A Quantized Inter-level Character in Quantum Systems

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For a quantum system subject to external parameters, the Berry phase is an intra-level property, which is gauge invariant module $2\pi$ for a closed loop in the parameter space and generally is non-quantized. In contrast, we define a inter-band character $\Theta$ for a closed loop, which is gauge invariant and quantized as integer values. It is a quantum mechanical analogy of the Euler characteristic numbers including the surface integral theorem says that there are two contributions to the Euler characteristic number for a manifold with boundary. The Gauss-Bonnet theorem says that there are two contributions to the Euler characteristic numbers including the surface integral of the Gaussian curvature and the loop integral of the geodesic curvature along the boundary. The quantum geometric potential plays the role of the geodesic curvature, and the Berry curvature difference between two levels is the analogy to the Gaussian curvature. We also generalized the quantum geometric potential to the case of degenerate quantum systems, and the quantized character $\Theta$ can be constructed accordingly.

Gauge invariant in non-degenerate quantum systems — For non-degenerate quantum systems, an inter-level gauge invariant, referred as “quantum geometric potential”, was introduced in literature [41]. Without loss of generality, we start with a non-degenerate $N$-level Hamiltonian $\hat{H}(\lambda(t))$ controlled by a real $l$-vector $\lambda(t) = \{\lambda_1(t), \lambda_2(t), \ldots, \lambda_i(t)\}$ as a function of time $t$. At each fixed $t$, a set of orthonormal eigenfunctions $|\phi_m(\lambda)\rangle$ associated with the eigenvalues $E_m(\lambda)$ are determined by $\hat{H}(\lambda)|\phi_m(\lambda)\rangle = E_m(\lambda)|\phi_m(\lambda)\rangle$, ($m = 1, 2, \ldots, N$). The Berry connection for each energy level is defined as $A_m^\mu = i\langle\phi_m(\lambda)|\partial_{\lambda^\mu}|\phi_m(\lambda)\rangle$ ($\mu = 1, 2, \ldots, l$). Consequently, quantum geometric potential arises as,

$$\Delta_{ND, mn} = A_m - A_n + \frac{d}{dt} \arg\langle\phi_m|\phi_n\rangle, \quad (1)$$

where $ND$ denotes the non-degenerate systems, and the “$\arg$” illustrates the time-derivative. In addition, $A_m = \sum_\mu A_m^\mu \lambda_\mu$ (in this paper the repeated indices imply the summation). The adiabatic solution to the time-dependent Schrödinger equation, $i\partial_t|\eta_m(\lambda(t))\rangle = \hat{H}(\lambda(t))|\eta_m(\lambda(t))\rangle$
is

\[ |\eta_m^n(t)⟩ = \exp \left( -i \int_0^t E_m(\tau) d\tau \right) |\tilde{\phi}_{m}^n(t)⟩, \tag{2} \]

with \( |\tilde{\phi}_{m}^n(t)⟩ = \exp \left( \int i A_m(t) d\tau \right) |\phi_{m}^n(t)⟩ \), if the initial state \( |\eta_m^n(0)⟩ = |\phi_{m}^n(0)⟩ \). Then \( \Delta_{ND, mn} \) can be also defined as

\[ \Delta_{ND, mn} = \frac{d}{dt} \arg(\phi_m^n |\phi_n). \tag{3} \]

\( \Delta_{ND, mn} \) is gauge invariant under an arbitrary local \( U(1) \otimes U(1) \) gauge transformation with \( |\phi_{m(n)}(t)⟩ → e^{i\alpha_{m(n)}(t)} |\phi_{m(n)}(t)⟩ \) where \( \alpha_{m(n)}(t) \) are smooth scalar functions. In the spin-\( \frac{1}{2} \) system coupled to an external time-dependent magnetic field, \( \Delta_{ND} \) is equivalent to the geodesic curvature of the path of the magnetic field orientation on the Bloch sphere, implying its geometric implications. When applying \( \Delta_{ND, mn} \) to the time-dependent system, an improved QAC for the non-degenerate system can be established for \( n \neq m \) \[1\].

\[ \left| \langle \phi_m |\phi_n \rangle / (E_m(t) - E_n(t) + \Delta_{ND, mn}(t)) \right| \ll 1, \tag{4} \]

which indicates \( E_m(t) - E_n(t) + \Delta_{ND, mn}(t) \) is more appropriate to describe the instantaneous energy gaps.

