Universal continuous bilinear forms for compactly supported sections of Lie algebra bundles and universal continuous extensions of certain current algebras

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Abstract

We construct a universal continuous invariant bilinear form for the Lie algebra of compactly supported sections of a Lie algebra bundle in a topological sense. Moreover we construct a universal continuous central extension of a current algebra $A \otimes g$ for a finite-dimensional Lie algebra $g$ and a certain class of topological algebras $A$. In particular taking $A = C^\infty_c(M)$ for a $\sigma$-compact manifold $M$ we obtain a more detailed justification for a recent result of Janssens and Wockel concerning a universal extension for the Lie algebra $C^\infty_c(M,g)$.

Introduction and Notation

A continuous invariant bilinear form $\gamma$ on a Lie algebra $g$ taking values in a locally convex space is called universal if we get all other continuous invariant bilinear forms on $g$ by composing $\gamma$ with a unique continuous linear map. Here invariance means that $\gamma([x,y],z) = \gamma(x,[y,z])$ for all $x,y,z \in g$. In [7] such a bilinear form was constructed in general. To this end, one considers the quotient of the symmetric square $S^2(g)$ by the closure of the subspace generated by elements of the form $[x,y] \vee z - x \vee [y,z]$. This quotient is called $V^T_g$ and $\kappa_g : g \times g \rightarrow V^T_g$, $(x,y) \mapsto [x \vee y]$ is a universal invariant bilinear form on $g$. We recall this general approach in Section [1]. Although any two universal invariant bilinear forms differ only by composition with an isomorphism of topological vector spaces, it is not enough to know the mere existence of a universal extension in general. Often, one would like to use more concrete realisations of universal invariant bilinear forms. This is the reason why in [7], Gündoğan constructed a concrete universal continuous
bilinear form for the Lie algebra $C^\infty(M, \mathfrak{g})$ of sections for a given Lie algebra bundle $\mathfrak{g}$. If $g$ is the finite-dimensional typical fiber of $\mathfrak{g}$ and we consider $V(\mathfrak{g})$ to be the vector bundle with base $M$ and fibers $V(\mathfrak{g}_p)$ for $p \in M$, Gündoğan showed that $C^\infty(M, \mathfrak{g}) \times C^\infty(M, \mathfrak{g}) \to V(\mathfrak{g}), (\eta, \zeta) \mapsto \kappa_g \circ (\eta, \zeta)$ is a universal continuous invariant symmetric bilinear form. In [8] Janssens and Wockel constructed a universal continuous central extension of the Lie algebra $C^\infty_c(M, \mathfrak{g})$ of compactly supported sections of a Lie algebra bundle $\mathfrak{g}$. In their proof, they implicitly use the concept of universal continuous invariant bilinear forms for these sections. But they do not discuss whether there exists a universal continuous invariant bilinear forms for the Lie algebra $C^\infty_c(M, \mathfrak{g})$. Hence the first aim of this paper is to construct a universal continuous invariant bilinear form on $C^\infty_c(M, \mathfrak{g})$. This will be done in Section 2.

To show the reader the application of this special universal invariant bilinear form we recall in 3.6 how Janssens and Wockel used this universal invariant bilinear form in [8]. With the help of this bilinear form they constructed a certain cocycle $\omega$ in $Z^2_c(C^\infty_c(M, \mathfrak{g}), E)$ for an appropriate locally convex space $E$, which they proved to be a universal cocycle. We also show the continuity of this cocycle, which was not discussed by Janssens and Wockel (see Lemma 3.4 and Theorem 3.5).

The second aim of this paper is to construct a universal continuous extension of certain so-called current algebras. In general these are algebras of the form $A \otimes \mathfrak{g}$, where $A$ is a locally convex topological algebra and $\mathfrak{g}$ a locally convex Lie algebra. In 2001, Maier constructed in [10] a universal continuous central extensions for current algebras of the form $A \otimes \mathfrak{g}$, where $A$ is a unital, commutative, associative, complete locally convex topological algebra and $\mathfrak{g}$ a finite-dimensional semi-simple Lie algebra. The canonical example for such a current algebra is given by the smooth functions from a manifold $M$ to $\mathfrak{g}$. To show the universality of the cocycle $\omega$ in [8], Janssens and Wockel used Maiers cocycle to construct a universal cocycle for the compactly supported smooth functions from a $\sigma$-compact manifold to the Lie algebra $\mathfrak{g}$ in [8, Theorem 7.2]. Gündoğan showed in [7, 5.1.14] that the ideas from [8] can be used to construct a universal cocycle for current algebras $A \otimes \mathfrak{g}$ with pseudo-unital, commutative and associative algebras $A$ that are inductive limits of unital Fréchet algebras. But this class of current algebras does not contain the compactly supported smooth maps $C^\infty_c(M, \mathfrak{g})$ from a $\sigma$-compact finite-dimensional manifold $M$ to the Lie algebra $\mathfrak{g}$. So in Section 4, we show that the cocycles constructed in [8] respectively [7, Theorem 5.1.14] work for locally convex associative algebras $A$ which are the inductive limit of complete locally convex algebras $A_n \subseteq A$, such that we can find an element $1_n \in A$ with $1_n \cdot a = a$ for all $a \in A_n$. Obviously, this class of algebras contains the
compactly supported smooth functions on a σ-compact manifold.

Throughout this paper we will use the following notations and conventions.

- We write \( \mathbb{N} \) for the set of integers \( \{1, 2, 3, \ldots\} \).
- All locally convex spaces considered are assumed Hausdorff.
- If \( E \) is finite-dimensional vector space and \( M \) a manifold, we write \( C^\infty_c(M, E) \) for the space of compactly supported smooth functions from \( M \) to \( E \).
- If \( M \) is a manifold and \( \pi: V \to M \) a vector bundle with base \( M \), we write \( C^\infty_c(M, V) \) for the space of compactly supported smooth sections of \( V \) and \( \Omega^k_c(M, V) \) for the space of compactly supported \( V \)-valued \( k \)-forms on \( M \).
- Let \( M \) be a \( \sigma \)-compact manifold, \( U \subseteq M \) open, \( V \) a vector space and \( \omega \) a vector bundle with base \( M \). For \( f \in C^\infty_c(U, V) \), \( X \in C^\infty_c(U, V) \) and \( \omega \in \Omega^k_c(U, V) \) respectively, we write \( f \), \( X \) and \( \omega \) respectively, for the extension of \( f \), \( X \) and \( \omega \) to \( M \) by 0 outside \( U \).
- If \( V \) and \( W \) are vector spaces, we write \( \text{Lin}(V, W) \) for the space of linear maps from \( V \) to \( W \) and in the case of topological vector spaces we write \( \text{Lin}_{ct}(V, W) \) for the space of continuous linear maps.
- If \( g \) and \( h \) are Lie algebras, we write \( \text{Hom}(g, h) \) for the space of Lie algebra homomorphisms from \( g \) to \( h \) and in the case of topological Lie algebras we write \( \text{Hom}_{ct}(g, h) \) for the space of continuous Lie algebra homomorphisms.
- We write \( A_1 \) for the unitalisation of an associative algebra \( A \).

1 Preliminaries

In this section, we recall the basic concepts of universal continuous invariant bilinear forms. These basic definitions and results can also be found in [10, Chapter 4].

**Definition 1.1.** Let \( g \) be a Lie algebra. A pair \((V, \beta)\) with a vector space \( V \) and a symmetric bilinear map \( \beta: g \times g \to V \) is called an **invariant symmetric bilinear form** on \( g \) if \( \beta([x, y], z) = \beta(x, [y, z]) \) for all \( x, y, z \in g \). The invariant symmetric bilinear form \((V, \beta)\) is called **algebraic universal** if for every other invariant symmetric bilinear form \((W, \gamma)\) on \( g \), there exists a unique linear map \( \psi: V \to W \) such that \( \gamma = \psi \circ \beta \). It is clear that if \( \beta \) is algebraic universal, then another invariant symmetric bilinear form \((W, \gamma)\) on \( g \) is algebraic universal if and only if we can find an isomorphism of vector spaces \( \varphi: V \to W \) with \( \gamma = \varphi \circ \beta \).
In the case that \( g \) is a locally convex Lie algebra and \( V \) is a locally convex space, the pair \((V, \beta)\) is called continuous invariant symmetric bilinear form on \( g \) if \( \beta \) is continuous and it is called topological universal or universal continuous invariant symmetric bilinear form if for every other continuous invariant symmetric bilinear form \((W, \gamma)\) on \( g \), there exists a unique continuous linear map \( \psi: V \to W \) such that \( \gamma = \psi \circ \beta \). It is clear that if \( \beta \) is topological universal, then another invariant symmetric bilinear form \((W, \gamma)\) on \( g \) is topological universal if and only if we can find an isomorphism \( \varphi: V \to W \) of topological vector spaces with \( \gamma = \varphi \circ \beta \).

Remark 4.1.5. and Proposition 4.1.7.] tell us that their always exists a universal continuous invariant symmetric bilinear form \((V_g, \kappa_g)\) for a given locally convex Lie algebra \( g \).

With [7, Proposition 4.3.3] we get directly the following Lemma 1.2.

