METAPLECTIC CATEGORIES, GAUGING AND PROPERTY F

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Abstract. $N$-Metaplectic categories, unitary modular categories with the same fusion rules as $SO(N)_2$, are prototypical examples of weakly integral modular categories generalizing the model for the Ising anyons. As such, a conjecture of the second author would imply that images of the braid group representations associated with metaplectic categories are finite groups, i.e. have property $F$. While it was recently shown that $SO(N)_2$ itself has property $F$, proving property $F$ for the more general class of metaplectic modular categories is an open problem. We verify this conjecture for $N$-metaplectic modular categories when $N$ is odd, exploiting their classification and enumeration to relate them to $SO(N)_2$. In another direction, we prove that when $N$ is divisible by 8 the $N$-metaplectic categories have 3 non-trivial bosons, and the boson condensation procedure applied to 2 of these bosons yields $\frac{N}{2}$-metaplectic categories. Otherwise stated: any $8k$-metaplectic category is a $\mathbb{Z}_2$-gauging of a $2k$-metaplectic category, so that the $N$ even metaplectic categories lie towers of $\mathbb{Z}_2$-gaugings commencing with $2k$- or $4k$-metaplectic categories with $k$ odd.

1. Introduction

$N$-Metaplectic categories are a major source of examples of weakly integral modular categories. As natural generalizations of the Ising anyons [20] they are important examples in the study of topological phases of matter and their applications [21] to quantum computation. They are defined as unitary modular categories with the same fusion rules as those obtained from the semisimple quotients $SO(N)_{21}$ of $\text{Rep}(U_q so_N)$ where $q = e^{\pi i/N}$ for $N$ even and $q = e^{\pi i/(2N)}$ for $N$ odd (see [25] for details of that construction). In general an $N$-metaplectic category has dimension $4N$ and has simple objects of dimension 1, 2 and $\sqrt{N}$ ($N$ odd) or $\sqrt{\frac{N}{2}}$ ($N$ even). In the case $N$ is odd $N$-metaplectic categories are relative centers of Tambara-Yamagami categories [19]. Recently, a complete classification and enumeration of $N$-metaplectic categories has been completed [1, 6, 7]. In addition, the $N$-metaplectic modular categories coming from quantum groups, i.e. $SO(N)_2$, have been shown to have finite braid group image [29] verifying the property $F$ conjecture for this subset of metaplectic categories (see [25]).

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1This notation is borrowed from conformal field theory. A more suitable notation might be $\text{Spin}(N)_2$ since the objects analogous to the spinor representations are included.
In this article we advance our understanding of $N$-metaplectic modular categories in two ways. First we extend the proof of property $F$ from $SO(N)_2$ with $N$ odd to all odd $N$-metaplectic categories. This is achieved as follows. In [1] it is shown that for $N$ odd there are precisely $2^{s+1}$ inequivalent $N$-metaplectic categories where $s$ is the number of prime factors of $N$. We show that each of these may be obtained from $SO(N)_2$ by Galois conjugation and twisting, which then allows us to describe the images of all $N$-metaplectic $B_n$-representations in terms of those obtained from $SO(N)_2$. Although we believe this technique should apply to the even $N$ cases as well, there are some further technicalities that have not been worked out yet. On the other hand our second result shows that even $N$-metaplectic categories appear in towers of gaugings. More precisely we show that if $8 \mid N$ then any $N$-metaplectic modular category is a $\mathbb{Z}_2$-gauging of an $\frac{N}{4}$-metaplectic modular category. Thus for each odd $k$ there are towers of even $N$-metaplectic categories starting with the $2k$- and $4k$-metaplectic categories.

2. Preliminaries

We assume the reader is familiar with the basic notions in the theory of fusion categories such as spherical and braiding structures and their properties. Good references for these details are: [14, 15, 2, 31].

2.1. Galois conjugation and twisting. It is well known that a fusion (or modular or ribbon) category $\mathcal{C}$ can be defined over a number field $\mathbb{F} = \mathbb{Q}(\alpha)$. That is, the data needed to construct $\mathcal{C}$ (6$j$-symbols, braiding isomorphisms, twists, mapping class group representations) all lie in a finite Galois extension of $\mathbb{Q}$. Moreover, if $\sigma$ is a Galois automorphism of $\mathbb{F}$ then twisting all data by $\sigma$ produces another category $\mathcal{C}^\sigma$. Now if $\mathcal{C}$ is a unitary category, or (possibly more generally) has dimension function taking values in $\mathbb{R}^+$ then $\mathcal{C}^\sigma$ may not have this property. Indeed, a Galois conjugate of a pseudo-unitary category is not generally pseudo-unitary.

On the other hand, any Galois conjugate of a weakly integral fusion category is pseudo-unitary [15, Proposition 8.24]. Thus, by [15, Propositions 8.23] any weakly integral fusion category admits a unique spherical structure $j_+$ with respect to which each object has positive dimension. Moreover, if $\mathcal{B}$ is the braided fusion category underlying a weakly integral modular category $\mathcal{C}$ (i.e. forgetting the spherical structure) then $\mathcal{B}$ equipped with any other choice of spherical structure is again modular (see [8, Lemma 2.4]). In particular, with respect to the unique spherical structure $j_+$ giving $\mathcal{B}^\sigma$ positive dimensions, $\mathcal{B}^\sigma_+ = (\mathcal{B}^\sigma, j_+)$ is modular. Note that while $\mathcal{C}^\sigma$ and $\mathcal{B}^\sigma_+$ have the same underlying braided fusion category $\mathcal{B}^\sigma$, their spherical structures (and therefore $S$ and $T$-matrices) may differ.

