NON-CYCLIC GRAPH OF A GROUP

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Abstract. We associate a graph $\Gamma_G$ to a non locally cyclic group $G$ (called the non-cyclic graph of $G$) as follows: take $G\setminus \text{Cyc}(G)$ as vertex set, where $\text{Cyc}(G) = \{x \in G \mid \langle x, y \rangle \text{ is cyclic for all } y \in G\}$, and join two vertices if they do not generate a cyclic subgroup. We study the properties of this graph and we establish some graph theoretical properties (such as regularity) of this graph in terms of the group ones. We prove that the clique number of $\Gamma_G$ is finite if and only if $\Gamma_G$ has no infinite clique. We prove that if $G$ is a finite nilpotent group and $H$ is a group with $\Gamma_G \cong \Gamma_H$ and $|\text{Cyc}(G)| = |\text{Cyc}(H)| = 1$, then $H$ is a finite nilpotent group. We give some examples of groups $G$ whose non-cyclic graphs are “unique”, i.e., if $\Gamma_G \cong \Gamma_H$ for some group $H$, then $G \cong H$. In view of these examples, we conjecture that every finite non-abelian simple group has a unique non-cyclic graph. Also we give some examples of finite non-cyclic groups $G$ with the property that if $\Gamma_G \cong \Gamma_H$ for some group $H$, then $|G| = |H|$. These suggest the question whether the latter property holds for all finite non-cyclic groups.

1. Introduction and results

Let $G$ be a group. Recall that the centralizer of an element $x \in G$ can be defined by

$$C_G(x) = \{y \in G \mid \langle x, y \rangle \text{ is abelian}\}$$

which is a subgroup of $G$. If, in the above definition, we replace the word “abelian” with the word “cyclic” we get a subset of the centralizer, called the cyclicizer (see [16, 15]). To be explicit, define the cyclicizer

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of an element \( x \in G \), denoted by \( CyC_G(x) \), by
\[
CyC_G(x) = \{ y \in G \mid \langle x, y \rangle \text{ is cyclic} \}.
\]
Also for a non-empty subset \( X \) of \( G \), we define the cyclicizer of \( X \) in \( G \), to be
\[
CyC_G(X) = \bigcap_{x \in X} CyC_G(x),
\]
when \( X = G \); we call \( CyC_G(G) \) the cyclicizer of \( G \), and denote it by \( CyC(G) \), so
\[
CyC(G) = \{ y \in G \mid \langle x, y \rangle \text{ is cyclic for all } x \in G \}
\]
It is a simple fact that for any group \( G \), \( CyC(G) \) is a locally cyclic subgroup of \( G \). As it is mentioned in [16], in general for an element \( x \) of a group \( G \), \( CyC_G(x) \) is not a subgroup of \( G \). For example, in the group \( H = \mathbb{Z}_2 \oplus \mathbb{Z}_4 \), we have
\[
CyC_H((0,2)) = \{(0,0), (0,1), (0,2), (0,3), (1,1), (1,3)\}
\]
which is not a subgroup of \( H \) (See also Theorem 5.4 below). One can associate a graph to a group in many different ways (see for example [1, 2, 8, 13, 14, 25]). The main idea is to study the structure of the group by the graph theoretical properties of the associated graph. Here we consider the following way to associate a graph with a non locally cyclic group.

Let \( G \) be a non locally cyclic group. We associate a graph \( \Gamma_G \) to \( G \) (called the non-cyclic graph of \( G \)) with vertex set \( V(\Gamma_G) = G \setminus CyC(G) \) and edge set
\[
E(\Gamma_G) = \{ \{ x, y \} \subseteq V(\Gamma_G) \mid \langle x, y \rangle \text{ is not cyclic} \}.
\]
Note that the degree of a vertex \( x \in V(\Gamma_G) \) in the non-cyclic graph \( \Gamma_G \) is equal to \( |G \setminus CyC_G(x)| \). We refer the reader to [6] for undefined graph theoretical concepts and to [18] for group theoretical ones.

The outline of this paper is as follows. In Section 2, we give some results on the cyclicizers which will be used in the sequel. In Section 3, we prove some general properties which hold for the non-cyclic graph of a group, e.g., the non-cyclic graph of any non locally cyclic group is always connected and its diameter is less than or equal to 3. In Section 4, we characterize groups whose non-cyclic graphs have no infinite clique. In fact we prove that such groups have finite clique numbers and in contrast there are groups \( G \) whose non-cyclic graphs have no infinite independent set and their independence numbers are not yet
finite. In Section 5, we characterize finite non-cyclic groups whose non-cyclic graphs are regular. In Section 6, we characterize finite non-cyclic abelian groups whose non-cyclic graphs have exactly two kind degrees. Section 7 contains some results on groups whose non-cyclic graphs are isomorphic. We give some groups $G$ with the property that if $\Gamma_G \cong \Gamma_H$ for some group $H$, then $|G| = |H|$. In Section 8 we give some groups $G$ whose non-cyclic graphs are “unique”, that is, if $\Gamma_G \cong \Gamma_H$ for some group $H$, then $G \cong H$. It will be seen that there are (many) groups whose non-cyclic graphs are not unique.

2. Some properties of cyclicizers

For a group $G$ and two non-empty subsets $X$ and $Y$ of $G$, we denote by $\text{Cyc}_X(Y)$ the set $\{x \in X \mid \langle x, y \rangle \text{ is cyclic for all } y \in Y\}$.

Lemma 2.1. Let $G$ be a group, $x \in G$, and $D = \text{Cyc}_G(x)$. Then

1. $D$ is the union of cosets of $\text{Cyc}(G)$. In particular, if $|D| < \infty$, then $|\text{Cyc}(G)| < \infty$ and divides $|D|$.
2. $\text{Cyc}_D(D)$ is a locally cyclic subgroup of $G$ containing $x$.

Proof. (1) First note that the union of a chain of locally cyclic subgroups is a locally cyclic subgroup. Thus, every element is contained in at least one maximal locally cyclic subgroup. Now it is easy to see that $D$ is the union of the maximal locally cyclic subgroups of $G$ which contain $x$. As each maximal locally cyclic subgroup must contain $\text{Cyc}(G)$, each of these subgroups is a union of cosets of $\text{Cyc}(G)$. Thus, so is $\text{Cyc}_G(x)$.

(2) It is clear that $x \in \text{Cyc}_D(D)$. Let $a, b \in \text{Cyc}_D(D)$ and suppose that $d \in D$. Now $\langle b, d \rangle = \langle c \rangle$ for some $c \in G$. As $\langle c, x \rangle \leq \langle b, d, x \rangle$ is cyclic, $\langle d, x \rangle$ is cyclic and contains $x$ so its generator must belong to $D$) it follows that $c \in D$.

Now $\langle ab^{-1}, d \rangle \leq \langle a, b, d \rangle = \langle a, c \rangle$ is cyclic. As $x \in D$, $ab^{-1} \in D$. It follows that $ab^{-1} \in \text{Cyc}_D(D)$. It follows that $\text{Cyc}_D(D) \leq G$. Now let $\{d_1, \ldots, d_n\} \subseteq \text{Cyc}_D(D)$. Then $\langle x, d_1 \rangle = \langle a_1 \rangle$ for some $a_1 \in G$. Since $x \in \langle a_1 \rangle$, $a_1 \in D$. Thus $\langle a_1, d_2 \rangle = \langle a_2 \rangle$ for some $a_2 \in D$. If we argue in this manner, then we find an element $a_n \in D$ such that $\langle d_1, \ldots, d_n \rangle \leq \langle a_n \rangle$. This implies that $\text{Cyc}_D(D)$ is locally cyclic. □

Proposition 2.2. (See [15]) Let $G$ be a finite $p$-group for some prime $p$. Then $\text{Cyc}(G) \neq 1$ if and only if $G$ is either a cyclic group or a generalized quaternion group.

Proof. Let $x$ be an element of order $p$ in $\text{Cyc}(G)$. If $A$ is a subgroup of order $p$ of $G$, then $A = \langle a \rangle$ for some $a \in A$. Thus $H = \langle a, x \rangle$ must be a
cyclic $p$-group and so $H$ has exactly one subgroup of order $p$. Therefore $A = \langle x \rangle$. It follows that $G$ has exactly one subgroup of order $p$. Now \cite[Theorem 5.3.6]{18} completes the proof. \hfill \Box

Lemma 2.3. Let $G$ be any group, $x \in G$, $\overline{G} = \frac{G}{Cyc(G)}$ and $\tilde{G} = \frac{G}{Z(G)}$. Then

1. $Cyc_{\overline{G}}(xCyc(G)) = \frac{Cyc_G(x)}{Cyc(G)}$.
2. (See \cite{16}) $Cyc(\overline{G}) = 1$.
3. (See \cite{16}) $Cyc(\tilde{G}) = 1$.
4. If $G$ is neither torsion nor torsion-free, then $Cyc(G) = 1$.
5. If $G$ is a torsion-free group such that $Cyc(G)$ is non-trivial, then $Cyc(G) = Z(G)$. Moreover, if $Z(G)$ is divisible, then $G$ is locally cyclic.

Proof. (1) Let $y \in G$ be such that $yCyc(G) \in Cyc_{\overline{G}}(xCyc(G))$. Then

$$\frac{\langle y, x \rangle Cyc(G)}{Cyc(G)}$$

is cyclic. Thus there exists an element $z \in \langle y, x \rangle$ and two elements $a_1$ and $a_2$ in $Cyc(G)$ such that $x = za_1$ and $y = za_2$. Now since

$$\langle x, y \rangle = \langle za_1, za_2 \rangle \leq \langle z, a_1, a_2 \rangle$$

and $\langle z, a_1, a_2 \rangle$ is cyclic, $y \in Cyc_G(x)$, as required.

(2) It follows from (1).

(3) It is straightforward.

(4) By hypothesis, $G$ has an element $x$ of infinite order and a non-trivial element $y$ of finite order. If $Cyc(G)$ contains a non-trivial element $c$ of finite order, then $\langle c, x \rangle$ must be cyclic, a contradiction; and if $Cyc(G)$ contains an element $d$ of infinite order, then $\langle y, d \rangle$ must be cyclic, a contradiction. Hence $Cyc(G) = 1$.

(5) Suppose, for a contradiction, that there exists a central element $x$ which is not in $Cyc(G)$. Then there exists $y \in G$ such that $\langle x, y \rangle$ is a non-cyclic abelian group. If $a$ is a non-trivial element of $Cyc(G)$, then $H = \langle a, x, y \rangle \cong A = Z \oplus Z$. Since $Cyc_A((1, 0)) = \langle (1, 0) \rangle$ and $Cyc_A((0, 1)) = \langle (0, 1) \rangle$, $Cyc(A) = 1$. It follows that $Cyc(H) = 1$, which is a contradiction, since $a \in Cyc(H)$. This proves that $Cyc(G) = Z(G)$. Now suppose that $Z(G)$ is divisible and let $y$ be any element of $G$. If $a$ is a non-trivial element of $Cyc(G)$, then $\langle a, y \rangle$ is cyclic, so $y^n \in \langle a \rangle \leq Z(G)$ for some non-zero integer $n$. It follows that $y^n = z^n$, for some $z \in Z(G)$, since $Z(G)$ is divisible. Hence $(yz^{-1})^n = 1$, and so $y \in Z(G)$. Thus $G = Z(G) = Cyc(G)$, as required. \hfill \Box
We end this section with the following question.

**Question 2.4.** Let $G$ be a torsion free group such that $\text{Cyc}(G)$ is non-trivial. Is it true that $G$ is locally cyclic?

3. Some properties of non-cyclic graph

For a simple graph $\Gamma$, we denote by $\text{diam}(\Gamma)$ the diameter of $\Gamma$.

**Proposition 3.1.** Let $G$ be a non locally cyclic group. Then $\text{diam}(\Gamma_G) = 1$ (or equivalently $\Gamma_G$ is complete) if and only if $G$ is an elementary abelian $2$-group.

