Non-Markovian Open Quantum Systems: Input-Output Fields, Memory, Monitoring

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(Dated: February 1, 2013)

Principles of monitoring non-Markovian open quantum systems are analyzed. We use the field representation of the environment (Gardiner and Collet, 1985) for the separation of its memory and detector part, respectively. We claim the system-plus-memory compound becomes Markovian, the detector part is tractable by standard Markovian monitoring. Because of non-Markovianity, only the mixed state of the system can be predicted, the pure state of the system can be retrodicted. We present the corresponding non-Markovian stochastic Schrödinger equation.

PACS numbers: 03.65.Yz, 42.50.Lc

In a seminal paper [1] Gardiner and Collett used quantum white-noise and the related Markovian quantum field to represent the dynamics of a quantum oscillator bath in the Markovian (memory-less) limit. This allowed the construction of exact stochastic differential equations to describe the influence of the bath B on the embedded (i.e.: open) quantum system S, the reaction of S on B, and the time-continuous monitoring of S. The theory became standard in quantum optics [2] and in many fields where a quantum system is open to natural or designed environmental influence [3]. If the memory of B cannot be ignored for S then Markovian tools become jeopardized. In non-Markovian (NM) case, S is coherently interacting with a finite part of B over a finite time. From different theoretical efforts [4]–[10] we distill a central question: how can we divide the environment B into the memory M and detector D? Part M is continuously entangled with S but the compound S+M becomes a Markovian open system, as we shall argue. Part D contains information on S and can be continuously disentangled, i.e.: monitored, without changing the dynamics of S.

As a matter of fact, the Markovian field representation [1] of B is capable to account for memory effects and leads to a natural separation between M and D. The local Markov field interacts with S in a finite range: this part makes the memory M. The output field carries away information on S, it makes the detector D. Most features of the Markovian theory [1] of monitoring apply invariably to the composite system S+M.

Earlier, Jack, Collett and Walls realized the role of a finite memory time in simulation [4] and in monitoring [5]. These authors calculated, for the first time, the retrodicted pure state overcomplete basis parametrized by the complex field $ξ(z)$. The (unnormalized) Bargman coherent states

$$|ξ⟩ = \exp \left( \int ξ(z)\hat{b}^†(z)dz \right)|0⟩$$

form an overcomplete basis:

$$M|ξ⟩⟨ξ'| = 1.$$
written in terms of the fields:

\[ b(t+T) \]

becomes non-vanishing, cf. (12) or (23).

modified by the B-S interaction. Typically, the mean \( \xi \) of the output field from range \( z \geq T \) propagates through the interaction range \( z \in [0, T) \) of non-zero coupling \( \kappa(z) \), gets modified by, and entangled with the system S, then it leaves to freely propagate away to left infinity as the output field. The interaction range makes the memory \( M \) and the output range \( z \leq 0 \) makes the detector \( D \) which can continuously be read out (monitored).

FIG. 1: The bath field \( \hat{b}(z,t) \), when free, is propagating from right to left without dispersion at velocity 1. The unperturbed input field from range \( z \geq T \) propagates through the interaction range \( z \in [0, T) \) of non-zero coupling \( \kappa(z) \), gets modified by, and entangled with the system S, then it leaves to freely propagate away to left infinity as the output field. The interaction range makes the memory \( M \) and the output range \( z \leq 0 \) makes the detector \( D \) which can continuously be read out (monitored).

\[ M \xi(z) = 0, \quad M \xi(z) \xi(z') = 0, \quad M \xi(z) \xi^*(z') = \delta(z - z'). \] (6)

If we perform the measurement, the state of B collapses on \( |\xi\rangle \) randomly, the complex field \( \xi(z) \) becomes the random read-out. But its statistics depends on the pre-measurement state. In the vacuum state \( |0\rangle \), the read-outs \( \xi(z) \) follow the statistics (6). This statistics gets modified by the B-S interaction. Typically, the mean becomes non-vanishing, cf. (12) or (23).

