COMPOSITION LAW AND NODAL GENUS-2 CURVES IN $\mathbb{P}^2$

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1. Introduction

Enumerative algebraic geometry is an old field of algebraic geometry. There are many fascinating problems going back more than a hundred years to the Italian school. The most famous one is perhaps the counting problem for the number of holomorphic curves in $\mathbb{P}^2$. There are in fact two different counting problems. Let $\tilde{N}_{g,d}$ be the number of irreducible, reduced, nodal, degree-$d$ genus-$g$ curves which pass through $3d + (g - 1)$ general points in $\mathbb{P}^2$. The integer $\tilde{N}_{g,d}$ is often referred to as the Severi number. These numbers have recently been computed in [C-H]. A companion number is the number $N_{g,d}$ of irreducible, reduced, nodal, degree-$d$ genus-$g$ curves whose normalization has a fixed complex structure and which pass through $m(d)$ general points in $\mathbb{P}^2$. Here $m(d)$ stands for $3d - 1$ when $g = 0$, 1 and $3d - 2(g - 1)$ when $g \geq 2$. Clearly, $\tilde{N}_{0,d} = N_{0,d}$.

The goal of this paper is to compute $N_{2,d}$ for $d \geq 4$ (see (1.1)). The lower degree cases for $N_{g,d}$ were known classically for many years. Not much progress has been made until the introduction of quantum cohomology theory. Based on ideas from physics, a recursion formula for $N_{0,d}$ was proved by [R-T, K-M]. The influence of physical ideas opens up entirely new directions in enumerative geometry. Roughly speaking, $N_{0,d}$ can be interpreted as a correlation function in a certain topological quantum field theory (topological sigma model [Wit]). All topological quantum field theories have a composition law, which in this instance gives the beautiful recursion formula for $N_{0,d}$. Furthermore, the composition law naively suggests a recursion formula for $N_{g,d}$ in terms of $N_{0,d}$. Unfortunately, a simple calculation of lower degree elliptic curves showed that the formula from physics always gives a wrong answer for higher genus case $g > 0$. It was showed in [R-T] that the correlation function $\Psi_{g,d}$ (Gromov-Witten invariants) of the topological sigma model counts the number of perturbed pseudo-holomorphic maps. Moreover, it satisfies the composition law physicists predicted and hence can be computed by $N_{0,d}$. However, $\Psi_{g,d} \neq N_{g,d}$ for $g > 0$. The original problem of computing $N_{g,d}$ remains to be solved. One obvious approach is to compute the error term $\Psi_{g,d} - N_{g,d}$ and then to use the formula of $\Psi_{g,d}$ to compute $N_{g,d}$. Such an approach involves some delicate obstruction analysis and was carried out in [Ion] for $N_{1,d}$. At the same time, a direct argument for $N_{1,d}$ was given independently in [Pa1].

To explain our approach, we have to explain the composition law of the topological sigma model. Recall that we fix a complex structure on a genus-$g$ curve

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$\Sigma_g$ to define $N_{g,d}$ (and $\Psi_{g,d}$). Roughly speaking, the composition law gives an explicit formula of $\Psi_{g,d}$ in terms of $\Psi_{g',d'}$ with $g' < g, d' \leq d$ when we degenerate $\Sigma_g$ to a stable curve. As we mentioned, such a composition law fails for $N_{g,d}$. Our observation is that if we degenerate $\Sigma_g$ to a stable curve $C_0$ with only rational components each of which contains exactly three nodal points, then an analogue of the composition law might still hold. In fact, we shall prove that for $d \geq 4$,

$$N_{2,d} = \frac{(d-1)(d-2)(d-3)}{2d}N_d + \sum_{d_1+d_2=d} \frac{d_1d_2(d_1d_2d - 6d + 18) - 4d(3d - 2)(3d_1 - 1)}{12d}d_1d_2N_{d_1}N_{d_2} \tag{1.1}$$

where for simplicity, we have used $N_d$ to stand for both $N_{0,d}$ and $\tilde{N}_{0,d}$. Our arguments are parallel to those of Pandharipande [Pa1]. We expect that the same method works for any $g$. The difficulty is a technical one which becomes harder as $g$ gets larger. However, we believe that $N_{g,d}$ is closely related to the number of irreducible, reduced, degree-$d$ plane rational curves which pass through certain general points in $\mathbb{P}^2$ and have certain types of singularities. Indeed, it is not difficult to see that one term involved in $N_{g,d}$ with $g > 2$ is

$$(3d - 2(g - 1))! \cdot \sum_{d_1+\ldots+d_{(g-1)}=d} \prod_{i=1}^{2(g-1)} \frac{d_i^2 \cdot N_{d_i}}{(3d_i - 1)!} \tag{1.2}$$

It is interesting to note that this term appears explicitly in $\Psi_{g,d}$. It would be very interesting to figure out the other terms in $N_{g,d}$.

This paper is organized as follows. In section 2, the formula (1.1) is proved. In the proof, we need to know the number of constraints on stable maps that are degenerations of maps on irreducible and smooth curves, and the number of irreducible, reduced, degree-$d$ rational plane curves that pass through certain general points in $\mathbb{P}^2$ and have exactly one triple point with all other singularities being nodes. These two numbers are studied in section 3.

2. Proof of (1.1)

First of all, we recall some definitions and notations for $g \geq 2$. Let $\overline{\mathcal{M}}_g$ be the Deligne-Mumford moduli space of stable genus-$g$ curves, and let

$$\overline{\mathcal{M}}_g(d) \overset{\text{def}}{=} \overline{\mathcal{M}}_{g,3d-2(g-1)}(\mathbb{P}^2, d) \tag{2.1}$$

be the moduli space of stable maps from (3d - 2(g - 1))-pointed genus-$g$ curves to $\mathbb{P}^2$ such that the homology class of the images is $d[\ell]$ where $[\ell]$ stands for the homology class of a line $\ell$ in $\mathbb{P}^2$. Then, there is a natural map $\pi : \overline{\mathcal{M}}_g(d) \to \overline{\mathcal{M}}_g$ obtained by forgetting the stable maps and all the marked points, and then contracting any unstable components. For a stable genus-$g$ curve $C$, let $\overline{\mathcal{M}}_C(d)$ be the moduli space of stable maps to $\mathbb{P}^2$ from curves $D$ stably equivalent to $C$ with (3d - 2(g - 1)) marked points such that the homology class of the images is $d[\ell]$. In the above, we say that $D$ is stably equivalent to $C$ if contracting the unstable components of $D$ yields $C$. By the universal properties of moduli spaces, there is a canonical bijection