*Proof.* Suppose that $\text{diam}(\Gamma_G) = 1$. If $x \neq x^{-1}$, for some $x \in G \setminus \text{Cyc}(G)$, then since $(x, x^{-1})$ is obviously cyclic, $x$ is not incident to $x^{-1}$, a contradiction. Hence $x^2 = 1$ for all $x \in G \setminus \text{Cyc}(G)$. If $z \in \text{Cyc}(G)$, then $xz \in G \setminus \text{Cyc}(G)$ for every $x \in G \setminus \text{Cyc}(G)$. Thus $(xz)^2 = 1$ and since $z \in \text{Cyc}(G) \leq Z(G)$, $x^2z^2 = 1$ from which it follows that $z^2 = 1$, since $x^2 = 1$. Hence $x^2 = 1$ for all $x \in G$ and so $G$ is an elementary abelian $2$-group.

The converse is clear. $\square$

**Proposition 3.2.** Let $G$ be a non locally cyclic group. Then $\Gamma_G$ is connected and $\text{diam}(\Gamma_G) \leq 3$. Moreover, if $Z(G) = \text{Cyc}(G)$, then $\text{diam}(\Gamma_G) = 2$.

*Proof.* Suppose that $x$ and $y$ are two vertices of $\Gamma_G$ such that there is no path of length at least 2 between them. It follows that $G = \text{Cyc}(x) \cup \text{Cyc}(y)$. If $Z(G) = \text{Cyc}(G)$, then $x$ and $y$ are non-central elements of $G$ and $G = \text{Cyc}(x) \cup \text{Cyc}(y) \subseteq C_G(x) \cup C_G(y)$.

It follows that either $G = C_G(x)$ or $G = C_G(y)$, which gives a contradiction, since $x$ and $y$ are not central elements. Now since $Z(G) = \text{Cyc}(G)$, $G$ is not elementary abelian, so $\text{diam}(\Gamma_G) = 2$, by Proposition 3.1.

Now consider the general case. We prove that for all $t_1 \in \text{Cyc}(x) \setminus \text{Cyc}(y)$ and for all $t_2 \in \text{Cyc}(y) \setminus \text{Cyc}(x)$, $t_1$ and $t_2$ are adjacent. Suppose, for a contradiction, that $\langle t_1, t_2 \rangle$ is cyclic for some $t_1$ and $t_2$ in $\text{Cyc}(x) \setminus \text{Cyc}(y)$ and $\text{Cyc}(y) \setminus \text{Cyc}(x)$, respectively. Thus $\langle t_1, t_2 \rangle = \langle t \rangle$ for some $t \in G$ and so $t \in \text{Cyc}(x)$ or $t \in \text{Cyc}(y)$. If $t \in \text{Cyc}(x)$, then $\langle t, x \rangle = \langle t_1, t_2, x \rangle$ is cyclic, so $\langle t_1, x \rangle$ is cyclic, a contradiction. Similarly the case $t \in \text{Cyc}(y)$ gives a contradiction. Hence $x - t_2 - t_1 - y$ is a path of length 3 between...
Lemma 3.4. Let $G$ be a finite non-cyclic nilpotent group. Then $\text{diam}(\Gamma_G) \leq 2$.

**Proof.** Let $x, y \in V(\Gamma_G)$ and $x \neq y$. Suppose that $p_1, \ldots, p_k$ are the prime divisors of $|G|$ and $G_i$ is the Sylow $p_i$-subgroup of $G$. Suppose that $x = x_1 \cdots x_k$ and $y = y_1 \cdots y_k$, where $x_i, y_i \in G_i$ for every $i \in \{1, \ldots, k\}$. Suppose that $x$ is not adjacent to $y$. Since $x$ and $y$ are not in $\text{Cyc}(G)$, Remark 3.3 implies that there exist $i, j \in \{1, \ldots, k\}$ such that $x_i \notin \text{Cyc}(G_i)$ and $y_j \notin \text{Cyc}(G_j)$. Thus there exist elements $z_i \in G_i$ and $z_j \in G_j$ such that $x_i$ and $y_j$ are incident to $z_i$ and $z_j$, respectively. If $i \neq j$, then since $\langle x, z_i z_j \rangle = \langle x_1, \ldots, x_k, z_i, z_j \rangle$ and $\langle y, z_i z_j \rangle = \langle y_1, \ldots, y_k, z_i, z_j \rangle$, we have that $z_i z_j$ is incident to both $x$ and $y$. Now assume that $i = j$, so $\langle x, y_j \rangle = \langle x, y_i \rangle = \langle a \rangle$ for some $a \in G_i$, since $x$ and $y$ are not adjacent. If $\text{Cyc}(G_i) = 1$, then let $z$ be an element of order $p_i$ in $\langle a \rangle$. Then $\langle z \rangle \leq \langle x_i \rangle \cap \langle y_i \rangle$. Since $\text{Cyc}(G_i) = 1$, there exists an element $b \in G_i$ such that $z$ is incident to $b$. Now since $\langle b, z \rangle \leq \langle b, x \rangle \cap \langle b, y \rangle$, $b$ is incident to both $x$ and $y$. So $x - b - y$ is a path of length 2, as required. Now assume that $\text{Cyc}(G_i) \neq 1$. Since $G_i$ is a finite $p_i$-group, by Proposition 3.2, $G_i$ is either cyclic or generalized quaternion. The former case is false, since $x_i \notin \text{Cyc}(G_i)$; and so $G_i$ is a generalized quaternion group. Thus $\text{Cyc}(G_i) = \mathbb{Z}(G_i)$ and so by Proposition 3.2, there exists an element $z \in G_i$ which is adjacent to both $x_i$ and $y_i$. It follows that $z$ is incident to both $x$ and $y$ so $x - z - y$ is a path of length 2 between $x$ and $y$, as required. Hence we have proved that the distance between any two vertices of $\Gamma_G$ is 1 or 2, which completes the proof. □

Proposition 3.5. Let $G$ be a group which is neither torsion nor torsion-free. Then $\text{diam}(\Gamma_G) = 2$.

**Proof.** In view of Proposition 3.1, it is enough to show that $\text{diam}(\Gamma_G) \leq 2$. By hypothesis $G$ has an element $x$ of infinite order and a non-trivial element $y$ of finite order. Let $a$ and $b$ be two non-trivial elements of $G$. If $a$ is of finite order and $b$ is of infinite order or vice versa, then $a$ is adjacent to $b$. If $a$ and $b$ are both of finite orders, then $a - x - b$ is a
path of length two between $a$ and $b$ and if $a$ and $b$ are both of infinite order, then $a - y - b$ is a desired path. This completes the proof. □

**Lemma 3.6.** If $G = \mathbb{Z} \oplus \mathbb{Z}$, then $\text{diam}(\Gamma_G) = 2$ and $\text{Cyc}(G) = 1$.

**Proof.** It follows from Lemma 2.3(5) that $\text{Cyc}(G) = 1$. To prove $\text{diam}(\Gamma_G) = 2$ suppose that $x$ and $y$ are two arbitrary distinct non-trivial elements of $G$. We prove that there exists a path of length 2 between $x$ and $y$, if $x$ is not incident to $y$. Thus $x = ta$ and $y = sa$ for some $a = (a_1, a_2) \in G$ and for non-zero integers $t$ and $s$. If $a_1 = 0$, then $x - (1, 0) - y$ is a path of length two in $\Gamma_G$, and if $a_2 = 0$, then $x - (0, 1) - y$ is such a path, so we may assume that $a_1$ and $a_2$ are non-zero. In this case, it is easy to see that $x - (ta_1, sa_2) - y$ is a path of length two in $\Gamma_G$ between $x$ and $y$. Now Proposition 3.1 completes the proof. □

**Proposition 3.7.** Let $G$ be a torsion-free non locally cyclic group. Then $\text{diam}(\Gamma_G) = 2$.

**Proof.** Let $x$ and $y$ be two distinct vertices of $\Gamma_G$. Suppose that $x$ and $y$ are not incident and suppose, for a contradiction, that there is no path of length 2 between $x$ and $y$. It follows that $G = \text{Cyc}_G(x) \cup \text{Cyc}_G(y)$ and so $G = \text{C}(x) \cup \text{C}(y)$. This implies that either $x \in \text{Z}(G)$ or $y \in \text{Z}(G)$. Assume that $x \in \text{Z}(G)$. Since $x$ is a vertex, there exists a vertex $z$ incident to $x$. If $z$ is incident to $y$, then $x - z - y$ is a path of length 2. Thus we may assume that $x$ is not incident to $y$. It follows that $H = \langle x, z, y \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$. Then, Lemma 3.6 implies that $x$ and $y$ are vertices of $\Gamma_H$ and there exists a path of length 2 between $x$ and $y$. Now Proposition 3.1 completes the proof. □

**Proposition 3.8.** Let $G$ be a non locally cyclic group. If $G$ is locally nilpotent, then $\text{diam}(\Gamma_G) \leq 2$.

**Proof.** Let $x$ and $y$ be two distinct vertices of $\Gamma_G$. Then there exist two vertices $c_1$ and $c_2$ in $\Gamma_G$ such that $x$ and $y$ are incident to $c_1$ and $c_2$, respectively. Let $H := \langle x, y, c_1, c_2 \rangle$. Then $H$ is a non-cyclic nilpotent group. Since $x$ and $y$ are also two vertices in $\Gamma_H$, it is enough to show that there is a path of length 2 in $\Gamma_H$ between $x$ and $y$. If $H$ is neither torsion nor torsion-free, then the proof follows from Proposition 3.5. So we may assume that $H$ is either torsion-free or torsion. If $H$ is torsion-free, then Proposition 3.7 completes the proof. If $H$ is torsion, since $H$ is a finitely generated nilpotent group, $H$ is finite. Now Lemma 3.4 completes the proof. □

**Proposition 3.9.** Let $S_3$ be the symmetric group of degree 3 and $G = \mathbb{Z}_6 \times S_3$. Then $\text{diam}(\Gamma_G) = 3$. 


Proof. It is easy to see that the shortest path between the elements 
\((3, e)\) and \((2, e)\) is of length 3, where \(e\) is the identity element of \(S_3\). 
Hence the proof follows from Proposition 3.2. □

We finish this section with the following questions.

**Question 3.10.** Is it possible to characterize all finite groups \(G\) having 
the property that \(\text{diam}(\Gamma_G) = 3\)?

In view of Propositions 3.2 and 3.8, the finite groups mentioned in 
Question 3.10 must be non-nilpotent with non-trivial center.

**Question 3.11.** What can be said about a finite non-cyclic group \(G\) 
whose cyclic graph namely, the complement of \(\Gamma_G\) is connected?

4. **Groups whose non-cyclic graphs have no infinite clique**

A subset \(X\) of the vertices of a simple graph \(\Gamma\) is called a *clique* if 
the induced subgraph on \(X\) is a complete graph. The maximum size 
of a clique (if exists) in a graph \(\Gamma\) is called the *clique number* of \(\Gamma\) and 
is denoted by \(\omega(\Gamma)\).

Let \(G\) be a non-abelian group and \(Z(G)\) be the center of \(G\). One can 
associate a graph \(\nabla_G\) with \(G\) (called the non-commuting graph of \(G\)),
whose vertex set is \(G \setminus Z(G)\) and two distinct vertices are joined if they 
do not commute. Note that if \(G\) is non-abelian, \(\nabla_G\) is a subgraph of \(\Gamma_G\).
This graph has been studied by many people (see e.g. [1, 13, 14, 17]). 
Paul Erdős, who was the first to consider the non-commuting graph of 
a group, posed the following problem in 1975 [14]: Let \(G\) be a group 
whose non-commuting graph \(\nabla_G\) has no infinite clique. Is it true that 
the clique number of \(\nabla_G\) is finite? B. H. Neumman [14] answered 
positively Erdős’ question as follows.

