Quasienergies and dynamics of superconducting qubit in time-modulated field

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We analyse dynamics of superconducting qubit and phenomenon of multi-order Rabi oscillations in the presence of time-modulated external field. Such field is also presented as a bichromatic field consisting of two spectral components, which are symmetrically detuned from the qubit resonance frequency. This approach lead to obtaining new qualitative quantum effects that were not considered in the case of monochromatic excitation of qubits. We calculate Floquet states and quasienergies of the composite system "superconducting qubit plus time-modulated field" for various resonant regimes. We analyse dependence of quasienergies from the amplitude of external field demonstrating the zeros of difference between quasienergies. We show that, as a rule, populations of qubit states exhibit aperiodic oscillations, but we demonstrate the specific important regimes in which dynamics of populations becomes periodically regular.

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I. INTRODUCTION

Superconducting circuits based on Josephson junctions are promising candidates for studying fundamental physics and implementing qubits, controllable quantum two-level systems, for quantum computing (see, for example, [1–4] for reviews). The simplest Josephson-junction qubit consists of a small superconducting island with \( n \) excess Cooper-pair charges connected by a tunnel junction with capacitance \( C_J \) and Josephson coupling energy \( E_J \) to a superconducting electrode and the single-electron charging energy \( E_C \). In the case of qubit only two charge states with \( n = 0 \) and \( n = 1 \) play a role while all other charge states, having a much higher energy, can be ignored. Thus, superconducting charge qubit [5] behaves as an artificial two-level atom in Cooper box which is well described by two charge states and the electrostatic energy difference between these states is controlled by the normalized gate charge.

When qubit is driven by an external periodically time-dependent electromagnetic field, it has given rise to new quantum effects such as Rabi oscillations and coherent control [6–8] which are the basis for quantum operations. In a series of experiments many fundamental effects from quantum optics have been demonstrated [9–15] including a lasing effect with a Josephson-junction charge qubit embedded in a superconducting resonator [11]. Superconducting qubits usually have short coherence time, therefore to decrease the time for performing gate operations a large-amplitude external fields should be applied. The dynamics of a qubit driven by large-amplitude external fields in the case of driving around the region of avoided level crossing has been also studied (see, [16] and [17] for reviews).

Most studies of qubit dynamics assume the driving field to be monochromatic or a single cavity mode. In the present paper we investigate dynamics of qubit and phenomenon of Rabi oscillations for an artificial two-level atom interacting with a monochromatic field with time-modulated amplitude. Such external field can be also presented as a bichromatic field that consists of two components of equal amplitudes which are symmetrically detuned from the qubit resonance frequency. In this case, the modulation frequency is displayed as the difference between frequencies of two spectral component. This approach can be applied also for investigation of a wide variety of interesting phenomena including tunneling dynamics of time-dependently driven nonlinear quantum systems. In addition, this problem offer an ideal testing ground for studying the fundamental interactions between qubits and multi-spectral component light.

Note, that investigations of bichromatically driven natural two-level systems has a long history in areas of laser physics, nonlinear optics and quantum optics. The corresponding Hamiltonians of such systems involve the coupling of a bichromatic field to the transition dipole moment between two states of atoms in contrast to the case of superconducting qubit in which an external field only drives the atomic energetic levels. The spectrum of resonance fluorescence (RF) of two-level atom in bichromatic field was calculated in [18–20]. The fluorescence spectrum for the general case of arbitrary detuning was obtained in Refs. [21, 22] and was observed experimentally in Ref. [23] in agreement with the theoretical results. Effects of cavity-modified dynamics were also found in Ref. [24] for two-level Rydberg atoms in a microwave cavity under the influence of a bichromatic field. In series of the papers it has been demonstrated that photon correlation and quadrature squeezing induced by bichromatic field are drastically different from the case of RF in monochromatic field [25–27]. We especially focus on unusually strong superbunching effects in the second-order correlation function as a result of strongly correlated two
photon emissions at the frequency of atomic transition \cite{25, 26} in applications for two-photon lasing. We believe that the results of forming atomic spectral lines with strongly different frequencies under bichromatic radiation are important also for the superconducting qubit inducing an additional Rabi oscillation on dressed states of new qubits. Indeed, we demonstrate below that description of bichromatically driven superconducting qubit differs drastically from the case of standard two-level atom in bichromatic field and unusual phenomena appears for the qubit.

