Limit Distributions for Smooth Total Variation and $\chi^2$-Divergence in High Dimensions

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Abstract—Statistical divergences are ubiquitous in machine learning as tools for measuring discrepancy between probability distributions. As these applications inherently rely on approximating distributions from samples, we consider empirical approximation under two popular estimating distributions from samples, we consider empirical approximation under two popular $f$-divergences: the total variation (TV) distance and the $\chi^2$-divergence. To circumvent the sensitivity of these divergences to support mismatch, the framework of Gaussian smoothing is adopted. We study the limit distributions of $\sqrt{n}d_{TV}(P_n \ast N_\sigma, P \ast N_\sigma)$ and $n\chi^2(P_n \ast N_\sigma \| P \ast N_\sigma)$, where $P_n$ is the empirical measure based on $n$ independently and identically distributed (i.i.d.) observations from $P$, $N_\sigma := N(0, \sigma^2 I_d)$, and $\ast$ stands for convolution. In arbitrary dimension, the limit distributions are characterized in terms of an integral operator of a centered Gaussian process on $\mathbb{R}^d$ with covariance operator that depends on $P$ and the isotropic Gaussian density of parameter $\sigma$. This, in turn, implies optimality of the $n^{-1/2}$ expected value convergence rates recently derived for $d_{TV}(P_n \ast N_\sigma, P \ast N_\sigma)$ and $\chi^2(P_n \ast N_\sigma \| P \ast N_\sigma)$. These strong statistical guarantees promote empirical approximation under Gaussian smoothing as a potent framework for learning and inference based on high-dimensional data.

I. INTRODUCTION

Statistical divergences are central to many fields, such as machine learning (ML), information theory and statistics. They quantify discrepancy between probability measures, which is central to those fields. Two fundamental statistical divergences are the total variation (TV) distance and the $\chi^2$-divergence, on which we focus herein. TV and $\chi^2$ fall under the broader framework of $f$-divergences \cite{2}. As such, they possess an array of important properties (data processing inequality, variational representation, etc.), making them appealing for analyzing and designing inference systems — see, e.g., \cite{3, 4} for recent applications of $f$-divergences to generative modeling. Focusing on statistical applications, where only samples of the underlying distributions are available, naturally leads to the question of empirical approximation under TV and $\chi^2$.

Suppose $P_n$ is the empirical measure induced by $n$ independently and identically distributed (i.i.d.) observations from a $d$-dimensional distribution $P$. We would like to consider the rate at which $P_n$ approaches $P$ under TV and $\chi^2$. However, in general, $P_n$ does not converge to $P$ under the TV topology (e.g., if $P$ is absolutely continuous with respect to (w.r.t.) Lebesgue) \cite{4}, while $\chi^2(P \| Q) = \infty$ whenever $P$ is not absolutely continuous w.r.t. $Q$. To obtain a well-posed empirical approximation setup, we adopt the Gaussian smoothing framework of \cite{5} (see also \cite{6, 7}). Accordingly, we define $d_{TV}^{(\sigma)}(P_n, P) := \delta_{TV}(P_n \ast N_\sigma, P \ast N_\sigma)$ and $\chi^2_\sigma := \chi^2(P_n \ast N_\sigma \| P \ast N_\sigma)$, where $P_n$ and $P$ are both convolved with an isotropic Gaussian measure $N_\sigma := N(0, \sigma^2 I_d)$. This alleviates the aforementioned pathologies since now both measures have densities supported on the entire space. Furthermore, \cite{5} showed that $\mathbb{E}[\delta_{TV}^{(\sigma)}(P_n, P)] = O(n^{-1/2})$ and $\mathbb{E}[\chi^2_\sigma(P_n \| P)] = O(n^{-1})$ for subgaussian $P$ in any dimension $d$\footnote{The full results have a prefactor of $c^d$ that quantifies the dependence on dimension. Still, $n$ and $d$ are decoupled and treating $d$ as fixed results in a convergence rate independent of it.}. Our objective is to upgrade these results by characterizing the limit distributions of their (normalized) versions and establish optimality of the above rates.

Building on empirical process theory and probability theory in Banach space\footnote{The reader is referred to, e.g., \cite{8, 9, 10} as useful references.} we characterize the limit distributions of $\sqrt{n}d_{TV}^{(\sigma)}(P_n, P)$ and $n\chi^2_\sigma(P_n \| P)$ as $n \to \infty$. Both limits are given in terms of an integral operator of a centered Gaussian process $B^{(\sigma)}_p := (B^{(\sigma)}_p(x))_{x \in \mathbb{R}^d}$. The covariance function of $B^{(\sigma)}_p$ depends on the data distribution $P$ and the noise parameter $\sigma$. Gaussian-smoothing is crucial here since it allows reasoning about TV and $\chi^2$ in terms of $L^p$ norms. With that perspective, the limit distribution results follow from the central limit theorem (CLT) in $L^1$ for TV and $L^2$ for $\chi^2$. A direct consequence of the limit distribution results is the optimality of the expected value convergence rates derived in \cite{5}, which was not established therein. A concentration inequality for $\delta_{TV}^{(\sigma)}$ via McDiarmid’s inequality is also derived. Our results hold under milder assumptions on $P$ than in \cite{5}.

