A nonlocal theory of fermion mixing

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We study a nonlocal generalization of two flavor fermion mixing and compute the transition probability by means of the path integral formalism. In our treatment, we delocalize only the mixing term, and consider a perturbative interaction with a small mixing angle. The oscillation formula derived reduces to the correct local form when the appropriate limit is considered. We apply our formalism to some string-inspired delocalization kernels and discuss the phenomenological deviations from the local theory.

INTRODUCTION

One of the most intriguing aspects of quantum mechanics is quantum nonlocality, which has no classical counterpart. Tests of the Bell inequality [1] and subsequent experiments have shown that the quantum theory is incompatible with local realism and that an intrinsic form of nonlocality characterizes phenomena such as the quantum entanglement. Nevertheless quantum nonlocality does not allow for faster than light communication and hence is compatible with special relativity. Quantum theory is indeed local in the strict sense defined by special relativity and, as such, the term “quantum nonlocality” is sometimes considered a misnomer. However, not all aspects of quantum mechanics are fully understood, and it cannot be excluded that some of these may hide new physics. It is then natural to ask whether strictly nonlocal effects may arise in the context of quantum field theory, which is usually constructed in terms of local interactions.

In fact, nonlocal field theory is a subject with a long history, born in 1950 thanks to Hideki Yukawa [2]. Despite the success of perturbative renormalization of quantum electrodynamics (QED) in the late forties, the idea of a non-local theory of interactions as a fundamental theory underlying the local theory remained prominent and motivated several studies [3]. Nonlocal field theories can indeed, at least in principle, improve on the ultraviolet (UV) behaviour of the local theory. The first nonlocal theories were still rudimentary and could not solve problems such as unitarity and causality. Later this new class of theories has developed and has overcome many, albeit not all, of the original problems. Nonlocality has been successfully introduced in the context of gravitational theories [4,11], and more recently a nonlocal generalization of the standard model has been proposed [12], with possible interesting consequences for the phenomenology of anomalous lepton moments [13]. Several theories beyond the standard model, including string theory [14,27], predict non-local interactions in the low energy effective Lagrangian.

In this paper, motivated by the previous considerations, we wish to investigate a nonlocal generalization of fermion mixing. The idea underpinning this research is that the local theory of fermion oscillations may be only a low energy approximation for a more fundamental (high energy) nonlocal theory. A signature of the nonlocal theory might appear in future high energy experiments. The extension of our treatment to boson mixing is also possible, eventually paving the way for a new line of research adjoining particle mixing and nonlocality. In particular we employ the “delocalization kernel” prescription (see e.g. [28]), which ensures, provided some basic regularity requirements for the kernel, that the resulting nonlocal theory respects unitarity and causality. The class of theories thus obtained is rather general, and it is possible to show that theories with modified kinetic term can be recast in the same form. The nonlocal theory of fermion mixing is introduced by delocalizing the standard mass term and, assuming a perturbative mixing term, a general oscillation formula in terms of the delocalization kernels is derived by employing a path integral approach. We show how the local theory is recovered in the appropriate limit. Finally, we analyze the nonlocal theory of fermion mixing for some specific kernels and discuss the possible deviations from the local theory.

Notice that, we consider a perturbative interaction controlled by a small parameter $\alpha$. In the case of two-flavor neutrino mixing, this parameter coincides with the off-diagonal element of the mass matrix $m_{e\mu}$ and the fact that $\alpha$ is small implies that $m_{e\mu} \ll m_e, m_\mu$ and leads to a small mixing angle $\theta \ll 1$. Then, the oscillation formula here derived has to be compared with the Pontecorvo-like formula, in which we consider the limit $\theta \to 0$.

The paper is structured as follows. In section I, we introduce the delocalization kernel prescription as applied to a scalar field theory. We briefly recall the main requirements on the delocalization kernel in order to preserve causality and unitarity. In section II, we introduce a nonlocal theory of fermion mixing. In section III, we derive general oscillation formulae in terms of the delocalization kernels. In section IV, we apply the formalism to some specific delocalization kernels. Section V is finally devoted to conclusions.
I NON LOCAL-SCALAR FIELD

The standard way to introduce nonlocality is to begin with the local Lagrangian:

\[ \mathcal{L} = \mathcal{L}_0[\phi] - V[\phi] = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \]  

(1)

and define a delocalized field \( \hat{\phi} \) as

\[ \hat{\phi}(x) = \int d^4y F(x-y) \phi(y) \]  

(2)

where \( F(x) \) is a scalar (density) function known as delocalization kernel (strictly speaking, \( F(x) \) is generally a distribution). Locality is restored for localization kernel (strictly speaking, \( F \)

(3)

The restrictions on \( F \) are usually stated as conditions upon the Fourier transform \( F(k) \). \( F(k) \) is required to belong to the space of functions of rapid decay and to be an analytic entire function in the whole \( k_\mu \) complex plane, for all \( \mu = 0, 1, 2, 3 \). The nonlocal Lagrangian is defined by plugging the delocalized field \( \hat{\phi} \) in the interaction potential

\[ \mathcal{L}_{nl} = \mathcal{L}_0[\phi] - \mathcal{V}[\hat{\phi}] . \]  

(5)

For instance, the nonlocal lagrangian corresponding to a \( \phi^4 \) interaction is

\[ \mathcal{L}_{nl} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \left( \int d^4y F(x-y) \phi(y) \right)^4 . \]  

(6)

Clearly the nonlocality introduced in this way only affects the interactions. The free lagrangian is left unaltered. This feature is common to the infinite derivative Lagrangians of the kind

\[ \mathcal{L}_{nl} = \phi G(\Box) \phi - \mathcal{V}[\phi] \]  

(7)

where nonlocality is introduced in the kinetic term (e.g. \( G(\Box) = e^{-F(\Box)} \)). It can further be shown that by an appropriate field redefinition the Lagrangians of the kind

(7) can be recast in the form (5).

In order for the nonlocality to respect causality and unitarity, some basic conditions on the delocalization kernel must be imposed. Unitarity can be assured by requiring that the Fourier transform of the delocalization kernel, \( F(k) \), be an entire analytic function i.e. analytic in each component \( k_\mu \) in the entire complex \( k_\mu \)-plane. A smooth UV behaviour of the amplitudes can be enforced by requiring that \( F(k) \) belongs to the space of functions of rapid decay. Some additional conditions may be needed to ensure causality, depending on the particular theory under consideration. It is important to remark, for the following section, that no choice a priori of the delocalization kernel is made. Apart from the regularity conditions mentioned above, we shall keep the delocalization kernel arbitrary, in order to develop the most general nonlocal theory of fermion mixing.