Note, that time-modulation of quantum dynamics for some systems allows effective control of dissipation and decoherence effects, essentially improving the quantum effects. Indeed, it has been shown that the time-modulation in an optical parametric oscillator leads to improvement of squeezing and continuous-variable entanglement of generated modes \cite{28, 29}, and application of such approach to anharmonic oscillator leads to preparation of oscillatory Fock states’ superpositions in the presence of decoherence \cite{30, 31}. Thus, we expect that this approach being applied to artificial atoms, particularly, superconducting qubits will lead to obtaining of new qualitative quantum effects involving control of superconducting qubits and improvement of decoherence. Nevertheless, in this paper, as the first part of these investigations, we only consider nondissipative dynamics of qubit in time-modulated field for short time intervals.

In this paper, we present an analytical results for non-trivial dynamics of qubit in time-modulated field (bichromatic field), particularly, considering in details time-dependent populations of qubit states. We calculate Floquet states (or quasienergetic states and quasienergies) of the composite system “superconducting qubit plus time-modulated field” using Floquet treatment of the interaction of quantum system with periodic on time external field.

Note, that at first, the quasienergetic states of atomic systems in time-periodic e.m. field have been considered in the papers \cite{32, 33}, including also a case of two-level atom in the bichromatic field \cite{15, 16, 25, 27}. Applications of Floquet states and quasienergies to Josephson qubits in monochromatic field have been done in several papers \cite{34, 35}.

The paper is arranged as follows. In Sec. II we derive the Hamiltonian of the system in the resonance approximation. In Sec. III we consider tunnelling amplitude of transitions between states of qubit in the presence of time-modulated (or bichromatic) field and calculate corresponding Floquet states as well as quasienergies. Then, in Sec. IV we investigate the Rabi-oscillations physics of the qubit driven by time-modulated field. We summarize our results in Sec. V.

II. FLOQUE PICTURE FOR QUBIT IN TIME-MODULATED FIELD

The qubit is realized if the charging energy of a superconducting electron box is much larger than the Josephson coupling energy. In the regime of low-level excitation the system is formed by two charge states: \(| \downarrow \rangle\) and \(| \uparrow \rangle\) which have either zero or one Cooper pairs. Thus, the system that we consider here is a qubit coupled to a time-modulated field (or bichromatic field) with the Hamiltonian

\[
\hat{H}(t) = \hat{H}_0 + \hat{H}_V, \tag{1}
\]

where

\[
\hat{H}_0 = -\frac{1}{2}(\varepsilon_0 + f(t))\hat{\sigma}_z, \quad \hat{H}_V = -\frac{\Delta}{2}\hat{\sigma}_x. \tag{2}
\]

Here, the external field reads as

\[
f(t) = 2A\cos(\omega_0 t)\cos(\delta t) \tag{3}
\]

with equal amplitudes of two spectral components at the frequencies \(\omega_1 = \omega_0 - \delta\) and \(\omega_2 = \omega_0 + \delta\).

Here, \(\varepsilon_0 = EQ(1 - 2n_p)\) is the electronic energy difference between the ground and excited states of the qubit and \(\Delta = E_J\) is the Josephson coupling energy or the tunneling amplitude between the basis states. The operators \(\hat{\sigma}_x, \hat{\sigma}_z\) denote the Pauli spin matrices: \(\hat{\sigma}_x = | \uparrow \rangle \langle \downarrow | + | \downarrow \rangle \langle \uparrow |\), \(\hat{\sigma}_z = | \uparrow \rangle \langle \uparrow | - | \downarrow \rangle \langle \downarrow |\). The Hamiltonian \(\hat{H}_V\) describes various physical systems in addition to JJ artificial atom. In general, it describes the tunneling dynamics of bichromatically driven nonlinear quantum two-level systems \cite{36}.