Most related to this work is \cite{7}, where the limit distribution of smooth empirical 1-Wasserstein distance was derived for all dimensions (the proof techniques employed therein are quite different from those in this work). A corresponding result for unsmoothed 1-Wasserstein is only known in the 1-dimensional case \cite{11}. Limit distributions under TV, $\chi^2$ and other $f$-divergences, were also studied before but under a framework different than ours. For example, \cite{12} and \cite{13} consider the TV distance and $\chi^2$-divergence between the distribution of a normalized sum of independent random variables and a Gaussian. They find conditions under which convergence holds and characterise the corresponding rates. Another related work is \cite{14} where bounds on the Poincaré constant of convolved...
measures were derived, with application to limit distribution questions under the same framework as [12, 13] (see also [15]). These results significantly differ from those presented herein, as we focus on (smooth) empirical TV and \( \chi^2 \), i.e., \( \sqrt{n}d_{TV}(\sigma) \)(\(P_n, P\)) and \( n\chi^2(P_n\|P) \), as the random variable sequence of interest (as opposed to distance measure between the distribution of a normalized sum and its Gaussian limit).

From a broader perspective, our results suggest that smooth statistical divergences may be particularly well-suited for high-dimensional learning and inference tasks. Statistical divergences can be used to formulate a variety of ML tasks, from generative modeling to outlier detection to ensemble methods. Such formulations lend themselves well for a theoretical analysis, but, their usefulness diminishes as the data dimension grows larger. This is because statistical divergences suffer from the curse of dimensionality (CoD) in empirical approximation, with error convergence rates effectively decaying as \( n^{-1/d} \), where \( n \) is the sample size and \( d \) is the data dimension (see, e.g., [16, 17] and [18] for CoD results for Wasserstein distances [19]. \( f \)-divergences and integral probability metrics [20], respectively). On the other hand, when smoothing is introduced, empirical convergence of TV and \( \chi^2 \) accelerates to dimension-free rates, posing then them as favorable figures of merit.

**Notation:** Let \( \| \cdot \| \) denote the Euclidean norm, and \( x, y \in \mathbb{R}^d \), denote their inner product. Let \( \mathcal{P}(\mathbb{R}^d) \) denote the class of Borel probability measures on \( \mathbb{R}^d \), and \( \mathcal{B}(\mathbb{R}^d) \) denote the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \). The isotropic Gaussian measure on \( \mathbb{R}^d \) with parameter \( \sigma > 0 \) is denoted as \( \mathcal{N}_\sigma := \mathcal{N}(0,\sigma^2 I_d) \) and its probability density function is denoted as \( \varphi_\sigma \), i.e., \( \varphi_\sigma(x) = (2\pi\sigma^2)^{-d/2}e^{-\|x\|^2/(2\sigma^2)} \), for \( x \in \mathbb{R}^d \). Given \( P, Q \in \mathcal{P}(\mathbb{R}^d) \), their convolution \( P * Q \in \mathcal{P}(\mathbb{R}^d) \) is defined as \( (P * Q)(A) := \int 1_A(x+y) dP(x) dQ(y) \) for \( A \in \mathcal{B}(\mathbb{R}^d) \), where \( 1_A \) is the indicator function of \( A \). The convolution between \( P \in \mathcal{P}(\mathbb{R}^d) \) and a measurable function \( f \) on \( \mathbb{R}^d \) is defined as \( P * f(x) := \int_{\mathbb{R}^d} f(x-y) dP(y) \), for \( x \in \mathbb{R}^d \), whenever the latter integral is well-defined for all \( x \in \mathbb{R}^d \) (cf. [21] p. 271).

Given \( P \in \mathcal{P}(\mathbb{R}^d) \) and \( X_1, \ldots, X_n \sim P \) i.i.d., the empirical distribution is defined by \( P_n := n^{-1} \sum_{i=1}^{n} \delta_{X_i} \), where \( \delta_{x} \) denotes the Dirac measure at \( x \). A stochastic process \( B = (B(t))_{t \in T} \) is called Gaussian if for any finite subset \( F \) of \( T \), \( (B(t))_{t \in F} \) is Gaussian. A version of \( B \) is another stochastic process with the same finite dimensional distributions.

II. LIMIT DISTRIBUTION RESULTS

We derive limit distributions of smooth TV distance and \( \chi^2 \)-divergence between \( P_n \) and \( P \). These results rely substantially on CLTs in \( L^p \) spaces, and for the reader’s convenience, we summarize basic CLT results in \( L^p \) spaces in Appendix A.