$$\overline{\mathcal{M}}_C(d) \to \pi^{-1}([C]) \tag{2.2}$$
which is an isomorphism when $[C] \in \mathcal{M}_g$ is general. Let $W_g(d) \subset \mathcal{M}_g(d)$ be the locus of stable maps whose domains are irreducible, and let $\mathcal{W}_g(d)$ be the closure of $W_g(d)$ in $\mathcal{M}_g(d)$. Then $W_g(d)$ is a reduced and irreducible open subset of dimension $6d - (g - 1)$. For $i = 1, \ldots, 3d - 2(g - 1)$, define the evaluation map $e_i : \mathcal{W}_g(d) \to \mathbb{P}^2$ by $[\mu : (D, p_1, \ldots, p_{3d-2(g-1)})] \mapsto \mu(p_i)$, and $L_i = e_i^*(\mathcal{O}_\mathbb{P}_2(1))$. Put $Z = e_1(L_1)^2 \cap \ldots \cap e_1(L_{3d-2(g-1)})^2 \cap \mathcal{W}_g(d)$. In the sequel, we will usually think of $Z$ as the cycle determined by the condition that $\mu(p_i)$ is a fixed general point of $\mathbb{P}^2$. Now for a general curve $[C] \in \mathcal{M}_g$, the intersection 

$$\pi^{-1}([C]) \cap [\mathcal{W}_g(d) - W_g(d)]$$

has codimension at least one in the irreducible and reduced subvariety $\pi^{-1}([C]) \cap \mathcal{W}_g(d)$. Since the linear series $e_i^*|\ell|$ are base-point-free,

$$\pi^{-1}([C]) \cap Z = [\pi^{-1}([C]) \cap \mathcal{W}_g(d)] \cap Z = [\pi^{-1}([C]) \cap W_g(d)] \cap Z.$$

Moreover, by Bertini’s Theorem, the intersection cycle $[\pi^{-1}([C]) \cap W_g(d)] \cap Z$ consists of finitely many reduced points in $\pi^{-1}([C]) \cap W_g(d)$. The number of these points is precisely $N_{g,d}$. Thus for a general curve $[C] \in \mathcal{M}_g$,

$$N_{g,d} = [\pi^{-1}([C]) \cap W_g(d)] \cap Z = [\pi^{-1}([C]) \cap Z] = \pi^{-1}([C]) \cdot Z.$$

It follows that for every stable curve $[C] \in \mathcal{M}_g$, we have

$$N_{g,d} = \pi^{-1}([C]) \cdot Z. \quad (2.3)$$

Next, we construct a special stable genus-$g$ curve $C_{0,g}$ in $\mathcal{M}_g$ by induction on $g$. First of all, $C_{0,2}$ consists of two smooth rational curves intersecting transversely at three points. To get $C_{0,3}$, we blow-up two nodal points in $C_{0,2}$ by adding two smooth rational curves which intersect transversely at one point. In general, to obtain $C_{0,g}$ from $C_{0,g-1}$, we blow-up two nodal points in $C_{0,g-1}$ by adding two smooth rational curves which intersect transversely at one point. Thus, $C_{0,g}$ consists of $2(g - 1)$ smooth rational curves which are denoted by $R_1, \ldots, R_{2(g-1)}$, and has $3(g - 1)$ nodal points. Moreover, if $g > 2$, then for each smooth rational curve $R_i$ in $C_{0,g}$, there exist three other smooth rational curves $R_{k_1}, R_{k_2}, R_{k_3}$ in $C_{0,g}$ such that $R_i$ and each $R_{k_j} (j = 1, 2, 3)$ intersect transversely at one point. For simplicity, we denote the curve $C_{0,g}$ by $C_0$. Let $\omega_{C_0}$ be the dualizing sheaf of $C_0$. Then the restriction of $\omega_{C_0}$ to each smooth rational curve $R_i$ in $C_0$ has degree 1. Thus if $L$ is a line bundle on $C_0$ such that $\text{deg}(L|_{R_i}) \geq 0$ for all $i$ with $1 \leq i \leq 2(g - 1)$, $\text{deg}(L|_{R_{i_1}}) > 1$ for at least one $i_1$, and $\text{deg}(L|_{R_{i_2}}) = 0$ for at most one $i_2$, then

$$H^1(C_0, L) \cong H^0(C_0, L^{-1} \otimes \omega_{C_0}) = 0. \quad (2.4)$$

Let $[C] \in \mathcal{M}_g$ be generic, and $|\text{Aut}(C)|$ be the order of the automorphism group of $C$. Then $|\text{Aut}(C)| = 2$ when $g = 2$, and $|\text{Aut}(C)| = 1$ when $g > 2$. By (2.3),

$$N_{g,d} = \pi^{-1}([C]) \cdot Z = \pi^{-1}([C_0]) \cdot Z = \frac{|\text{Aut}(C)|}{|\text{Aut}(C)|} (\mathcal{M}_g(d) \cdot Z).$$
Here $\overline{M}_{C_0}(d)$ is identified with $\pi^{-1}([C_0])$ but with the reduced scheme structure. So to prove (1.1), it suffices to show that for $g = 2$ and $d \geq 4$,

$$\overline{M}_{C_0}(d) \cdot Z = \frac{1}{|\text{Aut}(C_0)|} \left[ \frac{(d-1)(d-2)(d-3)}{d} N_d \right.$$

$$+ \sum_{d_1 + d_2 = d} \frac{d_1 d_2 (d_1 d_2 - 6d + 18) - 4d \left( 3d - 2 \right) (3d_1 - 1)}{6d} d_1 d_2 N_{d_1} N_{d_2} \right]. \tag{2.5}$$

Note that $\overline{M}_{C_0}(d) \cap Z \subset \overline{M}_{C_0}(d) \cap \overline{W}_g(d)$. Let $[\mu : (D, p_1, \ldots, p_{3d-2(g-1)})] \in \overline{M}_{C_0}(d) \cap Z$. Let $D_1, \ldots, D_m$ be all the irreducible components of $D$ such that $\mu|_{D_i}$ are not constant, and let $b_i = \deg(\mu|_{D_i})$ for $1 \leq i \leq m$. Then $m \leq 2(g-1)$; moreover, when $m = 2(g-1)$, $\mu(D_1), \ldots, \mu(D_m)$ have at most nodal singularities, intersect each other transversally at nonsingular points, and have degrees $b_1, \ldots, b_m$ respectively.

**Lemma 2.6.** Assume $[\mu : (D, p_1, \ldots, p_{3d-2(g-1)})] \in \overline{M}_{C_0}(d) \cap Z$. Let $D_1, \ldots, D_m$ be all the irreducible rational curves in $D$ which are identified with the $2(g-1)$ smooth rational curves $R_1, \ldots, R_{2(g-1)}$ in $C_0$ after $D$ is contracted to $C_0$. Dropping $R_1, \ldots, R_{2(g-1)}$ from $D$ results in a disjoint union $T_1 \bigcup \cdots \bigcup T_s$ of trees of smooth rational curves.