These arguments prove the following useful:

**Proposition 1.** Let $\mathcal{C}$ be any weakly integral modular category, $\mathcal{B}$ its underlying braided fusion category, and $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ a Galois automorphism. Then there is a unique choice of a spherical structure $j_+$ with respect to which $\mathcal{B}^\sigma_+ = (\mathcal{B}^\sigma, j_+)$ is a modular category with positive dimensions.
It is worth pointing out that distinct spherical structures on the braided fusion category $B$ underlying any modular category $C$ are in 1-1 correspondence with invertible self-dual objects of $C$ (see e.g. [14]).

A motivation for this paper is the following:

**Conjecture 1.** The braid group representations associated with any object in a weakly integral braided fusion category has finite image.

An object $X \in B$ so that the corresponding braid group representations all have finite image is called a **property F** object, and $B$ has property F if all objects are property F objects. It is conjectured (see [25]) that $\dim(X)^2 \in \mathbb{Z}$ if and only if $X$ has property F, so that $B$ has property F if and only if $B$ is weakly integral.

Suppose that every object in a modular $C$ has property F. Then the same is true of $C^{\sigma}$, since the relations defining a finite group are polynomials. Moreover, the braid group image only depends on the underlying braided fusion category $B$, i.e. is independent of the spherical structure. Thus if a weakly integral modular category $C$ has property F then for any Galois conjugation $\sigma$ the underlying braided fusion category, $B^{\sigma}$ equipped with the positive spherical structure $B_{+}$ also has property F.

Recently it was shown [29] that the integral modular categories $SO(N)_2$ obtained from quantum groups $U_q so_N$ at $q = e^{\pi i/N}$ ($N$ even) and $q = e^{\pi i/(2N)}$ ($N$ odd) have property F. The proof involves a detailed analysis of representations of these quantum groups, rather than categorical-level arguments. In particular the proof does not immediately imply that unitary modular categories with the same fusion rules as $SO(N)_2$ (i.e. metaplectic modular categories also have property F). On the other hand, metaplectic modular categories have now been classified and enumerated. This suggests that we can infer property F for those metaplectic modular categories with underlying braided fusion categories Galois conjugate to $SO(N)_2$.

**2.2. Boson Condensation and Gauging.** Two processes that we employ in our analysis are gauging and de-gauging (sometimes called anyon condensation), which may be interpreted physically as phase transitions for anyon systems [9]. First let us introduce the basic construction we call de-gauging (which was first described in [28] and subsequently rediscovered and developed in [23, 4, 12] under various conditions and under different names). Let $C$ be modular and $\text{Rep}(G) \cong D \subset C$ a Tannakian subcategory (here a Tannakian category is a symmetric braided fusion category equivalent to $\text{Rep}(G)$ for some finite group $G$). The $G$-de-equivariantization $C_G$ of $C$ is a faithfully $G$-graded category (in fact, a braided $G$-crossed category) with modular trivial component $[C_G]_e$ of dimension $\dim(C)/|G|^2$ and $[C_G]_e$ is the $G$-de-gauging of $C$ [12]. One does not need to understand the full $G$-de-equivariantization of $C$ to obtain $[C_G]_e$: in fact $[C_G]_e = (D')_G$, where

$$D' = \{ Y \in C : c_{X,Y} c_{Y,X} = \text{id}_{Y \otimes X} \text{ for all } X \in D \}$$

is the Müger centralizer of $D \subset C$ [12].

The simplest case of de-gauging is **boson condensation.** Whenever a modular category $C$ contains a **boson** $b$, i.e. a self-dual invertible object with twist $\theta_b = 1$, then the fusion
subcategory \( \langle b \rangle \) is equivalent to \( \text{Rep}(\mathbb{Z}_2) \). In this case, the de-equivariantization functor \( F : \mathcal{C} \to \hat{\mathcal{C}}_{G} \) is easier to understand. In particular, if \( X \in \mathcal{C} \) is a simple object and \( b \otimes X \not\cong X \), then \( F(X) \cong X^{(1)} \oplus X^{(2)} \) for simple objects \( X^{(1)}, X^{(2)} \). On the other hand, if \( b \otimes X \cong X \), then \( F(X) \) is a simple object. There is a trichotomy among self-dual invertible objects in a ribbon category: they are either bosons as above, semions with \( s = \pm i \) in which case the subcategory \( \langle s \rangle \) is modular or fermions \( f \) with \( \theta_f = -1 \) and \( \langle f \rangle \cong \text{sVec} \).

The reverse process, \( G \)-gauging, is more complicated \cite{BMR} \cite{GMR}. Here one starts with a modular category \( \mathcal{B} \) and an action of a finite group \( G \) by braided tensor autoequivalences: \( \rho : G \to \text{Aut}^{br}(\mathcal{B}) \). A \( G \)-gauging of \( \mathcal{B} \), when it exists, is a new modular category obtained by first constructing a \( G \)-graded fusion category \( \mathcal{D} \) with trivial component \( \mathcal{D}_e = \mathcal{B} \) and then equivariantizing to obtain a new modular category \( \mathcal{D}^G \). There are obstructions to the existence of a gauging, and when the obstructions vanish there can be many \( G \)-gaugings (see \cite{GMR}). A recent result of Natale \cite{Na} implies that any weakly group-theoretical modular category is a \( G \)-gauging of either a pointed modular category or a Deligne product of a pointed modular category and an Ising category. In \cite[Question 2]{GMR} they ask if every weakly integral modular category is weakly group-theoretical (the converse is known to be true). If the answer is “yes” (as many suspect) then to prove one direction of the property \( F \) conjecture it would be enough to prove that \( G \)-gauging preserves property \( F \).

3. Metaplectic Categories

We begin with the following definition:

**Definition 1.** A metaplectic modular category is a unitary modular category with the same fusion rules as \( SO(N)_2 \) for some \( N > 1 \).

The structure and properties of \( SO(N)_2 \) were studied in some detail in \cite{N}, from which much of the results we outline are taken. The fusion rules for \( SO(N)_2 \) (and hence \( N \)-metaplectic modular categories) naturally split into three cases, depending on the value of \( N \) mod 4.