**A. Smooth Total Variation Distance**

Consider \( \delta_{TV}(P, Q) := \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |P(A) - Q(A)| \) the TV distance between \( P, Q \in \mathcal{P}(\mathbb{R}^d) \), and define its smooth version as \( \delta_{TV}(\sigma)(P, Q) := \delta_{TV}(P * N_\sigma, Q * N_\sigma) \). Let \( B^{(\sigma)}_P = \left( B^{(\sigma)}_P(x) \right)_{x \in \mathbb{R}^d} \) denote a centered Gaussian process with covariance function

\[
\mathbb{E} \left[ B^{(\sigma)}_P(x) B^{(\sigma)}_P(y) \right] = \text{Cov}_P \left( \varphi_\sigma(x - \cdot), \varphi_\sigma(y - \cdot) \right),
\]

for all \( x, y \in \mathbb{R}^d \). The following theorem derives a limit distribution and moment bound for \( \delta_{TV}(\sigma)(P_n, P) \) under a certain moment condition on \( P \).

**Theorem 1** (Limit distribution for \( \delta_{TV}(\sigma) \)). If

\[
\int_{\mathbb{R}^d} \sqrt{\text{Var}_P(\varphi_\sigma(x - \cdot))} \, dx < \infty,
\]

then there exists a version of \( B^{(\sigma)}_P \) that is an \( L^1(\mathbb{R}^d) \)-valued random variable such that

\[
\sqrt{n}\delta_{TV}(\sigma)(P_n, P) \xrightarrow{\mathcal{D}} \frac{1}{2} \int_{\mathbb{R}^d} |B^{(\sigma)}_P(x)| \, dx. \tag{2a}
\]

In addition, we have

\[
\sqrt{n} \mathbb{E} \left[ \delta_{TV}(\sigma)(P_n, P) \right] \leq \frac{1}{2} \int_{\mathbb{R}^d} \sqrt{\text{Var}_P(\varphi_\sigma(x - \cdot))} \, dx. \tag{2b}
\]

The following lemma derives a sufficient condition for Condition (1) to hold.

**Lemma 1** (Sufficient condition for (1)). We have for \( X \sim P \),

\[
\int_{\mathbb{R}^d} \sqrt{\text{Var}_P(\varphi_\sigma(x - \cdot))} \, dx \leq \frac{\sigma^d/2 + \frac{2d}{\sigma^d(2d/2)}}{\sigma^d} \int_0^\infty t^{d/2-1} \sqrt{\mathbb{P}(\|X\| > t)} \, dt.
\]

Thus, if \( P \) has finite \((2d + \epsilon)\) moments for some \( \epsilon > 0 \), i.e., \( \mathbb{E}[\|X\|^{2d+\epsilon}] < \infty \), then Condition (1) holds.

Proposition 2 in [15] derives a moment bound on \( \delta_{TV}(\sigma)(P_n, P) \) assuming sub-Gaussian \( P \). This condition is substantially relaxed in Theorem 1 above (in addition to deriving a limit distribution). In fact, Condition (1) is sharp for the first moment of \( \delta_{TV}(\sigma)(P_n, P) \) to be of order \( n^{-1/2} \). The following proposition shows that if (1) does not hold, then \( \delta_{TV}(\sigma)(P_n, P) \) has a rate strictly slower than \( n^{-1/2} \). We say that a sequence of random variables \( Y_n \) is stochastically bounded (or tight) if for any \( \epsilon > 0 \), there exists a constant \( M = M_\epsilon \) such that \( \mathbb{P}(\|Y_n\| > M) \leq \epsilon \) for all \( n \).

**Proposition 1** (Sharpness of Condition (1)). If \( \sqrt{n}\delta_{TV}(\sigma)(P_n, P) \) is stochastically bounded, then Condition (1) holds.

The proof indeed shows that

\[
\liminf_n \sqrt{n} \mathbb{E} \left[ \delta_{TV}(\sigma)(P_n, P) \right] \geq \frac{1}{\sqrt{2\pi n}} \int_{\mathbb{R}^d} \sqrt{\text{Var}_P(\varphi_\sigma(x - \cdot))} \, dx
\]

without assuming Condition (1). The stochastic boundedness of \( \sqrt{n}\delta_{TV}(\sigma)(P_n, P) \) implies the boundedness of the first moment by Hoffmann-Jørgensen’s inequality, so the conclusion of the corollary follows. See Section III-C for details.

Finally, we state a concentration inequality for \( \delta_{TV}(\sigma)(P_n, P) \).
Corollary 1 (Concentration inequality for \( \delta_{TV}^{(\sigma)} \)). Under Condition [1], we have
\[
\mathbb{P}\left( \delta_{TV}^{(\sigma)}(P_n, P) \geq \mathbb{E}\left[ \delta_{TV}^{(\sigma)}(P_n, P) \right] + t \right) \leq e^{-nt^2/2}, \quad \forall t > 0.
\]
(4)

This result follows from a simple application of McDiarmid’s inequality (cf. \([22]\) or \([10]\) Theorem 3.3.14), together with the fact that \( \| \varphi_\sigma(\cdot - x) \|_1 = \int_{\mathbb{R}^d} \varphi_\sigma(x - X) \, dx = 1 \) (cf. the proof of Theorem 1). We omit the details for brevity.

