Vaught’s conjecture on analytic sets

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§0 Prehistory

In rough historical these are the groups for which we know the topological Vaught conjecture:

0.1 Theorem (Folklore) All locally compact Polish groups satisfy Vaught’s conjecture – that is to say, if $G$ is a locally compact Polish group acting continuously on a Polish space $X$ then either $|X/G| \leq \aleph_0$ or there is a perfect set of points with different orbits (and hence $|X/G| \geq 2^{\aleph_0}$).

0.2 Theorem (Sami) Abelian Polish groups satisfy Vaught’s conjecture.

0.3 Theorem (Hjorth-Solecki) Invariantly metrizable and nilpotent Polish groups satisfy Vaught’s conjecture.

0.4 Theorem (Becker) Complete left invariant metric and solvable Polish groups satisfy Vaught’s conjecture.

In each of these case the result was shortly or immediately after extended to analytic sets. For this purpose let us write TVC$(G, \Sigma^1_1)$ if whenever $G$ acts continuously on a Polish space $X$ and $A \subset X$ is $\Sigma^1_1$ (or analytic) then either $|A/G| \leq \aleph_0$ or there is a perfect set of orbit inequivalent points in $A$. Thus we have TVC$(G, \Sigma^1_1)$ for each of the group in the class mentioned in 0.1-0.4 above.

On the other hand, and in contrast to the usual topological Vaugh conjecture, that merely asserts that 0.1-0.4 hold for arbitrary Polish groups, it is known that TVC$(S_\infty, \Sigma^1_1)$ fails.

Here it is shown that the presence of $S_\infty$ is a necessary condition for TVC$(G, \Sigma^1_1)$ to fail:

0.5 Theorem If $G$ is a Polish group on which the Vaught conjecture fails on analytic sets then there is a closed subgroup of $G$ that has $S_\infty$ as a continuous homomorphic image.

The converse of 0.5 is known and by now considered trivial in light of 2.3.5 of [4]. Thus we have an exact characterization of TVC$(G, \Sigma^1_1)$. If as widely believed the Vaught conjecture should fail

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for $S_\infty$ then this would as well characterize the groups for which the topological Vaught conjecture holds.
§1 Preliminaries

All of this can be found in [4].

1.1 Theorem (Effros) Let $G$ be a Polish group acting continuously on a Polish space $X$ (in other words, let $X$ be a Polish $G$-space. For $x \in X$ we have $[x]_G \in \Pi^0_2$ if and only if

$$G \rightarrow [x]_G,$$

$$g \mapsto g \cdot x$$

is open.

1.2 Corollary Let $G$ be a Polish group and $X$ a Polish $G$-space. Suppose that $[x]_G$ is $\Pi^0_2$.

Then for all $V$ containing the identity we may find open $U$ such that for all $x' \in U \cap [x]_G$ and $U' \subset X$ open

$$[x]_G \cap U' \cap U \neq \emptyset$$

implies that there exists $g \in V$ such that

$$g \cdot x' \in U'.$$

1.3 Definition Let $X$ be a Polish space and $B$ a basis. Let $\mathcal{L}(B)$ be the propositional language formed from the atomic propositions $\dot{x} \in U$, for $U \in B$. Let $\mathcal{L}_{\infty,0}(B)$ be the infinitary version, obtained by closing under negation and arbitrary disjunction and conjunction. $F \subset \mathcal{L}_{\infty,0}(B)$ is a fragment if it is closed under subformulas and the finitary Boolean operations of negation and finite disjunction and finite conjunction.

For a point $x \in X$ and $\varphi \in \mathcal{L}_{\infty,0}(B)$, we can then define $x \models \varphi$ by induction in the usual fashion, starting with

$$x \models \dot{x} \in U$$

if in fact $x \in U$. In the case that $X$ is a Polish $G$-space and $V \subset G$ open we may also define $\varphi^{\Delta V}$ by induction on the logical complexity of $\varphi$ so that in any generic extension in which $\varphi$ is hereditarily countable

$$x \models \varphi^{\Delta V}$$

if and only if

$$\exists^* g \in V \ (g \cdot x \models \varphi)$$

(where $\exists^*$ is the categoricity quantifier “there exists non-meagerly many”).

