathbb{H}}^R \oplus ... \oplus V_{i-1,\mathbb{H}}^R \oplus V_{i+1,\mathbb{H}}^R \oplus ... \oplus V_{n,\mathbb{H}}^R \oplus N(P_{\sigma}).$$

As consequence, $V_{i,\mathbb{H}}^R$ and $W_{i,\mathbb{H}}^R$ are $T$-invariant and $\sigma_S(T|_{V_{i,\mathbb{H}}^R}) = [q_i]$. So, $q_i$ is a right eigenvalue of finite type.

Conversely, set $E_T^\sigma = \{q_1, q_2, ..., q_n\}$, where $q_i$ is a right eigenvalue of $T$ of finite type for all $i \in \{1, 2, ..., n\}$. Applying [12, Theorem 5.6], we have

$$\| R(P_{\sigma}) \| = \sum_{i=1}^n P_{[q_i]}|_{R(P_{\sigma})}.$$
Since \( P_{q_i} P_{q_j} = 0 \) for all \( i \neq j \), then
\[
R(P_\sigma) = R(P_{q_1}) \oplus R(P_{q_2}) \oplus ... \oplus R(P_{q_n}).
\]
This implies that
\[
\dim(R(P_\sigma)) = \sum_{i=1}^{n} \dim(R(P_{q_i})).
\]
In particular, we have \( P_\sigma \) which is a finite rank operator. \( \square \)

**Remark 3.18** In the complex spectral theory, much attention has been paid to eigenvalue of finite type, see [10, 21, 25, 26, 28]. It is useful for the study of the essential spectrum of certain operators-matrices. We refer to [10] for this point on the two-groupe transport operators. More precisely, let \( V_\mathbb{C} \) be a complex Banach space and let \( T \) be a closed operator in \( V_\mathbb{C} \). The Browder resolvent set of \( T \) is given by
\[
\rho_B(T) := \rho(T) \cup \sigma_d(T),
\]
where we use the notation \( \rho(.) \) for the resolvent set of \( T \) and \( \sigma_d(.) \) the set of eigenvalues of finite type of \( T \). In fact, the usual resolvent
\[
R_\lambda(A) := (A - \lambda)^{-1}
\]
can be extended to \( \rho_B(T) \), e.g. [28]. Motivated by this, [10] gives a version of the Frobenius-Schur factorization using the Browder resolvent. This makes it possible to study the essential spectrum of serval types operators-matrices. In this paper, we have described the discrete \( S \)-spectrum in quaternionic setting. In this regard, as in complex case, we can define the spherical Browder resolvent. Although, we avoided studying it in this paper, we will cover that in a future article.

### 4 Some Results on the Weyl \( S \)-spectrum

In this section, we develop a deeper understanding of the concept of the Weyl \( S \)-spectrum of the bounded right linear operator. More precisely, we describe the boundary of the \( S \)-spectrum. Likewise, we deal with the particular case of the spectral theorem. To begin with, we recall:

**Definition 4.1** [30] Let \( T \in B(V_\mathbb{H}^R) \). The Weyl \( S \)-spectrum is the set
\[
\sigma^S_W(T) = \bigcap_{K \in \mathcal{K}(V_\mathbb{H}^R)} \sigma_S(T + K).
\]

The study of the essential and the Weyl \( S \)-spectra are established using the Fredholm theory, see [29, 30]. We refer to [7] for the investigation of the Fredholm and Weyl elements with respect to a quaternionic Banach algebra homomorphism.
**Definition 4.2** A Fredholm operator is an operator $T \in \mathcal{B}(V^R_H)$ such that $N(T)$ and $V^R_H/R(T)$ are finite dimensional. We will denote by $\Phi(V^R_H)$ the set of all Fredholm operators.

From [29, 30], we have

$$\Phi(V^R_H) = \Phi_l(V^R_H) \cap \Phi_r(V^R_H)$$

where

$$\Phi_l(V^R_H) = \left\{ T \in \mathcal{B}(V^R_H) : \text{R}(T) \text{ is closed and } \dim(N(T)) < \infty \right\}$$

and

$$\Phi_r(V^R_H) = \left\{ T \in \mathcal{B}(V^R_H) : \text{R}(T) \text{ is closed and } \dim(N(T^\dagger)) < \infty \right\}.$$

Let $T \in \Phi_l(V^R_H) \cup \Phi_r(V^R_H)$. Then, the index of $T$ is given by

$$i(T) := \dim N(T) - \dim(V^R_H/R(T)).$$

**Theorem 4.3** [29, 30] Let $T \in \mathcal{B}(V^R_H)$. Then,

$$\sigma^S_e(T) = \mathbb{H}\setminus T \text{ and } \sigma^S_W(T) = \mathbb{H}\setminus W_T$$

where

$$\Phi_T := \left\{ q \in \mathbb{H} : Q_q(T) \in \Phi(V^R_H) \right\}$$

and

$$W_T := \left\{ q \in \mathbb{H} : Q_q(T) \in \Phi(V^R_H) \text{ and } i(Q_q(T)) = 0 \right\}.$$