B. Smooth \( \chi^2 \)-Divergence

The \( \chi^2 \)-divergence between \( P \) and \( Q \) is \( \chi^2(P||Q) := \int \frac{(dP - dQ)}{dQ} \, dQ \). We have the following limit distribution result for its smooth empirical version \( \chi^2_{TV}^{(\sigma)}(P||Q) := \chi^2(P * \mathcal{N}_\sigma|Q * \mathcal{N}_\sigma) \). Recall that \( P * \varphi_\sigma(x) := \int_{\mathbb{R}^d} \varphi_\sigma(x - y) \, dP(y) \).

**Theorem 2** (Limit distribution for \( \chi^2_{TV}^{(\sigma)} \)). If
\[
\int_{\mathbb{R}^d} \frac{\text{Var}_P(\varphi_\sigma(x - \cdot))}{P * \varphi_\sigma(x)} \, dx < \infty,
\]
then there exists a version of \( B_{P}^{(\sigma)} \) such that \( B_{P}^{(\sigma)} / \sqrt{P * \varphi_\sigma} \) is an \( L^2(\mathbb{R}^d) \)-valued random variable and
\[
n\chi^2_{TV}^{(\sigma)}(P_n||P) \overset{d}{\to} \int_{\mathbb{R}^d} \frac{|B_{P}^{(\sigma)}(x)|^2}{P * \varphi_\sigma(x)} \, dx.
\]
In addition, we have
\[
n\mathbb{E}\left[ \chi^2_{TV}^{(\sigma)}(P_n||P) \right] = \int_{\mathbb{R}^d} \frac{\text{Var}_P(\varphi_\sigma(x - \cdot))}{P * \varphi_\sigma(x)} \, dx.
\]
(6a)

Condition [5] holds if \( P \) is \( \beta \)-sub-Gaussian with \( \beta > \sigma / \sqrt{2} \).

**Definition 1** (Sub-Gaussian distribution). We call \( P \in \mathcal{P}(\mathbb{R}^d) \) \( \beta \)-sub-Gaussian, for \( \beta > 0 \), if \( X \sim P \) satisfies
\[
\mathbb{E}\left[ \exp(\alpha \cdot (X - \mathbb{E}[X])) \right] \leq e^{\alpha^2 \|\alpha\|^2/2}, \quad \forall \alpha \in \mathbb{R}^d.
\]

Let \( Z \sim \mathcal{N}(0, I_d) \). By Definition [4] for any \( \beta \)-sub-Gaussian \( X \) with mean zero and \( 0 \leq \eta < 1 / (2\beta^2) \), we have
\[
\mathbb{E}\left[ e^{\eta \|X\|^2} \right] = \mathbb{E}\left[ e^{\sqrt{2\eta}X \cdot Z} | X \right] = \mathbb{E}\left[ e^{\sqrt{2\eta}X \cdot Z} | Z \right] 
\leq \mathbb{E}\left[ e^{\beta^2 \eta \|Z\|^2} \right] = (1 - 2\beta^2 \eta)^{-d/2},
\]
(7)
where the last equality is because \( \|Z\|^2 \) has the \( \chi^2 \)-distribution with \( d \)-degrees of freedom (cf. \([23]\) Remark 2.3)).

**Lemma 2** (Sufficient condition for [5]). If \( P \) is \( \beta \)-sub-Gaussian for some \( \beta < \sigma / \sqrt{2} \), then Condition [5] holds.

The proof also derives an explicit bound on the integral in [5]. **Lemma 2** improves upon [5] Proposition 3) that shows that \( \mathbb{E}\left[ \chi^2_{TV}^{(\sigma)}(P_n||P) \right] = O(n^{-1}) \) for \( \beta \)-sub-Gaussian \( P \) with \( \beta < \sigma / 2 \). Proposition 4 in [5] shows that if \( \beta > \sqrt{2} \sigma \), then \( \mathbb{E}\left[ \chi^2_{TV}^{(\sigma)}(P_n||P) \right] \) need not be finite, so in general a sub-Gaussian condition on \( P \) is necessary to control \( \mathbb{E}\left[ \chi^2_{TV}^{(\sigma)}(P_n||P) \right] \).

Finally, we point out that if \( P = \mathcal{N}_\beta (\sigma) \), then [5] holds for any \( \beta > 0 \). More generally, the following holds.

**Lemma 3.** Suppose that \( P \) is \( \beta \)-sub-Gaussian for some \( \beta > 0 \), and \( X - \mathbb{E}[X] \), with \( X \sim P \), has a Lebesgue density bounded from below by \( ce^{-\|x\|^2/(2\gamma)} \) for some positive constants \( c, \gamma \). Then, Condition [5] holds if \( \beta < \sqrt{2\sigma^2 + 2\gamma} \). In particular, Condition [5] holds if \( P \) is Gaussian with covariance matrix \( \Sigma \) and \( \lambda_{\max} < \lambda_{\min} + \sigma^2 / 2 \), where \( \lambda_{\max} \) and \( \lambda_{\min} \) are the maximum and minimum eigenvalues of \( \Sigma \), respectively.