**Proof.** Note that $\sum_{i=1}^m b_i = d$. For $1 \leq i \leq m$, let $\hat{b}_i$ be the degree of $\mu(D_i)$ in $\mathbb{P}^2$. Then $\hat{b}_i \leq b_i$. On the one hand, since $[\mu : (D, p_1, \ldots, p_{3d-2(g-1)})] \in \overline{M}_{C_0}(d) \cap Z$, $\mu(D)$ has to pass $3d - 2(g-1)$ general points in $\mathbb{P}^2$. On the other hand, the degree $\hat{b}_i$ irreducible rational curve $\mu(D_i)$ can pass through at most $(3\hat{b}_i - 1)$ general points in $\mathbb{P}^2$. So $\mu(D)$ can pass through at most $\sum_{i=1}^m (3\hat{b}_i - 1) \leq (3d - m)$ general points in $\mathbb{P}^2$. Thus $m \leq 2(g-1)$. Moreover, if $m = 2(g-1)$, then $\hat{b}_i = b_i$ and $\mu(D_1), \ldots, \mu(D_m)$ have at most nodal singularities and intersect each other transversally. $\square$

**Proof of (1.1):** We shall now prove formula (1.1) by verifying (2.5). So let $g = 2$ and $d \geq 4$. Put $d_i = \deg(\mu|_{R_i})$ for $i = 1, 2$. Then, there are three cases:

(i) both $d_1$ and $d_2$ are positive;

(ii) exactly one of $d_1$ and $d_2$ is positive (so the other is zero);

(iii) $d_1 = d_2 = 0$.

Our strategy is the following. Fix the nonnegative integer $k$. In each of the above three cases, we shall estimate the number $n(k)$ of moduli of various points $[\mu : (D, p_1, \ldots, p_{3d-2})]$ in $\overline{M}_{C_0}(d) \cap \overline{W}_2(d)$. Since the linear systems $\epsilon_1^* [\ell]$ are base-point-free, it follows that if $n(k) < 6d - 4$, then the case will not contribute to the intersection number $\overline{M}_{C_0}(d) \cap Z$. We shall show that only cases (i) and (ii) with $k = 0$ may contribute. Furthermore, all cases (i) and (ii) with $k = 0$ actually contribute, i.e. any such map is actually in $\overline{M}_{C_0}(d) \cap \overline{W}_2(d)$. So formula (2.5) will be the sum of the contributions of cases (i) and (ii) with $k = 0$. Notice that by Lemma 2.6, we may assume that $m \leq 2$ and that if $m = 2$, then $\mu(D_1)$ and $\mu(D_2)$ have at most nodal singularities and intersect transversally.

First of all, we consider case (i), that is, both $d_1$ and $d_2$ are positive. By Lemma 2.6, $\mu$ is constant on the trees $T_i$. If $k = 0$, then the number of moduli of these points $[\mu : (D, p_1, \ldots, p_{3d-2})]$ in $\overline{M}_{C_0}(d)$ is

$$n(0) \leq (3d - 2) + (3d - 2) = 6d - 4$$
where the first $(3d-2)$ is the number of moduli of the stable map $\mu$, and the second $(3d-2)$ is the number of moduli of the $(3d-2)$ marked points. We claim that for arbitrary $k$, the number of moduli of these types of maps satisfies $n(k) \leq 6d-4-k$. To see this, consider the effect of adding an additional rational component $D'$ to $D$ on which $\mu$ is constant. If $D'$ meets the other components in one point, it must contain two marked points for stability, and these points do not have moduli. Since $D'$ can replace at most one point that has no moduli (the point $D' \cap D$), we have $n(k+1) \leq n(k) - 1$ for such types of maps. The other possibility is for $D'$ to meet the other components in two points. Then $D'$ must contain at least one point without moduli, but no marked points have been replaced, so we obtain $n(k+1) \leq n(k) - 1$ in this case as well. This proves the claim. Thus only the cases with $k = 0$ can occur, as claimed in the preceding paragraph. Furthermore, all cases (i) with $k = 0$ actually contribute. Indeed, suppose that we have a map $\mu : C_0 = (D) \to \mathbb{P}^2$ with $d_1 > 0$ and $d_2 > 0$. Consider a general flat family of curves $\eta : E \to \Delta$ with $\eta^{-1}(0) = C_0$ and $\eta^{-1}(t)$ smooth for $t \neq 0$. Consider the moduli functor of relative line bundles on families of curves. The obstruction space for this functor is 0, since it lies in an appropriate Ext functor of relative line bundles on families of curves. The obstruction space for this such stable maps $D$ (aside from $R_1$) with nonconstant $\mu \mid \mathbb{P}$. Let $T$ be the tree containing $\mathbb{P}$. Then $T \cap R_1$ is either empty or consists of precisely one point. If $T \cap R_1$ is empty, or if $T \cap R_1$ is nonempty but the unique point in $T \cap R_1$ is not identified with one of the three singular points of $C_0$, then $\mu(R_1)$ has at least a triple point. If the unique point in $T \cap R_1$ is one of the singular points, then the other two singular points of $C_0$, when identified with points of $R_1$, are mapped by $\mu$ to a double point of $\mu(R_1)$, through which $\mu(\mathbb{P})$ must pass. By Lemma 2.6, the points $[\mu : (D, p_1, \ldots , p_{3d-2})]$ can not be contained in $\mathbb{M}_{C_0}(d) \cap Z$. So the case $m = 2$ does not contribute.

Next, let $m = 1$. Then $\mu$ is constant on the closure $D \setminus R_1$ of $D \setminus R_1$ in $D$. So there exist at least three points $q_1, q_2, q_3$ in $R_1$ such that the images $\mu(q_1), \mu(q_2), \mu(q_3)$ are the same, i.e. every divisor in $(\mu \mid R_1)^* \ell$ which contains $p_1$ must contain both $p_2$ and $p_3$. This imposes 4 independent conditions in choosing the linear series $(\mu \mid R_1)^* \ell$. If $k = 0$, the number of moduli of these
points \([\mu : (D, p_1, \ldots, p_{3d-2})]\) in \(\overline{\mathcal{M}}_{C_0}(d) \cap \overline{\mathcal{W}}_2(d)\) is at most

\[n(0) = [(3d - 1) + (3 - 4)] + (3d - 2) = 6d - 4.\]