III FERMION MIXING AND NON LOCALITY

We now write down a nonlocal Lagrangian for fermion mixing. We closely follow the delocalization procedure outlined in the previous section for the scalar field. Our starting point is the local mixing Lagrangian

\[ \mathcal{L} = \bar{\psi}_e \left( i \partial - m_e \right) \psi_e + \bar{\psi}_\mu \left( i \partial - m_\mu \right) \psi_\mu - \alpha \left( \bar{\psi}_e \psi_\mu + \bar{\psi}_\mu \psi_e \right) \]  

(8)

where \( m_e, m_\mu, \alpha \) are positive parameters with dimensions of mass and the non locality is included in the interaction term via the definition of Eq. (2). In order to set up a perturbative scheme, \( \alpha \) should be regarded as a small coupling parameter \( \alpha \ll m_e, m_\mu \). The notation \( \alpha \) is used in place of the usual \( m_e \mu \) to remark that this coupling is not necessarily to be identified with the bare \( m_e \mu \) parameter. Any such identification is to be made only a posteriori. To simplify the treatment, a pure perturbative scheme is adopted; this approximation is valid only when the kinetic term dominates. If \( \alpha \) is considered of the same order as \( m_e \mu \), then the perturbation theory makes sense for typical momentum scales such that \( k \gg \alpha \).

The steps outlined previously for the \( \phi^4 \) theory are now performed in order to define the nonlocal Lagrangian. We introduce the the delocalized fields as

\[ \bar{\psi}_\beta(x) = \int d^4y F_\beta(x-y) \psi_\beta(y) \]  

(9)

where \( F_\beta(x) \) for \( \beta = e, \mu \) are (generally) flavor-dependent kernels. Consequently by definition:

\[ \bar{\psi}_\beta(x) = \bar{\psi}_\beta^1(x) \gamma^0 = \int d^4y F^*_\beta(x-y) \psi_\beta^1(y) \gamma^0 = \int d^4y F^*_\beta(x-y) \bar{\psi}_\beta(y) \]  

(10)

with \( F^* \) denoting the complex conjugate. Note that here we are considering the full kinetic term (derivative term plus diagonal elements of the mass Lagrangian) as the free Lagrangian. Only the mixing term, proportional
to $\alpha$, constitutes the interaction Lagrangian. It is also possible to consider the free Lagrangian as made up of the derivative term alone, and include the $m_e, m_\mu$ terms in the interaction Lagrangian. This scheme shall be explored in future works. The nonlocal Lagrangian is therefore

$$\mathcal{L}_{nl} = \bar{\psi}_e (i \partial - m_e) \psi_e + \bar{\psi}_\mu (i \partial - m_\mu) \psi_\mu$$

$$- \alpha \int d^4y \int d^4z \left( F_e^+(x-y)F_e(x-z)\bar{\psi}_e(y)\psi_e(z) + F_\mu^+(x-y)\bar{\psi}_\mu(y)\psi_e(z) \right)$$

(11)

The theory is specified by providing the form factors $F_\beta$. One special class of interactions is obtained by reducing one of the two kernels to a delta function. Consider for instance $F_e(x-y) = \delta^4(x-y)$. The nonlocal interaction term of Eq. (11) becomes

$$\mathcal{L}_{INT} = -\alpha \int d^4z F_\mu(x-z)\bar{\psi}_e(z)\psi_e(x) -\alpha \int d^4y F_\mu^+(x-y)\bar{\psi}_\mu(y)\psi_e(x) =$$

$$= -\alpha \int d^4y \left( F_\mu(x-y)\bar{\psi}_e(x)\psi_e(y) + F_\mu^+(x-y)\bar{\psi}_\mu(y)\psi_e(x) \right)$$

(12)

where the variable $z$ has been renamed. Consequently, the action is

$$S_{INT} = -\alpha \int d^4x \left( \int d^4y F_\mu(x-y)\bar{\psi}_e(x)\psi_e(y) + F_\mu^+(x-y)\bar{\psi}_\mu(y)\psi_e(x) \right)$$

$$= -\alpha \int d^4x \int d^4y \left( F_\mu(x-y)\bar{\psi}_e(x)\psi_e(y) + F_\mu^+(x-y)\bar{\psi}_\mu(x)\psi_e(y) \right)$$

(13)

where $x$ and $y$ in the last term have been swapped. Setting for instance

$$F_\mu(x-y) = i \left( \theta(x^0 - y^0) - \theta(y^0 - x^0) \right) \delta ((x-y)^2 - l^2)$$

(14)

one gets

$$S_{INT} = -\alpha \int d^4x \int d^4y \left[ i \left( \theta(x^0 - y^0) - \theta(y^0 - x^0) \right) \delta ((x-y)^2 - l^2) \bar{\psi}_e(x)\psi_e(y) 
- i \left( \theta(y^0 - x^0) - \theta(x^0 - y^0) \right) \delta ((x-y)^2 - l^2) \bar{\psi}_\mu(x)\psi_e(y) \right]$$

$$= -i\alpha \int d^4x \int d^4y \left[ \left( \theta(x^0 - y^0) - \theta(y^0 - x^0) \right) \left( \bar{\psi}_e(x)\psi_e(y) + \bar{\psi}_\mu(x)\psi_e(y) \right) \right]$$

which is the kind of action considered in [29].

**IV TRANSITION PROBABILITY**

In order to compute the transition probability, the central quantities are the 2-point functions defined by

$$S_{\alpha\beta}(x-y) = \langle 0 | T \left( \bar{\psi}_\alpha(x)\psi_\beta(y) \right) | 0 \rangle .$$

(15)

This kind of mixed propagators has been studied, for instance, in [30]. In the usual Minkowskian theory $T$ stands for the time ordering operation. The vacuum state $|0\rangle$, within the perturbative scheme, is referred to the free fields. Once the propagators are computed, we can define the transition probability as

$$P_{\alpha \rightarrow \beta}(t) = \mathcal{A} \text{ Tr} \left( S_{\alpha\beta}^{(ret)}(\vec{k},t)S_{\alpha\beta}^{(ret)\dagger}(\vec{k},t) \right)$$

(16)

where $(ret)$ denotes the retarded part of the propagator and it is understood that a Fourier transform on the space variables has been performed. We shall fix the value of the normalization constant $\mathcal{A}$ later on, by comparing our result with the Pontecorvo oscillation formula in the appropriate limit. The trace $\text{Tr}(\cdot)$ is understood over the 4-dimensional spinor space. The definition of Eq. (16)
was used, for instance, in the ref. [30]. There are in principle additional complications related to the choice of the vacuum state in Eq. (15) (see, e.g., [30]). These difficulties, however, are relevant only at low momenta. For large fermion momenta, as is the case that we shall consider, the formula (16) works fine with the standard choice of vacuum state in Eq. (15). The most convenient method to compute the Green’s functions, bypassing the canonical quantization, is to resort to the path integral approach. Among the other benefits, the path integral can be easily converted to Euclidean spacetime. The free generating functional is

\[ W_0[\eta_e, \bar{\eta}_e, \eta_\mu, \bar{\eta}_\mu] = \int D\bar{\psi}_e D\psi_\mu D\bar{\psi}_\mu D\psi_e e^{i(S_0[\psi, \bar{\psi}]+\bar{\psi}_e \eta_e + \bar{\eta}_e \eta_e + \bar{\psi}_\mu \eta_\mu + \bar{\eta}_\mu \eta_\mu)}. \]  