**III. PROOFS**

**A. Proof of Theorem 7**

Our argument relies on the CLT in \( L^1(\mathbb{R}^d) \); cf. Appendix A. Recall that \( P_n \sim \mathcal{N}_\sigma \) has density \( n^{-1} \sum_{i=1}^{n} Y_i(x) =: Y_n(x) \) with \( Y_i(x) := \varphi_\sigma(x - X_i) \). The process \( Y_i \) is jointly measurable and has paths in \( L^1(\mathbb{R}^d) \); indeed,
\[
\|Y_i\|_1 = \int_{\mathbb{R}^d} \varphi_\sigma(x - X_i) \, dx = 1.
\]
In addition, \( \mathbb{E}[Y_i] = P * \varphi_\sigma \). By Theorem 3 \( \sqrt{n}(Y_n - P * \varphi_\sigma) \) converges weakly to a Gaussian variable if and only if \( \int_{\mathbb{R}^d} \sqrt{\text{Var}(Y_i(x))} \, dx = \infty \).

One readily verifies that the limit Gaussian variable is \( B_{P}^{(\sigma)} \), and by the continuous mapping theorem, we have
\[
\sqrt{n}\delta_{TV}^{(\sigma)}(P_n, P) = \frac{1}{2} \int_{\mathbb{R}^d} \sqrt{n|\nabla Y_n(x) - P * \varphi_\sigma(x)|} \, dx 
\to \frac{1}{2} \int_{\mathbb{R}^d} \|B_{P}^{(\sigma)}(x)\| \, dx.
\]
(8)
In addition,
\[
\mathbb{E}\left[ \delta_{TV}^{(\sigma)}(P_n, P) \right] = \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}\left[ |\nabla Y_n(x) - P * \varphi_\sigma(x)|^2 \right] \, dx 
\leq \frac{1}{2} \int_{\mathbb{R}^d} \mathbb{E}\left[ |\nabla Y_n(x) - P * \varphi_\sigma(x)|^2 \right] \, dx 
= \frac{1}{2\sqrt{n}} \int_{\mathbb{R}^d} \sqrt{\text{Var}_P(\varphi(x - \cdot))} \, dx.
\]
(9)
This completes the proof.

**B. Proof of Lemma 7**

We first note that for \( X \sim P \),
\[
\text{Var}_P(\varphi_\sigma(x - \cdot)) \leq \mathbb{E}[\varphi_\sigma^2(x - X)] = \frac{1}{(2\pi\sigma^2)^d} \int_{\mathbb{R}^d} e^{-\|x-y\|^2/2} \, dP(y).
\]
Simplifying the integral over \( \mathbb{R}^d \) into \( \|y\| \leq \|x\|/2 \) and \( \|y\| > \|x\|/2 \), we have
\[
\int_{\mathbb{R}^d} e^{-\|x-y\|^2/\sigma^2} \, dP(y)
\leq \int_{\|y\| \leq \|x\|/2} e^{-\|x-y\|^2/\sigma^2} \, dP(y) + \mathbb{P}(\|X\| > \|x\|/2).
\]
(11)
Changing to polar coordinates, we have
\[
\int_{\mathbb{R}^d} \mathbb{P}(\|X\| > \|x\|/2) \, dx = \frac{2^{d+1}}{\Gamma(d/2)} \int_{0}^{\infty} e^{-t} \sqrt{\mathbb{P}(\|X\| > t)} \, dt.
\]
(12)
Second, recalling that \(\|x - y\|^2 \geq \|x\|^2/2 - \|y\|^2\), we have
\[
\int_{\|y\| \leq \|x\|/2} e^{-\|x - y\|^2/\sigma^2} \, dP(y) \leq e^{-\|x\|^2/(4\sigma^2)} \int_{\|y\| \leq \|x\|/2} dP(y) \leq e^{-\|x\|^2/(4\sigma^2)}. \tag{13}
\]
The square root of the RHS integrates to \((16\pi\sigma^2)^{d/2}\). Combining (10)–(13), we obtain inequality (5). Finally, if \(P\) has finite \((2d + \epsilon)\) moments, then by Markov’s inequality, we have
\[
e^{-t^{d-1}\sqrt{P(\|x\| > t)}} \leq \min\{t^{d-1}, \sqrt{E[\|x\|^{2d+\epsilon}/t^{1+\epsilon/2}]}\}.
\]
The RHS is integrable on \([0, \infty)\). This completes the proof.

\[\]

C. Proof of Proposition 2

We divide the proof into two steps.

**Step 1:** We show that if \(\sqrt{n}d^{[\sigma]}(P_n, P)\) is stochastically bounded, then its first moment is bounded (in \(n\)). Let \(S_n = \sum_{i=1}^n (\varphi(x - X_i) - P \ast \varphi(x))\). By Hoffmann-Jørgensen’s inequality (see [8] Proposition 6.8)), we have
\[
E[\|S_n\|_1] \leq \max_{1 \leq i \leq n} \|\varphi_\ast(x) - P \ast \varphi(x)\| + t_{n,0},
\]
where \(t_{n,0} = \inf\{t > 0 : \mathbb{P}(\max_{1 \leq m \leq n} \|S_m\|_1 > t) \leq \frac{1}{n}\} \). The first term on the RHS is bounded. In addition, by Montgomery-Smith’s inequality [24] Corollary 3), there exists a universal constant \(c\) such that
\[
t_{n,0} \leq \inf\{t > 0 : \mathbb{P}(\|S_n\|_1 > t) \leq c\}
\]

If \(d^{[\sigma]}(P_n, P) = \|S_n/\sqrt{n}\|_1\) is stochastically bounded, then \(t_{n,0}/\sqrt{n} \to 0\) as \(n \to \infty\), which implies \(\limsup \mathbb{E}[\|S_n\|_1] \leq \infty\).