Here, \([(3d - 1) + (3 - 4)]\) is an upper bound for the number of moduli of \(\mu(R_1)\), where the integer 3 in \((3d - 4)\) is the number of moduli of the three points \(q_1, q_2, q_3\) varying in \(R_1\). As in case (i) above, we see that for general \(k\), the number of moduli satisfies \(n(k) \leq 6d - 4 - k\). It follows that only the case \(k = 0\) can occur. Again by arguments similar to those in case (i), we see that all cases (ii) with \(k = 0\) actually contribute and that the contribution to \(\overline{\mathcal{M}}_{C_0}(d) \cap Z\) consists of finitely many reduced points. Furthermore, if \([\mu : (C_0, p_1, \ldots, p_{3d-2})]\) is such a reduced point in \(\overline{\mathcal{M}}_{C_0}(d) \cap Z\), then \(\mu(C_0) = \mu(R_1)\) is an irreducible and degree-\(d\) rational plane curve that passes through \((3d - 2)\) general points in \(\mathbb{P}^2\) and has at least one triple point. In fact, such a curve in \(\mathbb{P}^2\) must have exactly one triple point with all other singularities being nodes. Now there are only finitely many irreducible, reduced, degree-\(d\) rational plane curves that pass through \((3d - 2)\) general points in \(\mathbb{P}^2\) and have exactly one triple point with all other singularities being nodes. The number \(\tilde{N}_d\) of such curves is given by Lemma 3.2 which will be proved in the next section. Taking into account of the automorphism group of \(C_0\) and the symmetry between \(R_1\) and \(R_2\), we see that the contribution of case (ii) to \(\overline{\mathcal{M}}_{C_0}(d) \cdot Z\) is

\[
\frac{1}{|\text{Aut}(C_0)|} \cdot 2\tilde{N}_d. \quad (2.8)
\]

Next we consider case (iii), that is, both \(\mu|_{R_1}\) and \(\mu|_{R_2}\) are constant. We start with \(m = 1\). Let \(P\) be the unique smooth rational component of \(D\) with nonconstant \(\mu|_P\), and let \(T\) be the unique tree in \(T_1, \ldots, T_\ast\) such that \(P \subset T\). Here and in other subcases of case (iii), we will be able to assume without loss of generality that the tree \(T\) is actually a particularly simple chain. The recurring theme will be that if \(W\) is a subvariety of \(\overline{\mathcal{M}}_{C_0}(d)\) such that \(T\) is a certain type of chain and \(\dim(W \cap \overline{\mathcal{W}}_d(d)) < 6d - 4\), then the same conclusion will hold for subvarieties \(W'\) associated to more complicated trees, since in these situations \(W'\) will always be a subvariety of the closure \(\overline{W}\) obtained from contracting suitable \(\mu\)-constant curves. Returning to the subcase at hand, we may for the above reason assume that \(P = T\) has one component. If \(T\) intersects \(R_1\) or \(R_2\) but not both, then by Lemma 3.7 and Lemma 3.13, there exist 4 independent conditions in choosing the linear series \((\mu|_T^{-1})^\ast \ell\) from \(|(\mu|_T)^\ast \ell|\). So the number of moduli of the points \([\mu : (D, p_1, \ldots, p_{3d-2})]\) in \(\overline{\mathcal{M}}_{C_0}(d) \cap \overline{\mathcal{W}}_2(d)\) is at most

\[n(k) \leq [(3d - k) + (2 - 4)] + (3d - 2) = 6d - 4 - k \leq 6d - 5\]

where the integer 2 in \((2 - 4)\) is the number of the moduli of the point \(T \cap (R_1 \cup R_2)\) varying in both \(T\) and \(R_1 \cup R_2\). Thus this case makes no contribution to \(\overline{\mathcal{M}}_{C_0}(d) \cap Z\).

If \(T\) intersects both \(R_1\) and \(R_2\), then by Remark 3.12 (ii) and Lemma 3.7, there still exist 4 independent conditions in choosing \((\mu|_T)^\ast \ell\) from \(|(\mu|_T)^\ast \ell|\). So the number of moduli of the points \([\mu : (D, p_1, \ldots, p_{3d-2})]\) in \(\overline{\mathcal{M}}_{C_0}(d) \cap \overline{\mathcal{W}}_2(d)\) is at most

\[n(k) \leq [(3d - k) + (2 - 4)] + (3d - 2) = 6d - 4 - k \leq 6d - 5\]

where the integer 2 in \((2 - 4)\) is the number of the moduli of the two points \(T \cap R_3\) and \(T \cap R_2\) varying in \(T\). It follows that this case makes no contribution to \(\overline{\mathcal{M}}_{C_0}(d) \cap Z\).
We are left with case (iii) and $m = 2$. Then $k \geq 2$. Let $D_1, D_2$ be the two rational components in $D$ such that $\mu|_{D_1}, \mu|_{D_2}$ are nonconstant. Then $D_i \neq R_j$ for $i, j = 1, 2$. First, assume that $D_1$ and $D_2$ are contained in the same tree $T$ from $T_1, \ldots, T_s$. If $T$ intersects $R_1$ or $R_2$ but not both, then by Lemma 3.7, there exist 4 independent conditions in choosing the linear series $(\mu|_{D_1})^* \ell$ and $(\mu|_{D_2})^* \ell$ from the complete linear systems $[(\mu|_{D_1})^* \ell]$ and $[(\mu|_{D_2})^* \ell]$. So the number of moduli of the points $[\mu : (D, p_1, \ldots, p_{3d-2})]$ in $\overline{\mathcal{M}}_{C_0}(d) \cap W_2(d)$ is at most

$$n(k) \leq [(3d - k) + (2 - 4)] + (3d - 2) = 6d - 4 - k \leq 6d - 6$$

where the integer 2 in $(2 - 4)$ is the number of the moduli of the point $T \cap (R_1 \cup R_2)$ varying in both $T$ and $R_1 \cup R_2$, and this case makes no contribution to $\overline{\mathcal{M}}_{C_0}(d) \cap Z$. Similarly, by Remark 3.12 (ii) and Lemma 3.7, the case when $T$ intersects both $R_1$ and $R_2$ makes no contribution to $\overline{\mathcal{M}}_{C_0}(d) \cap Z$. Next, assume that $D_1$ and $D_2$ are contained in two different trees, say $T_1$ and $T_2$, from $T_1, \ldots, T_s$. If $T_i$ intersects both $R_1$ and $R_2$ and if $D_1$ intersects $D \setminus D_i$ at least twice, then $\mu(D_2)$ passes through a singular point of $\mu(D_1)$. By Lemma 2.6, this case makes no contribution to $\overline{\mathcal{M}}_{C_0}(d) \cap Z$. So we may assume that if $T_i$ intersects both $R_1$ and $R_2$, then $D_i$ intersects $D \setminus D_i$ once. By Lemma 3.7 and Remark 3.12 (ii), the number of moduli of the points $[\mu : (D, p_1, \ldots, p_{3d-2})]$ in $\overline{\mathcal{M}}_{C_0}(d) \cap W_2(d)$ is at most

$$n(k) \leq [(3d - k) + (4 - 4)] + (3d - 2) = 6d - 2 - k \leq 6d - 4.$$ 

If $n(k) = 6d - 4$, then $k = s = 2$ and $T_i = D_i$ ($i = 1, 2$) intersects $R_1$ and $R_2$ but not both. Since the number of the moduli for the two points $D_1 \cap (R_1 \cup R_2)$ and $D_2 \cap (R_1 \cup R_2)$ varying in $R_1 \cup R_2$ is 2, the pairs $(\mu(D_1), \mu(D_2))$ of curves form a codimension-2 subset in the $(3d - 2)$-dimensional variety $S(0, d_1) \times S(0, d_2)$ where $S(0, d_i)$ stands for the moduli space of irreducible, reduced, nodal, degree-$d_i$ rational plane curves. It follows that this case makes no contribution to $\overline{\mathcal{M}}_{C_0}(d) \cap Z$.

