Transformation brackets between $U(\nu + 1) \supset U(\nu) \supset SO(\nu)$ and $U(\nu + 1) \supset SO(\nu + 1) \supset SO(\nu)$

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We derive a general expression for the transformation brackets between the chains $U(\nu + 1) \supset U(\nu) \supset SO(\nu)$ and $U(\nu + 1) \supset SO(\nu + 1) \supset SO(\nu)$ for $\nu \geq 2$.

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I. INTRODUCTION

The properties of bound states in a large variety of physical systems can be described by writing the Hamiltonian and other operators in terms of a spectrum generating algebra, $G$. In many applications the spectrum generating algebra is taken to be the unitary algebra $G = U(\nu + 1)$, where $\nu$ denotes the dimension. Examples of this approach are: the description of the five quadrupole degrees of freedom of the interacting boson model in nuclear physics in terms of the algebra $U(6)$, the description of the three dipole degrees of freedom of the vibron model in molecular physics in terms of the algebra $U(4)$, and the description of the six degrees of freedom (two dipoles) of the valence quark model of baryons in hadronic physics in terms of the algebra $U(7)$. The algebra $U(\nu + 1)$ always admits (for $\nu \geq 2$) two subalgebra chains

$$
\begin{align*}
U(\nu) & \quad (I) \\
\uparrow & \quad \downarrow \\
U(\nu + 1) & \quad SO(\nu) \\
\uparrow & \quad \downarrow \\
SO(\nu + 1) & \quad (II)
\end{align*}
$$

in addition, eventually, to other chains. (The special case, $\nu = 1$, with $U(2) \supset U(1)$ and $U(2) \supset SO(2)$, in which the two subalgebras $U(1)$ and $SO(2)$ are isomorphic is treated in detail in Refs. 3 and 5 and it will not be discussed further.) In the applications mentioned above, the first chain has the physical meaning of a spherical oscillator in $\nu$ dimensions with $U(\nu)$ being the degeneracy algebra, while the second chain has the meaning of a displaced (or deformed) oscillator in $\nu$ dimensions. The best known example of the latter is the $SO(6)$ chain of the interacting boson model, which has played an important role in nuclear structure physics. In view of the fact that the algebraic method is presently being applied to a variety of problems in physics with different number of dimensions it is of interest to derive a general expression for the transformation brackets between the two chains given in the group lattice of Eq. (1), which includes the known cases $\nu = 5$ and $\nu = 3$, but extends the results to arbitrary $\nu \geq 2$. These transformation brackets are particularly useful if one wants to evaluate analytically certain quantities of physical interest, in particular matrix elements of operators, such as the electromagnetic transition operators, as discussed at length in Ref. 6. In this article, the general result for arbitrary $\nu \geq 2$ will be presented.

II. SPECTRUM GENERATING ALGEBRA

As mentioned above, many applications of the method of spectrum generating algebras (SGA) for bound-state problems in $\nu$ dimensions, have made use of the unitary algebra $U(\nu + 1)$. By generalizing the well known cases of $\nu = 5$ and $\nu = 3$, we introduce a realization of $U(\nu + 1)$ in terms of $\nu + 1$ boson operators,
divided into a set of $\nu$ operators, $b_j^\dagger$ ($j = 1, \ldots, \nu$), which transform as the fundamental representation of $U(\nu)$ and an additional boson operator, $b_0^\dagger = s^\dagger$, which transforms as a scalar under $U(\nu)$. The $\nu + 1$ boson operators, $b_j^\dagger$ ($j = 0, \ldots, \nu$), span the $(\nu + 1)$-dimensional space of $U(\nu + 1)$. The elements of $U(\nu + 1)$ can be written as the bilinear products

$$G \equiv U(\nu + 1) : \quad G_{jk} = b_j^\dagger b_k \quad (j, k = 0, 1, \ldots, \nu).$$

(2)

The states constructed by applying the boson creation operators to a vacuum state

$$B : \quad 1/N (b_j^\dagger)^n (b_k^\dagger)^n \ldots |0\rangle,$$

(3)