(17)

Here \( D\psi \) denotes a functional integration, \( \eta_e, \eta_\mu \) are spinor source terms and Grassmann (anticommuting) c-numbers. The free action term is

\[ S_0 = \int d^4x \left[ \bar{\psi}_e \left( i\partial - m_e \right) \psi_e + \bar{\psi}_\mu \left( i\partial - m_\mu \right) \psi_\mu \right]. \]  

(18)

The free generating functional can be conveniently rewritten as

\[ W_0[\eta_e, \bar{\eta}_e, \eta_\mu, \bar{\eta}_\mu] = e^{-i \int d^4x \int d^4y \bar{\eta}_e(x) S_e S_e \eta_e(y) + \bar{\eta}_\mu(x) S_\mu S_\mu \eta_\mu(y)}, \]  

(19)

with \( S_\beta(x - y) = (i\partial + m_\beta) \Delta_\beta(x - y) \)  

(20)

and the scalar Feynman propagator for mass \( m_\beta \) denoted \( \Delta_\beta(x - y) \). Two point functions are obtained by forming the functional derivatives with respect to the source fields:

\[ S_{\alpha\beta}(x - y) = -i \frac{\delta^2 W}{\delta \eta_\alpha(y) \delta \eta_\beta(x)} \bigg|_{\eta = 0 = \bar{\eta}}. \]  

(21)

For the non interacting theory one has:

\[ S_{ee} = S_e, \quad S_{\mu\mu} = S_\mu, \quad S_{e\mu} = 0 = S_{\mu e}. \]  

(22)

The generating functional for the interacting theory can be computed perturbatively, by writing

\[ W[\eta, \bar{\eta}] = \mathcal{N} e^{i S_{\text{INT}}[-i \frac{\delta}{\delta \eta_e} - i \frac{\delta}{\delta \eta_\mu}]} W_0[\eta, \bar{\eta}] \]  

(23)

where the trick of substituting fields with functional derivatives \( \psi \rightarrow -i \frac{\delta}{\delta \eta_e} \) has been used to take the interacting part out of the functional integral. \( \mathcal{N} \) is a normalization constant. The interaction action is given by:

\[ S_{\text{INT}} = \alpha \int d^4x \int d^4y \int d^4z \left[ F^*_e(x - y) F_\mu(x - z) \left( -i \frac{\delta}{\delta \eta_e} \right) \left( -i \frac{\delta}{\delta \eta_\mu} \right) + \frac{1}{2} \right] \]  

\[ F^*_e(x - x') F_e(x - z') \left( -i \frac{\delta}{\delta \eta_e} \right) \left( -i \frac{\delta}{\delta \eta_e} \right) \]  

\[ + \frac{1}{2} F^*_\mu(x - x') F_\mu(x - z') \left( -i \frac{\delta}{\delta \eta_\mu} \right) \left( -i \frac{\delta}{\delta \eta_\mu} \right) \]  

(24)
The generating functional for the non local action can be now computed at first order in $\alpha$:

$$W^{(1)}[\eta, \bar{\eta}] = N \left( 1 - i \alpha \int d^4x \int d^4x' \int d^4z' \right)$$

\[
\left\{ \left[ F_\mu^*(x - x') F_\mu(x - z') \right] \left( -i \frac{\delta}{\delta \eta_\mu(x')} \right) \left( -i \frac{\delta}{\delta \eta_\mu(z')} \right) \right\} W_0[\eta, \bar{\eta}].
\]

The two terms in this equation can be studied separately; for the first term, one has:

$$W_1^{(1)} = -i \alpha \int d^4x \int d^4x' \int d^4z' \left[ F_\mu^*(x - x') F_\mu(x - z') \right]$$

\[
\left( -i \frac{\delta}{\delta \eta_\mu(x')} \right) \left( -i \frac{\delta}{\delta \eta_\mu(z')} \right) W_0[\eta, \bar{\eta}]
\]

\[
= -i \alpha \int d^4x \int d^4x' \int d^4z' \left[ F_\mu^*(x - x') F_\mu(x - z') \right] \left( \int d^4z \eta_\mu(z) \right) W_0[\eta, \bar{\eta}].
\]

Performing the last functional derivative one arrives at the result:

$$W_1^{(1)} = -i \alpha \int d^4x \int d^4x' \int d^4z' \left[ F_\mu^*(x - x') F_\mu(x - z') \right]$$

\[
\left( \int d^4z \eta_\mu(z) \right) \left( \int d^4z \eta_\mu(z) \right) W_0[\eta, \bar{\eta}].
\]

And similarly for the second term:

$$W_2^{(1)} = -i \alpha \int d^4x \int d^4x' \int d^4z' \left[ F_\mu^*(x - x') F_\mu(x - z') \right]$$

\[
\left( \int d^4z \eta_\mu(z) \right) \left( \int d^4z \eta_\mu(z) \right) W_0[\eta, \bar{\eta}].
\]

i. e.

$$W_2^{(1)} = -i \alpha \int d^4x \int d^4x' \int d^4z' \left[ F_\mu^*(x - x') F_\mu(x - z') \right]$$

\[
\left( \int d^4z \eta_\mu(z) \right) \left( \int d^4z \eta_\mu(z) \right) W_0[\eta, \bar{\eta}].
\]
In these computations some care is required due to the anticommuting nature of the Grassman variables. For instance, the derivative

\[ \frac{\delta \left( \int d^4x d^4y \tilde{\eta}_\mu(x) S_\mu(x-y) \eta_\mu(y) \right)}{\delta \eta_\mu(x)} = - \int d^4x \int d^4y \tilde{\eta}_\mu(x) S_\mu(x-y) \frac{\delta \eta_\mu(y)}{\delta \eta_\mu(x)} \]  

involves a change of sign due to the anticommutation relation

\[ \left\{ \tilde{\eta}_\mu, \frac{\delta}{\delta \eta_\mu} \right\} = 0 \, . \]  

(31)