Summing up (2.7) and (2.8) and using (3.3), we obtain (2.5) and hence (1.1). \(\square\)

Finally, some remarks about the number $N_{g,d}$ with $g > 2$ follow. We expect that only the cases with $k = 0$ (that is, stable maps $[\mu : (D, p_1, \ldots, p_{3d-2} - 2(g-1))]$ with $D = C_0 = C_{0,g}$) contribute to $N_{g,d} = |\text{Aut}(C_0)| \cdot (\overline{\mathcal{M}}_{C_0}(d) \cdot Z)$. For instance, as in the proof of the above case (i), the case when $k = 0$ and $\deg(\mu|_{R_i}) > 0$ for all $1 \leq i \leq 2(g-1)$ contributes to $N_{g,d}$, and its contribution is

$$(3d - 2(g - 1))! \cdot \sum_{d_1 + \ldots + d_2 = 2d(g-1)} \prod_{i=1}^{2(g-1)} \frac{d_i^3 \cdot N_{d_i}}{(3d_i - 1)!}.$$ \hspace{1cm} (2.9)

It is interesting to notice that for $g > 2$, the term (2.9) is precisely a term that arises in the calculation of $\Psi_{g,d}$. In general, let $S$ be any nonempty subset of $\{1, \ldots, 2(g-1)\}$. We believe that the contribution of the cases with $k = 0$ and $\deg(\mu|_{R_i}) > 0$ if and only if $i \in S$ is closely related to the number of irreducible, reduced, degree-$d$ plane rational curves which pass through certain general points in $\mathbb{P}^2$ and have certain types of singularities.

3. Rational plane curves with triple points and constraint on stable maps
In this section, we prove the lemmas quoted in the previous section. We shall compute the number $N_d$ of irreducible, reduced, degree-$d$ rational plane curves that pass through $(3d - 2)$ general points in $\mathbb{P}^2$ and have exactly one triple point with all other singularities being nodes. Then we analyze the constraints on stable maps that are degenerations of maps whose domains are irreducible and smooth curves.

### 3.1. Rational plane curves with triple points

Fix $d \geq 3$. Intersections of $\mathbb{Q}$-divisors in the moduli space $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$ has been studied by Pandharipande. We recall some results from [Pa2, Pa3]. The boundary $\Delta$ of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$ is the locus corresponding to stable maps whose domains are reducible, and is of pure codimension-1. For $1 \leq i \leq \left\lceil \frac{d}{2} \right\rceil$, let $K_i$ be the irreducible component of $\Delta$ whose general elements are the form $\mu : D_1 \cup D_2 \to \mathbb{P}^2$ such that $\deg(\mu|D_1) = i$, and $D_1$ and $D_2$ are smooth rational curves meeting transversely at one point. Let $\mathcal{H}$ be the locus of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$ corresponding to maps whose images pass through a fixed point in $\mathbb{P}^2$. Then $\mathcal{H}$ is a Cartier divisor in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$, and $\text{Pic}(\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)) \otimes \mathbb{Q}$ is generated by $\mathcal{H}$ and $K_i$ with $1 \leq i \leq \left\lceil \frac{d}{2} \right\rceil$; moreover,

$$\mathcal{H}^{3d-1} = N_d, \quad K_i \cdot \mathcal{H}^{3d-2} = \begin{cases} \frac{(3d-2)!}{(3d-1)!} i(d-i)N_iN_{d-i}, & \text{if } i \neq \frac{d}{2}; \\ \frac{1}{2} \frac{(3d-2)!}{(3d-3)!} \cdot N_2^2, & \text{if } i = \frac{d}{2}. \end{cases} \quad (3.1)$$

Let $Z \subset \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$ be the subvariety consisting of degree-$d$ maps $\mu : \mathbb{P}^1 \to \mathbb{P}^2$ such that $\mu(\mathbb{P}^1)$ has exactly one triple point with all other singularities being nodes, and let $\overline{Z}$ be the closure of $Z$ in $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$. Then $Z$ is reduced and of pure codimension-1, and $\overline{Z}$ is a Weil divisor. Similar arguments as in section (3.4) of [Pa2] show that the intersection $\overline{Z} \cap \mathcal{H}^{3d-2}$ determined by $(3d - 2)$ general points in $\mathbb{P}^2$ consists of finitely many reduced points in $Z$. Thus $\tilde{N}_d = \overline{Z} \cdot \mathcal{H}^{3d-2}$.

**Lemma 3.2.** For $d \geq 3$, $\tilde{N}_d$ can be expressed in terms of the $N_d$:

$$\frac{(d-1)(d-2)(d-3)}{2d} N_d - \sum_{d_1 + d_2 = d} \frac{d_1d_2(d-6) + 2d(d-2)}{4d} d_1d_2N_{d_1}N_{d_2}. \quad (3.3)$$

**Proof.** Let $\overline{Z} = a\mathcal{H} + \sum_{i=1}^{\left\lceil \frac{d}{2} \right\rceil} a_iK_i$ where $a, a_i \in \mathbb{Q}$. By (3.1), it suffices to determine $a$ and $a_i$. Fix a nonsingular curve $C$. As in section (4.5) of [Pa2], we compute the intersection number $C \cdot \lambda^* \overline{Z}$ for some morphism $\lambda : C \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$ which is constructed as follows. Let $\pi_2 : S = \mathbb{P}^1 \times C \to C$ be the second projection, and let $\mathcal{N}$ be a line bundle on $S$ of degree type $(d, k)$ where $k$ is a very large integer. Let $z_0, z_1, z_2 \in H^0(S, \mathcal{N})$ determine a rational map $\phi : S \dashrightarrow \mathbb{P}^2$ such that

(i) every base point of $\phi$ is simple. Here a base point $s$ of $\phi$ is simple of degree $i$ with $1 \leq i \leq d$ if the blowing-up of $S$ at $s$ resolves $\phi$ locally at $s$ and the resulting map is of degree $i$ on the exceptional divisor;

(ii) there exist only finitely many points $c_1, \ldots, c_n \in C$ such that for each $i$, $c_i$ is not the projection of base points of $\phi$ and $\overline{\phi}^{-1}(c_i)$ contains one and exactly one triple point. Here $\overline{\phi}$ is the blow-up of $S$ at the base points, $\overline{\phi} : \overline{S} \to C$ is the projection to $C$, and $\phi : \overline{S} \to \mathbb{P}^2$ is the resolution of $\overline{\phi}$.

