Localization of heterotic anomalies on various hyper surfaces of \(T^6/\mathbb{Z}_4\)

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Abstract

We investigate the structure of local anomalies of heterotic \(E_8 \times E_8'\) theory on \(T^6/\mathbb{Z}_4\). We show that the untwisted states lead to anomalies in ten, six and four dimensions. At each of the six dimensional fixed spaces of this orbifold the twisted states ensure, that the anomalies factorize separately. As some of these twisted states live on \(T^2/\mathbb{Z}_2\), they give rise to four dimensional anomalies as well. At all four dimensional fixed points at worst a single Abelian anomaly can arise. Since the anomalies in all these dimensions factorize in a universal way, they can be canceled simultaneously. In addition, we show that for all U(1) factors at the four dimensional fixed points at least logarithmically divergent Fayet–Ilopoulos tadpoles are generated.

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1 Introduction

In this paper we investigate the structure of local anomalies on the orbifold $T^6/Z_4$ within the context of heterotic $E_8 \times E_8'$ string theory. Strings on orbifolds were discussed first by the authors of refs. [1, 2] and with the inclusion of non-trivial gauge field backgrounds, so-called Wilson lines, in [3, 4, 5]. More recently there has also been a lot of attention to compactifications on orbifolds in field theory. 

An important development was the investigation of the shape of anomalies on orbifolds. First in ref. [6] the anomalies on $S^1/Z_2$ were computed and it was found that they localize at the fixed points of this orbifold. Afterwards, various groups computed anomalies on the orbifolds $S^1/Z_2$, $S^1/Z_2 \times Z'_2$ [7, 8, 9, 10]. More general anomaly investigations, that apply to higher dimensional orbifolds, have been pursued in ref. [11, 12, 13].

The question of the shape of anomalies in the context of heterotic string theory compactified on $T^6/Z_3$ has been investigated in ref. [14]. This six dimensional orbifold has only zero dimensional fixed points, which may support twisted states. It was found that non–Abelian anomalies never arise at these four dimensional fixed points. However, at each fixed point a single anomalous U(1) is possible, not necessarily the same at each fixed point. In ref. [15] it was shown, that a local four dimensional remnant of the Green–Schwarz mechanism [16] cancels these Abelian anomalies.

The existence of a global anomalous U(1) is associated with the generation of a Fayet–Iliopoulos tadpole, which leads to spontaneous breaking of the global anomalous U(1) [14, 15, 18]. The existence of Fayet–Iliopoulos tadpoles on orbifolds, like $S^1/Z_2 \times Z'_2$, was realized in [20] and the shapes of these tadpoles over such orbifolds have been computed in refs. [21, 7, 8]. In the heterotic models on $T^6/Z_3$ local Fayet–Iliopoulos tadpoles are also generated [15]. However, still only a global one necessarily leads to spontaneous breaking. The full consequences of the local structure of these tadpoles have not been fully understood yet, but they may lead to dynamical instabilities as was discussed in five dimensional models on $S^1/Z_2$ [22, 10].

This paper continues the investigation of the papers [14, 15] of local anomalies in the context of the heterotic string. We have chosen to work on the non–prime orbifold $T^6/Z_4$, because it is the simplest six dimensional orbifold, which contains fixed hyper surfaces of various dimensions. We confirm the expectation that on both four and six dimensional fixed spaces anomalies localize. Furthermore, we show that the Green–Schwarz mechanism can cancel the local four, six and ten dimensional anomalies simultaneously. And in addition, we compute the Fayet–Iliopoulos tadpoles for U(1) factors at the four dimensional fixed points.

The paper is organized as follows: In section 2 we describe the geometry of the orbifold $T^6/Z_4$, focusing in particular on the fixed point structure. Next, we investigate the local spectra at these four and six dimensional fixed hyper surfaces: This includes both, the projections of untwisted states at the fixed points, as well as the twisted states that may live there. To calculate the local structure of anomalies, we first develop general orbifold traces in section 4 and explain how they can be applied to anomalies. After that, we collect the local anomaly contributions of both untwisted and twisted states at the four and six dimensional hyper surfaces. We describe the local version of the Green–Schwarz mechanism which cancels these local factorized anomalies. In section 6 we calculate the tadpoles associated with the (anomalous) U(1)’s at the four dimensional fixed points. Our conclusions have been collected in section 7. We have attached three appendices to this work: Appendix A is devoted to a description of spinors in various relevant dimensions using light–cone gauge. The next appendix gives some background on supergravity multiplets in six dimensions. In appendix C we describe how $Z_2$ and $Z_4$ gauge shifts can be classified.
Figure 1: An impression of the two dimensional fixed hyper surfaces within the orbifold $T^6/\mathbb{Z}_4$ are displayed: the bottom square represents a part of a two dimensional cross section (the $z_2$ plane) of the orbifold $T^4/\mathbb{Z}_4$. Above its fixed points $0$ and $\frac{1}{2}(1+i)$ in the $z_1$ direction one finds the orbifolds $T^2/\mathbb{Z}_2$. Because of the identification of the points $\frac{1}{2}$ and $\frac{1}{2}i$ the two–tori above them are mapped to each other via (7).

2 Geometry of $T^6/\mathbb{Z}_4$

We begin by reviewing the geometry of the orbifold $T^6/\mathbb{Z}_4$, a related discussion can be found in e.g. [23]. Let $\Gamma$ be the lattice generated by $z_j \sim z_j + R_j$, $z_j \sim z_j + i R_j$ for $j = 1, 2, 3$ on the coordinates $z = (z_1, z_2, z_3) \in \mathbb{C}^3$. This is the $\text{SO}(5) \times \text{SO}(5) \times \text{SU}(2) \times \text{SU}(2)$ lattice. (A general classification of orbifold compactification lattices can be found in [24, 25].) We obtain the torus $T^6 = \mathbb{C}^3/\Gamma$ by dividing out this lattice. The $\mathbb{Z}_4$ twist operator $\Theta$ acts on the complex coordinates as

$$\Theta(z_1, z_2, z_3) = (-z_1, iz_2, iz_3), \quad \Theta^4 = 1.$$  

(1)

For simplicity we have made the restriction to also only consider a square torus in the first complex direction, even though the orbifolding does not require this.

As the structure of fixed points is rather complicated, we introduce the following notation: $\zeta_{pq} = (p + i q) / 2$ for $p, q \in \{0, 1\}$. It is not hard to show that

$$i \zeta_{pq} = \zeta_{qp} - q, \quad i^2 \zeta_{pq} = \zeta_{pq} - p - i q, \quad i^3 \zeta_{pq} = \zeta_{qp} - i p.$$  

(2)

The fixed points of the twists $\Theta$ and $\Theta^3$ are zero dimensional. On these fixed points four dimensional states may arise in the heterotic theory, as will be discussed in 3.4

There are 16 different $\mathbb{Z}_4$ fixed points:

$$\{3^4_{pq}\} = \{(R_1 \zeta_{p_1q_1}, R_2 \zeta_{p_2q_2}, R_3 \zeta_{p_3q_3})\}.$$  

(3)

Using the identities (2) it is straightforward to work out which lattice shifts are needed to make these fixed points invariant within the covering space $\mathbb{C}^3$ under the orbifold twist:

$$\begin{align*}
\Theta 3^4_{pq} &= 3^4_{pq} - ((p_1 + i q_1)R_1, p_2 R_2, p_3 R_3), \\
\Theta^2 3^4_{pq} &= 3^4_{pq} - (0, (1 + i)p_2 R_2, (1 + i)p_3 R_3), \\
\Theta^3 3^4_{pq} &= 3^4_{pq} - ((p_1 + i q_1)R_1, ip_2 R_2, ip_3 R_3).
\end{align*}$$  

(4)
These shifts are important since they distinguish the different fixed points when Wilson lines are present: In section 3.4 we will see that these shifts determine the local spectra at the fixed points.

The $\mathbb{T}^2$ fixed hyper surfaces take the form of 16 disjoint $T^2$: $\{(z_1, \mathbb{Z}_{p,q}) \mid z_1 \in T^2\} = \{(z_1, R_2\zeta_{p2q2}, R_3\zeta_{p3q3}) \mid z_1 \in T^2\}$, \hspace{1cm} (5)

where the two–torus $T^2$ is defined by $z_1 \sim z_1 + R_1 \sim z_1 + iR_1$. Each two–torus $\{(z_1, \mathbb{Z}_{p,q}) \mid z_1 \in T^2\}$ can support six dimensional twisted states, which will be determined in section 3.3. The local shifts that bring the fixed points back to themselves in the covering space read

$$\Theta^2 \zeta^2_{p,q} = \zeta^2_{p,q} - (p_2 + iq_2)R_2, (p_3 + iq_3)R_3. \hspace{1cm} (6)$$

Since on the fixed space of $\Theta^2$ the $\mathbb{Z}_4$ twist acts non–trivially, the embedding of this fixed space in the orbifold $T^6/\mathbb{Z}_4$ is somewhat more involved. (In figure 11 we have given an artist’s impression of the configuration.) The actions of $\Theta$ and $\Theta^3$ on this space take the form

$$\Theta (z_1, \mathbb{Z}_{p,q}) = (-z_1, \mathbb{Z}_{p,q}) - (0, q_2R_2, q_3R_3), \hspace{1cm} (7)$$

$$\Theta^3 (z_1, \mathbb{Z}_{p,q}) = (-z_1, \mathbb{Z}_{p,q}) - i(0, p_2R_2, p_3R_3).$$

Notice that the order of $p$ and $q$ is interchanged, therefore it is important to distinguish between the $\mathbb{Z}_2$ fixed points with the vectors $p$ and $q$ equal or not, denoted by $\mathbb{Z}_{p=q}$ and $\mathbb{Z}_{p\neq q}$, respectively. The twist $\Theta$ leaves $\mathbb{Z}_{p=q}$ invariant, and hence the corresponding four two–tori are orbifolded to

$$\{(z_1, \mathbb{Z}_{p=q}) \mid z_1 \in T^2/\mathbb{Z}_2\} = \{(z_1, R_2\zeta_{p2q2}, R_3\zeta_{p3q3}) \mid z_1 \in T^2/\mathbb{Z}_2\}. \hspace{1cm} (8)$$

As each orbifold $T^2/\mathbb{Z}_2$ itself has four fixed points $R_i\zeta_{q_1}$, the fixed points of all four disjunct orbifolds together is precisely the same as all fixed points $\mathbb{Z}_{p,q}$ of the original orbifold $T^6/\mathbb{Z}_4$. On the other 12 two–tori the twist $\Theta$ acts freely; this leads to an identification of pairs of two–tori $(z_1, \mathbb{Z}_{p\neq q})$ and $(-z_1, \mathbb{Z}_{q\neq p})$ in the covering space $\mathbb{C}^3$ of the orbifold $T^6/\mathbb{Z}_4$. In other words, within the orbifold $T^6/\mathbb{Z}_4$ these spaces really only consist of six two–tori. In the covering space $T^6$ these two–tori are indicated by

$$\{(z_1, \mathbb{Z}_{p\neq q}) \oplus (-z_1, \mathbb{Z}_{q\neq p}) \mid z_1 \in T^2\}. \hspace{1cm} (9)$$

As this is a collection of two–tori, they do not have any orbifold singularities. Notice that non of these two–tori contain the fixed points $\mathbb{Z}_{p,q}$.

We close our discussion with a few comments concerning the orbifold $T^4/\mathbb{Z}_2$. To gain insight in some properties of heterotic string theory on $T^6/\mathbb{Z}_4$, the relation to the four dimensional orbifold $T^4/\mathbb{Z}_2$ turns out to prove very useful. For this reason we collect here the essential geometrical properties of this orbifold as well. We take $T^2 \times T^4$ to be described by the same lattice as $T^6$ above. (For a general $T^4/\mathbb{Z}_2$ orbifold of course we do not need to take a square lattice in order that the $\mathbb{Z}_2$ can act consistently on it. For comparison purposes between the theories on $T^6/\mathbb{Z}_4$ and $T^4/\mathbb{Z}_2$, we restrict ourselves to square $T^4$‘s only.) The $\mathbb{Z}_2$ orbifold twist acts on the $T^4$ as $\Theta^2(z_2, z_3) = (-z_2, -z_3)$. It follows that the fixed points of $T^4/\mathbb{Z}_2$ are given in 11. The local shifts needed to bring these fixed points back to themselves within the covering space $T^4$ are given in 11.
3 Local ten, six and four dimensional string spectra

The central purpose of this section is to determine the string spectrum of the heterotic $E_8 \times E_8'$ theory on $T^6/\mathbb{Z}_4$. These models may contain arbitrary gauge shift and Wilson lines. This makes the zero mode analysis of these models rather complicated. However, as our analysis here will show the investigation of the local spectra is relatively straightforward. In particular we are interested in the local twisted and untwisted states at the six and four dimensional fixed hyper surfaces discussed in section 2.