**Lemma 1.2.** For a \( \sigma \)-compact finite-dimensional manifold \( M \) and a finite-dimensional perfect Lie algebra \( g \) the map

\[
\kappa_g: C^\infty_c(M, g) \times C^\infty_c(M, g) \to C^\infty_c(M, V_g), (f, g) \mapsto \kappa^T_g \circ (f, g)
\]

is an algebraic universal symmetric invariant bilinear form. Notably the image of \( \kappa_g \) spans \( C^\infty_c(M, V_g) \).

In the case that \( M \) is connected, the preceding Lemma 1.2 can be found in [7, Corollary 4.3.4].

**Definition 1.3.** If \( g \) is a Lie algebra, we call the subalgebra

\[
\{ f \in \text{Lin}(g) : (\forall x, y \in g) f([x, y]) = [f(x), y]\}
\]

of the associative algebra \( \text{Lin}(g) \) the Centroid of \( g \).

The following Lemma 1.4 can be found in [7, Lemma 4.1.6].

**Lemma 1.4.** Let \( g \) be a Lie algebra, \( W \) a vector space and \( \beta: g \times g \to W \) an invariant bilinear map. Then \( \beta(f(x), y) = \beta(x, f(y)) \) for all \( x \in [g, g], y \in g \) and \( f \in \text{Cent}(g) \).

The next Lemma 1.5 comes from [7, Remark 4.2.7].

**Lemma 1.5.** The Lie algebra \( C^\infty_c(M, g) \) is perfect for every \( \sigma \)-compact manifold \( M \) and perfect finite-dimensional Lie algebra \( g \).
Definition 1.6. Let $M$ be a manifold, $\mathfrak{g}$ a Lie algebra and $\pi : \mathfrak{K} \to M$ a vector bundle with typical fiber $\mathfrak{g}$. If for every $m \in M$ the space $\pi^{-1}(\{m\})$ is endowed with a Lie algebra structure such that there exist an atlas of local trivialisations $\varphi : \pi^{-1}(U_\varphi) \to U_\varphi \times \mathfrak{g}$ of $\mathfrak{K}$ such that for every $p \in U_\varphi$ the map $\varphi(p, -) : \mathfrak{K}_m \to \mathfrak{g}$ is a Lie algebra homomorphism, then we call $\mathfrak{K}$ a Lie algebra bundle.

In Definition 1.7 we endow the vector space of sections as well as compactly supported sections into a given vector bundle with a locally convex topology. We follow the definitions from [5, Chapter 3].

Definition 1.7. Let $M$ be a finite-dimensional manifold, $V$ a finite-dimensional vector space and $\pi : V \to M$ a vector bundle with typical fiber $V$. If $\eta \in C^\infty(M, V)$ and $\varphi : \pi^{-1}(U) \to U_\varphi \times V$ is a local trivialisations of $V$ we write $\eta_\varphi := \text{pr}_2 \circ \varphi \circ \eta|_{U_\varphi} \in C^\infty(U_\varphi, V)$ for the local representation of $\eta$. Let $\mathcal{A}$ be an atlas of $V$. We give $C^\infty(M, V)$ the initial topology with respect to the maps $\sigma_\varphi : C^\infty(M, V) \to C^\infty(U_\varphi, V)$, $\eta \mapsto \eta_\varphi$. [5, Lemma 3.9] tells us that this topology does not depend on the choose of the atlas. Moreover [5, Lemma 3.7] tells us that the topological embedding $C^\infty(M, V) \to \prod_{\varphi \in \mathcal{A}} C^\infty(U_\varphi, V)\eta \mapsto (\eta_\varphi)_{\varphi \in \mathcal{A}}$ has closed image and so $C^\infty(M, V)$ becomes a locally convex space. Especially we see that $C^\infty(M, V)$ is a Fréchet space if we find a countable atlas of local trivialisations of $V$.

If $K \subseteq M$ is compact we write $C^\infty_K(M, V)$ for the closed subspace of sections from $V$, whose supports are contained in $K$. In the case of an countable atlas of local trivialisations of $V$ it is clear that $C^\infty_K(M, V)$ is a Fréchet space. We give $C^\infty_c(M, V)$ the topology such that it becomes the inductive limit of the spaces $C^\infty_K(M, V)$ in the category of locally convex spaces, where $K$ runs through all compact sets.

If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{K}$ a Lie algebra bundle with typical fiber $\mathfrak{g}$, we define the Lie bracket $[\_, \_] : C^\infty(M, \mathfrak{K}) \times C^\infty(M, \mathfrak{K}) \to C^\infty(M, \mathfrak{K})$ by $[\eta, \zeta](p) = [\eta(p), \zeta(p)]$ for $\eta, \zeta \in C^\infty(M, \mathfrak{K})$, where the latter Lie bracket, is taken in $\mathfrak{K}_p$. Together with this Lie bracket $C^\infty_c(M, \mathfrak{K})$ becomes a topological Lie algebra.

Lemma 1.8. Let $M$ be a finite-dimensional $\sigma$-compact manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\mathfrak{K}$ a Lie algebra bundle with typical fiber $\mathfrak{g}$. In this situation $C^\infty_c(M, \mathfrak{K})$ becomes a topological Lie algebra.

Proof. The map $\mathfrak{K} \oplus \mathfrak{K} \to \mathfrak{K}$ that maps $(v, w)$ to $[v, w]_{\mathfrak{K}_p}$ for $v, w \in \mathfrak{K}_p$ and $p \in M$ is continuous. With the $\Omega$-Lemma (see, e.g. [11, Theorem 8.7] or [4, F.24]) we see that $C^\infty_c(M, \mathfrak{K})$ is a topological Lie algebra. \qed

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2 Topological universal bilinear forms for compactly supported sections of Lie algebra bundles

The aim of this section is to construct a universal invariant continuous bilinear form for the space of compactly supported sections into a Lie algebra bundle. To this end, we first show the “local statement”, that means we construct a universal continuous invariant bilinear form for the compactly supported smooth functions on a $\sigma$-compact manifold into a Lie algebra $\mathfrak{g}$ (Theorem 2.4). Afterwards we glue the local solutions together to a global one (Theorem 2.11).

The following fact is well known.

**Lemma 2.1.** If $X$ is a locally compact space, $(K_n)_{n \in \mathbb{N}}$ a locally finite cover by compact sets $K_n$ then there exists an open neighbourhood $V_n$ of $K_n$ in $X$ for every $n \in \mathbb{N}$, such that $(V_n)_{n \in \mathbb{N}}$ is locally finite.

**Lemma 2.2.** If $M$ is a $\sigma$-compact finite-dimensional manifold, $E$ a finite-dimensional vector space and $(\rho_m)_{m \in \mathbb{N}}$ a partition of unity, then $\Phi : \bigoplus_{m \in \mathbb{N}} C_c^\infty(M,E) \to C_c^\infty(M,E), (f_m)_{m \in \mathbb{N}} \mapsto \sum_{m \in \mathbb{N}} \rho_m \cdot f_m$ is a quotient map.

**Proof.** First we show the continuity of $\Phi$. Because $\Phi$ is linear, it is enough to show that $C_c^\infty(M,E) \to C_c^\infty(M,E), f \mapsto \rho_m \cdot f$ is continuous for every $m \in \mathbb{N}$. The local convex space $C_c^\infty(M,E)$ is the inductive limit of spaces $C_{K_n}^\infty(M,E)$ with $n \in \mathbb{N}$, where $(K_n)_{n \in \mathbb{N}}$ is a compact exhaustion of $M$. Because the support of $\rho_m$ is compact we can find $n \in \mathbb{N}$ with $\text{supp}(\rho_m) \subseteq K_n$. We see that the map $C_c^\infty(M,E) \to C_{K_n}^\infty(M,E), f \mapsto \rho_m \cdot f$ takes its image in the subspace $C_{K_n}^\infty(M,E)$. Now we conclude that $\Phi$ is continuous, because $C_c^\infty(M,E) \to C_{K_n}^\infty(M,E), f \mapsto \rho_m \cdot f$ is continuous.

With Lemma 2.1 we find a locally finite cover $(V_n)_{n \in \mathbb{N}}$ of $M$, such that $V_n$ is a neighbourhood for $\text{supp}(\rho_n)$. For $n \in \mathbb{N}$ we choose a smooth function $\sigma_n : M \to [0,1]$, such that $\sigma_n|_{\text{supp}(\rho_n)} = 1$ and $\text{supp}(\sigma_n) \subseteq V_n$. Because a compact subset of $M$ is only intersected by finite many sets of the cover $(V_n)_{n \in \mathbb{N}}$ we can define the map $\Psi : C_c^\infty(M,E) \to \bigoplus_{n=1}^\infty C_c^\infty(M,E), \gamma \mapsto (\sigma_n \cdot \gamma)_{n \in \mathbb{N}}$, which is obviously a right-inverse for $\Phi$. If $K \subseteq M$ is compact, we find $N \in \mathbb{N}$, such that $K \cap V_n = \emptyset$ for $n \geq N$. We conclude $\Psi(C_{K_n}^\infty(M,E)) \subseteq \prod_{n=1}^N C_c^\infty(M,E) \times \{0\}$. Obviously the map $C_c^\infty(M,E) \to \prod_{n=1}^N C_c^\infty(M,E), \gamma \mapsto (\sigma_n \cdot \gamma)_{n=1,...,N}$ is continuous. We conclude that $\Psi$ is a continuous linear right-inverse for $\Phi$ and so we see, that $\Phi$ is an open linear map. \qed
Lemma 2.3. If $M$ is a finite-dimensional $\sigma$-compact manifold and $\mathfrak{g}$ a finite-dimensional Lie algebra, then $\kappa_{\mathfrak{g}*}: C^\infty_c(M, \mathfrak{g})^2 \to C^\infty_c(M, V_\mathfrak{g})$, $(f, g) \mapsto \kappa_{\mathfrak{g}} \circ (f, g)$ is continuous.

