A property of discriminants

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Abstract

For the family $P := x^n + a_1 x^{n-1} + \cdots + a_n$ of complex polynomials in the variable $x$ we study its discriminant $R := \text{Res}(P, P', x)$, $R \in \mathbb{C}[a]$, $a = (a_1, \ldots, a_n)$. When $R$ is regarded as a polynomial in $a_k$, one can consider its discriminant $\hat{D}_k := \text{Res}(R, \partial R/\partial a_k, a_k)$. We show that $\hat{D}_k = c_k(a_k) d(n,k) M_k^2 T_k^3$ where $c_k \in \mathbb{Q}^*$, $d(n,k) := \min(1, n-k) + \max(0, n-k-2)$, the polynomials $M_k, T_k \in \mathbb{C}[a_k]$ have integer coefficients, $a_k = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$, the sets $\{M_k = 0\}$ and $\{T_k = 0\}$ are the projections in the space of the variables $a_k$ of the closures of the strata of the variety $\{R = 0\}$ on which $P$ has respectively two double roots or a triple root. Set $P_k := P - x P'/ (n-k)$ for $1 \leq k \leq n-1$ and $P_n := P'$. One has $T_k = \text{Res}(P_k, P'_k, x)$ for $k \neq n-1$ and $T_{n-1} = \text{Res}(P_{n-1}, P'_{n-1}, x)/a_n$.

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1 Introduction

In the present paper we consider the general family of monic degree $n$ complex polynomials in one variable $P := x^n + a_1 x^{n-1} + \cdots + a_n$. (For $a_1 = 0$ this is the versal deformation of the $A_{n-1}$ singularity, see [2]). Its discriminant is the resultant $R := \text{Res}(P, P', x)$, i.e. the determinant of the Sylvester matrix $S(P, P', x)$. We remind that $S(P, P', x)$ is $(2n-1) \times (2n-1)$, its first (resp. $n$th) row equals

$$(1, a_1, \ldots, a_n, 0, \ldots, 0) \quad \text{(resp.} \quad (n, (n-1)a_1, \ldots, a_{n-1}, 0, \ldots, 0) \quad ),$$

its second (resp. $(n+1)$st) row is obtained by shifting the first (resp. the $n$th) one to the right by one position while adding 0 to the left etc. Set $a := (a_1, \ldots, a_n)$, $a^k := (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$ and $R_{ak} := \partial R/\partial a_k$. It is well-known that:

A) $R$ is a quasi-homogeneous polynomial in the coefficients $a_j$, where the quasi-homogeneous weight of $a_j$ equals $j$. It is a degree $n$ polynomial in each of the variables $a_j$, $1 \leq j \leq n-1$, and a degree $n-1$ polynomial in $a_n$.

B) The set $\{R = 0\}$ is the set of values of the coefficients $a_j$ for which $P$ has a multiple root. It contains the subsets $\Sigma$ and $\tilde{M}$ (the Maxwell stratum) such that for $a \in \Sigma$ (resp. $a \in \tilde{M}$) the polynomial $P$ has a root of multiplicity 3 (resp. has two different double roots). The semi-algebraic sets $\Sigma$ and $\tilde{M}$ are irreducible. Indeed, the closure of $\Sigma$ is the image of the map $\mathbb{C}^{n-2} \to \mathbb{C}^{n-2}$, $(z_1, z_4, z_5, \ldots, z_n) \mapsto a$, where in the computation of $(-1)^j a_j$ as $j$th elementary symmetric function of $z_1, \ldots, z_n$ one sets $z_2 = z_3 = z_1$; the closure of $\tilde{M}$ is the image of the map $\mathbb{C}^{n-2} \to \mathbb{C}^{n-2}$, $(z_1, z_3, z_5, z_6 \ldots, z_n) \mapsto a$, where in the computation of $a$ one sets $z_2 = z_1$ and
It is easy to see that the intersections of the sets $\Sigma$ and $\tilde{M}$ with each of the subspaces \( \{a_j = 0\} \) are proper subsets of $\Sigma$ and $\tilde{M}$.

One can consider $R$ as a polynomial in $a_k$, with coefficients in $\mathbb{C}[a^k]$. Thus one is led to consider the repeated resultants $\tilde{D}_k := \text{Res}(R, R_{a_k}, a_k)$. The following result is proved in [5] (see Proposition 7 there):

**Lemma 1.** Set $d(n, k) := \min(1, n - k) + \max(0, n - k - 2)$. The polynomial $\tilde{D}_k$ equals $(a_n)^{d(n, k)} \tilde{D}_k^0$, where $\tilde{D}_k^0 \in \mathbb{C}[a]$ is not divisible by any of the variables $a_i$, $1 \leq i \leq n$.