**Step 2:** We prove that if \(\sqrt{n}d^{[\sigma]}(P_n, P)\) is bounded, then Condition (1) holds. Let \(k\) be any positive integer. By Fubini’s theorem
\[
\sqrt{n}E[d^{(\sigma)}(P_n, P)] \geq \frac{1}{2} \int_{\mathbb{R}^d} E[\|\sqrt{n}Y_n(x) - P \ast \varphi(x)\|] \, dx.
\]
Since \(\|Y_i(x)\| \leq (2\pi\sigma^2)^{-d/2} = : C_{d,\sigma}\), the central limit theorem implies that
\[
\lim_n E\left[\left(\|\sqrt{n}Y_n(x) - P \ast \varphi(x)\|\right) \wedge k\right] = E\left[\|B^{(\sigma)}P(x)\| \wedge k\right]
\]
for any \(x \in \mathbb{R}^d\). Indeed, let \(Q_x\) denote the image measure of \(P\) under the map \(y \mapsto \varphi(y - x)\), and let \(Q = \{Q_x : x \in \mathbb{R}^d\}\). Each \(Q \in \mathcal{Q}\) is supported in \([-C_{d,\sigma}, C_{d,\sigma}]\). For each \(Q \in \mathcal{Q}\), let \(Q_1, \ldots, Q_n\) be i.i.d. with common distribution \(Q\). Let \(\sigma^Q_n\) denote the variance of \(Q\) (note that \(\sigma^Q_n = \varphi R P(x - \cdot) = \varphi(B^{(\sigma)}P(x))\)). Then, the central limit theorem implies that
\[
\lim_n E[d^{(\sigma)}(P_n \ast N_{\sigma}, P \ast N_{\sigma})] = \lim_n \sum_{i=1}^n \frac{\xi^Q_i - E[\xi^Q_i]}{\sqrt{n}} \leq C_{d,\sigma} N_{\sigma}^2.
\]
Since \(\|y\| \wedge k\) is bounded (by \(k\)) and (Lipschitz) continuous, the convergence (14) follows from the definition of weak convergence. Together with Fatou’s lemma, we have
\[
\liminf_n \sqrt{n}E[d_{TV}(P_n \ast N_{\sigma}, P \ast N_{\sigma})] \geq \frac{1}{2} \int_{\mathbb{R}^d} E[\|B^{(\sigma)}P(x)\| \wedge k] \, dx.
\]
Taking \(k \to \infty\), we conclude that
\[
\liminf_n \sqrt{n}E\left[d^{(\sigma)}_{TV}(P_n, P)\right] \geq \frac{1}{2} \int_{\mathbb{R}^d} E[\|B^{(\sigma)}P(x)\|] \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} \sqrt{\text{Var}_P(\varphi(x - \cdot))} \, dx,
\]
where the second equality is because \(E[\|\xi\|] = \sqrt{2\pi} \sqrt{E[\xi^2]}\) for a centered Gaussian variable \(\xi\). This completes the proof.

\[\]

D. Proof of Theorem 2

We will apply the CLT in \(L^2(P \ast N_{\sigma})\); cf. Appendix A. Let \(Z_i := \frac{\varphi_\ast(x - X_i)}{\sqrt{n}} - 1\). The process \(Z_i\) is jointly measurable and has paths almost surely in \(L^2(P \ast N_{\sigma})\); indeed,
\[
\mathbb{E}\left[\int_{\mathbb{R}^d} \|Z_i(x)\|^2 \, dP \ast N_{\sigma}(x)\right] = \int_{\mathbb{R}^d} \mathbb{E}\left[\frac{\varphi_\ast(x - \cdot)}{\sqrt{n}}\right] \, dx < \infty.
\]
Hence, we apply Theorem 3 to conclude that \(\sum_{i=1}^n Z_i/\sqrt{n}\) converges weakly in \(L^2(P \ast N_{\sigma})\) to a Gaussian variable. The limit Gaussian variable is \(B^{(\sigma)}P/P \ast \varphi_\ast\), and by the continuous mapping theorem, we have
\[
\lim_{n} n\chi^2(P_n, P) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{B^{(\sigma)}P(x)}{P \ast \varphi_\ast(x)} \, dx\right)^2 \, dP \ast N_{\sigma}(x)
\]
Finally, since \(L^2(P \ast N_{\sigma})\) is a Hilbert space, we have
\[
\mathbb{E} \left[\chi^2(P_n, P)\right] = \mathbb{E}\left[\sum_{i=1}^n \frac{Z_i}{\sqrt{n}}\right]^2 \leq n \int_{\mathbb{R}^d} \frac{B^{(\sigma)}P(x)}{P \ast \varphi_\ast(x)} \, dx.
\]
This completes the proof.