3.1. Fusion rules for odd \( N \). The \( N \)-metaplectic modular categories for odd \( N > 1 \) have 2 simple objects \( X_1, X_2 \) of dimension \( \sqrt{N} \), two simple objects \( 1, Z \) of dimension 1, and \( \frac{N-1}{2} \) objects \( Y_i, 1 \leq i \leq \frac{N-1}{2} \) of dimension 2. The fusion rules are [1]:

1. \( Z \otimes Y_i \cong Y_i, Z \otimes X_i \cong X_{i+1} \) (modulo 2), \( Z^\otimes 2 \cong 1 \),
2. \( X_i^\otimes 2 \cong 1 \oplus \bigoplus Y_i \),
3. \( X_1 \otimes X_2 \cong Z \oplus \bigoplus Y_i \),
4. \( Y_i \otimes Y_j \cong Y_{\min\{i+j,N-i-j\}} \oplus Y_{|i-j|}, \) for \( i \not= j \) and \( Y_i^\otimes 2 = 1 \oplus Z \oplus Y_{\min\{2i,N-2i\}} \).

It is shown in [1] that \( Z \) is always a boson, and \( N \)-metaplectic modular categories with \( N \) odd were classified and enumerated by condensing \( Z \): there are precisely \( 2^{s+1} \) inequivalent such categories, where \( s \) is the number of distinct primes dividing \( N \). The fusion rules for the (adjoint) subcategory generated by \( Y_1 \) with simple objects \( 1, Z \) and all \( Y_i \) are precisely
those of the dihedral group $D_N$ of order $2N$, and, moreover this subcategory coincides the centralizer of the Tannakian $\langle Z \rangle \cong \text{Rep}(\mathbb{Z}_2)$.

3.2. Fusion rules for $N \equiv 2 \pmod{4}$. The $N$-metaplectic modular categories for $N \equiv 2 \pmod{4}$ have rank $k + 7$, where $k = N/2$ (an odd number). We will denote by $SO(2)_2$ the pointed modular category $\mathcal{C}(\mathbb{Z}_8, Q)$ with twists $e^{i2\pi/16}$ for uniformity of notation so that there are 4 inequivalent 2-metaplectic modular categories (since there are 4 inequivalent non-degenerate symmetric quadratic forms on $\mathbb{Z}_8$ see [52]). Generally, [6] there are exactly $2^{s+1}$ inequivalent $N$-metaplectic modular categories in this case, where $s$ is the number of prime divisors of $N$. The group of isomorphism classes of invertible objects for $N \geq 6$ is isomorphic to $\mathbb{Z}_4$. Let $g$ be a generator of this group, so the (isomorphism classes of) invertible objects are $g^i$ for $0 \leq i \leq 3$. There are $k - 1$ self-dual simple objects, $X_i$ and $Y_i$ for $1 \leq i \leq \frac{k-1}{2}$, of dimension 2. The remaining four simples objects, $V_i$ for $1 \leq i \leq 4$, have dimension $\sqrt{k}$. The following fusion rules hold [6]:

- $g \otimes X_a \cong Y_{\frac{k+1}{2}-a}$, and $g^2 \otimes X_{a} \cong X_a$, and $g^2 \otimes Y_a \cong Y_a$ for $1 \leq a \leq (k - 1)/2$.
- $X_a \otimes X_a \cong 1 + g^2 \otimes X_{\min\{2a, k-2a\}}$; $X_a \otimes X_b \cong X_{\min\{a+b, k-a-b\}} \otimes X_{|a-b|}$ ($a \neq b$)
- $V_1 \otimes V_1 \cong g \oplus \bigoplus_{a=1}^{k-1} Y_a$.
- $g V_1 = V_3$, $g V_3 = V_4$, $g V_2 = V_1$, $g V_4 = V_2$ and $g^3 V_a = V_a^*$, $V_2 \cong V_1^*$, $V_4 \cong V_3^*$

Again adopting the same notion for simple objects in a general $N$-metaplectic category $\mathcal{C}$ with $N \equiv 2 \pmod{4}$ one finds that $g^2$ is always a boson and the classification of $N$-metaplectic modular categories with $N \equiv 2 \pmod{4}$ was obtained in [6] by condensing $\langle g^2 \rangle$, to obtain a pointed cyclic modular category. Indeed, the centralizer of $\langle g^2 \rangle \cong \text{Rep}(\mathbb{Z}_2)$ has simple objects $X_1, Y_1$ and the $g^i$ i.e all simple objects of dimension 1 or 2. The simple object $V_1$ generates this subcategory, which has the same fusion rules as $\text{Rep}(\mathbb{Z}_4 \times \mathbb{Z}_2)$ (with the generator of $\mathbb{Z}_4$ acting by inversion on $\mathbb{Z}_2$) see [25]. Remark 4.4 and Theorem 4.8). In this notation the $\mathbb{Z}_4$-grading on $\mathcal{C}$ has trivial component $\mathcal{C}_0$ with simple objects $1, g^2, X_1, \ldots, X_{k-1}$, component $\mathcal{C}_2$ with simple objects $g, g^3, Y_1, \ldots, Y_{\frac{k-1}{2}}$, and the other two components with simple objects $\{V_1, V_3\}$ and $\{V_2, V_4\}$ respectively. Obviously there are labeling ambiguities associated with $g \leftrightarrow g^3$ and $\{V_1, V_3\} \leftrightarrow \{V_2, V_4\}$.