**Theorem 4.1.** (B. H. Neumman [14]) The non-commuting graph of a 
group \(G\) has no infinite clique if and only if \(G/Z(G)\) is finite. In this 
case, the clique number of \(\nabla_G\) is finite.

Using Neumman’s theorem we prove the following similar result for 
the non-cyclic graph.

**Theorem 4.2.** The non-cyclic graph of a group \(G\) has no infinite clique 
if and only if \(G/Cyc(G)\) is finite. In this case, \(\omega(\Gamma_G)\) is finite.

To prove this theorem we need the following lemmas. 
For any prime number \(p\), we denote by \(\mathbb{Z}_{p^\infty}\) the \(p\)-primary component 
of \(\frac{\mathbb{Q}}{\mathbb{Z}}\).
Lemma 4.3. Let $G$ be a group whose non-cyclic graph has no infinite clique. Then $G$ does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_p$ for any prime number $p$, $\mathbb{Z} \oplus \mathbb{Z}$, or an infinite torsion abelian group $B$ with $\text{Cyc}(B) = 1$.

Proof. It is enough to show that the non-cyclic graph of each of these abelian groups contains an infinite clique. It is easy to see that
\[ \left\{ \left( p^n, 1 \right) \mid n \in \mathbb{N} \right\} \] and
\[ \left\{ \left( n, 1 \right) \mid n \in \mathbb{N} \right\} \]
are infinite cliques in $\mathbb{Z} \oplus \mathbb{Z}_p$ and $\mathbb{Z} \oplus \mathbb{Z}$, respectively.

For the last part first note that $G$ cannot contain a subgroup isomorphic to $\mathbb{Z}_p^\infty \oplus \mathbb{Z}_p$ for any prime number $p$, since
\[ \left\{ \left( 1/p^n + \mathbb{Z}, 1 \right) \mid n \in \mathbb{N} \right\} \]
is an infinite clique in $\mathbb{Z}_p^\infty \oplus \mathbb{Z}_p$. Now suppose, for a contradiction, that $G$ contains an infinite torsion abelian subgroup $B$ such that $\text{Cyc}(B) = 1$.

By the previous part and Remark 3.3, we have that $\text{Cyc}(C) = 1$ for every primary component $C$ of $B$. First assume that $B$ has infinitely many non-trivial primary components. Then for every $i \in \mathbb{N}$ there exist elements $a_i$ and $b_i$ in the same primary component of $B$ such that $\langle a_i, b_i \rangle$ is not cyclic and for any two distinct $i, j \in \mathbb{N}$, $\{a_i, b_i\}$ and $\{a_j, b_j\}$ lay in distinct primary components of $B$. Now define $x_1 := a_1$ and $x_i := b_1 + \cdots + b_{i-1} + a_i$ for all $i > 1$. Then it is easy to see that $\{x_n \mid n \in \mathbb{N}\}$ is an infinite clique in $B$, a contradiction. Thus we may assume that $B$ contains an infinite abelian $p$-subgroup $A$ for some prime $p$. It follows from [18, Theorem 4.3.11] that $A$ contains a subgroup isomorphic to a direct sum of an infinite family $\{A_i \mid i \in \mathbb{N}\}$ of non-trivial cyclic $p$-groups. If $a_i$ is a generator of $A_i$, then $\{a_i \mid i \in \mathbb{N}\}$ is an infinite clique in $\Gamma_G$, a contradiction. This completes the proof. □

Lemma 4.4. Let $G$ be a group whose non-cyclic graph has no infinite clique. Then every abelian subgroup of $G$ is either torsion-free locally cyclic, or isomorphic to a direct sum $\left( \bigoplus_{p \in T_1} \mathbb{Z}_p^\infty \right) \bigoplus \left( \bigoplus_{p \in T_2} \mathbb{Z}_p^{\alpha_p} \right) \bigoplus B$, where $T_1$ and $T_2$ are two (possibly empty) disjoint sets of prime numbers, $\alpha_p \geq 0$ integers, and $B$ is a finite abelian $(T_1 \cup T_2)'$-group with $\text{Cyc}(B) = 1$.

Proof. Let $A$ be an abelian subgroup of $G$. By Lemma 4.3, $A$ is either torsion-free or torsion. If $A$ is torsion-free, then, since by Lemma 4.3 it contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, $A$ must be locally cyclic. Now assume that $A$ is torsion. Then it follows easily from [18, Theorem 4.3.11] and Lemma 4.3 that $A$ is of the form stated in the lemma. □

Proof of Theorem 4.2. If $G$ is not abelian, then $\nabla_G$ is a subgraph of $\Gamma_G$. So $\nabla_G$ does not contain any infinite clique. Thus by Theorem 4.1 $G/Z(G)$ is finite. Therefore, in any case ($G$ is abelian or not), we
have that $G/Z(G)$ is finite. By Lemma 4.4, $Z(G)$ is either torsion-free locally cyclic or torsion. Suppose that $Z(G)$ is a non-trivial torsion-free locally cyclic group. Then for every $x \in G\backslash Z(G)$, $\langle x, Z(G) \rangle$ is abelian and since it is not torsion, it follows from Lemma 4.4 that $\langle x, Z(G) \rangle$ is a torsion-free locally cyclic group. Hence, in this case, $Z(G) = Cyc(G)$ and so $G/Cyc(G)$ is finite, as required.

Now assume that $Z(G)$ is an infinite torsion group. Since $G/Z(G)$ is finite, it follows that $G$ is a locally finite group. Thus there exists a finite subgroup $H$ of $G$ such that $G = Z(G)H$. By Lemma 4.4 there exists subgroups $Z_1, Z_2$ and $B$ of $Z(G)$ such that $Z(G) = Z_1 \oplus Z_2 \oplus B$, where

$$Z_1 \cong \bigoplus_{p \in T_1} \mathbb{Z}_{p^\infty}, Z_2 \cong \bigoplus_{p \in T_2} \mathbb{Z}_{p^\alpha p},$$

$T_1$ and $T_2$ are two (possibly empty) disjoint sets of prime numbers, $\alpha_p \geq 0$ integers, and $B$ is a finite abelian $(T_1 \cup T_2)'$-group with $Cyc(B) = 1$. Thus $BH$ is a finite subgroup of $G$ and so it is a $\pi'$-group for some finite set of primes $\pi$. Let $Z_0$ be the $(T_2 \backslash \pi)$-component of $Z_2$. We prove that $Z_1 \oplus Z_0 \leq Cyc(G)$ from which it follows that $G/Cyc(G)$ is finite and this completes the proof of the “only if” part. Let $x$ be an arbitrary element of $G$. Then $\langle x, Z_1 \rangle$ is an abelian group. Since $Z_1$ is divisible, $\langle Z_1, x \rangle = Z_1 \oplus \langle y \rangle$ for some $y \in G$. Now Lemma 4.4 implies that $y$ must be a $T_1'$-element, which implies that $\langle Z_1, x \rangle$ is locally cyclic. Thus $Z_1 \leq Cyc(G)$. Now let $x \in Z_0$ and $g \in G$. Then $g = z_1zz'bh$ for some elements $z_1 \in Z_1$, $b \in B$, $h \in H$, $z \in Z_0$ and $z'$ is a $(T_2 \cap \pi)$-element of $Z_2$. Then $\langle g, x \rangle$ is cyclic if and only if $\langle zz'bh, x \rangle$ is cyclic, since $z_1 \in Cyc(G)$. On the other hand $z'bh$ is a $\pi'$-element and $x, z$ are $\pi'$-elements that generate a cyclic group. It follows that $\langle zz'bh, x \rangle$ is cyclic. Therefore $Z_0 \leq Cyc(G)$ and this completes the proof as we mentioned.

For the converse, let $C$ be a subset of $G$ with $|C| > |G : Cyc(G)|$. By pigeon-hole principal there exists two distinct elements $x, y \in C$ which are in the same coset $aCyc(G)$, for some $a \in G$. Thus $x = ac_1$ and $y = ac_2$ for some $c_1, c_2 \in Cyc(G)$. Since $\langle x, y \rangle \leq \langle a, c_1, c_2 \rangle$ and this latter subgroup is cyclic, $\langle x, y \rangle$ is cyclic. This means that $\omega(\Gamma_G) \leq |G : Cyc(G)|. \quad \Box$

A subset $X$ of the vertices of a simple graph $\Gamma$ is called an independent set if the induced subgraph on $X$ has no edges. The maximum size of an independent set (if exists) in a graph $\Gamma$ is called the independence number of $\Gamma$ and denoted by $\alpha(\Gamma)$. Note that $\alpha(\Gamma) = \omega(\Gamma^c)$, where $\Gamma^c$ is the complement of $\Gamma$. So the following question may be posed as dual of Theorem 4.2.
Question 4.5. Let \( G \) be a group whose non-cyclic graph has no infinite independent set. Is it true that \( \alpha(\Gamma_G) \) is finite?

The answer of this question is negative, since for any prime \( p \), the non-cyclic graph of the direct sum \( \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_p \) has no infinite independent set and its independence number is not finite. More generally we have

**Proposition 4.6.** Let \( G \) be a non locally cyclic group.

1. The non-cyclic graph of \( G \) has no infinite independent set if and only if every abelian subgroup of \( G \) is a reduced torsion abelian group with finitely many primary components.
2. The independence number of \( G \) is finite if and only if the exponent of \( G \) is finite. In this case, \( \alpha(\Gamma_G) = \max\{|x| : x \in G\} - |\text{Cyc}(G)| \).

**Proof.** (1) Suppose that \( \Gamma_G \) has no infinite independent set. Then since \( G \) cannot have any infinite locally cyclic subgroup, it follows that every abelian subgroup of \( G \) is a reduced torsion abelian group with finitely many primary components. Now assume that every abelian subgroup of \( G \) is a reduced torsion abelian group with finitely many primary components and suppose, for a contradiction, that \( G \) has an infinite independent set \( X \). Then \( B = \langle X \rangle \), is an infinite torsion reduced abelian group with finitely many primary components. Since \( B \) has finitely many primary components, we may assume that elements of \( X \) lay in a \( p \)-primary component of \( B \), for some prime \( p \). Then \( B \) is a locally cyclic \( p \)-group, for if \( y_1, \ldots, y_n \in B \), then \( \langle y_1, \ldots, y_n \rangle = \langle x_1, \ldots, x_m \rangle \) for some \( x_1, \ldots, x_m \in X \) and since \( X \) is an independent set of \( p \)-elements, \( \langle x_i, x_j \rangle = \langle x_i \rangle \) or \( \langle x_j \rangle \). It follows that \( \langle y_1, \ldots, y_n \rangle = \langle x_\ell \rangle \) for some \( \ell \in \{1, \ldots, m\} \). Since \( X \) is infinite, \( B \) is an infinite locally cyclic \( p \)-group, and so \( B \) is divisible, a contradiction. This completes the proof of part (1).

(2) If \( \alpha(\Gamma_G) \) is finite, then the exponent of \( G \) will be finite and less than or equal to \( \alpha(\Gamma_G) \). Now suppose that the exponent of \( G \) is finite, \( e \) say. Let \( X \) be an independent set of \( G \). Then \( A = \langle X \rangle \) is an abelian group of exponent at most \( e \). Now it is easy to see that \( X \) lies in a cyclic subgroup of \( A \), so \( |X| \leq e \). This means that \( \alpha(\Gamma_G) \leq e \). The second part follows easily. \( \square \)

A *coloring partition* for a simple graph \( \Gamma \) is a partition of vertices of \( \Gamma \) whose members are independent sets of \( \Gamma \). For a positive integer \( k \), we say that a graph \( \Gamma \) is \( k \)-colorable if it has a coloring partition with \( k \) members. The least positive integer \( k \) (if exists) such that \( \Gamma \) is \( k \)-colorable, is called the *chromatic number* of \( \Gamma \) and we denote it by
the main theorem of [24], we have
\( \chi(\Gamma) \). It is clear that \( \omega(\Gamma) \leq \chi(\Gamma) \), for any finite graph \( \Gamma \). we prove that indeed for our non-cyclic graphs we always have “the equality”.