Proof. The lemma follows directly from [3, Corollary 4.17]. $\blacksquare$

Theorem 2.4. Let $\mathfrak{g}$ be a perfect finite-dimensional Lie algebra and $M$ a finite-dimensional $\sigma$-compact manifold. Then $\kappa_{\mathfrak{g}*}: C^\infty_c(M, \mathfrak{g})^2 \to C^\infty_c(M, V_\mathfrak{g})$ is topological universal.

Proof. We know that $\kappa_{\mathfrak{g}*}: C^\infty_c(M, \mathfrak{g})^2 \to C^\infty_c(M, V_\mathfrak{g})$ is an algebraic universal invariant form. Moreover $\kappa_{C^\infty_c(M, \mathfrak{g})}^T: C^\infty_c(M, \mathfrak{g})^2 \to V^T_{C^\infty_c(M, \mathfrak{g})}$ is a topological universal invariant form.

Because $\kappa_{\mathfrak{g}*}$ is a continuous invariant bilinear map, we find a continuous linear map $f: V^T_c(C^\infty_c(M, \mathfrak{g})) \to C^\infty_c(M, V_\mathfrak{g})$ such that $\kappa_{\mathfrak{g}*} = f \circ \kappa_{C^\infty_c(M, \mathfrak{g})}^T$ and because $\kappa_{C^\infty_c(M, \mathfrak{g})}^T$ is an invariant bilinear map, we find a linear map $g: C^\infty_c(M, V_\mathfrak{g}) \to V^T_{C^\infty_c(M, \mathfrak{g})}$ with $\kappa_{C^\infty_c(M, \mathfrak{g})}^T = g \circ \kappa_{\mathfrak{g}*}$. We get the commutative diagram

\[
\begin{array}{ccc}
C^\infty_c(M, \mathfrak{g})^2 & \xrightarrow{\kappa_{\mathfrak{g}*}} & C^\infty_c(M, V_\mathfrak{g}) \\
\downarrow & & \downarrow \\
V^T_{C^\infty_c(M, \mathfrak{g})} & \xrightarrow{g} & V^T_{C^\infty_c(M, \mathfrak{g})}
\end{array}
\]

With $f \circ g \circ \kappa_{\mathfrak{g}*} = \kappa_{\mathfrak{g}*}$ and the fact that $\kappa_{\mathfrak{g}*}$ is algebraic universal, we get

\[f \circ g = \text{id}_{C^\infty_c(M, V_\mathfrak{g})}.\] (1)

Let $(\rho_m)_{m \in \mathbb{N}}$ be a partition of unity of $M$. From Lemma 2.2 we know the quotient map $\Phi$ and get the commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{m \in \mathbb{N}} C^\infty_c(M, V_\mathfrak{g}) & \xrightarrow{h} & V^T_{C^\infty_c(M, \mathfrak{g})} \\
\downarrow & & \downarrow \\
C^\infty_c(M, V_\mathfrak{g}) & \xrightarrow{g} & C^\infty_c(M, V_\mathfrak{g})
\end{array}
\]

with $h: \bigoplus_{m \in \mathbb{N}} C^\infty_c(M, V_\mathfrak{g}) \to V^T_{C^\infty_c(M, \mathfrak{g})}$, $(\varphi_m)_{m \in \mathbb{N}} \mapsto \sum_{m \in \mathbb{N}} g(\varphi_m \cdot \rho_m)$. If we can show that $h$ is continuous, we get that also $g$ is continuous.

Because $h$ is linear it is enough to show that $C^\infty_c(M, V_\mathfrak{g}) \to V^T_{C^\infty_c(M, \mathfrak{g})}$, $\varphi \mapsto g(\varphi \cdot \rho_m)$ is continuous for all $m \in \mathbb{N}$. The space $V^T_\mathfrak{g}$ is finite-dimensional, because $\mathfrak{g}$ is finite-dimensional. Let $(v_i)_{i=1,\ldots,n}$ be a basis of $V_\mathfrak{g}$. We write $\varphi_i$ for the $i$-th component for a map $\varphi \in C^\infty_c(M, V_\mathfrak{g})$. We can find $\xi_{ij}, \zeta_{ij} \in$
\[ C^\infty_c(M, \mathfrak{g}), \] such that \( v_i \cdot \rho_m = \sum_{j=1}^{n_i} \kappa_{\mathfrak{g}_*}(\xi_{ij}, \zeta_{ij}) \), because \( \kappa_{\mathfrak{g}_*} : C^\infty_c(M, \mathfrak{g})^2 \to C^\infty_c(M, V_\mathfrak{g}) \) is algebraic universal and so \( \text{im}(\kappa_{\mathfrak{g}_*}) \) generates \( C^\infty_c(M, V_\mathfrak{g}) \). For \( \varphi \in C^\infty_c(M, V_\mathfrak{g}) \) we calculate

\[
g(\varphi \cdot \rho_m) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} g(\varphi_i \cdot \rho_m \cdot v_i) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} g \left( \varphi_i \cdot \kappa_{\mathfrak{g}_*}(\xi_{ij}, \zeta_{ij}) \right)
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n_i} \left( \kappa_{\mathfrak{g}_*}(\varphi_i \cdot \xi_{ij}, \zeta_{ij}) \right) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \kappa_{C^\infty_c(M, \mathfrak{g})}^T(\varphi_i \cdot \xi_{ij}, \zeta_{ij})
\]

Because \( C^\infty_c(M, \mathbb{R}) \to V_{C^\infty_c(M, \mathfrak{g})}^T, \psi \mapsto \kappa_{C^\infty_c(M, \mathfrak{g})}^T(\psi \cdot \xi_{ij}, \zeta_{ij}) \) is continuous we see that also \( g \) is continuous.

Now we have \( g \circ f \circ \kappa_{C^\infty_c(M, \mathfrak{g})}^T = \kappa_{C^\infty_c(M, \mathfrak{g})}^T \). Because \( g \) is continuous we get \( g \circ f = \text{id}_{V_{C^\infty_c(M, \mathfrak{g})}^T} \). And with (1), we see that \( f \) is an isomorphism of topological vector spaces. \( \square \)

**Remark 2.5.** If \( \mathfrak{g} \) and \( \mathfrak{h} \) are Lie algebras and \( f : \mathfrak{h} \to \mathfrak{g} \) is a Lie algebra homomorphism, then there exists a unique linear map \( f_\kappa : V_\mathfrak{h} \to V_\mathfrak{g} \) with \( f_\kappa(\kappa_{\mathfrak{h}}(x, y)) = \kappa_{\mathfrak{g}}(f(x), f(y)) \).

**Definition 2.6.** Let \( M \) be a manifold, \( \mathfrak{g} \) a finite-dimensional Lie algebra and \( \mathfrak{R} \) a Lie algebra bundle with base \( M \) and typical fiber \( \mathfrak{g} \). If \( \mathcal{A} \) is an atlas of local trivialisations of \( \mathfrak{R} \), we define \( V(\mathfrak{R}) := \bigcup_{m \in M} V(\mathfrak{R}_m) \) and the surjection \( \rho : V(\mathfrak{R}) \to M, v \mapsto m \) for \( v \in \mathfrak{R}_m \). For a local trivialisation \( \varphi : \pi^{-1}(U_\varphi) \to U_\varphi \times \mathfrak{g} \) we define the map \( \bar{\varphi} : \rho^{-1}(U_\varphi) \to U_\varphi \times V_\mathfrak{g}, v \mapsto (\rho(v), \varphi(\rho(v), -)) \). Together with the atlas of local trivialisations \( \{ \varphi : \varphi \in \mathcal{A} \} \) we get a vector bundle \( \rho : V(\mathfrak{R}) \to M \). In this article we will always write \( \bar{\varphi} \) for the trivialisation of \( V(\mathfrak{R}) \) that comes from a trivialisation \( \varphi \) of \( \mathfrak{R} \).

**Definition 2.7.** For a manifold \( M \), a Lie algebra \( \mathfrak{g} \) and a Lie algebra bundle \( \mathfrak{R} \) with base \( M \) and typical fiber \( \mathfrak{g} \), we define the map \( \kappa_{\mathfrak{R}} : C^\infty_c(M, \mathfrak{R})^2 \to C^\infty_c(M, V(\mathfrak{R})) \) by \( \kappa_{\mathfrak{R}}(X, Y)(m) = \kappa_{\mathfrak{R}_m}(X(m), Y(m)) \) for \( m \in M \).