A quantized character in non-degenerate system—

We introduce a new quantized gauge invariant character \( \Theta \) based on the quantum geometric potential as an analogy to Gauss-Bonnet theorem with boundary. For simplicity, we begin with a two-level system controlled by a real 3-vector \( \tilde{\lambda}(t) \). At each time \( t \), there exist a pair of eigenfunctions \( |\phi_{\pm}(\tilde{\lambda}(t))⟩ \) associated with the eigenvalues \( E_{\pm}^2(\tilde{\lambda}(t)) \). Define \( \omega = (\mathcal{A}_-^c - \mathcal{A}_-^a) d\lambda^\mu \) and \( \mathcal{F} = d\omega \) with \( d \) being the exterior derivative. Explicitly, \( \mathcal{F} \) is carried out as \( \mathcal{F} = F_- - F_+ \), where \( F_\pm = \frac{1}{2} F_{\mu\nu}^\pm d\lambda^\mu \wedge d\lambda^\nu \) with \( F_{\mu\nu}^\pm = \partial^\mu \mathcal{A}_{\pm}^\nu - \partial^\nu \mathcal{A}_{\pm}^\mu \). A novel quantized character \( \Theta \) is defined as

\[ 2\pi \Theta = \int_{\mathcal{M}} \mathcal{F} - \int_{\partial \mathcal{M}} \Delta dt = \Phi_+ - \Phi_- - \int_{\partial \mathcal{M}} d \arg(\phi_+ |\phi_-), \tag{5} \]

where \( \Delta \) is the gauge invariant in Eq. \[1\] for the non-degenerate systems, and \( \Phi_\pm = \int_{\partial \mathcal{M}} \mathcal{A}_{\pm}^\tau d\lambda^\tau - \int_{\mathcal{M}} \mathcal{F}_{\pm} \). Since \( \mathcal{F} \) and \( \Delta \) are both locally gauge invariant, \( \Theta \) is also gauge invariant.

To show the quantization of \( \Theta \), we first consider a simple example of a two-level problem with the Hamiltonian \( \hat{H}(t) = \hat{B} \hat{n}(t) \cdot \vec{\sigma} \). Here \( \hat{n} \) is a 3D unit vector, and the whole parameter space is the Bloch sphere. If \( \hat{n}(t) \) concludes a region \( \mathcal{M} \) on the Bloch sphere with a smooth boundary \( \partial \mathcal{M} \) (Fig. \[1\]), then \( \Theta \) is quantized. Consider the transition term \( \langle \phi_+ |\phi_- \rangle \) from the ground state to the excited state, which is a complex number. The corresponding \( \mathcal{F} \) is the Berry curvature difference between the ground and excited states. To explicitly calculate \( \Theta \), we can work in a give gauge that \( |\phi_-(\theta, \phi)⟩ = (\sin \frac{\theta}{2} e^{-i\phi}, -\cos \frac{\theta}{2})^T \) and \( |\phi_+(\theta, \phi)⟩ = (\cos \frac{\theta}{2} e^{-i\phi}, \sin \frac{\theta}{2})^T \). Under this gauge, \( \Phi_+ = 2\pi \) if \( \partial \mathcal{M} \) encloses the north pole, and \( \Phi_- = -2\pi \) if it encloses the south pole. Otherwise \( \Phi_\pm = 0 \). Meanwhile \( \arg(\phi_+ |\phi_-) = \arg \left( (\theta - i \sin \theta \phi) / 2 \right) \), when \( \tilde{\lambda}(t) \) completes a closed loop \( \partial \mathcal{M} \), correspondingly, \( z(t) = \langle \phi_+ |\phi_- \rangle \) defines a close curve in complex plane. The winding number of \( z(t) \) relative to the origin is defined as \( W[z] = \int_{\partial \mathcal{M}} d \arg(\phi_+ |\phi_-) \) as shown in Fig. \[2\]. If \( \partial \mathcal{M} \) does not enclose the north or south pole, \( \Phi_\pm \) do not contribute, and \( W[\langle \phi_+ |\phi_- \rangle] \) contributes \(-2\pi \), such that \( \Theta = 1 \). After a similar analysis for other situations, one can conclude that \( \Theta = 1 \) for any region \( \mathcal{M} \) on the sphere.

For a general non-degenerate model, we can define the quantized character \( \Theta \) between any two different energy levels \( E_\pm \) associated with a closed curve in the parameter.
FIG. 3: (a) A curve $\vec{X}(s)$ is plotted on a 2D manifold (shaded area) in the 3D real space. $\vec{V}(s)$ lives in the tangent space, and is parallelly transported along the curve. $\vec{T}(s) = \frac{d}{ds} \vec{X}(s)$ is the velocity vector, and $\theta$ is the angle between $\vec{V}$ and $\vec{T}$. The geodesic curvature $k_g = \frac{d\theta}{ds}$ of the curve from the local geodesics. Choose a vector function $\vec{V}(s)$ living in the tangent space at the position $\vec{X}(s)$ and is parallelly transported along the curve. Then $k_g = \frac{d\theta}{ds}$, where $\theta$ is the angle between the velocity vector $\vec{T} = d\vec{X}/ds$ and $\vec{V}(s)$.