(where $N$ is a normalization constant) transform as the symmetric representation $[N]$ of $U(\nu + 1)$, where $N$ is the total number of bosons,

$$N = \sum_{j=0}^{\nu} \tilde{n}_j = \sum_{j=0}^{\nu} b_j^\dagger b_j.$$

(4)

In the algebraic approach to bound state problems, the Hamiltonian (and other) operators are expressed as functions of the elements of $U(\nu + 1)$, i.e. they are in the enveloping algebra of $U(\nu + 1)$, and the basis states are the $[N]$ irreps of $U(\nu + 1)$.

In this article we discuss (i) the explicit construction of the basis states in terms of bosons operators for $U(\nu + 1) \supset U(\nu) \supset SO(\nu)$ and $U(\nu + 1) \supset SO(\nu + 1) \supset SO(\nu)$, and (ii) the transformation brackets relating the basis states in the two chains.

A. The chain $U(\nu + 1) \supset U(\nu) \supset SO(\nu)$

First we consider the chain

$$U(\nu + 1) \supset U(\nu) \supset SO(\nu).$$

(5)

The basis states of this chain are denoted by $[|N\rangle, n, \tau]$, where $N$ is the total number of bosons with $j = 0, 1, \ldots, \nu$, describing the irreps $[N] \equiv [N, 0, \ldots, 0]$ of $U(\nu + 1)$, $n$ is the number of bosons with $j = 1, \ldots, \nu$, describing the irreps $[n] \equiv [n, 0, \ldots, 0]$ of $U(\nu)$ and $\tau$ is the quantum number (boson seniority), describing the irreps $(\tau) \equiv (\tau, 0, \ldots, 0)$ of $SO(\nu)$. The branching rules are:

$$n = 0, 1, \ldots, N,$$

$$\tau = n, n - 2, \ldots, 1 \text{ or } 0 \quad (n \text{ odd or even, and } \nu > 2),$$

$$\tau = -n, -n + 2, \ldots, n \quad (\nu = 2).$$

(6)

The branching of irreps of $SO(\nu)$ into irreps of (eventual) subalgebras of $SO(\nu)$, is of no interest for the present problem and will not be discussed.
The elements (generators) of $U(\nu)$ and $SO(\nu)$ can be written as
\begin{align*}
U(\nu) & : \quad G_{jk} = b_j^\dagger b_k \quad (j, k = 1, \ldots, \nu) , \\
SO(\nu) & : \quad L_{jk} = i (b_j^\dagger b_k - b_k^\dagger b_j) \quad (j < k \text{ and } j, k = 1, \ldots, \nu) .
\end{align*}
(7)

The basis states of the chain (5) can be written in a compact form as
\begin{align*}
|[N], n, \tau\rangle &= \frac{1}{\sqrt{(N-n)!}} (s^\dagger)^{N-n} |[n], n, \tau\rangle , \\
|[n], n, \tau\rangle &= B_{n\tau} (I^\dagger_\nu)^{(n-\tau)/2} |[\tau], \tau, \tau\rangle .
\end{align*}
(8)

The normalization coefficient $B_{n\tau}$ can be derived by making use of the $SU(1,1)$ algebra, given in Eqs. (A 2) and (A 3) of Appendix A,
\begin{equation}
B_{n\tau} = (-1)^{(n-\tau)/2} \sqrt{\frac{(2\tau + \nu - 2)!!}{(n+\tau + \nu - 2)!!(n-\tau)!!}} .
\end{equation}
(9)

The operator $I^\dagger_\nu$ denotes the pair creation operator in $\nu$ dimensions
\begin{equation}
I^\dagger_\nu = \sum_{j=1}^{\nu} b_j^\dagger b_j^\dagger ,
\end{equation}
(10)
and commutes with the generators of $SO(\nu)$, $[I^\dagger_\nu, L_{jk}] = 0$ . The operator $s^\dagger (= b_0^\dagger)$ has been used in (8) to make it conform with the standard notation used in the literature.