**Lemma 2.8.** If \( M \) is a \( \sigma \)-compact, finite-dimensional manifold, \( \mathfrak{g} \) a finite-dimensional Lie algebra and \( \mathfrak{R} \) a Lie algebra bundle with base \( M \) and typical fiber \( \mathfrak{g} \), then \( \kappa_{\mathfrak{R}} : C^\infty_c(M, \mathfrak{R})^2 \to C^\infty_c(M, V(\mathfrak{R})) \) is continuous.

**Proof.** The map \( \mathfrak{R} \oplus \mathfrak{R} \to V(\mathfrak{R}) \) that maps \((v, w)\) to \( \kappa_{\mathfrak{R}_p}(v, w) \) for \( v, w \in \mathfrak{R}_p \) and \( p \in M \) is continuous. The assertion now follows from the \( \Omega \)-Lemma (see, e.g. (1) Theorem 8.7) or (1) F.24).

**Lemma 2.9.** The image of \( \kappa_{\mathfrak{R}} \) spans \( C^\infty_c(M, V(\mathfrak{R})) \), if \( \mathfrak{g} \) is a perfect finite-dimensional Lie algebra, \( M \) a \( \sigma \)-compact finite-dimensional manifold and \( \pi_{\mathfrak{R}} : \mathfrak{R} \to M \) a Lie algebra bundle with base \( M \) and typical fiber \( \mathfrak{g} \).
Proof. To show the assertion of the lemma, we only need to show that the global statement can be reduced to the local one, because the local statement follows from Lemma 2.9. Let $\eta \in C^\infty_c(M, V(\mathfrak{R}))$ and $K := \text{supp}(\eta)$. We find local trivialisations $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathfrak{g}$ for $i = 1, \ldots, k$ with $K \subseteq \bigcup_{i=1}^k U_i$. Let $(\lambda_i)_{i=0}^k$ be a partition of unity of $M$ that is subordinate to the open cover that consists of the sets $M \setminus K$ and $U_i$ for $i = 1, \ldots, k$. We get

$$\eta = \sum_{i=1}^k \lambda_i \cdot \eta \quad \text{and} \quad \lambda_i \cdot \eta \in C^\infty_c(M, V(\mathfrak{R})) \quad \text{with} \quad \text{supp}(\lambda_i \cdot \eta) \subseteq U_i.$$ 

The assertion now follows from the fact that $\tilde{\varphi}_i : \rho_{V(\mathfrak{R})}^{-1}(U_i) \to U_i \times \mathfrak{g}$ is a local trivialisation of $V(\mathfrak{R})$. □

**Lemma 2.10.** Let $M$ be a finite-dimensional $\sigma$-compact manifold, $\mathfrak{g}$ a finite-dimensional Lie algebra and $\mathfrak{R}$ a Lie algebra bundle with typical fiber $\mathfrak{g}$. If $\mathfrak{g}$ is perfect, then also $C^\infty_c(M, \mathfrak{R})$ is perfect.

Proof. Because the assertion holds for the local statement, it is enough to show that the global statement can be reduced to the local one. Let $\eta \in C^\infty_c(M, \mathfrak{R})$ and $K := \text{supp}(\eta)$. We choose $\varphi_i$ and $\lambda_i$ analogous to the proof of Lemma 2.9 and calculate $\eta = \sum_{i=1}^n \lambda_i \cdot \eta$. We have $\lambda_i \cdot \eta \in C^\infty_c(M, \mathfrak{R})$ with $\text{supp}(\lambda_i \cdot \eta) \subseteq U_i$. This finishes the proof. □

In [7, Theorem 4.4.4] a local statement for algebraic universal invariant bilinear forms is used to get an analogous global statement for the space of algebraic univerality of spaces of sections in a Lie algebra bundle. We now want to transfer this approach to a topological statement for compactly supported sections in a Lie algebra bundle in Theorem 2.11.

**Theorem 2.11.** For a perfect finite-dimensional Lie-algebra $\mathfrak{g}$, a $\sigma$-compact manifold $M$ and a Lie algebra bundle $\mathfrak{R}$ with base $M$ and typical fiber $\mathfrak{g}$, the map $\kappa_{\mathfrak{R}} : C^\infty_c(M, \mathfrak{R})^2 \to C^\infty_c(M, V(\mathfrak{R}))$ is topological universal.

Proof. Let $(\psi_i : \pi^{-1}(U_{\psi_i}) \to U_{\psi_i} \times \mathfrak{g})_{i \in I}$ be a bundle atlas of $\mathfrak{R}$ and $(\rho_i)_{i \in I}$ a partition of unity of $M$ with $\text{supp}(\rho_i) \subseteq U_{\psi_i}$. Let $\gamma : C^\infty_c(M, \mathfrak{R})^2 \to W$ be a continuous invariant bilinear form. For $i \in I$ we define

$$\gamma_i : C^\infty_c(U_{\psi_i}, \mathfrak{g})^2 \to W$$

$$(f, g) \mapsto \gamma ((\psi_i^{-1} \circ (\text{id}, f))_\cdot, (\psi_i^{-1} \circ (\text{id}, g))_\cdot).$$

The bilinear map $\gamma_i$ is an invariant symmetric bilinear form. We want to show that it is also continuous. Obviously it is enough to show that $C^\infty_c(U_{\psi_i}, \mathfrak{g}) \to C^\infty_c(M, \mathfrak{R})$, $f \mapsto (\psi_i^{-1} \circ (\text{id}, f))_\cdot$ is continuous. This follows
from the following diagram

\[
\begin{array}{c}
\xymatrix{
C^\infty_c(U_{\psi_i}, \mathfrak{g}) \ar[r]^{f-1(i_d, f)} \ar[d]^{\text{id}} & C^\infty_c(M, \mathfrak{K}) \ar[d] \\\n\bigoplus_{j \in I} C^\infty(U_{\psi_j}, \mathfrak{g})
}\end{array}
\]

So we can find a continuous linear map \( \beta: C^\infty_c(U_{\psi_i}, V_{\mathfrak{g}}) \rightarrow W \), such that the diagram

\[
\begin{array}{c}
\xymatrix{
C^\infty_c(U_{\psi_i}, \mathfrak{g})^2 \ar[r]^{\gamma_i} \ar[d]_{\kappa_{g \ast}} & W \\
C^\infty_c(U_{\psi_i}, V(\mathfrak{g})) \ar[u]_{\beta_i}
}\end{array}
\]

commutes.

For \( i \in I \) let \( \tilde{\psi}_i \) be the corresponding bundle-chart of \( V(\mathfrak{K}) \) that comes from \( \psi_i \). We define \( \beta: C^\infty_c(M, V(\mathfrak{K})) \rightarrow W \), \( X \mapsto \sum_{i \in I} \beta_i((\rho_i \cdot X)_{\psi_i}) \). To see that \( \beta \) is continuous it is enough to show that \( h: \bigoplus_{i \in I} C^\infty(U_{\psi_i}, V_{\mathfrak{g}}) \rightarrow W \), \( (f_i)_{i \in I} \mapsto \sum_{i \in I} \beta_i(\rho_i \cdot f_i) \) is continuous, because \( C^\infty_c(M, V(\mathfrak{K})) \rightarrow \bigoplus_{i \in I} C^\infty(U_{\psi_i}, V_{\mathfrak{g}}) \), \( X \mapsto X_{\psi_i} \) is a topological embedding. To check the continuity of \( h \) it is enough to show the continuity of \( C^\infty_c(U_{\psi_i}, V_{\mathfrak{g}}) \rightarrow C^\infty_c(U_{\psi_i}, V_{\mathfrak{g}}), f \mapsto \rho_i \cdot f \). The continuity of the latter map is clear, because it takes its image in a subspace \( C^\infty_c(U_{\psi_i}, V_{\mathfrak{g}}) \) for a compact set \( K \subseteq U_{\psi_i} \).

Let \( \zeta_i: M \rightarrow [0, 1] \) be a smooth map with \( \text{supp}(\zeta_i) \subseteq U_{\psi_i} \) and \( \zeta_i|_{\text{supp}(\rho_i)} = 1 \) for \( i \in I \). With Lemma 2.9 in mind we calculate for \( X, Y \in C^\infty_c(M, \mathfrak{K}) \)

\[
\beta(\kappa_\mathfrak{K}(X, Y)) = \sum_{i \in I} \beta_i((\rho_i \zeta_i \kappa_\mathfrak{K}(X, Y))_{\psi_i}) = \sum_{i \in I} \beta_i(\kappa_\mathfrak{K}(\rho_i X, \zeta_i Y)_{\psi_i})
\]

\[
= \sum_{i \in I} \beta_i(\rho_i X, \zeta_i Y) = \sum_{i \in I} \gamma_i((\rho_i X)_{\psi_i}, (\rho_i Y)_{\psi_i}) = \sum_{i \in I} \gamma(\rho_i X, \zeta_i Y)
\]

\[
= \sum_{i \in I} \gamma(\zeta_i (\rho_i X, Y) = \gamma(X, Y).
\]

Here we used that \( C^\infty_c(M, \mathfrak{K}) \rightarrow C^\infty_c(M, \mathfrak{K}), X \mapsto \zeta_i \cdot X \) is in \( \text{Cent}(C^\infty_c(M, \mathfrak{K})) \) and that \( C^\infty_c(M, \mathfrak{K}) \) is a perfect Lie algebra.

