Polytopal surfaces in Fuchsian manifolds

Roman Prosanov *

Abstract

Let $S_{g,n}$ be a surface of genus $g$ with $n$ punctures equipped with a complete hyperbolic cusp-metric. Then it can be uniquely realized as the boundary metric of a symmetric Fuchsian polytope. In the present paper we give a new variational proof of this result. Our proof is based on the discrete Hilbert-Einstein functional and on an interpretation of the Epstein–Penner convex hull construction.

1 Introduction

1.1 Theorems of Alexandrov and Rivin

Consider a convex polytope $P \subset \mathbb{R}^3$. Its boundary is homeomorphic to $S^2$ and carries a metric induced from the Euclidean metric on $\mathbb{R}^3$. What are the intrinsic properties of this metric?

A metric on $S^2$ is called polyhedral Euclidean if it is locally isometric to the Euclidean metric on $\mathbb{R}^2$ except finitely many points, which have neighborhoods isometric to an open subset of a cone (an exceptional point is mapped to the apex of this cone). If the angle at every exceptional point is less than $2\pi$, then this metric is called convex. It is clear that the induced metric on the boundary of a convex polytope is a convex polyhedral Euclidean metric. One can ask a natural question: is this description complete, in the sense that every convex polyhedral flat metric can be realized as the induced metric of a polytope? This question was answered positively by Alexandrov in 1942, see [1], [2].

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Theorem 1.1. For every convex polyhedral Euclidean metric \( d \) on \( S^2 \) there is a convex polytope \( P \subset \mathbb{R}^3 \) such that \( (S^2, d) \) is isometric to the boundary of \( P \). Moreover, such \( P \) is unique up to an isometry of \( \mathbb{R}^3 \).

Note that \( P \) can degenerate to a polygon. In this case \( P \) is doubly covered by the sphere.

The uniqueness part is easy and follows from the modified version of Cauchy’s global rigidity of convex polytopes. The original proof by Alexandrov of the existence part was not constructive. It was based on some topological properties of the map from the space of convex polytopes to the space of convex polyhedral Euclidean metrics. Another proof was done by Volkov in [23], a student of Alexandrov, by considering a discrete version of the total scalar curvature.

A new proof of Theorem 1.1 was proposed by Bobenko and Izmestiev in [3]. For a fixed metric they considered a space of singular polytopes realizing this metric at their boundary. In order to remove singularities they constructed a functional over this space and investigated its behavior. Such a proof can be turned into a practical algorithm of finding a polytopal realization of a given metric. It was implemented by Stefan Sechelmann. One should note that this algorithm is approximate as it uses numerical methods of solving variational problems, but it works well for all practical needs.

We turn our attention to hyperbolic metrics on surfaces. Let \( S_{g,n} \) be a surface of genus \( g \) endowed with a complete hyperbolic metric of a finite volume with \( n \) cusps. In [17] Rivin proved a version of Theorem 1.1 for cusp-metrics on a sphere \( S^2 \) with punctures.

Theorem 1.2. For every cusp-metric \( d \) on \( S_{0,n} \) there exists a convex ideal polytope \( P \subset \mathbb{H}^3 \) such that \( (S_{0,n}, d) \) is isometric to the boundary of \( P \). Moreover, such \( P \) is unique up to an isometry of \( \mathbb{H}^3 \).

Rivin gave a proof in the spirit of Alexandrov’s original proof. Very recently, in [21] Springborn gave a variational proof of Theorem 1.2.

1.2 Ideal Fuchsian Polyhedra and Alexandrov-type results

It is of interest how can we generalize these results to surfaces of higher genus. We restrict ourselves to the case \( g > 1 \) and to metrics with cusps. By \( G \) denote the fundamental group of \( S_g \). Let \( \rho : G \to \text{Iso}^+(\mathbb{H}^3) \) be a Fuchsian
representation: an injective homomorphism such that its image is discrete and there is a geodesic plane invariant under $\rho(G)$. Then $F := \mathbb{H}^3 / \rho(G)$ is a complete hyperbolic manifold, homeomorphic to $S_g \times (-\infty; +\infty)$. The image of the invariant plane is the so-called convex core of $F$ and is homeomorphic to $S_g$. The manifold $F$ is symmetric with respect to its convex core.

A subset of $F$ is called convex if it contains every geodesic between any two of its points. It is possible to consider convex hulls with respect to this definition. An ideal Fuchsian polytope $P$ is the convex hull of a finite point set in $\partial_F$ invariant under the reflection with respect to the convex core. The boundary of $P$ consists of two isometric copies of $S_{g,n}$. Now we can establish our main result.

**Theorem 1.3.** For every cusp-metric $d$ on $S_{g,n}$, $g > 1$, $n > 0$, there exists a Fuchsian manifold $F$ and an ideal Fuchsian polytope $P \subset F$ such that $(S_{g,n}, d)$ is isometric to each of two components of the boundary of $P$. Moreover, $F$ and $P$ are unique up to isometry.

This theorem was first proved by Schlenker in his unpublished manuscript [19]. Another proof was given by Fillastre in [8]. Both these proofs were non-constructive following the original approach of Alexandrov. The aim of the present paper is to give a variational proof of Theorem 1.3 in the spirit of papers [3], [9] and [21].

Several authors studied Alexandrov-type questions for hyperbolic surfaces of genus $g > 1$ in more general sense. They were collected in the following result of Fillastre [8]. Consider a complete hyperbolic metric $d$ on $S_{g,n}$ with cusps, conical points and complete ends of infinite area. Complete ends of infinite area are boundary components “at infinity”. One can see an example in the projective model as the intersection of a cone with the apex outside of $\mathbb{H}^3$ (such a point is called hyperideal point) with $\mathbb{H}^3$. Such a metric can be uniquely realized as the induced metric at the boundary of a generalized Fuchsian polytope. Some vertices of this polytope may be hyperideal, which corresponds to complete ends of infinite area. The case, when the surface is a sphere with punctures and holes, was proved by Schlenker in [18]. Some vertices may be in the interior of a Fuchsian manifold and correspond to conic points. The case of $g > 1$ with only conical singularities was first proved in an earlier paper of Fillastre [7] and with only cusps and infinite ends in the paper [20] by Schlenker. The torus case with only conical singularities was the subject of the paper [9] by Fillastre and Izmestiev. The last paper also followed the scheme of variational proof. All other mentioned works...
were done in the direction of the original Alexandrov approach. Recently another realization result of metrics on surfaces with conical singularities was obtained by Brunswic in [5].

We would like to thank Boris Springborn, who pointed that another proof of Theorem 1.3 follows from the paper [10]. The authors do not consider Alexandrov-type statements, but Theorem 1.3 can be deduced from their results using several geometric lemmas. The proof in this paper is also nonconstructive and investigates the correspondence between discrete Euclidean structures on surfaces up to discrete conformality and hyperbolic cusp-metrics up to isometry first noted in [4]. The variational approach is discussed in the end, but the functional is not provided directly.

There is an interesting interpretation of Theorem 1.3 in terms of the Teichmüller space. Let \( \tilde{T}_{g,n} \) be the cusped Teichmüller space, i.e. the space of all cusp-metrics on \( S_{g,n} \) up to isometry isotopic to the identity. Also, let \( T_{g,n} \) be the Teichmüller space with \( n \) marked points, i.e. the space of all complete hyperbolic metrics on \( S_{g} \) with \( n \) marked points up to isometry isotopic to the identity.

**Theorem 1.4.** There is a natural bijection between \( \tilde{T}_{g,n} \) and \( T_{g,n} \).

### 1.3 Overview of the proof

In general, we follow the road landmarked by the previous works [3], [9] and [21] contained variational proofs of Alexandrov-type problems. But in the new setting we encounter different obstacles. We highlight the connection (noted also in [21]) with Epstein–Penner decompositions (see [6], [14], [15]). For the definitions we refer to Subsection 2.4. In our paper for every Epstein–Penner decomposition of \( S_{g,n} \) we construct a Fuchsian polytope with singularities whose boundary structure coincides with the given decomposition. This is similar to the connection of the Euclidean case with the weighted Delaunay triangulations noted in [3] and may lead to a further comprehension of these objects.

We would like to highlight that our proof can be turned to an effective algorithm of finding a realization as a Fuchsian polytope of a given cusp-metric. This is a big difference comparing with the previous indirect proofs by Schlenker and Fillastre.

Our strategy of proof is as follows. A Fuchsian polytope \( P \) can be cut into two symmetric halves. Its boundary has a polytopal decomposition into...
faces. This decomposition can be projected to the convex core of $P$ and provides a decomposition of a half of $P$ into basic geometric objects: *semi-ideal rectangular prisms*. We go in the opposite direction: we glue several prisms altogether according to a geodesic triangulation of $S_{g,n}$. We obtain a complex that looks like a Fuchsian polytope, but has conical singularities on the inner edges of gluing. A variational argument shows that these singularities can be removed.

In Section 2 we overview and establish several necessary results from elementary hyperbolic geometry. In Section 3 we define our basic objects (called *prismatic complexes*) and study their properties. In Section 4 we introduce the space of all prismatic complexes. In Section 5 we represent our problem as a variational one and finish the proof of Theorem 1.3.

1.4 Related work and perspectives

It may be of interest to consider the following generalization of our statement. Define a *non-symmetric ideal Fuchsian polytope* $P$ as the convex hull of $n > 0$ ideal points belonging to one component of $\partial_{\infty} F$ and $m > 0$ ideal point belonging to the other one. The boundary of $P$ consists of $S_{g,n}$ with a cusp-metric and $S_{g,m}$ equipped with another cusp-metric. One can ask if we take two such metrics, is there a non-symmetric ideal Fuchsian polytope realizing both metrics at its boundary? The answer to this naive question is no. From the statement of Theorem 1.3 we can see that a cusp-metric $d$ on $S_{g,n}$ determines uniquely a metric on the convex core of the Fuchsian manifold $F$ such that $(S_{g,n}, d)$ can be convexly realized in $F$. If for two cusp metrics the corresponded metrics on the convex cores are different, then these cusp-metric can not be boundary components of the same non-symmetric ideal Fuchsian polytope (and, clearly, Theorem 1.3 implies that otherwise such a polytope exists). But we may consider polytopes in so-called *quasifuchsian manifolds*.

A representation $\rho$ of $G = \pi_1(S_g)$ in $\text{Iso}^+(\mathbb{H}^3)$ is called quasifuchsian if it is discrete, faithful and the limit set at the boundary at infinity of its action is a Jordan curve. A manifold $F$ is *quasifuchsian* if it is isometric to $\mathbb{H}^3/\rho(G)$. As in the Fuchsian case, $F$ is homeomorphic to $S_g \times \mathbb{R}$ and has the well-defined boundary at infinity. The *convex core* of $F$ is the image of the convex hull of the limit set under the projection of $\mathbb{H}^3$ onto $F$, which is 3-dimensional if $F$ is not Fuchsian. A *non-symmetric ideal quasifuchsian polytope* is the convex hull of $n > 0$ ideal points belonging to one component
of \( \partial_{\infty} F \) and \( m > 0 \) ideal point belonging to the other one.

