A generalized exponential-type estimator for population mean using auxiliary attributes

Sohail Ahmad\(^1\)*, Muhammad Arslan\(^2\), Aamna Khan\(^3\), Javid Shabbir\(^4\)

1 Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan, 2 School of Statistics, Shanxi University of Finance and Economics, Taiyuan, China, 3 Department of Statistics, Bahauddin Zakariya University, Multan, Pakistan, 4 Department of Statistics, Quaid-i-Azam University, Islamabad, Pakistan

* asohail@stat.qau.edu.pk

Abstract

In this paper, we propose a generalized class of exponential type estimators for estimating the finite population mean using two auxiliary attributes under simple random sampling and stratified random sampling. The bias and mean squared error (MSE) of the proposed class of estimators are derived up to first order of approximation. Both empirical study and theoretical comparisons are discussed. Four populations are used to support the theoretical findings. It is observed that the proposed class of estimators perform better as compared to all other considered estimator in simple and stratified random sampling.

Introduction

In survey sampling, we generally use the auxiliary information to increase precision of the estimators by taking the advantages of correlation between the study variable \(y\) and the auxiliary variable \(x\). When the relationship between \(y\) and \(x\) is positive, then the ratio estimator gives efficient results, and when the relationship is negative, then the product estimator performs better. However under different setup, the auxiliary attributes have been studied by different authors. In some situations, auxiliary information can be quantified in the form of auxiliary proportions to get better precision. For this reason, several authors have used one or more auxiliary proportions at the estimation stage to increase the efficiency of the estimators. The ratio, product and regression methods of estimation are the good examples in this context. These methods of estimation are more efficient than the usual mean per unit estimator under certain conditions. Different estimators have been suggested or modified by many authors when used mixed types estimators i.e using regression and ratio type estimators or exponential type estimator. These mixed estimators outperformed than the individual estimators. For this reason, several authors have used the auxiliary variables and auxiliary attributes at estimation stage to increase the efficiency of the estimators. For example, the expansiveness of a tree can be used as a key auxiliary variable while estimating the ordinary stature of trees in a forest and moreover the sort of a dairy creatures is a significant auxiliary characteristic while estimating typical milk yield. Additionally, to estimate the mean time-based compensations earned by the individuals, the auxiliary attribute can be utilized in type of the education and martial status etc.

Wynn [1] proposed an unbiased ratio estimator using the auxiliary characters in the form of known population proportion of the auxiliary variable when the population is divided into
two classes. Singh et al. [2] suggested ratio of proportion utilizing the auxiliary variables for further investigation. Naik and Gupta [3] introduced the idea of point bi-seral correlation coefficient. Using this idea, many authors have used the information on the auxiliary attribute for improving the precision of the estimators. Solanki and Singh [4] suggested a class of estimators for population mean of the study variable using known population proportion of the auxiliary attribute. The suggested class of estimators is more general and includes the usual unbiased sample mean estimator. Expressions of bias and mean square error (MSE) are obtained under large sample approximation. Malik and Singh [5] introduced the exponential type estimator using two auxiliary attributes. Zeng et al. [6] proposed Tobit model for the analysis of crash rates by injury severity for both correlation across injury severity and unobserved heterogeneity across road-segment observations are accommodated. The Tobit model is compared multivariate parameters in the context of Bayesian. In another study, Zeng et al. [7] researches multivariate spatio-transient examination for characterizing territory wide accident rates by injury seriousness. Likewise, Zeng et al. [8] investigated the connection between zone-level daytime and evening time crash frequencies and different elements identified with traffic, organization, and land use, including VHT, normal speed, street thickness, crossing point thickness, network example, and land use design.

Consider a finite population $U = \{U_1, U_2, \ldots, U_N\}$ of size $N$. We draw a sample of size $n$ by SRSWOR from a population $U$. Let $y_i$ be the study variable and $\phi_i$ be the characteristics of the auxiliary attribute i.e. $\phi_i = 1$ if the $i^{th}$ unit possess attribute and $\phi_i = 0$, otherwise. Let $A = \sum_{i=1}^{N} \phi_i$ be the total number of units in the population possessing attribute $\phi_i$ and $a = \sum_{i=1}^{n} \phi_i$ be the total number of units in the sample possessing attribute $\phi_i$. Let $P = (A/N)$ be the proportion of units in the population and $p = (a/n)$ be the proportion of units in the sample. Let $\bar{Y} = \frac{\sum_{i=1}^{N} y_i}{N}$ and $\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$ be the population mean and the sample mean respectively. Let $P_1$ and $P_2$ be the population proportion of auxiliary attributes and $p_1$ and $p_2$ be the sample proportion of auxiliary attributes. Let $S^2_{y} = \sum_{i=1}^{N} \frac{(y_i - \bar{Y})^2}{N-1}$ be the population variance of the study variable $y$. Let $S^2_{p_{1}} = \sum_{i=1}^{N} \frac{(p_{1i} - P_{1})^2}{N-1}$ and $S^2_{p_{2}} = \sum_{i=1}^{N} \frac{(p_{2i} - P_{2})^2}{N-1}$ respectively be the population variance of the auxiliary attributes $p_1$ and $p_2$. Let $C_y = \frac{S^2_{y}}{\bar{Y}}$ respectively be the co-efficient of variation of the study variable $y$. Let $C_{P_1} = \frac{S^2_{p_1}}{P_{1}}$ and $C_{P_2} = \frac{S^2_{p_2}}{P_{2}}$ be the co-efficients of variation of the auxiliary attributes $p_1$ and $p_2$. Let $S^2_{y/p_1} = \sum_{i=1}^{N} \frac{(y_i - \bar{y})\phi_i}{N-1}$ be the population covariance between the study variable $y$ and the auxiliary attribute $p_1$ and $S^2_{y/p_2} = \sum_{i=1}^{N} \frac{(y_i - \bar{y})\phi_i}{N-1}$ be the population covariance between the study variable $y$ and the auxiliary attribute $p_2$. Let $S^2_{p_1/p_2} = \sum_{i=1}^{N} \frac{(p_{1i} - P_{1})(p_{2i} - P_{2})}{N-1}$ be the population covariance between the auxiliary attributes $p_1$ and $p_2$. Let $\rho_{y/p_1} = \frac{S^2_{y/p_1}}{\sqrt{S^2_{y}S^2_{p_{1}}}}$ be the population point bi-serial correlation co-efficient between the study variable $y$ and the auxiliary attribute $p_1$. Let $\rho_{y/p_2} = \frac{S^2_{y/p_2}}{\sqrt{S^2_{y}S^2_{p_{2}}}}$ be the population point bi-serial correlation co-efficient between the study variable $y$ and the auxiliary attribute $p_2$. Let $\rho_{p_1/p_2} = \frac{S^2_{p_1/p_2}}{\sqrt{S^2_{p_{1}}S^2_{p_{2}}}}$ be the phi-correlation coefficient between the auxiliary attributes $p_1$ and $p_2$. Let $e_{i} = \frac{y_i - \bar{y}}{\bar{Y}}$, $e_{1} = \frac{p_{1i} - P_{1}}{P_{1}}$ and $e_{2} = \frac{p_{2i} - P_{2}}{P_{2}}$ be the error terms such that $E(e_{i}) = 0, (i = 0,1,2)$, $E(e_{0}e_{i}) = \lambda C_{y}, E(e_{0}e_{1}) = \lambda C_{p_{1}}, E(e_{0}e_{2}) = \lambda C_{p_{2}}, E(e_{1}e_{2}) = \lambda C_{p_{1}p_{2}}, E(e_{1}e_{1}) = \lambda \rho_{p_1p_2} C_{p_1} C_{p_2}$ and $E(e_{2}e_{2}) = \lambda \rho_{p_1p_2} C_{p_1} C_{p_2}$, where $\lambda = \frac{1}{(n-1)}$ and $f = \frac{n}{N}$.

**Existing estimators in simple random sampling**

We discuss the following estimators available in literature.
1. The usual mean per unit estimator in simple random sampling is:

\[ \bar{y}_0 = \bar{y} \]  

The MSE or variance of \( \bar{y}_0 \), is given by:

\[ \text{Var}(\bar{y}_0) = \lambda \bar{Y}^2 C_y^2 \]  

2. The ordinary ratio type estimator, is given by:

\[ \bar{y}_r = \bar{y} \left( \frac{P_1}{P} \right) \]  

3. The usual product type estimator, is given by:

\[ \bar{y}_p = \bar{y} \left( \frac{P}{P_1} \right) \]  

It is well known that the \( \bar{y}_r \) and \( \bar{y}_p \) are more precise than usual mean estimator \( \bar{y}_0 \) when

\[ \rho_{yp} > \frac{1}{2} \frac{C_y}{C_p} \] and \( \rho_{yp} < -\frac{1}{2} \frac{C_y}{C_p} \) respectively.

The bias and MSE of \( \bar{y}_r \), are given by:

\[ \text{Bias}(\bar{y}_r) \approx \bar{Y} \lambda \left[ C_y^2 - \rho_{yp} C_y C_p \right], \]  

and

\[ \text{MSE}(\bar{y}_r) \approx \bar{Y}^2 \lambda \left[ C_y^2 + C_p^2 - 2\rho_{yp} C_y C_p \right]. \]  

Similarly the bias and MSE of \( \bar{y}_p \), are given by:

\[ \text{Bias}(\bar{y}_p) \approx \bar{Y} \lambda \left[ \rho_{yp} C_y C_p \right], \]  

and

\[ \text{MSE}(\bar{y}_p) \approx \bar{Y}^2 \lambda \left[ C_y^2 + C_p^2 + 2\rho_{yp} C_y C_p \right]. \]  

4. Bahl and Tuteja [9] proposed ratio and product type estimators for estimating finite population mean using information on single auxiliary attribute.

\[ \bar{y}_{\text{exp}(r)} = \bar{y} \exp \left( \frac{P_1 - P}{P_1 + P} \right), \]  

and

\[ \bar{y}_{\text{exp}(p)} = \bar{y} \exp \left( \frac{P_1 - P}{P_1 + P} \right). \]  

The bias and MSE of \( \bar{y}_{\text{exp}(r)} \), are given by:

\[ \text{Bias}(\bar{y}_{\text{exp}(r)}) \approx \bar{Y} \lambda \left[ \frac{3}{8} C_p^2 - \frac{1}{2} \rho_{yp} C_y C_p \right], \]
and
\[
\text{MSE}\left(\hat{y}_{\text{exp}(R)}\right) \approx \bar{Y}^2 \lambda \left[ C_y^2 + \frac{1}{4} C_{p_i}^2 - \rho_{y_1,p} C_y C_{p_i} \right].
\] (12)

Similarly the bias and MSE of \(\hat{y}_{\text{exp}(R)}\) are given by:
\[
\text{Bias}\left(\hat{y}_{\text{exp}(R)}\right) = \frac{1}{2} \bar{Y} \lambda \left[ \rho_{y_1,p} C_y C_{p_i} - \frac{1}{4} C_{p_i}^2 \right],
\] (13)

and
\[
\text{MSE}\left(\hat{y}_{\text{exp}(P)}\right) = \bar{Y}^2 \lambda \left[ C_y^2 + \frac{1}{4} C_{p_i}^2 + \rho_{y_1,p} C_y C_{p_i} \right].
\] (14)

5. Kumar and Bhogal [10] proposed an exponential ratio-product type estimator, is given by:
\[
\hat{y}_{\text{KB}(RP)} = \bar{y} \left[ \exp \left( \frac{P_1 - P_2}{P_1 + P_2} \right) + (1 - \alpha) \exp \left( \frac{P_1 - P_1}{P_1 + P_1} \right) \right]
\] (15)

where \(\alpha\) is unknown constant.