Both the bath and interaction Hamiltonians can be written in terms of the fields:

\[ \hat{H}_B = \frac{i}{2} \int \hat{b}^\dagger(z) \partial_z \hat{b}(z)dz + \text{h.c.}, \] (7)

\[ \hat{H}_{SB} = is \int \hat{b}^\dagger(z) \kappa(z)dz + \text{h.c.}, \] (8)

where \( \kappa(z) \) is the Fourier transform of \( \kappa_\omega \).

The underlying picture (1)-(3) is that all B-modes are spatial excitations along a single direction \( z \). The coupling \( \kappa(z) \) is supposed to vanish outside the interaction range, say \( z \in [0, T] \), where \( T \) is the memory time. Memory effects are fully confined here. (If \( \kappa(z) \) decays only asymptotically to zero, finite memory time can still be a robust approximation (4, 5).) If \( \kappa \) were a delta-function, \( \kappa(z) \propto \delta(z) \), the interaction range would reduce to a single point \( z = 0 \), memory effects would be absent and S would be Markovian open system.

\[ \text{Heisenberg picture.} \quad \text{The solution of the Heisenberg field equation reads} \quad \hat{b}(z,t) = \hat{b}(z + t) + \int_0^t \hat{s}(t - \tau) \kappa(z + \tau) d\tau. \] (9)

The first term \( \hat{b}(z + t) \) on the r.h.s. corresponds to free dispersionless propagation along the line, from right to left (cf. Fig. 1). The free field plays the role of the ‘conveyor belt’ that carries information/perturbations one-way: from right to left, never the opposite! As usual, the free field will later be identified as the field (2) in interaction picture:

\[ \hat{b}_i(z) = \hat{b}(z + t). \] (10)

The second term on the r.h.s. of (9) represents the interaction with S, localized inside the interaction range \( z \in [0, T] \). In the input range \( z \geq T \) the vacuum field is freely propagating. In the output range \( z \leq 0 \) the field is freely propagating and carrying away the perturbations emerged in the interaction range. For \( t > 0 \), the input field does not depend on whatever happens at \( z < T \), and

\[ \text{FIG. 2: If we form a memory subsystem M from the local field oscillators of the interaction range then the system S and the memory M constitutes a Markovian open system. It is pumped by the standard Markovian quantum white-noise \( b(t + T) \) and monitored through the modified quantum white-noise \( \hat{b}_{\text{out}}(t) \) just like Markovian open quantum systems, apart form the delay \( T \) of read-out w.r.t. pump.} \]

\[ \text{FIG. 3: The system-plus-memory is pumped by the standard (external) quantum white-noise} \quad \hat{b}(t + T) \quad \text{and monitored through the modified quantum white-noise} \quad \hat{b}_{\text{out}}(t) \quad \text{just like Markovian open quantum systems, apart form the delay} \quad T \quad \text{of read-out w.r.t. pump.} \]
the dynamics of S remains undisturbed whatever happens
at \( z \leq 0 \) to the output field. Most importantly, we can
continuously observe the output field without altering the
dynamics of S. Accordingly, the memory M will consist of
the local field inside the interaction range and the detector
D will consist of the output field. We emphasize that
the coupling of M to the rest of B is Markovian hence
S+M becomes Markovian open system (Fig. 2). The full
armory of Markovian continuous measurement theories
[1–3], including the Ito-formalism, could be deployed—
with some peculiarities though.