The uniqueness of \( \beta \) follows from Lemma 2.9.

\[ \square \]

### 3 Continuity of a cocycle

In Theorem 3.6 we recall how the universal continuous invariant symmetric bilinear form on the Lie algebra of compactly supported sections is used in
to construct a certain cocycle $\omega$ in $Z^3_{\varepsilon}(C^\infty_c(M, \mathfrak{R}), E)$ for an appropriate locally convex space $E$. But as mentioned in the introduction the construction in [8] does not discuss the continuity of the cocycle $\omega$. We show that this cocycle is actually continuous with the help of Lemma 3.4 and Theorem 3.5.

In the following Definition 3.1 we equip the space of $k$-forms respectively the compactly supported $k$-forms on a $\sigma$-compact finite-dimensional manifold with the usual topology, as, e.g., in [7, Definition 5.2.6].

**Definition 3.1.** Let $M$ be a finite-dimensional $\sigma$-compact manifold, $\pi: V \to M$ a vector bundle with base $M$ and $k \in \mathbb{N}$. We give $\Omega^k(M, V)$ the induced topology of $C^\infty((TM)^k, V)$. The subspace $\Omega^k(M, V)$ is closed in $C^\infty((TM)^k, V)$, because convergence $C^\infty((TM)^k, V)$ implies pointwise convergence. Hence $\Omega^k(M, V)$ becomes a locally convex space. If the typical fiber of $V$ is a Fréchet-space, it even is a Fréchet-space. If $K \subseteq M$ is compact, we give $\Omega^k_K(M, V) := \{\omega \in \Omega^k(M, V) : \text{supp}(\omega) \subseteq K\}$ the induced topology of $\Omega^k(M, V)$, so that $\Omega^k_K(M, V) = \bigcap_{p \in M \setminus K, v \in (T_p M)^k} \text{ev}_{p,v}^{-1} \{0\}$ with the point evaluation $\text{ev}_{p,v} : \Omega^k(M, V) \to V_p, \omega \mapsto \omega_p(v)$, becomes a closed subspace of $\Omega^k(M, V)$ and by this a locally convex space. As usual we write $\Omega^k(M, V)$ for the space of compactly supported $k$-forms and equip it with the topology, such that it becomes the inductive limit of the spaces $\Omega^k_K(M, V)$.

We now want to fix our notation concerning $k$-forms and connections and recall some basic facts. All this is well known, for instance see [2] and [7, Chapter 2.2. and 2.3.].

**Definition 3.2.** Let $M$ be a finite-dimensional $\sigma$-compact manifold, $V$ a vector bundle with base $M$, $\mathfrak{R}$ a Lie-algebra-bundle with base $M$ and $k \in \mathbb{N}$.
(a) The space $\Omega^k_c(M, V)$ becomes a $C^\infty(M, \mathfrak{R})$-module by the multiplication $C^\infty(M, \mathfrak{R}) \times \Omega^k_c(M, V) \to \Omega^k_c(M, V), (f, \omega) \mapsto f \cdot \omega$ with $(f \cdot \omega)_p = f(p) \omega_p$.
(b) We get a bilinear map $\Omega^k_c(M, \mathbb{R}) \times C^\infty(M, V) \to \Omega^k_c(M, V) (\omega, \eta) \mapsto \omega \cdot \eta$ with $(\omega \cdot \eta)_p(v_1, ..., v_k) = \omega_p(v_1, ..., v_k) \cdot \eta(p)$ for $v_i \in T_p M$.
(c) We call a $\mathbb{R}$-linear map $d : C^\infty_c(M, V) \to \Omega^1_c(M, V)$ Koszul connection, if $d(f \eta) = f d\eta + \eta df$ for all $\eta \in C^\infty(M, V)$ and $f \in C^\infty(M, \mathbb{R})$.
(d) We define the continuous $C^\infty(M, \mathfrak{R})$-linear map $C^\infty_c(M, \mathfrak{R}) \times \Omega^1_c(M, \mathfrak{R}) \to \Omega^1_c(M, \mathfrak{R}), (\eta, \omega) \mapsto [\eta, \omega]$ with $[(\eta, \omega)]_p(v) = [\eta(p), \omega_p(v)]$. Moreover we set $[\omega, \eta] = -[\eta, \omega]$.
(e) We call a Koszul connection $D : C^\infty_c(M, \mathfrak{R}) \to \Omega^1_c(M, \mathfrak{R})$ a Lie connection, if $D[\eta, \tau] = [D\eta, \tau] + [\eta, D\tau]$ for $\eta, \tau \in C^\infty_c(M, \mathfrak{R})$.

The following fact in Lemma 3.3 should be part of the folklore. For instance see [7, Remark 2.3.14].
Lemma 3.3. For every finite-dimensional $\sigma$-compact manifold $M$ and Lie-algebra-bundle $\pi: \mathcal{F} \to M$, there exists a continuous Lie connection $D: C^\infty_c(M, \mathcal{F}) \to \Omega^1(M, \mathcal{F})$.

Lemma 3.4. If $M$ is a finite-dimensional $\sigma$-compact manifold and $\pi: \mathcal{F} \to M$ a Lie-algebra-bundle with finite-dimensional typical fiber, then we define the map

$$\bar{\kappa}_R: \Omega^1_c(M, \mathcal{F}) \times C^\infty_c(M, \mathcal{F}) \to \Omega^1_c(M, V(\mathcal{F}))$$

by $(\bar{\kappa}_R(\omega, \eta))_p(v) = \kappa_{\mathcal{F}_p}(\omega_p(v), \eta(p))$. The map $\bar{\kappa}_R$ is $C^\infty(M, \mathcal{F})$-bilinear and continuous. If moreover $D: C^\infty_c(M, \mathcal{F}) \to \Omega^1_c(M, \mathcal{F})$ is a continuous Lie connection, then

$$\beta: C^\infty_c(M, \mathcal{F})^2 \to \Omega^1_c(M, V(\mathcal{F})), \ (\zeta, \eta) \mapsto \bar{\kappa}_R(D\zeta, \eta) + \bar{\kappa}_R(\zeta, D\eta)$$

is a continuous, invariant, symmetric bilinear form.

Proof. To show the continuity of $\beta$ we only have to prove that $\bar{K}_R$ is continuous. The map

$$(T^*M \otimes \mathcal{F}) \oplus \mathcal{F} \to \text{Lin}(TM, V(\mathcal{F})), \ (\lambda \otimes v, w) \mapsto \kappa_{\mathcal{F}_p}(\lambda(\_ \cdot v, w)$$

is continuous. With the identifications $\Omega^1_c(M, \mathcal{F}) \cong C^\infty_c(M, T^*M \otimes \mathcal{F})$ and $\Omega^1_c(M, V(\mathcal{F})) \cong C^\infty_c(M, \text{Lin}(TM, V(\mathcal{F})))$ the continuity follows from the $\Omega$-Lemma (see, e.g., [11, Theorem 8.7] or [4, F.24]).

We show that $\beta$ is invariant:

$$\beta([\eta_1, \eta_2], \eta_3)_p(v) = \kappa'([D[\eta_1, \eta_2]]_p(v), \eta_3(p)) + \kappa([\eta_1(p), [\eta_2(p)]], (D\eta_3)_p(v))$$

$$+ \kappa([\eta_1(p), \eta_2(p)], (D\eta_3)_p(v))$$

$$= \kappa([D\eta_1, \eta_2](p), \eta_3(p)) + \kappa([\eta_1(p), D\eta_2](p), \eta_3(p))$$

$$+ \kappa([\eta_1(p), \eta_2(p)], (D\eta_3)_p(v))$$

$$= \kappa([D\eta_1, \eta_2](p), [\eta_3(p)] + \kappa([\eta_1(p), [\eta_2(p)]], (D\eta_3)_p(v))$$

$$+ \kappa([\eta_1(p), \eta_2(p)], (D\eta_3)_p(v))$$

The rest of the statement is clear. \qed

As in [8] we construct a Koszul connection for the vector bundle $C^\infty_c(M, V(\mathcal{F}))$. This is also the point where we apply Theorem 2.11 to see that the Koszul connection is actually continuous. Moreover we need the continuity of the map $\beta$ from Lemma 3.4.
Theorem 3.5. Let \( M \) be a finite-dimensional \( \sigma \)-compact manifold, \( \pi : \mathcal{R} \to M \) a Lie-algebra-bundle with perfect, finite-dimensional typical fiber \( g \), \( D : C_c^\infty(M, \mathcal{R}) \to \Omega^1_c(M, \mathcal{R}) \) a continuous Lie connection and \( \beta \) as in Lemma 3.4 a continuous, invariant, symmetric bilinear form. Then there exists a unique continuous Koszul connection \( d : C_c^\infty(M, V(\mathcal{R})) \to \Omega^1_c(M, V(\mathcal{R})) \) such that the diagram

\[
\begin{array}{ccc}
C_c^\infty(M, \mathcal{R})^2 & \xrightarrow{\beta} & \Omega^1_c(M, V(\mathcal{R})) \\
\downarrow{\kappa_R} & & \downarrow{d} \\
C_c^\infty(M, V(\mathcal{R})) & & \\
\end{array}
\]

commutes.