### 3.1 Ten dimensional states

The full ten dimensional spectrum of the heterotic $E_8 \times E_8'$ string theory appears on the interior of the orbifold $T^6/\mathbb{Z}_4$; away from the fixed hyper surfaces. At the fixed hyper surfaces a large number of states do not survive the local orbifold projections, as we will discuss in later subsections. As this ten dimensional spectrum is well-known, we will be brief at this point.

In table 1, we have summarized the ten dimensional zero mode spectrum of the string on the interior of the orbifold $T^6/\mathbb{Z}_4$ using light–cone gauge. In this table $|0\rangle_{\tilde{N}S}$ denotes the right–moving Neveu–Schwarz vacuum, and $|S_1,...,S_4\rangle_{\tilde{R}}$ the spinorial Ramond vacuum with positive chirality: $S_i = \pm 1/2$ and $\prod S_i > 0$. Here it is used, that the light–cone gauge automatically takes the ten dimensional Majorana condition into account. (See appendix A where we review how ten (and lower) dimensional spinors can be represented on the light–cone.) In table 1, $\alpha_{-1}^M \tilde{\psi}_{\frac{N}{2}}$ and $\tilde{\psi}_{\frac{M}{2}}$ denote the creation operators of string world sheet scalars and right–moving fermions, with $M,N$ spacetime indices. A light–cone spacetime vector, obtained from the $\tilde{N}S$ vacuum, is indicated by $|\pm 1,0,0,0\rangle_{\tilde{N}S} = \tilde{\psi}_{-1/2}^M |0\rangle_{\tilde{N}S}$. This notation will be extended to the twisted states in sections 3.3 and 3.4. And finally, $\text{Ad}_{[0]}$ denotes the adjoint representation of $E_8 \times E_8'$, which consists of the Cartan generators $H_I$ of $SO(16) \times SO(16)'$, the adjoint roots $w = (\pm 1,\pm 1,0^6)$ and spinorial roots $w = (\pm 1/2,...,\pm 1/2)$ (such that the product of the entries is positive) of $SO(16)$ and $SO(16)'$. For more details we refer to [26, 27]. (The notation with the subscript $[0]$ will become useful, since we will later define $\text{Ad}_{[v]}$.)

The $N = 1$ supergravity part of the spectrum consists of a graviton (i.e. the metric (perturbation)) $g_{MN}$, a scalar dilaton $\phi$, an antisymmetric tensor $B_{MN}$, and a left–handed gravitino $\tilde{\psi}^M$ and a right–handed dilatino $\lambda$. These states are simply the decomposition of the string states indicated in table 1 in irreducible $SO(8)$ representations. In particular, the gravitino constraint $\Gamma_M \psi^M = 0$ implies that

| $g_{MN}, B_{MN}, \phi$ : $\alpha_{-1}^M \tilde{\psi}_{\frac{N}{2}} |0\rangle_{\tilde{N}S}$ | $\psi^M, \lambda$ : $\alpha_{-1}^M |\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\rangle_{\tilde{R}}$ |
| $A_M$ : $|\text{Ad}_{[0]}\rangle \otimes |\pm 1,0,0,0\rangle_{\tilde{N}S}$ | $\chi$ : $|\text{Ad}_{[0]}\rangle \otimes |\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}\rangle_{\tilde{R}}$ |

Table 1: The ten dimensional zero mode spectrum of the heterotic string is identified using light–cone gauge $(M,N = 2,\ldots,9)$ in terms of string oscillators and vacua.
the chiralities of gravitino and dilatino are opposite. This supergravity multiplet is coupled to a ten
dimensional super Yang–Mills theory, which contains an \( \text{E}_8 \times \text{E}_8' \) gauge field \( A_M \) and a left–handed gaugino \( \chi \).

### 3.2 Shift and Wilson lines

The transformation properties of the ten dimensional supergravity sector are directly determined by specifying that the orbifold \( \Theta \) acts on an \( \text{SO}(8) \) representation as

\[
\Theta |S_1, \ldots, S_4\rangle = e^{2\pi i \phi} |S_1, \ldots, S_4\rangle, \quad \phi = \frac{1}{4} (0, -2, 1, 1).
\]

Notice that this is consistent with the action on the spacetime coordinates (also in light–cone gauge of course).

For the ten dimensional super Yang–Mills sector one can allow for more choices: gauge shift \( v \) and Wilson lines \( a_j \) and \( \tilde{a}_j \) are free up to certain requirements which we recall below. Let \( j \) denote the lattice vector of length \( R_j \) in the \( j \)th direction, we have for the \( \text{E}_8 \times \text{E}_8' \) gauge connection one–form:

\[
A_1(z + j) = T_j A_1(z) T_j^{-1}, \quad A_1(z + i j) = \tilde{T}_j A_1(z) \tilde{T}_j^{-1}, \quad A_1(\Theta z) = U A_1(z) U^{-1}.
\]

The group elements \( T_j, \tilde{T}_j \) and \( U \) are assumed to be commuting; they are all generated by the Cartan elements \( H_I \) of \( \text{SO}(16) \times \text{SO}(16)' \subset \text{E}_8 \times \text{E}_8' \). Their expressions in terms of gauge shift and Wilson lines read:

\[
T_j = e^{2\pi i a_j^I H_I}, \quad \tilde{T}_j = e^{2\pi i \tilde{a}_j^I H_I}, \quad U = e^{2\pi i v^I H_I}.
\]

The other generators of the gauge group \( \text{E}_8 \times \text{E}_8' \) are denoted by \( E_w \), where \( w \) are the roots of this algebra (described above). In the Cartan–Weyl basis we have the canonical commutation and conjugation relations

\[
[H_I, E_w] = w_I H_I, \quad e^{2\pi i t^I H_I} E_w e^{-2\pi i t^I H_I} = e^{2\pi i t^I w_I} E_w,
\]

for any \( t^I \in \mathbb{R} \). Because of the compatibility of the orbifold twist and the torus periodicities \([2, 4, 24, 25]\), we find that:

\[
\tilde{a}_j = a_j, \quad 4v^I w_I = 2a_j^I w_I = 2a_j^I w_I = 2\tilde{a}_j^I w_I = 0 \mod 1
\]

for \( j = 2, 3 \) and for all roots \( w \). There are additional conditions on gauge shift and Wilson lines coming from modular invariance of the string theory. We discuss them below (in equation \([17]\)), since there we have introduced sufficient ingredients to describe them naturally.

At the fixed hyper surfaces of the orbifold described in section\([2]\) the combination of gauge shift and Wilson lines can lead to different local projections of the untwisted gauge states, i.e. gauge fields and gauginos \([14]\). At the fixed points \( \mathcal{Z}_{pq}^4 \) and \( \mathcal{Z}_{pq}^2 \) the gauge field one–form satisfies

\[
A(\mathcal{Z}_{pq}^4) = R_{pq}^4 A(\mathcal{Z}_{pq}^4) (R_{pq}^4)^{-1}, \quad A(z_1, \mathcal{Z}_{pq}^2) = R_{pq}^2 A(z_1, \mathcal{Z}_{pq}^2) (R_{pq}^2)^{-1}.
\]

The second condition applies to the pairwise identified two–tori and the interior of the orbifolded two–tori. Since the fixed points of these orbifolded two–tori are the fixed points \( \mathcal{Z}_{pq}^4 \), there the first
requirement is again obtained. The local projection matrices $R^4_{pq} = \exp(2\pi i v^4_{pq} I H_I)$ and $R^2_{pq} = \exp(2\pi i v^2_{pq} I H_I)$ can be expressed in terms of the local shift vectors

$$v^4_{pq} = p_1 a_1 + q_1 a_1 + p_2 a_2 + p_3 a_3 + v, \quad v^2_{pq} = (p_2 + q_2) a_2 + (p_3 + q_3) a_3 + 2v. \quad (16)$$

These local shifts also determine the four and six dimensional twisted states as we describe below. At the two tori which are identified, the projections should of course be the same: This is indeed the case, since $v^2_{pq} = v^2_{q,p}$. Moreover, notice that because $2a_1^2 w_I = 2a_3^2 w_I = 0 \mod 1$, all local shifts at the four orbifolds $T^2/Z_2$ are equal: $v^2_{pq} = 2v$ (up to lattice shifts).

Because at all fixed (four and six dimensional) hyper surfaces the theory should correspond to a consistent string model, all these local shift vectors need to satisfy the modular invariance conditions

$$Z_4 : 4(\phi^2 - (v^4_{pq})^2) = 0 \mod 2, \quad Z_2 : 2(2\phi^2 - (v^2_{pq})^2) = 0 \mod 2. \quad (17)$$

Not all these conditions are independent: The $Z_2$ level matching conditions for $v^2_{pq} = q^2_{pq}$ are automatically satisfied, provided that all $v^4_{pq}$ fulfill the $Z_4$ conditions. However, the $Z_2$ conditions for $v^2_{p\neq q}$ give extra independent relations in general. Using that the conditions in (17) hold for all $p_i, q_i = 0, 1$ with $i = 1, 2, 3$ we find the requirements

$$2(\phi^2 - v^2) = a_2^2 = a_3^2 = 2a_1^2 = 2\tilde{a}_1^2 = 0 \mod 1,$$

$$2a_2a_3 = 4a_1a_2 = 4a_1a_3 = 4\tilde{a}_1a_2 = 4\tilde{a}_1a_3 = 0 \mod 1,$$

$$4v\tilde{a}_1 = 4va_1 = 4va_2 = 4va_3 = 0 \mod 1. \quad (18)$$

For the related orbifold models on $T^4/Z_2$ we take spacetime twist and gauge shifts to be $2\phi$ and $2v$, respectively. Furthermore the Wilson lines in the real and imaginary $z_2$ and $z_3$ directions, are respectively $a_2$ and $a_3$. Hence, we find the same local gauge shift $v^2_{pq}$ as given in (16) with the modular invariance requirement (17).

### 3.3 Local six dimensional spectra

The discussion of the local shift vectors in the preceding section, allows us to make an inventory of the local states at the fixed hyper surfaces of the orbifold $T^6/Z_4$. In this subsection we consider the six dimensional states. These states live on four dimensional Minkowski space times, either two identified two–tori, or the orbifold $T^2/Z_2$. However, as far as the classification of six dimensional states is concerned, the local spectra at the fixed points are completely determined by the local $Z_2$ shift vectors $v^2_{pq}$. Therefore, for a given fixed six dimensional hyper surface $Z^2_{pq}$ of $T^6/Z_4$, we can consider an equivalent pure orbifold model (i.e. without Wilson lines) on $T^4/Z_2$ with this gauge shift to determine the local spectrum there. The method of using equivalent models to determine the local spectra with Wilson lines present, was employed for the $T^6/Z_3$ orbifold in [13]. However, we should emphasize here, that in the present case we make identifications between spectra at fixed points of two theories on different orbifolds. Although for spectra this method works, one should be aware of important differences when computing anomalies (and other traces of local operators) on these different spaces. We will return to this important issue in section [14] where traces on these orbifolds are evaluated.