As we said, the D part of the field is the output field
\( b(z \geq 0, t) \). The earliest location of monitoring is \( z = 0 \)
and it is common to introduce the notation \( \hat{b}_{\text{out}}(t) =
\hat{b}(0, t) \) and it is common to call it the output field:

\[
\hat{b}_{\text{out}}(t) = \hat{b}(t) + \int_0^t \hat{s}(t - \tau) \kappa(\tau)d\tau.
\]

(11)

This is the famous input-output relationship which works
for the NM case as well. The equation expresses the vari-
able \( \hat{b}_{\text{out}}(t) \) which one can continuously monitor without
affecting the dynamics of S. Since \( \kappa(\tau) \) vanishes for \( \tau < 0 \),
the measured signal reflects delayed and coarse-grained
average of the S-variable \( \hat{s} \).

In particular, if we read out \( \hat{b}_{\text{out}}(t) \) in ideal heterodyne
measurement—which corresponds to the measurement in
the coherent state basis (4)—the resulting signal \( \hat{b}_{\text{out}}(t) \)
contains the standard complex white-noise (10):

\[
\hat{b}_{\text{out}}(t) = \xi(t) + \int_0^t \langle \hat{s}(t - \tau) \rangle \kappa(\tau)d\tau,
\]

(12)

where \( \langle \hat{s}(\tau) \rangle \) is the quantum expectation value of the
Heisenberg operator.

Markovian master equation. We construct the formal
Markovian reduced dynamics of S+M in Schrödinger pic-
ture. The Hamiltonian of M and the interaction are just
\( \hat{H}_B \) and \( \hat{H}_{SB} \), resp., restricted for the interaction range:

\[
\hat{H}_M = \frac{i}{2} \int_0^T \hat{b}^\dagger(z) \partial_z \hat{b}(z)dz + \text{h.c.},
\]

(13)

\[
\hat{H}_{SM} = i\hat{s} \int_0^T \hat{b}^\dagger(z) \kappa(z)dz + \text{h.c.}.
\]

(14)

We are not ready yet. The outer input field \( \hat{b}(z > T) \),
that we cut off, will be replaced by the time-dependent
vacuum white-noise \( \hat{b}(T + t) \) which is external w.r.t. M
since we take \( t > 0 \). This noise couples to \( \hat{b}(T) \) of the
upper edge \( z = T \) of M and pumps M via the following
Hamiltonian:

\[
\hat{H}_{Mt} = i\hat{b}^\dagger(T)\hat{b}(T + t) + \text{h.c.}.
\]

(15)

(This choice can be confirmed in Heisenberg picture: the
field equation \( d\hat{b}(z)/dt = i[\hat{H}_M + \hat{H}_{Mt}, \hat{b}(z)] \) yields the
correct solution (10) for \( z \in [0, T] \).) As to the output
field, we trace out the modes for \( z < 0 \) while we must
retain \( \hat{b}(z = 0) = \hat{b}_{\text{out}} \) if monitoring is included. The
total Hamiltonian is \( \hat{H}_S + \hat{H}_M + \hat{H}_{MT} + \hat{H}_{SM} \). We can
directly write down the corresponding master equation
for the density matrix of S+M:

\[
\dot{\rho}_{SM} = -i[\hat{H}_S + \hat{H}_M + \hat{H}_{SM}, \rho_{SM}]
\]

\[
+ \hat{b}(T)\rho_{SM}\hat{b}^\dagger(T) - \frac{1}{2}[\hat{b}^\dagger(T)\hat{b}(T)\rho_{SM} + \text{h.c.}].
\]

(16)

The non-Hamiltonian term on the r.h.s. is the typical
second-order contribution of the white-noise \( \hat{H}_{Mt} \).

We have thus transformed the original NM open sys-
tem S into a standard Markovian open system \( S + M \)
which is pumped by the vacuum white-noise \( \hat{b}(T + t) \) and
could be monitored through \( \hat{b}_{\text{out}} \), cf. Fig. 3. In principle,
the Markovian master equation (16) would be a possi-
ble starting point to include monitoring. Unfortunately,
the obtained equation is formal, its application would re-
quire further specifications on \( \hat{H}_M \) regarding the bound-
ary conditions. Rather we choose an alternative tool.

