Symmetric solutions of the dispersionless Toda hierarchy and associated conformal dynamics

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Abstract

Under certain reality conditions, a general solution to the dispersionless Toda lattice hierarchy describes deformations of simply-connected plane domains with a smooth boundary. The solution depends on an arbitrary (real positive) function of two variables which plays the role of a density or a conformal metric in the plane. We consider in detail the important class of symmetric solutions characterized by the density functions that depend only on the distance from the origin and that are positive and regular in an annulus $r_0 < |z| < r_1$. We construct the dispersionless tau-function which gives formal local solution to the inverse potential problem and to the Riemann mapping problem and discuss the associated conformal dynamics related to viscous flows in the Hele-Shaw cell.

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1 Introduction

In papers [1]-[8] it was shown that some classical problems of complex analysis in 2D, such as the inverse potential problem, the Dirichlet boundary value problem and related problems of viscous hydrodynamics have a hidden integrable structure. For simply-connected domains with a smooth enough boundary, it is the 2D Toda lattice (2DTL) hierarchy of Ueno-Takasaki [9] in the limit of zero dispersion [10] while in more general cases it is the universal Whitham hierarchy introduced by Krichever in [11, 12, 13].

In the hydrodynamical context, this integrable structure applies to viscous flows in the Hele-Shaw cell with negligible surface tension and, more generally, to a class of growth problems referred to as Laplacian growth (LG). They appear in different physical and mathematical contexts and have many important applications (see, e.g., [14]-[18] and references therein). In the 2D LG processes, the dynamics of a moving front or interface between two distinct phases (a closed curve in the plane) is driven by a harmonic scalar field in the domain bounded by the curve.

The hierarchical times $t_k, \bar{t}_k$ ($k \geq 1$) of the 2DTL hierarchy are suitably normalized harmonic moments of the domain and their complex conjugates, with $t_0$ being proportional to the area of the domain. The dispersionless tau-function $F = F(t_0, \{t_k\}, \{\bar{t}_k\})$, regarded as a function of the moments, is a kind of the master function for the above mentioned problems. In particular, it contains all the information about the conformal bijection of any domain with given moments to the unit disk.

The function $F$ is a particular solution to the dispersionless version of the Hirota equations for the 2DTL hierarchy. Although it admits a simple and explicit integral representation, it is a highly non-trivial function when regarded as a function of the $t_k$’s. Some recurrence combinatorial formulas for coefficients of its Taylor expansion are available [20, 21].

Further, in [22, 23] it was argued that any non-degenerate solution of the hierarchy, with certain reality conditions imposed, can be given a similar geometrical and hydrodynamical meaning. Such solutions are parameterized by a function $\sigma(z, \bar{z})$ of two variables which has the meaning of a background charge density, or conformal metric, in the complex plane. The moments should be now defined as integrals of powers of $z$ with this density. The integral representation for the dispersionless tau-function also changes accordingly but the formulas which express the conformal map through its second order derivatives do not depend on the choice of $\sigma$. In other words, the Toda dynamics encodes the shape dependence of the conformal mapping, which we call the conformal dynamics.

In [25], an important class of solutions to the dispersionless 2DTL hierarchy was distinguished. These solutions are characterized by the property that the derivatives $\partial F/\partial t_k$ restricted to the line $t_1 = t_2 = t_3 = \ldots = 0$ and $t_0 \neq 0$ are zero for all $k \geq 1$. In the context of the conformal dynamics, they correspond to axially symmetric functions $\sigma$ (i.e., the ones depending only on $|z|^2$). The corresponding dispersionless tau-functions admit recurrence combinatorial formulas for coefficients of their Taylor expansion which generalize those obtained in [20].

An important example ($\sigma = R/|z|^2$) was considered in [26]. It was shown that, on

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1 Degenerate solutions (known also as finite-component reductions of the infinite hierarchy) were shown in [24] to be related to conformal maps of slit domains.
one hand, this solution describes the LG on the surface of an infinite cylinder of radius \( R \) (or in a channel with periodic boundary conditions) and, on the other hand, it is closely related to enumerative algebraic geometry of ramified coverings of Riemann surfaces. Namely, the dispersionless tau-function \( F \) for this solution is a generating function for the double Hurwitz numbers which count connected genus 0 coverings of the Riemann sphere with prescribed ramification type at two points. See [27] for a review of the subject and [28]-[32] for various integrable properties of Hurwitz partition functions.

This paper is devoted to a more detailed exposition of symmetric solutions and their meaning in conformal dynamics. Contrary to the previous works, where the function \( \sigma \) was almost always assumed to be regular in the whole plane except maybe at infinity, our assumptions here are much weaker. We systematically study the case when the function \( \sigma \) is only assumed to be regular in an annulus \( A = \{ z \in \mathbb{C} \mid r_0 < |z| < r_1 \} \) with some \( r_0, r_1 \), and the boundary curve is within this annulus. In fact this does not bring any substantial changes because really important is only the local behavior of the function \( \sigma \) in a small neighborhood of the boundary curve. However, some formulas get modified in this more general setting because not all quantities remain to be well-defined and thus require a more accurate definition. As a result, some quantities may acquire dependence on the auxiliary parameter \( r_0 \) which, in physical terms, plays the role of a short-distance cut-off for divergent integrals. For example, the dispersionless tau-function for the domain \( D \) should be defined as

\[
F = -\frac{1}{\pi^2} \int_{A \cap D} \int_{A \cap D} \sigma(z, \bar{z}) \log |z^{-1} - \zeta^{-1}| \sigma(\zeta, \bar{\zeta}) \, d^2 z d^2 \zeta.
\]  

We also give a number of explicit examples containing all previously studied cases as well as some new ones. In fact all these examples belong to a rather general family

\[
\sigma(z, \bar{z}) \propto \frac{1}{zz} \left( C_1 \log(z\bar{z}) + C_0 \right)^{-\frac{k-3}{k-2}}
\]  

with integer \( k > 2 \) and some constants \( C_0, C_1 \). At \( k \to 2 \) with properly adjusted \( C_0, C_1 \) one gets the family of homogeneous densities \( \sigma(z, \bar{z}) \propto (zz)^{\alpha-1} \). Among them are the cases \( \sigma = 1 \) (\( \alpha = 1 \)) discussed in [1,2,4] and \( \sigma \propto 1/|z|^2 \) (\( \alpha = 0 \)) discussed recently in [25,26]. The latter is also the \( k = 3 \) case of the general family (1.2).

The paper is organized as follows. In section 2 we review the theory in the general (not necessarily symmetric) case, with the modifications caused by the cut-off at \( |z| = r_0 \). In section 3 we give a detailed analysis in the case of symmetric background densities \( \sigma \). We generalize some familiar results to non-zero values of \( r_0 \) and also present some statements and formulas which seem to be absent in the literature (Theorem 3.1 and Corollaries 3.1, 3.2). The explicit examples are given in section 3.3.

2 Deformations of plane domains and dispersionless integrability

The generic solutions to the dispersionless 2DTL hierarchy take on a geometric significance when the Toda times \( t_k, \bar{t}_k \) are identified with (complex conjugate) moments of
simply-connected domains in the complex plane with smooth boundary. In this case the Toda dynamics encodes the shape dependence of the conformal mapping of such a domain to some fixed reference domain.

2.1 Local coordinates in the space of simply-connected domains

Let \( D \subset \mathbb{C} \) be a compact simply-connected domain whose boundary is a smooth curve \( \gamma = \partial D \) and let \( D^c = \mathbb{C} \setminus D \) be its complement in the Riemann sphere \( \hat{\mathbb{C}} \).

Let \( B(r) = \{ z \in \mathbb{C} \mid |z| \leq r \} \) be the disk of radius \( r \) centered at the origin. Without loss of generality, we assume that \( B(r_0) \subset D \) and \( D \subset B(r_1) \) for some \( r_0 < r_1 \), i.e. the curve \( \gamma \) belongs to the annulus \( A = B(r_1) \setminus B(r_0) \). We will consider deformations of the domain \( D \) such that its boundary remains in the annulus (Fig. 1).

Fix a real-analytic and real-valued function \( U(z, \bar{z}) \) in \( A \) such that

\[
\sigma(z, \bar{z}) := \partial \bar{\partial} U(z, \bar{z}) > 0, \quad z \in A
\]

(we write \( \partial := \partial/\partial z, \bar{\partial} := \partial/\partial \bar{z} \)). The function \( \sigma \) plays the role of a background charge density in the complex plane and the function \( U \) is the electrostatic potential created by these charges. We introduce the set of harmonic moments as follows:

\[
t_k = \frac{1}{2 \pi i k} \oint_{\gamma} z^{-k} \partial \bar{\partial} U(z, \bar{z}) \, dz, \quad k \geq 1.
\]

(2.1)

Using the Green’s theorem, this contour integral can be represented as a 2D integral over \( B(r_1) \setminus D = A \cap D^c \) (or \( D \setminus B(r_0) = A \cap D \)) plus a domain-independent contour integral over \( \partial B(r_1) \):

\[
t_k = -\frac{1}{\pi k} \int_{A \cap D^c} z^{-k} \sigma(z, \bar{z}) \, d^2z + \frac{1}{2 \pi i k} \oint_{|z|=r_1} z^{-k} \partial \bar{\partial} U(z, \bar{z}) \, dz,
\]

(2.2)

where \( d^2z \equiv dx \, dy \). In general \( t_k \)'s are complex numbers. We claim that together with the real parameter

\[
t_0 = \frac{1}{2 \pi i} \oint_{\gamma} \partial U(z, \bar{z}) \, dz = \frac{1}{\pi} \int_{A \cap D} \sigma(z, \bar{z}) \, d^2z + \frac{1}{2 \pi i} \oint_{|z|=r_0} \partial \bar{\partial} U(z, \bar{z}) \, dz
\]

(2.3)
(the moment of constant function) they form a set of local coordinates in the space of domains $D$.