3.3. Fusion rules for $N \equiv 0 \pmod{4}$. The $N$-metaplectic modular categories with $N \equiv 0 \pmod{4}$ with $2k = N$ have rank $k + 7$ and dimension $4N$ [25]. The simple objects have dimension 1, 2 and $\sqrt{k}$ and are all self-dual. Setting $r = \frac{k}{2} - 1$, the $(2r + 1 = k - 1)$ simple objects $X_i$ for $0 \leq i \leq r - 1$ and $Y_j$ for $0 \leq j \leq r$ have dimension 2 and the simple objects $V_i, W_i$ have dimension $\sqrt{k}$. For $k > 2$ the key fusion rules are as follows [7]:

- $h \otimes Y_i \cong g \otimes Y_i \cong 1$, $h \otimes X_i \cong g \otimes X_i \cong X_{i-1}$ and $h \otimes Y_i \cong g \otimes Y_i \cong Y_{r-i}$
- $g \otimes V_1 \cong V_2$, $h \otimes V_1 \cong V_1$ and $h \otimes W_1 \cong W_2$, $g \otimes W_1 \cong W_1$
- $V_1 \otimes V_1 \cong 1 + h \oplus \bigoplus_{i=0}^{r-1} X_i$
- $W_1 \otimes W_1 \cong 1 + g \oplus \bigoplus_{i=0}^{r-1} X_i$
\[ W_1 \otimes V_1 \cong \bigoplus_{i=0}^r Y_i \]
\[ X_i \otimes X_j \cong \begin{cases} 
X_{i+j+1} \oplus X_{j-i-1} & i < j \leq \frac{r-1}{2} \\
1 \oplus hg \oplus X_{2i+1} & i = j - \frac{r-1}{2} < r - 1 \\
1 \oplus h \oplus g \oplus hg & i = j = \frac{r-1}{2} 
\end{cases} \]
\[ Y_i \otimes Y_j \cong \begin{cases} 
X_{i+j} \oplus X_{j-i-1} & i < j \leq \frac{r}{2} \\
1 \oplus hg \oplus X_{2i} & i = j \leq \frac{r-1}{2} \\
1 \oplus h \oplus g \oplus hg & i = j = \frac{r}{2} 
\end{cases} \]

Notice that all other fusion rules may be derived from the above by tensoring with \( h \) or \( g \) as needed. For example, \( V_1 \otimes V_2 \cong g \otimes V_1^{\otimes 2} \cong h \oplus hg \oplus \bigoplus_{i=0}^{r-1} X_i \). The \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) grading is clear from these rules, we denote the trivial component by \( C \). A Galois conjugate of the quantum group category \( \mathcal{C}(0,0) \) is always a boson. It is shown in \[7\] that, for \( N \geq 8 \) there are \( 3 \cdot 2^{s+1} \) inequivalent \( N \)-metaplectic modular categories where \( s \) is the number of distinct primes dividing \( N \). The degenerate case \( N = 4 \) is special: it has fusion rules like Ising\(^*\) \( \boxtimes \) Ising\(^*\) for which there are 20 inequivalent metaplectic modular categories, rather than 12.

The centralizer of the pointed subcategory \( \langle h, g \rangle \) is always the trivial component \( \mathcal{C}(0,0) \) with simple objects \( 1, h, g, hg \), and all \( X_i \), whereas \( \langle hg \rangle' \) also includes the component \( \mathcal{C}(1,1) \) with simple objects \( Y_j \) and the component with simple objects \( Y_j \) by \( \mathcal{C}(1,1) \). The classification of \( N \)-metaplectic modular categories with \( 4 \mid N \) was obtained in \[7\] by condensing \( hg \), which is always a boson. It is shown in \[7\] that, for \( N \geq 8 \) there are \( 3 \cdot 2^{s+1} \) inequivalent \( N \)-metaplectic modular categories where \( s \) is the number of distinct primes dividing \( N \). The degenerate case \( N = 4 \) is special: it has fusion rules as the representation category \( \text{Rep}(D_N) \) of the dihedral group of order \( N \).

4. Property F for \( N \)-Metaplectic Categories with \( N \) odd

**Theorem 1.** If \( \mathcal{C} \) is an \( N \)-metaplectic modular category with \( N := 2r + 1 \) odd, then \( \mathcal{C} \) has property F.

**Proof.** Let \( N = p_1^{a_1} \cdots p_s^{a_s} \) be the prime factorization of \( N \). From \[11\], we know that there are precisely \( 2^{s+1} \) \( N \)-metaplectic modular categories. We will show that Galois conjugation and twisting \[5\] produce all of these categories.

A Galois conjugate of the quantum group category \( SO(N)_2 \) is not necessarily unitary. However, it is pseudounitary, so there exists a choice of spherical structure on its underlying braided fusion category to make it unitary. This choice does not affect the braiding eigenvalues of the category.

Let \( \zeta = e^{\frac{2\pi i}{N}} \). There exists a simple object \( W \in SO(N)_2 \) of dimension \( \sqrt{N} \) such that the eigenvalues of the braiding \( R_{W,W} \) are \( \zeta^{n_j} \) for \( n_j = (4r+2)((r-j)(r-j+1)-j)+(2r+1)r+2j^2 \),
and $0 \leq j \leq r$ [21]. The non-isomorphic simple object $W'$ of dimension $\sqrt{N}$ has braiding eigenvalues $-\zeta^{n_j}$ for $0 \leq j \leq r$. 

The Galois group $\text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{N})^\times$ acts on the set of eigenvalue exponents $\{n_j : 0 \leq j \leq r\} \subset \mathbb{Z}/8\mathbb{N}$ by left translation. By the Chinese Remainder theorem, the Galois group acts on each factor of $\mathbb{Z}/8\mathbb{N} \cong \mathbb{Z}/p_1^{a_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_s^{a_s}\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ independently.

We first observe that $n_j = 2j^2 \pmod{N}$. Since $(\mathbb{Z}/2\mathbb{Z})^\times = \{0, 2\}$, we have $\{n_j : 0 \leq j \leq r\} = \{2j^2 : j \in \mathbb{Z}/N\mathbb{Z}\}$ as sets. Hence, for any $i$, we have $X := \{n_j (\text{mod } p_i): 0 \leq j \leq r\} = \{2j^2 : j \in \mathbb{Z}/p_i^{a_i}\mathbb{Z}\}$. The factor of the Galois group acting on $\mathbb{Z}/p_i^{a_i}\mathbb{Z}$ is $(\mathbb{Z}/p_i^{a_i})^\times$. Since $(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times$ is cyclic, the stabilizer subgroup $\text{Stab}_{(\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times}(X) = \{x^2 : x \in (\mathbb{Z}/p_i^{a_i}\mathbb{Z})^\times\}$ has index 2. Thus, we get two distinct sets of eigenvalues mod $p_i^{a_i}$ for each $i$.