Stochastic Schrödinger equation. We are interested in
the dynamics of the monitored quantum state. Non-
Markovian SSEs [11–13] are in tradition to describe open
system dynamics, whereas their role in monitoring either
was ignored [14–19] or urged for investigations [4], then
led to difficulties [6–9]. The difficulties, related to the
causal relationship between S and D, become transparent
in our new treatment.

We shall work in the interaction picture: according to
(10) we replace \( \hat{b}(z) \) by \( \hat{b}(z + t) \) and we replace \( \hat{s} \) by \( \hat{s}_t \). The interaction (14) becomes the functional of
the standard vacuum white-noise \( \hat{b}(t) \):

\[
\dot{\hat{H}}_t = i\hat{s}_t \int_0^T \hat{b}^\dagger(t + \tau)\kappa(\tau)d\tau + \text{h.c.}
\]

(17)

In interaction picture the separate pump Hamiltonian
(15) is not needed. To construct the Schrödinger dyna-
mics of S+M, let \( |\Psi_S(0)\rangle \) stand for the initial state of
S and \( |0\rangle \) for the initial vacuum state of M. We choose
an uncorrelated composite initial state:

\[
|\Psi_{SM}(0)\rangle = |\Psi_S(0)\rangle|0\rangle.
\]

(18)

Using (17), we get the following Schrödinger equation:

\[
\frac{d|\Psi_{SM}(t)\rangle}{dt} = \left( \hat{s}_t \int_0^T \hat{b}^\dagger(t + \tau)\kappa(\tau)d\tau - \text{h.c.} \right) |\Psi_{SM}(t)\rangle.
\]

(19)

Observe that the r.h.s. depends on the field \( \hat{b}(t + \tau) \) for
\( \tau \in [0, T] \), i.e., for later times than \( t \) itself.

Like in case of Markovian open systems, we have to
match the unitary evolution (19) with the continuous
read-out of \( \hat{b}(t) \). To this end, we project the M-part of
the composite state $|\Psi_{SM}(t)\rangle$ on the coherent state basis $\{|\xi\rangle\}$, cf. (12):

$$|\Psi_S(\xi^*)\rangle = \langle \xi^*|\Psi_{SM}(t)\rangle.$$  (20)

The Schrödinger equation (19) reads:

$$\frac{d|\Psi_S(\xi^*; t)\rangle}{dt} = \hat{s}_t \int_0^T d\tau \hat{\kappa}(\tau) \xi^*(t + \tau)|\Psi_S(\xi^*; t)\rangle$$

$$- \hat{s}_t \int_0^T d\tau' \hat{\kappa}^* (\tau') \frac{\delta |\Psi_S(\xi^*; t)\rangle}{\delta \xi^*(t + \tau')}.\quad(21)$$

This equation is just the Schrödinger equation (19) in different representation. But it is more than that if we consider the monitoring and read-out of $\hat{b}(t)$. Then $|\Psi_S(\xi^*)\rangle$ is the (unnormalized) conditional state vector of $S$, depending on the measured signal $b_{out} = \xi$. Since the signal is stochastic, we call (21) the non-Markovian SSE.

We have come to a landmark. The r.h.s. would contain the measured signal $\xi(t + \tau)$ at later times w.r.t. $t$, these data are not yet available at time $t$. We can still exploit the SSE in two ways. Either we propagate the conditional mixed state, or we propagate the retrodicted pure state. In both cases, we prepare the initial state (15) of $S+M$ at time $t=0$, let it go and start to read out the signal $b_{out}(t) = \xi(t)$. The field in $M$ becomes entangled with $S$ so we can never monitor the pure state $|\Psi_S\rangle$ of $S$. Nonetheless, at each time $t > 0$ we propagate (calculate) the solution of (21) by using the latest read-outs $\xi(t) = b_{out}(t)$ and by setting auxiliary values for $\xi(t + \tau)$ for $\tau \in (0, T)$. These latter data are not yet measured, we acknowledge our ignorance by tracing out the corresponding field degrees of freedom. Accordingly, we derive the following conditional mixed state from the pure state solution:

$$\hat{\rho}_S[b_{out}, b_{out}^*; t] = M|\Psi_S(\xi^*; t)\rangle \langle \Psi_S(\xi; t)|\xi(\sigma < t) = b_{out}(\sigma < t).\quad(22)$$