**Example 2.** For $n = 3$ one has $P := x^3 + ax^2 + bx + c$, $P' = 3x^2 + 2ax + b$ and

$$R := \text{Res}(P, P', x) = 4a^3c - a^2b^2 - 18abc + 4b^3 + 27c^2.$$ 

Set $\tilde{D}_a := \text{Res}(R, \partial R/\partial a, a)$ and similarly for $\tilde{D}_b$ and $\tilde{D}_c$. Hence

$$\tilde{D}_a = -64c(b^3 - 27c^2)^3, \quad \tilde{D}_b = -64c(a^3 - 27c)^3 \quad \text{and} \quad \tilde{D}_c = -432(-3b + a^2)^3.$$

**Example 3.** For $n = 4$ one has $P := x^4 + ax^3 + bx^2 + cx + d$, $P' = 4x^3 + 3ax^2 + 2bx + c$ and

$$R := \text{Res}(P, P', x) = -27a^4d^2 + 18a^3bcd - 4a^3c^3 + a^2b^2c^2 + 144a^2bd^2 - 4a^2b^3d$$

$$-6a^2c^2d - 80abc^2d + 18abc^3 - 192acd^2 + 16b^4d$$

$$-4b^3c^2 - 128b^2d^2 + 144bc^2d - 27c^4 + 256d^3.$$ 

One finds that

$$\tilde{D}_a = 6912d^2M_a^2T_a^3, \quad \tilde{D}_b = -4096dM_b^2T_b^3, \quad \tilde{D}_c = 6912dM_c^2T_c^3 \quad \text{and} \quad \tilde{D}_d = 4096M_d^2T_d^3,$$

where the factors $M_a, T_a, M_b, \ldots, T_d$ are irreducible:

$$M_a = 16b^2d^2 - 8bc^2d + c^4 - 64d^3, \quad T_a = 3b^4d - b^3c^2 + 72b^2d^2 - 108bc^2d + 27c^4 + 432d^3$$

$$M_b = a^2d - c^2, \quad T_b = 27a^4d^2 - a^3c^3 - 6a^2c^2d - 768acd^2 + 27c^4 + 4096d^3$$

$$M_c = a^4 - 8a^2b + 16b^2 - 64d, \quad T_c = 27a^4d^2 - a^2b^3 - 108a^2bd + 3b^4 + 72b^2d + 432d^2$$

$$M_d = a^3 - 4ab + 8c, \quad T_d = 27a^3c - 9a^2b^2 - 108abc + 32b^3 + 108c^2.$$ 

One can notice that the equation $M_b = 0$ defines the Whitney umbrella.

We prove the following theorem:

**Theorem 4.** For $n \geq 4$ the polynomial $\tilde{D}_k$ is of the form $c_k(a_n)^{d(n, k)}M_k^2T_k^3$, where $c_k \in \mathbb{Q}^*$, the degree $d(n, k)$ is defined in Lemma 7 and the polynomials $M_k, T_k \in \mathbb{C}[a^k]$ are with integer coefficients and irreducible. The zero sets of these polynomials are the closures of the projections in the space of the variables $a^k$ of the sets $\tilde{M}$ and $\Sigma$.

The proofs of Theorem 4, Lemma 7 and Lemma 8 are to be found in Section 3.

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2 Comments and lemmas

Theorem 4 is formulated for \( n \geq 4 \) because for \( n < 4 \) the set \( \tilde{M} \) does not exist. In Example 2 only the cubes of the factors \( T_k \) and the powers of \( a_n \) (i.e., of \( c \)) are present.

It is well-known that \( R = \prod_{1 \leq i < j \leq n} (z_i - z_j)^2 \). Denote by \( \Delta \) the union of hyperplanes \( \{ z_i = z_j \} \) in the space \( \mathbb{C}^n \) of the roots of the polynomial \( P \). In the last presentation of \( R \) as a product it is necessary to have the differences of roots \( z_i - z_j \) squared because when the roots change continuously along a loop avoiding the set \( \Delta \) so that in the end two of them are exchanged, then such an exchange should not change the value of \( R \).