Moreover, we have $n_j = r \pmod{8}$ for all $j$. If $r$ is relatively prime to 8, this gives 4 choices of Galois conjugates for $[n_j]_8$. If $r = 2$ or $r = 6 \pmod{8}$, we have 2 choices. If $r = 0$ or $r = 4 \pmod{8}$, there is only one choice. In all but the last ($r = 0$ or $r = 4$) case, we must divide by 2 to account for labelling ambiguity on the nonintegral objects. Thus, when $r$ is relatively prime to 8, we get $(2^s)(4)/2 = 2^{s+1}$ distinct categories from Galois conjugation. When $r = 2$ or $r = 6 \pmod{8}$, we get $(2)(2)/2 = 2^s$ distinct categories. When $r = 0$ or $r = 4 \pmod{8}$, we get $(2)(1) = 2^s$ distinct categories.

To construct the remaining metaplectic modular categories, we will use twisting in the sense of Bruillard et al. [3]. Let $\mathcal{D}$ be a modular category. Let $B \subset G(\mathcal{D})$ be a subgroup of the group of the invertibles of $\mathcal{D}$, and let $w \in Z^3(\hat{B}, U(1))$ be a 3-cocycle. The twisted category $\mathcal{D}_{(1,w)}$ is a $\hat{B}$-graded category with the same objects and tensor product as $\mathcal{D}$, but with an associator twisted by $w$. More explicitly, if $\sigma, \tau, \rho \in \hat{B}$, then we have

$$\hat{\alpha}_{X_{\sigma}, X_{\tau}, X_{\rho}} = w_{\sigma, \tau, \rho} \alpha_{X_{\sigma}, X_{\tau}, X_{\rho}},$$

where $\hat{\alpha}$ and $\alpha$ are the associators of $\mathcal{D}_{(1,w)}$ and $\mathcal{D}$, respectively.

Let $B \subset G(B)$ be a subgroup such that the induced map $U(G) \to \hat{G}(B) \to \hat{B} \cong \mathbb{Z}_2$ corresponds to the GN-grading. Let $w \in Z^3(\mathbb{Z}_2, U(1))$ be the normalized 3-cocycle given by $w(1,1,1) = -1$. Let $\alpha$ and $c$ denote the associator and braiding for some metaplectic modular category $\mathcal{D}$, and let $\hat{\alpha}$ and $\hat{c}$ denote the associator and braiding of the twisted category $\mathcal{D}_{(1,w)}$, respectively.

We claim that a solution to the hexagon equations is given by

$$\hat{c}_{X_{\sigma}, X_{\tau}} = c_{\sigma, \tau} c_{X_{\sigma}, X_{\tau}},$$

and

$$\hat{c}_{X_{\sigma}, X_{\tau}, X_{\rho}} = w_{\sigma, \tau, \rho} \alpha_{X_{\sigma}, X_{\tau}, X_{\rho}},$$

where $\hat{c}$ and $c$ are the braidings of $\mathcal{D}_{(1,w)}$ and $\mathcal{D}$, respectively.
where \( \epsilon_{\sigma, \tau} = i \) if \( \sigma = \tau = 1 \), and \( \epsilon_{\sigma, \tau} = 1 \) otherwise. Indeed, in diagrammatic composition order, we have

\[
\hat{\alpha}_{X_\sigma, X_\tau, X_\rho} \circ \hat{c}_{X_\sigma, X_\tau \otimes X_\rho} \circ \hat{\alpha}_{X_\tau, X_\rho, X_\sigma} = (w_{\sigma, \tau, \rho} \alpha_{X_\sigma, X_\tau, X_\rho}) \circ \epsilon_{\sigma, \tau, \rho} c_{X_\sigma, X_\tau \otimes X_\rho} \circ (w_{\tau, \rho, \sigma} \alpha_{X_\tau, X_\rho, X_\sigma}) \]
\[
= \epsilon_{\sigma, \tau, \rho} \cdot (c_{X_\sigma, X_\tau} \otimes \text{id}_{X_\rho}) \circ \alpha_{X_\tau, X_\rho, X_\sigma} \circ (\text{id}_{X_\tau} \otimes c_{X_\sigma, X_\rho}) \]
\[
= \epsilon_{\sigma, \tau, \rho} \epsilon_{\sigma, \rho}^{-1} w_{\tau, \sigma, \rho} \cdot (\hat{c}_{X_\sigma, X_\tau} \otimes \text{id}_{X_\rho}) \circ (\hat{\alpha}_{X_\tau, X_\rho, X_\sigma}) \circ (\text{id}_{X_\tau} \otimes \hat{c}_{X_\sigma, X_\rho}) \]
\[
= (\hat{c}_{X_\sigma, X_\tau} \otimes \text{id}_{X_\rho}) \circ (\hat{\alpha}_{X_\tau, X_\rho, X_\sigma}) \circ (\text{id}_{X_\tau} \otimes \hat{c}_{X_\sigma, X_\rho}),
\]
where the last equality follows from case analysis. The verification for the other hexagon equation is analogous.

The spherical structure on the twisted category \( \mathcal{D}_{(1, w)} \) is the same as the spherical structure on \( \mathcal{D} \). Since \( \epsilon \) and \( w \) are \( U(1) \)-valued, the modular category \( \mathcal{D}_{(1, w)} \) is also unitary.