This mixed state (with a normalizing factor) is the true conditional state of $S$ under monitoring. If we stick to the idea of a conditional pure state, we exploit the measured signal $b_{out}(t)$ differently. We use the SSE (21) at time $t$ to retrodict the state propagation at time $t - T$. Until time $t = T$, measured data are not sufficient to retrodict any pure state. From time $t = T$ on, we start to propagate the initial state $|\Psi_S(0)\rangle$, using the signal $b_{out} = \xi$ measured until time $t$. At each time $t > T$, we have $|\Psi_S(b_{out}^*; t - T)\rangle$ as the solution of the SSE. And this (with a normalizing factor) is our retrodicted conditional pure state for $S$. The pure state $|\Psi_S(b_{out}^*; t - T)\rangle$ looks a mere mathematical construction though it will appear—as it were the true state—in the expression (23) of the measured output signal.

So far we have not determined the statistics of the signal $b_{out}$. The candidate expression (12) does not resolve the selective evolution of $S$ under monitoring. This selection is only given by the SSE (21) together with its interpretation (22). As we said before, the signal $b_{out}$ would be the standard complex white-noise (6) of zero mean had we switched off the interaction. With the interaction on, the typical change is that the mean of $b_{out}$ will be non-vanishing. Lessons from the Markovian special case and the non-selective NM form (12) would suggest the following expression:

$$b_{out}(t) = \xi(t) + \int_0^t \langle \hat{s}_{t-\tau}\rangle_{t-\tau} \kappa(\tau) d\tau,$$  (23)

where $\langle \hat{s}_{t-\tau}\rangle_{t-\tau}$ is the quantum expectation value of $\hat{s}_{t-\tau}$ in the conditional mixed state $\hat{\rho}_S[b_{out}, b_{out}^*; t - \tau]$ or, alternatively, in the conditional pure state $|\Psi_S(b_{out}^*; t - \tau)\rangle$. We show later the second choice is the right one.

Structured bath. Non-Markovian open systems are often derived from Markovian coupling $\kappa_\omega = 1$ to a NM bath of non-flat spectral density $\alpha_\omega \geq 0$. A prototype of NM SSE was obtained in 1997 [11]:

$$\frac{d|\Psi_S(a^*; t)\rangle}{dt} = \hat{s}_t a^*_t |\Psi_S(a^*; t)\rangle - \hat{s}_t \int_0^t d\sigma a(t-\sigma) \frac{\delta |\Psi_S(a^*; t)\rangle}{\delta a^*_t}.$$  (24)

Here $a^*_t$ must be a (Gaussian) complex colored noise of zero mean and of correlation

$$\text{M} \alpha_\omega a^*_t = \alpha(t - \sigma),$$  (25)

where $\alpha(t)$ is the bath correlation function, i.e.: the Fourier transform of $\alpha_\omega$. The interpretation of this equation drew permanent attention. Gambetta and Wiseman showed [6, 8] that no monitoring process exists for $|\Psi_S(a^*; t)\rangle$ itself. If, however, the support of $\alpha(t)$ is finite (there is a finite memory time) then the SSEs like (24) can predict the mixed conditional state at $t$ and retrodict the pure conditional state at $t$ minus the memory time [3, 8]. Now we are in a position to unfold the causality structure of the SSE 1997: we rewrite it into the form of the SSE (21).