**Theorem 4.7.** Let \( G \) be a finite non-cyclic group. Then \( \omega(\Gamma_G) = \chi(\Gamma_G) = s \), where \( s \) is the number of maximal cyclic subgroups of \( G \). Moreover \( |G\_{Cyc(G)}| \leq \max\{(s - 1)^2(s - 3)!,(s - 2)^3(s - 3)!)\} \).

**Proof.** Let \( A_1, \ldots, A_s \) be all maximal cyclic subgroups of \( G \) and let \( A_i = \langle a_i \rangle \) for \( i \in \{1, \ldots, s\} \). Since \( \langle a_i, a_j \rangle \) is not cyclic for distinct \( i \) and \( j \), \( \{a_1, \ldots, a_s\} \) is a clique in \( \Gamma_G \). Thus \( s \leq \omega(\Gamma_G) \). On the other hand, since every element of \( G \) is contained in \( A_i \) for some \( i \), we have that \( G = \bigcup_{i=1}^{s} A_i \). If \( A_i \setminus Cyc(G) \) is not empty, then it is an independent set for \( \Gamma_G \), so it follows that \( \chi(\Gamma_G) \leq s \) and as we mentioned \( \omega(\Gamma_G) \leq \chi(\Gamma_G) \). This completes the proof of the first part. We have that \( G = \bigcup_{i=1}^{s} A_i \) is an irredundant covering for \( G \) (see [24] for definitions) and \( \bigcap_{i=1}^{s} A_i \) is clearly contained in \( Cyc(G) \). Thus by the main theorem of [24], we have \( |G\_{Cyc(G)}| \leq \max\{(s - 1)^2(s - 3)!,(s - 2)^3(s - 3)!)\} \). \( \square \)

5. Finite groups with regular non-cyclic graphs

In this section we prove the following.

**Theorem 5.1.** Let \( G \) be a non-cyclic finite group. Then the non-cyclic graph of \( G \) is regular if and only if \( G \) is isomorphic to one of the following groups:

1. \( Q_8 \times \mathbb{Z}_n \), where \( n \) is an odd integer and \( Q_8 \) is the quaternion group of order 8.
2. \( P \times \mathbb{Z}_m \), where \( P \) is a finite non-cyclic group of prime exponent \( p \) and \( m > 0 \) is an integer such that \( \gcd(m, p) = 1 \).

Throughout this section let \( G \) be a finite non-cyclic group whose non-cyclic graph is regular. Thus \( |G| - |Cyc_G(x)| = |G| - |Cyc_G(y)| \) for all \( x, y \in G \setminus Cyc(G) \), and so \( |Cyc_G(x)| = |Cyc_G(y)| \) for all \( x, y \in G \setminus Cyc(G) \). Note that, by Lemma 2.3, the non-cyclic graph of \( H = \frac{G}{Cyc(G)} \) is also regular and \( Cyc(H) = 1 \). Therefore we have \( |Cyc_G(x)| = |Cyc_H(x)| \) for any two non-trivial elements \( x, y \) of \( H \).

**Lemma 5.2.** For any two non-trivial elements \( x, y \in H \), either \( Cyc_H(x) \cap Cyc_H(y) = 1 \) or \( Cyc_H(x) = Cyc_H(y) \) and \( Cyc_H(x) \) is a subgroup of \( H \).

**Proof.** First note that if \( \langle m \rangle \) is a maximal cyclic subgroup of \( H \) such that \( x \in \langle m \rangle \), then \( Cyc_H(m) = \langle m \rangle \) and \( Cyc_H(x) \) contains \( \langle m \rangle \). Since all the cyclicizers of non-trivial elements have the same size, it follows
that \(Cyc_H(x) = \langle m \rangle\). Thus, for all \(1 \neq x \in H\), \(Cyc_H(x)\) is the unique maximal cyclic subgroup which contains \(x\).

Therefore, if \(1 \neq z \in Cyc_H(x) \cap Cyc_H(y)\), then \(z\) belongs to the unique maximal cyclic subgroup containing \(x\) and to the unique maximal cyclic subgroup containing \(y\). Since \(z\) is contained in a unique maximal cyclic subgroup, it follows that \(Cyc_H(x) = Cyc_H(y)\). □

In the proof of Theorem 5.1 we use the following result due to I.M. Isaacs [9] on equally partitioned groups.

**Theorem 5.3.** (see I.M. Isaacs [9]) Let \(A\) be a finite non-trivial group and let \(n > 1\) be an integer such that \(\{A_i \mid i = 1, \ldots, n\}\) is a set of subgroups of \(A\) with the property that \(A = \bigcup_{i=1}^{n} A_i\), \(|A_i| = |A_j|\) and \(A_i \cap A_j = 1\) for any two distinct \(i, j\). Then \(A\) is a group of prime exponent.

**Proof of Theorem 5.1** Since \(H = \bigcup_{x \in H \setminus \{1\}} Cyc_H(x)\), it follows from Lemma 5.2 and Theorem 5.3 that \(H\) is a finite group of exponent \(p\) for some prime \(p\). Assume that \(Cyc(G) = \langle x \rangle \times \langle y \rangle\), where \(x\) is a \(p'\)-element and \(y\) is a \(p\)-element. Since \(Cyc(G) \leq Z(G)\) and \(H\) is a \(p\)-group, \(G\) is nilpotent and \(G = P \times \langle x \rangle\), where \(P\) is the Sylow \(p\)-subgroup of \(G\). Note that \(y \in Cyc(P)\). If \(y = 1\), then \(P\) is a \(p\)-group of exponent \(p\) and if \(y \neq 1\), then \(P\) contains exactly one subgroup of order \(p\). Thus \(P\) is a generalized quaternion group or a cyclic group, by Proposition 2.2. But if \(P\) is a generalized quaternion group of order greater than 8, then the exponent of \(H\) is greater than 2 which is not possible. Thus \(P\) must be isomorphic to \(Q_8\), the quaternion group of order 8. This completes the proof of the “only if” part. The converse is straightforward. □

Let \(K\) be a group. There are cases in which \(Cyc_K(x)\) is a subgroup of \(K\) for every \(x \in K\); following [15], call such groups tidy. For example, if \(K\) is a group such that every non-identity element of \(K\) is of prime order then \(K\) is a tidy group. The next result shows that Theorem 5.1 cannot be improved in the following sense: If \(K\) is a finite non-cyclic group whose non-cyclic graph has two kind degrees, then \(K\) is not necessarily a tidy group, where for a given positive integer \(k\), we say that a simple graph has \(k\) kind degrees if the size of the set of degrees of vertices is \(k\). Note that a regular graph is a graph of one kind degree.

**Theorem 5.4.** Let \(p\) be a prime number, \(m \geq 1\) and \(n > 1\) be positive integers. Let \(G = \bigoplus_{i=1}^{n} Z_{p^{m}}\). Then \(Cyc(G) = \langle (0,0,\ldots,0) \rangle\) and if
$x \in G$ such that $|x| = p^\ell$ with $1 \leq \ell < m$, then

$$|Cyc_G(x)| = p^\ell + \sum_{i=\ell+1}^{m} (p^i - p^{i-1}) p^{(n-1)(m-i+\ell)},$$

and $|Cyc_G(y)| = p^m$ for every element $y$ of order $p^m$. It follows that $\Gamma_G$ is a graph having $m$ kind degrees and $G$ is not a tidy group.

Proof. Note that if $|y| = p^m$, then $Cyc_G(y) = \langle y \rangle$, since $|y|$ is equal to the exponent of $G$. Now suppose that $x'$ is another element of $G$ of order $p^\ell$, we first prove that $|Cyc_G(x)| = |Cyc_G(x')|$ and next, it is enough to prove that $|Cyc_G((p^{m-\ell}, 0, \ldots, 0))|$ is the number of the right hand side of $(*)$. There exist elements $z_1$ and $z'_1$ in $G$ of order $p^m$ such that $p^{m-\ell}z_1 = x$ and $p^{m-\ell}z'_1 = x'$ (we write $G$ additively). On the other hand, there exist elements $z_2, \ldots, z_n$ and $z'_2, \ldots, z'_n$ in $G$ such that the sets $\{z_1, z_2, \ldots, z_n\}$ and $\{z'_1, z'_2, \ldots, z'_n\}$ are both linearly independent in the sense of [LS pp. 95-96], and $G = \langle z_1, z_2, \ldots, z_n \rangle = \langle z'_1, z'_2, \ldots, z'_n \rangle$. Now define $\alpha: G \to G$ by

$$\alpha(\sum_{i=1}^{n} m_iz_i) = \sum_{i=1}^{n} m_iz'_i \text{ for all } m_i \in \{0, 1, \ldots, p^m - 1\}.$$ 

It is easy to see that $\alpha$ is an automorphism of $G$ with the property that $\alpha(x) = x'$. Thus $\alpha: Cyc_G(x) \to Cyc_G(x')$ is a bijection and so $|Cyc_G(x)| = |Cyc_G(x')|$. Now if

$$(a_1, a_2, \ldots, a_n) \in Cyc_G((p^{m-\ell}, 0, \ldots, 0)) \setminus \langle (p^{m-\ell}, 0, \ldots, 0) \rangle,$$

then $\langle (a_1, a_2, \ldots, a_n), (p^{m-\ell}, 0, \ldots, 0) \rangle$ is cyclic. It follows that $(a_1, a_2, \ldots, a_n)$ is of order $p^\ell$ for some $i > \ell$ and so

$$(p^{m-\ell}, 0, \ldots, 0) = p^{i-\ell}t(a_1, a_2, \ldots, a_n)$$

for some $t \in \{0, 1, \ldots, p^m - 1\}$ such that $\gcd(t, p) = 1$. Therefore $p^{i-\ell}ta_1 \equiv p^{m-\ell}$ and $p^{i-\ell}ta_j \equiv 0$ for each $1 < j \leq n$. Since $\gcd(t, p) = 1$, we have $p^{i-\ell}a_j \equiv 0$, so

$$a_j \in \{0, p^{i-\ell}, 2p^{i-\ell}, \ldots, (p^{m-i+\ell} - 1)p^{i-\ell}\} \text{ for each } j > 1.$$ 

Now let $a_1 = p^ks$ where $\gcd(p, s) = 1$ and $a_1 \in \{0, 1, \ldots, p^m - 1\}$. Thus $p^{i-\ell+k}st \equiv p^{m-\ell}$ from which it follows that $k = m - i$, since $\gcd(st, p) = 1$. Hence

$$a_1 \in \{p^{m-i}s \mid s \in \{1, \ldots, p^i - 1\} \text{ and } \gcd(s, p) = 1\}.$$
Therefore
\[ |Cyc_G((p^{m-\ell}, 0, \ldots, 0))| = |\langle (p^{m-\ell}, 0, \ldots, 0) \rangle| + \left| \{(a_1, a_2, \ldots, a_n) \in G : (a_1, a_2, \ldots, a_n) = p^i \text{ for some } \ell < i \leq m \text{ and } \right| \]
\[ (p^{m-\ell}, 0, \ldots, 0) \in \langle (a_1, a_2, \ldots, a_n) \rangle \right| \]
\[ = p^\ell + \sum_{i=\ell+1}^{m} (p^i - p^{i-1})p^{(n-1)(m-i+\ell)}. \]

This completes the proof. \( \square \)

6. Finite groups whose non-cyclic graphs have two kind degrees

Throughout this section let \( G \) be a finite non-cyclic group whose non-cyclic graph has two kind degrees and assume that \( H = \frac{G}{Cyc(G)} \). Then by Lemma 2.3 \( \Gamma_H \) has also two kind degrees.