Let $V_2$ be the gauge shift of a pure orbifold $T^4/Z_2$ model with spacetime shift $2\phi$. (We will denote the classifying shifts with capital letters: $V_2$ and $V_4$ for the $Z_2$ and $Z_4$ orbifolding, respectively.) The string spectrum can be divided in untwisted and twisted states. The untwisted states can be
Table 2: The local six dimensional spectrum of the heterotic string is given in terms of string oscillators and vacua. This spectrum is situated at the fixed points of $T^4/\mathbb{Z}_2$ and six dimensional fixed hyper surfaces of $T^6/\mathbb{Z}_4$. The following indices are used: light–cone: $m,n=2,...,5$, internal $T^4$: $a,b=i,\bar{i}=2,3,\bar{2},\bar{3}$, and spinor: $\alpha,\beta=\pm$. (To make the multiplet structure more manifest we have used the identification of internal space and spinor indices.)

| no. $2(V_2)^2$ | shift $2V_2$ | gauge group $\text{Ad}_{[V_2]}$ | untwisted matter $\text{R}_{[V_2]}$ |
|-----------------|--------------|-----------------|-----------------|
| 0               | (00000000)   | $\text{E}_8$    | nothing         |
| 1,3             | (11000000)   | $\text{E}_7 \times \text{SU}(2)$ | (56, 2)         |
| 2               | (20000000)   | $\text{SO}(16)$ | $128_s$         |

Table 3: The resulting gauge groups and six dimensional untwisted matter representation are given for representatives of the possible $\mathbb{Z}_2$ gauge shifts. General shifts are classified by computing $2V_2^2$ mod 4 as is discussed in appendix C.1. (Since gauge shifts with $2V_2^2=1,3$ mod 4 are just related to each other via some lattice shift, we will use only the symbol “1” for classification purposes.)
understood as those ten dimensional states that survive the $Z_2$ orbifolding, or can be obtained by a direct string calculation of untwisted six dimensional zero modes. In any case, these states are invariant under the orbifold action. Using the SO(8) light–cone representations given in section 3.3.1 and their six dimensional decompositions [3.1.1] of appendix B, we can classify the untwisted states according to the six dimensional supergravity multiplets reviewed in table [3.2.1] of the same appendix. For the states coming from the ten dimensional supergravity sector, this is rather straightforward. The ten dimensional gauge multiplet states give rise to six dimensional gauge multiplets in the adjoint $\text{Ad}_{[V_2]}$ of the gauge group $G_{[V_2]}$ unbroken by the orbifolding, and to charged hyper multiplets in representation $R_{[V_2]}$. These representations are defined by

$$\text{Ad}_{[V_2]} = \{ H_I \} \oplus \{ E_8 \text{ roots } w \ | \ V_2^I w_I = 0 \mod 1 \},$$

$$R_{[V_2]} = \{ E_8 \text{ roots } w \ | \ V_2^I w_I = \frac{1}{2} \mod 1 \}. \tag{19}$$

These definitions only apply to a single $E_8$, but can, of course, be easily extended to $E_8 \times E_8'$. (In the following we use this notation for both situations, assuming that the context makes clear whether one is concerned with a single or both $E_8$’s.) In table 2 we have summarized the full untwisted spectrum of the theory in six dimensions and indicated to which string states they correspond. Since the untwisted matter states contain 4 + 4 degrees of freedom, they naturally fall into hyper multiplets in gauge group representation $R_{[V_2]}$. To emphasize, this spectrum corresponds to the local spectrum at the fixed points of $T^4/Z_2$ but likewise to the spectrum at the $T^2$’s and $T^2/Z_2$’s within $T^6/Z_4$.

In addition to the untwisted spectrum, there exist twisted modes, which correspond to additional string states that are massless because of the orbifolding. As we will need similar formulae for the $Z_4$ twisted states, we describe here the masslessness conditions for a $k$th twisted sector of a $Z_N$ orbifold. (The 0th twisted sector gives the untwisted spectrum which we already characterized.) The masslessness conditions for the $k$th twisted sector of a $Z_N$ orbifold with gauge shift $V$ and spacetime twist $\phi$ read

$$\frac{1}{2} (w - k V)^2 + N_L^{(k)} + c(k) - 1 = 0, \quad \frac{1}{2} (\omega + k \phi)^2 + N_R^{(k)} + c(k) - \frac{1}{2} = 0, \tag{20}$$

with

$$c_k \equiv \frac{1}{2} \sum_{i=1}^4 \eta_i^i (1 - \eta_i^i), \quad \eta_i^i \equiv |k \phi^i| - \text{Int}|k \phi^i|. \tag{21}$$

The left and right–moving oscillator numbers $N_L^{(k)}$ and $N_R^{(k)}$ are fractionally quantized in terms of the modding of the various left– and right–moving world sheet fields. There are only a few possibilities for these oscillator numbers, since for massless states they are bounded from above by $1 - c(k)$ and $1/2 - c(k)$, respectively. The weights $w$ in the $E_8 \times E_8'$ root lattice, which satisfy the above requirements, define left–moving vacua $|w\rangle$. Likewise, the weights $\omega$ of the SO(8) root lattice (possibly shifted by the spinorial root $(1,1)$) determine the right–moving vacua $|\omega\rangle$. In addition the tensor products of the left– and right–moving states need to fulfill the generalized GSO projection:

$$\mathcal{P}^{(k)} \exp 2\pi i \left\{ \frac{k}{2} (V^2 - \phi^2) + (w - k V)V + (\omega + k \phi)\phi \right\} = 1, \tag{22}$$

where the $\mathcal{P}^{(k)}$ denote the phases due to (fractional) oscillator contributions.
Table 4: There are two modular invariant combinations of $Z_2$ gauge shifts. The labels $(V_2; V_2')$ correspond to the first column of table 3. The resulting gauge group and the twisted matter at a single fixed point is given.

Back to the six dimensional twisted states of $Z_2$. Since $c_{(1)} = 1/4$, the only possibility in the right–moving sector is $N_{L}^{(1)} = 0$ so that $(\omega + 2\phi)^2 = 1/2$. The GSO projection then only allows the vacuum states $|0, 0, 0, 0, 0, 0\rangle_{NS}$ and $|\alpha, \frac{\alpha}{2}, \frac{\alpha}{2}, 0, 0\rangle_{R}$. For the left–movers we have two options:

\[
N_{L}^{(1)} = 0 : \quad S_{[V_2]} = \{ w \mid \frac{1}{2}(w - V_2)^2 = \frac{3}{4} \}; \quad N_{L}^{(1)} = \frac{1}{2} : \quad D_{[V_2]} = \{ w \mid \frac{1}{2}(w - V_2)^2 = \frac{1}{4} \}. \tag{23}
\]

Since the latter states are obtained by acting with the creation operators $\alpha^{a}_{-1/2}$ on the vacuum, these states transform as $(2, 2) \otimes (2, 1) = (3, 2) \oplus (1, 2)$ under $SU(2)_R \times SU(2)_H$. In table 2 we have summarized this twisted spectrum as well.

Let us now discuss what the possible local six dimensional models are, by specifying modular invariant gauge shifts. In appendix C.1 we show that there are essentially three different $Z_2$ gauge shift vectors within a single $E_8$, which can be classified by the value of $2(V_2)^2$, and are listed in table 3. We label these possible gauge shifts by their value for $2(V_2)^2$. This could in principle lead to six different models, when the gauge shifts for both $E_8$’s are combined. However, the $Z_2$ level matching condition 14 only allows essentially two combinations, which are given in table 4. As the arising gauge group already determines the untwisted states according to table 3, we only have given the gauge representations of the twisted states in table 4. The twisted states in representation $S_{[V_2]}$ correspond only to $2 + 2$ degrees of freedom according to table 2, which would be too little to fill six dimensional hyper multiplets. But from table 4 we may read off that all representations $S_{[V_2]}$ are pseudo real, hence as discussed in appendix B these representation come with an anti–symmetric matrix, which allows one to form hyper multiplets with half their canonical degrees of freedom. For the other twisted states in representation $D_{[V_2]}$ there are two ways to form such half–hyper multiplets, since they fall in a doublet of the $SU(2)_H$ and the $SU(2)$ gauge group, see tables 2 and 4.

### 3.4 Local four dimensional spectra

After the discussion of the local spectra at the six dimensional fixed hyper surfaces of $T^6/Z_4$, the next task is to determine the four dimensional spectra at the fixed points $3^4_{pq}$ of $T^6/Z_4$. Our strategy will be essentially the same as before: we consider equivalent pure orbifold models, and use those to infer what the local spectra are in models with Wilson lines. However, there are a couple of additional complications now: We have to consider three sectors: ten and six dimensional states, which are projected at these fixed points, and genuine four dimensional twisted states. We would like to identify those four dimensional twisted states using the conditions for massless zero modes on $T^6/Z_4$. It is well–known that for a $Z_4$ there are both single and double twisted zero modes.
Table 5: The local four dimensional spectrum at the fixed points of $T^6/\mathbb{Z}_4$ of the heterotic string is given in terms of oscillators and vacua. The following indices are used: light–cone: $\mu, \nu = 2, \ldots, 3$, internal $T^6$: $1, j = 2, 3; \underline{1}, \underline{j} = 2, 3$, and spinor: $\alpha, \beta = \pm$.

| No. $8V_4^2$ | shift $4V_4$ | gauge group $A_\mu$ $Ad_{[V_4]}$ | untwisted matter $r_{[V_4]}$ | $R_{[V_4]}$ |
|--------------|-------------|---------------------------------|-----------------|------|
| 0            | (000000000) | $E_8$                            | nothing         | nothing |
| 1            | (110000000) | $E_7 \times U(1)$                | $\left(56\right)_1$ | $(1)_2 + (1)_2$ |
| 2            | (200000000) | $SO(14) \times U(1)$             | $(64)_1$        | $(14c)_2 + (14c)_2$ |
| 3            | (211000000) | $E_6 \times SU(2) \times U(1)$   | $(27, 2)_1 + (1, 2)_3$ | $(27, 1)_2 + (27, 1)_2$ |
| 4            | (220000000) | $E_7 \times SU(2)$               | nothing         | $(56, 2)$ |
| 4\_1         | (111111111) | $SU(8) \times U(1)$              | $(56)_1 + (8)_3$ | $(28)_2 + (28)_2$ |
| 5            | (310000000) | $SO(12) \times SU(2) \times U(1)$ | $(32_1 s)_1 + (12_1 v)_2$ | $(32c)_1 + (1, 1)_2 + (1, 1)_2$ |
| 6            | (222000000) | $SO(10) \times SU(4)$            | $(16, 4)$       | $(10, 6)$. |
| 7            | (311111100) | $SU(8) \times SU(2)$             | $(28, 2)$       | $(70, 1)$ |
| 8            | (400000000) | $SO(16)$                         | nothing         | $128c$. |

Table 6: The resulting gauge groups and six dimensional untwisted matter representations are given for representatives of the possible $\mathbb{Z}_4$ gauge shifts. General gauge shifts, which have been brought to their standard form, are classified by computing $8V_4^2$. For $8V_4^2 = 4$ there are two inequivalent gauge shifts that can be distinguished by $\sum_j V_4^j \mod 2 = 0, 1$, as is discussed in appendix C.4.3.
These double twisted zero modes are really obtained by compactification of the six dimensional twisted states to four dimensions. But as the geometrical analysis of $T^6/Z_4$ in section 2 taught us, $T^6/Z_4$ contains six two–tori and four orbifolds ($T^2/Z_2$'s), and only these orbifolds contain (as their fixed points) the fixed points of $T^6/Z_4$. Since we are interested in the local spectra at these fixed points, and we want to use the four dimensional zero modes to infer these spectra, we have to take care, that we do not count zero modes coming from the two–tori, as they do not live at those four dimensional fixed points. But there is an easy way to take this into account. Compactifications on $T^2$ and $T^2/Z_2$ are closely related: If the zero mode spectrum on $T^2/Z_2$ is in representation $T_2$, the zero mode spectrum on $T^2$ is $T_2 + \overline{T}_2$. As argued in section 2 the orbifold $T^6/Z_4$ contains six $T^2$ and four $T^2/Z_2$, hence we find the zero spectrum $10T_2 + 2\overline{T}_2$. This pattern has been found for $Z_4$ orbifold models in refs. [28, 29]. At the fixed points of $T^2/Z_2$, which coincide with four fixed points of $T^6/Z_4$, the double twisted states fall into representations $T_2$. Finally, these double twisted states are the local projections of six dimensional states, which live in the representation defined in [23]: The double twisted representation $T_2$ is obtained by the decomposition

$$S_{2V_4} + 4D_{2V_4} \rightarrow T_2[V_4] + \overline{T}_2[V_4],$$

associated with the branching $G_{V_4} \rightarrow G_{V_4}$ of the six dimensional gauge group to the four dimensional one. We use $V_4$ to denote a generic $Z_4$ gauge shift vector. The possible double twisted representations have been listed table 4.