Since any matrix in the twisted braid group representation differs from a matrix in the untwisted representation by a factor of the form \( i^n \), this twisting preserves Property F. By examining the exponents of the braiding eigenvalues \( \mod 8 \), we find that twisting accounts for another factor of 2 in our count when \( r \) is even, covering the remaining modular categories. \( \square \)

We illustrate the proof of the theorem with the following tables of braiding eigenvalues for 3- and 5-metaplectic categories.

3-metaplectic categories. The following table gives the exponents of the relevant braiding eigenvalues of the Galois conjugates of \( SO(3)_2 \). More explicitly, given \( \sigma \in (\mathbb{Z}/2\mathbb{Z})^\times \cong \text{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q}) \) and \( n \in \mathbb{Z}/24\mathbb{Z} \), we have the group action \( \sigma(n) = \sigma \cdot n \). Letting \( \zeta = e^{\pi i/12} \), the braiding eigenvalues of the \( \sigma \)-Galois conjugate of the first nonintegral object are \( \sigma(R_{V_1 V_1}^1) = \zeta^{\sigma(n_1)} \). The braiding values of the other nonintegral object are given by \( \sigma(R_{V_2 V_2}^1) = -\sigma(R_{V_1 V_1}^{12 + n_1}) = \zeta^{\sigma(12 + n_1)} \). Since \( n_0 = 9 \) and \( n_1 = 1 \), we have the following table of exponents of braiding eigenvalues of Galois conjugates.

| \( \sigma \)   | \( \sigma(n_0) \) | \( \sigma(n_1) \) | \( \sigma(12 + n_0) \) | \( \sigma(12 + n_1) \) |
|---|---|---|---|---|
| 1 | 9 | 1 | 21 | 13 |
| 5 | 21 | 5 | 9 | 17 |
| 7 | 15 | 7 | 3 | 19 |
| 11 | 3 | 11 | 15 | 23 |

Since we know there are precisely four 3-metaplectic categories, this table illustrates the fact that all four 3-metaplectic categories lie in the same orbit under the Galois conjugation action, since they are distinguished by these eigenvalues.

5-Metaplectic categories. Here \( \zeta = e^{\pi i/20} \). Similarly, we have the following table of exponents of braiding eigenvalues.
Since $r = 2$, we only have two distinct sets of braiding eigenvalues in the table, so that Galois conjugation only provides two of the four $5$-metaplectic categories. The other two categories are obtained by twisting: at the level of eigenvalues this is manifested by twisting by $i$, i.e. adding $10$ to each exponent in a row of the table.

5. A Sequence of Gaugings

$N$-metaplectic modular categories with $4 | N$ have $4$ self-dual invertible objects, are are therefore $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded. The $(0,0)$-graded component is the adjoint subcategory, and without loss of generality we may assume that the $(1,1)$-graded component contains all of the remaining $2$-dimensional simple objects. The $(1,0)$- and $(0,1)$-graded components each contain two isomorphism classes of $\sqrt{N/2}$-dimensional simple objects.

When $8 | N$ the $N$-metaplectic modular categories of have $3$ bosons $hg, h, g$, i.e. invertible, self-centralizing objects with trivial twists. The centralizer of each of these bosons consists of the $(0,0)$-graded (adjoint) component and one of the three other components. It was shown in [7] that condensing the boson $bgh$ that centralizes the (integral) $(1,1)$-graded component yields a cyclic modular category of the form $C(\mathbb{Z}_N, q)$, for some non-degenerate symmetric quadratic form $q$ on $\mathbb{Z}_N$. Except for the degenerate case $N = 8$, the bosons $h, g$ are uniquely determined (up to the labeling ambiguity $h \leftrightarrow g$) by the condition that they centralize a simple object of dimension $\sqrt{N/2}$. For $N = 8$ all non-invertible simple objects have dimension $2$, and the labels of all $3$ bosons are ambiguous, i.e. one cannot distinguish them by any of their properties. The follow shows that condensing either of the two bosons $h, g$ yields another metaplectic modular category.

**Theorem 2.** Let $\mathcal{C}$ be an $N$-metaplectic modular category with $8 | N$, and let $\mathcal{D}$ be the unitary modular category given by condensing the boson $h \in \mathcal{C}$ (or $g$) in the notation of subsection 3.3. Then $\mathcal{D}$ is an $\frac{N}{4}$-metaplectic modular category.

**Proof.** For the moment, assume that $N \geq 16$. It is relatively straightforward to verify that $\mathcal{D}$ has the right rank and dimensions of simple objects. Set $N = 2k$ and $r = \frac{k}{2} - 1$. $\mathcal{C}$ has rank $k + 7$, with $k - 1 = 2r + 1$ objects of dimension $2$: $X_0, \ldots, X_{r-1}$ and $Y_0, \ldots, Y_r$. By definition $\mathcal{D} = \langle (h) \rangle_{\mathbb{Z}_2}$ where $\text{Rep}(\mathbb{Z}_2) = \langle h \rangle$. From the discussion in Section 2 we see that
\[ \langle h \rangle' = \mathcal{C}_{(0,0)} \oplus \mathcal{C}_{(1,0)} \] with simple objects
\[ \{1, h, hg, g, X_0, X_1, \ldots, X_{r-1}, V_1, V_2\}. \]

Let \( F : \langle h \rangle' \to (\langle h \rangle')_{\mathbb{Z}_2} \) be the de-equivariantization functor. As \( h \otimes V_i \cong V_i \) and \( h \otimes X_i \cong X_{r-i-1} \) we have the following, where we set \( t = \frac{r-1}{2} = \frac{N}{8} - 1 \):

1. \( F(V_i) = V_i^{(0)} \oplus V_i^{(1)} \), where \( V_i^{(j)} \) are \( \sqrt{\frac{N}{8}} \)-dimensional objects.
2. \( \tilde{Y}_i := F(X_{2i}) \cong F(X_{r-2i-1}) \) are simple objects of dimension 2, for \( 0 \leq i < t/2 \) (provided \( N \geq 16 \), otherwise there are no \( \tilde{Y}_i \))
3. \( \tilde{X}_j := F(X_{2j+1}) \cong F(X_{r-2j-2}) \) are simple objects of dimension 2, for \( 0 \leq j < t/2 \) (provided \( N \geq 24 \), otherwise there are no \( \tilde{X}_j \))
4. \( F(X_i) = g_1 \oplus g_2 \) with \( g_1, g_2 \) invertible,
5. \( F(h) = F(1) = 1_\mathbb{D} \)
6. \( F(hg) = F(g) = Z \) an invertible object.