The 2-point function at first order in \( \alpha \) can be computed, using

\[ S^{[1]}_{\epsilon \mu}(x-y) = -i \frac{\delta^2 W^{[1]}}{\delta \eta_\epsilon(y) \delta \tilde{\eta}_\mu(x')} \Big|_{\eta=0=\tilde{\eta}} \, . \]  

(32)

Clearly \( S^{[0]}_{\epsilon \mu}(x-y) = 0 \), so the first term in (33) is proportional to \( W_0 \), yields no contribution. The first derivative gives:

\[ \frac{\delta W^{(1)}}{\delta \eta_\epsilon(y)} = -i \alpha \int d^4xd^4x'd^4z'd^z' \left[ F^*_\mu(x-x') F_\epsilon(x-z') \right] S_\mu(x''-z') S_\epsilon(x'-z) \eta_\epsilon(z) W_0[\eta, \tilde{\eta}] \]

\[-(-i \alpha) \int d^4xd^4x'd^4z'd^z' \left[ F^*_\mu(x-x') F_\epsilon(x-z') \right] \tilde{\eta}_\epsilon(y) S_\mu(y-z') S_\epsilon(x'-z) \frac{\delta \eta_\epsilon(z) W_0[\eta, \tilde{\eta}]}{\delta \tilde{\eta}_\mu(x'')} \]

\[-i \alpha \int d^4xd^4x'd^4z'd^z' \left[ F^*_\epsilon(x-x') F_\mu(x-z') \right] \tilde{\eta}_\epsilon(y) S_\epsilon(y-z) S_\mu(x'-z) \eta_\epsilon(z) \frac{\delta W_0[\eta, \tilde{\eta}]}{\delta \tilde{\eta}_\mu(x'')} \, . \]  

Upon acting with the second derivative, the second and third term of Eq. 33 contain at least one factor of \( \eta, \tilde{\eta} \), so that they vanish upon setting \( \eta = 0 = \tilde{\eta} \). Only the first term survives, yielding

\[ \frac{\delta^2 W^{(1)}}{\delta \eta_\epsilon(y') \delta \tilde{\eta}_\mu(x'')} = -i \alpha \int d^4xd^4x'd^4z' \left[ F^*_\mu(x-x') F_\epsilon(x-z') \right] S_\mu(x''-z') S_\epsilon(x'-y') \eta_\epsilon(z) W_0[\eta, \tilde{\eta}] \, . \]  

(34)

By noting that \( W_0[0,0] = 1 \) and that at first order, in absence of loops, the normalization factor is \( N = 1 \), we finally get

\[ S_{\epsilon \mu}(x''-y') = i \alpha \int d^4xd^4x'd^4z' \left\{ F^*_\mu(x-x') F_\epsilon(x-z') S_\mu(x''-z') S_\epsilon(x'-y') \right\} \, . \]  

(35)

We now compute the Fourier transform of \( S_{\epsilon \mu} \). As a first step we employ the inverse Fourier transforms

\[ S_\mu(x''-z') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x''-z')} S_\mu(k) ; \quad S_\epsilon(x'-y') = \int \frac{d^4k'}{(2\pi)^4} e^{-ik'(x'-y')} S_\epsilon(k') \, ; \]

\[ F^*_\mu(x-x') = \int \frac{d^4k''}{(2\pi)^4} e^{ik''(x-x')} F^*_\mu(k'') ; \quad F_\mu(x-z') = \int \frac{d^4k''}{(2\pi)^4} e^{-ik''(x-z')} F_\mu(k'') \]  

(36)

and insert them in equation (35) to get

\[ S_{\epsilon \mu}(x''-y') = \frac{i \alpha}{(2\pi)^{16}} \int d^4x' \int d^4z' \int d^4x \int d^4k' \int d^4k'' \int d^4k''' \left[ F^*_\epsilon(x-x') \right] F_\mu(x-z') S_\mu(k) S_\epsilon(k') \left\{ e^{-i(k''+k')x' - i(k''-k)x} e^{i(k+k'')z'} e^{-ika''x} e^{ika'y} \right\} \, . \]  

(37)

The integration over the spatial variables is immediate using the definition of the delta function \( \delta^{(4)}(k) = \int \frac{d^4x}{(2\pi)^4} e^{\pm ikx} \). This step reabsors 12 powers of \( \frac{1}{2\pi} \) and produces the three delta functions \( \delta^{(4)}(k''+k'), \delta^{(4)}(k''-k'), \delta^{(4)}(k+k'') \). Performing a final integration over the momenta, one obtains the result

\[ S_{\epsilon \mu}(z) = i \alpha \int \frac{d^4k}{(2\pi)^4} \left[ F^*_\epsilon(-k) F_\mu(-k) S_\mu(k) S_\epsilon(k) \right] e^{-ikz} \, . \]  

(38)
We can read the Fourier transform straight off Eq. (38)

\[ S_{\epsilon \mu}(k) = i\alpha F_{\epsilon}(-k) F_{\mu}(-k) S_{\epsilon \mu}(k) S_{\epsilon}(k). \]  

(39)

One final step is required to obtain \( S_{\epsilon \mu}^{(ret)}(k, t) \). We just need to compute the inverse Fourier transform with respect to the time component \( k_0 \) and multiply by the appropriate causal factor, the Heavyside function \( \Theta(t) \) which selects only positive time differences:

\[ S_{\epsilon \mu}^{(ret)}(k, t) = \Theta(t) \int \frac{dk_0}{2\pi} S_{\epsilon \mu}(k_0) e^{-ik_0 t}. \]  

(40)

Recalling that the free propagators are \( S_{\beta}(k) = \frac{k + m_\beta}{k - m_\beta} \), we can explicitly write

\[ S_{\epsilon \mu}^{(ret)}(k, t) = \Theta(t) \frac{i\alpha}{2\pi} \int dk_0 \chi(k) \left[ \frac{\gamma^0 k_0 - \gamma^i k^i + m_\mu}{k_0^2 - \omega_\mu^2} \right] \left[ \frac{\gamma^0 k_0 - \gamma^i k^i - m_e}{k_0^2 - \omega_e^2} \right] e^{-ik_0 t}, \]  

(41)

where we have used the notation \( \chi(k) = [F_{\epsilon}(-k) F_{\mu}(-k)] \), \( \omega_\mu = \sqrt{(k)^2 + m_\mu^2} \) and \( \omega_e = \sqrt{(k)^2 + m_e^2} \). Of course \( k^0 = k_0 \). In Eq. (41) the integral can be easily computed by using the residue theorem. In particular our integral presents four simple poles on the real axis for \( k_0 = \pm\omega_\mu \) and \( k_0 = \pm\omega_e \). Only two of them, of course, turn out to be relevant. We shift the poles by an infinitesimal imaginary amount \( \pm i\epsilon \), where, as usual, the positive poles are shifted downwards and the negative poles are shifted upwards.