1.4 Lemma Let $X$ be a Polish $G$-space. $P$ a forcing notion, $p \in P$ a condition, $\sigma$ a $P$-term. Suppose that $B$ is a countable basis for $X$ and $B_0$ a countable basis for $G$. Suppose that $G_0$ is a
countable dense subgroup of $G$ and $\mathcal{B}$ is closed under $G_0$ translation and that $\mathcal{B}_0$ is closed under left and right $G_0$ translation. Suppose that
\[
P \vDash_{\mathbb{P}} \sigma[\hat{G}] \in X
\]
and that $p$ decides the equivalence class of $\sigma$ in the sense that
\[
(p, p) \vDash_{\mathbb{P} \times \mathbb{P}} \sigma[\hat{G}_l]E_G\sigma[\hat{G}_r].
\]
Then there is a formula $\varphi_0$ and a fragment $F_0$ containing $\varphi_0$ so that:
1. for each $\alpha < \delta$
\[
(p_\alpha, p_\alpha) \vDash_{\mathbb{P}_\alpha \times \mathbb{P}_\alpha} \sigma_\alpha[\hat{G}_l]E_G\sigma_\alpha[\hat{G}_r];
\]
2. for each $\alpha < \beta < \delta$
\[
(p_\alpha, p_\beta) \vDash_{\mathbb{P}_\alpha \times \mathbb{P}_\beta} (\sigma_\alpha[\hat{G}_l]E_G\sigma_\beta[\hat{G}_r]).
\]

1.5 Lemma Let $G$ be a Polish group, $X$ a Polish $G$-space, $A \subset X$ a $\Sigma^1_1$ set displaying a counterexample to TVC($G, \Sigma^1_1$) – so that $A/G$ has uncountably many orbits, but no perfect set of $E_G$-inequivalent points.

Then for each ordinal $\delta$ there is a sequence $(\mathbb{P}_\alpha, p_\alpha, \sigma_\alpha)_{\alpha < \delta}$ so that:
1. for each $\alpha < \delta$
\[
(p_\alpha, p_\alpha) \vDash_{\mathbb{P}_\alpha \times \mathbb{P}_\alpha} \sigma_\alpha[\hat{G}_l]E_G\sigma_\alpha[\hat{G}_r];
\]
2. for each $\alpha < \beta < \delta$
\[
(p_\alpha, p_\beta) \vDash_{\mathbb{P}_\alpha \times \mathbb{P}_\beta} (\sigma_\alpha[\hat{G}_l]E_G\sigma_\beta[\hat{G}_r]).
\]
§2 Proof

2.1 Definition $U$ is a regular open set if

$$(U)^o = U$$

$U$ equals the interior of its closure. For $A$ a set let $RO(A) = (A)^o$.

Note then that $RO(A)$ is always a regular open set.

2.2 Lemma Let $G$ be a Polish group. For $V_0, V_1 \subset G$ regular open sets,

$$\{ g \in G : V_0 \cdot g = V_1 \}$$

is a closed subset of $G$. (□)

I need that the reader is willing to allow that we may speak of an $\omega$-model of set theory containing a Polish space, group, action, Borel set, and so on, provided suitable codes exist in the well founded part. Illfounded $\omega$-models are essential to the arguments below.

In what follows let ZFC* be some large fragment of ZFC, at the very least strong enough to prove all the lemmas of §1, but weak enough to admit a finite axiomatization.

2.3 Lemma Let $M$ be an $\omega$-model of ZFC*. Let $X, G, G_0, P, F_0$, and so on, be as in 1.4 inside $M$. Suppose

$$\pi : M \cong M$$

is an automorphism of $M$ fixing $X, G, G_0, P, F_0, \varphi_0$, and all elements of $B$ and $B_0$. Suppose $H \subset Coll(\omega, F_0)$ is $M$-generic and $x \in X^{M[H]}$ with

$$x \models \varphi_0.$$ 

Then there exists $g \in G$ so that for all $\psi \in F_0$ and $V \in B_0$

$$RO(\{ g \in G_0 : M[H] \models (g \cdot x \models \psi^{\Delta V}) \})g^{-1} = RO(\{ g \in G_0 : M[H] \models (g \cdot x \models \pi(\psi)^{\Delta V}) \}).$$

Proof. It suffices to find $g_0, g_1 \in G$ so that

$$RO(\{ g \in G_0 : M[H] \models (g \cdot x \models \psi^{\Delta V}) \})g_0^{-1} = RO(\{ g \in G_0 : M[H] \models (g \cdot x \models \pi(\psi)^{\Delta V}) \})g_1^{-1}$$

for all $\psi$ and $V$.