The similarity between $\Delta_{ND}$ and $k_g$ is illustrated in Fig. 3b. Following Eq. 3, the trajectory of $|\phi_{-}(t)\rangle$ is viewed as a curve in the Hilbert space. Due to the Berry phase, $|\phi_{-}(t)\rangle$ is actually orthogonal to $|\phi_{+}(t)\rangle$ for the two-level system. For the Berry connection, $|\phi_{+}(t)\rangle$ is just $|\phi_{-}(t)\rangle$ up to a complex factor. Consequently, the gauge invariant term $\Delta_{ND} = \frac{d\theta}{dt}$ is determined by the derivative of the angle $\theta = \arg(\phi_{+}|\phi_{-})$ over time. Therefore, therefore Eq. 3 can be viewed as a quantum analogy to the Gauss-Bonnet theorem described in Eq. 2.

There exist fundamental differences between the gauge invariant $\Delta_{ND}$ and the usual Berry connection. The integral of $\Delta_{ND}$ over a close loop is gauge invariant and single-valued. In contrast, the Berry connection is not gauge invariant locally, and the Berry phase for a closed loop evolution is gauge invariant but multiple valued modulo $2\pi$. The Berry connection and the Berry phase are intra-subspace quantities associated with one energy level, while $\Delta_{ND}$ is an inter-subspace property associated with two different energy levels.

A quantized character in degenerate systems — The gauge invariant quantized character $\Theta$ studied above can also be extended to the degenerate systems. For this purpose, the gauge invariant $\Delta_{ND}$ is generalized to the case with degeneracy, which is defined between two eigenspaces associated with two different degenerate energy levels. We first consider a special case that a Hamiltonian $\hat{H}(\hat{\lambda})$ possessing $N$ energy levels $E_m(\hat{\lambda})$ $(m = 1, 2, \cdots, N)$, each of which is $L$-fold degenerate. The situation for energy levels possessing different degeneracies is discussed in Appendix C.

For each energy level $m$, there is a set of instantaneous orthonormal eigenstates $|\phi_m^a(\hat{\lambda})\rangle$, satisfying $\hat{H}(\hat{\lambda})|\phi_m^a(\hat{\lambda})\rangle = E_m(\hat{\lambda})|\phi_m^a(\hat{\lambda})\rangle$ $(a = 1, 2, \cdots, L)$. If the system evolves adiabatically starting from the initial state $|\eta_m^a(\hat{\lambda}(0))\rangle = |\phi_m^a(\hat{\lambda}(0))\rangle$, then the adiabatic solution to the time-dependent Schrödinger equation, $i\partial_t|\eta_m^a(\hat{\lambda}(t))\rangle = \hat{H}(\hat{\lambda}(t))|\eta_m^a(\hat{\lambda}(t))\rangle$ is

$$|\eta_m^a(t)\rangle = \exp\{-i\int_0^t E_m(\tau)d\tau\}|\phi_m^a(\tau)\rangle$$

with $|\phi_m^a(\tau)\rangle = |\phi_m^b(\tau)\rangle [\Omega_m(t)]_{ab}$. The non-Abelian Berry phases $\Omega_m$ and the corresponding Berry connections $A_m^a$ explain this, we plot a curve $\vec{X}(s)$ on a 2D manifold in $\mathbb{R}^3$ as shown in the Fig. 3a, which is parameterized by the arc length $s$. $\vec{X}(s)$ represents the displacement vector for a point on the curve, then $k_g$ is a geometric quantity depending on both the manifold and the curve. The geodesic curvature $k_g$ reflects the deviation of the curve from the local geodesics. Choose a vector function $\vec{V}(s)$ living in the tangent space at the position $\vec{X}(s)$ and is parallelly transported along the curve. Then $k_g = \frac{d\theta}{ds}$, where $\theta$ is the angle between the velocity vector $\vec{T} = d\vec{X}/ds$ and $\vec{V}(s)$.