It should be noted that very often in areas “atom+laser” interaction the other Hamiltonian is used in which the coupling of a time-dependent electromagnetic field to the transition dipole moment between two states of atoms are taking place in contrast to the case of superconducting qubit where external field drives the atomic energetic levels \cite{11, 2}. The corresponding Hamiltonian \(\hat{H}_{at}\) describing interaction along the \(x\)-axis can be related to the Hamiltonian \(\hat{H}_V\) with the time-dependent component along the \(z\)-axis by a rotation around the \(y\)-axis. The result reads as

\[
\hat{H}_{at} = e^{-i\hat{\sigma}_y\hat{H}(t)}e^{i\hat{\sigma}_y} = -\frac{1}{2}\Delta\hat{\sigma}_z - \frac{1}{2}(\varepsilon_0 + f(t))\hat{\sigma}_x. \tag{5}
\]

The later Hamiltonian is typical for a natural two-level atom interacting with bichromatic field. In this case, the parameter \(\Delta\) describes an energy difference and the interaction term is responsible for the transitions between two atomic states.
We describe the dynamics of the system in the Floquet-state representation in which the equation for the vector state of the full system is
\[
\frac{\partial}{\partial t} |\Psi_U(t)\rangle = \hat{H}_I |\Psi_U(t)\rangle. \tag{6}
\]
The interaction Hamiltonian is given by
\[
\hat{H}_I(t) = \hat{U}^{-1}(t) \hat{H}_V \hat{U}(t) = -\frac{\Delta}{2} \hat{U}^{-1}(t) \hat{\sigma}_x \hat{U}(t), \tag{7}
\]
while the unitary operator \(\hat{U}(t)\) obeys the equation of motion
\[
i \frac{\partial}{\partial t} \hat{U}(t) = \hat{H}_0 \hat{U}(t). \tag{8}
\]
It is easy to realize that operator \(\hat{U}(t)\) has a simple form
\[
\hat{U}(t) = \exp \left[ -i \int_0^t \hat{H}_0(t') dt' \right] = \exp(i\varphi(t) \hat{\sigma}_z), \tag{9}
\]
where
\[
\varphi(t) = \frac{1}{2} \left[ \varepsilon_0 t + \frac{A}{\omega_1} \sin(\omega_1 t) + \frac{A}{\omega_2} \sin(\omega_2 t) \right]. \tag{10}
\]
Thus, the interaction Hamiltonian is calculated in the following form
\[
\hat{H}_I(t) = -\frac{\Delta}{2} e^{-i\varphi(t) \hat{\sigma}_x} e^{i\varphi(t) \hat{\sigma}_z}.
\]
For simplification of the Hamiltonian we use the following formulae with the Bessel functions
\[
\exp \left[ i \frac{A}{\omega_1} \sin(\omega_1 t) \right] = \sum_{n_1} J_{n_1} \left( \frac{A}{\omega_0 + \delta} \right) e^{i n_1(\omega_0 + \delta) t}, \tag{12}
\]
where \(J_n(x)\) is \(n\)-th order Bessel function of the first kind. In the limit we can obtain
\[
e^{2i\varphi(t)} = \sum_{n_1} \sum_{n_2} J_{n_1} \left( \frac{A}{\omega_0 + \delta} \right) J_{n_2} \left( \frac{A}{\omega_0 - \delta} \right) \times e^{i[\varepsilon_0 + n_1 n_2] w_0 + (n_1 - n_2) \delta] t}. \tag{13}
\]
We also add that \(e^{-2i\varphi} = (e^{2i\varphi})^*\). The resonance condition is formulated using the requirement that the oscillating terms in time are vanished. Thus, this condition is formulated for the central frequency \(\omega_0\) and the electronic energy difference as \(\varepsilon_0 - N_\omega = \Delta_N \ll \varepsilon_0\), where \(n_1 + n_2 = -N\). In this approximation we obtain
\[
e^{2i\varphi} = e^{i\Delta_N t} \sum_{n_1+n_2=-N} J_{n_1} \left( z_1 \right) J_{n_2} \left( z_2 \right) e^{i(n_1-n_2)\delta t} = e^{i(\Delta_N + \delta) t - iN\pi} \sum_{n_1} J_{n_1} \left( z_1 \right) J_{N+n_1} \left( z_2 \right) e^{i n_1 \gamma}. \tag{14}
\]
where \(z_1 = \frac{\Delta - \varepsilon}{\sqrt{\omega_0}}\), \(z_2 = \frac{\Delta + \varepsilon}{\sqrt{\omega_0}}\) and \(\gamma = 2\delta t + \pi\). In the following we use the well-known formulae of summing the Bessel functions for the further transformation of the Hamiltonian. The result reads as
\[
e^{2i\varphi} = e^{i(\Delta_N + N\delta) t - iN\pi} J_N(w(t)) \left( \frac{z_2 - z_1 e^{-i\gamma}}{z_2 - z_1 e^{i\gamma}} \right)^N. \tag{15}
\]
where \(w(t) = \left( z_1^2 + z_2^2 - 2z_1z_2 \cos(\gamma) \right)^{1/2}, |z_1 e^{i\gamma}| < z_2\). We rewrite the exponent in the following form
\[
e^{\pm 2i\varphi(t)} = J_N(w(t)) e^{\pm i\alpha(t)}, \tag{16}
\]
introducing the functions
\[
\alpha(t) = (\Delta_N + N\delta) t - N\pi - \frac{iN}{2} \ln \left[ \frac{z_2 - z_1 e^{-i\gamma}}{z_2 - z_1 e^{i\gamma}} \right]. \tag{17}
\]
Then, in the lowest approximation on \(\delta/\omega_0\) we obtain
\[
\alpha(t) = (N\Delta - N\pi) t - N\pi \approx 2 \frac{A}{\omega_0} |\cos(\delta t)|. \tag{18}
\]
In this approximation and for the case of exact resonance, \(\Delta_N = 0\), the interaction Hamiltonian is written in the following form
\[
\hat{H}_I(t) = (-1)^{N+1} \frac{\Delta}{2} J_N(w(t)) \hat{\sigma}_x. \tag{19}
\]
This Hamiltonian is non-stationary and describes effects of time-modulation on qubit dynamics. It is derived for the general case that involves one-quantum resonance process \(N = 1\) as well as high-order processes with \(N > 1\). Below we concentrate on consideration of two cases: \(N = 1\) and \(N = 2\) in details.

