SPARSE FILTERED NERVES
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ABSTRACT. We give an elementary approach to efficient sparsification of filtered Čech complexes and more generally sparsification of filtered Dowker nerves. For certain point clouds in high dimensional euclidean space the sparse nerves presented here are substantially smaller than the sparse Rips complex constructed by Sheehy in [12]. This means that is feasible to compute persistent homology of some point clouds and weighted undirected networks whose Rips complexes can not be computed by state of the art computational tools.

Our construction is inspired by [12]. We view it as consisting of two parts. In one part we replace a metric by a Dowker dissimilarity whose nerve is a small approximation of the Čech nerve of the original metric. In the other part we replace the nerve of a Dowker dissimilarity by a smaller homotopy equivalent filtered simplicial complex. This filtered simplicial complex is the smallest member of a class of sparsifications including the ones in [6] and [3].

1. Introduction

Given a subset $L$ of a metric space $W$, the ambient Čech complex $\check{C}_t(L, W)$ is the nerve of the set of $t$-balls in $W$ centred at points in $L$ considered as a cover of the union of such balls. Considering $L$ as a metric space with the induced metric, we call $\check{C}_t(L, L)$ the intrinsic Čech complex of $L$. Up to homotopy, another way to describe the ambient Čech complex $\check{C}_t(L, W)$ is as the filtration $t$ part of the Dowker nerve of the function

$$\Lambda: L \times W \to [0, \infty]$$

obtained by restricting the metric $d: W \times W \to [0, \infty)$ to the subset $L \times W$ of $W \times W$. From the perspective of computer implementation these relative Čech complexes and their Dowker counterparts have the defect that they grow rapidly when the size of $L$ increases. In order to mitigate this Sheehy, Botnan–Spreamann and Cavanna–Jahanseir–Sheehy proposed sparse approximations to $\check{C}_t(L, W)$ in the situation where $W = \mathbb{R}^d$ equipped with a convex metric and $L$ is a finite subset of $W$ [12] [4] [3]. Inspired by their work, in [3], we constructed
sparsifications of nerves of Dowker dissimilarities satisfying the triangle inequality. In this paper we construct sparsifications of arbitrary Dowker dissimilarities, that is, arbitrary functions of the form
\[ \Lambda : L \times W \to [0, \infty]. \]

Our construction is minimal among a class of sparsifications including the ones in \cite{6} and \cite{3}. In the situation where \( L \) and \( W \) are finite and all the values \( \Lambda(l, w) \) for \( (l, w) \in L \times W \) are stored in memory these sparsifications can be implemented on a computer in a direct way. In the situation where \( W = \mathbb{R}^d \) with Euclidean metric and \( L \) is a finite subset of \( W \) we use the miniball algorithm \cite{10} to implement sparse Čech nerves.

The intrinsic and the ambient Čech complexes are related by the inclusions
\[ \check{C}^t(L, L) \subseteq \check{C}^t(L, W) \subseteq \check{C}^{2t}(L, L), \]
so their corresponding persistent homologies are 2-interleaved. The ambient Čech complex has homotopy type given by the sublevel filtration for the function \( f : W \to [0, \infty) \) whose value on a point in \( W \) is its distance to \( L \). If \( L \) is contained in a metric subspace \( N \) of \( W \), then the persistent homologies of \( \check{C}(L, W) \) and \( \check{C}(N, W) \) are additively interleaved with interleaving factor given by the Hausdorff distance \( d_H(L, N) \) between \( L \) and \( N \). Moreover, the persistent homologies of \( \check{C}(L, L) \) and \( \check{C}(N, N) \) are interleaved with the factor \( 2d_H(L, N) \), that is, two times the Hausdorff distance between \( L \) and \( N \). Thus the persistent homologies of both \( \check{C}(L, L) \) and \( \check{C}(L, W) \) can be considered as approximations to the intrinsic Čech homology of \( N \). In particular if \( W \) is a Riemannian manifold with distance given by the geodesic metric, then the persistent homologies of \( \check{C}(L, W) \) and \( \check{C}(W, W) \) are additively interleaved with interleaving factor given by the Hausdorff distance \( d_H(L, W) \) between \( L \) and \( W \). At filtration values up to the convexity radius of \( W \) the persistent homology \( \check{C}(W, W) \) is isomorphic to the homology of \( W \). (See e.g. \cite[Section 6.5.3]{2})

This paper is organized as follows. Section \ref{2} introduces the reader to the basic concepts used throughout the remaining sections. In \cite{3} we did not explain how interleavings with respect to translation functions (see Definition \ref{2.3}) are related to matchings. Since this is crucial to the interpretation of persistence diagrams of sparse nerves we discuss this in Section \ref{3}. In Section \ref{4} we introduce truncation of Dowker nerves and give a direct argument showing that the truncated Dowker nerve is interleaved with the Dowker nerve of the original Dowker dissimilarity. In Section \ref{5} we sparsify Dowker nerves in a way that preserves homotopy
type. In particular, persistent homology does not change under sparsification. This sparsification is obtained via a function $R: L \to [0, \infty]$ having certain properties. Functions with these properties, we call restriction functions. With the concept of restriction functions at hand we display the smallest restriction function relative to a parent function $\varphi$, the $(\Lambda, \varphi)$-restriction. In Section 6, we give a short description of details behind python package for computation of persistent homology of sparsified Dowker nerves.

2. Preliminaries

2.1. Filtrations. We consider the interval $[0, \infty]$ as a category with the underlying set of the interval as objects and with a morphism $s \to t$ if and only if $s \leq t$.

**Definition 2.1.** Let $C$ be a category. The category of filtered objects in $C$ is the category of functors from $[0, \infty]$ to $C$. That is, a filtered object in $C$ is a functor $C: [0, \infty] \to C$ and a morphism $f: C \to C'$ of filtered objects in $C$ is a natural transformation.

Recall that a function $\alpha: [0, \infty) \to [0, \infty)$ is order preserving if $s \leq t$ implies $\alpha(s) \leq \alpha(t)$.

**Definition 2.2.** Let $\beta: [0, \infty) \to [0, \infty)$ be an order preserving function with $\lim_{t \to \infty} \beta(t) = \infty$. The generalized inverse of $\beta$ is the order preserving function $\beta^{-\leftarrow}: [0, \infty) \to [0, \infty)$, $\beta^{-\leftarrow}(t) = \inf\{s \in [0, \infty) \mid \beta(s) \geq t\}$ with the defining property $\beta^{-\leftarrow}(t) \leq s$ if and only if $t \leq \beta(s)$.

**Definition 2.3.** A translation function is an order preserving function $\alpha: [0, \infty) \to [0, \infty)$ with $t \leq \alpha(t)$ for every $t \in [0, \infty)$.

**Definition 2.4.** Given a filtered object $C: [0, \infty] \to C$ and a translation function $\alpha: [0, \infty) \to [0, \infty)$, the pull-back of $C$ along $\alpha$ is the filtered object $\alpha^*C = C \circ \alpha$ with $(\alpha^*C)(t) = C(\alpha(t))$. The $\alpha$-unit of $C$ is the morphism $\alpha_{\ast C}: C \to \alpha^*C$ with $\alpha_{\ast C}(t) = C(t \to \alpha(t))$: $C(t) \to (\alpha^*C)(t) = C(\alpha(t))$.

**Definition 2.5.** Let $k$ be a field. The category of persistence modules over $k$ is the category of filtered objects in the category of vector spaces over $k$.

**Definition 2.6.** Let $k$ be a field and let $\alpha: [0, \infty) \to [0, \infty)$ be a translation function. A persistence module $V$ over $k$ is $\alpha$-trivial if the $\alpha$-unit of $V$ is trivial, that is, if $\alpha_{\ast V} = 0$. 

2.2. **Dowker Dissimilarities.** Our presentation of this preliminary material closely follows [3].

**Definition 2.7** (Dowker [9]). The *nerve* of a relation $R \subseteq X \times Y$ is the simplicial complex

$$NR = \{ \text{finite } \sigma \subseteq X \mid \exists y \in Y \text{ with } (x, y) \in R \text{ for all } x \in \sigma \}.$$  

The following definition is inspired by the concept of networks as it appears in [7].