Let $P_0$ be the forcing notion $Coll(\omega, F_0)$. Fixing $d_G$ a complete metric on $G$ we also build

$h_i, h_i' \in G_0, \psi_i, \psi_i' \in F_0, W_i, W_i' \in B_0$ so that

(i) each $W_i$ is an open neighbourhood of the identity, $W_{i+1} \subset W_i$, $d_G(W_i) < 2^{-i}$.
(ii) \( \pi(\psi_i) = \psi'_i \);
(iii) \( h_{2i} = h_{2i+1} \); \( \forall g \in W_{2i+1} h_{2i}(d_G(g, h_{2i}) < 2^{-i}) \);
(iv) \( h'_{2i+1} = h'_{2i+2} \); \( \forall g \in W_{2i+2} h'_{2i+1}(d_G(g, h'_{2i+1}) < 2^{-i}) \);
(v) \( h_{i+1} \in W_i h_i; h'_{i+1} \in W_i h'_i \);
(vi) \( M[H] \models (h_i \cdot x \models (\psi_i)^{\Delta V_i}) \);
(vii) \( M[H] \models (h'_i \cdot x \models (\psi'_i)^{\Delta V_i}) \);
(viii) \( M^{\pi_0} \) satisfies that for all \( y_0, y_1 \in X \) all \( \psi \in F_0 \), and all \( V \in B_0 \), if

\[
y_0 \models \varphi_0 \land (\psi_i)^{\Delta V_i}
\]
\[
y_1 \models \varphi_0 \land (\psi'_i)^{\Delta V_i} \land \psi^{\Delta V}
\]

then

\[
y_0 \models ((\psi_i)^{\Delta V_i} \land \psi^{\Delta V})^{\Delta W_i}
\]

(ix) conversely \( M^{\pi_0} \) satisfies that for all \( y_0, y_1 \in X \), \( \psi \in F_0 \), \( V \in B_0 \), if

\[
y_0 \models \varphi_0 \land (\psi'_i)^{\Delta V_i}
\]
\[
y_1 \models \varphi_0 \land (\psi'_i)^{\Delta V_i} \land \psi^{\Delta V}
\]

then

\[
y_0 \models ((\psi'_i)^{\Delta V_i} \land \psi^{\Delta V})^{\Delta W_i}
\]

Note that (ix) actually follows from (viii), (ii), and the elementarity of \( \pi \).

Before verifying that we may produce \( h_i, h'_i \in G_0 \), \( \psi_i, \psi'_i \in F_0 \), \( W_i, V_i \in B_0 \) as above, let us imagine that it is already completed and see how to finish. Using (iii) and (iv) we may obtain \( g_0 = \lim h_i \) and \( g_1 = \lim h'_i \). It suffices to check that for all

\[
g \in RO(\{ h \in G_0 : M[H] \models (x \models \psi^{\Delta V}) \}) g_0^{-1}
\]

we have

\[
g \in (\{ h \in G_0 : M[H] \models (h \cdot x \models \pi(\psi)^{\Delta V}) \}) g_1^{-1}
\]

(since the converse implication will be exactly symmetric).

Then for sufficiently large \( i \) we may choose a sufficiently small open neighbourhood \( W \) of the identity and \( \hat{g} \in G_0 \) sufficiently close to \( g \) so that \( W \hat{g} W_i \) is an arbitrarily small neighbourhood of \( g \) and

\[
M[H] \models (\hat{g} h_i \cdot x \models \psi^{\Delta V})
\]
\[
\therefore M[H] \models (h_i \cdot x \models (\psi^{\Delta V})^{\Delta W \hat{g}})
\]
hence, as witnessed by \( y = h_i \cdot x \)

\[
M^{P_0} \models \exists y(y \models \varphi_0 \land (\psi_i)^{\Delta V_i} \land (\psi^{\Delta V})^{\Delta W \hat{g}}),
\]

\[
\therefore M^{P_0} \models \exists y(y \models \varphi_0 \land (\psi'_i)^{\Delta V_i} \land (\pi(\psi)^{\Delta V})^{\Delta W \hat{g}}),
\]

by elementarity of \( \pi \),

\[
\therefore M[H] \models (h'_i \cdot x \models (\pi(\psi)^{\Delta V})^{\Delta W \hat{g}})^{\Delta W_i}
\]

by (ix), and so there exists some \( \bar{g} \in W \hat{g}W_i \) so that

\[
M[H] \models (\bar{g}h'_i \cdot x \models \pi(\psi)^{\Delta V}).
\]

By letting \( dG(W \hat{g}W_i) \rightarrow 0 \) and \( h'_i \rightarrow g_1 \) we get

\[
g \in \{h \in G_0 : M[H] \models (x \models \pi(\psi)^{\Delta V})g_1^{-1},
\]

as required.