By analogy, the fact that the power of the factor \( T_k \) in the formula for \( \tilde{D}_k \) in Theorem 4 is a multiple of 3 can be explained like this. At a point \( a = a^* \in \Sigma \) (we assume that \( a^* \notin \Sigma \setminus \Sigma \)) three roots \( z_1, z_2, z_3 \) of \( P \) coalesce. For fixed nearby values of \( a^k \) the polynomial \( R \) (when considered as a polynomial in \( a_k \)) has two roots \( \zeta_1 \) and \( \zeta_2 \) that coalesce for \( a^k = a^{*k} \) (the projection of \( a^* \) in the space of the variables \( a^k \)). These roots correspond to equalities and inequalities between the roots of \( P \) of the form \( z_1 = z_2 \neq z_3 \) and \( z_1 \neq z_2 = z_3 \) for \( a^k \neq a^{*k} \), and to \( z_1 = z_2 = z_3 \) for \( a^k = a^{*k} \). When the \((n-1)\)-tuple of coefficients \( a^k \) circumvents the projection \( \Sigma_k \) of \( \Sigma \) in the space of the variables \( a^k \) along a generic loop, the three roots \( z_i \) of \( P \) undergo a cyclic permutation of order 3 and now the roots \( \zeta_1 \) and \( \zeta_2 \) of \( R \) correspond to other equalities and inequalities between the roots \( z_i \), namely, to \( z_3 = z_1 \neq z_2 \) and \( z_3 \neq z_1 = z_2 \). In order \( \tilde{D}_k \) to be invariant w.r.t. such permutations the power of \( T_k \) dividing the resultant \( \tilde{D}_k \) must be a multiple of 3.

For the power of \( M_k \) being even a similar explanation exists. To this end we remind first some facts about \( R \) for \( n = 4 \). The formula for \( R \) was obtained in Example 3. On Fig. 1 we show for real values of \( c \) and \( d \) the sets \( \{ R = 0 \}|_{a=0,b=-1}, \{ R = 0 \}|_{a=b=0} \) and \( \{ R = 0 \}|_{a=0,b=1} \) (from left to right) which are symmetric w.r.t. the \( d \)-axis. This figure can be compared with the well-known double point of the swallowtail catastrophe, see 7. Fig. 1 gives a sufficient idea about the set \( \{ R = 0 \}|_{a=0} \) because the set \( \{ R = 0 \} \) is invariant under the quasi-homogeneous dilatations \( a \mapsto ta, b \mapsto t^2b, c \mapsto t^3c, d \mapsto t^4d, t \neq 0 \).

At the points \( U \) and \( V \) the polynomial \( P \) has a triple real and a simple real root \( (U \) and \( V \) are ordinary 2/3-cusp points for the real curve \( \{ R = 0 \}|_{a=0,b=-1} \). One has

\[
\Sigma \cap \{ a = 0, b = -1 \} = \{ U, V \}, \quad \tilde{M} \cap \{ a = 0, b = -1 \} = \{ S \}.
\]

At the point \( S \) (with \( d \)-coordinate equal to 1/4) the curve \( \{ R = 0 \}|_{a=0,b=-1} \) has transversal self-intersection and the polynomial \( P \) has two double real roots. At the point \( T \) (which is an isolated double point of the real curve \( \{ R = 0 \}|_{a=0,b=1} \), with \( d \)-coordinate equal to 1/4) the polynomial \( P \) has a double complex conjugate pair. At the points \( I, J \) and \( K \) one has \( c = d = 0 \). The real curves \( \{ R = 0 \}|_{a=0,b=-1} \) and \( \{ R = 0 \}|_{a=0,b=1} \) are smooth at \( I \) and \( K \) respectively while \( \{ R = 0 \}|_{a=b=0} \) has a 4/3-type singularity at \( J \).

From now on we keep in mind that the set \( \{ R = 0 \} \) can be defined in both contexts – the ones of real or of complex variables \( x, a, b, c \) and \( d \). In this sense we make use of Fig. 1 as an illustration of the real case and as a hint for the complex one. Why for \( n = 4 \) the powers of the factors \( M_k \) should be even is suggested by the following lemma. For \( n > 4 \) the analogs of the loops \( \tilde{\gamma} \) and \( \Gamma \) of the lemma exist in a neighbourhood of any value of the parameters \( a_j \) for which the polynomial \( P \) has a quadruple root, but their explicit construction is harder to describe.

**Lemma 5.** In the complex case there exists a loop \( \tilde{\gamma} \) belonging to the space of variables \((b, c)\) which can be lifted to a loop \( \Gamma \subset \{ R = 0 \}|_{a=0} \) circumventing the set \( \Sigma \cup \tilde{M} \) such that any fibre
of the projection $\Gamma \rightarrow \gamma$ consists of two points and the monodromy defined on the fibre after one turn along $\gamma$ is nontrivial.