Now $\overline{\phi} : \overline{S} \to \mathbb{P}^2$ and $\overline{\phi} : \overline{S} \to C$ induce a morphism $\lambda : C \to \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d)$. A point of the intersection $C \cdot \lambda^* K_i$ can arise in two cases, that is, a simple base point of
degree $i$ or $(d - i)$ can be blown-up. Let $C \cdot \lambda^*K^i = x_i + y_i$ where $x_i$ and $y_i$ are
the number of instances of the first and second case respectively. On the one hand,
$$C \cdot \lambda^*H = 2dk - \sum_{i=1}^{|\mathcal{g}|} [i^2x_i + (d - i)^2y_i]$$
according to [Pa2]. Thus,
$$C \cdot \lambda^*Z = 2adk - \sum_{i=1}^{|\mathcal{g}|} a[i^2x_i + (d - i)^2y_i] + \sum_{i=1}^{|\mathcal{g}|} a_i(x_i + y_i). \quad (3.4)$$

On the other hand, the triple-point formula of [Kle] can be applied to the map
$(\overline{\phi}, \overline{\pi}) : \overline{S} \to \mathbb{P}^2 \times C$ to yield
$$C \cdot \lambda^*Z = (d - 1)(d - 2)(d - 3)k + \sum_{i=1}^{|\mathcal{g}|} (-\frac{1}{2}i^2d^2 + 3i^2d - \frac{1}{2}id - 1 + 3i - 5i^2)x_i$$
$$+ \sum_{i=1}^{|\mathcal{g}|} (-\frac{1}{2}(d - i)^2d^2 + 3(d - i)^2d - \frac{1}{2}(d - i)d - 1 + 3(d - i) - 5(d - i)^2)y_i. \quad (3.5)$$

Parts of this computation were performed using Schubert [K-S]. Comparing the
coefficients of $k, x_i, y_i$ in (3.4) and (3.5) leads to
$$a = \frac{(d-1)(d-2)(d-3)}{2d} \quad \text{and} \quad a_i = -\frac{i(d - i)(d - 6) + 2d}{2d}. \quad (3.6)$$

Therefore the formula for $\tilde{N}_d$ follows from (3.1), (3.6) and
$$\tilde{N}_d = Z \cdot H^{3d-2} = aH^{3d-1} + \sum_{i=1}^{|\mathcal{g}|} a_iK^i \cdot H^{3d-2}. \quad \square$$

### 3.2. Constraint on stable maps

Let $\eta : E \to \Delta_t$ be a flat family of stable genus-$g$ curves in $\overline{\mathcal{M}}_g$ satisfying

(a) $\eta^{-1}(t)$ is irreducible and smooth for all $t \neq 0$.

(b) for every smooth point $p \in C_0 = \eta^{-1}(0)$, there exists a basis $\Lambda_1, \ldots, \Lambda_g$

$H^0(C_0, \omega_{C_0})$ such that locally near $p$, we have $\Lambda_j = v^{j-1}f_j(v) \cdot dv$ for some

holomorphic functions $f_j(v)$ satisfying $f_j(0) \neq 0$ for $1 \leq j \leq g$ where $v$ is a

local coordinate of $C_0$ centered at $p$.

Let $\tilde{\eta} : E \to \Delta_t$ be the family obtained by blowing-up $E$ at points in $C_0$ (possibly

infinitely near) and by adding $(3d - 2(g - 1))$ markings. Let $D = \tilde{\eta}^{-1}(0)$, and

$T_1, \ldots, T_s$ be all the connected components of the closure $\overline{D \setminus C_0}$ of $D \setminus C_0$ in $D$.

Then each $T_i$ is a tree of smooth rational curves, and $D = C_0 \cup (\coprod_{i=1}^s T_i)$. Assume

that $\mu : E \to \mathbb{P}^2$ is a morphism such that $\mu, \tilde{\eta}$, and the $(3d - 2(g - 1))$ markings
determine a family of stable maps in $\overline{\mathcal{M}}_g(d)$. Furthermore, suppose that there exist

smooth rational components $D_1, \ldots, D_m$ contained in the trees $T_1, \ldots, T_s$ with

$\deg(\mu|_{D_i}) > 0$ for each $i$ and $\sum_{i=1}^m \deg(\mu|_{D_i}) = d$. 

Lemma 3.7. Let $g \geq 2$ and $d > 2(g-1)$. Assume that $\hat{E}$ is obtained from $E$ by a chain of blowups at smooth points of $C_0$ (possibly infinitely near). If $m \leq 2$, then there exist $2g$ independent conditions in choosing the $m$ linear series $(\mu|D_i)^*|\ell|$, $1 \leq i \leq m$ from the $m$ complete linear systems $(\mu|D_i)^*|\ell|$, $1 \leq i \leq m$.

Proof. There are three separate cases: (i) $m = 1$, (ii) $m = 2$, and $D_1, D_2$ are contained in the same tree in $T_1, \ldots, T_s$; (iii) $m = 2$, and $D_1, D_2$ are contained in the two different trees in $T_1, \ldots, T_s$. The proofs of (ii) and (iii) are very similar to the proof of (i) but need some extra preparation.

Case (i): We follow the approach in [Pa1]. For simplicity, we first assume that $\hat{E}$ is the blow-up of $E$ at a smooth point $p \in C_0$. Then $s = 1$, and $\mathbb{P} = T_1 = \mathbb{P}^1$ is the exceptional divisor. For each $j$ with $1 \leq j \leq d$, let $\mathcal{G}_j = \mathcal{H}_j \subset E$ be the open subset of $E$ on which the morphism $\eta$ is smooth. Put

$$X = \mathcal{G}_1 \times \Delta_1 \times \cdots \times \Delta_t \mathcal{G}_d \times \mathcal{H}_1 \times \Delta_1 \times \cdots \times \Delta_t \mathcal{H}_d.$$ 

Then $X$ is a smooth open subset of the $2d$-fold fiber product of $E$ over $\Delta_t$. Let $Y \subset X$ be the subset of points $y = (g_1, \ldots, g_d, h_1, \ldots, h_d)$ such that the two divisors $\sum_j g_j$ and $\sum_j h_j$ are linearly equivalent on the curve $\eta^{-1}(\eta(y))$.

