A quantum circuit for measuring the bi-expectation of an operator with applications to non-Hermitian winding numbers

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We propose a general quantum circuit inspired by the swap test for measuring the quantity $\langle \psi_1 | A | \psi_2 \rangle$, dubbed the bi-expectation, of an arbitrary operator $A$ with respect to two quantum states $|\psi_{1,2}\rangle$, a frequently encountered quantity in many fields of physics. We apply the circuit, in the field of non-Hermitian physics, to the measurement of bi-expectations with respect to left/right eigenstates, of a given non-Hermitian Hamiltonian. To efficiently prepare the left/right eigenstates as the input to the general circuit, we also develop a quantum circuit by effectively rotating the Hamiltonian in the complex plane. As an application, we demonstrate the validity of these circuits in the prototypical Su-Schrieffer-Heeger model with nonreciprocal hopping by measuring the Bloch and non-Bloch spin textures and the corresponding winding numbers under periodic and open boundary conditions (PBCs and OBCs), respectively. The numerical simulation shows that non-Hermitian spin textures building up these winding numbers can be well captured with high fidelity, and the distinct topological phase transitions between PBCs and OBCs are clearly characterized. We may expect that other non-Hermitian topological invariants composed of non-Hermitian spin textures, such as non-Hermitian Chern numbers, and even important bi-expectations in other branches of physics would also be measured by our general circuit, providing a new perspective to study novel properties in non-Hermitian as well as other physics realized in qubit systems.

I. INTRODUCTION

Since the theoretical prediction and experimental observation of unique features of non-Hermitian systems [1, 2], such as the parity-time-reversal symmetry breaking [3–8], the breakdown of conventional bulk-boundary correspondence [9–26], and the exceptional points (EPs) [2, 27, 28], the non-Hermitian physics has been attracting increasing attentions. Many non-Hermitian phenomena have been explored in various quantum platforms, including quantum optics [11, 29, 30], quantum spin systems [31–33], ultracold atoms [7, 8, 34–40], and so on. However, none of them discuss the direct measurement of the non-Hermitian generalization of expectation values, say $\langle \psi^L | A | \psi^R \rangle$ with $A$ being an arbitrary operator and $|\psi^L,R\rangle$ being a pair of left/right eigenstates, dubbed dual eigenstates, of a given non-Hermitian Hamiltonian, which is deeply involved in many definitions of non-Hermitian quantities as the straightforward generalization of the Hermitian counterparts [41]. The method of measuring the quantities of this form is urgent and may be a prerequisite for studying in a universal manner the exotic non-Hermitian phenomena in experiments.

One of the most interesting quantities involving $\langle \psi^L | A | \psi^R \rangle$ is the non-Hermitian topological invariant [1], such as the non-Hermitian winding number [19]. For example, the Su-Schrieffer-Heeger (SSH) model [42] with nonreciprocal hopping is a prototypical non-Hermitian topological model, the difference of which to its Hermitian counterpart is reflected by the winding number defined with the dual eigenstates [41]. Existing works try to establish relations between this non-Hermitian winding number and the experimentally measurable expectation values to figure it out indirectly. It was shown that the winding number can be calculated by the dynamic winding numbers, defined by the integral of long-time average of measurable expectation values [43]. On the other hand, the authors in Ref. [33] reparameterize the non-Hermitian Hamiltonian and use the measurable expectation values to fit the parameters; the winding number is reconstructed by the parameters. The limitation of these works is the lack of generality for measuring the quantity $\langle \psi^L | A | \psi^R \rangle$, and the relations they found may just be available in special cases.

Furthermore, the quantity $\langle \psi^L | A | \psi^R \rangle$ is not just restricted within the non-Hermitian physics if the dual eigenstates are relaxed to two arbitrary quantum states $|\psi_{1,2}\rangle$, i.e., $\langle \psi_1 | A | \psi_2 \rangle$, which is dubbed the bi-expectation of $A$ in the following. The quantities with the bi-expectation form are ubiquitous in quantum physics, endowed with distinct meanings in different fields of physics [44], such as the overlap integral of two states, matrix elements of an operator, scattering amplitudes, and the Green’s function or Feynman propagator in the quantum field theory. Since direct measurement of them is out of conventional formalism, e.g., the projective (von Neu-
mann) measurement [45], some indirect methods have been proposed, such as quantum fingerprinting [46], weak measurement [47], quantum circuits for simulating correlation function [48–50], etc. However, these proposals are still only valid for specific situations, i.e., \(|\psi_{1,2}\rangle\) cannot be arbitrary. A general scheme for measuring bi-expectations remains to be settled even in broader fields of quantum physics.