The point is that the said NM bath with Markovian coupling can equivalently be substituted by the Markovian B with the NM coupling satisfying $|\kappa_\omega|^2 = \alpha_\omega$. Precisely, if we solve

$$\alpha(t) = \int \kappa(t + \tau) \kappa^*(\tau) d\tau$$  (26)

for $\kappa(t)$ at condition $\kappa(\tau) = 0$ for $\tau < 0$ [20] then we can express $a_t$ through the standard complex white-noise (6):

$$a_t = \int \xi(t + \tau) \kappa^*(\tau) d\tau.$$  (27)

By inserting this into (24), the resulting equation coincides with the NM SSE (21). Therefore the discussion and resolution of the causality issue of monitoring, cf. our preceeding paragraph, can be directly adapted to the old form of the SSE (21).

Let’s verify the Girsanov transformation $\xi(t) \Rightarrow b_{out}(t)$ underlying our heuristic expression (23) of the output
signal. We exploit the Girsanov transformation \( a_t \Rightarrow \tilde{a}_t \) accomplished by (16) in [12]:
\[
\tilde{a}_t = a_t + \int_0^t \alpha(t-\tau)\langle \hat{s}_{t-\tau} \rangle_{t-\tau} d\tau
\]  
(28)
where \( a_t, \tilde{a}_t \) are related to \( \xi(t), \tilde{b}_{\text{out}}(t) \), resp., by the convolution [27]. Let’s arrange all terms on one side, apply [27] and insert [26], yielding
\[
\int \tilde{b}_{\text{out}}(t+\sigma) - \xi(t+\sigma) - \int_0^t \langle \hat{s}_{t-\tau} \rangle_{t-\tau} \kappa(t+\sigma) d\tau d\sigma = 0.
\]  
(29)
The removal of the outer convolution, legitimated at least when \( \kappa \) is nowhere zero, yields our result [23]. From [12] we know that \( \langle \hat{s}_{t-\tau} \rangle_{t-\tau} \) must be taken in the retrodicted pure state |\( \Psi_S^{b_{\text{out}}^*: t-\tau} \rangle.\) Since pure state retrodiction needs a minimum time delay \( T \), the theoretical prediction [23] of the output signal \( \tilde{b}_{\text{out}}(t) \) can only be calculated at time \( t + T \), i.e., at current time \( t \) the latest statistical retrodiction concerns \( \tilde{b}_{\text{out}}(t - T) \). This restriction is a typical quantum-non-Markovian feature. (However, the delayed access to the output signal is a common relativistic feature for any monitoring where the finite speed of light matters.)

**Summary.** We applied the well-known Markovian field representation of the environmental bath at non-Markovian coupling to the embedded open system. We argued that the field in the vicinity of the system plays the role of memory responsible for the non-Markovianity, far from this vicinity it remains Markovian and subject to standard Markovian theory of monitoring. We unfolded the abstract bath into the memory and the detector part. Our work should initiate further investigations along these principles.

A formal master equation has been derived just to confirm Markovianity of the system-plus-memory compound. We have derived a stochastic Schrödinger equation of the monitored system and pointed out its role in predicting the conditional mixed state and in retrodicting the conditional pure state—in accordance with recent discussions and anticipations about the phenomenological stochastic Schrödinger equation of Strunz and the author.

We are aware of two inevitable perspectives. First, a certain asymptotic Markovianity of the system-plus-memory reduced dynamics is readily seen for the Szegö class of couplings [22,23]. Investigations of asymptotic Markovianity should be extended for our class of couplings together with considering double-sided chain representations for both input and output regimes, respectively. (An independent method to treat the memory has appeared just recently [24].) Second, Itô differential and integral calculus should be deployed to improve our tentative derivations.

Support by the Hungarian Scientific Research Fund under Grant No. 75129, by the Bilateral Hungarian-South African R&D Collaboration Project, by EU COST Action MP100, and extensive discussions with Thomas Konrad and Francesco Petruccione are gratefully acknowledged.

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