We are left to hammer out the sequence.

Suppose that we have \( \psi, \psi', W, V, h, h' \) for \( j \leq 2i \). Immediately we may find \( W_{2i+1} \subset W_{2i} \) giving (iii), and then by 1.2 and 1.4(i) we can produce \( \psi_{2i+1}, V_{2i+1} \) satisfying (viii) and such that

\[
M[H] \models h_{2i} \cdot x = d f h_{2i+1} \cdot x \models (\psi_{2i+1})^{V_{2i+1}}.
\]

Then by considering that \( \pi \) is elementary

\[
M^{\pi_0} \models \exists y(y \models \varphi_0 \land (\psi_{2i})^{\Delta V_{2i}} \land (\psi_{2i+1})^{\Delta V_{2i+1}}).
\]

Thus by (ix) we may find \( h' \in G_0 \cap W_{2i} \) so that

\[
M[H] \models (h'h_{2i} \cdot x \models (\pi(\psi_{2i})^{\Delta V_{2i}} \land (\pi(\psi_{2i+1})^{\Delta V_{2i+1}}).
\]

In other words, by (ii), if we let \( \psi'_{2i+1} = \pi(\psi_{2i+1}) \) then

\[
M[H] \models (h'h_{2i} \cdot x \models (\psi'_{2i})^{\Delta V_{2i}} \land (\psi'_{2i+1})^{\Delta V_{2i+1}}).
\]

Taking \( h'_{2i+1} = h'h_{2i} \) we complete the transition from \( 2i \) to \( 2i + 1 \).

The further step of producing \( \psi_{2i+2}, \psi'_{2i+2}, W_{2j+2}, h_{2j+2}, V_{2j+2} \) and \( h'_{2j+2} \) is completely symmetrical. \( \square \)

2.4 Definition \( S_\infty \) divides a Polish group \( G \) if there is a closed subgroup \( H < G \) and a continuous onto homomorphism

\[
\pi : H \twoheadrightarrow S_\infty.
\]
(By Pettis’ lemma, any Borel homomorphism between Polish groups must be continuous.)

2.5 Lemma $S_{\infty}$ divides $\text{Aut}(\mathbb{Q},<)$, the automorphism group of the rationals equipped with the usual linear ordering.

2.6 Definition For $X, G, F_0, \text{ and so on},$ as in 1.4, $P_0 = \text{Coll}(\omega, F_0), \psi_0, \psi_1 \in F_0, V_0, V_1 \in B_0,$ set

$$(\psi_0, V_0)R(\psi_1, V_1)$$

if in $V^{P_0}$ for all $x \models \varphi_0$

$$RO(\{g \in G_0 : g \cdot x \models (\psi_0)^{\Delta V_0}\}) \cap RO(\{g \in G_0 : g \cdot x \models (\psi_1)^{\Delta V_1}\}) \neq \emptyset.$$ 

For $V \in B_0$ let $B(V)$ be the set of pairs $(\varphi, W)$ such that for all $\psi \in F_0$ and $W' \in B_0$

$$V^{P_0} \models \forall x_0 \models \varphi_0 \land \varphi^{\Delta W}(\exists x_1 \models \varphi_0 \land \varphi^{\Delta W} \land \psi^{\Delta W'}) \Rightarrow x_0 \models (\varphi^{\Delta W} \land \psi^{\Delta W'})^{\Delta V}.$$ 

In other words, $B(V)$ corresponds to the basic open sets witnessing 1.2 for $V$ in the topology $\tau_0(F_0)$.

The next lemma states that if the equivalence class corresponding to $\varphi_0$ requires large forcing to be introduced then the formulas $\{\psi^{\Delta V} : \psi \in F_0, V \in B_0\}$ have large $R$-discrete sets.

2.7 Lemma Let $X, G, F_0, \varphi, \varphi_0, \text{ and so on},$ be as in 1.4. Let $R$ be as in 2.6. Let $\kappa$ be a cardinal. Suppose no forcing notion of size less than $\kappa$ introduces a point in $X$ satisfying $\varphi_0$.