III. AMPLITUDES OF THE TRANSITIONS AND QUASIENERGIES

The different regimes of qubit dynamics in the presence of time-modulated field are formulated in the adiabatic and diabatic bases in analogy to the case of monochromatic field [17]. The diabatic basis states |↓⟩ and |↑⟩ are the eigenstates of the Hamiltonian, if \(\Delta\) and \(f(t)\) are vanished.

Let us consider the case \(\varepsilon \gg \Delta\). We assume that states of qubit are formed in the presence of a driving field and the tunnelling process is described by transitions between these states. Then, in the lowest order of the perturbation theory on the base of Eqs. (11) and (17) the tunnelling amplitude in the transition |↓⟩ → |↑⟩ reads as \(A_{1 \rightarrow 2} = \langle \uparrow | U^{-1} \hat{H}_V U | \downarrow \rangle \). Therefore, we obtain
\[
A_{1 \rightarrow 2} = (-1)^{N+1} \frac{\Delta}{2} J_N(w(t)). \tag{20}
\]
This amplitude describes the tunnelling transition in the presence of time-modulated external field that shifts the
energetic levels. It is interesting to compare this result with the analogous one for the case of external monochromatic field. It is known that in the later case the amplitude of the transition $|↓⟩ \to |↑⟩$ with parameters satisfying the resonance does not depend on time-intervals, while the amplitude \(20\) contains time dependent periodic oscillations at the modulation frequency. In Fig\(1\) and Fig\(2\) we depict the corresponding probabilities of the tunneling transition in dependence on dimensionless time for two resonant conditions: \(N = 1\) and \(N = 2\). As we see, the transition amplitudes are not constants and are periodic on time, while for the case of one-monochromatic driving field these quantities have constant values.

**FIG. 1.** Transition probabilities for first-order \((N = 1)\) resonance. The parameters are: (a) \(\Delta/\delta = 34, A/\omega_0 = 10^{-1}\); (b) \(\Delta/\delta = 34, A/\omega_0 = 4.5\).