**B. The chain $U(\nu + 1) \supset SO(\nu + 1) \supset SO(\nu)$**

Next we consider the chain
\begin{equation}
U(\nu + 1) \supset SO(\nu + 1) \supset SO(\nu) .
\end{equation}
(11)

The basis states of this chain are denoted by $|[N], \sigma, \tau\rangle$, where $[N]$ and $\tau$ label as before the symmetric irreps of $U(\nu + 1)$ and $SO(\nu)$, while $(\sigma) \equiv (\sigma, 0, \ldots, 0)$ labels the symmetric representation of $SO(\nu + 1)$.

The branching rules are:
\begin{align*}
\sigma &= N, N-2, \ldots, 1 \text{ or } 0 \quad (\text{Nodd or even}) , \\
\tau &= 0, 1, \ldots, \sigma \quad (\nu > 2) , \\
\tau &= -\sigma, -\sigma + 1, \ldots, \sigma \quad (\nu = 2) .
\end{align*}
(12)

The generators of $SO(\nu + 1)$ and $SO(\nu)$ can be written as
\begin{align*}
SO(\nu + 1) & : \quad L_{jk} = i (b_j^\dagger b_k - b_k^\dagger b_j) \quad (j < k \text{ and } j, k = 0, \ldots, \nu) , \\
SO(\nu) & : \quad L_{jk} = i (b_j^\dagger b_k - b_k^\dagger b_j) \quad (j < k \text{ and } j, k = 1, \ldots, \nu) .
\end{align*}
(13)

It is customary to separate the generators of $SO(\nu + 1)$ into two pieces
\begin{align*}
L_{jk} &= i (b_j^\dagger b_k - b_k^\dagger b_j) \quad (j < k \text{ and } j, k = 1, \ldots, \nu) , \\
D_j &= i (b_0^\dagger b_j - b_j^\dagger b_0) = i (s_1 b_j - b_1^\dagger s) \quad (j = 1, \ldots, \nu) .
\end{align*}
(14)
Using the same \( SU(1,1) \) algebra as discussed in Appendix A, but with the sum in (A 2) extending from \( j = 0 \) to \( j = \nu \), the basis states of the chain (11) can be written as

\[
|N, \sigma, \tau \rangle = A_N \sigma (I_{\nu+1}^\dagger)^{(N-\sigma)/2} |\sigma, \sigma, \tau \rangle .
\] (15)

Here \( A_N \sigma \) is a normalization coefficient

\[
A_N \sigma = (-1)^{(N-\sigma)/2} \sqrt{\frac{(2\sigma + \nu - 1)!!}{(N + \sigma + \nu - 1)!!}} ,
\] (16)

and \( I_{\nu+1}^\dagger \) represents the pair creation operator in \( \nu + 1 \) dimensions

\[
I_{\nu+1}^\dagger = \sum_{j=0}^{\nu} b_j^\dagger b_j^\dagger = s^\dagger s^\dagger + I_\nu^\dagger .
\] (17)

This pair creation operator commutes with the generators of \( SO(\nu + 1) \), \([I_{\nu+1}^\dagger, L_{jk}] = [I_{\nu+1}^\dagger, D_j] = 0 \). In Appendix B we show that the states \(|[\sigma, \sigma, \tau]\rangle\) can be written as

\[
|[\sigma, \sigma, \tau]\rangle = \sum_k F_k(\sigma, \tau) (s^\dagger)^{\sigma-\tau-2k} (I_{\nu+1}^\dagger)^k |[\tau, \tau, \tau]\rangle ,
\] (18)

where the expansion coefficients are given by

\[
F_k(\sigma, \tau) = \left[ \frac{(\sigma - \tau)!(2\tau + \nu - 2)!!}{(2\sigma + \nu - 3)!!(\sigma + \tau + \nu - 2)!!} \right]^{1/2} \left( \frac{1}{2} \right)^k \frac{(2\sigma + \nu - 3 - 2k)!!}{(\sigma - \tau - 2k)!} .
\] (19)