To directly deal with the measurement of the bi-expectation \(\langle \psi_1 | A | \psi_2 \rangle\), we propose a general circuit (Fig. 1) that can directly capture the real and imaginary parts of the bi-expectation with the aid of the SWAP test [46]. Meanwhile, to apply the general circuit to \(\langle \psi^L | A | \psi^R \rangle\) in non-Hermitian systems, a quantum circuit (Fig. 2) for efficiently preparing the dual eigenstates of a given non-Hermitian Hamiltonian as the input of the general circuit is also developed based on the dilation method [31].

By numerically simulating these circuits in the nonreciprocal SSH model, we successfully obtain the Bloch and non-Bloch spin textures and the corresponding winding numbers under periodic and open boundary conditions (PBCs and OBCs), respectively, which demonstrates the validity of our circuits.

The paper is organized as follows. The general quantum circuit for measuring bi-expectations is proposed in Sec. II. Specially for non-Hermitian systems, the quantum circuit for preparing the dual eigenstates of a non-Hermitian Hamiltonian is developed in Sec. III. In Sec. IV, we apply these circuits to the non-reciprocal SSH model and numerically simulate the measurement of Bloch and non-Bloch spin textures and the winding numbers. Sec. V comes to a conclusion.

II. A GENERAL MEASUREMENT CIRCUIT FOR BI-EXPECTATIONS

Our aim is to measure the quantity, \(\langle \psi_1 | A | \psi_2 \rangle\), of an arbitrary operator \(A\) with respect to two quantum states \(|\psi_1\rangle\) and \(|\psi_2\rangle\), dubbed a bi-expectation of \(A\), which reduces to the conventional expectation when the two states are identical, i.e., \(|\psi_1\rangle = |\psi_2\rangle\). Because any operator can be decomposed into Hermitian operators, \[ A = \left(\frac{A+A^T}{2}\right) + i \left(\frac{A-A^T}{2i}\right), \]
we just need to propose a quantum circuit to measure the bi-expectation of a Hermitian operator, i.e., \(\langle \psi_1 | O | \psi_2 \rangle\), where \(O\) represents an experimentally accessible Hermitian operator.

Figure 1 shows the quantum circuit for measuring \(\langle \psi_1 | O | \psi_2 \rangle\), which is the main result of this paper. Supposing that \(|\psi_{1,2}\rangle\) are obtained, this circuit consists of systems A and B, each of which is represented by \(n\) qubits, and an ancilla qubit. Firstly, the two states \(|\psi_{1,2}\rangle\) are put into systems A and B, respectively, and the ancilla qubit is initialized to \(|0\rangle\), yielding a product state

\[ |\Psi_0\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes |0\rangle, \]
as an initial state.

![FIG. 1. The general quantum circuit of measuring the bi-expectation \(\langle \psi_1 | O | \psi_2 \rangle\) scaled by \(\langle \psi_1 | O' | \psi_2 \rangle\). Two quantum states \(|\psi_{1,2}\rangle\) are assumed to be prepared by other methods as input respectively in systems A and B, each of which consists of \(n\) qubits. The ancilla qubit is initialized to \(|0\rangle\). The successive operations include a Hadamard gate (denoted by \(H\)) on the ancilla and a controlled-SWAP (Fredkin) gate with the ancilla as the control qubit. \(O\) and \(O'\) denote the experimentally accessible Hermitian operators. \(\sigma_{x,y}\) are two operators of Pauli matrices. The details of the readout process can be referred to Appendix B.](image)

Secondly, by applying a Hadamard gate to the ancilla, and then a controlled-SWAP (Fredkin) gate with the ancilla being the control qubit, we obtain successively

\[ |\Psi_1\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \]
\[ |\Psi_2\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |0\rangle + |\psi_2\rangle \otimes |\psi_1\rangle \otimes |1\rangle). \]

Finally, the operator \(O\) and an ancillary Hermitian operator \(O'\) are introduced respectively to systems A and B. Because of the Hermiticity of \(O\) and \(O'\), we obtain the following relations (see Appendix A for the detailed derivation):

\[ \frac{\langle \Psi_2 | O \otimes O' \otimes \sigma_x | \Psi_2 \rangle}{\langle \Psi_2 | O' \otimes O \otimes \sigma_x | \Psi_2 \rangle} = \text{Re} \left( \frac{\langle \psi_1 | O | \psi_2 \rangle}{\langle \psi_1 | O' | \psi_2 \rangle} \right), \]
\[ \frac{\langle \Psi_2 | O \otimes O' \otimes \sigma_y | \Psi_2 \rangle}{\langle \Psi_2 | O' \otimes O \otimes \sigma_y | \Psi_2 \rangle} = \text{Im} \left( \frac{\langle \psi_1 | O | \psi_2 \rangle}{\langle \psi_1 | O' | \psi_2 \rangle} \right), \]

where \(\sigma_{x,y}\) are two operators of Pauli matrices. Assuming that \(O\) and \(O'\) are experimentally accessible, from Eq. (4) the bi-expectation \(\langle \psi_1 | O | \psi_2 \rangle\) scaled by \(\langle \psi_1 | O' | \psi_2 \rangle\) can be figured out by measuring the conventional expectations [LHS of Eq. (4)] in experiment (see Appendix B for the details of measurement).