Then there is no infinite $\delta < \kappa$ such that each $B(V)$ for $V \in B_0$ has a maximal $R$-discrete set of size $\leq \delta$.

Proof. Suppose otherwise and choose large $\theta > \kappa$ so that $V_0 \models \text{ZFC}^*$ and choose an elementary substructure

$$A \prec V_0$$

so that

$$|A| = \delta,$$

$$\delta + 1 \subset A,$$

and $X, G, F_0, \varphi_0, \text{ and so on},$ in $A$. Let $N$ be the transitive collapse of $A$ and

$$\pi : N \to V_0$$

the inverse of the collapsing map. Set $\hat{\mathbb{P}} = \pi^{-1}(\mathbb{P}_0)$ (where $\mathbb{P}_0 = \text{Coll}(\omega, F_0)$), $\hat{\varphi_0} = \pi^{-1}(\varphi_0), \hat{F}_0 = \pi^{-1}(F_0),$ choose

$$\hat{H} \subset \hat{\mathbb{P}},$$
\[
H \subset \mathbb{P}_0
\]
to be \(V\)-generic, and choose \(\hat{x} \in N[\hat{H}] \) and \(x \in V[H]\) so that
\[
N[\hat{H}] \models (\hat{x} \models \varphi_0),
\]
\[
V[H] \models (x \models \varphi_0).
\]

It suffices to show
\[
\hat{x}E_Gx.
\]

As in the proof of 2.3 find \(h_i, h'_i \in G_0, \psi_i \in F_0, \psi'_i \in \hat{F}_0, V_i, V'_i \in B_0, W_i \in B_0 \) and \(U_i \subset X\) basic open so that:

(i) \(W_{i+1} \subset W_i, W_i = (W_i)^{-1}, d_G(W_i) < 2^{-i}, 1_G \in W_i; U_{i+1} \subset U_i, d_X(U_i) < 2^{-i};\)
(ii) \(\pi(\psi'_i) = \psi_i;\)
(iii) \(\forall g \in (W_{2i+1})^3 h_2(d_G(g, h_2) < 2^{-i}); h_{2i+1} = h_{2i};\)
(iv) \(\forall g \in (W_{2i+2})^3 h'_{2i+1}(d_G(g, h_{2i+1}) < 2^{-i}); h'_{2i+2} = h'_{2i+1};\)
(v) \(h_{i+1} \in (W_i)^3 h_i, h'_{i+1} \in (W_i)^3 h'_i;\)
(vi) \(V[H] \models (h_i \cdot x \models (\psi_i)^{\Delta V_i});\)
(vii) \(N[H] \models (h'_i \cdot \hat{x} \models (\psi'_i)^{\Delta V'_i});\)
(viii) \(V^{\varphi_0} \models (\psi_i, V_i) \in B(W_i);\)
(ix) \(N^{\varphi_0} \models (\psi'_i, V'_i) \in B(W_i);\)
(x) \((\pi(\psi'_i), V'_i)R(\psi_i, V_i);\)
(xi) \(h_i \cdot x, h'_i \cdot \hat{x} \in U_j.\)

Granting all this may be found we finish quickly. By (iii) and (iv) we get
\[
g_0 = \lim h_i \text{ and } g_1 = \lim h'_i,\]
whence
\[
g_0 \cdot x = g_1 \cdot \hat{x}
\]
by (xii). This would contradict \(\hat{p}\) being too small to introduce a representative of \([x]_G\).

So instead suppose we have built \(V_j, V'_j, \psi_j\) and so on for \(j < 2i\) and concentrate on trying to show that we may continue the construction up to \(2i + 2\).

First choose \(W_{2i+1} \subset W_{2i}\) in accordance with (i) and (iii) and then for (xi) and (i) choose \(U_{2i+1} \subset U_{2i}\) containing \(h_{2i} \cdot x(=d_{h_{2i+1}} h_{2i+1} \cdot x)\) with \(d_X(U_{2i+1}) < 2^{-2i-1}\). Then by 1.2 we may choose \(V_{2i+1}, \psi_{2i+1}\) as indicated at (vi) and (viii).

