The Homotopy Theory of Function Spaces: A Survey

Samuel Bruce Smith

Abstract. We survey research on the homotopy theory of the space $\text{map}(X, Y)$ consisting of all continuous functions between two topological spaces. We summarize progress on various classification problems for the homotopy types represented by the path-components of $\text{map}(X, Y)$. We also discuss work on the homotopy theory of the monoid of self-equivalences $\text{aut}(X)$ and of the free loop space $LX$. We consider these topics in both ordinary homotopy theory as well as after localization. In the latter case, we discuss algebraic models for the localization of function spaces and their applications.

1. Introduction.

In this paper, we survey research in homotopy theory on function spaces treated as topological spaces of interest in their own right. We begin, in this section, with some general remarks on the topology of function spaces. We then give a brief historical sketch of work on the homotopy theory of function spaces. This sketch serves to introduce the basic themes around which the body of the paper is organized.

By work of Brown [39, 1964] and Steenrod [260, 1967], the homotopy theory of function spaces may be studied in the “convenient category” of compactly generated Hausdorff spaces. Retopologizing is required, however. Given spaces $X$ and $Y$ in this category, let $Y^X$ denote the space of all continuous functions with the compact-open topology. Define

$$\text{map}(X, Y) = k(Y^X)$$

to be the associated compactly generated space. Then $\text{map}(X, Y)$ satisfies the desired exponential laws and is a homotopy invariant of $X$ and $Y$. The space $\text{map}(X, Y)$ is generally disconnected with path-components corresponding to the set of free homotopy classes of maps. We write $\text{map}(X, Y; f)$ for the path-component containing a given map $f : X \to Y$. Important special cases include: $\text{map}(X, Y; 0)$, the space of null-homotopic maps; $\text{map}(X, X; 1)$, the identity component; $\text{aut}(X)$,
Concrete results on the path-components of \( \text{map}(X,Y) \) often require much more restrictive hypotheses on \( X \) and \( Y \). By Milnor \cite{121}, when \( X \) is a compact, metric space and \( Y \) is a CW complex, the components \( \text{map}(X,Y; f) \) are of CW homotopy type. A natural case to consider then is when \( X \) is a finite CW complex and \( Y \) is any CW complex. By Kahn \cite{153}, \( \text{map}(X,Y) \) is also of CW type when \( X \) is any CW complex and \( Y \) has finitely many homotopy groups.

The space \( \text{map}(X,Y) \) has two close relatives. If \( X \) and \( Y \) are pointed spaces, we have \( \text{map}_*(X,Y) \) the space of basepoint preserving functions, with components \( \text{map}_*(X,Y; f) \) for \( f \) a based map. Given a fibration \( p: E \to X \), we have \( \Gamma(p) \) the space of sections with components \( \Gamma(p; s) \) for \( s \) a fixed section. Of course, \( \text{map}(X,Y) \simeq \Gamma(p) \) when \( p \) fibre-homotopy trivial with fibre \( Y \). Many theorems about \( \text{map}(X,Y) \) generalize to \( \Gamma(p) \) and many have related versions for \( \text{map}_*(X,Y) \).

For the sake of brevity, when possible we state theorems for the free function space and omit extensions and restrictions. Theorems stated for the based function space are then results that do not apply to \( \text{map}(X,Y) \).

1.1. A Brief History. Function spaces are at the foundations of homotopy theory and appear in the literature dating back, at least, to Hurewicz’s definition of the homotopy groups in the 1930s. Work focusing explicitly on the homotopy theory of a function space first appears in the 1940s. Whitehead \cite{286}, 1946 introduced the problem of classifying the homotopy types represented by the path-components of a function space, focusing on the case \( \text{map}(S^2, S^2) \). Hu \cite{148}, 1946 showed

\[ \pi_1(\text{map}(S^2, S^2; \iota_m)) \cong \mathbb{Z}/2|m|, \]

where \( \iota_m \) is the map of degree \( m \) thus distinguishing components of different absolute degree.

A decade later, papers of Thom \cite{270}, 1957 and Federer \cite{88}, 1956 appeared giving dual methods for computing homotopy groups of components of \( \text{map}(X,Y) \). Thom used a Postnikov decomposition of \( Y \) to indicate a method of calculation. Federer constructed a spectral sequence converging to these homotopy groups using a cellular decomposition of \( X \). Both authors obtained the following basic identity:

\[ \pi_q(\text{map}(X,K(\pi,n); 0)) \cong H^{n-q}(X; \pi) \]

for \( X \) a CW complex and \( \pi \) an abelian group.

In the 1960s, the monoid \( \text{aut}(X) \) of all homotopy self-equivalences of \( X \) emerged as a central object for the theory of fibrations. Stasheff \cite{259}, 1963 constructed a universal fibration for CW fibrations with fibre of the homotopy type of a fixed finite CW complex \( X \), building on work of Dold-Lashof \cite{72}, 1957. His result implied the universal \( X \)-fibration is obtained, up to homotopy, by applying the Dold-Lashof classifying space functor to the evaluation fibration \( \omega: \text{map}(X,X;1) \to X \). In this same period, Gottlieb \cite{108}, 1965 introduced and studied the evaluation subgroups or Gottlieb groups:

\[ G_n(X) = \text{im}\{\omega_2: \pi_n(\text{map}(X,X;1)) \to \pi_n(X)\} \subseteq \pi_n(X) \]

initiating a rich literature on the evaluation map. Among many other properties, he showed the Gottlieb groups correspond to the image of the linking homomorphism in the long exact sequence of homotopy groups of the universal \( X \)-fibration. Thus
the vanishing of a Gottlieb group $G_n(X)$ is equivalent to the vanishing of the linking homomorphism in degree $n$ for every CW fibration with fibre $X$.

In the 1970s, Hansen [129] 1974 began a systematic study of the homotopy classification problem for the path components of $\text{map}(X,Y)$. He completed the classification for $\text{map}(S^n,S^n)$ building on the methods of Whitehead, mentioned above. He and other authors obtained complete results in many special cases involving spheres, suspensions, projective spaces and certain manifolds.

The space of holomorphic maps $\text{Hol}(M,N)$ between two complex manifolds offers a deep refinement of the homotopy classification problem for continuous maps with important interdisciplinary connections. Segal [244] 1979 initiated the study of the space $\text{Hol}(M,N)$ in homotopy theory proving the inclusion

$$\text{Hol}^*_k(S^2, CP^m) \hookrightarrow \text{map}_*(S^2, CP^m; t_k)$$

induces a homology equivalence through a range of degrees. Here $\text{Hol}^*_k(S^2, CP^m)$ denotes the space of based holomorphic maps of degree $k$. In fundamental work in complex geometry, Gromov [110] 1989 identified the class of elliptic manifolds and proved they satisfy the “Oka Principle”. As a consequence, he identified a large class of manifolds for which the inclusion $\text{Hol}(M,N) \hookrightarrow \text{map}(M,N)$ is a weak equivalence. Cohen-Cohen-Mann-Milgram [56] 1991 described the full stable homotopy type of $\text{Hol}^*_k(S^2, CP^m)$, their description given in terms of configuration spaces. A related problem of stabilization for moduli spaces of connections is the subject of the famous “Atiyah-Jones conjecture” in mathematical physics Atiyah-Jones [15] 1978.

The gauge groups provide a connection between the homotopy theory of function spaces and the theory of principal bundles. Let $P: E \to X$ be a principal $G$-bundle for $G$ a connected topological group classified by a map $h: X \to BG$. The gauge group $\mathcal{G}(P)$ of $P$ is defined to be the group of $G$-equivariant homeomorphisms $f: E \to E$ over $X$. Atiyah-Bott [14] 1983 used the gauge group in their celebrated study of Yang-Mills equations and principal bundles over a Riemann surface. They made use of Thom’s theory and a multiplicative equivalence originally due to Gottlieb [112] 1972

$$\mathcal{G}(P) \simeq \Omega \text{map}(X,BG; h)$$

to study the homotopy theory of $BG(P)$. Gottlieb’s identity, in turn, links the classification of gauge groups up to $H$-homotopy type, for fixed $G$ and $X$, to the homotopy classification problem for $\text{map}(X,BG)$. Crabb-Sutherland [67] 2000 proved that the gauge groups $\mathcal{G}(P)$ represent only finitely many homotopy types for $G$ a compact Lie group and $X$ a finite complex. In contrast, the path-components of $\text{map}(X,BG)$ may represent infinitely many distinct homotopy types in this case by Masbaum [199] 1991.

The advent of localization techniques introduced new depth to the study of function spaces while opening up a wide range of fundamental problems and applications. In his seminal paper on rational homotopy theory, Sullivan [263] 1977 sketched a construction for an algebraic model for components of $\text{map}(X,Y)$ for $X$ and $Y$ simply connected CW complexes with $X$ finite, as an extension of Thom’s ideas. Sullivan’s construction was completed by Haefliger [123] 1982. Sullivan also identified the rational Samelson Lie algebra of $\text{aut}(X)$ for $X$ a finite, simply connected CW complex via an isomorphism:

$$\pi_*(\text{aut}(X)) \otimes \mathbb{Q}, [ , ] \cong H_*(\text{Der}(\mathcal{M}_X)), [ , ]$$
Here the latter space is the homology of the Lie algebra of degree lowering derivations of the Sullivan minimal model of $X$ with the commutator bracket.

One of the early applications of Sullivan’s rational homotopy theory was the proof by Vigué-Poirrier-Sullivan [282, 1976] of the unboundedness of the Betti numbers of the free loop space $LX = \text{map}(S^1, X)$ for certain simply connected CW complexes $X$. Combined with a famous result of Gromoll-Meyer [115, 1969] in geometry, this calculation solved the “closed geodesic problem” for many manifolds. The calculation was deduced from a Sullivan model constructed for $LX$.

The $p$-local homotopy theory of a function space featured in a landmark result in algebraic topology, the proof of the Sullivan conjecture. Miller [213, 1984] proved

$$\pi_n(\text{map}_*(B\pi, X; 0)) = 0 \quad \text{for all } n \geq 0$$

where $\pi$ is any finite group and $X$ any finite CW complex. Among many applications, this result was used by McGibbon-Neisendorfer [209, 1984] to affirm Serre’s conjecture: $\pi_m(X)$ contains a subgroup of order $p$ for infinitely many $m$.

Lannes [180, 1987] constructed the $T$-functor which is left adjoint to the tensor product in the category of unstable modules over the Steenrod algebras. His construction provided a model for the mod $p$ cohomology of the space $\text{map}(BV, X)$ where $V$ is a $p$-group. Lannes’ construction was adapted to the rational homotopy setting by Bousfield-Peterson-Smith [30, 1989] and, later, Brown-Szczarba [37, 1997] to give another model for the rational homotopy type of $\text{map}(X, Y; f)$. Fresse [102, preprint] recently constructed a version of Lannes’ functor in a category of operadic algebras giving a model for the integral homotopy type of certain function spaces.

The free loop space recently re-emerged as a central object for study in homotopy theory with the appearance of work of Chas-Sullivan [50, preprint]. They constructed a product on the regraded homology

$$H_*(LM^m) = H_{*+m}(LM^m)$$

for a simply connected, closed, oriented $m$-manifold $M^m$ using intersection theory. They also defined a bracket on the equivariant homology of $LM^m$ and a degree $+1$ operator giving $H_*(LM^m)$ the structure of Batalin-Vilkovisky algebra. These structures have incarnations in diverse other settings. Their study, known as string topology, is now an active subfield in the intersection of homotopy theory and geometry.

1.2. Organization. In Section 2 we discuss work on the ordinary and stable homotopy theory, as opposed to the local homotopy theory of function spaces. We focus on the areas introduced above, namely: (i) the general path component $\text{map}(X, Y; f)$; (ii) the monoid $\text{aut}(X)$; and (iii) the free loop space $LX$. We also discuss work on the stable homotopy theory of these spaces and on spectral sequence calculations of their invariants. In Section 3 we discuss the localization of function spaces. We describe the algebraic models of Sullivan, and of later authors, for the general component, the monoid of self-equivalences and the free loop space in rational homotopy theory, and survey their applications. We also discuss the $p$-local homotopy theory of function spaces including the work of Miller, Lannes and others on the space of maps out of a classifying space, and algebraic models for function spaces in tame homotopy theory. The paper includes a rather extensive
bibliography gathering together both papers directly focused on function spaces and papers giving significant applications and extensions.

2. Ordinary and Stable Homotopy Theory of Function Spaces.

We divide our discussion in this section according to the cases (i), (ii) and (iii) above. We then discuss some general results in stable homotopy theory and spectral sequence constructions for function spaces.

2.1. General Components. As mentioned in the introduction, the following open problem lies at the historical roots of the study of function spaces as objects in their own right.

**Problem 2.1.** Given spaces $X$ and $Y$ classify the path-components $\text{map}(X,Y; f)$ up to homotopy type for homotopy classes $f: X \to Y$.

We consider a variety of cases here beginning with the most classical, mentioning progress on Problem 2.1 when appropriate.

2.1.1. Maps from Spheres and Suspensions. The components of $\text{map}(S^p, Y)$ correspond to the homotopy classes in $\pi_p(Y)$. The coproduct on $S^p$ gives rise to an equivalence $\text{map}_*(-; Y; \alpha) \simeq \text{map}_*(S^p, Y; 0)$, for any class $\alpha: S^p \to Y$. By adjointness, $\pi_n(\text{map}_*(S^p, Y; 0)) \cong \pi_{n+p}(Y)$. These observations were made by Whitehead [286] who gave the first algebraic method for computation. Whitehead identified the linking homomorphism in the long exact homotopy sequence for the evaluation fibration $\text{map}_*(S^p, Y; \alpha) \to \text{map}(S^p, Y; \alpha) \to Y$ obtaining:

\[
\pi_{n+1}(Y) \xrightarrow{\partial} \pi_n(\text{map}_*(S^p, Y; \alpha)) \xrightarrow{W(\alpha)} \pi_n(\text{map}(S^p, Y; \alpha)) \xrightarrow{\cong} \pi_{n+p}(Y)
\]

where $W(\alpha)(\beta) = -[\alpha, \beta]_w$ denotes the Whitehead product map. Using this sequence, he proved $\text{map}(S^2, S^2; \iota) \not\cong \text{map}(S^2, S^2; 0)$ by comparing homotopy groups. Hu [148] and Koh [160] computed $\pi_{2m-1}(\text{map}(S^{2m}, S^{2m}; \alpha))$ for small values of $m$. In these cases, the order of $\pi_{2m-1}(\text{map}(S^{2m}, S^{2m}; \alpha))$ depends on the absolute value of the degree of $\alpha$ and so distinguishes components with different absolute order. Since clearly $\text{map}(S^{2m}, S^{2m}; \alpha) \simeq \text{map}(S^{2m}, S^{2m}; \alpha)$ the classification in these cases was complete with these calculations.

Hansen [129] 1974 obtained the complete classification for self-maps of $S^n$. For even spheres, he proved

\[
\text{map}(S^{2m}, S^{2m}; \alpha) \simeq \text{map}(S^{2m}, S^{2m}; \beta) \iff [\alpha, \iota]_w = \pm [\beta, \iota]_w.
\]

Here $\iota \in \pi_{2m}(S^{2m})$ is the fundamental class. For odd spheres, the components of $\text{map}(S^{2m-1}, S^{2m-1})$ are all homotopy equivalent for $m = 1, 2, 4$ due to the existence of a multiplication on $S^{2m-1}$ in these cases. For $m \neq 1, 2, 4$, Hansen showed $\text{map}(S^{2m-1}, S^{2m-1}; \iota) \not\cong \text{map}(S^{2m-1}, S^{2m-1}; 0)$ and

\[
\text{map}(S^{2m-1}, S^{2m-1}; \alpha) \simeq \begin{cases} \text{map}(S^{2m-1}, S^{2m-1}; \iota) & \text{if } \deg(\alpha) = \text{odd} \\ \text{map}(S^{2m-1}, S^{2m-1}; 0) & \text{if } \deg(\alpha) = \text{even}. \end{cases}
\]

Problem 2.1 remains open for $\text{map}(S^m, S^n)$ for $m > n$. Yoon [305] 1995 observed a connection between the Gottlieb group $G_m(Y)$ and the homotopy classification problem for $\text{map}(S^m, Y)$ showing $\text{map}(S^m, Y; \alpha) \simeq \text{map}(S^m, Y; 0)$ if and
only if $\alpha \in G_m(Y)$. Lupton-Smith [193] 2008 extended this to a surjection of sets

$$\pi_m(Y)/G_m(Y) \xrightarrow{\text{homotopy equivalence}} \{\text{components } \text{map}(S^n, Y; f)\}.$$ 

Thus the complexity of the classification problem for $\text{map}(S^n, S^n)$ is roughly that of computing Gottlieb groups $G_m(S^n)$. Extensive, low-dimensional calculations of this group were recently made by Golasiński-Mukai [106] 2009. Lee-Mimura-Woo [184] 2004 calculated the Gottlieb groups for certain homogeneous spaces.

When $X = \Sigma A$ is a suspension, the fibres $\text{map}_*(\Sigma A, Y; f)$ of the various evaluation fibrations $\omega_f: \text{map}(\Sigma A, Y; f) \to Y$ are all homotopy equivalent to the space $\text{map}_*(\Sigma A, Y; 0)$ with homotopy groups

$$\pi_q(\text{map}_*(\Sigma A, Y; 0)) = [\Sigma^{q+1} A, Y].$$

Lang [178] 1973 extended Whitehead’s exact sequence to this case replacing the Whitehead product in $\pi_n(Y)$ by the generalized Whitehead product in $[\Sigma^* A, Y]$. It is natural to consider, as Whitehead did, a stronger version of Problem 2.1, namely, the classification of the evaluation fibrations $\omega_f: \text{map}(X, Y; f) \to Y$ up to fibre homotopy type for homotopy classes $f: X \to Y$. Hansen [128] 1974 defined $\omega_f: \text{map}(\Sigma A, \Sigma B; f) \to Y$ to be strongly fibre homotopy equivalent to $\omega_g: \text{map}(\Sigma A, \Sigma B; g) \to \Sigma B$ if the fibre homotopy equivalence is homotopic to the identity after (fixed) identification of the fibres with $\text{map}_*(\Sigma A, \Sigma B; 0)$. He proved:

$$\omega_f \text{ is strongly fibre homotopic to } \omega_g \iff [f, 1_{\Sigma B}] = [g, 1_{\Sigma B}]$$

where $[\cdot , \cdot]$ here denotes the generalized Whitehead product in $[\Sigma A, \Sigma B]$. McClen- don [206] 1981 showed that the evaluation fibrations $\omega_f: \text{map}(\Sigma A, Y; f) \to Y$ behave as principal fibrations and, in particular, are classified by maps $s: Y \to \text{map}(A, Y)$ determined by generalized Whitehead products.