For convenience, \(O'\) can be set as an identity operator and the scaling factor \(\langle \psi_1 | O' | \psi_2 \rangle\) reduces to \(\langle \psi_1 | \psi_2 \rangle\) that is usually not equal to 0. Sometimes, if the two states are orthogonal, i.e., \(\langle \psi_1 | \psi_2 \rangle = 0\), \(O'\) should be properly selected such that \(\langle \psi_1 | O' | \psi_2 \rangle \neq 0\) and the bi-expectation with respect to these two orthogonal states can also be measured only up to a constant.
Ⅲ. PREPARATION FOR DUAL EIGENSTATES OF A NON-HERMITIAN HAMILTONIAN

For different purposes, we are interested in various quantities with the form of bi-expectation. In the context of non-Hermitian physics, the bi-expectation of a Hermitian operator $O$ with respect to a pair of dual eigenstates $|\psi^{R,L}\rangle$ is defined as

$$\langle O \rangle_{\text{NH}} \equiv \frac{\langle \psi^L | O | \psi^R \rangle}{\langle \psi^L | \psi^R \rangle},$$

which usually emerges to characterize important non-Hermitian quantities. Here, $|\psi^R\rangle$ is one right eigenstate of a given non-Hermitian Hamiltonian $H$ with the eigenenergy $E$, and $|\psi^L\rangle$ is the corresponding left eigenvector of $H$, i.e., $H|\psi^R\rangle = E|\psi^R\rangle$ and $H^\dagger|\psi^L\rangle = E^*|\psi^L\rangle$; $|\psi^{R,L}\rangle$ form a pair of dual eigenstates of $H$. To measure $\langle O \rangle_{\text{NH}}$ according to Eq. (4), we just need prepare $|\psi^{1,2}\rangle = |\psi^{L,R}\rangle$ as the input states of the general circuit in Fig. 1 and $O'$ is set as the identity operator.

Contrary to the adiabatic-evolution method of generating eigenstates of a given Hermitian Hamiltonian, the amplifying/decaying feature, imprinted in complex eigenenergies, of non-Hermitian Hamiltonians offers a more convenient principle to prepare eigenstates [33, 51]: If some eigenenergies of a non-Hermitian Hamiltonian $H$ have nonvanishing imaginary parts, the long-time non-Hermitian system, several methods are proposed [31, 32] with the demanded $\alpha H$ and $-\alpha^* H^\dagger$ respectively determining $H$ and $-H^\dagger$, especially the selection of the complex multiplier $\alpha$ depends on the specific case. $H^{R,L}(t)$ are dilated Hermitian Hamiltonians in the dilation method, respectively determining $\alpha H$ and $-\alpha^* H^\dagger$. $P_0 = |0\rangle\langle 0|$ is a projection operator on state $|0\rangle$ in the ancillas for postselection. Other notations are explained in the main text.

$$|\psi^{R,L}(t \to \infty)\rangle \rightarrow |\psi^{R,L}\rangle$$

are just the demanded pair of dual eigenstates when the ancillas are measured to be $|0\rangle$.

IV. APPLICATIONS TO NON-HERMITIAN WINDING NUMBERS

In this section, we take the pedagogical SSH model with nonreciprocal hopping [9, 16, 19] as an example, and apply our proposed circuits of Figs. 1 and 2 to the measurement of non-Hermitian spin textures and winding numbers, including Bloch spin textures and winding numbers under PBCs and non-Bloch spin textures and winding numbers under OBCs, which are much different from their Hermitian counterparts.

A. Bloch winding numbers

The Hamiltonian of the nonreciprocal SSH model under PBCs in $k$ space reads

$$H(k) = \begin{bmatrix}
0 & t_1 - \delta + t_2 e^{-ik} \\
t_1 + \delta + t_2 e^{ik} & 0
\end{bmatrix}$$

where $t_1 \pm \delta$ are the nonreciprocal intra-cell hopping amplitudes and $t_2$ is the reciprocal inter-cell hopping amplitude; all the parameters are real. In terms of Pauli matrices, $H(k) = d(k) \cdot \sigma$, where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of Pauli matrices and $d(k) = [d_x(k), d_y(k), d_z(k)] = (t_1 + t_2 \cos k, t_2 \sin k - i\delta, 0)$ can be regarded as an effective complex magnetic field.