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$$|\eta_m^a(t)\rangle = \exp\{-i\int_0^t E_m(\tau)d\tau\}|\phi_m^a(\tau)\rangle$$

with $|\phi_m^a(\tau)\rangle = |\phi_m^b(\tau)\rangle [\Omega_m(t)]_{ab}$. The non-Abelian Berry phases $\Omega_m$ and the corresponding Berry connections $A_m^a$
are defined as
\[ \Omega_m(t) = \mathcal{P} \left\{ \exp \left\{ i \int_{\bar{\lambda}(0)}^{\bar{\lambda}(t)} A_m^\alpha \, d\lambda^\mu \right\} \right\}, \]  
(8)
\[ A_m^\alpha(\bar{\lambda}) = i \langle \phi_m^a(\bar{\lambda}) | \partial_\nu | \phi_m^b(\bar{\lambda}) \rangle, \]  
(9)
where \( \mathcal{P} \) means path-ordering \(^{39}\). The exact time-dependent solution can be expanded as \( |\psi(t)\rangle = c_m^a(t) |\eta_m^a(t)\rangle \), then one obtains
\[ c_m^a(t) = - \sum_{n \neq m} \exp \left\{ i \int_0^t \epsilon_{mn}(\tau) \, d\tau \right\} \left( \Omega_m^\dagger T_{mn} \Omega_n \right)^{ab} b_n^b(t), \]  
(10)
where \( \epsilon_{mn}(\tau) = E_m(\tau) - E_n(\tau) \) (details in the Appendix A). The transition matrices \( T_{mn} \) are followed by
\[ T_{mb}^{ab} = \langle \phi_m^a(\bar{\lambda}) | \partial_t | \phi_n^b(\bar{\lambda}) \rangle, \]  
(11)
where \( a \) and \( b \) denote the row and column indices of the matrix \( T_{mn} \), respectively, with \( m \) and \( n \) being energy level labels.

To figure out the gauge invariant in the degenerate case, we extract the “phase” from \( \Omega_m^\dagger T_{mn} \Omega_n \), i.e., the counterpart of \( \Delta_{ND,mn} \) (in Eq. 1). The “phase” of \( T \) is defined as \( \theta_T = \frac{1}{2} \text{Tr} \{ F_\pm \} \), where \( U \) and \( V \) are unitary matrices from \( T \)’s singular value decomposition, \( T_{mn} = U_{mn} S_{mn} V_{mn}^\dagger \), and \( S_{mn} \) is a diagonal real matrix with non-negative elements. We assume all the singular values of \( T \) are positive (The details are in the Appendix B). The “phase” of \( \Omega_m \) is \( \frac{1}{n} \text{Tr} \{ f(A_m) \} \) where \( A_m = A_m^\mu \lambda^\mu \), i.e., because \( \Omega_m \) can be expressed as \( \exp \{ f(A_m) \} \Omega_m \) where \( d \Omega_m = 1 \). Then the gauge invariant in the degenerate systems is defined as
\[ \Delta_{D,mn} = \frac{1}{L} \text{Tr} \left\{ A_n - A_m - i \frac{d}{dt} \ln(U_{mn} V_{mn}^\dagger) \right\}, \]  
(12)
or, in a compact form
\[ \Delta_{D,mn} = - i \frac{1}{L} \text{Tr} \{ \bar{X}_{mn} X_{mn}^\dagger \} \]  
(13)
with \( X_{mn}(\bar{\lambda}(t)) = \Omega_m^\dagger U_{mn} V_{mn}^\dagger \Omega_n \) (here “D” denotes the degenerate systems). The “phase” of \( \Omega_m^\dagger T_{mn} \Omega_n \) is defined as \( i \Delta_{D,mn} d\tau \), and Eq. (10) can be rewritten as
\[ c_m^a = - \sum_{n \neq m} \exp \left\{ i \int_0^t \left( \epsilon_{mn}(\tau) + \Delta_{D,mn}(\tau) \right) d\tau \right\} \times \left( \Omega_m^\dagger U_{mn} S_{mn} V_{mn}^\dagger \Omega_n \right)^{ab} b_n^b(t). \]  
(14)

Similar as \( \Delta_{ND,mn} \) in non-degenerate situations, \( \Delta_{D,mn} \) provides a proper correction for the instantaneous energy gaps for the degenerate systems. With the introduction of \( \Delta_{D,mn} \), a modified QAC is discussed in Appendix A.

\[ \Delta_{D,mn} \] is \( U(L) \otimes U(L) \) gauge invariant under any two independent \( U(L) \) gauge transformations \( W_m \) and \( W_n \) (details in the Appendix C):
\[ |\phi_m^a(\bar{\lambda})\rangle \rightarrow |\phi_m^a(\bar{\lambda})\rangle (W_m(\bar{\lambda}))^{ba}, \]
\[ |\phi_n^b(\bar{\lambda})\rangle \rightarrow |\phi_n^b(\bar{\lambda})\rangle (W_n(\bar{\lambda}))^{ba}. \]  
(15)