To state an analog of the uniqueness part of Theorem 1.3 we need a way to connect Teichmüller spaces for surfaces with different number of punctures. A marked cusp-metric is a cusp-metric on \( S_{g,n} \) together with a marking monomorphism \( \pi_1(S_g) \hookrightarrow \pi_1(S_{g,n}) \). A quasifuchsian manifold \( F \) has a canonical identification \( \pi_1(F) \cong \pi_1(S_g) \). For every quasifuchsian polytope \( P \) it induces monomorphisms \( \iota_+ \) and \( \iota_- \) of \( \pi_1(S_g) \) to the fundamental groups of the upper and lower boundary components of \( P \) respectively.

**Conjecture 1.5.** Let \( d_1 \) and \( d_2 \) be two marked cusp-metrics on \( S_{g,n} \) and \( S_{g,m} \) respectively, \( n,m > 0 \). Then, there is a unique non-symmetric ideal quasifuchsian polytope \( P \) such that one component of its boundary is isometric to \( (S_{g,n},d_1) \), the other one is isometric to \( (S_{g,m},d_2) \) and the compositions of marking monomorphisms with the maps induced by these isometries coincide with \( \iota_+ \) and \( \iota_- \).

We think that our proof of Theorem 1.3 can be adapted to a proof of this conjecture. It is a perspective direction of a further research.

In order to remove singularities we use the so-called discrete Hilbert–Einstein functional. Another perspective direction for further research is determining its signature for triangulations of Euclidean and hyperbolic manifolds. This may lead to new proofs of various geometrization and rigidity results. We refer the reader to [11] for a survey of mentioned ideas.

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## 2 Some hyperbolic geometry

### 2.1 Hyperbolic and de-Sitter spaces and duality

In this section we fix some notation and mention results from basic hyperbolic geometry that will be used below.

For several proofs we deal with the hyperboloid model for \( \mathbb{H}^3 \). Consider \( \mathbb{R}^{1,3} \) with the scalar product

\[
\langle \bar{x}, \bar{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4.
\]

By letters with lines above like \( \bar{x} \) we denote points of \( \mathbb{R}^{1,3} \). Define

\[
\mathbb{H}^3 = \{ \bar{x} \in \mathbb{R}^{1,3} : \langle \bar{x}, \bar{x} \rangle = -1; x_1 > 0 \}.
\]
By \( \mathbb{R}^{1,2} \) we denote the plane \( \{ \overline{x} : x_4 = 0 \} \) and by \( \mathbb{H}^2 \) denote \( \mathbb{H}^3 \cap \mathbb{R}^{1,2} \). Define the three-dimensional de Sitter space

\[
\mathbb{dS}^3 = \{ \overline{x} \in \mathbb{R}^{1,3} : \langle \overline{x}, \overline{x} \rangle = 1 \}
\]

and a half of the cone of light-like vectors

\[
\mathbb{L} = \{ \overline{x} \in \mathbb{R}^{1,3} : \langle \overline{x}, \overline{x} \rangle = 0; x_1 > 0 \}.
\]

There is a natural correspondence between ideal points of \( \mathbb{H}^3 \) and generatrices of \( \mathbb{L} \). Horospheres are intersections of \( \mathbb{H}^3 \) with affine planes parallel to generatrices of \( \mathbb{L} \). Slightly abusing the notation we will use the same letters both for these planes and for the horosphere defined by them. For such a plane \( L \) define its polar dual \( \overline{l} \in \mathbb{L} \) by the equation

\[
\langle \overline{l}, \overline{x} \rangle = -1,
\]

for all \( x \in L \).

A hyperbolic plane \( M \) in \( \mathbb{H}^3 \) is the intersection of \( \mathbb{H}^3 \) with a two-dimensional subspace of \( \mathbb{R}^{1,3} \) such that its normal \( \overline{m} \) is space-like. Again, in our notation we will not distinguish these planes in \( \mathbb{R}^{1,3} \) from the corresponding planes in \( \mathbb{H}^3 \). (The same also holds for hyperbolic lines in \( \mathbb{H}^2 \).) For such a plane \( M \) there are two unit normals satisfying the equation

\[
\langle \overline{m}, \overline{x} \rangle = 0,
\]

for all \( x \in M \). These two normals naturally correspond to halfspaces defined by the plane \( M \). If \( \overline{m} \in \mathbb{dS}^3 \) is chosen, then denote these halfspaces by

\[
M_+ = \{ \overline{x} \in \mathbb{H}^3 : \langle \overline{x}, \overline{m} \rangle \geq 0 \}, \quad M_- = \{ \overline{x} \in \mathbb{H}^3 : \langle \overline{x}, \overline{m} \rangle \leq 0 \}.
\]

### 2.2 Computations in hyperbolic space

We will need the following interpretation of scalar products between vectors of \( \mathbb{R}^{1,3} \) in terms of distances (see \([12], [16], [22]\))

**Lemma 2.1.** 1. If \( \overline{a} \in \mathbb{H}^3 \) and \( \overline{m} \in \mathbb{dS}^3 \), then

\[
\langle \overline{a}, \overline{m} \rangle = \sinh \text{dist}(x, M),
\]

where the distance is signed: it is nonnegative if \( \overline{a} \in M_+ \) and negative otherwise.
2. If $\bar{a} \in H^3$ and $\bar{l} \in L$, then

$$\langle \bar{a}, \bar{l} \rangle = e^{-\text{dist}(x,L)},$$

where the distance is also signed: it is positive if $\bar{a}$ is outside the horoball bounded by $L$ and negative otherwise.

3. If $\bar{m} \in dS^3$ and $\bar{l} \in L$, then

$$\langle \bar{m}, \bar{l} \rangle = \pm e^{\text{dist}(M,L)},$$

where the distance between a plane and a horosphere is the length of the common perpendicular taken with the minus sign if the line intersects the horosphere. The sign of the right hand side depends on at which halfspace with respect to $M$ the center of $L$ lies.

4. If $\bar{l}_1 \in L$ and $\bar{l}_2 \in L$, then

$$\langle \bar{l}_1, \bar{l}_2 \rangle = -2e^{\text{dist}(L_1,L_2)},$$

where the distance between two horospheres is the length of the common perpendicular taken with the minus sign if these horospheres intersect.

Now we establish some formulas.

Lemma 2.2. Let $M_1$ and $M_2$ be two ultraparallel planes in $H^3$ and $\bar{a} \in M_1$ be the nearest point from $M_1$ to $M_2$. Then for any $\bar{x} \in M_1$ we have

$$\sinh(\text{dist}(\bar{x},M_2)) = \cosh(\text{dist}(\bar{x},\bar{a})) \sinh(\text{dist}(M_1,M_2)).$$

The proof is a straightforward computation.

Since this point we will always suppose that they are equipped with horospheres. Under this agreement, it is always possible to define the distance between two points. For two ideal points the distance is the distance between corresponding horospheres. This distance is signed: we write it with the minus sign if horospheres intersect. For one ideal and one point in $H^3$, the distance means the signed distance from the point in $H^3$ to the corresponded horosphere at ideal point.

Lemma 2.3. Let $ABC$ be an ideal hyperbolic triangles, all vertices of which are equipped with horospheres. By $a$, $b$ and $c$ denote the distances between respective horospheres and by $\alpha_A$ denote the length of the part of the horosphere centered at $A$ inside the triangle. Then

$$\alpha_A^2 = e^{a-b-c}.$$
A proof can be found in [14]. We will need a semi-ideal version of this lemma. It is less known, hence, we provide a full proof.

**Lemma 2.4.** Let $ABC$ be a hyperbolic triangle with ideal vertices $A$ and $B$ equipped with horospheres. By $a$ and $b$ denote the distances from $C$ to the horospheres at $B$ and $A$ respectively, by $c$ denote the distance between these horospheres, by $\alpha_A$ denote the length of the part of the horocycle centered at $A$ that is inside this triangle. Then

$$\alpha_A^2 = e^{a-b-c} - e^{-2b}.$$ 

**Proof.** Consider the hyperboloid model. Let $\tilde{C}$ be an intersection of the ray $AC$ with boundary at infinity and put a horocycle at $\tilde{C}$ such that it passes through $C$ (see Figure 1). Denote the side lengths of this new ideal decorated triangle by $\tilde{a}$, $\tilde{b} = b$ and $\tilde{c} = c$. From Lemma 2.3 it follows that $\alpha_A^2 = e^{\tilde{a}-\tilde{b}-\tilde{c}} = e^{\tilde{a}-b-c}$. Hence, we need to calculate $\tilde{a}$.

By $\tilde{l}_A$, $\tilde{l}_B$, $\tilde{l}_C$ denote the polar light-like vectors corresponded to all horocycles; by $\overline{x}_C$ denote the vector of the point $C$ at hyperboloid. Then

$$\tilde{l}_C = \lambda \overline{x}_C + \mu \tilde{l}_A.$$
We have
\[ \langle \tilde{l}_C, I_A \rangle = -\lambda e^b = -2e^b. \]
Hence, we obtain that \( \lambda = 2. \) Now calculate
\[ \langle \tilde{l}_C, x_C \rangle = -1 = -\lambda - \mu e^b. \]
We obtain \( \mu = -e^{-b}. \) We need only to evaluate
\[ \langle \tilde{l}_C, I_B \rangle = -2e^\tilde{a} = -2e^a + 2e^{c-b}. \]
So, we get \( e^\tilde{a} = e^a - e^{c-b}. \)
Finally, \( \alpha_A^2 = e^{a-b-c} - e^{-2b}. \)

2.3 Fuchsian manifolds and polytopes

In this subsection we briefly discuss Fuchsian polytopes and different approaches to them. Consider a surface \( S_g \) of genus \( g > 1. \) Now we formulate precisely some concepts touched in the introduction.

Definition 2.5. A homomorphism \( \rho : G = \pi_1(S_g) \to \text{Iso}^+(\mathbb{H}^3) \) is called a Fuchsian representation if it is discrete, faithful and there is a geodesic plane \( P \) fixed by \( \rho(G). \)

Definition 2.6. A 3-dimensional hyperbolic manifold \( F \) is called Fuchsian if it is isometric to \( \mathbb{H}^3/\Gamma, \) where \( \Gamma \) is the image of a Fuchsian representation of \( \pi_1(S_g). \)

The action of such \( \Gamma \) can be extended to the boundary at infinity of \( \mathbb{H}^3. \) The factor of this action can be naturally seen as the boundary at infinity of \( F. \) It is homeomorphic to two copies of \( S_g \) and inherits a conformal structure. The plane \( P \) projects in \( F \) onto a geodesic surface homeomorphic to \( S_g. \) As it was noted in the introduction, it is called the convex core of \( F. \) The manifold \( F \) has a natural isometric involution preserving its convex core coming from the symmetry of \( \mathbb{H}^3 \) with respect to \( P. \)

Definition 2.7. A set \( P \subset F \) is called convex if it contains every geodesic between any pair of points of \( P. \) The convex hull of a set \( Q \) is the inclusion-minimal convex set containing \( Q. \)
Definition 2.8. A set $P \subset F$ is called a Fuchsian polytope if it is the convex hull of a finite point set in $F$ that is symmetric with respect to the convex core. It is called ideal if all its vertices belong to the boundary at infinity of $F$.