In particular, the modular category \( \mathcal{D} = F(\langle h \rangle') \) has the same dimensions (1 with multiplicity 4, 2 with multiplicity \( t = \frac{N}{8} - 1 \) and \( \sqrt{\frac{N}{8}} \) with multiplicity 4), global dimension \( (N) \) and rank \( \langle \frac{N}{8} + 7 \rangle \) as an \( \frac{N}{8} \)-metaplectic modular category. It is important to point out that when \( 16 \mid N \) we have \( \frac{N}{4} \equiv 0 \pmod{4} \) so that \( t \) is odd, while \( \frac{N}{4} \equiv 2 \pmod{4} \) so that \( t \) is even otherwise, so these cases correspond to either the self-dual fusion rules of subsection 3.3 or the non-self-dual fusion rules of subsection 3.2. Here are a few useful observations that can be deduced from the fusion rules of \( \mathcal{C} \):

- \( \mathcal{D} \) is graded by a group of order 4, with each component of dimension \( \frac{N}{4} \).
- If \( 16 \mid N \) (so that \( t \) is odd) the trivial component \( \mathcal{D}_0 \) contains all 1-dimensional simple objects and \( \frac{N}{16} \) simple objects of dimension 2, otherwise (i.e. \( t \) is even) the trivial component contains \( 1_\mathbb{D} \) and \( Z \) but not \( g_1 \) or \( g_2 \).
- The object \( \tilde{Y}_0 \) generates the subcategory with simple objects \( 1_\mathbb{D}, Z, g_1, g_2 \) and all \( \tilde{X}_j, \tilde{Y}_i \).
- The objects \( Z, \tilde{Y}_i, \tilde{X}_j \) are self-dual.
- The subcategory generated by \( \tilde{X}_0 \) is the adjoint subcategory \( \mathcal{D}_0 \). In particular no \( \tilde{Y}_i \) lie in the adjoint subcategory.
- The 4 objects \( V_i^{(j)} \) appear in two distinct graded components, in pairs.

One may directly show that \( \mathcal{D} \) has the same fusion rules as \( SO(\frac{N}{4})_{\mathbb{Z}_2} \) using standard techniques, however this is a somewhat tedious task. We will instead make use of [25, Theorem 4.2 and Remark 4.4] and the descriptions in Section 3 to derive the result. The first step is to verify that the fusion rules for the \( \frac{N}{4} \)-dimensional subcategory \( \langle \tilde{Y}_0 \rangle \) with simple objects of dimensions 1 and 2 has the fusion rules of either \( \text{Rep}(D_{\frac{N}{4}}) \) for \( t \) odd or \( \text{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_4) \) for \( t \) even. Then we must verify the fusion rules involving the \( V_i^{(j)} \) are also as expected. We will do these tasks simultaneously.

For \( t \) odd, the observations above reduce the verification of the hypotheses of Theorem 4.2 of [25] to showing that the \( g_i \) are self-dual, from which we can conclude that \( \langle \tilde{Y}_0 \rangle \) has the same
fusion rules as $\text{Rep}(D_N)$. For $t$ even, we must show that $g_1 \cong g_2^t$ to verify the hypotheses of Remark 4.4 of [25] to conclude that $\langle \hat{Y}_0 \rangle$ has the same fusion rules as $\text{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_4)$. We calculate:

$$F(V_1^{\otimes 2}) = \left( V_1^{(0)} \oplus V_1^{(1)} \right)^{\otimes 2} \cong g_1 \oplus g_2 \oplus 2(1_D \bigoplus_{i=0}^{\infty} \hat{Y}_i \oplus \bigoplus_{j=0}^{\infty} \hat{X}_j). \tag{1}$$

Since

$$\left( V_1^{(0)} \oplus V_1^{(1)} \right)^{\otimes 2} \cong \left( V_1^{(1)} \right)^{\otimes 2} \oplus \left( V_1^{(2)} \right)^{\otimes 2} \oplus 2(V_1^{(1)} \otimes V_1^{(2)})$$

it is clear that the $g_1, g_2$ cannot be subobjects of $(V_1^{(1)} \otimes V_1^{(2)})$ and $V_1^{(j)}$ for $j = 0, 1$ are either self-dual or dual to each other. As we have a labeling choice we may assume $g_j \subset \left( V_1^{(j)} \right)^{\otimes 2}$ for $j = 0, 1$.

Now observe that $\left( V_1^{(j)} \right)^{\otimes 2}$ is odd-dimensional when $t$ is even so that in this case $1_D$ is not a subobject of $\left( V_1^{(j)} \right)^{\otimes 2}$ and hence the $V_1^{(j)}$ are non-self-dual, i.e. are dual to each other for $j = 0, 1$. Moreover, the $g_i$ are not in the trivially graded component for $t$ even so that we can conclude that the grading is by $\mathbb{Z}_4$ in this case, so that the group of (isomorphism classes of) invertible objects is isomorphic to $\mathbb{Z}_4$ and hence $g_1 \cong g_2^t$. Thus we can conclude that the fusion rules are the same as those of $\text{Rep}(\mathbb{Z}_4 \ltimes \mathbb{Z}_4)$. Since the adjoint subcategory $\mathcal{D}_0$ contains only simple objects $1_D, Z$ and all $\hat{X}_j$, the fusion rules involving $V_1^{(j)}$ (and similarly $V_2^{(j)}$) are completely determined.