The two eigenenergies of $H(k)$ read $E_{\pm}(k) = \pm d(k) = \pm \sqrt{d_x^2(k) + d_y^2(k) + d_z^2(k)}$ and the corresponding right and left eigenstates are $|\psi^{R,L}_{\pm}(k)\rangle$. With the pair of dual eigenstates labeled by $+$, the Bloch spin texture in this non-Hermitian model can be defined as

$$n(k) \equiv \langle \sigma(k) \rangle_{\text{NH}} = \frac{\langle \psi^L_{+}(k) | \sigma | \psi^R_{+}(k) \rangle}{\langle \psi^L_{+}(k) | \psi^R_{+}(k) \rangle},$$

FIG. 2. The quantum circuit of preparing a pair of dual eigenstates of a given non-Hermitian Hamiltonian $H$, as the input of the general circuit in Fig. 1. The target right and left eigenstates of $H$ can be reached by long-time evolutions under $\alpha H$ and $-\alpha^* H^\dagger$, respectively. The selection of the complex multiplier $\alpha$ depends on the specific case. $H^{R,L}(t)$ are dilated Hermitian Hamiltonians in the dilation method, respectively determining $\alpha H$ and $-\alpha^* H^\dagger$. $P_0 = |0\rangle\langle 0|$ is a projection operator on state $|0\rangle$ in the ancillas for postselection. Other notations are explained in the main text.

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which can also be expressed in terms of another pair of dual eigenstates $|\psi_{R,L}^{+}(k)\rangle$. This formula follows the structure of the bi-expectation, so we can measure it using our proposed measurement circuit of Fig. 1 with the replacement of $O$ by $\sigma$ and $O'$ by the identity matrix in Eq. (4), as long as the dual eigenstates $|\psi_{R,L}^{+}(k)\rangle$ as the input are well prepared, and $(\psi_{R}^{+}(k)|\psi_{L}^{+}(k)\rangle \neq 0$ that excludes the EPs; the cases for EPs and points nearby are discussed in the end of this subsection.

For the non-Hermitian model (6), the pair of dual eigenstates $|\psi_{R,L}^{+}(k)\rangle$ at each $k$ can be obtained following the recipe in Sec. III by evolving an initial state $|\psi_{R,L}(t = 0)\rangle = |0\rangle$ under respective $H(k)$ and $-H^\dagger(k)$ ($\alpha = 1$), if $\text{Im}\{E_+(k)\} > \text{Im}\{E_-(k)\}$, and otherwise, under respective $-H(k)$ and $H^\dagger(k)$ ($\alpha = -1$), if $\text{Im}\{E_+(k)\} < \text{Im}\{E_-(k)\}$; when $\text{Im}\{E_+(k)\}$ is close to 0, the measurement accuracy of the spin texture $n(k)$ can be improved by adjusting $\alpha$ to ensure $\text{Im}\{\alpha E_+(k)\} > \text{Im}\{\alpha E_-(k)\}$. With the prepared dual eigenstates, the general circuit of Fig. 1 is applied, wherein the details of readout process can be referred to Appendix B. Thus, according to Eq. (4), the non-Hermitian spin textures can be obtained.

With the measured spin texture $n(k)$, the non-Hermitian Bloch winding number, defined as [19]

$$w_B = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \phi(k) \, dk,$$

(8)
can be recast by the complex angle $\phi(k) = \tan^{-1}[d_y(k)/d_x(k)] = \tan^{-1}[n_y(k)/n_x(k)]$. For the Hermitian case ($\delta = 0$), this winding number classifies a topological phase with $w_B = 1$ and a topologically trivial phase with $w_B = 0$ [55]. However, the non-Hermitian case ($\delta \neq 0$) has three topologically distinct phases [19]:

1. $w_B = 1$, for $|t_1 + |\delta|| < |t_2|$;
2. $w_B = 1/2$, for $|t_1 - |\delta|| < |t_2| < |t_1 + |\delta||$, which has no counterpart in Hermitian systems;
3. $w_B = 0$, for $|t_1 - |\delta|| > |t_2|$.

To demonstrate the validity of our scheme, we numerically simulate our quantum circuits and calculate the Bloch winding numbers. The results are shown in Fig. 3. When the evolution time takes the order of $T = 10/t_2$, the dual eigenstates $|\psi_{R,L}^{+}(k)\rangle$ are well captured by the final states $|\psi_{R,L}^{+}(t = T)\rangle$ with the same population [Fig. 3(a)] and the high fidelity [Fig. 3(b)]. The spin textures $n_{y,x}(k)$ calculated using the final states coincide very well with the analytical results [Fig. 3(c)], and thus, the Bloch winding numbers $w_B$ approach to the expected values when the sampling number $N$ becomes larger and larger [Fig. 3(d)]. The topological phase transitions are also reflected near the points $t_1/t_2 = 0.5$ and 1.5.