Intrinsically, an ideal Fuchsian polytope is a complete hyperbolic 3-manifold with an isometric involution with respect to a geodesic embedding of $S_g$ and with convex piecewise-geodesic boundary isometric to two copies of $S_{g,n}$ equipped with a cusp-metric. This description is full as shown by the following lemma.

Lemma 2.9. Let $Q$ be a manifold satisfying the description above. Then there is a Fuchsian manifold $F$ and an ideal Fuchsian polytope $P$ isometric to $Q$.

We will give a proof of this lemma in Subsection 3.1 after introducing the necessary machinery. Our proof can be easily extended to non-ideal case, but we do not need it. Now we are able to speak about Fuchsian polytopes without referring to ambient Fuchsian manifolds.

Consider a Fuchsian polytope $P$ in $F = \mathbb{H}^3/\Gamma$. A boundary component of $P$ can be lifted to $\mathbb{H}^3$. This lift is a polytopal surface in $\mathbb{H}^3$ that is invariant under the action of $\Gamma$. It is clear that this construction also works backwards.

Lemma 2.10. There is 1:1 correspondence (up to isometry) between Fuchsian polytopes and polytopal surfaces in $\mathbb{H}^3$ invariant under the action of $\Gamma$, which is the image of a Fuchsian representation of $\pi_1(S_g)$.

2.4 Epstein–Penner decompositions

We need to remind the concept of Epstein-Penner ideal polygonal decomposition of a decorated cusped hyperbolic surface $S_{g,n}$.

Fix a decoration of $S_{g,n}$, i.e. a horosphere at every cusp. Then, the space of all decorations of $S_{g,n}$ can be identified with $\mathbb{R}^n$. A point $r \in \mathbb{R}^n$ corresponds to the choice of horospheres at the distances $r_1, \ldots, r_n$ from the fixed ones.

Consider the hyperboloid model of $\mathbb{H}^2$. Develop $S_{g,n}$ as $\mathbb{H}^2/\Gamma$ in $\mathbb{H}^2 \subset \mathbb{R}^{1,2}$, where $\Gamma$ is a discrete subgroup of $\text{Iso}^+(\mathbb{H}^2)$ isomorphic to $\pi_1(S_g)$. Take $r \in \mathbb{R}^n$ and the corresponding decoration. By $L_1^i, L_2^i, \ldots$ denote the horoballs in the orbit of the horoball at $i$-th cusp under the action of $\Gamma$. As in Section 2 we
denote by $L^k_i$ also the affine plane such that the corresponded horoball is the intersection of $\mathbb{H}^2$ and this plane. Also, recall that $\tilde{L}^k_i$ denotes the polar vector to the plane $L^k_i$. By $\mathcal{L}$ denote the union of all vectors $\tilde{L}^k_i$.

Let $C$ be the convex hull of the set $\{L^k_i\}$. Its boundary $\partial C$ is divided in two parts $\partial_l C \sqcup \partial_t C$ consisting of lightlike points and timelike points. Below we describe well-known properties of this construction. For proofs we refer to [13].

Lemma 2.11.

- The convex hull $C$ is 3-dimensional.
- The set $C \cap \mathcal{L}$ is the set of points $\alpha L^k_i$ for some $\alpha \geq 1$.
- Every time-like ray intersects $\partial_t C$ exactly once.
- The boundary $\partial_t C$ is decomposed into countably many Euclidean polygons. This decomposition is $\Gamma$-invariant and projects to a $\Gamma$-invariant decomposition of $\mathbb{H}^2$ which provides a decomposition of $S_{g,n}$ into finitely many ideal polygons.

Definition 2.12. The Epstein–Penner decomposition is the decomposition of decorated $S_{g,n}$ obtained in the above way.

This decomposition depends on a choice of decorations.

Definition 2.13. An Epstein–Penner triangulation is a triangulation that refines the Epstein–Penner decomposition.

The space $\mathbb{R}^n$ is subdivided into cells corresponding to Epstein–Penner decompositions. Each $n$-dimensional cell corresponds to a decomposition that is a triangulation. Epstein–Penner triangulations and cells are well studied, see [14], [15].

3 Prisms and complexes

In this section we are going to introduce our main objects of study: prismatic complexes. They are metric spaces glued from basic building blocks. Hence, first we define these blocks and study their properties.
3.1 Prisms and their properties

Definition 3.1. A rectangular prism is the convex hull of a triangle \( A_1A_2A_3 \) (with possibly ideal vertices) and its orthogonal projection to a plane such that \( A_1A_2A_3 \) does not intersect this plane.

By \( B_1, B_2 \) and \( B_3 \) denote the images of \( A_1, A_2 \) and \( A_3 \) under the projection. Such a prism has nine edges. We call the edges \( A_1A_2, A_2A_3 \) and \( A_3A_1 \) upper edges, the edges \( B_1B_2, B_2B_3 \) and \( B_3B_1 \) lower edges and edges \( A_1B_1, A_2B_2 \) and \( A_3B_3 \) lateral edges. In the same way, we call the face \( A_1A_2A_3 \) the upper face, the face \( B_1B_2B_3 \) the lower face and other faces the lateral faces.

The dihedral angles of edges \( B_1B_2, B_2B_3 \) and \( B_3B_1 \) are equal \( \pi/2 \). The dihedral angles \( A_1A_2, A_2A_3 \) and \( A_3A_1 \) are denoted by \( \phi_3, \phi_1 \) and \( \phi_2 \) respectively. The dihedral angles \( A_1B_1, A_2B_2 \) and \( A_3B_3 \) are denoted by \( \omega_1, \omega_2 \) and \( \omega_3 \). Note that the points \( A_i \) may be ideal, but \( B_i \) may not. An important special case is a singular rectangular prism, when the points \( B_1, B_2 \) and \( B_3 \) are collinear. For the sake of brevity, until the end of the article we will use the word prism instead of rectangular prism.

Some of the points \( A_i \) can be ideal. In this case as we agreed in Section 2 we will always suppose that they are equipped with horospheres. Then the lengths of upper edges and lateral edges are defined as in Section 2. Mainly we will deal with such prisms that all \( A_i \) are ideal.

Definition 3.2. A semi-ideal prism (see Figure 2) is a prism with an ideal upper face.

It is clear that in a semi-ideal prism the lines \( A_1A_2, A_2A_3 \) and \( A_3A_1 \) are ultraparallel to the plane \( B_1B_2B_3 \).

Lemma 3.3. Let \( A_1A_2A_3 \subset \mathbb{H}^2 \) be a triangle and \( r_1, r_2, r_3 \) be three real numbers. If \( A_i \in \mathbb{H}^3 \), we assume \( r_i > 0 \). If \( A_i \) is an ideal point, then it is equipped with a horosphere. Then there exists at most one prism up to isometry such that its upper face is isometric to \( A_1A_2A_3 \) and the lengths of the corresponding lateral edges are equal to \( r_i \).

Proof. Consider \( A_1A_2A_3 \) embedded in \( \mathbb{H}^3 \) in the hyperboloid model. Suppose that the plane \( A_1A_2A_3 \) is the plane \( \{ \tau: x_4 = 0 \} = \mathbb{R}^{1,2} \). If \( A_i \) is a non-ideal point, then also denote it by \( \tau_i \in \mathbb{R}^{1,2} \). It is a time-like vector.

If \( A_i \) is ideal, then it is equipped with a horosphere \( L_i \). As in Section 2 denote by \( l_i \) its polar dual.
We need to find three points $B_1$, $B_2$ and $B_3$ such that $A_1A_2A_3B_3B_2B_1$ will be a prism with lateral edges equal to $r_1$, $r_2$ and $r_3$. Equivalently, we need to find a 2-plane $M$ such that $\text{dist}(A_i, M) = r_i$ (the distance from an ideal vertex to hyperplane is naturally the distance from corresponding horosphere).

By Lemma 2.1 we can see that the existence of the desired prism is equivalent to the existence of a vector $\overline{m} \in dS^3$ such that $\langle m, l_i \rangle = e^{r_i}$ for every ideal $A_i$ and $\langle m, \overline{a}_i \rangle = \sinh(r_i)$ otherwise. For ideal points we take plus signs because we can look only for such $\overline{m}$ that $A_1A_2A_3 \subset M_+$.

Every $\overline{a}_i$ and $\overline{l}_i$ is in $\mathbb{R}^{1,2}$. Therefore, it has the last coordinate equal to zero. Hence, the system of equations is a linear system for the first three coordinates. The vectors $\overline{a}_i$ are linearly independent which implies that this system has a unique solution. Now we need to find the last coordinate for $\overline{m}$ using the equation

$$\langle \overline{m}, \overline{m} \rangle = 1.$$  

This is a quadratic equation. It is clear that if it has only one solution, then we obtain a singular prism, and if it has two solutions, then we have two prisms corresponding to two possible choices of orientation (and differing by
Figure 3: A semi-ideal trapezoid.

the symmetry with respect to the plane $A_1A_2A_3$.

A lateral face of a prism is a special hyperbolic quadrilateral, which we naturally call a trapezoid.

**Definition 3.4.** A trapezoid is the convex hull of a segment $A_1A_2 \subset \mathbb{H}^2$ and its orthogonal projection to a line such that $A_1A_2$ does not intersect this line.

By $B_1$ and $B_2$ denote the images of $A_1$ and $A_2$ under the projection. The notions of lateral, upper and lower edges are similar. We need also the following planar version of Lemma 3.3.

**Lemma 3.5.** Let $A_1A_2 \subset \mathbb{H}^2$ be a segment, $r_1$ and $r_2$ be two real numbers. If $A_i$ is an ideal point, then we suppose that it is equipped with a horocycle. If $A_i \in \mathbb{H}^2$, we assume that $r_i > 0$. Then there exists at most one trapezoid up to isometry such that its upper edge is isometric to $A_1A_2$ and the lengths of the corresponding lateral edges are equal to $r_i$. (See Figure 3)

The proof goes along the lines of the proof of Lemma 3.3 and is quite straightforward, hence, we omit it.

Now let us establish several formulas for trapezoids.