The contour used for the integration is then shown in the Figure (1). The \( \Theta(t) \) forces us to close the contour in the lower half plane, implying the clockwise direction and a \( -2\pi i \) factor in the final result. It is straightforward to compute the residues as:

\[ \lim_{k_0 \to \omega_\mu} (k_0 - \omega_\mu) f(k_0) = \lim_{k_0 \to \omega_\mu} (k_0 - \omega_\mu) \chi \left[ \frac{\gamma^0 k_0 - \gamma^i k^i + m_\mu}{k_0^2 - \omega_\mu^2} \right] \left[ \frac{\gamma^0 k_0 - \gamma^i k^i - m_e}{k_0^2 - \omega_e^2} \right] e^{-ik_0 t} \]

(42)

and

\[ \lim_{k_0 \to -\omega_e} (k_0 - \omega_e) f(k_0) = \lim_{k_0 \to -\omega_e} (k_0 - \omega_e) \chi \left[ \frac{\gamma^0 k_0 - \gamma^i k^i + m_\mu}{k_0^2 - \omega_\mu^2} \right] \left[ \frac{\gamma^0 k_0 - \gamma^i k^i - m_e}{k_0^2 - \omega_e^2} \right] e^{-ik_0 t} \]

(43)

where \( \chi \) and \( \chi' \) are shorthand notations for \( \chi \left( (\omega_\mu, \bar{k}) \right) = F_{\epsilon}(-\omega_\mu, \bar{k}) F_{\mu}(-\omega_\mu, \bar{k}) \) and \( \chi' = \chi \left( (\omega_e, \bar{k}) \right) = F_{\epsilon}(-\omega_e, \bar{k}) F_{\mu}(-\omega_e, \bar{k}) \) respectively. Putting all together we have

\[ S_{\epsilon \mu}^{(ret)}(k, t) = \Theta(t) \alpha (A e^{-i\omega_\mu t} + B e^{-i\omega_e t}). \]  

(44)

Here

\[ A = \frac{\chi}{2\omega_\mu (\omega_\mu^2 - \omega_e^2)} [\gamma_\mu k_\mu + m_\mu] [\gamma_e k_e + m_e]. \]  

(45)
and explicitly:

\[ A = C[\gamma^\rho \gamma^\nu \kappa^\mu_k \kappa^\nu + m_e \gamma^\rho \kappa^\nu + m_\mu \gamma^\nu \kappa^\rho + m_e m_\mu] \]  (46)

Note that the first term can be symmetrized and one obtains \( \gamma^\rho \gamma^\nu \kappa^\mu_k \kappa^\nu = k^2 = m_\mu^2 \). Moreover:

\[ B = \frac{-\chi'}{2\omega_\nu(\omega_\mu^2 - \omega^2)} [\gamma_\mu p^\mu + m_\mu][\gamma_\nu p^\nu + m_e] \]  (47)

and in a similar way:

\[ B = D[m_e^2 + m_e m_\mu + (m_e + m_\mu)\gamma_\nu p_\nu] \]  (48)

where

\[ \chi = F^*_e(-k)F_\mu(-k) \]
\[ \chi' = F^*_e(-p)F_\mu(-p) \]  (49)

with

\[ k^\mu = (\omega_\mu, \vec{k}) \]
\[ p^\mu = (\omega_e, \vec{k}) \]  (50)

We have also set \( C = \frac{\chi}{2\omega_\nu(\omega_\mu^2 - \omega^2)} \) and \( D = \frac{-\chi'}{2\omega_\nu(\omega_\mu^2 - \omega^2)} \) for compactness.

Now we are able to compute:

\[ S^{(ret)}_{\chi\mu}(\vec{k}, t) S^{(ret)*}_{\chi\mu}(\vec{k}, t) = |\alpha|^2 \left[ (AA^\dagger + BB^\dagger) + AB^\dagger e^{-i(\omega_\mu - \omega_e)t} + BA^\dagger e^{i(\omega_\mu - \omega_e)t} \right] \]  (51)

Each term can be computed separately. Starting with \( AA^\dagger \) we have

\[ AA^\dagger = |C|^2[m^2_\mu + m_e m_\mu + (m_e + m_\mu)\gamma_\nu p_\nu][m^2_\mu + m_e m_\mu + (m_e + m_\mu)(\gamma^0 p^0 + \gamma_\rho p_\rho)] = |C|^2[m^2_\mu + m_e m_\mu + (m_e + m_\mu)\gamma_\nu p_\nu][m^2_\mu + m_e m_\mu + (m_e + m_\mu)(2k^0 \gamma_0 - \gamma_\rho p_\rho)] = D^2[m^2_\mu + 2m_e m_\mu + m_\mu^2 + 2m_e^2(m_\mu^2 + m_\mu)(2k^0 \gamma_0 - \gamma_\rho p_\rho)] = 2m_e m_\mu + m_\mu^2 \gamma_\nu p_\nu - m_e m_\mu \gamma_\nu p_\nu + m_\mu^2 m_\mu \gamma_\rho p_\rho\]  (52)

and its trace

\[ \text{Tr}(AA^\dagger) = |C|^2[4m_\mu^4 + 8m_e m_\mu^3 + 4m_\mu^2 m_\mu^2 + 8(m_e + m_\mu)^2(k^0)^2 - 4k^2(m_e + m_\mu)^2 = 8|C|^2\omega_\nu^2 (m_e + m_\mu)^2] \]  (53)

which after some trivial manipulations becomes

\[ \text{Tr}(AA^\dagger) = \frac{2|\chi|^2}{(m^2_\mu - m_e)^2}. \]  (54)

In a similar way:

\[ BB^\dagger = |D|^2[m^2_\mu + m_e m_\mu + (m_e + m_\mu)\gamma_\nu p_\nu][m^2_\mu + m_e m_\mu + (m_e + m_\mu)(\gamma_\rho p_\rho + \gamma^0 p^0)] = |D|^2[m^2_\mu + m_e m_\mu + (m_e + m_\mu)\gamma_\nu p_\nu][m^2_\mu + m_e m_\mu + (m_e + m_\mu)(2k^0 \gamma_0 - \gamma_\rho p_\rho)] = D^2[4m_\mu^4 + 2m_\mu^2 m_\mu + m_\mu^2 m_\mu + 2m_\mu^2(m_e + m_\mu)(2k^0 \gamma_0 - \gamma_\rho p_\rho)] = 4m_\mu^2 m_\mu + m_\mu^2 m_\mu \gamma_\rho p_\rho - m_\mu^2 m_\mu \gamma_\rho p_\rho + m_\mu^2 m_\mu \gamma_\rho p_\rho\]  (55)

and its trace:

\[ \text{Tr}(BB^\dagger) = |D|^2[4m_\mu^4 + 8m_e m_\mu^3 + 4m_\mu^2 m_\mu^2 + 8(m_e + m_\mu)^2(p^0)^2 - 4(m_e + m_\mu)^2 p^0 = 4|D|^2\omega_\nu^2 (m_e + m_\mu)^2] \]  (56)

or

\[ \text{Tr}(BB^\dagger) = \frac{2|\chi'|^2}{(m_e - m_\mu)^2}. \]