Then the quantized character \( \Theta \) can be defined between any two eigenspaces associated with eigenvalues \( E_+ \). \( \Delta \) in Eq. (15) is replaced by \( \Delta_D \), and \( F \) is defined as \( \frac{1}{2} \text{Tr} \{ F_+ - F_- \} \), where \( F_\pm = \frac{1}{2} F_{\pm}^{\mu\nu} d\lambda^\mu \wedge d\lambda^\nu \) with \( F_{\pm}^{\mu\nu} = \partial^\mu A_\nu^\pm - \partial^\nu A_\mu^\pm - i [A_\mu^\pm, A_\nu^\pm] \) being the non-Abelian Berry curvatures. \( z(t) = \exp \left\{ \frac{1}{2} \text{Tr} \ln(U V^\dagger) \right\} \) defines a closed curve in the complex plane, when \( \bar{\lambda} \) completes a close loop. Therefore, \( W[z] = \int \frac{1}{2} \text{Tr} \ln(U V^\dagger) \) is a winding number of \( z \) relative to the origin of the complex plane, which is quantized and play the counterpart of \( \oint \bar{X}_{mn} \, d\arg \langle \phi_+ | \phi_- \rangle \) in non-degenerate case. Therefore Eq. (5) still holds for degenerate system.

**Discussion and conclusions** — Based on the gauge invariant quantum geometric potential, we define a new quantized character \( \Theta \) for both non-degenerate and degenerate quantum systems. It is a quantum analogy to the Gauss-Bonnet theorem for a manifold with boundary. This character is fundamentally different from the Chern number which is quantized for the bundle based on the origin of the complex plane, which is quantized and play the counterpart of \( \oint \bar{X}_{mn} \, d\arg \langle \phi_+ | \phi_- \rangle \) in non-degenerate case. Therefore Eq. (5) still holds for degenerate system.

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Appendix A: Time Evolving Equation for Degenerate system

As discussed in the article, the solution to the time dependent Schrödinger equation can be expanded by $|\eta^m_n\rangle$ defined in Eq. (7) in the main text as

$$|\psi(t)\rangle = c_m^a(t)|\eta^m_n\rangle,$$

(A1)

or with $|\tilde{\phi}^a_m\rangle = \langle \phi^b_m(t) | \Omega_m(t) \rangle^b a$ as

$$|\psi(t)\rangle = c_m^a(t) \exp\{-i \int_0^t E_m(\tau)d\tau\} |\tilde{\phi}^a_m\rangle,$$

(A2)

where $\Omega_m$ is defined in Eq. (9) in the main text. It can be shown that $\langle \tilde{\phi}^a_m | \tilde{\phi}^a_m \rangle = 0$, because

$$\langle \tilde{\phi}^a_m | \tilde{\phi}^a_m \rangle = \langle \Omega^a_m \rangle^{a c} \langle \phi^c_m | \phi^b_m \rangle (\Omega^b_m)^{b a} + \langle \Omega^a_m \rangle^{a c} \langle \phi^c_m | \phi^b_m \rangle (\Omega^b_m)^{b a}$$

$$= \langle \Omega^a_m \rangle^{a c} (i \mathcal{A}_m)^{a b} (\Omega^b_m)^{b a} + \langle \Omega^a_m \rangle^{a c} (i \mathcal{A}_m)^{a b} (\Omega^b_m)^{b a} = 0.$$

(A3)

Solving the time dependent Schrödinger equation $i \partial_t |\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$:

$$i \{c_m^a |\tilde{\phi}^a_m\rangle - i E_m(t)c_m^a|\tilde{\phi}^a_m\rangle + c_m^b |\tilde{\phi}^b_m\rangle \} \exp\{-i \int_0^t E_m(\tau)d\tau\} = E_m^a c_m^a \exp\{-i \int_0^t E_m(\tau)d\tau\} |\tilde{\phi}^a_m\rangle.$$

(A4)

Left multiply $\langle \tilde{\phi}^a_m |$ to the equation above, one obtains

$$i \{c_m^a - i E_m(t) \} \exp\{-i \int_0^t E_m(\tau)d\tau\} + \sum_{b,n,n\neq m} i \phi^b_n(t) \langle \tilde{\phi}^a_m | \tilde{\phi}^b_n \rangle \exp\{-i \int_0^t E_n(\tau)d\tau\} = E_m(t)c_m^a(t) \exp\{-i \int_0^t E_m(\tau)d\tau\}.$$

(A5)
Then one arrives at
\[ \dot{c}_m^a(t) = - \sum_{n,n \neq m} \left\{ \exp\{i \int_0^t \epsilon_{mn}(\tau) d\tau \} \langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle \right\} c_n^b(t). \]  \hspace{1cm} (A6)

