Estimates for the ratio of the first two eigenvalues of the Dirichlet-Laplace operator with a drift

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Abstract

Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Consider the eigenvalue problem of the Laplace operator with a drift term $-\Delta u - x \cdot \nabla u = \lambda u$ in $\Omega$ subject to the homogeneous Dirichlet boundary condition ($u = 0$ on $\partial \Omega$). Denote by $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ the first two eigenvalues of the problem. We show that $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} - 1 \leq 1 + \frac{4}{N}$. In particular, we complement a similar result obtained by Thompson [Stud. Appl. Math. 48 (1969) 281–283] for the classical eigenvalue problem of the Laplace operator.

Keywords: Laplace operator; drift; eigenvalue; eigenfunction.

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1. Motivation and main result

For each positive integer $N$ denote by $\mathbb{R}^N$ the $N$-dimensional Euclidean space and by $| \cdot |_N$ the Euclidean norm in $\mathbb{R}^N$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set.

1.1. The eigenvalue problem of the Dirichlet-Laplace operator

The eigenvalue problem for the Dirichlet-Laplace operator on $\Omega$ reads as follows

$$
\begin{cases}
-\Delta u = \lambda u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1)

It is well-known that the spectrum of problem (1) consists of an increasing and unbounded sequence of positive real numbers (see, e.g. [10, Theorem 8.2.1]). Denote by $\mu_1(\Omega)$ and $\mu_2(\Omega)$ the first two eigenvalues of problem (1). Let us also denote by $\Omega^*$ a ball from $\mathbb{R}^N$ which has the same volume as $\Omega$ (i.e., $|\Omega^*| = |\Omega|$). In 1955-1956, Payne, Pólya and Weinberger [7] showed that

$$
\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \leq 3,
$$

when $N = 2$ and conjectured that the right-hand side could be replaced by

$$
\frac{\mu_2(\Omega^*)}{\mu_1(\Omega^*)} \approx 2.539.
$$

This result was extended to all dimensions in 1969 by Thompson [8], who showed that

$$
\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \leq 1 + \frac{4}{N},
$$

and, again, it was conjectured that the right-hand side could be replaced by

$$
\frac{\mu_2(\Omega^*)}{\mu_1(\Omega^*)}.
$$

In the case $N = 2$ important advances on this problem were obtained by Brands [3], de Vries [5], Chiti [4], and the conjecture was finally settled positively by Asbaugh and Benguria [1] in 1992. The above historical pieces of information are mainly taken from the book by Kesavan [6, Section 4.4, pp. 98-99].
1.2. The eigenvalue problem of the Dirichlet-Laplace operator with a drift

Consider the eigenvalue problem of the Dirichlet-Laplace operator with a drift term

\[
\begin{align*}
-\Delta u - x \cdot \nabla u &= \lambda u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where \( \lambda \) is a real parameter. We say that \( \lambda \) is an eigenvalue of problem (2) if there exists \( u_\lambda \in X_0 \setminus \{0\} \) such that

\[
\int_{\Omega} \nabla u_\lambda \cdot \nabla \varphi \, dm_N = \lambda \int_{\Omega} u_\lambda \varphi \, dm_N ,
\]

where \( dm_N := e |x|^2/2 \, dx \) and \( X_0 := H^1_0(\Omega; dm_N) \). Function \( u_\lambda \) from the above definition is called an eigenfunction corresponding to the eigenvalue \( \lambda \).