**Remark 4.4** Let $V^R_H$ be a quaternionic space and $T \in \mathcal{B}(V^R_H)$.

(1) Note that, in general, we have

$$\sigma^S_e(T) \subset \sigma^S_W(T) = \sigma^S_{1,W}(T) \cup \sigma^S_{2,W}(T) \subset \sigma_S(T) \setminus \sigma_d(T).$$

where

$$\sigma^S_{1,W}(T) := \mathbb{H}\setminus \left\{ q \in \mathbb{H} : Q_q(T) \in \Phi_l(V^R_H) \text{ and } i(Q_q(T)) \leq 0 \right\}$$

and

$$\sigma^S_{2,W}(T) := \mathbb{H}\setminus \left\{ q \in \mathbb{H} : Q_q(T) \in \Phi_r(V^R_H) \text{ and } i(Q_q(T)) \geq 0 \right\}.$$
In particular, \( \sigma^S_W(T) \) does not contain eigenvalues of finite type.

(2) In [7], one proves that \( q \mapsto -\overrightarrow{i}(T) \) is constant on any component of \( \Phi_T \). In this way, we see that if \( \Phi_T \) is connected, then

\[
\sigma_e^S(T) = \sigma^S_W(T).
\]

The first result in this section is the next theorem.

**Theorem 4.5** Let \( T \in \mathcal{B}(V^R_H) \). Then,

\[
\partial \sigma_e^S(T) \subset \sigma^S(W(T)).
\]

In particular, if \( \Phi_T \) is connected, then

\[
\partial \sigma_e^S(T) = \partial \sigma^S_W(T) \subset \left\{ q \in \mathbb{H} : Q_q(T) \notin \Phi_1(V^R_H) \right\}.
\]

To prove Theorem 4.5, we first study the concept of the minimum modulus. Let \( V^R_H \) be a separable right Hilbert space and \( T \in \mathcal{B}(V^R_H) \). The minimum modulus of \( T \) is given by

\[
\mu(T) := \inf_{\|x\|=1} \|Tx\|.
\]

To begin with, we give the following lemma.

**Lemma 4.6** Let \( T \) and \( S \) be two bounded right linear operators on a right quaternionic Hilbert space. Then,

1. If \( \|T - S\| < \mu(T) \), then \( \mu(S) > 0 \) and \( \overline{R(S)} \) is not a proper subset of \( \overline{R(T)} \).
2. If \( \|T - S\| < \frac{\mu(T)}{2} \), then \( \overline{R(S)} \) is not a proper subset of \( \overline{R(T)} \) and \( \overline{R(T)} \) is not a proper subset of \( \overline{R(S)} \).

**Proof** The proof is the same as for the complex Banach space, see Lemma 2.3 and lemma 2.4 in [20] for a complex proof.

For \( T \in \mathcal{B}(V^R_H) \), \( q \in \mathbb{H} \) and \( \varepsilon > 0 \) we set:

\[
O(T, q, \varepsilon) := \left\{ q' \in \mathbb{H} : 2|\text{Re}(q) - \text{Re}(q')|\|T\| + ||q'||^2 - |q|^2 < \varepsilon \right\}.
\]

It is clear that \( O(T, q, \varepsilon) \) is an open set in \( \mathbb{H} \).

**Corollary 4.7** Let \( T \in \mathcal{B}(V^R_H) \) and \( q_0 \in \rho_S(T) \). Then, \( q \in \rho_S(T) \) for each \( q \in O(T, q_0, \mu(Q_{q_0}(T))) \).

**Proof** Let \( q \in O(T, q_0, \mu(Q_{q_0}(T))) \). Then,

\[
\|Q_q(T) - Q_{q_0}(T)\| = \|2(\text{Re}(q_0) - \text{Re}(q))T + |q|^2 - |q_0|^2\| \\
\leq 2(|(\text{Re}(q_0) - \text{Re}(q))\|T\| + ||q||^2 - |q_0|^2| \\
< \mu(Q_{q_0}(T)).
\]
We can apply Lemma 4.6 to conclude that
\[ \mu(Q_q(T)) > 0 \text{ and } R(Q_q(T)) = R(Q_{q_0}(T)) = V^R_H. \]

By [29, Proposition 3.5], \( R(Q_q(T)) \) is closed. Hence, \( q \in \rho_S(T) \). \( \square \)

We recall:

**Lemma 4.8** [3, Lemma 7.3.9] *Let \( n \in \mathbb{N} \) and \( q, s \in \mathbb{H} \). Set
\[ P_{2n}(q) = q^{2n} - 2\text{Re}(s^n)q^n + |s^n|^2. \]
Then, \( P_{2n}(q) = Q_{2n-2}(q)(q^2 - 2\text{Re}(s)q + |s|^2) = (q^2 - 2\text{Re}(s)q + |s|^2)Q_{2n-2}(q) \), where \( Q_{2n-2}(q) \) is a polynomial of degree \( 2n - 2 \) in \( q \).*