**Definition 2.8.** A *Dowker dissimilarity* $\Lambda$ consists of two sets $L$ and $W$ and a function $\Lambda : L \times W \to [0, \infty]$. Given $t \in [0, \infty]$, we let $\Lambda_t \subseteq L \times W$ be the relation

$$\Lambda_t = \{(l, w) \in L \times W \mid \Lambda(l, w) < t\}.$$  

**Definition 2.9.** Let $\Lambda : L \times W \to [0, \infty]$ be a Dowker dissimilarity. The *Dowker Nerve* $N\Lambda$ of $\Lambda$ is the filtered simplicial complex with vertex set $L$ and the nerve $N\Lambda_t$ of the relation $\Lambda_t$ in filtration degree $t \in [0, \infty]$. 

**Definition 2.10.** A *morphism* $C : \Lambda \to \Lambda'$ of Dowker dissimilarities $\Lambda : L \times W \to [0, \infty]$ and $\Lambda' : L' \times W' \to [0, \infty]$ consists of a relation $C \subseteq L \times L'$ so that for every $t \in [0, \infty]$ and for every $\sigma \in N\Lambda_t$, the set

$$NC(\sigma) = \{l' \in L' \mid (\sigma \times \{l'\}) \cap C \text{ is non-empty}\}$$

is non-empty and contained in $N\Lambda'_t$. If $C \subseteq L \times L'$ and $C' \subseteq L' \times L''$ are morphisms $C : \Lambda \to \Lambda'$ and $C' : \Lambda' \to \Lambda''$ of Dowker dissimilarities, then the composition

$$C'C : \Lambda \to \Lambda''$$

is the subset of $L \times L''$ defined by

$$C'C = \{(l, l''') \mid \text{there exists } l' \in L' \text{ with } (l, l') \in C \text{ and } (l', l'') \in C'\}.$$  

The identity morphism $\Delta_L : N\Lambda \to N\Lambda$ is

$$\Delta_L = \{(l, l) \mid l \in L\} \subseteq L \times L.$$  

**Proposition 2.11.** The Dowker nerve is functorial in the sense that it induces a functor $N$ from the category of Dowker dissimilarities to the category of functors from $[0, \infty]$ to the category of topological spaces.

**Proof.** Let $C \subseteq L \times L'$ and $C'' \subseteq L' \times L''$ be morphisms $C : \Lambda \to \Lambda'$ and $C' : \Lambda' \to \Lambda''$ of Dowker dissimilarities. Given $t \in [0, \infty]$, the functions

$$NC : N\Lambda_t \to N\Lambda'_t \quad \text{and} \quad NC' : N\Lambda'_t \to N\Lambda''_t$$
are order preserving. Thus, they induce morphisms of geometric realizations of barycentric subdivisions. The identity morphism $\Delta_L: \Lambda \to \Lambda$ induces the identity function $N\Delta_L: N\Lambda \to N\Lambda$. In order to finish the proof we show that $NC'(NC(\sigma)) = N(C'C)(\sigma)$ for every $\sigma \in N\Lambda_t$. If $l'' \in NC'(NC(\sigma))$, then there exists $l' \in NC(\sigma)$ so that $(l', l'') \in C'$. Since $l' \in NC(\sigma)$ there exists $l \in \sigma$ so that $(l, l') \in C$. We conclude that $(l, l'') \in C'C$ and thus $l'' \in N(C'C)(\sigma)$. Conversely, if $l'' \in N(C'C)(\sigma)$, then there exists $l \in \sigma$ so that $(l, l'') \in C'.C$. By definition of $C'C$ this means that there exists $l' \in L'$ so that $(l, l') \in C$ and $(l', l'') \in C'$. We conclude that $l' \in NC(\sigma)$ and that $l'' \in NC'(NC(\sigma))$. \hfill \square

**Corollary 2.12.** Let $k$ be a field. The persistent homology $H_*(N\Lambda)$ of $N\Lambda$ with coefficients in $k$ is functorial in the sense that it is a functor from the category of Dowker dissimilarities to the category of persistence modules over $k$.

2.3. **Interleaving.** Here we present a notion of interleaving inspired by Bauer and Lesnik [1].

**Definition 2.13.** Let $C$ and $C'$ be filtered objects in a category $\mathcal{C}$ and let $\alpha: [0, \infty) \to [0, \infty)$ be a translation function.

1. A morphism $G: C \to C'$ is an $\alpha$-interleaving if for every $t \in [0, \infty]$ there exists a morphism $F_t: C'(t) \to C(\alpha(t))$ in $\mathcal{C}$ such that

$$\alpha_{*C}(t) = F_t \circ G(t) \quad \text{and} \quad \alpha_{*C'}(t) = G(\alpha(t)) \circ F_t.$$

2. We say that $C$ and $C'$ are $\alpha$-interleaved if there exists an $\alpha$-interleaving $G: C \to C'$.

Suppose we are in the situation that we have an inclusion $K \subseteq L$ of filtered simplicial complexes and that we are able to compute the filtration value of simplices in $L$, but we have no constructive way of computing the filtration value of simplices in $K$. Then we can construct at filtered simplicial sub-complex $K'$ of $L$ with $K'_\infty = K_\infty$. If know that the inclusion $K \subseteq L$ is an $\alpha$-interleaving, then the following lemma implies that also the inclusion $K' \subseteq L$ is an $\alpha$-interleaving. This situation occurs for example when $L$ is a Čech complex.

**Lemma 2.14.** Let $C$, $C'$ and $C''$ be filtered objects in a category $\mathcal{C}$ and let $\alpha: [0, \infty) \to [0, \infty)$ be a translation function. Let $G: C \to C'$ and $G': C' \to C''$ be morphisms of filtered objects. Suppose that $G'(t): C'(t) \to C''(t)$ is a monomorphism for every $t \in [0, \infty]$ and that the composition $G'G: C \to C''$ is an $\alpha$-interleaving. Then also $G': C' \to C''$ is an $\alpha$-interleaving.
Proof. Let \( t \in [0, \infty] \), and pick \( E_t : C''(t) \to C(\alpha(t)) \) such that
\[
\alpha_{sC}(t) = E_t \circ (G'G)(t) \quad \text{and} \quad \alpha_{sC''}(t) = (G'G)(\alpha(t)) \circ E_t.
\]
Defining \( F_t = E_t G'(t) : C'(t) \to C(\alpha(t)) \) the above relations imply that
\[
\alpha_{sC}(t) = E_t \circ (G'G)(t) = (E_t G'(t)) \circ G(t) = F_t \circ G(t)
\]
and
\[
G'(\alpha(t)) \circ \alpha_{sC''}(t) = (G'G)(\alpha(t)) \circ E_t \circ G'(t).
\]
Then also \( \alpha_{sC''}(t) = G(\alpha(t)) \circ F_t \), since \( G'(\alpha(t)) \) is a monomorphism. \( \square \)

The following results are analogues of [5, Proposition 2.2.11 and Proposition 2.2.13].

Lemma 2.15 (Functoriality). Let \( C \) and \( C' \) be filtered objects in a category \( \mathcal{C} \), let \( \alpha : [0, \infty) \to [0, \infty) \) be a translation function and let \( H: \mathcal{C} \to \mathcal{D} \) be a functor. If \( C \) and \( C' \) are \( \alpha \)-interleaved, then the filtered objects \( HC \) and \( HC' \) are \( \alpha \)-interleaved.

Lemma 2.16 (Triangle inequality). Let \( G: C \to C' \) be an \( \alpha \)-interleaving and let \( G': C' \to C'' \) be an \( \alpha' \)-interleaving of filtered objects in a category \( \mathcal{C} \). Then the composition \( G'' = G'G : C \to C'' \) is an \( \alpha \alpha' \)-interleaving.

Proof. Let \( \alpha'' = \alpha \alpha' \) and let \( t \in [0, \infty] \). By definition there exist morphisms \( F_{\alpha'(t)} : C''(\alpha'(t)) \to C(\alpha(\alpha'(t))) \) and \( F_t' : C'(t) \to C'(\alpha'(t)) \) so that
\[
\alpha_{sC}(\alpha'(t)) = F_{\alpha'(t)} \circ G(\alpha'(t)) \quad \text{and} \quad \alpha_{sC''}(\alpha'(t)) = G(\alpha(\alpha'(t))) \circ F_{\alpha'(t)}.
\]
and
\[
\alpha'_{sC'}(t) = F_t' \circ G'(t) \quad \text{and} \quad \alpha'_{sC''}(t) = G'(\alpha'(t)) \circ F_t'.
\]
Let \( F''_t = F_{\alpha'(t)} F'_t : C'' \to C_{\alpha\alpha'(t)} \).