In the case of weak driving field \(A \ll \varepsilon_0\) and \(\omega_0 \sim \varepsilon_0\), we can use the following approximation for the Bessel function:

\[
J_n(x) \sim \frac{x^n}{2^n n!}, \quad x \ll 1,
\]

therefore,

\[
J_N(w(\tau)) = \frac{1}{N!} \left( \frac{A}{\omega_0} \right)^N |\cos(\delta t)|^N.
\]

Thus, the amplitude of tunnelling for a weak driving field is calculated as

\[
A_{1\to 2} = (-1)^{N+1} \frac{\Delta}{2} \frac{1}{N!} \left( \frac{A}{\omega_0} \right)^N |\cos(\delta t)|^N.
\]

This result is in accordance with the results of numerical calculations corresponding to first-order and second-order resonances presented in Fig\(1\) \(a\) and Fig\(2\) \(a\).

Below we turn to the general case of qubit dynamics considering the state of the full system in \(|↓⟩, |↑⟩\) basis as

\[
|\Psi_U(t)⟩ = C_1(t)|↓⟩ + C_2(t)|↑⟩.
\]

In this case, the Schrodinger equation is reduced to two coupled first-order equations for the amplitudes in the following form

\[
i\dot{C}_1(t) = -\frac{\Delta}{2} J_N(w(t)) e^{-i\alpha(t)} C_2(t), \quad (24a)
\]

\[
i\dot{C}_2(t) = -\frac{\Delta}{2} J_N(w(t)) e^{i\alpha(t)} C_1(t). \quad (24b)
\]

The coefficients of these equation have non trivial dependence on time, nevertheless we demonstrate that for the resonance case \(\Delta_N = 0\) the solution of these equations can be found in a simple analytical form as following

\[
C_1(t) = \cos(\gamma_N(t)), \quad (25a)
\]

\[
C_2(t) = i e^{i\alpha} \sin(\gamma_N(t)), \quad (25b)
\]

while the function \(\gamma_N(t)\) is calculated from equations \(24a\), \(24b\) as

\[
\gamma_N(t) = \frac{\Delta}{2} \int_0^t J_N(w(\tau)) d\tau \quad (26)
\]

and \(\alpha(t) = -N \pi\).

This solution is presented for the concrete initial conditions assuming that the system is initially in the lower state, therefore \(C_1(0) = 1\) and \(C_2(0) = 0\). The populations of the initial and excited states (if the system was initially in the lower state) as a function of time is then given by

\[
P_1(t) = |C_1(t)|^2 = \cos^2(\gamma_N(t)), \quad (27a)
\]

\[
P_2(t) = |C_2(t)|^2 = \sin^2(\gamma_N(t)). \quad (27b)
\]

To calculate these quantities further we need to analyse the function \(\gamma_N(t)\) that involves integration of a periodic
function $J_N(w(\tau))$ with period $T = \pi/\delta$. It is easy to represent the function $\gamma_N(t)$ as

$$\gamma_N(t) = \frac{\Delta}{2} J_N t + \Phi_N(t),$$

(28)

where

$$J_N \equiv J_N(w(t)) = \frac{1}{T} \int_{t_0}^{t_0+T} J_N(w(\tau))d\tau,$$

(29)

$$\Phi(t)$$ is a periodic function defined for $t \in [t_0, t_0 + T]$ as

$$\Phi_N(t) = \frac{\Delta}{2} \int_{t_0}^{t} (J_N(w(\tau)) - \bar{J}_N)d\tau,$$

(30)

and for other $t \in [0, \infty]$ through periodicity relation $\Phi(t+T) = \Phi(t)$ (see, Appendix).