Since apart from the double twisted states at the fixed points, the method working with an equivalent model proceeds as discussed in ref. [13], we only quote our definitions here and give the spectra in similar tables as tables 4 and 7. Unlike the discussion of the previous section, here we do not include any gravitational induced states. The main reason for this is, that in the end we are interested in (local) anomalies, but gravitational states do not give rise to anomalies in four dimensions. Hence we can safely ignore them here. The full local four dimensional spectrum has been collected in table 5.

The possible different gauge shifts in a single $E_8$ are listed in table 6 and table 7 gives the modular invariant combinations.

The (gauge part of the) untwisted spectrum falls into three categories, which we can describe using similar notation as in the six dimensional case: there is a four dimensional gauge multiplet in the adjoint $Ad_{V_4}$ (corresponding to the gauge group $G_{V_4}$), and a single chiral multiplet in representation $R_{V_4}$. As can be seen from table 6 the latter representation is never complex. Additionally, one encounters two chiral multiplets in the representation

$$r_{V_4} = \{w \mid V_4^I w_I = \frac{3}{4} \mod 1\}. \tag{25}$$

The relevant four dimensional $N = 1$ super multiplets for the untwisted sector have been given in table 5. The gauge group $G_{V_4}$ may contain U(1) factor(s). The generators of these U(1)'s are proportional to the gauge shift embedded in the Cartan subalgebra:

$$q_{V_4} = V_4^I H_I, \quad q'_{V_4} = V_4^I H'_I \tag{26}$$

in the Cartan subalgebra of both $E_8$'s. These U(1)'s are normalized such that the smallest U(1) charge appearing in the untwisted sectors $R_{V_4}$ and $r_{V_4}$ has absolute value 1. The U(1) charges of the untwisted states can be found in table 6.

To complete the local four dimensional spectrum, we mention the single twisted states. Here we can follow the same analysis as in section 3.3 again we find $N_R^{(1)} = 0$ leading to $\omega + \phi = (0, \pm 2, \pm 1, \pm 1)/4$
| shift $V_4$ | gauge group $G_{|V_4|}$ | single twisted $T_{1|V_4|}$ | double twisted $T_{2|V_4|}$ |
|-----------|-----------------|-----------------|-----------------|
| (3:0)     | $E_6 \times SU(2) \times U(1) \times E_8'$ | $(27, 1)_{-1/2}(1)' + 2(1, 2)_{-3/2}(1)' + 5(1, 1)_{3/2}(1)'$ | $(27, 1)_1(1)' + (1, 1)_{-3}(1)' + 2(1, 2)_0(1)'$ |
| (3:40)    | $E_6 \times SU(2) \times U(1) \times E_8' \times SU(2)'$ | $(1, 2)_{-3/2}(1, 2)' + 2(1, 1)_{3/2}(1, 2)'$ | $(27, 1)_{-1}(1)' + (1, 1)_{3}(1)' + 2(1, 2)_0(1, 1)'$ |
| (3:41)    | $E_6 \times SU(2) \times U(1) \times SU(8)' \times U(1)'$ | $(1, 1)_{3/2}(8)'_1 + (1, 2)_{-3/2}(1)' + 2(1, 1)_{3/2}(1)'_2$ | $(1, 2)_0(8)'_1$ |
| (3:8)     | $E_6 \times SU(2) \times U(1) \times SU(8)' \times U(1)'$ | $(1, 1)_{3/2}(16)'_4$ | $(27, 1)_1(1)' + (1, 1)_{-3}(1)' + 2(1, 2)_0(1)'$ |
| (7:0)     | $SU(8) \times SU(2) \times \times E_8'$ | $(8, 2)(1)' + 2(8, 1)(1)'$ | $(28, 1)(1)' + 2(1, 2)(1)'$ |
| (7:40)    | $SU(8) \times SU(2) \times E_8' \times SU(2)'$ | $(8, 1)(1, 2)'$ | $(28, 1)(1, 1)' + 2(1, 2)(1, 1)'$ |
| (7:41)    | $SU(8) \times SU(2) \times SU(8)' \times U(1)'$ | $(8, 1)(1)'_2$ | $(1, 2)(8)'_1$ |
| (7:8)     | $SU(8) \times SU(2) \times SU(8)' \times U(1)'$ | nothing | $(28, 1)(1)' + 2(1, 2)(1)'$ |
| (2:1)     | $SO(14) \times U(1) \times E_7' \times U(1)'$ | $(14_v)_1(1)'_{1/2} + (1, 1)_{1/2} + 5(1, 1)_{3/2}$ | $(14_v)_{0, 1}(1)' + (1, 2)(1)'_{-1} + (1)_{-2}(1)'_1$ |
| (2:5)     | $SO(14) \times U(1) \times SO(12)' \times SU(2)' \times U(1)'$ | $(1)_1(12, 1)_1/2 + 2(1, 1)_{1/2}$ | $(14_v)_{0, 1}(1)' + (1, 2)(1)'_{-1} + 1(1, 2)(1, 1)'_1$ |
| (6:1)     | $SO(10) \times SU(4) \times E_7' \times U(1)'$ | $(16_v, 1)(1)'_{1/2} + 2(1, 4)(1)'_{1/2}$ | $(10_v, 1)(1)'_{1} + (1, 6)(1)'_1$ |
| (6:5)     | $SO(10) \times SU(4) \times SO(12)' \times SU(2)' \times U(1)'$ | $(1, 4)(1)'_{1/2}$ | $(10_v, 1)(1)'_{1} + (1, 6)(1)'_1$ |

Table 7: There are 12 modular invariant combinations of $Z_4$ gauge shifts, which are listed in table $\text{[6]}$ (The numbers $(n; n')$ correspond to the first column of that table.) The resulting gauge group and the single and double twisted matter at a single fixed point is given.
since \(c_{(1)} = 5/16\). Invoking the GSO projection and requiring a phase \(i\) under the twist, gives the bosonic vacuum state \(|0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle_{NS}\). For the left–moving sector we find the gauge representations

\[
T_{1}[V_{4}] = \{ w \mid \frac{1}{2}(w - V_{4})^2 = \frac{1}{16} - N_{L}^{(1)} \}, \quad N_{L}^{(1)} = 0, \frac{1}{4}, \frac{1}{2}.
\]  

(27)

In addition to the single twisted states there are the triple twisted states. However, it is not hard to show, that they have opposite chirality and are in the complex conjugate representation \(T_{1}\). This just means, that the single and triple twisted states combine into chiral multiplets, as given in table 6. The possible representations \(T_{1}\) of within the 12 different orbifold models are collected in table 7.

4 Orbifold traces and anomaly calculations

In the previous section we saw, that the ten dimensional heterotic string states are projected at the six and four dimensional fixed surfaces of \(T^{6}/Z_{4}\). Similarly, the six dimensional states at the orbifolds \(T^2/Z_{2}\) within \(T^{6}/Z_{4}\) give rise to projected states at the four dimensional fixed points. In particular, we have computed the representations in which all these states reside. To determine these spectra we used equivalent pure orbifold models. In particular, in section 3.3 we used pure \(T^{4}/Z_{2}\) orbifold models to determine the local six dimensional spectra on \(T^{6}/Z_{4}\). However, as we warned at the beginning of section 3.3 this method does not directly apply to the computation of anomalies and the computation of traces of operators on these orbifolds in general. The task of this section is to obtain exact expressions for such traces on the orbifolds \(T^{6}/Z_{4}\) and \(T^{4}/Z_{2}\). Moreover, since some of the six dimensional twisted states at \(T^{6}/Z_{4}\) live on orbifolds \(T^2/Z_{2}\), we also give a trace formula for that case. The general machinery for such calculations has been collected in ref. 13, on which this section is based heavily. We first give the general trace formulae, and we apply them to compute anomalies in subsection 4.2.

4.1 General orbifold trace formulae

Example 4.2 of ref. 13 gives the expression for the trace \(\text{Tr}_{\mathbb{R}^{4} \times T^{6}/Z_{4}, R}[O]\) over the \(T^{6}/Z_{4}\) orbifold Hilbert space of an arbitrary operator \(O(x, z; \partial)\). The orbifold twist operator \(R = \exp(2\pi i \phi^{4} S_{4})\) acts on both, gauge and spacetime indices. Hence here we may simply copy the result:

\[
\text{Tr}_{\mathbb{R}^{4} \times T^{6}/Z_{4}, R}[O] = \frac{1}{4} \text{Tr}_{\mathbb{R}^{4} \times T^{6}/Z_{4}}[O] + \frac{1}{4} \cdot \frac{1}{16} \sum_{p,q} \text{Tr}_{(\mathbb{R}^{4}, \mathbb{Z}_{p,q}^{2})}[\mathcal{R}_{p,q}^{4}O_{\Theta} + (\mathcal{R}_{p,q}^{4})^{3}O_{\Theta^{3}}] + \frac{1}{4} \cdot \frac{1}{16} \sum_{p \neq q} \text{Tr}_{(\mathbb{R}^{4}, \mathbb{Z}_{p,q}^{2})}[\mathcal{R}_{p,q}^{2}O_{\Theta^{2}}] + \frac{1}{4} \cdot \frac{1}{16} \sum_{p = q} \text{Tr}_{(\mathbb{R}^{4}, \mathbb{Z}_{p,q}^{2})}[\mathcal{R}_{p,q}^{2}O_{\Theta^{2}}],
\]

(28)

where the operators \(O_{\Theta^{i}}, \ i = 1, 2, 3\) are defined by

\[
O_{\Theta} = O(x, z; \frac{1}{2} \partial_{1}, \frac{1+i}{2} \partial_{2}, \frac{1+i}{2} \partial_{3}), \quad O_{\Theta^{2}} = O(x, z; \partial_{1}, \frac{1}{2} \partial_{2}, \frac{1}{2} \partial_{3}), \quad O_{\Theta^{3}} = O(x, z; \frac{1}{2} \partial_{1}, \frac{1-i}{2} \partial_{2}, \frac{1-i}{2} \partial_{3}).
\]  

(29)

From the first term we learn, that one quarter of the states on \(\mathbb{R}^{4} \times T^{6}/Z_{4}\) behaves like ten dimensional ones, but without any orbifold twist acting on their spacetime or gauge indices. The second term on
the first line of (28) is evaluated at the four dimensional fixed points, and the terms on the second line give the contributions at the fixed $T^2$'s and $T^2/Z_2$'s. The presence of Wilson lines in the form of periodicity matrices $T$ are taken into account by the local matrices

$$
R^4_{p,q} = e^{2\pi i (\alpha^4_{p,q} H_1 + \phi \cdot S_i)},
$$

$$
R^2_{p,q} = e^{2\pi i (\alpha^2_{p,q} H_1 + 2 \phi \cdot S_i)},
$$

(30)

where $S_i$ denotes the spins in light–cone gauge and the local shifts are given in (16). It should be stressed, that these traces on the six and four dimensional fixed surfaces are still taken over the ten dimensional gauge and spacetime representation space. To obtain traces over the gauge and spacetime representations of lower dimensional fixed spaces, one needs to take into account the phase factors coming from (50). In the discussion of the application of these formulae to anomalies in section 4.2 we explain how this works.

We have seen in section 3.3 that the local six dimensional spectra of $T^6/Z_4$ and $T^4/Z_2$ can be identified locally. However, when one computes traces over the Hilbert space of $T^4/Z_2$ one obtains a somewhat different result than the one for $T^6/Z_4$ given above:

$$
\text{Tr}_{\mathbb{R}^4 \times T^2 \times T^2 / Z_2, R} \left[ O \right] = \frac{1}{2} \text{Tr}_{\mathbb{R}^4 \times T^2 \times T^2 / Z_2} \left[ O \right] + \frac{1}{2} \cdot \frac{1}{16} \sum_{p,q} \text{Tr}_{(\mathbb{R}^4 \times T^2, \tilde{\Gamma}_p, \tilde{\Gamma}_q)} \left[ R^2_{p,q} O_{\phi^2} \right],
$$

(31)

here $O_{\phi^2}$ is again given by (29). (This result is obtained by the methods discussed in ref. [13].) The two main differences with (28) are, that the traces at the four dimensional fixed points are absent, of course, and that there are now factors of $1/2$ instead of $1/4$.