2.1.2. Maps into Eilenberg-Mac Lane Spaces. The weak homotopy type of the space $\text{map}(X, K(\pi, n); f)$ may be described for any $f: X \to K(\pi, n)$ for $\pi$ abelian. The ideas are due to Thom [270] 1957 with a refinement by Haefliger [123] 1982. First, observe that these components are all homotopy equivalent since $K(\pi, n)$ has the homotopy type of a topological group. A homotopy class $\alpha \in \pi_p(\text{map}(X, K(\pi, n); 0))$ corresponds, by adjointness, to a map $A: S^p \times X \to K(\pi, n)$.

On cohomology,

$$A^*(x_n) = 1 \otimes a_n + u_p \otimes a_{n-p}$$

where $a_n, a_{n-p} \in H^*(X; \pi)$ with subscripts indicating degree while $u_p \in H^p(S^p; \pi)$ and $x_n \in H^n(K(\pi, n); \pi)$ are the fundamental classes. Since $A$ restricts to the constant map on $S^p \times \ast$ we see $a_n = 0$. The assignment $\alpha \mapsto a_{n-p}$ gives the identification

$$\pi_p(\text{map}(X, K(\pi, n); f)) \cong H^{n-p}(X; \pi),$$

mentioned in the introduction and leads to directly to a weak equivalence

$$\text{map}(X, K(\pi, n); f)) \simeq \bigwedge_{p \geq 1} K(H^{n-p}(X; \pi), p).$$

Thom also indicated how the homotopy groups $\pi_p(\text{map}(X, Y; f))$ for $Y$ a finite Postnikov piece are determined, up to extensions, by the $k$-invariants of $Y$ and the groups $H^{n-p}(X, \pi_n(Y))$. This approach was encoded in Haefliger’s construction of a Sullivan model for $\text{map}(X, Y; f)$, as discussed below.
Gottlieb [110] 1969] extended Thom’s result to the case $n = 1$ and $\pi$ any group. Here

$$\text{map}(X, K(\pi, 1); f) \approx_w K(C(f_2), 1)$$

where $C(f_2)$ denotes the centralizer of the image of $f_2: \pi_1(X) \to \pi$. Möller [218] 1987] showed that when $Y$ is a twisted Eilenberg-Mac Lane space, then $\text{map}(X, Y; f)$ is one also with homotopy groups determined by the cohomology groups of $X$ with twisted coefficients in the homotopy groups of $Y$. Note that the weak equivalences above are homotopy equivalences by Whitehead’s Theorem, when $\text{map}(X, K(\pi, n))$ is of CW type, e.g., when $X$ is compact or a CW complex. In general, the study of the homotopy type of $\text{map}(X, Y; f)$ when $Y$ has at least two nonvanishing homotopy groups is a difficult, open problem.

2.1.3. Maps between Manifolds. The homotopy theory of $\text{map}(M^m, N^n)$ for $M^m$ and $N^n$ closed manifolds is a topic of wide-ranging interest. In this case, important variations have been considered. Below we consider one such variation with direct ties to Problem 2.1 namely spaces of holomorphic maps. We begin with the space $\text{map}(M^m, N^n)$.

If $T_g$ is an orientable surface, then $T_g \simeq K(\pi_1(T_g), 1)$ and the classification problem for $\text{map}(X, T_g)$ reduces to the computation of centralizers of homomorphisms into $\pi_1(T_g)$. For $g \geq 2$ this group is highly nonabelian and the only possible nontrivial centralizers are isomorphic to $\mathbb{Z}$ by Hansen [135] 1983]. Hansen [131] 1974] earlier considered the space $\text{map}(T_g, S^2)$. As a generalization of Whitehead’s exact sequence, he showed an exact sequence

$$0 \to \mathbb{Z}/2|m| \to \pi_1(\text{map}(T_g, S^2; \iota_m)) \to \mathbb{Z}^{2g} \to 0$$

which gives the classification, in terms of degree, in this case. The fundamental group $\pi_1(\text{map}(T_g, S^2; \iota_m))$ was later completely determined by Larmore-Thomas [182] 1980].

Hansen [133] 1981] extended his classification result for the space of self-maps of spheres to the case $\text{map}(M^m, S^n)$ where $M^m$ is closed, oriented, connected manifold with vanishing first Betti number. Note that, by Hopf’s Theorem, $[M^m, S^n] = \mathbb{Z}$ with maps classified by degree. When $m$ is even and $\geq 4$, Hansen showed the homotopy types of $\text{map}(M^m, S^n; \alpha)$ are classified by the absolute values of the degrees of the $\alpha$. When $m$ is odd and $m \neq 1, 4, 7$ there are two homotopy types corresponding to the distinct types $\text{map}(M^m, S^n; 0)$ and $\text{map}(M^m, S^n; \iota)$ where $\iota$ is of degree 1. Again in this case, components are classified by the parity of degree of the class $\alpha$.

Sutherland [264] 1983] extended Hansen’s work eliminating the restriction on the first Betti number and dealing with the case $M^m$ nonorientable. Note that in the latter case there are only two distinct classes $\alpha: M^m \to S^m$ and so the problem reduces to distinguishing these components for $m \neq 1, 4, 7$. Sutherland observed that the components of $\text{map}(M^m, S^n)$ all have the same homotopy type if there is a map $\iota: M^m \to \text{map}(S^m, S^n; \iota)$ making the diagram

$$\begin{align*}
\text{map}(S^m, S^n; 1) & \\
M^m & \xrightarrow{\iota} S^m
\end{align*}$$
the homotopy theory of the space $\text{Hol}(M, N)$ is a difficult open problem. Recently, Maruyama-Oshima complete classification for the components of $\text{map}(M^m, S^n)$ are all of the same homotopy type if there exists a map $f : M^m \to \mathbb{R}P^n$ of odd degree giving examples with $m \neq 1, 4, 7$ for which all the components are homotopy equivalent.

Sasao [235] 1974 studied the homotopy type of components of $\text{map}(\mathbb{C}P^m, \mathbb{C}P^n; i)$ for $m \leq n$ and $i : \mathbb{C}P^m \to \mathbb{C}P^n$ the inclusion. He constructed a map 
$$\alpha_{m,n} : U(n+1)/\Delta(m+1) \times U(n-m) \to \text{map}(\mathbb{C}P^m, \mathbb{C}P^n; i)$$

where $\Delta(m+1) \subset U(m+1)$ denotes scalar multiplies of the identity. He proved $\alpha_{m,n}$ induces an isomorphism on rational homotopy groups and on ordinary homotopy groups through degree $4n - 4m + 1$. Yamaguchi [289] 1983 extended Sasao’s analysis to the case of quaternionic projective spaces. Möller [215] 1984 gave the complete classification for the components $\text{map}(\mathbb{C}P^m, \mathbb{C}P^n)$ showing the homotopy types are classified by the absolute value of the degree of a representative class. The result is a direct consequence of his calculation

$$H_{2n-2m+1}(\text{map}(\mathbb{C}P^m, \mathbb{C}P^n; i)) \cong \mathbb{Z}/d$$

where $d = (n+1)/m$.

Yamaguchi [292] 2006 studied maps between real projective spaces. He defined the analogue of Sasao’s map, here of the form

$$\beta_{m,n} : O(n+1)/\Delta(m+1) \times O(n-m) \to \text{map}(\mathbb{R}P^m, \mathbb{R}P^n; i)$$

and proved $\beta_{m,n}$ is an equivalence on ordinary and rational homotopy groups through certain ranges of degrees.

When $G$ is a topological group (or group-like space) the path-components of $\text{map}(X, G)$ are all of the same homotopy type. Problem 2.1 thus reduces, in this case, to the study of the homotopy theory of the null-component $\text{map}(X, G; 0)$. Given Lie groups $G$ and $H$, the calculation of homotopy invariants of $\text{map}(G, H)$ is a difficult open problem. Recently, Maruyama-Oshima [198] 2008 computed the homotopy groups of $\text{map}_*(G, G)$ for $G = SU(3), Sp(2)$ in degrees $\leq 8$.

2.1.4. Spaces of Holomorphic Maps. Segal [244] 1979 proved a basic result on the homotopy theory of the space $\text{Hol}(M, N)$. His work launched a vital subfield of research on the “stability” of the inclusion $\text{Hol}(M, N) \hookrightarrow \text{map}(M, N)$. Segal proved

$$\text{Hol}_k^*(T_g, \mathbb{C}P^n) \hookrightarrow \text{map}_*(T_g; i)$$

induces a homology isomorphism through dimension $(k - 2g)(2n - 1)$ where $T_g$ is a Riemann surface of genus $g$. Specializing to the case of the sphere, he proved

$$\text{Hol}_k^*(S^2, \mathbb{C}P^n) \hookrightarrow \text{map}_*(S^2, \mathbb{C}P^n; i)$$

induces a homotopy equivalence up to degree $2n - 1$.

Segal’s work was extended by many authors. Guest [119] 1984 proved the corresponding stability result on homology for $\text{Hol}_k^*(S^2, F) \hookrightarrow \text{map}(S^2, F; i)$ for certain complex flag manifolds $F$. His proof involved developing the analogue of a Morse-theoretic result for the case of the energy functional on the space $C^\infty(S^2, F)$ of smooth maps. Kirwan [158] 1986 extended Segal’s result to the case the target is the complex Grassmannian manifold $G(n, n + m)$ of $n$-planes in $n + m$-space proving $\text{Hol}_k^*(S^2, G(n, n + m)) \hookrightarrow \text{map}(S^2, G(n, n + m); i)$ induces a homology isomorphism in degrees depending on $k, n$ and $m$. Mann-Milgram [196] 1991
considered this case as well, constructing a spectral sequence to analyze the homology of $\text{Hol}_k^*(S^2, G(n, n + m))$. Gravesen [113, 1989] studied holomorphic maps into space $\Omega^G$ for $G$ a complex, compact Lie group.

Cohen-Cohen-Mann-Milgram [56, 1991] and Cohen-Shimamota [64, 1991] described the stable homotopy type of $\text{Hol}_k^*(S^2, \mathbb{C}P^n)$. They proved

$$\text{Hol}_k^*(S^2, \mathbb{C}P^n) \simeq C_k(\mathbb{R}^2, S^{2n-1})$$

where $C_k(\mathbb{R}^2, S^{2n-1})$ is the configuration space of distinct points in $\mathbb{R}^2$ with labels in $S^{2n-1}$ of length at most $k$. Cohen-Cohen-Mann-Milgram also computed the homology of $\text{Hol}_k^*(S^2, \mathbb{C}P^n)$ with $\mathbb{Z}_p$-coefficients in terms of Dyer-Lashof operations. Mann-Milgram [197] used the stable homotopy decomposition above to prove the homology stability of the inclusion $\text{Hol}_k^*(S^2, F) \hookrightarrow \text{map}_*(S^2, F)$ for $F$ an $\text{SL}(n, \mathbb{C})$-flag-manifold. Boyer-Mann-Hurtubise-Milgram [33, 1994] and Hurtubise [149, 1996] proved homology stabilization theorems for the space $\text{Hol}_k^*(S^2, G/P)$ for certain complex homogeneous spaces $G/P$. Boyer-Hurtubise-Milgram gave a configuration space description of $\text{Hol}_k(T_g, M)$ for certain complex manifolds admitting nice Lie group actions extending the approach of Gravesen.

Segal’s stabilization problem has deep interdisciplinary connections. Gromov [116, 1989] obtained general stability results as a consequence of his work on the Oka Principle in complex geometry. A complex manifold $S$ satisfies the Oka principle if every continuous map $f : S \rightarrow M$ is homotopic to a holomorphic map where $S$ is a Stein manifold. Gromov identified the class of “elliptic” manifolds and proved elliptic manifolds satisfied the Oka principle. Consequently, he obtained the inclusion

$$\text{Hol}(S, M) \hookrightarrow \text{map}(S, M)$$

is a weak homotopy equivalence for $S$ Stein and $M$ elliptic. The class of elliptic manifolds includes complex Lie groups and their homogeneous spaces.

The problem of stabilization also has a famous incarnation in Yang-Mills theory and mathematical physics. Atiyah-Jones [15, 1978] constructed a map

$$\theta_k : \mathcal{M}_k \rightarrow \text{map}_*(S^3, SU(2); \iota_k)$$

where $\mathcal{M}_k$ is a moduli space of connections on a principal $SU(2)$-bundle $P_k$ over $S^4$ corresponding to a map $S^4 \rightarrow BSU(2)$ of degree $k$. They proved $\theta_k$ induces a homology surjection through a range of degrees and conjectured $\theta_k$ induces an equivalence in both homotopy and homology through a range depending on $k$.

Work on the Atiyah-Jones conjecture includes Taubes [268, 1989], Gravesen [113, 1989] and Boyer-Hurtubise-Mann-Milgram [31, 1993].

Many authors have studied related spaces of maps. Vassilev [277, 1992] proved a stable equivalence

$$\text{Hol}_k^*(S^2, \mathbb{C}P^n) \simeq \text{SP}_n^k(\mathbb{C})$$

where the latter is the space of monic complex polynomials of degree $k$ with all roots of multiplicity $< n$. Guest-Kozlowski-Yamaguchi [122, 1994] extended Segal’s result in a different direction, proving the inclusion

$$\text{Hol}_k^*(S^2, X_n) \hookrightarrow \text{map}_*(S^2, X_n)$$

is a homotopy equivalence up to degree $k$ where $X_n \subseteq \mathbb{C}P^{n-1}$ is the subspace of points with at most one coordinate zero. The cohomology of the space of basepoint-free holomorphic maps $\text{Hol}_1(S^2, S^2)$ was studied by Havlicek [140, 1995] while the
homotopy groups of $\text{Hol}_k(S^2, S^2)$ were studied by Guest-Kozlowski-Murrayama-Yamaguchi [121] 1995. Kallel-Milgram [154] 1997 gave a complete calculation of the homology of $\text{Hol}^*_k(T_g, CP^n)$ for $T_g$ an elliptic Riemann surface. The space of real rational functions was recently studied by Kamiyama [157] 2007.

Kallel-Salvatore [155] 2006 applied techniques from string topology to the study of spaces of maps between manifolds. Set

$$\mathbb{H}_n(\text{map}(M^m, N^n)) = H_{n+n}(\text{map}(M^m, N^n))$$

and similarly for $\text{Hol}(S^2, N^n)$. When $M^m, N^n$ are closed, compact and orientable, they proved $\mathbb{H}_n(\text{map}(S^m, N))$ has a ring structure corresponding to an intersection product and $\mathbb{H}_n(\text{map}(M^m, N^n))$ is a module over this ring. They used this structure to compute $\mathbb{H}_n(\text{map}(S^2, CP^n; t_k))$ and $\mathbb{H}_n(\text{Hol}(S^2, CP^n))$ with $\mathbb{Z}_p$-coefficients. They also studied $\mathbb{H}_n(\text{map}(T_g, CP^n; t_k); \mathbb{Z}_p)$ for $T_g$ a compact Riemann surface proving, among other results, that these groups are isomorphic for all $k$ when $p$ divides $n$.

2.1.5. Maps into a Classifying Space and Gauge Groups. Let $X$ be a space and $G$ a connected topological group. Suppose $P: E \to X$ is a principal $G$-bundle. The gauge group $G(P)$ is the topological group of bundle automorphisms of $P$. The gauge group featured in important work of Atiyah-Bott [14] 1983 in mathematical physics. They considered the action of $G(P)$ on the moduli space $A$ of Yang Mills connections on a principal $U(n)$-bundle $P: E \to M$ for $M$ a Riemann manifold. Among other results, they proved $H^*(BG(P))$ is torsion free and computed its Poincaré series. Their calculation depends on the identity:

$$G(P) \simeq \Omega \text{map}(X, BG; h),$$

where $h: X \to BG$ is the classifying map of $P$, a result originally due to Gottlieb [112] 1972). Here $X$ is a finite CW complex. Thus $BG(P) \simeq \text{map}(X, BG; h)$. By Bott periodicity, the loops and double loops on $BU(n)$ are torsion free. Atiyah-Bott used this fact and Thom’s theory to make their calculations.

The classification of gauge groups for fixed $X$ and $G$ up to $H$-equivalence or, alternately, up to ordinary equivalence is the subject of active research. Gottlieb’s identity implies the homotopy classification problem for $\text{map}(X, BG)$ refines the gauge group classification problem. Masbaum [199] 1991 studied the homology of the components of the space $\text{map}(X, BSU(2))$ for $X$ a 4-dimensional CW complex obtained by attaching a single 4-cell to a bouquet of 2-spheres. This case includes oriented, simply connected 4-dimensional manifolds. Using a cofibre sequence for $X$, he obtained, in particular, that the components of $\text{map}(S^4, BSU(2))$ represent infinitely many homotopy types. Using a related analysis, Sutherland [264] 1992 considered the classification of components of $\text{map}(T_g, BU(n))$ for $T_g$ an orientable surface of genus $g$. He obtained the calculation

$$\pi_{2n-1}(\text{map}(T_g, BU(n); t_k)) \cong \mathbb{Z}^g \oplus \mathbb{Z}/d$$

where $d = (n - 1)!/(k, n)$

thus distinguishing the components corresponding to maps $k$ and $l$ with $(k, n) \neq (l, n)$. Tsukuda [276] 2001 classified the homotopy types represented by the components of $\text{map}(S^4, BSU(2))$ showing $\text{map}(S^4, BSU(2); t_k) \simeq \text{map}(S^4, BSU(2); t_l)$ if and only if $k = \pm l$. Kono-Tsukuda [163] 2000 generalized this result from $X = S^4$ to $X$ a simply connected 4-dimensional manifold.