The bias and minimum MSE of \(\hat{y}_{\text{KB}(RP)}\) are given by:
\[
\text{Bias}\left(\hat{y}_{\text{KB}(RP)}\right) \approx \bar{Y} \lambda \left[ -\frac{1}{2} (4\alpha - 1) C_{p_i}^2 - \left( \alpha - \frac{1}{2} \right) \rho_{y_1,p} C_y C_{p_i} \right]
\] (16)

and
\[
\text{MSE}\left(\hat{y}_{\text{KB}(RP)}\right)_{\text{min}} \approx \bar{Y}^2 \lambda \left[ 1 - \rho_{y_1,p}^2 \right].
\] (17)

The optimum value of \(\alpha\), is given by:
\[
\alpha_{opt} \equiv \frac{1}{2} + \rho_{y_1,p} \frac{C_y}{C_{p_i}}.
\] (18)

6. Singh and Kumar [11] suggested double ratio and product type estimators, are given by:
\[
\hat{y}_{\text{SK}(DR)} = \bar{y} \left( \frac{p_1}{p_1} \right) \left( \frac{p_2}{p_2} \right),
\] (19)

and
\[
\hat{y}_{\text{SK}(DP)} = \bar{y} \left( \frac{p_1}{p_1} \right) \left( \frac{p_2}{p_2} \right).
\] (20)

The bias and MSE of \(\hat{y}_{\text{SK}(DR)}\) are given by:
\[
\text{Bias}\left(\hat{y}_{\text{SK}(DR)}\right) \approx \bar{Y} \lambda \left[ C_y^2 + C_{p_i}^2 + \rho_{p_1,p_2} C_p C_{p_i} - \rho_{y_1,p} C_y C_{p_i} - \rho_{y_2,p} C_y C_{p_i} \right],
\] (21)

and
\[
\text{MSE}\left(\hat{y}_{\text{SK}(DR)}\right) \approx \bar{Y}^2 \lambda \left[ C_y^2 + C_{p_i}^2 + C_{p_2}^2 - 2(\rho_{y_1,p} C_y C_{p_i} - \rho_{p_1,p_2} C_p C_{p_i} + \rho_{y_2,p} C_y C_{p_i}) \right].
\] (22)

Similarly the bias and MSE of \(\hat{y}_{\text{SK}(DP)}\) are given by:
\[
\text{Bias}\left(\hat{y}_{\text{SK}(DP)}\right) \approx \bar{Y} \lambda \left[ \rho_{p_1,p_2} C_p C_{p_i} + \rho_{y_1,p} C_y C_{p_i} + \rho_{y_2,p} C_y C_{p_i} \right],
\] (23)
and

\[
\text{MSE}(\hat{y}_{\text{SK}(p)}) \cong \bar{Y}^2 \left[ \sum_j \left( C_j^2 + C_{p_j}^2 + C_{p_j}^2 \right) + 2 \left( \rho_{p_j} C_j C_{p_j} + \rho_{p_j} C_j C_{p_j} + \rho_{p_j} C_j C_{p_j} \right) \right].
\]  

(24)

**Proposed class of estimators**

In the lines of Shukla et al. [12], we proposed a generalized class of factor type estimators for mean estimator. The proposed estimator, is given by:

\[
\hat{y}_{\text{prop}} = \bar{Y} \left[ \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right) \right],
\]  

(25)

where,

\[
S_1 = (A_1 + C_1)P_1 + fB_1 P_1, \quad S_2 = (A_2 + C_2)P_2 + fB_2 P_2, \quad M_1 = (A_1 + fB_1)P_1 + C_1 P_1,
\]

\[
M_2 = (A_2 + fB_2)P_2 + C_2 P_2, \quad A_1 = (K_i - 1)(K_i - 2), \quad B_i = (K_i - 1)(K_i - 4) \text{ and } C_i = (K_i - 2)(K_i - 3)(K_i - 4).
\]

Substituting different values of \(K_i (i = 1,2,3,4)\) in (25), we can generate many more different types of estimators from our general proposed class of estimators (see Table 1).

Solving \(\hat{y}_{\text{prop}}\) given in Eq (25) in terms of errors, we have

\[
\hat{y}_{\text{prop}} \cong \bar{Y} \left( 1 + \frac{1}{2} \sigma_x \epsilon_i - \frac{1}{4} \sigma_x \epsilon_i^2 + \frac{1}{8} \sigma_x^2 \epsilon_i^2 + \ldots \right) \left( 1 + \frac{1}{2} \sigma_y \epsilon_i - \frac{1}{4} \sigma_y \epsilon_i^2 + \frac{1}{8} \sigma_y^2 \epsilon_i^2 + \ldots \right),
\]  

(26)

**Table 1. Family members of proposed class of estimators.**

| S.No. | \(K_1\) | \(K_2\) | Estimators |
|-------|--------|--------|-----------|
| 1     | 1      | 1      | \(\hat{y}_{\text{prop} 1} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 2     | 1      | 2      | \(\hat{y}_{\text{prop} 2} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 3     | 1      | 3      | \(\hat{y}_{\text{prop} 3} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 4     | 1      | 4      | \(\hat{y}_{\text{prop} 4} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 5     | 2      | 1      | \(\hat{y}_{\text{prop} 5} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 6     | 2      | 2      | \(\hat{y}_{\text{prop} 6} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 7     | 2      | 3      | \(\hat{y}_{\text{prop} 7} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 8     | 2      | 4      | \(\hat{y}_{\text{prop} 8} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 9     | 3      | 1      | \(\hat{y}_{\text{prop} 9} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 10    | 3      | 2      | \(\hat{y}_{\text{prop} 10} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 11    | 3      | 3      | \(\hat{y}_{\text{prop} 11} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 12    | 3      | 4      | \(\hat{y}_{\text{prop} 12} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 13    | 4      | 1      | \(\hat{y}_{\text{prop} 13} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 14    | 4      | 2      | \(\hat{y}_{\text{prop} 14} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 15    | 4      | 3      | \(\hat{y}_{\text{prop} 15} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |
| 16    | 4      | 4      | \(\hat{y}_{\text{prop} 16} = \bar{Y} \exp \left( \frac{S_1 - M_1}{S_1 + M_1} \right) \exp \left( \frac{S_2 - M_2}{S_2 + M_2} \right)\) |

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where

\[
\sigma_1 = \frac{fB_1 - C_1}{A_1 + fB_1 + C_1}, \quad \nu_1 = \frac{fB_1 + C_1}{A_1 + fB_1 + C_1},
\]

and

\[
\sigma_2 = \frac{fB_2 - C_2}{A_2 + fB_2 + C_2}, \quad \nu_2 = \frac{fB_2 + C_2}{A_2 + fB_2 + C_2}.
\]

To first order approximation, we have

\[
\left( \bar{y}_{\text{prop}} - \bar{Y} \right) = \bar{Y} \left[ e_0 + \frac{1}{2} \rho_1 \rho_{1p1} C_1 C_{p1} \rho_{2p2} + 2 \rho_2 \rho_{2p2} C_2 C_{p2} + \frac{1}{8} \rho_2 C_2 \left( \sigma_1^2 - 2 \sigma_1 \nu_1 \right) + \frac{1}{8} \rho_2 C_2 \left( \sigma_2^2 - 2 \sigma_2 \nu_2 \right) \right],
\]

(27)

Using (27), the bias and MSE of \( \bar{y}_{\text{prop}} \) are given by:

\[
\text{Bias} \left( \bar{y}_{\text{prop}} \right) \approx \bar{Y} \left[ \frac{1}{2} \rho_1 \rho_{1p1} C_1 C_{p1} + \frac{1}{2} \rho_2 \rho_{2p2} C_2 C_{p2} + \frac{1}{8} \rho_2 C_2 \left( \sigma_1^2 - 2 \sigma_1 \nu_1 \right) + \frac{1}{8} \rho_2 C_2 \left( \sigma_2^2 - 2 \sigma_2 \nu_2 \right) \right],
\]

(28)

and

\[
\text{MSE} \left( \bar{y}_{\text{prop}} \right) = \bar{Y}^2 \left[ \frac{1}{4} \sigma_1^2 C_1^2 + \frac{1}{4} \sigma_2^2 C_2^2 + \rho_1 \rho_{1p1} C_1 C_{p1} + \rho_2 \rho_{2p2} C_2 C_{p2} + \frac{1}{2} \sigma_1 \rho_1 \rho_{1p1} C_1 C_{p1} + \frac{1}{2} \sigma_2 \rho_1 \rho_{2p2} C_2 C_{p2} \right].
\]

(29)

Differentiate Eq (29) with respect to \( \sigma_1 \) and \( \sigma_2 \), we get the optimum values of \( \sigma_1 \) and \( \sigma_2 \) i.e.

\[
\sigma_1(\text{opt}) = \frac{2C_1 (\rho_{1p1} - \rho_{1p2} \rho_{2p2})}{C_1 (\rho_{1p2} - 1)},
\]

and

\[
\sigma_2(\text{opt}) = \frac{2C_2 (\rho_{2p2} - \rho_{1p2} \rho_{1p1})}{C_2 (\rho_{1p2} - 1)}.
\]

Substituting the optimum values i.e \( \sigma_1(\text{opt}) \) and \( \sigma_2(\text{opt}) \) in (29), we get minimum MSE of \( \bar{y}_{\text{prop}} \),

is given by:

\[
\text{MSE} \left( \bar{y}_{\text{prop}} \right)_{\text{min}} \approx \bar{Y}^2 C_2 \left[ 1 - R_{\text{yp2p2}}^2 \right],
\]

(30)

where

\[
R_{\text{yp2p2}}^2 = \frac{\rho_{1p1}^2 + \rho_{2p2}^2 - 2 \rho_{1p1} \rho_{2p2} \rho_{1p2}}{1 - \rho_{1p2}^2},
\]

is the multiple correlation coefficient of \( y \) on \( p_1 \) and \( p_2 \).

Now by putting different values of \( K_i \) in Eq (25) some members of the proposed class of estimators can be obtained as:

1. For \( K_1 = 1 \) and \( K_2 = 1 \)

\[
\bar{y}_{\text{prop}}_1 = \bar{y} \exp \left( \frac{P_1 - P_2}{P_1 + P_2} \right) \exp \left( \frac{P_2 - P_1}{P_2 + P_1} \right).
\]
The bias and MSE of $\tilde{y}_{\text{prop }1}$ are given by:

$$\text{Bias}(\tilde{y}_{\text{prop }1}) \approx \lambda \bar{Y} \left[ \frac{3}{8} C_p^2 + \frac{3}{8} C_p^2 - \frac{1}{2} \rho_{yp_1} C_y C_p - \frac{1}{2} \rho_{yp_2} C_y C_p \right]$$

and

$$\text{MSE}(\tilde{y}_{\text{prop }1}) \approx \lambda \bar{Y}^2 \left[ C_p^2 + \frac{1}{4} C_p^2 + \frac{1}{4} C_p^2 - \rho_{yp_1} C_y C_p + \rho_{yp_2} C_y C_p + \frac{1}{2} \rho_{yp_2} C_y C_p \right].$$

2. For $K_1 = 1$ and $K_2 = 2$

$$\tilde{y}_{\text{prop }2} = \tilde{y} \exp \left( \frac{P_1 - P_1}{P_1 + P_1} \right) \exp \left( \frac{P_2 - P_2}{P_2 + P_2} \right)$$

The bias and MSE of $\tilde{y}_{\text{prop }2}$ are given by:

$$\text{Bias}(\tilde{y}_{\text{prop }2}) \approx \lambda \bar{Y} \left[ -\frac{1}{8} C_p^2 - \frac{1}{8} C_p^2 - \frac{1}{2} \rho_{yp_1} C_y C_p \right]$$

and

$$\text{MSE}(\tilde{y}_{\text{prop }2}) \approx \lambda \bar{Y}^2 \left[ C_p^2 + \frac{1}{4} C_p^2 + \frac{1}{4} C_p^2 - \rho_{yp_1} C_y C_p + \rho_{yp_2} C_y C_p - \frac{1}{2} \rho_{yp_2} C_y C_p \right].$$

3. For $K_1 = 1$ and $K_2 = 3$

$$\tilde{y}_{\text{prop }3} = \tilde{y} \exp \left( \frac{P_1 - P_1}{P_1 + P_1} \right) \exp \left( \frac{n(P_2 - P_2)}{2NP_2 - n(P_2 + P_2)} \right)$$

The bias and MSE of $\tilde{y}_{\text{prop }3}$ are given by:

$$\text{Bias}(\tilde{y}_{\text{prop }3}) \approx \lambda \bar{Y} \left[ \frac{3}{8} C_p^2 + \frac{1}{8} C_p^2 - \frac{1}{4} \left( \frac{f}{1-f} \right)^2 C_p^2 - \frac{1}{2} \rho_{yp_1} C_y C_p - \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{yp_2} C_y C_p \right]$$

and

$$\text{MSE}(\tilde{y}_{\text{prop }3}) \approx \lambda \bar{Y}^2 \left[ C_p^2 + \frac{1}{4} C_p^2 + \frac{1}{4} \left( \frac{f}{1-f} \right)^2 C_p^2 - \rho_{yp_1} C_y C_p - \left( \frac{f}{1-f} \right) \rho_{yp_2} C_y C_p + \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{yp_2} C_y C_p \right].$$

4. For $K_1 = 2$ and $K_2 = 1$

$$\tilde{y}_{\text{prop }5} = \tilde{y} \exp \left( \frac{P_1 - P_1}{P_1 + P_1} \right) \exp \left( \frac{P_2 - P_2}{P_2 + P_2} \right)$$