Proof. In what follows an additional index $d$ denotes the projection of a given set in the space of variables $(b, c, d)$ ($a$ is presumed equal to 0) into the space of variables $(b, c)$. Consider the point $A$ on Fig. 1. We are going to construct a continuous path $\gamma \subset \{R = 0\}_{a=0}$ leading from $A$ to $G$, one of the two points of $\{R = 0\}_{a=0}$ which share with $A$ the same $b$- and $c$-coordinates as shown on Fig. 1. As $b$ increases from $-1$ to 1, the point $A$ becomes the point $B$ for $b = 0$ and then $C$ for $b = 1$. Then we decrease $c$ by keeping the same value of $b$ – this gives the arc $CKD$. Then we fix $c$ and decrease $b$ – this gives the arc $DEF$. Finally we add the arc $FG$. The thus constructed path is real. Three remarks will be needed for what follows:

1) The path $\gamma$, in its part between the points $A$ and $F$, can be constructed as symmetric w.r.t. the plane $\{c = 0\}$.

2) The projection $\Sigma_d$ of $\Sigma$ is defined by $32b^3 + 108c^2 = 0$, i.e. $8b^3 + 27c^2 = 0$; the equation of this semi-cubic parabola is obtained from the equation $T_d = 0$ by setting $a = 0$, see Example 3. There exists a unique number $b_0 \in (-1, 0)$ such that for $b = b_0$ the projection $\gamma_d$ of $\gamma$ intersects $\Sigma_d$ at two points $(b_0, \pm c_0)$.

3) In the real case the path $\gamma$ has to pass through the point $S \in \tilde{M}$, but in the complex one $\gamma$ can be modified so that it circumvent $S$. The points of the modified path $\gamma$ which are close to $S$ do not have all their coordinates real.

Now we construct (in the complex case) a path $\gamma^1 \subset \{R = 0\}_{a=0}$ leading from $G$ to $A$ and satisfying the condition $\gamma^1_a = \gamma_d$. At the same time we modify the path $\gamma$ in order to have this condition. If the path $\gamma_1$ is defined such that $\gamma^1_d = \gamma_d$, then for $b = b_0$, $\gamma_1$ will intersect the set $\Sigma$. Therefore for $b$ close to $b_0$ we modify $\gamma_1$ and $\gamma$ so that $\gamma^1$ avoid the set $\Sigma$. (We make two such modifications, corresponding to points of $\gamma_d$ and $\gamma^1_d$ close to $(b_0, c_0)$ and to $(b_0, -c_0)$. The modifications can be made symmetrically w.r.t. the plane $\{c = 0\}$.)

For the values of $b$ close to $b_0$ the points of $\gamma$ do not have all their coordinates real. As for $\gamma^1$, its points do not have all coordinates real not only for $b$ close to $b_0$, but also for $b \in [b_0, 1]$ (recall the construction of the arcs $ABC$ and $DEF$ of $\gamma$) and for $b = 1, c \neq 0$ (recall the construction of its arc $CKD$). Indeed, as $R$ is a degree 3 polynomial in $d$, then in the real case it has either three real roots (see for instance the vertical line on the left part of Fig. 1 which intersects the set $\{R = 0\}$ at three points two of which are $A$ and $G$) or one real and two complex conjugate ones; this is, in particular, the case of any vertical line different from the $d$-axis for $b = 1$, see

![Figure 1: The sets $\{R = 0\}_{a=0,b=-1}$, $\{R = 0\}_{a=b=0}$ and $\{R = 0\}_{a=0,b=1}$ for $n = 4$.](image)
the right part of Fig. 1. (The d-axis on the right part of the figure corresponds to one simple root at 0 and a double one at $1/4$. One simple and one double real root is also the situation observed on the vertical lines passing through the points $U$ and $V$.)

To obtain the proof of the lemma one sets $\gamma = \gamma_d = \gamma_1^1$ and one defines the loop $\Gamma$ as the concatenation of $\gamma$ and $\gamma_1^1$. For points of $\gamma$ and $\gamma_1^1$ close to the point $S$ one has $\gamma_d = \gamma_1^1$ and no self-intersection of $\Gamma$ takes place. □

Remarks 6. (1) To prove Theorem 4 we need to recall some notation and results from [5]. Suppose that $G_1$ and $G_2$ are polynomials in several variables one of which is denoted by $y$. By $S(G_1, G_2, y)$ we denote the Sylvester matrix of $G_1$ and $G_2$ when considered as polynomials in $y$. We set $P_k := P - xP'/(n - k)$ for $1 \leq k \leq n - 1$ and $P_n := P'$.