The above formulae allow us to introduce the Floquet states or the quasienergetic states of the qubit in time-modulated driving field. Indeed, it is easy to check that the solution of Eq. (9) with periodic on time Hamiltonian (11) can be expressed in the adiabatic basis as

$$|\phi_{N,\pm}(t)\rangle = e^{\pm i(-1)^{N}\gamma_N(t)}|\varphi_{\pm}\rangle,$$

(31)

where

$$|\varphi_{\pm}\rangle = |\downarrow\rangle \pm |\uparrow\rangle.$$  

Then, by using the formula (28), these states can be presented in the form of the Floquet states

$$|\phi_{N,\pm}(t)\rangle = e^{E^\pm_N t}|U_{N,\pm}(t)\rangle,$$

(33)

where the states

$$|U_{N,\pm}(t)\rangle = e^{\pm i(-1)^{N}\Phi(t)}|\varphi_{\pm}\rangle$$

(34)

are periodic on time, $|U_{N,\pm}(t+T)\rangle = |U_{N,\pm}(t)\rangle$, and $E^\pm_N = \pm E_N$,

$$E_N = (-1)^N \frac{\Delta}{2} J_N$$

(35)

are quasienergies. The sum of two quasienergies obeys the relation $E^+_N + E^-_N = 0$ due to the use of the truncated Hamiltonian (11), while the difference between them reads as $E^+_N - E^-_N = 2E_N$ in this case. The analogous relation also takes place for a two-level atom in monochromatic field. For rigorous consideration of such relation, see (37).

Dependence of the quasienergy $E^\pm_N$ on the parameter $A/\omega_0$ as a function $E^\pm_N = E^\pm_N(A/\omega_0)$ for two types of the resonances: for the first-order as well as for the second-order are shown in Fig.3. As we demonstrate, the quasienergies $E_1$ and $E_2$ for both types of resonances have zeros for definite values of $A/\omega_0$. The lower zeros are at $A/\omega_0 = 3.13, 6.3, 9.45$ for $N = 1$ and are at $A/\omega_0 = 3.8, 7.05, 10.2$ for $N = 2$. Below we analyse the quasienergies for the regime of a weak external fields.

![Graph](image)

**FIG. 3.** Quasienergy for $(N = 1)$ first-order resonance (solid curve); $(N = 2)$ second-order resonance (dashed curve).

### A. Quasienergies and phase-function at the regime of weak driving

In this subsection we derive approximative analytical results for the quasienergies and the phase-function using the formula (28) which describes the weak driving limit. Integration on the formula (29) leads to

$$J_N = \frac{\delta}{\pi} \int_{0}^{\pi/\delta} J_N(w(\tau))d\tau = \frac{F_{[0,\pi],N}(\pi/\delta)}{N\pi} \left(\frac{A}{\omega_0}\right)^N$$

$$= 2\Gamma\left(\frac{1+N}{2}\right) \left[\frac{(1+N)\Gamma\left(\frac{1+N}{2}\right)}{2\sqrt{\pi}N!(1+N)\Gamma\left(\frac{3+N}{2}\right)} \left(\frac{A}{\omega_0}\right)^N \right]$$

(36)

In this formula, we introduce a function, which also determines the periodic part of the phase-function (30),

$$F_{[0,\pi],N}(t) = \frac{\pi}{\delta} \int_{0}^{T} J_N(w(\tau))d\tau = \frac{F_{[0,\pi],N}(\pi/\delta)}{N\pi} \left(\frac{A}{\omega_0}\right)^N$$

$$= 2\Gamma\left(\frac{1+N}{2}\right) \left[\frac{(1+N)\Gamma\left(\frac{1+N}{2}\right)}{2\sqrt{\pi}N!(1+N)\Gamma\left(\frac{3+N}{2}\right)} \left(\frac{A}{\omega_0}\right)^N \right]$$

(37)

which is defined on $[0; \pi/\delta]$. Here, $2F_1(a, b; c; z)$ is a hypergeometric function. The final result for the quasienergy reads as following

$$E_N = (-1)^N \frac{\Delta}{2} \int_{0}^{T} (J_N(w(\tau)) - \bar{J}_N)d\tau$$

(38)

while the periodic part of the phase-function is calculated as

$$\Phi_N(t) = \frac{\Delta}{\delta} \int_{0}^{T} (J_N(w(\tau)) - \bar{J}_N)d\tau$$

(39)

Note, that the results of this section on the quasienergetic states of the qubit in time-modulated driving field are essentially different from the analogous results for the states obtained for two-level atomic system driven by bichromatic field with the Hamiltonian (15). The main peculiarities of the quasienergetic states.
calculated for the Hamiltonian (5) is that the corresponding quasienergies are equal to zero for all ranges of the parameters in rotating wave approximation. It should be noted that the Floquet basis derived here for qubit in bicromatic field is useful for studying the Rabi oscillations physics as well as for writing the master equation governing the dynamics of the reduced density matrix of a driven system, which is in contact with an external environment.