As the discussion in section 2 revealed, the orbifold $T^6/Z_4$ contains fixed orbifolds $T^2/Z_2$, which support six dimensional twisted states as given in section 3.3; therefore we also give the final trace formula for this case. The trace formula at such an orbifold $(\mathbb{R}^4 \times T^2 / Z_2, \tilde{\Sigma}^2_{p=q})$ for an operator $O$ that acts on these twisted states reads:

$$
\text{Tr}_{(\mathbb{R}^4 \times T^2 / Z_2, \tilde{\Sigma}^2_{p=q})} \left[ O \right] = \frac{1}{2} \text{Tr}_{(\mathbb{R}^4 \times T^2 / Z_2, \tilde{\Sigma}^2_{p=q})} \left[ O \right] + \frac{1}{2} \cdot \frac{1}{4} \sum_{p_1,q_1} \text{Tr}_{(\mathbb{R}^4 \times R_1 \times \tilde{\Sigma}^2_{p=q})} \left[ R^4_{p=q} \tilde{O}_{\phi \theta} \right],
$$

(32)

where $\tilde{O}_{\phi \theta} = O(x, z; \frac{1}{2} \partial_1)$. The pattern of this expression is similar to (31): There is a six dimensional part, where the trace is evaluated on the orbifold $T^2/Z_2$ without any orbifold twist in the spacetime or gauge sector. And in addition, for the second term we can recognize the factor $1/4$ that arises because $T^2/Z_2$ has four fixed points. (The expression here is consistent, because for $p_2 = q_2$ and $p_3 = q_3$ the matrix $R^4_{p,q}$ squares to the identity.)

### 4.2 Anomaly calculations using trace formulae

We now briefly describe how these general trace formulae for $T^6/Z_4$, $T^4/Z_2$ and $T^2/Z_2$ can be applied to the evaluation of anomalies. Anomalies correspond to the formal Hilbert space trace in $2N$ dimensions

$$
A_{2N}(\Lambda) = 2\pi i \text{Tr}(\tilde{\Gamma} \Lambda),
$$

(33)

where $\Lambda$ corresponds to a gauge or local Lorentz transformation, and $\tilde{\Gamma}$ is the chirality operator. To evaluate this trace one needs to regularize this expression, for example, by employing the heat kernel regularization following Fujikawa [30, 31]. As the treatment of this method to orbifolds has been
recently discussed in refs. 14, 11, 12, we here only focus on some important consequences that can be read off from the trace formulae given in the previous subsection.

Let us start with applying the result \( \frac{1}{28} \) for \( T^6/\mathbb{Z}_4 \) to anomalies. We see that there is a ten dimensional anomaly but with a relative normalization factor of 1/4 w.r.t. the result on a smooth ten dimensional manifold. Next, the two expressions on the second line refer to trace contributions at the six dimensional fixed hyper surfaces, \( \mathbb{R}^4 \times T^2 \) and \( \mathbb{R}^4 \times T^2/\mathbb{Z}_2 \) of \( T^6/\mathbb{Z}_4 \), respectively. As stressed below that equation, the traces are still taken over gauge and spacetime representations in ten dimensions, while we would like to express the result for the anomaly in six dimensional representations only. To do this we need to keep track of the phase factors resulting from \( \mathcal{R}_{p,q}^2 \), given in \( \frac{1}{28} \). We will not give this calculation for all states, but just illustrate the method for the ten dimensional gaugino. This state can be decomposed as

\[
\left( |A_{d(v^2_{p,q})}| + |R_{v^2_{p,q}}| \right) \begin{pmatrix} \frac{1}{2} \alpha_2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{pmatrix}_{NS}.
\]

(34)

where we have taken into account that the ten–dimensional gaugino is left–handed. The six dimensional chirality of these states is given by \( \alpha_1 \alpha_2 = \alpha_3 \alpha_4 \), and their phase factors are computed easily:

\[
\mathcal{R}_{p,q}^2 |A_{d(v^2_{p,q})}| \begin{pmatrix} \frac{1}{2} \alpha_2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{pmatrix}_{NS} = e^{2\pi i \left[ \frac{1}{4} + \frac{1}{4} (\alpha_3 + \alpha_4 - 2\alpha_2) \right]} |A_{d(v^2_{p,q})}| \begin{pmatrix} \frac{1}{2} \alpha_2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{pmatrix}_{NS},
\]

\[
\mathcal{R}_{p,q}^2 |R_{v^2_{p,q}}| \begin{pmatrix} \frac{1}{2} \alpha_2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{pmatrix}_{NS} = e^{2\pi i \left[ \frac{1}{4} + \frac{1}{4} (\alpha_3 + \alpha_4 - 2\alpha_2) \right]} |R_{v^2_{p,q}}| \begin{pmatrix} \frac{1}{2} \alpha_2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{pmatrix}_{NS}.
\]

(35)

The factor \( e^{2\pi i \frac{1}{4} \alpha_2} \) leads to the opposite sign between the states in \( \mathcal{A}_{d(v^2_{p,q})} \) and \( \mathcal{R}_{v^2_{p,q}} \). This reflects the opposite six dimensional chirality of vector and hyper multiplets. Computing the remaining sum, noting that \( e^{2\pi i \frac{1}{2} \alpha_2} = -1 \) gives

\[
\sum_{\alpha_3, \alpha_4 = \pm} (\alpha_3 \alpha_4) e^{2\pi i \left[ \frac{1}{4} (\alpha_3 + \alpha_4 - 2\alpha_2) \right]} = 4,
\]

(36)

shows that the factor 1/4 of \( \frac{1}{28} \) is canceled. Notice that this leaves only \( \alpha_2 = \pm \) to give two degrees of freedom. However, since six dimensional chiral spinors contain four degrees of freedom (on-shell) we have to introduce a factor 1/2 to normalize the anomaly to the anomaly of a chiral six dimensional fermion; this result is given in equation \( \frac{1}{28} \) below.

Similar arguments can be used to evaluate the anomaly at the four dimensional fixed points. From the second (two) terms of \( \frac{1}{28} \) we obtain the symmetrization factors due to \( \mathcal{R}_{p,q}^4 \) and \( (\mathcal{R}_{p,q}^4)^{-1} \):

\[
\sum_{\alpha_3, \alpha_4 = \pm} \left\{ e^{2\pi i \left[ \frac{1}{4} + \frac{1}{4} (\alpha_3 + \alpha_4 - 2\alpha_3 \alpha_4) \right]} + e^{-2\pi i \left[ \frac{1}{4} + \frac{1}{4} (\alpha_3 + \alpha_4 - 2\alpha_3 \alpha_4) \right]} \right\} = 4,
\]

(37)

since all ten dimensional states \( |R_{v^4_{p,q}}| \otimes |\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle_{NS} \) in this representation contribute (only the four dimensional chirality has already been fixed). This cancels one 1/4, and the factor 1/16 remains. The other four dimensional representations \( \mathcal{A}_{d(v^4_{p,q})} \) and \( \mathcal{R}_{v^4_{p,q}} \) are not complex, and hence do not give rise to a four dimensional anomaly.

Putting all this together, we find that the final result for the gaugino anomaly is given by

\[
\mathcal{A}_{\mathbb{R}^4 \times T^6/\mathbb{Z}_4} \left( \Lambda^2 \right) = \int \left\{ \frac{1}{4} \cdot \frac{1}{2} I_{10} |A_{d(v^2_{p,q})}| + \sum_{p,q} \frac{1}{16} \cdot \frac{1}{2} \left( I_{16} |A_{d(v^2_{p,q})}| - I_{16} |R_{v^2_{p,q}}| \right) \delta^4(z - 3_{p,q}) I^4 z \right. \\
+ \left. \sum_{p,q} \frac{1}{16} \cdot 2 I_{4} |R_{v^2_{p,q}}| \delta^6(z - 3_{p,q}) I^6 z \right\}.
\]

(38)
Here $I_{2N}$ denote the anomalies associated with the closed and gauge invariant anomaly polynomials $I_{2N+2}$ via the descent equations

$$dI_{2N+1} = I_{2N+2}, \quad \delta A I_{2N+1} = dI_{2N}; \quad \text{with} \quad I_{2N+2|r} = \hat{A}(R)\text{ch}_r(iF),$$

(39)

where in the last equation the general $(2N + 1)$–form anomaly polynomial for a chiral fermion in representation $r$ is defined. Here $\hat{A}(R)$ is the roof genus of the curvature two form $R$, and $\text{ch}_r(iF)$ the Chern character of the field strength two form $F$, for their definitions see e.g. [32, 33, 34, 35]. (Throughout this work we assume wedge products between forms to be understood.) In ten dimensions, spinors are both chiral and Majorana, this means that only half of the states contribute to the anomalies; this explains the appearance of the factor $1/2$ of the first term of (38). As discussed above, the factor $1/2$ in front of the second and third terms is due to the normalization of chiral anomalies in six dimensions. The factor $2$ in front of the last term is due to the fact that states in representation $r|V_4$ have multiplicity 2, see table 5.

It should be noted that the calculation presented in (36), where all traces have been expressed in a six dimensional form, applies equally well to the case of the orbifold $T^4/Z_2$. Using the trace formula (31) for that orbifold we find for the gaugino anomaly

$$A_{\mathbb{R}^4 \times T^2 \times T^4/\mathbb{Z}_2}^{\hat{A}(\Lambda)} = \int \left\{ \frac{1}{2} \cdot \frac{1}{2} (I_{10}^{\text{Ad}_{10}} + \sum_{p,q} \frac{1}{16} (I_6^{\text{Ad}_{1+p,q}} - I_6^{\text{R}_{1+p,q}}) \delta^4(z - 3^2_{p,q})d^4z \right\}. \quad (40)$$

Observe that there is a difference of a factor of two between this expression and (38) for both the ten and six dimensional anomalies.

The computation of the anomaly for six dimensional states on $T^2/\mathbb{Z}_2$ is straightforward and similar to the results presented above. Using (32) we find for the half–hyperinos

$$A_{\mathbb{R}^4 \times T^2/\mathbb{Z}_2}^{\hat{A}(\Lambda)} = \int \left\{ \frac{1}{2} \cdot \frac{1}{2} (I_6^{\text{S}_{p,q}} + I_6^{\text{D}_{p,q}}) + \sum_{p,q} \frac{1}{16} I_4^{\text{T}_{2[p,q]}} \delta^2(z_1 - R_1 \zeta_{p,q})d^2z_1 \right\}. \quad (41)$$

For the hyperinos that live on the two $T^2$'s which are identified, the anomaly is given by the first two terms in this equation, because like for the orbifold only half of the states exist at a two–torus.