The above relations imply that the right hand triangles in the diagram
commute. The quadrangle in the above diagram commutes since \( G \) is a natural transformation and commutativity of the left hand triangle follows directly from the definition of the \( \alpha'' \)-unit. We conclude that

\[
\alpha''_{aC}(t) = F''_t \circ G''(t).
\]

The above relations also imply that the upper triangles in the diagram commute. The quadrangle in the above diagram commutes since \( G' \) is a natural transformation and commutativity of the left hand triangle follows directly from the definition of the \( \alpha'' \)-unit. We conclude that

\[
\alpha''_{aC^\prime}(t) = G''(\alpha''(t)) \circ F''_t.
\]

We find that the following lemma justifies our definition of \( \alpha \)-interleaving.

**Lemma 2.17.** Let \( \alpha : [0, \infty) \to [0, \infty) \) be a translation function and let \( V \) and \( V' \) be persistence modules. A morphism \( G : V \to V' \) of persistence modules is an \( \alpha \)-interleaving if and only if both \( \text{ker} \ G \) and \( \text{coker} \ G \) are \( \alpha \)-trivial.

**Proof.** Suppose first that \( G \) is an \( \alpha \)-interleaving. Fix \( t \) and pick \( F_t : V'(t) \to V'(\alpha(t)) \) so that \( \alpha_{V*}(t) = F_t \circ G(t) \) and \( \alpha_{V'*}(t) = G(\alpha(t)) \circ F_t \). Then \( v \in \text{ker} \ G(t) \) implies

\[
\alpha_{\text{ker} G*} v = \alpha_{V*} v = F_t(G(t)(v)) = 0.
\]

Similarly, if \( v' + \text{im} \ G(t) \in \text{coker} \ G(t) \), then

\[
\alpha_{\text{coker} G*} (v' + \text{im} \ G(t)) = \alpha_{V'*} (v') + \text{im} \ G(\alpha(t)) = 0
\]

since \( \alpha_{V'*}(v') = G(\alpha(t))(F_t v') \in \text{im} \ G(\alpha(t)) \). Thus \( \text{ker} \ G \) and \( \text{coker} \ G \) are \( \alpha \)-trivial.

Conversely, suppose that \( \text{ker} \ G \) and \( \text{coker} \ G \) are \( \alpha \)-trivial and fix \( t \). Choose a basis \( e_1, \ldots, e_a \) for \( \text{ker} \ G(t) \) and choose \( f_1, \ldots, f_b \) so that

\[
e_1, \ldots, e_a, f_1, \ldots, f_b
\]
is a basis for $C(t)$. Note that $G(t)(f_1), \ldots, G(t)(f_b)$ are linearly independent in $C'(t)$ and choose $g_1, \ldots, g_c$ so that

$$G(t)(f_1), \ldots, G(t)(f_b), g_1, \ldots, g_c$$

is a basis for $C'(t)$. We use this basis to define $F_t : C'(t) \to C(\alpha(t))$ as follows: On basis elements of the form $G(t)(f_i)$ we define

$$F_t(G(t)(f_i)) = \alpha C^*_t(f_i).$$

Now consider basis elements of the form $g_i$. Since $\alpha_{\ker G} = 0$ we know that $\alpha_{C^*_t}(t)(g_i) \in \text{im} G(\alpha(t))$. We choose $c_i \in C(\alpha(t))$ so that

$$G(\alpha(t))(c_i) = \alpha_{C^*_t}(t)(g_i)$$

and define

$$F_t(g_i) = c_i.$$ 

Since $\alpha_{\ker G} = 0$ we have

$$\alpha_{C^*_t}(t)(e_i) = 0 = F_t(G(e_i)),$$

so $F_t G(t) = \alpha_{C^*_t}(t)$. On the other hand, the equation

$$G(\alpha t)(F_t(G(t)(e_i))) = G(\alpha t)(\alpha_{C^*_t}(e_i)) = \alpha_{C^*_t}(G(t)(e_i))$$

shows that $\alpha_{C^*_t}(t) = G(\alpha t) F_t$. \hfill \Box

3. Matchings

Our presentation of matchings follows [11].

Definition 3.1. The set of persistence intervals is the set $E$ of intervals in $[0, \infty]$.

We write $\overline{a}$ for the closure of an interval $a \in E$. Note that $\overline{a}$ is determined by the end points of the interval $a$.

Definition 3.2. A persistence diagram consists of a set $X$ and a function $p : X \to E$ from $X$ to the set of persistence intervals. We refer to the elements of $X$ as persistence classes.

Definition 3.3. A matching $R$ of two persistence diagrams $p : X \to E$ and $p' : X' \to E$ consists of a relation $R \subseteq X \times X'$ with the property that the compositions

$$\pi_1 : R \to X, \quad \pi_1(x, x') = x$$

and

$$\pi_2 : R \to X', \quad \pi_2(x, x') = x'$$

with the inclusion of $R$ in $X \times X'$ and the projections to $X$ and $X'$ respectively are injective with $p \circ \pi_1 = p' \circ \pi_2$. We say that a persistence class $x \in X$ is matched by $R$ if there exists a persistence class $x' \in X'$
so that \((x, x') \in R\). Similarly we say that a persistence class \(x' \in X'\) is matched by \(R\) if there exists a persistence class \(x \in X\) so that \((x, x') \in R\).

**Definition 3.4.** Let \(J\) be a subset of \([0, \infty]\). The persistence module \(k(J)\) has values 
\[
k(J)(t) = \begin{cases} k & \text{if } t \in J \\ 0 & \text{otherwise} \end{cases}
\]
and structure maps equal to identity maps whenever possible.

**Definition 3.5.** Let \(\alpha : [0, \infty) \to [0, \infty)\) be a translation function and let \(p : X \to E\) be a persistence diagram. We say that a persistence class \(x \in X\) is \(\alpha\)-trivial if the persistence module \(k(p(x))\) is \(\alpha\)-trivial. Otherwise we say that \(x\) is \(\alpha\)-nontrivial.

Note that if \(p(x)\) has end points \(b < d\), then \(p(x)\) is \(\alpha\)-trivial if and only if \(d \leq \alpha(b)\).

**Definition 3.6.** Let \(V\) be a persistence module over a field \(k\). We say that \(p : X \to E\) is a persistence diagram of \(V\) if there exists an isomorphism of the form 
\[
V \cong \bigoplus_{x \in X} k(p(x)).
\]

**Definition 3.7.** The category of pointwise finite dimensional persistence modules over the field \(k\) is the full subcategory of the category of persistence modules \(V\) over \(k\) with \(V_t\) finite dimensional for every \(t \in [0, \infty]\).

We restate the decomposition theorem for pointwise finite-dimensional persistence modules [8, Theorem 1.1] in our notation.

**Theorem 3.8.** Let \(k\) be a field. Every pointwise finite dimensional persistence module over \(k\) has a persistence diagram.

We now state the generalized induced matching theorem [1, Theorem 6.1] and [11, Theorem 3.2]. In order to do this we use the generalized inverse function of a translation function from Definition 2.2.

**Theorem 3.9.** There exists a function \(\chi : \text{Mor}(\text{Pers}) \to \text{Match}\) from the set of morphisms of pointwise finite dimensional persistence modules over the field \(k\) to the set of matchings with the following properties: Let \(f : V \to V'\) be a morphism of pointwise finite persistence modules and let \(\chi(f)\) be of the form 
\[
\chi(f) : (X \xrightarrow{p} E) \to (X' \xrightarrow{p'} E),
\]
that is,
\[ \chi(f) \subseteq X \times X'. \]

Assume that \( f \) is an \( \alpha \)-interleaving and that \((x, x') \in \chi(f)\) with \( p(x) = [b, d] \) and \( p'(x') = [b', d'] \). Then the following holds:

1. \( b' \leq b < d' \leq d \)
2. \( b \leq \alpha(b') \)
3. \( \alpha^{-1}(d) \leq d' \).