(2) It is shown in [5] that for $k \neq n - 1$ the polynomial $V_k := \text{Res}(P_k, P', x)$ is irreducible and that the polynomial $\text{Res}(P_{n-1}, P_n', x)$ is the product of $a_n$ and an irreducible polynomial in $a_n^{-1}$. We set $V_{n-1} := \text{Res}(P_{n-1}, P_n', x)/a_n$. It follows from Theorem 12 of [5] that $V_k = T_k$, $k = 1, \ldots, n$. Theorem 4 allows to find the polynomials $M_k$ and $T_k$; however the definition of $T_k$ as $T_k = V_k$ is an easier way to find $T_k$.

(3) We denote by $\text{QHD}(U)$ the quasi-homogeneous degree of a quasi-homogeneous polynomial $U \in \mathbb{C}[a]$, where the quasi-homogeneous weight of $a_k$ is $k$.

(4) Set $Q_k := (n - k)P_k = (n - k)P - xP'$, $k \leq n - 1$, $Q_n := P'$. When we compare polynomials $P_k$, $Q_k$, $R$ or $V_k$ for two consecutive values of $n$ (i.e. for $n$ and $n + 1$) we write $P_n$, $P_{n+1}$, $Q_n$, $Q_{n+1}$, $R_n$, $R_{n+1}$ or $V_n$, $V_{n+1}$. Notice that as $Q_k = -kx^n + \sum_{j=1}^{n}(j - k)a_jx^{n-j}$, one has

$$Q_k^{n+1} = xQ_k^n + (n + 1 - k)a_{n+1} \quad \text{and} \quad (Q_k^{n+1})' = x(Q_k^n)' + Q_k^n. \quad (1)$$

In the following lemma and its proof $\Omega$ denotes unspecified nonzero rational numbers.

**Lemma 7.** (1) One has $V_* := V_k^{n+1}|_{a_{n+1} = 0} = \Omega(a_n)^2V_k^n$ for $1 \leq k \leq n - 2$, $V_* = \Omega(a_n)^3V_k^n$ for $k = n - 1$ and $V_* = \Omega(a_{n-1})^3V_k^n$ for $k = n$.

(2) One has $R_{n+1}|_{a_{n+1} = 0} = \pm a_n^2R_n$.

The following lemma announces the quasi-homogeneous degrees of certain polynomials that appear in this text:

**Lemma 8.** For $n \geq 4$ one has the following quasi-homogeneous degrees of polynomials:

(1) $\text{QHD}(R) = \text{QHD}(V_k) = n(n - 1)$, $1 \leq k \leq n - 2$.

(2) $\text{QHD}(V_{n-1}) = n(n - 2)$.

(3) $\text{QHD}(V_n) = (n - 1)(n - 2)$.

(4) $\text{QHD}(R_{a_n}) = n(n - 1) - k$, $1 \leq k \leq n - 2$, $\text{QHD}(R_{a_{n-1}}) = n^2 - 3n + 1$, $\text{QHD}(R_{a_n}) = n^2 - 4n + 2$.

(5) $\text{QHD}(\tilde{D}_k) = n(n - 1)^2 + n^2(n - k - 1)$, $1 \leq k \leq n - 1$, $\text{QHD}(\tilde{D}_n) = n(n - 1)(n - 2)$.

(6) $\text{QHD}(M_k) = n^3 - 3n^2 + 2n - (n^2 - n)(k + 1)/2$, $1 \leq k \leq n - 2$, $\text{QHD}(M_{n-1}) = n(n - 2)(n - 3)/2$.

(7) $\text{QHD}(M_n) = (n - 1)(n - 2)(n - 3)/2$.

### 3 Proofs

**Proof of Lemma 7.** The equality $A = [B]_{i, r}$ means that the matrix $A$ is obtained from the matrix $B$ by deleting its $i$th row and $r$th column. Prove part (1). In the proof of the lemma we use the polynomials $Q_k$ instead of $P_k$. For $1 \leq k \leq n - 2$ set $Q_* := Q_k^{n+1}|_{a_{n+1} = 0} = xQ_k^n$. Consider the $(2n + 1, 2n + 1)$-Sylvester matrix $S_* := S(Q_*, Q'_*, x)$. The only nonzero entry in its last column
is $\Omega a_n$ in position $(2n + 1, 2n + 1)$. Hence when finding its determinant $\Omega V_n$ one can develop it w.r.t. the last column to obtain $V_n = \Omega a_n V_*$, where $V_* = \det S_{**}$, $S_{**} = [S_j]_{2n+1,2n+1}$.