The bias and MSE of $\tilde{y}_{\text{prop }5}$ are given by:

$$\text{Bias}(\tilde{y}_{\text{prop }5}) \approx \lambda \bar{Y} \left[ -\frac{1}{8} C_p^2 + \frac{3}{8} C_p^2 + \frac{3}{8} C_p^2 - \frac{1}{2} \rho_{yp_1} C_y C_p - \frac{1}{2} \rho_{yp_2} C_y C_p \right]$$

and

$$\text{MSE}(\tilde{y}_{\text{prop }5}) \approx \lambda \bar{Y}^2 \left[ C_p^2 + \frac{1}{4} C_p^2 + \frac{1}{4} C_p^2 - \rho_{yp_1} C_y C_p + \rho_{yp_2} C_y C_p - \frac{1}{2} \rho_{yp_2} C_y C_p \right].$$
5. For $K_1 = 2$ and $K_2 = 2$

$$\bar{y}_{prop 6} = \bar{y} \exp\left(\frac{p_1 - P_1}{P_1 + P_1}\right) \exp\left(\frac{p_2 - P_2}{P_2 + P_2}\right)$$

The bias and MSE of $\bar{y}_{prop 6}$ are given by:

$$\text{Bias}\left(\bar{y}_{prop 6}\right) \approx \lambda \bar{Y} \left[-\frac{1}{8} C_1^2 - \frac{1}{8} C_2^2 + \frac{1}{2} \rho_{y_1} C_y C_1 + \frac{1}{2} \rho_{y_2} C_y C_2\right]$$

and

$$\text{MSE}\left(\bar{y}_{prop 6}\right) \approx \lambda \bar{Y}^2 \left[C_1^2 + \frac{1}{4} C_1^2 + \frac{1}{4} C_2^2 + \rho_{y_1} C_y C_1 + \rho_{y_2} C_y C_2 + \frac{1}{2} \rho_{y_1} C_y C_2\right].$$

6. For $K_1 = 2$ and $K_2 = 3$

$$\bar{y}_{prop 7} = \bar{y} \exp\left(\frac{p_1 - P_1}{P_1 + P_1}\right) \exp\left(\frac{n(p_2 - P_2)}{2NP_2 - n(p_2 + P_2)}\right)$$

The bias and MSE of $\bar{y}_{prop 7}$ are given by:

$$\text{Bias}\left(\bar{y}_{prop 7}\right) \approx \lambda \bar{Y} \left[-\frac{1}{8} C_1^2 - \frac{1}{8} \left(\frac{f}{1 - f}\right) C_2^2 + \frac{1}{2} \rho_{y_1} C_y C_1 - \frac{1}{2} \left(\frac{f}{1 - f}\right) \rho_{y_2} C_y C_2\right]$$

and

$$\text{MSE}\left(\bar{y}_{prop 7}\right) \approx \lambda \bar{Y}^2 \left[C_1^2 + \frac{1}{4} C_1^2 + \frac{1}{4} \left(\frac{f}{1 - f}\right)^2 C_2^2 + \rho_{y_1} C_y C_1 - \left(\frac{f}{1 - f}\right) \rho_{y_2} C_y C_2 - \frac{1}{2} \left(\frac{f}{1 - f}\right) \rho_{y_1} C_y C_2\right].$$

7. For $K_1 = 3$ and $K_2 = 1$

$$\bar{y}_{prop 8} = \bar{y} \exp\left(\frac{p_1 - P_1}{P_1 + P_1}\right) \exp\left(\frac{n(p_2 - P_2)}{2NP_2 - n(p_2 + P_2)}\right)$$

The bias and MSE of $\bar{y}_{prop 8}$ are given by:

$$\text{Bias}\left(\bar{y}_{prop 8}\right) \approx \lambda \bar{Y} \left[-\frac{1}{8} \left(\frac{f}{1 - f}\right)^2 C_1^2 + \frac{3}{8} C_2^2 - \frac{1}{2} \left(\frac{f}{1 - f}\right) \rho_{y_1} C_y C_1 + \frac{1}{2} \rho_{y_2} C_y C_2\right]$$

and

$$\text{MSE}\left(\bar{y}_{prop 8}\right) \approx \lambda \bar{Y}^2 \left[C_1^2 + \frac{1}{4} \left(\frac{f}{1 - f}\right)^2 C_1^2 + \frac{1}{4} C_2^2 - \left(\frac{f}{1 - f}\right) \rho_{y_1} C_y C_1 - \rho_{y_2} C_y C_2 + \frac{1}{2} \left(\frac{f}{1 - f}\right) \rho_{y_1} C_y C_2\right].$$

8. For $K_1 = 3$ and $K_2 = 2$

$$\bar{y}_{prop 10} = \bar{y} \exp\left(\frac{n(p_1 - P_1)}{2NP_2 - n(p_2 + P_2)}\right) \exp\left(\frac{p_2 - P_2}{p_2 + P_2}\right)$$

The bias and MSE of $\bar{y}_{prop 10}$ are given by:

$$\text{Bias}\left(\bar{y}_{prop 10}\right) \approx \lambda \bar{Y} \left[-\frac{1}{8} \left(\frac{f}{1 - f}\right)^2 C_1^2 - \frac{1}{8} C_2^2 - \frac{1}{2} \left(\frac{f}{1 - f}\right) \rho_{y_1} C_y C_1 + \frac{1}{2} \left(\frac{f}{1 - f}\right) \rho_{y_2} C_y C_2\right]$$
and

$$\text{MSE} \left( \bar{y}_{\text{prop} 10} \right) \approx \lambda \bar{Y}^2 \left[ C_y + 1 \left( \frac{-f}{1-f} \right)^2 C_{p_1} + 1 \left( \frac{-f}{1-f} \right)^2 C_{p_2} - \left( \frac{f}{1-f} \right) \rho_{p_1} C_y C_{p_1} + \rho_{p_2} C_y C_{p_2} - \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{p_1} \rho_{p_2} C_y C_{p_1} C_{p_2} \right].$$

9. For $K_1 = 3$ and $K_2 = 3$

$$\bar{y}_{\text{prop} 11} = \bar{y} \exp \left( \frac{n(P_1 - p_1)}{2NP_1 - n(p_1 + P_1)} \right) \exp \left( \frac{n(P_2 - p_2)}{2NP_2 - n(p_2 + P_2)} \right)$$

The bias and MSE of $\bar{y}_{\text{prop} 11}$ are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop} 11} \right) \approx \lambda \bar{Y} \left[ -1 \left( \frac{-f}{1-f} \right)^2 C_{p_1} - 1 \left( \frac{-f}{1-f} \right)^2 C_{p_2} - \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{p_1} C_y C_{p_1} - \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{p_2} C_y C_{p_2} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop} 11} \right) \approx \lambda \bar{Y}^2 \left[ C_y + 1 \left( \frac{-f}{1-f} \right)^2 C_{p_1} + 1 \left( \frac{-f}{1-f} \right)^2 C_{p_2} - \left( \frac{f}{1-f} \right) \rho_{p_1} C_y C_{p_1} - \left( \frac{f}{1-f} \right) \rho_{p_2} C_y C_{p_2} + \frac{1}{2} \left( \frac{f}{1-f} \right)^2 \rho_{p_1} \rho_{p_2} C_y C_{p_1} C_{p_2} \right].$$

10. For $K_1 = 3$ and $K_2 = 4$

$$\bar{y}_{\text{prop} 12} = \bar{y} \exp \left( \frac{n(P_1 - p_1)}{2NP_1 - n(p_1 + P_1)} \right)$$

The bias and MSE of $\bar{y}_{\text{prop} 12}$ are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop} 12} \right) \approx \lambda \bar{Y} \left[ -1 \left( \frac{-f}{1-f} \right)^2 C_{p_1} - \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{p_1} C_y C_{p_1} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop} 12} \right) \approx \lambda \bar{Y}^2 \left[ C_y + \left( \frac{-f}{1-f} \right)^2 C_{p_1} - \left( \frac{f}{1-f} \right) \rho_{p_1} C_y C_{p_1} \right].$$

11. For $K_1 = 4$ and $K_2 = 1$

$$\bar{y}_{\text{prop} 13} = \bar{y} \exp \left( \frac{P_2 - p_2}{P_2 + p_2} \right)$$

The bias and MSE of $\bar{y}_{\text{prop} 13}$ are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop} 13} \right) \approx \lambda \bar{Y} \left[ \frac{3}{8} C_{p_2} - \frac{1}{2} \rho_{p_2} C_y C_{p_2} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop} 13} \right) \approx \lambda \bar{Y}^2 \left[ C_y + \frac{1}{4} C_{p_2} - \rho_{p_1} C_y C_{p_2} \right].$$
12. For $K_1 = 4$ and $K_2 = 2$

$$\bar{y}_{\text{prop} \ 14} = \bar{y} \exp \left( \frac{p_2 - P}{p_2 + P} \right)$$

The bias and MSE of $\bar{y}_{\text{prop} \ 14}$, are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop} \ 14} \right) \approx \lambda \bar{Y} \left[ -\frac{1}{8} C_{p_2}^2 + \frac{1}{2} \rho_{yp_2} C_{y} C_{p_2} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop} \ 14} \right) \approx \lambda \bar{Y}^2 \left[ C_{y}^2 + \frac{1}{4} C_{p_2}^2 + \rho_{yp_2} C_{y} C_{p_2} \right].$$

13. For $K_1 = 4$ and $K_2 = 3$

$$\bar{y}_{\text{prop} \ 15} = \bar{y} \exp \left( \frac{n(P_2 - p_2)}{2NP_2 - n(p_2 + P_2)} \right)$$

The bias and MSE of $\bar{y}_{\text{prop} \ 15}$, are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop} \ 15} \right) \approx \lambda \bar{Y} \left[ -\frac{1}{8} \left( \frac{f}{1-f} \right)^2 C_{p_2}^2 - \frac{1}{2} \left( \frac{f}{1-f} \right) \rho_{yp_2} C_{y} C_{p_2} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop} \ 15} \right) \approx \lambda \bar{Y}^2 \left[ C_{y}^2 + \frac{1}{4} \left( \frac{f}{1-f} \right)^2 C_{p_2}^2 - \left( \frac{f}{1-f} \right) \rho_{yp_2} C_{y} C_{p_2} \right].$$

**Theoretical comparison**

In this section, we compare our proposed generalized exponential type estimator with other estimator is given by:

1. From (2) and (30),

$$\text{MSE}\left( \bar{y}_{\text{prop} \ 14} \right)_{\text{min}} < \text{MSE}(\bar{y}_o) \text{ if }$$

$$\text{MSE}(\bar{y}_o) - \text{MSE}\left( \bar{y}_{\text{prop} \ 14} \right)_{\text{min}} > 0 \text{ or if }$$

$$\lambda \bar{Y}^2 C_{y}^2 - \bar{Y}^2 \lambda C_{y}^2 \left[ 1 - R_{yp_2}^2 \right] > 0, \text{ or if }$$

$$R_{yp_2}^2 > 0.$$  

2. From (6) and (30),

$$\text{MSE}\left( \bar{y}_{\text{prop} \ 15} \right)_{\text{min}} < \text{MSE}(\bar{y}_R) \text{ if }$$

$$\text{MSE}(\bar{y}_R) - \text{MSE}\left( \bar{y}_{\text{prop} \ 15} \right)_{\text{min}} > 0 \text{ or if }$$

$$\bar{Y}^2 \lambda \left[ C_{y}^2 + C_{p_1}^2 - 2 \rho_{yp_1} C_{y} C_{p_1} \right] - \bar{Y}^2 \lambda C_{y}^2 \left[ 1 - R_{yp_1}^2 \right] > 0, \text{ or if }$$
\[ C_{p_1}^2 - 2\rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} + C_{jy_{p_2}p_2}^2 \geq 0. \]

3. From (8) and (30),

\[ \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\tilde{y}_p) \text{ if} \]

\[ \text{MSE}(\tilde{y}_p) - \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}^2 \lambda \left[ C_j^2 + C_{p_1}^2 + 2\rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} \right] - \bar{Y}^2 \lambda C_{jy_{p_2}}^2 \left[ 1 - R_{y_{p_1}p_2}^2 \right] > 0, \text{ or if} \]

\[ \left[ C_{p_1}^2 + 2\rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} + C_{jy_{p_2}p_2}^2 \right] > 0. \]

4. From (12) and (30),

\[ \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\tilde{y}_{\text{exp}(R)}) \text{ if} \]

\[ \text{MSE}(\tilde{y}_{\text{exp}(R)}) - \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}^2 \lambda \left[ C_j^2 + \frac{1}{4} C_{p_1}^2 - \rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} \right] - \bar{Y}^2 \lambda C_{jy_{p_2}}^2 \left[ 1 - R_{y_{p_1}p_2}^2 \right] > 0, \text{ or if} \]

\[ \left[ \frac{1}{4} C_{p_1}^2 - \rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} + C_{jy_{p_2}p_2}^2 \right] > 0. \]

5. From (14) and (30),

\[ \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\tilde{y}_{\text{KB}(RP)}) \text{ if} \]

\[ \text{MSE}(\tilde{y}_{\text{KB}(RP)}) - \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}^2 \lambda \left[ C_j^2 + \frac{1}{4} C_{p_1}^2 + \rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} \right] - \bar{Y}^2 \lambda C_{jy_{p_2}}^2 \left[ 1 - R_{y_{p_1}p_2}^2 \right] > 0, \text{ or if} \]

\[ \left[ \frac{1}{4} C_{p_1}^2 + \rho_{y_{p_1}} C_{y_{p_1}} C_{p_1} + C_{jy_{p_2}p_2}^2 \right] > 0. \]