When $t$ is odd $\left( V_1^{(j)} \right)^{\otimes 2}$ is even-dimensional so we must have both $1_D$ and $g_j$ as subobjects. In particular, the grading is by $\mathbb{Z}_2 \times \mathbb{Z}_2$ so that the $g_i$ are self-dual. Now we can conclude that the fusion rules of the subcategory $\langle \hat{Y}_0 \rangle$ are the same as $\text{Rep}(D_N)$ and the fusion rules involving $V_1^{(j)}$ (and similarly $V_2^{(j)}$) are determined from equation (1).

Condensing the boson $h$ in an 8-metaplectic modular category produces a pointed category of dimension 8, with the same fusion rules as $\mathbb{Z}_8$, which we have conveniently identified with a 2-metaplectic modular category.

6. Conclusions and Speculations

We have obtained two results on metaplectic modular categories. For odd $N$, we extend the results of [29] proving property $F$ for $SO(N)_2$ to all $N$-metaplectic modular categories. This provides some insight into the relationships among (certain) braided fusion categories with the same fusion rules. A recent paper of Nikshych [27] explores the different braidings that a fixed fusion category may have. One consequence (see [27, Remark 4.2]) is that if a modular category has property $F$ then any braiding on the underlying fusion category has property $F$ as well (whether the braiding in non-degenerate or not). Of course, a fixed unitary fusion category has a unique unitary braiding by results of [18], so for metaplectic categories
this does not help. On the other hand, it seems to be often the case that all (finitely many, by Ocneanu rigidity [15]) fusion categories with a fixed set of fusion rules are related to each other by some type of twisting of associativity constraints (see [22, 30], for example). One conceptual step towards proving property \( F \) would be to extend the results of [27] to prove that braided fusion categories with a fixed set of fusion rules either all have property \( F \) or all do not.

In a related direction, we have shown that the \( N \)-metaplectic modular categories for \( 8 \mid N \) are obtained from \( 2k \)- and \( 4k \)-metaplectic modular categories (with \( k \geq 1 \) odd) by iteratively gauging by a non-trivial \( \mathbb{Z}_2 \)-action. Physically, this can be interpreted to mean that the systems modeled by \( 2^k \)-metaplectic modular categories for all \( s \geq 1 \) of the same parity are just different phases of the same topological order [9]. It is interesting to note that the number of \( 2^t \)-metaplectic modular categories stabilizes for \( t \geq 2 \), so that the choices in the \( \mathbb{Z}_2 \)-gauging process are eventually unique. Of course it is already known that any \( N \)-metaplectic modular category is a \( \mathbb{Z}_2 \)-gauging of a pointed category ([1, 6, 7]), but this result provides an infinite sequence of categories with non-trivial Picard group (see [17]), i.e. non-trivial braided tensor autoequivalences. Notice this is in contrast to the odd \( N \)-metaplectic modular categories: for example \( N = 3 \) we see that 3-metaplectic modular categories admit no non-trivial braided tensor autoequivalences. This can be deduced from [13]: the Brauer-Picard group of \( SO(3) = SU(2)_1 \) is \( \mathbb{Z}_2 \), with the non-trivial element corresponding to interchanging the two \( \sqrt{3} \)-dimensional objects. Since the twists of these two objects are distinct, this action does not give a braided tensor autoequivalence. Of course, failing to have a non-trivial Picard group does not preclude a category from having non-trivial (i.e. not a Deligne product) gaugings: the pointed category \( \text{Sem} \) has trivial Picard group and yet prime modular categories of the form \( \mathcal{C}(\mathbb{Z}_8, Q) \) can be obtained as \( \mathbb{Z}_2 \)-gaugings of \( \text{Sem} \) [3].

In the special case when \( N = 2^k \) we encounter degenerate (in the sense of Lie algebras) categories: an 8-metaplectic modular category with fusion rules like \( SO(8)_2 \) has 3 non-trivial bosons, but they cannot be distinguished. Condensing any of them yields a category with fusion rules like \( \mathbb{Z}_8 \) which is a 2-metaplectic category. If we condense the boson in any of the four \( \mathcal{C}(\mathbb{Z}_8, Q) \) theories we obtain either \( \text{Sem} \) or \( \overline{\text{Sem}} \), which we could call \( 1/2 \)-metaplectic. It is worth pointing out that \( SO(8)_2 \) should have an \( S_3 \) action by braided tensor autoequivalences.

For \( N = 16 \) if we condense either of the two bosons that centralize a simple object of dimension \( \sqrt{8} \) we obtain a 4-metaplectic modular category, of the form Ising\( \kappa \)Ising\( \mu \) e.g. \( SO(4)_2 \). It is known ([7]) that there are 12 inequivalent 16-metaplectic modular categories, whereas there are 20 with the same fusion rules as Ising\( \kappa \)Ising\( \mu \). Which of the 20 can appear in this way? In this case we find that only the 12 that are \( \mathbb{Z}_2 \)-gaugings of the 4 pointed categories \( \mathcal{C}(\mathbb{Z}_4, Q_s) \) have the correct central charge \( e^{(2s+1)\pi i/4} \). We could call these \( \mathcal{C}(\mathbb{Z}_4, Q_s) \) theories 1-metaplectic categories as they are obtained by condensing a boson in \( SO(4)_2 \). More generally, let \( k \geq 3 \) be an odd number with precisely \( s \) distinct prime factors. Then there are \( 2^{s+2} \) inequivalent \( 2k \)-metaplectic categories [6] and \( 3 \cdot 2^{s+2} \) inequivalent \( 2^a k \)-metaplectic categories for \( a \geq 2 \) [7]. In particular we find that the cohomological choices in the gauging
process from a $2^k$-metaplectic category to a $2^{k+2}$-metaplectic category does not increase the number of such categories, rather their cardinality stabilizes.

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