Moreover all \( \alpha \)-nontrivial persistence classes of \( X \) and \( X' \) are matched by \( \chi(f) \).

In the above situation, if \( \alpha \) is bijective, then (3) is equivalent to
\[ d \leq \alpha(d'). \]

If we further assume that \( x' \) is \( \alpha \)-nontrivial, then \( \alpha(b') < d' \) and the point \((b, d)\) lies in the box with corners \((b', d')\) and \((\alpha(b'), \alpha(d'))\). Conversely, if \( \alpha \) is bijective and \( x \) is \( \alpha \)-nontrivial, then \( \alpha(b) < d \) and the point \((b', d')\) lies in the box with corners \((\alpha^{-1}b, \alpha^{-1}d)\) and \((b, d)\).

4. Truncated Nerves

**Definition 4.1.** Let \( \Lambda: L \times W \to [0, \infty] \) be a Dowker dissimilarity and let \( \alpha: [0, \infty) \to [0, \infty) \) be a translation function. We say that a function \( T: L \to [0, \infty] \) is an \( \alpha \)-truncation function for \( \Lambda \) if for all \( t \in [0, \infty] \) and all \( l \in L \) there exists \( l' \in L \) so that for all \( w \in M \) with \( \Lambda(l, w) < t \) we have that \( \Lambda(l', w) < \alpha(t) \) and \( \Lambda(l', w) < T(l') \).

**Definition 4.2.** Let \( \Lambda: L \times W \to [0, \infty] \) be a Dowker dissimilarity, let \( \alpha: [0, \infty) \to [0, \infty) \) be a translation function and let \( T: L \to [0, \infty] \) be an \( \alpha \)-truncation function for \( \Lambda \). The \( T \)-truncation of \( \Lambda \) is the Dowker dissimilarity \( \Gamma: L \times W \to [0, \infty] \) defined by
\[
\Gamma(l, w) = \begin{cases} 
\Lambda(l, w) & \text{if } \Lambda(l, w) < T(l) \\
\infty & \text{otherwise}.
\end{cases}
\]

**Proposition 4.3.** Let \( \Lambda: L \times W \to [0, \infty] \) be a Dowker dissimilarity, let \( \alpha: [0, \infty) \to [0, \infty) \) be a translation function and let \( T: L \to [0, \infty] \) be an \( \alpha \)-truncation function for \( \Lambda \). Let \( \Gamma \) be the \( T \)-truncation of \( \Lambda \). Then the inclusion of the nerve \( N\Gamma \) of \( \Gamma \) in the nerve \( N\Lambda \) of \( \Lambda \) is an \( \alpha \)-interleaving.
Proof. It suffices, for every \( t \in [0, \infty] \), to find a map \( f_t: N\Lambda_t \to N\Gamma_{\alpha(t)} \) so that the following diagrams commute up to homotopy:

\[
\begin{array}{ccc}
N\Gamma_t & \longrightarrow & N\Lambda_t \\
\downarrow & & \downarrow \ f_t \\
N\Gamma_{\leq \alpha(t)} & \longrightarrow & N\Lambda_{\alpha(t)}
\end{array}
\]

and

\[
\begin{array}{ccc}
N\Lambda_t & \longrightarrow & N\Gamma_{\alpha(t)} \\
\downarrow & & \downarrow \\
N\Lambda_{\leq \alpha(t)} & \longrightarrow & N\Lambda_{\alpha(t)}
\end{array}
\]

Fix \( t \) and choose a function \( f_t: L \to L \) so that for every \( l \in L \) with \( \Lambda(l, w) < t \) the inequalities \( \Lambda(f_t(l), w) < \alpha(t) \) and \( \Lambda(f_t(l), w) < T(f_t(l)) \) hold.

Below we first show that \( f_t \) induces a simplicial map

\( f_t: N\Lambda_t \to N\Gamma_{\alpha(t)} \).

That is, we show that if \( \sigma \in N\Lambda_t \), then \( f_t(\sigma) \in N\Gamma_{\alpha(t)} \). Next we show that \( f_t(\sigma) \cup \sigma \in N\Lambda_{\alpha(t)} \) so that the lower of the above displayed diagrams commutes up to homotopy. We will finish by showing that if \( \sigma \in N\Gamma_t \), then \( f_t(\sigma) \cup \sigma \in N\Gamma_{\alpha(t)} \) so that also the upper of the above displayed diagrams commutes up to homotopy.

Let \( \sigma \in N\Lambda_t \) and pick \( w \in W \) so that \( \Lambda(l, w) < t \) for every \( l \in \sigma \). Then, for every \( l \in \sigma \) we have \( \Lambda(f_t(l), w) < \alpha(t) \) and \( \Lambda(f_t(l), w) < T(f_t(l)) \) so in particular \( \Gamma(f_t(l), w) = \Lambda(f_t(l), w) < \alpha(t) \). This implies both that \( f_t(\sigma) \in N\Gamma_{\alpha(t)} \) and that \( f_t(\sigma) \cup \sigma \in N\Lambda_{\alpha(t)} \). Finally, if \( \sigma \in N\Gamma_t \) and we pick \( w \in W \) so that \( \Gamma(l, w) = \Lambda(l, w) < t \) for every \( l \in \sigma \), then the above argument also implies that \( f_t(\sigma) \cup \sigma \in N\Gamma_{\alpha(t)} \). □

**Example 4.4.** Let \( \Lambda: L \times W \to [0, \infty] \) be a Dowker dissimilarity and let \( \alpha: [0, \infty) \to [0, \infty) \) be a translation function. Given \( l, l' \in L \), let

\[
P(l', l) = \{ \Lambda(l', w) \mid w \in W \text{ with } \alpha(\Lambda(l, w)) \leq \Lambda(l', w) \}.
\]

Given a base point \( l_0 \in L \) the **cover dissimilarity** \( \Lambda^\alpha: L \times L \to [0, \infty] \) for \( \Lambda \) with respect to \( \alpha \) by

\[
\Lambda^\alpha(l', l) = \begin{cases} 
0 & \text{if } l = l' \\
\infty & \text{if } l \neq l' \text{ and } l' = l_0 \\
\sup(P(l', l) \cup \{0\}) & \text{if } l \neq l' \text{ and } l' \neq l_0.
\end{cases}
\]
If $L$ is finite and $<$ is a total order on $L$ with $l_0$ as minimal element, we define the \textit{\(\alpha\)-insertion radius} $\lambda^\alpha(l)$ of $l \in L$ as

$$
\lambda^\alpha(l) = \begin{cases} 
\infty & \text{if } l = l_0 \\
\sup_{k \geq l} \inf_{l \leq k} \Lambda^\alpha(l', k) & \text{if } l \neq l_0 
\end{cases}
$$

Given $l \in L$ with $l \neq l_0$ and $t \in [0, \infty]$ with $\alpha(t) > 0$, pick $l' \in L$ minimal with $\Lambda^\alpha(l', l) < \alpha(t)$. (Such an $l'$ exists since $\Lambda^\alpha(l, l) = 0$ and $\Lambda^\alpha(l_0, l) = \infty$). Let $w \in W$ with $\Lambda(l, w) < t$. Then either $\Lambda(l', w) > \Lambda^\alpha(l', l)$ or $\Lambda(l', w) \leq \Lambda^\alpha(l', l) < \alpha(t)$. If $\Lambda(l', w) > \Lambda^\alpha(l', l)$, then

$$
\Lambda(l', w) < \alpha(\Lambda(l, w)) \leq \alpha(t).
$$

Thus, $\Lambda(l, w) < t$ implies $\alpha(t) > \Lambda(l', w)$. Since $\infty = \lambda^\alpha(l_0) \geq \alpha(t)$ and

$$
\lambda^\alpha(l') = \sup_{k \geq l} \inf_{l' < k} \Lambda^\alpha(l'', k) \geq \inf_{l' < k} \Lambda^\alpha(l'', l) \geq \alpha(t),
$$

the function $\lambda^\alpha : L \to [0, \infty]$ is an $\alpha$-truncation function for $\Lambda$.