Subtract for $j = 1, \ldots, n$ the $j$th row of $S_{**}$ from its $(n + j)$th row. This doesn’t change $V_*$. Hence the terms $\Omega a_n$ disappear in the $(n + 1)$st, $\ldots$, $(2n)$th rows of $S_{**}$, see (1). The only nonzero entry of the new matrix (denoted by $S_{***}$) in its last column is $\Omega a_n$ in position $(n, 2n)$. It is easy to see that $[S_{***}]_{n,2n} = S(Q^{n+1}_k, (Q^{n+1}_k)', x)$ (this can be deduced from (1)). Hence $V_* = \det S_{**} = \Omega a_n V_k^*$ and $V_* = \Omega(a_n)^2 V_k^*$.

For $k = n - 1$ the above reasoning differs only in the end – one defines $V_{n-1}^n$ not as $\det([S_{***}]_{n,2n})$ (the latter is divisible by $a_n$), but as $\det([S_{***}]_{n,2n})/a_n$. Hence $V_* = \Omega(a_n)^2 V_{n-1}^n$.

For $k = n$ consider the $(2n + 1) \times (2n + 1)$-matrix $S^0 := S(Q^{n+1}_n, (Q^{n+1}_n)', x)$. Its last column contains a single nonzero entry (in position $(n, 2n + 1)$). By definition $S^0 = \Omega a_{n+1} V_{n+1}^n$. Hence $V_* = \Omega \det S^1$, where $S^1 = ([S^0]_{n,2n+1})|_{a_n=1}=0$.

The last column of $S^1$ contains a single nonzero entry ($\Omega a_{n-1}$ in position $(2n, 2n)$), so to find $\det S^1$ one can develop it w.r.t. the last column. This gives $V_* = \Omega a_{n-1} \det S^1$, where $S^1 = [S^1]_{2n,2n}$.

Subtract the $j$th row of $S^1$ from its $(n - 1 + j)$th one, $j = 1, \ldots, n - 1$; hence the terms $\Omega a_{n-1}$ disappear in the $n$th, $\ldots$, $(2n - 2)$nd rows (see (1)). This gives the matrix $S^1$ such that $\det S^1 = \det S^0$.

The only nonzero entry in the last column of $S^1$ is $\Omega a_{n-1}$ in position $(2n - 1, 2n - 1)$. Hence $\det S^1 = \Omega a_{n-1} \det S^1$, where $S^1 = [S^1]_{2n-1,2n-1}$. The only nonzero entry of $S^1$ in its last column is in position $(n - 1, 2n - 2)$ and equals $\Omega a_{n-1}$. Thus $V_* = \Omega(a_{n-1})^2 \det S^1$, where $S^1 = [S^1]_{n-1,2n-2}$. The $(2n - 3) \times (2n - 3)$-matrix $S^1$ contains $S(Q_n^a/x, (Q_n^a/x)', x)$, i.e. $\Omega S((P^n)^a, (P^n)^a)'$.

To prove part (2) one notices that for $a_{n+1} = 0$ one has $P^{n+1}_a = x P^n_a$ and the Sylvester matrix $S^1 := S(x P^n_a, (x P^n_a)', x)$ contains a single nonzero entry in its last column, namely $a_n$ in position $(2n + 1, 2n + 1)$. Set $S^2 := [S^1]_{2n+1,2n+1}$. Hence $R^{n+1}_1|_{a_{n+1}=0} = \det S^1 = a_n \det S^2$. For $j = 1, \ldots, n$ subtract the $j$th row of $S^2$ from its $(n + j)$th one. The newly obtained matrix (denoted by $S^3$) has a single nonzero entry in its last column. This is $a_n$ in position $(n, 2n)$. Set $S^3 := [S^2]_{n,2n}$. Hence $\det S^2 = \pm a_n \det S^3$, i.e. $R^{n+1}_1|_{a_{n+1}=0} = \pm a_n^2 \det S^3$. On the other hand $S^3 = S(P^n, (P^n)', x)$ from which part (2) follows.

\[\square\]