## 5 Anomalies

Anomaly investigations of ten dimensional theories have played an important role in the development of string theory. In particular the possibility of canceling factorisable anomalies via the so–called Green–Schwarz mechanism [16] paved the way for heterotic string theory. On orbifolds the ten dimensional factorization is still at work. The 12–form anomaly polynomial is given by

$$I_{12} = \left[ \text{tr}R^2 - \frac{1}{30} \text{tr}(iF)^2 - \frac{1}{30} \text{tr}(iF')^2 \right] \frac{1}{4} X_{8GS}, \quad (42)$$

where $X_{8GS}$ is the standard Green–Schwarz eight form [16, 36, 37]. Here we have explicitly given the factor $1/4$ that arises because we have to evaluate the anomalies on the orbifold $T^6/\mathbb{Z}_4$. From the first term of (42), we see that we only get $1/4$ of the anomaly on a smooth manifold. Next we investigate the anomalies at the fixed spaces of the orbifold $T^6/\mathbb{Z}_4$. 
5.1 Local six dimensional anomalies

Six dimensional anomalies arising from compactification of ten dimensional supergravity coupled to $E_8 \times E_8'$ super Yang–Mills have first been considered in [37]. The application to heterotic orbifold models can be found e.g. in [38]. The relevant representations for the local six dimensional anomaly investigation have been given in section 3.3. We denote the six dimensional anomaly polynomial for a fermion with positive chirality transforming in representation $r$ by

$$I_{8|r} = -\frac{i}{(2\pi)^3} \left[ \frac{1}{24} \text{tr}_r(iF)^4 - \frac{1}{96} \text{tr}_r(iF)^2 \text{tr}R^2 + \frac{\text{dim} r}{128} \left( \frac{1}{45} \text{tr}R^4 + \frac{1}{36} (\text{tr}R^2)^2 \right) \right].$$  (43)

(This formula also applies to gauge singlets, in which case all traces $\text{tr}_r$ are of course zero and $\text{dim} r$ denotes the number of these gauge singlets. For a single gauge singlet we denote its anomaly polynomial by $I_{8|1/2}$.) This equation directly applies to gauginos, because they have positive chirality. Since the chirality of the hyperinos and the dilatino is negative, the overall sign of the anomaly polynomial is opposite. (We give this sign explicitly to avoid confusion.) In addition, the anomaly contribution of the gravitino reads

$$I_{8|3/2} = -\frac{i}{(2\pi)^3} \frac{1}{128} \left( \frac{245}{45} \text{tr}R^4 - \frac{43}{36} (\text{tr}R^2)^2 \right).$$  (44)

Combining these ingredients, the total anomaly polynomial for the heterotic $T^6/\mathbb{Z}_4$ orbifold model at a six dimensional fixed point $\mathbb{Z}_{pq}^2$ is given by

$$I_{8|\mathbb{Z}_{pq}^2} = \frac{1}{32} I_{8|3/2} - \frac{1}{32} (1 + 4) I_{8|1/2} + \frac{1}{32} I_{8|\text{Ad}_{v_{pq}^2}} - \frac{1}{32} I_{8|\text{R}_{v_{pq}^2}} - \frac{1}{4} \left( I_{8|S_{v_{pq}^2}} + I_{8|D_{v_{pq}^2}} \right).$$  (45)

The different contributions are due to the gravitino, the dilatino and the 4 neutral hyperinos, the gaugino, the untwisted matter, and the twisted matter, respectively (see table 2). Here we used the anomaly results [38] and [41] obtained in section 4.2. (Only the gauginos are treated there, but the discussion extends to gravitino and dilatino as well.) We should note that if one does this analysis for $T^4/\mathbb{Z}_2$ one obtains two times the result of (45), which follows from (40) and the fact that in that case the (half–)hyperinos on the six dimensional spaces are not orbifolded.

Since both six dimensional models we encounter in this work (see table 4) do not contain Abelian subgroups, we are concerned with semi–simple gauge groups only. The non–Abelian anomalies do not have to vanish identically, but can be canceled by a six dimensional Green–Schwarz mechanism instead. For this it is crucial that the anomaly polynomial factorizes, which means in particular that all irreducible anomalies have to vanish identically. For both models the irreducible gravitational anomaly $\text{tr}R^4$ vanishes because of the relation between the number of vector and hyper multiplets

$$\frac{1}{16} \left( 245 - 1 - 4 \right) + \frac{1}{16} \text{dim} \text{Ad}_{v_{pq}^2} - \frac{1}{16} \text{dim} \text{R}_{v_{pq}^2} - \frac{1}{2} \left( \text{dim} S_{v_{pq}^2} + \text{dim} D_{v_{pq}^2} \right) = 0,$$  (46)
at a given six dimensional hyper surface (This is nothing but the local version of the well–known zero–mode statement, that the total number of hyper– minus vector multiplets must be 244). Of the groups appearing in table 4 only SO(16)′ has a non–vanishing fourth order Casimir and may therefore lead to irreducible gauge anomalies. However, by virtue of the SO(16)′ trace identities \[31, 38\] for the adjoint and spinor representation

\[
\text{tr}_{120}(iF)^4 = 8 \text{tr}_{16}(iF)^4 + 3(\text{tr}_{16}(iF)^2)^2, \quad \text{tr}_{128}(iF)^4 = -8 \text{tr}_{16}(iF)^4 + 6(\text{tr}_{16}(iF)^2)^2,
\]

also these irreducible anomaly contributions of the gaugino and the untwisted and twisted matter cancel each other: \((1/16)8 = (1/16)(-8) = (1/2)2 = 0\). The remaining six dimensional reducible anomaly at a given fixed point \(Z_{pq}^2\) always factorizes into the form

\[
I_{8|Z_{pq}^2} = -\left[\text{tr} R^2 - \sum_i c_i \text{tr}(iF_i)^2\right] - \frac{i}{2} \sum_i d_i \text{tr}(iF_i)^2 = -\frac{i}{2} \sum_i d_i \text{tr}(iF_i)^2,
\]

where \(i, j\) runs over the (semi–simple) gauge group factors and the traces are taken in the corresponding fundamental representations. The factors \(c_i\) give universal normalization of the quadratic traces in the sense, that they only depend on the gauge groups: \(c_i = 2/I(G_i)\), where \(I(G_i)\) is the index of the group \(G_i\). On the other hand, the coefficients \(d_i\) are model dependent, as can be seen from table 8 where we have listed \(c_i\) and \(d_i\) for both models of table 4.

We do not give the details for the derivation of these coefficients from \[18\] and \[15\]; they are obtained using trace identities, like \[17\], which can be found e.g. in \[39, 38\]. However, we would like to remark that the relation between these coefficients is rather delicate: For example, the mixed gauge anomalies, that appear in the last term in the second line of \[48\], can only arise if there is matter charged under two gauge group factors \(G_i\) and \(G_j\) simultaneously. Therefore, if two group factors do not have any matter that is charged under both, their coefficients satisfy \(c_i d_j + d_i c_j = 0\). In model \((1, 0)\) there is no mixed matter that is charged under \(E_8′\), hence this relation is fulfilled for \(j = E_8′\), as is readily verified from table 8. Model \((1, 2)\) does not contain any matter charged under both \(E_7\) and \(SO(16)′\), so a similar conclusion holds.

### 5.2 Local four dimensional anomalies

The relevant representations for the local four dimensional anomaly investigation have been given in section 3.4. We denote the four dimensional anomaly polynomial for a fermion with positive chirality transforming in representation \(r\) by

\[
I_{0|r} = -\frac{i}{(2\pi)^2} \left[\frac{1}{3!} \text{tr}_r (iF)^3 - \frac{1}{48} \text{tr} R^2 \text{tr}_r (iF)\right].
\]

The general form of the anomaly polynomial at the four dimensional fixed point \(Z_{pq}^4\) of \(T^6/\mathbb{Z}_4\) is given by

\[
I_{0|Z_{pq}^4} = \frac{1}{16} \cdot 2 I_{0|r}^{[v_{pq}^4]} + 4 I_{0|t_2[v_{pq}^4]} + 3 I_{0|t_1[v_{pq}^4]},
\]

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Table 9: This table shows that all non–Abelian anomalies, which may arise in the eight models with non–automatically anomaly free groups, are canceled locally at each four dimensional fixed point of $T^6/\mathbb{Z}_4$.

This result is obtained by combining (38) and (41). The first term is due to the part of the ten dimensional gauginos that resides in representation $r_{v_{p,q}}'$. The other ten dimensional states do not give rise to anomalies since their representations $A_{v_{p,q}}$ and $R_{v_{p,q}}$ are by definition (19) never complex. The six dimensional twisted states at the orbifold $T^2/\mathbb{Z}_2$ (that contains $3_{p,q}^4$) give rise to the second term in (50).

We show that there are no non–Abelian anomalies at any of the four dimensional fixed points. The relevant groups and representations for the local anomaly analysis have been given in tables $\S6$ and $\S7$. Of all possible local non–Abelian gauge groups only SU(8), SU(4) and SU(8)$'$ are not automatically anomaly free. (Here we refer to table $\S4$ to identify these groups.) In table $\S9$ we verify explicitly that all models, that contain these gauge groups, do not suffer from non–Abelian anomalies. For the SU(4) cases this is straightforward, for some of the SU(8) and SU(8)$'$ cases, trace identities are needed (cf. e.g. $\S56$ $\S68$).

Models that contain U(1) factors may be anomalous, but like in ten and six dimensions the anomaly polynomial factorizes $\S10$:

$$I_{6|3_{p,q}} = \left[ tr R^2 - \sum c_i tr(iF_i)^2 \right] \frac{-i}{(2\pi)^2} \frac{1}{48} \left[ F_{U(1)'} tr(q_{v_{p,q}}') + F_{U(1)} tr(q_{v_{p,q}}) \right] $$

(51)

where the coefficients $c_i$ are again related to the indices of the various groups that exist at this four dimensional fixed point. Because factorization only allows a single field strength 2–form to appear on the right hand side of (51), only U(1) factors may have pure and mixed anomalies. Here the charges $q_{v_{p,q}}$ and $q_{v_{p,q}}'$ are defined as in (26). In fact, it is simply the local sum of U(1) charges that decide whether a given U(1) factor is anomalous or not. In table $\S10$ we have computed the sum of charges for all models with U(1) factors, listed in table $\S4$. The models (3;0) and (3;4) do not have an anomalous U(1) even though they contain U(1) factors. All other models considered in table $\S10$ have only one anomalous U(1) in one of the two $E_8$ factors, except for model (2;5), which seems to have two. However, as observed in ref. $\S11$ one can always find two other linear combinations of the charges $q_{v_{p,q}}$ and $q_{v_{p,q}}'$, such that only one of them is anomalous.
### 5.3 Local Green–Schwarz mechanisms

To conclude the discussion of anomalies, we return to the Green–Schwarz mechanism to cancel the leftover and factorized anomalies locally at all the fixed hyper surfaces of $T^6/\mathbb{Z}_4$. Since the essence of this mechanism on orbifolds has recently been discussed in ref. [13], we here only quote the particularities of the investigation on $T^6/\mathbb{Z}_4$.

The theory of $N = 1$ supergravity in ten dimensions can be formulated using a two form $B_2$ (alternatively one can use a six form potential [42, 43, 44]) described by the action

$$S_{GS} = \int \left( -\frac{1}{2} \ast dB_2 \ast dB_2 + \ast (X_3 + X_7) dB_2 - \frac{1}{2} * X_3 X_3 \right),$$

which is invariant under the natural gauge transformations of the 2–form $\delta \Lambda B_2 = d\Lambda_1$. Here the 3– and 7–forms $X_3, X_7$ are derived from arbitrary closed and gauge invariant 4– and 8–forms, $X_4, X_8$, by Poincaré’s lemma (i.e. locally $dX_3 = X_4$ and $dX_7 = X_8$). The gauge variation of the 2–form and its action reads

$$\delta \Lambda B_2 = X_3^1, \quad \delta \Lambda S_{GS} = \int X_7 \delta \Lambda X_3,$$

where $\Lambda$ may refer to either a gauge transformation $\delta \Lambda A_1 = d\Lambda + [\Lambda, A_1]$ or a local Lorentz transformation $\delta_L \omega_1 = dL + [L, \omega_1]$ of the spin connection one form $\omega_1$.

In the preceding subsections we have observed that the ten, six and four dimensional anomalies factorize, see equations (12), (48) and (51). The anomalous variation of the Green–Schwarz action (53) can cancel all these anomalies simultaneously, by taking

$$X_4 = \text{tr} R^2 - \frac{1}{30} \text{tr} (iF)^2 - \frac{1}{30} \text{tr} (iF')^2$$

$$X_8 = \frac{1}{8} X_8 GS + \frac{i}{(2\pi)^2} \int \left[ \text{tr} R^2 - \sum_i \text{tr} (iF_i)^2 \right] 3pq \delta^4(z - 3p^2) d^4 z$$

$$+ \frac{1}{(2\pi)^2} \int \left[ F_{U(1)} \text{tr} (q) + F_{U(1)'} \text{tr} (q') \right] 3p^2 q \delta^6(z - 3p^2 q) d^6 z.$$

Here the notation $[\ldots]_{3p^2 q}$ signifies that the expression in between the brackets is evaluated at fixed hyper surface $3p^2 q$, and so on. The reason that these forms can cancel the anomalies in the various dimensions is, that when $X_4$ is restricted to a lower dimensional hyper surface, the terms
(1/30)\text{tr}(iF)^2 + (1/30)\text{tr}(iF')^2 \text{ always reduce to } \sum c_i \text{tr}(iF_i)^2, \text{ which appears in the factorizations (48) and (51) in six and four dimensions, respectively. Therefore, the universality of the coefficients } c_i, \text{ as noted in sections 5.1 and 5.2 is essential to ensure this.}

Let us close with some comments why the local Green–Schwarz mechanism will always work within heterotic orbifold models; i.e. explain why the local factorization of the orbifolds \( T^6/\mathbb{Z}_3 \) or \( T^6/\mathbb{Z}_4 \) were no accidents. As discussed in this paper and in refs. [14, 15], the local shifts at the orbifold fixed points should satisfy the appropriate modular invariance requirements of string theory. However, for the zero mode anomalies, it was demonstrated in ref. [10] that modular invariance implies factorization. Since for pure orbifold models, there is a direct identification between the zero mode anomaly and the local anomalies at the fixed points, this implies that the factorization holds at all fixed points separately. Naturally, this local factorization continues to hold when Wilson lines are present, since the factorization only depends on the modular invariance of the local gauge shift at a given fixed point.