Here, we directly perform the non-unitary evolutions in our numerical calculations without using the dilation method, because the dilated Hamiltonians $H^{R,L}[k(t)]$ need be fine-tuned case by case for thousands of sampling points, but the validity of the dilation method is demonstrated for some instances in Appendix C. Because the measurement up to now does not involve EPs except the transition points, we can safely remove $O'$ in Eq. (4) due to $\langle \psi_{R}^{+}|\psi_{R}^{+}\rangle \neq 0$. At or near EPs, the orthogonality of dual eigenstates may fail the measurement due to the vanishing of denominator in Eq. (5). For these scenarios, one way is to find a simple operator $O'$ such that $\langle \psi_{R}^{+}|O'|\psi_{R}^{+}\rangle \neq 0$ and the measured spin texture is the same up to a scaling factor; the other way is to set $O$ and $O'$ as being $\sigma_y$ and $\sigma_x$, respectively, to directly measure the angle $\phi(k)$ determined by the ratio $n_y(k)/n_x(k)$ that bypasses the orthogonal condition.
B. Non-Bloch winding numbers

The existence of topological edge states localized at the ends of the non-Hermitian SSH chain under OBCs cannot be correctly predicted by the Bloch winding number calculated under PBCs, which is the phenomenon of breakdown of bulk-boundary correspondence in non-Hermitian systems [9, 12, 13, 16]. This is due to the non-Hermitian skin effect [17] that has no Hermitian counterpart. Yao et al. [16] proposed a non-Bloch winding number defined in the generalized Brillouin zone (GBZ) under OBCs, successfully restoring the correspondence. Using our proposed circuit, we can also measure this important non-Hermitian quantity.

The generalized Brillouin zone is defined by a complex variable \( \beta \) with a modulus \( |\beta| = \frac{1}{2} \sqrt{t_1^2 + t_2^2} \) and \( k = \frac{\pi}{t_1} \). Like the Bloch Hamiltonian, the non-Bloch Hamiltonian (9) can also be written as

\[
H(\beta) = \begin{bmatrix}
0 & t_1 - \delta + \beta^{-1} t_2 \\
t_1 + \beta + t_2 & 0
\end{bmatrix}, \tag{9}
\]

For this nonreciprocal SSH model, we can make the parameterization \( \beta = re^{it} \) with \( r = \sqrt{\frac{t_1^2 + t_2^2}{t_1^2 - \delta^2}} \) and \( k \) being a real parameter that is similar to the Bloch wave number. Thus, \( \beta \) can take values in a non-unit circle in the complex plane. Like the Bloch Hamiltonian, the non-Bloch Hamiltonian (9) can also be written as \( H(\beta) = \mathbf{d}(\beta) \cdot \mathbf{\sigma} \), where \( \mathbf{d}(\beta) = [d_x(\beta), d_y(\beta), d_z(\beta)] = (t_1 + \frac{\beta + \beta^{-1}}{2} t_2, -\frac{\beta - \beta^{-1}}{2} t_2 - i\delta, 0) \). The non-Bloch winding number shares the same form of the Bloch one, yielding

\[
\psi_N = \frac{1}{2\pi} \int_{C_{\beta}} \partial_z \phi(\beta) \, d\beta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \phi(\beta(k)) \, dk, \tag{10}
\]

where \( \phi(\beta) \equiv \frac{1}{t_1^2 - \delta^2} [d_x(\beta)/d_x(\beta) - d_y(\beta)/d_y(\beta)] \) is determined by the non-Bloch spin texture \( \mathbf{n}(\beta) \) that is of the same form as Eq. (7) with only the replacement of \( k \) by \( \beta \). In the view of this non-Bloch winding number, this nonreciprocal SSH model has two topologically distinct phases: \( \psi_N = 1 \) for \( t_1^2 - \delta^2 < t_2^2 \), which supports topological edge states, and otherwise, \( \psi_N = 0 \), where there is no edge state.

Similar circuits are used as in the previous Bloch part for preparing dual eigenstates \( |\psi_{+}^{RL}(\beta)\rangle \) and measuring the spin textures \( \mathbf{n}(\beta) \), and then, non-Bloch winding numbers can be recast likewise. The results are shown in Fig. 4 with the same system parameters as those in Fig. 3(c) and 3(d). Other parameters are also the same as those in Fig. 3.

V. CONCLUSION

We propose a quantum circuit (Fig. 1) for measuring the quantity, \( \langle \psi_1 | O | \psi_2 \rangle \), of a Hermitian operator \( O \) with respect to two given quantum states \( |\psi_1\rangle \) and \( |\psi_2\rangle \), and thus the general quantity, \( \langle \psi_1 | A | \psi_2 \rangle \), dubbed the bi-expectation, of an arbitrary operator \( A \). With the aid of the operators \( \sigma_x \) and \( \sigma_y \) acting on the ancilla of the circuit, both real and imaginary parts of the bi-expectation can be obtained via the experimentally accessible conventional expectations, i.e., the LHS of Eq. (4), which are the main results of this paper.