Proof. With Theorem 2.11 we find a unique continuous \( \mathbb{R} \)-linear map \( d : C_c^\infty(M, V(\mathcal{R})) \to \Omega^1_c(M, \mathcal{R}) \) such that the above diagram commutes. Now we have to show that \( d(f \cdot \eta) = df \cdot \eta + \eta \cdot df \eta \) holds for all \( f \in C_c^\infty(M, \mathbb{R}) \) and \( \eta \in C_c^\infty(M, V(\mathcal{R})) \). Because the image of \( \kappa_R \) spans \( C_c^\infty(M, V(\mathcal{R})) \), it is sufficient to show the assertion for \( \eta = \kappa_R(\xi, \zeta) \) with \( \xi, \zeta \in C_c^\infty(M, V(\mathcal{R})) \).

\[
d(f \cdot \kappa_R(\xi, \zeta))(v) = (d(\kappa_R(f\xi, \zeta)))(v) = d\kappa(D(f \cdot \xi)(v), \zeta(p)) + \kappa(f(p) \cdot \xi(p), (D\zeta)_p(v)) \\
= \kappa(df(v) \cdot \xi(p), \zeta(p)) + \kappa(f(p)(D\xi)_p(v), \zeta(p)) + \kappa(f(p)\xi(p), (D\zeta)_p(v)) \\
= df(v) \cdot \kappa(\xi(p), \zeta(p)) + f(p)\kappa(D\xi_p(v), \zeta(p)) + f(p)\kappa(\xi(p), D\zeta_p(v)) \\
= (df \cdot \kappa(\xi, \zeta))(v) + f(p) \cdot (\beta(\xi, \zeta))p(v) \\
= (df \cdot \kappa(\xi, \zeta))(v) + f(p) \cdot (\kappa(\xi, \zeta))p(v).
\]

\( \square \)

In Theorem 3.6 we now repeat the construction of the cocycle from 3.5 and find with Theorem 3.5 that this cocycle is actually continuous.

Theorem 3.6. Let \( M \) be a finite-dimensional \( \sigma \)-compact manifold, \( \pi : \mathcal{R} \to M \) a Lie-algebra-bundle with perfect, finite-dimensional typical fiber \( g \) and \( D : C_c^\infty(M, \mathcal{R}) \to \Omega^1_c(M, \mathcal{R}) \) a continuous Lie connection. We use the Koszul connection \( d : C_c^\infty(M, V(\mathcal{R})) \to \Omega^1_c(M, V(\mathcal{R})) \) constructed in Theorem 3.5 write \( \overline{\Omega}^1_c(M, V(\mathcal{R})) := \Omega^1_c(M, V(\mathcal{R}))/dC_c^\infty(M, V(\mathcal{R})) \) and define the map

\[
\omega : C_c^\infty(M, \mathcal{R})^2 \to \overline{\Omega}^1_c(M, V(\mathcal{R})), \ (\eta, \zeta) \mapsto [\kappa_R(D\eta, \zeta)]
\]

with the help of \( \kappa_R \) of Lemma 3.4. Then \( \omega \) is a cocycle and also continuous, i.e., \( \omega \in H^2_c(C_c^\infty(M, \mathcal{R}), \overline{\Omega}^1_c(M, V(\mathcal{R}))) \).
Proof. The continuity of $\omega$ follows from Lemma \ref{lemma}. The $\mathbb{R}$-bilinearity is clear. That $\omega$ is alternating follows from $\tilde{\kappa}(D\eta, \xi) + \tilde{\kappa}(D\xi, \eta) \in dC^\infty(M, V(\mathbb{R}))$.

Moreover, we have:

\[
\sum_{\sigma \in A_3} \omega([\eta_{\sigma(1)}, \eta_{\sigma(2)}], \eta_{\sigma(3)}) = \kappa([\eta_1, \eta_2], D\eta_3) + \kappa([\eta_2, \eta_3], D\eta_1) + \kappa([\eta_3, \eta_1], D\eta_2).
\]

On the other hand we have:

\[
\kappa([\eta_1, \eta_2], D\eta_3) = -\kappa(\eta_3, [D\eta_1, \eta_2]) = -\kappa(\eta_3, \eta_2) - \kappa(\eta_3, D\eta_2) = -\kappa([\eta_3, \eta_1], D\eta_2) = -\kappa([\eta_2, \eta_3], D\eta_1) - \kappa([\eta_3, \eta_1], D\eta_2).
\]

This finishes the proof.

4 Universal continuous extensions of certain current algebras

Maier constructed in \cite{Maier} a universal cocycle for current algebras with a unital complete locally convex algebra. In \cite[Theorem II.7]{JanssensWockel} Janssens and Wockel showed that this cocycle works also for the algebra of compactly supported functions on a $\sigma$-compact finite-dimensional manifold. Gündoğan showed in \cite{Gundogan} that this approach also works for a certain class of locally convex pseudo-unital algebras. But the class of locally convex algebras he considers does not contain the compactly supported functions on a $\sigma$-compact finite-dimensional manifold. Our aim in this section is to use the ideas from \cite{Gundogan} to show that the cocycle constructed in \cite{JanssensWockel} respectively \cite{Maier} actually works, in a topological sense for a class of locally convex algebras without unity that contains the compactly supported smooth functions on a $\sigma$-compact algebra.

In Definition \ref{def} we first recall the basic concept of current algebras that should be part of the folklore.

**Definition 4.1.** If $A$ is a commutative pseudo unital algebra and $\mathfrak{g}$ a finite-dimensional Lie algebra, we endow the tensor product $A \otimes \mathfrak{g}$ with the unique Lie bracket that satisfies $[a \otimes x, b \otimes y] = ab \otimes [x, y]$ for $a, b \in A$ and $x, y \in \mathfrak{g}$. We endow $A \otimes \mathfrak{g}$ with the topology of the projective tensor product of locally convex spaces. This Lie algebra is even a locally convex algebra as one can see in \cite[Remark 2.1.9.]{Gundogan}. Moreover \cite[Remark 4.2.7]{Gundogan} tells us that $A \otimes \mathfrak{g}$ is perfect if $\mathfrak{g}$ is so.
Definition 4.2. Let \(g\) be a locally convex Lie algebra and \(V\) a complete locally convex space that is a trivial \(g\)-module and \(\omega \in Z^2_c(g, V)\) a cocycle. We call \(\omega\) weakly universal for complete locally convex spaces if for every complete locally convex space \(W\) considered as trivial \(g\)-module, the map \(\delta_W : \text{Lin}_{ct}(V, W) \to H^2_{ct}(g, W), \theta \mapsto [\theta \circ \omega]\) is bijective.

The following Lemma 4.3 can be found in [12, Lemma 1.12 (iii)].

Lemma 4.3. Let \(g\) be a locally convex Lie algebra and \(V\) a complete locally convex space that is a trivial \(g\)-module and \(\omega \in Z^2_c(g, V)\) a weak universal cocycle. If \(g\) is topological perfect, then \(\omega\) is universal for all complete locally convex spaces in the classic sense.

Remark 4.4. Actually the lemma in [12] requires the condition that the considered extension is weak universal for the underlying field \(K\) of the vector spaces. But this condition is only used in the proof of statement 1.12 (ii). The proof of statement (iii) neither requires statement 1.12 (ii) nor this condition.

We remind the reader of the concept of topological universal differential module. This concept is for example developed in [7, Chapter 5.2] or [10].

Definition 4.5. For a complete locally convex associative commutative unital algebra \(A\), a pair \((E, D)\) with a complete locally convex \(A\)-module \(E\) and continuous derivation \(D : A \to E\) of \(E\) is called universal topological differential module of \(A\) if we find a unique continuous linear map \(\varphi : E \to F\) such that

\[
\begin{array}{c}
A \xrightarrow{T} F \\
D \downarrow \searrow \varphi \\
E & \uparrow \nearrow \end{array}
\]

commutes for every complete locally convex \(A\)-module \(F\) and continuous \(F\)-derivation \(T : A \to F\). [7, Chapter 5.2] or [10] tell us that there always exists a universal topological differential module \((\Omega(A), d_A)\) for a given complete locally convex associative commutative unital algebra \(A\).

Now we recall the cocycle for current algebras of the form \(A \otimes g\) with unital complete locally convex algebra \(A\) that Maier proved in [10] to be universal.

Definition 4.6. If \(g\) is a finite-dimensional semisimple Lie algebra, \(A\) a commutative, associative unital complete locally convex algebra, then we define the cocycle

\[
\omega_{g,A} : A \otimes g \times A \otimes g \to V_g \otimes (\Omega(A)/d_A(A))
\]
For convenience we write \( V_{g,A} := V_g \otimes (\Omega(A)/d_A(A)) \).

In [10] Theorem 16] we directly get the following Lemma 4.7.

**Lemma 4.7.** Let \( g \) be a finite-dimensional semisimple Lie algebra, \( A \) a commutative, associative unital complete algebra. If \( W \) is a complete locally convex space considered as trivial \( g \)-module, then the map \( \delta_W : \text{Lin}(V_{g,A}, W) \to H^2(A \otimes g, W) \), \( \theta \mapsto [\theta \circ \omega_{g,A}] \) is bijective.