As regards the homotopy type of the gauge group, Kono [161] 1991 proved $G(P_k) \simeq G(P_l) \iff (12, k) = (12, l)$
for \( k, l \in \pi_4(\text{BSU}(2)) \cong \mathbb{Z} \). Thus the infinitely many distinct homotopy types represented by the path-components of \( \text{map}(S^4, \text{BSU}(2)) \) loop to only 6 distinct homotopy types. Kono’s proof included the calculation

\[
\pi_2(\mathcal{G}(P_k)) = \mathbb{Z} / (120, k)
\]

using, essentially, Whitehead’s exact sequence mentioned above. Kono-Tsukuda [163, 1996] extended this result from \( X = S^4 \) to \( X \) a closed, simply connected manifold using a cofibre sequence for \( X \) to make the corresponding calculation. Hamanaka-Kono [127, 2006] obtained a corresponding classification for \( SU(3) \)-bundles over \( S^4 \). They proved

\[
\mathcal{G}(P_k) \simeq \mathcal{G}(P_l) \iff (120, k) = (120, l)
\]

where \( P_k \) and \( P_l \) are principal \( SU(3) \)-bundles over \( S^6 \) with third Chern class equal to \( 2k \) and \( 2l \), respectively.

Crabb-Sutherland [67, 2000] obtained a global result on the classification of gauge groups. They proved that, for any fixed finite CW complex \( X \) and compact Lie group \( G \), there are only finitely many \( H \)-homotopy types represented by the gauge groups \( \mathcal{G}(P) \) for \( P \) a principal \( G \)-bundle over \( X \). Their proof is based on an alternate description of the gauge group:

\[
\mathcal{G}(P) \cong \Gamma(\text{Ad}(P))
\]

where \( \text{Ad}(P) = E \times_G G^{ad} \to X \) is the adjoint bundle associated to \( P \): \( E \to X \) and the space of sections has multiplication induced by \( G \). A key step in their finiteness result is the proof that the fibrewise rationalization of \( \text{Ad}(P) \) is equivariantly trivial. They also classified the \( H \)-homotopy types of gauge groups of \( SU(2) \)-principal bundles over \( S^4 \) complementing Masbaum and Kono’s work. Here

\[
\mathcal{G}(P_k) \simeq_H \mathcal{G}(P_l) \iff (180, k) = (180, l).
\]

Summarizing, the infinitely many distinct homotopy types represented by the components of \( \text{map}(S^4, \text{BSU}(2)) \) loop to 6 distinct homotopy types and 18 distinct \( H \)-homotopy types.

2.2. Spaces of Self-Equivalences. The space of equivalences \( \text{aut}(X) \) of a space \( X \) with some additional structure admits many important refinements. When \( M \) is a Riemannian manifold, we have the chain of subspaces

\[
\text{Isom}(M) \hookrightarrow \text{Diff}(M) \hookrightarrow \text{Homeo}(M) \hookrightarrow \text{aut}(M)
\]

given by spaces of isometries, diffeomorphisms and homeomorphisms, respectively. Each of these spaces is the subject of active research in homotopy theory and geometric topology. Smale [247, 1959] proved the inclusion

\[
\text{Isom}(S^2) \hookrightarrow \text{Diff}(S^2)
\]

is a homotopy equivalence. Since \( \text{Isom}(S^2) \simeq O(3) \) this determines the homotopy type of \( \text{Diff}(S^2) \). The (Generalized) Smale Conjecture asserts that \( \text{Isom}(M) \simeq \text{Diff}(M) \) for \( M \) a 3-manifold of constant, positive curvature. The Smale Conjecture was affirmed for \( M = S^3 \) by Hatcher [137, 1983]. Gabai [104, 2003] proved the corresponding result for \( M \) a closed, hyperbolic 3-dimensional manifold.

The first result on the \( H \)-homotopy type of \( \text{aut}(X) \) is due, essentially, to Thom [270, 1957]. By his results mentioned above, we have

\[
\text{aut}(K(\pi, n)) \simeq_H \text{aut}(\pi) \times K(\pi, n)
\]
for $\pi$ abelian. When $n = 1$, Gottlieb’s extension of Thom’s results leads to an identification
\[ aut(K(\pi, 1)) \simeq_H Out(\pi) \rtimes K(\pi, 1) \]
where $Out(\pi)$ denotes the group of outer automorphisms. Note that these results include a description of $\pi_0(aut(X))$, the group of free homotopy self-equivalences of $X$. This group is, in general, quite complicated even for simple $X$. See Arkowitz [12], 1990 and Rutter [233], 1997 for surveys of the extensive literature on this group.

Since the path-components of $aut(X)$ are all of the same homotopy type, we focus on the component of the identity which we denote $aut(X)_0$. Thus
\[ aut(X)_0 = map(X, X; 1) \]
is the identity component in the space of self-maps.

Hansen [136], 1990 identified the homotopy type of $aut(S^2)_0$ by comparing the evaluation fibration for this space with the fibre sequence $SO(2) \to SO(3) \to S^2$. He proved
\[ aut(S^2)_0 \simeq_H SO(3) \times \widetilde{aut}(S^2)_0 \]
where $\widetilde{Z}$ denotes the universal cover. Combined with Smale’s result, this shows $Diff(S^2) \leftrightarrow aut(S^2)$ is not a homotopy equivalence. Yamanoshita [304], 1993 obtained a related result proving
\[ aut(\mathbb{R}P^2)_0 \simeq_H SO(3) \times (aut_*(\mathbb{R}P^2)_0/O(2)) \]
This result implies $Diff(\mathbb{R}P^2) \simeq O(3)$ is not homotopy equivalent to $aut(\mathbb{R}P^2)$. Yamanoshita [301], 1985 also obtained a general result
\[ aut(X \times Y) \simeq aut(X) \times aut(Y) \times map_*(Y, aut(X)) \times map_*(X, aut(Y)) \]
provided the dimension of $X$ is less than the connectivity of $Y$. In particular:
\[ aut(S^1 \times Y) \simeq O(2) \times aut(Y) \times \Omega aut_*(Y) \]
for simply connected $Y$.

McCullough [207], 1981 computed $\pi_q(aut(M)_0)$ for $1 \leq q \leq n - 3$ for $M$ a connected sum of closed, aspherical manifolds of dimension $\leq 3$ proving the groups $\pi_{n-2}(aut(M)_0)$ are not finitely generated. He used this result to give examples of closed 3-manifolds $M$ such that the fundamental group of $Homeo(M)$ is not finitely generated. Didierjean [70], 1990 and [71], 1992 used a Postnikov decomposition of $X$ to construct a spectral sequence converging to the homotopy groups of $aut(X)$. She determined low degree homotopy groups of $aut(X)$ for $X = SU(3)$ and $X = Sp(2)$ up to extensions.

Given a fibration $p: E \to B$, we may consider the monoid $aut(p)$ of fibre-homotopy equivalences $f: E \to E$ covering the identity of $B$. Booth-Heath-Morgan-Piccinni [26], 1984 extended Gottlieb’s result for the gauge group to prove an $H$-equivalence
\[ aut(p) \simeq_H \Omega map(B, Baut(F) ; h) \]
where $F$ is the fibre of $p$ and $h: B \to Baut(F)$ is the classifying map. A simplicial version of this result was earlier obtained by Dror-Dwyer-Kan [73], 1980. Didierjean [69], 1987 extended Thom’s result to the fibrewise setting, proving
\[ \pi_q(aut(p)) \cong H^{n-q}(B; \pi) \]
for \( p: E \to B \) a principal fibration with fibre \( K(\pi, n) \) and made calculations of the homotopy groups of \( aut(p) \) when \( F \) has two nonzero homotopy groups. She constructed a spectral sequence converging to the homotopy groups of \( aut(p) \), expanding on work of Legrand [185, 1983].

2.3. The Free Loop Space. The space of maps 
\[
\text{LX} = \text{map}(S^1, X)
\]
is the subject of intensive research in diverse branches of mathematics. Given a compact Lie group \( G \), the space of smooth loops on \( G \) is an infinite-dimensional Lie group, called the “loop group” of \( G \). The representation theory of loop groups has deep connections to mathematical physics (cf. Pressley-Segal [230, 1986]).

Gromoll-Meyer [115, 1969] linked the closed geodesic problem for a compact Riemannian manifold \( M \) to the homotopy theory of the free-loop space \( \text{LM} \). They proved that \( M \) admits infinitely many closed, prime geodesics if the Betti numbers of \( \text{LM} \) grow without bound. Vigué-Poirrier-Sullivan [282, 1976] proved that the Betti numbers of \( \text{LM} \) are unbounded when the rational cohomology of \( M \) requires at least two generators. Their calculation was facilitated by a Sullivan model for \( \text{LM} \), described in Section 3, below.

More recently, Chas-Sullivan [50, preprint] and [51, 2004] unearthed a wealth of structure on the (regraded) homology of \( \text{LM} \) of a closed, oriented, smooth \( m \)-manifold \( M \). Setting \( H^\ast(\text{LM}; F) = HH^\ast(S^\ast(M), S^\ast(M); F) \) where the latter is the Hochschild cohomology of the algebra of singular cochains of \( M \). Here \( F \) is a field. The Cohen-Jones construction was extended to more general ring spectra by Gruher-Salvatore [118, 2008]. Cohen-Klein-Sullivan [63, 2008], Crabb [65, 2008], and Gruher-Salvatore independently proved the homotopy invariance of the loop product and bracket, a significant advance since the original constructions depended on the smooth structure of \( M \). Chataur [52, 2005], Hu [147, 2006] and Kallel-Salvatore [155, 2006] considered generalizations of string operations from \( \text{LM} = \text{map}(S^1, M) \) to \( \text{map}(S^n, M) \). The work of these various authors include constructions of the string topology operations in the frameworks of ring spectra (Cohen-Jones and Gruher-Salvatore), bordism theory (Chataur) and fibrewise homotopy theory (Crabb).

As regards the ordinary homotopy theory of the free loop space, Hansen [130, 1974] gave an example of an aspherical space \( X \) with \( \pi_1(\text{LX}) \) not finitely generated. Note that when \( X \) is an \( H \)-space \( \text{LX} \simeq X \times \Omega X \). Aguadé [2, 1987] made a general study of spaces \( X \), called \( T \)-spaces, for which the evaluation fibration 
\[
\Omega X \to \text{LX} \to X
\]
is fibre-homotopically trivial. He obtained a refinement of the notion of $H$-space via a sequence of classes $T = T_1 \subset T_2 \subset \ldots \subset T_\infty = H$-spaces with separating examples. Woo-Yoon \cite{288} 1995 proved that when $X$ is a $T$-space the components of $map(\Sigma A, X)$ are all homotopy equivalent. Fadell-Husseini \cite{87} 1989 proved $LX$ has infinite L.S. category for $X$ a simply connected CW complex with finitely generated, nontrivial rational cohomology.

Smith \cite{248} 1981 and \cite{249} 1984 constructed an Eilenberg-Moore spectral sequence for the cohomology of a free loop space. Starting with the pull-back square

$$
\begin{array}{ccc}
LX & \xrightarrow{\omega} & X \\
\downarrow & & \downarrow \Delta \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
$$

he obtained a spectral sequence

$$E_2^{*,*} = Tot_{H^*(X \times X; \mathbb{F})}(H^*(X; \mathbb{F}), H^*(X; \mathbb{F})) \implies H^*(LX; \mathbb{F}).$$

He proved collapsing results for this spectral sequence and obtained calculations of $H^*(LM; \mathbb{F})$ for $\mathbb{F}$ of characteristic 2 and 0. Kuribayashi \cite{169} 1991 used the Eilenberg-Moore spectral sequence to prove the fibre is totally noncohomologous to zero in the fibration $\Omega M \rightarrow LM \rightarrow M$ for $M$ a Grassmann or Stiefel manifold and mod $p$ cohomology for certain primes $p$.

McCleary-Ziller \cite{205} 1987 proved the Betti numbers of $LM$ are unbounded for $M$ a compact, simply connected homogeneous space not equivalent to a rank one symmetric space using spectral sequence methods and extending earlier work of Ziller \cite{307} 1977 who used Morse theory. Roos \cite{232} 1988 studied the Poincaré-Betti series for $LX$ for $X$ a wedge of spheres using local algebra. He proved the series for $X = S^2 \vee S^2$ is not rational. Halperin-Vigué-Poirrier \cite{126} 1991 proved the $\mathbb{F}$-Betti numbers are unbounded for a field $\mathbb{F}$ of positive characteristic $k$ provided $H^*(X; \mathbb{F})$ requires at least 2 generators and under certain restrictions on $k$ and $X$. McCleary-McLaughlin \cite{204} 1992 studied the free loop space in the context of Morava $K$-theory while Ottenso\text{sen} \cite{226}, 2003 considered the Borel cohomology of the free loop space. Lambrecht \cite{177}, 2001 proved the Betti numbers of the free loop space are unbounded for certain connected sums. Burghelea-Fiedorowicz \cite{44} 1984 and Goodwillie \cite{107}, 1995 proved an isomorphism of graded spaces

$$H^*(LX) \cong HH^*(S_*(\Omega X), S_*(\Omega X))$$

where the latter space is the Hochschild cohomology. Menichi \cite{211}, 2001, Dupont-Hess \cite{77}, 2002 and Ndombol-Thomas \cite{223}, 2002 independently proved this is an isomorphism of algebras. Menichi also made calculations of the graded algebra $H^*(LX; \mathbb{Z}_p)$ for $X$ a suspension and $X = \mathbb{C}P^m$ while Ndombol-Thomas used Hochschild cohomology to make calculations of $H^*(LX; \mathbb{Z}_p)$ for $X = S^m, \mathbb{C}P^m$ and $\Sigma \mathbb{C}P^m$. Kuribayashi-Yamaguchi \cite{173}, 1997 made complete calculations of $H^*(LX; \mathbb{Z}_p)$ when $X$ is simply connected with mod $p$ cohomology an exterior algebra on few generators using Hochschild cohomology to obtain information on the $E_2$-term in the Eilenberg-Moore spectral sequence. Recently, Seeliger \cite{243}, 2008 used the Serre spectral sequence applied to the evaluation fibration to make calculations of $H^*(L\mathbb{C}P^m)$. 


Since the appearance of the paper of Chas-Sullivan, the structure of \( \mathbb{H}_*(LM) \) has seen an explosion of research with many partial descriptions of the loop product, the Gerstenhaber algebra and the BV algebra structure in special cases. We mention a sampling of these results here. COHEN-JONES-YAN [62] 2004] constructed a spectral sequence of algebras

\[
E^2_{p,q} = H^q(M; H_*(\Omega M)) \implies \mathbb{H}_*(LM)
\]

and used this to calculate the loop product for \( M = S^2, CP^n \). TAMANOI [266] 2006] computed the BV algebra structure of \( \mathbb{H}_*(LM) \) for \( M = SU(n) \) and \( M \) a complex Stiefel manifold. GUGHER-SALVATORE [118] 2008] extended the string operations to the case \( M = BG \) for \( G \) a compact Lie group. MENICHI [212] 2009] proved the Cohen-Jones isomorphism

\[
\mathbb{H}_*(LS^2; \mathbb{F}_2) \cong HH^*(S^2; \mathbb{F}_2), S^*(S^2; \mathbb{F}_2))
\]

mentioned above is an isomorphism of Gerstenhaber algebras but, surprisingly, not an isomorphism of BV algebras.

### 2.4. Spectral Sequences and Stable Decompositions.

We here discuss some general results on homotopy invariants and the stable homotopy type of function spaces not covered by the preceding discussion.

FEDERER [88] 1956] constructed a spectral sequence converging to the homotopy groups of map\((X,Y; f)\) for \( X \) any finite CW complex and \( Y \) a simple CW complex. He defined an exact couple from the long exact homotopy sequence of the restriction maps \( \rho_n: map(X_n,Y; f_n) \to map(X_{n-1}, Y; f_{n-1}) \). He identified the homotopy groups of the fibre of \( \rho_n \) with cellular cochain groups of \( X \) with coefficients in \( \pi_*(Y) \) and obtained a spectral sequence

\[
E^2_{p,q} = H^q(X; \pi_{p+q}(Y)) \implies \pi_p(map(X,Y; f)).
\]

DYER [86] 1966] applied the Federer spectral sequence to calculate low degree homotopy and homology group of components of map\((X,Y)\) when dim\(X\) is less than the connectivity of \( Y \). SCHULTZ [242] 1973] and MÖLLER [219] 1990] constructed equivariant versions of this spectral sequence.

BORSUK [27] 1952] proved that if \( X \) is a finite CW complex of dimension \( k \) with nonzero \( k \)th Betti number then map\((X,S^n)\) has nonzero \((n-k)\)th Betti number. MOORE [222] 1956] extended this analysis and asked for a spectral sequence relating the cohomology of \( X \) to the homology of map\((X,S^n)\). SPANIER [258] 1959] showed the functor

\[
F_n(X) = \lim_k map(X, \Omega^k SP^{n+k} S^{n+k}),
\]

converts cohomology groups to homotopy groups. Here \( SP^{n+k} \) is the symmetric product functor. ANDERSON [6] 1972] constructed an Eilenberg-Moore spectral sequence for the cohomology of map\((X,Y)\) using the cobar construction in the category of cosimplicial spaces. LEGRAND [186] 1986] constructed a spectral sequence in the spirit of Moore’s problem for map\((X,Y)\) using a Postnikov decomposition of \( Y \). PATRAS-THOMAS [229] 2003] proved the Anderson spectral sequence converges when the dimension of \( X \) is no bigger than the connectivity of \( Y \). CHATAUR-THOMAS [54] 2004] gave a related \( E_{\infty} \)-model for function spaces. Applied to the free loop space, their model gives an operadic version of Hochschild cohomology.

The stable homotopy type of the based function space map\((X,Y)\) has been described in many cases. When \( X = S^n \), this is just the loop space \( \Omega^n Y \). Stable
decompositions of this space are central structural results in homotopy theory. Snaith \cite{257} 1974] proved a stable decomposition for \( \Omega^n Y \) in terms of configuration spaces of \( j \)-tuples of distinct points of \( \mathbb{R}^n \). His proof was obtained by analyzing the stable homotopy type of approximations due to May \cite{200} 1972] for iterated loop spaces. Bödigheimer \cite{23} 1987] proved a generalization of Snaith’s splitting result. He showed

\[
\Omega^n \Sigma^\infty map_*(M, \Sigma^n Y) \simeq \Omega^n \Sigma^\infty (C(M, \partial M; n) \wedge_{\Sigma_n} Y^n)
\]

where \( M \) is a compact manifold with boundary and \( C(M, \partial M; n) \) is the configuration space. Arone \cite{13} 1999] described the Goodwillie tower of \( \Omega^n \Sigma^\infty map_*(X, \Sigma^n Y) \) for more general \( X \) recovering the previous splittings. He obtained an Eilenberg-Moore spectral sequence from this description. Ahearn-Kuhn \cite{5} 2002] and Kuhn \cite{167} 2006] studied this spectral sequence proving it a spectral sequence of graded algebras and studying functorial properties.