Then, we apply the general circuit to the measurement of bi-expectations in non-Hermitian systems, \( \langle O \rangle_{NH} \), where dual eigenstates of non-Hermitian Hamiltonians...
are usually used; therefore, we also propose an efficient circuit (Fig. 2), in light of the dilution method [31, 32], to prepare the dual eigenstates by effectively rotating the Hamiltonian in the complex plane. As an application, we take the nonreciprocal SSH model as an example and numerically simulate the measurement using the prepared circuits, obtaining the Bloch and non-Bloch spin textures and the corresponding winding numbers under PBCs and OBCs, respectively. The results are as good with high fidelity as expected by the theories (Figs. 3 and 4).

In principle, other non-Hermitian topological invariants composed of non-Hermitian spin textures, e.g., non-Hermitian Chern numbers [56, 57] and Wilson loops [58], can also be measured following our schemes. In broader settings of physics, the specific meaning of the bi-correlation function, scattering amplitude, etc., endows our circuits more potential applications.

Appendix A: Derivation of Eq. (4)

In this section, we give the details of derivation of Eq. (4) in the main text.

\[
\langle \Psi_2 | O \otimes O' \otimes \sigma_x | \Psi_2 \rangle = \frac{\text{Re} \left[ \langle \psi_1 | O | \psi_2 \rangle \langle \psi_2 | O' | \psi_1 \rangle \right]}{\langle \psi_1 | O | \psi_2 \rangle^2} = \frac{\text{Re} \left( \langle \psi_1 | O | \psi_2 \rangle \right)}{\langle \psi_1 | O | \psi_2 \rangle},
\]

(A5)

Substituting \( | \Psi_2 \rangle \) of Eq. (3) in the main text into the following quantities, we have

\[
\langle \Psi_2 | O \otimes O' \otimes \sigma_x | \Psi_2 \rangle = \frac{1}{2} \left( \langle \psi_1 | O | \psi_2 \rangle \langle \psi_2 | O' | \psi_1 \rangle + \langle \psi_2 | O | \psi_1 \rangle \langle \psi_1 | O' | \psi_2 \rangle \right),
\]

(A1)

\[
\langle \Psi_2 | O \otimes O' \otimes \sigma_y | \Psi_2 \rangle = \frac{1}{2i} \left( \langle \psi_1 | O | \psi_2 \rangle \langle \psi_2 | O' | \psi_1 \rangle - \langle \psi_2 | O | \psi_1 \rangle \langle \psi_1 | O' | \psi_2 \rangle \right),
\]

(A2)

When \( O = O' \), it is found that

\[
\langle \Psi_2 | O' \otimes O' \otimes \sigma_x | \Psi_2 \rangle = | \langle \psi_1 | O' | \psi_2 \rangle |^2 \geq 0,
\]

(A3)

\[
\langle \Psi_2 | O' \otimes O' \otimes \sigma_y | \Psi_2 \rangle = 0.
\]

(A4)

We can choose \( O' \) such that \(| \langle \psi_1 | O' | \psi_2 \rangle |^2 \neq 0 \), and then Eq. (4) is obtained as follows:

\[
\langle \Psi_2 | O' \otimes O' \otimes \sigma_x | \Psi_2 \rangle = | \langle \psi_1 | O' | \psi_2 \rangle |^2 - | \langle \psi_2 | O' | \psi_1 \rangle |^2.
\]

Appendix B: Details of measurement

In this section, we demonstrate how to measure \( \langle \Psi_2 | O \otimes O' \otimes \sigma_{x,y} | \Psi_2 \rangle \), which is the target observable of the general circuit (Fig. 1) in the main text.

Define the projection operators \( P_0 = |0\rangle \langle 0 | \) and \( P_1 = |1\rangle \langle 1 | \), where \( |0\rangle \) and \( |1\rangle \) are the two eigenstates of the Pauli operator \( \sigma_z \). The expectation of \( \sigma_z \) can be written as

\[
\langle \sigma_z \rangle \equiv \langle \psi | \sigma_z | \psi \rangle = \langle \psi | P_0 - P_1 | \psi \rangle = | \langle 0 | \psi \rangle |^2 - | \langle 1 | \psi \rangle |^2 \approx (N_0 - N_1)/N,
\]

(B1)

where we use \( N_s \) \( (s = 0, 1) \) to represent the number of times the eigenstate \( |s \rangle \) is detected when measuring the state \( |\psi \rangle \), and \( N = N_0 + N_1 \) is the total measurement times. This equation means that the expectation of \( \sigma_z \) can be measured by counting \( N_0 \) and \( N_1 \).