6. From (17) and (30),

\[ \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\tilde{y}_{\text{KB}(RP)}) \text{ if} \]

\[ \text{MSE}(\tilde{y}_{\text{KB}(RP)}) - \text{MSE}(\tilde{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}^2 \lambda C_j^2 \left[ 1 - \rho_{y_{p_1}}^2 \right] - \bar{Y}^2 \lambda C_{jy_{p_2}}^2 \left[ 1 - R_{y_{p_1}p_2}^2 \right] > 0, \text{ or if} \]

\[ \left[ R_{y_{p_1}p_2}^2 - \rho_{y_{p_1}}^2 \right] > 0. \]
7. From (22) and (30),

\[ \text{MSE}(\hat{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\bar{y}_{\text{UKS(K)}}) \] if

\[ \text{MSE}(\bar{y}_{\text{UKS(K)}}) - \text{MSE}(\hat{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{y}^2 [C_{p_1}^2 + C_{p_2}^2 + 2(\rho_{y,p_1} C_{p_1} - \rho_{y,p_2} C_{p_2}) - \bar{y}^2 C_{p_1}^2 [1 - R_{y,p_1}^2] > 0, \text{ or if} \]

\[ [C_{p_1}^2 + C_{p_2}^2 - 2(\rho_{y,p_1} C_{p_1} - \rho_{y,p_2} C_{p_2}) + \bar{y}^2 C_{p_1}^2 [1 - R_{y,p_2}^2] > 0]. \]

8. From (24) and (30),

\[ \text{MSE}(\hat{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\bar{y}_{\text{UKS(P)}}) \] if

\[ \text{MSE}(\bar{y}_{\text{UKS(P)}}) - \text{MSE}(\hat{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{y}^2 [C_{p_1}^2 + C_{p_2}^2 + 2(\rho_{y,p_1} C_{p_1} - \rho_{y,p_2} C_{p_2}) - \bar{y}^2 C_{p_1}^2 [1 - R_{y,p_1}^2] > 0, \text{ or if} \]

\[ [C_{p_1}^2 + C_{p_2}^2 - 2(\rho_{y,p_1} C_{p_1} - \rho_{y,p_2} C_{p_2}) + \bar{y}^2 C_{p_1}^2 [1 - R_{y,p_2}^2] > 0]. \]

**Numerical comparison under simple random sampling**

To observe the performance of our proposed generalized class of estimators with respect to other considered estimators under simple random sampling, we use the following data sets, which earlier used by many authors in literature.

**Population 1.** [Source: Koyuncu and Kadilar [13]]

Let \( y \) be the number of teachers, \( p_1 \) be the number of students both primary and secondary schools in Turkey in 2007 for 923 districts in six regions which is greater than 11440.5 and \( p_2 \) be the number of students both primary and secondary schools in Turkey in 2008 for 923 districts in six regions which is greater than 333.1647. We use the proportional allocation.

\[ N = 923, n = 180, \bar{y} = 436.4345, P_1 = 2.6625, P_2 = 3.125, \rho_{y,p_1} = 0.6904898, C_{p_1} = 1.826732, \rho_{y,p_2} = 0.8465885, C_{p_2} = 1.641621, C_y = 1.718333 \text{ and } \rho_{y,p} = 0.652149. \]

**Population 2.** [Source: Singh [14]]

Let \( y \) be the estimated number of fish caught by marine recreational fisherman in year 1995, \( p_1 \) be the proportion of fishes caught greater than 1000 in 1993 and \( p_2 \) be the proportion of fishes caught greater than 2000 in 1994.

\[ N = 69, n = 14, \bar{y} = 4514.89, P_1 = 0.7391304, P_2 = 0.5507246, C_{p_1} = 0.5984409, \rho_{p_1} = 0.6577519, C_{p_1} = 0.9098277, C_y = 1.350, \rho_{y,p_2} = 0.538047 \text{ and } \rho_{y,p} = 0.3966081. \]

**Population 3.** [Source: www.pbs.gov.pk]

Let \( y \) be the tobacco area production in hectares during the year 2009, \( p_1 \) be the proportion of farms with tobacco cultivation area greater than 500 hectares during the year 2007 and \( p_2 \) be the proportion of farms with tobacco cultivation area greater than 800 hectares during the year 2008 for 47 districts of Pakistan.

\[ N = 47, n = 10, \bar{y} = 1004.447, P_1 = 0.4255319, P_2 = 0.3829787, C_{p_1} = 1.174456, \rho_{p_1} = 0.9153857, C_{p_1} = 1.283018, C_y = 2.341245, \rho_{y,p_2} = 0.4661508 \text{ and } \rho_{y,p} = 0.4395989. \]

**Population 4.** [Source: www.pbs.gov.pk]
Let \( y \) be the cotton production in hectares during the year 2009, \( p_1 \) be the proportion of farms with cotton cultivation area greater than 37 hectares during the year 2007 and \( p_2 \) be the proportion of farms with cotton cultivation area greater than 35 hectares during the year 2008 for 52 districts of Pakistan.

\[
N = 52, \quad n = 11; \quad \bar{Y} = 50.03846, \quad P_1 = 0.3846154, \quad P_2 = 0.4423077, \quad C_{p_1} = 1.277252, \quad \rho_{p_1p_2} = 0.8877181, \quad C_{p_2} = 1.13384, \quad C_y = 1.421524, \quad \rho_{yp_2} = 0.6935718 \quad \text{and} \quad \rho_{yp_1} = 0.7369579.
\]

We use the following expression to obtain the Percentage Relative Efficiency (PRE):

\[
PRE = \frac{\text{MSE}(\bar{y}_0)}{\text{MSE}(\bar{y}_i) \text{or} \text{MSE}(\bar{y}_i)_{(\text{min})}} \times 100,
\]

where \( i = 0, R, P, \exp(R), \exp(P), KB(RP), SK(DR), SK(DP) \) and \( \bar{y}_{\text{prop}} \).

The results based on data sets (1–4) are given in Table 2. It is observed in Table 2, that the proposed estimators (\( \bar{y}_{\text{prop}} \)) is more efficient than its competitors in SRS.

Table 2 clearly shows that our generalized proposed class of estimators is better than all existing estimators. The product estimators \( \bar{y}_P \), \( \bar{y}_{\exp(P)} \) and \( \bar{y}_{SK(DP)} \) perform poor in all four populations because of negative correlation. It is also observed that \( \bar{y}_{SK(DR)} \) also perform poor in populations 1, 3 and 4. From this results, we conclude that the generalized proposed class of estimators is more efficient than other estimators. Some members of the proposed family of estimators based on PRE are given in Table 3.

Table 3 gives the PRE of the proposed family of estimators in simple random sampling. It is obsessed that the proposed family of estimators perform poorly because of poor correlation between study and auxiliary variables.

### Existing estimators in stratified random sampling

The auxiliary information is used in reduction of MSE of various estimators for estimating different population parameters that is mean, variance, ratio of two population means and variances etc. To increase the precision we divide the population into homogeneous groups with respect to some characteristic of interest. Many Statisticians have used the auxiliary information in the estimation of population parameters in stratified random sampling for improving the efficiency of estimators.

Dalabehera and Sahoo [15] proposed different regression type estimators in stratified random sampling with two auxiliary variables. Later Kadilar and Cingi [16] also proposed ratio type estimators in stratified random sampling to get efficient results by extending the

| Estimator | Population 1 | Population 2 | Population 3 | Population 4 |
|-----------|--------------|--------------|--------------|--------------|
| \( \bar{y}_0 \) | 100 | 100 | 100 | 100 |
| \( \bar{y}_R \) | 151.0459 | 118.3598 | 123.3652 | 207.0429 |
| \( \bar{y}_P \) | 27.79133 | 64.59406 | 59.07796 | 31.9321 |
| \( \bar{y}_{\exp(R)} \) | 182.3193 | 114.5063 | 118.7098 | 185.2997 |
| \( \bar{y}_{\exp(P)} \) | 49.58877 | 81.63674 | 77.91625 | 53.64828 |
| \( \bar{y}_{KB(RP)} \) | 191.1228 | 118.6659 | 123.9537 | 218.8696 |
| \( \bar{y}_{SK(DR)} \) | 48.82051 | 103.427 | 90.63836 | 77.81008 |
| \( \bar{y}_{SK(DP)} \) | 13.37497 | 32.04538 | 33.25377 | 16.26898 |
| \( \bar{y}_{\text{prop}} \) | 551.1775 | 141.3846 | 127.9299 | 222.428 |

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Table 3. Percentage relative efficiency of proposed family estimators.

|                | Population 1 | Population 2 | Population 3 | Population 4 |
|----------------|--------------|--------------|--------------|--------------|
| $\hat{y}_{\text{prop1}}$ | 171.3685     | 138.4056     | 126.9301     | 215.6989     |
| $\hat{y}_{\text{prop2}}$ | 103.1149     | 79.9349      | 95.5017      | 107.0865     |
| $\hat{y}_{\text{prop3}}$ | 102.8836     | 112.9439     | 123.0256     | 204.9024     |
| $\hat{y}_{\text{prop4}}$ | 83.90536     | 113.9393     | 102.3297     | 86.82563     |
| $\hat{y}_{\text{prop5}}$ | 30.32412     | 55.9706      | 57.4785      | 34.5501      |
| $\hat{y}_{\text{prop6}}$ | 30.2428      | 89.68737     | 84.3266      | 60.913388    |
| $\hat{y}_{\text{prop7}}$ | 83.54702     | 136.1699     | 125.2152     | 189.3723     |
| $\hat{y}_{\text{prop8}}$ | 30.23099     | 70.94277     | 80.54632     | 68.31584     |
| $\hat{y}_{\text{prop9}}$ | 32.99369     | 114.0681     | 112.8626     | 141.189      |
| $\hat{y}_{\text{prop10}}$ | 49.40173     | 104.3372     | 105.8206     | 119.4923     |
| $\hat{y}_{\text{prop11}}$ | 165.2506     | 133.1671     | 122.0072     | 165.0596     |
| $\hat{y}_{\text{prop12}}$ | 54.01867     | 67.74307     | 75.15793     | 58.40237     |
| $\hat{y}_{\text{prop13}}$ | 53.8342      | 109.283      | 106.7871     | 115.8713     |

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Estimators Upadhyaya and Singh [17]. Singh and Kumar [11] used transformed variables in stratified random sampling. Koyuncu and Kadilar [13] also used ratio and product types estimators using two auxiliary variables in stratified sampling.

Let $U = \{U_1, U_2, \ldots, U_N\}$ be a finite population of size $N$ and let $y$, $p_1$, and $p_2$ respectively be the study and two auxiliary attributes associated with each unit $U_j = (j = 1, 2, \ldots, N)$. Assume that a population is stratified into $L$ homogeneous strata with the $h$th stratum containing $N_h$ units, where $h = 1, 2, \ldots, L$ such that $\sum_{h=1}^{L} N_h = N$. A simple random sample of size $n_h$ is drawn without replacement from the $h$th stratum such that $\sum_{h=1}^{L} n_h = n$. Let $(y_{ih}, p_{ih1}, p_{ih2})$ be the observed values of $y$, $p_1$, and $p_2$ on the $i$th unit of the $h$th stratum, where $i = 1, 2, \ldots, n_h$. Moreover, let $\tilde{y}_h = \sum_{i=1}^{n_h} \frac{y_{ih}}{n_h}$, $\tilde{y}_{ih} = \sum_{h=1}^{L} W_h \tilde{y}_h$, and $\bar{Y} = \sum_{h=1}^{L} \frac{N_h}{N} \tilde{y}_h$ be the sample and population means of $y$ respectively, where $W_h = \frac{N_h}{n_h}$ is the known stratum weight.