Therefore the time evolving equation of Eq. (10) in the main text can be obtained,
\[ \dot{c}_m^a(t) = - \sum_{n \neq m} \exp\{i \int_0^t \epsilon_{mn}(\tau) d\tau \} \Omega_m^\dagger T_{mn} \Omega_n \right) \langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle c_n^b(t), \]  \hspace{1cm} (A7)

with \( \epsilon_{mn}(\tau) = E_+(\tau) - E_-(\tau). \) Therefore
\[ \dot{c}_m^a = - \sum_{n \neq m} \exp\left\{ i \int_0^t (\epsilon_{mn}(\tau) + \Delta_{D, mn}(\tau)) d\tau \right\} \left( \Omega_m^\dagger \hat{U}_m S_{mn} \hat{V}_n^\dagger \Omega_n \right) \langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle c_n^b(t). \]  \hspace{1cm} (A8)

With the gauge invariant \( \Delta_D \) in the degenerate systems Eq. (12), we can further revise the QAC for the quantum degenerate systems. For an adiabatic process, all the \( c_m^b(t) \)'s are nearly time-independent, because \( |\eta_m^b(t)\rangle \) are already the adiabatic evolution states. If one further assumes \( \epsilon_{mn}(t), \Delta_{D, mn}(t), S_{mn} \) and \( (\Omega_m^\dagger \hat{U}_m S_{mn} \hat{V}_n^\dagger \Omega_n)^{ab} \) are slow varying variables, and the system is initially prepared in the states \( |\eta_m^b(0)\rangle \), then the time-evolving part is approximately controlled by \( \exp\{i(\epsilon_{mn} + \Delta_{D, mn})t\} \). With these conditions, the QAC for the degenerate systems can be expressed as \( \forall m \neq n \)
\[ \max(S_{mn}) < 1, \]  \hspace{1cm} (A9)
where \( \max(S_{mn}) \) is the maximum value of the singular values of the transition matrix \( T_{mn} \). Physically, the \( \max(S_{mn}) \) represents the most possible channel in the process of transition.

To illustrate how the degenerate QAC Eq. (A9) works, we construct a two-level toy model as follows:
\[ H(t) = \begin{bmatrix} \vec{n}_1(t) \cdot \vec{\sigma} & i \vec{n}_2(t) \cdot \vec{\sigma} \end{bmatrix}. \]  \hspace{1cm} (A10)

If \( \vec{n}_1 = \vec{n}_2 = (\sin \theta \cos(\omega t), \sin \theta \sin(\omega t), \cos \theta) \), then \( H \) is simply a double copy of Rabi model. \( \Delta_D \) can be calculated by Eq. (12), and the result is \( (1 - 2 \cos^2(\theta/2))\omega \). After extracting the phase term \( i \int_0^t \Delta_D(\tau) d\tau \), the remaining part \( \Omega_m^\dagger \hat{U}_m S_{mn} \hat{V}_n^\dagger \Omega_n \) is a constant, and \( S \) is also a contant matrix \( \sin(\theta)\omega/2 \cdot 2 \times 2 \), so that we can use Eq. (A9) to judge the adiabaticity as
\[ \frac{|\sin(\theta)\omega/2|}{2 + (1 - 2 \cos^2(\theta/2))\omega} \ll 1. \]  \hspace{1cm} (A11)

When \( \theta \to 0^+ \), Eq. (A11) breaks down if \( \omega \simeq 2 \), because the denominator goes to zero. This is expected since when \( \omega \) matches the energy gap, the resonance happens so that the system is not adiabatic anymore.

Besides \( \Delta_D \), one can also define other gauge invariants within the general time-dependent problem described above. Every single element of the matrix, \( (\Omega_m^\dagger T_{mn} \Omega_n)^{ab} \) can be evaluated as \( \langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle \), and it is also gauge invariant as long as the initial basis are fixed. Similar as what we do in the non-degenerate case, we can separate the phase factor from \( \langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle \) as
\[ \langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle = \exp\{i \int_0^t \Delta^{ab}_{mn} d\tau\} |\langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle| \]  \hspace{1cm} (A12)
with \( \Delta^{ab}_{mn} = \frac{d}{dt} \arg(\langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle) \). Then Eq. (10) can be rewritten by using \( \Delta^{ab}_{mn} \) as
\[ \dot{c}_m^a(t) = - \sum_{n \neq m} \exp\{i \int_0^t (\epsilon_{mn}(\tau) + \Delta^{ab}_{mn}(\tau)) d\tau\} |\langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle| c_n^b(t). \]  \hspace{1cm} (A13)

If one further assume \( \epsilon_{mn}(t) = \epsilon_{mn}, |\langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle| \) and \( \Delta^{ab}_{mn} \) are slow varying variables, the adiabatic condition can be deduced as
\[ \frac{|\langle \hat{\phi}_m^a | \hat{\phi}_n^b \rangle|}{|\epsilon_{mn} + \Delta^{ab}_{mn}|} \ll 1 \quad \forall a, b, m \neq n. \]  \hspace{1cm} (A14)
Under the gauge transformations above, $A$ with $C$ have to be equal. Define $\tilde{\Delta}$ with unitary matrices, but there is neither ambiguity of the singular values nor ambiguity of the product of two adiabatically evolved basis with different energies are all very weak, so that this degenerate system can evolve adiabatically.