This means, first, that any small deformation of a given domain that preserves all its moments is trivial (local uniqueness of a domain with given moments [15, 8]).

**Proposition 2.1** Any one-parameter deformation $D(t)$ of $D = D(0)$ with some real parameter $t$ such that all $t_k$ are preserved, $\partial_t t_k = 0$, $k \geq 0$, is trivial.

The proof is a modification of the one presented in [8] for the case $\sigma(z, \bar{z}) = 1$. We omit the proof because it is almost literally the same as in [25], where it was assumed, in addition, that $\sigma$ is a regular function in the whole plane. But this assumption is actually irrelevant because what really matters is the behavior of $\sigma$ in a vicinity of the curve. In fact it is enough to require that $\sigma$ is regular and $\sigma \neq 0$ in some strip-like neighbourhood of $\gamma$.

Second, the set of moments is not overcomplete, i.e., they are independent parameters. This fact follows from the explicit construction of vector fields in the space of domains that change real or imaginary part of any moment keeping all the others fixed (see below). These arguments allow one to prove the following theorem.

**Theorem 2.1** The real parameters $t_0$, $\text{Re} t_k$, $\text{Im} t_k$, $k \geq 1$, form a set of local coordinates in the space of simply-connected plane domains with smooth boundary.

This statement allows one to identify functionals on the space of domains $D$ with functions of infinitely many independent variables $t_0, \{t_k\}, \{\bar{t}_k\}$.

### 2.2 The Green’s function and special deformations

#### 2.2.1 The Green’s function and the Poisson formula

According to the Riemann mapping theorem, there exists a conformal map $w(z)$ from $D^c$ onto the exterior of the unit circle. It is convenient to normalize it by the conditions $w(\infty) = \infty$ and $w'(\infty)$ is real positive. The Laurent expansion of $w(z)$ at infinity has the form $w(z) = pz + \sum_{j \geq 0} p_j z^{-j}$, where $p > 0$.

If the conformal map $w(z)$ is known, one can construct the Green’s function of the Dirichlet boundary value problem in the domain $D^c$:

$$G(z, \xi) = \log \left| \frac{w(z) - w(\xi)}{w(z)w(\xi) - 1} \right|. \quad (2.4)$$

This function solves the Dirichlet boundary value problem through the use of the Poisson formula

$$u^H(z) = -\frac{1}{2\pi} \oint_{\gamma} u(\xi) \partial_{\xi} G(z, \xi) |d\xi|. \quad (2.5)$$

Here $\partial_{\xi}$ denotes the derivative along the outward normal vector to the boundary of $D$ with respect to the second variable and $|d\xi|$ is an infinitesimal element of length along
the boundary. The Poisson formula provides the (unique) harmonic continuation \( u^H \) of any function \( u \) from the curve \( \gamma \) to its exterior (i.e., the harmonic function in \( D^c \) regular at \( \infty \) such that \( u^H|_{\gamma} = u \)).

The Green’s function has the following properties: a) it is symmetric under permutation of the arguments, b) it is harmonic in each variable everywhere in \( D^c \) except \( z = \xi \) where it has the logarithmic singularity \( G(z, \xi) = \log |z - \xi| + \ldots \) as \( z \to \xi \), c) \( G(z, \xi) = 0 \) for any \( z \in D^c \) and \( \xi \in \gamma \).

### 2.2.2 Special deformations induced by the Green’s function

We will describe infinitesimal deformations of the domain \( D \) by the normal displacement of the boundary \( \delta n(z) \) at any point \( z \in \gamma \), positive if directed outward \( D \).

Fix a point \( a \in D^c \) and consider a special infinitesimal deformation defined by the normal displacement

\[
\delta_a n(z) = -\frac{\varepsilon}{2\sigma(z, \bar{z})} \partial_{n_a} G(a, z), \quad z \in \gamma, \quad \varepsilon \to 0. \tag{2.6}
\]

Equivalently, one may speak about the normal “velocity” of the boundary deformation which is \( V_n(z) = \lim_{\varepsilon \to 0} (\delta n_a(z)/\varepsilon) \), with \( \varepsilon \) playing the role of time. Note that \( \partial_{n_a} G(a, z) < 0 \), so at positive \( \varepsilon \) the domain expands. For any sufficiently smooth initial boundary this deformation is well-defined as \( \varepsilon \to 0 \). By \( \delta_a \) we denote the variation of any quantity under this deformation.

Let us introduce the differential operator

\[
\nabla(z) = \partial_0 + D(z) + \bar{D}(\bar{z}), \tag{2.7}
\]

where \( D(z), \bar{D}(\bar{z}) \) are given by

\[
D(z) = \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k, \quad \bar{D}(\bar{z}) = \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k. \tag{2.8}
\]

Hereafter we abbreviate \( \partial_k = \partial/\partial t_k, \bar{\partial}_k = \partial/\partial \bar{t}_k \).

**Lemma 2.1** Let \( X \) be any functional on the space of domains \( D \) regarded as a function of \( t_0, \{t_k\}, \{\bar{t}_k\} \), then for any \( z \in D^c \) we have \( \delta_z X = \varepsilon \nabla(z) X \).

**Proof.** From (2.2), (2.3) it is easy to see that

\[
\delta_z t_0 = -\frac{\varepsilon}{2\pi} \oint_{\gamma} \partial_{n_z} G(z, \xi) d\xi = \varepsilon, \quad \delta_z t_k = -\frac{\varepsilon}{2\pi k} \oint_{\gamma} \xi^{-k-1} \partial_{n_z} G(z, \xi) d\xi = \frac{\varepsilon}{k} z^{-k}
\]

by virtue of the Poisson formula (2.3). Therefore by Theorem 2.1 we have:

\[
\delta_z X = \frac{\partial X}{\partial t_0} \delta_z t_0 + \sum_{k \geq 1} \frac{\partial X}{\partial t_k} \delta_z t_k + \sum_{k \geq 1} \frac{\partial X}{\partial \bar{t}_k} \delta_z \bar{t}_k = \varepsilon \left( \partial_0 + \sum_{k \geq 1} \frac{z^{-k}}{k} \partial_k + \sum_{k \geq 1} \frac{\bar{z}^{-k}}{k} \bar{\partial}_k \right) X.
\]
Lemma 2.2 Let $X$ be a functional of the form $X = \int_{A \cap D} \Psi(\zeta, \bar{\zeta}) \sigma(\zeta, \bar{\zeta}) d^2 \zeta$ with an arbitrary domain-independent integrable function $\Psi$ regular on the boundary, then

$$\nabla(z)X = \pi \Psi^H(z),$$

where $\Psi^H(z)$ is the (unique) harmonic extension of the function $\Psi$ from the boundary to the domain $D^c$.

Proof. The variation of $X$ under the special deformation (2.6) is

$$\delta_z X = \oint_{\gamma} \Psi(\zeta, \bar{\zeta}) \sigma(\zeta, \bar{\zeta}) \delta n_z(\zeta) |d\zeta| = -\frac{\varepsilon}{2} \oint_{\gamma} \Psi(\zeta, \bar{\zeta}) \partial_n G(z, \zeta) |d\zeta|.$$  

The assertion obviously follows from Lemma 2.1 and the Poisson formula (2.5).  

Now we can explicitly define the deformations that change only either $x_k = \text{Re} t_k$ or $y_k = \text{Im} t_k$ keeping all other moments fixed. From the proof of Lemma 2.1 it follows that the normal displacements $\delta n(\xi) = \varepsilon \text{Re} (\partial_{n_\xi} H_k(\xi))$ and $\delta n(\xi) = \varepsilon \text{Im} (\partial_{n_\xi} H_k(\xi))$, where

$$H_k(\xi) = \frac{1}{2\pi i} \oint_{\infty} z^k \partial_z G(z, \xi) dz$$

(the contour integral goes around infinity) change the real and imaginary parts of $t_k$ by $\pm \varepsilon$ respectively keeping all other moments unchanged. In particular, the deformation

$$\delta_{\infty} n(\xi) = -\frac{\varepsilon}{2\sigma(\xi, \xi)} \partial_{n_\xi} G(\infty, \xi) = \frac{\partial_n \log |w(\xi)|}{2\sigma(\xi, \xi)}$$  

changes $t_0$ only. Therefore, the vector fields $\partial/\partial t_0, \partial/\partial x_k, \partial/\partial y_k$ in the space of domains are locally well-defined and commute. Existence of such vector fields means that the variables $t_k$ are independent and $\partial_k = \frac{1}{2}(\partial_{x_k} - i \partial_{y_k}), \bar{\partial}_k = \frac{1}{2}(\partial_{x_k} + i \partial_{y_k})$ can be understood as partial derivatives.

2.2.3 Deformations of the domain with given moments induced by small changes of the potential

Given a variation of the potential $U \to U + \delta U$, one can consider a simultaneous deformation of the boundary curve $\gamma$ such that all the moments $t_k$ remain fixed.

Proposition 2.2 (cf. [33]) Let $U \to U + \delta U$ be a variation of the potential, then the deformation of the domain given by

$$\delta n(z) = -\frac{\partial_n (\delta U(z, \bar{z}) - \delta U^H(z, \bar{z}))}{4\sigma(z, \bar{z})}$$  

preserves all the moments $t_k, k \geq 0$. 