Let

$$e_{ih} = \frac{\tilde{y}_h - \bar{Y}}{\bar{Y}} = \frac{p_{ih} - P_{ih}}{P_{ih}}, \quad e_{ih} = \frac{p_{ih} - P_{ih}}{P_{ih}}.$$ 

such that $E(e_{ih}) = 0$, $(i = 0, 1, 2)$,

$$1E(e_{ih}^2) = \hat{\lambda}_h C_{P_{ih}}, \quad E(e_{ih}^2) = \hat{\lambda}_h C_{P_{ih}}, \quad E(e_{ih}^2) = \hat{\lambda}_h C_{P_{ih}}, \quad E(e_{ih}^2) = \hat{\lambda}_h C_{P_{ih}},$$

$$E(e_{ih}^2) = \hat{\lambda}_h C_{P_{ih}}, \quad \text{and} \quad E(e_{ih}^2) = \hat{\lambda}_h C_{P_{ih}},$$

where,

$$\hat{\lambda}_h = \left(1 - \frac{1 - \frac{1}{n_h}}{1 - \frac{1}{N_h}}\right), \quad \hat{\lambda}_h = \frac{n_h}{N_h}, \quad C_{P_{ih}} = \frac{S_{P_{ih}}}{P_{ih}}, \quad C_{P_{ih}} = \frac{S_{P_{ih}}}{P_{ih}}, \quad C_{P_{ih}} = \frac{S_{P_{ih}}}{P_{ih}}, \quad C_{P_{ih}} = \frac{1}{N_h - 1} \sum_{i=1}^{n_h} (y_{ih} - \bar{Y})^2,$$

$$S_{P_{ih}} = \frac{1}{N_h - 1} \sum_{i=1}^{n_h} (p_{ih} - P_{ih})^2 \quad \text{and} \quad S_{P_{ih}} = \frac{1}{N_h - 1} \sum_{i=1}^{n_h} (p_{ih} - P_{ih})^2.$$

Now we discuss the same existing estimators in stratified random sampling.
1. The usual mean per unit estimator and its MSE in stratified random sampling, are given by:

\[
\bar{y}_u = \sum_{h=1}^{l} W_h \bar{y}_h
\]

and

\[
\text{MSE}(\bar{y}_u) = \sum_{h=1}^{l} W_h^2 \bar{y}_h^2 C^2_{y,h}.
\]

2. The usual ratio estimator under stratified random sampling, is given by:

\[
\bar{y}_{ph} = \sum_{h=1}^{l} W_h \bar{y}_h \left( \frac{P_{1h}}{P_{1h}} \right)
\]

3. The usual product estimator under stratified random sampling, is given by:

\[
\bar{y}_{ph} = \sum_{h=1}^{l} W_h \bar{y}_h \left( \frac{P_{1h}}{P_{1h}} \right)
\]

\[
\text{Bias}(\bar{y}_{ph}) \approx \sum_{h=1}^{l} W_h \bar{y}_h \bar{y}_h \left[ \rho_{y,h} C_{y,h} C_{P_{1h}} \right],
\]

and

\[
\text{MSE}(\bar{y}_{ph}) \approx \sum_{h=1}^{l} W_h^2 \bar{y}_h^2 \left[ C^2_{y,h} + 2 \rho_{y,h} C_{y,h} C_{P_{1h}} \right].
\]

Similarly the bias and MSE of the \( \bar{y}_{ph} \), are given by:

\[
\text{Bias}(\bar{y}_{ph}) \approx \sum_{h=1}^{l} W_h \bar{y}_h \bar{y}_h \left[ \rho_{y,h} C_{y,h} C_{P_{1h}} \right],
\]

and

\[
\text{MSE}(\bar{y}_{ph}) \approx \sum_{h=1}^{l} W_h^2 \bar{y}_h^2 \left[ C^2_{y,h} + 2 \rho_{y,h} C_{y,h} C_{P_{1h}} \right].
\]

4. Bahl and Tuteja [9] estimator in stratified random sampling, are given by:

\[
\bar{y}_{exp(ph)} = \sum_{h=1}^{l} W_h \bar{y}_h \exp \left[ \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right],
\]

and

\[
\bar{y}_{exp(ph)} = \sum_{h=1}^{l} W_h \bar{y}_h \exp \left[ \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right].
\]

The bias and MSE of \( \bar{y}_{exp(ph)} \), are given by:

\[
\text{Bias}(\bar{y}_{exp(ph)}) \approx \sum_{h=1}^{l} W_h \bar{y}_h \exp \left[ \frac{3}{8} C^2_{y,h} - \frac{1}{2} \rho_{y,h} C_{y,h} C_{P_{1h}} \right],
\]

and

\[
\text{MSE}(\bar{y}_{exp(ph)}) \approx \sum_{h=1}^{l} W_h^2 \bar{y}_h^2 \exp \left[ \frac{1}{4} C^2_{y,h} + C^2_{P_{1h}} - \rho_{y,h} C_{y,h} C_{P_{1h}} \right].
\]

Similarly the bias and MSE of \( \bar{y}_{exp(ph)} \), are given by:

\[
\text{Bias}(\bar{y}_{exp(ph)}) \approx \frac{1}{2} \sum_{h=1}^{l} W_h \bar{y}_h \exp \left[ \rho_{y,h} C_{y,h} C_{P_{1h}} - \frac{1}{4} C^2_{y,h} \right],
\]

and

\[
\text{MSE}(\bar{y}_{exp(ph)}) \approx \sum_{h=1}^{l} W_h^2 \bar{y}_h^2 \exp \left[ \rho_{y,h} C_{y,h} C_{P_{1h}} - \frac{1}{4} C^2_{y,h} \right].
\]
and

\[
\text{MSE}(\tilde{y}_\text{exp(h)}) \cong \sum_{h=1}^{l} W_h^2 \tilde{y}_h^2 \left[ \frac{C^2_{\text{bh}}}{\bar{C}_{\text{bh}}} + \frac{1}{4} C^2_{\text{p_h}} + \rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}} \right]. \tag{44}
\]

5. Kumar and Bhogal [10] proposed exponential ratio-product type estimator for the population mean in stratified random sampling, is given by:

\[
\tilde{y}_{KB(RP_1)} = \sum_{h=1}^{l} W_h \tilde{y}_h \left[ \frac{p_{1_h} - p_{1_h}}{p_{1_h} + p_{1_h}} + (1 - \alpha_h) \exp \left( \frac{p_{1_h} - p_{1_h}}{p_{1_h} + p_{1_h}} \right) \right] \tag{45}
\]

where \(\alpha_h\) is the unknown constant.

The bias and minimum MSE of \(\tilde{y}_{KB(RP_1)}\), are given by:

\[
\text{Bias}(\tilde{y}_{KB(RP_1)}) \cong \sum_{h=1}^{l} W_h \tilde{y}_h \left[ \frac{1}{8} (4\alpha_h - 1) C^2_{\text{p_h}} \right] - \left( 1 - \frac{1}{2} \right) \rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}}, \tag{46}
\]

and

\[
\text{MSE}(\tilde{y}_{KB(RP_1)})_{\text{min}} = 2 \tilde{y}^2 C_{\text{p_h}} \left[ 1 - \rho_{\text{p_hp_h}}^2 \right]. \tag{47}
\]

The optimum value of \(\alpha_h\), is given by:

\[
\alpha_{\text{opt}} \cong \frac{1}{2} + \rho_{\text{p_hp_h}} C_{\text{p_h}}. \tag{48}
\]

6. Singh and Kumar [11] suggested double ratio and product type estimators in stratified random sampling, are given by:

\[
\tilde{y}_{SK(DR_1)} = \sum_{h=1}^{l} W_h \tilde{y}_h \left( \frac{p_{1_h}}{p_{1_h}} \right) \left( \frac{p_{2_h}}{p_{2_h}} \right), \tag{49}
\]

and

\[
\tilde{y}_{SK(DP_1)} = \sum_{h=1}^{l} W_h \tilde{y}_h \left( \frac{p_{1_h}}{p_{1_h}} \right) \left( \frac{p_{2_h}}{p_{2_h}} \right). \tag{50}
\]

The bias and MSE of \(\tilde{y}_{SK(DR_1)}\), are given by:

\[
\text{Bias}(\tilde{y}_{SK(DR_1)}) \cong \sum_{h=1}^{l} W_h \tilde{y}_h \left[ \frac{C^2_{\text{p_h}}}{C_{\text{p_h}}} + \frac{C^2_{\text{p_h}}}{C_{\text{p_h}}} + \rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}} - \rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}} \right], \tag{51}
\]

and

\[
\text{MSE}(\tilde{y}_{SK(DR_1)}) \cong \sum_{h=1}^{l} W_h^2 \tilde{y}_h^2 \left[ C^2_{\text{p_h}} + C^2_{\text{p_h}} + C^2_{\text{p_h}} - 2(\rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}}) \right]. \tag{52}
\]

Similarly the bias and MSE of \(\tilde{y}_{SK(DP_1)}\), are given by:

\[
\text{Bias}(\tilde{y}_{SK(DP_1)}) \cong \sum_{h=1}^{l} W_h \tilde{y}_h \left[ \rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}} + \rho_{\text{p_hp_h}} C_{\text{p_h}} C_{\text{p_h}} + \rho_{\text{p_hp_h}} \right], \tag{53}
\]
Proposed class of estimators

In the lines of Shukla et al. [12], we proposed a generalized class of estimators in stratified random sampling, is given by:

$$\bar{y}_{\text{prop}} = \sum_{i=1}^{k} W_i \bar{y}_i \left[ \exp \left( \frac{S_{ih} - M_{ih}}{S_{ih} + M_{ih}} \right) \right].$$  (54)

where,

$$S_{ih} = (A_{ih} + C_{ih})P_{ih} + f_i B_{ih} P_{ih}, \quad S_{ih} = (A_{ih} + C_{ih})P_{ih} + f_i B_{ih} P_{ih},$$

$$M_{ih} = (A_{ih} + f_i B_{ih})P_{ih} + C_{ih} P_{ih}, \quad M_{ih} = (A_{ih} + f_i B_{ih})P_{ih} + C_{ih} P_{ih},$$

$$A_{ih} = (K_{ih} - 1)(K_{ih} - 2), \quad B_{ih} = (K_{ih} - 1)(K_{ih} - 4) \text{ and}$$

$$C_{ih} = (K_{ih} - 2)(K_{ih} - 3)(K_{ih} - 4).$$

Substituting different values of $K_{ih}$ ($i = 1, 2, 3, 4$) in (54), we can generate many more different types of estimators from our general proposed class of estimators, given in Table 4.

| S.No | $K_{ih}$ | $K_{ih}$ | Estimators |
|------|---------|---------|------------|
| 1    | 1       | 1       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 2    | 1       | 2       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 3    | 1       | 3       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 4    | 1       | 4       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 5    | 2       | 1       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 6    | 2       | 2       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 7    | 2       | 3       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 8    | 2       | 4       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 9    | 3       | 1       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 10   | 3       | 2       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 11   | 3       | 3       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 12   | 3       | 4       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 13   | 4       | 1       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 14   | 4       | 2       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 15   | 4       | 3       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |
| 16   | 4       | 4       | $\bar{y}_{\text{prop}} = \sum_{k=1}^{K} W_k \bar{y}_k \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right) \exp \left( \frac{2a - a_i}{2a - a_i + P_k} \right)$ |

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Solving $\bar{y}_{\text{prop}}$ given in Eq (54) in terms of errors, we have
\[
\bar{y}_{\text{prop}} = \sum_{i=1}^{L} W_i \bar{y}_h \left[ (1 + e_{ih}) \left( 1 + \frac{1}{2} \sigma_{1h} e_{ih} - \frac{1}{4} \sigma_{1h} \sigma_{1h} e_{ih}^2 + \frac{1}{8} \sigma_{1h} \sigma_{1h} e_{ih}^2 + \ldots \right) \right],
\]
where
\[
\sigma_{1h} = \frac{f_h B_{ih} - C_{1h}}{A_{ih} + f_h B_{ih} + C_{1h}}, \quad \gamma_{1h} = \frac{f_h B_{ih} + C_{1h}}{A_{ih} + f_h B_{ih} + C_{1h}}.
\]
and
\[
\sigma_{2h} = \frac{f_h B_{2h} - C_{2h}}{A_{2h} + f_h B_{2h} + C_{2h}}, \quad \gamma_{2h} = \frac{f_h B_{2h} + C_{2h}}{A_{2h} + f_h B_{2h} + C_{2h}}.
\]
To first order approximation, we have
\[
\left( \bar{y}_{\text{prop}} - \bar{Y}_h \right) \approx \sum_{i=1}^{L} W_i \bar{y}_h \left[ e_{ih} + \frac{1}{2} \sigma_{1h} e_{ih} + \frac{1}{2} \sigma_{2h} e_{2h} + \frac{1}{2} \sigma_{1h} \gamma_{1h} e_{ih} + \frac{1}{2} \sigma_{2h} \gamma_{2h} e_{2h} \right],
\]
Taking squaring and expectation of Eq (56) to first order of approximation, we get the bias and MSE:
\[
\text{Bias} \left( \bar{y}_{\text{prop}} \right) \approx \sum_{i=1}^{L} W_i \bar{y}_h \left[ \frac{1}{2} \sigma_{1h} \rho_{\text{prop}p} \sigma_{1h} C_p + \frac{1}{2} \sigma_{2h} \rho_{\text{prop}p} \sigma_{2h} C_p \right] + \frac{1}{8} \sigma_{1h} \gamma_{1h} \left( \sigma_{1h} - 2 \sigma_{1h} \gamma_{1h} + \frac{1}{8} \sigma_{2h} \gamma_{2h} \right),
\]
and
\[
\text{MSE} \left( \bar{y}_{\text{prop}} \right) \approx \sum_{i=1}^{L} W_i \bar{y}_h \left[ C_p + \frac{1}{4} \sigma_{1h} \sigma_{1h} C_p + \frac{1}{4} \sigma_{2h} \sigma_{2h} C_p + \sigma_{1h} \rho_{\text{prop}p} \sigma_{1h} C_p \right] + \frac{1}{8} \sigma_{1h} \gamma_{1h} \left( \sigma_{1h} - 2 \sigma_{1h} \gamma_{1h} + \frac{1}{8} \sigma_{2h} \gamma_{2h} \right),
\]
Differentiate Eq (58) with respect to $\sigma_{1h}$ and $\sigma_{2h}$, we get the optimum values of $\sigma_{1h}$ and $\sigma_{2h}$ i.e.
\[
\sigma_{1h(\text{opt})} = \frac{2 \sigma_{1h} \rho_{\text{prop}p} - \rho_{\text{prop}p} \rho_{\text{prop}p}}{C_p (\rho_{\text{prop}p}^2 - 1)} \quad \text{and} \quad \sigma_{2h(\text{opt})} = \frac{2 \sigma_{2h} \rho_{\text{prop}p} - \rho_{\text{prop}p} \rho_{\text{prop}p}}{C_p (\rho_{\text{prop}p}^2 - 1)}.
\]
Substituting the optimum values of $\sigma_{1h(\text{opt})}$ and $\sigma_{2h(\text{opt})}$ in Eq (58), we get minimum MSE of $\bar{y}_{\text{prop}}$ is given by:
\[
\text{MSE} \left( \bar{y}_{\text{prop}} \right)_{\text{min}} \approx \sum_{i=1}^{L} W_i \bar{y}_h \sigma_{1h} \sigma_{2h} \left[ 1 - R^2_{\gamma_{1h} \gamma_{2h}} \right],
\]
where
\[
R^2_{\gamma_{1h} \gamma_{2h}} = \frac{\rho_{\text{prop}p}^2 + \rho_{\text{prop}p}^2 - 2 \rho_{\text{prop}p} \rho_{\text{prop}p} \rho_{\text{prop}p} \rho_{\text{prop}p}}{1 - \rho_{\text{prop}p}^2}
\]
is the multiple correlation coefficient of $y_h$ on $p_{1h}$ and $p_{2h}$.
Now by putting different values of $K_{1h}$ in Eq (54), some member of the proposed class of estimators can be obtained as:

1. For $K_{1h} = 1$ and $K_{2h} = 1$

$$\bar{y}_{\text{prop 1h}} = \sum_{k=1}^{l} W_h \bar{y}_h \exp \left( \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right) \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)$$

The bias and MSE of $\bar{y}_{\text{prop 1h}}$ are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop 1h}} \right) \approx \sum_{k=1}^{l} W_h \hat{y}_h \bar{y}_h \left[ \frac{3}{8} C_{1h}^2 + \frac{3}{8} C_{2h}^2 - \frac{1}{2} \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} - \frac{1}{2} \rho_{\hat{y}y_{2h}} C_{1h} C_{2h} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop 1h}} \right) \approx \sum_{k=1}^{l} W_h^2 \hat{y}_h^2 \bar{y}_h^2 \left[ C_{1h}^2 + \frac{1}{4} C_{2h}^2 + \frac{1}{4} C_{2h}^2 - \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} - \rho_{\hat{y}y_{2h}} C_{1h} C_{2h} \right] \left[ \frac{1}{4} \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} \right].$$

2. For $K_{1h} = 1$ and $K_{2h} = 2$

$$\bar{y}_{\text{prop 2h}} = \sum_{k=1}^{l} W_h \bar{y}_h \exp \left( \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right) \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)$$

The bias and MSE of $\bar{y}_{\text{prop 2h}}$ are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop 2h}} \right) \approx \sum_{k=1}^{l} W_h \hat{y}_h \bar{y}_h \left[ -\frac{1}{8} C_{1h}^2 - \frac{1}{8} C_{2h}^2 - \frac{1}{2} \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop 2h}} \right) \approx \sum_{k=1}^{l} W_h^2 \hat{y}_h^2 \bar{y}_h^2 \left[ C_{1h}^2 + \frac{1}{4} C_{2h}^2 + \frac{1}{4} C_{2h}^2 - \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} + \rho_{\hat{y}y_{2h}} C_{1h} C_{2h} \right] \left[ -\frac{1}{2} \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} \right].$$

3. For $K_{1h} = 1$ and $K_{2h} = 3$

$$\bar{y}_{\text{prop 3h}} = \sum_{k=1}^{l} W_h \bar{y}_h \exp \left( \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right) \exp \left( \frac{n(P_{2h} - P_{2h})}{2NP_{2h} - n(P_{2h} + P_{2h})} \right)$$

The bias and MSE of $\bar{y}_{\text{prop 3h}}$ are given by:

$$\text{Bias} \left( \bar{y}_{\text{prop 3h}} \right) \approx \sum_{k=1}^{l} W_h \hat{y}_h \bar{y}_h \left[ \frac{3}{8} C_{1h}^2 + \frac{1}{8} C_{2h}^2 - \frac{1}{4} \left( \frac{f_{1h}}{1 - f_{1h}} \right)^2 C_{1h}^2 - \frac{1}{2} \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} - \frac{1}{2} \left( \frac{f_{1h}}{1 - f_{1h}} \right)^2 \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} \right]$$

and

$$\text{MSE} \left( \bar{y}_{\text{prop 3h}} \right) \approx \sum_{k=1}^{l} W_h^2 \hat{y}_h^2 \bar{y}_h^2 \left[ C_{1h}^2 + \frac{1}{4} C_{1h}^2 + \frac{1}{4} \left( \frac{f_{1h}}{1 - f_{1h}} \right)^2 C_{1h}^2 - \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} - \rho_{\hat{y}y_{2h}} C_{1h} C_{2h} + \frac{1}{2} \left( \frac{f_{1h}}{1 - f_{1h}} \right)^2 \rho_{\hat{y}y_{1h}} C_{1h} C_{2h} \right].$$
4. For $K_{1h} = 2$ and $K_{2h} = 1$
\[
\bar{y}_{\text{prop 5h}} = \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{\left( P_{1h} - P_{1h} \right) \exp\left( \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right)}{P_{1h} + P_{1h}} \right\}
\]

The bias and MSE of $\bar{y}_{\text{prop 5h}}$, are given by:
\[
\text{Bias}\left( \bar{y}_{\text{prop 5h}} \right) \approx \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{C^2_{P_{1h}}}{2} + \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} - \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} \right\}
\]
and
\[
\text{MSE}\left( \bar{y}_{\text{prop 5h}} \right) \approx \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{C^2_{P_{1h}}}{2} + \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} - \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} \right\}.
\]

5. For $K_{1h} = 2$ and $K_{2h} = 2$
\[
\bar{y}_{\text{prop 6h}} = \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{\left( P_{1h} - P_{1h} \right) \exp\left( \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right)}{P_{1h} + P_{1h}} \right\}
\]

The bias and MSE of $\bar{y}_{\text{prop 6h}}$, are given by:
\[
\text{Bias}\left( \bar{y}_{\text{prop 6h}} \right) \approx \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{C^2_{P_{1h}}}{2} + \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} - \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} \right\}
\]
and
\[
\text{MSE}\left( \bar{y}_{\text{prop 6h}} \right) \approx \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{C^2_{P_{1h}}}{2} + \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} - \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} \right\}.
\]

6. For $K_{1h} = 2$ and $K_{2h} = 3$
\[
\bar{y}_{\text{prop 7h}} = \sum_{h=1}^{L} W_h \bar{y}_h \exp\left( \frac{P_{1h} - P_{1h}}{P_{1h} + P_{1h}} \right) \exp\left( \frac{n(P_{1h} - P_{2h})}{2NP_{2h} - n(P_{2h} + P_{2h})} \right)
\]

The bias and MSE of $\bar{y}_{\text{prop 7h}}$, are given by:
\[
\text{Bias}\left( \bar{y}_{\text{prop 7h}} \right) \approx \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{C^2_{P_{1h}}}{2} + \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} - \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} \right\}
\]
and
\[
\text{MSE}\left( \bar{y}_{\text{prop 7h}} \right) \approx \sum_{h=1}^{L} W_h \bar{y}_h \left\{ \frac{C^2_{P_{1h}}}{2} + \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} - \frac{1}{2} \rho_{P_{1h}P_{1h}} C_{P_{1h}} C_{P_{1h}} \right\}.
\]
7. For $K_{1h} = 3$ and $K_{2h} = 1$

$$
\bar{y}_{prop \ 9h} = \sum_{h=1}^{l} W_h \tilde{y}_h \exp \left( \frac{n(P_{1h} - P_{2h})}{2NP_{1h} - (nP_{1h} + P_{2h})} \right) \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)
$$

The bias and MSE of $\bar{y}_{prop \ 9h}$, are given by:

$$
\text{Bias} \left( \bar{y}_{prop \ 9h} \right) \approx \sum_{h=1}^{l} W_h \tilde{y}_h \left[ -\frac{1}{2} \left( 1 - P_{2h} \right) C_{P_{1h}}^2 + \frac{3}{8} C_{P_{2h}}^2 - \frac{1}{2} \left( 1 - P_{2h} \right) \rho_{yxh} C_{yxh} + \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} \right]
$$

and

$$
\text{MSE} \left( \bar{y}_{prop \ 9h} \right) \approx \sum_{h=1}^{l} W_h \tilde{y}_h \left[ C_{y1h}^2 + \frac{1}{2} \left( 1 - P_{2h} \right) C_{y2h}^2 - \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} \right].
$$

8. For $K_{1h} = 3$ and $K_{2h} = 2$

$$
\bar{y}_{prop \ 10h} = \sum_{h=1}^{l} W_h \tilde{y}_h \exp \left( \frac{n(P_{1h} - P_{2h})}{2NP_{1h} - (nP_{1h} + P_{2h})} \right) \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)
$$

The bias and MSE of $\bar{y}_{prop \ 10h}$, are given by:

$$
\text{Bias} \left( \bar{y}_{prop \ 10h} \right) \approx \sum_{h=1}^{l} W_h \tilde{y}_h \left[ -\frac{1}{2} \left( 1 - P_{2h} \right) C_{P_{1h}}^2 + \frac{3}{8} C_{P_{2h}}^2 - \frac{1}{2} \left( 1 - P_{2h} \right) \rho_{yxh} C_{yxh} + \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} \right]
$$

and

$$
\text{MSE} \left( \bar{y}_{prop \ 10h} \right) \approx \sum_{h=1}^{l} W_h \tilde{y}_h \left[ C_{y1h}^2 + \frac{1}{2} \left( 1 - P_{2h} \right) C_{y2h}^2 - \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} \right].
$$

9. For $K_{1h} = 3$ and $K_{2h} = 3$

$$
\bar{y}_{prop \ 11h} = \sum_{h=1}^{l} W_h \tilde{y}_h \exp \left( \frac{n(P_{1h} - P_{2h})}{2NP_{1h} - (nP_{1h} + P_{2h})} \right) \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)
$$

The bias and MSE of $\bar{y}_{prop \ 11h}$, are given by:

$$
\text{Bias} \left( \bar{y}_{prop \ 11h} \right) \approx \sum_{h=1}^{l} W_h \tilde{y}_h \left[ -\frac{1}{2} \left( 1 - P_{2h} \right) C_{P_{1h}}^2 + \frac{3}{8} C_{P_{2h}}^2 - \frac{1}{2} \left( 1 - P_{2h} \right) \rho_{yxh} C_{yxh} + \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} \right]
$$

and

$$
\text{MSE} \left( \bar{y}_{prop \ 11h} \right) \approx \sum_{h=1}^{l} W_h \tilde{y}_h \left[ C_{y1h}^2 + \frac{1}{2} \left( 1 - P_{2h} \right) C_{y2h}^2 - \frac{1}{2} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} + \frac{1}{4} \left( \frac{f_h}{1 - f_h} \right) \rho_{yxh} C_{yxh} \right].
$$

10. For $K_{1h} = 3$ and $K_{2h} = 4$

$$
\bar{y}_{prop \ 12h} = \sum_{h=1}^{l} W_h \tilde{y}_h \exp \left( \frac{n(P_{1h} - P_{2h})}{2NP_{1h} - (nP_{1h} + P_{2h})} \right)
$$
The bias and MSE of $\bar{y}_{prop\ 12b}$ are given by:

$$\text{Bias}(\bar{y}_{prop\ 12b}) \cong \sum_{h=1}^{L} W_h \hat{y}_h \left[ -\frac{1}{8} \left( \frac{f_h}{1-f_h} \right)^2 c_{P_2h}^2 - \frac{1}{2} \left( \frac{f_h}{1-f_h} \right) \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right]$$

and

$$\text{MSE}(\bar{y}_{prop\ 12b}) \cong \sum_{h=1}^{L} W_h^2 \hat{y}_h \left[ c_{y_2h}^2 + \frac{1}{4} \left( \frac{f_h}{1-f_h} \right)^2 c_{P_2h}^2 - \left( \frac{f_h}{1-f_h} \right) \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right].$$