Another realization of \( SO(\nu + 1) \) which is used frequently in physical applications, is by the generators \( \bar{D}_j = s^\dagger b_j + b_j^\dagger s \) with \( j = 1, \ldots, \nu \) and \( L_{jk} = i(b_j^\dagger b_k^\dagger - b_k b_j^\dagger) \) with \( j < k \) and \( j, k = 1, \ldots, \nu \). The corresponding pair creation operator differs from \( I_{\nu+1}^\dagger \) in Eq. (17) by a relative sign

\[
\bar{I}_{\nu+1}^\dagger = s^\dagger s^\dagger - I_{\nu}^\dagger .
\] (20)

**III. TRANSFORMATION BRACKETS**

The transformation brackets between the two chain are obtained by taking the overlap between the two sets of basis states. Since both are written explicity in terms of the states \(|[\tau, \tau, \tau]\rangle\), the overlap is straightforward and yields the result

\[
c_{n}\sigma^\tau = \langle [N], n, \tau | [N], \sigma, \tau \rangle = \sqrt{(N-n)!} \frac{A_{N}\sigma}{B_{n}\tau} \sum_{k=k_0}^{\left\lceil (n-\sigma)/2 \right\rceil} F_k(\sigma, \tau) \left( k + \frac{N-\sigma}{n-\tau} \right) ,
\] (21)

with \( k_0 = \max(0, \frac{1}{2}(n-\tau-N+\sigma)) \). For the second realization of the \( SO(\nu + 1) \) with the pair creation operator of Eq. (20) the transformation brackets have an additional sign \((-1)^{(n-\tau)/2}\). The transformation
brackets of Eq. (21) can be obtained by inserting the expressions for the coefficients of Eqs. (9,16,19)

\[ c^\tau_{n\sigma} = (-1)^{(N-\sigma-n+\tau)} \left[ \frac{(N-n)!(n+\tau+\nu-2)!(\sigma-\tau)!(2\sigma+\nu-1)}{(N+\sigma+\nu-1)!(N-\sigma)!((\sigma+\tau+\nu-2)!(n-\tau))} \right]^{1/2} \]

\[
\sum_{k=k_0}^{[(\sigma-\tau)/2]} (-1)^k \frac{(2\sigma+\nu-3-2k)!(N-\sigma+2k)!}{(\sigma-\tau-2k)!(2k)!(N-\sigma-n+\tau+2k)!} ,
\]

or by introducing Pochhammer’s symbol \((a)_k = \Gamma(a+k)/\Gamma(a)\) as

\[ c^\tau_{n\sigma} = \left( -\frac{1}{2} \right)^{(N-\sigma-n+\tau)/2} (2\sigma+\nu-1)! \]

\[
\frac{(N-n)!(N-\sigma)!(n+\tau+\nu-2)!(\sigma+\tau+\nu-2)!(2\sigma+\nu-1)}{(\sigma-\tau)!((n-\tau)!!(\sigma+\tau+\nu-2)!!(2\sigma+\nu-1)!!)}^{1/2} \sum_{k=k_0}^{[(\sigma-\tau)/2]} \frac{1}{k!} \frac{((N-\sigma)/2+1)_k((\sigma-\tau)/2)_k((\tau-\sigma+1)/2)_k}{((N-\sigma-n+\tau)/2+k)!!(-\sigma-(\nu-3)/2)_k} .
\]

Eqs. (22) and (23) reduce for \(\nu = 5\) and \(\nu = 3\) to the expressions derived in Refs. 5, 8 and 9.

For the lowest \(SO(\nu+1)\) representation \(\sigma = N\) the sum appearing in the general expression for the transformation bracket can be carried out explicitly to give

\[ c^\tau_{nN} = \left[ \frac{(N-\tau)!(N+\tau+\nu-2)!}{(N-n)!(n+\tau+\nu-2)!(n-\tau)!((2\nu+3)!!)} \right]^{1/2} ,
\]

in agreement with the results obtained previously \(^6\) for \(\nu = 5\) and \(\nu = 3\).