**Definition 4.8.** For a locally convex algebra \( A \) and a finite-dimensional Lie algebra \( g \) the Lie algebra \( A \otimes g \) is locally convex. For \( y \in g \), we get a continuous bilinear map \( A \times g \to A \otimes g \) \( (c, x) \to c \otimes [y, x] \) which induces a continuous linear map \( \delta_y : A \otimes g \to A \otimes g \) with \( \delta_y(c \otimes x) = c \otimes [y, x] \). We get a linear map \( \delta : g \to \text{End}_V(A \otimes g) \), \( y \mapsto \delta_y \). Moreover \( \delta \) is a Lie algebra homomorphism, because \( \delta_{[y_1, y_2]}(c \otimes x) \) becomes a locally convex \( A \otimes g \times g \) with the Lie-bracket \( [[z_1, y_1], (z_2, y_2)] = ([z_1, z_2] + \delta_{y_2}(z_2) - \delta_{y_1}(z_1), [y_1, y_2]) = ([z_1, z_2] + [y_1, z_2] - [y_2, z_1], [y_1, y_2]) \) for \( z_i \in A \otimes g \) and \( y_i \in g \), where we wrote \( [y_,_] := \delta_y \) for \( y \in g \). \( A \otimes g \times g \) becomes a locally convex Lie algebra.

We identify \( A \otimes g \) with the ideal \( \text{im}(i) \), where \( i : A \otimes g \to A \otimes g \times g \), \( z \mapsto (z, 0) \) is a topological embedding that is a Lie algebra homomorphism. The image is closed as kernel of the projection \( A \otimes g \times g \to g \).

Moreover we identify \( g \) with the subalgebra \( \text{im}(i_g) \), where \( i_g : g \to (A \otimes g) \times g \), \( x \mapsto (0, x) \) is a topological embedding with closed image that is a Lie algebra homomorphism.

For \( (z, x) \in A \otimes g \times g \) we can write \( (z, x) = (z, 0) + (0, x) = z + x \).

By Remark 5.1.7 and Lemma 5.1.8] lead to the following Lemma 4.9.

**Lemma 4.9.** For a locally convex algebra \( A \) and a finite-dimensional Lie algebra \( g \) we have an isomorphism of locally convex Lie algebras \( \varphi : A_1 \otimes g \to (A \otimes g) \times g \) with \( (\lambda, a) \otimes w \mapsto (a \otimes w, \lambda w) \) for all \( \lambda \in \mathbb{K} \), \( a \in A \) and \( w \in g \).

**Lemma 4.10.** If \( g \) and \( h \) are locally convex Lie algebras, \( V \) a locally convex space and \( \varphi : g \to h \) a continuous Lie algebra homomorphism, then \( H^2_{cl}(\varphi) : H^2_{cl}(h, V) \to H^2_{cl}(g, V) \), \( [\omega] \mapsto [\omega \circ (\varphi, \varphi)] \) is a well-defined and linear map.
Proof. For \( \eta \in \text{Lin}(\mathfrak{h}, V) \) and \( \omega \in Z^2_A(\mathfrak{h}, V) \) with \( \omega = \eta \circ [\cdot, \cdot] \) we have

\[
(\varphi, \varphi)^*(\omega) = \eta \circ \varphi \circ [\cdot, \cdot].
\]

We will use the concept of neutral triple evolved in [7, Definition 5.1.3] and recall it in the next Definition [4.11].

**Definition 4.11.** Let be \( A \) a pseudo-unital associative commutative locally convex algebra \( A \) and \( g \) a finite-dimensional perfect Lie algebra.

(a) We get an \( A \)-module structure \( \cdot : A \times (A \otimes g) \rightarrow A \otimes g \) with \( a \cdot (b \otimes y)(a \cdot b) \otimes y \)

for \( a, b \in A \) and \( y \in g \). Actually \( A \otimes g \) is a \( A \)-module in the category of locally convex spaces, because \( g \) is finite-dimensional. In this situation we call \( \nu \in A \) neutral for \( f \in A \otimes g \), if \( \nu \cdot f = f \).

(b) For \( f \in A \otimes g \) (resp. \( \varphi \in A \)) we call \((\lambda, \nu, \mu) \in A^3 \) a neutral triple for \( f \) (resp. \( \varphi \)), if \( \mu \cdot f = f \) (resp. \( \mu \cdot \varphi = \varphi \)), \( \nu \cdot \mu = \mu \) and \( \lambda \cdot \nu = \nu \).

(c) Let \((v_i)_{i=1, \ldots, n}\) be a basis of \( g \). For \( f = \sum_{i=1}^n \varphi_i \otimes v_i \in A \otimes g \) with \( \varphi \in A \) we choose \( \mu_f \in A \) with \( \mu \) neutral for all \( \varphi_i \). Moreover we choose \( \nu_f \) neutral for \( \mu_f \) and \( \lambda_f \) neutral for \( \nu_f \). It is clear that \((\lambda_f, \nu_f, \mu_f)\) is a neutral triple for \( f \) and for all \( \varphi_i \). For the rest of this section we will use this notation to talk about this element. For every \( \varphi \in A \) we choose a neutral triple \((\lambda, \nu, \mu)\). Obviously \((\lambda, \nu, \mu)\) is a neutral triple for \( \varphi \otimes v \) for every \( v \in g \).

From the proof of [7, Theorem 5.1.10] we can extract the following Lemma [4.12].

**Lemma 4.12.** Let \( A \) be a pseudo-unital associative commutative locally convex algebra, \( g \) a finite-dimensional perfect Lie algebra and \( V \) a locally convex space, \( \omega \in Z^2_A(A \otimes g, V) \), \( f \in A \otimes g \) and \( y \in g \). If \((\lambda_1, \nu_1, \mu_1)\) and \((\lambda_2, \nu_2, \mu_2)\) are neutral triples for \( f \), then \( \omega(f, \lambda_1 \otimes y) = \omega(f, \lambda_2 \otimes y) \).

We now prove that [7, Theorem 5.1.14] also holds for a class of locally convex algebras that contains the compactly supported smooth functions on a \( \sigma \)-compact finite-dimensional manifold.

**Theorem 4.13.** Let \( A \) be a complete pseudo-unital algebra that is the strict inductive limit of locally convex subalgebras \( A_m \subseteq A \) for which we can find an element \( 1_m \in A \) with \( 1_m \cdot a = a \) for all \( a \in A_m \). Moreover, let \( g \) be a semisimple finite-dimensional Lie algebra, \( V \) a locally convex space and \( i : A \otimes g \rightarrow A \otimes g \times g \) the natural inclusion. Then \( H^2_{A}(i) : H^2_{A}(A \otimes g \times g, V) \rightarrow H^2_{A}(A \otimes g, V) \) is bijective.

**Proof.** Surjectivity: Let \((v_i)_{i=1, \ldots, n}\) be a basis of \( g \). We use the notation of Definition [4.11]. Given \( \omega_0 \in Z^2_A(A \otimes g, V) \), we define \( \omega : A \otimes g \times g \rightarrow V \) \((f_1, y_1), (f_2, y_2) \mapsto \omega_0(f_1, f_2) + \omega_0(f_1, \lambda_{f_1} \otimes y_1) - \omega_0(f_2, \lambda_{f_2} \otimes y_1)\).
For \( f, g \in A \otimes g, r \in \mathbb{K} \) and \( y \in g \) we can choose a neutral triple \((\lambda, \nu, \mu)\) that is neutral for \( f \) and for \( g \). Especially this triple is also neutral for \( rf + g \). Because we now get
\[
\omega_0(rf + g, \lambda_f \otimes y) = \omega_0(rf + g, \lambda \otimes y) = r\omega_0(f, \lambda \otimes y) + \omega_0(g, \lambda_f \otimes y) = \omega_0(g, \lambda_g \otimes y),
\]
we can easily proof that \( \omega \) is bilinear. Obviously \( \omega \) is anti-symmetric.

The argument that \( \omega \) is a cocycle works exactly like in the proof [7, 5.10.]. For the convenience of the reader we will recall this argument in the appendix.