\textbf{Definition 4.5.} Given a Dowker dissimilarity of the form $\Lambda : L \times L \to [0, \infty]$ with $L$ finite a \textit{farthest point sample} for $\Lambda$ is a total order $<$ on $L$ with minimal element $l_0$ so that for $l \neq l_0$ we have

$$
\inf_{l' < l} \Lambda(l', l) = \sup_{l'' \geq l} \inf_{l' < l} \Lambda(l', l'').
$$

The \textit{insertion radius} of $l \in L$ with respect to the total order $<$ is

$$
\lambda(l) = \begin{cases} 
\infty & \text{if } l = l_0 \\
\inf_{l' < l} \Lambda(l', l) & \text{otherwise.}
\end{cases}
$$

For $\Lambda$ as in Definition 4.5 a farthest point sample $L = \{l_0 < \cdots < l_n\}$ can be produced recursively starting from an initial point $l_0$. When $l_0, \ldots, l_k$ have been produced, we choose $l_{k+1}$ so that

$$
\inf_{l' \in \{l_0, \ldots, l_k\}} \Lambda(l', l_{k+1}) = \sup_{l'' \not\in \{l_0, \ldots, l_k\}} \inf_{l' \in \{l_0, \ldots, l_k\}} \Lambda(l', l'').
$$

Note that

$$
\lambda(l) = \begin{cases} 
\infty & \text{if } l = l_0 \\
\sup_{k \geq l} \inf_{l' < l} \Lambda(l', k) & \text{otherwise.}
\end{cases}
$$

\textbf{Example 4.6.} Let $d : W \times W \to [0, \infty]$ be a metric, let $L$ be a finite subset of $W$ and let $\Lambda : L \times W \to [0, \infty]$ be the restriction of $d$ to the subset $L \times W$ of $W \times W$. Let $\Lambda^\varepsilon : L \times L \to [0, \infty]$ be the restriction of $\Lambda$ to the subset $L \times L$ of $L \times W$ and let $L = \{l_0 < \cdots < l_n\}$ be a farthest point sampling for $\Lambda^\varepsilon$. We write $\lambda^\varepsilon(l) = \lambda(l)$ for the corresponding insertion radius. Let $c > 1$ and let $\alpha : [0, \infty) \to [0, \infty)$ be the translation function $\alpha(t) = ct$. 

If $\alpha\Lambda(l, w) \leq \Lambda(l', w)$, then the triangle inequality for $d$ implies that $\Lambda(l', w) \leq \Lambda^L(l', l) + \Lambda(l, w)$ and therefore $\Lambda(l, w) \leq \Lambda^L(l', l)/(c - 1)$. This, together with the triangle inequality for $d$ implies that

$$\Lambda(l', w) \leq \Lambda^L(l', l) + \Lambda(l, w) \leq \Lambda^L(l', l) + \frac{\Lambda^L(l', l)}{c - 1} = \frac{c\Lambda^L(l', l)}{c - 1}.$$  

From this consideration we can conclude that

$$\Lambda^\alpha(l', l) \leq \frac{c\Lambda^L(l', l)}{c - 1}$$

and that

$$\lambda^\alpha(l) \leq \frac{c\lambda^L(l)}{c - 1}.$$  

Since $\lambda^\alpha$ is an $\alpha$-truncation function of $\Lambda$, so is the function $T(l) = \frac{c\lambda^L(l)}{c - 1}$.

There exist many truncation functions for a given translation function $\alpha$. We have not succeeded in finding a class of truncation functions for $\alpha$ that are practical to implement and produces a smallest possible simplicial complex under this constraint. We leave this as a problem for further investigation. If the goal is merely to construct a Dowker dissimilarity whose Dowker nerve is small the amount of possibilities is even bigger.

5. Sparse Filtered Nerves

**Definition 5.1.** Let $\Lambda: L \times W \to [0, \infty]$ be a Dowker dissimilarity and let $R: L \to [0, \infty]$ and $\varphi: L \to L$ be functions. We say that $l \in L$ is a slope point if $R(l) > R(l')$ for every $l' \in \varphi^{-1}(l)$. The $(R, \varphi)$-nerve of $\Lambda$ is the filtered simplicial complex $N(\Lambda, R, \varphi)$ with $N(\Lambda, R, \varphi)(t)$ consisting of all $\sigma \in N\Lambda_t$ such that there exists $w \in W$ satisfying:

1. $\Lambda(l, w) < t$ for all $l \in \sigma$,
2. $\Lambda(l, w) \leq R(l')$ for all $l, l' \in \sigma$ and
3. $\Lambda(l, w) < R(l)$ for all slope points $l$ in $\sigma$.

**Definition 5.2.** A function $\varphi: L \to L$ is a parent function if $\varphi^n(l) = l$ for $n > 0$ implies $\varphi(l) = l$.

Note that $\varphi: L \to L$ is a parent function if and only if the directed graph with $L$ as set of nodes and $E(\varphi) = \{(\varphi(l), l) \mid l \in L, \varphi(l) \neq l\}$ as set of edges is acyclic.

**Definition 5.3.** Let $\Lambda: L \times W \to [0, \infty]$ be a Dowker dissimilarity and let $\varphi: L \to L$ be a parent function. We say that a function $R: L \to [0, \infty]$ is a restriction function for $\Lambda$ relative to $\varphi$ if the following holds:
(1) For all \((l, w) \in L \times W\) with \(\Lambda(l, w) < \Lambda(\varphi(l), w)\) we have
\(\Lambda(\varphi(l), w) \leq R(l)\).
(2) For every \(l \in L\) with we have \(R(\varphi(l)) \geq R(l)\).
(3) If \(\varphi(l) = l\), then \(R(l) = \infty\).

**Definition 5.4.** Let \(\Lambda: L \times W \to [0, \infty]\) be a Dowker dissimilarity and let \(\varphi: L \to L\) be a parent function. Given \(l, l' \in L\) let
\[ P(l, l') = \{\Lambda(l', w) \mid w \in W \text{ with } \Lambda(l, w) < \Lambda(l', w)\} \]
and define
\[ \rho(l, l') = \begin{cases} \sup P(l, l') & \text{if } P(l, l') \text{ is non-empty} \\ 0 & \text{if } P(l, l') = \emptyset. \end{cases} \]
The \((\Lambda, \varphi)\)-restriction \(R(\Lambda, \varphi): L \to [0, \infty]\) is defined in several steps. First define \(R': L \to [0, \infty]\) by
\[ R'(l) = \begin{cases} \rho(l, \varphi(l)) & \text{if } \varphi(l) \neq l \\ \infty & \text{if } \varphi(l) = l. \end{cases} \]
Given \(l \in L\), let \(D(l)\) be the set of descendants of \(l\), that is, \(l' \in D(l)\) if and only if there exists \(m \geq 0\) so that \(l = \varphi^m(l')\). Next, we define \(R(\Lambda, \varphi): L \to L\) by
\[ R(\Lambda, \varphi)(l) = \max_{l' \in D(l)} R'(l'). \]
Then, for every \(l \in L\) we have \(R(\Lambda, \varphi)(\varphi(l)) \geq R(\Lambda, \varphi)(l)\), and \(\varphi(l) = l\) implies \(R(\Lambda, \varphi)(l) = \infty\). Also, \(\Lambda(l, w) < \Lambda(\varphi(l), w)\) implies
\[ \Lambda(\varphi(l), w) \leq R'(l') \leq R(\Lambda, \varphi)(l). \]

**Proposition 5.5.** Let \(\Lambda: L \times W \to [0, \infty]\) be a Dowker dissimilarity with \(L\) finite and let \(\varphi: L \to L\) be a parent function. Then the \((\Lambda, \varphi)\)-restriction \(R(\Lambda, \varphi)\) is the minimal restriction function for \(\Lambda\) relative to \(\varphi\): If \(R\) is another restriction function for \(\Lambda\) relative to \(\varphi\), then \(R(\Lambda, \varphi)(l) \leq R(l)\) for every \(l \in L\).