Eqs. (22) and (23) conclude the derivations of the transformation brackets for arbitrary \(\nu \geq 2\),

\[ ||N, \sigma, \tau|| = \sum_n c^\tau_{n\sigma} ||N, n, \tau|| .
\]

IV. CONCLUSIONS

In this article, we have reported a closed expression for the transformation brackets between the chains of Eq. (1) for an arbitrary number of dimensions \(\nu \geq 2\). These transformation brackets are useful in a variety of problems which are being investigated at the present time within the framework of the algebraic method. For example, the case \(\nu = 2\) is of interest in the treatment of bending vibrations of linear molecules, while the case \(\nu = 9\) is of interest in the treatment of rotations and vibrations of non-planar tetratomic molecules.

The transformation brackets derived here can be used to evaluate matrix elements of an operator \(\hat{T}\) in the ‘deformed’ chain (of great physical interest) by a two-step process, i.e. by first evaluating them in the ‘spherical’ chain (which is a relatively easy calculation) and subsequently transforming the results to the ‘deformed’ chain

\[ \langle [N], \sigma', \tau' | \hat{T} | [N], \sigma, \tau \rangle = \sum_{n',n} c^\tau_{n',\sigma'} c^\tau_{n\sigma} \langle [N], n', \tau' | \hat{T} | [N], n, \tau \rangle ,
\]

where the coefficients \(c\) are the transformation brackets derived in this article.
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APPENDIX A: THE REDUCTION $U(\nu) \supseteq SO(\nu)$

The normalization coefficients $B_{n\tau}$ and $A_{N\sigma}$ of Eqs. (9) and (16) can be found by making use of the properties of the $SU(1,1)$ ‘quasi-spin’ algebra for a system of bosons. Consider a system of bosons in $\nu$ dimensions with the algebraic structure

$$ U(\nu) \supseteq SO(\nu). \quad (A\ 1) $$

The corresponding basis states are characterized by $|n, \tau\rangle$. The operators

$$ \hat{Q}_+ = \frac{1}{2} \sum_{j=1}^{\nu} b_j^\dagger b_j^\dagger, $$
$$ \hat{Q}_- = \frac{1}{2} \sum_{j=1}^{\nu} b_j b_j, $$
$$ \hat{Q}_0 = \frac{1}{4} \sum_{j=1}^{\nu} (b_j^\dagger b_j + b_j b_j^\dagger) = \frac{1}{2}(n + \frac{1}{2} \nu), \quad (A\ 2) $$

satisfy the commutation relations

$$ [\hat{Q}_+, \hat{Q}_-] = -2 \hat{Q}_0, $$
$$ [\hat{Q}_0, \hat{Q}_\pm] = \pm \hat{Q}_\pm, \quad (A\ 3) $$

of the $SU(1,1)$ algebra. The basis states are labeled by $|q, q_0\rangle$. The generators of the $SU(1,1)$ algebra defined in (A 2) commute with the generators of $SO(\nu)$, $L_{jk}$ of Eq. (7). The relation between the two sets of basis states $|q, q_0\rangle$ and $|n, \tau\rangle$ can be found by using the generators of (A 2). First, $\hat{Q}_0$ is diagonal in the basis states

$$ \hat{Q}_0 |q, q_0\rangle = q_0 |q, q_0\rangle, $$
$$ \hat{Q}_0 |n, \tau\rangle = \frac{1}{2}(n + \frac{1}{2} \nu) |n, \tau\rangle, \quad (A\ 4) $$

and hence $q_0 = \frac{1}{2}(n + \frac{1}{2} \nu)$. Furthermore, $\hat{Q}_-$ annihilates the highest-weight state

$$ \hat{Q}_- |q = q_0, q_0\rangle \equiv \hat{Q}_- |n = \tau, \tau\rangle = 0, \quad (A\ 5) $$

which gives $q = \frac{1}{2}(\tau + \frac{1}{2} \nu)$. The basis states can be expanded as

$$ |q, q_0\rangle = A_{q,q_0} (\hat{Q}_+)^{q_0-q} |q, q_0 = q\rangle, $$
$$ A_{q,q_0} = (-1)^{q_0-q} \sqrt{\frac{(2q-1)!}{(q_0-q)!(q_0+q-1)!}}, \quad (A\ 6) $$