\[\]

E. Proof of Lemma 2

We first note that
\[
\int_{\mathbb{R}^d} \varphi R P(x - \cdot) \, dx \leq \int_{\mathbb{R}^d} \frac{\varphi(x - y)^2}{P \ast \varphi_\ast(x)} \, dx \, dP(y) = \int_{\mathbb{R}^d} e^{-\|y\|^2/\sigma^2} \frac{1}{(2\pi\sigma^2)^{d/2}} \int_{\mathbb{R}^d} e^{2y \cdot x/\sigma^2 - \|x\|^2/\sigma^2} \, dP(y).
\]
Since \(\|x - y\|^2 \leq (1+\eta)\|x\|^2 (1+1/\eta)\|y\|^2, \forall \eta \in (0, 1)\), we have
\[
P \ast \varphi_\ast(x) \geq e^{-(1+\eta)\|x\|^2/(2\sigma^2)} \frac{1}{(2\pi\sigma^2)^{d/2}} \int_{\mathbb{R}^d} e^{-(1+\eta)\|y\|^2/(2\sigma^2)} \, dP(y) = C_{P, \eta},
\]
so that
\[
\frac{1}{(2\pi\sigma^2)^{d/2}} \int_{\mathbb{R}^d} e^{2y \cdot x/\sigma^2 - \|x\|^2/\sigma^2} \, dx \leq \frac{1}{(1-\eta)^{d/2} C_{P, \eta}} \int_{\mathbb{R}^d} e^{2y \cdot x/\sigma^2} \varphi_\ast(x) \, dx \frac{1}{(1-\eta)^{d/2} C_{P, \eta}} \exp\left(\frac{2\|y\|^2}{(1-\eta)\sigma^2}\right).
\]
Conclude that
\[
\int_{\mathbb{R}^d} \frac{\text{Var}_{P}(\varphi_\beta(x - \cdot))}{P * \varphi_\beta(x)} \, dx \\
\leq \frac{1}{(1 - \eta)d^2/C_{P, \eta}} \int_{\mathbb{R}^d} \exp\left(\frac{1}{(1 - \eta)\sigma^2} - \frac{1}{2\sigma^2}\right) \, dP(y) \\
= \frac{1}{(1 - \eta)d^2/C_{P, \eta}} \left[ 1 - \frac{2(1 + \eta)^{\beta/2} - 1}{(1 - \eta)\sigma^2}\right]^{-d/2}
\]
by (7), provided that \((1 + \eta)^{\beta/2} < \frac{1}{2\sqrt{\eta}}\), i.e., \(\beta < \sigma\sqrt{\frac{1 - \eta}{1 + \eta}}\).
Since \(\eta \in (0, 1)\) is arbitrary, the desired result follows. \(\square\)

**F. Proof of Lemma 3**

By translation invariance of Lebesgue measure, we may assume \(P\) has zero mean. The proof is similar to that of Lemma 3 so we only outline required modifications. Observe that
\[
P * \varphi_\beta(x) \geq e^{-\frac{||x||^2}{2\sigma^2}} \int_{\mathbb{R}^d} e^{\frac{-y^2}{2\sigma^2}} \varphi_\beta \left( \frac{y}{\sqrt{\sigma^2 + \gamma}} \right) \, dy = e^{-\frac{||x||^2}{2(\sigma^2 + \gamma)}},
\]
so that
\[
\int_{\mathbb{R}^d} \frac{\varphi_\beta(x)}{P * \varphi_\beta(x)} \, dx \leq \int_{\mathbb{R}^d} \exp\left(\frac{2x \cdot y - \sigma^2 + 2\gamma}{2\sigma^2(\sigma^2 + \gamma)} \right) \frac{||x||^2}{dx} \\
\leq \exp\left(\frac{2(\sigma^2 + \gamma)}{\sigma^2} \frac{||y||^2}{dx}\right).
\]
Multiplying \(e^{-\frac{||y||^2}{2\sigma^2}}\) to the RHS leads to \(\exp\left(\frac{||y||^2}{\sigma^2(\sigma^2 + \gamma)}\right)\), whose integration w.r.t. \(P\) is finite as soon as \(\frac{\sigma^2 + 2\gamma}{\sigma^2} < \frac{1}{\gamma}\), i.e., \(\beta < \sqrt{\frac{\sigma^2}{2\gamma}}\). If \(P\) is Gaussian with covariance matrix \(\Sigma\), then we may take \(\beta = \sqrt{\lambda_{\max}}\) and \(\gamma = \lambda_{\min}\). \(\square\)

**APPENDIX A**

**CENTRAL LIMIT THEOREM IN L^p SPACES**

We summarize basic CLT results in \(L^p\) spaces with \(1 \leq p < \infty\). Let \(\mu\) be a \(\sigma\)-finite measure on a measurable space \((S, S)\) with \(S\) being countably generated, and let \(L^p = L^p(S, S, \mu), 1 \leq p < \infty\) be the space of real-valued measurable functions \(f\) on \(S\) such that \(\|f\|_p := (\int_S |f(s)|^p \, d\mu(s))^{1/p} < \infty\). As usual, we identify functions \(f, g\) on \(S\) if \(f = g\) \(\mu\)-a.e.; under this identification, the space \((L^p, \|\cdot\|_p)\) is separable Banach space (and thus Polish). Recall that any Borel measurable random variable with values in a Polish space is tight (Radon) by Ulam’s theorem (Theorem 7.1.3). A Borel measurable random variable with values in \(L^p\) is called an \(L^p\)-valued random variable.