Campbell-Cohen-Peterson-Selick \cite{45} 1987] studied \( map_*(P_m(2^r), S^n) \) where \( P_m(2^r) = S^{m-1} \cup e^m \) is a Moore space with attaching map of degree \( 2^r \). They gave a partial description of the mod \( 2 \) Steenrod operations and proved that \( map_*(P_3(2), S^n) \) is not decomposable as a product except, perhaps, for finitely many \( n \). Westerland \cite{284} 2006] obtained a stable splitting for components of the space \( map_*(T_g, S^2) \) where \( T_g \) is a surface of genus \( g > 0 \). Cohen \cite{60} 1987] and Bödigheimer \cite{23} 1987] independently obtained a stable splitting of the free loop space \( \Sigma^{\infty} X \) in terms of configuration spaces. Other results on the stable homotopy of the free loop space include splitting results for \( LR^Pn \) by Bauer-Crabb-Spreafico \cite{16} 2001] and Yamaguchi \cite{294} 2005].

3. Localization of Function Spaces.

In this section, we survey work on function spaces after localization. Recall a nilpotent space \( X \) is a connected CW complex such that \( \pi_1(X) \) is a nilpotent group and the standard action of \( \pi_1(X) \) on the higher homotopy groups of \( X \) is a nilpotent action. By Sullivan \cite{262} 1971] and Hilton-Mislin-Roitberg \cite{142} 1975], a nilpotent space \( X \) admits a \( P \)-localization \( \ell_X : X \to X_P \) which is a map inducing \( P \)-localization on homotopy groups. When \( P = \{ p \} \) we write \( X_p \) for the \( p \)-localization and when \( P \) is empty we write \( X_\mathbb{Q} \) for the rationalization of \( X \).

Under reasonable hypotheses on \( X \) and \( Y \), function spaces behave well with respect to localization. Hilton-Mislin-Roitberg proved that if \( X \) is a finite CW complex and \( Y \) is a nilpotent space then the path-components of \( map(X,Y) \) are nilpotent spaces. The components of \( map(X,Y) \) are of CW type in this case by Milnor \cite{214} 1959]. Hilton-Mislin-Roitberg also proved that composition by \( \ell_Y \) gives a \( P \)-localization map

\[
(\ell_Y)_* : map(X,Y;f) \to map(X,Y_P;\ell_Y \circ f).
\]

Below we write \( f_P = \ell_Y \circ f \). Hilton-Mislin-Roitberg-Steiner \cite{143} 1978] obtained the same results if, alternately, \( X \) is a finite type CW complex and \( Y \) is a nilpotent Postnikov piece. Here the CW type result is due to Kahn \cite{153} 1984].
\[ \begin{array}{c}
A \xrightarrow{u} E \\
\downarrow i \quad \downarrow p \\
B \xrightarrow{f} X
\end{array} \]

where here \( i \) is closed cofibration and \( p \) is a fibration. In this case, \( P \)-localization is obtained by passing from \( p: E \to X \) to its fibrewise \( P \)-localization \( p(P): E(P) \to X \) as constructed by May [201] 1980. In particular, fibrewise localization induces \( P \)-localization \( \Gamma(p; s) \to \Gamma(p(P); s') \) for \( X \) finite CW and \( F = p^{-1}(s) \) a nilpotent space. Möller [219, 1990] proved the corresponding nilpotence and localization results for certain equivariant function spaces. Klein-Schochet-Smith [159, 2009] extended the nilpotence and localization results from the case when \( X \) is finite CW to the case \( X \) is compact metric provided the corresponding function or section space is known \textit{a priori} to be nilpotent.

Bousfield-Kan [29, 1972] introduced a more general localization theory for subrings \( R \subseteq Q \) including homotopy completions \( Y \to R_\infty Y \). They proved that \( R \)-completion (respectively, \( R \)-localization) induces \( R \)-completion (respectively, \( R \)-localization) on the based function spaces \( map_*(X, Y; 0) \) when \( X \) is a finite CW complex and \( Y \) is nilpotent. Further significant results on the behavior of function spaces under Bousfield-Kan localization and completion are discussed below.

### 3.1. Rational Homotopy Theory of Function Spaces

Quillen [231, 1969] constructed an equivalence between the homotopy category of simply connected rational CW complexes and a homotopy category of connected, differential graded Lie algebras (DGLAs) over \( Q \) initiating rational homotopy theory. The Quillen minimal model of a simply connected space \( X \) is a \textit{minimal} DGLA \( (\mathcal{L}(X), d_X) \) which means \( \mathcal{L}(X) = L(V) \) is a free GLA and \( d_X \) satisfies \( d_X(V) \subseteq [L(V), L(V)] \). The rational homology and homotopy Lie algebra of \( X \) are recovered via isomorphisms

\[ V \cong s^{-1} \mathcal{H}(X; Q) \quad \text{and} \quad \pi_*(\Omega X) \otimes Q, [\cdot, \cdot] \cong H_*(\mathcal{L}(X)), [\cdot, \cdot]. \]

Sullivan [263, 1977] constructed another categorical equivalence, here between the homotopy theory of simply connected CW complexes and a homotopy category of connected differential graded algebras (DGAs) over \( Q \). The Sullivan minimal model of a space \( X \) is a minimal DGA \( (\mathcal{M}(X), d_X) \) where \( \mathcal{M}(X) = \Lambda V \) is a free DGA with \( d_X(V) \subseteq \Lambda^+ V \cdot \Lambda^+ V \) with

\[ V \cong \text{Hom}(\pi_*(X), Q) \quad \text{and} \quad H_*(\mathcal{M}(X)) \cong H^*(X; Q). \]

More generally, a Sullivan model \((\mathcal{A}(X), d)\) for \( X \) is a DGA admitting a chain equivalence to the de Rahm complex of rational PL forms on \( X \). In particular, \( H_*(\mathcal{A}(X)) \cong H^*(X; Q) \) and \((\mathcal{A}(X), d) \simeq (\mathcal{M}(X), d_X)\) are homotopy equivalent in Sullivan’s DGA category. Comprehensive treatments of the subject were given by Tanré [267, 1983] and Félix-Halperin-Thomas [94, 2001].

Sullivan described separate models for the general path-component of a function space, \( map(X, Y; f) \), the space of self-equivalences \( aut(X)_n \) and the free loop space \( LX \) each within his framework of DGAs. We discuss these models and their extensions and applications now.
3.1.1. General Components. Following the sketch by Sullivan, 
Haefliger [123] 1982] constructed a (non-minimal) Sullivan model for the rational homotopy type of \( \text{map}(X, Y; f) \) where \( f: X \to Y \) is a map of nilpotent spaces with \( X \) finite. The construction builds on the ideas of Thom, described above. Let \( p_r: Y_r \to Y_{r-1} \) with fibre \( K(G_r, n_r) \) be a term in the principal refinement of the Postnikov tower of \( Y \) with \( k \) invariant \( k_{r-1}: Y_{r-1} \to K(G_r, n_r + 1) \). We then obtain a pullback diagram \[
\begin{array}{ccc}
\text{map}(X, Y_r; (f_Q)_r) & \xrightarrow{\pi_r} & P\text{map}(X, K(G_r, n_r + 1); 0) \\
\downarrow & & \downarrow \\
\text{map}(X, Y_{r-1}; (f_Q)_{r-1}) & \xrightarrow{(k_{r-1} \circ (f_Q)_{r-1})_*} & \text{map}(X, K(G_r, n_r + 1); 0)
\end{array}
\]
where the right fibration is the path/loop fibration. Let \( V = \bigoplus_n \text{Hom}(G_r, Q) \). Since \( X \) is finite, \( X \) admits a finite model \( (A, d) \). Write \( A = \text{Hom}(A, Q) \) for the dual to \( A \) and grade \( A \) in negative degrees. Thom’s calculation of \( \pi_q(\text{map}(X, K(G, n); 0)) \) is then reflected in the grading on the ordinary tensor product \( A \otimes V \). Let \( I \) denote the ideal of \( A(A \otimes V) \) generated by elements of degree \( \leq 0 \). Haefliger described a differential \( d \) on \( A(A \otimes V)/I \) in terms of the “\( k \)-invariants” \( (k_{r-1} \circ (f_Q)_{r-1})_* \) above and proved the result is a Sullivan model for \( \text{map}(X, Y; f) \).

Bousfield-Peterson-Smith [30] 1989] gave an alternate construction, motivated by seminal work of Lannes [130] 1987 in \( p \)-local homotopy theory, discussed below. The construction makes use of the fact that \( \text{map}(X, \_ \_ \_) \) defines a functor on topological spaces that is right adjoint to the product functor \( \_ \_ \_ \times X \). In the category of DGAs, this corresponds to the fact that \( \text{Hom}(A, \_ \_ \_) \) is right adjoint to the tensor product functor \( \_ \_ \_ \otimes A \). In this setting, \( \_ \_ \_ \otimes A \) has an left adjoint, as well, provided \( A \) is finite. The construction is a version of Lannes’ \( T \)-functor. It is conveniently written here as \( (\_ \_ \_: A) \). Given a map \( \psi: B \to A \) define \( (B : A)_\psi \) to be the connected DGA determined by \( \psi \). Assume \( X \) and \( Y \) are nilpotent spaces with \( X \) finite. Let \( A \) be a finite Sullivan model for \( X \). Bousfield-Peterson-Smith proved \( (\mathcal{M}(Y) : A)_\psi \) is a Sullivan model for \( \text{map}(X, Y; f) \) where \( \psi: \mathcal{M}(Y) \to A \) is a Sullivan model for \( f: X \to Y \).

Brown-Szczarba [37] 1998] and [38] 1998] expanded on the work of Haefliger and Bousfield-Peterson-Smith. They constructed a model \( (AW, d) \) for \( \text{map}(X, Y; f) \) where \( W^q = \left( \sum_n \pi_n(Y) \otimes H_{n-q}(X; Q) \right) / K^q \) for certain subspaces \( K^q \). The differential \( d \) was described explicitly in terms of the coproduct on \( H_*(X; Q) \). Here \( X \) is a finite CW complex and \( Y \) is nilpotent. They deduced descriptions of the rational homotopy groups of \( \text{map}(X, Y; f) \). When \( f \) is trivial they proved \( K_q = 0 \) thus obtaining an isomorphism of graded spaces \( \pi_* (\text{map}(X, Y; 0)) \cong (H_*(X; Q)) \otimes (\pi_*(Y) \otimes Q) \).

Again, the space \( H_*(X; Q) \) is assumed to be negatively graded. This last result was earlier proved by Smith [250] 1994].

Applications of the Haefliger model include the following results: Vigué-Poirrier [280] 1986] identified the rational homotopy Lie algebra of \( \text{map}(X, Y; 0) \) for \( X \) nilpotent of dimension strictly less than the degree of the first nontrivial
homotopy group of $Y$ via an isomorphism

$$\pi_n(\Omega map(X, Y; 0)) \otimes \mathbb{Q}, [\cdot, \cdot] \cong (H^*(X; \mathbb{Q})) \otimes (\pi_*(\Omega Y) \otimes \mathbb{Q}), [\cdot, \cdot].$$

Here $H^*(X; \mathbb{Q})$ is negatively graded and the tensor product has the GLA structure induced by the product and bracket on the terms. Møller-Raussen [221] 1986 studied the rational homotopy classification problem for components of $map(X, Y)$ with $Y = S^n, CP^n$ for $X$ nilpotent and suitably rationally co-connected. They obtained complete classifications in these cases including descriptions of the rational homotopy types. Félix [89] 1990 proved the rational L.S. category of $map(X, Y; 0)$ is often infinite. Smith [253] 1997 studied the rational homotopy classification problem for $map(G_1/T_1, G_2/T_2)$, where $G_1, G_2$ are classical compact Lie groups and $T_1, T_2$ maximal tori, identifying the rational type of certain components as generalized flag manifolds. Smith [254] 1999 gave an explicit description of the Haefliger model for $X$ and $Y$ elliptic spaces (simply connected spaces having finite-dimensional rational homotopy and homology) with evenly graded rational cohomology obtaining examples of components of $map(X, Y)$ of finite L.S. category.

Kotani [165] 2004 used the Brown-Szczarba model to give necessary and sufficient conditions for the space $map(X, Y)$ to be a rational $H$-space for $X$ a formal, nilpotent CW complex of dimension $\leq$ the connectivity of $Y$. Félix-Tanré [98] 2005 generalized this result replacing the formality of $X$ by a condition involving L.S. category. Buijs-Murillo [43] 2006 constructed the Brown-Szczarba model within the simplicial category framework for rational homotopy theory due to Bousfield-Gugenheim [28] 1976. They obtained a functorial version of the Brown-Szczarba model in this setting and used this model, in Buijs-Murillo [43] 2008, to identify the rational homotopy Lie algebra of components $map(X, Y; f)$ with $X, Y$ restricted, as usual, to nilpotent spaces with $X$ finite. Kuribayashi-Yamaguchi [174] 2006 combined the Haefliger and Brown-Szczarba approaches to obtain a rational splitting of $map_*(X \cup_\alpha e^{k+1}, Y; 0)$ where $\alpha$ is an attaching map. Under certain restrictions on $X, Y$ and $\alpha$ they proved

$$map_*(X \cup_\alpha e^{k+1}, Y; 0) \simeq_\mathbb{Q} map_*(X, Y; 0) \times \Omega^{k+1}Y.$$ 

Hirato-Kuribayashi-Oda [146] 2008 applied the Brown-Szczarba model to the study of the rational evaluation subgroups, i.e., the image of the map induced on rational homotopy groups by the evaluation map $\omega_f: map(X, Y; f) \rightarrow Y$. Buijs-Félix-Murillo [41] 2009 used the Brown-Szczarba model to study the rational homotopy type of the homotopy fixed point of a circle action.

The higher rational homotopy groups of $map(X, Y; f)$ for suitable $X$ and $Y$ can be described directly in terms the homology of certain DG space of derivations. In the DGA setting, given a map $\psi: (A, d) \rightarrow (B, d)$ of DGAs let $\text{Der}_n(A, B; \psi)$ denote the space of linear maps $\theta: A^* \rightarrow B^{*-n}$ satisfying, for $x, y \in A$, the identity $\theta(xy) = \theta(x)\psi(y) + (-1)^{|x||y|}\psi(x)\theta(y)$. In the DGLA setting, let $\text{Der}_*(L, K; \psi)$ denote the space of degree raising linear maps satisfying the corresponding identity. In both cases, a degree $-1$ differential is given by $D(\theta) = d \circ \theta - (-1)^n \theta \circ d$. When $X$ and $Y$ nilpotent spaces with $X$ finite

$$\pi_n(map(X, Y; f)) \cong H_n(\text{Der}(\mathcal{M}(Y), \mathcal{M}(X); \mathcal{M}(f))).$$

for $n \geq 2$. This result is due to Sullivan for the case $f = 1$ as discussed below. The general result was proved, independently, by Block-Lazarev [22]
2005], Lupton-Smith [191, 2007] and Bulis-Murillo [43, 2008]. The rationaliza-
tion of the fundamental group $\pi_1(map(X, Y; f))$ is, in general, nonabelian. 
Lupton-Smith [190, 2007] proved the rank of $\pi_1(map(X, Y; f))_Q$ is the dimension 
of $H_1(Der(M(Y), M(X); M(f)))$. Bulis-Murillo [43, 2008] extended this to an 
identification of the Malcev completion of $\pi_1(map(X, Y; f))_Q$.

Within the framework of Quillen minimal models, we have an isomorphism

$$\pi_n(map(X, Y; f)) \otimes \mathbb{Q} \cong H_n(\text{Rel}(ad_{L(f)}))$$

for $n \geq 2$ for $X$ and $Y$ simply connected CW with $X$ finite. Here $\text{Rel}_*(ad_{L(f)})$ is 
the mapping cone of the chain map

$$ad_{L(f)} : L(Y) \rightarrow \text{Der}_*(L(X), L(Y); L(f))$$

given by $ad_{L(f)}(y|x) = [L(f)(x), y]$ for $x \in L(X), y \in L(Y)$. This result was proved 
for the identity component by Tanrê [267, 1983] and Schlessinger-Stasheff [241 preprint]. The result for the general component was proved by Lupton-
Smith [192, 2007]. Lupton-Smith [194, 2010] identified rational Whitehead products 
in terms of this identification. Bulis-Félix-Murillo [40, 2009] described a Quillen model for function spaces and obtained a result on the exponential growth 
of rational homotopy groups of function spaces.

The homotopy classification problem for gauge groups corresponding to principal 
$G$-bundles $P : E \rightarrow X$ is trivial after rationalization for $X$ finite CW and $G$ 
a compact Lie group. In this case, $BG$ is a rational $H$-space. As mentioned above, 
Crabb-Sutherland [67, 2000] used this fact to prove the fibrewise localization of 
the universal $G$-adjoint bundle $Ad(P_G) : E_G \times_G G^{\text{ad}}$ is equivariantly trivial. Their 
result implies a rational $H$-equivalence

$$(G(P)_0)_Q \simeq_H \text{map}(X, G_0; 0).$$

The rational homotopy groups of $G(P)$ may thus be computed as

$$\pi_* (G(P)) \otimes \mathbb{Q} \cong H_*(X; \mathbb{Q}) \otimes (\pi_*(G) \otimes \mathbb{Q})$$

since $G_0$ is a product of Eilenberg-Mac Lane spaces where here, as above, $H_*(X; \mathbb{Q})$ 
is negatively graded.

Lupton-Phillips-Schochet-Smith [189, 2009] proved a related result in the 
context of commutative Banach algebras. Let $A$ be unital, commutative Banach 
algebra and $GL_n(A)$ the group of $n \times n$ invertible matrices with coefficients in $A$. Then

$$GL_n(A)_0 \simeq_{\mathbb{Q}} \prod K(V_n, n) \text{ where } V = \hat{H}_*(\text{Max}(A); \mathbb{Q}) \otimes \Lambda(s_1, \ldots, s_{2n-1}).$$

Here Max($A$) is the maximal ideals space and $s_{2i-1}$ is of degree $2i - 1$. Klein-
Schochet-Smith [159, 2009] extended this result to the group of unitaries $UA_\zeta$ 
where $A_\zeta$ is the $C^*$-algebra sections of a complex $n$-matrix bundle $\zeta$ over a compact 
metric space $X$. The latter result is based on an extension of the result $(G(P)_0)_Q \simeq_H \text{map}(X, G_0; 0)$ from the case $X$ finite CW to the case $X$ compact metric.