We now want to show that \( \omega \) is also continuous. We just have to show that the bilinear map \( \psi : A \otimes g \times g \to V, (f, y) \mapsto \omega_0(f, \lambda_f \otimes y) \) is continuous. Because we can identify \((A \otimes g) \otimes g \) with \((A \otimes g)^n\), it is sufficient to prove the continuity of \( (A \otimes g)^n \to V, (f_i)_{i=1,...,n} \mapsto \sum_{i=1}^n \omega_0(f_i, \lambda_{f_i} \otimes v_i) \). To show the continuity of \( A \otimes g \to V, f \mapsto \omega_0(f, \lambda_f \otimes v) \), we again identify \( A \otimes g \) with \( A^n \) and prove the continuity of \( A^n \to V, (\varphi_i)_{i=1,...,n} \mapsto \sum_{i=1}^n \omega_0(\varphi_i \otimes v_i, \lambda_{f_i} \otimes y) \) with \( f = \sum_{i=1}^n \varphi_i \otimes v_i \) and an arbitrary \( y \in g \). But because of the construction of the neutral triple \((\lambda_f, \nu_f, \mu_f)\) we get \( \omega_0(\varphi_i \otimes v_i, \lambda_f \otimes y) = \omega_0(\varphi_i \otimes v_i, \lambda_{f_i} \otimes y) \) for \( i \in \{1,...,n\} \). It remains to show that the linear map \( A \to V, \varphi \mapsto \omega_0(\varphi \otimes x, \lambda_x \otimes y) \) is continuous for \( x, y \in g \). But for \( m \in \mathbb{N} \) we find an element \( 1_m \in A \) with \( 1_m \cdot a = a \) for all \( a \in A_m \). We choose an element \( \tilde{1}_m \in A \) that is unital for \( 1_m \) and an element \( \tilde{1}_m \) that is unital for \( \tilde{1}_m \) and see that \((\tilde{1}_m, \tilde{1}_m, 1_m)\) is a unital triple for every \( \varphi \in A_m \). We see that \( A_m \to V, \varphi \mapsto \omega_0(\varphi \otimes x, \lambda_{\varphi} \otimes y) \) is continuous and conclude that also the map \( A \to V, \varphi \mapsto \omega_0(\varphi \otimes x, \lambda_{\varphi} \otimes y) \) is continuous, because \( A_m \) is the inductive limit of the subalgebras \( A_m \subseteq A \).

The equation \( H^2_{\alpha}(i)([\omega]) = [\omega_0] \) is easily checked, because for \( f, g \in A \otimes g \) we have \( \omega \circ (i, i)(f,g) = \omega(f,g) = \omega_0(f,g) \).

The argument that \( H^2_{\alpha}(i) \) is injective works exactly like in the proof [7, 5.10.]. For the convinience of the reader we will recall this argument in the appendix. \( \square \)

**Remark 4.14.** It is clear that Theorem [7 Theorem 5.1.14] is a direct consequence of Theorem 4.13. Moreover we see that the class of locally convex algebras \( A \) considered in 4.13 contains the compactly supported functions on a finite-dimensional \( \sigma \)-compact manifold \( M \).

**Remark 4.15.** Although the class of algebras in 4.13 contains the compactly supported smooth functions on a \( \sigma \)-compact manifold \( M \) considered in [8, Theorem 2.7.], it contains other interesting algebras, e.g. the compactly supported continuous functions on a \( \sigma \)-compact manifold. So Theorem 4.13 can also be understood as a generalisation of [8, Theorem 2.7.]. Also in [8, Theorem 2.7.] Janssens and Wockel do not discuss if the constructed cocycle \( \omega \) that is mapped to \( \omega_0 \) by \( H^2_{\alpha}(i) \) is actually continuous. For this point an argument is given in the above proof.
The application of Theorem 4.13 is not very difficult.

**Corollary 4.16.** Let \( A \) be a locally convex commutative and associative algebra, such that it is the inductive limit of complete subalgebras \( A_n \subseteq A \) with \( n \in \mathbb{N} \), such that we find for every \( n \in \mathbb{N} \) an element \( 1_n \in A \) with \( 1_n \cdot a = a \) for all \( a \in A_n \). Moreover let \( \mathfrak{g} \) be a finite-dimensional perfect Lie algebra, then \( \omega_{\mathfrak{g},A} : A \otimes \mathfrak{g} \times A \otimes \mathfrak{g} \rightarrow V_{\mathfrak{g},A} \) with \( (a \otimes x, b \otimes y) \mapsto \kappa_{\mathfrak{g}}(x, y) \otimes [a \cdot d_{A_1}(b)] \) is a universal cocycle for \( A \otimes \mathfrak{g} \).

**Proof.** The assertion follows directly from Lemma 4.3 and Theorem 4.13. \( \Box \)

The transition of the fact stated in Corollary 4.16 for current algebras of the form \( C_e^\infty(M, \mathfrak{g}) \) can be done like in [7, Chapter 5.2.] or [8, Theorem II.7.]. We recall this approach in Remark 4.17.

**Remark 4.17.** [10] Theorem 11] tells us that if \( M \) is a \( \sigma \)-compact manifold, \( \Omega^1(C_e^\infty(M)) \) the universal \( C_e^\infty(M) \)-module in the category of complete locally convex spaces, then \( d \circ pr_1 : C_e^\infty(M) \rightarrow \Omega^0(M) \) induces an isomorphism of topological \( C_e^\infty(M)_1 \)-modules \( \Omega^1(C_e^\infty(M)) \rightarrow \Omega^1(M) \).

Let be \( \mathfrak{g} \) a semi simple Lie algebra. For \( g \in C_e^\infty(M, \mathfrak{g}) \) and \( \eta \in \Omega^0(M, \mathfrak{g}) \) we define the 1-form \( \kappa_g(f, dg) \in \Omega^1(M, \mathfrak{g}) \), by \( \kappa_g(f, \eta)_p(v) := \kappa_g(f(p), \eta_p(v)) \).

Because \( d(C_e^\infty(M, \mathbb{R}^n)) \) is closed in \( \Omega^1(M, \mathbb{R}^n) \) the map

\[
C_e^\infty(M, \mathfrak{g}) \times C_e^\infty(M, \mathfrak{g}) \rightarrow \Omega_*(M, \mathfrak{g})/d(C_e^\infty(M, \mathfrak{g}))
\]

\[
(f, g) \mapsto [\kappa_g(f, dg)].
\]

is a universal cocycle for all complete locally convex spaces.

**Appendix**

**Details for the proof of 4.13:** The following argument follows the proof of [7 Theorem 5.1.10]. We use the notation from the proof of Theorem 4.13.

In order to show that \( \omega \in Z^2(A \otimes \mathfrak{g}, \mathfrak{g}, V) \) we choose \( f, g, h \in A \otimes \mathfrak{g} \) and \( x, y, z \in \mathfrak{g} \). First we mention the trivialities \( d\omega(f, g, h) = d\omega_0(f, g, h) = 0 \) and \( d\omega(x, y, z) = 0 \). We can choose a trivial \( (\lambda, \nu, \mu) \) that is neutral for \( f \) and \( g \), and we can write \( f = \sum_{i=1}^n f_i \otimes v_i \) as well as \( g = \sum_{j=1}^n g_j \otimes v_j \). We calculate

\[
\lambda \cdot [f, g] = \sum_{i,j} \lambda f_i g_j \otimes [v_i, v_j] = [\lambda f, g] = [f, g]
\]

and see that \( (\lambda, \nu, \mu) \) is a neutral triple for \( [f, g] \). Now we calculate

\[
d\omega(f, g, y) = \omega([f, g], y) + \omega([g, y], f) + \omega([y, f], g)
\]
\[ \omega_0([f, g], \lambda \otimes y) + \omega_0([g, y], f) + \omega_0([y, f], g) = \omega_0([f, g], \lambda \otimes y) = \omega([g, \lambda \otimes y], f) + \omega([\lambda \otimes y, f], g) = dw_0(f, g, \lambda \otimes y) = 0. \]

To check that \( \omega \) is a cocycle we calculate
\[
d\omega((f, x), (g, y)(h, z)) = d\omega(f, g, h) + d\omega(f, g, z) + d\omega(f, y, h) + d\omega(f, y, z) + d\omega(x, g, h) + d\omega(x, y, h) + d\omega(x, y, z) = 0.
\]

It remains to show the injectivity of the map \( H^2_{cl}(i) \). Let \( \omega \in Z^2_{cl}(A \otimes g \times g, V) \) with \( \omega \circ (i, i) = \eta \circ [\_, \_] \) for \( \eta \in \text{Lin}_{cl}(A \otimes g, V) \). We define the continuous linear map \( \eta' : A \otimes g \times g \to V \), \((f, v) \mapsto \eta(f)\). We define the cocycle \( \omega' := \omega - \eta' \circ [\_, \_] \) on \( A \otimes g \times g \). For all \( f, g, h \in A \otimes g \) we have
\[
\omega'((f, g), (h, z)) = \omega'((h, z), (f, g)) = \omega'([g, y], f) + \omega'([y, f], g) = \omega'([f, g], y).
\]

Because \( A \otimes g \) is perfect, we get that \( \omega' \) vanishes \( 0 \) on \( A \otimes g \times g \) in terms of the natural identifications. For \( f_1, f_2 \in A \otimes g \) and \( y_1, y_2 \in g \) we have
\[
\omega'((f_1, y_1), (f_2, y_2)) = \omega'((f_1, f_2)) + \omega'((y_1, f_2)) + \omega'((f_1, y_2)) + \omega'((y_1, y_2)).
\]

Because \( g \) is a subalgebra of \( A \otimes g \times g \) we get \( \omega|_{g \times g} \in Z^2_{cl}(g, V) \) and because \( g \) is semisimple, we get with the Whitehead theorem for locally convex spaces, stated in [7, Corollary A.2.9], that \( H^2_{cl}(g, V) = \{0\} \). Therefore, we find \( \eta'' \in \text{Lin}_{cl}(g, V) \) with \( \omega|_{g \times g} = \eta'' \circ [\_, \_] \). Finally we see \( \omega' = \eta'' \circ [\_, \_] \) with \( \eta'' : A \otimes g \times g \to V \).

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