IV. APERIODIC AND PERIODIC RABI OSCILLATIONS

In this section, time-dependent populations of states are investigated for various regimes. We investigate dynamics of the driven qubit in time domain for various resonance conditions. Thus, we consider the occupation probability as a function of time in dimensionless units, assuming that the system was initially in the state $| \downarrow \rangle$ and the Rabi frequency is given by

$$\Omega_N(t) = \gamma N(t).$$  \hspace{1cm} (40)

According to the formulas (25), (27) this dynamics is determined by the function $\gamma N(t)$ that involves both the quasienergy and the periodic function $\Phi_N(t)$ with period $T = \pi/\delta$. To present this statement in a clear form we rewrite the formula (28) as

$$\gamma N(t) = (-1)^N E_N t + \Phi_N(t).$$  \hspace{1cm} (41)

The function $\gamma N(t)$ is an increasing function on time but it grows also periodically due to its "linear+periodic" structure. Therefore, the dynamics of populations seems to be aperiodic on time. Indeed, the typical results for the phase-function as well as the populations are depicted on Figs. 4 and 5. The dynamics of populations for the case of weak external field is shown on Fig. 4 for two resonance regimes. On Fig. 4(a) we compare two curves of the occupation probabilities for $N = 1$ (solid curve) and for $N = 2$ (dashed curve). We can see here fast oscillations of the population for the regime $N = 1$ and slow oscillations for the case of $N = 2$ for large time intervals in Fig. 4(b)). The results for the second-order resonance regime are also demonstrated in Fig 4(c) for the other parameter $\Delta/\delta$. Analysing these results, we note that dynamics of populations strongly depends on the value of the ratio $\Delta/\delta$. It can be seen from the formulae (36), (39), that population behaviour shown on Fig. 4(b) for $N = 2$ is mainly governed by the linear on time term in the phase function (39), thus we can see that the dynamics looks like cosinusoidal oscillations. The periodic on time part $\Phi_N(t)$ only slightly modulated these oscillations. This part of phase-function increases with increasing the parameter $\Delta/\delta$ that leads to increasing the role of periodic modulations giving rise a nontrivial time-dependence of occupation probability (see, Fig 4(c) for the case $N = 2$).

The typical examples of occupation probabilities corresponding to the large-amplitude regime are depicted in Figs. 5, 6. In order to illustrate the role of phase-function in the development of aperiodic dynamics we also show here the time-dependence of this function.

![Fig. 4. Populations of the ground state for both resonance conditions: (N = 1) and (N = 2). The parameters are: (a) first (solid curve) and second (dashed curve) order resonances, $\Delta/\delta = 40, A/\omega_0 = 10^{-1}$; (b) second order resonance (N = 2) for the same values of $\Delta/\delta$ and $A/\omega_0$ as in (a); (c) second order resonance (N = 2), $\Delta/\delta = 370, A/\omega_0 = 10^{-1}$.](image-url)
sin(x) in the formulas [27]. Such consideration leads to
the following formula

\[ m\delta = n|E_N|, \]

(43)

where \( m \) and \( n \) are positive integers. The physical means
of this formula is very simple. The population of the
states depends on \( \gamma_N(t) \) as a square of cosine, for example
\( P_1 = \cos^2(\gamma_N(t)) \). Thus, if during \( m \) periods its growth
is equal to any period of \( \cos^2(x) \), which can be written
as \( n\pi \), the population will repeat its behaviour.