The mixed terms are just slightly more involved and yield

\[ AB^\dagger = CD^*[m^2_\mu + m_e m_\mu + (m_e + m_\mu)\gamma_\nu k_\nu][m^2_\mu + m_e m_\mu + (m_e + m_\mu)(\gamma_\rho p_\rho + \gamma^0 p^0)] = CD^*[m^2_\mu + m_e m_\mu + (m_e + m_\mu)\gamma_\nu k_\nu][m^2_\mu + m_e m_\mu + 2(m_e + m_\mu)p^0 \gamma_0 - (m_e + m_\mu)p_\rho \gamma_\rho\]  (57)

or

\[ \text{Tr}(AB^\dagger) = \frac{2|\chi|^2}{(m_e - m_\mu)^2}. \]

The mixed terms are just slightly more involved and yield
and the trace
\[ \text{Tr}(AB^\dagger) = CD^\dagger \{ 4m_e m_\mu^3 + 8m_e^2 m_\mu^2 + 4m_e m_\mu + 8(m_e + m_\mu)^2 p \cdot k - 4(m_e + m_\mu)^2 k \cdot p \} \]
\[ = CD^\dagger \{ 4m_e m_\mu^3 + 8m_e^2 m_\mu^2 + 4m_e^3 m_\mu + 4\omega_1 \omega_\mu (m_e + m_\mu)^2 + 4k^2 (m_e + m_\mu)^2 \}. \] 

Moreover
\[ BA^\dagger = DC^\dagger [m_e^2 + m_e m_\mu + (m_e + m_\mu) \gamma^\nu p_\mu] [m_\mu^2 + m_e m_\mu + (m_e + m_\mu) (\gamma^0 \gamma^\rho \gamma^\nu k_\rho)] = \]
\[ DC^\dagger \{ m_\mu^2 m_e^2 + 2m_e^2 (m_e + m_\mu) k^0 \gamma_0 - m_e (m_e + m_\mu) k^0 \gamma_\rho + \]
\[ + m_e^2 (m_e + m_\mu) p_\nu \gamma^\nu + m_e m_\mu (m_e + m_\mu) p_\nu \gamma^\nu + 2 (m_e + m_\mu)^2 p_\nu k^0 \gamma_0 - (m_e + m_\mu)^2 p_\nu k^0 \gamma^\nu \gamma^\rho \}
\]

with
\[ \text{Tr}(BA^\dagger) = DC^\dagger \{ 4m_e^3 m_\mu + 8m_e^2 m_\mu^2 + 4m_e m_\mu^3 + 8(m_e + m_\mu)^2 p \cdot k - 4(m_e + m_\mu)^2 k \cdot p \mu \}
\[ = DC^\dagger \{ 4m_e^3 m_\mu + 8m_e^2 m_\mu^2 + 4m_e m_\mu^3 + 4\omega_e \omega_\mu (m_e + m_\mu)^2 + 4k^2 (m_e + m_\mu)^2 \} \]

Inserting Eqs. \ref{eq:54}, \ref{eq:55}, \ref{eq:56} and \ref{eq:58} into the definition \ref{eq:16}, we finally obtain the transition probability of two flavor fermion oscillation in non local-quantum field theory
\[ P_{e \rightarrow \mu}(t) = A \text{Tr} (S_{e\mu}^{\text{ret}}(t) S_{e\mu}^{\text{ret} \dagger}(t)) \]
\[ = \frac{2A \alpha^2}{(m_\mu - m_e)^2} \left\{ (|\chi|^2 + |\chi'|^2) + \frac{[\omega_\mu \omega_e + m_e m_\mu + k^2]}{2\omega_\mu \omega_e} \right\} \]
\[ \text{(61)} \]

Where we have used that \( \alpha \) is real and the equality \( m_e m_\mu (m_e + m_\mu)^2 = m_e^3 m_\mu + 2m_e^2 m_\mu^2 + m_e m_\mu^3 \).

Note that this probability is real, as it should. Furthermore one has to observe that this new two flavor oscillation formula (for fermions fields) is totally general. Except for some minimal regularity requirements on \( F_e(k) \) and \( F_\mu(k) \), the latter are completely arbitrary.

To obtain a definite result it remains to determine the normalization constant \( A \). In order to do so we evaluate the local limit of equation \ref{eq:61} and compare it with the standard result (The Pontecorvo oscillation formulae).

This will also provide a consistency check for our formula.

Before proceeding to the computation we remark the conditions under which the formula \ref{eq:61} constitutes a non-local generalization of the usual oscillation formula. First of all, we are dealing, from the outset, with a perturbative interaction controlled by the small parameter \( \alpha \). The quantity that corresponds to \( \alpha \) in two-flavor neutrino mixing is the off-diagonal element of the mass matrix \( m_{e\mu} \). The smallness of \( \alpha \) then amounts to the condition \( m_{e\mu} \ll m_e, m_\mu \). We shall see that this implies a small mixing angle \( \theta \ll 1 \). Then our result (Eq. \ref{eq:61}) is to be compared with the perturbative equivalent of the Pontecorvo formula, in which the limit \( \theta \rightarrow 0 \) is taken.

The second condition has to do with the fermion momentum. In order to avoid complications related to the choice of the vacuum state in Eq. \ref{eq:15}, we have to consider the ultrarelativistic limit \( |\vec{k}| \gg m_e, m_\mu \). In this case the choice of the vacuum state becomes irrelevant, since both the mass and the flavor vacuum give essentially the same result up to vanishingly small corrections.