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In the same way as in (2.2), we can represent them in the form

\[ \delta I = \delta \left( \int_{\mathcal{A} \cap \mathcal{D}} z^{-k} \sigma d^2 z \right) = - \oint_{\gamma} z^{-k} \sigma \delta n(z) |dz| + \int_{\mathcal{A} \cap \mathcal{D}^c} z^{-k} \delta \sigma d^2 z \]

\[ = \frac{1}{4} \oint_{\gamma} z^{-k} \partial_n \left( \delta U(z, \bar{z}) - \delta U^H(z, \bar{z}) \right) |dz| + \frac{1}{4} \int_{\mathcal{A} \cap \mathcal{D}^c} z^{-k} \Delta \left( \delta U(z, \bar{z}) - \delta U^H(z, \bar{z}) \right) d^2 z. \]

Here \( \Delta = 4 \partial \bar{\partial} \) is the Laplace operator and \( \delta U^H \) in the last integral can be added because \( \Delta(\delta U^H) = 0 \). By the Green theorem, the sum of the two integrals in the last line yields

\[ \delta I_1 = \frac{1}{4} \oint_{|z|=r_1} \left[ z^{-k} \partial_n \left( \delta U(z, \bar{z}) - \delta U^H(z, \bar{z}) \right) - \left( \delta U - \delta U^H \right) \bar{\partial}_n(z^{-k}) \right] |dz| = \frac{1}{2i} \oint_{|z|=r_1} z^{-k} \partial U d\bar{z} \]

(the last equality comes out as a result of some simple transformations). But this is equal to \( \delta I_2 = \frac{1}{2i} \oint_{|z|=r_1} z^{-k} \delta \partial U d\bar{z} \). Therefore, \( \delta t_k = 0 \) for \( k \geq 1 \). A similar calculation for \( t_0 \) gives \( \delta t_0 = 0 \).

2.3 Complimentary moments

The set of complimentary moments can be introduced by the contour integrals

\[ v_k = \frac{1}{2\pi i} \oint_{\gamma} z^k \partial U(z, \bar{z}) |dz|, \quad k \geq 1. \tag{2.11} \]

In the same way as in (2.2), we can represent them in the form

\[ v_k = \frac{1}{\pi} \int_{\mathcal{A} \cap \mathcal{D}} z^k \sigma(z, \bar{z}) d^2 z + \frac{1}{2\pi i} \oint_{|z|=r_0} z^k \partial U(z, \bar{z}) |dz|. \tag{2.12} \]

The moment \( v_k \) is “dual” to the moment \( t_k \) in the sense which will be clarified below. Dual to \( t_0 \) is the logarithmic moment

\[ v_0 = \frac{1}{\pi} \int_{\mathcal{A} \cap \mathcal{D}} \log |z|^2 \sigma(z, \bar{z}) d^2 z \tag{2.13} \]

which can be also represented through contour integrals:

\[ v_0 = \frac{1}{2\pi i} \left( \oint_{\gamma} - \oint_{|z|=r_0} \right) \left( \log |z|^2 \partial U(z, \bar{z}) |dz + U(z, \bar{z}) d\log \bar{z} \right). \tag{2.14} \]

The moments \( v_k \) are functions of the moments \( t_0, \{t_k\}, \{\tilde{t}_k\} \).

Consider the function

\[ \phi(z, \bar{z}) = -\frac{1}{\pi} \int_{\mathcal{A} \cap \mathcal{D}} \log |z^{-1} - \zeta^{-1}|^2 \sigma(\zeta, \bar{\zeta}) d^2 \zeta \tag{2.15} \]

which has the meaning of 2D Coulomb potential created by the charge distributed in \( \mathcal{A} \cap \mathcal{D} \) with density \( \sigma \) and a point-like charge at the origin. This function is known to be continuous across the boundary together with its first order partial derivatives: if we write

\[ \phi(z, \bar{z}) = \Theta_{\mathcal{A} \cap \mathcal{D}}(z) \phi^{(+)}(z, \bar{z}) + \Theta_{\mathcal{D}^c}(z) \phi^{(-)}(z, \bar{z}), \]

Proof. The proof is straightforward. At \( k \geq 1 \) we write (see (2.2)): \( \pi k \delta t_k = -\delta I_1 + \delta I_2 \), where

\[ \delta I_1 = \delta \left( \int_{\mathcal{A} \cap \mathcal{D}} z^{-k} \sigma d^2 z \right) = - \oint_{\gamma} z^{-k} \sigma \delta n(z) |dz| + \int_{\mathcal{A} \cap \mathcal{D}^c} z^{-k} \delta \sigma d^2 z \]

\[ = \frac{1}{4} \oint_{\gamma} z^{-k} \partial_n \left( \delta U(z, \bar{z}) - \delta U^H(z, \bar{z}) \right) |dz| + \frac{1}{4} \int_{\mathcal{A} \cap \mathcal{D}^c} z^{-k} \Delta \left( \delta U(z, \bar{z}) - \delta U^H(z, \bar{z}) \right) d^2 z. \]
where $\Theta_D(z)$ is the characteristic function of the domain ($\Theta_D(z) = 1$ if $z \in D$ and 0 otherwise), then

$$
\phi^{(+)}(z, \bar{z}) \big|_{z \in \gamma} = \phi^{(-)}(z, \bar{z}) \big|_{z \in \gamma}, \quad \partial \phi^{(+)}(z, \bar{z}) \big|_{z \in \gamma} = \partial \phi^{(-)}(z, \bar{z}) \big|_{z \in \gamma}.
$$

(2.16)

Equivalently, using the Green theorem, we can represent the function $\phi$ as follows:

$$
\phi(z, \bar{z}) = -U(z, \bar{z})\Theta_{A \cap \overline{D}}(z)
$$

$$
- \frac{1}{2\pi i} \left( \oint_{\gamma} \frac{f - f}{\|z\| = r_0} \right) \left( \log|z^{-1} - \zeta^{-1}|^2 \partial_U(\zeta, \bar{\zeta}) d\zeta + \frac{U(\zeta, \bar{\zeta})}{(\zeta - \bar{\zeta})} d\bar{\zeta} \right).
$$

(2.17)

In this form the continuity across the boundary is implicit but the function $\phi$ becomes ready for expanding it in a series both inside and outside $A \cap D$:

$$
\phi^{(+)}(z, \bar{z}) = -U(z, \bar{z}) - u_0 + t_0 \log|z|^2 + \sum_{k \geq 1} (t_k z^k + \bar{t}_k \bar{z}^k) + \psi(z, \bar{z}), \quad z \in A \cap D
$$

(2.18)

$$
\phi^{(-)}(z, \bar{z}) = v_0 + \sum_{k \geq 1} \frac{1}{k} (u_k z^{-k} + \bar{u}_k \bar{z}^{-k}) + \psi(z, \bar{z}), \quad z \in D^c
$$

(2.19)

Here

$$
\psi(z, \bar{z}) = \frac{1}{2\pi i} \oint_{|\zeta| = r_0} \left[ \log\left(1 - \frac{\zeta}{z}\right) \partial_U \xi d\zeta - \log\left(1 - \frac{\bar{\zeta}}{\bar{z}}\right) \partial_{\bar{U}} \bar{\xi} d\bar{\zeta} \right]
$$

(2.20)

$$
u_0 = \frac{1}{2\pi i} \oint_{|\zeta| = r_0} \log|\zeta|^2 \partial_U \xi d\zeta + U d\log\zeta.
$$

(2.21)

Note that $u_0$ is a real number and $\psi$ is a harmonic function in $\mathbb{C} \setminus B(r_0)$. This function can be expanded in a series in negative powers of $z, \bar{z}$ but here we do not need this expansion in the explicit form. We see that the moments $t_k$ determine the harmonic part of the potential inside $A \cap D$ that depends on the shape of its exterior boundary $\gamma$ while the complimentary moments are coefficients of the multipole expansion of the potential outside it.

A holomorphic generating function for the moments is given by the integral of Cauchy type

$$
C(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\partial U(\zeta, \bar{\zeta})}{\zeta - z} d\zeta.
$$

It defines a function holomorphic in $D$ and $D^c$ with a jump across $\gamma$. Let $C^{\pm}(z)$ be the holomorphic functions defined by this integral in $D$ and $D^c$ respectively. By the Sokhotski-Plemelj formula, the jump of the function $C(z)$ across the contour $\gamma$ is equal to $\partial U$:

$$
(C^{+}(z) - C^{-}(z)) \big|_{z \in \gamma} = \partial U(z, \bar{z}).
$$

(2.22)

This relation is nothing else than the equation of the curve $\gamma$ in the $z$-plane. Expanding $C^{+}(z)$ in the Taylor series for small enough $|z|$, we see that it is the generating function of the moments $t_k$ with $k \geq 1$:

$$
C^{+}(z) = \frac{1}{2\pi i} \oint_{\gamma} \frac{\partial U(\zeta, \bar{\zeta})}{\zeta - z} = \sum_{k \geq 1} kt_k z^{k-1}.
$$
Similarly, $C^-(z)$ is the generating function for the complimentary moments $v_k$ with $k \geq 1$:

$$C^-(z) = \frac{1}{2\pi i} \oint_\gamma \frac{\partial U(\zeta, \bar{\zeta})}{\bar{\zeta} - z} d\zeta = -\frac{t_0}{z} - \sum_{k \geq 1} v_k z^{-k-1}, \quad z \in D^c.$$  