Using [2, relation (2.10) on page 715] (see also [9, Théorème 8.7] with \( H = X_0 \) applied for the particular case induced by problem (2)), we deduce that the first eigenvalue of problem (2) has the following variational characterization

\[
\lambda_1(\Omega) := \inf_{u \in X_0 \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2_N \, dm_N}{\int_{\Omega} u^2 \, dm_N},
\]

and there exists a corresponding eigenfunction corresponding to \( \lambda_1(\Omega), e_1 \in X_0 \setminus \{0\} \) such that

\[
\int_{\Omega} |\nabla e_1|^2_N \, dm_N = \lambda_1(\Omega) \quad \text{and} \quad \int_{\Omega} e_1^2 \, dm_N = 1 .
\]

Moreover, the second eigenvalue of problem (2) has the following variational characterization

\[
\lambda_2(\Omega) := \inf_{u \in X_0 \setminus \{0\} : \int_{\Omega} u e_1 \, dm_N = 0} \frac{\int_{\Omega} |\nabla u|^2_N \, dm_N}{\int_{\Omega} u^2 \, dm_N}.
\]

The goal of this paper is to prove the following theorem:

**Theorem 1.1.** The following estimate holds true

\[
\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq 1 + \frac{4}{N}.
\]

Consequently, we show that the result by Thompson [8] established in the case of the Laplace operator continues to hold true in the case of the Laplace operator with the drift \( x \cdot \nabla \cdot \). Note that even if the two cases seem to be very similar we can point out differences between them. For example, it is easy to check that the ratio \( \frac{\mu_2(\Omega)}{\mu_1(\Omega)} \) is invariant on rescaled domains. More precisely, if for some \( t > 0 \) we denote \( \Omega_t := t \Omega = \{tx : x \in \Omega\} \) then

\[
\frac{\mu_2(\Omega)}{\mu_1(\Omega)} = \frac{\mu_2(\Omega_t)}{\mu_1(\Omega_t)}, \quad \forall t > 0.
\]

This equality holds since a simple change of variable shows that \( \mu_i(\Omega_t) = t^{-2} \mu_i(\Omega) \) for \( i \in \{1, 2\} \). Such a relation fails to hold true in the case of \( \lambda_i(\Omega_t) \) and consequently in general

\[
\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \neq \frac{\lambda_2(\Omega_t)}{\lambda_1(\Omega_t)} .
\]

In other words, we want to point out the fact that in the case of the Laplace operator with the drift \( x \cdot \nabla \cdot \) the dependence of the ratio \( \frac{\lambda_i(\Omega)}{\lambda_1(\Omega)} \) on the domain \( \Omega \) is more involved than in the case of the ratio \( \frac{\mu_i(\Omega)}{\mu_1(\Omega)} \). Despite this fact, we can establish the same bound from above \( (1 + 4N^{-1}) \) which depends only on the dimension of the Euclidean space and not on the domain \( \Omega \) on which the eigenvalue problem is analysed.

2. Proof of Theorem 1.1

For each \( i \in \{1, \ldots, N\} \), define

\[
A_i := \int_{\Omega} \left( \frac{\partial e_1}{\partial x_i} \right)^2 \, dm_N .
\]
Assume that $A_1 = \min_{i \in \{1, \ldots, N\}} A_i$. Then we have

$$A_1 \leq \frac{1}{N} \sum_{i=1}^{N} A_i.$$  

(5)

Next, note that taking $t_i := \int_{\Omega} x_i e_i^2 \, dm_N$ we have

$$\int_{\Omega} (x_i - t_i) e_i^2 \, dm_N = 0, \quad \forall \, i \in \{1, \ldots, N\}$$  

(6)

since function $e_i$ satisfies (4).

By the definition of $\lambda_2(\Omega)$ and the fact that $u \in H^1_0(\Omega; dm_N)$ if and only if $ue^{\lambda_i/2} \in H^1_0(\Omega; dx)$ and $|x|^u ue^{\lambda_i/2} \in L^2(\Omega)$ (see [2, Proposition 2.3]) we deduce

$$\lambda_2(\Omega) \leq \frac{\int_{\Omega} |\nabla [(x_i - t_i) e_i]|^2_N \, dm_N}{\int_{\Omega} (x_i - t_i)^2 e_i^2 \, dm_N}, \quad \forall \, i \in \{1, \ldots, N\}.$$  

(7)