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or alternatively as
\[
\begin{align*}
[n, \tau] &= B_{n\tau} (I^1_{\nu})^{(n-\tau)/2} \left\{ \begin{array}{c}
\tau \\
\tau 
\end{array} \right\}, \\
B_{n\tau} &= (-1)^{(n-\tau)/2} \sqrt{(2\tau + \nu - 2)!!/(n + \tau + \nu - 2)!!(n - \tau)!!}.
\end{align*}
\] (A 7)

The choice of phase in (A 6) is conventional.

**APPENDIX B: THE REDUCTION \(SO(\nu + 1) \supset SO(\nu)\)**

The \(SU(1,1)\) algebra of (A 2) can be used for the reductions \(U(\nu) \supset SO(\nu)\) and \(U(\nu + 1) \supset SO(\nu + 1)\). For the chain (11) we need the further reduction \(SO(\nu + 1) \supset SO(\nu)\). This is by far more complex. We use here Ref. 12 and a generalization of the method discussed on pages 152-157 of Ref. 5. The defining equations are
\[
\begin{align*}
\hat{N} \left\{ \begin{array}{c}
N \\
N \\
N \\
N \\
\end{array} \right\}, \sigma, \tau \rangle &= N \left\{ \begin{array}{c}
N \\
N \\
N \\
N \\
\end{array} \right\}, \sigma, \tau \rangle, \\
\hat{C}_{SO(\nu + 1)} \left\{ \begin{array}{c}
N \\
N \\
N \\
N \\
\end{array} \right\}, \sigma, \tau \rangle &= \sigma(\sigma + \nu - 1) \left\{ \begin{array}{c}
N \\
N \\
N \\
N \\
\end{array} \right\}, \sigma, \tau \rangle, \\
\hat{C}_{SO(\nu)} \left\{ \begin{array}{c}
N \\
N \\
N \\
N \\
\end{array} \right\}, \sigma, \tau \rangle &= \tau(\tau + \nu - 2) \left\{ \begin{array}{c}
N \\
N \\
N \\
N \\
\end{array} \right\}, \sigma, \tau \rangle.
\end{align*}
\] (B 1)

The notation for the states is the same as in Section II.A. \(\hat{C}_G\) represents the quadratic Casimir invariant of \(G\). These equations can be expressed in terms of a set of separable differential equations by introducing hyperspherical coordinates
\[
(x_1, \ldots, x_{\nu+1}) \rightarrow (r, \phi, \theta_{\nu-1}, \ldots, \theta_1),
\] (B 2)

by
\[
\begin{align*}
x_1 &= r \sin \phi \sin \theta_{\nu-1} \cdots \sin \theta_2 \cos \theta_1, \\
x_2 &= r \sin \phi \sin \theta_{\nu-1} \cdots \sin \theta_2 \sin \theta_1, \\
x_3 &= r \sin \phi \sin \theta_{\nu-1} \cdots \cos \theta_2, \\
&\vdots \\
x_{\nu} &= r \sin \phi \cos \theta_{\nu-1}, \\
x_{\nu+1} &= r \cos \phi,
\end{align*}
\] (B 3)

with \(0 \leq r < \infty\) and \(0 \leq \phi, \theta_{\nu-1}, \ldots, \theta_2 < \pi\) and \(0 \leq \theta_1 < 2\pi\). The volume element is given by
\[
dx_1 \cdots dx_{\nu+1} = r^\nu (\sin \phi)^{\nu-1} (\sin \theta_{\nu-1})^{\nu-2} \cdots \sin \theta_2 \sin \theta_{\nu-1} \cdots d\phi \sin \theta_2 d\theta_{\nu-1} \cdots d\theta_2 \sin \theta_1.
\] (B 4)
The Casimir invariants can be obtained from the Laplacian in $\nu + 1$ dimensions and a recursion relation between the Casimir operators of the orthogonal groups (see page 493 of Ref. 12)