### 2.4 The dispersionless tau-function

Consider the following functional on the space of domains $D$:

$$F = -\frac{1}{\pi^2} \iint_{A \cap D} \int_{A \cap \dot{D}} \sigma(z, \bar{z}) \log \left| z^{-1} - \zeta^{-1} \right| \sigma(\zeta, \bar{\zeta}) d^2zd^2\zeta. \quad (2.23)$$

**Theorem 2.2** (cf. [1]-[4], [22]) The following relations hold:

$$v_0 = \partial_0 F, \quad v_k = \partial_k F, \quad \bar{v}_k = \bar{\partial}_k F, \quad k \geq 1. \quad (2.24)$$

**Proof.** The variation of $F$ under the special deformation $\delta z$ is

$$\delta z F = -\varepsilon \frac{1}{4\pi} \oint_\gamma \phi(z, \bar{z}) \partial_\zeta G(z, \zeta) d\zeta | + \frac{1}{2\pi} \iint_{A \cap \dot{D}} \delta \phi(z, \bar{\zeta}) \sigma(z, \bar{\zeta}) d^2\zeta. \quad (2.25)$$

The first term is equal to $\varepsilon \phi(z, \bar{z})/2$ by the Poisson formula with taking into account that $\phi(z, \bar{z})$ is harmonic in $D^c$. In the second term $\delta \phi(z, \bar{\zeta}) = -\varepsilon \log |z^{-1} - \zeta^{-1}|^2$ because the function $\log |z^{-1} - \zeta^{-1}|^2$ for $\zeta \notin D^c$ is harmonic as a function of $z \in D^c$ (see the proof of Lemma 2.2). Hence the second term is $\varepsilon \phi(z, \bar{z})/2$, the same as the first one. Therefore, by Lemma 2.1 we get

$$\nabla(z) F = \phi(z, \bar{z}) = -\frac{1}{\pi} \iint_{A \cap D} \log \left| z^{-1} - \zeta^{-1} \right| \sigma(\zeta, \bar{\zeta}) d^2\zeta, \quad z \in D^c. \quad (2.25)$$

The expansion of both sides in powers of $z, \bar{z}$ yields (2.24).

**Remark.** The assertion of the theorem means that the function $F$ gives a formal local solution to the inverse potential problem in 2D: given $t_k$, one can find the complimentary moments by means of (2.21) and then the shape of the domain from the equation of the boundary curve (2.22). The theorem also justifies the interpretation of the moment $v_k$ as dual to $t_k$. In fact one could choose the $v_k$’s as independent coordinates on the space of domains. In these variables, one should work with the Legendre transformation of the function $F$.

**Theorem 2.3** ([4]-[6], [22]) It holds

$$G(z, \zeta) = \log |z^{-1} - \zeta^{-1}| + \frac{1}{2} \nabla(z) \nabla(\zeta) F. \quad (2.26)$$

**Proof.** The proof consists in the application of Lemma 2.2 to (2.23) and using the characteristic properties of the Green’s function. Applying the lemma to (2.25), we conclude that $\nabla(\zeta) \nabla(z) F$ is the harmonic continuation of the function $-2 \log |z^{-1} - \zeta^{-1}|$ from the boundary to the domain $D^c$. This function is harmonic everywhere in $D^c$ except at $\zeta = z$, where it has the logarithmic singularity. It can be canceled, without changing the boundary value, by adding the function $2G(z, \zeta)$.
Corollary 2.1 The conformal map \( w(z) \) is given by

\[
w(z) = z \exp \left( -\frac{1}{2} \partial_0^2 - \partial_0 D(z) \right) F .
\] (2.27)

Proof. From equation (2.4) it follows that \( G(z, \infty) = -\log |w(z)| \). Tending \( \xi \to \infty \) in (2.26) and separating holomorphic and antiholomorphic parts in \( z \), we get the result. \( \blacksquare \)

Note that the limit \( z \to \infty \) in (2.27) yields \( \log p = -\frac{1}{2} \partial_0^2 F \).

The following theorem establishes the connection with integrability by identifying \( F \) with the tau-function of the dispersionless 2DTL hierarchy.

Theorem 2.4 ([1], [4], [23]) The function \( F \) satisfies the equations

\[
(z - \zeta)e^{D(z)D(\zeta)F} = ze^{-\partial_0 D(z)F} - \zeta e^{-\partial_0 D(\zeta)F},
\]

\[
(z - \bar{\zeta})e^{D(z)D(\bar{\zeta})F} = \bar{z}e^{-\partial_0 D(z)F} - \bar{\zeta} e^{-\partial_0 D(\bar{\zeta})F},
\]

\[
1 - e^{-D(z)D(\bar{\zeta})F} = \left( z\bar{\zeta} \right)^{-1} e^{\partial_0 (\partial_0 + D(z) + \bar{D}(\zeta))} F .
\] (2.28)

The proof is the same as in [8]. Namely, combining (2.4) and (2.26), we get

\[
\log \left| \frac{w(z) - w(\zeta)}{1 - w(z)\bar{w}(\bar{\zeta})} \right| = \log \left| \frac{1}{z} - \frac{1}{\bar{\zeta}} + \frac{1}{2} \nabla(z) \nabla(\zeta) F .
\]

Next, substituting here \( w(z) \) from (2.27) and separating holomorphic and antiholomorphic parts in \( z, \zeta \), we obtain the result.

Equations (2.28) comprise the dispersionless 2DTL hierarchy in the Hirota form [10]. Note that although the definitions of the harmonic moments and the function \( F \) essentially depend on the background density \( \sigma \), the formulas for the Green’s function and the conformal map \( (2.26), (2.27) \) are \( \sigma \)-independent. This means that the conformal dynamics is described by any solution to the dispersionless 2DTL hierarchy of this class.

Theorem 2.5 The third order derivatives of the function \( F \) are given by

\[
\nabla(z_1) \nabla(z_2) \nabla(z_3) F = -\frac{1}{4\pi} \oint_{\gamma} \partial_{n_3} G(z_1, z) \partial_{n_2} G(z_2, z) \partial_{n_1} G(z_3, z) \frac{|dz|}{\sigma(z, \bar{z})} .
\] (2.29)

Proof. This assertion follows from the Hadamard variational formula [34] for the Green’s function:

\[
\delta G(z_1, z_2) = \frac{1}{2\pi} \oint_{\gamma} \partial_{n_2} G(z_1, z) \partial_{n_1} G(z_2, z) \delta n(z) |dz|
\]

applied to the special deformation \( \delta n(z) = \delta z_3 n(z) \) and from Lemma 2.1 \( \blacksquare \)

For \( \sigma(z, \bar{z}) = \text{const} \) this formula is equivalent to the residue formula from [13]. We also mention the particular case of this formula as \( z_i \to \infty \):

\[
\partial_0^3 F = \frac{1}{4\pi} \oint_{\gamma} \frac{|dw(z)|^3}{\sigma(z, \bar{z}) dz d\bar{z}} .
\] (2.30)
2.5 Conformal dynamics

Consider deformations such that the moments \( t_k \) with \( k \geq 1 \) are kept fixed and the only deformation parameter is \( t = t_0 \) (“time”). Equivalently, such deformations can be defined by the requirement that the normal velocity of the boundary curve \( \gamma \), \( V_n(z) = \lim_{\varepsilon \to 0} \delta_{\infty} n(z)/\varepsilon \), at any point \( z \in \gamma \) and at any time \( t \in [t_1, t_2] \) with some \( t_1 < t_2 \) is given by

\[
V_n(z) = \frac{\partial_n \log |w(z)|}{2\sigma(z, \bar{z})} = \frac{|w'(z)|}{2\sigma(z, \bar{z})},
\]

(see (2.29)). If \( \sigma(z, \bar{z}) = \text{const} \), then equation (2.31) states that the normal velocity of the interface \( \gamma \) is proportional to the gradient of the Green function of the Laplace operator. This is the Darcy law for the dynamics of interface between viscous and non-viscous fluids confined in the Hele-Shaw cell, assuming vanishing surface tension at the interface: \( V_n(z) \propto |w'(z)| \). Such process is also called Laplacian growth (LG); see, e.g., [14]-[18]. For a non-constant \( \sigma \) we have the LG in a non-uniform background\(^2\).

The case when \( \sigma(z, \bar{z}) \) is the squared modulus of a holomorphic function is special and important. In this case the problem can be mapped to another LG problem, in some other plane which we call the \( Z \)-plane, with uniform background but with different boundary conditions. Namely, consider a map \( z \mapsto Z(z) \) which is conformal in the annulus \( A \). The boundary curve \( \gamma \) is mapped to a curve \( \Gamma \) in the \( Z \)-plane. It is clear that the normal velocities, \( V^{(Z)}_n(z) \) and \( V^{(Z)}(Z) \), at the corresponding points of the curves in the two planes are related as

\[
V^{(Z)}_n(Z)\big|_{Z \in \Gamma} = \frac{dZ}{dz} V^{(z)}_n(z)\big|_{z \in \gamma}.
\]

Using (2.31) and writing \( |w'(z)| = \left| \frac{dw}{dz} \right| \cdot \left| \frac{dz}{d\bar{z}} \right| \), we get

\[
V^{(Z)}_n(Z) = \left| \frac{dZ/dz}{\sigma(z, \bar{z})} \right|^2 |w'(Z)|, \quad \omega(Z) := w(z(Z)).
\]

Hence at \( \sigma(z, \bar{z}) \propto |dZ/dz|^2 \) we have the LG problem in the \( Z \)-plane with the uniform background: \( V^{(Z)}_n(Z) \propto |w'(Z)|. \) Some examples are given in the next section.