**Proof.** In the notation of Definition 5.4 it suffices to show that \(\rho(l, \varphi(l)) \leq R(l)\) for all \(l \in L\). We can assume that \(\varphi(l) \neq l\) because otherwise \(\rho(l, \varphi(l)) = \infty = R(l)\). Given \(l \in L\), if there exists a \(w \in W\) with
\[ \Lambda(l, w) < \Lambda(\varphi(l), w) \]
we have \(\Lambda(\varphi(l), w) \leq R(l)\). By construction of \(\rho\), this implies that
\[ \rho(l, \varphi(l)) \leq R(l). \]
If no such \(w \in W\) with
\[ \Lambda(l, w) < \Lambda(\varphi(l), w) \]
exists, then $\rho(l, \varphi(l)) = 0 \leq R(l)$. □

Proposition 5.5 shows that the $(\Lambda, \varphi)$-restriction function is the minimal restriction function for $\Lambda$ relative to $\varphi$. In the following two examples we show that the sparsifications from [3, 12] also are $(R, \varphi)$-nerves and that therefore the $(\Lambda, \varphi)$-restriction results in smaller nerves.

**Example 5.6.** Let $\Lambda: L \times W \rightarrow [0, \infty]$ be a Dowker dissimilarity with $L$ finite. As in Definition 5.4, given $l, l' \in L$ let

$$P(l, l') = \{\Lambda(l', w) \mid w \in W \text{ with } \Lambda(l, w) < \Lambda(l', w)\}$$

and define

$$\rho(l, l') = \begin{cases} 
\sup P(l, l') & \text{if } P(l, l') \text{ is non-empty} \\
0 & \text{if } P(l, l') = \emptyset.
\end{cases}$$

Define $R_0: L \rightarrow [0, \infty]$ by

$$R_0(l) = \inf \{\rho(l, l') \mid l' \in L \text{ and } l' \neq l\}$$

and let $<$ be a total order on $L$ with minimal element $l_0$ so that $R_0(l') > R_0(l)$ implies $l' < l$. Given $l \in L$ let

$$Q_0(l) = \{l' \in L \mid l' < l \text{ and } \rho(l, l') = R_0(l)\}.$$

If $Q_0(l)$ is non-empty we define

$$\varphi(l) = \min Q_0(l).$$

Otherwise, that is, if $Q_0(l)$ is empty, we let

$$R_1(l) = \inf \{\rho(l, l') \mid l' < l\}.$$

and

$$Q_1(l) = \{l' \in L \mid l' < l \text{ and } \rho(l, l') = R_1(l)\}.$$

If $l$ is the minimal element of $L$, then $Q_1(l)$ is empty and we define $\varphi(l) = l$. Otherwise $Q_1(l)$ is non-empty and we define

$$\varphi(l) = \min Q_1(l).$$

Since $\varphi(l) \leq l$ for every $l \in L$ and $<$ is a total order on $L$, the function $\varphi: L \rightarrow L$ is a parent function. We define $R: L \rightarrow [0, \infty]$ to be the restriction function for $\Lambda$ relative to $\varphi$ constructed in Definition 5.4.

**Example 5.7 (Parent restriction).** In [3] we constructed the sparse filtered nerve $N\Lambda$ of a Dowker dissimilarity $\Lambda: L \times W \rightarrow [0, \infty]$ with $L$ finite. In this example we describe a function $\varphi: L \rightarrow L$ and a restriction $R$ for $\Lambda$ relative to $\varphi$ so that the $(R, \varphi)$-nerve of $\Lambda$ is equal to the sparse Dowker nerve in [3, Definition 38]. Given $l \in L$ we let

$$W_l = \{w \in W \mid \Lambda(l, w) < \infty\}$$
and

\[ \tilde{\tau}(l) = \sup \{ \Lambda(l, w) \mid w \in W_l \}. \]

Let \( l_0 \in L \) and define \( \tau : L \to [0, \infty) \) as the function

\[ \tau(l) = \begin{cases} \infty & \text{if } \{l = l_0\} \\ \tilde{\tau}(l) & \text{otherwise.} \end{cases} \]

Given \( l \in L \) we let

\[ Q(l) = \{ l' \mid \tau(l') > \tau(l) \}. \]

If \( Q(l) \) is non-empty we pick \( l' \in Q(l) \) and define \( \varphi(l) = l' \). Otherwise we define \( \varphi(l) = l_0 \). The parent restriction \( R : L \to [0, \infty) \) is defined by

\[ R(l) = \tau(\varphi(l)). \]

It is readily verified that the above structure satisfies is a sparsification function for \( \Lambda \) with respect to \( \varphi \). The \( R \)-nerve \( N(\Lambda, R, \varphi) \) is the sparse Dowker nerve introduced in [3, Definition 38].

**Example 5.8 (Sheehy restriction).** Let \( \Lambda : L \times W \to [0, \infty] \) be a Dowker dissimilarity with \( \Lambda(l, w) < \infty \) for all \( (l, w) \in L \times W \) and satisfying the triangle inequality. The specific case we have in mind is \( L = W \) and \( d : L \times L \to (0, \infty) \) a metric. Let \( \alpha : [0, \infty) \to [0, \infty) \) be a translation function of the form \( \alpha = \text{id} + \beta \) for an order preserving function \( \beta : [0, \infty) \to [0, \infty) \) so that the function \( [0, \infty) \to [\beta(0), \infty) \) taking \( t \) to \( \beta(t) \) is bijective. Note that this implies that \( \beta \circ \beta = \text{id} \).

Let \( \lambda : L \to [0, \infty) \) be the canonical insertion function for \( \Lambda \) as defined in [3] and let \( \varphi : L \to L \) be the associated parent function. That is, for \( L = \{l_0, \ldots, l_n\} \), the function \( \lambda : L \to [0, \infty) \) is defined by

\[ \lambda(l) = \begin{cases} \infty & \text{if } l = l_0 \\ \sup_{w \in W} \inf_{l' \in \{l_0, \ldots, l_k\}} \Lambda(l', w) & \text{if } l = l_k \end{cases} \]

and \( \varphi(l) \) is the smallest element in \( L \) such that there exists \( w \in W \) with \( \Lambda(\lambda(l), w) = \lambda(l) \). Recall that the \( (\lambda, \beta) \)-truncation of \( \Lambda \) is the Dowker dissimilarity \( \Gamma : L \times W \to [0, \infty] \) defined in [3] by

\[ \Gamma(l, w) = \begin{cases} \Lambda(l, w) & \text{if } \Lambda(l, w) \leq \alpha \beta^{-1} \lambda(l) \text{ and } \beta(0) \leq \lambda(l) \\ \infty & \text{otherwise.} \end{cases} \]

Since elements \( l \in L \) with \( \lambda(l) < \beta(0) \) do not contribute to the Dowker nerve of \( \Gamma \) we assume without loss of generality that \( \beta(0) \leq \lambda(l) \) for every \( l \in L \).
This truncation is not a truncation of \( \Lambda \) as defined in Definition 4.2. However, the Dowker dissimilarity \( \Gamma': L \times W \to [0, \infty] \) described by the formula

\[
\Gamma'(l, w) = \begin{cases} 
\Lambda(l, w) & \text{if } \Lambda(l, w) < \alpha \beta \downarrow \lambda(l) \\
\infty & \text{otherwise}
\end{cases}
\]

is a truncated Dowker dissimilarity and is smaller than \( \Gamma \). In our implementation we use this description.

The Sheehy restriction function is

\[
S: L \to [0, \infty], \quad S(l) = \alpha^2 \beta \downarrow \lambda(l)
\]

Our assumption on \( l_0 \) implies that \( S(l_0) = \infty \). The Sheehy parent function \( \varphi: L \to L \) is defined as follows: first we define \( \varphi(l_0) = l_0 \) for the minimal element \( l_0 \) in \( L \). Next, given \( l \neq l_0 \), choose \( w' \in W \) so that \( (l, w') \in T \) and use that \( \lambda \) is a \( \beta \)-insertion function to choose \( l' \in L \) so that

\[
\Lambda(l', w') \leq \beta \alpha \beta \downarrow \lambda(l) < \lambda(l')
\]

and define \( \varphi(l) = l' \). Since

\[
\lambda(l) \leq \beta \beta \downarrow \lambda(l) \leq \beta \alpha \beta \downarrow \lambda(l) < \lambda(l')
\]

we have \( S(\varphi(l)) > S(l) \) for every \( l \in L \) with \( S(l) < \infty \).