**Theorem 6.1.** If \( G \) is nilpotent, then \( H \) is a \( p \)-group for some prime \( p \).

**Proof.** Suppose, for a contradiction, that \( |H| \) is divisible by at least two distinct prime numbers \( p_1 \) and \( p_2 \). Thus \( H = P_1 \times P_2 \times P_3 \), where \( P_3 \) is a Hall \( \{p_1, p_2\}'-\)subgroup of \( H \). If \( x \in P_i \), then \( |Cyc_H(x)| = |Cyc_{P_i}(x)|\Pi_{j \neq i}|P_j| \). Note that since \( Cyc(H) = 1 \) and
\[ Cyc(H) = Cyc(P_1) \times Cyc(P_2) \times Cyc(P_3), \]
\( Cyc(P_i) = 1 \) for every \( i \in \{1, 2, 3\} \). It follows that
\[ |Cyc_H(x_1)|, |Cyc_H(x_2)| \text{ and } |Cyc_H(x_3)| \]
are all distinct for every \( x_i \in P_i \\backslash \{1\} \). This implies that \( \Gamma_H \) has at least 3 kind degrees, if \( P_3 \neq 1 \). Therefore \( P_3 = 1 \). Now take \( x_i \in P_i \) such that \( |x_i| = exp(P_i) \) for \( i = 1, 2 \). Then \( Cyc_H(x_1x_2) = \langle x_1x_2 \rangle \), because \( exp(P_i) = |x_1x_2| \). Since \( Cyc(P_i) = 1 \), we have that \( |P_i| > exp(P_i) \) for \( i = 1, 2 \). Hence \( |Cyc_H(x_1)|, |Cyc_H(x_2)| \text{ and } |Cyc_H(x_1x_2)| \) are all distinct. This contradiction completes the proof. \( \square \)

**Theorem 6.2.** The group \( G \) is abelian if and only if \( G \cong \mathbb{Z}_m \bigoplus (\oplus_{i=1}^n \mathbb{Z}_{p^i}) \), where \( p \) is a prime number, \( n > 1 \) is an integer and \( m \) is a positive integer with \( \gcd(m, p) = 1 \).

**Proof.** By Theorem 6.1 \( H \) is an abelian \( p \)-group for some prime \( p \). Thus \( H \cong M = \bigoplus_{i=1}^{\alpha_1} (\oplus_{j=1}^{t_i} \mathbb{Z}_{p^j}) \) for some positive integers \( t_i, \ell \) and \( \alpha_1 < \cdots < \alpha_\ell \). If we prove \( \ell = 1 \), then Theorem 5.4 implies that \( \alpha_1 = 2 \), and since \( G \) is abelian, by Remark 3.3 we have that \( G \cong Cyc(G) \bigoplus H \).
Therefore it is enough to show that $\ell = 1$. Suppose, for a contradiction, that $\ell > 1$. Assume first $\ell > 2$ and take $x_i$ to be a generator of a direct summand $\mathbb{Z}_{p^{\alpha_i}}$ for $i = 1, 2, 3$. Then $\text{Cyc}_M(x_i) = \langle x_i \rangle$ for $i = 1, 2, 3$. Since $\alpha_1 < \alpha_2 < \alpha_3$, we have $|x_1| < |x_2| < |x_3|$, which gives a contradiction. Thus we may assume that $\ell = 2$. Hence

$$H \cong M = \left( \oplus_{j=1}^{t_1} \mathbb{Z}_{p^{\alpha_1}} \right) \bigoplus \left( \oplus_{j=1}^{t_2} \mathbb{Z}_{p^{\alpha_2}} \right).$$

Now take $x_i$ to be a generator of a direct summand $\mathbb{Z}_{p^{\alpha_i}}$ for $i = 1, 2$. Then, since $\alpha_1 < \alpha_2$, $\langle x_2 \rangle \cup \langle x_1, x_2 \rangle \subseteq \text{Cyc}_M(x_2^{\alpha_2})$. Since $\langle x_2 \rangle \cap \langle x_1, x_2 \rangle = \langle x_2^{\alpha_2} \rangle$, we have $|\text{Cyc}_M(x_2^{\alpha_2})| \geq 2p^{\alpha_2} - p^{\alpha_1}$. Hence $|\text{Cyc}_M(x_2^{\alpha_1})|$, $|\text{Cyc}_M(x_1)|$ and $|\text{Cyc}_M(x_2)|$ are all distinct, a contradiction. Therefore $\ell = 1$.

The converse follows from Theorem 5.4, Lemma 2.3 and Remark 3.3.

\[ \square \]

7. Groups with the same non-cyclic graph

In this section the following question is of interest. What is the relation between two non locally cyclic groups $G$ and $H$ if $\Gamma_G \cong \Gamma_H$?

More precisely one may propose the following question:

**Question 7.1.** For which group property $\mathcal{P}$ if $G$ and $H$ are two non locally cyclic groups such that $\Gamma_G \cong \Gamma_H$, and $G$ has the group property $\mathcal{P}$, then $H$ has also $\mathcal{P}$?

First we consider Question 7.1 when $\mathcal{P}$ is the property of being finite. In this case Question 7.1 has positive answer, as we see

**Proposition 7.2.** Let $G$ be a finite non-cyclic group such that $\Gamma_G \cong \Gamma_H$ for some group $H$. Then $H$ is a finite non-cyclic group. Moreover $|\text{Cyc}(H)|$ divides

$$\gcd \left( |G| - |\text{Cyc}(G)|, |G| - |\text{Cyc}(G)| : g \in G \setminus \text{Cyc}(G) \right).$$

**Proof.** By the hypothesis $|H \setminus \text{Cyc}(H)| = |G| - |\text{Cyc}(G)| \neq 0$. As $\text{Cyc}(H) \leq H$ and $H \setminus \text{Cyc}(H)$ is a finite set, we have that $H$ is finite and since $H \setminus \text{Cyc}(H)$ is non-empty, $H$ is non-cyclic, as required. Since $\Gamma_G \cong \Gamma_H$, we have

$$\{\deg(v), |V(\Gamma_G)| : v \in V(\Gamma_G)\} = \{\deg(w), |V(\Gamma_H)| : w \in V(\Gamma_H)\}.$$

But $\deg(w) = |H| - |\text{Cyc}_H(w)|$ for every $w \in V(\Gamma_H)$ and so it follows from Lemma 2.3 that $|\text{Cyc}(H)|$ divides $\deg(w)$ for every $w \in V(\Gamma_H)$ and clearly $|\text{Cyc}(H)|$ divides $|H| - |\text{Cyc}(H)| = |V(\Gamma_H)|$. This completes the proof. \[ \square \]
When $\mathcal{P}$ is the property of being nilpotent we have no negative answer to Question 7.1; however under an additional condition we give a positive answer. In fact we have

**Theorem 7.3.** Let $G$ be a finite non-cyclic nilpotent group. If $H$ is a group such that $|\text{Cyc}(G)| = |\text{Cyc}(H)| = 1$ and $\Gamma_G \cong \Gamma_H$, then $H$ is a finite nilpotent group. Furthermore, for every prime number $p$, if $P$ and $Q$ are Sylow $p$-subgroups of $G$ and $H$, respectively, then $\Gamma_P \cong \Gamma_Q$.

**Proof.** First note that by Proposition 7.2, $H$ is finite. Let $p$ and $q$ be two distinct prime divisors of $|H|$. It is enough to show that every $p$-element $x$ of $H$ commutes with every $q$-element $y$ of $H$, or equivalently $\langle x, y \rangle$ is cyclic. Since for every prime divisor $r$ of $|H|$, each $r$-element of $H$ is contained in a cyclic $r$-subgroup of $H$ which is maximal among cyclic $r$-subgroups of $H$, we may assume that $\langle x \rangle$ and $\langle y \rangle$ are maximal cyclic $p$-subgroup ($q$-subgroup, respectively) of $H$. Now let $\phi$ be a graph isomorphism from $\Gamma_H$ to $\Gamma_G$ and note that, since $|\text{Cyc}(G)| = |\text{Cyc}(H)| = 1$, we may consider $\phi$ as a bijection from $H$ onto $G$, by defining $\phi(1_G) = 1_H$. Now if we show that $\phi(x)$ and $\phi(y)$ are $p$-element and $q$-element in $G$, respectively, then as $G$ is nilpotent, $\phi(x)$ and $\phi(y)$ are not adjacent, and so $x$ and $y$ are not adjacent, as required. So finally, by the symmetry between $x$ and $y$, it is enough to show that $\phi(x)$ is a $p$-element of $G$.

First we show that, for any $p$-element of $a$ of $G$, $\psi(a)$ is a $p$-element of $H$, where $\psi$ is the inverse of $\phi$. Let $D = \text{Cyc}_G(a)$. Then as $G$ is the direct product of its Sylow subgroups,

$$D = \text{Cyc}_P(a) \times \prod_{\ell \neq \ell_1} P_\ell,$$

where $P_\ell$ is the Sylow $p_\ell$-subgroup of $G$ and $p = p_i$. Now since $\text{Cyc}(G) = 1$, $\text{Cyc}(P_\ell) = 1$ for every $\ell \in \{1, \ldots, k\}$. It follows that $\text{Cyc}_D(D) = \text{Cyc}_C(C)$, where $C = \text{Cyc}_P(a)$. Now Lemma 2.1 shows that $\text{Cyc}_D(D)$ is a cyclic $p$-subgroup of $G$. Since $\psi$ is a graph isomorphism from $\Gamma_G$ to $\Gamma_H$, we have that $\psi(C \text{yc}_D(D)) = \psi(C \text{yc}_D(D'))$, where $D' = \text{Cyc}_H(\psi(a))$. Thus $|\text{Cyc}_D(D')|$ is a $p$-power number and so, by Lemma 2.1, $\psi(a)$ is a $p$-element of $H$.

Now suppose that $\phi(x) = x_1 x_2 \cdots x_k$, where $x_\ell$ is a $p_\ell$-element of $G$. Let $M = \text{Cyc}_G(\phi(x))$. Then $M = \prod_{\ell=1}^k \text{Cyc}_{P_\ell}(x_\ell)$. Now we prove that

$$\text{Cyc}_{P_\ell}(x_\ell) = \phi(\langle x \rangle). \quad (*)$$
Let $z \in Cyc_P(x_i)$. Then $\langle z, \phi(x) \rangle$ is cyclic and so $A = \langle \psi(z), x \rangle$ is cyclic. By the previous part $\psi(z)$ is a $p$-element and so $A$ is a cyclic $p$-subgroup of $H$ containing $x$. Now as $\langle x \rangle$ is a maximal cyclic $p$-subgroup of $H$, $\psi(z) \in \langle x \rangle$ and so $z \in \phi(\langle x \rangle)$. On the other hand, by Lemma 2.1, $|x|$ divides $|Cyc_\psi(M)| = |Cyc(M)| = \prod_{i=1}^{k}|Cyc_{D_i}(D_i)|$, where $D_\ell = Cyc_P(x_\ell)$ for every $\ell \in\{1, \ldots, k\}$. Now since $|x|$ is a $p$-power number, Lemma 2.1 implies that $|x|$ must divide $|Cyc_{D_i}(D_i)| \leq |D_i|$. Therefore $|x| \leq |Cyc_P(x_i)|$ and also we have $Cyc_P(x_i) \subseteq \phi(\langle x \rangle)$. Now since $|\phi(\langle x \rangle)| = |x|$, we have proved $(\ast)$. This implies that $\phi(x) \in Cyc_P(x_i) \subseteq P_i$ and so $\phi(x)$ is a $p$-element. This completes the proof of the first part.

By the proof of the first part, $H$ and $G$ are both finite nilpotent of the same size and

$$|Cyc(P_i)| = |Cyc(Q_\ell)| = 1 \text{ and } |P_\ell| = |Q_\ell|,$$

where $Q_\ell$ is the Sylow $p_\ell$-subgroup of $H$. By the proof of the previous part we have that, $\psi(P_\ell)$ is the Sylow $p_\ell$-subgroup of $H$, for we have proved that every $p_\ell$-element of $G$ maps under $\psi$ to a $p_\ell$-element of $H$, and since $H$ is now nilpotent, every $p_\ell$-element of $H$ maps under $\phi$ to a $p_\ell$-element of $G$. Hence $\phi$ induces a graph isomorphism from $\Gamma_{P_\ell}$ to the non-cyclic graph of $Q_\ell$, the Sylow $p_\ell$-subgroup of $H$. This completes the proof.