6  Fayet–Iliopoulos tadpoles on \( T^6/\mathbb{Z}_4 \)

The appearance of anomalous \( U(1) \)'s in a four dimensional supersymmetric gauge theory is associated with Fayet–Iliopoulos tadpoles for the auxiliary \( D \) fields of the \( U(1) \) vector multiplets being generated. For the heterotic string compactified on \( T^6/\mathbb{Z}_4 \) the local structure of such tadpoles were calculated in [15]. Moreover, it was explicitly shown there that these tadpoles for auxiliary field components are associated with tadpoles for the internal gauge fields. Here we refrain from giving a complete discussion of these different tadpoles on \( T^6/\mathbb{Z}_4 \). We simply give the expressions for the \( D \) term tadpoles, as they can straightforwardly be obtained from the general trace formulae presented in section 4.

Using the four dimensional off–shell formulation of ten dimensional super Yang–Mills [15], it is not difficult to see that the possible diagrams that contribute to tadpoles are the ones given in figure 2. In the loop they contain the ten dimensional gauge fields and the six and four dimensional twisted states.

Since the only possible local anomalous \( U(1) \) generators are \( q^1_{v^q} \) or \( q^2_{v^q} \) defined in [20], we only have to give tadpole expressions for those generators for fixed point \( 3^4_{p^q} \):

\[
L_{FI} = D^I \int \frac{d^4p}{(2\pi)^4} \left\{ \frac{2}{p^2} \text{tr}_{[v^q_{p^q}]} (H_I) + \frac{1}{p^2 + \frac{1}{4} \Delta_1 + \frac{1}{2} \Delta_23} + \frac{\text{tr}_{T[v^q_{p^q}]} (H_I)}{p^2} + \frac{\text{tr}_{T[v^q_{p^q}]} (H_I)}{p^2} \right\} \delta^6(z - 3^4_{p^q}).
\]
The factors $1/4$ and $1/2$ in front of the internal Laplacian $\Delta_1 = \bar{\partial}_1 \partial_1$ and $\Delta_{23} = \bar{\partial}_2 \partial_2 + \bar{\partial}_3 \partial_3$ are a consequence of the trace formulae, see (28) and (32). We have written this expression in such a way, that the relative contributions of the different terms at each of the fixed points of $T^6/Z_4$, can directly be read off from table 10.

If one uses a cut-off scheme to regularize the divergences here, one finds quadratically divergent contributions for the anomalous U(1)'s. However, for all U(1) factors, anomalous or not, at least logarithmically divergent tadpoles are generated. Due to supersymmetry similar tadpoles arise for the Cartan directions of the internal gauge fields as well. Their background will be similar to the one obtained in ref. [15] for the $T^6/Z_3$ orbifold.

7 Conclusions

In this paper we have investigated the local properties of heterotic $E_8 \times E_8'$ theory compactified on $\mathbb{R}^4 \times T^6/Z_4$. Because this orbifold contains both, four and six dimensional fixed hyper surfaces, the zero mode spectrum of the theory can be rather complicated; certainly when Wilson lines are present. However, at the fixed points locally the structure of the heterotic string is very transparent: At the six dimensional hyper surfaces all spectra are equivalent to one of two possible $T^4/Z_2$ models, while at the four dimensional fixed spaces there are essentially 12 different spectra possible.

We derived the local anomalies on the orbifolds $T^6/Z_4$, $T^4/Z_2$ and $T^2/Z_2$ by applying a general method to compute traces over orbifold Hilbert spaces, developed in [13]. The calculation of the anomalies on $T^4/Z_2$ confirms the expectation that the anomaly structure of this orbifold and of the six dimensional hyper surfaces of $T^6/Z_4$ are closely related: However, an important numerical difference of a half was found to be the result that one is a $Z_2$ and the other a $Z_4$ orbifold. Some of the six dimensional hyper surfaces within $\mathbb{R}^4 \times T^6/Z_4$ are orbifolds themselves, namely $T^2/Z_2$'s. The twisted states there give rise to anomalies in four and six dimensions.

Collecting the anomaly contributions from the various sectors of heterotic string theory on this orbifold, we found the following results: The ten dimensional anomaly has a normalization factor of $1/4$ w.r.t. the standard ten dimensional heterotic theory. The non–Abelian anomalies factorize at each of the six dimensional fixed hyper surfaces separately. (Six dimensional Abelian anomalies are absent for all models.) Similarly, the Abelian anomalies factorize at the four dimensional fixed points, while non–Abelian anomalies never arise there. These conclusions were obtained by using the fact, that at each four or six dimensional fixed point only a finite number of equivalent models can arise. Because of the universal factorization involving the restriction of $\text{tr} R^2 - \frac{1}{30} \text{tr}(iF)^2 - \frac{1}{30} \text{tr}(iF)^2$ to the respective local gauge groups of the various fixed surfaces, the Green–Schwarz mechanism is able to cancel all factorized anomalies, in four, six and ten dimensions, locally at the same time. Using the modular invariance of the local shifts, it can be understood that factorization is implied [40].

We computed the local Fayet–Iliopoulos tadpoles at the four dimensional fixed points by using the general trace formulae of [13] again. For all local models involving U(1) factors, such tadpoles are generated, even if those U(1) factors are not anomalous themselves. However, in that case the divergence of the Fayet–Iliopoulos tadpole is only logarithmically instead of quadratically.

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A Spinors in ten dimensions

This appendix provides some useful background for ten dimensional spinors which are used in the main text of this work. (More details can be found in [46, 26].) We take the (1, 9) dimensional Clifford algebra generated by

\[
\begin{align*}
\Gamma_0 &= (i \sigma_1, 1, 1, 1, 1), & \Gamma_5 &= (\sigma_3, \sigma_3, \sigma_2, 1, 1), \\
\Gamma_1 &= (\sigma_2, 1, 1, 1, 1), & \Gamma_6 &= (\sigma_3, \sigma_3, \sigma_3, \sigma_1, 1), \\
\Gamma_2 &= (\sigma_3, \sigma_1, 1, 1, 1), & \Gamma_7 &= (\sigma_3, \sigma_3, \sigma_3, \sigma_2, 1), \\
\Gamma_3 &= (\sigma_3, \sigma_2, 1, 1, 1), & \Gamma_8 &= (\sigma_3, \sigma_3, \sigma_3, \sigma_3, \sigma_1), \\
\Gamma_4 &= (\sigma_3, \sigma_3, \sigma_1, 1, 1), & \Gamma_9 &= (\sigma_3, \sigma_3, \sigma_3, \sigma_3, \sigma_2).
\end{align*}
\] (A.1)

Here we have introduce a short hand notation for the tensor product of five times the two dimensional Clifford space. These two dimensional Clifford spaces are generated by the Pauli matrices \(\sigma_1\) and \(\sigma_2\). The two dimensional chirality operator is \(\sigma_3\). An explicit representation of these matrices reads

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & \sigma_2 &= \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, & \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\] (A.2)

The matrices \(\sigma_1\) and \(\sigma_2\) can also be used as the charge conjugation matrices \(s_+ = \sigma_1\) and \(s_- = \sigma_2 = -i \sigma_3 s_+\) in two dimensions

\[
s_{\pm}^{-1} \sigma_i s_{\pm} = \pm \sigma_i^T, \quad i = 1, 2, \quad s_{\pm}^{-1} \sigma_3 s_{\pm} = -\sigma_3^T, \quad s_{\pm}^{-1} = s_{\pm} = s_+^{\dagger} = -s_-^T.
\] (A.3)

Notice that in two Euclidean dimensions one can only define Majorana fermions w.r.t. the charge conjugation \(s_+\), because \((\eta^s)^{s_\pm} = \pm \eta\), where \(\eta^{s_\pm} = s_\pm (\eta^{\dagger})^T\). Now, let \(\eta\) be a Majorana fermion and denote two dimensional chiral spinors as \(\eta_{\kappa}\), where \(\sigma_3 \eta_{\kappa} = \kappa \eta_{\kappa}\). The \(s_+\) charge conjugates of these spinors have opposite chirality: \((\eta_{\kappa})^{s_+} = \eta_{-\kappa}\).

Using the basis of generators for the (1, 9) dimensional Clifford algebra and the charge conjugations in two dimensions, it is not hard to show that the charge conjugation matrices \(C_{\pm} = (s_\pm, s_\mp, s_\pm, s_\mp, s_\pm)\) in ten dimensions have the properties

\[
(C_{\pm})^{-1} \Gamma_M C_{\pm} = \pm \Gamma_M^T, \quad (C_{\pm})^{-1} \tilde{\Gamma} C_{\pm} = -\tilde{\Gamma}^T, \quad (C_{\pm})^{-1} = (C_{\pm})^\dagger = C_{\pm} = \pm (C_{\pm})^T.
\] (A.4)
Here we have introduced the $(1, 9)$ dimensional chirality operator $\tilde{\Gamma} = (\sigma_3, \sigma_3, \sigma_3, \sigma_3)$. A basis for ten dimensional spinors is given by

$$|S\rangle_{(1,9)} = \eta_2 s_0 \otimes \ldots \otimes \eta_2 s_4, \quad \mathbb{1}^i \otimes \sigma_3 \otimes \mathbb{1}^{4-i} |S\rangle_{(1,9)} = 2S_i |S\rangle_{(1,9)},$$

(A.5)

with $S = (S_i)$, $i = 0, \ldots, 4$ and $S_i = \pm \frac{1}{2}$. The Majorana conjugation of such a spinor reads

$$\left( |S\rangle_{(1,9)} \right)^C_\pm = \alpha_\pm |\bar{S}\rangle_{(1,9)},$$

(A.6)

with $\alpha_+ = -(2S_1)(2S_3)$ and $\alpha_- = i(2S_0)(2S_2)(2S_4)$. However, a general spinor build out of this basis is not irreducible. In fact, from the properties above, it follows that in $(1, 9)$ dimensions Majorana–Weyl fermions exist:

$$\tilde{\Gamma} \chi = \beta \chi, \quad \chi^{C\pm} = C^{\pm} \chi^T = \chi,$$

(A.7)

with chirality $\beta = \pm$. Since we encounter spinors of both chiralities in the main text, we only solve the Majorana condition explicitly: By going to light–cone gauge (with the assumption that the spatial momentum vector is in the 1 direction) only the last four spinor indices of $|S\rangle_{(1,9)}$ are relevant, and we may fix $S_0 = +1/2$, since by Majorana conjugation (A.6) we can always obtain $S_0 = -1/2$. This leads to the introduction of a spinorial basis for SO(8)

$$|S\rangle = \eta_2 s_1 \otimes \ldots \otimes \eta_2 s_4.$$

(A.8)

The relation with the $(1, 9)$ dimensional spinors is therefore $|\pm \frac{1}{2}, \pm S\rangle_{(1,9)} = \eta_\pm \otimes |\pm S\rangle$. Since we may fix $S_0 = 1/2$ by the Majorana condition, it follows that a $(1, 9)$ spinor of chirality $\pm$ is represented by SO(8) spinors $|S\rangle$ with $\prod(2S_i) = \pm$. (The definition of $|S\rangle$ can be naturally extended to any irreducible representation of SO(8). For example, the vector representation is denoted by $|\pm 1, 0, 0, 0\rangle$.)