Given \( w \in W \) with \( \Gamma'(l, w) < \infty \) we have \( \Lambda(l, w) < \alpha \beta \downarrow \lambda(l) \), so for \( l' \) and \( w' \) as above the triangle inequality gives

\[
\Lambda(l', w) \leq \Lambda(l', w') + \Lambda(l, w) < \beta \alpha \beta \downarrow \lambda(l) + \alpha \beta \downarrow \lambda(l) = \alpha^2 \beta \downarrow \lambda(l).
\]

The inequality \( \Lambda(l', w) \leq \alpha^2 \beta \downarrow \lambda(l) \) implies

\[
\beta \alpha \downarrow \Lambda(l', w) \leq \beta \alpha \downarrow \alpha^2 \beta \downarrow \lambda(l) \leq \beta \alpha \beta \downarrow \lambda(l) < \lambda(l').
\]

Since \( \varphi(l) = l' \) we can conclude that

\[
\Gamma'(\varphi(l), w) = \Lambda(\varphi(l), w) \leq \alpha^2 \beta \downarrow \lambda(l) = S(l)
\]

for every \( l \in L \). We conclude that \( \Gamma'(l, w) < \infty \) implies \( \Gamma'(\varphi(l), w) \leq S(l) \).

**Theorem 5.9.** Let \( \Lambda: L \times W \to [0, \infty] \) be a Dowker dissimilarity and let \( R \) be a restriction function for \( \Lambda \) relative to \( \varphi: L \to L \). If \( L \) is finite, then for every \( t \in [0, \infty] \) the geometric realization of the inclusion

\[
u: N(\Lambda, R, \varphi)(t) \to N\Lambda_t
\]

is a homotopy equivalence.
Proof. Since \( R(l) = \infty \) whenever \( \varphi(l) = l \), the two complexes agree when \( L \) is of cardinality 1, and thus the result holds in this case. Let \( t \in [0, \infty] \) and let \( n > 1 \). Below we will show that if \( \Lambda: L \times W \to [0, \infty] \) is a Dowker dissimilarity with \( L \) a set of cardinality \( n \) and \( R \) is a restriction function for \( \Lambda \) relative to \( \varphi \) so that the inclusion\

\[
i: N(\Lambda, R)(t) \to N\Lambda_t
\]

is not a homotopy equivalence, then there exists a Dowker dissimilarity \( \Lambda': L' \times W \to [0, \infty] \) with \( L' \) a set of cardinality \( n-1 \) and \( R' \) a restriction function for \( \Lambda' \) relative to a function \( \varphi': L' \to L' \) so that the inclusion\

\[
i: N(\Lambda', R')(t) \to N\Lambda'_t
\]

is not a homotopy equivalence. Negating this we obtain the inductive step implying that the result holds for all finite sets \( L \).

As above, let \( \Lambda: L \times W \to [0, \infty] \) be a Dowker dissimilarity with \( L \) a set of cardinality \( n > 1 \) and let \( R \) be a restriction function for \( \Lambda \) relative to \( \varphi \). Fix \( t \in [0, \infty] \) and pick \( l_n \in L \) so that firstly \( R(l_n) \leq R(l) \) for every \( l \in L \) and secondly \( l_n \) is not in the image of \( \varphi: L \to L \). This is possible since \( L \) is finite and \( R(\varphi(l)) \leq R(l) \) for every \( l \in L \). Suppose that\

\[
i: N(\Lambda, R)(t) \to N\Lambda_t
\]

is not a homotopy equivalence. Then there exists \( l \in L \) with \( R(l) < t \) since otherwise the two complexes are obviously equal. In particular \( R(l_n) < t \).

Let \( L' = L \setminus \{l_n\} \) and let \( \Lambda': L' \times W \to [0, \infty] \) be the restriction of \( \Lambda \) to \( L' \times W \subseteq L \times W \). Further, let \( R': L' \to [0, \infty] \) be the restriction of \( R \) to \( L' \) and let \( \varphi': L' \to L \) be the restriction of \( \varphi \) to \( L' \). Since \( l_n \) is not in the image of \( \varphi \) we can consider \( \varphi' \) as a function \( \varphi': L' \to L' \). Clearly \( R' \) is a restriction function for \( \Lambda' \) relative to \( \varphi' \).

We define \( f_t: L \to L' \) by\

\[
f_t(l) = \begin{cases} 
\varphi(l_n) & \text{if } l = l_n \\
\varphi(l) & \text{otherwise.}
\end{cases}
\]

Given \( \sigma \in N\Lambda_t \) we claim that \( \sigma \cup f_t(\sigma) \in N\Lambda_t \). If \( l_n \notin \sigma \), then this claim is trivially satisfied. In order to justify the claim when \( l_n \in \sigma \) we pick \( w \in W \) with \( \Lambda(l, w) < t \) for every \( l \in \sigma \). If \( \Lambda(l_n, w) \geq \Lambda(\varphi(l_n), w) \) then \( \Lambda(\varphi(l_n), w) < t \) and \( \sigma \cup f_t(\sigma) \in N\Lambda_t \). Otherwise by part (1) of Definition 5.3 the inequalities \( \Lambda(l_n, w) < t \) and \( \Lambda(l_n, w) < \Lambda(\varphi(l_n), w) \) imply that \( \Lambda(\varphi(l_n), w) \leq R(l_n) < t \). We conclude that \( \sigma \cup f_t(\sigma) \in N\Lambda_t \) also in this situation.
Next we claim that
\[ \sigma \in N(\Lambda, R, \varphi)(t) \quad \text{implies} \quad \sigma \cup f_t(\sigma) \in N(\Lambda, R, \varphi)(t). \]
Again we only need to consider the case \( l_n \in \sigma \). We have already shown that \( \sigma \cup f_t(\sigma) \in N(\Lambda, R, \varphi)(t) \).

If \( \Lambda(l_n, w) > t \) implies \( \Lambda(\varphi(l_n), w) \leq \Lambda(l_n, w) \) for all \( l_n \in \sigma \). Note in particular that \( \Lambda(l_n, w) \leq \Lambda(\varphi(l_n), w) \).

We can now conclude that the function \( f_t: L \to L' \) defines simplicial maps
\[ f_t: N\Lambda_t \to N\Lambda'_t \]
and
\[ f_t: N(\Lambda, R, \varphi)(t) \to N(\Lambda', R', \varphi')(t). \]

On the other hand, the inclusion \( \iota: L' \to L \) defines simplicial maps
\[ \iota: N\Lambda'_t \to N\Lambda_t \]
and
\[ \iota: N(\Lambda', R', \varphi')(t) \to N(\Lambda, R, \varphi)(t). \]

Moreover the above claims imply that the compositions
\[ N\Lambda_t \xrightarrow{f_t} N\Lambda'_t \xrightarrow{\iota} N\Lambda_t \]
and
\[ N(\Lambda, R, \varphi)(t) \xrightarrow{f_t} N(\Lambda', R', \varphi')(t) \xrightarrow{\iota} N(\Lambda, R, \varphi)(t) \]
are contiguous to identity maps. Since \( f_t \iota \) is the identity this implies that geometric realizations of the inclusions
\[ N\Lambda'_t \xrightarrow{\iota} N\Lambda_t \]
and
\[ N(\Lambda', R', \varphi')(t) \xrightarrow{\iota} N(\Lambda, R, \varphi)(t) \]
are homotopy equivalences. Since we have assumed that the geometric realization of the inclusion
\[ N(\Lambda, R, \varphi)(t) \xrightarrow{\iota} N\Lambda_t \]
is not a homotopy equivalence, we can conclude that the geometric realization of the inclusion
\[ N(\Lambda', R', \varphi')(t) \xrightarrow{\iota} N\Lambda'_t \]
is not a homotopy equivalence, as desired.

6. Implementation

In this section we will explain how to implement the computation of persistent homology of sparse approximations to the ambient- and intrinsic filtered Čech complexes of a finite subset of Euclidean space as well as general finite Dowker dissimilarities. Our implementation is available on [github].

6.1. Interleaving Lines. Our approximations to Čech- and Dowker nerves are interleaved with the original Čech- and Dowker nerves. As a consequence their persistence diagrams are interleaved with the persistence diagrams of the original filtered complexes. In order to visualize where the points may lie in the original persistence diagrams, we can draw the matching boxes from Theorem 3.9. However, this result in messy graphics with lots of overlapping boxes. Instead of drawing these matching boxes we draw a single interleaving line. Points strictly above the line in the persistence diagram of the approximation match points strictly above the diagonal in the persistence diagram of the original filtered simplicial complex. More precisely, the matching boxes of points above the interleaving line do not cross the diagonal, while the matching boxes of all points below the diagonal have a non-empty intersection with the diagonal.