Proof of Lemma 3 We denote by $W$ any of the polynomials $R$, $V_k$, $k \leq n - 2$, or $a_n V_{n-1}$ and we remind that $T_k = V_k$, see Remarks 3. Any polynomial $W$ contains a monomial $\beta a_n^{n-1}$, $\beta \neq 0$. Indeed, the only positions in which the matrix $S(W, W', x)$ contains the variable $a_n$ are $(i, n + i)$, $i = 1, \ldots, n - 1$; in these positions the matrix has terms of the form $\eta a_n$, $\eta \neq 0$. When $\det(S(W, W', x))$ is computed, these terms are multiplied by the constant nonzero terms in positions $(n - 1 + j, j)$, $j = 1, \ldots, n$ to give the only monomial of the form $\beta a_n^{n-1}$ in $\det(S(W, W', x))$. Hence $QHD(R) = QHD(V_k) = QHD(a_n V_{n-1}) = n(a_n - 1)$ which proves parts (1) and (2). The proof of part (3) is analogous (one considers polynomials $W$ of degree $n - 1$ instead of $n$ and $a_{n-1}$ plays the role of $a_n$).

Part (4) follows from parts (1), (2) and (3) – when $R$ is differentiated w.r.t. $a_k$, its quasi-homogeneous degree decreases by $k$.

Prove part (5). For $a_i = 0$, $k \neq i \neq n$, $k < n$, one has $R = \Omega a_k a_n^{n-k-1} + \Omega_2 a_n^{n-1}$, $\Omega_1 \neq 0 \neq \Omega_2$, see Statement 8 in [3]. Therefore the Sylvester matrix $S(R, R_{a_k}, a_k)$ has only the following nonzero entries, in the following positions:

\[\square\]
Suppose that for some $P_k \leq \{ m \}$ intersection of $P_n$ double roots and $x$ not smooth (see Theorem 4 in [5]). It is not smooth also at points for which $a_i = 0$ (i.e. points of $\partial R^a$, $\partial R^b$, $\partial R^c$, $\partial R^d$). Further we prove the theorem by induction on $n$. Hence a priori the polynomial $\tilde{\Omega}'(2) = \tilde{\Omega}'(2)$ has only the following nonzero entries, in the following positions:

$$
\begin{align*}
\Omega_3 \text{ at } (i, i) , \\
\Omega_4 a_{n-1} \text{ at } (i, n-1 + i) , \\
\text{and } n \Omega_1 a_{n-k-1} \text{ at } (n-1 + j, j) , \\
\end{align*}
$$

Hence its determinant equals $\tilde{\Omega}' a_{n-1}^{(n-2)}$, $\tilde{\Omega}' \neq 0$. Part (5) is proved.

Part (6) follows from the previous parts, from Lemma [1] and from Theorem [4]. Indeed, for $k \leq n-2$ one has

$$
\begin{align*}
\text{QHD}(M_k) &= (\text{QHD}(\tilde{D}_k) - 3\text{QHD}(V_k - n(n-k-1)))/2 \\
&= (n(n-1)^2 + n^2(n-k-1) - 3n(n-1) - n(n-k-1))/2 \\
&= n^3 - 3n^2 + 2n - (n^2 - n)(k+1)/2 \\
\end{align*}
$$

For $k = n-1$ one obtains

$$
\begin{align*}
\text{QHD}(M_{n-1}) &= (\text{QHD}(\tilde{D}_{n-1}) - 3\text{QHD}(V_{n-1}) - n)/2 \\
&= (n(n-1)^2 - 3n(n-2) - n)/2 = n(n-2)(n-3)/2 \\
\end{align*}
$$

Finally for $k = n$ one gets

$$
\begin{align*}
\text{QHD}(M_n) &= (\text{QHD}(\tilde{D}_n) - 3\text{QHD}(V_n))/2 = (n-1)(n-2)(n-3)/2 \\
\end{align*}
$$

Proof of Theorem [4]. At a point of the set $\{ R = 0 \}$, where $P$ has one double nonzero root and $n-2$ simple roots, this set is locally the graph of a function analytic in the variables $a^k$, for any $1 \leq k \leq n$; if the double root is at $0$, then this property holds for $k = n$ and fails for $1 \leq k \leq n-1$; at a point of this set for which $P$ has a root of multiplicity $\geq 3$ the set is not smooth (see Theorem 4 in [5]). It is not smooth also at points for which $P$ has $m \geq 2$ double roots and $n-2m$ simple ones; at such points the set $\{ R = 0 \}$ is locally the transversal intersection of $m$ smooth hypersurfaces (see part (1) of Remarks 6 in [5]).