**Lemma 3.6.** Let $A_1A_2B_2B_1$ be a trapezoid (all vertices are finite, $A_1B_1$ and $A_2B_2$ are orthogonal to the segment $B_1B_2$), $l_{12}$ be the length of the upper edge, $a_{12}$ be the length of the lower edge and $r_1$, $r_2$ be the lengths of lateral edges. Then

$$\cosh(a_{12}) = \frac{\sinh(r_1) \sinh(r_2) + \cosh(l_{12})}{\cosh(r_1) \cosh(r_2)}.$$
Proof. Direct computation using the cosine law for a de-Sitter triangle. □

**Lemma 3.7.** Let $A_1A_2B_2B_1$ be a semi-ideal trapezoid, $l_{12}$ be the length of the upper edge, $a_{12}$ be the length of the lower edge and $r_1$, $r_2$ be the lengths of lateral edges. Then

$$\cosh(a_{12}) = 1 + 2e^{l_{12}-r_1-r_2}.$$  

Proof. Assume that $B_1$ and $B_2$ are outside of horospheres at $A_1$ and $A_2$ and these horospheres are disjoint. Consider a sequence of points $\{A_1(i)\} \subset A_1B_1$ tending to $A_1$ and a sequence of points $\{A_2(i)\} \subset A_2B_2$ tending to $A_2$. Let $C_1$ and $C_2$ be the intersection points of $A_1B_1$ and $A_2B_2$ with horospheres at $A_1$ and $A_2$ respectively and $D_{1(i)}$ and $D_{2(i)}$ be the intersection points of $A_1(i)A_2(i)$ with these horospheres. Observe that $|A_1(i)C_1| \to 1$ and $|A_2(i)C_2| \to 1$ as $i$ grows, $|C_1B_1| = r_1$, $|C_2B_2| = r_2$ and $|D_{1(i)}D_{2(i)}|$ tends to $l_{12}$. Using it and the fact that $e^{-x} \to 1$ as $x$ grows we can see that our formula follows from Lemma 3.6 as a limiting case. □

**Lemma 3.8.** Let $A_1A_2B_2B_1$ be an ideal trapezoid, $l_{12}$ be the length of the upper edge and $r_1$, $r_2$ be the lengths of lateral edges. Denote by $\alpha_{12}$ the length of the part of the horocycle centered at $A_1$ that is inside this trapezoid. Then

$$\alpha_{12}^2 = e^{r_2-r_1-l_{12}} + e^{-2r_1}.$$  

Proof. A direct application of Lemma 3.7 and Lemma 2.4 □

**Lemma 3.9.** Let $A_1A_2B_2B_1$ be an ideal trapezoid, $a_{12}$ be the length of the lower edge and $\rho_{12}$ be the distance from the upper edge to the lower edge. Then

$$\cosh(a_{12}) = 1 + \frac{2}{\sinh^2(\rho_{12})}.$$  

Proof. Direct computation. □

### 3.2 Prismatic complexes

Let $S_{g,n}$ be a surface of genus $g$ with $n$ punctures equipped with a complete hyperbolic cusp-metric $d$. Consider a decoration of $S_{g,n}$. Let $T$ be an ideal geodesic triangulation of $S_{g,n}$. By $E(T)$ and $F(T)$ denote its sets of edges and faces respectively. Then $d$ can be fully described in Penner coordinates:
the lengths of decorated ideal edges of $T$. Cusps are denoted by $A_1, \ldots, A_n$. Note that $E(T)$ may contain loops and multiple edges. It is also possible that is some triangles there are edges glued together. But without loss of generality, when we consider a particular triangle (or a pair of distinctive adjacent triangles), we will denote it as $A_iA_jA_h$ ($A_iA_jA_h$ and $A_jA_hA_g$).

Suppose that to every cusp $A_i$ some real weight $r_i$ is assigned. Denote the weight vector by $r \in \mathbb{R}^n$.

**Definition 3.10.** A pair $(T, r)$ is called admissible if for every decorated ideal triangle $A_iA_jA_h \in F(T)$ there exists a semi-ideal prism with the lengths of lateral edges $A_iB_i$, $A_jB_j$, $A_hB_h$ equal to $r_i$, $r_j$ and $r_h$.

Let $(T, r)$ be an admissible pair. For each ideal triangle $A_iA_jA_h \in F(T)$ consider a prism from the last definition.

**Definition 3.11.** A prismatic complex $K(T, r)$ is a metric space obtained by identifying all these prisms via isometries of lateral faces: if two triangles of $T$ have a common edge $A_iA_j$, then we isometrically identify the faces $A_iA_jB_jB_i$ of corresponded prisms.

This definition is correct because of Lemma 3.5. Note that the decorations at ideal vertices are identified with decorations. For the sake of brevity, we will omit the word prismatic. We also write $K$ instead of $K(T, r)$ when it does not bring an ambiguity.

Every prismatic complex is a complete cone-manifold with polyhedral boundary. The boundary consists of two components. The union of upper faces forms the upper boundary isometric to $S_{g,n}$ with $d$. The union of lower faces forms the lower boundary which is isometric to $S_g$ equipped with a hyperbolic metric with cone-singularities at points $B_i$. We can consider $T$ as a triangulation of both components. There are well-defined total dihedral angles of edges of triangulations $A_iA_j$ and $B_iB_j$ equal to the sum of corresponding dihedral edges in both glued prisms. In the same way we can define the total dihedral angle of every inner edge $A_iB_i$ as the sum of corresponding dihedral angles of all prisms containing $A_iB_i$.

**Definition 3.12.** A prismatic complex $K$ is called convex if for every upper edge $A_iA_j \in E(T)$ its dihedral angle is at most $\pi$. If $K = K(T, r)$, then the pair $(T, r)$ is also called convex.
Note that in every prism either the plane containing the upper face intersects the plane containing the lower face, or they are asymptotically parallel, or they are ultraparallel. The following lemma will be important.

**Lemma 3.13.** Let $K$ be a convex prismatic complex. Then for every prism $A_iA_jA_hB_hB_jB_i \subset K$ its upper face is ultraparallel to its lower face.

**Proof.** Embed the prism $A_iA_jA_hB_hB_jB_i \subset K$ in $\mathbb{H}^3$. First, we will show that the plane $A_iA_jA_h$ (denote it by $M_1$) can not intersect the plane $B_iB_jB_h$ (denote it by $M_2$).

Assume the contrary. Let these two planes intersect and $l$ is the line of intersection.

The intersection of $M_1$ with $\partial_\infty \mathbb{H}^3$ is a circle. The line $l$ divides it into two arcs. All points $A_i, A_j$ and $A_h$ belong to the same arc and one of them lies between the two others. Suppose that this point is $A_i$. Then we call the edge $A_jA_h$ “heavy” and two other edges “light” (see Figure 4).

Let $\chi$ be the dihedral angle between $M_1$ and $M_2$. For every $\overline{x} \in M_1$, we have

$$\sinh \text{dist}(\overline{x}, M_2) = \sinh \text{dist}(\overline{x}, l) \sin(\chi),$$

by the law of sines in a right-angled hyperbolic triangle.
It follows that the distances from the light edges to \(M_2\) are both strictly bigger than the distance from the heavy edge.

For the dihedral angles of the upper edges we have \(\phi_i > \pi/2\) and \(\phi_j, \phi_h < \pi/2\).

Indeed, let \(x \in A_jA_h\) be the nearest point from this edge to \(M_2\); \(x' \in M_2\) and \(x'' \in l\) be the bases of perpendiculars from \(x\) to \(M_2\) and \(l\). Then \(\angle xx'x'' = \pi/2\), \(\angle x'xx'' < \pi/2\) and \(\phi_i = \pi - \angle x'xx'' > \pi/2\). Next, we can consider the ideal vertex \(A_j\). Using that the sum of three dihedral angles at one vertex is equal \(\pi\) we obtain

\[
\omega_j + \phi_i + \phi_h = \pi.
\]

It implies that \(\phi_h < \pi/2\). Similarly, \(\phi_j < \pi/2\).

Edge \(A_jA_h\) can not be glued in \(T\) neither with the edge \(A_iA_j\) nor with \(A_iA_h\) because these edges have different distances to the lower face. Therefore, there is another triangle \(A_jA_hA_g \in T\) containing \(A_jA_h\). Embed the corresponding prism \(A_jA_hA_gB_gB_hB_j \subseteq K\) in \(\mathbb{H}^3\) in such a way that it is glued with the former prism over the face \(A_jA_hB_hB_j\) via an orientation-reversing isometry. Then \(B_j\) belongs to the plane \(B_iB_jB_h\).

The total dihedral angle at \(A_jA_h\) is less or equal than \(\pi\). Since that, the line \(l\) and the triangle \(B_jB_hB_g\) lie on opposite sides with respect to the plane \(A_jA_hA_g\) and hence the plane \(A_jA_hA_g\) also intersects \(M_2\). Therefore, the light edges and the heavy edge are well-defined for the new prism. Moreover, it is clear that in this prism \(A_jA_h\) is light. Hence, we can see that the distance from the new heavy edge to \(M_2\) is strictly less than the distance from \(A_jA_h\). Now for this edge we can choose the next prism containing it and continue in the same way. The distances from the heavy edges to \(M_2\) are strictly decreasing. But the number of edges in \(K\) is finite. We get a contradiction.

Now we need to consider the case when the upper face is asymptotically parallel to the lower face. The proof is very similar. We also have one heavy edge, two light edges and all other details remain the same.

\[
\square
\]

**Corollary 3.14.** Let \(K\) be a convex prismatic complex. Then all prisms in \(K\) are non-singular.

Assume that for some complex \(K\) all total dihedral angles at inner edges \(A_iB_i\) equal to \(2\pi\). Double this complex and glue two copies together along their lower boundaries. We obtain a complete hyperbolic manifold \(P\) with
convex ideal polyhedral boundary. Each component of the boundary is isometric to \((S_{g,n}, d)\), there is a canonical embedding of \(S_g\) in \(P\) and an isometric involution permuting components of the boundary and preserving the canonical embedding. We call it the canonical surface of \(P\). This is precisely the definition of an ideal Fuchsian polytope in intrinsic terms as discussed in Subsection 2.3. Now we are ready to prove Lemma 2.9 from that subsection.

**Proof.** The manifold \(P\) can be decomposed into semi-ideal prisms according to the polyhedral structure of its boundary. The canonical surface \(S_g\) in \(P\) inherits the induced complete hyperbolic metric \(d'\) on \(S_g\) and a triangulation \(T\) on it. There is a unique up to isometry Fuchsian manifold \(F\) such that \((S_g, d')\) is isometric to the convex core of \(F\). The canonical surface of \(P\) is isometric to the convex core of \(F\). Consider such an isometry. It induces the triangulation \(T\) on the convex core of \(F\). Let \(B_1, \ldots, B_n\) be its vertices. There is a unique line passing through every point \(B_i\) orthogonal to the convex core. Let \(A_1, \ldots, A_n\) be the intersections of these lines with one of the components of the boundary at infinity. Choose a horosphere at \(A_i\) such that the distance between this horosphere and the convex core is equal to \(r_i\) - the respective distance in \(P\). Now to finish the proof we need to note that every semi-ideal prism is uniquely defined by its lower boundary and the lengths of lateral edges. We omit the proof of this claim as it follows the proof of Lemma 3.3 line by line.

This lemma shows that to prove Theorem 1.3 it is enough to show that there exists a complex \(K\) with the total dihedral angle of every inner edge \(A_iB_i\) equal to \(2\pi\).

### 4 The space of convex complexes

Denote by \(\mathcal{K}\) the set of all convex prismatic complexes \(K\) up to isometry such that the upper boundary of \(K\) is isometric to \((S_{g,n}, d)\). In this section we are going to completely describe it.