Appendix B: Ambiguity of the Singular Value Decomposition (SVD)

For a general $l \times l$ matrix $C$, when applying SVD to it, one will obtain $(C')^{ab} = (U)^{ad}(\Lambda)^{dl}(V^\dagger)^{lb}$, with $l$ non-negative singular values $\Lambda_d$ ($a$, $b$, and $d$ vary from $1 \to l$) and $U$ and $V$ being unitary matrices. SVD has its intrinsic ambiguity that comes from the unitary matrices $U$ and $V$. In the case that all the singular values are positive, one can insert two diagonal matrices as:

$$(C')^{ab} = (U)^{ad}(\Lambda)^{dl}(V^\dagger)^{lb} = (U)^{ad}e^{i\lambda_d}(\Lambda)^{dl}e^{-i\lambda_d}(V^\dagger)^{lb}$$

with $\lambda_d$ being any real numbers. After the insertion, one can define $(U')^{ad} = (U)^{ad}e^{i\lambda_d}$ and $(V')^{ad} = (V)^{ad}e^{i\lambda_d}$, so that $C = U'AV'^\dagger$ which is also a valid SVD of $C$. Therefore SVD has its intrinsic ambiguity of the choice of the unitary matrices, but there is neither ambiguity of the singular values nor ambiguity of the product of $U$ and $V^\dagger$ in this case.

When the singular values of a matrix $C$ contain a zero or multiple zeros, there are further ambiguities. For example, if $C$ is decomposed as $C = UAV^\dagger$ and the $n^{th}$ singular value is zero, then one can also insert two diagonal matrices as:

$$(C')^{ab} = (U)^{ad}(\Lambda)'^{dl}(V^\dagger)^{lb} = (U)^{ad}e^{i\lambda_d}(\Lambda)'^{dl}e^{-i\lambda_d}(V^\dagger)^{lb}$$

with $\lambda_d$ and $\lambda'_d$ being any real numbers and $\lambda_d = \lambda'_d$ if $d \neq n$. Because the $n^{th}$ singular value is zero, $\lambda_n$ and $\lambda'_n$ do not have to be equal. Define $(U')^{ad} = (U)^{ad}e^{i\lambda_d}$ and $(V')^{ad} = (V)^{ad}e^{i\lambda'_d}$, so that $C = U'AV'^\dagger$, however $UV^\dagger \neq U'V'^\dagger$.

Appendix C: Proof of the Gauge Invariance of the Quantum Geometric Potential $\Delta_D$

As mentioned in the article, $\Delta_D$ is gauge invariant under any independent $U(L)$ gauge transformations $W_m$

$$|\phi^a_m(\tilde{\lambda})\rangle \rightarrow |\phi^a_m(\tilde{\lambda})\rangle(W_m(\tilde{\lambda}))^b.$$  

(C1)

Under the gauge transformations above, $A_m$, $T_{mn}$ and $U_{mn}V_{mn}$ transform as follows:

$$A^\mu_m \rightarrow W_m A^\mu_m W_n^\dagger + iW_m^\dagger \partial_{\lambda^*} W_m$$

(C2)

$$T_{mn} \rightarrow W_m^\dagger T_{mn} W_n$$

(C3)

$$U_{mn}V_{mn}^\dagger \rightarrow W_m^\dagger U_{mn}V_{mn}^\dagger W_n.$$  

(C4)

$(U_{mn}$ and $V_{mn}^\dagger$ are the unitary matrices come from the SVD of $T_{mn}$; $A_m$ and $T_{mn}$ are introduced in Eq. (9) and Eq. (11) in the main text). $\Delta_D$ is carried out as

$$\Delta_{D,mn} = \frac{1}{L} \text{Tr}\{[A_n - A_m] + i\frac{d}{dt}(-\ln(U_{mn}V_{mn}^\dagger))\},$$

(C5)

and under the gauge transformations $W_m$

$$\Delta_{D,mn} \rightarrow \frac{1}{L} \text{Tr}\{(W_n^\dagger A_n W_n + iW_n^\dagger \partial_{\lambda^*} W_n) - W_m^\dagger A_m W_m - iW_m^\dagger \partial_{\lambda^*} W_m) + i\frac{d}{dt}(-\ln(U_{mn}V_{mn}^\dagger))\}.$$  

(C6)