Simple computations imply that for each $i \in \{1, \ldots, N\}$ we have

$$\nabla [(x_i - t_i) e_i] = \left( (x_i - t_i) \frac{\partial e_i}{\partial x_1}, \ldots, (x_i - t_i) \frac{\partial e_i}{\partial x_i} + e_i, \ldots, (x_i - t_i) \frac{\partial e_i}{\partial x_N} \right),$$

and, thus, we find

$$|\nabla [(x_i - t_i) e_i]|^2_N = (x_i - t_i)^2 |\nabla e_i|^2_N + 2(x_i - t_i) e_i \frac{\partial e_i}{\partial x_i} + e_i^2.$$  

Multiplying the last equality by $e^{\lambda_i/2}$ and then integrating on $\Omega$ we get

$$\int_{\Omega} |\nabla [(x_i - t_i) e_i]|^2_N \, dm_N = \int_{\Omega} (x_i - t_i)^2 |\nabla e_i|^2_N \, dm_N + 2 \int_{\Omega} (x_i - t_i) e_i \frac{\partial e_i}{\partial x_i} \, dm_N + \int_{\Omega} e_i^2 \, dm_N.$$  

(8)

On the other hand, since $\lambda_1(\Omega)$ is the first eigenvalue of problem (2) with the corresponding eigenfunction $e_1$ satisfying (4), by (3) with $\lambda = \lambda_1(\Omega)$, $u_\lambda = e_1$ and $\varphi = (x_i - t_i)^2 e_1$ we know that

$$\int_{\Omega} \nabla e_1 \cdot \nabla [(x_i - t_i)^2 e_1] \, dm_N = \lambda_1(\Omega) \int_{\Omega} (x_i - t_i)^2 e_1^2 \, dm_N, \quad \forall \, i \in \{1, \ldots, N\}.$$  

(9)

Simple computations imply that for each $i \in \{1, \ldots, N\}$ we have

$$\nabla [(x_i - t_i)^2 e_1] = \left( (x_i - t_i)^2 \frac{\partial e_i}{\partial x_1}, \ldots, (x_i - t_i)^2 \frac{\partial e_i}{\partial x_i} + 2(x_i - t_i) e_i \frac{\partial e_i}{\partial x_i}, \ldots, (x_i - t_i)^2 \frac{\partial e_i}{\partial x_N} \right),$$

and, thus, we find

$$\nabla e_1 \cdot \nabla [(x_i - t_i)^2 e_1] = (x_i - t_i)^2 |\nabla e_i|^2_N + 2(x_i - t_i) e_i \frac{\partial e_i}{\partial x_i}.$$  

Multiplying the last equality by $e^{\lambda_i/2}$ and then integrating on $\Omega$ and using (9) we find

$$\int_{\Omega} (x_i - t_i)^2 |\nabla e_i|^2_N \, dm_N + 2 \int_{\Omega} (x_i - t_i) e_i \frac{\partial e_i}{\partial x_i} \, dm_N = \lambda_1(\Omega) \int_{\Omega} (x_i - t_i)^2 e_i^2 \, dm_N, \quad \forall \, i \in \{1, \ldots, N\}.$$  

(10)

By (7), (8) and (10) we infer

$$\lambda_2(\Omega) \leq \frac{\int_{\Omega} |\nabla [(x_i - t_i) e_i]|^2_N \, dm_N}{\int_{\Omega} (x_i - t_i)^2 e_i^2 \, dm_N} = \frac{\int_{\Omega} (x_i - t_i)^2 |\nabla e_i|^2_N \, dm_N + 2 \int_{\Omega} (x_i - t_i) e_i \frac{\partial e_i}{\partial x_i} \, dm_N + \int_{\Omega} e_i^2 \, dm_N}{\int_{\Omega} (x_i - t_i)^2 e_i^2 \, dm_N}$$  

$$= \lambda_1(\Omega) + \frac{\int_{\Omega} e_i^2 \, dm_N}{\int_{\Omega} (x_i - t_i)^2 e_i^2 \, dm_N}, \quad \forall \, i \in \{1, \ldots, N\}.$$  