11. For $K_{1h} = 4$ and $K_{2h} = 1$

$$\bar{y}_{prop\ 13b} = \sum_{h=1}^{L} W_h \hat{y}_h \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)$$

The bias and MSE of $\bar{y}_{prop\ 13b}$ are given by:

$$\text{Bias}(\bar{y}_{prop\ 13b}) \cong \sum_{h=1}^{L} W_h \hat{y}_h \left[ \frac{3}{8} c_{P_2h}^2 - \frac{1}{2} \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right]$$

and

$$\text{MSE}(\bar{y}_{prop\ 13b}) \cong \sum_{h=1}^{L} W_h^2 \hat{y}_h \left[ c_{y_2h}^2 + \frac{1}{4} c_{P_2h}^2 - \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right].$$

12. For $K_{1h} = 4$ and $K_{2h} = 2$

$$\bar{y}_{prop\ 14b} = \sum_{h=1}^{L} W_h \hat{y}_h \exp \left( \frac{P_{2h} - P_{2h}}{P_{2h} + P_{2h}} \right)$$

The bias and MSE of $\bar{y}_{prop\ 14b}$ are given by:

$$\text{Bias}(\bar{y}_{prop\ 14b}) \cong \sum_{h=1}^{L} W_h \hat{y}_h \left[ -\frac{1}{8} c_{P_2h}^2 + \frac{1}{2} \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right]$$

and

$$\text{MSE}(\bar{y}_{prop\ 14b}) \cong \sum_{h=1}^{L} W_h^2 \hat{y}_h \left[ c_{y_2h}^2 + \frac{1}{4} c_{P_2h}^2 + \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right].$$

13. For $K_{1h} = 4$ and $K_{2h} = 3$

$$\bar{y}_{prop\ 15b} = \sum_{h=1}^{L} W_h \hat{y}_h \exp \left( \frac{n(p_{2h} - P_{2h})}{2n p_{2h} - n(p_{2h} + P_{2h})} \right)$$

The bias and MSE of $\bar{y}_{prop\ 15b}$ are given by:

$$\text{Bias}(\bar{y}_{prop\ 15b}) \cong \sum_{h=1}^{L} W_h \hat{y}_h \left[ -\frac{1}{8} \left( \frac{f_h}{1-f_h} \right)^2 c_{P_2h}^2 - \frac{1}{2} \left( \frac{f_h}{1-f_h} \right) \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right]$$

and

$$\text{MSE}(\bar{y}_{prop\ 15b}) \cong \sum_{h=1}^{L} W_h^2 \hat{y}_h \left[ c_{y_2h}^2 + \frac{1}{4} \left( \frac{f_h}{1-f_h} \right)^2 c_{P_2h}^2 - \left( \frac{f_h}{1-f_h} \right) \rho_{y_2p_2h} c_{y_2h} c_{P_2h} \right].$$
Theoretical comparison

1. From (32) and (59),

\[ \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\bar{y}_{\text{st}}) \text{ if} \]

\[ \text{MSE}(\bar{y}_{\text{st}}) - \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \sum_{k=1}^{m} W_k^2 \frac{Y_k^2}{h_k} C_k^2 - \sum_{k=1}^{m} W_k^2 \frac{Y_k}{h_k} C_k^2 [1 - R^2_{\gamma_{\text{st} P_k} P_k}] > 0, \text{ or if} \]

\[ [R^2_{\gamma_{\text{st} P_k} P_k}] > 0 \]

2. From (36) and (59),

\[ \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\bar{y}_{\text{Rh}}) \text{ if} \]

\[ \text{MSE}(\bar{y}_{\text{Rh}}) - \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}_{h}^2 \frac{h_k}{Y_k} [C_k^2 + C_{P_k}^2 - 2 \rho_{\gamma_{P_k} \gamma_{P_k}} C_k C_{P_k}] - \bar{Y}_{h_k}^2 C_k^2 = [1 - R^2_{\gamma_{P_k} P_k}] > 0, \text{ or if} \]

\[ [C_{P_k}^2 - 2 \rho_{\gamma_{P_k} \gamma_{P_k}} C_k C_{P_k} + C_{j_k}^2 Y_{P_k}^2] > 0 \]

3. From (38) and (59),

\[ \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\bar{y}_{\text{Ph}}) \text{ if} \]

\[ \text{MSE}(\bar{y}_{\text{Ph}}) - \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}_{h}^2 \frac{h_k}{Y_k} [C_k^2 + C_{P_k}^2 + 2 \rho_{\gamma_{P_k} \gamma_{P_k}} C_k C_{P_k}] - \bar{Y}_{h_k}^2 C_k^2 [1 - R^2_{\gamma_{P_k} P_k}] > 0, \text{ or if} \]

\[ [C_{P_k}^2 + 2 \rho_{\gamma_{P_k} \gamma_{P_k}} C_k C_{P_k} + C_{j_k}^2 Y_{P_k}^2] > 0 \]

4. From (42) and (59),

\[ \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} < \text{MSE}(\bar{y}_{\text{exp(Rh)}}) \text{ if} \]

\[ \text{MSE}(\bar{y}_{\text{exp(Rh)}}) - \text{MSE}(\bar{y}_{\text{prop}})_{\text{min}} > 0 \text{ or if} \]

\[ \bar{Y}_{h}^2 \frac{h_k}{Y_k} \left[ C_k^2 + \frac{1}{4} C_{P_k}^2 - \rho_{\gamma_{P_k} \gamma_{P_k}} C_k C_{P_k} \right] - \bar{Y}_{h_k}^2 C_k^2 \left[ 1 - R^2_{\gamma_{P_k} P_k} \right] > 0, \text{ or if} \]

\[ \left[ \frac{1}{4} C_{P_k}^2 - \rho_{\gamma_{P_k} \gamma_{P_k}} C_k C_{P_k} + C_{j_k}^2 Y_{P_k}^2 \right] > 0 \]
5. From (44) and (59),

\[
\text{MSE}(\bar{y}_{\text{prop}})_{\min} < \text{MSE}(\bar{y}_{\text{exp}(ph)}) \quad \text{if}
\]

\[
\text{MSE}(\bar{y}_{\text{exp}(ph)}) - \text{MSE}(\bar{y}_{\text{prop}})_{\min} > 0 \quad \text{or if}
\]

\[
\bar{Y}_{h}^{2}C_{h}^{2}[1 - \rho_{p_h}^{2}C_{p_h}^{2}] - \bar{Y}_{h}^{2}C_{h}^{2}[1 - R_{p_h}^{2}] > 0, \text{ or if}
\]

\[
\left[1 - \frac{1}{4}C_{p_h}^{2} + \rho_{p_h}C_{p_h} + C_{p_h}^{2}R_{h}^{2} + C_{p_h}^{2}R_{h}^{2} \right] > 0
\]

6. From (47) and (59),

\[
\text{MSE}(\bar{y}_{\text{prop}})_{\min} < \text{MSE}(\bar{y}_{\text{xk}(ph)}) \quad \text{if}
\]

\[
\text{MSE}(\bar{y}_{\text{xk}(ph)}) - \text{MSE}(\bar{y}_{\text{prop}})_{\min} > 0 \quad \text{or if}
\]

\[
\bar{Y}_{h}^{2}C_{h}^{2}[1 - \rho_{p_h}^{2}C_{p_h}^{2}] - \bar{Y}_{h}^{2}C_{h}^{2}[1 - R_{p_h}^{2}] > 0, \text{ or if}
\]

\[
\left[R_{p_h}^{2} - \rho_{p_h}C_{p_h} \right] > 0
\]

7. From (51) and (59),

\[
\text{MSE}(\bar{y}_{\text{prop}})_{\min} < \text{MSE}(\bar{y}_{\text{xk}(ph)}) \quad \text{if}
\]

\[
\text{MSE}(\bar{y}_{\text{xk}(ph)}) - \text{MSE}(\bar{y}_{\text{prop}})_{\min} > 0 \quad \text{or if}
\]

\[
\bar{Y}_{h}^{2}C_{h}^{2}[1 - \rho_{p_h}^{2}C_{p_h}^{2}] - \bar{Y}_{h}^{2}C_{h}^{2}[1 - R_{p_h}^{2}] > 0, \text{ or if}
\]

\[
\left[C_{p_h}^{2} + C_{p_h}^{2} - 2(\rho_{p_h}C_{p_h}C_{p_h} + \rho_{p_h}C_{p_h}C_{p_h} + \rho_{p_h}C_{p_h}C_{p_h}) \right]
\]

\[
-\bar{Y}_{h}^{2}C_{h}^{2}[1 - R_{p_h}^{2}] > 0, \text{ or if}
\]

\[
C_{p_h}^{2} + C_{p_h}^{2} - 2(\rho_{p_h}C_{p_h}C_{p_h} + \rho_{p_h}C_{p_h}C_{p_h} + \rho_{p_h}C_{p_h}C_{p_h})
\]

\[+C_{p_h}^{2}R_{p_h}^{2} > 0\]

8. From (53) and (58),

\[
\text{MSE}(\bar{y}_{\text{prop}})_{\min} < \text{MSE}(\bar{y}_{\text{xk}(ph)}) \quad \text{if}
\]

\[
\text{MSE}(\bar{y}_{\text{xk}(ph)}) - \text{MSE}(\bar{y}_{\text{prop}})_{\min} > 0 \quad \text{or if}
\]
A generalized exponential-type estimator for population mean using auxiliary attributes

\[ \hat{Y}_n = \hat{Y}_n \left[ C_{p1}^2 + C_{p2}^2 + C_{p3}^2 \right] - \hat{Y}_n \left[ C_{p1}^2 \left( 1 - R_{p1}^2 \right) \right] > 0, \text{ or if} \]

\[ C_{p1}^2 + C_{p2}^2 + 2 \left( \rho_{y_{p1}p} \cdot C_{p1} \cdot C_{p2} + \rho_{y_{p1}p} \cdot C_{p1} \cdot C_{p3} + \rho_{y_{p2}p} \cdot C_{p2} \cdot C_{p3} \right) \]

Numerical comparison under stratified random sampling

To observe the performance of our proposed generalized class of estimators with respect to other considered estimators under stratified random sampling, we use the following data sets, which earlier used by many authors.

**Population 1.** [Source: Kadilar and Cingi [16]]

Let \( y \) be the apple production amount in 1999, \( p_1 \) be the proportion of number of apple trees greater than 20,000 in 1999 and \( p_2 \) be the proportion of number of apple production amount greater than 25000 in 1998.

In Table 5, the population size is 854 and the sample size is 200 by the use of different strata. Also, find out the coefficient of variation and correlation coefficient and conclude that all the correlation is positive among the variables.

**Population 2.** [Source: Sarndal et al. [18]]

Let \( y \) be the population in thousands during 1985, \( p_1 \) be the population proportion during 1975 which is less than 60 and \( p_2 \) be the proportion of number of seats in municipal council less than 100. We use proportional allocation.

In Table 6, the population size is 284 and the sample size is 68 by the use of different strata. Also, find out the sample mean, coefficient of variation and correlation coefficient. We conclude that some of the correlation is positive or negative among the variables.

**Population 3.** [Source: Koyuncu and Kadilar [13]]

Let \( y \) be the number of teachers, \( p_1 \) be the number of students both primary and secondary schools in Turkey in 2007 for 923 districts in six regions which is less then 1000 and \( p_2 \) be the number of students both primary and secondary schools in Turkey in 2008 for 923 districts in six regions which is less then 200. We use proportional allocation.

In Table 7, the population size is 923 and the sample size is 180 for different strata. Also, find out the sample mean, coefficient of variation and correlation coefficient. We conclude that all of the correlation is positive among the variables.