We have used the fact that $\text{Tr}\{\ln(A)B\} = \text{Tr}\{\ln(A)\} + \text{Tr}\{\ln(B)\}$ if $A, B \in U(L)$ in the equation above. $W_\mu^m A_m W_\mu$ are similarity transformations, so that the trace remains the same as before. If $A \in U(L)$, then $\text{Tr}\{\ln(A)\} = \ln(\det(A))$, so that $\frac{d}{dt}\text{Tr}\{\ln(A)\} = \frac{d}{dt}\text{Tr}\{\ln(A)\}$ with $VAV^\dagger = A$ and $A$ being diagonal. If $VAV^\dagger = A \in U(L)$, then

$$\text{Tr}\{A^\dagger \dot{A}\} = \text{Tr}\{V^\dagger \dot{V}\}A^\dagger V^\dagger + V^\dagger \dot{V}A^\dagger = \text{Tr}\{A^\dagger \dot{A}\} = \text{Tr}\{\Lambda^{-1} \dot{\Lambda}\} = \frac{d}{dt}\text{Tr}\{\ln(A)\},$$

(C7)

so that if $A \in U(L)$, $\frac{d}{dt}\text{Tr}\{\ln(A)\} = \text{Tr}\{A^\dagger \dot{A}\}$. Then Eq. (C6) can be simplified as

$$\Delta_D \rightarrow \Delta_D + \frac{1}{L} \text{Tr}\{-i\frac{d}{dt}([\ln(W_n^\dagger) + \ln(W_n) + \ln(W_m^\dagger) - \ln(W_m))]) = \Delta_D.$$  

(C8)
Therefore $\Delta_D$ is gauge invariant under a $U(L) \times U(L)$ gauge transformation.

As for the case that the degeneracies of these two eigenspaces are different, one can still define the gauge invariant as

$$\Delta_{D, mn} = -\frac{i}{\min(L_m, L_n)} \text{Tr}\{\hat{X}_{mn} X_{mn}^\dagger\}, \quad \text{(C9)}$$

where $X$ is $\Omega_m^\dagger U_{mn} V_{mn}^\dagger \Omega_n$, and $L_m$ and $L_n$ are the degeneracies of these two eigenspaces (suppose $L_m < L_n$). As mentioned in the main text of this article, $V^\dagger V = I_{L_m \times L_m}$ is an identity matrix, while $VV^\dagger$ is not. $\mathcal{A}_m$, $T_{mn}$, and $U_{mn} V_{mn}^\dagger$ transform as same as Eq. (C2), Eq. (C3) and Eq. (C4), so that under the gauge transformation:

$$\begin{align*}
\Delta_{D, mn} &\to -\frac{i}{L_m} \text{Tr}\{-iA_m + \frac{d}{dt}(U_{mn} V_{mn}^\dagger)(V_{mn} U_{mn}^\dagger) + iU_{mn} V_{mn}^\dagger A_n V_{mn} U_{mn}^\dagger\} \\
&\quad + \frac{i}{L_m} \text{Tr}\{-iW_{mn}^\dagger A_m W_m + W_m^\dagger W_m + \frac{d}{dt}(U_{mn} V_{mn}^\dagger)(V_{mn} U_{mn}^\dagger)\}
\end{align*} \quad \text{(C10)}$$

$$\Delta_{D, mn} = -\frac{i}{L_m} \text{Tr}\{-iW_{mn}^\dagger A_m W_m + W_m^\dagger W_m + \frac{d}{dt}(U_{mn} V_{mn}^\dagger)(V_{mn} U_{mn}^\dagger)\}
\quad + \frac{i}{L_m} \text{Tr}\{-iW_{mn}^\dagger A_m W_m + W_m^\dagger W_m + \frac{d}{dt}(U_{mn} V_{mn}^\dagger)(V_{mn} U_{mn}^\dagger)\}
\quad + U_{mn} W_{mn}^\dagger W_{mn} U_{mn}^\dagger W_{mn} + U_{mn} V_{mn}^\dagger V_{mn} U_{mn}^\dagger W_{mn} (iW_{mn}^\dagger A_n W_n - W_{mn}^\dagger W_{mn}^\dagger A_n V_{mn} U_{mn}^\dagger W_{mn})
\quad + U_{mn} V_{mn}^\dagger W_{mn} V_{mn} U_{mn}^\dagger W_{mn}
\quad + U_{mn} V_{mn}^\dagger W_{mn} V_{mn} U_{mn}^\dagger W_{mn}
\quad = \Delta_{D, mn}.
\quad \text{(C13)}$$

Therefore the gauge invariance is verified. Note that some terms in the equations above, like $U_{mn} V_{mn}^\dagger A_n V_{mn} U_{mn}^\dagger$ and $U_{mn} V_{mn}^\dagger W_{mn} V_{mn} U_{mn}^\dagger W_{mn}$ are in fact not similarity transformations of $\mathcal{A}_n$ and $\tilde{W}_n W_{mn}^\dagger$. 