**Remark.** In [20], for the particular case of the background density \( \sigma = R/|z|^2 \) corresponding to the LG process on the surface of a cylinder \( \{ Z \in \mathbb{C} \mid 0 \leq \text{Im}Z \leq 2\pi R \} \) in the \( Z \)-plane, the \( z \)-plane was called the auxiliary physical plane while the \( Z \)-plane was called the physical plane.

3 Symmetric solutions

The case when the background density function \( \sigma \) (and the function \( U \)) is axially symmetric, i.e., depends only on \( |z| \), is of a special interest. We study it in the rest of the paper. In this case we will write \( U(z, \bar{z}) = U(z\bar{z}), \sigma(z, \bar{z}) = \sigma(z\bar{z}), \) etc and will sometimes

\(^2\)For example, the viscous flow in the Hele-Shaw cell with a non-uniform spacing between the glass plates.
denote the argument of these functions of one variable by \( x \) \( (x = |z|^2) \). The corresponding solutions \( F \) of equations (2.28) are called symmetric [25]. They are characterized by the property that the derivatives \( \partial_k F \) restricted to zero values of the \( t_k \)'s with \( k \geq 1 \) vanish.

### 3.1 Some general relations

For any axially symmetric background it holds:

\[
\begin{align*}
    z \partial_z U(z \bar{z}) &= z \bar{z} U''(z \bar{z}) = \bar{z} \partial_{\bar{z}} U(z \bar{z}), \\
    \sigma(x) &= x U''(x) + U'(x) = (x U'(x))', \\
    \frac{1}{\pi} \int_{A \cap D} \sigma \, d^2 z &= t_0 - r_0^2 U''(r_0^2).
\end{align*}
\]

It is also easy to see that in expansions (2.18), (2.19) \( \psi(z, \bar{z}) = 0 \) and \( u_0 = r_0^2 \log(r_0^2 U'(r_0^2)) \), so the expansions simplify and acquire the form

\[
\phi(z, \bar{z}) = \begin{cases}
    -U(z \bar{z}) - u_0 + t_0 \log(z \bar{z}) + \sum_{k \geq 1} (t_k z^k + \bar{t}_k \bar{z}^k), & z \in A \cap D \\
    v_0 + \sum_{k \geq 1} \frac{1}{k} (v_k z^{-k} + \bar{v}_k \bar{z}^{-k}), & z \in D^c.
\end{cases}
\]

For symmetric solutions, formulas (2.2) and (2.12) for the moments also simplify because the contour integrals in their right hand sides vanish:

\[
\begin{align*}
    t_k &= -\frac{1}{\pi k} \int_{D \cap B(r_1)} z^{-k} \sigma(z \bar{z}) \, d^2 z, & v_k &= \frac{1}{\pi} \int_{D \cap B(r_0)} z^k \sigma(z \bar{z}) \, d^2 z, & k \geq 1.
\end{align*}
\]

Note that these integrals do not depend on \( r_1, r_0 \) provided that \( \gamma \subset A = B(r_1) \setminus B(r_0) \) and the function \( \sigma \) is regular everywhere in the annulus \( A \). In particular, if \( \sigma \) is allowed to have singularities at 0 and \( \infty \) only, then the integration can be extended to the whole domains \( D^c \) and \( D \) with the prescription that the angular integration is performed first. Let us also mention that the contour integral representation (2.14) for \( v_0 \) in the symmetric case can be written in the form

\[
v_0 = \frac{1}{2\pi i} \left( \oint_{\gamma} - \oint_{|z|=r_0} \right) \left( \log |z|^2 \partial U - z^{-1} U \right) \, dz.
\]

**Proposition 3.1** Suppose that only a finite number of the moments \( t_k \) are different from 0. Then the tau-function (2.23) for a symmetric background can be represented as

\[
2F = -\frac{1}{\pi} \int_{A \cap D} U \sigma \, d^2 z + t_0 v_0 + \sum_{k \geq 1} (t_k v_k + \bar{t}_k \bar{v}_k) - u_0 t_0 + u_0 r_0^2 U'(r_0^2).
\]
Proof. Under the assumption of the proposition expansion (3.4) is valid everywhere in \( \mathbb{A} \cap \mathbb{D} = \mathbb{D} \cap \mathbb{B}(r_0) \). Substituting it into (2.23) written in the form \( 2F = \frac{1}{\pi} \int_{\mathbb{A} \cap \mathbb{D}} \phi \sigma d^2z \), performing the termwise integration and using the definition of the complimentary moments, we get (3.7). \( \blacksquare \)

Remark. For any, not necessarily symmetric, background the formula is basically the same; only the \( r_0 \)-dependent terms change (the last two terms in (3.7)).

**Theorem 3.1** Let \( U = U(z\bar{z}; \lambda) \) be a symmetric background potential depending on a parameter \( \lambda \). Then the partial \( \lambda \)-derivative of \( F \) taken at constant \( \{t_k\}_0^\infty \) is given by

\[
\frac{\partial F}{\partial \lambda} \bigg|_{\{t_k\}} = -\frac{1}{\pi} \int_{\mathbb{A} \cap \mathbb{D}} \partial_\lambda U \sigma d^2z - \left( t_0 - r_0^2 U'(r_0^2) \right) \partial_\lambda u_0, \tag{3.8}
\]

where \( \partial_\lambda u_0 = r_0^2 \log r_0^2 \partial_\lambda U'(r_0^2) - \partial_\lambda U(r_0^2) \).

**Proof.** We should find the variation of \( F = \frac{1}{2\pi} \int_{\mathbb{A} \cap \mathbb{D}} \phi \sigma d^2z \) under the deformation of the potential function \( U \to U + \delta U \). \( \delta U = \partial_\lambda U \delta \lambda \), and a simultaneous deformation of the domain that preserves the moments \( t_k \) (see Proposition [2.2]). We have:

\[
\delta F = \frac{1}{2\pi} \int_{\mathbb{A} \cap \mathbb{D}} \delta \phi \sigma d^2z + \frac{1}{2\pi} \int_{\mathbb{A} \cap \mathbb{D}} \phi \delta \sigma d^2z + \frac{1}{2\pi} \oint_{\gamma} \phi \delta n \sigma |dz|,
\]

where \( \delta \phi = -\delta U - \delta u_0 \) (this follows from (3.4) at \( \delta t_k = 0 \)), \( \delta \sigma = \frac{1}{r_0} \Delta \delta U \) and \( \delta n \) is given by equation (2.10). Plugging all this into the right hand side and using the Green theorem in the form \( \int_{\mathbb{A} \cap \mathbb{D}} (f \Delta g - g \Delta f) d^2z = \oint_{\gamma} (f \partial_n g - g \partial_n f) |dz| \), we arrive at the expression

\[
\delta F = -\frac{1}{2} \left( t_0 - r_0^2 U'(r_0^2) \right) \delta u_0 - \frac{1}{\pi} \int_{\mathbb{A} \cap \mathbb{D}} \delta U \sigma d^2z
\]

\[
\quad + \frac{1}{8\pi} \oint_{\gamma} (\delta U \partial_n \phi - \phi \partial_n \delta U) |dz| + \frac{1}{8\pi} \oint_{\gamma} (\phi \partial_n \delta U^H - \delta U \partial_n \phi) |dz|.
\]

The last integral is equal to zero because we can write \( \oint_\gamma \phi \partial_n \delta U^H |dz| = \oint_\gamma \delta U^H \partial_n \phi |dz| \) (since both functions \( \phi \) and \( \delta U^H \) are harmonic in \( \mathbb{D}^c \)) and then the integrand vanishes on \( \gamma \) since \( \delta U^H = \delta U \) there. One can also note that the terms like \( \delta t_k z^k \) and \( \delta t_k \bar{z}^k \) that are present in the expansion of \( \phi \), owing to the symmetry, do not contribute to the integral over the circle \(|z| = r_0\). The remaining terms give the right hand side of equation (3.8). \( \blacksquare \)

**Corollary 3.1** Let \( U(z\bar{z}) \) be a symmetric background potential of the form

\[
U(z\bar{z}) = \sum_j \lambda_j U_j(z\bar{z})
\]

with some functions \( U_j \) and parameters \( \lambda_j \). Then

\[
\sum_j \lambda_j \frac{\partial F}{\partial \lambda_j} = -\frac{1}{\pi} \int_{\mathbb{A} \cap \mathbb{D}} U \sigma d^2z - u_0 t_0 + u_0 r_0^2 U'(r_0^2). \tag{3.9}
\]
This directly follows from equations (3.8) written for $\partial \lambda_j F$.

**Corollary 3.2** The function $F$ is a homogeneous function of $t_0, \{t_k\}_1^\infty, \{\bar{t}_k\}_1^\infty$ and $\lambda_j$’s of degree two:

$$2F = \sum_j \lambda_j \partial \lambda_j F + t_0 \partial_0 F + \sum_{k \geq 1} (t_k \partial_k F + \bar{t}_k \bar{\partial}_k F).$$  \hspace{1cm} (3.10)

This is obtained by combining (3.9), (3.7) and (2.24).

### 3.2 The restriction to the $t_0$-line

Let us call the line $t_k = 0$ for all $k \geq 1$ (but $t_0 \neq 0$) the $t_0$-line in the space of moments. For symmetric solutions, it is natural to consider the restriction of any quantity depending on the domain to the $t_0$-line. Below we will denote such restriction by \((\ldots)|_{t_0}\).