Hence a priori the polynomial $\tilde{D}_k$ is of the form $(a_n)^s M^a_k T^b_k$, where $s_k \in \mathbb{N} \cup 0$, $\alpha_k, \beta_k \in \mathbb{N}$, $\{M_k = 0\}$ (resp. $\{T_k = 0\}$) is the projection of the set $\bar{M}$ (resp. of $\bar{T}$) in the space of the variables $a^k$. The equality $s_k = d(n, k)$ follows from Lemma [1].

Further we prove the theorem by induction on $n$. For $n = 4$ its proof follows from Example [3].

Suppose that for some $a \in \mathbb{C}^{n+1}$ the polynomial $P^{n+1}$ has a simple root $h \in \mathbb{C}$. Set $x \mapsto x + h$. The new polynomial $P^{n+1}$ has a simple root at $0$ hence $a_{n+1} = 0$. The discriminant $R^{n+1}$ depends only on the differences between the roots of $P^{n+1}$ hence it remains invariant under shifts of the variable $x$. For $a_{n+1} = 0$ one can apply Lemma [7]. The lemma implies that for $k \leq n-1$ the discriminant $\text{Res}(R^{n+1}, \partial R^{n+1}/\partial a_k, a_k)$ is of the form $a^k_n M^2_k T^3_k$, $t_k \in \mathbb{N}$, i.e. one
has $\alpha_k = 2$ and $\beta_k = 3$ for $k \leq n - 1$, $a_n \neq 0$ and $a_{n-1} \neq 0$. The sets $\tilde{M}$ and $\Sigma$ are irreducible and their intersections with each of the subspaces \( \{ a_j = 0 \} \) are their proper subsets. Therefore the restriction $a_n \neq 0$ and $a_{n-1} \neq 0$ can be lifted and one concludes that $\alpha_k = 2$ and $\beta_k = 3$ for $k \leq n - 1$. The number $h \in \mathbb{C}$ is arbitrary and for $n > 4$ the set of polynomials $P^n$ without simple roots is a variety in the space of variables $a$ of codimension $\geq 3$. Hence the above reasoning is the proof that for $n + 1$ the claim of the theorem is true if $k \leq n - 1$.

To perform the induction also for $k = n$ and $k = n + 1$ we consider the discriminant of the family of polynomials $P^{n+1}_r := a_0 x^{n+1} + a_1 x^n + \cdots + a_n$. For its discriminant (denoted also by $R^{n+1}$) one has $R^{n+1} = (a_0)^{2n} \prod_{1 \leq i < j \leq n+1} (z_i - z_j)^2$ ($z_i$ being the roots of $P^{n+1}_r$, see [S]). Consider the polynomial $P_r^{n+1} := x^{n+1} P^{n+1}_r(1/x)$ (the index $r$ stands for “reverted”). Its roots equal $1/z_i$. Hence its discriminant $R^{n+1}_r$ equals

$$(a_{n+1})^{2n} \prod_{1 \leq i < j \leq n+1} (1/z_i - 1/z_j)^2 = (a_0)^{2n} \prod_{1 \leq i < j \leq n+1} (z_i - z_j)^2 = R^{n+1}.$$ 

For $P^{n+1}$ the coefficient $a_0$ plays the same role as $a_{n+1}$ plays for $P^{n+1}_r$. Denote by $\tilde{\alpha}_k$, $\tilde{\beta}_k$ the quantities $\alpha_k$, $\beta_k$ when defined for the polynomial $P^{n+1}_r$ instead of $P^{n+1}$. Hence one can make a shift $x \mapsto x + \tilde{h}$, where $\tilde{h}$ is a simple root of $P^{n+1}_r$, and in the same way as above conclude that $\tilde{\alpha}_k = 2$ and $\tilde{\beta}_k = 3$ for $k \leq n - 1$. This is tantamount to $\alpha_k = 2$ and $\beta_k = 3$ for $k \geq 2$. As $n \geq 4$, this means in particular that $\alpha_n = \alpha_{n+1} = 2$ and $\beta_n = \beta_{n+1} = 3$.

The polynomials $\tilde{D}_k$ and $V_k$ are determinants of Sylvester matrices defined after polynomials with integer coefficients. Hence $\tilde{D}_k$ and $V_k$ have also integer coefficients. Hence the polynomials $M_k$ can also be chosen with integer coefficients which implies $c_k \in \mathbb{Q}^*$.

\[ \square \]

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