3.1.2. Spaces of Self-Equivalences. The rational homotopy type of a connected 
grouplike space $G$ is completely determined by isomorphism type of the Samelson 
Lie algebra $\pi_*(G) \otimes \mathbb{Q}, [\ , \ ]$ (c.f. Scheerer [237, 1985]). Sullivan [263, 1977] 
identified the rational Samelson Lie algebra of the space $aut(X)_0$ for $X$ a simply 
connected finite CW complex via an isomorphism:

$$\pi_*(aut(X)_0) \otimes \mathbb{Q}, [\ , \ ] \cong H_*(\text{Der}(\mathcal{M}(X))), [\ , \ ].$$
A corresponding identity in Quillen’s DGLA framework for rational homotopy theory was given by TANRÉ [267] 1983 and Schlessinger-Stasheff [241] preprint:

\[ \pi_*(\text{aut}(X)_\circ) \otimes \mathbb{Q}, [\cdot, \cdot] \cong H_*(\text{Rel}(\text{ad}_{\mathcal{L}(X)})), [\cdot, \cdot]. \]

Here the bracket on the mapping cone of \( \text{ad}_{\mathcal{L}(X)} \) is induced from that on \( \mathcal{L}(X) \).

Sullivan’s identity connects the monoid \( \text{aut}(X) \) to a fundamental open conjecture in rational homotopy theory. Let \( X \) be a simply connected elliptic CW complex with evenly graded rational cohomology. We refer to such spaces as \( F_0 \)-spaces. The class includes (products of) spheres, complex projective spaces and homogeneous spaces \( G/H \) with \( G \) a compact Lie group and \( H \subset G \) a closed subgroup of maximal rank. Motivated by this last case, HALPERIN [124] 1978 conjectured that the rational Serre spectral sequence collapses at the \( E_2 \) term for all orientable fibrations with fibre an \( F_0 \)-space. THOMAS [271] 1981 and MEIER [210] 1981 independently proved that Halperin’s conjecture is equivalent to the condition \( H_{\text{even}}(\text{Der}(\mathcal{M}(X))) = 0 \) for an \( F_0 \)-space \( X \). Thus, by Sullivan’s identity, Halperin’s conjecture holds for \( X \) if and only if \( \text{aut}(X)_\circ \) is rationally equivalent to a product of odd spheres.

The Halperin conjecture has been affirmed in several special cases including for Kähler manifolds by Meier, for homogeneous spaces of maximal rank pairs by Shiga-Tezuka [234] 1991; for the rational Samelson Lie algebra of the monoid \( \text{aut}(p)_\circ \) of fibre-homotopy self-equivalences of a fibration \( p: E \to B \) of simply connected finite CW complexes:

\[ \pi_*(\text{aut}(p)_\circ) \otimes \mathbb{Q}, [\cdot, \cdot] \cong H_*(\text{Der}_\Lambda(\mathcal{M}(X))) \]

and used this result to give a formula for \( \pi_{\text{odd}}(\text{aut}(X)_\circ) \otimes \mathbb{Q} \).

SALVATORE [234] 1997 proved the nilpotency of the Lie algebra \( H_*(\text{Der}(\mathcal{M}(X))) \) coincides with the rational homotopical nilpotency of the monoid \( \text{aut}(X)_\circ \) — the least integer \( n \) such that the \( n \)-fold commutator for \( \text{aut}(X)_\circ \) is rationally trivial. He calculated the rational homotopical nilpotency of \( \text{aut}(X)_\circ \) for \( X \) a rational two-stage Postnikov system and proved the monoid \( \text{aut}(S^{2n-1} \vee S^{2n-1})_\circ \) is not rationally homotopy nilpotent. SMITH [255] 2001 computed the rational homotopy nilpotency of \( \text{aut}(X)_\circ \) for certain spaces \( X \) admitting a two-stage Sullivan model. Recently, FÉLIX-LUPTON-SMITH [96] preprint obtained a formula, in the spirit of Sullivan’s above, for the rational Samelson Lie algebra of the monoid \( \text{aut}(p)_\circ \) of fibre-homotopy self-equivalences of a fibration \( p: E \to B \) of simply connected finite CW complexes:

\[ \pi_*(\text{aut}(p)_\circ) \otimes \mathbb{Q}, [\cdot, \cdot] \cong H_*(\text{Der}_\Lambda(\mathcal{M}(X))) \]

Here \( (\Lambda W, d_B) \to (\Lambda W \otimes \Lambda V, D) \) is the Koszul-Sullivan model of the fibration and \( \text{Der}_\Lambda(\mathcal{M}(X)) \) denotes the DGLA of derivations vanishing on \( \Lambda V \).

As for the rational homotopy of Gottlieb group \( G_n(X) \) and the evaluation map \( \omega: \text{aut}(X)_\circ \to X \), LANG [179] 1975 proved \( G_*(X) \otimes \mathbb{Q} \cong G_*(X_\mathbb{Q}) \) for \( X \) a finite simply connected CW complex. FÉLIX-HALPERIN [93] 1982 identified the rationalized Gottlieb groups \( G_n(X) \otimes \mathbb{Q} \) for these \( X \) in terms of the Sullivan identification \( \pi_*(X) \otimes \mathbb{Q} \cong \mathcal{M}(X) = \Lambda V \). Here an element \( v \in V_n \) corresponds to a rational Gottlieb element if the dual map \( v \mapsto 1 \) extends to a derivation cycle in
They used this result to prove two global results on the rationalized Gottlieb groups of a simply connected CW complex \(X\) of finite rational L. S. category:

\[
G_{\text{even}}(X) \otimes \mathbb{Q} = 0 \quad \text{and} \quad \dim(G_{\text{odd}}(X) \otimes \mathbb{Q}) \leq \text{cat}_\mathbb{Q}(X).
\]

A Quillen model description of the rationalized Gottlieb group was given by Tanré [267] in 1983:

\[
G_n(X) \otimes \mathbb{Q} \cong \ker\{H(\text{ad}_\mathcal{L}(X)) : \mathcal{L}(X) \to \text{Der}(\mathcal{L}(X))\}.
\]

Rationalized Gottlieb groups have been calculated by many authors using various means including Smith [251] in 1996, Lupton-Smith [191] in 2007 and [192] in 2007, Hirato-Kuribayashi-Oda [146] in 2008 and Yamaguchi [298] in 2008. Félix-Lupton [95] in 2007 proved the evaluation map \(\omega : \text{aut}(X)_0 \to X\) is rationally homotopy trivial if and only if it is trivial on rational homotopy groups for \(X\) a finite, simply connected CW complex.

3.1.3. The Free Loop Space. Vigué-Poirrier-Sullivan [282] in 1976 constructed a Sullivan model for \(LX\) when \(X\) is a simply connected CW complex. Let \((\Lambda V, d)\) be the minimal model for \(X\). Their model for \(LX\) is given by

\[
(\Lambda V \otimes \Lambda sV, \delta) \quad \text{with} \quad \delta(v) = dv \quad \text{and} \quad \delta(sv) = -sd(v) \quad \text{for} \ v \in V
\]

where \(s\) is the degree \(-1\) derivation of \(\Lambda V \otimes \Lambda sV\) defined by setting \(s(v) = sv\) and \(s(sv) = 0\). Halperin [125] in 1981 constructed a related model in the non-simply-connected case under restrictions on the component. As discussed above, Vigué-Poirrier-Sullivan used their model to prove the Betti numbers of \(LX\) are unbounded if \(H^*(X; \mathbb{Q})\) requires at least two generators. In fact, they showed the Betti numbers of \(LX\) grow exponentially in this case. Vigué-Poirrier [279] in 1984 proved that the same is true for wedges of spheres and manifolds of L.S. category less than 2. She conjectured the Betti numbers of \(LX\) grow exponentially for all finite, simply connected \(X\) with infinite dimensional rational homotopy. She gave examples of spaces \(X\) with finite-dimensional rational homotopy for which

\[
\lim\sup \frac{\log \sum_{i=1}^n \beta_i}{\log n} = \dim(\pi_{\text{odd}}(X) \otimes \mathbb{Q}).
\]

Lambrechts [177] in 2001 affirmed Vigué-Poirrier’s conjecture for the class of simply connected, coformal, finite complexes.

Dupont-Vigué-Poirrier [79] in 1998 proved a basic result concerning the question of formality for the free loop space. Given a simply connected CW complex \(X\) with Noetherian rational cohomology, the space \(LX\) is formal if and only if \(X\) is a rational \(H\)-space. Yamaguchi [296] in 2000 generalized this to the function space \(\text{map}(X, Y; 0)\) for \(X\) simply connected CW of dimension less than the connectivity of \(Y\) an elliptic space. He showed that, again, \(\text{map}(X, Y; 0)\) is formal if and only if \(Y\) is a rational \(H\)-space. Vigué-Poirrier [281] in 2007 proved a further result in this vein showing, with the same dimension/connectivity hypotheses, that \(\text{map}(X, Y; 0)\) is formal if and only if \(Y\) is a rational \(H\)-space provided the odd rational Hurewicz homomorphism of \(X\) is nontrivial.

Félix-Thomas-Vigué-Poirrier [101] in 2007 studied the string topology operations on \(\mathbb{H}_*(LM; \mathbb{Q})\) within the framework of Sullivan minimal models. They gave a description of the loop-product and the string-bracket in this setting making explicit computations. They also proved the isomorphism of graded spaces due to
is an isomorphism of Gerstenhaber algebras. Félix-Thomas [100] 2008 extended this last result proving the above is an isomorphism of BV-algebras.

3.2. Function Spaces and $p$-Localization. Function spaces are central to the theory of homotopy localizations and completions with respect to subrings $R \subset \mathbb{Q}$. As mentioned above, Bousfield-Kan [29] 1972 proved that their $R$-completion functor induces $R$-completion on the based function space $\text{map}_*(X,Y;0)$ for $X$ finite CW and $Y$ nilpotent. This result was used in the proof of the fracture lemma for $R$-completions. A first reduction in Miller’s proof of the Sullivan conjecture is a weak equivalence

$$\text{map}_*(X,Y;0) \simeq_w \text{map}_*(X,R_\infty Y;0)$$

where $R = \mathbb{F}_p$ and $R_\infty$ is the $R$-completion functor. Here $Y$ is a nilpotent space and $X$ is a connected, $\mathbb{Z}[\frac{1}{p}]$-acyclic space. Function spaces also feature in the theory of homotopy localization and cellularization. Given a map $f: X \to Y$, a space $Z$ is defined to be $f$-local if the induced map of function spaces $f^*: \text{map}_*(Y,Z) \to \text{map}_*(X,Z)$ is a weak homotopy equivalence. Dror-Farjoun [75] 1996 constructed $f$-localizations and showed the known localization functors all occur as special cases, for suitable choices of the map $f$. As usual, we consider work here which focuses explicitly on the homotopy type of function spaces.

3.2.1. Maps out of a Classifying Space. The function spaces $\text{map}(B\pi,X)$ and $\text{map}_*(B\pi,X)$ for $\pi$ a finite group appear in major developments in homotopy theory in the $p$-local category. In celebrated work, Miller [213] 1984 affirmed the Sullivan conjecture, proving the weak triviality of spaces $\text{map}_*(B\mathbb{Z}_p,R_\infty Y)$ where $R_\infty$ is the Bousfield-Kan $p$-completion functor. Miller proved the latter fact by establishing the vanishing of certain Ext-sets in a category of unstable modules over the mod $p$ Steenrod algebra.

Miller’s Theorem had many important consequences for function spaces. Zabrodsky [306] 1991 connected the result to the study of phantom maps. He also obtained the following extension. Let $W$ be a connected CW complex with finitely many, locally finite homotopy groups. Then

$$\pi_n(\text{map}_*(B\pi,X)) = 0 \text{ for all } n \geq 0,$$

for $\pi$ a locally finite group and $X$ a connected, finite CW complex. Using the $R$-completion theorem for $\text{map}_*(X,Y)$ mentioned above, the problem reduces to proving the weak triviality of spaces $\text{map}_*(B\mathbb{Z}_p,R_\infty Y)$ where $R_\infty$ is the Bousfield-Kan $p$-completion functor. Miller proved the latter fact by establishing the vanishing of certain Ext-sets in a category of unstable modules over the mod $p$ Steenrod algebra.

Miller’s Theorem had many important consequences for function spaces. Zabrodsky [306] 1991 connected the result to the study of phantom maps. He also obtained the following extension. Let $W$ be a connected CW complex with finitely many, locally finite homotopy groups. Then

$$\pi_n(\text{map}_*(W,X)) = 0 \text{ for all } n \geq 0,$$

for $X$ any connected, finite CW complex. Zabrodsky also proved that, if $P: E \to B$ is a principal $G$-bundle with $\text{map}_*(G,Y)$ contractible, then $\text{map}_*(B,Y) \to \text{map}_*(E,Y)$ is a homotopy equivalence. Miller’s result had equivariant generalizations in terms of fixed point and homotopy fixed point set. In particular, if $X$ is a $\pi$-space for $\pi$ a $p$-group then

$$R_\infty(X^\pi) = (R_\infty X)^{h\pi}$$

where $R = \mathbb{Z}/p$ and $X^\pi$ is the fixed point set while $X^{h\pi}$ is the homotopy fixed point sets (c.f. Carlsson [47] 1991]). Dwyer-Zabrodsky [80] 1986 applied this
latter result to obtain a mod $p$ decomposition

$$\text{map}(B\pi, BG) \simeq_p \coprod_{\rho} BC(\rho).$$

Here $\pi$ is a finite $p$-group, $G$ is a compact Lie group and the disjoint union is over $G$-conjugacy classes of homomorphisms $\rho: \pi \to G$. As usual, $C(\rho)$ denotes the centralizer of the image of $\rho$ in $G$. Friedlander-Mislin [103] 1986 gave conditions on a Lie group $G$ such that $\text{map}_*(BG, R_\infty X)$ is weakly trivial. Here $X$ is nilpotent and $R_\infty$ is $p$-completion. McGibbon [208] 1996 proved $\text{map}_*(W, R_\infty X)$ is weakly contractible for $W$ a connected infinite loop space with torsion fundamental group. Strom [261] 2005 proved that if $\text{map}_*(X, S^n)$ is weakly contractible for all sufficiently large $n$ then $\text{map}_*(X, Y)$ is actually weakly contractible for any nilpotent, finite CW complex $Y$. He thus obtained a method for recognizing spaces $X$ satisfying the conclusion of Miller’s Theorem.

Lannes [180] 1987 and [181] 1992 complemented and extended Miller’s work. Let $\mathcal{U}$ and $\mathcal{K}$ denote, respectively, the category of unstable modules and algebras over $A$, the mod $p$ Steenrod algebra. Lannes constructed the $T$-functor which is left-adjoint to the tensor product functor $- \otimes_{\mathcal{U}} H^*(B\mathbb{Z}_p; \mathbb{Z}_p)$ on $\mathcal{U}$. He showed, among other properties, that $T$ is exact, preserves tensor products and restricts to a functor on $\mathcal{K}$. These results are used to prove the natural map

$$\Theta_X: T(H^*(X; \mathbb{Z}_p)) \to H^*(\text{map}(B\mathbb{Z}_p, X; \mathbb{Z}_p))$$

is an isomorphism of unstable $A$-algebras whenever $T(H^*(X; \mathbb{Z}_p))$ is trivial in degree 1 and of finite type.

Aguadé [3] 1989 computed $T(M)$ for certain $A$-algebras $M$ including subalgebras of $\mathbb{Z}_p[x_1, \ldots, x_n]$ invariant under the general linear action. Here $p$ is an odd prime. Dror-Smith [74] 1990 constructed an Eilenberg-Moore spectral sequence for computing $\Theta_X$ above and gave a geometric interpretation of the $T$-functor. Dwyer-Wilkerson [85] 1990 proved if $\pi$ is a locally finite group and $X$ is a simply connected $p$-complete space with $H^*(X; \mathbb{Z}_p)$ finitely generated as an algebra, then $\text{map}_*(B\pi, X; 0)$ is weakly contractible. Kuhn-Winstead [168] 1996 proved

$$H^*(X; \mathbb{Z}_p) = 0 \implies \tilde{H}^*(\text{map}(B\mathbb{Z}_p, X); \mathbb{Z}_p) = 0.$$ 

More generally, they showed the image of $\Theta_X$ consists of $\mathbb{Z}_p^\wedge$-integral classes if $H^*(X; \mathbb{Z}_p)$ does where a $\mathbb{Z}_p^\wedge$-integral class in $H^*(\mathbb{Z}_p^\wedge)$ is a class in the reduction from $H^*(\mathbb{Z}_p^\wedge)$. Dehon-Lannes [68] 2000 proved that if $X$ is $p$-complete and $H^*(X; \mathbb{Z}_p)$ is Noetherian and generated in even degrees then $H^*(\text{map}(B\mathbb{Z}_p, X); \mathbb{Z}_p)$ is Noetherian and $\text{map}(B\mathbb{Z}_p, X)$ is $p$-complete. Aguadé-BROTO-Saumell [4] 2004 introduced the notion of $T$-representability for a space $X$ and proved it a sufficient condition for $\Theta_X$ to be an isomorphism. They gave an example of a $p$-complete space $X$ for which $\Theta_X$ is not an isomorphism.

We mention some further results falling under the current heading. Dwyer-Mislin [84] 1987 identified the homotopy type of the nontrivial components of $\text{map}_*(BS^3, BS^3)$. The null component is contractible by Zabrodsky [306] 1991, as mentioned above. Jackowski-McClure-Oliver [151] 1992 determined the homology with coefficients in a finite group of the components of $\text{map}(BG, BG)$ for $G$ a simple compact, connected Lie group. Building on these results, Andersen-Grodal [7] 2009 expressed the classification of $p$-compact groups with component
group a $p$-group $\pi$ in terms of the homotopy types of the components of a space of maps out of $B\pi$.