For \(1 \leq p < \infty\), let \(q\) be its conjugate index, i.e., \(\frac{1}{p} + \frac{1}{q} = 1\) \((q = \infty\) if \(p = 1\)). For an \(L^p\)-valued random variable \(X\), let
\[
\tilde{X}(f) = \int_S f(s) X(s) \, d\mu(s), \ f \in L^q,
\]
which is a stochastic process indexed by \(L^q\), the dual space of \(L^p\). An \(L^p\)-valued random variable \(G\) is Gaussian if \(\{G(f) : f \in L^q\}\) is a Gaussian process.

Let \(X\) be an \(L^p\)-valued random variable such that \(E[X(f)] = 0\) and \(E[X(f)^2] < \infty\) for all \(f \in L^q\). \(X\) is said to be pre-Gaussian if there exists a centered \(L^p\)-valued Gaussian random variable \(G\) with the same covariance function as \(X\), i.e., \(E[G(f)G(g)] = E[X(f)X(g)]\) for all \(f, g \in L^q\).

Note that if \(X\) is a jointly measurable (Gaussian) process with paths in \(L^p\) then \(X\) can be identified to be an \(L^p\)-valued (Gaussian, resp.) random variable and vice versa; cf. [26].

**Theorem 3** (Proposition 2.1.11 in [9].) Let \(1 \leq p < \infty\), and \(X, X_1, \ldots, X_n\) be i.i.d. \(L^p\)-valued random variables with zero mean (in the sense of Bochner). The following are equivalent:

(i) There exists a centered Gaussian variable \(G\) in \(L^p\) with the same covariance function as \(X\) such that \(S_n := \sum_{i=1}^n X_i/\sqrt{n}\) converges weakly in \(L^p\) to \(G\).

(ii) \(\int_S (E[|X(s)|^2])^{p/2} \, d\mu(s) < \infty\) and \(P(||X||_p > \tau) = o(t^{-2})\) as \(t \to \infty\).

**Proof.** This is [9, Proposition 2.1.11] but it does not contain a proof. The proposition cites Theorem 10.10 in [8], but we shall complement some arguments for the reader’s convenience. Theorem 10.10 in [8] and the discussion following it imply that \(S_n\) converges weakly in \(L^p\) if and only if \(X\) is pre-Gaussian and \(P(||X||_p > \tau) = o(t^{-2})\). So we only have to verify that \(X\) is pre-Gaussian if and only if \(\int_S (E[|X(s)|^2])^{p/2} \, d\mu(s) < \infty\).

If \(X\) is pre-Gaussian, then
\[
\infty > E[||G||_p^p] = \int_S E[|G(s)|^p] \, d\mu(s) = c_p \int_S (E[|X(s)|^2])^{p/2} \, d\mu(s)
\]
with \(c_p = E[|Z|^p]\) for \(Z \sim N(0, 1)\). For the “if” part, let \(B = L^p\) and \(B' = L^q\). Since \(B\) is separable, we may assume that \(X\) is defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \(\mathcal{F}\) countably generated. Then \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) is separable and so there exists a complete orthonormal system \((h_i)_{i=1}^\infty\) of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\). Define a Gaussian variable \(G_n(\cdot) = \sum_{i=1}^n Z_i[E[h_i(X(\cdot))\cdot]\) (defined as a \(B\)-valued random variable), where \((Z_i)_{i=1}^\infty\) is a sequence of independent \(N(0, 1)\) random variables. We show that \(G_n\) converges in \(L^p(B) = L^p(\Omega, \mathcal{F}, \mathbb{P}; B)\). To this end, observe that for \(n > m\),
\[
E[||G_n - G_m||_p^p] = \int_S E \left[ \left| \sum_{i=m+1}^n Z_i[E[h_i(X(\cdot))]\right|^p \right] \, d\mu(s)
\]
\[
= c_p \int_S \left\{ \sum_{i=m+1}^n \{E[h_i(X(\cdot))]\}^2 \right\}^{p/2} \, d\mu(s).
\]
Hence, \(G_n\) is Cauchy in \(L^p(B)\) and so there exists a \(B\)-valued random variable \(G\) such that \(E[||G_n - G||_p^p] \to 0\).

It is not difficult to see that \(G\) is Gaussian with \(E[G(f)] = \lim_m E[G_n(f)] = 0\) and \(E[G(f)^2] = \lim_m E[G_n(f)^2] = E[X(f)^2]\) for every \(f \in B'\). Thus, \(X\) is pre-Gaussian. \(\square\)
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