Every \(K \in \mathcal{K}\) can be represented as \(K(T, r)\). Clearly, if \(K' = K(T', r')\), \(K'' = K(T'', r'')\) and \(r' \neq r''\), then complexes \(K'\) and \(K''\) are not isometric. This defines a map which we denote by \(r : \mathcal{K} \to \mathbb{R}^n\) abusing the notation.

The plan of this section as follows. In Subsection 4.1 we prove

**Lemma 4.1.** Let \((T', r)\) and \((T'', r)\) be two convex pairs. Then the corresponding complexes \(K'\) and \(K''\) are isometric.

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Corollary 4.2. The map \( r : K \to \mathbb{R}^n \) is injective.

Hence, \( K \) can be identified with a subset of \( \mathbb{R}^n \). In Subsection 4.2 we show that

Lemma 4.3. The image \( r(K) = \mathbb{R}^n \).

4.1 The proof of Lemma 4.1

First, we need to introduce some machinery.

For every convex prismatic complex \( K = (T, r) \) we can define a function \( \rho_K : S_{g,n} \to \mathbb{R}_{>0} \).

Here \( S_{g,n} \) is identified with the upper boundary of \( K \). For \( x \in A_iA_jA_k \) \((A_iA_jA_k \in F(T))\), let \( \rho(x) \) be the distance from \( x \) to the plane containing the lower face of the prism \( A_iA_jA_kB_kB_jB_i \) after some (and any) embedding in \( \mathbb{H}^3 \).

Definition 4.4. This function is called the distance function of \( K \).

We omit the subscript \( K \) where it is redundant.

It follows from Lemma 2.2 that for any \( A_iA_jA_h \in F(T) \) if we embed the corresponding prism in \( \mathbb{H}^3 \), then for some \( b \in \mathbb{R} \) and a point \( \pi \) from the plane \( A_iA_jA_h \) the restriction of \( \rho \) to this triangle has the form

\[
\rho(x) = \arcsinh(b \cosh(\text{dist}(\pi, \pi)))
\]  

(1)

Moreover, let \( s : [x^0; x^1] \to S_{g,n} \) be a geodesic segment with an arc-length parametrization such that its image is contained in \( A_iA_jA_h \). Then \( \rho \circ s \) has a form \( \arcsinh(b \cosh(x - a)) \) for some real numbers \( a \) and \( b \).

Now consider any geodesic segment \( s : [x^0; x^1] \to S_{g,b} \) with natural parametrization that is transversal to every edge of \( T \). Let

\[
[x^0; x^1] = [x_0; x_1] \cup \ldots \cup [x_{k-1}; x_k]
\]

\((x^0 = x_0 \text{ and } x_k = x^1)\) be a subdivision induced by intersections with the edges of \( T \). The restriction of \( \rho \circ s \) to \([x_l; x_{l+1}]\) is

\[
\arcsinh(b_l \cosh(x - a_l)).
\]  

(2)
Figure 5: Graphics of distance and piecewise distance functions in the projection to a geodesic.

The points $x_l$ corresponding to strictly convex edges of $T$ are singular points of $\rho \circ s$ in the sense that $\rho \circ s$ is not differentiable at these points, but both the left derivative and the right derivative exist. It is clear that convexity of $K$ means that at every singular point $x_l$ the left derivative of $\rho \circ s$ is greater or equal than the right derivative.

**Lemma 4.5.** Let $\rho(x) = \text{arcsinh}(b \cosh(x - a))$ be a function $[x^0; x^1] \to \mathbb{R}$. Let $\tilde{\rho}$ be a function such that

(i) $\rho(x^0) = \tilde{\rho}(x^0); \rho(x^1) = \tilde{\rho}(x^1);

(ii) there is a subdivision $[x^0; x^1] = [x_0; x_1] \cup \ldots \cup [x_k; x_{k+1}]$ ($x_0 = x_0$ and $x_{k+1} = x^1$) such that the restriction of $\tilde{\rho}$ to $[x_l; x_{l+1}]$ is equal to $\text{arcsinh}(b_l \cosh(x - a_l));$

(iii) at every point $x_l$ the left derivative of $\tilde{\rho}(x)$ is greater or equal than the right derivative.

Then for all $x \in [x^0; x^1]$, $\tilde{\rho}(x) \geq \rho(x)$ (See Figure 5).

**Proof.** We can assume without loss of generality that at $[x_0; x_1]$ and $[x_k; x_{k+1}]$ $\rho(x)$ and $\tilde{\rho}$ do not coincide and for every $l$ the pairs $(a_l; b_l)$ and $(a_{l+1}; b_{l+1})$ are different. Under these assumptions we will prove a stronger statement: for every $x \in (x^0; x^1)$, $\rho(x) > \tilde{\rho}(x)$.
Claim 1: if \( a_1 \neq a_2 \), then the function \( \frac{\cosh(x-a_1)}{\cosh(x-a_2)} \) is strictly monotone.

Indeed, just compute its derivative.

Claim 2: if the pairs \((a_1; b_1)\) and \((a_2; b_2)\) are different, then the functions \( \arcsinh(b_1 \cosh(x-a_1)) \) and \( \arcsinh(b_2 \cosh(x-a_2)) \) can not coincide at more than one point.

It is a direct implication of Claim 1.

Claim 3: let \( f_l(x) := \arcsinh(b_l \cosh(x-a_l)) \) be the extensions to \( \mathbb{R} \) of the restrictions of \( \tilde{\rho} \). If \( l < m \), then for all \( x > x_l \) we have \( f_l(x) > f_m(x) \).

Induction over \( m - l \).

The base case \( m = l + 1 \) follows from Claim 2 and the fact that in the singular point \( x_{l+1} \) the derivative of \( f_l \) is greater or equal than the derivative of \( f_{l+1} \).

The inductive step is obvious.

Claim 4: for all \( x \in (x_0; x_1] \) we have \( \tilde{\rho}(x) \geq \rho(x) \).

Claim 2 and the assumption \( \rho(x^0) = \tilde{\rho}(x^0) \) imply that the sign of the difference \( \tilde{\rho}(x) - \rho(x) \) is constant on \( (x_0; x_1] \). Assume that it is negative. Then for every \( x > x^0 \) we have \( f_0(x) < \rho(x) \). Substituting \( x = x^1 \) and using Claim 3 we obtain that \( \tilde{\rho}(x^1) = f_k(x^1) < f_0(x^1) = \rho(x^1) \) which contradicts the statement.

Claim 5: Suppose that for some \( x' \in (x^0; x^1) \) we have \( \tilde{\rho}(x') = \rho(x') \). Clearly for all \( x > x' \) we have \( \tilde{\rho}(x) < \rho(x) \). Again, we obtain a contradiction.

It implies that the difference \( \tilde{\rho}(x) - \rho(x) \) is strictly positive over the interval \( (x^0; x^1) \).

\boxed{\square}

Now we can prove Lemma 4.1.

**Proof.** Let \( A \) be the point of intersection of an edge \( e' \) of \( T' \) with an edge \( e'' \) of \( T'' \). The edge \( e'' \) is a geodesic in \( (S_g, d) \), we can consider it in the upper
boundary of $K'$ and look at the restriction of the distance function $\rho'$ of the complex $K'$ at $e''$. Since $K'$ is convex, $A$ is a singular point for this function and the left derivative is greater or equal than the right derivative. Consider also the restriction of the distance function $\rho''$ of the complex $K''$. It has the form $[2]$. From Lemma 4.5 we infer that that $\rho''(A) \geq \rho'(A)$, where $A$.

Similarly, we obtain that $\rho'(A) \geq \rho''(A)$. Therefore, $\rho'(A) = \rho''(A)$.

Let $A$ be the union of the set of all cusps and of the set of all intersection points of the edges of $T'$ with the edges of $T''$. Edges of $T \cup T'$ decompose $S_{g,n}$ into simply-connected geodesic polygons. We subdivide every polygon into geodesic triangles and obtain a triangulation $T$ with the vertex set $A$ such that $T$ refines both $T'$ and $T''$. This triangulation induces a subdivision of both $K'$ and $K''$ into prisms. Two corresponding prisms are isometric because of Lemma 3.3. It follows that $K'$ is isometric to $K''$.

\[\square\]

We would like to note one easy corollary. For a convex pair $(T, r)$ denote by $E_s(T, r)$ the union of all strictly convex edges of the corresponding complex $K$.

**Corollary 4.6.** If $(T, r)$ and $(T', r)$ are two convex pairs then $E_s(T, r) = E_s(T', r)$. Hence, we can denote it by $E_s(r)$.

It means that two different triangulations of the same convex complex may be different only in “flat” edges.

Let $(T, r)$ be a convex pair and $K$ be the corresponding convex prismatic complex.

**Definition 4.7.** For a triangle $\Delta \in F(T)$, the face of $K$ containing $\Delta$ is the union of all triangles $\Delta'$ such that there exists a path from an interior point of $\Delta$ to an interior point of $\Delta'$ that intersects only edges of $T$ with dihedral angles equal to $\pi$.

Clearly, for two triangles the relation “to be in one face” is an equivalence relation. Hence, we obtain a decomposition of the upper boundary of $K$ into faces.

A face $\Pi$ may not be simply-connected. Below we prove that in this case some inner edges of $\Pi$ are strictly convex and if we delete the union of all such edges, then we obtain a simply-connected set (see Figure 6).

**Definition 4.8.** An open face of $K$ is a face minus $E_s(r)$.
Figure 6: An example of non simply-connected face. The dotted edges are flat. If we delete other inner edges, we obtain a simply-connected set.

An alternative analytic definition: an open face is a maximal connected open set such that the distance function \( \rho_K \) is of the form (2) over it.

**Lemma 4.9.** Every open face is a simply-connected set.

*Proof.* Let \( \Pi \) be an open face. We prove that if \( \Pi \) is not simply-connected, then there is a closed geodesic in \( \Pi \) that does not intersect strictly convex edges.

Consider a simple homotopically nontrivial closed curve \( \psi \) in \( \Pi \) that does not intersect strictly convex edges and is transversal to every edge. Lift \( \psi \) to \( \mathbb{H}^2 \) and develop all triangles of \( T \) that intersect \( \psi \). We obtain an ideal polygon \( P \). Some edges of \( P \) are glued. The triangulation \( T \) is lifted to a triangulation of \( P \). All inner edges \( P \) are lifts of flat edges of \( \Pi \) (the dihedral angles are equal to \( \pi \)).

Let \( \tau : P \to \Pi \) be a projection. Suppose that \( AB \) and \( CD \) are two glued ideal edges: \( \tau(AB) = \tau(CD), \tau(A) = \tau(C) \) and \( \tau(B) = D \) (note that \( A \) may coincide with \( C \) and \( B \) may coincide with \( D \)). We remind that ideal points are decorated and the decoration defines the gluing map between \( AB \) and \( CD \). For a point \( X \in AB \) there is \( Y \in CD \) such that \( \tau(Y) = \tau(X) \). A hyperbolic segment \( XY \) corresponds to a geodesic loop in \( \Pi \). It can have a singular point at \( \tau(X) = \tau(Y) \). Clearly, \( \tau(XY) \) is a closed geodesic if and only if \( \angle BXY + \angle XYD = \pi \). It is clear that as \( X \) tends to \( B \), the point \( Y \) tends to \( D \) and this sum tends to \( 2\pi \). Similarly, as \( X \) tends to \( C \), this sum tends to 0. Therefore, for some \( X \) this sum will be equal to \( \pi \). In this case
τ(XY) is a closed geodesic ψ′ ⊂ Π. It intersects only edges of T that were lifted to inner edges of P. Therefore, it does not intersect any strictly convex edges.