Blanc-Notbohm [21] 1993] proved

$$\text{map}(BG, BH; f)_p^\wedge \simeq \text{map}(BG, (BH)_p^\wedge; f^\wedge_p)$$

for $G, H$ compact Lie groups. Broto-Levi [35] 2002] proved

$$\text{aut}((B\pi)_p^\wedge) \simeq K(Z(\pi/O^p(\pi), 1)$$

for $\pi$ a finite group. Here $O^p(\pi)$ denotes the maximal normal $p'$-subgroup of $\pi$. The proof uses the Bousfield-Kan spectral sequence to prove asphericality and the $Z^*$-theorem from group theory to compute the fundamental group.

3.2.2. Algebraic Models. In this final section, we discuss results on modeling the $p$-local homotopy theory of function spaces.

Dwyer [51] 1979] proved that a version of Quillen’s rational homotopy theory extends to the homotopy theory of tame spaces $X$ which are $(r - 1)$-connected CW complexes, $r \geq 3$, with $\pi_{r+k}(X)$ uniquely $p$-divisible for all primes $p$ with $2p - 3 \leq k$. He proved the homotopy category of tame spaces endowed with an appropriate model structure is equivalent to a homotopy category of $(r - 1)$-reduced integral DGLAs. Tame homotopy theory admits a version in the spirit of Sullivan’s approach to rational homotopy theory by Cenkl-Porter [76] 1983]. Scheerer-Tanré [239] 1988] identified homotopy invariants, e.g., homology and the homotopy Lie algebra, in Dwyer’s framework. Anick [10] 1989] and [11] 1990] gave an alternate approach using a classical construction of Adams-Hilton [11] 1955] on the Pontryagin algebra $H_*(\Omega X; R)$ for $R \subset \mathbb{Q}$. He constructed a DGLA model over $R$ for the category $\text{CW}^m_r$ consisting of $(r - 1)$-connected complexes of dimension $m$ when all primes $p$ with $m > pr$ are invertible in $R$.

The description of function spaces in tame homotopy theory has been undertaken in several works. Anick-Dror-Farjoun [9] 1990] described a simplicial skeleton of the space $\text{map}_r(X, Y)$ for $R$-local spaces $X, Y \in \text{CW}_r^m$. Given suitably reduced DGLAs $L$ and $K$ over $R$ they constructed a function complex $\text{Hom}(L, K)$ giving an explicit description through a range of dimensions. Scheerer-Tanré [240] 1992] described $R$-local homotopy theory in a suitable category of DG coalgebras of $R$. They constructed an adjoint to the wedge functor in this context and used it to give a model for the space $\text{map}_r(X, Y)$ for $R$-local $X, Y \in \text{CW}_r^m$. As an example, they described a model for the $R$-localization of $\text{map}_r(\mathbb{H}P^2, M_t)$, where $M_t$ denotes a tamed Moore space and $R$ is suitably chosen. Félix-Thomas [90] 1993] obtained a $p$-local decomposition

$$\text{map}_r(\Sigma X, Y) \simeq_p \prod_{i=2}^n (\Omega^{i+1} Y)^{\beta_i(X)}$$

for $X$ simply connected with torsion-free homology of dimension $n < 2p$ and $Y$ $(r - 1)$-connected with $r > n + 1$. Scheerer [238] 1994] recovered this result as a special case of a corresponding decomposition for $\text{map}_r(C, Y)$ for $C$ a co-H-space. Dupont-Hess [76] 1999], [77] 2002] and [78] 2003] used Anick’s framework to obtain a model for the mod $p$ cohomology of the free loop space $LX$ for a simply connected space $X \in \text{CW}_r^m$ and prime $p$ with $m \leq pr$. They constructed a DGA over $\mathbb{Z}_p$, built from Anick’s model, and proved the homology of this algebra is isomorphic to $H^*(LX; \mathbb{Z}_p)$. 


Dwyer-Kan [82] 1980] identified function complexes in a simplicial homotopy theory category $L^H(M)$ of a general Quillen model category $M$. Here $L^H$ is their “hammock” localization functor. They showed $L^H(M(X,Y))$ for $X,Y \in M$ behaves properly with respect to simplicial and cosimplicial resolutions. This approach yielded, in particular, a good model for the monoid of self-equivalences for arbitrary model categories. In Dwyer-Kan [83] 1983, they identified the homotopy type of the function complex as a homotopy inverse limit for $X$ cofibrant and $Y$ fibrant.

Recently, Fresse [102] [preprint] constructed an algebraic model for the homotopy type of $map(X,Y)$ for $X$ a finite complex and $\pi_n(Y)$ a finite $p$-group for all $n$. Let $N^*(Y;\mathbb{Z}_p)$ denote the normalized cochain complex where $\mathbb{Z}_p$ is the algebraic closure of the field of $p$-elements. Then $N^*(Y;\mathbb{Z}_p)$ is an $E_{\infty}$-algebra and, more generally, an algebra over the Barrett-Eccles operad $E_{\infty}$. Mandell [195] 2001] proved this algebra structure determines the homotopy type of $Y$. Fresse constructed a Lannes $T$-functor left adjoint to the tensor product in the category of algebras over $E_{\infty}$. As a consequence, he obtained a model $(N^*(Y;\mathbb{Z}_p) : N^*(X;\mathbb{Z}_p))$ for $map(X,Y)$.

Chataur-Kuribayashi [53] 2007] extended this result to the case $Y$ is connected and nilpotent of finite type and made calculations with the resulting spectral sequence.

References
1. J. F. Adams and P. J. Hilton, On the chain algebra of a loop space, Comment. Math. Helv. 30 (1956), 305–330. MR0077929 (17,1119b)
2. Jaume Aguadé, Decomposable free loop spaces, Canad. J. Math. 39 (1987), no. 4, 938–955. MR915024 (88m:55012)
3. □, Computing Lannes $T$-functor, Israel J. Math. 65 (1989), no. 3, 303–310. MR1005014 (90h:55026)
4. Jaume Aguadé, Carles Broto, and Laia Saumell, The functor $T$ and the cohomology of mapping spaces, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 191, Birkhäuser, Basel, 2004, pp. 1–20. MR2039756 (2005e:55033)
5. Stephen T. Ahearn and Nicholas J. Kuhn, Product and other fine structure in polynomial resolutions of mapping spaces, Algebr. Geom. Topol. 2 (2002), 591–647 (electronic). MR1917068 (2003j:55009)
6. D. W. Anderson, A generalization of the Eilenberg-Moore spectral sequence, Bull. Amer. Math. Soc. 78 (1972), 784–786. MR0310889 (46 #9987)
7. Kasper K. S. Andersen and Jesper Grodel, The classification of 2-compact groups, J. Amer. Math. Soc. 22 (2009), 387-436. MR2476779 (2010d:55022)
8. M. H. Andrade Claudio and M. Spreafico, Homotopy type of gauge groups of quaternionic line bundles over spheres, Topology Appl. 156 (2009), no. 3, 643–651. MR2492312
9. D. J. Anick and E. Dror Farjoun, On the space of maps between $R$-local CW complexes, Astérisque (1990), no. 191, 5, 15–27, International Conference on Homotopy Theory (Marseille-Luminy, 1988). MR1098963 (92c:55008)
10. David J. Anick, Hopf algebras up to homotopy, J. Amer. Math. Soc. 2 (1989), no. 3, 417–453. MR991015 (90c:16007)
11. □, $R$-local homotopy theory, Homotopy theory and related topics (Kinosaki, 1988), Lecture Notes in Math., vol. 1418, Springer, Berlin, 1990, pp. 78–85. MR1048177 (91c:55008)
12. Martin Arkowitz, The group of self-homotopy equivalences—a survey, Groups of self-equivalences and related topics (Montreal, PQ, 1988), Lecture Notes in Math., vol. 1425, Springer, Berlin, 1990, pp. 170–203. MR1070585 (91i:55001)
13. Greg Arone, A generalization of Stasheff-type filtrations, Trans. Amer. Math. Soc. 351 (1999), no. 3, 1123–1150. MR1638238 (99j:55011)
14. M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A 308 (1983), no. 1505, 523–615. MR702806 (85k:14006)
15. M. F. Atiyah and J. D. S. Jones, Topological aspects of Yang-Mills theory, Comm. Math. Phys. 61 (1978), no. 2, 97–118. MR503187 (80j:58021)

16. Sven Bauer, Michael Crabb, and Mauro Spreako, The space of free loops on a real projective space, Groups of homotopy self-equivalences and related topics (Gargnano, 1999), Contemp. Math., vol. 274, Amer. Math. Soc., Providence, RI, 2001, pp. 33–38. MR1817001 (2001m:55028)

17. H. J. Baues, Algebraic homotopy, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989. MR985099 (90i:55016)

18. H. J. Baues and J.-M. Lemaire, Minimal models in homotopy theory, Math. Ann. 225 (1977), no. 3, 219–242. MR0431172 (55 #4174)

19. Martin Bendersky and Sam Gitler, The cohomology of certain function spaces, Trans. Amer. Math. Soc. 326 (1991), no. 1, 423–440. MR1010881 (92d:55005)

20. David Blanc, Mapping spaces and $M$-CW complexes, Forum Math. 9 (1997), no. 3, 367–382. MR1441926 (98i:55015)

21. David Blanc and Dietrich Notbohm, Mapping spaces of compact Lie groups and $p$-adic completion, Proc. Amer. Math. Soc. 117 (1993), no. 1, 251–258. MR1112487 (93c:55016)

22. J. Block and A. Lazarev, André-Quillen cohomology and rational homotopy of function spaces, Adv. Math. 193 (2005), no. 1, 18–39. MR2132759 (2006a:55014)

23. C.-F. Bödigheimer, Stable splittings of mapping spaces, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 174–187. MR922926 (89c:55011)

24. C.-F. Bödigheimer, F. R. Cohen, and M. D. Peim, Mapping class groups and function spaces, Homotopy methods in algebraic topology (Boulder, CO, 1999), Contemp. Math., vol. 271, Amer. Math. Soc., Providence, RI, 2001, pp. 17–39. MR1831345 (2002f:55036)

25. Marcel Bökstedt and Iver Ottosen, A splitting result for the free loop space of spheres and projective spaces, Q. J. Math. 56 (2005), no. 4, 443–472. MR2182460 (2006f:55008)

26. Peter Booth, Philip Heath, Chris Morgan, and Renzo Piccinini, $H$-spaces of self-equivalences of fibrations and bundles, Proc. London Math. Soc. (3) 49 (1984), no. 1, 111–127. MR743373 (85k:55013)

27. Karol Borsuk, Concerning the homological structure of the functional space $S^X_{St}$, Fund. Math. 29 (1952), 25–37 (1953). MR0056285 (15,51g)

28. A. K. Bousfield and V. K. A. M. Gugenheim, On PL de Rham theory and rational homotopy type, Mem. Amer. Math. Soc. 8 (1976), no. 179, ix+94. MR0425956 (54 #13906)

29. A. K. Bousfield and D. M. Kan, Homotopy limits, completions and localizations, Lecture Notes in Mathematics, Vol. 304, Springer-Verlag, Berlin, 1972. MR0365573 (51 #1825)

30. A. K. Bousfield, C. Peterson, and L. Smith, The rational homology of function spaces, Arch. Math. (Basel) 52 (1989), no. 3, 275–283. MR989883 (90d:55020)

31. C. P. Boyer, J. C. Hurtubise, B. M. Mann, and R. J. Milgram, The topology of instanton moduli spaces. I. The Atiyah-Jones conjecture, Ann. of Math. (2) 137 (1993), no. 3, 561–609. MR1217348 (94h:55010)

32. C. P. Boyer, J. C. Hurtubise, and R. J. Milgram, Stability theorems for spaces of rational curves, Internat. J. Math. 12 (2001), no. 2, 223–262. MR1823576 (2002g:55010)

33. C. P. Boyer, B. M. Mann, J. C. Hurtubise, and R. J. Milgram, The topology of the space of rational maps into generalized flag manifolds, Acta Math. 173 (1994), no. 1, 61–101. MR1294670 (95h:55007)

34. Charles P. Boyer, H. Blaine Lawson, Jr., Paulo Lima-Filho, Benjamin M. Mann, and Marie-Louise Michelsohn, Algebraic cycles and infinite loop spaces, Invent. Math. 113 (1993), no. 2, 373–388. MR1228130 (95a:55021)

35. Carles Broto and Ran Levi, Loop structures on homotopy fibers of self maps of spheres, Amer. J. Math. 122 (2000), no. 3, 547–580. MR1759888 (2001c:55005)

36. ———, On spaces of self-homotopy equivalences of p-completed classifying spaces of finite groups and homotopy group extensions, Topology 41 (2002), no. 2, 229–255. MR1876889 (2002j:55013)

37. Edgar H. Brown, Jr. and Robert H. Szczarba, On the rational homotopy type of function spaces, Trans. Amer. Math. Soc. 349 (1997), no. 12, 4931–4951. MR1407482 (98c:55015)

38. ———, Some algebraic constructions in rational homotopy theory, Topology Appl. 80 (1997), no. 3, 251–258. MR1473920 (98h:55017)
39. R. Brown, *Function spaces and product topologies*, Quart. J. Math. Oxford Ser. (2) **15** (1964), 238–250. MR0165497 (29 #2779)
40. Urzzi Buijs, Yves Félix, and Aniceto Murillo, *Lie models for the components of sections of a nilpotent fibration*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5601–5614. MR2515825
41. **Rational homotopy of the (homotopy) fixed point sets of circle actions**, Adv. Math. **222** (2009), no. 1, 151–171. MR2531370
42. Urzzi Buijs and Aniceto Murillo, *Basic constructions in rational homotopy theory of function spaces*, Ann. Inst. Fourier (Grenoble) **56** (2006), no. 3, 815–838. MR2244231 (2007h:55009)
43. *The rational homotopy Lie algebra of function spaces*, Comment. Math. Helv. **83** (2008), no. 4, 723–739. MR2442961 (2009f:55009)
44. D. Burghelea and Z. Fiedorowicz, *Hermitian algebraic K-theory of topological spaces*, Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982), Lecture Notes in Math., vol. 1046, Springer, Berlin, 1984, pp. 32–46. MR750675 (86c:18005)
45. H. E. A. Campbell, F. R. Cohen, F. P. Peterson, and P. S. Selick, *The space of maps of Moore spaces into spheres*, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 72–100. MR921473 (89a:55012)
46. G. E. Carlsson, R. L. Cohen, T. Goodwillie, and W. C. Hsiang, *The free loop space and the algebraic K-theory of spaces*, *K*-Theory **1** (1987), no. 1, 53–82. MR899917 (88i:55002)
47. Gunnar Carlsson, *Equivariant stable homotopy and Sullivan’s conjecture*, Invent. Math. **103** (1991), no. 3, 497–525. MR1091616 (92g:55007)
48. Carles Casacuberta and José L. Rodríguez, *On weak homotopy equivalences between mapping spaces*, Topology **37** (1998), no. 4, 709–717. MR1607716 (98k:55006)
49. Bohumil Cenkl and Richard Porter, *Differential forms and torsion in the fundamental group*, Adv. in Math. **48** (1983), no. 2, 189–204. MR700985 (85d:57005)
50. Moira Chas and Dennis Sullivan, *String topology*, preprint, arXiv:math.GT/9911159.
51. *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 771–784. MR2077595 (2005f:55007)
52. David Chataur, *A bordism approach to string topology*, Int. Math. Res. Not. (2005), no. 46, 2829–2875. MR2180465 (2007b:55009)
53. David Chataur and Katsuhiko Kuribayashi, *An operadic model for a mapping space and its associated spectral sequence*, J. Pure Appl. Algebra **210** (2007), no. 2, 321–342. MR2320001 (2008h:55017)
54. David Chataur and Jean-Claude Thomas, *$E_{\infty}$-model for a mapping space*, Topology Appl. **145** (2004), no. 1-3, 191–204. MR2100872 (2005k:55011)
55. Younggi Choi, *Homology of gauge groups associated with special unitary groups*, Topology Appl. **155** (2008), no. 12, 1340–1349. MR2423972 (2009f:55014)
56. F. R. Cohen, R. L. Cohen, B. M. Mann, and R. J. Milgram, *The topology of rational functions and divisors of surfaces*, Acta Math. **166** (1991), no. 3-4, 163–221. MR1097023 (92k:55011)
57. *The homotopy type of rational functions*, Math. Z. **213** (1993), no. 1, 37–47. MR1217669 (94a:55006)
58. F. R. Cohen and L. R. Taylor, *The homology of function spaces*, Proceedings of the Northwestern Homotopy Theory Conference (Evanston, Ill., 1982) (Providence, RI), Contemp. Math., vol. 19, Amer. Math. Soc., 1983, pp. 39–50. MR711041 (85d:55016)
59. *Homology of function spaces*, Math. Z. **198** (1988), no. 3, 299–316. MR946606 (89d:55017)
60. Ralph L. Cohen, *A model for the free loop space of a suspension*, Algebraic topology (Seattle, Wash., 1985), Lecture Notes in Math., vol. 1286, Springer, Berlin, 1987, pp. 193–207. MR922928 (89d:55018)
61. Ralph L. Cohen and John D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. **324** (2002), no. 4, 775–798. MR1942249 (2004c:55019)
62. Ralph L. Cohen, John D. S. Jones, and Jun Yan, *The loop homology algebra of spheres and projective spaces*, Categorical decomposition techniques in algebraic topology (Isle of Skye, 2001), Progr. Math., vol. 215, Birkhäuser, Basel, 2004, pp. 77–92. MR2039760 (2005c:55016)
63. Ralph L. Cohen, John R. Klein, and Dennis Sullivan, *The homotopy invariance of the string topology loop product and string bracket*, J. Topol. **1** (2008), no. 2, 391–408. MR2399136 (2009h:55004)
64. Ralph L. Cohen and Don H. Shimamoto, *Rational functions, labelled configurations, and Hilbert schemes*, J. London Math. Soc. (2) **43** (1991), no. 3, 509–528. MR1113390 (93c:55009)

65. M. C. Crabb, *Loop homology as fibrewise homology*, Proc. Edinb. Math. Soc. (2) **51** (2008), no. 1, 27–44. MR2391632 (2009c:55018)

66. M. C. Crabb and W. A. Sutherland, *Function spaces and Hurwitz-Radon numbers*, Math. Scand. **55** (1984), no. 1, 67–90. MR769026 (86d:58014)