B Supersymmetric multiplets in six dimensions

This appendix is devoted to a brief exposition of important supersymmetric multiplets in six dimensions. Following ref. [47] we classify the multiplets using the little group $\text{SU}(2)_+ \times \text{SU}(2)_- = \text{SO}(4) \subset \text{SO}(1, 5)$ and the $R$–symmetry group $\text{SU}(2)_R$ of supersymmetry. (The subscript $\pm$ refers to the Spin(4) chiralities.) On the light–cone the following multiplets can be found:

- **SUGRA** = $(3, 3; 1) + (3, 1; 1) + (3, 2; 2)$
- **tensor** = $(1, 3; 1) + (1, 1; 1) + (1, 2; 2)$
- **vector** = $(2, 2; 1)$ + $(2, 1; 2)$
- **half–hyper** = $(1, 1; 2)$ + $(1, 2; 1)$

in terms of irreducible $\text{SU}(2)_+ \times \text{SU}(2)_- \times \text{SU}(2)_R$ representations. Here the last terms of each row always contain the fermionic super partners of the multiplet. The bosonic content of the supergravity multiplet is the graviton $g_{mn}$ (with $m, n = 2, \ldots, 5$ spacetime indices in light–cone gauge) plus the anti–self–dual part of the anti–symmetric tensor $B_{mn}$. The bosonic constituents of the tensor multiplet are the self–dual part of the anti–symmetric tensor $B_{mn}$ and a dilaton $\sigma$. From this we see that in supergravity models the supergravity multiplet comes together with a tensor multiplet.
As some of these multiplets may come from ten dimensions upon compactification, we briefly describe how these representations can be described using the SO(8) weights discussed in section 3.1 and appendix A. From the branching rule

\[
\text{SO}(8) \rightarrow \text{SO}(4)_1 \times \text{SO}(4)_2 \rightarrow \text{SU}(2)_+ \times \text{SU}(2)_- \times \text{SU}(2)_R \times \text{SU}(2)_H
\]

we get the following roots corresponding to SU(2)'s: \(\alpha_+ = (1, 1, 0, 0), \) \(\alpha_- = (-1, 0, 0, 0), \) \(\alpha_R = (0, 0, 1, 1), \) and \(\alpha_H = (0, 0, 1, -1).\) The holonomy group \(\text{SU}(2)_H\) just represents an internal symmetry from the six dimensional point of view. (Since the holonomy \(\text{SU}(2)_H\) is broken to \(\mathbb{Z}_2\) in our orbifold limit, the representations under \(\text{SU}(2)_H\) give merely rise to phase factors and multiplicities.)

For a detailed description of the construction of supergravity, tensor, vector and hyper multiplet systems in six dimensions, see refs. [48, 49] for example. Here we mainly focus on the fermionic properties of these multiplets as they are important for the computation of six dimensional anomalies.

To describe these properties we take the gamma matrices \(\Gamma_m, m = 0, \ldots, 5,\) that generate the Clifford algebra in six dimensions. Using the basis given in \(\text{A.1},\) the corresponding chirality and charge conjugation matrices can be represented by

\[
\tilde{\Gamma}_6 = (\sigma_3, \sigma_3, \sigma_3, 1, 1), \quad C_6 = (s_-, s_+, s_-, 1, 1).
\]

Their properties can be summarized as

\[
(C_6)^{-1} \Gamma_m C_6 = -\Gamma_m^T, \quad (C_6)^{-1} \tilde{\Gamma}_6 C_6 = -\tilde{\Gamma}_6^T, \quad (C_6)^{-1} = C_6 = +(C_6)^T.
\]

We can define six dimensional chiral spinors as those having a definite eigenvalue under \(\tilde{\Gamma}_6.\) One cannot define Majorana fermions because for any spinor \(\zeta\)

\[
(\zeta C_6)^C_6 = -\zeta.
\]

However, we can define symplectic Majorana fermions

\[
\zeta C_6 = \rho \zeta, \quad (\rho^\dagger)^T \rho = -\mathbb{1}, \quad \rho^T = -\rho,
\]

in terms of a matrix \(\rho.\) These two properties follow by demanding consistency with \(\text{B.4}\) and the requirement that the kinetic Lagrangian for the fermions \(\zeta = (\zeta^\alpha)\) is a scalar quantity. It follows that the number of indices \(\alpha\) is always even. For the smallest choice of two, the matrix \(\rho\) then takes the form

\[
\epsilon = -i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

In general the matrix \(\rho\) can always be brought to the form \(\rho = \mathbb{1} \otimes \epsilon.\) The essential properties of six dimensional spinors are therefore encoded in their chirality and the corresponding matrix \(\rho.\) For the relevant multiplets of this work, we have listed them in table \(\text{B.1}.\)

The reality condition of the hyper multiplets appearing in table \(\text{B.1}\) is of special interest in section 3.3. If the fermions are in a real or complex representation, then the number of components has to be doubled before the reality condition can be imposed: Let \(\psi\) be a chiral fermion is such a representation, then the corresponding chiral symplectic Majorana spinor is given by

\[
\zeta = \begin{pmatrix} \psi \\ -\psi C_6 \end{pmatrix}, \quad \text{with} \quad \zeta C_6 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \zeta.
\]
six dimensional supermultiplets

| multiplet  | content        | reality  | chirality |
|------------|----------------|----------|-----------|
| SUGRA      | $g_{mn}$, $B_{mn}$, $\phi$, $(\psi_m)^C = \epsilon \psi_m$, $\tilde{\Gamma}_6 \psi_m = \psi_m$ | $\lambda^C = \epsilon \lambda$, $\tilde{\Gamma}_6 \lambda = -\lambda$ |
| + tensor   | $\psi^i_m$, $\lambda^i$ | $\chi^C = \epsilon \chi$, $\tilde{\Gamma}_6 \chi = \chi$ |
| vector     | $A_m$, $\chi^i$ | $q^\alpha = -\rho q \epsilon$, $\zeta^C = \rho \zeta$, $\tilde{\Gamma}_6 \zeta = -\zeta$ |
| hyper      | $q^\alpha$, $\zeta^\alpha$ |

Table B.1: The most common six dimensional supergravity multiplets are given together with the reality and chirality properties. The indices $i$ and $\alpha$ are $\text{SU}(2)_R$ and $\text{USp}(2N)$ indices, respectively.

However, if the fermion $\psi$ is in a pseudo real representation, then it can be directly used to form a special kind of hyper multiplet: the so-called half–hyper multiplet. The reason for this is that a pseudo real representation comes with a real anti–symmetric matrix $\rho$ such that

$$T^* = -\rho T \rho^{-1}, \quad \rho^T = -\rho,$$

for the representation matrix $T$ of the group.

## C E$_8$ Weyl reflections and classification of E$_8$ shifts

In the main text we have used extensively that the local shifts are equivalent to only a limited set of standard shifts of both $\mathbb{Z}_2$ as well as $\mathbb{Z}_4$ twists. Here we give a classification of the possible shifts. The material presented in this appendix is related to the $\mathbb{Z}_3$ shift classification presented in [14, 52]. In section 3.1 we have given the E$_8$ roots as roots and weights of the positive chiral spinor representation of SO(16). Since a gauge shift $v$ of a $\mathbb{Z}_N$ orbifold has to fulfill $N v^I w_I \equiv 0$ for all roots $w$, it follows that $N v$ is an element of the E$_8$ root lattice, as this lattice is self–dual. Two E$_8$ gauge shifts $v$ and $v'$ are said to be equivalent, $v \simeq v'$, if

$$v' = v + u, \quad u \in \Gamma_8 \quad \text{or} \quad v' = W_{\alpha}(v) = v - (\alpha, v)\alpha.$$

where $W_{\alpha}(v)$ denotes the Weyl reflection in root $\alpha$ of E$_8$. Since all E$_8$ roots have length 2, it follows that for a Weyl reflection $v'^2 = v^2$.

A useful application are the Weyl reflections at the SO(16) roots

$$(v_1, v_2, v_3, \ldots) \simeq W_{(1, \pm 1, 0^6)}(v_1, v_2, v_3, \ldots) = (\mp v_2, \mp v_1, v_3, \ldots).$$

Hence we see that by interchanging two shift elements, or replacing two shift elements by minus those elements equivalent shifts are obtained. In particular, if a shift has at least one zero, the sign of all other entries is irrelevant.

### C.1 Classification of $\mathbb{Z}_2$ gauge shifts

First of all, if $V_2$ is an element of the $E_8$ root lattice, the gauge group will not be broken, and hence be equal to E$_8$. Since the E$_8$ lattice is even, such a lattice vector fulfills $2(V_2)^2 = 0 \mod 4$. Assume
that \( V_2 \) is not an \( E_8 \) root lattice vector. To classify these \( Z_2 \) gauge shifts we note that for all \( E_8 \) roots \( \alpha \) we have \( (V_2 + \alpha)^2 = V_2^2 \) mod 1, since \( (\alpha, V_2) = 0, 1/2 \) mod 1 and \( \alpha^2 = 2 \). From this it follows that for equivalent \( Z_2 \) gauge shifts \( V_2^2 \) mod 1 is equal, since the sum of squared entries is always invariant under Weyl reflections. This completes the description of the \( Z_2 \) gauge shift classification, the three possibilities are given in table \( \text{[table]} \). A standard shift can be defined by a shift with maximal number of zeros and all entries positive, which is the form we used in that table.

C.2 Classification of \( Z_4 \) gauge shifts

For the classification of \( Z_4 \) gauge shifts, we first need to bring them to a standard form: The entries of \( 4v \) can either all be half-integer, or all be integer. Since all inequivalent models can be computed using only the latter type of gauge shifts, we will not consider half-integer entries here. Since \( (2, 0^7) \) and its permutations are the sums of two roots of SO(16), we infer that the integer valued entries of \( 4v \) can be restricted to \( 4v^I = 1, -1, 0 \). In fact, we may even assume that either \( 4v = (4, 0^7) \) or that no entry \( 4v^I \) is equal to 4. If there are two or more entries equal to 4, then by adding the SO(16) root with \(-1\) at two of these entries, they become zero. If there is just one entry equal to 4, then either \( 4v = (4, 0^7) \) or there is another entry of \( 4v^I = -3, \ldots, 3 \), for which, by adding a SO(16) root again, we can make the 4 entry 0, and map \( 4v^I \) back into \( 4v^I = (-3, \ldots, 3) \). (We have assumed, that if the shift contains more than one non–vanishing entries, this procedure has been applied throughout the paper to set all entries \( 4v^I \in \{-3, -2, -1, 0, 1, 2, 3\} \).)

From now on, we would like to exploit the following conditions: 1) Since \( 4v \) is in the \( E_8 \) root lattice, the number of odd entries of \( 4v \) is always even. 2) If there is at least one entry \( 4v^I = 0 \) it follows from \( \text{[table]} \), that signs do not matter anymore; we take them positive. 3) A pair of entries \( \pm 3 \) can always be mapped to a pair \( \pm 1 \) by addition of an SO(16) root, so that we can restrict to at most one entry of 3. 4) If there is a entry of 2 in a gauge shift \( 4v \) with at least one zero, by adding a SO(16) root with \(-1\) at this entry, the 2 gets mapped to \(-2\) which is with the help of 2) equivalent to 2 again, so that all other non–vanishing entries of \( 4v \) are equivalent modulo 4. The standard form of a gauge shift is defined to be that form with the least possible numbers of 3’s and 2’s in \( 4v \). And we require that all entries are positive. This is always possible except when all entries of \( 4v \) are \( \pm 1 \). If the number of minus signs is even, they can be made all positive, while in the odd case, one has to keep one \(-1\). The number of 3’s and 2’s can often be reduced by subtracting a spinorial root, for example we have \( (3, 1^7)/4 \simeq (1^7, -1)/4 \).

Once in this standard form, we can again consider the sum of the square of the entries of the shift vector. This classifies the shifts almost uniquely, expect when \( (4v)^2 = 8 \). In this case there are two possibilities, but the sum of the entries of \( 2v \) mod 2 can be used to distinguish between both possibilities. The results are summarized in table \( \text{[table]} \).

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