(11)
Further, note that, integrating by parts, we get
\[
2 \int_\Omega (x_i - t_i) e_1 \frac{\partial e_1}{\partial x_i} \, dm_N = 2 \int_\Omega (x_i - t_i) e_1 \frac{\partial e_1}{\partial x_i} e_1 |x|^2 \, dx
\]
\[
= \int_\Omega (x_i - t_i) \frac{\partial e_1^2}{\partial x_i} e_1 |x|^2 \, dx
\]
\[
= \int_{\partial \Omega} (x_i - t_i) \frac{\partial e_1^2}{\partial x_i} e_1 |x|^2 \sigma_i \, d\sigma(x) - \int_\Omega \frac{\partial}{\partial x_i} \left((x_i - t_i) e_1 |x|^2 \right) e_1^2 \, dx
\]
\[
= -\int_\Omega \left[e_1 |x|^2 + (x_i - t_i) x_i e_1 |x|^2 \right] e_1^2 \, dx
\]
\[
= -\int e_1^2 \, dm_N - \int_\Omega (x_i - t_i) x_i e_1^2 \, dm_N, \quad \forall i \in \{1, \ldots, N\}.
\]

Since relation (6) holds true, we have
\[
\int_\Omega (x_i - t_i) x_i e_1^2 \, dm_N = \int_\Omega (x_i - t_i)^2 e_1^2 \, dm_N + t_i \int_\Omega (x_i - t_i) e_1^2 \, dm_N = \int_\Omega (x_i - t_i)^2 e_1^2 \, dm_N, \quad \forall i \in \{1, \ldots, N\}
\]
and then we deduce
\[
\int_\Omega e_1^2 \, dm_N = -\int_\Omega (x_i - t_i)^2 e_1^2 \, dm_N - 2 \int_\Omega (x_i - t_i) e_1 \frac{\partial e_1}{\partial x_i} \, dm_N, \quad \forall i \in \{1, \ldots, N\}.
\]
The above equality implies
\[
\int_\Omega e_1^2 e_1 |x|^2 \, dx \leq -2 \int_\Omega (x_i - t_i) e_1 \frac{\partial e_1}{\partial x_i} e_1 |x|^2 \, dx, \quad \forall i \in \{1, \ldots, N\},
\]
or, using Hölder’s inequality, we get
\[
\left( \int_\Omega e_1^2 e_1 |x|^2 \, dx \right)^2 \leq 4 \left( \int_\Omega (x_i - t_i) e_1 \frac{\partial e_1}{\partial x_i} \, dx \right)^2 \left( \int_\Omega e_1^2 |x|^2 \, dx \right)^2, \quad \forall i \in \{1, \ldots, N\}.
\]
Equivalently, we have
\[
\frac{\int_\Omega e_1^2 \, dm_N}{\int_\Omega (x_i - t_i)^2 e_1^2 \, dm_N} \leq 4 \frac{\int_\Omega \left( e_1 \frac{\partial e_1}{\partial x_i} \right)^2 \, dm_N}{\int_\Omega e_1^2 \, dm_N}, \quad \forall i \in \{1, \ldots, N\}.
\]
Relations (11) and (12) yield
\[
\lambda_2(\Omega) \leq \lambda_1(\Omega) + 4 \frac{\int_\Omega \left( e_1 \frac{\partial e_1}{\partial x_i} \right)^2 \, dm_N}{\int_\Omega e_1^2 \, dm_N} = \lambda_1(\Omega) + 4 A_i, \quad \forall i \in \{1, \ldots, N\}.
\]
Letting \( i = 1 \) above and using (5) we conclude that
\[
\lambda_2(\Omega) \leq \lambda_1(\Omega) + 4 \frac{\int_\Omega |\nabla e_1|^2 \, dm_N}{\int_\Omega e_1^2 \, dm_N} = \lambda_1(\Omega) + \frac{4}{N} \lambda_1(\Omega).
\]
The proof of Theorem 1.1 is now complete.

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