Preliminarily we analyze the quantity
\[ U_k = \sqrt{\left( \frac{\omega_{k,1} + m_1}{2\omega_{k,1}} \right) \left( \frac{\omega_{k,2} + m_2}{2\omega_{k,2}} \right) \left(1 + \frac{k^2}{(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)} \right)} \]
\[ \text{(62)} \]

which is one of the Bogoliubov coefficients in the flavor Fock space theory of neutrino mixing \cite{mbers,bb}. Here \( m_1 \) and \( m_2 \) are the eigenvalues of the mass Hamiltonian, and are not to be confused with \( m_e \) and \( m_\mu \), which are just its diagonal elements. The energies are \( \omega_{k,j} = \sqrt{k^2 + m_j^2} \) for \( j = 1, 2 \). It is straightforward to see that \( U_k \rightarrow 1 \) in the ultrarelativistic limit \( |\vec{k}| \gg m_j \). Its squared modulus is
\[ |U_k|^2 = \frac{[\omega_1 \omega_2 + m_1 m_2 + k^2]}{2\omega_1 \omega_2} \]
\[ \text{(63)} \]
which is strongly reminiscent of the factor in front of the oscillating term in Eq. (61). Indeed, for a perturbative mixing angle \( \theta \to 0 \), the two almost coincide. To see this, let us Taylor expand the diagonal elements of the mass matrix with respect to \( \theta \)

\[
\begin{align*}
  m_e &= \cos^2 \theta m_1 + \sin^2 \theta m_2 \\
  m_\mu &= \sin^2 \theta + \cos^2 \theta m_2
\end{align*}
\]

\Rightarrow \begin{align*}
  m_e &\simeq m_1 \left( 1 - \theta^2 \right) + \theta^2 m_2 \\
  m_\mu &\simeq \theta^2 m_1 + m_2 \left( 1 - \theta^2 \right)
\end{align*} \Rightarrow m_e m_\mu \simeq m_1 m_2 + \theta^2 (m_2 - m_1)^2 + O(\theta^4)

(64)

Similarly

\[ \omega_e \omega_\mu = \omega_1 \omega_2 + O(\theta) \]

(65)

so that, evidently

\[
\frac{1}{2 \omega_\mu \omega_e} \left[ \omega_\mu \omega_e + m_e m_\mu + \vec{k}^2 \right] = \frac{1}{2 (\omega_1 \omega_2 + O(\theta))} \left[ \omega_1 \omega_2 + O(\theta) + m_1 m_2 + O(\theta^2 + \vec{k}^2) \right]
\]

\[ \simeq |U_k|^2 + O(\theta). \]

The factor in front of the parentheses in Eq. (61) is

\[
\frac{2 A \alpha^2}{(m_\mu - m_e)^2} = \frac{2 A m_{e\mu}^2}{(m_\mu - m_e)^2} = \frac{A}{2} \tan^2 2\theta \simeq 2 A \theta^2 \simeq \frac{A}{2} \sin^2 2\theta.
\]

(66)

Here we have identified \( \alpha = m_{e\mu} \) and used the standard relation in two flavor neutrino mixing \( \tan 2\theta = \frac{2 m_{e\mu}}{m_\mu - m_e} \). The last equalities hold approximately for \( \theta \to 0 \). From equation (67) one also sees that the \( \alpha \to 0 \) limit corresponds to the \( \theta \to 0 \) limit, as we have previously claimed. We can now work out the local ultrarelativistic limit of Eq. (61).

Locality means that the non-local kernels are delta functions \( F_{e,\mu}(x) = \delta^{(4)}(x) \), so that, according to our conventions, their transforms are just unity \( F_{e,\mu}(k) = 1 \). Then \( \chi, \chi' \to 1 \) and the probability reads

\[
P_{e\to\mu}(t) = A \sin^2 2\theta \left\{ 1 - (|U_k|^2 + O(\theta)) \cos ((\omega_\mu - \omega_e) t) \right\}.
\]

(68)

Notice that

\[
\cos ((\omega_\mu - \omega_e) t) = \frac{e^{i (\omega_\mu - \omega_e) t} + e^{-i (\omega_\mu - \omega_e) t}}{2} = \frac{2 \sin \Delta \omega t + O(\theta)}{2} \approx \frac{2 \sin \Delta \omega t + O(\theta)}{2}
\]

(69)

where the equalities are understood in the \( \theta \to 0 \) limit and we have introduced the energy difference \( \Delta \omega = \omega_2 - \omega_1 \). Finally, discarding all the terms higher than second order in \( \theta \) we obtain the local ultrarelativistic limit as

\[
P_{e\to\mu} = 2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}.
\]

As shown above, in the ultrarelativistic limit the factor in front of the oscillating part tends to one. Then we have

\[
P_{e\to\mu}(t) = \frac{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}}{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}} \left\{ \left( |\chi|^2 + |\chi'|^2 \right) - 2 \chi \chi^* e^{i (\omega_\mu - \omega_e) t} - \chi' \chi^{*'} e^{-i (\omega_\mu - \omega_e) t} \right\}.
\]

(70)

Let us consider the \( t = 0 \) expression

\[
P_{e\to\mu}(t) = \frac{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}}{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}} \left\{ \left( |\chi|^2 + |\chi'|^2 \right) - 2 \chi \chi^* - \chi' \chi^{*'} \right\}.
\]

At \( t = 0 \) the transition probability should vanish:

\[
P_{e\to\mu}(t) = \frac{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}}{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}} \left\{ \left( |\chi|^2 + |\chi'|^2 \right) - 2 \chi \chi^* \right\}
\]

\[
= \frac{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}}{2 A \sin^2 2\theta \sin^2 \frac{\Delta \omega t}{2}} |\chi - \chi'|^2.
\]
which is only possible if \( \chi = \chi' \). This is a non-trivial requirement for the non-local kernels of the theory. When such constraint is imposed, the transition probability of Eq. (71) becomes

\[
P_{e\rightarrow\mu}(t) = \frac{4\alpha^2|\chi|^2}{(m_\mu - m_e)^2} \sin^2 \left( \frac{(|\omega_\mu - \omega_e|) t}{2} \right), \tag{72}
\]

with \( \chi = F_e^*(\mu)F_\mu(-k) = \chi' \).

V APPLICATIONS

Having verified the correct local limit, let us apply our formalism to some specific non-local kernels. An often recurring form is the string-inspired exponential function\[ F(k) = e^{-\frac{k^2}{M^2}} \] which introduces a parameter with dimensions of mass, the non-locality scale \( M \). A nonlocal generalization of the standard model using this kernels has been proposed in\[ 12 \] and it has been shown that it may have interesting phenomenological consequences on the anomalous magnetic moment of charged leptons\[ 13 \].

A prudent estimate for the nonlocality scale \( M \) is beyond the energies currently reached in particle accelerators, \( M \gtrsim 10 \text{ TeV} \). The condition of vanishing transition probability at \( t = 0 \) and thus Eq. (72) cannot be satisfied by this kernels. It is however instructive to study the resulting oscillation formula. Let us consider two cases

- **Flavor blind** - we assume that the same non-locality scale \( M \) characterizes both the kernels for \( e, \mu \), so that

\[
F_e(k) = F_\mu(k) = e^{-\frac{k^2}{M^2}}. \tag{73}
\]

Obviously, since the kernels depend only on the relativistic invariant \( k^2 \), we have \( F_\beta(k) = F_\beta(-k) \) for each \( \beta = e, \mu \). The \( \chi, \chi' \) functions are evaluated by plugging the 4-vectors \( k^\mu \equiv (\omega_\mu, \vec{k}) \) and \( p^\mu \equiv (\omega_\mu, \vec{k}) \), selected by the poles, as arguments.