\[
\nabla^2_{\nu+1} = \frac{1}{r^{\nu}} \frac{\partial}{\partial r} \left( r^{\nu} \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \hat{C}_{\nu+1}(\phi, \theta),
\]

\[
\hat{C}_{\nu+1}(\phi, \theta) = -\frac{1}{(\sin \phi)^{\nu-1}} \frac{\partial}{\partial \phi} \left( (\sin \phi)^{\nu-1} \frac{\partial}{\partial \phi} \right) + \frac{1}{(\sin \phi)^2} \hat{C}_{\nu}(\theta),
\]

Here $(\theta) = (\theta_{\nu-1}, \ldots, \theta_1)$. The number operator expressed in hyperspherical coordinates is given by

\[
\hat{N} = \frac{1}{2} \left[ -\frac{1}{r^{\nu}} \frac{\partial}{\partial r} \left( r^{\nu} \frac{\partial}{\partial r} \right) + r^2 + \frac{1}{r^2} \hat{C}_{\nu+1}(\phi, \theta) - (\nu + 1) \right].
\]

The eigenvector equations can be solved by separation of variables and have solutions in terms of products of Laguerre and Gegenbauer polynomials

\[
\psi_{N\sigma\tau\alpha}(r, \phi, \theta) = f_{N\sigma}(r) g_{\sigma\tau}(\phi) \Phi_{\tau\alpha}(\theta),
\]

\[
= A_{N\sigma\tau} r^\sigma e^{-r^2/2} L_{(N-\sigma)/2}^{(2\sigma+\nu-1)/2}(r^2) (\sin \phi)\tau C_{\sigma-\tau}^{(2\sigma+\nu-1)/2}(\cos \phi) \Phi_{\tau\alpha}(\theta),
\]

with

\[
A_{N\sigma\tau} = (-1)^{(N-\sigma)/2}(2\tau + \nu - 3)!! \left[ \frac{(2\sigma+\nu+1)/2(2\sigma + \nu - 1)(N-\sigma)!!(\sigma-\tau)!}{\pi(N+\sigma+\nu-1)!(\sigma+\tau+\nu-2)!} \right]^{1/2}.
\]

Next we apply Dragt’s theorem to the highest weight state with $N = \sigma$

\[
\psi_{\sigma\sigma\sigma\alpha}(r, \phi, \theta) = \frac{A_{\sigma\sigma\tau}}{A_{\tau\tau\tau}} r^{\sigma-\tau} C_{\sigma-\tau}^{(2\sigma+\nu-1)/2}(\cos \phi) \psi_{\tau\tau\alpha}(r, \phi, \theta),
\]

by replacing

\[
r \rightarrow (I_{\nu+1}^l/2)^{1/2},
\]

\[
\cos \phi \rightarrow t^l = s^l/(I_{\nu+1}^l)^{1/2},
\]

to obtain

\[
||\sigma, \sigma, \tau\rangle = \frac{A_{\sigma\sigma\tau}}{A_{\tau\tau\tau}} (I_{\nu+1}^l)^{(\sigma-\tau)/2} C_{\sigma-\tau}^{(2\sigma+\nu-1)/2}(t^l) ||\tau, \tau\rangle,
\]

\[
= \left[ \frac{(\sigma-\tau)!(2\tau + \nu - 2)!!}{(2\sigma + \nu - 3)!!(\sigma + \tau + \nu - 2)!!} \right]^{1/2} \sum_{k=0}^{[\sigma-\tau]/2} \left( -\frac{1}{2} \right)^k \frac{(2\sigma + \nu - 3 - 2k)!!}{k!(\sigma-\tau-2k)!} (s^l)^{\sigma-\tau-2k} (I_{\nu+1}^l)^k ||\tau, \tau\rangle.
\]

By comparing (B 11) with (18), one finds the expression of Eq. (19) for the expansion coefficients $F_k(\sigma, \tau)$.
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