Now let us consider Question 7.1 when $P$ is the property of being a fixed finite size, that is, in view of Proposition 7.2, we are asking the following question:

**Question 7.4.** Let $G$ and $H$ be two non-cyclic finite groups such that $\Gamma_G \cong \Gamma_H$. Is it true that $|G| = |H|$?

We were unable to find a negative answer for Question 7.4. In the following we give certain groups $G$ for which Question 7.4 has positive answer. First let us examine some cases in which we can use Theorem 5.1.

**Proposition 7.5.** Let $n > 0$ be an odd integer and let $G$ be a group such that its non-cyclic graph is isomorphic to the non-cyclic graph of $H = Q_8 \times \mathbb{Z}_n$. Then $G \cong Q_8 \times \mathbb{Z}_n$.

**Proof.** We have

$$|H| - |Cyc(H)| = |G| - |Cyc(G)|. \quad (I)$$
By hypothesis $\Gamma_G$ is also regular and so by Theorem 5.1, $G$ is isomorphic to $G_1 = Q_8 \times \mathbb{Z}_{n'}$ for some odd integer $n'$ or $G_2 = P \times \mathbb{Z}_m$, where $P$ is a finite non-cyclic group of exponent $p$ for some prime $p$ and $m > 0$ is an integer such that
\[
\gcd(p, m) = 1.
\] (II)

If $G \cong G_1$, then $|\text{Cyc}(G)| = 2n'$ and by (I) we have $8n - 2n = 8n' - 2n'$ and so $n = n'$, thus $G \cong Q_8 \times \mathbb{Z}_n$. Now suppose, for a contradiction, that $G \cong G_2$. Then $|\text{Cyc}(G)| = m$ and $|\text{Cyc}_G(x)| = pm$ for all $x \in G \setminus \text{Cyc}(G)$. Therefore by (I) we have
\[
6n = |P|m - m
\] (1)
and by the hypothesis there exists an element $a \in H \setminus \text{Cyc}(H)$ such that $|\text{Cyc}_H(a)| - |\text{Cyc}(H)| = |\text{Cyc}_G(x)| - |\text{Cyc}(G)|$. Therefore
\[
4n - 2n = pm - m.
\] (2)

Now it follows from (1) and (2) that
\[
\frac{p^m - 1}{p - 1} = \frac{q^s - 1}{q - 1} \quad \text{and} \quad n(p - 1) = t(q - 1).
\] (G)

**Proof.** It is easy to see that $\Gamma_G$ ($\Gamma_H$, respectively) is a complete $(\frac{p^m - 1}{p - 1})$-partite graph ($(\frac{q^s - 1}{q - 1})$-partite graph, respectively) whose parts have equal size $(p - 1)n$ ($(q - 1)t$, respectively). These completes the proof. \qed

**Proposition 7.6.** Let $p$ and $q$ be two prime numbers and $m$ and $t$ be positive integers with $\gcd(p, m) = \gcd(q, t) = 1$. Suppose that $G = P \times \mathbb{Z}_n$ and $H = Q \times \mathbb{Z}_t$, where $P$ and $Q$ are finite non-cyclic groups of exponents $p$ and $q$, respectively and $|P| = p^m$ and $|Q| = q^s$ for some integers $m > 1$ and $s > 1$. Then $\Gamma_G \cong \Gamma_H$ if and only if
\[
\frac{p^m - 1}{p - 1} = \frac{q^s - 1}{q - 1} \quad \text{and} \quad n(p - 1) = t(q - 1).
\]

**Proof.** It is easy to see that $\Gamma_G$ ($\Gamma_H$, respectively) is a complete $(\frac{p^m - 1}{p - 1})$-partite graph ($(\frac{q^s - 1}{q - 1})$-partite graph, respectively) whose parts have equal size $(p - 1)n$ ($(q - 1)t$, respectively). These completes the proof. \qed

**Proposition 7.7.** Let $G$ be a group. Then for a positive odd integer $n$ and an integer $m > 1$, $\Gamma_G \cong \Gamma_H$ if and only if $G \cong (\oplus_{i=1}^m \mathbb{Z}_2) \bigoplus \mathbb{Z}_n$.

**Proof.** Since the non-cyclic graph of $(\oplus_{i=1}^m \mathbb{Z}_2) \bigoplus \mathbb{Z}_n$ is regular, $\Gamma_G$ is also regular. Now Theorem 5.1 and Proposition 7.5 imply that $G \cong Q \times \mathbb{Z}_t$, where $Q$ is a finite non-cyclic group of exponent $q$ and of order $q^s$, for some prime number $q$ and integer $s > 1$, and $t > 0$ is an integer such that $\gcd(q, t) = 1$. Now Proposition 7.6 implies that
\[
2^m - 1 = \frac{q^s - 1}{q - 1} \quad \text{and} \quad n = t(q - 1).
\]
Since \( n \) is odd, the equality \( n = t(q - 1) \) yields that \( q \) cannot be odd and so \( q = 2 \). Therefore \( n = t, \ p = q = 2 \) and \( m = s \). Since the exponent of \( Q \) is 2 and \(|Q| = 2^s = 2^m\), \( Q \) is isomorphic to \( \oplus_{i=1}^m \mathbb{Z}_2 \). This completes the proof of the “only if” part.

The converse follows from Proposition 7.6. \( \square \)

**Proposition 7.8.** Let \( G \) be a group. Then for a prime number \( p \) and an integer \( n > 0 \) such that \( \gcd(n, p) = 1 \), \( \Gamma_G \cong \Gamma_{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_n} \) if and only if \( G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_n \).

**Proof.** Since the non-cyclic graph of \( \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_n \) is regular, so is \( \Gamma_G \).

Now Theorem 5.1 and Proposition 7.5 imply that \( G \cong Q \times \mathbb{Z}_t \), where \( Q \) is a finite group of exponent \( q \) and of order \( q^s \), for some prime number \( q \) and integer \( s > 1 \), and \( t > 0 \) is an integer such that \( \gcd(q, t) = 1 \).

Now Proposition 7.6 implies that
\[
\frac{p^2 - 1}{p - 1} = \frac{q^s - 1}{q - 1} \tag{1}
\]
and \( n(p - 1) = t(q - 1) \). From (1) it follows that \( p = q(q^{s-2} + \cdots + 1) \) which gives \( p = q \) and \( s = 2 \), since \( p \) and \( q \) are prime numbers. Therefore \( n = t \) and this completes the proof of “only if” part.

The converse follows from Proposition 7.6. \( \square \)

**Remark 7.9.** It was conjectured by Goormaghtigh [7] that the Diophantine equation
\[
\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \tag{GO}
\]
for \( x, y, m, n \in \mathbb{N}, x > y > 1 \) and \( n > m > 2 \) has only the solutions \((x, y, m, n) = (5, 2, 3, 5)\) and \((90, 2, 3, 13)\). This conjecture has not been solved so far. By Proposition 7.6 Question 7.4 is true for finite groups with regular non-cyclic graphs, if the equation \((G)\) which is a certain case of Goormaghtigh’s conjecture has no solution. This shows that how “hard” it is to settle Question 7.4 for certain groups, such as elementary abelian groups. In the following we use a result of M. Le [11] on this conjecture to prove another certain case of Question 7.4.

**Theorem 7.10.** (See M. Le [11] Theorem)

If \((x, y, 3, n) \notin \{(5, 2, 3, 5), (90, 2, 3, 13)\}\) is a solution of the equation \((GO)\) with \( m = 3 \), then we have \( \gcd(x, y) > 1 \) and \( y \nmid x \).

**Proposition 7.11.** Let \( G \) be a group. Then for a prime number \( p > 2 \) and an integer \( n > 0 \) such that \( \gcd(n, p) = 1 \), \( \Gamma_G \cong \Gamma_{\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_n} \) if and only if \( G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_n \) or \( T \times \mathbb{Z}_n \), where \( T \) is the only non-abelian group of order \( p^3 \) and exponent \( p \).
Proof. Note that $\Gamma_G$ is regular. Now Theorem 5.1 and Proposition 7.5 imply that $G \cong Q \times \mathbb{Z}_t$, where $Q$ is a finite group of exponent $q$ and of order $q^s$, for some prime number $q$ and integer $s > 1$, and $t > 0$ is an integer such that $\gcd(q,t) = 1$. It follows from Proposition 7.6 that

$$\frac{p^3 - 1}{p - 1} = \frac{q^s - 1}{q - 1}$$

and $n(p - 1) = t(q - 1)$. Now Theorem 7.10 implies that $p = q$ and $s = 3$. Therefore $n = t$ and $Q$ is a finite group of exponent $p$ and of order $p^3$. This completes the proof of the “only if” part. The converse follows from Proposition 7.6. \qed

Remark 7.12. (1) Proposition 7.11 shows that Question 7.1 is not true for the group property of being abelian.
(2) Proposition 7.11 shows that Question 7.1 is not true for the group property of being isomorphic to a fixed group and by Proposition 7.6 one can find further examples of non-isomorphic groups whose non-cyclic graphs are isomorphic.

We end this section with the following proposition which may be considered as an starting point toward settling Question 7.4 for abelian $p$-groups with non-regular non-cyclic graphs.

Proposition 7.13. Let $p$ be a prime number and $G$ be a finite non-cyclic $p$-group of order $p^n$ for some $n \geq 3$ and let $H$ be a group such that $\Gamma_G \cong \Gamma_H$. Suppose that $x$ is an element in $G$ such that $\text{Cyc}_G(x) = \langle x \rangle$.

(1) If $\text{Cyc}(G) \neq 1$, then both $G$ and $H$ are isomorphic to $Q_{2^n}$, the generalized quaternion group of order $2^n$.
(2) If $|x| = p = 2$, then $|G| = |H|$. This is the case, if $G$ has a direct factor of order 2.
(3) If $|x| \in \{ p^{n-1}, p^{n-2} \}$, then $|G| = |H|$. This is the case, if $G$ is of exponent $p^{n-1}$ or $p^{n-2}$, or $G$ has a cyclic direct factor of order $p^{n-2}$.

Proof. (1) By Proposition 2.2, $G \cong Q_{2^n}$. Thus $\Gamma_H \cong \Gamma_{Q_{2^n}}$. We have $2^n - 2 = |H| - |\text{Cyc}(H)|$, since $|\text{Cyc}(Q_{2^n})| = 2$. As the exponent of $Q_{2^n}$ is $2^{n-1}$, there exists an element $a \in Q_{2^n}$ such that $D = \text{Cyc}_{Q_{2^n}}(a) = \langle a \rangle$. It follows that $\text{Cyc}_D(D) = D$. If $\phi : \Gamma_{Q_{2^n}} \to \Gamma_H$ is a graph isomorphism, then

$$|C \setminus \text{Cyc}_C(C)| = |D \setminus \text{Cyc}_D(D)|,$$

where $C = \text{Cyc}_H(\phi(a))$. Hence it follows from Lemma 2.1 that there exists an element $b \in H \setminus \text{Cyc}(H)$ such that $\text{Cyc}_H(b)$ is a cyclic subgroup of $H$ and $2^n - 2^{n-1} = |H| - |\text{Cyc}_H(b)|$. Now Proposition 7.2
implies that $|Cyc(H)|$ must divide $\gcd(2^{n-1}, 2^n - 2) = 2$. Therefore $|Cyc(H)| \in \{1, 2\}$. If $|Cyc(H)| = 1$, then $|H| = 2^n - 1$ and $|Cyc_H(b)| = 2^{n-1} - 1$. Since $Cyc_H(b) \leq H$, we have that $2^{n-1} - 1$ divides $2^n - 1$, which is impossible, since $n \geq 3$. Thus $|Cyc(H)| = 2$ and so $|H| = 2^n$. Now Proposition 2.2 completes the proof of part (1).