Since \( k^2 = m_e^2 \) and \( p^2 = m_\mu^2 \), we find \( \chi = e^{-\frac{2m_e^2}{M^2}} \) and \( \chi' = e^{-\frac{2m_\mu^2}{M^2}} \). Inserting in the general formula of Eq. (70) we obtain

\[
P_{e\rightarrow\mu}(t) = \frac{\alpha^2}{m_\mu^2 - m_e^2} \left\{ e^{-\frac{4m_e^2}{M^2}} + e^{-\frac{4m_\mu^2}{M^2}} \right\} - 2e^{-\frac{m_\mu^2 + m_e^2}{M^2}} \frac{\omega_\mu \omega_e + m_em_\mu + \vec{k}^2}{2\omega_\mu \omega_e} \cos \left( (\omega_\mu - \omega_e) t \right), \tag{74}
\]

and, in particular, in the ultrarelativistic limit

\[
P_{e\rightarrow\mu}(t) = \frac{\alpha^2}{m_\mu^2 - m_e^2} \left\{ e^{-\frac{4m_e^2}{M^2}} + e^{-\frac{4m_\mu^2}{M^2}} \right\} - 2e^{-\frac{m_\mu^2 + m_e^2}{M^2}} \cos (\Delta \omega t). \tag{75}
\]

Due to the relative size of the masses \( m_e, m_\mu \) and the nonlocality scale, there is hardly any difference between the formula (75) and its local counterpart.

This is neatly seen in Figure (2), where the formula of Eq. (75) is plotted for sample values of the parameters. For reasonable values of \( M \) the resulting formula is practically indistinguishable from the corresponding Pontecorvo-like result. The nonlocal curves of Figure (2) converge to the local result for values as low as \( M = 10^{-1} \text{eV} \).

- **Flavor-dependent** - in this case there are two non-locality scales \( M_e \) and \( M_\mu \). The kernels read

\[
F_\beta(k) = e^{-\frac{k^2}{M_\beta^2}}, \quad \beta = e, \mu \tag{76}
\]

and we get \( \chi = e^{-\frac{m_e^2}{M_e^2}}, \chi' = e^{-\frac{m_\mu^2}{M_\mu^2}} \).

The resulting probability is

\[
P_{e\rightarrow\mu}(t) = \frac{\alpha^2}{(m_\mu - m_e)^2} \left\{ e^{-2m_e^2\left(\frac{1}{M_e^2} + \frac{1}{M_\mu^2}\right)} + e^{-2m_\mu^2\left(\frac{1}{M_e^2} + \frac{1}{M_\mu^2}\right)} - 2e^{-\frac{2m_e^2 + m_\mu^2}{M_e^2}} \frac{\omega_\mu \omega_e + m_em_\mu + \vec{k}^2}{2\omega_\mu \omega_e} \cos (\Delta \omega t) \right\}. \tag{77}
\]

Just like the flavor-blind case, such a probability is essentially indistinguishable from the Pontecorvo-
like result for reasonable values of the nonlocality scales.

A more interesting application is represented by the class of kernels considered in the ref. [35] which show an explicit breakdown of Lorentz symmetry. These are functions of the spatial momentum alone $F(|\vec{k}|^2)$. Not only do these kernels produce a more interesting phenomenology, but they can also satisfy the condition $\chi = \chi'$, and thus Eq. (72) in a trivial way. It is indeed sufficient to consider real and flavor-independent kernels $F_\alpha(k) = F_{\mu}(k) = F(|\vec{k}|^2)$. Since the momenta determined by the poles $k^\mu$ and $p^\mu$ have the same spatial part $\vec{k}$, this automatically ensures that $\chi = \chi'$.

As a simple example consider the exponential form

$$F_\mu(k) = F_{\mu}(k) = e^{-\frac{|\vec{k}|^2}{M^2}}.$$  \hspace{1cm} (78)

Insertion in Eq. (72) gives

$$P_{e\rightarrow\mu}(t) = \frac{4\alpha^2 e^{-\frac{|\vec{k}|^2}{M^2}}}{(m_\mu - m_e)^2} \sin^2 \left( \frac{(\omega_\mu - \omega_e)t}{2} \right). \hspace{1cm} (79)$$

The behaviour of Eq. (79) as a function of $|\vec{k}|$ for a fixed value of time and sample values of the parameters is shown in the Figures (3) and (4). The parameters in Figure (3) are chosen for the purpose of exhibiting the effect of the nonlocal kernel and are not necessarily realistic. For a fixed time, we can clearly see the damping due to the exponential $e^{-\frac{4|\vec{k}|^2}{M^2}}$ superposed to the oscillations in momentum. The parameters of Figure (4) are more realistic, with nonlocality scale of order $M \approx 1 \text{ TeV}$. As it could be expected from Eq. (79), the effect of nonlocality shows up only in the high energy $|\vec{k}| > 200 \text{ GeV}$ tail of the oscillations.

**VI DISCUSSION AND CONCLUSIONS**

In this article we have developed a nonlocal generalization of two flavor fermion mixing. We have used the path...
integral formalism to compute the transition probability and we have demonstrated that it reduces to the correct local form when the appropriate limit is considered. The formula we have derived is rather general and, apart from minimal regularity requirements, can be applied to any delocalization kernel. We have considered a perturbative interaction and a small mixing angle. We have then applied our formalism to some string-inspired delocalization kernel and have discussed the possible phenomenological deviations from the local theory. We have then delocalized only the mixing term, yet another possible scheme is to delocalize the full mass term. A study of this scheme shall be pursued in forthcoming works. We have here delocalized only the momenta large enough. We have here delocalized only the full mass term. A study of this scheme shall be pursued in forthcoming works. We have then applied our formalism to some string-inspired delocalization kernel and have discussed the possible phenomenological deviations from the local theory. We have considered a perturbative interaction and a small mixing angle \( \theta \ll 1 \). In this case, the oscillation formula here presented has to be compared with the Pontecorvo-like formula with \( \theta \rightarrow 0 \).

In future works we shall explore the natural generalization of this treatment in the more realistic case of large mixing angle and in the case of three flavor neutrino oscillation of this treatment in the more realistic case of large mixing angle and in the case of three flavor neutrino oscillation, as well as the analogous analysis for mixed boson fields.

ACKNOWLEDGEMENTS

Partial financial support from MIUR and INFN is acknowledged. A.C. also acknowledges the COST Action CA1511 Cosmology and Astrophysics Network for Theoretical Advances and Training Actions (CANTATA).