Consider a distance function ρK. Its restriction to ψ′ must be periodic, because ψ′ is closed. On the other hand ψ′ intersects no strictly convex edges. Therefore, the restriction of ρ to ψ′ has a form (2) which is not periodic. We obtain a contradiction. □

Lemma 4.9 implies that for every face there are at most finitely many triangulations of this face containing all strictly convex edges.

**Corollary 4.10.** For every r ∈ ℝn there are finitely many triangulations T such that the pair (T, r) is convex.

More precisely, we have the following.

**Corollary 4.11.** If r ∈ ℝn is such that at least one convex complex (T, r) exists, then r determines a decomposition of Sg,n into faces. Every such T can be obtained as a refinement of this decomposition.

For a triangulation T denote by K(T) ⊂ ℝn the set of all r ∈ ℝn such that the pair (T, r) is convex. This defines a subdivision of K into cells corresponding to different triangulations. An inner points of a cell K(T) to r such that the decomposition in Corollary 4.11 is the triangulation T itself. Boundary points of K(T) have the property that there are ideal polygons in this decomposition that are not triangles.

### 4.2 Proof of Lemma 4.3

We prove the following

**Lemma 4.12.** The pair (T, r) is convex if and only if T is an Epstein–Penner triangulation for r ∈ ℝn.

Clearly, this lemma implies Lemma 4.3. Moreover, the decomposition described in Corollary 4.11 is exactly the Epstein–Penner decomposition for r and the subdivision K = ⋃K(T) is the Epstein–Penner subdivision of ℝn.

**Proof.** Let r ∈ ℝn and let T be one of its Epstein–Penner triangulations. Further, consider Sg,n as ℍ2/Γ and let ˜T be the lift of T to a triangulation of ℍ2 and ˜Δ = A_iA_jA_h ∈ F(˜T) be a triangle of T. Let L_i, L_j and L_h be the
horospheres at $A_i$, $A_j$ and $A_h$ (we omit upper indices as redundant). The
affine plane $M = M(\tilde{\Delta})$ spanned by the points $\tilde{l}_i$, $\tilde{l}_j$ and $\tilde{l}_h$ is a supporting
plane of $C$. Clearly, $M$ is spacelike which means that its normal $\tilde{m}$ (in the
direction of $C$) is timelike. Let $L_M = M \cap \mathcal{L}$. By the construction of the
convex hull, for $\tilde{l} \in \mathcal{L}$ we have

$$\langle \tilde{m}, \tilde{l} \rangle = \begin{cases} -1 & \text{if } \tilde{l} \in L_M, \\ < -1 & \text{otherwise}. \end{cases}$$

Now assume that $\mathbb{R}^{1,2} \hookrightarrow \mathbb{R}^{1,3}$ as $\{ \overline{x} \in \mathbb{R}^{1,3} : x_4 = 0 \}$. Let $\overline{m}'$ be the
intersection of $dS^3$ with the ray $\overline{m} + \lambda e_4, \lambda > 0$.

By construction we have

$$\langle \overline{m}', \tilde{l} \rangle = \begin{cases} -1 & \text{if } \tilde{l} \in L_M, \\ < -1, & \text{otherwise}. \end{cases}$$

Let $M' \subset \mathbb{H}^3$ be the plane orthogonal to $\overline{m}'$. From Lemma 2.1 we see that $M$ is tangent to the horosphere $L$ if and only if $\tilde{l} \in L_M$. Moreover, if $\tilde{l} \notin L_M$, then the horosphere $L$ does not intersect $M$ and lies in the same halfspace as other horospheres $L$ for $\tilde{l} \notin L_M$. We proved the following description of the Epstein–Penner decomposition (we proved only in one direction, but the converse direction is clear).

**Lemma 4.13.** A triangle $A_iA_jA_h$ is contained in a face of the Epstein–
Penner decomposition if and only if all decorating horospheres are on one
side from the common tangent plane to the horospheres $L_i$, $L_j$ and $L_h$.

By $B_i$, $B_j$ and $B_h$ denote the tangent points of $M'$ with $L_i$, $L_j$ and $L_h$
respectively. We see that the prism $A_iA_jA_hB_hB_jB_i$ is a semi-ideal prism
with lateral edges $r_i$, $r_j$ and $r_h$. It follows that the pair $(T, \mathbf{r})$ is possible.
From this construction we obtain the prismatic complex $K$.

Now we should check the convexity. Take two adjacent triangles $\Delta' = A_iA_jA_h$, $\Delta'' = A_jA_hA_g \in F(\overline{T})$ and corresponding semi-ideal prisms. Geometrically, it is clear that $\phi_i + \phi_g \leq \pi/2$. Indeed, we need to bend this two prisms around the edge $A_jA_h$. Lemma 4.13 implies that we bend in the right
direction and will obtain angle less or equal than $\pi$ in the end. Below we
give more rigorous analytical proof of this statement.

Clearly, $\tilde{l}_g \in L_M(\tilde{\Delta}_i)$ if and only if the plane $M(\tilde{\Delta}_1)$ coincides with the
plane $M(\tilde{\Delta}_2)$ which is equivalent to the condition $\phi_i + \phi_g = \pi/2$ (edge $A_jA_h$
is “flat”). Also, $\phi_i + \phi_g < \pi/2$ is equivalent to the condition that for some (and hence for every) geodesic $\psi$ intersecting $A_jA_h$ transversely at a point $X$, the left derivative of $\rho_K(X)|_\psi$ is strictly greater than the right.

Assume that it $\tilde{f}_g \notin \mathcal{L}_{M(\Delta_1)}$. Take $\psi := A_iA_g$ and parametrize it by length over $\mathbb{R}$. Let $X = A_jA_h \cap A_iA_g$ and $x \in \mathbb{R}$ be its coordinate. The distance from $X$ to $M(\Delta_1)$ is equal to the distance from $X$ to $M(\Delta_2)$ (see Figure 7). The distance function $\rho_K|_\psi$ is equal to $f_1(x') = \operatorname{arcsinh}(b_1 \cosh(x' - a_1))$ for $x' \leq x$ and $f_2(x') = \operatorname{arcsinh}(b_2 \cosh(x' - a_2))$ for $x' \geq x$. The point $x$ is a kink point of $\rho_K|_\psi$, hence $(a_1, b_1) \neq (a_2, b_2)$ and by Claim 2 of Lemma 4.5 the sign of $f_1(x') - f_2(x')$ is constant over the halfrays $(-\infty, x)$ and $(x, +\infty)$. Consider $x'$ tending to $+\infty$. Take a sphere $S(x')$ centered at the corresponded point $X' \in XA_g$ tangent to the plane $B_jB_hB_g$. The tangent point tends to $B_g$ and the sphere tends to the horosphere centered at $A_g$. This horosphere does not intersect the plane $B_iB_hB_h$ hence for some sufficiently large $x'$, $S(x')$ does not intersect $B_iB_jB_h$ and in turn it implies that the distance from $X'$ to $B_iB_jB_h$ is greater than the distance to $B_iB_hB_g$. It implies that $f_1(x') - f_2(x') > 0$ over $(x, +\infty)$ and the left derivative of $\rho_K(x)|_\psi$ is greater than the right. Therefore, $\phi_i + \phi_g < \pi/2$.

We proved the “if” part. If $T$ is an Epstein–Penner triangulation for $r$, $K$ is the corresponding complex and it can be represented as $(T', r)$, then according to Corollary 4.6 it can be different only in flat edges. By Lemma 4.9 faces of $K$ are simply-connected ideal polygons, hence, $T$ and $T'$ can be connected with a sequence of flips of flat edges. If $T_k$ and $T_{k+1}$ are two consequent triangulations in this sequence and $T_k$ is an Epstein–Penner triangulation, then $T_{k+1}$ also is another Epstein–Penner triangulation. Indeed, for $T_k$ we consider the construction as in proof of the “if” part. Let $\Delta_1 = A_iA_jA_h$ and $\Delta_2 = A_jA_hA_g$ are the triangles, where the flip were made. It was made over a flat edge, therefore, four points $B_i, B_j, B_h$ and $B_g$ are in the same plane. Then, $M(\Delta_1) = M(\Delta_2)$, which means that $T_{k+1}$ is just another triangulation refining the Epstein–Penner decomposition. $\square$

5 The variational approach

In this section we show that there is $r \in \mathbb{R}^n$ such that the total dihedral angles around all inner edges of the complex corresponding to $r$ are equal $2\pi$. Such a point is a critical point of some functional over $\mathbb{R}^n$ that we will introduce.
Figure 7: The section orthogonal to the geodesic $A_iA_g$ and its translation in terms of the graphic of distance function.

5.1 The discrete Hilbert–Enstein functional

For a complex $K = (T, r)$ define $\tilde{\omega}_i$ to be the total dihedral angle around the $i$-th edge and $\kappa_i = 2\pi - \tilde{\omega}_i$. For $e \in E(T)$ let $\tilde{\phi}_e$ be the total dihedral angle at $e$ and $\theta_e = \pi - \tilde{\phi}_e$. Introduce the discrete Hilbert–Einstein functional over the space of all complexes

$$S(r) := -2\text{vol}(K) + \sum_{1 \leq i \leq n} r_i \kappa_i + \sum_{e \in E(T)} l_e \theta_e. \quad (3)$$

Let us show that $S(r)$ is well defined. Indeed, if $M$ can be represented as $(T, r)$ and $(T', r)$, then $T$ and $T'$ can be different only in flat edges, so for such edges $\theta_e = 0$. 

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Lemma 5.1. For every $r \in \mathbb{R}^n$, $S(r)$ is twice differentiable and
\[
\frac{\partial S}{\partial r_i} = \kappa_i.
\]

Proof. Assume that $r$ is an inner point of $K(T)$ for some triangulation $T$ and $K$ is the corresponding complex. Then $T$ is precisely the face-decomposition of the upper boundary of $K$. The same holds for every $r'$ that is sufficiently close to $r$. Hence, combinatorics of complexes does not change in some neighborhood of $r$ and every total angle can be written as the sum of dihedral angles in the same prisms. Clearly, every dihedral angle in every prism is differentiable. Moreover, by Schlaffli’s differential formula for a prism $P = A_iA_jA_hB_hB_jB_i \subset K$ we have
\[
-\frac{1}{2}d\text{vol}(P) = r_id\omega_i + r_jd\omega_j + r_Hd\omega_h + l_{jh}d\phi_i + l_{ih}d\phi_j + l_{ij}d\phi_h.
\]

Summing these equalities over all prisms we obtain (4). Since dihedral angles are differentiable we obtain that $S$ is twice differentiable at $r$.