An arbitrary Hermitian operator acting on a single qubit can be written as \( u \cdot \sigma + u_0 \sigma_0 \), where \( u = (u_x, u_y, u_z) \) is a real-valued vector with the norm defined by \( u \), and \( \sigma = (\sigma_x, \sigma_y, \sigma_z) \) is the vector of Pauli operators, and \( \sigma_0 \) represents the identity operator. With the help of \( U(\theta) = e^{-i \frac{\theta}{2} u \cdot \sigma} \) representing the counterclockwise rotation of qubit by \( \theta \) about the \( \hat{u} \) axis, the Hermitian operator can be rewritten as \( u \cdot \sigma + u_0 \sigma_0 = U^\dagger (u \sigma + u_0 \sigma_0) U \).

Without loss of generality, an experimentally accessible Hermitian operator acting on \( n \) qubits can be written in a compact form as

\[
O = \bigotimes_{i=1}^n \left( u \cdot \sigma + u_0 \sigma_0 \right)_i = \bigotimes_{i=1}^n \left( U^\dagger (u \sigma + u_0 \sigma_0) U \right)_i,
\]

(B2)

where the subscript \( i \) labels the operators acting on the \( i \)th qubit, and the expectation of \( O \) with respect to \( n \)-qubit state \( | \Psi \rangle \) can be measured as

\[
\langle \Psi | O | \Psi \rangle = \left\langle \Psi \left| \bigotimes_{i=1}^n (u \sigma + u_0 \sigma_0)_i \right| \Psi \right\rangle = \prod_{i=1}^n \langle u \sigma + u_0 \sigma_0 \rangle_i \approx \prod_{i=1}^n \left( u \sigma + u_0 \sigma_0 \right)_i,
\]

(B3)

where \( | \Psi \rangle = \bigotimes_{i=1}^n U_i \Psi \) and the superscript "r" denotes the quantities with respect to \( | \Psi \rangle \).
For the readout state $|\Psi_2\rangle$ of Eq. (3) in the main text, which consists of $n$ qubits in each of systems A and B, and one ancilla qubit labeled by $a$, applying Eq. (B3) yields
\[
\langle \Psi_2 | O \otimes O' \otimes \sigma_{x,y} | \Psi_2 \rangle \\
\approx \prod_{i=1}^{n} \left( u \frac{N_0' - N_1'}{N'} + u_0 \right)_{i,A} \left( u \frac{N_0' - N_1'}{N'} + u_0 \right)_{i,B} \\
\times \left( u \frac{N_0' - N_1'}{N'} + u_0 \right)_{a}, \quad (B4)
\]
where $|\Psi_2\rangle = \bigotimes_{i=1}^{n} U_{i,A} \bigotimes_{i=1}^{n} U_{i,B} \otimes U_a |\Psi_2\rangle$.

In the case of nonreciprocal SSH model as an application in the main text, for the numerators of LHS of Eq. (4), we have $n = 1$, $O = \sigma$, and $O' = \sigma_0$, and Eq. (B4) reduces to
\[
\langle \Psi_2 | \sigma \otimes \sigma_0 \otimes \sigma_{x,y} | \Psi_2 \rangle \\
\approx \left( \frac{N_0' - N_1'}{N'} \right)_{i,A} \left( \frac{N_0' - N_1'}{N'} \right)_{i,B} \left( \frac{N_0' - N_1'}{N'} \right)_{a}, \quad (B5)
\]
where $|\Psi_2\rangle = U_A \otimes U_a |\Psi_2\rangle$ with $U_{A,a}$ depending on the implementing operators, i.e., $U_{A,a}(\theta) = Y(-\pi/2)$ for $\sigma_x$, $X(\pi/2)$ for $\sigma_y$, and 1 (no rotation) for $\sigma_z$.

For the denominators of LHS of Eq. (4), we have $n = 1$ and $O = O' = \sigma_0$, and Eq. (B5) reduces to
\[
\langle \Psi_2 | \sigma_0 \otimes \sigma_0 \otimes \sigma_{x,y} | \Psi_2 \rangle \\
\approx \left( \frac{N_0' - N_1'}{N'} \right)_{i,A} \left( \frac{N_0' - N_1'}{N'} \right)_{i,B} \left( \frac{N_0' - N_1'}{N'} \right)_{a}, \quad (B6)
\]
where $|\Psi_2\rangle = U_a |\Psi_2\rangle$ with $U_a$ taking the same form as for the numerators.

In a word, after obtaining the final state $|\Psi_2\rangle$, we need first rotate it to $|\Psi'_2\rangle$ in the $(2n+1)$-qubit space according to the expected measurement operators; then, count the times of detected $|0\rangle$ and $|1\rangle$ for each qubit; finally, we can obtain the target bi-expectation through Eq. (4) in the main text.