(2) By part (1), we may assume that $Cyc(G) = 1$. Then $|H| - |Cyc(H)| = 2^n - 1$ and $|H| - |Cyc_H(h)| = 2^n - 2$ for some vertex $h$ in $\Gamma_H$. Thus $|Cyc(H)|$ divides $\gcd(2^n - 1, 2^n - 2) = 1$ and so $|H| = |G|$.

(3) By part (1), we may assume that $Cyc(G) = 1$. If $\phi : \Gamma_G \to \Gamma_H$ is a graph isomorphism, then

$$|D_1 \setminus Cyc_{D_1}(D_1)| = |D_2 \setminus Cyc_{D_2}(D_2)|,$$

where $D_1 = Cyc_G(a)$, $D_2 = Cyc_H(\phi(a))$ and $a$ is any vertex of $\Gamma_G$. Assume that $|x| = p^i$. It follows from Lemma 2.1 that there exists a vertex $s$ in $\Gamma_G$ such that $Cyc_H(s)$ is a cyclic subgroup of $H$,

$$|x| - |Cyc_G(x)| = |H| - |Cyc_H(s)| \quad \text{and} \quad |G| - |Cyc(G)| = |H| - |Cyc(H)|.$$

Therefore $|H| - |Cyc(H)| = p^n - 1$, $|H| - |Cyc_H(s)| = p^n - p^i$. It follows that $|Cyc(H)|$ divides $\gcd(p^i - 1, p^n - 1) = p^d - 1$, where $d = \gcd(i, n)$. Thus there exists an integer $\ell$ such that

$$p^d - 1 = |Cyc(H)| \ell. \quad (*)$$

Since $\frac{|Cyc_H(s)|}{|Cyc(H)|} = \frac{p^{d-1}}{p^d - 1} \ell + 1$, $\frac{|H|}{|Cyc(H)|} = \frac{p^n - 1}{p^d - 1} \ell + 1$ and $Cyc_H(s)$ is a subgroup of $H$, we have that $\frac{p^{d-1}}{p^d - 1} \ell + 1$ divides $\frac{p^n - 1}{p^d - 1} \ell + 1$. It follows that $\frac{p^{d-1}}{p^d - 1} \ell + 1$ divides $\frac{p^n - 1}{p^d - 1} \ell + 1 - \frac{p^i}{p^d - 1} \ell - 1$, so $\frac{p^{d-1}}{p^d - 1} \ell + 1$ divides $\frac{p^i p^{d-1}}{p^d - 1} \ell$. Now since $\gcd\left(\frac{p^{d-1}}{p^d - 1} \ell + 1, \ell\right) = 1$, we have $\frac{p^{d-1}}{p^d - 1} \ell + 1$ divides $p^d \frac{p^{d-1}}{p^d - 1} \ell$. We must prove that $Cyc(H) = 1$ and so by $(*)$, it is enough to prove $\ell = p^d - 1$.

If $i = n - 1$, then $d = 1$ and so we have that $\frac{p^{n-1}}{p-1} \ell + 1$ divides $p^{n-1}$. This implies that $\ell = p - 1$, as required.

If $i = n - 2$, then $d \in \{1, 2\}$. If $d = 1$, then $\frac{p^{n-2}}{p-1} \ell + 1$ divides $p^{n-2}(p + 1)$. Now suppose, for a contradiction, that $\ell < p - 1$. It follows that $\gcd\left(\frac{p^{n-2}}{p-1} \ell + 1, p\right) = 1$. Therefore $\frac{p^{n-2}}{p-1} \ell + 1$ divides $p + 1$, which implies that $n - 2 < 1$, contrary to $n \geq 3$. Hence we have that $\ell = p - 1$, as required. Now assume that $d = 2$. Therefore $n$ is even and we have $\frac{p^{n-2}}{p-1} \ell + 1$ divides $p^{n-2}$. Since $p^2 \leq \frac{p^{n-2}}{p-1} \ell + 1$ and
3 ≤ n is even, we have that \( p^2 \) must divide \( \frac{p^{n-2}-1}{p^2-1} \ell + 1 \) which implies that \( p^2 \mid \ell + 1 \). Hence \( \ell + 1 = p^2 \), as required.

\[ \square \]

8. Groups whose non-cyclic graphs are unique

In this section we study Question 7.1 for the group property of being isomorphic to a fixed group. As we mentioned in Remark 7.12(2), in general, Question 7.1 is not true for this property. But we saw in Section 7 that there are groups \( G \) whose non-cyclic graphs are “unique”, that is, if \( \Gamma_G \cong \Gamma_H \) for some group \( H \), then \( G \cong H \). Here we give some other groups whose non-cyclic graphs are unique. Before this, we need to state some notations.

For any finite group \( G \), we denote by \( \pi_e(G) \) the set of orders of elements of \( G \). There is a uniquely characterized subset \( \mu(G) \) of \( \pi_e(G) \) with the following properties:

1. For every element \( x \) of \( G \), there exists \( t \in \mu(G) \) such that \( |x| \) divides \( t \).
2. If \( t, s \) are two distinct elements of \( \mu(G) \), then \( t \) does not divide \( s \) and vice versa.

It is clear that

\[ \pi_e(G) = \{ s \in \mathbb{N} \mid s \text{ divides some member of } \mu(G) \}. \]

In fact \( \mu(G) \) is the set of elements of \( \pi_e(G) \) which are maximal under the divisibility relation.

**Lemma 8.1.** If \( G \) is a non-cyclic finite group, then

\[ \mu(G) \cap \pi_e(Cyc(G)) = \emptyset. \]

**Proof.** Suppose, for a contradiction, that \( n \in \mu(G) \cap \pi_e(Cyc(G)) \). It follows that there exists an element \( g \in Cyc(G) \) such that \( |g| = n \). Since \( G \) is not cyclic, there exists an element \( x \in G \setminus Cyc(G) \). Thus \( \langle x, g \rangle = \langle y \rangle \) for some \( y \in G \) such that \( |y| > n \). Now by the definition of \( \mu(G) \), there is \( m \in \mu(G) \) such that \( |y| \) divides \( m \). Therefore \( n \mid m \) and since \( n, m \in \mu(G) \), we have that \( n = m \). Hence \( |y| = n \), a contradiction. This completes the proof. \( \square \)

**Lemma 8.2.** Let \( g \) be an element of a finite group \( G \) such that \( |g| \in \mu(G) \). Then \( Cyc_G(g) = \langle g \rangle \).

**Proof.** Suppose, for a contradiction, that \( x \in Cyc_G(g) \setminus \langle g \rangle \). Then

\[ \langle g \rangle \not\subseteq \langle g, x \rangle = \langle y \rangle \]
for some $y \in G$. It follows that $|g| < |y|$. On the other hand, by the definition of the set $\mu(G)$, there exists $s \in \mu(G)$ such that $|y|$ divides $s$. Therefore $|g|$ divides $s$, so $|g| = s$. It follows that $|g| = |y|$, a contradiction. This completes the proof. □

**Theorem 8.3.** Let $G$ and $H$ be two finite non-cyclic groups such that $\Gamma_G \cong \Gamma_H$. If $|G| = |H|$, then $\pi_e(G) = \pi_e(H)$.

**Proof.** Let $t \in \mu(G)$. Then by Lemma 8.1 there exists an element $g \in G\setminus \text{Cyc}(G)$ such that $|g| = t$, and by Lemma 8.2 $\text{Cyc}_G(g) = \langle g \rangle$. Since $\Gamma_G \cong \Gamma_H$, there exists an element $h \in H\setminus \text{Cyc}(H)$ such that $\text{Cyc}_D(D) = D$ where $D = \text{Cyc}_H(h)$. Now Lemma 2.1 implies that $\text{Cyc}_H(h)$ is a cyclic subgroup. Also since $\Gamma_G \cong \Gamma_H$ and $|G| = |H|$, we have that $|\text{Cyc}_G(g)| = |\text{Cyc}_H(h)|$ from which it follows that $H$ has an element of order $t$. Hence we have proved that $\mu(G) \subseteq \pi_e(H)$ which implies that $\pi_e(G) \subseteq \pi_e(H)$. By the symmetry between $G$ and $H$, we have that $\pi_e(H) \subseteq \pi_e(G)$, completing the proof. □

Now we give some groups with unique non-cyclic graphs. For an integer $n > 2$, we denote by $D_{2n}$ the dihedral group of order $2n$.

**Proposition 8.4.** Let $n > 2$ be an integer. If $G$ is a group with $\Gamma_G \cong \Gamma_{D_{2n}}$, then $|G| = 2n$ and $G$ contains a cyclic subgroup of order $n$. In particular, if $n$ is odd, then $G \cong D_{2n}$.

**Proof.** We have $2n - 1 = |G| - |\text{Cyc}(G)|$, since $\text{Cyc}(D_{2n}) = 1$. The dihedral group $D_{2n}$ has an element $a$ of order $n$, such that $D = \text{Cyc}_{D_{2n}}(a) = \langle a \rangle$. It follows that $\text{Cyc}_D(D) = D$. It follows from the hypothesis and Lemma 2.1 that there exists an element $x \in G\setminus \text{Cyc}(G)$ such that $\text{Cyc}_G(x)$ is a cyclic subgroup of $G$ and $2n - n = |G| - |\text{Cyc}_G(x)|$. Now by Proposition 7.2 that $|\text{Cyc}(G)|$ divides $\gcd(2n - 1, n) = 1$. Hence $|G| = 2n$ and so by Theorem 8.3 $G$ contains an element of order $n$. Now assume that $n$ is odd. By Theorem 8.3 $\mu(G) = \{n, 2\}$. Since $|G| = 2n$ and $n$ is odd, it follows that $\text{Cyc}_G(b) = \langle b \rangle$ for every involution $b \in G$. Let $A$ be a cyclic subgroup of order $n$ in $G$ which is clearly normal in $G$. Thus $G = A \langle b \rangle$ for every involution $b \in G$. On the other hand, since $C_G(b) = \langle b \rangle$, $b$ acts fixed point freely on $A$, thus $c^b = c^{-1}$ for all $c \in A$. This implies that $G \cong D_{2n}$, as required. □

Thus the non-cyclic graphs of generalized quaternion groups are unique. What about the other finite $p$-groups with a maximal cyclic subgroup? We know the complete classification of these groups (see [18, Theorem 5.3.4]) and in the following we show that their non-cyclic graphs are not unique, in general, but Question 7.1 has positive answer
Proposition 8.5. Let \( p \) be a prime number and \( n \geq 3, m \geq 4 \) be integers. Suppose that \( G(p^n) = \left\{ a, x \mid x^p = a^{p^{n-1}} = 1, a^x = a^{1+p^{n-2}} \right\} \), \( H = \left\{ a, x \mid x^2 = a^{2m-1} = 1, a^x = a^{2m-2-1} \right\} \) and \( K(p^n) = \mathbb{Z}_{p^{n-1}} \oplus \mathbb{Z}_p \).

(1) If \( S \) is a group such that \( \Gamma_S \cong \Gamma_{K(8)} \), then \( S \cong K(8) \).
(2) If either \( n > 3 \) or \( p > 2 \), then \( \Gamma_{G(p^n)} \cong \Gamma_{K(p^n)} \).
(3) If \( S \) is a group such that \( \Gamma_S \cong \Gamma_H \), then \( S \cong H \).
(4) If \( S \) is a group such that \( \Gamma_S \cong \Gamma_{D_{2p}} \), then \( S \cong D_{2p} \).
(5) Let \( S \) be a group. Then \( \Gamma_S \cong \Gamma_{K(p^n)} \) if and only if \( S \cong G(p^n) \) or \( K(p^n) \).