**Proof of Theorem 4.5** Set:
\[ f(T) := \sup_{K \in \mathcal{K}(V^R_H)} \mu(T + K). \]

Similar proof in the complex case, we have \( f(T) > 0 \) if and only if
\[ T \in \Phi(T) \text{ and } \dim N(T) \leq \dim(V^R_H / R(T)). \]

Now, Since \( \sigma^S_e(T) \) is not empty (e.g., [29, Proposition 7.14]) and \( \sigma^S_e(T) \subset \sigma^S_{\Phi}(T) \), then \( \partial \sigma^S_{\Phi}(T) \) is not empty. Let us then take an element \( p \) in \( \partial \sigma^S_{\Phi}(T) \). Assume that \( p \notin \sigma^S_{1,w}(T) \). Then,
\[ f(Q_p(T)) > 0. \]

So, there exists \( K_0 \in \mathcal{K}(V^R_H) \) such that
\[ \mu(Q_p(T) + K_0) > 0. \]

Since \( \mathcal{O}(T, p, \frac{\mu(Q_p(T) + K_0)}{2}) \) is an open neighborhood of \( p \) and
\[ p \in \left\{ q \in \mathbb{H} : Q_q(T) \in \Phi(V^R_H) \text{ and } i(Q_q(T)) = 0 \right\}, \]
then there exists \( p_0 \in \mathcal{O}(T, p, \frac{\mu(Q_p(T) + K_0)}{2}) \) such that
\[ p_0 \in W_T. \]
On the other hand,
\[
\|Q(p(T) + K_0 - Q_p(T) - K_0\| \leq \|p_0\|^2 - |p|^2 + 2Re(p) - Re(p_0)\|T\| \leq \frac{\mu(Q_p(T) + K_0)}{2}.
\]
Applying Lemma 4.6, we obtain
\[
R(Q_p(T) + K_0) = V_{RH}.
\]
Indeed, since \(\mu(Q_{p_0} + K_0) > 0\) and \(p_0 \in W_T\), then
\[
\dim(V_{RH}/R(Q_{p_0}(T) + K_0)) = 0.
\]
In this way, we see that \(Q_p(T) + K_0 \in \Phi(V_{RH})\) and \(i(Q_p(T) + K_0) = 0\).

This implies that, \(p \notin \sigma_{\tilde{W}}^S(T)\). The rest of the proof follows immediately from [7, Theorem 5.13].

We will now deal with the particular spectral theorem for the essential S-spectra.

**Theorem 4.9** Let \(T \in \mathcal{B}(V_{RH})\). Then,
\[
\sigma_e^S(T^n) = \left\{ q^n \in \mathbb{H} : q \in \sigma_e^S(T) \right\} = (\sigma_e^S(T))^n.
\]

**Proof** According to [3, Lemma 3.10] and the proof of [3, Theorem 7.3.11] we have
\[
T^{2n} - 2Re(q^n)T^n + |q^n|^2I_{V_{RH}} = \prod_{j=0}^{n-1}(T^2 - 2Re(q_j)T + |q_j|^2I_{V_{RH}}),
\]
where \(q_j, j = 0, ..., n - 1\) are the solutions of \(p^n = q\) in the complex plane \(\mathbb{C}_{I_q}\). Let \(q \in \sigma_e^S(T^n)\). Then, \(Q_q(T^n) \notin \Phi(V_{RH})\). We can apply [29, Theorem 6.13], we infer that there exists \(i \in \{0, 1, ..., n - 1\}\) such that
\[
Q_{q_i}(T) \notin \Phi(V_{RH}).
\]
Therefore, \(q_i \in \sigma_e^S(T)\). In this way, we see that \(q = q_i^n \in (\sigma_e^S(T))^n\). To prove the inverse inclusion, we consider \(p = q^n\), where \(q \in \sigma_e^S(T)\). By Lemma 4.8 and [3, Theorem 7.3.7], we get
\[
T^{2n} - 2Re(q^n)T^n + |q^n|^2I_{V_{RH}} = Q_{2n-2}(T)(T^2 - 2Re(q)T + |q|^2I_{V_{RH}})
= (T^2 - 2Re(q)T + |q|^2I_{V_{RH}})Q_{2n-2}(T).
\]
Since $Q_q(T) \notin \Phi(V_{R})$, we can apply [29, Corollary 6.14], we deduce that

$$T^{2n} - 2\text{Re}(q^n)T^n + |q^n|^2 \mathbb{I}_{V_{R}} \notin \Phi(V_{R}).$$

So, $p \in \sigma^S(T^n)$. \qed

Acknowledgements We would like to thank the reviewers for taking the time and effort necessary to review the manuscript. We sincerely appreciate all valuable comments and suggestions, which helped us to improve the quality of the manuscript.

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