The symmetry $U(z, \bar{z}) = U(z \bar{z})$ implies that when all moments $t_k$ at $k \geq 1$ are equal to 0, the domain $D$ is a disk of radius $r$ such that

$$t_0 = \frac{1}{2\pi i} \oint_{|z|=r} z \partial U \frac{dz}{z} = r^2 U''(r^2)$$ \hspace{1cm} (3.11)

(see (3.1)). Differentiating this equality w.r.t. $t_0$, we get:

$$\partial_0 r^2(t_0) \sigma(r^2(t_0)) = 1 \quad \text{or} \quad \partial_0 \log r^2|_{t_0} = F'''(t_0) = \frac{1}{r^2 \sigma(r^2)},$$ \hspace{1cm} (3.12)

where $F(t_0) := F|_{t_0}$ is the restriction of the function $F$ to the $t_0$-line.

The complimentary moments $v_k$ with $k \geq 1$ vanish, $v_k|_{t_0} = \partial_k F|_{t_0} = 0$ for all $k \geq 1$, while $v_0$ is given by

$$v_0|_{t_0} = \int_{r_0^2}^{r^2} \log x \sigma(x) \, dx = t_0 \log r^2 - U(r^2) - u_0.$$ \hspace{1cm} (3.13)

The potential $\phi$ is

$$\phi(z, \bar{z})|_{t_0} = \begin{cases} -U(z \bar{z}) - u_0 + t_0 \log(z \bar{z}), & r_0 \leq |z| \leq r \\ v_0, & |z| > r. \end{cases}$$ \hspace{1cm} (3.14)

Since it depends on $|z|^2$ only, in this subsection we will write $\phi(z, \bar{z})|_{t_0} = \phi(z \bar{z})$.

There are several integral formulas for the function $F(t_0)$. One of them is obtained by a direct calculation of the integral (2.23) in polar coordinates using the integral

$$\int_0^{2\pi} \log \left( a^2 + b^2 - 2ab \cos \varphi \right) d\varphi = \pi \log \left[ \max (a^2, b^2) \right].$$

This leads to the double integral formula

$$F(t_0) = \int_{r_0^2}^{r^2} \sigma(x) \, dx \int_{r_0^2}^{x} \log x' \sigma(x') \, dx',$$ \hspace{1cm} (3.15)

\[\text{depending on the function $\sigma$, the values of $t_0$ corresponding to domains in the $z$-plane may belong to some interval of this line.}\]
which can be further simplified by taking into account that \( \log x \sigma(x) \) is a full derivative: 
\[ \log x \sigma(x) = \partial_x \left( U'(x) x \log x - U(x) \right) . \]
Then we obtain from (3.15):
\[ F(t_0) = \int_{r_0}^{r_2} \left( U'(x) x \log x - U(x) \right) \sigma(x) dx - \left( t_0 - r_0^2 U'(r_0^2) \right) u_0 . \]

Here \( r^2 \) should be understood as a function of \( t_0 \) implicitly given by (3.1). From this formula one can easily see, using (3.12), that \( F'(t_0) = v_0 \) and \( \partial v_0 / \partial t_0 = F''(t_0) = \log r^2 \), as it should be. Another integral formula follows from (2.15) and (2.23) and from (3.14), with taking into account (3.3):
\[ 2F(t_0) = \int_{r_0}^{r_2} F(x) \sigma(x) dx 
= - \int_{r_0}^{r_2} U(x) (xU'(x))' dx + t_0(v_0 - u_0) + u_0 r_0^2 U'(r_0^2) 
= \int_{r_0}^{r_2} x(U'(x))^2 dx + t_0^2 \log r^2 - 2t_0 U(r^2) - 2u_0 t_0 + r_0^4 (U'(r_0^2))^2 \log r_0^2 . \]

It is easy to check directly that (3.16) and (3.17) are equivalent.

**Remark.** We would like to mention that on the \( t_0 \)-line the 2D inverse potential problem for the Laplace operator \( \Delta = 4 \partial^2 / \partial X^2 \) becomes one-dimensional (for the operator \( \partial^2 / \partial X^2 \)). To see this, we introduce the new variable \( X = \log (z \bar{z} / r_0^2) \geq 0 \) and denote \( U(X) = U(r_0^2 e^X) \). Then equation (3.17) can be rewritten as
\[ 2F(t_0) = - \int_{\log r_0^2}^{\log r_2} U(X) U''(X) dX + t_0 v_0 - \left( t_0 - U'(\log r_0^2) \right) u_0 . \]

Set \( U(X) = \sum_{k \geq 2} \tau_k X^k \), then condition (3.11) (equivalent to the continuity of the derivative of the potential \( \phi \) at \( X = \log r^2 \), i.e., \( \partial_X (t_0 X - U(X)) \bigg|_{X=\log r^2} = 0 \)) reads
\[ t_0 = - \sum_{k \geq 2} k \tau_k (\log r^2)^{k-1} . \]

which means that \( u = \log r^2 (t_0, \tau_2, \tau_3, \ldots) \) satisfies equations of the dispersionless KdV hierarchy
\[ \frac{\partial u}{\partial \tau_k} + ku^{k-1} \frac{\partial u}{\partial t_0} = 0, \quad k \geq 2 . \]

### 3.3 Examples

#### 3.3.1 The homogeneous density \( \sigma(z\bar{z}) = c(z\bar{z})^{\alpha-1}, \alpha > 0 \)

Our first example is
\[ U(z\bar{z}) = \frac{c}{\alpha^2} (z\bar{z})^\alpha, \quad \sigma(z\bar{z}) = c(z\bar{z})^{\alpha-1} \]
with some real positive $c, \alpha$. We also have

$$z\partial U(z\bar{z}) = \frac{c}{\alpha}(z\bar{z})^\alpha = \alpha U(z\bar{z}). \quad (3.21)$$

We start with the case of vanishing all the $t_k$’s except $t_0$:

$$t_0 = \frac{c}{\alpha} r^{2\alpha}, \quad v_0|_{t_0} = \frac{t_0}{\alpha} \log \left( \frac{\alpha t_0}{c} \right) - \frac{t_0}{\alpha} - u_0 \quad (3.22)$$

and

$$F(t_0) = \frac{t_0^2}{2\alpha} \log \left( \frac{\alpha t_0}{c} \right) - \frac{3t_0^2}{4\alpha} - u_0 t_0 + c_0, \quad (3.23)$$

where

$$u_0 = \frac{c}{\alpha} r^{2\alpha} \log r^2 - \frac{c}{\alpha^2} r^{2\alpha}, \quad c_0 = \frac{c^2 r^{4\alpha}}{4\alpha^3} (2\alpha \log r^2 - 1).$$

One can check that $F''(t_0) = \frac{1}{\alpha} \log \left( \frac{\alpha t_0}{c} \right) = \log r^2$.

Now let us consider the case of non-zero moments $t_k$. At $\alpha > 0$ all integrals converge at the origin and one can put $r_0 = 0$ but we will keep it non-zero for illustrative purposes.

**Proposition 3.2** Assuming that only a finite number of the moments $t_k$ are different from 0, the dispersionless tau-function for this solution is quasi-homogeneous, that is it obeys the relation

$$2F = t_0 \partial_0 F + \sum_{k \geq 1} \left(1 - \frac{k}{2\alpha}\right) \left(t_k \partial_k F + \bar{t}_k \bar{\partial}_k F\right) + Q(t_0), \quad (3.24)$$

where $Q(t_0) = -\frac{t_0^2}{2\alpha} - u_0 t_0 + 2c_0$.

**Proof.** We use equation (3.7). In the integral term we write $U\sigma = \bar{\partial}(U\partial U) - \partial U \bar{\partial} U$ and notice that for the particular function $U$ we have $U\sigma = \partial U \bar{\partial} U$, and also $z\partial U = \alpha U$, so $U\sigma = \frac{1}{2} \bar{\partial}(U\partial U)$. This allows us to transform the 2D integral to a contour integral:

$$\frac{1}{\pi} \int_{\Gamma \cap D} U\sigma \, dz = \frac{1}{4\pi i \alpha} \left( \oint_{\gamma} - \oint_{|z| = r_0} \right) (z\partial U)^2 \frac{dz}{z} = \frac{1}{4\pi i \alpha} \oint_{\gamma} (z\partial U)^2 \frac{dz}{z} - \frac{c^2 r^{4\alpha}}{2\alpha^3}. \quad (3.25)$$

Now recall (2.22) and represent $\partial U = C^+ - C^-$ (on $\gamma$), with $C^+$ being a polynomial. Shrinking the integration contour to $\infty$, we obtain:

$$\frac{1}{2\pi i} \oint_{\gamma} (z\partial U)^2 \frac{dz}{z} = t_0^2 + 2 \sum_{k \geq 1} k t_k v_k. \quad (3.25)$$

Since the initial integral is obviously real, we conclude that $\sum_{k \geq 1} kt_k v_k$ is a real quantity, i.e., $\sum_{k \geq 1} kt_k v_k = \sum_{k \geq 1} \bar{t}_k \bar{v}_k$. The quasi-homogeneity relation (3.24) follows.

Note that the function $\tilde{F} = F - F(t_0)$ satisfies the same relation (3.24) without the last term $Q(t_0)$.
The conformal dynamics with \( \sigma = c(z\bar{z})^{\alpha - 1} \) is mapped to the Laplacian growth in the cone with the angle \( 2\pi\alpha \).