Now consider the case when $r$ belongs to the boundary of some $K(T)$ (see the end of Subsection 4.1 for the definition of $K(T)$). First, we show that this boundary is piecewise-analytic. Remind that $K'(T)$ is the set of all $r'$ for which the pair $(T, r')$ is admissible and $r$ is an inner point of $K'(T)$ (as $K$ does not contain singular prisms). Then for every $e \in E(T)$ consider the total dihedral angle of a complex $(T, r')$ as a function of $r'$. It is analytic over the interior of $K'(T)$. Hence, the condition $\bar{\phi}_e = \pi$ is analytic in a neighborhood of $r$. The boundary of $K(T)$ consists of different pieces corresponded to different flat edges of $T$ and is piecewise-analytic.

Consider a coordinate vector $e_i$. As every boundary is piecewise-analytic, we have $r + \lambda e_i$ is in the interior of some $K(T)$ for small enough $\lambda$. Therefore, we can compute the directional derivative of $S(r)$ in the direction $e_i$ using the formula (4). For every coordinate direction they are continuous, hence $S$ is differentiable. Below we compute the derivatives of $\kappa_i$ and show that they are also continuous, which finishes the proof that $S$ is twice differentiable in this case.

Lemma 5.1 implies that if $r$ is a critical point of $S$, then every inner dihedral angle is equal to $2\pi$. In order to find such a point we should consider the second partial derivatives of $S$. We saw that it is sufficient to calculate them for a fixed triangulation $T$.

Define $X_{ij} := \frac{\partial^2 F}{\partial r_i \partial r_j} = \frac{\partial^2}{\partial r_i \partial r_j}$. 

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Lemma 5.2. For every $1 \leq i \leq n$

(i) $X_{ii} < 0$,

(ii) for $i \neq j$, $X_{ij} > 0$,

(iii) for every $1 \leq i \leq n$, $\sum_{1 \leq j \leq n} X_{ij} < 0$,

(iv) the second derivatives are continuous at every point $\tau \in \mathbb{R}^n$. In particular, it implies that $X_{ij} = X_{ji}$.

Note that a matrix satisfying the properties (i)–(iii) is a particular case of so-called diagonally dominated matrices.

Corollary 5.3. The function $S$ is strictly concave over $\mathbb{R}^n$.

Proof. We prove that the Hessian $X$ of $S$ is negatively definite over $\mathbb{R}^n$. Indeed,

$$r^T X r = \sum_{i=1}^{n} X_{ii} r_i^2 + \sum_{1 \leq i < j \leq n} 2 X_{ij} r_i r_j = - \sum_{1 \leq i < j \leq n} X_{ij} (r_i - r_j)^2 + \sum_{i=1}^{n} r_i^2 \sum_{j=1}^{n} X_{ij} < 0.$$ 

Let $A_1 A_2 A_3 B_3 B_2 B_1$ be a semi-ideal prism. The intersection of the trihedral angle at the vertex $A$ and the horosphere centered at it is a Euclidean triangle with side lengths equal to $\alpha_{12}$, $\alpha_{13}$ and $\lambda$; corresponding angles are $\phi_{12}$, $\phi_{13}$ and $\omega_1$. Then by the cosine theorem we have

$$\cos(\omega_1) = \frac{\alpha_{12}^2 + \alpha_{13}^2 - \lambda^2}{2 \alpha_{12} \alpha_{13}}.$$ 

We calculate the derivatives of $\omega_1$

$$\frac{\partial \omega_1}{\partial \alpha_{12}} = - \frac{\cot(\phi_{12})}{\alpha_{12}},$$

$$\frac{\partial \omega_1}{\partial \alpha_{13}} = - \frac{\cot(\phi_{13})}{\alpha_{13}}.$$ 

Calculate the derivatives of $\alpha_{12}$ from Lemma 3.8

$$\frac{\partial \alpha_{12}}{\partial r_1} = - \frac{\alpha_{12}^2 - e^{-2r_1}}{2 \alpha_{12}}.$$
\[ \frac{\partial \alpha_{12}}{\partial r_2} = \frac{\alpha_{12}^2 - e^{-2r_1}}{2\alpha_{12}}. \]

If the ideal face is fixed then this prism is uniquely determined by lengths \( r_1, r_2 \) and \( r_3 \). Consider a deformation of this prism with preserved upper face. Then

\[
\frac{\partial \omega_1}{\partial r_1} = \frac{\partial \omega_1}{\partial \alpha_{12}} \frac{\partial \alpha_{12}}{\partial r_1} + \frac{\partial \omega_1}{\partial \alpha_{13}} \frac{\partial \alpha_{13}}{\partial r_1} = \tag{5}
\]

\[ = \frac{\cot(\phi_{12})}{2\alpha_{12}^2} (\alpha_{12}^2 + e^{-2r_1}) + \frac{\cot(\phi_{13})}{2\alpha_{13}^2} (\alpha_{13}^2 + e^{-2r_1}), \]

\[
\frac{\partial \omega_1}{\partial r_2} = \frac{\partial \omega_1}{\partial \alpha_{12}} \frac{\partial \alpha_{12}}{\partial r_2} = \frac{\cot(\phi_{12})}{2\alpha_{12}^2} (-\alpha_{12}^2 + e^{-2r_1}), \tag{6}
\]

\[
\frac{\partial \omega_1}{\partial r_3} = \frac{\partial \omega_1}{\partial \alpha_{13}} \frac{\partial \alpha_{13}}{\partial r_3} = \frac{\cot(\phi_{13})}{2\alpha_{13}^2} (-\alpha_{13}^2 + e^{-2r_1}).
\]

Now consider a complex \( K = (T, r) \). Consider the set \( E^{or}(T) \) of oriented edges of \( T \). Every edge \( e \in E(T) \) gives rise to two oriented edges in \( E^{or}(T) \). By \( E^{orp}_i(T) \subset E^{or}(T) \) denote the set of oriented edges starting at \( A_i \), but ending not in \( A_i \). By \( E^{ort}_i(T) \subset E^{or}(T) \) denote the set of oriented loops from \( A_i \) to \( A_i \) (every non-oriented loop is counted twice). By \( E^{or}_i \) denote the union \( E^{orp}_i(T) \cup E^{ort}_i(T) \) For an oriented edge \( \vec{e} \in E^{or}_i(T) \) denote by \( \alpha_{\vec{e}} \) the length of the arc of horosphere at \( A_i \) between \( A_iB_i \) and \( \vec{e} \). To calculate \( \frac{\partial \bar{\omega}_i}{\partial r_i} \) we consider \( \bar{\omega}_i \) as the sum of angles in all prisms incident to \( A_i \) and take their derivatives. If there are no loops among the upper edges of a prism, then this prism makes a contribution of the form (5). If there are some loops, we should also add contributions of the form (6). Combining the summands containing the terms \( \alpha_{\vec{e}} \) for the same \( \vec{e} \) we get

\[
\frac{\partial \bar{\omega}_i}{\partial r_i} = -X_{ii} = \sum_{e \in E^{orp}_i(T)} \frac{\alpha_{\vec{e}}^2 + e^{-2r_i}}{2\alpha_{\vec{e}}^2} (\cot \phi_{\vec{e}+} + \cot \phi_{\vec{e}-}) + \sum_{e \in E^{ort}_i(T)} \frac{e^{-2r_i}}{\alpha_{\vec{e}}^2} (\cot \phi_{\vec{e}+} + \cot \phi_{\vec{e}-}),
\]

where \( \phi_{\vec{e}+} \) and \( \phi_{\vec{e}-} \) are the dihedral angles at \( \vec{e} \) in two prisms containing \( \vec{e} \).
For every $e \in E(T)$ we have

$$\phi_{\bar{e}+} + \phi_{\bar{e}-} = \tilde{\phi}_{\bar{e}} \leq \pi,$$

where $e$ is $\bar{e}$ forgetting orientation. Hence $(\cot \phi_{\bar{e}+} + \cot \phi_{\bar{e}-}) \geq 0$ and $\frac{\partial \tilde{\omega}_i}{\partial r_i} = -X_{ii} \geq 0$. Also, equality here means that the total dihedral angle of every edge starting at $A_i$ is equal to $\pi$. But in this case we obtain a non simply-connected open face of $K$, which is impossible by Lemma 4.9. Similarly, for $i \neq j$ denote by $E_{ij}^{or}(T) \subset E_{ij}^{or}$ the set of all oriented edges starting at $A_i$ and ending at $A_j$. Then,

$$\frac{\partial \tilde{\omega}_i}{\partial r_j} = -X_{ij} = \sum_{e \in E_{ij}^{or}(T)} \frac{e^{-2r_i} - \alpha_{\bar{e}}^2}{2\alpha_{\bar{e}}^2} (\cot \phi_{\bar{e}+} + \cot \phi_{\bar{e}-}) =$$

$$= - \sum_{e \in E_{ij}^{or}(T)} \frac{e^{-2r_i} - \alpha_{\bar{e}}^2}{2\alpha_{\bar{e}}^2} (\cot \phi_{\bar{e}+} + \cot \phi_{\bar{e}-}) < 0$$

From this we obtain for every $i$

$$\sum_{1 \leq j \leq n} \frac{\partial \tilde{\omega}_i}{\partial r_j} = \sum_{1 \leq j \leq n} -X_{ij} = \sum_{e \in E_i^{or}(T)} \frac{e^{-2r_i}}{2\alpha_{\bar{e}}^2} (\cot \phi_{\bar{e}+} + \cot \phi_{\bar{e}-}),$$

which is greater than zero for similar reasons. It finishes the proof of Lemma 5.2.

We know that $S(r)$ is strictly concave over $\mathbb{R}^n$. Therefore, it has at most one maximum point. We want to prove that such a point exist. To do it we need to study what happens with complexes when the absolute values some coordinates are large. The plan is as follows. First, we study the case when all coordinates are sufficiently negative. Second, we deal with the case when there is at least one sufficiently positive coordinate. Then we combine these results and get the desired conclusion.

### 5.2 The behavior of $S$ near infinity

**Lemma 5.4.** For every $\varepsilon > 0$ there exists $C > 0$ such that if in $K = (T, r)$ we have $r_i < -C$ for some $i$, then $\tilde{\omega}_i < \varepsilon$.

**Proof.** Fix $\varepsilon > 0$. Remind that for $\bar{e} \in E_i^{or}(T)$ ending at $A_j$ (not necessarily different from $A_i$) Lemma 3.8 gives the following expression of $\alpha_{\bar{e}}$

$$\alpha_{\bar{e}}^2 = e^{r_j - r_i - l_i} + e^{-2r_i}.$$
Hence,
\[ \alpha_e^2 \geq e^{-2r_i}. \]

Consider two consecutive edges \( \vec{e}_1 \) and \( \vec{e}_2 \in E_i^\sigma(T) \). Together with the line \( A_iB_i \) they cut a triangle at the horosphere of \( A_i \) with the side length \( \alpha_{\vec{e}_1}, \alpha_{\vec{e}_2} \) and \( \lambda \). If \( r_i < -C \), then both the lengths \( \alpha_{\vec{e}_1} \) and \( \alpha_{\vec{e}_2} \) are at least \( e^C \) and \( \lambda \) is bounded from above by the total length of the horocycle at \( A_i \) on \( S_{g,n} \). The angle \( \omega \) between \( \alpha_{\vec{e}_1} \) and \( \alpha_{\vec{e}_2} \) in this triangle is sufficiently small.