**Population 4.** [Source: Gerard et al. [19]]

Table 5. Summary statistics of population 1.

| \( N_1 \) | \( N_2 \) | \( N_3 \) | \( N_4 \) | \( N_5 \) | \( N_6 \) |
|---|---|---|---|---|---|
| 106 | 106 | 94 | 171 | 204 | 173 |
| \( n_1 \) = 13 | \( n_2 \) = 24 | \( n_3 \) = 55 | \( n_4 \) = 95 | \( n_5 \) = 10 | \( n_6 \) = 3 |
| \( \hat{Y}_1 \) = 1536.774 | \( \hat{Y}_2 \) = 212.594 | \( \hat{Y}_3 \) = 9384.309 | \( \hat{Y}_4 \) = 5588.02 | \( \hat{Y}_5 \) = 966.960 | \( \hat{Y}_6 \) = 404.39 |
| \( C_{p1} \) = 4.181 | \( C_{p1} \) = 5.221 | \( C_{p1} \) = 3.187 | \( C_{p1} \) = 5.126 | \( C_{p1} \) = 2.472 | \( C_{p1} \) = 2.339 |
| \( C_{p1} \) = 1.702 | \( C_{p1} \) = 1.563 | \( C_{p1} \) = 1.095 | \( C_{p1} \) = 1.069 | \( C_{p1} \) = 1.358 | \( C_{p1} \) = 2.773 |
| \( C_{p1} \) = 1.857 | \( C_{p1} \) = 1.677 | \( C_{p1} \) = 1.220 | \( C_{p1} \) = 1.265 | \( C_{p1} \) = 1.483 | \( C_{p1} \) = 2.942 |
| \( \rho_{y_{p1}p} \) = 0.35 | \( \rho_{y_{p2}p} \) = 0.266 | \( \rho_{y_{p3}p} \) = 0.332 | \( \rho_{y_{p4}p} \) = 0.198 | \( \rho_{y_{p5}p} \) = 0.396 | \( \rho_{y_{p6}p} \) = 0.673 |
| \( \rho_{y_{p1}p} \) = 0.377 | \( \rho_{y_{p2}p} \) = 0.281 | \( \rho_{y_{p3}p} \) = 0.363 | \( \rho_{y_{p4}p} \) = 0.229 | \( \rho_{y_{p5}p} \) = 0.421 | \( \rho_{y_{p6}p} \) = 0.677 |
| \( \rho_{y_{p1}p} \) = 0.949 | \( \rho_{y_{p2}p} \) = 0.932 | \( \rho_{y_{p3}p} \) = 0.897 | \( \rho_{y_{p4}p} \) = 0.846 | \( \rho_{y_{p5}p} \) = 0.893 | \( \rho_{y_{p6}p} \) = 0.883 |

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Table 6. Summary statistics of population 2.

| N_1 | N_2 | N_3 | N_4 | N_5 | N_6 | N_7 | N_8 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 25  | 48  | 32  | 38  | 56  | 41  | 15  | 29  |

| n_1 | n_2 | n_3 | n_4 | n_5 | n_6 | n_7 | n_8 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 6   | 11  | 8   | 9   | 13  | 10  | 4   | 7   |

| \bar{Y}_1 = 62.44 | \bar{Y}_2 = 29.60 | \bar{Y}_3 = 24.06 | \bar{Y}_4 = 31.51 | \bar{Y}_5 = 29.41 | \bar{Y}_6 = 20.83 | \bar{Y}_7 = 26.67 | \bar{Y}_8 = 17.52 |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|

| \rho_{y_1y_2} = -0.61 | \rho_{y_1y_3} = -0.93 | \rho_{y_1y_4} = -0.77 | \rho_{y_1y_5} = -0.72 | \rho_{y_1y_6} = -0.77 | \rho_{y_1y_7} = -0.82 | \rho_{y_1y_8} = -0.91 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

| \rho_{y_2y_3} = -0.12 | \rho_{y_2y_4} = -0.52 | \rho_{y_2y_5} = -0.36 | \rho_{y_2y_6} = -0.35 | \rho_{y_2y_7} = -0.56 | \rho_{y_2y_8} = -0.62 | \rho_{y_3y_4} = -0.76 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

| \rho_{y_3y_5} = 0.10 | \rho_{y_3y_6} = 0.38 | \rho_{y_3y_7} = 0.22 | \rho_{y_3y_8} = 0.29 | \rho_{y_4y_5} = 0.24 | \rho_{y_4y_6} = 0.20 | \rho_{y_4y_7} = 0.36 | \rho_{y_4y_8} = 0.50 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

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Table 7. Summary statistics of population 3.

| N_1 | N_2 | N_3 | N_4 | N_5 | N_6 | N_7 | N_8 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 127 | 117 | 103 | 170 | 205 | 201 |

| n_1 | n_2 | n_3 | n_4 | n_5 | n_6 | n_7 | n_8 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 31  | 21  | 29  | 38  | 22  | 39  |

| \bar{Y}_1 = 703.74 | \bar{Y}_2 = 413 | \bar{Y}_3 = 573.17 | \bar{Y}_4 = 424.67 | \bar{Y}_5 = 267.03 | \bar{Y}_6 = 393.84 |
|------------------|----------------|----------------|----------------|----------------|----------------|

| \rho_{y_1y_2} = 0.27 | \rho_{y_1y_3} = 0.19 | \rho_{y_1y_4} = 0.18 | \rho_{y_1y_5} = 0.25 | \rho_{y_1y_6} = 0.25 | \rho_{y_1y_7} = 0.16 | \rho_{y_1y_8} = 0.16 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

| \rho_{y_2y_3} = 0.56 | \rho_{y_2y_4} = 0.48 | \rho_{y_2y_5} = 0.43 | \rho_{y_2y_6} = 0.53 | \rho_{y_2y_7} = 0.58 | \rho_{y_2y_8} = 0.40 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

| \rho_{y_3y_4} = 0.43 | \rho_{y_3y_5} = 0.30 | \rho_{y_3y_6} = 0.37 | \rho_{y_3y_7} = 0.41 | \rho_{y_3y_8} = 0.33 | \rho_{y_4y_7} = 0.34 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

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Table 8. Summary statistics of population 4.

| N_1 | N_2 | N_3 | N_4 | N_5 | N_6 | N_7 | N_8 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 70  | 111 | 64  | 65  | 69  | 84  | 44  | 44  |

| n_1 | n_2 | n_3 | n_4 | n_5 | n_6 | n_7 | n_8 |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 18  | 28  | 16  | 17  | 18  | 21  | 11  | 10  |

| \bar{Y}_1 = 81845269 | \bar{Y}_2 = 77647833 | \bar{Y}_3 = 53163090 | \bar{Y}_4 = 72678044 | \bar{Y}_5 = 45901248 | \bar{Y}_6 = 33436792 | \bar{Y}_7 = 52906990 | \bar{Y}_8 = 33313100 |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|

| \rho_{y_1y_2} = -0.30 | \rho_{y_1y_3} = -0.68 | \rho_{y_1y_4} = -0.68 | \rho_{y_1y_5} = -0.42 | \rho_{y_1y_6} = -0.63 | \rho_{y_1y_7} = -0.58 | \rho_{y_1y_8} = -0.72 | \rho_{y_2y_3} = -0.62 | \rho_{y_2y_4} = -0.66 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

| \rho_{y_2y_5} = 0.10 | \rho_{y_2y_6} = 0.12 | \rho_{y_2y_7} = 0.15 | \rho_{y_2y_8} = 0.07 | \rho_{y_3y_4} = 0.18 | \rho_{y_3y_5} = -0.17 | \rho_{y_3y_6} = -0.62 | \rho_{y_3y_7} = -0.02 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

| \rho_{y_3y_8} = 0.11 | \rho_{y_4y_5} = -0.18 | \rho_{y_4y_6} = 0.35 | \rho_{y_4y_7} = -0.16 | \rho_{y_4y_8} = -0.16 | \rho_{y_5y_6} = -0.11 | \rho_{y_5y_7} = -0.10 | \rho_{y_5y_8} = 0.38 | \rho_{y_6y_7} = 0.05 |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|

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Let \( y \) be the Total Taxation in Euros in 2001, \( p_1 \) be the Total Taxable income in Euros in 2001 which less then from mean and \( p_2 \) be the Total average income in Euros in 2001 which is less then from mean. We use proportional allocation.

In Table 8, the population size is 589 and the sample size is 150 for different strata. Also, find out the sample mean, coefficient of variation and correlation coefficent. We conclude that all of the correlation is negative among the variables.

We use the following expression to obtain the Percentage Relative Efficiency(PRE):

\[
PRE = \frac{\text{MSE}(\bar{y}_i)}{\text{MSE}(\bar{y}_i)_{\text{or MSE}(\bar{y}_i)_{\text{min}}}} \times 100
\]

where \( i = 0, Rh, Ph, \exp(Rh), \exp(Ph), KB_{(Rh)}, SK_{(Rh)}, SK_{(Ph)} \) and \( \bar{y}_{\text{pre}} \).

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The results based on population 1–4 are given Table 9.

The PRE values of different estimators with respect to $\bar{y}_u$ is given Table 9. Some members of the proposed class of estimators show poor performance because of negative correlation particularly product type estimators. Overall the performance of the proposed estimators $\bar{y}_{prop}$ outperforms as compared to all other considered estimators. The PRE values of some members of the proposed class of estimators are given in Table 10.

From Table 10, we observed that the PRE values of some members of the proposed class of estimators perform poorly. These are expected results because product type estimators are performing poorly.

**Conclusion**

In this paper, we proposed an improved class of estimators of finite population mean by utilizing data sets on two auxiliary attributes in both simple random sampling (SRS) and Stratified random sampling (StRS) schemes. Bias and MSE expressions of proposed class of estimators $\bar{y}_{prop}$ and $\bar{y}_{prop_h}$ are acquired upto first order of approximation. It can be seen that, both

### Table 9. Percentage relative efficiency (PRE) with respect to $\bar{y}_u$.

| Estimator          | Population 1 | Population 2 | Population 3 | Population 4 |
|--------------------|--------------|--------------|--------------|--------------|
| $\bar{y}_u$        | 100          | 100          | 100          | 100          |
| $\bar{y}_{st}$     | 109.3217     | 74.75077     | 105.0303     | 55.80278     |
| $\bar{y}_{rh}$     | 73.55255     | 132.0828     | 85.96212     | 121.2923     |
| $\bar{y}_{st(\text{ini})}$ | 108.3332 | 86.4283      | 103.9904     | 75.81773     |
| $\bar{y}_{mph(\text{ini})}$ | 87.29826 | 115.3814     | 93.70092     | 119.7365     |
| $\bar{y}_{mph(\text{ratio})}$ | 110.3059 | 189.0565     | 105.0827     | 129.4229     |
| $\bar{y}_{mph(\text{p})}$ | 92.04641  | 23.73677     | 118.2915     | 46.85948     |
| $\bar{y}_{mph(\text{r})}$ | 48.70935  | 39.4912      | 42.66078     | 70.30191     |
| $\bar{y}_{prop}$   | 112.2599     | 196.8767     | 133.2979     | 131.4789     |

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### Table 10. Percentage relative efficiency of proposed family estimator.

| Population          | Population 1 | Population 2 | Population 3 | Population 4 |
|---------------------|--------------|--------------|--------------|--------------|
| $\bar{y}_{prop_{1k}}$ | 111.015      | 49.66966     | 129.2438     | 72.86185     |
| $\bar{y}_{prop_{2k}}$ | 97.38008     | 70.62802     | 73.5152      | 65.27101     |
| $\bar{y}_{prop_{3k}}$ | 110.8836     | 76.4984      | 112.1906     | 76.51009     |
| $\bar{y}_{prop_{4k}}$ | 101.1683     | 59.68287     | 121.5799     | 108.65973    |
| $\bar{y}_{prop_{5k}}$ | 72.25059     | 85.23734     | 65.48686     | 98.36143     |
| $\bar{y}_{prop_{6k}}$ | 95.2416      | 99.81591     | 101.7252     | 119.892      |
| $\bar{y}_{prop_{7k}}$ | 103.0881     | 54.31054     | 126.207      | 93.1835      |
| $\bar{y}_{prop_{8k}}$ | 86.46328     | 78.39059     | 70.33049     | 82.51838     |
| $\bar{y}_{prop_{9k}}$ | 106.4946     | 83.8941      | 109.7694     | 92.26266     |
| $\bar{y}_{prop_{10k}}$ | 105.0036     | 95.51832     | 101.3751     | 91.55516     |
| $\bar{y}_{prop_{11k}}$ | 109.8506     | 54.56871     | 127.6962     | 93.51715     |
| $\bar{y}_{prop_{12k}}$ | 85.38252     | 77.96911     | 69.99571     | 83.52082     |
| $\bar{y}_{prop_{13k}}$ | 106.2032     | 87.51429     | 108.4386     | 100.6563     |

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theoretically and mathematically the proposed class of estimators consistently performs better than the existing estimators considered here under SRS and StRS. Based on these findings, we suggest the utilization of the proposed estimators for efficient estimation of population mean in presence of the auxiliary attributes in SRS and (StRS) schemes, are preferable for future study.

**Author Contributions**

**Conceptualization:** Sohail Ahmad, Javid Shabbir.

**Data curation:** Sohail Ahmad.

**Formal analysis:** Sohail Ahmad, Muhammad Arslan, Aamna Khan.

**Funding acquisition:** Sohail Ahmad.

**Investigation:** Sohail Ahmad.

**Methodology:** Sohail Ahmad, Muhammad Arslan.

**Project administration:** Sohail Ahmad.

**Resources:** Sohail Ahmad, Aamna Khan.

**Software:** Sohail Ahmad.

**Supervision:** Sohail Ahmad, Aamna Khan.

**Validation:** Sohail Ahmad, Muhammad Arslan.

**Visualization:** Sohail Ahmad.

**Writing – original draft:** Sohail Ahmad, Javid Shabbir.

**Writing – review & editing:** Sohail Ahmad, Javid Shabbir.

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