Appendix C: Dilation method

In this section, we briefly outline the idea of the dilation method developed by Wu et al. [31] through two instances of the nonreciprocal SSH model under PBCs and OBCs. This method can be implemented in several quantum systems, such as NV centers [31, 32], superconducting qubits systems [39].

To find a state $|\psi(t)\rangle$ of a time-dependent non-Hermitian Hamiltonian $H(t)$, i.e.,
\[
i \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle, \quad (C1)
\]
we can integrate it into a dilated state $|\Psi(t)\rangle$ of a composite system by introducing an ancilla qubit as follows,
\[
|\Psi(t)\rangle = |\psi(t)\rangle \otimes |\rangle - \rangle - e^{t\eta(t)} |\psi(t)\rangle \otimes |+\rangle, \quad (C2)
\]
where $\eta(t)$ is an appropriate linear operator, $b$ is a coefficient to offset the amplitude of $|\psi(t)\rangle$, $|\rangle -\rangle = ((0) - (1))/\sqrt{2}$ and $|+\rangle = (-i(0) + (1))/\sqrt{2}$ form an orthonormal basis of the ancilla qubit. If we can attain the dilated state $|\Psi(t)\rangle$ in experiment, the aimed state $|\psi(t)\rangle$ can be easily measured by post-selecting the ancilla state $|\rangle -\rangle$.

The dilated state $|\Psi(t)\rangle$ can be evolved by a Hermitian time-dependent Hamiltonian $H(t)$ in the composite system, i.e.,
\[
i \frac{d}{dt} |\Psi(t)\rangle = H(t) |\Psi(t)\rangle. \quad (C3)
\]

To make $H(t)$ accessible in experiment (in the case of NV centers for example), we can choose
\[
H(t) = \Lambda(t) \otimes \sigma_0 + \Gamma(t) \otimes \sigma_z, \quad (C4)
\]
where
\[
\Lambda(t) = \left\{ H'(t) + \left[ i \frac{d}{dt} \eta(t) + \eta(t) H'(t) \right] \eta(t) \right\} M^{-1}(t),
\]
\[
\Gamma(t) = i \left[ H'(t) \eta(t) - \eta(t) H'(t) - i \eta(t) \frac{d}{dt} \eta(t) \right] M^{-1}(t), \quad (C5)
\]
with
\[
H'(t) = H(t) - ibI, \quad \eta(t) = \sqrt{M(t) - I},
\]
\[
M(t) = T e^{-i \int_0^t H'(t) dt} M(0) T e^{i \int_0^t H'(t) dt}. \quad (C6)
\]
We choose $M(0) = \eta(0)^2 + I$, where $I$ is the identity operator. The setting of $\eta(0) = \eta_0 I$ ensures $\text{det}[M(t) - I] > 0$. 

\[
\begin{align*}
\text{FIG. 5.} & \quad \text{The comparison of the dilated method governed by} \quad H(t) \quad \text{and the non-unitary dynamics by} \quad H(t). \quad \text{The upper and lower panels correspond to the circled states II in Fig. 3 (d) and III in Fig. 4 (d) of the main text with parameters} \quad \{\eta_0, b\} = \{0.8, 0.23\} \quad \text{and} \quad \{0.7, 0.35\}, \quad \text{respectively. The left (middle) panels show the components of the dilated Hamiltonians} \quad H^R(t) \quad \text{and} \quad H^L(t). \quad \text{The right panels compare the dilation method (triangles) with the effective non-Hermitian ones (lines) through the evolution of population} \quad |\langle 0|\psi^R(t)\rangle|^2.
\end{align*}
\]
during the whole evolution. Using \( \dot{H}(t) = H(t) - i b I \) to generate \( H(t) \) can reduce the experimental difficulty [32], where \( H(t) \) is just our target non-Hermitian Hamiltonian.

To demonstrate the validity of the dilation method in our examples of the main text, we generate the dilated Hamiltonian with \( t_1 / t_2 = 1.8 \) (1.6) under PBCs (OBCs), \( \delta / t_2 = 1/2 \), and \( k = \pi / 2 \) as the testbed. Choosing \( n_0 = 0.8 \) and \( b = 0.23 \) under PBCs (\( n_0 = 0.7 \), \( b = 0.35 \), and \( \alpha = e^{i \pi / 16} \) under OBCs), each component of \( H^{R,L}(t) \), defined as \( \Lambda(t) / t_2 = \sum_i \Lambda_i(t) \sigma_i \), \( \Gamma(t) / t_2 = \sum_i \Gamma_i(t) \sigma_i \), are plotted in the left and middle panels of Fig. 5. The right panels of Fig. 5 compare the population evolutions \( |(0| \psi(t,L) (t))|^2 \) respectively generated by the dilation method of \( H^{R,L}(t) \) (dots) and by the effective non-Hermitian method of \( H(t) \) (lines), which shows the dilation method is reliable.

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