Thus, the populations could be made periodic by
choosing the values of parameters \( \delta/\Delta \) and \( A/\omega_0 \) sat-
ifying the following condition

\[ \delta/\Delta = \frac{n}{m} \frac{1}{\pi} \int_0^\pi J_N \left( \frac{A}{\omega_0} |\cos(\tau)| \right) d\tau, \]

(44)

that follows from the formulae [35] and [43]. For the
case of weak driving field this condition is simplified and
reads as

\[ \frac{1}{N!} \frac{\Delta}{2\delta} \left( \frac{A}{\omega_0} \right)^N \sqrt{\pi} \frac{\Gamma \left( \frac{1+N}{2} \right)}{\Gamma \left( \frac{1+N}{2} + \frac{N}{2} \right)} = \frac{\pi m}{n}. \]

(45)

The typical results for Rabi oscillations with regular,
periodic dynamics are depicted in Fig.7 for \( N = 2 \) res-
onance condition. Here, the parameters \( A/\omega_0 \) and two
used parameters: \( \Delta/\delta = 401 \) (see, Figs.7(a)), \( \Delta/\delta = 31 \)
(see, Figs.7(b)) satisfying the periodicity condition [43].
We compare the results shown in Fig.7(a) with the result
depicted in Fig.7(c). Both results are obtained for the
second-order resonance condition and for the same pa-
rameter \( A/\omega_0 = 10^{-1} \); however using the parameter \( \Delta/\delta \)
satisfying the condition of periodicity [43] in Fig.7(a)
leads to the periodic dynamics of the populations. These
regimes in which quantum dynamics of occupation prob-
abilities becomes periodically regular can be useful, for
example, in applications where one is dealing with logic
operations on qubits.


V. CONCLUSION

In conclusion, we have analysed dynamics of superconducting qubit interacting with electromagnetic wave with time-modulated amplitude (or bichromatic field). We have considered time-dependence of occupation probabilities of qubit states and Rabi physics for both first-order ($N = 1$) and second-order ($N = 2$) resonance regimes, when the central frequency $\omega_0$ and the electronic energy difference obey to rules $\varepsilon_0 = N\omega_0$. We have calculated Floquet states and quasienergies of the composite system "superconducting qubit plus time-modulated field" considering numerically arbitrary intensities of external field and analytically in details the regime of weak driving. Considering the dependence of quasienergies from the parameter $A/\omega_0$ we have shown oscillation-type behaviour of quasienergies for the case of strong field. In this way, the results obtained for superconducting qubit in bichromatic field differ drastically from the Floquet states describing standard two-level atom in bichromatic field in which the coupling of the bichromatic field to the transition dipole moment between the two levels of atoms is taking place in contrast to the case of superconducting qubit in which the external field drives the atomic levels. Particularly, for the standard two-level model in bichromatic field the quasienergies are equal to zero for all ranges of the parameters $\{18, 19, 25–27\}$. Considering Rabi oscillations between qubit states we have shown that these oscillations are apperiodic on time due to effects of time-dependent modulation. Nevertheless, further we have demonstrated new regimes in which dynamics of populations becomes periodically regular. These regimes can be realized if the ratio of quasienergy to the detuning is positive integer $E_N/\delta = r$ for arbitrary order of resonances.

Appendix

The formula (28) can be derived by using the Fourier expansion

$$J_N(w(t)) = \sum_{n=-\infty}^{\infty} G(n) e^{i2\pi n t}. \quad (A.1)$$

Then, the integral from the equation (20) is transformed to

$$\int_{0}^{t} J_N(w(\tau)) d\tau = \sum_{n=-\infty}^{\infty} G(n) \int_{0}^{t} e^{-i2\pi n t'} dt'$$

$$= \sum_{|n|=1}^{\infty} G(n) \frac{T}{i2\pi n}(e^{i2\pi n t} - 1) + G(0)t. \quad (A.2)$$

Here, it is easy to realize that

$$G_N(0) = \frac{\Phi_N(w(t))}{T} = \frac{\Phi_N}{T}, \quad (A.3)$$

thus the formula (28) is obtained. In this representation the second term of the function (28) is written in the following form

$$\Phi_N(t) = \frac{\Delta}{2} \sum_{|n|=1}^{\infty} G(n) \frac{T}{i2\pi n}(e^{i2\pi n t} - 1). \quad (A.4)$$

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