Proof. (1) Since \( Cyc_{K(8)}((1,0)) = \langle (1,0) \rangle \) and \( Cyc_{K(8)}((0,1)) = \langle (0,1) \rangle \), it follows from Proposition 7.13 that \( |S| = 8 \). Now Propositions 7.13 and 7.11 imply that \( S \cong D_8 \) or \( S \cong K(8) \). But \( \Gamma_{D_8} \not\cong \Gamma_{K(8)} \), since (2, 0) has degree 2 in \( \Gamma_{K(8)} \) while every vertex in \( \Gamma_{D_8} \) has degree equal to 6 or 4. This completes the proof of part (1).
(2) It is easy to see that if \( \gcd(j, p) = 1 \), then \( Cyc_{G(p^n)}(a^i x^j) = \langle a^i x^j \rangle \), and if \( p^{n-1} \nmid i \), then
\[
Cyc_{G(p^n)}(a^i) = \{ bc \mid b \in \langle a \rangle, |b| > |a^i|, c \in \langle x \rangle \setminus \{1\} \} \cup \langle a \rangle.
\]

Also if \( \gcd(j, p) = 1 \), then \( Cyc_{K(p^n)}((i, j)) = \langle (i, j) \rangle \), and if \( p^{n-1} \nmid i \), then
\[
Cyc_{K(p^n)}((i, 0)) = \{ (b, c) \mid b \in \mathbb{Z}_{p^{n-1}}, |b| > |i|, c \in \mathbb{Z}_p \setminus \{0\} \} \cup \langle (1, 0) \rangle.
\]

Now using these information, it is easy to see that \( \phi : V(\Gamma_{G(p^n)}) \to V(\Gamma_{K(p^n)}) \) defined by \( \phi(a^i x^j) = (i, j) \) is a graph isomorphism from \( \Gamma_{G(p^n)} \) to \( \Gamma_{K(p^n)} \).
(3) We have that
\[
Cyc_H(a^i x^j) = \begin{cases} 
\langle a^i x \rangle & \text{if } j = 1 \\
\langle a \rangle & \text{if } j = 0 \text{ and } 2m-2 \nmid i \\
\{ a^i x : 2 \nmid j \} \cup \langle a \rangle & \text{if } j = 0 \text{, } 2m-2 \nmid i \text{ and } 2m-2 \mid i
\end{cases}
\]

Thus, it follows from Proposition 7.13 that \( |S| = |H| = 2^n \). Now Theorem 8.3 implies that \( S \) has an element of order \( 2m-1 \). Therefore \[18\] Theorem 5.3.4] and Proposition 7.13(1) yield that \( S \cong H, D_{2m}, K(2^m) \) or \( G(2^m) \). But \( \Gamma_H \not\cong \Gamma_{K(2^m)} \), since the set of degrees of vertices in \( \Gamma_{K(2^m)} \) is \( \{2, 2^m - 2^j \mid j \in \{1, \ldots, m-1\}\} \), while the corresponding set in \( \Gamma_H \) is \( \{2, 4, 2^{m-2} + 2^{m-1}\} \). So it follows from part (2) that \( S \cong H \).
or \( S \cong D_{2m} \). Now for \( D_{2m} = \langle b, y \mid b^{2^{m-1}} = y^2 = 1, by = yb^{-1} \rangle \), we have

\[
\text{Cyc}_H(b^iy^j) = \begin{cases} 
\langle b^i \rangle & \text{if } j = 1 \\
\langle b \rangle & \text{if } j = 0 \text{ and } 2^{m-1} \nmid i.
\end{cases}
\]

This implies that \( \Gamma_H \not\cong \Gamma_{D_{2m}} \), since the set of degrees of vertices in \( \Gamma_{D_{2m}} \) is \( \{2, 2^{m-1}\} \). Hence \( S \cong H \), as required.

(4) By Proposition 8.4, \( |S| = 2^n \) and \( S \) contains an element of order \( 2^n-1 \). Now if \( n > 3 \), it follows from Theorem \[18, \text{Theorem 5.3.4}\], Proposition 7.13-(1) and the proof of part (3) that \( S \cong D_8 \). If \( n = 3 \), then Propositions 7.13-(1) and 7.11 imply that \( S \cong D_8 \) or \( S \cong K(8) \).

Now part (1) completes the proof.

(5) It follows from parts (1)-(4), Proposition 7.13 and \([18, \text{Theorem 5.3.4}]\). \( \Box \)

There is a conjecture due to Shi and Bi (see Conjecture 1 of \[21\]) saying that if \( M \) is a finite non-abelian simple group such that \( |G| = |M| \) for some group \( G \) with \( \pi_e(G) = \pi_e(M) \), then \( G \cong M \). This conjecture has been proved for many non-abelian finite simple groups.

\textbf{Theorem 8.6.} (See \[4, 19, 20, 21, 22, 23, 26\]) Let \( G \) be a finite group and \( M \) one of the following finite simple groups: (1) A cyclic simple group \( \mathbb{Z}_p \); (2) An alternating group \( A_n \), \( n \geq 5 \); (3) A sporadic simple group; (4) A Lie type group except \( B_n, C_n, D_n \) (\( n \) even); or (5) A simple group with order \( < 10^8 \). Then \( G \cong M \) if and only if (a) \( \pi_e(G) = \pi_e(M) \), and (b) \( |G| = |M| \).

This conjecture has been also proved for some non-simple groups. The following is an example.

\textbf{Theorem 8.7.} (See \[3\]) Let \( G \) be a group. Then \( G \cong S_n \), \( n \geq 3 \) (\( S_n \) denotes the symmetric group of degree \( n \)) if and only if (a) \( \pi_e(G) = \pi_e(S_n) \), and (b) \( |G| = |S_n| \).

\textbf{Theorem 8.8.} Let \( G \) be a finite simple sporadic group. If \( \Gamma_G \cong \Gamma_H \) for some group \( H \), then \( G \cong H \).

\textit{Proof.} We first prove that \( |G| = |H| \). It is easy to see (e.g., from Table III in \[5\] or Tables 1a-1c in \[12\]) that there is a prime divisor \( p \) of \( |G| \) such that

\[ p - 1 \text{ divides } |G| \quad (I) \]

and \( C_G(g) = \langle g \rangle \) for every element \( g \in G \) of order \( p \). Since \( \Gamma_G \cong \Gamma_H \), \( |G| - 1 = |H| - |\text{Cyc}(H)| \). It follows that

\[ |\text{Cyc}(H)| \text{ divides } |G| - 1. \quad (*) \]
Now if \( g \in G \) and \( |g| = p \), then there exists an element \( h \in H \setminus \text{Cyc}(H) \) such that \( |G| - |\text{Cyc}_G(g)| = |H| - |\text{Cyc}_H(h)| \). But \( C_G(g) = \langle g \rangle \) implies that \( \text{Cyc}_G(g) = \langle g \rangle \). Thus \( |G| - p = |H| - |\text{Cyc}_H(h)| \). Now Proposition 7.2 yields that
\[
|\text{Cyc}(H)| \text{ divides } |G| - p. \tag{**}
\]
Now it follows from (*) and (**) that \( |\text{Cyc}(H)| \text{ divides } p - 1 \) and so by (I) it must divide \( |G| \). Therefore \( |\text{Cyc}(H)| \text{ divides } \gcd(|G|, |G| - 1) = 1 \), and from this it follows that \( |G| = |H| \). Hence by Theorem 8.6-(3), we have \( \pi_e(G) = \pi_e(H) \). Now Theorem 8.6-(3), completes the proof. □

**Theorem 8.9.** If \( G \) is a group and \( n > 2 \) is an integer, then the following hold:
1) If \( \Gamma_G \cong \Gamma_{S_n} \), then \( G \cong S_n \).
2) If \( n > 3 \) and \( \Gamma_G \cong \Gamma_{A_n} \), then \( G \cong A_n \).

**Proof.** 1) Let \( a \) and \( b \) be the cycles \((1,2,\ldots,n)\) and \((1,2,\ldots,n-1)\), respectively. Then \( C_{S_n}(a) = \langle a \rangle \) and \( C_{S_n}(b) = \langle b \rangle \). Thus \( C_{\text{Cyc}_{S_n}}(a) = \langle a \rangle \) and \( C_{\text{Cyc}_{S_n}}(b) = \langle b \rangle \). Now since \( \Gamma_G \cong \Gamma_{S_n} \), we have that \( |\text{Cyc}(G)| \text{ divides } n - (n - 1) = 1 \). Thus \( |\text{Cyc}(G)| = 1 \) and so \( |G| = |S_n| \). Now Theorems 8.3 and 8.7 implies that \( G \cong S_n \).
2) Suppose first that \( n \) is odd. Then \( a \in A_n \) and \( C_{A_n}(a) = \langle a \rangle \). Thus \( C_{\text{Cyc}_{A_n}}(a) = \langle a \rangle \). Since \( \Gamma_G \cong \Gamma_{A_n} \), we have \( |\text{Cyc}(G)| \text{ divides } \gcd(n - 1, \frac{n!}{2} - 1) = 1 \), since \( n > 3 \). Hence \( \text{Cyc}(G) = 1 \) and so, in this case, \( |G| = |A_n| \).

Now assume that \( n \) is even. Then \( b \in A_n \) and \( C_{A_n}(b) = \langle b \rangle \) and so \( C_{\text{Cyc}_{A_n}}(b) = \langle b \rangle \). It follows that \( |\text{Cyc}(G)| \text{ divides } \gcd(n - 2, \frac{n!}{2} - 1) = 1 \), since \( n > 3 \). Therefore \( \text{Cyc}(G) = 1 \) and so \( |G| = |A_n| \). Therefore in any case, \( |G| = |A_n| \) and so by Theorem 8.3 we have \( \pi_e(G) = \pi_e(A_n) \). Therefore Theorem 8.6-(2) implies that \( G \cong A_n \) for \( n \geq 5 \) and since \( A_4 \) is the only group of order 12 having no element of order 6, for \( n = 4 \) we have also \( G \cong A_n \). This completes the proof. □

Let \( G \) be a finite non-trivial group. The Gruenberg-Kegel graph (or prime graph) of \( G \) is the graph whose vertices are prime divisors of \( |G| \), and two distinct primes \( p \) and \( q \) are adjacent if \( G \) contains an element of order \( pq \). Denote by \( s(G) \) the number of connected components of the Gruenberg-Kegel graph of \( G \). The finite simple groups with non-connected Gruenberg-Kegel graph were classified by Williams [25] and Kondrat’ev [10]. A list of these groups can be found in [12].

**Theorem 8.10.** Let \( M \) be a finite non-abelian simple group of Lie type with \( s(M) \geq 2 \). If \( \Gamma_G \cong \Gamma_M \) for some group \( G \), then \( |G| = |M| \).
Proof. Note that if \( T \) is a cyclic subgroup of a group \( H \) such that \( C_H(T) = T \), then \( Cyc_H(T) = T \). Now the proof is \textit{mutatis mutandis} the proof of [13, Theorem 3], except that instead of Lemma 2(a) of [13], we may use Proposition 7.2.

\[ \square \]

**Theorem 8.11.** Let \( M \) be a finite non-abelian simple group of Lie type except \( B_n, C_n, D_n \) \((n \text{ even})\) with \( s(M) \geq 2 \). If \( \Gamma_M \cong \Gamma_G \) for some group \( G \), then \( G \cong M \).

Proof. It follows from Theorems 8.10, 8.3 and 8.6-(4).

\[ \square \]

**Remark 8.12.** Professor V.D. Mazurov told the first author in Antalya Algebra Days VII (May 2005), that the validity of Shi-Bi conjecture will be completed within the next two years for all finite non-abelian simple groups, and so far its validity for all finite simple groups except those of Lie types \( B_n \) and \( C_n \) have been proved. So in view of Theorems 8.10 and 8.3 it will be possible to say that the non-cyclic graph of a non-abelian finite simple group is unique. Anyway we close this paper by putting forward the following conjecture.

**Conjecture 8.13.** Let \( M \) be a finite non-abelian simple group. If \( \Gamma_M \cong \Gamma_G \) for some group \( G \), then \( G \cong M \).

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