67. *Counting homotopy types of gauge groups*, Proc. London Math. Soc. (3) **81** (2000), no. 3, 747–768. MR1781154 (2001m:55024)

68. Françoise Derhoun and Jean Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d’un groupe de Lie compact commutatif*, Inst. Hautes Études Sci. Publ. Math. (1999), no. 89, 127–177 (2000). MR1793415 (2001m:55038)

69. Geneviève Didierjean, *Homotopie de l’espace des équivalences d’homotopie fibré es*, Ann. Inst. Fourier (Grenoble) **35** (1985), no. 3, 33–47. MR810666 (87e:55008)

70. *Homotopie des espaces d’équivalences*, Groups of self-equivalences and related topics (Montreal, PQ, 1988), Lecture Notes in Math., vol. 1425, Springer, Berlin, 1990, pp. 32–39. MR1070573 (91j:55008)

71. *Homotopie de l’espace des équivalences d’homotopie*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 153–163. MR986023 (92f:55017)

72. Albrecht Dold and Richard Lashof, *Principal quasi-fibrations and fibre homotopy equivalence of bundles.*, Illinois J. Math. **3** (1959), 285–305. MR0101521 (21 #331)

73. E. Dror, W. G. Dwyer, and D. M. Kan, *Automorphisms of fibrations*, Proc. Amer. Math. Soc. **80** (1980), no. 3, 491–494. MR581012 (81h:55012)

74. E. Dror Farjoun and J. Smith, *A geometric interpretation of Lannes’ functor T*, Astérisque (1990), no. 191, 6, 87–95, International Conference on Homotopy Theory (Marseille-Luminy, 1988). MR1089868 (92h:55013)

75. Emmanuel Dror Farjoun, *Cellular spaces, null spaces and homotopy localization*, Lecture Notes in Mathematics, vol. 1622, Springer-Verlag, Berlin, 1996. MR1392221 (98f:55010)

76. Nicolas Dupont and Kathryn Hess, *How to model the free loop space algebraically*, Math. Ann. **314** (1999), no. 3, 469–490. MR1704545 (2000e:18011)

77. *An algebraic model for homotopy fibers*, Homology Homotopy Appl. **4** (2002), no. 2, part 1, 117–139 (electronic). The Roos Festschrift volume, 1. MR1918186 (2003e:55012)

78. *Commutative free loop space models at large primes*, Math. Z. **244** (2003), no. 1, 1–34. MR1981874 (2004c:55016)

79. Nicolas Dupont and Micheline Vigué-Poirrier, *Formalité des espaces de lacets libres*, Bull. Soc. Math. France **126** (1998), no. 1, 141–148. MR1651384 (99i:55018)

80. W. Dwyer and A. Zabrodsky, *Maps between classifying spaces*, Algebraic topology, Barcelona, 1986, Lecture Notes in Math., vol. 1298, Springer, Berlin, 1987, pp. 106–119. MR928826 (90b:55018)

81. W. G. Dwyer, *Tame homotopy theory*, Topology **18** (1979), no. 4, 321–338. MR551014 (81a:55020)

82. W. G. Dwyer and D. M. Kan, *Function complexes in homotopical algebra*, Topology **19** (1980), no. 4, 427–440. MR584566 (81m:55018)

83. *Function complexes for diagrams of simplicial sets*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 2, 139–147. MR705421 (85e:55038)

84. W. G. Dwyer and G. Mislin, *On the homotopy type of the components of map∗(BS, BS)*, Algebraic topology, Barcelona, 1986, Lecture Notes in Math., vol. 1298, Springer, Berlin, 1987, pp. 82–89. MR928824 (90b:55018)

85. William G. Dwyer and Clarence W. Wilkerson, *Spaces of null homotopic maps*, Astérisque (1990), no. 191, 6, 97–108, International Conference on Homotopy Theory (Marseille-Luminy, 1988). MR1098969 (92b:55004)

86. Micheal N. Dyer, *The influence of π1(Y) on the homology of M(X, Y)*, Illinois J. Math. **10** (1966), 648–651. MR0203726 (34 #3577)

87. E. Fadell and S. Husseini, *A note on the category of the free loop space*, Proc. Amer. Math. Soc. **107** (1989), no. 2, 527–536. MR984789 (90a:55008)

88. Herbert Federer, *A study of function spaces by spectral sequences*, Trans. Amer. Math. Soc. **82** (1956), 340–361. MR0079265 (18,59b)

89. Y. Félix, *Rational category of the space of sections of a nilpotent bundle*, Comment. Math. Helv. **65** (1990), no. 4, 615–622. MR1078101 (91j:55012)
90. Y. Félix and Jean-Claude Thomas, *Homology and homotopy groups of mapping spaces at large primes*, Topology 32 (1993), no. 1, 1–4. MR1204401 (93m:55018)
91. Y. Félix, Jean-Claude Thomas, and M. Vigué-Poirrier, *Free loop spaces of finite complexes have infinite category*, Proc. Amer. Math. Soc. 111 (1991), no. 3, 869–875. MR1025277 (91f:55003)
92. Y. Félix, Jean-Claude Thomas, and M. Vigué-Poirrier, *Free loop spaces of finite complexes*, Trans. Amer. Math. Soc. 332 (1992), no. 1, 1–38. MR664027 (84h:55011)
93. Yves Félix and Stephen Halperin, *Rational LS category and its applications*, Trans. Amer. Math. Soc. 273 (1982), no. 1, 1–38. MR664027 (84h:55011)
94. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR1802847 (2002d:55014)
95. Yves Félix and Gregory Lupton, *Evaluation maps in rational homotopy*, Topology 46 (2007), no. 5, 493–506. MR2337558 (2008e:55014)
96. Yves Félix, Gregory Lupton, and Samuel B. Smith, *The rational homotopy type of the space of self-equivalences of a fibration*, Preprint, arXiv:0903.1470v1.
97. Yves Félix and John Oprea, *Rational homotopy of gauge groups*, Proc. Amer. Math. Soc. 137 (2009), no. 4, 1519–1527. MR2465678 (2009h:55011)
98. Yves Félix and Daniel Tanré, *H-space structure on pointed mapping spaces*, Algebr. Geom. Topol. 5 (2005), 713–724 (electronic). MR2153111 (2006a:55020)
99. Yves Félix and Jean-Claude Thomas, *Monoid of self-equivalences and free loop spaces*, Proc. Amer. Math. Soc. 132 (2004), no. 1, 305–312 (electronic). MR2021275 (2004k:55007)
100. Yves Félix, Jean-Claude Thomas, and Micheline Vigué-Poirrier, *Rational string topology*, J. Eur. Math. Soc. (JEMS) 9 (2007), no. 1, 123–156. MR2283106 (2007k:55009)
101. Yves Félix, Jean-Claude Thomas, and Micheline Vigué-Poirrier, *Rational string topology*, J. Eur. Math. Soc. (JEMS) 9 (2007), no. 1, 123–156. MR2283106 (2007k:55009)
102. Benoit Fresse, *Derived division functors and mapping spaces*, Preprint, arXiv:math/0208091.
103. Eric M. Friedlander and Guido Mislin, *Locally finite approximation of Lie groups. I*, Invent. Math. 83 (1986), no. 3, 425–436. MR827361 (87i:55038)
104. David Gabai, *The Smale conjecture for hyperbolic 3-manifolds: Isom(M^3) \cong Diff(M^3)*, J. Differential Geom. 58 (2001), no. 1, 113–149. MR1895350 (2003c:57016)
105. Tudor Ganea, *Lusternik-Schnirelmann category and cocategory*, Proc. London Math. Soc. (3) 10 (1960), 623–639. MR0126278 (23 #A3574)
106. Marek Golasiński and Juno Mukai, *Gottlieb groups of spheres*, Topology 47 (2008), no. 6, 399–430. MR2427733 (2009b:55017)
107. Thomas G. Goodwillie, *Cyclic homology, derivations, and the free loopspace*, Topology 24 (1985), no. 2, 187–215. MR793184 (87c:18009)
108. Daniel Henry Gottlieb, *A certain subgroup of the fundamental group*, Amer. J. Math. 91 (1969), 493–510. MR0264551 (41 #9143)
109. Karsten Grove, Stephen Halperin, and Micheline Vigué-Poirrier, *The rational homotopy theory of certain path spaces with applications to geodesics*, Acta Math. 140 (1978), no. 3-4, 277–303. MR496895 (80g:58024)
Homotopy type of gauge groups of $SU(3)$. Hiroaki Hamanaka and Akira Kono, The homology of a free loop space. Vagn Lundsgaard Hansen, Equivalence of evaluation fibrations. Peter Hilton, Guido Mislin, and Joseph Roitberg, Localization of nilpotent groups and spaces. John W. Havlicek, The cohomology of holomorphic self-maps of the Riemann sphere. Kate Gruher and Paolo Salvatore, Generalized string topology operations. Stephen Halperin and Micheline Vigué-Poirrier, The homotopy type of the space of rational functions. J. Math. Kyoto Univ. 35 (1995), no. 4, 631–638. MR1365252 (96j:58020)

Stephen Halperin, Rational fibrations, minimal models, and fibrings of homogeneous spaces. Trans. Amer. Math. Soc. 273 (1982), no. 2, 609–620. MR667163 (84a:55010)

André Haefliger, Rational homotopy of the space of sections of a nilpotent bundle. Trans. Amer. Math. Soc. 244 (1978), 199–224. MR0515558 (58 #24264)

Stephen Halperin and Micheline Vigué-Poirrier, The homotopy of a free loop space, Pacific J. Math. 147 (1991), no. 2, 311–324. MR1084712 (92e:55012)

Hiroaki Hamanaka and Akira Kono, Homotopy type of gauge groups of $SU(3)$-bundles over $S^n$, Topology Appl. 154 (2007), no. 7, 1377–1380. MR2310471 (2008c:57047)

Vagn Lundsgaard Hansen, Equivalence of evaluation fibrations, Invent. Math. 23 (1974), 163–171. MR0368000 (51 #4242)

Stephen Halperin, Rational fibrations, minimal models, and fibrings of homogeneous spaces. Trans. Amer. Math. Soc. 244 (1978), 199–224. MR0515558 (58 #24264)

Kevin P. Knudson, A proof of the Smale conjecture, Trans. Amer. Math. Soc. 293 (1986), no. 1, 79–81. MR816308 (87a:57049)

Knut Keeton, Deformations and the rational homotopy type of the monoid of fiber homotopy equivalences, Illinois J. Math. 37 (1993), no. 4, 537–560. MR1226781 (94h:55017)

Fibrations, self homotopy equivalences and related topics (Gargnano, 1999), Contemp. Math., vol. 244, Amer. Math. Soc., Providence, RI, 2001, pp. 169–182. MR1817009 (2001m:55033)

John W. Havlicek, The cohomology of holomorphic self-maps of the Riemann sphere, Math. Z. 218 (1995), no. 2, 179–190. MR1318152 (96c:58029)

Kathryn Hess and Ran Levi, An algebraic model for the loop space homotopy of a homotopy fiber, Algebr. Geom. Topol. 7 (2007), 1699–1765. MR2366176 (2008j:57056)

Peter Hilton, Guido Mislin, and Joseph Roitberg, Localization of nilpotent groups and spaces, North-Holland Publishing Co., Amsterdam, 1975, North-Holland Mathematics Studies, No. 15, Notes on Mathematics, No. 55. [Notes on Mathematics, No. 55. MR0478146 (57 #17635)

Peter Hilton, Guido Mislin, Joseph Roitberg, and Richard Steiner, On free maps and free homotopies into nilpotent spaces, Algebraic topology (Proc. Conf., Univ. British Columbia, Vancouver, B.C., 1977), Lecture Notes in Math., vol. 673, Springer, Berlin, 1978, pp. 202–218. MR517093 (80c:55007)
144. Yasumasa Hirashima and Nobuyuki Oda, *Pairings of function spaces*, Topology Appl. **154** (2007), no. 12, 2412–2424. MR2333796 (2008g:54021)

145. Koichi Hirata and Kohhei Yamaguchi, *Spaces of polynomials without 3-fold real roots*, J. Math. Kyoto Univ. **42** (2002), no. 3, 509–516. MR1967220 (2004c:55013)

146. Yoshihiro Hirato, Katsuhiko Kuribayashi, and Nobuyuki Oda, *A function space model approach to the rational evaluation subgroups*, Math. Z. **258** (2008), no. 3, 521–555. MR2369043 (2008j:55012)

147. Po Hu, *Higher string topology on general spaces*, Proc. London Math. Soc. (3) **93** (2006), no. 2, 515–544. MR2251161 (2007f:55007)

148. Sze-tsen Hu, *Concerning the homotopy groups of the components of the mapping space YSP*, Nederl. Akad. Wetensch., Proc. **49** (1946), 1025–1031 = Indagationes Math. **8**, 623–629 (1946). MR0019920 (8,481a)

149. J. C. Hurtubise, *Stability theorems for moduli spaces*, Canadian Mathematical Society. 1945–1995, Vol. 3, Canadian Math. Soc., Ottawa, ON, 1996, pp. 153–171. MR1661615 (2000c:14016)

150. Norio Iwase, *On the splitting of mapping spaces between classifying spaces. I*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 3, 445–453. MR905020 (89i:55004)

151. Stefan Jackowski and James McClure and Bob Oliver, *Homotopy classification of self-maps of BG via G-actions. I and II*, Ann. of Math. **135** (1992), 183-270. MR1147962 (93e:55019a,b)

152. John D. S. Jones, *Cyclic homology and equivariant homology*, Invent. Math. **87** (1987), no. 2, 403–423. MR870737 (88f:18016)

153. Peter J. Kahn, *Some function spaces of CW type*, Proc. Amer. Math. Soc. **90** (1984), no. 4, 599–607. MR733413 (85j:55027)

154. Sadok Kallel and R. James Milgram, *The geometry of the space of holomorphic maps from a Riemann surface to a complex projective space*, J. Differential Geom. **47** (1997), no. 2, 321–375. MR1601616 (98m:58014)

155. Sadok Kallel and Paolo Salvatore, *Rational maps and string topology*, Geom. Topol. **10** (2006), 1579–1606 (electronic). MR2284046 (2007k:58018)

156. Sadok Kallel and Denis Sjerve, *On the topology of fibrations with section and free loop spaces*, Proc. London Math. Soc. (3) **83** (2001), no. 2, 419–442. MR1839460 (2002c:55021)

157. Yasuhiko Kamiyama, *Remarks on spaces of real rational functions*, Rocky Mountain J. Math. **37** (2007), no. 1, 247–257. MR2316447 (2008c:55012)

158. Frances Kirwan, *On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles*, Ark. Mat. **24** (1986), no. 2, 221–275. MR884188 (88h:14014)

159. John R. Klein, Claude L. Schochet, and Samuel B. Smith, *Continuous trace C∗-algebras, gauge groups and rationalization*, J. Topol. Anal. **1** (2009), no. 3, 261–288. MR2574026

160. S. S. Koh, *Note on the homotopy properties of the components of the mapping space XSP*, Proc. Amer. Math. Soc. **11** (1960), 896–904. MR0119201 (22 #9967)

161. Akira Kono, *A note on the homotopy type of certain gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A **117** (1991), no. 3-4, 295–297. MR1103296 (92b:55005)

162. Akira Kono and Kazumoto Kozima, *The adjoint action of a Lie group on the space of loops*, J. Math. Soc. Japan **45** (1993), no. 3, 495–510. MR1219882 (94h:57053)

163. Akira Kono and Shuichi Tsukuda, *A remark on the homotopy type of certain gauge groups*, J. Math. Kyoto Univ. **36** (1996), no. 1, 115–121. MR1381542 (97k:57040)

164. Akira Kono and Shinichi Tsuchimichi, *A remark on the homotopy type of certain gauge groups*, J. Pure Appl. Algebra **151** (2000), no. 3, 227–237. MR1776430 (2001k:55020)

165. Yasuhiro Kotani, *Note on the rational cohomology of the function space of based maps, Homology Homotopy Appl. 6* (2004), no. 1, 341–350 (electronic). MR2084591 (2005e:55014)

166. Andrzej Kozlowski and Kohhei Yamaguchi, *Topology of complements of discriminants and resultants*, J. Math. Soc. Japan **52** (2000), no. 4, 499–595. MR1774637 (2001e:55024)

167. Nicholas J. Kuhn, *Mapping spaces and homology isomorphisms*, Proc. Amer. Math. Soc. **134** (2006), no. 4, 1237–1248 (electronic), With an appendix by Greg Arone and the author. MR2196061 (2006b:55010)

168. Nicholas J. Kuhn and Mark Winstead, *On the torsion in the cohomology of certain mapping spaces*, Topology **35** (1996), no. 4, 875–881. MR1404914 (97d:55013)
169. Katsuhiko Kuribayashi, On the mod p cohomology of the spaces of free loops on the Grassmann and Stiefel manifolds, J. Math. Soc. Japan 43 (1991), no. 2, 331–346. MR1096437 (92c:55010)

170. , Module derivations and non triviality of an evaluation fibration, Homology Homotopy Appl. 4 (2002), no. 1, 87–101 (electronic). MR1937960 (2003b:55025)

171. , Eilenberg-Moore spectral sequence calculation of function space cohomology, Manuscripta Math. 114 (2004), no. 3, 305–325. MR2075968 (2005d:55025)

172. , A rational model for the evaluation map, Georgian Math. J. 13 (2006), no. 1, 127–141. MR2242331 (2007b:55015)

173. Katsuhiko Kuribayashi and Toshihiro Yamaguchi, The cohomology algebra of certain free loop spaces, Fund. Math. 154 (1997), no. 1, 57–73. MR1472851 (98j:55007)

174. , A rational splitting of a based mapping space, Algebr. Geom. Topol. 6 (2006), 309–327 (electronic). MR2220679 (2007g:55011)

175. Katsuhiko Kuribayashi and Masaaki Yokotani, Iterated cyclic homology, Kodai Math. J. 30 (2007), no. 1, 127–141. MR2242333 (2007f:55025)

176. Pascal Lambrechts, The Betti numbers of the free loop space of a connected sum, J. London Math. Soc. (2) 64 (2001), no. 1, 205–228. MR1840780 (2002f:55022)

177. , On the Betti numbers of the free loop space of a coformal space, J. Pure Appl. Algebra 161 (2001), no. 1-2, 177–192. MR1834084 (2002d:55015)