On the \(N\) side we use the assumption on \(R\) to find \(V''_{2i+1}\) and \(\psi''_{2i+1}\) in \(N\) so that
\[
N^{\varphi_0} \models (\psi''_{2i+1}, V''_{2i+1}) \in B(W_{2i+1})
\]
and
\[
(\pi(\psi''_{2i+1}), V''_{2i+1})R(\psi_{2i+1}, V_{2i+1}).
\]
Unwinding the definitions gives

\[ V^p_0 \models (y \models \varphi_0 \land \pi(\psi'_2)_{V_2}) \Rightarrow y \models ((\psi_2)_{V_2} \land \pi(\psi'_2)_{V_2})_{W_2}, \]

\[ V^p_0 \models (y \models \varphi_0 \land (\psi_2)_{V_2}) \Rightarrow y \models ((\psi_2)_{V_2} \land (\psi_2+1)_{V_2})_{W_2}, \]

\[ V^p_0 \models (y \models \varphi_0 \land (\psi_2+1)_{V_2}) \Rightarrow y \models ((\psi_2+1)_{V_2} \land \pi(\psi'_2)_{V_2})_{W_2}. \]

In particular, assuming without loss of generality that

\[ (\psi_2+1)_{V_2} \Rightarrow \hat{x} \in U_{2i+1} \]

we have

\[ V^p_0 \models (y \models \varphi_0 \land \pi(\psi'_2)_{V_2}) \Rightarrow y \models ((\psi_2+1)_{V_2} \land \hat{x} \in U_{2i+1})_{W_2}. \]

Thus by elementarity of \( \pi \) we may find \( h' \in (W_2)^3 \cap G_0 \) so that \( h'h'_2 \cdot \hat{x} \in U_{2i+1} \) and

\[ N[\hat{H}] \models (h'h'_2 \cdot \hat{x} = (\psi'_2+1)_{V_2}). \]

Then setting \( h_{2i+1} = h'h_2 \) completes the transition from \( 2i \) to \( 2i + 1 \).

The step from \( 2i + 1 \) to \( 2i + 2 \) is similar.

We need a fact from infinitary model theory.

**2.8 Theorem** Let \( \varphi \in \mathcal{L}_{\omega_1,\omega} \) and suppose

\[ N \models \varphi, \]

and \( P \) is a predicate in the language of \( N \) with

\[ |(P)^N| \geq \aleph_1. \]

Then \( \varphi \) has a model with generating indiscernibles in \( P \).

More precisely there is a model \( M \) with language \( \mathcal{L}^* \supset \mathcal{L}(N) \), \( \mathcal{L}^* \) having a new symbol \( < \), along with new function symbols of the form \( f_{\varphi} \) for \( \varphi \) in the fragment of \( \mathcal{L}(N)_{\omega_1,\omega} \) generated by \( \varphi \), and distinguished elements \( (c_i)_{i \in \mathbb{N}} \), so that:

(i) \( (<)^M \) linearly orders \( (P)^M \);

(ii) each \( f_{\varphi} \) is a Skolem function for \( \varphi \);

(iii) \( M \) is the Skolem hull of \( \{c_i : i \in \mathbb{N}\} \) (under the functions of the form \( f_{\varphi} \);

(iv) each \( c_i \in (P)^M \);

(v) for all \( \psi \) in the fragment of \( \mathcal{L}^*_{\omega_1,\omega} \) generated by \( \varphi \) and \( i_1 < i_2 < \ldots < i_n, j_1 < \ldots < j_n \) in \( \mathbb{N} \)

\[ M \models \psi(c_{i_1}, c_{i_2}, \ldots, c_{i_n}) \Leftrightarrow \psi(c_{j_1}, c_{j_2}, \ldots, c_{j_n}); \]
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(vi) \( M \models \varphi \).

See [5].

2.9 Theorem Let \( G \) be a Polish group for which \( \text{TVC}(G, \mathcal{S}_1^1) \) fails. Then \( S_\infty \) divides \( G \).