The following formulas are direct consequences of Corollary 3.2 and equation (3.24):

\[
2F = c\partial_c F + t_0 \partial_0 F + \sum_{k \geq 1} (t_k \partial_k F + \bar{t}_k \bar{\partial}_k F),
\]

(3.26)

\[
c\partial_c F = -\frac{1}{\alpha} \sum_{k \geq 1} kt_k \partial_k F + Q(t_0).
\]

(3.27)

The case \( \alpha = 1 \) (\( \sigma(z, \bar{z}) = \text{const} \)) was considered in [6, 2] in connection with the Laplacian growth (the Hele-Shaw problem) in the plane with a sink at infinity. As is mentioned in [3], at arbitrary \( 0 < \alpha \leq 1 \) the conformal dynamics can be mapped to a LG process in a sector with angle \( 2\pi\alpha \) and periodic conditions at the boundary rays (a cone), see Fig. 2. Some details are given below.

We map the whole \( z \)-plane punctured at \( z = 0 \) to the sector

\[
S_\alpha = \{ Z \in \mathbb{C} \setminus \{0\} \mid 0 \leq \arg Z < 2\pi\alpha \}
\]

in the \( Z \)-plane by the map \( Z(z) = z^\alpha \). Let \( D^\pm_\alpha \) be the images of \( D^c \) and \( D \) respectively. The image of the curve \( \gamma \) is a curve \( \Gamma \subset S_\alpha \) such that its endpoints on the boundary rays of the cone are at the same distance from the origin. The moments \( t_k \) and \( v_k \) can be represented as integrals in the \( Z \)-plane:

\[
t_0 = \frac{c}{2\pi i \alpha^2} \int_{\Gamma} \bar{Z} dZ = \frac{c}{\pi \alpha^2} \text{Area}(D^-_\alpha), \quad t_k = -\frac{1}{\pi k} \int_{D^+_\alpha} Z^{-k/\alpha} d^2Z,
\]

\[
v_0 = \frac{c}{\pi \alpha^3} \int_{D^-_\alpha} \log |Z|^2 d^2Z, \quad v_k = \frac{c}{\pi \alpha^2} \int_{D^+_\alpha} Z^{k/\alpha} d^2Z.
\]

At \( \alpha > 0 \) we put \( r_0 = 0 \) and \( r_1 \to \infty \) because all 2D integrals are either convergent or can be made such by using the simple additional prescription that the angular integration is made first.
The LG problem in $S_\alpha$ is the following moving boundary value problem:

$$
\begin{align*}
\Delta \Phi(Z, \bar{Z}) &= 0 \quad \text{in} \quad D_+ \\
\Phi(Ze^{2\pi i \alpha}, \bar{Z}e^{-2\pi i \alpha}) &= \Phi(Z, \bar{Z}) \\
\Phi(Z, \bar{Z}) &= 0, \quad Z \in \Gamma \\
\Phi(Z, \bar{Z}) &= -\frac{1}{2} \log |Z| + \ldots \quad \text{as} \quad |Z| \to +\infty,
\end{align*}
$$

(3.28)

with the normal velocity of the curve $\Gamma$ given by $V_n(Z) = -\partial_n \Phi(Z, \bar{Z})$, $Z \in \Gamma$. The conformal map from the exterior of the unit circle in the $w$-plane onto $D^\alpha_+$ is

$$
\omega(Z) = w(Z^{1/\alpha})
$$

(see section 2.5). Then $\Phi(Z, \bar{Z}) = -\frac{\alpha}{2} \log |\omega(Z)|$ meets all the requirements (3.28) and hence is the (unique) solution. The normal velocity is then

$$
V_n(Z) = \frac{\alpha}{2} \partial_n \log |\omega(Z)| = \frac{\alpha}{2} |\omega'(Z)|.
$$

The moment $t_0$ is proportional to time while the moments $t_k$ at $k \geq 1$ are Richardson’s conserved quantities for this problem [19].

### 3.3.2 The homogeneous density $\sigma(z\bar{z}) = c(z\bar{z})^{\alpha-1}$, $\alpha < 0$

With small modifications, the formulas given above hold in the case $\alpha < 0$ (and, in particular, $\alpha = -1$). The conformal dynamics in the $z$-plane can be mapped by the map $Z(z) = z^\alpha$ to the Laplacian growth in the compact interior domain bounded by the curve $\Gamma$ and the rays $\arg Z = 0$, $\arg Z = 2\pi \alpha$ which is now the image of the exterior domain $D^-_\alpha$ in the $z$-plane.

At negative $\alpha$ one can not put $r_0 = 0$. Formulas for $t_k$ and $v_k$ remain the same at $k \geq 1$ but change at $k = 0$:

$$
t_0 = -\frac{c}{\pi \alpha^2} \text{Area}(D^+_\alpha), \quad v_0 = \frac{c}{\pi \alpha^3} \int\int_{D^-_\alpha} \log |Z|^2 d^2 Z.
$$

The domain $\tilde{D}^-_{\alpha_0} \subset S_\alpha$ is bounded by the curve $\Gamma$ and the arc $2\pi \alpha \leq \arg Z < 0$. This “cut-off” is necessary because the domain $D^-_\alpha$ is non-compact and the integral over the whole $D^-_\alpha$ diverges. Note that $t_0$ becomes negative and the physical time is $t \propto -t_0$.

### 3.3.3 The homogeneous density $\sigma(z\bar{z}) = R/(z\bar{z})$

Formally, this is the limiting case of the previous family of solutions with $\alpha \to 0$. However, because the limit is not easy to perform and, most important, because this case is very interesting by itself due to the connection with Hurwitz numbers, it deserves a separate consideration. It is convenient to choose the function $U(z\bar{z})$ in such a way that $U(r_0) = U'(r_0) = 0$:

$$
U = \frac{R}{2} \left[ \log \frac{z\bar{z}}{r_0^2} \right]^2, \quad z\partial U = R \log \frac{z\bar{z}}{r_0^2}, \quad \bar{z}\partial U = \sigma = \frac{R}{z\bar{z}}
$$

(3.29)
where $R$ is a parameter of the solution. Sometimes we will also use the parameter $\beta = 1/R$.

Again, we start with the case of the vanishing $t_k$’s except $t_0$ (the $t_0$-line):

$$
t_0 = R \log \frac{r^2}{r_0^2}, \quad v_0|_{t_0} = t_0 \log r^2 - \frac{R}{2} \left[ \log \frac{r^2}{r_0^2} \right]^2 = \frac{t_0^2}{2R} + t_0 \log r_0^2
$$

(3.30)

and $u_0 = 0$, so

$$
F(t_0) = \frac{t_0^3}{6R} + \frac{t_0^2}{2} \log r_0^2.
$$

(3.31)

One can check that $F''(t_0) = \beta t_0 + \log r_0^2 = \log r^2$.

Now we address the general case of non-zero $t_k$’s.

**Proposition 3.3** The dispersionless tau-function for the solution determined by the data (3.29) obeys the following homogeneity relation:

$$
2F = R \partial_R F + t_0 \partial_0 F + \sum_{k \geq 1} \left( t_k \partial_k F + \bar{t}_k \bar{\partial}_k F \right)
$$

(3.32)

(the $R$-derivative is taken at constant $t_0$, $t_1$, $t_2$, ...)

This is a particular case of (3.10) (see Corollary 3.2).

**Proposition 3.4** The dispersionless tau-function (3.32) satisfies the relations

$$
\frac{\partial F}{\partial \log r_0^2} = \frac{t_0^2}{2} + \sum_{k \geq 1} kt_k \partial_k F,
$$

$$
\frac{\partial F}{\partial \beta} = \frac{t_0^3}{6} + t_0 \sum_{k \geq 1} kt_k \partial_k F + \frac{1}{2} \sum_{k,l \geq 1} (klt_{t_l} \partial_{k+l} F + (k+l)t_{k+l} \partial_k F \partial_l F),
$$

(3.33)

where the derivatives are taken at constant $t_0, t_1, t_2, \ldots$

*Proof.* The first formula can be proved by a direct variation of the function $F$ under a small change $r_0^2 \to r_0^2 + \delta r_0^2$ similar to the one done in the proof of Theorem 3.1. On the one hand, the calculation in the present case is somewhat simpler because the corresponding $\delta U$ is harmonic in $A \cap D$ but, on the other hand, one should take into account that the interior boundary of the domain (the circle $|z| = r_0$) also moves. An accurate calculation which we omit here gives:

$$
\frac{\partial F}{\partial \log r_0^2} = Rv_0 - Rt_0 \log r_0^2.
$$

(see also [26] for a different proof). To proceed, we use (3.6) and the fact that our particular potential $U$ satisfies the relation $U = (z \partial U)^2/(2R)$, so we can write

$$
v_0 = \frac{1}{2\pi i} \left( \oint_{\gamma} - \oint_{|z|=r_0} \right) \left( \log |z|^2 z \partial U - U \right) \frac{dz}{z} = t_0 \log r_0^2 + \frac{1}{4\pi i R} \oint_{\gamma} (z \partial U)^2 \frac{dz}{z}.
$$
Figure 3: The conformal dynamics with \( \sigma = R/|z|^2 \) is mapped to the Laplacian growth on the cylinder of radius \( R \). (In the \( Z \)-plane, two copies of the curve \( \Gamma \) are shown.)

Then, using the same argument as in the proof of Proposition 3.2 (see equation (3.25)), we get the first equation in (3.33).