We can choose large enough \( C > 0 \) such that if \( r_i < -C \), then the angle at \( A_iB_i \) in every such triangle, which is the dihedral angle of \( A_iB_i \) in the prism containing \( \vec{e}_1 \) and \( \vec{e}_2 \), is less than \( \varepsilon/(12(n + g - 1)) \).

Note that the number of triangles incident to one cusp is bounded from above by the total number of triangles which can be calculated from the Euler characteristic and is equal to \( 4(n + g - 1) \). Therefore, the total angle \( \tilde{\omega}_i < \varepsilon \).

**Lemma 5.5.** For every \( \varepsilon > 0 \) there exists \( C > 0 \) such that if in \( K = (T, \mathbf{r}) \) for some \( i \) and every \( j \) we have \( r_i + r_j \geq C \), then at every point \( x \in S_{g,n} \) the value of distance function \( \rho_K(x) \geq \varepsilon \).

**Proof.** Let \( B_1, \ldots, B_n \) be the intersections of our fixed open horoballs with \( S_{g,n} \) and \( G_1, \ldots, G_n \) be the intersections of corresponding horospheres with \( S_{g,n} \). Denote \( \hat{B} = \bigcup_{j \neq i} B_j \) and \( \hat{G} = \bigcup_{j \neq i} G_j \). Also denote \( r_m = \min_{j \neq i} r_j \).

For any \( t \in \mathbb{R} \) consider the following two sets (remind that we measure distances from horospheres with signs):

\[
D_1(t) = \{ x \in S_{g,n} \mid \text{dist}(x, G_i) \leq t \} \cup B_i,
\]

\[
D_2(t) = \{ x \in \hat{B} \mid \text{dist}(x, \hat{G}) \leq -t \}.
\]

Note \( D_1(t) \) is a horoball centered at the \( i \)-th cusp and containing \( B_i \) for \( t \geq 0 \). \( D_2(t) \) is the union of horoballs centered at other cusps and is contained in \( \hat{B} \).

If \( x \in D_1(t) \) then \( \rho_K(x) \geq r_i - t \). Indeed, we can connect \( x \) with point \( y \in G_i \) by a geodesic such that the length of this geodesic is at most \( t \). \( \rho_K(y) \geq r_i \) and \( \rho_K(x) \) cannot differ from \( \rho_K(y) \) more than by the length of this geodesic.

If \( x \in D_2(t) \) then \( \rho_K(x) \geq r_m + t \) because for some \( j \), \( x \) belongs to the horoball at the \( j \)-th cusp which is at the distance \( t \) from our fixed cusp.
Take \( t_1 = r_1 - \varepsilon \) and \( t_2 = \varepsilon - r_m \). We have that if \( x \in D(t_1) \cup D(t_2) \), then \( \rho_K(x) \geq \varepsilon \).

Consider \( H = S_{g,n} \setminus \bigcup_{1 \leq j \leq n} B_j \). It is compact. Define

\[
p = \max_{x \in H} \text{dist}(x, B_i) < \infty.
\]

It is clear that for every \( x \in \tilde{B} \), then

\[
\text{dist}(x, G_i) \leq \text{dist}(x, \tilde{G}) + p.
\]

Therefore, if \( t_1 \geq t_2 + p \) then \( S_{g,n} = D(t_1) \cup D(t_2) \).

The inequality \( t_1 \geq t_2 + p \) is equivalent to the inequality \( r_i + r_m \geq 2\varepsilon + p \). Take \( C = 2\varepsilon + p \). Hence, the condition \( r_i + r_m \geq C \) implies that \( \rho_K(x) \geq \varepsilon \) for every \( x \in S_{g,n} \) as desired. \( \square \)

Next two lemmas are straightforward.

**Lemma 5.6.** For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if in a hyperbolic triangle every edge length is less than \( \delta \), then its sum of angles is bigger than \( \pi - \varepsilon \).

From Lemma 3.9 we see that

**Lemma 5.7.** For every \( \varepsilon > 0 \) there exists \( C > 0 \) such that if the distance \( \rho_{\tilde{e}} \) from an oriented edge \( \tilde{e} \) to the lower boundary is greater than \( C \), then \( \alpha_{\tilde{e}}^2 < \varepsilon \).

**Corollary 5.8.** For every \( \varepsilon > 0 \) there exists \( C > 0 \) such that if in \( K = (T, \mathbf{r}) \) for some \( i \) and every \( j \) we have \( r_i + r_j > C \), then

\[
\sum_{1 \leq i \leq n} \omega_i \geq 4\pi(n + g - 1) - \varepsilon.
\]

Now we are able to prove that \( S \) attains its maximal point at \( \mathbb{R}^n \).

**Lemma 5.9.** Consider a cube \( T \) in \( \mathbb{R}^n \): \( T = \{ \mathbf{r} \in \mathbb{R}^n : \max(|r_i|) \leq q \} \). Then for sufficiently large \( q \), the maximum of \( S(\mathbf{r}) \) over \( T \) is attained at an interior point of \( T \).
Proof. Indeed, consider $q > C_1 + C_2$, where $C_1$ is taken from Lemma 5.4 for $\varepsilon = 2\pi$ and $C_2$ is taken from Corollary 5.8 for some small enough $\varepsilon = \varepsilon_0$. The cube $T$ is convex and compact, $S$ is concave, therefore $S$ reaches its maximal value over $T$ at some point $r^0 \in T$. Assume that $r^0 \in \text{bd}T$. Then there are two possibilities: either there is $i$ such that $r_i^0 < -C_1 < 0$ or for every $i$ we have $r_i^0 \geq -C_1$.

In the first case, by Lemma 5.4, $\omega_i < 2\pi$. Therefore, $\kappa_i = \frac{\partial S}{\partial r_i}\big|_{r=r^0} > 0$. Let $\mathbf{v}_i$ be the $i$-th coordinate vector. We can see that for small enough $\mu > 0$, $S(r^0 + \mu \mathbf{v}_i) > S(r^0)$ and $r^0 + \mu \mathbf{v}_i \in T$, which is a contradiction.

In the second case, consider $i$ such that $|r_i^0| = q$. Then $r_i^0 = q > 0$ (because otherwise the first case holds) and for every $j$ we have $r_i^0 + r_j^0 > C_2$. Therefore, by Corollary 5.8 we have

$$\sum_{1 \leq i \leq n} \omega_i \geq 4\pi (n + g - 1) - \varepsilon_0.$$  

Consider two sets $I = \{i : 1 \leq i \leq n, r_i^0 = q\}$ and $J = [n] \setminus I$. Clearly, if $j \in J$, then $\frac{\partial S}{\partial r_j}\big|_{r=r^0} = 0$. Therefore, $\omega_j = 2\pi$. Let $k = |J|$. Then we have

$$\sum_{i \in I} \omega_i \geq 4\pi (n + g - 1) - \varepsilon_0 - 2k\pi.$$  

Hence, for some $i \in I$ and small enough $\varepsilon_0$ we obtain $\omega_i > 2\pi$ and so $\kappa_i < 0$. Therefore, for small enough $\mu > 0$, $S(r^0 - \mu \mathbf{v}_i) > S(r^0)$ and $r^0 - \mu \mathbf{v}_i \in T$, which is a contradiction. \qed

Bibliography

[1] A. Alexandrov. Existence of a convex polyhedron and of a convex surface with a given metric. Rec. Math. [Mat. Sbornik] N.S., 11(53):15–65, 1942.

[2] A. D. Alexandrov. Convex polyhedra. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[3] A. I. Bobenko and I. Izmestiev. Alexandrov’s theorem, weighted Delaunay triangulations, and mixed volumes. Ann. Inst. Fourier (Grenoble), 58(2):447–505, 2008.
[4] A. I. Bobenko, U. Pinkall, and B. A. Springborn. Discrete conformal maps and ideal hyperbolic polyhedra. *Geom. Topol.*, 19(4):2155–2215, 2015.

[5] L. Brunswic. *Surfaces de Cauchy polyédrales des espaces-temps plats singuliers*. PhD thesis, Université d’Avignon, 2017.

[6] D. B. A. Epstein and R. C. Penner. Euclidean decompositions of non-compact hyperbolic manifolds. *J. Differential Geom.*, 27(1):67–80, 1988.

[7] F. Fillastre. Polyhedral realisation of hyperbolic metrics with conical singularities on compact surfaces. *Ann. Inst. Fourier (Grenoble)*, 57(1):163–195, 2007.

[8] F. Fillastre. Polyhedral hyperbolic metrics on surfaces. *Geom. Dedicata*, 134:177–196, 2008.

[9] F. Fillastre and I. Izmestiev. Hyperbolic cusps with convex polyhedral boundary. *Geom. Topol.*, 13(1):457–492, 2009.

[10] X. Gu, R. Guo, F. Luo, J. Sun, and T. Wu. A discrete uniformization theorem for polyhedral surfaces II. *ArXiv e-prints*, Jan. 2014.

[11] I. Izmestiev. Variational properties of the discrete Hilbert-Einstein functional. *Actes des rencontres du CIRM*, 3(1):151–157, 11 2013.

[12] I. Izmestiev. Spherical and hyperbolic conics. *ArXiv e-prints*, Feb. 2017.

[13] B. Martelli. An introduction to geometric topology. *ArXiv e-prints*, Oct. 2016.

[14] R. C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.*, 113(2):299–339, 1987.

[15] R. C. Penner. *Decorated Teichmüller theory*. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012. With a foreword by Yuri I. Manin.

[16] J. G. Ratcliffe. *Foundations of hyperbolic manifolds*, volume 149 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2006.
[17] I. Rivin. Intrinsic geometry of convex ideal polyhedra in hyperbolic 3-space. In Analysis, algebra, and computers in mathematical research (Luleå, 1992), volume 156 of Lecture Notes in Pure and Appl. Math., pages 275–291. Dekker, New York, 1994.

[18] J.-M. Schlenker. Métriques sur les polyèdres hyperboliques convexes. J. Differential Geom., 48(2):323–405, 1998.

[19] J.-M. Schlenker. Hyperbolic manifolds with polyhedral boundary. ArXiv e-prints, Sept. 2002.

[20] J.-M. Schlenker. Hyperideal polyhedra in hyperbolic manifolds. ArXiv e-prints, Dec. 2002.

[21] B. Springborn. Hyperbolic polyhedra and discrete uniformization. ArXiv e-prints, July 2017.

[22] W. P. Thurston. Three-dimensional geometry and topology. Vol. 1, volume 35 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy.

[23] J. A. Volkov and E. G. Podgornova. Existence of a convex polyhedron with a given evolute. Taškent. Gos. Ped. Inst. Učen. Zap., 85:3–54, 83, 1971.