Let $\gamma : \Delta_t \rightarrow E$ be a local holomorphic section of $\eta$ such that $\gamma(0) = p$. Let $V$ be a nonvanishing local holomorphic field of vertical tangent vectors to $E$ on an open subset containing $p$. The section $\gamma$ and the vertical vector field $V$ together determine local holomorphic coordinates $(t, v)$ on $E$ at $p$. Let $\phi_V : E \times \mathbb{C} \rightarrow E$ be the holomorphic flow of $V$ defined locally near $(p, 0) \in E \times \mathbb{C}$. Then the coordinate map $\psi : \mathbb{C}^2 \rightarrow E$ is defined by $\psi(t, v) = \phi_V(\gamma(t), v)$. Since $p \in C_0$ is a smooth point in $C_0$, $p \in \mathcal{G}_j$ and $p \in \mathcal{H}_j$ for $1 \leq j \leq d$. Put $x_p = (p, \ldots, p, p, \ldots, p) \in X$. Then the local coordinates on $X$ near $x_p$ are given by $(t, v_1, \ldots, v_d, w_1, \ldots, w_d)$, and the coordinate map $\psi_X$ is defined by putting $\psi_X(t, v_1, \ldots, v_d, w_1, \ldots, w_d)$ to be

$$(\psi(t, v_1), \ldots, \psi(t, v_d), \psi(t, w_1), \ldots, \psi(t, w_d)) \in X.$$ 

By our assumption, there exists a basis $\Lambda_1, \ldots, \Lambda_g$ of $H^0(C_0, \omega_{C_0})$ such that near $p$, we have $\Lambda_k(v) = v^{k-1}f_k(v) \cdot dv$ and $f_k(0) \neq 0$ for $1 \leq k \leq g$. Let $\Lambda_k(t, v)$ ($1 \leq k \leq g$) be a local holomorphic extension of $\Lambda_k(v)$ at $0 \in \Delta_t$ such that for a fixed $t$, $\Lambda_1(t, v), \ldots, \Lambda_g(t, v)$ form a basis of $H^0(\eta^{-1}(t), \omega_{\eta^{-1}(t)})$ using the coordinate map $\psi$. This basis may be chosen so that

$$\Lambda_k(t, v) = v^{k-1}f_k(t, v) \cdot dv$$

with $f_k(0, 0) = f_k(0) \neq 0$. Now the local equations of $Y \subset X$ at the point $x_p$ are

$$\sum_{j=1}^d \left( \int_0^{v_j} \Lambda_k(t, v) - \int_0^{w_j} \Lambda_k(t, v) \right) = 0, \quad 1 \leq k \leq g. \quad (3.8)$$

Let $L_1$ and $L_2$ be general divisors in $\mu^*|\ell|$ such that each intersects $\mathbb{P}$ at $d$ distinct points. For $1 \leq \alpha \leq 2$, $L_\alpha$ determines local holomorphic sections $s_{\alpha,1} + \ldots + s_{\alpha,d}$ of $\mathcal{H}_\alpha$ at $0 \in \Delta_t$. These sections $s_{\alpha,j}$ with $1 \leq \alpha \leq 2$ and $1 \leq j \leq d$ determine a map $\lambda : \Delta_t \rightarrow Y$ locally at $0 \in \Delta_t$. Let an affine coordinate on $\mathbb{P}^1$ be given by $\xi$ corresponding to the normal direction

$$\frac{d\gamma}{dt}|_{t=0} + \xi \cdot V(p).$$
Let \( s_{1,j}(0) = \nu_j \in \mathbb{C}^1 \subset \mathbb{P}^1 \) and \( s_{2,j}(0) = \omega_j \in \mathbb{C}^1 \subset \mathbb{P}^1 \) be given in terms of the affine coordinates \( \xi \). Then the map \( \lambda \) has the form

\[
\lambda(t) = (t, \nu_1(t), \ldots, \nu_d(t), \omega_1(t), \ldots, \omega_d(t))
\]

(3.9)

where

\[
\nu_j(t) = \nu_j t + O(t^2), \quad \omega_j(t) = \omega_j t + O(t^2), \quad 1 \leq j \leq d.
\]

(3.10)

Since \( Y \) is defined by the equations (3.8), we obtain

\[
\sum_{j=1}^{d} \left( \int_{0}^{\nu_j(t)} v^{k-1} f_k(t, v) \cdot dv - \int_{0}^{\omega_j(t)} v^{k-1} f_k(t, v) \cdot dv \right) = 0, \quad 1 \leq k \leq g. \quad (3.11)
\]

Differentiating (3.11) \( k \)-times with respect to \( t \) and evaluating at \( t = 0 \) results in

\[
\sum_{j=1}^{d} ((k - 1)! f_k(0, 0) \cdot \nu_j^k - (k - 1)! f_k(0, 0) \cdot \omega_j^k) = 0, \quad 1 \leq k \leq g.
\]

Since \( f_k(0, 0) = f_k(0) \neq 0 \), we have \( \sum_{j=1}^{d} \nu_j^k = \sum_{j=1}^{d} \omega_j^k \) where \( 1 \leq k \leq g \). Let \( \beta_k \) be the \( k \)-th elementary symmetric function in \( d \) variables. Then, \( \beta_k(\nu_1, \ldots, \nu_d) = \beta_k(\omega_1, \ldots, \omega_d) \) for \( 1 \leq k \leq g \). Put \( \beta_k' = (-1)^k \cdot \beta_k(\nu_1, \ldots, \nu_d) \) for \( 1 \leq k \leq g \). Then the divisors in \( (\mu|\nu)^*\ell \) correspond to degree-\( d \) polynomials of the form

\[
K(\xi^d + \beta_1' \cdot \xi^{d-1} + \ldots + \beta_g' \cdot \xi^{d-g} + \ldots)
\]

where \( K \) stands for constants. It follows that the linear series \( (\mu|\nu)^*\ell \) has vanishing sequence \( \{0, \geq (g + 1), *\} \) at the point \( \xi = \infty \) which is the intersection \( C_0 \cap \mathbb{P} \). Since the complete linear system \( |(\mu|\nu)^*\ell| \) is base-point-free, the existence of a vanishing sequence of the form \( \{0, \geq (g + 1), *\} \) for the linear series \( (\mu|\nu)^*\ell \) imposes \( 2g \) independent conditions in choosing \( (\mu|\nu)^*\ell \) from \( |(\mu|\nu)^*\ell| \).

The general case arises when \( n \) blowups are needed to obtain \( \mathbb{P} \). In this situation, using automorphisms of the rational components in \( T_i \), we may assume that the form of \( \lambda \) is again given by (3.9) with (3.10) replaced by

\[
\nu_j(t) = \nu_j t^n + O(t^{n+1}), \quad \omega_j(t) = \omega_j t^n + O(t^{n+1}), \quad 1 \leq j \leq d.
\]

Then the calculation concludes as before. (Compare with [Pa1].)