For the proof of the second formula we note that in the case (3.29) \( U \sigma = \frac{1}{3} \bar{\partial}(U \partial U) \) and thus

\[
\frac{1}{\pi} \int_{A \cap D} U \sigma d^2z = \frac{\beta}{12 \pi i} \int_\gamma (z \partial U)^3 \frac{dz}{z}
\]

(the integral over the interior boundary is absent because \( \partial U = 0 \) there). Proceeding as in the proof of Proposition 3.2, we obtain

\[
\frac{1}{12 \pi i} \int_\gamma (z \partial U)^3 \frac{dz}{z} = \frac{t_0^3}{6} + t_0 \sum_{k \geq 1} k t_k v_k + \frac{1}{2} \sum_{k,l \geq 1} (k l t_k t_l v_{k+l} + (k + l) t_{k+l} v_k v_l).
\]

Then from Proposition 3.1 we have:

\[
2F = t_0 v_0 + \sum_{k \geq 1} (t_k v_k + \bar{t}_k \bar{v}_k) - \frac{\beta t_0^3}{6} - \beta t_0 \sum_{k \geq 1} k t_k v_k
\]

\[
- \frac{\beta}{2} \sum_{k,l \geq 1} (k l t_k t_l v_{k+l} + (k + l) t_{k+l} v_k v_l).
\]

Combining this with the homogeneity property (Proposition 3.3), we arrive at the second formula in (3.33).

\[
\text{Remark.} \text{ Combinatorial formulas for Taylor expansion coefficient of the function } F \text{ have been suggested in } [25]. \text{ These coefficients are essentially the double Hurwitz numbers for connected genus 0 coverings of the Riemann sphere. The double sum in the second equation in (3.33) is the genus 0 part of the celebrated cut-and-join operator } [35], \text{ see}
\]
also [30]. At the same time, this function is closely connected with the Laplacian growth problem on a cylinder, see below and [26].

As is shown in [26], the conformal dynamics in the $z$-plane punctured at $z = 0$ with
\[
\sigma = \frac{R}{|z|^2}
\]
can be mapped to a LG process on the surface of an infinite cylinder of radius $R$, see Fig. 3. We map the whole $z$-plane to the strip
\[
C_R = \{ Z \in \mathbb{C} \setminus \{0, \infty\} | 0 \leq \text{Im} Z < 2\pi R \}
\]
in the $Z$-plane by the map
\[
Z(z) = R \log(z/r_0).
\]
Let $D_\pm$ be the images of $D^c$ and $D$ respectively. The image of the curve $\gamma$ is a curve $\Gamma \subset C_R$ such that the endpoints on the two sides of the strip have the same real parts. The moments $t_k$ and $v_k$ can be represented as integrals in the $Z$-plane:
\[
t_0 = \frac{1}{2\pi i R} \int_\Gamma (Z + \bar{Z}) dZ = \frac{1}{\pi R} \int_\Gamma X dY = \frac{\text{Area}(D_0^0)}{\pi R},
\]
\[
v_0 = \frac{R}{\pi} \iint_{D \setminus B(r_0)} \frac{\log(|z\bar{z}|)}{z\bar{z}} d^2z = 2t_0 \log r_0 + \frac{2}{\pi R^2} \iint_{D_0^0} X d^2Z,
\]
\[
t_k = -\frac{r_0^{-k}}{\pi k R} \iint_{D_+} e^{-kZ/R} d^2Z, \quad v_k = \frac{t_0^k}{\pi R} \iint_{D_-} e^{kZ/R} d^2Z.
\]
Here $D_0^0$ is the domain bounded by the curve $\Gamma$ and the section $\text{Re} Z = 0$. Note that $D_0^0$ is the image of $B(r_0)$ under the map from the auxiliary physical plane.

The LG problem in $C_R$ is the following moving boundary value problem:
\[
\begin{cases}
\Delta \Phi(Z, \bar{Z}) = 0 & \text{in } D_+ \\
\Phi(Z + 2\pi i R, \bar{Z} - 2\pi i R) = \Phi(Z, \bar{Z}) \\
\Phi(Z, \bar{Z}) = 0, \quad Z \in \Gamma \\
\Phi(Z, \bar{Z}) = -\frac{1}{2} \text{Re } Z + \ldots & \text{as } \text{Re } Z \to +\infty,
\end{cases}
\]
with the normal velocity of the curve $\Gamma$ given by
\[
V_n(Z) = -\frac{1}{2} \frac{\partial_n \Phi(Z, \bar{Z})}{|d\Phi(Z, \bar{Z})|}, \quad Z \in \Gamma.
\]
The conformal map from the exterior of the unit circle in the $w$-plane onto $D^+$ is $\omega(Z) = w(r_0e^{Z/R})$. Then $\Phi(Z, \bar{Z}) = -\frac{R}{2} \log |\omega(Z)|$ meets all the requirements and hence is the (unique) solution. The normal velocity is then
\[
V_n(Z) = \frac{R}{2} \frac{\partial_n \log |\omega(Z)|}{|d\omega(Z)|} = \frac{R}{2} |\omega'(Z)|.
\]
The moment $t_0$ is proportional to time while the moments $t_k$ at $k \geq 1$ are Richardson’s conserved quantities for this problem [19].

### 3.3.4 A more general family of examples

All previous examples can be unified in a broader family by setting
\[
U(z\bar{z}) = (C_1 \log(z\bar{z}) + C_0)\nu
\]
(3.36)
with some constants \( C_1, C_0 \) and \( \nu = \frac{k-1}{k-2} \) with integer \( k > 2 \), so
\[
z \partial U = C_1 \nu (C_1 \log(z \bar{z}) + C_0)^{\nu-1}, \quad \sigma = \frac{C_1^2 \nu (\nu-1)}{z \bar{z}} (C_1 \log(z \bar{z}) + C_0)^{\nu-2}. \tag{3.37}
\]
All functions \( U \) of the form \( (3.36) \) satisfy the differential equation
\[
U''(x) - \frac{1}{k-1} \left( \frac{U''(x)}{U(x)} \right)^2 + \frac{1}{x} U'(x) = 0,
\tag{3.38}
\]
with the first integral \( C_1 \nu U^{\frac{k-1}{k}} = U' \) which can be also written as
\[
U = (C_1 \nu)^{1-k} (z \partial U)^{k-1}. \tag{3.39}
\]
We assume that \( r_0^2 > e^{-C_0/C_1} \), so these functions are non-singular in \( A \).

The case \( k = 3 \) at \( C_1 = \sqrt{R/2}, C_0 = -\sqrt{R/2} \log r_0^2 \) is the potential \( (3.29) \). Formally, the case \( k = 2 \) also belongs to this family and corresponds to potential \( (3.20) \) since
\[
\lim_{k \to 2} \left( \alpha(k-2) \log(z \bar{z}) + 1 \right)^{\frac{k-1}{k-2}} = (z \bar{z})^\alpha.
\]
At \( k = 4 \) the potential is \( U(z \bar{z}) = (C_1 \log(z \bar{z}) + C_0)^{3/2} \).

At the \( t_0 \)-line we have:
\[
t_0 = C_1 \nu \left( C_1 \log r^2 + C_0 \right)^{\nu-1},
\]
\[
v_0 |_{t_0} = (\nu - 1) \left( \frac{t_0}{C_1 \nu} \right)^{\frac{2}{2\nu-1}} - \frac{C_0}{C_1} t_0 - u_0,
\]
\[
u_0 = \left( C_1 (\nu - 1) \log r_0^2 - C_0 \right) \left( C_1 \log r_0^2 + C_0 \right)^{\nu-1}.
\]
The calculation of \( F(t_0) \) by means of \( (3.17) \) gives:
\[
F(t_0) = C_1 \nu (\nu-1)^2 \left( \frac{t_0}{C_1 \nu} \right)^{2\nu-1} - \frac{C_0}{2C_1} r_0^2 - u_0 t_0 + K_0,
\]
where
\[
K_0 = \frac{C_1^2 \nu^2}{2\nu - 1} \left( C_1 \log r_0^2 + C_0 \right)^{2\nu-2} \left[ (\nu - 1) \log r_0^2 - \frac{C_0}{2C_1} \right]
\]
or
\[
F(t_0) = \frac{a_k}{C_1^{k-1}} t_0^k - \frac{C_0}{2C_1} t_0^2 - u_0 t_0 + K_0, \quad a_k = \frac{1}{k} (k-2)^{k-2} \frac{1}{k(k-1)^{k-1}} \tag{3.40}
\]
It is easy to check that \( F''(t_0) = k(k-1) a_k \frac{t_0^{k-2}}{C_1^{k-1}} - \frac{C_0}{C_1} = \log r^2 \) as it should be.

In order to make this family of examples closer to the formulas for the case \( k = 3 \) we set \( R_k := (k-1) C_1^{k-1}, C_0 = -C_1 \log r_0^2 \) and
\[
U_k(z \bar{z}) = \left( \frac{R_k}{k-1} \right)^{\frac{1}{k-1}} \left( \log \frac{z \bar{z}}{r_0^2} \right)^{\frac{k-1}{k-2}},
\]

24
so that $U_3$ coincides with $U$ from equation (3.29) with $R = R_3$. We also have $U_k(r_0^2) = U'_k(r_0^2) = 0$. Equation (3.40) acquires the form

$$F(t_0) = \frac{(k-1)a_k}{R_k} t_0^k + \frac{t_0^2}{2} \log r_0^2.$$  \hspace{1cm} (3.41)

In the general case of $t_k \neq 0$ we can write, taking into account (3.39):

$$\frac{1}{\pi} \int \int_{A \cap D} U_k \sigma_k d^2z = \frac{(k-1)(k-2)a_k}{2\pi i R_k} \oint_\gamma (z \partial U_k)^k \frac{dz}{z}.$$

This relation allows one to obtain analogs of equation (3.34) with higher $W_k$-operators instead of the cut-and-join operator $W_3$.

One can also consider the potential $\sum_k U_k(z \bar{z})$. The corresponding function $F$ is presumably the dispersionless limit of the tau-function (or rather the free energy) for the Gromov-Witten theory of $\mathbb{C}P^1$.

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