178. George E. Lang, Jr., The evaluation map and ehp sequences, Pacific J. Math. 44 (1973), 201–210. MR0341484 (49 #6235)

179. , Localizations and evaluation subgroups, Proc. Amer. Math. Soc. 50 (1975), 489–494. MR0367986 (51 #4228)

180. Jean Lannes, Sur la cohomologie modulo p des p-groupes abéliens élémentaires, Homotopy theory (Durham, 1985), London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge, 1987, pp. 97–116. MR932261 (89e:55037)

181. , Sur les espaces fonctionnels dont la source est le classifiant d’un p-groupe abélien élémentaire, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman. MR1179079 (93j:55019)

182. L. L. Larmore and E. Thomas, On the fundamental group of a space of sections, Math. Scand. 47 (1980), no. 2, 232–246. MR612697 (82h:55014)

183. A. Lazarev, The Stasheff model of a simply-connected manifold and the string bracket, Proc. Amer. Math. Soc. 136 (2008), no. 2, 735–745 (electronic). MR2358516 (2008k:55025)

184. Kee-Young Lee, Mamoru Mimura, and Moo Ha Woo, Gottlieb groups of homogeneous spaces, Topology Appl. 145 (2004), no. 1-3, 147–155. MR2100869 (2005g:55024)

185. Claude Legrand, Sur l’homologie des espaces fonctionnels, C. R. Acad. Sci. Paris Sér. I Math. 297 (1983), no. 7, 429–432. MR732851 (86a:55037)

186. , Sur les espaces fonctionnels dont la source est le classifiant d’un p-groupe abélien élémentaire, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 135–244, With an appendix by Michel Zisman. MR1179079 (93j:55019)

187. Jiayuan Lin, Rational homotopy stability for the spaces of algebraic maps, New Zealand J. Math. 38 (2008), 179–186. MR2515373

188. Gregory Lupton, Note on a conjecture of Stephen Halperin’s, Topology and combinatorial group theory (Hanover, NH, 1986/1987; Enfield, NH, 1988), Lecture Notes in Math., vol. 1440, Springer, Berlin, 1990, pp. 148–163. MR1082989 (92a:55012)

189. Gregory Lupton, N. Christopher Phillips, Claude L. Schochet, and Samuel B. Smith, Banach algebras and rational homotopy theory, Trans. Amer. Math. Soc. 361 (2009), no. 1, 267–295. MR2439407 (2009k:46086)

190. Gregory Lupton and Samuel Bruce Smith, Rank of the fundamental group of any component of a function space, Proc. Amer. Math. Soc. 135 (2007), no. 8, 2649–2659 (electronic). MR2302588 (2008b:55027)

191. , Rationalized evaluation subgroups of a map. I. Sullivan models, derivations and G-sequences, J. Pure Appl. Algebra 209 (2007), no. 1, 159–171. MR2292124 (2008c:55017)

192. , Rationalized evaluation subgroups of a map. II. Quillen models and adjoint maps, J. Pure Appl. Algebra 209 (2007), no. 1, 173–188. MR2292125 (2008c:55018)

193. , Criteria for components of a function space to be homotopy equivalent, Math. Proc. Cambridge Philos. Soc. 145 (2008), no. 1, 95–106. MR2431641 (2009k:55009)

194. , Whitehead products in function spaces: Quillen model formulae, J. Japan. Math. Soc. 62 (2010), no. 1, 49–81.
195. Michael A. Mandell, $E_{\infty}$ algebras and $p$-adic homotopy theory, Topology 40 (2001), no. 1, 43–94. MR1791268 (2001m:55025)

196. Benjamin M. Mann and R. James Milgram, Some spaces of holomorphic maps to complex Grassmann manifolds, J. Differential Geom. 33 (1991), no. 2, 301–324. MR1094457 (93e:55022)

197. Michael A. Mandell, algebras and $p$-adic homotopy theory, Topology 40 (2001), no. 1, 39–103. MR1231702 (95c:58031)

198. Ken-ichi Maruyama and Hideaki Oshima, Homotopy groups of the spaces of self-maps of Lie groups, J. Math. Soc. Japan 60 (2008), no. 3, 767–792. MR2440413 (2009h:55014)

199. Gregor Masbaum, On the moduli space of $SU(n)$ monopoles and holomorphic maps to flag manifolds, J. Differential Geom. 38 (1993), no. 1, 39–103. MR1231702 (95c:58031)

200. John McCleary, On the mod $p$ Betti numbers of loop spaces, Invent. Math. 87 (1987), no. 3, 643–654. MR874040 (88i:57018)

201. John McCleary and Dennis A. McLaughlin, Morava $K$-theory and the free loop space, Proc. Amer. Math. Soc. 114 (1992), no. 1, 243–250. MR1079897 (92e:55011)

202. John McCleary and Wolfgang Ziller, On the free loop space of homogeneous spaces, Amer. J. Math. 109 (1987), no. 4, 765–781. MR900038 (88k:58023)

203. John McCleary and Martin Raussen, Rational homotopy of spaces of maps between complex projective spaces and complex projective bundles, Pacific J. Math. 116 (1985), no. 1, 143–154. MR769828 (86k:55019)

204. Jesper Michael Møller, On spaces of maps between complex projective spaces, Proc. Amer. Math. Soc. 91 (1984), no. 3, 471–476. MR744651 (86k:55004)

205. David McDuff, The fundamental group of a symplectic manifold, J. Diff. Geom. 20 (1984), no. 2, 223–257. MR788665 (86k:55024)

206. John Milnor, On spaces having the homotopy type of a $CW$-complex, Trans. Amer. Math. Soc. 90 (1959), 272–280. MR0100267 (20 #6700)

207. Jesper Michael Møller, Nilpotent spaces of sections, Trans. Amer. Math. Soc. 303 (1987), no. 2, 733–741. MR902794 (88j:55007)

208. Spaces of sections of $Eilenberg-Mac$ $Lane$ fibrations, Pacific J. Math. 130 (1987), no. 1, 171–186. MR910659 (89d:55048)

209. Equivariant function spaces, Pacific J. Math. 142 (1990), no. 1, 103–119. MR1036731 (91a:55024)

210. Samelson products in spaces of self-homotopy equivalences, Can. J. Math. 42 (1990), no. 1, 95–108. MR1043153 (92a:55015)

211. Jesper Michael Møller and Martin Raussen, Rational homotopy of spaces of maps into spheres and complex projective spaces, Trans. Amer. Math. Soc. 292 (1985), no. 2, 721–732. MR808750 (86m:55019)
222. J. C. Moore, *On a theorem of Borsuk*, Fund. Math. **43** (1956), 195–201. MR0083123 (18,662d)

223. Bitjong Ndombol and Jean-Claude Thomas, *On the cohomology algebra of free loop spaces*, Topology **41** (2002), no. 1, 85–106. MR1871242 (2002h:57052)

224. Joseph Neisendorfer, *Lie algebras, coalgebras and rational homotopy theory for nilpotent spaces*, Pacific J. Math. **74** (1978), no. 2, 429–460. MR1494641 (80b:55010)

225. Dietrich Notbohm and Larry Smith, *Rational homotopy of the space of homotopy equivalences of a flag manifold*, Algebraic topology (San Feliu de Guíxols, 1990), Lecture Notes in Math., vol. 1509, Springer, Berlin, 1992, pp. 301–312. MR1185980 (94c:55012)

226. Iver Ottosen, *On the Borel cohomology of free loop spaces*, Math. Scand. **93** (2003), no. 2, 185–220. MR2090581 (2004h:55004)

227. Iver Ottosen and Marcel Bökstedt, *String cohomology groups of complex projective spaces*, Algebr. Geom. Topol. **7** (2007), 2165–2238. MR2366191 (2008m:55015)

228. James Michael Parks, *A note on the monoid of self-equivalences*, Houston J. Math. **7** (1981), no. 3, 403–406. MR640982 (83c:55009)

229. Frédéric Patras and Jean-Claude Thomas, *Cochain algebras of mapping spaces and finite group actions*, Topology Appl. **128** (2003), no. 2-3, 189–207. MR1956614 (2004b:55028)

230. Andrew Pressley and Graeme Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications. MR900587 (88i:22049)

231. Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR0258031 (41 #2678)

232. Jan-Erik Roos, *Homology of free loop spaces, cyclic homology and nonrational Poincaré-Betti series in commutative algebra*, Algebra—some current trends (Varna, 1986), Lecture Notes in Math., vol. 1352, Springer, Berlin, 1988, pp. 173–189. MR981826 (90f:55020)

233. John W. Rutter, *Spaces of homotopy self-equivalences*, Lecture Notes in Mathematics, vol. 1662, Springer-Verlag, Berlin, 1997, A survey. MR1474967 (98f:55005)

234. Paolo Salvatore, *Rational homotopy nilpotency of self-equivalences*, Topology Appl. **77** (1997), no. 1, 37–50. MR1443426 (98d:55011)

235. Seiya Sasao, *The homotopy of Map(CP^m, CP^n)*, J. London Math. Soc. (2) **8** (1974), 193–197. MR0346783 (49 #11507)

236. *On the homotopy of certain mapping spaces*, Kodai Math. J. **11** (1988), no. 2, 306–315. MR949137 (89k:55006)

237. Hans Scheerer, *On rationalized H- and co-H-spaces. With an appendix on decomposable H- and co-H-spaces*, Manuscripta Math. **51** (1985), no. 1-3, 63–87. MR836673 (86m:55016)

238. *An application of algebraic R-local homotopy theory*, J. Pure Appl. Algebra **91** (1994), no. 1-3, 329–332. MR1255936 (95b:55008)

239. Hans Scheerer and Daniel Tanré, *Exploration de l’homotopie modérée de W. G. Dwyer. I*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 14, 783–785. MR972081 (98f:55020)

240. *Homotopie modérée et tempérée avec les coalgébres. Applications aux espaces fonctionnels*, Arch. Math. (Basel) **59** (1992), no. 2, 130–145. MR1170636 (93f:55016)

241. M. Schlessinger and J. Stasheff, *Deformation theory and rational homotopy type*, preprint.

242. Reinhard Schultz, *Homotopy decompositions of equivariant function spaces. I*, Math. Z. **131** (1973), 49–75. MR0407866 (53 #11636)

243. Nora Seeliger, *The homotopy of Map(CP^m, CP^n)*, J. London Math. Soc. (2) **8** (1974), 193–197. MR0346783 (49 #11507)

244. Graeme Segal, *The topology of spaces of rational functions*, Acta Math. **143** (1979), no. 1-2, 37–72. MR533892 (81c:55013)

245. Katsuyuki Shibata, *On Haefliger’s model for the Gel’fand-Fuchs cohomology*, Japan. J. Math. (N.S.) **7** (1981), no. 2, 37–415. MR836673 (86m:55016)

246. H. Shiga and M. Tezuka, *Rational fibrations, homogeneous spaces with positive Euler characteristics and Jacobians*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 1, 81–106. MR894562 (89g:55019)

247. Stephen Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626. MR0112149 (22 #3004)

248. Larry Smith, *On the characteristic zero cohomology of the free loop space*, Amer. J. Math. **103** (1981), no. 5, 887–910. MR630771 (83k:57035)
249. Samuel B. Smith, *Rational homotopy of the space of self-maps of complexes with finitely many homotopy groups*, Trans. Amer. Math. Soc. **342** (1994), no. 2, 895–915. MR1225575 (94j:55012)

250. Samuel B. Smith, *Rational evaluation subgroups*, Math. Z. **221** (1996), no. 3, 387–400. MR1381587 (97a:55014)

251. Samuel B. Smith, *A based Federer spectral sequence and the rational homotopy of function spaces*, Manuscripta Math. **93** (1997), no. 1, 59–66. MR1446191 (98f:55011)

252. Samuel B. Smith, *Rational classification of simple function space components for flag manifolds*, Canad. J. Math. **49** (1997), no. 4, 895–915. MR1471062 (97a:55014)

253. Samuel B. Smith, *Rational L.S. category of function space components for F0-spaces*, Bull. Belg. Math. Soc. Simon Stevin **6** (1999), no. 2, 295–304. MR1705124 (2000g:55016)

254. Samuel B. Smith, *The rational homotopy Lie algebra of classifying spaces for formal two-stage spaces*, J. Pure Appl. Algebra **160** (2001), no. 2-3, 333–343. MR1836007 (2002f:55031)

255. Jaka Smrekar, *Compact open topology and CW homotopy type*, Topology Appl. **130** (2003), no. 3, 291–304. MR1978893 (2004c:55015)

256. V. P. Snaith, *A stable decomposition of ΩnSnX*, J. London Math. Soc. (2) **7** (1974), 577–583. MR0339155 (49 #3918)

257. E. Spanier, *Infinite symmetric products, function spaces, and duality*, Ann. of Math. (2) **69** (1959), 142–198. MR0105106 (21 #3851)

258. James Stasheff, *A classification theorem for fibre spaces*, Topology **2** (1963), 239–246. MR0154286 (27 #4235)

259. N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152. MR0210075 (35 #970)

260. Jeffrey Strom, *Miller spaces and spherical resolvability of finite complexes*, Fund. Math. **178** (2003), no. 2, 97–108. MR2029919 (2005b:55026)

261. Dennis Sullivan, *Geometric topology. Part I*, Massachusetts Institute of Technology, Cambridge, Mass., 1971, Localization, periodicity, and Galois symmetry, Revised version. MR0494074 (58 #13006a)

262. W. A. Sutherland, *Path-components of function spaces*, Quart. J. Math. Oxford Ser. (2) **34** (1983), no. 134, 223–233. MR698208 (84g:55019)

263. Shuichi Tsukuda, *A remark on the homotopy type of the classifying space of certain gauge groups*, J. Math. Kyoto Univ. **36** (1996), no. 1, 123–128. MR1381543 (97h:55017)

264. Svjetlana Terzić, *The rational topology of gauge groups and of spaces of connections*, Compos. Math. **141** (2005), no. 1, 262–270. MR2099779 (2005h:55011)

265. R. Thom, *L’homologie des espaces fonctionnels*, Colloque de topologie algébrique, Louvain, 1956, Georges Thone, Liège, 1957, pp. 29–39. MR0089408 (19,669h)

266. Jean-Claude Thomas, *Comparing the homotopy types of the components of Map(S4, BSU(2)),* J. Pure Appl. Algebra **161** (2001), no. 1-2, 235–243. MR1834088 (2002g:55013)
277. V. A. Vassiliev, *Topology of complements to discriminants and loop spaces*. Theory of singularities and its applications, Adv. Soviet Math., vol. 1, Amer. Math. Soc., Providence, RI, 1990, pp. 9–21. MR1089669 (92e:32019)

278. Micheline Vigué-Poirrier, *Dans le fibré de l’espace des lacets libres, la fibre n’est pas, en général, totalement non cohomologique à zéro*, Math. Z. **181** (1986), no. 2, 177–191. MR860369 (87f:55009)

279. _, *Rational formality of function spaces*, J. Homotopy Relat. Struct. **2** (2007), no. 1, 99–108 (electronic). MR2326935 (2008e:55015)

280. Micheline Vigué-Poirrier and Dennis Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geometry **11** (1976), no. 4, 633–644. MR0455028 (56 #13269)

281. _, *Sur l’homotopie rationnelle des espaces fonctionnels*, Manuscripta Math. **56** (1986), no. 2, 177–191. MR850369 (87h:55009)

282. Craig Westerland, *Dyer-Lashof operations in the string topology of spheres and projective spaces*, Math. Z. **250** (2005), no. 3, 711–727. MR2179618 (2006h:55017)

283. _, *Stable splittings of surface mapping spaces*, Topology Appl. **153** (2006), no. 15, 2834–2865. MR2248387 (2007g:55006)

284. _, *String homology of spheres and projective spaces*, Algebr. Geom. Topol. **7** (2007), 309–325. MR238367 (2002f:55024)

285. George W. Whitehead, *On products in homotopy groups*, Ann. of Math (2) **47** (1946), 460–475. MR0016672 (8,50b)

286. C. Wockel, *The Samelson product and rational homotopy for gauge groups*, Abh. Math. Sem. Univ. Hamburg **77** (2007), 219–228. MR2379340 (2008k:57064)

287. Moo Ha Woo and Yeon Soo Yoon, *T-spaces by the Gottlieb groups and duality*, J. Austral. Math. Soc. Ser. A **59** (1995), no. 2, 193–203. MR1346627 (96e:55009)

288. Kohhei Yamaguchi, *On the rational homotopy of \( \text{Map}(H^m, H^n) \)*, Kodai Math. J. **6** (1983), no. 3, 279–288. MR717319 (85j:55024)

289. _, *Spaces of polynomials with real roots of bounded multiplicity*, J. Math. Kyoto Univ. **42** (2002), no. 2, 249–259. MR1838367 (2002f:55024)

290. _, *Spaces of free loops on real projective spaces*, Kyushu J. Math. **59** (2005), no. 1, 145–153. MR2124058 (2006a:55011)

291. _, *Fundamental groups of spaces of holomorphic maps and group actions*, J. Math. Kyoto Univ. **44** (2004), no. 3, 479–492. MR2103778 (2005h:55011)

292. _, *Spaces of holomorphic maps with bounded multiplicity*, Q. J. Math. **52** (2001), no. 2, 249–259. MR1838367 (2002f:55024)

293. _, *Spaces of self-homotopy equivalences for fibre spaces*, Publ. Res. Inst. Math. Sci. **22** (1986), no. 1, 43–56. MR834347 (87g:55002)

294. _, *On the genus of free loop fibrations over \( F_0 \)-spaces*, Int. J. Math. Math. Sci. (2004), no. 65-68, 3617–3619. MR2128776 (2006i:55007)

295. _, *On the space of self-homotopy equivalences of the projective plane*, J. Math. Soc. Japan **45** (1993), no. 3, 489–494. MR1219881 (94f:55007)
305. Yeon Soo Yoon, *Decomposability of evaluation fibrations*, Kyungpook Math. J. 35 (1995), no. 2, 361–369. MR1369453 (97a:55018)

306. Alex Zabrodsky, *On spaces of functions between classifying spaces*, Israel J. Math. 76 (1991), no. 1-2, 1–26. MR1177330 (93i:55020)

307. Wolfgang Ziller, *The free loop space of globally symmetric spaces*, Invent. Math. 41 (1977), no. 1, 1–22. MR0649625 (58 #31198)

Department of Mathematics, Saint Joseph’s University, Philadelphia, PA 19131

E-mail address: smith@sju.edu