6.2. Truncation of Dowker dissimilarities. Let $\Lambda$ be a Dowker dissimilarities of the form $\Lambda: L \times W \to [0, \infty]$ obtained by restricting a metric $d: W \times W \to [0, \infty)$ on $W$ to the subset $L \times W$ of $W \times W$. Given $t \in [0, \infty]$, the Dowker nerve $N_t \Lambda$ can be described as the nerve of the set of $s$-balls in $W$ centred at points in $L$ for $s \leq t$. The Čech complex $\hat{C}_t(L, W)$ is the nerve of the set of $t$-balls in $W$ centred at points in $L$. Since, every ball in $W$ of radius $s \leq t$ centred at a point in $L$ is contained in a $t$-ball in $W$ centred at a point in $L$ the geometric realization of the inclusion $\hat{C}_t(L, W) \subseteq N_t \Lambda$ is a homotopy equivalence [3, Example 7.11]. It follows that the persistent homologies of $\hat{C}(L, W)$ and $N \Lambda$ are isomorphic.

More generally, let the Dowker dissimilarity $\Lambda$, the insertion function $\lambda$ and the order preserving function $\beta: [0, \infty] \to [0, \infty]$ be as required in Example 5.8. In particular $\Lambda$ can be obtained from a metric as above, and $\lambda$ can be a farthest point sampling as in Definition 4.5. We have implemented the truncated Dowker dissimilarity $\Gamma: L \times W \to [0, \infty]$ given in Example 5.8. By Proposition 4.3 the Dowker nerve of $\Gamma$ is $\alpha = \text{id} + \beta$-interleaved with the Dowker nerve of $\Lambda$. Using the
restriction function $R^σ_Γ$ and the Sheehy restriction function $S$ we obtain sparse approximations $N(Γ, R_Γ)$ and $N(Γ, S)$ to the Dowker nerve of $Λ$. Below we elaborate on the implementation of these.

6.3. Sparsification of finite Dowker dissimilarities. Let $Γ: L \times W → [0, ∞]$ be a Dowker dissimilarity. Suppose that $L$ is finite and let $R$ be a truncation function for $Γ$ relative to $ϕ: L → L$.

The underlying simplicial complex $N_Γ$ of the Dowker nerve $N_Γ$ consists of those subsets $σ$ of $L$ whose filtration value

$$\inf_{w \in W} \max_{l \in σ} Γ(l, w)$$

is finite. If there exist $(l, w) ∈ L \times W$ with $Γ(l, w) = ∞$, then $N_Γ$ may not be full simplex, and it makes sense to ask for its maximal simplices.

Choose a total order $≤$ on $L$ so that $ϕ(l) ≤ l$ for every $l ∈ L$. Every maximal simplex of $N_Γ$ will be of the form

$$σ(l, w) = \{l′ ∈ L \mid Γ(l′, w) < ∞ \text{ and } l′ ≤ l\}.$$ 

This means that the maximal simplices of $N_Γ$ can be found by computing $σ(l, w)$ for every $(l, w) ∈ L × W$ with $Γ(l, w) < ∞$. This strategy can be elaborated to find the maximal simplices of $N_∞(Γ, R, ϕ)$. Recall that given $t ∈ [0, ∞]$, the simplicial complex $N_t(Γ, R, ϕ)$ consists of all subsets $σ$ of $L$ such that there exists $w ∈ W$ satisfying:

1. $Λ(l, w) < t$ for all $l ∈ σ$,
2. $Λ(l, w) ≤ R(l′)$ for all $l, l′ ∈ σ$ and
3. $Λ(l, w) < R(l)$ for all slope points $l$ in $σ$.

Given $l ∈ L$ let

$$W_l = \{w ∈ W \mid Γ(l, w) < ∞\}$$

and

$$σ(l) = \{l′ ∈ L \mid R(l′) ≤ R(l)\}.$$ 

Given $w ∈ W$ we define

$$σ(l, w) = \{l′ ∈ σ(l) \mid Γ(l′, w) ≤ R(l) \text{ and } Γ(l′, w) < R(l′) \text{ if } l′ \text{ is a slope point}\}.$$ 

Then every maximal simplex of $N_∞(Γ, R, ϕ)$ is of the form $σ(l, w)$ for some $(l, w) ∈ L × W$ with $w ∈ W_l$. In our implementation we have used this strategy to construct the set of maximal simplices in $N_∞(Γ, R, ϕ)$.
6.4. **Sparsification of ambient Čech filtrations.** Let now $\Lambda$ be a Dowker dissimilarity of the form $\Lambda: L \times W \to [0, \infty]$ obtained by restricting the Euclidean metric on $W = \mathbb{R}^d$ to the subset $L \times W$ of $W \times W$. We will modify the above strategy in order to obtain a sparse approximation of the Dowker nerve of $\Lambda$. Let the insertion function $\lambda$ and the order preserving function $\beta: [0, \infty] \to [0, \infty]$ be as required in Example 5.8 and let $\Gamma$ be the truncation of $\Lambda$ as in Example 5.8. Let $R$ be a restriction function for $\Gamma$ relative to $\varphi: L \to L$.

Unfortunately, in general, we do not understand the geometry well enough to be able to construct the maximal simplices in $N_\infty(\Gamma, R, \varphi)$. Instead we will construct a filtered simplicial complex $N^a(\Lambda, R, \varphi)$ such that

$$N(\Gamma, R, \varphi) \subseteq N^a(\Lambda, R, \varphi) \subseteq N\Lambda$$

and such that we are able to construct the maximal simplices of $N^a_\infty(\Lambda, R, \varphi)$. Let $\alpha = \beta + \text{id}$. Lemma 2.14 implies that since the inclusion $N(\Gamma, R, \varphi) \subseteq N\Lambda$ is an $\alpha$-interleaving, also the inclusion $N^a(\Lambda, R, \varphi) \subseteq N\Lambda$ is an $\alpha$-interleaving. In order to construct $N^a(\Lambda, R, \varphi)$, we specify a simplicial complex $N^a_\infty(\Lambda, R, \varphi)$ containing $N_\infty(\Lambda, R, \varphi)$ and define

$$N^a_t(\Lambda, R, \varphi) = N^a_\infty(\Lambda, R, \varphi) \cap N\Lambda_t$$

for $t \in [0, \infty]$. Thus, we can use the minball algorithm to compute filtration values in $N^a_\infty(\Lambda, R, \varphi)$. Working out the definition, we see that a subset $\sigma$ of $L$ is in $N_\infty(\Gamma, R, \varphi)$ if and only if there exists $w \in \mathbb{R}^d$ so that $\Lambda(l, w) < \alpha \beta^{-} \lambda(l)$ and $\Lambda(l, w) \leq R(l')$ for all $l, l' \in L$ and $\Lambda(l', w) < R(l')$ for all slope points $l'$ in $\sigma$. Note that if $\sigma \in N_\infty(\Gamma, R, \varphi)$ and $w$ are as above and if $l'' \in \sigma$ satisfies $\lambda(l'') \leq \lambda(l)$ for every $l \in \sigma$, then by the triangle inequality

$$\Lambda(l, l'') \leq \Lambda(l, w) + \Lambda(l'', w) \leq \beta^{-} \alpha \lambda(l) + \min_{l' \in \sigma} \beta^{-} \alpha \lambda(l')$$

and

$$\Lambda(l, l'') \leq \Lambda(l, w) + \Lambda(l'', w) \leq 2R(l')$$

for $l, l' \in L$. This leads us to define $N^a_\infty(\Lambda, R, \varphi)$ as the simplicial complex consisting of all $\sigma \subseteq L$ so that there exists $l'' \in L$ satisfying

$$\Lambda(l, l'') < \beta^{-} \alpha \lambda(l) + \min_{l' \in \sigma} \beta^{-} \alpha \lambda(l')$$

and

$$\Lambda(l, l'') \leq 2R(l').$$
for every $l, l' \in L$. The above discussion shows that $N_{\infty}(\Gamma, R, \varphi) \subseteq N_{\infty}^{a}(\Gamma, R, \varphi)$ as desired. The parent function we use in praxis is obtained from a farthest point sampling.

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