**Case (ii):** For simplicity, we assume that \( \mathcal{E} \) is the 2-fold blow-up of \( \mathcal{E} \) at a smooth point \( p \in C_0 \). Then \( s = 1 = i \), and \( T_1 = D_1 \cup D_2 \) is the union of the two exceptional divisors. Let \( d_i = \deg(\mu|D_i) \) for \( i = 1, 2 \). Then, \( d_1 > 0, d_2 > 0, \) and \( d_1 + d_2 = d \). Let \( L_1 \) and \( L_2 \) be general divisors in \( \mu^*\ell \) such that each intersects \( D_i \) at \( d_i \) distinct points. Let other notations be as in Case (i). Then locally at \( 0 \in \Delta_t \), \( L_1 \) and \( L_2 \) induce a map \( \lambda : \Delta_t \to Y \) sending \( t \in \Delta_t \) to the following point in \( Y \):

\[
\lambda(t) = (t, \nu_{1,1}(t), \ldots, \nu_{1,d_1}(t), \nu_{2,1}(t), \ldots, \nu_{2,d_2}(t),
\omega_{1,1}(t), \ldots, \omega_{1,d_1}(t), \omega_{2,1}(t), \ldots, \omega_{2,d_2}(t))
\]
where for $1 \leq i \leq 2$ and $1 \leq j \leq d_i$, we may assume that $\nu_{i,j}(t) = \nu_{i,j} t^i + O(t^{i+1})$ and $\omega_{i,j}(t) = \omega_{i,j} t^i + O(t^{i+1})$ for some constants $\nu_{i,j}$ and $\omega_{i,j}$. A similar argument as in Case (i) shows that the linear series $(\mu|D_1)^*|\ell|$ has vanishing sequence $\{0, \geq (g+1), *\}$ at the point $C_0 \cap D_1$. So there exist $2g$ independent conditions in choosing $(\mu|D_1)^*|\ell|$ from the complete linear system $|(\mu|D_1)^*|\ell|$.

**Case (iii):** Again for simplicity, we assume that $\hat{E}$ is the 2-fold blow-up of $E$ at two smooth points $p_1, p_2 \in C_0$. Then $s = 2$, and $\{T_1, T_2\} = \{D_1, D_2\}$ is the set of the two exceptional divisors. Let $d_i = \text{deg}(\mu|D_i)$ for $i = 1, 2$. Then, $d_1 > 0, d_2 > 0$, and $d_1 + d_2 = d$. Since $d > 2(g-1)$, we may assume that $d_1 \geq g$. As in Case (i), we construct local coordinates $(t, v_i)$ and coordinate map $\psi_i$ on $E$ at each point $p_i$. Let $X$ and $Y$ be as in Case (i). Define a point $x_{p_1, p_2} \in X$ by

\[
\begin{align*}
x_{p_1, p_2} = & \left( p_1, \ldots, p_1, p_2, \ldots, p_2, \ldots, p_1, p_2, \ldots, p_2 \right).
\end{align*}
\]

The local coordinates on $X$ near $x_{p_1, p_2}$ are given by $(t, v_1, \ldots, v_d, w_1, \ldots, w_d)$, and the coordinate map $\psi_X$ is defined by putting $\psi_X(t, v_1, \ldots, v_d, w_1, \ldots, w_d)$ to be

\[
\begin{align*}
\psi_1(t, v_1), \ldots, \psi_1(t, v_1), & \psi_2(t, v_2, v_{d+1}), \ldots, \psi_2(t, v_d), \\
\psi_1(t, v_1), \ldots, \psi_1(t, v_1), & \psi_2(t, w_d, w_{d+1}), \ldots, \psi_2(t, w_d).
\end{align*}
\]

Note that $x_{p_1, p_2} \in Y$ and the local equations of $Y \subset X$ at $x_{p_1, p_2}$ are given by (3.8) where $\Lambda_1(t, v), \ldots, \Lambda_2(t, v)$ are chosen so that $\Lambda_k(t, v_1) = v_1^{k-1} f_k(t, v_1) \cdot dv_1$ with $f_k(0, 0) \neq 0$. Let $L_1$ and $L_2$ be general divisors in $\mu^*|\ell|$ such that each intersects $D_i$ at $d_i$ distinct points. Then locally at $0 \in \Delta_i$, $L_1$ and $L_2$ induce $\lambda: \Delta_i \rightarrow Y$ by

\[
\lambda(t) = \left( t, v_1, \ldots, v_{d_1}, t, v_2, \ldots, v_{d_2} \right), \\
\omega_1(t), \ldots, \omega_{d_1}(t), & \omega_2(t), \ldots, \omega_{d_2}(t)
\]

where for $1 \leq i \leq 2$ and $1 \leq j \leq d_i$, we have $\nu_{i,j}(t) = \nu_{i,j} t + O(t^2)$ and $\omega_{i,j}(t) = \omega_{i,j} t + O(t^2)$ for some constants $\nu_{i,j}$ and $\omega_{i,j}$. Now a similar argument as in Case (i) shows that for $1 \leq k \leq g, \sum_{j=1}^{d_i} \nu_{i,j}^k - \sum_{j=1}^{d_i} \omega_{i,j}^k$ is a homogeneous polynomial in $v_2, \ldots, v_{d_2}$ and in $\omega_2, \ldots, \omega_{d_2}$. Therefore for a fixed linear series $(\mu|D_2)^*|\ell|$, there exist $2g$ independent conditions in choosing $(\mu|D_1)^*|\ell|$ from $|(\mu|D_1)^*|\ell|$, i.e. there exist $2g$ independent conditions in choosing the linear series $(\mu|D_1)^*|\ell|$ and $(\mu|D_2)^*|\ell|$ from the complete linear systems $|(\mu|D_1)^*|\ell|$ and $|(\mu|D_2)^*|\ell|$.

This completes the proof of Case (iii) and hence the proof of the lemma. \( \square \)

**Remark 3.12.** (i) It is reasonable to expect that Lemma 3.7 holds for any $m$.

(ii) In Lemma 3.7, we have assumed that $\hat{E}$ is the blow-up of $E$ at smooth points of $C_0$. However, a slight modification of its proof shows that the conclusion is still true if $g = 2$ and the blowing-up $\hat{E} \rightarrow E$ also takes place at nodal points of $C_0$.

Finally, we show that the stable genus-2 curve $C_0$ constructed in Section 2 satisfies hypothesis (b) leading up to the statement of Lemma 3.7.

**Lemma 3.13.** For every smooth point $p \in C_0$, there exists a basis $\Lambda_1, \Lambda_2$ for $H^0(C_0, \omega_{C_0})$ such that $\Lambda_1 = v^{-1} f_j(v) \cdot dv$ and some holomorphic functions $f_j(v)$ such that $f_j(0) \neq 0$ for $1 \leq j \leq 2$ where $v$ is a local coordinate of $C_0$ centered at $p$.

**Proof.** Assume that $p \in R_1 = \mathbb{P}^1$. Choose an affine coordinate $z$ for $R_1$ such that the three nodal points in $C_0$ are identified with $0, 1, \infty$ in $R_1$. Then a basis
for $H^0(C_0, \omega_{C_0})$ can be identified with $\Lambda'_1 = \frac{1}{\zeta} \cdot dz, \Lambda'_2 = \frac{1}{z-1} \cdot dz$. Let $z_0$ be the coordinate of $p \in R_1$, and let $v = z - z_0$. Then the desired basis consists of

\[
\Lambda_1 = \frac{1}{z} \cdot dz = \frac{1}{v + z_0} \cdot dv, \quad \Lambda_2 = \frac{(z - z_0)}{z(z - 1)} \cdot dz = \frac{v}{(v + z_0)(v + z_0 - 1)} \cdot dv. \quad \square
\]

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