Proof. Choose some Polish \( G \)-space \( X \) witnessing the failure of \( \text{TVC}(G, \mathcal{S}_1^1) \). Following 1.5 we may find some \((p, p, \sigma)\) introducing an equivalence class as in 1.4 that may not be produced by a forcing notion of size less than \( \mathfrak{V} \). Fix \( \varphi_0, B, B_0, F_0, G_0, \) and so on, as in 1.4, so that in all generic extensions \( V[H] \) of \( V \)

\[
V[H] \models p \models \forall y \in X(yE_G \sigma[G] \iff y \models \varphi_0).
\]

Let \( V_0 \) be large enough to contain \( X, G, \varphi_0, \) and so on, and satisfy \( \text{ZFC}^* \). By 2.7 choose \( P \subset V_0 \) to be of size \( \mathfrak{V}, \) and \( R \)-discrete (or more precisely, so for all \((\psi, V) \neq (\psi', V') \in P \) we have for any \( V \)-generic \( H \subset \text{Coll}(\omega, F_0) \) that \( V[H] \models \neg((\psi, V)R(\psi', V')) \)). Applying 2.8 to \( N = (V_0; \in, P, X, G, G_0, \varphi_0, \ldots) \) and we may obtain an \( \omega \)-model with indiscernibles \((\psi_q, V_q)_{q \in \mathbb{Q}} \) in \( B^M \). Let \( H \subset \text{Coll}(\omega, (F_0)^M) \) be \( M \)-generic. Choose \( x \in M[H] \) so that

\[
M[H] \models (x \models \varphi_0).
\]

All this granted we may define \( G_1 \) to be the set of \( \bar{g} \in G \) so that for all \( q \in \mathbb{Q} \) there exists \( r \in \mathbb{Q} \) with

\[
\text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q}) \} \bar{g}^{-1} = \text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r}) \}
\]

and for \( q \in \mathbb{Q} \) there exists \( r \in \mathbb{Q} \) with

\[
\text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q}) \} \bar{g} = \text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r}) \}.
\]

\( G_1 \) is \( \Pi^0_2 \) in \( G \), by 2.2 and since \( \bar{g} \) is in \( G_1 \) if and only if the following four conditions hold:

(i) for all \( q \in \mathbb{Q} \) there exists \( r \in \mathbb{Q} \) with

\[
\text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q}) \} \bar{g}^{-1} \cap \text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r}) \} \neq \emptyset,
\]

(ii) for all \( q, r \in \mathbb{Q} \)

\[
\text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q}) \} \bar{g}^{-1} \cap \text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r}) \} \neq \emptyset
\]

implies

\[
\text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q}) \} \bar{g}^{-1} = \text{RO} \{ g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r}) \}.
\]
(iii) for all $q \in \mathbb{Q}$ there exists $r \in \mathbb{Q}$ with
\[ \mathcal{R}O(\{g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q})\}) \cap \mathcal{R}O(\{g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r})\}) \neq \emptyset, \]

(iv) for all $q, r \in \mathbb{Q}$
\[ \mathcal{R}O(\{g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q})\}) \cap \mathcal{R}O(\{g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r})\}) \neq \emptyset \]
implies
\[ \mathcal{R}O(\{g \in G_0 : M[H] \models (g \cdot x \models (\psi_q)^{\Delta V_q})\})^{-1} = \mathcal{R}O(\{g \in G_0 : M[H] \models (g \cdot x \models (\psi_r)^{\Delta V_r})\}). \]

Since $G_1$ is a $\Pi^0_2$ subgroup of $G$ it must be closed.

For $g \in G_1$ we may define the permutation $\hat{\pi}(g)$ of $\mathbb{Q}$ by the specification that for all $q \in \mathbb{Q}$
\[ (\hat{\pi}(g))(q) = r \]
if and only if $r$ is as above in the definition of $G_1$. This is well defined by the $R$-discreteness of the set $(P)^M$.\[ \]

Now let $G_2$ be the set of $g \in G_1$ such that $\hat{\pi}(g)$ defines an automorphism of the structure $(\mathbb{Q}, <)$. $G_2$ is a closed subgroup of $G_1$ and hence $G$. Since every order preserving permutation of the indiscernibles induces an automorphism of $M$ the map
\[ \hat{\pi} : G_2 \to \text{Aut}(\mathbb{Q}, <) \]
is onto by 2.3. Then by 2.5 $S_\infty$ divides $G$. \[ \square \]

**2.10 Conjecture** Assume AD$_{L(R)}$. Let $G$ be a Polish group, $X$ a Polish $G$-space, $A \subset X$ in $\Sigma^1_1$, and suppose in $L(R)$ there is an injection
\[ i : A/G \leftrightarrow 2^{<\omega_1}. \]

Then there is a Polish $S_\infty$-space $Y$ and a $\Sigma^1_1$ set $B \subset Y$ and a bijection
\[ \pi : A/G \cong B/S_\infty. \]
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